PROPER HOLOMORPHIC DISKS IN THE COMPLEMENT OF VARIETIES IN $\mathbb{C}^2$

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Abstract. We prove that for any complete pluripolar set $X \subset \mathbb{C}^2$ (in particular for any analytic subset) there exists a proper holomorphic embedding $\varphi: \Delta \hookrightarrow \mathbb{C}^2$ of the open unit disk $\Delta \subset \mathbb{C}$ such that $\varphi(\Delta) \cap X = \emptyset$. It follows that the same holds true in $\mathbb{C}^n$ for any $n > 1$.

1. Introduction

In [4] the authors proved that there are proper holomorphic disks in $\mathbb{C}^2$ that avoid the set $\{zw = 0\}$. Furthermore they said that it would be interesting to know whether there could be such disks avoiding any finite set of complex lines. This question was solved in [3] where it was proved that in any Stein manifold there are proper holomorphic disks in the complement of any closed complete pluripolar set.

We show that there exist properly embedded disks satisfying this more general property:

Theorem 1. Let $X$ be any closed complete pluripolar subset of $\mathbb{C}^2$ and let $\Delta$ denote the unit disk in $\mathbb{C}$. Then there exists a proper holomorphic embedding $\varphi: \Delta \hookrightarrow \mathbb{C}^2$ such that $\varphi(\Delta) \cap X = \emptyset$.

Since any analytic subset of a Stein manifold is a complete pluripolar set, this proves the existence of properly embedded disks in the complement of analytic subsets of $\mathbb{C}^2$. It should be remarked that the corresponding result for $\mathbb{C}$ instead of the disk is false since Kobayashi hyperbolicity of $\mathbb{C}^2 \setminus X$ is an obstruction. In fact it is known that any analytic subset $X$ of $\mathbb{C}^2$ can be embedded in a different way $f: X \hookrightarrow \mathbb{C}^2$ into $\mathbb{C}^2$ such that the complement $\mathbb{C}^2 \setminus f(X)$ is Kobayashi hyperbolic (see [1, 2]). In such a situation there is not even a non-constant holomorphic map from $\mathbb{C}$ into that complement. An easier example is the following:

Example 2. Let $X$ be the union of the following three lines in $\mathbb{C}^2_{z,w}$:

$$l_1 = \{w = 0\}, \quad l_2 = \{w = 1\}, \quad l_3 = \{z = w\}.$$  

If a holomorphic map $\varphi: \mathbb{C} \rightarrow \mathbb{C}^2$ avoids $l_1$ and $l_2$ it is of the form $\varphi(\theta) = (f(\theta), c)$ since the projection $\pi_w \circ \varphi$ is a map from $\mathbb{C}$ into $\mathbb{C} \setminus \{0, 1\}$ and thus constant $= c$. For $\varphi$ to be an embedding means that $f(\theta) = a\theta + b$, $a \neq 0$. Therefore the image of $\varphi$ meets $l_3$.

Note that in this example $\mathbb{C}^2 \setminus X$ is not Kobayashi hyperbolic. The maps $\theta \mapsto (\exp \theta + c, c)$ provide non-degenerate holomorphic maps from $\mathbb{C}$ into $\mathbb{C}^2 \setminus X$ if $c \neq 0, 1$.

Received by the editors July 19, 2007.
Hyperbolicity of $C^2 \setminus X$ is the reason why additional interpolation on discrete (or even finite) sets is not possible in general for embeddings as in our theorem.

2. Construction

Recall the following (simplified) definition from [6]:

**Definition 3.** Given a smooth real curve $\Gamma = \{ \gamma(t); t \in [0, \infty) \text{ or } t \in (-\infty, \infty) \}$ in $C^2$ without self-intersection, we say that $\Gamma$ has the nice projection property if there is a holomorphic automorphism $\alpha \in \text{Aut}_{hol}(C^2)$ of $C^2$ such that, if $\beta(t) = \alpha(\gamma(t))$, $\Gamma' = \alpha(\Gamma)$, and $\pi_1: C^2 \to C$ denotes the projection onto the first coordinate, then the following hold:

(i) $\lim_{|t| \to \infty} |\pi_1(\beta(t))| = \infty$, and

(ii) There is an $M \in \mathbb{R}$ such that for all $R \geq M$ we have that $C \setminus (\pi_1(\Gamma') \cup \overline{B_R})$ does not contain any relatively compact connected components.

