STABILITY FOR SEMILINEAR PARABOLIC PROBLEMS IN $L_2$, $W^{1,2}$, AND INTERPOLATION SPACES

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Abstract. An asymptotic stability result for parabolic semilinear problems in $L_2(\Omega)$ and interpolation spaces is shown. In particular, folklore results in $W^{1,2}(\Omega)$ are improved. These results are consequences of a general theory which we develop for operators satisfying Kato’s square root property. The approach is based on fractional powers of extensions of operators to functional spaces. As a side result, some relations to Kato’s square root problem are obtained.

1. Introduction

All results known to the authors dealing with linear stability of semilinear equations $u_t + Au = f(u)$ make use of semigroup techniques. In the simplest of these results [26], the nonlinearity $f$ is assumed to act (and be e.g. differentiable) in the same Banach space $H$ in which the semigroup acts. In the case of heat equations or reaction-diffusion systems, i.e., when the semigroup is (essentially) given by the Laplace operator, the classical choices of the space $H$ are e.g. $W^{1,p}(\Omega)$ (or subspaces taking some boundary conditions into account) or $L_p(\Omega)$. However, in these cases, the nonlinearity given by a superposition operator is differentiable if and only if it is affine, see e.g. [17].

One possible solution of this problem is to work in spaces of continuous functions, see [20]. However, this is not possible if one wants to consider Sobolev or $L_p$ spaces. In this case, another approach can be found in [11], where the nonlinearity is assumed to act only from a space $H_\alpha$ with $\alpha \in [0,1)$ into $H$ with $H_\alpha$ being the domain of a fractional power of the (negative of the) generator of the semigroup. This idea can be extended to somewhat more general interpolation spaces, which in some cases avoids the problem that the space depends on the operator (which is important for quasilinear problems), see [3]. The classical folklore way to apply this result is to work in $H = L_p(\Omega)$, and one obtains that $H_\alpha$ is for sufficiently large $p$ embedded into $C(\Omega)$, hence differentiability of the nonlinearity is not an issue anymore. However, one obtains asymptotic stability only in the space $H_\alpha$ with large $\alpha > 0$ since otherwise one ends up with very restrictive (or in case $\alpha = 0$ even degenerate) hypotheses about the nonlinearity $f$.

Results obtained in this way are usually not comparable with stability or instability results for e.g. obstacle problems. In fact, the methods known for variational inequalities often require a Hilbert space setting (hence, we need $p = 2$) to which we restrict ourselves from now on. Moreover, the only general results known for stability or instability of obstacle problems involve the $W^{1,2}(\Omega)$ or $L_2(\Omega)$ topology. In order to compare the problems with and without obstacles, we should thus know something about their linear stability in $W^{1,2}(\Omega)$.

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and $L_2(\Omega)$. Now the folklore way to do this is rather suboptimal. For a stability result for the Laplace operator with Neumann boundary conditions in the $W^{1,2}(\Omega)$ topology, we would need to consider $H = L_2(\Omega)$ and get $H_{1/2} = W^{1,2}(\Omega)$, hence our nonlinearity has to act from $W^{1,2}(\Omega)$ into $L_2(\Omega)$ which (in space dimension $N > 1$) amounts to a certain growth hypothesis on the function generating the superposition operator; a corresponding result for a reaction diffusion system was formulated e.g. in [30]. Moreover, to get a stability result in the $L_2(\Omega)$-topology in this way, one would have to choose $\alpha = 0$, that is, one would need to consider the nonlinearity acting from $H$ into itself. As mentioned above, this means that one cannot consider differentiable nonlinearities of superposition type at all.

Note that, in contrast, if one is interested in stationary solutions, i.e. in the corresponding elliptic problem, a natural approach is to consider the superposition operator acting from $W^{1,2}(\Omega)$ into the dual space with respect to the $L_2$-scalar product, i.e. into the antidual space $W^{1,2}(\Omega)'$. Since $L_p(\Omega) \subseteq W^{1,2}(\Omega)'$ for some $p < 2$, this approach requires a milder growth condition than if the nonlinearity acts from $W^{1,2}(\Omega)$ into $L_2(\Omega)$.

The approach we present in this paper gives a result about linear stability in $W^{1,2}(\Omega)$ under the “natural” acting conditions as in the elliptic problem (that is, for subcritical growth of the nonlinearity), thus relaxing the growth hypothesis supposed in e.g. [30]. Moreover, simultaneously it gives a stability result in the $L_2(\Omega)$ topology which is really applicable for superposition operators. To the authors knowledge this is the first result in this direction.

Since we are mainly interested in Hilbert spaces, we restrict our considerations to operators generated by sesquilinear forms. Nevertheless, we point out that for particular parabolic equations results allowing similar approaches in $L_p(\Omega)$ with $p$ close to 2 have been studied by K. Gröger, J. Rehberg, and others. The authors thank J. Rehberg for pointing out references to corresponding abstract results (personal communication).

The plan of the paper is as follows. In Section 2 we introduce the abstract setting and recall (slight extensions of) the classical results related to stability from [11]. In Section 3 we formulate our main results in the abstract setting. The machinery to prove these results requires extending operators to functional spaces and studying fractional powers thereof. This machinery is developed in Section 4. The proofs of the main results of Section 5 (in “half” of the cases) together with some generalization of these results are provided in Section 5. By “iterating” the argument, we can also treat the other “half” of the cases in Section 6. Finally, in Section 7 we apply the main results to partial differential equations, giving results in the $L_2(\Omega)$ topology which to our knowledge cannot be obtained as corollaries of standard methods, and also improving the “folklore” results in $W^{1,2}(\Omega)$ by requiring only a “natural” subcritical growth condition on the nonlinearity. Somewhat surprisingly, the machinery developed in Section 4 is closely related to a characterization of operators satisfying Kato’s square root problem; we summarize these observations in Appendix A. In Appendix B some auxiliary assertions concerning compactness and restrictions of fractional powers are obtained which we need in several places.

2. Abstract Setting and a Summary of Classical Results

2.1. Abstract Setting. Our abstract setting is the following: Let $(H, (\cdot, \cdot), |\cdot|)$ be a complex Hilbert space, and $(V, \|\cdot\|)$ a complex Banach space which is densely embedded into $H$. Let $(V', \|\cdot\|_{V'})$ and $H'$ denote the antidual of $V$ and $H$, respectively, endowed with the corresponding functional norm. The (Banach space) adjoint of the given embedding $i: V \to H$ defines the embedding $i': H' \to V'$ which has automatically a dense range, since $i$ is one-to-one. Identifying $H$ with $H'$ and $i'(u)$ with an element of $V'$, we thus have a
Gel’fand triple

\[ V \subseteq H \subseteq V'. \]

As customary, we denote the pairing of \( V' \) and \( V \) also by \((\cdot, \cdot)\) (which on \( H \times V \subseteq H \times H \) coincides with the scalar product of \( H \) by definition of the adjoint, so that the notation is actually unique).

Throughout, let \( a: V \times V \to \mathbb{C} \) be a sesquilinear form on \( V \) which is continuous, that is, there is \( C \in [0, \infty) \) with

\[ |a(u, v)| \leq C \|u\| \|v\|, \quad (2.2) \]

and which is strongly accretive in the sense that there is \( c > 0 \) with

\[ \text{Re} a(u, u) \geq c \|u\|^2 \quad \text{for all} \ u \in V. \quad (2.3) \]

Note that (2.3) implies that \( a \) is uniformly accretive in \( H \), that is, there is \( c_0 > 0 \) with

\[ \text{Re} a(u, u) \geq c_0 \|u\|^2 \quad \text{for all} \ u \in V. \quad (2.4) \]

The hypotheses (2.2) and (2.3) mean that \( u \mapsto (\text{Re} a(u, u) + |u|^2)^{1/2} \) defines an equivalent norm on \( V \) so that \( a \) is a closed form on \( H \times H \) with domain \( D(a) = V \) in the sense of \([12,11,23]\). Moreover, \( a \) is sectorial in the sense of \([14,23]\) (called regular in \([12]\)), that is, there is a constant \( \delta \in (0, \infty) \) with

\[ |\text{Im} a(u, u)| \leq \delta \text{Re} a(u, u) \quad \text{for all} \ u \in V. \quad (2.5) \]

(Indeed, (2.5) holds with \( \delta := C/c. \))

We associate with \( a \) the linear operator \( A: D(A) \to H \), defined by the duality \((Au, \cdot) = a(u, \cdot)\), that is, \( D(A) \) is the set of all \( u \in V \) for which there is some (uniquely determined) \( Au \in H \) with

\[ (Au, \varphi) = a(u, \varphi) \quad \text{for all} \ \varphi \in V. \]

**Remark 2.1.** For every \( M \geq 0 \), the sesquilinear form

\[ b_M(u, v) := \frac{1}{2}(a(u, v) + \overline{a(v, u)} + M \cdot (u, v)) \quad (2.6) \]

is symmetric, that is, \( b_M(v, u) = \overline{b_M(u, v)} \), and \( b_M \) satisfies estimates of the type (2.2) and (2.3). Hence, \( b_M \) becomes a scalar product on \( V \), and the norm induced by this scalar product is equivalent to the norm on \( V \). Thus, a form \( a \) satisfying (2.2) and (2.3) exists if and only if \( V \) is (isomorphic to) a Hilbert space.

We summarize some well-known facts about \( A \), providing references to the proofs for the reader’s convenience.

**Proposition 2.2.** The operator \( A \) is closed and densely defined in \( H \) with spectrum contained in the closure of the numerical range \( \{a(u, u) : u \in V, \ |u| = 1 \} \) which in turn is contained in the intersection of the sector

\[ \{z \in \mathbb{C} : |\text{Im} z| \leq \delta \text{Re} z\} \quad (2.7) \]

with the closed half-plane

\[ \{z \in \mathbb{C} : \text{Re} z \geq c_0\}. \quad (2.8) \]

In particular, \( A^{-1}: H \to H \) is bounded. Moreover, \( A^{-1}: V \to V \) is bounded, and \( D(A) \) is dense in \( V \). The operator \( A: D(A) \to H \) is positive (of positive type) in the sense of \([27]\) (or \([4]\), respectively) and sectorial in the sense of \([11,24]\), and \( -A \) generates an analytic contraction \( C_0 \) semigroup in \( H \).

If \( a \) is symmetric, then \( A: D(A) \to H \) is selfadjoint in \( H \) with spectrum and numerical range contained in \([c_0, \infty)\).
Proof. According to [14, Theorem VI.2.1], we find that $D(A)$ is dense in $V$ and that $A$ is $m$-sectorial and thus quasi-$m$-accretive in the sense of [14, Section VI.10]. Hence, the assertions about the spectrum follow together with the resolvent estimates required for sectorial and positive operators from [14, Theorem VI.3.2]. It follows (see e.g. [23, Proposition 1.51 and Theorem 1.52]), that $-A$ generates an analytic contraction $C_0$ semigroup. The boundedness of $A^{-1}: V \to V$ follows from the estimate
\[ c||u||^2 \leq \text{Re}(Au, u) \leq ||Au||u| \leq C_0^2||Au||u| \quad \text{for all } u \in D(A), \]
where $C_0$ denotes the embedding constant of $i: V \to H$. The assertions about symmetric $a$ are contained in [13, Theorem VI.2.6]. \hfill \Box

Since $A$ is of positive type, we can define powers $A^\alpha$ ($\alpha \geq 0$) of $A$ in the standard way (see e.g. [11, 12, 24, 27]) on certain domains $H_\alpha := D(A^\alpha) \subseteq H$, which we equip with the norm
\[ ||u||_{H_\alpha} := |A^\alpha u|, \]
equivalent to the graph norm of $A^\alpha$ in $H$. Recall that one can also define $A^{-\alpha}$ ($\alpha > 0$), see e.g. [11, 12, 24, 27]. The operators $A^{-\alpha}: H \to H$ are bounded, $D(A^{-\alpha}) = H$, and $A^{-\alpha} = (A^\alpha)^{-1}$. It will be convenient to define $H_{-\alpha}$ as the completion of $D(A^{-\alpha}) = H$ with respect to the norm
\[ ||u||_{H_{-\alpha}} := |A^{-\alpha} u|. \]

For a closed operator $B: D(B) \to H$ with dense $D(B) \subseteq H$, we denote by $B^*$ the Hilbert-space adjoint (in $H$). In particular, for $\alpha \geq 0$, we define $(A^\alpha)^*$ as the Hilbert-space adjoint of $A^\alpha$, the latter considered as an unbounded operator in $H$ with domain $D(A^\alpha)$. We actually have $(A^\alpha)^* = (A^*)^\alpha$, see Proposition [13, 3]. We define $H_\alpha^* := D((A^*)^\alpha) = D((A^\alpha)^*)$, equipped with the norm
\[ ||u||_{H_\alpha^*} := |(A^*)^\alpha u| = |(A^\alpha)^* u|, \]
which is equivalent to the graph norm of $(A^*)^\alpha$ in $H$.

We collect some well-known facts about $H_\alpha$. We denote by $[\cdot, \cdot]_\theta$ the complex interpolation functor of order $\theta$, see e.g. [27].

Proposition 2.3. $H_0 = H$, $A^\alpha$ is the $n$th iterate of $A$ for $n = 1, 2, \ldots$, and $H_{(1-\theta)+\beta\theta} \cong [H_\alpha, H_\beta]_\theta$ for every $\alpha, \beta \geq 0$ and $\theta \in (0, 1)$. Moreover,
\[ H_\alpha^* \cong H_\alpha \quad \text{if } \alpha \in [0, 1/2). \quad (2.9) \]
If $a$ is symmetric, then $H_{1/2}^* = H_{1/2} \cong V$.

Here and in the following, notations like $H_{1/2} \cong V$ are to be understood that the spaces coincide as sets and carry equivalent norms.

Proof. According to [13, Theorem 5] there holds $||A^t|| \leq e^{\pi|t|/2}$ for real $t$, and so the formula $H_{(1-\theta)+\beta\theta} \cong [H_\alpha, H_\beta]_\theta$ follows from [27, Theorem 1.15.3]. For the crucial case $\alpha = 0$, $\beta = 1$, a more direct proof of this formula was given in [18]; for symmetric $a$, a very simple proof of this formula can also be found in [27, Theorem 1.18.10]. The assertion (2.9) follows from [12, Theorem 1.1]. For symmetric $a$, we have $A = A^*$, hence $H_{1/2} = H_{1/2}^*$, and the assertion $H_{1/2} \cong V$ is shown in [12] (see also [23, Theorem 8.1]). \hfill \Box
2.2. Summary of Classical Results. Our main interest lies in some dynamical assertions about stability of equilibria for semilinear parabolic equations, which we formulate now. We start by summarizing (slight extensions of) well-known results which can be found in e.g. [11].

From now on, we keep a choice \( \alpha \in [0, 1) \) and the corresponding space \( H_\alpha \) fixed; the case \( \alpha = 0 \), that is, \( H_\alpha = H \) is explicitly admissible.

Given a subset \( U \subseteq \mathbb{R} \times H_\alpha \) and a function \( f : U \to 2^H \) (we include multi-valued \( f \) for completeness), we consider the problem

\[
u'(t) + Au(t) \in f(t, u(t)).
\]

**Definition 2.4.** We call \( u : [t_0, t_1) \to H \) a strong/mild solution of (2.10) if there is a function \( f_0 : (t_0, t_1) \to H \) with \( f_0 \in L_1((t_0, \tau), H) \) for every \( \tau < t_1 \) such that the following holds for every \( t \in (t_0, t_1) \): \( (t, u(t)) \in U \), \( f_0(t) \in f(t, u(t)) \), and

- **(strong solution):** \( u \in C([t_0, t_1), H) \), \( u'(t) \in H \) exists in the sense of the norm of \( H \), \( u(t) \in D(A) \), and \( u'(t) + Au(t) = f_0(t) \).
- **(mild solution):**

\[
u(t) = e^{-(t-t_0)A}u(t_0) + \int_{t_0}^t e^{-(t-s)A}f_0(s) \, ds.
\]

**Theorem 2.5** (Classical Regularity). Every strong solution is a mild solution, and the converse holds if \( f_0 \) in Definition 2.4 is locally Hölder continuous. Moreover, if \( u : [t_0, t_1) \to H \) satisfies (2.11) for all \( t \in (t_0, t_1) \), and if for every \( \tau < t_1 \) there is \( p > 1/(1-\alpha) \) with \( f_0 \in L_p((t_0, \tau), H) \), then \( u : (t_0, t_1) \to H_\alpha \) is locally Hölder continuous, and \( u \in C([t_0, t_1), H_\alpha) \) if and only if \( u(t_0) \in H_\alpha \).

**Proof.** The first assertions can be found as e.g. [24 Corollary 4.2.2]. The remaining assertions follow by a standard calculation for weakly singular integrals (see e.g. [5 Satz 6.12] for the scalar case) by using that \( e^{-tA} : H \to H_\alpha \) is bounded for \( t > 0 \) by \( C_0/t^\alpha \) with \( C_0 \) independent of \( t \geq 0 \), that the function \( g_u : [0, \infty) \to H_\alpha \), \( g_u(t) := e^{-tA}u_0 \) is locally Hölder continuous on \( (0, \infty) \) if \( u_0 \in H \) by [24, Theorem 2.6.3], and continuous at 0 if \( u_0 \in H_\alpha \), because for \( u_1 := A^\alpha u_0 \) there holds \( A^\alpha g(t) = e^{-tA}u_1 \), see e.g. [24, Theorem 2.6.13(b,c)].

Concerning existence results, we will for simplicity only consider single-valued \( f \) in which case we also get uniqueness and regularity. We say that \( f \) satisfies a right local Hölder-Lipschitz condition if for each \( (t_0, u_0) \in U \) there is a (relative) neighborhood \( U_0 \subseteq [t_0, \infty) \times H_\alpha \) of \( (t_0, u_0) \) with \( U_0 \subseteq U \) such that there are constants \( L < \infty \) and \( \sigma > 0 \) with

\[
|f(t, u) - f(s, v)| \leq L \cdot (|t - s|^{\sigma} + \|u - v\|_{H_\alpha}) \quad \text{for all} \ (t, u), (s, v) \in U_0.
\]

We call \( f \) left-locally bounded into \( H \) if for each \( t_1 > t_0 \) and each bounded \( M \subseteq H_\alpha \) there is some \( \varepsilon > 0 \) such that \( f(U \cap [(t_1 - \varepsilon, t_1) \times M]) \) is bounded in \( H \).

**Theorem 2.6** (Classical Uniqueness, Existence, Maximal Interval). (1) If \( f : U \to H \) satisfies a right local Hölder-Lipschitz condition, then for every \( (t_0, u_0) \in U \) and \( t_1 \in (t_0, \infty] \) there is at most one mild solution \( u \in C([t_0, t_1), H_\alpha) \) of (2.10) satisfying \( u(t_0) = u_0 \).

(2) Moreover, such a strong solution exists with some \( t_1 > t_0 \), and if \( f \) is left-locally bounded into \( H \), then some maximal \( t_1 > t_0 \) can be chosen such that either \( t_1 = \infty \) or \( \|u(t)\|_{H_\alpha} \to \infty \) as \( t \to t_1 \) or the limit \( u_1 = \lim_{t \to t_1^-} u(t) \) exists in \( H_\alpha \) with \( (t_1, u_1) \notin U \).
Theorem 2.6 is only the motivation for the subsequent classical asymptotic stability result. We formulate this result even for multi-valued $f: U \to 2^H$, since the proof is practically the same as in the classical single-valued case. We call $u_0 \in D(A)$ an equilibrium of (2.10) if $0 \in Au_0 + f(t, u_0)$ for all $t > 0$ and make the following hypothesis:

(B): Let $u_0$ be an equilibrium, $U_1 \subseteq H_\alpha$ an open neighborhood of $u_0$ and $[0, \infty) \times U_1 \subseteq U$. Assume that there is a bounded map $B: H_\alpha \to H$ such that the function $g(t, u) := f(t, u_0 + u) + Au_0 - Bu$ satisfies

$$\lim_{\|u\|_{H_\alpha} \to 0} \sup \{ \|v\| : v \in g((0, \infty) \times \{u\}) \} = 0.$$  

(Here, we use the convention $\sup \emptyset := 0$.)

