Matsushima-Lichnerowicz type theorems of Lie algebras of automorphisms of generalized Kähler manifolds of symplectic type

Ryushi Goto

November 25, 2021

Abstract

In Kähler geometry, Fujiki–Donaldson show that the scalar curvature arises as the moment map for Hamiltonian diffeomorphisms. In generalized Kähler geometry, one does not have good notions of Levi-Civita connection and curvature, however there still exists a precise framework for the moment map and the scalar curvature is defined as the moment map [14]. Then a fundamental question is to understand the existence or non-existence of generalized Kähler structures with constant scalar curvature. In the paper, we study the Lie algebra of automorphisms of a generalized complex manifold \((M, J)\). We assume that \(H^1(M, \mathbb{C}) = 0\). Then we show that the Lie algebra of the automorphisms is a reductive Lie algebra if \((M, J)\) admits a generalized Kähler structure of symplectic type with constant scalar curvature. This is a generalization of Matsushima and Lichnerowicz theorem in Kähler geometry. We explicitly calculate the Lie algebra of the automorphisms of a generalized complex structure given by a cubic curve on \(\mathbb{C}P^2\). Cubic curves are classified into nine cases (see Figure.1 ∼ 9 in Section 7). In the three cases as in Figures. 7, 8 and 9, the Lie algebra of the automorphisms is not reductive and there is an obstruction to the existence of generalized Kähler structures of symplectic type with constant scalar curvature in the three cases. We also discuss deformations starting from an ordinary Kähler manifold \((X, \omega)\) with constant scalar curvature and show that nontrivial generalized Kähler structures of symplectic type with constant scalar curvature arise as deformations if the Lie algebra of automorphisms of \(X\) is trivial. We show the Hessian formula of generalized extremal Kähler structures and obtain the decomposition theorem of the Lie algebra of the reduced automorphisms of a generalized extremal Kähler manifold.

1 Introduction

It is known that the isometry group of a compact Riemannian manifold is a compact Lie group of finite dimension [28] and the automorphism group Aut\((X)\) of a compact complex manifold \(X\) is a finite dimensional complex Lie group [24]. The reduced automorphisms \(\text{Aut}_0(X)\) is defined to be the subgroup of \(\text{Aut}(X)\) which acts trivially on the Albanese torus \(H^0(X, \Omega^1)^\ast/H_1(X, \mathbb{Z})\). Matsushima and Lichnerowicz show that if a compact complex manifold \(X\) admits a Kähler metric with constant scalar curvature, then the Lie algebra of the reduced automorphisms \(\text{Aut}_0(X)\) is a reductive Lie algebra which
is the complexification of the isometry group \([26], [27]\). A generalized Kähler manifold is a successful generalization of the ordinary Kähler manifold. A generalized Kähler structure on a manifold is a triple \((g, I, J)\) consisting of a Riemannian metric \(g\) compatible with two complex structures \(I, J\) satisfying the certain conditions, which has an origin in a non-linear sigma model in Mathematical physics. However, a generalized Kähler structure has a natural description using the language of Hitchin’s generalized complex geometry \([21]\), which is a commutative pair \((J_1, J_2)\) of generalized complex structures equipped with a positivity condition \([17]\). The deformation-stability theorem of generalized Kähler structures shows that every holomorphic Poisson structure on a compact Kähler manifold gives rise to nontrivial deformations of generalized Kähler structures \([10], [11], [12], [13], [14], [20], [22]\).

In our previous paper \([14]\), the scalar curvature of a generalized Kähler manifold of symplectic type is defined as the moment map for Hamiltonian diffeomorphisms, which is a natural generalization of the moment map framework due to Fujiki and Donaldson in Kähler Geometry. In this paper, the existence and non-existence problems of generalized Kähler structures with constant scalar curvature are discussed. In order to obtain results of the non-existence, we define the Lie algebra of automorphisms of a generalized complex manifold and we introduce the Lie algebra of the reduced automorphisms of a generalized Kähler manifold. We show that the Lie algebra of reduced automorphisms is a reductive Lie algebra if there exists a generalized Kähler structure of symplectic type with constant scalar curvature (see Definition 2.8 and Section 6). This is a generalization of Matsushima-Lichnerowicz theorem, which gives an obstruction to the existence.

Regarding the existence questions, we discuss deformations starting from an ordinary Kähler manifold \((X, \omega)\) with constant scalar curvature and show that nontrivial generalized Kähler structures of symplectic type with constant scalar curvature arise as deformations if the Lie algebra of automorphisms of \(X\) is trivial (see Theorem 8.2) cf. \([25]\).

This paper is organized as follows. In Section 2, notations and preliminary results on generalized complex structures and generalized Kähler structures are explained. There are many good references and lecture notes on them (see for instance \([17], [19], [6], [23], [11]\)). In Section 3, we introduce generalized Hamiltonian diffeomorphisms for an arbitrary generalized complex structure. In particular, if a generalized complex structure comes from a real symplectic structure and a \(b\)-field, then the group of generalized Hamiltonian diffeomorphisms coincides with the group of Hamiltonian diffeomorphisms twisted by the \(b\)-field. In Section 4, we show that the set of compatible almost generalized complex structures with a fixed \(J_\psi\) is an infinite dimensional Kähler manifold. If the generalized complex structure \(J_\psi\) is locally given by \(d\)-closed nondegenerate, pure spinor, then there exists a moment map for the action of generalized Hamiltonian diffeomorphisms with respect to \(J_\psi\). Then we define the scalar curvature as the moment map. In Subsection 5.1, we introduce the Lie algebra \(\mathfrak{g}_J\) of automorphisms of a generalized complex manifold and the Lie algebra of the reduced automorphisms \(\mathfrak{g}_0\). In Subsection 5.2, we introduce a real Lie algebra \(\mathfrak{g}_R^\circ\), which can be regarded as the Lie algebra of generalized isometry group of a generalized Kähler manifold. We show that \(\mathfrak{g}_R^\circ\) is always a reductive Lie algebra. In Subsection 5.3, we show that the condition (5.13) implies \(\mathfrak{g}_0\) is the complexification of \(\mathfrak{g}_R^\circ\), which is then reductive. In Subsection 5.4, we obtain a structure theorem of \(\mathfrak{g}_J\) (see Theorem 5.12). In particular, we see that \(\mathfrak{g}_J \cong \mathfrak{g}_0\) if \(H^1(M, \mathbb{C}) = 0\) (Corollary 5.13). In Subsection 5.5, we show if \(H^1(M, \mathcal{O}) = 0\), then the Lie algebra \(\mathfrak{g}_J\beta\) is given by the Lie algebra of holomorphic vector fields preserving a holomorphic Poisson structure \(\beta\). In Section 6, we
prove one of our main theorems:

**Theorem 6.5.** Let \((M, J)\) be a \(2n\) dimensional compact generalized complex manifold. We assume that \(H^1(M, \mathbb{C}) = 0\). If \(M\) admits a generalized Kähler structure \((J, J_\psi)\) of symplectic type with constant scalar curvature, then the Lie algebra \(\mathfrak{g}_J\) is a reductive Lie algebra.

In fact, the condition \(H^1(M, \mathbb{C}) = 0\) implies that \(\mathfrak{g}_J = \mathfrak{g}_0\) and the existence of generalized Kähler structure of symplectic type with constant scalar curvature implies that \(\mathfrak{g}_0\) is the complexification of \(\mathfrak{g}_{\mathbb{R}}^0\) which is reductive (Theorem 6.4). Applying Theorem 6.5 to the most important cases of Poisson deformations, we obtain

**Theorem 6.6.** Let \((M, I, \omega)\) be a compact Kähler manifold with a holomorphic Poisson structure \(\beta \neq 0\). We assume \(H^1(M, \mathbb{C}) = 0\). We denote by \((M, J_\beta t, J_\psi t)\) a generalized Kähler manifold given by Poisson deformations. Then if the scalar curvature \(S(J_\beta t, J_\psi t)\) is a constant, the Lie algebra of the automorphisms \(\mathfrak{g}_{J_\beta t}\) is a reductive Lie algebra.

In Section 7, on \(\mathbb{C}P^2\), a holomorphic Poisson structure \(\beta\) is given by a section of the anticanonical line bundle with the zero locus given by a cubic curve. Cubic curves of \(\mathbb{C}P^2\) are classified into nine cases. An explicit calculation of the Lie algebra \(\mathfrak{g}_{J_\beta}\) is shown for each case. In Section 8, the results of the existence are discussed by using deformations. In particular, Del Pezzo surfaces with trivial automorphisms admit generalized Kähler structures with constant scalar curvature. In Section 9, we introduce a generalized extremal Kähler manifold and calculate the Hessian of the Calabi type functional. We obtain the decomposition of the Lie algebra of automorphisms of a generalized extremal Kähler manifold (cf. [3], [9], [32]).

## 2 Generalized complex structures and generalized Kähler structures

### 2.1 Generalized complex structures and nondegenerate, pure spinors

Let \(M\) be a differentiable manifold of real dimension \(2n\). The bilinear form \((\ , \ )_{T \oplus T^*}\) on the direct sum \(T_M \oplus T^*_M\) over a differentiable manifold \(M\) of \(\text{dim}= 2n\) is defined by

\[
(v + \xi, u + \eta)_{T \oplus T^*} = \frac{1}{2} (\xi(u) + \eta(v)), \quad u, v \in T_M, \xi, \eta \in T^*_M.
\]

Let \(\text{SO}(T_M \oplus T^*_M)\) be the fibre bundle over \(M\) with fibre \(\text{SO}(2n, 2n)\) which is a subbundle of \(\text{End}(T_M \oplus T^*_M)\) preserving the bilinear form \((\ , \ )_{T \oplus T^*}\). An almost generalized complex structure \(J\) is a section of \(\text{SO}(T_M \oplus T^*_M)\) satisfying \(J^2 = -\text{id}\). Then as in the case of almost complex structures, an almost generalized complex structure \(J\) yields the eigenspace decomposition:

\[
(T_M \oplus T^*_M)^C = \mathcal{L}_J \oplus \overline{\mathcal{L}}_J,
\]

(2.1)
where $\mathcal{L}_J$ is $-\sqrt{-1}$-eigenspace and $\overline{\mathcal{L}}_J$ is the complex conjugate of $\mathcal{L}_J$. The Courant bracket of $T_M \oplus T_M^*$ is defined by

$$[u + \xi, v + \eta]_w = [u, v] + \mathcal{L}_u \eta - \mathcal{L}_v \xi - \frac{1}{2}(du \eta - dv \xi),$$

where $u, v \in T_M$ and $\xi, \eta$ is $T_M^*$ if $\mathcal{L}_J$ is involutive with respect to the Courant bracket, then $J$ is a generalized complex structure, that is, $[e_1, e_2]_w \in \Gamma(\mathcal{L}_J)$ for any two elements $e_1 = u + \xi$, $e_2 = v + \eta \in \Gamma(\mathcal{L}_J)$. Let $\text{CL}(T_M \oplus T_M^*)$ be the Clifford algebra bundle which is a fibre bundle with fibre the Clifford algebra $\text{CL}(2n, 2n)$ with respect to $(\cdot, \cdot)_{T \oplus T^*}$ on $M$. Then a vector $v$ acts on the space of differential forms $\oplus_{p=0}^{2n} \wedge^p T_M^*$ by the interior product $i_v$ and a 1-form $\theta$ acts on $\oplus_{p=0}^{2n} \wedge^p T_M^*$ by the exterior product $\theta \wedge$, respectively. Thus $(T_M \oplus T_M^*)^C$ acts on differential forms. (Note that by using (2.1), $\mathcal{L}_J \oplus \overline{\mathcal{L}}_J$ also acts on differential forms.) Then the space of differential forms gives a representation of the Clifford algebra $\text{CL}(T_M \oplus T_M^*)$ which is the spin representation of $\text{CL}(T_M \oplus T_M^*)$. Thus the spin representation of the Clifford algebra arises as the space of differential forms

$$\wedge^* T_M^* = \oplus_p \wedge^p T_M^* = \wedge^{\text{even}} T_M^* \oplus \wedge^{\text{odd}} T_M^*.$$

The inner product $\langle \cdot, \cdot \rangle_s$ of the spin representation is given by

$$\langle \alpha, \beta \rangle_s := (\alpha \wedge \beta)_{[2n]},$$

where $(\alpha \wedge \beta)_{[2n]}$ is the component of degree $2n$ of $\alpha \wedge \beta \in \oplus_p \wedge^p T_M^*$ and $\sigma$ denotes the Clifford involution which is given by

$$\sigma \beta = \begin{cases} +\beta & \text{deg } \beta \equiv 0, 1 \mod 4 \\ -\beta & \text{deg } \beta \equiv 2, 3 \mod 4 \end{cases}$$

We define $\ker \Phi := \{ e \in (T_M \oplus T_M^*)^C \mid e \cdot \Phi = 0 \}$ for a complex differential form $\Phi \in \wedge^{\text{even/odd}} T_M^*$. If $\ker \Phi$ is maximal isotropic, i.e., $\dim \ker \Phi = 2n$, then $\Phi$ is called a pure spinor of even/odd type. A pure spinor $\Phi$ is nondegenerate if $\ker \Phi \cap \ker \overline{\Phi} = \{0\}$, i.e., $(T_M \oplus T_M^*)^C = \ker \Phi \oplus \ker \overline{\Phi}$. Then a nondegenerate, pure spinor $\Phi \in \wedge^* T_M^*$ gives an almost generalized complex structure $J_\Phi$ which satisfies

$$J_\Phi e = \begin{cases} -\sqrt{-1} e, & e \in \ker \Phi \\ +\sqrt{-1} e, & e \in \ker \overline{\Phi} \end{cases}$$

Conversely, an almost generalized complex structure $J$ locally arises as $J_\Phi$ for a nondegenerate, pure spinor $\Phi$ which is unique up to multiplication by non-zero functions. Thus an almost generalized complex structure yields the canonical line bundle $K_J := \mathbb{C} \langle \Phi \rangle$ which is a complex line bundle locally generated by a nondegenerate, pure spinor $\Phi$ satisfying $J = J_\Phi$. A generalized complex structure $J_\Phi$ is integrable if and only if $d\Phi = \eta \cdot \Phi$ for a section $\eta \in (T_M \oplus T_M^*)^C$. The type number of $J = J_\Phi$ is defined as the minimal degree of the differential form $\Phi$. Note that type number $\text{Type} J$ is a function on a manifold which is not a constant in general.

**Example 2.1.** Let $J$ be a complex structure on a manifold $M$ and $J^*$ the complex structure on the dual bundle $T_M^*$ which is given by $J^* \xi(v) = \xi(Jv)$ for $v \in T_M$ and $\xi \in T_M^*$. Then a generalized complex
structure $\mathcal{J}_f$ is given by the following matrix

$$
\mathcal{J}_f = \begin{pmatrix}
J & 0 \\
0 & -J^*
\end{pmatrix}.
$$

Then the canonical line bundle is the ordinary one which is generated by complex forms of type $(n,0)$. Thus we have $\text{Type } \mathcal{J}_f = n$.

\textbf{Example 2.2.} Let $\omega$ be a symplectic structure on $M$ and $\hat{\omega}$ the isomorphism from $T_M$ to $T_M^*$ given by $\hat{\omega}(v) := i_v \omega$. We denote by $\hat{\omega}^{-1}$ the inverse map from $T_M^*$ to $T_M$. Then a generalized complex structure $\mathcal{J}_\psi$ is given by the following

$$
\mathcal{J}_\psi = \begin{pmatrix}
0 & -\hat{\omega}^{-1} \\
\hat{\omega} & 0
\end{pmatrix}, \quad \text{Type } \mathcal{J}_\psi = 0.
$$

Then the canonical line bundle is given by the differential form $\psi = e^{-\sqrt{-1}\omega}$. Thus $\text{Type } \mathcal{J}_\psi = 0$.

\textbf{Example 2.3 (b-field action).} A real $d$-closed 2-form $b$ acts on a generalized complex structure by the adjoint action of Spin group $e^b$ which provides a generalized complex structure $Ad_{e^b} \mathcal{J} = e^b \circ \mathcal{J} \circ e^{-b}$.

\textbf{Example 2.4 (Poisson deformations).} Let $\beta$ be a holomorphic Poisson structure on a complex manifold. Then the adjoint action of Spin group $e^\beta$ gives deformations of new generalized complex structures $\mathcal{J}_\beta x = Ad_{e^\beta} \mathcal{J}$. Then $\text{Type } \mathcal{J}_\beta x = n - 2$ (rank of $\beta_x$) at $x \in M$, which is called the Jumping phenomena of type number.

Let $(M, \mathcal{J})$ be a generalized complex manifold and $L_{\mathcal{J}}$ the eigenspace of eigenvalue $\sqrt{-1}$. Then we have the Lie algebroid complex $\bigwedge \cdot L_{\mathcal{J}}$ (cf. [19]):

$$
0 \rightarrow \bigwedge^0 L_{\mathcal{J}} \rightarrow \bigwedge^1 L_{\mathcal{J}} \rightarrow \bigwedge^2 L_{\mathcal{J}} \rightarrow \bigwedge^3 L_{\mathcal{J}} \rightarrow \cdots
$$

Since the Lie algebroid complex is an elliptic complex, the cohomology $H^p(\bigwedge \cdot L_{\mathcal{J}})$ of the Lie algebroid complex is finite dimensional if $M$ is compact. Let $\{e_i\}_{i=1}^n$ be a local basis of $L_{\mathcal{J}}$ for an almost generalized complex structure $J$, where $\langle e_i, \bar{e}_j \rangle = \delta_{i,j}$. The almost generalized complex structure $\mathcal{J}$ is written as an element of Clifford algebra,

$$
\mathcal{J} = \frac{\sqrt{-1}}{2} \sum_i e_i \cdot \bar{e}_i,
$$

where $\mathcal{J}$ acts on $T_M \oplus T_M^*$ by the adjoint action $[\mathcal{J}, \cdot]$. Thus we have $[\mathcal{J}, e_i] = -\sqrt{-1} e_i$ and $[\mathcal{J}, \bar{e}_i] = \sqrt{-1} e_i$. An almost generalized complex structure $\mathcal{J}$ acts on differential forms by the Spin representation which gives the decomposition into eigenspaces:

$$
\bigwedge \cdot T_M = U_{\mathcal{J}}^{-n} \oplus U_{\mathcal{J}}^{-n+1} \oplus \cdots U_{\mathcal{J}}^n,
$$

where $U_{\mathcal{J}}^i (= U_{\mathcal{J}}^i)$ denotes the $i$-eigenspace. Then $K_{\mathcal{J}} = U_{\mathcal{J}}^{-n}$ and $U_{\mathcal{J}}^{-n+p}$ is given by $\bigwedge^p L_{\mathcal{J}} \cdot K_{\mathcal{J}}$ which denotes the spin action of $\bigwedge^p L_{\mathcal{J}}$ on $K_{\mathcal{J}}$. Since $\mathcal{J}$ is integrable, the exterior derivative $d$ is decomposed into $\delta_{\mathcal{J}} + \delta_{\mathcal{J}}$, where $\delta_{\mathcal{J}} : U_{\mathcal{J}}^i \rightarrow U_{\mathcal{J}}^{i-1}$ and $\delta_{\mathcal{J}} : U_{\mathcal{J}}^i \rightarrow U_{\mathcal{J}}^{i+1}$.
2.2 Generalized Kähler structures

**Definition 2.5.** A generalized Kähler structure is a pair \((J_1, J_2)\) consisting of two commuting generalized complex structures \(J_1, J_2\) such that \(\hat{G} := -J_1 \circ J_2 = -J_2 \circ J_1\) gives a positive definite symmetric form \(G := \langle \hat{G}, \cdot, \cdot \rangle\) on \(T_M \oplus T_M^*\). We call \(G\) a generalized metric.

**Example 2.6.** Let \(X = (M, J, \omega)\) be a Kähler manifold. Then the pair \((J_1, J_2)\) is a generalized Kähler where \(\psi = \exp(\sqrt{-1}\omega)\).

**Example 2.7.** Let \((J_1, J_2)\) be a generalized Kähler structure. Then the action of \(b\)-fields gives a generalized Kähler structure \((\text{Ad}_{b\cdot} J_1, \text{Ad}_{b\cdot} J_2)\) for a real \(d\)-closed 2-form \(b\).

**Definition 2.8.** A generalized Kähler structure of symplectic type is a generalized Kähler structure \((J, J_\psi)\), where \(J_\psi\) is a general complex structure induced from a \(d\)-closed, nondegenerate, pure spinor \(\psi = e^{b-\sqrt{-1}\omega}\) for a \(d\)-closed 2-form \(b\) and a symplectic structure \(\omega\).

Let \((J_1, J_2)\) be a generalized Kähler structure. Then each \(J_i\) gives the decomposition \((T_M \oplus T_M^*)^C = \mathcal{L}_{J_i} \oplus \mathcal{L}_{J_i^*}\) for \(i = 1, 2\). Since \(J_1\) and \(J_2\) are commutative, we have the simultaneous eigenspace decomposition
\[
(T_M \oplus T_M^*)^C = (\mathcal{L}_{J_1} \cap \mathcal{L}_{J_2}) \oplus (\mathcal{L}_J \cap \mathcal{L}_{J_2}) \oplus (\mathcal{L}_{J_1} \cap \mathcal{L}_{J_2}) \oplus (\mathcal{L}_{J_1^*} \cap \mathcal{L}_{J_2}).
\]
Since \(\hat{G}^2 = +\text{id}\), the generalized metric \(\hat{G}\) also gives the eigenspace decomposition: \(T_M \oplus T_M^* = C_+ \oplus C_-\), where \(C_\pm\) denote the eigenspaces of \(\hat{G}\) of eigenvalues \(\pm 1\). We denote by \(\mathcal{L}_{J_1}^\pm\), the intersection \(\mathcal{L}_{J_1} \cap C_\pm\). Then it follows
\[
\mathcal{L}_{J_1} \cap \mathcal{L}_{J_2} = \mathcal{L}_{J_1}^+, \quad \mathcal{L}_{J_1} \cap \mathcal{L}_{J_2} = \mathcal{L}_{J_1}^-
\]
\[
\mathcal{L}_{J_1^*} \cap \mathcal{L}_{J_2} = \mathcal{L}_{J_1}^+, \quad \mathcal{L}_{J_1^*} \cap \mathcal{L}_{J_2} = \mathcal{L}_{J_1}^-.\]

Then \((\wedge^i \mathcal{L}_{J_1}^+) \wedge (\wedge^j \mathcal{L}_{J_1}^-)\) acts on \(K_J\) by the spin action to yield \(U^{-n+i+j, j-i} := (\wedge^i \mathcal{L}_{J_1}^+) \wedge (\wedge^j \mathcal{L}_{J_1}^-) \cdot K_J\).

