Discrete convolutions of BV functions in quasiopen sets in metric spaces

Panu Lahti

Received: 3 January 2019 / Accepted: 31 October 2019 / Published online: 8 January 2020
© Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract

We study fine potential theory and in particular partitions of unity in quasiopen sets in the case \( p = 1 \). Using these, we develop an analog of the discrete convolution technique in quasiopen (instead of open) sets. We apply this technique to show that every function of bounded variation (BV function) can be approximated in the BV and \( L^\infty \) norms by BV functions whose jump sets are of finite Hausdorff measure. Our results seem to be new even in Euclidean spaces but we work in a more general complete metric space that is equipped with a doubling measure and supports a Poincaré inequality.

Mathematics Subject Classification

30L99 · 31E05 · 26B30

1 Introduction

In Euclidean spaces, a standard and very useful method for approximating a function of bounded variation (BV function) by smooth functions in a weak sense is to take convolutions with mollifier functions. In the setting of a more general doubling metric measure space, an analog of this method is given by so-called discrete convolutions. These are constructed by means of Lipschitz partitions of unity subordinate to Whitney coverings of an open set, and they possess most of the good properties of standard convolutions. Discrete convolutions and their properties have been considered e.g. in [25,26,36]. Whitney coverings and related partitions of unity were originally developed in [13,37,42].

In open sets, it is of course easy to pick Lipschitz cutoff functions that are then used in constructing a partition of unity. On the other hand, being limited to open sets is also a drawback of (discrete) convolutions; sometimes one may wish to smooth out a function in a finer way. In potential theory, one sometimes works with the concept of quasiopen sets. For nonlinear potential theory and its history in the Euclidean setting, in the case \( 1 < p < \infty \), see especially the monographs [1,23,38]. Nonlinear fine potential theory in metric spaces has been studied in several papers in recent years, see [7–9]. The typical assumptions on a metric space, as well as many of its properties, are encapsulated in the following:
space, which we make also in the current paper, are that the space is complete, equipped with a doubling measure, and supports a Poincaré inequality; see Sect. 2 for definitions.

Much less is known (even in Euclidean spaces) in the case $p = 1$, but certain results of fine potential theory when $p = 1$ have been developed by the author in metric spaces in [28–30]. In quasiopen sets, the role of Lipschitz cutoff functions needs to be taken by Sobolev functions (often called Newton–Sobolev functions in metric spaces). A theory of Newton–Sobolev cutoff functions in quasiopen sets when $p = 1$ was developed in [29], analogously to the case $1 < p < \infty$ studied previously in [7]. In the current paper we apply this theory to construct partitions of unity in quasiopen sets, and then we develop an analog of the discrete convolution technique in such sets. This is given in Theorem 4.6 and is, as far as we know, new even in Euclidean spaces. We point out that the partitions of unity that we construct depend on the function itself, and so our discrete convolutions are nonlinear, contrary to the usual ones in open sets.

As an application, we prove a new approximation result for BV functions. The jump set of a BV function is always $\sigma$-finite, but not necessarily finite, with respect to the codimension one (in the Euclidean setting, $n - 1$-dimensional) Hausdorff measure. On the other hand, in the study of minimization problems one often considers subclasses of BV functions for which the jump set is of finite Hausdorff measure. Approximation results for this kind of BV functions by means of piecewise smooth functions were studied recently in [14]. In the current paper, we prove that it is possible to approximate an arbitrary BV function by BV functions whose jump sets are of finite Hausdorff measure, in the following sense.

**Theorem 1.1** Suppose $(X, d, \mu)$ is a complete metric space such that $\mu$ is doubling and the space supports a $(1, 1)$-Poincaré inequality. Let $\Omega \subset X$ be an open set and let $u \in \text{BV}(\Omega)$. Then there exists a sequence $(u_i) \subset \text{BV}(\Omega)$ such that $\|u_i - u\|_{\text{BV}(\Omega)} + \|u_i - u\|_{L^\infty(\Omega)} \to 0$, and $\mathcal{H}(S_{u_i}) < \infty$ for each $i \in \mathbb{N}$.

This is given (with more details) in Theorem 5.3. Note that here the approximation is not only in the usual weak sense but in the BV norm. Yet the most subtle problem seems to be to obtain approximation simultaneously in the $L^\infty$ norm; for this the usual (discrete) convolution method seems too crude, demonstrating the need for the “quasiopen version”.

### 2 Definitions and assumptions

In this section we present the notation, definitions, and assumptions used in the paper.

Throughout the paper, $(X, d, \mu)$ is a complete metric space that is equipped with a metric $d$ and a Borel regular outer measure $\mu$ satisfying a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B(x, r) := \{ y \in X : d(y, x) < r \}$. When we want to state that a constant $C$ depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$. When a property holds outside a set of $\mu$-measure zero, we say that it holds almost everywhere, abbreviated a.e.

All functions defined on $X$ or its subsets will take values in $[-\infty, \infty]$. As a complete metric space equipped with a doubling measure, $X$ is proper, that is, every closed and bounded set is compact. Given a $\mu$-measurable set $A \subset X$, we define $L^1_{\text{loc}}(A)$ as the class of functions $u$ on $A$ such that for every $x \in A$ there exists $r > 0$ such that $u \in L^1(A \cap B(x, r))$. Other local spaces of functions are defined similarly. For an open set $\Omega \subset X$, a function is in the
class $L^1_{\text{loc}}(\Omega)$ if and only if it is in $L^1(\Omega')$ for every open $\Omega' \subseteq \Omega$. Here $\Omega' \subseteq \Omega$ means that $\overline{\Omega'}$ is a compact subset of $\Omega$.

For any set $A \subset X$ and $0 < R < \infty$, the restricted Hausdorff content of codimension one is defined by

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i \in I} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i \in I} B(x_i, r_i), r_i \leq R \right\},$$

where $I \subset \mathbb{N}$ is a finite or countable index set. The codimension one Hausdorff measure of $A \subset X$ is then defined by

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into $X$. A nonnegative Borel function $g$ on $X$ is an upper gradient of a function $u$ on $X$ if for all nonconstant curves $\gamma$, we have

$$|u(x) - u(y)| \leq \int_{\gamma} g \, ds,$$

where $x$ and $y$ are the end points of $\gamma$ and the curve integral is defined by means of an arc-length parametrization, see [24, Section 2] where upper gradients were originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|, |u(y)|$ is infinite.

We say that a family of curves $\Gamma$ is of zero 1-modulus if there is a nonnegative Borel function $\rho \in L^1(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho \, ds$ is infinite. A property is said to hold for 1-almost every curve if it fails only for a curve family with zero 1-modulus. If $g$ is a nonnegative $\mu$-measurable function on $X$ and (2.1) holds for 1-almost every nonconstant curve, we say that $g$ is a 1-weak upper gradient of $u$. By only considering curves $\gamma$ in a set $A \subset X$, we can talk about a function $g$ being a (1-weak) upper gradient of $u$ in $A$.

For a $\mu$-measurable set $H \subset X$, we define

$$\|u\|_{N^1,1(H)} := \|u\|_{L^1(H)} + \inf \|g\|_{L^1(H)},$$

where the infimum is taken over all 1-weak upper gradients $g$ of $u$ in $H$. The substitute for the Sobolev space $W^{1,1}$ in the metric setting is the Newton–Sobolev space

$$N^{1,1}(H) := \{ u : \|u\|_{N^1,1(H)} < \infty \},$$

which was first introduced in [41]. We also define the Dirichlet space $D^1(H)$ consisting of $\mu$-measurable functions $u$ on $H$ with an upper gradient $g \in L^1(H)$ in $H$. Both spaces are clearly vector spaces and by [5, Corollary 1.20] (or its proof) we know that each is also a lattice, so that

if $u, v \in D^1(X)$, then $\min\{u, v\}, \max\{u, v\} \in D^1(X).$  \hspace{1cm} (2.2)

For any $H \subset X$, the space of Newton–Sobolev functions with zero boundary values is defined as

$$N^{1,1}_0(H) := \{ u|_H : u \in N^{1,1}(X) \text{ and } u = 0 \text{ on } X \setminus H \}.$$  

This space is a subspace of $N^{1,1}(H)$ when $H$ is $\mu$-measurable, and it can always be understood to be a subspace of $N^{1,1}(X)$. The class $N^{1,1}_c(H)$ consists of those functions $u \in N^{1,1}(X)$ that have compact support in $H$, i.e. $\text{spt } u \subset H$. 
Note that we understand Newton–Sobolev functions to be defined at every $x \in H$ (even though $\| \cdot \|_{N^{1,1}(H)}$ is then only a seminorm). It is known that for any $u \in N^{1,1}_0(H)$ there exists a minimal 1-weak upper gradient of $u$ in $H$, always denoted by $g_u$, satisfying $g_u \leq g$ a.e. in $H$, for any 1-weak upper gradient $g \in L^1_{\text{loc}}(H)$ of $u$ in $H$, see [5, Theorem 2.25]. Sometimes we also use the notation $g_{u,H}$ to specify that we mean the minimal 1-weak upper gradient of $u$ in $H$, even though $u$ may be defined in a larger set.

We will assume throughout the paper that $X$ supports a $(1, 1)$-Poincaré inequality, meaning that there exist constants $C_p > 0$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L^1_{\text{loc}}(X)$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_p r \int_{B(x,\lambda r)} g \, d\mu,$$

where

$$u_{B(x,r)} := \int_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

The 1-capacity of a set $A \subset X$ is defined by

$$\text{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)},$$

where the infimum is taken over all functions $u \in N^{1,1}(X)$ such that $u \geq 1$ in $A$. If a property holds outside a set $A \subset X$ with $\text{Cap}_1(A) = 0$, we say that it holds 1-quasieverywhere, or 1-q.e. We know that for any $\mu$-measurable set $H \subset X$,

$$u = 0 \text{ 1-q.e. in } H \text{ implies } \|u\|_{N^{1,1}(H)} = 0,$$

see [5, Proposition 1.61].

The variational 1-capacity of a set $A \subset H$ with respect to a set $H \subset X$ is defined by

$$\text{cap}_1(A, H) := \inf \int_H g_u \, d\mu,$$

where the infimum is taken over functions $u \in N^{1,1}_0(H)$ such that $u \geq 1$ in $A$, and where $g_u$ is the minimal 1-weak upper gradient of $u$ (in $X$). By truncation, we can alternatively require that $u = 1$ in $A$. For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see e.g. [5]. By [20, Theorem 4.3, Theorem 5.1] we know that for $A \subset X$,

$$\text{Cap}_1(A) = 0 \text{ if and only if } H(A) = 0.$$

We say that a set $U \subset X$ is 1-quasiopen if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $U \cup G$ is open. Given a set $H \subset X$, we say that a function $u$ is 1-quasi (lower/upper semi-)continuous on $H$ if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $u|_{H \setminus G}$ is finite and (lower/upper semi-)continuous. We do not mention $H$ when $H = X$. Note that we always take the 1-capacity with respect to functions in $N^{1,1}(X)$ and not $N^{1,1}(H)$, but if $H$ is 1-quasiopen, this does not even make a difference, see [10, Proposition 3.4].

