A BIRMAN-SERIES TYPE RESULT FOR GEODESICS WITH INFINITELY MANY SELF-INTERSECTIONS

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Abstract. Given a hyperbolic surface $S$, a classic result of Birman and Series states that for each $K$, all complete geodesics with at most $K$ self-intersections can only pass through a certain nowhere dense, Hausdorff dimension 1 subset of $S$. We define a self-intersection function for each complete geodesic, which bounds the number of self-intersections in finite length subarcs. We then extend the Birman-Series result to sets of complete geodesics with certain bounds on their self-intersection functions. In fact, we get the same conclusion as the Birman-Series result for sets of complete geodesics whose self-intersection functions are in $o(l^2)$, where $l$ measures arclength.

1. Introduction

Let $S$ be a genus $g$ surface with $n$ boundary components, and let $X$ be a hyperbolic metric on $S$ in which each boundary component is geodesic. Consider the set $\mathcal{G}$ of complete geodesics on $S$. In particular, geodesics in $\mathcal{G}$ never hit the boundary of $S$.

Given any subset $\mathcal{H} \subset \mathcal{G}$, we define the image of $\mathcal{H}$ in $S$, denoted $\text{Im} \mathcal{H}$, to be the set of points in $S$ that lie on some curve in $\mathcal{H}$. Birman and Series showed that, for each $K$, if $\mathcal{H}$ is the set of all complete geodesics with at most $K$ self-intersections, then $\text{Im} \mathcal{H}$ is nowhere dense and has Hausdorff dimension 1 [BS85].

In this paper, we find much weaker conditions on subsets $\mathcal{H} \subset \mathcal{G}$ that give the same conclusion. We get these conditions by studying the self-intersection function of each $\gamma \in \mathcal{G}$, which is defined as follows. Take a complete geodesic $\gamma : \mathbb{R} \to S$ parameterized by arclength. Let $\gamma_l = \gamma|_{[-\frac{l}{2}, \frac{l}{2}]}$ be the length $l$ subarc of $\gamma$ centered at $\gamma(0)$. Then $f(l) = i(\gamma_l, \gamma_l)$ is the self-intersection function of $\gamma$.

This function depends on the parameterization of $\gamma$, so we choose a parameterization for each $\gamma \in \mathcal{G}$ so that its self-intersection function is as small as possible. That is, if $\gamma : \mathbb{R} \to S$ and $\gamma' : \mathbb{R} \to S$ are two parameterizations by arclength of the same complete geodesic, then $\gamma$ has a smaller self-intersection function than $\gamma'$ if $i(\gamma_l, \gamma_l) \lesssim i(\gamma'_l, \gamma'_l)$. Note that we write $A(l) \lesssim B(l)$ if $\limsup_{l \to \infty} \frac{A(l)}{B(l)} \leq 1$. There need not be a parameterization of $\gamma$ with the least self-intersection function, so we make an arbitrary choice of parameterization for each $\gamma \in \mathcal{G}$.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is any function. Let

$$\mathcal{G}(f) = \{ \gamma \in \mathcal{G} \mid i(\gamma_l, \gamma_l) \lesssim f(l) \}$$

Theorem 1.1. For any $k > 0$, suppose $f(l)$ is a function with $f(l) \leq (kl)^2$ for all $l$ large enough. Then the Hausdorff dimension of $\text{Im} \mathcal{G}(f)$ is at most $\mu(k)$, where $\lim_{k \to \infty} \mu(k) = 1$. In particular, if $f(l) = o(l^2)$, then $\text{Im} \mathcal{G}(f)$ has Hausdorff dimension 1.
On the other hand, \cite{LS15} implies that $\text{Im} \mathcal{G}(f)$ is dense whenever $f(l)$ is superlinear in $l$ (see Section \ref{sec:1.2}). Nevertheless, we get a nowhere density result once we get more control on the self-intersection function of our geodesics. So if $f : \mathbb{R} \to \mathbb{R}$ is any function, then let

$$
\mathcal{G}(f, L) = \{ \gamma \in \mathcal{G} \mid i(\alpha, \alpha) \leq f(l), \forall \alpha \subset \gamma \text{ s.t. } l(\alpha) \geq L \}
$$

be the set of $\gamma \in \mathcal{G}$ such that all length $l$ subarcs have at most $f(l)$ self-intersections, whenever $l \geq L$.

**Theorem 1.2.** There is a $k_0 > 0$ so that if $f(l) \leq (k_0 l)^2$ for all $l$, then $\text{Im} \mathcal{G}(f, L)$ is nowhere dense for all $L \geq 0$.

**Remark 1.3.** Both the original result of Birman and Series, as well as Theorems \ref{thm:1.1} and \ref{thm:1.2} still hold when $S$ has a negatively curved metric with curvature bounded away from zero and infinity. However, the function $\mu$ and constant $k_0$ will depend on the metric. This is because the results below that use hyperbolic geometry can be proven in the general negative curvature case, but with different constants.

1.1. **Previous results for complete geodesics.** Complete geodesics on $S$ satisfy the following dichotomy. On the one hand, when $X$ is a complete, finite volume metric, then $\text{Im} \mathcal{G} = S$. Even when $X$ has geodesic boundary, $\text{Im} \mathcal{G}$ can have Hausdorff dimension greater than 1, and points of Lebesgue density. In the case of a pair of pants, this is proven in \cite{HJJL12}.

Moreover, when $X$ has finite volume, any “typical” geodesic in $\mathcal{G}$ has dense image in $S$ in the following sense. Let $T_1S$ be the unit tangent bundle of $S$. Then we can choose a vector $v \in T_1S$ at random with respect to Lebesgue measure, and consider the complete geodesic $\gamma$ with tangent vector $\gamma'(0) = v$. By mixing of the geodesic flow, $\text{Im} \gamma$ will be dense with probability 1. Note that in this case, mixing of the geodesic flow also implies that the self-intersection function of $\gamma$ will grow asymptotically like $\kappa l^2$, with probability 1.

On the other hand, the classical result of Birman and Series that we reference above shows that when $f(l) = K$ is constant, meaning that $\mathcal{G}(f)$ consists of complete geodesics with at most $K$ self-intersections, then $\text{Im} \mathcal{G}(f)$ has Hausdorff dimension 1 and is nowhere dense \cite{BS85}.

So we can think of self-intersection functions on a sliding scale. On one end of the scale, we have functions with $f(l) = O(1)$, and on the other side, we have functions with $f(l) = O(l^2)$. Theorems \ref{thm:1.1} and \ref{thm:1.2} allow us to interpolate between these two extremes. They indicate that the transition from Hausdorff dimension 1 to Hausdorff dimension 2, and from being nowhere dense to being dense, occur at the far end of the scale, among functions with $f(l) = O(l^2)$.

1.2. **Contrast with results for closed geodesics.** There is an analogous story for closed geodesics. Let $\mathcal{G}^c$ be the set of closed geodesics on $S$. When $X$ has finite volume (and no boundary), $\text{Im} \mathcal{G}^c$ is dense in $S$ by the closing lemma and mixing of the geodesic flow. On the other hand, the set of simple closed geodesics has nowhere dense image by \cite{BS85}.

