On the continuity of bending

CHRISTOS KOUROUNIOTIS

Abstract We examine the dependence of the deformation obtained by bending quasi-Fuchsian structures on the bending lamination. We show that when we consider bending quasi-Fuchsian structures on a closed surface, the conditions obtained by Epstein and Marden to relate weak convergence of arbitrary laminations to the convergence of bending cocycles are not necessary. Bending may not be continuous on the set of all measured laminations. However we show that if we restrict our attention to laminations with non negative real and imaginary parts then the deformation depends continuously on the lamination.

AMS Classification 30F40; 32G15

Keywords Kleinian groups, quasi-Fuchsian groups, geodesic laminations

The deformation of hyperbolic structures by bending along totally geodesic submanifolds of codimension one was introduced by Thurston in his lectures on The Geometry and Topology of 3–manifolds. The geometric and algebraic properties of the deformation were studied in [4] and [3]. Epstein and Marden [2] introduced the notion of a bending cocycle and used it to describe bending a hyperbolic surface along a measured geodesic lamination. The same notion was used in [5] to extend bending to a holomorphic family of local biholomorphic homeomorphisms of quasi-Fuchsian space $Q(S)$.

Epstein and Marden [2] give a careful analysis of the dependence of the bending cocycle on the measured lamination. They consider the set of measured laminations on $H^2$ consisting of geodesics that intersect a compact subset $K \subset H^2$. This is a subset of the space of measures on the space $G(K)$ of geodesics in $H^2$ intersecting $K$, with the topology of weak convergence of measures. In this topology, the bending cocycle does not depend continuously on the lamination. One reason for this is the behaviour of the laminations near the endpoints of the segment over which we evaluate the cocycle. For example, consider the geodesic segment $[e^{i\theta}, i]$ in $H^2$, for suitable $\theta$ in $[0, \pi/2]$, and the measured laminations $\mu_n$, with weight 1 on the geodesic $(1/n, n)$ and weight $-1$ on the geodesic $(-1/n, -n)$. Then $\{\mu_n\}$ converges weakly to the zero lamination, but
the cocycle of $\mu_n$ relative to $[e^{i\theta}, i]$ is approximately a hyperbolic isometry of translation length 1. Epstein and Marden find conditions under which a sequence of measured laminations gives a convergent sequence of cocycles relative to a given pair of points.

In this article we show that when the lamination is invariant by a discrete group and we only consider cocycles relative to points in the orbit of a suitable point $x \in \mathcal{H}^2$, any sequence of measured laminations $\{\mu_n\}$ which converges weakly gives rise to cocycles which converge up to conjugation. We show further that the same conjugating elements can be used for the cocycles for $\mu_n$ corresponding to the different generators of the group. Hence the laminations $\mu_n$ determine bending homomorphisms which, after conjugation by suitable isometries, converge to the bending homomorphism determined by $\mu_0$. This implies that the deformations converge in $Q(S)$.

**Theorem 1** Let $S$ be a closed hyperbolic surface and $Q(S)$ its space of quasi-Fuchsian structures. Let $\{\mu_n\}$ be a sequence of complex measured geodesic laminations, converging weakly to a lamination $\mu_0$. Then the bending deformations

$$B_{\mu_n} : \mathcal{D}_{\mu_n} \to Q(S)$$

converge to the deformation $B_{\mu_0}$, uniformly on compact subsets of $\mathcal{D} = \mathcal{D}_{\mu_0} \cap (\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{D}_{\mu_n})$.

We also state an infinitesimal version of the Theorem.

**Theorem 2** Let $S$ be a closed hyperbolic surface and $Q(S)$ its space of quasi-Fuchsian structures. Let $\{\mu_n\}$ be a sequence of complex measured geodesic laminations, converging weakly to a lamination $\mu_0$. Then the holomorphic bending vector fields $T_{\mu_n}$ on $Q(S)$ converge to $T_{\mu_0}$, uniformly on compact subsets of $Q(S)$.

These results do not necessarily imply the continuous dependence of the deformation on the bending lamination, because the space of measured laminations is not first countable. If however we restrict our attention to the subset of measured laminations with non negative real and imaginary parts, then we can apply results in [6] to obtain the following Theorem.

**Theorem 3** The mapping $\mathcal{ML}^{++}(S) \times Q(S) \to T(Q(S)) : (\mu, [\rho]) \mapsto T_{\mu}([\rho])$ is continuous, and holomorphic in $[\rho]$.
On the continuity of bending

The proof of Theorem 1 is based on the observation that, when the lamination is invariant by a discrete group and we are considering cocycles with respect to points $x$ and $g(x)$, for some $g$ in the group, the effect of a lamination near the endpoints of the segment $[x, g(x)]$ is controlled by its effect near $x$, provided that the lamination does not contain geodesics very close to the geodesic carrying $[x, g(x)]$. This last condition can be achieved by choosing $x$ to be a point not on the axis of a conjugate of $g$ (see Corollary 2.12).

In Section 1 we describe the space of measured laminations and we recall the definition of bending. In the beginning of Section 2 we recall or modify certain results from [2] and [5] which provide bounds for the effect of bending along nearby geodesics. Lemma 2.11 and the results following it examine the consequences of the above condition on the choice of $x$.

The proof of Theorems 1, 2 and 3 is given in Section 3. The laminations $\mu_n$ are replaced by finite approximations. The main result is Lemma 3.1, which gives the basic estimate for the difference between the bending homomorphism of $\mu_0$ and a conjugate of the bending homomorphism of $\mu_n$. Then a diagonal argument is used to obtain the convergence of bending.

1 The setting

We consider a closed surface $S$ of genus greater than 1. We fix a hyperbolic structure on $S$, and let $\rho_0: \pi_1(S) \to PSL(2, \mathbb{R})$ be an injective homomorphism with discrete image $\Gamma_0 = \rho_0(\pi_1(S))$, such that $S$ is isometric to $H^2/\Gamma_0$.

We consider the space $R$ of injective homomorphisms $\rho: \Gamma_0 \to PSL(2, \mathbb{C})$ obtained by conjugation with a quasiconformal homeomorphism $\phi$ of $\hat{C}$: if $g \in \Gamma_0$, acting on $\hat{C}$ as Möbius transformations, then $\rho(g) = \phi \circ g \circ \phi^{-1}$.

$PSL(2, \mathbb{C})$ acts on the left on $R$ by inner automorphisms. The quotient of $R$ by this action is the space $Q(S)$ of quasi-Fuchsian structures on $S$, or quasi-Fuchsian space of $S$. We denote the equivalence class in $Q(S)$ of a homomorphism $\rho \in R$ by $[\rho]$. Then $[\rho]$ is a Fuchsian point if there is a circle in $\hat{C}$ left invariant by $\rho(\Gamma_0)$, so that $\rho(\Gamma_0)$ is conjugate to a Fuchsian group of the first kind. The subset of Fuchsian points in $Q(S)$ is the Teichmüller space of $S$, $T(S)$.

