Quark-Gluon Bags with Surface Tension

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(Dated: February 8, 2020)

I. INTRODUCTION

Investigation of the strongly interacting matter properties observed in relativistic nuclear collisions has reached the stage when the predictions of the lattice quantum chromodynamics (QCD) can be checked experimentally on the existing data and future measurements at BNL RHIC, CERN SPS, and GSI FAIR. However, a comparison of the theoretical results with the experimental data is not straightforward because during the collision process the matter can have several phase transformations which are difficult to model. The latter reason stimulated the development of a wide range of phenomenological models of the strongly interacting matter equation of state which are used in dynamical simulations.

One of these models is the famous bag model [1] which treats the hadrons as the bags of QGP confined inside a hadron with help of bag pressure. The bag model is able to simultaneously describe the hadron mass spectrum, i.e. the hadron masses and their proper volumes, and the properties of the deconfined phase [2]. This success led to a development of the statistical model of QGP, the gas of bags model (GBM) [3, 4, 5], which itself contains two well-known models of deconfined and confined phases: the bag model of QGP [2] and the hadron gas model [6]. There were hopes [7] that an exact analytical solution of the GBM found in [3] could be helpful in understanding the properties of strongly interacting matter. However, this solution does not allow one to introduce the critical end point of the strongly interacting matter phase diagram. Also, a complicated construction of the line, along which the phase transition order gradually increases, suggested in [7], does look too artificial. Therefore, the present GBM formulation lacks an important physical input and is interesting only as a toy example which can be solved analytically.

On the other hand, the models, which can correctly reproduce the expectation [8, 8, 10] that the end point of the 1st order phase transition (PT) line to QGP should be the 2nd order critical point, are indeed necessary for heavy ion phenomenology. In addition, such phenomenological models can provide us with the information about the phase structure and equation of state of strongly interacting matter which is located between the critical endpoint and the region of the color superconductivity because such an information is unavailable otherwise. Therefore, the present work is devoted to the extension of the GMB. We think that the GMB can be drastically improved by the inclusion of such a vitally important element as the surface tension of the quark-gluon bags.

The dynamical surface tension of the quark-gluon bags was estimated long ago [11], but it was never used in statistical description of the equation of state. Moreover, the estimate of the bag surface tension made in [11] is negligible for \( u \) and \( d \) quarks of and, hence, can be safely neglected in our treatment. Furthermore, although the influence of the surface energy of the QGP bag properties in the vacuum was discussed recently [12], the surface tension of large bags was not included into statistical description of QGP. Therefore, the present paper is devoted to the investigation and analysis of the critical properties of the model of quark-gluon bags with surface tension (QGBST model hereafter).

In statistical mechanics there are several exactly solvable cluster models with the 1st order PT which describe the critical point properties very well. These models are built on the assumptions that the difference of the bulk part (or the volume dependent part) of free energy of two phases disappears at phase equilibrium and that, in addition, the difference of the surface part (or the surface tension) of free energy vanishes at the critical point. The most famous of them is the Fisher droplet model (FDM) [13, 14] which has been successfully used to analyze the condensation of a gaseous phase (droplets of all sizes) into a liquid. The FDM has been applied to many different systems, including nuclear multifragmentation [13], nucleation of real fluids [16], the compressibility factor of real fluids [17], bags of the Ising model [18] and percolation bags [19].
On the basis of the statistical multifragmentation model (SMM) [21] commonly used to study nuclear multifragmentation, there was recently formulated a simplified SMM version which was solved analytically both for infinite [21, 22] and for finite [23, 24, 25] volumes of the system. In the SMM the surface tension temperature dependence differs from that of the FDM, but it was shown [22] that the value of Fisher exponent $\tau_{SMM} = 1.825 \pm 0.025$, which contradicts to the FDM value $\tau_{FDM} \approx 2.16$, is consistent with ISiS Collaboration data [26] and EOS Collaboration data [27]. Lately, our analytical results [22] were confirmed by the numerical studies [28, 29].

Such an experimentally obtained range of the $\tau$ index is of a principal importance because it gives a very strong evidence that the SMM, and, thus, the nuclear matter, has a tricritical endpoint rather than a critical endpoint [21, 22].

This success of the SMM initiated the studies of the surface partitions of large clusters within the Hills and Dales Model [30, 31] and led to a discovery of the origin of the temperature independent surface entropy similar to the FDM. As a consequence, the surface tension coefficient of large clusters consisting of the discrete constituents should linearly depend on the temperature of the system [30] and must vanish at the critical endpoint. However, the present formulation of the Hills and Dales Model [30, 31], which successfully estimates the upper and lower bounds of the surface deformations of the discrete physical clusters, does not look suitable for quark-gluon bags. Therefore, in this work we assume a certain dependence of the surface tension coefficient on temperature and baryonic chemical potential, and concentrate on the impact of surface tension of the quark-gluon bags on the properties of the deconfinement phase diagram and the QCD critical endpoint. A discussion of the origin of the surface tension is a subject of our future work.

Here we will show that the existence of a cross-over at low values of the baryonic chemical potential along with the 1st order deconfinement PT at high baryonic chemical potentials leads to the existence of an additional PT of the 2nd or higher order along the curve where the surface tension coefficient vanishes. Thus, it turns out that the QGBST model predicts the existence of the tricritical rather than critical endpoint.

The paper is organized as follows. Sect. II contains the formulation of the basic ingredients of the GBM. In Sect. III we formulate the QGBST model and analyze all possible singularities of its isobaric partition for vanishing baryonic densities. This analysis is generalized to non-zero baryonic densities in Sect. IV. Sect. V is devoted to the analysis of the surface tension induced PT which exists above the deconfinement PT. The conclusions and research perspectives are summarized in Sect. V.