Recently, Lenzhen and Souto have considered the sets of closed geodesics between these two extremes \cite{LS15}. In particular, for any function $f : \mathbb{R} \to \mathbb{R}$, they consider the set

$$
\{ \gamma \in \mathcal{G}^c \mid i(\gamma, \gamma) \leq f(l(\gamma)) \}
$$
They show that the image of this set is dense whenever \( \lim_{l \to \infty} f(l)/l = \infty \), and that its lift to \( T_1 S \) has Hausdorff dimension strictly smaller than 3 if \( f(l) = o(l) \).

The first part of their result can be combined with our theorems to get the following corollary.

**Corollary 1.4** (Consequence of [LS15] and Theorem 1.1). If \( \lim_{l \to \infty} f(l)/l = \infty \), then \( \text{Im} \mathcal{G}(f) \) is dense. In particular, there is some \( k_0 > 0 \) so that for any \( k < k_0 \), \( \text{Im} \mathcal{G}(k^2 l^2) \) is dense, but does not have full Hausdorff dimension.

**Proof.** Suppose \( \lim_{l \to \infty} f(l)/l = \infty \). This corollary follows from the fact that if \( \gamma \) is a closed geodesic with \( l(\gamma) = L \) and \( i(\gamma, \gamma) \leq f(L) \), then \( \gamma \in \mathcal{G}(f) \).

To see this, note that we can view any closed geodesic \( \gamma \) as a complete geodesic \( \gamma : \mathbb{R} \to S \) parameterized by arclength. If \( l(\gamma) = L \), then this parameterization has period \( L \). We can define \( \gamma_t \) as above to be the length \( l \) subarc of \( \gamma \) centered at \( \gamma(0) \), where \( \gamma_t \) is defined for any \( l \in \mathbb{R} \). Then\[
\frac{i(\gamma_t, \gamma)}{l} \lesssim \frac{f(L)}{L}.
\]

Because \( \lim_{l \to \infty} f(l)/l = \infty \), it is trivially true that \( \lim_{l \to \infty} \frac{f(L)}{L} \cdot \frac{l}{f(l)} = 0 \), since \( L \) is a constant. Therefore \( i(\gamma_t, \gamma_t) \lesssim f(l) \). In other words, \( \gamma \in \mathcal{G}(f) \).

Thus, \( \text{Im} \mathcal{G}(f) \) contains a dense set by [LS15], and so it is dense. \( \square \)

On the other hand, a closed geodesic \( \gamma \) with \( l(\gamma) = L \) and \( i(\gamma, \gamma) \leq f(L) \) does not necessarily belong to \( \mathcal{G}(f, L') \), if \( L' < L \). In fact, to determine whether \( \gamma \in \mathcal{G}(f, L') \), one would have to examine how the self-intersections of \( \gamma \) are distributed along its length. So [LS15] does not contradict Theorem 1.2.

It is interesting to note that the transition away from full Hausdorff dimension occurs when \( f(l) = (k_0 l)^2 \) for complete geodesics, while it occurs around \( f(l) = O(l) \) for closures of sets of closed geodesics.

1.3. **Reduction to closed surfaces.** It is enough to prove Theorems 1.1 and 1.2 for closed surfaces \( S \) (without boundary). In fact, if \( X \) has geodesic boundary, then we can double \( S \) across this boundary to get a \( S' \) with finite volume metric \( X' \).

There is a natural inclusion \( S \hookrightarrow S' \) along which \( X' \) pulls back to \( X \). Any \( \gamma \subset S \) gets sent to a geodesic on \( S' \) with the same self-intersection function. So if \( \text{Im} \mathcal{G}(f) \) has Hausdorff dimension \( h \), or is nowhere dense, on \( S' \), then the same is true on \( S \).

1.4. **Structure of the paper.** In Section 2 we show how to approximate any complete geodesic \( \gamma \) by a sequence of closed geodesics \( \gamma_n \) that also approximate the self-intersection function of \( \gamma \). In particular, given any geodesic arc \( \alpha \) of length \( L \), which can be thought of as a subarc of \( \gamma \), we show how to find a nearby closed geodesic whose length and self-intersection number are not much larger than those of \( \alpha \) (Lemma 2.2).

In Section 3 we apply Lemma 2.2 to construct covers for \( \mathcal{G}(f) \) and \( \mathcal{G}(f, L) \) that consist of regular neighborhoods of closed geodesics. In particular, for each function \( f \), we get a sequence of finite covers \( \{C_n\} \). In Lemma 3.1 we show that \( C_n \) covers \( \mathcal{G}(f, L) \) for all \( n \) large enough (depending on \( L \)). Moreover, we show that any infinite union of these covers is, in fact, a cover for \( \mathcal{G}(f) \) (Lemma 3.2).

In Section 4 we approximate the number of open sets in the cover \( C_n \), for each \( n \). The set \( C_n \) is a collection of regular neighborhoods of closed geodesics that lie...
in a certain set. So to approximate the size of $C_n$, we need to approximate the size of this set of closed geodesics. We do this in Lemma 4.1.

In Sections 5 and 6, we prove Theorems 1.2 and 1.1, respectively, given the above lemmas. We do this by getting upper bounds the Lebesgue and Haussdorff measures of each cover $C_n$, where the measure of a cover is defined to be the measure of the union of elements of that cover.

1.5. Notation. There are several points in this paper where we only need coarse estimates. We use the following notation. If two functions $A(x)$ and $B(x)$ satisfy $A(x) \leq cB(x)$ where $c$ is a constant depending only on some quantity $D$, then we write

$$A(x) \lesssim B(x)$$

and say that the constants depend only on $D$. We will also say that $A(x)$ is coarsely bounded by $B(x)$.

Furthermore, given two curves $\alpha$ and $\beta$, $i(\alpha, \beta)$ denotes the least number of self-intersections between all curves freely homotopic to $\alpha$ and $\beta$. On the other hand, $\# \alpha \cap \beta$ denotes the number of transverse intersections between the curves $\alpha$ and $\beta$ themselves. In particular, we use the notation $\# \alpha \cap \beta$ when $\alpha$ and $\beta$ are arcs rather than closed curves.

2. Lemmas about arcs

It is well-known that one can approximate any complete geodesic with a sequence of closed geodesics. For example, closed geodesics are dense in the space of geodesic currents, which also contain the set of complete geodesics [Bon88]. So given any complete geodesic $\gamma_\infty$, we can find a sequence $\{\gamma_i\} \subset \mathcal{G}$ so that $\lim \gamma_i = \gamma_\infty$. Note that this limit holds in the measure-theoretic sense of geodesic currents, but it must also hold as a Hausdorff limit of the geodesics themselves. See [Bon88] for more details.

Suppose $\gamma_\infty \in \mathcal{G}(f)$. Then not only do we want to approximate $\gamma_\infty$ by a sequence $\{\gamma_i\}$ of closed geodesics, we want the self-intersection numbers of the closed geodesics to eventually be coarsely bounded by the self-intersection function of $\gamma_\infty$. That is, if $l(\gamma_i) = l_i$, we want $i(\gamma_i, \gamma_i) \lesssim f(l_i)$, where the constant is independent of $i$.

