Smooth Neighborhood Structures in a Smooth Topological Spaces

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Abstract: Problem Statement: Various concepts related to a smooth topological spaces have been introduced and relations among them studied by several authors (Chattopadhyay, Ramadan, etc).

Conclusion/Recommendations: In this study, we presented the notions of three sorts of neighborhood structures of a smooth topological spaces and give some of their properties which are results by Ying extended to smooth topological spaces.

Key words: Fuzzy smooth topology, smooth neighborhood structures

INTRODUCTION

Šostak (1985) introduced the fuzzy topology as an extension of Chang (1968) fuzzy topology. It has been developed in many directions (Ramadan, 1992; Chattopadhyay and Samanta, 1993; EL Gayyar et al., 1994; Höhle and Rodabaugh, 1998; Kubiak and Šostak, 1997; Demirici, 1997; Ramadan et al., 2001; 2009; Abdel-Sattar, 2006).

Ying (1994) studied the theory of neighborhood systems in fuzzy topology with the method used to develop fuzzifying topology (Ying, 1991) by treating the membership relation as a fuzzy relation. In this study, we generate the structures of neighborhood systems in a smooth topology with the method used in (Ying, 1991), by using fuzzy sets and fuzzy points.

Notions and preliminaries: The class of all fuzzy sets on a universal set X will be denote by L^X, where L is the special lattice and L = ([0,1], ≤). Also, L_0 = (0,1] and L_1 = [0, 1). Notions and preliminaries: The class of all fuzzy sets on a universal set X will be denote by L^X, where L is the special lattice and L = ([0,1], ≤). Also, L_0 = (0,1] and L_1 = [0, 1).

Definition 1: Pu and Liu (1980) a fuzzy set in X is called a fuzzy point iff it takes the value 0 for all y ∈ X, except one, say x ∈ X. If its value at x is λ (0 < λ ≤ 1) we denote this fuzzy point by x_λ, where the point x is called its support. The fuzzy point is said to be contained in a fuzzy set A, or belong to A, denoted by x_λ ∈ A, iff λ ≤ A(x). Evidently, every fuzzy set A can be expressed as the union of all fuzzy points which belong to A.

Definition 2: Ying (1991) Let X be a non-empty set. Let x_λ be a fuzzy point in X and let A be a fuzzy subset of X. Then the degree to which x_λ belongs to A is:

\[ m(x_\lambda, A) = \text{sup}\{m(x, B) : B \in \tau, B \subseteq A \} \]

Obviously, we have the following properties:

1. \( m(x, A) = A(x) \)
2. \( m(x_\lambda, A) = 1 \) iff \( x_\lambda \in A \), \( m(x_\lambda, A) = 0 \) iff \( \lambda = 1 \) and \( A(x) = 0 \)
3. \( m(x_\lambda, \bigcup_{\mu \in \Gamma} A_\mu) = \bigvee_{\mu \in \Gamma} m(x_\lambda, A_\mu) \), (generalized multiple choice principles)

Definition 3: Ying (1991) let \((X, \tau)\) be a fuzzy topological space (fts, for short), let e be a fuzzy point in X and let A be a fuzzy subset of X. Then the degree to which A is a neighborhood of e is defined by:

\[ N_e(A) = \text{sup}\{m(e, B) : B \in \tau, B \subseteq A \} \]

Thus \( N_e(A) \in L^{X} \) is called the fuzzy neighborhood system of e in \((X, \tau)\).

Definition 4: Ying (1991) let \((X, \tau)\) be a fts, e a fuzzy point in X and A a fuzzy subset of X. Then the degree to which e is an adherent point of A is given as:

\[ \text{ad}(e, A) = \inf_{B \in \tau} (1 - N_e(B)) \]

where, \( A^c \) is the complement of A.

