Different manifestations of S-matrix poles

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Abstract

Making use of the analytical properties of the S-matrix and a theorem of Mittag-Leffler, model independent non-relativistic expressions for cross sections in single channel elastic scattering, scattering phase shifts and survival probabilities of resonances are derived. Provided certain conditions are satisfied by the poles, the residues can also be estimated analytically. Considerations of the low energy behaviour of the S-matrix and cross sections reveal additional conditions on the residues of the poles appearing in the Mittag-Leffler expansions. The exact expressions for the resonant cross section and phase shift are shown to reduce to the commonly used Breit-Wigner formula plus corrections. The latter is shown to approach the exact result with the example of a meson and a baryon resonance. Finally, a comparison of the exact expressions with some realistic examples is presented. The calculations of survival probabilities in particular reveal the reason behind the non-observability of non-exponential decay.

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I. INTRODUCTION

The occurrence of resonances and resonant phenomena is ubiquitous in nature. Indeed the only stable elementary objects in nature are the proton and electron. As a result, resonance production, propagation and its decay play an important role in almost all branches of physics. For example, the existence of a $^{12}\text{C}$ resonance first predicted by Fred Hoyle \cite{1} and then found experimentally is crucial in order to explain the $^{12}\text{C}$ abundance in the universe. Indeed, resonances appearing in nuclear processes play an important role in stellar nucleosynthesis \cite{2}. Many of the current research topics in particle and hadron physics also revolve around the searches \cite{3} and understanding of different exotic unstable states \cite{4,5}. Some decays such as the positronium for example, even play an important role in advances in medical technology \cite{6}. In the scattering of particles and nuclei, resonances usually show up as an enormous increase in the cross sections which are most commonly described by a Breit-Wigner form \cite{7} proposed several decades ago. Indeed, analytical forms of the Breit-Wigner propagators were studied for decades along with their consistency with gauge invariance arguments (see \cite{8} and references therein). Though the latter works quite well in the case of most narrow resonances \cite{9,10} (unstable states with large lifetimes), it should be remembered that it is only an approximate model. One can derive it for example from the argument that the exponential decay law (which is by itself an approximation) is encoded in the wave function, $|\Psi(r, t)|^2 = |\Psi(r, 0)|^2 e^{-\Gamma t} \Rightarrow \Psi(r, t) = \Psi(r, 0)e^{-\Gamma t/2}$. Expressing the wave function as a superposition of components having different energies, it is easy to obtain (after a Fourier transform) a Lorentzian or the Breit-Wigner form for the probability of finding the unstable state at an energy $E$.

Characterization of particle resonances from data is in principle a very complex undertaking \cite{11} and there exist both model dependent approaches (with free parameters of the model fitted to data) or model independent approaches \cite{12} where the poles and residues in the expansion of the scattering amplitudes are fitted to data. Given the complexity of the analyses, model independent analytic expressions giving a physical insight into the time evolution and decay of resonances are not always available. In the present work, making use of the analytical properties of the $S$-matrix and a theorem of Mittag-Leffler (ML) \cite{13}, an expansion of the $S$-matrix and thereby a general expression for the cross section depending on all the possible poles (bound, virtual, resonant) of the $S$-matrix is derived. The energy
derivative of the phase shift (which, as shown by the Beth-Uhlenbeck formula \[14\] corresponds to the density of states) is also obtained from the ML expansion and is used to derive analytical expressions for the survival amplitude within the Fock-Krylov method \[15, 16\]. The cross sections and phase shifts in single channel resonant elastic scattering derived using the above approach are shown to reduce to the commonly used Breit-Wigner forms plus corrections. We find that the derivation of all the above expressions near threshold requires special considerations and dedicate separate subsections to discuss the calculations of cross sections and survival probabilities for the resonances occurring close to threshold. The analytic S-matrix is a powerful tool which encodes the information of the bound, virtual, quasibound and resonant states through its poles. In what follows, we shall see its various manifestations in scattering and decay which enable us to obtain analytic expressions for the quantities mentioned above.

II. MITTAG-LEFFLER BASED ANALYSIS OF CROSS SECTIONS

Let us begin by recalling some basic facts \[17\] on which the rest of this work will be based. Starting with the centre of mass momentum, \(k\), in a two-body system where \(k^2 = E\) (i.e., using \(2m = \hbar = 1\)):

i. If the Hamiltonian that describes an elastic scattering process is invariant under parity and time reversal, and if the partial wave S-matrix element, \(S_l(k = k_o) = S_o\), then

\[
S_l(k = k_o^*) = \frac{1}{S_o}, \quad S_l(k = -k_o^*) = S_o^*; \quad S_l(k = -k_o) = \frac{1}{S_o}.
\]

This property implies that if \(k = k_o\) is a pole of \(S_l(k)\), then \(k = -k_o^*\) is also one, while \(k = k_o^*\) and \(k = -k_o\) are zeros of \(S_l(k)\).

ii. Taking all the properties of the S-matrix into account \[17\] allows us to characterize the poles and zeros of S if the poles on the imaginary axis (both virtual and bound) and those in the third or fourth quadrant of the complex momentum \(k\) plane are known.

The convention used for the poles and zeros of the S-matrix will be as follows:

iii. \(\{k_{ln}\}, n = 1, 2, \ldots\) are simple poles of \(S_l(k)\) in the fourth quadrant, i.e., \(\text{Re} \: k_{ln} > 0\) and \(\text{Im} \: k_{ln} < 0\) for all \(n\).
iv. \( \{i\zeta_{lm}\} \) are simple poles of \( S_l(k) \) on the imaginary axis. For \( m = 1, 2, \ldots, \zeta_{lm} > 0 \) and they represent bound states. For \( m = -1, -2, \ldots, \zeta_{lm} < 0 \) and they represent virtual states.

The \( S \) matrix will then have additional simple poles \( \{-k_{ln}^*\} \) and simple zeros at \( \{-k_{ln}\}, \{k_{ln}^*\} \) and \( \{-i\zeta_{lm}\} \). We remind that for mathematical simplicity, we set, \( 2m = h = 1 \) and hence \( k^2 = E \). To clarify further the notation used, let us note that the resonance pole occurring at an energy \( E_{lr} \) and width \( \Gamma_{lr} \) in the \( l \)th partial wave is given for example by

\[
k_{lr}^2 = \epsilon_{lr} - i\frac{\Gamma_{lr}}{2}
\]

where \( \epsilon_{lr} = E_{lr} - E_{th} \), with \( E_{th} \) being the threshold energy (or the sum of the masses of the decay products of the resonance).

A. Expansion of the \( S \)-matrix

A consequence of the above points (i) and (ii) is that \( S_l(k) \) is a meromorphic function. Assuming that the poles of these functions are known, it is natural to ask if the information of the poles is sufficient to determine the function itself. The answer lies in a theorem of Mittag-Leffler (ML) \[13\].

If the only singularities of a meromorphic function \( f(z) \) are the simple poles \( z = a_1, a_2, \ldots \) such that \( |a_1| \leq |a_2| \leq \ldots \), with residues \( b_1, b_2, \ldots \), respectively, and if \( C_N \) is a circumference of radius \( R_N \) which contains \( N \) poles of the function \( f(z) \) (and does not pass through any of the remaining poles), i.e., \( |a_N| < R_N < |a_{N+1}| \), and in addition the function \( f(z) \) satisfies \( |f(z)| < M \), where \( M \) is not dependent on \( N \), then, when \( N \to \infty \),

\[
f(z) = f(0) + \lim_{N \to \infty} \sum_{n=1}^{N} b_n \left\{ \frac{1}{z - a_n} + \frac{1}{a_n} \right\} + \lim_{N \to \infty} \frac{z}{2\pi i} \oint_{C_N} \frac{f(\zeta)}{\zeta(\zeta - z)} \, d\zeta = f(0) + \sum_{n=1}^{\infty} \frac{b_n z}{a_n(z - a_n)}. \tag{2}
\]

To derive the expressions for the cross sections, we begin by writing the Mittag-Leffler expansion of the \( S \)-matrix. In order to use the ML theorem, we must know the value of the function at the origin. To evaluate \( S \) at \( k = 0 \), we use the property \[17\]: \( S_l(k)S_l(-k) = 1 \), as a consequence of which and using the definition of \( S_l = \exp(2i\delta_l) \), we get, \( \delta_l(k) = -\delta_l(-k) \).

The phase shift, \( \delta_l = 0 \) for \( k = 0 \) and hence \( S_l(0) = 1 \). With the convention mentioned in
the beginning for the $S$-matrix poles and using the ML theorem, we obtain,

$$S_t(k) = 1 + k \sum_n \left[ \frac{b_{tn}}{k_{tn}(k - k_{tn})} - \frac{c_{tn}}{k_{tn}(k + k_{tn}^*)} \right] + k \sum_m \frac{d_{tm}}{i \zeta_{tm}(k - i \zeta_{tm})},$$  \hspace{1cm} (3)

where, $b_{tn} = \text{Res} \left[ S_t(k), k = k_{tn} \right]$, $c_{tn} = \text{Res} \left[ S_t(k), k = -k_{tn}^* \right]$ and $d_{tm} = \text{Res} \left[ S_t(k), k = i \zeta_{tm} \right]$.

1. **Conditions imposed on the $S$-matrix expansion**

The analytic properties of the $S$-matrix impose the following conditions on (3):

i) If $k$ is purely imaginary, $S_t(k)$ should be real: If $k = i \gamma$, with $\gamma$ real, Eq. (3) can be written as

$$S_t(i \gamma) = 1 + i \gamma \sum_n \left[ \frac{b_{tn}}{k_{tn}(i \gamma - k_{tn})} - \frac{c_{tn}}{k_{tn}(i \gamma + k_{tn}^*)} \right] + i \gamma \sum_m \frac{d_{tm}}{i \zeta_{tm}(i \gamma - i \zeta_{tm})}.$$  \hspace{1cm} (4)

Taking the complex conjugate,

$$S_t^*(i \gamma) = 1 - i \gamma \sum_n \left[ \frac{b_{tn}^*}{k_{tn}^*(i \gamma - k_{tn}^*)} - \frac{c_{tn}^*}{k_{tn}^*(i \gamma + k_{tn}^*)} \right] - i \gamma \sum_m \frac{d_{tm}^*}{-i \zeta_{tm}(i \gamma + i \zeta_{tm})}$$

$$= 1 + i \gamma \sum_n \left[ \frac{b_{tn}^*}{k_{tn}^*(i \gamma + k_{tn}^*)} - \frac{c_{tn}^*}{k_{tn}^*(i \gamma - k_{tn}^*)} \right] + i \gamma \sum_m \frac{-d_{tm}^*}{i \zeta_{tm}(i \gamma - i \zeta_{tm})},$$  \hspace{1cm} (5)

and comparing the two expressions, we deduce that $c_{tn} = -b_{tn}^*$ and $d_{tm} = -d_{tm}^*$. Considering the properties of the residues of the imaginary poles, we rewrite:

$$D_{lm} \equiv \frac{d_{lm}}{i} = \frac{1}{i} \text{Res} \left[ S_t(k), k = i \zeta_{lm} \right],$$  \hspace{1cm} (6)

and since $d_{tm} = -d_{tm}^*$, $D_{lm}$ is real. Finally, the expansion for the $S$-matrix can be written in the following form:

$$S_t(k) = 1 + k \sum_n \left[ \frac{b_{tn}}{k_{tn}(k - k_{tn})} + \frac{b_{tn}^*}{k_{tn}(k + k_{tn}^*)} \right] + k \sum_m \frac{D_{lm}}{\zeta_{lm}(k - i \zeta_{lm})},$$  \hspace{1cm} (7)

ii) $\{-k_{tn}\}$, $\{k_{tn}^*\}$ and $\{-i \zeta_{lm}\}$ are zeros of the $S$ matrix: In this case, the expansion (7) must satisfy the following equations:

$$S_t(-k_{tn}) = 1 - k_{tn} \sum_p \left[ \frac{b_{lp}}{k_{lp}(-k_{tn} - k_{lp})} + \frac{b_{lp}^*}{k_{lp}(-k_{tn}^* + k_{lp}^*)} \right]$$

$$- k_{tn} \sum_q \frac{D_{lq}}{\zeta_{lq}(-k_{tn} - i \zeta_{lq})} = 0,$$  \hspace{1cm} (8)
\[ S_l(k_{in}) = 1 + k_{in}^* \sum_p \left[ \frac{b_{lp}}{k_{lp}(k_{in}^* - k_{lp})} + \frac{b_{lp}^*}{k_{lp}(k_{in}^* + k_{lp})} \right] \]

\[ \quad + k_{in}^* \sum_q \frac{D_{lq}}{\zeta_{lq}(k_{in}^* - i\zeta_{lq})} = 0, \tag{9} \]

\[ S_l(i\zeta_{lm}) = 1 + i\zeta_{lm} \sum_p \left[ \frac{b_{lp}}{k_{lp}(i\zeta_{lm} - k_{lp})} + \frac{b_{lp}^*}{k_{lp}(i\zeta_{lm} + k_{lp})} \right] \]

\[ \quad + i\zeta_{lm} \sum_q \frac{D_{lq}}{\zeta_{lq}(i\zeta_{lm} - i\zeta_{lq})} = 0. \tag{10} \]