It is a well-known fact that Newton–Sobolev functions are 1-quasicontinuous on open sets, see [11, Theorem 1.1] or [5, Theorem 5.29]. In fact, by [10, Theorem 1.3] we know that more generally

$$\text{for a 1-quasiopen } U \subset X, \text{ any } u \in N^{1,1}_{\text{loc}}(U) \text{ is 1-quasicontinuous on } U.$$
By [5, Proposition 5.23] we also know that for a 1-quasiopen $U \subset X$ and functions $u, v$ that are 1-quasicontinuous on $U$,

$$\text{if } u = v \text{ a.e. in } U, \text{ then } u = v \text{ 1-q.e. in } U. \quad (2.7)$$

Here again by “1-q.e.” we mean that the 1-capacity is taken with respect to the metric space $X$; actually the above result is given in [5] in terms of a version of $\text{Cap}_1$ defined by considering $U$ as the metric space, but [10, Proposition 4.2] and [40, Remark 3.5] guarantee that this does not make a difference.

Next we present the definition and basic properties of functions of bounded variation on metric spaces, following [39]. See also e.g. the monographs [3, 15, 16, 19, 43] for the classical theory in the Euclidean setting. Given an open set $\Omega \subset X$ and a function $u \in L^1_{\text{loc}}(\Omega)$, we define the total variation of $u$ over $\Omega$ by

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in N^{1,1}_{\text{loc}}(\Omega), \ u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\}, \quad (2.8)$$

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $\Omega$. (In [39], local Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition.) We say that a function $u \in L^1(\Omega)$ is of bounded variation, and denote $u \in \text{BV}(\Omega)$, if $\|Du\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf \{\|Du\|(W) : A \subset W, \ W \subset X \text{ is open}\}. \quad (2.9)$$

Note that if we defined $\|Du\|(A)$ simply by replacing $\Omega$ with $A$ in (2.8), we would get a different quantity compared with the definition given in (2.9). However, in a 1-quasiopen set $U$ these give the same result; we understand the expression $\|Du\|(U) < \infty$ to mean that there exists some open set $\Omega \supset U$ such that $u \in L^1_{\text{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$.

**Theorem 2.1** ([33, Theorem 4.3]) *Let $U \subset X$ be 1-quasiopen. If $\|Du\|(U) < \infty$, then*

$$\|Du\|(U) = \inf \left\{ \liminf_{i \to \infty} \int_{U} g_{u_i} \, d\mu : u_i \in N^{1,1}_{\text{loc}}(U), \ u_i \to u \text{ in } L^1_{\text{loc}}(U) \right\},$$

*where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $U$.*

Note that 1-quasiopen sets are $\mu$-measurable by [6, Lemma 9.3]. We also have the following lower semicontinuity.

**Theorem 2.2** ([33, Theorem 4.5]) *Let $U \subset X$ be a 1-quasiopen set. If $\|Du\|(U) < \infty$ and $u_i \to u$ in $L^1_{\text{loc}}(U)$, then*

$$\|Du\|(U) \leq \liminf_{i \to \infty} \|Du_i\|(U).$$

If $u \in L^1_{\text{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$, then $\|Du\|$ is a Radon measure on $\Omega$ by [39, Theorem 3.4], and we call it the variation measure. The BV norm is defined by

$$\|u\|_{\text{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

A $\mu$-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where $\chi_E$ is the characteristic function of $E$. The measure-theoretic interior of a set $E \subset X$ is defined by

$$I_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\}, \quad (2.10)$$
and the measure-theoretic exterior by

\[ O_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}. \]

The measure-theoretic boundary \( \partial^* E \) is defined as the set of points \( x \in X \) at which both \( E \) and its complement have strictly positive upper density, i.e.

\[
\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.
\]

For an open set \( \Omega \subset X \) and a \( \mu \)-measurable set \( E \subset X \) with \( \|D\chi_E\|/\Omega < \infty \), we know that for any Borel set \( A \subset \Omega \),

\[ \|D\chi_E\|(A) = \int_{\partial^* E \cap A} \theta_E \, d\mathcal{H}, \tag{2.11} \]

where \( \theta_E : X \to [\alpha, C_d] \) with \( \alpha = \alpha(C_d, C_P, \lambda) > 0 \), see [2, Theorem 5.3] and [4, Theorem 4.6].

For any \( u, v \in L^1_{\text{loc}}(\Omega) \) and any \( A \subset \Omega \), it is straightforward to show that

\[ \|D(u + v)\|(A) \leq \|Du\|(A) + \|Dv\|(A). \tag{2.12} \]

The lower and upper approximate limits of a function \( u \) on an open set \( \Omega \) are defined respectively by

\[ u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\} \]

and

\[ u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\} \]

for \( x \in \Omega \). Always \( u^\wedge \leq u^\vee \), and the jump set of \( u \) is defined by

\[ S_u := \{u^\wedge < u^\vee\} := \{x \in \Omega : u^\wedge(x) < u^\vee(x)\}. \]

Note that since we understand \( u^\wedge \) and \( u^\vee \) to be defined only on \( \Omega \), also \( S_u \) is understood to be a subset of \( \Omega \). For \( u \in L^1_{\text{loc}}(\Omega) \), we have \( u = u^\wedge = u^\vee \) a.e. in \( \Omega \) by Lebesgue’s differentiation theorem (see e.g. [22, Chapter 1]). Unlike Newton–Sobolev functions, we understand BV functions to be \( \mu \)-equivalence classes. To consider fine properties, we need to consider the pointwise representatives \( u^\wedge \) and \( u^\vee \).

Recall that Newton–Sobolev functions are quasicontinuous; BV functions have the following quasi-semicontinuity property, which follows from [34, Corollary 4.2], which in turn is based on [36, Theorem 1.1]. The property was first proved in the Euclidean setting in [12, Theorem 2.5].

**Proposition 2.3** Let \( \Omega \subset X \) be open and let \( u \in L^1_{\text{loc}}(\Omega) \) with \( \|Du\|(\Omega) < \infty \). Then \( u^\wedge \) is 1-quasi lower semicontinuous and \( u^\vee \) is 1-quasi upper semicontinuous on \( \Omega \).
and jump parts $\|Du\|^c$ and $\|Du\|^j$, which have the following form. Given an open set $\Omega \subset X$ and $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$, we have for any Borel set $A \subset \Omega$

$$\|Du\|(A) = \|Du\|^d(A) + \|Du\|^s(A) = \|Du\|^d(A) + \|Du\|^c(A) + \|Du\|^j(A) = \int_A a \, d\mu + \|Du\|^c(A) + \int_{A \cap S_u} \int_{u^+(x)} \theta_{[u > t]}(x) \, dt \, d\mathcal{H}(x),$$

where $a \in L^1(\Omega)$ is the density of the absolutely continuous part and the functions $\theta_{[u > t]} \in [\alpha, C_d]$ are as in (2.11). In [4] it is assumed that $u \in \text{BV}(\Omega)$, but the proof is the same for the slightly more general $u$ that we consider here.

Next we define the fine topology in the case $p = 1$.

**Definition 2.4** We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r \to 0} r \cap_{p}(A \cap B(x, r), B(x, 2r)) \mu(B(x, r)) = 0.$$ 

We also say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on $X$.

We denote the 1-fine interior of a set $H \subset X$, i.e. the largest 1-finely open set contained in $H$, by fine-int $H$. We denote the 1-fine closure of $H$, i.e. the smallest 1-finely closed set containing $H$, by fine-$\overline{H}$.

We say that a function $u$ defined on a set $U \subset X$ is 1-finely continuous at $x \in U$ if it is continuous at $x$ when $U$ is equipped with the induced 1-fine topology on $U$ and $[-\infty, \infty]$ is equipped with the usual topology.

See [30, Section 4] for discussion on this definition, and for a proof of the fact that the 1-fine topology is indeed a topology. By [28, Lemma 3.1], 1-thinness implies zero measure density, i.e.

$$\text{if } A \text{ is 1-thin at } x, \text{ then } \lim_{r \to 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} = 0. \quad (2.14)$$

**Theorem 2.5** ([35, Corollary 6.12]) A set $U \subset X$ is 1-quasiopen if and only if it is the union of a 1-finely open set and a $\mathcal{H}$-negligible set.

**Theorem 2.6** ([29, Theorem 5.1]) A function $u$ on a 1-quasiopen set $U$ is 1-quasicontinuous on $U$ if and only if it is finite 1-q.e. and 1-finely continuous 1-q.e. in $U$.

Throughout this paper we assume that $(X, d, \mu)$ is a complete metric space that is equipped with a doubling measure $\mu$ and supports a $(1, 1)$-Poincaré inequality.

### 3 Preliminary results

In this section we prove and record some preliminary results needed in constructing the discrete convolutions in 1-quasiopen sets. The first lemma states that in the definition of the total variation, we can consider convergence in $L^1(\Omega)$ instead of convergence in $L^1_{\text{loc}}(\Omega)$.

**Lemma 3.1** ([27, Lemma 5.5]) Let $\Omega \subset X$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then there exists a sequence $(u_i) \subset \text{Lip}_{\text{loc}}(\Omega)$ with $u_i \to u$ in $L^1(\Omega)$ and $\int_{\Omega} g_{u_i} \, d\mu \to \|Du\|(\Omega)$, where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $\Omega$.
Note that we cannot write $u_i \to u$ in $L^1(\Omega)$, since the functions $u_i, u$ are not necessarily in the class $L^1(\Omega)$.

Now we consider preliminary approximation results for BV functions. We have the following approximation result for BV functions whose jumps remain bounded.

