Finite Element Technique for Solving the Stream Function Form of a Linearized Navier-Stokes Equations Using Argyris Element

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Abstract

The numerical implementation of finite element discretization method for the stream function formulation of a linearized Navier-Stokes equations is considered. Algorithm 1 is applied using Argyris element. Three global orderings of nodes are selected and registered in order to conclude the best banded structure of matrix and a fluid flow calculation is considered to test a problem which has a known solution. Visualization of global node orderings, matrix sparsity patterns and stream function contours are displayed showing the main features of the flow.

Key words: Finite element method, Navier-Stokes equations, stream function form, Argyris element.

1 Introduction

The numerical treatment of nonlinear problems that arise in areas such as fluid mechanics often requires solving large systems of nonlinear equations. Many methods have been proposed that attempt to solve these systems efficiently; one such class of methods is the finite element method, the most widely used technique for engineering design and analysis.

The Navier-Stokes equations may be solved using either the primitive variable or stream function formulation. Here, we use the stream function formulation. The attractions of the stream function formulation are that the incompressibility constraint is automatically satisfied, the pressure is not present in the weak form, and there is only one scalar unknown to be determined. The standard weak formulation of the stream function first appeared in 1979 in [9]; a general analysis of convergence for this formulation has been done in [4, 5].

The goal of this paper is to demonstrate that the method can be implemented to approximate solutions for incompressible viscous flow problems.
2 Governing Equations

Consider the Navier-Stokes equations describing the flow of an incompressible fluid

\[ -\text{Re}^{-1} \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p = \vec{f} \quad \text{in} \quad \Omega, \quad (2.1) \]
\[ \nabla \cdot \vec{u} = 0 \quad \text{in} \quad \Omega, \quad (2.2) \]
\[ \vec{u} = 0 \quad \text{on} \quad \partial \Omega, \quad (2.3) \]

where \( \vec{u} = (u_1, u_2) \) and \( p \) denotes the unknown velocity and pressure field, respectively, in a bounded, simply connected polygonal domain \( \Omega \subseteq \mathbb{R}^2 \); \( \vec{f} \) is a given body force; and \( \text{Re} \) is the Reynolds number.

The introduction of a stream function \( \psi(x, y) \) defined by

\[ u_1 = -\frac{\partial \psi}{\partial y}, \quad u_2 = \frac{\partial \psi}{\partial x} \]

means that the continuity equation (2.2) is satisfied identically. The pressure may then be eliminated from (2.1) to give

\[ \text{Re}^{-1} \Delta^2 \psi - \psi_y \Delta \psi_x + \psi_x \Delta \psi_y = \nabla \times \vec{f} \quad \text{in} \quad \Omega \quad (2.4) \]
\[ \psi = 0 \quad \text{on} \quad \partial \Omega \quad (2.5) \]
\[ \frac{\partial \psi}{\partial \hat{n}} = 0 \quad \text{on} \quad \partial \Omega, \quad (2.6) \]

where \( \hat{n} \) represents the outward unit normal to \( \Omega \). Equation (2.4) is a nonlinear fourth-order partial differential equation which turns into the known linear biharmonic fourth-order partial differential equation by omitting the second and third terms in the left-hand side of the equation. In order to write (2.4) – (2.6) in a variational form, we define the Sobolev spaces

\[ H^1(\Omega) = \{ v : v \in L^2(\Omega), \nabla v \in L^2(\Omega) \}, \quad (2.7) \]
\[ H^2_0(\Omega) = \{ v : v \in H^1(\Omega), v = 0 \quad \text{on} \quad \partial \Omega \}, \quad (2.8) \]
\[ H^2(\Omega) = \{ v : v \in L^2(\Omega), Dv \in L^2(\Omega), D^2v \in L^2(\Omega) \}, \quad (2.9) \]
\[ H^2_0(\Omega) = \{ v : v \in H^2(\Omega) : v = \frac{\partial v}{\partial n} = 0, \quad \text{on} \quad \partial \Omega \}, \quad (2.10) \]

where \( L^2(\Omega) \) is the space of square integrable functions on \( \Omega \) and \( D \) represents differentiation with respect to \( x \) or \( y \). For each \( \varphi \in H^1(\Omega) \), define \( \nabla \times \varphi = \begin{pmatrix} \varphi_y \\ -\varphi_x \end{pmatrix} \). The following linearized weak form of equations (2.4) – (2.6) is considered:

Find \( \psi \in H^2_0(\Omega) \) such that for all \( \varphi \in H^2_0(\Omega) \),

\[ a(\psi, \varphi) + b(\psi^*; \psi, \varphi) = \ell(\varphi), \quad (2.11) \]
where

\[ a(\psi, \varphi) = \text{Re}^{-1} \int_\Omega \Delta \psi \cdot \Delta \varphi d\Omega, \]

\[ b(\xi; \psi, \varphi) = \int_\Omega \Delta \xi(\psi_y \varphi_x - \psi_x \varphi_y) d\Omega, \]

\[ \ell(\varphi) = (\vec{f}, \text{curl} \varphi) = \int_\Omega \vec{f} \cdot \text{curl} \varphi d\Omega, \]

where \( \psi^* \) is a fixed given function (a primitive approximation for \( \psi \)).

