A Note on the SPICE Method

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Abstract—In this article, we analyze the SPICE method developed in [1], and establish its connections with other standard sparse estimation methods such as the Lasso and the LAD-Lasso. This result positions SPICE as a computationally efficient technique for the calculation of Lasso-type estimators. Conversely, this connection is very useful for establishing the asymptotic properties of SPICE under several problem scenarios and for suggesting suitable modifications in cases where the naive version of SPICE would not work.

I. INTRODUCTION

SPECTRAL line estimation, or the problem of estimating the amplitudes and frequencies of a signal composed of a sum of sinusoids contaminated by Gaussian white noise, is a ubiquitous and well studied area in the field of signal processing [2]. Many classes of methods have been devised to solve this problem under several different scenarios like, e.g., uniformly/non-uniformly spaced samples, a priori known/unknown number of sinusoids, homoscedastic/heteroscedastic (constant/varying variance) samples, parametric/non-parametric model-based, and so on [2, 3, 4].

Recently, SPICE (SemiParametric/SParse Iterative Covariance-based Estimator), a new technique for spectral line estimation inspired by ideas from sparse estimation, has also been developed for array signal processing [5], a white noise, is a ubiquitous and well studied area in the field of signal processing [2].

In this paper, we establish the connection between SPICE and standard sparse estimation methods such as the Lasso [6] and the LAD-Lasso [2]. This connection, based on the so-called Elfving theorem from optimal experiment design [8], puts the SPICE method into perspective, allowing us to examine the asymptotic properties of SPICE under several scenarios by simply applying the existing theory for the Lasso and its variants (see, e.g., the recent book [9]). Conversely, the relationship between SPICE and Lasso-type estimators suggests that SPICE may be used as a (new) numerically efficient technique for computing Lasso estimates.

The manuscript is organized as follows. Section II describes the spectral line estimation problem and the SPICE method. Section III establishes the relation between SPICE and Lasso-type sparse estimation methods. In Section IV a simulation example illustrating the equivalence between SPICE and a version of Lasso is presented. Finally, Section V concludes the paper.

Notation: Vectors and matrices are written in bold lowercase and uppercase fonts, respectively. $^T$ and $^H$ denote transposition and complex conjugate transposition, respectively. $\Re z$ and $\Im z$ stand for the real and imaginary parts of the complex number $z$, and $j$ is the square root of $-1$. $\mathbb{R}^p_+$ is the set of non-negative real numbers, and $\mathbb{C}$ is the complex plane. $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_F$ and $|\cdot|$ correspond to the 1-norm, Euclidean norm, Frobenius norm and absolute value, respectively. $\text{diag}(a_1,\ldots,a_n)$ is a diagonal matrix whose diagonal is given by $a_1,\ldots,a_n$, $I$ is the identity matrix. $\mathbb{E}\{\cdot\}$ denotes mathematical expectation.

II. PROBLEM FORMULATION AND SPICE METHOD

Consider the following problem: Let $y \in \mathbb{C}^{N \times 1}$ be given, satisfying the equation

$$y = \sum_{k=1}^{K} a_k s_k + \epsilon,$$  \hspace{1cm} (1)

where $\epsilon \in \mathbb{C}^{N \times 1}$ is a complex Gaussian random vector of zero mean and covariance matrix $\text{diag}(\sigma_1,\ldots,\sigma_N)$, and $\{a_k\}_{k=1}^K \subset \mathbb{C}^{N \times 1}$ are known complex vectors. $\{s_k\}_{k=1}^K \subset \mathbb{C}$ are unknown complex quantities, of the form $s_k = |s_k|e^{j\phi_k}$, where the phases $\{\phi_k\}_{k=1}^K \subset [0,2\pi)$ are independent random variables uniformly distributed in $[0,2\pi)$, and the magnitudes $\{|s_k|\}_{k=1}^K \subset \mathbb{R}^+_0$ are deterministic parameters to be estimated.

The spectral line estimation problem considers a particular case of (1), where the $a_k$’s are vectors of imaginary exponents of the form $e^{j\omega t}$.

