Dual Gravitation

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Abstract
We propose a canonical relation between gravity and space-time noncommutativity.

1 Introduction

The subject of space-time noncommutativity is now being the focus of considerable interest. Space-time noncommutativity was first clearly noticed in effective field theories obtained from string theory [1]. In this context it involves a kind of constant noncommutativity which violates Lorentz covariance. This is an unattractive feature and caused the interest in this type of space-time noncommutativity to be rather limited.

A different kind of noncommutativity for the space-time coordinates appeared in connection with generalizations of special relativity [2,3]. These generalizations are based on principles of relativity which include, in addition to the constant speed of light $c$, some other universal invariant, like a minimum length or a constant radius of curvature. The space-time noncommutativity associated to these generalizations of special relativity is contained in generalizations of the Poincaré algebra, but it is not clear if these generalized space-time algebras have a physical meaning.

The first Lorentz-covariant noncommutative space-time was proposed by H. S. Snyder [4] in 1947. Because the Snyder commutators are based on a projective geometry approach in momentum space to de Sitter space, this motivated the desire to understand the relation between gravitation and space-time noncommutativity. The search for this relation was further motivated by the theoretical verification that, when quantum measurement processes involve energies of the order of the Planck scale, the fundamental assumption of locality is no longer a good approximation in theories containing gravity [5]. The measurements alter the space-time metric in a fundamental manner governed by the flat space-time commutation relation

$$[x_\mu, p_\nu] = i\eta_{\mu\nu}$$

and the classical field equations of gravitation [5]. This change in the space-time metric destroys the locality, and hence the commutativity, of position mea-
measurements operators [5]. Despite knowledge of these results, a clear relation between gravitation and space-time noncommutativity was lacking until now. Researchers in the field usually take the working hypothesis [6] that there is one physical property which at large scales manifests itself as gravity, and at small scales as noncommutativity.

In this work we present formal manipulations which indicate that there is a canonical relation between gravitation and space-time noncommutativity. According to this canonical relation, in a noncommutative space-time, usual gravitational fields, which depend on the space-time positions only, can not exist. In a noncommutative space-time, only momentum-dependent gravitational fields can exist. This new and surprising picture leads us to the concept of dual gravitation. As we show here, the usual picture of a position-dependent gravitational field defined over a commutative space-time can be obtained from this new picture by performing a canonical duality transformation. The results of this work therefore elucidate at least one of the many possible relations between gravity and noncommutativity.

The paper is divided as follows. In section two we review the basics of massless relativistic particle theory and show how the classical analogue of Snyder’s noncommutative quantized space-time can be constructed in particle theory. We then discuss how the classical canonical brackets we found lead to the concept of dual gravitation. In section three we extend massless particle theory to a more symmetric theory in a higher dimensional space-time and show how two gauge-equivalent sets of canonical brackets can be constructed for this extended theory. These two dual sets of canonical brackets give a clear picture of the relation between gravitation and space-time noncommutativity in the extended theory and confirm the idea of dual gravitation. Some concluding remarks appear in section four.

2 Massless Relativistic Particles

A massless relativistic particle in a $d$-dimensional Minkowski space-time with signature $(d - 1, 1)$, where $d$ is the number of space-like dimensions, is described by the action

$$S = \frac{1}{2} \int d\tau \lambda^{-1} \dot{x}^2$$

where a dot denotes derivatives with respect to the parameter $\tau$. Action (2.1) is invariant under the local infinitesimal reparametrizations

$$\delta x_\mu = \epsilon(\tau) \dot{x}_\mu$$

$$\delta \lambda = \frac{d}{d\tau}[\epsilon(\tau) \lambda]$$

and therefore describes gravity on the world-line. Action (2.1) is also invariant under the global Poincaré transformations

$$\delta x^\mu = a^\mu + \omega_\nu^\mu x^\nu$$
\( \delta \lambda = 0 \) \hspace{1cm} (2.3b)

where \( \omega_{\nu \mu} = -\omega_{\mu \nu} \), under the global scale transformations

\[
\delta x^\mu = \alpha x^\mu \\
\delta \lambda = 2\alpha \lambda
\] \hspace{1cm} (2.4a)