3 Finite Element Discretization

For the standard finite element discretization of (2.11), we choose conforming finite element subspace \( X \subset H^2_0(\Omega) \). We then seek \( \psi^h \in X \) such that for all \( \varphi^h \in X \),

\[ a(\psi^h, \varphi^h) + b(\psi^{*h}, \psi^h, \varphi^h) = \ell(\varphi^h). \quad (3.1) \]

The existence and uniqueness for the solution of the discrete problem (3.1) has been proved in [1]. Once the finite element spaces are prescribed, problem (3.1) reduces to solving a system of algebraic equations. Various iterative methods can be used to solve problem (3.1). In this paper the following algorithm has been applied to solve problem (3.1) for a fixed mesh of size \( h \). In each iteration in Algorithm 1, we need to solve a linear system. The resulting linear system is nonsymmetric whose symmetric part is positive definite. Moreover, the resulting matrix is a sparse matrix. We choose the conjugate gradient stabilized (BICGSTAB) method, which requires two matrix-vector products and four inner products in each iteration.

**Algorithm 1** (the finite element algorithm)

Given Max-iteration & Tolerance

Given \( \psi_0 \) as a starting guess

For \( i = 1 : \text{Max-iteration} \)

Solve the linear system on the mesh \( X \subset H^2_0(\Omega) \) for \( \psi_i; \)

\[ a(\psi_i, \varphi) + b(\psi_{i-1}, \psi_i, \varphi) = \ell(\varphi) \quad \forall \varphi \in X \]

If \( \|\psi_i - \psi_{i-1}\| \leq \text{Tolerance} \) & Residual \( \leq \text{Tolerance} \)

Stop

End

4 Finite Element Space

The inclusion \( X \subset H^2_0(\Omega) \) requires the use of finite element functions that are continuously differentiable over \( \Omega \). We choose Argyris triangle as a finite element space for the stream function formulation. We will impose boundary conditions by setting all the degrees of freedom at the boundary nodes to be zero and the normal derivative equal to zero at all vertices and nodes on the boundary. In Argyris triangle, the functions are quintic polynomials within
each triangle and the 21 degrees of freedom are chosen to be the function value, the first and second derivatives at the vertices, and the normal derivative at the midsides.

5 Global Node Orderings

For linear systems derived from a partial differential equation, each unknown corresponds to a node in the discretization mesh. Different orderings of unknowns correspond to permutations of the coefficient matrix. Since the convergence speed of iterative method may depend on the ordering used, the next three options have been considered.

1. Start numbering the nodes at the vertices with six successive numbers at each node. Then the numbering of the midpoint nodes is performed starting with the horizontal sides, followed by the vertical and oblique side. This is illustrated in Figure (5.1).

2. Start numbering the function values of the nodes at vertices followed by the derivatives, and then go back to the midpoint nodes using the same ordering as above. This is illustrated in Figure (5.2).

3. Start numbering the nodes at the vertices with six successive numbers by skipping a node alternatively, and then go back to the midpoint nodes of horizontal, vertical and oblique sides using the same orderings as above. This is illustrated in Figure (5.3).
It can be concluded that the first option is the most appropriate since it has the best matrix property, being a banded structure, as shown in Figures 5.4, 5.5, 5.6 which visualize, respectively, the location of the nonzero elements for the three selected global orderings. Therefore this ordering has been used in the next section.
Furthermore, the following table illustrates the cpu-time, the number of arithmetic operations, and the number of bicgstab iterations which have been computed by performing the suggested global numberings in a specified problem:

|   | cpu-time | n.c.o.    | bicgstab iter. |
|---|----------|-----------|----------------|
| 1 | 28.45    | 227800974 | 282.5          |
| 2 | 29.77    | 231648608 | 289.0          |
| 3 | 29.06    | 239406016 | 297.5          |

6 Numerical Example

In this section we describe some numerical results obtained by implementing the finite element algorithm, for which we have an exact solution. The region $\Omega$ is the unit square $\{0 < x < 1, \; 0 < y < 1\}$ and for the finite element discretization, we use the Argyris elements. The biharmonic equation along with equation (2.11) was solved in the same way. They were solved until the norm of the difference in successive iterates and the norm of the residual were within a fixed tolerance.

