Fermionic zero energy modes are predicted to exist in defects in a variety of condensed matter systems from one dimensional (1d) chains of polyacetylene to the vortices of certain two dimensional (2d) superconductors \[1\] \[2\]. These modes typically occur in systems with a bulk gap which are in a topologically non-trivial phase. The zero energy modes are often topologically protected in the sense that their existence is only tied to the particular topological phase of the system and to some global properties of the system such as the winding number of the phase of the order parameter in a superconductor.

Unpaired fermionic zero energy modes which are localized in the vortices of certain superconductors such as a spinless \(p_x + ip_y\) superconductor are of special interest. These modes, also called Majorana modes, have attracted a fair bit of recent theoretical attention due to the fact that they can obey non-abelian statistics and have potential applications in decoherence free quantum computation \[3\] \[6\]. The existence of these modes has traditionally been demonstrated through explicit analytic solutions of the Bogoliubov de-Gennes (BdG) equations \[1\] \[4\] \[7\] \[9\].

In 1d, Jackiw and Rebbi showed that the Dirac equation interacting with a scalar field with a topologically interesting configuration such as a soliton has zero energy solutions whose number was related to an integer which characterizes the non-trivial topology \[10\]. Similar zero energy states also occur in polyacetylene \[2\]. Recently many works have attempted to explain the Majorana modes in 2d by reducing the problem to a 1d problem \[11\] \[13\].

Since the vortex zero energy states occur in 2d systems, a more general and inherently 2d explanation of the Majorana zero modes is desirable. In the current work, this is achieved through a flux insertion argument similar to Laughlin’s gauge argument \[14\]. The argument is not tied to any particular model and also applies to disordered systems where translational invariance is broken and is particularly useful in the context of recent efforts to find Majorana modes in systems other than a \(p_x + ip_y\) superconductor \[15\] \[18\]. One of our main results is a prediction about precisely what systems in all the different symmetry classes of the classification introduced by Altland and Zirnbauer \[19\] support zero energy states.

Using a flux insertion argument, we first show that a certain class of insulators with \(\pi\) flux inserted through a plaquette have exact zero energy eigenstates which are topologically protected in the sense that their existence is connected to the Hall conductance of the insulator. We then use these results to study superconductors with and without time reversal and spin rotational symmetry and predict precisely which systems have protected zero modes.

The insulator Hamiltonians studied below have the same symmetries as the mean field Bogoliubov-de-Gennes Hamiltonians of superconductors. Though such insulators are possibly of little physical significance by themselves, we study them since their spectra are identical to that of the BdG Hamiltonians of superconductors. Further since they are insulators rather than superconductors, they do not expel flux, making a flux insertion argument possible. Consider a 2d tightbinding insulator on a lattice with an even number of orbitals, \(2s\) per site. The basis states are written in the form \(|i,\alpha\rangle\) where \(i\) is an index for the position and \(\alpha\) for the orbital. We first study infinite sized systems where the energy of the topologically protected vortex states is exactly zero and later consider corrections that would occur in a finite sized system.

The Hamiltonian can be written in the form \(\mathcal{H} = \sum_{i,j} \Psi_i^\dagger H_{ij} \Psi_j\) where \(H_{ij}\) is a \(2s \times 2s\) dimensional matrix \(\Psi_j = (\psi_{j,\alpha})^T, \Psi_i^\dagger = (\psi_{i,\alpha}^\dagger)\) where \(i,j\) are the lattice position indices and \(\alpha,\beta\) the orbital and spin indices. The Fermi energy is set to zero and is assumed to lie in a gap.

In the first instance, we study systems with neither spin rotational nor time reversal symmetry, though we will analyze systems with these symmetries later on. We further restrict our study to Hamiltonians \(\mathcal{H}\) which posses a symmetry analogous to that of Bogoliubov-de-Gennes Hamiltonians. In other words, we assume the existence of an anti-unitary operator, \(S\), such that

\[
S\mathcal{H}S^{-1} = -\mathcal{H}.
\] (1)

Further, we assume that \(S\) acting on single particle position eigenkets in the Hilbert space produces a linear
combination of kets at the same position, i.e., \( S \Psi_j S^{-1} = U_j \Psi_j \) and \( U_j \) is a \( 2S \times 2S \) dimensional unitary matrix. We may therefore conclude that the Hamiltonian of the system despite being that of an insulator, belongs to the symmetry class D in the classification introduced by Altland and Zirnbauer. The symmetry under \( S \) also implies that \( U_j H_i U_j^{-1} = -H_i \).