We have the decomposition of differential forms:
\[
\wedge^* T_M^* = \oplus U^{p,q}
\]
The exterior differential \(d\) is also decomposed into \(\delta_+ + \delta_- + \delta_+ + \delta_-\), where \(\delta_+ = \delta_+ + \delta_-\) and \(\delta_- = \delta_+ + \delta_-\), and \(\delta^+ : U^{p,q} \to U^{p-1,q-1}\), \(\delta^- : U^{p,q} \to U^{p+1,q+1}\) and \(\delta^- : U^{p,q} \to U^{p+1,q+1}\), \(\delta^- : U^{p,q} \to U^{p+1,q-1}\). The generalized metric \(G\) gives the formal adjoint operators \(d^*, \delta^*_+, \delta^*_-, \delta^*_+, \delta^*_-, \delta^*_+\). Then the generalized Kähler identity holds: \(\delta_+ = -\delta_-\), \(\delta_- = -\delta_+\). We denote by \(\delta := dd^* + d^*d\) the Laplacian of \(d\) and \(\Box_{\delta J} := \delta_{\delta J} \delta_{\delta J} + \delta_{\delta J} \delta_{\delta J}\) the Laplacian of \(\delta_{\delta J}\). We also define the Laplacians \(\Box_{\delta J} := \delta_{\delta J} \delta_{\delta J} + \delta_{\delta J} \delta_{\delta J}\) and \(\Box_{\delta J} := \delta_{\delta J} \delta_{\delta J} + \delta_{\delta J} \delta_{\delta J}\). Then we have
\[
\delta = 2\Box_{\delta J} = 4\Box_{\delta J} = 4\Box_{\delta J}
\]
Thus we have the generalized Hodge decomposition:

**Proposition 2.9** (Gualtieri, [18]).
\[
H^* (M, \mathbb{C}) = H^{p,q} (M, J_1, J_2),
\]

where \(H^* (M, \mathbb{C}) = \bigoplus_{i=0}^{\dim M} H^i (M, \mathbb{C})\) and \(H^{p,q} (M, J_1, J_2) := \ker \delta \cap U^{p,q}\).

**Remark 2.10.** The decomposition does hold only when we consider cohomologies of all degrees.
2.3 The deformation-stability theorem of generalized Kähler manifolds

It is known that the deformation-stability theorem of ordinary Kähler manifolds holds

**Theorem 2.11** (Kodaira-Spencer). Let $X = (M, J)$ be a compact Kähler manifold and $X_t$ small deformations of $X = X_0$ as complex manifolds. Then $X_t$ inherits a Kähler structure.

The following deformation-stability theorem of generalized Kähler structures provides many interesting examples of generalized Kähler manifolds of symplectic type.

**Theorem 2.12** (Goto, [11]). Let $X = (M, J, \omega)$ be a compact Kähler manifold and $(\mathcal{J}_J, \mathcal{J}_0)$ the induced generalized Kähler structure, where $\psi = e^{-\sqrt{-1}\omega}$. If there are analytic deformations $\{\mathcal{J}_t\}$ of $\mathcal{J}_0 = \mathcal{J}_J$ as generalized complex structures, then there are deformations of $d$-closed nondegenerate, pure spinors $\{\psi_t\}$ such that pairs $(\mathcal{J}_t, \mathcal{J}_{\psi_t})$ are generalized Kähler structures, where $\psi_0 = \psi$

Then we have the following:

**Corollary 2.13.** Let $X = (M, J, \omega)$ be a compact Kähler manifold with a nontrivial holomorphic Poisson structure $\beta$. Then there exist nontrivial deformations of generalized Kähler structures $(\mathcal{J}_J, \mathcal{J}_{\psi_t})$ such that $\{\mathcal{J}_t\}$ is the Poisson deformations given by Example 2.4 and $\{\psi_t\}$ is a family of $d$-closed nondegenerate, pure spinors and $\psi_0 = e^{-\sqrt{-1}\omega}$.

3 Generalized Hamiltonian diffeomorphisms

Let $\mathcal{J}$ be a generalized complex structure on a manifold $M$. Then $\mathcal{J}$ acts on an exact 1-form $du$ to give $\mathcal{J}du \in T_M \oplus T_M^*$ for a real function $u$ on $M$. Then we define $\mathfrak{ham}_\mathcal{J}(M)$ by

$$\mathfrak{ham}_\mathcal{J}(M) := \{ \mathcal{J}du \mid u \in C^\infty(M, \mathbb{R}) \}$$

The Courant bracket on $T_M \oplus T_M^*$ does not satisfy the Jacobi identity in general. However if we restrict the Courant bracket to $\mathfrak{ham}_\mathcal{J}(M)$, the Jacobi identity does hold and we obtain a Lie algebra.

**Proposition 3.1.** $\mathfrak{ham}_\mathcal{J}(M)$ is a Lie algebra with respect to the Courant bracket.

**Proof.** Since $\langle \mathcal{J}du_1, \mathcal{J}du_2 \rangle_{T \oplus T^*} = \langle du_1, du_2 \rangle_{T \oplus T^*} = 0$, it follows $\mathfrak{ham}_\mathcal{J}(M)$ is isotropic. Since $\mathcal{J}$ is integrable, the Nijenhuis tensor vanishes,

$$[\mathcal{J}du_1, \mathcal{J}du_2]_\omega = [du_1, du_2]_\omega + \mathcal{J}[du_1, \mathcal{J}du_2]_\omega + \mathcal{J}[\mathcal{J}du_1, du_2]_\omega$$

(3.1)

$$= \mathcal{J}[du_1, \mathcal{J}du_2]_\omega + \mathcal{J}[\mathcal{J}du_1, du_2]_\omega$$

(3.2)

From the definition of the Courant bracket, we have

$$[\mathcal{J}du_1, du_2]_\omega = L_{\mathcal{J}du_1}(du_2) = dL_{\mathcal{J}du_1}u_2$$

We denote by $\{u_1, u_2\}_\mathcal{J}$ a real function $L_{\mathcal{J}du_1}u_2 - \mathcal{J}fu_1$, which reduces to the usual $\omega$-Poisson bracket if $\mathcal{J}$ is given by a nondegenerate, pure spinor $e^{-\sqrt{-1}\omega}$. Then we obtain

$$[\mathcal{J}du_1, \mathcal{J}du_2]_\omega = \mathcal{J}d\{u_1, u_2\}_\mathcal{J} \in \mathfrak{ham}_\mathcal{J}(M)$$

Thus $\mathfrak{ham}_\mathcal{J}(M)$ is closed under the courant bracket and isotropic. Hence $\mathfrak{ham}_\mathcal{J}(M)$ is a Lie algebra. $\square$
Then $\mathfrak{ham}_J(M)$ is identified with $C^\infty(M, \mathbb{R})_0 := C^\infty(M, \mathbb{R})/\{\text{constants}\}$.

**Definition 3.2.** The Lie algebra $\mathfrak{ham}_J(M)$ defines a connected Lie group $\text{Ham}_J$ which is called a *generalized Hamiltonian diffeomorphisms* with respect to $J$.

**Remark 3.3.** Let $\omega$ be a symplectic structure on $M$. Then $e^{\sqrt{-1}\omega}$ is a $d$-closed nondegenerate, pure spinor. If $J$ is induced from the structure $e^{\sqrt{-1}\omega}$, then $\mathfrak{ham}_J(M)$ coincides with the Lie algebra of the ordinary Hamiltonian diffeomorphisms.

### 4 Generalized scalar curvature as moment map

Let $\mathcal{B}(M)$ be the set of almost generalized complex structures on a differentiable compact manifold $M$ of dimension $2n$, that is,

$$\mathcal{B}(M) := \{ J : \text{almost generalized complex structure on } M \}.$$ 

We also define $\mathcal{B}^{\text{int}}(M)$ as the set of generalized complex structures on $M$, i.e., integral ones

$$\mathcal{B}^{\text{int}}(M) := \{ J : \text{generalized complex structure on } M \}.$$ 

We fix a generalized complex structure $J_\psi$ which is defined by a set of nondegenerate, pure spinors $\psi := \{ \psi_\alpha \}$ relative to a cover $\{ U_\alpha \}$ of $M$. Then we have

$$d\psi_\alpha = \zeta_\alpha \cdot \psi_\alpha,$$

where we take $\zeta_\alpha \in \sqrt{-1}(T_M \oplus T^*_M)$. We can take $\{ \psi_\alpha \}$ which satisfies

$$\langle \psi_\alpha, \overline{\psi}_\alpha \rangle_s = \langle \psi_\beta, \overline{\psi}_\beta \rangle_s$$

if $U_\alpha \cap U_\beta = \emptyset$. Then we define a volume form $\text{vol}_M$ to be $(\sqrt{-1})^n \langle \psi_\alpha, \overline{\psi}_\alpha \rangle_s$ for each $\alpha$ which is globally defined. An almost generalized complex structure $J$ is $J_\psi$-compatible if and only if the pair $(J, J_\psi)$ is an almost generalized Kähler structure. Let $\mathcal{B}_{J_\psi}(M)$ be the set of $J_\psi$-compatible almost generalized complex structure, that is

$$\mathcal{B}_{J_\psi}(M) := \{ J \in \mathcal{B}(M) : (J, J_\psi) \text{is an almost generalized Kähler structure} \}.$$ 

We assume that $\mathcal{B}_{J_\psi}(M)$ is not an empty set through this paper. We also define $\mathcal{B}^{\text{int}}_{J_\psi}(M)$ to be the set of $\psi$-compatible generalized complex structures. For each point $x \in M$, we define $\mathcal{B}_{J_\psi}(M)_x$ to be the set of $\psi_x$-compatible almost generalized complex structures on $T_xM \oplus T^*_xM$, that is,

$$\mathcal{B}_{J_\psi}(M)_x := \{ J_x : (J_x, J_{\psi,x}) : \text{almost generalized Kähler structure at } x \}.$$ 

Then we see that $\mathcal{B}_{J_\psi}(M)_x$ is given by the Riemannian Symmetric space of type AIII

$$U(n, n)/U(n) \times U(n)$$

which is biholomorphic to the complex bounded domain

$$\{ h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0 \},$$

where $M_n(\mathbb{C})$ denotes the set of complex matrices of $n \times n$. 

8
REMARK 4.1. In Kähler geometry, the set of almost complex structures compatible with a symplectic structure $\omega$ is given by the Riemannian symmetric space $\text{Sp}(2n)/U(n)$ which is biholomorphic to the Siegel upper half plane

$$\{ h \in \text{GL}_n(\mathbb{C}) \mid 1_n - h^*h > 0, h^t = h \}$$

Let $P_{\mathcal{J}^\circ}$ be the fibre bundle over $M$ with fibre $\mathcal{B}_{\mathcal{J}^\circ}(M)_x$, that is,

$$P_{\mathcal{J}^\circ} := \bigcup_{x \in M} \mathcal{B}_{\mathcal{J}^\circ}(M)_x \to M,$$

Then $\mathcal{B}_{\mathcal{J}^\circ}(M)$ is given by smooth sections $\Gamma(M, P_{\mathcal{J}^\circ})$ which contains the integral ones $\mathcal{B}_{\mathcal{J}^\circ}^\text{int}(M)$. We can introduce a Sobolev norm on $\mathcal{B}_{\mathcal{J}^\circ}(M)$ such that $\mathcal{B}_{\mathcal{J}^\circ}(M)$ becomes a Banach manifold in the standard method. The tangent bundle of $\mathcal{B}_{\mathcal{J}^\circ}(M)$ at $\mathcal{J}$ is given by

$$T_{\mathcal{J}}\mathcal{B}_{\mathcal{J}^\circ}(M) = \{ \dot{\mathcal{J}} \in \text{so}(T_M \oplus T_M^\ast) : \dot{\mathcal{J}} + \mathcal{J} \dot{\mathcal{J}} = 0, \dot{\mathcal{J}} \mathcal{J}_\psi = \mathcal{J}_\psi \dot{\mathcal{J}} \},$$

where $\text{so}(T_M \oplus T_M^\ast)$ denotes the set of sections of Lie algebra bundle of $\text{SO}(T_M \oplus T_M^\ast)$. Then it follows that there exists an almost complex structure $\mathcal{J}_B$ on $\mathcal{B}_{\mathcal{J}^\circ}(M)$ which is given by

$$\mathcal{J}_B(\dot{\mathcal{J}}) := \mathcal{J} \dot{\mathcal{J}}, \quad (\dot{\mathcal{J}} \in T_{\mathcal{J}}\mathcal{B}_{\mathcal{J}^\circ}(M) )$$

We also have a Riemannian metric $g_B$ and a 2-form $\Omega_B$ on $\mathcal{B}_{\mathcal{J}^\circ}(M)$ by

$$g_B(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) := \int_M \text{tr}(\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \text{vol}_M \quad (4.2)$$

$$\Omega_B(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) := - \int_M \text{tr}(\mathcal{J} \dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \text{vol}_M \quad (4.3)$$

for $\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2 \in T_{\mathcal{J}}\mathcal{B}_{\mathcal{J}^\circ}(M)$.

**Proposition 4.2.** $\mathcal{J}_B$ is integrable almost complex structure on $\mathcal{B}_{\mathcal{J}^\circ}(M)$ and $\Omega_B$ is a Kähler form on $\mathcal{B}_{\mathcal{J}^\circ}(M)$.

**Proof.** Let $\mathcal{J}_V$ be an almost generalized complex structure on a real vector space $V$ of dimension $2n$. We denote by $X_n$ the Riemannian symmetric space $U(n,n)/U(n) \times U(n)$ which is identified with the set of almost generalized complex structures compatible with $\mathcal{J}_V$. We already see that $\mathcal{B}_{\mathcal{J}_V}(M)$ is the set of global sections of the fibre bundle $P_{\mathcal{J}_V}$ over a manifold $M$ with fibre $X_n$ which is biholomorphic to the bounded domain $\{ h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0 \}$. Let $\mathcal{J}_0$ be an element of $\mathcal{B}_{\mathcal{J}_V}(M)$. Then a generalized Kähler structure $(\mathcal{J}_0, \mathcal{J}_B)$ gives the decomposition of $(T_M \oplus T_M^\ast)^\mathbb{C}$ as in (2.3)

$$(T_M \oplus T_M^\ast)^\mathbb{C} = \mathcal{L}_{\mathcal{J}_B}^\ast \oplus \mathcal{L}_{\mathcal{J}_V} \oplus \mathcal{T}_{\mathcal{J}_B}^\ast \oplus \mathcal{T}_{\mathcal{J}_V},$$

where $\mathcal{L}_{\mathcal{J}_V} = \mathcal{L}_{\mathcal{J}_V} \cap \mathcal{L}_{\mathcal{J}_0}$ and $\mathcal{T}_{\mathcal{J}_V} = \mathcal{T}_{\mathcal{J}_V} \cap \mathcal{L}_{\mathcal{J}_0}$. Note that the adjoint action of the group $\text{SO}(T_M \oplus T_M^\ast)$ on the set of almost generalized complex structures is transitive. An element of $\text{SO}(T_M \oplus T_M^\ast)$ preserves $\mathcal{J}_V$ if and only if it preserves $\mathcal{L}_{\mathcal{J}_V}$. Then every real element $g$ of $\text{SO}(T_M \oplus T_M^\ast)$ preserving $\mathcal{J}_V$ is given by the following

$$g = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \text{GL}(\mathcal{L}_V),$$

for $A, B, C, D \in \mathbb{R}$ with

$$g \cdot \mathcal{L}_V \subseteq \mathcal{L}_V.$$
where $A \in \text{End}(\mathcal{L}^+_{\mathcal{J}_0})$ and $B \in \text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^+_{\mathcal{J}_0})$ and $C \in \text{Hom}(\mathcal{L}^-_{\mathcal{J}_0}, \mathcal{L}^+_{\mathcal{J}_0})$ and $D \in \text{End}(\mathcal{L}^-_{\mathcal{J}_0})$ satisfy

\begin{align}
A^* A - B^* B &= \text{id}_{\mathcal{L}^+_{\mathcal{J}_0}}, \quad -C^* C + D^* D = -\text{id}_{\mathcal{L}^-_{\mathcal{J}_0}} \quad (4.4) \\
A^* D &= B^* C \quad (4.5)
\end{align}

Thus it follows that $g$ is a section of the fibre bundle whose fibre is $U(n, n)$. Let $h := BA^{-1} \in \text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^-_{\mathcal{J}_0})$. Then $h$ satisfies

\[ \text{id}_{\mathcal{L}^+_{\mathcal{J}_0}} - h^* h > 0 \in \text{End}(\mathcal{L}^+_{\mathcal{J}_0}) \quad (4.6) \]

Thus it follows that the fibre bundle $P_{\mathcal{J}_0}$ is identified with an open fibre bundle with fibre the bounded domain, which is a open subbundle of the complex vector bundle $\text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^-_{\mathcal{J}_0})$. Thus $\mathcal{B}_{\mathcal{J}_0}(M)$ is the space of global sections of an fibre bundle over $M$ which is an open subbundle of a complex vector bundle $\text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^-_{\mathcal{J}_0})$ over $M$. In general, the set of global sections of the complex vector bundle is a complex manifold, cf. [30]. (We choose a Sobolev norm $L^k$ to consider the set of $L^k$-sections.) Since $\mathcal{B}_{\mathcal{J}_0}(M)$ is an open set of the global sections of the complex vector bundle, $\mathcal{B}_{\mathcal{J}_0}(M)$ is a complex submanifold with a complex structure which is given by

\[ \text{id}_{\mathcal{L}^+_{\mathcal{J}_0}} - h^* h > 0 \in \text{End}(\mathcal{L}^+_{\mathcal{J}_0}) \quad (4.6) \]

Thus it follows that the fibre bundle $P_{\mathcal{J}_0}$ is identified with an open fibre bundle with fibre the bounded domain, which is a open subbundle of the complex vector bundle $\text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^-_{\mathcal{J}_0})$. Thus $\mathcal{B}_{\mathcal{J}_0}(M)$ is integrable almost complex structure on $\mathcal{B}_{\mathcal{J}_0}(M)$. We denote by $g_{\mathcal{J}_n}$ the Riemannian metric on $\mathcal{J}_n$ and by $\omega_{\mathcal{J}_n}$ the Kähler form which are respectively given by

\[ g_{\mathcal{J}_n}(\hat{J}_1, \hat{J}_2) = \text{tr}(\hat{J}_1 \hat{J}_2) \]

\[ \omega_{\mathcal{J}_n}(\hat{J}_1, \hat{J}_2) = -\text{tr}(\hat{J}_1 \hat{J}_2), \]

where $\hat{J}_1, \hat{J}_2 \in T_{\mathcal{J}} \mathcal{J}_n$. The complex bounded domain $\{ h \in \text{GL}_n(\mathbb{C}) \mid 1_n - h^* h > 0 \}$ admits a Kähler structure which is given by

\[ 4\sqrt{-1} \partial \bar{\partial} \log \det(1_n - h^* h). \]

Then under the identification $\mathcal{J}_n \cong \{ h \in M_n(\mathbb{C}) \mid 1_n - h^* h > 0 \}$ by using $\mathcal{J}_n$, we have $\omega_{\mathcal{J}_n} = 4\sqrt{-1} \partial \bar{\partial} \log \det(1_n - h^* h)$. Since $\mathcal{B}_{\mathcal{J}_0}(M)$ is the set of global section of the open subbundle, the tangent bundle $T\mathcal{B}_{\mathcal{J}_0}(M)$ is canonically identified with the trivial bundle $\mathcal{B}_{\mathcal{J}_0}(M) \times \Gamma(M, \text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^-_{\mathcal{J}_0}))$. The complex manifold $\mathcal{B}_{\mathcal{J}_0}(M)$ inherits a Riemannian metric $g_{\mathcal{B}}$ and a Hermitian 2-form $\Omega_{\mathcal{B}}$ which are given by

\[ g(\hat{J}_1, \hat{J}_2) := \int_M \text{tr}(\hat{J}_1 \hat{J}_2) \text{vol}_M \quad (4.7) \]

\[ \Omega_{\mathcal{B}}(\hat{J}_1, \hat{J}_2) := -\int_M \text{tr}(\hat{J}_1 \hat{J}_2) \text{vol}_M \quad (4.8) \]

Since the tangent bundle of $\mathcal{B}_{\mathcal{J}_0}(M)$ is canonically identified with the trivial bundle, each global section $a \in \text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^-_{\mathcal{J}_0})$ gives a vector field $\hat{J}_a$ of $\mathcal{B}_{\mathcal{J}_0}(M)$ such that $\hat{J}(\hat{J}) = a$ for all $\hat{J} \in \mathcal{B}_{\mathcal{J}_0}(M)$. By using $L^k$-metric, we obtain a basis of vector fields of $\mathcal{B}_{\mathcal{J}_0}(M)$ by using global sections of $\text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^-_{\mathcal{J}_0})$. We also denote by $\hat{J}_{a,x}$ the vector field on the fibre at $x \in M$. For global sections $a_1, a_2 \in \text{Hom}(\mathcal{L}^+_{\mathcal{J}_0}, \mathcal{L}^-_{\mathcal{J}_0})$ and for each $x \in M$, we have

\[ 4\sqrt{-1} \partial \bar{\partial} \log \det(1_n - h^* h)(\hat{J}_{a_1,x}, \hat{J}_{a_2,x}) = -\text{tr}(\hat{J}_{a_1,x} \hat{J}_{a_2,x}). \]
Then it follows
\[
\Omega_B(\dot{J}_{a_1}, \dot{J}_{a_2}) = \int_M 4\sqrt{-1} \partial \bar{\partial} \log \det(1 - h^*h)(\dot{J}_{a_1}, \dot{J}_{a_2}) \, \text{vol}_M
\] (4.9)

Since \( \mathcal{B}_{\bar{J}_0}(M) \) is given by the set of global sections of \( \text{Hom}(\mathcal{L}^+_{\bar{J}_0}, \mathcal{L}^-_{\bar{J}_0}) \) satisfying (4.6), \( h \mapsto \int_M \log \det(\text{id}_{\mathcal{L}^+_{\bar{J}_0}} - h^*h) \text{vol}_M \) is regarded as a function on \( \mathcal{B}_{\bar{J}_0}(M) \). Let \( \partial_B \) be the \( \partial \)-operator of the complex manifold \( \mathcal{B}_{\bar{J}_0}(M) \) and \( \partial_B \) the complex conjugate of \( \partial_B \). Then
\[
\left( \partial_B \bar{\partial}_B \int_M \log \det(\text{id}_{\mathcal{L}^+_{\bar{J}_0}} - h^*h) \text{vol}_M \right)(\dot{J}_{a_1}, \dot{J}_{a_2})
\]
is given by (1,1)-component of
\[
\frac{d}{dt_1} \frac{d}{dt_2} \bigg|_{t_1,t_2=0} \int_M \log \det(\text{id}_{\mathcal{L}^+_{\bar{J}_0}} - h_{t_1,t_2}^*h_{t_1,t_2}) \text{vol}_M,
\]
where \( h_{t_1,t_2} = h + a_1 t_1 + a_2 t_2 \in \Gamma(M, \text{Hom}(\mathcal{L}^+_{\bar{J}_0}, \mathcal{L}^-_{\bar{J}_0})) \) and \( t_1, t_2 \) are parameters of small deformations. Then it follows
\[
\left( \frac{d}{dt_1} \frac{d}{dt_2} \bigg|_{t_1,t_2=0} \int_M \log \det(\text{id}_{\mathcal{L}^+_{\bar{J}_0}} - h_{t_1,t_2}^*h_{t_1,t_2}) \text{vol}_M \right) = \int_M \frac{d}{dt_1} \frac{d}{dt_2} \bigg|_{t_1,t_2=0} \log \det(\text{id}_{\mathcal{L}^+_{\bar{J}_0}} - h_{t_1,t_2}^*h_{t_1,t_2}) \text{vol}_M,
\] (4.10)
\[
\left( \frac{d}{dt_1} \frac{d}{dt_2} \bigg|_{t_1,t_2=0} \log \det(\text{id}_{\mathcal{L}^+_{\bar{J}_0}} - h_{t_1,t_2}^*h_{t_1,t_2}) \right)
\]
From (4.9), we have
\[
\Omega_B(\dot{J}_{a_1}, \dot{J}_{a_2}) = 4\sqrt{-1} \partial_B \bar{\partial}_B \int_M \log \det(\text{id}_{\mathcal{L}^+_{\bar{J}_0}} - h^*h) \text{vol}_M \bigg)(\dot{J}_{a_1}, \dot{J}_{a_2})
\]
Thus \( \Omega_B \) is \( \partial_B \bar{\partial}_B \)-exact. Hence \( \Omega_B \) is closed. Thus \( (\mathcal{B}_{\bar{J}_0}(M), J_B, \Omega_B) \) is a Kähler manifold.

