Repulsive Interactions in the Microstructure of Regular Hayward Black Hole in Anti-de Sitter Spacetime

Naveena Kumara A.,∗ Ahmed Rizwan C.L.,† Kartheek Hegde,‡ and Ajith K.M.§

Department of Physics, National Institute of Technology Karnataka, Surathkal 575 025, India

Abstract

We study the interaction between the microstructures of regular AdS Hayward black hole using Ruppeiner geometry. Our investigation shows that the dominant interaction between the black hole molecules is attractive in most of the parametric space, as in van der Waals system. However, in contrast to the van der Waals fluid, there exists a weak dominant repulsive interaction for small black hole phase in some parameter domain. This result clearly distinguishes the interactions in a magnetically charged black hole from that of van der Waals fluid. However, the interactions are universal for charged black holes since they do not dependent on magnetic charge or temperature.

PACS numbers:

Keywords: Black hole thermodynamics, Hayward AdS black hole, Extended phase space, van der Waals fluid, Ruppeiner geometry, Black hole microstructure, Repulsive interactions

∗Electronic address: naviphysics@gmail.com
†Electronic address: ahmedrizwancl@gmail.com
‡Electronic address: hegde.kartheek@gmail.com
§Electronic address: ajithkm@gmail.com
I. INTRODUCTION

In recent years the subject of black hole chemistry has become an attractive window to probe the properties of AdS black holes. In black hole chemistry, the negative cosmological constant of the AdS spacetime is identified as the thermodynamic variable pressure to study the phase transition of the AdS black holes [1, 2]. Interestingly the phase transition of certain AdS black hole analytically resembles that of the van der Waals system [3–5]. Recently by studying the phase transitions, attempts were made to investigate of the underlying microscopic properties of the AdS black holes [6–21]. In these researches, the geometric methods were the key tools in probing the microscopic details of the black holes. Contrast to the statistical investigation in ordinary thermodynamics the approach here is upside down, the macroscopic thermodynamic details are ingredients for the microscopic study [22]. The technique is inspired by the applications of thermodynamic geometry in ordinary thermodynamic systems [23–27].

Recently, a general Ruppeiner geometry framework is developed from the Boltzmann entropy formula, to study the black hole microstructure [7]. The fluctuation coordinates are taken as the temperature and volume, and a universal metric was constructed in that scheme. When this methodology is applied to the van der Waals fluid only a dominant attractive interaction was observed, as it should be. However, when the same methodology is used for the RN AdS black hole, a different result was obtained. In a small parameter range, the repulsive interaction is also found in addition to the dominant attractive interaction between the black hole molecules [7, 8]. Even so, in the case of five-dimensional neutral Gauss-Bonnet black hole only a dominant attractive interaction was discovered, which is similar to van der Waals fluid [12]. Therefore, in general, the nature of the black hole molecular interactions are not universal. In our recent work [14], we have observed that there exists a repulsive interaction in regular Hayward black hole, like that of RN AdS case. In the present work, we will make a detailed study of the previously observed repulsive interaction.

The primary motivation for our research is due to the great interest on the regular black holes in black hole physics since they do not possess singularities. Wide variety of regular black holes exists, ranging from the first solution given by Bardeen [28], the later versions [29, 30], to the one on which we are interested, the Hayward black hole [31]. (We suggest the readers to go through our previous article [14] for the chronological discussion on this).
Hayward black hole is the solution to Einstein gravity non-linearly coupled to an electromagnetic field, which carries a magnetic charge. In this article, we probe the phase structure and repulsive interactions in the microstructure of this magnetically charged AdS black hole.

The article is organised as follows. After a brief introduction, we discuss the phase structure of the black hole in section II. Then the Ruppeiner geometry for the black hole is constructed for microstructure scrutiny (section III). Then we present our findings in section IV.

