On an enhancement of the category of shifted $L_\infty$-algebras

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Abstract

We construct a symmetric monoidal category $\mathcal{S}^\text{Lie}_\infty^{\text{MC}}$ whose objects are shifted $L_\infty$-algebras equipped with a complete descending filtration. Morphisms of this category are “enhanced” infinity morphisms between shifted $L_\infty$-algebras. We prove that any category enriched over $\mathcal{S}^\text{Lie}_\infty^{\text{MC}}$ can be integrated to a simplicial category whose mapping spaces are Kan complexes. The advantage gained by using enhanced morphisms is that we can see much more of the simplicial world from the $L_\infty$-algebra point of view. We use this construction in a subsequent paper [3] to produce a simplicial model of a $(\infty,1)$-category whose objects are homotopy algebras of a fixed type.

1 Introduction

Let $L$ be a $\mathbb{Z}$-graded $k$-vector space\footnote{In this paper, we assume that $\text{char}(k) = 0$.}. A shifted $L_\infty$-algebra structure on $L$ is a degree 1 coderivation $Q$ of the cocommutative coalgebra

$$\mathcal{S}(L) := L \oplus S^2(L) \oplus S^3(L) \oplus \ldots$$

satisfying the Maurer-Cartan (MC) equation

$$[Q, Q] = 0.$$ 

An $\infty$-morphism $F : (L, Q) \to (\tilde{L}, \tilde{Q})$ is a homomorphism of dg cocommutative coalgebras

$$F : (\mathcal{S}(L), Q) \to (\mathcal{S}(\tilde{L}), \tilde{Q}).$$ \hspace{1cm} (1.1)

It is often convenient to extend any such coalgebra homomorphism (1.1) to the homomorphism of coalgebras $S(L), S(\tilde{L})$ with counits by requiring that

$$F(1) = 1.$$ 

It is known that the category of shifted $L_\infty$-algebras is naturally a symmetric monoidal category: given two shifted $L_\infty$-algebras $L_1, L_2$ their “tensor product” is defined as the direct sum $L_1 \oplus L_2$; given two $\infty$-morphisms

$$F_1 : \mathcal{S}(L_1) \to \mathcal{S}(\tilde{L}_1), \quad F_2 : \mathcal{S}(L_2) \to \mathcal{S}(\tilde{L}_2),$$
their tensor product\(^2\) is defined as

\[ F_1 \otimes F_2 := \left. F_1 \otimes F_2 \right|_{S(L_1 \oplus L_2)} : S(L_1 \oplus L_2) \to S(\tilde{L}_1 \oplus \tilde{L}_2). \]

It is clear that \(0\) is the unit object of this symmetric monoidal category.

In this paper, we construct a useful enhancement \(\mathfrak{S}\text{Lie}^{\text{MC}}\) of the symmetric monoidal category of shifted \(L_\infty\)-algebras. Objects of the category \(\mathfrak{S}\text{Lie}^{\text{MC}}\) are shifted \(L_\infty\)-algebras equipped with a complete descending filtration. A morphism from an object \(L\) to an object \(\tilde{L}\) of \(\mathfrak{S}\text{Lie}_{\infty}^{\text{MC}}\) is a pair

\[(\alpha, F)\]

where \(\alpha\) is a MC element of \(\tilde{L}\), \(F\) is a continuous \(\infty\)-morphism from \(L\) to \(\tilde{L}^\alpha\), and the shifted \(L_\infty\)-algebra \(\tilde{L}^\alpha\) is obtained from \(\tilde{L}\) via twisting by \(\alpha\).

We use the Getzler-Hinich construction \([9], [11]\) to show that every \(\mathfrak{S}\text{Lie}_{\infty}^{\text{MC}}\)-enriched category can be “integrated” to a simplicial category with mapping spaces being Kan complexes. This result illuminates the main reason for introducing enhanced morphisms. Indeed, to an ordinary \(\infty\)-morphism \(F : L_1 \to L_2\), the Getzler-Hinich construction assigns a map between simplicial sets which must preserve the canonical base point corresponding to the trivial MC element \(0\). This is too restrictive, since it prevents us from modeling more general simplicial maps. By using enhanced morphisms, we see more of the simplicial hom-set at the level of \(\mathfrak{S}\text{Lie}_{\infty}\)-algebras. We can, for example, model simplicial maps from the point \(\Delta^0\) as \(\mathfrak{S}\text{Lie}_{\infty}^{\text{MC}}\) morphisms originating from the unit object \(0\).

In a subsequent paper \([3]\), we show that homotopy algebras of a fixed type form a \(\mathfrak{S}\text{Lie}_{\infty}^{\text{MC}}\)-enriched category and use this fact to produce a simplicial enrichment of the category of homotopy algebras. Furthermore, we prove that this simplicial category is a model for the \((\infty, 1)\)-category of homotopy algebras.