Let \( \widetilde{\text{Diff}}(M) \) be an extension of diffeomorphisms of \( M \) by 2-forms which is defined as
\[
\widetilde{\text{Diff}}(M) := \{ e^b F : F \in \text{Diff}(M), \, b : 2\text{-form} \}.
\]
Note that the product of \( \widetilde{\text{Diff}}(M) \) is given by
\[
(e^{b_1} F_1)(e^{b_2} F_2) := e^{b_1 + F_1^*(b_2)} F_1 \circ F_2,
\]
where \( F_1, F_2 \in \text{Diff}(M) \) and \( b_1, b_2 \) are real 2-forms. The action of \( \widetilde{\text{Diff}}(M) \) on \( \mathcal{G}C(M) \) by
\[
e^b F_\# \circ J \circ \overline{F_\#}^{-1} e^{-b},
\] (4.12)
where \( F \in \text{Diff}(M) \) acts on \( J \) by \( F_\# \circ J \circ \overline{F_\#}^{-1} \) and and \( e^b \) is regarded as an element of \( \text{SO}(T_M \oplus T_M^*) \) and \( F_\# \) denotes the bundle map of \( T_M \oplus T_M^* \) which is the lift of \( F \).
REMARK 4.3. An element \( v + \theta \in T_M \oplus T_M^* \) generates a 1-parameter family of \( \Diff(M) \). An element of the Lie algebra of \( \tilde{\Diff}(M) \) is a pair \( (v, d\theta) \) which consists of a vector field \( v \) and a \( d \)-exact 2-form \( d\theta \). Then the Lie bracket is given by

\[
[v_1 + d\theta_1, v_2 + d\theta_2] = [v_1, v_2] + L_{v_1} \theta_2 - L_{v_2} \theta_1
\]

Since \( \mathfrak{ham}_J(M) \) is an isotropic subspace of \( T_M \oplus T_M^* \), we have a homomorphism from the Lie algebras \( \mathfrak{ham}_J(M) \) to the Lie algebra of \( \tilde{\Diff}(M) \).

For a (integral) generalized complex structure \( J_\psi \), we define \( \tilde{\Diff}(M)_{J_\psi} \) to be a subgroup consists of elements of \( \tilde{\Diff}(M) \) which preserves \( J_\psi \),

\[
\tilde{\Diff}_{J_\psi}(M) = \{ e^b F \in \tilde{\Diff}(M) : e^b F_# \circ J_\psi \circ F_#^{-1} e^{-b} = J_\psi \}.
\]

Then from (4.2), we have the following,

**PROPOSITION 4.4.** The symplectic structure \( \Omega_B \) is invariant under the action of \( \psi \)-preserving group \( \tilde{\Diff}_{J_\psi}(M) \).

**Proof.** The result follows from (4.8) and (4.12) since \( \vol_M \) is invariant under the action of \( \tilde{\Diff}_{J_\psi}(M) \).

**PROPOSITION 4.5.** Let \( \Ham_{J_\psi} \) be the generalized Hamiltonian diffeomorphisms whose Lie algebra is \( \mathfrak{ham}_{J_\psi}(M) \). Then \( \Ham_{J_\psi} \) also preserves \( \Omega_B \).

**Proof.** The Lie algebra \( \mathfrak{ham}_{J_\psi} \) of the Lie group \( \Ham_{J_\psi} \) is given by \( \{ e := J_\psi du \mid u \in C^\infty(M, \mathbb{R}) \} \) as before. Then the action of \( J_\psi du \) on \( \psi_\alpha \) is given by the Lie derivative \( L_e \psi_\alpha = d(J_\psi du) \cdot \psi_\alpha + (J_\psi du) d\psi_\alpha \). Since \( \sqrt{-1} du + J_\psi (du) \in L_{J_\psi} \), we have \( (\sqrt{-1} du + J_\psi (du)) \cdot \psi_\alpha = 0 \). From (4.1), we have

\[
L_e \psi_\alpha = - \sqrt{-1} d((du) \psi_\alpha) + (J_\psi du) \cdot \psi_\alpha
= \sqrt{-1} (du) \wedge \psi_\alpha + (J_\psi du) \cdot \psi_\alpha
= \sqrt{-1} (du) \wedge \psi_\alpha + (J_\psi du) \cdot \psi_\alpha
= (\sqrt{-1} (du) + (J_\psi du)) \cdot \psi_\alpha
\]

Since \( (\sqrt{-1} du + (J_\psi du)) \in L_{J_\psi} \), we see that the component

\[
\pi_{\tilde{\mathcal{L}}_{J_\psi}} \mathcal{L}_e \psi_\alpha = \psi_\alpha = 0.
\]

Thus \( \mathcal{L}_e \psi_\alpha \) is in \( K_{J_\psi} \). Hence \( \mathcal{L}_e \) preserves the canonical line bundle \( K_{J_\psi} \) and then it follows that \( \mathfrak{ham}_{J_\psi}(M) \) preserves \( J_\psi \). Thus \( \Ham_{J_\psi} \) also preserves \( J_\psi \) and \( \vol_M \). The infinitesimal action of \( (v, \theta) \in \mathfrak{ham}_{J_\psi}(M) \) on \( B_{J_\psi} \) is given by \( L_v + d\theta \) which is the infinitesimal action of \( (v, d\theta) \) of the Lie algebra of \( \tilde{\Diff}_{J_\psi}(M) \). From Proposition 4.4, it follows that \( \Omega_B \) vanishes by the infinitesimal action of \( (v, \theta) \in \mathfrak{ham}_{J_\psi}(M) \). Thus one see that the action of \( \Ham_{J_\psi} \) preserves \( \Omega_B \).

As is shown before, the Lie algebra \( \mathfrak{ham}_{J_\psi}(M) \) is given by \( C_0^\infty(M) \), where \( C_0^\infty(M) = \{ f \in C^\infty(M) \mid \int_M f \vol_M = 0 \} \). Then \( e := J_\psi (df) \in T_M \oplus T_M^* \) is called a generalized Hamiltonian element. Note that we have \( e \cdot \psi_\alpha = - \sqrt{-1} df \cdot \psi_\alpha \).

In order to show the existence of the moment map, we shall restrict our attention to generalized Kähler manifolds \( (J, J_\psi) \), where \( J_\psi \) is induced from a set of locally defined \( d \)-closed nondegenerate, pure spinors \( \psi := \{ \psi_\alpha \} \).
Theorem 4.6. We assume that $J_{\psi}$ is induced from a set of $d$-closed, nondegenerate, pure spinors $\psi := \{\psi_\alpha\}$. Then there exists a moment map $\mu : B_{J_{\psi}}(M) \to C_0^\infty(M, \mathbb{R})^*$ for the action of the generalized Hamiltonian diffeomorphisms $\operatorname{Ham}_{J_{\psi}}$, which is explicitly written in terms of pure spinors.

Remark 4.7. In the previous paper [14], the existence of the moment map was shown in the rather restricted cases of generalized Kähler manifolds of symplectic type. Thus our theorem is a generalization of the previous one and the method of our proof is improved.

Remark 4.8. Boulanger also obtained the moment map in the cases of toric generalized Kähler manifolds of symplectic type. Thus our theorem is a generalization of the one in our paper (see also [14], for generalized Kähler structures of type (0,0)). The Calabi-Yau type problem of generalized Kähler manifolds of type (0,0) was discussed by Apostolov and Streets in [4].

In order to show Theorem 4.6, we need several Lemmata. Let $J \in B_{J_{\psi}}(M)$ be an almost generalized complex structure which is induced from a set of nondegenerate, pure spinors $\phi = \{\phi_\alpha\}$. We normalize $\{\phi_\alpha\}$ such that $\langle \phi_\alpha, \overline{\phi}_\alpha \rangle_s = \operatorname{vol}_M$ for each $\alpha$. Then one has

Lemma 4.10. $d\phi_\alpha$ is given by

$$d\phi_\alpha = (\eta_\alpha + N_\alpha) \cdot \phi_\alpha,$$  \hspace{1cm} (4.13)

where $\eta_\alpha \in \sqrt{-1}(T_M \oplus T_M^*)$ and $N_\alpha \in (\wedge^3 \mathcal{L}_J \oplus \wedge^3 \overline{\mathcal{L}}_J)^\mathbb{R}$. Note that $\eta_\alpha$ and $N_\alpha$ are uniquely determined.

Proof. It suffices to show that $d\phi_\alpha \in U^{-n+1}_J \oplus U^{-n+3}_J$. In fact, one has

$$e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot d\phi_\alpha = e_1 \cdot e_2 \cdot [e_3, e_4]_{co} \cdot \phi_\alpha,$$

for any $e_1, e_2, e_3, e_4 \in \mathcal{L}_J$. Since $[e_3, e_4]_{co}$ is given by $e_5 + \overline{e}_6$, for some $e_5 \in \mathcal{L}_J$ and $\overline{e}_6 \in \overline{\mathcal{L}}_J$, it follows from $e_2 \cdot e_6 + \overline{e}_6 \cdot e_2 = 2(e_2, e_6)_{T \oplus \overline{T}^*}$ and and $\mathcal{L}_J = \ker \phi_\alpha$ that

$$e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot d\phi_\alpha = e_1 \cdot e_2 \cdot e_6 \cdot \phi_\alpha = 2(e_2, e_6)_{T \oplus \overline{T}^*} \cdot e_1 \cdot \phi_\alpha = 0.$$

Thus one has $d\phi_\alpha = \eta'_\alpha \cdot \phi_\alpha + N'_\alpha \cdot \phi_\alpha \in U^{-n+1}_J \oplus U^{-n+3}_J$, where $\eta'_\alpha \in \overline{\mathcal{L}}_J$ and $N'_\alpha \in \wedge^3 \overline{\mathcal{L}}_J$. Then $\eta_\alpha$ is the imaginary element $\eta'_\alpha - \overline{\eta'_\alpha}$ and $N_\alpha$ is the real one $N'_\alpha + \overline{N'_\alpha}$. \hfill $\Box$

Remark 4.11. Note that $N_\alpha$ is a real element and $N_\alpha = N_\beta$ for all $\alpha, \beta$. Then $N_\alpha$ defines a global element $N$, which is called Nijenhuis tensor. In fact, $J$ is integrable if and only if $N$ vanishes.

Lemma 4.12. $N \cdot \psi_\alpha = 0$
Proof. Since $N$ is uniquely defined by (4.13), for $e_1, e_2, e_3 \in \mathcal{L}_J$, we have
\begin{align}
N(e_1, e_2, e_3)\langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s &= \langle d\phi_\alpha, e_1 \cdot e_2 \cdot e_3 \cdot \bar{\phi}_\alpha \rangle_s = -\langle e_1 \cdot e_2 \cdot d\phi_\alpha, e_3 \cdot \bar{\phi}_\alpha \rangle_s \\
&= -\langle [e_1, e_2]_s \cdot \phi_\alpha, e_3 \cdot \bar{\phi}_\alpha \rangle_s
\end{align}
(4.14)
Thus we have
\begin{equation}
N(e_1, e_2, e_3) = 2\langle [e_1, e_2]_s, e_3 \rangle_{T \otimes T^*}
\end{equation}
(4.16)
This implies that $N = 0$ if and only if $J$ is integrable. By using $\mathcal{J}_\psi$, we have the decomposition $\mathcal{L}_J = \mathcal{L}_J^+ \oplus \mathcal{L}_J^-$ and $\mathcal{T}_J = \mathcal{T}_J^+ \oplus \mathcal{T}_J^-$. Since $\ker \psi_\alpha = \mathcal{L}_J^+ \oplus \mathcal{T}_J^-$ and $N \in (\wedge^3 \mathcal{L}_J \oplus \wedge^3 \mathcal{T}_J)^R$, we have $N \cdot \psi = (\mathcal{N}^+ + \mathcal{N}^-) \cdot \psi$, where $\mathcal{N}^+ \in \wedge^3 \mathcal{T}_J^+$ and $\mathcal{N}^- \in \wedge^3 \mathcal{T}_J^-$. From (4.16), we see
\begin{align}
N(e_1, e_2, e_3) &= \langle [e_1, e_2]_s, e_3 \rangle_{T \otimes T^*}, \\
(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3) &= \mathcal{L}_J.
\end{align}
Since $\mathcal{J}_\psi$ is integrable, it follows that $[e_1, e_2]_s \in \mathcal{L}_\mathcal{J}_\psi$. Since $e_3 \in \mathcal{L}_\mathcal{J}_\psi$, we have $N(e_1, e_2, e_3) = 0$. Then it follows $\mathcal{N}^+ = 0$. We also have $\mathcal{N}^- = 0$. Hence $N \cdot \psi = 0$. \hfill \Box

**Lemma 4.13.** Let $\mathcal{J}_t$ be deformations of $\mathcal{J}$ such that $(\mathcal{J}_t, \mathcal{J}_\psi)$ is an almost generalized Kähler structures. We denote by $\{\phi_\alpha(t)\}$ a family of nondegenerate, pure spinors which gives $\mathcal{J}_t$ and $\delta\phi_\alpha(t) = (\eta_\alpha(t) + N(t)) \cdot \phi_\alpha(t)$, where $\eta_\alpha(t) \in \sqrt{-1}(T_M \oplus T_M)$ and $N \in (\wedge^3 \mathcal{L}_J \oplus \wedge^3 \mathcal{T}_J)^R$. Let $\tilde{N} = \frac{d}{dt} N(t)|_{t=0}$. Then we have
\begin{equation}
\tilde{N} \cdot \psi_\alpha = 0
\end{equation}

**Proof.** From Lemma 4.12, we have $N(t) \cdot \psi_\alpha = 0$ for all $t$. Then we have the result. \hfill \Box

**Lemma 4.14.** $\langle e \cdot \phi_\alpha, \tilde{N} \cdot \bar{\phi}_\alpha \rangle_s = 0$.

**Proof.** The space $\wedge^4(T_M \oplus T_M)$ is decomposed into $\wedge^4 T_M \oplus (\wedge^3 T_M \otimes T_M^*) \oplus (\wedge^2 T_M \otimes \wedge^2 T_M^*) \oplus (T_M \otimes \wedge^3 T_M^*) \oplus \wedge^4 T_M^*$. We denote by Cont$^{2,2}$ the contraction of the component $(\wedge^2 T_M \otimes \wedge^2 T_M^*)$ which yields a map from $\wedge^4(T_M \oplus T_M^*)$ to $C^\infty(M)$. Then it follows
\begin{align}
\langle e \cdot \phi_\alpha, \tilde{N} \cdot \bar{\phi}_\alpha \rangle_s &= -\langle \phi_\alpha, e \cdot \tilde{N} \cdot \bar{\phi}_\alpha \rangle_s \\
&= -\text{Cont}^{2,2}(e \cdot \tilde{N})(\phi_\alpha, \bar{\phi}_\alpha)_s
\end{align}
(4.17)
Since $\langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = \langle \psi_\alpha, \bar{\psi}_\alpha \rangle_s$, we have
\begin{align}
\langle e \cdot \phi_\alpha, \tilde{N} \cdot \bar{\phi}_\alpha \rangle_s &= -\text{Cont}^{2,2}(e \cdot \tilde{N})(\psi_\alpha, \bar{\psi}_\alpha)_s \\
&= \langle e \cdot \psi_\alpha, \tilde{N} \cdot \bar{\psi}_\alpha \rangle_s
\end{align}
(4.18)
Since $\tilde{N}$ is real, it follows from Lemma 4.13 that $\tilde{N} \cdot \bar{\psi}_\alpha = 0$. Hence we have $\langle e \cdot \phi_\alpha, \tilde{N} \cdot \bar{\phi}_\alpha \rangle_s = 0$. \hfill \Box

**Proof.** of Theorem 4.6. Every infinitesimal deformation of $\mathcal{J}$ is written by the adjoint action of $h$
\begin{equation}
\mathcal{J}_h := [h, \mathcal{J}]
\end{equation}
where $h$ denotes a real element $(\wedge^2 \mathcal{L}_J \oplus \wedge^2 \mathcal{T}_J)^R$. Then the corresponding infinitesimal deformation of $\phi$ is given by the Clifford action of $h$ on each $\phi_\alpha$,
\begin{equation}
\dot{\phi}_\alpha = h \cdot \phi_\alpha
\end{equation}
An Hamiltonian element $e = \mathcal{J}_c df$ gives the infinitesimal deformation $\mathcal{L}_c \mathcal{J}$ of $\mathcal{J}$. Then the corresponding infinitesimal deformation of $\phi$ is given by

$$
\mathcal{L}_c \phi_\alpha = d(e \cdot \phi_\alpha) + e \cdot d\phi_\alpha = d(e \cdot \phi_\alpha) + e \cdot (\eta_\alpha + N) \cdot \phi_\alpha.
$$

(For simplicity, $d(e \cdot \phi_\alpha)$ is also denoted by $de \cdot \phi_\alpha$.) Then in order to show the existence of a moment map, we shall calculate $\Omega_B(\mathcal{L}_c \mathcal{J}, \dot{J}_h)$. In [14], the following description of $\Omega_B$ in terms of pure spinors is already given. (see Lemma 7.1 and (7.1) in Section 7 of [14], where $\text{vol}_M = i^{-n} \langle \psi, \overline{\psi} \rangle_s$, and $\rho_\alpha = 1$.)

$$
\Omega_B(\dot{J}_{h_1}, \dot{J}_{h_2}) = c_n \text{Im} \left( i^{-n} \int_M \langle h_1 \cdot \phi_\alpha, h_2 \cdot \overline{\phi_\alpha} \rangle_s \right),
$$

(4.21)

where $c_n$ is a constant depending only on $n$. Applying (4.21), we obtain

$$
\Omega_B(\mathcal{L}_c \mathcal{J}, \dot{J}_h) = c_n \text{Im} \left( i^{-n} \int_M \langle \mathcal{L}_c \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_s \right)
$$

(4.22)

$$
= c_n \text{Im} \left( i^{-n} \int_M \langle de \cdot \phi_\alpha + e \cdot (\eta_\alpha + N) \cdot \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_s \right)
$$

(4.23)

Since $h \in (\wedge^2 \mathcal{L}_c \mathcal{J} \oplus \wedge^2 \overline{\mathcal{L}_c \mathcal{J}})^R$, we have $\langle \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle = 0$. Since $e \cdot \eta_\alpha + \eta_\alpha \cdot e = 2(e, \eta_\alpha)_{\text{tr}^c}$, we have

$$
\langle e \cdot \eta_\alpha \cdot \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_s = - \langle \eta_\alpha \cdot e \cdot \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_s
$$

Then we have

$$
\Omega_B(\mathcal{L}_c \mathcal{J}, \dot{J}_h) = c_n \text{Im} \left( i^{-n} \int_M \langle (d - \eta_\alpha + N)e \cdot \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_s \right)
$$

(4.24)

It follows $d\langle e \cdot \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_{[2n-1]} = \langle d(e \cdot \phi_\alpha), h \cdot \overline{\phi_\alpha} \rangle_s - \langle (e \cdot \phi_\alpha), d(h \cdot \overline{\phi_\alpha}) \rangle_s$, where $\langle e \cdot \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_{[2n-1]}$ is the component of $(2n-1)$-form of $(e \cdot \phi_\alpha) \wedge \sigma(h \cdot \overline{\phi_\alpha})$. Recall that we assume the normalization $i^{-n} \langle \phi_\alpha, \overline{\phi_\alpha} \rangle_s = \text{vol}_M$ for all $\alpha$. Thus it follows $\phi_\alpha = e^{ip_{\alpha, \beta}} \phi_\beta$, where $p_{\alpha, \beta}$ denotes a real function. Then it follows $(e \cdot \phi_\alpha) \wedge \sigma(h \cdot \overline{\phi_\alpha}) = (e \cdot \phi_\beta) \wedge \sigma(h \cdot \overline{\phi_\beta})$. Thus $(e \cdot \phi_\alpha) \wedge \sigma(h \cdot \overline{\phi_\alpha})$ gives a globally defined $(2n-1)$-form. From the Stokes Theorem, we have

$$
\int_M \langle d(e \cdot \phi_\alpha), h \cdot \overline{\phi_\alpha} \rangle_s = \int_M \langle e \cdot \phi_\alpha, d(h \cdot \overline{\phi_\alpha}) \rangle_s
$$

Since $\eta_\alpha$ is in $\sqrt{-1}(T_M \oplus T_M^*)$, we also have

$$
\langle \eta_\alpha \cdot e \cdot \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_s = - \langle e \cdot \phi_\alpha, \eta_\alpha \cdot h \cdot \overline{\phi_\alpha} \rangle_s.
$$

Since $N$ is $\wedge^3(T_M \oplus T_M^*)^R$, it follows

$$
\langle N \cdot e \cdot \phi_\alpha, h \cdot \overline{\phi_\alpha} \rangle_s = + \langle e \cdot \phi_\alpha, N \cdot h \cdot \overline{\phi_\alpha} \rangle_s.
$$

Substituting them, we obtain

$$
\Omega_B(\mathcal{L}_c \mathcal{J}, \dot{J}_h) = c_n \text{Im} \left( i^{-n} \int_M \langle e \cdot \phi_\alpha, (d + \eta_\alpha - N) \cdot (h \cdot \overline{\phi_\alpha}) \rangle_s \right)
$$

15
Let $\phi(t) = \{\phi_\alpha(t)\}$ be a family of nondegenerate, pure spinors which gives

$$\frac{d}{dt} \phi_\alpha(t)_{|t=0} = h \cdot \phi_\alpha$$

Then we have

$$d\phi_\alpha(t) = (\eta_\alpha(t) + N(t)) \cdot \phi_\alpha(t).$$

Taking time derivative of both sides at $t = 0$, we have

$$d(h \cdot \phi_\alpha) = (\dot{\eta}_\alpha + \dot{N}) \cdot \phi_\alpha + (\eta_\alpha + N) \cdot (h \cdot \phi_\alpha)$$

Since $\eta_\alpha$ is pure imaginary and $N$ is real, we have

$$d(h \cdot \phi_\alpha) = (-\dot{\eta}_\alpha + \dot{N}) \cdot \overline{\phi_\alpha} + (-\eta_\alpha + N) \cdot (h \cdot \overline{\phi_\alpha})$$

Thus we obtain

$$\langle e \cdot \phi_\alpha, (d + \eta - N) \cdot (h \cdot \overline{\phi_\alpha})\rangle_s = - \langle e \cdot \phi_\alpha, (\dot{\eta}_\alpha - \dot{N}) \cdot \overline{\phi_\alpha}\rangle_s$$

