POLYHEDRAL PRODUCTS FOR SHIFTED COMPLEXES AND HIGHER WHITEHEAD PRODUCTS

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Abstract. This paper studies the map between polyhedral products $Z_K(CX, X) \to Z_K(\Sigma X, \ast)$ induced from the pinch maps $(CX_i, X_i) \to (\Sigma X_i, \ast)$, which is the higher order Whitehead product if $K$ is the boundary of a simplex. When $K$ is a shifted complex, a wedge decomposition of $Z_K(CX, X)$ is given in [IK1]. Based on this decomposition, when $K$ is shifted, the induced pinch map is explicitly described as a wedge of iterated Whitehead products each of which includes at most one higher product. As a corollary, the Jacobi identity of Whitehead products including higher products due to Hardie [H1] is generalized.

1. Introduction

To a simplicial complex and a collection of pairs of spaces indexed by vertices of the simplicial complex, there is associated a space called a polyhedral product. It is first studied by Porter [P] to define higher order Whitehead products, in which the only simplicial complex considered is the boundary of a simplex. There are some works expanding this idea. See [AA, H1, H2] for example. Besides the homotopy theoretical importance, in a seminal work of Davis and Januszkiewicz [DJ], several interesting aspects of polyhedral products such as connection with combinatorial commutative algebra are found, which broaden the directions of the study of polyhedral products [BP, DS, D, GW, IK1, IK2, IK3, N]. In this paper, we go back to the origin and study the aspect of polyhedral products in homotopy theory, which of course influences many other aspects of polyhedral products.

We set up our problem. Let us first recall the precise definition of polyhedral products. Throughout the paper, let $K$ be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$, where we assume that $[m]$ is equipped with the usual ordering. For a collection of pairs of spaces $(X, A) = \{(X_i, A_i)\}_{i \in [m]}$ indexed by the vertex set of $K$, the polyhedral product is defined as

$$Z_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma \quad (\subset X_1 \times \cdots \times X_m)$$

where $(X, A)^\sigma = Y_1 \times \cdots \times Y_m$ for $Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma \end{cases}$. Notice that we implicitly use the ordering of $[m]$ in this definition, i.e. $Z_K(X, A)$ depends on the ordering of the vertex set of $K$.

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Let us next recall the definition of higher order Whitehead products. We set notation. Hereafter, let $X$ be a collection of based spaces $\{X_i\}_{i \in [m]}$ indexed by the vertex set of $K$. We put $(CX, X) = \{(CX_i, X_i)\}_{i \in [m]}$ and $(X, *) = \{(X_i, *)\}_{i \in [m]}$. For a finite set $F$, let $\partial F$ denote the boundary of the simplex on the vertex set $F$, and let $|K|$ denote the geometric realization of $K$. When $m = 2$, we have

$$Z_{\partial \Delta^{|m|}}(CX, X) = X_1 \ast X_2 \quad \text{and} \quad Z_{\partial \Delta^{|m|}}(\Sigma X, *) = \Sigma X_1 \lor \Sigma X_2,$$

and the map

$$X_1 \ast X_2 = Z_{\partial \Delta^{|m|}}(CX, X) \to Z_{\partial \Delta^{|m|}}(\Sigma X, *) = \Sigma X_1 \lor \Sigma X_2$$

induced from the pinch maps $(CX_1, X_i) \to (\Sigma X_i, *)$ is the universal Whitehead product, where $X_1 \ast X_2$ is the join of $X_1$ and $X_2$, e.g. $X_1 \ast X_2 \simeq \Sigma X_1 \land X_2$. Generalizing this, higher order Whitehead products are defined. Porter [P] constructs a homotopy equivalence

$$\Sigma |\partial \Delta^{|m|}| \land \tilde{X}^{|m|} \simeq Z_{\partial \Delta^{|m|}}(CX, X)$$

which we here fix to the one of Theorem 1.3, where $\tilde{X}^I = X_{i_1} \land \cdots \land X_{i_k}$ for a subset $I = \{i_1 < \cdots < i_k\} \subset [m]$. Then the $(m - 1)^{st}$ order Whitehead product is defined as the composite

$$\Sigma |\partial \Delta^{|m|}| \land \tilde{X}^{|m|} \simeq Z_{\partial \Delta^{|m|}}(CX, X) \to Z_{\partial \Delta^{|m|}}(\Sigma X, *)$$

which we denote by $\omega_{|m|}$, where the target $Z_{\partial \Delta^{|m|}}(\Sigma X, *)$ is the fat wedge of $X_1, \ldots, X_m$. Then it is now natural to ask:

**Problem 1.1.** For which simplicial complex $K$, can the induced pinch map

$$\omega : Z_K(CX, X) \to Z_K(\Sigma X, *)$$

be described by (higher order) Whitehead products?