where \( \alpha \) is a constant, and under the conformal transformations

\[
\delta x^\mu = (2x^\mu x^\nu - \eta^{\mu \nu} x^2) b_\nu \\
\delta \lambda = 4\lambda x.b
\] \hspace{1cm} (2.5a)

where \( b_\mu \) is a constant vector. As a consequence of the presence of these global invariances we can define in space-time the following field

\[
V = a^\mu p_\mu - \frac{1}{2} \omega^{\mu \nu} M_{\mu \nu} + \alpha D + b^\mu K_\mu
\] \hspace{1cm} (2.6)

with the generators

\[
p_\mu \\
M_{\mu \nu} = x_\mu p_\nu - x_\nu p_\mu \\
D = x.p \\
K_\mu = 2x_\mu x.p - x^2 p_\mu
\] \hspace{1cm} (2.7a)

\( p_\mu \) generates translations in space-time, \( M_{\mu \nu} \) generates space-time rotations, \( D \) is the generator of space-time dilatations and \( K_\mu \) generates conformal transformations. These generators define the algebra

\[
\{ p_\mu , p_\nu \} = 0 \\
\{ p_\mu , M_{\nu \lambda} \} = \eta_{\mu \nu} p_\lambda - \eta_{\mu \lambda} p_\nu \\
\{ M_{\mu \nu} , M_{\lambda \rho} \} = \eta_{\mu \lambda} M_{\rho \nu} + \eta_{\mu \rho} M_{\lambda \nu} - \eta_{\mu \nu} M_{\lambda \rho} - \eta_{\mu \lambda} M_{\rho \nu} \\
\{ D , D \} = 0 \\
\{ D , p_\mu \} = p_\mu \\
\{ D , M_{\mu \nu} \} = 0 \\
\{ D , K_\mu \} = -K_\mu \\
\{ p_\mu , K_\nu \} = -2\eta_{\mu \nu} D + 2M_{\mu \nu} \\
\{ M_{\mu \nu} , K_\lambda \} = \eta_{\mu \lambda} K_\nu - \eta_{\nu \lambda} K_\mu \\
\{ K_\mu , K_\nu \} = 0
\] \hspace{1cm} (2.8h)

computed in terms of the Poisson brackets

\[
\{ p_\mu , p_\nu \} = 0
\] \hspace{1cm} (2.9a)
\[ \{x_\mu, p_\nu\} = \eta_{\mu\nu} \] (2.9b)
\[ \{x_\mu, x_\nu\} = 0 \] (2.9c)

The algebra (2.8) is the conformal space-time algebra. The massless particle theory defined by action (2.1) is a conformal theory in \( d \) dimensions.

As is well known, conformal invariance in \( d \) dimensions is equivalent to Lorentz invariance in \( d + 2 \) dimensions. By defining [7]

\[ L_{\mu\nu} = M_{\mu\nu} \] (2.10a)
\[ L_{\mu d} = \frac{1}{2}(p_\mu + K_\mu) \] (2.10b)
\[ L_{\mu(d+1)} = \frac{1}{2}(p_\mu - K_\mu) \] (2.10c)
\[ L_{d(d+1)} = D \] (2.10d)

the conformal algebra (2.8) can be put in the standard form

\[ \{L_{MN}, L_{RS}\} = \delta_{MR}L_{NS} + \delta_{NS}L_{MR} - \delta_{MS}L_{NR} - \delta_{NR}L_{MS} \] (2.11)

with \( M, N = 0, 1, ..., d, d + 1 \) and \( \eta_{MN} = \text{diag}(-1, +1, ..., +1, -1) \). This shows that there are hidden dimensions in massless particle theory. In the next section we will use these hidden dimensions to generalize the world-line gravity action (2.1) to a more symmetric theory in a \( (d + 2) \)-dimensional space-time.

