Asymptotics for the Taylor coefficients of certain infinite products

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Abstract

Let \((m_1, \ldots, m_J)\) and \((r_1, \ldots, r_J)\) be two sequences of \(J\) positive integers satisfying \(1 \leq r_j < m_j\) for all \(j = 1, \ldots, J\). Let \((\delta_1, \ldots, \delta_J)\) be a sequence of \(J\) nonzero integers. In this paper, we study the asymptotic behavior of the Taylor coefficients of the infinite product

\[
\prod_{j=1}^{J} \left( \prod_{k \geq 1} \left( 1 - q^{r_j + m_j(k-1)}(1 - q^{-r_j + m_j k}) \right) \right)^{\delta_j}.
\]

Our work generalizes many known results, including an asymptotic formula due to Lehner for the partition function arising from the first Rogers–Ramanujan identity. The main technique used here is based on Rademacher’s circle method.

Keywords Infinite product · Taylor coefficient · Asymptotics · Circle method

Mathematics Subject Classification 11P55

1 Introduction

1.1 Motivations

For complex variables \(\alpha\) and \(q\) with \(|q| < 1\), we denote

\[
(\alpha; q)_n := \prod_{k=0}^{n-1} (1 - \alpha q^k) \quad \text{and} \quad (\alpha; q)_\infty := \prod_{k \geq 0} (1 - \alpha q^k).
\]

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We also use the notation
\[(\alpha, \beta, \ldots, \gamma; q)_\infty := (\alpha; q)_{\infty} (\beta; q)_{\infty} \cdots (\gamma; q)_{\infty}.\]

Let \(p(n)\) be the number of partitions of \(n\); that is, the number of representations of \(n\) written as a sum of a nonincreasing sequence of positive integers. It is well known that \(p(n)\) has generating function
\[
\sum_{n \geq 0} p(n)q^n = \frac{1}{(q; q)_\infty}.
\]
The asymptotic behavior of \(p(n)\) was first studied by Hardy and Ramanujan [9]. A couple of decades later, Rademacher [21] further proved the following formula
\[
p(n) = \frac{1}{2\sqrt{2\pi}} \sum_{k \geq 1} A_k(n) \sqrt{k} \frac{d}{dn} \left( \frac{2}{\sqrt{n - \frac{1}{24}}} \sinh \left( \frac{\pi}{k} \sqrt{\frac{2}{3} \left( n - \frac{1}{24} \right)} \right) \right),
\]
where
\[
A_k(n) = \sum_{0 < h < k \quad \gcd(h,k)=1} e^{\pi i (s(h,k) - 2nh/k)}
\]
with \(s(h, k)\) being the Dedekind sum defined in (1.5).

Apart from ordinary partitions, partitions under symmetrical congruence conditions also attract broad research interest. Here, quoting the definition of Grosswald [7], a set \(\mathcal{S}\) of distinct least positive residues modulo some fixed \(m\) is called symmetrical, if \(m - r\) is in \(\mathcal{S}\) whenever \(r\) is an element of \(\mathcal{S}\). We further say that a partition is under a symmetrical congruence condition if all parts in this partition are congruent to some \(r\) modulo \(m\) where \(r\) is in a symmetrical set with respect to the modulus \(m\). The most famous examples arise from the Rogers–Ramanujan identities (Rogers [25], Ramanujan [23]). Here the first Rogers–Ramanujan identity states that (cf. [3, Corollary 7.67])
\[
\frac{1}{(q, q^4; q^5)_\infty} = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n}.
\]
Using the language in partition theory, the above identity can be restated as follows. The number of partitions of \(n\) such that each part is congruent to \(\pm 1\) modulo 5 equals the number of partitions of \(n\) such that the adjacent parts differ by at least two. Let \(p_{5, \pm 1}(n)\) be the number of partitions of \(n\) such that each part is congruent to \(\pm 1\) modulo 5. Its asymptotic formula was shown by Lehner [15]:
\[
p_{5, \pm 1}(n) \sim \frac{\csc(\pi/5)}{4 \cdot 3^{1/4} \cdot 5^{1/4}} n^{-3/4} \exp \left( 2\pi \sqrt{\frac{n}{15}} \right).
\]
The interested reader may also refer to Niven [18], Livingood [16], Petersson [19, 20], Grosswald [7], Iseki [12–14], Hagis Jr. [8], Subrahmanyasastri [26] and so forth for the asymptotic behaviors of other partition functions under symmetrical congruence conditions.

As we have seen, the generating function of $p_{5, \pm 1}(n)$ is indeed an infinite product under a symmetrical congruence condition. Further, similar infinite products are also of number-theoretic interest. One example is the Rogers–Ramanujan continued fraction defined by

$$R(q) := \frac{q^{1/5}}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}} = \frac{q^{1/5} (q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty}.$$ 

Let us focus on the infinite product part in $R(q)$ and write

$$\sum_{n \geq 0} C(n) q^n = \frac{(q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty}.$$ 

It is known from Richmond and Szekeres [24] that

$$C(n) \sim 2^{1/2} 5^{3/4} \cos \left( \frac{4\pi}{5} \left( n + \frac{3}{20} \right) \right) n^{-3/4} \exp \left( \frac{4\pi}{5} \sqrt{\frac{n}{5}} \right). \quad (1.2)$$

Hence for sufficiently large $n$, $C(5n + 0, 2) > 0$ and $C(5n + 1, 3, 4) < 0$. We also remark that in [24], Richmond and Szekeres indeed studied the asymptotic behavior of the Taylor coefficients of the general infinite product

$$\prod_{j=1}^{m-1} (q^j; q^m)_{-x(j)},$$

where $m$ is a positive fundamental discriminant, $x(j) = (m \mid j)$ is the Kronecker symbol and $x$ is either 1 or $-1$.

Recently, there are a number of papers [2, 4, 11, 17, 27] studying vanishing Taylor coefficients of certain infinite products. For instance, Tang [27] showed that the Taylor coefficients of

$$\sum_{n \geq 0} B(n) q^n = (-q^2, -q^3; q^5)^2 (q^2, q^8; q^{10})_\infty = \frac{(q^2, q^8; q^{10})_\infty}{(q^2, q^3; q^5)_\infty} \frac{(q^4; q^6; q^{10})_\infty^2}{(q^2, q^3; q^5)_\infty^2}$$

satisfy $B(5n + 1) = 0$ for all $n \geq 0$. At the end of Tang’s paper, he also provided numerical evidence of the inequalities $B(5n + 0, 2, 3) > 0$ and $B(5n + 4) < 0$ for sufficiently large $n$. Similar numerical evidences are also provided for inequalities of Taylor coefficients of other infinite products.
Motivated by these work, it is natural to investigate a broad family of infinite products. Let $m = (m_1, \ldots, m_J)$ and $r = (r_1, \ldots, r_J)$ be two sequences of $J$ positive integers satisfying $1 \leq r_j < m_j$ for all $j = 1, \ldots, J$. Let $\delta = (\delta_1, \ldots, \delta_J)$ be a sequence of $J$ nonzero integers. In this paper, we shall study the asymptotics for the Taylor coefficients of the following infinite product

$$
\sum_{n \geq 0} g(n)q^n = \prod_{j=1}^{J} (q^{r_j}, q^{m_j-r_j}, q^{m_j})_{\infty}^{\delta_j}.
$$

### 1.2 Notation and main result

Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{H}$ be the upper half complex plane. Let gcd and lcm be the greatest common divisor function and least common multiple function, respectively. For a positive integer $n$, we accept the convention that $\gcd(0, n) = n$.

We define the big-$O$ notation as usual: $f(x) = O(g(x))$ means that $|f(x)| \leq C g(x)$ where $C$ is an absolute constant. Furthermore, $f(x) \ll g(x)$ means that $f(x) = O(g(x))$. Throughout this paper, we always assume that the constant $C$ depends on $m$, $r$ and $\delta$ unless otherwise stated.

Below we assume that $0 \leq h < k$ are integers such that $\gcd(h, k) = 1$. Also, $m$ is a positive integer. Let us define auxiliary functions

$$
\lambda_{m,r}(h, k) := \left\lceil \frac{rh}{\gcd(m, k)} \right\rceil
$$

and

$$
\lambda_{m,r}^*(h, k) := \lambda_{m,r}(h, k) - \frac{rh}{\gcd(m, k)}.
$$

We denote by $h_m(h, k)$ an integer such that

$$
h_m(h, k) \cdot \frac{mh}{\gcd(m, k)} \equiv -1 \left( \mod \frac{k}{\gcd(m, k)} \right).
$$

Notice that one may always find such an integer since $\gcd(h, k) = 1$ and $m$ is nonzero.