II. PHASE STRUCTURE OF THE HAYWARD ADS BLACK HOLE

The Hayward black hole solution in the four dimensional AdS background is given by [32, 33] (see [14] for a brief explanation),

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, \]

where \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \) and the metric function,

\[ f(r) = 1 - \frac{2Mr^2}{g^3 + r^3} + \frac{8\pi Pr^2}{3}. \]

We study the phase structure in the extended phase space where the cosmological constant \( \Lambda \) gives the pressure term \( P = -\Lambda/8\pi \). The parameter \( g \) is related to the total magnetic charge of the black hole \( Q_m \) as,

\[ Q_m = \frac{g^2}{\sqrt{2\alpha}}. \]

where \( \alpha \) a free integration constant. The thermodynamic quantities temperature, volume and entropy of the black hole are easily obtained to be,

\[ T = \frac{f'(r_+)}{4\pi} = \frac{2Pr^4}{g^3 + r^3} - \frac{g^3}{2\pi r (g^3 + r^3)} + \frac{r^2}{4\pi (g^3 + r^3)}; \]

\[ V = \frac{4}{3}\pi(g^3 + r^3) \quad \text{and} \quad S = 2\pi\left(\frac{r^2}{2} - \frac{g^3}{r}\right). \]

These results are consistent with the first law

\[ dM = T dS + \Psi dQ_m + V dP + \Pi d\alpha, \]

and the Smarr relation,

\[ M = 2(TS - VP + \Pi \alpha) + \Psi Q_m. \]
The heat capacity of the black hole system at constant volume is,

\[ C_V = T \left( \frac{\partial S}{\partial T} \right)_V = 0. \tag{8} \]

Inverting the expression for the Hawking temperature (4) we get the equation of state,

\[ P = \frac{g^3}{4\pi r^5} + \frac{g^3 T}{2r^4} - \frac{1}{8\pi r^2} + \frac{T}{2r}. \tag{9} \]

From the state equation one can see that the black hole shows critical behaviour similar to van der Waals system. This often interpreted as the transition between a small black hole and a large black hole phases. In our earlier studies [14], we have shown that an alternate interpretation is possible, using Landau theory of continuous phase transition, where the phase transition is between the black hole phases at different potentials. In this alternate view the black hole phases, namely high potential, intermediate potential and low potential phases, are determined by the magnetic charge. In either of these interpretations the phase transition can be studied by choosing a pair of conjugate variables like \((P - V)\) or \((T - S)\). With the conjugate pair \((P, V)\), the Maxwell’s equal area law has the form,

\[ P_0(V_2 - V_1) = \int_{V_1}^{V_2} P dV. \tag{10} \]

Since there exists no analytical expression for the coexistence curve for Hayward AdS black hole we seek numerical solutions most of the time. For that, we obtain the key ingredient from the Maxwell’s equal area law. Using the equation (10) and expressions for \(P_0(V_1)\) and \(P_0(V_2)\) from equation of state we get,

\[ r_2 = g \left[ \frac{x (x^3 + 6x^2 + 6x + 1) + \sqrt{y}}{x^4} \right]^{1/3}, \tag{11} \]

\[ P_0 = \frac{3 \left[ \sqrt[3]{\frac{\sqrt{y} + x(x^3 + 6x^2 + 6x + 1)}{x^4}} \right]^{1/3} \left[ (-2x^4 - 11x^3 - 20x^2 - 11x - 2) \sqrt{y} + z \right]}{16\pi g^2 x (x^2 + 4x + 1) (3x^2 + 4x + 3)^2}, \tag{12} \]

\[ T_0 = \frac{\left[ \frac{\sqrt[3]{\sqrt{y} + x(x^3 + 6x^2 + 6x + 1)}}{x^4} \right]^{2/3} \left[ u = (x^3 + 4x^2 + 4x + 1) \sqrt{y} \right]}{4\pi gx (3x^4 + 16x^3 + 22x^2 + 16x + 3)}. \tag{13} \]

Where

\[ y = x^2 (x^6 + 12x^5 + 54x^4 + 82x^3 + 54x^2 + 12x + 1), \tag{14} \]

\[ z = x (2x^7 + 23x^6 + 104x^5 + 213x^4 + 213x^3 + 104x^2 + 23x + 2), \tag{15} \]
\[ u = x \left( x^6 + 10x^5 + 37x^4 + 54x^3 + 37x^2 + 10x + 1 \right). \]  

(16)

