Does the electromagnetic field of an accelerated charge satisfy Maxwell equations?

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(January 10, 2022)

Abstract

We considered the electromagnetic field of a charge moving with a constant acceleration along an axis. We found that this field obtained from the Liénard-Wiechert potentials does not satisfy Maxwell equations.

PACS numbers: 03.50.-z, 03.50.De
I. INTRODUCTION

It is well-known that the electromagnetic field created by an arbitrarily moving charge

\[ E(r,t) = q \left\{ \frac{(R - R\frac{V}{c})(1 - \frac{V^2}{c^2})}{(R - R\frac{V}{c})^3} \right\}_{t_0} + q \left\{ \frac{[R \times ((R - R\frac{V}{c}) \times \frac{V}{c})]}{(R - R\frac{V}{c})^3} \right\}_{t_0}, \]

\[ B(r,t) = \left\{ \frac{R \times E}{R} \right\}_{t_0} \]

was obtained directly from Liénard-Wiechert potentials:

\[ \varphi(r,t) = \left\{ \frac{q}{R - R\frac{V}{c}} \right\}_{t_0}, \quad A(r,t) = \left\{ \frac{qV}{c(R - R\frac{V}{c})} \right\}_{t_0}. \]

Usually, the first terms of the right-hand sides (rhs) of (1) and (2) are called “velocity fields” and the second ones are called “acceleration fields”.

It was recently claimed by E.Comay that “... Acceleration fields by themselves do not satisfy Maxwell’s equations. Only the sum of velocity fields and acceleration fields satisfies Maxwell’s equations.” We wish to argue that this sum does not satisfy Maxwell’s equations:

\[ \nabla \cdot E = 4\pi \rho, \]  
\[ \nabla \cdot B = 0, \]  
\[ \nabla \times H = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial E}{\partial t}, \]  
\[ \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}. \]

First, let us recall the usual way of deriving the formulas (1), (2) for the electric (E) and magnetic (B) fields.

To obtain the values of \( \varphi, A \) (see Eq.(3)) and \( E, B \) (see Eqs. (1) and (2)) at the instant \( t \) one has to take the values of \( V, \dot{V} \) and \( R \) at instant \( t_0 \). Here \( t_0 = t - \tau, \tau \) is the so called “retarded time”, \( R \) is the vector connecting the site \( r_0(x_0, y_0, z_0) \) of the charge \( q \) at instant \( t_0 \) with the point of observation \( r(x, y, z) \). The instant \( t_0 \) is determined from the condition (see Eq.(63.1) of Ref.1):
\[ t_0 = t - \tau = t - \frac{R(t_0)}{c}. \]  

The rhs of (3) contain functions of \( t_0 \), which, in turn, depends on \( x, y, z, t \):

\[ t_0 = f(x, y, z, t). \]  

To calculate the fields \( \mathbf{E} \) and \( \mathbf{B} \) one has to substitute \( \varphi \) and \( \mathbf{A} \) from (3) in the following expressions\(^1\):

\[ \mathbf{E} = -\nabla \varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = [\nabla \times \mathbf{A}]. \]  

Substituting \( \varphi, A_x, A_y, A_z \) given by (3) in Eq.(10), one ought to calculate \( \partial \{ \} / \partial t \) and \( \partial \{ \} / \partial x \) \((x_i \text{ are } x, y, z)\) using the following scheme:

\[
\begin{align*}
\frac{\partial \varphi}{\partial x_i} &= \frac{\partial \varphi}{\partial t_0} \frac{\partial t_0}{\partial x_i}, \\
\frac{\partial \mathbf{A}}{\partial t} &= \frac{\partial \mathbf{A}}{\partial t_0} \frac{\partial t_0}{\partial t}, \\
\frac{\partial A_k}{\partial x_i} &= \frac{\partial A_k}{\partial t_0} \frac{\partial t_0}{\partial x_i},
\end{align*}
\]  

and as a result one obtains the formulas (1) and (2).