In order to estimate the magnitudes $|s_k|$, let

$$R := \mathbb{E}\{yy^H\} = A^H PA,$$

where

$$A^H := [a_1 \cdots a_K I],$$

$$P := \text{diag}(|s_1|^2,\ldots,|s_K|^2,\sigma_1,\ldots,\sigma_N) =: \text{diag}(p_1,\ldots,p_{K+N}).$$

The SPICE estimate [1] of the $|s_k|$’s is an iterative procedure of the form:

$$R(i) = A^H \text{diag}(p_1(i),\ldots,p_{K+N}(i))A,

\rho_k(i) = p_k(i) |a_k^H R^{-1}(i)y|/\|y\|_2^2,$$

$$w_k(i) = \frac{d_k}{\|y\|_2^2},$$

where $i$ is the iteration number, and $p_k(i)$ is the estimate of $p_k$ at iteration $i$. This method is initialized by any initial estimate.
of the $p_k$'s, and its estimate $\mathbf{R}(i)$ converges to the matrix $\mathbf{R}$ minimizing
\[
    f(\mathbf{R}) := \| \mathbf{R}^{-1/2} (\mathbf{y} \mathbf{y}^H - \mathbf{R}) \|^2_F.
\] (3)
The $p_k$'s that give $\mathbf{R}$ correspond to the limits $\lim_{i \to \infty} p_k(i)$.

Remark 1: The presence of the inverse of $\mathbf{R}(i)$ in the SPICE method may in principle lead to complications if such a matrix becomes singular. However, if the $p_k(0)$'s are chosen to be strictly positive, then $\mathbf{R}(i+1)$ is generically non-singular (since $\mathbf{a}_k$ is generically in the column range of $\mathbf{R}(i)$, and $\mathbf{y}$ is a Gaussian random vector which lies in the null space of $\mathbf{R}(i)$ with probability 0). Because of this, here and in the sequel we will implicitly assume for the derivations that $\mathbf{R}$ is non-singular.

Remark 2: In [5], SPICE was defined based on a slightly different $f(\mathbf{R})$. We will not consider that version of SPICE, because such a version can only be defined in a multi-snapshot case. However, similar steps as the ones described in the following sections can be applied to the method in [5] to arrive at an equivalent Lasso-type formulation.

III. ANALYSIS OF SPICE

The first version of SPICE in [11] allows the variances $\sigma_k$ to be different, while a variant of the method imposes the constraint that $\sigma_1 = \cdots = \sigma_N =: \sigma$ [11] Section III.D]. We will treat these cases separately, starting with the case where the variances can be different.

A. Different variances

As shown in [11], the function $f$ in (3) can be written as
\[
    f(\mathbf{R}) = \text{tr} \left\{ \left[ \mathbf{R}^{-1/2} (\mathbf{y} \mathbf{y}^H - \mathbf{R}) \right]^H \mathbf{R}^{-1/2} (\mathbf{y} \mathbf{y}^H - \mathbf{R}) \right\}
    = \| \mathbf{y} \|^2_F \mathbf{R}^{-1} \mathbf{y} - 2 \| \mathbf{y} \|^2_F + \text{tr} \mathbf{R},
\]
hence minimizing $f(\mathbf{R})$ is equivalent to minimizing
\[
    g(\mathbf{R}) := \| \mathbf{y} \|^2_F \mathbf{R}^{-1} \mathbf{y} + \frac{1}{\| \mathbf{y} \|^2_F} \text{tr} \mathbf{R}
    = \| \mathbf{y} \|^2_F \mathbf{R}^{-1} \mathbf{y} + \sum_{k=1}^{K+N} \| \mathbf{a}_k \|^2_F w_k
    = \| \mathbf{y} \|^2_F \mathbf{R}^{-1} \mathbf{y} + \sum_{k=1}^{K+N} w_k p_k,
\] (4)
subject to $p_k \geq 0$, where
\[
    w_k := \frac{\| \mathbf{a}_k \|^2_F}{\| \mathbf{y} \|^2_F},
\]
To further simplify the problem, in [5] Appendix B] it is argued that the minimization of $g(\mathbf{R})$ is equivalent to solving
\[
    \min_{\tilde{p}_1, \ldots, \tilde{p}_{K+N}} \sum_{k=1}^{K+N} \| \mathbf{a}_k \|^2_F \tilde{p}_k
    \text{ s.t. } \sum_{k=1}^{K+N} w_k \tilde{p}_k = 1
\] (5)
Equation (5) will be our starting point for the analysis of SPICE. A slight simplification can be achieved by defining $\tilde{p}_k := w_k p_k$ and $\tilde{a}_k := w_k^{-1/2} a_k$ for all $k = 1, \ldots, K + N$. This gives the re-parameterized problem
\[
    \min_{\tilde{p}_1, \ldots, \tilde{p}_{K+N}} \sum_{k=1}^{K+N} \| \mathbf{a}_k \|^2_F \tilde{p}_k
    \text{ s.t. } \sum_{k=1}^{K+N} \tilde{a}_k \tilde{a}_k^H \tilde{p}_k = \mathbf{R}.
\]