Eq. (9) is the conjugate of (8). Therefore, these conditions reduce to

\[ 1 + k_{in} \sum_p \left[ \frac{b_{lp}}{k_{lp}(k_{in}^* + k_{lp})} + \frac{b_{lp}^*}{k_{lp}(k_{in}^* - k_{lp})} \right] + k_{in} \sum_q \frac{D_{lq}}{\zeta_{lq}(k_{in}^* + i\zeta_{lq})} = 0, \tag{11} \]

\[ 1 + i\zeta_{lm} \sum_p \left[ \frac{b_{lp}}{k_{lp}(i\zeta_{lm} - k_{lp})} + \frac{b_{lp}^*}{k_{lp}(i\zeta_{lm} + k_{lp})} \right] + \zeta_{lm} \sum_q \frac{D_{lq}}{\zeta_{lq}(\zeta_{lm} - i\zeta_{lq})} = 0. \tag{12} \]

iii) If \( k \) is real, \( S_l(k) \) must be unimodular \[17]. This implies that the expansion (7) must satisfy the condition: \( S_l(k)S_l^*(k) = 1. \) The proof for this is given in Appendix A.

2. Residues in the S-matrix expansion

To determine the S-matrix completely from the Mittag-Leffler expansion, we must now calculate the residues in (7). We mention here already that the complete expression for the residue of a single pole will be found to depend on all other (bound, virtual and resonant if any) poles and difficult to use. Hence, we shall derive an approximate expression which can be used under certain conditions. The S-matrix in literature and text books \[18\] can be found to be written either in the context of potential scattering or otherwise. We write,

\[ S_l(k) = D(k) \prod_n \frac{(k + k_{in})(k - k_{in}^*)}{(k - k_{in})(k + k_{in}^*)} \prod_m \frac{i\zeta_{lm} + k}{i\zeta_{lm} - k^*}, \tag{13} \]

where (considering the properties of the S-matrix) the function \( D(k) \) is unity for \( k = 0 \) and describes the case of a potential of finite range \( a \) or otherwise with the following form:

\[ D(k) = \begin{cases} 
1 & \text{General} \\
 e^{-2iak} & \text{Potential picture}
\end{cases} \tag{14} \]
The $S$-matrix from the above considerations satisfies the following: $S_i(0) = 1$, $S$ is real when $k$ is purely imaginary and is modular when $k$ is real. The residue of $S_i(k)$ at $k = k_{ln}$ is calculated as,

$$b_n = \text{Res} \left[ S_i(k), k = k_{ln} \right] = \lim_{k \to k_{ln}} (k - k_{ln})S_i(k)$$

$$= 2k_{ln} \frac{k_{ln} - k_{ln}^*}{k_{ln} + k_{ln}^*} D(k_{ln}) \prod_{p \neq n} \frac{(k_{ln} + k_{lp})(k_{ln} - k_{lp}^*)}{(k_{ln} - k_{lp})(k_{ln} + k_{lp}^*)} \prod_{m} \frac{i\zeta_{ln} + k_{ln}}{i\zeta_{ln} - k_{ln}}. \quad (15)$$

If we define the function,

$$T_i^{(n)}(k) = D(k_{ln}) \prod_{p \neq n} \frac{(k + k_{lp})(k - k_{lp}^*)}{(k - k_{lp})(k + k_{lp}^*)} \prod_{m} \frac{i\zeta_{ln} + k}{i\zeta_{ln} - k}, \quad (16)$$

the residue can be written as,

$$b_{ln} = 2ik_{ln} \frac{\text{Im} \left[ k_{ln} T_i^{(n)}(k_{ln}) \right]}{\text{Re} \left[ k_{ln} T_i^{(n)}(k_{ln}) \right]} = 2ik_{ln} \tan \left( \text{Arg} k_{ln} \right) T_i^{(n)}(k_{ln}). \quad (17)$$

The function $T_i^{(n)}(k)$, by definition is unity at $k = 0$ and hence we can write

$$T_i^{(n)}(k) = 1 + G_i^{(n)}(k), \quad (18)$$

where $G_i^{(n)}(k)$ is defined such that $G_i^{(n)}(k) = 0$ for $k = 0$. Inserting (18) in (17),

$$b_{ln} = 2ik_{ln} \tan \left( \text{Arg} k_{ln} \right) T_i^{(n)}(k_{ln}) \left[ 1 + G_i^{(n)}(k_{ln}) \right]. \quad (19)$$

The above expression can be split into two parts: one which depends solely on the pole (of which we are evaluating the residue) and another which depends on the remaining poles. This is to say,

$$b_{ln} = b_{ln}^{(1)} + b_{ln}^{(2)}, \quad (20)$$

where

$$b_{ln}^{(1)} = 2ik_{ln} \tan \left( \text{Arg} k_{ln} \right), \quad (21)$$

$$b_{ln}^{(2)} = 2ik_{ln} \tan \left( \text{Arg} k_{ln} \right) G_i^{(n)}(k_{ln})$$

$$= 2ik_{ln} \tan \left( \text{Arg} k_{ln} \right) D(k_{ln}) \left\{ \prod_{p \neq n} \frac{(k_{ln} + k_{lp})(k_{ln} - k_{lp}^*)}{(k_{ln} - k_{lp})(k_{ln} + k_{lp}^*)} \prod_{m} \frac{i\zeta_{ln} + k_{ln}}{i\zeta_{ln} - k_{ln}} - 1 \right\}. \quad (22)$$

Using the above decomposition, we can now try to get an approximate estimate of the residues depending only on the resonance poles. This is to say, we shall now verify if the
second term is negligible and if not, under what conditions can it be neglected. Let us begin by writing,

\[ b_{ln} \approx b_{ln}^{(1)} = 2ik_{ln} \tan (\text{Arg } k_{ln}). \tag{23} \]

Note that this estimate is independent of the interaction involved. In order to derive the condition mentioned above, we start by expanding the function \( T_l^{(n)}(k) \) about \( k = 0 \). In the case of potential scattering, from (16), we get,

\[ \left. \frac{dT_l^{(n)}(k)}{dk} \right|_{k=0} = T_l^{(n)}(0) \frac{d}{dk} \ln [T_l^{(n)}(k)] \bigg|_{k=0} = 2i \left[ 2 \text{Im} \sum_{p \neq n} \frac{1}{k_{lp}} - \left( a + \sum_m \frac{1}{\zeta_{lm}} \right) \right] + O(k^2). \tag{24} \]

Thus

\[ T_l^{(n)}(k) = 1 + 2ik \left[ 2 \text{Im} \sum_{p \neq n} \frac{1}{k_{lp}} - \left( a + \sum_m \frac{1}{\zeta_{lm}} \right) \right] + O(k^2). \tag{25} \]

The \( S \)-matrix residue can be written using (17) as,

\[ b_{ln} = 2ik_{ln} \tan (\text{Arg } k_{ln}) \left\{ 1 + 2ik_{ln} \left[ 2 \text{Im} \sum_{p \neq n} \frac{1}{k_{lp}} - \left( a + \sum_m \frac{1}{\zeta_{lm}} \right) \right] \right\} + O(k_{ln}^2). \tag{26} \]

The above equation implies that the estimate of the residue (of the resonant pole) given by (21) is correct if,

\[ \left| 2ik_{ln} \left[ 2 \text{Im} \sum_{p \neq n} \frac{1}{k_{lp}} - \left( a + \sum_m \frac{1}{\zeta_{lm}} \right) \right] \right| \ll 1. \tag{27} \]

This condition can be written in another way:

\[ \left| 2 \text{Im} \sum_{p \neq n} \frac{1}{k_{lp}} - \left( a + \sum_m \frac{1}{\zeta_{lm}} \right) \right| \ll \frac{1}{2|k_{ln}|}. \tag{28} \]

If the above is satisfied, it is possible to use (23), otherwise we must use (22). In a later section, we shall use the expression (23) in connection with the discussion of an isolated resonance. However, if one performs a complete analysis of the cross sections including all possible poles (not necessarily one resonant pole), then the above condition proves useful.

In the absence of a potential picture, the condition (28) is obtained in a similar way. The condition in this case is,

\[ \left| 2 \text{Im} \sum_{p \neq n} \frac{1}{k_{lp}} - \sum_m \frac{1}{\zeta_{lm}} \right| \ll \frac{1}{2|k_{ln}|}. \tag{29} \]

We end this subsection by noting that the above expressions give us a means to determine the residues if the poles of the \( S \)-matrix are known. An extension of such an analysis to the
multichannel case can be quite useful for partial wave analyses of cross sections. In [12], for example, the authors perform expansions such as those in the present work. However, the residues are fitted as free parameters from experimental data. Analytical expressions such as the above can be used as consistency criteria on the fitted residues.

B. Single channel cross sections

Having defined the $S$-matrix using the Mittag-Leffler expansion, we now apply this result to the calculation of the angle integrated cross section, $\sigma$, which in terms of the scattering amplitude, $F(\theta, k)$, can be written using the optical theorem as:

$$\sigma = \frac{4\pi}{k} \text{Im} f(0, k),$$  \hspace{1cm} (30)

where

$$f(\theta, k) = \frac{1}{k} \sum_{l=0}^{\infty} (2l + 1) e^{i\delta_l} \sin \delta_l P_l(\cos \theta).$$  \hspace{1cm} (31)

In terms of $S$, it can be written as,

$$f(\theta, k) = \sum_{l=0}^{\infty} (2l + 1) \frac{S_l(k) - 1}{2ik} P_l(\cos \theta).$$  \hspace{1cm} (32)

Thus,

$$\sigma = -\frac{2\pi}{k^2} \sum_{l=0}^{\infty} (2l + 1) \text{Re} \left[ S_l(k) - 1 \right].$$  \hspace{1cm} (33)

Using (34), $\text{Re} \left[ S_l(k) - 1 \right]$ is given by,

$$2 \text{Re} \left[ S_l(k) - 1 \right] = \left[ S_l(k) - 1 \right] + \left[ S_l(k) - 1 \right]^*$$  \hspace{1cm} (34)

$$= 4 \text{Re} \sum_n b_{ln} \frac{k^2}{k_{ln}^2 - k^2} + \sum_m D_{lm} \frac{2k^2}{\zeta_{lm}^2 + \zeta_{lm}^2}. $$

Replacing (34) in (33) leads to,

$$\sigma = -4\pi \text{Re} \sum_{n,l} (2l + 1) b_{ln} \frac{1}{k_{ln}^2 - k^2} - 2\pi \sum_{m,l} D_{lm} \frac{2l + 1}{\zeta_{lm}^2 + \zeta_{lm}^2}. $$  \hspace{1cm} (35)

The above equation gives the total cross section simply in terms of the poles of the $S$-matrix. Indeed, the cross section for a given value of $l$ is given by,

$$\sigma^{(l)} = -4\pi(2l + 1) \text{Re} \sum_n b_{ln} \frac{1}{k_{ln}^2 - k^2} - 2\pi \sum_m D_{lm} \frac{2l + 1}{\zeta_{lm}^2 + \zeta_{lm}^2},$$  \hspace{1cm} (36)

where

$$\sigma = \sum_{l=0}^{\infty} \sigma^{(l)}. $$  \hspace{1cm} (37)
C. Threshold considerations: S-matrix and cross sections

The expressions for the S-matrix and total cross sections in the above discussions were derived without paying any special attention to the behaviour of these quantities near threshold. The behaviour of the S-matrix and total cross section for a given value of the orbital angular momentum \( l \) can be found in standard text-books to be given by (19):

\[
S_l(k) \sim 1 + 2idi k^{2l+1}, \quad d_l \in \mathbb{R}
\]

\[
\sigma^{(l)} \sim k^{4l}.
\]

Though the expressions obtained for \( S_l \) and \( \sigma^{(l)} \) as such are correct, if we perform a Taylor expansion of these expressions near threshold, we would not obtain the threshold behaviour mentioned above. The latter implies that if we wish to study the threshold behaviour explicitly using the Mittag-Leffler expansion, the poles and residues must satisfy certain conditions which have been derived explicitly in Appendix B.