We fix a point $[\rho] \in Q(S)$, represented by the homomorphism $\rho: \Gamma_0 \to PSL(2, \mathbb{C})$ obtained by conjugation with the quasiconformal homeomorphism $\phi: \hat{C} \to \hat{C}$. We denote the image of $\rho$ by $\Gamma$. The limit set of $\Gamma_0$ is $\mathbb{R}$. Then
\(\phi(\mathbb{R})\) is the limit set of \(\Gamma\). If \(\gamma\) is a geodesic in \(\mathcal{H}^2\) with endpoints \(u, v \in \mathbb{R}\), we denote by \(\phi_*(\gamma)\) the geodesic in \(\mathcal{H}^3\) with endpoints \(\phi(u), \phi(v)\) in \(\phi(\mathbb{R})\). In this way, geodesics on the surface \(S \cong \mathcal{H}^2/\Gamma_0\) are associated to geodesics in the hyperbolic 3-manifold \(\mathcal{H}^3/\Gamma\).

We want to study the deformation of quasi-Fuchsian structures by bending, [4], [2], [5]. Bending is determined by a geodesic lamination on \(S\) with a complex valued transverse measure.

A measured geodesic lamination on \(S\) lifts to a measured geodesic lamination on \(\mathcal{H}^2\). The space \(G(\mathcal{H}^2)\) of unoriented geodesics in \(\mathcal{H}^2\) is homeomorphic to a Möbius strip without boundary. Let \(K\) be a compact subset of \(\mathcal{H}^2\), projecting onto \(\mathcal{H}^2/\Gamma_0\). The set \(G(K)\) of geodesics in \(\mathcal{H}^2\) intersecting \(K\) is a compact metrizable space.

A measured geodesic lamination on \(\mathcal{H}^2\) determines a complex valued Borel measure \(\mu\) on \(G(K)\), with the property that if \(\gamma_1\) and \(\gamma_2\) are distinct geodesics in the support of \(\mu\), then they are disjoint. The set of measured geodesic laminations on \(S\) can be considered as a subset of \(\mathcal{M}(G(K))\), the set of complex valued Borel measures on \(G(K)\). The set \(\mathcal{M}(G(K))\) has a norm, defined by

\[
\|\mu\| = \sup \left\{ \left| \int f \mu \right| , f \text{ continuous complex valued function on } G(K), |f| \leq 1 \right\}
\]

We shall use the weak* topology on \(\mathcal{M}(G(K))\), with basis the sets of the form

\[
U(\mu, \varepsilon, f_1, \ldots, f_m) = \left\{ \nu \in \mathcal{M}(G(K)) : \left| \int f_i \mu - \int f_i \nu \right| < \varepsilon, i = 1, \ldots, m \right\}
\]

where \(\mu \in \mathcal{M}(G(K))\), \(f_i, i = 1, \ldots, m\) are continuous functions on \(G(K)\), and \(\varepsilon\) is a positive number.

A measured geodesic lamination \(\mu\) on \(S\) is called finite if it is supported on a finite set of simple closed geodesics in \(S\). Then, for any compact subset \(K\) of \(\mathcal{H}^2\), the measure on \(G(K)\) determined by the lift of \(\mu\) to \(\mathcal{H}^2\) has finite support.

Given a finite measured geodesic lamination \(\mu\) on \(S\), we define bending the quasi-Fuchsian structure \([p]\) on \(S\) as follows.

Let \(g_1, \ldots, g_k\) be a set of generators of \(\Gamma_0\). Choose a point \(x\) on \(\mathcal{H}^2\) and, for each \(g_j\), consider the geodesic segment \([x, g_j(x)]\). Let \(\gamma_1, \ldots, \gamma_m\) be the geodesics in the support of \(\mu\) intersecting \([x, g_j(x)]\), and let \(z_1, \ldots, z_m\) be the corresponding measures. If \(\gamma_1\) (or \(\gamma_m\)) go through \(x\) (or \(g_j(x)\) respectively), we replace \(z_1\) (or \(z_m\)) by \(\frac{1}{2} z_1\) (or \(\frac{1}{2} z_m\)).

If \(\gamma\) is an oriented geodesic in \(\mathcal{H}^3\) and \(z \in \mathbb{C}\), we denote by \(A(\gamma, z)\) the element of \(PSL(2, \mathbb{C})\) with axis \(\gamma\) and complex displacement \(z\). We will use the same

\(\text{Geometry \\& Topology Monographs, Volume 1 (1998)}\)
notation for one of the matrices in $SL(2, \mathbb{C})$ corresponding to $A(\gamma, z)$. In such cases either the choice of the lift will not matter, or there will be an obvious choice.

We orient the geodesics $\gamma_1, \ldots, \gamma_m$ so that they cross the segment $[x, g_j(x)]$ from right to left, and define the isometry

$$C_{t\mu}(x, g_j(x)) = A(\phi_*(\gamma_1), tz_1) \cdots A(\phi_*(\gamma_m), tz_m).$$

For each generator $g_j$, $j = 1, \ldots, k$, define

$$\rho_{t\mu}(g_j) = C_{t\mu}(x, g_j(x)) \rho(g_j).$$

For $t$ in an open neighbourhood of 0 in $\mathbb{C}$, the representation $[\rho_{t\mu}]$ is quasi-Fuchsian, [4].

Any measured geodesic lamination $\mu$ on $S$ can be approximated by finite laminations so that the corresponding bending deformations converge, [2], [5]. In this way, we obtain for any measured geodesic lamination on $S$ a deformation $B_\mu$ defined on an open set $D_\mu \subset Q(S) \times \mathbb{C}$,

$$B_\mu : D_\mu \to Q(S) : ([\rho], t) \mapsto [\rho_{t\mu}].$$

$B_\mu$ is a holomorphic mapping.

2 The lemmata

In the vector space $\mathbb{C}^2$ we introduce the norm

$$\| (z_1, z_2) \| = \max\{ |z_1|, |z_2| \}.$$

A complex matrix $A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ acts on $\mathbb{C}^2$ and has norm

$$\| A \| = \max\{ |a| + |b|, |c| + |d| \}.$$

We will use this norm on $SL(2, \mathbb{C})$.

**Lemma 2.1** ([2], 3.3.1) Let $X$ be a set of matrices in $SL(2, \mathbb{C})$ and $c = (0, 0, 1) \in \mathcal{H}^3$. Then the following are equivalent.

i) The closure of $X$ is compact.

ii) There is a positive number $M$ such that if $A \in X$ then $\| A \| \leq M$.

iii) There is a positive number $M$ such that if $A \in X$ then $\| A \| \leq M$ and $\| A^{-1} \| \leq M$.
iv) There is a positive number $R$ such that if $A \in X$ then $d(c, A(c)) \leq R$. 

Let $\Lambda$ be a maximal geodesic lamination on $S$, and $\psi : S \to \mathcal{H}^3/\Gamma$ the pleated surface representing the lamination $\Lambda$ [1]. Let $\tilde{\psi} : \mathcal{H}^2 \to \mathcal{H}^3$ be the lift of $\psi$.