The strategy now is to consider a derivation similar to Elving's theorem, from optimal experiment design [8], to obtain an optimization problem equivalent to (6). First notice that
\[
    (\mathbf{y}^H \mathbf{R}^{-1} \mathbf{y})_{\mathbf{R}} = \min_{c_1, \ldots, c_{K+N}} \sum_{k=1}^{K+N} \| c_k \|^2_{\tilde{p}_k}
    \text{ s.t. } \tilde{\mathbf{A}}^H \mathbf{c} = \mathbf{y},
\] (7)
where $\tilde{\mathbf{A}}^H := [\tilde{a}_1 \cdots \tilde{a}_{K+N}]$ and $\mathbf{c} := [c_1 \cdots c_{K+N}]^T$. Here the ' symbol in the summation sign indicates that the values of $k$ for which $\tilde{p}_k = 0$ should be omitted from the sum. The proof of (7) is given in the appendix.

The combination of (6) and (7) gives a minimization problem in $\{\tilde{p}_k\}$ and $\{c_k\}$, i.e.,
\[
    \min_{\tilde{p}_1, \ldots, \tilde{p}_{K+N}, c_1, \ldots, c_{K+N}} \sum_{k=1}^{K+N} \| c_k \|^2_{\tilde{p}_k}
    \text{ s.t. } \sum_{k=1}^{K+N} \tilde{p}_k = 1
    \text{ and } \tilde{\mathbf{A}}^H \mathbf{c} = \mathbf{y},
\] (8)
where the order of the minimizing variables can be exchanged. Now, when the $c_k$'s are kept fixed, the minimization of the cost in (8) with respect to $\{\tilde{p}_k\}$ can be done explicitly. To see this, notice that by the Cauchy-Schwarz inequality we have
\[
    \sum_{k=1}^{N+k} \frac{|c_k|^2}{\tilde{p}_k} = \left( \sum_{k=1}^{N+k} \frac{|c_k|^2}{\tilde{p}_k} \right) \left( \sum_{k=1}^{K+N} \tilde{p}_k \right) \geq \left( \sum_{k=1}^{N+k} \frac{|c_k|^2}{\sqrt{\tilde{p}_k}} \right) \left( \sum_{k=1}^{K+N} \frac{1}{\sqrt{\tilde{p}_k}} \right)^2
    = \left( \sum_{k=1}^{N+k} |c_k|^2 \right)^2,\]
where the lower bound is attained if and only if there is an $\alpha \in \mathbb{C}$ such that
\[
    \frac{|c_k|^2}{\tilde{p}_k} = \alpha \tilde{p}_k, \quad k = 1, \ldots, K + N,
\]
or
\[
    \tilde{p}_k = \frac{|c_k|}{\sqrt{\alpha}}, \quad k = 1, \ldots, K + N.
\]
The proportionality constant $\alpha$ can be determined from the condition $\sum_{k=1}^{K+N} \tilde{p}_k = 1$, giving
\[
    \tilde{p}_k = \frac{|c_k|}{\sum_{i=1}^{K+N} |c_i|}, \quad k = 1, \ldots, K + N.
\] (9)
Putting this expression in \( \Phi \) gives the reduced problem

\[
\min_{c_1, \ldots, c_{K+N}} \left( \sum_{k=1}^{K+N} |c_k| \right)^2
\]

s.t.
\[
\tilde{A}^H c = y,
\]

or, equivalently,

\[
\min_{c_1, \ldots, c_{K+N}} \sum_{k=1}^{K+N} |c_k|
\]

s.t.
\[
\tilde{A}^H c = y.
\]

This is a complex-valued \( l_1 \)-optimization problem, hence it can be expected to give a sparse solution in \( \{ \hat{c}_k \} \) through \( \Phi \). This, in turn, gives a sparse solution in \( \{ \tilde{p}_k \} \) through \( \tilde{p}_k \).