We have taken \( x = r_1/r_2 \), where \( r_1 \) and \( r_2 \) are the radii of black holes for first order phase transition points. The critical values are readily obtained by setting \( x = 1 \),

\[ T_c = \frac{(5\sqrt{2} - 4\sqrt{3}) (3\sqrt{6} + 7)^{2/3}}{4 \sqrt[5/6]{\pi g}}, \]  

(17)

\[ P_c = \frac{3 (\sqrt{6} + 3)}{16 \sqrt[2/3]{(3\sqrt{6} + 7)^{5/3}} \pi g^2}, \]  

(18)

and

\[ V_c = 4 (2\sqrt{6} + 5) \pi g^3. \]  

(19)

The reduced thermodynamic variables are defined as,

\[ T_r = \frac{T}{T_c}, \quad P_r = \frac{P}{P_c}, \quad V_r = \frac{V}{V_c}. \]  

(20)

Using these we can write the equation of state in the reduced parameter space,

\[ P_r = \frac{2^{2/3} (3\sqrt{6} + 7)^{5/3}}{(\sqrt{6} + 3) \left[ 3 (2\sqrt{6} + 5) V_r - 1 \right]^{5/3}} \left[ V_r \left( (6\sqrt{6} + 14) \sqrt[2/3]{(3\sqrt{6} + 7)^{2/3}} T_r \left( 3 (2\sqrt{6} + 5) V_r - 1 \right)^{1/3} - 4\sqrt{6} - 10 \right) + 2 \right]. \]  

(21)

The reduced equation of state is independent of the magnetic charge parameter \( g \). From the reduced state equation we obtain the spinodal curve, which separates metastable phases from the unstable phase, using the condition,

\[ (\partial_{V_r} P_r)_{T_r} = 0. \]  

(22)

The explicit form of spinodal curve is,

\[ T_{rsp} = \frac{2^{4/3} (2\sqrt{6} + 5) \left[ (2\sqrt{6} + 5) V_r - 2 \right]}{3 \sqrt[2/3]{(2\sqrt{6} + 5) V_r + 1} \left[ 3 (2\sqrt{6} + 5) V_r - 1 \right]^{1/3}}. \]  

(23)

Solving this for \( V_r \) and substituting in equation (21) we obtain the curve in \( P-V \) plane. The spinodal curve along with the coexistence curve display the stable, unstable and metastable phases of the black hole. The coexistence curve is obtained numerically using the equations (11), (12) and (13). The spinodal and coexistence curves are shown together in fig 1. Actually, by fitting the coexistence curve in \( P-T \) plane we have obtained the following expression,

\[ P_r = 5.622 \times 10^{-7} - 5.539 \times 10^{-5} T_r + 0.693 T_r^2 + 0.1365 T_r^3 + 0.1966 T_r^4 \\
- 0.4255 T_r^5 + 1.134 T_r^6 - 1.698 T_r^7 + 1.621 T_r^8 - 0.8651 T_r^9 + 0.2085 T_r^{10}. \]  

(24)
Figure 1: Phase structure of Hayward AdS black hole in \(P-T\) and \(T-V\) diagrams. The coexistence curve is shown in red (solid) colour and the spinodal curves are shown in blue (dashed) colour.

The small black hole, large black hole and supercritical black hole phases are depicted in fig 1(a). The coexistence curve separates the small black hole and large black hole phases. It terminates at the critical point, after which the distinction between the SBH and LBH states is not possible, hence corresponds to the supercritical black holes. The region between the coexistence curve and spinodal curve corresponds to the metastable states, namely the supercooled LBH and the superheated SBH, which are shown in the \(T-V\) diagram. An observable feature in these diagrams is that the spinodal and coexistence curves meet each other at the critical point. Another important property associated with the spinodal curve is that the Ruppeiner scalar curvature diverges at that curve (section III).

