EQUIVARIANT LS-CATEGORY OF TORUS MANIFOLDS

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ABSTRACT. We study the equivariant LS-category in terms of fixed point set and also compute LS-category and equivariant LS-category of torus manifolds over nice manifolds with corner. Moreover we compute equivariant LS-category of lens spaces.

1. INTRODUCTION

Let $G$ be a compact, Hausdorff, topological group, acting on a Hausdorff topological space $X$. In most cases $G$ is a Lie group acting on a compact manifold $X$. The equivariant LS-category of $X$, denoted by $\text{cat}_G(X)$ was introduced by Marzantowicz in [17], as a generalization of classical category of a space [16], which is called Lusternik-Schnirelmann category [15]. Marzantowicz showed that for a compact Lie group $G$, classical $\text{cat}$ of orbit space is a lower bound for $\text{cat}_G$,

$$\text{cat}(X/G) \leq \text{cat}_G(X).$$

Colman studied the $\text{cat}_G(X)$ for finite group $G$ in [5] and gave an upper bound in terms of the dimension of orbit space and $\text{cat}_G$ of the singular set for the action. In [14], Hurder and Töben proved that for a manifold $M$ with a proper $G$-action, where $G$ is a Lie group, the number of components of the fixed point set is a lower bound for $\text{cat}_G(M)$. Later $\text{cat}_G(X)$ is studied by Colman and Grant [6], for a compact Hausdorff topological group $G$, acting continuously on a Hausdorff space $X$.

Similar to definition of classical $\text{cat}$, $\text{cat}_G(X)$ is defined to be the least number of open subsets of $X$, which form a covering for $X$ and each open subset is equivariantly contractible to an orbit, rather than a point (see Definition 2.2). In this paper we study $\text{cat}_G(X)$, particularly for locally standard torus manifolds, which are even dimensional smooth manifolds with locally standard action by half-dimensional compact torus action (see Definition 3.2). In Section 2 we study $\text{cat}_G(X)$ in terms of $X^G$ and $\text{cat}_G(X^G)$, and some lower bound and upper bound for $\text{cat}_G(X)$ is given. In Section 3 some elementary results on locally standard torus manifolds are discussed. In Section 4 some results on classical $\text{cat}$ of quasitoric manifolds are given. We show that the equivariant connected sum in quasitoric manifolds does not

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affect on value of classical $\text{cat}$, i.e. for $2n$-dimensional quasitoric manifolds $M_1$ and $M_2$,  
\[ \text{cat}(M_1 \#_{\mathbb{T}^k} M_2) = \text{cat}(M_1) = \text{cat}(M_2) = n + 1 , \]
for any $k, n$ except $k = n = 2$. Besides we examine the situations that for 4-dimensional locally standard torus manifold $M$, the equality holds, means $\text{cat}(M) = 3$. Moreover the explicit construction of categorical covering for $M$ is given. In Section 5, $\text{cat}_{\mathbb{T}^n}$ of quasitoric manifolds, their equivariant connected sum, and their product with diagonal action are computed. Moreover we examined the exact value for $\text{cat}_{\mathbb{T}^2}$ of 4-dimensional locally standard torus manifolds, and the equivariant LS-category of lens spaces is computed in Section 6.

Some of the results in this paper are relevant to the work of Colman and Grant [6] in the following way. In their paper there are two statements on $\text{cat}_G$ of product, one with diagonal action, Theorem 3.15, and another with product action, Theorem 3.16. However the hypotheses are not sufficient and counterexamples may be found in Section 6.

2. Equivariant LS-category

In this section we prove a number of results for $\text{cat}_G(X)$ in terms of the fixed point set $X^G$. We begin by recalling some definitions and fixing some notations. Let $G$ be a compact Hausdorff topological group, acting continuously on a Hausdorff topological space $X$. In this case $X$ is called a $G$-space. For each $x \in X$, the set  
\[ O(x) = \{ g.x \mid g \in G \} \]
is called the orbit of $x$, and  
\[ G_x = \{ g \in G \mid g.x = x \} \]
is called the isotropy group or stabilizer of $x$. The set $X/G$ of all equivalence classes determined by the action, and equipped with the quotient topology is called the orbit space. The set  
\[ X^G = \left\{ x \in X \mid \forall g \in G, \, g.x = x \right\} \]
is called the fixed point set of $X$. Here $X^G$ is endowed with subspace topology.

**Definition 2.1.** Let $X$ be a topological space, and $G$ be a topological group acting on $X$.

1. An open subset $U$ of $X$, is called $G$-open set (or $G$-invariant) if $U$ is stable under $G$-action; i.e. $GU \subseteq U$.
2. Let $U$ be a $G$-invariant subset of $X$, the homotopy $H : U \times I \to X$ is called $G$-homotopy, if for every $g \in G$, $x \in U$, and $t \in I$,  
\[ gH(x, t) = H(gx, t). \]
(3) Let $U$ be a $G$-invariant subset of $X$, then $U$ is called $G$-categorical if there exists a $G$-homotopy $H : U \times I \to X$ such that $H(x,0) = x$ for each $x \in U$, and $H(U, 1)$ is a subset of an orbit.

**Definition 2.2.** A $G$-categorical covering for a $G$-space $X$ is a finite number of $G$-categorical subsets $\{ U_i \}_{i=1}^n$ that form a covering for $X$. The least value of $n$ for which such a covering exists, is called the equivariant category of $X$, denoted $\text{cat}_G(X)$. If no such covering exist, we write $\text{cat}_G(X) = \infty$.

**Lemma 2.3.** Let $U$ be a $G$-categorical subset of $G$-space $X$, which contains a fixed point $x_0 \in X^G$. Then $U$ is equivariantly contractible to $x_0$. In this case $U$ is called $G$-contractible, and denoted by $U \simeq_G x_0$.

**Proof.** Let $H : U \times I \to X$ be a $G$-homotopy, where $H(x,0) = x$, $H(x,1) \in O(z)$ for some $z \in X$. Since $gH(x,0,t) = H(gx_0,t) = H(x_0,t)$, it is easy to see that for all $t \in I$, $H(x_0,t) \in X^G$. Therefore $H(x_0,1) \in O(z)$, which implies $O(z) = \{ H(x_0,1) \}$. Define $H' : U \times I \to X$, by

$$H'(x,t) = \begin{cases} H(x,2t) : & 0 \leq t \leq \frac{1}{2} \\ H(x_0,2 - 2t) : & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Clearly $H'$ is a $G$-homotopy. The lemma follows. \qed

Note that for a $G$-categorical set $U$, which contains a fixed point $x_0$, the following diagrams commutes:

$$
\begin{array}{ccc}
I \cong \{ x_0 \} \times I & \longrightarrow & X \\
\downarrow & & \downarrow \\
U \times I & \longrightarrow & X
\end{array}
$$

So there exists a path $\Phi : I \to X^G$, defined by $\Phi(t) = H(x_0,t)$.

**Definition 2.4.** $x_0 \in X^G$ is called an isolated fixed point if there exists a neighborhood $U$ of $x_0$ that does not contain any other fixed points.

**Lemma 2.5.** Let $X$ be a Hausdorff space, and $U$ be a $G$-categorical subset that contains an isolated fixed point $x_0$. Then the $G$-homotopy $H : U \times I \to X$ fixes $x_0$, and $x_0$ is the only fixed point of $U$.

**Proof.** Let $V$ be an open neighborhood of $x_0$ that does not contain any other fixed points, and $\Phi : I \to X^G$ where $\Phi(t) = H(x_0,t)$.

First we show that for all $t \in I$, $\Phi(t) \in U \cap V$, and therefore $\Phi(t) = x_0$.

Let

$$\mathcal{A} = \left\{ t \in I \mid \Phi(s) = x_0, \quad \text{for all } s \leq t \right\}.$$

$\mathcal{A}$ is non empty, since $0 \in \mathcal{A}$. If $\mathcal{A} = I$, then we are done. If $\mathcal{A} \neq I$, because of being bounded, $\sup(\mathcal{A})$ exists. Let $a = \sup(\mathcal{A})$, so by definition

$$\forall t < a, \quad \Phi(t) = x_0.$$  \hspace{1cm} (2.1)
• If Φ(a) = x₀, because U ∩ V is open, there exists an open neighborhood W₀, such that Φ(a) ∈ W₀ ⊂ U ∩ V. Since Φ⁻¹(W₀) is open in I, therefore
  
  \[ a ∈ (p_a, q_a) ⊂ Φ⁻¹(W₀) \]
  
  for some interval (p_a, q_a) in I. So for any a < b < q_a, we have
  
  \[ b ∈ Φ⁻¹(W₀) \implies Φ(b) ∈ W₀ ⊂ U ∩ V \implies Φ(b) = x₀. \]
  
  So b ∈ A, that contradicts to a = sup(A).
  