We make this precise in terms of subarcs of complete geodesics. In particular, given a point $x \in \text{Im} \gamma_\infty$, we can take a nested sequence of subarcs $\alpha_l \subset \gamma_\infty$ centered at $x$ so that $\alpha_l$ has length $l$. Then the following lemma says we can find a closed geodesic close to $\alpha_l$ that does not have too much more length or to many more self-intersections.

Definition 2.1. We say a closed geodesic $\gamma$ $r$-fellow travels a geodesic arc $\alpha$ if there are some lifts $\hat{\gamma}$ and $\hat{\alpha}$ of $\gamma$ and $\alpha$, respectively, to the universal cover $\tilde{S}$ of $S$ so that $\hat{\alpha}$ lies in a $r$-neighborhood of $\hat{\gamma}$.

Let

$$\mathcal{G}_L(K) = \{ \gamma \in \mathcal{G}^c \mid l(\gamma) \leq L, i(\gamma, \gamma) \leq K \}$$

be the set of closed geodesics with length at most $L$ and with at most $K$ self-intersections.
Lemma 2.2. There is a constant $d$ depending only on the metric $X$ so that the following holds. Let $\alpha$ be a geodesic arc of length $L \geq d$ with $\# \alpha \cap \alpha = K$. Then there exists a closed geodesic

$$\gamma \in G^{2L}(K + dL)$$

that 1-fellow-travels $\alpha$.

This Lemma is a direct consequence of Claims 2.4 and 2.3 below. The proof of Lemma 2.2 given these claims is at the end of this section.

Claim 2.3. For any geodesic arc $\alpha$ with $l(\alpha) \geq 3$ there is a $\gamma \in G^c$ so that $\gamma$ 1-fellow travels $\alpha$ and $l(\gamma) \leq l(\alpha) + R$, where $R$ is a constant depending only on $X$.

We would like to thank Chris Leininger for suggesting the idea for this claim and its proof.

Proof. Suppose $\beta$ is a geodesic arc so that $\alpha$ and $\beta$ can be concatenated into a closed curve $\gamma'$. Then $\gamma'$ is a piecewise geodesic closed curve with corners at the endpoints of $\alpha$. Suppose the angle deficit at each corner is at most $\epsilon$ (Figure 1).

Let $\gamma$ be the geodesic representative of $\gamma'$. Then $\gamma$ must $D(\epsilon)$-fellow travel $\gamma'$, for a function $D(\epsilon)$ depending only on the maximal angle deficit $\epsilon$ with

$$\lim_{\epsilon \to 0} D(\epsilon) = 0$$

To see this, we will round the corners of $\gamma'$ to get a nearby, piecewise $C^2$ curve $\gamma_c$ that $\epsilon \sinh(1)$-fellow travels $\gamma$. The curve $\gamma_c$ will have geodesic curvature bounded by a function $g(\epsilon)$ at each point, where $\lim_{\epsilon \to 0} g(\epsilon) = 0$. So by [Lei06], we can conclude that $\gamma$ must $f(\epsilon)$-fellow travel $\gamma_c$, where $f(\epsilon)$ is a continuous function in $\epsilon$, with $f(0) = 0$.

First, if $l(\beta) \leq 2$, we need to modify $\gamma'$ slightly: Replace $\gamma'$ by the curve $\tilde{\gamma}''$ that is freely homotopic to it relative one of the endpoints of $\alpha$. There are lifts $\tilde{\gamma}'$ and $\tilde{\gamma}''$ to the universal cover $\tilde{S}$ of $S$ that form a geodesic triangle $\triangle abc$ where $c$ is the vertex opposite $\tilde{\gamma}''$ and has angle at least $\pi - \epsilon$ (Figure 2). Applying the hyperbolic law of sines, we see that the distance from $\tilde{\gamma}'$ to $\tilde{\gamma}''$ is at most $\epsilon \sinh(2)$.

The rest of the proof is essentially the same, whether we deal with $\gamma'$ or $\gamma''$. So assume for what follows that $l(\beta) > 2$, so we deal with $\gamma'$. Take a bi-infinite lift $\tilde{\gamma}'$ of $\gamma'$ to $\tilde{S}$.

The curve $\tilde{\gamma}'$ is a piecewise geodesic with angle deficit at most $\epsilon$ at its corners. We will now find a nearby piecewise $C^2$ curve $\tilde{\gamma}_c$, whose curvature is bounded above by $g(\epsilon)$ at each point, where $\lim_{\epsilon \to 0} g(\epsilon) = 0$.

For this, we use the upper half plane model for $\mathbb{H}^2$. Since $X$ is a hyperbolic metric, we can view $\tilde{S}$ as a subset of $\mathbb{H}^2$. Applying a hyperbolic isometry, we can
assume that $\tilde{\gamma}'$ has a geodesic segment from some point $iy$ to the point $i$, and that it then turns by an angle $\theta < \epsilon$ and has a length 1 geodesic segment from $i$ to some point $b$. Note that this is possible since we assume that $l(\alpha), l(\beta) \geq 2$. Then the segment from $i$ to $b$ lies on a circle with center $a \in \mathbb{R}$ and radius $r$, where

$$a = \frac{1}{\tan \theta}, \quad r = \frac{1}{\sin \theta}.$$ 

In particular, $b = a + re^{i(\pi - \theta - \Delta \theta)}$ where $\Delta \theta$ goes to zero as $\theta$ goes to zero (left side of Figure 3).

For each $\theta$, there is a unique $y = y(\theta)$ so that there is a Euclidean circle with center inside $H^2$ that is tangent to $\tilde{\gamma}'$ at both $iy$ and at $b$. Moreover, we can compute that as $\theta$ goes to zero, $y(\theta)$ approaches $\frac{1}{2e}$. So for all $\epsilon$ small enough, the hyperbolic distance between $iy$ and $i$ is smaller than 1.

We will replace the subarc from $iy$ to $b$ by a smooth arc with the same initial and final tangent vectors. As long as the distance from $iy$ to $i$ is at most 1, the fact that $l(\alpha), l(\beta) \geq 2$ means we can do this to both corners of $\gamma'$ at the same time to get the nearby $C^2$ curve $\gamma_c$.

For each $\theta$, there is a unique $y = y(\theta)$ so that there is a Euclidean circle with center inside $H^2$ that is tangent to $\tilde{\gamma}'$ at both $iy$ and at $b$. Moreover, we can compute that as $\theta$ goes to zero, $y(\theta)$ approaches $\frac{1}{2e}$. So for all $\epsilon$ small enough, the hyperbolic distance between $iy$ and $i$ is smaller than 1.

Suppose the circle has center at $A$ and Euclidean radius $\rho$. If it meets the real axis at angle $\phi$, then its hyperbolic curvature is $|\cos(\phi)|$ [GRS65, Lemma 3]. So we see that its curvature is $\frac{y}{\rho}$ at each point. By explicitly computing $y$ and $\rho$, one can show that the curvature goes to zero as $\theta$ (and $\epsilon$) go to zero.

