Schwarzian derivative
and
Numata Finsler structures

C. DUVAL

Centre de Physique Théorique, CNRS, Luminy, Case 907
F-13288 Marseille Cedex 9 (France)

Abstract
The flag curvature of the Numata Finsler structures is shown to admit a non-trivial prolongation to the one-dimensional case, revealing an unexpected link with the Schwarzian derivative of the diffeomorphisms associated with these Finsler structures.

Mathematics Subject Classification 2000: 58B20, 53A55

1 Finsler structures in a nutshell
1.1 Finsler metrics
A Finsler structure is a pair \((M, F)\) where \(M\) is a smooth, \(n\)-dimensional, manifold and \(F: TM \to \mathbb{R}^+\) a given function whose restriction to the slit tangent bundle \(TM \setminus M = \{(x,y) \in TM \mid y \in T_x M \setminus \{0\}\}\) is strictly positive, smooth, and positively homogeneous of degree one, i.e., \(F(x, \lambda y) = \lambda F(x, y)\) for all \(\lambda > 0\); one furthermore demands that the \(n \times n\) vertical Hessian matrix with entries \(g_{ij}(x, y) = \frac{1}{2} F^2 \frac{\partial^2}{\partial y_i \partial y_j}\) be positive definite, \((g_{ij}) > 0\). See [1]. These quantities are (positively) homogeneous of degree zero, and the fundamental tensor

\[ g = g_{ij}(x, y) dx^i \otimes dx^j \]

defines a sphere’s worth of Riemannian metrics on each \(T_x M\) parametrized by the direction of \(y\). See [2].

The distinguished “vector field”

\[ \ell = \ell^i(\frac{\partial}{\partial x^i}), \quad \text{where} \quad \ell^i(x, y) = \frac{y^i}{F(x, y)}, \]

actually a section of \(\pi^*(TM)\) where \(\pi: TM \setminus M \to M\) is the natural projection, is such that \(g(\ell, \ell) = 1\).
There is a wealth of Finsler structures, apart from the special case of Riemannian structures \((M, g)\) for which \(F(x, y) = \sqrt{g_{ij}(x)y^iy^j}\). For instance, the so-called Randers metrics
\[
F(x, y) = \sqrt{a_{ij}(x)y^iy^j + b_i(x)y^i}
\]  
(1.3)
satisfy all previous requirements if \(a = a_{ij}(x)dx^i \otimes dx^j\) is a Riemann metric and if the 1-form \(b = b_i(x)dx^i\) is such that \(a^{ij}(x)b_i(x)b_j(x) < 1\) for all \(x \in M\).

### 1.2 Flag curvature

Unlike the Riemannian case, there is no canonical linear Finsler connection on \(\pi^*(TM)\). An example, though, is provided by the Chern connection \(\omega^i_j = \Gamma^i_{jk}(x, y)dx^k\) which is uniquely defined by the following requirements [1]: (i) it is symmetric, \(\Gamma^i_{jk} = \Gamma^i_{kj}\), and (ii) it almost transports the metric tensor, i.e., \(dg_{ij} - \omega^k_i g_{jk} - \omega^k_j g_{ik} = 2C_{ijk} \delta y^k\), with \(\delta y^i = dy^i + N^i_j(x)dx^j\), where the \(N^i_j(x,y) = \Gamma^i_{jk}y^k\) are the components of the non linear connection associated with the Chern connection, and the \(C_{ijk}(x,y) = \frac{1}{2} (g_{ij}) y^k\) those of the Cartan tensor, specific to Finsler geometry.

Using the “horizontal covariant derivatives” \(\delta / \delta x^i = \partial / \partial x^i - N^j_i \partial / \partial y^j\), one expresses the (horizontal-horizontal part of the) Chern curvature by
\[
R^k_{jkl} = \frac{\delta}{\delta x^k} \Gamma^i_{jl} + \Gamma^i_{mk} \Gamma^m_{jl} - (k \leftrightarrow l),
\]  
(1.4)
and the flag curvature (associated with the flag \(\ell \wedge v\) defined by \(v \in T_xM\)) by
\[
K(x, y, v) = R^i_{ik}v^j v^k / g(v, v) - g(\ell, v)R^j_{jkl} \ell^l.
\]  
(1.5)

One says that a Finsler structure is of scalar curvature if \(K(x, y, v)\) does not depend on the vector \(v\), i.e., if
\[
R^k_{ik} = K(x, y)h_{ik},
\]  
(1.6)
with \(h_{ik} = g_{ik} - \ell_i \ell_k\) the components of the “angular metric”, where \(\ell_i = g_{ij} \ell^j(= F_{y^j})\).

### 2 Numata Finsler structures

#### 2.1 The Numata metric

Numata [4] has proved that metrics of the form \(F(x, y) = \sqrt{q_{ij}(y)y^iy^j + b_i(x)y^i}\), on TM where \(M \subset \mathbb{R}^n\), with \((q_{ij}) > 0\) and \(db = 0\) are, indeed, of scalar curvature. See [2].