In case of the S-matrix, note that \( S_l - 1 \) has a zero of order \( 2l + 1 \) at \( k = 0 \). From (B11), its poles and residues must satisfy:

\[
\sum_n b_n \frac{\ln k^{2l+1}}{k^{\nu+1}} + (1)^{\nu} \sum_n \frac{b_n^*}{(k_n^*)^{\nu+1}} + \sum_m iD_{lm} \frac{\ln (i\zeta_{lm})^{2l+1}}{(i\zeta_{lm})^{\nu+1}} = 0, \quad \nu = 1, 2, \ldots, 2l; \quad (40)
\]

and, from (B5), \( S_l(k) \) can be written as,

\[
\frac{S_l(k) - 1}{k^{2l+1}} = \sum_n b_n \frac{k_n^*}{(k - k_{ln})} + \sum_n \frac{b_n^*}{(k_n^*)^{2l+1}} \frac{\ln (k_n^*)^{2l+1}}{(k + k_n^*)} + \sum_m \frac{iD_{lm}}{(i\zeta_{lm})^{2l+1}} \frac{\ln (i\zeta_{lm})^{2l+1}}{(k - i\zeta_{lm})}. \quad (41)
\]

The partial cross section has poles at \( k = \pm k_{ln}, \pm k_n^*, \pm i\zeta_{lm} \). The residues are given by,

\[
\text{Res} \left( \sigma^{(l)}, k = \pm k_{ln} \right) = \lim_{k \to \pm k_{ln}} (k \mp k_{ln}) \sigma^{(l)} = \mp \pi (2l + 1) \frac{b_{ln}}{k_{ln}}, \quad (42)
\]

\[
\text{Res} \left( \sigma^{(l)}, k = \pm k_n^* \right) = \lim_{k \to \pm k_n^*} (k \mp k_n^*) \sigma^{(l)} = \mp \pi (2l + 1) \frac{b_n^*}{k_n^*}, \quad (43)
\]

\[
\text{Res} \left( \sigma^{(l)}, k = \pm i\zeta_{lm} \right) = \lim_{k \to \pm i\zeta_{lm}} (k \mp i\zeta_{lm}) \sigma^{(l)} = \mp \pi (2l + 1) \frac{D_{lm}}{i\zeta_{lm}^2}. \quad (44)
\]

The condition (B11) in this case is:

\[
\text{Re} \left( \sum_n \frac{b_{ln}}{k_{ln}^{2\mu+3}} \right) = \frac{1}{2} \sum_m (-1)^\mu \frac{D_{lm}}{\zeta_{lm}^{2\mu+3}}, \quad \mu = 1, 2, \ldots, 2l - 1. \quad (45)
\]

and the cross section, using (B5), is given as,

\[
\sigma^{(l)} = -4\pi \frac{(2l + 1)}{k^2} k^{2(2l+1)} \left( \text{Re} \sum_n \frac{b_{ln}}{k_{ln}^{4l+1}} \frac{1}{k^2 - k_{ln}^2} + \frac{1}{2} \sum_m \frac{D_{lm}}{\zeta_{lm}^{4l+1}} \frac{1}{k^2 + \zeta_{lm}^2} \right). \quad (46)
\]
$S_l$ and $\sigma^{(l)}$, given by (41) and (46), contain the correct threshold behaviour which is obtained only if the poles and residues satisfy the conditions (40) and (45).

It can be seen from (46) that the partial cross section diverges for large $k$. This is expected since the above expressions were derived with threshold considerations. In other words, for a given value of $L$, all partial wave cross sections for $L + 1$ and bigger can be neglected when $k \to 0$ (see also eqs. (B5)-(B7) in the Appendix B). Thus,

$$\sigma^{(l+1)} = O(\sigma^{(l)}), \quad k \to 0.$$  

(47)

This further implies that the total cross section must be written in the following manner:

$$\sigma = \sum_{l=0}^{L} \sigma^{(l)} + O(\sigma^{(L+1)}), \quad k \to 0,$$

(48)

where $L = 0, 1, 2, \ldots$ Comparing (35) and (48), we can see that they lead to the same expression for $l = 0$ and $k \to 0$. Even if one considers more terms for $k \to 0$, both the expressions will display the same behaviour for $k \to 0$. Finally, we close by mentioning that for large values of $k$, independent of the value of $l$, one must use (35).

III. RESONANCES

In the previous section, we obtained formulae for the cross sections in terms of all the poles and residues of the $S$-matrix which can be evaluated analytically. We shall now focus only on the resonant cross sections and obtain generalized expressions in terms of the pole values, i.e. the central value of the energy (or mass) and width of the resonance. It is gratifying to find that the generalized formula can be expressed in terms of the commonly used Breit-Wigner formula plus corrections. As in the previous section, we will derive the expressions with threshold considerations explicitly.

A word of caution is in order here. In what follows, we are going to approximate the general partial wave cross section formula (36) by disregarding all possible virtual and bound state poles as well as other resonances by picking only one term from the infinite sum. This is a reasonable approximation if all poles other than the resonant one under consideration lie much farther away and do not have large residues. Unfortunately, this is not always the case. In realistic hadron spectroscopy, for example, the resonance widths are often of the same order of magnitude as the radial and/or angular splittings. Even if the latter is true,
simplistic approximations such as the above provide an ease in mathematical manipulations and as we shall see below, also lead to some useful conclusions.

To start with, we note that by “generalized expression” here, we mean the case of a resonance pole, $k_{lr}^2 = \epsilon_{lr} - i\Gamma_{lr}/2$, where $\Gamma_{lr}/2\epsilon_{lr} \ll 1$ is not necessarily true but $\Gamma_{lr}/2\epsilon_{lr} < 1$. If $\Gamma_{lr}/2\epsilon_{lr} \ll 1$, however, we recover the Breit-Wigner resonance formula. Recall that $\epsilon_{lr} = E_{lr} - E_{th}$, with $E_{th}$ being the threshold energy and $E_{lr}$ the real part of the pole in the complex energy plane. In what follows, we derive the resonant cross section formula depending on the parameter

$$x_{lr} \equiv \frac{\Gamma_{lr}}{2\epsilon_{lr}},$$

(49)

which characterizes this ratio. Using the general expression (35) for the cross section derived earlier and neglecting all other (such as the virtual and bound state) poles except the resonant one, we can write for an isolated resonance and a given orbital angular momentum $l$:

$$\sigma^{(l)} = -4\pi (2l + 1) \text{Re} \left( \frac{b_{lr}}{k_{lr}} \frac{1}{k^2 - k_{lr}^2} \right) = -4\pi \frac{2l + 1}{|k_{lr}|^2} \text{Re} \left[ \frac{b_{lr}k_{lr}^* (k^2 - k_{lr}^* 2)}{|k^2 - k_{lr}^* 2|^2} \right].$$

(50)

In terms of $k_{lr}^2 = \epsilon_{lr} - i\Gamma_{lr}/2$, we have,

$$\sigma^{(l)} = -4\pi \frac{2l + 1}{|k_{lr}|^2} \frac{\Gamma_{lr}}{2} \left[ 1 + \frac{\text{Im} (b_{lr}k_{lr}^*) \left( k^2 - \epsilon_{lr} \right)}{\text{Re} (b_{lr}k_{lr}^*) \left( \Gamma_{lr}/2 \right)} \right] \left[ 1 + \left( \frac{k^2 - \epsilon_{lr}}{\Gamma_{lr}/2} \right)^2 \right].$$

(51)

From $b_{lr} = 2ik_{lr} \tan (\text{Arg } k_{lr})$, we find that $b_{lr}k_{lr}^* = 2|k_{lr}|^2 \tan (\text{Arg } k_{lr})$ with its real and imaginary parts given respectively by, 0 and $2|k_{lr}|^2 \tan (\text{Arg } k_{lr})$. Therefore,

$$\frac{\text{Re} (b_{lr}k_{lr}^*)}{\text{Im} (b_{lr}k_{lr}^*)} = 0,$$

(52)

and

$$\sigma^{(l)} = -4\pi (2l + 1) \frac{2 \tan (\text{Arg } k_{lr})}{\Gamma_{lr}/2} \frac{1}{\left[ 1 + \left( \frac{k^2 - \epsilon_{lr}}{\Gamma_{lr}/2} \right)^2 \right]}.$$

(53)

On the other hand, the argument of $k_{lr}$ is:

$$\text{Arg } (k_{lr}) = \text{Arg } \sqrt{\epsilon_{lr} - i\frac{\Gamma_{lr}}{2}} = -\frac{1}{2} \text{arctan } (x_{lr}).$$

(54)
The identity \( \tan \frac{1}{2} z = \csc z - \cot z \) allows us to write the tangent of this argument as,

\[
\tan (\text{Arg } k_{lr}) = \frac{2\epsilon_{lr}}{\Gamma_{lr}} \left[ 1 - \sqrt{1 + (x_{lr})^2} \right],
\]

(55)

and on expanding in powers of \( \Gamma_{lr}/2\epsilon_{lr} \),

\[
\tan (\text{Arg } k_{lr}) = -\frac{1}{2} x_{lr} + \frac{1}{8} (x_{lr})^3 + \cdots
\]

(56)

Substituting (56) in (53), we find that

\[
\sigma^{(l)} = \frac{4\pi (2l + 1)}{\epsilon_{lr}} \frac{(\Gamma_{lr}/2)^2}{(k^2 - \epsilon_{lr})^2 + (\Gamma_{lr}/2)^2} \left[ 1 - \frac{1}{4} \left( \frac{\Gamma_{lr}}{2\epsilon_{lr}} \right)^2 + \frac{1}{8} \left( \frac{\Gamma_{lr}}{2\epsilon_{lr}} \right)^4 - \frac{5}{64} \left( \frac{\Gamma_{lr}}{2\epsilon_{lr}} \right)^6 + \cdots \right].
\]

(57)

where \( \frac{\Gamma_{lr}}{2\epsilon_{lr}} < 1 \) but \( \Gamma_{lr}/2\epsilon_{lr} \ll 1 \) is not necessarily true.

FIG. 1: Comparison of the exact cross section expression (53) (black solid line) with the commonly used Breit-Wigner cross section (70) (black dashed line). The cross section has been scaled by a factor to plot the dimensionless quantity, \( \xi = \frac{\sigma^{(l)}}{4\pi (2l + 1) / \epsilon_{lr}} \). The red lines indicate the cross sections obtained after adding the corrections given in (57) one by one to the Breit-Wigner cross section. The left panel shows the case of a broad resonance with the ratio \( x = \Gamma_r/(2\epsilon_r) \) close to 1 and the right panel displays the case of the baryon resonance \( \Delta(1232) \) with a smaller \( x \).

In Fig. 1, we show a comparison of the exact expression (53) with the Breit-Wigner (BW) formula (the first term in (57)) for a broad (\( \sigma \) meson) and a not so broad (\( \Delta(1232) \)) resonance.
We choose here an average mass and width of the $\sigma$ meson such that $x = \Gamma_r/(2\epsilon_r) < 1$. The idea of this exercise is to simply demonstrate the comparison of the exact and the BW formula in the case of a broad and narrow resonance. In this respect, we must clarify to the reader that the intricate physics of the $\sigma$ meson cannot be contained in a naive single-pole description. This involves quark confinement, quark pair creation, coupled channels and an Adler zero which was shown to be crucial for the description of the $\sigma$ and $\kappa$ mesons. The choice of the $\sigma$ and $\Delta(1232)$ may not be the best realistic choice but serves the purpose of our exercise.