We replace each corner of $\tilde{\gamma}'$ in this way. This gives us our piecewise $C^2$ curve $\tilde{\gamma}_c$ whose curvature at each point goes to zero uniformly as $\epsilon$ goes to zero. Note that $\tilde{\gamma}_c$ projects down to a piecewise $C^2$ closed curve $\gamma_c$ in $S$. By [Lei06], the distance from $\tilde{\gamma}_c$ to $\tilde{\gamma}$ is at most $f(\epsilon)$, where $f(\epsilon)$ is a continuous function with $f(0) = 0$.

By construction, the distance from $\tilde{\gamma}_c$ to $\tilde{\gamma}'$ is at most $\epsilon \sinh(1)$, as the circle segment is contained in the geodesic triangle with vertices at $iy, i$ and $b$. Thus, for
all $\epsilon$ small enough, the distance from $\tilde{\gamma}'$ to $\tilde{\gamma}$ is at most
\[ d(\tilde{\gamma}, \tilde{\gamma}') \leq \epsilon \sinh(\rho) + f(\epsilon) \]
In particular, this distance approaches 0 as $\epsilon$ goes to 0. Note that if $l(\beta) < 2$, then the same argument implies that $d(\tilde{\gamma}, \tilde{\gamma}') \leq \epsilon(\sinh(\rho) + \sinh(2)) + f(\epsilon)$. This quantity also goes to zero with $\epsilon$. So in either case, there is a function $D(\epsilon)$ so that $\gamma$ must $D(\epsilon)$ fellow travel $\gamma'$, with $\lim_{\epsilon \to 0} D(\epsilon) = 0$.

We now show the following: For any $\epsilon > 0$ there is an $R > 0$ so that for any geodesic arc $\alpha$ there exists a geodesic arc $\beta$ so that $l(\beta) \leq R$ and we can form a closed curve $\gamma' = \alpha \circ \beta$ with angle deficit at most $\epsilon$ at each corner.

Take a unit speed parameterization $\alpha : [0, L] \to S$. Let $v = \alpha'(L)$ be its tangent vector in the unit tangent bundle $T_1(S)$ of $S$. Let $f_t$ denote geodesic flow on $T_1(S)$ for time $t$, and let $r_\theta$ denote rotation by angle $\theta \in [-\pi, \pi]$.

We get coordinates in a small neighborhood about $v$ by assigning each vector $z$ the triple $(\theta, t, \phi)$, with $\theta, t, \phi$ the coordinates $(\theta, t, \phi)$ near $v$.

Let $N(\alpha, \epsilon)$ be the set of all vectors with coordinates $(\theta, t, \phi)$ so that $|\theta|, |\phi| < \frac{\epsilon}{2}$ and $0 < t < \frac{1}{2} inj(X)$, where $inj(X)$ denotes the injectivity radius of $X$. (Figure 4).

![Figure 4](image)

**Figure 4.** The set $N(v, \epsilon)$ contains all vectors $z$ with $|\theta|, |\phi| < \frac{\epsilon}{2}$ and $t < \frac{1}{2} inj(X)$.

Set $w = \alpha'(0) \in T_1(S)$. Fix any $\epsilon > 0$. Since $N(v, \epsilon)$ is a set of positive Lebesgue measure, the mixing of the geodesic flow implies that there is some $z \in N(v, \epsilon)$ and $T > 0$ so that
\[ w \in N(f_T(z), \epsilon) \]
(as in Figure 5).

Then if $z$ has coordinates $(\theta, t, \phi)$ near $v$ and $w$ has coordinates $(\theta', t', \phi')$ near $f_T(z)$, we can let $\eta$ be the piecewise geodesic arc whose lift to $T_1(S)$ is given by the parameterization
\[ \eta'(t) = \begin{cases} 
  f_s \cdot r_\theta(v) & \text{if } 0 \leq s < t \\
  f_{s-t} \cdot r_\theta(f_T(z)) & \text{if } t \leq s < T + t \\
  f_{s-t-T} \cdot r_\theta(f_T(z)) & \text{if } T + t \leq s \leq T + t + t' 
\end{cases} \]
(Figure 5). That is, we start at $v$ and flow in the direction $r_\theta(v)$ for time $t$. Then we flow in the direction of $z$ for time $T$, and lastly, in the direction of $r_\theta(f_T(z))$ for time $t'$. Note that $r_\theta(v)$ and $z$ lie over the same point in $S$, as do $f_T(z)$ and $r_\theta(f_T(z))$. Thus, this defines a closed curve in $S$.

Let $\beta$ be the arc freely homotopic to $\eta$ relative its endpoints. Then $|\theta|, |\theta'|, |\phi|, |\phi'| < \frac{\epsilon}{2}$ implies that the angle deficit at each of the two points where $\alpha$ meets $\beta$ is at most $\epsilon$. By the triangle inequality, $l(\beta) \leq T + 2$.

In fact, there is a continuous function $T(\cdot, \cdot, \cdot) : T_1(S) \times T_1(S) \times \mathbb{R}^+ \to \mathbb{R}^+$ so that $T \leq T(v, w, \epsilon)$. Moreover, because $T_1(S)$ is compact, there is a continuous function

\[ T(\cdot, \cdot, \cdot) : T_1(S) \times T_1(S) \times \mathbb{R}^+ \to \mathbb{R}^+ \]

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so that $T \leq T(v, w, \epsilon)$. Moreover, because $T_1(S)$ is compact, there is a continuous function.
function $T : \mathbb{R}^+ \to \mathbb{R}^+$ so that
\[ T(v, w, \epsilon) \leq T(\epsilon) \]
This is the only place where we use that $S$ is compact. In particular, for any $\alpha$ and $\epsilon$, there is an arc $\beta$ of length at most $R(\epsilon) = T(\epsilon) + 2$ so that we can concatenate $\alpha$ and $\beta$ into a closed curve with angle deficit at most $\epsilon$ at its corners.

Choose $\epsilon$ small enough so that $D(x) \leq 1$, where $D(x)$ is the function we defined in the first part of this proof. Let $R = R(\epsilon)$. Then for every arc $\alpha$, there is a closed geodesic $\gamma$ so that
\[ l(\gamma) \leq l(\alpha) + R \]
and $\gamma$ 1-fellow travels $\alpha$. \hfill \Box

The previous claim says we can approximate any arc $\alpha$ by a closed geodesic $\gamma$ of roughly the same length. The next claim allows us to estimate the self-intersection number of $\gamma$.

**Claim 2.4.** If $\alpha$ and $\beta$ are two geodesic arcs with $l(\alpha) \leq L_\alpha$ and $l(\beta) \leq L_\beta$, then
\[ \# \alpha \cap \beta \leq \kappa L_\alpha L_\beta \]
where we require $L_\alpha, L_\beta \geq 1$, and $\kappa$ is a constant that depends only on $X$.