We consider as a test example the two-dimensional Navier-Stokes equations (2.1) – (2.3) on the unit square $\Omega = (0, 1)^2$, where we define the right-hand side by $f := -\text{Re}^{-1} \Delta \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + \nabla p$ with the following prescribed exact solution:

$$ u = \left( \begin{array}{c} \psi_y \\ -\psi_x \end{array} \right) \quad \text{with} \quad \psi(x, y) = x^2(x - 1)^2y^2(y - 1)^2, $$

$$ p = x^3 + y^3 - 0.5. $$
For this test problem, all requirements of the theory concerning the geometry of the domain and the smoothness of the data are satisfied. Moreover, the stream function $\psi(x, y)$ satisfies the boundary conditions of the stream function equation of the Navier-Stokes equations.

In all numerical calculations in this example, we have used the Argyris elements with $\text{Re} = 1$ and $\text{tol} = 10^{-5}$. We pick three values of $h$: $1/3$, $1/5$, and $1/9$. The cpu-time, the number of PCG iterations, and the error $|\psi_c - \psi_e|$ for different values of $h$ are tabulated in Table 6.1 based on four quadrature points, and Table 6.2 based on six quadrature points for the biharmonic equation. The cpu-time, the number of BICGSTAB iterations, errors, and the number of the mathematical operations are tabulated in Table 6.3 based on six quadrature points for the linearized equation (2.11), where $\psi^*$ is computed by solving the biharmonic equation as initial guess, and where

- $n.q.p.$ = number of quadrature points,
- $n.c.o.$ = number of operations,
- pcg-itr. = number of PCG iterations,
- bicgstab itr. = number of BICGSTAB iterations,
- $\psi_c$ = The computed solution of the function values,
- $\psi_e$ = The exact solution of the function values.

| $h$  | n.q.p | cpu time | n.c.o. | Error $\psi_c - \psi_e$ | pcg-itr |
|------|-------|----------|--------|------------------------|---------|
| $1/3$ | 4     | 1.3200   | 17601719 | $0.1644 \times 10^{-3}$ | 72      |
| $1/5$ | 4     | 6.7500   | 246112068| $0.1899 \times 10^{-3}$ | 211     |
| $1/9$ | 4     | 64.4300  | $3.7401 \times 10^9$ | $0.1376 \times 10^{-3}$ | 437     |

| $h$  | n.q.p | cpu time | n.c.o. | Error $\psi_c - \psi_e$ | pcg-itr |
|------|-------|----------|--------|------------------------|---------|
| $1/3$ | 6     | 1.8200   | 16476551 | $0.1984 \times 10^{-3}$ | 72      |
| $1/5$ | 6     | 7.4700   | 249166334| $0.1935 \times 10^{-3}$ | 209     |
| $1/9$ | 6     | 64.7100  | $3.6482 \times 10^9$ | $0.1379 \times 10^{-3}$ | 454     |

| $h$  | cpu time | n.c.o. | $|\psi_c - \psi_e|_0$ | $|\psi_c - \psi_e|_1$ | $|\psi_c - \psi_e|_2$ | bicgstab itr. |
|------|----------|--------|-----------------|-----------------|-----------------|---------------|
| $1/3$ | 13.24    | 17528466 | $0.2589 \times 10^{-3}$ | $0.1294 \times 10^{-2}$ | $0.1692 \times 10^{-1}$ | 100.5        |
| $1/5$ | 35.32    | 227800974| $0.2148 \times 10^{-3}$ | $0.1062 \times 10^{-2}$ | $0.1048 \times 10^{-1}$ | 282.5        |
| $1/9$ | 179.88   | $3.50366282 \times 10^9$ | $0.1423 \times 10^{-3}$ | $0.6986 \times 10^{-3}$ | $0.6016 \times 10^{-2}$ | 567.5        |
Fig. 6.1: Streamlines for $h = 1/3, 1/5, 1/9$ with $\text{Re} = 1$ using finite element method on the linear problem with four quadrature points as shown in Table 6.1.
Fig. 6.2: Streamlines for $h = 1/3, 1/5, 1/9$ with $Re = 1$ using finite element method on the linear problem with six quadrature points as shown in Table 6.2.
Fig. 6.3: Streamlines for $h = 1/3, 1/5, 1/9$ with $Re = 1$ using finite element method on the linear problem with six quadrature points as shown in Table 6.3
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