Ising model on the Apollonian network with node dependent interactions

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This work considers an Ising model on the Apollonian network, where the exchange constant $J_{i,j} \sim 1/(k_i k_j)^\mu$ between two neighboring spins $(i, j)$ is a function of the degree $k$ of both spins. Using the exact geometrical construction rule for the network, the thermodynamical and magnetic properties are evaluated by iterating a system of discrete maps that allows for very precise results in the thermodynamic limit. The results can be compared to the predictions of a general framework for spins models on scale-free networks, where the node distribution $P(k) \sim k^{-\gamma}$, with node dependent interacting constants. We observe that, by increasing $\mu$, the critical behavior of the model changes, from a phase transition at $T = \infty$ for a uniform system ($\mu = 0$), to a $T = 0$ phase transition when $\mu = 1$: in the thermodynamic limit, the system shows no exactly critical behavior at a finite temperature. The magnetization and magnetic susceptibility are found to present non-critical scaling properties.

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I. INTRODUCTION

Magnetic models on complex networks have quite distinct behavior from those on regular lattices [1]. Their properties are of far greater importance than just a mathematical curiosity, since they establish landmarks for the behavior of many systems, like social, economic, and communication networks. For such systems, the understanding of the conditions leading to a phase transition, or a sudden collective change in the behavior of the agents, is of utmost importance to avoid a breakdown of social structures or collective current day technological facilities [2, 3].

The absence of a finite temperature phase transition in the thermodynamic limit $T \to \infty$, where $N$ is the number of nodes [4, 5], concomitant with the presence of a finite degree of magnetic ordering, stays among the first results that have been obtained for Ising models on the standard Barabasi-Albert (BA) scale-free network [6], where the exponent of the node distribution $P(k) \sim k^{-\gamma}$ assumes the value $\gamma = 3$. It was also observed that finite temperature critical behavior is found when $\gamma \in (3, 5]$, while, for $\gamma > 5$ the critical behavior collapses at $T = 0$. Later, an interesting interplay between critical behavior and node dependent interaction constants has been evidenced [7, 8]: if the strength of interactions in a BA network, with a given value $\gamma$, is non-uniformly reduced according to

$$J_{i,j} = J_0/(k_i k_j)^\mu,$$

where $k_\ell$ is the degree of node $\ell$, the critical behavior moves into the universality class of the uniform model

with a different value $\gamma'$. This makes it possible, for instance, to devise models in the standard BA network that undergo finite temperature phase transition. An analytic expression

$$\gamma' = (\gamma - \mu)/(1 - \mu)$$

has been derived based on scaling arguments but, although it has been numerically verified for BA networks, it is not known whether its validity extends to other networks.

The purpose of this work is to investigate the effect of a node dependent coupling constant on the properties of an Ising model on the Apollonian network (AN) [8, 10]. This network has very special features [11, 12], including presenting a power law distribution of node degrees, with exponent $\gamma \simeq 2.58$. Previous results for Ising models on the AN have shown that, for a variety of situations where both ferro- and antiferromagnetic interactions are allowed, phase transition in the thermodynamic limit occur only at $T = \infty$ [8, 13]. AN’s are constructed according to precise geometrical rules, which lead to exact self similar patterns and scaling properties. They are also amenable to mathematical analysis based on renormalization or inflation methods, as the transfer matrix (TM) formalism we will use here, which allow for the evaluation of its properties in the thermodynamic limit.

These facts turn this model particularly suited for testing the existence of a finite temperature phase by modulating the coupling constants according to Eq. (1). On the other hand, since the AN geometric rules lead to a well defined value of $\gamma$, there is no general free parameter we can use in the study to verify the validity of Eq. (2). Further, it must be stressed that, despite the fact that AN displays power law distribution of node degree, it differs substantially from BA network with respect to other topological properties, as the existence of many closed
loops. This is expressed, among other measures, by the clustering coefficient $C$, which is very high ($\sim 0.85$) for AN and very small ($\sim 1/N$) for the BA \cite{10,14}.

The rest of this paper is organized as follows: Section 2 introduces the basic properties of AN networks and of the proposed model; details of the used TM scheme to evaluate the thermodynamical properties are discussed in Section 3. We discuss our main results in Section 4, emphasizing the emergence of a cross-over in $\mu$. Finally, Section 5 closes the paper with our concluding remarks.