Consider the effect of a vector potential which arises from flux insertion through an infinitesimal tube. The Hamiltonian of the system with a flux tube somewhere in the sample may be derived by multiplying each hopping term in the matrix \( H_{ij} \) by the phase factors \( \exp \left[ i(e/h) \int_j^i \mathbf{dr} \cdot \mathbf{A} \right] \) where the integral is along the hopping path and \( \mathbf{A} \) is the vector potential. The Hamiltonian of the system with the vector potential, which we call \( \mathcal{H}(\mathbf{A}) \) then has the following property:

\[
S \mathcal{H}(\mathbf{A}) S^{-1} = S \left( \sum_{i,j} \Psi_j^i H_{ij} \Psi_j e^{i(e/h) \int_j^i \mathbf{dr} \cdot \mathbf{A}} \right) S^{-1} = - \mathcal{H}(\mathbf{-A}).
\]

Consider an infinite sample of the above system and let us adiabatically thread flux through an infinitesimal flux tube through a plaquette at the center of the sample. If \( \mathcal{H}(\phi) \) is the Hamiltonian in the presence of a flux \( \phi \), then on the basis of the above, we conclude that

\[
S \mathcal{H}(\phi) S^{-1} = - \mathcal{H}(-\phi).
\]

Thus if \( \mathcal{H}(\phi) \) has an eigenstate with eigenvalue \( E \), then \( \mathcal{H}(-\phi) \) has an eigenstate with eigenvalue \(-E\).

Now suppose the system has a quantized Hall conductance of \( pe^2/2\pi h \). Then as a flux of \( 2\pi h/e \) is adiabatically inserted through the flux tube, a total charge of \( pe \) flows in from infinity towards the flux tube \([13]\). In an infinite sample, the total spectral flow, i.e., the total number of states which cross the gap at the Fermi surface from below minus the number of states which cross it from above is equal to \( p[14][20] \).

Let \( n(\phi) \) be the number of eigenstates of the Hamiltonian whose energy is zero at the flux \( \phi \). A schematic example of the energy spectrum of extended states close to the Fermi energy as a function of flux is plotted in Fig. 1. The function \( n(\phi) \) is non-zero only for \( \phi \in \{ \phi_a, \phi_b, 2\pi - \phi_a, \phi_d, 2\pi - \phi_d \} \) where \( \phi_a = 2\pi - \phi_a \) and \( \phi_d = 2\pi - \phi_d \).

From Eq. (3),

\[
n(2\pi - \phi) = n(\phi).
\]

Apart from the states which traverse the gap and thus cross the Fermi energy an odd number of times, there might also be states which cross the Fermi energy an even number of times as shown in the figure.

Thus,

\[
\sum_{\phi_i \ni n(\phi_i) \neq 0} n(\phi_i) = p + 2m,
\]

where \( p \) is the total Chern number of the ground state.

It follows from Eqs. (4) and (5) that when \( p \) is odd,

\[
n(0) + n(2\pi h/2e) = 2k + 1
\]

where \( k \) is some integer.

At zero flux, the Fermi energy lies in a gap. Thus, \( n(0) = 0 \) which in turn implies that \( n(2\pi h/2e) \) is an odd integer when \( p \) is odd. Since the integer \( p \) is a topological invariant, which cannot change under small transformations of the Hamiltonian, the zero mode is topologically protected.

We now use this result to study superconductors. In the remainder of the paper, we are frequently going to consider Hamiltonians with \( 2s \) degrees of freedom per site and we write such matrices in the form:

\[
\begin{pmatrix}
M_{11} & M_{12} \\
M_{12}^\dagger & M_{22}
\end{pmatrix}.
\]

