DIMENSION-LIKE FUNCTIONS AND SPECTRUMS OF FINSLER MANIFOLDS

ZHONGMIN SHEN AND WEI ZHAO

Abstract. In this paper, we study the spectral problem on a compact Finsler manifold by dimension-like functions. First, the existences of solutions for both the closed eigenvalue problem and Dirichlet eigenvalue problem have been proved under a weak but natural assumption on dimension-like functions. Secondly, the spectrums induced by the Lusternik-Schnirelmann category, the Krasnoselskii genus and the essential dimension are investigated and some unexpected examples of trivial spectrum are found. Thirdly, we give a Cheng type upper bound estimate for eigenvalues.

1. Introduction

Throughout this paper, a triple \((M,F,\text{d}m)\) denotes a compact reversible Finsler manifold with or without smooth boundary equipped with a smooth measure \(\text{d}m\). Set

\[ X := \begin{cases} H^{1,2}(M) & \text{if } \partial M = \emptyset, \\ \{ f \in H^{1,2}(M) : f|_{\partial M} = 0 \} & \text{if } \partial M \neq \emptyset, \end{cases} \]

Then the canonical energy functional (i.e., Rayleigh quotient) on \(X \setminus \{0\}\) is defined by

\[ E(u) := \frac{\int_M F^{*}2(du)\text{d}m}{\int_M u^2\text{d}m}. \]

Since \(E\) is 0-homogenous, i.e., \(E(\alpha u) = E(u)\) for all \(\alpha \neq 0\), it is convenient to consider the restriction of \(E\) on the "\(L^2\)-unit sphere"

\[ S := \left\{ u \in \mathcal{X} : \int_M u^2\text{d}m = 1 \right\}. \]

According to [11], \(u \in S\) (resp., \(\lambda = E(u)\)) is called an eigenfunction (resp., eigenvalue) if it is a critical point of \(E\) on the "\(L^2\)-unit sphere" that is characterized by the following Euler equation:

\[ 0 = (v, D E(u)) = -2 \int_M (\Delta u + \lambda u)v\text{d}m, \forall v \in \mathcal{X}. \]

The problem arising from [11] is called a closed problem (resp., Dirichlet problem) if \(\partial M = \emptyset\) (resp., \(\partial M \neq \emptyset\)).

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If \( F \) is Riemannian, then (1.1) always has solutions and these eigenvalues can be ordered

\[
0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \rightarrow +\infty.
\]

In particular, each of them can be obtained by Courant’s minimax principle, i.e.,

\[
\lambda_k = \min_{V \in \mathcal{H}_k} \max_{u \in V \setminus \{0\}} E(u),
\]

where

\[
\mathcal{H}_k = \{ V \subset \mathcal{F} : V \text{ is a linear subspace with } \dim_C(V) = k \},
\]

and \( \dim_C \) is the Lebesgue covering dimension.

However, if \( F \) is non-Riemannian, the situation is much different, because that the Laplacian \( \Delta \) is a non-linear elliptic operator and usually \( \Delta u \) cannot be defined on \( M \) in the strong sense even if \( u \in C^\infty(M) \), which makes it hard to study the existence of eigenvalue from (1.1). Thus, a natural problem is how to define a spectrum on a Finsler manifold \((M, F, dm)\) such that each eigenvalue satisfies (1.1) and it is equivalently given by (1.2) when \( F \) is Riemannian.

Inspired by Gromov [8], we study this spectral problem in this paper by so-called dimension-like functions. More precisely, let \( \mathcal{F} \) be a certain collection of subsets of \( S \). A \textit{dimension-like function} \( \dim : \mathcal{F} \to \mathbb{N} \cup \{\pm \infty\} \) is a function satisfying

\[
\dim(A) \leq \dim(B), \text{ for any } A, B \in \mathcal{F} \text{ with } A \subset B.
\]

Given \( \lambda \in \mathbb{R}^+ \cup \{\pm \infty\} \), the \textit{dimension} of the level \( E^{-1}[0, \lambda] \) is defined as

\[
\dim E^{-1}[0, \lambda] := \sup\{ \dim(A) : A \in \mathcal{F}, A \subset E^{-1}[0, \lambda] \}.
\]

The \textit{spectrum} \( \{\lambda_k\}_{k=1}^{\infty} \) of \( E \) induced by \( (\mathcal{F}, \dim) \) is defined as follows:

\[
\lambda_k = \sup\{ \lambda \in \mathbb{R}^+ \cup \{\pm \infty\} : \dim E^{-1}[0, \lambda] < k \}.
\]

For such a spectrum, we find that each eigenvalue has a corresponding eigenfunction, i.e., there exists a solution for (1.1), if \( \dim \) satisfies a monotonicity condition.

**Theorem 1.1.** Suppose \( \dim \) satisfies one of the following conditions:

(i) \( \dim(A) \leq \dim(H(A, 1)) \), for any \( A \in \mathcal{F} \) and any \( C^0 \)-map \( H : S \times [0, 1] \to S \) with \( H(\cdot, 0) = \text{Id} \);

(ii) \( \dim(A) \leq \dim(h(A)) \), for any \( A \in \mathcal{F} \) and any homeomorphism \( h : S \to S \).

Then for each finite eigenvalue \( \lambda_k \), there exists \( u \in S \) such that

\[
\int_M (\Delta u + \lambda_k u) v dm = 0, \forall v \in \mathcal{F}.
\]

We remark that the monotonicity condition of \( \dim \) in Theorem 1.1 can be weakened. In fact, if \( \dim \) is increasing under some special homotopies or homeomorphisms, the theorem still holds. See Theorem 5.3 below for more details.

Next we construct some specific dimension-like functions on \( S \). Set \( \mathcal{F}^S := \{ A \subset S : A \text{ is compact} \} \). For any \( A \in \mathcal{F}^S \), we define the \textit{Lusternik-Schnirelmann dimension} and the \textit{essential dimension} by

\[
\dim_{LS}(A) := \text{cat}_{\mathcal{F}}(p(A)), \quad \dim_{ES}(A) := \text{ess}(p(A)) + 1,
\]

where \( p : S \to \mathbb{P}(\mathcal{F}) := S/\mathbb{Z}_2 \) is a 2-fold covering, \( \text{cat}_{\mathcal{F}}(\cdot) \) and \( \text{ess}(\cdot) \) are the relative Lusternik-Schnirelmann category [5] and the essential dimension [8] defined on \( \mathbb{P}(\mathcal{F}) \), respectively. In this paper, the spectrum induced by \( (\mathcal{F}^S, \dim_{LS}) \) (resp., \( (\mathcal{F}^S, \dim_{ES}) \)) is called the \textit{strong} \( \dim_{LS}-\text{spectrum} \) (resp., the \textit{strong} \( \dim_{ES}-\text{spectrum} \), which is denoted by \( \{\lambda_k^{LS}\} \) (resp., \( \{\lambda_k^{ES}\} \)). Inspired by [1] [21], we
also consider the spectrum induced by the Krasnoselskii genus. More precisely, let 
\( S^F := \{ A \in S : A = -A \} \) and for any \( A \in S^F \), set \( \gamma(A) := \dim_K(A) \), where \( \gamma(\cdot) \) denotes the Krasnoselskii genus (cf. [15]). The spectrum induced by 
\((S^F, \dim_K)\) is called the strong \( \dim_K \)-spectrum.

Then we have the following result.

**Theorem 1.2.** Let \((M,F,dm)\) be a compact reversible Finsler manifold. Then
\[
\lambda^{\text{SES}}_k \leq \lambda^{\text{SLS}}_k = \lambda^S_k, \quad \forall k \in \mathbb{N}^+.
\]
Moreover, for \( \lambda_k = \lambda^{\text{SES}}_k, \lambda^{\text{SLS}}_k \) or \( \lambda^S_k \), we have

(i) (Monotonicity)
\[
0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots < +\infty, \text{ if } \partial M = \emptyset,
\]
\[
0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots < +\infty, \text{ if } \partial M \neq \emptyset,
\]
In particular, the multiplicity of each eigenvalue \( \lambda_k \) is always finite.

(ii) (First positive eigenvalue)
\[
\begin{cases}
\lambda_2 = \inf_{u \in \mathcal{F}_0} E(u), & \text{if } \partial M = \emptyset; \\
\lambda_1 = \inf_{u \in \mathcal{F}_0} E(u), & \text{if } \partial M \neq \emptyset;
\end{cases}
\]
where
\[
\mathcal{F}_0 := \left\{ f \in H^{1,2}(M) : \int_M f dm = 0 \right\} \quad \text{if } \partial M = \emptyset,
\]
\[
\left\{ f \in C^\infty(M) : f|_{\partial M} = 0 \right\} \quad \text{if } \partial M \neq \emptyset.
\]

(iii) (Existence of eigenfunction) For each \( k \in \mathbb{N}^+ \), there exists \( u = (m(M))^{-\frac{1}{2}} \) or 
\( u \in \mathcal{F}_0 \cap C^{1,\alpha} \) with
\[
\int_M (\Delta u + \lambda_k u)vdm = 0, \quad \forall v \in \mathcal{F}.
\]

(iv) (Riemannian case) \( \{\lambda_k\}_{k=1}^\infty \) is the standard spectrum when \( F \) is Riemannian.

We remark that (i)-(iv) in Theorem 1.1 remain valid for "weaker" spectrums which are induced by the collection of closed subsets instead of the collection compact subsets. See Sec. 5 below for more details. But, unfortunately, we yet do not know whether \( \lambda^{\text{SES}}_k < \lambda^{\text{SLS}}_k \) or \( \lambda^{\text{SES}}_k = \lambda^{\text{SLS}}_k \). This problem will be discussed somewhere else.

On the other hand, it is easy to check that these three dimension-like functions satisfy the following property:

Given \( k \in \mathbb{N}^+ \), for an arbitrary \( k \)-dimensional linear subspace \( V \subset \mathcal{F} \),
\[
V \cap \mathcal{S} \in \mathcal{F} \text{ with } \dim(V \cap \mathcal{S}) \geq k.
\]
(1.3)
The importance of this property is that it furnishes the following Cheng type upper bound estimate for eigenvalues.

**Theorem 1.3.** Let \((M,F,dm)\) be a compact convex reversible Finsler \( n \)-manifold with diameter \( D \), and let \( \{\lambda_k\}_{k=1}^\infty \) be the spectrum induced by \((\mathcal{F}, \dim)\) which satisfies \([1,3]\) for all \( k \in \mathbb{N}^+ \). If for some \( N \in [n, +\infty) \), the \( N \)-Ricci curvature satisfies \( \text{Ric}_N \geq (N - 1)K \), then
\[
\lambda_k \leq \frac{(N - 1)^2}{4} |K| + C(N) \left( \frac{k}{D} \right)^2, \quad \forall k \in \mathbb{N}^+
\]
where \( C(N) \) is a constant only dependant on \( N \).
It is noticeable that not every dimension-like function is suitable to define the spectrum. In fact, there exist trivial spectrums induced by some dimension-like functions which have nice properties. For example, the spectrum induced by \((\mathcal{F}^S, \text{cat}_S)\) on a closed Finsler manifold satisfies
\[
\lambda_k = \begin{cases} 
0, & \text{if } k = 1; \\
+\infty, & \text{if } k \geq 2,
\end{cases}
\]
where \(\text{cat}_S\) is the relative Lusternik-Schnirelmann category on \(S\). That is the reason why we define \(\text{dim}_{LS}\) by \(\text{cat}_{P(X)}\) instead of \(\text{cat}_S\). See Sec. 5.4 for more examples.

Recently the Krasnoselskii spectrum on \(L^2(M)\) and it convergence under the Gromov-Hausdorff distance have been studied in an excellent reference [1] due to Ambrosio-Honda-Portegies. We remark that not only their interest but the method are much different from ours. In fact, instead of the Rayleigh quotient, the Cheeger energy is their main tool. Refer to [1] for more details.

Last but not least, although we focus on reversible Finsler manifolds, lots of results in this paper, for instance, Theorem 1.1 and 1.3 can be extended to the irreversible case.

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**2. Preliminaries**

In this section, we recall some definitions and properties about Finsler manifolds. See [2] [20] for more details.

A reversible Finsler \(n\)-manifold \((M, F)\) is an \(n\)-dimensional differential manifold \(M\) equipped with a (reversible) Finsler metric \(F\) which is a nonnegative function on \(TM\) satisfying the following two conditions:

1. \(F\) is absolutely homogeneous, i.e., \(F(\lambda y) = |\lambda| F(y)\), for any \(\lambda \in \mathbb{R}\) and \(y \in TM\);
2. \(F\) is smooth on \(TM \setminus \{0\}\) and the Hessian \(\frac{1}{2}[F^2]_{y'y'}(x, y)\) is positive definite, where \(F(x, y) := F(y', \frac{dy'}{dx'})|_x\).

Note that if \(F\) is only positive homogeneous but not absolutely homogeneous, i.e., \(F(\lambda y) = \lambda F(y)\), only for \(\lambda > 0\), then \(F\) is called a Finsler metric which might be nonreversible. In this paper, we mainly consider the reversible case.
Let $\pi : PM \to M$ and $\pi^*TM$ be the projective sphere bundle and the pullback bundle, respectively. Then a Finsler metric $F$ induces a natural Riemannian metric $g = g_{ij}(x, [y]) \, dx^i \otimes dx^j$, which is the so-called fundamental tensor, on $\pi^*TM$, where

$$g_{ij}(x, [y]) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad dx^i = \pi^* \, dx^i.$$ 

The Euler theorem yields that $F^2(x, y) = g_{ij}(x, [y])y^i y^j$, for $(x, y) \in TM \setminus \{0\}$. It also should be remarked that $g_{ij}$ can be viewed as a local function on $TM \setminus \{0\}$, but it cannot be defined on $y = 0$ unless $F$ is Riemannian.