Now, we would like to study the change in volume at the black hole phase transition as a function of temperature and pressure. Using equation (11), we make the functional change \(V(r) \rightarrow V(x)\) to obtain a parametric expression for \(\Delta V_r\). The parametric expression of \(\Delta V_r\) along with that of \(T_r\) and \(P_r\) (equations 13 and 12) are used to plot fig 2, which gives the behaviour of \(\Delta V_r\). From the figure 2 it is clear that, \(\Delta V_r\) decreases with increase in both temperature and pressure. It approaches zero at the critical point \((T_r = 1\) and \(P_r = 1\)). The behaviour near the critical point is,

\[
\Delta V_r \sim (1 - T_r)^{1/2} \sim (1 - P_r)^{1/2}.
\]  
(25)

This suggests that the change in volume \(\Delta V_r\) can serve as the order parameter to characterise the black hole phase transition, with the universal critical exponent \(1/2\).
III. MICROSTRUCTURE OF THE HAYWARD ADS BLACK HOLE

In this section we examine the microstructure of the black hole using Ruppeiner geometry in which $T$ and $V$ are taken as fluctuation coordinates. The line element in these parameter space has the form [7],

$$dl^2 = \frac{C_V}{T^2}dT^2 - \frac{(\partial_V P)_T}{T}dV^2.$$  \hfill (26)

The heat capacity $C_V$ vanishes for the Hayward AdS black hole (equation 8). This makes the line element (26) singular, and hence the corresponding geometry will not give the information regarding the microstructure of the black hole. Therefore the normalised scalar curvature is used for studying the microscopic interactions,

$$R_N = C_V R.$$  \hfill (27)

From a straightforward calculation, for the Hayward AdS black hole we obtain,

$$R_N = \frac{(8\pi g^3 - V) \left\{ 8g^3 \left[ \pi^{5/3} T (6V - 8\pi g^3)^{1/3} + \pi \right] + V \left[ 2\pi^{2/3} T (6V - 8\pi g^3)^{1/3} - 1 \right] \right\}}{2 \left\{ 4\pi g^3 \left[ \pi^{2/3} T (6V - 8\pi g^3)^{1/3} + 2 \right] + V \left[ \pi^{2/3} T (6V - 8\pi g^3)^{1/3} - 1 \right] \right\}^2}.$$

\hfill (28)

In terms of the reduced parameters,

$$R_N = \frac{4 \left[ (2\sqrt{6} + 5) V_r - 2 \right] \left[ -A(T_r, V_r) + 2 \left( 2\sqrt{6} + 5 \right) V_r - 4 \right]}{[A - 4 \left( 2\sqrt{6} + 5 \right) V_r + 8]^2},$$

\hfill (29)
where,

$$A(T_r, V_r) = 2^{1/6} \left(3\sqrt{6} + 7\right)^{2/3} T_r \left(\sqrt{2}V_r + 5\sqrt{2} - 4\sqrt{3}\right) \left(3 \left(2\sqrt{6} + 5\right) V_r - 1\right)^{1/3}. \quad (30)$$

Similar to the case of charged AdS black hole and Gauss Bonnet black hole $R_N$ is independent of $g$. The behaviour of $R_N$ with reduced volume $V_r$ for fixed temperature is studied in fig (3).

Figure 3: The behaviour of the normalised curvature scalar $R_N$ against the reduced volume $V_r$ at constant temperature.

For $T_r < 1$, below critical temperature, $R_N$ has two negative divergence points. They come nearer as the temperature increases and merge together at $V_r = 1$ for $T_r = 1$. These divergences do not exist for temperatures greater than the critical value. We see that always there exist small regions where the curvature scalar is positive (shown in inlets). We need
to examine whether these regions are thermodynamically stable. Setting $R_N = 0$ we get,

$$T_0 = \frac{T_{r_{sp}}}{2} = \frac{25/6 \left[ (2\sqrt{6} + 5) V_r - 2 \right]}{(3\sqrt{6} + 7)^{2/3} \left( \sqrt{2}V_r + 5\sqrt{2} - 4\sqrt{3} \right) \left[ 3 \left( 2\sqrt{6} + 5 \right) V_r - 1 \right]}.$$  \hspace{1cm} (31)

This is the sign-changing temperature, which is half of the spinodal curve temperature as in vdW system, RN AdS and Gauss-Bonnet black holes. Another solution for this is,

$$V_r = \frac{2}{2\sqrt{6} + 5} \equiv V_0.$$  \hspace{1cm} (32)