Coming back to Fig. 1, we see that the BW formula overestimates the cross sections. Addition of the correction terms in (57), alternately decreases and increases the cross sections, bringing them closer to the exact result. The broad $\sigma$ meson which decays dominantly by $\sigma \to \pi\pi$ has a mass ($M_\sigma$) and width ($\Gamma_r$) of about 500 and 400 MeV respectively. Thus the value of $\epsilon_r = M_\sigma - 2M_\pi$ is 220 MeV and $x = \Gamma_r/(2\epsilon_r)$ is about 0.9. In case of the $\Delta$ resonance which decays by emitting a pion and a nucleon, the ratio, $x = \Gamma_r/(2\epsilon_r) = 0.375$. A good agreement with the exact expression in case of the $\sigma$ is obtained after adding five corrections terms, whereas, for the $\Delta$, only two correction terms suffice. The insets in the figures display how the cross sections with BW+corrections approach the exact result. Since we are interested in comparing the exact results with the BW in the region around the peak of the resonance, we do not worry about the correct threshold behaviour of the cross section in Fig. 1.

A. Estimating the goodness of the Taylor expansion

Let us begin with the following expression derived earlier:

$$\sigma^{(l)} = -4\pi(2l + 1) \frac{2\tan (\text{Arg } k_{lr})}{\Gamma_{lr}/2} \frac{1}{1 + \left(\frac{k^2 - \epsilon_{lr}}{\Gamma_{lr}/2}\right)^2},$$

where

$$\tan (\text{Arg } k_{lr}) = \frac{2\epsilon_{lr}}{\Gamma_{lr}} \left[1 - \sqrt{1 + x_{lr}^2}\right] = \frac{1 - \sqrt{1 + x_{lr}^2}}{x_{lr}}.$$  \hspace{1cm} (59)

Substituting $z = x_{lr}^2$ and $\alpha = 1/2$ in the expansion

$$(1 + z)^\alpha = \sum_{q=0}^{\infty} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - q)} \frac{z^q}{q!},$$

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we get:

\[
\sqrt{1 + x_{lr}^2} = \sum_{q=0}^{\infty} \frac{\Gamma\left(\frac{3}{2} - q\right)}{\Gamma\left(\frac{1}{2} - q\right)} \frac{x_{lr}^{2q}}{q!} = 1 + \frac{1}{2} x_{lr}^2 + \frac{\sqrt{\pi}}{2} \sum_{q=1}^{\infty} \frac{1}{\Gamma\left(\frac{1}{2} - q\right)} \frac{x_{lr}^{2q+2}}{(q+1)!}.
\]  

(61)

From the properties of the Gamma function, the last equation can be written as

\[
\sqrt{1 + x_{lr}^2} = 1 + \frac{x_{lr}^2}{2} \sum_{q=0}^{\infty} \frac{(-1)^q (2q)!}{q + 1} \left(\frac{x_{lr}}{2}\right)^{2q}.
\]  

(62)

Hence,

\[
\frac{\sqrt{1 + x_{lr}^2}}{1 + K^2} = \frac{1}{\epsilon_{lr}} \frac{1}{1 + K^2} \sum_{q=0}^{\infty} \frac{(-1)^q (2q)!}{q + 1} \left(\frac{x_{lr}}{2}\right)^{2q}.
\]  

(63)

and the cross section is given by:

\[
\sigma^{(l)} = \frac{4\pi(2l + 1)}{\epsilon_{lr}} \frac{1}{1 + K^2} \frac{1}{1 + K^2} \sum_{q=0}^{\infty} \frac{(-1)^q (2q)!}{q + 1} \left(\frac{x_{lr}}{2}\right)^{2q} = \frac{4\pi(2l + 1)}{\epsilon_{lr}} \frac{1}{1 + K^2} \left(1 - \frac{1}{4} x_{lr}^2 + \frac{1}{8} x_{lr}^4 - \frac{5}{64} x_{lr}^6 + \frac{7}{128} x_{lr}^8 + \cdots\right),
\]  

(64)

where \( K = \frac{k^2 - \epsilon_{lr}}{\Gamma_{lr}/2} \). Let us now estimate the error in computing \( \sigma^{(l)} \) when we take \( N - 1 \) terms of the last expansion. Since the series is alternating, the error is less than the absolute value of the next term omitted. Calling this error \( \eta_N \) in units of \( \frac{4\pi(2l + 1)}{\epsilon_{lr}} \),

\[
\eta_N = \left| \frac{\sigma^{(l)}}{4\pi(2l + 1)/\epsilon_{lr}} - \frac{1}{1 + K^2} \sum_{q=0}^{N-1} \frac{(-1)^q (2q)!}{q + 1} \left(\frac{x_{lr}}{2}\right)^{2q} \right| \leq \left[ \frac{1}{1 + K^2} \right] \frac{1}{N + 1} \frac{(2N)!}{(N!)^2} \left(\frac{x_{lr}}{2}\right)^{2N}.
\]  

(65)

The term in square brackets is always less than one for any value of \( k \), hence,

\[
\eta_N \leq \frac{1}{N + 1} \frac{(2N)!}{(N!)^2} \left(\frac{x_{lr}}{2}\right)^{2N}.
\]  

(66)

Applying this result to the examples discussed before, in case of the broad \( \sigma \) meson we find that the error in calculating the cross section using 5 correction terms is less than or equal to \( 1, 4 \times 10^{-2} \) and the same in case of the \( \Delta \) (1232) using only 2 terms is less than \( 2.6 \times 10^{-3} \).

To illustrate the usefulness of the formula derived above, in the following table, we show the number of terms necessary in order to calculate the cross section using (64) such that the error is less than different powers of 10 shown in the table.
TABLE I: Number of terms necessary to calculate the cross section using the expansion (64) with an error less than or equal to $\eta_N$.

| $x_{lr}$ | 10^{-1} | 10^{-2} | 10^{-3} | 10^{-4} | 10^{-5} | 10^{-6} |
|---------|---------|---------|---------|---------|---------|---------|
| 0.1     | 1       | 1       | 2       | 2       | 3       | 3       |
| 0.2     | 1       | 2       | 2       | 3       | 3       | 4       |
| 0.3     | 1       | 2       | 3       | 3       | 4       | 5       |
| 0.4     | 1       | 2       | 3       | 4       | 5       | 6       |
| 0.5     | 1       | 2       | 4       | 5       | 6       | 8       |
| 0.6     | 1       | 3       | 4       | 6       | 8       | 10      |
| 0.7     | 2       | 3       | 6       | 8       | 11      | 14      |
| 0.8     | 2       | 4       | 8       | 12      | 16      | 20      |
| 0.9     | 2       | 6       | 12      | 20      | 29      | 38      |

B. Threshold considerations

In order to incorporate the correct threshold behaviour in the cross section expression, we repeat the above exercise with the cross section expression (46) valid close to threshold, i.e., starting with,

$$\sigma(l) = -4\pi \text{Re} \left\{ (2l + 1) \frac{b_{lr}}{k_{lr}} \left( \frac{k}{k_{lr}} \right)^4 \frac{1}{k^2 - k_{lr}^2} \right\},$$

and replacing the residues as before and after some lengthy algebra, we find,

$$\sigma(l) = -4\pi \frac{2l + 1}{k^2} k^{2(2l+1)} 2 \tan(\text{Arg} k_{lr}) \frac{\text{Re}(k_{lr}^{*4l})}{\Gamma_{lr}/2} \frac{1 - \frac{\text{Im}(k_{lr}^{*4l})}{\text{Re}(k_{lr}^{*4l})} \left( k^2 - \epsilon_{lr} \right)}{1 + \left( \frac{k^2 - \epsilon_{lr}}{\Gamma_{lr}/2} \right)^2}.$$  \hspace{1cm} (68)

Once again, performing expansions in terms of the variable $x_{lr}$, one can show,

$$\sigma(l) = 4\pi \frac{(2l + 1)}{k^2} k^{2(2l+1)} \frac{k_{lr}^{2(2l+1)}}{\epsilon_{lr}^2} \left[ \frac{\Gamma_{lr}/2}{(k^2 - \epsilon_{lr})^2 + (\Gamma_{lr}/2)^2} \left( \frac{\Gamma_{lr}}{2\epsilon_{lr}} \right) \right] \left[ \frac{\Gamma_{lr}/2}{(k^2 - \epsilon_{lr})^2} - \frac{8l^2 + 4l + 1}{4} \left( \frac{\Gamma_{lr}}{2\epsilon_{lr}} \right) + \cdots \right].$$

\hspace{1cm} (69)
Finally, some comments regarding (57) and (69) are in order.

1) Considering the first term in the expansion of \( \sigma^{(l)} \) in Eq. (57), i.e. for the case when \( \Gamma_{lr}/2\varepsilon_{lr} \ll 1 \):

\[
\sigma^{(l)} = \frac{4\pi(2l+1)}{\varepsilon_{lr}} \frac{(\Gamma_{lr}/2)^2}{(k^2 - \varepsilon_{lr})^2 + (\Gamma_{lr}/2)^2},
\]

and we obtain the standard Breit-Wigner distribution for the total cross section.

2) The first term of \( \sigma^{(l)} \) given by (69):

\[
\sigma^{(l)} = \frac{4\pi(2l+1)}{k^2} \left( \frac{k^2}{\text{Re} k_{lr}^2} \right)^{2l+1} \frac{(\Gamma_{lr}/2)^2}{(k^2 - \varepsilon_{lr})^2 + (\Gamma_{lr}/2)^2},
\]

explicitly displays the threshold behaviour which is similar for example to that used in the partial wave analysis of baryon resonances [34]. The above equation reduces to Eq. (57) for the \( l = 0 \) case. It is gratifying to recover the commonly used Breit-Wigner distributions for narrow resonances from the generalized expressions (53) and (68).

IV. ENERGY DERIVATIVE OF THE PHASE SHIFT

In the Fock-Krylov method [35], the survival amplitude, \( A_l(t) \), of an unstable state is given by the Fourier transform of the density of states \( \rho_l(E) \) (usually taken to be of a Lorentzian form) with a threshold factor and a form factor \( f(E) \) to ensure that the energy distribution \( \rho_l(E) \to 0 \) for large \( E \). The latter, with the use of the Beth-Uhlenbeck formula for the difference in the density of states with and without interaction, can be written in terms of the energy derivative of the phase shift \( (\delta_l) \) [14, 15]. While calculating the second virial coefficients \( B, C \) in the equation of an ideal gas, \( PV = RT(1 + B/V + C/V^2 + ...) \), Beth and Uhlenbeck found that the difference in the density of states (of scattered particles) with interaction \( dn_l(E)/dE \) and \( dn_l^{(0)}(E)/dE \) without interaction is

\[
\rho_l(E) = - \frac{dn_l(E)}{dE} + \frac{dn_l^{(0)}(E)}{dE} = \frac{2l+1}{\pi} \frac{d\delta_l(E)}{dE}.
\]

In resonant scattering, this is the density of states of a resonance (in terms of the decay products) [36] and is useful in writing the survival amplitude of the resonance as,

\[
A_l(t) = \int_0^\infty f(E) \left( \frac{d\delta_l}{dE} \right) e^{-iEt} dE.
\]
Here, $E = E_{CM} - E_{th}$, where $E_{CM}$ is the energy available in the centre of mass of the decay products and hence the lower limit of integration corresponding to $E_{CM} = E_{th}$ is $E = 0$. The form factor $f(E)$ is commonly taken to be, $f(E) = e^{-cE}, \quad c > 0$. The amplitude can further be expressed as,

$$A_l(t) = \int_0^\infty \left( \frac{d\delta_l}{dE} \right) e^{-(c+it)E} dE = \int_0^\infty \left( \frac{d\delta_l}{dE} \right) e^{-sE} dE,$$

(73)

where $s \equiv c + it$, and Re $s > 0$. The probability amplitude written in this manner is the Laplace transform of $d\delta_l/dE$. The energy derivative of the phase shift in elastic scattering was also shown to represent the time delay introduced in the scattering process due to the propagation of a resonance, by Wigner [38] in 1955.

We shall now calculate the derivative of the phase shift with respect to $k$ (here $k^2 = E$) and eventually relate it to the energy derivative. In terms of the $S$-matrix, $S_l(k) = e^{2i\delta_l(k)}$ and the phase shift derivative is given by,

$$\frac{d\delta_l}{dk} \equiv h_l(k) = \frac{1}{2i} \frac{1}{S_l(k)} \frac{dS_l}{dk}.$$  \hspace{1cm} (74)

In order to evaluate (73), knowledge of the analytical properties of $d\delta_l/dk$, which depend on the properties of $S_l(k)$ are required. The poles of $h_l(k)$ correspond either to the zeros or poles of $S_l(k)$. With $S_l(k)$ being a meromorphic function (as observed with the points in the beginning), we shall now attempt a Mittag-Leffler expansion of $h_l(k)$. To this end, let us first evaluate the residues of $h_l(k)$.