Consider a tightbinding BCS Hamiltonian for fermions hopping on a lattice which can be written in the form: \( \mathcal{H}_S = \sum_{i,j} \Psi_i^N H_{ij} \Psi_j^N \) where \( \Psi_i^N = (\psi_{i,\uparrow}, \psi_{i,\downarrow}^\dagger) \) is a Nambu spinor. Here, \( i,j \) stand for positions, and \( \gamma \) is an index for the orbitals and spin which runs from 1 to \( s \). The matrix \( H^{\gamma} \) is the BdG Hamiltonian which is of the form of Eq. (6) with \( M_{11} = h, M_{12} = \Delta \) and \( M_{22} = -h^T \) where \( h \) is the single particle Hamiltonian and \( \Delta \) is the gap matrix. We can map the Hamiltonian \( \mathcal{H}_S \), to the Hamiltonian of an insulator, \( \mathcal{H}_I \) (which we call the associated insulator) given by \( \mathcal{H}_I = \sum_{i,j} \Psi_i^N H_{ij}^T \Psi_j^N \) where \( \Psi_j = (\psi_{i,\alpha}, \psi_{i,\beta}^\dagger) \).

We first study BdG Hamiltonians with neither time reversal nor spin rotational symmetry. Let us imagine inserting an infinitesimal flux tube containing a flux of \( 2\pi h/2e \) at the origin of our coordinates which is placed at the center of one of the plaquettes of the system. We

\[\text{FIG. 1: A schematic plot of the energy, } E \text{ versus } \phi, \text{ where } \phi \text{ is the flux inserted through a plaquette of the insulator with Bogoliubov symmetry. Only states which lie close to the Fermi energy at } \phi = 0 \text{ are shown. The dashed line is the single energy curve which traverses the gap as the flux is changed from 0 to } 2\pi h/e.\]
imagine that this flux does not leak into the rest of the superconductor and that the flux is inserted in such a way that the low energy configuration where the phase of the superconductor winds by $2\pi$ around the flux tube is attained.

In the presence of the flux, the single particle Hamiltonian gets transformed as follows. Let $r_i, \theta_i$ be respectively the distance from the origin and the polar angle of site $i$ in a coordinate system with the origin located at the position of the flux tube. Then, $h_{ij} \rightarrow h'_{ij} = h_{ij}e^{i(\theta_i/\pi)}f \cdot \mathbf{r} \cdot \mathbf{A}$ where $\mathbf{A}$ is chosen to be $\frac{\hbar}{2\pi} \nabla(\theta)$. The gap matrix transforms as: $\Delta_{ij} \rightarrow \Delta'_{ij} = \Delta e^{i(\theta_i + \theta_j)/2}$ in the presence of the flux tube.

The BdG eigenvalue equation in the presence of the vortex is thus: $H'\psi = E\psi$, where $\psi = (u, v)^T$ and $H'$ has the form of Eq. (8) with $M_{11} = h', M_{12} = \Delta'$ and $M_{22} = -(h')^T$. Let $\bar{u}_i = u_i, \bar{v}_i = v_i$. Then $\psi = (\bar{u}, \bar{v})^T$ satisfies the eigenvalue equation: $H''\bar{\psi} = E\bar{\psi}$, where $H''$ can be written in the form of Eq. (8) with $M_{11} = h', M_{12} = \Delta''$, $M_{22} = -(h'')^T$, $h''_{ij} = h_{ij}e^{i(\theta_i - \theta_j)/2}$ and $\Delta''_{ij} = \Delta e^{i(\theta_i - \theta_j)/2}$.

We now replace the superconductor with the associated insulator, $\mathcal{H}_I$ with the same flux configuration, i.e., half a quantum of flux inserted at the origin, which lies at the center of a plaquette. It is easy to verify that $\mathcal{H}_I(\pi)$ can be written as $\sum_{ij} \bar{\psi}_i^\dagger H''_{ij} \bar{\psi}_j$ with $H''_{ij}$ as given above. Further, $\mathcal{H}_I$ satisfies the symmetry in Eq. (1) since it is derived from a BdG Hamiltonian and the analysis for the zero modes for insulators made previously can therefore be used.

The necessary and sufficient condition for the existence of an exact zero-energy mode which is localized around the flux tube is therefore that the Hall conductance of $\mathcal{H}_I$ is $pe^2/2\pi\hbar$ where $p$ is odd. If this condition is met, it follows that there is a zero energy mode localized around the plaquette containing the tube. Since the Hall conductance is a robust topological invariant, the existence of the zero mode for the superconductor is also topologically protected.