In the transition to the Hamiltonian formalism action (2.1) gives the canonical momenta

\[ p_\lambda = 0 \] (2.12)
\[ p_\mu = \dot{x}_\mu \] (2.13)

and the canonical Hamiltonian

\[ H = \frac{1}{2} \lambda p^2 \] (2.14)

Equation (2.12) is a primary constraint [8]. Introducing the Lagrange multiplier \( \xi(\tau) \) for this constraint we can write the Dirac Hamiltonian

\[ H_D = \frac{1}{2} \lambda p^2 + \xi p_\lambda \] (2.15)

Requiring the dynamical stability of constraint (2.12), \( \dot{p}_\lambda = \{p_\lambda, H_D\} = 0 \), we obtain the secondary constraint

\[ \phi = \frac{1}{2} p^2 \approx 0 \] (2.16)

Constraint (2.16) has a vanishing Poisson bracket with constraint (2.12), being therefore a first-class constraint [8]. Constraint (2.12) generates translations in the arbitrary variable \( \lambda(\tau) \) and can be dropped from the formalism.
In equation (2.16) we have introduced the **weak equality symbol** \( \approx \). This is to emphasize that constraint \( \phi \) is numerically restricted to be zero in the subspace of phase space where the canonical coordinates \((x^\mu, p^\mu)\) satisfy equation (2.16), but it does not identically vanish throughout phase space. In particular, it has nonzero Poisson brackets with the canonical positions. More generally, two functions \( F \) and \( G \) that coincide on the submanifold of phase space defined by the constraint \( \phi \approx 0 \) are said to be **weakly equal** and one writes \( F \approx G \).

On the other hand, an equation that holds throughout phase space and not just on the submanifold \( \phi \approx 0 \) is called **strong**, and the usual equality symbol is used in that case. It can be demonstrated that, in general [9]

\[
F \approx G \iff F - G = c_i(x, p)\phi_i
\]

Now we point out that the massless particle Hamiltonian (2.14) is invariant under the local scale transformations

\[
p_\mu \to \tilde{p}_\mu = \exp\{-\beta\}p_\mu
\]

\[
\lambda \to \exp\{2\beta\}\lambda
\]

where \( \beta \) is an arbitrary function of \( x \) and \( p \). From the equation (2.13) for the canonical momentum we find that \( x^\mu \) transforms as

\[
x^\mu \to \tilde{x}^\mu = \exp\{\beta\} x^\mu
\]

when \( p_\mu \) transforms as in (2.18a). The local scale invariance (2.18) of the massless particle Hamiltonian (2.14) is the residue of a broken gauge invariance of action (2.1). The Lagrangian is not invariant because the kinetic term \( \dot{x}p \) in the Legendre transformation, \( L = \dot{x}p - H \), is not invariant under transformation (2.18). Perhaps the notion of broken local scale invariance may be the clue for the quantum mechanics of the gravitational field.

Consider now the bracket structure that transformations (2.18a) and (2.18c) induce in the massless particle phase space. Retaining only the linear terms in \( \beta \) in the exponentials, we find that the new transformed canonical variables \((\tilde{x}_\mu, \tilde{p}_\mu)\) obey the brackets

\[
\{\tilde{p}_\mu, \tilde{p}_\nu\} = (\beta - 1)[\{p_\mu, \beta\}p_\nu + \{\beta, p_\nu\}] + \{\beta, \beta\}p_\mu p_\nu
\]

\[
\{\tilde{x}_\mu, \tilde{p}_\nu\} = (1 + \beta)[\delta_{\mu\nu}(1 - \beta) - x_\mu\{\beta, x_\nu\}]
\]

\[
+ (1 - \beta)x_\mu\{\beta, p_\nu\} - \{\beta, \beta\}x_\mu p_\nu
\]

\[
\{\tilde{x}_\mu, \tilde{x}_\nu\} = (1 + \beta)[x_\mu\{\beta, x_\nu\} - x_\nu\{\beta, x_\mu\}] + \{\beta, \beta\}x_\mu x_\nu
\]

If we choose \( \beta = \phi = \frac{1}{2}p^2 \approx 0 \) in equations (2.19) and compute the brackets on the right side in terms of the Poisson brackets (2.9), we find the expressions

\[
\{\tilde{p}_\mu, \tilde{p}_\nu\} = 0
\]

\[
\{\tilde{x}_\mu, \tilde{p}_\nu\} = (1 + \frac{1}{2}p^2)[\eta_{\mu\nu}(1 - \frac{1}{2}p^2) - p_\mu p_\nu]
\]
\[
\{ \tilde{x}_\mu, \tilde{x}_\nu \} = -(1 + \frac{1}{2} p^2) (x_\mu p_\nu - x_\nu p_\mu) \quad (2.20c)
\]

