Horospherical coordinates of lattice points in hyperbolic space: effective counting and equidistribution

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Abstract

We establish effective counting and equidistribution results for lattice points in families of domains in hyperbolic spaces, of any dimension and over any field. The domains we focus on are defined as product sets with respect to the Iwasawa decomposition. Several classical Diophantine problems can be reduced to counting lattice points in such domains, including distribution of shortest solution to the gcd equation, and angular distribution of primitive vectors in the plane. We give an explicit and effective solution to these problems, and extend them to imaginary quadratic number fields. Further applications include counting lifts of closed horospheres to hyperbolic manifolds and establishing an equidistribution property of integral solutions to the Diophantine equation defined by a Lorentz form.

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1 Introduction and statement of main results

A lattice in a Lie group is a discrete subgroup whose fundamental domain has finite Haar measure. Our goal in the present paper is to establish effective counting and equidistribution results for Iwasawa components of lattice elements in real rank one Lie groups that are simple up to a finite center; namely, isometry groups of hyperbolic spaces. These problems are instances of hyperbolic counting problems, in which one seeks to study the asymptotic behavior of the number of lattice orbit points in some expanding family of regions in hyperbolic space, and generalize the classical question of counting in hyperbolic balls.

A natural extension of the much studied class of counting problems in the euclidean space, hyperbolic counting problems typically have the property that the number of lattice points inside sufficiently regular domains is asymptotic to the volume of these domains. In both settings there is a special interest in estimating the error term, i.e. the difference between the volume of a domain and the number of lattice points inside it. Unlike the euclidean setting, in which one can produce a bound for the error term in terms of the volume of a thin neighborhood of the boundary of suitable domains, the hyperbolic setting presents a special challenge; this is due to the fact that a fixed proportion of the volume, and of the lattice points, is concentrated near the boundary.

Counting the points of a lattice orbit in a hyperbolic space can be easily deduced from counting the elements of the lattice subgroup itself inside the group of isometries $G$; the approach we take is the one of counting in the actual group.

The domains that we consider are product sets in the Iwasawa coordinates on $G$: $G = NAK$, where $K$ is maximal compact, $A \cong \mathbb{R}$, and $N$ is the unipotent subgroup that stabilizes the ideal boundary point $\{\infty\}$. The map $N \times A \times K \to G$ given by $(n, a, k) \mapsto nak$ is a diffeomorphism, so these are indeed coordinates on $G$. For example, $\text{SL}_2(\mathbb{R})$ decomposes into

$$
N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}, \\
A = \left\{ \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} : t \in \mathbb{R} \right\}, \\
K = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} : 0 \leq \theta < 2\pi \right\} = \text{SO}(2)
$$

(1.1)

Let $G$ denote a non-exceptional simple Lie group of real rank one with finite center; namely, locally isomorphic to one of the following: $\text{SO}(1, n)$, $\text{SU}(1, n)$, or $\text{SP}(1, n)$ for some $n \geq 1$. The corresponding rank 1 symmetric spaces $G/K$ are, respectively: the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ and the quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{H}}$. Every $G$ acts on the corresponding space by isometries of the Riemannian distance, which we will refer to as the “hyperbolic distance” and denote by $d(\cdot, \cdot)$. The remaining rank one simple Lie group is $\text{F}_4(-20)$, which corresponds to the octonionic hyperbolic plane $\mathbb{H}^2_O$; we shall not consider this case.

A Haar measure $\mu$ on $G$ is given in the Iwasawa coordinates as follows. As in the above example, $A$ is parametrized such that $A = \{a_t : t \in \mathbb{R}\}$, and $d(a_t \cdot i, a_s \cdot i) = |t - s|$, where $i$ is
Figure 1: The domains $R_T(\Psi, \Phi)$ projected to: (a) real hyperbolic plane in upper half plane model, (b) real hyperbolic 3-space in upper half space model, (c) real hyperbolic plane in unit disc model.

the point that $K$ stabilizes in the symmetric space. Let $\mu_K$ denote a Haar measure on $K$. The subgroup $N$ is parametrized by a euclidean space of the appropriate dimension (see table 1 for the different cases), and a Haar measure on $N$ is the Lebesgue measure on this underlying euclidean space. A Haar measure on $G$ w.r.t. the Iwasawa coordinates is given by

$$\mu = \mu_N \times \frac{dt}{e^{2\rho t}} \times \mu_K,$$

where $\rho$ is a parameter that depends on the group $G$. The Iwasawa subgroups, symmetric spaces and Haar measure of the rank one groups are summarized in table 1. We will mainly focus on lattices $\Gamma < G$ that are non-cocompact; without loss of generality we may assume that such $\Gamma$ has a cusp at $\infty$. We consider lattice points whose $N$ and $K$ components lie in given bounded subsets $\Psi \subset N$ and $\Phi \subseteq K$, and study their asymptotic behavior as their $A$-components tend to $\infty$. When the lattice has a cusp at $\infty$, there are only finitely many lattice points in $\Psi A\Phi$ whose $A$-coordinate is positive. This finite number of points clearly does not affect the asymptotics, and it is therefore sufficient to consider lattice points in the family $\{R_T(\Psi, \Phi)\}_{T>0}$, where

$$R_T(\Psi, \Phi) := \Psi A_{[-T,0]} \Phi = \{na_t k : n \in \Psi, t \in [-T,0], k \in \Phi\}$$

(Figure 1) as $T \to \infty$. According to (1.2) the volume of these domains equals

$$\mu(R_T(\Psi, \Phi)) = \frac{1}{2\rho} \cdot \mu_N(\Psi) \mu_K(\Phi) (e^{2\rho T} - 1).$$

We shall require that the domains $\Psi \subset N$ and $\Phi \subseteq K$ are nice: bounded, full dimension embedded submanifolds whose boundaries are piecewise smooth — namely, a finite union of
submanifolds of co-dimension 1. We allow the case where only some of these submanifolds are included in the nice set, while others are not, e.g. a half open rectangle that two of its edges are included and the remaining two are not included.

**Theorem 1.1.** Let \( \Psi \subset N \) and \( \Phi \subset K \) be nice domains, and consider the family \( R_T (\Psi, \Phi) \) as defined above. For any lattice \( \Gamma < G \), there exists a parameter \( \kappa = \kappa (\Gamma) < 1 \) (defined explicitly in 4.3) such that for \( T > 0 \):

\[
\# (R_T (\Psi, \Phi) \cap \Gamma) = \frac{\mu (R_T (\Psi, \Phi))}{\mu (G/\Gamma)} + O \left( \log \left( \frac{\mu (R_T (\Psi, \Phi))}{\mu (G/\Gamma)} \right) \right) \kappa
\]

\[
= \frac{\mu_N (\Psi) \mu_K (\Phi)}{\mu (G/\Gamma)} \cdot \frac{e^{2\rho T}}{2\rho} + O \left( T \left( e^{2\rho T} \right)^\kappa \right).
\]

The implicit constant depends on \( \Psi \) and \( \Phi \).

For example, for the lattice \( \text{SL}_2 (\mathbb{Z}) \subset \text{SL}_2 (\mathbb{R}) \) \( \kappa (\text{SL}_2 (\mathbb{Z})) = 7/8 \), and Theorem 1.1 produces the best known error estimate for this case. This particular case has received considerable attention, which we will briefly detail at the end of the next section. We note that while (as noted above) the domains \( R_T (\Psi, \Phi) \) are natural to consider in the context of lattices with a cusp, Theorem 1.1 applies for any lattice \( \Gamma < G \). However, when the lattice in question is co-compact, the cuspidal strip \( \Psi A (0, \infty) \Phi \) may contain infinitely many lattice points, despite its bounded volume. The irregularity caused by this cuspidal strip is the reason why the domains \( R_T \) must be truncated at height \( t = 0 \). In order to study the co-compact case, one should consider the sets \( \Psi A_{[-T,T]} \Phi \), which we will do elsewhere.

For \( H \in \{ N, A, K \} \), we denote the projection to the \( H \)-component by \( \pi_H : G \to H \).

**Corollary 1.2.** Let \( \Psi, \Psi' \subset N \) and \( \Phi, \Phi' \subset K \) be nice, and let \( \Gamma < G \) be any lattice. For \( 0 < T \),

\[
\frac{\# (\Gamma \cap R_T (\Psi', \Phi'))}{\# (\Gamma \cap R_T (\Psi, \Phi))} = \frac{\mu_N (\Psi') \mu_K (\Phi')}{\mu_N (\Psi) \mu_K (\Phi)} + O \left( T \left( e^{2\rho T} \right)^{-1-\kappa} \right)
\]

where the implied constant depends on \( \Psi, \Psi', \Phi, \Phi' \) and \( \kappa = \kappa (\Gamma) < 1 \) is the exponent associated with \( \Gamma \) appearing in Theorem 1.1.

1. The set of \( N \)-components \( \{ \pi_N (\gamma) : \gamma \in \Gamma \cap \Psi A_{[-T,0]} \Phi \} \) become effectively equidistributed in \( \Psi \) w.r.t. \( \mu_N \) as \( T \to \infty \). Namely, for every compactly supported Lipschitz function \( f \) on \( N \),

\[
\left| \frac{1}{\# (\Gamma \cap \Psi A_{[-T,0]} \Phi)} \sum_{\gamma \in \Psi A_{[-T,0]} \Phi} f (\pi_N (\gamma)) - \frac{1}{\mu_N (\Psi)} \int_{\Psi} f \, d\mu_N \right| \leq \text{const} \cdot T e^{-2\rho T (1-\kappa)},
\]

where the constant depends on the function \( f \).
The set of $K$-components $\{\pi_K(\gamma): \gamma \in \Gamma \cap \Psi A_{[-T,0]})\Phi\}$ become effectively equidistributed in $\Phi$ w.r.t. $\mu_K$ as $T \to \infty$ (implying an analogous statement to the one in [2] for a compactly supported Lipschitz function on $K$).

The proofs for Theorem 1.1 and Corollary 1.2 are in Section 4.

Remark 1.3. Iwasawa decomposition of a Lie group is used in one of two conventions: $G = NAK$ or $G = KAN$. Our results are phrased with respect to the first option, but the corresponding statements with respect to the $KAN$ decomposition may be easily deduced. Indeed, the $KAN$ coordinates of $g \in G$ are obtained from the $NAK$ coordinates of $g^{-1}$: $g^{-1} = nak$ implies $g = k^{-1}a^{-1}n^{-1}$. In particular, the Haar measure with respect to the $KAN$ coordinates is $\mu_K \times e^{2\rho t} dt \times \mu_N$, and the statement of Theorem 1.1 is replaced by

$$\# \Gamma \cap (\Phi A_{[0,T]} \Psi) = \frac{\mu_N(\Psi) \mu_K(\Phi)}{\mu(G/\Gamma)} \cdot \frac{e^{2\rho T}}{2\rho} + O\left(T^{(e^{2\rho T})^\kappa}\right)$$

for $\Phi \subset K$, $\Psi \subset N$ and $\kappa$ as in Theorem 1.1, and $T > 0$.