A more realistic description of a vortex would include a finite region larger than a single plaquette where the magnetic field is non-zero rather than the situation considered above where the flux is confined to a single plaquette. In the general case, the gap parameter and the single particle Hamiltonians may be modeled as: $\Delta_{ij} = \Delta f((r_i + r_j)/2)e^{i(\theta_i + \theta_j)/2}$; $h'_{ij} = h_{ij}e^{i\phi_{ij}}$. Here, $f(r)$ is some function which is zero at the origin and goes to one for distances larger than the coherence length and $\phi_{ij} = \int (e/h)\mathbf{r} \cdot \mathbf{A}$ where $\mathbf{A}$ is the vector potential which corresponds to a smeared out flux and which approaches the value $(\hbar/e)\nabla(\theta)/2$ for distances much larger than the penetration depth.

The corresponding BdG Hamiltonian may therefore be written as $\mathcal{H} = \mathcal{H}_0 + V$ where $\mathcal{H}_0$ is the Hamiltonian which corresponds to the idealized vortex discussed above and $V$ is a local perturbation in the sense that $V_{ij} = 0$ sufficiently far from the vortex. If there is an odd number of Majorana zero energy states, then the Hilbert space of the idealized vortex has one unpaired fermionic mode, while every other eigenstate of energy, $E$, can be paired with a state of energy, $-E$. Any local perturbation cannot alter the Hilbert space structure. It follows that the perturbation, $V$, does not alter the existence of a zero energy mode and that the zero mode persists in the more realistic configuration.

We now intend to study superconductors with time reversal symmetry using the same technique and therefore first study the corresponding insulators. Insulators with time reversal symmetry are classified by a $Z_2$ invariant. Any Hamiltonian, $\mathcal{H}_I$ of this class can be continuously deformed, without closing the gap or breaking time reversal symmetry, to a Hamiltonian $\mathcal{H}'$ which is diagonal in the basis of the $z$-component of spin and can thus be written in the form $\mathcal{H}' = \mathcal{H}_I' + \mathcal{H}_I''$. The Hamiltonians $\mathcal{H}_I'$ and $\mathcal{H}_I''$ can be analyzed separately for studying their zero modes. The sum of the Hall conductance of $\mathcal{H}_I'$ and $\mathcal{H}_I''$ must be zero. If $|\sigma_{xy}(\mathcal{H}_I')/e| < 1$ of $\mathcal{H}_I''$ is an odd number, then the insulator belongs to the non-trivial class. By the previous analysis, in this case, when a flux of $\pi$ is inserted through the central plaquette the system has an odd number of Kramers pairs at zero energy, one member of each pair associated with $\mathcal{H}_I'$ and the other with $\mathcal{H}_I''$. Kramers theorem then prevents a gap from opening up at $\pi$ flux when the Hamiltonian is continuously deformed without breaking time reversal symmetry. This implies that the original Hamiltonian, $\mathcal{H}_I$ must also have an odd number of pairs of zero modes at $\pi$ flux. When $\sigma_{xy}(\mathcal{H}_I')$ is an even multiple of $e^2/2\pi\hbar$ on the other hand, the number of pairs of zero modes must be even and is therefore not protected.

Superconductors with time reversal symmetry fall in the class DIII of the classification scheme \[19\]. These superconductors are classified by a $Z_2$ invariant \[21\] and can be mapped onto insulators with time reversal symmetry which satisfy Eq. (1) and which have the same spectrum, exactly as in the case of superconductors without time reversal or spin rotational symmetry. We have shown in the previous paragraph that when flux is inserted through a plaquette in these associated insulators, they have an odd number of Kramers pairs of zero energy modes if and only if they belong to the non-trivial topological class. The zero eigenmodes of the equation that determines the spectra of the insulator with half a quantum of flux inserted through a plaquette may be related to the zero eigenmodes of the superconductor with a vortex following the discussion after Eq. (8). Thus, superconductors with time reversal symmetry have an odd number of Kramers pairs of zero energy modes in their vortex cores whenever the superconductors are in the non-trivial topological class. A pair of zero-energy modes may be regarded as a single Dirac mode.
These Hamiltonians may be regarded as the sum of $H_\uparrow = (\psi_\uparrow, \psi_\downarrow)H_\uparrow(\psi_\uparrow, \psi_\downarrow)^T$ and $H_\downarrow = (\psi_\downarrow, \psi_\uparrow)H_\downarrow(\psi_\downarrow, \psi_\uparrow)^T$, where $H_\uparrow$ and $H_\downarrow$ have the form of Eq. (6) with $M_{1\uparrow} = h, M_{1\downarrow} = \Delta, M_{2\uparrow} = -h^T$ and $M_{1\downarrow} = h, M_{2\downarrow} = -\Delta, M_{2\downarrow} = -h^T$ respectively.

