ANISOTROPIC KELVIN TRANSFORM AND THE SOBOLEV ANISOTROPIC CRITICAL EQUATION

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Abstract. We introduce and study a Kelvin-type transform in the anisotropic setting. Then we exploit it to study symmetry properties of solution to the anisotropic Sobolev critical equation in the entire space.

1. Introduction

The aim of this paper is to consider elliptic partial differential equations involving the Finsler Laplace operator:

\[-\Delta^H u := -\text{div}(H(\nabla u)\nabla H(\nabla u)),\]

where $H$ is a Finsler norm (see assumption $(h_H)$ below), and for $H(\xi) = |\xi|$, the operator $\Delta^H$ reduces to the classical Laplacian. Elliptic equations involving the anisotropic operators arise as Euler-Lagrange equations of Wulff-type energy functionals.

We investigate qualitative properties of the anisotropic Sobolev critical equation:

\[(1.1)\quad -\Delta^H u = u^{2^*-1} \quad \text{in } \mathbb{R}^N,\]

where $N \geq 3$ and $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent.

We remark that in the Euclidean framework, equation (1.1) reduces to the classical Sobolev critical problem, for which the classification of positive solutions $u \in D^{1,2}(\mathbb{R}^N)$ was started in the seminal paper of [19]. Later on, Caffarelli, Gidas and Spruck [8], using the Kelvin transformation, classified all the solutions of the Sobolev critical equation without any a priori assumption on the energy. When $p \neq 2$ the situation is much more involved, since the Kelvin transformation doesn’t work. In [30] and [34], when $p \neq 2$, positive solutions in $\mathbb{R}^N$ belonging to $D^{1,p}(\mathbb{R}^N)$ have been completely characterized. The approach used to achieve this classification requires a careful application of the moving plane method, and asymptotic estimates on $u$ and on $\nabla u$ both from above and below.

In the anisotropic setting, Ciraolo, Figalli and Roncoroni, see [11], recently, obtained a complete classification result, that holds also in the quasilinear case $p \neq 2$, but under the a priori assumption that the solutions are in $D^{1,p}(\mathbb{R}^N)$. This is the same a priori assumption that appears also in [30, 34], and it is caused by the fact that is not possible, in general, to deduce a Kelvin-type transformation in the anisotropic setting neither in the quasilinear setting.

We address here this deep issue and we succeed in providing a suitable Kelvin-type transformation in the Riemannian case,

\[(h_M)\quad H(\xi) = \sqrt{\langle M\xi, \xi \rangle}\]
for some symmetric and positive definite matrix $M$. In this setting, we define the anisotropic Kelvin-type transformation by

$$T_H : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}, \quad T_H(\xi) := \frac{\nabla H(\xi)}{H(\xi)},$$

and consequently

$$\hat{u} := \frac{u \circ T_H}{H^{N-2}} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

see Section 3 for more details.

**Remark 1.1.** Although our main results will be proved in the Riemannian case ($h_M$), a part of our construction holds for a general Finsler norm. This is why many preliminary results will be stated in full generality and also we shall try to use the language of Finsler geometry, where it is possible. This will also highlight the geometric idea that is behind our construction.

With such construction at hand, we prove the following.

**Theorem 1.2.** Let $H$ be given by ($h_M$). If $u \in H^1_{\text{loc}}(\mathbb{R}^N)$ is a locally bounded solution of $-\Delta^H u = f(x)$ in $\mathbb{R}^N$, with $f \in L^2_{\text{loc}}(\mathbb{R}^N)$, then $\hat{u}$ belongs to $D^{1,2}$ far from the origin, and weakly solves the dual equation

$$(1.2) \quad -\Delta^{H^\circ} \hat{u} = \frac{f \circ T_H}{H^{N+2}} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

where $H^\circ$ is the dual norm.

As observed by Ciraolo, Figalli and Roncoroni in [11], a Kelvin-type approach doesn’t work, in general, in the anisotropic context. This also happen in the Euclidean framework when $p \neq 2$. As we shall show adding an example in Remark 3.4, a pure invariance of the equation is not expected for general Finsler norms, even in the semilinear case $p = 2$. Actually, by our Theorem 1.2 and direct computation, it follows that the critical Sobolev equation is invariant meaning that the Riemannian norm is replaced by the dual one, see Section 2 for more details.

To the best of our knowledge this is the first result in this direction in the anisotropic framework. Let us also mention the work of Monti and Morbidelli [24] for a non-trivial application of the Kelvin transformation for equations involving the Grushin operators.

Exploiting an anisotropic version of the moving plane method of Alexandrov-Serrin [1, 27], for the transformed equation, we prove the following:

**Theorem 1.3.** Let $H$ be given by ($h_M$) and let $\Gamma$ be a closed subset of $\Pi_0^\nu := \{x \in \mathbb{R}^N \mid \langle x, \nu \rangle = 0\}$ such that

$$(1.3) \quad \text{Cap}_2(\mathbb{R}^N \setminus \{0\}) = 0,$$

and $u \in H^1_{\text{loc}}(\mathbb{R}^N \setminus \Gamma)$ is a non-negative locally bounded solution to

$$(1.4) \quad -\Delta^H u = u^{2^*-1} \quad \text{in} \quad \mathbb{R}^N \setminus \Gamma$$

that has a non-removable singularity on $\Gamma$, then $u$ is symmetric with respect to the hyperplane $\Pi_0^\nu$, where $\nu$ is an eigenvector of $M$.

Theorem 1.3 is proved exploiting the Kelvin-type transformation introduced above. This is helpful to study the problem far from the origin avoiding a priori assumptions on the solutions. On the contrary, the transformed solution may exhibit a singularity at the origin. To overcome this difficulty we will exploit the moving plane technique in the way developed in [16, 31]. If, in Theorem 1.3, $\Gamma = \emptyset$, our result reduces to:
Theorem 1.4. Let $H$ be given by $(h_M)$. Then any non-negative and locally bounded solution $u \in H^1_{loc}(\mathbb{R}^N)$ to (1.1) is symmetric with respect to some affine hyperplane $\Pi_{\nu} := \{x \in \mathbb{R}^N \mid \langle x, \nu \rangle = \lambda_0\}$, where $\nu$ is an eigenvector of $M$ and $\lambda_0 \in \mathbb{R}$.

Concluding the introduction, let us mention that anisotropic operators arise in Finsler geometry, see [7, 10, 22, 28, 29], as modelization e.g. of material science [9, 20], relativity [4], biology [2], image processing [15, 25].

In the following we devote Section 2 to the introduction of the main geometrical notion used in the paper. In Section 3 we study the Kelvin-type transformation, defined above, and we study the invariance of the critical equation. In Section 4 we prove our symmetry results, namely we prove Theorem 1.3 and Theorem 1.4.

2. Notation and preliminary results

The aim of this section is to introduce some notation and to recall technical results about anisotropic elliptic operator that will be involved in the proof of the main results.