II. BASIC INGREDIENTS OF THE GBM

To remind the basic ingredients of the GBM let us consider the Van der Waals gas consisting of $n$ hadronic species, which are called bags in what follows, at zero baryonic chemical potential. Its grand canonical partition (GCP) is given by [3]

$$
Z(V, T) = \sum_{\{N_k\}_{k=1}^{n}} \left[ \prod_{k=1}^{n} \frac{[(V - v_1 N_1 - ... - v_n N_n) \phi_k(T)]^{N_k}}{N_k!} \right] \times \theta (V - v_1 N_1 - ... - v_n N_n) ,
$$

(1)

where the function $\phi_k(T) \equiv g_k \phi(T, m_k)$

$$
\phi_k(T) \equiv \frac{g_k}{2\pi^2} \int_0^\infty p^2 dp e^{-\frac{(p^2 + m_k^2)^{1/2}}{T}} = g_k \frac{m_k^2 T}{2\pi^2} K_2 \left(\frac{m_k}{T}\right)
$$

is the particle density of bags of mass $m_k$ and eigen volume $v_k$ and degeneracy $g_k$. Using the standard technique of the Laplace transformation [3, 21] with respect to volume, one obtains the isobaric partition:

$$
\hat{Z}(s, T) \equiv \int_0^\infty dv \exp(-sv) Z(V, T) = \frac{1}{[s - F(s, T)]}
$$

(2)

with

$$
F(s, T) \equiv \sum_{j=1}^{n} \exp(-v_j s) g_j \phi(T, m_j).
$$

(3)

From the definition of pressure in the grand canonical ensemble it follows that, in the thermodynamic limit, the GCP of the system behaves as $Z(V, T) \approx \exp[pV/T]$. An exponentially increasing $Z(V, T)$ generates the rightmost singularity $s^* = p/T$ of the function $\hat{Z}(s, T)$ in variable $s$. This is because the integral over $V$ in Eq. (2) diverges at its upper limit for $s < p/T$. Therefore, the rightmost singularity $s^*$ of $\hat{Z}(s, T)$ gives us the system pressure:

$$
p(T) = T \lim_{V \to \infty} \frac{\ln Z(V, T)}{V} = T s^*(T).
$$

(4)

The singularity $s^*$ of $\hat{Z}(s, T)$ [2] can be calculated from the transcendental equation [3, 21]:

$$
s^*(T) = F(s^*, T).
$$

(5)

As long as the number of bags, $n$, is finite, the only possible singularities of $\hat{Z}(s, T)$ are simple poles. For example, for the ideal gas ($n = 1; v_1 = 0$ in Eq. (6)) $s^* = g_1 \phi(T, m_1)$ and thus from Eq. (5) one gets $p = T g_1 \phi(T, m_1)$ which corresponds to the grand canonical ensemble ideal gas equation of state for the particles of mass $m_1$ and degeneracy $g_1$.

However, in the case of an infinite number of sorts of bags an essential singularity of $Z(s, T)$ may appear. This property is used in the GBM: to the finite sum over different bag states in [2] the integral $\int_{M_0}^{\infty} dm \, dv ... p(m, v)$ is added with the bag mass-volume spectrum, $\rho(m, v)$,
which defines the number of bag states in the mass-volume region $[m, v; m+dm, v+dv]$. In this case the function $F(s, T)$ in Eqs. (2) and (4) should be replaced by

$$\begin{align*}
F(s, T) &= F_H(s, T) + F_Q(s, T) = \sum_{j=1}^{n} g_{j} e^{-v_{j} s} \phi(T, m_{j}) \\
&+ \int_{V_{0}}^{\infty} d\nu \int_{M_{\nu}+Bv}^{\infty} \rho(m, v) \exp(-sv)\phi(T, m). \quad (6)
\end{align*}$$

The first term of Eq. (4), $F_H$, represents the contribution of a finite number of low-lying hadron states. This function has no $s$-singularities at any temperature $T$ and can generate a simple pole of the isobaric partition, whereas the mass-volume spectrum of the bags $F_Q(s, T)$ can be chosen to generate an essential singularity $s_Q(T) \equiv p_Q(T)/T$ which defines the QGP pressure $p_Q(T)$ at zero baryonic densities [33, 34]. The usage of the grand canonical description for the exponential mass spectrum introduced by Hagedorn is irrelevant to the present model.

There are several possibilities to parameterize the mass-volume spectrum $\rho(m, v)$. Thus, in the simplest case one can assume that for heavy resonances their mass and eigen volume are proportional, i.e. the spectrum $\rho(m, v)$ contains the function $\delta(m-v\text{Const})$. An alternative choice was suggested in [32], but in either case the resulting expression for the continuum spectrum of the GBM $F_Q(s, T)$ can be cast as

$$F_Q(s, T) = u(T) \int_{V_{0}}^{\infty} dv \frac{\exp[-v(s-s_Q(T))]}{\nu^{\tau}}, \quad (7)$$

where $u(T), \tau > 0$ are the model parameters. The QGP pressure $p(T) = Ts_Q(T)$ can be parameterized in many ways. For instance, the MIT bag model equation of state corresponds to $s_Q(T) = \frac{1}{3} \sigma_Q T^3 - \frac{\tau}{2}$ and $u(T) = C_{\pi^{-1}}^{2} \sigma_{Q}^{2} T^{1/2 + \delta} (\sigma_{Q} T^{4} + B)^{\delta/2}$. Here $B$ denotes the bag constant, $\sigma_Q = \frac{3}{2} \frac{95}{90}$ is the Stefan-Boltzmann constant counting gluons (spin, color) and (anti-)quarks (spin, color and $u, d, s$-flavor) degrees of freedom; and the constants $C, \delta < 0$, $V_{0} \approx 1$ fm$^{3}$ and $M_{0} \approx 2$ GeV are the parameters of the mass-volume spectrum.