If $f(t, \cdot)$ is single-valued in a neighborhood of $u_0$, then $Au_0 = -f(t, u_0)$ so that hypothesis (B) means that $f(t, \cdot)$ is Fréchet differentiable at $u_0$ with derivative $B$, uniformly with respect to $t \in [0, \infty)$.

Theorem 2.7 (Classical Asymptotic Stability). Under hypothesis (B), assume that there is $\lambda_0 > 0$ such that every element $\lambda$ of the spectrum of $A - B$ in $H$ satisfies $\text{Re} \lambda > \lambda_0$.

Then there exist $M_1, M_2 > 0$ such that if $t_1 > t_0 \geq 0$ and if $u \in C([t_0, t_1), H_\alpha)$ is a mild solution of (2.10) on $[t_0, t_1]$ with $\|u(t_0) - u_0\|_{H_\alpha} \leq M_1$, then $u$ satisfies the asymptotic stability estimate

$$\|u(t) - u_0\|_{H_\alpha} \leq M_2 e^{-\lambda_0(t-t_0)}\|u(t_0) - u_0\|_{H_\alpha} \quad \text{for all } t \in [t_0, t_1). \quad (2.13)$$

If $f$ satisfies in addition the hypotheses of part (11) of Theorem 2.6 then additionally for every $t_0 \geq 0$ and every $u_1 \in H_\alpha$ with $\|u_1 - u_0\| \leq M_1$ a unique strong solution $u \in C([t_0, \infty), H_\alpha)$ with $u(t_0) = u_1$ exists and satisfies (2.13) with $t_1 = \infty$.

Proof. The result is proved analogously to [11, Theorem 5.1.1].

Remark 2.8. Theorems 2.5, 2.7 hold actually even if $A$ is not associated with a sesquilinear form. For these results, it is only used that $A$ is sectorial in the sense of [11] (and of positive type to properly define $A^\alpha$ and $H_\alpha$) or, equivalently (see [24]), that $-A$ generates an analytic semigroup.

The above classical results have several disadvantages. In the lack of a local Hölder-Lipschitz condition or, even more, in the multi-valued case, there may be solutions of (2.10) in a weaker sense which are not covered in Theorem 2.7. Moreover, in the most important case $H = L_2(\Omega)$ and when $f$ is generated by a superposition operator, the choice $\alpha = 0$ is not possible, that is, one cannot obtain a nontrivial stability criterion in $H_0 = L_2(\Omega)$ by Theorem 2.7. Indeed, it is well known that any differentiable (single-valued) superposition operator $f$ in $L_2(\Omega)$ is actually affine, see e.g. [17].

In addition, even just the acting condition $f: U \to H$ in the spaces $H_\alpha = V = W^{1,2}(\Omega)$ and $H = L_2(\Omega)$ leads to a growth condition on $f$ which appears unnecessarily restrictive. In the study of stationary solutions, one typically only requires that $f: V \to V'$ is continuous (and usually compact) which is satisfied under a much milder growth condition.

Therefore, our aim is to replace the image space $H$ in Theorems 2.6 and 2.7 by a larger space with a weaker topology. This turns out to be possible for a wide class of operators associated with sesquilinear forms.
3. New Results

**Definition 3.1.** We call \( A \) a **Kato** operator if it is the operator associated with a sesquilinear form \( a: V \times V \to \mathbb{C} \) satisfying (2.2) and (2.3) and \( D(A^{1/2}) = H_{1/2} \cong V \).

**Proposition 3.2.** If \( A \) is the operator associated with a sesquilinear form \( : V \times V \to \mathbb{C} \) satisfying (2.2) and (2.3), then \( A \) is a Kato operator if and only if \( H_{1/2}^* \cong H_{1/2} \). In particular, \( A \) is a Kato operator if and only if
\[
H_{\alpha}^* \cong H_{\alpha} \quad \text{for all } \alpha \in [0, 1/2].
\]

**Proof.** The first assertion is a special case of [13, Theorem 1], and (3.1) follows in view of (2.9). The last assertion follows from the previous assertion and the observation that \( A^* \) is generated by the form \( a^*(u, v) := \overline{a(v, u)} \), see e.g. [23, Proposition 1.24]. \( \square \)

For the rest of this section, we consider Kato operators. The name is motivated by Kato’s famous square root problem originally posed in [12]: to characterize the forms \( a \) for which \( A \) is a Kato operator. According to Proposition 2.3, \( A \) is a Kato operator if \( a \) is symmetric. However, also many elliptic differential operators (even nonsymmetric) induce Kato operators, see e.g. [23, Chapter 8] and [3, 25]. So the requirement that \( A \) is a Kato operator is rather mild from the viewpoint of applications we have in mind.

Fixing numbers \( \alpha, \gamma \) with the properties
\[
0 \leq \alpha < 1 - \gamma \leq 1,
\]
we relax now the acting condition of \( f \) by replacing \( H \) by \( (H_*^\gamma)' \) in the above results. We require only \( f: U \to 2(H_*^\gamma)' \). Recall in this connection that \( (H_*^\gamma)' \) is defined as the antidual space of \( H_*^\gamma \), endowed with the functional norm. The following result gives a more instructive characterization of \( (H_*^\gamma)' \). Note that the result implies in particular that, if \( A \) is a Kato operator and \( \gamma \leq 1/2 \), then the space \( (H_*^\gamma)' \cong H_*^\gamma \) depends only on \( V \) and \( H \) (and their embedding) but not on \( A \).

**Proposition 3.3.** Let \( A \) be a Kato operator.

1. If (3.2) holds, we have continuous dense embeddings
\[
D(A) \subseteq H_{1-\gamma} \subseteq H_{\alpha} \subseteq H_0 = H
\]
and in case \( \gamma \in [0, 1/2] \)
\[
D(A) \subseteq H_{1-\gamma} \subseteq H_{1/2} \cong V \subseteq H_{\gamma} \cong H_*^\gamma \subseteq H_0 = H = H' \subseteq (H_*^\gamma)' \cong H_*^\gamma \subseteq V',
\]
while in case \( \gamma \in [1/2, 1] \)
\[
D(A^*) \subseteq H_*^\gamma \subseteq H_{1/2} \cong V \subseteq H_{1-\gamma} \cong H_*^{1-\gamma} \subseteq H_0 = H = H' \subseteq V' \subseteq (H_*^{1-\gamma})'.
\]
2. If \( \gamma \in (0, 1/2) \), then
\[
(H_*^\gamma)' \cong H_*^\gamma \cong [H, V]_{2,\gamma} \cong [V', V]_{\frac{1}{2},\gamma} \cong [V', V]_{1-\gamma} \cong [V', H]_{1-\gamma},
\]
3. For all \( \gamma \in [0, 1] \) there is a canonical identification \( (H_*^\gamma)' = H_{-\gamma} \) induced on the dense subset \( H' = H \) by the scalar product \( (\cdot, \cdot) \). For all \( \gamma_1, \gamma_2 \in [0, 1] \) and \( \theta \in (0, 1) \) there holds
\[
H_{-\gamma_1(1+\theta)-\gamma_2\theta} = (H_*^{\gamma_1(1-\theta)+\gamma_2\theta})' \cong [(H_*^{\gamma_1})', (H_*^{\gamma_2})']_{\theta} = [H_{-\gamma_1}, H_{-\gamma_2}]_{\theta}.
\]
(4) For $\gamma = 1$, we have $D(A^\gamma)' = (H_\gamma^*)' \cong H_{-1}$, while for $\gamma \in (0, 1)$

\[
(H_\gamma^*)' \cong [D(A^\gamma)', H]_{1-\gamma} \cong [H_{-1}, H]_{1-\gamma}.
\]

(3.7)

If $\gamma \in (1/2, 1)$, there holds

\[
(H_\gamma^*)' \cong [D(A^\gamma)', V']_{2-\gamma} \cong [V, D(A^\gamma)]_{2\gamma-1}.
\]

(3.8)

Proof. (1): $H'$ is dense in $H_\gamma'$ by the same argument that we gave for $V'$. Now (1) follows easily in view of (3.1).

(2): The first two equalities in (3.5) follow from Proposition 2.3 and from $H_{1/2} \cong V$, and the remaining ones will be shown in Remark 4.1.

(3): We show $(H_\gamma^*')' = H_{-\gamma}$ if $\gamma \in (0, 1]$. The discussion preceeding (A.1) implies that the Hilbert-space adjoint $(A^{-\gamma})^*$ is a bounded operator, and its inverse is $(A^\gamma)^*$. For $u \in H$, we calculate

\[
\|u\|_{H_{-\gamma}} = |A^{-\gamma}u| = \sup_{|v| \leq 1} |(A^{-\gamma}u, v)| = \sup_{|v| \leq 1} |(u, (A^{-\gamma})^*v)| = \sup_{|(A^\gamma)^*v| \leq 1} |(u, w)| = \|u\|_{(H_\gamma')'}.
\]

Since $H = H'$ is dense in both spaces $(H_\gamma^*')'$ and $H_{-\gamma}$, we obtain $(H_\gamma^*')' = H_{-\gamma}$.

The identity (3.6) will be shown in Remark 6.3.

(4): The identity (3.7) and the first equality in (3.8) are special cases of (3.6) with the choices $\gamma_2 = 1$ and $\gamma_1 = 0$ or $\gamma_1 = 1/2$, respectively. Since $A^\gamma$ is a Kato operator by Proposition 3.2, we obtain from Proposition 2.3 that $H_\gamma^* \cong [H_{1/2}^*, H_1^*]_{2\gamma-1}$ for $\gamma \in (1/2, 1)$, which implies the last equality of (3.8).

Definition 3.4. Let $\gamma \in [0, 1/2]$. We call $u \in C((t_0, t_1), H_\gamma')$ a $H_\gamma'$-weak solution of (2.10) if there is some $f_0: (t_0, t_1) \to H_\gamma'$ with $f_0 \in L_1((t_0, \tau), H_\gamma')$ for every $\tau \in (t_0, t_1)$ such that the following holds for every $t \in (t_0, t_1)$: $f_0(t) \in f(t, u(t)), u'(t) \in H_\gamma'$ exists in the sense of the norm of $H_\gamma'$, $u(t) \in V$, and

\[
(u'(t), \varphi) + a(u(t), \varphi) = (f_0(t), \varphi) \quad \text{for all } \varphi \in V.
\]

In Section 5 we will also introduce and discuss a corresponding notion of mild $H_\gamma'$-weak solutions and a regularity result similar to Theorem 2.5 (Theorem 5.4).

The reader should be aware that our notion of $H_\gamma'$-weak solution depends on $\gamma$: the smaller $\gamma$ is, the more restrictive the requirements are. In the most restrictive case $\gamma = 0$, that is $H' \cong H$, the definition of $A$ implies that $H$-weak solutions and strong solutions coincide.

It will turn out that the case $\gamma \in [0, 1/2]$ is actually the “natural” condition if one is interested in stability in the space $V = H_{1/2}$. This also gives a reasonable condition if one is interested in stability in the space $H = H_0$, but in this case, conditions which we have to impose will be weaker if one considers larger values of $\gamma < 1$.

Thus, in order to get rid of the restriction $\gamma \leq 1/2$ in Definition 3.4, we consider for $\gamma \in [0, 1]$ the operator $A$ as an operator from $H_{1-\gamma}$ into $H_{-\gamma} = (H_\gamma^*)'$ with domain $D(A)$. Proposition 6.2 will imply that this operator is bounded. Since $D(A)$ is dense in $H_{1-\gamma}$, the operator $A$ has a unique bounded linear extension

\[
A_{\gamma}: H_{1-\gamma} \to H_{-\gamma} = (H_\gamma^*)'
\]

with full domain $D(A_{\gamma}) = H_{1-\gamma}$. Actually, Proposition 6.2 will imply that $A_{\gamma}$ is an isomorphism of $H_{1-\gamma}$ onto $H_{-\gamma}$.
Summarizing, for $\gamma \in [0, 1/2]$, we have the following densely embedded spaces and isomorphisms $A_\gamma$ between them, cf. (3.3):

\[
\begin{align*}
D(A) &= H_1 & V = H_{1/2} & H = H' = H_0 & H_{-\gamma} = (H^*_\gamma)' \\
A = A_0 & & & & \leq H_{1/2} \\
A_1 & & & & \Rightarrow H_{1/2} \\
A_{1/2} & = H_{-1/2} & & & \geq H_{1/2}
\end{align*}
\]

In case $\gamma \in [1/2, 1]$, we extend our view to the right, cf. (3.4):

\[
\begin{align*}
V & = H_{1/2} & H_{1-\gamma} & = H' & H = H' = H_0 & H_{-\gamma} = (H^*_\gamma)' \\
V' & \leq H_{-1/2} & \geq H_{1/2}' & \geq H^*_1 & \Rightarrow H_{1/2}'
\end{align*}
\]

**Remark 3.5.** If $0 \leq \tilde{\gamma} \leq \gamma \leq 1$, we have continuous embeddings $H_{1-\gamma} \subseteq H_{1-\gamma}$ and $(H^*_\gamma)' \subseteq (H^*_\gamma)'$, and so $A_{\tilde{\gamma}}$ is also bounded as an operator from a dense subset of $H_{1-\gamma}$ into $(H^*_\gamma)'$. It follows that $A_{\gamma}$ is the $(H^*_\gamma)'$-realization of $A_{\gamma}$, that is

\[
A_{\gamma} = A_{\gamma}|_{D(A_{\gamma})} \quad \text{and} \quad D(A_{\gamma}) = H_{1-\gamma} = A_{\gamma}^{-1}((H^*_\gamma)').
\] (3.9)

In particular, $A_{\gamma}$ is the $(H^*_\gamma)'$-realization of $A_1$ for every $\gamma \in [0, 1]$, that is

\[
A_{\gamma} = A_1|_{D(A_{\gamma})} \quad \text{and} \quad D(A_{\gamma}) = H_{1-\gamma} = A_1^{-1}((H^*_\gamma)').
\] (3.10)

Since $(H^*_\gamma)' \cong H$, also $A = A_0$ is the $H$-realization of $A_{\gamma}$, that is

\[
A = A_{\gamma}|_{D(A)} \quad \text{and} \quad D(A) = H_1 = A_{\gamma}^{-1}(H) \quad \text{for all} \ \gamma \in [0, 1].
\] (3.11)

Since $A$ is a Kato operator, we have in case $\gamma = 1/2$

\[
(A_{1/2}u, \varphi) = a(u, \varphi) \quad \text{for all} \ u, \varphi \in V,
\] (3.12)

because $a$ is continuous in the sense (2.2). Moreover, in view of (3.11) for $\gamma \in [0, 1/2]$ the operators $A_{\gamma}$ are $H^*_\gamma$-realizations of $A_{1/2}$, that is

\[
A_{\gamma} = A_{1/2}|_{D(A_{\gamma})} \quad \text{and} \quad D(A_{\gamma}) = H_1 = A_{1/2}(H^*_\gamma) \quad \text{for all} \ \gamma \in [0, 1/2].
\] (3.13)

**Definition 3.6.** Let $\gamma \in [0, 1]$. We call $u \in C([t_0, t_1], (H^*_\gamma)')$ a $(H^*_\gamma)'$-weak solution of (2.10) if there is some $f_0: (t_0, t_1) \to (H^*_\gamma)'$ with $f_0 \in L_1((t_0, \tau), (H^*_\gamma)')$ for every $\tau \in (t_0, t_1)$ such that the following holds for every $t \in (t_0, t_1)$: $f_0(t) \in f(t, u(t))$, $u'(t) \in (H^*_\gamma)'$ exists in the sense of the norm of $(H^*_\gamma)'$, $u(t) \in D(A_{\gamma})$, and $u'(t) + A_{\gamma}u(t) = f_0(t)$.

If $\gamma \leq 1/2$, then (3.13) and the density of $H_{\gamma}$ in $V$ imply that the notion of $(H^*_\gamma)'$-weak solutions coincides with the notion of $(H^*_\gamma)'$-weak solutions.

The crucial point of relaxing the acting condition of $f$ is that we can also relax the corresponding continuity hypotheses. We replace (2.12) by

\[
\|f(t, u) - f(s, v)\|_{(H^*_\gamma)'} \leq L \cdot |t - s|^\sigma + \|u - v\|_{H_{\gamma}} \quad \text{for all} \ \{t, u\} \in U_0.
\] (3.14)

Similarly, we call $f$ left-locally bounded into $(H^*_\gamma)'$ if for each $t_1 > t_0$ and each bounded $M \subseteq H_a$ there is some $\varepsilon > 0$ such that $f(U \cap ([t_1 - \varepsilon, t_1] \times M))$ is bounded in $(H^*_\gamma)'$.

Now we can formulate our first main result (generalization of Theorem 2.6).
Theorem 3.7 (Uniqueness, Existence, Maximal Interval). Let $A$ be a Kato operator, and assume (3.2).

1. If $f: U \to (H_\gamma^*)'$ satisfies a right local Hölder-Lipschitz condition in the sense (3.14), then for every $(t_0, u_0) \in U$ and $t_1 \in (t_0, \infty]$ there is at most one $(H_\gamma^*)'$-weak solution $u \in C([t_0, t_1], H_\alpha)$ of (2.10) satisfying $u(t_0) = u_0$.

2. Moreover, such a $(H_\gamma^*)'$-weak solution exists with some $t_1 > t_0$, and if $f$ is left-locally bounded into $(H_\gamma^*)'$, then some maximal $t_1 > t_0$ can be chosen such that either $t_1 = \infty$ or $\|u(t)\|_{H_\alpha} \to \infty$ as $t \to t_1$ or the limit $u_1 = \lim_{t \to t_1} u(t)$ exists in $H_\alpha$ with $(t_1, u_1) \notin U$.

To generalize Theorem 2.7 we note that we assume now $f: U \to 2(H_\gamma^*)'$ so that we have to generalize some notions.

Definition 3.8. In case $\gamma \in [0, 1/2]$, an element $u_0 \in V$ is called a $\gamma$-weak equilibrium of (2.10) if $u_0 \in H_{1-\gamma}$ and if there is $v_0 \in H_{\gamma}^*$ such that $v_0 \in f(t, u_0)$ for every $t > 0$ and

$$a(u_0, \varphi) = (v_0, \varphi) \quad \text{for all } \varphi \in V.$$ 

More generally, in case $\gamma \in [0, 1/2]$, an element $u_0 \in H_{1-\gamma}$ is called a $\gamma$-weak equilibrium of (2.10) if $v_0 := A_\gamma u_0 \in f(t, u_0)$ for every $t > 0$.

If $\gamma \in [0, 1/2]$, the two definitions are equivalent in view of (3.13), since $V$ is dense in $H_\gamma^*$.

Remark 3.9. In view of (3.9), we have for all $0 \leq \tilde{\gamma} \leq \gamma \leq 1$ that each $\tilde{\gamma}$-weak equilibrium is a $\gamma$-weak equilibrium. Conversely, if $u_0$ is a $\gamma$-weak equilibrium and if the above element $v_0 \in f(t, u_0)$ belongs to $(H_{\gamma}^*)'$ then $u_0$ is a $\tilde{\gamma}$-weak equilibrium. Moreover, in view of (3.10), we have for $\tilde{\gamma}$ that “$0$-weak equilibrium” means the same as “equilibrium”. In particular, each equilibrium $u_0$ is a $\gamma$-weak equilibrium, and the converse holds if $v_0 \in (H_{\gamma}^*)' = H$.

We will make the following hypothesis:

$$(B_\gamma): \text{Let } u_0 \text{ be a } \gamma\text{-weak equilibrium, } U_1 \subseteq H_\alpha \text{ an open neighborhood of } u_0 \text{ and } [0, \infty) \times U_1 \subseteq U. \text{ Assume that there is a bounded linear map } B: H_\alpha \to (H_{\gamma}^*)' \text{ such that the function } g(t, u) := f(t, u) + a(u_0, \cdot)|_{H_\gamma} - Bu \text{ satisfies }$$

$$\lim_{\|u\|_{H_\alpha} \to 0} \sup \left\{ \frac{\|v\|_{(H_{\gamma}^*)'} : v \in g((0, \infty) \times \{u\})} \right\} = 0.$$ 

Note that Proposition 3.3(1) implies that $B|_{H_{1-\gamma}} : H_{1-\gamma} \to (H_{\gamma}^*)'$ is bounded.