Remark 1.4. Note that Theorem 1.1 was formulated for a family of domains in $G$ itself, rather than in the symmetric space; this enables us to analyze the distribution of the $K$-components of the lattice elements. As we shall see below, equidistribution of the $K$-components plays a key role in a number of applications, including angular equidistribution of shortest solutions to the gcd equation in $\mathbb{Z}^2$. The connection between the problem of equidistribution of the norms of the shortest solutions and the equidistribution of Iwasawa N-components in $\text{SL}_2(\mathbb{Z})$ was first pointed out by Risager and Rudnick [18], and has motivated the approach pursued in the present paper. We will first formulate and prove our results and then comment further on the history of this problem.

## 2 Iwasawa components and diophantine problems

### 2.1 Distribution of shortest solutions of the gcd equation

We now turn to some consequences of Corollary 1.2 for certain integral lattices in a real hyperbolic space of small dimension. In what follows, the norm we refer to is the euclidean norm on $\mathbb{R}^2$ or $\mathbb{C}^2$, denoted by $\|\cdot\|$.

For every primitive integral vector $v = (a,b)$, let $w_v$ denote the shortest integral vector that completes $v$ to a (positively oriented) basis of $\mathbb{Z}^2$, namely, the shortest solution to the gcd equation $bx - ay = 1$. Let $\theta_v$ denote the angle from $w_v$ to $v$ (anticlockwise). We say that $v$ is \textit{positive} if $\theta_v$ is acute, and \textit{negative} if $\theta_v$ is obtuse (Figure 3).

In the case of the lattice $\text{SL}_2(\mathbb{Z})$, Corollary 1.2 has the following geometric interpretation.

**Theorem 2.1.** Let $\Theta \subseteq S^1$ an arc in the unit circle, and let $S_{\Theta}$ be the corresponding sector of the plane $\mathbb{R}^2$ (see Figure 2). For every primitive integral vector $v = (a,b)$, let $w_v$ and $\theta_v$ as above. For $v \in S_{\Theta}$, $\|v\| \to \infty$:
Figure 2: $\mathbb{Z}^2$-points contained in the sector $S_{\Theta}$.

1. The ratios $\|w_v\| / \|v\|$ of the length of the shortest solution relative to the length of $v$ become effectively equidistributed in $[0, 1/2]$;

2. The values $v / \|v\|$ become effectively equidistributed in $\Theta$;

3. The values $w_v / \|w_v\|$ become effectively equidistributed in $\Theta$ when $v$ is restricted to positive vectors, and in $-\Theta$ when $v$ is restricted to negative vectors. In particular, when $|\Theta| \leq \pi$, the values $w_v / \|w_v\|$ become effectively equidistributed in $\Theta$, in $-\Theta$ and in $\Theta \cup -\Theta$;

4. Parts 1 and 2 hold when $v$ is restricted to positive vectors only, or to negative vectors only.

In all the above effective equidistribution statements, the rate of convergence is $O \left( \|v\|^{-1/4} \cdot \log \|v\| \right)$.

Remark 2.2. Note that part 2 asserts that the directions of primitive integral vectors in any sector in the plane converge to uniform distribution on the corresponding arc of the circle at a rate given by the radius to the power of $-1/4$. This result may well be known, but we have not been able to find it in the literature.

Proof of Theorem 2.1. If $v = (a, b) \in \mathbb{Z}^2$ is primitive, it can be completed to countably many matrices in $\text{SL}_2(\mathbb{Z})$, representing the different integral solutions to the equation $bx - ay = 1$; The NAK components of these integral matrices encode the vector $v$ and the different solutions to $bx - ay = 1$ as follows. Recall that the Iwasawa decomposition of $\text{SL}_2(\mathbb{R})$ given in 1.1 if $(x, y)$ is such a solution, the corresponding matrix in $\text{SL}_2(\mathbb{Z})$ has NAK decomposition

$$
\begin{bmatrix}
  x & y \\
  a & b
\end{bmatrix} = \left( \begin{array}{c}
  1 \\
  \frac{za+yb}{a^2+b^2} \\
  \frac{1}{\sqrt{a^2+b^2}} \\
  \frac{1}{\sqrt{a^2+b^2}} \\
\end{array} \right) \left( \begin{array}{c}
  b & -a \\
  a & b
\end{array} \right).
$$

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Note that the $A$ and $K$ components depend only on the vector $v$: the $A$-component $\begin{bmatrix} 1/\|v\| & 0 \\ 0 & \|v\| \end{bmatrix}$ encodes the norm of $v$, and the $K$-component $\begin{bmatrix} v^1/\|v\| \\ v/\|v\| \end{bmatrix}$ encodes the angle of $v = (a, b)$ w.r.t. the positive real axis. The $N$-component depends on the specific solution $(x, y)$, namely the upper row of the matrix; if $w := (x, y)$, then the $N$-component is $\begin{bmatrix} 1 \langle w, v \rangle/\|v\|^2 \\ 0 \end{bmatrix}$ (the projection of $w$ to the line span $\{v\}$, divided by the norm of $v$).

Throughout the rest of the proof we shall identify the subgroups $N, A, K$ and $\Gamma$ of $\text{SL}_2(\mathbb{R})$ given in $\ref{1}$ with $\mathbb{R}, \mathbb{R},$ and $S^1$ respectively through $\begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix} \leftrightarrow x, \begin{bmatrix} e^{i/2} & 0 \\ 0 & e^{-i/2} \end{bmatrix} \leftrightarrow t$, and $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \leftrightarrow \theta$.

The different solutions to $bx - ay = 1$ are $\{(x + ma, y + mb) : m \in \mathbb{Z}\}$, and they correspond to matrices $\begin{bmatrix} x+ma & y+mb \\ a & b \end{bmatrix}$ whose $N$-coordinates are

$$\frac{(x + ma)a + (y + mb)b}{a^2 + b^2} = m + \frac{xa + yb}{a^2 + b^2} = m + \frac{\langle w, v \rangle}{\|v\|^2}$$

(namely, all the integral translations of the real number $\langle w, v \rangle/\|v\|^2$). Observe that, among all the integral matrices that correspond to $v$, the one whose $N$-coordinate is minimal — i.e., in the interval $[-1/2, 1/2]$ — is the one that corresponds to the shortest solution to $bx - ay = 1$, namely, the one whose upper row has minimal norm. This is because the integral solutions are the integral points on the affine line span $\{v\} + w$ (where $w$ is any solution) which is parallel to span $\{v\}$; hence when decomposing $\mathbb{R}^2$ as span $\{v\} \oplus \text{span} \{v^\perp\}$, all of these solutions have the same $v^\perp$ component, and the shortest integral solution is the one with the shortest $v^\perp$-component (namely, the shortest projection on span $\{v\}$). The shortest integral solution $w_v$ corresponds to the matrix $\begin{bmatrix} w_v \\ v \end{bmatrix} = \begin{bmatrix} x_v & y_v \\ a & b \end{bmatrix}$, which we denote by $\gamma_v$.

We conclude that the set $\{\gamma \in \text{SL}_2(\mathbb{Z}) : \pi_N(\gamma) \in [-1/2, 1/2]\}$ is in one-to-one correspondence $\gamma_v \leftrightarrow v$ with the set of primitive integral vectors, where for each $\gamma_v$, $\pi_K(\gamma_v)$ is the angle of the corresponding primitive vector $v$ and $e^{-\pi A(\gamma_v)/2}$ is the length of $v$. Thus, applying part $\ref{2}$ of Corollary $\ref{1}$ with $\Gamma = \text{SL}_2(\mathbb{Z}), e^{T/2} \geq \|v\|,$ and $\Phi = \Theta$ in $S^1$ (these will remain fixed throughout the proof), as well as $\Psi = (1, 1, 1/2)$, proves statement $\ref{2}$ of the theorem; indeed, as we shall see in Section $\ref{4.1}$ $\kappa (\text{SL}_2(\mathbb{Z})) = 7/8$. We also remark that since this is an equidistribution argument, it does not matter if we replace the half-closed interval $[-1/2, 1/2]$ by the closed interval $[-1/2, 1/2]$.

Recall that $\theta_v$ denotes the angle from $w_v$ to $v$ (anticlockwise), hence the $N$-component of $\gamma_v$ is given by

$$\pi_N(\gamma_v) = \frac{x_v a + y_v b}{a^2 + b^2} = \frac{\langle w_v, v \rangle}{\|v\|^2} = \frac{\|w_v\| \cos(\theta_v)}{\|v\|}.$$  \hfill (2.1)

Since

$$1 = \det \begin{bmatrix} x_v & y_v \\ a & b \end{bmatrix} = \det \begin{bmatrix} w_v \\ v \end{bmatrix} = \|w_v\| \|v\| |\sin(\theta_v)|,$$

and $\|w_v\| \geq 1$, it follows that $|\sin(\theta_v)| = O(\|v\|^{-1})$, or equivalently $1 - |\cos(\theta_v)| = O(\|v\|^{-2})$. From part $\ref{1}$ of Corollary $\ref{1.2}$ applied to $\Psi = [-1/2, 1/2]$, we have that for primitive vectors $v$
in $\mathcal{S}_\Theta$, the $N$-components of $\gamma_v$ become uniformly equidistributed in $[-1/2,1/2]$ as $\|v\| \to \infty$, at rate $O\left(||v||^{-1/4} \cdot \log \|v\|\right)$. Clearly, this means that their absolute values become uniformly equidistributed in $[0,1/2]$ at the same rate. These absolute values are

$$|\pi_N(\gamma_v)| = \frac{\|w_v\|}{\|v\|} \cdot |\cos(\theta_v)| = \frac{\|w_v\|}{\|v\|} \left(1 + O\left(||v||^{-2}\right)\right) = \frac{\|w_v\|}{\|v\|} + O\left(||v||^{-2}\right);$$

thus, the values $\|w_v\| / \|v\|$ are also uniformly equidistributed in $[0,1/2]$ at rate $O\left(||v||^{-1/4} \cdot \log \|v\|\right)$ (since $\|v\|^{-2} < ||v||^{-1/4} \cdot \log \|v\|$), which proves part 1.

By the computation 2.1 of the $N$-component of $\gamma_v$, the vector $v$ is positive if and only if $\cos \theta_v \geq 0$, i.e. if and only if $\pi_N(\gamma_v) \geq 0$. Alternatively, $v$ is negative if and only if $\pi_N(\gamma_v) \leq 0$.

Thus, by applying Corollary 1.2 to $\Psi = [0,1/2]$, we obtain part 2 for the positive vectors, and similarly when $\Psi = [-1/2,0]$, we obtain part 3 for the negative vectors.

We now restrict attention to positive vectors $v$, where $\theta_v$ is acute and therefore $\theta_v \approx \sin \theta_v = O\left(||v||^{-1}\right)$. In particular, the direction of the vector $w_v$ approaches the direction of $v$ at rate $O\left(||v||^{-1}\right)$. Since the directions of the positive primitive vectors $v$ are uniformly equidistributed in $\Theta$ with rate $O\left(||v||^{-1/4} \cdot \log \|v\|\right)$ (by part 4), then so do the directions of the vectors $w_v$.

