Extraction of density-layered fluid from a porous medium

J. Jose · G. C. Hocking · D. E. Farrow

1 Introduction

Flow of fluids with free boundaries or interfaces in porous media is very important across a range of applications. Oil recovery from underground, pumping fresh water from aquifers and mineral leaching in mining applications are obvious examples [1–3]. A characteristic of the fluids in many of these applications is that they are stratified in density, either due to fluid properties (oil water), salt content (fresh water, salt water) or temperature [1,4,5].

In particular, understanding withdrawal is very important for determining water quality or monitoring oil recovery. Muskat and Wyckoff [6], Bear and Dagan [7], Giger [8], McCarthy [9], Lucas et al. [10], Zhang et al. [11,12], Hocking and Zhang [13] and others have considered the withdrawal into line and point sinks in layered fluids with...
interfaces or free boundaries using hodograph and boundary element techniques. In all cases, it was found that there was a limit on the steady flow rate beyond which single-layered flows could not be found. This limiting maximal flow is known as coning \cite{14}, in which the fluid surface (or interface in the case of a two-layer fluid) is drawn up sharply and enters the sink directly. List \cite{15} and Yih \cite{16} showed that there is a distinct layering of the flow for linear density stratification. At flow rates higher than the critical coning value, solutions include flows with multiple flowing layers, such as those described in \cite{5,17–19}. These solutions are analogous to similar supercritical flows in surface water hydrodynamics \cite{20,21}.

Here we examine the case of withdrawal through a point sink in a fluid that has either a continuous density stratification or a distinct layering, and consider the limit as one approaches the other. An eigenfunction expansion \cite{16} is used to solve for the flow if the stratification is linear, and the results are used to verify a finite difference method that is then used to consider nonlinear density strata. After that a spectral method is used to examine the flow into a point sink from a single layer of constant density, separated from a second layer by a sharp density interface. A brief investigation of this problem with the point sink situated on the top of the region was considered in Jose et al. \cite{22} and this work is extended here to consider all elevations of the sink.

2 Problem formulation

Consider the axisymmetric flow of a density stratified fluid into a point sink within a vertically confined aquifer. In cylindrical coordinates, the sink is situated at \( \hat{r} = 0 \) and at some height, \( \hat{z} = H_s \), within a vertically confined region with height \( d \) (see Fig. 1) and coordinates \((\hat{r}, \hat{z})\). The flow is assumed to be radially symmetric towards the sink, and the fluid to have a density \( \hat{\rho}(\hat{r}, \hat{z}) \).

Following the work of Yih \cite{16}, consider the cylindrical coordinates \((\hat{r}, \hat{\phi}, \hat{z})\) with \( \hat{z} \) increasing vertically upward. In axisymmetric flow, the flow is a function of \((\hat{r}, \hat{z})\) only, and the pseudo-velocity \((\hat{u}', \hat{w}')\) is defined by

\[
(\hat{u}', \hat{w}') = \frac{\mu}{\mu_0}(\hat{u}, \hat{w}),
\]

where \( \mu_0 \) is a constant reference viscosity, and \( \hat{u} \) and \( \hat{w} \) are the velocity components in the \( \hat{r} \) and \( \hat{z} \) directions, respectively. In many situations, with a single fluid, the viscosity \( \mu \) will be approximately constant at \( \mu_0 \), so that the actual velocity and pseudo-velocity will be the same. It is only if the fluid has different properties that this transformation is required. In terms of the pseudo-velocity, the equations of motion invoking Darcy’s law become

\[
\frac{\mu_0}{k} \hat{u}' = -\frac{\partial p}{\partial \hat{r}} \quad \text{and} \quad \frac{\mu_0}{k} \hat{w}' = -\frac{\partial p}{\partial \hat{z}} - g \hat{\rho}.
\]

Neglecting diffusive effects, the viscosity \( \mu \) does not change along a streamline in steady flow. The pseudo-velocities \( \hat{u}' \) and \( \hat{w}' \) satisfy the continuity equation and so we define a stream function \( \psi' \) which is defined by

\[
\hat{u}' = -\frac{1}{\hat{r}} \frac{\partial \psi'}{\partial \hat{z}} \quad \text{and} \quad \hat{w}' = \frac{1}{\hat{r}} \frac{\partial \psi'}{\partial \hat{r}}.
\]

![Fig. 1](image-url) Sketch defining the variables for flow into a point sink located at \((0, H_s)\). Arrows indicate the direction of flow and \( \hat{\rho} = f(\hat{z}) \) is the variable density gradient. The region has total height \( d \). The radial coordinate is \( \hat{r} \) and the vertical coordinate is \( \hat{z} \).
After some work \cite{16}, we reach the equation

\[ \left( \frac{\partial^2}{\partial \hat{z}^2} + \frac{\partial^2}{\partial \hat{r}^2} - \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \right) \psi' = -\frac{k g \hat{r}}{\mu_0} \frac{d \rho}{d \hat{r}} \frac{\partial \psi'}{\partial \hat{r}}. \]  

(4)

If the fluid is incompressible, then in steady flow \( \hat{\rho} \) is a function of \( \psi' \) only.