A. Residues of the phase shift derivative

The expansion of the derivative of the phase shift, $h_l(k)$ in (74), can be done with the knowledge of the residues of the corresponding poles. If $k = P$ is a pole of $S_l(k)$, using the Laurent theorem, the $S$-matrix can be written as,

$$S_l(k) = \frac{B}{k - P} + F_l(k),$$

(75)

where $B$ is the residue of $S_l(k)$ at the given pole and $F_l(k)$ is an analytic function in the vicinity of $k = P$. $h_l(k)$ can be written as follows:

$$h_l(k) = \frac{1}{2i} \frac{1}{S_l(k)} \frac{dS_l}{dk} = \frac{1}{2i} \frac{B}{(k - P)^2} + F_l \left. \right| \frac{1}{2i} \frac{1}{k - P} \frac{1}{B} \frac{F_l}{F_l}.$$  \hspace{1cm} (76)

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The rightmost ratio in the above expression is analytic at $k = P$ and is unity for $k = P$. On performing a Taylor series expansion about $k = P$, the first term in the expansion does not change the coefficient $i/2$ (the residue of $h_t(k)$ at $k = P$) of $(k - P)^{-1}$.

Let us now assume that $k = Z$ is a simple zero of $S_t(k)$. In this case, we can write $S_t(k)$ as,

$$S_t(k) = (k - Z)\left[b_1 + b_2(k - Z) + \cdots\right]$$  \hfill (77)

such that, $h_t(k)$ is given by,

$$h_t(k) = \frac{1}{2i} \frac{1}{S_t(k)} \frac{dS_t(k)}{dk} = \frac{1}{2i} \frac{1}{k - Z} \frac{b_1 + 2b_2(k - Z) + \cdots}{b_1 + b_2(k - Z) + \cdots}$$  \hfill (78)

Once again, the rightmost ratio is analytic at $k = Z$ and its value at this point is 1. Expanding once again about $k = Z$, the coefficient of $(k - Z)^{-1}$, namely, $-i/2$, which is the residue of $h_t(k)$ at $k = Z$ is not altered. To summarize the above:

If $k = P$ is a pole of the $S$-matrix, the residue of $h_t(k) = d\delta_t / dk$ at $k = P$ is:

$$\text{Res} \left( \frac{d\delta_t}{dk}, P \right) = \lim_{k \to k_{in}} (k - P) \frac{d\delta_t}{dk} = + \frac{i}{2},$$  \hfill (79)

and if $k = Z$ is a zero of the $S$-matrix, the residue of $h_t(k) = d\delta_t / dk$ at $k = Z$ is:

$$\text{Res} \left( \frac{d\delta_t}{dk}, Z \right) = \lim_{k \to k_{in}} (k - Z) \frac{d\delta_t}{dk} = - \frac{i}{2},$$  \hfill (80)

Note that these residues are independent of the location and the kind of poles as well as the potential involved.

**B. Mittag-Leffler expansion of the phase shift derivative**

We are now in a position to write the Mittag-Leffler expansion of the phase shift derivative. In what follows, we shall neglect virtual and bound state poles. Considering the properties of $h_t(k)$ discussed so far, if $k_{in}$ are poles of the $S$ matrix in the fourth quadrant, then they are also poles of $h_t(k)$, just as $-k_{in}$ and $\pm k_{in}^*$. Before applying the Mittag-Leffler theorem, we recall that in the vicinity of $k = 0$, $h_t(k) = d\delta_t / dk \sim A_t k^{2l}$, where $A_t$, for example, in case of potential scattering, is a constant given by [19]

$$A_t = a_{2l+1} \frac{a^{2l+1} + l - a_{2l}^*}{(2l + 1)!!(2l - 1)!!} \frac{l + 1 + a_{2l}}{l + 1 + a_{2l}},$$  \hfill (81)
with $a$ being the range of the potential and $\hat{\gamma}_l$ the derivative of the radial wave function at $r = a$. As a result, at $k = 0$ the latter is $A_l\delta_{l0}$, and the Mittag-Leffler theorem leads us to

$$h_l(k) = A_l\delta_{l0} + \sum_n \frac{i}{2} \frac{k}{k_{l n}^2} \left( \frac{1}{k - k_{l n}} + \frac{1}{k + k_{l n}} \right) - \sum_n \frac{i}{2} \frac{k}{k_{l n}^*} \left( \frac{1}{k - k_{l n}^*} + \frac{1}{k + k_{l n}^*} \right)$$

(82)

$$= A_l\delta_{l0} - 2k^2 \text{Im} \sum_n \frac{1}{k_{l n} (k^2 - k_{l n}^2)} .$$

The derivative with respect to energy is hence given by

$$\frac{d\delta_l}{dE} = \frac{d\delta_l}{dk} \frac{dk}{dE} = \frac{h_l(\sqrt{E})}{2\sqrt{E}} = \frac{A_l}{2} \delta_{l0} E^{-1/2} - \text{Im} \sum_n \frac{\sqrt{E}}{k_{l n} (E - k_{l n}^2)} .$$

(83)

A brief discussion of the physical meaning of the above relation at this point is in order. Let us recall that the energy derivative of the phase shift in single channel elastic scattering also represents Wigner’s time delay [38] due to interaction. This “phase” time delay, in the $l = 0$ case displays a singularity near threshold. If one considers the phase time in one dimensional tunneling, the singularity can be shown to arise due to the interference between the incident and reflected waves in front of the barrier. Subtraction of the singular term leads to the dwell time delay which was shown in [39] to reproduce the correct behaviour near threshold. Apart from this, in [40] the authors demonstrated that the dwell time indeed has a physical meaning and gives the half lives of radioactive nuclei. In the above expression for the energy derivative of the phase shift, we can see using Eq. [38], that the interference term is proportional to

$$\frac{\text{Im} S_0(k) dk}{k} \frac{dE}{dE} = \frac{d_0}{2\sqrt{E}} = \frac{1}{2} d_0 E^{-1/2} .$$

Since the energy derivative of the phase shift also corresponds to the density of states, (83) is expected to give the correct density for $S$-waves only after a subtraction of the singular term. Thus, the corrected density is given by

$$\left( \frac{d\delta_l}{dE} \right)_C = -\text{Im} \sum_n \frac{\sqrt{E}}{k_{l n} (E - k_{l n}^2)} .$$

(84)

Integrating the above expression, Eq. (84) for the derivative of the phase shift, we obtain the phase shift:

$$\delta_l(E) = \int_0^E \left( \frac{d\delta_l}{dE} \right)_C dE = \text{Im} \sum_n \frac{1}{k_{l n}} \int_0^E \frac{E^{1/2}}{E_{l n}^2 - E} dE .$$

(85)
With a change of variable \( E = k_{ln}^2 \tanh^2 z \) and integrating,

\[
\delta_l(E) = 2 \text{Im} \sum_n \left[ \tanh^{-1} \left( \frac{\sqrt{E}}{k_{ln}} \right) - \frac{\sqrt{E}}{k_{ln}} \right]
\]  

(86)

To end this section, we investigate the behaviour of the phase shift derivative in case of an isolated resonance. Given a pole at \( E_{lr} = k_{lr}^2 = \epsilon_{lr} - i \Gamma_{lr}/2 \), we begin from (85) by writing it as:

\[
\frac{d\delta_l}{dE} = \text{Im} \frac{1}{k_{lr}} \left( \frac{E^{1/2}}{k_{lr}^2 - E} \right).
\]  

(87)

Expanding \( E^{1/2} \) about \( E = k_{lr}^2 \) leads us to

\[
\frac{d\delta_l}{dE} = \text{Im} \frac{1}{k_{lr}^2 - E} + \cdots
\]  

(88)

Evaluating the imaginary part,

\[
\frac{d\delta_l}{dE} = \text{Im} \frac{(k_{lr}^2)^2 - E}{|k_{lr}^2 - E|^2} + \cdots = \frac{\Gamma_{lr}/2}{(E - \epsilon_{lr})^2 + (\Gamma_{lr}/2)^2} + \cdots
\]  

(89)

and we notice that the phase shift derivative around an isolated resonance pole can be described by a Breit-Wigner distribution plus corrections. Integrating Eq. (89) with respect to \( E \), we get

\[
\delta_l(E) = \arctan \left( \frac{\Gamma_{lr}/2}{\epsilon_{lr} - E} \right) + \cdots
\]  

(90)

which is once again the standard expression for a Breit-Wigner phase shift as found in textbooks \[19\]. Eqs (89) and (90) are valid in the vicinity of an isolated resonance.

C. Threshold behaviour of the phase shift derivative

Extending the argument of the interference term discussed in the previous subsection for non-zero orbital angular momenta, such a term is proportional to \[39, 41\],

\[
\text{Im} \frac{S_l(k)}{k} \frac{dk}{dE} = \frac{1}{2} d_l E^{l-1/2}.
\]

This means if we perform a Taylor series expansion of \( \frac{dk}{dE} \) about \( E = 0 \):

\[
\frac{d\delta_l}{dE} = d_l E^{l-1/2} \left( 1 + a_1 E + a_2 E^2 + \cdots \right),
\]

the first term should be eliminated in order to obtain the correct behaviour of the phase shift derivative at low energies. Indeed, as expected, we do obtain the correct threshold
behaviour $E^{l+1/2}$ [15, 37]. As derived in a previous section, the poles and residues of $d\delta_l/dk$ must satisfy the condition (B11):

$$
\sum_n \frac{i}{k_{ln}^{\eta+1}} + \sum_n \frac{i}{(-k_{ln}^*)^{\eta+1}} - \sum_n \frac{i}{k_{ln}^{\eta+1}} - \sum_n \frac{i}{(-k_{ln}^*)^{\eta+1}} \quad (91)
$$

$$
= 2i \sum_n [1 + (-1)^\eta] \text{Im} \left( \frac{1}{k_{ln}^{\eta+1}} \right)
$$

$$
= 0, \quad \eta = 1, 2, \ldots, 2l + 1.
$$

For odd $\eta$, we obtain a null condition from (91). Substituting $\eta \rightarrow 2\eta$ in the previous equation,

$$
\text{Im} \sum_n \left( \frac{1}{k_{ln}^{2\eta+1}} \right) = 0, \quad \eta = 1, 2, \ldots, l.
$$

Thus, the expression for the phase shift derivative near threshold is:

$$
\frac{d\delta_l}{dE} = \frac{d\delta_l}{dk} \frac{dk}{dE} = \sum_n \left[ \frac{i}{2k_{ln}^{2l+1}} \left( \frac{E^{l+1/2}}{E - k_{ln}^2} \right) - \frac{i}{2(k_{ln}^*)^{2l+1}} \left( \frac{E^{l+1/2}}{E - k_{ln}^*^2} \right) \right]
$$

$$
= \text{Im} \sum_n \frac{1}{k_{ln}^{2l+1}} \frac{E^{l+1/2}}{k_{ln}^2 - E^*}, \quad l = 0, 1, 2, \ldots
$$

V. SURVIVAL PROBABILITY OF AN UNSTABLE STATE

For a resonance formed as an intermediate unstable state in a scattering process, the survival amplitude (using the Fock-Krylov method) can be expressed as seen before, as a Laplace transform of the energy derivative of the scattering phase shift. Using the Mittag-Leffler expansions derived for the phase shift derivatives we shall now try to derive analytical expressions for the survival probabilities and discuss their long time behaviour.