• If Φ(a) ≠ x₀, because X is Hausdorff, there exist open neighborhoods W and W′ such that
  
  \[ Φ(a) ∈ W', \quad x₀ ∈ W, \quad \text{and} \quad W' ∩ W = ∅. \]
  
  Since Φ⁻¹(W′) is open, there exists an open interval (p′, q′), where
  
  \[ a ∈ (p′, q′) ⊂ Φ⁻¹(W′). \]
  
  So for any p′ < b′ < a, we have
  
  \[ Φ(b′) ∈ W' \quad \text{and} \quad x₀ ∉ W' \implies Φ(b′) ≠ x₀ \]
  
  which contradicts to (2.1).

Thus for all t ∈ I, Φ(t) = x₀. So H fixes x₀.

If U contains another fixed point z₀, then there exists a path Ψ : I → X^G, where Ψ(0) = z₀ and Ψ(1) = x₀. Similarly one can show that for all t ∈ I, Ψ(t) = x₀, and therefore x₀ = z₀.

\[ \Box \]

**Corollary 2.6.** If X^G ≠ ∅ and cat_G(X) = 1, then X is G-contractible to a point.

Note that in general case if cat_G(X) = 1, X may not be necessarily contractible. As for G = S^1, which acts on X = S^1, by product, cat_G(X) = 1, while X is not contractible.

**Lemma 2.7.** Let (X, x₀) and (Y, y₀) be pointed G-spaces. By pointed G-space, it means a G-space with base point such that the base point is fixed by G. Then

\[ cat_G(X ∨ Y) ≤ cat_G(X) + cat_G(Y) - 1. \]

**Proof.** Let \{U_i\}_{i=1}^n and \{V_j\}_{j=1}^m be G-categorical covering for X and Y respectively. Let x₀ ∈ U_i and y₀ ∈ V_j for some i and j. By Lemma 2.3, U_i ~_G x₀ and V_j ~_G y₀. By identifying x₀ = y₀, one can show that U_i ∪ V_j is G-contractible to x₀ in X ∨ Y. \[ \Box \]

**Lemma 2.8.** Let U be a G-categorical subset in X. If U' = U ∩ X^G ≠ ∅, then U' is a G-categorical subset in X^G.

**Proof.** It is clear that U' is G-invariant. Since U' ≠ ∅, it contains a fixed point α and by Lemma 2.3, there exits a G-homotopy H : U × I → X, such that for all x ∈ U we have H(x, 0) = x and H(x, 1) = α. Take the restriction of H to U'

\[ H_{|U'} = H' : U' × I → X^G, \quad H'(x, t) = H(x, t). \]
$H'$ is well-defined because for every $x \in U' = U \cap X^G$, we have
\[ g.H'(x,t) = g.H(x,t) = H(g.x,t) = H(x,t) = H'(x,t) \]
for all $g \in G$ and $t \in I$. Therefore the inclusion of $U'$ in $X^G$ is $G$-contractible to $O(\alpha) = \{\alpha\}$. □

**Corollary 2.9.** Suppose $\{U_i\}_{i=1}^n$ is a $G$-categorical covering of $X$. Then $\{U_i \cap X^G\}_{i=1}^n$ is a $G$-categorical covering of $X^G$ and therefore
\[ |\pi_0(X^G)| \leq \text{cat}(X^G) = \text{cat}_G(X^G) \leq \text{cat}_G(X). \]

Note that the previous corollary also follows from [14].

**Lemma 2.10.** If $|X^G| < \infty$, then every $G$-categorical set contains at most one fixed point. So all fixed points are isolated fixed points and we have $|X^G| = \text{cat}_G(X^G) = \text{cat}(X^G)$.

**Proof.** Since $X$ is Hausdorff and $|X^G| < \infty$, every $x \in X^G$ is an isolated fixed point. Thus the statement follows from Lemma 2.5. □

**Lemma 2.11.** Let $\alpha$ and $\beta$ be two distinct fixed points belong to a path-component of $X^G$. If $U$ and $W$ be two disjoint subsets of $X$ which are $G$-contractible to $\alpha$ and $\beta$ respectively, then $U \cup W$ is $G$-contractible to $\alpha$.

**Proof.** Since $U$ and $W$ are disjoint, it is enough to show that $W$ is $G$-contractible to $\alpha$. Let $F : W \times I \to X$ be a $G$-homotopy such that for all $w \in W$, $F(w,0) = w$ and $F(w,1) = \beta$. Let $\phi : I \to X^G$, be a path from $\beta$ to $\alpha$. Define $H : W \times I \to X$, by
\[ H(w,t) = \begin{cases} F(w,2t), & 0 \leq t \leq \frac{1}{2} \\ \phi(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases} \]
Clearly $H$ is well-defined. Also it is a $G$-homotopy because $F$ is a $G$-homotopy and for all $t \in I$ and $g \in G$ we have $g.\phi(t) = \phi(t)$. So $U$ and $W$ are two disjoint $G$-categorical subsets that are $G$-contractible to $\alpha$. Therefore $U \cup W$ is also $G$-contractible to $\alpha$. □

**Definition 2.12.** Let $G$ be a topological group acting on a topological space $X$. The sequence
\[ \emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n = X \]
of open sets in $X$ is called $G$-categorical sequence or simply $G$-cat sequence of length $n$ if
- each $A_i$ is $G$-invariant, and
- for each $1 \leq i \leq n$, there exists a $G$-categorical subset $U_i$ of $X$, such that
\[ A_i - A_{i-1} \subset U_i. \]
A $G$-cat sequence of length $n$ is called minimal if there exists no $G$-sequence with smaller length in $X$.

**Lemma 2.13.** Let $G$ be a topological group acting on a topological space $X$. Then there exists a minimal $G$-cat sequence of length $n$ in $X$, if and only if $\text{cat}_G(X) = n$.

**Proof.** Suppose $\text{cat}_G(X) = n$, so there exist a $G$-categorical covering, $\{U_i\}_{i=1}^n$, such that each $U_i$ is $G$-categorical. Take $A_0 = \emptyset$ and $A_i = \bigcup_{k=1}^i U_k$.

Note that each $A_i$ is $G$-invariant, and $A_i - A_{i-1} \subset U_i$. Therefore there exist a $G$-cat sequence in $X$. To show that this sequence is minimal, it is enough to show that for any minimal $G$-cat sequence in $X$, there exists a $G$-categorical covering with $n$ elements.

Suppose
\[ \emptyset = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_m = X \]
is a minimal $G$-cat sequence, so $m \leq n$. By definition, for any $1 \leq i \leq m$, there exists a $G$-categorical subset of $X$, $V_i$, such that $B_i - B_{i-1} \subset V_i$.

Obviously $\{V_i\}_{i=1}^m$ is a $G$-categorical covering for $X$, and therefore $n = \text{cat}_G(X) \leq m$. Thus $n = m$. $\square$

**Definition 2.14.** A $G$-path from an orbit $O(x)$ to an orbit $O(y)$ is a $G$-homotopy $H : O(x) \times I \to X$ such that the following hold:

1. $H_0$ is the inclusion of $O(x)$ in $X$.
2. $H_1(O(x)) \subseteq O(y)$

**Lemma 2.15.** (Lemma 3.2, [14]) Let $H : O(x) \times I \to X$ be a $G$-path in $X$ and $x_t = H(x,t)$. Then $G_x \subseteq G_{x_t}$ for all $0 \leq t \leq 1$.

**Lemma 2.16.** Let $O(x)$ and $O(y)$ be two distinct orbits in a $G$-space $X$. If $O(x)$ and $O(y)$ both sit inside a $G$-categorical subset, then there exist an orbit $O(z)$ such that there are $G$-paths from $O(x)$ to $O(z)$ and $O(y)$ to $O(z)$.

**Proof.** It is clear from the definition of $G$-categorical open subset. $\square$

**Definition 2.17.** A $G$-space $X$ is called $G$-connected if for every closed subgroup $H$ of $G$, $X^H$ is path-connected.

**Lemma 2.18.** ([6] Lemma 3.14) Let $X$ be a $G$-connected, and let $x, y \in X$ such that $G_x \subset G_y$. Then there exists a $G$-homotopy $F : O(x) \times I \to X$ such that $F_0 = \text{Id}_{O(x)}$ and $F_1(O(x)) \subset O(y)$.

**Lemma 2.19.** Let $X$ and $Y$ be $G$-connected. Then $X \times Y$ with diagonal action is $G$-connected.
Proof. Let \( H \) be a closed subgroup of \( G \), for all \((x, y) \in X \times Y \) and \( h \in H \) one have
\[
h(x, y) = (x, y) \iff (h.x, h.y) = (x, y) \iff h.x = x \text{ and } h.y = y.
\]
So \((X \times Y)^H = X^H \times Y^H\), and the lemma follows. \( \square \)

**Lemma 2.20.** Let \( X \) be a \( G \)-connected space with \( \alpha \in X^G \neq \emptyset \). Then every \( G \)-categorical subset \( U \) in \( X \) is equivariantly contractible to \( \alpha \).