We now solve (4) together with appropriate boundary conditions

\[ \psi' = \begin{cases} 
0, & \hat{r} = 0, \ 0 < \hat{z} < H_s; \\
0, & \hat{z} = 0, \ \hat{r} > 0; \\
Q, & \hat{r} = 0, \ H_s < \hat{z} < d; \\
Q, & \hat{z} = d, \ \hat{r} > 0;
\end{cases} \]  

(5)

where \( Q \) is the discharge into the sink. The discontinuity of \( \psi' \) at \( r = 0 \) and \( z = H_s \) represents the sink.

We define the upstream (\( \hat{r} \rightarrow \infty \)) density as \( \hat{\rho} \rightarrow f(\hat{z}) \), where \( f(\hat{z}) \) defines the density as a function of height far upstream. To reduce the number of parameters in the problem, we use the dimensionless quantities

\[ (\xi, \zeta) = \left( \frac{\hat{r}}{d}, \frac{\hat{z}}{d} \right), \ \Psi = \frac{2 \pi \psi'}{Q}, \ \rho = \frac{\hat{\rho}}{\rho_0}, \]  

in which case (4) becomes

\[ \left( \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \zeta^2} \right) \Psi = -\left( \frac{2 \pi k g \rho_0 d^2}{Q \mu_0} \right) \xi \frac{d \rho}{d \Psi} \frac{\partial \Psi}{\partial \zeta}, \]  

(6)

with the boundary conditions

\[ \psi = \begin{cases} 
0, & 0 < \zeta < h_s, \ \xi = 0; \\
0, & \zeta = 0, \ \xi > 0; \\
1, & h_s < \zeta < 1, \ \xi = 0; \\
1, & \zeta = 1, \ \xi > 0;
\end{cases} \]  

(7)

where \( h_s = \frac{H_s}{d} \). If \( h_s = 1 \), the sink is located on the top surface.

### 3 Linear density profile

A long way upstream (\( \xi \rightarrow \infty \)) we assume a density profile of the form \( \rho = f(\zeta) \), so that \( \frac{d \rho}{d \Psi} = f'(\zeta) \) is the density gradient. In the case of a linear gradient to be considered in this section, \( \rho = f(\zeta) = 1 - \beta \zeta \) is the appropriate density function and \( \Psi \rightarrow \zeta \) as the flow velocity approaches zero. In incompressible flow, the density along each streamline remains the same, so that it is true everywhere that

\[ \frac{d \rho}{d \Psi} = -\beta. \]

The nondimensional form of Eq. (4) with the linear density profile is

\[ \left( \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \zeta^2} \right) \Psi = \left( \frac{2 \pi k g \beta \rho_0 d^2}{Q \mu_0} \right) \xi \frac{\partial \Psi}{\partial \xi} = \lambda^2 \xi \frac{\partial \Psi}{\partial \xi}, \]  

(8)

where

\[ \lambda^2 = \frac{2 \pi k g \rho_0 d^2}{Q \mu_0} \]  

(9)

is proportional to the density gradient or the inverse of the flux into the sink \( Q \). A large value of the nondimensional parameter \( \lambda \) means either a strong density stratification or a low rate of flow into the outlet. The quantity \( \lambda \) is the same as that defined by Yih \cite{16} and we will use it here for consistency. However, we note that if we define a new
quantity \( \gamma - 1 = \rho_1 / \rho_0 - 1 \) where \( \rho_0 \) is the density at the top of the region and \( \rho_1 \) is at the bottom, then \( \beta = \gamma - 1 \) and we can define a slightly more general quantity

\[
G^2 = \frac{2\pi k g' \rho_0 d^2}{Q \mu_0},
\]

(10)

where \( g' = g(\gamma - 1) \) is a reduced gravity. The quantities \( \lambda \) and \( G \) relate the buoyancy of the fluid to the suction at the sink.

Thus, the problem is to solve (8) with the boundary conditions (7) along with \( \Psi \to \zeta \) as \( \xi \to \infty \). The quantity \( G \) will allow us to compare with flows later in this work. In the case of a linear density profile \( G = \lambda \).

3.1 Series solutions

Following Yih [16], and taking \( h_s = 1 \), i.e. the sink on the top surface, and in accordance with boundary conditions (7), we seek a solution as an eigenfunction expansion in the form

\[
\Psi = \zeta + \sum_{n=1}^{\infty} A_n g_n(\xi) \sin n\pi \zeta.
\]

(11)

The differential equation satisfied by \( g_n(\xi) \) is

\[
g_n'' - \left( \lambda^2 \xi + \frac{1}{\xi} \right) g_n' - n^2 \pi^2 g_n = 0,
\]

where \( g_n(0) \neq 0 \), and \( g_n \to 0 \) as \( n \to \infty \).

