CERTAIN PROPERTIES OF A NEW SUBCLASS OF ANALYTIC AND $p$-VALENTLY CLOSE-TO-CONVEX FUNCTIONS

SERAP BULUT

Abstract. In the present paper we introduce and investigate an interesting subclass $K_s^{(k)}(\gamma, p)$ of analytic and $p$-valently close-to-convex functions in the open unit disk $U$. For functions belonging to this class, we derive several properties as the inclusion relationships and distortion theorems. The various results presented here would generalize many known recent results.

1. Introduction

Let $A_p$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}z^{n+p} \quad (p \in \mathbb{N} := \{1, 2, \ldots\}) \quad (1.1)$$

which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$ 

In particular, we write $A_1 = A$.

For two functions $f$ and $\Theta$, analytic in $U$, we say that the function $f$ is subordinate to $\Theta$ in $U$, and write

$$f(z) \prec \Theta(z) \quad (z \in U),$$

if there exists a Schwarz function $\omega$, which is analytic in $U$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U)$$

such that

$$f(z) = \Theta(\omega(z)) \quad (z \in U).$$

Indeed, it is known that

$$f(z) \prec \Theta(z) \quad (z \in U) \Rightarrow f(0) = \Theta(0) \quad \text{and} \quad f(U) \subset \Theta(U).$$

Furthermore, if the function $\Theta$ is univalent in $U$, then we have the following equivalence

$$f(z) \prec \Theta(z) \quad (z \in U) \Leftrightarrow f(0) = \Theta(0) \quad \text{and} \quad f(U) \subset \Theta(U).$$

2010 Mathematics Subject Classification. Primary 30C45; Secondary 30C80.

Key words and phrases. Analytic functions, $p$-valently close-to-convex functions, $p$-valently starlike functions, inclusion relationships, distortion and growth theorems, subordination principle.
A function \( f \in A_p \) is said to be \( p \)-valently starlike of order \( \gamma \) \((0 \leq \gamma < p)\) in \( \mathbb{U} \) if it satisfies the inequality
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \gamma \quad (z \in \mathbb{U})
\]
or equivalently
\[
\frac{zf'(z)}{f(z)} < \frac{p + (p - 2\gamma)z}{1 - z} \quad (z \in \mathbb{U}).
\]
The class of all \( p \)-valent starlike functions of order \( \gamma \) in \( \mathbb{U} \) is denoted by \( S^*_p(\gamma) \). Also, we denote that \( S^*_p(0) = S^*_p \), \( S^*_1(\gamma) = S^*_1 \) and \( S^*_1(0) = S^* \).

A function \( f \in A_p \) is said to be \( p \)-valently close-to-convex of order \( \gamma \) \((0 \leq \gamma < p)\) in \( \mathbb{U} \) if \( g \in S^*_p(\gamma) \) and satisfies the inequality
\[
\Re \left( \frac{zf'(z)}{g(z)} \right) > \gamma \quad (z \in \mathbb{U})
\]
or equivalently
\[
\frac{zf'(z)}{g(z)} < \frac{p + (p - 2\gamma)z}{1 - z} \quad (z \in \mathbb{U}).
\]
The class of all \( p \)-valent close-to-convex functions of order \( \gamma \) in \( \mathbb{U} \) is denoted by \( K_p(\gamma) \). Also, we denote that \( K_p(0) = K_p \), \( K_1(\gamma) = K(\gamma) \) and \( K_1(0) = K \).