**Proof.** Let \( F : U \times I \to X \) be a \( G \)-homotopy such that \( F(x, 0) = x \) and \( F(x, 1) \in \mathcal{O}(z) \), for some \( z \in X \). Since \( G_z \) is a subset of \( G_\alpha = G \), and \( X \) is \( G \)-connected, by Lemma 2.18, there exists a \( G \)-homotopy \( E : \mathcal{O}(z) \times I \to X \) so that \( E(y, 0) = y \) and \( E(y, 1) = \alpha \). Define the desire \( G \)-homotopy \( H : U \times I \to X \) by
\[
H(x, t) = \begin{cases} 
F(x, 2t), & 0 \leq t \leq \frac{1}{2} \\
E(F(x, 1), 2t - 1), & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
and the lemma follows. \( \square \)

By using Lemma 2.20 one can show that if \( X \) is a \( G \)-connected space with \( \alpha \in X^G \neq \emptyset \). Then for every two disjoint \( G \)-categorical subset \( U \) and \( W \) in \( X \), \( U \cup W \) is equivariantly contractible to \( \alpha \). Also for every \( G \)-categorical subset \( V \) in \( Y \), where \( Y \) is another \( G \)-connected space with \( \beta \in Y^G \neq \emptyset \), \( U \times V \) is equivariantly contractible to \( (\alpha, \beta) \).

**Definition 2.21.**

- A topological space \( X \) is called completely normal if for every two subsets \( A \) and \( B \) of \( X \) with
\[
\overline{A} \cap B = \emptyset , \quad A \cap \overline{B} = \emptyset ,
\]
there exist two disjoint open subsets containing \( A \) and \( B \).
- A \( G \)-space \( X \) is called \( G \)-completely normal if for every two \( G \)-invariant subsets \( A \) and \( B \) of \( X \) with
\[
\overline{A} \cap B = \emptyset , \quad A \cap \overline{B} = \emptyset 
\]
there exist two disjoint \( G \)-invariant subsets containing \( A \) and \( B \).

Note that every metric space is completely normal.

**Lemma 2.22.** ([6] Lemma 3.12 ) If \( X \) is a completely normal \( G \)-space, then \( X \) is \( G \)-completely normal.

**Theorem 2.23.** Let \( X \) and \( Y \) be \( G \)-connected such that \( X \times Y \) is completely normal. If \( X^G \neq \emptyset \) and \( Y^G \neq \emptyset \), then
\[
cat_G(X \times Y) \leq cat_G(X) + cat_G(Y) - 1,
\]
where \( X \times Y \) is given the diagonal \( G \)-action.
Proof. The idea of proof is similar to the proof for classical cat. Let \( \alpha \in X^G \), \( \beta \in Y^G \), \( \text{cat}_G(X) = n \), and \( \text{cat}_G(Y) = m \). So by Lemma 2.13 there exist \( G \)-cat sequences of length \( n \) and \( m \):

\[
\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = X,
\emptyset = B_0 \subset B_1 \subset \cdots \subset B_m = Y.
\]

Denote the \( G \)-categorical subsets containing the differences by

\[
A_i - A_{i-1} \subset U_i \quad \text{and} \quad B_j - B_{j-1} \subset W_j.
\]

Define subsets of \( X \times Y \) by

\[
C_0 = \emptyset, \quad C_1 = A_1 \times B_1, \quad C_k = \bigcup_{i=1}^{k} A_i \times B_{k+1-i}, \quad C_{n+m-1} = A_n \times B_m = X \times Y,
\]

where \( A_i = \emptyset \) if \( i > n \), and \( B_j = \emptyset \) if \( j > m \). Note that \( C_k \) is \( G \)-invariant and

\[
C_k - C_{k-1} = \bigcup_{t=1}^{k} (A_t - A_{t-1}) \times (B_{k+1-t} - B_{k-t}).
\]

Also for any \( k \) and \( t \),

\[
(A_t - A_{t-1}) \times (B_{k+1-t} - B_{k-t}) \subset U_k \times W_{k+1-t},
\]

where \( U_k \times W_{k+1-t} \) is a \( G \)-categorical subset of \( X \times Y \) contracting to \((\alpha, \beta)\). Although for a fixed \( k \) and varying \( t \) there may be intersections among these sets, but this issue can be resolve by using the assumption that \( X \times Y \) is \( G \)-completely normal. Denote

\[
\Sigma_i = (A_i - A_{i-1}) \times (B_{i+1-t} - B_{i-t}).
\]

Since for \( i \neq j \) we have

\[
\Sigma_i \cap \Sigma_j = \emptyset \quad \text{and} \quad \Sigma_i \cap \Sigma_j = \emptyset,
\]

and \( X \times Y \) is \( G \)-completely normal, there exist disjoint \( G \)-invariant neighborhoods about \( \Sigma_i \) and \( \Sigma_j \). By taking the intersection of those disjoint neighborhoods with \( U_i \times W_{i+1-t} \) and \( U_j \times W_{j+1-t} \), we obtain disjoint \( G \)-categorical neighborhoods of \( \Sigma_i \) and \( \Sigma_j \), for \( i \neq j \). So each \( C_k - C_{k-1} \) sits inside a \( G \)-categorical subset of \( X \times Y \), and therefore

\[
\emptyset = C_0 \subset C_1 \subset \cdots \subset C_{m+n-1} = X \times Y
\]

is a \( G \)-sequence for \( X \times Y \). Thus

\[
\text{cat}_G(X \times Y) \leq n + m - 1.
\]

We remark that in \([6]\) the authors have a similar statement (Theorem 3.15), however there the assumption on fixed point set is not enough and leads to counterexample (See Example 6.4).
3. **Locally Standard Torus Manifolds**

An $n$-dimensional manifold with corners is a Hausdorff second-countable topological space together with a maximal atlas of local charts onto open subsets of $\mathbb{R}_{\geq 0}^n$ such that the overlap maps are homeomorphisms which preserve codimension function. Codimension function at the point $x = (x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n$ is the number of $x_i$ which are zero (see Section 6 of [8]). The boundary of an $n$-dimensional manifold with corners is the correspondent set of points in local charts for which the codimension function is atleast one. An $n$-dimensional *simple* polytope in $\mathbb{R}^n$ is a convex polytope where exactly $n$ bounding hyperplanes meet at each vertex. Clearly a codimension-$k$ face is the intersection of unique collection of $k$ many codimension-1 faces. By polytope we mean a subset of $\mathbb{R}^n$ which is diffeomorphic as manifold with corners to a convex hull of finite number of points in $\mathbb{R}^n$. The codimension one faces of a manifold with corners are called *facets*. For the rest of the paper $P$ is an $n$-dimensional nice (in the sense of Davis [8]) manifold with corners. We denote the set of vertices of $P$ by $V(P)$ and the set of facets of $P$ by $\mathcal{F}(P)$.

**Definition 3.1.** A smooth action of $T^n$ on a $2n$-dimensional smooth manifold $M$ is said to be locally standard if every point $y \in M$ has a $T^n$-invariant open neighborhood $U_y$ and a diffeomorphism $\psi : U_y \to V$, where $V$ is a $T^n$-invariant open subset of $\mathbb{C}^n$, and an isomorphism $\delta_y : T^n \to T^n$ such that $\psi(t \cdot x) = \delta_y(t) \cdot \psi(x)$ for all $(t, x) \in T^n \times U_y$.

Modifying the definition of quasitoric manifold and torus manifold in [2] and [13], we consider the following. More general torus actions are discussed in [21] by Yoshida.

**Definition 3.2.** A closed, connected, oriented, and smooth $2n$-dimensional $T^n$-manifold $M$ is called a locally standard torus manifold over $P$ if the following conditions are satisfied:

1. The $T^n$-action is locally standard.
2. $\partial P \neq \emptyset$, where $\partial P$ is the boundary of $P$.
3. There is a projection map $q : M \to P$ constant on orbits which maps every $l$-dimensional orbit to a point in the interior of an $l$-dimensional face of $P$.

In the case that $P$ is a polytope, $M$ is called a quasitoric manifold.

Note that according to the Definition 3.2, the orbit space $P$ is path-connected. Also we remark that for the definition of torus manifolds in [13], the authors assume that the torus action has fixed points. But here we do not have such restrictions.

**Example 3.3.** Consider the natural $T^n$-action on

$$S^{2n} = \{ (z_1, \ldots, z_n, x) \in \mathbb{C}^n \times \mathbb{R} : |z_1|^2 + \cdots + |z_n|^2 + x^2 = 1 \}.$$
which is defined by

\[(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n, x) \mapsto (t_1z_1, \ldots, t_nz_n, x).\]

The orbit space is given by 

\[Q = \{(x_1, \ldots, x_n, x) \in S^n : x_1, \ldots, x_n \geq 0\}\] and the number of fixed points is 2.

This action is a locally standard action, so \(S^{2n}\) is a locally standard torus manifold. Note that \(S^{2n}\) is not a quasitoric manifold if \(n \geq 2\). When \(n = 2\) the orbit space is eye shape.