We will actually present two generalizations of Theorem 2.7. The first generalization looks formally rather analogous to Theorem 2.7, but one has to replace $A$ by $A_\gamma$ and thus obtains a result with a somewhat abstract spectral hypothesis. In order to formulate it, we consider from now on $A_\gamma - B: H_{1-\gamma} \to (H_{\gamma}^*)'$ as an unbounded operator in $(H_{\gamma}^*)'$ with domain $H_{1-\gamma} \subseteq (H_{\gamma}^*)'$.

Theorem 3.10 (Asymptotic Stability). Let $A$ be a Kato operator, and assume (3.2). Let hypothesis $(B_\gamma)$ be satisfied. Suppose that there is $\lambda_0 > 0$ such that every spectral value $\lambda \in \mathbb{C}$ of $A_\gamma - B$ in $(H_{\gamma}^*)'$ satisfies $\text{Re } \lambda > \lambda_0$. Then there exist $M_1, M_2 > 0$ such that if $t_1 > t_0 \geq 0$ and $u \in C([t_0, t_1], H_\alpha)$ is a $(H_{\gamma}^*)'$-weak solution of (2.10) with $\|u(t_0) - u_0\|_{H_\alpha} \leq M_1$, then $u$ satisfies the asymptotic stability estimate (2.13).

If $f$ satisfies in addition the hypotheses of part (I) of Theorem 2.6, then additionally for every $t_0 \geq 0$ and every $u_1 \in H_\alpha$ with $\|u_1 - u_0\| \leq M_1$ a unique $(H_{\gamma}^*)'$-weak solution $u \in C([t_0, \infty), H_\alpha)$ with $u(t_0) = u_1$ exists and satisfies (2.13) with $t_1 = \infty$. 

In a second generalization of Theorem 2.7, we assume that \( B|_{H_{1-\gamma}} : H_{1-\gamma} \to (H^*_\gamma)' \) is compact. In this case, the abstract spectral hypothesis of Theorem 3.10 can be reformulated as a hypothesis about “\( \gamma \)-weak” eigenvalues. The compactness assumption is actually rather natural.

**Remark 3.11.** If the embedding \( i : V \to H \) is compact, then \( B|_{H_{1-\gamma}} : H_{1-\gamma} \to (H^*_\gamma)' \) is compact. Indeed, in this case the embedding \( H_{1-\gamma} \subseteq H_\alpha \) is compact by Proposition 3.14.

We have to define an appropriate notion of eigenvalues.

**Definition 3.12.** We call \( \lambda \in \mathbb{C} \) a \( \gamma \)-weak eigenvalue of \( A-B \) if \( \lambda \) is an eigenvalue of \( A_{\gamma} - B \). In case \( \gamma \in [0,1/2] \), \( \lambda \) is a \( \gamma \)-weak eigenvalue of \( A-B \) with corresponding eigenvector \( u \neq 0 \) if and only if \( u \in H_{1-\gamma} \) and

\[
a(u, \varphi) - (Bu, \varphi) = \lambda(u, \varphi) \quad \text{for all } \varphi \in V.
\]

(3.15)

Indeed, the characterization (3.15) follows from (3.13), since \( V \) is dense in \( (H^*_\gamma)' \) for \( \gamma \in [0,1/2] \). Analogously to Remark 3.9, we obtain from Remark 3.5:

**Remark 3.13.** If \( 0 \leq \tilde{\gamma} \leq \gamma \leq 1 \) and \( \lambda \) is a \( \tilde{\gamma} \)-weak eigenvalue of \( A-B \) then \( \lambda \) is a \( \gamma \)-weak eigenvalue of \( A-B \).

Conversely, if \( \lambda \) is a \( \gamma \)-weak eigenvalue of \( A-B \) with eigenvector \( u \) (hence \( u \in H_{1-\gamma} \)), and if \( Bu \in (H^*_\gamma)' \), then actually \( u \in H_{1-\gamma} \), and thus \( \lambda \) is a \( \tilde{\gamma} \)-weak eigenvalue of \( A-B \).

Moreover, “\( 0 \)-weak eigenvalue” means the same as “eigenvalue”. In particular, each eigenvalue \( \lambda \) of \( A-B \) is a \( \gamma \)-weak eigenvalue of \( A-B \); conversely, if \( \lambda \) is a \( \gamma \)-weak eigenvalue of \( A-B \) with an eigenvector \( u \in H_{1-\gamma} \) satisfying \( Bu \in H \), then actually \( u \in H \) and \( \lambda \) is an eigenvalue of \( A-B \).

Remark 3.13 implies in particular:

**Proposition 3.14.** If \( \gamma \in [0,1] \) and \( B(H_{1-\gamma}) \subseteq H \), then \( \lambda \) is a \( \gamma \)-weak eigenvalue of \( A-B \) with eigenspace \( E \) if and only if \( \lambda \) is an eigenvalue of \( A-B \) with the same eigenspace \( E \), and moreover, \( E \subseteq D(A) \).

**Remark 3.15.** The hypothesis of Proposition 3.14 is in particular satisfied if \( \gamma \in [0,1/2] \) and \( B(V) \subseteq H \), because \( H_{1-\gamma} \subseteq V \).

Now we are in a position to formulate a variant of Theorem 3.10 in terms of eigenvalues instead of spectral values. The constant \( c_0 \) in this result is that from (2.4).

**Theorem 3.16** (Asymptotic Stability with Eigenvalues). Let \( A \) be a Kato operator, and assume (3.2). Let hypothesis (B\( \gamma \)) be satisfied, and suppose that \( B|_{H_{1-\gamma}} : H_{1-\gamma} \to (H^*_\gamma)' \) is compact. Suppose that there is \( \lambda_0 \in (0,c_0) \) such that all \( \gamma \)-weak eigenvalues \( \lambda \in \mathbb{C} \) of \( A-B \) satisfy \( \text{Re} \lambda > \lambda_0 \). Then the hypotheses of Theorem 3.10 are satisfied with that value \( \lambda_0 \).

In particular, there exist \( M_1,M_2 > 0 \) such that if \( t_1 > t_0 \geq 0 \) and \( u \in C([t_0,t_1],H_\alpha) \) is a \( (H^*_\gamma)' \)-weak solution of (2.10) with \( \|u(t_0) - u_0\|_{H_\alpha} \leq M_1 \), then it satisfies the asymptotic stability estimate (2.0).

If \( f \) satisfies in addition the hypotheses of part (1) of Theorem 2.6 then additionally for every \( t_0 \geq 0 \) and every \( u_1 \in H_\alpha \) with \( \|u_1 - u_0\| \leq M_1 \) a unique \( (H^*_\gamma)' \)-weak solution \( u \in C([t_0,\infty),H_\alpha) \) with \( u(t_0) = u_1 \) exists and satisfies (2.13) with \( t_1 = \infty \).
Moreover, these in turn to the fractional power space satisfying (2.2), we have indeed that \( \gamma \) results in the space \( H^{\gamma} \). We do this in two steps. First, we replace \( \gamma \) to (3.11) implies that \( A \) is actually the operator \( A_{1/2} \) of Section 3 but it is more standard and more convenient to denote this operator by \( \mathcal{A} \) in the following. Later on, we define \( \mathcal{A}_\beta \) as an appropriate restriction of \( \mathcal{A} \). It will turn out that actually \( \mathcal{A}_\beta = A_{1-\gamma} \) for \( \beta = \frac{1}{2} - \gamma \in [0,1/2] \), but this is not clear at the moment.

The basic idea for the proof of the main results is rather simple: We just apply the classical results in the space \( H^{\gamma}_\gamma (\subseteq V') \) instead of \( H \). Recall that \( H^{\gamma}_\gamma \cong (H^2_\gamma)' \) for \( \gamma \in [0,1/2] \). Moreover, \( H^{1/2}_\gamma \cong (H^1_\gamma)' \) if \( A \) is a Kato operator.

To implement this idea, we must replace \( A : D(A) \to H \) by some operator \( \mathcal{A}_\beta : D(\mathcal{A}_\beta) \to H^{\beta}_\gamma \). We do this in two steps. First, we replace \( A \) by the operator \( \mathcal{A} : V \to V' \), where for \( v \in V \), the value \( \mathcal{A} v \) is defined by

\[
(\mathcal{A} v, \varphi) = a(v, \varphi) \quad \text{for all } \varphi \in V.
\]  

Note that (3.11) implies that \( \mathcal{A} \) is actually the operator \( A_{1/2} \) of Section 3 but it is more standard and more convenient to denote this operator by \( \mathcal{A} \) in the following. Later on, we define \( \mathcal{A}_\beta \) as an appropriate restriction of \( \mathcal{A} \). It will turn out that actually \( \mathcal{A}_\beta = A_{1-\gamma} \) for \( \beta = \frac{1}{2} - \gamma \in [0,1/2] \), but this is not clear at the moment.

However, the classical results then make only assertions in the corresponding fractional power spaces \( X_{\beta,\alpha} := D(\mathcal{A}_\beta^\alpha) \) while our main results make assertions in the space \( H_\alpha = D(A^\alpha) \). Therefore, we will relate \( X_{\beta,\alpha} \) to the fractional power spaces \( X_\alpha \) to the spaces \( H_\alpha \). Finally, in Section 4.3 we introduce \( \mathcal{A}_\beta \) and relate the spaces \( X_{\beta,\alpha} \) to \( X_\alpha \) and \( H_\alpha \).

4. Extensions of \( A \) and Fractional Powers in Case \( \gamma \in [0,1/2] \)

In this and the next sections, we will only consider the case \( \gamma \in [0,1/2] \). The case \( \gamma \in [1/2,1] \) will be reduced to the former case only in Section 6.

The basic idea for the proof of the main results is rather simple: We just apply the classical results in the space \( H^{\gamma}_\gamma (\subseteq V') \) instead of \( H \). Recall that \( H^{\gamma}_\gamma \cong (H^2_\gamma)' \) for \( \gamma \in [0,1/2] \). Moreover, \( H^{1/2}_\gamma \cong (H^1_\gamma)' \) if \( A \) is a Kato operator.

4.1. Association of \( A \) in \( V' \) with a Form \( a \). Let \( A \) be the operator associated with a sesquilinear form \( a : V \times V \to \mathbb{C} \) satisfying (2.2) and (2.3). Unless stated otherwise, we will not assume in this section that \( A \) is a Kato operator.

Let \( \mathcal{A} : V \to V' \) be the extended operator (1.1). Since \( a \) is a sesquilinear form on \( V \times V \) satisfying (2.2), we have indeed that \( \mathcal{A} : V \to V' \) is linear and bounded by \( C \). Moreover,
the operator $A$ is by its definition exactly the $H$-realization of $A$, that is $D(A) = A^{-1}(H)$, and $A = A|_{D(A)} : D(A) \to H$.

The constants $c$ and $c_0$ in the following assertion are that from (2.3) and (2.4).

**Proposition 4.1.** For every $\lambda$ from the half-plane $\{ z \in \mathbb{C} : \text{Re} \, z < c_0 \}$ the map $L_u := Au - \lambda u$ is an isomorphism of $V$ onto $V'$, the inverse being bounded by $M_\lambda = c^{-1}c_0/(c_0 - \text{Re} \, \lambda)$.

**Proof.** Putting $\varepsilon := c_0 - \text{Re} \, \lambda$, we obtain from (2.4) that $\varepsilon |u|^2 \leq \text{Re} \,(Lu, u)$. Together with (2.3), we find

$$c\|u\|^2 \leq \text{Re} \,(u, u) + c_0|u|^2 \leq (1 + c^{-1}\varepsilon)\text{Re} \,(Lu, u).$$

Since $\text{Re} \,(Lu, u) \leq \|Lu\|_{V'}\|u\|$, it follows that the bounded map $L : V \to V'$ is an isomorphism onto its range $R \subseteq V'$ with inverse bounded by $M_\lambda$. In particular, $R$ is isomorphic to $V$ and thus a Banach space, hence closed in $V'$. Since $R$ contains the range of $A - \lambda id$, which is $H$ by Proposition 2.2 and since $H$ is dense in $V'$, we conclude that $R = V'$. \( \square \)

Unless stated otherwise, we will from now on consider $A$ as an unbounded operator in $V'$ with domain $D(A) = V$. We equip $D(A)$ with the norm

$$\|u\|_{D(A)} := \|Au\|_{V'},$$

which is equivalent to the graph norm of $A$ in $V'$.

In Remark 1.8 we will give a rather short proof of the following result if $A$ is a Kato operator. For the general case, we refer to the literature, instead. The constants $c_0$ and $\delta$ in the following result are that from (2.4) and (2.5), respectively.

**Proposition 4.2.** The operator $A$ is closed and densely defined in $V'$ with $D(A) \cong V$. The operator $-A$ is the generator of an analytic uniformly bounded $C_0$ semigroup in $V'$. The restriction of this semigroup to $H$ is the semigroup generated by $-A$. The spectrum of $A$ in $V'$ is contained in the intersection of the sector (2.7) with the half-plane (2.8). Here, $\delta$ is from (2.5), and $c_0$ is from (2.4). Moreover, $A$ is positive (of positive type) in the sense of [27] or [7], respectively, and sectorial in the sense of [11][24].

**Proof.** Since the embeddings in (2.1) are dense and continuous and $A : V \to V'$ is bounded, it follows that $A$ is densely defined and that the graph norm of $A$ is equivalent to the norm of $V$, hence $A$ is closed in $V'$ by [3] Lemma I.1.1.2. Proposition 1.1 implies that $A$ is of positive type and has its spectrum contained in (2.8). The assertions concerning the semigroup and the spectrum of $A$ are contained in [23] Theorem 1.55 and subsequent remarks]. \( \square \)

Our aim is now to show that $A$ is the operator associated with a strongly accretive continuous sesquilinear form $a$ on $H$. In order to give a meaning to this, we first have to equip $V'$ with an appropriate scalar product.

Our idea for this is to fix a scalar product $b$ on $V$ which generates a norm $\|u\|_b := \sqrt{b(u, u)}$ on $V$ equivalent to $\| \cdot \|$. Note that Proposition 1.1 implies that $\|u\|_X := \|A^{-1}u\|_b$ then defines an equivalent norm in $V'$. We denote by $X^{a,b}$ the Hilbert space which we obtain from $V'$ when we pass to this equivalent norm which is induced by the scalar product

$$[u, v]_{a,b} := b(A^{-1}u, A^{-1}v) \quad \text{for all } u, v \in X^{a,b}. \quad (4.2)$$

Note that for any choice of $(a, b)$ as above we have $X^{a,b} \cong V'$. However, it is crucial for our approach to distinguish the various scalar products.

Indeed, the following result characterizes those scalar products $b$ on $V$ for which a form $a$ on $H$ with the required properties exists:
Proposition 4.3. The following assertions are equivalent for every $c_1 > 0$.

1. There exists a sesquilinear form $a: H \times H \to \mathbb{C}$ such that there are constants $c_2, c_3 \geq 0$ with
   $$\Re a(u, u) \geq c_1|u|^2, \quad |a(u, v)| \leq c_2 |u||v|, \quad |a(u, u)| \leq c_3|u|^2$$
   for all $u, v \in H$, and
   
   $$a(u, v) = [Au, v]_{a,b} \quad \text{for all } u \in V, v \in H.$$  \hspace{1cm} (4.3)

2. There is $c_2 \geq 0$ with
   $$\Re b(u, A^{-1}u) \geq c_1|u|^2 \quad \text{and} \quad |b(u, A^{-1}v)| \leq c_2 |u||v| \quad \text{for all } u, v \in D(A).$$
   \hspace{1cm} (4.5)

3. There is $c_3 \geq 0$ with
   $$\Re b(u, A^{-1}u) \geq c_1|u|^2 \quad \text{and} \quad |b(u, A^{-1}u)| \leq c_3|u|^2 \quad \text{for all } u \in D(A).$$
   \hspace{1cm} (4.6)

The smallest possible constants $c_2, c_3$ in the above assertions are respectively the same. If $\text{(1)}$ holds, then $a$ is the unique continuous function $a: H \times H \to \mathbb{C}$ satisfying (4.4), and $A$ is the operator associate to $a$ in the Hilbert space $X^{a,b}$, that is, $u \in H$ belongs to $D(A) = V$ if and only if there is some $w \in X^{a,b}$ with $a(u, \varphi) = [w, \varphi]_{a,b}$ for all $\varphi \in H$.

Proof. \textbf{“(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (2)”}: Let $a_1: V \times V \to \mathbb{C}$ be the sesquilinear form defined by $a_1(u, v) = [Au, v]_{a,b}$. Due to (4.4), we have $a_{1V \times V} = a_1$. Since $V$ is dense in $H$, $a$ is uniquely determined by its restriction to $V \times V$, and moreover, there is at most one continuous function $a: H \times H \to \mathbb{C}$ with $a|_{V \times V} = a_1$.

The operator $\mathcal{L}$ associated with the form $a$ is the Friedrichs extension of $A$. Since $A$ is sectorial in the sense of [11], it follows from [14, Theorem VI.2.9] that $\mathcal{L} = A$.

Using (4.2), the fact that $A$ is the $H$-realization of $\mathcal{A}$, and (4.1), we calculate

$$a_1(u, v) = b(u, A^{-1}v) = b(u, A^{-1}v) \quad \text{for all } u, v \in V.$$  \hspace{1cm} (4.7)

Hence, (4.3) implies (4.5) and (4.6) even for all $u, v \in V$ (with the same constants $c_2, c_3$). Clearly, (4.5) implies (4.6) with $c_3 := c_2$. Conversely, if (4.6) holds, then $|a_1(u, u)| \leq c_3|u|^2$ for all $u \in D(A)$, and an application of the polarization identity (14) (VI.1.1) for the sesquilinear form $a_1$ shows that $a_1$ is bounded with respect to the norm $|\cdot|$. This means that (4.5) holds with some $c_2 \geq 0$.

\textbf{“(2) $\Rightarrow$ (1)”}: If (4.5) holds then, since $D(A)$ is dense in $V$ and since the left-hand side is continuous in $V$, we obtain that (4.5) holds even for all $u, v \in V$. Hence, by (4.7),

$$\Re a_1(u, u) \geq c_1|u|^2 \quad \text{and} \quad |a_1(u, v)| \leq c_2 |u||v| \quad \text{for all } u, v \in V.$$  \hspace{1cm} (4.7)

Since $V$ is dense in $H$, it follows that the sesquilinear form $a_1$ has a continuous extension $a: H \times H \to \mathbb{C}$, and $a$ satisfies (4.3) (with the same constants $c_1, c_2 > 0$).

Proposition 4.3 motivates the following definition.

Definition 4.4. We call a scalar product $b$ on $V$ an $A$-Kato scalar product if the norm induced by $b$ is equivalent to $\| \cdot \|$, and if (4.5) or, equivalently, (4.6) are satisfied with some $c_1 > 0$.

Corollary 4.5. If $a$ is symmetric, then $b := a$ is an $A$-Kato scalar product on $V$ for which $a$ in Proposition 4.3 is the scalar product $a(u, v) = (u, v)$ in $H$.

Proof. It suffices to prove that assertion (1) of Proposition 4.3 holds. We calculate for $a_1$ from the proof of Proposition 4.3 for all $u, v \in V$, using (4.2) and (4.1), that

$$a_1(u, v) = [v, Au]_{a,a} = a(A^{-1}v, A^{-1}Au) = (A^2u, A^{-1}v) = (u, v).$$
Hence, \( a_1(u, v) = (u, v) \) for all \( u, v \in V \), and by continuity and density \( a_1 \) has a unique continuous extension \( a(u, v) = (u, v) \) for all \( u, v \in H \).

\[ \square \]

**Remark 4.6.** Corollary [15] may appear rather surprising: The form \( a \) generating \( A \) (and thus also defining \( A \)) is actually independent of \( a \)! However, the explanation for this apparent contradiction is that the scalar product in \( X^{\alpha, \beta} \cong V' \) heavily depends on \( a \), of course (although the generated topology is independent of \( a \)).

We will see that, in the general case where \( a \) is not necessarily symmetric, the existence of an \( A \)-Kato scalar product \( b \) on \( V \) is actually equivalent to the assertion that \( A \) is a Kato operator. Let us first show that if \( A \) is a Kato operator, then such a scalar product \( b \) does exist (the converse is shown in Theorem [A.2]).