We shall consider solutions to the equation

$$-\Delta^H u = f(x) \quad \text{in } \mathbb{R}^N, \quad N \geq 2,$$

which is the Euler-Lagrange equation of the Wulff-type energy functional

$$\mathcal{W}(u) = \int_{\mathbb{R}^N} \left[ \frac{H(\nabla u)^2}{2} - f(x)u \right] \mathrm{d}x.$$  \hspace{1cm} (2.6)

As usual, $u \in H^1_{loc}(\mathbb{R}^N)$ is a weak solution of (2.5) if

$$\int_{\mathbb{R}^N} H(\nabla u) \langle \nabla H(\nabla u), \nabla \varphi \rangle \mathrm{d}x = \int_{\mathbb{R}^N} f(x)\varphi \mathrm{d}x, \quad \forall \varphi \in C^1_c(\mathbb{R}^N).$$  \hspace{1cm} (2.7)

Under suitable assumption on $f$, due to the results in [3, 14, 23, 32], it is possible to assume that solutions of (2.5) are of class $C^{1,\alpha}$. This will be the case when considering the critical problem in the entire space.

In all the paper generic fixed and numerical constants will be denoted by $C$ (with subscript in some cases) and they will vary from line to line. For $a, b \in \mathbb{R}^N$ we denote by $a \otimes b$ the matrix whose entries are $(a \otimes b)_{ij} = a_ib_j$. We remark that for any $v, w \in \mathbb{R}^N$ it holds that:

$$\langle a \otimes b \, v, w \rangle = \langle b, v \rangle \langle a, w \rangle.$$

Moreover, we denote with $I$ the identity matrix of order $N$.

Let $H$ be a function belonging to $C^2(\mathbb{R}^N \setminus \{0\})$. $H$ is said a “Finsler norm” if it satisfies the following assumptions:

$$(h_H) \quad \text{(i)} \quad H(\xi) > 0 \quad \forall \xi \in \mathbb{R}^N \setminus \{0\};$$

$$\text{(ii)} \quad H(s\xi) = |s|H(\xi) \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \forall s \in \mathbb{R};$$

$$\text{(iii)} \quad H \text{ is uniformly elliptic, that means the set } \mathcal{B}_1^H := \{\xi \in \mathbb{R}^N : H(\xi) < 1\} \text{ is uniformly convex}$$

$$\exists \Lambda > 0 : \quad \langle D^2H(\xi)v, v \rangle \geq \Lambda |v|^2 \quad \forall \xi \in \partial \mathcal{B}_1^H, \forall v \in \nabla H(\xi) \perp.$$  \hspace{1cm} (2.8)

Now we provide some remarks on Finsler norms.

Remark 2.1. A set is said uniformly convex if the principal curvatures of its boundary are all strictly positive. We point out that these kind of sets are also called “strongly convex”, which is stronger a condition than the usual “strict convexity”. Moreover, assumption (iii) is equivalent to assume that $D^2(H^2)$ is definite positive.
Remark 2.2. Actually, in literature the definition of “Finsler norm” is not unique. In differential geometry, it is usual to define this class of norms satisfying directly the assumption that $D^2(H^2)$ is definite positive. The last one allows to define the fundamental tensor, that is a very useful tool in order to show suitable properties of geometrical objects like curvatures, distances and so on. However, in many framework, it is enough to require that $B^H_1$ is only strictly convex, and not strongly convex. Therefore, it is usual to give a definition of Finsler norm without the requirement (iii) (see for instance [6, 7, 18]), but with the only assumption that $H^2$ has to be strictly convex. Moreover, for $p \neq 2$, the $p$-norm in $\mathbb{R}^N$ is not uniformly elliptic, because its unit ball $B^H_1$ is not strongly convex but only strictly convex. For our computations we need the stronger condition (iii) and we choose a particular class of Finsler norms satisfying it.

Since $H$ a norm in $\mathbb{R}^N$, we immediately get that there exists $c_1, c_2 > 0$ such that:

$$c_1|\xi| \leq H(\xi) \leq c_2|\xi|, \quad \forall \xi \in \mathbb{R}^N.$$  \hfill (2.9)

The dual norm $H^\circ : \mathbb{R}^N \to [0, +\infty)$ is defined as:

$$H^\circ(x) = \sup\{\langle \xi, x \rangle : H(\xi) \leq 1\}.$$  

It is easy to prove that $H^\circ$ is also a Finsler norm and it has the same regularity properties of $H$. Moreover it follows that $(H^\circ)^\circ = H$. For $r > 0$ and $\xi \in \mathbb{R}^N$ we define:

$$B^H_r(x) = \{x \in \mathbb{R}^N : H(x - \xi) \leq r\}$$

and

$$B^H_r(\xi) = \{x \in \mathbb{R}^N : H^\circ(x - \xi) \leq r\}.$$  

For simplicity, when $\xi = 0$, we set: $B^H_r = B^H_r(0)$, $B^H_r = B^H_r(0)$. In literature $B^H_r$ and $B^H_r$ are also called “Wulff shape” and “Frank diagram” respectively.

We recall that, since $H$ is a differentiable and 1-homogeneous function, it holds the Euler’s characterization result, i.e.

$$\langle \nabla H(\xi), \xi \rangle = H(\xi) \quad \forall x \in \mathbb{R}^N,$$  \hfill (2.10)

and

$$\nabla H(t\xi) = \text{sign}(t)H(\xi) \quad \forall \xi \in \mathbb{R}^N, \forall t \in \mathbb{R},$$  \hfill (2.11)

and the same is true for $H^\circ$. Moreover, there holds following identities:

$$H(\nabla H^\circ(x)) = 1 = H^\circ(\nabla H(x))$$  \hfill (2.12)

and

$$H(x)\nabla H^\circ(\nabla H(x)) = x = H^\circ(x)\nabla H(\nabla H^\circ(x)),$$  \hfill (2.13)

and we refer the reader to [7] for a complete proof. For more details on Finsler geometry see for instance [5, 7, 21].

As pointed out in Remark 1.1, a part of our construction hold for generic Finsler norms, but the main results are proved in the Riemannian case $(h_M)$. In this case

$$H^\circ(\xi) = \sqrt{\langle M^{-1}\xi, \xi \rangle}.$$  

We refer to [12, 13, 18] for a discussion on the geometry of the Riemannian case and for interesting related results.
3. Anisotropic Kelvin transform

The aim of this section is to prove Theorem 1.2. In order to achieve our claim, we introduce the formal definition of Kelvin transformation in the context of Finsler geometry.

**Definition 3.1.** Let $H$ be a Finsler norm. We define the map $T_H : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}$ as

$$T_H(\xi) = \frac{\nabla H(\xi)}{H(\xi)}.$$  

(3.14)

Note that, when $H(\xi) = |\xi|$, $T_H$ coincides with the usual Euclidean Kelvin transform. Hence, we observe that it holds the following result.

**Proposition 3.2.** The map $T_H$ is a global diffeomorphism with inverse given by $T_{H^o}$.

**Proof.** We show that, if $y = T_H(x)$, then $x = T_{H^o}(y)$. By (2.11),(2.12) and (2.13) we have:

$$T_H(T_{H^o}(y)) = \nabla H \left( \frac{H^o(y)}{H(\xi)} \right) H^{-1} \left( \frac{H^o(y)}{H(\xi)} \right) = \nabla H \left( \nabla H^o(y) \right) H^o(y) H^{-1} \left( \nabla H^o(y) \right) = H^o(y) \nabla H \left( \nabla H^o(y) \right) = y.$$  

(3.15)

Now, we denote by $DT_H$ the Jacobian matrix of $T_H$, namely:

$$DT_H = \frac{1}{H^2} \left[ HD^2H - \nabla H \otimes \nabla H \right] = \frac{1}{H^2} \left[ D \left( \frac{1}{2} \nabla (H^2) \right) - 2 \nabla H \otimes \nabla H \right].$$

In particular, if $H$ satisfies $(h_M)$, we have

$$DT_H(x) = \frac{1}{H(x)^2} \left[ M - \frac{2}{H(x)^2} Mx \otimes Mx \right].$$  

(3.16)

Moreover, we set $J(x) = |\det DT_H(x)|$. Using these notations, we are ready to prove the following key lemma.