### III. THE ROLE OF SURFACE TENSION

At the moment the particular choice of function $F_Q(s, T)$ is not important. The key point for our study is that it should have the form of Eq. (7) which has a singularity at $s = s_Q$ because for $s < s_Q$ the integral over $dv$ diverges at its upper limit. Note that the exponential in (7) is nothing else, but a difference of the bulk free energy of a bag of volume $v$, i.e. $-Ts_{v}$, which is under external pressure $T/\nu$ and the bulk free energy of the same bag filled with QGP, i.e. $-Ts_{Qv}$. At phase equilibrium this difference of the bulk free energies vanishes. Despite all positive features, Eq. (7) lacks the surface part of free energy of bags, which will be called a surface energy hereafter. In addition to the difference of the bulk free energies the realistic statistical models which demonstrated their validity, the FDM [13] and SMM [20], have the contribution of the surface energy which plays an important role in defining the phase diagram structure [21, 22]. Therefore, we modify Eq. (7) by introducing the surface energy of the bags in a general fashion [22]:

$$F_Q = u(T) \int_{V_{0}}^{\infty} dv \frac{\exp[(s_Q(T) - s)v - \sigma(T) \nu^{\tau}]}{\nu^{\tau}}, \quad (8)$$

where the ratio of the temperature dependent surface tension coefficient to $T$ (the reduced surface tension coefficient hereafter) which has the form $\sigma(T) = \frac{\nu^{\tau}}{\nu_{\text{cep}}} \cdot \frac{T_{\text{cep}} - T}{T_{\text{cep}}}^{2k+1}$ $(k = 0, 1, 2, \ldots)$. Here $\nu_{\text{cep}} > 0$ can be a smooth function of the temperature, but for simplicity we fix it to be a constant. For $k = 0$ the two terms in the surface (free) energy of a $v$-volume bag have a simple interpretation [13]: thus, the surface energy of such a bag is $\sigma_{v} \nu^{\nu_{v}}$, whereas the free energy, which comes from the surface entropy $\sigma_{\nu} T_{\text{cep}}^{-1} \nu^{\nu_{v}}$, is $-T \sigma_{\nu} T_{\text{cep}}^{-1} \nu^{\nu_{v}}$. Note that the surface entropy of a $v$-volume bag counts its degeneracy factor or the number of ways to make such a bag with all possible surfaces. This interpretation can be extended to $k > 0$ on the basis of the Hills and Dales Model [31, 32].

In choosing such a simple surface energy parameterization we follow the original Fisher idea [13] which allows one to account for the surface energy by considering some mean bag of volume $v$ and surface $v^{\nu_{v}}$. The consideration of the general mass-volume-surface bag spectrum we leave for the future investigation. The power $\nu_{v} < 1$ which describes the bag’s effective surface is a constant which, in principle, can differ from the typical FDM and SMM value $\frac{1}{3}$. This is so because near the deconfinement PT region QGP has low density and, hence, like in the low density nuclear matter [29], the non-spherical bags (spaghetti-like or lasagna-like [38]) can be favorable. A similar idea of “polymerization” of gluonic quasiparticles was introduced recently [33].

The second essential difference between the FDM and SMM surface tension parameterization is that we do not require the vanishing of $\sigma(T)$ above the CEP. As will be shown later, this is the most important assumption which, in contrast to the GBM, allows one to naturally describe the cross-over from hadron gas to QGP. Note that negative value of the reduced surface tension coefficient $\sigma(T)$ above the CEP does not mean anything wrong. As we discussed above, the surface tension coef-
icient consists of energy and entropy parts which have opposite signs \[12, 30, 31\]. Therefore, \(\sigma(T) < 0\) does not mean that the surface energy changes the sign, but it rather means that the surface entropy, i.e. the logarithm of the degeneracy of bags of a fixed volume, simply exceeds their surface energy. In other words, the number of non-spherical bags of a fixed volume becomes so large that the Boltzmann exponent, which accounts for the energy "costs" of these bags, cannot suppress them anymore.

Finally, the third essential difference with the FDM and SMM is that we assume that the surface tension in the QGBST model happens at some line in \(\mu_B - T\) plane, i.e. \(T_{\text{exp}} = T_{\text{exp}}(\mu_B)\). However, in the subsequent sections we will consider \(T_{\text{exp}} = \text{Const}\) for simplicity, and in Sect. V we will discuss the necessary modifications of the model with \(T_{\text{exp}} = T_{\text{exp}}(\mu_B)\).

The surface energy should, in principle, be introduced into a discrete part of the mass-volume spectrum \(F_H\), but a successful fitting of the particle yield ratios \[6\] with the experimentally determined hadronic spectrum \(F_H\) does not indicate such a necessity.

According to the general theorem \[6\] the analysis of PT existence of the GCP is now reduced to the analysis of the rightmost singularity of the isobaric partition \[3\]. Depending on the sign of the reduced surface tension coefficient, there are three possibilities. \(\text{(I) The first possibility corresponds to } \sigma(T) > 0. \) Its treatment is very similar to the GBM choice \[7\] with \(\tau > 2 \[6\]. \) In this case at low temperatures the QGP pressure \(T_s q(T)\) is negative and, therefore, the rightmost singularity is a simple pole of the isobaric partition \(s^* = s_H(T) = F(s_H(T), T) > s_Q(T)\), which is mainly defined by a discrete part of the mass-volume spectrum \(F_H(s, T)\). The last inequality provides the convergence of the volume integral in \[8\] (see Fig. 1). On the other hand at very high \(T\) the QGP pressure dominates and, hence, the rightmost singularity is the essential singularity of the isobaric partition \(s^* = s_Q(T)\). The phase transition occurs, when the singularities coincide:

\[
\frac{p_H(T_c)}{T_c} = s_Q(T_c) = \frac{p_Q(T_c)}{T_c}, \tag{9}
\]

which is nothing else, but the Gibbs criterion. The graphical solution of Eq. \[5\] for all these possibilities is shown in Fig. 1. Like in the GBM \[3, 6\], the necessary condition for the PT existence is the finiteness of \(F_Q(s_Q(T), T)\) at \(s = s_Q(T)\). It can be shown that the sufficient conditions are the following inequalities: \(F_Q(s_Q(T), T) > s_Q(T)\) for low temperatures and \(F(s_Q(T), T) < s_Q(T)\) for \(T \to \infty\). These conditions provide that at low \(T\) the rightmost singularity of the isobaric partition is a simple pole, whereas for high \(T\) the essential singularity \(s_Q(T)\) becomes its rightmost one (see Fig. 1 and a detailed analysis of case \(\mu_B \neq 0\)).

