Stochastic Porous Media Equations and Self-Organized Criticality

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Abstract: The existence and uniqueness of nonnegative strong solutions for stochastic porous media equations with noncoercive monotone diffusivity function and Wiener forcing term is proven. The finite time extinction of solutions with high probability is also proven in 1-D. The results are relevant for self-organized criticality behavior of stochastic nonlinear diffusion equations with critical states.

1. Introduction

The phenomenon of self-organized criticality is widely studied in Physics from different perspectives. (We refer to [1,2,8–10,13–19,23] for various studies). Roughly speaking it is the property of systems to have a critical point as attractor and to reach spontaneously a critical state.

In [2] Bantay and Janosi beautifully explained that the continuum limit of the sand pile model of Bak-Tang-Wiesenfeld in [1] ("BTW model"), which was based on a cellular automaton algorithm, can be interpreted as a solution of an anomalous (singular) diffusion equation of the type

\[ dX(t) = \frac{\Delta}{\Delta(H(X(t) - x_c))} dt, \tag{1.1} \]

where \( H \) is the Heaviside function and \( x_c \) is the critical value. In [13] (see also [14]) Diaz-Guilera pointed out that for this and a similar model due to Zhang [24] given by

\[ dX(t) = (X(t) - x_c) \Delta(H(X(t) - x_c)) dt, \tag{1.2} \]

it is more realistic to consider Eqs. (1.1) and (1.2) perturbed by (an additive) noise to model a random amount of energy put into the system varying all over the underlying domain. The resulting equations are then stochastic partial differential equations (SPDE) of evolution type, however, with very singular (non-continuous) coefficients which mathematically can only be treated as multi-valued functions.
The purpose of this paper is to analyze such type of equations within the framework of multi-valued stochastic evolution equations with (1.1) and (1.2) as the underlying motivating examples. To the best of our knowledge this is the first time this is done in the presence of a stochastic force and in such generality in a mathematically strict way. Let us introduce our framework.

Let $\mathcal{O}$ be an open bounded domain of $\mathbb{R}^d$, $d = 1, 2, 3$, with smooth boundary $\partial \mathcal{O}$. We shall study here the nonlinear stochastic diffusion equation with linear multiplicative noise,

$$
\begin{aligned}
&dX(t) - \Delta \Psi(X(t))dt \ni \sigma(X(t))dW(t), \quad \text{in } (0, \infty) \times \mathcal{O}, \\
&\Psi(X(t)) \ni 0, \quad \text{on } (0, \infty) \times \partial \mathcal{O}, \\
&X(0, x) = x \quad \text{on } \mathcal{O},
\end{aligned}
$$

where $x$ is an initial datum and $\Psi : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a maximal monotone (possibly multivalued) graph with polynomial growth and random forcing term

$$
\sigma(X)dW = \sum_{k=1}^{\infty} \mu_k X d\beta_k \ e_k, \quad t \geq 0,
$$

which is linear in $X$. Here $\{e_k\}$ is an orthonormal basis in $L^2(\mathcal{O})$, $\{\mu_k\}$ is a sequence of positive numbers and $\{\beta_k\}$ a sequence of independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

We note that the linear operator $\sigma(X)$ is defined by

$$
\sigma(X)h = \sum_{k=1}^{\infty} \mu_k X \langle h, e_k \rangle_2 e_k, \quad \forall \ h \in L^2(\mathcal{O}),
$$

where $\langle \cdot, \cdot \rangle_2$ is the scalar product in $L^2(\mathcal{O})$.

Apart from the self-organized criticality phenomena mentioned above, Eq. (1.3) models the dynamics of flows in porous media and more generally the phase transition (including melting and solidification processes) in the presence of a random forcing term $\sigma(X)dW$.

Existence for stochastic equations of the form (1.3) with additive and multiplicative noise was studied in [6] under the main assumption that $\Psi$ is monotonically increasing, continuous and such that

$$
\begin{aligned}
\Psi(0) = 0, \quad \Psi'(r) &\leq \alpha_1 |r|^{m-1} + \alpha_2, \quad \forall \ r \in \mathbb{R}, \\
\int_0^r \Psi(s) ds &\geq \alpha_3 |r|^{m+1} + \alpha_4, \quad \forall \ r \in \mathbb{R},
\end{aligned}
$$

where $\alpha_1 \geq 0, \alpha_3 > 0, \alpha_2, \alpha_4 \geq 0$ and $m \geq 1$. (See also [7] and [22] for general growth conditions on $\Psi$.)

Here we shall study Eq. (1.3) under the following assumptions.

**Hypothesis 1.1.** (i) $\Psi$ is a maximal monotone multivalued function from $\mathbb{R}$ into $\mathbb{R}$ such that $0 \in \Psi(0)$. 