**Lemma 3.3.** If $H$ is a Finsler norm that satisfy $(h_M)$, then

$$J(x) = \frac{\det M}{H(x)^{2N}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$  

(3.17)

**Proof.** Let $L$ be a diagonal matrix with entries $d_1, \ldots, d_N$ and denote by $\sqrt{L}$ the diagonal matrix whose entries are $\sqrt{d_1}, \ldots, \sqrt{d_N}$. Suppose $H(x) = \sqrt{\langle Lx, x \rangle}$. In view of (3.16) we have:

$$DT_H(x) = \frac{1}{H(x)^2} \left( L - \frac{2}{\langle Lx, x \rangle} Lx \otimes Lx \right) = \frac{1}{H(x)^2} \sqrt{L} \left( I - \frac{2}{\langle Lx, \sqrt{L}x \rangle} \sqrt{L}x \otimes \sqrt{L}x \right) \sqrt{L}$$

(3.18)

Since

$$\det \left( I - \frac{2}{|y|^2} y \otimes y \right) = 1, \quad \forall y \in \mathbb{R}^N \setminus \{0\},$$

we get (3.17) in the case $L$ diagonal. Let $H$ be of the form $(h_M)$. The spectral theorem ensures that it holds the decomposition $C^tMC = L$, where $C$ and $D$ are respectively an orthogonal and diagonal matrices. Having in mind this fact, we have

$$C^t DT_H(x) C = \frac{1}{H(x)^2} \sqrt{L} \left( I - \frac{2}{\langle LC^t x, \sqrt{LC}^t x \rangle} \sqrt{LC}^t x \otimes \sqrt{LC}^t x \right) \sqrt{L}.$$  

(3.20)
By (3.19) we get the thesis. \[\square\]

Assumption \( (h_M) \) is crucial in the previous lemma, and this result is essential also in the proof of Theorem 1.2 and Theorem 1.3. Indeed, our approach doesn’t work without Lemma 3.3, as pointed out in the next remark.

**Remark 3.4.** In the proof of Theorem 1.2 it will be crucial the fact that \( H(x)^{2N} J(x) \) is constant, see (3.17). For a generical Finsler norm this property is not true. For example, if we consider the norm \( H(x) = \sqrt{x_1^4 + 3x_1^2 x_2^2 + x_2^4} \) for \( x = (x_1, x_2) \in \mathbb{R}^2 \), then it follows that \( H(x)^{2N} J(x) \) is not constant, even if \( H \) satisfies all the conditions (i)-(ii)-(iii) stated above in the definition of a Finsler norm.

Now, in the same spirit of the seminal paper of Caffarelli, Gidas and Spruck [8], given a function \( u(x) \), we define the action of the Kelvin transform on \( u \) as follows

\[
\hat{u}(y) := \frac{1}{H(y)^{N-2}} u(T_H(y)),
\]

for every \( y \in \mathbb{R}^N \setminus \{0\} \).

**Remark 3.5.** It is easy to check that

\[
u(x) := \frac{1}{H^o(x)^{N-2}} \hat{u}(T_{H^o}(x)),
\]

for every \( x \in \mathbb{R}^N \setminus \{0\} \). In fact, using (2.12), by (3.21) we have:

\[
u(x) = \hat{u}(T_{H^o}(x)) H (T_{H^o}(x))^{N-2} = \hat{u}(T_{H^o}(x)) H \left( \frac{\nabla H^o(x)}{H^o(x)} \right)^{N-2} = \hat{u}(T_{H^o}(x)) \frac{1}{H^o(x)^{N-2}} H(\nabla H^o(x))^{N-2} = \frac{\hat{u}(T_{H^o}(x))}{H^o(x)^{N-2}}.
\]

Now we are ready to prove the first result of our paper.

**Proof of Theorem 1.2.** We use \( \psi \in C^1_c(\mathbb{R}^N \setminus \{0\}) \) as test function in (2.7). Then \( \varphi(x) := \psi(T_{H^o}(x)) \) can also be used as test function in (2.7), obtaining:

\[
\int_{\mathbb{R}^N} H(\nabla u(x)) \nabla H(\nabla u(x)) (D T_{H^o}(x) \nabla \psi(T_{H^o}(x))) \, dx = \int_{\mathbb{R}^N} f(x) \psi(T_{H^o}(x)) \, dx.
\]

Using the change of variable \( y = T_{H^o}(x) \), (3.24) becomes

\[
\int_{\mathbb{R}^N} H(\nabla u(T_H(y))) (D T_{H^o}(T_H(y)) \nabla H(\nabla u(T_H(y)))) \nabla \psi(y) J(y) \, dy = \int_{\mathbb{R}^N} f(T_H(y)) \psi(y) J(y) \, dy.
\]

Under our assumption \( (h_M) \) we deduce that

\[
H(\nabla u(T_H(y))) D T_{H^o}(T_H(y)) \nabla H(\nabla u(T_H(y))) = H(y)^4 H^o(D T_H(y) \nabla u(T_H(y))) \nabla H^o(D T_H(y) \nabla u(T_H(y))).
\]

If we set \( u^*(y) := u(T_H(y)) \) and \( f^*(y) := f(T_H(y)) \), by using (3.26) and Lemma 3.3, (3.25) becomes:

\[
\int_{\mathbb{R}^N} \frac{1}{H(y)^{2N-4}} H^o(\nabla u^*(y)) (\nabla H^o(\nabla u^*(y))) \nabla \psi(y) \, dy = \int_{\mathbb{R}^N} \frac{1}{H(y)^{2N}} f^*(y) \psi(y) \, dy.
\]
Let $\phi \in C^1_c(\mathbb{R}^N \setminus \{0\})$ be such that $\psi(y) = H(y)^{N-2} \phi(y)$. Then $\psi$ can be used as test function in (3.27), getting:

$$
\int_{\mathbb{R}^N} \frac{1}{H(y)^{N-2}} H^\circ (\nabla u^*(y)) \langle \nabla H^\circ (\nabla u^*(y)), \nabla \phi(y) \rangle \, dx
$$

(3.28)

$$
- \int_{\mathbb{R}^N} H^\circ (\nabla u^*(y)) \langle \nabla H^\circ (\nabla u^*(y)), \nabla \left( \frac{1}{H(y)^{N-2}} \right) \phi(y) \rangle \, dx
$$

$$
= \int_{\mathbb{R}^N} \frac{1}{H(y)^{N+2}} f^\ast(y) \phi(y) \, dy
$$

Recalling (2.11), the fact that $H$ is a 1-homogeneous function and that

$$
\nabla \tilde{u} = \nabla \left( \frac{1}{H^{-1}} \right) u^* + \frac{1}{H^{-1}} \nabla u^*,
$$

we get

$$
\int_{\mathbb{R}^N} H^\circ \left( \nabla \tilde{u}(y) - \nabla \left( \frac{1}{H(y)^{N-2}} \right) u^* \right) \langle \nabla H^\circ \left( \nabla \tilde{u}(y) - \nabla \left( \frac{1}{H(y)^{N-2}} \right) u^* \right), \nabla \phi(y) \rangle \, dy
$$