Definition 5: Ramadan (1992) A smooth topological space (sts, for short) is an ordered pair (X, τ), where X is a non-empty set and τ: \( L^X \rightarrow L \) is a mapping satisfying the following properties:
(O1) $\tau(1) = \tau(0) = 1$

(O2) For all $A_1, A_2 \in L^X, \tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$

(O3) $\forall \tau, (\bigwedge_{\tau \in \tau} \tau(A) \geq \bigwedge_{\tau \in \tau} \tau(A)$

Definition 6: EL Gayyar et al. (1994) let $(X, \tau)$ be a sts and $\alpha \in \mathbb{L}_0$. Then the family: $\tau_{\alpha} = \{A \in L^X : \tau(A) \geq \alpha\}$, which is clearly a fuzzy topology Chang (1968) sense.

Definition 7: Demirici (1997) Let $(X, \tau)$ be a sts and $A \in L^X$. Then the $\tau$-smooth interior of $A$, denoted by:

$$A^0 = \bigcup \{B \in L^X : \tau(B) > 0, B \subseteq A\}$$

Remark 1: Demirici (1997) let $\tau$ be a Chang’s fuzzy topology (CFT, for short) on the non-empty set $X$. Then the smooth topology and smooth cotopology $\tau_{\alpha}, \tau^*_{\alpha} : L^X \to L$, defined by:

$$\tau(A) = \begin{cases} 1, & \text{if } A \in \tau \\ 0, & \text{if } A \notin \tau \end{cases}$$

and $\tau^*_{\alpha}(A) = \tau(A^\alpha)$ for each $A \in L^X$, identify the CFT $\tau$ and corresponding fuzzy cotopology for it. Thus the $\tau_{\alpha}$- smooth interior of $A$ is:

$$A^0 = \bigcup \{B \in L^X : \tau(B) > 0, B \subseteq A\} = \bigcup \{B \in L^X : B \in \tau, B \subseteq A\}$$

This show that $A^0$ is exactly the interior of $A$ with respect to $\tau$ in Chang (1968) sense.

Lemma 1: Ramadan (1992) $\sup_{\alpha \in \mathbb{L}} \sup \{A(x) \wedge B(x) : A(x) \geq \alpha\} = \sup_{\alpha \in \mathbb{L}} \sup \{\alpha \wedge B(x) : A(x) \geq \alpha\}$.

Smooth neighborhood systems of a fuzzy set: Here, we build a smooth neighborhood systems of a fuzzy set in a sts and we give some of its properties.

For a mapping $M : L^X \to L^{0X}$ and $A \in L^X, \alpha \in [0; 1]$; let us define the family $M_{\alpha}^\nu = \{B \in L^X : M_{\alpha}(B) > \alpha\}$; which will play an important role in this part.

Definition 8: Let $(X, \tau)$ be a sts and $A \in L^X$. Then a mapping $N_A : L^X \to L^X$ is called the smooth neighborhood (nbd, for short) of $A$ with respect to the st $\tau$ iff for each $\alpha \in [0; 1)$:

$$N_{\alpha}^\nu = \{B \in L^X : \exists C \in \tau^\nu(A \subseteq C \subseteq B)\}$$

where, $\tau^\nu = \{A \in L^X : \tau(A) > \alpha\}$ the strong $\alpha$- level of $\tau$.

Remark 2:

- The real number $N_{\alpha}(B)$ is called the degree of nbdness of the fuzzy set $B$ to the fuzzy set $A$. If the smooth nbd system of a fuzzy set $A$ has the following property: $N_{\alpha}(L^X) \subseteq [0, 1]$, then $N_{\alpha}$ is called the fuzzy nbd system of $A$.

- We say that the family $(N_{\alpha})_{\alpha} = \{B : N_{\alpha}(B) > \alpha\}$ is a fuzzy nbd system of $A$ for each $\alpha \in [0,1)$ and $(N_{\alpha})_{\alpha}$ is called the strong $\alpha$-level fuzzy nbd of $A$.