Consider the negative vectors $v$, where $\theta_v$ is obtuse and therefore $\pi - \theta_v \approx \sin \theta_v = O\left(||v||^{-1}\right)$. Now the direction of the vector $w_v$ approaches the direction of $-v$ at rate $O\left(||v||^{-1}\right)$. Since the directions of the negative primitive vectors $v$ are uniformly equidistributed in $\Theta$ with rate $O\left(||v||^{-1/4} \cdot \log \|v\|\right)$, the directions of their negatives $-v$ become uniformly equidistributed in $-\Theta$ at the same rate; it follows that the directions of the vectors $w_v$ become uniformly equidistributed in $-\Theta$ with rate $O\left(||v||^{-1/4} \cdot \log \|v\|\right)$, which concludes the proof of part 3.

Theorem 2.1 extends to rings of integers in imaginary quadratic number fields as follows. Let $d$ be a positive square free integer, and let $\mathcal{O}_d$ denote the ring of integers in the quadratic number field $\mathbb{Q}\left[\sqrt{-d}\right]$. The ring $\mathcal{O}_d$ is a lattice in $\mathbb{C}$, and has a fundamental parallelogram

$$\mathcal{P}_d = \left\{ z : \frac{1}{2} \leq \Re(z) < \frac{1}{2}, \quad -\sqrt{\text{Disc}(d)} / 4 \leq \Im(z) < \sqrt{\text{Disc}(d)} / 4 \right\}$$

(e.g. [7]), where Disc($d$) is the discriminant of $\mathbb{Q}\left[\sqrt{-d}\right]$. The rectangle $\mathcal{P}_d$ is symmetric w.r.t. the origin, hence all its vertices have the same norm, which we denote by $\rho_d$. We let $\nu_d$ denote the probability measure on $[0,\rho_d]$ which is the distribution of the norm of a random point in $\mathcal{P}_d$. Note that $\nu_d$ is not Lebesgue measure. We refer to $v = (\alpha, \beta)$ in $\mathcal{O}_d^2$ as primitive if the ideals $\langle \alpha \rangle$ and $\langle \beta \rangle$ are co-prime; namely, if there exists a solution $(\xi, \eta)$ in $\mathcal{O}_d^2$ to $\alpha \xi - \beta \eta = 1$. 

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Figure 3: $v$, $w_v$ and $\theta_v$. This Figure also depicts the lines $W_m = \{ w : \det (w^T) = m \}$ for $m \in \mathbb{Z}$, where $W_0 = \text{span} \{ v \}$ and $w_v$ is the shortest integral vector in $W_1$.

**Theorem 2.3.** Let $\Theta \subseteq S^3$ be a spherical cap in the unit sphere, and let $S_\Theta$ be the corresponding sector of $\mathbb{R}^4 \cong \mathbb{C}^2$. For every primitive vector $v = (\alpha, \beta) \in O_d^2$, let $w_v$ denote the shortest vector that completes $v$ to a basis of $O_d^2$, namely, the shortest $O_d$-integral solution to the equation $\alpha \xi - \beta \eta = 1$. For $v \in S_\Theta$, $\|v\| \to \infty$:

1. The values $v / \|v\|$ become effectively equidistributed in $\Theta$;
2. The ratios $\|w_v\| / \|v\|$ of the length of the shortest $O_d$-integral solution relative to the length of $v$ become effectively equidistributed in $[0, \rho_d]$ with respect to $\nu_d$.

In all the above effective equidistribution statements, the rate of convergence is $O \left( \|v\|^{4(1-\kappa_d)} \cdot \log \|v\| \right)$, where $\kappa_d$ is the exponent that corresponds to the lattice $\text{PSL}_2(O_d)$ of $\text{PSL}_2(\mathbb{C})$ in Theorem 1.1.

Observe that when $O_d$ is a euclidean domain, i.e. when $d \in \{1, 2, 3, 7, 11\}$, $\alpha$ and $\beta$ are coprime and the equation $\alpha \xi - \beta \eta = 1$ is their gcd equation. Thus, $w_v$ is the shortest solution to the gcd equation defined by $v$, as in the case of $\mathbb{Z}$ which was discussed in Theorem 2.1.

All the arguments in the proof of Theorem 2.1 carry through to the proof of Theorem 2.3, where this time Corollary 1.2 is applied for the Iwasawa components of the lattice $\text{SL}_2(O_d)$ in $\text{SL}_2(\mathbb{C})$. We briefly describe the necessary adjustments.
Proof. Recall the Iwasawa decomposition of $\text{SL}_2(\mathbb{C})$ consists of the subgroups:

- $N = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} : z \in \mathbb{C} \right\}$
- $A = \left\{ \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} : t \in \mathbb{R} \right\}$
- $K = \left\{ \begin{bmatrix} b & -\pi \\ a & b \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\} = \text{SU}(2)$.

Clearly, $K$ is isomorphic to the unit sphere $S^3$ in $\mathbb{C}^2$.

A primitive pair $(\alpha, \beta) \in O_d^2$ can be completed to a matrix $\begin{bmatrix} \xi & \eta \\ \alpha & \beta \end{bmatrix}$ in $\text{SL}_2(O_d)$, and the Iwasawa coordinates of such a matrix are

$$\begin{bmatrix} \xi & \eta \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} 1 & \frac{\xi \pi + \eta \beta}{\|\alpha\|^2 + \|\beta\|^2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\|\alpha\|^2 + \|\beta\|^2} \\ 0 \end{bmatrix}^{-1} \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}.$$  

The $A$ and $K$ components encode the vector $v$: the $A$-component encodes its norm, and the $K$-component encodes its projection to the sphere $S^3$. The $N$-component corresponds to the upper row: if $w = (\xi, \eta)$, this component equals $\langle w, v \rangle / \|v\|^2$. The set of solutions to $\beta \xi - \alpha \eta = 1$ is $\{ (\xi + m\beta, \eta + m\alpha) : m \in O_d \}$, and the matrices in $\text{SL}_2(O_d)$ that correspond to these solutions differ only by their $N$-components; these components are

$$\begin{bmatrix} 1 & m + \frac{\xi \pi + \eta \beta}{\|\alpha\|^2 + \|\beta\|^2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m + \frac{(w, v)}{\|v\|^2} \\ 1 \end{bmatrix},$$

where $m \in O_d$. By the same Pythagorean argument that was used in the real case (Theorem 2.1), the shortest $O_d$-integral solution $w_v = (\xi_v, \eta_v)$ to $\beta \xi - \alpha \eta = \text{det}(w_v) = 1$ corresponds to the matrix $\gamma_v \in \text{SL}_2(O_d)$ whose $N$-coordinate is minimal.

Clearly, $\{ m + \frac{(w, v)}{\|v\|^2} : m \in O_d \}$ is a coset of the lattice $O_d$ in $\mathbb{C}$, hence there is a unique element from this coset in every $O_d$-integral translation of the fundamental domain $P_d$. The representative which is of minimal norm is the one that lies in $P_d$ itself. Thus, $\gamma_v$ is the unique matrix in $\text{SL}_2(O_d)$, among the matrices that correspond to $v$, whose $N$-component lies in $P_d$.

Let $s(v)$ and $c(v)$ be such that

$$1 = \det \left( \begin{bmatrix} \xi_v & \eta_v \\ \alpha & \beta \end{bmatrix} \right) = \det \left( \begin{bmatrix} w_v \\ v \end{bmatrix} \right) = \|w_v\| \|v\| \cdot s(v),$$

and

$$\langle w_v, v \rangle = \|w_v\| \|v\| \cdot c(v).$$
(these are the analogs for \( \sin (\theta_v) \) and \( \cos (\theta_v) \) from the proof of Theorem 2.3). It can be verified that
\[
|s(v)|^2 + |c(v)|^2 = 1,
\]
and in particular, when \( s(v) \) is small, \( |s(v)|^2 \approx 1 - |c(v)| \). Since \( s(v) = O \left( \|v\|^{-1} \right) \), we have \( 1 - |c(v)| = O \left( \|v\|^{-2} \right) \), and now the proof proceeds analogously to the one of Theorem 2.3 by applying Corollary 1.2 to \( \Gamma = \text{SL}_2(\mathcal{O}_d) \), \( e^{T/2} \geq \|v\| \), \( \Psi = \mathcal{P}_d \) and \( \Phi = \Theta \).

### 2.2 Counting and equidistribution of Iwasawa coordinates: history of the problem

**From the gcd equation in \( \mathbb{Z}^2 \) to equidistribution of real parts of lattice orbits.** The problem of analyzing the distribution of the shortest solution to the gcd equation in \( \mathbb{Z}^2 \) was considered by Dinaburg and Sinai [5] who measured the size of the shortest solution by the maximum norm, and used the theory of continued fractions. It was subsequently noted by Risager and Rudnick [18] that when the size of the smallest solution is measured using the Euclidean norm, equidistribution of shortest solutions is equivalent to the problem of equidistribution of real parts of the points in the orbit of \( i \) under \( \text{SL}_2(\mathbb{Z}) \) in the upper half plane, and the latter result has already been established by Good [10]. Truelsen [20] has established, using estimates of exponential sums, a quantitative form for the equidistribution of real parts for any lattice with a standard cusp in \( \text{SL}_2(\mathbb{R}) \). In particular this establishes a rate of convergence in the equidistribution of shortest solutions of the gcd equation, and this rate is slightly improved upon in Theorem 2.1.

**Counting above fixed intervals in the upper half plane.** Truelsen’s result is based on establishing the existence of the right number of points in the orbit of \( \Gamma \) inside \( N_I A_{[-T,0]} \cdot i \) for any interval \( I \) contained in \([-\frac{1}{2}, \frac{1}{2}]\) up to an error of lower order. We note that establishing equidistribution depends on solving the lattice point counting problem associated with **every** interval \( I \). In the specific case of \( I = [-\frac{1}{2}, \frac{1}{2}] \), better error terms were established by Good for general non-cocompact lattices in \( \text{SL}_2(\mathbb{R}) \), and by Chamizo [1] for the specific lattice orbit \( \text{SL}_2(\mathbb{Z}) \cdot i \). Observe that this particular case is equivalent to the **primitive circle problem**. Chamizo, and later on Truelsen, have established some further lattice point counting results for a variety of other families in the upper half plane.

**Lifts of horospheres in hyperbolic space.** Eskin and McMullen [8] have raised the problem of counting the number of lifts of a closed horosphere \( \mathcal{H} \) in \( G/\Gamma \) which intersect a ball of radius \( T \) in hyperbolic space, and have established the main term for this counting problem. As we shall see below, in the case of the hyperbolic space, the problem amounts to counting the points of a non-cocompact lattice lying in the sets \( R_T (\Psi, K) \), where \( \Psi \subset \mathbb{R}^{n-1} \) is the fundamental domain of \( \Gamma \cap N \). The problem can be formulated for an arbitrary symmetric space, and the main term
of the asymptotics has been established by [16]. Further work on the subject has been recently carried out in [1] and [19].