From Lemma 4.14, we have $\langle e \cdot \phi_\alpha, \overline{N} \cdot \overline{\phi_\alpha}\rangle_s = 0$. Hence we obtain

$$\langle e \cdot \phi_\alpha, (d + \eta - N) \cdot (h \cdot \overline{\phi_\alpha})\rangle_s = - \langle e \cdot \phi_\alpha, \dot{\eta}_\alpha \cdot \overline{\phi_\alpha}\rangle_s$$

We decompose $e$ as $e^{1,0} + e^{0,1}$, where $e^{1,0} \in \mathcal{L}_\bar{\mathcal{J}}$ and $e^{0,1} \in \mathcal{L}_\mathcal{J}$. We also decompose $\dot{\eta}_\alpha = \dot{\eta}_\alpha^{1,0} + \dot{\eta}_\alpha^{0,1}$, where $\dot{\eta}_\alpha^{1,0} \in \mathcal{L}_\mathcal{J}$ and $\dot{\eta}_\alpha^{0,1} \in \mathcal{L}_{\bar{\mathcal{J}}}$. Then we have

$$-\text{Im} \left( i^{-n} \langle e \cdot \phi_\alpha, \dot{\eta}_\alpha \cdot \overline{\phi_\alpha}\rangle_s \right) = - \text{Im} \left( i^{-n} \langle e^{1,0} \cdot \phi_\alpha, \dot{\eta}_\alpha^{1,0} \cdot \overline{\phi_\alpha}\rangle_s \right)$$

$$= \text{Im} \left( i^{-n} \langle e^{0,1} \cdot \phi_\alpha, \dot{\eta}_\alpha^{0,1} \cdot \overline{\phi_\alpha}\rangle_s \right)$$

$$= \text{Im} \left( i^{-n} 2 \langle e^{0,1}, \dot{\eta}_\alpha^{1,0} \rangle_{T \otimes T^*} \langle \phi_\alpha, \overline{\phi_\alpha}\rangle_s \right)$$

Since $e$ is real and $\eta_\alpha$ is pure imaginary, we have

$$\langle \dot{\eta}_\alpha, e \rangle_{T \otimes T^*} = \langle \dot{\eta}_\alpha^{1,0}, e^{1,0} \rangle_{T \otimes T^*} + \langle \dot{\eta}_\alpha^{0,1}, e^{0,1} \rangle_{T \otimes T^*}$$

$$= \langle \dot{\eta}_\alpha^{1,0}, e^{0,1} \rangle_{T \otimes T^*} + \langle -\dot{\eta}_\alpha^{1,0}, e^{1,0} \rangle_{T \otimes T^*}$$

$$= \langle \dot{\eta}_\alpha^{1,0}, e^{0,1} \rangle_{T \otimes T^*} - \langle \dot{\eta}_\alpha^{1,0}, e^{0,1} \rangle_{T \otimes T^*}$$

$$= 2 \sqrt{-1} \text{Im} \left( \langle \dot{\eta}_\alpha^{1,0}, e^{1,0} \rangle_{T \otimes T^*} \right)$$

Since $i^{-n} \langle \phi_\alpha, \overline{\phi_\alpha}\rangle_s = \text{vol}_M$, we have

$$-\text{Im} \left( i^{-n} \langle e \cdot \phi_\alpha, \dot{\eta}_\alpha \cdot \overline{\phi_\alpha}\rangle_s \right) = \text{Im} \langle \dot{\eta}_\alpha, e \rangle_{T \otimes T^*} \text{vol}_M$$

Since $i^{-n} \langle \psi_\alpha, \overline{\psi_\alpha}\rangle_s = \text{vol}_M$, we have

$$\text{Im} \langle \dot{\eta}_\alpha, e \rangle_{T \otimes T^*} \text{vol}_M = \text{Im} \langle \dot{\eta}_\alpha, e \rangle_{T \otimes T^*} i^{-n} \langle \psi_\alpha, \overline{\psi_\alpha}\rangle_s$$

Then as before, we obtain

$$\text{Im} \langle \dot{\eta}_\alpha, e \rangle_{T \otimes T^*} \text{vol}_M = \text{Im} \langle \dot{\eta}_\alpha, e \rangle_{T \otimes T^*} i^{-n} \langle \psi_\alpha, \overline{\psi_\alpha}\rangle_s$$

$$= -\text{Im} \left( i^{-n} \langle e \cdot \psi_\alpha, \dot{\eta}_\alpha \cdot \overline{\psi_\alpha}\rangle_s \right)$$
Since \( e \) is a generalized Hamiltonian element, we have
\[
e \cdot \psi_\alpha = -\sqrt{-1} \text{Id} \cdot \psi_\alpha.
\]
Applying \( d\psi_\alpha = \zeta_\alpha \cdot \psi_\alpha \), we have
\[
e \cdot \psi_\alpha = -(\sqrt{-1} \text{Id}) \cdot \psi_\alpha = -\sqrt{-1}(d(f\psi_\alpha) - f\zeta_\alpha \cdot \psi_\alpha)
= -\sqrt{-1}(d - \zeta_\alpha)(f\psi_\alpha)
\]
Then we obtain
\[
\Omega_B(\mathcal{L}_e \mathcal{J}, \dot{J}_h) = -\text{Im} \left( i^{-n} \int_M (e \cdot \psi_\alpha, \eta_\alpha \cdot \bar{\psi}_\alpha)_s \right) = \text{Im} \left( \int_M i^{-n+1} ((d - \zeta_\alpha)f\psi_\alpha, \eta_\alpha \cdot \bar{\psi}_\alpha)_s \right)
\]  
(4.25)
We already see that \( \phi_\alpha = e^{i\psi_{\alpha \beta}} \phi_\beta \). Thus it follows \( \eta_\alpha = \eta_\beta + ip_{\alpha \beta} \) where \( p_{\alpha \beta} \) denotes a real function which does not change under small deformations. Thus \( \dot{\eta}_\alpha = \dot{\eta}_\beta \). (Note that \( \eta_\alpha \) is regarded as a generalized connection form and then infinitesimal deformations of connections is given by \( \dot{\eta}_\alpha \) which is a globally defined section of \( \sqrt{-1}(T_M \oplus T_M^*) \).) From the normalization \( i^{-n}(\psi_\alpha, \bar{\psi}_\alpha)_s = \text{vol}_M \), it follows that \( f\psi_\alpha \wedge \sigma(\eta_\alpha \cdot \bar{\psi}_\alpha) = f\psi_\beta \wedge \sigma(\eta_\beta \cdot \bar{\psi}_\beta) \) for all \( \alpha, \beta \). Thus the component of \((2n-1)\)-form of \( f\psi_\alpha \wedge \sigma(\eta_\alpha \cdot \bar{\psi}_\alpha) \) is globally defined. Applying the Stokes theorem again, we obtain
\[
\Omega_B(\mathcal{L}_e \mathcal{J}, \dot{J}_h) = \text{Im} \left( i^{-n+1} \int_M \langle f\psi_\alpha, (d + \zeta_\alpha) \cdot (\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s \right)
\]  
(4.27)
Then we shall show that \( \langle \psi_\alpha, (d + \zeta_\alpha) \cdot (\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s \) defines a globally defined \( 2n \)-form on \( M \). At first, we see
\[
(d + \zeta_\alpha) \cdot (\eta_\alpha \cdot \bar{\psi}_\alpha) = d(\eta_\alpha \cdot \bar{\psi}_\alpha) + \eta_\alpha d\bar{\psi}_\alpha - \eta_\alpha d\bar{\psi}_\alpha + \zeta_\alpha \cdot \eta_\alpha \cdot \bar{\psi}_\alpha
= \mathcal{L}_{\eta_\alpha} \bar{\psi}_\alpha + \eta_\alpha \cdot \zeta_\alpha \cdot \bar{\psi}_\alpha + \zeta_\alpha \cdot \eta_\alpha \cdot \bar{\psi}_\alpha
= \mathcal{L}_{\eta_\alpha} \bar{\psi}_\alpha + 2\langle \zeta_\alpha, \eta_\alpha \rangle_{T \otimes T} \bar{\psi}_\alpha
\]
Note that \( d\bar{\psi}_\alpha = -\zeta_\alpha \cdot \bar{\psi}_\alpha \), since \( \zeta_\alpha \) is pure imaginary. Both \( \phi_\alpha \) and \( \psi_\alpha \) satisfy
\[
i^{-n}(\phi_\alpha, \bar{\phi}_\alpha)_s = i^{-n}(\psi_\alpha, \bar{\psi}_\alpha)_s = \text{vol}_M
\]
for all \( \alpha \). Thus if \( U_\alpha \cap U_\beta \) is not empty, we have \( \phi_\alpha = e^{ip\alpha \beta} \phi_\beta \) and \( \psi_\alpha = e^{iq\alpha \beta} \psi_\beta \) for real functions \( p_{\alpha \beta} \) and \( q_{\alpha \beta} \). Then we have \( \eta_\alpha = \eta_\beta + ip_{\alpha \beta} \) and \( \zeta_\alpha = \zeta_\beta + id_{\alpha \beta} \). Since \( \mathcal{L}_{dp_{\alpha \beta}} = 0 \) and \( \langle dp_{\alpha \beta}, dq_{\alpha \beta} \rangle_{T \otimes T^*} = 0 \), we see
\[
\mathcal{L}_{\eta_\alpha} \bar{\psi}_\alpha + 2\langle \zeta_\alpha, \eta_\alpha \rangle_{T \otimes T} \bar{\psi}_\alpha = \mathcal{L}_{\eta_\alpha} \bar{\psi}_\alpha + 2\langle \zeta_\alpha, \eta_\beta \rangle_{T \otimes T} \bar{\psi}_\alpha + 2\langle \zeta_\beta, id_{\alpha \beta} \rangle_{T \otimes T} \bar{\psi}_\alpha
= e^{-iq_{\alpha \beta}} (\mathcal{L}_{\eta_\beta} \bar{\psi}_\beta + 2\langle \zeta_\beta, \eta_\beta \rangle_{T \otimes T} \bar{\psi}_\beta)
+ e^{-iq_{\alpha \beta}} (2\langle \zeta_\beta, id_{\alpha \beta} \rangle_{T \otimes T} \bar{\psi}_\beta)
\]
Since $\mathcal{J}_\psi$ is given by a set of locally closed nondegenerate, pure spinors, every $\zeta_\alpha$ is a one form. Since $\zeta_\beta$ is a one form for all $\beta$, we have $\langle \zeta_\beta, id\eta_{\alpha,\beta} \rangle_{T \otimes T^*} = 0$. Thus it follows

$$\mathcal{L}_{\eta_\alpha} \bar{\psi}_\alpha + 2\langle \zeta_\alpha, \eta_\alpha \rangle_{T \otimes T^*} \bar{\psi}_\alpha = e^{-i\eta_\alpha,\beta} (\mathcal{L}_{\eta_\beta} \bar{\psi}_\beta + 2\langle \zeta_\beta, \eta_\beta \rangle_{T \otimes T^*} \bar{\psi}_\beta)$$

Hence we obtain

$$\langle \psi_\alpha, (d + \zeta_\alpha) \cdot (\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s = \langle \psi_\beta, (d + \zeta_\beta) \cdot (\eta_\beta \cdot \bar{\psi}_\beta) \rangle_s$$

Hence $\langle \psi_\alpha, (d + \zeta_\alpha) \cdot (\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s$ defines a globally defined 2-form on $M$ which is denoted by $i^n \mu(\mathcal{J})\text{vol}_M$,

$$i^n \mu(\mathcal{J})\text{vol}_M := \langle \psi_\alpha, (d + \zeta_\alpha) \cdot (\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s,$$  \hspace{1cm} (4.28)

where $\mu(\mathcal{J})$ is a function on $M$. For an infinitesimal deformation $\{\mathcal{J}_\theta\}$ in $\mathcal{B}_{\mathcal{J}_\psi}(M)$, we have

$$\Omega_B(\mathcal{L}_c \mathcal{J}, \dot{\mathcal{J}}) = \text{Im} \left( i^{-n+1} \int_M \langle f \psi_\alpha, (d + \zeta_\alpha) \cdot (\eta_\alpha \cdot \bar{\psi}_\alpha) \rangle_s \right)$$

$$= \text{Im} \left( i^{-n+1} \int_M f \left. \frac{d}{dt} \langle \psi_\alpha, (d + \zeta_\alpha)(\eta_\alpha(t) \cdot \bar{\psi}_\alpha) \rangle_s \right|_{t=0} \right)$$

$$= \text{Im} \left( i \int_M f \left. \frac{d}{dt} \mu(\mathcal{J})\text{vol}_M \right|_{t=0} \right)$$

$$= \int_M f \left. \frac{d}{dt} \mu(\mathcal{J})\text{vol}_M \right|_{t=0}$$  \hspace{1cm} (4.29)

Thus we have $\Omega_B(\mathcal{L}_c \mathcal{J}, \dot{\mathcal{J}}) = d(\mu, f)(\dot{\mathcal{J}})$, by using the coupling in terms of the integration over $M$. Hence $\mu : \mathcal{B}_{\mathcal{J}_\psi}(M) \to C^\infty(M, \mathbb{R})$ is a moment map of $\mathcal{B}_{\mathcal{J}_\psi}(M)$ for the action of $\text{Ham}_{\mathcal{J}_\psi}$, which is explicitly given by (4.28) in terms of pure spinors.

**Theorem 4.15.** Let $X = (M, J, \omega)$ be a compact Kähler manifold with a holomorphic Poisson structure $\beta$ and $(\mathcal{J}_\beta, \mathcal{J}_\psi)$ a generalized Kähler manifold which is given by Poisson deformation. Then there is a moment map $\mu : \mathcal{B}_{\mathcal{J}_\psi}(M) \to C^\infty(M, \mathbb{R})$ for the action of the group $\text{Ham}_{\mathcal{J}_\psi}$.

**Proof.** Since $(\mathcal{J}_\beta, \mathcal{J}_\psi)$ is a generalized Kähler manifold of symplectic type, $\mathcal{J}_\psi$ is given by a $d$-closed differential form. Thus the result follows form Theorem 4.6. \hfill $\Box$

**Definition 4.16.** We define the scalar curvature $S(\mathcal{J})$ of a generalized Kähler manifold $(M, J, \mathcal{J}_\psi)$ to be the moment map $\mu(\mathcal{J})$. Since the scalar curvature depends on both $\mathcal{J}$ and $\mathcal{J}_\psi$, we also denote the scalar curvature by $S(\mathcal{J}, \mathcal{J}_\psi)$.

### 5 Lie algebras of generalized complex manifolds and generalized Kähler manifolds

#### 5.1 The Lie algebra $\mathfrak{g}_J$ and the reduced Lie algebra $\mathfrak{g}_0(\mathcal{J}, \mathcal{J}_\psi)$

Let $\mathcal{J}$ be a generalized complex structure on $M$ which gives the decomposition $(T_M \oplus T_M^*)^\mathbb{C} = \mathcal{L}_\mathcal{J} \oplus \overline{\mathcal{L}_\mathcal{J}}$. Then we have the Lie algebroid complex:

$$0 \to \Lambda^0 \overline{\mathcal{L}_\mathcal{J}} \overset{\mathcal{S}_J}{\to} \Lambda^1 \overline{\mathcal{L}_\mathcal{J}} \overset{\mathcal{S}_J}{\to} \Lambda^2 \overline{\mathcal{L}_\mathcal{J}} \overset{\mathcal{S}_J}{\to} \cdots \overset{\mathcal{S}_J}{\to} \Lambda^n \overline{\mathcal{L}_\mathcal{J}} \to 0$$

We denote by $H^\bullet(\Lambda^\bullet \overline{\mathcal{L}_\mathcal{J}})$ the cohomology groups of the Lie algebroid complex $\Lambda^\bullet \overline{\mathcal{L}_\mathcal{J}}$. 18
Lemma 5.1. The first cohomology group $H^1(\wedge^*\mathcal{L}_J)$ inherits a Lie algebra structure which is induced from the Courant bracket $[,]_{\text{co}}$.

Proof. Since $\mathcal{L}_J$ is isotropic, the Courant bracket gives the Lie algebra structure on $\mathcal{L}_J$, that is, the Jacobi identity holds. Since $\mathcal{L}_J$ is a Lie bialgebroid, we have

$$\overline{\mathcal{J}}_J[e_1, e_2]_{\text{co}} = [\overline{\mathcal{J}}_J e_1, e_2]_{\text{Sch}} + [e_1, \overline{\mathcal{J}}_J e_2]_{\text{Sch}}$$

for $e_1, e_2 \in \mathcal{L}_J$, where $[,]_{\text{Sch}}$ denotes the Schouten bracket. Note that the Courant bracket restricted to $\mathcal{L}_J$ coincides with the Schouten bracket on $\mathcal{L}_J$. We also have $\overline{\mathcal{J}}_J[e, f]_{\text{Sch}} = [\overline{\mathcal{J}}_J e, f]_{\text{Sch}} + [e, \overline{\mathcal{J}}_J f]_{\text{Sch}}$, for a function $f$ and $e \in \mathcal{L}_J$. Thus the Courant bracket induces the Lie algebra structure on the first cohomology group $H^1(\wedge^*\mathcal{L}_J)$.

Then we have

**Definition 5.2.** The Lie algebra $H^1(\wedge^*\mathcal{L}_J)$ is denoted by $\mathfrak{g}_J$ which is called the Lie algebra of automorphisms of $(M, J)$.

We have the following lemma:

**Lemma 5.3.** If $e_1, e_2 \in \mathcal{L}_J$ and satisfy $\overline{\mathcal{J}}_J e_1 = \overline{\mathcal{J}}_J e_2 = 0$. Then the Courant bracket of $e_1$ and the conjugate $\overline{e}_2$ is given by

$$[e_1, \overline{e}_2]_{\text{co}} = (\partial_J - \overline{\partial}_J)(e_1, \overline{e}_2)_{\wedge T^*}.$$

Proof. Let $\{A, B\}$ be the anti-bracket $AB + BA$ for operators $A, B$ in general. Then we have the super-Jacobi identity

$$[A, \{B, C\}] = [[A, B], C] + [A, [C, B]].$$

Thus we have

$$[d, \{e_1, \overline{e}_2\}] = [[d, e_1], \overline{e}_2] + [d, [\overline{e}_2], e_1]$$

Since $\{e_1, \overline{e}_2\} = 2(e_1, \overline{e}_2)_{\wedge T^*}$, we obtain

$$2d(e_1, \overline{e}_2)_{\wedge T^*} = [[d, e_1], \overline{e}_2] + [d, [\overline{e}_2], e_1]$$

From the definition of the Courant bracket, we have $[e_1, \overline{e}_2]_{\text{co}} = \frac{1}{2}[[d, e_1], \overline{e}_2] - \frac{1}{2}[d, [\overline{e}_2], e_1]$. Since $\overline{\mathcal{J}}_J e_1 = \overline{\mathcal{J}}_J e_2 = 0$, we also have

$$[[d, e_1], \overline{e}_2] + [d, [\overline{e}_2], e_1] = [[\partial_J, e_1], \overline{e}_2] + [[\overline{\partial}_J, \overline{e}_2], e_1]$$

Since $\partial_J \overline{e}_2 = \overline{\partial}_J e_1 = 0$ and $\overline{\partial}_J e_1 = \overline{\partial}_J e_1 = 0$, applying the super-Jacobi identity again, we have

$$[[\partial_J, e_1], \overline{e}_2] - [[\overline{\partial}_J, \overline{e}_2], e_1] = \partial_J [e_1, \overline{e}_2] - \overline{\partial}_J [\overline{e}_2, e_1]$$

(5.1)

(5.2)

Hence we have

$$[e_1, \overline{e}_2]_{\text{co}} = (\partial_J - \overline{\partial}_J)(e_1, \overline{e}_2)_{\wedge T^*}.$$
We define a map $F : L_J \rightarrow (T_M \oplus T^*_M)$ by taking the real part of $e \in L_J$,

$$F(e) := e + \bar{e} \quad (5.3)$$

Then $F$ restricted to $\ker \partial_J$ yields the map from $\ker \partial_J$ to the real part $(\ker \partial_J + \ker \partial_J)^R$. Taking the quotient, we have the map from $g_J$ to $(\ker \partial_J + \ker \partial_J)^R/(\Im \partial_J + \Im \partial_J)^R$. By the abuse of notation, we denote by $F$ the map to the quotient. Then we have

**Proposition 5.4.** The quotient $(\ker \partial_J + \ker \partial_J)^R/(\Im \partial_J + \Im \partial_J)^R$ is a Lie algebra with respect to the Courant bracket and

$$F : g_J \rightarrow (\ker \partial_J + \ker \partial_J)^R/(\Im \partial_J + \Im \partial_J)^R$$

is an isomorphism between Lie algebras.

