A note on symmetric orderings

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Abstract

Let $\hat{A}_n$ be the completion by the degree of a differential operator of the $n$-th Weyl algebra with generators $x_1, \ldots, x_n, \partial^1, \ldots, \partial^n$. Consider $n$ elements $X_1, \ldots, X_n$ in $\hat{A}_n$ of the form

$$X_i = x_i + \sum_{K=1}^{\infty} \sum_{l=1}^{n} \sum_{j=1}^{n} x_l p^{K-1,l}_{ij}(\partial) \partial^j,$$

where $p^{K-1,l}_{ij}(\partial)$ is a degree $(K-1)$ homogeneous polynomial in $\partial^1, \ldots, \partial^n$, antisymmetric in subscripts $i, j$. Then for any natural $k$ and any function $i: \{1, \ldots, k\} \rightarrow \{1, \ldots, n\}$ we prove

$$\sum_{\sigma \in \Sigma(k)} X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(k)}} \triangleright 1 = k! x_{i_1} \cdots x_{i_k},$$

where $\Sigma(k)$ is the symmetric group on $k$ letters and $\triangleright$ denotes the Fock action of the $\hat{A}_n$ on the space of (commutative) polynomials.

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1 Introduction and motivation

In an earlier article \[3\], we derived a universal formula for an embedding of the universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra $\mathfrak{g}$ with underlying rank $n$ free module over a commutative ring $k$ containing the field $\mathbb{Q}$ of rational numbers into a completion $\hat{A}_{n,k}$ of the $n$-th Weyl algebra over $k$. 
Definition 1. $n$-th Weyl algebra $A_{n,k}$ over a commutative ring $k$ is the associative $k$-algebra defined by generators and relations

$$A_{n,k} = k\langle x_1, \ldots, x_n, \partial^1, \ldots, \partial^n \rangle / \langle [x_i, x_j], [\partial^i, \partial^j], [x_i, \partial^j] - \delta^j_i, i, j = 1, \ldots, n \rangle.$$ 

We use the “contravariant” notation ($[7]$, 1.1). The reader should recall the usual interpretation of the Weyl algebra elements as regular differential operators ($[2, 3]$). In other words, the physicist’s Fock action (here denoted $\bowtie$) of $A_{n,k}$ on the symmetric algebra $S(g) \cong k[x_1, \ldots, x_n]$ consisting of commutative polynomials, with unit polynomial 1 interpreted as the vacuum state.

Complete $A_{n,k}$ along the filtration given by the degree of differential operator ($[3, 7, 8]$); the completion will be denoted $\hat{A}_{n,k}$. Thus, the elements in $\hat{A}_{n,k}$ can be represented as arbitrary power series in $\partial^1, \ldots, \partial^n$ with coefficients (say on the left) in the polynomial ring $k[x_1, \ldots, x_n]$.

For a fixed basis $X^g_1, \ldots, X^g_n$ of $g$, denote by $C^g_{ij} \in k$ for $i, j, k \in \{1, \ldots, n\}$ the structure constants defined by

$$[X^g_i, X^g_j] = \sum_{k=1}^{n} C^g_{ij} X^g_k. \tag{1}$$

Constants $C^g_{ij}$ are antisymmetric in lower indices and satisfy a quadratic relation reflecting the Jacobi identity in $g$. Then, there is a unique monomorphism of $k$-algebras $\iota : U(g) \to \hat{A}_{n,k}$ extending the formulas ($[3]$)

$$X^g_i \mapsto \iota(X^g_i) = \sum_{l=1}^{n} x_l \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} B_N (C^N)^{il}, \tag{2}$$

where $B_N$ is the $n$-th Bernoulli number and $C$ is an $n \times n$ matrix with values in $k$, defined by

$$C^i_j = \sum_{k=1}^{n} C^g_{ik} \partial^k.$$ 

Monomorphism $\iota$ does not depend on the choice of the basis; over $\mathbb{R}$ and $\mathbb{C}$ the formula ($[2]$) appeared to be known much before ($[11, 5]$) and, suitably interpreted, corresponds to the Gutt’s star product ($[4]$). A simple differential geometric derivation of the formula ($[2]$) over $\mathbb{R}$ is explained in detail in ($[7]$), 1.2. Similarly, sections 7-9 of ($[3]$) provide a geometrical derivation in formal geometry over any ring containing rationals. See also ($[6]$) for another point of
view. It can also be interpreted as coming from the part of Campbell-Baker-Hausdorff series linear in the first argument ([3], Sections 7-9). Denote by