**Local statistics of the Iwasawa N-component.** Marklof and Vinogradov [15] have considered, among other things, the projection of lattice orbit points to a neighborhood of a horizontal horosphere tending to the boundary, namely the sets given by \( R_T(\Psi, K) \setminus R_{T-c}(\Psi, K) \). They have analyzed the local statistics of the Iwasawa N-components in \( \Psi \) as \( T \to \infty \); this problem is more delicate than just the equidistribution of the N-component, and was established in real hyperbolic space of any dimension, but not in an effective form.

**Contribution of the current paper.** We note that the counting and equidistribution results in the present paper are effective, namely include an error estimate. In the case of the lattice \( \text{SL}_2(\mathbb{Z}) \), for which quantitative results have been established in [20], our error estimate reduces from the previous \( O\left(e^{\frac{7}{8}}T + \epsilon\right) \) for every \( \epsilon > 0 \), to \( O\left(Te^{\frac{7}{8}}\right) \). The method that we utilize uses only the size of the spectral gap in the automorphic representation, and avoids the detailed spectral analysis of eigenfunctions of the Laplacian, and any use of Kloosterman sums which appeared in previous arguments. This fact is what allows an easy generalization to any dimension and any group of real rank one, and it is also responsible for elimination of the \( \epsilon \) in the error exponent, reducing the main spectral estimate to known estimates of the Harish-Chandra \( \Xi \) function. Our consideration of lattice points and equidistribution of their Iwasawa components in the group itself, rather than a lattice orbit in the symmetric space, allows us to obtain equidistribution of the \( K \)-components of the elements of a lattice, in addition to their \( N \)-components. This fact, along with our consideration of dimensions greater than 2, enables us to extend the results of Risager and Rudnick to include angular equidistribution, and rings of integers in \( \mathbb{C} \).

### 3 Extensions and applications

Recall that \( 2\rho \) is the exponent that appears in the Haar measure of \( G \) when given in Iwasawa coordinates.

**3.1 Difference of domains of the form \( R_T \)**

Theorem [1.1] can be extended to include the case where the horosphere that delimits the domains \( R_T(\Psi, \Phi) \) from above is not fixed at height 1. We let \( 0 \leq S \leq T \) and consider the domains \( \Psi A_{[-T,-S]} \Phi \), whose volume equals \( \frac{1}{2\rho} \cdot \mu_N(\Psi) \mu_K(\Phi) (e^{2\rho T} - e^{2\rho S}) \). Note that when \( S \) is increasing as a function of \( T \), e.g. when \( S = \alpha T \) with \( 0 < \alpha < 1 \), the domains in the family \( \{\Psi A_{[-T,-S(T)]} \Phi\}_{T>0} \) may not be contained in one another.
Corollary 3.1. Let $0 \leq S \leq T$. For any lattice $\Gamma < G$ and $\Psi, \Phi, \kappa$ as in Theorem 1.1,

$$
\# \left( \Psi A_{[-T,-S]} \Phi \cap \Gamma \right) = \frac{\mu(\Psi A_{[-T,-S]} \Phi)}{\mu(G/\Gamma)} + O (\log(\mu(R_T(\Psi, \Phi))) \cdot \mu(R_T(\Psi, \Phi))^{\kappa})
$$

$$
= \frac{\mu_N(\Psi) \mu_K(\Phi)}{\mu(G/\Gamma)} \cdot \frac{e^{2\rho T} - e^{2\rho S}}{2\rho} + O \left( T \left( e^{2\rho T} \right)^{\kappa} \right).
$$

Note that the error term does not depend on $S$, and that when $S, T$ are such that $e^{2\rho T} - e^{2\rho S} = O \left( T \left( e^{2\rho T} \right)^{\kappa} \right)$, i.e. when the interval $[-T,-S]$ is of length $O \left( e^{-2\rho(1-\kappa)T} \right)$, the main term may be smaller than the error term, so this result is only an upper bound.

Proof. Let $\Gamma < G$ and $\kappa = \kappa(\Gamma)$ as in Theorem 1.1. According to this theorem, the function

$$
E(T) := \# (R_T(\Psi, \Phi) \cap \Gamma) - \mu(R_T(\Psi, \Phi))
$$

is bounded by $O \left( T e^{2\rho T \kappa} \right)$. Namely, there exist $C,T_0 > 0$ such that

$$
|E(T)| \leq C T e^{2\rho T \kappa} \text{ for all } T \geq T_0.
$$

It follows that there exists $C' > 0$ such that

$$
|E(T)| \leq C' T e^{2\rho T \kappa} \text{ for all } T \geq 0.
$$

In particular, for $0 \leq S \leq T$

$$
|E(S)| \leq C' S e^{2\rho S \kappa} \leq C' T e^{2\rho T \kappa},
$$

and therefore $E(S) = O \left( T e^{2\rho T \kappa} \right)$. Now,

$$
\# \left( \Psi A_{[-T,-S]} \Phi \cap \Gamma \right) = \# (R_T(\Psi, \Phi) \cap \Gamma) - \# (R_S(\Psi, \Phi) \cap \Gamma)
$$

$$
= \frac{\mu(R_T(\Psi, \Phi))}{\mu(G/\Gamma)} + E(T) - \left( \frac{\mu(R_S(\Psi, \Phi))}{\mu(G/\Gamma)} + E(S) \right)
$$

$$
= \frac{\mu(R_T(\Psi, \Phi))}{\mu(G/\Gamma)} - \frac{\mu(R_S(\Psi, \Phi))}{\mu(G/\Gamma)} + O \left( T e^{2\rho T \kappa} \right)
$$

$$
= \frac{\mu(\Psi A_{[-T,-S]} \Phi)}{\mu(G/\Gamma)} + O \left( \log(\mu(R_T(\Psi, \Phi))) \cdot \mu(R_T(\Psi, \Phi))^{\kappa} \right).
$$

\qed
3.2 Lifts of horospheres

Let $\Gamma$ be a non-co-compact lattice in $G$, with a cusp at the point $\sigma$ at the boundary of the associated hyperbolic space. Let $H_\sigma$ be the unipotent subgroup in $G$ which stabilizes $\sigma$ (in particular, it is conjugated to $N$). We consider the case in which $\Gamma \cap H_\sigma$ is a lattice in $H_\sigma$. Let $H$ be a horosphere in the hyperbolic space of $G$ which is based at $\sigma$; in other words, $H$ is an orbit of $H_\sigma$. Observe that $H$ projects to a closed horosphere $\bar{H}$ in the space $\Gamma \backslash G$. Let $B_T(z)$ denote a hyperbolic ball of radius $T$ that is centered at $z$, and let $N(T)$ denote the number of horospheres of the form $\gamma H, \gamma \in \Gamma$, that meet the ball $B_T(z)$. Eskin and McMullen [8, Theorem 7.2] have considered the counting function $N(T)$ and discussed the case of $G = \text{PSL}_2(\mathbb{R})$. This problem can be formulated for a Lie group of any real rank, see [16]; we will provide an effective estimate for real rank 1.

**Theorem 3.2.** Let $\Gamma < G$ a non co-compact lattice, and let $\sigma$, $H_\sigma$, $H$ as above. If $\Gamma \cap H_\sigma$ is a lattice in $H_\sigma$, then

$$N(T) = \frac{\text{Vol}_{\Gamma \backslash G} (\bar{H})}{\mu(\Gamma \backslash G)} \cdot \frac{e^{2\rho T}}{2\rho} + O \left( T \left( e^{2\rho T} \right)^\kappa \right),$$

where $\kappa = \kappa(\Gamma)$ is the exponent associated with $\Gamma$.

**Proof.** By conjugation, we may assume that $z = i$ (the point stabilized by $K$) and that $\sigma = \infty$ (the point stabilized by $N$), namely $H_\sigma = N$. Then $H$ is a horizontal horosphere, i.e. it is orthogonal to the geodesic $A \cdot i$, and we may write $H = Na_y \cdot i$ for some $y \in \mathbb{R}$. Then the number of horospheres $\gamma H$ that meet the ball $B_T(i)$ are in one to one correspondence with the elements of the following set:

$$\{\gamma N : d(i, \gamma H) < T\} = \{\gamma N : d(i, \gamma Na_y \cdot i) < T\}.$$

We write the elements of $\Gamma$ in their $KAN$ coordinates, and denote $\gamma = k_\gamma a_t(\gamma) n_\gamma$.

$$= \{\gamma N : d(i, k_\gamma a_t(\gamma) n_\gamma Na_y \cdot i) < T\} = \{\gamma N : d(i, a_t(\gamma) Na_y \cdot i) < T\} = \{\gamma N : d(i, a_t(\gamma) Na_y \cdot i) < T\} = \{\gamma N : d(i, Na_y \cdot i) < T\} = \{\gamma N : d(i, a_t(\gamma) Na_y \cdot i) < T\},$$

since the horosphere $Na_y + t(\gamma) \cdot i$ is orthogonal to the geodesic $A \cdot i$, thus the point nearest to $i$ on this horosphere is its meeting point with the geodesic, $a_y + t(\gamma) \cdot i$.

Now, $d(i, a_t(\gamma) Na_y \cdot i) = |t(\gamma) + y|$, so $d(i, a_t(\gamma) Na_y \cdot i) < T$ if and only if $-T - y \leq t(\gamma) \leq T - y$. Moreover, the cosets $\gamma N$ are in one to one correspondence with the lattice elements $\gamma = k_\gamma a_t(\gamma) n_\gamma$ such that $n_\gamma \in \Psi(\Gamma)$, for a choice $\Psi(\Gamma)$ of a fundamental domain for $\Gamma \cap N$ in $N$. Then,

$$N(T) = \# \{ \gamma = k_\gamma a_t(\gamma) n_\gamma : n_\gamma \in \Psi(\Gamma), -T - y \leq t(\gamma) \leq T - y\} = \# \Gamma \cap (KA_{[-T-y, T-y]} \Psi(\Gamma)) .$$
Now the desired result follows from Theorem 1.1 and Remark 1.3:

\[ N(T) = \frac{\mu_N(\Psi(\Gamma))}{\mu(G/\Gamma)} \cdot \frac{e^{2\rho(T-y)}}{2\rho} + O\left((T-y)\left(e^{2\rho(T-y)}\right)^\kappa\right) \]

\[ = \frac{\mu_N(\Psi(\Gamma))}{\mu(G/\Gamma)} \cdot e^{-2\rho y} \cdot \frac{e^{2\rho T}}{2\rho} + O\left(T\left(e^{2\rho(T-y)}\right)^\kappa\right) \]

\[ = \frac{\text{Vol}_{\Gamma \backslash G}(\mathcal{H})}{\mu(\Gamma \backslash G)} \cdot \frac{e^{2\rho T}}{2\rho} + O\left(T\left(e^{2\rho T}\right)^\kappa\right). \]

3.3 Diophantine equation associated with the Lorentz form

When \( G = \text{SO}^0(1,n), \text{SU}(1,n), \text{or SP}(1,n) \) (not just locally isomorphic to it), then the elements of the subgroups \( A \) and \( N \) of \( G \) can be written explicitly as

\[
a_t = \begin{pmatrix}
\cosh t & 0 & \sinh t \\
0 & I_{n-2} & 0 \\
\sinh t & 0 & \cosh t
\end{pmatrix}
\]  

(3.1)

and

\[
n_{v,z} = \begin{pmatrix}
1 + z + \frac{1}{2} \|v\|^2 & v^* & -z - \frac{1}{2} \|v\|^2 \\
v^* & I_{n-2} & -v \\
z + \frac{1}{2} \|v\|^2 & v^* & 1 - z - \frac{1}{2} \|v\|^2
\end{pmatrix}
\]  

(3.2)

(e.g. [9, p.373 and p.375]).