There is a homotopy fibration

$$Z_K(C\Omega X, \Omega X) \to Z_K(X, *) \xrightarrow{\text{incl}} X_1 \times \cdots \times X_m$$

which plays the fundamental role in many works (see [BP, DJ, IK2]), where the fiber inclusion is the composite

$$Z_K(C\Omega X, \Omega X) \xrightarrow{\text{incl}} Z_K(\Sigma \Omega X, *) \to Z_K(X, *).$$

where the second arrow is induced from the evaluation maps $\Sigma \Omega X_i \to X_i$. So through this homotopy fibration, the above problem is related to many other aspects of polyhedral products.

As a first step to consider the above problem, we choose shifted complexes since the homotopy type of the polyhedral product $Z_K(CX, X)$ is explicitly described in [IK1] for a shifted complex $K$ as follows. We now recall the definition of shifted complexes, where as in [IK1] we use the reverse ordering of the usual definition of shifted complexes.

**Definition 1.2.** $K$ is called shifted if $i \in \sigma \in K$ and $i < j$, then $(\sigma - i) \cup j \in K$. 

For example, any skeleton of a simplex is shifted, and any induced subcomplex of a shifted complex is shifted. One can find more examples in [IK1]. To state the needed result of [IK1], we set notation. For \( \emptyset \neq I \subseteq [m] \), let \( K_I \) denote the induced subcomplex of \( K \) on \( I \), i.e. \( K_I = \{ \sigma \in K \mid \sigma \subset I \} \). Let \( m(K) \) be the set of minimal non-faces of \( K \) including the maximum vertex \( m \), i.e. \( m(K) = \{ F \subset [m] \mid F \not\in K, \partial \Delta^F \subset K \text{ and } m \in F \} \).

**Theorem 1.3.** [IK1] If \( K \) is shifted, there is a natural homotopy equivalence

\[
Z_K(CX, X) \simeq \bigvee_{\emptyset \neq I \subseteq [m]} \bigvee_{F \in m(K_I)} \Sigma|\partial \Delta^F| \land \hat{X}^I.
\]

**Remark 1.4.** Grbić and Theriault [GT1] also studied the homotopy type of \( Z_K(CX, X) \) for a shifted complex \( K \), but it includes serious errors. For example, the closedness of the class \( W_n \) by retractions claimed in the proof of the main theorem is false.

The naturality of the homotopy equivalence will be explained in the next section. Using this homotopy equivalence, we describe the induced pinch map \( \omega \) by (higher order) Whitehead products. To do this, we set notation for alternation of the ordering. For \( \emptyset \neq F \subset I \subseteq [m] \), let \( \sigma(F, I) \) be the permutation

\[
\begin{pmatrix}
(i_1 & \cdots & i_k & j_1 & \cdots & j_{m-k-\ell} & j_{m-k-\ell+1} & \cdots & j_{m-k}) \\
i_1 & \cdots & i_k & b_1 & \cdots & b_{m-k-\ell} & a_1 & \cdots & a_\ell
\end{pmatrix}
\]

on \([m]\), where \([m] - I = \{ i_1 < \cdots < i_k \} \), \( I = \{ j_1 < \cdots < j_{m-k-\ell} \} \), \( F = \{ a_1 < \cdots < a_\ell \} \) and \( I - F = \{ b_1 < \cdots < b_{m-k-\ell} \} \). For a permutation \( \sigma \) on \([m]\), we denote the chain \([m]\) with the ordering altered by \( \sigma \) by \([m]^{\sigma}\). We also denote by \( Z_K(X, A)^{\sigma} \) the polyhedral product constructed from \( K \) and \( (X, A) \) with the ordering of \([m]^{\sigma}\). Then we have the canonical homeomorphism \( Z_K(X, A) \rightarrow Z_K(X, A)^{\sigma} \) which we denote by \( \tilde{\sigma} \). When \( \tau \) is a permutation on the subset \( \emptyset \neq I \subset [m] \), let \( \tilde{\tau} : \hat{X}^I \rightarrow \hat{X}^{I^{\tau}} \) be the canonical permutation map. We now state our main theorem.