We now introduce the following subclass of analytic functions:

**Definition 1.** Let the function \( f \) be analytic in \( \mathbb{U} \) and defined by (1.1). We say that \( f \in K^{(k)}(\gamma, p) \), if there exists a function \( g \in S^*_p \left( \frac{(k-1)p}{k} \right) \) \((k \in \mathbb{N} \) is a fixed integer) such that
\[
\Re \left( \frac{z^{(k-1)p+1}f'(z)}{g_k(z)} \right) > \gamma \quad (z \in \mathbb{U}; \ 0 \leq \gamma < p),
\]
where \( g_k \) is defined by the equality
\[
g_k(z) = \prod_{\nu=0}^{k-1} \varepsilon^{-\nu p}g(\varepsilon^\nu z), \quad \varepsilon = e^{2\pi i/k}.
\]

By simple calculations we see that the inequality (1.2) is equivalent to
\[
\left| \frac{z^{(k-1)p+1}f'(z)}{g_k(z)} - p \right| < \frac{z^{(k-1)p+1}f'(z)}{g_k(z)} + p - 2\gamma.
\]

**Remark 1.** (i) For \( p = 1 \), we get the class \( K^{(k)}_s(\gamma, 1) = K^{(k)}_s(\gamma) \) \((0 \leq \gamma < 1)\) recently studied by Şeker [8].

(ii) For \( p = 1 \) and \( k = 2 \), we have the class \( K^{(2)}_s(\gamma, 1) = K_s(\gamma) \) \((0 \leq \gamma < 1)\) introduced and studied by Kowalczyk and Leś-Bomba [5].
NEW SUBCLASS OF $p$-VALENTLY CLOSE-TO-CONVEX FUNCTIONS

(iii) For $p = 1$, $k = 2$ and $\gamma = 0$, we have the class $K^{(2)}_s(0, 1) = K_s$ introduced and studied by Gao and Zhou [2].

In this work, by using the principle of subordination, we obtain inclusion theorem and distortion theorems for functions in the function class $K^{(k)}_s(\gamma, p)$. Our results unify and extend the corresponding results obtained by Şeker [8], Kowalczyk and Leś-Bomba [5], and Gao and Zhou [2].

2. Preliminary Lemmas

We assume throughout this paper that $P$ denote the class of functions $p$ of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in U)$$

which are analytic in $U$ and $k \in \mathbb{N}$ is a fixed integer.

In order to prove our main results for the functions class $K^{(k)}_s(\gamma, p)$, we need the following lemmas.

Lemma 1. If

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \in S_p^* \left( \frac{1}{k} \right),$$

then

$$G_k(z) = \frac{g_k(z)}{z^{(k-1)p}} = z^p + \sum_{n=1}^{\infty} B_{n+p} z^{n+p} \in S_p^*,$$

where $g_k$ is given by (1.3).

Proof. Suppose that

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \in S_p^* \left( \frac{1}{k} \right).$$

By (1.3), we have

$$G_k(z) = \frac{g_k(z)}{z^{(k-1)p}} = \prod_{\nu=0}^{k-1} \epsilon^{-\nu p} g(\epsilon^\nu z).$$

Differentiating (2.4) logarithmically, we obtain

$$\frac{zG'_k(z)}{G_k(z)} = \sum_{\nu=0}^{k-1} \frac{\epsilon^\nu z g'(\epsilon^\nu z)}{g(\epsilon^\nu z)} - (k - 1) p.$$  (2.5)

From (2.5) together with (2.3), we get

$$\Re \left( \frac{zG'_k(z)}{G_k(z)} \right) = \sum_{\nu=0}^{k-1} \Re \left( \frac{\epsilon^\nu z g'(\epsilon^\nu z)}{g(\epsilon^\nu z)} \right) - (k - 1) p > 0,$$

which completes the proof of our theorem. \qed
Lemma 2. Let the function
\[ H(z) = p + h_1 z + h_2 z^2 + \cdots \quad (z \in \mathbb{U}) \]
be analytic in the unit disk \( \mathbb{U} \). Then, the function \( H \) satisfies the condition
\[ \left| \frac{H(z) - p}{(p - 2\gamma) + H(z)} \right| < \beta \quad (z \in \mathbb{U}) \]
for some \( \beta \) (\( 0 < \beta \leq 1 \)), if and only if there exists an analytic function \( \varphi \) in the unit disk \( \mathbb{U} \), such that \( |\varphi(z)| \leq \beta \) (\( z \in \mathbb{U} \)), and
\[ H(z) = \frac{p - (p - 2\gamma) z \varphi(z)}{1 + z \varphi(z)} \quad (z \in \mathbb{U}). \]

