Latest advances in the semiclassical theory of the Stark broadening of spectral lines in plasmas

Eugene Oks
Physics Department, 206 Allison Lab, Auburn University, Auburn, AL 36849, USA
goks@physics.auburn.edu

Abstract. Spectral lines continue to be used as an important practical tool for diagnosing various laboratory and astrophysical plasmas. As experiments move to very high electron densities of plasmas or to strongly magnetized plasmas, new physical phenomena have to be taken into account while calculating Stark broadening of hydrogen spectral lines. As for spectral lines having dipole-forbidden components (such as, He-, He-like-, Li-, Li-like-lines), it is shown that they provide the best way for diagnosing Langmuir solitons in plasmas.

1. Stark broadening of hydrogen lines in plasmas of densities up to or more than $N_e \sim 10^{20} \text{ cm}^{-3}$

Kielkopf & Allard (KA) recently presented benchmark measurements of the Full Width at Half Maximum (FWHM) of the H$_\alpha$ line, reaching the electron densities $N_e$ by two orders of magnitude greater than the corresponding previous benchmark experiments, namely up to $N_e = 1.4 \times 10^{20} \text{ cm}^{-3}$ [1]. Their experiment was performed in a laser-produced hydrogen plasma of the average $T = 28000 \text{ K}$. Their experimental FWHM are presented by dots in figure 1.

![Figure 1](image.jpg)

**Figure 1.** Comparison of the experimental FWHM of the H$_\alpha$ line from Kielkopf-Allard experiment [1] (dots) with the corresponding FWHM yielded by our analytical theory from paper [2] (solid line). The FWHM is measured in Angstrom, while $N_e$ – in cm$^{-3}$. 
At this range of $N_e$, no theoretical calculations of the FWHM of the $H_\alpha$ line existed, except the calculation by KA; however, the latter is incomplete because of the neglect of the contribution by plasma electrons (as noted by KA). In paper of 2015 [2] we developed a consistent analytical theory that is relevant to the range of the electron densities reached in KA experiment. At this range of $N_e$, a new factor becomes significant – the factor never taken into account in any previous simulations or analytical theories. This new factor is a rising contribution of the electrostatic plasma turbulence at the thermal level of its energy density. This contribution becomes comparable to the corresponding contribution by electron and ion microfields at this range of $N_e$.

According to Bohm and Pines [3], the number of collective degrees of freedom in a unit volume of a plasma is $N_{coll} = 1/(6\pi^2 r_D^3)$, where $r_D$ is the Debye radius. Therefore the energy density of the oscillatory electric fields at the thermal level is $E_r^2/(8\pi) = N_{coll}/T/2$, so that $E_r^2 = 16\pi^4 e^4 N_e^2/3(3T)^{1/2}$. At the absence of the magnetic field, the thermal energy density of the collective degrees of freedom is distributed in equal parts between two branches: $E_\theta^2 = E_0^2 = E_{tot}^2/2$, where $E_0$, $E_\theta$, and $E_{tot}$ are the root-mean-square thermal electric fields of the ion acoustic turbulence, the Langmuir turbulence, and the total turbulence, respectively.

The contribution of the thermal ion acoustic turbulence was taken into account by calculating the convolution of Sholin-Oks’ distribution of the turbulent field $F_i$ [4] with the APEX distribution of the ion microfield $W_i(u)$. The thermal Langmuir turbulence was allowed for as an additional contribution to the dynamical Stark width according to Oks-Sholin paper [5]. (More details can be found in [2].)

The resulting FWHM of the $H_\alpha$ line yielded by this analytical theory is shown in figure 1 by the solid line. It is seen that the agreement is very good. Even at the highest density point of KA experiment [1] ($N_e = 1.39\times10^{20}$ cm$^{-3}$, $T = 34486$ K), the theoretical FWHM differs by just 4.5% from the most probable experimental value and is well within the experimental error margin. This theory can be also used for calculating Stark profiles and the FWHM of other hydrogen spectral lines.

2. Influence of magnetic-field-caused modifications of trajectories of plasma electrons on spectral line shapes: application to magnetic fusion and white dwarfs

A general framework for calculating shapes of hydrogen (or deuterium) spectral lines in strongly-magnetized plasmas with the allowance for spiraling trajectories of perturbing electrons has been developed in paper [6] of 2016. It has been shown that in this situation the first order term $\Phi^{(1)}(B)$ of the Dyson expansion of the electron broadening operator does not vanish – in distinction to the case of rectilinear trajectories.