TABLE I: Conditions for superconductors in the various symmetry classes to support protected zero modes in vortex cores, expressed as conditions on $H_\uparrow$, the insulator associated with the superconductor. The last column indicates whether there is a single protected Majorana mode (M) or a protected pair of modes (D). No protected modes exist for the class CI.

| Class | Time-rev | Spin-rot | Condn. on $H_\uparrow$ | Mode |
|-------|----------|----------|------------------------|------|
| D     | No       | No       | $\sigma_{xy}2\pi\hbar/e^2 = 2k - 1$ | M    |
| C     | No       | Yes      | $\sigma_{xy}2\pi\hbar/e^2 = 2(2k - 1)$ | D    |
| DIII  | Yes      | No       | non-trivial $Z_2$     | D    |
| CI    | Yes      | Yes      | -                      | -    |

The spectra of $H_\uparrow$ and $H_\downarrow$ are identical. If $(u, v)^T$ written in the particle hole basis, is a zero mode of $H_\uparrow$, then $(u, -v)^T$ is a zero mode of $H_\downarrow$. The superconductor may be mapped onto an insulator, $H_\downarrow$, which is the sum of two single particle Hamiltonians, $H_{1,\uparrow}$ and $H_{1,\downarrow}$, and which may be regarded as separate systems. The condition that in the presence of a vortex, the matrix $H_\uparrow$ has an odd number of zero modes is, as deduced in the study of Hamiltonians of class D, that the Hall conductance of the corresponding insulator, $\sigma_{xy}(H_{1,\uparrow})2\pi\hbar/e^2$ is an odd integer. The net Hall conductance, $\sigma_{xy}(H_\downarrow)$ is twice that of $H_{1,\downarrow}$ and is thus always an even integer. Thus, when the Hall conductance of $H_\downarrow$ has the form $2p\pi e^2/2\pi\hbar$ where $p$ is an odd integer, the system has an odd number of pairs of zero modes in its vortices, while when $p$ is even, the system has an even (or zero) number of pairs of zero modes.

Superconductors with both time reversal and spin rotational symmetry, which belong to the class CI, may be regarded as belonging to the trivial $Z_2$ class of superconductors with time reversal symmetry. These superconductors thus have no topologically protected zero modes in their vortex cores.

Our results are summarized in Table I. In the notation used in Ref. [24], superconductors in the class D have protected Majorana modes in their vortex cores when the integer invariant for this class is odd, superconductors in class C and DIII have protected Dirac modes when the invariants for these classes are odd and non-trivial respectively. Most continuum models can be simulated to an arbitrary degree of accuracy by a series of lattice models. Since the results derived above are not limited to a particular tightbinding model, one expects that the analysis presented above extends also to continuum models [24].

Our analysis thus far has been restricted to a single vortex in an infinite superconductor. A finite superconductor with a vortex is topologically equivalent to a system with two vortices and periodic boundary conditions. One may then start with the associated insulator with no flux inserted and gradually insert flux into two tubes in such a way that the flux entering one tube is equal to the flux leaving the other leaving the total flux through the system zero. At $2\pi\hbar/2e$ flux, the states localized at the two flux tubes will in general hybridize, giving rise to a finite splitting. The magnitude of the splitting when the two vortices are a certain distance $d$ is proportional to the overlap between two zero-energy eigenstates of the infinite system placed at the same distance and therefore falls exponentially as the distance between the vortices.

Finally, we note that the arguments made above would be also applicable in the case when there is a mobility gap in the absence of flux rather than a gap to all states since the states within the mobility gap have zero Chern number and their energies return to their initial values when the flux inserted varies from 0 to $2\pi\hbar/e$.

In summary, we have provided a simple and general argument which shows that certain topological classes of superconductors have topologically protected, robust zero modes, which can either be unpaired Majorana modes, or come in pairs. We applied this analysis to the various symmetry classes of superconducting Hamiltonians.

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