Proof. We will employ the technique similar with those of Padamanabhan [7]. Assume that the function
\[ H(z) = p + h_1 z + h_2 z^2 + \cdots \quad (z \in \mathbb{U}) \]
satisfies the condition
\[ \left| \frac{H(z) - p}{(p - 2\gamma) + H(z)} \right| < \beta \quad (z \in \mathbb{U}). \]

Setting
\[ h(z) = \frac{p - H(z)}{(p - 2\gamma) + H(z)}, \]
we see that the function \( h \) analytic in \( \mathbb{U} \), satisfies the inequality \( |h(z)| < \beta \) for \( z \in \mathbb{U} \) and \( h(0) = 0 \). Now, by using the Schwarz’s lemma, we get that the function \( h \) has the form \( h(z) = z \varphi(z) \), where \( \varphi \) is analytic in \( \mathbb{U} \) and satisfies \( |\varphi(z)| \leq \beta \) for \( z \in \mathbb{U} \). Thus, we obtain
\[ H(z) = \frac{p - (p - 2\gamma) h(z)}{1 + h(z)} = \frac{p - (p - 2\gamma) z \varphi(z)}{1 + z \varphi(z)}. \]

Conversely, if
\[ H(z) = \frac{p - (p - 2\gamma) z \varphi(z)}{1 + z \varphi(z)} \]
and \( |\varphi(z)| \leq \beta \) for \( z \in \mathbb{U} \), then \( H \) is analytic in the unit disk \( \mathbb{U} \). So we get
\[ \left| \frac{H(z) - p}{(p - 2\gamma) + H(z)} \right| = |z \varphi(z)| \leq \beta |z| < \beta \quad (z \in \mathbb{U}) \]
which completes the proof of our lemma. \( \square \)

Lemma 3. [4] A function \( p \in \mathcal{P} \) satisfies the following condition:
\[ \Re(p(z)) > 0 \quad (z \in \mathbb{U}) \]
if and only if
\[ p(z) \neq \frac{\zeta - 1}{\zeta + 1} \quad (z \in \mathbb{U}; \, \zeta \in \mathbb{C}; \, |\zeta| = 1). \]
Lemma 4. A function \( f \in A_p \) given by \( (1.1) \) is in the class \( K_s^{(k)}(\gamma, p) \) if and only if
\[
1 + \sum_{n=1}^{\infty} A_{n+p} z^n \neq 0 \quad (z \in \mathbb{U}),
\]
where
\[
A_{n+p} = \frac{(\zeta + 1)(n + p) a_{n+p} + (p - 2\gamma - p\zeta) B_{n+p}}{2(p - \gamma)} \quad (\zeta \in \mathbb{C}; \; |\zeta| = 1).
\]