II. APOLLONIAN NETWORK AND MODEL

AN’s have been recently introduced in the complex network framework \cite{9,10}, although the original concepts can be traced back to ancient Greece, where the problem of optimally filling two and three dimensional spaces with circles and spheres has been studied by Apollonius of Perga \cite{14}. The complex solution to this problem, which amounts to placing tangent structures with well defined radii at precise centers, suggests the far simpler problem of constructing the AN. In this case, one just has to put a node in each circle center, and a network edge between the centers of each pair of tangent circles. This process can be followed in a recursive way in terms of the generation $g$ in which new circles are added to the structure. In this work, we consider that, at the zeroth generation $g = 0$, three tangent circles with the same radius occupy the centers of an equilateral triangle (see Fig. 1). For the $g + 1$-th generation, the network construction consists in putting a node within each triangle of the $g$-th generation, and connecting it to each of the triangle corners. It is a simple matter to verify that the number of network nodes $N(g)$ and edges $B(g)$ increase according to, respectively, $N(g) = (3^g + 5)/2$ and $B(g) = (3^{g+1} + 3)/2$. The average number of neighbors per node equals 6, since $B(g)/N(g) \to 3$ in the limit $g \to \infty$.

For a given generation $g > 0$, the largest node degree is $k_c(g) = 3 \times 2^{g-1}$, where the subscript $c$ indicates that such node occupies the central network position. The second largest degree nodes, with $k_e(g) = 2^g + 1$, occupy the external corners. At any generation $g$, there will be nodes with degree $k = k_e(g)$, $g = 1, \ldots, g$ and $k_c$. The degree dependent node multiplicity is $m(g) = 3^g - 3$ for the internal nodes, and $m(g) = 3 \forall g$, for the nodes at outer network corners.

As already quoted, the resulting AN is scale free. However, it also has other properties that are typical for other complex network classes, as being small world (mean minimal path $(\ell) \sim \ln N$), hierarchical (the clustering coefficient of individual nodes $c(k)$ has a power law dependence on $k$), and having a large clustering coefficient $C$. Because of this, systematic network clustering analysis based on several independent measures \cite{15} shows that AN does not belong to the same class as the most studied network sets, generated by the algorithms proposed by Watts and Strogatz \cite{16} and Barabasi and Albert \cite{5}.

We consider the Ising model with spins $\sigma_i = \pm 1$ placed on each site of the Apollonian network. Pairs of spins $(i, j)$, which are neighbors on the network, interact with coupling constants $J_{i,j}$. Thus, the Hamiltonian for the system can be written as

$$H_g = -\sum_{(i,j)} J_{i,j} \sigma_i \sigma_j - h \sum_i \sigma_i,$$

where $J_{i,j}$ is given by Eq. (1). In our previous studies, we have considered inhomogeneous models, in which the constants $J_{i,j}$ depend on the generation $g$ at which the edge, hence the second spin in the pair, was introduced into the network. Due to the fact that, at each generation, the newly introduced nodes are connected to nodes that were introduced in previous generations, the scheme introduced in Ref. \cite{13} does not assign the values of $J_{i,j}$ according to the rule of Eq. (1). In the following Section we discuss how to implement the interaction constants of Eq. (1) in connection with the TM method used to evaluate the model properties.

III. TM RECURRENCE MAPS

The basic steps to implement the TM method we use to evaluate the thermodynamic properties have been presented, with some detail, in one of our previous works \cite{13}. However, the method needs to be adapted to the specific situation introduced by the more complex interaction given in Eq. (1). Thus, let us briefly recall that the TM scheme amounts to write down the partition function $Z(T, h, N(g))$, where $H$ denotes the magnetic field, for any value of $g$ in terms of a TM that describes the interactions between any two of the outer AN sites. In this process, it is necessary to perform a partial trace over all interaction dependent configurations. Due to the exact
geometric AN construction rule, it is possible to express the TM matrix elements at generation \( g + 1 \) in terms of the corresponding elements at generation \( g \). In this framework, we basically work with a set of \( 2 \times 2 \) square matrices

\[
M_g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix},
\]

and a set of non-square auxiliary matrices

\[
L_g = \begin{pmatrix} p_g & q_g & r_g & s_g \\ t_g & u_g & v_g & w_g \end{pmatrix},
\]

which explicitly include the dependence of the third outer node spin variable. As the \( L_g \) matrix elements are numbered according to the lexicographic order, the following relations hold: \( a_g = p_g + q_g \), \( b_g = r_g + s_g \), \( c_g = t_g + u_g \), \( d_g = v_g + w_g \). For more symmetrical models, and field independent situations, the number of independent variables can be reduced.