The explicit \( N \) and \( A \) components of a given \( g \in G \) are extracted in the following claim.

**Claim 3.3.** Let

\[
g = \begin{pmatrix}
g_{0,0} & \cdots & g_{0,n} \\
\vdots & \ddots & \vdots \\
g_{n,0} & \cdots & g_{n,n}
\end{pmatrix} \in G.
\]

If \( g = n_{v,z} a_t k \), then

\[
e^t = (g_{0,0} - g_{n,0})^{-1}
\]

\[
v = \frac{1}{g_{0,0} - g_{n,0}} \begin{pmatrix}
g_{1,0} \\
\vdots \\
g_{n-1,0}
\end{pmatrix}
\]

\[
z = \frac{1}{2} \left( \frac{g_{0,0} + g_{n,0}}{g_{0,0} - g_{n,0}} - \frac{1 + \sum_{j=1}^{n-1} |g_{j,0}|^2}{(g_{0,0} - g_{n,0})^2} \right).
\]
Proof. On the one hand,
\[
g \cdot i = \begin{pmatrix} g_{0,0} & \cdots & g_{0,n} \\ \vdots & \ddots & \vdots \\ g_{n,0} & \cdots & g_{n,n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} g_{0,0} \\ \vdots \\ g_{n,0} \end{pmatrix}.
\]

On the other hand,
\[
g \cdot i = n_{v,z} a_t k \cdot i = n_{v,z} a_t \cdot i,
\]
where
\[
n_{v,z} a_t \cdot i = \begin{pmatrix} 1 + z + \frac{1}{2} \|v\|^2 & v^* & -z - \frac{1}{2} \|v\|^2 \\ v & I_{n-2} & -v \\ z + \frac{1}{2} \|v\|^2 & v^* & 1 - z - \frac{1}{2} \|v\|^2 \end{pmatrix} \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-2} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]
\[
= \begin{pmatrix} \cosh t + e^{-t} \left( z + \frac{1}{2} \|v\|^2 \right) \\ e^{-t} \cdot v \\ \sinh t + e^{-t} \left( z + \frac{1}{2} \|v\|^2 \right) \end{pmatrix}.
\]

Namely,
\[
\begin{pmatrix} g_{0,0} \\ \vdots \\ g_{n,0} \end{pmatrix} = \begin{pmatrix} \cosh t + e^{-t} \left( z + \frac{1}{2} \|v\|^2 \right) \\ e^{-t} \cdot v \\ \sinh t + e^{-t} \left( z + \frac{1}{2} \|v\|^2 \right) \end{pmatrix}.
\]

In particular,
\[
g_{0,0} - g_{n,0} = \cosh t - \sinh t = e^{-t},
\]
and
\[
\begin{pmatrix} g_{1,0} \\ \vdots \\ g_{n-1,0} \end{pmatrix} = e^{-t} \cdot v.
\]

Clearly \( e^t \) and \( v \) may be extracted from the above, and \( z \) can be extracted from:
\[
g_{0,0} + g_{n,0} = \cosh t + \sinh t + 2e^{-t} \left( z + \frac{1}{2} \|v\|^2 \right)
\]
\[
= e^t + 2e^{-t} \left( z + \frac{1}{2} \|v\|^2 \right).
\]
\[\square\]
Let us focus on the case \( G = \text{SO}^0(1, n) \). Clearly, if \( g = (g_{i,j})_{0 \leq i,j \leq n} \) is in \( \text{SO}^0(1, n) \), then 
\[ g_{0,0}^2 - g_{1,0}^2 - \cdots - g_{n,0}^2 = 1, \]
namely the first column of \( g \) satisfies the equation 
\[ x_0^2 - x_1^2 - \cdots - x_n^2 = 1. \quad (3.3) \]

By Claim 3.3 above, the \( N \) and \( A \) components of \( g \) depend only on the first column of \( g \); hence, Corollary 1.2 concerning the equidistribution of the \( N \)-components as the \( A \)-components approach \( \infty \), can be used to study the behavior of the corresponding parameters of equation (3.3).

For every \( x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \) that satisfies equation (3.3) define the height function 
\[ h(x) = \log \left( \frac{1}{x_0-x_n} \right) \] (corresponds to the \( A \)-component) and the vector 
\[ v(x) = \frac{1}{x_0-x_n} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \] \( \in \mathbb{R}^{n-1} \) (corresponds to the \( N \)-component). Assume \( \Psi \subset \mathbb{R}^{n+1} \) is nice. By applying Corollary 1.2 (part 1) to the lattice \( \text{SO}^0(1, n) \mathbb{Z} \) in \( \text{SO}^0(1, n) \), we conclude:

**Corollary 3.4.** Consider the integral solutions \( x \) for \( x_0^2 - x_1^2 - \cdots - x_n^2 = 1 \). The rational vectors \( v(x) \) become effectively equidistributed in \( \Psi \) as \( -h(x) \to \infty \), at rate \( O \left( e^{-2h(x)(1-\gamma)} \cdot h(x) \right) \).

In the case where the lattice \( \text{SO}^0(1, n) \mathbb{Z} \) is non-cocompact, i.e. when the Lorentz form is isotropic \([14]\), the equidistribution also occurs when \( |h(x)| \to \infty \).

In analogy with the discussion in Section 2 we may consider equidistribution of rational vectors \( v(x) \) that correspond to shortest integral representatives to cosets of \( N \mathbb{Z} \). Consider the discrete subgroup of \( \mathbb{R}^{n-1} \): \( \Lambda = \{ (x_1, \ldots, x_{n-1}) \in \mathbb{Z}^{n-1} : \sum x_i \in 2\mathbb{Z} \} \). Then \( N \mathbb{Z} := N \cap \text{SO}^0(1, n) \mathbb{Z} \) satisfies \( N \mathbb{Z} = \{ n_v : v \in \Lambda \} \) by Formula 3.2, and is therefore isomorphic to \( \Lambda \). One possible choice for a fundamental domain for \( \Lambda \) in \( \mathbb{R}^{n-1} \) is the unit ball with respect to the \( \| \cdot \|_1 \) norm, \( \Psi_0 := \{ (x_1, \ldots, x_{n-1}) : \sum |x_i| \leq 1 \} \). This fundamental domain has the property that it contains the shortest (with respect to the 2-norm) representative of every coset of \( \Lambda \).

For every \( h \in \mathbb{R} \), let \( X_h \) denote the set of solutions with height \( h \); for every \( h \) such that \( X_h \cap \mathbb{Z}^{n+1} \neq \emptyset \), let \( x_h \) denote the unique integral solution for which \( v(x_h) \) is the shortest, namely lies in \( \Psi_0 \). By applying Corollary 1.2 to the lattice \( \text{SO}^0(1, n) \mathbb{Z} \) in \( \text{SO}^0(1, n) \), \( \Psi = \Psi_0 \) and \( \Phi = K \), we conclude:

**Corollary 3.5.** The rational vectors \( v(x_h) \) associated with shortest solutions, become effectively equidistributed (at rate as above) in \( \Psi_0 \) as \( |h| \to \infty \).

Compare to part 1 of Theorem 2.1. Of course, this can be done for any Lorentz form defined over \( \mathbb{Q} \).

## 4 Proof of the main theorem

### 4.1 A spectral method for counting lattice points

In the following discussion, \( G \) is an almost simple Lie group, not necessarily of rank 1. The lattice point counting method in family of domains \( \{ \mathcal{B}_T \} \subset G \) that we will use \([12], [11] \) has
two ingredients: a spectral estimate and a regularity property. The crucial spectral estimate requires bounding the norm of the averaging operators defined by $B_T$ in the representation on $L^2_0(\Gamma \backslash G)$. Let us recall the fact that there exists $m \in \mathbb{N}$ such that the unitary representation of $G$ in $L^2_0(\Gamma \backslash G)$, when taken to the $m$-th tensor power, is weakly contained in the regular representation of $G$. The essential property of such $m$ is that $m \geq p/2$, where $p$ satisfies that the $K$-finite matrix coefficients of $\pi^0_{\Gamma \backslash G}$ are in $L^{p+\epsilon}(G)$ for every $\epsilon > 0$. We define $m(\Gamma)$ to be the least even integer with this property if $p > 2$, or 1 if $p = 2$ (see [12, Definition 3.1]). One of the remarkable features of harmonic analysis on simple Lie groups is that then for any measurable set of positive finite measure $B$ in $G$, if we denote by $\beta$ the Haar uniform measure on $B_T$, the following estimate holds [17]:

$$\|\pi^0_{\Gamma \backslash G}(\beta)\| \leq C_{G,\epsilon} \cdot m_G(B)^{-1/2m(\Gamma)+\epsilon}.$$  (4.1)

Thus, $m(\Gamma)$ measures the size of the spectral gap in $L^2(\Gamma \backslash G)$. The lattice $\Gamma$ is called tempered if the representation $\pi^0_{\Gamma \backslash G}$ is already weakly contained in regular representation, namely if $m(\Gamma) = 1$.

We now turn to the second ingredient, which is the Lipschitz property of the domains $B_T$.

**Definition 4.1 ([12]).** Let $G$ be a Lie group with Haar measure $m_G$. Assume $\{B_T\} \subset G$ is a family of bounded domains of positive-measure such that $m_G(B_T) \rightarrow \infty$ as $T \rightarrow \infty$. Let $O_\epsilon \subset G$ be the image of a ball of radius $\epsilon$ in the Lie algebra under the exponential map. Denote

$$B_T^+(\epsilon) := O_\epsilon B_T O_\epsilon = \bigcup_{u,v \in O_\epsilon} u B_T v,$$

$$B_T^-(\epsilon) := \bigcap_{u,v \in O_\epsilon} u B_T v$$

(Figure 4). The family $\{B_T\}$ is Lipschitz well-rounded if there exist $\epsilon_0 > 0$ and $T_0 \geq 0$ such that for every $0 < \epsilon \leq \epsilon_0$ and $T \geq T_0$:

$$m_G(B_T^+(\epsilon)) \leq (1 + C\epsilon) m_G(B_T^-(\epsilon)),$$

where $C > 0$ is a constant that does not depend on $\epsilon$ or $T$.

The concept of well-roundedness appeared first in [6], and later formulated in [8]. It has also been used in [13]. The conditions in Definition 4.1 generalize those that occurred in the aforementioned papers.