**Lemma 4.7.** The following two assertions are equivalent:

1. \( A \) is a Kato operator.
2. \( A^{1/2} \) is an isomorphism of \( V \) onto \( H \).

If they are satisfied, then \( b(u, v) := (A^{1/2}u, A^{1/2}v) \) defines an \( A \)-Kato scalar product on \( V \).

**Proof.** Since \( A \) is one-to-one and onto, it follows that also \( A^{1/2} : D(A^{1/2}) \to H \) is one-to-one and onto, hence an isomorphism. Now the equivalence of (1) and (2) is straightforward. If (1) and (2) hold, then there are constants \( C_1, C_2 > 0 \) with

\[
C_1|A^{1/2}v| \leq ||v|| \leq C_2|A^{1/2}v| \quad \text{for all} \quad v \in V \cong D(A^{1/2}).
\]

Hence, in this case we calculate for every \( u \in V \), noting that \( u \in H \) and thus \( v := A^{-1/2}u \in D(A) \), that

\[
b(u, A^{-1}u) = (A^{1/2}u, A^{1/2}A^{-1}u) = (Av, v) = a(v, v)
\]

and \( |A^{1/2}v| = |u| \). This implies on the one hand that

\[
\text{Re} \ b(u, A^{-1}u) \geq c||v|| \geq C_1c|A^{1/2}v| = C_1c|u|,
\]

and on the other hand

\[
|b(u, A^{-1}u)| \leq C||v||^2 \leq C_2^2C|A^{1/2}v| = C_2^2C|u|^2.
\]

Hence, [4.10] holds with \( c_1 := C_1c \) and \( c_3 := C_2^2C \). \[ \square \]

**Remark 4.8.** If \( A \) is a Kato operator, then Lemma [4.7] and Proposition [4.3] imply that \( A \) is induced by a strongly accretive continuous sesquilinear form \( a \) in \( H \). Hence, if \( A \) is a Kato operator a rather simple proof of Proposition [4.2] follows by just applying Proposition [2.2] with \((A, a, V, H)\) replaced by \((A, a, H, X^{\alpha, \beta})\). Moreover, in this case, we can even conclude that the semigroup generated by \(-A\) is even a (strictly) contraction semigroup when \( X^{\alpha, \beta} \) is equipped with the norm corresponding to the scalar product [4.2].

#### 4.2. Fractional Powers of \( A \)

It follows from Proposition [14] that, for every \( \beta > 0 \), one can define \( A^\beta \) on a corresponding domain \( D(A^\beta) \subseteq V' \). We equip \( X_\beta := D(A^\beta) \) with the norm \( ||u||_{X_\beta} = ||A^\beta||_{V'} \) which is equivalent to the graph norm of \( A^\beta \) in \( V' \). The next step of our approach is to relate the spaces \( H_\alpha \) and \( X_\beta \). We can do this for Kato operators.

Thus, from now on, we assume that \( A \) is a Kato operator and thus that there is an \( A \)-Kato scalar product \( b \) on \( V \) (Lemma [17]). We equip \( X^{\alpha, \beta} \cong V' \) with the corresponding scalar product [4.2].

**Proposition 4.9.** We have \( X_0 = V' \), \( X_1 = D(A) \cong V \), \( A^n \) is the nth iterate of \( A \) for \( n = 1, 2, \ldots \), and if \( A \) is a Kato operator, then the following holds:

1. \( X_{\alpha(1-\theta)+\beta} \cong [X_\alpha, X_\beta]_\theta \) for every \( \alpha, \beta \geq 0 \) and \( \theta \in (0, 1) \).
(2) $X_{1/2} \cong H$.

(3) For every $\beta \in [0, 1/2]$ the embeddings $V \subseteq X_{1-\beta} \subseteq H$ are continuous, and there is a canonical identification $X_{1-\beta}' = X_{\beta}$ induced on the dense subset $V$ by the scalar product $(\cdot, \cdot)$.

**Proof.** The first assertions follow from Proposition 2.3 with the form $a$ of Proposition 4.3 and with the spaces $(V, H)$ replaced by $(H, X_{a,b})$.

For the remaining assertions (2) and (3), it thus suffices to show that $[V', V]_{1/2} \cong H$, that for every $\beta \in [0, 1/2]$ the embeddings $V \subseteq [V', V]_{1-\beta} \subseteq H$ are dense and continuous, and that $[V', V]_{1-\beta} \cong [V', V]_\beta$ for every $\beta \in [0, 1/2]$. Since these assertions are actually independent of our choice of $a$, we can assume without loss of generality that $a$ is symmetric, since otherwise we can replace $a$ by the symmetric form $(2.6)$ (with $M = 0$).

According to Corollary 4.5 we can in this case choose $b := a$ and obtain that $a$ is symmetric. Applying Proposition 2.3 with the symmetric form $a$ (and with the spaces $(V, H)$ replaced by $(H, X_{a,b})$), we obtain that the induced operator $A$ satisfies $X_\beta = D(A^\beta) \cong [V', V]_\beta$ for every $\beta \in (0, 1)$ and $X_{1/2} = D(A^{1/2}) \cong H$. In particular, for $\beta \in [0, 1/2]$ we have continuous dense embeddings $V \cong X_1 \subseteq X_{1-\beta} \subseteq X_{1/2} \cong H$. Since $a$ is symmetric, Proposition 2.3 applied with the spaces $(V, H)$ replaced by $(H, X_{a,b} \cong V')$ implies that $A$ is selfadjoint in $X_{a,b}$. In particular, we have for all $u, v \in V$ that

$$(u, v) = [a(u, v)]_{a,b} = [A^{1-\beta} u, A^\beta v]_{a,b}.$$ 

Taking absolute values on both sides and the supremum over all $v \in V$ from the unit ball of $X_\beta$, we find, since $V \cong X_1 \subseteq X_\beta$ is densely embedded, that $\|u\|_{X_\beta} = \|u\|_{X_{1-\beta}}$ for all $u \in V$.

Using that the embeddings $V \cong X_1 \subseteq X_{1-\beta}$ and $V \subseteq X_\beta \subseteq H = H' \subseteq X'_\beta$ are dense (by the same arguments as in Proposition 3.3), we obtain the assertion. \qed

The first assertion of Proposition 4.9 was obtained independently in [3] by using a somewhat different argument, based on the result of [6, Section 5.5.2].

**Proposition 4.10.** If $A$ is a Kato operator, then $X_{\alpha+\frac{1}{2}} \cong H_\alpha$ for all $\alpha \geq 0$.

**Proof.** The case $\alpha = 0$ is contained in Proposition 4.9. For $\alpha = n = 1, 2, \ldots$, the set $X_{\alpha+\frac{1}{2}}$ consists of all $\varphi \in X_{a,b}$ such that $A^n \varphi = A^{n+1} \varphi = A^{n+1/2} \varphi$ is defined, that is, $A^n \varphi \in X_{1/2} \cong H$. Since $A$ is the $H$-realization of $A$, this means $\varphi \in D(A^n) = H_\alpha$, and taking norms in the previous expressions, we see that also the corresponding (graph) norms are equivalent. Hence, the assertion is proved for all integer numbers $\alpha \geq 0$. For non-integer $\alpha \geq 0$, let $n \geq 0$ denote the unique integer with $\theta := \alpha - n \in (0, 1)$. Then also $m := n + 1 \geq 0$ is an integer, and $(n + r)(1 - \theta) + (m + r)\theta = \alpha + r$ for every $r$. Hence, Proposition 4.9 the integer case just proved, and Proposition 2.3 imply

$$X_{\alpha+\frac{1}{2}} \cong [X_{n+\frac{1}{2}}, X_{n+\frac{1}{2}}]' \cong [H_n, H_m]' \cong H_\alpha,$$

so that the assertion follows also in the non-integer case. \qed

**Remark 4.11.** For Propositions 4.9 and 4.10 we actually did not use that $A$ is a Kato operator but only that there is an $A$-Kato scalar product on $V$.

**Remark 4.12.** If $0 < \gamma < 1/2$ and $A$ is a Kato operator, then Propositions 4.9 and 4.10 imply

$$H'_\gamma \cong X'_{\frac{1}{2}+\gamma} \cong X_{\frac{1}{2}-\gamma}, \quad X_{\frac{1}{2}+\gamma} \cong [X_0, X_{\frac{1}{2}+\gamma}], \quad X_{\frac{1}{2}-\gamma} \cong [X_0, X_{\frac{1}{2}-\gamma}],$$

$$X_{\frac{1}{2}-\gamma} \cong [X_0, X_{1/2}]_{1-2\gamma}, \quad X_1 \cong V, \quad X_0 \cong V', \quad X_{1/2} \cong H.$$

Combining these equalities, we obtain the remaining assertions of (3.5).
4.3. The Operator $\mathcal{A}_\beta$ and its Powers. The previous information about the operator $\mathcal{A}$ is already sufficient to prove the assertions of Section 3 in the case $\gamma = 1/2$, that is, when $H'_\gamma = V'$. However, in order to treat the general case, we need some more information about the fractional powers of $\mathcal{A}$ and $A$.

For $\beta \geq 0$, we denote by $\mathcal{A}_\beta$ the $X_\beta$-realization of $\mathcal{A}$, that is, $\mathcal{A}_\beta = \mathcal{A}|_{D(\mathcal{A}_\beta)} : D(\mathcal{A}_\beta) \to X_\beta$ where $D(\mathcal{A}_\beta) := A^{-1}(X_\beta)$ is equipped with the norm

$$\|u\|_{D(\mathcal{A}_\beta)} = \|\mathcal{A}_\beta u\|_{X_\beta}$$

which is equivalent to the graph norm of $\mathcal{A}_\beta$ in $X_\beta$ (and in case $\beta = 0$ it is equivalent to the norm previously defined for $D(A) = D(A_0)$).

Our first aim is to determine $D(\mathcal{A}_\beta)$.

**Lemma 4.13.** $\mathcal{A}_\alpha|_{H_\alpha} = A^\alpha$ for all $\alpha \geq 0$.

**Proof.** $A$ is the $H$-realization of $\mathcal{A}$: $V \to V', \ V \subseteq H$. Hence, the assertion follows from Proposition 13.2.  

**Lemma 4.14.** Let $A$ be a Kato operator. Then $A^{1/2}A^{1/2} = A$, and for all $\beta \in [0, 1/2]$ we have $D(\mathcal{A}_\beta) \cong H_{\beta + \frac{1}{2}}$ and $\mathcal{A}_\beta = A^{1/2}A^{1/2}|_{D(\mathcal{A}_\beta)}$ is the composition of the two isomorphisms $A^{1/2}|_{D(\mathcal{A}_\beta)} : D(\mathcal{A}_\beta) \to H_\beta$ and $A^{1/2}|_{H_\beta} : H_\beta \to X_\beta$.

**Proof.** Since $A$ is a Kato operator, $H_{1/2} \cong V$, and so Lemma 4.13 implies $A^{1/2} = \mathcal{A}|_{H_{1/2}}$. Hence, $\mathcal{A}^{1/2}A^{1/2} = \mathcal{A}^{1/2}A^{1/2}|_{V} = A$ by [27, Theorem 1.15.2(c)]. By [27, Theorem 1.15.2(e)], the operators $\mathcal{L}_0 := A^{1/2}|_{H_{\beta + \frac{1}{2}}} : H_{\beta + \frac{1}{2}} \to H_\beta$ and $\mathcal{L}_1 := A^{1/2}|_{X_{\beta + \frac{1}{2}}} : X_{\beta + \frac{1}{2}} \to X_\beta$ are isomorphisms. Since $H_\beta \cong X_{\beta + \frac{1}{2}}$ by Proposition 4.10 we obtain that $\mathcal{L}_1\mathcal{L}_0 = \mathcal{A}|_{H_{\beta + \frac{1}{2}}}$ is an isomorphism of $H_{\beta + \frac{1}{2}}$ onto $X_\beta$. Hence, $\mathcal{L}_1\mathcal{L}_0 = \mathcal{A}_\beta$, and thus $H_{\beta + \frac{1}{2}} \cong D(\mathcal{A}_\beta)$.

Our next aim is to prove that $\mathcal{A}_\beta$ is an operator associated with some strongly accretive continuous sesquilinear form.

To this end, we assume again that $A$ is a Kato operator, and so we can equip $V$ with an $A$-Kato scalar product $b$ (Lemma 4.7). We equip $X^{a,b} \cong V'$ with the corresponding scalar product (4.2). Moreover, we let $a : V \to V \to \mathbb{C}$ be the sesquilinear form of Proposition 4.3.

For $\beta \in [0, 1/2]$, we equip $X_\beta$ with the scalar product

$$b_\beta(u,v) := \langle A^\beta u, A^\beta v \rangle_{a,b},$$

and we consider the particular sesquilinear form

$$a_\beta(u,v) := a(A^\beta u, A^\beta v).$$

These are indeed the required scalar product and the required sesquilinear form.

**Proposition 4.15.** Let $A$ be a Kato operator and $\beta \in [0, 1/2]$. Then (4.8) defines a scalar product on $X_\beta$ which generates an equivalent norm. Moreover, $a_\beta := H_\beta \times H_\beta \to \mathbb{C}$ is a sesquilinear form satisfying

$$|a_\beta(u,v)| \leq C_1\|u\|_{H_\beta}\|v\|_{H_\beta} \quad \text{and} \quad \text{Re}a_\beta(u,u) \geq C_2\|u\|_{H_\beta}^2,$$

where $C_1, C_2 > 0$ are independent of $u, v \in H_\beta$. The operator associated with $a_\beta$ in the Hilbert space $X_\beta$ is $\mathcal{A}_\beta$.

**Proof.** Recall that Proposition 4.10 implies $Z := X_{\beta + \frac{1}{2}} \cong H_\beta$. According to [27, Theorem 1.15.2(e)], $A^\beta$ is an isomorphism of $X_\beta$ onto $V' \cong X^{a,b}$, and $A^\beta|_Z$ is an isomorphism of $Z$ onto $X_{1/2} \cong H$. The former implies that $b_\beta$ is well-defined and that the induced norm is
equivalent to the norm of $X_\beta$. The latter implies that $\|u\|_Z := |A^2u|$ defines an equivalent norm on $Z \cong H_\beta$. For $u, v \in Z$, we obtain with $c_1, c_2$ as in Proposition 4.3 that

$$\text{Re } a_\beta(u, u) \geq c_1 \|u\|_Z \quad \text{and} \quad |a_\beta(u, v)| \leq c_2 \|u\|_Z \|v\|_Z$$

for all $u, v \in Z$, and so the assertions about $a_\beta$ follow.

It remains to show that $A_\beta$ coincides with the operator $L$ associated with the form $a_\beta$ in the Hilbert space $X_\beta$. The domain $D(L)$ consists by definition of all $u \in H_\beta$ for which there is $v = Lu \in X_\beta$ with

$$a_\beta(u, \varphi) = b_\beta(v, \varphi) \quad \text{for all } \varphi \in H_\beta. \quad (4.9)$$

By definition of $a_\beta$ and $b_\beta$, $(4.9)$ is equivalent to

$$a(A^\beta u, A^\beta \varphi) = [A^2 v, A^2 \varphi]_{a,b} \quad \text{for all } \varphi \in H_\beta.$$

Since $A^\beta$ is an isomorphism of $H_\beta$ onto $H$ by [27, Theorem 1.15.2(e)], and since $A^\beta = A^\beta|_{H_\beta}$ by Lemma 4.13, we find that $(4.9)$ is actually equivalent to

$$a(A^\beta u, \psi) = [A^2 v, \psi]_{a,b} \quad \text{for all } \psi \in H.$$

Since $A$ is the operator induced by $a$, we thus find that

$$D(L) = \{u \in H_\beta : A^\beta u \in D(A) \cong V\} \quad \text{and} \quad A^\beta Lu = AA^\beta u \quad \text{for all } u \in D(L). \quad (4.10)$$

Since $A^\beta|_{H_{\beta+\frac{1}{2}}} = A^\beta|_{H_{\beta+\frac{1}{2}}}$ is an isomorphism of $H_{\beta+\frac{1}{2}} \cong D(A^\beta)$ onto $V$ by [27, Theorem 1.15.1(e)], the first equality of $(4.10)$ implies $D(L) = D(A^\beta)$. Moreover, since $A^\beta$ is an isomorphism and $AA^\beta u = A^{1+\beta} u = A^\beta Au$ by [27, Theorem 1.15.1(c)], the second equality of $(4.10)$ implies $Lu = A^\beta u$ for all $u \in D(L)$. Summarizing, we have shown $L = A_\beta$. \[\square\]

**Proposition 4.16.** Let $A$ be a Kato operator and $\beta \in [0, 1/2]$. Then $A_\beta$ is a closed densely defined operator in $X_\beta$ with spectrum contained in the spectrum of $A$ in $V'$. Moreover, $A_\beta$ is positive (of positive type) in the sense of [27] or [4], respectively, and sectorial in the sense of [17][24], and $-A_\beta$ is the generator of an analytic (strict) contraction $C_0$ semigroup $e^{-tA_\beta}$ in $X_\beta$ (equipped with the norm generated by the scalar product $(4.3)$). Moreover, $e^{-tA_\beta} = e^{-tA}|_{X_\beta}$ for all $t \geq 0$.

**Proof.** In view of Proposition 4.15 most assertions follow from Proposition 2.2. The only nontrivial assertion is that the spectrum of $A_\beta$ is contained in the spectrum of $A$. Thus, assume that $\lambda id - A$ has a bounded inverse $R$ in $V'$. Then $R|_{X_\beta}$ is the inverse of $A_\beta$. Indeed, if $u = Rv$ for some $v \in X_\beta$, then $u \in V \subseteq X_\beta$ and thus $Au = v + \lambda u \in X_\beta$ which implies $u \in D(A_\beta)$. Since $\lambda id - A_\beta$ is closed, it follows that also the inverse $R|_{X_\beta}$ is closed and thus bounded by the closed graph theorem. The last assertion follows from [11, Theorem 1.3.4] and the definition of $A_\beta$. \[\square\]

**Remark 4.17.** A rather different approach to prove that a restriction of $-A$ generates an analytic semigroup in $X_\beta$ is by considering $A_\beta$ as an operator interpolating between $A = A|_{D(A)}$ in $H$ and $A$ in $V'$. In fact, the inequalities needed for the required resolvent estimates follow rather straightforwardly from the corresponding estimates for $-A$ and $-A$ in view of the interpolation inequalities, see e.g. the proof of [10, Lemma 5.3].

This approach has the advantage that it carries over to Banach spaces and was therefore successfully employed in e.g. [10] to obtain regularity results in (non-Hilbert) Sobolev spaces. In the Hilbert space case, however, our above approach gives more insight, e.g., it shows that $A_\beta$ comes from an appropriate sesquilinear form. Nevertheless, parts of such insight might also be obtained differently: For instance, one might combine [7, Theorem 3.3]
with \cite{9} and \cite{19} Theorem 1.15.1] to find that there is some equivalent scalar product such that the interpolation operator $A_{\beta}$ is associated with an appropriate sesquilinear form.

In view of Proposition 4.16 we can define fractional powers $A_{\beta}^\alpha$ for every $\alpha \geq 0$. We equip $X_{\beta,\alpha} := D(A_{\beta}^\alpha)$ with the norm
\[
\|u\|_{X_{\beta,\alpha}} := \|A_{\beta}^\alpha u\|_{X_{\beta}}
\]
which is equivalent to the graph norm of $A_{\beta}^\alpha$ in $X_{\beta}$.

**Proposition 4.18.** Let $A$ be a Kato operator, and $\beta \in [0, 1/2]$. Then we have for all $\alpha \geq 0$ that $X_{\beta,\alpha} \cong X_{\beta+\alpha}$ and $A_{\beta}^\alpha = A^{\alpha}|_{X_{\beta,\alpha}}$. For all $\alpha \geq \frac{1}{2} - \beta$ there holds $X_{\beta,\alpha} \cong H_{\beta+\alpha-\frac{1}{2}}$.

