Abstract

Monopoles and solitons have important topological aspects like quantized fluxes, winding numbers and curved target spaces. Naive discretizations which substitute a lattice of points for the underlying manifolds are incapable of retaining these features in a precise way. We study these problems of discrete physics and matrix models and discuss mathematically coherent discretizations of monopoles and solitons using fuzzy physics and noncommutative geometry. A fuzzy $\sigma$-model action for the two-sphere fulfilling a fuzzy Belavin-Polyakov bound is also put forth.
A fuzzy space \([1–8]\) is obtained by quantizing a manifold, treating it as a phase space. An example is the fuzzy two-sphere \(S_2^F\). It is described by operators \(x_i\) subject to the relations \(\sum_i x_i^2 = 1\) and \([x_i, x_j] = (i/\sqrt{l(l+1)})\epsilon_{ijk}x_k\). Thus \(L_i = \sqrt{l(l+1)}x_i\) are \((2l+1)\)-dimensional angular momentum operators while the canonical classical two-sphere \(S^2\) is recovered for \(l\to\infty\). Planck’s work shows that quantization creates a short distance cut-off, therefore quantum field theories (QFT’s) on fuzzy spaces are ultraviolet finite. If the classical manifold is compact, it gets described by a finite-dimensional matrix model, the total number of states being finite too. Noncommutative geometry \([1, 9–14]\) has an orderly prescription for formulating QFT’s on fuzzy spaces so that these spaces indeed show us an original approach to discrete physics.

In this note, we focus attention on \(S_2^F\) and discuss certain of its remarkable aspects, entirely absent in naive discretizations. Quantum physics on \(S_2^F\) is a mere matrix model, all the same they can coherently describe twisted topologies like those of monopoles and solitons. Traditional attempts in this direction based on naive discrete physics have at best been awkward having ignored the necessary mathematical structures (projective modules and cyclic cohomology). Not all our results are new, our construction of monopoles being a reformulation of the earlier important work of Grosse et al. \([4]\).

1 Classical Monopoles and \(\sigma\)-Models

There is an algebraic formulation of monopoles and solitons suitable for adaptation to fuzzy spaces. We first outline it using the case of \(S^2\) \([13]\).

Let \(\mathcal{A}\) be the commutative algebra of smooth functions on \(S^2\). Vector bundles on \(S^2\) can be described by projectors \(\mathcal{P}\). \(\mathcal{P}\) is a matrix with coefficients in \(\mathcal{A}(\mathcal{P}_{ij}\in\mathcal{A})\), and fulfills \(\mathcal{P}^2 = \mathcal{P}\) and \(\mathcal{P}^\dagger = \mathcal{P}\). If the points of \(S^2\) are described by unit vectors \(\vec{n}\in\mathbb{R}^3\), the projector for unit monopole charge is \(\mathcal{P}^{(1)} = (1 + \vec{\tau}.\hat{n})/2\) where \(\tau_i\) are Pauli matrices and \(\hat{n}_i\) are coordinate functions, \(\hat{n}_i(\vec{n}) = n_i\). If \(\mathcal{A}^{2N} = \mathcal{A}\otimes\mathbb{C}^{2N}\) consists of \(2^N\)-component vectors \(\xi = (\xi_1, \xi_2, ..., \xi_{2N})\), \(\xi_i\in\mathcal{A}\), then the sections of vector bundles for monopole charge 1 are \(\mathcal{P}^{(1)}\mathcal{A}^2\). For monopole charge \(\pm N\) \((N > 0)\), the corresponding projectors are \(\mathcal{P}^{(\pm N)} = \prod_{i=1}^{2N}(1 \pm \vec{\tau}^{(i)}\cdot\hat{n})/2\) where \(\vec{\tau}^{(i)}\) are commuting sets of Pauli matrices. They give sections of vector bundles \(\mathcal{P}^{(\pm N)}\mathcal{A}^{2N}\) with \(\vec{\tau}^{(i)}\cdot\hat{n}\ \mathcal{P}^{(\pm N)}\xi = \pm\mathcal{P}^{(\pm N)}\xi\), \(\vec{\tau}^{(i)}\) acting on the \(i\)th \(\mathbb{C}^2\) factor. For the trivial
bundle, we can use $\mathcal{P}^0 = (1 + \tau_3)/2$ (or $(1 - \tau_3)/2$), or more simply just the identity.

The self-same projectors also describe nonlinear $\sigma$-models. To see this, consider the projector $\mathcal{P}^{(0)}$ and its orbit $\langle h\mathcal{P}^{(0)}h^{-1} : h \in SU(2) \rangle = S^2$. If now we substitute for $h$ a field $g$ on $S^2$ with values in $SU(2)$, each $g\mathcal{P}^{(0)}g^{-1}$ describes a map $S^2 \to S^2$. It is a $\sigma$-model field on $S^2$ with target space $S^2$ and zero winding number. (Winding number is zero as $g$ can be deformed to a constant map). For winding number 1, it is appropriate to consider the orbit of $\mathcal{P}^{(1)}$ under $g$. For fixed $\vec{n}$, as $g(\vec{n})$ is varied, $g(\vec{n})^{1 + \vec{\sigma} \cdot \hat{n}(\vec{n})}g(\vec{n})^{-1}$ is still $S^2$ so that as $\vec{n}$ is varied, we get a map $S^2 \to S^2$. (More correctly we get the section of an $S^2$ bundle over $S^2$). For winding number $\pm N$, we can consider the orbit of $\mathcal{P}^{(\pm N)}$ under conjugation by $g^{\hat{n}}$s where $g^{\hat{n}}(\vec{n}) = g(\vec{n}) \otimes g(\vec{n}) \otimes \cdots \otimes g(\vec{n})$ ($N$ factors). Here the $i$th $g(\vec{n})$ acts only on $\vec{r}^{(i)}$.