For the atomic electron to experience the spiraling nature of the trajectories of perturbing electrons, it requires $\rho_d/\rho_B > 1$, where $\rho_B$ is the electron Debye radius. It is equivalent to $B > B_{cr} = c(4m_e N_e)^{1/2}$, $B_{cr}(\text{Tesla}) = 1.81\times10^{-7}[N_d(\text{cm}^{-3})]^{1/2}$. This condition is fulfilled modern tokamaks, in many DA white dwarfs, as well as in capacitor-produced plasmas and in plasmas produced by high-intensity lasers.

We consider the situation where the temperature $T$ of the radiators satisfies the condition $T << (11.12 \text{ keV/n})(M/M_H)^2$, where $M$ is the radiator mass, $M_H$ is the mass of hydrogen atoms, $n$ is the principal quantum number of the energy levels, from which the spectral line originates. For plasmas of tokamaks and DA white dwarfs this condition is fulfilled for the head lines (i.e., the most intense lines) of spectral series. Under this condition, the Lorentz field effects can be disregarded compared to the “pure” magnetic field effects.

The first order of the nondiagonal elements of the electron broadening operator $\Phi^{(1)}(u)$ can be represented in the form (the diagonal elements of $\Phi^{(1)}$ vanish):

$$
\Phi^{(1)}_{\text{non}} = -i \sum_{r} (e^2/\hbar) \rho_{D\alpha N_e} \int_{0}^{\infty} dv_{\rho} W_{\rho}(v_{\rho}) \int_{-\infty}^{\infty} dv_{2} W_{2}(v_{2}) f_{\delta}(u), \quad u = B/B_{cr}. \tag{2.1}
$$

The universal function $f(u)$ in equation (2.1) that controls the phenomenon under consideration is expressed in [6] analytically via elliptic integrals. Figure 2 presents its plot.
Examples related to several tokamaks, such as T-10 in Russia, EAST in China, and the future ITER in France, show the following. The primary effect of spiraling trajectories of plasma electrons is that the ratio of the intensity of the central peak in the Zeeman triplet to the intensity of any of the two lateral peaks increases by up to a factor of two. For DA white dwarfs this ratio increases by over 20%. The secondary effect – both for tokamaks and DA white dwarfs – is an additional shift of the lateral components of the Zeeman triplet. More details can be found in [6].

3. Role of Lorentz-Doppler Broadening of Hydrogen/Deuterium Spectral Lines in Magnetized Plasmas: Applications to Magnetic Fusion and Solar Physics

Under the condition $T > (11.12 \text{ keV/n})(M/M_{\text{H}})^2$, which is the opposite to the situation considered in the previous section, the Lorentz field effects dominate over the “pure” Zeeman effect. This condition is fulfilled for highly-excited hydrogen/deuterium lines observed in tokamaks ($n \sim 10$) and in the quiet Sun ($n \sim 30$). Also the Lorentz field exceeds the most probable ion microfield when the magnetic field $B$ exceeds the following critical value: $B_c(\text{Tesla}) = 4.69 \times 10^{-7} N_e^{2/3}/[T(\text{K})]^{1/2}$. This condition is also met in tokamaks and in the quiet Sun. In this situation the broadening is controlled by the Lorenz(field)- and Doppler-effects, and the combination of these two effects cannot be taken into account via a convolution, as noted in paper [7]: they entangle in a more complicated way.

In papers [8] of 2013 and [9] of 2015 were obtained, in particular, analytical expressions for Lorentz(field)-Doppler profiles of hydrogen/deuterium lines – for arbitrary values of the scaled magnetic field $b = (\text{Lorentz shift})/\text{Doppler shift}$. For observations along the magnetic field $\mathbf{B}$ these expressions were obtained for the first time. For observations perpendicular $\mathbf{B}$ these expressions were obtained for the first time for $b > 1$ and $b = 1$. For $b < 1$ we corrected errors in analytical results from paper [9].

For the lack of space we present here only one result from [8, 9]: for observation perpendicular to $\mathbf{B}$, the width of the Lorentz-Doppler profiles is a non-monotonic function of the magnetic field. As $b$ increases from zero, the width first decreases, then reaches a minimum at $b = 1$, and then increases.

