Path Coalescence in Spatially Correlated Random Walks

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A particle subject to successive, random displacements is said to execute a random walk (in position or some other coordinate). The mathematical properties of random walks have been very thoroughly investigated, and the model is used in many areas of science and engineering as well as other fields such as finance and the life sciences. This letter describes a phenomenon occurring in a natural extension of this model: we consider the motion of a large number of particles subject to successive random displacements which are correlated in space, but not in time. If these random displacements are smaller than their correlation length, the trajectories coalesce onto a decreasing number of trails. This surprising effect is explained and quantitative results are obtained. Various possible realisations are discussed, ranging from coalescence of the tracks of water droplets blown off a windshield to migration patterns of animals.

The phenomenon we have discovered is illustrated in figure 1. It shows the positions $x$ of 20 particles as a function of time $t$. Each particle undergoes a random walk, but the displacements of nearby particles are correlated, such that particles which are very close experience almost identical displacements. The paths of the particles are seen to coalesce onto a decreasing number of trails, along which the...
particles follow almost the same trajectory. This effect is surprising because random walks normally reduce rather than accentuate inhomogeneities in the density of particles. In the following we explain this effect quantitatively and suggest some contexts in the physical sciences in which the effect can certainly be observed, and areas in biology where it may find interesting applications.

We consider a system of particles labelled by an index $i$, having positions $x_i(t)$ at time $t$. The particles are subjected to random displacements at times which are integer multiples of a small increment $\delta t$. At time $t = n \delta t$ the displacement is given by a random function $f_n(x)$ evaluated at the position $x_i(t)$:

$$x_i(t + \delta t) = x_i(t) + f_n(x_i(t)) .$$

(1)

The random displacements $f_n$ satisfy $\langle f_n(x) \rangle = 0$ and $\langle f_n(x)f_{n'}(x') \rangle = \delta_{nn'}c(x - x')$. Here $\delta_{nn'}$ is unity if $n = n'$ and zero otherwise, and we write $\langle A \rangle$ for the ensemble average of a quantity $A$ (that is, the average over different trials of the random process generating $A$). The correlation function $c(X)$ is even in $X = x - x'$, decays rapidly as $X \to \infty$, and is small when $|X| \gg \xi$. A suitable choice (adopted in figure 1 and in the following) is $c(X) = \varepsilon^2 \exp(-X^2/2\xi^2)$, $\varepsilon$ is the typical magnitude of each displacement and $\xi$ is the correlation length. Figure 1 used the values $\varepsilon^2 \approx 8 \times 10^{-6}$ and $\xi = 10^{-1}$.

As an example of a situation where the effect could be observed, consider the motion of liquid droplets on a surface, moving in one direction under a constant force (rain blown off a perspex windshield is an example of this situation). If the surface is randomly contaminated, the wetting angle will be different on opposite sides of each drop, and the trajectory of the droplet will be randomly deflected. We are concerned with the case where the surface contaminants are smeared over an area large compared to the droplets (perhaps resulting from cleaning the windshield with a waxy polish), so that nearby droplets are deflected in the same direction. There is no interaction between the drops unless they are close enough to combine due to surface tension: we stress that the coalescence is that of the paths taken by
different drops, not of the droplets themselves. We model the motion of a droplet across the surface by a particle of mass \( m \). At position \( \mathbf{r} = (x, y) \) on the surface, the drop is subject to a force \( \mathbf{F}(\mathbf{r}) + F_0 \mathbf{j} \), where \( F_0 \) is the magnitude of a steady force acting in the direction of the unit vector \( \mathbf{j} \) defining the \( y \)-axis and \( \mathbf{F}(\mathbf{r}) \) is a homogeneous and isotropic random force with correlation length \( \xi \). We assume that the particles are subjected to a viscous resistive force proportional to their velocity across the surface, such that the equations of motion are

\[
\frac{m}{\gamma} \frac{d\mathbf{r}}{dt} = \mathbf{p} , \quad \frac{d\mathbf{p}}{dt} = F_0 \mathbf{j} + \mathbf{F}(\mathbf{r}) - \gamma \mathbf{p} \tag{2}
\]

(where \( \mathbf{p} \) is the momentum of the drop). When the fluctuating force is weak, the trajectories are locally approximated by straight lines, with \( x \) approximately constant and with \( y \) increasing at a rate \( v_y = F_0/m\gamma \). Furthermore, when the damping \( \gamma \) is large, the velocity is given by the local value of the force: \( \mathbf{v}(t) \sim \mathbf{p}(t)/m = v_y \mathbf{j} - \mathbf{F}(x(t), v_y t)/m\gamma \). Under these circumstances the displacement \( x(t) \) perpendicular to the steady force satisfies

\[
\frac{dx(t)}{dt} = \frac{1}{m\gamma} F_x(x(t), v_y t) . \tag{3}
\]

When \( v_y \) is large, the velocity of the particle fluctuates very rapidly, and in this limit it is reasonable to model equation (2) by the stochastic equation (1).