2 Winding Numbers for the Classical Sphere

What about formulas for invariants like Chern character and winding number? The ideal way is to follow Connes (\cite{9-14}) and introduce the Dirac and chirality operators

$$
\mathcal{D} = \epsilon_{ijk}\sigma_i \hat{n}_j \hat{J}_k, \\
\Gamma = \sigma. \hat{n}
$$

where $\sigma_i$ are Pauli matrices, $\hat{J} = -i(\vec{r} \times \vec{\nabla}) + \vec{\sigma}/2$ is the total angular momentum and $\hat{n} = \vec{r}/|\vec{r}|$. The important points to keep in mind here are the following:

i) $\Gamma$ commutes with elements of $\mathcal{A}$ and anti-commutes with $\mathcal{D}$.

ii) $\Gamma^2 = 1$ and $\Gamma^\dagger = \Gamma$.

The Chern numbers (or the quantized fluxes) for monopoles then are

$$
\pm N = \frac{1}{4\pi} \int d(\cos \theta)d\phi \, \text{Tr} \, \Gamma \mathcal{P}^{(\pm N)} \left[ \mathcal{D}, \mathcal{P}^{(\pm N)} \right] \left[ \mathcal{D}, \mathcal{P}^{(\pm N)} \right](\vec{n}). \quad (2)
$$

They do not change if $\mathcal{P}^{(\pm N)}$ are conjugated by $g^{\hat{n}}$ and are also the soliton winding numbers.
3  Fuzzy Monopoles

The algebra $A$ generated by $x_i$ is the full matrix algebra of $(2l+1) \times (2l+1)$ matrices. Fuzzy monopoles are described by projectors $p^{(\pm N)}$ ($p^{(\pm N)}_{ij} \in A$), which as $l \to \infty$ approach $P^{(\pm N)}$. We can find them as follows: For $N = 1$, we can try $(1 + \vec{\tau}.x)/2$, but that is not an idempotent as the $x$'s do not commute. We can fix that through: since $(\vec{\tau}.\vec{L})^2 = l(l+1) - \vec{\tau}.\vec{L}$, $\gamma_\tau = \frac{1}{l+1/2}(\vec{\tau}.\vec{L} + 1/2)$ squares to 1 as first remarked by Watamuras ( ). Hence $p^{(1)} = (1 + \gamma_\tau)/2$.

There is a simple interpretation of $p^{(1)}$. We can combine $\vec{L}$ and $\vec{\tau}/2$ into the $SU(2)$ generator $\vec{K}^{(1)} = \vec{L} + \vec{\tau}/2$ with spectrum $k(k+1)$ (with $k = l \pm 1/2$) for $K^{(1)} = k\vec{K}^{(1)}$. The projector to the space with the maximum $k$, namely $p^{(1)} = \frac{K^{(1)} - (l-1/2)(l+1/2)}{(l+1/2)(l+3/2)-(l-1/2)(l+1/2)}$, is just $p^{(1)}$.

This last remark shows the way to fuzzify $P^{(N)}$. We substitute $\vec{K}^{(N)} = \vec{L} + \sum_{i=1}^{N} (\vec{\tau}^{(i)})/2$ for $K^{(1)}$ and consider the subspace where $K^{(N)} = k\vec{K}^{(N)}$. The projector to the space with the maximum $k$, namely $p^{(N)} = \frac{K^{(N)} - (l-1/2)(l+1/2)}{(l+1/2)(l+3/2)-(l-1/2)(l+1/2)}$, is just $p^{(N)}$.

$p^{(-N)}$ comes similarly from the least value $k_{\text{min}} = l - N/2$ of $k$. [We assume that $2l \geq N$.]

We remark that the limits as $l \to \infty$ of $p^{(\pm N)}$ are exactly $P^{(\pm N)}$, and not say $P^{(\pm N)}$ times another projector. That is because if $\vec{\tau}^{(i)} \vec{L}$ are all $l$, then $\vec{\tau}^{(i)} \vec{\tau}^{(j)} = 1$ for all $i \neq j$ and hence $k_{\text{max}} = l + N/2$. A proof goes as follows. Vectors with $\vec{L}^2 = l(l+1)1$ can be represented as symmetric tensor products of $2l$ spinors, with components $T_{a_1...a_2l}$. The vectors with $\vec{\tau}^{(i)} \vec{L} = 1$ as well have components $T_{a_1...a_{2l} b_1...b_N}$ with symmetry under exchange of any $a_i$ with $a_j$ or $b_k$. So they are symmetric under all exchanges of $b_i$ and $b_j$ and have $(\vec{\tau}^{(i)} + \vec{\tau}^{(j)})^2 = 2$, $\vec{\tau}^{(i)} \vec{\tau}^{(j)} = 1$.

Having obtained $p^{(\pm N)}$, we can also write down the analogues of $P^{(\pm N)} A^{2N}$: they are the “projective modules” $p^{(\pm N)} A^{2N}$, $A^{2N} = \langle (a_1, a_2, ..., a_{2N}) : a_i \in A \rangle$, and are the noncommutative substitutes for sections of vector bundles.

If $(a_1, a_2, ..., a_{2N})$ is regarded as a column, then column dimension of $p^{(\pm N)} A^{2N}$ is $L + 1 = 2(l + N/2) + 1$ and its row dimension is $M + 1 = 2l + 1$. 