This counterintuitive result is illustrated in figure 3. The narrowing effect is the most pronounced when the Lorentz-field shift is equal to the Doppler shift.
4. Inhibition of the ion dynamical broadening by a strong magnetic field: questionable notions in the literature

In magnetic fusion machines, a strong magnetic field $B$ partially inhibits the ion dynamical broadening compared to the $B=0$ case. Physically this is because a relatively large Zeeman splitting $\omega_B/2$ diminishes the range of ion impact parameters $\rho$, for which the characteristic frequency of the variation of the ion field $v_i/\rho$ exceeds $\omega_B/2$. This effect was described analytically in year 1994 by Derevianko and Oks [11] using an advanced Stark broadening formalism (called GT). Figure 4 shows the corresponding Stark width reducing factor for the Ly$\beta$ line versus $B$ at $N_e=10^{14}$ cm$^{-3}$ and $T=5$ eV. At $B=0$, the exact analytical solution for Ly$\beta$ was given by Derevianko and Oks in paper [12].

15 years after Derevianko-Oks paper [11], Rosato et al published paper [13] describing the same effect using a less advanced Stark broadening formalism (called ST). Essentially Rosato et al repeated Sholin-Demura-Lisitsa’s version of the ST [14] with the substitution of the magnetic splitting $\omega_B$ of hydrogen/deuterium energy levels instead of the electric splitting $\omega_F$.

In our paper [15] and in our review [16] it was shown that the analytical results by Rosato et al [13] yield a very significant inaccuracy – up to orders of magnitude. In a later paper [17], Rosato et al made the second attempt to show that for magnetized plasmas the ST is allegedly better than the GT. The results from Rosato et al 2012 [17] cannot be trusted (just like those from Rosato et al 2009 [13]) for the following reasons.
First and foremost, it is known since the publication of paper by Sholin-Demura-Lisitsa in 1973 [14] that the impact shift of the spectral components of hydrogen/deuterium lines of the Ly series is proportional to the following combination of the parabolic quantum numbers $n(n_1-n_2)$, so that for the $\sigma$-components of the Ly$_a$ line $n_1-n_2=0$ and this shift vanishes, which is a rigorous analytical result for rectilinear trajectories of the perturbers. Despite this, computations of Rosato et al 2012 [17] yielded a non-zero impact shift of the $\sigma$-components of the Ly$_a$ line (for rectilinear trajectories of the perturbers), which is obviously incorrect.

Second, it is very important that for the magnetic fields typical for the edge plasmas of tokamaks, used as an example by Rosato et al both in 2009 and in 2012, the so-called nonadiabatic contribution to the broadening becomes much smaller than the primary, adiabatic contribution (i.e., the contribution of the component of the ion microfield parallel to the magnetic field). The primary, adiabatic contribution is calculated in the GT exactly (not by perturbation theory), while in the ST by Rosato et al – only by the order of magnitude (using the 2$^{nd}$ order of the perturbation theory). This is an undisputable fact. As a result, the primary, adiabatic contribution was significantly overestimated by Rosato et al both in 2009 [13] and in 2012 [17]. Some details are presented below.

In our paper [15] and in our review [16], we calculated the ratio of the adiabatic broadening cross-section by Rosato et al [13] to the adiabatic broadening cross-section by the GT: $\kappa = \sigma_{\text{adRos}}/\sigma_{\text{adGT}}$. While calculating $\sigma_{\text{adRos}}$, we had to set to unity a certain coefficient of the order of unity. After Rosato et al pointed this out in 2012 [17], we have calculated the ratio of the adiabatic broadening cross-sections $\kappa = \sigma_{\text{adRos}}/\sigma_{\text{adGT}}$ more accurately and found the following [6] (below the components of a particular line are identified by the parabolic quantum numbers: $(n_1, n_2)$ or $(n_1, n_2, m)$; the ratio $\kappa$ is calculated at $T = 4$ eV and $N_e = 10^{13}$ cm$^{-3}$).