**Example 3.4.** Let \(M_1\) and \(M_2\) be two quasitoric manifolds of dimension \(2n\), and \(\mathbb{T}^k\) be the \(k\)-dimensional torus, \(0 \leq k \leq n\). Let \(\phi_i : \mathbb{T}^k \to M_i\) be the embedding onto \(k\)-dimensional orbit of \(M_i\), and let \(\tau_i\) be the invariant tubular neighborhood of \(\phi_i(\mathbb{T}^k)\) for \(i = 1, 2\). Identifying the boundary of \(\tau_1\) in \(M_1\) and \(\tau_2\) in \(M_2\) via an equivariant diffeomorphism, we get a smooth \(\mathbb{T}^n\)-manifold, which is called an equivariant connected sum of \(M_1\) and \(M_2\), denoted \(M_1 \#_{\tau_1} M_2\). Clearly \(M_1 \#_{\tau_1} M_2\) is a torus manifolds, and it is not a quasitoric manifold if \(k \geq 1\).

A more general equivariant connected sum of smooth manifolds with torus action is described in [12]. Equivariant connected sum of quasitoric manifolds at a fixed point and along a principal orbit is discussed in [3] and [20] respectively.

**Definition 3.5.** A function \(\lambda : F(P) \to \mathbb{Z}^n\) is called characteristic function if the submodule generated by \(\{\lambda(F_{j_1}), \ldots, \lambda(F_{j_l})\}\) is an \(l\)-dimensional direct summand of \(\mathbb{Z}^n\) whenever the intersection of the facets \(F_{j_1}, \ldots, F_{j_l}\) is nonempty.

The vectors \(\lambda_j = \lambda(F_j)\) are called characteristic vectors and the pair \((P, \lambda)\) is called a characteristic pair.

In [18] the authors show that given a torus manifold with locally standard action one can associate a characteristic pair to it up to the choice of sign of characteristic vectors. They also constructed a torus manifold with locally standard action from the pair \((P, \lambda)\). Following [18] we write the construction briefly. A more general construction is done in [21].

Let \(P\) be a nice manifold with corner and \((P, \lambda)\) be a characteristic pair. A codimension-\(k\) face \(F\) of \(P\) is a connected component of the intersection \(F_{j_1} \cap \ldots \cap F_{j_k}\) of unique collection of \(k\) facets \(F_{j_1}, \ldots, F_{j_k}\) of \(P\). Let \(Z(F)\) be the submodule of \(\mathbb{Z}^n\) generated by the characteristic vectors \(\lambda_{j_1}, \ldots, \lambda_{j_k}\). Then \(Z(F)\) is a direct summand of \(\mathbb{Z}^n\). Therefore the torus \(\mathbb{T}_F := (\mathbb{Z}(F) \otimes \mathbb{R})/\mathbb{Z}(F)\) is a direct summand of \(\mathbb{T}^n\). Define \(Z(P) = (0)\) and \(\mathbb{T}_P\) to be the proper trivial subgroup of \(\mathbb{T}^n\). If \(p \in P\), then \(p\) belongs to the relative interior of a unique face \(F(p)\) of \(P\).

Define an equivalence relation \(\sim\) on the product \(\mathbb{T}^n \times P\) by

\[(t, p) \sim (s, q) \iff p = q\] and \(s^{-1}t \in \mathbb{T}_{F(p)}\).

Let

\[M(P, \lambda) = (\mathbb{T}^n \times P) / \sim\]
be the quotient space. The group operation in $\mathbb{T}^n$ induces a natural $\mathbb{T}^n$-action on $M(P, \lambda)$. The projection onto the second factor of $\mathbb{T}^n \times P$ descends to the quotient map

$$q : M(P, \lambda) \to P, \quad q([t, p]) = p$$

where $[t, p]$ is the equivalence class of $(t, p)$. So the orbit space of this action is $P$. One can show that the space $M(P, \lambda)$ has the structure of a locally standard torus manifold.

**Definition 3.6.** Two $\mathbb{T}^n$-actions on $2n$-dimensional torus manifolds $M_1$ and $M_2$ are called equivalent if there is a homeomorphism $f : M_1 \to M_2$ such that

$$f(t \cdot x) = t \cdot f(x), \quad \forall (t, x) \in \mathbb{T}^n \times M_1.$$ 

**Definition 3.7.** Let $\delta : \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism. Two torus manifolds $M_1$ and $M_2$ over the same manifold with corners $P$ are called $\delta$-equivariantly homeomorphic if there is a homeomorphism $f : M_1 \to M_2$ such that

$$f(t \cdot x) = \delta(t) \cdot f(x), \quad \forall (t, x) \in \mathbb{T}^n \times M_1.$$ 

When $\delta$ is the identity automorphism, $f$ is called an equivariant homeomorphism.

**Proposition 3.8.** Let $M$ be a $2n$-dimensional locally standard torus manifold over $P$, and $\lambda : \mathcal{F}(P) \to \mathbb{Z}^n$ be its associated characteristic function. Let $M(P, \lambda)$ be the locally standard torus manifold constructed from the pair $(P, \lambda)$, and $H^2(P, \mathbb{Z}) = 0$. Then there is an equivariant homeomorphism $f : M(P, \lambda) \to M$ covering the identity on $P$.

This proposition is a particular case of Theorem 6.2 in [21]. We remark that this result is proved for quasitoric manifolds in [9], for torus manifolds with locally standard action in [18], and for specific 4-dimensional manifolds with effective $\mathbb{T}^2$-action in [19].

**Lemma 3.9.** Let $M_1$ and $M_2$ be $2n$-dimensional quasitoric manifolds, then $M_1 \#_{\mathbb{T}^k} M_2$ is simply connected for all $n$ and $k$ except $k = n = 2$.

**Proof.** We adhere the notations of Example 3.4. Let $q_i : M_i \to P_i$ be the orbit map, and $Q_i = P_i - q_i(\tau_i)$ for $i = 1, 2$. Then $Q_i$ is contractible and $M_i - \tau_i = q_i^{-1}(Q_i)$. By Proposition 3.8 we have

$$M_i - \tau_i \cong (\mathbb{T}^n \times Q_i) / \sim$$

where $\sim$ is defined in (3.1).

Let $g_i : \mathbb{T}^n \times Q_i \to M_i - \tau_i$ be the quotient map, for $i = 1, 2$. By definition of the equivalence relation $\sim$, $g_i^{-1}(x)$ is connected for all $x \in M_i - \tau_i$. Also $\mathbb{T}^n \times Q_i$ is locally path-connected and $M_i - \tau_i$ is semi-locally simply connected. Thus by Theorem 1.1 in [4], we get a surjective map

$$\pi_1(g_i) : \pi_1(\mathbb{T}^n \times Q_i) \to \pi_1(M_i - \tau_i).$$
Since $Q_i$ is contractible,

$$\pi_1(\mathbb{T}^n \times Q_i) = \pi_1(\mathbb{T}^n).$$

Existence of fixed point in $M_i - \tau_i$ implies that all generator of $\pi_1(\mathbb{T}^n)$ maps to zero. So $\pi_1(M_i - \tau_i)$ is trivial. Hence $\pi_1(M_1 \#_{\mathbb{T}^k} M_2)$ is trivial by Van-Kampen theorem.

□

More generally we have,

**Theorem 3.10.** Let $M$ be a locally standard torus manifold with orbit space $P$. If $M$ has a fixed point and $P$ is simply connected, then $M$ is simply connected.

**Proof.** Since $M$ is a smooth locally standard torus manifold with fixed point, the orbit space $P$ is a nice manifold with corners and $\partial P \neq \emptyset$ (see Section 4 in [21]).

By result of Yoshida [21], $M$ is equivariantly homeomorphic to $T_P/\sim$, where $T_P$ is a principal $\mathbb{T}^n$-bundle over $P$ and $\sim$ is defined in Definition 4.9 in [21]. Since $P$ is simply connected, the fibration

$$\mathbb{T}^n \to T_P \to P$$

induces a surjective map $i_* : \pi_1(\mathbb{T}^n) \to \pi_1(T_P)$. Let $f : T_P \to T_P/\sim : I$ be the quotient map. From Section 4 of [21], the fiber $f^{-1}(x)$ of each point $x \in T_P/\sim$ is a connected subset of $\mathbb{T}^n$. Hence by Theorem 1.1 in [4],

$$q_* : \pi_1(T_P) \to \pi_1(T_P/\sim : I) = \pi_1(M)$$

is surjective and therefore $q_* \circ i_*$ is surjective. Since $\mathbb{T}^n$-action has a fixed point, all generators of $\pi_1(\mathbb{T}^n)$ maps to identity via $q_* \circ i_*$. Thus $\pi_1(M)$ is trivial. □

4. **LS-category of Locally Standard Torus Manifolds**

The Lusternik-Schnirelmann category of a space $X$, denoted $\text{cat}(X)$, is the least integer $n$ such that there exists an open covering $U_1, \ldots, U_n$ of $X$ with each $U_i$ contractible to a point in the space $X$. If no such integer exists, we write $\text{cat}(X) = \infty$.