**Theorem 1.5.** If \( K \) is shifted and \( \emptyset \neq F \subset I \subset [m] \) with \( F \in m(K_I) \), then the composite

\[
\Sigma|\partial \Delta^F| \land \hat{X}^I \xrightarrow{\text{incl}} Z_K(CX, X) \xrightarrow{\omega} Z_K(\Sigma X, \ast)
\]

is homotopic to

\[
\Sigma|\partial \Delta^F| \land \hat{X}^I \xrightarrow{\text{sgn}(\sigma(F, I)) \land \sigma(F, I)} \Sigma|\partial \Delta^F| \land \hat{X}^{I^{\sigma(F, I)}} \rightarrow Z_K(\Sigma X, \ast)^{\sigma(F, I)} \xrightarrow{\sigma(F, I)^{-1}} Z_K(\Sigma X, \ast)
\]

where the second arrow is the iterated Whitehead product

\[
[e_{i_1}, [\cdots [e_{i_{t-1}}, [e_{i_t}, j \circ \omega_F] \cdots]]],
\]

\( e_i : \Sigma X_i \rightarrow Z_K(\Sigma X, \ast)^{\sigma(F, I)} \) and \( j : Z_{\partial \Delta^F}(\Sigma X^F, \ast) \rightarrow Z_K(\Sigma X, \ast)^{\sigma(F, I)} \) are the inclusions and \( I - F = \{ i_1 < \cdots < i_\ell \} \).
Remark 1.6. In Theorem 1.5, we omit the shift of the suspension parameters for defining Whitehead products, but it is easy to write it explicitly. We will often omit the homeomorphism \( \sigma(F, I) \) but no confusion will occur.

Remark 1.7. Grbić and Theriault [GT2] also considered Problem 1.1 for MF-complexes which they introduced. But there are serious mistakes: if \( K \) is a simplicial complex obtained from a square by attaching an unfilled triangle to each edge, then \( K \) is a MF-complex but \( \mathcal{Z}_K(CX, X) \) does not decompose as in Theorem 1.2, so the starting point is false; several arguments in the crucial part is false such as the one at the end of the proof of Theorem 6.4.

As a first corollary, we immediately have the following by (1.3).

**Corollary 1.8.** If \( K \) is shifted, the fiber inclusion \( \mathcal{Z}_K(C\Omega_X, \Omega_X) \to \mathcal{Z}_K(X, \ast) \) of the homotopy fibration (1.2) can be identified with a wedge of iterated Whitehead products

\[
[e_{i_1}, \cdots [e_{i_{t-1}}, [e_{i_t}, \hat{\omega}]] \cdots ]
\]

where \( e_i : \Sigma \Omega X_i \to X_i \) are the inclusions and \( \hat{\omega} \) is the (higher order) Whitehead product of \( e_i \).

As a second corollary, we generalize the Jacobi identity of higher order Whitehead products in [H1], where Hardie [H1] considered the case that all \( X_i \) are spheres.

**Corollary 1.9 ([H1]).** Let \( \sigma_i \) be the permutation \( \sigma([m] - i, [m]) \) for \( i \in [m] \). Then for the inclusions \( e_i : \Sigma X_i \to \mathcal{Z}_{(\Delta[m])m-3}(\Sigma X, \ast) \), it holds that

\[
\sum_{i=1}^{m} (-1)^{i-1}[e_i, j_i \circ \omega_{[m]-i}] \circ (1 \wedge \hat{\sigma}_i) = 0
\]

where \( j_i : \mathcal{Z}_{(\Delta[m]-i)m-3}(\Sigma X_{[m]-i}, \ast) \to \mathcal{Z}_{(\Delta[m])m-3}(\Sigma X, \ast) \) is the inclusion and \( K^\ell \) means the \( \ell \)-skeleton of \( K \).

From Corollary 1.9, we can derive the Jacobi identity of usual Whitehead products.

**Corollary 1.10 (cf. [NT]).** For \( \alpha \in \pi_p(S^n), \beta \in \pi_q(S^n) \) and \( \gamma \in \pi_r(S^n) \) with \( p, q, r > 1 \),

\[
(-1)^{p(r-1)}[\alpha, [\beta, \gamma]] + (-1)^{q(p-1)}[\beta, [\gamma, \alpha]] + (-1)^{r(q-1)}[\gamma, [\alpha, \beta]] = 0.
\]