Proof. Upon setting
\[
p(z) = \frac{z^{(k-1)p+1} f'(z)}{g_k(z)} - \gamma \quad (f \in K_s^{(k)}(\gamma, p)),
\]
we find that
\[
p(z) \in \mathcal{P} \quad \text{and} \quad \Re(p(z)) > 0 \quad (z \in \mathbb{U}).
\]
Using Lemma 3, we have
\[
\left. \frac{z^{(k-1)p+1} f'(z)}{g_k(z)} - \gamma \right|_{z=0} \neq \frac{\zeta - 1}{\zeta + 1} \quad (z \in \mathbb{U}; \; \zeta \in \mathbb{C}; \; |\zeta| = 1). \tag{2.6}
\]
For \( z = 0 \), the above relation holds, since
\[
\left. \frac{z^{(k-1)p+1} f'(z)}{g_k(z)} - \gamma \right|_{z=0} = 1 \neq \frac{\zeta - 1}{\zeta + 1} \quad (\zeta \in \mathbb{C}; \; |\zeta| = 1).
\]
For \( z \neq 0 \), the relation (2.6) is equivalent to
\[
(\zeta + 1) z^{(k-1)p+1} f'(z) + (p - 2\gamma - p\zeta) g_k(z) \neq 0 \quad (f \in K_s^{(k)}(\gamma, p); \; \zeta \in \mathbb{C}; \; |\zeta| = 1).
\]
Thus from (1.1) and (2.2) we find that
\[
(\zeta + 1) \left( p z^p + \sum_{n=1}^{\infty} (n + p) a_{n+p} z^{n+p} \right) + (p - 2\gamma - p\zeta) \left( z^p + \sum_{n=1}^{\infty} B_{n+p} z^{n+p} \right) \neq 0,
\]
that is,
\[
2 (p - \gamma) z^p + \sum_{n=1}^{\infty} [(\zeta + 1)(n + p) a_{n+p} + (p - 2\gamma - p\zeta) B_{n+p}] z^{n+p} \neq 0.
\]
Now, dividing both sides of above relation by \( 2(p - \gamma) z^p \) \((z \neq 0)\), we obtain
\[
1 + \sum_{n=1}^{\infty} \frac{(\zeta + 1)(n + p) a_{n+p} + (p - 2\gamma - p\zeta) B_{n+p}}{2(p - \gamma)} z^n \neq 0
\]
which completes the proof of Lemma 4. \( \square \)

Lemma 5. \([6]\) Let \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\). Then
\[
1 + \frac{A_1 z}{1 + B_1 z} < 1 + \frac{A_2 z}{1 + B_2 z}.
\]
3. Main Results

We now state and prove the main results of our present investigation.

**Theorem 1.** Let \( f \) be an analytic function in \( U \) given by (1.1). Then \( f \in K^s_k(\gamma, p) \) if and only if there exists a function \( g \in S_p^* \left( \frac{(k-1)p}{k} \right) \) such that

\[
\frac{z^{(k-1)p+1}f'(z)}{g_k(z)} < \frac{p + (p - 2\gamma)z}{1 - z} \quad (z \in U),
\]

where \( g_k \) is given by (1.3).

**Proof.** Let \( f \in K^s_k(\gamma, p) \). Then, there exists a function \( g \in S_p^* \left( \frac{(k-1)p}{k} \right) \) such that

\[
\Re \left( \frac{z^{(k-1)p+1}f'(z)}{g_k(z)} \right) > \gamma \quad (z \in U; \ 0 \leq \gamma < p)
\]

or equivalently

\[
\frac{z^{(k-1)p+1}f'(z)}{g_k(z)} < \frac{p + (p - 2\gamma)z}{1 - z}.
\]

Conversely, we assume that the subordination (3.1) holds. Then, there exists an analytic function \( w \) in \( U \) such that \( w(0) = 0 \), \( |w(z)| < 1 \) and

\[
\frac{z^{(k-1)p+1}f'(z)}{g_k(z)} = \frac{p + (p - 2\gamma)w(z)}{1 - w(z)}.
\]

Hence, by using condition \( |w(z)| < 1 \) we get (1.4), which is equivalent to (1.2), so \( f \in K^s_k(\gamma, p) \).

**Remark 2.** Letting \( p = 1 \) in Theorem [1] we have [8, Theorem 1].

**Theorem 2.** We have

\( K^s_k(\gamma, p) \subset K_p \).