To explore the behavior of SPICE in more detail, we can notice, by denoting first \( K \) components of the \( k \)-th row of \( \tilde{A}^H \) as \( \varphi_k^H \), i.e., \( \varphi_k^H := (\tilde{a}_1)_k \cdots (\tilde{a}_K)_k \), and observing that the constraints in \( 10 \) read \( c_{k+j} = y_j - \varphi_k^H \tilde{c} \) for \( j = 1, \ldots, N \), that \( 10 \) is equivalent to

\[
\min_{\epsilon} \| \mathbf{y} - \Phi \epsilon \|_1 + \| \epsilon \|_1,
\]

where \( \Phi := [\varphi_1 \cdots \varphi_N] \), i.e., \( \Phi \) corresponds to the first \( K \) columns of \( \tilde{A}^H \). Equation \( 11 \) is essentially a simplified (complex-valued) version of the LAD-Lasso \( 7 \) or the RLAD \( 10 \), where \( \epsilon \) takes the role of a parameter vector, and the regressors have been scaled by \( \| \mathbf{y} \|_2 / \| \mathbf{a}_k \|_2 \), so that their Euclidean norms are equal to \( \| \mathbf{y} \|_2 \). The fact that the cost function in \( 11 \) considers the \( l_1 \) norm of the residuals \( \| \mathbf{y} - \Phi \epsilon \| \) instead of their \( l_2 \) norm suggests that SPICE might be a robust estimator against outliers or errors with heavy-tailed distributions (since, heuristically speaking, it does not penalize large deviations of the residuals from zero, due mainly to outliers, as much as the \( l_2 \) norm); in fact, this is the reason why some authors have proposed the use of the LAD-Lasso instead of the standard Lasso in the presence of outliers \( 7 \).

We can summarize these results in the following theorem:

**Theorem 1:** The limit value of the SPICE iterations (allowing for different \( \sigma_k \)), which corresponds to the minimizer of \( 3 \), is also given by the minimizer of \( 11 \), by performing the following change of variables:

\[
p_k = \frac{\| \mathbf{y} \|_2 |c_k|}{\| \mathbf{a}_k \|_2 \left( \sum_{i=1}^K |c_i| + \sum_{k=1}^N |y_k - \varphi_k^H \tilde{c}| \right)},
\]

\[k = 1, \ldots, K + N.\]

**B. Equal variances**

Now we will analyze the variant of SPICE where the variances are constrained to be equal. The development in this case is exactly as in Section III-A until equation \( 8 \). At this point, the constraint \( \sigma_1 = \cdots = \sigma_N = \sigma \) implies that \( \tilde{p}_K = \cdots = \tilde{p}_{K+N} \), which allows us to simplify \( 8 \) as

\[
\min_{c_1, \ldots, c_{K+N}} \sum_{k=1}^{K+N} |c_k|^2
\]

s.t.
\[
\tilde{A}^H c = y.
\]

This is a simplification of the problem, and the solution can be found by performing the following change of variables:

\[
p_k = \frac{\| \mathbf{y} \|_2 |c_k|}{\| \mathbf{a}_k \|_2 \left( \sum_{i=1}^K |c_i| + \sum_{k=1}^N |y_k - \varphi_k^H \tilde{c}| \right)},
\]

\[k = 1, \ldots, K + N.\]
Remark 4: Even though the equivalent Lasso formulations are not given in the same variables as the SPICE method, the required variables transformations (between the \(c_k\)’s and the \(p_k\)’s) are simple scalings. This means that the sparsity properties of SPICE are essentially the same as the ones for the equivalent Lasso estimators.