In this section we discuss the LS-category of locally standard torus manifolds for the following cases:

- Quasitoric manifolds.
- Locally standard torus manifold over $P$, where $P$ is simply connected and a connected component of $\partial P$ is a simple polytope.
- 4-dimensional locally standard torus manifold over $P$, where a connected component of $\partial P$ is a boundary of a polygon.

**Lemma 4.1.** Let $M$ be a $2n$-dimensional quasitoric manifold over a simple polytope $P$. Then $\text{cat}(M) = n + 1$. 

Proof. By Proposition 3.10 in [9], each generator of degree 2n in the cohomology group of $M$ is a product of $n$ cohomology classes of lowest dimension 2. Since $\dim(M) = 2n$, cuplength of $M$ (see Definition 1.4 of [7]) is $n$,

$$\text{cup}_Z(M) = n.$$ 

Thus by Proposition 1.5 in [7],

$$\text{cat}(M) \geq n + 1.$$ 

By Corollary 3.9 of [9], $M$ is simply connected. Therefore by Proposition 27.5 in [10],

$$\text{cat}(M) \leq n + 1.$$ 

□

Lemma 4.2. Let $M$ be a 2n-dimensional locally standard torus manifold over $P$. If a connected component of $\partial P$ is a boundary of an $n$-dimensional simple polytope $Q$, then

$$\text{cat}(M) \geq n + 1.$$ 

Proof. Let $v$ be a vertex of $Q$ and $v = F_{i_1} \cap \cdots \cap F_{i_n}$, where $F_{i_1}, \cdots, F_{i_n}$ are unique $n$-many facets of $Q$. Let $x_v = \pi^{-1}(v)$ and $X_j = \pi^{-1}(F_{i_j})$, for $j = 1, 2, \cdots, n$. Since $T^n$-action on $M$ is locally standard, $x_v$ is a fixed point and the intersection $X_1 \cap \cdots \cap X_n (= x_v)$ is transversal. Therefore the Poincaré dual of $X_j$ represents a non-zero cohomology class in $H^2(X, \mathbb{Z})$ (see Section 0.4 in [11]). So by definition of cup-length, $\text{cup}_Z(M) \geq n + 1$. □

Note that Lemma 4.2 is not true for every locally standard torus manifold, see the Example 6.6.

Theorem 4.3. Let $M$ be a 2n-dimensional locally standard torus manifold with a simply connected orbit space $P$. If a connected component of $\partial P$ is the boundary of a simple polytope $Q$, then

$$\text{cat}(M) = n + 1.$$ 

Proof. By Theorem 3.10 $M$ is simply connected, so $\text{cat}(M) \leq n + 1$. On the other hand by Lemma 4.2 $\text{cat}(M) \geq n + 1$. □

Corollary 4.4. Let $M_1$ and $M_2$ be quasitoric manifolds. Then for any $k$ and $n$ except $k = n = 2$, we have

$$\text{cat}(M_1 \#_{\mathbb{T}_k} M_2) = n + 1.$$ 

Proof. Let $P$ be the orbit space of locally standard $T^n$-action on $M_1 \#_{\mathbb{T}_k} M_2$. Since $M_1$ and $M_2$ are quasitoric manifolds, $\partial P$ contains the boundary of
a simple polytope. Also by Lemma 3.9, \( M_1 \# T_k M_2 \) is simply connected. Therefore by Theorem 4.3
\[
\text{cat}(M_1 \# T_k M_2) = n + 1.
\]

\[\square\]

**Lemma 4.5.** Let \( M \) be a 4-dimensional locally standard torus manifold with a fixed point \( x_0 \). Then any orbit is contractible to \( x_0 \).

**Proof.** Let \( P \) be the orbit space and \( \pi: M \to P \) be the orbit map. By Proposition 3.8, we may assume that \( M = M(P, \lambda) \) where \( \lambda \) is the characteristic function of \( M \). Let \( \theta \) be an orbit such that \( \pi(\theta) = x \in P \). We can choose a path \( \alpha: [0,1] \to P \) from \( x \) to \( x_0 \) such that \( \alpha \) is injective and \( \alpha(0,1) \cap P \subset P^0 \). We denote the image of \( \alpha \) by \([x, x_0]\). Then
\[
(T^2 \times [x, x_0]) / \sim \subset M.
\]
Let \( T^2_x \) be the isotropy group of \( x \). Then
\[
\theta = \pi^{-1}(x) = (T^2 \times x) / \sim \cong T^2 / T^2_x.
\]
Since the \( T^2 \)-action is locally standard, we have \( T^2 \cong T^2_x \oplus (T^2 / T^2_x) \). Observe that \((T^2 / T^2_x \times [x, x_0]) / \sim \) gives a homotopy. \[\square\]

**Theorem 4.6.** Let \( M \) be a 4-dimensional locally standard torus manifold over \( P \), such that a connected component of \( \partial P \) is the boundary of a polygon. Then
\[
\text{cat}(M) = 3.
\]

**Proof.** By Lemma 4.2 \( \text{cat}(M) \geq 3 \). Since \( T^2 \)-action on \( M \) is locally standard, \( P \) is a nice 2-dimensional manifold with corners. So every component of \( \partial P \) is either boundary of a polygon, a circle, or an eye shape (see Figure 1). Note that \( P \) can be obtained from a closed surface by removing the interior points of a finite number of non-intersecting polygons, or polygons and eye shapes, or polygons and circles, or polygons and eye shapes and circles. Thus by [1] \( P \) has a triangulation \( \Sigma \) such that vertices of \( P \) belong to the vertex set of \( \Sigma \). Let
- \( \{x_1, \ldots, x_l\} \) be the vertices of \( \Sigma \),
- \( \{E_1, \ldots, E_m\} \) be the edges of \( \Sigma \), and
- \( \{F_1, \ldots, F_n\} \) be the faces of \( \Sigma \).
Suppose \( y_j \) and \( z_k \) are interior point of \( E_j \) and \( F_k \) respectively, for \( j = 1, \ldots, m \) and \( k = 1, \ldots, n \). Regarding to the Figure 2 one can choose the neighborhoods \( X_i, Y_i, Z_k \) of \( x_i, y_j, z_k \) in \( P \) respectively such that

![Figure 1. An eye shape](image-url)
Figure 2. Choosing neighborhood $X_i$, $Y_j$, and $Z_k$

(1) $X_i \cap X_j = \emptyset$, $Y_i \cap Y_j = \emptyset$ and $Z_i \cap Z_j = \emptyset$ if $i \neq j$.
(2) $y_i, z_i \notin X_j$, $x_i, z_i \notin Y_j$ and $x_i, y_i \notin Z_j$ for all $i, j$.
(3) $X_i \cap Y_j \cap X_{i2}$ is an open neighborhood of $E_j$ in $P$ if $x_{i1}$ and $x_{i2}$ are vertices of $E_j$.
(4) $Z_k \cup Y_{k1} \cup Y_{k2} \cup Y_{k3} \cup X_{i1} \cup X_{i2} \cup X_{i3}$ is an open neighborhood of $F_k$ in $P$ if $E_{i1}, E_{i2}, E_{i3}$ are edges of $F_k$ and $x_{i1}, x_{i2}, x_{i3}$ are vertices of $F_k$.
(5) $Z_k \subset F_k^0$ where $F_k^0$ is the interior of $F_k$.
(6) Each $X_i$ is either homeomorphic to $\mathbb{R}^2_{\geq 0}$, or $\mathbb{R}_{\geq 0} \times \mathbb{R}$, or $\mathbb{R}^2$.
(7) Each $Y_j$ is either homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}$, or $\mathbb{R}^2$.
(8) Each $Z_k$ is homeomorphic to $\mathbb{R}^2$.

(See Figure 3).

Suppose $\pi : M \to P$ is the orbit map. Let $U_i = \pi^{-1}(X_i)$, $V_j = \pi^{-1}(Y_j)$ and $W_k = \pi^{-1}(Z_k)$ for $i = 1, \ldots, l$, $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Then $U_i, V_j$ and $W_k$ are equivariant contractible to the orbit $\pi^{-1}(x_i), \pi^{-1}(y_j)$, and $\pi^{-1}(z_k)$ respectively. By hypothesis $M$ has a fixed point say $\hat{x}_0$. By Lemma 4.5 $\pi^{-1}(x_i), \pi^{-1}(y_j)$, and $\pi^{-1}(z_k)$ are contractible to $\hat{x}_0$. Thus $U_i, V_j$ and $W_k$ are equivariant contractible to $\hat{x}_0$. Let

$$A = \bigcup_{i=1}^{l} U_i, \quad B = \bigcup_{j=1}^{m} V_j \quad \text{and} \quad C = \bigcup_{k=1}^{n} W_k.$$ 

By the choice of $X_i$, $Y_j$ and $Z_k$ we get that $A, B$ and $C$ are contractible to $\hat{x}_0$. Clearly $M = A \cup B \cup C$. Therefore $\text{cat}(M) \leq 3$.