Next, we define

$$
\Omega := \sum_{j=1}^{J} \delta_j \left( 2m_j - 12r_j + \frac{12r_j^2}{m_j} \right),
$$

$$
\Delta(h, k) := -\sum_{j=1}^{J} \delta_j \left( \frac{2 \gcd^2(m_j, k)}{m_j} \right.
$$

$$
+ \left. \frac{12 \gcd^2(m_j, k)}{m_j} (\lambda_{m,j,r_j}^*(h, k) - \lambda_{m,j,r_j}^*(h, k)) \right).
$$
and

$$\omega_{h,k} := \exp \left( -\pi i \sum_{j=1}^{J} \delta_j \cdot s \left( \frac{m_j h}{\gcd(m_j, k)}, \frac{k}{\gcd(m_j, k)} \right) \right), \tag{1.4}$$

where $s(d, c)$ is the Dedekind sum defined by

$$s(d, c) := \sum_{n \mod c} \left( \left\langle \frac{dn}{c} \right\rangle \left\langle \frac{n}{c} \right\rangle \right) \tag{1.5}$$

with

$$\langle x \rangle := \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

We also define

$$\mathcal{D}_{h,k} := \exp \left( \pi i \sum_{j=1}^{J} \delta_j \left( \frac{r_j h}{k} - \frac{r_j \gcd(m_j, k)}{m_j} + \frac{2r_j \gcd(m_j, k)\lambda^*_{m_j,r_j}(h, k)}{m_j} \right) + \frac{h_{m_j}(h, k) \gcd(m_j, k)}{k} \left( \lambda^2_{m_j,r_j}(h, k) - \lambda_{m_j,r_j}(h, k) \right) \right).$$

One readily verifies that the choice of $\hbar_{m}(h, k)$ does not affect the value of $\mathcal{D}_{h,k}$. At last, we define

$$\Pi_{h,k} := \prod_{j: \lambda^*_{m_j,r_j}(h,k)=0} \left( 1 - \exp \left( 2\pi i \frac{r_j \gcd(m_j, k) + r_j h_{m_j}(h, k)m_jh}{m_jk} \right) \right)^{\delta_j}$$

if there exists $j$ such that $\lambda^*_{m_j,r_j}(h,k) = 0$ and $\Pi_{h,k} := 1$ otherwise. Remark 6.1 tells us that the choice of $\hbar_{m}(h, k)$ also does not affect the value of $\Pi_{h,k}$. Also, Proposition 6.3 indicates that for any $j$ with $\lambda^*_{m_j,r_j}(h, k) = 0$, we have

$$1 - \exp \left( 2\pi i \frac{r_j \gcd(m_j, k) + r_j h_{m_j}(h, k)m_jh}{m_jk} \right) \neq 0.$$ 

Hence the value $\Pi_{h,k}$ is well defined and $\Pi_{h,k} \neq 0$.

Given a real $0 \leq x < 1$, we define

$$\Upsilon(x) := \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } 0 < x \leq 1/2, \\ 1 - x & \text{if } 1/2 < x < 1. \end{cases}$$

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Let $L = \text{lcm}(m_1, \ldots, m_J)$. We define two disjoint sets:

\[
\mathcal{L}_{>0} := \{(\nu, \ell) : 1 \leq \ell \leq L, 0 \leq \nu < \ell, \ \Delta(\nu, \ell) > 0\}, \\
\mathcal{L}_{\leq0} := \{(\nu, \ell) : 1 \leq \ell \leq L, 0 \leq \nu < \ell, \ \Delta(\nu, \ell) \leq 0\}.
\]

Our main result is stated as follows.

**Theorem 1.1** If the inequality

\[
\min_{1 \leq j \leq J} \left( \nu \left( \lambda_{m_j, r_j}(\nu, \ell) \right) \frac{\gcd^2(m_j, \ell)}{m_j} \right) \geq \frac{\Delta(\nu, \ell)}{24} \tag{1.6}
\]

holds for all $1 \leq \ell \leq L$ and $0 \leq \nu < \ell$, then for positive integers $n > -\Omega/24$, we have

\[
g(n) = E(n) + 2\pi i \sum_{j=1}^{J} \delta_j \sum_{1 \leq \ell \leq L} \sum_{0 \leq \nu < \ell} \left( \frac{24n + \Omega}{\Delta(\nu, \ell)} \right)^{-\frac{1}{2}}
\]

\[
\times \left\{ \sum_{1 \leq k \leq N^*} \frac{1}{k} I_{-1} \left( \frac{\pi}{6k} \sqrt{\Delta(\nu, \ell)(24n + \Omega)} \right) \\
\times \left( \sum_{0 \leq h < k} \frac{\omega_{h,k}^2}{\gcd(h,k)} e^{-2\pi i nh} (-1)^{\sum_{j=1}^{J} \delta_j \lambda_{m_j, r_j}(h,k) \omega_{h,k}} D_{h,k} \ell \right) \right\}., \tag{1.7}
\]

where

\[
N^* = \left\lfloor \frac{2\pi}{\sqrt{2\pi} \left( n + \frac{\Omega}{24} \right)} \right\rfloor, \tag{1.8}
\]

$I_s(x)$ is the modified Bessel function of the first kind, and

\[
E(n) \ll_{m,r,\delta} 1. \tag{1.9}
\]

**Remark 1.1** To better understand the asymptotic behavior of $g(n)$, one may apply the asymptotic expansion of $I_s(x)$ (cf. [1, p. 377, (9.7.1)]): for fixed $s$, when $|\arg x| < \pi/2$,

\[
I_s(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left( 1 - \frac{4s^2 - 1}{8x} + \frac{(4s^2 - 1)(4s^2 - 9)}{2!(8x)^2} - \cdots \right). \tag{1.10}
\]
2 Applications of the main result

Before moving to the proof of the main result, we first give some applications. In the first two examples, we reproduce the asymptotic formulas (1.1) and (1.2), respectively. We then confirm Tang’s inequalities in [27] in the asymptotic sense. In this section, we always expand the infinite product as \( \sum_{n \geq 0} g(n) q^n \).

In general, to obtain an explicit asymptotic formula of \( g(n) \), we first compute \( L > 0 \). Next, we find the largest number among \( \sqrt{\Delta(\kappa, \ell)} / k \) with \( (\kappa, \ell) \in L > 0 \) and \( k \equiv \ell \pmod{L} \). Now we need to check if the corresponding \( I \)-Bessel function vanishes for this choice. If it is nonvanishing, then the asymptotic formula shall be obtained from this \( I \)-Bessel term. Otherwise, we find the second largest number among \( \sqrt{\Delta(\kappa, \ell)} / k \) and carry out the same procedure. Notice that if there are multiple choices of \( \kappa, \ell \) and \( k \) giving the same value of \( \sqrt{\Delta(\kappa, \ell)} / k \), we should sum up all such \( I \)-Bessel terms and check if the summation vanishes or not.

2.1 Partitions into parts congruent to \( \pm 1 \) modulo 5

Let

\[
\sum_{n \geq 0} g(n) q^n = \frac{1}{(q, q^4; q^5)_{\infty}}.
\] (2.1)

Then \( m = \{5\}, r = \{1\} \) and \( \delta = \{-1\} \). Hence \( L = 5 \) and \( \Omega = -2/5 \). We now compute that

\[
L_{>0} = \{(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3),
(0, 4), (1, 4), (2, 4), (3, 4), (0, 5), (1, 5), (4, 5)\}.
\]

First, we verify that assumption (1.6) is satisfied. We next find that the largest number among \( \sqrt{\Delta(\kappa, \ell)} / k \) with \( (\kappa, \ell) \in L_{>0} \) and \( k \equiv \ell \pmod{L} \) is \( \sqrt{2/5} \). Here we have two choices:

\[
(\kappa, \ell, k) = (0, 1, 1), (0, 5, 5).
\]

When \( k = 1 \), the admissible \((h, k)\) is \((0, 1)\). We compute that the \( I \)-Bessel term is

\[
\frac{\pi \csc \left( \frac{\pi}{5} \right)}{2 \sqrt{15}} \left( n - \frac{1}{60} \right)^{-1/2} I_{-1} \left( \frac{2\pi}{\sqrt{15}} \sqrt{n - \frac{1}{60}} \right).
\]

When \( k = 5 \), noticing that \( \gcd(0, 5) = 5 \neq 1 \), there is no admissible \((h, k)\). Hence,

\[
g(n) \sim \frac{\pi \csc \left( \frac{\pi}{5} \right)}{2 \sqrt{15}} \left( n - \frac{1}{60} \right)^{-1/2} I_{-1} \left( \frac{2\pi}{\sqrt{15}} \sqrt{n - \frac{1}{60}} \right)
\]

\[
\sim \frac{\csc \left( \frac{\pi}{5} \right)}{4 \cdot 3^{1/4} \cdot 5^{1/4}} n^{-3/4} \exp \left( \frac{2\pi}{\sqrt{15}} \sqrt{n} \right).
\]
2.2 The Rogers–Ramanujan continued fraction

Let

\[ \sum_{n \geq 0} g(n)q^n = \frac{(q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty}. \]  

(2.2)

Then \( m = \{5, 5\}, \ r = \{1, 2\} \) and \( \delta = \{1, -1\} \). Hence \( L = 5 \) and \( \Omega = 24/5 \). We compute that

\[ L > 0 = \{(2, 5), (3, 5)\}. \]

First, we verify that assumption (1.6) is satisfied. We next find that the largest number among \( \{\sqrt{\Delta(\tau, \ell)/k}\} \) with \( (\tau, \ell) \in L > 0 \) and \( k \equiv \ell \text{ (mod } L) \) is \( \frac{2\sqrt{6}}{5\sqrt{5}} \). Here we have two choices:

\[ (\tau, \ell, k) = (2, 5, 5), (3, 5, 5). \]

When \( k = 5 \), the admissible \( (h, k) \) are \( (2, 5) \) and \( (3, 5) \). We compute that, in total, the \( I \)-Bessel term is

\[ 4\pi \sqrt{5} \cos \left( \frac{4\pi}{5} \left( n + \frac{3}{20} \right) \right) \left( n + \frac{1}{5} \right)^{-1/2} I_{-1} \left( \frac{4\pi}{5\sqrt{5}} \sqrt{n + \frac{1}{5}} \right). \]

Notice that \( \cos \left( \frac{4\pi}{5} (n + \frac{3}{20}) \right) \) does not vanish for all \( n \). Hence,

\[ g(n) \sim 4\pi \sqrt{5} \cos \left( \frac{4\pi}{5} \left( n + \frac{3}{20} \right) \right) \left( n + \frac{1}{5} \right)^{-1/2} I_{-1} \left( \frac{4\pi}{5\sqrt{5}} \sqrt{n + \frac{1}{5}} \right) \]

\[ \sim \frac{2^{1/2}}{5^{3/4}} \cos \left( \frac{4\pi}{5} \left( n + \frac{3}{20} \right) \right) n^{-3/4} \exp \left( \frac{4\pi}{5\sqrt{5}} \sqrt{n} \right). \]

