HAUSDORFF DIMENSION OF LEVEL SETS IN BETA-EXPANSIONS FOR PSEUDO-GOLDEN RATIOS

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ABSTRACT. We obtain an exact formula of the Hausdorff dimension of the level sets in beta-expansions for pseudo-golden ratios by using a variation formula. Before this, we prove that the Hausdorff dimension of an arbitrary set in the shift space is equal to its projection in [0, 1], and we clarify that for calculating the Hausdorff dimension of the level sets, one only needs to focus on the Markov measures when \( \beta \in (1, 2) \) and the beta-expansion of 1 is finite.

1. INTRODUCTION

Let \( \beta > 1 \) be a real number. Given \( x \in [0, 1) \), the most common way to \( \beta \)-expand \( x \) as

\[
x = \sum_{n=1}^{\infty} \frac{w_n}{\beta^n}
\]

is to use the greedy-algorithm. It is known as \( \beta \)-expansion, introduced by Rényi [12] in 1957 and studied in detail by Parry [10] in 1960.

We are interested in the digit frequency of \( \beta \)-expansions. More precisely, fix some \( a \in [0, 1) \) and let \( F_a \) be the set of those \( x \) with digit frequency of 0’s equal to \( a \) in its \( \beta \)-expansion, i.e., the level set

\[
F_a := \{ x \in [0, 1) : \lim_{n \to \infty} \frac{\#\{1 \leq k \leq n : \varepsilon_k(x, \beta) = 0\}}{n} = a \},
\]

where \( \varepsilon_k(x, \beta) \) is the \( k \)th digit in the \( \beta \)-expansion of \( x \). We use the Hausdorff dimension (see [3] for definition) to describe the size of \( F_a \).

For the case that \( \beta = \frac{\sqrt{5}+1}{2} \) is the golden ratio (i.e., the \( \beta \)-expansion of 1 is \( \varepsilon(1, \beta) = 110^\infty \)) and \( \frac{1}{2} \leq a \leq 1 \), it is well known (see for examples [6, 8]) that

\[
\dim_H F_a = \frac{a \log a - (2a - 1) \log(2a - 1) - (1 - a) \log(1 - a)}{\log \beta}.
\]

One of our main result Theorem 5.1 is for the case that \( \beta \) is a pseudo-golden ratio (i.e., there exists integer \( m \geq 3 \) such that \( \varepsilon(1, \beta) = 1^m0^\infty \) and \( \frac{1}{m} \leq a \leq 1 \), we have

\[
\dim_H F_a = \frac{1}{\log \beta} \cdot \max_{x_1, \ldots, x_{m-2}} f_a(x_1, \ldots, x_{m-2})
\]

where

\[
f_a(x_1, \ldots, x_{m-2}) = a \log a - (a - x_1) \log(a - x_1) - (x_1 - x_2) \log(x_1 - x_2) - \cdots - (x_{m-3} - x_{m-2}) \log(x_{m-3} - x_{m-2}) - (1 - a - x_1 - \cdots - x_{m-2}) \log(1 - a - x_1 - \cdots - x_{m-2}) - (x_1 + \cdots + x_{m-3} + 2x_{m-2} + a - 1) \log(x_1 + \cdots + x_{m-3} + 2x_{m-2} + a - 1) \text{ with } a \geq x_1 \geq
\]
\[ x_2 \geq \cdots \geq x_{m-2} \geq 0 \quad \text{and} \quad x_1 + \cdots + x_{m-3} + x_{m-2} \leq 1 - a \leq x_1 + \cdots + x_{m-3} + 2x_{m-2}, \]
i.e., all terms in the brackets in \( f_a \) are non-negative.

In particular for \( m = 3 \), \( \varepsilon(1, \beta) = 1110^\infty \) and \( \frac{1}{2} \leq a \leq 1 \), we have

\[
\dim_H F_a = \frac{1}{\log \beta} \left( a \log a - \frac{10a - 3 - \sqrt{-8a^2 + 12a - 3}}{6} \log \frac{10a - 3 - \sqrt{-8a^2 + 12a - 3}}{6}
- \frac{-2a + 3 - \sqrt{-8a^2 + 12a - 3}}{3} \log \frac{-2a + 3 - \sqrt{-8a^2 + 12a - 3}}{3}
- \frac{-a + \sqrt{-8a^2 + 12a - 3}}{3} \log \frac{-a + \sqrt{-8a^2 + 12a - 3}}{3} \right).
\]

This paper is organized as follows. In Section 2, we introduce some notations and preliminaries. In Section 3, we establish Theorem 3.1, which means that the Hausdorff dimension of an arbitrary set in the shift space is equal to its projection in \( [0, 1] \). In Section 4, we present the exact variation formula we are going to use, which clarifies a fact that for calculating the Hausdorff dimension of the level sets, one only needs to focus on the Markov measures when \( \beta \in (1, 2) \) and the beta-expansion of 1 is finite. In Section 5, we give the statement and the proof of the exact formula of the Hausdorff dimension of the level sets for pseudo-golden ratios.

Throughout this paper, \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{N} \) denotes the set of positive integers \( \{1, 2, 3, \cdots \} \).