In the next section we will consider a charge moving with a constant acceleration along the \( X \) axis and we will show that the Eq.(7) is not satisfied if one substitutes \( \mathbf{E} \) and \( \mathbf{B} \) from Eqs.(1) and (2) in Eq.(7). To verify this we have to find the derivatives of \( x-, y-, z-\) components of the fields \( \mathbf{E} \) and \( \mathbf{B} \) with respect to the time \( t \) and the coordinates \( x, y, z \). The functions \( \mathbf{E} \) and \( \mathbf{B} \) depend on \( x, y, z, t \) through \( t_0 \) from the conditions (8)-(9). In other words, we will show that these fields \( \mathbf{E} \) and \( \mathbf{B} \) do not satisfy the Maxwell equations if the differentiation rules that were applied to \( \varphi \) and \( \mathbf{A} \) (to obtain \( \mathbf{E} \) and \( \mathbf{B} \)) are applied identically to \( \mathbf{E} \) and \( \mathbf{B} \).
II. DOES THE ELECTROMAGNETIC FIELD OF A CHARGE MOVING WITH A CONSTANT ACCELERATION SATISFY MAXWELL EQUATIONS?

Let us consider a charge $q$ moving with a constant acceleration along the $X$ axis. In this case its velocity and acceleration have only $x$-components, respectively $V(x,0,0)$ and $a(a,0,0)$. Now we will rewrite the Eqs. (1) and (2) by components:

$$E_x(x, y, z, t) = q \left\{ \frac{(V^2 - c^2)[RV - c(x - x_0)]}{[(cR - V(x - x_0))^3]} \right\}_{t_0} + q \left\{ \frac{ac[(x - x_0)^2 - R^2]}{[(cR - V(x - x_0))^3]} \right\}_{t_0}, \quad (12)$$

$$E_y(x, y, z, t) = -q \left\{ \frac{CV^2 - c^2(y - y_0)}{[(cR - V(x - x_0))^3]} \right\}_{t_0} + q \left\{ \frac{ac(x - x_0)(y - y_0)}{[(cR - V(x - x_0))^3]} \right\}_{t_0}, \quad (13)$$

$$E_z(x, y, z, t) = -q \left\{ \frac{c(V^2 - c^2)(z - z_0)}{[(cR - V(x - x_0))^3]} \right\}_{t_0} + q \left\{ \frac{ac(x - x_0)(z - z_0)}{[(cR - V(x - x_0))^3]} \right\}_{t_0}, \quad (14)$$

$$B_x(x, y, z, t) = 0, \quad (15)$$

$$B_y(x, y, z, t) = q \left\{ \frac{V(V^2 - c^2)(z - z_0)}{[(cR - V(x - x_0))^3]} \right\}_{t_0} - q \left\{ \frac{acR(z - z_0)}{[(cR - V(x - x_0))^3]} \right\}_{t_0}, \quad (16)$$

$$B_z(x, y, z, t) = -q \left\{ \frac{V(V^2 - c^2)(y - y_0)}{[(cR - V(x - x_0))^3]} \right\}_{t_0} + q \left\{ \frac{acR(y - y_0)}{[(cR - V(x - x_0))^3]} \right\}_{t_0}, \quad (17)$$

Obviously, these components are functions of $x, y, z, t$ through $t_0$ from the conditions (8)-(9). This means that when substituting the field components given by Eqs.(12)-(17) in the Maxwell equations (4)-(7), we still have to use the differentiation rules as in (11):

$$\frac{\partial E \{or\, B\}^k}{\partial t} = \frac{\partial E \{or\, B\}^k}{\partial t_0} \frac{\partial t_0}{\partial t},$$

$$\frac{\partial E \{or\, B\}^k}{\partial x_i} = \frac{\partial E \{or\, B\}^k}{\partial t_0} \frac{\partial t_0}{\partial x_i},$$

(18)
where \( k \) and \( x \) are \( x, y, z \).