The uniformity constant $\Lambda_F$ (\[5\]) is defined by

$$\Lambda_F := \sup_{X,Y,Z \in SM} \frac{g_X(Y,Y)}{g_Z(Y,Y)},$$

where $S_xM := \{y \in T_xM : F(x, y) = 1\}$ and $SM := \cup x \in SM$. Clearly, $\Lambda_F \geq 1$, with equality iff $F$ is Riemannian.

The average Riemannian metric $\hat{g}$ on $M$ induced by $F$ is defined as

$$\hat{g}(X,Y) := \frac{1}{\nu(S_xM)} \int_{S_xM} g_y(X,Y) \, d\nu_x(y), \quad \forall X,Y \in T_xM,$$

where $\nu(S_xM) = \int_{S_xM} d\nu_x(y)$, and $d\nu_x$ is the Riemannian volume form of $S_xM$ induced by $F$. It is noticeable that

$$(2.1) \quad \Lambda_F^{-1} \cdot F^2(X) \leq \hat{g}(X,X) \leq \Lambda_F \cdot F^2(X),$$

with equality iff $F$ is Riemannian.

The dual Finsler metric $F^*$ on $M$ is defined by

$$F^*(\eta) := \sup_{x \in T_xM \setminus \{0\}} \eta(X) \frac{g_X(X,\cdot)}{F(X)}, \quad \forall \eta \in T^*_xM,$$

which is a Finsler metric on $T^*M$. The Legendre transformation $\mathcal{L} : TM \to T^*M$ is defined by

$$\mathcal{L}(X) := \begin{cases} \frac{g_X(X,\cdot)}{F(X)} & X \neq 0, \\ 0 & X = 0. \end{cases}$$

In particular, $F^*(\mathcal{L}(X)) = F(X)$. Given $f \in C^1(M)$, the gradient of $f$ is defined as $\nabla f = \mathcal{L}^{-1}(df)$. Thus, $df(X) = g_{\mathcal{L}(X)}(\nabla f, X)$.

The Finsler metric induces a vector field on $TM \setminus \{0\}$

$$\textbf{G} = y^i \frac{\partial}{\partial x^i} - G^i \frac{\partial}{\partial y^i}$$

where $G^i$ are given by

$$G^i(y) := \frac{1}{4} g^{ij}(y) \left\{ 2 \frac{\partial g_{ij}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^i}(y) \right\} y^j y^k.$$

The Riemannian curvature $R_y$ of $F$ is a family of linear transformations on tangent spaces. More precisely, set $R_y := R^i_k(y) \frac{\partial}{\partial x^i} \otimes dx^k$, where

$$R^i_k(y) := \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. $$

The Ricci curvature of $y$ is defined by

$$\text{Ric}(y) := \frac{R^i_k(y)}{F^2(y)}.$$
Given a Lipschitz continuous path \( \gamma : [0, 1] \to M \), the length of \( \gamma \) is defined by

\[
L_F(\gamma) := \int_0^1 F(\dot{\gamma}(t))dt.
\]

Define the distance function \( d : M \times M \to [0, +\infty) \) by \( d(p, q) := \inf L_F(\gamma) \), where the infimum is taken over all Lipschitz continuous paths \( \gamma : [a, b] \to M \) with \( \gamma(a) = p \) and \( \gamma(b) = q \). Given \( R > 0 \), the metric ball \( B_p(R) \) is defined by

\[
B_p(R) := \{ x \in M : d(p, x) < R \}.
\]

**3. Basic properties of \( H^{1,2}(M) \)**

Let \( (M, F, dm) \) be a compact (reversible) Finsler \( n \)-manifold with or without boundary \( \partial M \). Define a norm \( \| \cdot \|_H \) on \( C^\infty(M) \) by

\[
\| f \|_H := \| f \|_{L^2} + \| F^*(df) \|_{L^2},
\]

where

\[
\| f \|_{L^2} := \left( \int_M f^2dm \right)^{\frac{1}{2}}, \quad \| F^*(df) \|_{L^2} := \left( \int_M F^{-2}(df)dm \right)^{\frac{1}{2}}.
\]

It is easy to check that

1. \( \| f \|_H \geq 0 \), with equality if \( f = 0 \);
2. \( \| \lambda \cdot f \|_H = |\lambda| \cdot \| f \|_H \), \( \forall \lambda \in \mathbb{R} \);
3. \( \| f_1 + f_2 \|_H \leq \| f_1 \|_H + \| f_2 \|_H \).

Let \( H^{1,2}(M) \) be the completeness of \( C^\infty(M) \) under the norm \( \| \cdot \|_H \). Let \( \hat{g} \) be the average Riemannian metric induced by \( F \) and let \( \mathcal{H}^{1,2}(M) \) be the standard Sobolev space induced by \( \hat{g} \) on \( M \).

**Proposition 3.1.** \( H^{1,2}(M) = \mathcal{H}^{1,2}(M) \). Hence, \( H^{1,2}(M) \) is separable, reflexive and can be compactly embedded into \( L^2(M) \).

**Proof.** Since \( M \) is compact, \( \Lambda_F < \infty \) and there exists a positive constant \( C_m \geq 1 \) such that

\[
(3.1) \quad C_m^{-1} \cdot d\text{vol}_\hat{g} \leq dm \leq C_m \cdot d\text{vol}_\hat{g}.
\]

Hence, \( (2.1) \) then yields

\[
(C_m \cdot \Lambda_F)^{-1} \int_M \| df \|_{\hat{g}}^2 d\text{vol}_\hat{g} \leq \int_M F^{-2}(df)dm \leq C_m \cdot \Lambda_F \int_M \| df \|_{\hat{g}}^2 d\text{vol}_\hat{g},
\]

which implies the Sobolev norm induced by \( \hat{g} \) is equivalent to \( \| \cdot \|_{H^{1,2}} \). \( \square \)

In the following, we use \( \| \cdot \|_H \) to study \( H^{1,2}(M) \). Now define

\[
\mathcal{X} := \left\{ \begin{array}{ll}
H^{1,2}(M) & \text{if } \partial M = \emptyset, \\
\{ f \in C^\infty(M) : f|_{\partial M} = 0 \} & \text{if } \partial M \neq \emptyset.
\end{array} \right.
\]

Clearly, \( \mathcal{X} \) is a Banach space contained in \( H^{1,2}(M) \). And it is easy to check that for any \( f \in \mathcal{X} \), we have

\[
\int_M (X, df)dm = -\int_M f \text{div} X dm, \quad \forall X \in C^1(TM).
\]
Given any $f \in H^{1,2}(M)$, the definition of $H^{1,2}(M)$ yields that $df$ is a $L^2$-section of $T^*M$. Now we define $\nabla f := \mathcal{L}^{-1}(df)$, where $\mathcal{L}$ is the Legendre transformation.

**Proposition 3.2.** There exists a finite constant $C = C(M) > 0$ such that

$$
\|F(\nabla u - \nabla v)\|_{L^2} \leq C \cdot \|F^*(du - dv)\|_{L^2}, \quad \forall u, v \in H^{1,2}(M).
$$

Hence, if \{f_n\} is a sequence convergent to $f$ in $H^{1,2}(M)$, then

$$
\lim_{n \to +\infty} \|F(f_n - f)\|_{L^2} = 0,
$$

and therefore,

$$
\lim_{n \to +\infty} \|F(f_n)\|_{L^2} = \|F(f)\|_{L^2}.
$$

**Proof.** The proof is inspired by [11]. Given a local coordinate system $(x')$, let $(\eta_i)$ denote the coordinates of $T^*M$ and set

$$
A^i(x, \eta) := \frac{1}{2} \frac{\partial F^{*2}}{\partial \eta_i}(x, \eta).
$$

Then $\nabla f = A^i(x, df) \cdot \partial_i$. Given any $x \in M$, one can find a small neighbourhood $U_x$ such that $U_x$ is compact and there is a coordinate system defined on $\overline{U}_x$. There exists a constant $C_x \geq 1$ with (see [11] (11))

$$
|(A^i(q, \xi) - A^i(q, \zeta)) \eta_i| \leq C_x \cdot \|\eta\|_E \cdot \|\xi - \zeta\|_E, \quad \forall (q, \xi, q, \zeta) \in T^*(U_x) \setminus \{0\},
$$

where $\| \cdot \|_E$ is the standard Euclidean norm. Since $U_x$ is compact, there exists a positive constant $C'_x \geq 1$ such that

$$
(C'_x)^{-1} \|\eta\|_E \leq F^*(q, \eta) \leq C'_x \|\eta\|_E, \quad \forall (q, \eta) \in T^*(U_x).
$$

Now on $U_x$, we have

$$
F^2(q, \nabla u - \nabla v) \leq (C'_x)^2 \sum_{i=1}^n |A^i(q, du) - A^i(q, dv)|^2
$$

$$
\leq n(C_x C'_x)^2 \|du - dv\|_E^2 \leq n(C_x)^2 (C'_x)^4 \cdot F^{*2}(q, du - dv).
$$

Since $M$ is compact, there are only finitely many $U_x$’s, say $\{U_x\}_{i=1}^N$ covering $M$. Set $C = \max\{(C_x)^2(C'_x)^4, i = 1, \ldots, N\}$. Then we have

$$
\int_M F^2(\nabla u - \nabla v)dm \leq \sum_{i=1}^N \int_{U_i} F^2(\nabla u - \nabla v)dm \leq nNC \int_M F^{*2}(du - dv)dm.
$$

\[ \square \]

**Definition 3.3.** Given $f \in \mathcal{X}$, define $\Delta f$ by

$$
\int_M \varphi \Delta f dm := - \int_M \langle \nabla f, d\varphi \rangle dm, \quad \forall \varphi \in C^\infty(M).
$$

An interesting fact is that $\int_M \Delta f dm = 0$ for any $f \in \mathcal{X}$. If $f \in C^\infty$ and $df \neq 0$ on an open subset $\Omega$, then $(\Delta f)|_\Omega = \text{div}(\nabla f)|_\Omega$. And it is easy to check the following fact.

**Proposition 3.4.** **Definition 3.3** can be extended to $H^{1,2}(M)$. That is, for any $f \in \mathcal{X}$,

$$
\int_M \varphi \Delta f dm := - \int_M \langle \nabla f, d\varphi \rangle dm, \quad \forall \varphi \in H^{1,2}(M),
$$

$$
\int_M \nabla \varphi \cdot df dm = 0, \quad \forall \varphi \in C^\infty(M).
$$
is well-defined. In particular,
\[ \int_M f \Delta f \, dm = - \int_M F^* (df) \, dm, \quad \forall f \in \mathcal{X}. \]

Set
\[ \mathcal{X}_0 := \begin{cases} \{ f \in H^{1,2}(M) : \int_M f \, dm = 0 \} & \text{if } \partial M = \emptyset, \\ \{ f \in C^\infty(M) : f|_{\partial M} = 0 \} & \text{if } \partial M \neq \emptyset. \end{cases} \]

Since \( M \) is compact, it is easy to check that \( \mathcal{X}_0 \) is a closed subspace of \( \mathcal{X} \) and therefore, it is a complete Banach space as well.

4. Rayleigh Quotient

Let \( \mathcal{X} \) be defined as before. Define the energy functional (i.e., Rayleigh quotient) on \( \mathcal{X} \) by
\[ E(u) := \int_M F^* (dv) \, dm = \left( \frac{\|F^*(dv)\|_{L^2}}{\|u\|_{L^2}} \right)^2, \quad \forall u \in \mathcal{X} \setminus \{0\}. \]

Clearly, \( E \) is also continuous on \( \mathcal{X} \setminus \{0\} \). Given \( u \in \mathcal{X} \setminus \{0\} \), for any \( v \in \mathcal{X} \), we have
\[ \langle v, DE(u) \rangle = \frac{d}{dt} \bigg|_{t=0} E(u + tv) = -2 \frac{\int_M v(\Delta u + E(u) \cdot u) \, dm}{\int_M u^2 \, dm}. \]

According to [21], \( DE(u) \) is a linear functional on \( \mathcal{X} \cong T \mathcal{X} \). And \( u \) is a critical point and \( E(u) \) is a critical value of \( E \) if \( DE(u) = 0 \) (i.e., \( \langle v, DE(u) \rangle = 0 \) for all \( v \in \mathcal{X} \)). For convenience, in this paper, we say
\[ \Delta u + E(u) \cdot u = 0 \text{ in the weak sense} \]
if \( DE(u) = 0 \). On the other hand, \( DE(u) \) is a continuous and bounded functional. In fact, we have
\[ \|DE(u)\| = \sup_{v \neq 0} \frac{\|v, DE(u)\|}{\|v\|_{H}} \leq 2 \frac{\|F^*(dv)\|_{L^2} \|F(\nabla u)\|_{L^2} + E(u) \cdot \|u\|_{L^2} \|v\|_{L^2}}{\|u\|_{L^2}^2 \|v\|_{H}} \leq 2 \frac{\max\{\sqrt{E(u)}, E(u)\}}{\|u\|_{L^2}}. \]

Note that the equality does not hold, because the Hölder inequality together with the Cauchy-Schwarz inequality implies \( du = \alpha \cdot dv \) and \( u = -\beta v \), for some \( \alpha, \beta > 0 \). However, this cannot happen. A direct calculation together with Proposition 3.2 yields the following result, i.e., \( E \in C^1(\mathcal{X} \setminus \{0\}) \).

**Proposition 4.1.** Let \( \{u_m\} \) be a sequence in \( \mathcal{X} \setminus \{0\} \) converging to \( u \neq 0 \). Then
\[ \lim_{m \to +\infty} \|DE(u_m) - DE(u)\| = 0. \]

That is, \( DE(u) \) is continuous in \( u \).