A. General expression for the survival amplitude

Replacing Eq. (84) in (73), we get,

$$
A_l(t) = \sum_n \frac{i}{2k_{ln}} \int_0^\infty \frac{E^{1/2}}{E - k_{ln}^2} e^{-tE} dE - \sum_n \frac{i}{2k_{ln}^*} \int_0^\infty \frac{E^{1/2}}{E - k_{ln}^*^2} e^{-tE} dE \quad (94)
$$

If we define $J(\alpha, \sigma, s)$ as

$$
J(\alpha, \sigma, s) \equiv \int_0^\infty \frac{E^{\alpha}}{E + \sigma} e^{-sE} dE, \quad \text{Re } s > 0,
$$

22
then the survival amplitude can be written as,

$$A_l(t) = \sum_n \frac{i}{2k_in} J\left(\frac{1}{2}, -k_{in}^2, s\right) - \sum_n \frac{i}{2k^*_in} J\left(\frac{1}{2}, -k_{in}^2, s\right).$$  \hspace{1cm} (96)$$

In order to ensure that the survival probability, $|A_l(t)|^2$ should be unity at $t = 0$, we evaluate the normalization factor $A = A_l(t = 0)$ and write,

$$A_l(t) = \frac{1}{A} \sum_n \frac{i}{2k_in} J\left(\frac{1}{2}, -k_{in}^2, s\right) - \frac{1}{A} \sum_n \frac{i}{2k^*_in} J\left(\frac{1}{2}, -k_{in}^2, s\right),$$  \hspace{1cm} (97)$$

where $A$ is equal to

$$A = \sum_n \frac{i}{2k_in} J\left(\frac{1}{2}, -k_{in}^2, c\right) - \sum_n \frac{i}{2k^*_in} J\left(\frac{1}{2}, -k_{in}^2, c\right).$$  \hspace{1cm} (98)$$

The calculation of the survival amplitude reduces simply to the evaluation of the integral \[95\]. We start with

$$\frac{1}{E + \sigma} = \int_0^\infty e^{-(E+\sigma)F} dF, \quad E > -\text{Re} \sigma,$$

and replacing in $J$, we get

$$J(\alpha, \sigma, s) = \int_0^\infty E^\alpha e^{-sE} \left[ \int_0^\infty e^{-(E+\sigma)F} dF \right] dE = \int_0^\infty e^{-sF} \left[ \int_0^\infty E^\alpha e^{-(s+F)E} dE \right] dF$$

$$= \Gamma(\alpha + 1) \int_0^\infty \frac{e^{-sF}}{(s+F)^{\alpha+1}} dF = \Gamma(\alpha + 1) e^{\sigma s} \int_s^\infty \frac{e^{-sF}}{F^{\alpha+1}} dF$$

$$= \Gamma(\alpha + 1) e^{\sigma s} \sigma^\alpha \int_0^\infty \frac{e^{-F}}{F^{\alpha+1}} dF. \hspace{1cm} (100)$$

As a result of the uniform convergence of the integral \[99\] over the interval of $E$ integration, the order of integration can be changed whenever $\text{Re} \sigma > 0$. The last integral can be expressed in terms of the incomplete Gamma functions $\Gamma(\beta, z)$ \[43, 44\]:

$$\Gamma(\beta, z) = \int_z^\infty e^{-t^{\beta-1}} dt, \quad |\text{Arg} \, z| < \pi.$$  \hspace{1cm} (101)$$

Therefore,

$$J(\alpha, \sigma, s) = \Gamma(\alpha + 1) e^{\sigma s} \sigma^\alpha \Gamma(-\alpha, \sigma s).$$  \hspace{1cm} (102)$$

The integral exists if, $\text{Re} \, s > 0$, $\text{Re} \, \alpha > -1$, $\text{Re} \, \sigma > 0$ and $|\text{Arg} \, (\sigma s)| < \pi$. The last two conditions can be changed to $|\text{Arg} \, \sigma| < \pi$. Since $\alpha = l + \frac{1}{2}$, with $l$ being a positive integer, it

\[1\] The integral can be found in \[42\], Chapter 4, page 137.
is possible to express $J$ in terms of error functions. The latter is very useful in the evaluation of the survival amplitudes numerically. In order to do this, we begin with

$$\frac{E^l}{E + \sigma} = \sum_{p=0}^{l-1} (-1)^p \sigma^p E^{l-p-1} + (-1)^l \frac{\sigma^l}{E + \sigma}. \quad (103)$$

Replacing in the definition of $J$, we get

$$J(l + \frac{1}{2}, \sigma, s) = \sum_{p=0}^{l-1} (-1)^p \sigma^p \int_0^\infty E^{l-p-1/2} e^{-sE} dE + (-1)^l \sigma^l \int_0^\infty \frac{E^{1/2}}{E + \sigma} e^{-sE} dE$$

$$= (-1)^l \sigma^{l+1/2} \left[ \sum_{p=0}^{l} (-1)^p \frac{\Gamma(p + \frac{1}{2})}{\sigma^{l+1/2} s^{l+1/2}} - \pi s e^{\sigma s} \text{erfc} \left( \frac{\sigma^{1/2} s^{1/2}}{2} \right) \right], \quad (104)$$

where $|\text{Arg} \sigma| < \pi$ and Re $s > 0$. The above was done by using the following integral [42]:

$$\int_0^\infty \frac{E^{1/2}}{E + \sigma} e^{-sE} dE = \sqrt{\frac{\pi}{s}} - \pi s^{1/2} e^{\sigma s} \text{erfc} \left( \sigma^{1/2} s^{1/2} \right), \quad |\text{Arg} \sigma| < \pi, \text{Re } s > 0.$$ 

Eq. (104) can also be deduced from (102) using recurrence relations of the incomplete gamma functions. This allows us to write the results as follows:

$$A_l(t) = -\frac{1}{2A} \Gamma\left(\frac{3}{2}\right) \sum_n \left[ e^{-k_{in}^2 s} \Gamma\left(-\frac{1}{2}, -k_{in}^2 s\right) - e^{-k_{in}^2 s} \Gamma\left(-\frac{1}{2}, -k_{in}^2 s\right) \right]. \quad (105)$$

Also, the factor $A$ can be calculated in a similar way:

$$A = -\frac{1}{2} \Gamma\left(\frac{3}{2}\right) \sum_n \left[ e^{-k_{in}^2 c} \Gamma\left(-\frac{1}{2}, -k_{in}^2 c\right) - e^{-k_{in}^2 c} \Gamma\left(-\frac{1}{2}, -k_{in}^2 c\right) \right]. \quad (106)$$

The above $A_l(t)$ gives us a general, model independent expression for the survival amplitude of resonances occurring in the $l^{th}$ partial wave.

**B. Large time behaviour of the survival probability**

It is well known by now that the exponential decay is only an approximation and quantum mechanics predicts a non-exponential decay (actually power laws) at very short and large times [45] in the time evolution of an unstable state. Hence, using the results obtained so

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2 An expression for the survival amplitude of a Breit-Wigner resonance, in terms of hypergeometric functions is derived in [46].
far, we shall now examine the behaviour of the survival amplitude at large times. In order
to examine the survival probability at large times, we begin with
\[ e^z \Gamma(-\alpha, z) \sim \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(\alpha + 1 + r)}{\Gamma(\alpha + 1)} z^{-(\alpha+1+r)}, \quad \alpha > 0. \]

Replacing the above expression in Eq. (105), one obtains
\[ A_l(t) \sim -\frac{1}{2A} \sum_{n} \frac{\Gamma(r + \frac{3}{2})}{(r+3/2)^{\alpha+1+3}} \left( \frac{1}{(-k_{ln}^2 s)^{r+3/2}} - \frac{1}{(-k_{ln}^2 s)^{r+3/2}} \right) \]
\[ \sim \frac{1}{A} \sum_{r=0}^{\infty} \frac{\Gamma(r + \frac{3}{2})}{s^{r+3/2}} \text{Im} \left( \sum_{n} \frac{1}{k_{ln}^{2r+3}} \right). \] (107)

Using the condition (92), and noting that \( s \sim it \), we have
\[ A_l(t) \sim \frac{1}{A} \sum_{n} \frac{\Gamma(l + \frac{3}{2})}{i^{l+3/2}} \text{Im} \left( \sum_{n} \frac{1}{k_{ln}^{2l+3}} \right) t^{-l-3/2}. \] (108)

The dominant contribution is given by the term for which \( r = l \). Therefore,
\[ A_l(t) \sim \frac{1}{A} \frac{\Gamma(l + \frac{3}{2})}{i^{l+3/2}} \text{Im} \left( \sum_{n} \frac{1}{k_{ln}^{2l+3}} \right) t^{-l-3/2}, \] (109)

and the survival probability, \( P_l(t) = |A_l(t)|^2 \) at large times is proportional to \( t^{-2l-3} \). This is
indeed in accordance with literature [15, 37].

C. Survival amplitude for a Breit-Wigner resonance plus corrections

If we begin with a Breit-Wigner distribution for the density of states, the survival ampli-
tude would be proportional to
\[ A_{l}^{BW}(t) \propto \int_{0}^{\infty} \frac{E^{1/2}e^{-cE}E^{-iEt}}{(E - \epsilon_{tr})^2 + (\Gamma_{tr}/2)^2} \, dE. \] (110)

Since \( k_{tr}^2 = \epsilon_{tr} - i\Gamma_{tr}/2 \), the above integral can be written as
\[ \int_{0}^{\infty} \frac{E^{1/2}e^{-sE}}{(E - k_{tr}^2)(E - k_{tr}^*2)} \, dE = \frac{i}{\Gamma_{tr}} \int_{0}^{\infty} \left( \frac{E^{1/2}e^{-sE}}{E - k_{tr}^2} - \frac{E^{1/2}e^{-sE}}{E - k_{tr}^*2} \right) \, dE. \] (111)

Expressing it further in terms of \( J \), one obtains
\[ A_{l}^{BW}(t) \propto \left[ J\left(\frac{1}{2}, -k_{tr}^2, s\right) - J\left(\frac{1}{2}, -k_{tr}^*2, s\right) \right]. \] (112)
Thus the survival amplitude for a Breit-Wigner resonance would be proportional to the difference of the functions $J\left(\frac{1}{2}, -k_{lr}^2, s\right)$ and $J\left(\frac{1}{2}, -k_{lr}^2, s\right)$. The latter suggests that the above result can also be obtained if we start with the general expression obtained earlier if we expand $k_{ln}^{-1}$ and $k_{ln}^{* -1}$ in powers of $x_{ln} = \Gamma_{ln}/2\epsilon_{ln}$. From the expansion

$$(1 + z)^\nu = \Gamma(\nu + 1) \sum_{q=0}^{\infty} \frac{z^q}{q!\Gamma(\nu + 1 - q)}, \quad \nu \neq -1, -2, \ldots \quad (113)$$

with $\nu = -1/2$ and $z = x_{ln}$, we can write $k_{ln}^{-1}$ and $k_{ln}^{* -1}$ as

$$k_{ln}^{-1} = \frac{1}{\sqrt{\epsilon_{ln}}} (1 - i x_{ln})^{-1/2} = \frac{1}{\sqrt{\epsilon_{ln}}} \Gamma\left(\frac{1}{2}\right) \sum_{q=0}^{\infty} \frac{e^{-iq\pi/2}}{q!\Gamma\left(\frac{1}{2} - q\right)} x_{ln}^q, \quad (114)$$

$$k_{ln}^{* -1} = \frac{1}{\sqrt{\epsilon_{ln}}} (1 + i x_{ln})^{-1/2} = \frac{1}{\sqrt{\epsilon_{ln}}} \Gamma\left(\frac{1}{2}\right) \sum_{q=0}^{\infty} \frac{e^{iq\pi/2}}{q!\Gamma\left(\frac{1}{2} - q\right)} x_{ln}^q. \quad (115)$$

Replacing the above in Eq. (97), we obtain

$$A_l(t) = \frac{i}{2A} \sum_{q=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)}{q!\Gamma\left(\frac{1}{2} - q\right)} \sum_{n} \frac{1}{\sqrt{\epsilon_{ln}}} \left[ e^{-iq\pi/2} J\left(\frac{1}{2}, -k_{ln}^2, s\right) - e^{iq\pi/2} J\left(\frac{1}{2}, -k_{ln}^{*2}, s\right) \right] x_{ln}^q. \quad (116)$$

Writing the first few terms, the survival amplitude is given as,

$$A_l(t) = \frac{i}{2A} \sum_{n} \frac{1}{\sqrt{\epsilon_{ln}}} \left[ J\left(\frac{1}{2}, -k_{ln}^2, s\right) - J\left(\frac{1}{2}, -k_{ln}^{*2}, s\right) \right] $$

$$+ \frac{i}{2A} \sum_{n} \frac{3}{8\sqrt{\epsilon_{ln}}} \left[ J\left(\frac{1}{2}, -k_{ln}^2, s\right) + J\left(\frac{1}{2}, -k_{ln}^{*2}, s\right) \right] \left(\frac{\Gamma_{ln}}{2\epsilon_{ln}}\right)^2 $$

$$- \frac{i}{2A} \sum_{n} \frac{5i}{16\sqrt{\epsilon_{ln}}} \left[ J\left(\frac{1}{2}, -k_{ln}^2, s\right) + J\left(\frac{1}{2}, -k_{ln}^{*2}, s\right) \right] \left(\frac{\Gamma_{ln}}{2\epsilon_{ln}}\right)^3 + \ldots \quad (117)$$

Note that the first term here is similar to (112) and the next terms are corrections suppressed by powers of $\Gamma_{ln}/2\epsilon_{ln}$.