\[ e^g : k[x_1, \ldots, x_n] \to U(g), \quad x_{\alpha_1} \cdots x_{\alpha_k} \mapsto \frac{1}{k!} \sum_{\sigma \in \Sigma(k)} X_{\alpha_{\sigma(1)}}^g \cdots X_{\alpha_{\sigma(k)}}^g \]  

(3)

the standard symmetrization (or coexponential) map (of vector spaces), where the symmetric group on \( k \) letters is denoted \( \Sigma(k) \). The expression on the right-hand side of (3) can be embedded into \( \hat{A}_{n,k} \) and then we can act with the resulting element on 1 by the (extension to the completion of) the Fock action. We recover back the left-hand side (3). In other words ([3, 8]),

\[ ((\iota \circ e^g)(g)) \triangleright 1 = q, \quad q \in k[x_1, \ldots, x_n], \]  

(4)

where \( \triangleright \) denotes the Fock action by differential operators.

In this paper, it is proven that already the tensorial form of the universal formula (2),

\[ X_i \mapsto \tilde{X}_i = \sum_{l=1}^{n} x_l \sum_{N=0}^{\infty} A_N (C_N^l), \]  

(5)

even with arbitrary coefficients \( A_N \) instead of \( \frac{(-1)^N}{N!} B_N \) (for \( N > 0 \)), and with generators \( X^g_i \) of \( U(g) \) replaced by generators \( X_i \) of an arbitrary finitely generated associative \( k \)-algebra \( U \), guarantees that in characteristics 0 precisely the symmetrically ordered noncommutative expressions

\[ \frac{1}{k!} \sum_{\sigma \in \Sigma(k)} X_{\alpha_{\sigma(1)}} \cdots X_{\alpha_{\sigma(k)}}, \]

interpreted via our embedding, and after acting upon the vacuum, recover back the commutative product \( x_{\alpha_1} \cdots x_{\alpha_k} \).

More generally, we can even replace \( A_N (C_N^l) \) in (5) by an arbitrary expression of the form \( p_{ij}^{N-1,l}(\partial^1, \ldots, \partial^n)\partial^j \) where \( p_{ij}^{N-1,l} = p_{ij}^{N-1,l}(\partial^1, \ldots, \partial^n) \) is an arbitrary homogeneous polynomial of degree \( (N - 1) \) in \( \partial^1, \ldots, \partial^n \), antisymmetric under interchange of \( i \) and \( j \). Note that the previous case involving \( U(g) \) is recovered by setting

\[ p_{ij}^{N-1,l} = \frac{(-1)^N B_N}{N!} \sum_{s=1}^{n} (C_N^{N-1})^s_{ij} C^s_{ij}. \]
We are not discussing in this article when formula (5), and its further generalization involving $p^{N-1,l}_{ij}$, extend to a homomorphism $U \to A_{n,k}$ of algebras (in physics literature also called a realization of $U$). If $U$ is tautologically defined as the subalgebra of $A_{n,k}$ generated by the expressions $\tilde{X}_i$ in $A_{n,k}$, we warn the reader that generically the PBW type theorem does not hold and the dimension of the space of degree $k$ noncommutative polynomials of $\tilde{X}_i$ is bigger than the space of symmetric polynomials of degree $k$.