The differential equation (12) has a regular singular point at \( \xi = 0 \) and so we use the Frobenius method in powers of \( \xi \) [23]. Following Yih [16], we let \( g_n = e^{\sigma/2} h(\sigma) \) where \( \sigma = \frac{\lambda^2 \xi^2}{2} \), and then

\[
h''(\sigma) + \left( -\frac{1}{4} + \frac{k_n}{\sigma} \right) h(\sigma) = 0,
\]

(13)

where

\[
k_n = -\frac{n^2 \pi^2}{2\lambda^2}, \quad n = 1, 2, 3 \ldots.
\]

Equation (13) has solutions in the form of a Whittaker function [24], and the appropriate solution is

\[
g_n(\xi) = \int_0^\infty t^{-k_n} \left( \frac{\lambda^2 \xi^2}{2} + t \right)^{k_n/2} e^{-t} \, dt.
\]

(14)

At \( \xi = 0 \), \( g_n(0) = 1 \) from (14), and coefficients \( A_n \) are chosen to satisfy (7), so that the stream function

\[
\Psi = \zeta + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} g_n(\xi) \sin n\pi \zeta.
\]

Equally spaced streamlines for \( \lambda = 1 \) and \( \lambda = 8 \) are shown in Fig. 2. For \( \lambda = 1 \), the flow is stronger nearer to the sink due to the higher flow rate, and so the streamlines ‘fill’ more of the channel. The flow coming from the bottom part of the region is relatively slow when \( \lambda = 8 \) and so the streamlines fill less of the depth. Thus, near the bottom, a long way from the sink, the horizontal flow is much slower. This effect is clearly seen in the velocity profiles shown in Fig. 3. For the case \( \lambda = 8 \), there is a region near the bottom of the aquifer that is almost stagnant. Note that for a linear density profile upstream, the streamlines are also contours of density, and so the local stratification can be seen from the location of the streamlines.

Therefore, as \( \lambda \) increases, the density stratification has the effect of restricting the flow so that it is much greater close to the level of the sink. In what follows we will investigate this further for cases in which the point sink is no longer on the top of the region and also for the situation in which the stratification is not linear but takes on a more layered structure as is found in oil reservoirs or coastal aquifers.
Fig. 2 Equally spaced streamlines $\psi = 0, 0.1, \ldots, 1$ of a stratified fluid flow due to a point sink with $\lambda = 1$ (left) and $\lambda = 8$ (right). Larger $\lambda$ indicates lower flow rate, so at higher flow, the streamlines fill more of the channel. Note the streamlines in this case are also equally spaced contours of density.

Fig. 3 Velocity profiles at horizontal locations $\xi = 0.75, 1, 1.5, 3$ (left to right curves in each plot) with $h_x = 1$. Here the dots represents the eigenfunction solution and the lines represent the finite difference solutions with $\lambda = 1$ (left) and $\lambda = 8$ (right). At the lower flow rate ($\lambda = 8$), flow is faster in a narrow layer at the level of the sink, and the fluid is almost stagnant near $\zeta = 0$.

3.2 Finite difference solution

To compute solutions for flows in which the density stratification is not linear or the sink is located in a different location, we approach the problem using the finite difference method. To proceed, we will compare the solution with that computed by Yih [16] for the case of a linear density variation with the sink on top of the region, presented in Section 3.1, and then consider cases in which the sink is in an arbitrary location, and the density profile is not linear. In particular we are interested in cases in which the stratification approaches two layers of different density separated by a thin interface.

If we make a guess for $\Psi(\xi, \zeta)$, at the grid points $(\xi_i, \zeta_j)$, $i = 2, 3, \ldots, N_R - 1$ and $j = 2, 3, \ldots, N_Z - 1$, separated by $\Delta \xi$ and $\Delta \zeta$, respectively, then the errors, $E_{i,j}$, at the corresponding points for the finite difference form of (8) are
\[ E_{i,j} = \Psi_{i+1,j} \left[ \Delta \zeta^2 - \frac{1}{2} \left( \frac{1}{\xi_i} + \lambda^2 \xi_i \right) \Delta \xi \Delta \zeta^2 \right] + \Psi_{i,j} \left( -2 \Delta \xi^2 - 2 \Delta \zeta^2 \right) + \Psi_{i-1,j} \left[ \Delta \zeta^2 + \frac{1}{2} \left( \frac{1}{\xi_i} + \lambda^2 \xi_i \right) \Delta \xi \Delta \zeta^2 \right] + \Psi_{i,j+1} \Delta \xi^2 + \Psi_{i,j-1} \Delta \zeta^2, \]

\[ i = 2, \ldots, N_R - 1, \quad j = 2, \ldots, N_Z - 1. \]  

(15)

The values of \( \Psi_{i,1}, \Psi_{i,N_Z} \) and \( \Psi_{1,j} \) are given by the boundary conditions (7). This gives \((N_R - 2) \times (N_Z - 2)\) equations in the same number of unknowns which can be solved in Octave [25] using the nonlinear equation solver \texttt{fsolve}. While these equations are linear in the case of a linear stratification, we use a nonlinear solver for ease of translating to the later nonlinear problem. Convergence in this linear case takes only one iteration. Solutions were computed using \( \lambda = 1.8 \) for comparison with the series solutions. Values of \( \Delta \xi = 0.05 \) and \( \Delta \zeta = 0.02 \) gave solutions for velocity profiles and streamlines to graphical accuracy. The contours (not shown) are very similar to those shown in Fig. 2 and the comparative velocity profiles of the two cases are given in Fig. 3 with good agreement.