Why do the trajectories in figure 1 merge? The simplest argument is based upon linear stability analysis. The separation \( \delta x(t) \) of two nearby trajectories varies exponentially in time, with rate \( \lambda = t^{-1} \langle \log |\delta x(t)|/|\delta x(0)| \rangle = \langle \log |1 + f'_n|/\delta t \rangle \). Provided \( \varepsilon/\xi \) is small, the magnitudes of the derivatives \( f'_n = df_n/dx \) are small compared to unity. Taylor expansion of the logarithm gives \( \lambda \sim -\frac{1}{2} \langle f'_n^2 \rangle/\delta t \), so that \( \lambda \) is negative. In this case, most nearby trajectories approach each other with an exponentially decreasing separation, implying coalescence. This argument also indicates that if \( \varepsilon/\xi \) is large, the coalescence effect disappears: if the function \( f_n \) has a Gaussian distribution, the exponent \( \lambda \) becomes positive when \( \langle f'_n^2 \rangle \) exceeds 2.421....
A more complete and concrete understanding of the coalescence effect is obtained by considering statistics of the density of particles, \( \rho(x,t) = \sum_i \delta(x(t) - x) \). Translational invariance implies that an initially uniform density remains uniform, \( \langle \rho(x,t) \rangle = \rho_0 \). Path coalescence is revealed by the density-density correlation function \( \mathcal{K}(x,x';t) = \langle \rho(x,t) \rho(x',t) \rangle - \rho_0 \delta(x - x') \). Because of translational invariance, the correlation function \( \mathcal{K} \) is a function of \( X = x - x' \) only: we write \( \mathcal{K}(x,x';t) = K(x-x',t) \). A tendency for particles to cluster is demonstrated by \( K(X,t) \) becoming large for \( X \) small, in the limit \( t \to \infty \). When the typical magnitude of the jumps is small compared to the correlation length \( \xi \), we find that the correlation function satisfies a generalised diffusion equation, or Fokker-Planck equation

\[
\frac{\partial K(X,t)}{\partial t} = \frac{\partial^2}{\partial X^2} \left[ D(X) K(X,t) \right] .
\] (4)

The diffusion constant \( D(X) = \frac{[c(0) - c(X)]/\delta t}{\delta t} \) approaches zero quadratically at the origin: \( D(X) \sim \kappa X^2 \) for \( X \ll \xi \), where \( \kappa = -\frac{1}{2} c''(0)/\delta t \). When \( X \gg \xi \) the diffusion constant approaches a constant value, \( D_0 = \varepsilon^2/\delta t \). Equation (4) was obtained from the stochastic model, equation (1), but in the appropriate limits the density-density correlation function of (3) also satisfies equation (4).

Now consider the properties of solutions of equation (4). We note that equation (4) is in the form of a continuity equation, \( \partial K/\partial t + \partial J/\partial x = 0 \), so that the integral of the correlation function over all \( X \) is a conserved quantity. The flux of the correlation function passing the separation parameter \( X \) at time \( t \) is

\[
J(X,t) = -\frac{\partial}{\partial X} [D(X)K(X,t)] .
\] (5)

Consider an initially uniform distribution of density, with value \( \rho_0 \) (corresponding to \( K(X,0) = \rho_0^2 \)). For \( X \ll \xi \) the diffusion constant is an increasing function of \( X \). Together with (3) this implies an initial flux of correlation towards \( X = 0 \). At large times, \( K(X,t) \) is thus sharply peaked at the origin. For \( X \gg \xi \), on the other hand,
the approximate solution of (4) is

\[ K(X, t) \sim \varrho_0^2 \text{erf} \left( \frac{|X|}{\sqrt{4D_0t}} \right). \]  

(6)

Using the fact that \( K(X, t) \) satisfies a conservation law, we deduce that the average number of particles condensing into each trail at time \( t \) is

\[ N(t) \sim \frac{4}{\sqrt{\pi}} \varrho_0 \sqrt{D_0t}. \]  

(7)

When \( t/\delta t \) is large, and \( X \ll \xi \) (but not too close to zero) the flux \( J(X, t) \) is found to be approximately uniform. This implies

\[ K(X, t) \sim \varrho_0^2 \frac{D_0}{\kappa \sqrt{\pi}} \frac{1}{X^1} \sqrt{\frac{D_0}{\pi}} \sqrt{t}. \]  

(8)

The \( 1/X \) divergence of (8) is non-integrable, so that this expression must fail near the origin. An exact calculation shows that \( K(X, t) \) is finite at \( X = 0 \), the height of the peak is \( K(0, t) = \exp(2\kappa t) \). In summary, at large times the correlation function \( K(X, t) \) develops a correlation hole for \( \xi \ll X \ll \sqrt{4D_0t} \) reflecting the path coalescence effect. The results of numerical simulations shown in figure 2 confirm these predictions.