Proposition 1: Let $(X, \tau)$ be a sts and $A \in L^X$. Then a mapping $N_A : L^X \to L^{0X}$ is the smooth nbd system of $A$ with respect to the st $\tau$ iff:

$$N_{\alpha}(B) \begin{cases} \sup \{\tau(C) : A \subseteq C \subseteq B\}, & \text{if } A \subseteq B \\ 0, & \text{if } A \not\subseteq B \end{cases}$$

Proof:

(1) Suppose that the mapping $N_A : L^X \to L^{0X}$ is the smooth nbd systems of $A$ with respect to the st $\tau$. Consider the following two cases:

- For the case $A \not\subseteq B$, suppose that $N_{\alpha}(B) > 0$. From Definition 1, there exists $C \in \tau^\nu$ such that $A \subseteq C \subseteq B$, i.e., $A \subseteq B$, a contradiction. Thus $N_{\alpha}(B) = 0$.

- For the case $A \subseteq B$. We may have $N_{\alpha}(B) = 0$ or $N_{\alpha}(B) > 0$. If $N_{\alpha}(B) = 0$, then $N_{\alpha}(B) = 0 \leq \sup \{\tau(C) : A \subseteq C \subseteq B\}$, if $\sup \{\tau(C) : A \subseteq C \subseteq B\} = \lambda > 0$, then $\exists C \in L^X$ such that $\tau(C) > 0$ and $A \subseteq C \subseteq B$: We obtain $N_{\alpha}(B) > 0$, a contradiction.

Therefore:

$$N_{\alpha}(B) = 0 = \sup \{\tau(C) : A \subseteq C \subseteq B\}$$

Now suppose that $N_{\alpha}(B) = \lambda > 0$. For an arbitrary $0 < \varepsilon \leq \lambda$, we have $N_{\alpha}(B) = \lambda - \varepsilon$, i.e., $B \in N_{\alpha}^{\lambda - \varepsilon}$. Since the mapping: $N_A : L^X \to L^{0X}$ is a smooth nbd system of $A$, $\exists \varepsilon \in L^X$ such that $C \in \tau^{\varepsilon}$ and $A \subseteq C \subseteq B$, i.e., $\sup \{\tau(C) : A \subseteq C \subseteq B\} = \lambda - \varepsilon$. Since $\varepsilon > 0$ is arbitrary we have:

$$\sup \{\tau(C) : A \subseteq C \subseteq B\} \geq \lambda = N_{\alpha}(B)$$

On the other hand, let $\sup \{\tau(C) : A \subseteq C \subseteq B\} = \gamma > 0$. Then for every $0 < \varepsilon \leq \gamma$, $\exists \varepsilon \in L^X$ such that $\tau(C) = \gamma - \varepsilon$ and $A \subseteq C \subseteq B$. Therefore $B \in N_{\alpha}^{\gamma - \varepsilon}$, i.e., $N_{\alpha}(B) > \gamma - \varepsilon$. Since $\varepsilon$ is an arbitrary we have:
\[ N_A(B) \geq \gamma = \sup \{ \tau(C) : A \subseteq C \subseteq B \} \]

Hence the inequality follows:

(2) For \( \alpha \in [0, 1) \), let \( B \in N^\alpha_A \), i.e., \( N_A(B) > \alpha \). Then we can write \( \alpha < N_A(B) = \sup \{ \tau(C) : A \subseteq C \subseteq B \} \), i.e., \( \exists C \in L^X \) such that \( \tau(C) > \alpha \), \( A \subseteq C \subseteq B \). Then we have:

\[ N_A^\alpha = \{ B \in L^X : (\exists C \in \tau^\alpha)(A \subseteq C \subseteq B) \} \]

By the same way we can show that:

\[ \{ B \in L^X : (\exists C \in \tau^\alpha)(A \subseteq C \subseteq B) \} \subseteq N_A^\alpha \]

Hence:

\[ N_A^\alpha = \{ B \in L^X : (\exists C \in \tau^\alpha)(A \subseteq C \subseteq B) \} \]