(3.29)

$$
- \int_{\mathbb{R}^N} H^\circ (\nabla u^*(y)) \langle \nabla H^\circ (\nabla u^*(y)), \nabla \left( \frac{1}{H(y)^{N-2}} \right) \phi(y) \rangle \, dy
$$

$$
= \int_{\mathbb{R}^N} \frac{1}{H(y)^{N+2}} f^\ast(y) \phi(y) \, dy
$$

By $(h_M)$, we deduce that $H^\circ \nabla H^\circ$ is linear and symmetric with respect to the Euclidean inner product of $\mathbb{R}^N$, i.e.

$$
\langle H^\circ(x) \nabla H^\circ(x), y \rangle = \langle x, H^\circ(y) \nabla H^\circ(y) \rangle \quad \forall x, y \in \mathbb{R}^N.
$$

Hence, we have

$$
\int_{\mathbb{R}^N} H^\circ (\nabla \tilde{u}(y)) \langle \nabla H^\circ (\nabla \tilde{u}(y)), \nabla \phi(y) \rangle \, dy
$$

(3.30)

$$
- \int_{\mathbb{R}^N} H^\circ \left( \nabla \left( \frac{1}{H(y)^{N-2}} \right) \right) \langle \nabla H^\circ \left( \nabla \left( \frac{1}{H(y)^{N-2}} \right) \right), \nabla (u^*(y) \phi(y)) \rangle \, dy
$$

$$
= \int_{\mathbb{R}^N} \frac{1}{H(y)^{N+2}} f^\ast(y) \phi(y) \, dy.
$$

Integrating by parts (3.30), we get

$$
\int_{\mathbb{R}^N} H^\circ (\nabla \tilde{u}(y)) \langle \nabla H^\circ (\nabla \tilde{u}(y)), \nabla \phi(y) \rangle \, dy
$$

(3.31)

$$
+ \int_{\mathbb{R}^N} \text{div} \left( H^\circ \left( \nabla \left( \frac{1}{H(y)^{N-2}} \right) \right) \nabla H^\circ \left( \nabla \left( \frac{1}{H(y)^{N-2}} \right) \right) \right) u^*(y) \phi(y) \, dy
$$

$$
= \int_{\mathbb{R}^N} \frac{1}{H(y)^{N+2}} f^\ast(y) \phi(y) \, dy.
$$

Since $w(x) := H(x)^{2-N}$ is an $H^\circ$-harmonic function, namely $\Delta H^\circ w = 0$, we get the thesis.

\[\square\]

### 4. The anisotropic critical equations

The aim of this section is to prove the symmetry result stated in Theorem 1.3. The proof is based on the joint use of an adaptation of the well-known moving plane method in the anisotropic framework and the decay information of the solution at infinity provided by the application of
the Kelvin transform defined in the previous section. In view of Theorem 1.2, if \( u \) is a solution of (1.1), then \( \hat{u} \) solves

\[
-\Delta^{H^0} \hat{u} = \hat{u}^{2^*-1} \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\]

In fact, we have

\[
-\Delta^{H^0}(\hat{u}(y)) = \frac{u(T(y))^{N+2}}{H(y)^{N+2}} = \frac{(\hat{u}(y)H(y)^{N-2})^{N+2}}{H(y)^{N+2}} = \hat{u}(y)^{2^*-1}.
\]

Let us fix some notations typical in the context of the moving plane method. For \( \nu \in \mathbb{S}^{N-1} \) and \( \lambda \in \mathbb{R} \) we set

\[
\Sigma^{\nu}_\lambda = \{ x \in \mathbb{R}^n : \langle x, \nu \rangle < \lambda \} \quad \text{and} \quad \Pi^{\nu}_\lambda = \{ x \in \mathbb{R}^n : \langle x, \nu \rangle = \lambda \}.
\]

Moreover, we define the reflection of any point \( x \in \mathbb{R}^N \) through the hyperplane \( \Pi^{\nu}_\lambda \) as

\[
R^{\nu}_\lambda(x) = x + 2(\lambda - \langle x, \nu \rangle)\nu = 2\lambda \nu + (I - 2\nu \otimes \nu)x,
\]

so that

\[
DR^{\nu}_\lambda(x) = I - 2\nu \otimes \nu.
\]

For simplicity of exposition, in the sequel we set

\[
\mathcal{R}^\nu = I - 2\nu \otimes \nu.
\]

Furthermore, we give the following remarks.

**Remark 4.1.** We observe that the \((\mathcal{R}^\nu)^T = (\mathcal{R}^\nu)^{-1} = \mathcal{R}^\nu\). Moreover, it is easy to check that \( |\det(\mathcal{R}^\nu)| = 1 \).

**Remark 4.2.** Let \( H \) be a generic Finsler norm. If \( H \) is symmetric with respect to \( \nu \in \mathbb{S}^{n-1} \), then also \( H^0 \) satisfies the same property. In fact, we have \( H(\mathcal{R}^\nu x) = H(x) \) and by definition of \( H^0 \) it follows that

\[
H^0(\mathcal{R}^\nu x) = \sup_{y \neq 0} \langle \mathcal{R}^\nu x, \frac{y}{H(y)} \rangle = \sup_{y \neq 0} \langle x, \mathcal{R}^\nu \frac{y}{H(y)} \rangle = \sup_{y \neq 0} \langle x, \frac{\mathcal{R}^\nu y}{H(\mathcal{R}^\nu y)} \rangle = H^0(x).
\]

In the following proposition we observe that a norm satisfying \((h_M)\) is symmetric with respect to the hyperplane orthogonal to each eigenvector of \( M \).

**Proposition 4.3.** Let us assume that \( H \) is a Finsler norm such that \((h_M)\) holds. Then \( H(\mathcal{R}^\nu x) = H(x) \) for all \( x \in \mathbb{R}^N \) if and only if \( \nu \) is an eigenvector of \( M \).

**Proof.** Let \( \nu \in \mathbb{S}^{N-1} \) be such that \( M\nu = \mu \nu \), with \( \mu > 0 \) (we recall that \( M \) is definite positive).

We have:

\[
H(\mathcal{R}^\nu x) = \sqrt{\langle M(I - 2\nu \otimes \nu)x, (I - 2\nu \otimes \nu)x \rangle}
\]

\[
= \sqrt{\langle Mx - 2\mu(\nu, x)\nu, x - 2(\nu, x)\nu \rangle}
\]

\[
= \sqrt{\langle Mx, x \rangle - 2(\nu, x)\langle Mx, \nu \rangle + 2\mu(\nu, x)^2}.
\]

Decomposing \( x = v_1 + v_2 \) with \( v_1 \) parallel to \( \nu \) and \( v_2 \) orthogonal to \( \nu \), it is immediate to see that the quantity \(-2(\nu, x)\langle Mx, \nu \rangle + 2\mu(\nu, x)^2\) vanishes and hence we get \( H(\mathcal{R}^\nu x) = H(x) \). The counterpart follows by simple algebraic computations evaluating (4.35) on the eigenvectors of \( M \).