## 2. Notations and Preliminaries

Define the \( \beta \)-transformation \( T_\beta : [0, 1] \rightarrow [0, 1] \) by

\[ T_\beta(x) := \beta x - \lfloor \beta x \rfloor \quad \text{for} \quad x \in [0, 1] \]

where \( \lfloor y \rfloor := \max \{ i : \text{integer} \ i \leq y \} \). We use \( A_\beta \) to denote \( \{0, 1, \cdots, \beta - 1\} \) and \( \{0, 1, \cdots, \lfloor \beta \rfloor\} \) for \( \beta \in \mathbb{N} \) and \( \beta \notin \mathbb{N} \) respectively. For any \( n \in \mathbb{N} \) and \( x \in [0, 1] \), define

\[ \varepsilon_n(x, \beta) := [\beta T_\beta^{n-1}(x)] \in A_\beta. \]

Then we can write

\[ x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x, \beta)}{\beta^n} \]

and call the sequence \( \varepsilon(x, \beta) := \varepsilon_1(x, \beta) \varepsilon_2(x, \beta) \cdots \varepsilon_n(x, \beta) \cdots \) the beta-expansion of \( x \).

We say that \( \varepsilon(x, \beta) \) is infinite if there are infinitely many \( n \in \mathbb{N} \) such that \( \varepsilon_n(x, \beta) \neq 0 \). Otherwise, there exists a smallest \( m \in \mathbb{N} \) such that for any \( j > m, \varepsilon_j(x, \beta) = 0 \) but \( \varepsilon_m(x, \beta) \neq 0 \) and we say that \( \varepsilon(x, \beta) \) is finite with length \( m \).

The modified \( \beta \)-expansion of 1 defined by

\[ \varepsilon^*(1, \beta) := \left\{ \begin{array}{ll}
\varepsilon(1, \beta) & \text{if} \ (1, \beta) \text{ is infinite} \\
(\varepsilon(1, \beta) \cdots \varepsilon_{m-1}(1, \beta)(\varepsilon_m(1, \beta) - 1))^\infty & \text{if} \ (1, \beta) \text{ is finite with length } m
\end{array} \right. \]

is very useful for showing the admissibility of a sequence (see for example Lemma 2.2).

Let \( d_\beta \) be the usual metric on \( A_\beta^\mathbb{N} \) defined by

\[ d_\beta(w, v) := \beta^{-\inf\{k \geq 0 : w_{k+1} \neq v_{k+1}\}} \quad \text{for} \ w, v \in A_\beta^\mathbb{N} \]

and \( \sigma : A_\beta^\mathbb{N} \rightarrow A_\beta^\mathbb{N} \) be the shift defined by

\[ \sigma(w_1w_2 \cdots) = w_2w_3 \cdots \quad \text{for} \ w \in A_\beta^\mathbb{N}. \]
Then $\sigma$ is continuous.

**Definition 2.1** (Admissibility). Let $w$ be a sequence in $\mathcal{A}_\beta^\mathbb{N}$. If there exists $x \in [0, 1)$ such that $\varepsilon(x, \beta) = w$, we say that $w$ is admissible. The set of admissible sequences is denoted by $\Sigma_\beta$.

We use $S_\beta$ to denote the (topological) closure of $\Sigma_\beta$ in $(\mathcal{A}_\beta^\mathbb{N}, d_\beta)$ and $\pi_\beta : S_\beta \to [0, 1]$ to denote the continuous projection map defined by

$$\pi_\beta(w) := \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n}{\beta^n} + \cdots \text{ for } w \in S_\beta.$$  

The following criterion due to Parry is well known.

**Lemma 2.2** ([10]). Let $w$ be a sequence in $\mathcal{A}_\beta^\mathbb{N}$. Then

$$w \in \Sigma_\beta \iff \sigma^k(w) \prec \varepsilon^*(1, \beta) \text{ for all } k \geq 0$$

and

$$w \in S_\beta \iff \sigma^k(w) \preceq \varepsilon^*(1, \beta) \text{ for all } k \geq 0$$

where $\prec$ and $\preceq$ denote the lexicographic order in $\mathcal{A}_\beta^\mathbb{N}$.

**Definition 2.3** (Cylinder). Let $x \in [0, 1)$ and $n \in \mathbb{N}$. The cylinder of order $n$ containing $x$ is defined by

$$I_n(x) := \{ y \in [0, 1) : \forall k \text{ with } 1 \leq k \leq n, \varepsilon_k(y, \beta) = \varepsilon_k(x, \beta) \}.$$  

The following covering property given by Bugeaud and Wang can be deduced from the length and distribution of full cylinders (see [2, 5, 9] for definition and more details).

**Proposition 2.4** ([2]). Let $\beta > 1$. For any $x \in [0, 1)$ and any positive integer $n$, the ball $B(x, \beta^{-n})$ intersected with $[0, 1)$ can be covered by at most $4(n + 1)$ cylinders of order $n$.

**Definition 2.5** (Hausdorff measure and dimension in metric space). Let $(X, d)$ be a metric space. For any $U \subset X$, denote the diameter of $U$ by $\|U\| := \sup_{x, y \in U} d(x, y)$. For any $A \subset X, s \geq 0$ and $\delta > 0$, write

$$\mathcal{H}^s_\delta(A, d) := \inf \left\{ \sum_i |U_i|^s : A \subset \bigcup_i U_i, |U_i| \leq \delta \right\}.$$  

We define the $s$-dimensional Hausdorff measure of $A$ in $(X, d)$ by

$$\mathcal{H}^s(A, d) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A, d)$$

and the Hausdorff dimension of $A$ in $(X, d)$ by

$$\dim_H(A, d) := \sup\{ s \geq 0 : \mathcal{H}^s(A, d) = \infty \}.$$  

In $\mathbb{R}$ (equipped with usual metric), we use $\mathcal{H}^s(A)$ and $\dim_H A$ to denote the $s$-dimensional Hausdorff measure and the Hausdorff dimension of $A$ respectively for simplification.
3. Dimension equality

For any $\beta > 1$, the projection map $\pi_\beta : S_\beta \to [0, 1]$ is Lipschitz continuous, which implies that the Hausdorff dimension of a set in the shift space $(S_\beta, d_\beta)$ is larger than or equal to its projection in $[0, 1]$. But even if omitting countable many points, the inverse of the projection is not continuous (which implies not Lipschitz). It means that the inverse inequality is not so obvious. But we can still establish the following theorem as the main result in this section.