2.3 Tang’s inequalities

Let

\[ \sum_{n \geq 0} g(n)q^n = \frac{(q^2, q^8; q^{10})_\infty (q^4; q^6; q^{10})_\infty^2}{(q^2, q^3; q^5)_\infty^2}. \]  

(2.3)

Then \( m = \{5, 10, 10\}, \ r = \{2, 2, 4\} \) and \( \delta = \{-2, 1, 2\} \). Hence \( L = 10 \) and \( \Omega = -8 \). We compute that

\[ L > 0 = \{(0, 1), (0, 3), (1, 3), (2, 3), (0, 5), (2, 5), (3, 5), (0, 7), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7), (0, 9), (1, 9), (2, 9), (3, 9), (4, 9), (5, 9), (6, 9), (7, 9), (8, 9), (1, 10), (2, 10), (3, 10), (4, 10), (6, 10), (7, 10), (8, 10), (9, 10)\}. \]
First, we verify that assumption (1.6) is satisfied. We next find that the largest number among \( \sqrt{\Delta(\kappa, \ell)/k} \) with \((\kappa, \ell) \in \mathcal{L}_{>0} \) and \( k \equiv \ell \pmod{L} \) is \( \frac{1}{\sqrt{15}} \). Here we have four choices:

\[
(\kappa, \ell, k) = (0, 1, 1), (0, 5, 5), (2, 5, 5), (3, 5, 5).
\]

When \( k = 1 \), the admissible \((h, k)\) is \((0, 1)\). We compute that the \( I \)-Bessel term is

\[
\frac{\sqrt{2\pi}}{\sqrt{15}} \sin \left( \frac{\pi}{5} \right) \left( n - \frac{1}{3} \right)^{-1/2} I_{-1} \left( \frac{\sqrt{2\pi}}{\sqrt{15}} \sqrt{n - \frac{1}{3}} \right).
\]

When \( k = 5 \), the admissible \((h, k)\) are \((2, 5)\) and \((3, 5)\). We compute that, in total, the \( I \)-Bessel term is

\[
\frac{\sqrt{2\pi}}{\sqrt{15}} \left( \sin \left( \frac{2\pi}{5} (2n + 1) \right) - \frac{1}{3} \right)^{-1/2} I_{-1} \left( \frac{\sqrt{2\pi}}{\sqrt{15}} \sqrt{n - \frac{1}{3}} \right).
\]

In total, we have

\[
\sqrt{\frac{2\pi}{15}} \left( \sin \left( \frac{\pi}{5} \right) + \sin \left( \frac{2\pi}{5} (2n + 1) \right) \right) \left( n - \frac{1}{3} \right)^{-1/2} I_{-1} \left( \frac{\sqrt{2\pi}}{\sqrt{15}} \sqrt{n - \frac{1}{3}} \right).
\]

Notice that \( \sin \left( \frac{\pi}{5} \right) + \sin \left( \frac{2\pi}{5} (2n + 1) \right) \) vanishes only if \( n \equiv 1 \pmod{5} \). Hence, for \( n \equiv 1 \pmod{5} \),

\[
g(n) \sim \sqrt{\frac{2\pi}{15}} \left( \sin \left( \frac{\pi}{5} \right) + \sin \left( \frac{2\pi}{5} (2n + 1) \right) \right) \left( n - \frac{1}{3} \right)^{-1/2} I_{-1} \left( \frac{\sqrt{2\pi}}{\sqrt{15}} \sqrt{n - \frac{1}{3}} \right) \\
\sim \frac{1}{30^{1/4}} \left( \sin \left( \frac{\pi}{5} \right) + \sin \left( \frac{2\pi}{5} (2n + 1) \right) \right) n^{-3/4} \exp \left( \frac{\sqrt{2\pi}}{\sqrt{15}} \sqrt{n} \right).
\]

It follows that \( g(5n + 0, 2, 3) > 0 \) and \( g(5n + 4) < 0 \) for sufficiently large \( n \). If we further compute a number of lower \( I \)-Bessel terms, we still encounter the same vanishing phenomenon for \( n \equiv 1 \pmod{5} \). This highly suggests that \( g(5n + 1) = 0 \), which is, indeed, proved by Tang using elementary techniques in [27].

All other inequalities conjectured by Tang can be proved for sufficiently large \( n \) in the same manner. We therefore omit the details here. It is notable that in recent papers of Tang and Xia [29], Xia and Zhao [30] and Tang [28], exact dissection formulas of several infinite products including (2.3) were obtained, from which one is also able to deduce the aforementioned inequalities of the Taylor coefficients.
3 Dedekind eta function and Jacobi theta function

In this section, we introduce the Dedekind eta function and Jacobi theta function. All results here are standard, which can be found in, for example, [5] or [31].

Let $\tau \in \mathbb{H}$ and $\varsigma \in \mathbb{C}$. The Dedekind eta function is defined by

$$\eta(\tau) := q^{1/24}(q; q)_{\infty}$$

with $q := e^{2\pi i \tau}$. Further, the Jacobi theta function reads

$$\vartheta(\varsigma; \tau) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} e^{2\pi i \nu (\varsigma + \frac{1}{2}) + \pi i \nu^2 \tau}.$$ 

Notice that if we let $\zeta := e^{2\pi i \varsigma}$, then the Jacobi triple product identity implies that

$$\vartheta(\varsigma; \tau) = -i q^{1/8} \zeta^{-1/2} (\zeta, \zeta^{-1} q, q; q)_{\infty}.$$ 

It follows immediately that

$$\mathcal{K}(\varsigma; \tau) := (\zeta, \zeta^{-1} q; q)_{\infty} = i e^{-\frac{\pi i}{6}} e^{\pi i \varsigma} \frac{\vartheta(\varsigma; \tau)}{\eta(\tau)}. \quad (3.1)$$

The Dedekind eta function and Jacobi theta function are of broad interest due to their transformation properties. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ where we assume that $c > 0$. Recall that the Möbius transformation for $\tau \in \mathbb{H}$ is defined by

$$\gamma(\tau) := a \tau + b \quad \frac{c \tau + d}{c \tau + d}.$$ 

Further, for the $\gamma$ given above, we write for convenience

$$\gamma^*(\tau) := \frac{1}{c \tau + d}.$$ 

If

$$\chi(\gamma) = \exp \left( \pi i \left( \frac{a + d}{12c} - s(d, c) - \frac{1}{4} \right) \right),$$

where, again, $s(d, c)$ is the Dedekind sum, then

$$\eta(\gamma(\tau)) = \chi(\gamma)(c \tau + d)^{1/2} \eta(\tau) \quad (3.2)$$
and
\[ \vartheta(q \gamma^*(\tau); \gamma(\tau)) = \chi(\gamma)^3 (e^{\tau} + d)^{1/2} e^{\frac{\pi i q^2}{\tau + d}} \vartheta(q; \tau). \] (3.3)

Further, let \( \alpha \) and \( \beta \) be integers. The Jacobi theta function also satisfies
\[ \vartheta(q + \alpha \tau + \beta; \tau) = (-1)^{\alpha + \beta} e^{-\pi i \alpha^2 \tau} e^{-2\pi i \alpha \xi} \vartheta(q; \tau). \] (3.4)

4 Farey arcs and a transformation formula

To study the asymptotics for the Taylor coefficients of \( G(q) \), we turn to the celebrated circle method due to Rademacher [21,22], whose idea originates from Hardy and Ramanujan [9]. Noticing that \( G(q) \) is holomorphic inside the unit disk, we may directly apply Cauchy’s integral formula to deduce
\[ g(n) = \frac{1}{2\pi i} \oint_{\mathcal{C}_{|q|=r}} \frac{G(q)}{q^{n+1}} \, dq, \]
where the contour integral is taken counter-clockwise. Now one lets \( r = e^{-2\pi q} \) with \( q = 1/N^2 \) where \( N \) is a sufficiently large positive integer.

Next, we dissect the circle \( \mathcal{C} \) by Farey arcs. Let \( h/k \) with \( \gcd(h,k) = 1 \) be a Farey fraction of order \( N \). If we denote by \( \xi_{h,k} \) the interval \([-\theta_{h,k}', \theta_{h,k}''] \) with \( \theta_{h,k}' \) and \( \theta_{h,k}'' \) being the positive distances from \( h/k \) to its neighboring mediants, then \( \mathbb{R}/\mathbb{Z} \) can be covered by intervals \( \bigcup_{h,k} \xi_{h,k} \) where \( 0 \leq h < k \leq N \) and \( \gcd(h,k) = 1 \). For each \( q \) on the circle \( \mathcal{C} \), we may find a Farey fraction \( h/k \) such that \( \arg(q) = 2\pi(h/k + \phi) \) with \( \phi \in \xi_{h,k} \). Thus, we have \( q = e^{2\pi i(h/k + i\phi)} \) and hence,
\[ g(n) = \sum_{1 \leq h < k \leq N} \sum_{\gcd(h,k)=1} e^{-\frac{2\pi i nh}{k}} \int_{\xi_{h,k}} G(e^{2\pi i(h/k + i\phi)}) e^{-2\pi in\phi} e^{2\pi n\phi} \, d\phi. \]

Let \( z = k(q - i\phi) \). Making the change of variables \( \tau = (h + iz)/k \) yields
\[ g(n) = \sum_{1 \leq h < k \leq N} \sum_{\gcd(h,k)=1} e^{-\frac{2\pi i nh}{k}} \int_{\xi_{h,k}} G(e^{2\pi i\tau}) e^{-2\pi in\phi} e^{2\pi n\phi} \, d\phi. \] (4.1)

Let \( r < m \) be positive integers. Our next task is to apply the transformation properties of the Dedekind eta function and Jacobi theta function to reformulate each \( \mathcal{K}(r\tau; m\tau) \). To do so, we need to construct a suitable matrix in \( SL_2(\mathbb{Z}) \).