Remark 3: In Proposition 3, the fuzzy subsets \( A \) of \( X \) can be replaced by the fuzzy points on \( X \), that is, by the special fuzzy subsets \( e \), in this case:

\[ N_e(A) = \sup \{ \tau(C) : e \subseteq C \subseteq A \} \]

if \( e \subseteq A \), \( if \ e \not\subseteq A \)

Proposition 2: Let \( (X, \tau) \) be a sts and \( A \in L^X \). If the mapping \( N_A : L^X \rightarrow L^X \) is the smooth nbd system of \( A \) with respect to the st \( \tau \), then the following properties hold:

(N1) \( N_A(\emptyset) = N_A(A) = 1 \) and \( N_A(B) > 0 \) \( \Rightarrow A \subseteq B \)

(N2) If \( A_1 \subseteq A \) and \( B \subseteq B_1 \), then \( N_A(B) \leq N_{A_1}(B_1) \)

(N3) \( N_A(B_1 \cap B_2) \leq N_A(B_1 \cap B_2) \)

(N4) \( N_A(B) = \sup_{A \subseteq C \subseteq B} (N_A(C) \wedge N_C(B)) \), \( \forall A, B, C \in L^X \)

Proof: (N1) and (N2) follow directly from Definition 1 and Proposition 3. (N3) Suppose that \( N_A(B_1) = \alpha_1 > 0 \) and \( N_A(B_2) > 1 \). Then for a fixed \( \varepsilon > 0 \) such that: \( \varepsilon \leq \alpha_1 \wedge \alpha_2 \Rightarrow N_A(B_1) > \alpha_1 - \varepsilon \geq 0 \) and \( N_A(B_2) > \alpha_2 - \varepsilon > 0 \).

From Definition 1, it is clear that there exists \( C_1, C_2 \in L^X \) such that:

\[ \tau(C_1) > \alpha_1 - \varepsilon, \tau(C_2) > \alpha_2 - \varepsilon \]

\[ A \subseteq C_1 \subseteq B_1, A \subseteq C_2 \subseteq B_2 \]

Therefore, \( \tau(C_1 \cap C_2) \geq \tau(C_1) \wedge \tau(C_2) > (\alpha_1 - \varepsilon) \wedge (\alpha_2 - \varepsilon) = (\alpha_1 \wedge \alpha_2) - \varepsilon \) and \( A \subseteq C_1 \cap C_2 \subseteq B_1 \cap B_2 \). Thus \( N_A(B_1 \cap B_2) \geq (\alpha_1 \wedge \alpha_2) - \varepsilon \). Since \( \varepsilon \) is arbitrary, we find that \( N_A(B_1 \cap B_2) \geq N_A(B_1) \wedge N_A(B_2) \).

(N4) \( N_A(B) = \sup \{ \tau(C) : A \subseteq C \subseteq B \} \).

Thus, \( \tau(C) \leq N_A(C) \) and \( \tau(C) \leq N_C(B) \).

Thus, \( \sup \{ \tau(C) : A \subseteq C \subseteq B \} \leq \sup \{ N_A(C) \wedge N_C(B) \} \).

Hence:

\[ N_A(B) \leq \sup_{A \subseteq C \subseteq B} \{ N_A(C) \wedge N_C(B) \} \]

Smooth neighborhood systems of a fuzzy points:

Definition 9: Let \( (X, \tau) \) be a sts, \( e \) a fuzzy point in \( X \) and \( A \) be a fuzzy subset of \( X \). Then the degree to which \( A \) is a NBD of \( e \) is defined by:

\[ N_e(A) = \begin{cases} \sup_{B \subseteq A} \{ m(e, B) \wedge \tau(B) \} & : \tau(B) > 0, \ if \ m(e, A) > 0 \\ 0, & : \ otherwise \end{cases} \]

Thus \( N_e \in L^X \) is called the smooth NBD system of \( e \) in \( (X, \tau) \).