**Proposition 3.2** ([32, Proposition 5.2]) Let $U \subset \Omega$ such that $U$ is 1-quasiopen and $\Omega$ is open, and let $u \in \text{BV}(\Omega)$ and $\beta > 0$ such that $u^\vee - u^\wedge < \beta$ in $U$. Then for every $\varepsilon > 0$ there exists $v \in N^{1,1}(U)$ such that $\|v - u\|_{L^\infty(U)} \leq 4\beta$ and

$$\int_U g_v \, d\mu < \|Du\|(U) + \varepsilon.$$ 

In fact, in the proof of the above proposition in [32], the $L^\infty$-bound is stated in the following slightly more precise way (note that $v, u^\wedge, u^\vee$ are all pointwise defined functions):

$$u^\vee - 4\beta \leq v \leq u^\wedge + 4\beta \quad \text{in } U. \quad (3.1)$$

By [5, Corollary 2.21] we know that if $H \subset X$ is a $\mu$-measurable set and $v, w \in N^{1,1}_\text{loc}(H)$, then

$$g_v = g_w \text{ a.e. in } \{x \in H : v(x) = w(x)\}, \quad (3.2)$$

where $g_v$ and $g_w$ are the minimal 1-weak upper gradients of $v$ and $w$ in $H$.

**Proposition 3.3** Let $U \subset \Omega \subset X$ be such that $U$ is 1-quasiopen and $\Omega$ is open, and let $u \in \text{BV}(\Omega)$ and $\beta > 0$ such that $u^\vee - u^\wedge < \beta$ in $U$. Then there exists a sequence $(u_i) \subset N^{1,1}(U)$ such that $u_i \to u$ in $L^1(U)$, $\sup_u |u_i - u^\wedge| \leq 9\beta$ for all $i \in \mathbb{N}$, and

$$\lim_{i \to \infty} \int_U g_{u_i,U} \, d\mu = \|Du\|(U).$$

Recall that $g_{u_i,U}$ denotes the minimal 1-weak upper gradient of $u_i$ in $U$.

The proof reveals that we also have $\sup_u |u_i - u^\wedge| \leq 9\beta$, that is, we can replace the pointwise representative $u^\vee$ by $u^\wedge$.

**Proof** By Proposition 3.2 and (3.1) we find a function $v \in N^{1,1}(U)$ such that $u^\vee - 4\beta \leq v \leq u^\wedge + 4\beta$ in $U$ and

$$\int_U g_{v,U} \, d\mu \leq \|Du\|(U) + 1.$$ 

Define $v_1 := v - 5\beta$ and $v_2 := v + 5\beta$, so that $v_1, v_2 \in N^{1,1}_\text{loc}(U)$ with $u^\vee - 9\beta \leq v_1 \leq u^\wedge - \beta$ and $u^\vee + \beta \leq v_2 \leq u^\wedge + 9\beta$ in $U$, and

$$\int_U g_{v_j,U} \, d\mu \leq \|Du\|(U) + 1 \quad (3.3)$$

for $j = 1, 2$.

Take open sets $\Omega_i$ such that $U \subset \Omega_i \subset \Omega$ and $\|Du\|(\Omega_i) < \|Du\|(U) + 1/i$, for each $i \in \mathbb{N}$. By Lemma 3.1 we find functions $w_i \in \text{Lip}_\text{loc}(\Omega_i) \subset \text{Lip}_\text{loc}(U)$ such that $\|w_i - u\|_{L^1(\Omega_i)} < 1/i$ and

$$\int_{\Omega_i} g_{w_i,\Omega_i} \, d\mu < \|Du\|(\Omega_i) + 1/i.$$
It follows that \( w_i \to u \) in \( L^1(U) \) and
\[
\limsup_{i \to \infty} \int_U g_{w_i, U} \, d\mu \leq \limsup_{i \to \infty} \int_U g_{u_i, \Omega_1} \, d\mu \leq \|Du\|(U),
\]
and then by Theorem 2.1 we must in fact have
\[
\lim_{i \to \infty} \int_U g_{w_i, U} \, d\mu = \|Du\|(U).
\]
By passing to a subsequence (not relabeled), we can assume that also \( w_i \to u \) a.e. in \( U \). Then define \( u_i := \min\{v_2, \max\{v_1, w_i\}\} \). By Lebesgue’s differentiation theorem, we have also \( u - 9\beta \leq v_1 \leq u - \beta \) and \( u + \beta \leq v_2 \leq u + 9\beta \) a.e. in \( U \), whence
\[
\|u_i - u\|_{L^1(U)} \leq \|w_i - u\|_{L^1(U)} \to 0,
\]
that is \( u_i \to u \) in \( L^1(U) \), as desired. Moreover, \( \sup_U |u_i - u|^\epsilon \leq 9\beta \) for all \( i \in \mathbb{N} \). In addition, by (3.2) we have for each \( i \in \mathbb{N} \)
\[
\int_U g_{u_i, U} \, d\mu = \int_{\{w_i > v_2\}} g_{v_2, U} \, d\mu + \int_{\{w_i < v_1\}} g_{v_1, U} \, d\mu + \int_{\{v_1 \leq w_i \leq v_2\}} g_{w_i, U} \, d\mu
\]
\[
\leq \int_{\{w_i > v_2\}} g_{v_2, U} \, d\mu + \int_{\{w_i < v_1\}} g_{v_1, U} \, d\mu + \int_{\{v_1 \leq w_i \leq v_2\}} g_{w_i, U} \, d\mu.
\]
Since \( \int_U g_{v_2, U} \, d\mu < \infty \) by (3.3) and since \( w_i \to u < v_2 \) a.e. in \( U \), by Lebesgue’s dominated convergence theorem we get \( \int_{\{w_i > v_2\}} g_{v_2, U} \, d\mu \to 0 \). Treating the integral involving \( v_1 \) similarly, we get
\[
\limsup_{i \to \infty} \int_U g_{u_i, U} \, d\mu \leq \limsup_{i \to \infty} \int_U g_{w_i, U} \, d\mu = \|Du\|(U),
\]
and then in fact \( \lim_{i \to \infty} \int_U g_{u_i, U} \, d\mu = \|Du\|(U) \) by Theorem 2.1.

The variation measure is always absolutely continuous with respect to the 1-capacity, in the following sense.

**Lemma 3.4** ([34, Lemma 3.8]) Let \( \Omega \subset X \) be an open set and let \( u \in L^1_{\text{loc}}(\Omega) \) with \( \|Du\|(\Omega) < \infty \). Then for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( A \subset \Omega \) with \( \text{Cap}_1(A) < \delta \), then \( \|Du\|(A) < \epsilon \).

The following proposition describes the weak* convergence of the variation measure; recall that we understand the expression \( \|Du\|(U) < \infty \) to mean that there exists some open set \( \Omega \supset U \) such that \( u \in L^1_{\text{loc}}(\Omega) \) and \( \|Du\|(\Omega) < \infty \).

**Proposition 3.5** ([31, Proposition 3.9]) Let \( U \subset X \) be 1-quasiopen. If \( \|Du\|(U) < \infty \) and \( u_i \to u \) in \( L^1_{\text{loc}}(U) \) such that
\[
\|Du\|(U) = \lim_{i \to \infty} \int_U g_{u_i} \, d\mu,
\]
where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( U \), then
\[
\int_U \eta \, d\|Du\| \geq \limsup_{i \to \infty} \int_U \eta g_{u_i} \, d\mu
\]
for every nonnegative bounded 1-quasi upper semicontinuous function \( \eta \) on \( U \).
Recall the definition of 1-quasi upper semicontinuity: for every \( \varepsilon > 0 \) there is an open set \( G \subset X \) such that \( \text{Cap}_1(G) < \varepsilon \) and \( \eta|_{U \setminus G} \) is finite and upper semicontinuous.

The following lemma will be applied later to functions \( \eta \) that form a partition of unity in a 1-quasiopen set. Recall that we understand \( N_0^{1,1}(H) \) to be a subspace of \( N^{1,1}(X) \).

**Lemma 3.6** Let \( H \subset X \) be \( \mu \)-measurable, let \( u \in N^{1,1}(H) \) be bounded, and let \( \eta \in N_0^{1,1}(H) \) with \( 0 \leq \eta \leq 1 \) on \( X \). Then \( \eta u \in N_0^{1,1}(H) \) with a 1-weak upper gradient \( \eta g_u + |u|g_\eta \) (in \( X \), with the interpretation that an undefined function times zero is zero).

**Proof** We have \( |u| \leq M \) in \( H \) for some \( M \geq 0 \). By the Leibniz rule, see [5, Theorem 2.15], we know that \( \eta u \in N^{1,1}(H) \) with a 1-weak upper gradient \( \eta g_u + |u|g_\eta \) in \( H \). Moreover, \( -M\eta \leq \eta u \leq M\eta \in N_0^{1,1}(H) \), and so by [5, Lemma 2.37] we conclude that \( \eta u \in N_0^{1,1}(H) \), with \( g_{\eta u} = 0 \) in \( X \setminus H \) by (3.2). Finally, by [6, Proposition 3.10] we know that \( \eta g_u + |u|g_\eta \) (with \( g_\eta = 0 \) a.e. in \( X \setminus H \) by (3.2)) is a 1-weak upper gradient of \( \eta u \) in \( X \).

Next we observe that convergence in the BV norm implies the following pointwise convergence; this follows from [36, Lemma 4.2].

**Lemma 3.7** Let \( u_i, u \in BV(X) \) with \( u_i \to u \) in \( BV(X) \). By passing to a subsequence (not relabeled), we have \( u_i^\land \to u^\land \) and \( u_i^\lor \to u^\lor \mathcal{H}\text{-a.e. in } X \).

We have the following result for BV functions whose variation measure has no singular part; recall the decomposition (2.13).

**Theorem 3.8** Let \( \Omega \subset X \) be open and let \( v \in L^1_{\text{loc}}(\Omega) \) with \( \|Dv\|(\Omega) < \infty \) and \( \|Dv\|^+(U) = 0 \) for a \( \mu \)-measurable set \( U \subset \Omega \). Then a modification \( \tilde{v} \) of \( v \) in a \( \mu \)-negligible subset of \( U \) satisfies \( \tilde{v} \in N_{\text{loc}}^{1,1}(U) \) such that for every \( \mu \)-measurable \( H \subset U \),

\[
\int_H g_{\tilde{v}} \, d\mu \leq C_0 \|Dv\|(H)
\]

where \( g_{\tilde{v}} \) is the minimal 1-weak upper gradient of \( \tilde{v} \) in \( U \) and \( C_0 \geq 1 \) is a constant depending only on the doubling constant of \( \mu \) and the constants in the Poincaré inequality.

**Proof** This result is given in [21, Theorem 4.6], except that there it is assumed that \( v \in BV(\Omega) \) (that is, \( v \) is in \( L^1(\Omega) \) and not just in \( L^1_{\text{loc}}(\Omega) \)). However, by exhausting \( \Omega \) with relatively compact open sets and applying [21, Theorem 4.6] in these sets, we obtain the result (note that by (2.7) and (2.4) we know that we do not need to keep redefining \( \tilde{v} \) in this construction).