In the remainder we discuss a number of possible realisations of the path coalescence effect. Unlike the case of liquid droplets described above, these are speculative, but they show that the path coalescence effect is likely to have a very broad range of applications.

There are many potential applications in the physical sciences involving fast particles interacting with a random potential. Two examples of this type are the motion of rocks rolling down a scree slope, and the motion of highly energetic electrons through disordered crystalline solids. There are also possible connections between the coalescence effect and a ‘streaming’ observed in the flow of electrons away from a constriction in a two-dimensional electron gas with very low scattering, and we shall discuss this case in some detail. The experiment shows regions of markedly increased current density persisting to some distance from the constriction. This
was explained by showing similarity to simulations of independent electron motion in the smoothly varying random potential of the doping atoms. This model is essentially (2), with $\gamma = 0$. Theoretical discussion of this system (5) has emphasised that caustics are important in understanding the empirical results. We remark that these experiments might show an even more pronounced effect if dissipation were introduced. Equations (2) can show various types of behaviour; figure 3 shows numerically computed trajectories for one choice of parameters. These trajectories initially show evidence of the streaming effect discussed in references (7, 8) with the associated caustics. For larger times the trajectories coalesce, in contrast to the dissipationless model where the streams eventually disperse. We note that dissipation of the electron motion could be increased by increasing the temperature of the system.

There are also potential applications in the biological sciences, involving the movement of organisms in response to small random fluctuations in their environment. Our model provides a mechanism through which large numbers of organisms can congregate without communicating.

One example of this type is the migration of animals across a nearly homogeneous smooth terrain. Thus figure 1 could be thought of as a map showing paths of animals on an Eastward migration. The paths of the animals will be deflected by small random fluctuations of topography or vegetation. Our calculations show that the animals can be drawn together onto the same paths, even if there is no communication between them, and no gross features in the terrain favouring particular routes.

A second example applies to simple organisms such as plankton which can move in response to changes in their environment, such as nutrient concentration. In cases where there are small, spatially correlated random fluctuations of the nutrient concentration, the path coalescence effect could lead to unexpectedly large concentrations of organisms. Such a mechanism could be utilised by evolution, en-
abling simple organisms which cannot communicate directly to congregate for sexual reproduction.

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[5] The condition defining this overdamped situation is $\gamma \tau \gg 1$, where $\tau$ is the timescale of fluctuations of the potential experienced by a particle moving with speed $v_y$, namely $\tau = \xi / v_y$.

[6] Equation (1) is the Itô stochasticisation of equation (3). If we use the arguably more appropriate Stratonovich stochasticisation, we still arrive at the same equation (6) for the correlation function.

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Figure 1: Positions $x_i(t)$ for 20 particles performing spatially correlated random walks. The trajectories coalesce.

Figure 2: Statistics of the particle density $\rho(x, t)$. a The density-density correlation function $K(X, t)$ of the motion illustrated in figure 1 derived from simulations ($\circ$), compared with the limiting theoretical forms (6), full line, and (8), dashed line. Parameter values: $\varepsilon^2 \approx 1.25 \times 10^{-8}$, $\xi \approx 6.4 \times 10^{-3}$, and $t = 5 \times 10^5\delta t$. The inset shows numerical results verifying that $K(0, t) = \exp(2\kappa t)$ for the same parameter values. b The mean number of particles in a cluster, $N(t)$, for the process shown in figure 1. The results of the simulation ($\circ$) are compared with theory (7), full line. Parameter values: $\varepsilon^2 \approx 2 \times 10^{-7}$, $\xi \approx 1.6 \times 10^{-3}$. Particles are considered to be part of a cluster of $N$ if all of their positions are within an interval of length $\xi$.

Figure 3: Trajectories (red) for damped motion in a smooth random potential described by equation (2). Particles are introduced with uniform density on the left; here the $x$ axis is vertical. The force is a sum of two terms: a random force derived from the gradient of a potential, $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$ with $\langle V(\mathbf{r})V(\mathbf{r}')\rangle = \varepsilon^2 \exp[(\mathbf{r} - \mathbf{r}')^2/2\xi^2]$, and a steady force $F_0\mathbf{j}$ acting in the $y$-direction. The potential is shown in green, higher values correspond to darker colours. Parameter values: $F_0 = 1.5$, $\varepsilon^2 = 1.25 \times 10^{-4}$, $\xi \approx 6 \times 10^{-2}$, $m = 1$, and $\gamma = 4$. 
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