**Theorem 3.1.** Let $\beta > 1$. For any $Z \subset S_\beta$, we have

$$\dim_H(Z, d_\beta) = \dim_H \pi_\beta(Z).$$

**Proof.** $\geq$ follows from the fact that $\pi_\beta : S_\beta \to [0, 1]$ is Lipschitz continuous.

$\leq$ follows from Lemma 3.2. In fact, for any $t < \dim_H(Z, d_\beta)$, there exists $s$ such that $t < s < \dim_H(Z, d_\beta)$. By $H^s(Z, d_\beta) = \infty$ and Lemma 3.2, we get $H^t(\pi_\beta(Z)) = \infty$. Thus $t \leq \dim_H \pi_\beta(Z)$. □

**Lemma 3.2.** Let $\beta > 1$, $s > 0$ and $Z \subset S_\beta$. Then for any $\varepsilon \in (0, s)$, $H^s(Z, d_\beta) \leq H^{s-\varepsilon}(\pi_\beta(Z))$.

**Proof.** Fix $\varepsilon \in (0, s)$.

(1) Choose $\delta_0 > 0$ small enough as below.

Since $\beta^{(n+1)\varepsilon} \to \infty$ much faster than $8\beta^n \to \infty$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $8\beta^n \leq \beta^{(n+1)\varepsilon}$. By $-\frac{\log \delta_0}{\log \beta} - 1 \to \infty$ as $\delta \to 0^+$, there exists $\delta_0 > 0$ small enough such that $-\frac{\log \delta_0}{\log \beta} - 1 > n_0$. Then for any $n > -\frac{\log \delta_0}{\log \beta} - 1$, we will have $8\beta^n \leq \beta^{(n+1)\varepsilon}$.

(2) In order to get the conclusion, it suffices to prove that for any $\delta \in (0, \delta_0)$, we have $H^s_\delta(Z, d_\beta) \leq H^{s-\varepsilon}_\delta(\pi_\beta(Z))$.

Fix $\delta \in (0, \delta_0)$. Let $\{U_i\}$ be a $\delta$-cover of $\pi_\beta(Z)$, i.e., $0 < |U_i| \leq \delta$ and $\pi_\beta(Z) \subset \cup_i U_i$. Then for each $U_i$, there exists $n_i \in \mathbb{N}$ such that $\frac{1}{\beta^{n_i}} < |U_i| \leq \frac{1}{\beta^{n_i+1}}$. By Proposition 2.4, $U_i$ can be covered by at most $8n_i$ cylinders $I_{i,1}, I_{i,2}, \ldots, I_{i,8n_i}$ of order $n_i$. Noting that

$$|\pi^{-1}_\beta I_{i,j}| = \frac{1}{\beta^{n_i}} < \beta |U_i| \leq \beta \delta$$

and $Z \subset \bigcup_i \pi^{-1}_\beta U_i \subset \bigcup_i \bigcup_{j=1}^{8n_i} \pi^{-1}_\beta I_{i,j}$,

we get

$$H^s_\delta(Z, d_\beta) \leq \sum_i \sum_{j=1}^{8n_i} |\pi^{-1}_\beta I_{i,j}|^s = \sum_i \frac{8n_i}{\beta^{n_i s}} \leq \sum_i \frac{1}{\beta^{(n_i+1)(s-\varepsilon)}} < \sum_i |U_i|^{s-\varepsilon}.$$

Taking inf on the right, we conclude that $H^s_\delta(Z, d_\beta) \leq H^{s-\varepsilon}_\delta(\pi_\beta(Z))$.

(\text{*}) is because $\frac{1}{\beta^{n_i+1}} < |U_i| < \delta_0$ implies $n_i > -\frac{\log \delta_0}{\log \beta} - 1$ and then by (1), $8n_i \beta^s \leq \beta^{(n_i+1)\varepsilon}$.) □

4. Variation formula

First we need the following concept (see Section 6.2 in [7] for more details).

**Definition 4.1** ($k$-step Markov measure). Let $k \in \mathbb{N}$ and $\mu \in M_\mu(S_\beta)$ (the set of $\sigma$-invariant Borel probability measures on $S_\beta$). We call $\mu$ a $k$-step Markov measure if there exists an $1 \times 2^k$ probability vector $p = (p_{i_1,\ldots,i_k})_{i_1,\ldots,i_k=0,1}$ (i.e., $\sum_{i_1,\ldots,i_k=0,1} p_{i_1,\ldots,i_k} = 1$). If $\mu$ is a $k$-step Markov measure and $\pi_\beta : S_\beta \to [0, 1]$ is Lipschitz continuous, then $H^{s-\varepsilon}_\delta(\pi_\beta(Z)) \leq H^s_\delta(Z, d_\beta) \leq H^{s-\varepsilon}_\delta(\pi_\beta(Z))$. Therefore, $H^s_\delta(Z, d_\beta) = H^{s-\varepsilon}_\delta(\pi_\beta(Z))$.