Recall the following (P.-S.) condition.
**Proposition 4.2** (\[4.2\]). Given any $0 < D < \infty$, if $\{u_n\}$ is a sequence in $\mathcal{X} \setminus \{0\}$ with

\[
\|u_n\|_{L^2} = 1, \quad E(u_n) \leq D, \quad \|DE(u_n)\| \to 0,
\]

then there exists a (strongly) convergent subsequence in $\mathcal{X} \setminus \{0\}$.

**Corollary 4.3.** For any $\beta \geq 0$, the following eigenset, $K_\beta$, is compact

\[ K_\beta := \{ u \in \mathcal{X} : \|u\|_{L^2} = 1, \ E(u) = \beta, \ DE(u) = 0 \}. \]

**Proof.** Given a sequence $\{u_m\} \subset K_\beta$, (P.-S.) yields that a subsequence $\{u_{mk}\}$ converges to $u \in \mathcal{X}$ strongly. Now Proposition \[4.1\] yields that

\[
\|u\|_{L^2} = 1, \ E(u) = \beta, \ DE(u) = 0 \Rightarrow u \in K_\beta.
\]

Hence, $K_\beta$ is compact. \qed

Set

\[
\mathcal{S} := \{ u \in \mathcal{X} : \|u\|_{L^2} = 1 \}.
\]

Let $\| \cdot \|$ be the norm on $T \mathcal{X}$ induced by $\| \cdot \|_H$. Then $(\mathcal{X}, \| \cdot \|)$ is a $C^\infty$-Finsler manifold in the sense of Palais (cf. \[13\] Def. 2.10). On the other hand, consider the function $h : \mathcal{X} \to \mathbb{R}$

\[
h(u) := \|u\|^2_{L^2}.
\]

It is easy to see that

\[
Dh(u)(\phi) = \left. \frac{d}{dt} \right|_{t=0} h(u + t\phi) = 2 \int_M u\phi dm;
\]

\[
D^2h(u)(\phi_1, \phi_2) = \left. \frac{d}{dt} \right|_{t=0} Dh(u + t\phi_2)(\phi_1) = 2 \int_M \phi_1 \phi_2 dm;
\]

\[
D^3h(u)(\phi_1, \phi_2, \phi_3) = \left. \frac{d}{dt} \right|_{t=0} D^2h(u + t\phi_3)(\phi_1, \phi_2) = 0.
\]

Using the Hölder inequality and $M$ is compact, one can see that $h \in C^\infty(\mathcal{X})$. Moreover, if for some $u \in \mathcal{X}$ with $Dh(u) = 0$, it is easy to see that $u = 0 \in \mathcal{X}$. Thus, $h(u)$ has no critical point in $\mathcal{S} = h^{-1}(1)$, i.e., $1$ is a regular value. According to Preimage Theorem (\[22\] Theorem 73.C), $\mathcal{S}$ is a closed $C^\infty$-submanifold of $\mathcal{X}$ (also refer to \[18\] Theorem 5.9). Thus, according to \[18\] Theorem 3.6, \[22\] Definition 73.38 and Proposition \[4.4\] we have

**Proposition 4.4.** $(\mathcal{S}, \| \cdot \|_{|T\mathcal{S}})$ is a complete $C^\infty$-Finsler manifold in the sense of Palais and $i^*E$ is a $C^1$-function on $\mathcal{S}$, where $i : \mathcal{S} \hookrightarrow \mathcal{X}$ is the inclusion.

This result yields that $\mathcal{S}$ is a paracompact Banach manifold and hence, an ANR (i.e., absolute neighborhood retract) (cf. \[17\] Corollary, p.3). In Appendix \[A\] we show that it is an AR (i.e., absolute retract) (see Proposition \[A.1\]).

Note that if $u \in \mathcal{S}$ is a critical point of $E$, then $DE(u) = 0 \Rightarrow D(i^*E)(u) = 0$. That is, $u$ is a critical point of $i^*E$. On the other hand, we have

**Lemma 4.5.** If $u \in \mathcal{S}$ is a critical point of $i^*E$, then either $u = (m(M))^{-\frac{1}{2}}$ or $u \in \mathcal{X}_0 \cap C^{1,\alpha}$ and moreover,

\[
\int_M v(\Delta u + E(u)u) dm = 0, \quad \forall v \in \mathcal{X} \Rightarrow DE(u) = 0.
\]
Proof. Consider the smooth curve $\gamma(t) := \frac{u+tv}{\|u+tv\|_2} \subset \mathcal{S}$. If $u$ is a critical point of $i^*E$, then

$$0 = \frac{d}{dt} \bigg|_{t=0} E(\gamma(t)) = \frac{d}{dt} \bigg|_{t=0} E(u + tv) = -2 \int_M v(\Delta u + E(u)u)dm.$$ 

Choosing $v \equiv 1$ if $\partial M = \emptyset$, one can see either $u = (m(M))^{-\frac{1}{2}}$ or $u \in \mathcal{X}_0$. Now $u \in C^{1,\alpha}$ follows from [11, Theorem 1.1].

Thus, from the point of view of critical points, there is no difference between $E$ and $i^*E$. So in the following, by abuse of notation, we use $E$ to denote $i^*E$.

Given $\beta > 0$, set

$$E_\beta := \{ u \in \mathcal{S} : E(u) < \beta \}.$$ 

Proposition 4.4 together with [21, Chapter II, Theorem 3.11] and Proposition 4.2 yield

**Theorem 4.6.** Let $\beta \geq 0$, $\theta > 0$ and let $N \subset \mathcal{S}$ be any open neighborhood of the eigenset $K_\beta$ (see [4,7]). Then there exist a numbers $\vartheta \in (0, \theta)$ and a continuous 1-parameter family of homeomorphisms $\Phi(\cdot, t)$ of $\mathcal{S}$, $0 \leq t < \infty$, with the following properties:

1. $\Phi(u, t) = u$, if $t = 0$, or $DE(u) = 0$, or $|E(u) - \beta| \geq \theta$;
2. $E(\Phi(u, t))$ is non-increasing in $t$ for any $u \in \mathcal{S}$;
3. $\Phi(E_{\beta+\vartheta} \setminus N, 1) \subset E_{\beta-\vartheta}$, and $\Phi(E_{\beta+\vartheta}, 1) \subset E_{\beta-\vartheta} \cup N$.

Moreover, $\Phi : \mathcal{S} \times [0, +\infty) \to \mathcal{S}$ has the semi-group property. In particular, $\Phi(-u, t) = -\Phi(u, t)$, for all $t \geq 0$ and $u \in \mathcal{S}$.

Let $\mathbb{P}(\mathcal{X})$ denote the quotient space $\mathcal{S}/\mathbb{Z}_2$. Thus, $p : \mathcal{S} \to \mathbb{P}(\mathcal{X})$ is a 2-fold covering as $\mathbb{Z}_2$ act freely and properly discontinuously on $\mathcal{S}$. In particular, $\mathbb{P}(\mathcal{X})$ is a normal ANR (see Proposition A.2).

**Proposition 4.7.** $\mathbb{P}(\mathcal{X})$ is (homeomorphic to) the projective space $(\mathcal{X} \setminus \{0\})/\sim$, where $u \sim v$ if there exists $\alpha \neq 0$ such that $u = \alpha \cdot v$.

**Proof.** Define $f : \mathcal{X} \setminus \{0\} \to \mathcal{S}$ as $f(u) := u/\|u\|_2$. Let $q : (\mathcal{X} \setminus \{0\}) \to (\mathcal{X} \setminus \{0\})/\sim$ be the projection and $i : \mathcal{S} \to \mathcal{X} \setminus \{0\}$ be the inclusion. Thus, using the the following commutative diagram, one can define two continuous maps $l_1 : (\mathcal{X} \setminus \{0\})/\sim \to \mathbb{P}(\mathcal{X})$ and $l_2 : \mathbb{P}(\mathcal{X}) \to (\mathcal{X} \setminus \{0\})/\sim$ with $l_1 \circ l_2 = \text{Id}_{\mathbb{P}(\mathcal{X})}$ and $l_2 \circ l_1 = \text{Id}_{(\mathcal{X} \setminus \{0\})/\sim}$.

$\begin{array}{cccccc}
\mathcal{X} \setminus \{0\} & \xrightarrow{f} & \mathcal{S} & \xrightarrow{i} & \mathcal{X} \setminus \{0\} & \xrightarrow{f} & \mathcal{S} \\
\downarrow q & & \downarrow p & & \downarrow q & & \downarrow p \\
(\mathcal{X} \setminus \{0\})/\sim & \xrightarrow{l_1} & \mathbb{P}(\mathcal{X}) & \xrightarrow{l_2} & (\mathcal{X} \setminus \{0\})/\sim & \xrightarrow{l_1} & \mathbb{P}(\mathcal{X})
\end{array}$

$\square$

In the following, we use $\mathbb{P}(\mathcal{X})$ to denote the projective space $(\mathcal{X} \setminus \{0\})/\sim$ for convenience. Moreover, given a $k$-dimensional linear (closed) subspace $V$ of $\mathcal{X}$, $\mathbb{P}(V) := p(V \cap \mathcal{S})$ is also utilized to denote the projective space $q(V \setminus \{0\})$ induced by $V$. 
5. Dimension-like functions and spectrums

In general, it is hard to define the high order eigenvalues for a non-Riemannian Finsler manifold by the traditional method as the Laplacian is nonlinear. In [S], Gromov suggested that the eigenvalues of an arbitrary energy function could be defined by a dimension-like function. Inspired by this idea, we give the following definition.

**Definition 5.1.** Let \( S \) be a certain collection of subsets of \( S \). A dimension-like function \( \dim : S \to \mathbb{N} \cup \{ \pm \infty \} \) is a monotone increasing function on \( S \), that is, if \( A_1, A_2 \in S \) with \( A_1 \subset A_2 \), then

\[
\dim(A_1) \leq \dim(A_2).
\]

Given \( \lambda \in \mathbb{R}^+ \cup \{ +\infty \} \), the "dimension" of the level \( E^{-1}[0, \lambda] \) is defined as

\[
\dim E^{-1}[0, \lambda] := \sup\{ \dim(A) : A \in S, A \subset E^{-1}[0, \lambda] \}.
\]

The \( \dim \)-spectrum \( \{ \lambda_k \} \) of \( E \) is defined as

\[
\lambda_k = \sup\{ \lambda \in \mathbb{R}^+ \cup \{ +\infty \} : \dim E^{-1}[0, \lambda] < k \}.
\]

First we point out that the eigenvalue defined above can be obtained by the min-max principle.

**Proposition 5.2.** For any \( k \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\} \), set

\[
\mathcal{F}_k := \{ A \in S : \dim(A) \geq k \}.
\]

Then we have

\[
\lambda_k = \begin{cases} 
\inf_{A \in \mathcal{F}_k} \sup_{u \in A} E(u) & \text{if } \mathcal{F}_k \neq \emptyset, \\
+\infty & \text{if } \mathcal{F}_k = \emptyset.
\end{cases}
\]

**Proof.** (1) Suppose \( \mathcal{F}_k = \emptyset \). Then for all \( A \in S \), \( \dim(A) < k \), which implies that \( \dim E^{-1}[0, \lambda] < k \), for any \( \lambda \in \mathbb{R}^+ \cup \{ +\infty \} \). Hence, \( \lambda_k = +\infty \).

(2) Suppose \( \mathcal{F}_k \neq \emptyset \). Thus, for any \( \lambda \in \mathbb{R}^+ \cup \{ +\infty \} \) with \( \dim E^{-1}[0, \lambda] \geq k \), it is easy to see \( \lambda \geq \lambda_k \). Now let \( \hat{\lambda}_k := \inf\{ \lambda \in \mathbb{R}^+ \cup \{ +\infty \} : \dim E^{-1}[0, \lambda] \geq k \} \).

Clearly, \( \lambda_k \leq \hat{\lambda}_k \).

On the other hand, if \( \lambda_k = +\infty \), then \( \lambda_k \geq \hat{\lambda}_k \). Now suppose \( \lambda_k < +\infty \). Thus, for any \( \epsilon > 0 \), \( \dim E^{-1}[0, \lambda_k + \epsilon] \geq k \), which means \( \lambda_k \geq \hat{\lambda}_k \). Therefore, we have

\[
\lambda_k = \hat{\lambda}_k = \inf\{ \lambda \in \mathbb{R}^+ \cup \{ +\infty \} : \dim E^{-1}[0, \lambda] \geq k \}
\]

\[
= \inf\{ \lambda \in \mathbb{R}^+ \cup \{ +\infty \} : \exists A \in S \text{ with } A \subset E^{-1}[0, \lambda] \text{ and } \dim(A) \geq k \}
\]

\[
= \inf\left\{ \lambda \in \mathbb{R}^+ \cup \{ +\infty \} : \exists A \in \mathcal{F}_k \text{ with } \sup_{u \in A} E(u) \leq \lambda \right\} = \inf_{A \in \mathcal{F}_k} \sup_{u \in A} E(u).
\]

\( \square \)

In this paper, the \( \dim \)-spectrum is called *degenerate* from some \( k \in \mathbb{N}^+ \) if \( \mathcal{F}_k = \emptyset \). In particular, if \( \mathcal{F}_2 = \emptyset \), the \( \dim \)-spectrum is *strictly degenerate*. On the other hand, the \( \dim \)-spectrum is said to be *trivial* from some \( k \in \mathbb{N}^+ \) if all \( \lambda_{k+l} \)'s are same for \( l \geq 0 \). Particularly, the \( \dim \)-spectrum is *strictly trivial* if all the eigenvalues coincide.

Clearly, the degenerateness implies the trivialness. See Sec. [5.3](#) for some interesting examples which are strictly trivial but not degenerate.
Additional Assumption. All the dimension-like functions in this paper satisfy
(5.1) \( \dim(\emptyset) \leq 0 \implies \emptyset \notin \mathcal{F}_k, \ \forall k \in \mathbb{N}^+ \).