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Notation and conventions

The ground field \(k\) has characteristic zero. For most of the algebraic structures considered here, the underlying symmetric monoidal category is the category \(\text{Ch}_k\) of unbounded cochain complexes of \(k\)-vector spaces. We will frequently use the ubiquitous combination “\(\text{dg}\)” (differential graded) to refer to algebraic objects in \(\text{Ch}_k\). For a cochain complex \(V\), we denote by \(sV\) (resp. by \(s^{-1}V\)) the suspension (resp. the desuspension) of \(V\). In other words,

\[(sV)^\bullet = V^{\bullet-1}, \quad (s^{-1}V)^\bullet = V^{\bullet+1}.
\]

\(^2\)Even though the “tensor product” of (shifted) \(L_\infty\)-algebras is \(\oplus\), it is more convenient to use the notation \(\otimes\) for the tensor product of \(\infty\)-morphisms of (shifted) \(L_\infty\)-algebras.
Any \(\mathbb{Z}\)-graded vector space \(V\) is tacitly considered as the cochain complex with the zero differential.

For a pair \(V, W\) of \(\mathbb{Z}\)-graded vector spaces, we denote by
\[
\text{Hom}(V, W)
\]
the corresponding inner-hom object in the category of \(\mathbb{Z}\)-graded vector spaces, i.e.
\[
\text{Hom}(V, W) := \bigoplus_m \text{Hom}_m^k(V, W),
\]
where \(\text{Hom}_m^k(V, W)\) consists of \(k\)-linear maps \(f: V \to W\) such that
\[
f(V^\bullet) \subset W^{\bullet+m}.
\]

The notation \(S_n\) is reserved for the symmetric group on \(n\) letters and \(S_{p_1,...,p_k}\) denotes the subset of \((p_1, \ldots, p_k)\)-shuffles in \(S_n\), i.e. \(S_{p_1,...,p_k}\) consists of elements \(\sigma \in S_n, n = p_1 + p_2 + \cdots + p_k\) such that
\[
\sigma(1) < \sigma(2) < \cdots < \sigma(p_1),
\]
\[
\sigma(p_1 + 1) < \sigma(p_1 + 2) < \cdots < \sigma(p_1 + p_2),
\]
\[
\cdots
\]
\[
\sigma(n - p_k + 1) < \sigma(n - p_k + 2) < \cdots < \sigma(n).
\]

We tacitly assume the Koszul sign rule. In particular,
\[
(-1)^{\varepsilon(\sigma;v_1,\ldots,v_m)}
\]
will always denote the sign factor corresponding to the permutation \(\sigma \in S_m\) of homogeneous vectors \(v_1, v_2, \ldots, v_m\). Namely,
\[
(-1)^{\varepsilon(\sigma;v_1,\ldots,v_m)} := \prod_{(i<j)} (-1)^{|v_i||v_j|},
\]
where the product is taken over all inversions \((i < j)\) of \(\sigma \in S_m\).

For a finite group \(G\) acting on a cochain complex (or a graded vector space) \(V\), we denote by
\[
V^G \quad \text{and} \quad V_G,
\]
respectively, the subcomplex of \(G\)-invariants in \(V\) and the quotient complex of \(G\)-coinvariants. Using the advantage of the zero characteristic, we often identify \(V_G\) with \(V^G\) via this isomorphism
\[
v \mapsto \sum_{g \in G} g(v): V_G \to V^G.
\]

For a graded vector space (or a cochain complex) \(V\), the notation \(S(V)\) (resp. \(S(V)\)) is reserved for the underlying vector space of the symmetric algebra (resp. the truncated symmetric algebra) of \(V\):
\[
S(V) = \mathbb{k} \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \ldots,
\]
\[ S(V) = V \oplus S^2(V) \oplus S^3(V) \oplus \ldots, \]

where

\[ S^n(V) = (V^{\otimes n})_{S_n}. \]

We denote by \( \text{Com} \) (resp. \( \text{Lie} \)) the operad governing commutative (and associative) algebras without unit (resp. the operad governing Lie algebras). Furthermore, we denote by \( \text{coCom} \) the cooperad which is obtained from \( \text{Com} \) by taking the linear dual. The coalgebras over \( \text{coCom} \) are cocommutative (and coassociative) coalgebras without counit.

The notation Cobar is reserved for the cobar construction \([5, \text{Section 3.7}]\).