**Proof.** Since $A_{\beta}$ is the $X_{\beta}$-realization of $A$ and since $D(A) \cong X_1 \subset X_{\beta}$, we obtain from Proposition 3.2 that $A_{\beta}^\alpha = A^{\alpha}|_{X_{\beta,\alpha}}$ and $X_{\beta,\alpha} \cong (A^{\alpha})^{-1}(X_{\beta})$. According to \cite[Theorem 1.15.2(e)\textcopyright]{27}, $B := A^{\alpha}|_{X_{\beta+\alpha}}$ is an isomorphism of $X_{\beta+\alpha}$ onto $X_{\beta}$, and $A_{\beta}^\alpha$ is an isomorphism of $X_{\beta,\alpha}$ onto $X_{\beta}$. Hence, $B^{-1} A_{\beta}^\alpha = i d_{X_{\beta,\alpha}}$ is an isomorphism of $X_{\beta,\alpha}$ onto $X_{\beta+\alpha}$. This proves the first assertion, and the second follows in view of Proposition 4.10. \qed

## 5. Proofs of the Main Results in the Case $\gamma \in [0, 1/2]$

Throughout this section, we use the notation of Section 4. For $\alpha, \gamma$ satisfying (3.2) and $\gamma \leq 1/2$, and when $A$ is a Kato operator, we put in the following

\[
\beta := \frac{1}{2} - \gamma \in [0, 1/2],
\]

and consider the problem
\[
u'(t) + A_{\beta} u(t) \in f(t, u(t))
\]

in the space $H'_{\gamma} \cong X'_{1-\gamma} \cong X_{\beta}$ (recall Propositions 4.9 and 4.10).

**Remark 5.1.** In view of Lemma 4.14 and (5.1), we have $D(A_{\beta}) \cong H_{\beta+\frac{1}{2}} = H_{1-\gamma}$, and $A_{\beta}$ is an isomorphism of $H_{1-\gamma}$ onto $X_{\beta} \cong H'_{\gamma}$.

Strong and mild solutions of (5.2) are defined analogously to Definition 2.4 (with $(A, H)$ replaced by $(A_{\beta}, X_{\beta})$).

**Remark 5.2.** By Proposition 4.16, $u$ is a mild solution of (5.2) on $[t_0, t_1]$ if and only if there is $f_0 \colon (t_0, \tau) \to H'_{\gamma}$ with $f_0 \in L_1((t_0, \tau), H'_{\gamma})$ for every $\tau \in (t_0, t_1)$ such that the following holds for every $t \in (t_0, t_1)$: $f_0(t) \in f(t, u(t))$, and
\[
u(t) = e^{-(t-t_0)A_{\beta} u(t)} + \int_{t_0}^{t} e^{-(t-s)A_{\beta}} f_0(s) \, ds.
\]

**Proposition 5.3.** A function $u$ is a $H'_{\gamma}$-weak solution of (2.10) if and only if it is a strong solution of (5.2) in $H'_{\gamma}$.

**Proof.** If $u$ is a strong solution of (5.2), then it is a $H'_{\gamma}$-weak solution of (2.10) by definition of $A_{\beta} = A|_{D(A_{\beta})}$. Conversely, if $u$ is a $H'_{\gamma}$-weak solution of (2.10) on $(t_0, t_1)$, then it is a $V'$-weak solution, and for all $t \in (t_0, t_1)$ we have that $u'(t)$ exists (with the same value) even in $H'_{\gamma}$. In particular, $A u(t) = f_0(t) - u'(t) \in H'_{\gamma} \cong X_{\beta}$ and thus $u(t) \in A^{-1}(X_{\beta}) = D(A_{\beta})$, and $A_{\beta} u(t) = A u(t) = f_0(t) - u'(t)$ for all $t \in (t_0, t_1)$. \qed

Hence, the following generalization of Theorem 2.5 makes sense:
Theorem 5.4 (Regularity). Let \( A \) be a Kato operator, and assume (3.2) and \( \gamma \leq 1/2 \). Then every \( H'_{\gamma} \)-weak solution of (2.10) is a mild solution of (5.2), and the converse holds if \( f_0 \in L^p((t_0,t_1), H'_{\gamma}) \), with \( p > 1/(1-\alpha-\gamma) \) and \( f_0 \in L^p((t_0,t_1), H'_{\gamma}) \) for \( p > 1/(1-\alpha-\gamma) \). Moreover, if \( u : [t_0,t_1] \rightarrow V' \) satisfies (5.3) for all \( t \in (t_0,t_1) \), and if for every \( \tau < t_1 \) there is \( p > 1/(1-\alpha-\gamma) \) with \( f_0 \in L^p((t_0,\tau), H'_{\gamma}) \), then \( u : (t_0, t_1) \rightarrow H_{\alpha} \) is locally H"older continuous, and \( u \in C([t_0,t_1], H_{\alpha}) \) if and only if \( u(t_0) \in H_{\alpha} \).

Theorem 5.5. Theorems 3.7 and 3.10 in case \( \gamma \leq 1/2 \) hold also when we replace in the assertions "\( H'_\gamma \)-weak solution" by "mild solution of (5.2)". An analogous assertion holds also for Theorem 3.16.

Proof of Theorems 3.7, 3.10, 5.4 and of the first part of Theorem 5.5 in case \( \gamma \leq 1/2 \). Applying Theorem 2.5, we find that every strong solution of (5.2) is a mild solution of (5.2). In view of Proposition 5.3, this proves the first assertion of Theorem 5.4.

For the remaining assertions, we put \( \alpha_0 := \alpha + \frac{1}{2} - \beta = \alpha + \gamma \). The result follows by applying Theorems 2.5, 2.6, and 2.7 to the problem (5.2) instead of (2.10), that is, with \((A,a,V,H,H_a)\) replaced by \((A_\beta,a_\beta,H_\beta,X_\beta,X_\beta,\alpha_0)\). Note that we have indeed \( \alpha_0 \in [0,1) \), \( H'_{\gamma} \cong X_\beta \) by Proposition 4.10, and \( X_{\beta,\alpha_0} \cong H_{\alpha} \) by Proposition 4.18.

Proof of Theorem 3.10 and of the second part of Theorem 5.5 in case \( \gamma \leq 1/2 \). We show that under the hypotheses about \( B \) and \( \lambda_0 \) in Theorem 3.10 the spectral hypothesis of Theorem 5.10 is satisfied. In other words, we have to show that \( \text{Re} \lambda \leq \lambda_0 \) implies that \( \lambda \) is not a spectral value of \( A_\beta - B \) in \( X_\beta \).

Since \( \lambda_0 < \alpha_0 \), \( \lambda \) does not belong to the half-plane (2.5), and so \( \mathcal{L} := \lambda id - A_\beta \) has a bounded inverse in \( X_\beta \) by Propositions 4.10 and 4.1. Since \( \mathcal{L} \) is a bounded injective operator of \( Y := D(A_\beta) \) onto \( X_\beta \cong H'_{\gamma} \), the bounded inverse theorem implies that \( \mathcal{L}^{-1} : H'_{\gamma} \rightarrow Y \) is bounded. Since \( Y \cong H_{\beta+\frac{1}{2}} = H_{1-\gamma} \) by Lemma 4.14, we obtain that \( \mathcal{L}^{-1} B : Y \rightarrow Y \) is compact. Hence, \( \mathcal{S} := id_Y + \mathcal{L}^{-1} B \) is a Fredholm operator of index zero in \( V \). It is one-to-one, since if \( u \neq 0 \) would belong to its null space, then \( Lu = -Bu \) and thus \( Au - Bu = \lambda u \) which means that \( u \in Y \) satisfies (3.15). Consequently, \( \mathcal{S} \) is an isomorphism of \( Y \), and so \( \mathcal{L} + B = \mathcal{L} \mathcal{S} \) is an isomorphism of \( Y \) onto \( X_\beta \). In particular, \( (\mathcal{L} + B)^{-1} : X_\beta \rightarrow Y \subseteq X_\beta \) exists and is bounded. This means that \( \lambda \) is not a spectral value of \( A_\beta - B \) in \( X_\beta \).

If \( A \) fails to be Kato, we have nevertheless that the operator \(-A\) generates an analytic \( C_0\)-semigroup by Proposition 4.2, and that we can define fractional powers \( A^\beta \). Hence, we obtain by considering directly

\[ u_t + Au \in f(t, u(t)) \tag{5.4} \]

in \( V' \) and taking Remark 2.8 into account:

Theorem 5.6. In case \( \gamma \in [0,1/2] \), Theorems 3.7, 3.10, 5.4, and 5.5 are valid if \( A \) is not necessarily a Kato operator, provided that one replaces in their assertions (5.2) by (5.4), \( H'_{\gamma} \), and \( H_{1-\gamma} \) by \( V \), \( H_{\alpha} \) by \( X_\beta \) with \( \beta \in [0,1) \), and the requirement for \( p \) in Theorem 5.4 by \( p > 1/(1-\beta) \).

If \( A \) is a Kato operator and \( \gamma = 1/2 \), then \( A_\beta = A \), \( H_{\gamma} = H_{1-\gamma} = V \), and one can replace \( H_{\alpha} \) by \( H_{\alpha} ' \) with some \( \alpha \in [0,1/2] \) and the requirement for \( p \) in Theorem 5.4 correspondingly by \( p > 2/(1 + 2\alpha) \).

Proof. The last assertion of Theorem 5.6 follows from the first by observing that if \( A \) is a Kato operator, then \( X_\beta \cong X_{1-\beta} ' \cong H_{\alpha} ' \) for \( \beta = \frac{1}{2} - \alpha \in [0,1/2] \) by Propositions 4.9 and 4.10.

In particular, Theorem 5.6 implies Remark 3.17.
6. The Case $\gamma \in [1/2, 1)$

Our construction of $\mathcal{A}$ and the related operators can be repeated by replacing $(A, a, V, H)$ throughout by $(\mathcal{A}, a, H, X_{a,b})$ in all previous considerations, because $\mathcal{A}$ becomes again a Kato operator if $A$ is a Kato operator, see Theorem [A.1].

To avoid notational confusion, we will denote the antidual of an embedded Hilbert space $U \subseteq X_{a,b}$ as $U^!$ when we mean the dual pairing $U^! \times U \to \mathbb{C}$ which is compatible with the scalar product $[\cdot, \cdot]_{a,b}$ in $X_{a,b}$. In particular, the role which in our previous results was played by the space $V'$ is now played by the space $H^!$. Moreover, the role which was played previously by $A$ is now played by an extension $\mathcal{A}$ of $A$ which is an isomorphism of $H$ onto $H^!$. According to our previous considerations, there is an $\mathcal{A}$-Kato scalar product $\beta$ on $H$, and $H^!$ can be equipped with a corresponding scalar product $[\cdot, \cdot]_{a,b}$. The role which was previously played by the spaces $X := X_{a,b} \cong V'$ is now played by $Y := Y_{a,b} \cong H^!$, and the role played by $X_\alpha = D(\mathcal{A}^\alpha)$ is now played by $Y_\alpha := D(\mathcal{A}^\alpha)$. If we denote by $\mathcal{A}_\beta$ the $Y_\beta$-realization of $\mathcal{A}$, the role previously played by $X_{\beta,\alpha} = D(\mathcal{A}_\beta^\alpha)$ is now played by the spaces $Y_{\beta,\alpha} := D(\mathcal{A}_\beta^\alpha)$.

The following diagram gives an overview of some densely embedded spaces and isomorphisms between them:

\[
\begin{array}{ccc}
D(\mathcal{A}) = H_2 & \cong & V \cong H_1 \\
\cong & \cong & \cong \\
A = A_0 & \cong & A = A_{1/2} \\
\cong & \cong & \cong \\
H = H_0 = H' & \cong & H^! \cong Y = Y_0 \\
\cong & \cong & \cong \\
V' \cong H_{-1/2} & \cong & X \cong X' \\
\cong & \cong & \cong \\
A = A_1 & \cong & A = A_{1/2} \\
\end{array}
\]

(6.1)

In (6.1), we already inserted the identities which follow from the subsequent considerations and Proposition [3.3].

Remark 6.1. The above identities and assertions are actually somewhat ambiguous. In fact, the two simultaneous identifications $H \cong H'$ and $X \cong X'$ are incompatible with the attempt to consider $H = H'$ as a subset of $X'$. Indeed on the one hand, we identify an element $u \in H$ with the functional $u_H(v) := (u, v)$ over $H$. On the other hand, we identify the functional $u_H$, and hence the element $u \in H$ itself, with the functional $u_X(f) := [u_H, f]_{a,b}$ over $X$. Now the inconsistency arises if we evaluate $u_X$ on an element $v \in H$, which according to our agreement is identified with the functional $v_H = (v, \cdot) \in H' \subseteq X$:

$$u_X(v) = [u_H, v_H]_{a,b}.$$ 

Thus, in general, $u_H(v) \neq u_X(v)$ for $v \in H$, and we simultaneously identify $u$ with two different functionals over $H$.

Therefore, let us do the above considerations more rigorously. Consider the map $i_H: H \to X$ such that $i_H u$ denotes the functional $(u, \cdot) \in X$.

Then immediately a further difficulty becomes visible: To apply the results of the previous sections for $\mathcal{A}$, and to define the interpolation spaces $X_{\beta}$ correctly, it is necessary to have the diagram

\[
\begin{array}{ccc}
D(\mathcal{A}) & \subseteq & V \\
\subseteq & \subseteq & \subseteq \\
\mathcal{A} & \mathcal{A} & \mathcal{A} \\
\end{array}
\]

\[
\begin{array}{ccc}
D(\mathcal{A}) = V & \subseteq & H \\
\subseteq & \subseteq & \subseteq \\
\mathcal{A} & \mathcal{A} & \mathcal{A} \\
\end{array}
\]

(6.2)

It is important that the first two inclusions are in the set-theoretical sense, not only in the sense of some identifications/inclusions. However, the operator $\mathcal{A}: D(\mathcal{A}) = V \to X$ gives
when we apply the results of the previous sections as described above, we actually have to replace \((A,a,V,H)\) by \((\tilde{A},\tilde{a},\tilde{H},X)\), where \(\tilde{H} := i_H(H)\) is endowed with the norm \(\|i_H u\|_{\tilde{H}} = |u|\), and \(\tilde{A}: \tilde{V} \to X\) is defined on \(D(\tilde{A}) = \tilde{V} := i_H(V) \subseteq \tilde{H}\) by

\[
\tilde{A}(i_H u) = Au \quad \text{for all } u \in V,
\]

and \(\tilde{a}(i_H u,i_H v) := a(u,v)\) for all \(u,v \in H\). Hence, strictly speaking, we consider the diagram

\[
D(\tilde{A}) \subseteq \tilde{H} \subseteq X \xrightarrow{ix} Y,
\]

where \(i_X: X \to Y \cong \tilde{H}'\) is the map given by \(i_X v = [v, \cdot]_{a,b} \in Y \cong \tilde{H}'\). This diagram has a structure matching \(6.2\). The definitions of \(A\) and \(\tilde{A}\) imply

\[
i_H Au = Au \quad \text{for all } u \in D(A),
\]

\[
i_X \tilde{A} v = Av \quad \text{for all } v \in D(\tilde{A}) = \tilde{V},
\]

and together with \(6.3\) and \(v := i_H u\), we thus find

\[
i_X i_H Au = i_X Av = A_i_H u \quad \text{for all } u \in D(A). \tag{6.4}
\]

In particular, when we now return to our previous ambiguous notation in \(6.1\), where we suppress the embeddings \(i_H\) and \(i_X\), we see that \(A\) is an extension of the operator \(A\).

**Proposition 6.2.** Let \(A\) be a Kato operator. Then we have with the above notations:

1. \(Y \cong H_{-1}\) with a canonical identification on \(H\).
2. \(Y = Y_0, Y_{1/2} \cong X \cong V', Y_1 \cong H\). For all \(\alpha, \beta \geq 0, \theta \in (0,1)\) there holds \(Y_{(1-\theta)+\beta \theta} \cong [Y_{\alpha}, Y_{\beta}]\). Moreover, \(Y_{1+\alpha} \cong X_{\alpha}, Y_{1+\alpha} \cong H_{\alpha}\), and \(Y_{\beta,\alpha} \cong Y_{\beta+\alpha}\) if \(\beta \in [0,1/2]\).
3. \(A\) induces an isomorphism of \(H_{\alpha}\) onto \(Y_{\alpha}\) for every \(\alpha \geq 0\). More precisely, this isomorphism is \(A = A_0\) in case \(\alpha = 1\), a restriction of \(A\) in case \(\alpha > 1\), or a restriction of the unique bounded linear extension \(A_1: H \to H_{-1}\) in case \(\alpha < 1\).
4. \(A\) induces a norm-preserving isomorphism of \(H_{\alpha}\) onto \(H_{\alpha-1}\) for all \(\alpha \geq 0\).
5. \(Y_{\alpha} \cong H_{\alpha-1}\) for all \(\alpha \geq 0\).

**Proof.** \([1]\): We show first that the norms are equivalent on the subspaces \(H = H'\). Thus, let \(u \in H\) be given. We denote as previously the antilinear functional \((u, \cdot) \in H' \cong X^{a,b}\) by \(u\). The norm of this functional, considered as an element of \(Y\), is by our construction given by

\[
\|u\|_Y = \sup_{|v| \leq 1} |(u,v)_{a,b}|.
\]

Using \(4.4\) and that \(A\) is the \(H\)-realization of \(A\), we calculate for \(u,v \in H\) that

\[
[u,v]_{a,b} = a(A^{-1}u,v)_{a,b} = a(A^{-1}u,v).
\]
Using the first two inequalities in (4.3), the second with the choice \( v = A^{-1}u / |A^{-1}u| \) in case \( u \neq 0 \), we thus obtain
\[
c_1 |A^{-1}u| \leq \| u \|_Y \leq c_2 |A^{-1}u|.
\]
This shows that on the dense subset \( H = H' \) of \( Y \) the norm of \( Y \) is equivalent to the norm of \( H_{-1} \), hence by definition the completion \( H_{-1} \) coincides with \( Y \).

(2): The assertions are just reformulations of assertions of Propositions 4.9, 4.10 and 4.18 for the exchanged data; the only exception is the assertion \( Y_{1+\alpha} \cong H_{\alpha} \) which in view of \( X_{1+\alpha} \cong H_{\alpha} \) follows from \( Y_{1+\alpha} \cong X_{1+\alpha} \).

(3): It follows from the definition of \( H_{-1} \) and (1) that the unique bounded linear extension \( A_1 \) of \( A \) is an isomorphism of \( H = H_0 \) onto \( H_{-1} \cong Y = Y_0 \). Since \( A = A_1|_{D(A)} \) is also an isomorphism of \( D(A) \cong H_1 \) onto \( H \cong Y_1 \), we have shown (3) in the cases \( \alpha = 0 \) and \( \alpha = 1 \). Since \( [\cdot, \cdot]_\theta \) is an interpolation functor of order \( \theta \) for every \( \theta \in (0,1) \), see e.g. [27, Theorem 1.9.3(a)], it follows that a restriction of \( A_1 \) is an isomorphism of \( [H_0,H_1]_\theta \cong H_\theta \) onto \( [Y_0,Y_1]_\theta \equiv Y_\theta \) for every \( \theta \in (0,1) \). This proves (3) for the case \( \alpha = \theta \in (0,1) \). For \( \alpha > 1 \), we have by [27, Theorem 1.15.2(e)] that a restriction of \( A \) is an isomorphism of \( D(A^\alpha) \cong H_\alpha \) onto \( D(A^{\alpha-1}) \cong H_{\alpha-1} \cong Y_\alpha \) by (2). Thus, we have shown (3) for every \( \alpha \geq 0 \).

(4): We find by [27, Theorem 1.15.2(a)] for every \( \alpha \geq 0 \) that
\[
A^{\alpha-1}Au = A^\alpha u \quad \text{for all } u \in D(A^{2m}),
\]
if \( m > \max\{1, \alpha - 1\} \). Since \( D(A^{2m}) \) is dense in \( D(A^\alpha) = H_\alpha \), see e.g. [11, Theorem 1.4.8], this implies that \( A \) induces a norm-preserving isomorphism of \( H_\alpha \) onto a subspace of \( H_{\alpha-1} \).

By a similar argument, we find
\[
A^\alpha A^{-1}u = A^{\alpha-1}u \quad \text{for all } u \in D(A^{2m})
\]
if \( m > \alpha \) and conclude from the density of \( D(A^{2m}) \) in \( H \) and thus in \( H_{\alpha-1} \) that \( A^{-1} \) induces a norm-preserving isomorphism of \( H_{\alpha-1} \) onto a subspace of \( H_\alpha \). Combining both, we have shown (4).