□
1 and \( p_{(i_1 \cdots i_k)} \geq 0 \) for all \( i_1, \cdots, i_k \in \{0, 1\} \) and a \( 2^k \times 2^k \) stochastic matrix \( P = (P_{(i_1 \cdots i_k)(j_1 \cdots j_k)})_{i_1, \cdots, i_k, j_1, \cdots, j_k = 0, 1} \) (i.e., \( \sum_{j_1, \cdots, j_k = 0, 1} P_{(i_1 \cdots i_k)(j_1 \cdots j_k)} = 1 \) for all \( i_1, \cdots, i_k \in \{0, 1\} \) and \( P_{(i_1 \cdots i_k)(j_1 \cdots j_k)} \geq 0 \) for all \( i_1, \cdots, i_k, j_1, \cdots, j_k \in \{0, 1\} \)) with \( pP = p \) such that

\[
\mu[i_1 \cdots i_k] = p_{(i_1 \cdots i_k)}
\]

for all \( i_1, \cdots, i_k \in \{0, 1\} \) and

\[
\mu[i_1 \cdots i_n] = p_{(i_1 \cdots i_k)} p_{(i_2 \cdots i_{k+1})} P_{(i_{k+2} \cdots i_{k+2})} \cdots P_{(i_{n-k+1} \cdots i_{n-k+1})}
\]

for all \( i_1, \cdots, i_n \in \{0, 1\} \) and \( n > k \), where

\[
[w] := \{ v \in S_\beta : v \text{ begins with } w \}
\]

for any finite word \( w \) on \( \{0, 1\} \).

In this section, we are going to establish the following variation formula which will be used to prove our main result in the next section.

**Theorem 4.2.** Let \( \beta \in (1, 2) \) such that \( \varepsilon(1, \beta) = \varepsilon_1(1, \beta) \cdots \varepsilon_m(1, \beta)0^\infty \) for some integer \( m \geq 2 \) with \( \varepsilon_m(1, \beta) = 1 \) and let \( a \in [0, 1] \). Then

\[
\dim_H F_a = \frac{1}{\log \beta} \sup \{ h_\mu(\sigma) : \mu \in \mathcal{M}_a(S_\beta), \mu[0] = a, \mu \text{ is an } (m-1)\text{-step Markov measure} \}
\]

where \( \sup \emptyset := 0 \), \( F_a \) is the lever set defined in Section 1 and \( h_\mu(\sigma) \) is the measure-theoretic entropy of \( \sigma \) with respect to \( \mu \) (see [16] for definition).

Compared to the following variation formula which is essentially from [11], Theorem 4.2 means that for the dimension of the level set, we only need to focus on the Markov measures when \( \beta \in (1, 2) \) and \( \varepsilon(1, \beta) \) is finite.

**Proposition 4.3.** Let \( \beta > 1 \) and \( a \in [0, 1] \). Then

\[
\dim_H F_a = \frac{1}{\log \beta} \sup \{ h_\mu(\sigma) : \mu \in \mathcal{M}_a(S_\beta), \mu[0] = a \}
\]

where \( \sup \emptyset := 0 \).

For the convenience of the readers, we recall some definitions and show how Proposition 4.3 comes from [11].

**Definition 4.4.** Let \( \beta > 1 \).

1. For any \( w \in S_\beta \) and \( n \in \mathbb{N} \), the empirical measure is defined by

\[
\mathcal{E}_n(w) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\sigma^i w}
\]

where \( \delta_w \) is the Dirac probability measure concentrated on \( w \).

2. Let \( \mathcal{A} \) be an arbitrary non-empty parameter set. Write

\[
\mathcal{F} := \{(f_\alpha, c_\alpha, d_\alpha) : \alpha \in \mathcal{A}\}
\]

where \( f_\alpha : S_\beta \to \mathbb{R} \) is continuous and \( c_\alpha \leq d_\alpha \in \mathbb{R} \) for all \( \alpha \in \mathcal{A} \). Define

\[
S_{\beta, \mathcal{F}} := \{ w \in S_\beta : \forall \alpha \in \mathcal{A}, c_\alpha \leq \lim_{n \to \infty} \int f_\alpha d\mathcal{E}_n(w) \leq \lim_{n \to \infty} \int f_\alpha d\mathcal{E}_n(w) \leq d_\alpha \}
\]

and

\[
\mathcal{M}_{\beta, \mathcal{F}} := \{ \mu \in \mathcal{M}_a(S_\beta) : \forall \alpha \in \mathcal{A}, c_\alpha \leq \int f_\alpha d\mu \leq d_\alpha \}.
\]
Combining Theorem 5.2 and Theorem 5.3 in [11], we get the following.

**Lemma 4.5.** Let $\beta > 1$. If $\mathcal{M}_{\beta, F}$ is a non-empty closed connected set, then

\[
    h_{\text{top}}(S_{\beta, F}, \sigma) = \sup \{ h_\mu(\sigma) : \mu \in \mathcal{M}_{\beta, F} \}
\]

where $h_{\text{top}}(S_{\beta, F}, \sigma)$ is the topological entropy of $S_{\beta, F}$ in the dynamical system $(S_\beta, d_\beta, \sigma)$. (See [1] for the definition of the topological entropy for non-compact sets.)

For more results about variation formulas, see for examples [4], [13] and [14].

For $\beta > 1$ and $0 \leq a \leq 1$, let

\[
    S_a := \{ w \in S_\beta : \lim_{n \to \infty} \frac{\# \{ 1 \leq k \leq n : w_k = a \} }{n} = a \}.
\]

In Definition 4.4 (2), let $\mathcal{F}$ be the singleton $\{(\mathbb{1}_0, a, a)\}$ where the characteristic function $\mathbb{1}_0 : S_\beta \to \mathbb{R}$ is continuous. (Here we note that another characteristic function $\mathbb{1}_{(0, \frac{1}{2})} : [0, 1] \to \mathbb{R}$ is not continuous, which means that some other similar variation formulas corresponding to the dynamical systems in $[0,1]$ can not be applied directly in our case.) We get the following lemma as a special case of the above one.