**Theorem 4.2 ([12]).** Let $G$ be an almost simple Lie group with Haar measure $m_G$, and let $\Gamma < G$ be a lattice. Assume $\{B_T\} \subset G$ is a family of finite-measure domains which satisfies $m_G(B_T) \rightarrow \infty$ as $T \rightarrow \infty$. If the family $\{B_T\}$ is Lipschitz well-rounded, then

$$\# (B_T \cap \Gamma) = \frac{1}{m_G(G/\Gamma)m_G(B_T)} + O \left( m_G(B_T) \cdot E(T)^{\frac{1}{2+\text{dim}(G)}} \right).$$
as $T \to \infty$, where $m_G(\Gamma) \geq 0$ is the measure of a fundamental domain of $\Gamma$ in $G$, and $E(T)$ is (a bound on) the rate of decay of operator norm $\|\pi_{\Gamma \setminus G}^0(\beta_T)\|$. 

Note that the above theorem applies to every lattice $\Gamma$.

When plugging the estimation $4.1$ for $\|\pi_{\Gamma \setminus G}^0(\beta_T)\|$, the obtained error term in Theorem 4.2 is:

$$O\left(m_G(B_T)^{\kappa(\Gamma)+\epsilon}\right)$$

for every $\epsilon > 0$

where

$$\kappa(\Gamma) = 1 - \frac{1}{2m(\Gamma)\left(1 + \dim(G)\right)} \in (0, 1). \tag{4.2}$$

In our case, where $G$ is of real rank one and the family of domains is $R_T(\Psi, \Phi)$, the estimation $4.1$ may be improved so that the error term is reduced to

$$O\left((\log(m_G(R_T(\Psi, \Phi))) \cdot m_G(R_T(\Psi, \Phi)))^{\kappa(\Gamma)}\right),$$

as we now explain. Assume that a set $B \subset G$ of positive finite measure satisfies that

$$\mu(K \cdot B \cdot K) \leq \text{const} \cdot \mu(B);$$

this property is called $K$-radializability ([11, Def. 3.21]). When $B$ is radializable, then it is a consequence of the spectral transfer principle [17] and of estimates on the Harish-Chandra function in real rank one that

$$\|\pi_{\Gamma \setminus G}^0(\beta)\| \leq C_G \cdot (\log(\mu(B)))^{\frac{1}{m(\Gamma)}} \cdot (\mu(B))^{-\frac{1}{2m(\Gamma)}}.$$

([11] Prop. 5.9]). The sets $R_T(\Psi, \Phi)$ are indeed radializable, with constant that does not depend on $T$. Thus, if $\beta_T$ are the probability measures that corresponds to $R_T = R_T(\Psi, \Phi)$, then

$$E(T) = \|\pi_{\Gamma \setminus G}^0(\beta_T)\| \leq C_G \cdot (\log(\mu(R_T))) \cdot (\mu(R_T))^{-\frac{1}{2m(\Gamma)}},$$
As claimed.

From the above discussion it follows that in order to prove Theorem 1.1, it suffices to show that the family \( \{ R_T (\Psi, \Phi) \} \) is Lipschitz well rounded.

### 4.2 Lipschitz property for Iwasawa coordinates in the negative direction of \( A \)

In order to show that the family \( R_T (\Psi, \Phi) \) is Lipschitz well-rounded, it will be convenient to introduce coordinates on \( N \) as well, in addition to the parametrization we have already set for \( A \); recall \( A = \{ a_t : t \in \mathbb{R} \} \) such that \( d (a_t \cdot i, a_s \cdot i) = |t - s| \). Let \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \} \) be the “field” over which the matrices in \( G \) are defined, and \( n \) the dimension (over \( \mathbb{K} \)) of the corresponding hyperbolic space. The group \( N \) is of Heisenberg type (see \( [2], [3] \)), and in particular it is parametrized by the space \( \mathbb{K}^n \oplus \mathbb{I} (\mathbb{K}) \), where \( \mathbb{I} (\mathbb{K}) \) is the subspace of “pure imaginary” numbers in \( \mathbb{K} \), namely of elements \( w \) such that \( w + \bar{w} = 0 \). A parametrization may be chosen such that

\[
N = \{ n_{v,z} : v \in \mathbb{K}^n, z \in \mathbb{I} (\mathbb{K}) \},
\]

with the group multiplication

\[
n_{v_1,z_1} n_{v_2,z_2} = n_{v_1+v_2,z_1+z_2+\langle v_2,v_1 \rangle}
\]

(where \( \langle v_2,v_1 \rangle = v_1^* v_2 \)). The subspaces \( \mathbb{K}^n \) and \( \mathbb{I} (\mathbb{K}) \) correspond to subsets of \( N \) that are invariant under conjugation by \( A \), and specifically,

\[
a_t n_{v,z} a_{-t} = n_{e^{i2t}v,e^{2t}z}.
\]

As a result, if \( p := \dim_{\mathbb{R}} (\mathbb{K}^n) \) and \( q := \dim_{\mathbb{R}} (\mathbb{I} (\mathbb{K})) = \dim_{\mathbb{R}} (\mathbb{K}) - 1 \), then \( \mu_N \) is the Lebesgue measure on \( \mathbb{R}^{p+q} \), and the parameter \( \rho \) that appears in Formula 1.2 for the Haar measure equals \( \frac{1}{2} (p + 2q) \).

Let \( \mathcal{N} \) denote the opposite unipotent group, namely the one that corresponds to the negative roots:

\[
a_t \mathcal{N}_{v,z} a_{-t} = n_{e^{-i2t}v,e^{-2t}z}.
\]
On the subgroups $H \in \{A,K\}$ we consider the metric $d_H$ induced by the Riemannian metric on $G$. We denote by $K_{(\phi,\delta)}$ a ball in $K$ with center $\phi \in K$ and radius $\delta$, and by $A_{(t,\delta)}$ a ball in $A$, with center $t$ and radius $\delta$ (these are simply the elements that correspond to the interval $(t-\delta,t+\delta)$, since $d_A$ is the euclidean metric on $\mathbb{R}$). We let $d_N$ denote the product of euclidean metrics on $\mathbb{R}^n \cong \mathbb{R}^p$ and $\mathbb{R}^q$, and let $N_{(v,\delta_1)\times(z,\delta_2)}$ be the domain in $N$ parametrized by the product of euclidean balls in $\mathbb{R}^n \cong \mathbb{R}^p$ and $\mathbb{R}^q$ with centers $v, z$ and radii $\delta_1, \delta_2$ respectively. When a ball is centered at the identity we omit the center and denote $K_{\delta}$, $A_{\delta}$, and $N_{(\delta_1)\times(\delta_2)}$.

In what follows, $\|\cdot\|_{ck}$ is the Cartan-Killing norm on the Lie algebra $\text{Lie}(G)$ of $G$, and $\|\cdot\|_{op}$ is the norm on the space of linear operators on $\text{Lie}(G)$.

**Lemma 4.3.** Let $G$ be a semi-simple linear Lie group. Let $B_\epsilon = \{X \in \text{Lie}(G) : \|X\| \leq \epsilon\}$, and let $\mathcal{O}_\epsilon = \exp(B_\epsilon)$. For every $g \in G$,

$$g^{-1}\mathcal{O}_\epsilon g \subseteq \mathcal{O}_\epsilon \|\text{Ad}g\|_{op} = \exp\left\{X \in \text{Lie}(G) : \|X\|_{ck} \leq \epsilon \cdot \|\text{Ad}g\|_{op}\right\}.\,$$

**Proof.** Recall

$$\|\text{Ad}g\|_{op} = \max_{X \in B_1} \|\text{Ad}g(X)\|_{ck} = \max_{X \in B_1} \|g^{-1}Xg\|_{ck}.\,$$

Observe that $\text{Ad}g(B_\epsilon) \subseteq \text{Lie}(G)$ is contained in a ball of radius $\epsilon$.

$$\max_{X \in B_\epsilon} \|\text{Ad}g(X)\|_{ck} = \max_{X \in B_\epsilon} \|g^{-1}Xg\|_{ck} = \max_{X \in B_1} \|g^{-1}Xg\|_{ck} = \epsilon \|\text{Ad}g\|_{op}.\,$$

Now,

$$g^{-1}\mathcal{O}_\epsilon g = g^{-1} \exp(B_\epsilon) g = \exp(g^{-1} B_\epsilon g) = \exp(\text{Ad}g(B_\epsilon))\,$$

$$\subseteq \exp\left(B_\epsilon \|\text{Ad}g\|_{op}\right) = \mathcal{O}_\epsilon \|\text{Ad}g\|_{op}.\,$$

Let $M$ denote the centralizer of $A$ in $K$. We will use the following: there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, there are positive constants $c_1,c_2$ such that

$$\mathcal{O}_\delta \subseteq N_{(c_1\delta)\times(c_1\delta)}A_{(c_1\delta)}K_{(c_1\delta)},\,$$

$$\mathcal{O}_\delta \subseteq N_{(c_2\delta)\times(c_2\delta)}A_{(c_2\delta)}M_{(c_2\delta)}N_{(c_2\delta)\times(c_2\delta)}\,$$

(4.6)

(the latter are the Bruhat coordinates on a neighborhood of the identity in $G$).
Proposition 4.4 (Effective Iwasawa decomposition). Let \( n_{v,z} \in N, \phi \in K, a_t \in A \) with \( t \leq 0 \). There exists \( \epsilon_1 > 0 \) such that for every \( 0 < \epsilon \leq \epsilon_1 \) there are positive constants \( C'_{N}, C''_{N}, C_A, C_K \) that depend only on \( n_{v,z} \) and \( \phi \) (in particular, independent of \( t \)) such that

\[
\mathcal{O}_\epsilon \cdot \mathcal{O}_\epsilon \subseteq N_{(v,c_1\epsilon)} \times (z,c_2\epsilon) A^{(t,c_3\epsilon)} K_{(\phi,c_4\epsilon)}.
\]

Furthermore, when \( n_{v,z} \) varies over a compact set \( \Psi \), and \( \phi \) varies over \( K \), these constants can be taken to be uniform.