(5): Since \( A \) induces an isomorphism \( H_\alpha \rightarrow H_{\alpha-1} \) by (4) and its inverse induces an isomorphism \( Y_\alpha \rightarrow H_\alpha \) by (3), we obtain \( Y_\alpha \cong H_{\alpha-1} \).

Remark 6.3. For \( \gamma_1, \gamma_2 \in [0,1] \), the assertions (2) an (5) of Proposition 6.2 imply with the choice \( \alpha := 1 - \gamma_1 \) and \( \beta := 1 - \gamma_2 \) in view of
\[
\alpha(1 - \theta) + \beta \theta = 1 - (\gamma_1(1 - \theta) + \gamma_2(1 - \theta))
\]
the interpolation equality (3.5).

Remark 6.4. Since \( A \) is an extension of \( A \) in the sense (6.4), the restriction \( A|_{Y_{\alpha+1}} : Y_{\alpha+1} \rightarrow Y_\alpha \) is an isomorphism for all \( \alpha \geq 0 \), and \( D(A) \) is dense in \( Y_{\alpha+1} \cong H_\alpha \), we find that the isomorphism \( A_1|_{H_\alpha} \) of Proposition 6.2 is actually \( A|_{Y_{\alpha+1}} \).

It follows from Remark 6.3 and the just proved canonical identity \( H_{-\gamma} \cong Y_{1-\gamma} \), that the map \( A_\gamma \) introduced in Section 3 can (in the sense of Remark 6.1) be considered as the \( Y_{1-\gamma} \)-realization of \( A \), that is, we have \( A_1 = A \) and
\[
A_\gamma = A|_{D(A_\gamma)} \quad \text{with} \quad D(A_\gamma) = A^{-1}(Y_{1-\gamma}) = H_{1-\gamma}.
\]
In particular, \( A_\gamma \) plays exactly the role which in our previous results (with the unexchanged data) was played by \( A_\beta \) with \( \beta = 1 - \gamma \), that is \( A_\gamma = A_{1-\beta} \). Thus, if \( \gamma \in [1/2,1] \), we have \( \tilde{\gamma} := 1 - \gamma \in [0,1/2] \), and applying the results proved in Section 3 with \( (A,a,V,H,\gamma) \) replaced throughout by \( (A,a,H,X^{a,b},\tilde{\gamma}) \), we obtain in particular the results of Section 3 for the case \( \gamma \in [1/2,1] \).
7. Applications to Semilinear Parabolic Problems

7.1. Superposition Operators in $L_2$ and Sobolev Spaces. Let $\Omega \subseteq \mathbb{R}^d$ be open and $H := L_2(\Omega, \mathbb{C}^n)$. In the following, we use the scalar product (and respective dual pairing)

$$(u, v) := \int_{\Omega} u(x) \overline{v(x)} \, dx.$$ 

In case $d \geq 3$, we put $p_* := \frac{2d}{d-2}$; in case $d \leq 2$, we fix an arbitrary $p_* \in (2, \infty)$. Let $V \subseteq W^{1,2}(\Omega, \mathbb{C}^n)$ be a closed subspace which is dense in $H$. We assume that $\Omega$ is such that Sobolev’s embedding theorem is valid in the sense that there is a continuous embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$.

Remark 7.1. For the case that $\Omega$ is such that the dense embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$ holds only with some smaller power $p_* \in (2, \infty)$, all subsequent considerations hold as well with this choice of $p_*$. 

Lemma 7.2. Let $A$ be a Kato operator.

1. Let $\gamma \in [0, 1/2]$.
   a. Then we have a continuous embedding $L_{q_{\gamma}}(\Omega, \mathbb{C}^n) \subseteq H_{\gamma}^* \equiv (H_{\gamma}^*)'$ with
   $q_{\gamma} := \left(1 + \frac{2\gamma}{p_*}\right)^{-1} \left(\gamma = \frac{2d}{d+4} \in \left[\frac{2d}{d+2}, 2\right] \text{ if } d \geq 3\right)$, \hspace{1cm} (7.1)
   b. and if we have a continuous embedding $D(A) \subseteq L_{p}(\Omega, \mathbb{C}^n)$ $(1 \leq p < \infty)$, then we also have a continuous embedding $H_{1-\gamma} \subseteq L_{p_\gamma}(\Omega, \mathbb{C})$ with
   $p_\gamma := \left(\frac{2\gamma + 1}{p} - \frac{2\gamma}{p_*}\right)^{-1}$. \hspace{1cm} (7.2)

2. Let $\gamma \in [1/2, 1]$.
   a. We have a continuous embedding $H_{1-\gamma} \subseteq L_{p_\gamma}(\Omega, \mathbb{C})$ with
   $p_\gamma := \left(\gamma - \frac{1}{2} + \frac{2 - 2\gamma}{p_*}\right)^{-1} \left(\gamma = \frac{2d}{4\gamma + d - 2} \in \left[\frac{2d}{d+2}, 2\right] \text{ if } d \geq 3\right)$, \hspace{1cm} (7.3)
   b. If we have a continuous embedding $D(A^*) \subseteq L_{p}(\Omega, \mathbb{C}^n)$ $(1 \leq p < \infty)$, then we also have a continuous embedding $L_{q_{\gamma}}(\Omega, \mathbb{C}^n) \subseteq (H_{\gamma}^*)'$ with
   $q_{\gamma} := \left(1 - \frac{2\gamma - 1}{p} - \frac{2 - 2\gamma}{p_*}\right)^{-1}$. \hspace{1cm} (7.4)

Proof. By hypothesis, we have a continuous dense embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$. Hence, with $rac{1}{p_*} + rac{1}{p_*} = 1$ also the (Banach space) adjoint embedding $L_{p_*}^*(\Omega, \mathbb{C}^n) \subseteq V'$ is continuous and dense. In case $\gamma = 1/2$ we have $q_{\gamma} = p_\gamma = p'_\gamma$, and thus the assertion (2) follows. In case $\gamma = 0$ we have $q_{\gamma} = 2$ and $p_\gamma = p$, and the assertion (1) is trivial. In case $\gamma \in (0, 1/2)$, we use [27, Theorem 1.18.4], the fact that $\left[\cdot, \cdot\right]_\theta$ is an interpolation functor of order $\theta$ (see e.g. [27, Theorem 1.9.3(a)]), and (3.3). Then we have a continuous embedding

$L_{q_{\gamma}}(\Omega, \mathbb{C}^n) \cong [L_{p_\gamma}(\Omega, \mathbb{C}^n), L_2(\Omega, \mathbb{C}^n)]_{1-2\gamma} \subseteq [V', H]_{1-2\gamma} \cong H_{\gamma}^*$,

which proves (1a).

Defining $p'$ by $\frac{1}{p} + \frac{1}{p'} = 1$, we find in view of $H_1^* = D(A^*) \subseteq L_{p'}(\Omega, \mathbb{C}^n)$ that $L_{p'}(\Omega, \mathbb{C}^n) = L_{p}(\Omega, \mathbb{C}^n)' \subseteq (H_{\gamma}^*)'$. This shows (2) for $\gamma = 1$, since $q_{\gamma} = p'$ and $p_\gamma = 2$. Moreover, for $\gamma \in (1/2, 1)$, we find similarly as above with (3.5) the continuous embedding

$L_{q_{\gamma}}(\Omega, \mathbb{C}^n) = [L_{p'}(\Omega), L_{p_\gamma}(\Omega, \mathbb{C}^n)]_{2-2\gamma} \subseteq [D(A^*)', V']_{2-2\gamma} \cong (H_{\gamma}^*)'$,
which implies (2a). A similar argument shows with Proposition 2.3 that in case $\gamma \in (0, 1/2)$

$$H_{1-\gamma} \cong [H_{1/2}, H_{1-2\gamma}] \subseteq [L_p, (\Omega, C^n), L_p(\Omega, C^n)]_{1-2\gamma} \cong L_p, (\Omega, C^n),$$

proving (1b), while in case $\gamma \in (1/2, 1)$

$$H_{1-\gamma} \cong [H_0, H_{1/2}], 2-2\gamma] \subseteq [L_2(\Omega, C^n), L_2(\Omega, C^n)], 2-2\gamma] \cong L_p, (\Omega, C^n),$$

proving (2a) (all embeddings being countinous).

We assume also that the nonlinearity $f(t, \cdot)$ is given by a superposition operator induced by a function $\tilde{f} : [0, \infty) \times \Omega \times C^n \rightarrow 2^{C^n}$, that is, for each $t \in [0, \infty)$

$$f(t, u) := \{ v : \Omega \rightarrow C^n \mid v \text{ measurable and } v(x) \in \tilde{f}(t, x, u(x)) \text{ for almost all } x \in \Omega \}. (7.5)$$

For the stability result, without loss of generality, we will consider only the case $u_0 = 0$ and assume that $\tilde{f}$ is uniformly linearizable at $u = 0$ in the following sense. There are $r \in (1, \infty]$, a measurable $\tilde{B} : \Omega \rightarrow C^{n \times n}$, and a function $\tilde{g} : (0, \infty) \times \Omega \times C^n \rightarrow 2^{C^n}$ with

$$\tilde{f}(t, x, u) = \tilde{B}(x)u + \tilde{g}(t, x, u) \text{ for all } (t, x, u) \in (0, \infty) \times \Omega \times C^n$$

such that

$$\lim_{|u| \rightarrow 0} \sup_{|v|} \left\{ |v| : v \in \tilde{g}((0, \infty) \times \{x\} \times \{u\}) \right\} = 0 \text{ (7.6)}$$

for almost all $x \in \Omega$. Moreover, we assume that there is $C_0 \in (0, \infty)$ such that

$$\sup_{|v|} \left\{ |v| : v \in \tilde{g}((0, \infty) \times \{x\} \times \{u\}) \right\} \leq C_0 \cdot (|u| + |u|^\sigma) \text{ for all } u \in C^n \text{ (7.7)}$$

for almost all $x \in \Omega$ and some $\sigma \in (1, \infty)$. We define a corresponding multiplication operator $B$ by

$$Bu(x) := \tilde{B}(x)u(x) \text{ for all } x \in \Omega. \text{ (7.8)}$$

With this notation, the following holds.

**Proposition 7.3.** Let $A$ be a Kato operator and $u_0 = 0$. Suppose that

$$\Omega \text{ has finite (Lebesgue) measure} \text{ (7.9)}$$

and that $r \in [1, \infty]$ and $\sigma \in (0, \infty)$ are such that $\tilde{B} \in L_r(\Omega, C^{n \times n})$ and (7.6) and (7.7) hold.

1. Let $\alpha = 0$. Assume

$$\begin{cases}
  r = 2 & \text{if } d = 1, \\
  r > 2 & \text{if } d = 2, \\
  r = \frac{2p_*}{p_* - 2} = d & \text{if } d \geq 3,
\end{cases} \text{ (7.10)}$$

and

$$\begin{cases}
  \sigma = 2 & \text{if } d = 1, \\
  \sigma < 2 & \text{if } d = 2, \\
  \sigma = 2 - \frac{2}{p_*} = 1 + \frac{2}{d} & \text{if } d \geq 3.
\end{cases} \text{ (7.11)}$$

Then for every $\gamma \in [1/2, 1)$ there holds $f : [0, \infty) \times H_\alpha \rightarrow 2^{(H_\gamma)'},$ and the hypothesis $(B_{r_\gamma})$ of Theorems 3.10 and 3.17 is satisfied with $H_\alpha = L_2(\Omega, C^n)$.
(2) Let $\alpha = 0$. Assume that the embedding $D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n)$ is continuous for some $p \in (p_*, \infty)$, and
\[ r > \frac{2p}{p-2} \quad \text{and} \quad \sigma < 2 - \frac{2}{p}, \quad (7.12) \]
Then
\[ \gamma_0 := \frac{(\sigma - 2)p_* - 2p_2 + 4p}{4(p-p_*)} < 1, \quad \gamma_1 := \frac{2pp_* - r(pp_* + 2p_* - 4p)}{4(p-p_*)r} < 1, \quad (7.13) \]
and for every $\gamma \in [\max\{\gamma_0, \gamma_1, 1/2\}, 1)$ the same conclusion as in (1) is valid.

(3) Let $\alpha = 1/2$. Suppose that $B \in L_r(\Omega, \mathbb{C}^{n \times n})$ with some
\[ r > \frac{p_*}{p_* - 2}, \quad (7.14) \]
and that (7.6) and (7.7) hold with some
\[ \sigma < p_* - 1, \quad (7.15) \]
Then
\[ \gamma_0 := \frac{2\sigma - p_*}{2p_* - 4} \quad \text{and} \quad \gamma_1 := \frac{p_*}{r(p_* - 2)} - \frac{1}{2} < \frac{1}{2}, \quad (7.16) \]
and for every $\gamma \in [\max\{0, \gamma_0, \gamma_1\}, 1/2)$ we have $f: [0, \infty) \times H_{\alpha} \to 2^{H_\gamma}$, and the hypothesis (B.) of Theorems 3.10 and 3.16 is satisfied with $H_{\alpha} = V$.

**Proof.** In case (1), it is no loss of generality to assume $\gamma = 1/2$, and we assume first $d \geq 3$. In cases (1) and (2), we put $\tilde{p} = 2$ and define $q_\gamma$ by (7.4), while in case (3), we put $\tilde{p} = p_*$ and define $q_\gamma$ by (7.11). Then we put $U := L_{\tilde{p}}(\Omega, \mathbb{C}^n)$ and $V_\gamma := L_{q_\gamma}(\Omega, \mathbb{C}^n)$. Letting $r$ satisfy (7.10), (7.12), or (7.14), and requiring $r \geq \gamma_1$ with $\gamma_1$ as in (7.13) or (7.16) in the respective cases, we find
\[ \frac{1}{q_\gamma} \geq \frac{1}{\tilde{p}} + \frac{1}{r}, \]
and so we obtain from the (generalized) Hölder inequality that $B: U \to V_\gamma$ is bounded. Since we have a bounded embedding $H_{\alpha} \subseteq U$, we obtain from Lemma 7.2 that $B: H_{\alpha} \to (H_\gamma')'$ is bounded.

Moreover, letting $\sigma$ satisfy (7.11), (7.12), or (7.15), and requiring $\gamma \geq \gamma_0$ with $\gamma_0$ as in (7.13) or (7.16) in the respective cases, we find $\sigma \leq \tilde{p}/q_\gamma$. Hence, the superposition operator $g$ generated by $\tilde{g}$ satisfies $g: [0, \infty) \times U \to 2^{V_\gamma}$ and
\[ \lim_{\|u\|_U \to \infty} \sup_{v \in g((0, \infty) \times \{u\})} \frac{\|v\|_{V_\gamma}}{\|u\|_U} = 0, \]
see [29] Theorem 4.14]. Since we have continuous embeddings $H_{\alpha} \subseteq U$ and $V_\gamma \subseteq (H_\gamma')'$ (Lemma 7.2), the condition (B.) is proved.

Case (1) with $d = 2$ is treated in a similar way (with a sufficiently large $p_*$), and for $d = 1$ we can put $q_\gamma = 1$ in the above calculation, since in this case we have still a continuous embedding $V_\gamma \subseteq (H_\gamma')'$ by the continuity of the embedding $(H_{\gamma})' \subseteq H_{1/2} \subseteq L_\infty(\Omega, \mathbb{C}^n)$. □

**Remark 7.4.** The last observation in the proof extends to a more general situation: If $\gamma \in [0, 1/2]$ is such that the embedding $H_\gamma \subseteq L_\infty(\Omega, \mathbb{C}^n)$ is continuous, then the conclusion of Proposition 7.3(1) is valid with $r = \sigma = 2$ (we put $q_\gamma = 1$ in the proof).
Remark 7.5. For $d \geq 3$ assertion (2) of Proposition 7.3 requires strictly less about $r$ and $\sigma$ than assertion (1), because in view of $p > p_*$ there holds
\[
\frac{2p}{p - 2} < \frac{2p_*}{p_* - 2} \quad \text{and} \quad 2 - \frac{2}{p} > 2 - \frac{2}{p^*}.
\]

Remark 7.6. In case $d \geq 3$ the quantities $\gamma_0$ and $\gamma_1$ in (7.10) have the form
\[
\gamma_0 = \frac{(d - 2)\sigma - d}{4} \quad \text{and} \quad \gamma_1 = \frac{1}{2} \left( \frac{d}{r} - 1 \right).
\] (7.17)

Remark 7.7. Proposition 7.3(3) holds also with $\gamma = 1/2$. Moreover, for $\gamma = 1/2$ one does not have to require that the inequalities in (7.14) or (7.15) are strict. However, the choice $\gamma = 1/2$ violates the hypothesis (3.2) of Theorems 3.10 and 3.16 if $\alpha = 1/2$.

Remark 7.8. Hypothesis (7.9) is obviously needed for the assertion (3) of Proposition 7.3. However, we used this hypothesis also for the assertion (1) when we applied [29, Theorem 4.14]. If hypothesis (7.9) fails, one can apply other criteria for the differentiability of superposition operators like e.g. [29, Theorem 4.9], but we do not formulate corresponding results here.

While Proposition 7.3 gives a sufficient condition for the hypothesis $(B_\gamma)$, this is not sufficient to apply Theorem 3.16. For the latter, one also has to estimate all $\gamma$-weak eigenvalues of $A - B$, and the latter in turn is usually simpler if one knows that all $\gamma$-weak eigenvalues of $A - B$ are eigenvalues of $A - B$. For the operator $B$ from (7.8), this is the content of the following result.

Proposition 7.9. Suppose (7.9). Let $B$ have the form (7.8) with some $\tilde{B} \in L_r(\Omega, \mathbb{C}^n)$, $r \in [1, \infty]$.

1. If $r$ satisfies (7.10) then $B|_V : V \to H$ is bounded.
2. If $A$ is a Kato operator, $\gamma \in [1/2, 1)$, and
\[
\gamma \leq \tilde{\gamma}_0 := \begin{cases} 
1 - \frac{p_*}{(p_* - 2)r} & \text{if } r < \infty, \\
1 & \text{if } r = \infty,
\end{cases}
\] (7.18)

then $B|_{H_{1-\gamma}} : H_{1-\gamma} \to H$.
3. If $A$ is a Kato operator,
\[
2 < r < \frac{2p_*}{p_* - 2} \quad (= d \text{ if } d \geq 3),
\] (7.19)

and if there is $p \geq \frac{2r}{r - 2}$ ($> p_*$) with $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$, then
\[
\tilde{\gamma}_p := \frac{1}{2} \left( \frac{1}{2} - \frac{1}{r} - \frac{1}{p} \right) \cdot \left( \frac{1}{p_*} - \frac{1}{p} \right)^{-1} \in [0, 1/2),
\] (7.20)

and for all $\gamma \leq \tilde{\gamma}_p$ the operator $B|_{H_{1-\gamma}} : H_{1-\gamma} \to H$ is bounded.

If $A$ is a Kato operator, the hypotheses of either (1), (2), or (3) are satisfied and $\gamma \leq 1/2$, $\gamma \leq \tilde{\gamma}_0$, or $\gamma \leq \tilde{\gamma}_p$, respectively, then $\lambda$ is a $\gamma$-weak eigenvalue of $A - B$ if and only if $\lambda$ is an eigenvalue of $A - B$.

Proof. In case (1) with $d \geq 3$, we apply in view of
\[
\frac{1}{2} = \frac{1}{p_*} + \frac{1}{r}
\]
the (generalized) Hölder inequality to obtain that \( B : L_{p_\ast}(\Omega, \mathbb{C}^n) \to L_2(\Omega, \mathbb{C}^n) \) is bounded and thus \( B : V \to H \) is bounded. Case (1) with \( d \geq 2 \) is similar (with sufficiently large \( p_\ast \)), and for \( d = 1 \) one can formally put \( p_\ast = \infty \) by the continuity of the embedding \( H_{1/2} \subset C(\Omega, \mathbb{C}^n) \).