**Proof.** For $e_1, e_2 \in \ker \partial_J$, from Lemma 5.3 and taking the complex conjugate, one has

$$[e_1 + \bar{e}_1, e_2 + \bar{e}_2] = [e_1, e_2] + [\bar{e}_1, e_2] + [e_1, \bar{e}_2] + [\bar{e}_1, \bar{e}_2] \quad (5.4)$$

$$= [e_1, e_2] + [\bar{e}_1, \bar{e}_2] \quad (5.5)$$

since $(\partial_J - \bar{\partial}_J)((e_1, e_2)_{T\oplus T^*} - (\bar{e}_1, \bar{e}_2)_{T\oplus T^*}) \in (\Im \partial_J + \Im \partial_J)^R$. Hence $(\ker \partial_J + \ker \partial_J)^R/(\Im \partial_J + \Im \partial_J)^R$ is a Lie algebra with respect to the Courant bracket. Since we see

$$[F(e_1), F(e_2)] = F([e_1, e_2]) \mod (\Im \partial_J + \Im \partial_J)^R,$$

thus $F$ is an isomorphism between Lie algebras. \qed

Let $(M, J, J_\psi)$ be a generalized Kähler manifold. Then we define a subspace $\tilde{g}_0(J_\psi)$ of $\ker \partial_J$ by

$$\tilde{g}_0(J, J_\psi) := \{ J_\psi(\partial_J u) | \partial_J J_\psi \partial_J u = 0, u \in C^\infty(M, \mathbb{C}) \} \subset \overline{L_J}$$

For simplicity, we also denote by $\tilde{g}_0$ the subspace $\tilde{g}_0(J, J_\psi)$ . Since $\tilde{g}_0$ is a subspace of $\ker \partial_J$, we have the following diagram:

$$\tilde{g}_0 \xrightarrow{i} \ker \partial_J \xrightarrow{j} g_J := \ker \partial_J/\Im \partial_J$$

Then we have

**Proposition 5.5.** If $M$ is compact, then the map $j : \tilde{g}_0 \rightarrow g_J$ is injective.

**Proof.** It suffices to show that the intersection $\tilde{g}_0 \cap \Im \partial_J = \{0\}$. We assume that there exist two functions $u$ and $v$ such that

$$J_\psi(\partial_J u) = \partial_J v.$$
Since $\overline{\partial}_J u - \sqrt{-1} J_\psi (\overline{\partial}_J u) \in \mathcal{L}_{\mathcal{J}_0}$, we have $\overline{\partial}_J u - \sqrt{-1} \overline{\partial}_J v = \overline{\partial}_J (u - \sqrt{-1} v) \in \mathcal{L}_{\mathcal{J}_0}$. We have the decomposition $\mathcal{L}_J = \mathcal{L}_J^+ \oplus \mathcal{L}_J^-$, where $\mathcal{L}_J^+ = \mathcal{L}_J \cap \mathcal{L}_{\mathcal{J}_0}$ and $\mathcal{L}_J^- = \mathcal{L}_J \cap L_{\mathcal{J}_0}$. Thus $\overline{\partial}_J (u - \sqrt{-1} v) \in \mathcal{L}_J^-$. Hence

$$\overline{\partial}_J^-(u - \sqrt{-1} v) = 0,$$

where $\overline{\partial}_J = \overline{\partial}_J^- + \overline{\partial}_J^+$. By $\overline{\partial}_J u + \sqrt{-1} J_\psi (\overline{\partial}_J u) \in \mathcal{L}_{\mathcal{J}_0}$, we have $\overline{\partial}_J u + \sqrt{-1} \overline{\partial}_J v = \overline{\partial}_J (u + \sqrt{-1} v) \in \mathcal{L}_{\mathcal{J}_0}$. Thus we also have $\overline{\partial}_J^+ (u + \sqrt{-1} v) \in \mathcal{L}_J^+$. Hence

$$\overline{\partial}_J^+(u + \sqrt{-1} v) = 0.$$  

Thus we have

$$\overline{\partial}_J^- (u - \sqrt{-1} v) \cdot \overline{\psi} = \delta_-( (u - \sqrt{-1} v) \overline{\psi}) = 0$$
$$\overline{\partial}_J^+ (u + \sqrt{-1} v) \cdot \psi = \delta_+ ((u + \sqrt{-1} v) \psi) = 0$$

Since the generalized Kähler identity, the Laplacian $(\overline{\partial}_J \overline{\partial}^*)^\ast \overline{\partial} + \overline{\partial}_J (\overline{\partial} J^-)^\ast$ of the operator $\overline{\partial}_J$ is $\frac{1}{2} \Delta$, where $\Delta$ denotes the ordinary Laplacian $dd^* + d^*d$. Since $(u \pm \sqrt{-1} v)$ is a function, then it follows from (5.7) and (5.8) that $\Delta (u \pm \sqrt{-1} v) = 0$. Thus $u + \sqrt{-1} v$ and $u - \sqrt{-1} v$ are constants. Thus $u$ and $v$ are constants also. Hence $\mathcal{J}_\psi (\overline{\partial}_J u) = \overline{\partial}_J v = 0$. Thus we have $g_0 \cap \text{Im} \overline{\partial}_J = \{0\}$.  

**Definition 5.6.** We define $g_0$ to be the image $j(\overline{g}_0)$ in $g_J$.

**Proposition 5.7.** $g_0 \subset g_J := H^1 (\Lambda^* \mathcal{L}_J)$ is a Lie subalgebra.

**Proof.** Since $\mathcal{J}_\psi$ is integrable, the Nijenhuis tensor vanishes,

$$[\mathcal{J}_\psi e_1, \mathcal{J}_\psi e_2]_{\psi} = [e_1, e_2]_{\psi} + \mathcal{J}_\psi [\mathcal{J}_\psi e_1, e_2]_{\psi} + \mathcal{J}_\psi [e_1, \mathcal{J}_\psi e_2]_{\psi},$$  

(5.9)

where $e_1, e_2 \in \mathcal{L}_J$. For simplicity, we denote by $\mathcal{J}$ the operator $\overline{\partial}_J$. For $u, v \in C^\infty (M, \mathbb{C})$, we assume $\mathcal{J}_\psi (\overline{\partial}_J u)$ and $\mathcal{J}_\psi (\overline{\partial}_J v) \in \mathcal{L}_J$ satisfy $\mathcal{J}_\psi (\overline{\partial}_J u) = 0$ and $\mathcal{J}_\psi (\overline{\partial}_J v) = 0$, respectively. Then from (5.9), we have

$$[\mathcal{J}_\psi (\overline{\partial}_J u), \mathcal{J}_\psi (\overline{\partial}_J v)_{\psi} = [\overline{\partial}_J u, \overline{\partial}_J v]_{\psi} + \mathcal{J}_\psi [\mathcal{J}_\psi (\overline{\partial}_J u), \overline{\partial}_J v]_{\psi} + \mathcal{J}_\psi [\overline{\partial}_J u, \mathcal{J}_\psi (\overline{\partial}_J v)_{\psi}$$  

(5.10)

Since $\mathcal{L}_J$ is a Lie bialgebroid, we have $[\overline{\partial}_J e_1, e_2]_{\psi} = [\overline{\partial}_J e_1, e_2]_{\psi} + [e_1, \overline{\partial}_J e_2]_{\psi}$, for $e_1, e_2 \in \mathcal{L}_J$. Thus we have $[\overline{\partial}_J u, \overline{\partial}_J v]_{\psi} = \overline{\partial}_J [u, \overline{\partial}_J v]_{\psi}$, which vanishes as an element of $H^1 (\Lambda^* \mathcal{L}_J)$. From our assumption $\mathcal{J}_\psi (\overline{\partial}_J u) = 0$, $\mathcal{J}_\psi (\overline{\partial}_J v) = 0$, we have

$$\mathcal{J}_\psi (\overline{\partial}_J u), \mathcal{J}_\psi (\overline{\partial}_J v)_{\psi} = \mathcal{J}_\psi (\mathcal{J}_\psi (\overline{\partial}_J u), \overline{\partial}_J v)_{\psi}$$  

(5.11)

$$\mathcal{J}_\psi (\overline{\partial}_J u), \mathcal{J}_\psi (\overline{\partial}_J v)_{\psi} = \mathcal{J}_\psi (\overline{\partial}_J u, \mathcal{J}_\psi (\overline{\partial}_J v)_{\psi}$$  

(5.12)

Thus we obtain

$$[\mathcal{J}_\psi (\overline{\partial}_J u), \mathcal{J}_\psi (\overline{\partial}_J v)]_{\psi} = \overline{\partial}_J [u, \overline{\partial}_J v]_{\psi} + \mathcal{J}_\psi [\overline{\partial}(\mathcal{J}_\psi (\overline{\partial}_J u), \overline{\partial}_J v)]_{\psi}$$

where $[\mathcal{J}_\psi (\overline{\partial}_J u)]_{\psi} := \mathcal{L}_J (\mathcal{J}_\psi (\overline{\partial}_J u)) u$ and $[\mathcal{J}_\psi (\overline{\partial}_J v)]_{\psi} := -\mathcal{L}_J (\mathcal{J}_\psi (\overline{\partial}_J v)) u$ and

$$[\mathcal{J}_\psi (\overline{\partial}_J u), v]_{\psi} + [u, \mathcal{J}_\psi (\overline{\partial}_J v)]_{\psi} \in C^\infty (M, \mathbb{C})$$

We denote by $\{ u, v \}_{J, \psi}$ the complex function $[\mathcal{J}_\psi (\overline{\partial}_J u), \overline{\partial}_J v]_{\psi}$. Then we have

$$[\mathcal{J}_\psi (\overline{\partial}_J u), \mathcal{J}_\psi (\overline{\partial}_J v)]_{\psi} \cong \mathcal{J}_\psi (\overline{\partial}_J u, \mathcal{J}_\psi (\overline{\partial}_J v)) \in H^1 (\Lambda^* \mathcal{L}_J)$$

Hence the result follows.  

Then $g_0$ is called the Lie algebra of reduced automorphisms of $(M, \mathcal{J}, \mathcal{J}_\psi)$.  

21
5.2 The real Lie algebra $g_0^\mathbb{R}$ of the Lie algebra of the reduced automorphisms

In this section, we assume that $(M, J, \mathcal{J}_\psi)$ is a compact generalized Kähler manifold of symplectic type, i.e., $\mathcal{J}_\psi$ is given by a $d$-closed, nondegenerate, pure spinor $\psi = e^{H - \sqrt{-1} \omega}$, where $\omega$ denotes a real symplectic 2-form on $M$ and $B$ is a real $d$-closed 2-form on $M$. Since the map $j$ is injective, $j$ gives an isomorphism between $g_0^\mathbb{R}$ and $\mathfrak{g}_0$. Thus $g_0$ is identified with $\mathfrak{g}_0$.

**Definition 5.8.** Consider a real Lie subalgebra $g_0^\mathbb{R}$ of $g_0$ by

$$g_0^\mathbb{R} := \{ \mathcal{J}_\psi \overline{\partial}_J u \in g_0 | u \in C^\infty(M, \mathbb{R}) \}$$

A Lie algebra is called a reductive Lie algebra if the radical of the Lie algebra equals the center, where the radical is the maximal solvable ideal. A reductive Lie algebra is the direct sum of a semisimple Lie algebra and an abelian Lie algebra. It is known that a Lie algebra is reductive if the associated Lie group of the Lie algebra is a compact Lie group.

**Proposition 5.9.** The real sub Lie algebra $g_0^\mathbb{R}$ is a reductive Lie algebra if a compact generalized Kähler manifold $(M, J, \mathcal{J}_\psi)$ is of symplectic type.

**Proof.** By using the $B$-field transformation, Proposition 5.9 reduces to the case $B = 0$. Thus it suffices to show Proposition in the case $B = 0$. By using the map $F$ as in (5.3), it follows from $u \in C^\infty(M, \mathbb{R})$ that $F(\mathcal{J}_\psi \overline{\partial}_J u) = \mathcal{J}_\psi du$. Since $B = 0$, $\mathcal{J}_\psi du$ is an ordinary Hamiltonian vector field with respect to $\omega$. Thus the real Lie algebra $g_0^\mathbb{R}$ is isomorphic to $F(\mathfrak{g}_0^\mathbb{R})$ which is a subgroup of the Lie algebra of Hamiltonian vector fields

$$\{ \mathcal{J}_\psi du \in T_M | u \in C^\infty(M, \mathbb{R}) \}$$

Since a Hamiltonian vector field acts on $M$ preserving $\mathcal{J}_\psi$ and $g_0^\mathbb{R}$ also preserves $\mathcal{J}$, it follows that $g_0^\mathbb{R}$ preserves the generalized metric $G$ of $(M, J, \mathcal{J}_\psi)$. The generalized metric $G$ consists of a Riemannian metric $g$ and a 2-form $b$ which satisfies $d^*_\psi \omega_{1+} = -d^*_\psi \omega_{1-} = db$. Thus a Hamiltonian vector field $\mathcal{J}_\psi du \in F(\mathfrak{g}_0^\mathbb{R})$ is a Killing vector field with respect to $g$ which preserves $b$. Since $L_e \mathcal{J} = 0$ is equivalent to $(\overline{\partial}_J e) = 0$ for $e \in \mathcal{J}$, if a Hamiltonian vector field $\mathcal{J}_\psi du$ is a Killing vector field preserving $b$, then $\mathcal{J}_\psi du \in g_0^\mathbb{R}$. Thus $\mathcal{J}_\psi du \in g_0^\mathbb{R}$ if and only if $\mathcal{J}_\psi du \in g_0^\mathbb{R}$ is a Killing vector field which preserves $b$. We denote by $C_0^\mathbb{R}$ the associated Lie group with $g_0^\mathbb{R}$. Then $G_0^\mathbb{R}$ is a subgroup of the isometry group $\text{Isom}(M, g)$ of the Riemannian manifold $(M, g)$. It is known that $\text{Isom}(M, g)$ is a compact Lie group of finite dimension. Let $\text{Symp}_0(M, \omega)$ be the identity component of diffeomorphisms which preserves $\omega$. We denote by $\text{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms. Then the following theorem is known as the Flux conjecture which is affirmatively solved.

**Theorem 5.10.** [29] $\text{Ham}(M, \omega)$ is a closed subgroup of $\text{Symp}_0(M, \omega)$ with respect to $C^1$-topology.

Thus it follow from Theorem 5.10 that the intersection $\text{Ham}(M, \omega) \cap \text{Isom}(M, g)$ is a compact Lie group. The group $G_0^\mathbb{R}$ is a subgroup of $\text{Ham}(M, \omega) \cap \text{Isom}(M, g)$ which preserves $b$. Let $\{ f_i \}$ be a set of $G_0^\mathbb{R}$ which converges to a function $f_\infty \in \text{Ham}(M, \omega) \cap \text{Isom}(M, g)$ with respect to $C^r$-topology for $r \geq 1$. Then we have $\lim_{i \to \infty} f_i^* b = f_\infty^* b$. Since $f_i^* b = b$, we have $f_\infty^* b = b$. Thus $G_0^\mathbb{R}$ is a closed subgroup of a compact Lie group $\text{Ham}(M, \omega) \cap \text{Isom}(M, g)$, which is also a compact Lie group. Hence $g_0^\mathbb{R}$ is a Lie algebra of a compact Lie group. Thus $g_0^\mathbb{R}$ is a reductive Lie algebra. □
5.3 Reductivity of \( g_0 \)

Let \( g_0 \) be the Lie algebra of reduced automorphisms of \((M, J, J_\psi)\) as before. A complex function \( u \) is denoted by \( u_R + \sqrt{-1} u_{1\text{m}} \), where \( u_R \) is the real part of \( u \) and \( u_{1\text{m}} \) is the imaginary part of \( u \). We consider the following condition (5.13) on a generalized Kähler manifold \((M, J, J_\psi)\):

If a complex function \( u \) satisfies \( \bar{\partial} J_\psi \bar{\partial} u = 0 \), then both \( u_R \) and \( u_{1\text{m}} \) also satisfy

\[
\bar{\partial} J_\psi \bar{\partial} u_R = 0, \quad \bar{\partial} J_\psi \bar{\partial} u_{1\text{m}} = 0
\]

(5.13)

**Proposition 5.11.** Let \((M, J, J_\psi)\) be a compact generalized Kähler manifold of symplectic type which satisfies the condition (5.13). Then \( g_0 \) is the complexification of the Lie algebra of a compact Lie group, that is, \( g_0 \) is a reductive Lie algebra.

**Proof.** The condition (5.13) implies that \( g_0 \) is the complexification of \( g_0^R \). Then the result follows from Proposition 5.9. \( \square \)

5.4 The structure theorem of the Lie algebra \( g_0 \) and the Lie algebra \( g_J \)

Let \((M, J, J_\psi)\) be a compact generalized Kähler manifold of symplectic type. Then we have the decomposition \( \bar{\mathcal{L}}_J = \bar{\mathcal{L}}_J^+ \oplus \bar{\mathcal{L}}_J^- \) and \( \bar{\partial} J_\psi = \bar{\partial}_+ + \bar{\partial}_- \). We define \( \bar{\mathcal{L}}^{p,q} := \wedge^p \bar{\mathcal{L}}_J^+ \otimes \wedge^q \bar{\mathcal{L}}_J^- \). Then we have the double complex \((\bar{\mathcal{L}}^{\bullet, \bullet}, \bar{\partial}_+ , \bar{\partial}_-)\). In generalized Kähler manifold, the space of differential forms is decomposed into \( \bigoplus_{p,q} U^{p,q} \), where \(-n \leq p + q \leq n \) and \(-n \leq -p + q \leq n \). The exterior derivative \( d \) is decomposed into \( \delta_+ + \delta_- + \bar{\partial}_+ + \bar{\partial}_- \) and it is known that the generalized Kähler identity does holds. Then the double complex \((U^{\bullet, \bullet}, \bar{\partial}_+ , \bar{\partial}_-)\) defines the cohomology groups \( H^{p,q}(M, J, J_\psi) \) and the generalized Hodge decomposition holds:

\[
\oplus_{i=0}^{2n} H^i(M, \mathbb{C}) = \oplus_{p,q} H^{p,q}(M, J, J_\psi)
\]

The isotropic space \( \bar{\mathcal{L}}^{p,q}_J \) acts on \( \psi \) by the Spin action which is given by the interior product and the \( \bar{\partial}_+ \)-closed element of \( \bar{\mathcal{L}}_J = \bar{\mathcal{L}}_J^+ + \bar{\mathcal{L}}_J^- \). We define \( U^{p,q} := \wedge^p \bar{\mathcal{L}}_J^+ \otimes \wedge^q \bar{\mathcal{L}}_J^- \). We denote by \( H^{odd}(M, \mathbb{C}) \) the direct sum \( \bigoplus_{i=0}^{2n} H^{2i+1}(M, \mathbb{C}) \) of the de Rham cohomology groups of odd degree. Let \([a]\) be a class in \( g_J = H^1(\wedge^\bullet \bar{\mathcal{L}}_J) \). Then the representative \( a = a_+ + a_- \) is a \( \bar{\partial}_+ \)-closed element of \( \bar{\mathcal{L}}_J = \bar{\mathcal{L}}_J^+ + \bar{\mathcal{L}}_J^- \), where \( a_+ \in \bar{\mathcal{L}}_J^+ \) and \( a_- \in \bar{\mathcal{L}}_J^- \). The condition \( \bar{\partial}_J a_+ = 0 \) is equivalent to \( \bar{\partial}_+ a_+ = 0, \bar{\partial}_- a_- = 0, \bar{\partial}_- a_+ + \bar{\partial}_+ a_- = 0 \). Then \( a_+ \) acts on \( \psi \) to obtain \( a_+ \cdot \psi \in U^{1,-n+1} \) and \( a_- \) also acts on \( \bar{\psi} \) to get \( a_- \cdot \bar{\psi} \in U^{1,n-1} \). Since \( \psi \) is \( d \)-closed, we obtain

\[
\bar{\partial}_+(a_+ \cdot \psi) = (\bar{\partial}_+ a_+) \cdot \psi = 0, \quad \bar{\partial}_-(a_- \cdot \bar{\psi}) = (\bar{\partial}_- a_-) \cdot \bar{\psi} = 0.
\]

If \( a = \bar{\partial}_J u = \bar{\partial}_+ u + \bar{\partial}_- u \), then we have

\[
a_+ \cdot \psi = (\bar{\partial}_+ u) \cdot \psi = \bar{\partial}_+(u \psi), \quad a_- \cdot \bar{\psi} = (\bar{\partial}_- u) \cdot \bar{\psi} = \bar{\partial}_-(u \bar{\psi}).
\]

Thus \( a_+ \psi \) defines a class \( [a_+ \psi] \in H^{1,-n+1}(M, J, J_\psi) \) and \( a_- \) also defines a class \( [a_- \bar{\psi}] \in H^{1,n-1}(M, J, J_\psi) \). Thus we have a map

\[
\begin{align*}
\phi_J : H^{1,-n+1}(M, J, J_\psi) &\oplus H^{1,n-1}(M, J, J_\psi) \\
&\rightarrow H^1(M, J, J_\psi)
\end{align*}
\]
Since $\psi$ is a differential form $e^{b-\sqrt{-1}\omega}$, it follows from the generalized Hodge decomposition that $H^{1,-n+1}(M, J, J_\psi) \oplus H^{1,n-1}(M, J, J_\psi)$ is isomorphic to $H^1(M, \mathbb{C})$. In fact, $(a_+ \cdot \phi) \wedge e^{b-\sqrt{-1}\omega} + (a_- \cdot \phi) \wedge e^{b-\sqrt{-1}\omega}$ gives a representative of $H^1(M, \mathbb{C})$. Thus we obtain a map $j : g_J \to H^1(M, \mathbb{C})$.

**Theorem 5.12** (the structure theorem of $g_J$). Then the following exact sequence of Lie algebras holds

$$0 \to g_0 \xrightarrow{i} g_J \xrightarrow{j} H^1(M, \mathbb{C}) \to 0,$$

where $g_0$ is the Lie algebra of reduced automorphisms of $(M, J, J_\psi)$, which is a Lie subalgebra of $g_J$, and $g_0$ has a Lie subalgebra $g_0^R$ which is a real reductive Lie algebra. Moreover, $H^1(M, \mathbb{C})$ is a commutative Lie algebra.

**Proof.** First we shall show that $\ker j = i(g_J)$. If a class $[a] = [a_+ + a_-] \in g_J$ satisfies $j([a]) = 0 \in H^1(M, \mathbb{C})$, then there exist two functions $u, v$ such that

$$a_+ \cdot \psi = \overline{\partial}_+ (u \psi) = (\overline{\partial}_+ u) \cdot \psi, \quad a_- \cdot \overline{\psi} = \overline{\partial}_- (v \overline{\psi}) = (\overline{\partial}_- v) \cdot \overline{\psi}.$$ 

Thus we see $a_+ = \overline{\partial}_+ u$, $a_- = \overline{\partial}_- v$. Since $\overline{\partial}_\pm = \frac{1}{2} (\partial_J + \sqrt{-1} J_\psi \overline{\partial}_J)$, we have

$$a = a_+ + a_- = \frac{1}{2} (\partial_J - \sqrt{-1} J_\psi \overline{\partial}_J) u + \frac{1}{2} (\partial_J + \sqrt{-1} J_\psi \overline{\partial}_J) v = \frac{1}{2} \overline{\partial}_J (u + v) - \frac{1}{2} \sqrt{-1} J_\psi \overline{\partial}_J (u - v).$$

Thus the class $[a] = \overline{\partial}_J (u - v)$ is represented by $-\frac{1}{2} \sqrt{-1} J_\psi \overline{\partial}_J (u - v) \in g_0$. Hence $\ker j \subset i(g_0)$. Conversely, a class $i(g_0)$ is represented by $J_\psi \overline{\partial}_J u = \sqrt{-1} \overline{\partial}_+ u - \sqrt{-1} \overline{\partial}_- u$. Then we see $i(g_0) \subset \ker j$. Hence $\ker j = i(g_0)$. From the generalized Hodge decomposition theorem, it follows that $j$ is surjective. By using the generalized $\partial_J \overline{\partial}_J$-lemma, we obtain $[a, b]_{g_0} \in g_0$ for all $a, b \in g_J$ satisfying $\overline{\partial}_J a = \overline{\partial}_J b = 0$. Thus the quotient $g_J / g_0 = H^1(M, \mathbb{C})$ is a commutative Lie algebra. \qed

**Corollary 5.13.** Let $(M, J, J_\psi)$ be a compact generalized Kähler manifold of symplectic type. If $H^1(M, \mathbb{C}) = 0$, then $g_J \cong g_0$.

**Proof.** The result follows from Theorem 5.12. \qed

### 5.5 The Lie algebras $g_J$ and $g_0$ of generalized Kähler manifolds which are given by Poisson deformations

Let $X = (M, J)$ be a compact complex manifold with a Kähler form $\omega$ and $\beta$ a holomorphic Poisson structure. We assume that $H^1(X, \mathcal{O}) = 0$ in this section. We denote by $\{(M, J_\beta^t, J_\psi^t)\}$ a family of generalized Kähler manifolds which is given by Poisson deformations, where $\psi_t = e^{b_t-\sqrt{-1}\omega_t}$ is the closed, nondegenerate, pure spinor, where $t$ is a parameter of deformations. Note that $\omega_t$ is a symplectic form which is not of type $(1, 1)$ with respect to the ordinary complex structure $J$. Then we have the Lie algebroid complex :

$$0 \to C^\infty(M, \mathbb{C}) \xrightarrow{\overline{\partial}_J^t} \mathcal{L}_{J_\beta^t} \xrightarrow{\overline{\partial}_J^t} \wedge^2 \mathcal{L}_{J_\beta^t} \xrightarrow{\overline{\partial}_J^t} \ldots$$

Then we have the Lie algebra $H^1(\wedge^* \mathcal{L}_{J_\beta^t})$ and we see that the Lie algebra $H^1(\wedge^* \mathcal{L}_{J_\beta})$ does not depend on $t \neq 0$. For simplicity, we denote by $g_{J_\beta}$ the Lie algebra $H^1(\wedge^* \mathcal{L}_{J_\beta})$. 