This assumption is not only natural but also helpful in studying the \( \dim \)-spectrum. Now Proposition 5.2 yields
\[ 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \]
Moreover, we have the following universal properties of \( \lambda_k \).

**Theorem 5.3.** (1) If for any \( f \in \mathcal{S} \), the singleton \( \{f\} \in \mathcal{F}_1 \), then
\[ \lambda_1 = \inf_{u \in \mathcal{X}\setminus\{0\}} E(u). \]
In particular, \( \lambda_1 = 0 \) if \( \partial M = \emptyset \) while \( \lambda_1 > 0 \) if \( \partial M \neq \emptyset \).

(2) Given some \( k \in \mathbb{N}^+ \), suppose that \( S \cap V \in \mathcal{F}_k \) for any \( k \)-dimensional linear subspace \( V \subset \mathcal{F} \). If \( F \) is Riemannian, then \( \lambda_k \leq \lambda_k^A \), where \( \lambda_k^A \) is the standard eigenvalue of the Laplacian \( \Delta \).

(3) Suppose that \( \dim(A) \leq \dim(\Phi(A, 1)) \) for any \( A \in \mathcal{F} \), where \( \Phi(\cdot, 1) \) is defined in Theorem 5.4. Thus, for any \( k \in \mathbb{N}^+ \) with \( \lambda_k < +\infty \), \( \lambda_k \) is a critical value of \( E \) and hence, there exists \( u = (m(M))^{-\frac{1}{2}} \) or \( u \in \mathcal{H}_0 \cap C^{1, \alpha} \) such that
\[ \Delta u + \lambda_k u = 0 \text{ in the weak sense.} \]

**Proof.** (1) For convenience, set \( \lambda_1^* := \inf_{u \in \mathcal{X}\setminus\{0\}} E(u) = \inf_{u \in \mathcal{S}} E(u) \).

Then Proposition 5.2 implies that \( \lambda_1 \leq \lambda_1^* \). On the other hand, for any \( A \in \mathcal{F}_1 \), (5.1) implies \( A \neq \emptyset \) and hence, Proposition 5.2 yields
\[ \sup_{u \in A} E(u) \geq \lambda_1^* \implies \lambda_1 \geq \lambda_1^* \implies \lambda_1 = \lambda_1^*. \]

If \( \partial M = \emptyset \), consider \( A = \{(m(M))^{-\frac{1}{2}}\} \). Thus, \( A \in \mathcal{F}_1 \) and hence,
\[ 0 \leq \lambda_1 \leq \sup_{u \in A} E(u) = 0. \]

Now suppose \( \partial M \neq \emptyset \). Consider the average Riemannian metric \( \hat{g} \). The spectral theory in Riemannian geometry then yields
\[ \lambda_1 = \inf_{u \in \mathcal{X}\setminus\{0\}} \frac{\int_M F^{\alpha 2}(du) \, dm}{\int_M u^2 \, dm} \geq \frac{1}{\Lambda_F \cdot C_m^2} \inf_{u \in \mathcal{X}\setminus\{0\}} \frac{\int_M ||du||_g^2 \, dvol_g}{\int_M u^2 \, dvol_g} > 0, \]
where \( C_m \) is defined in (3.1).

(2) Let \( \dim_c(V) \) denote the Lebesgue covering dimension of a linear space \( V \). Since \( F \) is Riemannian, Courant’s minimax principle yields
(5.2) \[ \lambda_k^A = \min_{V \in \mathcal{H}_k} \max_{u \in V \setminus\{0\}} E(u), \]
where \( \mathcal{H}_k = \{V \subset \mathcal{X} : V \text{ is a linear subspace with } \dim_c(V) = k \} \).

Thus, for any \( \epsilon > 0 \), there exists a \( k \)-dimensional linear space \( V \) with \( \max_{u \in V} E(u) < \lambda_k^A + \epsilon \). Since \( S \cap V \in \mathcal{F}_k \) and \( S \cap V = \{u \in V : ||u||_{L^2} = 1\} \),
\[ \lambda_k \leq \sup_{u \in S \cap V} E(u) = \max_{u \in V \setminus\{0\}} E(u) < \lambda_k^A + \epsilon \implies \lambda_k \leq \lambda_k^A. \]
(3) Since $\lambda_k < +\infty$, Proposition 5.2 implies $\mathcal{F}_k \neq \emptyset$. Suppose that $\lambda_k$ is a regular value of $E$. That is, if $u \in \mathcal{S}$ with $E(u) = \lambda_k$, then $DE(u) \neq 0$. Hence, the eigenset $K_{\lambda_k} = \emptyset$ (see (3.1)). Now Theorem 4.6 ($N = 0$ and $\theta = 1$) yields that there exists $\vartheta > 0$ and a family of homeomorphisms $\Phi(\cdot, t) : \mathcal{S} \rightarrow \mathcal{S}$, $t \in [0, 1]$ such that $\Phi(E_{\lambda_k + \vartheta}, 1) \subset E_{\lambda_k - \vartheta}$. For this $\vartheta$, there is $A \in \mathcal{F}_k$ with

$$\sup_{u \in A} E(u) < \lambda_k + \vartheta \Rightarrow A \subset E_{\lambda_k + \vartheta}.$$ 

Thus, $E(\Phi(A, 1)) < \lambda_k - \vartheta$. Since $\dim(\Phi(A, 1)) \geq \dim(A) \geq k$, $\Phi(A, 1) \in \mathcal{F}_k$ and hence,

$$\lambda_k \leq \sup_{u \in \Phi(A, 1)} E(u) \leq \lambda_k - \vartheta,$$

which is a contradiction. This means $\lambda_k$ is a critical value of $E$. Now the result follows from Lemma 4.5 directly.

**Remark 5.4.** Clearly, (3) in the above theorem implies Theorem 1.1. Furthermore, Theorem 5.3 factually holds in the irreversible case. In particular, (3) points out closed eigenvalue problem and Dirichlet eigenvalue problem in general Finsler case always have solutions if the dimension-like function satisfies the assumption in (3). Note that $\Phi(\cdot, t)$ in Theorem 4.6 is a homeomorphism for all $t$ and hence, this assumption is quite natural. In fact, all the dimension-like functions constructed in this paper satisfy this condition.

Since $F$ is reversible, the proof of (1) in Theorem 5.3 also yields the following result.

**Corollary 5.5.** If $V \cap \mathcal{S} \in \mathcal{F}_1$ for any 1-dimensional linear space $V$, then $\lambda_1 = \inf_{u \in \mathcal{F} \setminus \{0\}} E(u)$.

**Theorem 5.6.** Suppose the following two conditions hold:

(1) the definitions of $\mathcal{F}_k$, $\forall k \in \mathbb{N}^+$ are independent of the Finsler metric $F$;

(2) $\lambda_k = \lambda_k^{\Delta}$ when $F$ is Riemmannian.

Thus, for each $k \in \mathbb{N}^+$, $\lambda_k$ is finite, but $\lim_{k \rightarrow +\infty} \lambda_k = +\infty$. In particular, $\lambda_2 > 0$ and the multiplicity of each eigenvalue is finite.

**Proof.** Let $\hat{g}$ be the average Riemannin metric induced by $F$. Condition (1)-(2) imply that one can use $\mathcal{F}_k$ to define the standard eigenvalues of $\hat{g}$, which are denoted by $\lambda_k^{\Delta}$'s. Thus, by (3.1) one gets

$$\frac{\lambda_k^{\Delta}}{\Lambda_F \cdot C_2^2} = \frac{1}{\Lambda_F \cdot C_2^2} \inf_{A \in \mathcal{F}_k} \sup_{u \in A} \frac{\int_M ||du||^2_{\hat{g}} d\text{vol}_{\hat{g}}}{\int_M u^2 d\text{vol}_{\hat{g}}} \leq \inf_{A \in \mathcal{F}_k} \sup_{u \in A} E(u) = \lambda_k.$$

Now $\lambda_2 > 0$ and $\lambda_k \rightarrow +\infty$ follow from the spectral theory in Riemannian geometry. On the other hand, the similar argument yields $\lambda_k \leq \Lambda_F \cdot C_2^2 \cdot \lambda_k^{\Delta} < +\infty$ for all $k$, which together with $\lim_{k \rightarrow +\infty} \lambda_k = +\infty$ implies the finiteness of the multiplicity.

**Remark 5.7.** In fact, the above theorem remains valid for general compact Finsler manifolds, since the proof is independent of the reversibleness of $F$.

In the following, we investigate some specific dimension-like functions.
5.1. Lusternik-Schnirelmann category.
In this subsection, we consider a dimension-like function induced by the Lusternik-
Schnirelmann category. First, we recall the relative Lusternik-Schnirelmann (LS) category on $\mathbb{P}(\mathcal{X})$.

**Definition 5.8 ([7] 21).** Given a closed subset $A \subset \mathbb{P}(\mathcal{X})$, the LS category of $A$
(relative to $\mathbb{P}(\mathcal{X})$) $\text{cat}_{\mathbb{P}(\mathcal{X})}(A)$, is the smallest possible integer value $k$
such that $A$ is covered by $k$ closed sets $A_j$, $1 \leq j \leq k$, which are contractible in $\mathbb{P}(\mathcal{X})$.
If no such finite covering exists we write $\text{cat}_{\mathbb{P}(\mathcal{X})}(A) = +\infty$.

Then we define the Lusternik-Schnirelmann dimension as follows.

**Definition 5.9.** Set $\mathcal{F}^{LS} := \{A \subset \mathcal{S}: A$ is closed $\}$. For each $A \in \mathcal{F}^{LS}$, we define
the Lusternik-Schnirelmann dimension of $A$ by
\[
\dim_{LS}(A) := \text{cat}_{\mathbb{P}(\mathcal{X})}(p(A)).
\]

Note that $p: S \to \mathbb{P}(\mathcal{X})$ is an open and closed map and hence, the definition
above is well-defined. In this paper, we say a continuous map $h: A \times I \to \mathcal{S}$ is a homotopy
$\text{h}(-, 0) = \text{Id}|_{A}$. The following observation is clear but important.

**Lemma 5.10.** Given any $A \subset $, suppose that $h: A \times I \to S$ is an odd homotopy,
that is, $h(-u, t) = -h(u, t)$. Then
\[
H : p(A) \times I \to p(\mathcal{X}), ([u], t) \mapsto [h(u, t)]
\]
is a homotopy of $p(A)$.

**Proof.** Since $h_t := h(\cdot, t)$ is odd, $H$ is well-defined. Moreover, since
\[
H \circ p = p \circ h,
\]
and $p$ is a quotient map, $H$ is continuous and hence, is a homtopy. \hfill \Box

$\dim_{LS}$ satisfies all the properties of the LS category (cf. [21] Chapter II, Proposition 5.13). We prove the following proposition in Appendix A.

**Proposition 5.11.** Let $A, B \in \mathcal{F}^{LS}$ and let $h: A \times I \to S$ be an odd homotopy of
$A$. Then the following properties hold:

(i) $\dim_{LS}(A) \geq 0$; In particular, $\dim_{LS}(A) = 0$ if and only if $A = \emptyset$.

(ii) $A \subset B$ implies $\dim_{LS}(A) \leq \dim_{LS}(B)$.

(iii) $\dim_{LS}(A \cup B) \leq \dim_{LS}(A) + \dim_{LS}(B)$.

(iv) $\dim_{LS}(A) \leq \dim_{LS}(h(A, I))$.

(v) If $A$ is compact, then $\dim_{LS}(A) < \infty$ and there is a neighborhood $N$ of $A$ in $\mathcal{S}$
such that $N \in \mathcal{F}^{LS}$ and $\dim_{LS}(A) = \dim_{LS}(N)$.

(vi) For a singleton $A = \{u\}$, $\dim_{LS}(A) = 1$. Given any $k$-dimensional linear space
$V \subset \mathcal{X}$, $\dim_{LS}(\mathcal{S} \cap V) = k$ always holds.

Now we define the $\dim_{LS}$-spectrum as follows.

**Definition 5.12.** Given $k \in \mathbb{N}^+$, set
\[
\mathcal{F}^{LS}_k := \{A \in \mathcal{F}^{LS}: \dim_{LS}(A) \geq k\},
\]
and
\[
\gamma^k_{LS} := \inf_{A \in \mathcal{F}^{LS}_k} \sup_{u \in A} E(u).
\]
It follows from Proposition 5.11 (v) that $\mathcal{F}_k^{LS} \neq \emptyset$ for each $k \in \mathbb{N}^+$. Thus, Proposition 5.2 implies the definition above is well-defined. And it is easy to see that Theorem 5.3 holds for $\dim_{LS}$. Moreover, we have

**Proposition 5.13.** If for some $k \in \mathbb{N}^+$,

$$0 \leq \lambda_k^{LS} = \lambda_{k+1}^{LS} = \cdots = \lambda_{k+l-1}^{LS} = \beta < \infty,$$

i.e., the multiplicity of the eigenvalue $\beta$ is $l$, then $\dim_{LS}(K_\beta) \geq l$ (see (4.4)). In particular, there exist (at least) $l$ linearly independent functions $u \in \mathcal{X}$ which satisfy the equation

$$\Delta u + \beta \cdot u = 0 \text{ in the weak sense.}$$

Moreover, if $l > 1$, then $K_\beta$ is a infinite set.

The proof is given in the next subsection. The importance of Proposition 5.13 is that it implies Theorem 5.6 also holds for the $\dim_{LS}$-spectrum.