Corollary 4.7. Let $M$ be a $2n$-dimensional locally standard torus manifold over $P$, such that a connected component of $\partial P$ is the boundary of a polygon.
If there exists a triangulation for $P$, then

$$\text{cat}(M) = n + 1.$$ 

Note that Lemma 4.6 is not true for every locally standard torus manifold, see Examples 4.8 and 6.6.

**Example 4.8.** Consider the annulus $P$ and characteristic function $\lambda$ as in the Figure 4. Note that $P \cong C \times I$ where $C$ is a circle and $I$ is the closed interval $[0, 1]$. Then the following is an equivariant homeomorphism

$$\left(\mathbb{T}^2 \times C \times I\right)/\sim \cong C \times \left(\mathbb{T}^2 \times I\right)/\sim$$

where $\sim$ is defined in (3.1). By Section 2 in [19],

$$\left(\mathbb{T}^2 \times I\right)/\sim \cong \mathbb{RP}^3.$$ 

Therefore

$$M(P, \lambda) \cong \left(\mathbb{T}^2 \times C \times I\right)/\sim \cong C \times \left(\mathbb{T}^2 \times I\right)/\sim \cong \mathbb{S}^1 \times \mathbb{RP}^3.$$ 

Since $\text{cat} \left(\mathbb{RP}^3\right) = 4$ and $\text{cat} \left(\mathbb{S}^1\right) = 2$, using categorical sequence (see Section 1.5 in [7]), one can show that

$$\text{cat} \left(\mathbb{S}^1 \times \mathbb{RP}^3\right) \leq 5.$$ 

On the other hand by Künneth theorem,

$$H^* \left(\mathbb{S}^1 \times \mathbb{RP}^3, \mathbb{Z}_2\right) = H^* \left(\mathbb{S}^1, \mathbb{Z}_2\right) \otimes \mathbb{Z}_2 \otimes H^* \left(\mathbb{RP}^3, \mathbb{Z}_2\right)$$
Therefore $\cup_{Z_2}(S^1 \times \mathbb{RP}^3) = 5$. Thus by Proposition 1.5 in [7],
\[
cat(S^1 \times \mathbb{RP}^3) = 5.
\]

![Figure 4. An annulus in $\mathbb{R}^2$.](image)

5. **Equivariant LS-category of Torus Manifolds**

In this section, we compute equivariant LS-category of some locally standard torus manifolds.

**Theorem 5.1.** Let $M$ be a $2n$-dimensional quasitoric manifold with $k$ fixed points. Then

\[
cat_{T^n}(M) = k.
\]

**Proof.** Since the fixed points are isolated, by Corollary 3.9 of [14] we have

\[
cat_{T^n}(M) \geq k.
\]

So it is enough to show that for any $v \in M^{T^n}$, there is a $T^n$-categorical subset $X_v$, such that

\[
M = \bigcup_{v \in M^{T^n}} X_v.
\]

Let $q : M \rightarrow P$ be the orbit map. Then $P$ is a simple $n$-polytope and also $M^{T^n}$ corresponds bijectively to $V(P)$, the vertex set of $P$. So we may assume

\[
M^{T^n} = V(P).
\]

For $v \in V(P)$, let

\[
C_v = \bigcup_{v \notin F} F, \quad U_v = P - C_v, \quad \text{and} \quad X_v = q^{-1}(U_v)
\]

where $F$ is a face of $P$. Clearly $X_v$ is $T^n$-invariant. Since $U_v$ is a convex subset of $P$, it is contractible to $v$. So there exists a homotopy $h : U_v \times I \rightarrow P$
such that for all $x \in U_v$, $h(x, 0) = x, h(x, 1) = v$, and also for any face $F$ of $U_v$ we have

$$h(x, t) \in F, \quad \forall x \in F, t \in I.$$ 

By Lemma 1.8 of [9],

$$M \cong M(P, \lambda) \quad \text{and} \quad X_v \cong (\mathbb{T}^n \times U_v)/\sim,$$

where $\lambda, M(P, \lambda)$, and $\sim$ are recalled in Section 2. Therefore $h$ induces a homotopy

$$Id \times h : \mathbb{T}^n \times U_v \times I \to \mathbb{T}^n \times P$$

defined by $((t, x), r) \mapsto (t, h(x, r))$. Since for each face $F$ of $U_v$, we have

$$x \in F \implies h(x, r) \in F, \quad \forall r \in I,$$

$Id \times h$ induces a homotopy $H : X_v \times I \to M$, with $([t, x], r) \mapsto [t, h(x, r)]$. Since

$$sH([t, x], r) = s[t, h(x, r)] = [st, h(x, r)] = H([st, x], r) = H(s[t, x], r),$$

therefore $H$ is $\mathbb{T}^n$-homotopy. Also

$$H(x, 0) = x, \quad H(x, 1) = q^{-1}(v) = v, \quad \forall x \in X_v.$$ 

Thus $X_v$ is $\mathbb{T}^n$-categorical subset of $M$. Clearly $\{X_v : v \in V(P)\}$ covers $M$, therefore $cat_{\mathbb{T}^n}(M) = |V(P)| = k$. \hfill $\square$

**Lemma 5.2.** Let $M_i$ be a $2n$-dimensional quasitoric manifold over $P_i$, for $i=1,2$. Then

$$cat_{\mathbb{T}^n}(M_1 \#_{\mathbb{T}^k} M_2) = |V(P_1)| + |V(P_2)|, \quad \text{for } k \geq 1.$$ 

**Proof.** We adhere the notations of Example 3.4 and Theorem 5.1. By the construction of equivariant connected sum we have $M_1 \#_{\mathbb{T}^k} M_2$ is a locally standard torus manifold. Let $k \geq 1$. Then the number of fixed points of $\mathbb{T}^n$-action on $M_1 \#_{\mathbb{T}^k} M_2$ is $|V(P_1)| + |V(P_2)|$. So by Corollary 3.9 in [14], we have

$$cat_{\mathbb{T}^n}(M_1 \#_{\mathbb{T}^k} M_2) \geq |V(P_1)| + |V(P_2)|.$$ 

Let $q_i : M_i \to P_i$ be the orbit map and $q_i(\mathbb{T}^k) = x_i$, so $x_i$ belongs to the relative interior of a $k$-dimensional face $E_i$ of $P_i$ for $i = 1, 2$. Let $L(P_i)$ be the face lattice of $P_i$ and $v \in V(P_i)$. Define

$$C_v = \bigcup_{v \notin F \in L(P_i)} F, \quad U_v = P_i - C_v \quad \text{and} \quad X_v = q_i^{-1}(U_v).$$ 

Let $S_1 = \{v_{11}, \ldots, v_{1p}\}$ and $S_2 = \{v_{21}, \ldots, v_{2q}\}$ be the vertices of $E_1$ and $E_2$ respectively. For $i \in \{1, 2\}$, let

$$\alpha_{ij} : I \to P_i$$

be the simple path from $x_i$ to $v_{ij}$ such that:

- $\alpha_{ij}(I^0) \subset E_i^0$, where $E_i^0$ is the relative interior of $E_i$, and
- $\alpha_{i1}(I^0) \cap \alpha_{i2}(I^0) = \emptyset$, 

where $1 \leq j \leq p$ for $i = 1$ and $1 \leq j \leq q$ for $i = 2$. Let

$$V_v = \begin{cases} U_v - q_i(\tau_i) & \text{if } v \in V(P_i) - S_i \text{ for } i \in \{1, 2\} \\ U_v - \{q_i(\tau_i) \cup \alpha_i(L^0)\} & \text{if } v \in S_i \text{ and } v \neq v_{il} \end{cases}$$

(5.1)

If $v \in V(P_i)$, then $Y_v = q_i^{-1}(V_v)$ is a $\mathbb{T}^n$-invariant subset of $M_i$ which is equivariantly contractible to the fixed point $q_i^{-1}(v)$. From the definition of equivariant connected sum, there is a $\mathbb{T}^n$-homotopy from $Y_v$ with a $\mathbb{T}^n$-homotopy from $Y_v$ to $Y_v$. Then the collection

$$\left\{ \hat{Y}_v : v \in V(P_1) \cup V(P_2) \right\}$$

is a $\mathbb{T}^n$-categorical covering of $M_1 \#_{\mathbb{T}^n} M_2$. Thus

$$\text{cat}_{\mathbb{T}^n}(M_1 \#_{\mathbb{T}^n} M_2) \leq |V(P_1)| + |V(P_2)|.$$

\qed

**Remark 5.3.** If $k = 0$, then $M_1 \#_{\mathbb{T}^n} M_2$ is a quasitoric manifold, therefore we can apply Lemma 5.1.

**Example 5.4.** Let $M_1$ and $M_2$ be 4-dimensional quasitoric manifolds over triangle $P_1$ and rectangle $P_2$ respectively. Let $x_i$ be the interior point of $P_i$, $i = 1, 2$. Then $q_i(\tau_i)$ is a neighborhood of $x_i$ with the boundary $C_i$ for $i = 1, 2$. Regarding to Lemma 5.2 here $E_1 = P_1$ and $E_2 = P_2$. So