\( \square \)

Finally, we have the following two simple results for 1-quasiopen sets.

**Lemma 3.9** Let \( U \subset X \) be 1-quasiopen. Then \( \chi_U \) is 1-quasi lower semicontinuous.

**Proof** Let \( \varepsilon > 0 \). We find an open set \( G \subset X \) such that \( \text{Cap}_1(G) < \varepsilon \) and \( U \cup G \) is open. Thus \( U \) is open in the subspace topology of \( X \setminus G \), and so \( \chi_U|_{X \setminus G} \) is lower semicontinuous.

\( \square \)

Conversely, it is easy to see that the super-level sets \( \{ u > t \} \), \( t \in \mathbb{R} \), of a 1-quasi lower semicontinuous function \( u \) are 1-quasiopen; see e.g. the proof of [10, Proposition 3.4]. We will use this fact, or its analog for 1-quasi (upper semi-)continuous functions, without further notice.
Lemma 3.10 Let \( U \subset X \) be 1-quasiopen, let \( v \in N^{1,1}(U) \) with \( \|Dv\|(U) < \infty \), and let \( A \subset U \) with \( \mu(A) = 0 \). Then \( \|Dv\|(A) = 0 \).

Note that \( v \in N^{1,1}(U) \) does not automatically imply \( \|Dv\|(U) < \infty \), since the latter involves an extension to an open set.

**Proof** We find open sets \( W_j \supset A \), \( j \in \mathbb{N} \), such that \( \mu(W_j) \to 0 \). Then the sets \( W_j \cap U \) are easily seen to be 1-quasiopen, and so by Theorem 2.1 we get

\[
\|Dv\|(A) \leq \|Dv\|(W_j \cap U) \leq \int_{W_j \cap U} g_v \cdot d\mu \to 0 \quad \text{as} \quad j \to \infty,
\]

where \( g_v \) is the minimal 1-weak upper gradient of \( v \) in \( U \).

\[\square\]

4 The discrete convolution method

In this section we study partitions of unity in 1-quasiopen sets and then we use these to develop the discrete convolution method in such sets. To construct the partitions of unity, we first need suitable cutoff functions in quasiopen sets. These cannot be taken to be Lipschitz functions, but we can use Newton–Sobolev functions instead. The following definition and proposition are analogs of the theory in the case \( 1 < p < \infty \), which was studied in the metric setting in [7].

**Definition 4.1** A set \( A \subset D \) is a 1-strict subset of \( D \) if there is a function \( \eta \in N_{0,1}^1(D) \) such that \( \eta = 1 \) in \( A \).

A countable family \( \{U_j\}_{j=1}^\infty \) of 1-quasiopen sets is a quasicovering of a 1-quasiopen set \( U \) if \( \bigcup_{j=1}^\infty U_j \subset U \) and \( \text{Cap}_1 \left( U \setminus \bigcup_{j=1}^\infty U_j \right) = 0 \). If every \( U_j \) is a 1-finely open 1-strict subset of \( U \) and \( U_j \Subset U \), then \( \{U_j\}_{j=1}^\infty \) is a 1-strict quasicovering of \( U \).

**Proposition 4.2** ([29, Proposition 5.4]) If \( U \subset X \) is 1-quasiopen, then there exists a 1-strict quasicovering \( \{U_j\}_{j=1}^\infty \) of \( U \). Moreover, the associated Newton–Sobolev functions can be chosen to be compactly supported in \( U \).

We will need a 1-strict quasicovering that is adapted to a given BV function. Recall that the class \( N_{c,1}^1(U) \) consists of those functions \( u \in N^{1,1}(X) \) that have compact support in \( U \), i.e. \( \text{spt } u \Subset U \).

**Proposition 4.3** Let \( U \subset \Omega \subset X \) be such that \( U \) is 1-quasiopen and \( \Omega \) is open, and let \( u \in \text{L}^1_{\text{loc}}(\Omega) \) with \( \|Du\|(\Omega) < \infty \). Then there exists a 1-strict quasicovering \( \{U_j\}_{j=1}^\infty \) of \( U \), and associated Newton–Sobolev functions \( \{\rho_j \in N_{c,1}^1(U)\}_{j=1}^\infty \) such that \( -\infty < \inf_{\text{spt} \rho_j} u^\wedge \leq \sup_{\text{spt} \rho_j} u^\vee < \infty \) for all \( j \in \mathbb{N} \).

**Proof** Define \( V_j := \{x \in U : -j < u^\wedge(x) \leq u^\vee(x) < j\} \) for each \( j \in \mathbb{N} \). By Proposition 2.3 and the fact that the intersection of two 1-quasiopen sets is 1-quasiopen (see e.g. [18, Lemma 2.3]), each of these sets is 1-quasiopen. By [26, Lemma 3.2] we know that \( \mathcal{H}(U \setminus \bigcup_{j=1}^\infty V_j) = 0 \). For each \( j \in \mathbb{N} \), apply Proposition 4.2 to find a 1-strict quasicovering \( \{U_{j,k}\}_{k=1}^\infty \) of \( V_j \), and the associated Newton–Sobolev functions \( \rho_{j,k} \in N_{c,1}^1(V_j) \). Then
\[ \{U_j,k\}_{j,k=1}^{\infty} \] is a 1-strict quasicovering of \( U \) with the associated Newton–Sobolev functions \( \rho_{j,k} \in N^{1,1}_c(U) \), such that

\[ -\infty < -j \leq \inf_{\text{spt}_\rho_{j,k}} u^\wedge \leq \sup_{\text{spt}_\rho_{j,k}} u^\vee \leq j < \infty \]

for all \( j, k \in \mathbb{N} \).

By truncating if necessary, we can always assume that the Newton–Sobolev functions take values between 0 and 1.

Now we construct the partition of unity.

**Proposition 4.4** Let \( U \subset X \) be 1-quasiopen and let \( \{U_j\}_{j=1}^{\infty} \) be a 1-strict quasicovering of \( U \) with the associated Newton–Sobolev functions \( \rho_j \in N^{1,1}_c(U) \), \( 0 \leq \rho_j \leq 1 \). Then we can find functions \( \eta_j \in N^{1,1}_c(U) \) such that \( \eta_1 = \rho_1 \), \( 0 \leq \eta_j \leq \rho_j \) for all \( j \in \mathbb{N} \), \( \sum_{j=1}^{\infty} \eta_j = 1 \) 1-q.e. in \( U \), and 1-q.e. \( x \in U \) has a 1-fine neighborhood where \( \eta_j \neq 0 \) for only finitely many \( j \in \mathbb{N} \).

We describe the last two conditions by saying that \( \{\eta_j\}_{j=1}^{\infty} \) is a 1-finely locally finite partition of unity on \( U \).

**Proof** Define recursively for each \( j \in \mathbb{N} \)

\[ \eta_j := \min \left\{ \left( 1 - \sum_{l=1}^{j-1} \eta_l \right)_+, \rho_j \right\}. \]

It is clear that \( 0 \leq \eta_j \leq \rho_j \) for all \( j \in \mathbb{N} \), and then by the lattice property (2.2) we get \( \eta_j \in N^{1,1}_c(U) \). Moreover, for 1-q.e. \( x \in U \) there is \( k \in \mathbb{N} \) such that \( x \in U_k \), and since \( \rho_k = 1 \) in \( U_k \), we get \( \sum_{j=1}^{k} \eta_j = 1 \) in \( U_k \) and then \( \eta_j = 0 \) in \( U_k \) for all \( j \geq k + 1 \). Thus \( \eta_j \neq 0 \) for only finitely many \( j \in \mathbb{N} \) in a 1-fine neighborhood of \( x \), and \( \sum_{j=1}^{\infty} \eta_j = 1 \) 1-q.e. in \( U \). \( \square \)

**Remark 4.5** In an open set \( \Omega \), we can pick a Whitney covering consisting of balls \( B_j = B(x_j, r_j) \) that have radius comparable to the distance from \( x_j \) to \( X \setminus \Omega \), and then we can pick a Lipschitz partition of unity \( \{\eta_j\}_{j=1}^{\infty} \) subordinate to this covering. Then the discrete convolution approximation of a function \( u \in BV(\Omega) \) is defined by

\[ v := \sum_{j=1}^{\infty} u_{B_j} \eta_j \in Lip_{\text{loc}}(\Omega). \]

Using the Poincaré inequality (2.3), which is formulated for balls and thus lends itself well to estimating quantities involving integral averages such as \( u_{B_j} \), it can be shown that \( v \) has a 1-weak upper gradient of the form

\[ C \sum_{j=1}^{\infty} \frac{\|Du\|(B(x_j, 5\lambda r_j))}{\mu(B(x_j, 5\lambda r_j))} \chi_{B(x_j, r_j)}, \]

see e.g. the proof of [26, Proposition 4.1]. However, when \( \{\eta_j\}_{j=1}^{\infty} \) is instead a partition of unity in a 1-quasiopen set, the situation is more complicated, because we cannot take the functions \( \eta_j \) to be Lipschitz cutoff functions corresponding to balls \( B_j \). The above definition of the discrete convolution used the integral averages \( u_{B_j} \), but in the quasiopen case it is not clear how to use integral averages and how to then apply the Poincaré inequality when estimating the upper gradient. Instead we will make use of the preliminary approximation results and other machinery developed in Sect. 3.
The following theorem gives the discrete convolution technique in 1-quasiopen sets.

**Theorem 4.6** Let $U \subset \Omega \subset X$ be such that $U$ is 1-quasiopen and $\Omega$ is open, and let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(|\Omega|) < \infty$. Let $0 < \varepsilon < 1$. Then we find a partition of unity $\{\eta_j \in N_{c,1}(U)\}_{j=1}^{\infty}$ in $U$ and functions $u_j \in N_{1,1}(\{\eta_j > 0\})$ such that the function

$$v := \sum_{j=1}^{\infty} \eta_j u_j$$

satisfies $\|v - u\|_{L^1(U)} < \varepsilon$, $\int_U g v \, d\mu < \|Du\|(U) + \varepsilon$, and

$$\sup_{U} |v - u|^{\vee} \leq 9 \sup_{U} (u^{\vee} - u^{\wedge}) + \varepsilon. \quad (4.2)$$

Moreover, understanding $v - u$ to be zero extended to $X \setminus U$, we have

$$\|D(v - u)(X) < 2 \|Du\|(U) + \varepsilon \quad \text{and} \quad \|D(v - u)(X \setminus U) = 0, \quad (4.3)$$

and $|v - u|^{\vee} = 0$ $\mathcal{H}$-a.e. in $X \setminus U$.