Proof. Observe that

\[
N_{(\delta_1) \times (\delta_2)} N_{(\rho_1) \times (\rho_2)} \subseteq N_{(\delta_1 + \rho_1) \times (\delta_2 + \rho_2 + \rho_1 \delta_1)}
\]

and

\[
n_{v,z} N_{(\rho_1) \times (\rho_2)} \subseteq N_{(v,\rho_2 + \|v\| \rho_1)}.
\]  \hspace{1cm} (4.7)

In particular,

\[
n_{v,z} N_{(\delta_1) \times (\delta_2)} N_{(\rho_1) \times (\rho_2)} \subseteq n_{v,z} N_{(\delta_1 + \rho_1) \times (\delta_2 + \rho_2 + \rho_1 \delta_1)}
\]

\[
\subseteq N_{(v,\delta_1 + \rho_1) \times (z,\delta_2 + \rho_2 + \rho_1 \delta_1 + \|v\| (\delta_1 + \rho_1))} \hspace{1cm} (4.8)
\]

Finally, note that

\[
K_{(\delta)} \phi \subseteq K_{(\phi,\delta)}.
\]  \hspace{1cm} (4.9)

Step 1: Right perturbations. We show that

\[
n_{v,z} a_t \phi \cdot \mathcal{O}_\epsilon \subseteq N_{(v,r_1) \times (z,r_2)} A^{(t,r_3)} K_{(\phi,r_4)}
\]

where \( r_i = r_i (v, z) \) is independent of \( t \leq 0 \). Recall \( \|\text{Ad} \phi\| = 1 \). By Lemma 4.3

\[
n_{v,z} a_t \phi \cdot \mathcal{O}_\epsilon \subseteq n_{v,z} a_t \mathcal{O}_\epsilon \phi.
\]

By 4.5

\[
\subseteq n_{v,z} a_t \left( N_{(c_1\epsilon)} \times (c_1\epsilon) A^{(c_1\epsilon)} K_{(c_1\epsilon)} \right) \phi.
\]

By 4.3

\[
\subseteq n_{v,z} N_{(e^t c_1 \epsilon)} \times (e^{2t c_1 \epsilon}) a_t A^{(c_1\epsilon)} K_{(c_1\epsilon)} \phi,
\]

and by 4.7 and 4.9

\[
\subseteq N_{(v,c_1 \epsilon) \times (z,c_1 \epsilon^{2t} + c_1 \|v\| e^t \epsilon)} A^{(t,c_1 \epsilon)} K_{(\phi,c_1 \epsilon)}.
\]

Since \( e^t \leq 1 \),

\[
\subseteq N_{(v,c_1 \epsilon) \times (z,c_1 \epsilon + c_1 \|v\| \epsilon)} A^{(t,c_1 \epsilon)} K_{(\phi,c_1 \epsilon)}.
\]
Step 2: Left perturbations. We show that
\[ \mathcal{O}_\epsilon \cdot n_{v,z} a_t \phi \subseteq N_{(v,\ell_1) \times (z,\ell_2\epsilon)} A(t,\ell_3\epsilon) K(\phi,\ell_4\epsilon) \]
where \( \ell_i = \ell_i(v, z) \) is independent of \( t \leq 0 \).
Denote \( \eta = \|\text{Ad} n_{v,z}\|_{\text{op}} \). By Lemma 4.3,
\[ \mathcal{O}_\epsilon \cdot n_{v,z} a_t \phi \subseteq n_{v,z} \mathcal{O}_\eta a_t \phi. \]
Set \( \epsilon_1 = \min \{1, \delta_0/\eta\} \). Then for \( \epsilon \leq \epsilon_1 \), Lemma 4.6 implies
\[ \subseteq n_{v,z} \left( N_{(c_2\eta)e} A(c_2\eta) M(c_2\eta) \overline{N}_{(c_2\eta)e} \right) a_t \phi \]
by 4.4.
\[ \subseteq n_{v,z} N_{(c_2\eta)e} A(c_2\eta) a_t M(c_2\eta) \overline{N}_{(c_2\eta)e} \phi. \]
Since \( M \overline{N}_{\delta_1, \delta_2} \subseteq K \mathcal{O}_{\max\{\delta_1, \delta_2\}} = \mathcal{O}_{\max\{\delta_1, \delta_2\}} \) and \( \epsilon' \geq \epsilon^{2t} \),
\[ \subseteq n_{v,z} N_{(c_2\eta)e} A(c_2\eta) a_t \left( \mathcal{O}_{c_2\eta} \right) \phi. \]
By 4.5
\[ \subseteq n_{v,z} N_{(c_2\eta)e} A(t, c_2\eta) \left( N_{(c_2\eta)\epsilon} A(c_2\eta) K(c_2\eta) \right) \phi 
\]
and by 4.3 and 4.9,
\[ \subseteq n_{v,z} N_{(c_2\eta)e} N_{(c_2\eta)\epsilon} \left( A(t, c_2\eta) A(c_2\eta) K(c_2\eta) \right) \phi. \]
Since \( \epsilon' \leq 1 \) and \( \epsilon \leq \epsilon_1 \leq 1, \)
\[ \subseteq N_{(v, (1+c_1\epsilon^2+c_2\eta)e)} A(t, (1+c_1\epsilon^2+c_2\eta) K(\phi, c_1\epsilon^2+c_2\eta). \]
Step 3: Combining left and right perturbations. Let \( g := n_{v,z} a_t \phi \) with \( t \leq 0 \) and let \( \epsilon \leq \epsilon_1 \). Choose uniform (independent of \( t \)) constants \( \overline{\ell}_i = \max \{ \ell_i(v', z') : n_{v', z'} \in \pi_N (g \cdot \mathcal{O}_1) \} \).
Since \( g \cdot \mathcal{O}_\epsilon \subset g \cdot \mathcal{O}_1 \), it follows from Step 2 that for every
\[ g_0 = n_{v_0, z_0} a_t \phi_0 \in g \cdot \mathcal{O}_\epsilon, \]
it holds that
\[ \mathcal{O}_\epsilon \cdot g_0 \subset N_{(v_0, \overline{\ell}_1 \epsilon) \times (z_0, \overline{\ell}_2 \epsilon)} A(t_0, \overline{\ell}_3 \epsilon) K(\phi_0, \overline{\ell}_4 \epsilon). \]
But, as was shown in Step 1, \( d_N (v_0, v) \leq r_1 \epsilon, d_N (z_0, z) \leq r_2 \epsilon, d_A (t_0, t) \leq r_3 \epsilon \) and \( d_K (\phi_0, \phi) \leq r_4 \epsilon \). Then by the triangle inequality,
\[ \mathcal{O}_\epsilon \cdot g \cdot \mathcal{O}_\epsilon \subset N_{(v, r_1 \epsilon + \overline{\ell}_1 \epsilon) \times (z, r_2 \epsilon + \overline{\ell}_2 \epsilon)} A(t, r_3 \epsilon + \overline{\ell}_3 \epsilon) K(\phi, r_4 \epsilon + \overline{\ell}_4 \epsilon). \]
4.3 Lipschitz-Regularity of the domains $R_T (\Psi, \Phi)$

Recall that we wish to show that the family $\{ R_T (\Psi, \Phi) \}_{T > 0}$ is Lipschitz well-rounded (Definition 4.1). Since we have already established the Lipschitz property for the Iwasawa coordinates in the negative direction of $A$, all that remains is to bound the quotient of the measures of $R_T (\Psi, \Phi)^+ (\epsilon)$ and $R_T (\Psi, \Phi)^- (\epsilon)$, which we perform below.

Proof of Theorem 1.1 Throughout this proof, it will be convenient to parametrize $N$ as $\mathbb{R}^{p+q}$ instead of $\mathbb{R}^p \oplus \mathbb{R}^q$. We will write $n_x$ instead of $n_{\nu,x}$, and $N(x, \delta)$ for a ball of radius $\delta$ centered at $x$.

For convenience, let us denote $\mu_A = \frac{dt}{e^{2\pi t}}$. Then $\mu = \mu_N \times \mu_A \times \mu_K$ and therefore it is sufficient to show that there exist $\epsilon_0 , T_0 > 0$ such that for every $H \in \{ N, A, K \}$ there exists a positive constant $c_H$ satisfying

$$\frac{\mu_H (\pi_H (R_T (\Psi, \Phi)^+ (\epsilon)))}{\mu_H (\pi_H (R_T (\Psi, \Phi)^- (\epsilon)))} \leq 1 + c_H \epsilon$$

for every $0 < \epsilon \leq \epsilon_0 , T \geq T_0$. Alternatively,

$$\frac{\mu_H (\pi_H (R_T (\Psi, \Phi)^+ (\epsilon))) - \mu_H (\pi_H (R_T (\Psi, \Phi)^- (\epsilon)))}{\mu_H (\pi_H (R_T (\Psi, \Phi)^- (\epsilon)))} \leq c_H \epsilon$$

for every $0 < \epsilon \leq \epsilon_0 , T \geq T_0$. Since this is a property of the measures of $\pi_H (R_T (\Psi, \Phi)^\pm (\epsilon))$, we may assume that the nice sets $\Psi$ and $\Phi$ are compact.

Recall that for every $H \in \{ N, A, K \}$, $\xi \in H$ and $\delta > 0$, $H(\xi, \delta)$ denotes the (closed) ball of radius $\delta$ centered at $\xi$ w.r.t. the metric $d_H$ on $H$. We let $H^0(\xi, \delta)$ denote the corresponding open ball. By Proposition 4.4 there exist positive constants $C_N, C_A, C_K$ that depend on $\Psi$ and $\Phi$ alone such that for every $x \in \Psi$, $\phi \in \Phi$, $0 < \epsilon \leq \epsilon_1$ and $t < 0$,

$$O_\epsilon \cdot n_x a_t k_\phi \cdot O_\epsilon \subset N(\xi, C_N \epsilon) A(\xi, t a_t) K(\phi, C_K \epsilon).$$

It follows that for every $H \in \{ N, A, K \}$ and the corresponding $\Xi \in \{ \Psi, [-T, 0], \Phi \}$ in $H$,

$$\pi_H (R_T (\Psi, \Phi)^+ (\epsilon)) \subseteq \bigcup_{\xi \in \Xi} H(\xi, C_H \epsilon)$$

and

$$\pi_H (R_T (\Psi, \Phi)^- (\epsilon)) \supseteq \bigcup_{\xi \in \Xi} H^0(\xi, C_H \epsilon) \setminus \bigcup_{\xi \in \Xi} H^0(\xi, C_H \epsilon).$$

Note that the set on the right-hand side of (4.10) is the union of all $C_H \epsilon$-balls that are centered at a point in $\pi_H (R_T (\Psi, \Phi))$, where the set on the right-hand side of (4.11) is the set of points whose (closed) $C_H \epsilon$-ball is fully contained in $\pi_H (R_T (\Psi, \Phi))$. (4.10) is obvious; to see (4.11) we note that

$$g \in R_T (\Psi, \Phi)^- (\epsilon) \iff g \in u R_T (\Psi, \Phi)v, \forall u, v \in O_\epsilon$$

$$\iff ugv \in R_T (\Psi, \Phi), \forall u, v \in O_\epsilon$$

$$\iff \pi_H (ugv) \in \pi_H (R_T (\Psi, \Phi)), \forall u, v \in O_\epsilon,$$
since $O_\epsilon = O_\epsilon^{-1}$ and since $R_T (\Psi, \Phi)$ is a product set. But for every $g$ such that $\pi_H (g) \in \bigcup_{\xi \in \Xi} H_{(\xi, C_H \epsilon)}^0 \setminus \bigcup_{\xi \in \partial \Xi} H_{(\xi, C_H \epsilon)}^0$,

$$\pi_H (ugv) \in H_{(\pi_H(g), C_H \epsilon)} \subset \pi_H (R_T (\Psi, \Phi)).$$

Thus every such $g$ is contained in $R_T (\Psi, \Phi)^- (\epsilon)$ by 4.12, and in particular $\pi_H (g) \in \pi_H (R_T (\Psi, \Phi)^- (\epsilon))$ for every $H$.