**Lemma 2.2** ([5], 2.5) Let $K$ be a compact disc of radius $R$ about $c = (0,0,1) \in \mathcal{H}^3$, and $M$ a positive number. There is a positive number $N$ with the following property. If $[x,y]$ is a geodesic segment in $\mathcal{H}^2$ such that $\tilde{\psi}([x,y]) \subset K$ and $\gamma_i, z_i \in \Lambda$, $i = 1, \ldots, m$ is a finite measured lamination with support contained in $\Lambda$, whose leaves all intersect $[x,y]$ and are numbered in order from $x$ to $y$, and such that $\sum_{i=1}^m |\text{Re} z_i| < M$, then

$$\|A(\gamma_1, z_1) \cdots A(\gamma_m, z_m)\| \leq N.$$

**Lemma 2.3** ([2], 3.4.1, [5], 2.4) Let $K$ be a compact subset of $SL(2, \mathbb{C})$, $M$ a positive number, and let $\gamma$ be the geodesic $(0, \infty)$. Then there is a positive number $N$ with the following property. For any $B, C \in K$, and $z \in \mathbb{C}$ with $|z| \leq M$, we have

$$\|BA(\gamma,z)B^{-1} - CA(\gamma,z)C^{-1}\| \leq N \|B - C\| |z|.$$

In order to examine the effect of bending along nearby geodesics, in Lemma 2.5 and 2.6, we shall use the notion of a solid cylinder in hyperbolic space. A solid cylinder $C$ over a disk $D$ in $\mathcal{H}^n$ is the union of all geodesics orthogonal to a $(n - 1)$–dimensional hyperbolic disc $D$ in $\mathcal{H}^n$. The radius of the cylinder is the hyperbolic radius of the disc $D$. If $x$ is the centre of $D$, we say that $C$ is a solid cylinder based at $x$. The boundary of $C$ at infinity consists of two discs $D_1$ and $D_2$ in $\partial \mathcal{H}^n$. We say that the solid cylinder $C$ is supported by $D_1$ and $D_2$. The geodesic orthogonal to $D$ through its centre is the core of the solid cylinder $C$. We shall denote the cylinder with core $\gamma$, basepoint $x \in \gamma$ and radius $r$ by $C(\gamma,x,r)$.

**Lemma 2.4** ([5], 2.6) Let $L$ be a compact set in $\mathcal{H}^3$. Then there exists a positive number $M$ with the following property. If $D$ is a disc of radius $r$, contained in $L$, and $\alpha, \beta$ are two geodesics contained in the solid cylinder over $D$, then there is an element $A \in SL(2, \mathbb{C})$ such that $A(\alpha) = \beta$ and $\|A - I\| \leq Mr$.

If $C$ is a solid cylinder supported on the discs $D_1$ and $D_2$, with $D_1 \cap D_2 = \emptyset$, and $\gamma_1, \gamma_2$ are two geodesics, each having one end point in $D_1$ and one in $D_2$, we say that $\gamma_1$ and $\gamma_2$ are concurrently oriented in $C$ if their origins lie in the same component of $D_1 \cup D_2$. 

*Geometry & Topology Monographs, Volume 1 (1998)*
Lemma 2.5 Let \( m \) be a positive number and \( L \) a compact subset of \( \mathcal{H}^3 \). Then there are positive numbers \( M_1 \) and \( M_2 \) with the following property. If \( \gamma_1, \gamma_2 \) are concurrently oriented geodesics contained in a cylinder of radius \( r \), based at a point in \( L \), and \( z_1, z_2 \) are complex numbers such that \( |z_i| \leq m \), then there are lifts of \( A(\gamma_i, z_i) \) to \( SL(2, \mathbb{C}) \) such that

\[
\|A(\gamma_1, z_1) - A(\gamma_2, z_2)\| \leq M_1 r \min\{|z_1|, |z_2|\} + M_2 |z_1 - z_2|.
\]

Proof We assume that \( |z_1| \leq |z_2| \). We have

\[
\|A(\gamma_1, z_1) - A(\gamma_2, z_2)\| \leq \|A(\gamma_1, z_1) - A(\gamma_2, z_1)\| + \|A(\gamma_2, z_1) - A(\gamma_2, z_2)\|.
\]

Let \( B \in SL(2, \mathbb{C}) \) be an element mapping the geodesic \((0, \infty)\) to \( \gamma_2 \), and mapping the point \( c = (0, 0, 1) \) to a point in \( L \). Then, by Lemma 2.1, there is a constant \( K_1 \) depending only on \( L \), such that \( \|B\| \leq K_1 \). By Lemma 2.4 there is an element \( C \in SL(2, \mathbb{C}) \) such that \( C(\gamma_2) = \gamma_1 \), and \( \|C - I\| \leq K_2 r \) for some constant \( K_2 \) depending only on \( L \).

By Lemma 2.3 there is a constant \( K_3 \) such that

\[
\|A(\gamma_1, z_1) - A(\gamma_2, z_1)\| \leq K_3 \|CB - B\| |z_1| \leq K_1 K_2 K_3 r |z_1|.
\]

On the other hand,

\[
\|A(\gamma_2, z_1) - A(\gamma_2, z_2)\| \leq \|B\| \|A((0, \infty), z_1 - z_2) - I\| \|B^{-1}\| \|A((0, \infty), z_2)\|.
\]

By Lemma 2.1 and the fact that the entries of \( A((0, \infty), z_1 - z_2) \) depend analytically on \( z_1 - z_2 \), there is a constant \( K_4 \), depending on \( L \) and \( m \) such that

\[
\|A(\gamma_2, z_1) - A(\gamma_2, z_2)\| \leq K_4 |z_1 - z_2|.
\]

Lemma 2.6 ([5], 2.7) Let \( m \) be a positive number and \( L \) a compact subset of \( \mathcal{H}^3 \). Then there is a positive number \( M \) with the following property. Let \( C \) be a solid cylinder of radius \( r \) based at a point in \( L \). Let \( \gamma_1, \ldots, \gamma_k \) be geodesics in \( C \) and \( z_1, \ldots, z_k \) complex numbers with \( \sum_{i=1}^{k} |\text{Re}(z_i)| \leq m \). Then

\[
\left\| A(\gamma_1, z_1) \cdots A(\gamma_k, z_k) - A\left(\gamma_1, \sum_{i=1}^{k} z_i\right) \right\| \leq Mr \sum_{i=1}^{k} |z_i|.
\]

We want to show that if two geodesics on \( S \) are sufficiently close, then the corresponding geodesics in \( \mathcal{H}^3/T \) will also be close, (Lemma 2.10).
Lemma 2.7  Let $K$ be a compact subset of $\mathcal{H}^2$, and $\phi: \partial \mathcal{H}^2 \to \partial \mathcal{H}^3$ a homeomorphism onto its image. Then there is a compact subset $L$ of $\mathcal{H}^3$ such that if $\gamma$ is a geodesic of $\mathcal{H}^2$ intersecting $K$, then $\phi_*(\gamma)$ intersects $L$, i.e. $\phi_*(G(K)) \subset G(L)$.

Proof  We consider the Poincaré disk model of hyperbolic space. There, it is clear that if $K$ is a compact subset of $B^2$, then there is a positive number $m$ such that if $\gamma$ is a geodesic in $G(K)$ with end-points $u,v$, then $|u-v| \geq m$. Since $\phi^{-1}$ is uniformly continuous, there is a positive number $M$ such that $|\phi(u) - \phi(v)| \geq M$, and hence there is a compact subset of $B^3$ intersecting $\phi_*(\gamma)$.