Now, we are ready to define the reflected function of \( u \)

\[
u^{\nu}_\lambda(x) := u(R^{\nu}_\lambda(x)),
\]

whose gradient is

\[
\nabla \nu^{\nu}_\lambda(x) = (\mathcal{R}^\nu)^T \nabla u(R^{\nu}_\lambda(x)) = (I - 2\nu \otimes \nu)^T \nabla u(R^{\nu}_\lambda(x)) = (I - 2\nu \otimes \nu) \nabla u(R^{\nu}_\lambda(x)),
\]
and hence
\[ \nabla u(R_\lambda^+(x)) = (R_\nu)^{-1}\nabla u_\lambda^+(x) = R_\nu \nabla u_\lambda^+(x). \]

We remark that, if \( u \) satisfies the anisotropic critical Sobolev equation associated to \( H \) (or \( H^\circ \)), also \( u_\lambda^+ \) satisfies the same equation. In fact we have:
\[ \int_\Omega H(\nabla u(y))(\nabla H(\nabla u(y)), \nabla \psi(y)) \, dy = \int_\Omega u^{2^*-1}(y)\psi(y) \, dy \]
and, taking into account the change of variable \( y = R_\lambda^+(x) \), by Remark 4.1 and using (4.36), we get:
\[ \int_\Omega H(\nabla u_\lambda^+(x)) \langle R_\nu \nabla H(\nabla u_\lambda^+(x)), R_\nu \nabla \psi(R_\lambda^+(x)) \rangle \, dx = \int_\Omega u^{2^*-1}(R_\lambda^+(x))\psi(R_\lambda^+(x)) \, dx. \]

Having in mind Proposition 4.3, if \( \nu \) is an eigenvector of \( M \), we have
\[ H(\nabla^\nu \xi) = H(\xi) \quad \forall \xi \in \mathbb{R}^N \]
and by Remark 4.1 we deduce that
\[ \nabla^\nu \nabla H(\nabla^\nu \xi) = \nabla H(\xi) \quad \forall \xi \in \mathbb{R}^N. \]

Now, if we set \( \varphi(x) = \psi(R_\lambda^+(x)) \), taking into account (4.39) and (4.40) in (4.38), we obtain
\[ \int_\Omega H(\nabla u_\lambda^+(x)) \langle \nabla H(\nabla u_\lambda^+(x)), \nabla \varphi(x) \rangle \, dx = \int_\Omega (u_\lambda^+(x))^{2^*-1}\varphi(x) \, dx. \]

Now we are ready to prove our symmetry result.

**Proof of Theorem 1.3.** We first note that the solution \( u \) is smooth in \( \mathbb{R}^N \setminus \Gamma \). Furthermore we observe that it is enough to prove the theorem for the special case when the origin does not belong to \( \Gamma \). Indeed, if the result is true in this special case, then we can apply it to the function \( u_z(x) := u(x + z) \), where \( z \in \Pi_0^+ \setminus \Gamma \neq \emptyset \), which satisfies the equation (1.1) with \( \Gamma \) replaced by \( -z + \Gamma \) (note that \( -z + \Gamma \) is a closed and proper subset of \( \Pi_0^+ \) with \( \text{Cap}_2(-z + \Gamma) = 0 \) and such that the origin does not belong to it).

For our purpose we focus the attention on sets with vanishing 2-capacity, whose definition can be found for example in [17].

Given a solution \( u \) to (1.1), let \( \hat{u} \) be defined as in (3.21) for every \( x \in \mathbb{R}^N \setminus \{\Gamma^* \cup \{0\} \} \), where \( \Gamma^* = T_H(\Gamma) \). We observed above that \( \hat{u} \) satisfies the dual equation of (1.1) in \( \mathbb{R}^N \setminus \{\Gamma^* \cup \{0\} \} \). Moreover, \( \Gamma^* \subset \Pi_0^+ \) since, by assumption, \( \Gamma \subset \Pi_0^+ \). Furthermore, by Lemma 4.1 and 4.2 in [16] we have that \( \Gamma^* \) is bounded (not necessarily closed), since we assumed that \( 0 \notin \Gamma \) and it has vanishing 2-capacity.

In order to show Theorem 1.3, we prove now the following lemma.

**Lemma 4.4.** Under the assumption of Theorem 1.3, if \( w_\lambda(x) = \hat{u}(x) - \hat{u}_\lambda^+(x) \), then for every \( \lambda < 0 \) we have that
\[ \int_{\Sigma^\lambda_\nu} |\nabla w_\lambda^+|^2 \, dx \leq C, \]
where \( C \) is positive constant depending only on \( N \), \( \|\hat{u}\|_{L^\infty(\Sigma^\lambda_\nu)} \), \( \|\hat{u}\|_{L^{2^*}(\Sigma^\lambda_\nu)} \). Consequently it is possible to deduce that \( w_\lambda^+ \in D^{1,2}(\Sigma^\lambda_\nu) \).
Proof. We immediately see that $w_\lambda^+ \in L^2(\Sigma^\nu_\lambda)$, since $0 \leq w_\lambda^+ \leq \hat{u} \in L^2(\Sigma^\nu_\lambda)$. The rest of the proof is inspired to [8] and on a truncation argument related to [16, 31]. Since $\text{Cap}(\Gamma^*) = 0$, for every $\varepsilon > 0$, it is easy to construct a cut-off function $\psi_\varepsilon \in C^{0,1}(\mathbb{R}^N, [0,1])$ such that

$$
\int_{\Sigma^\nu_\lambda} |\nabla \psi_\varepsilon|^2 < \varepsilon
$$

and $\psi_\varepsilon = 0$ in an open neighborhood $B_\varepsilon$ of $R_\lambda(\Gamma^* \cup \{0\})$, with $B_\varepsilon \subset \Sigma_\lambda$ (see [16, 31] for further details).

Fix $R_0 > 0$ such that $R_\lambda(\Gamma^* \cup \{0\}) \subset B_{R_0}^{H^\circ}$ and, for every $R > R_0$, let $\varphi_R \in C_c^\infty(\mathbb{R}^N)$ be a standard cut-off function such that

$$
\begin{cases}
0 \leq \varphi_R \leq 1 & \text{in } \mathbb{R}^N \\
\varphi_R \equiv 0 & \text{in } (B_{R_0}^{H^\circ})^c \\
\varphi_R \equiv 1 & \text{in } B_{R_0}^{H^\circ} \\
|\nabla \varphi_R| \leq \frac{2}{R} & \text{in } B_{2R}^{H^\circ} \setminus B_R^{H^\circ}.
\end{cases}
$$

(4.43)

Finally, we define

$$
\varphi := \begin{cases}
w_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 & \text{in } \Sigma^\nu_\lambda, \\
0 & \text{in } \mathbb{R}^N \setminus \Sigma^\nu_\lambda.
\end{cases}
$$

Now, it is easy to check that $\varphi \in C_c^{0,1}(\mathbb{R}^N)$ with $\text{supp}(\varphi)$ contained in $\Sigma_\lambda \cap \overline{B_{2R_0}^{H^\circ}} \setminus R_\lambda(\Gamma^* \cup \{0\})$ and

$$
\nabla \varphi = \psi_\varepsilon^2 \varphi_R^2 \nabla w_\lambda^+ + 2w_\lambda^+ (\psi_\varepsilon^2 \varphi_R \nabla \varphi_R + \psi_\varepsilon \varphi_R^2 \nabla \psi_\varepsilon).
$$

(4.44)