**Proposition 5.14.** If $F$ is Riemannian, then $\lambda_k^{LS}$ is the standard $k$th-eigenvalue of the Laplacian.

**Proof.** Firstly, Proposition 5.11 (vi) together with Theorem 5.3 (2) yields $\lambda_k \leq \lambda_\Delta < +\infty$, for all $k \in \mathbb{N}^+$. Hence, for each $k \geq 1$,

$$0 \leq \lambda_1^{LS} \leq \cdots \leq \lambda_k^{LS} < +\infty,$$

which together with Theorem 5.3 implies that for each $j$ with $1 \leq j \leq k$, there exists $u_j \in C^\infty(M)$ such that $E(u_j) = \lambda_j^K$ and $\Delta u_j + \lambda_j^K u_j = 0$. If $\lambda_i^{LS} \neq \lambda_j^{LS}$, then the standard theory in Riemannian geometry yields

$$(5.3) \quad \int_M u_i \cdot u_j \, dvol_g = 0, \quad \int_M \langle \nabla u_i, \nabla u_j \rangle \, dvol_g = 0, \quad \text{if } i \neq j,$$

where $g$ is the Riemannian metric induced by $F$. If $\lambda_i^{LS} = \lambda_{i+1}^{LS} = \cdots = \lambda_{i+l-1}^{LS} = \beta$, i.e., the multiplicity of the eigenvalue $\beta$ is $l$, Proposition 5.13 yields that there exist $l$ linearly independent eigenfunctions $\{u_s\}_{s=1}^l$ corresponding to $\beta$, which still satisfy (5.3) (since $\Delta$ is linear). Hence, one always obtains $k$ eigenfunctions $\{u_j\}_{j=1}^k$ such that $\{u_j\}$ are mutually orthogonal (in the sense of (5.3)) and $u_j$ corresponds to $\lambda_j^{LS}$. Now let

$$V_k := \text{Span}\{u_1, \ldots, u_k\} \subset \mathcal{X}.$$ 

Clearly, $\dim_C(V_k) = k$ and then Courant’s minimax principle (5.2) together with (5.3) yields

$$\lambda_k^\Delta \leq \sup_{u \in V_k} E(u) = \lambda_k^{LS}. \quad \Box$$

From above, we have the following conclusion.

**Theorem 5.15.** Let $(M,F,\text{d}m)$ be a compact reversible Finsler manifold and let $\{\lambda_k^{LS}\}_{k=1}^\infty$ denote the $\dim_{LS}$-spectrum. Then

$$0 = \lambda_1^{LS} \leq \lambda_2^{LS} \leq \cdots \leq \lambda_k^{LS} \leq \cdots \nearrow \infty, \text{ if } \partial M = \emptyset,$$

$$0 < \lambda_1^{LS} \leq \lambda_2^{LS} \leq \cdots \leq \lambda_k^{LS} \leq \cdots \nearrow \infty, \text{ if } \partial M \neq \emptyset.$$
where the first positive eigenvalue satisfies

\begin{equation}
\begin{cases}
\lambda_2^{LS} = \inf_{u \in \mathcal{X}_0} E(u), & \text{if } \partial M = \emptyset; \\
\lambda_1^{LS} = \inf_{u \in \mathcal{X}_0} E(u), & \text{if } \partial M \neq \emptyset.
\end{cases}
\end{equation}

For each \( k \in \mathbb{N}^+ \), there exists \( u = (m(M))^{-\frac{1}{2}} \) or \( u \in \mathcal{X}_0 \cap C^{1,\alpha} \) with

\[ \Delta u + \lambda_k^{LS} u = 0 \text{ in the weak sense}. \]

The multiplicity \( m(\lambda_k^{LS}) \) of each eigenvalue \( \lambda_k^{LS} \) is always finite. In particular, there exist \( m(\lambda_k^{LS}) \) linearly independent eigenfunctions corresponding to \( \lambda_k^{LS} \).

Moreover, \( \{\lambda_k^{LS}\}_{k=1}^\infty \) is the standard spectrum of the Laplacian when \( F \) is Riemannian.

**Proof.** We only need to prove (5.4) in the case of \( \partial M = \emptyset \). Theorem 5.6 together with Proposition 5.14 yields \( \lambda_2^{LS} > 0 \). Let \( f \) be the eigenfunction corresponding to \( \lambda_2^{LS} \). Since \( f \in \mathcal{X}_0 \), \( \lambda_2^{LS} = E(f) \geq \inf_{u \in \mathcal{X}_0} E(u) \). On the other hand, for each \( u \in \mathcal{X}_0 \), set \( V_u = \text{Span}\{1, u\} \). Then Proposition 5.11 yields \( A_u := \mathcal{S} \cap V_u \in \mathcal{F}_K^2 \) and hence,

\[ \lambda_2^K \leq \sup_{v \in A_u} E(v) = \sup_{\alpha \in [0, 2\pi]} \cos^2 \alpha \cdot E(u) = E(u) \implies \lambda_2^{LS} \leq \inf_{u \in \mathcal{X}_0} E(u). \]

\[ \square \]

**Remark 5.16.** It is unsuitable to use the relative LS category of closed sets in \( \mathcal{S} \) to define the spectrum, since \( \mathcal{S} \) is contractible (see Proposition A.1). In fact, for any nonempty closed set \( A \subset \mathcal{S} \), \( \text{cat}_S(A) = 1 \), in which case the spectrum is strictly degenerate.

On the other hand, it is not hard to see that the \( \dim_{LS} \)-spectrum can be defined by the closed sets on \( \mathbb{P}(\mathcal{X}) \) directly, since \( F \) is reversible. However, we prefer to utilize Definition 5.9 because it still works in the irreversible case.

### 5.2. Krasnoselskii genus.

In this subsection, we use the Krasnoselskii genus to define eigenvalues. Also refer to [1] for the corresponding spectrum defined on \( L^2(M) \), in which the Cheeger energy, instead of the Rayleigh quotient, is the main tool to study eigenvalues.

Now set

\[ \mathcal{K}^K := \{A \subset \mathcal{X}: A \text{ is closed and } A = -A\}, \]

\[ \mathcal{K} := \{A \subset \mathcal{S}: A \text{ is closed and } A = -A\} \subset \mathcal{K}^K. \]

According to [10, 21], the **Krasnoselskii genus** \( \dim_K : \mathcal{K}^K \to \mathbb{N} \cup \{+\infty\} \) is defined by

\[ \dim_K(A) := \gamma(A) = \begin{cases} 
\inf \{m \in \mathbb{N} : \exists h \in C^0(A; \mathbb{R}^m \setminus \{0\}), h(-u) = -h(u)\} \\
\infty, & \text{if } \{\cdots\} = \emptyset, \text{ or } 0 \in A.
\end{cases} \]

And [21, Chapter II, Proposition 5.2, Proposition 5.4, Observation 5.5] yields the following result.

**Proposition 5.17.** Let \( A, B \in \mathcal{K}^K \) and let \( h : \mathcal{X} \to \mathcal{X}^\ast \) be continuous and odd. Then the following properties hold:

(i) \( \dim_K(A) \geq 0 \); In particular, \( \dim_K(A) = 0 \) if and only if \( A = \emptyset \).

(ii) \( A \subset B \) implies \( \dim_K(A) \leq \dim_K(B) \).
Given an arbitrary compact reversible Finsler manifold, we always have \( \dim_K(A) \leq \dim_K(h(A)) \).

(v) If \( A \) is compact and \( \emptyset \notin A \), then \( \dim_K(A) < \infty \) and there is a neighborhood \( N \) of \( A \) in \( \mathcal{K} \) such that \( \overline{N} \in \mathcal{K} \) and \( \dim_K(A) = \dim_K(\overline{N}) \).

(vi) If \( A \) is a finite collection of antipodal pairs \( u_i, -u_i \), then \( \dim_K(A) = 1 \).

(vii) Given \( k \in \mathbb{N}^+ \), for any \( k \)-dimensional linear space \( V \subset \mathcal{K} \), \( \dim_K(\mathcal{S} \cap V) = k \).

Now the Krasnoselskii genus-spectrum is defined as follows.

**Definition 5.18.** Given \( k \in \mathbb{N}^+ \), set

\[
\mathcal{F}_k^K := \{ A \in \mathcal{F}_K : \dim_K(A) \geq k \},
\]

and

\[
\lambda_k^K := \inf_{A \in \mathcal{F}_k^K} \sup_{u \in A} E(u).
\]

Proposition 5.17 (iv) yields the following equivalent definition directly.

**Proposition 5.19.** Set \( \mathcal{G}_k^K := \{ A \in \mathcal{F}_K : \dim_K(A) \geq k, 0 \notin A \} \) and

\[
\lambda_k := \inf_{A \in \mathcal{G}_k^K} \sup_{u \in A} E(u).
\]

Thus, \( \lambda_k = \lambda_k^K \), for all \( k \in \mathbb{N}^+ \).

Using Proposition 5.17 and similar arguments, one can check that Theorem 5.3 remains valid for \( \dim_{LS} \). Now we point out an important fact between the \( \dim_K \)-spectrum and the \( \dim_{LS} \)-spectrum.

**Theorem 5.20.** Given an arbitrary compact reversible Finsler manifold, we always have \( \lambda_k^{LS} = \lambda_k^K \), for all \( k \in \mathbb{N}^+ \).

**Proof.** According to Proposition A.1 and 4.4, \( \mathcal{S} \) is a contractible, paracompact, and \( \mathbb{Z}_2 \)-free space. Given \( A \in \mathcal{F}_k^K \), \( A \) is \( \mathbb{Z}_2 \)-invariant and \( p(A) = A/\mathbb{Z}_2 \). Thus, [7, Theorem, (3), p.34] yields

\[
k \leq \dim_K(A) = \text{cat}_{\mathcal{S}}(A/\mathbb{Z}_2) = \dim_{LS}(A),
\]

which implies \( A \in \mathcal{F}_k^{LS} \) and hence, \( \lambda_k^{LS} \leq \lambda_k^K \).

On the other hand, given any \( A \in \mathcal{F}_k^{LS} \), consider \( A' := A \cup -A \). [7, Theorem, (3), p.34] again yields

\[
\dim_K(A') = \text{cat}_{\mathcal{S}}(A'/\mathbb{Z}_2) = \text{cat}_{\mathcal{S}}(p(A')) = \dim_{LS}(A) \geq k.
\]

Note that \( F \) is reversible and thus,

\[
\lambda_k^K \leq \sup_{u \in A'} E(u) = \sup_{u \in A} E(u) = \lambda_k \leq \lambda_k^{LS}.
\]

By Theorem 5.20, we can give an easy proof for Proposition 5.13.

**Proof of Proposition 5.13.** By Theorem 5.20, it suffices to show the \( \dim_K \)-spectrum satisfies Proposition 5.13. According to Corollary 4.3, \( \dim_K(K_\beta) \geq l \) follows from [21, Chapter II, Lemma 5.6] directly. Recall that \( (\mathcal{S}, \cdot, \cdot) \) is a complete Hilbert space, where \( (\cdot, \cdot) \) is defined by (A.1). Since the topology of \( \mathcal{S} \) does not change (see Proposition 3.1), \( K_\beta \) is still compact in \( (\mathcal{S}, \cdot, \cdot) \) with \( \dim_K(K_\beta) \geq l \). Now [21, Chapter II, Proposition 5.3] yields that \( K_\beta \) contains at least \( l \) mutually orthogonal vectors \( \{u_j, j = 1, \ldots, l\} \) w.r.t \( (\cdot, \cdot) \). The last statement follows from Proposition 5.17 (vi) directly.
Remark 5.21. We compare $\lambda^K$ with the eigenvalues defined on $L^2(M)$ (cf. [1]).

Set $S^{L^2} := \{u \in L^2(M) : ||u||_{L^2} = 1\}$ and $\mathcal{S}^{L^2}(M) := \{A \subset S^{L^2} : A \text{ is compact and } A = -A\}$. Let $\dim^{L^2}_K$ denote the Krasnoselskii genus defined on $\mathcal{S}^{L^2}(M)$ and $\mathcal{F}^{L^2}_k := \{A \in \mathcal{S}^{L^2} : \dim^{L^2}_K(A) \geq k\}$. The Krasnoselskii spectrum on $L^2(M)$ is defined by

$$\lambda^{L^2}_k := \inf_{A \in \mathcal{F}^{L^2}_k} \sup_{u \in A} \text{Ch}(u),$$

where the Cheeger energy

$$\text{Ch}(u) := \begin{cases} \int_M F^*(du) dm, & \text{if } u \in H^{1,2}(M), \\ +\infty, & \text{if } u \in L^2(M) \setminus H^{1,2}(M). \end{cases}$$

Then using the compact embedding theorem and the lower semi-continuousness of Ch, one can show $\lambda^{L^2}_k \leq \lambda^K$ for all $k \in \mathbb{N}^+$.

On the other hand, one can define the spectrum on $L^2(M)$ by the relative LS category on $S^{L^2}$ and prove that Theorem 5.20 still holds.

5.3. Essential dimension.

In [8], Gromov introduced the essential dimension to the projective space. In this section, we utilize this dimension-like function to define the spectrum of a Finsler manifold.

In the following, let $\dim_C$ denote the Lesbesgue covering dimension. A subset $A \subset \mathbb{P}(\mathcal{X})$ is said to be contractible in $\mathbb{P}(\mathcal{X})$ onto a subset $B \subset \mathbb{P}(\mathcal{X})$ if there exists a continuous map (i.e., homotopy) $h : A \times [0,1] \to \mathbb{P}(\mathcal{X})$ with $h(\cdot,0) = \text{Id}_A$ and $h(A,1) = B$.

**Definition 5.22** (Gromov [8]). Given a closed set $A \subset \mathbb{P}(\mathcal{X})$ and $A \neq \emptyset$, the essential dimension of $A$ is defined by

$$\text{ess}(A) := \text{the smallest integer } i \text{ such that } A \text{ is contractible in } \mathbb{P}(\mathcal{X}) \text{ onto}$$

$$\text{a subset } B \subset \mathbb{P}(\mathcal{X}) \text{ with } \dim_C(B) = i,$$

And set $\text{ess}(\emptyset) := -1$.