24
Proposition 5.14. We assume that $H^1(X, O) = 0$. Then the Lie algebra $\mathfrak{g}_{J_\beta}$ of automorphisms with respect to $J_\beta$ is given by the Lie algebra of holomorphic vector fields preserving the holomorphic Poisson structure $\beta$, i.e.,

$$\mathfrak{g}_{J_\beta} = \{ V \in H^0(X, T^1,0_J) \mid L_V \beta = 0 \}.$$ 

Proof. For simplicity, we denote by $\mathfrak{g}_\beta$ the Lie algebra $\mathfrak{g}_{J_\beta} \cong H^1(\Lambda \mathcal{T}_{J_\beta})$. The cohomology $H^1(\Lambda \mathcal{T}_{J_\beta})$ is the total cohomology of the double complex $(\Lambda^p T_J^1 \otimes \wedge^q J_\beta)$, where $\Lambda_{\beta 1} = \{ \theta + [\bar{\theta}, \theta] \mid \theta \in \wedge_{\beta 1} \}$ and $\mathcal{G}_\beta = e^{-\beta} \circ \mathcal{O} \circ e^\beta$ and $\Lambda_{\beta 0} = \Lambda_{\beta 1}$ and $\delta_\beta : \Lambda^p T_J^1 \rightarrow \Lambda^{p+1} T_J^1$ denotes the Poisson complex. The complex $(\Lambda^p T_J^1 \otimes \wedge^q J_\beta)$ for each $p$ is quasi-isomorphic to the ordinary Dolbeault complex $(\wedge^p T_J^1 \otimes \wedge^q, \partial)$. Thus taking the cohomologies by using $\mathcal{G}_\beta$ at first, we have the $E_1$-terms in terms of the ordinary Dolbeault cohomology groups,

$$E_1^{p,q} = H^0(\Lambda \mathcal{T}_{J_\beta})$$

Since $H^1(X, O) = 0$, we have

$$E_1^{0,0} = 0, \quad E_1^{1,0} = H^0(X, T^1,0_J)$$

Thus the $E_2$-terms are given by

$$E_2^{0,0} = 0, \quad E_2^{1,0} = \text{Ker} \delta_\beta : H^0(X, T^1,0_J) \rightarrow H^0(X, T^2,0_J).$$

Thus the total cohomology $H^1(\Lambda \mathcal{T}_{J_\beta})$ is $E_2^{1,0}$. Since $\delta_\beta V$ is given by the Lie derivative $L_V \beta$ by $\beta$ by $V$, we have $H^1(\Lambda \mathcal{T}_{J_\beta}) = \{ V \in H^0(X, T^1,0_J) \mid L_V \beta = 0 \}$. Hence we see that $\mathfrak{g}_\beta = \{ V \in H^0(X, T^1,0_J) \mid L_V \beta = 0 \}$. Thus the result follows. 

We also have the Lie algebra of the reduced automorphisms $\mathfrak{g}_0(\mathcal{J}_\beta, \mathcal{J}_\psi, t) \subset H^1(\Lambda \mathcal{T}_{J_\beta,t})$ of a generalized Kähler manifold $(M, J_\beta, \mathcal{J}_\psi)$ as in Section 5.1.

Proposition 5.15. We assume that $H^1(M, \mathbb{C}) = 0$. Then the Lie algebra $\mathfrak{g}_{J_\beta}$ of automorphisms coincides with the Lie algebra $\mathfrak{g}_0 := \mathfrak{g}_0(\mathcal{J}_\beta, \mathcal{J}_\psi)$ of the reduced automorphisms of a generalized Kähler manifold $(M, \mathcal{J}_\beta, \mathcal{J}_\psi)$.

Proof. The result follows from Corollary 5.13. 

6 The Lie algebra of automorphisms of generalized Kähler manifolds with constant scalar curvature (Matsushima-Lichnerowicz type theorems)

Let $(M, \mathcal{J}, \mathcal{J}_\psi)$ be a generalized Kähler manifold of symplectic type, where $\psi = e^{b - \sqrt{-1} \omega}$ and $\omega$ is a symplectic form. Then the generalized metric $G = -\mathcal{J} \circ \mathcal{J}_\psi$ gives a Hermitian metric on $\Lambda \mathcal{T}_{\mathcal{J}}$. The operator $\overline{\partial}_\mathcal{J} : \Lambda^* \mathcal{T}_{\mathcal{J}} \to \Lambda^* + 1 \mathcal{T}_{\mathcal{J}}$ together with $\mathcal{J}_\psi : \mathcal{T}_{\mathcal{J}} \to \mathcal{T}_{\mathcal{J}}$ gives a second order differential operator $\overline{\partial}_\mathcal{J} \mathcal{J}_\psi \overline{\partial}_\mathcal{J} : C^\infty(M, \mathbb{C}) \to \Lambda^2 \mathcal{T}_{\mathcal{J}}$. The adjoint operator of $\overline{\partial}_\mathcal{J} \mathcal{J}_\psi \overline{\partial}_\mathcal{J}$ is denoted by $(\overline{\partial}_\mathcal{J} \mathcal{J}_\psi \overline{\partial}_\mathcal{J})^* : \Lambda^2 \mathcal{T}_{\mathcal{J}} \to C^\infty(M, \mathbb{C})$. Then we define the fourth order differential operator $L : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$ by the composition

$$L = (\overline{\partial}_\mathcal{J} \mathcal{J}_\psi \overline{\partial}_\mathcal{J})^* \circ (\overline{\partial}_\mathcal{J} \mathcal{J}_\psi \overline{\partial}_\mathcal{J}). \quad (6.1)$$

25
We denote by $\overline{L}$ the complex conjugate of the operator $L$, i.e.,

$$L = (\partial J \overline{J} \partial J)^* \circ (\partial J \overline{J} \partial J) \quad (6.2)$$

For simplicity, we also denote by $\overline{J}$ the operator $\overline{J}$ in this section.

**Definition 6.1.** For $u \in C^\infty(M, \mathbb{C})$, we define $X_u$ to be $J_u \overline{\partial} u \in \Omega^u$. Let $X_u$ denotes a real element $X_u + \overline{X}_u = T_M \oplus T_M$. Then the real $X_u = \nu + \theta$ is a family of diffeomorphisms $F_t^\nu$ generated by the vector field $v$ and $d\theta$ is a $d$-exact $2$-form.

Let $\{J_t^u\}$ be deformations of generalized complex structures which are given by $J_t^u = J$ and $J_t^u := (F_t^\nu)\# J$, where $(F_t^\nu)\# J$ denotes the action of $F_t^\nu \in \text{Diff}(M)$ on $J$. Then infinitesimal deformations of $J_t^u$ is given by $\varepsilon_u := \overline{\partial} J_u \partial u \in \Lambda^2 \Omega^u$. From the moment map formula, we already know the formula of derivation of generalized scalar curvature under deformations of $J$ preserving $J$. For $w \in C^\infty(M, \mathbb{R})$, we have

$$\frac{d}{dt} \int_M S(J_t^u) w \text{vol}_M \bigg|_{t=0} = \Omega_B(\mathring{J}_u, \mathring{J}_w),$$

where $\Omega_B$ is the Kähler form of $B_J(M)$ as before and $\mathring{J}_u$ and $\mathring{J}_w$ are infinitesimal deformations given by $\varepsilon_u$ and $\varepsilon_w$, respectively. Then $\Omega_B(\mathring{J}_u, \mathring{J}_w)$ is given by the Imaginary part of the integration of $h(\varepsilon_u, \varepsilon_w)$ over $M$,

$$\Omega_B(\mathring{J}_u, \mathring{J}_w) = \text{Im} \int_M h(\varepsilon_u, \varepsilon_w) \text{vol}_M = \frac{1}{2\sqrt{-1}} \{ \int_M h(\varepsilon_u, \varepsilon_w) - h(\varepsilon_u, \varepsilon_w) \} \text{vol}_M,$$

and $h(\cdot, \cdot)$ denotes the Hermitian metric on $\Lambda^2 \Omega^u$ which is given by

$$h(\varepsilon_1, \varepsilon_2) := 4 \text{tr} (\text{ad}_{\varepsilon_1}, \text{ad}_{\varepsilon_2}),$$

where $\text{ad}_{\varepsilon_1} := [\varepsilon_1, \cdot] \in \text{Hom}(\mathbb{C}J, \Omega^u)$ and $\text{ad}_{\varepsilon_2} := [\varepsilon_2, \cdot] \in \text{Hom}(\Omega^u, \mathbb{C}J)$ (see also Section 7 in [14]). Since $h(\varepsilon_u, \varepsilon_w) = 4 h(\overline{\partial} J_u \partial u, \overline{\partial} J_w \partial u)$, $h(\varepsilon_u, \varepsilon_w) = 4 h(\partial J_u \partial u, \partial J_w \partial u)$, applying (6.1) and (6.2), we obtain

$$\frac{d}{dt} \int_M S(J_t^u) w \text{vol}_M \bigg|_{t=0} = \frac{2}{\sqrt{-1}} \left\{ \int_M (L^u) w \text{vol}_M - \int_M (\overline{L}^u) w \text{vol}_M \right\} \quad (6.3)$$

A complex function $u$ is written as $u = u_\mathbb{R} + \sqrt{-1} u_\mathbb{I}$. If $u$ is a real function, i.e., $u = u_\mathbb{R}$, then we have $X_u := J_u \overline{\partial} u + J_u \partial u = J_u \partial u$. Thus $X_u$ is a Hamiltonian element of $u$ with respect to $J_u$. We denote by $\{F_t^{\mathbb{R}}\}$ the corresponding family of $\text{Diff}(M)$ which gives $J_t^u := (F_t^{\mathbb{R}})\# J$. Since $(F_t^{\mathbb{R}})$ also preserves the volume form, we obtain

$$S(J_t^u) w \text{vol}_M = (F_t^{\mathbb{R}})\# (S(J_0) \text{vol}_M) w$$

$$= (F_t^{\mathbb{R}})^* (S(J_0) \text{vol}_M) w.$$

(Note that $F_\# = F_*^{-1} \circ F^*$ acts on a differential form $\alpha$ by $F_\# \alpha = F^* \alpha$.) Since $F_t^{\mathbb{R}}$ is a Hamiltonian element of $u_\mathbb{R}$, we have

$$\frac{d}{dt} \int_M S(J_t^u) w \text{vol}_M \bigg|_{t=0} = + \int_M \{u_\mathbb{R}, S(J_0)\} J_u \partial u \text{vol}_M.$$

(26)
where $\{,\rangle_{\mathcal{J}_\psi}$ denotes the Lie derivative $\mathcal{L}_{X_u} S(\mathcal{J}_0)$ which is a generalization of the Poisson bracket. Thus if both $u$ and $w$ are real functions, i.e., $u = u_R, w = w_R$, then from (6.3) we have

$$
\int_M \{u_R, S(\mathcal{J}_0)\}_{\mathcal{J}_\psi} w_R \text{vol}_M = \frac{2}{\sqrt{-1}} \int_M (L u_R - \mathcal{L} u_R) w_R \text{vol}_M
$$

(6.7)

Then we obtain

**Proposition 6.2.**

$$
\{u, S(\mathcal{J})\}_{\mathcal{J}_\psi} = \frac{2}{\sqrt{-1}} (L - \mathcal{L}) u
$$

for every complex function $u$

**Proof.** From (6.7), the formula holds for a real function $u$. Then it follows that the formula holds for every complex function since both sides are $C$-linear with respect to $u$. \qed

Next we shall show the derivation formula of the generalized scalar curvature in the case of a pure imaginary function $u = \sqrt{-1} u_{\text{Im}}$. Then $\mathcal{J}_\psi \partial u = \sqrt{-1} \mathcal{J}_\psi \partial u_{\text{Im}}$ gives a family $\{F_t^u\}$ of $\tilde{\text{Diff}}(M)$ which yields deformations $\mathcal{J}_t^u := (F_t^u)_\# \mathcal{J}$ as in Definition 6.1. Then we have

**Proposition 6.3.**

$$
\frac{d}{dt} S(\mathcal{J}_t^u)|_{t=0} = 2(L + \mathcal{L}) u_{\text{Im}}
$$

**Proof.** Since $\mathcal{J}_\psi \partial u = 2\sqrt{-1} \partial_- \mathcal{J}_\psi u_{\text{Im}} \in \mathcal{T}_\mathcal{J}^+ \wedge \mathcal{T}_\mathcal{J}^-$, we see that $\mathcal{J}_\psi \partial u$ is an infinitesimal tangent of generalized complex structures at $\mathcal{J}$ preserving $\mathcal{J}_\psi$. Thus we can apply the formula of derivation of the moment map. Since the derivation of the generalized scalar curvature is given by the Moment map formula as before, we have

$$
\frac{d}{dt} \int_M S(\mathcal{J}_t^u) w \text{vol}_M |_{t=0} = \Omega_B(\mathcal{J}_t^u, \mathcal{J}_t^w) = \text{Im} \int_M h(\varepsilon_u, \varepsilon_w) \text{vol}_M,
$$

for a real function $w \in C^\infty(M, \mathbb{R})$. Since $u = \sqrt{-1} u_{\text{Im}}$, we have

$$
\begin{align*}
    h(\varepsilon_u, \varepsilon_w) &= 4 h(\sqrt{-1} \partial_- \mathcal{J}_\psi u_{\text{Im}}, \partial \mathcal{J}_\psi \partial w), \\
    h(\varepsilon_u, \varepsilon_w) &= 4 h(-\sqrt{-1} \partial \mathcal{J}_\psi u_{\text{Im}}, \partial \mathcal{J}_\psi \partial w).
\end{align*}
$$

(6.8)

(6.9)

Thus from (6.3), we also have

$$
\frac{d}{dt} \int_M S(\mathcal{J}_t^u) w \text{vol}_M |_{t=0} = 2 \int_M (L u_{\text{Im}}) w \text{vol}_M + 2 \int_M (\mathcal{L} u_{\text{Im}}) w \text{vol}_M
$$

(6.10)

Thus we obtain the result. \qed

Then we obtain

**Theorem 6.4.** If the scalar curvature $S(\mathcal{J}, \mathcal{J}_\psi)$ is a constant, then the Lie algebra of the reduced automorphisms $g_0$ is reductive.
Proof. Since $S(J)$ is a constant, we have $\{u, S(J)\}_{\mathcal{J}_c} = 0$. Then from Proposition 6.2, we have $L = \overline{\mathcal{T}}$. Thus if a complex function $u$ satisfies $Lu = 0$, then $\overline{\mathcal{T}}u = 0$. Then the real part $u_R$ of $u$ also satisfies $Lu_R = 0$. Hence if a complex function $u$ satisfies $\overline{\mathcal{J}_c}u = 0$, then both $u_R$ and $u_{\text{Im}}$ also satisfy

$$\overline{\mathcal{J}_c}u_R = 0, \quad \overline{\mathcal{J}_c}u_{\text{Im}} = 0$$

(6.11)

Thus the condition (5.13) holds. Then it follows from Proposition 5.11 that $\mathfrak{g}_0$ is reductive.

Theorem 6.5. Let $(M, J)$ be a $2n$ dimensional compact generalized complex manifold. We assume that $H^1(M, \mathbb{C}) = 0$. If $M$ admits a generalized Kähler structure $(J, J_\varphi)$ of symplectic type with constant scalar curvature, then the Lie algebra $\mathfrak{g}_J$ is a reductive Lie algebra.

Proof. From Corollary 5.13, we see $\mathfrak{g}_J = \mathfrak{g}_0$. Then the result follows from Theorem 6.4

Theorem 6.6. Let $(M, I, \omega)$ be a compact Kähler manifold with a holomorphic Poisson structure $\beta \neq 0$. We assume $H^1(M, \mathbb{C}) = 0$. We denote by $(M, J_\beta, J_\psi)$ a generalized Kähler manifold given by Poisson deformations. Then if the scalar curvature $S(J_\beta, J_\psi)$ is a constant, the Lie algebra of the automorphisms $\mathfrak{g}_{J_0}$ is a reductive Lie algebra.

Proof. It follows from Theorem 6.4 that $\mathfrak{g}_0$ is reductive. Then from Proposition 5.15, we have $\mathfrak{g}_0 = \mathfrak{g}_{J_\beta}$. Thus the result follows.

7 The Lie algebra $\mathfrak{g}_\beta$ of automorphisms of $(\mathbb{C}P^2, J_\beta)$

Let $X = (M, J, \omega)$ be the complex projective surface $\mathbb{C}P^2$ and $\beta$ a Poisson structure on $X$, where $M$ denotes the underlying differentiable manifold and $J$ is a complex structure and $\omega$ is a Kähler structure on $M$. Since $\beta$ is a holomorphic section of $K_X^{-1}$, $\beta$ is given by a homogeneous polynomial $f(z_0, z_1, z_2)$ of degree 3 (a cubic curve). It is known that cubic curves are classified into nine cases as shown in the following figures:
From the deformations-stability theorem, there exists a family of generalized Kähler structures \((J_{\beta t}, J_{\psi t})\). Since \(H^1(X, \mathbb{C}) = 0\), it follows from Proposition 5.15 that the Lie algebra of automorphisms \(g_{J_{\beta t}}\) coincides with the Lie algebra of reduced automorphisms \(g_0\). Since \(g_{J_{\beta t}}\) is the same for \(t \neq 0\), for simplicity, we denote by \(g_{\beta}\) the Lie algebra \(g_{J_{\beta t}}\) for \(t \neq 0\). From Theorem 6.5, if the scalar curvature \(S(J_{\beta t}, J_{\psi t})\) is constant, the Lie algebra \(g_{\beta}\) is reductive. Thus we have an obstruction to the existence of constant scalar curvature on a generalized complex manifold \((M, J_{\beta t})\) for \(t \neq 0\). From Proposition 5.14, we already know that the Lie algebra \(g_{\beta}\) is given by the Lie algebra of holomorphic vector fields preserving the holomorphic Poisson structure \(\beta\), i. e., \(g_{\beta} = \{V \in H^0(X, T^1,0) \mid L_V \beta = 0\}\). The Lie algebra \(sl(3, \mathbb{C})\) acts on \(\mathbb{C}P^2\) linearly. Let \(f := f(z_0, z_1, z_2)\) be a homogeneous polynomial of degree 3 given by \(\beta\). Then \(g_{\beta}\) is given by

\[ g_{\beta} = \{ a \in sl(3, \mathbb{C}) \mid a^* f = 0 \}, \]

where \(a^* f\) denotes the action of the Lie algebra \(sl(3, \mathbb{C})\) on the space of homogeneous polynomials of degree 3.

Then we have the following explicit calculations:

**Figure 1**: If an anticanonical divisor is a smooth elliptic curve \(C\), then we see \(g_{\beta} = 0\).

**Figure 2**: In the case of a nodal curve, we also have \(g_{\beta} = 0\).

**Figure 3,4,5,6**: If an anticanonical divisor is a curve given in Figure 3, 4, 5, 6, then it follows that \(g_{\beta}\) is abelian. Thus \(g_{\beta}\) is reductive.

**Figure 7**: In the cases where an anticanonical divisor \(f\) is given by three lines intersecting at one point
(Figure 7), $f$ can be taken a following form $z_0z_1(z_0 + z_1)$. Then $g_β$ is generated by matrices

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
* & * & 0
\end{pmatrix}
$$

Then $g_β$ has a nonabelian solvable ideal. Thus $g_β$ is not reductive.

**Figure 8:** If the anticanonical divisor $f$ is $z_0^2z_1$ which consists of a double line $z_0^2$ and another line $z_1$ (Figure 8). Then we see $g_β$ is generated by

$$
\begin{pmatrix}
-2t & 0 & 0 \\
0 & t & 0 \\
* & * & t
\end{pmatrix}, \quad (t \in \mathbb{C}).
$$

Thus $g_β$ is not reductive also.

**Figure 9:** If an anticanonical divisor $f$ is a triple line $z_0^3$ (figure. 9), then $g_β$ is generated by the following elements

$$
\begin{pmatrix}
0 & 0 & 0 \\
* & t & * \\
* & * & -t
\end{pmatrix}, \quad (t \in \mathbb{C}).
$$

Then $g_β$ has a nonabelian solvable ideal which is generated by

$$
\begin{pmatrix}
0 & 0 & 0 \\
* & 0 & 0 \\
* & 0 & 0
\end{pmatrix}
$$

Thus the radical of $g_β$ is not abelian. Thus $g_β$ is not reductive.

Thus we see that $g_β$ is reductive for the cases as in Figures 1, 2, 3, 4, 5, 6. However $g_β$ is not reductive for the cases as in figures 7, 8, 9. Then we have

**Proposition 7.1.** Let $(M, J_β)$ be a generalized complex manifold which $β$ is given by three cases as in Figures 7, 8, 9. Then $(M, J_β)$ does not admits a generalized Kähler structure with constant scalar curvature.

**Proof.** Since $g_β$ is not reductive in these three cases, the result follows Theorem 6.5.

We have previously shown that the existence of generalized Kähler structures with constant scalar curvature in the cases of Figure 6.