Note that if a curve $\Gamma$ has the nice projection property and $\Theta \in \text{Aut}_{hol}(C^2)$, then $\Theta(\Gamma)$ has the nice projection property. To see this let $\alpha$ be as in the definition and consider the composition $\alpha' := \alpha \circ \Theta^{-1}$.

The reason for introducing this notion is the following lemma from [7]:

**Lemma 4.** Let $\Gamma$ be a curve having the nice projection property, let $K \subset C^2 \setminus \Gamma$ be a polynomially convex compact set, and let $\varepsilon > 0$. Then for any $R \in \mathbb{R}$ there exists a $\Phi \in \text{Aut}_{hol}(C^2)$ such that:

(i) $\|\Phi(z) - z\| < \varepsilon$ for all $z \in K$, and

(ii) $\Phi(\Gamma) \subset C^2 \setminus B_R$.

Starting with an embedded surface in $C^2$ with such a boundary $\Gamma$, one can apply the lemma to create a proper embedding of the surface by carrying the boundary inductively to infinity (see also [6]).

Let $W$ denote the set $W := \overline{\Delta \setminus \{1\}}$ and let $\Gamma$ denote the set $\Gamma := \{ z \in W; |z| = 1 \}$. We will say that a subset $\tilde{W} \subset W$ is $b$-nice if $\tilde{W}$ has a smooth boundary, and if there is a disk $D$ centered at 1 such that $\tilde{W} \cap D = W \cap D$. We let $\tilde{\Gamma}$ denote $\partial W \cap W$. Note that if $\varphi(W)$ is an embedding such that $\varphi(\Gamma)$ has the nice projection property, then $\varphi(\tilde{\Gamma})$ has the nice projection property. This is because the two embedded curves are the same near infinity.

The following lemma will provide us with the inductive step in our construction:

**Lemma 5.** Let $X$ be a closed complete pluripolar subset of $C^2$ and let $\varphi: W \hookrightarrow C^2$ be a smooth embedding, holomorphic on the interior, such that the following hold for some integer $N$:

(i) $\lim_{j \to \infty} \|\varphi(z_j)\| = \infty$ for all $\{z_j\} \subset W$ with $z_j \to 1$,

(ii) $\varphi(\Gamma)$ has the nice projection property,

(iii) $\varphi(W) \cap X \cap \overline{B_N} = \emptyset$,

(iv) $\varphi(\Gamma) \subset C^2 \setminus \overline{B_{N+1}}$, and

(v) $\varphi(W)$ intersects $\partial B_N$ transversally.
Let $S$ denote the set $S := \varphi^{-1}(\varphi(W) \cap \overline{\mathcal{B}_N})$, let $V$ be a connected component of $S$ and let $\varepsilon > 0$ (by (iv) we have that $S \subset \Delta$). Then there exists a $b$-nice subset $\hat{W}$ of $W$ with $V \subset \subset \hat{W}$, $W \cap (S \setminus V) = \emptyset$, and an embedding $\hat{\varphi} : \hat{W} \hookrightarrow \mathbb{C}^2$ (smooth, and holomorphic on the interior) such that:

(a) $\lim_{j \to \infty} \|\hat{\varphi}(z_j)\| = \infty$ for all $\{z_j\} \subset \hat{W}$ with $z_j \to 1$,

(b) $\hat{\varphi}(\Gamma)$ has the nice projection property,

(c) $\hat{\varphi}(\hat{W}) \cap X \cap \overline{\mathcal{B}_{N+1}} = \emptyset$,

(d) $\hat{\varphi}(\hat{W}) \subset \mathbb{C}^2 \setminus \overline{\mathcal{B}_{N+2}}$,

(e) $\hat{\varphi}(\hat{W})$ intersects $\partial \overline{\mathcal{B}_{N+1}}$ transversally,

(f) $\|\hat{\varphi} - \varphi\|_V < \varepsilon$, and

(g) $\hat{\varphi}(\hat{W} \setminus V) \subset \mathbb{C}^2 \setminus \overline{\mathcal{B}_{N-\varepsilon}}$.