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Their difference is ±N. This means that $p^{(±N)}A^{2N}$ can be identified with $\hat{H}_{L,M}$ of ref. [4] where of course $L - M = ±N$. In particular angular momentum acts on $p^{(±N)}A^{2N}$ via $\vec{K}^{(N)}$ on left (they commute with $p^{(±N)}$) and $-\vec{L}$ on right, while there are similar actions of angular momentum on $\hat{H}_{L,M}$ [see ref. [4]].

4 Fuzzy $\sigma$-models

In the fuzzy versions of $\sigma$-models on $S^2$ with target $S^2$, $g$ becomes a $2 \times 2$ unitary matrix $u$ with $u_{ij} \in A$. Therefore $u \in U(2(2l + 1))$. (We can impose $\det u = 1$, that makes no difference). An appropriate generalization $u^{\tilde{N}}$ of $g^{\tilde{N}}$ can be constructed as follows. If $C$ and $D$ are $2 \times 2$ matrices with entries $C_{ij}, D_{ij} \in A$, we can define $Ca$ and $aD$ for $a \in A$ by $(Ca)_{ij} = C_{ij}a$ and $(aD)_{ij} = aD_{ij}$. Let $C \otimes_A D$ denote the tensor product of $C$ and $D$ over $A$ where by definition $Ca \otimes_A D = C \otimes_A aD$. This definition can be extended to more factors. For example, $C \otimes_A D \otimes_A E$ has the properties $Ca \otimes_A D \otimes_A E = C \otimes_A aD \otimes_A E$, $C \otimes_A Da \otimes_A E = C \otimes_A D \otimes_A aE$. Then:

$$u^{\tilde{N}} = u \otimes_A u \otimes_A \ldots \otimes_A u \ (N \text{ factors}). \quad (4)$$

We can understand this construction in familiar terms by writing $u = 1_{2 \otimes 2} a_0 + \tau_j a_j = \tau_\mu a_\mu (a_\mu \in A)$ where $\tau_0 = 1_{2 \otimes 2}$. [Greek subscripts run from 0 to 3, Roman ones from 1 to 3]. Unitarity requires that

$$\tau_\mu \tau_\nu a_\mu^* a_\nu = 1, \quad a_\mu^* \equiv a_\mu^\dagger. \quad (5)$$

In this notation, $u \otimes_A u = \tau_\mu \otimes_\tau a_\mu a_\nu$, $\otimes(\equiv \otimes_C)$ denoting Kronecker product. It is also $\tau_\mu^{(1)} a_\mu \tau_\nu^{(2)} a_\nu$ in an evident notation. Proceeding in this way, we find,

$$u \otimes_A u \otimes_A \ldots \otimes_A u = \tau_\mu^{(1)} a_\mu \tau_\nu^{(2)} a_\nu \ldots \tau_\mu^{(N)} a_\mu^N. \quad (6)$$

It is unitary in view of (5).

The significant point here is that $u^{\tilde{N}}$ is a matrix with coefficients in $A$ and not $A \otimes A \otimes \ldots \otimes A$ as is the case for $u \otimes u \otimes u \ldots \otimes u$. We remark that $g^{\tilde{N}}$ can also be written as $g \otimes_A g \otimes_A g \ldots \otimes_A g \ (N \text{ factors})$. It is then a function only of $\tilde{n}$ and has the meaning stated earlier.
The orbits of \( p^{(\pm N)} \) under conjugation by \( \tilde{u}^{\pm N} \) are fuzzy matrix versions of \( \sigma \)-model fields with winding numbers \( \pm N \). [Here we take \( p^{(0)} \) to be \((1 + \tau_3)/2 \) say and its \( \tilde{u}^{\pm N} \) to be \( u \) itself. Henceforth our attention will be focused on \( N \neq 0 \).]

5 “Winding Numbers” for the Fuzzy Sphere

The fuzzy Dirac operator \( D \) and chirality operator \( \gamma \) are important for writing formulae for the invariants of projectors. There are proposals for \( D \) and \( \gamma \) in [2–7], we briefly describe those in [6,7]. There is a left and right action (“left” and “right” “regular representations” \( A^L \) and \( A^R \)) of \( A \) on \( A \): \( b^L a = ba \) and \( b^R a = ab, (b, a \in A, b^L \in A^L, b^R \in A^R) \) with corresponding angular momentum operators \( L^L_i \) and \( L^R_i \) and fuzzy coordinates \( x^L_i \) and \( x^R_i \).

\[
D = \epsilon_{ijk} \sigma_i x^L_j L^R_k, \\
\gamma = -\frac{\sigma \cdot \vec{L}_R - 1/2}{l + 1/2}.
\]  
(7)

Identifying \( A^L \) as the representation of the fuzzy version of \( A \), we have as before,

\[
\gamma b^L = b^L \gamma, \\
\gamma D + D \gamma = 0, \\
\gamma^2 = 1, \\
\gamma^\dagger = \gamma.
\]  
(8)

The carrier space of \( A^L, D \) and \( \gamma \) is \( A^2 \). When \( p^{(\pm N)} \) are also included, it gets expanded to \( A^{2N+1} \) as \( \vec{\tau}^{(i)} \) commute with \( \vec{\sigma} \). Note that \( p^{(\pm N)} \) commute with \( \gamma \), as the \( x \)'s they contain are now being identified with \( x^L \)'s.