For the Paschen-alpha line, specifically its component (102) – (101), which is one of the two most intense $\sigma$-components: $\kappa = 14$. (Here and below the components are labeled by the parabolic quantum numbers $(n_1, n_2, m)$.) For the Paschen-beta line, specifically its component (121) – (102), which is the $\sigma$-component closest to the line center: $\kappa = 22$. For the Balmer-alpha line, specifically the component (101) – (100), which is one of the two most intense $\sigma$-components: $\kappa = 9$. Thus, more accurate calculations lead qualitatively to the same conclusion: Rosato et al overestimated the primary, adiabatic contribution to the dynamical Stark broadening by ions in magnetic fusion plasmas up to an order of magnitude.

After producing the above erroneous results for the primary, adiabatic contribution, Rosato et al 2012 [17] focused at the much smaller nonadiabatic contribution. After attempting to numerically calculate the nonadiabatic contribution by the GT, they claimed that it violates unitarity at small impact parameters. However, the violation of the unitarity at small impact parameters for both the adiabatic and nonadiabatic contributions is actually the characteristic feature of the ST by Rosato et al for all hydrogen/deuterium lines (resulting, after some rather arbitrary cutoff, in the inability of the ST from Rosato et al to calculate the adiabatic contribution better than by the order of magnitude). As for the GT, Touma et al [18] rigorously showed analytically that for the overwhelming majority of hydrogen/deuterium spectral lines, the nonadiabatic contribution does not violate unitarity at small impact parameters, as illustrated also in figures A.1 – A.3 from [6].

As for the Ly$_a$ line, where the secondary, nonadiabatic contribution of GT does require the unitarity cutoff at small impact parameters $p$ (which was mentioned by us already in 1995 [19]), the width function for the $\sigma$-components $A(p)$ is proportional to $p$ at small $p$, which is a rigorous analytical result: $A(p) = \text{const}\, p$. In distinction, the corresponding attempted “GT computations” by Rosato et al 2012 for the Ly$_a$ line yielded that $A(0)$ is infinite (?), which is again obviously incorrect.

Thus, all of the above shows that the results from Rosato et al 2012 [17] are incorrect. One of the possible reasons was actually noted by Rosato et al 2012 [17]: they wrote that “the broadening function used in the GT involves a multiple integral of strongly oscillating functions, not suitable for fast numerical evaluation”. So, their computations failed to properly handle the multiple integral of strongly oscillating functions. This by itself already shows that the results by Rosato et al 2012 [17] cannot be trusted – just like the results from Rosato et al 2009 [13].
5. Manifestations of Langmuir solitons in satellites of dipole-forbidden spectral lines of helium, lithium, and of the corresponding ions

In our paper [20] there was calculated analytically the shape of satellites of dipole-forbidden lines in a spectrum spatially-integrated through a Langmuir soliton (or through a sequence of Langmuir solitons separated by a distance L). The Langmuir solitons have the following form in space according to book [21]: \( F(x, t) = E(x) \cos \omega_0 t, E(x) = E_0/\sqrt{\cosh^2(x/\lambda) - 1} \), \( \lambda << L \). Here \( \omega_0 = \omega_{pe} - 3T_e/(2m_0e^2\lambda^2) \), where \( \omega_{pe} \) is the plasma electron frequency. For diagnosing solitons it is necessary not only to find experimentally an electric field oscillating at the frequency \( \sim \omega_{pe} \), but also to make sure that the spatial distribution of the amplitude corresponds to the formfactor \( E(x) \).

Under any quasimonochromatic electric field, dipole-forbidden spectral lines of helium, lithium, and of the corresponding ions can exhibit satellites. In cases of a relatively large separation between the forbidden and allowed lines, or a relatively weak amplitudes, the intensities of the far (+) and near (−) satellites are \( S_+ = a_+ E_0^2(x) \), where \( a_+ \) does not depend on \( x \).

The spatially-integrated profile of the satellites, calculated with the allowance for the quadratic shift of their frequencies \( \Delta \omega = \pm bE_0^2(x) \), where \( b \) does not depend on \( x \), has the form:

\[
S_\pm(\Delta \omega) = (1/L) \int_{L/2}^{L/2} dx \, a_\pm E_0^2(x) \, \delta[f(x)], \quad f(x) = \Delta \omega \pm \omega_0 - bE_0^2(x). \tag{5.1}
\]

After calculating the integral in Eq. (5.1), in paper [20] it was obtained:

\[
S_\pm(\Delta \omega) = (\lambda/L)[a_\pm/(2|b|)]/[1 - (\Delta \omega \pm \omega_0)/(bE_0^2)]^{1/2}. \tag{5.2}
\]