Let \( d = \gcd(m,k) \). For convenience, we write \( m = dm' \) and \( k = dk' \). Recalling that \( h_m(h,k) \) satisfies \( h_m(h,k)m'h \equiv -1 \pmod{k'} \), we put \( b_{m'} = (h_m(h,k)m'h + 1)/k' \).
It is straightforward to verify that the following matrix is in $SL_2(\mathbb{Z})$:

$$
\gamma(m,h,k) = \begin{pmatrix} h_m(h,k) & -b_m' \\
 k' & -m'h \end{pmatrix}.
$$

Since $\tau = (h + iz)/k = (h + iz)/dk'$, one may compute

$$
\gamma(m,h,k)(m\tau) = \frac{h_m(h,k) \cdot m \frac{h+iz}{dk'} - b_m'}{k' \cdot m \frac{h+iz}{dk'} - m'h} = \frac{h_m(h,k)m'h + ih_m(h,k)m'h - (h_m(h,k)m'h + 1)}{m'kh' + ik'm'h - m'kh'}
$$

Thus,

$$
\gamma(m,h,k)(m\tau) = \frac{h_m(h,k) \gcd(m,k)}{k} + \frac{\gcd^2(m,k)}{mkz} i. \quad (4.2)
$$

On the other hand, we have

$$
\gamma^*(m,h,k)(m\tau) = \frac{1}{k' \cdot m \frac{h+iz}{dk'} - m'h} = -\frac{\gcd(m,k)}{mz} i
$$

and hence,

$$
r\tau\gamma^*(m,h,k)(m\tau) = \frac{r \gcd(m,k)}{mk} - \frac{rh \gcd(m,k)}{mkz} i. \quad (4.3)
$$

Further,

$$
r\tau\gamma^*(m,h,k)(m\tau) + \lambda_{m,r}(h,k)\gamma(m,h,k)(m\tau)
$$

$$
= \frac{r \gcd(m,k)}{mk} + \lambda_{m,r}(h,k) \frac{h_m(h,k) \gcd(m,k)}{k} + \lambda^*_{m,r}(h,k) \frac{\gcd^2(m,k)}{mkz} i. \quad (4.4)
$$

Recalling from (3.1) that

$$
\gamma_K(r\tau; m\tau) = ie^{-\frac{\pi imr}{6}} e^{\pi irr} \frac{\vartheta(r\tau; m\tau)}{\eta(m\tau)},
$$

one has, from (3.2), (3.3), (3.4) and the fact $s(-m'h, k') = -s(m'h, k')$, that

$$
\gamma_K(r\tau; m\tau) = ie^{-\frac{\pi imr}{6}} e^{\pi irr} \chi(\gamma(m,h,k))^{-2} e^{-\frac{\pi i k'r^2}{k'mr-m'h}}
$$

$$
\times \frac{\vartheta(r\tau\gamma^*(m,h,k)(m\tau); \gamma(m,h,k)(m\tau))}{\eta(\gamma(m,h,k)(m\tau))}
$$

$$
= ie^{-\frac{\pi imr}{6}} e^{\pi irr} \chi(\gamma(m,h,k))^{-2} e^{-\frac{\pi i k'r^2}{k'mr-m'h}} (-1)^{\lambda_{m,r}(h,k)}
$$

$$
\times e^{\pi i k^2_{m,r}(h,k)\gamma(m,h,k)(m\tau)} e^{2\pi i \lambda_{m,r}(h,k) r\tau\gamma^*(m,h,k)(m\tau)}
$$

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\[
\frac{\partial (r \tau y^*_{(m,h,k)}(m \tau) + \lambda_{m,r}(h,k) \gamma_{(m,h,k)}(m \tau))}{\eta(\gamma_{(m,h,k)}(m \tau))} \\
= i (-1)^{\lambda_{m,r}(h,k)} e^{-2\pi i x(m'h,k')} \\
\times \exp \left( \pi i \left( \frac{rh}{k} - \frac{rd}{mk} \frac{2d \lambda^*_{m,r}(h,k)}{mk} \\
+ \frac{h_m(h,k)d}{k} (\lambda^2_{m,r}(h,k) - \lambda_{m,r}(h,k)) \right) \right) \\
\times \exp \left( \frac{\pi}{12k} \left( \frac{2m - 12r}{m} + \frac{12r^2}{m} \right) z \\
- \left( \frac{2d^2}{m} + \frac{12d^2}{m} (\lambda^2_{m,r}(h,k) - \lambda_{m,r}(h,k)) \frac{1}{z} \right) \right) \\
\times \mathcal{K}(r \tau y^*_{(m,h,k)}(m \tau) + \lambda_{m,r}(h,k) \gamma_{(m,h,k)}(m \tau)) \gamma_{(m,h,k)}(m \tau)).
\]

Consequently, we deduce the following transformation formula.

\[
G(e^{2\pi i \tau}) = \prod_{j=1}^{J} \mathcal{K}^{\delta_j}(r_j \tau; m_j \tau) \\
= i \sum_{j=1}^{J} \delta_j (-1) \sum_{j=1}^{J} \delta_j \lambda_{m_j,r_j}(h,k) \omega_{h,k}^{2} [\Pi_{h,k} \\
\times \exp \left( \frac{\pi}{12k} (\Omega z + \Delta(h,k) z^{-1}) \right) \\
\times \prod_{j=1}^{J} \mathcal{K}^{\delta_j}(r_j \tau y^*_{(m_j,h,k)}(m_j \tau) + \lambda_{m_j,r_j}(h,k) \gamma_{(m_j,h,k)}(m_j \tau)) \gamma_{(m_j,h,k)}(m_j \tau)).
\]

\[4.5\]

**Remark 4.1** It follows from (4.4) that for all \( j = 1, 2, \ldots, J, \)

\[0 \leq \Im(r_j \tau y^*_{(m_j,h,k)}(m_j \tau) + \lambda_{m_j,r_j}(h,k) \gamma_{(m_j,h,k)}(m_j \tau) < \Im(\gamma_{(m_j,h,k)}(m_j \tau)).\]

This inequality will be used in the sequel; see, for example, (7.1).

**5 Some auxiliary results**

**5.1 Necessary bounds**

Now we are going to present some useful bounds, which were obtained in previous work; see, for example, [21].
First, it is well known (cf. Chapter 3 in [10]) that for a Farey fraction \(h/k\) of order \(N\), one has
\[
\frac{1}{2kN} \leq \theta'_{h,k}, \theta''_{h,k} \leq \frac{1}{kN}.
\] (5.1)

Let \(|\xi_{h,k}|\) be the length of the interval \(\xi_{h,k}\). By noticing that \(|\xi_{h,k}| = \theta'_{h,k} + \theta''_{h,k}\), one has
\[
\frac{1}{kN} \leq |\xi_{h,k}| \leq \frac{2}{kN}.
\]

Next, since \(z = k(\varrho - i\phi)\), it follows that
\[
\Re(z) = k\varrho = \frac{k}{N^2}.
\] (5.2)

Further, one has
\[
\Re \left( \frac{1}{z} \right) \geq \frac{k}{2},
\] (5.3)
since
\[
\Re \left( \frac{1}{z} \right) = \frac{1}{k \varrho^2 + \phi^2} \geq \frac{1}{k \frac{N^2}{N^4 + 2N^2 + 1}} = \frac{k}{1 + \frac{1}{2}} = \frac{k}{2},
\]
where we use the fact \(k \leq N\) in the last inequality.

### 5.2 A partition-theoretic result

Let \(\eta\) be a positive integer. Let \(p^*_{\eta}(s, t; n)\) denote the number of 2-colored (say, red and blue) partition \(\eta\)-tuples of \(n\) with \(s\) parts in total colored by red and \(t\) parts in total colored by blue. Here we allow 0 as a part. Let \(q, \zeta\) and \(\xi\) be such that \(|q| < 1, |\zeta| < 1\) and \(|\xi| < 1\). The following infinite triple summation
\[
\sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} p^*_{\eta}(s, t; n) \xi^s \xi^t q^n = \left( \frac{1}{(\zeta, \xi; q)^\eta} \right)
\]
is absolutely convergent. Further, considering another absolutely convergent infinite triple summation
\[
\sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} d^*_{\eta}(s, t; n) \xi^s \xi^t q^n = (\zeta, \xi; q)_{\infty}^\eta.
\]
we have that \((-1)^{s+t} d^*_{\eta}(s, t; n)\) denotes the number of 2-colored (again, red and blue) distinct partition (in which 0 is still allowed as a part) \(\eta\)-tuples of \(n\) with \(s\) parts in total colored by red and \(t\) parts in total colored by blue. An easy partition-theoretic argument indicates that \(|d^*_{\eta}(s, t; n)| \leq p^*_{\eta}(s, t; n)\) for all \(s, t, n \geq 0\). Also, we have \(d^*_{\eta}(0, 0; 0) = p^*_{\eta}(0, 0; 0) = 1\).
In general, for a nonzero integer $\delta$, if we write
\[
\sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} a_\delta(s, t; n) \zeta^s \xi^t q^n := (\zeta, \xi; q)_\infty^\delta,
\]
then
\[
a_\delta(s, t; n) = \begin{cases} p_\delta^s(s, t; n) & \text{if } \delta < 0, \\ d_\delta^s(s, t; n) & \text{if } \delta > 0, \end{cases}
\]
and hence $|a_\delta(s, t; n)| \leq p_\delta^s(s, t; n)$ for all $s, t, n \geq 0$. Trivially, we also have
\[
|(\zeta, \xi; q)_\infty^\delta| = \left| \sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} a_\delta(s, t; n) \zeta^s \xi^t q^n \right| \leq \sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} p_\delta^s(s, t; n) |\zeta|^s |\xi|^t |q|^n.
\]