- $V_{11} = P_1 - \{q_1(\tau_1) \cup [v_{12}, v_{13}] \cup \alpha_{12}(L^0)\}$.
- $V_{12} = P_1 - \{q_1(\tau_1) \cup [v_{11}, v_{13}] \cup \alpha_{11}(L^0)\}$.
- $V_{13} = P_1 - \{q_1(\tau_1) \cup [v_{11}, v_{12}] \cup \alpha_{11}(L^0)\}$.
- $V_{21} = P_2 - \{q_2(\tau_2) \cup [v_{22}, v_{23}] \cup [v_{23}, v_{24}] \cup \alpha_{22}(L^0)\}$.
- $V_{22} = P_2 - \{q_2(\tau_2) \cup [v_{23}, v_{24}] \cup [v_{21}, v_{24}] \cup \alpha_{21}(L^0)\}$.
- $V_{23} = P_2 - \{q_2(\tau_2) \cup [v_{21}, v_{22}] \cup [v_{21}, v_{24}] \cup \alpha_{21}(L^0)\}$.
- $V_{24} = P_2 - \{q_2(\tau_2) \cup [v_{21}, v_{22}] \cup [v_{22}, v_{23}] \cup \alpha_{21}(L^0)\}$.
Here \([v_{ij}, v_{kl}]\) is the edge joining the vertices \(v_{ij}\) and \(v_{kl}\). Clearly \(Y_{ij} = q_i^{-1}(V_{ij})\) is \(T^2\)-invariant and equivariantly contractible to the fixed point \(q_i^{-1}(v_{ij})\). Note

\[ M_1 \#_{T^2} M_2 = Y_{11} \cup Y_{12} \cup Y_{13} \cup Y_{21} \cup \cdots \cup Y_{24}. \]

Thus \(\text{cat}_{T^2}(M_1 \#_{T^2} M_2) = 3 + 4 = 7\).

**Lemma 5.5.** Let \(M\) and \(N\) be two \(2n\)-dimensional quasitoric manifolds with \(p\) and \(q\) many fixed points respectively. Then \(\text{cat}_{T^n}(M \times N) = pq\), where \(T^n\)-action on \(M \times N\) is diagonal.

**Proof.** We adhere the notations of Theorem 5.1. First observe that the diagonal \(T^n\)-action on \(M \times N\) has \(pq\) many fixed points. By Corollary 3.9 of [14],

\[ \text{cat}_{T^n}(M \times N) \geq pq. \]

Let \(X_u\) and \(Y_v\) be \(T^n\)-categorical open subsets of \(M\) and \(N\) respectively (as constructed in Theorem 5.1), where \(u \in M^{T^n}\) and \(v \in N^{T^n}\). Let

\(H : X_u \times I \to X_u\) and \(K : Y_v \times I \to Y_v\)

be the respective \(T^n\)-homotopy such that

\(H(x, 0) = x, H(x, 1) = u, \forall x \in X_u\) and \(K(y, 0) = y, K(y, 1) = v, \forall y \in Y_v\).

Then the \(T^n\)-homotopy

\(L : X_u \times Y_v \times I \to X_u \times Y_v\)

defined by \(L(x, y, r) = (H(x, r), K(y, r))\) implies that \(X_u \times Y_v \subset M \times N\) is \(T^n\)-categorical. Since

\[ M \times N = \bigcup_{u \in M^{T^n}, v \in N^{T^n}} X_u \times Y_v, \]

\(\text{cat}_{T^n}(M \times N) \leq pq\). Thus \(\text{cat}_{T^n}(M \times N) = pq\). \(\square\)

**Corollary 5.6.** Let \(M_i\) be a \(2n\)-dimensional quasitoric manifold with \(p_i\) many fixed points for \(i = 1, \ldots, l\). Then \(\text{cat}_{T^n}(M_1 \times \cdots \times M_l) = p_1 \cdots p_l\), where \(T^n\) acts on \(M_1 \times \cdots \times M_l\) diagonally.

**Proof.** The argument is similar to the proof of Lemma 5.5. So left as an exercise. \(\square\)

**Lemma 5.7.** Let \(M\) be a 4-dimensional locally standard torus manifold over \(P\), and \(l\) be the number of circles in \(\partial P\) (see proof of Theorem 4.5). Then \(\text{cat}_{T^2} M \geq |M^{T^2}| + 2l\).

**Proof.** By Corollary 3.9 of [14]

\[ \text{cat}_{T^2}(M) \geq |M^{T^2}|. \]

Let \(q : M \to P\) be the orbit map, and

\[ X = q^{-1}(\bigcup_{i=1}^l C_i) = \bigcup_{i=1}^l q^{-1}(C_i), \]
where $C_1, \ldots, C_l$ are the circles in $\partial P$. We claim that if a $T^2$-categorical open subset $U$ contains a fixed point, then $U \cap X = \emptyset$. Suppose there is $z \in U \cap X$ and $U$ contains a fixed point $v$. So $O(z) \subset U$. Since $z \in X$, $q(z) \in C_i$ for some $i \in \{1, \ldots, l\}$. Since $T^2$-action on $M$ is locally standard and $C_i \subset \partial P$, $O(z)$ is homeomorphic to a circle and isotropy of $z$ is a circle subgroup of $T^2$.

Suppose $H: T^2 \times I \to M$ be a $T^2$-path from $O(z)$ to $O(v) = v$. Then $q \circ H: z \times I \to P$ is a path from $q(z)$ to $q(v)$. Observe that $Im(q \circ H) \cap P^0 \neq \emptyset$. Since isotropy group over the interior of $P^0$ is trivial, it is a contradiction to Lemma 2.15. This proves our claim.

On the other hand for each $i \in \{1, \ldots, l\}$, $q^{-1}(C_i)$ is homeomorphic to $C_i \times S^1$, for some circle subgroup $S^1$ of $T^2$. So there is no equivariant homotopy from $q^{-1}(C_i)$ to any orbit, and therefore it cannot be covered by a $T^2$-categorical open set. Hence

$$cat_{T^2}(M) \geq \left| M^{T^2} \right| + 2l.$$  

6. Examples

Example 6.1. Consider the natural $T^2$-action on

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\},$$

which is defined by

$$(t_1, t_2) \cdot (z_1, z_2) \to (t_1 z_1, t_2 z_2).$$

The orbits $O(1, 0)$ and $O(0, 1)$ satisfy the condition of Lemma 2.16, because all the isotropy groups $T^2_x$, are trivial except for $x = (1, 0)$ and $x = (0, 1)$. Hence $O(1, 0)$ and $O(0, 1)$ can not belong to a same $T^2$-categorical subset of $S^3$ and therefore $cat_{T^2}(S^3) \geq 2$. Let

$$U_1 = S^3 - O(1, 0) \text{ and } U_2 = S^3 - O(0, 1).$$

Let $B^2$ be the open disk. Since $U_1$ and $U_2$ are equivariantly homeomorphic to $S^1 \times B^2$, there are $T^2$-homotopies from $U_1$ and $U_2$ onto the orbits $O(0, 1)$ and $O(1, 0)$ respectively. Thus $cat_{T^2}(S^3) = 2$.

Example 6.2. Consider the natural $T^2$-action on

$$S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\},$$

which is defined by

$$(t_1, t_2) \cdot (z_1, z_2, z_3) \to (t_1 z_1, t_2 z_2, z_3).$$

An orbit of this action is either a point, circle, or torus; And $S^5$ is not contractible to any of them. So $cat_{T^2}(S^5) \geq 2$. Let

$$V_1 = S^5 - \{(0, 0, -1)\} \text{ and } V_2 = S^5 - \{(0, 0, 1)\}.$$
Clearly $V_1$ and $V_2$ are equivariantly contractible to the fixed points $(0,0,1)$ and $(0,0,-1)$ respectively. So they make a $T^2$-categorical covering of $S^5$. Thus $\text{cat}_{T^2}(S^5) = 2$.

**Lemma 6.3.** Consider the $T^2$-actions defined in the Examples 6.1 and 6.2. For any subgroup $H$ of $T^2$, the fixed point sets $(S^3)^H$ and $(S^5)^H$ are path-connected. Hence $S^3$ and $S^5$ are $T^2$-connected.

**Proof.** If $H = \{(1,1)\}$ is the trivial subgroup of $T^2$, then $(S^3)^H = S^3$, and it is path-connected.

- Assume $H$ is non-trivial and there exist $\alpha \neq 1 \neq \beta$ such that $p_0 = (\alpha, \beta) \in H$. In this case
  $$(S^3)^H \subset (S^3)^{\{p_0\}} = \emptyset.$$

- Assume $H$ is non-trivial and for all elements $(\alpha, \beta)$ in $H$, whether $\alpha = 1$ or $\beta = 1$. If all elements of $H$ look like $(1, \beta)$, then
  $$(S^3)^H = \left\{ (z_1,0) \in S^3 \mid |z_1|^2 = 1 \right\} \cong S^1.$$

Similarly if all elements of $H$ look like $(\alpha,1)$, then $(S^3)^H \cong S^1$.