**Proof.** Apply \( \alpha, \beta, \gamma \) to Corollary 1.9, we get the identity

\[
(-1)^p[\alpha, [\beta, \gamma]] - (-1)^q(-1)^p[\beta, [\alpha, \gamma]] + (-1)^r(-1)^{p+q}[\gamma, [\alpha, \beta]] = 0
\]

where the signs \( (-1)^p, (-1)^q, (-1)^r \) correspond to the shift of suspension parameters as noted in Remark 1.6. Then the proof is completed by the identity \( [\alpha, \gamma] = (-1)^{pr+1}[\gamma, \alpha] \) and by multiplying \( (-1)^{pr} \).
2. Homotopy equivalence of theorem 1.3 and vertex permutation

In this section, we recall the construction of the homotopy equivalence of Theorem 1.3 for special $K$, and show its behavior on vertex permutations of $K$ which is used to prove Theorem 1.5. We put $K = |K|/\text{star}(m)$ which has the same homotopy type of $|K|$, and put also

$$W_K(X) = \bigvee_{\emptyset \neq I \subseteq [m]} \Sigma K_I \wedge \tilde{X}^I.$$ 

Then $W_K(X)$ is natural with respect to $X$ and the inclusion of a subcomplex of $K$ on the same vertex set $[m]$. We state the precise statement of the homotopy decomposition of $Z_K(CX, X)$ for a shifted complex $K$.

**Theorem 2.1** ([IK2]). If $K$ is shifted, there is a homotopy equivalence

$$\epsilon : Z_K(CX, X) \xrightarrow{\sim} W_K(X).$$

Moreover, this is natural with respect to $X$ and the inclusion of a shifted subcomplex of $K$ on the vertex set $[m]$.

The naturality of $\epsilon$ will play the key role in considering vertex permutation. We next recall a convenient description of the homotopy type of shifted complexes given in [IK1].

**Proposition 2.2** ([IK1]). If $K$ is shifted, the wedge of inclusions

$$\bigvee_{F \in m(K)} |\partial \Delta F| \to |K|$$

is a homotopy equivalence.

Combining Theorem 2.1 and Proposition 2.2, we get Theorem 1.3.

We next recall from [IK1] the construction of the homotopy equivalence $\epsilon$ of Theorem 2.1 for a shifted complex. Every simplicial complex is constructed by iterated use of two operations of adding vertices; one is to add a discrete vertex and the other is to add a vertex as the cone point by attaching a cone to a subcomplex. The homotopy equivalence $\epsilon$ is defined inductively on the number of vertices by applying these two operations. The important point is that we construct $\epsilon$ by the decreasing induction for the first operation and by the increasing order for the second step. We explicitly describe how to construct $\epsilon$ for a special shifted complex as follows. Let $F = \{d_1 < \cdots < d_i\}$ be a subset of $[m]$, and put $d_{i+1} = m + 1$ for convenience. We define $\Delta(F, [m])$ as the minimum shifted complex on the vertex set $[m]$ containing $\Delta F$. Put $F_1 = \{d_1 < \cdots < d_i\}$ and

$$d(c, k) = \begin{cases} i - 1 & \text{if } d_i \leq c < d_{i+1}, \ i \leq k + 1 \\ k & \text{otherwise.} \end{cases}$$
Observe that $\Delta(F_i, [d_1, c])^k = \Delta(F_i, [d_1, c])^{d(c,k)}$. Then one can easily see that $\Delta(F, [m])^k$ can be constructed by the following induction steps.

1. Start from a single point $d_2 - 1$ and add disjoint vertices $d_2 - 2, \ldots, d_1$ in this order to get $\Delta(F_1, [d_1, d_2 - 1])^k = \Delta(F_1, [d_1, d_2 - 1])^0$.
2. For $d_i \leq c < d_{i+1}$ with $2 \leq i \leq \ell$,
   $$\Delta(F_1, [d_1, c])^k = \Delta(F_1, [d_1, c])^{d(c,k)}$$
   $$= \Delta(F_i - c, [d_1, c - 1])^{d(c,k)} \cup \Delta(F_{i-1}[d_1, c-1])^{d(c,k)-1} (\Delta(F_i - c, [d_1, c - 1])^{d(c,k)-1} * c).$$
3. Add disjoint vertices $d_1 - 1, \ldots, 1$ in this order to $\Delta(F, [d_1, m])^k$, then we finally obtain $\Delta(F, [m])^k$.