Remark 4: It is clear that when a fuzzy point \( e \in B \subseteq L^X \), then \( m(e, B) = 1 \) and

\[ N_e(A) = \begin{cases} \sup_{B \subseteq A} \{ m(e, B) \wedge \tau(B) \} & : \tau(B) > 0, \ if \ e \subseteq A \\ 0, & : \ otherwise \end{cases} \]

is the NBD systems in the sense of Demirici (1997)

Remark 5: For any crisp point \( x \) in \( X \), we have:

\[ N_e(A) = \sup_{B \subseteq A} \{ m(e, B) \wedge \tau(B) \} : \tau(B) > 0, B(x) \neq 0. \]

Proposition 3: The NBD systems \( N_e \) of \( e \) in \( X \) can be constructed from the cuts \( \tau_\alpha, \alpha \in (0,1] \), by using the equality:

\[ N_e(A) = \sup_{\alpha \in (0,1]} \{ [N_e(A)]^\alpha \wedge \alpha \} \]

where, \( [N_e(A)]^\alpha = \sup_{B \subseteq A} \{ m(e, B) : B \subseteq A, B \subseteq \tau_\alpha \} \), is the NBD systems in the sense of Ying (1994; Theorem 1).

Proof: By using Definition 9, we have:

\[ N_e(A) = \sup_{B \subseteq A} \{ m(e, B) \wedge \tau(B) \} : \tau(B) > 0 \]

\[ = \sup_{\alpha \in (0,1]} \sup_{B \subseteq A} \{ m(e, B) \wedge \alpha : \tau(B) > \alpha \} \]

\[ = \sup_{\alpha \in (0,1]} \{ \sup_{B \subseteq A} \{ m(e, B) : B \subseteq \tau_\alpha \wedge \alpha \} \} \]

\[ = \sup_{\alpha \in (0,1]} \{ [N_e(A)]^\alpha \wedge \alpha \} \]

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Remark 6: For any crisp point \( x \) in \( X \); we have:

\[
N_x(\alpha) = \sup_{A \subseteq X} \{N(A)^\alpha \wedge \alpha \}
\]

where, \([N_x(\alpha)]^\alpha = \sup_{B \subseteq A} \{B(x) : B \in \tau_x \} \).

Theorem 1: Let \((X, \tau)\) be a sts and \( e \) a fuzzy point of \( X \). If the mapping \( N_e: L^X \rightarrow L \) is the smooth NBD systems of \( e \) with respect to \( \tau \), then the following properties hold:

(N1) \( N_e(A) \leq m(e, B) \)

(N2) If \( A \subseteq B \) and \( A, B \in L^\tau \), then \( N_e(A) \leq N_e(B) \)

(N3) For all \( A, B \in L^\tau \), \( N_e(A \cap B) \leq N_e(A) \wedge N_e(B) \)

(N4) \( N_e(A) = \sup_{\lambda \in \sigma} \{[N_e(B)]^\lambda \wedge \alpha : \quad \lambda \leq \alpha \} \)

Proof: (N1) and (N2) follows directly from Remark 2.

(N3): \( m(e, C) \wedge \tau(C) : \tau(C) > 0 \)\]

\( \leq \sup_{C \subseteq A} \{\min(\sup_{C \subseteq A} \{m(e, C) \wedge \tau(C) : \tau(C) > 0\}) \}) \}

\( \leq \sup_{B \subseteq A} \{\min(\sup_{B \subseteq A} \{m(e, C) \wedge \tau(C) : \tau(C) > 0\}) \} \)

\( = N_e(A \cap B) \)

(N4) Combining axiom (4) in Theorem 1, in (Ying, 1994) and Proposition 4, (N4) follows.