Note that we may have $\sup_j (u^{\vee} - u^{\wedge}) = \infty$ and then (4.2) is vacuous. The conditions $\|D(v - u)(X \setminus U) = 0$ and $|v - u|^{\vee} = 0$ $\mathcal{H}$-a.e. in $X \setminus U$ essentially say that $v$ and $u$ have the same “boundary values”. This is the crucial new property that we obtain compared with Proposition 3.3, because it says that $v$ can always be “glued” nicely with $u$, in the sense of (5.5) in the next section.

Since the proof will be long, we first give an outline. We start out with a quasicovering of $U$ given by Proposition 4.3, adapted to the function $u$, and then we collect sufficiently (but finitely) many of the sets in the covering so that the $\|Du\|$-measure of the remainder is small. We take the union of this finite collection to be the first set in a new quasicovering, keeping the remaining sets as they are. Then we use Proposition 4.4 to obtain a corresponding partition of unity $\{\eta_j\}_{j=1}^{\infty}$ which, it should be noted, now depends on the function $u$, contrary to the case of the usual discrete convolutions in open sets. Then we use Proposition 3.3 to obtain suitable functions $u_j$ approximating $u$ in $\{\eta_j > 0\}$. Finally, via some careful analysis that utilizes the results of Sect. 3, we prove that $v = \sum_{j=1}^{\infty} \eta_j u_j$ satisfies all of the desired properties.

**Proof** First we choose a suitable partition of unity in $U$. By Proposition 4.3 we find a 1-strict quasicovering $\{\tilde{U}_j\}_{j=1}^{\infty}$ of $U$, and associated Newton–Sobolev functions $\{\tilde{\rho}_j \in N_{c,1}(U)\}_{j=1}^{\infty}$, $0 \leq \tilde{\rho}_j \leq 1$, such that

$$-\infty < \inf_{\text{spt} \tilde{\rho}_j} u^{\wedge} \leq \sup_{\text{spt} \tilde{\rho}_j} u^{\vee} < \infty$$

for all $j \in \mathbb{N}$. Since $\tilde{\rho}_j = 1$ in the 1-finely open set $\tilde{U}_j$ for each $j \in \mathbb{N}$, we have $\bigcup_{j=1}^{k} \tilde{U}_j \subset \text{fine-int} \{\max_{j \in \{1,\ldots,k\}} \tilde{\rho}_j = 1\}$ for each $k \in \mathbb{N}$. Now by the fact that $\{\tilde{U}_j\}_{j=1}^{\infty}$ is a 1-quasicovering of $U$ and by Lemma 3.4,
\[ \|Du\|\left(U \setminus \text{fine-int}\left\{ \max_{j \in \{1, \ldots, k\}} \tilde{\rho}_j = 1 \right\} \right) \leq \|Du\|\left(U \setminus \bigcup_{j=1}^{k} \tilde{U}_j \right) \]

\[ \overset{k \to \infty} \|Du\|\left(U \setminus \bigcup_{j=1}^{\infty} \tilde{U}_j \right) = 0; \]

note that 1-quasiopen sets are easily seen to be \(1\)-measurable by using Lemma 3.4, see [31, Lemma 3.5]. Thus for some \(N \in \mathbb{N}\), we have

\[ \|Du\|\left(U \setminus \text{fine-int}\left\{ \max_{j \in \{1, \ldots, N\}} \tilde{\rho}_j = 1 \right\} \right) < \frac{\varepsilon}{8C_0}, \]

where \(C_0\) is the constant from Theorem 3.8. Now define \(U_1 := \bigcup_{j=1}^{N} \tilde{U}_j\), \(U_j := \tilde{U}_{N+1+j}\) for \(j = 2, 3, \ldots\), \(\rho_1 := \max_{j \in \{1, \ldots, N\}} \tilde{\rho}_j\), and \(\rho_j := \tilde{\rho}_{N+1+j}\) for \(j = 2, 3, \ldots\). Then \(\{U_j\}_{j=1}^{\infty}\) is another 1-strict quasicovering of \(U\) with associated Newton–Sobolev functions \(\rho_j \in N^{1,1}_c(U)\), such that \(0 \leq \rho_j \leq 1\) and

\[ -\infty < \inf_{\text{spt} \rho_j} u^{\wedge} \leq \sup_{\text{spt} \rho_j} u^{\vee} < \infty \]

for all \(j \in \mathbb{N}\). Moreover,

\[ \|Du\|\left(U \setminus \text{fine-int}\{\rho_1 = 1\} \right) < \frac{\varepsilon}{8C_0}. \]

Then by Proposition 4.4 we find a nonnegative, 1-finely locally finite partition of unity \(\{\eta_j \in N^{1,1}_c(U)\}_{j=1}^{\infty}\) in \(U\) such that

\[ -\infty < \inf_{\text{spt} \eta_j} u^{\wedge} \leq \sup_{\text{spt} \eta_j} u^{\vee} < \infty \] (4.4)

for all \(j \in \mathbb{N}\). Moreover, \(\eta_1 = \rho_1\) and so

\[ \|Du\|\left(U \setminus \text{fine-int}\{\eta_1 = 1\} \right) < \frac{\varepsilon}{8C_0}. \] (4.5)

(In the rest of the proof, any other partition of unity satisfying the properties mentioned in this paragraph would also work.)

For each \(j \in \mathbb{N}\), since we have \(\text{spt} \eta_j \subseteq \Omega\), there exists an open set \(\Omega_j\) with \(\text{spt} \eta_j \subseteq \Omega_j \subseteq \Omega\), and then \(u \in BV(\Omega_j)\). Since every function \(\eta_j \in N^{1,1}_c(U) \subseteq N^{1,1}(X)\) is 1-quasicontinuous, every set \(\{\eta_j > 0\}\) is 1-quasiopen. Now by Proposition 3.3 we find sequences \((u_{j,i}) \subset N^{1,1}(\{\eta_j > 0\})\) such that \(u_{j,i} \to u\) in \(L^1(\{\eta_j > 0\})\),

\[ \sup_{\{\eta_j > 0\}} |u_{j,i} - u^{\vee}| \leq 9 \sup_{\{\eta_j > 0\}} (u^{\vee} - u^{\wedge}) + \varepsilon < \infty \text{ (by (4.4))} \] (4.6)

for all \(i \in \mathbb{N}\), and

\[ \lim_{i \to \infty} \int_{\{\eta_j > 0\}} g_{u_{j,i}} d\mu = \|Du\|(\{\eta_j > 0\}), \]

where each \(g_{u_{j,i}}\) denotes (here and later) the minimal 1-weak upper gradient of \(u_{j,i}\) in \(\{\eta_j > 0\}\). By passing to subsequences (not relabeled), we can also assume that \(u_{j,i} \to u\) a.e. in \(\{\eta_j > 0\}\). For any set \(W \subset X\), the function \(\chi_{\overline{W}}\) is 1-quasi upper semicontinuous by Theorem 2.5.
and Lemma 3.9, and then the function $\eta_j x_{W^1}$ is also 1-quasi upper semicontinuous. Thus by Proposition 3.5 we get

$$\limsup_{i \to \infty} \int_{\{\eta_j > 0\} \cap W^1} \eta_j g u_{j,i} \, d\mu \leq \int_{\{\eta_j > 0\} \cap W^1} \eta_j \|Du\|$$  \hspace{1cm} (4.7)

for each $j \in \mathbb{N}$. By a suitable choice of indices $i(j) \in \mathbb{N}$, for each $j \in \mathbb{N}$ we have with $u_j := u_{j,i(j)}$ that $u_j \in N^{1,1}(\{\eta_j > 0\})$,

$$\sup_{\{\eta_j > 0\}} |u_j - u^\vee| \leq 9 \sup_{\{\eta_j > 0\}} (u^\vee - u^\wedge) + \varepsilon < \infty,$$

$$\|u_j - u\|_{L^1(\{\eta_j > 0\})} < 2^{-j} \varepsilon,$$

and

$$\int_{\{\eta_j > 0\} \cap W_k^1} |u_j - u| g_{\eta_j} \, d\mu < \frac{2^{-j-2}\varepsilon}{C_0},$$  \hspace{1cm} (4.8)

where the last inequality is achieved by Lebesgue’s dominated convergence theorem, exploiting the boundedness of $\sup_{\{\eta_j > 0\}} |u_j,i - u^\vee|$. Here $g_{\eta_j}$ is the minimal 1-weak upper gradient of $\eta_j$ in $X$. Define $W_0 := X$, $W_1 := X \setminus \{\eta_1 = 1\}$, and

$$W_k := X \setminus \bigcup_{j=1}^k \text{spt } \eta_j \quad \text{for } k = 2, 3, \ldots$$

By (4.7) we can also assume for each $j \in \mathbb{N}$

$$\int_{\{\eta_j > 0\} \cap W_k^1} \eta_j g u_{j,i} \, d\mu < \int_{\{\eta_j > 0\} \cap W_k^1} \eta_j \|Du\| + \frac{2^{-j-2}\varepsilon}{C_0}$$  \hspace{1cm} (4.10)

for the (finite number of) choices $k = 0, \ldots, j$. Using Lebesgue’s dominated convergence theorem as above, we have

$$\lim_{i \to \infty} \int_{\{\eta_j > 0\}} |u_j - u_{j,i}| g_{\eta_j} \, d\mu = \int_{\{\eta_j > 0\}} |u_j - u| g_{\eta_j} \, d\mu.$$  \hspace{1cm} (4.11)