For the homogeneous systems, it was possible to write down a single set of recurrence relations between matrix elements in successive generations. Although the basic idea of the method remains the same, for the current model, it is necessary to track the way the nodes are reconnected when they go from \( g \) to \( g + 1 \). This influences the change in their degrees, so that the same node will contribute differently for distinct values of \( g \).

We start the discussion of the changes in the TM scheme by pointing out that, besides knowing the set of node degrees and corresponding degree (node) multiplicity, it is necessary to go one step further, and identify each of the \( P(g) = (g^2 - g + 2)/2 \) different triangles in which the \( g \) network can be disassembled. In this respect, each triangle is characterized by the node degrees \( k_i, k_j, \) and \( k_\ell \) of the nodes \( i, j, \) and \( \ell \), respectively. For any triangle and any \( g > 1 \), there is always (only) one node with \( k = 3 \). Note that \( P(g) \) grows only with the square of \( g \), so that, even for a complex interaction structure, there is practically no constraint to numerically compute these matrix elements for very large values of \( g \).

Once this set has been identified, we evaluate the model properties at generation \( g \) by computing the contribution to the partition function from each of these \( P(g) \) triangles, storing them in corresponding TM’s \( L_g^i \), \( i = 1, \ldots, P(g) \).

To proceed further, we must consider that the \( g \) evaluation is equivalent to the \( g - 1 \) one, provided we start with triangular units with a fourth node added at the central position. This way, it is possible to compute the contribution of the new \( P(g-1) \) triangles, by performing partial trace over the contributions from the central node of each of these structures. The new form of the general recurrence relations for the matrix elements

\[
(L_{g+1})^\alpha_{i,k} = \sum_\ell (L_g)^\eta_{i,j\ell}(L_g)^\epsilon_{i,\ell k}(L_{g+1})^\kappa_{j,\ell i},
\]

is quite similar to that of the uniform model. The difference refers to the superscripts \( \eta, \epsilon, \) and \( \kappa \), which identify which three TM’s (corresponding triangles) have been put together. The same arguments can be used again, until we obtain one single TM that accounts for the contributions of all network nodes.

The results we present in the next Section consider \( g \leq 50 \) which, for the largest value, is roughly of the order of magnitude of the Avogadro number. The adaptation of the uniform TM procedure to take into account the node dependent interaction constant depends basically on the identification of the basic triangular units and the assembling rules that combine them when one goes from \( g \) to \( g + 1 \). A summary of the implementation of the details is provided in the Appendix.

Finally, it is important to note that the map iterations can be more conveniently performed if we rewrite the set of recurrence maps given by Eq. 6 in terms of the free energy and the ratio of the \( L_g \) matrix elements to the largest one \( (q_g) \). Indeed, this avoids numerical divergences, in the low temperature region, when \( g \) increases, as conveniently discussed in Ref. 13.

**IV. RESULTS**

According to the previous Section, we present results for fixed number \( g \) of generations, usually \( g = 10, 20, 30, 40, \) and \( 50 \), corresponding to networks with \( 2.953 \times 10^4, 1.743 \times 10^5, 1.030 \times 10^{14}, 6.079 \times 10^{19} \) and \( 3.590 \times 10^{23} \) sites, respectively. The precise numerical evaluation of the free energy \( f(T, h) \) allows to obtain the entropy \( s(T) \), specific heat \( c(T) \), magnetization \( m(T, h = 0) \), and susceptibility \( \chi(T, h = 0) \). It is also possible to calculate the ratio \( \lambda_1/\lambda_2 \) of the two TM eigenvalues. For models on Euclidian lattice, as well as on hierarchical and several fractal structures, this quantity is directly related to the correlation length \( \xi \). In the case of complex networks and, in particular, of the AN, the connection between these quantities is not so obvious, as the distance between two outer nodes remains always 1.
in any generation. Therefore, we will discuss the behavior of $1/\ln(\lambda_1/\lambda_2)$, although we refrain ourselves from calling it $\xi$.