Further, for real $0 \leq \alpha, \beta, x < 1$, we have
\[
\sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} p_\delta^s(s, t; n) \alpha^s \beta^t x^n = \frac{1}{(\alpha, \beta; x)_\infty} = \exp \left( - \sum_{k \geq 0} \log(1 - \alpha x^k) - \sum_{\ell \geq 0} \log(1 - \beta x^\ell) \right) \leq \exp \left( \frac{\alpha}{1 - \alpha} + \frac{\alpha x}{(1 - x)^2} + \frac{\beta}{1 - \beta} + \frac{\beta x}{(1 - x)^2} \right).
\]
\[(5.4)\]

For the last inequality, we use the facts that for real $0 \leq x, y < 1$,
\[
- \log(1 - x) = \sum_{m \geq 1} \frac{x^m}{m} \leq \sum_{m \geq 1} x^m = \frac{x}{1 - x}
\]
and
\[
- \sum_{k \geq 1} \log(1 - y x^k) = \sum_{k \geq 1} \sum_{m \geq 1} \frac{x^m y^m}{m} = \sum_{n \geq 1} x^n \sum_{d | n} \frac{y^d}{d} \leq y \sum_{n \geq 1} nx^n = \frac{xy}{(1 - x)^2}.
\]
6 Outline of the proof

We know from (4.1) and (4.5) that

\[
g(n) = \sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop \gcd(h,k) = 1} e^{-\frac{2\pi i n h}{k}} \int_{\xi_{h,k}} G(e^{2\pi i \tau}) e^{-2\pi i n \phi} e^{2\pi n \phi} d\phi
\]

\[
= i \sum_{j=1}^J \delta_j \sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop \gcd(h,k) = 1} e^{-\frac{2\pi i n h}{k}} (-1)^j \delta_j \lambda_{m_j, r_j}(h,k) \omega_{h,k}^2 \Pi_{h,k}
\]

\[
\times \int_{\xi_{h,k}} \exp \left( \frac{\pi}{12k} (\Omega z + \Delta(h, k) z^{-1}) \right)
\]

\[
\times \prod_{j=1}^J \gamma_j^\delta \left( r_j \tau \gamma^*_{(m_j, h, k)}(m_j \tau) + \lambda_{m_j, r_j}(h, k) \gamma_{(m_j, h, k)}(m_j \tau); \gamma_{(m_j, h, k)}(m_j \tau) \right)
\]

\[
\times e^{-2\pi i n \phi} e^{2\pi n \phi} d\phi.
\]

Let us fix a Farey fraction $h/k$. We first find integers $1 \leq \ell \leq L$ and $0 \leq \varkappa < \ell$ such that $k \equiv \ell \pmod{L}$ and $h \equiv \varkappa \pmod{\ell}$. For convenience, we write $\rho(h, k) := (\varkappa, \ell)$. It is not hard to observe that for all $j = 1, 2, \ldots, J$,

\[
gcd(m_j, k) = \gcd(m_j, \ell) \quad \text{and} \quad \lambda_{m_j, r_j}^*(h, k) = \lambda_{m_j, r_j}^*(\varkappa, \ell).
\]

This implies that $\Delta(h, k) = \Delta(\varkappa, \ell)$. We now split $g(n)$ as follows.

\[
g(n) = i \sum_{j=1}^J \delta_j \sum_{1 \leq \ell \leq L} \sum_{0 \leq \varkappa < \ell \atop \ell \equiv \varkappa \pmod{L}} \sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop \gcd(h,k) = 1} e^{-\frac{2\pi i n h}{k}}
\]

\[
\times (-1)^j \lambda_{m_j, r_j}(h,k) \omega_{h,k}^2 \Pi_{h,k}
\]

\[
\times \int_{\xi_{h,k}} \exp \left( \frac{\pi}{12k} (\Omega z + \Delta(\varkappa, \ell) z^{-1}) \right)
\]

\[
\times \prod_{j=1}^J \gamma_j^\delta \left( r_j \tau \gamma^*_{(m_j, h, k)}(m_j \tau) + \lambda_{m_j, r_j}(h, k) \gamma_{(m_j, h, k)}(m_j \tau); \gamma_{(m_j, h, k)}(m_j \tau) \right)
\]

\[
\times e^{-2\pi i n \phi} e^{2\pi n \phi} d\phi
\]

\[
=: i \sum_{j=1}^J \delta_j \sum_{1 \leq \ell \leq L} \sum_{0 \leq \varkappa < \ell} S_{\varkappa, \ell}.
\]

The minor arcs are those with respect to $h/k$ with $\rho(h, k) \in \mathcal{L}_{\leq 0}$. We have the following bound.
Theorem 6.1  Let $(\varkappa, \ell) \in \mathcal{L}_{\leq 0}$. For positive integers $n > -\Omega/24$, we have

$$S_{\varkappa, \ell} \ll m, r, \delta \exp \left( \frac{2\pi}{N^2} \left( n + \frac{\Omega}{24} \right) \right).$$

In particular, if we take $N = \left\lceil \sqrt{2\pi \left( n + \frac{\Omega}{24} \right)} \right\rceil$, then $S_{\varkappa, \ell} \ll m, r, \delta \ 1$.

The arcs with respect to $h/k$ with $\rho(h, k) \in \mathcal{L}_{>0}$ give us the main contribution.

Theorem 6.2  Let $(\varkappa, \ell) \in \mathcal{L}_{>0}$. If the inequality

$$\min_{1 \leq j \leq J} \left( \Upsilon(\lambda_{m_j, r_j}(\varkappa, \ell)) \frac{\gcd^2(m_j, \ell)}{m_j} \right) \geq \frac{\Delta(\varkappa, \ell)}{24} \quad (6.1)$$

holds, then for positive integers $n > -\Omega/24$, we have

$$S_{\varkappa, \ell} = E_{\varkappa, \ell} + \sum_{1 \leq k \leq N} \sum_{0 < h < k \atop k \equiv \ell \mod L} e^{-\frac{2\pi i h}{k}} (-1)^{\sum_{j=1}^{J} \delta_{\lambda_{m_j, r_j}(h,k)}} \omega_{h,k}^2 J_{h,k} \Pi_{h,k}$$

$$\times \frac{2\pi}{k} \left( \frac{24n + \Omega}{\Delta(\varkappa, \ell)} \right)^{-\frac{1}{2}} I_{-1} \left( \frac{\pi}{6k} \sqrt{\Delta(\varkappa, \ell)(24n + \Omega)} \right),$$

where

$$E_{\varkappa, \ell} \ll m, r, \delta \frac{2\pi}{N^2} \left( n + \frac{\Omega}{24} \right) + \frac{N^2 e^{\frac{2\pi}{N^2} \left( n + \frac{\Omega}{24} \right)}}{n + \frac{\Omega}{24}}.$$

In particular, if we take $N = \left\lceil \sqrt{2\pi \left( n + \frac{\Omega}{24} \right)} \right\rceil$, then $E_{\varkappa, \ell} \ll m, r, \delta \ 1$.

Theorems 6.1 and 6.2 immediately imply the main result. Before presenting proofs of the two results in Sects. 7 and 8, respectively, we make the following preparations.

For fixed $\varkappa$ and $\ell$ with $1 \leq \ell \leq L$ and $0 \leq \varkappa < \ell$, one may split the indices \{1, 2, \ldots, J\} into two disjoint parts:

$$\mathcal{J}_{\varkappa, \ell} = \{j_1^*, \ldots, j_a^*\} \quad \text{and} \quad \mathcal{J}_{\varkappa, \ell}^{**} = \{j_1^{**}, \ldots, j_b^{**}\}$$

so that for $j^* \in \mathcal{J}_{\varkappa, \ell}^*$ we have $\lambda_{m_j^*, r_j^*}(\varkappa, \ell) = 0$ and for $j^{**} \in \mathcal{J}_{\varkappa, \ell}^{**}$ we have $\lambda_{m_{j^{**}}, r_{j^{**}}}(\varkappa, \ell) \neq 0$.

Proposition 6.3  Let $j^* \in \mathcal{J}_{\varkappa, \ell}^*$. For any Farey fraction $h/k$ such that $k \equiv \ell \ (\text{mod } L)$ and $h \equiv \varkappa \ (\text{mod } \ell)$, we have that

$$r_{j^*} \gamma_{(m_{j^*}, h,k)}(m_{j^*} \tau) + \lambda_{m_{j^*}, r_{j^*}}(h, k) \gamma_{(m_{j^*}, h,k)}(m_{j^*} \tau)$$

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\[
\frac{r_j \gcd(m_j^*, k) + r_j \tau_{m_j^*, (h, k)}(m_j^* h)}{m_j^* k} = \frac{\tau \gamma_{(m, h, k)}(m \tau) + \lambda_{m, r}(h, k) \gamma_{(m, h, k)}(m \tau)}{m \tau}
\]

is a real noninteger. Further,

\[
\left| 1 - e^{\frac{2\pi i}{m_j^*}} \right| \leq 1 \left| 1 - e^{\frac{2\pi i}{m_j^*}} \left( r_j \tau_{m_j^*, (h, k)}(m_j^* h) + \lambda_{m, r}(h, k) \gamma_{(m_j^*, h, k)}(m_j^* h) \right) \right| \leq 2. \quad (6.2)
\]

**Proof** In this proof, we write for short \( m = m_j^* \) and \( r = r_j^* \). We also write \( d = \gcd(m, k) \), \( m = dm' \) and \( k = dk' \). Since \( j^* \in J^*_{\ell, \ell} \), we have \( \lambda_{m, r}(h, k) = \lambda_{m, r}(\ell, \ell) = 0 \). Hence \( d \) divides \( rh \) and \( \lambda_{m, r}(h, k) = rh/d \). We know from (4.4) that

\[
\frac{r \tau \gamma_{(m, h, k)}(m \tau) + \lambda_{m, r}(h, k) \gamma_{(m, h, k)}(m \tau)}{m \tau} = \frac{r d}{mk} + \lambda_{m, r}(h, k) \frac{h_m(h, k) d}{k} + \frac{\lambda_{m, r}(h, k)}{mk} d^2 i
\]

where as in Sect. 4, we have put \( b_{m'} = (h_m(h, k)m'h + 1)/k' \). Hence it is a real number.