By the definition of $v$ given in (4.1) and by (4.8), (4.9), we clearly have

$$\sup_U |v - u^\vee| \leq 9 \sup_U (u^\vee - u^\wedge) + \varepsilon$$

and

$$\|v - u\|_{L^1(U)} = \left\| \sum_{j=1}^\infty \eta_j (u_j - u) \right\|_{L^1(U)} \leq \sum_{j=1}^\infty \|u_j - u\|_{L^1(\{\eta_j > 0\})} < \varepsilon,$$

as desired. Similarly,

$$\left\| \sum_{j=1}^\infty \eta_j |u_j - u| \right\|_{L^1(U)} \leq \sum_{j=1}^\infty \|u_j - u\|_{L^1(\{\eta_j > 0\})} < \varepsilon,$$

and so in particular $\sum_{j=1}^\infty \eta_j |u_j - u| \in L^1(U)$ and thus we have

$$\sum_{j=1}^l \eta_j (u_j - u) \to v - u \quad \text{in } L^1(U) \quad \text{as } l \to \infty$$  \hspace{1cm} (4.12)

by Lebesgue’s dominated convergence theorem.
Moreover, for every \( j \in \mathbb{N} \) we have \( \eta_j(u_j - u_{j,i}) \to \eta_j(u_j - u) \) in \( L^1(X) \). By (4.4), (4.6), and (4.8), we know that
\[
\eta_j(u_j - u_{j,i}) \quad \text{and} \quad \eta_j(u_j - u) \quad \text{are bounded in} \quad \{ \eta_j > 0 \}. \tag{4.13}
\]
Thus by the lower semicontinuity of the total variation with respect to \( L^1 \) convergence and by Lemma 3.6, we get for any open set \( W \subset X \) (in fact any 1-quasiopen set, see comment below)
\[
\| D(\eta_j(u_j - u)) \|(W) \leq \liminf_{i \to \infty} \int_W g_{\eta_j(u_j - u_{j,i})} \, d\mu
\]
\[
\leq \liminf_{i \to \infty} \left( \int_{\{ \eta_j > 0 \}} |u_j - u_{j,i}| g_{\eta_j} \, d\mu + \int_{W \cap \{ \eta_j > 0 \}} \eta_j(g_{u_j} + g_{u_{j,i}}) \, d\mu \right) \tag{4.14}
\]
\[
\leq \int_{\{ \eta_j > 0 \}} |u_j - u| g_{\eta_j} \, d\mu + \int_{W \cap \{ \eta_j > 0 \}} \eta_j g_{u_j} \, d\mu + \int_{W \cap \{ \eta_j > 0 \}} \| \eta_j \| d\| Du \|
\]
by (4.11) and (4.7). Note that with \( W = X \), all the terms on the right-hand side are finite, and so \( \| D(\eta_j(u_j - u)) \|(X) < \infty \) and then by Theorem 2.1 the above holds also for 1-quasiopen \( W \). For \( k \in \mathbb{N} \), note that
\[
\eta_j = 0 \quad \text{for} \quad j = 1, \ldots, k - 1 \quad \text{in} \quad W_k \tag{4.15}
\]
and that the set \( W_1 \) is 1-quasiopen by the quasicontinuity of \( \eta_1 \), while the sets \( W_2, W_3, \ldots \) are open. Using (2.12), we get for all \( k, l \in \mathbb{N}, l \geq k, \)
\[
\| D\left( \sum_{j=1}^l \eta_j(u_j - u) \right) \|(W_k) \leq \sum_{j=1}^l \| D(\eta_j(u_j - u)) \|(W_k)
\]
\[
= \sum_{j=k}^l \| D(\eta_j(u_j - u)) \|(W_k) \quad \text{by} \quad (4.15)
\]
\[
\leq \sum_{j=k}^l \int_{\{ \eta_j > 0 \}} |u_j - u| g_{\eta_j} \, d\mu + \sum_{j=k}^l \int_{W_k \cap \{ \eta_j > 0 \}} \eta_j g_{u_j} \, d\mu \quad \text{by} \quad (4.14)
\]
\[
+ \sum_{j=k}^l \int_{W_k \cap \{ \eta_j > 0 \}} \eta_j d\| Du \|
\]
\[
< \frac{1}{C_0} \sum_{j=k}^l 2^{-j-2\varepsilon} + 2 \sum_{j=k}^l \int_{W_k \cap \{ \eta_j > 0 \}} \eta_j d\| Du \| + \frac{1}{C_0} \sum_{j=k}^l 2^{-j-2\varepsilon} \quad \text{by} \quad (4.9), \tag{4.10}
\]
\[
= 2 \sum_{j=k}^l \int_{W_k \cap \{ \eta_j > 0 \}} \eta_j d\| Du \| + \frac{2^{-k\varepsilon}}{C_0}.
\]

For \( k = 0 \) (recall that \( W_0 = X \)) and any \( 1 \leq m \leq l \) we get by essentially the same calculation
\[
\left\| D\left( \sum_{j=m}^l \eta_j(u_j - u) \right) \right\|(X) < 2 \sum_{j=m}^l \int_U \eta_j d\| Du \| + 2^{-m\varepsilon}. \tag{4.17}
\]
By (4.12) we had \( \sum_{j=1}^{l} \eta_j(u_j - u) \rightarrow v - u \) in \( L^1(U) \), so understanding \( v - u \) to be zero extended to \( X \setminus U \), we now get by lower semicontinuity of the total variation with respect to \( L^1 \)-convergence,

\[
\|D(v-u)\|(X) \leq \liminf_{l \to \infty} \left\| D\left( \sum_{j=1}^{l} \eta_j(u_j - u) \right) \right\|(X)
\]

\[
\leq 2 \sum_{j=1}^{\infty} \int_{U} \eta_j \|Du\| + 2^{-1} \epsilon
\]

\[
= 2 \|Du\|(U) + 2^{-1} \epsilon,
\]

proving the first inequality in (4.3). Now by Theorem 2.2 and (4.16), we have for each \( k \in \mathbb{N} \)

\[
\|D(v-u)\|(W_k) \leq \liminf_{l \to \infty} \left\| D\left( \sum_{j=1}^{l} \eta_j(u_j - u) \right) \right\|(W_k)
\]

\[
\leq 2 \sum_{j=k}^{\infty} \int_{W_k \cap \{\eta_j > 0\}} \eta_j \|Du\| + \frac{2^{-k} \epsilon}{C_0}
\]

\[
\leq 2 \sum_{j=k}^{\infty} \int_{U \setminus \text{fine-int}\{\eta_1 = 1\}} \eta_j \|Du\| + \frac{2^{-k} \epsilon}{C_0}.
\]

Note that \( \sum_{j=k}^{\infty} \eta_j \rightarrow 0 \) 1-q.e. in \( U \) as \( k \rightarrow \infty \), and then also \( \|Du\|\)-a.e. in \( U \) by Lemma 3.4. Since \( W_k \supset X \setminus U \) for all \( k \in \mathbb{N} \), by Lebesgue’s dominated convergence theorem we now get \( \|D(v-u)\|(X \setminus U) = 0 \), proving the second inequality in (4.3).

Moreover, \( h_l := \sum_{j=1}^{l} \eta_j(u_j - u) \) is a Cauchy sequence in \( BV(X) \), since by (4.17) we get for any \( 1 \leq m < l \)

\[
\left\| D\left( \sum_{j=m}^{l} \eta_j(u_j - u) \right) \right\|(X) < 2 \sum_{j=m}^{\infty} \int_{U} \eta_j \|Du\| + 2^{-m} \epsilon \rightarrow 0
\]

as \( m \rightarrow \infty \). Thus \( h_l \rightarrow v - u \) in \( BV(X) \) (and not just in \( L^1(X) \) as noted in (4.12)). Since each \( h_l \) has compact support in \( U \) and thus \( h_l^\wedge = 0 = h_l^\gamma \) in \( X \setminus U \), by Lemma 3.7 it follows that \( (v-u)^\wedge(x) = 0 = (v-u)^\gamma(x) \) for \( \mathcal{H}\)-a.e. \( x \in X \setminus U \), and so also \( |v-u|^\gamma(x) = 0 \) for \( \mathcal{H}\)-a.e. \( x \in X \setminus U \), as desired.

Since the partition of unity \( \{\eta_j\}_{j=1}^{\infty} \) is 1-finely locally finite, the sets \( V_k := \text{fine-int} \left\{ \sum_{j=1}^{k} \eta_j = 1 \right\} \) cover 1-quasi all of \( U \). Moreover, \( v \in N^{1,1}(V_k) \) for all \( k \in \mathbb{N} \); this follows from the fact that \( v \) in \( V_k \) is the finite sum \( \sum_{j=1}^{k} \eta_j u_j \), which is in \( N^{1,1}(X) \) by Lemma 3.6 and (4.13). Let \( A \subset U \) such that \( \mu(A) = 0 \). By Theorem 2.5, each \( V_k \) is 1-quasiopen and then by Lemma 3.10 we have \( \|Dv\|(A \cap V_k) = 0 \) for all \( k \in \mathbb{N} \) (note that \( \|Dv\|(\Omega) < \infty \) by the first inequality in (4.3), understanding \( v \) to be extended to \( \Omega \cup U \) as \( u \)). Thus using also Lemma 3.4,

\[
\|Dv\|(A) \leq \|Dv\|(A \cap \bigcup_{k=1}^{\infty} V_k) + \|Dv\|(A \setminus \bigcup_{k=1}^{\infty} V_k) = 0.
\]

Thus \( \|Dv\| \) is absolutely continuous with respect to \( \mu \) in \( U \), and so by Theorem 3.8 we know that a modification \( \widehat{v} \) of \( v \) in a \( \mu \)-negligible subset of \( U \) satisfies \( \widehat{v} \in N^{1,1}_{loc}(U) \) such that
\( \int_H g_\hat{v} \, d\mu \leq C_0 \| Dv \|(H) \) for every \( \mu \)-measurable \( H \subset U \), where \( g_\hat{v} \) is the minimal 1-weak upper gradient of \( \hat{v} \) in \( U \). Now for each \( k \in \mathbb{N} \), \( v \) and \( \hat{v} \) are both 1-quasicontinuous on the 1-quasiopen set \( V_k \) by (2.6), with \( v = \hat{v} \) a.e. in \( V_k \), and so by (2.7) we have in fact \( v = \hat{v} \) 1-q.e. in \( V_k \). Thus \( v = \hat{v} \) 1-q.e. in \( U \) and then by (2.4) we can in fact let \( \hat{v} = v \) everywhere in \( U \).