We see from the above equations that, on the constraint surface defined by equation (2.16), the brackets (2.20) reduce to

\[
\begin{align*}
\{ \tilde{p}_\mu, \tilde{p}_\nu \} &= 0 \quad (2.21a) \\
\{ \tilde{x}_\mu, \tilde{p}_\nu \} &= \eta_{\mu\nu} - p_\mu p_\nu \quad (2.21b) \\
\{ \tilde{x}_\mu, \tilde{x}_\nu \} &= - (x_\mu p_\nu - x_\nu p_\mu) \quad (2.21c)
\end{align*}
\]

To impose \( \phi = \frac{1}{2} p^2 \approx 0 \) strongly at the end of the computation of brackets (2.20), the expression for the corresponding Dirac brackets [8] on the right side should be used in place of the Poisson brackets. However, for the special case \( \beta = \phi = \frac{1}{2} p^2 \approx 0 \) we can use the property [9] of the Dirac bracket that, on the first-class constraint surface,

\[
\{ \tilde{G}, \tilde{F} \}_D \approx \{ \tilde{G}, \tilde{F} \} \quad (2.22)
\]

when \( G \) is a first-class constraint and \( F \) is an arbitrary function of the canonical variables. This justifies the use of Poisson brackets to arrive at (2.21).

Now, keeping the same order of approximation used to arrive at brackets (2.19), that is, retaining only the linear terms in \( \beta \), the transformation equations (2.18a) and (2.18c) read

\[
\begin{align*}
\tilde{p}_\mu &= \exp\{-\beta\} p_\mu = (1 - \beta) p_\mu \quad (2.23a) \\
\tilde{x}_\mu &= \exp\{\beta\} x_\mu = (1 + \beta) x_\mu \quad (2.23b)
\end{align*}
\]

Using again the same function \( \beta = \phi = \frac{1}{2} p^2 \approx 0 \) in equations (2.23), we write them as

\[
\begin{align*}
\tilde{p}_\mu &= p_\mu - \frac{1}{2} p^2 p_\mu \quad (2.24a) \\
\tilde{x}_\mu &= x_\mu + \frac{1}{2} p^2 x_\mu \quad (2.24b)
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
\tilde{p}_\mu - p_\mu &= c_\mu(x, p) \phi \quad (2.25a) \\
\tilde{x}_\mu - x_\mu &= d_\mu(x, p) \phi \quad (2.25b)
\end{align*}
\]

where \( c_\mu(x, p) = -p_\mu \) and \( d_\mu(x, p) = x_\mu \). Equations (2.25) are in the form (2.17) and so we can write

\[
\begin{align*}
\tilde{p}_\mu &\approx p_\mu \quad (2.26a) \\
\tilde{x}_\mu &\approx x_\mu \quad (2.26b)
\end{align*}
\]

Using these weak equalities in brackets (2.21) we rewrite them as

\[
\begin{align*}
\{ p_\mu, p_\nu \} &\approx 0 \quad (2.27a) \\
\{ x_\mu, p_\nu \} &\approx \eta_{\mu\nu} - p_\mu p_\nu \quad (2.27b)
\end{align*}
\]
\{x_\mu, x_\nu \} \approx - (x_\mu p_\nu - x_\nu p_\mu) \quad (2.27c)

to emphasize that these brackets are valid only on the constraint surface defined by equation (2.16). In the transition to the quantum theory the brackets (2.27) will reproduce the structure of the Snyder commutators.