Eq. (1) indicates that the coupling constants $J_{i,j}$ linearly depend on $J_0$. According to Section 2, the number of connections in the AN at generation $g$ is $L_g = (3^{g+1} + 3)/2$. If we take $J_0 = 1$ when $\mu = 0$, the free energy per spin in the $g \to \infty$ limit is $u_0 = -3$ at $T = 0$. If we fix $J_0 = 1$ and let $\mu$ increase, the value of $u_0$ decreases and, besides that, all thermodynamic effects will occur at a lower value of $T$. Thus, to avoid choosing an adequate temperature scale to work with at each value of $\mu$, we find it more convenient to choose a $\mu$ dependent value $J_0(\mu)$, by requiring that $u_0 = -3, \forall \mu$. In Fig. 2, we show the dependence of $J_0(\mu)$ on $\mu$, which shows that $J_0 \sim \exp(\mu)$. As a consequence of this choice, all maxima of the specific heat occur roughly at the same value of $T$.

Fig. 3 shows, for $g = 40$ and 50 and $\mu = 0, 0.5, 1,$ and 3, that the curves for $c$ are rather insensitive to the values of $g$ and $\mu$. Moreover, they are completely smooth, with a Schottky like maximum at a temperature $T_s$. This constitutes a main difference to the results for the $\gamma = 3$ BA networks [7]. There is reported the presence of a finite critical temperature, identified by a jump in the specific heat when $\mu > 0.5$, which changes into having a diverging slope when $0.5 > \mu > 0.33$. According to Eq. (2) and to the AN known value of $\gamma \approx 2.58$, similar critical behaviors should emerge for $\mu > 0.61$ and $0.61 > \mu > 0.47$, if the AN were to fall within the BA universality class. This clearly shows that the validity of Eq. (2) can not, in general, be extended from the BA to other network classes, even if they are scale free as the AN.

The same calculations reveal that the resulting patterns for $m, \chi$ and $1/\ln(\lambda_1/\lambda_2)$ depend, first, on the generation $g$, and further on whether $0 < \mu < 1$, $\mu = 1$ and $\mu > 1$. So it is adequate to discuss them separately.

**A. $0 < \mu < 1$**

Within this parameter interval, we observe that, like for the uniform model, $1/\ln(\lambda_1/\lambda_2)$ numerically diverges ($> 10^{100}$) for a non-zero temperature $T_d(g)$, which increases linearly with $g$. Since $g \sim \ln N$, $T_d$ depends in a logarithmic way on the system size. If we write $T_d(g) \sim A(\mu)\ln N$, we find that $A(\mu)$ decreases with $\mu$ (see Fig. 4).

In Fig. 5, the behavior of the zero field magnetization $m(T, h = 0, N(g))$, which is exactly one when $T = 0$, slowly decreases when $T$ increases. Its behavior when $\mu = 0.5$, at larger values of $T$, is a bit more complex than that for $\mu = 0$ (Fig. 5a). There it is clear that $m$ suffers a first cross-over to an exponential decay at $T_s$, which is followed by a transition to a second exponential decay, mediated by a larger constant, at $T_m(g)$. The magnetization curves for different $g$ collapse during the first and second regimes. The third regime will later on be interrupted again by a smoother decay. As observed with $T_d(g), T_m(g) \sim B(\mu)g$, with $B(\mu) \sim A(\mu)$. However, $T_d$ and $T_m$ do not coincide. The second part of the magnetization curves, where $m(T, 0, N(g))$ overlaps for different values of $g$, extends over wider $T$ intervals when $g$ increases. This shows that, in the thermodynamical limit $g \to \infty$, the value of $m$ will follow the second exponential decay when $T \to \infty$. Nevertheless, as observed for $T_d$, this region grows logarithmically with the network size.