Notice that \( d = \gcd(m, k) \). Since \( \gcd(h, k) = 1 \), \( d \mid rh \) implies that \( d' \mid r \). Further, \( b_{m'} = (h_m(h, k)m'h + 1)/k' \) implies that \( \gcd(m', b_{m'}) = 1 \). Hence, if \( \frac{b_{m'} r}{m'd} \) is an integer, then \( m' \mid \frac{r}{d} \) so that \( m = dm' \mid r \). This violates the assumption that \( 1 \leq r \leq m - 1 \). Hence \( r \tau \gamma_{(m, h, k)}(m \tau) + \lambda_{m, r}(h, k) \gamma_{(m, h, k)}(m \tau) \) is not an integer and (6.2) follows immediately. \( \square \)

**Remark 6.1** Recall that \( h_m(h, k) \) is defined to be an integer such that

\[
h_m(h, k) \equiv -1 \left( \text{mod} \frac{k}{\gcd(m, k)} \right).
\]

Let \( n \) be an integer. It turns out that

\[
\exp \left( 2\pi i \frac{r_j \gcd(m_j^*, k) + r_j \tau_{m_j^*, (h, k)}(m_j^* h) + n \frac{h_m(h, k)}{\gcd(m_j^*, k)} m_j^* h}{m_j^* k} \right)
\]

\[
= \exp \left( 2\pi i \frac{r_j \gcd(m_j^*, k) + r_j \tau_{m_j^*, (h, k)}(m_j^* h) + 2n \tau \frac{\gcd(m_j^*, k)}{m_j^* k} h}{m_j^* k} \right)
\]

\[
= \exp \left( 2\pi i \frac{r_j \gcd(m_j^*, k) + r_j \tau_{m_j^*, (h, k)}(m_j^* h)}{m_j^* k} \right),
\]

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since from the above proof we have \( \gcd(m_{j*}, k) | r_{j*} \). Hence the choice of \( h_{m_{j*}}(h, k) \) does not affect the value of

\[
\exp\left(2\pi i \left(r_{j*} \tau \gamma_{(m_{j*},h,k)}(m_{j*} \tau) + \lambda_{m_{j*},r_{j*}}(h, k) \gamma_{(m_{j*},h,k)}(m_{j*} \tau)\right)\right)
\]

\[
= \exp\left(2\pi i \frac{r_{j*} \gcd(m_{j*}, k) + r_{j*} h_{m_{j*}}(h, k) m_{j*} h}{m_{j*} k}\right).
\]

7 Minor arcs

Let \((\kappa, \ell) \in \mathcal{L}_{\leq 0}\), namely, \(\Delta(\kappa, \ell) \leq 0\). We write \(J^* = J^*_{\kappa, \ell}\) and \(J^{**} = J^{**}_{\kappa, \ell}\).

Notice that

\[
|S_{\kappa, \ell}| \leq \sum_{1 \leq \kappa \leq N} \sum_{k \equiv \ell \mod L} \gcd(h, k) = 1 \int_{\xi_{h,k}} \exp\left(\frac{\pi}{12k}(\Omega_1 h(z) + \Delta(\kappa, \ell) \Re(z^{-1}))\right)
\]

\[
\times \prod_{j=1}^{J} \mathcal{H}^j\left( r_{j} \tau \gamma_{(m_{j},h,k)}(m_{j} \tau) + \lambda_{m_{j*},r_{j}}(h, k) \gamma_{(m_{j},h,k)}(m_{j} \tau)\right) \gamma_{(m_{j},h,k)}(m_{j} \tau)
\]

\[
\times e^{2\pi n \theta} d\phi.
\]

We now consider the Farey arcs with respect to \(h/k\) with \(k \equiv \ell \mod L\) and \(h \equiv \kappa \mod \ell\). Since \(\Delta(\kappa, \ell) \leq 0\), it follows from (5.2) and (5.3) that

\[
\exp\left(\frac{\pi}{12k}(\Omega_1 h(z) + \Delta(\kappa, \ell) \Re(z^{-1}))\right) \leq \exp\left(\frac{\pi}{12k}\left(\Omega \frac{k}{N^2} + \Delta(\kappa, \ell) \frac{k}{2}\right)\right)
\]

\[
= \exp\left(\frac{\pi \Omega}{12}\right) \exp\left(\frac{\pi \Delta(\kappa, \ell)}{24}\right).
\]

For convenience, now we write \(\lambda_j = \lambda_{m_{j},r_{j}}(h, k)\) and \(\lambda^*_j = \lambda_{m_{j*},r_{j}}(h, k)\). We also write for short \(\tilde{\xi}_j = r_{j} \tau \gamma_{(m_{j},h,k)}(m_{j} \tau) + \lambda_{m_{j*},r_{j}}(h, k) \gamma_{(m_{j},h,k)}(m_{j} \tau)\) and \(\tilde{\tau}_j = \gamma_{(m_{j},h,k)}(m_{j} \tau)\). We know from (4.2) and (4.4) that

\[
\Im(\tilde{\tau}_j) = \frac{\gcd^2(m_{j}, k)}{m_{j} k} \Re(z^{-1}) = \frac{\gcd^2(m_{j}, \ell)}{m_{j} k} \Re(z^{-1})
\]

and

\[
\Im(\tilde{\xi}_j) = \frac{\lambda^*_j \gcd^2(m_{j}, k)}{m_{j} k} \Re(z^{-1}) = \frac{\lambda^*_j \gcd^2(m_{j}, \ell)}{m_{j} k} \Re(z^{-1}).
\]
Notice that by Remark 4.1,

$$0 \leq \Im(\xi_j) < \Im(\tilde{\xi}_j). \quad (7.1)$$

We write

$$\prod_{j=1}^{J} \mathcal{H}^\delta_j(\xi_j; \tilde{\xi}_j) = \prod_{j^* \in \mathcal{J}^*} (1 - e^{2\pi i \xi_{j^*}})^{\delta_{j^*}}$$
$$\times \prod_{j^* \in \mathcal{J}^*} (e^{2\pi i (\tilde{\xi}_{j^*} + \xi_{j^*})}, e^{2\pi i (\tilde{\xi}_{j^*} - \xi_{j^*})}, e^{2\pi i \tilde{\xi}_{j^*}})_\infty^{\delta_{j^*}}$$
$$\times \prod_{j^{**} \in \mathcal{J}^{**}} (e^{2\pi i \xi_{j^{**}}}, e^{2\pi i (\tilde{\xi}_{j^{**}} - \xi_{j^{**}})}, e^{2\pi i \tilde{\xi}_{j^{**}}})_\infty^{\delta_{j^{**}}}.$$

First, it follows from Proposition 6.3 that

$$\prod_{j^* \in \mathcal{J}^*} (1 - e^{2\pi i \xi_{j^*}})^{\delta_{j^*}} \ll 1.$$ 

Further, as we have seen in Sect. 5.2, for $j^* \in \mathcal{J}^*$ (hence $\lambda_{j^*} = 0$),

$$\left| e^{2\pi i (\tilde{\xi}_{j^*} + \xi_{j^*})}, e^{2\pi i (\tilde{\xi}_{j^*} - \xi_{j^*})}, e^{2\pi i \tilde{\xi}_{j^*}} \right|^{\delta_{j^*}}$$
$$\leq \sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} p_{\delta_{j^*}}^{*}(s, t; n) e^{2\pi i (\tilde{\xi}_{j^*} + \xi_{j^*})} |s| e^{2\pi i (\tilde{\xi}_{j^*} - \xi_{j^*})} |t| e^{2\pi i \tilde{\xi}_{j^*}} |n$$
$$= \sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} p_{\delta_{j^*}}^{*}(s, t; n) e^{-2\pi \Im(\tilde{\xi}_{j^*} + \xi_{j^*})} s e^{-2\pi \Im(\tilde{\xi}_{j^*} - \xi_{j^*})} t e^{-2\pi \Im(\tilde{\xi}_{j^*})} n$$
$$= \sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} p_{\delta_{j^*}}^{*}(s, t; n) \exp \left( -2\pi \frac{\gcd^2(m_{j^*}, \ell)}{m_{j^*} k} \Im(z^{-1}) s \right)$$
$$\times \exp \left( -2\pi \frac{\gcd^2(m_{j^*}, \ell)}{m_{j^*} k} \Im(z^{-1}) t \right) \exp \left( -2\pi \frac{\gcd^2(m_{j^*}, \ell)}{m_{j^*} k} \Im(z^{-1}) n \right)$$
$$\leq \sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} p_{\delta_{j^*}}^{*}(s, t; n) \exp \left( -\pi \frac{\gcd^2(m_{j^*}, \ell)}{m_{j^*} s} \right)$$
$$\times \exp \left( -\pi \frac{\gcd^2(m_{j^*}, \ell)}{m_{j^*} t} \right) \exp \left( -\pi \frac{\gcd^2(m_{j^*}, \ell)}{m_{j^*} n} \right),$$

where we use (5.3). It follows from (5.4) that

$$(e^{2\pi i (\tilde{\xi}_{j^*} + \xi_{j^*})}, e^{2\pi i (\tilde{\xi}_{j^*} - \xi_{j^*})}, e^{2\pi i \tilde{\xi}_{j^*}})_\infty^{\delta_{j^*}} \ll 1.$$
Likewise, for $j^{**} \in \mathcal{J}^{**}$,

\[
\left| e^{2\pi i \xi_{j^{**}}} \right| \leq \sum_{n \geq 0} \sum_{s \geq 0} \sum_{t \geq 0} p^{*}_{|j^{**}|}(s, t; n) \exp \left( \frac{-\pi \lambda_{j^{**}}}{m_{j^{**}}} \gcd^2(m_{j^{**}}, \ell) s \right) \times \exp \left( \frac{-\pi (1 - \lambda_{j^{**}})}{m_{j^{**}}} t \right) \exp \left( \frac{-\pi \gcd^2(m_{j^{**}}, \ell)}{m_{j^{**}}} n \right)
\]

\[< 1.\]