Although the space-time coordinates now have non-vanishing classical brackets, which will correspond to non-vanishing commutators in the quantized theory, we can not say that an effective gravitational field appears on the right side of bracket (2.27b), as would be expected from the results in [5]. This is because, according to the current point of view, a physical gravitational field should depend only on the particle’s position. We propose here that this point of view should be enlarged to contain also the notion of momentum-dependent gravitational fields. This is because bracket (2.27c) will unavoidably lead to space-time quantization, and a position-dependent gravitational field could therefore never be a continuous field. If we take the point of view that the resulting space-time geometry can be determined from the gravitational contributions to the flat commutator (1.1), we have to admit the possibility that in a noncommutative space-time the gravitational field can only depend on the particle’s momentum, as is suggested by bracket (2.27b). Momentum which, according to bracket (2.27a), remains continuous, giving therefore a continuous momentum-dependent gravitational field. In the next section we will confirm this interpretation by constructing the dual picture, that is, in a commutative space-time the gravitational field must be position-dependent because momenta are quantized.

As an initial step for the developments in the next section, we rewrite (2.1) in the form

\[ S = \int d\tau (\dot{x} \cdot p - \frac{1}{2} \lambda p^2) \quad (2.28) \]

If we solve the equation of motion for \( p_\mu \) that follows from action (2.28) and insert the solution back into it, we recover action (2.1).

### 3 Two-time physics

The higher-dimensional extension of the massless particle action (2.28) is a gauge theory with two time-like dimensions, usually referred to as “two-time physics” [10-16]. The construction of this theory is based on the introduction of a new gauge invariance in phase space, by gaugeing the duality of the canonical commutator (1.1). This procedure leads to a symplectic Sp(2,\( R \)) gauge theory.

To remove the distinction between position and momentum we rename them \( X_1^M = X^M \) and \( X_2^M = P^M \) and define the doublet \( X_i^M = (X_1^M, X_2^M) \). The local \( Sp(2, R) \) symmetry acts as

\[ \delta X_i^M (\tau) = \epsilon_{ik} \omega^{kl}(\tau) X_i^L (\tau) \quad (3.1) \]

\( \omega^{ij}(\tau) \) is a symmetric matrix containing three local parameters and \( \epsilon_{ij} \) is the Levi-Civita symbol that serves to raise or lower indices. The \( Sp(2, R) \) gauge
field $A^{ij}$ is symmetric in $(i, j)$ and transforms as
\[ \delta A^{ij} = \partial_r \omega^{ij} + \omega^{ik} \epsilon_{kl} A^{lj} + \omega^{jk} \epsilon_{kl} A^{il} \] (3.2)

The covariant derivative is
\[ D_r X_i^M = \partial_r X_i^M - \epsilon_{ik} A^{kl} X_l^M \] (3.3)

An action invariant under the $Sp(2, \mathbb{R})$ gauge symmetry is
\[ S = \frac{1}{2} \int d\tau (D_r X_i^M) \epsilon^{ij} X_j^N \eta_{MN} \] (3.4a)

After an integration by parts this action can be written as
\[ S = \int d\tau (\partial_r X_i^M X_2^N - \frac{1}{2} A^{ij} X_i^M X_j^N) \eta_{MN} \]
\[ = \int d\tau [X.P - (\frac{1}{2} \lambda_3 P^2 + \lambda_2 X.P + \frac{1}{2} \lambda_1 X^2)] \] (3.4b)

where $A^{11} = \lambda_3$, $A^{12} = A^{21} = \lambda_2$, $A^{22} = \lambda_1$ and the canonical Hamiltonian is
\[ H = \frac{1}{2} \lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2} \lambda_3 X^2 \] (3.5)

The equations of motion for the $\lambda$’s give the primary constraints
\[ \phi_1 = \frac{1}{2} P^2 \approx 0 \] (3.6)
\[ \phi_2 = X.P \approx 0 \] (3.7)
\[ \phi_3 = \frac{1}{2} X^2 \approx 0 \] (3.8)

Constraints (3.6)-(3.8), as well as evidences of two-time physics, were independently obtained in [7].

If we consider the Minkowski metric as the background space-time, we find that the surface defined by the constraint equations (3.6)-(3.8) is trivial. The only metric giving a non-trivial surface, preserving the unitarity of the theory, and avoiding the ghost problem is the flat metric with two time-like dimensions [10-16]. Following [10-16] we introduce another space-like dimension and another time-like dimension and work in a Minkowski space-time with signature $(d, 2)$. Action (3.4b) is the $(d + 2)$-dimensional generalization of the $d$-dimensional massless particle action (2.28). Action (3.4b) describes conformal gravity on the world-line.