The profiles \( S_\pm(\Delta \omega) \), formally calculated by Eq. (5.2), have singularities at \( \Delta \omega \pm \omega_0 = bE_0^2 \). From the physical point of view, for obtaining a finite result at \( \Delta \omega \pm \omega_0 = bE_0^2 \), it is necessary to replace the \( \delta \)-function in Eq. (5.1) by a real profile, e.g., by the Lorentz profile representing the dynamical Stark broadening by electrons and by some ions:

\[
\delta[f(x_0)] \to \frac{1}{(\pi\gamma)^{1/2}} \frac{1}{\gamma^2 + [f(x_0)]^2}, \tag{5.3}
\]

where \( x_0 \) is the root of the equation \( f(x) = 0 \). Using the Taylor expansion of \( f(x) \) at \( x = x_0 \) and taking into account the first derivative \( df/dx \) vanishes at \( x = x_0 \), the right side of Eq. (5.3) can be approximated as \( (1/\pi\gamma)^{1/2} \left[ \gamma^2 + \left( \frac{df(x_0)}{dx} \right)^2 (x-x_0)^2 \right] \). Then using the integral

\[
\left( \frac{1}{\pi} \right) \int_{-\infty}^{\infty} dz \, \frac{\gamma}{(\gamma^2 + z^2)} = 1/(2\gamma)^{1/2}, \tag{5.4}
\]

we find

\[
S_{\text{soliton}}^{\text{max}} = S_\pm(\Delta \omega) \pm \omega_0 - bE_0^2 = [\lambda a_\pm/(2^{1/2}\pi Lb^{1/2})]E_0^2/\gamma^{1/2}, \tag{5.5}
\]

where \( S_{\text{soliton}}^{\text{max}} \) is the peak intensity of the satellites in the case of solitons.

In the case of non-solitonic Langmuir waves, the peak intensity of the satellites is \( S_{\text{nonsoliton}}^{\text{max}} = a_\pm E_0^2/(\pi\gamma) \). So, for the ratio of the peak intensity of the satellite under solitons to the peak intensity of the same satellite under nonsolitons we get

\[
S_{\text{soliton}}^{\text{max}}/S_{\text{nonsoliton}}^{\text{max}} = [\lambda/(2^{1/2}L)] \left[ \gamma/(E_0^2) \right]^{1/2} \gg 1 \tag{5.6}
\]

for the typical situation where \( \gamma >> bE_0^2 \), i.e., where the dynamical Stark width \( \gamma \) is much greater than the quadratic shift \( bE_0^2 \) of the satellites.
Thus, in the case of Langmuir solitons, the peak intensity of the satellites can be significantly enhanced – by orders of magnitude – compared to the case of non-solitonic Langmuir waves. This distinctive feature of satellites under Langmuir solitons allows distinguishing them from non-solitonic Langmuir waves.

We note that for a more general case, where both the dynamical Stark broadening and the Doppler broadening are taken into account, the δ-function in Eq. (5.1) should be substituted by the Voigt profile resulting in

$$\frac{S_{\text{soliton max}}}{S_{\text{non-soliton max}}} \approx \frac{\Delta \omega_{\text{Voigt}}}{(bE_0^2)^{1/2}}$$

where \(\Delta \omega_{\text{Voigt}}\) is the halfwidth of the Voigt profile. Again, in the typical situation where \(\Delta \omega_{\text{Voigt}} \gg bE_0^2\), the peak intensity of the satellites is significantly enhanced – by orders of magnitude – compared to the case of non-solitonic Langmuir waves.

Finally we mention that Hannachi et al [22] performed simulations for finding the effect of Langmuir solitons on the hydrogen Ly\(\alpha\) line. The effect was an additional broadening. However, even at the low electron density \(N_e = 10^{14} \text{cm}^{-3}\), the effect was very small compared to the Stark broadening by plasma microfields. Moreover, the additional broadening rapidly diminished with the increase of \(N_e\), so that there would be practically no additional broadening at \(N_e > 10^{15} \text{cm}^{-3}\). Therefore, it seems that the results by Hannachi et al [22] could not be useful for the experimental diagnostics of Langmuir solitons.

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