The PT order can be found from the \(T\)-derivatives of \(s_H(T)\). Thus, differentiating \[5\] one finds

\[
\frac{d s_H}{d T} = \frac{G + u K_{\tau-1}(\Delta, -\sigma) - s'_Q}{1 + u K_{\tau-1}(\Delta, -\sigma)}, \tag{10}
\]

where the functions \(G\) and \(K_{\tau-a}(\Delta, -\sigma)\) are defined as

\[
G \equiv F_H^0 + \frac{u'}{u} F_Q + \frac{(T_{\text{exp}} - 2K_{\tau-1}(\Delta, -\sigma))}{(T_{\text{exp}} - 2K_{\tau-1}(\Delta, -\sigma))} u K_{\tau-a}(\Delta, -\sigma), \tag{11}
\]

\[
K_{\tau-a}(\Delta, -\sigma) \equiv \int_{V_0}^{\infty} \frac{d v \exp[-\Delta v - \sigma(T) v^\sigma]}{v^{\tau-a}}, \tag{12}
\]

where \(\Delta \equiv s_H - s_Q\).

Now it is easy to see that the transition is of the 1st order, i.e. \(s'_Q(T_c) > s'_H(T_c)\), provided \(\sigma(T) > 0\) for any \(\tau\). The 2nd or higher order phase transition takes place provided \(s'_Q(T_c) = s'_H(T_c)\) at \(T = T_c\). The latter condition is satisfied when \(K_{\tau-1}\) diverges to infinity at \(T \to (T_c - 0)\), i.e. for \(T\) approaching \(T_c\) from below. Like for the GBM choice \[11\], such a situation can exist for \(\sigma(T_c) = 0\) and \(\frac{4}{3} < \tau \leq 2\). Studying the higher \(T\)-derivatives of \(s_H(T)\) at \(T_c\), one can show that for \(\sigma(T) \equiv 0\) and \((n + 1)/n < \tau < n/(n - 1)\) \((n = 3, 4, 5, \ldots)\) there is a \(n^{th}\) order phase transition

\[
\frac{d^n s_H(T_c)}{d T^n} = s^{(n)}_H(T_c) = s^{(n)}_Q(T_c), \tag{13}
\]

with \(s^{(n)}_H(T_c) = \infty\) for \((n + 1)/n < \tau < n/(n - 1)\) and with a finite value of \(s^{(n)}_H(T_c)\) for \(\tau = (n + 1)/n\).
(II) The second possibility, $\sigma(T) \equiv 0$, described in the preceding paragraph, does not give anything new compared to the GBM [3, 2]. If the PT exists, then the graphical picture of singularities is basically similar to Fig. 1. The only difference is that, depending on the PT order, the derivatives of $F(s, T)$ function with respect to $s$ should diverge at $s = s_Q(T)$. On the other hand, the partial derivatives $\frac{\partial F_H(s, T)}{\partial s} < 0$ and $\frac{\partial F_Q(s, T)}{\partial s} < 0$ are always negative. Therefore, the function $F(s, T) = F_H(s, T) + F_Q(s, T)$ is a monotonically decreasing function of $s$, which vanishes at $s \to \infty$. Since the left hand side of Eq. (5) is a monotonically increasing function of $s$, then there can exist a single intersection $s^*$ of $s$ and $F(s, T)$ functions. Moreover, for finite $s_Q(T)$ values this intersection can occur on the right hand side of the point $s = s_Q(T)$, i.e. $s^* > s_Q(T)$ (see Fig. 2). Thus, in this case the essential singularity $s = s_Q(T)$ can become the rightmost one for infinite temperature only. In other words, the pressure of the pure QGP can be reached at infinite $T$, whereas for finite $T$ the hadronic mass spectrum gives a non-zero contribution into all thermodynamic functions.

Note also that all these nice properties would vanish, if the reduced surface tension coefficient is zero or positive above $T_{cexp}$. This is one of the crucial points of the present model which puts forward certain doubts about the vanishing of the reduced surface tension coefficient in the FDM [13] and SMM [20]. These doubts are also supported by the first principle results obtained by the Hills and Dales Model [31, 32], because the surface entropy simply counts the degeneracy of a cluster of a fixed volume and it does not physically affect the surface energy of this cluster.

IV. GENERALIZATION TO NON-ZERO BARYONIC DENSITIES

The possibilities (I)-(III) discussed in the preceding section remain unchanged for non-zero baryonic numbers. The latter should be included into consideration to make our model more realistic. To keep the presentation simple, we do not account for strangeness. The inclusion of the baryonic charge of the quark-gluon bags does not change the two types of singularities of the isobaric partition [4] and the corresponding equation for them [5], but it leads to the following modifications of the $F_H$ and $F_Q$ functions:

$$F_H(s, T, \mu_B) = \sum_{j=1}^{n} g_j e^{\frac{\phi(T, m_j)}{v_{ij}}},$$  \hspace{1cm} (14)$$

$$F_Q(s, T, \mu_B) = u(T, \mu_B) \times \int_{V_0}^{\infty} dv \exp \left[(s_Q(T, \mu_B) - s) v - \sigma(T)v^2\right].$$  \hspace{1cm} (15)$$

Here the baryonic chemical potential is denoted as $\mu_B$, the baryonic charge of the $j$-th hadron in the discrete part of the spectrum is $b_j$. The continuous part of the spectrum, $F_Q$ can be obtained from some spectrum $\rho(m, v, b)$ in the spirit of Ref. [32], but this will lead us away from the main subject.

The QGP pressure $p_Q = T s_Q(T, \mu_B)$ can be also chosen in several ways. Here we use the bag model pressure

$$p_Q = \frac{\pi^2 T^4}{90} \left[ \frac{95}{2} + \frac{10}{\pi^2} \left( \frac{\mu_B}{T} \right)^2 + \frac{5}{3} \frac{\left( \frac{\mu_B}{T} \right)^4}{4} \right] - B,$$  \hspace{1cm} (16)$$

but the more complicated model pressures, even with the PT of other kind like the transition between the color superconducting QGP and the usual QGP, can be, in principle, used.