**Proof.** The proof is almost exactly the same as the proof of [Bas13, Theorem 1.1], where Basmajian proves this result in the case where $\alpha$ and $\beta$ are the same (non-simple) closed geodesic. We recreate it here for completeness.

Take a pants decomposition $\Pi$ of $S$. Further cut each pair of pants into two congruent right-angled hexagons. So each hexagon has boundary edges that lie on curves in $\Pi$, and seam edges that join curves in $\Pi$.

The hexagon decomposition cuts the arc $\alpha$ into segments, which are maximal subarcs that lie in a single hexagon, and the same is true for $\beta$. Note that if a hexagon $h$ has $n$ $\alpha$-segments and $m$ $\beta$-segments, then the total number of intersections between $\alpha$ and $\beta$ in $h$ is at most $nm$. This is because each hexagon is simply connected and convex, so any pair of segments intersects at most once. Therefore, if $\alpha$ has $N_\alpha$ total segments, and if $\beta$ has $N_\beta$ total segments, then
\[ \# \alpha \cap \beta \leq N_\alpha N_\beta \]
So we just need to bound $N_\alpha$ and $N_\beta$ in terms of $l(\alpha)$ and $l(\beta)$, respectively. We will say a full segment is any segment of $\alpha$ or $\beta$ that does not contain an endpoint of that arc. Take three consecutive full segments $\sigma_1, \sigma_2, \sigma_3$ of $\alpha$. Then by the argument in [Bas13, Step 2, Section 3], we have that
\[ l(\sigma_1) + l(\sigma_2) + l(\sigma_3) \geq C \]
where $C$ depends only on the metric $X$. The total length of all full segments of $\alpha$ is at most $l(\alpha)$. Thus, its number of full segments is at most $\frac{3l(\alpha)}{C}$. The arc $\alpha$ only has two segments that are not full: they are the ones that contain its endpoints. So we have

$$N_\alpha \leq 2 + \frac{3l(\alpha)}{C}$$

and likewise,

$$N_\beta \leq 2 + \frac{3l(\beta)}{C}$$

Therefore,

$$\#\alpha \cap \beta \leq \left(2 + \frac{3l(\alpha)}{C}\right) \left(2 + \frac{3l(\beta)}{C}\right)$$

If $l(\alpha) \leq L_\alpha$ and $l(\beta) \leq L_\beta$, and if we assume $L_\alpha, L_\beta \geq 1$, we have that

$$\#\alpha \cap \beta \leq (2 + \frac{3}{C})^2 L_\alpha L_\beta$$

Thus, the claim holds for $\kappa = (2 + \frac{3}{C})^2$, which is a constant depending only on $X$. □

**Proof of Lemma 2.2.** Assume that $l(\alpha) > 3$. Then Claim 2.3 says there is a closed geodesic $\gamma$ that $1$-fellow travels $\alpha$. By construction, $\gamma$ is freely homotopic to a concatenation $\alpha \circ \beta$, where $\beta$ is a geodesic arc of length at most $R$, for $R$ depending only on the metric $X$. So Claim 2.4 allows us to estimate the self-intersection number of $\gamma$.

Assuming $R \geq 1$, Claim 2.4 implies that

$$\#\alpha \cap \beta \leq \kappa R l(\alpha)$$

and

$$\#\beta \cap \beta \leq \kappa R^2$$

Thus, if we assume $l(\alpha) < L$,

$$i(\gamma, \gamma) \leq i(\alpha, \alpha) + i(\alpha, \beta) + i(\beta, \beta) \leq i(\alpha, \alpha) + \kappa R l(\alpha) + \kappa R^2 \leq K + \kappa R(L + R)$$

So if $L \geq R$, then $i(\gamma, \gamma) \leq K + 2\kappa RL$. Setting $d = 2\kappa R$, we get Lemma 2.2. □

3. COVERING BY NEIGHBORHOODS OF CLOSED CURVES

The proofs of Theorems 1.1 and 1.2 come from covering our sets of complete geodesics with neighborhoods of closed geodesics. For any function $f(l)$, we give an infinite collection of finite open covers $\{C_n(f) = C_n\}$. Then for any $L$, $C_n$ is a cover of $\text{Im} G(f, L)$ for all $n$ large enough. Moreover, the union of any infinite subsequence over these covers gives a cover of $\text{Im} G(f)$.

Specifically, for each $\gamma \in G^c$, let $N_{\epsilon}(\gamma)$ be an $\epsilon$-neighborhood of $\text{Im} \gamma$. Then if $H \subset G^c$ is any collection of closed geodesics, we let

$$N_{\epsilon}(H) = \bigcup_{\gamma \in H} N_{\epsilon}(\gamma)$$

We define the cover $C_n$ by

$$C_n = N_{\epsilon(n)}(G^c_n(c_X \cdot f(n)))$$
where $\epsilon(n) = 2e^{-n/4}$, and $c_X = 2 + d/2$, for the constant $d$ defined in Lemma 2.2. That is, $C_n$ is a finite collection of $\epsilon(n)$-neighborhoods of closed geodesics of length at most $n$, with at most $c_X f(n)$ self-intersections.

3.1. Finite covers. Recall that

$$\mathcal{G}(f, L) = \{ \gamma \in \mathcal{G} \mid \# \gamma|_{[a,a+t]} \cap \gamma|_{[a,a+t]} \leq f(l), \forall l \geq L \}$$

In other words, this is the set of complete geodesics $\gamma$ so that all length $l$ subarcs have self-intersection number at most $f(l)$, for all $l \geq L$. Because we impose this regularity on the self-intersection function of geodesics in $\text{Im} \mathcal{G}(f, L)$, we can show that each $C_n$ is a cover of $\mathcal{G}(f, L)$, as long as $n$ is large enough.

Observe that if $f(x) \leq g(x)$, then $\mathcal{G}(f) \subset \mathcal{G}(g)$, and in fact, $\mathcal{G}(f, L) \subset \mathcal{G}(g, L)$ for all $L$. So in this section, we assume without loss of generality that $f(l) \geq l$ for all $l$.

**Lemma 3.1.** Suppose $f(l) \geq l$ for all $l$. Then for each $L$, there is an $N > 0$ so that for all $n \geq N$, $C_n$ is a cover of $\text{Im} \mathcal{G}(f, L)$.

**Proof.** Let $x \in \text{Im} \mathcal{G}(f, L)$. Then there is a $\gamma \in \mathcal{G}(f, L)$ parameterized by $\gamma : \mathbb{R} \to S$ so that $x = \gamma(t_x)$ for some time $t_x \in \mathbb{R}$. Choose $l \geq L$. Let $\alpha_{x,l} = \gamma|_{[t_x-l/2,t_x+l/2]}$ be the length $l$ subarc of $\gamma$ centered at $x$. Then because $\gamma \in \mathcal{G}(f, L)$,

$$\# \alpha_{x,l} \cap \alpha_{x,l} \leq f(l)$$

By Lemma 2.2, as long as $l \geq d$, there is a closed geodesic $\delta \in \mathcal{G}_{\mathbb{R}}^0(f(l) + dl)$ that 1-fellow travels $\alpha_{x,l}$, where $d$ is a constant depending only on $x$. Since $f(l) \geq l$, we have, in fact, that $\delta \in \mathcal{G}_{\mathbb{R}}^0(c_X f(l))$, where $c_X = 1 + d$.