Lemma 2.8  ([5], 2.2) Let $\varepsilon$ and $\eta$ be two positive numbers. Then there is a positive number $\delta$ with the following property. If $D_1$ and $D_2$ are discs in $S^2$, with spherical radius $\leq \delta$, and the spherical distance between $D_1$ and $D_2$ is $\geq \eta$, then the solid cylinder supported by $D_1$ and $D_2$ has hyperbolic radius $r \leq \varepsilon$.

Lemma 2.9  Let $K$ be a compact subset of $B^n$, and $d$ a positive number. Then there is a positive number $\delta$ with the following property. If $C$ is a solid cylinder in $B^n$, over a disc with radius $r \leq \delta$ and centre at a point in $K$, then the spherical radius of each of the discs supporting $C$ is $\leq d$.

Proof  The radii of the supporting discs are given by continuous functions of the core geodesic, the base point and the radius of the cylinder. For a fixed base point, they tend to zero with the radius of the cylinder. The result follows by compactness.

Lemma 2.10  Let $[\rho]$ be a quasi-Fuchsian structure on $S$, $K$ a compact subset of $\mathcal{H}^2$, and $L$ a compact subset of $\mathcal{H}^3$ such that $\phi_*(G(K)) \subset G(L)$. Let $r$ be a positive number. Then there is a positive number $\delta$ with the following property. If $\gamma \in G(K)$, $x \in \gamma \cap K$ and $0 \leq r_1 \leq \delta$, then there is some point $x' \in L$ such that for any geodesic $\alpha$ contained in the solid cylinder $C(\gamma, x, r_1)$, the geodesic $\phi_*(\alpha)$ is contained in the solid cylinder $C(\phi_*(\gamma), x', r) \subset \mathcal{H}^3$.

Proof  We work in the Poincaré disc model of the hyperbolic plane and space, $B^2$ and $B^3$. Since $L$ is a compact subset of $B^3$, there is a number $\eta_2 > 0$ such that if $u$ and $v$ are the endpoints of any geodesic in $B^3$ intersecting $L$, then the spherical distance between $u$ and $v$ is $\geq \eta_2$. Then, by Lemma 2.8, there is a
positive number $\delta_2$, such that any solid cylinder with core a geodesic $\gamma \in G(L)$ and supported on discs of spherical radius $\leq \delta_2$, has hyperbolic radius $\leq r$.

Since $\phi : S^1 \to S^2$ is uniformly continuous, there is a positive number $\delta_1$, such that any arc in $S^1$ of length $\leq \delta_1$ is mapped into a disc in $S^2$, of radius $\leq \delta_2$. Then, by Lemma 2.9, there is a positive number $\delta$ such that any solid cylinder of radius $\leq \delta$ and based at a point in $K$, is supported on two arcs of length $\leq \delta_1$.

Recall that, if $X$ is a subset of $H^2$, we denote by $G(X)$ the set of geodesics in $H^2$ which intersect $X$. To simplify notation, we will write $G(x)$ for the set of geodesics through the point $x \in H^2$, and $G(x,y)$ for the set of geodesics intersecting the open geodesic segment $(x,y)$.

If $\Gamma$ is a group of isometries of $H^2$, we denote by $G'_\Gamma$ the set of geodesics in $H^2$ which do not intersect any of their translates by $\Gamma$:

$$G'_\Gamma = \{ \gamma \in G(H^2) : \forall g \in \Gamma, g(\gamma) \cap \gamma = \emptyset \text{ or } g(\gamma) = \gamma \}.$$ 

In the following Lemma we consider the angle between unoriented geodesics to lie in the interval $[0, \frac{\pi}{2}]$.

**Lemma 2.11** Let $\ell$ and $\theta$ be positive numbers. Then there is a positive number $\zeta$ with the following property. Let $x,y \in H^2$, $\gamma$ the geodesic carrying the segment $[x,y]$, $g \in PSL(2,\mathbb{R})$ and $\gamma' \in G'_g$, such that:

i) The hyperbolic distance $d(x,y) \leq \ell$.

ii) The geodesic segments $[x,y]$ and $[g(x),g(y)]$ intersect, and the angle between $\gamma$ and $g(\gamma)$ is $\alpha \geq \theta$.

iii) $\gamma'$ intersects the segment $[x,y]$ and the angle between $\gamma$ and $\gamma'$ is $\beta$.

Then $\beta \geq \zeta$.

**Proof** Without loss of generality, we may assume that $x = i \in H^2$ and $y = ti$. The angle of intersection between the geodesics $\delta$ and $g(\delta)$ is a continuous function of $\delta$. Hence there is a neighbourhood $U$ of $\gamma \in G(H^2)$ disjoint from $G'_g$, that is consisting of geodesics $\delta$ such that $g(\delta)$ intersects $\delta$.

There is a positive number $r$ such that the (two dimensional) solid cylinder $C(\gamma, i\sqrt{t}, r)$ has the property: if $\delta \subset C(\gamma, i\sqrt{t}, r)$ then $\delta \subset U$. Then it is easy to show, using hyperbolic trigonometry, that there is a positive number $\zeta$ such that any geodesic $\delta$ intersecting $[x,y]$ at an angle $\leq \zeta$ is contained in $C(\gamma, i\sqrt{t}, r)$, and hence $\delta \notin G'_g$. 

**Geometry & Topology Monographs, Volume 1 (1998)**
Corollary 2.12  If \( g \) is a hyperbolic isometry of \( \mathcal{H}^2 \) and \( x \in \mathcal{H}^2 \) does not lie on the axis of \( g \), then there is a positive number \( \zeta \) with the following property. If \( \mu \) is any geodesic lamination invariant by \( g \), then no leaf of the lamination intersects the geodesic segment \([x,g(x)]\) at an angle smaller than \( \zeta \).

Lemma 2.13 Let \( \ell, \theta \) and \( \varepsilon \) be positive numbers. Then there is a positive number \( r \) with the following property. Let \( x, y \in \mathcal{H}^2 \) with \( d(x, y) \leq \ell \), and let \( \gamma \) be the geodesic carrying the segment \([x, y] \). Let \( g \in \text{PSL}(2, \mathbb{R}) \) be such that \([x, y]\) intersects \([g(x), g(y)]\) at the point \( x_0 \), and at an angle \( \alpha \geq \theta \). If \( \delta \in G'(g) \cap G(D(x_0, r)) \), then \( \delta \) intersects both \( \gamma \) and \( g(\gamma) \), and the points of intersection lie in \( D(x_0, \varepsilon) \).

Proof Since \( g^{-1}(x_0) \in [x, y] \), we have \( d(g^{-1}(x_0), x_0) \leq \ell \). We consider the geodesic segment \([x', y']\) of length \( 3\ell \) on the geodesic \( \gamma \), centred at \( x_0 \).