The sufficient conditions for a PT existence are

$$F(s_Q(T, \mu_B = 0), T, \mu_B = 0) > s_Q(T, \mu_B = 0),$$  \hspace{1cm} (17)$$

$$F(s_Q(T, \mu_B), T, \mu_B) < s_Q(T, \mu_B),$$  \hspace{1cm} (18)$$

The condition (17) provides that the simple pole singularity $s^* = s_H(T, \mu_B = 0)$ is the rightmost one at vanishing $\mu_B = 0$ and given $T$, whereas the condition (18) ensures that $s^* = s_Q(T, \mu_B)$ is the rightmost singularity of the isobaric partition for all values of the baryonic chemical potential above some positive constant $\mu_A$. This can be seen in Fig. 1 for $\mu_B$ being a variable. Since $F(s, T, \mu_B)$, where it exists, is a continuous function of its parameters, one concludes that, if the conditions (17) and (18), are fulfilled, then at some chemical potential $\mu_B^*(T)$ the
both singularities should be equal. Thus, one arrives at the Gibbs criterion (9), but for two variables

\[ s_H(T, \mu_B(T)) = s_Q(T, \mu_B(T)). \] (19)

It is easy to see that the inequalities (17) and (18) are the sufficient conditions of a PT existence for more complicated functional dependencies of \( F_H(s, T, \mu_B) \) and \( F_Q(s, T, \mu_B) \) than the ones used here.

For our choice (14), (15) and (16) of \( F_H(s, T, \mu_B) \) and \( F_Q(s, T, \mu_B) \) functions the PT exists at \( T < T_{cep} \), because the sufficient conditions (17) and (18) can be easily fulfilled by a proper choice of the bag constant \( B \) and the function \( u(T, \mu_B) > 0 \) for the interval \( T \leq T_{up} \) with the constant \( T_{up} > T_{cep} \). Clearly, this is the 1st order PT, since the surface tension is finite and it provides the convergence of the integrals (11) and (12) in the expression (10), where the usual \( T \)-derivatives should be now understood as the partial ones for \( \mu_B = \text{const} \).

Assuming that the conditions (17) and (18) are fulfilled by the correct choice of the model parameters \( B \) and \( u(T, \mu_B) > 0 \), one can see now that at \( T = T_{cep} \) there exists a PT as well, but its order is defined by the value of \( \tau \). As was discussed in the preceding section for \( \frac{3}{2} < \tau \leq 2 \) there exists the 2nd order PT. For \( 1 < \tau \leq \frac{3}{2} \) there exist the PT of higher order, defined by the conditions formulated in Eq. (13). This is a new possibility, which, to our best knowledge, does not contradict to any general physical principle (see Fig. 3).

The case \( \tau > 2 \) can be ruled out because there must exist the first order PT for \( T \geq T_{cep} \), whereas for \( T < T_{cep} \) there exists the cross-over. Thus, the critical endpoint in \( T - \mu_B \) plane will correspond to the critical interval in the temperature-baryonic density plane. Since such a struc-

ture of the phase diagram in the variables temperature-density has, to our knowledge, never been observed, we conclude that the case \( \tau > 2 \) is unrealistic (see Fig. 4). Note that a similar phase diagram exists in the FDM with the only difference that the boundary of the mixed and liquid phases (the latter in the QGBST model corresponds to QGP) is moved to infinite particle density.

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The case \( \tau > 2 \) can be ruled out because there must exist the first order PT for \( T \geq T_{cep} \), whereas for \( T < T_{cep} \) there exists the cross-over. Thus, the critical endpoint in \( T - \mu_B \) plane will correspond to the critical interval in the temperature-baryonic density plane. Since such a struc-

ture of the phase diagram in the variables temperature-density has, to our knowledge, never been observed, we conclude that the case \( \tau > 2 \) is unrealistic (see Fig. 4). Note that a similar phase diagram exists in the FDM with the only difference that the boundary of the mixed and liquid phases (the latter in the QGBST model corresponds to QGP) is moved to infinite particle density.
and $\mu_B < \mu_B^c(T_{cusp})$, one can calculate $s_H(T, \mu_B)$ for the values of $T > T_{cusp}$ which are inside the convergence radius of the Taylor expansion.

The other situation is for $\mu_B \geq \mu_B^c(T_{cusp})$ and $T > T_{cusp}$, namely in this case above the deconfinement PT there must exist a weaker PT induced by the disappearance of the reduced surface tension coefficient. To demonstrate this we have solve Eq. (5) in the limit, when $T$ approaches the curve $T = T_{cusp}$ from above, i.e. for $T \to T_{cusp} + 0$, and study the behavior of $T$ derivatives of the solution of Eq. (5) for fixed values of $\mu_B$. For this purpose we have to evaluate the integrals $K_+(\Delta, \gamma^2)$ introduced in Eq. (12). Here the notations $\Delta \equiv \gamma^2 - s_Q(T, \mu_B)$ and $\gamma^2 = -\sigma(T) > 0$ are introduced for convenience.

To avoid the unpleasant behavior for $\tau \leq 2$ it is convenient to transform Eq. (12) further on by integrating by parts:

$$K_+(\Delta, \gamma^2) = g_+(V_0) - \frac{\Delta}{(\tau - 1)} K_{\tau - \nu}(\Delta, \gamma^2) + \frac{\gamma^2}{(\tau - 1)} K_{\tau - \nu}(\Delta, \gamma^2),$$

where the regular function $g_+(V_0)$ is defined as

$$g_+(V_0) \equiv \frac{1}{(\tau - 1) V_0^{-1}} \exp \left[ -\Delta V_0 + \gamma^2 V_0^\nu \right].$$

For $\tau - a > 1$ one can change the variable of integration $v \to z/\Delta$ and rewrite $K_{\tau - a}(\Delta, \gamma^2)$ as

$$K_{\tau - a}(\Delta, \gamma^2) = \Delta^{\tau - a - 1} \int V_0 \Delta \frac{\exp \left[ -z + \gamma^2 z^{\nu} \right]}{z^{\tau - a}} dz,$$

This result shows that in the limit $\gamma \to 0$, when the right-most singularity must approach $s_Q(T, \mu_B)$ from above, i.e. $\Delta \to 0^+$, the function (22) behaves as $K_{\tau - a}(\Delta, \gamma^2) \sim \Delta^{\tau - a - 1} + O(\Delta^{-\nu})$. This is so because for $\gamma \to 0$ the ratio $\gamma^2 \Delta^{-\nu}$ cannot go to infinity, otherwise the function $K_{\tau - a}(1, \gamma^2 \Delta^{-\nu})$, which enters into the right hand side of Eq. (20), diverges exponentially and makes impossible the existence of the solution of Eq. (5) for $T = T_{cusp}$. The analysis shows that for $\gamma \to 0$ there exist two possibilities: either $\nu \equiv \gamma^2 \Delta^{-\nu} \to Const$ or $\nu \equiv \gamma^2 \Delta^{-\nu} \to 0$.