Proof. It is not hard to see that (i) and (iv) implies that there exist positive real numbers $0 < r, \delta < 1$ such that the set $A_r := \{z \in W; |z| \geq r\}$ satisfies

(*)

\[ \varphi(A_r) \subset \mathbb{C}^2 \setminus \overline{\mathcal{B}_{N+1+\delta}}. \]

It follows that the set $P := \varphi^{-1}(\varphi(W) \cap (X \cap \overline{\mathcal{B}_{N+1}}))$ has a connected complement since the total intersection set $Z := \varphi^{-1}(\varphi(W) \cap X)$ is a complete pluripolar set in $W$ and since $P \subset Z \cap (W \setminus A_r)$. The set $S$ is clearly also contained in $W \setminus A_r$, and by (v) it is a finite disjoint union of smoothly bounded sets.

Now for an arbitrarily small neighborhood $N$ of $P$ we have that $\text{dist}(\varphi((\hat{W} \setminus A_r) \setminus N), X \cap \overline{\mathcal{B}_{N+1}}) > 0$. Since we also have (*) we get that

\[ \text{dist}(\varphi(W \setminus N), X \cap \overline{\mathcal{B}_{N+1}}) > 0. \]

This means that we may choose a $b$-nice domain $\tilde{W} \subset W \setminus (P \cup (S \setminus V))$ such that $V \subset \subset \tilde{W}$ and such that $\text{dist}(\varphi(\tilde{W}), X \cap \overline{\mathcal{B}_{N+1}}) > 0$. Note that $\varphi(\tilde{W}) \subset \mathbb{C}^2 \setminus \mathcal{B}_N$ and that $\varphi(\tilde{W})$ has the nice projection property since $\tilde{W}$ is the same as $W$ near 1.

Since $K := \overline{\mathcal{B}_N} \cup (X \cap \overline{\mathcal{B}_{N+1}})$ is polynomially convex (for the proof remark that the plurisubharmonic convex hull and the polynomial convex hull in $\mathbb{C}^n$ are the same and use the same idea of proof as in [5, Lemma 2]) there is an open neighborhood $\Omega$ of $K$ such that $\Omega$ is polynomially convex and such that $\Omega \cap \varphi(\Gamma) = \emptyset$. Thus by Lemma 4 (see also [6]) there exists a $\Phi \in \text{Aut}_{hod}(\mathbb{C}^2)$ such that $\|\Phi - \text{Id}\|_\Omega < \varepsilon$ and such that $\Phi(\varphi(\tilde{W})) \subset \mathbb{C}^2 \setminus \overline{\mathcal{B}_{N+2}}$. By possibly having to decrease $\varepsilon$ we may assume that $\Phi(\varphi(\tilde{W})) \cap (X \cap \overline{\mathcal{B}_{N+1}}) = \emptyset$ and so we may put $\hat{\varphi} := \Phi \circ \varphi$. The conditions (a), (c), (d), and (f) are then immediate. Since $\Phi$ is an automorphism we have that $\hat{\varphi}(\hat{W})$ has the nice projection property and so we get (b). Condition (g) follows since we chose $\tilde{W}$ such that $\varphi(\tilde{W} \setminus V) \subset \mathbb{C}^2 \setminus \mathcal{B}_N$ and because $\|\Phi - \text{Id}\|_\Omega < \varepsilon$. Finally, consider the case where the intersection of $\hat{\varphi}(\hat{W})$ with any sphere $\partial \mathcal{B}_\rho$ is not transversal. In that case there is a point $z \in \hat{W}$ with $\|\hat{\varphi}(z)\| = \rho$ and $\langle \hat{\varphi}(z), d\hat{\varphi}(z) \rangle = 0$. Hence the set of problematic points is analytic and thus discrete in $\hat{W}$. So there exist $\rho$'s arbitrarily close to 1 such that the intersection of $\hat{\varphi}(\hat{W})$ with $\partial \mathcal{B}_{\rho(N+1)}$ is transversal. Thus there are arbitrarily small linear perturbations of $\hat{\varphi}$ that give us (e), and the other properties are clearly preserved. \qed
Proof of Theorem 1. We will inductively construct an increasing sequence of simply connected sets in the unit disk along with a corresponding sequence of holomorphic embeddings.