We now construct a certain generalization of (2) for the fuzzy sphere. It looks like (2), or rather the following expression:

\[
\pm N = -Tr_\omega \left( \frac{1}{|D|^2} \Gamma [\mathcal{D}, \mathcal{P}^{(\pm N)}] [\mathcal{D}, \mathcal{P}^{(\pm N)}] \right), \\
|\mathcal{D}| = \text{Positive square root of } D^\dagger D.
\]  
(9)

where \( \mathcal{F} = D/|\mathcal{D}| \) [3–4]. It is equivalent to (2). It involves a Dixmier trace \( Tr_\omega \) and furthermore the inverse of \( |\mathcal{D}| \).
But the massless Dirac operator on $A^{2N+1}$ has zero modes and $|D|$ has no inverse. An easy proof is as follows. We can write the elements of $A^{2N+1}$ as rectangular matrices with entries $\xi_{\lambda j} \in A$ ($\lambda = 1, 2, ..., 2N; j = 1, 2$) where $\lambda$ carries the action of $\vec{\tau}$'s and $j$ carries the action of $\vec{\sigma}$. The dimensions of the subspaces $U_\pm$ of $A^{2N+1}$ with $\gamma = \pm 1$ are $\left[2(l \pm 1/2) + 1\right][2(l + 1)2N]$. The first factor is the row dimension of $U_\pm$ and is deduced from the fact that $(-\vec{L}^R + \vec{\sigma}/2)^2$ has the definite values $(l + 1/2)(l + 3/2)$ and $(l - 1/2)(l + 1/2)$ on $U_\pm$. The second factor is the column dimension of $U_\pm$. $D$ anticommutes with $\gamma$. So if $D^{(+)}$ is the restriction of $D$ to the domain $U_+: D^{(+)} = D|_{U_+: U_+ \rightarrow U_-}$, its index is $\dim U_+ - \dim U_- = 2[(2l+1)2N]$. This is the minimum number of zero modes of $D$ in $U_+$. Calculations [6,7] show this to be the exact number of zero modes, $D$ having no zero mode in $U_-$. In any case, $D^{(+)}$ and $D^{(-)}$ have no inverse. So we work instead with the massive Dirac operator $D_m = D + m\gamma$ ($m \neq 0$) with the strictly positive square $D_m^2 = D^2 + m^2$ and form the operator

$$f_m = \frac{D_m}{|D_m|}$$

(10)

where

$$|D_m| = \text{Positive square root of } D_m^\dagger D_m, \quad f_m^\dagger = f_m, \quad f_m^2 = 1.$$  

(11)

Consider $\frac{1}{2}p^{(N)}f_m p^{(N)}\frac{1+\gamma}{2}$ where we pick $p^{(N)}$ and not $p^{(-N)}$ for specificity. It anticommutes with $\gamma$. Let $\hat{V}_\pm = p^{(N)}U_\pm$. It then follows that the index of the operator

$$\hat{f}_m^{(+)} = \frac{1 - \gamma}{2} p^{(N)} f_m p^{(N)} \frac{1+\gamma}{2}$$

(12)

(restricted to $\hat{V}_+$, such restrictions are hereafter to be understood) is dimension of $\hat{V}_+$ — that of $\hat{V}_- = 2[2l+1+N]$. The index of its adjoint

$$\hat{f}_m^{(+)*} = \hat{f}_m^{(-)} = \frac{1 + \gamma}{2} p^{(N)} f_m p^{(N)} \frac{1-\gamma}{2}$$

(13)

is $-2[2l+1+N]$.

We may try to associate the index of $\hat{f}_m^{(+)}$ say with the winding number $N$. But that will not be correct: this index is not zero for $N = 0$. The source of
This unpleasant feature is also a set of unwanted zero modes. Their presence can be established by looking at \( \hat{f}_m^{(\pm)} \) more closely. \( \hat{f}_m^{(\pm)} \) and \( \gamma \) commute with “total angular momentum” \( \vec{J} = \vec{L} - \vec{L}^R + \sum_i \vec{\omega}^{(i)} + \frac{d}{2} \) while \( \gamma \) anticommutes with \( \hat{f}_m^{(\pm)} \). So if an irreducible representation (IRR) of \( \vec{J} \) with \( \hat{J}^2 = j(j+1) \mathbf{1} \) occurs an odd number of times in \( \hat{V}_+ + \hat{V}_- \), \( \hat{f}_m^{(+)} + \hat{f}_m^{(-)} \) must vanish on at least one of the \((2j+1)\) - dimensional eigenspaces. The remaining \((2j+1)\) - dimensional eigenspaces can pair up so as to correspond to eigenvalues \( \pm \lambda \neq 0 \) and get interchanged by \( \gamma \). There are two such \( j \), both in \( \hat{V}_+ \). They label IRR’s with multiplicity 1 and are its maximum and minimum \( j^{(N)} = 2l + \frac{N+1}{2} \) and \( \frac{N-1}{2} \). We can see that their eigenspaces have \( \gamma = +1 \) as follows: the angular momentum value of \( \vec{L} + \sum_i \vec{\omega}^{(i)} \) in \( p^{(N)} A^{2N+1} \) is \( l + N/2 \) so that the angular momentum value of \( -\vec{L}^R + \frac{d}{2} \) must be \( l + 1/2 \) to attain the \( j \)-values \( j^{(N)} \) and \( \frac{N-1}{2} \). A further point is that since \((2j^{(N)} + 1) + [2(\frac{N-1}{2}) + 1] \) is the index of \( \hat{f}_m^{(\pm)} \) found earlier, we can conclude that there are no other obligatory zero modes. Indeed every other \( j \) labels IRR’s of multiplicity 2, one with \( \gamma = +1 \) and the other with \( \gamma = -1 \).