3.3 Changing sink elevation

Henceforth we will assume the sink location to be more general at \( z = h_s \) as indicated by the boundary conditions (7). Since the sink is now below the top surface, the nondimensional quantity \( G = \lambda \) is changed very slightly because the flux into the sink is now doubled (since the full sink is exposed to the fluid, rather than just the lower half). Therefore, \( G \) is now given by

\[ G^2 = \frac{4\pi k g' \rho_0 d^2}{Q \mu_0} \]  

(16)

to reflect this doubling of the inward flux.

The finite difference method was modified to solve the problem with a linear density gradient for arbitrary sink height and the results are shown in Figs. 4, 5, and 6. In Fig. 4, we can see that when the withdrawal rate is larger the streamlines pull closer towards the sink, reflecting the velocity profiles seen in Fig. 5, in which the higher withdrawal rate gives a more even spread over the vertical region. In the case of linear stratification, the streamlines are also contours of density, and so the contours are also evenly spaced in density. When \( \lambda = 8 \), for low flow, the velocity

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**Fig. 4** Streamlines of a stratified fluid flow due to a point sink located at the middle height \( h_s = 0.5 \) of an aquifer with \( \lambda = 1 \) (left) representing the faster flow and \( \lambda = 8 \) (right) representing the lower flow. For the high flow, streamlines have a tendency to fill the aquifer as before. In the weaker flow, again the layering evident in the earlier work is apparent, with a very slow flow near both the top and the bottom. Note that since the density is constant along each streamline, these lines are also equally spaced contours of density.
profile is more confined to a narrow layer in the middle of the region. Note that the quantity $\lambda$ is proportional to the stratification value $\beta$. Thus large $\lambda$ also corresponds to stronger stratification. This means that for a given flow rate, a stronger density gradient restricts the vertical movement of the fluid leading to the narrow flowing “layer”. This layering was identified and studied by List [15] and Yih [16]. In cases where the sink is not at the middle height, i.e. $h_s \neq 0.5$, as shown in Fig. 6, the same effect becomes apparent but the profiles are no longer symmetric.

4 Nonlinear density profile

Now that we have verified the finite difference method we can consider different density strata. In particular, we are interested in the behaviour as the linear gradient transitions into a two-layer strata with a region of lighter fluid sitting on top of a region of heavier fluid. In order to consider the case of general stratification of the form $\rho(\zeta) = f(\zeta)$, we use the density profile considered by Farrow and Hocking [26] in nondimensional form as

\[
\rho = 1 - \frac{1}{2d_2} \left( \arctan \left[ d_1(2\zeta - 1) \right] + d_2 \right),
\]  

(17)
where the scaling has been to make the density vary by a single unit from top to bottom so that the density at the base is $\rho = 1$ and at the top $\rho = 0$ and the nondimensional density perturbation is one unit. The values of $d_1$ and $d_2$ determine the shape of the density profile, as seen in Fig. 7, and are related by the expression $d_2 = \arctan(d_1)$.

After differentiating $\rho$ with respect to $\Psi$, we obtain

$$\frac{\partial \rho}{\partial \Psi} = \frac{-d_1/d_2}{[1 + d_1(2\Psi - 1)]^2}.$$  \hfill (18)

Substituting (18) into (6), we find

$$\frac{\partial^2 \Psi}{\partial \xi^2} - \left(\frac{1}{\xi} - G^2\xi f(\Psi)\right) \frac{\partial \Psi}{\partial \xi} + \frac{\partial^2 \Psi}{\partial \xi^2} = 0.$$ \hfill (19)

We now treat (19) as a boundary value problem with the boundary conditions (5) plus $\frac{d\Psi}{d\xi} \to 0$ as $\xi \to \infty$ and $0 < \zeta < 1$, which dictates the flow becomes horizontal, and solve using the finite difference method.

For lower $G$, discharge into the sink is very high, while for higher $G$, discharge into the sink is low. Figure 8 shows streamlines for $d_2 = 0.4\pi$ in which the density stratification is close to linear (see Fig. 7a), for $G = 2$ and $G = 8$ and the behaviour is very much like that shown in Fig. 2 as might be expected. Figure 9 shows the discharge profiles for the same cases and again is very similar to the linear gradient case. In the slower flow of Fig. 9b, the top layers are moving quite quickly towards the point sink while the bottom region is almost stagnant.

Figure 10 clearly shows the effect of the different density stratification. In (a), $d_2 = 0.4\pi$ is an almost linear stratification while in (b), $d_2 = 0.49\pi$ is almost two distinct layers (see Fig. 7b). Discharge profiles are shown for the case $G=1$, which is a relatively strong flow, and clearly shows that the density layers in the step stratification are restricting the flow. Discharge above the interface is much stronger and this is also reflected in the streamlines in Fig. 8, where the greater separation near the base is indicative of the slower flow.