**Lemma 4.6.**

\[
    h_{\text{top}}(S_a, \sigma) = \sup \{ h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(S_\beta), \mu[0] = a \}
\]

where $\sup \emptyset := 0$.

Hence, Proposition 4.3 follows from

\[
    \dim_H F_a \begin{array}{c} \frac{\pi_\beta(S_a) \cap F_a}{\pi_\beta(S_a)} \end{array} \text{dim}_H \pi_\beta(S_a) \begin{array}{c} \text{by Theorem 3.1} \end{array} \text{dim}_H(S_a, d_\beta) \begin{array}{c} \text{by Lemma 4.7} \end{array} \frac{1}{\log \beta} \cdot h_{\text{top}}(S_a, \sigma).
\]

**Lemma 4.7.** [15, Lemma 5.3] Let $\beta > 1$. For any $Z \subset S_\beta$, we have

\[
    \text{dim}_H(Z, d_\beta) = \frac{1}{\log \beta} \cdot h_{\text{top}}(Z, \sigma).
\]

We give the following to end this section.

**Proof of Theorem 4.2.** $\geq$ follows immediately from Proposition 4.3.

$\leq$ (We still use Proposition 4.3.) Let $\mu \in \mathcal{M}_\sigma(S_\beta)$ with $\mu[0] = a$. For any $i_1, \ldots, i_m \in \{0, 1\}$, let $p_{(i_1 \cdots i_{m-1})} := \mu[i_1 \cdots i_{m-1}]$. Then $p = (p_{(i_1 \cdots i_{m-1})})_{i_1, \ldots, i_{m-1} = 0, 1}$ is an $1 \times 2^{m-1}$ probability vector. We define a $2^{m-1} \times 2^{m-1}$ stochastic matrix

\[
    P = (P_{(i_1 \cdots i_{m-1} j_2 \cdots j_m)}: i_1, \ldots, i_{m-1}, j_2, \ldots, j_m = 0, 1)
\]

as follows.

i) If there exists integer $k$ with $2 \leq k \leq m - 1$ such that $i_k \neq j_k$, let

\[
    P_{(i_1 i_2 \cdots i_{m-1} j_2 \cdots j_{m-1})} := 0;
\]

ii) If $\mu[i_1 \cdots i_{m-1}] \neq 0$, let

\[
    P_{(i_1 \cdots i_{m-1} i_2 \cdots i_m)} := \frac{\mu[i_1 \cdots i_m]}{\mu[i_1 \cdots i_{m-1}]} \quad \text{for } i_m = 0, 1;
\]

iii) If $\mu[i_1 \cdots i_{m-1}] = 0$, let

\[
    P_{(i_1 \cdots i_{m-1} i_2 \cdots i_{m-1} 0)} := 1 \quad \text{and} \quad P_{(i_1 \cdots i_{m-1} i_2 \cdots i_{m-1} 1)} := 0.
\]
(In fact, it suffices to define \( P_{(i_1 \cdots i_{m-1})} + P_{(i_1 \cdots i_{m-1})} = 1 \) in this case.) Then \( \sum_{j_2, \cdots, j_m=0,1} P_{(i_1 \cdots i_{m-1})} = 1 \) for all \( i_1, \cdots, i_{m-1} \in \{0,1\} \) and \( pP = p \). For any \( i_1, \cdots, i_n \in \{0,1\} \), define
\[
\mu_m[i_1 \cdots i_n] := \left\{ \begin{array}{ll}
\mu[i_1 \cdots i_n] & \text{if } n \leq m; \\
\mu_P[i_1 \cdots i_n] P_{(i_1 \cdots i_{m-1})} P_{(i_2 \cdots i_{m-1})} \cdots P_{(i_{m-1} \cdots i_{m-1})} & \text{if } n > m.
\end{array} \right.
\]

It is not difficult to check
\[
\sum_{j=0}^{n-1} \mu_m[j] = 1, \quad \sum_{j=0}^{n-1} \mu_m[i_1 \cdots i_n j] = \mu_m[i_1 \cdots i_n] \quad \text{and} \quad \sum_{j=0}^{n-1} \mu_m[j i_1 \cdots i_n] = \mu_m[i_1 \cdots i_n]
\]
for any \( n \geq 1 \), and \( \mu_m \) can be extended to become an \((m-1)\)-step Markov measure in \( \mathcal{M}_\mu(S_\beta) \) with \( \mu_m[0] = a \). Using \( \mathcal{P} := \{[0], [1]\} \) as a partition generator of the Borel sigma-algebra on \((S_\beta, d_\beta)\), by classical calculation, we know that \( h_{\mu_m}(\sigma) \) is equal to the conditional entropy of \( \mathcal{P} \) given \( \bigvee_{k=1}^{m-1} \sigma^{-k} \mathcal{P} \) with respect to \( \mu \), i.e.,
\[
h_{\mu_m}(\sigma) = H_{\mu}(\mathcal{P} | \bigvee_{k=1}^{m-1} \sigma^{-k} \mathcal{P}).
\]

Since \( H_{\mu}(\mathcal{P} | \bigvee_{k=1}^{m-1} \sigma^{-k} \mathcal{P}) \) decreases as \( n \) increases, by [16, Theorem 4.14], we get
\[
h_{\mu}(\sigma) \leq h_{\mu_m}(\sigma).
\]
Then the conclusion follows from Proposition 4.3.

5. Hausdorff dimension of level sets

**Theorem 5.1** (Hausdorff dimension of level sets for pseudo-golden ratios). Let \( \beta \in (1, 2) \) such that \( \varepsilon(1, \beta) = 1^m0^\infty \) for some integer \( m \geq 3 \).