Let \( U \) be a neighbourhood of \( \gamma \in G(\mathcal{H}^2) \) disjoint from \( G'_g \). There is \( r_1 \) such that any geodesic which intersects \( D(x_0, r_1) \) and does not intersect \([x', y']\), lies in \( U \), and hence it is not in \( G'_g \). So, if \( \delta \in G'_g \cap G(D(x_0, r_1)) \), \( \delta \) intersects the segment \([x', y']\). Similarly, there is \( r_2 \) such that if \( \delta \in G'_g \cap G(D(x_0, r_2)) \), \( \delta \) intersects the segment \([g(x'), g(y')]\).

By Lemma 2.11, the angle at the points of intersection is greater than a constant \( \zeta \). If \( r \) satisfies \( 0 < r < \min(r_1, r_2) \) and \( \sinh r < \sin \zeta \sinh \varepsilon \), then it has the required property.

The following Lemma shows that, under certain conditions, taking integrals along geodesic segments describes weak convergence of measures.

Lemma 2.14 Let \( \{\mu_n\} \) be a sequence of measured geodesic laminations on \( \mathcal{H}^2 \), invariant by \( g \in \text{PSL}(2, \mathbb{R}) \), and assume that \( \mu_n \) converge weakly to a measured lamination \( \mu \). Let \( \gamma \) be a geodesic in \( \mathcal{H}^2 \), such that \( \gamma \) and \( g(\gamma) \) intersect at one point. Then, for every geodesic segment \([u, v]\) on \( \gamma \) and for every continuous function \( f : [u, v] \to [0, 1] \), with \( f(u) = f(v) = 0 \), the sequence \( \int_{[u,v]} f \mu_n \) converges to \( \int_{[u,v]} f \mu \).

Proof Since \( \gamma \) intersects \( g(\gamma) \) at one point, there is a neighbourhood \( U \) of \( \gamma \) in \( G(\mathcal{H}^2) \) which is disjoint from \( G'_g \). We define a continuous function \( \tilde{f} : G(\mathcal{H}^2) \to [0, 1] \) by letting \( \tilde{f}(\delta) = f(y) \) if \( y \in [u, v] \) and \( \delta \in G(y) - U \), and extending continuously to the rest of \( G(\mathcal{H}^2) \). Then, for any measured geodesic lamination \( \nu \) invariant by \( g \),

\[ \tilde{f} \nu(G(u, v)) = \int_{[u,v]} f \nu. \]
3 The theorems

We fix a reference point \([\rho_0] \in T(S)\), and we consider a point \([\rho] \in Q(S)\). Let \(g_1, \ldots, g_k \in PSL(2, \mathbb{R})\) be a set of generators for \(\Gamma_0 = \rho_0(\pi_1(S))\). Let \(x \in \mathcal{H}^2\) be a point which does not lie on the axis of any conjugate of the generators \(g_j\).

Let \(\theta\) be the minimum of the angles between the geodesics carrying the segments \([g_j^{-1}(x), x]\) and \([x, g_j(x)]\), for \(j = 1, \ldots, k\). Let \(d\) and \(d'\) be the maximum and the minimum, respectively, of the distances between \(x\) and \(g_j(x)\), for \(j = 1, \ldots, k\).

Let \(K\) be a compact disc in \(\mathcal{H}^2\) containing in its interior the points \(x, g_j(x), g_j^{-1}(x)\), for \(j = 1, \ldots, k\), and projecting onto \(S_0 = \mathcal{H}^2/\Gamma_0\). Let \(L\) be a compact disc in \(\mathcal{H}^3\) such that \(\phi_*(G(K)) \subset G(L)\).

We consider a positive integer \(m\), and a positive number \(r(m)\) such that \(d/m\) is less than the number \(\delta(K, L, r(m))\) given by Lemma 2.10.

Let \(\mu\) be a complex measured geodesic lamination on \(\mathcal{H}^2\), invariant by the group \(\Gamma_0\), with \(||\mu|| < M_0\). We consider one of the generators \(g_j\), \(j = 1, \ldots, k\), and to simplify notation we drop the suffix \(j\) for the time being. Let \(\gamma\) denote the geodesic carrying the segment \([x, g(x)]\). We divide the segment \([x, g(x)]\) into \(m\) equal subsegments, by the points

\[ x = x_0, x_1, \ldots, x_{m-1}, x_m = g(x). \]

If \([x, y]\) is a geodesic segment in \(\mathcal{H}^2\) and \(\nu\) is a measure on a set of geodesics in \(\mathcal{H}^2\), we introduce the notation

\[ \int'_{[x, y]} \nu = \frac{1}{2} \nu(G(x)) + \nu(G(x, y)) + \frac{1}{2} \nu(G(y)) \]

We define two new measures on the set \(G(\mathcal{H}^2)\) of geodesics in \(\mathcal{H}^2\) in the following way. For every \(i = 1, \ldots, m\), let \(\tilde{\gamma}_i\) be a geodesic in \(\text{supp } \mu\), intersecting \(\gamma\) in \([x_{i-1}, x_i]\). We define, for \(i = 1, \ldots, m\),

\[ \tilde{\mu}(\tilde{\gamma}_i) = \int'_{[x_{i-1}, x_i]} \mu. \]

For every \(i = 1, \ldots, m - 1\), let \(\gamma'_i\) be the geodesic in \(\text{supp } \mu\) intersecting the open segment \((x_{i-1}, x_{i+1})\) as near as possible to \(x_i\). Let \(\lambda_i: [x_0, x_m] \to [0, 1]\), \(i = 1, \ldots, m - 1\), be continuous functions satisfying

(1) \(\text{supp } (\lambda_i) \subset [x_{i-1}, x_{i+1}]\) and
Then, by Lemma 2.2, there is a positive number \( \delta \) such that, if

\[
\mu'((\gamma_i')) = \int_{[x_{i-1}, x_{i+1}]} \lambda_i \mu
\]

we put

\[
i = 1, \ldots, m - 1,
\]

and

\[
C_i = A(\phi_*(\tilde{\gamma}_i), \hat{\mu}(\tilde{\gamma}_i)) \quad \text{for } i = 1, \ldots, m
\]

and

\[
D_i = A(\phi_*(\gamma_i'), \hat{\mu}'(\gamma_i')) \quad \text{for } i = 1, \ldots, m - 1.
\]

We want to bound the norm \( \|C_1 C_2 \cdots C_m - D_1 D_2 \cdots D_{m-1}\| \).