The most straightforward way to analyze these possibilities for $\gamma \to 0$ is to assume the following behavior

$$\Delta = A \gamma^\alpha + O(\gamma^{\alpha + 1}),$$

and find out the $\alpha$ value by equating (24) with the $T$ derivative (11). Indeed, using (10), (11) and (12), one can write

$$\frac{\partial \Delta}{\partial T} = \frac{G_2 + u K_{\tau - \nu}(\Delta, \gamma^2) 2 \gamma^\alpha}{1 + u K_{\tau - 1}(\Delta, \gamma^2)} \approx \frac{\Delta^{2 - \nu} G_2}{u K_{\tau - 1}(1, \nu) + 2 \gamma^\alpha \Delta^{1 - \nu} [\nu K_{\tau - 2}(1, \nu) - K_{\tau - 1 - \nu}(1, \nu)]},$$

where the prime denotes the partial $T$ derivative. Note that the function $G_2 \equiv F^* + u K_{\tau}(\Delta, \gamma^2) - s_Q$ can vanish for very few values of $\mu_B$ only. In the last step of deriving (25) we used the identities (21) and (22) and dropped the non-singular terms. As we discussed above, in the limit $\gamma \to 0$ the function $\nu$ either remains a constant or vanishes, then the term $\nu K_{\tau - 2}(1, \nu)$ in (25) is either the same order as the constant $K_{\tau - 1 - \nu}(1, \nu)$ or vanishes. Thus, to reveal the behavior of (25) for $\gamma \to 0$ it is sufficient to find a leading term out of $\Delta^{2 - \nu}$ and $\gamma^\alpha \Delta^{1 - \nu}$ and compare it with the assumption (24).

The analysis shows that for $\Delta^{2 - \nu} \leq \gamma^\alpha \Delta^{1 - \nu}$ the last term in the right hand side of (25) is the leading one. Consequently, equating the powers of $\gamma$ of the leading terms in (24) and (25), one finds

$$\gamma^{\alpha - 2} \sim \Delta^{1 - \nu} \Rightarrow \alpha = 2 \quad \text{for} \quad \tau \leq 1 + \frac{\nu}{2k + 1},$$

where the last inequality follows from the fact that the term $\gamma^2 \Delta^{1 - \nu}$ in (25) is the dominant one. Similarly, for $\Delta^{2 - \nu} \geq \gamma^\alpha \Delta^{1 - \nu}$ one obtains $\gamma^{\alpha - 1} \gamma \sim \Delta^{2 - \nu}$ and, consequently,

$$\alpha = \frac{2}{(\tau - 1)(2k + 1)} \quad \text{for} \quad \tau \geq 1 + \frac{\nu}{2k + 1}.$$

Summarizing our results for $\gamma \to 0$ as

$$\frac{\partial \Delta}{\partial T} \sim \frac{T - T_{cusp}}{T_{cusp}}^{\gamma^\alpha},$$

we can also write the expression for the second derivative of $\Delta$ as

$$\frac{\partial^2 \Delta}{\partial T^2} \sim \frac{T - T_{cusp}}{T_{cusp}}^{\gamma^\alpha - 2},$$

The last result shows us that, depending on $\nu$ and $k$ values, the second derivatives of $s_H(T, \mu_B)$ can differ from each other for $\frac{1}{2} < \tau < 2$ or can be equal for $1 < \tau \leq \frac{3}{2}$. In other words, we found that at the line $T = T_{cusp}$ there exists the 2nd order PT for $\frac{3}{2} < \tau < 2$ and the higher order PT for $1 < \tau \leq \frac{3}{2}$, which separates the pure QGP phase from the region of a cross-over, i.e. the mixed states of hadronic and QGP bags. Since it exists at the line of a zero surface tension, this PT will be called the surface induced PT.
FIG. 5: A schematic picture of the deconfinement phase transition diagram (full curve) in the plane of baryonic chemical potential $\mu_B$ and $T$ for the 2nd order PT at the tricritical endpoint (CEP). The model predicts an existence of the surface induced PT of the 2nd or higher order (depending on the model parameters). This PT starts at the CEP and goes to higher values of $T$ and/or $\mu_B$. Here it is shown by the dashed curve CEP-A, if the phase diagram is endless, or by the dashed-dot curve CEP-B, if the phase diagram ends at $T = 0$. Below (above) each of the dashed curves the reduced surface tension coefficient is positive (negative).

Since the analysis performed in the present section did not include any $\mu_B$ derivatives of $\Delta$, it remains valid for the $\mu_B$ dependence of the reduced surface tension coefficient, i.e. for $T_{cep}(\mu_B)$. Only it is necessary to make a few comments on a possible location of the surface tension null line $T_{cep}(\mu_B)$. In principle, such a null line can be located anywhere, if its location does not contradict to the sufficient conditions (17) and (18) of the 1st deconfinement PT existence. Thus, the surface tension null line must cross the deconfinement line in the $\mu_B - T$ plane at a single point which is the tricritical endpoint $(\mu_{cep}^B; T_{cep}(\mu_{cep}^B))$, whereas for $\mu_B > \mu_{cep}^B$ the null line should have higher temperature for the same $\mu_B$ than the deconfinement one, i.e. $T_{cep}(\mu_B) > T_c(\mu_B)$ (see Fig 5). Clearly, there exist two distinct cases for the surface tension null line: either it is endless, or it ends at zero temperature. But recalling that at low temperatures and high values of the baryonic chemical potential there may exist the Color-Flavor-Locked (CFL) phase [40], it is possible that the null line may also cross the boundary of the CFL phase and, perhaps, it may create another special point at this intersection.