Remark 5: The relations between the \(c_k\)’s and the \(p_k\)’s given by Theorems 1 and 2 have a nontrivial structure, which comes from the fact that SPICE considers the (unknown) noise variances as parameters to be estimated, and puts them in the same footing as the amplitudes of the spectral lines.

Remark 6: The cost function \(g(R)\) minimized by SPICE in (4) can be interpreted as follows: The first term of \(g(R)\), \(y^H R^{-1} y\), is a model fit measure, while the second term, \(\|y\|_2^2 \text{tr} R\), can be interpreted as a trace heuristic or nuclear norm regularization (since \(R = R^H \geq 0\), so the trace and nuclear norm coincide) [11]. This regularization term is known to encourage low rank matrices \(R\), which, due to its structure, \(R = A^H P A\), enforces the vector \([p_1, \ldots, p_{K+N}]^T\) to be sparse. This interpretation thus provides an alternative heuristic justification for the sparsity-inducing behavior of SPICE.

Remark 7: Theorems 1 and 2 have been presented for the complex-valued versions of SPICE. However, the derivations in this section apply almost unaltered to real valued problems. This means that Theorems 1 and 2 establish Lasso-type equivalences for the real-valued versions of SPICE as well. Notice, however, that the complex Lasso versions can be seen as real-valued Group Lasso estimators, as explained next.

Remark 8: The complex-valued nature of SPICE is inherited by its Lasso equivalents. Thus, for example problem (13) does not behave as the standard (real-valued) Lasso, but as the (real-valued) Group Lasso [12]. To see this, let us define

\[
\begin{align*}
y_R &:= \begin{bmatrix} \text{Re} y \\ \text{Im} y \end{bmatrix}, & \tilde{c}_R &:= \begin{bmatrix} \text{Re} \tilde{c} \\ \text{Im} \tilde{c} \end{bmatrix} \\
\Phi_R &:= \begin{bmatrix} \text{Re} \Phi & -\text{Im} \Phi \\ \text{Im} \Phi & \text{Re} \Phi \end{bmatrix}
\end{align*}
\]

Based on this notation, (13) can be written as

\[
\min_{\tilde{c}_R} \sqrt{N} \|y_R - \Phi_R \tilde{c}_R\|_2 + \sum_{k=1}^K \|\left(\tilde{c}_R\right)_k \|_2.
\]

The second term in (14) is a sum of Euclidean norms, which promotes group sparsity, i.e., it tries to enforce that both the real and imaginary parts of individual entries of \(\tilde{c}\) become zero simultaneously. Similarly, (11) corresponds to a grouped version of the LAD-Lasso.

Remark 9: Recently, a re-weighted version of SPICE, called LIKES, has been proposed in [15]. We will not address here the relation between LIKES and standard sparse estimators (such as Sparse Bayesian Learning (SBL) and Automatic Relevance Determination (ARD) [14]), because this has partly been discussed in [15], and the equivalence to Lasso-type estimators can be formally studied along the lines of [14].

IV. SIMULATION EXAMPLE

In this section, a numerical example, based on Section IV, is used to illustrate the equivalence between SPICE and the LAD-Lasso, formally established in Theorem 1.

Let \(y_k = y(t_k), k = 1, \ldots, N\), be the \(k\)-th sample, where the \(t_k\)’s are irregular time samples, drawn independently from a uniform distribution on \([0, 200]\). The basis functions considered here are of the form

\[
\alpha_k = [e^{j\omega_1 t_1} \ldots e^{j\omega_T t_N}]^T,
\]

where \(\omega_k := 2\pi k/1000\). Following [1], we take \(N = 100\), and \(y\) to be given by [1] with \(K = 3\), \(s_{145} = 3e^{j\phi_1}\), \(s_{310} = 10e^{j\phi_2}\) and \(s_{315} = 10e^{j\phi_3}\), and \(s_k = 0\) otherwise. The phases \(\phi_1, \phi_2\) and \(\phi_3\) are independent random variables, uniformly distributed in \([0, 2\pi]\). The noise \(\epsilon\) is assumed to have a covariance matrix \(0.25I\).