Therefore, by a standard density argument we can use $\varphi$ as test function in the weak formulation of the dual equation of (1.1) and in the dual equation of (4.41). Hence, subtracting, recalling (2.10) and using the linearity of $H^\circ \nabla H^\circ$ (due to assumption $(h_M)$), we have:

$$
\begin{aligned}
\int_{\Sigma^\nu_\lambda} H^\circ(\nabla w_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx &= -2 \int_{\Sigma^\nu_\lambda} H^\circ(\nabla w_\lambda^+) (\nabla H^\circ(\nabla w_\lambda^+), \nabla \psi_\varepsilon) w_\lambda^+ \psi_\varepsilon \varphi_R^2 \, dx \\
&\quad - 2 \int_{\Sigma^\nu_\lambda} H^\circ(\nabla w_\lambda^+) (\nabla H^\circ(\nabla w_\lambda^+), \nabla \varphi_R) w_\lambda^+ \varphi_R \psi_\varepsilon^2 \, dx \\
&\quad + \int_{\Sigma^\nu_\lambda} (\hat{u}^2 - \hat{u}_\lambda^{2*}) w_\lambda^+ \psi_\varepsilon^2 \varphi_R^2 \, dx \\
&= : \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}
$$

(4.45)

Exploiting the weighted Young’s inequality, recalling that $0 \leq w_\lambda^+ \leq \hat{u}$ and since $\nabla H^\circ$ is bounded in $\mathbb{R}^N$, we get:

$$
|\mathcal{I}_1| \leq \delta \int_{\Sigma^\nu_\lambda} H^\circ(\nabla w_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + C_\delta \int_{\Sigma^\nu_\lambda} |\nabla \psi_\varepsilon|^2 (w_\lambda^+)^2 \varphi_R^2 \, dx
$$

(4.46)

$$
\leq \delta \int_{\Sigma^\nu_\lambda} H^\circ(\nabla w_\lambda^+)^2 \psi_\varepsilon^2 \varphi_R^2 \, dx + C_\delta \| \hat{u} \|^2_{L^\infty(\Sigma^\nu_\lambda)}.
$$
Furthermore we have:

\[
|I_2| \leq \sigma \int_{\Sigma_\lambda^\circ} H^\circ(\nabla w_\lambda^+)^2\psi_\varepsilon^2\varphi^2 \, dx + C_\sigma \int_{\Sigma_\lambda^\circ \cap (B_{2R}^H \setminus B_R^H)} |\nabla \varphi_R|^2 (w_\lambda^+)^2\psi_\varepsilon^2 \, dx \\
\leq \sigma \int_{\Sigma_\lambda^\circ} H^\circ(\nabla w_\lambda^+)^2\psi_\varepsilon^2\varphi^2 \, dx \\
+ C_\sigma \left( \int_{\Sigma_\lambda^\circ \cap (B_{2R}^H \setminus B_R^H)} |\nabla \varphi_R|^N \, dx \right)^{\frac{2}{N}} \left( \int_{\Sigma_\lambda^\circ \cap (B_{2R}^H \setminus B_R^H)} \tilde{u}^{2^*} \, dx \right)^{\frac{N-2}{N}} \\
\leq \sigma \int_{\Sigma_\lambda^\circ} H^\circ(\nabla w_\lambda^+)^2\psi_\varepsilon^2\varphi^2 \, dx + C_{\sigma, N} \left( \int_{\Sigma_\lambda^\circ \cap (B_{2R}^H \setminus B_R^H)} \tilde{u}^{2^*} \, dx \right)^{\frac{N-2}{N}}.
\]

(4.47)

We now estimate \( I_3 \). Since \( \hat{u}(x), \hat{u}_\lambda(x) > 0 \), by the convexity of the function \( t \mapsto t^{2^*-1} \) for \( t > 0 \) and the monotonicity of \( t \mapsto t^{2^*-2} \) for \( t > 0 \), using the definition of \( w_\lambda^+ \), we have that there exists \( C_N > 0 \) such that

\[
(\hat{u}^{2^*-1} - \hat{u}_\lambda^{2^*-1})w_\lambda^+ \leq C_N \hat{u}^{2^*-2}(\hat{u} - \hat{u}_\lambda)w_\lambda^+ \leq C_N \hat{u}^{2^*-2}(w_\lambda^+)^2.
\]

Therefore

\[
I_3 \leq C_N \int_{\Sigma_\lambda^\circ} \hat{u}^{2^*-2}(w_\lambda^+)^2\psi_\varepsilon^2\varphi^2 \, dx \leq C_N \int_{\Sigma_\lambda^\circ} \hat{u}^{2^*} \, dx = C_N \|\hat{u}\|_{L^{2^*}(\Sigma_\lambda^\circ)}^2,
\]

(4.48)

where we also used that \( 0 \leq w_\lambda^+ \leq \hat{u} \). Taking into account the estimates on \( I_1, I_2, I_3 \) and fixing \( \delta, \sigma > 0 \) sufficiently small such that \( 1 - \sigma - \delta > 0 \), by (4.45) we deduce that

\[
\int_{\Sigma_\lambda^\circ} H^\circ(\nabla w_\lambda^+)^2\varphi^2 \varphi^2 \, dx \leq C \|\hat{u}\|_{L^\infty(\Sigma_\lambda^\circ)}^2 + C_N \left( \int_{\Sigma_\lambda^\circ \cap (B_{2R}^H \setminus B_R^H)} \hat{u}^{2^*} \, dx \right)^{\frac{N-2}{N}} + C_N \|\hat{u}\|_{L^{2^*}(\Sigma_\lambda^\circ)}^2.
\]

(4.49)

By Fatou’s Lemma, as \( \varepsilon \) tends to zero and \( R \) tends to infinity, we infer:

\[
\int_{\Sigma_\lambda^\circ} |\nabla w_\lambda^+|^2 \, dx \leq C,
\]

(4.50)

where \( C := C(N, \|\hat{u}\|_{L^\infty(\Sigma_\lambda^\circ)}, \|\hat{u}\|_{L^{2^*}(\Sigma_\lambda^\circ)}) > 0 \). To prove that \( w_\lambda^+ \in D^{1,2}(\Sigma_\lambda^\circ) \) we refer to [17].

We complete the proof of our symmetry result using the moving plane procedure in the whole space \( \mathbb{R}^N \). In the same spirit of the Euclidean framework, we split the proof into three steps.

Step 1: there exists \( \mathcal{M} > 1 \) such that \( \hat{u} \leq \tilde{u}_\lambda^\circ \) in \( \Sigma_\lambda^\circ \setminus R_\lambda(\Gamma^* \cup \{0\}) \), for all \( \lambda < -\mathcal{M} \).

Arguing as in the proof of Lemma 4.4 and using the same notations and the same construction for \( \psi_\varepsilon, \varphi_R \) and \( \varphi \), we get

\[
\int_{\Sigma_\lambda^\circ} H^\circ(\nabla w_\lambda^+)^2\psi_\varepsilon^2\varphi^2 \, dx = -2 \int_{\Sigma_\lambda^\circ} H^\circ(\nabla w_\lambda^+)(\nabla^2 H^\circ(\nabla w_\lambda^+), \nabla \psi_\varepsilon)w_\lambda^+\psi_\varepsilon\varphi^2 \, dx \\
-2 \int_{\Sigma_\lambda^\circ} H^\circ(\nabla w_\lambda^+)(\nabla^2 H^\circ(\nabla w_\lambda^+), \nabla \varphi_R)w_\lambda^+\varphi_R\psi_\varepsilon^2 \, dx \\
+ \int_{\Sigma_\lambda^\circ} (\hat{u}^{2^*-1} - \hat{u}_\lambda^{2^*-1})w_\lambda^+\psi_\varepsilon^2\varphi^2 \, dx \\
=: I_1 + I_2 + I_3.
\]