The normalised curvature scalar $R_N$ diverges along the spinodal curve. The regions under the spinodal curve for $V_r > V_0$, $R_N$ is positive. This region corresponds to the coexistence phase of SBH and LBH, similar to van der Waals fluid’s coexistence phase. Everywhere below $V_0$, $R_N$ is positive, including region below and above the coexistence curve. The region under the coexistence curve is the same as the previous case, a coexistence phase. However in the region above the curve, small black hole phase, we can safely say that the black hole molecules possess repulsive interaction. Therefore in Hayward black hole for a small parameter range there exist dominant repulsive interaction. This result is similar to RN-AdS black hole and in contrast to five-dimensional neutral Gauss-Bonnet, where there is no repulsive interaction in the latter case.

![Figure 4](image)

**Figure 4:** 4(a): The sign changing curve of $R_N$ along with the coexistence and spinodal curves. 4(b): The behaviour of normalised curvature scalar $R_N$ along the coexistence line. The red (solid) line and blue (dashed) line corresponds to large black hole and small black hole, respectively. The region where the SBH branch takes positive $R_N$ value is zoomed in in the inlet.
Finally, we consider the behaviour of the scalar curvature $R_N$ along the coexistence curve. Since there exists no analytical expression for the coexistence curve, the analytic study of the curvature scalar behaviour near the critical point is not possible. The numerical solution is obtained and shown in fig 4(b). Both the SBH branch and LBH branch of $R_N$ have the divergence near the critical point. For a large black hole, the sign of $R_N$ is always negative and hence the microscopic interaction is always attractive. Interestingly, for the small black hole, there is a lower temperature range where $R_N$ is positive (zoomed-in in the inlet). This indicates a repulsive interaction between the black hole molecules. From this, we can conclude that in the low-temperature regime the microstructure, as well as microscopic interaction of the black hole, changes drastically during the phase transition. Whereas in the high-temperature range only microstructure changes and the nature of interaction remains attractive in both phases. These results are strikingly different from that of van der Waals fluid, where the dominant interaction among the molecules is always attractive.

IV. DISCUSSIONS

In this paper, we have studied the phase transitions and microstructure of the Hayward AdS black hole. The microscopic properties are analysed from the behaviour of Ruppeiner curvature scalar along the coexistence curve. Since an analytical expression for the coexistence curve is not feasible we have carried out our investigation numerically. In the first part of the paper, we probed the phase structure of the black hole using the coexistence curve in $P_r - T_r$ and $T_r - V_r$ planes. Along with this the spinodal curve also displayed, which enable us to identify the metastable phases of the black holes, namely the superheated small black hole and the supercooled large black hole. It is shown that the change in volume $\Delta V_r$ during the small black hole - large black hole phase transition can serve an order parameter to describe the same. The behaviour of $\Delta V_r$ has a critical exponent $1/2$ which is universal.

In the second part of this article, we have focused on the Ruppeiner geometry of the black hole. We have adopted the definition of curvature scalar given in the ref. [7], where the fluctuation coordinates are temperature and volume. The normalised curvature scalar diverges to the negative infinity at the critical point. Even though the black hole shows van der Waals like phase transition, the microstructure properties differ in some aspects. In van der Waals fluid the dominant interaction among the constituent molecules is always
attractive, which does not change during the phase transition. The change in microstructure does not lead to any change in the nature of microscopic interaction. However, in Hayward black hole there exists a domain, low-temperature range for the small black hole, where the dominant interaction between the black hole molecules is repulsive. This is inferred from the positive sign of the normalised curvature scalar. During the phase transition, in this temperature range, the microscopic interaction of the black hole changes significantly. This result is similar to what is observed in RN AdS black hole and in contrast to the five-dimensional neutral Gauss-Bonnet black hole, where the interaction is always attractive like van der Waals fluid. To conclude, the magnetic charge in the Hayward black hole plays a similar role as the electric charge in RN AdS black hole in contributing to the microstructure. We believe that this is another significant step in understanding the black hole microstructure properties.

Acknowledgments

Author N.K.A. and A.R.C.L. would like to thank U.G.C. Govt. of India for financial assistance under UGC-NET-SRF scheme.

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