We now recall the construction of $\epsilon$ for $\Delta(F, [m])^k$ following the above inductive construction of $\Delta(F, [m])^k$. Put

\[ Z_{a,b}^k = Z_{\Delta(F \cap [a,b], [a,b])}^k (CX_{[a,b]}, X_{[a,b]}) \text{ and } W_{a,b}^k = W_{\Delta(F \cap [a,b], [a,b])}^k \left( X_{[a,b]} \right). \]

(1) We start from the projection $\epsilon : Z_{d_2 - 1, d_2 - 1}^0 = CX_{d_2 - 1} \rightarrow \ast = W_{d_2 - 1, d_2 - 1}^0$, and then define $\epsilon : Z_{c,d_2 - 1}^0 \rightarrow W_{c,d_2 - 1}^0$ for $d_1 \leq c < d_2 - 1$ by the composite

\[ Z_{c,d_2 - 1}^0 \xrightarrow{\text{proj}} (X_c \times Z_{d_1 + 1, d_2 - 1}^0) \cup (CX_c \times (X_{d_2 - 1} \times \cdots \times X_{d_2 - 1})) \]

\[ \xrightarrow{(1 \times \epsilon) \cup (1 \times \epsilon)} (X_c \times W_{c,d_2 - 1}^0) \cup (\Sigma X_c \wedge (X_{d_2 - 1} \times \cdots \times X_{d_2 - 1})) \]

\[ \xrightarrow{\mathcal{X}_{J \cup c}} (X_c \times W_{c,d_2 - 1}^0) \cup \bigvee_{\emptyset \neq J \subseteq [c+1, d_2 - 1]} \Sigma \mathcal{X}_{J \cup c} \xrightarrow{\ast} W_{c,d_2 - 1}^0 \]

where we identify $\Sigma \mathcal{X}_{J \cup c}$ with $\Sigma |\partial \Delta_{\max(J)}| \wedge \mathcal{X}_{J \cup c}$ for $\emptyset \neq J \subset [c + 1, d_2 - 1]$.

(2) For $d_i \leq c < d_{i+1}$ with $2 \leq i \leq \ell$,

\[ Z_{d_1,c}^{d(c,k)} = (Z_{d_1,c-1}^{d(c,k)} \times CX_c) \cup (Z_{d_1,c-1}^{d(c,k)} \times X_c) \]

and the homotopy equivalence $\epsilon : Z_{d_1,c}^{d(c,k)} \rightarrow W_{d_1,c}^{d(c,k)}$ is the composite

\[ Z_{d_1,c}^{d(c,k)} \xrightarrow{\text{proj}} (Z_{d_1,c-1}^{d(c,k)} \times CX_c) \cup (Z_{d_1,c-1}^{d(c,k)} \times X_c) \]

\[ \xrightarrow{(c \times 1) \cup (c \times 1)} (W_{d_1,c-1}^{d(c,k)} \wedge \Sigma X_c) \cup (W_{d_1,c-1}^{d(c,k)} \wedge X_c) \xrightarrow{\ast} W_{d_1,c}^{d(c,k)} \]

where we identify $W_{d_1,c-1}^{d(c,k)-1} \wedge \Sigma X_c$ with $\bigvee_{E \in m(\Delta(F - c, [d_1, c-1])^{d(c,k)})} \Sigma |\partial \Delta_{E \cup c}| \wedge \mathcal{X}_{[d_1,c]}$.

(3) For $c < d_1$, as in the step (1) the homotopy equivalence $\epsilon : Z_{c,m}^k \rightarrow W_{c,m}^k$ is the composite

\[ Z_{c,m}^k \xrightarrow{\text{proj}} (X_c \times Z_{c+1,m}^k) \cup (CX_c \times (X_{c+1} \times \cdots \times X_m)) \]

\[ \xrightarrow{(1 \times c) \cup (1 \times c)} (X_c \times W_{c+1,m}^k) \cup (\Sigma X_c \wedge (X_{c+1} \times \cdots \times X_m)) \]

\[ \xrightarrow{\mathcal{X}_{J \cup c}} (X_c \times W_{c+1,m}^k) \cup \bigvee_{\emptyset \neq J \subseteq [c+1, m]} \Sigma \mathcal{X}_{J \cup c} \xrightarrow{\ast} W_{c,m}^k \]
where we identify $\Sigma \hat{X}^{J,lc}$ with $\Sigma |\partial \Delta^{(c, \max(J))}| \land \hat{X}^{J,lc}$ for $\emptyset \neq J \subset [c+1,m]$.

We consider vertex permutation on $\Delta(F,[m])$. Of course, $\Delta(F,[m])$ depends on the ordering of $[m]$. Then if $\sigma$ is a permutation on $[m]$, we can also define $\Delta(F,[m]^\sigma)$ by using the ordering of $[m]^\sigma$.