By [6, Proposition 3.5] and [40, Remark 3.5] we know that \( g_{v, [\eta_1 > 0]} = g_v \) a.e. in \( [\eta_1 > 0] \), that is, it does not make a difference whether we consider the minimal 1-weak upper gradient of \( v \) a.e. in \( U \) or in the smaller 1-quasiopen set \( [\eta_1 > 0] \). Then by (3.2) we have \( g_{u_1} = g_{v, [\eta_1 > 0]} = g_v \) a.e. in \( [\eta_1 = 1] \). It follows that

\[
\int_U g_v \, d\mu \leq \int_{\{\eta_1 = 1\}} g_v \, d\mu + \int_{U \setminus \{\eta_1 = 1\}} g_v \, d\mu
\leq \int_{\{\eta_1 = 1\}} g_{u_1} \, d\mu + C_0 \| Dv \|(U \cap W_1)
\leq \int_{\{\eta_1 = 1\}} g_{u_1} \, d\mu + C_0 \| D(v - u) \|(W_1) + C_0 \| Du \|(U \cap W_1)
\leq \int_{\{\eta_1 = 1\}} g_{u_1} \, d\mu + 3C_0 \| Du \|(U \setminus \text{fine-int}\{\eta_1 = 1\}) + 2^{-1}\varepsilon \quad \text{by (4.18)}
\leq \| Du \|(U) + \varepsilon
\]

by (4.10) with the choices \( W_k = X \) and \( j = 1, \) and (4.5).

\[\square\]

Remark 4.7 Note that in the usual discrete convolution technique described in Remark 4.5, we only get the estimate \( \int_{\Omega} g_v \, d\mu \leq C \| Du \|(\Omega) \) for some constant \( C \geq 1 \) depending on the doubling and Poincaré constants, whereas in Theorem 4.6 we obtained \( \int_{\Omega} g_v \, d\mu \leq \| Du \|(\Omega) + \varepsilon \). Thus our technique may seem to be an improvement on the usual discrete convolution technique already in open sets, but in fact the (usual) discrete convolutions have other good properties, in particular the uniform integrability of the upper gradients in the case where \( \| Du \| \) is absolutely continuous with respect to \( \mu \), see [17, Lemma 6]. The uniform integrability is seemingly more difficult to obtain in the quasiopen case, but it is also perhaps not interesting for the following reason: if \( \| Du \| \) is absolutely continuous in a 1-quasiopen set \( U \), then Theorem 3.8 (whose proof is based on discrete convolutions) already tells that \( u \in N_{\text{loc}}^{1, 1}(U) \), and so it is not interesting to approximate \( u \) with functions \( v \in N_{\text{loc}}^{1, 1}(U) \) given by Theorem 4.6.

5 An approximation result

In this section we apply the discrete convolution technique of the previous section to prove a new approximation result for BV functions, given in Theorem 1.11 of the introduction. In this result we approximate a BV function in the BV and \( L^\infty \) norms by BV functions whose jump sets are of finite Hausdorff measure.

First we note that without the requirement of approximation in the \( L^\infty \) norm, the theorem could be proved by using standard discrete convolutions. Indeed, if \( \Omega \subset X \) is an open set and \( u \in BV(\Omega) \), we can take a suitable open set \( W \subset \Omega \) containing the part of the jump set \( S_u = \{ u^{-} > u^{+} \} = \{ x \in \Omega : u^{-}(x) > u^{+}(x) \} \) where the size of the jump \( u^{-} - u^{+} \) is small, and then we can take a discrete convolution of \( u \) in \( W \). By gluing this with the function \( u \) in \( \Omega \setminus W \), we get the desired approximation; we omit the details but the essential aspects of this kind of technique are given in [36, Corollary 3.6]. However, the open set \( W \) may unavoidably
contain also large jumps of $u$, and so it seems impossible to obtain approximation in the $L^\infty$ norm with this method. We sketch this problem in the following example.

**Example 5.1** Let $X = \mathbb{R}^2$ (unweighted) and $\Omega := (0, 1) \times (0, 1)$. Define the strips

$$A_j := \{x = (x_1, x_2) \in \mathbb{R}^2 : 2^{-j} \leq x_1 < 2^{-j+1}, 0 < x_2 < 1\}, \quad j \in \mathbb{N},$$

and the function

$$v := \sum_{j=1}^{\infty} j^{-1} \chi_{A_j} \in BV(\Omega).$$

Also take a function $w \in BV(\Omega)$, $0 \leq w \leq 1$, for which

$$S_w = \{x \in \Omega : w^\vee(x) - w^\wedge(x) = 1\}$$

is dense in $\Omega$; such a function can be obtained as the characteristic function of a set that is constructed roughly as follows. Take a fat Sierpinski carpet having finite perimeter. In each of the removed squares, add the complement of a fat Sierpinski carpet with finite perimeter. In each of the thus added squares, remove the complement of a fat Sierpinski carpet with finite perimeter, and so on. If the perimeters go to zero fast enough, at the limit we obtain a set of finite perimeter of the desired type.

Then let $u := v + w \in BV(\Omega)$. Denote by $\mathcal{H}^1$ the 1-dimensional Hausdorff measure; note that this is comparable to the codimension one Hausdorff measure. Since $\mathcal{H}^1(S_w) = \infty$ and $\mathcal{H}^1(S_w^c) < \infty$ (otherwise $\|Dw\|((\Omega) = \infty$ by (2.13)), clearly $\mathcal{H}^1(S_u) = \infty$. Suppose we take an open set $W \subseteq \Omega$ containing the set $\{u^\vee - u^\wedge < \delta\}$ for some (small) $\delta > 0$.

Then $W$ is nonempty and so contains a point $x \in \{w^\vee - w^\wedge = 1\}$, and then clearly also $x \in \{u^\vee - u^\wedge \geq 1/2\}$. If $h$ is a continuous function in $W$ (for example if $h$ is a discrete convolution of $u$), then it is straightforward to check that $\|h - u\|_{L^\infty(W)} \geq 1/4$ and thus we do not have approximation in the $L^\infty$ norm.

To prove the approximation result, we need the following lemma; recall the definition of the measure-theoretic interior from (2.10).

**Lemma 5.2** Let $U \subset X$ be 1-quasiopen. Then $\mathcal{H}(U \setminus I_U) = 0$.

**Proof** By Theorem 2.5 we find a 1-finely open set $V \subset U$ such that $\mathcal{H}(U \setminus V) = 0$. By (2.14), $V \subset I_V$, and then obviously $V \subset I_U$. \hfill $\square$

First we give the approximation result in the following form containing more information than Theorem 1.1, whose proof is then given afterwards. The symbol $\Delta$ denotes the symmetric difference.

**Theorem 5.3** Let $\Omega \subset X$ be open, let $u \in BV(\Omega)$, and let $\varepsilon, \delta > 0$. Then we find $w \in BV(\Omega)$ such that $\|w - u\|_{L^1(\Omega)} < \varepsilon$,

$$\|D(w - u)\|((\Omega)) < 2\|Du\|(\{0 < u^\vee - u^\wedge < \delta\}) + \varepsilon,$$

$$\|w - u\|_{L^\infty(\Omega)} \leq 10\delta, \quad \mathcal{H}(S_w \Delta \{u^\vee - u^\wedge \geq \delta\}) = 0,$$

and

$$\|Du\|(\{0 < u^\vee - u^\wedge < \delta\}) + \varepsilon (5.1)$$

$$\mu(\{0 < u^\vee - u^\wedge < \delta\}) + \varepsilon (5.2)$$

and

$$\mu(\{0 < u^\vee - u^\wedge < \delta\}) < \varepsilon.$$
Proof Take an open set $W$ such that $\{0 < u^\vee - u^\wedge < \delta\} \subset W \subset \Omega$,
\[ \|Du\|(W) < \|Du\|(\{0 < u^\vee - u^\wedge < \delta\}) + \varepsilon/4, \]
and $\mu(W) < \varepsilon$; recall that by the decomposition (2.13), the jump set $S_u$ is $\sigma$-finite with respect to $\mathcal{H}$ and thus $\mu(S_u) = 0$. By Proposition 2.3, the set $\{u^\vee - u^\wedge < \delta\}$ is 1-quasiopen, and then so is $U := W \cap \{u^\vee - u^\wedge < \delta\}$. Moreover,
\[ \|Du\|(U) \leq \|Du\|(W) < \|Du\|(\{0 < u^\vee - u^\wedge < \delta\}) + \varepsilon/4. \tag{5.3} \]
By Theorem 4.6 we find a function $v \in N^{1,1}(U)$ satisfying $\|v - u\|_{L^1(U)} < \varepsilon$,
\[ \sup_U |v - u^\vee| \leq 9 \sup_U (u^\vee - u^\wedge) + \delta \leq 10\delta, \]
and, understanding $v - u$ to be zero extended to $X \setminus U$,
\[ \|D(v - u)\|(X) < 2\|Du\|(U) + \varepsilon/2. \tag{5.4} \]
By Lebesgue’s differentiation theorem, now also $\|v - u\|_{L^\infty(U)} \leq 10\delta$. Define
\[ w := \begin{cases} v & \text{in } U, \\ u & \text{in } \Omega \setminus U. \end{cases} \tag{5.5} \]
Then $\|w - u\|_{L^1(\Omega)} < \varepsilon$ and $\|w - u\|_{L^\infty(\Omega)} \leq 10\delta$. From (5.4), (5.3) we get
\[ \|D(w - u)\|(\Omega) < 2\|Du\|(U) + \varepsilon/2 < 2\|Du\|(\{0 < u^\vee - u^\wedge < \delta\}) + \varepsilon, \]
as desired. The function $v$ is 1-quasicontinuous on the 1-quasiopen set $U$ by (2.6), and then also 1-finely continuous 1-q.e. in $U$ by Theorem 2.6. By Lemma 5.2 we also have $x \in I_U$ for $\mathcal{H}$-a.e. $x \in U$. By (2.5), $\mathcal{H}$-a.e. $x \in U$ satisfies both these properties, and then by (2.14) we find that $w^\wedge(x) = w^\vee(x)$. Thus $\mathcal{H}(S_u \cap U) = 0$.

By definition of $U$ we have $\{u^\vee - u^\wedge \geq \delta\} = S_u \setminus U$. Since $|w - u|^\vee = 0$ $\mathcal{H}$-a.e. in $\Omega \setminus U$ by Theorem 4.6, we have $u^\wedge = w^\wedge$ and $u^\vee = w^\vee$ $\mathcal{H}$-a.e. in $\Omega \setminus U$, and so the sets $\{u^\vee - u^\wedge \geq \delta\}$ and $S_u \setminus U$ coincide outside a $\mathcal{H}$-negligible set. In total, $\mathcal{H}(S_u \Delta \{u^\vee - u^\wedge \geq \delta\}) = 0$, as desired.