**Proof.** Let \( f \in K^s_k(\gamma, p) \) be an arbitrary function. From Definition [1] we obtain

\[
\Re \left( \frac{zf'(z)}{g_k(z)} \right) > \gamma.
\]

Note that the condition (3.2) can be written as

\[
\Re \left( \frac{zf'(z)}{G_k(z)} \right) > \gamma,
\]

where \( G_k \) is given by (2.2). By Lemma [1] since \( G_k \in S_p^* \), from the above inequality, we deduce that \( f \in K_p \).
Theorem 3. Suppose that \( \gamma \in S_p^* \left( \frac{(k-1)p}{k} \right) \), where \( g_k \) is given by (1.3). If \( f \) is an analytic function in \( U \) of the form (1.1), such that

\[
2 \sum_{n=1}^{\infty} (n + p) |a_{n+p}| + (|p - 2\gamma| + p) \sum_{n=1}^{\infty} |B_{n+p}| < 2(p - \gamma), \tag{3.3}
\]

where the coefficients \( B_{n+p} \) are given by (2.2), then \( f \in K_{s}^{(k)}(\gamma,p) \).

Proof. For \( f \) given by (1.1) and \( g_k \) defined by (1.3), we set

\[
\Lambda = \left| z f'(z) - p \frac{g_k(z)}{z^{(k-1)p}} \right| - \left| z f'(z) + \frac{(p - 2\gamma) g_k(z)}{z^{(k-1)p}} \right|
\]

\[
= \sum_{n=1}^{\infty} (n + p) a_{n+p} z^{n+p} - p \sum_{n=1}^{\infty} B_{n+p} z^{n+p}
\]

\[
- \left| 2(p - \gamma) z^p + \sum_{n=1}^{\infty} (n + p) a_{n+p} z^{n+p} + (p - 2\gamma) \sum_{n=1}^{\infty} B_{n+p} z^{n+p} \right|.
\]

Thus, for \( |z| = r \) (0 \( \leq r < 1 \)), we get

\[
\Lambda \leq \sum_{n=1}^{\infty} (n + p) |a_{n+p}| |z|^{n+p} + p \sum_{n=1}^{\infty} |B_{n+p}| |z|^{n+p}
\]

\[
- \left( 2(p - \gamma) |z|^p - \sum_{n=1}^{\infty} (n + p) |a_{n+p}| |z|^{n+p} - |p - 2\gamma| \sum_{n=1}^{\infty} |B_{n+p}| |z|^{n+p} \right)
\]

\[
= -2(p - \gamma) |z|^p + 2 \sum_{n=1}^{\infty} (n + p) |a_{n+p}| |z|^{n+p} + (|p - 2\gamma| + p) \sum_{n=1}^{\infty} |B_{n+p}| |z|^{n+p}
\]

\[
= - \left( 2(p - \gamma) + 2 \sum_{n=1}^{\infty} (n + p) |a_{n+p}| + (|p - 2\gamma| + p) \sum_{n=1}^{\infty} |B_{n+p}| \right) |z|^p.
\]

From inequality (3.3), we obtain that \( \Lambda < 0 \). Thus we have

\[
\left| z f'(z) - p \frac{g_k(z)}{z^{(k-1)p}} \right| < \left| z f'(z) + \frac{(p - 2\gamma) g_k(z)}{z^{(k-1)p}} \right|
\]

which is equivalent to (1.4). Hence \( f \in K_{s}^{(k)}(\gamma,p) \). This completes the proof of Theorem 3. \( \square \)

Remark 3. Letting \( p = 1 \) in Theorem 3 we have \( \delta \) Theorem 2. \( \delta \)