As $G$ increases and the discharge decreases, the stratification begins to have a greater effect. Figure 11 shows the density contours about the interface with $d_2 = 0.4\pi$. In (a), $G=2$ and the flow is quite strong and so fluid is withdrawn over the full height, simply pulling the interface up into the outlet. However, in Fig. 11b, where $G = 8$, the flow is quite weak and it turns out that the upstream density gradient adjusts to having the interface much higher. Attempts to impose the condition that the interface remained at the mid-height resulted in spurious oscillations in the solutions even when the domain of the computations was greatly increased. This is an indication that there is no solution with the centre of the density interface at the middle height, thus violating the upstream condition. The reason for this is that the flow suction is not strong enough to overcome the potential energy of such a stratification.
Fig. 8 The streamlines $\psi = 0, 0.1, \ldots, 1$ for $G = 2$ (left) and $G = 8$ (right) with density profile value $d_2 = 0.4\pi$. Stronger flow has smaller $G$. These correspond to the density profile in the left panel of Fig. 7. Since the density is constant along a streamline, the colours are also indicative of the density. However, the density values fit the functions shown in Fig. 7 rather than being evenly spaced (see Fig. 11).

Fig. 9 Velocity profiles at locations, $\xi = 0.75, 1, 1.5, 3$ (left to right curves in each plot) with density profile $d_2 = 0.4\pi$. $G = 2$ (left) indicates strong flow and $G = 8$ (right) indicates weak flow. The weaker flow starts to exhibit a strong two-layer velocity structure.

It is most likely that flows with that strength and an upstream interface at the mid-level will become single-layer flow as described later in this work. However, solutions here assume that the streamlines all enter the sink directly. The greater separation of the streamlines as $\xi \to \infty$ is a reflection of the fact that the flows near the bottom are slowing. The case in which the bottom layer becomes separate is considered in the next section where a dividing streamline forms leaving a stagnant region near the bottom.

In cases where the stratification is more strongly layered, that is for $d_2 > 0.4\pi$ say, there is a distinct layering in the flow velocity, see Fig. 10. Therefore, if the flow rate is sufficiently small, then there is no longer a solution in which the initially prescribed density structure exists upstream, and the numerical scheme adjusts to a new upstream condition in which the centre of the interface is no longer located at $\zeta = 0.5$.

Given the adjustment of the upstream level of the interface, it is appropriate to define $G^* = G(1 - H_{mid})$, where $G^*$ represents the value of $G$ when the length scale adjusted to be the distance to the middle of the interface, and $H_{mid}$ is defined as the location of the density isoline with nondimensional density $\rho = 0.5$. $H_{mid}$ also coincides with the point at which the density gradient is steepest. This value will provide an estimate of the effective maximum $G$ value. Figure 12 shows $G$ versus $G^*$ for several different values of $d_2$. Clearly as $G$ increases, $G^*$ levels off.
Fig. 10 Velocity profiles at locations, $\xi = 0.75, 1, 1.5, 3$ (left to right curves in each plot) with $G = 2$ and with different densities obtained from $d_2 = 0.4\pi$ (left) and $d_2 = 0.49\pi$ (right). The sharper density step ($d_2 = 0.49\pi$) leads to a distinct step in the velocity profile and lower velocity in the bottom layer.

Fig. 11 Density contours $\rho = 0, 0.25, 0.75, 1$ for $d_2 = 0.4\pi$ with $G = 2$ (left), which shows a high flow rate and water flowing over the whole depth, and $G = 8$ (right), in which a narrow layer develops near the top.

Fig. 12 $G$ vs. $G^*$ for $d_2 = 0.45\pi, 0.4\pi, 0.35\pi$, bottom to top. As $G$ increases, $G^*$ levels off at around $G^* \approx 2$, and as $d_2$ increases the maximum $G^*$ decreases slightly, i.e. a sharper interface results in a lower minimum value of $G^*$. 
indicating that for any density stratification and sink height, there is a flow rate beneath which the withdrawal pressure is not large enough to pull the water beneath the interface upward into the outlet. The result of this is that for $G^*$ greater than this value, the water will almost exclusively come from this layer adjacent to the sink. In this example, $G^*$ approaches $G^* = 2$ as $G$ increases. This will be investigated further later in this work.

4.1 Changing sink elevation—nonlinear density profile

Thus far we have only considered situations with nonlinear stratification for which the point sink is situated on the upper surface of the region. However, it is also of interest to consider cases in which the sink is situated at some arbitrary height. The flow will still be axisymmetric, and so the only change required to the method is to modify the values of $\psi$ on $\xi = 0$ so that boundary conditions are given in (7) along with $\psi_\xi = 0$ as $\xi \to \infty$ to ensure the streamlines remain horizontal.

Figure 13 shows the streamlines for the case with $h_s = 0.5$, i.e. mid-height of the region, for two cases of flow rates, $G = 2$ and $G = 8$, all with $d_2 = 0.4\pi$. The relatively broad interface in this example is centred at $\zeta = 0.5$, so that the withdrawal point is in the middle of this interface. At the higher value of $G$ (lower flow rate), the narrowing of the flowing velocity profile can be seen.

Figure 14 shows the velocity profiles for these two cases. At low flow rates, $G = 8$, the profiles are very narrow in the vertical direction as the interface acts like a very strongly stratified region, so the withdrawal layer is very narrow with a relatively high velocity centre. The higher flow, $G = 2$, looks more like the earlier results as the flow is much stronger over the whole height of the channel and the strength of the flow overcomes the stratification.