1. If \( 0 \leq a < \frac{1}{m} \), then \( F_a = \emptyset \) and \( \dim_H F_a = 0 \).

2. If \( \frac{1}{m} \leq a \leq 1 \), then
\[
\dim_H F_a = \frac{1}{\log \beta} \max_{x_1, \cdots, x_{m-2}} f_a(x_1, \cdots, x_{m-2})
\]
where \( f_a(x_1, \cdots, x_{m-2}) = a \log a - (a - x_1) \log(a - x_1) - (x_1 - x_2) \log(x_1 - x_2) - \cdots - (x_{m-3} - x_{m-2}) \log(x_{m-3} - x_{m-2}) - (1 - a - x_1 - \cdots - x_{m-2}) \log(1 - a - x_1 - \cdots - x_{m-2}) - (x_1 + \cdots + x_{m-3} + 2x_{m-2} + a - 1) \log(x_1 + \cdots + x_{m-3} + 2x_{m-2} + a - 1) \) with \( a \geq x_1 \geq x_2 \geq \cdots \geq x_{m-2} \geq 0 \) and \( x_1 + \cdots + x_{m-3} + x_{m-2} \leq 1 - a \leq x_1 + \cdots + x_{m-3} + 2x_{m-2} \), i.e., all terms in the brackets in \( f_a \) are non-negative.

In particular, \( \dim_H F_{\frac{1}{m}} = \dim_H F_1 = 0 \).

**Remark 5.2.** For \( m = 3 \), \( \varepsilon(1, \beta) = 1110^\infty \) and \( \frac{1}{3} \leq a \leq 1 \), by a simple calculation, we get
\[
\dim_H F_a = \frac{1}{\log \beta} \left( a \log a - \frac{10a - 3 - \sqrt{-8a^2 + 12a - 3}}{6} \log \frac{10a - 3 - \sqrt{-8a^2 + 12a - 3}}{6} - \frac{2a + 3 - \sqrt{-8a^2 + 12a - 3}}{6} \log \frac{2a + 3 - \sqrt{-8a^2 + 12a - 3}}{6} - \frac{-a + \sqrt{-8a^2 + 12a - 3}}{3} \log \frac{-a + \sqrt{-8a^2 + 12a - 3}}{3} \right).
\]

In particular, \( \dim_H F_{\frac{1}{3}} = \dim_H F_1 = 0 \).

We need the following lemma which follows immediately from the convexity of the function \( x \log x \).
Lemma 5.3. Let \( \varphi : [0, \infty) \rightarrow \mathbb{R} \) be defined by
\[
\varphi(x) = \begin{cases} 
0 & \text{if } x = 0; \\
-x \log x & \text{if } x > 0.
\end{cases}
\]
Then for all \( x, y \in [0, \infty) \) and \( a, b \geq 0 \) with \( a + b = 1 \),
\[
a \varphi(x) + b \varphi(y) \leq \varphi(ax + by).
\]
The equality holds if and only if \( x = y, a = 0 \) or \( b = 0 \).

Proof of Theorem 5.1.
(1) By \( \varepsilon^+(1, \beta) = (1, m^{-1}0)^\infty \) and Lemma 2.2, we know that for any \( x \in [0, 1) \), every \( m \) consecutive digits in \( \varepsilon(x, \beta) \) must contain at least one 0. Thus
\[
\lim_{n \to \infty} \frac{\sharp\{1 \leq k \leq n : \varepsilon_k(x, \beta) = 0\}}{n} \geq \frac{1}{m}
\]
for any \( x \in [0, 1) \). If \( 0 \leq a < \frac{1}{m} \), we get \( F_a = \emptyset \).
(2) When \( \frac{1}{m} \leq a \leq 1 \), \( f_a \) is a continuous function on its domain of definition
\[
D_{m,a} := \{ (x_1, x_2, \ldots, x_{m-2}) : a \geq x_1 \geq x_2 \geq \cdots \geq x_{m-2} \geq 0, \\
x_1 + \cdots + x_{m-3} + x_{m-2} \leq 1 - a \leq x_1 + \cdots + x_{m-3} + 2x_{m-2} \}
\]
which is closed and non-empty since
\[
\begin{cases} 
(a, \frac{2a}{m-2}, \ldots, \frac{2a}{m-2}) \in D_{m,a} & \text{if } \frac{1}{m} \leq a < \frac{1}{2}; \\
(1 - a, 0, \ldots, 0) \in D_{m,a} & \text{if } a \geq \frac{1}{2}.
\end{cases}
\]
Therefore \( \max_{(x_1, \ldots, x_{m-2}) \in D_{m,a}} f_a(x_1, \ldots, x_{m-2}) \) exists.

In order to get our conclusion, by Theorem (4.2), it suffices to prove
\[
\sup\{h_\mu(\sigma) : \mu \in M_\sigma(S\beta), \mu[0] = a, \mu \text{ is an } (m - 1)-\text{step Markov measure}\} = \max_{(x_1, \ldots, x_{m-2}) \in D_{m,a}} f_a(x_1, \ldots, x_{m-2}).
\]
(The skills in the following proof are enlightened by drawing figures of the cylinders in \([0, 1)\) and understanding their relations.)