Theorem 4. Suppose that an analytic function \( f \) given by (1.1) and \( g \in S_p^* \left( \frac{(k-1)p}{k} \right) \) given by (2.1) are such that the condition (1.2) holds. Then, for \( n \geq 1 \), we have

\[
| (n + p) a_{n+p} - pB_{n+p} |^2 - 4(p - \gamma)^2 \leq 2(p - \gamma) \sum_{m=p+1}^{n+p-1} \left\{ 2m |a_m B_m| + (|p - 2\gamma| + p) |B_m|^2 \right\}, \tag{3.4}
\]

where the coefficients \( B_{n+p} \) are given by (2.2).
Proof. Suppose that the condition \([1, 2]\) is satisfied. Then from Lemma 2, we have
\[
\frac{zf'(z)}{G_k(z)} = \frac{p - (p - 2\gamma) z\varphi(z)}{1 + z\varphi(z)} \quad (z \in \mathbb{U}),
\]
where \(\varphi\) is an analytic functions in \(\mathbb{U}\), \(|\varphi(z)| \leq 1\) for \(z \in \mathbb{U}\), and \(G_k\) is given by \([2, 2]\). From the above equality, we obtain that
\[
[zf'(z) + (p - 2\gamma) G_k(z)] z\varphi(z) = pG_k(z) - zf'(z). \tag{3.5}
\]
Now, we put
\[
z\varphi(z) := \sum_{n=1}^{\infty} t_n z^n \quad (z \in \mathbb{U}).
\]
Thus from \((3.5)\) we find that
\[
\left( 2(p - \gamma) z^p + \sum_{n=1}^{\infty} (n + p) a_{n+p} z^{n+p} + (p - 2\gamma) \sum_{n=1}^{\infty} B_{n+p} z^{n+p} \right) \sum_{n=1}^{\infty} t_n z^n
\]
Equating the coefficient of \(z^{n+p}\) in \((3.6)\), we have
\[
pB_{n+p} - (n + p) a_{n+p} = 2(p - \gamma) t_n + [(p + 1) a_{p+1} + (p - 2\gamma) B_{p+1}] t_{n-1} + \cdots + [(n + p - 1) a_{n+p-1} + (p - 2\gamma) B_{n+p-1}] t_1.
\]
Note that the coefficient combination on the right side of \((3.6)\) depends only upon the coefficients combinations:
\[
[(p + 1) a_{p+1} + (p - 2\gamma) B_{p+1}], \ldots, [(n + p - 1) a_{n+p-1} + (p - 2\gamma) B_{n+p-1}].
\]
Hence, for \(n \geq 1\), we can write as
\[
\left( 2(p - \gamma) z^p + \sum_{m=p+1}^{n+p-1} [ma_m + (p - 2\gamma) B_m] z^m \right) z\varphi(z) = \sum_{m=p+1}^{n+p} [pB_m - ma_m] z^m + \sum_{m=n+p+1}^{\infty} c_m z^m.
\]
Using the fact that \(|z\varphi(z)| \leq |z| < 1\) for all \(z \in \mathbb{U}\), this reduces to the inequality
\[
\left| 2(p - \gamma) z^p + \sum_{m=p+1}^{n+p-1} [ma_m + (p - 2\gamma) B_m] z^m \right| > \left| \sum_{m=p+1}^{n+p} [pB_m - ma_m] z^m + \sum_{m=n+p+1}^{\infty} c_m z^m \right|.
\]
Then squaring the above inequality and integrating along \(|z| = r < 1\), we obtain
\[
\int_0^{2\pi} \left| 2(p - \gamma) r^p e^{i\theta} + \sum_{m=p+1}^{n+p-1} [ma_m + (p - 2\gamma) B_m] r^m e^{im\theta} \right|^2 d\theta
\]
Thus from (3.7),

\[
\int_0^{2\pi} \left| \sum_{m=n+1}^{n+p} [pB_m - ma_m] r^m e^{im\theta} + \sum_{m=n+p+1}^{\infty} c_m r^m e^{im\theta} \right|^2 d\theta.
\]

Using now the Parseval’s inequality, we obtain

\[
4(p - \gamma)^2 r^{2p} + \sum_{m=n+1}^{n+p} |ma_m + (p - 2\gamma) B_m|^2 r^{2m} + \sum_{m=n+p+1}^{\infty} |ma_m - pB_m|^2 r^{2m} + \sum_{m=n+p+1}^{\infty} |c_m|^2 r^{2m}.
\]