Lift $\delta$ and $\alpha_{x,l}$ to curves $\tilde{\delta}$ and $\tilde{\alpha}_{x,l}$ in the universal cover such that the endpoints of $\tilde{\alpha}_{x,l}$ are at most distance 1 away from $\tilde{\delta}$. The midpoint of $\tilde{\alpha}_{x,l}$ is a lift $\tilde{x}$ of $x$. Thus,

$$d(\tilde{x}, \tilde{\delta}) < 2e^{-l/2}$$

To see this, drop perpendiculars from $\tilde{x}$ and from an endpoint of $\tilde{\alpha}_{x,l}$ down to $\tilde{\delta}$. This forms a quadrilateral with 3 right angles (called a Lambert quadrilateral.) By properties of Lambert quadrilaterals, $d(\tilde{x}, \tilde{\delta}) \leq \sinh(1)/\cosh(\frac{l}{2})$. Since $\sinh(1) < 2$ and $\cosh(\frac{l}{2}) > e^{l/2}$, we have our inequality (Figure 6).

![Figure 6](image.png)

Let $N = \max\{2L, d\}$. Then for each $n \geq N$, and for each $x \in \text{Im} \mathcal{G}(f, L)$, there is a $\delta \in \mathcal{G}_{\mathbb{R}}^0(c_X f(n))$ so that $x \in N_{\epsilon(n)}(\delta)$ for $\epsilon(n) = 2e^{-n/4}$. In other words, for all $n \geq N$, $C_n$ is a cover of $\text{Im} \mathcal{G}(f, L)$.

$\square$
3.2. Infinite covers. To cover \textbf{Im} \( G(f) \), we need to take the union of infinitely many covers in the sequence \( \{C_n\} \). Once again, observe that if \( f(l) \leq g(l) \) for two functions \( f \) and \( g \), then \( \mathcal{G}(f) \subset \mathcal{G}(g) \). So we can assume without loss of generality that \( f(l) \geq l \) for all \( l \), and that \( f(l) \) is increasing in \( l \).

**Lemma 3.2.** Suppose \( f(l) \) is an increasing function so that \( f(l) \geq l \). Then for each \( N > 0 \), \( \bigcup_{n=N}^\infty C_n \) is a cover of \textbf{Im} \( \mathcal{G}(f) \).

**Proof.** Let \( x \in \textbf{Im} \mathcal{G}(f) \). Then there is a \( \gamma \in \mathcal{G}(f) \) parameterized by \( \gamma : \mathbb{R} \to S \) so that \( x = \gamma(t_x) \) for some time \( t_x \in \mathbb{R} \).

Recall that \( \gamma_l = \gamma|_{[-\frac{1}{2}, \frac{1}{2}]} \) is the length \( l \) subarc of \( \gamma \) centered at \( \gamma(0) \). Because \( \gamma \in \mathcal{G}(f) \), there is some length \( l_0 \) depending on \( \gamma \) so that for all \( l \geq l_0 \),

\[
\#\gamma_l \cap \gamma_l \leq 2f(l)
\]

For each \( l \), let \( \alpha_{x,l} = \gamma|_{[t_x - \frac{d}{2}, t_x + \frac{d}{2}]} \) be the length \( l \) subarc of \( \gamma \) centered at \( x \). Then \( \alpha_{x,l} \subset \gamma_{t_x + l} \) for each \( l \).

Let \( l \geq \max\{l_0, t_x, d\} \), where \( d \) is the constant from Lemma 2.2. Then \( l + t_x \leq 2l \). Since we assume that \( f(l) \) is increasing, this means that \( f(l + t_x) \leq f(2l) \). Thus,

\[
\#\alpha_{x,l} \cap \alpha_{x,l} \leq 2f(2l)
\]

By the same argument as in the proof of Lemma 3.1, there is a closed geodesic \( \delta \in \mathcal{G}^c_{2l}(2f(2l) + dl) \) so that

\[
d(x, \delta) \leq 2e^{-l/2}
\]

Since we assume \( f(l) \geq l \), we have, in fact, that \( \delta \in \mathcal{G}^c_{2l}(c_X f(2l)) \), where \( c_X = 2 + d/2 \).

Then for each \( x \), and for each \( n \geq \frac{1}{2} \max\{l_0, t_x, d\} \), there is a \( \delta \in \mathcal{G}^c_n(c_X f(n)) \) so that \( x \in N_{c_X f(n)}(\mathcal{G}_n) \) for \( c_X = 2e^{-n/4} \). In other words, \( \bigcup_{n \geq N} C_n \) is a cover of \textbf{Im} \( \mathcal{G}(f) \) for each \( N \).

\[\blacksquare\]

4. Counting the approximating closed curves

We will define the Lebesgue (or Hausdorff) measure of any cover \( C_n \) to be the measure of the union of the open sets in \( C_n \). We will eventually wish to show that the measures of these covers go to zero as \( n \) goes to infinity. Recall that we defined each cover \( C_n \) as the collection of open neighborhoods about geodesics in \( \mathcal{G}^c_n(c_X f(n)) \). So to show that these measure go to zero, we need to bound the number of closed geodesics in these sets.

**Lemma 4.1.** If \( f(n) \leq (kn)^2 \), then

\[
\#\mathcal{G}_n^c(c_X f(n)) = o\left(\frac{1}{n} e^{\mu(k)n}\right)
\]

where \( \lim_{k \to \infty} \mu(k) = 0 \), and \( c_X \) is the constant depending only on \( X \) defined in Section 3.

In fact, we will show that \( \mu(k) = a_X k \log(a_X (1 + \frac{1}{k})) \), where \( a_X \) depends only on \( X \). First we will bound \( \#\mathcal{G}_n^c(K) \) for any \( L \) and \( K \), and then we will set \( L = n \) and consider the case where \( K(n) \leq (kn)^2 \).

**Claim 4.2.** Let \( S \) be a closed, genus \( g \) surface. For any \( L \) and \( K \), we have

\[
\#\mathcal{G}_n^c(K) \leq p(L) \left( a_X \frac{L}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}}
\]
where $p(L)$ is a polynomial in $L$, and $a_X$, as well as the coefficients of $p(L)$, depend only on the metric $X$.

In fact, one can look carefully at the argument in [Mir16, Lemma 5.6] to see that $p(L)$ can be replaced by a polynomial in $K$ times $L^{6g-6}$. However, the formula in Lemma 4.1 is easier to prove, and suffices for Lemma 4.1.