This effect is greatly exaggerated in Fig. 15, in which the density interface is the case for $d_2 = 0.45\pi$ and hence is much sharper. In this example, the withdrawal layer is narrow even in the higher flow, $G = 2$ case, but for $G = 8$, the low flow case, the layer is very thin with a very sharp looking velocity profile.

If the sink height, $h_s$, is moved away from the mid-height of the density interface, $H_{\text{mid}}$, then the flow begins to separate more strongly into the two different layers: above the interface and below the interface. Figure 16b shows the case with the sink located at $h_s = 0.7$, i.e. near the middle of the upper layer, and a sharper interface with $d_2 = 0.49\pi$. It is clear that the flow coming from below the interface is very low and for $h_s = 0.7$, there is a clear separation in velocity between the two layers, and within each layer, the velocity profiles are quite flat, indicating almost a potential flow. In the case of the withdrawal from the middle of the interface at $h_s = 0.5$, the flow is evenly spread over the whole region.

![Fig. 13 Streamlines of a stratified fluid flow due to a point sink located at $h_s = 0.5$ with density profile with $d_2 = 0.4\pi$. $G = 2$ (left) represents the faster flow and $G = 8$ (right) represents the lower flow. For the high flow, streamlines are more evenly spread over the aquifer when compared to the weaker flow.](image-url)
Fig. 14 Velocity profiles at horizontal locations, $\xi = 1, 1.5, 2, 3$ (left to right). Density profile ($d_2 = 0.4\pi$) with the height $h_s = 0.5$ and $G = 2$ (left) indicates the strong flow, and $G = 8$ (right) is the weak flow.

Fig. 15 Velocity profiles at horizontal locations, $\xi = 1, 1.5, 2, 3$ (left to right) with different densities, $d_2 = 0.4\pi$ (on left) and $d_2 = 0.45\pi$ (on right). The height of the sink $h_s = 0.5$ and $G = 8$ for both cases. We can see that in Fig. 15b, the velocity profiles are narrower and centred around the interface between the two regions.

These results again show that the nature of the stratification has a profound impact on the level from which the fluid is drawn. The presence of an interface at any level will limit the extent of withdrawal from a part of the flow domain, and therefore needs to be kept in mind during extraction of water from aquifers, or oil from reservoirs.

When the sink is not on the top, i.e. $h_s \neq 1$, $H_{\text{mid}}$ will eventually rise up to the level of $h_s$ as $G$ increases, and so an appropriate calculation for the maximum $G$ for each $h_s$ is to compute when the middle of the interface lifts off from the height $\zeta = 0.5$, i.e. the point at which the solution is no longer valid as a flow with the upstream level of the middle of the interface at $\zeta = 0.5$.

The correct boundary condition upstream for flows with the interface centred around $h = 0.5$ is that the mid-level density line and the equivalent streamline remain at this height as $\xi \to \infty$. As $G$ increases, this condition is violated.
as the flow slows. In other words, the solution is modified to one in which the interface is no longer centred at \( \zeta = 0.5 \) but at some higher level. An examination of solutions with truncation of the region at larger values of \( \xi \) indicated that the streamlines do level off and the error is not due to the finite domain. Attempting to enforce the correct boundary condition resulted in the method failing to converge. The conclusion is that solutions with this upstream condition do not exist for larger values of \( G \), i.e. slower flows.

Figure 17 shows the maximum \( G \) for each \( h_s \) at different \( d_2 \). These values can be considered as a maximum for each sink height. It is interesting that the value gets lower as the interface becomes sharper. This figure is important for comparison with the single-layer flows. It is likely that at values of \( G \) larger than this, a region develops that is stagnant and not drawn towards the sink.

5 Axisymmetric withdrawal with flow in only one layer

In the first part of this paper, we have considered the situation of a variable density gradient and determined that if the flow rate is not large enough then solutions with flow across the full height of the channel do not exist. Therefore, we now seek solutions for steady withdrawal through a point sink in the upper layer of a two-layer fluid with an extremely thin interface in which the lower layer is stagnant. The goal is to determine the flow rate at which coning [17] occurs, in which the interface is drawn up directly into the point sink. It is to be expected that this may coincide
with the point at which the strongly layered solutions in the previous section fail as $G$ increases. A sketch of such a flow is shown in Fig. 18, where the distance from the free interface to the top of the region is $D$ and the flow into the sink is $Q$ as before and $H_s^*$ is the sink height above the interface.

Define piezometric head functions $\phi_0 = \frac{p}{\rho_0 g} + \hat{z}$ and $\phi_1 = \frac{p}{\rho_1 g} + \hat{z}$ for the upper and lower layers in coordinates $(\hat{r}, \hat{z})$. Matching the pressure across the interface and nondimensionalising the $\phi$ values with respect to $g'D$, where $g' = (\rho_1 - \rho_0)/\rho_0 g$, and the lengths with respect to $D$ gives us similar parameters to the earlier sections. Implementing Darcy’s law (2) and conservation of mass for the (nondimensional) functions $\Phi_0$ and $\Phi_1$ in the flowing layer in nondimensional coordinates $(r, z)$, leads to a requirement that

$$\nabla^2 \Phi_0 = \nabla^2 \Phi_1 = 0$$

throughout the regions above and below the interface, respectively. If the lower layer is stagnant, then $\Phi_1 = 0$ throughout, so that our only concern is the value of $\Phi_0$ in the upper layer, subject to conditions on the upper boundary and the interface between the two layers. Consequently, we drop the subscript and use $\Phi$ henceforth to represent $\Phi_0$.