We use the Poisson brackets
\[ \{ P_M, P_N \} = 0 \] (3.9a)
\[ \{ X_M, P_N \} = \eta_{MN} \] (3.9b)
\{X_M, X_N\} = 0 \quad (3.9c)

where \(M, N = 0, ..., d+1\), and verify that constraints (3.6)-(3.8) obey the algebra

\begin{align*}
\{\phi_1, \phi_2\} &= -2\phi_1 \quad (3.10a) \\
\{\phi_1, \phi_3\} &= -\phi_2 \quad (3.10b) \\
\{\phi_2, \phi_3\} &= -2\phi_3 \quad (3.10c)
\end{align*}

These equations show that all constraints \(\phi\) are first-class. Equations (3.10) represent the symplectic \(Sp(2, \mathbb{R})\) gauge algebra of two-time physics.

Action (3.4) also has a global symmetry under Lorentz transformations \(SO(d, 2)\) with generator \([10-16]\)

\[L_{MN} = \epsilon^{ij} X_i^M X_j^N = X^M P^N - X^N P^M \quad (3.11)\]

It satisfies the space-time algebra (2.11) and is gauge invariant because it has vanishing brackets with the first-class constraints (3.6)-(3.8), \(\{L_{MN}, \phi_i\} = 0\).

Now, Hamiltonian (3.5) is invariant under the local scale transformations

\begin{align*}
X^M &\rightarrow \tilde{X}^M = \exp\{\beta\} X^M \quad (3.12a) \\
P_M &\rightarrow \tilde{P}_M = \exp\{-\beta\} P_M \quad (3.12b) \\
\lambda_1 &\rightarrow \exp\{2\beta\} \lambda_1 \quad (3.12c) \\
\lambda_2 &\rightarrow \lambda_2 \quad (3.12d) \\
\lambda_3 &\rightarrow \exp\{-2\beta\} \lambda_3 \quad (3.12e)
\end{align*}

where \(\beta\) is an arbitrary function of \(X^M(\tau)\) and \(P_M(\tau)\). Keeping only the linear terms in \(\beta\) in transformation (3.12), we can write the brackets

\begin{align*}
\{\tilde{P}_M, \tilde{P}_N\} &= (\beta - 1)[\{P_M, \beta\} P_N + \{\beta, P_N\} P_M] + \{\beta, \beta\} P_M P_N \quad (3.13a) \\
\{\tilde{X}_M, \tilde{P}_N\} &= (1 + \beta)[\eta_{MN}(1 - \beta) - \{X_M, \beta\} P_N] \\
&\quad + (1 - \beta)X_M \{\beta, P_N\} - X_M P_N \{\beta, \beta\} \quad (3.13b) \\
\{\tilde{X}_M, \tilde{X}_N\} &= (1 + \beta)[X_M \{\beta, X_N\} - X_N \{\beta, X_M\}] + X_M X_N \{\beta, \beta\} \quad (3.13c)
\end{align*}

for the transformed canonical variables. If we choose \(\beta = \phi_1 = \frac{1}{2}P^2 \approx 0\) in equations (3.13) and compute the brackets on the right side using the Poisson brackets (3.10), we find the expressions

\begin{align*}
\{\tilde{P}_M, \tilde{P}_N\} &= 0 \quad (3.14a) \\
\{\tilde{X}_M, \tilde{P}_N\} &= (1 + \frac{1}{2}P^2)[\eta_{MN}(1 - \frac{1}{2}P^2) - P_MP_N] \quad (3.14b) \\
\{\tilde{X}_M, \tilde{X}_N\} &= -(1 + \frac{1}{2}P^2)(X_M P_N - X_N P_M) \quad (3.14c)
\end{align*}
We see from the above equations that, on the constraint surface defined by the first-class constraints (3.6)-(3.8), brackets (3.14) reduce to

\[
\{ \tilde{P}_M, \tilde{P}_N \} = 0 \quad (3.15a)
\]

\[
\{ \tilde{X}_M, \tilde{P}_N \} = \eta_{MN} - P_M P_N \quad (3.15b)
\]

\[
\{ \tilde{X}_M, \tilde{X}_N \} = -(X_M P_N - X_N P_M) \quad (3.15c)
\]

where, as in the massless particle case, the property (2.22) of the Dirac bracket was used.