The zero modes for \( j^{(N)} \) are unphysical as discussed by Watamuras [3,4]: there are no similar modes in the continuum. If we can project them out, the index will shrink to \( 2\frac{N-1}{2} + 1 = N \), just what we want. So let \( \pi^{(j^{(N)})} \) be the projection operator for \( j^{(N)} \), constructed in the same fashion as \( p^{(N)} \). It commutes with \( p^{(N)} \) since \( p^{(N)} \) commutes with \( \vec{J} \). In fact, \( p^{(N)} \pi^{(j^{(N)})} = \pi^{(j^{(N)})} p^{(N)} \) since if \( j \) is maximum, then so is \( k \). We thus find that

\[
\Pi^{(N)} = p^{(N)}[1 - \pi^{(j^{(N)})}] = p^{(N)} - \pi^{(j^{(N)})}
\]  

(14)

is a projector. It commutes with \( \gamma \) too. Let

\[
V_\pm = \Pi^{(N)} U_\pm,
\]

\[
f_m^{(\pm)} = \frac{1 \pm \gamma}{2} \Pi^{(N)} f_m \Pi^{(N)} \frac{1 \pm \gamma}{2}
\]  

(15)

where \( f_m^{(+)} \dagger = f_m^{(-)} \). Then \( f_m^{(+)} \) (restricted to \( V_+ \)) has the index \( N \) we want.

The eigenvalues of \( f_m^{(-, +)} f_m^{(+, -)} \) are the same (not counting degeneracy) and for nonzero eigenvalues, the dimensions of the corresponding eigenspaces
are also identical. (We omit the elementary proofs.) Therefore,
\begin{align*}
Tr \frac{1 + \gamma}{2} \Pi(N)[1 - f_m^+ f_m^+] & - \\
Tr \frac{1 - \gamma}{2} \Pi(N)[1 - f_m^- f_m^-] & = \\
Tr \frac{1 + \gamma}{2} \Pi(N) - Tr \frac{1 - \gamma}{2} \Pi(N) & = N \\
& = \text{Index of } f_m^+.
\end{align*}

We want to be able to write (16) as a cyclic cocycle coming from a Fredholm module [9–14]. The latter for us is based on a representation \( \Sigma \) of \( A_L \otimes A_R \) on a Hilbert space, and operators \( F \) and \( \epsilon \) with the following properties:

\begin{align*}
(i) \quad & F^\dagger = F, \quad F^2 = 1, \\
(ii) \quad & \epsilon^\dagger = \epsilon, \quad \epsilon^2 = 1, \quad \epsilon \Sigma(\alpha) = \Sigma(\alpha) \epsilon, \quad \epsilon F = -F \epsilon
\end{align*}

where \( \alpha \in A_L \otimes A_R \). [This gives an \textit{even} Fredholm module, there need be no \( \epsilon \) in an odd one.] We choose for \( \Sigma \) the representation

\begin{equation}
\Sigma : \alpha \rightarrow \Sigma(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}
\end{equation}

on \( A_{2N+1} \otimes A_{2N+1} \) and set

\begin{equation}
F = \begin{pmatrix} 0 & f_m \\ f_m & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{equation}

Introduce the projector

\begin{equation}
P^{(N)} = \begin{pmatrix} \frac{1 + \gamma}{2} \Pi(N) & 0 \\ 0 & \frac{1 - \gamma}{2} \Pi(N) \end{pmatrix}.
\end{equation}

Then

\begin{equation}
(P^{(N)} F P^{(N)})^2 = \begin{pmatrix} f_m^+ f_m^+ & 0 \\ 0 & f_m^- f_m^- \end{pmatrix}.
\end{equation}

Therefore,

\begin{align*}
\text{Index of } f_m^+ & = Tr \epsilon [P^{(N)} - (P^{(N)} F P^{(N)})^2].
\end{align*}
But since \[9\]
\[
P^{(N)} - (P^{(N)} F P^{(N)})^2 = -P^{(N)}[F, P^{(N)}]^2 P^{(N)},
\]
Index of \(f_m^{(+)}\) = \(N = -\text{Tr} \epsilon P^{(N)} [F, P^{(N)}] [F, P^{(N)}].\) \(24\)

This is the formulation of \([14]\) we aimed at and is the analogue of \([4]\). It is worth remarking that we can replace \(\epsilon\) by \(\left(\begin{array}{cc}
\gamma & 0 \\
0 & \gamma
\end{array}\right)\) here since \(\epsilon P^{(N)} =
\[
\left(\begin{array}{cc}
\gamma & 0 \\
0 & \gamma
\end{array}\right) P^{(N)}.
\]

In \(p^{(-N)} A^{2N+1}\) as well, the unwanted zero modes correspond to the top value \(j^{(-N)} = 2\ell - \frac{N+1}{2}\) of “total angular momentum ”. Once they are suppressed, the remaining obligatory zero modes are readily shown to have the \(j\)– value \(\frac{N-1}{2}\), multiplicity \(N\) and \(\gamma = -1\). Let \(\pi(j^{(-N)})\) be the projector for the top angular momentum. Then it can be projected out by replacing \(p^{(-N)}\) by

\[
\Pi^{(-N)} = p^{(-N)}[1 - \pi(j^{(-N)})],
\]

but now \(p^{(-N)} \pi(j^{(-N)}) \neq \pi(j^{(-N)})\). Substituting \(\Pi^{(-N)}\) for \(\Pi^{(N)}\) in \([24]\), we define \(P^{(-N)}\) and then by using \([24]\) can associate \(N\) too with an index.