The behavior of $\chi$ is strongly correlated with that of $m$. It vanishes when $T \to 0$, then it grows with $T$, shows a first maximum at a $g$ independent $T_s$, and a second $g$ dependent maximum at $T_m$. As for $m$, the curves for larger values of $g$ overlap for much larger distances. The maxima of the $\chi$ curves are described by an universal
function, as can be observed in the very precise re-scaled curves in Fig. 6a, which shows that the scaling exponents increase with $\mu$. Note that only the value of $\chi$ needs to be scaled by the corresponding maxima, while the location at the temperature axis is corrected by shifting the scale by $T_m(g)$. This excludes any possibility of having a critical phenomenon associated with susceptibility maxima.

### B. $\mu = 1$

This value of $\mu$ determines a crossover in the behavior of the system, which is reflected both in $m$ and $\chi$. This change can be noticed in Fig. 3, which shows that $A(\mu = 1) = B(\mu = 1) = 0$, i.e., the temperatures associated with the maxima of the susceptibility and the divergence of $1/\ln(\lambda_1/\lambda_2)$ become independent of the system size. The precise value of $T_d$ depends, of course, on the threshold value of the numerical divergence. However, by plotting the value of $1/\ln(\lambda_1/\lambda_2)$ as function of $1/T$, we notice a linear dependence in the $T \to 0$ limit, suggesting that $T_d = 0$.

However, this new behavior cannot be associated with the emergence of criticality. First we recall that Fig. 3 does not indicate any change in the Schottky profile and, second, we see that $T_m > T_s$. Finally, the $\chi$ curves in the region around $T_m$, which shows a perfect scaling with respect to $g$ with scaling exponent 1, are completely smooth (see Fig. 6b). Note that the horizontal axis indicates that it is not necessary to shift temperature as in Fig. 6a. Note that the two maxima, which were observed when $\mu < 1$, have merged together, and that the large temperature side is characterized by an exponential decay.

As for the previous $\mu$ interval, the behavior of $m$ is strongly correlated with that for $\chi$. It is characterized by a single exponential decay after $T_m$, with a very large constant, as shown in Fig. 5b.

### C. $\mu > 1$

In the last range of parameter values, $T_d$ and $T_m$ decrease with respect to $g$. As shown by Fig. 4c, both values converge exponentially to 0 with respect to $g$. Fig. 4c also shows that the rate in the exponential increases with $\mu$.

Therefore, the behavior of $1/\ln(\lambda_1/\lambda_2)$ is different from that one observed for $\mu = 1$, the divergence when $T \to 0$ becoming slower at increasing values of $g$. This suggests that, when $g \to \infty$, any collective spin ordering is weaker than that of an Ising chain, rather typical for a paramagnetic situation.

The shape of the $m$ curves becomes completely different. The stable plateau at $m = 1$ for a finite temperature interval, which survived until $\mu = 1$, disappears as $g$ increases, indicating that no spontaneous magnetization occurs.
exists for a $T > 0$. This suggests that, in the thermodynamic limit, $m \equiv 0 \forall T > 0$. Our results do not allow to assert that $m \equiv 0$ also for $T = 0$.

The behavior of $\chi$ supports the conclusion of a low temperature paramagnetic phase. It increases very rapidly from 0 to a maximum value at $T_m$, followed by a $1/T$ law. Since $T_m$ goes exponentially to zero, a Curie law prevails for large $g$. This is shown by the scaled curves in Fig. 6c, which indicate that the Curie constant $C$ depends on $g$.

V. DISCUSSION AND CONCLUSIONS

The results we obtained for the magnetic behavior of the Ising model with node dependent interaction constants reveal a quite rich picture, although no critical behavior at a finite temperature has been identified. The properties of specific heat show that the $g$ dependent curves converge very rapidly to a well defined value in the thermodynamic limit. On the other hand, magnetization and susceptibility indicate a much more complex behavior which, for certain temperature intervals, are heavily dependent on the value of $g$.

The TM method allows for the comparison of $m$ and $\chi$ for different values of $g$, which leads to the identification that part of the results are due to finite size events. The curves showing such effects are amenable to very precise collapsing by adequate scaling expressions, similar to critical points in magnetic models on Euclidian lattices. This includes the dependence of characteristic values of the temperature ($T_d$ and $T_m$).