**Remark 5.23.** In [8], $\text{ess}(\emptyset) := -\infty$. We use the above assumption for convenience.

A standard argument yields the following result.

**Proposition 5.24.** If $f : \mathbb{P}(\mathcal{X}) \to \mathbb{P}(\mathcal{X})$ is a homeomorphism, then $\text{ess}(A) = \text{ess}(f(A))$. That is, ess is a topological invariant.

Recall that $p : \mathcal{S} \to \mathbb{P}(\mathcal{X})$ is a closed map. Now we introduce the following definition.

**Definition 5.25.** Set $\mathcal{F}^{ES} := \{A \subset \mathcal{S} : A \text{ is closed}\}$. For each $A \in \mathcal{F}^{ES}$, we define the *essential dimension* of $A$ by

$$\dim^{ES}(A) := \text{ess}(p(A)) + 1.$$

According to [8] 0.4B, we have the following result.

**Proposition 5.26.** Let $A, B \in \mathcal{F}^{ES}$ and $h : \mathcal{S} \to \mathcal{S}$ be an odd homeomorphism.

(i) $\dim^{ES}(A) \geq 0$ with equality iff $A = \emptyset$.

(ii) If $A \subset B$, then $\dim^{ES}(A) \leq \dim^{ES}(B)$.

(iii) $\dim^{ES}(A \cup B) \leq \dim^{ES}(A) + \dim^{ES}(B)$. 

(iv) \( \dim_{ES}(A) = \dim_{ES}(h(A)) \).

(v) For a point set \( A = \{u\} \), \( \dim_{ES}(A) = 1 \). Given any \( k \)-dimensional linear space \( V \subset \mathbb{R}^\alpha \), \( \dim_{ES}(S \cap V) = k \) always holds.

Proof. (i) and (ii) follows from the definition directly and (iii) follows from [8 0.4B]. We now prove (iv). According to Proposition 5.24, it suffices to show that \( H([u]) := p \circ h(u), G([u]) := p \circ h^{-1}(u) \) are two homeomorphisms on \( \mathbb{P}(\mathcal{X}) \). Firstly, it is easy to see that \( H, G \) are well-defined and surjective. And the proof of Lemma 5.10 yields \( H \) and \( G \) are continuous. Note that

\[
H \circ G = \text{Id}, \quad G \circ H = \text{Id},
\]

which imply \( H \) and \( G \) are injective and hence, homeomorphisms. Thus, Proposition 5.24 yields

\[
\dim_{ES}(h(A)) = \text{ess}(H(p(A))) + 1 = \text{ess}(p(A)) + 1 = \dim_{ES}(A).
\]

In order to show (v), consider the "unit sphere" \( A := V \cap S \) in a \( k \)-dimensional linear space \( V \). Then \( p(A) = \mathbb{P}(V) \), which together with [8 0.4B] yields

\[
\dim_{ES}(A) = \text{ess}(\mathbb{P}(V)) + 1 = k.
\]

\[\square\]

Definition 5.27. Given \( k \in \mathbb{N}^+ \), set

\[
\mathcal{F}^ES_k := \{ A \in \mathcal{F}^ES : \dim_{ES}(A) \geq k \},
\]

and

\[
\lambda^ES_k := \inf_{A \in \mathcal{F}^ES_k} \sup_{u \in A} E(u).
\]

Remark 5.28. It is unsuitable to define the spectrum by extending Def. 5.22 to the closed sets of \( S \) directly, since \( S \) is contractible (Proposition A.1). Otherwise, for any nonempty closed set \( A \subset S \), \( \text{ess}(A) = 1 \) and then the spectrum is strictly degenerate. On the other hand, Def. 5.27 still works in the nonreversible case.

Using Proposition 5.26 and Theorem 4.6, one can check Theorem 5.3 is valid for \( \dim_{ES} \). Moreover, we have the following result.

Proposition 5.29. If \( F \) is Riemannian, then \( \lambda^ES_k = \lambda^\Delta_k \).

Proof. First, Theorem 5.3 yields that \( \lambda^ES_k \leq \lambda^\Delta_k < +\infty \). Now we show \( \lambda^\Delta_k \leq \lambda^ES_k \).

Step 1. In this step, we show that if \( \beta \) is an eigenvalue w.r.t the essential dimension, there exists an open subset \( N \subset S \) such that \( \mathcal{N} \supset K_\beta \) and \( \dim_{ES}(\mathcal{N}) = \dim_{ES}(K_\beta) < \infty \) (see (4.1)).

Since \( F \) is Riemannian, the eigenspace of \( \beta \), say \( W_\beta \), is finitely \( k \)-dimensional which is spanned by \( \{u_1, \ldots, u_k\} \) satisfies (5.3). Since \( \Delta \) is linear, \( K_\beta = S \cap W_\beta \) and hence, \( p(K_\beta) = \mathbb{P}(W_\beta) \), which together with [8 0.4B] yields

\[
\dim_{ES}(K_\beta) = \text{ess}(p(K_\beta)) + 1 = \text{ess}(\mathbb{P}(W_\beta)) + 1 = k < \infty.
\]

On the other hand, since \( (\mathcal{X}, \langle \cdot, \cdot \rangle) \) is a sparable Hilbert space (see (A.1)), for a fixed \( \epsilon \in (0, 1) \), define

\[
N' := \{ u + \rho : u \in W_\beta, \rho \in W_\beta^1, \sqrt{\langle \rho, \rho \rangle} < \epsilon \}
\]

Clearly, using a complete orthonormal basis, it is easy to check that \( N' \) is an open neighbourhood of \( W_\beta \). Since the expression of \( u + \rho \in N' \) is unique, we can define a homotopy \( H : N' \times [0, 1] \rightarrow W_\beta, H(u + \rho, t) = u + (1 - t)\rho \). Hence, \( N' \) is
contractible onto $W_\beta$. On the other hand, $N' \cap S$ is an open neighbourhood of $K_\beta$. Since $H(u + \rho, t) = 0$ implies $u = 0$ and $\|\rho\|_{L^2} \leq \sqrt{\langle \rho, \rho \rangle} \leq \epsilon < 1$, we define a homotopy $H' : N' \cap S \to S$ by

$$H'(\cdot, t) = \frac{H(\cdot, t)}{\|H(\cdot, t)\|_{L^2}}.$$  

It is easy to see that $H'(\cdot, 0) = \text{Id}$ and $H'(N' \cap S, 1) = K_\beta$. That is, $N' \cap S$ is contractible onto $K_\beta$. Now set $N := p(N' \cap S)$. Clearly, $N$ is an open neighbourhood of $p(K_\beta)$. Since $H'$ is odd, Lemma 5.10 yields a homotopy $H'' : \overline{N} \times [0, 1] \to \mathbb{P}(\mathbb{Z})$ such that

$$H'' \circ p = p \circ H',$$

which implies $N$ is contractible onto $p(K_\beta)$ and hence, $\text{ess}(N) \leq \text{ess}(p(K_\beta))$. However, $N' \supset p(K_\beta)$, which implies

$$\text{ess}(p(K_\beta)) \leq \text{ess}(N) \leq \text{ess}(p(K_\beta)) = k - 1.$$

Now we are done by setting $N := N' \cap S$.

**Step 2.** In this step, we prove that if for some $s \in \mathbb{N}^+$,

$$0 \leq \lambda_{k}^{ES} = \lambda_{k+1}^{ES} = \cdots = \lambda_{k+s-1}^{ES} = \beta < \infty,$$

i.e., the multiplicity of the eigenvalue $\beta$ is $l$, then there exist $l$ eigenfunctions $u_i$ corresponding to $\beta$ such that $\|u_i\|_{L^2} = 1$ and (5.3) holds.

All the notations are the same as in Step 1. Suppose $\text{dim}_{ES}(K_\beta) = k$. Let $\theta = 1$ and $N$ be the open set defined in Step 1. Thus, Theorem 4.6 yields that $\Phi(E_{k+l-1}) \subset E_{\beta-\theta} \cup N$.

On the other hand, there exists $A \subset \mathcal{F}_{k+l-1}$ such that $\sup_{u \in A} E(u) < \beta + \vartheta$, i.e., $A \subset E_{\beta+\vartheta}$. Now set $B := \Phi(A, 1)$. Thus, $\text{dim}_{ES}(B) = \text{dim}_{ES}(A) \geq s + l - 1$. From above, $B \subset E_{\beta-\theta} \cup N$ and $\text{dim}_{ES}(E_{\beta-\theta}) < s$. Now (3) in Proposition 5.26 yields

$$k = \text{dim}_{ES}(K_\beta) = \text{dim}_{ES}(N) \geq \text{dim}_{ES}(E_{\beta-\theta} \cup N) - \text{dim}_{ES}(E_{\beta-\theta})$$

$$> \text{dim}_{ES}(B) - s \geq \text{dim}_{ES}(A) - s \geq l - 1.$$

Now the results follows from the definition of $k$ in Step 1 (also see Proposition 5.26 (4)).

**Step 3.** For each $k \geq 1$, we see that

$$0 \leq \lambda_1^{ES} \leq \cdots \leq \lambda_k^{ES} < +\infty.$$  

A similar argument to the proof of Proposition 5.14 together with Step 2 furnishes $\lambda_k^{ES} \leq \lambda_k^{ES}$. Thus, we are done. \qed

By the same argument as in the proof of Theorem 5.13, we have the following conclusion.

**Theorem 5.30.** Let $(M, F, dm)$ be a compact reversible Finsler manifold and let $\{\lambda_k^{ES}\}_{k=1}^{\infty}$ denote the $\text{dim}_{ES}$-spectrum. Then

$$0 = \lambda_1^{ES} \leq \lambda_2^{ES} \leq \cdots \leq \lambda_k^{ES} \leq \cdots \nearrow +\infty, \text{ if } \partial M = \emptyset,$$

$$0 < \lambda_1^{ES} \leq \lambda_2^{ES} \leq \cdots \leq \lambda_k^{ES} \leq \cdots \nearrow +\infty, \text{ if } \partial M \neq \emptyset,$$
where the first positive eigenvalue satisfies

\[
\begin{align*}
\lambda_2^{ES} &= \inf_{u \in \mathcal{F}_0} E(u), \quad \text{if } \partial M = \emptyset; \\
\lambda_1^{ES} &= \inf_{u \in \mathcal{F}_0} E(u), \quad \text{if } \partial M \neq \emptyset.
\end{align*}
\]

For each \( k \in \mathbb{N}^+ \), there exists \( u = (m(M))^{-\frac{1}{2}} \) or \( u \in \mathcal{F}_0 \) with

\[\Delta u + \lambda_k^{ES} u = 0 \text{ in the weak sense.}\]

The multiplicity \( \text{m}(\lambda_k^{ES}) \) of each eigenvalue \( \lambda_k^{ES} \) is always finite. Moreover, \( \{\lambda_k^{ES}\}_{k=1}^{\infty} \) is exactly the standard spectrum when \( F \) is Riemannian.

Usually there is no connection between the \( \text{dim}_{LS} \)-spectrum and \( \text{dim}_{ES} \)-spectrum as the definition of \( \mathcal{F} := \mathcal{F}^{LS} = \mathcal{F}^{ES} \) is quite weak. In the following, we consider a "stronger" subfamily of \( \mathcal{F} \).

**Definition 5.31.** Let \( \mathcal{F}^S := \{A \subseteq S: A \text{ is compact}\} \subseteq \mathcal{F} \). Given \( k \in \mathbb{N}^+ \), set

\[\mathcal{F}_k^{SES} := \{A \in \mathcal{F}^S: \text{dim}_{ES}(A) \geq k\}, \quad \mathcal{F}_k^{SLS} := \{A \in \mathcal{F}^S: \text{dim}_{LS}(A) \geq k\},\]

and

\[\lambda_k^{SES} := \inf_{A \in \mathcal{F}_k^{SES}} \sup_{u \in A} E(u), \quad \lambda_k^{SLS} := \inf_{A \in \mathcal{F}_k^{SLS}} \sup_{u \in A} E(u).\]

The spectrum \( \{\lambda_k^{SES}\}_{k=1}^{\infty} \) (resp., \( \{\lambda_k^{SLS}\}_{k=1}^{\infty} \)) is called the strong \( \text{dim}_{ES} \)-spectrum (resp., strong \( \text{dim}_{LS} \)-spectrum).

For any \( k \in \mathbb{N}^+ \), let \( V \) be a \( k \)-dimensional linear subspace in \( \mathcal{F} \). Clearly, \( V \cap S \in \mathcal{F}^S \) with \( \text{dim}_{ES}(V \cap S) = k = \text{dim}_{LS}(V \cap S) \). Thus, it is easy to see that Theorem 5.15 (resp., Theorem 5.30) remains valid for the strong \( \text{dim}_{LS} \)-spectrum (resp., strong \( \text{dim}_{LS} \)-spectrum). Thus, the second part of Theorem 1.2 follows. Furthermore, we have the following relationship, which together with Theorem 5.20 implies the first part of Theorem 1.2.