Let \( \mu \in M_\sigma(S\beta) \) be an \((m - 1)\)-step Markov measure such that \( \mu[0] = a \). By the same classical calculation as in the proof of Theorem 4.2, we get
\[
h_\mu(\sigma) = -\sum_{i_1, \ldots, i_m = 0, 1} \mu[i_1 \cdots i_m] \log \frac{\mu[i_1 \cdots i_m]}{\mu[i_1 \cdots i_{m-1}]}
\]
(where we regard \( 0 \log 0 \) and \( 0 \log 0 \) as \( 0 \)). By Lemma 2.2 and the definition of \([w]\), we get \([1^m] = \emptyset\). It follows from \( \mu[1^m] = 0 \) and \( \mu[1^m-1] = \mu[1^m-1] \) that
\[
h_\mu(\sigma) = -\sum_{\substack{i_1, \ldots, i_m \in \{0, 1\} \atop i_2, \ldots, i_{m-1} \neq 1^m-2}} \mu[i_1 \cdots i_m] \log \frac{\mu[i_1 \cdots i_m]}{\mu[i_1 \cdots i_{m-1}]} - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}.
\]
For $i_2 \cdots i_{m-1} \neq 1^{m-2}$ and $i_m \in \{0, 1\}$, we have

\[-\mu[0i_2 \cdots i_m] \log \frac{\mu[0i_2 \cdots i_m]}{\mu[i_2 \cdots i_m]} - \mu[1i_2 \cdots i_m] \log \frac{\mu[1i_2 \cdots i_m]}{\mu[i_2 \cdots i_m]}
\]

\[= \mu[i_2 \cdots i_{m-1}](\mu[0i_2 \cdots i_{m-1}] \left( -\mu[0i_2 \cdots i_m] \log \frac{\mu[0i_2 \cdots i_m]}{\mu[0i_2 \cdots i_{m-1}]} \right) + \mu[1i_2 \cdots i_{m-1}] \left( -\mu[1i_2 \cdots i_m] \log \frac{\mu[1i_2 \cdots i_m]}{\mu[1i_2 \cdots i_{m-1}]} \right)) \leq -\mu[i_2 \cdots i_m] \log \frac{\mu[i_2 \cdots i_m]}{\mu[i_2 \cdots i_{m-1}]}
\]

where the last inequality follows from Lemma 5.3. Thus

\[h_\mu(\sigma) \leq - \sum_{i_2 \cdots i_m \in \{0, 1\}} \mu[i_2 \cdots i_m] \log \frac{\mu[i_2 \cdots i_m]}{\mu[i_2 \cdots i_{m-1}]}
\]

\[-\mu[01^{m-2}0] \log \frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}
\]

\[= - \sum_{i_1 \cdots i_{m-1} \in \{0, 1\}} \mu[i_1 \cdots i_{m-1}] \log \frac{\mu[i_1 \cdots i_{m-1}]}{\mu[i_1 \cdots i_{m-2}]}
\]

\[-\mu[01^{m-2}0] \log \frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}
\]

\[= - \sum_{i_1 \cdots i_{m-1} \in \{0, 1\}} \mu[i_1 \cdots i_{m-1}] \log \frac{\mu[i_1 \cdots i_{m-1}]}{\mu[i_1 \cdots i_{m-2}]}
\]

\[-\mu[01^{m-3}0] \log \frac{\mu[01^{m-3}0]}{\mu[01^{m-3}]} - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-3}]}
\]

\[-\mu[01^{m-2}0] \log \frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}
\]

For $i_2 \cdots i_{m-2} \neq 1^{m-3}$ and $i_{m-1} \in \{0, 1\}$, we have

\[-\mu[0i_2 \cdots i_{m-1}] \log \frac{\mu[0i_2 \cdots i_{m-1}]}{\mu[0i_2 \cdots i_{m-2}]} - \mu[1i_2 \cdots i_{m-1}] \log \frac{\mu[1i_2 \cdots i_{m-1}]}{\mu[1i_2 \cdots i_{m-2}]}
\]

\[= \mu[i_2 \cdots i_{m-2}](\mu[0i_2 \cdots i_{m-2}] \left( -\mu[0i_2 \cdots i_{m-1}] \log \frac{\mu[0i_2 \cdots i_{m-1}]}{\mu[0i_2 \cdots i_{m-2}]} \right) + \mu[1i_2 \cdots i_{m-2}] \left( -\mu[1i_2 \cdots i_{m-1}] \log \frac{\mu[1i_2 \cdots i_{m-1}]}{\mu[1i_2 \cdots i_{m-2}]} \right)) \leq -\mu[i_2 \cdots i_{m-1}] \log \frac{\mu[i_2 \cdots i_{m-1}]}{\mu[i_2 \cdots i_{m-2}]}
\]
where the last inequality follows from Lemma 5.3. Thus

\[
\begin{align*}
\mu_h(\sigma) & \leq - \sum_{i_2 \cdots i_{m-1} \in \{0, 1\}} \mu[i_2 \cdots i_{m-1}] \log \frac{\mu[i_2 \cdots i_{m-1}]}{\mu[i_2 \cdots i_{m-2}]} \\
& \quad - \mu[01^{m-3}] \log \frac{\mu[01^{m-3}]}{\mu[01^{m-3}]} - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
& \quad - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]} \\
& = - \sum_{i_2 \cdots i_{m-2} \in \{0, 1\}} \mu[i_2 \cdots i_{m-2}] \log \frac{\mu[i_2 \cdots i_{m-2}]}{\mu[i_2 \cdots i_{m-3}]} \\
& \quad - \mu[01^{m-3}] \log \frac{\mu[01^{m-3}]}{\mu[01^{m-3}]} - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
& \quad - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]} \\
& = - \sum_{i_2 \cdots i_{m-2} \in \{0, 1\}} \mu[i_2 \cdots i_{m-2}] \log \frac{\mu[i_2 \cdots i_{m-2}]}{\mu[i_2 \cdots i_{m-3}]} \\
& \quad - \mu[01^{m-4}] \log \frac{\mu[01^{m-4}]}{\mu[01^{m-4}]} - \mu[01^{m-3}] \log \frac{\mu[01^{m-3}]}{\mu[01^{m-4}]} \\
& \quad - \mu[01^{m-3}] \log \frac{\mu[01^{m-3}]}{\mu[01^{m-3}]} - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
& \quad - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]} \\
& \quad - \mu[01^{m-3}] \log \frac{\mu[01^{m-3}]}{\mu[01^{m-3}]} - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
& \quad - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}
\end{align*}
\]