To calculate \( \partial t_0/\partial t \) and \( \partial t_0/\partial x_i \) one ought to use differentiation rules for implicit functions:

\[
\frac{\partial t_0}{\partial t} = -\frac{\partial F/\partial t}{\partial F/\partial t_0}, \quad \frac{\partial t_0}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial t_0},
\]

where

\[
F(x, y, z, t, t_0) = t - t_0 - \frac{R}{c} = 0, \quad R = \left( \sum_i [(x_i - x_{0i}(t_0))^2] \right)^{1/2}.
\]

In this case one obtains:

\[
\frac{\partial t_0}{\partial t} = \frac{R}{R - (x - x_0)V/c} \quad \text{and} \quad \frac{\partial t_0}{\partial x_i} = -\frac{x_i - x_{0i}}{c[R - (x - x_0)V/c]}.
\]

Remember that we are considering the case with \( V = (V, 0, 0) \) here. There is a different way to calculate the derivatives (19)\(^4\), which gives the expressions (21).

Let us rewrite Eq.(7) by components taking into account the rules (18) and Eq.(15):

\[
\frac{\partial E_z}{\partial t_0} \frac{\partial t_0}{\partial y} - \frac{\partial E_y}{\partial t_0} \frac{\partial t_0}{\partial z} = 0, \quad (22)
\]

\[
\frac{\partial E_x}{\partial t_0} \frac{\partial t_0}{\partial z} - \frac{\partial E_z}{\partial t_0} \frac{\partial t_0}{\partial x} + \frac{1}{c} \frac{\partial B_y}{\partial t_0} \frac{\partial t_0}{\partial t} = 0, \quad (23)
\]

\[
\frac{\partial E_y}{\partial t_0} \frac{\partial t_0}{\partial x} - \frac{\partial E_x}{\partial t_0} \frac{\partial t_0}{\partial y} + \frac{1}{c} \frac{\partial B_z}{\partial t_0} \frac{\partial t_0}{\partial t} = 0. \quad (24)
\]

In order to calculate the derivatives \( \partial E(\text{or } B)_k/\partial t_0 \) we have to know the values of the expressions \( \partial V/\partial t_0 \), \( \partial x_0/\partial t_0 \) and \( \partial R/\partial t_0 \). Note that to find \( \partial \phi/\partial t_0 \) from (3) one uses the following values of these expressions\(^5\):

\[
\frac{\partial R}{\partial t_0} = -c, \quad \frac{\partial \mathbf{R}}{\partial t_0} = -\frac{\partial \mathbf{r}_0}{\partial t_0} = -\dot{\mathbf{V}}(t_0) \quad \text{and} \quad \frac{\partial \mathbf{V}}{\partial t_0} = \ddot{\mathbf{V}}. \quad (25)
\]

So, in our case we have to use

\[
\frac{\partial R}{\partial t_0} = -c, \quad \frac{\partial x_0}{\partial t_0} = V \quad \text{and} \quad \frac{\partial V}{\partial t_0} = a. \quad (26)
\]
Now, using Eqs. (21) and (26), we want to verify the validity of Eqs. (22)-(24). The result of the verification is as follows:

\[
\frac{\partial E_z}{\partial t_0} \frac{\partial t_0}{\partial y} - \frac{\partial E_y}{\partial t_0} \frac{\partial t_0}{\partial z} = 0,
\]

(27)

\[
\frac{\partial E_x}{\partial t_0} \frac{\partial t_0}{\partial z} - \frac{\partial E_z}{\partial t_0} \frac{\partial t_0}{\partial x} + \frac{1}{c} \frac{\partial B_y}{\partial t_0} \frac{\partial t_0}{\partial t} = \frac{ac(z - z_0)}{[cR - V(x - x_0)]^3},
\]

(28)

\[
\frac{\partial E_y}{\partial t_0} \frac{\partial t_0}{\partial x} - \frac{\partial E_x}{\partial t_0} \frac{\partial t_0}{\partial y} + \frac{1}{c} \frac{\partial B_z}{\partial t_0} \frac{\partial t_0}{\partial t} = \frac{ac(y - y_0)}{[cR - V(x - x_0)]^3}.
\]

(29)

The verification shows that Eq.(22) is valid. But instead of Eq.(23) and Eq.(24) we have Eq.(28) and Eq.(29) respectively.