We put \( a_i = \int_{[x_{i-1}, x_i]} \lambda_i \mu \) and \( b_i = \int_{[x_i, x_{i+1}]} \lambda_i \mu \). Then \( \mu'(\gamma_i') = a_i + b_i \), for \( i = 1, \ldots, m - 1 \), and \( \hat{\mu}(\tilde{\gamma}_1) = a_1, \hat{\mu}(\tilde{\gamma}_m) = b_{m-1} \), and for \( i = 2, \ldots, m - 1 \),

\[
\hat{\mu}(\tilde{\gamma}_i) = b_{i-1} + a_i.
\]

We put

\[
D_i^r = A(\phi_*(\gamma_i'), a_i) \quad \text{and} \quad D_i^r = A(\phi_*(\gamma_i'), b_i)
\]

With this notation we have

\[
\|C_1 \cdots C_m - D_1 \cdots D_{m-1}\| \leq \|C_1 \cdots C_{m-1}\| \|C_m - D_{m-1}\|^r
\]

\[
+ \|C_1 \cdots C_{m-2}\| \|C_{m-1} - D_{m-2}D_{m-1}\| \|D_{m-1}\|^r
\]

\[
+ \cdots + \|C_1 \cdots C_{s-1}\| \|C_s - D_{s-1}D_s\| \|D_s D_{s+1} \cdots D_{m-1}\| \|D_{m-1}\|^r
\]

Then, by Lemma 2.2, there is a positive number \( M_1 \), depending on \( L \) and \( M_0 \), which is an upper bound for the norm of the factors of the form \( C_1 \cdots C_s, D_s^r D_{s+1} \cdots D_{m-1} \). By Lemma 2.6, there is a positive number \( M_2 \), depending on \( L \) and \( M_0 \), such that each factor of the form \( C_s - D_{s-1}D_s^r \) has norm bounded by \( M_2 r(m) \hat{\mu}(\tilde{\gamma}_s) \). Then

\[
\|C_1 \cdots C_m - D_1 \cdots D_{m-1}\| \leq M_0 M_1^2 M_2 r(m).
\]

In the following we want to examine the behaviour of \( D_1 \cdots D_{m-1} \) as \( m \to \infty \) and as the lamination \( \mu \) changes. For this we must consider more carefully the leaves of the lamination near \( x \).

By Lemma 2.13, there is an open set \( U \subset G(K) \), depending on \( d, \theta \) and \( d'/m \) such that, if \( \delta \) is any geodesic in \( U \cap \text{supp } \mu \), then \( \delta \) intersects the geodesics.
γ and g(γ) at a distance less than \(d'/m\) from \(x\). Let \(\chi: G(K) \to [0,1]\) be a continuous function, with \(\text{supp} \chi \subset U\) and \(\chi|_{G(x)} = 1\). We introduce the notation

\[
a' = \int_{[x_0,x_1]} \chi \mu \quad \quad a'' = \int_{[x_0,x_1]} (1 - \chi) \mu
\]

\[
b' = \int_{[x_{m-1},x_m]} (\chi \circ g^{-1}) \mu \quad \quad b'' = \int_{[x_{m-1},x_m]} (1 - \chi \circ g^{-1}) \mu
\]

\[
P = A(\phi_*(\gamma'_1), a') \quad \quad Q = A(\phi_*(\gamma'_1), a'')
\]

\[
R = A(\phi_*(\gamma'_{m-1}), b'') \quad \quad S = A(\phi_*(\gamma'_{m-1}), b'),
\]

and we have

\[
D_1 = P Q D_1^r \quad \quad D_{m-1} = D_{m-1}^l RS.
\]

Let \(\{\mu_n\}\) be a sequence of complex measured geodesic laminations on the surface \(S_0\), converging weakly in \(\mathcal{M}(G(K))\) to a measured lamination \(\mu_0\). Then, by the Uniform Boundedness Principle, there is a positive number \(M_0\) such that \(\|\mu_n\| \leq M_0\) for all \(n \geq 0\).

For each positive integer \(m\), for each \(i = 1, \ldots, m-1\), for each \(j = 1, \ldots, k\) and for each measured lamination \(\mu_n\), \(n \geq 0\), we define as above the points \(x_{j,m,i}\), the geodesics \(\gamma'_{n,j,m,i}\), the functions \(\lambda_{j,m,i}\), the quantities \(a_{n,j,m,i}\), \(b_{n,j,m,i}\), \(a'_{n,j,m}\), \(b'_{n,j,m}\) and the isometries \(D_{n,j,m,i}\), \(P_{n,j,m}\), \(Q_{n,j,m}\), \(R_{n,j,m}\), \(S_{n,j,m}\).

Let \(B_{n,j,m} = D_{n,j,m,1} \cdots D_{n,j,m,m-1}\). We want to find a bound for the norm of the difference between \(B_{0,j,m} g_j\) and some conjugate of \(B_{n,j,m} g_j\).

**Lemma 3.1** With the above notation, there exist positive numbers \(N_1, N_2\) and functions \(r: \mathbb{N} \to \mathbb{R}, \varepsilon: \mathbb{N} \times \mathbb{N} \to \mathbb{R}\) such that

\[
\lim_{m \to \infty} r(m) = 0, \quad \lim_{n \to \infty} \varepsilon(m, n) = 0 \quad \text{for each } m \in \mathbb{N}
\]

and

\[
\left\| P_{0,1,m}^{-1} B_{n,j,m} g_j P_{n,1,m}^{-1} - B_{0,j,m} g_j \right\| \leq N_1 r(m) + N_2 \varepsilon(m, n).
\]
Proof To simplify notation, we drop the index \( m \) for the time being, and write, for example, \( D_{n,j;i} \) for \( D_{n,j,m,i} \). We have
\[
\left\| P_{0,1}^{-1} B_{n,j} g_j P_{n,1} P_{0,1}^{-1} - B_{0,j} g_j \right\| \leq
\left\| P_{0,1}^{-1} B_{n,j} g_j P_{n,1} P_{0,1}^{-1} - P_{0,j} P_{0,j}^{-1} B_{n,j} g_j P_{n,j} P_{0,j}^{-1} \right\|
+ \left\| P_{0,j} P_{n,j}^{-1} B_{n,j} g_j P_{0,j}^{-1} g_j - P_{0,j} P_{n,j}^{-1} B_{n,j} S_{n,j}^{-1} S_{0,j} \right\| \left\| g_j \right\|
+ \left\| P_{0,j} P_{n,j}^{-1} B_{n,j} S_{n,j}^{-1} S_{0,j} - B_{0,j} \right\| \left\| g_j \right\|.
\]

We will find upper bounds for the three terms of the right hand side of the above inequality.

The first term of (2) is bounded above by
\[
\left\| P_{0,1}^{-1} P_{n,1}^{-1} - P_{0,j} P_{n,j}^{-1} \right\| \left\| B_{n,j} g_j P_{n,1} P_{0,1}^{-1} \right\|
+ \left\| P_{0,j} P_{n,j}^{-1} B_{n,j} g_j \right\| \left\| P_{n,j} P_{0,j}^{-1} - P_{n,j} P_{0,j}^{-1} \right\|.
\]

By Lemma 2.2, the factors containing \( g_j \) are bounded above by \( M_1 \). We consider the other factor in each term. Recall that \( P_{n,j} = A(\phi_*(\gamma_{n,j;1}), a'_{n,j}) \). We have
\[
\left\| P_{0,j} P_{n,j}^{-1} - P_{0,1} P_{n,1}^{-1} \right\| \leq
\left\| P_{0,j} \right\| \left\| P_{n,j}^{-1} - A(\phi_*(\gamma_{0,j;1}), -a'_{n,j}) \right\|
+ \left\| A(\phi_*(\gamma_{0,j;1}), a'_{0,j} - a'_{n,j}) - A(\phi_*(\gamma_{0,1;1}), a'_{0,1} - a'_{n,1}) \right\|
+ \left\| P_{0,1} \right\| \left\| A(\phi_*(\gamma_{0,1;1}), -a'_{n,1}) - P_{n,1}^{-1} \right\|.
\]

By Lemma 2.5, there is a positive constant \( M' \) such that the first and the third term of the right hand side of (3) are bounded by \( M_0 M_1 M' r(m) \). To find a bound for the second term we consider two cases.