2 Results

Theorem 2. Assume $k$ is a field of characteristic different from 2. Let

$$X_i = x_i + \sum_{l=1}^{n} x_l \sum_{N=1}^{\infty} p^{N-1,l}_{ij}(\partial^1, \ldots, \partial^n) \partial^j, \quad i = 1, \ldots, n,$$

be $n$ distinguished elements of $\hat{A}_{n,k}$ of the given form, where $p^{N-1,l}_{ij}(\partial^1, \ldots, \partial^n)$ are arbitrary homogeneous polynomials of degree $(N-1)$ in $\partial^1, \ldots, \partial^n$, antisymmetric in lower indices $i, j$. Let $\alpha : \{1, \ldots, k\} \to \{1, \ldots, n\}$ be any function. Then, in the index notation, $\alpha_i = \alpha(i)$,

$$\sum_{\sigma \in \Sigma(k)} X_{\alpha_\sigma(1)} \cdots X_{\alpha_\sigma(k)} x_{\alpha_{1}} \cdots x_{\alpha_{k}}$$

Dokaz. We prove the theorem by induction on degree $k$. For $k = 1$ all terms with $N \geq 1$ vanish, because we apply at least one derivative to 1.

For general $k$, we write the sum (7) over all permutations in $\Sigma(k)$ another way. We use the fact that the set of permutations of $n$ elements $\Sigma(n)$ is in the bijection with the set of pairs $(i, \rho)$ where $0 \leq i \leq k$ and $\rho \in \Sigma(k-1)$. This can be done in many ways, but we use this concrete simple-minded bijection

$$(i, \rho) \mapsto \sigma, \quad \sigma(k) := \begin{cases} i, & k = 1, \\ \rho(k-1), & k > 1 \text{ and } \rho(k-1) < i, \\ \rho(k-1) + 1, & k > 1 \text{ and } \rho(k-1) \geq i. \end{cases}$$

For example, $(3, (2, 3, 1, 5, 4)) \mapsto (3, 2, 4, 1, 6, 5)$.

Define a bijection $\Theta_i : \{1, \ldots, k-1\} \to \{1, \ldots, i-1, i+1, \ldots, k\}$ by

$$\Theta_i(j) := \begin{cases} j, & j < i, \\ j + 1, & j \geq i. \end{cases}$$
Clearly now \( \sigma(j + 1) = \Theta_i(\rho(j)) \) for \( 1 \leq j < k \).

We may thus renumber the sum
\[
\sum_{\sigma \in \Sigma(k)} X_{\alpha_{\sigma(2)}} \cdots X_{\alpha_{\sigma(k)}}
\]
as the double sum
\[
\sum_{i=1}^{k} X_{\alpha(i)} \cdot \sum_{\rho \in \Sigma(k-1)} X_{(\alpha \circ \Theta_i)(\rho(1))} \cdots X_{(\alpha \circ \Theta_i)(\rho(k-1))}
\]
By the assumption of induction,
\[
\sum_{\rho \in \Sigma(k-1)} X_{(\alpha \circ \Theta_i)(\rho(1))} \cdots X_{(\alpha \circ \Theta_i)(\rho(k-1))} \triangleright 1 = (k - 1)! x_{(\alpha \circ \Theta_i)(1)} \cdots x_{(\alpha \circ \Theta_i)(k-1)}
\]
The function \( \Theta_i \) takes all values between 1 and \( k \) except \( i \) exactly once.

Therefore, (7) may be rewritten as
\[
(k - 1)! \sum_{i=1}^{k} X_{\alpha(i)} \triangleright (x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}).
\]
Thus (7) equals
\[
(k - 1)! \sum_{i=1}^{k} \sum_{l} x_{l} \sum_{ij} p_{\alpha(i)\alpha(j)}^{N-1,l} \partial^{i} \triangleright (x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}).
\]
For \( N = 0 \) we have the contribution
\[
(k - 1)! \sum_{i=1}^{k} \sum_{r} x_{r} \partial^{r}_{\alpha(i)} \triangleright (x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}) = x_{\alpha(1)} \cdots x_{\alpha(k)}.
\]
Thus, it is sufficient now to show that for any \( N > 0 \),
\[
(k - 1)! \sum_{i=1}^{k} \sum_{l} x_{l} \sum_{ij} p_{\alpha(i)\alpha(j)}^{N-1,l} \partial^{i}(x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}) = 0.
\]
But this is clear. Namely, for \( i \neq j \),
\[
\partial^{i}(x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}) = \prod_{r \in \{1, \ldots, k\} \setminus \{i, j\}} x_{\alpha(r)}.
\]
Clearly, this expression is symmetric under the interchange of \( i \) and \( j \); hence the contracting it with the antisymmetric tensor \( p^{N-1,l}_{\alpha(i)\alpha(j)} \) gives zero. More precisely, already

\[
\sum_{ij} p^{N-1,l}_{\alpha(i)\alpha(j)} \otimes \partial^{a(j)}(x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}) = 0,
\]

hence after applying the evaluation of \( p^{N-1,l}_{\alpha(i)\alpha(j)} \) on the other tensor factor, the sum stays zero.