The behavior of the system in the region $\mu < 1$ is close to that observed for magnetic system with uniform interactions on BA networks: only an ordered phase is observed at any value of $T$. $\mu = 1$ characterizes a crossover in the behavior of the system, as for $\mu < 1$ the magnetization vanishes, for any value of $T$, when $g \to \infty$. This region reveals a typical behavior of a genuine paramagnetic system. This picture is corroborated by the behavior of $\chi$, as one finds that a Curie law is valid in a limited region close to $T = 0$. For larger values of $T$, the decay is characterized by an exponential decay.

In the context of complex networks our most important finding is that the relation of Eq.(1) between effective topology and interaction strength proposed in Refs. [7,8] does not have general validity for all scale-free networks since the Apollonian case behaves differently.

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VII. APPENDIX

As discussed in the Section II, for any generation $g$, the node occupying the central position of the AN has the largest degree $k_c(g)$, the value of which results from the difference equation relating the values of $k_c$ at two successive generations: $k_c(g + 1) = 2k_c(g)$. The degree of the nodes at the external corners obeys a similar equation, namely: $k_c(g + 1) = 2k_c(g) - 1$.

The AN can be disassembled in triangles, in such a way that each node $i$ of degree $k_i$ belongs to $k_i$ triangles. The only exception refers to the nodes at the external corners, which have degree $k_c(g)$ but belong to $k_c(g) - 1$ triangles. Each such triangle can be characterized by the degree of its three nodes. The number $P(g)$ of different triangles at generation $g$ can be expressed in terms of $\tau(g)$ and $\rho(g)$, respectively the number of triangles that does not include (includes) an external node: $P(g) = \rho(g) + \tau(g)$. Since they obey the relations $\rho(g + 1) = \rho(g) + \tau(g) - 1$ and $\rho(g + 1) = \tau(g) + 1$, we obtain $\rho(g) = (g^2 - 3g + 2)/2$ and $\tau(g) = g$, from which the expression for $P(g)$ anticipated in Section III follows. The number $\rho(g)$ can be further decomposed in terms of $\nu(g)$ and $\kappa(g)$, respectively the number of different triangles that includes (does not include) the central node at generation $g$. It is a simple matter of inspection to see that $\nu(g) = g - 2$ and $\kappa(g) = (g^2 - 5g + 6)/2, \forall g \geq 2$.

For the purpose of computing the TM’s, it is necessary to identify the distinct triangles present in the AN. This proceeds by the collection $T_{g,\zeta}(k_1, k_2, k_3)$, where $g$ indicates the generation, $\zeta$ is a number in $[1, P(g)]$, and $k_i$ indicate the degrees of the nodes at the vertices of the triangle. $T_{g,\zeta}(k_1, k_2, k_3)$ are recursively defined according to the following rules:

1) $k_3 = 3, \forall g, \forall \zeta$.

2) Since $\tau(g) = g$, one single new triangle containing an external node is introduced into the network, $\forall g$. We use $\zeta = P(g - 1) + 1$ to characterize it, and note further that $k_1 = k_c(g), k_2 = k_c(g)$. Any such triangle retains the $k_c$ values for all further generations, i.e.:

$$T_{g,\zeta}(k_1, k_2, k_3) = T_{g,\zeta'}(k_1, k_2, k_3), \forall \zeta' \geq \zeta.$$  (7)

3) $\forall g$, there are $\rho(g)$ different triangles, among which $\kappa(g)$ have been introduced in previous generations. They will be characterized by the same values of $k_i$, so that Eq. (7) also holds for this subset. The remaining $\nu(g)$ new triangles are numbered according to the rule: $\zeta = P(g - 1) + \ell, \ell = 2, 3, ..., g - 1$. For each value of $\ell$, we set $k_1 = k_c(g), k_2 = k_c(\ell)$.