Now, keeping the same order of approximation used to arrive at brackets (3.13), transformation equations (3.12a) and (3.12b) can be written as

\[
\tilde{X}^M - X^M = C_1^M (X, P) \phi_i \quad (3.16a)
\]

\[
\tilde{P}_M - P_M = D_1^M (X, P) \phi_i \quad (3.16b)
\]

with \( C_1^M = X^M, C_2^M = C_3^M = 0 \) and \( D_1^M = -P_M, D_2^M = D_3^M = 0 \). Equations (3.16) are again in the form (2.14) and so we can write

\[
\tilde{X}^M \approx X^M \quad (3.17a)
\]

\[
\tilde{P}_M \approx P_M \quad (3.17b)
\]

Using these weak equalities in brackets (3.15) we rewrite them as

\[
\{ P_M, P_N \} \approx 0 \quad (3.18a)
\]

\[
\{ X_M, P_N \} \approx \eta_{MN} - P_M P_N \quad (3.18b)
\]

\[
\{ X_M, X_N \} \approx -(X_M P_N - X_N P_M) \quad (3.18c)
\]

Brackets (3.18) are the \((d + 2)\)-dimensional extensions of the \(d\)-dimensional brackets (2.27) we found for the massless particle. But now we have a larger gauge invariance and so we can explicitly check the observations we made at the end of the previous section about dual gravitational fields.

We can now perform the gauge duality transformation

\[
X_M \rightarrow P_M \quad (3.19a)
\]

\[
P_M \rightarrow -X_M \quad (3.19b)
\]

\[
\lambda_1 \rightarrow \lambda_3 \quad (3.19c)
\]

\[
\lambda_2 \rightarrow -\lambda_2 \quad (3.19d)
\]

\[
\lambda_3 \rightarrow \lambda_1 \quad (3.19e)
\]

under which, after an integration by parts, the Lagrangian in action (3.4b) transforms as \( \delta L = -\partial_t (X \cdot P) \). Transformation (3.19) therefore leaves action (3.4b) invariant. But transformation (3.19b) changes the gauge function \( \beta = \phi_1 = \frac{1}{2} P^2 \approx 0 \) we used to arrive at brackets (3.18) into the new gauge function.
\( \beta = \phi_3 = \frac{1}{2} X^2 \approx 0 \). Repeating the same steps as before for this choice of \( \beta \), and using again properties (2.14) and (2.22), we arrive at the brackets

\[
\{ P_M, P_N \} = X_M P_N - X_N P_M \tag{3.20a}
\]
\[
\{ X_M, P_N \} = \eta_{MN} + X_M X_N \tag{3.20b}
\]
\[
\{ X_M, X_N \} = 0 \tag{3.20c}
\]

We now clearly see that when the space-time coordinates have a vanishing classical bracket, which will correspond to a commutative continuous space-time in the quantized theory, a position-dependent effective gravitational field given by

\[
G_{MN} = \eta_{MN} + X_M X_N \tag{3.21}
\]

appears on the right side of equation (3.20b). This is the only possibility for a physical gravitational field in a commutative space-time since, according to equation (3.20a), the momenta will be noncommutative and therefore discontinuous in the quantized theory.

We see from brackets (3.18) and (3.20) that the Minkowski metric tensor \( \eta_{MN} \) plays the role of the unit tensor in both the space of the \( G_{MN}(X) \) and the space of the \( G_{MN}(P) \).

\section{4 Concluding remarks}

In this work we proposed a canonical relation between gravity and space-time noncommutativity. According to this canonical relation, in a noncommutative space-time a physical gravitational field can not depend on the particle’s positions because this would imply sudden discontinuities in the field. In this kind of space-time a physical gravitational field can depend only on the particle’s momenta, which are the continuous canonical variables. This situation is inverted in a commutative space-time because now the momenta are the discontinuous canonical variables.

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