**VI. APPLICATIONS AND LIMITATIONS**

In the previous sections, we derived analytical, model independent expressions for phase shifts, energy derivatives of phase shift (which can be related to the density of states in a resonance) and cross sections in single channel elastic scattering solely by using the analyticity properties of the $S$-matrix and a theorem of Mittag-Leffler. The latter also allowed us
to derive the survival amplitudes with the Fock-Krylov method and confirm their quantum mechanical behaviour which is non-exponential at large times. We shall now apply all the above “exact” expressions to realistic examples. However, it is important to note that within the formalism developed here, we cannot be too ambitious in reproducing the experimental results due to the following reasons:

(i) the present formalism restricts to elastic scattering and most (hadronic) resonances can decay to multiple channels. The S-matrix considered here does not take into account the effect of inelasticities.

(ii) Non-resonant background can distort the shape of the purely resonant cross sections and the extracted phase shifts may not necessarily be described simply by the pole structure (see for example [34] where the S-matrix is parametrized with a resonant and non-resonant part).

Nevertheless, in what follows, we shall try to test how far the present approach succeeds within the limitations mentioned above.

A. Low energy neutron-proton scattering

We start with a pedagogical example of the total cross section for low energy neutron proton (np) scattering where one is aware of the existence of the bound state, namely, the deuteron with a binding energy of 2.2245 MeV. It is also well known that the total np cross section cannot be reproduced unless one considers the existence of a virtual (singlet) state [48] around 0.1 MeV below the np threshold. Beginning with the cross section expression derived in this work and including one bound state and one virtual state pole, we evaluate the cross section for np scattering with \( l = 0 \). The deuteron in principle is a spin 1 nucleus and can be described as an admixture of the \(^3S_1-^3D_1\) states. The virtual state is \(^1S_0\). Without entering into the details of the amount of the above admixture (which can range between 3 - 7 % [49]) and nuclear physics aspects, we attempt to calculate the np total cross section using the above formula for s-wave scattering. In Fig. 2, we see the results of the calculations as compared to data [50]. The agreement with data is reasonable and indeed the magnitude of the cross section is sensitive to the existence of the virtual state as expected. The theoretical curve shown in Fig. 2 is evaluated using (36) with a bound state at -2.2245 MeV and a virtual state at -0.14 MeV.
B. Phase shifts in pion nucleon scattering

Our next example is that of scattering phase shifts extracted from the data on pion nucleon (\(\pi N\)) scattering. In Fig. 3, we present the single energy values extracted from a partial wave analysis of the \(\pi N\) data corresponding to the resonance regions of the \(\Delta(1232)\) and N(1520) resonances in the \(P_{33}\) and \(D_{13}\) (notation: \(l_{2I,2J}\)) waves respectively. The agreement of the calculated phase shifts with the single energy values is not very good. However, the energy derivative of the phase shift does reproduce the peak structure at the pole position. The agreement was not expected to be very good due to the reasons (such as inelasticities and non-resonant processes) already mentioned before.

C. Survival probabilities and critical times

The decay law of an unstable particle (or nucleus) can be shown classically to be of an exponential nature but according to quantum mechanics, this law is an approximation which fails for short and large times. The non-exponential behaviour predicted by quantum mechanics has intrigued experimental nuclear and particle physicists who performed experiments (see and references therein) with nuclei such as \(^{222}\text{Rn}, {^{60}\text{Co}, {^{56}\text{Mn and measured}}

FIG. 2: Comparison of the theoretical cross section expression with the neutron-proton low energy total cross section data.

\[\sigma_{\text{tot}} \text{[b]} \quad \text{ neutron − proton} \]
Using the expressions derived in the present work, we shall now evaluate the decay law (the survival probability) for a broad and a narrow resonance and try to understand the possible reasons behind not observing the non-exponential decay law at large times. We consider the case of the $\sigma$ meson as a typical broad resonance and that of the lowest excited state of the $^8\text{Be}$ nucleus as the narrow resonance. In Fig. 4 we notice the typical exponential decay law for $^8\text{Be}$ (which has 100 \% decay to two $\alpha$’s), followed by an oscillatory transition region (see inset) and a power law at large times. The broad resonance does not show the same characteristics. Indeed, it does not show an exponential behaviour at all (consistent with previous theoretical calculations [16]) at any time. We show the behaviour for 4 different pole values taken from literature [20]. Though the non-exponential behaviour in the beginning is different for the different pole values, the power law sets in at roughly the same time and all curves exhibit the same power law behaviour of $t^{-3}$ (as expected for $l = 0$).

The number of half-lives after which the power law starts for the $^8\text{Be}(0^+)$ resonance is about 60. A similar calculation for a narrow nuclear resonance such as $^{56}\text{Mn}$ leads to a critical time of 300 half-lives for the power law to set in. The experiment in case of $^{56}\text{Mn}$ [52] was carried out only up to 45 half-lives and the non-exponential was not observed. However, performing measurements up to 300 half-lives would be practically impossible.
since the exponential decay law would destroy almost all the sample by the time the narrow resonance reaches the power law. Broad resonances such as the $\sigma$ meson reach the power law much earlier, however, the lifetime is too short making the experimental observation once again difficult.

VII. SUMMARY AND CONCLUSIONS

Performing a Mittag-Leffler (ML) based analysis of the $S$-matrix and related quantities such as the phase shifts and cross sections, it is shown that a knowledge of the poles is sufficient to determine these quantities. We summarize the main findings of the present work below:

1. The ML expansion of the $S$-matrix in terms of its poles and the corresponding residues, along with the conditions imposed due to the analytic properties of the $S$-matrix is presented. Analytic expressions for the residues are derived.

2. Making use of the optical theorem and the ML expansion of the $S$-matrix, expressions for the cross sections with the corresponding poles are derived. Restricting these expressions to the case of resonant poles, generalized formulae for resonances in single channel elastic scattering are obtained.
3. The generalized cross sections are shown to reduce to the standard Breit-Wigner (BW) cross sections plus corrections. The BW formula overestimates the cross sections. However, with the examples of a broad meson and a baryon resonance, we show that adding the correction terms, the cross sections decrease and increase alternately, getting closer to the exact result. The number of correction terms required depends on the ratio, $\Gamma_r/(2\epsilon_r)$, of the half width and the energy ($\epsilon_r = E_r - E_{th}$) above threshold.

4. Performing the Mittag-Leffler based analysis with a focus on the correct threshold behaviour, we find that additional conditions must be imposed on the residues. It is gratifying to find that the generalized formulae near threshold also reduce to the Breit-Wigner forms plus corrections.

5. A Mittag-Leffler based analysis of the energy derivative of the phase shift is also presented and further used in order to derive analytical expressions for the survival amplitudes of resonances. Thus, it is shown that a knowledge of the pole value is sufficient to calculate the survival probability of an unstable state analytically within the Fock-Krylov method. The large time (power law) behaviour of these survival amplitudes is consistent with expectations from literature.

6. Applications of the analytical expressions derived are presented in the last section. Calculations of the survival probabilities of broad and narrow resonances shed some light on the reason for the non-observability of the non-exponential decay law predicted by quantum mechanics at large times.

Having summarized the main findings, we must add a few words of caution. The present work is a first step in the derivation of non-relativistic cross sections and survival probabilities in a model independent way and hence deals only with the case of single channel elastic scattering. A realistic scattering amplitude must of course take into account the effects due to multiple channels and background processes. Apart from this, the non-relativistic framework may not be the best as far as hadronic resonances are concerned. However, a relativistic extension to obtain analytical formulae as in the present work can be quite challenging. For a more realistic approach, we refer the reader to [11, 12], where, using the Laurent (Mittag-Leffler) expansion, the authors fitted data to extract the pole positions and residues. We hope to perform an extension to consider the above points in
future.

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Appendix A: Unimodularity of the Mittag-Leffler expansion of the $S$-matrix

The $S$-matrix, when $k$ is real, is unimodular \[17\]:

\[ S_l(k)S_l^*(k) = 1. \] (A1)

From \[17\], we can see that the poles and zeros of the $S$-matrix are all poles of $S_l(k)S_l^*(k)$. Thus, it is possible to apply the ML theorem to this function. The value of the latter at $k = 0$ is 1, and its residues are computed with the help of (11) and (12):

a) Residue at $k = k_{in}$:

\[
\text{Res}\left[ S_l(k)S_l^*(k), k = k_{in} \right] = b_{in} \left[ 1 + k_{in} \sum_p \left( \frac{b_{ip}^*}{k_{ip}(k_{in} - k_{ip}^*)} + \frac{b_{ip}^*}{k_{ip}(k_{in} + k_{ip})} \right) + k_{in} \sum_q \frac{D_{iq}}{\zeta_{iq}(k_{in} + i\zeta_{iq})} \right] = 0.
\]

b) Residue at $k = -k_{in}^*$:

\[
\text{Res}\left[ S_l(k)S_l^*(k), k = -k_{in}^* \right] = b_{in}^* \left[ 1 + k_{in}^* \sum_p \left( \frac{b_{ip}}{k_{ip}(k_{in} + k_{ip})} + \frac{b_{ip}}{k_{ip}(k_{in} - k_{ip})} \right) + k_{in} \sum_q \frac{D_{iq}}{\zeta_{iq}(k_{in} + i\zeta_{iq})} \right]^* = 0.
\]

c) Residue at $k = -k_{in}$:

\[
\text{Res}\left[ S_l(k)S_l^*(k), k = -k_{in} \right] = -b_{in} \left[ 1 + k_{in} \sum_p \left( \frac{b_{ip}}{k_{ip}(k_{in} + k_{ip})} + \frac{b_{ip}}{k_{ip}(k_{in} - k_{ip}^*)} \right) + k_{in} \sum_q \frac{D_{iq}}{\zeta_{iq}(k_{in} + i\zeta_{iq})} \right] = 0.
\]
d) Residue at $k = k_{ln}^*$:

$$\text{Res} \left[ S_l(k)S^*_l(k), k = k_{ln}^* \right] = b_{ln}^* \left[ 1 + k_{ln} \sum_p \left( \frac{b_{lp}^*}{k_{lp}(k_{ln} - k_{lp}^*)} + \frac{b_{lp}}{k_{lp}(k_{ln} + k_{lp})} \right) + i \sum_q \frac{D_{lq}}{\zeta_{lq}(k_{ln} + i \zeta_{lq})} \right]^* = 0.$$  

e) Residue at $k = i\zeta_{lm}$:

$$\text{Res} \left[ S_l(k)S^*_l(k), k = i\zeta_{lm} \right] = iD_{lm} \left[ 1 + i\zeta_{lm} \sum_p \left( \frac{b_{lp}^*}{k_{lp}(i\zeta_{lm} - k_{lp}^*)} + \frac{b_{lp}}{k_{lp}(i\zeta_{lm} + k_{lp})} \right) + \zeta_{lm} \sum_q \frac{D_{lq}}{\zeta_{lq}(\zeta_{lm} - \zeta_{lq})} \right] = 0.$$  

f) Residue at $k = -i\zeta_{lm}$:

$$\text{Res} \left[ S_l(k)S^*_l(k), k = -i\zeta_{lm} \right] = -iD_{lm} \left[ 1 - i\zeta_{lm} \sum_p \left( \frac{b_{lp}^*}{k_{lp}(i\zeta_{lm} + k_{lp}^*)} + \frac{b_{lp}}{k_{lp}(i\zeta_{lm} - k_{lp})} \right) + \zeta_{lm} \sum_q \frac{D_{lq}}{\zeta_{lq}(\zeta_{lm} - \zeta_{lq})} \right] = 0.$$  

Since all residues of $S_l(k)S^*_l(k)$ are zero, from the ML theorem, $S_l(k)S^*_l(k) = 1$ and hence we can say that Eq. 7 satisfies the unimodularity condition.