For an operad (resp. a cooperad) \( P \) and a cochain complex \( V \) we denote by \( P(V) \) the free \( P \)-algebra (resp. the cofree \( P \)-coalgebra) generated by \( V \):

\[
P(V) := \bigoplus_{n \geq 0} \left( P(n) \otimes V^{\otimes n} \right)_{S_n}. \tag{1.5}
\]

For example,

\[
\text{Com}(V) = \text{coCom}(V) = S(V)
\]

as vector spaces. We denote by \( \mathcal{S} \) the underlying collection of the endomorphism operad

\[
\text{End}_{s^{-1}k}
\]

of the 1-dimensional space \( s^{-1}k \) placed in degree \(-1\). The \( n \)-the space of \( \mathcal{S} \) is

\[
\mathcal{S}(n) = \text{sgn}_n \otimes s^{n-1}k,
\]

where \( \text{sgn}_n \) denotes the sign representation of the symmetric group \( S_n \). Recall that \( \mathcal{S} \) is naturally an operad and a cooperad.

For a (co)operad \( P \), we denote by \( \mathcal{S}P \) the (co)operad which is obtained from \( P \) by tensoring with \( \mathcal{S} \):

\[
\mathcal{S}P := \mathcal{S} \otimes P.
\]

It is clear that tensoring with

\[
\mathcal{S}^{-1} := \text{End}_{s^{-1}k}
\]

gives us the inverse of the operation \( P \mapsto \mathcal{S}P \).

For example, the dg operad \( \text{Lie}_\infty := \text{Cobar}(\mathcal{S}^{-1}\text{coCom}) \) governs \( L_\infty \)-algebras and the dg operad

\[
\mathcal{S}\text{Lie}_\infty = \text{Cobar}(\text{coCom}) \tag{1.6}
\]

governs \( \mathcal{S}\text{Lie}_\infty \)-algebras.

In this paper, we often call \( \mathcal{S}\text{Lie}_\infty \)-algebras shifted \( L_\infty \)-algebras. Although a \( \mathcal{S}\text{Lie}_\infty \)-algebra structure on a cochain complex \( V \) is the same thing as an \( L_\infty \) structure on \( sV \), working with \( \mathcal{S}\text{Lie}_\infty \)-algebras has important technical advantages. This is why we prefer to deal with shifted \( L_\infty \)-algebras (a.k.a. \( \mathcal{S}\text{Lie}_\infty \)-algebras) from the outset.

The abbreviation “MC” is reserved for the term “Maurer-Cartan”.

\[3\]In this paper we only consider nilpotent coalgebras.
2 Filtered $\mathcal{G}\text{Lie}_\infty$ algebras, Maurer-Cartan (MC) elements, and twisting

Let $V$ be a cochain complex and $\mathcal{C}$ be a coaugmented dg cooperad. We recall\(^4\) that $\text{Cobar}(\mathcal{C})$-algebra structures on $V$ are in bijection with coderivations $Q$ of the $\mathcal{C}$-coalgebra $\mathcal{C}(V)$ satisfying the condition

$$Q \mid V = 0$$

and the MC equation

$$[\partial, Q] + \frac{1}{2}[Q, Q] = 0,$$

where $\partial$ comes from the differential on $V$ and $\mathcal{C}$ (if $\mathcal{C}$ has a non-zero differential).

Thus, since $\mathcal{G}\text{Lie}_\infty$-algebras are $\text{Cobar}(\text{coCom})$-algebras, a $\mathcal{G}\text{Lie}_\infty$-structure on a cochain complex $(L, \partial)$ is a degree 1 coderivation $Q$ of the cofree cocommutative coalgebra

$$\text{coCom}(L) = S(L)$$

(2.1)

satisfying the MC equation

$$[\partial, Q] + \frac{1}{2}[Q, Q] = 0$$

(2.2)

and the additional condition

$$Q \mid L = 0.$$ 

(2.3)

Introducing a degree 1 coderivation $Q$ on (2.1) satisfying (2.3) is equivalent to introducing the infinite collection of degree 1 operations

$$\{\cdot, \cdot, \ldots, \cdot\} : S^m(L) \to L$$

for all $m \geq 2$. Namely,

$$\{v_1, v_2, \ldots, v_m\} := p_L \circ Q(v_1 v_2 \ldots v_m),$$

(2.5)

where $p_L$ is the canonical projection $p_L : S(L) \to L$.