(1) The segment \([x_0, x_{j;1}]\) intersects the same geodesics in \( \text{supp}(\chi \mu_n) \) as does the segment \([x_0, x_{1,1}]\).

(2) The two segments intersect different sets of geodesics in \( \text{supp}(\chi \mu_n) \).

Let \( z_{n,i} = \int_{[x_0, x_{i;1}]} \chi(\mu_0 - \mu_n) = a'_{0,i} - a'_{n,i} \).

In case (1), \( z_{n,j} = z_{n,1} \), and the geodesics \( \gamma_{0,j;1}, \gamma_{0,1;1} \) lie in a (2-dimensional) solid cylinder of radius \( d/m \) based at \( x_0 \). The segments \([x_0, x_{j;1}]\) and \([x_0, x_{1;1}]\)
induce concurrent orientations on the geodesics $\gamma_{0,j;1}'$ and $\gamma_{0,1;1}'$ respectively. So, by Lemma 2.5,

$$\|A(\phi_*(\gamma_{0,j;1}'), z_{n,j}) - A(\phi_*(\gamma_{0,1;1}'), z_{n,1})\| \leq M_0M'r(m).$$

Note that if $\mu_n$ satisfies the conditions of case (1) for large enough $n$, then $\mu_0$ also satisfies these conditions.

In case (2), the orientations induced by the segments $[x_0, x_{j;1}]$ and $[x_0, x_{1,1}]$ on the geodesics $\gamma_{0,j;1}'$ and $\gamma_{0,1;1}'$ respectively, are not concurrent. Hence, by Lemma 2.5,

$$\|A(\phi_*(\gamma_{0,j;1}'), z_{n,j}) - A(\phi_*(\gamma_{0,1;1}'), z_{n,1})\| \leq M_0M'r(m) + M'n|z_{n,j} + z_{n,1}|.$$

Note that, in this case,

$$a_{0,j}' + a_{0,1}' = \int_{[x_0, x_{j;1}]} \chi\mu_0 + \int_{[x_0, x_{1,1}]} \chi\mu_0 = \chi\mu_0(G)$$

and similarly for $\mu_n$. Hence $z_{n,j} + z_{n,1} = \chi\mu_0(G) - \chi\mu_n(G)$. Let

$$\varepsilon_0(m, n) = \sup_{s \geq n} |\chi_{m, \mu_0} - \chi_{m, \mu_n}|.$$

Now we turn our attention to the second term of equation (2). This term involves only the generator $g_j$, so we drop the subscript $j$ from the notation.

We have

$$\|P_n^{-1}B_n g P_n P_0^{-1}g^{-1} - P_0 P_n^{-1}B_n S_n^{-1}S_0\| \leq \|P_n^{-1}B_n\| \|S_n^{-1}\| \|S_n g P_n^{-1}g^{-1} - S_0 g P_0 g^{-1}\| \|g P_0^{-1}g^{-1}\|.$$

We consider the term $S_n g P_n^{-1}g^{-1}$, which is equal to

$$A\left(\phi_*(\gamma_{m,m-1}'), \int_{[x_{m-1}, x_m]} (\chi \circ g^{-1})\mu_n\right) A\left(\phi_*(g(\gamma_{n;1}'), \int_{[x_0,x_1]} \chi\mu_n)\right).$$

Since $\mu_n$ is invariant by $g$, and $x_m = g(x_0)$, we have

$$\int_{[x_{m}, g(x_1)]} (\chi \circ g^{-1})\mu_n = \int_{[x_0,x_1]} \chi\mu_n.$$

We have to consider two cases:

1. The segments $[x_{m-1}, x_m]$ and $[x_{m}, g(x_1)]$ intersect the same geodesics in $\text{supp} \left((\chi \circ g^{-1})\mu_n\right)$.
2. The segments $[x_{m-1}, x_m]$ and $[x_{m}, g(x_1)]$ intersect different sets of geodesics in $\text{supp} \left((\chi \circ g^{-1})\mu_n\right)$.
In case (1), we let \( z_n = \int_{[x_{m-1}, x_m]} (\chi \circ g^{-1}) \mu_n = \int_{[x_m, g(x_1)]} (\chi \circ g^{-1}) \mu_n \). The geodesics \( \gamma_{n;m-1} \) and \( g(\gamma_{n;1}) \) lie in a solid cylinder of radius \( d/m \), based at \( x_m \), and the orientations induced by the segments \([x_{m-1}, x_m]\) and \([x_m, g(x_1)]\) are not concurrent. Hence, by Lemma 2.6, \( \| S_n g P_n g^{-1} - I \| \leq M_0 M_2 r(m) \). As before, if \( \mu_n \) satisfies the conditions of case (1) for large enough \( n \), then \( \mu_0 \) also satisfies these conditions. Hence
\[
\| S_n g P_n g^{-1} - S_0 g P_0 g^{-1} \| \leq 2M_0 M_2 r(m).
\]

In case (2), since \( \mu_n \) is invariant by \( g \), and \( x_m = g(x_0) \), we have
\[
\int_{[x_{m}, g(x_1)]} (\chi \circ g^{-1}) \mu_n + \int_{[x_{m-1}, x_m]} (\chi \circ g^{-1}) \mu_n = \chi \mu_n(G)
\]
and if \( n \) is large enough, the same is true of \( \mu_0 \). Then
\[
\| S_n g P_n g^{-1} - S_0 g P_0 g^{-1} \| \leq \\
\| S_n g P_n g^{-1} - A(\phi_n, (\gamma_{n;m-1})_1, \chi \mu_n(G)) \| + \| A(\phi_n, (\gamma_{n;m-1})_1, \chi \mu_n(G)) - A(\phi_0, (\gamma_{0;m-1})_1, \chi \mu_0(G)) \| + \| A(\phi_0, (\gamma_{0;m-1})_1, \chi \mu_0(G)) - S_0 g P_0 g^{-1} \|.
\]

By Lemma 2.5 and Lemma 2.6, this is bounded above by \( M' r(m) + M'' \varepsilon(m, n) \).