**Appendix B: Threshold conditions for the Mittag-Leffler expansions**

We want to study the behavior of a function with a zero (not necessarily simple) at the origin of the complex momentum plane through its ML expansion. The $S$-matrix, total cross section and density of states are examples of such functions. Let $g(z)$ be the function which satisfies the ML theorem. Also, $g(z)$ has a zero of multiplicity $m$ at $z = 0$. Consider the integral

$$\frac{1}{2\pi i} \oint_{C_N} \frac{g(\zeta)}{\zeta^m(\zeta - z)} d\zeta,$$

where $z \neq a_n$ is different from zero and is inside the contour of integration $C_N: |\zeta| = R_N$, which contains $N$ poles of $g(\zeta)$. From its definition, this radius is such that

$$|a_N| < R_N < |a_{N+1}|.$$

The integrand has simple poles at $\zeta = a_n$ and $\zeta = z$. Using the residue theorem, we get:

$$\frac{1}{2\pi i} \oint_{C_N} \frac{g(\zeta)}{\zeta^m(\zeta - z)} d\zeta = \frac{g(z)}{z^m} + \sum_{n=1}^{N} \frac{b_n}{a_n^m(a_n - z)},$$

where

- $\zeta = a_n$: $\text{Res} \left[ S_l(k)S^*_l(k), k = a_n \right] = b_n$.
- $\zeta = z$: $\text{Res} \left[ S_l(k)S^*_l(k), k = z \right] = \frac{g(z)}{z^m}$.

The expression on the right-hand side is an ML expansion of the function $g(z)$.
where the sum is over all poles closed by $C_N$. Since $|g(z)| < M$ on $C_N$, with $M$ constant,

$$\left| \frac{1}{2\pi i} \oint_{C_N} \frac{g(\zeta)}{S_m(\zeta - z)} d\zeta \right| \leq \frac{1}{2\pi} \frac{M \cdot 2\pi R_N}{R_N^m (R_N - |z|)} = \frac{M}{R_N^m \left(1 - \frac{|z|}{R_N}\right)}. \quad (B4)$$

If $N \to \infty$, $R_N \to \infty$ and the integral tends to zero if $m > 0$. Thus,

$$g(z) = \sum_{n=1}^{\infty} \frac{b_n}{a_n^m (z - a_n)}. \quad (B5)$$

If $z \to 0$,

$$\lim_{z \to 0} \left| \frac{g(z)}{z^m} \right| = \left| \sum_{n=1}^{\infty} \frac{b_n}{a_n^{m+1}} \right| \leq \sum_{n=1}^{\infty} \left| \frac{b_n}{a_n^{m+1}} \right|. \quad (B6)$$

If the last series converges absolutely, $g(z) = O(z^m)$ for $z \to 0$. On the other hand, if $z \to \infty$,

$$\lim_{z \to \infty} \left| \frac{g(z)}{z^m} \right| = 0. \quad (B7)$$

We have, for $z \to \infty$, $g(z) = o(z^m)$.

If we apply the ML theorem to $g(z)$:

$$g(z) = \sum_{n=1}^{\infty} \frac{b_n z}{a_n (z - a_n)}, \quad (B8)$$

and we compare this expansion with (B5), it is natural to ask ourselves how the two are connected; what we mean is, how do we deduce (B5) from (B8)? The key is to expand in a Taylor series (B8) and see what conditions should we impose on the residues and poles of $g(z)$ for obtaining the zero at $z = 0$ with the correct multiplicity.

Since

$$\frac{1}{z - a_n} = -\frac{1}{a_n} \sum_{s=0}^{\infty} \left( \frac{z}{a_n} \right)^s, \quad |z| < |a_n|, \quad (B9)$$

thus

$$g(z) = -\sum_{n=1}^{\infty} \left( \frac{b_n}{a_n} \right) \sum_{s=0}^{\infty} \left( \frac{z}{a_n} \right)^{s+1} = -\sum_{n=1}^{\infty} z^s \left( \sum_{n=1}^{\infty} \frac{b_n}{a_n^{s+1}} \right), \quad |z| < \min_n |a_n| = |a_1|. \quad (B10)$$

The first $m - 1$ terms of the last expansion must be zero, since $g(z)$ has a zero at $z = 0$ of order $m$. This means that

$$\sum_{n} \frac{b_n}{a_n^{s+1}} = 0, \quad s = 1, 2, \ldots, m - 1. \quad (B11)$$
Conditions (B11) provide a bridge between the expansions (B8) and (B5) in the following way: from identity \(^3\)

\[
\frac{1}{z - w} = \left( \frac{z}{w} \right)^p \frac{1}{z - w} - \frac{1}{w} \sum_{s=0}^{p-1} \left( \frac{z}{w} \right)^s, \quad p = 1, 2, \ldots
\]  

(B12)

with \(w = a_n\) and \(p = m - 1\); we have \(f(z)\), given by (2), is equal to

\[
g(z) = \sum_n b_n z \left[ \left( \frac{z}{a_n} \right)^{m-1} \frac{1}{z - a_n} - \frac{1}{a_n} \sum_{s=0}^{m-2} \left( \frac{z}{a_n} \right)^s \right]
\]

\[
= z^m \sum_n \frac{b_n}{a_n} \frac{1}{z - a_n} - \sum_{s=0}^{m-2} z^{s+1} \left( \sum_n \frac{b_n}{a_n^{s+2}} \right)
\]

\[
= z^m \sum_n \frac{b_n}{a_n} \frac{1}{z - a_n} - \sum_{s=1}^{m-1} z^{s} \left( \sum_n \frac{b_n}{a_n^{s+1}} \right).
\]  

(B13)

Using (B11), we obtain the expansion (B5).

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[1] F. Hoyle, ApJS 1, 121 (1954).

[2] C. A. Bertulani, Nuclei in the Cosmos, World Scientific, Singapore, 2013.

[3] N. G. Kelkar, K. P. Khemchandani, N. J. Upadhyay and B. K. Jain, Rep. Prog. Phys. 76, 066301 (2013); P. Moskal, M. Skurzok and W. Krzemien, AIP Conf. Proc. 1753, 030012 (2016).

[4] G. Rupp and Eef van Beveren, Chin. Phys. C 41, 053104 (2017); G. Rupp, S. Coito and Eef van Beveren, Acta Phys. Pol. Supp. 5, 1007 (2012).

[5] K. P. Khemchandani et al., Phys. Rev. D 97, 034005 (2018); A. Martinez Torres and K. P. Khemchandani, Phys. Rev. D 94, 076007 (2016).

[6] S. Niedzwiecki et al., Acta Phys. Pol. B 48, 1567 (2017).

[7] G. Breit and E. Wigner, Phys. Rev. 49, 519 (1936).

[8] M. Nowakowski and A. Pilaftsis, Z. Phys. C 60, 121 (1993).

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\(^3\) This can be deduced if we write \(\frac{1}{z - w}\) as:

\[
\frac{1}{z - w} = \frac{w^n - z^n + z^n}{w^n(z - w)} = \left( \frac{z}{w} \right)^n \frac{1}{z - w} - \left( \frac{1}{w} \right) \frac{1 - (z/w)^n}{1 - z/w} = \left( \frac{z}{w} \right)^n \frac{1}{z - w} - \frac{1}{w} \sum_{s=0}^{n-1} \left( \frac{z}{w} \right)^s
\]

Here, \(n \geq 1\).
[9] J. M. Blatt and V. M. Weisskopf, “Theoretical Nuclear Physics”, Dover Publications, New York (2010).
[10] C. Iliadis, “Nuclear Physics of Stars”, Wiley-VCH, Weinheim (2007).
[11] A. Švarc et al., Phys. Lett. B 755, 452 (2016).
[12] A. Švarc et al., Phys. Rev. C 88, 035206 (2013).
[13] E. T. Copson, “An Introduction to the Theory of Functions of a Complex Variable”, Oxford University Press (1935).
[14] E. Beth and G. E. Uhlenbeck, Physica 4, 915 (1937); K. Huang, “Statistical Mechanics”, Wiley, New York (1987).
[15] N. G. Kelkar, M. Nowakowski and K. P. Khemchandani, Phys. Rev. C 70, 024601 (2004).
[16] N. G. Kelkar and M. Nowakowski, J. Phys. A 43, 385308 (2010).
[17] A. I. Baz, Ya. B. Zeldovich and A. M. Perelomov, “Scattering, Reactions and Decay in Nonrelativistic Quantum Mechanics”, Israel Program for Scientific Translations, Springfield (1969).
[18] R. G. Newton, “Scattering Theory of Waves and Particles”, Dover Publications, 2nd ed., New York (2013).
[19] C. J. Joachain, “Quantum Collision Theory”, North-Holland Publishing Company (1975).
[20] C. Patrignani et al., Chin. Phys. C 40, 100001 (2016).
[21] N. A. Törnqvist, Z. Phys. C 68, 647 (1995).
[22] N. A. Törnqvist and M. Roos, Phys. Rev. Lett. 76, 1575 (1996)
[23] N. Isgur and J. Speth, Phys. Rev. Lett. 77, 2332 (1996).
[24] N. A. Törnqvist and M. Roos, Phys. Rev. Lett. 77, 2333 (1996).
[25] M. Harada, F. Sannino and J. Schechter, Phys. Rev. Lett. 78, 1603 (1997).
[26] N. Tornqvist and M. Roos, Phys. Rev. Lett. 78, 1604 (1997).
[27] E. van Beveren and G. Rupp, Eur. Phys. J. C 10, 469 (1999).
[28] I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. 96, 132001 (2006).
[29] J. R. Pelaez, Phys. Rept. 658, 1 (2016).
[30] D.V. Bugg, Phys. Lett. B 572, 1 (2003). Erratum-ibid B 595, 556 (2004).
[31] M. M. Nagels, T. A. Rijken and J. J. de Swart, Phys. Rev. D 12, 744 (1975).
[32] M. M. Nagels, T. A. Rijken and J. J. de Swart, Phys. Rev. D 15, 2547 (1977).
[33] M. M. Nagels, T. A. Rijken and J. J. de Swart, Phys. Rev. D 20, 1633 (1979).
[34] R. A. Arndt, I. I. Strakovsky, Ron L. Workman and M. M. Pavan, Phys. Rev. C 52, 2120
(1995).

[35] V. Fock and N. Krylov, JETP 17, 93 (1947).

[36] M. Nowakowski and N. G. Kelkar, AIP Conf. Proc. 1030, 250 (2008).

[37] L. Fonda, G. C. Ghirardi and A. Rimini, Rep. Prog. Phys. 41, 587 (1978).

[38] E. P. Wigner, Phys. Rev. 98, 145 (1955).

[39] N. G. Kelkar, Phys. Rev. Lett. 99, 210403 (2007).

[40] N. G. Kelkar, M. Nowakowski and H. M. Castañeda, Eur. Phys. Lett. 85, 200006 (2009).

[41] H. G. Winful, Phys. Rev. Lett. 91, 260401 (2003).

[42] A. Erdélyi, “Table of Integral Transforms, Vol. I”, McGrawHill (1954).

[43] M. Abramowitz and I. A. Stegun, “Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables”, National Bureau of Standards Applied Mathematics (1972).

[44] N. N. Lebedev, “Special Functions and their Applications”, Dover Publications (1975).

[45] K. Urbanowski, Acta Phys. Pol. B 48 1847 (2017); ibid, Eur. Phys. J D 71, 118 (2017).

[46] A. Brzeski and J. Lukierski, Acta Physica Polonia, Vol B6, 577 (1975)

[47] E. T. Copson, “Asymptotic Expansions”, Cambridge University Press (1965).

[48] J. Schwinger and E. Teller, Phys. Rev. 52, 286 (1937).

[49] N. G. Kelkar and D. Bedoya Fierro, Phys. Lett. B 772, 159 (2017).

[50] Data available at http://www.nndc.bnl.gov/sigma/.

[51] N. G. Kelkar, M. Nowakowski, K. P. Khemchandani and S. R. Jain, Nucl. Phys. A 730, 121 (2004); single energy values provided by R.A. Arndt, I.I. Strakovsky, et al., (private communication).

[52] E. B. Norman, S. B. Gazes, S. G. Crane and D. A. Bennett, Phys. Rev. Lett. 60, 2246 (1988).

[53] A. R. Bohm and Y. Sato, Phys. Rev. D 71, 085018 (2005).

[54] A. R. Bohm and M. J. Mithaiwala, J. Phys. A. 35, 8479 (2002).