There is the topic of fuzzy \(\sigma\)-fields yet to be discussed in this section. We first note that in \(u\) defined earlier, \(a_\mu\) is to be identified with \(a_\mu^{L}\). Let us extend \(u^{\tilde{\sigma}N}\) and \(g^{\tilde{\sigma}N}\) from \(A^{2N}\) to \(A^{2N+1} = A^{2N} \otimes \mathbb{C}^2\) and \(A^{2N+1} = A^{2N} \otimes \mathbb{C}^2\) so that they act as identity on the last \(\mathbb{C}^2\)’s. We also extend them further to \(A^{2N+1} \oplus A^{2N+1} \equiv A^{2N+1} \otimes \mathbb{C}^2\) and \(A^{2N+1} \oplus A^{2N+1} \equiv A^{2N+1} \otimes \mathbb{C}^2\) so that they act as identity on these last \(\mathbb{C}^2\)’s. Define also \(Q(u) = u^{\tilde{\sigma}N} Q(u^{\tilde{\sigma}N})^{-1}\) for an operator \(Q(= Q(1))\) on \(A^{2N+1} \oplus A^{2N+1}\). The right hand side of \([24]\) is invariant under the substitution \(P^{(N)} \rightarrow P^{(N)}(u)\) without changing \(F\). So \(P^{(N)}(u)\) is a candidate for a fuzzy winding number \(N\) \(\sigma\)-field in the present context whereas previously it was \(p^{(N)}(u)\). But we must justify this candidacy by looking at the continuum limit. In that limit, \(u^{\tilde{\sigma}N} \rightarrow g^{\tilde{\sigma}N}, \pi(j^{(N)}) \rightarrow \pi^{(N)}\) say and \(\Pi^{(N)} \rightarrow \Pi^{(N)}_\infty = P^{(N)} - \pi(j^{(N)})\). The stability group of \(P^{(N)}\) under conjugation by \(g^{\tilde{\sigma}N}\) is as before \(U(1)\) at each \(\tilde{n}\). Now \(\pi^{(j^{(N)})}\) projects out states where any one of \((\vec{L}^L + \overline{\pi(j^{(N)})})^2, (\vec{L}^L + \overline{\pi(j^{(N)})} + \vec{r})^2, (\vec{L}^L - \vec{r})^2\) has the maximum value. (Then any other pair of angular momenta also adds up to maximum value as we saw in Section 3.) So \(\vec{r}, \vec{L}^N\pi(j^{(N)}) = \vec{r}, \vec{L}^N\pi(j^{(N)}) = \pi(j^{(N)})\),
\[ \vec{x}_R \pi^{(j(N))} = -\pi^{(j(N))} \]. The last condition is just a rule telling us that as \( l \to \infty \), \( \vec{x}_R \) becomes \( -\hat{n} \) on vectors projected by \( \pi^{(j(N))} \). It will not show up in the continuum projector. This establishes that \( \pi^{(j(N))} \) can be identified with \( \mathcal{P}^{(N+1)} = (\prod_{i=1}^{N} \frac{1+\pi^{(j(N))}_i}{2}) \frac{1+\hat{n}}{2} \) while of course \( \gamma \to \Gamma \). So \( \mathcal{P}^{(N+1)} \) and \( \frac{1+\hat{n}}{2} \) have \( U(1) \) stability groups, \( \Pi_N^{(N)}(g) \) is a \( \sigma \)-field on \( S^2 \) and \( \mathcal{P}^{(N)}(u) \) is a good choice for the fuzzy \( \sigma \)-field.

For winding number \( -N \), we propose \( \mathcal{P}^{(-N)}(u) \) as the fuzzy \( \sigma \)-field. We can check its validity also by going to the continuum limit. As \( l \to \infty \), \( \mathcal{P}^{(-N)} \to \mathcal{P}^{(-N)} = \prod_{i=1}^{N} \frac{1-\pi^{(j(N))}_i}{2} \) and has the \( U(1) \) stability group at each \( \vec{n} \).

Next consider the product \( \mathcal{P}^{(-N)} \pi^{(j(N))} \). The presence of \( \mathcal{P}^{(-N)} \) allows us to assume that \( (\vec{K}^{(N)})^2 = k_{\text{min}}(k_{\text{min}}+1) \), \( k_{\text{min}} = l-N/2 \). Also we can substitute for \( \pi^{(j(N))} \) the projector coupling \( \vec{K}^{(N)} \) with \( -\vec{L}^R + \frac{\hat{n}}{2} \) to give maximum angular momentum. This projector for \( l \to \infty \) gives the projector \( \frac{1+\hat{n}}{2} \), the \( N \) giving no contribution. (We also get the condition \( \vec{x}_R \to -\hat{n} \)). So \( \mathcal{P}^{(-N)} \pi^{(j(-N))} \) as \( l \to \infty \) can be identified with \( \mathcal{P}^{(-N)} \frac{1+\hat{n}}{2} \) and that too has the \( U(1) \) stability group at each \( \vec{n} \). This shows that \( \mathcal{P}^{(-N)}(u) \) is a good fuzzy \( \sigma \)-field for winding number \( -N \).

## 6 Dynamics and Continuum Limit for Fuzzy \( \sigma \)-Models

The simplest action for the \( O(3) \) nonlinear \( \sigma \)-model on \( S^2 \) is

\[
S = \frac{\beta}{2} \int \frac{d\cos \theta d\phi}{4\pi} (\mathcal{L}_i \Phi_a)(\vec{n})(\mathcal{L}_i \Phi_a)(\vec{n}),
\]

\[
\sum_{a=1}^{3} \Phi_a(\vec{n})^2 = 1, \quad \beta > 0
\] (26)

where \(-i\mathcal{L}_i\) are the angular momentum operators on \( S^2 \). It fulfills the important bound [16]

\[
S \geq \beta N
\] (27)

where \( N(\geq 0) \) or \( -N \) is as usual the winding number of the map \( \vec{\Phi} : S^2 \to S^2 \):

\[
\text{Winding number of } \vec{\Phi} = \frac{1}{2} \int_{S^2} \frac{d\cos \theta d\phi}{4\pi} \epsilon_{ijk} n_i \epsilon_{abc} \Phi_a(\mathcal{L}_j \Phi_b)(\mathcal{L}_k \Phi_c).
\] (28)
This bound is obtained by integrating the inequality
\[
(L_i \Phi_a \pm \epsilon_{ijk} n_j \epsilon_{abc} \Phi_b \Phi_k) \geq 0
\] (29)
and is saturated if and only if
\[
L_i \Phi_a \pm \epsilon_{ijk} n_j \epsilon_{abc} \Phi_b \Phi_k = 0
\] (30)
for one choice of sign. The solutions of (30) can be thought of as two-dimensional instantons [16].