We begin with the $N$-component. Since $\Psi$ is assumed to be nice, and since an $\epsilon$-ball in $N$ has $\mu_N$-volume which is proportional to $\epsilon^{\dim N}$, there exists a constant $\alpha_1$ which depends on $\partial \Psi$ and $C_N$ such that

$$\mu_N \left( \bigcup_{x \in \partial \Psi} N_{(x, C_N \epsilon)} \right) \leq \alpha_1 \epsilon^{\dim N} \leq \alpha_1 \epsilon.$$

Thus, by 4.10

$$\mu_N \left( \pi_N \left( R_T (\Psi, \Phi)^+ (\epsilon) \right) \right) \leq \mu_N \left( \bigcup_{x \in \Psi} N_{(x, C_N \epsilon)} \right) \leq \mu_N (\Psi) + \mu_N \left( \bigcup_{x \in \partial \Psi} N_{(x, C_N \epsilon)} \right) \leq \mu_N (\Psi) + \alpha_1 \epsilon,$$

and by 4.11

$$\mu_N \left( \pi_N \left( R_T (\Psi, \Phi)^- (\epsilon) \right) \right) \geq \mu_N \left( \bigcup_{x \in \Psi} N_{(x, C_N \epsilon)} \right) - \mu_N \left( \bigcup_{x \in \partial \Psi} N_{(x, C_N \epsilon)}^0 \right)$$

$$= \mu_N \left( \bigcup_{x \in \Psi} N_{(x, C_N \epsilon)} \right) - \mu_N \left( \bigcup_{x \in \partial \Psi} N_{(x, C_N \epsilon)} \right)$$

$$\geq \mu_N (\Psi) - \mu_N \left( \bigcup_{x \in \partial \Psi} N_{(x, C_N \epsilon)} \right)$$

$$\geq \mu_N (\Psi) - \alpha_1 \epsilon.$$

By assuming $\epsilon$ is small enough such that $\alpha_1 \epsilon \leq \frac{1}{2} \mu_N (\Psi)$, the last two inequalities imply

$$\frac{\mu_N \left( \pi_N \left( R_T (\Psi, \Phi)^+ (\epsilon) \right) \right) - \mu_N \left( \pi_N \left( R_T (\Psi, \Phi)^- (\epsilon) \right) \right)}{\mu_N \left( \pi_N \left( R_T (\Psi, \Phi)^- (\epsilon) \right) \right)} \leq \frac{\mu_N (\Psi) + \alpha_1 \epsilon - (\mu_N (\Psi) - \alpha_1 \epsilon)}{\frac{1}{2} \mu_N (\Psi)} = \frac{2 \alpha_1}{\frac{1}{2} \mu_N (\Psi)} \epsilon.$$

The same considerations apply for $\Phi \subseteq K$, since it is also assumed to be nice, and the $\mu_K$-volume of $\epsilon$-balls in $K$ is proportional to $\epsilon^{\dim K}$. Therefore, there exists $\alpha_2 > 0$ that depends on $\partial \Phi$ and $C_K$ such that

$$\mu_K \left( \bigcup_{k \in \partial \Phi} K_{(k, C_K \epsilon)} \right) \leq \alpha_2 \epsilon^{\dim K} \leq \alpha_2 \epsilon,$$

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and, similarly to the $N$ case, by assuming $\alpha_2 \epsilon \leq \frac{1}{2} \mu_N (\Phi)$:

$$\frac{\mu_K (\pi_K (R_T (\Psi, \Phi)^+ (\epsilon))) - \mu_K (\pi_K (R_T (\Psi, \Phi)^- (\epsilon)))}{\mu_K (\pi_K (R_T (\Psi, \Phi)^- (\epsilon)))} \leq \frac{2 \alpha_2}{\frac{1}{2} \mu_K (\Phi)} \cdot \epsilon.$$  

Finally, for the $A$-component, it follows from 4.10 and 4.11 that

$$\pi_A (R_T (\Psi, \Phi)^+ (\epsilon)) \subseteq [-T - C_A \epsilon, 0 + C_A \epsilon]$$

and

$$\pi_A (R_T (\Psi, \Phi)^- (\epsilon)) \supseteq [-T + C_A \epsilon, 0 - C_A \epsilon].$$

Thus,

$$\mu_A (\pi_A (R_T (\Psi, \Phi)^+ (\epsilon))) \leq \int_{t=-T-C_A \epsilon}^{t=0+C_A \epsilon} \frac{dt}{e^{2 \rho t}} = \frac{1}{2 \rho} \left( e^{2 \rho (T+C_A \epsilon)} - e^{-2 \rho C_A \epsilon} \right)$$

and

$$\mu_A (\pi_A (R_T (\Psi, \Phi)^- (\epsilon))) \geq \int_{t=-T+C_A \epsilon}^{t=0-C_A \epsilon} \frac{dt}{e^{2 \rho t}} = \frac{1}{2 \rho} \left( e^{2 \rho (T-C_A \epsilon)} - e^{2 \rho C_A \epsilon} \right).$$

As a result,

$$\frac{\mu_A (\pi_A (R_T (\Psi, \Phi)^+ (\epsilon))) - \mu_A (\pi_A (R_T (\Psi, \Phi)^- (\epsilon)))}{\mu_A (\pi_A (R_T (\Psi, \Phi)^- (\epsilon)))} \leq \frac{e^{2 \rho (T+C_A \epsilon)} - e^{-2 \rho C_A \epsilon} - (e^{2 \rho (T-C_A \epsilon)} - e^{2 \rho C_A \epsilon})}{e^{2 \rho (T-C_A \epsilon)} - e^{2 \rho C_A \epsilon}}$$

$$= \frac{(e^{2 \rho T} + 1)}{e^{2 \rho T}} \cdot \frac{e^{2 \rho C_A \epsilon} - e^{-2 \rho C_A \epsilon}}{e^{-2 \rho C_A \epsilon} - e^{-2 \rho T} e^{2 \rho C_A \epsilon}}.$$

For $\epsilon \leq (4 \rho C_A)^{-1}$ and $T \geq 2 \rho^{-1}$ it holds that $e^{2 \rho C_A \epsilon} - e^{-2 \rho C_A \epsilon} \leq 3 \cdot 2 \rho C_A \epsilon$ and $e^{-2 \rho C_A \epsilon} - e^{-2 \rho T} e^{2 \rho C_A \epsilon} \geq 1/2$; therefore,

$$\frac{\mu_A (\pi_A (R_T (\Psi, \Phi)^+ (\epsilon))) - \mu_A (\pi_A (R_T (\Psi, \Phi)^- (\epsilon)))}{\mu_A (\pi_A (R_T (\Psi, \Phi)^- (\epsilon)))} \leq 2 \cdot \frac{6 \rho C_A \epsilon}{1/2} = 24 \rho C_A \epsilon.$$

By choosing $T_0 = 2 \rho^{-1}$ and $\epsilon_0 = \min \left\{ \epsilon_1, \frac{\mu_N (\Psi)}{2 \alpha_1}, \frac{\mu_N (\Phi)}{2 \alpha_2}, \frac{1}{4 \rho C_A} \right\}$ we conclude that the family $\{ R_T (\Psi, \Phi) \}_{T>0}$ is Lipschitz well-rounded, and by Theorem 4.2 (and the discussion in Section 4.1) we are done.  

\[\square\]

**Proof of Corollary 1.2** Let $\Psi, \Psi', \Phi, \Phi'$ and $\kappa$ as in the statement of the corollary. By Theorem 1.1,

$$\frac{\# (\Gamma \cap R_T (\Psi', \Phi'))}{\# (\Gamma \cap R_T (\Psi, \Phi))} = \frac{\mu_N (\Psi') \mu_K (\Phi') e^{2 \rho T} + O \left( T e^{2 \rho T} \right)}{\mu_N (\Psi) \mu_K (\Phi) e^{2 \rho T} + O \left( T e^{2 \rho T} \right)}$$

$$= \frac{\mu_N (\Psi') \mu_K (\Phi')}{\mu_N (\Psi) \mu_K (\Phi)} + O \left( T (e^{2 \rho T})^{-(1-\kappa)} \right)$$

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The proof of part 2 is analogous.

Let \( \psi \) and \( \phi \) be non-negative compactly supported Lipschitz functions with positive integral, with \( \psi \) supported on \( N \), and \( \phi \) supported on \( K \). Let \( R_T (\psi, \phi) \) be the measure on \( G \) whose density with respect to Haar measure on \( G \) (written in Iwasawa coordinates as in \([1,2]\)) is given by the function \( D_T (na_tk) = \psi(n) \chi_{[-T,0]}(a_t) \phi(k) \). Equivalently, the measure is given by the following formula: for \( F \in C_c (G) \),

\[
R_T (\psi, \phi) (F) = \int_N \int_0^\infty \int_K F(na_tk)\psi(n)\phi(k)\mu_N(n)\frac{dt}{e^{2\rho t}} d\mu_K (k) .
\]

The family of measures \( R_T (\psi, \phi) \) is Lipschitz well-rounded, in the following sense. Defining

\[
D_T^+ (g) = \sup_{u,v \in O_x} D_T (ugv) , \quad D_T^- (g) = \inf_{u,v \in O_x} D_T (ugv)
\]

we have

\[
\int_G D_T^+ (g) d\mu (g) \leq (1 + C\epsilon) \int_G D_T^- (g) d\mu (g) .
\]

Under these assumption, the family \( R_T (\psi, \phi) \) satisfies a weighted version of the lattice point counting result which the sets \( R_T (\Psi, \Phi) \) satisfy, namely

\[
\sum_{\gamma \in \Gamma} D_T (\gamma) = \int_G D_T (g) d\mu (g) + O \left( \left( \int_G D_T (g) d\mu (g) \right)^{\kappa (\Gamma)} \cdot \log \int_G D_T (g) d\mu (g) \right)
\]

so that in the present case

\[
\sum_{\gamma \in \Gamma} \psi (\pi_N (\gamma)) \chi_{[-T,0]} (\pi_A (\gamma)) \phi (\pi_K (\gamma)) =
\]

\[
eq e^{2\rho T} \int_N \psi (n) d\mu_N (n) \cdot \int_K \phi (k) d\mu_K (k) + O \left( T e^{-2\rho T \kappa (\Gamma)} \right) .
\]

The proof of the weighted version of the lattice point problem stated above under the assumption of Lipschitz well-roundedness is a straightforward modification of the arguments that appear in \([1,2]\). The fact that when \( \psi \) and \( \phi \) are Lipschitz functions on \( N \) and \( K \) the measures \( R_T (\psi, \phi) \) defined above are Lipschitz well-rounded is a straightforward modification of the arguments in the present paper. Note that it suffices to consider non-negative Lipschitz functions on \( N \) and \( K \), and the case of general Lipschitz functions follows. Finally, the statement of Corollary \([1,2]\) part \( \text{[1]} \) follows by considering a Lipschitz function \( \psi \) on \( N \) supported in \( \Psi \), fixing a nice subset \( \Phi \subset K \) and letting \( \phi \) be its characteristic function, defining \( D_T \) using \( \psi \) and \( \phi \), and estimating the ratios as follows

\[
\frac{\sum_{\gamma \in \Gamma} D_T (\gamma)}{\sum_{\gamma \in \Gamma} \chi_{RT (\Psi, \Phi)} (\gamma)} = \frac{\int_N \psi (n) d\mu_N (n)}{\mu_N (\Psi)} + O \left( T e^{-(1-\kappa)T} \right)
\]

The proof of part \( \text{[2]} \) is analogous.
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