The results of applying 100 iterations of SPICE, (2), and its LAD-Lasso equivalent (11), solved using the CVX package [16], are presented in Figure 1. As the figure shows, both estimators practically coincide, their differences being mainly due to numerical implementations. Notice also that these estimators correctly detect the location of the peaks of the true spectrum, even though the estimated amplitudes do not approach their true values; this observation is consistent with theoretical results regarding the bias of the Lasso and its variants [9]. On a PC with an 2.53 GHz Intel Core Duo CPU and 4 Gb RAM, 100 iterations of SPICE take 23.0 s, while the implementation of LAD-Lasso using CVX only takes 14.6 s. However, if \(N\) is further increased to 1000, CVX is incapable of solving the LAD-Lasso problem, while SPICE can still provide a good (and numerically reliable) estimate.

V. CONCLUSION

In this manuscript, the recently proposed SPICE method for sparse estimation has been studied, and its relation to Lasso-type estimators has been established. This connection may enable the use of existing theoretical results for the Lasso to predict the behavior of SPICE in diverse problem settings,
and, at the same time, the application of the computationally efficient algorithm developed for SPICE to sparse estimation problems where the Lasso algorithms are currently impractical.

As a interesting future line of research, the relation between SPICE and the Group Lasso suggests that the former method could be modified to deal with general group sparsity problems (instead of only groups with two real variables). In addition, from this relation it is easy to modify SPICE in order to compensate for deficiencies already detected in standard Lasso estimators, such as lack of consistency in sparse support recovery, which can be fixed by adding re-weighting steps (see, e.g., [16]).

APPENDIX
PROOF OF EQUATION (5)

In this Appendix we prove (7). Without loss of generality we can assume that the values of $k$ for which $\hat{p}_k = 0$ have been removed from the sum. We start by rewriting (7) as

$$ y^H (\hat{A}^H \hat{P} \hat{A})^{-1} y = \min_{c \in \mathbb{C}^N} e^H \hat{P}^{-1} c \quad \text{s.t.} \quad \hat{A}^H c = y, \quad (15) $$

where $\hat{P} := \text{diag}(\hat{p}_1, \ldots, \hat{p}_{K+N})$. We will proceed by establishing the minimum value of the right hand side of (15) and showing that it coincides with its left hand side. To this end, notice that since that optimization problem is convex, $c$ is an optimal solution of the right hand side of (15) if and only if there is a Lagrange multiplier $\lambda \in \mathbb{C}^N$ such that

$$ \frac{\partial}{\partial c}\left[e^H \hat{P}^{-1} c + \lambda^H (\hat{A}^H c - y)\right] = 0, \quad \hat{A}^H c = y, $$

or, equivalently,

$$ 2\hat{P}^{-1} c + \hat{A} \lambda = 0, \quad \hat{A}^H c = y. $$

From this set of equations we obtain

$$ \lambda = -2(\hat{A}^H \hat{P} \hat{A})^{-1} y $$

and the optimal cost of right hand side of (15) gives

$$ e^H \hat{P}^{-1} c = y^H (\hat{A}^H \hat{P} \hat{A})^{-1} \hat{A}^H \hat{P}^{-1} \hat{P} \hat{A} (\hat{A}^H \hat{P} \hat{A})^{-1} y = y^H (\hat{A}^H \hat{P} \hat{A})^{-1} y, $$

which corresponds to the left hand side of (15). This concludes the proof of (7).

Remark 10: Equation (7) is closely related to the so-called Gauss-Markov theorem, which states that, in a linear regression framework, the least squares estimator is the minimum variance unbiased estimator [17]. In fact, let $z = \hat{A} \theta + e$, where $\theta \in \mathbb{C}^{K+N}$, $e \sim \mathcal{CN}(0, \hat{P}^{-1})$. Furthermore, suppose we are interested in estimating $x = y^H \theta$. Then, the cost function in the right hand side of (7) can be interpreted as the variance of an estimate $\hat{x} = e^H z$ of $x$, and the corresponding constraint $\hat{A}^H c = y$ restricts $\hat{x}$ to be unbiased, while the left hand side of (7) corresponds to the minimum achievable variance, according to the Gauss-Markov theorem.

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