The nondimensional variables $(r, z)$ are slightly different to those in the earlier section, $(\xi, \zeta)$ because in the earlier section the whole aquifer height was scaled to be one, while here it is the distance from the top to the density interface as $r \to \infty$. The free surface is defined as $z = \eta(r)$, and the boundary and surface conditions are

$$\Phi = \eta; \quad z = \eta(r),$$

$$\Phi_r \eta_r = \Phi_z; \quad z = \eta(r),$$

$$\Phi_z = 0; \quad z = 1,$$

where the first two conditions are applied on the free interface to ensure constant pressure in the stagnant lower fluid and that there is no flow through the surface, while the third stipulates there can be no flow through the impermeable upper surface.

The function $\Phi$ must satisfy the condition that

$$\Phi_0 \to \frac{M}{4\pi (r^2 + (z - h_s^*)^2)^{1/2}} \quad \text{as} \quad (r, z) \to (0, h_s^*),$$

where $M$ is representative of the sink strength. This places a mathematical point sink at $(r, z) = (0, h_s^*)$, where $h_s^* = H_s^*/D$, to model the withdrawal behaviour. The '*' indicates the slight change in definition of sink height to be from the level of the interface.

The quantity $M$ is related to $G$ in Eq. (10) by the relation

$$\frac{M}{4\pi} = \left( \frac{d}{DG} \right)^2,$$

where $d$ is the total height of the confined aquifer as defined in Fig. 1, and in general will be $d \approx 2D$ depending on the level of the interface as $r \to \infty$.

Care must be taken in comparing results because in the single-layer flow, all of the flux remains above the interface, while in the earlier sections, the mid-point of the interface enters the sink directly, so that only half of the flux into the point sink is from above the interface.

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To find a solution to this single-layer problem, an appropriate choice for $\Phi$ that satisfies (20) and (23) is

$$\Phi(r, z) = \Phi_s + \int_0^\infty C(\theta) \cosh[\theta(z - 1)]J_0(\theta r) \, d\theta,$$

where $J_0(\theta r)$ is a Bessel function of the first kind, and $C(\theta)$ is a weighting function to be determined so that $\Phi$ will satisfy the surface conditions, and

$$\Phi_s = -\frac{M}{4\pi} \left[ \frac{1}{(r^2 + (z - h_s^*)^2)^{1/2}} + \frac{1}{(r^2 + (z - (2 - h_s^*))^2)^{1/2}} \right]$$

defines a point sink at height, $z = h_s^*$, and also satisfies (23).

The integral in Eq. (26) is not amenable to a numerical solution of the nonlinear problem, and so we choose to replace it with a sum, so that we represent

$$\Phi(r, z) = \Phi_s(r, z) + \sum_{k=0}^\infty C_k \cosh[\theta_k(z - 1)]J_0(\theta_k r),$$

where $\theta_k, k = 0, 1, 2, \ldots$ are appropriately chosen eigenvalues.

To ensure that the interface becomes horizontal as $r \to \infty$ in the discrete form (28), it is necessary that $\Phi_z = 0$ at $r = L$ and so $\theta_k$ are the zeros of $J_0(\theta_k L) = 0$, for $k = 1, 2, 3, \ldots$, and $L$ is a large value of $r$ at which the domain must be truncated.

5.1 Linear solution

If the flow rate is comparatively small, then (22) can be approximated with

$$\Phi_r \eta_r \approx 0 \Rightarrow \Phi_z \approx 0,$$

on $z = 0$ and also from (21)

$$\eta(r) \approx \Phi(r, 0).$$

Then noting

$$\Phi_{zz} \big|_{z=0} = -\frac{M}{4\pi} \left[ \frac{h_s^*}{(r^2 + h_s^{*2})^{3/2}} + \frac{(2 - h_s^*)}{(r^2 + (2 - h_s^{*2})^2)^{3/2}} \right]$$

and applying the linear condition (29) in (28), gives

$$\sum_{j=0}^\infty C_j \theta_j \sinh \theta_j J_0(\theta_j r) = -\frac{M}{4\pi} \left[ \frac{h_s^*}{(r^2 + h_s^{*2})^{3/2}} + \frac{(2 - h_s^*)}{(r^2 + (2 - h_s^{*2})^2)^{3/2}} \right].$$

To compute $C_k$, multiply (32) by $J_0(\theta_k r) r \, dr$ and integrate from 0 to $L$ and thus by orthogonality of the eigenfunctions (see Abramowitz and Stegun [24]),

$$C_k = -\frac{M L}{2\pi \theta_k \sinh(\theta_k)} \left( \frac{h_s^*}{(r^2 + h_s^{*2})^{3/2}} + \frac{(2 - h_s^*)}{(r^2 + (2 - h_s^{*2})^2)^{3/2}} \right) J_0(\theta_k r) r \, dr,$$

Once the $C_k$s are known, we can use (28) and (30), i.e. $\eta(r) \approx \Phi(r, 0)$ to obtain approximate surface shapes for different $M$ and $h_s^*$. Some examples are shown in Fig. 19 for the case $h_s^* = 0.5$ and with $M = 0.05, 0.1$ and 0.15. Experiments with the numerical calculations showed that the value of truncation $L$ had almost no influence on the solution once $L$ was greater than $L \approx 15$, and this was also found to be the case in the nonlinear solutions below.
5.2 Nonlinear solution for single-layer flow

It is possible to extend this method to solve the full nonlinear problem for the shape of the interface. Choosing the potential $\Phi$ to take the form given in (26), we must now implement the surface conditions directly on $z = \eta(r)$ rather than on $z = 0$. As above, an appropriate choice for $\Phi$ is given by (28).