Hence,

\[
S_{\kappa, \ell} \ll \sum_{1 \leq k \leq N} \sum_{0 \leq h < k} \frac{1}{kN} \int_{\xi_{h,k}} \frac{e^{\pi \varphi \delta} e^{2\pi n \varphi}}{k} d\phi \ll \sum_{1 \leq k \leq N} \sum_{0 \leq h < k} \frac{e^{2\pi \varphi(n + \frac{\varphi}{2})}}{kN} \ll e^{2\pi \varphi(n + \frac{\varphi}{2})} = e^{\frac{2\pi}{N^2}}(n + \frac{\varphi}{2}).
\]

### 8 Major arcs

Let $(\kappa, \ell) \in \mathcal{L}_{>0}$, namely, $\Delta(\kappa, \ell) > 0$. Again, we write $\mathcal{J}^{*} = \mathcal{J}_{\kappa, \ell}^{*}$ and $\mathcal{J}^{**} = \mathcal{J}_{\kappa, \ell}^{**}$. Let us consider the Farey arcs with respect to $h/k$ with $k \equiv \ell \pmod{L}$ and $h \equiv \kappa \pmod{\ell}$. For convenience, we write $\tilde{\xi}_j(h, k) = r_j \tau \gamma^{*}_{(m_j, h, k)}(m_j \tau) + \lambda_{m_j, r_j}(h, k) \gamma_{(m_j, h, k)}(m_j \tau)$ and $\tilde{\tau}_j(h, k) = \gamma_{(m_j, h, k)}(m_j \tau)$.

Recall that

\[
S_{\kappa, \ell} = \sum_{1 \leq k \leq N} \sum_{0 \leq h < k} \frac{\omega_{h,k}^2}{k} \prod_{h \equiv \kappa \pmod{\ell}} \int_{\xi_{h,k}} \exp \left( \frac{\pi}{12k} (\Omega z + \Delta(\kappa, \ell) z^{-1}) \right) \prod_{j=1}^J \text{Li}(\xi_j(h, k); \tilde{\tau}_j(h, k)) \times e^{-2\pi i n \phi} e^{2\pi n \phi} d\phi.
\]

We split $S_{\kappa, \ell}$ into two parts $\Sigma_1$ and $\Sigma_2$ where

\[
\Sigma_1 := \sum_{1 \leq k \leq N} \sum_{0 \leq h < k} \frac{\omega_{h,k}^2}{k} \prod_{h \equiv \kappa \pmod{\ell}} \int_{\xi_{h,k}} \exp \left( \frac{\pi}{12k} (\Omega z + \Delta(\kappa, \ell) z^{-1}) \right) \prod_{j=1}^J \text{Li}(\xi_j(h, k); \tilde{\tau}_j(h, k))
\]
\[
\times \int_{\xi_{h,k}} \exp \left( \frac{\pi}{12k} (\Omega z + \Delta(x, \ell)z^{-1}) \right) \Pi_{h,k} e^{-2\pi i n_\phi} e^{2\pi n_\phi} d\phi
\]

and

\[
\Sigma_2 := \sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop k \equiv \ell \mod L} e^{-\frac{2\pi i n_\phi}{k}} (-1)^{\sum_{j=1}^{J} \delta_j \lambda_{m_j, r_j} (h, k) \omega_{h, k}^2} \chi_{h, k}
\]

\[
\times \int_{\xi_{h,k}} \exp \left( \frac{\pi}{12k} (\Omega z + \Delta(x, \ell)z^{-1}) \right) \left( \prod_{j=1}^{J} \chi^{\delta_j} (\tilde{\zeta}_j (h, k); \tilde{\tau}_j (h, k)) - \Pi_{h,k} \right)
\]

\[
\times e^{-2\pi i n_\phi} e^{2\pi n_\phi} d\phi.
\]

We first show that \( \Sigma_2 \) is negligible. Notice that by (5.2),

\[
|\Sigma_2| \leq \sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop k \equiv \ell \mod L} e^{2\pi \varphi (n + \frac{\Omega}{2\pi})} |\Pi_{h,k}|
\]

\[
\times \int_{\xi_{h,k}} \exp \left( \frac{\pi \Delta(x, \ell)}{12k} R(z^{-1}) \right) \left| \frac{1}{\Pi_{h,k}} \prod_{j=1}^{J} \chi^{\delta_j} (\tilde{\zeta}_j (h, k); \tilde{\tau}_j (h, k)) - 1 \right| d\phi.
\]

Let us fix \( h \) and \( k \) and write \( \tilde{\zeta}_j = \tilde{\zeta}_j (h, k) \) and \( \tilde{\tau}_j = \tilde{\tau}_j (h, k) \). We also write \( \lambda_j^* = \lambda_{m_j, r_j}^* (h, k) \). Recalling the definition of \( \Pi_{h,k} \) and Proposition 6.3, we have

\[
\frac{1}{\Pi_{h,k}} \prod_{j=1}^{J} \chi^{\delta_j} (\tilde{\zeta}_j; \tilde{\tau}_j) - 1 = \prod_{j^* \in J^*} \left( e^{2\pi i (\tilde{\tau}_{j^*} + \tilde{\zeta}_{j^*})}, e^{2\pi i (\tilde{\tau}_{j^*} - \tilde{\zeta}_{j^*})}; e^{2\pi i \tilde{\tau}_{j^*}} \right)_{\infty}^{\delta_j^*}
\]

\[
\times \prod_{j^{**} \in J^{**}} \left( e^{2\pi i \tilde{\zeta}_{j^{**}}}, e^{2\pi i (\tilde{\tau}_{j^{**}} - \tilde{\zeta}_{j^{**}})}; e^{2\pi i \tilde{\tau}_{j^{**}}} \right)_{\infty}^{\delta_{j^{**}}} - 1.
\]

Let us write for short

\[
\tilde{\zeta}_{j}^{\text{New}} = \begin{cases} 
\tilde{\tau}_j + \tilde{\zeta}_j & \text{if } j \in J^*, \\
\tilde{\zeta}_j & \text{if } j \in J^{**}.
\end{cases}
\]
Asymptotics for the Taylor coefficients...

It follows again from (4.2) and (4.4) that

$$\Im(\tilde{\tau}_j) = \frac{\gcd^2(m_j, \ell)}{m_j k} \Re(z^{-1}),$$

$$\Im(\tilde{\varsigma}_j) = \lambda^*_j \frac{\gcd^2(m_j, \ell)}{m_j k} \Re(z^{-1})$$

and

$$\Im(\tilde{\varsigma}^\text{New}_j) = \Phi(\lambda^*_j) \frac{\gcd^2(m_j, \ell)}{m_j k} \Re(z^{-1}),$$

where for real $0 \leq x < 1$,

$$\Phi(x) := \begin{cases} 1 & \text{if } x = 0, \\ x & \text{otherwise}. \end{cases}$$

We have

$$\left| \frac{1}{\prod_{h,k} J \prod_{j=1}^J \mathcal{K}^\delta_j(\tilde{\varsigma}_j; \tilde{\tau}_j)} - 1 \right|$$

$$= \left| \prod_{j=1}^J \left( e^{2\pi i \tilde{\varsigma}^\text{New}_j}, e^{2\pi i (\tilde{\tau}_j - \tilde{\varsigma}_j)}; e^{2\pi i \tilde{\tau}_j} \right)_\infty - 1 \right|$$

$$\leq \sum_{n:=(n_1, \ldots, n_J) \in \mathbb{Z}_{\geq 0}^J} \sum_{s:=(s_1, \ldots, s_J) \in \mathbb{Z}_{\geq 0}^J} \sum_{t:=(t_1, \ldots, t_J) \in \mathbb{Z}_{\geq 0}^J} \prod_{j=1}^J p^*_{[\delta_j]}(s_j, t_j; n_j) e^{2\pi i \tilde{\varsigma}^\text{New}_j} |s_j| e^{2\pi i (\tilde{\tau}_j - \tilde{\varsigma}_j)} |t_j| e^{2\pi i \tilde{\tau}_j} |n_j| - 1$$

$$= \sum_{n \times s \times t \in (\mathbb{Z}_{\geq 0}^J \setminus \{0, \ldots, 0\})^3 \setminus \{0, \ldots, 0\}^3} \prod_{j=1}^J p^*_{[\delta_j]}(s_j, t_j; n_j) e^{2\pi i \tilde{\varsigma}^\text{New}_j} |s_j| e^{2\pi i (\tilde{\tau}_j - \tilde{\varsigma}_j)} |t_j| e^{2\pi i \tilde{\tau}_j} |n_j|$$

$$= \sum_{n \times s \times t \in (\mathbb{Z}_{\geq 0}^J \setminus \{0, \ldots, 0\})^3 \setminus \{0, \ldots, 0\}^3 \setminus \{0, \ldots, 0\}^3} \prod_{j=1}^J p^*_{[\delta_j]}(s_j, t_j; n_j) e^{-2\pi \Im(\tilde{\varsigma}^\text{New}_j) s_j} e^{-2\pi \Im(\tilde{\tau}_j - \tilde{\varsigma}_j) t_j} e^{-2\pi \Im(\tilde{\tau}_j) n_j}$$

$$= \sum_{n \times s \times t \in (\mathbb{Z}_{\geq 0}^J \setminus \{0, \ldots, 0\})^3 \setminus \{0, \ldots, 0\}^3 \setminus \{0, \ldots, 0\}^3} \left( \prod_{j=1}^J p^*_{[\delta_j]}(s_j, t_j; n_j) \right)$$

$$\times \exp \left( -2\pi \frac{\Re(z^{-1})}{k} \sum_{j=1}^J \frac{\gcd^2(m_j, \ell)}{m_j} (\Phi(\lambda^*_j) s_j + (1 - \lambda^*_j) t_j + n_j) \right).$$
Hence,

\[
\exp\left(\frac{\pi \Delta(z, \ell)}{12k} \Re(z^{-1})\right) \left| \frac{1}{\Pi_{h,k}} \prod_{j=1}^{J} \chi_{\delta_j}^j(\tilde{z}_j; \tilde{\tau}_j) - 1 \right| \\
\leq \sum_{n \times s \times t \in (\mathbb{Z}_{\geq 0})^J \setminus (0, \ldots, 0)^3} \left( \prod_{j=1}^{J} p_{\delta_j}^*(s_j, t_j; n_j) \right) \\
\times \exp\left( - \frac{2\pi}{k} \Re(z^{-1}) \left( - \frac{\Delta(z, \ell)}{24} + \sum_{j=1}^{J} \frac{\gcd^2(m_j, \ell)}{m_j} (\Phi(\lambda_j^*) s_j + (1 - \lambda_j^*) t_j + n_j) \right) \right).
\]