For the present we refrain from any comment regarding this result. However, we would like to cite the following phrase from the recent work: “Maxwell equations may not be an adequate description of nature”.

**APPENDIX**

To obtain Eqs. (1) and (2), let us rewrite Eqs.(10) taking into account Eqs.(11):

\[
E = -\nabla \varphi - \frac{1}{c} \frac{\partial A}{\partial t} = \frac{\partial \varphi}{\partial t_0} \nabla t_0 - \frac{1}{c} \frac{\partial A}{\partial t_0} \frac{\partial t_0}{\partial t},
\]

(30)

\[
B = [\nabla \times A] = \left[ \nabla t_0 \times \frac{\partial A}{\partial t_0} \right].
\]

(31)

From Eqs.(3) we obtain:

\[
\frac{\partial \varphi}{\partial t_0} = -\frac{q}{(R - R\beta)^2} \left( \frac{\partial R}{\partial t_0} \beta - R \frac{\partial \beta}{\partial t_0} \right),
\]

(32)

where \( \beta = V/c \). Hence, taking into account Eqs.(25), we have (after some algebraic simplification):

\[
\frac{\partial \varphi}{\partial t_0} = \frac{qc(1 - \beta^2 + R\dot{\beta}/c)}{(R - R\beta)^2}.
\]

(33)

In turn
\[
\frac{\partial A}{\partial t_0} = \frac{\partial \varphi}{\partial t_0} \beta + \varphi \dot{\beta}.
\] (34)

Putting \( \varphi \) from Eqs. (3), Eq. (33) and Eq. (34) together, we obtain (after simplification):

\[
\frac{\partial A}{\partial t_0} = qc \frac{\beta (1 - \beta^2 + R \dot{\beta}/c) + (\dot{\beta}/c)(R - R \beta)}{(R - R \beta)^2}.
\] (35)

Finally, substituting Eqs. (21)\(^9\), (33) and (35) in Eq. (30) we obtain:

\[
E = \frac{qc(1 - \beta^2 + R \dot{\beta}/c)}{(R - R \beta)^2} \left( - \frac{R}{c(R - R \beta)} \right) - q \frac{\beta (1 - \beta^2 + R \dot{\beta}/c) + (\dot{\beta}/c)(R - R \beta)}{(R - R \beta)^2} \left( \frac{R}{R - R \beta} \right) =
\]

\[
= q \frac{R(1 - \beta^2 + R \dot{\beta}/c) - R \beta(1 - \beta^2 + R \dot{\beta}/c) - (R \dot{\beta}/c)(R - R \beta)}{(R - R \beta)^3}.
\] (36)

Grouping together all terms with acceleration together, one can reduce this expression to

\[
E = q \frac{(R - R \beta)(1 - \beta^2)}{(R - R \beta)^3} + q \frac{(R \dot{\beta}/c)(R - R \beta) - (R \dot{\beta}/c)(R - R \beta)}{(R - R \beta)^3}.
\] (37)

Now, using the formula of double vectorial product\(^10\), it is not worth reducing the numerator of the second term of Eq. (37) to \([R \times [(R - R \beta) \times \dot{\beta}/c]]\). As a result we have Eq. (1).

Analogically, substituting Eqs. (21) and (35) in Eq. (31) we obtain

\[
B = \left[ \frac{R}{R} \times q \frac{R \beta(1 - \beta^2 + R \dot{\beta}/c) - (R \dot{\beta}/c)(R - R \beta)}{(R - R \beta)^3} \right].
\] (38)

If we add \( R(1 - \beta^2 + R \dot{\beta}/c) \) to the numerator of the second term of the vectorial product (38)\(^11\) we obtain Eq. (2) (comparing with Eq. (36)).