The final step consists in establishing the rule to combine the contributions to the partition function from three distinct triangles at generation $g$ to obtain the partition function at generation $g + 1$ according to Eq. (10). If the properties of the systems are to be computed until a chosen value $g$, we are required to start with $P(g)$ distinct triangles, precisely identified as discussed above. Then, as discussed in Section III, it is necessary to define a map that selects the proper values of $\eta, \epsilon, \delta \in [1, P(g)]$ used to perform the trace over the common central node of the triangle $\alpha \in [1, P(g - 1)]$. So let us note that

$$[1, P(g)] = [1] \bigcup \left\{ g \left| \begin{array}{c} \cup \{P(g - 1) + 1, P(g - 1) + j - 1 \} \end{array} \right. \right\}. \quad (8)$$

Then, the values of $\eta, \epsilon, \delta$ are given, as function of $\alpha$, by the following expressions:

$$\alpha = 1 :$$
$$\eta = 1, \epsilon = \delta = 2;$$
$$\alpha = P(j - 1) + 1, j \in [2, g - 1] :$$
$$\eta = 2, \epsilon = P(j - 1) + 1, \delta = P(j) + 2;$$
$$\alpha \in [P(j - 1) + 2, P(j - 1) + j - 1], j \in [2, g - 1] :$$
$$\eta = P(\alpha - P(j - 1)) + 2, \epsilon = P(j) + 2, \delta = \alpha + j.$$  (9)

As the AN is self similar, these maps are also valid for all forthcoming partial trace operations, until only one single triangle is left. At this step, the remaining TM contains the contribution form all spin configurations, from which the thermodynamical properties follow.

With the help of these relations, a set of recurrence maps can derived from Eq. (10), which allow for the evaluation of the free energy and its derivatives:

$$f_{g+1}^{\alpha} = \frac{\sum_{\nu(g)}^{T} \{f_{g}^{\nu(g)} + f_{g}^{\nu(g) + 2} - f_{g}^{\nu(g) + 2} \} - \sum_{\kappa(g)}^{T} \ln(1 + x^{\nu(g)}_{1, g} x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h))}{N_{g+1}}; \quad (10)$$

$$x_{1, g+1}^{\alpha} = \frac{x^{\nu(g)}_{2, g} x^{\nu(g)}_{2, g} + x^{\nu(g)}_{1, g} x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h))}{1 + x^{\nu(g)}_{1, g} x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h));} \quad (11)$$

$$x_{2, g+1}^{\alpha} = \frac{x^{\nu(g)}_{2, g} x^{\nu(g)}_{2, g} + x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h))}{1 + x^{\nu(g)}_{1, g} x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h));} \quad (12)$$

$$x_{3, g+1}^{\alpha} = \frac{x^{\nu(g)}_{2, g} x^{\nu(g)}_{2, g} + x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h))}{1 + x^{\nu(g)}_{1, g} x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h));} \quad (13)$$

$$x_{4, g+1}^{\alpha} = \frac{x^{\nu(g)}_{2, g} x^{\nu(g)}_{2, g} + x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h))}{1 + x^{\nu(g)}_{1, g} x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h));} \quad (14)$$

$$x_{5, g+1}^{\alpha} = \frac{x^{\nu(g)}_{2, g} x^{\nu(g)}_{2, g} + x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h))}{1 + x^{\nu(g)}_{1, g} x^{\nu(g) + 2}_{1, g} x^{\nu(g) + 2}_{1, g} \exp(-2\beta h));} \quad (15)$$
\[ x_{7,g}^{\alpha+1} = \frac{x_{6,g} x_{6,g} x_{6,g} + x_{7,g} x_{7,g} x_{7,g} \exp(-2\beta h))}{1 + x_{1,g} x_{1,g} x_{1,g} \exp(-2\beta h))}. \]  

(16) \[ x_{4,g} = \frac{t_g}{p_g}; \quad x_{5,g} = \frac{u_g}{p_g}; \quad x_{6,g} = \frac{v_g}{p_g}; \]  

(18) \[ x_{7,g} = \frac{w_g}{p_g}. \]  

(19) In the above relations, the following variables have been used:

\[ x_{1,g} = \frac{q_g}{p_g}; \quad x_{2,g} = \frac{r_g}{p_g}; \quad x_{3,g} = \frac{s_g}{p_g}; \]  

(17) \[ x_{1,g} = \frac{q_g}{p_g}; \quad x_{2,g} = \frac{r_g}{p_g}; \quad x_{3,g} = \frac{s_g}{p_g}; \]  

(17)

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