**Lemma 2.3.** Let $m \in F \subset [m]$. If $c \in F$ and $c+1 \in I$, then for the transposition $\tau = (d \ d+1)$ there is a homotopy commutative diagram

$$
\begin{array}{ccc}
|\Sigma \partial \Delta^{F}| \land \hat{X}^{[m]} & \xrightarrow{\text{incl}} & \mathcal{W}_{\Delta(F,[m])}(X) \\
-1 \land \hat{X} & \xrightarrow{\epsilon^{-1}} & \mathcal{Z}_{\Delta(F,[m])}(CX_{c},X) \\
|\Sigma \partial \Delta^{(F-c)\cup(c+1)}| \land \hat{X}^{[m]^\tau} & \xrightarrow{\text{incl}} & \mathcal{W}_{\Delta((F-c)\cup(c+1),[m]^\tau)}(X) \\
& \xrightarrow{\epsilon^{-1}} & \mathcal{Z}_{\Delta(F,[m])}(CX_{c},X)^\tau.
\end{array}
$$

**Proof.** We prove the analogous homotopy commutative diagram for the $k$-skeleton of $\Delta(F,[m])$ by inductive steps in the construction of $\Delta(F,[m])^k$. We first consider the case $c+1 \in F$. In this case, $\epsilon$ restricts to

$$
Z^{(d,k)-2}_{f,c} \times CX_{c} \times CX_{c+1} \xrightarrow{\text{proj}} Z^{(d,k)-2}_{f,c-1} \land \Sigma X_{c} \land \Sigma X_{c+1} \xrightarrow{\epsilon \times \epsilon^{-1}} W^{(d,k)-2}_{f,c-1} \land \Sigma X_{c} \land \Sigma X_{c+1}
$$

where $f$ is the minimum of $F$, and the complement $Z^{d(c,k)}_{f,c} - Z^{d(c,k)-2}_{f,c-1} \times CX_{c} \times CX_{c+1}$ maps to other wedge summands. Then we have the desired homotopy commutative diagram.

We next consider the case $c+1 \not\in F$. When $c \geq d_{2}$, we consider the second step of the construction of $\epsilon$. In this case, $\epsilon$ restricts to the map

$$
Z^{d(c,k)-1}_{f,c} \times ((CX_{c} \times CX_{c+1}) \cup (X_{c} \times CX_{c+1})) \xrightarrow{\epsilon \times \pi} W^{d(c,k)-1}_{f,c-1} \land \Sigma (X_{d} \land X_{d+1})
$$

where $\pi$ sends $(t_{c},x_{c},t_{c+1},x_{c+1}) \in (CX_{c} \times CX_{c+1}) \cup (X_{c} \times CX_{c+1})$ to $(t_{c},x_{c},x_{c+1}) \in \Sigma (X_{c} \land X_{c+1})$. The complement $Z^{d(c,k)}_{f,c+1} - Z^{d(c,k)-1}_{f,c} \times ((CX_{c} \times CX_{c+1}) \cup (X_{c} \times CX_{c+1}))$ maps onto other wedge summands. Then we have the desired homotopy commutative diagram. When $c < d_{2}$, we consider the first and the third step, and $\epsilon$ restricts to

$$
((CX_{c} \times CX_{c+1}) \cup (X_{c} \times CX_{c+1})) \times (X_{c+2} \times \cdots \times X_{m}) \xrightarrow{\Sigma \hat{X}^{(F-c)\cup(c+1)} \land X_{c} \land X_{c+1}}
$$

which is given by the map $(CX_{c} \times CX_{c+1}) \cup (X_{c} \times CX_{c+1}) \rightarrow \Sigma (X_{c+1})$, $(t_{c},x_{c},t_{c+1},x_{c+1}) \mapsto (t_{c},x_{c},x_{c+1})$ and the projection $X_{d+2} \times \cdots \times X_{m} \rightarrow \hat{X}^{F-c}$. Since the complement maps onto other wedge summands as well as the above case, it is easy to see that one has the desired homotopy commutative diagram.