We now propose a fuzzy $\sigma$-action using these properties of $S$ as our guide. Consider the inequality
\[
([F, P(u)] \frac{1 \pm \epsilon}{2} P(u))^\dagger ([F, P(u)] \frac{1 \pm \epsilon}{2} P(u)) \geq 0
\] (31)
where $P(u)$ can be $P^{(N)}(u)$ or $P^{(-N)}(u)$ and $Q \geq 0$ here means that $Q$ is a nonnegative operator. This is the analogue of (29). Taking trace, we get the analogue of (27),
\[
s_F \equiv Tr P(u)[F, P(u)] [F, P(u)] \geq N.
\] (32)
The bound is saturated if and only if
\[
[F, P(u)] \frac{1 \pm \epsilon}{2} P(u) = 0
\] (33)
for one choice of sign, just like in (30). All this suggests the novel fuzzy $\sigma$-action
\[
S_F = \beta_F s_F.
\] (34)

Qualitative remarks about the approach to continuum of $S_F$ will now be made. The first is that $\beta_F$ and $m$ must be scaled as $l \to \infty$. As regards the scaling of $\beta_F$, we conjecture that (33) has no solution for finite $l$ and that (32) is a strict inequality. Choose
\[
\Lambda(l) = \frac{1}{N} \times (\text{Minimum of } s_F)
\] (35)
so that $\frac{s_F}{\Lambda(l)}$ is $N$ at minimum. Then we suggest that we should set
\[
\beta_F = \frac{\beta}{\Lambda(l)}.
\] (36)
It is our conjecture too that $\Lambda(l)$ diverges as $l \to \infty$ in such a way that (upto factors)

$$
S_F \to S_\infty = \beta Tr_{\omega} P_\infty(g)[F, P_\infty(g)][F, P_\infty(g)],
$$

$$
\mathcal{F} = \begin{pmatrix} 0 & \mathcal{D}/|\mathcal{D}| \\ \mathcal{D}/|\mathcal{D}| & 0 \end{pmatrix},
$$

$$
g = \lim_{l \to \infty} u,
$$

$$
P_\infty(g) = \lim_{l \to \infty} P(u)
$$

(37)

where we have let $m$ become zero as $\mathcal{D}$ has no zero mode. An alternative form of $S_\infty$ is

$$
S_\infty = \beta \int \frac{d \cos \theta d \phi}{4\pi} Tr P_\infty(g) \left[ \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{pmatrix}, P_\infty(g) \right] \left[ \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{pmatrix}, P_\infty(g) \right]
$$

(38)

where the trace $Tr$ is only over the internal indices.

We now argue that $P(u)$ itself must be corrected by cutting off all high angular momenta (and not just the top one) while passing to continuum. Thus it was mentioned before that state vectors with top “total” angular momentum $j^{(\pm N)}$ are unphysical. Their characteristic feature is their divergence as $l \to \infty$. That means that once normalized these vectors become weakly zero in the continuum limit. In fact any sequence of vectors with a linearly divergent $j$ as $l \to \infty$ is unphysical. Such $j$ contribute eigenvalues to the Dirac operator which are nonexistent in the continuum, as one can verify using the results of [6, 7] for $N = 0$: the spectrum of $\mathcal{D}$ then is $\pm (j + \frac{1}{2})[1 + (1 - (j + \frac{1}{2})^2/(4l(l + 1))]^{1/2}$ while that of $\mathcal{D}$ is $\pm (j + \frac{1}{2})$, $j$ being total angular momentum. The corresponding eigenvectors too if normalized are weakly zero in the $l \to \infty$ limit. It seems necessary therefore to eliminate them in a suitable sense during the passage to the limit.

One way to do so may be to use a double limit which we now describe. Let $\pi^{(J)}$ be the projection operator for all states with $j \geq J$. Let us define

$$
P^{(\pm N)(J)} = \begin{pmatrix} 1+\gamma \frac{1}{2} p^{(\pm N)}(1-\pi^{(J)}) & 0 \\ 0 & 1-\gamma \frac{1}{2} p^{(\pm N)}(1-\pi^{(J)}) \end{pmatrix},
$$

$$
P^{(\pm N)(J)}(u) = u^{\tilde{\otimes}N} P^{(\pm N)(J)}[u^{\tilde{\otimes}N}]^{-1}.
$$

(39)

We then consider the fuzzy $\sigma$-model with $P^{(\pm N)(J)}(u)$ replacing $P^{(\pm N)}(u)$ and thereby cutting off angular momenta $\geq J$. That would
not affect index theory arguments so long as \( J > \frac{N-1}{2} \) as the important zero modes will then be left intact. We are thus led to the cut-off action

\[
S_F^{(J)} = \frac{\beta}{\Lambda^{(J)}(l)} s_F^{(J)},
\]

\[
s_F^{(J)} = Tr P^{(\pm N)(J)}(u)[F, P^{(\pm N)(J)}(u)][F, P^{(\pm N)(J)}(u)],
\]

\[
\Lambda^{(J)}(l) = \frac{\text{Minimum of } s_F^{(J)}}{N},
\]

and the following suggestion: A good way to define the continuum partition function is to let \( l \) and \( J \to \infty \) in that order in the partition function of \( S_F^{(J)} \).