Letting \( r \to 1 \) in this inequality, we get

\[
\sum_{m=n+1}^{n+p} |ma_m - pB_m|^2 \leq 4(p - \gamma)^2 + \sum_{m=n+1}^{n+p-1} |ma_m + (p - 2\gamma) B_m|^2.
\]

Hence we deduce that

\[
|n + p) a_{n+p} - pB_{n+p}|^2 - 4(p - \gamma)^2 \leq 2(p - \gamma) \sum_{m=p+1}^{n+p} \left\{2m |a_mB_m| + (|p - 2\gamma| + p) |B_m|^2 \right\},
\]

and thus we obtain the inequality (3.4), which completes the proof of Theorem 4. \( \square \)

**Remark 4.** Letting \( p = 1 \) in Theorem 4, we have [8, Theorem 3].

**Theorem 5.** If \( f \in \mathcal{K}_s^{(k)}(\gamma, p) \), then for \( |z| = r \) \((0 \leq r < 1)\), we have

(i) \[
\frac{|p - (p - 2\gamma) r|}{(1 + r)^{2p+1}} \leq |f'(z)| \leq \frac{|p + (p - 2\gamma) r|}{(1 - r)^{2p+1}},
\]

(ii) \[
\int_0^r \frac{|p - (p - 2\gamma) \tau|}{(1 + \tau)^{2p+1}} d\tau \leq |f(z)| \leq \int_0^r \frac{|p + (p - 2\gamma) \tau|}{(1 - \tau)^{2p+1}} d\tau,
\]

**Proof.** If \( f \in \mathcal{K}_s^{(k)}(\gamma, p) \), then there exists a function \( g \in \mathcal{S}_s^{(k)}(\frac{(k-1)p}{k}) \) such that (1.2) holds.

(i) From Lemma [1], it follows that the function \( G_k \) given by (2.2) is \( p \)-valently starlike function. Hence from [1, Theorem 1] we have

\[
\frac{r^p}{(1 + r)^{2p}} \leq |G_k(z)| \leq \frac{r^p}{(1 - r)^{2p}} \quad (|z| = r \quad (0 \leq r < 1)).
\]

Let us define \( \Psi(z) \) by

\[
\Psi(z) := \frac{zf'(z)}{G_k(z)} \quad (z \in \mathbb{U}).
\]

Then by using a similar method [3, p. 105], we have

\[
\frac{p - (p - 2\gamma) r}{1 + r} \leq |\Psi(z)| \leq \frac{p + (p - 2\gamma) r}{1 - r} \quad (z \in \mathbb{U}).
\]

Thus from (3.9) and (3.10), we get the inequalities (3.7).
Theorem 6. Let \( z = re^{i\theta} \) (\( 0 < r < 1 \)). If \( \ell \) denotes the closed line-segment in the complex \( \zeta \)-plane from \( \zeta = 0 \) and \( \zeta = z \), i.e. \( \ell = [0, re^{i\theta}] \), then we have

\[
f(z) = \int_{\ell} f'(\zeta) d\zeta = \int_0^r f'(\tau e^{i\theta}) e^{i\theta} d\tau \quad (|z| = r \quad (0 \leq r < 1)).
\]

Thus, by using the upper estimate in (3.7), we have

\[
|f(z)| = \left| \int_{\ell} f'(\zeta) d\zeta \right| \leq \int_0^r |f'(\tau e^{i\theta})| d\tau \leq \int_0^r \left[ p + (p - 2\gamma) \tau \right] \tau^{p-1} d\tau \quad (|z| = r \quad (0 \leq r < 1)),
\]

which yields the right-hand side of the inequality in (3.8). In order to prove the lower bound in (3.8), let \( z_0 \in U \) with \( |z_0| = r \quad (0 < r < 1) \), such that

\[
|f(z_0)| = \min \{|f(z)| : |z| = r \}.
\]