**Theorem 5.32.** For any \( k \in \mathbb{N}^+ \),

\[\min\{\lambda_k^{LS}, \lambda_k^{ES}\} \leq \lambda_k^{SES} \leq \lambda_k^{SLS}.\]

**Proof.** It suffices to show that \( \lambda_k^{SES} \leq \lambda_k^{SLS} \). Given \( A \in \mathcal{F}_k^{SLS} \), let \( B \) be the homotopic image of \( p(A) \) with \( \text{dim}_{C}(B) = \text{ess}(p(A)) \). Note that \( B \) is compact (closed) and \( P(\mathcal{F}) \) is a normal ANR (see Proposition 1.2). Then [5] Remark 1.12 (2)] yields that

\[\text{dim}_{ES}(A) = \text{dim}_{C}(B) + 1 \geq \text{cat}_{\mathcal{F}}(B).\]

On the other hand, it follows from [5] Lemma 1.13.(3)] that

\[\text{cat}_{\mathcal{F}}(B) \geq \text{cat}_{\mathcal{F}}(p(A)) = \text{dim}_{LS}(A).\]

Hence, \( \text{dim}_{ES}(A) \geq \text{dim}_{LS}(A) \geq k \) and \( A \in \mathcal{F}_k^{SES} \), which implies \( \lambda_k^{SES} \leq \lambda_k^{SLS} \).

**Remark 5.33.** Theorem 5.32 still holds for irreversible Finsler manifolds.

5.4. **Some flawed dimension-like functions.**

In the above sections, we have seen some strictly degenerate spectrums. Now we point out that the spectrums induced by the Lebesgue covering dimension and the projective dimension are strictly trivial but not degenerate.
5.4.1. Lebesgue covering dimension.
Set $\mathcal{F}^C := \{A \subseteq S : A \text{ is closed}\}$. Given any $A \subseteq \mathcal{F}^C$, define the modified Lebesgue covering dimension $\dim_{MC}(A) := \dim_C(A) + 1$, where $\dim_C$ is the standard Lebesgue covering dimension. Since $\mathcal{F}$ is a separable metric space, $\dim_C(A) = \text{ind}(A)$ for any subset $A \subseteq S \subseteq \mathcal{F}$, where $\text{ind}(\cdot)$ denotes the (small) inductive dimension. Thus, it is easy to show the following result.

**Proposition 5.34.** Let $A, B \in \mathcal{F}^C$ and $h : S \to S$ be an homeomorphism.
(1) $\dim_{MC}(A) \geq 0$ with equality iff $A = \emptyset$.
(2) If $A \subseteq B$, then $\dim_{MC}(A) \leq \dim_{MC}(B)$.
(3) $\dim_{MC}(A \cup B) \leq \dim_{MC}(A) + \dim_{MC}(B)$.
(4) $\dim_{MC}(A) = \dim_{MC}(h(A))$.
(5) For a point set $A = \{u\}$, $\dim_{MC}(A) = 1$. Given any $k$-dimensional linear space $V \subset \mathcal{F}$, $\dim_{MC}(S \cap V) = k$ always holds.

Now we define the $\dim_{MC}$-spectrum similarly as before. Given $k \in \mathbb{N}^+$, set
$$\mathcal{F}_{k}^{MC} := \{A \in \mathcal{F}^C : \dim_{MC}(A) \geq k\},$$
and
$$\lambda_{k}^{MC} := \inf_{A \in \mathcal{F}_{k}^{MC}} \sup_{u \in A} E(u).$$

Clearly, $\mathcal{F}_{k}^{MC} \neq \emptyset$ for each $k \in \mathbb{N}^+$ and Theorem 5.3 holds for the $\dim_{MC}$-spectrum. However, we now show the spectrum is strictly trivial in the Riemannian case.

**Proposition 5.35.** Let $(M, g)$ be a compact Riemannian manifold. Thus,
$$\lambda_{1}^{MC} = \lambda_{2}^{MC} = \cdots = \lambda_{k}^{MC} = \cdots.$$

**Proof.** First note that $\lambda_{1}^{MC} = \lambda_{1}^{1}$. For $k \geq 2$, let $e_{i}, i = 1, \ldots, k$ be the eigenfunctions corresponding to the standard eigenvalues $\lambda_{i}^{1}$ with $\|e_{i}\|_{L^2} = 1$. Set $W_{k} := \text{Span}(e_{1}, \ldots, e_{k})$. Thus, $\{e_{i}\}$ is an orthonormal basis for $W_{k}$, i.e., $(e_{i}, e_{j})_{L^2} := \int_{M} e_{i} e_{j} \, dm = \delta_{ij}$. Let $(\cos \alpha_{1}, \ldots, \cos \alpha_{k})$ denote the direction cosines of a vector in $(W_{k}, (\cdot, \cdot)_{L^2})$ w.r.t $\{e_{i}\}$. That is, $\cos \alpha_{i} := (\cdot, e_{i})_{L^2} / \|\cdot\|_{L^2}$.

Given $\theta_{n} := \pi / 2^{n}$, consider the set
$$A_{n} := \left\{ \sum_{i=1}^{k} \cos \alpha_{i} \cdot e_{i} : -\theta_{n} \leq \alpha_{1} \leq \theta_{n} \right\}.$$ 

Clearly, $A_{n} \in \mathcal{F}_{k}^{MC}$ and hence,
$$\lambda_{1}^{MC} \leq \lambda_{k}^{MC} \leq \sup_{u \in A_{n}} E(u) = \cos^{2} \alpha_{1} \cdot \lambda_{1}^{1} + \sum_{i=2}^{k} \cos^{2} \alpha_{i} \cdot \lambda_{i}^{1}$$
$$\leq \cos^{2} \alpha_{1} \cdot \lambda_{1}^{1} + \sin^{2} \alpha_{1} \cdot \lambda_{k}^{1} \rightarrow \lambda_{1}^{1} = \lambda_{k}^{MC}.$$ 

This implies that $\lambda_{1}^{MC} = \cdots = \lambda_{k}^{MC}$. 

\[ \square \]

5.4.2. Projective dimension.
The following dimension-like function is inspired by \[8\].

**Definition 5.36.** Given $A \in \mathcal{P}(\mathcal{F})$, set
$$\text{pro}(A) := \max\{\dim_{C}(B) : B \subset A, B \text{ is homeomorphic to some projective subspace}\}.$$
Proposition 5.38. Let $\mathcal{P}$ be a compact Riemannian manifold. Thus, 

$$\dim_p(A) := \text{pro}(p(A)) + 1.$$ 

Given $k \in \mathbb{N}^+$, set 

$$\mathcal{P}_k := \{ A \in \mathcal{P} : \dim_p(A) \geq k \},$$ 

and 

$$\lambda^p_k := \inf_{A \in \mathcal{P}_k} \sup_{u \in A} E(u).$$

Remark 5.37. It is noticeable that the original definition of $\text{pro}(A)$ in [8] is the maximal dimension of projective subspaces contained in $A$. We modified it such that $\text{pro}(A)$ in Def. 5.36 is a topological invariant.

The definition yields $\dim_p$ satisfies all the properties in Proposition 5.34 except (3). Thus, $\dim_p$-spectrum satisfies Theorem 5.3. In particular, $\mathcal{P}_k \neq \emptyset$, $\forall k \in \mathbb{N}^+$. However, we now show that the eigenvalues are still strictly trivial in the Riemannian case.

Proposition 5.38. Let $(M, g)$ be a compact Riemannian manifold. Thus, 

$$\lambda_1^\Delta = \lambda_1^p = \cdots = \lambda_k^p = \cdots.$$ 

Proof. Clearly, $\lambda^p_i = \lambda^\Delta$. Given $k \geq 2$, Let $e_i$, $i = 1, \ldots, k$ be the unit eigenfunctions corresponding to the standard eigenvalue $\lambda^\Delta_i$. Let $W_k$ and $A_n$ be defined as in Proposition 5.35. On the other hand, use $(x_1, \ldots, x_k)$ to denote the coordinates of $(W_k, \langle \cdot, \cdot \rangle_{L^2})$. Thus, the unit sphere in $W_k$ is $S_W := \{(x_1, \ldots, x_k) \in W_k : \sum_{i=1}^k x_i^2 = 1\}$. It is easy to show that there exists an odd homeomorphism $h_n : A_n \to S_W^+ := \{ (x_1, \ldots, x_k) \in S_W : x_1 \geq 0 \}$. Now the proof of Proposition 5.26 (iv) yields that $p(A_n)$ is homeomorphic to $p(S_W^+) = \mathcal{F}(W_k)$, which implies that $A_n \in \mathcal{P}_k^p$. Thus, (5.5) holds and the statement follows.

6. A UNIVERSAL UPPER BOUNDED FOR EIGENVALUES

In this section, we prove Theorem 1.3. In the following, let $\dim^*$ be a dimension-like function with (1.3) and let $\lambda^*_n$ denote the (strong) $\dim^*$-spectrum.

Given a precompact domain $D$ in a Finsler manifold $M$, set 

$$H(D) := \{ u \in H^{1,2}(M) : u|_{M \setminus D} = 0 \}.$$ 

Let $\lambda^*_1(D)$ denote the first eigenvalue defined on $H(D)$ w.r.t $\dim^*$. According to Corollary 5.5, one has 

$$\lambda^*_1(D) = \inf_{u \in H(D) \setminus \{0\}} E(u).$$

On the other hand, let $B^*_K(r_0)$ denote a geodesic ball with radius $r_0$ in the $N$-dimensional Riemannian space form with constant sectional curvature $K$, and let $\lambda_1(B^*_K(r_0))$ be the standard first eigenvalue defined on $B^*_K(r_0)$.

Next, we recall the Laplacian comparison obtained by Obata [16].

Lemma 6.1 (16 Theorem 5.2). Let $(M, F, dm)$ be a forward complete Finsler $n$-manifold. If for some $N \geq [n, \infty)$, the weighted Ricci curvature satisfies $\text{Ric}_N \geq (N - 1)K$, then the Laplacian of the distance function $r(x) = d_F(p, x)$ from any given point $p \in M$ can be estimated as follows:

$$\Delta r \leq \frac{d}{dr} \left( \log \mathcal{E}_K^{N-1}(r) \right),$$
which holds pointwise on $M - \text{Cut}_p \cup \{p\}$ and in the sense of distributions on $M - \{p\}$.

Thus, inspired by [4], we obtain the following lemma.

**Lemma 6.2.** Let $(M, F, dm)$ be a complete reversible Finsler $n$-manifold. If for some $N \in [n, +\infty)$, the $N$-Ricci curvature satisfies $\text{Ric}_N \geq (N - 1)K$, then for any $p \in M$,

$$\lambda_1^*(B_p(r_0)) = \lambda_1(B_K(N)(r_0)).$$

Proof. Let $\varphi$ be the nonnegative eigenfunction w.r.t. $\lambda_1(B_K(N)(r_0))$, which is always a radial function. Let $r(x) := d(p, x)$ be the distance function from $p$ to $x$ in $M$. Thus, $f(x) := \varphi \circ r(x) \in H(B_p(r_0))$. Consider compact sets $A := \{ \pm f/\|f\|_{L^2} \} \subset S$. Since $\dim^+(A) = 1$,

$$\lambda_1^*(B_p(r_0)) \leq \int_{B_p(r_0)} F^s(\varphi)^2 \, dm \int_{B_p(r_0)} f^2 \, dm.$$

Let $(t, y)$ be the polar coordinate system at $p$. Then $f(t, y) = \varphi(t)$ for all $0 < t < \min\{r_0, r\} =: a(t)$ and hence, $\partial f/\partial t = df/\partial t < 0$. Clearly, one has

$$\int_{B_p(r_0)} F^s(\varphi)^2 \, dm = \int_{S_p M} dV_p(y) \int_0^{a(t)} \frac{d\varphi}{dt}^2 \, \sigma_p(t, y) \, dt;$$

(6.1)

$$\int_{B_p(r_0)} f^2 \, dm = \int_{S_p M} dV_p(y) \int_0^{a(t)} \varphi^2(t) \, \sigma_p(t, y) \, dt,$$

where $dm(t, y) := \sigma_p(t, y) \, dt$ and $dV_p(y)$ is the Riemannian volume form of $S_p M$ induced by $F$ (cf. [23]). Note that

$$\int_0^{a(t)} \left( \frac{d\varphi}{dt} \right)^2 \, \sigma_p(t, y) \, dt$$

(6.2)

$$= \varphi \frac{d\varphi}{dt} \sigma_p(t, y) \bigg|_0^{a(t)} - \int_0^{a(t)} \sigma_p(t, y) \left[ \frac{d\varphi}{dt} \right] \, dt \sigma_p(t, y) \, dt.$$

Now Lemma 6.1 yields

$$\Delta r = \frac{\partial}{\partial t} \log \sigma_p(t, y) \leq \frac{d}{dt} \log s_K^{N-1}(t),$$

(6.3)

which implies

$$\frac{1}{\sigma_p(t, y)} \frac{d}{dt} \left[ \frac{d\varphi}{dt} \sigma_p(t, y) \right] = \frac{d^2 \varphi}{dt^2} + \frac{d\varphi}{dt} \frac{\partial}{\partial t} \log \sigma_p(t, y)$$

$$\geq \frac{d^2 \varphi}{dt^2} + \frac{d\varphi}{dt} \left( \log s_K^{N-1}(t) \right) = -\lambda_1(B_K(N)(r_0)) \varphi.$$

Note that $\lim_{t \to 0^+} \sigma_p(t, y) = 0$ (cf. [23 Lemma 3.1]). Thus, the above inequality together with (6.2) yields

$$\int_0^{a(t)} \left( \frac{d\varphi}{dt} \right)^2 \sigma_p(t, y) \, dt \leq \lambda_1(B_K(N)(r_0)) \int_0^{a(t)} \varphi^2(t) \, dt.$$

(6.4)

Integrating (6.3) on $S_p M$, one can see that

$$\int_{B_p(r_0)} F^s(\varphi)^2 \, dm \leq \lambda_1(B_K(N)(r_0)) \int_{B_p(r_0)} f^2 \, dm.$$
Remark 6.3. Suppose that $\lambda_1^*(B_p(r_0)) = \lambda_1^*(B_K^N(t_0))$. Thus, $N = n$ follows from (6.3) and $\sigma_p(t, y) \sim t^{n-1}e^{-\tau(n)}$ ([23, Lemma 3.1]). And the definition of $\text{Ric}_N$ yields that $S = 0$ and hence, $S = 0$. Thus, $\text{Ric}_N = \text{Ric}$ and therefore, $\text{Ric} = (n-1)K$. Now (6.3) together with [23, Theorem 3.4] yields that $K(\frac{t}{r_0}; \cdot) = K$, $0 \leq t \leq r_0$.