Repeat the above process a finite number of times. Finally we get

\[
\begin{align*}
\mu_h(\sigma) & \leq - \mu[00] \log \frac{\mu[00]}{\mu[0]} - \mu[01] \log \frac{\mu[01]}{\mu[0]} \\
& \quad - \mu[010] \log \frac{\mu[010]}{\mu[01]} - \mu[011] \log \frac{\mu[011]}{\mu[01]} \\
& \quad - \mu[01^{m-3}] \log \frac{\mu[01^{m-3}]}{\mu[01^{m-3}]} - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
& \quad - \mu[01^{m-2}] \log \frac{\mu[01^{m-2}]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}
\end{align*}
\]
Combining \([w0] \cup [w1] = [w], [0w] \cup [1w] = \sigma^{-1}[w]\) for any word \(w\) and the fact that \(\mu\) is \(\sigma\)-invariant, we get

\[
\begin{align*}
\mu[0] + \mu[1] &= 1, \\
\mu[00] + \mu[01] &= \mu[0], \\
\mu[010] + \mu[011] &= \mu[01], \\
\mu[01m-30] + \mu[01m-2] &= \mu[01m-3], \\
\mu[01m-20] + \mu[01m-1] &= \mu[01m-2], \\
\mu[01m-1] &= \mu[01m-1] + \mu[1m] = \mu[1m-1] \quad \text{(where } \mu[1m] = 0).\]
\end{align*}
\]

Let \(y_1 := \mu[01], y_2 := \mu[011], \cdots, y_{m-2} := \mu[01m-2]\). Then by

\[
\begin{align*}
\mu[0] &= a, \mu[00] = a - y_1, \mu[010] = y_1 - y_2, \mu[0110] = y_2 - y_3, \cdots, \mu[01m-30] = y_{m-3} - y_{m-2}, \\
\mu[1] &= 1 - a, \mu[11] = 1 - a - y_1, \cdots, \mu[1m-1] = 1 - a - y_1 - y_2 - \cdots - y_{m-2}, \\
\mu[01m-1] &= 1 - a - y_1 - y_2 - \cdots - y_{m-2}, \mu[01m-20] = y_1 + y_2 + \cdots + y_{m-3} + 2y_{m-2} + a - 1
\end{align*}
\]

and a simple calculation, we get

\[h_\mu(\sigma) \leq f_a(y_1, \cdots, y_{m-2}).\]

It follows from \(\mu[00], \mu[010], \cdots, \mu[01m-30], \mu[01m-20], \mu[01m-1] \geq 0\) that \((y_1, \cdots, y_{m-2}) \in D_{m,a}\). Therefore

\[h_\mu(\sigma) \leq \max_{(x_1, \cdots, x_{m-2}) \in D_{m,a}} f_a(x_1, \cdots, x_{m-2}).\]

\[\leq \]

Let \((y_1, \cdots, y_{m-2}) \in D_{m,a}\) such that

\[f_a(y_1, \cdots, y_{m-2}) = \max_{(x_1, \cdots, x_{m-2}) \in D_{m,a}} f_a(x_1, \cdots, x_{m-2}).\]

Define

\[
\begin{align*}
\mu[0] &= a, & \mu[1] &= 1 - a, \\
\mu[00] &= a - y_1, & \mu[10] &= a, \\
\mu[010] &= y_1 - y_2, & \mu[110] &= y_2, \\
\mu[01m-30] &= y_{m-3} - y_{m-2}, & \mu[1m-20] &= y_{m-2}, \\
\mu[01m-20] &= y_1 + y_2 + \cdots + y_{m-3} + 2y_{m-2} + a - 1, & \mu[01m-1] &= 1 - a - y_1 - y_2 - \cdots - y_{m-2}, \\
\mu[1m] &= 0
\end{align*}
\]

and

\[\mu[uvw] := \frac{\mu[uw] \cdot \mu[vw]}{\mu[w]} \quad \text{for } u, v \in \{0, 1\}, w \in \bigcup_{k=1}^{m-2} \left(\{0, 1\}^k \setminus \{1^k\}\right) \quad (5.1)\]

where we regard \(0^0\) as 0. Then \(\mu\) is well defined on all the cylinders with order \(\leq m\).

Let \(p\) be the vector and \(P\) be the matrix defined as in the proof of Theorem 4.2. For any \(n \geq m + 1\) and \(i_1, \cdots, i_n \in \{0, 1\}\), define

\[\mu[i_1 \cdots i_n] := p(i_1 \cdots i_{m-1})P_{(i_1 \cdots i_{m-1})(i_2 \cdots i_m)}P_{(i_2 \cdots i_m)(i_3 \cdots i_{m+1})} \cdots P_{(i_{m-1} \cdots i_{m-1})(i_m \cdots i_n)}.\]

Then \(\mu\) can be extended to become an \((m - 1)\)-step Markov measure in \(M_\sigma(S_0)\) with \(\mu[0] = a\). By (5.1) and Lemma 5.3, it is not difficult to check that in the proof of \(\leq\), all the “\(\leq\)” in the upper bound estimation of \(h_\mu(\sigma)\) can take “\(=\)” and then

\[h_\mu(\sigma) = f_a(y_1, \cdots, y_{m-2}) = \max_{(x_1, \cdots, x_{m-2}) \in D_{m,a}} f_a(x_1, \cdots, x_{m-2}).\]

□
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