The third term of equation (2) is bounded by
\[
\| P_0 \| \| P_n^{-1} B_n S_n^{-1} - P_0^{-1} B_0 S_0^{-1} \| \| S_0 \| \| g \|.
\]

But
\[
\| P_n^{-1} B_n S_n^{-1} - P_0^{-1} B_0 S_0^{-1} \| = \\
\| Q_n D_{n;1} D_{n;2} \cdots D_{n;m-2} D_{n;m-1} R_n - Q_0 D_{0;1} D_{0;2} \cdots D_{0;m-2} D_{0;m-1} R_0 \|
\]
and by Lemma 2.2, this is bounded by
\[
M^2_t \left( \| D_{n;m-1} R_n - D_{0;m-1} R_0 \| + \sum_{i=2}^{m-2} \| D_{n;i} - D_{0;i} \| + \right.
\]
\[
+ \| Q_n D_{n;1} - Q_0 D_{0;1} \| \right).
\]

Note that \( Q_n D_{n;1} = A \left( \phi_n(\gamma_{n;1}), \int_{[x_0, x_1]} \lambda_1(1 - \chi) \mu_n \right) \) and hence
\[
\| Q_n D_{n;1} - Q_0 D_{0;1} \| \leq M' r(m) + M'' \varepsilon_1(m, n)
\]
where \( \varepsilon_1(m, n) = \sup_{\varepsilon \geq n} \left| \int_{[x_0, x_1]} \lambda_1(1 - \chi_m)(\mu_s - \mu_0) \right|, \) and similarly for the other terms of (4), for suitable \( \varepsilon_i, i = 2, \ldots, m - 1 \).
To complete the proof of Lemma 3.1 we must show that \( r(m) \) and \( \varepsilon(m, n) = \sum_{i=0}^{m-1} \varepsilon_i(m, n) \) have the required properties. It is clear that we can choose a sequence \( r(m) \), with \( \lim_{m \to \infty} r(m) = 0 \), such that the pair \( r = r(m), \delta = d/m \) satisfy the conditions of Lemma 2.10. Lemma 2.14 implies that, for each \( m \), \( \lim_{n \to \infty} \varepsilon(m, n) = 0 \).

We let \( E_{n,j,m} = C_{n,j,m,1} \cdots C_{n,j,m,m} \) and \( H_{n,m} = P_{0,1,m}^{-1} P_{n,1,m} \). Then, combining the above result with (1), we have

\[
\| H_{n,m}E_{n,j,m}g_jH_{n,m}^{-1} - E_{0,j,m}g_j \| \leq M(r(m) + \varepsilon(m, n)). \tag{5}
\]

If \( g_1, \ldots, g_k \) is a set of generators for \( \Gamma_0 \), the space \( R \) of homomorphisms \( \rho: \Gamma_0 \to PSL(2, \mathbb{C}) \) with quasi-Fuchsian image is a subspace of \( PSL(2, \mathbb{C})^k \), and \( Q(S) \) is a subspace of the quotient by the adjoint action on the left, \( PSL(2, \mathbb{C})^k / PSL(2, \mathbb{C}) \). Let

\[
\rho_{n,m} = (H_{n,m}E_{n,j,m}g_jH_{n,m}^{-1}, \ j = 1, \ldots, k)
\]

and let \([\rho_{n,m}]\) denote the equivalence class of \( \rho_{n,m} \) in \( PSL(2, \mathbb{C})^k / PSL(2, \mathbb{C}) \).

Let \( n(m) \) be a sequence such that \( n(m) \geq m \) and \( \varepsilon(n(m), m) \leq 1/m \). Then \( \lim_{m \to \infty} \rho_{n(m),m} = \rho_{00} \). As \( m \to \infty \), \([\rho_{n,m}]\) converge, uniformly in \( n \), to the bending deformation \([\rho_{\mu_0}]\), [5]. Hence, \( \lim_{m \to \infty}[\rho_{n(m),m}] = \lim_{n \to \infty}[\rho_{n(m)}] = \lim_{n \to \infty}[\rho_{\mu_0}] \), and we have

\[
\lim_{n \to \infty}[\rho_{\mu_0}] = [\rho_{\mu_0}]. \tag{6}
\]

To complete the proof of Theorem 1, it remains to show that the convergence is uniform in compact subsets of \( D \). If \( ([\rho], t) \in D \), each bound used in the proof of (6) depends at most linearly on \( t \), while it depends on \( \rho \) only in terms of the endpoints of a finite number of geodesics \( \phi_\kappa(\gamma) \). The endpoints of the geodesic \( \phi_\kappa(\gamma) \) are, for each \( \gamma \), holomorphic functions of \([\rho]\). Hence each bound can be chosen uniformly on each compact subset of \( D \).

Note that \( D \) contains in its interior the set \( Q(S) \times \{0\} \). If the laminations \( \mu_n \) are real for all but a finite number of \( n \), then \( D \) also contains the set \( Q(S) \times \mathbb{R} \), but this is not true in the general case.

To prove Theorem 2 we recall that the bending vector field \( T_\mu \) is defined by

\[
T_\mu([\rho]) = \frac{\partial}{\partial t} B_\mu([\rho], t).
\]
The vector fields \( T_{\mu_n} \) are holomorphic, and \( B_{\mu_n}((\rho), t) \) converge to \( B_{\mu_0}((\rho), t) \) for \((\rho), t) \in D\). It follows that \( T_{\mu_n} \) converge to \( T_{\mu_0} \), uniformly on compact subsets of \( Q(S) \).

We conclude with the proof of Theorem 3. We consider the subset of \( \mathcal{ML}(S) \) consisting of measured laminations with non negative real and imaginary parts, and we denote it by \( \mathcal{ML}^{++}(S) \). We identify \( \mathcal{ML}^{++}(S) \) with a subset of the set of pairs of positive measured laminations \( \mathcal{ML}_R^+(S) \times \mathcal{ML}_R^+(S) \). If \( \nu \in \mathcal{ML}^{++}(S) \), then \( \text{Re}\nu \) and \( \text{Im}\nu \) are in \( \mathcal{ML}_R^+(S) \) and they satisfy the condition

\[
\text{supp}(\text{Re}\nu) \cup \text{supp}(\text{Im}\nu) \text{ is a geodesic lamination. (7)}
\]

Conversely, any pair \( \nu_1, \nu_2 \) of positive measured laminations satisfying (7) define a measure \( \nu = \nu_1 + i\nu_2 \in \mathcal{ML}^{++}(S) \). The mapping is a homeomorphism of \( \mathcal{ML}^{++}(S) \) onto a subset of \( \mathcal{ML}_R^+(S) \times \mathcal{ML}_R^+(S) \). But \( \mathcal{ML}_R^+(S) \) is homeomorphic to \( \mathbb{R}^{6g-6} \), [6]. Thus \( \mathcal{ML}^{++}(S) \) is first countable, and Theorem 2 implies that \( \mu \mapsto T_\mu \) is continuous. Theorem 3 then follows by the continuity of the evaluation map.

References

[1] R D Canary, D B A Epstein, P Green, Notes on notes of Thurston, from: “Analytical and Geometric Aspects of Hyperbolic Space”, D B A Epstein (editor), LMS Lecture Note Series 111, Cambridge University Press (1987) 3–92

[2] D B A Epstein, A Marden, Convex Hulls in Hyperbolic Space, a Theorem of Sullivan, and Measured Pleated Surfaces, from: “Analytical and Geometric Aspects of Hyperbolic Space”, D B A Epstein (editor), LMS Lecture Note Series 111, Cambridge University Press (1987) 113–253

[3] D Johnson, J J Millson, Deformation spaces associated to compact hyperbolic manifolds, from: “Discrete Groups in Geometry and Analysis”, Progress in Math. 67, Birkhaüser (1987) 48–106

[4] C Kourouniotis, Deformations of hyperbolic structures, Math. Proc. Camb. Phil. Soc. 98 (1985) 247–261

[5] C Kourouniotis, Bending in the space of quasi-Fuchsian structures, Glasgow Math. J. 33 (1991) 41–49

[6] R C Penner, J L Harer Combinatorics of train tracks, Annals of Math. Studies 125, Princeton University Press (1992)