To start the induction we embed the disk into \( \mathbb{C}^2 \) as follows: Start by letting \( f_1: \overline{\Delta} \to \mathbb{C}^2 \) be the map \( z \mapsto (3z, 0) \). We may of course assume that \( f(\overline{\Delta}) \cap X = \emptyset \). For \( \delta > 0 \) let \( f_\delta \) denote the rational map \( f_\delta: \mathbb{C}^2 \to \mathbb{C}^2 \) given by \( (z, w) \mapsto (z, w + \frac{\delta}{z^2}) \).

Let \( W \) be as in Lemma 5. If we put \( \varphi_1 := f_\delta \circ f_1 : W \to \mathbb{C}^2 \) for a small enough \( \delta \) it is not hard to verify that all conditions in Lemma 5 are satisfied with \( N = 1 \) (to get the nice projection property, project to the \( w \)-axis). Let \( U_1 := \varphi_1(W) \cap \overline{\mathbb{B}}_1 \) — a set we may assume to be connected and (automatically) simply connected — and choose \( \varepsilon_1 > 0 \) such that if \( \psi: U_1 \to \mathbb{C}^2 \) is any holomorphic map with \( \|\psi - \varphi_1\|_{\mathbb{B}^1} < \varepsilon_1 \) then \( \psi \) is an embedding and \( \psi(U_1) \cap X = \emptyset \). Choose \( \varepsilon_1 \) such that \( \varepsilon_1 < 2^{-2} \).

Assume that we have constructed/chosen the following objects with the listed properties:

1. Smoothly bounded simply connected domains \( U_j \subset \Delta \) and \( b \)-nice domains \( W_j \) such that \( U_1 \subset U_2 \subset \ldots \subset U_N \subset W_N \subset W_{N-1} \subset \ldots \subset U_1 \subset \overline{\Delta} \).
2. Holomorphic embeddings \( \varphi_j: W_j \to \mathbb{C}^2 \) such that \( \varphi_j(U_j) \subset \mathbb{B}_j \) and \( \|\varphi_j(z)\| = j \) for all \( z \in \partial U_j \).
3. \( \varphi_j(U_j \setminus U_{j-1}) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_{j-1} \).
4. \( \varphi_j(U_j) \cap X = \emptyset \), and
5. The pair \( (\varphi_N, W_N) \) satisfies the condition in Lemma 5.

(Technically \( W_N \) is not the same as in Lemma 5, but by the Riemann Mapping Theorem it does not make a difference.) Additionally, assume that we have inductively chosen a sequence \( \varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_N > 0 \) with \( \varepsilon_j < 2^{-j-1} \) and assured that

6. If \( \psi: U_j \to \mathbb{C}^2 \) is a holomorphic map with \( \|\psi - \varphi_j\|_{\mathbb{B}^j} < \varepsilon_j \) then \( \psi \) is an embedding and \( \psi(U_j) \cap X = \emptyset \), and
7. \( \|\varphi_j - \varphi_{j-1}\|_{\mathbb{B}^{j-1}} < \varepsilon_{j-1} 2^{-j} \).

We now show how to get \( U_{N+1}, \varphi_{N+1}, W_{N+1}, \) and \( \varepsilon_{N+1} \) so that we have (1)–(7) with \( N + 1 \) in place of \( N \).

To apply Lemma 5 we let \( \varphi := \varphi_N, W := W_N, V := U_N, \) and \( \varepsilon := \varepsilon_N 2^{-N-1} \). Let \( \varphi_{N+1} \) and \( W_{N+1} \) denote the objects corresponding to \( \varphi \) and \( W \) in the conclusion of the lemma. We get immediately then that the pair \( (\varphi_{N+1}, W_{N+1}) \) satisfy the conditions in the lemma, i.e. we have (5). In particular this means that

\[
\varphi_{N+1}(W_{N+1}) \cap (\overline{\mathbb{B}}_{N+1} \cap X) = \emptyset.
\]

Since \( U_N = V \) by assumption we also get that \( \|\varphi_{N+1} - \varphi_N\|_{\mathbb{B}^N} < \varepsilon = \varepsilon_N 2^{-N-1} \), i.e. we get (7). To define \( U_{N+1} \) we consider the set \( S := \varphi_{N+1}^{-1}(\varphi_{N+1}(W_{N+1}) \cap \overline{\mathbb{B}}_{N+1}) \).