Since $|w - u|^\vee = 0$ $\mathcal{H}$-a.e. in $\Omega \setminus U$, this holds also $\mu$-a.e. and $\|Du\|$-a.e. in $\Omega \setminus U$ (recall (2.5) and Lemma 3.4). Thus we get estimates (5.1) and (5.2). □

Proof of Theorem 1.1 For each $i \in \mathbb{N}$, choose the function $u_i$ to be $w \in BV(\Omega)$ as given by Theorem 5.3 with the choices $\varepsilon = 1/i$ and $\delta = 1/i$. Then $\|u_i - u\|_{L^1(\Omega)} < 1/i$ and
\[ \|D(u_i - u)\|(\Omega) < 2\|Du\|(\{0 < u^\vee - u^\wedge < 1/i\}) + 1/i \to 0 \text{ as } i \to \infty, \]
and so $\|u_i - u\|_{BV(\Omega)} \to 0$ as $i \to \infty$. Also, $\|u_i - u\|_{L^\infty(\Omega)} \to 10/i \to 0$ as desired. By the decomposition (2.13) we find that $\mathcal{H}(\{u^\vee - u^\wedge \geq 1/i\}) < \infty$ for all $i \in \mathbb{N}$ and so
\[ \mathcal{H}(S_{u_i}) = \mathcal{H}(\{u^\vee - u^\wedge \geq 1/i\}) < \infty \]
for all $i \in \mathbb{N}$. □

We observe that the proofs of Theorems 5.3 and 1.1 were quite straightforward, because most of the hard work was already done in the proof of the discrete convolution technique, Theorem 4.6. Since Theorem 4.6 can be applied rather easily in any 1-quasiopen set, we expect that it will be useful also in the context of other problems, for example if one considers minimization problems in 1-quasiopen domains.
We say that $u \in BV(\Omega)$ is a *special function of bounded variation*, and denote $u \in SBV(\Omega)$, if the Cantor part of the variation measure vanishes, i.e. $\|Du\|c(\Omega) = 0$. The following approximation result was proved (with some more details) in [32, Corollary 5.15].

**Theorem 5.4** Let $\Omega \subset X$ be open and let $u \in BV(\Omega)$. Then there exists a sequence $(u_i) \subset SBV(\Omega)$ such that

- $u_i \to u$ in $L^1(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$,
- $\lim_{i \to \infty} \|D(u_i - u)\|(\Omega) = 2\|Du\|c(\Omega)$,
- $\limsup_{i \to \infty} \|Du\|(\{|u_i - u|\neq 0\}) \leq \|Du\|c(\Omega)$ and 
  $\lim_{i \to \infty} \mu(\{|u_i - u|\neq 0\}) = 0$,
- $\lim_{i \to \infty} \|u_i - u\|_{L^\infty(\Omega)} = 0$, and
- $\mathcal{H}(S_{u_i} \setminus S_u) = 0$ for all $i \in \mathbb{N}$.

Combining this with Theorem 5.3, we get the following corollary.

**Corollary 5.5** Let $\Omega \subset X$ be open and let $u \in BV(\Omega)$. Then there exists a sequence $(u_i) \subset SBV(\Omega)$ with $\mathcal{H}(S_{u_i}) < \infty$ for all $i \in \mathbb{N}$, such that

- $u_i \to u$ in $L^1(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$,
- $\lim_{i \to \infty} \|D(u_i - u)\|(\Omega) = 2\|Du\|c(\Omega)$,
- $\limsup_{i \to \infty} \|Du\|(\{|u_i - u|\neq 0\}) \leq \|Du\|c(\Omega)$ and 
  $\lim_{i \to \infty} \mu(\{|u_i - u|\neq 0\}) = 0$, and
- $\lim_{i \to \infty} \|u_i - u\|_{L^\infty(\Omega)} = 0$.

The first condition in the corollary is often expressed by saying that the $u_i$’s converge to $u$ in the *strict sense*, whereas the second condition describes closeness in the BV norm. The third condition describes approximation in the Lusin sense. In all, the corollary states that we can always approximate a BV function in a rather strong sense with functions that have neither a Cantor part of the variation measure nor a large jump set.

**Acknowledgements** Part of the research for this paper was conducted while the author was visiting Aalto University and the University of Cincinnati; he wishes to thank Juha Kinnunen and Nageswari Shanmugalingam for the kind invitations. The author also wishes to thank the referee for giving detailed and helpful feedback.

**References**

1. Adams, D., Hedberg, L.I.: Function spaces and potential theory. Grundlehren der Mathematischen Wissenschaften, p 314. Springer, Berlin (1996)
2. Ambrosio, L.: Fine properties of sets of finite perimeter in doubling metric measure spaces, calculus of variations, nonsmooth analysis and related topics. Set Valued Anal. 10(2–3), 111–128 (2002)
3. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs. The Clarendon Press, New York (2000)
4. Ambrosio, L., Miranda, M., Jr., Pallara, D.: Special functions of bounded variation in metric measure spaces, calculus of variations: topics from the mathematical heritage of E. De Giorgi, 1–45, Quad. Mat., 14, Dept. Math., Seconda Univ. Napoli, Caserta (2004)
5. Björn, A., Björn, J.: Nonlinear potential theory on metric spaces, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich (2011)
6. Björn, A., Björn, J.: Obstacle and Dirichlet problems on arbitrary nonopen sets in metric spaces, and fine topology. Rev. Mat. Iberoam. 31(1), 161–214 (2015)
7. Björn, A., Björn, J., Latvala, V.: Sobolev spaces, fine gradients and quasicontinuity on quasiopen sets. Ann. Acad. Sci. Fenn. Math. 41(2), 551–560 (2016)
8. Björn, A., Björn, J., Latvala, V.: The Cartan, Choquet and Kellogg properties for the fine topology on metric spaces. J. Anal. Math. 135(1), 59–83 (2018)
9. Björn, A., Björn, J., Latvala, V.: The weak Cartan property for the p-fine topology on metric spaces. Indiana Univ. Math. J. 64(3), 915–941 (2015)
10. Björn, A., Björn, J., Malý, J.: Quasiopen and p-path open sets, and characterizations of quasicontinuity. Potential Anal. 46(1), 181–199 (2017)
11. Björn, A., Björn, J., Shanmugalingam, N.: Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces. Houston J. Math. 34(4), 1197–1211 (2008)
12. Carriero, M., Dal Maso, G., Leaci, A., Pascali, E.: Relaxation of the nonparametric plateau problem with an obstacle. J. Math. Pures Appl. (9) 67(4), 359–396 (1988)
13. Coifman, R.R., Weiss, G.: Analyse harmonique non-commutative sur certaines espaces homogènes. Étude de certaines intégrales singulières. Lecture Notes in Mathematics, Vol. 242. Springer, Berlin (1971)
14. De Philippis, G., Fusco, N., Pratelli, A.: On the approximation of SBV functions. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 28(2), 369–413 (2017)
15. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics Series. CRC Press, Boca Raton (1992)
16. Federer, H.: Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, vol. 153. Springer, New York (1969)
17. Franchi, B., Hajłasz, P., Koskela, P.: Definitions of Sobolev classes on metric spaces. Ann. Inst. Fourier (Grenoble) 49(6), 1903–1924 (1999)
18. Fuglede, B.: The quasi topology associated with a countably subadditive set function. Ann. Inst. Fourier 21(1), 123–169 (1971)
19. Giusti, E.: Minimal Surfaces and Functions of Bounded Variation, Monographs in Mathematics, vol. 80. Birkhäuser Verlag, Basel (1984)
20. Hakkarainen, H., Kinnunen, J.: The BV-capacity in metric spaces. Manuscr. Math. 132(1–2), 51–73 (2010)
21. Hakkarainen, H., Kinnunen, J., Lahti, P., Lehtelä, P.: Relaxation and integral representation for functionals of linear growth on metric measure spaces. Anal. Geom. Metr. Spaces 4, 13 (2016)
22. Heinonen, J.: Lectures on Analysis on Metric Spaces, Universitext. Springer, New York (2001)
23. Heinonen, J., Kilpeläinen, T., Martio, O.: Nonlinear potential theory of degenerate elliptic equations, Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY (2006)
24. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. 181(1), 1–61 (1998)
25. Heikkinen, T., Koskela, P., Tuominen, H.: Sobolev-type spaces from generalized Poincaré inequalities. Studia Math. 181(1), 1–16 (2007)
26. Kinnunen, J., Korte, R., Shanmugalingam, N., Tuominen, H.: Pointwise properties of functions of bounded variation in metric spaces. Rev. Mat. Complut. 27(1), 41–67 (2014)
27. Korte, R., Lahti, P., Li, X., Shanmugalingam, N.: Notions of Dirichlet problem for functions of least gradient in metric measure spaces. Rev. Mat. Iberoam. 35(6), 1603–1648 (2019)
28. Lahti, P.: A Federer-style characterization of sets of finite perimeter in metric spaces. Calc. Var. Partial Differ. Equ. 56(5), 22 (2017), Art. 150
29. Lahti, P.: A new Cartan-type property and strict quasicoverings when $p = 1$ in metric spaces. Ann. Acad. Sci. Fenn. Math. Volumen 43, 1027–1043 (2018)
30. Lahti, P.: A notion of fine continuity for BV functions on metric spaces. Potential Anal. 46(2), 279–294 (2017)
31. Lahti, P.: A sharp Leibniz rule for BV functions in metric spaces. Rev. Mat. Complut. (2019). https://doi.org/10.1007/s13163-019-00341-y
32. Lahti, P.: Approximation of BV by SBV functions in metric spaces (2018). https://arxiv.org/abs/1806.04647
33. Lahti, P.: Quasiopen sets, bounded variation and lower semicontinuity in metric spaces. Potential Anal. (to appear)
34. Lahti, P.: Strong approximation of sets of finite perimeter in metric spaces. Manuscr. Math. 155(3–4), 503–522 (2018)
35. Lahti, P.: The Choquet and Kellogg properties for the fine topology when $p = 1$ in metric spaces. J. Math. Pures Appl. 126(9), 195–213 (2019)
36. Lahti, P., Shanmugalingam, N.: Fine properties and a notion of quasicontinuity for BV functions on metric spaces. Journal de Mathématiques Pures et Appliquées 107(2), 150–182 (2017)
37. Macías, R.A., Segovia, C.: A decomposition into atoms of distributions on spaces of homogeneous type. Adv. Math. 33(3), 271–309 (1979)
38. Malý, J., Ziemer, W.: Fine Regularity of Solutions of Elliptic Partial Differential Equations, Mathematical Surveys and Monographs, vol. 51. American Mathematical Society, Providence (1997)
39. Miranda Jr., M.: Functions of bounded variation on “good” metric spaces. J. Math. Pures Appl. (9) 82(8), 975–1004 (2003)
40. Shanmugalingam, N.: Harmonic functions on metric spaces. Ill. J. Math. 45(3), 1021–1050 (2001)
41. Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoam. 16(2), 243–279 (2000)
42. Whitney, H.: Analytic extensions of differentiable functions defined in closed sets. Trans. Am. Math. Soc. 36, 63–89 (1934)
43. Ziemer, W.P.: Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation, Graduate Texts in Mathematics, vol. 120. Springer, New York (1989)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.