(4.51)
where \( I_1, I_2 \) and \( I_3 \) can be estimated exactly as in (4.46), (4.47) and (4.48). The latter yield (4.52)
\[
\int_{\Sigma_\lambda^\nu} H^\nu(\nabla w_\lambda^+)^2 \varphi^2 dx \leq C_\varepsilon \|\hat{u}\|_{L^\infty(\Sigma_\lambda^\nu)}^2 + C_N \left( \int_{\Sigma_\lambda^\nu \cap (B_{2R}^H \setminus B_R^H)} \hat{u}^{2^*} dx \right)^{\frac{\lambda - 2}{\lambda N}} + C_N \|\hat{u}\|_{L^{2^*}(\Sigma_\lambda^\nu)}^{2^*}.
\]
Taking the limit in the latter as \( \varepsilon \) tends to zero and \( R \) tends to infinity, we get:
\[
\int_{\Sigma_\lambda^\nu} H^\nu(\nabla w_\lambda^+)^2 dx \leq C,
\]
which, using Lemma 4.4, gives
\[
\int_{\Sigma_\lambda^\nu} H^\nu(\nabla w_\lambda^+)^2 dx \leq C_N \left( \int_{\Sigma_\lambda} \hat{u}^{2^*-2}(w_\lambda^+)^2 dx \right)
\leq C_N \left( \int_{\Sigma_\lambda} \hat{u}^{2^*} dx \right)^{\frac{\lambda - 2}{\lambda N}} \left( \int_{\Sigma_\lambda} (w_\lambda^+)^2 dx \right)^{\frac{\lambda}{\lambda N}}
\leq C_NC_S^2 \left( \int_{\Sigma_\lambda} \hat{u}^{2^*} dx \right)^{\frac{\lambda}{\lambda N}} \int_{\Sigma_\lambda^\nu} H^\nu(\nabla w_\lambda^+)^2 dx.
\]
Recalling that \( \hat{u} \in L^{2^*}(\Sigma_\lambda^\nu) \), we deduce the existence of \( \mathcal{M} > 1 \) such that
\[
C_NC_S^2 \left( \int_{\Sigma_\lambda} \hat{u}^{2^*} dx \right)^{\frac{\lambda}{\lambda N}} < 1
\]
for every \( \lambda < -\mathcal{M} \), where \( C_S \) is the best Sobolev constant in the embedding. The latter and (4.54) lead to
\[
\int_{\Sigma_\lambda^\nu} H^\nu(\nabla w_\lambda^+)^2 dx = 0.
\]
This implies that \( w_\lambda^+ = 0 \) and the claim is proved.

Then we define
\[
\Lambda_0 = \{ \lambda < 0 : \hat{u} \leq \hat{u}_t^\nu \text{ in } \Sigma_t^\nu \setminus R_t^\nu(\Gamma^* \cup \{0\}) \text{ for all } t \in (-\infty, \lambda]\}
\]
and
\[
\lambda_0 = \sup \Lambda_0.
\]

Step 2: \( \lambda_0 = 0 \). We argue by contradiction assuming \( \lambda_0 < 0 \). By continuity we know that \( \hat{u} \leq \hat{u}_{\lambda_0}^\nu \) in \( \Sigma_{\lambda_0}^\nu \setminus R_{\lambda_0}^\nu(\Gamma^* \cup \{0\}) \). By the strong maximum principle [26, 33] (here we have a linear elliptic operator) we deduce that \( \hat{u} < \hat{u}_{\lambda_0}^\nu \) in \( \Sigma_{\lambda_0}^\nu \setminus R_{\lambda_0}^\nu(\Gamma^* \cup \{0\}) \). Indeed, \( \hat{u} = \hat{u}_{\lambda_0}^\nu \) in \( \Sigma_{\lambda_0}^\nu \setminus R_{\lambda_0}^\nu(\Gamma^* \cup \{0\}) \) is not possible if \( \lambda_0 < 0 \), since in this case \( \hat{u} \) would be singular on the set \( R_{\lambda_0}^\nu(\Gamma^* \cup \{0\}) \).

One can easily check that for every \( \delta > 0 \) there are \( \bar{\tau}(\delta, \lambda_0) > 0 \) and a compact set \( \mathcal{K} \) (depending on \( \delta \) and \( \lambda_0 \)) such that
\[
\mathcal{K} \subset \Sigma_{\lambda}^\nu \setminus R_{\lambda}^\nu(\Gamma^* \cup \{0\}), \quad \int_{\Sigma_{\lambda}^\nu \setminus \mathcal{K}} \hat{u}^{2^*} < \delta, \quad \forall \lambda \in [\lambda_0, \lambda_0 + \bar{\tau}].
\]

Moreover, by uniform continuity of the functions \( \hat{u}, \hat{u}_{\lambda_0+\bar{\tau}}^\nu \) we also deduce that \( \hat{u} \leq \hat{u}_{\lambda_0+\bar{\tau}}^\nu \) in \( \mathcal{K} \).

Now we repeat verbatim the arguments used in the proof of Lemma 4.4 but using the test function
\[
\varphi := \begin{cases}
\frac{w_{\lambda_0+\tau}^+}{\lambda_0+\tau} \psi^2 \varphi_R & \text{in } \Sigma_{\lambda_0+\tau}^\nu \setminus \Sigma_{\lambda_0+\tau}^\nu \\
0 & \text{in } \mathbb{R}^N \setminus \Sigma_{\lambda_0+\tau}^\nu.
\end{cases}
\]
Thus we recover the first inequality in (4.54), which, for any $0 \leq \tau < \bar{\tau}$, immediately gives:

$$
\int_{\Sigma_{\lambda_0+\tau}} H^\circ (\nabla w_{\lambda_0+\tau}^+) \, dx \leq C_N \int_{\Sigma_{\lambda_0+\tau}} \tilde{u}^{2^*-2} (w_{\lambda_0+\tau}^+)^2 \, dx
$$

(4.55)

$$
\leq C_N \left( \int_{\Sigma_{\lambda_0+\tau}} \tilde{u}^{2^*} \, dx \right)^{\frac{2}{N}} \left( \int_{\Sigma_{\lambda_0+\tau}} (w_{\lambda_0+\tau}^+)^{2^*} \, dx \right)^{\frac{2^*}{N}}
$$

$$
\leq C_N C_S^2 \left( \int_{\Sigma_{\lambda_0+\tau}} \tilde{u}^{2^*} \, dx \right)^{\frac{2}{N}} \left( \int_{\Sigma_{\lambda_0+\tau}} H^\circ (\nabla w_{\lambda_0+\tau}^+) \, dx \right).
$$

Fix $\delta < [2C_N C_S^2]^{-\frac{N}{2}}$ and we observe that:

$$
C_N C_S^2 \left( \int_{\Sigma_{\lambda_0+\tau}} \tilde{u}^{2^*} \, dx \right)^{\frac{2}{N}} < \frac{1}{2}, \quad \forall \, 0 \leq \tau < \bar{\tau}
$$

which, plugged into (4.55), implies that

$$
\int_{\Sigma_{\lambda_0+\tau}} H^\circ (\nabla w_{\lambda_0+\tau}^+) \, dx = 0,
$$

for every $0 \leq \tau < \bar{\tau}$. Hence

$$
\int_{\Sigma_{\lambda_0+\tau}} H^\circ (\nabla w_{\lambda_0+\tau}^+) \, dx = 0 \text{ for every } 0 \leq \tau < \bar{\tau}, \text{ since } \nabla w_{\lambda_0+\tau}^+ \text{ is zero on an open neighborhood of } K. \text{ Hence by Lemma 4.4 we infer that } w_{\lambda_0+\tau}^+ = 0 \text{ on } \Sigma_{\lambda_0+\tau}^\nu \text{ for every } 0 \leq \tau < \bar{\tau} \text{ and thus } \tilde{u} \leq \tilde{u}_{\lambda_0+\tau}^\nu \text{ in } \Sigma_{\lambda_0+\tau}^\nu \setminus R_{\lambda_0+\tau}^\nu (\Gamma^* \cup \{0\}) \text{ for every } 0 \leq \tau < \bar{\tau}, \text{ which gives a contradiction with the fact that } \lambda_0 < 0. \text{ This fact immediately prove that } \lambda_0 = 0.