Since $z = \eta(r)$ is unknown, we require an expression that is consistent with $\Phi$ in (28), so we let

$$
\eta(r) = \sum_{k=0}^{\infty} B_k J_0(\theta_k r)
$$

and therefore

$$
\eta'(r) = -\sum_{k=0}^{\infty} B_k \theta_k J_1(\theta_k r),
$$

where the $\theta_k, k = 0, 1, 2, 3, \ldots$ are defined as above. To solve the nonlinear problem, we take these forms for $\Phi$ and $\eta$ and substitute into Eqs. (21) and (22). Note that (23) is satisfied automatically by the choice of the functions for $\Phi$.

To find the coefficients $B_k$ and $C_k$, we need to generate a set of equations from (21) to (22). Discretising the equations leads to an ill-conditioned system of nonlinear equations and so instead we follow the procedure in the linear solution of multiplying by $J_0(\theta_k r)r dr$ for each $k = 1, 2, \ldots$ and again integrating from 0 to $L$, giving

$$
\int_0^L (\Phi_r \eta_r - \Phi_z) J_0(\theta_k r)r dr = 0, \quad \forall k = 0, 1, 2, \ldots, N - 1
$$

and

$$
\int_0^L (\eta - \Phi) J_0(\theta_k r)r dr = 0, \quad \forall k = 0, 1, 2, \ldots, N - 1.
$$

Truncating the series to $N$ terms gives $2N$ equations for the $B_k$ and $C_k, k = 0, 1, \ldots, N - 1$. Integration was performed using highly accurate Gaussian quadrature. Using the iteration routine fsolve in Octave [25], the values of the coefficients can be found.

Some examples of interface shapes are given in Fig. 19, which shows a comparison between the approximate linear solutions and the full nonlinear solutions. It is clear that for small values of flow rate $M$ agreement is very good, but as $M$ increases, the two solutions begin to differ as we would expect. However, the excellent comparison of the two for small values of $M$ is confirmation of the numerical scheme.

As the value of $M$ increases, the surface pulls up towards the sink point. At some point, coning will occur, meaning that the interface will pull directly into the sink (draw up). Figure 20 shows the height of the middle of
the interface \((r = 0)\) as flux \(M\) increases for different numbers of coefficients, \(N = 50, 60\). The value of \(M\) at which draw-up occurs is independent of the number of coefficients for \(N > 50\), to 3 decimal places. As the point of draw-up is reached the height of the middle of the interface increases rapidly. Computations were performed for a range of values of \(h_s^*\) to determine the value of the flux, \(M_{\text{max}}\), as this event occurs. Results are shown in Fig. 21. As expected, the value of \(M_{\text{max}}\) increases as the sink moves further above the level of the interface, as in the stratified case considered in the previous sections.

That two-layer solutions with a stagnant lower layer exist for values of flow rate \(M\) below \(M_{\text{max}}\) is consistent with the existence of solutions flowing over the full depth for \(G^2\) only below some value in the stratified fluid, i.e. above some high flow rate. However, calculations indicate that the qualitative values are not particularly close, with the value of \(G\) computed as the minimum of the single-layer flow being quite a bit larger than that obtained as the maximum of the stratified computations, suggesting further work is needed to consider the transition of the smoothly stratified case to the two-layer case.

6 Summary

We have considered axisymmetric flow of a stratified fluid into a point sink in a vertically confined aquifer. First we considered the case where the density stratification was linear. Two methods of solution were used, an eigenfunction expansion in Whittaker functions and then a finite difference method. A comparison showed good agreement between the two methods, verifying the numerical approach.

The finite difference method was extended to consider nonlinear density strata and different sink locations. Velocity profiles were plotted to understand the effects of variations in flow rate and density. The differences in flow behaviour for nonlinear density profiles were identified. Regions in parameter space were found in which there were no steady solutions with streamlines filling the entire solution domain. In particular, low flow rates with strongly nonlinear density gradients seemed to be limited in the scope of possible solutions. It would seem likely that there
are solutions in which the streamfunction $\Psi = 0$ does not go along the base of the region, but instead separates from the wall with a stagnant region below. However, this calculation would be difficult using this finite difference method. Another method would be necessary to consider the formation of this stagnant region in situations with a general density stratification.

This suggestion that there is such a stagnant region led us to consider the case of flow in only a single layer above a sharp interface, and we were able to show not only that such a layer exists, but also identify the maximum flow rates beneath which they can be found. The next step is to identify such solutions with a variable density strata.

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