In case (2) and (3), we define \( p_\gamma \) by (7.3) or (7.2), respectively, and observe that, due to (7.18) or (7.20), respectively, we have the estimate

\[
\frac{1}{2} \geq \frac{1}{p_\gamma} + \frac{1}{r}.
\]

Hence, by the (generalized) Hölder inequality, \( B : L_{p_\gamma}(\Omega, \mathbb{C}^n) \to L_2(\Omega, \mathbb{C}^n) \) is bounded, and thus also \( B : H_{1-\gamma} \to H \) is bounded by Lemma (7.2). The last assertion follows from Proposition 3.14 and Remark 3.15.

If one is interested in stability in \( H \) (the case \( \alpha = 0 \)), one should consider Proposition 7.3 part (1) or (2). In the former case, Proposition 7.9(1) is automatically satisfied, and in the latter case one would like to apply Proposition 7.9(2). In the latter case, \( \gamma \in [1/2, 1) \) has to satisfy \( \gamma_i \leq \gamma \leq \tilde{\gamma}_0 \) for \( i = 0, 1 \) with \( \gamma_i \) from (7.13). Obviously, \( \gamma_1 \) and \( \tilde{\gamma}_0 \) depend monotonically on \( r \), and \( \gamma_1 < \tilde{\gamma}_0 \) if \( r \) is sufficiently large, and then \( \gamma_0 < \tilde{\gamma}_0 \) if \( \sigma \) is sufficiently large, so that Proposition 7.3(2) and Proposition 7.9(2) apply simultaneously for all \( \gamma \) from some proper interval (if \( r \) is sufficiently large).

If one is interested in stability in \( V \) (the case \( \alpha = 1/2 \)), one should consider Proposition 7.3(3). In this case, the hypothesis of Proposition 7.9(1) means an additional requirement for \( r \). The purpose of Proposition 7.9(3) is to relax this requirement. However, it is not immediately clear whether this relaxed requirement applies in the situation of Proposition 7.3(3), since then \( \gamma \in [0, 1/2) \) needs to satisfy \( \gamma_i \leq \gamma \leq \tilde{\gamma}_p \) for \( i = 0, 1 \) with \( \gamma_i \) from (7.13). Although \( \gamma_1 \) and \( \tilde{\gamma}_p \) depend monotonically on \( r \) and satisfy \( \gamma_1 < \tilde{\gamma}_p \) if \( r \) is sufficiently large, one cannot choose \( r \) arbitrarily large in view of (7.19): Otherwise already the additional requirement of Proposition 7.9(1) is satisfied. In fact, the following observation may be somewhat discouraging at a first glance.

**Remark 7.10.** If (7.19) holds, then the term \( \tilde{\gamma}_p \) in (7.20) is strictly increasing with respect to \( p \geq \frac{2r}{r-2} \). In particular,

\[
\tilde{\gamma}_p = \lim_{p \to \infty} \tilde{\gamma}_p = \frac{r - 2}{4r} p_\ast.
\]

Thus, even if we know that \( D(A) \subset L_{p_\ast}(\Omega, \mathbb{C}^n) \) for every \( p \in (1, \infty) \), we still have \( \gamma < \tilde{\gamma}_\infty \), and the latter can be arbitrarily small if \( r \) is sufficiently close to 2.

Nevertheless we will show in the following remark that Proposition 7.3(3) and Proposition 7.9(3) apply simultaneously with the same \( \gamma \) provided that \( r \) is not “too” small and \( \sigma \) is not “too” large.

**Remark 7.11.** Suppose that Sobolev’s embedding theorem holds in the sense described earlier and, moreover, that we have a continuous embedding \( D(A) \subset L_{p_\ast}(\Omega, \mathbb{C}^n) \) with \( p = \frac{2d}{d-1} \) in case \( d \geq 5 \) and any \( p \in (p_\ast, \infty) \) in case \( d \leq 4 \). For instance, by standard Sobolev embedding theorems (see [21, Theorem 1.4.5]), this is the case if \( D(A) \subset W^{2,2}(\Omega, \mathbb{C}^n) \). Proposition 7.9(3) applies with

\[
\begin{align*}
\text{if } d \geq 5, & \quad \frac{d}{2} - \frac{d}{2d/(d-4)} = 1 - \frac{d}{2d/(d-4)}; \\
\text{if } d = 3, 4, & \quad \gamma < \tilde{\gamma}_\infty = \frac{r - 2}{4r} p_\ast.
\end{align*}
\]
In view of (7.17) it follows that if
\[
\begin{cases}
r \in \left[\frac{3}{2}d, d\right) \text{ and } \sigma \leq \frac{(d+1)r-2d}{(d-2)r} & \text{if } d \geq 5, \\
r \in \left(\frac{d^2}{d-2}, d\right] \text{ and } \sigma < \frac{d^2r-4d}{(d-2)^2r} & \text{if } d = 3, 4,
\end{cases}
\]
then Proposition 7.3(3) applies with
\[
\max\{\gamma_0, \gamma_1\} \leq \tilde{\gamma}_{2d/(d-4)} \quad \text{if } d \geq 5,
\]
\[
\max\{\gamma_0, \gamma_1\} < \tilde{\gamma}_\infty \quad \text{if } d = 3, 4.
\]
Hence, in these cases there exists \( \gamma \in [0, 1/2) \) for which Proposition 7.3(3) and Proposition 7.3(4) apply simultaneously.

A result similar to Proposition 7.3 holds for a Lipschitz condition. We assume that \( \tilde{f} : [0, \infty) \times \Omega \times \mathbb{C}^n \to \mathbb{C}^n \) is single-valued. Let \( \tilde{p} \geq 1, \sigma > 0, \) and \( \gamma \in [0, 1/2] \). We define \( q_\gamma \) by (7.11). We assume that for each \( t_0 \in [0, \infty) \) there are \( L_{t_0} \geq 0, \sigma_{t_0} > 0, \) and a neighborhood \( I \subseteq [0, \infty) \) of \( t_0 \) such that for each \( t \in I \) there are measurable \( a_t, b_t : \Omega \to [0, \infty) \) with
\[
\int_{\Omega} a_t(x)^p \, dx \leq 1 \quad \text{and} \quad \int_{\Omega} b_t(x)^{q_\gamma} \, dx \leq 1
\]
such that for almost all \( x \in \Omega \) the uniform (for all \( u, v \in \mathbb{C}^n \)) estimate
\[
|\tilde{f}(t, x, u) - \tilde{f}(t, x, v)| \leq L_{t_0} \cdot (a_t(x) + |u| + |v|)^{-1} |u - v| \tag{7.21}
\]
holds and such that for each \( t, s \in I \) we have for almost all \( x \in \Omega \) the uniform (for all \( u \in \mathbb{C}^n \)) estimate
\[
|\tilde{f}(t, x, u) - \tilde{f}(s, x, u)| \leq L_{t_0} (b_t(x) + b_s(x) + |u|^\sigma)|t - s|^{\gamma_0}. \tag{7.22}
\]
Finally, we assume that
\[
\tilde{f}(t, \cdot, u) \text{ is measurable for all } (t, u) \in [0, \infty) \times \mathbb{C}^n, \quad \text{and} \quad \tilde{f}(0, \cdot, 0) \in L_{q_\gamma}(\Omega, \mathbb{C}^n). \tag{7.23}
\]

**Proposition 7.12.** Let \( A \) be a Kato operator, and assume (7.9). Assume one of the following:

1. Let \( \alpha = 0 \) and \( \gamma \in [1/2, 1) \). Suppose that conditions (7.21), (7.22), and (7.23) hold with \( \tilde{p} = 2 \) and with \( \sigma \) from (7.11).
2. Let \( \alpha = 0 \), and assume that the embedding \( D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n) \) is continuous for some \( p \in (p_\sigma, \infty) \). Let \( \sigma \) satisfy (7.12), and thus \( \gamma_0 \) from (7.13) satisfies \( \gamma_0 < 1 \). Let \( \gamma \in [\max\{\gamma_0, 0\}, 1) \), and suppose that conditions (7.21), (7.22), and (7.23) hold with \( \tilde{p} = 2 \).
3. Let \( \alpha = 1/2 \). Let \( \sigma \) satisfy (7.15), and thus \( \gamma_0 \) from (7.16) satisfies \( \gamma_0 < 1/2 \). Let \( \gamma \in [\max\{\gamma_0, 0\}, 1/2) \), and suppose that conditions (7.21), (7.22), and (7.23) hold with \( \tilde{p} = p_\sigma \).

Then \( f \) maps \([0, \infty) \times H_\alpha \) into \((H^\alpha)^\prime\) and satisfies a right local Hölder-Lipschitz condition (8.11) and is left-locally bounded into \((H^\alpha)^\prime\).

**Proof.** We use the notation of the proof of Proposition 7.3. Note that (7.23) implies in view of (7.22) by a straightforward estimate that \( f(t, 0) \in V_\gamma \) for every \( t > 0 \). From [15, Appendix] we obtain together with (7.21) that for each \( t \in I \) the function \( f(t, \cdot) \) maps \( U_\gamma \) into \( V_\gamma \) and satisfies a Lipschitz condition on every bounded set \( M \subseteq U \) with Lipschitz constant being independent of \( t \in I \). Using (7.22), we find by a straightforward estimate that
\[
\|f(t, u) - f(s, u)\|_{V_\gamma} \leq C_{M,t_0}|t - s|^{\gamma_0} \quad \text{for all } t, s \in I, u \in M,
\]
where $C_{M,t_0}$ is independent of $t,s \in I$ and $u \in M$. Combining both assertions and the triangle inequality, we obtain that $f : [0, \infty) \times U \to V_\gamma$ satisfies a right Hölder-Lipschitz condition and is left-locally bounded into $V_\gamma$. Since we have bounded embeddings $H_\alpha \subseteq U$ and $V_\gamma \subseteq (H_\gamma_*)'$, the assertion follows. \hfill $\Box$

**Remark 7.13.** If $\alpha = 0$ and $\gamma \in [0,1)$ is such that the embedding $H_\gamma \subseteq L_\infty(\Omega, \mathbb{C}^n)$ is continuous, then the conclusion of Proposition 7.12 is also valid (with the same proof and $q_\gamma = 1$, cf. Remark 7.4).

7.2. **Semilinear Parabolic PDEs.** Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial \Omega$. Let $\Gamma_D, \Gamma_N \subseteq \partial \Omega$ be disjoint and measurable (with respect to the $(d-1)$-dimensional Hausdorff measure) and such that

$$\overline{(\partial \Omega) \setminus (\Gamma_D \cup \Gamma_N)}$$

is a null set. \hfill (7.24)

It is explicitly admissible that $\Gamma_D = \emptyset$ or $\Gamma_N = \emptyset$. Given $a_{j,k}, b_j \in L_\infty(\Omega, \mathbb{C}^{n \times n})$ ($j,k = 1, \ldots, d$) and $\tilde{f} : [0, \infty) \times \Omega \times \mathbb{C}^n \to 2^\mathbb{C}^n$, we consider the semilinear PDE

$$\frac{\partial u}{\partial t} = Pu + \tilde{f}_0(t,x,u) \quad \text{on } \Omega,$$

where

$$Pw := \sum_{j,k=1}^d \frac{\partial}{\partial x_j} \left( a_{j,k}(x) \frac{\partial w(x)}{\partial x_k} \right) + \sum_{j=1}^d b_j(x) \frac{\partial w(x)}{\partial x_j}.$$  \hfill (7.25)

We impose the mixed boundary condition

$$\begin{cases}
  u = 0 & \text{on } \Gamma_D, \\
  \sum_{j,k=1}^d \nu_j a_{j,k} \frac{\partial u}{\partial x_k} = 0 & \text{on } \Gamma_N,
\end{cases}$$

(7.26) where $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ denotes the outer normal at $x \in \partial \Omega$.

We put $H := L_2(\Omega, \mathbb{C}^n)$ and

$$V := \{ v \in W^{1,2}(\Omega, \mathbb{C}^n) : v|_{\Gamma_D} = 0 \text{ in the sense of traces} \},$$

equipping $V$ with the norm of $W^{1,2}(\Omega, \mathbb{C}^n)$. For $M \geq 0$, we introduce the form

$$a(u,v) := \int_{\Omega} \left( \sum_{j,k=1}^d \left( a_{j,k}(x) \frac{\partial u(x)}{\partial x_k} \right) \cdot \frac{\partial v(x)}{\partial x_j} + \sum_{j=1}^d \left( b_j(x) \frac{\partial u(x)}{\partial x_j} +Mu(x) \right) \cdot v(x) \right) dx.$$  \hfill (7.27)

Our main assumption is as follows.

**C:** Let $a(u,v)$ satisfy \hfill (2.3) with some $M \geq 0$ and one of the following holds:

1. the matrices $\text{Re} \left( \sum_{j,k=1}^d a_{j,k}(x) \xi_j \xi_k \right)$ are positive definite, uniformly with respect to all $x \in \Omega$ and $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, $a_{j,k} \in C^1(\overline{\Omega}, \mathbb{C}^{n \times n}), b_j \in \text{Lip}(\overline{\Omega}, \mathbb{C}^{n \times n})$ for all $j,k = 1, \ldots, d$, $\Gamma_D$ and $\Gamma_N$ are open in $\partial \Omega$ domains, and the set \hfill (7.24) is a Lipschitz manifold of dimension $d - 2$.

2. $a_{j,k}(x) = (a_{k,j}(x))^*$ and $b_j(x) = 0$ for almost all $x \in \Omega$ and all $j,k = 1, \ldots,d$.

For a discussion of various algebraic conditions that are sufficient for $a(u,v)$ to satisfy \hfill (2.3), we refer the reader, to e.g. \hfill [22].

Now we introduce the function

$$\tilde{f}(t,x,u) := \tilde{f}_0(t,x,u) + Mu$$
and define strong (weak, mild) solutions of (7.25), (7.26) as strong (weak, mild) solutions of (2.10) with the superposition operator (7.5). A connection between solutions of (7.25), (7.26) and (2.10) is described in e.g. [28] Theorem 4.4.4.

Theorem 7.14. Assume that hypothesis (C) holds. Then the operator $A$ associated with $a$ is a Kato operator.

Moreover, let $(f, \alpha, \gamma)$ satisfy the hypotheses of Proposition 7.3 part (1) or (2) or (3), and suppose that there is some $\lambda_0 > 0$ such that every $\gamma$-weak eigenvalue $\lambda$ of $A - B$ satisfies $\text{Re} \lambda \geq \lambda_0$. Then $u_0 = 0$ is asymptotically stable in $H_1 = L_2(\Omega, \mathbb{C}^n)$ or $H_2 = W^{1,2}(\Omega, \mathbb{C}^n)$ in the sense that for every $\varepsilon > 0$ there is $\delta > 0$ such that any $(H_1^*)'$-weak solution $u \in C(\mathbb{R}, H_2)$ of (7.25) with $\|u(0, \cdot)\|_{H_1} \leq \delta$ satisfies $\|u(t, \cdot)\|_{H_1} \leq \varepsilon$ for all $t \geq 0$, and $\|u(t, \cdot)\|_{H_1} \to 0$ exponentially fast as $t \to \infty$.

If in addition $f$ satisfies the hypothesis of Proposition 7.12 part (1) or (2), then for every $u_0 \in H_1$ there is a unique $(H_1^*)'$-weak solution $u \in C(\mathbb{R}, H_0)$ of (7.25), (7.26) with $u(0, \cdot) = u_0$.

Remark 7.15. We emphasize that under the additional assumptions mentioned in Proposition 7.3 it suffices to consider eigenvalues of $A - B$ instead of $\gamma$-weak eigenvalues.

Proof. In case (C), part (1), the main result of [3] implies in view of [11] that $A$ is a Kato operator. In case (C), part (2), $a$ is symmetric, and so $A$ is a Kato operator by Proposition 2.2 or by Theorem A.7. Note that if the hypothesis of Proposition 7.3 part (2) is satisfied, then also the hypothesis of Proposition 7.9 part (3) is satisfied. Hence, the assertion follows from Theorem 3.16.

Remark 7.16. According to Theorem 5.5, the assertions (uniqueness and stability estimates) of Theorem 7.14 hold even in the possibly larger class of solutions in the sense of Remark 5.2.

Remark 7.17. In Theorem 7.14, the hypotheses of Proposition 7.3 part (1) or (2) can also be replaced by the hypothesis of Remark 7.4.

Example 7.18. Let $\Omega \subseteq \mathbb{R}^d$ be bounded with a Lipschitz boundary, $\Gamma_D, \Gamma_N \subseteq \partial \Omega$ be measurable with (7.24). Let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{C}$ be continuous with $f_i(0) = 0$, and suppose that there are $L \geq 0$ and $\rho > 0$ with

$$|f_i(u) - f_i(v)| \leq L(1 + |u| + |v|)^\rho|u - v|$$

(7.27)

for all $u \in \mathbb{C}^2$. Assume that $(b_{11}, b_{22}) = f_i'(0)$ exist for $i = 1, 2$, are real, and satisfy the sign conditions

$$b_{11} > 0, \quad b_{11} + b_{22} < 0, \quad b_{11}b_{22} - b_{12}b_{21} > 0.$$ 

For $d_1, d_2 > 0$, we consider the reaction-diffusion system

$$\frac{\partial u_j}{\partial t} = d_j \Delta u_j + f_j(u_1, u_2) \quad \text{on } \Omega \text{ for } j = 1, 2,$$

(7.28)

with mixed boundary conditions (for $u = (u_1, u_2)$)

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N.$$ 

(7.29)

Let $\kappa_k > 0$ $(k = 1, 2, \ldots)$ denote the nonzero eigenvalues of $\Delta$ with boundary conditions (7.29); if $\Gamma_D$ is a null set, do not include the trivial eigenvalue $\kappa_0 = 0$ into this sequence. Suppose $(d_1, d_2)$ lies to the right/under the envelope of the hyperbolas

$$C_k = \{(d_1, d_2) : d_1, d_2 > 0 \text{ and } (\kappa_k d_1 - b_{11})(\kappa_k d_2 - b_{22}) = b_{12}b_{21}\}.$$
that is, \((d_1, d_2)\) belongs to
\[
\bigcap_{k=1}^{\infty} \left\{(d_1, d_2) : d_1 \geq \kappa_k^{-1} b_{11} \text{ or } d_2 < \frac{\kappa_k^{-2} b_{12} b_{21}}{d_1 - \kappa_k^{-1} b_{11}} + \frac{b_{22}}{\kappa_k}\right\},
\]
(7.30)
see Figure 7.1. Then the following holds in each of the following two cases.

\[\text{Figure 7.1. The hyperbolas } C_k\]

(1) \(H_\alpha = L_2(\Omega, \mathbb{C}^2)\) and one of the following holds:
(a) \(\gamma \in [1/2, 1)\) and either \(d = 1, \rho \leq 1\), or \(d = 2, \rho < 1\), or \(d \geq 3, \rho \leq 2/d\);
(b) \(D(A)\) is continuously embedded into \(L_\rho(\Omega, \mathbb{C})\), \(\rho < 1 - \frac{2}{p}\), and \(\gamma \in (0, 1)\) is sufficiently large;
(c) \(\rho \leq 1, \gamma \in (1/2, 1)\), and \(H_\gamma\) is continuously embedded into \(L_\infty(\Omega, \mathbb{C})\);
(d) \(D(A)\) is continuously embedded into \(W^{2,2}(\Omega, \mathbb{C})\), and either \(d \leq 3, \rho = 2, \gamma \in (d/4, 1)\), or \(d \geq 4, \rho < 4/d\), and \(\gamma \in (0, 1)\) is sufficiently large.

(2) \(H_\alpha = W^{1,2}(\Omega, \mathbb{C}^2)\) and one of the following holds:
(a) \(d \leq 2, \rho > 0, \gamma \in [0, 1/2)\);
(b) \(d \geq 3, \rho < 4/(d-2)\), \(\gamma \in [\max\{0, \gamma_0\}, 1/2)\), where \(\gamma_0\) is defined in (7.16) with \(\sigma = \rho + 1\).

For each \(\varepsilon > 0\) there is \(\delta > 0\) such that for each \(u_0 \in H_\alpha\) with \(\|u_0\| \leq \delta\) there is a unique \((H_\alpha)^{\gamma}\)'-weak solution \(u \in C([0, \infty), H_\alpha)\) of (7.28), (7.29) with \(u(0, \cdot) = u_0\), \(\|u(t, \cdot)\|_{H_\alpha} \leq \varepsilon\) for all \(t > 0\) and \(\|u(t, \cdot)\|_{H_\alpha} \to 0\) exponentially as \(t \to \infty\).