We need the following special vertex permutation.
Proposition 2.4. Let $\emptyset \neq I \subset [m]$ and $F \in m(K_I)$. If $K$ is shifted, there is a homotopy commutative diagram

$$
\begin{array}{ccc}
|\Sigma \partial F| \cap \hat{X}_{I^\sigma(F,I)} & \xrightarrow{\text{incl}} & W_{\Delta(F,I^\sigma(F,I))}(X_{I^\tau}) \\
\downarrow \text{sgn}(\sigma(F,I) \wedge \sigma(F,I)) & & \downarrow \Sigma_{\Delta(F,I)}(CX_I, X_I) \\
|\Sigma \partial F| \cap \hat{I} & \xrightarrow{\text{incl}} & W_K(X) \\
& & \downarrow \epsilon^{-1} \\
& & Z_K(CX, X)
\end{array}
$$

where $\Delta(F, I^\sigma(F,I)) = \partial \Delta F \cup I$.

Proof. If $d \in F$ and $d + 1 \notin F$, we use Lemma 2.3 and then by the naturality of $\epsilon$ with respect to the inclusion $\Delta((F-d) \sqcup d+1, I^\tau) \to \Delta(F, I^\tau)$, we can replace $\Delta(F, I)$ with $\Delta((F-d) \sqcup d+1, I^\tau)$, where $\tau$ is the transposition $(d d+1)$. Iterating this process, we then can replace $\Delta(F, I)$ with $\partial \Delta F \cup I$. We next use the naturality of $\epsilon$ with respect to the inclusion $\Delta(F, I) \to K_I$ to obtain the proposition for $K_I$. So we finally use the naturality of $\epsilon$ with respect to the inclusion $Y \to X$ to complete the proof, where $Y = \{Y_i\}_{i \in [m]}$ with $Y_i = \begin{cases} X_i & i \in I \\ \ast & i \notin I \end{cases}$. \[\square\]

3. Proof of Theorem 1.5 and Corollary 1.9

In this section, Theorem 1.5 and Corollary 1.9 are proved. We first prepare a technical lemma. Define a map $\rho : A \ast B \to \Sigma(A \wedge B)$ by $\rho(s, a, t, b) = (t, a, b)$ for $s, t \in [0, 1], a \in A, b \in B$.

Lemma 3.1. The composite

$$
A \ast B \xrightarrow{\text{proj}} (CA \times *) \cup (A \times \Sigma B) \xrightarrow{\text{proj}} A \times \Sigma B \\
\Sigma (A \times \Sigma B) \vee (A \times \Sigma B) \xrightarrow{\text{proj}} \Sigma B \vee \Sigma (A \wedge B)
$$

is homotopic to $i_2 \circ \rho$, where $\nabla$ is the comultiplication and $i_2 : \Sigma(A \wedge B) \to \Sigma B \vee \Sigma(A \wedge B)$ is the inclusion.

Proof. The composite in the proposition is identified with the composite

$$
A \ast B \xrightarrow{\text{proj}} (CA \wedge B) \cup (A \times CB) \xrightarrow{1 \cup (1 \times \nabla)} (CA \wedge B) \cup (A \times (CB \vee \Sigma B)) \\
= ((CA \wedge B) \cup (A \times CB)) \vee A \times \Sigma B \xrightarrow{\text{proj}} CB \vee (A \times \Sigma B) \xrightarrow{\text{proj}} \Sigma B \vee \Sigma (A \wedge B)
$$

which is homotopic to $i_2 \circ \rho$ since $CB$ is contractible. \[\square\]

We consider the special case of Theorem 1.5.
Proposition 3.2. For $k < m$, the composite
\[ \Sigma \partial \Delta^{[k, m]} \cap \widehat{X}^{[m]} \xrightarrow{\text{incl}} W_{\partial \Delta^{[k, m] \cup [m]}}(X) \xrightarrow{-1} Z_{\partial \Delta^{[k, m] \cup [m]}}(CX, X) \xrightarrow{\omega} Z_{\partial \Delta^{[k, m] \cup [m]}}(\Sigma X, \ast) \]
is homotopic to the iterated Whitehead product
\[ [e_1, \cdots, e_{k-2}, [e_{k-1}, \omega_{[k, m]}]]]. \]