Thus in any case $(S^3)^H$ is path-connected. Similarly one can show that $(S^5)^H$ is path-connected. \(\square\)

Note that since every compact metric space is completely normal, so by Lemma 2.22 $S^3$, $S^5$ and $S^3 \times S^5$ are $T^2$-completely normal spaces.

**Example 6.4** (Counterexample of Theorem 3.15 in [6]). We adhere notations of Examples 6.1 and 6.2. Let $X = S^3 \times S^5$. Consider the diagonal $T^2$-action on $X$, which is defined by

$$t \cdot (p,q) \rightarrow (t \cdot p, t \cdot q).$$

Let $A_0 = \emptyset$, $A_1 = U_1$, $A_2 = S^3$ and $B_0 = \emptyset$, $B_1 = V_1$, $B_2 = S^5$. Clearly $A_0 \subset A_1 \subset A_2$ and $B_0 \subset B_1 \subset B_2$ are $T^2$-categorical sequences for $S^3$ and $S^5$ respectively. Consider the sequence

$$(*) \quad C_0 \subset C_1 \subset C_2 \subset C_3$$

where

$C_0 = \emptyset$, $C_1 = A_1 \times B_1$, $C_2 = A_2 \times B_1 \cup A_1 \times B_2$, and $C_3 = A_2 \times B_2 = X$.

However $S^3$, $S^5$ and $X$ satisfy the conditions in Theorem 3.15 in [6], we show that

$$C_2 - C_1 = (A_2 - A_1) \times B_1 \cup A_1 \times (B_2 - B_1)$$

does not sit in any $T^2$-categorical set of $X$, and therefore $(* \star)$ is not a $T^2$-categorical sequence.

Let $S^1_1$ and $S^1_2$ be the circle subgroups of $T^2$ determined by the standard vectors $e_1$ and $e_2$ in $\mathbb{Z}^2$ respectively. Let $x = ((1,0),(0,0,1))$ and $y = ((0,1),(0,0,-1))$. Note that

$$O(x) \subset (A_2 - A_1) \times B_1 \quad \text{and} \quad O(y) \subset A_1 \times (B_2 - B_1).$$
Also for isotropy groups we have, \( T^2 \) and \( T^2_y \). Suppose there exists \( z \in X \) with \( T^2 \)-paths from \( O(x) \) to \( O(z) \) and from \( O(y) \) to \( O(z) \). By Lemma \[2.15\] \( S_1 \) and \( S_1 \) are subgroups of \( T^2 \). Thus \( z \) is a fixed point. But \( T^2 \)-action on \( X \) has no fixed point, therefore by Lemma \[2.16\] there is no \( T^2 \)-categorical subset in \( X \) containing \( C_2 - C_1 \). This contradicts the arguments in the proof of Theorem 3.15 in \[6\].

Here we show that \( \text{cat}_{T^2}(S^3 \times S^5) = 4 \). Clearly \( U_1 \times V_1, U_1 \times V_2, U_2 \times V_1 \), and \( U_2 \times V_2 \) form a \( T^2 \)-categorical cover for \( S^3 \times S^5 \). Hence \( \text{cat}_{T^2}(S^3 \times S^5) \leq 4 \). On the other hand according to orbit types of \( T^2 \)-action on \( S^3 \times S^5 \), one can show that the isotropy groups are whether trivial or homeomorphic to \( T \). So by using Theorem 3.7 in \[14\], it is enough to show that

\[
\text{cat}_{T^2}(S^1 \times S^3) \geq 2.
\]

By looking at homology groups, it is clear that \( S^1 \times S^3 \) cannot contract to an orbit. Hence \( \text{cat}_{T^2}(S^1 \times S^3) \) cannot be one. Thus

\[
\text{cat}_{T^2}(S^3 \times S^5) = \text{cat}_{T^2}(S^1 \times S^3) + \text{cat}_{T^2}(S^1 \times S^3) \geq 4.
\]

**Example 6.5** (Counterexample of Theorem 3.16 in \[6\]). Let \( M \) and \( N \) be \( 2m \) and \( 2n \) dimensional quasitoric manifolds over the polytopes \( P \) and \( Q \) respectively. Then \( M \times N \) is a \( 4mn \)-dimensional quasitoric manifold over \( P \times Q \). By Theorem \[5.1\]

\[
\text{cat}_{T^m \times T^n}(M \times N) = |V(P \times Q)| = |V(P)| \times |V(Q)| = \text{cat}_{T^m}(M) \times \text{cat}_{T^n}(N^{2n})
\]

Note that \( M \) is a \( T^m \)-manifold, \( N \) is a \( T^n \)-manifold, and \( M \times N \) is a \( T^m \times T^n \)-manifold. Also \( M \times N \) is a compact metrizable space, so it is completely normal.

**Example 6.6.** We adhere the notation of Example \[3.3\]. Let

\[
V_1 = S^{2n} - \{(0,0, -1)\}, \quad V_2 = S^{2n} - \{(0,0, 1)\}.
\]

Since \( V_1 \) and \( V_2 \) are equivariantly contractible to the fixed points \( (0,0,1) \) and \( (0,0,-1) \) respectively, so they are \( T^n \)-categorical subset of \( S^{2n} \). Thus \( \text{cat}_{T^n}(S^{2n}) = 2 \). In particular \( \text{cat}(S^{2n}) = 2 \).

**Example 6.7.** Let \( p > 0, q_1, \ldots, q_n \) be integers such that \( p \) and \( q_i \) are relatively prime for all \( i = 1, \ldots, n \). Consider

\[
S^{2n+1} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid |z_1|^2 + \cdots + |z_{n+1}|^2 = 1\}.
\]

The \((2n+1)\)-dimensional lens space \( L = L(p; q_1, \ldots, q_n) \) is the orbit space \( S^{2n+1}/\mathbb{Z}_p \) where \( \mathbb{Z}_p \)-action on \( S^{2n+1} \) is defined by

\[
\theta: \mathbb{Z}_p \times S^{2n+1} \rightarrow S^{2n+1},
\]

\[
([k], (z_1, \ldots, z_n)) \mapsto (e^{2kq_1\pi\sqrt{-1}/p}z_1, \ldots, e^{2kq_n\pi\sqrt{-1}/p}z_n, e^{2k\pi\sqrt{-1}/p}z_{n+1}).
\]

The equivalence class of \((z_1, \ldots, z_n)\) is denoted by \([z_1, \ldots, z_{n+1}]\). The \((n+1)\)-dimensional compact torus \( T^{n+1} \) acts on \( L \) by:

\[
(t_1, \ldots, t_{n+1}) \times [z_1, \ldots, z_{n+1}] \rightarrow [t_1z_1, \ldots, t_{n+1}z_{n+1}].
\]
Let \( e_1, \ldots, e_{n+1} \) be the standard vectors in \( \mathbb{C}^{n+1} \), and \([e_i]\) be the equivalence class of \( e_i \) in \( L \). The orbit of \([e_i]\) is \( O_i = \{[0, \ldots, 0, z_i, 0, \ldots, 0] : |z_i| = 1\} \). From the action in Equation (6.1) \( O_1, \ldots, O_{n+1} \) are the only orbits of dimension one and there is no orbit of dimension less than one. Suppose there are \( T^{n+1} \)-paths from \( O_i \) to \( O(z) \) and from \( O_j \) to \( O(z) \) for some \( z \in L \) with \( i \neq j \). So we get inclusion of isotropy groups,

\[
T^{n+1}_{e_i} \subset T^{n+1}_{e_j} \quad \text{and} \quad T^{n+1}_{e_j} \subset T^{n+1}_{e_i}.
\]

Thus \( T^{n+1}_{e_i} = T^{n+1} \), since \( i \neq j \). This contradicts the fact that \( T^{n+1} \)-action on \( L \) has no fixed point. By Lemma 2.16, \( O_i \) and \( O_j \) can not belong to same \( T^{n+1} \)-categorical subset of \( L \). Thus

\[
\text{cat}_{T^{n+1}}(L) \geq n + 1.
\]

Let

\[
U_i = \{[z_1, \ldots, z_{n+1}] \in L : z_i \neq 0\}, \quad \text{for} \quad i = 1, \ldots, n+1.
\]

Then \( U_i \) is invariant open subset of \( L \). It is not difficult to show that \( U_i \) is a \( T^{n+1} \)-categorical set containing \( O_i \). Since \( U_1, \ldots, U_{n+1} \) covers \( L \),

\[
\text{cat}_{T^{n+1}}(L) \leq n + 1. \Rightarrow \text{cat}_{T^{n+1}}(L) = n + 1.
\]

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