One may wonder why this surface induced PT was not observed so far. The main reason is that the lattice QCD calculations at non-zero $\mu_B$ are very difficult, and because of this the identification of the precise location of the critical endpoint is highly nontrivial task [8, 9, 10]. Therefore, the identification of the 2nd or higher order PT which might be located in the vicinity of the deconfinement PT could be a real challenge. In addition, the surface induced PT may lie so close to the deconfinement line that it would be extremely difficult to observe it at the present lattices.

VI. CONCLUSIONS AND PERSPECTIVES

Here we suggested an analytically solvable statistical model which simultaneously describes the 1st and 2nd order PTs with a cross-over. The approach is general and can be used for more complicated parameterizations of the hadronic mass-volume spectrum, if in the vicinity of the deconfinement PT region the discrete and continuous parts of this spectrum can be expressed in the form of Eqs. (14) and (15), respectively. Also the actual parameterization of the QGP pressure $p = T s_Q(T, \mu_B)$ was not used so far, which means that our result can be extended to more complicated functions, that can contain other phase transformations (chiral PT, or the PT to color superconducting phase) provided that the sufficient conditions (17) and (18) for the deconfinement PT existence are satisfied.

In this model the desired properties of the deconfinement phase diagram are achieved by accounting for the temperature dependent surface tension of the quark-gluon bags. As we showed, it is crucial for the cross-over existence that at $T = T_{cep}$ the reduced surface tension coefficient vanishes and remains negative for temperatures above $T_{cep}$. Then the deconfinement $\mu_B - T$ phase diagram has the 1st PT at $\mu_B > \mu_{cep}^B(T_{cep})$ for $\frac{2}{3} < \tau < 2$, which degenerates into the 2nd order PT (or higher order PT for $\tau \geq 1$) at $\mu_B = \mu_{cep}^B(T_{cep})$, and a cross-over for $0 \leq \mu_B < \mu_{cep}^B(T_{cep})$. These two ingredients drastically change the critical properties of the GBM [3] and resolve the long standing problem of a unified description of the 1st and 2nd order PTs and a cross-over, which, despite all claims, was not resolved in Ref. [7]. In addition, we found that at the null line of the surface tension there must exist the surface induced PT of the 2nd or higher order, which separates the pure QGP from the mixed states of hadrons and QGP bags, that coexist above the cross-over region (see Fig 5). Thus, the QGBST model predicts that the QCD critical endpoint is the tricritical endpoint. It would be interesting to verify this prediction with the help of the lattice QCD analysis. For this one will need to study the behavior of the bulk and surface contributions to the free energy of the QGP bags and/or the string connecting the static quark-antiquark pair.

In contrast to popular mean-field models the PT mechanism in the present model is clear: it happens due to the competition of the rightmost singularities of the isobaric partition function. Since the GCP function of the QGBST model does not depend on any (baryonic or entropy or energy) density, but depends exclusively on $T, \mu_B$ and $V$, its phase diagram does not contain any back bending and/or spinodal instabilities [41] which are typical for the mean-field (= classical) models. The found exact analytical solution does not require a complicated and
artificial procedure of conjugating the two parts of the equation of state in the vicinity of the critical endpoint like it is done by hands in Refs. [12, 13] because all this is automatically included in the statistical description.

Also in the QGBST model the pressure of the deconfined phase is generated by the infinite bag, whereas the discrete part of the mass-volume spectrum plays an auxiliary role even above the cross-over region. Therefore, there is no reason to believe that any quantitative changes of the properties of low lying hadronic states generated by the surrounding media (like the mass shift of the $\omega$ and $\rho$ mesons [44]) would be the robust signals of the deconfinement PT. On the other hand, the QGP bags created in the experiments have finite mass and volume and, hence, the strong discontinuities which are typical for the 1st order PT should be smeared out which would make them hardly distinguishable from the cross-over. Thus, to seriously discuss the signals of the 1st order deconfinement PT and/or the tricritical endpoint, one needs to solve the finite volume version of the QGBST model like it was done for the SMM [23] and the GBM [24]. This, however, is not sufficient because, in order to make any reliable prediction for experiments, the finite volume equation of state must be used in hydrodynamic equations which, unfortunately, are not suited for such a purpose. Thus, we are facing a necessity to return to the foundations of heavy ion phenomenology and to modify them according to the requirements of the experiments. The present model can be considered as the next step in this direction.

Although the present model has a great advantage compared to other models because, in principle, it can be formulated on the basis of the experimental data on the degeneracies, masses and eigen volumes of hadronic resonances in the spirit of Ref. [32], a lot of additional work is necessary to properly study the issues addressed in [15]. Thus, above the surface tension null line the hadrons can coexist with QGP at high temperatures. Consequently, the nonrelativistic consideration of hard core repulsion in the present model should be modified to its relativistic treatment for light hadrons like it is suggested in [46, 47]. This can lead to some new effects discussed recently in [47]. Also, the realistic equation of state requires the inclusion of the temperature and mass dependent width of heavy resonances into a continuous part of the mass-volume spectrum which may essentially modify our understanding of the cross-over mechanism [48].

Finally, a precise temperature dependence of the surface tension coefficient of the bags should be investigated and its relation to the interquark string tension should be studied in detail. For this it will be necessary to modify the Hills and Dales Model [30, 31] in order to include the surface deformations with the base of arbitrary size whereas its present formulation is suited for discrete clusters and, hence, for discrete bases of surface deformations.

Acknowledgments. I am thankful to A. Blokhin for important comments and K. Rajagopal for pointing out Ref. [11]. A warm hospitality of the Frankfurt Institute for Advanced Studies, where an essential part of this work was done, is appreciated. The financial support of the Alexander von Humboldt Foundation is acknowledged.