**ACKNOWLEDGMENTS** We are grateful to Prof. V. Dvoeglagov and Dr. D.W.Ahluwalia for many stimulating discussions. We acknowledge paper of Professor E.Comay, which put an idea into us to make present work.
1L.D.Landau and E.M.Lifshitz, *Teoria Polia* (Nauka, Moscow, 1973) [English translation: *The Classical Theory of Field* (Pergamon, Oxford, 1975), pp. 158-160].
2E.Comay, “Decomposition of electromagnetic fields into radiation and bound components”, Am.J.Phys.65, 862-867(1997). [See p.863].
3C.Teitelboim, D.Villarroel, and Ch.G.van Weert, “Classical Electrodynamics of Retarded Fields and Point Charges”, Riv. Nuovo Cimento 3, 1-64(1980). [See (3.25) on p.13].
4One can calculate $\frac{\partial R}{\partial t}$ and $\frac{\partial R}{\partial x_i}$ following Ref.1, p. 159:

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t_0} \frac{\partial t_0}{\partial t} = -\frac{RV}{R} \frac{\partial t_0}{\partial t} = c \left( 1 - \frac{\partial t_0}{\partial t} \right),$$

and

$$\nabla t_0 = -\frac{1}{c} \nabla R(t_0) = -\frac{1}{c} \left( \frac{\partial R}{\partial t_0} \nabla t_0 + \frac{R}{R} \right).$$

As a result one obtains

$$\frac{\partial t_0}{\partial t} = \frac{1}{1 - \frac{RV}{Rc}} \quad \text{and} \quad \frac{\partial t_0}{\partial x_i} = -\frac{x_i - x_{0i}}{c \left( R - \frac{RV}{c} \right)}.$$

5This follows from expressions $R = c(t - t_0)$ and $R = r - r_0(t_0)$. See e.g. I.V.Saveliev, *Foundation of Theoretical Physics (Osnovy Teoreticheskoi Fiziki)* (Nauka, Moscow 1975), Vol 1, ch. XIV, §78, p. 278 (in Russian). A detailed derivation of the formulas (1) and (2) can be found in this book or the Appendix of the present paper. We have found an interesting recent work by A.Gupta and T.Padmanabhan “Radiation from a charged particle and radiation reaction - revisited” where the authors have obtained the formulas (1) and (2) by solving Maxwell’s equations in the rest frame of the charged particle (which is a noninertial frame) and transforming the results to the inertial frame (see hep-physics/9710036).

6The expressions (22)-(24) were calculated using the program “Mathematica, Version 2.2”, therefore it is easy to check these calculations.

7D.W.Ahluwalia,”A New Type of Massive Spin-One Boson: And Its Relation with Maxwell Equations” in *The Present Status of the Quantum Theory of Light*, Eds. S.Jeffers at al (Kluwer, 1997), p.p. 443-457. A reader can also find similar speculations in the following works:

S.Weinberg, “Feynmann Rules for any Spin. II Massless Particles”, Phys. Rev.B 134 (1964), p.p. 882-896. [see p.B888, the first statement after Eqs. (4.21) and (4.22)];

D.W.Ahluwalia and D.J.Ernst, “Paradoxical Kinematic Acausality in Weinberg’s Equations for Massless Particles of arbitrary Spin”, Modern Phys. Lett. A 7 (1992), p.p. 1967-1974;

V.V.Dvoeglazov, “Use of the Quasipotential Method for Describing the Vector Particle Interactions”, Russian Phys. J., 37, N 9 (1994), p.p. 898-902;

V.V.Dvoeglazov, “Can the 2(2j+1) Components Weinberg-Tucher-Hammer Equations Describe the Electromagnetic Field?”, Preprint hep-th/9410174, Zacatecas, Oct. 1994.

8In Eq.(31) we used a well-known formula of vectorial analysis:

$$[\nabla \times f] = \left[ \nabla \xi \times \frac{\partial f}{\partial \xi} \right]$$
where \( f = f(\xi) \) and \( \xi = \xi(x, y, z) \).

In Eq.(21) \( (x - x_0)V = RV \) in general.

\[ [a \times [b \times c]] = b(a \cdot c) - c(a \cdot b). \]

The meaning of Eq.(38) does not change because of \([R \times R] = 0\).