**Proof.** Let $\operatorname{Mod}_S$ denote the mapping class group of $S$. Then $\operatorname{Mod}_S$ acts on $G^c$, preserving self-intersection number. For each $\gamma \in G^c$ let $\operatorname{Mod}_S \cdot \gamma$ denote its orbit. Let $\mathcal{O}(\gamma, K)$ be the set of orbits of curves with at most $K$ self-intersections:

$$\mathcal{O}(\gamma, K) = \{ \operatorname{Mod}_S \cdot \gamma \mid i(\gamma, \gamma) \leq K \}$$

If $\gamma$ has $K$ self-intersections, then the shortest curve in $\operatorname{Mod}_S \cdot \gamma$ has length between $c_1 \sqrt{K}$ and $c_2 K$ for some constants $c_1$ and $c_2$. In fact, there exist such constants, depending only on $X$, for which these bounds are tight [AGPS16, Bas13, Gas16]. Thus, if $L$ is small enough, then not all $\operatorname{Mod}_S$ orbits contain curves of length at most $L$. So we let

$$\mathcal{O}(L, K) = \{ \operatorname{Mod}_S \cdot \gamma \mid \operatorname{Mod}_S \cdot \gamma \cap G^c_\ell(K) \neq \emptyset \}$$

be those orbits that contain curves of length at most $L$.

In [Sap16b], we show that

$$\# \mathcal{O}(L, K) \leq \left( a_X \frac{L}{\sqrt{K}} + a_X \sqrt{K} \right)^{a_X \sqrt{K}}$$

for a constant $a_X$ depending only on $X$.

Since we have a bound on the number of orbits, we just need a bound on the number of curves in each orbit. For each $\gamma$, let

$$s(L, \gamma) = \{ \gamma' \in \operatorname{Mod}_S \cdot \gamma \mid l(\gamma') \leq L \}$$

In [Mir10], Mirzakhani shows that $s(L, \gamma)$ grows asymptotically like $a_{\gamma, X} L^{6g-6}$, where the constant $a_{\gamma, X}$ depends on the $\operatorname{Mod}_g$ orbit of $\gamma$, and on $X$. The dependence of $a_{\gamma, X}$ on $\gamma$ is difficult to determine. As mentioned above, a careful analysis of [Mir16, Lemma 5.6] should imply that $s(L, \gamma) \leq p(K) L^{6g-6}$, where $p(K)$ is a polynomial in $K$ whose coefficients depend only on the metric $X$. However, there is a faster way to see that

$$\# s(L, \gamma) \ll L^{30g-12}$$

where the constant depends only on $X$.

First, suppose $\gamma$ is a filling curve on a closed, genus $g$ surface. Then we the proof of [Sap16a, Lemma 2.2] implies that

$$\# s(L, \gamma) \ll L^{6g-6}$$

where the constant depends only on $X$.

Now suppose $\gamma$ only fills a proper subsurface $T \subset S$. Then by [Sap16a, Proposition 2.7], $l(\partial T) \leq 2l(\gamma)$. So for any $\gamma \in \operatorname{Mod}_S$ with $l(g \cdot \gamma) \leq L$ we have $l(g \cdot \partial T) \leq 2L$. The number of simple closed curves of length at most $2L$ on $S$ is at most $b_X (2L)^{6g-6}$, where $b_X$ is a constant depending only on $X$.

So we can now fix a subsurface $T$ of $S$, and count all curves in $\operatorname{Mod}_S \cdot \gamma$ that fill $T$:

$$s(L, \gamma, T) = \{ \gamma' \in \operatorname{Mod}_S \cdot \gamma \mid l(\gamma') \leq L, \gamma' \text{ fills } T \}$$
Double $T$ across its boundary. This gives a new surface $Q = T \cup \bar{T}$, where $\bar{T}$ is the complement of $T$ in $Q$. The metric on $Q$ is obtained by doubling the metric on $T$. Moreover, if $\gamma' \in \text{Mod}_g \cdot \gamma$ lies in $T$, then it has a mirror image $\gamma'$ that lies in $\bar{T}$. Finally, for each component $\alpha$ of $\partial T$, there is a curve $\beta$ that intersects $\alpha$ minimally with $l(\beta) \leq l(\gamma')$, where the constant depends only on $X$. Let $\eta$ be the union of all such curves $\beta$ together with $\partial T$. Then consider the curve $\delta = \gamma' \cup \gamma' \cup \eta$.

By construction, $\delta$ fills the closed surface $Q$. If $S$ had genus $g$, then the genus of $Q$ is at most $4g$. (Really, the genus of $Q$ is at most $4g - 1$, but this slight improvement in the upper bound leads to messier formulae, which are still not tight.)

So we can count curves in $s(L, \gamma, T)$ on $S$ by counting curves in $s(L, \delta)$ on $Q$, instead. In fact, since $\delta$ fills a closed surface, we get

$$s(L, \gamma, T) \leq s(L, \delta) \leq L^{24g - 6}$$

Thus,

$$s(L, \gamma) \leq L^{6g - 6} \cdot L^{24g - 6} = L^{30g - 12}$$

for constants that depends only on the metric $X$.

Combined with the orbit counting result, we get that

$$\#G^c_L(K) \leq p(L) \left( a_L \frac{L}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}}$$

where $p(L)$ is a polynomial in $L$ of degree $30g - 12$, whose constants depend only on $X$, and $a_X$ depends only on $X$. \hfill \square

Proof of Lemma 4.7. Now set $L = n$ and suppose $K \leq (kn)^2$ for some $k$. Then we are ready to show that

$$\#G^c_n(\epsilon_X f(n)) = o(\frac{1}{n} e^{\mu(k) n})$$

where

$$\mu(k) = \frac{1}{2} a_X k \ln \left( \frac{a_X}{k} + a_X \right)$$

Because $p(n) = o(\frac{1}{n} e^{\frac{1}{2} \mu(k)n})$ for any polynomial $p(n)$ and any positive $\mu(k)$, we only need to find a function $\mu(k)$ so that

$$\left( a_X n \sqrt{K} + a_X \right)^{a_X \sqrt{K}} = o(e^{\frac{1}{2} \mu(k)n})$$

whenever $K \leq (kn)^2$.

For any fixed $n \geq 1$, we have that $\left( a_X n \sqrt{K} + a_X \right)^{a_X \sqrt{K}}$ is an increasing function in $K$, since we may assume that $a_X \geq e$. Since we assume that $K \leq (kn)^2$, this means

$$\left( a_X \frac{n}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}} \leq \left( a_X \frac{1}{k} + a_X \right)^{a_X \cdot kn} = e^{n - a_X k \ln \left( \frac{a_X}{k} + a_X \right)}$$

In other words, if $K = K(n) \leq (kn)^2$, then

$$\left( a_X \frac{n}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}} = o(e^{1/2 \mu(k)n})$$
for $\mu(k) = 4a_Xk \ln(\frac{2n}{k} + a_X)$. Lastly, note that $\lim_{k \to 0} \mu(k) = 0$. 

\section{Nowhere density}

A set $U \subset S$ is nowhere dense if its closure has an empty interior. In particular, $U$ is nowhere dense if, for any open ball $B$, $B \setminus U$ contains a non-empty open set.