We first note that (1d) is actually a special case of (1b) and (1c) by the Sobolev embedding theorems and [24, Theorem 1.6.1], respectively. Since \(f_i\) is independent of \(x\) and \(t\), hypothesis (7.14) follows with \(\sigma = \rho + 1\) from (7.27) and from the definition of \(f_i'\). Note also that the symmetry of \(A\) implies \(D(A) = D(A')\) and \(H_\sigma = H_\sigma^\\ast\). The existence and uniqueness assertion follows from Proposition 7.12 or Remark 7.13 in case (1c).

For the stability assertion, we apply Theorem 7.14 or Remark 7.17 in case (1c) with \(r = \infty\) and \(\sigma = 1 + r\). In view of Proposition 7.9, it thus suffices to verify that there is \(\lambda_0 > 0\) such that every eigenvalue \(\lambda\) of \(A - B\) satisfies \(\text{Re } \lambda \geq \lambda_0\). Under condition (7.30) the latter was verified in [30]. It can be shown by a similar calculation that if \(d_i > 0\) violate (7.30) then there is an eigenvalue \(\lambda\) of \(A - B\) with \(\text{Re } \lambda < 0\) (\(\lambda = 0\) if \((d_1, d_2) \in \bigcup_{k=1}^{\infty} C_k\)). In this sense, the domain of stability sketched in Figure 7.1 is maximal.

Note that (1d) involves a strictly weaker requirement concerning \(\rho\) than (1a) for every \(d \geq 2\). The embedding required for (1d) holds in case \(\Gamma_D = \emptyset\) or \(\Gamma_N = \emptyset\) if \(\partial \Omega\) is sufficiently smooth.
The result obtained in [30] concerning Example 7.18 did not cover the case $H_\alpha = L_2(\Omega, \mathbb{C}^2)$. Moreover, even in the case $H_\alpha = W^{1,2}(\Omega, \mathbb{C}^2)$ and $d \geq 3$, the result in [30] essentially needed the more restrictive hypothesis $\rho \leq 2/(d-2)$ which is (almost) by the factor 2 worse than our above requirement for that case.

**Appendix A. On the Characterization of Kato Operators**

Throughout this section, let $a: V \times V \to \mathbb{C}$ be a continuous sesquilinear form satisfying (2.3). We use the notations of Section 4.

We start with a curious characterization of Kato operators which is a side result of our theory in Section 4:

**Theorem A.1.** (1) If $A$ is a Kato operator, then $A$ is a Kato operator; that is, there exists an equivalent scalar product on $V'$ and a continuous strongly accretive sesquilinear form $a: H \times H \to \mathbb{C}$ such that $A$ in $V'$ is associated with $a$ and $D(A^{1/2}) \cong H$.

(2) If $D(A^{1/2}) \cong H$, then $A$ and $A$ are Kato operators.

**Proof.** If $A$ is a Kato operator, then Lemma 4.7 and Proposition 4.3 imply that $A$ is associated with a corresponding form $a: H \times H \to \mathbb{C}$, and Proposition 4.3 implies $X_{1/2} \cong H$. Conversely, let $X_{1/2} \cong H$. Then [27, Theorem 1.15.2(e)] implies that $L := A^{1/2}|_V: V \to H$ is an isomorphism. In particular, $L$ is the $H$-realization of $A^{1/2}$, and so Proposition 13.2 implies that $A^{1/2} = L$. Hence, $D(A^{1/2}) = D(L) \cong V$ with the corresponding graph norm. □

As another side result, our theory implies also the following characterization of Kato operators which appears to be rather useful:

**Theorem A.2.** The following assertions are equivalent:

(1) $A$ is a Kato operator.

(2) $A^{1/2}$ is an isomorphism of $V$ onto $H$.

(3) There exists an $A$-Kato scalar product on $V$.

**Proof.** According to Lemma 4.7, it suffices to show that if there is an $A$-Kato scalar product on $V$, then $A$ is a Kato operator. Indeed, if there is an $A$-Kato scalar product on $V$, then Proposition 4.10 implies in view of Remark 4.11 for $\alpha = 1/2$ that $H_{1/2} \cong X_1$. Since $X_1 \cong V$, this means that $A$ is a Kato operator. □

We note that our approach has some similarities with [3], but the idea to consider appropriate scalar products on $V$ appears to be new.

As an application of Theorem A.2, we obtain now a sufficient criterion for Kato operators. In fact, in the following we give a necessary and sufficient condition under which the particular scalar product (2.6) is $A$-Kato.

Recall that Proposition 2.2 implies in particular that $A^{-1}: H \to H$ is bounded. It is well known (see e.g. [14, Theorem III.5.30]) that this implies that also $(A^*)^{-1}: H \to H$ exists and is bounded and is actually the (bounded) Hilbert-space adjoint $(A^{-1})^*$, i.e.

$$(A^*)^{-1} = (A^{-1})^*.$$  

(A.1)

**Definition A.3.** We call $A$ quasi-symmetric if there are constants $\alpha > -1$, $\beta$, $M \in [0, \infty)$ with

$$\text{Re}((A^*)^{-1}(Au + Mu), u) \geq \alpha|u|^2 \quad \text{and} \quad |(A^*)^{-1}Au| \leq \beta|u| \quad \text{for all} \quad u \in D(A).$$  

(A.2)

If $M \geq 0$ is given, we call $A$ $M$-quasi-symmetric if there are constants $\alpha > -1$, $\beta \geq 0$ with (A.2).
Remark A.4. The larger \( M \) is, the less restrictive condition (A.2) becomes. Indeed, (A.1) implies

\[
\text{Re}( (A^*)^{-1} u, u ) = \text{Re}( u, A^{-1} u ) = \text{Re}( A(A^{-1} u), A^{-1} u ) \\
\geq \max\{ c\| A^{-1} u \|^2, c_0|A^{-1} u|^2 \} \geq 0 \quad \text{for all } u \in H.
\]

Remark A.5. If \( A \) is symmetric, then (A.3) implies that \( A \) is \( M \)-quasi-symmetric with every \( M \geq 0 \).

Roughly speaking, estimates (A.2) mean indeed that \( A \) is quantitatively almost symmetric in the sense that \( (A^*)^{-1} A \) does not differ too much from the identity in a quantitative manner, namely that it is “almost” accretive and bounded in \( H \) (on the subspace \( D(A) \)). The restriction \( \alpha > -1 \) may appear very strange at a first glance, but it is the correct hypothesis for the following result:

Proposition A.6. For every \( M \geq 0 \) the following assertions are equivalent.

1. \( A \) is \( M \)-quasi-symmetric.
2. There are \( \alpha, \beta > 0 \) with

\[
\text{Re}(Au + Mu, A^{-1}u) \geq \alpha |u|^2 \quad \text{and} \quad |(Au, A^{-1}u)| \leq \beta |u|^2
\]

for all \( u \in D(A) \).
3. The formula (2.6) defines an \( A \)-Kato scalar product on \( V \).

The relation of the largest possible constants \( \alpha \) in (A.2) and (A.4) and \( c_1 \) in Proposition 4.3 is given by \( 2c_1 = 1 + \alpha \).

Proof. For every \( u \in D(A) \), we obtain from (2.6), the definition of \( A \), and (A.1) that

\[
2b_M(u, A^{-1}v) = a(u, A^{-1}v) + a(A^{-1}u, v) + M \cdot (u, A^{-1}v) \\
= (Au + Mu, A^{-1}v) + (u, v) = ((A^*)^{-1}(Au + Mu), v) + (u, v).
\]

Hence, if (A.5) or (A.6) hold, then (A.2) or (A.3) hold with \( \alpha := 2c_1 - 1 \) and some \( \beta, \beta' > 0 \), respectively. Conversely, if (A.2) or (A.3) holds, then (A.5) shows that (A.5) or (A.6) hold with \( c_1 := (\alpha + 1)/2 \) and some \( c_2, c_3 > 0 \), respectively. \( \square \)

Theorem A.7. If \( A \) is quasi-symmetric, then \( A \) is a Kato operator.

Proof. In view of Proposition A.6, the assertion follows from Theorem A.2. \( \square \)

Appendix B. Some Assertions on Fractional Power Spaces

The following result holds in the setting of Section 4.

Proposition B.1. If the embedding \( i: V \to H \) is compact, then also the embedding \( H' \subseteq V' \)
and each of the embeddings \( H_{1/2} \subseteq H_\beta, X_\alpha \subseteq X_\beta \) is compact for every \( \alpha > \beta \geq 0 \). If additionally \( A \) is a Kato operator, then also each of the embeddings \( H_{1/2} \subseteq V \subseteq H_\gamma \subseteq H \subseteq H_{1/2} \subseteq V' \)
and \( X_\alpha \subseteq H \subseteq X_\gamma \) is compact for \( \alpha > 1/2 > \gamma > 0 \).

Proof. The adjoint of any compact operator is compact by Schauder’s theorem, see e.g. [8 Theorem VI.4.2]. Hence, \( H' \subseteq V' \) is compactly embedded, and since the embeddings \( D(A) \subseteq V \) and \( D(A) \subseteq H \) are bounded, we have compactness of the embeddings \( D(A) \subseteq H \) and \( D(A) \subseteq V' \). It follows that \( A \) and \( A \) have compact resolvents, and so [11 Theorem 1.4.8] implies the first assertion. If \( A \) is Kato, then \( V \cong H_{1/2}, H \cong X_{1/2}, H_\gamma \cong X_{1/2+\gamma}, H_\gamma \cong X_{1/2+\gamma}, V' \cong X_0, \) with \( \beta = 1/2 - \gamma \) by Theorem A.1 and Propositions 4.9 and 4.10.

Hence, the second assertion follows from the first. \( \square \)
For the following result, we consider a more general setting: Let \((E, \| \cdot \|)\) be a Banach space and \(\mathcal{L} : D(\mathcal{L}) \to E\) be a densely defined closed linear operator of positive type. Then \(\mathcal{L}^\alpha\) are defined for every \(\alpha \in \mathbb{C}\) on their natural domain \(D(\mathcal{L}^\alpha) \subseteq E\). Here, we use Komatu’s characterization of fractional power operators \([16]\) (see also \([27, \text{Section 1.15.1}]\)). We equip \(D(\mathcal{L}^\alpha)\) with the graph norm of \(\mathcal{L}^\alpha\) in \(E\).

**Proposition B.2.** Let \(\mathcal{L} : D(\mathcal{L}) \to E\) be a closed densely defined operator of positive type in a Banach space \((E, \| \cdot \|)\). Let \(U \subseteq E\) be a continuously embedded Banach space with \(D(\mathcal{L}) \subseteq U\). Let \(L : D(L) \to U\) denote the \(U\)-realization of \(\mathcal{L}\), that is, \(D(L) = \mathcal{L}^{-1}(U)\) and \(L = \mathcal{L}|_{D(L)}\). Suppose that \(L\) is closed, densely defined and of positive type. Then we have for any \(\alpha \in \mathbb{C}\) that \(L^\alpha = \mathcal{L}^\alpha|_{D(L^\alpha)}\), and in case \(\Re \alpha > 0\), \(L^\alpha\) is the \(U\)-realization of \(\mathcal{L}^\alpha\), that is, \(D(L^\alpha) = (\mathcal{L}^\alpha)^{-1}(U)\).

**Proof.** We note first that \(D(\mathcal{L}) \subseteq U\) implies

\[
(\mathcal{L} + \lambda \text{id}_E)^{-1}|_U = (L + \lambda \text{id}_U)^{-1} \quad \text{for all } \lambda \geq 0. 
\]

(B.1)

Indeed, if \(Lu + \lambda u = v \in U\), then \(\mathcal{L}u = v - \lambda u \in U\), and so \(u \in D(L)\) and \(Lu = \mathcal{L}u\) since \(L\) is the \(U\)-realization of \(\mathcal{L}\). Hence, (B.1) is established.

By definition, \(L^\alpha u\) and \(\mathcal{L}^\alpha u\) is (for certain values of \(u\)) an integral in \(U\) (in \(E\)) of an integrand containing only iterates of (B.1) and of \(L = \mathcal{L}|_{D(L)}\) (and \(\mathcal{L}\)). Since due to the continuous embedding \(U \subseteq E\) integrals in \(U\) have the same value as integrals in \(E\), we obtain thus that \(L^\alpha u = \mathcal{L}^\alpha u\) for all \(u\) on a subset \(M \subseteq D(L^\alpha)\) such that \(L^\alpha\) is the closure of \(L^\alpha|_M\). Hence, for every \(u \in D(L^\alpha)\) there is a sequence \(u_n \in M\) with \(\|u_n - u\|_U \to 0\) and \(\|\mathcal{L}^\alpha u_n - L^\alpha u\|_U \to 0\). It follows that \(\|u_n - u\| \to 0\) and \(\|\mathcal{L}^\alpha u_n - L^\alpha u\| \to 0\) which implies \(u \in D(L^\alpha)\) and \(L^\alpha u = \mathcal{L}^\alpha u\).

In case \(\Re \alpha > 0\), we obtain by [27, Theorem 1.15.2(e)] that \(\mathcal{L}^\alpha\) is a bijection of \(D(\mathcal{L}^\alpha)\) onto \(E\) and that \(L^\alpha\) is bijection of \(D(L^\alpha)\) onto \(U\). Hence, \(\text{id}_{D(L^\alpha)} = (\mathcal{L}^\alpha)^{-1}L^\alpha\) is a bijection of \(D(L^\alpha)\) onto \((\mathcal{L}^\alpha)^{-1}(U)\), that is, these two sets coincide. \(\square\)

**Proposition B.3.** Let \(A : D(A) \to H\) be a closed densely defined operator in a Hilbert space \(H\) such that the left open half-plane is contained in the resolvent set of \(A\), and such that there is \(C > 0\) with \(\|(A + \lambda \text{id})^{-1}\| \leq C/ \Re \lambda \) whenever \(\Re \lambda > 0\); in particular, \(A\) is of positive type. Then also the Hilbert-space adjoint \(A^* : D(A^*) \to H\) has the same properties, and we have for every \(\alpha \in (-1, 1)\) that \((A^*)^\alpha = (A^\alpha)^*\).

**Proof.** \(A^*\) is closed and densely defined by [14, Theorem III.5.29]. Since (A.1) implies that

\[
(A^* + \lambda \text{id})^{-1} = ((A + \lambda \text{id})^{-1})^* 
\]

(B.2)

are defined for \(\lambda\) on the resolvent set of \(A\), and \(\|(A^* + \lambda \text{id})^{-1}\| = \|(A + \lambda \text{id})^{-1}\|\), the first assertions are immediate. In case \(\alpha = 0\), there is nothing more to prove. In case \(\alpha \in (-1, 0)\), we use the representation formula

\[
A^\alpha = -\frac{\sin \pi \alpha}{\pi} \int_0^\infty s^\alpha(s \text{id} + A)^{-1} ds, 
\]

see e.g. [14, Remark III.3.50]. We obtain from (B.2) and the same formula with \(A\) replaced by \(A^*\) that

\[
(A^*)^\alpha = -\frac{\sin \pi \alpha}{\pi} \int_0^\infty s^\alpha(s \text{id} + A^*)^{-1} ds = (A^\alpha)^*. 
\]

In view of (A.1), this implies also the assertion in case \(\alpha \in (0, 1)\), since \(A^\alpha\) and \((A^\alpha)^*\) are the inverses of \(A^{-\alpha}\) and \((A^*)^{-\alpha}\), respectively. \(\square\)
References

[1] Agranovich, M. S., Mixed problems on a Lipschitz domain for strongly elliptic second-order systems, Funkcional. Anal. i Priložen. 45 (2011), no. 2, 1–22, Engl. transl.: Funct. Anal. Appl. 45 (2011), no. 2, 81–98.

[2] ________, Spectral problems in Lipschitz domains, Sovrem. Mat. Fundam. Napravl. 39 (2011), 11–35, Engl. transl.: J. Math. Sci. 190 (2013), no. 1, 8–33.

[3] Agranovich, M. S. and Selitskii, A. M., Fractional powers of operators corresponding to coercive problems in Lipschitz domains, Funkcional. Anal. i Priložen. 47 (2013), no. 2, 2–17, Engl. transl.: Funct. Anal. Appl. 47 (2013), no. 2, 83–95.

[4] Amann, H., Linear and quasilinear parabolic problems, vol. I, Birkhäuser, Basel, Boston, Berlin, 1995.

[5] Appell, J. and Väth, M., Elemente der Funktionalanalysis, Vieweg & Sohn, Braunschweig, Wiesbaden, 2005.

[6] Arendt, W., Semigroups and evolution equations: Functional calculus, regularity and kernel estimates, Handbook of Differential Equations (Dafermos, C. M. and Feireisl, E., eds.), vol. 1, Elsevier, Amsterdam, 2004, 1–85.

[7] Arendt, W., Bu, S., and Haase, M., Functional calculus, variational methods, and Lyapunov’s theorem, Arch. Math. (Basel) (2001), no. 77, 65–75.

[8] Banach, S., Introduction to linear operations, North-Holland, Amsterdam, New York, Oxford, 1987, translation of Théorie des Opérations Linéaires, Polish Scientific Publ., Warsaw, 1979.

[9] Dore, G., IF∞ functional calculus in real interpolation spaces, Studia Math. 137 (1999), no. 2, 162–166.

[10] Haller-Dintelmann, R. and Rehberg, J., Maximal parabolic regularity for divergence operators including mixed boundary conditions, J. Differential Equations 247 (2009), 1354–1396.

[11] Henry, D., Geometric theory of semilinear parabolic equations, Lect. Notes Math., no. 840, Springer, Berlin, New York, 1981.

[12] Kato, T., Fractional powers of dissipative operators, J. Math. Soc. Japan 13 (1961), 246–274.

[13] ________, Fractional powers of dissipative operators II, J. Math. Soc. Japan 14 (1962), 242–248.

[14] ________, Perturbation theory for linear operators, Springer, New York, 1966.

[15] Kim, I.-S. and Väth, M., The Krasnoselskii-Quittner formula and instability of a reaction-diffusion system with unilateral obstacles, (submitted).

[16] Komatsu, H., Fractional powers of operators, II Interpolation spaces, Pacific J. Math. 21 (1967), no. 1, 89–111.

[17] Krasnoselskii, M. A., Topological methods in the theory of nonlinear integral equations in Russian, Gostehizdat, Moscow, 1956, Engl. transl.: Pergamon Press, Oxford 1964.

[18] Lions, J.-L., Espaces d’interpolation et domaine de puissances fractionnaires d’opérateurs, J. Math. Soc. Japan 14 (1962), 233–241.

[19] Lions, J.-L. and Magenes, E., Non-homogeneous boundary value problems and applications I, Springer, Berlin, Heidelberg, New York, 1972.

[20] Lunardi, A., Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser, Basel, Boston, Berlin, 1994.

[21] Mazja, V. G., Sobolev spaces, Springer, Berlin, Heidelberg, 1985.

[22] McLean, W., Strongly elliptic systems and boundary integral equations, Univ. Press, Cambridge, 2000.

[23] Ouhabaz, E. M., Analysis of heat equations on domains, Princeton Univ. Press, Princeton, Oxford, 2005.

[24] Pazy, A., Semigroups of linear operators and applications to partial differential equations, Springer, New York, Berlin, Heidelberg, 1992.

[25] Shamin, R. V., Spaces of initial data for differential equations in Hilbert spaces and the Kato problem, Ulmer Seminar. Funktionalanalysis und Differentialgleichungen. 7 (2002), 375–388.

[26] Smoller, J., Shock waves and reaction diffusion equations, Springer, New York, 1983.

[27] Triebel, H., Interpolation theory, function spaces, differential operators, North-Holland, Amsterdam, New York, Oxford, 1978.

[28] Väth, M., Ideal spaces, Lect. Notes Math., no. 1664, Springer, Berlin, Heidelberg, 1997.

[29] ________, Continuity and differentiability of multivalued superposition operators with atoms and parameters I, J. Anal. Appl. 31 (2012), 93–124.

[30] ________, Instability of Turing type for a reaction-diffusion system with unilateral obstacles modeled by variational inequalities, Math. Bohem. (Prague, 2013), Proceedings of Equadiff 13, 2013.

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