[1] A. Chodos et al., Phys. Rev. D 9 (1974) 3471.
[2] E.V. Shuryak, Phys. Rep. 61 (1980) 71; J. Cleymans, R. V. Gavai, and E. Suhonen, Phys. Rep. 130 (1986) 217.
[3] M. I. Gorenstein, V. K. Petrov and G. M. Zinovjev, Phys. Lett. B 106 (1981) 327.
[4] R. Hagedorn and J. Rafelski, Phys. Lett. B 97 (1980) 136.
[5] J. I. Kapusta, Phys. Rev. D 23 (1981) 2444.
[6] see, for instance, J. Cleymans and H. Satz, Z. Phys. C 57 (1993) 135; G. D. Yen, M. I. Gorenstein, W. Greiner, and S. N. Yang, Phys. Rev. C 56 (1997) 2210; P. Braun-Munzinger, I. Happe and J. Stachel, Phys. Lett. B 465 (1999) 15; F. Becattini et. al., Phys. Rev. C 69 (2004) 024905.
[7] M. I. Gorenstein, M. Gaźdicki and W. Greiner, Phys. Rev. C 72, 024909 (2005) and references therein.
[8] for qualitative arguments see M. Stephanov, Acta Phys. Polon. B 35, 2939 (2004).
[9] Z. Fodor and S. D. Katz, JHEP 0203, 014 (2002).
[10] F. Karsch et. al., Nucl. Phys. Proc. Suppl. 129, 614 (2004).
[11] M. S. Berger and R. L. Jaffe, Phys. Rev. C 35, 213 (1987); and Erratum-ibid. C 44, 566 (1991).
[12] L. G. Moretto, K. A. Bugaev, J. B. Elliott and L. Phair, LBNL preprint 59103; arXiv:nucl-th/0511180 15 p.
[13] M. E. Fisher, Physics 3 (1967) 255.
[14] for a review on Fisher scaling see J. B. Elliott, K. A. Bugaev, L. G. Moretto and L. Phair, arXiv:nucl-ex/0608022 (2006) 36 p. and references therein.
[15] L. G. Moretto et. al., Phys. Rep. 287 (1997) 249.
[16] A. Dillmann and G. E. A. Meier, J. Chem. Phys. 94 (1991) 3872.
[17] C. S. Kiang, Phys. Rev. Lett. 24 (1970) 47.
[18] C. M. Mader et al., Phys. Rev. C 68 (2003) 064601.
[19] D. Stauffer and A. Aharony, “Introduction to Percolation”, Taylor and Francis, Philadelphia (2001).
[20] J. P. Bondorf et al., Phys. Rep. 257 (1995) 131.
[21] K. A. Bugaev, M. I. Gorenstein, I. N. Mishustin and W. Greiner, Phys. Rev. C62 (2000) 044320; arXiv:nucl-th/0007062 (2000); Phys. Lett. B 498 (2001) 144; arXiv:nucl-th/0103075 (2001).
[22] P. T. Reuter and K. A. Bugaev, Phys. Lett. B 517 (2001) 233.
[23] K. A. Bugaev, Acta. Phys. Polon. B 36 (2005) 3083.
[24] K. A. Bugaev, arXiv:nucl-th/0507028 7 p.
[25] K. A. Bugaev, arXiv:nucl-th/0511031 21 p. (to appear in Phys. Part. Nucl. Lett.)
(26) L. Beaulieu et al., Phys. Lett. B 463, 159 (1999).
(27) J. B. Elliott et al., (The EOS Collaboration), Phys. Rev. C 62, 064603 (2000).
(28) V. A. Karnaukhov et al., Phys. Rev. C 67, 011601R (2003).
(29) N. Buyukcizmeci, R. Ogul and A. S. Botvina, arXiv:nucl-th/0506017 and references therein.
(30) K. A. Bugaev, L. Phair and J. B. Elliott, Phys. Rev. E 72 (2005) 047106;
(31) K. A. Bugaev and J. B. Elliott, arXiv:nucl-th/0501080, 7p. (to appear in Ukr. J. of Phys.)
(32) M. I. Gorenstein, G.M. Zinovjev, V.K. Petrov, and V.P. Shelest Teor. Mat. Fiz. (Russ) 52 (1982) 346.
(33) R. Hagedorn, Nuovo Cimento Suppl. 3 (1965) 147.
(34) R. Hagedorn and J. Ranft, Suppl. Nuovo Cimento 6, (1968) 169.
(35) L. G. Moretto, K. A. Bugaev, J. B. Elliott and L. Phair, Europhys. Lett. 76 (2006) 402.
(36) K. A. Bugaev, J. B. Elliott, L. G. Moretto and L. Phair, LBNL preprint 57363; arXiv:hep-ph/0504011, 5p.
(37) L. G. Moretto, K. A. Bugaev, J. B. Elliott and L. Phair, arXiv:nucl-th/0601010, 4p.
(38) D. G. Ravenhall, C. J. Pethick and J. R. Wilson, Phys. Rev. Lett. 50, 2066 (1983).
(39) J. Liao and E. V. Shuryak, Phys. Rev. D 73, 014509 (2006) arXiv:hep-ph/0510110.
(40) M. G. Aflord, K. Rajagopal and F. Wilczek, Nucl. Phys. B 537, 443 (1999) hep-ph/9804403.
(41) P. Chomaz, M. Colonma and J. Randrup, Phys. Rep. 389 (2004) 263.
(42) C. Nonaka and M. Asakawa, Phys. Rev. C 71 (2005) 044904 and references therein.
(43) N. G. Antoniou, F. K. Diakonos and A. S. Kapoyannis, Nucl. Phys. A 759 (2005) 417.
(44) E. Shuryak, arXiv:hep-ph/0504048.
(45) E. Shuryak, arXiv:nucl-th/0609011, 12 p.
(46) K. A. Bugaev, M. I. Gorenstein, H. Stöcker and W. Greiner, Phys. Lett. B 485 (2000) 121; G. Zeeb, K. A. Bugaev, P. T. Reuter and H. Stöcker, arXiv:nucl-th/0209011, 16 p.
(47) K. A. Bugaev, arXiv:nucl-th/0611102, 18 p.
(48) for a discussion and alternative cross-over mechanism see D. B. Blaschke and K. A. Bugaev, Fizika B 13 (2004) 491; Phys. Part. Nucl. Lett. 2 (2005) 305