It is sufficient to prove that the left-hand side inequality holds for this point \( z_0 \). Moreover, we have

\[
|f(z)| \geq |f(z_0)| \quad (|z| = r \quad (0 \leq r < 1)).
\]

The image of the closed line-segment \( \ell_0 = [0, f(z_0)] \) by \( f^{-1} \) is a piece of arc \( \Gamma \) included in the closed disk \( \mathbb{U}_r \) given by

\[
\mathbb{U}_r = \{ z : z \in \mathbb{C} \quad \text{and} \quad |z| \leq r \quad (0 \leq r < 1) \},
\]

that is, \( \Gamma = f^{-1}(\ell_0) \subset \mathbb{U}_r \). Hence, in accordance with (3.7), we obtain

\[
|f(z_0)| = \int_{\ell_0} |dw| = \int_{\Gamma} |f'(\zeta)| |d\zeta| \geq \int_0^r \left[ p - (p - 2\gamma) \tau \right] \tau^{p-1} d\tau.
\]

This finishes the proof of the inequality (3.8). \( \square \)

Remark 5. Letting \( p = 1 \) in Theorem 5, we have \[8, Theorem 4\].

Theorem 6. Let \( 0 \leq \gamma_2 \leq \gamma_1 < p \). Then we have

\[
K_s^{(k)}(\gamma_1, p) \subset K_s^{(k)}(\gamma_2, p).
\]

Proof. Suppose that \( f \in K_s^{(k)}(\gamma_1, p) \). By Theorem 1, we have

\[
\frac{z^{(k-1)p+1} f'(z)}{g_k(z)} < \frac{p + (p - 2\gamma_1) z}{1 - z} \quad (z \in U).
\]

Since \( 0 \leq \gamma_2 \leq \gamma_1 < p \), we get

\[
-1 < 1 - \frac{2\gamma_1}{p} \leq 1 - \frac{2\gamma_2}{p} \leq 1.
\]

Thus by Lemma 5, we have

\[
\frac{z^{(k-1)p+1} f'(z)}{g_k(z)} < \frac{p + (p - 2\gamma_2) z}{1 - z} \quad (z \in U),
\]

that is \( f \in K_s^{(k)}(\gamma_2, p) \). Hence the proof is complete. \( \square \)

Remark 6. Letting \( p = 1 \) in Theorem 6, we have \[8, Theorem 5\].
References

[1] M.K. Aouf, On a class of \( p \)-valent starlike functions of order \( \alpha \), Internat. J. Math. Math. Sci. 10 (4) (1987), 733–744.

[2] C.-Y. Gao and S.-Q. Zhou, On a class of analytic functions related to the starlike functions, Kyungpook Math. J. 45 (2005), 123–130.

[3] A.W. Goodman, Univalent Functions, vol. I, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[4] T. Hayami, S. Owa and H.M. Srivastava, Coefficient inequalities for certain classes of analytic and univalent functions, J. Ineq. Pure Appl. Math. 8 (4) (2007), Article 95, 1–10.

[5] J. Kowalczyk and E. Leś-Bomba, On a subclass of close-to-convex functions, Appl. Math. Lett. 23 (2010), 1147–1151.

[6] M.S. Liu, On a subclass of \( p \)-valent close-to-convex functions of order \( \beta \) and type \( \alpha \), J. Math. Study 30 (1997), 102–104.

[7] K.S. Padmanabhan, On a certain classes of starlike functions in the unit disc, J. Indian Math. Soc. 32 (1968), 89–103.

[8] B. Şeker, On certain new subclass of close-to-convex functions, Appl. Math. Comput. 218 (2011), 1041–1045.

Kocaeli University, Faculty of Aviation and Space Sciences, Arslanbey Campus, 41285 Kocaeli, Turkey

E-mail address: serap.bulut@kocaeli.edu.tr