Proof. It is sufficient to prove the case $1 \not\in F$. Since $Z = Z_{\partial \Delta^{[k, m] \cup [2, m]}}(CX_{[2, m]}, X_{[2, m]})$ and $W = W_{\partial \Delta^{[k, m] \cup [2, m]}}(X_{[2, m]}, X_{[2, m]})$ are suspensions, we can consider $\Sigma^{-1}Z$ and $\Sigma^{-1}W$. Then there is a homotopy commutative diagram
\begin{align*}
X_1 \ast \Sigma^{-1}W & \xrightarrow{\text{proj}} (CX_1 \times \ast) \cup (X_1 \times W) \xrightarrow{\text{incl}} W_{\partial \Delta^{[k, m] \cup [m]}}(X) \\
X_1 \ast \Sigma^{-1}Z & \xrightarrow{\text{proj}} (CX_1 \times \ast) \cup (X_1 \times Z) \xrightarrow{\text{incl}} Z_{\partial \Delta^{[k, m] \cup [m]}}(CX, X) \\
\Sigma X_1 \vee Z & \xrightarrow{1 \vee \omega} \Sigma X_1 \vee Z_{\partial \Delta^{[k, m] \cup [2, m]}}(\Sigma X_{[2, m]}, \ast) \xrightarrow{\omega} Z_{\partial \Delta^{[k, m] \cup [m]}}(\Sigma X, \ast).
\end{align*}
where the upper left arrow is $1 \wedge \epsilon$ through the homotopy equivalence $A \ast B \simeq \Sigma(A \wedge B)$. By the construction of $\epsilon$, the composite of the top row is identified with the composite in Lemma 3.1 which is homotopic to the inclusion
\[ X_1 \wedge W_{\partial \Delta^{[k, m] \cup [2, m]}}(X_{[2, m]}) = \bigvee_{\emptyset \neq I \subset [2, m]} \Sigma \partial(\Delta^{[k, m] \cup [2, m]} \cap [2, m]) \wedge \widehat{X}^{[1, 2]} \rightarrow W_{\partial \Delta^{[k, m] \cup [m]}}(X). \]
This completes the proof. \hfill \square

Proof of Theorem 1.5. Combine Proposition 2.4 and 3.2. \hfill \square

Proof of Corollary 1.9. By Theorem 1.5, the composite
\[ (\Delta^{[m]})^{m-3}_{[m-1]} \wedge X_i \wedge \widehat{X}^{[m-1]} \xrightarrow{\text{sgn}(\sigma_i) \wedge \hat{\sigma}_i} (\Delta^{[m]})^{m-3}_{[m-1]} \wedge \widehat{X}^{[m]} \xrightarrow{\text{incl}} W_{(\Delta^{[m]})^{m-3}}(X) \xrightarrow{-1} Z_{(\Delta^{[m]})^{m-3}}(CX, X) \xrightarrow{\omega} Z_{(\Delta^{[m]})^{m-3}}(\Sigma X, \ast) \]
is equal to $(-1)^{i-1}[e_i, \omega_{[m-1]}] \circ (1 \wedge \hat{\sigma}_i)$ for $i \neq m$. For $i = m$, by combining Theorem 1.5 and Proposition 2.4 in the case $(d, d + 1) = (m - 1, m)$, we also have that the above composite is equal to $(-1)^{m-1}[e_m, \omega_{[m-1]}] \circ (1 \wedge \hat{\sigma}_m)$. Since for the inclusions $\iota_i : (\Delta^{[m]})^{m-3}_{[m-1]} \rightarrow (\Delta^{[m]})^{m-3}$ the sum $\sum_{i=1}^{m} (-1)^{i-1}\iota_i = 0$ in $\pi_{m-3}((\Delta^{[m]})^{m-3})$, the proof is completed. \hfill \square

4. Further problems

In this section we propose further problems concerning the map (1.1) and homotopy operations. In this paper, we consider the map (1.1) based on the homotopy decomposition of Theorem 1.3 which is the result of [IK1]. Generalizing this result, it is shown in [IK2] that
there is an analogous homotopy decomposition for a much broader class of simplicial complexes called dual sequentially Cohen-Macaulay over $\mathbb{Z}$, where the naturality of Theorem 2.1 regarding the inclusions of subcomplexes is not proved, implying that the same proof as this paper does not seem to work. So we further investigate Problem 1.1 as follows.

**Problem 4.1.** If $K$ is dual sequentially Cohen-Macaulay over $\mathbb{Z}$, can one describe the map (1.1) by (higher order) Whitehead products?

As a corollary of the main result Theorem 1.5, we prove the Jacobi identity of higher order Whitehead products. But there must be more identities concerning higher order Whitehead products. Then we ask:

**Problem 4.2.** Can one derive more Jacobi identities of higher order Whitehead products from Theorem 1.5?

Even if the homotopy decomposition of Theorem 2.1 is not given, the map 1.1 is still important as is mentioned in Section 1. So we finally ask:

**Problem 4.3.** Which homotopy theoretical structure does the map (1.1) detect?

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