Note first that \( U_N \subset S \) since we just established (7). This means that we may define \( U_{N+1} \) to be the interior of the connected component of \( S \) that contains \( U_N \).

By (e) we have that \( U_{N+1} \) is smoothly bounded, and we get (1), (2), and (4). Since \( U_{N+1} \setminus U_N \subset W_{N+1} \setminus U_N \) we get (3) from Lemma 5 (g). Finally we choose \( \varepsilon_{N+1} \) small enough to get (6).
To finish the proof we construct a sequence \((U_j, \varphi_j)\) according to the above procedure. We define \(U := \bigcup_{j=1}^{\infty} U_j\). Then \(U\) is an increasing union of simply connected domains and so \(U\) is itself simply connected. By the Riemann Mapping Theorem, \(U\) is conformally equivalent to the unit disk. We define a map \(\psi: U \to \mathbb{C}^2\) by

\[
\psi(z) = \lim_{j \to \infty} \varphi_j(z).
\]

To see that this is well defined we consider a point \(z \in \bigcup_{k=1}^{\infty} U_k\): For \(m > n \geq k\) we have by (7) that

\[
\|\varphi_m(z) - \varphi_n(z)\| \leq \sum_{i=n+1}^{m} \|\varphi_i(z) - \varphi_{i-1}(z)\| \leq \sum_{i=n+1}^{m} \epsilon_{i-1} 2^{-i} < \epsilon_n < 2^{-n-1}.
\]

This shows that \(\{\varphi_j(z)\}\) is a Cauchy sequence and so \(\psi\) defines a holomorphic map from \(U\) into \(\mathbb{C}^2\). It also shows that

\[
(\star \star) \quad \|\psi - \varphi_k\|_{\overline{U}_k} < \epsilon_k,
\]

so it follows from (6) that \(\psi\) is an embedding and that \(\psi(U) \cap X = \emptyset\).

To see that \(\psi\) is proper, consider a point \(z \in U_{k+1} \setminus \overline{U}_k\) for some \(k\). By (3) we have that \(\|\varphi_{k+1}(z)\| \geq k - 2^{-k-1}\), and so by (\(\star \star\)) we get that \(\|\psi(z)\| \geq k - 2^{-k-1} - \epsilon_k > k - 2^{-k}\). This means that \(\|\psi(z)\| > k - 2^{-k}\) for all \(z \in U \setminus U_k\), hence \(\psi\) is proper. \(\square\)

3. Concluding remarks

We note that if \(n \geq 2\) the analogous result to our main theorem holds with \(\mathbb{C}^n\) instead of \(\mathbb{C}^2\) (if the codimension of \(X\) is bigger than one, this is an easy consequence of transversality).

**Corollary 6.** Let \(X\) be any closed complete pluripolar subset of \(\mathbb{C}^n\), \(n \geq 2\), and let \(\triangle\) denote the unit disk in \(\mathbb{C}\). Then there exists a proper holomorphic embedding \(\varphi: \triangle \hookrightarrow \mathbb{C}^n\) such that \(\varphi(\triangle) \cap X = \emptyset\).

The construction in the proof of Theorem 1 works in this case also. On the other hand one can simply take a 2-dimensional \(\mathbb{C}\)-linear subspace \(C\) of \(\mathbb{C}^n\) such that \(X \cap C \neq \emptyset\) and apply Theorem 1 to embed \(\triangle\) into \(\mathbb{C}^n \setminus (X \cap C)\).

As pointed out in the introduction there is no way to add interpolation conditions at two or more points due to hyperbolicity obstructions, but it is a trivial addition to the proof in order to assure interpolation at one point, i.e. we can assure that the origin of the disk passes through a prescribed point in the complement of the pluripolar set. In fact, we believe that it is possible to to make the image of the embedding containing a prescribed discrete subset \(A\) of \(\mathbb{C}^n\) (with \(A \cap X = \emptyset\)).

**Acknowledgements**

In the original version of Theorem 1, the result was formulated for the special case of analytic subsets. We thank the referee for pointing out the paper [3] to us and asking whether Theorem 1 could be proven for complete pluripolar sets as well, thus suggesting the present result.

All authors supported by Schweizerische Nationalfonds grant 200021-116165/1.
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