Theorem 2: Let the mapping \( N_e: L^X \rightarrow L \) satisfy the conditions (N1)-(N4), then the mapping \( \tau: L^X \rightarrow L \) defined by:

\( \tau(A \cap B) = \inf_{\alpha} \{m(e, A \cap B) \wedge N_e(A \cap B) \} \)

\( \geq \inf_{\alpha} \{m(e, A \cap B) \wedge (\min(N_e(A), N_e(B))) \} \)

\( = \inf_{\alpha} \{m(e, A) \wedge (\min(N_e(A), N_e(B))) \} \)

\( \leq \inf_{\alpha} \{m(e, A) \wedge N_e(A) \wedge \tau(A) \} \)

\( \tau(U_{\mu} A) = \inf_{\alpha} \{m(e, U_{\mu} A) \wedge N_e(U_{\mu} A) \} \)

\( \leq \inf_{\alpha} \{m(e, A) \wedge \tau(A) \} \)

Proof: (O1) Obvious.
Hence, the equality \( N_e = M_e \) follows at once from (1) and (2).

**Definition 10:** Let \((X, \tau)\) be a sts, \( e \) a fuzzy point in \( X \) and \( A \) a fuzzy subset of \( X \). Then the degree to which \( e \) is an adherent point of \( A \) is given as:

\[
\text{ad}(e, A) = \inf_{B \subseteq X} \left( 1 - N_e(B) \right)
\]

where, \( A^c \) is the complement of \( A \).

**Remark 6:** For any crisp point \( x \) in \( X \), we have:

\[
\text{ad}(x, A) = \inf_{B \subseteq X} \left( 1 - N_e(B) \right)
\]

**Proposition 4:**

\[
\text{ad}(e, A) = \inf_{B \subseteq X} \left( 1 - N_e(B) \right)
\]

**Proof:** Follows from Proposition 4.

**Proposition 5:**

\[
N_{s_e}(A) \leq \sup_{\alpha > 0} \left\{ \min(1, 1 - \lambda + [N_e(A)]^\alpha) \right\}
\]

**Proof:**

\[
N_{s_e}(A) = \sup_{\alpha > 0} \left\{ \min(1, 1 - \lambda + [N_e(A)]^\alpha) \right\}
\]

Fuzzy smooth r-neighborhood:

**Definition 11:** Let \((X, \tau)\) be a sts, \( A \in L^X \), \( e \) a fuzzy point in \( X \) and \( r \in L_0 \). Then the degree to which \( A \) is a fuzzy smooth r-nbd system of \( e \) is defined by:

\[
N_{s_e}(A, r) = \sup_{B \subseteq X} \left\{ \min(1, 1 - \lambda + [N_e(A)]^\alpha) \right\}
\]

A mapping \( N_{s_e} : L^X \times L_0 \to L \) is called the fuzzy smooth r-nbd system of \( e \).

**Theorem 2:** Let \((X, \tau)\) be a sts and \( N_e \) the fuzzy smooth r-nbd system of \( e \). For \( A, B \in L^X \) and \( r, s \in L_0 \), it satisfies the following properties:

1. \( N_{s_e}(A, r) \leq m(e, A) \) for each \( r \in L_0 \)
2. \( N_{s_e}(A, r) \leq N_{s_e}(B, r) \), if \( A \subseteq B \)
3. \( N_{s_e}(A, r) \leq N_{s_e}(A \cap B, r) \)
4. \( N_{s_e}(A, r) \leq \sup\{N_{s_e}(B, r) : B \subseteq A, m(d, B) \leq N_{s_e}(B, r) \} \) for all fuzzy point \( d \) in \( X \)
5. \( N_{s_e}(A, r) \geq N_{s_e}(A, s) \), if \( r \leq s \)
6. \( N_{s_e}(A, r) = \min(1, 1 - t + N_{s_e}(A, r)) \)

**Proof:** (2) and (5) are easily proved.

(1) It is proved from the following:

\[
N_{s_e}(A, r) = \sup\{m(e, \cup B_j) : B_j \subseteq A, \tau(B_j) \geq r\}
\]

Suppose there exist \( A, B \in L^X \) and \( r \in L_0 \) such that:

\[
N_{s_e}(A, r) > t \quad \text{and} \quad N_{s_e}(B, r) > t
\]

Since \( N_{s_e}(A, r) > t \) and \( N_{s_e}(B, r) > t \), there exist \( C_1, C_2 \in L^X \) with:

\[
C_1 \subseteq A, \tau(C_1) \geq r, \quad C_2 \subseteq B, \tau(C_2) \geq r
\]

Such that:

\[
m(e, C_1) \wedge m(e, C_2) = m(e, C_1 \cap C_2) > t
\]

On the other hand, since:

\[
C_1 \cap C_2 \subseteq A \cap B, \tau(C_1 \cap C_2) \geq r
\]

We have:

\[
N_{s_e}(A \cap B, r) \geq m(e, C_1 \cap C_2) > t
\]

It is a contradiction.