Of some interest is the special case \(q_{ij} = \delta_{ij}\) and \(b = df\) with \(f \in C^\infty(M)\), viz.,
\[
F(x, y) = \sqrt{\delta_{ij}y^iy^j + f_x y^i},
\]  
(2.1)
where
\[
M = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^n f_{x^i}^2 < 1 \}.
\]  
(2.2)
The computation of the flag curvature of this particular Randers metric (1.3) can be found in [1] and yields

\[ K(x, y) = 3 \frac{1}{4} F^4 \left( f_{x^i x^j y^k} \right)^2 - \frac{1}{2} \frac{1}{F^3} f_{x^i x^j x^k y^i y^j y^k}. \]  

(2.3)

2.2 Flag curvature & Schwarzian derivative

The expression (2.3) of the flag curvature of the Numata metric (2.1) holds for \( n \geq 2 \).

If \( n = 1 \), the left-hand side of (1.6) vanishes along with the curvature (1.4), while its right-hand side vanishes as well since the angular metric has rank zero. For this particular dimension, Equation (1.6) trivially holds true, but tells, however, nothing about the flag curvature \( K(x, y) \).

At this stage, it is worth noting that (2.3) indeed admits a prolongation to the one-dimensional case; it is therefore tempting to specialize its expression for \( n = 1 \).

Suppose, thus, that \( M \subset S^1 \) is a nonempty open subset (2.2), so that we have \( TM \setminus M = T_+ M \cup T_- M \), where \( T_\pm M = M \times \mathbb{R}_\pm^+ \). The metric (2.1) then reads

\[ F(x, y) = |y| + f'(x)y, \]  

(2.4)

using an affine coordinate, \( x \), on \( S^1 \), with \(-1 < f'(x) < +1\) (see (2.2)); its restrictions to \( T_\pm M \) are given by \( F_\pm(x, y) = \varphi'_\pm(x)y > 0 \), where

\[ \varphi'_\pm(x) = f'(x) \pm 1, \]  

(2.5)

implying \( \varphi_\pm \in \text{Diff}_{\pm}(S^1) \), with \( |\varphi'_\pm(x)| < 2 \) (all \( x \in M \)).

The Numata metric (2.4) on \( T_+ M \), say, is thus associated, via (2.5), to orientation-preserving diffeomorphisms \( \varphi \) of \( S^1 \) such that \( 0 < \varphi'(x) < 2 \) (all \( x \in M \)). Given such a \( \varphi \in \text{Diff}_{+}(S^1) \), the fundamental tensor (1.1) retains the form \( g = \varphi'(x)^2 dx^2 \) and is, naturally, a Riemannian metric on \( M \).

Rewriting Equation (2.3) for \( T_+ M \), and bearing in mind that \( y = F(x, y)/\varphi'(x) \), we readily find that \( K(x, y) \) is actually independent of \( y \), namely

\[ K(x) = -\frac{1}{2} \frac{1}{\varphi'(x)^2} S(\varphi)(x), \]  

(2.6)

where

\[ S(\varphi)(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2} \left( \frac{\varphi''(x)}{\varphi'(x)} \right)^2 \]  

(2.7)

denotes the Schwarzian derivative [5] of the diffeomorphism \( \varphi \) of \( S^1 \). The argument clearly still holds, mutatis mutandis, for orientation-reversing diffeomorphisms of \( S^1 \).
We have thus proved the

**Theorem 2.1.** The Numata Finsler structure \((M, F)\), with metric \(F\) given by (2.4) where \(M \subset S^1\) is defined by (2.2), induces a Riemannian metric, \(g(\varphi) = \varphi^*(dx^2)\), where \(\varphi \in \text{Diff}(S^1)\) is as in (2.5). The flag curvature (2.3) admits a prolongation to this one-dimensional case and retains the form

\[
K = -\frac{1}{4} \frac{S(\varphi)}{g(\varphi)},
\]

(2.8)

where \(S(\varphi) = S(\varphi)(x)dx^2\) is the Schwarzian quadratic differential of \(\varphi \in \text{Diff}(S^1)\).

As an illustration, the one-dimensional Numata Finsler structures of constant flag curvature are associated, through (2.5), to the solutions \(\varphi\) of (2.8) for \(K \in \mathbb{R}\), viz.,

\[
\varphi_{\pm}(x) = K^{-\frac{1}{2}} \arctan\left(\frac{K^{\frac{1}{2}}(ax + b)/(cx + d)}{\text{where } a, b, c, d \in \mathbb{R} \text{ with } ad - bc = \pm 1.}
\]

Let us mention another instance where the Schwarzian derivative is associated with curvature, namely the geometry of curves in Lorentzian surfaces of constant curvature [3].

Discussions with P. Foulon are warmly acknowledged.

**References**

[1] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, GTM 200, Springer, New York, 2004.

[2] D. Bao, and C. Robles, “Ricci and Flag Curvatures in Finsler Geometry”, in *A Sampler of Riemann-Finsler Geometry*, D. Bao, R. L. Bryant, S.-S. Chern, and Z. Shen (Editors), MSRI Publications 50, Cambridge University Press, 2004.

[3] C. Duval, and V. Ovsienko, “Lorentzian Worldlines and the Schwarzian Derivative”, Funct. Anal. Applic. 34:2 (2000), 135–137.

[4] S. Numata, “On the torsion tensors \(R_{ijk}\) and \(P_{ijk}\) of Finsler spaces with a metric \(ds = (g_{ij}(dx^i)dx^j)^{1/2} + b_i(x)dx^i\)”, Tensor (N.S.) 32 (1978), 27–32.

[5] V. Ovsienko, and S. Tabachnikov, *Projective Differential Geometry Old And New: From The Schwarzian Derivative To The Cohomology Of Diffeomorphism Group*, Cambridge University Press, 2005, and References therein.