We will show this is the case for $\text{Im} \mathcal{G}(f, L)$. In particular, Lemma 5.4 gives us a family $\{C_n\}$ of covers of $\text{Im} \mathcal{G}(f, L)$, where each $C_n$ is a finite collection of regular neighborhoods about closed geodesics. Below we show that these covers have arbitrarily small Lebesgue measure. (The Lebesgue measure of $\text{Im} \mathcal{G}(f, L)$ is nowhere dense.

**Proof of Theorem 1.2.** By Lemma 5.4, there is an $N$ depending only on $L$ so that for all $n \geq N$, $C_n$ is a cover of $\text{Im} \mathcal{G}(f, L)$. We wish to estimate the Lebesgue measure of $C_n$. Recall that $C_n$ is the set of $\epsilon(n)$-neighborhoods of the closed geodesics in $\mathcal{G}_n^0(\gamma, x)$, where $\epsilon(n) = 2e^{-n/4}$.

Let $\lambda(A)$ denote the Lebesgue measure of any subset $A \subset S$. If $\gamma \in \mathcal{G}_n^0(\gamma, x)$, then $l(\gamma) \leq n$. So for all $\epsilon(n)$ small enough, the measure of $N_{\epsilon(n)}(\gamma)$ is bounded above by

$$\lambda(N_{\epsilon(n)}(\gamma)) \leq 5ne^{-n/4}$$

By Lemma 4.1 if $f(n) \leq (kn)^2$, then $\#\mathcal{G}_n^0(\gamma, x) = o\left(\left(\frac{1}{n}e^{\mu(k)n}\right)\right)$. So

$$\lambda(C_n) = o\left(e^{\mu(k) - \frac{1}{4}n}\right)$$

We have that $\lim_{k \to 0} \mu(k) = 0$. So there is some $k_0$ so that for all $k < k_0$, $\mu(k) < \frac{1}{4}$. Then for all $k \leq k_0$,

$$\lim_{n \to \infty} \lambda(C_n) = 0$$

Suppose $k \leq k_0$. Choose any open ball $B \subset S$. Choose $n$ so that $\lambda(C_n) < \frac{1}{4}\lambda(B)$. Then $B$ is crossed by finitely many elements of $C_n$. The elements of $C_n$ are regular neighborhoods of closed geodesics, so our choice of $n$ guarantees that $B \setminus C_n$ has non-empty interior. But $C_n$ is an open cover of $\text{Im} \mathcal{G}(f, L)$. So $B \setminus \text{Im} \mathcal{G}(f, L)$ has non-empty interior, as well. Therefore, $\text{Im} \mathcal{G}(f, L)$ is nowhere dense for all $L$ and all functions $f$ with $f(l) \leq (kl)^2$.

\section{Hausdorff dimension}

**Proof of Theorem 1.1.** The Hausdorff dimension of a set is defined as follows. Given a subset $G$ of a metric space $X$, let $\mathcal{C} = \{B(x_i, r_i)\}$ be a countable cover of $G$ by metric balls centered at $x_i$ and of radius $r_i$, for each $i$. We define the $h$-dimensional Hausdorff measure of $\mathcal{C}$ to be $\nu_h(\mathcal{C}) = \sum r_i^h$. The $h$-dimensional Hausdorff measure of a set $G$ is defined as

$$\nu_h(G) = \inf_{\mathcal{C}} \nu_h(\mathcal{C})$$

where the infimum is taken over all such covers of $G$. Then the Hausdorff dimension of $G$ is defined to be

$$\dim_H(G) = \inf \{h \mid \nu_h(G) = 0\}$$
By Lemma 3.2 infinite unions of the covers $C_1, \ldots, C_n, \ldots$ cover $\text{Im} \mathcal{G}(f)$. These covers are by regular neighborhoods of closed geodesics, but we can use them to build covers of $\text{Im} \mathcal{G}(f)$ by metric balls. In fact, for each $n$, we can build new cover $C_n^H$, which is a collection of balls whose union contains the union of open sets in $C_n$. Note that for all $\epsilon(n)$ small enough, we can cover the $\epsilon(n)$-regular neighborhood of any $\gamma \in \mathcal{G}_n^c(c_X f(n))$ by $2 e^{\epsilon(n)}$ balls of radius $2 \epsilon(n)$. So let $C_n^H$ be the union of all these balls for each open set in $C_n$. We will use the collection $\{C_n^H\}$ of these covers to bound the Hausdorff dimension of $\text{Im} \mathcal{G}(f)$.

Then Lemmas 3.2 and 4.1 allow us to estimate the Hausdorff $h$-volume of $\text{Im} \mathcal{G}(f)$ by estimating the volume of $C_n^H$. (The volume of a cover is defined to be the volume of the union of all elements of the cover.)

Let $C_n^H$ be the collection of metric balls defined above. First, we find a condition on $h$ so that

$$\lim_{n \to \infty} \nu_h(C_n^H) = 0$$

Each ball in $C_n^H$ has radius $2 \epsilon(n) = 4 e^{-\frac{n}{h}}$. Each closed geodesic $\gamma \in \mathcal{G}_n^c(c_X f(n))$ has length $n$, so it contributes $2 e^{\epsilon(n)} = n e^{-\frac{n}{h}}$ balls to the cover. So the total Hausdorff $h$-volume of $C_n^H$ is bounded above by

$$\nu_h(C_n^H) = \sum_{\mathcal{G}_n^c(c_X f(n))} n e^{-\frac{n}{h}} (4 e^{-\frac{n}{h}})^h$$

If $f(n) \leq (kn)^2$, then by Lemma 4.1 the number of these closed geodesics in $\mathcal{G}_n^c(c_X f(n))$ grows like $o \left( \frac{4^h}{n} e^{\mu(k)n} \right)$, where $\lim_{n \to 0} \mu(k) = 0$. Since $4^h$ is a constant for each $h$,

$$\nu_h(C_n^H) = o \left( \frac{4^h}{n} e^{\mu(k)n} \right)$$

In particular, $\lim_{n \to \infty} \nu_h(C_n^H) = 0$ whenever $h > 4 \mu(k) + 1$.

Suppose $h > 4 \mu(k) + 1$. Then there is a subsequence $\{n_i\}$ so that $\nu_h(C_{n_i}^H) \leq 2^{-i}$ for each $i$. By Lemma 3.2 any infinite subsequence of $\{C_n^H\}$ covers $\text{Im} \mathcal{G}(f)$. In particular, for any $N \geq 0$, $\bigcup_{n=N}^\infty C_n^H$ covers $\text{Im} \mathcal{G}(f)$. Thus, whenever $h > 4 \mu(k) + 1$,

$$\nu_h(\text{Im} \mathcal{G}(f)) \leq 2^{-N}$$

for any $N$. In other words, the Hausdorff dimension of $\text{Im} \mathcal{G}(f)$ is at most $4 \mu(k) + 1$.

Furthermore, suppose $f(l) = o(l^2)$. Then $\mathcal{G}(f) \subset \cap_{k=1}^\infty \mathcal{G}(f_k)$, for $f_k = kl^2$. So in this case, the Hausdorff dimension of $\text{Im} \mathcal{G}(f)$ is 1.

□



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