(4) If \( \tau(B) \geq r \), then \( N_{s_e}(B, r) = m(d, B) \); for each fuzzy point \( d \) in \( X \). It implies:

\[
N_{s_e}(A, r) = \sup\{m(e, B) : B \subseteq A, \tau(B) \geq r\}
\]

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Theorem 3: Let \( N_e \) be the fuzzy smooth \( r \)-nbd system of \( e \) satisfying the above conditions (1)-(5), the function \( \tau_N : \mathbb{L}^X \rightarrow \mathbb{L}^0 \) defined by:

\[
\tau_N(A) = \bigvee \{ r \in \mathbb{L}_0 : m(e, A) = N_e(A, r) \text{ for all fuzzy point } e \text{ in } X \}
\]

has the following properties:

1. \( \tau_N \) is a st. on \( X \)
2. If \( N_e \) is the fuzzy nbd systems of \( e \) induced by \( (X, \tau) \), then \( \tau_N = \tau \)
3. If \( N_e \) satisfy the conditions (1)-(6), then:

\[
\tau_N(A) = \bigvee \{ r \in \mathbb{L}_0 : m(x, A) = N_e(x, r), x \in X \}
\]

Proof: (1) We will show that \( \tau_N(B_1 \cap B_2) \geq \tau_N(B_1) \land \tau_N(B_2) \), for any \( B_1, B_2 \in \mathbb{L}^X \).

Suppose there exist \( B_1, B_2 \in \mathbb{L}^X \) and \( r \in \mathbb{L}_0 \) such that:

\[
\tau_N(B_1 \cap B_2) < r < \tau_N(B_1) \land \tau_N(B_2)
\]

For each \( i \in \{1, 2\} \) there exists \( r_i \in \mathbb{L}_0 \) with:

\[
m(e, B_i) = N_e(B_i, r_i); \text{ for all fuzzy point } e \text{ in } X
\]

Such that: \( \tau_N(B_i) \geq r_i > r \).

From (I), (II) and (5), we have:

\[
m(e, B_i) = N_e(B_i, r_i) \leq m(e, B_i)
\]

It implies \( m(e, B_i) = N_e(B_i, r) \): Furthermore:

\[
m(e, B_i \cap B_2) = N_e(B_i, r) \land N_e(B_2, r)
\]

Thus, \( N_e(B_i \cap B_2, r) = m(e, B_i \cap B_2) \), i.e., \( \tau N_e(B_i \cap B_2) \geq r \). It is a contradiction for the Eq. I.

Suppose there exists \( B = \bigcup_{i \in \Gamma} B_i \in \mathbb{L}_X \) and \( r_0 \in \mathbb{L}_0 \) such that:

\[
\tau_N(B) < r_0 < \bigwedge_{i \in \Gamma} \tau_N(B_i)
\]

For each \( i \in \Gamma \), there exists \( r_i \in \mathbb{L}_0 \) with

\[
m(e, B_i) = N_e(B_i, r_i); \text{ for all fuzzy point } e \in X
\]

Such that: \( \tau_N(B_i) \geq r_i > r \)

From (I), (IV) and (5), we have:

\[
m(e, B_i) = N_e(B_i, r_i) \leq N_e(B_i, r) \leq m(e, B_i)
\]

It implies \( m(e, B_i) = N_e(B_i, r) \): Furthermore:

\[
m(e, \bigcup_{i \in \Gamma} B_i) = \bigvee_{i \in \Gamma} m(e, B_i)
\]

\[
\leq \bigvee_{i \in \Gamma} N_e(B_i, r_i)
\]

\[
\leq m(e, \bigcup_{i \in \Gamma} B_i).
\]

Thus, \( N_e(\bigcup_{i \in \Gamma} B_i, r) = m(e, \bigcup_{i \in \Gamma} B_i) \), i.e., \( \tau N_e(\bigcup_{i \in \Gamma} B_i) \geq r_0 \). It is a contradiction for the Eq. III.