**Proposition 7.2.** If $β$ is given by three lines in general position, then there exists generalized Kähler structures with constant scalar curvature

**Proof.** In the cases when the Poisson tensor is given by an action of 2-dimensional torus on $\mathbb{C}P^2$, we can apply the result in [14] (see Proposition 12.3).
8 Deformations of generalized Kähler manifolds with constant scalar curvature

Let \((\mathcal{J}, \mathcal{J}_\psi)\) be a generalized Kähler manifold of symplectic type on a compact manifold \(M\) and \(S(\mathcal{J}, \mathcal{J}_\psi)\) the scalar curvature of \((\mathcal{J}, \mathcal{J}_\psi)\). For simplicity, \((\mathcal{J}, \mathcal{J}_\psi)\) is denoted by \((\mathcal{J}, \psi)\). We assume that the scalar curvature \(S(\mathcal{J}, \psi)\) is a constant \(\bar{S}\), which is topologically given by the 1-st Chern class of the canonical line bundle \(K_\mathcal{J}\) together with the class \([\psi]\). We will consider a natural deformation problem for generalized Kähler structures of symplectic type with constant scalar curvature under fixing the cohomology class \([\psi]\). If we fix the class \([\psi]\) and \(\mathcal{J}\) and deform \(\psi\) such that \((\mathcal{J}, \psi_s)\) are generalized Kähler structures parametrized by \(s \in (-\varepsilon, \varepsilon)\), then it follows from the \(\bar{\partial}\overline{\partial}\)-lemma for generalized Kähler manifolds that the derivative \(\dot{\psi} := \frac{d}{dt}\psi|_{s=0}\) is given by

\[
\dot{\psi} := d(\bar{\partial}_\mathcal{J}\psi) \in \mathcal{U}^{0,-n} \oplus \mathcal{U}^{0,-n+2}, \tag{8.1}
\]

for a complex function \(u\). However, we only consider deformations \(\psi_s\) given by a real function \(u\) to apply the implicit function theorem later. If we have deformations \(\{\mathcal{J}_t\}\) of generalized complex structures with \(\mathcal{J}_0 = \mathcal{J}\) and \(t\) denotes a parameter of deformations satisfying \(|t| < \varepsilon\). Then deformation-stability theorem provides deformations of generalized Kähler structures \((\mathcal{J}_t, \mathcal{J}_\psi_s)\) such that \(\mathcal{J}_{\psi_0} = \mathcal{J}_\psi\). Further it turns out that deformation-stability theorem yields 2-parameter deformations \((\mathcal{J}_t, \psi_{t,u})\) of generalized Kähler manifolds of symplectic type which are smoothly parametrized by \(t\) and a real function \(u\). We denote by \(S(\mathcal{J}_t, \psi_{t,u})\) the scalar curvature of \((\mathcal{J}_t, \psi_{t,u})\). We normalize a function \(u\) such that \(\int_M u\text{vol}_M = 0\). Let \(L^2_k(M)\) be the Sobolev space of real functions on \(M\) whose first \(k\) derivatives are square integrable. The Sobolev embedding theorem states that \(L^2_k(M) \subset C^l(M)\) if \(k > n + l\), where \(2n = \dim\, M\) and \(C^l(M)\) denotes the space of continuous functions whose derivative of order at most \(l\) are also continuous. Note that \(L^2_k(M)\) is a Banach algebra if \(k > n\). We denote by \(L^2_k(M)/\mathbb{R}\) the space of normalized functions of \(L^2_k(M)\). We shall define the scalar curvature operator \(S\) as a non-linear differential operator. First assume that there exist deformations \(\{\mathcal{J}_t\}\) of generalized complex structures parametrized by \(t\) in a neighborhood of the origin of \(\mathbb{C}\). We need to take \((t, u)\) in a small open set \(D_\varepsilon \times \mathcal{U}\) of the origin of \(\mathbb{C} \times L^2_{k+4}(M)/\mathbb{R}\) such that \((\mathcal{J}_t, \psi_{t,u})\) is a generalized Kähler structure, where \(D_\varepsilon = \{t \in \mathbb{C} \mid -\varepsilon < |t| < \varepsilon\}\). Then we define the map

\[\mathcal{S} : D_\varepsilon \times \mathcal{U} \rightarrow L^2_k(M)/\mathbb{R}\]

which is given by \(\mathcal{S}(\mathcal{J}_t, \psi_{t,u}) := S(\mathcal{J}_t, \psi_{t,u}) - \bar{S}\). Then we have

**Theorem 8.1.** For \(k > n\), the map \(\mathcal{S}\) is well-defined and the derivative of \(\mathcal{S}\) at the origin along the direction of the function space \(\mathcal{U}\) is given by

\[
d\mathcal{S}_{(0,0)} : L^2_{k+4}(M)/\mathbb{R} \rightarrow L^2_k(M)/\mathbb{R}
\]

\[
u \mapsto 2Lu, \tag{8.2}
\]

where \(L\) is the fourth order differential operator \(L = (\bar{\partial}_\mathcal{J} \mathcal{J}_\psi \bar{\partial}_\mathcal{J})^* \circ (\bar{\partial}_\mathcal{J} \mathcal{J}_\psi \bar{\partial}_\mathcal{J})\) as in (6.1).

**Proof.** Since \((\bar{\partial}_\mathcal{J} \psi) = (\bar{\partial}_+ u) \cdot \psi = -\sqrt{-1}(\mathcal{J}_\psi \bar{\partial}_\mathcal{J} u) \cdot \psi\), from (8.1), we have

\[
\dot{\psi} = d(\bar{\partial}_\mathcal{J} u \cdot \psi) = -\sqrt{-1}d(\mathcal{J}_\psi \bar{\partial}_\mathcal{J} u) \cdot \psi.
\]
We denote by \( v \) the pure imaginary function \(-\sqrt{-1} u\). Let \( F_s^v \) be a family of \( \text{Diff}(M) \) for the pure imaginary function \( v \) as in Definition 6.1, where \( s \) denotes a parameter. Define \( \psi_s \) by \((F_s^v)^* \psi \). Since \( \frac{d}{ds} (F_s^v)^* (\psi) \big|_{s=0} \) is given by the Lie derivative \( \{d, (\mathcal{J} \vartheta \mathcal{J} + \mathcal{J} \vartheta \mathcal{T})\}\) \( \psi \), then it follows that

\[
\frac{d}{ds}(F_s^v)^*(\psi) \big|_{s=0} = d((\mathcal{J} \vartheta \mathcal{J} + \mathcal{J} \vartheta \mathcal{T}) \cdot \psi) = -\sqrt{-1} d((\mathcal{J} \vartheta \mathcal{J} u) \cdot \psi).
\]  

(8.4)

Thus we have

\[
\psi := \frac{d}{ds}(F_s^v)^*(\psi) \big|_{s=0}
\]

and we have \( \frac{d}{ds} S(\mathcal{J}_t, \psi_{t,us}) \big|_{s=0} = \frac{d}{ds} S(\mathcal{J}_t, (F_s^v)^* \psi) \big|_{s=0} \). Since the scalar curvature is equivalent under the action of \( \text{Diff}(M) \), we have

\[
S \left( (F_s^v)^* \circ (F_s^{v'})^* \mathcal{J}_t, (F_s^{v'})^* \psi \right) = (F_s^{v'})^* S \left( (F_s^{v'})^* \mathcal{J}_t, \psi \right)
\]

(8.5)

From our assumption \( S(\mathcal{J}, \psi) = 0 \), then we have

\[
\frac{d}{ds} S(\mathcal{J}_t, \psi_{t,us}) \big|_{s,t=0} = \frac{d}{ds} S((F_s^{v'})^* \mathcal{J}_t, \psi) \big|_{s,t=0}
\]

(8.6)

Since \( S(\mathcal{J}, \psi) \) is a constant, Proposition 6.2 shows \( L = \mathcal{L} \). Applying Proposition 6.3 to a pure imaginary function \(-v = \frac{\sqrt{-1}}{2} u\), we obtain

\[
\frac{d}{ds} S((F_s^{v'})^* \mathcal{J}_t, \psi) \big|_{s=0} = (L + \mathcal{L}) u = 2Lu
\]

Thus the differential of \( S \) at \((0,0)\) is given by \( dS_{0,0}(u) = 2Lu \).

\[ \square \]

**Theorem 8.2.** Let \((\mathcal{J}, \psi)\) be a generalized Kähler structure of symplectic type on a compact manifold \( M \) with constant scalar curvature \( S(\mathcal{J}, \psi) \). We assume that the Lie algebra of the reduced automorphisms \( \mathfrak{g}_0 \) is trivial. Then for deformations generalized complex structures \( \{\mathcal{J}_t\}, (-\epsilon < |t| < \epsilon) \), there exist deformations of generalized Kähler structures \((\mathcal{J}_t, \psi_{t,us})\) with constant scalar curvature for sufficiently small \( t \).

**Proof.** From Theorem 8.1, \( \ker dS_{(0,0)} \) is given by \( \ker L \). Since \( \mathfrak{g}_0 = 0 \), it follows \( \ker L = 0 \). Since \( L \) is a self-dual operator, it follows that \( dS_{(0,0)} \) is surjective and has a right inverse operator. Thus applying the implicit function theorem of Hilbert spaces, we obtain deformations of generalized Kähler structures \( \{\mathcal{J}_t, \psi_{t,us}\} \) such that \( S(\mathcal{J}_t, \psi_{t,us}) = 0 \). Hence we obtain the result. \( \square \)

**Example 8.3.** Let \((M, J, \omega)\) be a compact Kähler manifold with constant scalar curvature. We assume that the reduced automorphisms of \((M, J)\) is trivial. If there exists a nonzero holomorphic Poisson structure \( \beta \) on \((M, J)\), then there exist deformations of generalized Kähler structures \((\mathcal{J}_\beta, \psi_t)\) with constant scalar curvature. In particular, del Pezzo surfaces with trivial automorphisms admit generalized Kähler structures with constant scalar curvature.
9 Generalized extremal Kähler manifolds and Hessian formula

Our moment map framework naturally leads us to define a generalized extremal Kähler manifold. We denote by $\mathcal{J}_\psi$ a generalized complex structure given by $\psi$ which is fixed in this section.

**Definition 9.1.** Let $(M, \mathcal{J}, \mathcal{J}_\psi)$ be a generalized Kähler manifold and $S(\mathcal{J})$ the generalized scalar curvature of $(M, \mathcal{J}, \mathcal{J}_\psi)$. If $\overline{\partial}_J\mathcal{J}_\psi\overline{\partial}S(\mathcal{J}) = 0$, then $(\mathcal{J}, \mathcal{J}_\psi)$ is a generalized extremal Kähler structure and $(M, \mathcal{J}, \mathcal{J}_\psi)$ is called a generalized extremal Kähler manifold.

**Proposition 9.2.** Let $\mathcal{E}$ be a functional on the space

$$\{\mathcal{J} | (\mathcal{J}, \mathcal{J}_\psi) \text{ is a generalized Kähler structure}\}$$

which is defined by

$$\mathcal{E}(\mathcal{J}) = \frac{1}{2} \int_M S(\mathcal{J})^2 \text{vol}_M$$

Then a critical point of the functional $\mathcal{E}$ is attained by a generalized extremal Kähler structure.

**Proof.** Let $\varepsilon \in \wedge^2 \mathbb{L}_J$ is an infinitesimal deformation which is given by a one-parameter deformation $\{\mathcal{J}_\varepsilon\}$. Then the differential of $\mathcal{E}$ is given by

$$\frac{d}{dt} \mathcal{E}(\mathcal{J}_\varepsilon)|_{t=0} = \int_M \mathcal{J}_\varepsilon\overline{\partial}S(\mathcal{J})\text{vol}_M$$

Since $S(\mathcal{J})$ gives the moment map, we have

$$\frac{d}{dt} \mathcal{E}(\mathcal{J}_\varepsilon)|_{t=0} = \Omega_B(\mathcal{J}_{S(\mathcal{J})}, \mathcal{J}_\varepsilon),$$

where $\mathcal{J}_{S(\mathcal{J})}$ denotes the infinitesimal deformation corresponding to $\overline{\partial}_J\mathcal{J}_\psi\overline{\partial}S(\mathcal{J})$. Since $\Omega_B$ is nondegenerate, $\mathcal{J}$ is a critical point of $\mathcal{E}$ if and only if $\mathcal{E}_{S(\mathcal{J})} = \overline{\partial}_J\mathcal{J}_\psi\overline{\partial}S(\mathcal{J}) = 0$. Thus the result follows. \qed

In order to calculate the Hessian of the functional $\mathcal{E}$, we need several lemmas. A diffeomorphism $f \in \text{Diff}(M)$ gives rise to a bundle map $f_\# : T_M \oplus T^*_M \rightarrow T_M \oplus T^*_M$ which is defined by $f_\#(v, \theta) = (f^{-1}_*(v), f^*(\theta))$, where $v \in T_M$ and $\theta \in T^*_M$. A 2-form $b$ acts on $T_M \oplus T^*_M$ by $\text{Ad}_b$. Then $F = e^bf \in \text{Diff}(M)$ also gives a bundle map which is the composition $F_\#(v, \theta) = \text{Ad}_b \circ f_\#(v, \theta) = \text{Ad}_b(f^{-1}_*(v), f^*(\theta)) = (f^{-1}_*(v) + b(f^{-1}_*(v)), f^*(\theta))$. Thus $F \in \text{Diff}(M)$ acts on $\mathcal{J}$ by the adjoint $F_\# \circ \mathcal{J} \circ F^{-1}_\#$. For simplicity we denote by $F_\# \mathcal{J}$ the adjoint $F_\# \circ \mathcal{J} \circ F^{-1}_\#$.

**Lemma 9.3.** We denote by $P := P(\mathcal{J}, \mathcal{J}_\psi)$ the operator $\overline{\partial}_J\mathcal{J}_\psi\overline{\partial}_J$ and $L := L(\mathcal{J}, \mathcal{J}_\psi)$ the 4-th order differential operator $(\overline{\partial}_J\mathcal{J}_\psi\overline{\partial}_J)^*(\overline{\partial}_J\mathcal{J}_\psi\overline{\partial}_J)$. For $F = e^{df} \in \text{Diff}(M)$, we have

$$P(F_\#\mathcal{J}, F_\#\mathcal{J}_\psi) = F_\# \circ P(\mathcal{J}, \mathcal{J}_\psi) \circ F^{-1}_\#$$

Further we have

$$L(F_\#\mathcal{J}, F_\#\mathcal{J}_\psi) = F_\# \circ L(\mathcal{J}, \mathcal{J}_\psi) \circ F^{-1}_\#$$
Proof. $F \in \text{Diff}(M)$ acts on a differential form $\alpha$ by $F_#\alpha := e^b \wedge f^* \alpha$, where $f^*$ denotes the pull back of $\alpha$ by $f \in \text{Diff}(M)$ and $e^b \wedge$ is the wedge product of $e^b$. The we have $d \circ F_# = F_# \circ d$. Since $F_# J = F_# \circ J \circ F_#^{-1}$, we have

$$2\overline{\partial} F_#(\mathcal{J}) = d + \sqrt{-1} F_#(\mathcal{J}) d = F_# \circ d \circ F_#^{-1} + \sqrt{-1} F_# \circ J \circ d \circ F_#^{-1}$$

$$= 2F_# \circ (\overline{\partial} \mathcal{J}) \circ F_#^{-1}.$$ 

Since $F_#(\mathcal{J}_\psi) = F_# \circ \mathcal{J}_\psi \circ F_#^{-1}$, we also have

$$(\overline{\partial} F_#\mathcal{J} \circ F_# \mathcal{J}_\psi \circ \overline{\partial} F_# \mathcal{J}) = \left( F_# \circ \overline{\partial} \mathcal{J} \circ \mathcal{J}_\psi \circ \overline{\partial} \mathcal{J} \circ F_#^{-1} \right)$$

(9.1)

Hence we obtain

$$P(F_#(\mathcal{J}), F_#(\mathcal{J}_\psi)) = F_# \circ P(\mathcal{J}, \mathcal{J}_\psi) \circ F_#^{-1}$$

We also have $(F_# \circ P \circ F_#^{-1})^* = F_# \circ P^* \circ F_#^{-1}$. Then we obtain

$$L(F_#(\mathcal{J}), F_#(\mathcal{J}_\psi)) = (F_# \circ P^* \circ F_#^{-1}) \circ (F_# \circ P \circ F_#^{-1})$$

(9.3)

$$= F_# \circ P^* P \circ F_#^{-1}$$

(9.4)

$$= F_# \circ L \circ F_#^{-1}$$

(9.5)

Hence

$$L(F_#(\mathcal{J}), F_#(\mathcal{J}_\psi))(F_# S) = F_#(L(\mathcal{J}, \mathcal{J}_\psi) S)$$

Thus we obtain

$$L(F_# \mathcal{J}, F_# \mathcal{J}_\psi) = F_# \circ L(\mathcal{J}, \mathcal{J}_\psi) \circ F_#^{-1}$$

\[\square\]

Let $(M, \mathcal{J}, \mathcal{J}_\psi)$ be a generalized extremal Kähler manifold as before. For a real function $u$, we define a real element $e_u \in T_M \oplus T_M^*$ by

$$(du) \cdot \psi = \sqrt{-1} e_u \cdot \psi.$$  

(9.6)

Applying $\mathcal{J}$ to the both sides, we see that (9.6) is equivalent to $\mathcal{J}_\psi(du) = -e_u$.

Let $\{F_s\}$ be the family of the extended diffeomorphisms $\text{Diff}(M)$ which is generated by $-\mathcal{J} e_u \in T_M \oplus T_M^*$, that is, \( \frac{d}{ds} F_s \big|_{s=0} = -\mathcal{J} e_u \), and $F_0 = \text{id}$ and $\varepsilon < s < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Since $\mathcal{J}_\psi = \mathcal{J}_\psi \mathcal{J}$, we have $-\mathcal{J} e_u = \mathcal{J}_\psi \mathcal{J} du = \mathcal{J}_\psi \mathcal{J} du = \mathcal{J}_\psi(\sqrt{-1} \partial u - \sqrt{-1} \bar{\partial} u)$. Thus $F_s$ coincides with $F_s \sqrt{-1} u$ which are deformations given by a pure imaginary function $\sqrt{-1} u$ as in Definition 6.1. Since $-\mathcal{J} e_u$ is written as $v + \eta \in T_M \oplus T_M^*$, $F_s$ is given by $F_s = e^{du} f_s$, where $du$ is the exact 2-form and $f_s$ is a diffeomorphism of $M$. Then $F_s$ acts on $\psi$ by $(F_s)_#^* \psi := e^{du} \wedge f_s^* \psi$. We also denote by $\psi_s$ the nondegenerate, pure spinor $e^{du} \wedge f_s^* \psi$. For simplicity, we denote by $S$ the scalar curvature $S(\mathcal{J})$.

Lemma 9.4. Let $\{\psi_s\}$ be a family which is given by $(F_s)_#^* \psi$. Then we have

$$\frac{d}{ds} \psi_s \big|_{s=0} = \mathcal{L}_{\mathcal{J}(e_u)} \psi = d(\overline{\partial}_+ u - \partial_- u) \psi.$$ 

The projection of $\mathcal{L}_{\mathcal{J}(e_u)} \psi$ to the component $U_{\mathcal{J}_\psi}^{-n+2}$ is given by

$$\pi_{U_{\mathcal{J}_\psi}^{-n+2}} \mathcal{L}_{\mathcal{J}(e_u)} \psi = -2(\overline{\partial}_+ \partial_- u) \psi.$$
Proof. From (9.6), we have
\[ d\mathcal{J}(du) \cdot \psi = +\sqrt{-1}d\mathcal{J}(e_u) \cdot \psi = +\sqrt{-1}\mathcal{L}_{\mathcal{J}(e_u)}\psi. \]
where \( \mathcal{L}_{\mathcal{J}(e_u)} \) denotes the differential operator \( d \circ \mathcal{J}(e_u) + \mathcal{J}(e_u) \circ d \). Then we have
\[ \mathcal{L}_{\mathcal{J}(e_u)}\psi = -\sqrt{-1}(\mathcal{J}(du)\psi) \]
\[ = -\sqrt{-1}(\sqrt{-1}\overline{\mathcal{J}}_+ u - \sqrt{-1}\partial_- u) \cdot \psi \]
\[ = d(\overline{\partial}_+ u - \partial_- u)\psi \]
(9.7)
(9.8)
(9.9)

Then we have
\[ \pi_{\mathcal{J}_{\psi}^{-n+2}}\mathcal{L}_{\mathcal{J}(e_u)}\psi = \pi_{\mathcal{J}_{\psi}^{-n+2}}(d(\overline{\partial}_+ u - \partial_- u)\psi) \]
\[ = 2(\partial_- \overline{\partial}_+ u)\psi = -2(\overline{\partial}_+ \partial_- u)\psi \]
(9.10)

\[ \square \]

Note that \( (\overline{\partial}_+ \partial_- u) \in \mathcal{L}_{\mathcal{J}}^+ \cap \mathcal{L}_{\mathcal{J}}^- \subset \wedge^2 \mathcal{L}_{\mathcal{J}} \psi \) which is not a differential operator but a tensor. Let \( \{ \mathcal{J}_{\psi_n} \} \) be a family of generalized complex structures which are given by \( \psi_{-s} \). We denote by \( \hat{\mathcal{J}}_\psi \) the differential of \( \frac{d}{dE} \mathcal{J}_{\psi_{-s}} \mid_{s=0} \).

**Lemma 9.5.** Let \( E_{-s} \in \mathcal{L}_{\psi_{-s}} \) be a smooth family with \( E = E_0 \in \mathcal{L}_{\psi} \). We denote by \( \hat{E} \) the differential \( \frac{d}{dE_{-s}} \mid_{s=0} \) and \( \pi_{\mathcal{L}_{\psi}} \) the projection to \( \mathcal{L}_{\psi} \). Then we have
\[ \hat{\mathcal{J}}_\psi E = -2\sqrt{-1}\pi_{\mathcal{L}_{\psi}^+} \hat{E} = +4\sqrt{-1}[\overline{\partial}_+ \partial_- u, E], \]
for \( E \in \mathcal{L}_{\mathcal{J}_\psi} \),
\[ \hat{\mathcal{J}}_\psi \mathcal{E} = +2\sqrt{-1}\pi_{\mathcal{L}_{\psi}^-} \mathcal{E} = -4\sqrt{-1}[\partial_+ \overline{\partial}_- u, \mathcal{E}], \]
for \( \mathcal{E} \in \mathcal{L}_{\mathcal{J}_\psi} \).
(9.11)
(9.12)

Proof. Since \( E_{-s} \in \mathcal{L}_{\psi_{-s}} \), we have \( E_{-s} \cdot \psi_{-s} = 0 \) Since \( E \in \mathcal{L}_{\psi} \), we also have \( E \cdot \psi = 0 \). Thus we have \( E \cdot \pi_{\mathcal{J}_{\psi}^{-n}}(\mathcal{L}_{\mathcal{J}(e_u)}\psi) = 0 \). Since \( \psi = \mathcal{L}_{\mathcal{J}(e_u)}\psi \), it follows from Lemma 9.4 that the differential of both sides \( E_{-s} \cdot \psi_{-s} = 0 \) gives
\[ 0 = \hat{E} \cdot \psi + E \cdot \hat{\psi} = \hat{E} \cdot \psi + E \cdot \mathcal{L}_{\mathcal{J}(e_u)}\psi \]
\[ = \hat{E} \cdot \psi + E \cdot \pi_{\mathcal{J}_{\psi}^{-n+2}}(\mathcal{L}_{\mathcal{J}(e_u)}\psi) \]
\[ = \hat{E} \cdot \psi - 2[E, \overline{\partial}_+ \partial_- u] \cdot \psi \]
\[ = E \cdot \psi + 2[\overline{\partial}_+ \partial_- u, E] \cdot \psi, \]
(9.13)
(9.14)
(9.15)
(9.16)
(9.17)
(9.18)

where \([, , \) denotes the commutator of the Clifford algebra. Hence we have
\[ \hat{E} \cdot \psi = -2[\overline{\partial}_+ \partial_- u, E] \cdot \psi \]
(9.19)

The \( \mathcal{L}_{\psi} \)-component of \( \hat{E} \) is denoted by \( \pi_{\mathcal{L}_{\psi}} \hat{E} \). Then we have
\[ \pi_{\mathcal{L}_{\psi}} \hat{E} = -2[\overline{\partial}_+ \partial_- u, E]. \]
(9.20)
Since $E_{-s} \in L_{J_{\psi}}$, we have

$$J_{\psi} E_{-s} = -\sqrt{-1} E_{-s}.$$  

The differential of the both sides yields

$$\dot{J}_{\psi} E = -\sqrt{-1} E$$

Then from (9.20), we have

$$\dot{J}_{\psi} E = -2\sqrt{-1} \pi_{L_{\psi}} \dot{E} = 4\sqrt{-1} \langle \partial_{+} \partial_{-} u, E \rangle$$  \hspace{1cm} (9.21)