**Corollary 3.** Under the assumptions of Theorem 1 the \( k \)-linear map \( k[x_1, \ldots, x_n] \to \hat{A}_{n,k} \) extending the formulas

\[
\tilde{e} : x_{\alpha_1} \cdots x_{\alpha_k} \mapsto \sum_{\sigma \in \Sigma(k)} X_{\alpha_{\sigma(1)}} \cdots X_{\alpha_{\sigma(k)}}, \quad (9)
\]

for all \( k \geq 0 \) and all (nonstrictly) monotone \( \alpha : \{1, \ldots, k\} \to \{1, \ldots, n\} \) satisfies

\[
\tilde{e}(P_k) \triangleright 1 = k! P_k, \quad (10)
\]

for each (commutative) polynomial \( P_k = P_k(x_{\alpha_1}, \ldots, x_{\alpha_n}) \) homogeneous of degree \( k \). In particular, \( \tilde{e} \) is injective iff \( k \) is of characteristic 0. In that case, the elements \( e(x_{\alpha_1} \cdots x_{\alpha_n}) \) are linearly independent. In characteristics zero, the version of the map normalized on homogenous components,

\[
e : x_{\alpha_1} \cdots x_{\alpha_k} \mapsto \frac{1}{k!} \sum_{\sigma \in \Sigma(k)} X_{\alpha_{\sigma(1)}} \cdots X_{\alpha_{\sigma(k)}}, \quad (11)
\]

is an injection.

Formula (11) can be restated as \( \tilde{e}(\cdot) \triangleright 1 = k! \text{id} \). Note that the expressions (9) do not span an associative subalgebra, but only a subspace \( e(k[x_1, \ldots, x_n]) \) of the subalgebra of \( \hat{A}_{n,k} \) generated by \( X_1, \ldots, X_n \), in general. In characteristics 0, the map \( e \) can be viewed as a vector space section of the projection map \( \pi \) (the restriction of the Fock action on vacuum vector 1) from \( k\langle X_1, \ldots, X_n \rangle \subset A_{n,k} \) onto \( k[x_1, \ldots, x_n] \), in particular \( e \) is an isomorphism onto its image and \( \text{Ker} \ \pi \oplus \text{Im} \ e = k\langle X_1, \ldots, X_n \rangle \).

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Literatura

[1] F. A. Berezin, Some remarks about the associated envelope of a Lie algebra, Funkc. Anal. Pril 1:2 (1967) 1-14 (in Russian); Engl. transl. Funct. Anal. Appl. 1:2 (1967) 91–102.

[2] S. C. Coutinho, A primer of algebraic D-modules, Cambr. UP 1995.

[3] N. Durov, S. Meljanac, A. Samsarov, Z. Škoda, A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra, Journal of Algebra 309:1, (2007) 318–359, math.RT/0604096.

[4] S. Gutt, An explicit $\star$-product on the cotangent bundle of a Lie group, Lett. Math. Phys. 7 (1983) 249–258.

[5] M. Karasev, V. Maslov, Nonlinear Poisson brackets, Moskva, Nauka 1991 (in Russian); Engl.: AMS Transl. Math. Monogr. 119 (1993)

[6] V. Kathotia, Kontsevich’s universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula, Internat. J. Math. 11 (2000); no. 4, 523–551; math.QA/9811174

[7] S. Meljanac, Z. Škoda, M. Stojić, Lie algebra type noncommutative phase spaces are Hopf algebroids, Lett. Math. Phys. 107:3, 475–503 (2017) arXiv:1409.8188.

[8] S. Meljanac, Z. Škoda, Leibniz rules for enveloping algebras and a diagrammatic expansion, www2.irb.hr/korisnici/zskoda/scopr8.pdf (old version: arXiv:0711.0149).

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