Since at least one coordinate of \( n \times s \times t \) is nonzero, under the condition (6.1), we know that

\[
- \frac{\Delta(z, \ell)}{24} + \sum_{j=1}^{J} \frac{\gcd^2(m_j, \ell)}{m_j} (\Phi(\lambda_j^*) s_j + (1 - \lambda_j^*) t_j + n_j) \\
\geq - \frac{\Delta(z, \ell)}{24} + \min_{1 \leq j \leq J} \left( \Upsilon(\lambda_j^*) \frac{\gcd^2(m_j, \ell)}{m_j} \right) \geq 0
\]

for all \( n \times s \times t \in (\mathbb{Z}_{\geq 0})^J \setminus (0, \ldots, 0)^3 \). Recalling (5.3), it follows that

\[
\exp\left(\frac{\pi \Delta(z, \ell)}{12k} \Re(z^{-1})\right) \left| \frac{1}{\Pi_{h,k}} \prod_{j=1}^{J} \chi_{\delta_j}^j(\tilde{z}_j; \tilde{\tau}_j) - 1 \right| \\
\leq \exp\left(\frac{\pi \Delta(z, \ell)}{24} \right) \sum_{n \times s \times t \in (\mathbb{Z}_{\geq 0})^J \setminus (0, \ldots, 0)^3} \left( \prod_{j=1}^{J} p_{\delta_j}^*(s_j, t_j; n_j) \right) \\
\times \exp\left( - \pi \sum_{j=1}^{J} \frac{\gcd^2(m_j, \ell)}{m_j} (\Phi(\lambda_j^*) s_j + (1 - \lambda_j^*) t_j + n_j) \right) \\
\ll 1.
\]
Together with the fact \( \prod_{h,k} \ll 1 \) which follows from (6.2), we conclude that

\[
\Sigma_2 \ll \sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop k \equiv \ell \mod L \gcd(h,k) = 1} e^{2\pi \varphi(n + \frac{a}{k} \varphi)} \int_{\xi_{h,k}} 1 \, d\phi
\]

\[
\ll \sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop k \equiv \ell \mod L \gcd(h,k) = 1} e^{2\pi \varphi(n + \frac{a}{k} \varphi)} \frac{1}{kN}
\]

\[
\ll e^{2\pi \varphi(n + \frac{a}{k} \varphi)} = e^{\frac{2\pi}{N^2}(n + \frac{a}{k} \varphi)}.
\]

Finally, we estimate the main contribution \( \Sigma_1 \). To do so, we need the following evaluation of an integral, which is a special case of Lemma 2.4 in [6]. For the sake of completeness, we sketch a brief proof.

**Lemma 8.1** Let \( a \in \mathbb{R}_{>0} \) and \( b \in \mathbb{R} \). Let \( \gcd(h, k) = 1 \). Define

\[
I := \int_{\xi_{h,k}} e^{\frac{\pi}{12k} \left( \frac{a}{z} + \frac{b}{z^2} \right)} e^{-2\pi i \varphi} e^{2\pi n \varphi} \, d\phi.
\]  

(8.1)

Then for positive integers \( n \) with \( n > -b/24 \), we have

\[
I = \frac{2\pi}{k} \left( \frac{24n + b}{a} \right)^{-\frac{1}{2}} I_{-1} \left( \frac{\pi}{6k} \sqrt{a(24n + b)} \right) + E(I),
\]

(8.2)

where

\[
E(I) \ll a \frac{e^{2\pi \varphi(n + \frac{b}{24})}}{n + \frac{b}{24}}.
\]

(8.3)

**Proof** Putting \( w = z/k = \varphi - i\phi \) and reversing the integral order, one has

\[
I = \frac{1}{2\pi i} \int_{e^{-i\theta''_{h,k}}}^{e^{i\theta''_{h,k}}} 2\pi e^{\frac{\pi a}{12k^2 w}} e^{2\pi w \left( n + \frac{b}{24} \right)} \, dw.
\]

We now split the integral into three parts:

\[
I = \frac{1}{2\pi i} \left( \int_{-\infty}^{\theta''_{h,k}} - \int_{-\infty}^{-i\theta''_{h,k}} + \int_{\theta''_{h,k}}^{\theta'_{h,k}} \right) 2\pi e^{\frac{\pi a}{12k^2 w}} e^{2\pi w \left( n + \frac{b}{24} \right)} \, dw
\]

=: \( J_1 - J_2 + J_3 \),

where
\[
\Gamma := (-\infty - i\theta_{h,k}^\prime) \to (\varrho - i\theta_{h,k}^\prime) \to (\varrho + i\theta_{h,k}^\prime) \to (-\infty + i\theta_{h,k}^\prime)
\]
is a Hankel contour.

The dominant contribution to \(I\) comes from \(J_1\). We make the following change of variables \(t = wk \sqrt{(24n + b)/a}\). Then

\[
J_1 = \frac{2\pi}{k} \left( \frac{24n + b}{a} \right)^{-\frac{1}{2}} \frac{1}{2\pi i} \int_{\tilde{\Gamma}} e^{\frac{\pi}{12k} \sqrt{a(24n + b)}(t + \frac{1}{2})} dt,
\]
in which the new contour \(\tilde{\Gamma}\) is still a Hankel contour. Recalling the contour integral representation of \(I_s(x)\):

\[
I_s(x) = \frac{1}{2\pi i} \int_{\Gamma} t^{-x-1} e^{\frac{\pi}{2k} \left(t + \frac{1}{2}\right)} dt \quad (\Gamma \text{ is a Hankel contour}),
\]
we conclude that

\[
J_1 = \frac{2\pi}{k} \left( \frac{24n + b}{a} \right)^{-\frac{1}{2}} I_{-1} \left( \frac{\pi}{6k} \sqrt{a(24n + b)} \right).
\]

We next bound the error term \(E(I)\), coming from \(J_2\) and \(J_3\). Let \(w = x + i\theta\) with \(-\infty \leq x \leq \varrho\) and \(\theta \in \{\theta_{h,k}^\prime, -\theta_{h,k}^\prime\}\). We know that

\[
\left| e^{2\pi w(n + \frac{b}{24})} \right| = e^{2\pi x(n + \frac{b}{24})}
\]
and

\[
\left| e^{\frac{\pi a}{12k} \vartheta} \right| = e^{\frac{\pi a}{12k} \vartheta} \leq e^{\frac{\pi a}{12k} \frac{x}{x + \theta}} \leq e^{\frac{\pi a}{12k} \frac{1}{\theta}} \leq e^{\frac{\pi a}{12k} \varphi(2kN)^2} = e^{\frac{\pi a}{k}} \ll a,
\]
where we use the fact \(\frac{1}{2kN} \leq |\theta| \leq \frac{1}{kN}\). Hence for \(j = 2\) and \(3\), we have

\[
|J_j| \ll a \int_{-\infty}^{\varrho} e^{2\pi x(n + \frac{b}{24})} dx \ll_a \frac{e^{2\pi \varphi(n + \frac{b}{24})}}{n + \frac{b}{24}}.
\]

This implies that

\[
|E(I)| = | - J_2 + J_3 | \leq |J_2| + |J_3| \ll_a \frac{e^{2\pi \varphi(n + \frac{b}{24})}}{n + \frac{b}{24}},
\]
which gives (8.3). \(\square\)
Recall that
\[
\Sigma_1 = \sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop \gcd(h,k)=1} e^{-2\pi i n h \over k} (-1)^{\sum_{j=1}^{J} \delta_j \lambda_m j \omega_j (h,k)} \omega_h^{2} \mathcal{I}_{h,k} \Pi_{h,k}
\]
\[
\times \int_{\mathbb{C}} \exp \left( \frac{\pi}{12k} (\Omega z + \Delta(\zeta, \ell) z^{-1}) \right) e^{-2\pi i n \phi} e^{2\pi n \varphi} d\phi.
\]

The main contribution to $\Sigma_1$ is
\[
\sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop \gcd(h,k)=1} e^{-2\pi i n h \over k} (-1)^{\sum_{j=1}^{J} \delta_j \lambda_m j \omega_j (h,k)} \omega_h^{2} \mathcal{I}_{h,k} \Pi_{h,k}
\]
\[
\times \frac{2\pi}{k} \left( 24n + \Omega \over \Delta(\zeta, \ell) \right)^{-\frac{1}{2}} I_{-1} \left( \frac{\pi}{6k} \sqrt{\Delta(\zeta, \ell)(24n + \Omega)} \right).
\]

The error term in $\Sigma_1$ is bounded by
\[
\sum_{1 \leq k \leq N} \sum_{0 \leq h < k \atop \gcd(h,k)=1} \frac{e^{2\pi \varphi \left( n + \Omega \over 24 \right)}}{n + \Omega \over 24} \ll \frac{N^2 e^{2\pi n \varphi \over 24}}{n + \Omega \over 24}.
\]

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