Taking the complex conjugate, we also have

$$\dot{J}_{\psi} E = 2\sqrt{-1} \bar{E} = -4\sqrt{-1} \langle \partial_{+} \partial_{-} u, E \rangle,$$  \hspace{1cm} (9.22)

for $\bar{E} \in L_{\psi}$. \hfill $\Box$

Then applying (9.21) and (9.22) to $\partial_{\pm} S$ respectively, we have

$$\dot{J}_{\psi} (\partial S) = \dot{J}_{\psi} (\partial_{+} S + \partial_{-} S)$$
$$= -4\sqrt{-1} \langle \partial_{+} \partial_{-} u, \partial_{+} S \rangle + 4\sqrt{-1} \langle \partial_{+} \partial_{-} u, \partial_{-} S \rangle$$
$$= +4\sqrt{-1} \langle \partial_{-} \partial_{+} u, \partial_{+} S \rangle + 4\sqrt{-1} \langle \partial_{+} \partial_{-} u, \partial_{-} S \rangle$$  \hspace{1cm} (9.23)

**Lemma 9.6.** If $\partial_{\pm} \partial_{\pm} S = 0$, then we have

$$2\partial_{+} (\partial_{-} u, \partial_{-} S)_{T \otimes T^*} = \langle \partial_{+} \partial_{-} u, \partial_{-} S \rangle$$  \hspace{1cm} (9.24)

$$2\partial_{-} (\partial_{+} u, \partial_{+} S)_{T \otimes T^*} = \langle \partial_{-} \partial_{+} u, \partial_{+} S \rangle$$  \hspace{1cm} (9.25)

**Proof.** From $2(\partial_{-} u, \partial_{-} S)_{T \otimes T^*} = (\partial_{-} u) \cdot (\partial_{-} S) + (\partial_{-} S) \cdot (\partial_{-} u)$, we obtain

$$2\partial_{+} (\partial_{-} u, \partial_{-} S)_{T \otimes T^*} = (\partial_{+} \partial_{-} u)(\partial_{-} S) - (\partial_{-} u)(\partial_{+} \partial_{-} S).$$

Since $\partial_{\pm} \partial_{\pm} S = 0$, we obtain (9.24). By the same method, we obtain the result. \hfill $\Box$

**Proposition 9.7.** $\partial_{\pm} \dot{J}_{\psi} (\partial S)$ is given by

$$\partial_{\pm} \dot{J}_{\psi} (\partial S) = 8\partial_{+} \dot{J}_{\psi} (\partial u, J_{\psi}(\partial S))_{T \otimes T^*}$$  \hspace{1cm} (9.26)

**Proof.** Applying Lemma 9.6 to (9.23), we obtain

$$\dot{J}_{\psi} (\partial S) = 8\sqrt{-1} \partial_{-} (\partial_{+} u, \partial_{+} S)_{T \otimes T^*} + 8\sqrt{-1} \partial_{+} (\partial_{-} u, \partial_{-} S)_{T \otimes T^*}$$  \hspace{1cm} (9.27)

Thus we have

$$\partial_{\pm} \dot{J}_{\psi} (\partial S) = 8\sqrt{-1} \partial_{\pm} \partial_{\pm} (\partial_{+} u, \partial_{+} S)_{T \otimes T^*} - 8\sqrt{-1} \partial_{\pm} \partial_{\pm} (\partial_{-} u, \partial_{-} S)_{T \otimes T^*}$$
$$= 8\sqrt{-1} \partial_{\pm} \partial_{\pm} ((\partial_{+} u, \partial_{+} S)_{T \otimes T^*} - (\partial_{-} u, \partial_{-} S))_{T \otimes T^*}$$
$$= 8\partial_{\pm} \partial_{\pm} ((\partial_{+} u, J_{\psi}(\partial_{\pm} S))_{T \otimes T^*} + (\partial_{-} u, J_{\psi}(\partial_{\pm} S))_{T \otimes T^*})$$
$$= 8\partial_{\pm} \partial_{\pm} ((\partial_{+} u, J_{\psi}(\partial S))_{T \otimes T^*} + (\partial_{-} u, J_{\psi}(\partial S))_{T \otimes T^*})$$
$$= 8\partial_{\pm} \partial_{\pm} (\partial u, J_{\psi}(\partial S))_{T \otimes T^*}.$$ \hfill $\Box$
Proposition 9.8. Let $S_s$ be a family of real functions which smoothly depends on a parameter $s$ and satisfies $S_0 = S(J)$ and

$$
\dot{S} = -4\sqrt{-1}\langle J\psi \bar{S}, \partial u \rangle_{T \oplus T^*}.
$$

Then the family $\{S_s\}$ satisfies the following

$$
\frac{d}{ds} \left( \partial J\psi \bar{S} \right)_{s=0} = 0,
$$

where $J_{\psi-s}$ is the generalized complex structure induced from $\psi-s$.

Proof. (9.29) is written as

$$
0 = \frac{d}{ds} \partial J\psi \bar{S} \bigg|_{s=0} = \partial \dot{J}\psi \partial S + \partial J\psi \partial \dot{S}.
$$

From $\partial \dot{J}\psi \partial S = -2\sqrt{-1} \partial_+ \partial_- \dot{S}$ and Proposition 9.7, we obtain

$$
\partial \dot{J}\psi \partial S + \partial J\psi \partial \dot{S} = 8\partial_+ \partial_- \langle \partial u, J\psi(\partial S) \rangle_{T \oplus T^*} - 2\sqrt{-1} \partial_+ \partial_- \dot{S} \tag{9.30}
$$

Since $\dot{S}$ satisfies

$$
\dot{S} = -4\sqrt{-1}\langle \partial u, J\psi(\partial S) \rangle_{T \oplus T^*},
$$

(9.29) holds.

Let $L(J, J_{\psi-s})$ be the fourth order differential operator $\left( \partial J_{\psi-s} \bar{\partial} \right)^*(\partial J_{\psi-s} \bar{\partial})$, where $(\partial J_{\psi-s} \bar{\partial})^*$ denotes the adjoint of $(\partial J_{\psi-s} \bar{\partial})$.

Proposition 9.9. Let $F_s$ be deformations given by $X\sqrt{-1}u_s := \sqrt{-1} J\psi \bar{u}_s - \sqrt{-1} J\psi \partial u$. We denote by $L_s$ the operator $L(F_s \# J, J\psi)$. We assume that $\partial J\psi \bar{S} = 0$. Then we have

$$
\left( \frac{d}{ds} L_s \right) S_{s=0} = 2L(I - L)u
$$

Proof. Since $\partial J\psi \bar{S} = 0$, we have $L(J, J\psi)S = 0$. We can take a smooth family of functions $\{S_s\}$ which satisfies $S_0 = S$ and (9.29). (It is not necessary that $S_s$ arises as scalar curvature.) Then (9.29) which is equivalent to

$$
\frac{d}{ds} \left( L(J, J_{\psi-s}) S_s \right)_{s=0} = 0 \tag{9.31}
$$

From (9.3), we have

$$
L(F_s \# J, J\psi) f_s^* S_s = L(F_s \# J, F_s \# (F^{-s} \# \psi)) f_s^* S_s = f_s^* \left( L(J, F^{-s} \# \psi) S_s \right).
$$

Then it follows that

$$
\frac{d}{ds} L(F_s \# J, J\psi) f_s^* S_s = \frac{d}{ds} f_s^* \left( L(J, F^{-s} \# \psi) S_s \right).
$$

37
Thus (9.31) is equivalent to
\[ \frac{d}{ds} L(F_{\#} \mathcal{J}, \mathcal{J}_\psi) f_s^* S \bigg|_{s=0} = 0 \]  
(9.32)

We denote by \(L_s\) the operator \(L(F_{\#} \mathcal{J}, \mathcal{J}_\psi)\). Then (9.32) is equivalent to
\[ \left( \frac{d}{ds} L_s \right) S \bigg|_{s=0} + L_s ((\mathcal{J} e_u) S + \tilde{S}) \bigg|_{s=0} = 0, \]
(9.33)

where \(-(\mathcal{J} e_u) S = \frac{d}{ds} f_s^* S \bigg|_{s=0} = -2 \langle \mathcal{J} e_u, dS \rangle_{T_T^*}\). Since \(-\mathcal{J}_\psi (du) = e_u\), we have
\[ -\mathcal{L}_{\mathcal{J}(e_u)} S = -2 \langle \mathcal{J} e_u, dS \rangle_{T_T^*} = 2 \langle \mathcal{J} \mathcal{J}_\psi du, dS \rangle_{T_T^*} \]
(9.34)
\[ = 2 \langle \mathcal{J}_\psi J du, dS \rangle_{T_T^*} = -2 \langle \mathcal{J} du, \mathcal{J}_\psi dS \rangle_{T_T^*} \]
(9.35)
\[ = -2 \langle \sqrt{-1} \partial u - \sqrt{-1} \partial u, \mathcal{J}_\psi (du + \bar{d}S) \rangle_{T_T^*} \]
(9.36)
\[ = -2 \langle \sqrt{-1} \partial u, \mathcal{J}_\psi dS \rangle_{T_T^*} + 2 \langle \sqrt{-1} \partial u, \mathcal{J}_\psi \bar{d}S \rangle_{T_T^*} \]
(9.37)

Then from (9.33), (9.28) and (9.34), we obtain
\[ \left( \frac{d}{ds} L_s \right) S \bigg|_{s=0} = -L_s (-\mathcal{L}_{\mathcal{J}(e_u)} S + \tilde{S}) \bigg|_{s=0} \]
\[ = -L (2 \langle \sqrt{-1} \partial u, \mathcal{J}_\psi dS \rangle_{T_T^*} + 2 \langle \sqrt{-1} \partial u, \mathcal{J}_\psi \bar{d}S \rangle_{T_T^*}) \]
\[ -L (-4 \sqrt{-1} \langle \partial u, \mathcal{J}_\psi \bar{d}S \rangle_{T_T^*}) \]
\[ = 2L (\langle \sqrt{-1} \partial u, \mathcal{J}_\psi dS \rangle_{T_T^*} + \langle \sqrt{-1} \partial u, \mathcal{J}_\psi \bar{d}S \rangle_{T_T^*}) \]
\[ = -2 \sqrt{-1} L (\langle \mathcal{J}_\psi du, dS \rangle_{T_T^*}) \]
\[ = -2 \sqrt{-1} L \mu \{ u, S \} \mathcal{J}_\psi \]

Applying Proposition 6.2, i.e., \(\frac{2}{\sqrt{-1}} (L - \mathcal{L}) u = \{ u, S(\mathcal{J}) \} \mathcal{J}_\psi\), we obtain
\[ \left( \frac{d}{ds} L_s \right) S \bigg|_{s=0} = 2L(L - \mathcal{L}) S \]

\[ \square \]

**Proposition 9.10.** Let \((M, \mathcal{J}, \mathcal{J}_\psi)\) be a generalized extremal Kähler manifold. Then we have
\[ LL = \mathcal{L} L \]

**Proof.** Two pure imaginary functions \(\sqrt{-1} u_1, \sqrt{-1} u_2\) gives \(\mathcal{X}_{\sqrt{-1} u_1}\) and \(\mathcal{X}_{\sqrt{-1} u_2}\), respectively. Then \(\mathcal{X}_{\sqrt{-1} u_1}\) and \(\mathcal{X}_{\sqrt{-1} u_2}\) gives rise to 2-parameter deformations \((\mathcal{J}_{t_1}, t_2)\). We shall calculate the Hessian of the functional \(E(\mathcal{J}) := \int_M S(\mathcal{J})^2 \text{vol}_M\) under the deformations \((\mathcal{J}_{t_1}, t_2)\). From Proposition 6.3, the differential of \(\Phi(\mathcal{J}_{t_1}, t_2)\) with respect to \(t_1\) is given by
\[ \frac{1}{2} \frac{\partial}{\partial t_1} \Phi(\mathcal{J}_{t_1}, t_2) = \frac{1}{2} \int_M S(\mathcal{J}_{t_1}, t_2) \left( \frac{d}{dt_1} S(\mathcal{J}_{t_1}, t_2) \right) \bigg|_{t_1=0} \]
\[ = \int_M S(\mathcal{J}_{t_1}, t_2) \left( (L_{t_1, t_2} + \mathcal{L}_{t_1, t_2}) u_1 \right) \text{vol}_M \]
\[ = \int_M u_1 \left( (L_{t_1, t_2} + \mathcal{L}_{t_1, t_2}) S(\mathcal{J}_{t_1}, t_2) \right) \text{vol}_M \]
From Proposition 6.2, we also have

\[
(L_{t_1, t_2} - J_{t_1, t_2}) S(J_{t_1, t_2}) = \{ S(J_{t_1, t_2}), S(J_{t_1, t_2}) \}
\]

Thus

\[
\frac{1}{2} \frac{\partial}{\partial t_1} \Phi(J_{t_1, t_2}) = 2 \int_M u_1 L_{t_1, t_2} S(J_{t_1, t_2}) = 2 \int_M u_1 T_{t_1, t_2} S(J_{t_1, t_2})
\]

(9.41)

From Proposition 9.9, i.e., \( \dot{L} = 2L(L - L)u \) and Proposition 6.3, i.e., \( \frac{d}{dt} S(J^u) |_{t=0} = 2(L + L)u_{t=0} \), the differential of \( \frac{\partial}{\partial t_1} \Phi \) with respect to \( t_2 \) is given by

\[
\frac{1}{2} \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_1} \Phi(J_{t_1, t_2}) \Big|_{t_1, t_2=0} = 2 \int_M u_1 \frac{\partial}{\partial t_2} L_{t_1, t_2} S(J_{t_1, t_2}) \Big|_{t_1, t_2=0} = 2 \int_M u_1 L_{t_1, t_2} S(J_{t_1, t_2}) \Big|_{t_1, t_2=0} = 2 \int_M u_1 \frac{\partial}{\partial t_2} L_{t_1, t_2} S(J_{t_1, t_2}) \Big|_{t_1, t_2=0} = 4 \int_M u_1 L(L - L)u_2 + u_1 L(L + L)u_2
\]

(9.42)

\[
= 8 \int_M u_1 (L^2 \text{vol}_M)
\]

(9.46)

From (9.41), the similar calculation gives

\[
\frac{1}{2} \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_1} \Phi(J_{t_1, t_2}) \Big|_{t_1, t_2=0} = 2 \int_M u_1 \frac{d}{dt_2} T_{t_1, t_2} S(J_{t_1, t_2}) \Big|_{t_1, t_2=0} = 4 \int_M u_1 T(L - L)u_2 + u_1 T(L + L)u_2
\]

(9.47)

\[
= 8 \int_M u_1 (T^2 \text{vol}_M)
\]

(9.49)

Hence we obtain \( L^2 u = T^2 u \). \( \square \)

Let \( (M, J, J_\psi) \) be a generalized Kähler manifold with generalized scalar curvature \( S(J) \). The generalized metric \( G = -J \circ J_\psi \) defines the generalized isometry group \( I_G(M) \) which is the subgroup of \( \text{Diff}(M) \) preserving \( G \). Then it turns out that \( I_G(M) \) is a compact Lie group. The Lie algebra of \( I_G(M) \) is denoted by \( i_G(M) \). If \( (M, J, J_\psi) \) is a generalized extremal Kähler manifold, then the scalar curvature \( S := S(J) \) gives the class \( [J_\psi \circ \bar{J}S] \in H^1(\Lambda^\bullet \bar{J}J) \) and the adjoint action of \( [J_\psi \bar{J}S] \) on \( g_0 \) gives the decomposition into \( \lambda \)-eigenspaces

\[
g(\lambda) := \{ a \in g_0 | \text{ad}_{[J_\psi \bar{J}S]} a = \lambda a \}
\]

**Theorem 9.11.** Let \( (M, J, J_\psi) \) be a generalized extremal Kähler manifold with generalized scalar curvature \( S(J) \). Then the Lie algebra of the reduced automorphisms \( g_0 \) of \((M, J, J_\psi)\) admits the following decomposition as Lie algebra:

\[
g_0 = g(0) \oplus \sum_{\lambda \neq 0} g(\lambda),
\]

where \( g(0) \) is the maximal reductive subalgebra \( (i_G(M) \cap g_0) \otimes \mathbb{C} \).

39
Proof. Since the Lie algebra $g_0$ of the reduced automorphisms is identified with the space of complex functions which are annihilated by the action of $L$,

$$\{ u \in C_0^\infty(M, \mathbb{C}) \mid Lu = 0 \},$$

where $Lu = (\bar{\partial}J\psi \partial)u$. From Proposition 9.10, we have $LL = L$. Thus the action of $L$ preserves the kernel space $\{ u \in C_0^\infty(M, \mathbb{C}) \mid Lu = 0 \}$ of $L$ and then we have the eigenspace decomposition of the action of $L$ under the identification,

$$g_0 = \bigoplus \lambda V_{-\sqrt{-1}\lambda}$$

Then from Proposition 6.2, i.e., $\sqrt{-1}(L - L)u = \{ u, S(\mathcal{J}) \}_{Poi}$, it follows that $-\sqrt{-1}\lambda$-eigenfunction $u$ satisfies the following:

$$-\sqrt{-1}\lambda u = Lu = (L - L)u \quad (9.50)$$

$$= -\sqrt{-1}\{ S(\mathcal{J}), u \}_{Poi} \quad (9.51)$$

Thus we have

$$[\mathcal{J}_\psi \bar{\partial}S(\mathcal{J}), \mathcal{J}_\psi \bar{\partial}u]_{\text{co}} = \{ S(\mathcal{J}), u \}_{Poi} = \lambda u.$$}

Hence we have $g(\lambda) = V_{-\sqrt{-1}\lambda}$. Since $V_0 = \ker L \cap \ker L$, we have $V_0 = (i_G(M) \cap g_0) \otimes \mathbb{C}$. Thus we have the result. \qed

References

[1] T. Abdelgadir, S. Okawa, K. Ueda, Compact moduli of noncommutative projective planes, arXiv:1411.7770.

[2] V. Apostolov, P. Gauduchon, G. Grantcharov, Bihermitian structures on complex surfaces, Proc. London Math. Soc. 79 (1999), 414-429 + Erratum in Proc. London Math. Soc. 92 (2006), 200-202.

[3] V. Apostolov, G. Maschler, Gideon, Conformally Kähler, Einstein-Maxwell geometry, J. Eur. Math. Soc. (JEMS) 21 (2019), no. 5, 1319-1360.

[4] V. Apostolov, J. Streets, The nondegenerate generalized Kähler Calabi-Yau problem, arXiv:1703.08650.

[5] L. Boulanger, Toric generalized Kähler structures, J. Symplectic Geom. 17 (2019), no. 4, 973-1019.

[6] Gil. R. Cavalcanti, Introduction to generalized complex geometry, IMPA, 2007.

[7] S. K. Donaldson, Remarks on gauge theory, complex geometry and 4-manifold topology, Fields Medallists’ lectures, 384-403, World Sci. Ser. 20th Century Math., 5, World Sci. Publ., River Edge, NJ, 1997.

[8] A. Fujiki, Moduli space of polarized algebraic manifolds and Kähler metrics [translation of Sugaku 42 (1990), no. 3, 231-243; MR1073369]. Sugaku Expositions. Sugaku Expositions 5 (1992), no. 2, 173-191.
[9] A. Futaki, Akito, H. Ono, *Conformally Einstein-Maxwell Kähler metrics and structure of the automorphism group*, Math. Z. 292 (2019), no. 1-2, 571-589.

[10] R. Goto, *Poisson structures and generalized Kähler structures*, J. Math. Soc. Japan, 61 (2009) no. 1, 107-132, arXiv:0712.2685.

[11] R. Goto, *Deformations of generalized complex and generalized Kähler structures*, J. Differential Geom. 84 (2010), no. 3, 525–560, arXiv:0705.2495.

[12] R. Goto, *Unobstructed K-deformations of Generalized Complex Structures and Bihermitian Structures*, Adv. Math. 231 (2012) 1041-1067, arXiv:0911.2958.

[13] R. Goto, *Unobstructed deformations of generalized complex structures induced by $C^\infty$ logarithmic symplectic structures and logarithmic Poisson structures*, Geometry and Topology of Manifolds 10th China-Japan Conference 2014, pp. 159-183, arXiv:1501.03398.

[14] R. Goto, *Scalar curvature as moment map in generalized Kahler geometry*, J. Symplectic Geom. 18 (2020), no. 1, 147-190, arXiv:1612.08190.

[15] R. Goto, *Moduli spaces of Einstein-Hermitian generalized connections over generalized Kähler manifolds of symplectic type*, arXiv:1707.03143.

[16] R. Goto, *Kobayashi-Hitchin correspondence of generalized holomorphic vector bundles over generalized Kähler manifolds of symplectic type*, arXiv:1903.07425.

[17] M. Gualtieri, *Generalized complex geometry*, Oxford University DPhil thesis (2003),

[18] M. Gualtieri, *Generalized geometry and the Hodge decomposition*, Lecture at the String Theory and Geometry workshop, August 2004, Oberwolfach. 7, . arXiv:math/0409093.

[19] M. Gualtieri, *Generalized complex geometry*, Ann. of Math. (2) 174 (2011), no. 1, 75-123.

[20] M. Gualtieri, *Generalized Kähler metrics from Hamiltonian deformations*, Geometry and physics. Vol. II, 551-579, Oxford Univ. Press, Oxford, 2018.

[21] N. J. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. 54 (2003), no. 3, 281-308.

[22] N. J. Hitchin, *Bihermitian metrics on Del Pezzo surfaces*, J. Symplectic Geom. 5 (2007), 1-7.

[23] N.J. Hitchin, *Lectures on generalized geometry*, Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, 79-124, Surv. Differ. Geom., 16, Int. Press, Somerville, MA, 2011.

[24] S. Kobayashi, *Transformation groups in differential geometry* Reprint of the 1972 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. viii+182 pp. ISBN: 3-540-58659-8

[25] C. LeBrun, R.S. Simanca, *Extremal Kähler metrics and complex deformation theory*, Geom. Funct. Anal. 4, 298-336 (1994).
[26] A. Lichnerowicz, *Géométrie des groupes de transformations* (French) Travaux et Recherches Mathématiques, III. Dunod, Paris 1958 ix+193 pp.

[27] Y. Matsushima, *Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kählérienne* (French) Nagoya Math. J. 11 (1957), 145-150.

[28] S. B. Myers, N. E. Steenrod, *The group of isometries of a Riemannian manifold* Ann. of Math. (2) 40 (1939), no. 2, 400-416.

[29] K. Ono, *Floer-Novikov cohomology and the flux conjecture*, Geom. Funct. Anal. 16 (2006), no. 5, 981-1020.

[30] Richard. S. Palais, *Foundations of global non-linear analysis*, W. A. Benjamin, Inc., New York-Amsterdam 1968.

[31] J. Streets, *Generalized Kähler-Ricci flow and the classification of nondegenerate generalized Kähler surfaces*, arXiv:1601.02981.

[32] Lijiang. Wang, *Hessians of the Calabi functional and the norm function*, Ann. Global Anal. Geom. 29 (2006), no. 2, 187-196.

[33] Wang, Yicao, *Toric generalized Kähler structures. III*, J. Geom. Phys. 151 (2020).

E-mail address: goto@math.sci.osaka-u.ac.jp
Department of Mathematics, Graduate School of Science,
Osaka University Toyonaka, Osaka 560-0043, JAPAN