Step 3: conclusion. Performing the moving plane method in the opposite direction, we get that $\tilde{u}$ is symmetric w.r.t. the hyperplane $\Pi_0^\nu$, namely $\tilde{u}(R^\nu y) = \tilde{u}(y)$. This implies that also $u$ has the same property. In fact, recalling that

$$
u(x) = \frac{1}{H^\circ(x)^{N-2}} \tilde{u} \left( \nabla H^\circ(x) \right),$$

we have:

$$
u(R^\nu x) = \frac{1}{H^\circ(R^\nu x)^{N-2}} \tilde{u} \left( \frac{\nabla H^\circ(R^\nu x)}{H^\circ(R^\nu x)} \right)$$

$$= \frac{1}{H^\circ(x)^{N-2}} \tilde{u} \left( \frac{\nabla \nabla H^\circ(x)}{H^\circ(x)} \right)$$

$$= \left( \frac{\nabla H^\circ(x)}{H^\circ(x)} \right) = u(x),$$

where we used that the symmetry of $H^\circ$ implies $\nabla H^\circ(R^\nu x) = R^\nu \nabla H^\circ(x)$.

Proof of Theorem 1.4. The proof follows arguing as done in Theorem 1.3, and it is actually simpler. It is only important to remark that $\tilde{u}$ may exhibit a non-removable singularity at 0 or a removable singularity at 0. The latter case implies that, actually, the solution $u$ to (1.1) already had suitable behaviour at infinity, in order to apply the standard technique. The proof, therefore, follows as in [8] (and exploiting our technique with $\Gamma = \emptyset$), just observing that in this case the symmetry hyperplane is not a priori fixed.
References

[1] A.D. Alexandrov. A characteristic property of the spheres. *Ann. Mat. Pura Appl.*, 58, 1962, pp. 303–354.
[2] P.L. Antonelli, R.S. Ingarden and M. Matsumoto. The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, *Fundamental Theories of Physics* 58, 1993, Kluwer Academic Publishers Group, Dordrecht.
[3] C.A. Antonini, G. Ciraolo and A. Farina. Interior regularity results for inhomogeneous anisotropic quasilinear equations. *arxiv.org/abs/2112.09087*.
[4] G.S. Asanov. Finsler Geometry, Relativity and Gauge Theories, *Fundamental Theories of Physics*, 12, 1985, D. Reidel Publishing Co., Dordrecht.
[5] B. A. Caffarelli, B. Gidas and J. Spruck. Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42(3), 1989, pp. 271–297.
[6] J.W. Cahn and D.W. Hoffmann. A vector thermodynamics for anisotropic surfaces. I. Fundamentals and applications to plane surface junctions. *Surf. Sci.*, 1972, pp. 31–368.
[7] A. Cianchi and P. Salani. Overdetermined anisotropic elliptic problems. *Math. Ann.*, 345(4), 2009, pp. 859–881.
[8] G. Ciraolo, A. Figalli and A. Roncoroni. Symmetry results for critical anisotropic p-Laplacian equations in convex cones. *Geom. Funct. Anal.*, 30(3), 2020, pp. 770–803.
[9] M. Cozzi, A. Farina and E. Valdinoci. Gradient bounds and rigidity results for singular, degenerate, anisotropic partial differential equations. *Comm. Math. Phys.*, 331, 2014, pp. 189–214.
[10] M. Cozzi, A. Farina and E. Valdinoci. Monotonicity formulae and classification results for singular, degenerate, anisotropic PDEs. *Adv. Math.*, 293, 2016, pp. 343–381.
[11] E. Di Benedetto. $C^{1+\alpha}$-local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.*, 7(8), 1983, pp. 827–850.
[12] S. Esedoglu and S.J. Osher. Decomposition of images by the anisotropic Rudin-Osher-Fatemi model. *Commun. Pure Appl. Math.*, 57, 2004, pp. 1609–1626.
[13] F. Esposito, A. Farina and B. Sciunzi. Qualitative properties of singular solutions to semilinear elliptic problems. *J. of Differential Equations*, 265(5), 2020, pp. 1962–1983.
[14] L.C. Evans and R.F. Gariepy. Measure theory and fine properties of functions. *Studies in Advanced Mathematics*. CRC Press, Boca Raton, FL, 1992.
[15] V. Ferone and B. Kawohl. Remarks on a Finsler-Laplacian. *Proc. Amer. Math. Soc.*, 137, 2009, pp. 247–253.
[16] B. Gidas, W. M. Ni and L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$. Math. Anal. and Applications Part A, *Advances in Math. Suppl. Studies 7A*, Academic Press, 1981, pp. 369–402.
[17] M.E. Gurtin. Thermomechanics of evolving phase boundaries in the plane. *Claredon Press*, Oxford, 1993.
[18] M.A. Javaloyes and M. Sanchez. Remarks on a Finsler-Laplacian. *Anal. Mach. Intell.*, 12(7), 1990, pp. 629-639.
[19] J.M. Lee and T.H. Parker. The Yamabe problem. *Bull. Am. Math. Soc.*, 17, 1987, pp. 37–91.
[20] G.M. Lieberman. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.*, 12(11), 1988, pp. 1203–1219.
[21] R. Monti and D. Morbidelli. Kelvin transform for Grushin operators and critical semilinear equations. *Duke Math. J.*, 131(1), 2006, pp. 167–202.
[22] P. Perona and J. Malik. Scale-space and edge-detection using anisotropic diffusion. *IEEE Trans. Pattern Anal. Mach. Intell.*, 12(7), 1990, pp. 629-639.
[23] Z. Shen. A symmetry problem in potential theory. *Arch. Rational Mech. Anal.*, 43, 1971, pp. 304–318.
[24] Z. Shen. Finsler spaces of constant positive curvature. In: Finsler Geometry. *Contemporary Mathematics*, 196, 1996, pp. 83–92.
[25] B. Sciunzi. Classification of positive $D^\alpha p(\mathbb{R}^N)$-solutions to the critical $p$-Laplace equation in $\mathbb{R}^N$. *Adv. Math.*, 291, 2016, pp. 12–23.
[26] B. Sciunzi. On the moving plane method for singular solutions to semilinear elliptic equations. *J. Math. Pures Appl.*, 108(9), 2017, pp. 111–123.
[32] P. Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *J. Differ. Equ.*, 51(1), 1984, pp. 126–150.

[33] J.L. Vazquez. A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.*, 12(3), 1984, pp. 191–202.

[34] J. Vétois. A priori estimates and application to the symmetry of solutions for critical $p$-Laplace equations. *J. Differ. Equ.*, 260(1), 2016, pp. 149–161.

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