Thus we propose the continuum partition function

\[
Z = \lim_{J \to \infty} \lim_{l \to \infty} \int d\mu \exp(-S_F^{(J)}),
\]

\( d\mu \) denoting the functional measure. The inner limit recovers the continuum where the contributions of vectors with divergent \( J \) should not matter, for this reason this method may eliminate the influence of unwanted modes from \( Z \). Perhaps an equivalent limiting procedure would be let \( l, J \to \infty \) with \( J/l \to 0 \).

Taking the limit \( l \to \infty \) with fixed \( J \) is compatible with the continuum description of the \( \sigma \)-field. In that limit, \( p^{(\pm N)} \) becomes \( P^{(\pm N)} \). Next consider the vectors projected by \( p^{(\pm N)}[1 - \pi^{(J)}] \). The effect of the last factor on the projected vectors is as follows: For \( \gamma = 1 \) say, we must combine the angular momentum value \( l \pm \frac{N}{2} \) of \( \vec{K}^{(N)} \) with the value \( l + 1/2 \) of \( -\vec{L}^R + \frac{\pi}{2} \) to produce an allowed value \( j < J \) of any such vector. So \( [\vec{K}^{(N)} + (-\vec{L}^R + \frac{\pi}{2})]^2 = j(j+1), \]

\[
[\vec{K}^{(N)}]^2 = (l+\frac{N}{2})(l+\frac{N}{2}+1) \quad \text{and} \quad (-\vec{L}^R + \frac{\pi}{2})^2 = (l+\frac{1}{2})(l+\frac{3}{2}).
\]

Letting \( l \to \infty \), we find that \( \vec{x}^L, \vec{x}^R \to 0 \) due to the factor \([1 - \pi^{(J)}]\), where we have used the fact that \( \frac{\pi^{(i)}}{l} \) and \( \frac{\pi^{(i)}}{l} \to 0 \) as \( l \to \infty \). But this is just a rule instructing us to set \( \vec{x}^R = \hat{n} \) for large \( l \) for these vectors, and will not show up in the continuum projector.

The \( \gamma = -1 \) case is no different in the continuum limit. Thus for \( l \to \infty \), \( p^{(\pm N)}(u)[1 - \pi^{(J)}(u)] \) can be interpreted as \( P^{(\pm N)}(g) = g^{\otimes N} P^{(\pm N)}[g^{\otimes N}]^{-1} \), the continuum \( \sigma \)-fields.

Let

\[
P^{(\pm N)(J)}_\infty (g) = \lim_{l \to \infty} P^{(\pm N)(J)}(u) = \begin{pmatrix}
\frac{1+\Gamma}{2} P^{(\pm N)}(g) & 0 \\
0 & \frac{1-\Gamma}{2} P^{(\pm N)}(g)
\end{pmatrix}.
\]

Then the naive \( l \to \infty, m \to 0 \) limit of \( S_F^{(J)} \) is expected to be (upto factors)

\[
S_\infty = \beta Tr_\omega P^{(\pm N)(J)}_\infty (g)[\mathcal{F}, P^{(\pm N)(J)}_\infty (g)][\mathcal{F}, P^{(\pm N)(J)}_\infty (g)]
\]

(43)
which can be simplified to
\[
S_{\infty} = \beta \int \frac{d\cos \theta d\phi}{4\pi} Tr \mathcal{D}^{(\pm)N}(g)[\mathcal{D}, \mathcal{P}^{(\pm)N}(g)][\mathcal{D}, \mathcal{P}^{(\pm)N}(g)].
\] (44)

It seems to correspond to (\[26\]).

7 Remarks

- The Dirac operators in (1) and (7) differ from those of [6,7] by unitary transformations generated by \(\Gamma\) and \(\gamma\).

- Nonabelian monopoles such as the elementary \(U(2)\) monopoles and their fuzzy versions can very likely be accommodated in our approach using different projectors.

- In the same manner, there seems to be no big barrier to studying the case of Grassmannians \(G_{n,k}(\mathbb{C}) = U(n+k)/[U(n)\times U(k)]\) as target spaces in \(\sigma\)-models as they are orbits of rank \(n\) projectors under \(U(n+k)\). We can also imagine treating other target spaces by considering orbits under subgroups of \(U(n+k)\), the previous choice of \(\{g^{\otimes N}(\vec{n})\}\subset U(2N)\) being an example.

- We have managed to generalize the approach here to fuzzy manifolds like fuzzy \(\mathbb{C}P^2\) (\[17\]) or more generally to fuzzy versions of orbits of simple Lie groups in the adjoint representation. We will report on this work elsewhere.

- Further discussion of fuzzy quantum physics of monopoles and solitons is needed to better reveal the implications of fuzzy quantum physics in its topological aspects.

- The fuzzy Dirac operator \(D'\) used in \[3,4\] gives a much better approximation to the spectrum of the continuum Dirac operator. Its eigenvalue for the top angular momentum state vector is largest in modulus, recedes to infinity with \(l\) and is not a zero mode as in the case of \(D\). The contribution of this vector therefore tends to be suppressed in functional integrals. In \[18\], a chirality operator \(\gamma'\) for \(D'\) (with the correct continuum limit) has been constructed after projecting out this vector.
The contents of the present paper can be easily recast using $D'$ and $\gamma'$. Ref. [19] discusses $\theta$-states and chiral anomalies in gauge theories of fuzzy physics using these new operators.

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