(2) Suppose there exists \( A \in \mathbb{L}^X \) such that:

\[
\tau_N(A) > \tau(A)
\]

From the Definition of \( \tau_N \), there exists \( r_0 \in \mathbb{L}_0 \) with

\[
m(e, A) = N_e(A, r_0) \text{ such that}
\]

\[
\tau_N(A) \geq r_0 > \tau(A)
\]

Since:

\[
m(e, A) = N_e(A, r_0) = \sup \{ m(e, B_i) : B_i \subseteq A, \tau(B_i) \geq r_0 \}
\]

Then, for each \( x \in X \):

\[
(\bigcup_{B_i}(x)) = \sup \{ m(x, B_i) : B_i \subseteq A \} = m(x, A) = A(x)
\]

Thus, \( A = \bigcup_{i \in \Gamma} B_i \in \mathbb{L}_X \). It is a contradiction. Suppose there exists \( A \in \mathbb{L}^X \) such that:

\[
\tau_N(A) < \tau(A)
\]

There exists \( r_1 \in \mathbb{L}_0 \) such that:

\[
\tau_N(A) < r_1 \leq \tau(A)
\]

Since \( \tau(A) \geq r_1 \), we have:

\[
N_e(A, r_1) = \sup \{ m(e, B) : B \subseteq A, \tau(B) \geq r_1 \} = m(e, A)
\]
Hence $\tau_n(A) \geq \tau_1$. It is a contradiction.

**CONCLUSION**

(3) We only show that $m(x, A) = N_x(A, r)$, for all fuzzy point $x_1$ in $X$ iff $m(x, A) = A(x) = N_x(A, r)$, $\forall x \in X$:

$(\Rightarrow)$ It is trivial.

$(\Leftarrow)$ From the condition (6):

$$
N_x (A, r) = \min(1, 1 - t + N_x (A, r)) \\
= \min(1, 1 - t + m(x, A)) \\
= \min(1, 1 - t + A(x)) \\
= m(x, A).
$$

**Example 1**: Let $X = \{a, b\}$ be a set, $N$ a natural number set and $B \in L^X$ as follows:

$B(a) = 0.3$, $B(b) = 0.4$

We define a smooth fuzzy topology:

$$
\tau(A) = \begin{cases} 
1, & \text{if } A = 0 \text{ or } 1, \\
\frac{1}{2}, & \text{if } A = B, \\
0, & \text{otherwise}
\end{cases}
$$

From Definition 1, $N_a$, $N_b$: $L^X \times L_0 \to L$ as follows:

$$
N_a(A) = \begin{cases} 
1, & \text{if } A = 1, \quad r \in L_0 \\
0.3, & \text{if } 1 \neq A \supseteq B, \quad 0 < r \leq \frac{1}{2} \\
0, & \text{otherwise}
\end{cases}
$$

$$
N_b(A) = \begin{cases} 
1, & \text{if } A = 1, \quad r \in L_0 \\
0.4, & \text{if } 1 \neq A \supseteq B, \quad 0 < r \leq \frac{1}{2} \\
0, & \text{otherwise}
\end{cases}
$$

From Theorem 2 and Theorem 3 (3), we have:

$$
\tau_n(A) = \begin{cases} 
1, & \text{if } A = 0 \text{ or } 1, \\
\frac{1}{2}, & \text{if } A = B, \\
0, & \text{otherwise}
\end{cases}
$$

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