According to [14], we introduce convex Finsler manifolds.

Definition 6.4. Let $(M, F)$ be a forward complete Finsler manifold (without boundary) and let $C$ be a subset in $M$. $C$ is called convex in $M$, if for any $p, q \in \overline{C}$, there exists a minimal geodesic in $M$ from $p$ to $q$, which is contained in $\overline{C}$.

A compact Finsler $n$-manifold $M$ with or without boundary $\partial M$ is said to be convex if there are a forward complete Finsler $n$-manifold $W$ (without boundary) and isometric imbedding $i: M \hookrightarrow W$ such that $i(M)$ is a convex subset of $W$.

Clearly, a closed Finsler manifold is always convex. Then we have the following result, which implies Theorem 1.3

Theorem 6.5. Let $(M, F, dm)$ be a compact convex reversible Finsler $n$-manifold. If for some $N \in [n, +\infty)$, the $N$-Ricci curvature satisfies $\text{Ric}_N \geq (N-1)K$, then for any $k \in \mathbb{N}^+$,

$$\lambda_k^* \leq \lambda_1^*(B_K^N\left(\frac{D}{2k}\right)),$$

where $D$ is the diameter of $M$. Hence,

(1) if $K \geq 0$, then

$$\lambda_k^* \leq \frac{2k^2 N(N+4)}{D^2}.$$

(2) if $K < 0$, then

$$\lambda_k^* \leq \left\{ \begin{array}{ll}
\frac{(2m+1)^2}{4} |K| + \frac{4\pi^2 k^2 (1+2m)^2}{D^2}, & \text{if } N = 2(m+1), m = 0, 1, \ldots, \\
\frac{(2m+2)^2}{4} |K| + \frac{4k^2 (1+\pi^2)(1+2m^2)^2}{D^2}, & \text{if } N = 2m+3, m = 0, 1, \ldots.
\end{array} \right.$$ 

Proof. Since $\text{diam}(M) = D < +\infty$, there exist $k$ distinct points $\{p_i\}_{i=1}^k$ such that $B_i := B_{p_i}(D/(2k))$ are disjoint. Let $\varphi$ be the first eigenfunction on $B_K^N(D/(2k))$ and set $f_i = \varphi \circ r_i$, where $r_i(x) := d(p_i, x)$. And extend $f_i$ on $M$ by $f_i(x) = 0$, if $x \in M \setminus B_i$. Thus, $f_i \in \mathcal{X}$. Define

$$V_k = \text{Span}\{f_1, \ldots, f_k\}, \quad A := \{f \in V_k : \|f\|_{L^2} = 1\} = V_k \cap \mathcal{S}.$$ 

Thus, $\dim^*(A) \geq k$ and hence,

$$\lambda_k^* \leq \sup_{u \in A} E(u) = \sup_{u \in V_k \setminus \{0\}} E(u).$$

Now, the proof of Lemma 6.2 yields that for each $i \in \{1, \ldots, k\}$,

$$\int_M F^{u^2}(df_i)dm \leq \lambda_1^*(B_K^N\left(\frac{D}{2k}\right)) \int_M f_i^2 dm.$$
On the other hand, since the support sets $B_i$ of $f_i$ are disjoint, then for any $f = \sum_{i=1}^{k} a_i f_i \in V_k$ ($a_i$ are constants), we have

$$\int_M F^2(df) d\mathbf{m} = \int_M \sum_{i,j=1}^{k} a_i a_j g_{ij}(df_i, df_j) d\mathbf{m} = \sum_{i=1}^{k} a_i^2 \int_M F^2(df_i) d\mathbf{m}$$

$$\leq \lambda_1 \left( B_K^N \left( \frac{D}{2k} \right) \right) \sum_{i=1}^{k} a_i^2 \int_M f_i^2 d\mathbf{m} = \lambda_1 \left( B_K^N \left( \frac{D}{2k} \right) \right) \int_M f^2 d\mathbf{m}.$$
Proof. Since \( \mathcal{S} \) is a Banach manifold and \( p : \mathcal{S} \to \mathbb{P}(\mathcal{X}) \) is a 2-fold covering, a standard argument yields that \( \mathbb{P}(\mathcal{X}) \) is a topological Banach manifold.

Now we show \( \mathbb{P}(\mathcal{X}) \) is paracompact. Given any open covering \( \{U_\alpha\} \) of \( \mathbb{P}(\mathcal{X}) \), we can obtain a refinement \( \{V_\beta\} \) and an open covering \( \{V_\beta, -V_\beta\} \) of \( \mathcal{S} \) such that \( p|_{\pm V_\beta} : \pm V_\beta \to V_\beta \) are homeomorphisms. Recall that \( \mathcal{S} \) is a metric space and hence, paracompact. Thus, there exists a locally finite refinement \( \{O_\gamma\} \) of \( \{V_\beta, -V_\beta\} \) and thus, \( p|_{O_\gamma} : O_\gamma \to p(O_\gamma) \) is homeomorphism. So, \( \{p(O_\gamma)\} \) is a covering of \( \mathbb{P}(\mathcal{X}) \). In particular, since \( O_\gamma \subset V_\beta \) or \(-V_\beta\), we have

\[
p(O_\gamma) \subset p(V_\beta) = V_\beta \subset U_\alpha,
\]

and hence, \( \{p(O_\gamma)\} \) is a refinement of \( \{U_\alpha\} \). On the other hand, for each \( [u] \in \mathbb{P}(\mathcal{X}) \), there are two open neighbourhoods \( N_\pm \subset \mathcal{S} \) of \( \pm u \) such that each of them intersects \( \{O_\gamma\} \) finitely times. Let \( N_{[u]} := p(N_+) \cap p(N_-) \), which is an open set since \( p \) is open. Thus, if \( N_{[u]} \) intersects some \( p(O_\gamma) \), then \( O_\gamma \) must intersect at least one of \( N_\pm \) but not vice versa, which implies that the number of \( p(O_\gamma) \) intersecting \( N_{[u]} \) must be not greater than the number of \( O_\gamma \) intersecting \( N_\pm \) and therefore, it is finite. That is, \( \{p(O_\gamma)\} \) is locally finite. Hence, \( \mathbb{P}(\mathcal{X}) \) is paracompact and therefore, normal. Now it follows from [17] Corollary, p.3 that \( \mathbb{P}(\mathcal{X}) \) is an ANR.

The proof of Proposition 5.11 (i), (ii) and (iii) follow from the definition directly. Now we show (iv). Suppose that \( \dim_{LS}(A) = k \). Using Lemma 5.10 one can define a homotopy \( H_t : p(A) \to \mathbb{P}(\mathcal{X}) \) such that \( H_t \circ p = p \circ h_t \). According to [5] Lemma 1.13, (3)], we have

\[
\dim_{LS}(h_1(A)) \geq \dim_{LS}(h_1(A)) = \text{cat}_{\mathcal{P}(\mathcal{X})}(H_1 \circ p(A)) \geq \text{cat}_{\mathcal{P}(\mathcal{X})}(p(A)) = \dim_{LS}(A).
\]

(v): If \( A \) is compact, \( p(A) \) is compact and hence, \( \dim_{LS}(A) = \text{cat}_{\mathcal{P}(\mathcal{X})}(p(A)) < +\infty \). Since \( \mathbb{P}(\mathcal{X}) \) is a normal ANR (see Proposition A.2, [5] Lemma 1.13 (4)) yields that there exists an open neighbourhood \( \mathcal{N} \) of \( p(A) \) such that \( \text{cat}_{\mathcal{P}(\mathcal{X})}(\mathcal{N}) = \text{cat}_{\mathcal{P}(\mathcal{X})}(p(A)) \). Now we claim that

\[
\overline{p^{-1}(\mathcal{N})} = p^{-1}(\mathcal{N}).
\]

If \( \overline{p^{-1}(\mathcal{N})} \) is true, \( N := p^{-1}(\mathcal{N}) \) is an open neighbourhood of \( A \) with

\[
\dim_{LS}(N) = \text{cat}_{\mathcal{P}(\mathcal{X})}(N) = \text{cat}_{\mathcal{P}(\mathcal{X})}(p(A)) = \dim_{LS}(A).
\]

Now we show \( \overline{A} \). Clearly, \( p^{-1}(\mathcal{N}) \subset \overline{p^{-1}(\mathcal{N})} \). On the other hand, given any \( u \in p^{-1}(\mathcal{N}) \), since \( p \) is a 2-fold covering, there exists an open neighbourhood \( O_u \) (resp., \( O_{p(u)} \)) of \( u \) (resp., \( p(u) \)) such that \( p : O_u \to O_{p(u)} \) is a homeomorphism.

For any open neighbourhood \( V \) of \( u \), consider \( V \cap O_u \). Clearly, \( p(V \cap O_u) \) is an open neighbourhood of \( p(u) \). Since \( p(u) \in \mathcal{N} \), there exists \( y \in \mathcal{N} \cap p(V \cap O_u) \neq \emptyset \) and hence, \( p^{-1}(y) \cap (V \cap O_u) \neq \emptyset \). Choosing \( z \in p^{-1}(y) \cap (V \cap O_u) \), we have \( z \in p^{-1}(\mathcal{N}) \cap (V \cap O_u) \subset p^{-1}(\mathcal{N}) \cap V \), that is, \( p^{-1}(\mathcal{N}) \cap V \neq \emptyset \). Hence, \( u \in p^{-1}(\mathcal{N}) \), i.e., \( \overline{p^{-1}(\mathcal{N})} \subset \overline{p^{-1}(\mathcal{N})} \). Now \( \overline{A} \) follows.

(vi): The first statement follows from the definition directly. On the other hand, for each \( k \in \mathbb{N} \), \( A := V \cap S \) is the "unit sphere" in a \( k \)-dimensional linear space \( V \). Thus, \( p(A) = \mathbb{P}(V) \) and hence, \( \dim_{LS}(A) = \text{cat}_{\mathcal{P}(\mathcal{X})}(\mathbb{P}(V)) = k \).

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References

[1] L. Ambrosio, S. Honda and J. W. Portegies, Continuity of nonlinear eigenvalues in CD(K,∞) spaces with respect to measured Gromov-Hausdorff convergence, Calc. Var. (2018) 57: 34. https://doi.org/10.1007/s00526-018-1315-0.
[2] D. Bao, S. S. Chern and Z. Shen, An introduction to Riemannian-Finsler geometry, GTM 200, Springer-Verlag, 2000.
[3] M. Craioveanu, M. Puta, and Th.M. Rassias, Old and new aspects in spectral geometry, volume 534 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 2001.
[4] Cheng, S. Eigenvalue comparison theorems and its geometric applications. Math. Z. 143 (1975), 289-297.
[5] O. Cornea, G. Lupton, J. Oprea and D. Tanr é, Lusternik-Schnirelmann Category, Mathematical Surveys and Monographs, Volume 103, 2003.
[6] D. Egloff, Uniform Finsler Hadamard manifolds, Ann. Inst. Henri Poincaré, 66(1997), 323-357.
[7] E. Fadell, The relationship between Lusternik-Schnirelmann category and the concept of genus, Pacific J. Math., 89 (1980), 33-42.
[8] M. Gromov, Dimension, nonlinear spectra and width, Geometric aspects of functional analysis, Israel seminar (1986-87), Lecture Notes in Math., 1317, Springer, Berlin (1988), 132-184.
[9] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces. Based on the 1981 French original. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates. Progress in Mathematics, 152. Birkhauser Boston, Inc., Boston, MA, 1999. xx+585 pp.
[10] N. Gigli, A. Mondino, and G. Savaré. Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows. Proc. Lond. Math. Soc. (3), 111(5) (2015), 1071-1129.
[11] Y. Ge and Z. Shen, Eigenvalues and eigenfuncitons of metric measure manifolds, Proc. London Math. Soc. (3) 82(2001), 725-746.
[12] E. Hebey, Sobolev spaces on Riemannian manifolds, Lecture Notes in Math. 1635, Springer, 1996.
[13] S. Kakutani, Topological properties of the unit sphere of a Hilbert space, Proc. Imp. Acad. Tokyo, 19(1943), 269-271.
[14] S. Kronwith, Convex manifolds of nonnegative curvature, Journal of Differential Geometry 14(1979), 621-628.
[15] M. A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, MacMillan, N. Y., (1965).
[16] S.-I. Ohta and K.-T. Sturm, Heat flow on Finsler manifolds, Comm. Pure Appl. Math., 62(2009), 1386-1433.
[17] R. S. Palais, Homotopy theory of infinite dimensional manifolds, Topology 5(1966), 1-16.
[18] R. S. Palais, Lusternik-Schnirelman theory on Banach manifolds, Topology 5(1966), 115-132.
[19] H. Rademacher, Nonreversible Finsler metrics of positive ag curvature, A sampler of Riemann-Finsler geometry, Cambridge Univ. Press, Cambridge, 2004, pp. 261-302.
[20] Z. Shen, Lectures on Finsler geometry, World Sci., Singapore, 2001.
[21] M. Struwe, Variational methods, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, fourth edition, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.
[22] E. Zeidler, Nonlinear Functional Analysis and Its Applications IV: Applications to Mathematical Physics, Springer-Verlag, Berlin, Germany, 1997.
[23] W. Zhao and Y. Shen, A Universal Volume Comparison Theorem for Finsler Manifolds and Related Results, Can. J. Math., 65(2013), 1401-1435.
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