MC equation (2.2) is equivalent to the following sequence of relations on the brackets $\{\cdot, \cdot, \ldots, \cdot\}:

$$\partial\{v_1, v_2, \ldots, v_m\} + \sum_{i=1}^{m} (-1)^{|v_i|+\cdots+|v_{i-1}|}\{v_1, \ldots, v_i-1, \partial v_i, v_{i+1}, \ldots, v_m\}$$

$$+ \sum_{k=2}^{m-1} \sum_{\sigma \in \text{Sh}_k,m-k} (-1)^{\varepsilon(\sigma,v_1,\ldots,v_m)}\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}, v_{\sigma(k+1)}, \ldots, v_{\sigma(m)}\} = 0, \quad (2.6)$$

where $(-1)^{\varepsilon(\sigma,v_1,\ldots,v_m)}$ is the Koszul sign factor (see eq. (1.3)).

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\(^4\)See, for example, [5, Corollary 5.3].
An $\infty$-morphism $F$ from a $\mathcal{GLie}_\infty$-algebra $(L, \partial, Q)$ to a $\mathcal{GLie}_\infty$-algebra $(\tilde{L}, \tilde{\partial}, \tilde{Q})$ is a homomorphism

$$F : (\mathcal{G}(L), \partial + Q) \to (\mathcal{G}(\tilde{L}), \tilde{\partial} + \tilde{Q})$$

of the corresponding dg cocommutative coalgebras. Recall that any such coalgebra homomorphism $F$ is uniquely determined by its composition $F'$ with the projection $p_L : \mathcal{G}(\tilde{L}) \to \tilde{L}$:

$$F' := p_L \circ F : \mathcal{G}(L) \to \tilde{L}$$

More precisely, given a degree zero map $F' : \mathcal{G}(L) \to \tilde{L}$, the homomorphism of coalgebras $F$ is restored by the formula

$$F(v_1v_2\ldots v_n) = \sum_{t \geq 1} \sum_{k_1+\ldots+k_t=n} \sum_{\sigma \in \mathfrak{S}_{k_1,k_2,\ldots,k_t}} F'(v_{\sigma(1)} \ldots v_{\sigma(k_1)}) F'(v_{\sigma(k_1+1)} \ldots v_{\sigma(k_1+k_2)}) \ldots F'(v_{\sigma(n-k_t+1)} \ldots v_{\sigma(n)}), \quad (2.7)$$

where $\mathfrak{S}_{k_1,k_2,\ldots,k_t}$ is the subset of permutations in $\mathfrak{S}_{k_1,k_2,\ldots,k_t}$ satisfying the condition

$$\sigma(1) < \sigma(k_1 + 1) < \sigma(k_1 + k_2 + 1) < \cdots < \sigma(n - k_t + 1).$$

The compatibility of $F$ with the differentials $\partial + Q$ and $\partial + \tilde{Q}$ is equivalent to the sequence of equations

$$\tilde{\partial}F'(v_1, v_2, \ldots, v_m) = \sum_{i=1}^{m-1} (-1)^{|v_1| + \cdots + |v_{i-1}|} F'(v_1, \ldots, v_{i-1}, \partial v_i, v_{i+1}, \ldots v_m)$$

$$- \sum_{p=2}^{m-1} \sum_{\sigma \in \mathfrak{S}_{p,m-p}} (-1)^{\varepsilon(\sigma; v_1, \ldots, v_m)} F'\{v_{\sigma(1)}, \ldots, v_{\sigma(p)}\}, v_{\sigma(p+1)}, \ldots, v_{\sigma(m)}\} =$$

$$\sum_{k \geq 2} \sum_{\begin{array}{c} p_1 + p_2 + \cdots + p_k = m \\
\quad p_j \geq 1 \end{array}} (-1)^{\varepsilon(\tau; v_1, \ldots, v_m)} \{F'(v_{\tau(1)}, \ldots, v_{\tau(p_1)}), F'(v_{\tau(p_1+1)}, \ldots, v_{\tau(p_1+p_2)}), \ldots$$

$$\ldots, F'(v_{\tau(m-p_k+1)}, \ldots, v_{\tau(m)})\} \cdots. \quad (2.8)$$

**Definition 2.1** We say that a $\mathcal{GLie}_\infty$ algebra $(L, \partial, \{\cdot, \cdot\}, \{\cdot, \cdot, \cdot\}, \ldots)$ is filtered if the underlying complex $(L, \partial)$ is equipped with a complete descending filtration,

$$L = F_1L \supset F_2L \supset F_3L \cdots \quad (2.9)$$

$$L = \lim_k \frac{L}{F_kL}, \quad (2.10)$$

which is compatible with the brackets, i.e.

$$\{F_{i_1}L, F_{i_2}L, \ldots, F_{i_m}L\} \subseteq F_{i_1+i_2+\cdots+i_m}L \quad \forall \ m > 1.$$
The filtration \((2.9)\) induces a natural descending filtration and hence a topology on \(S(L)\). In this paper, we tacitly assume that \(\infty\)-morphisms of filtered \(\mathfrak{GLie}_\infty\)-algebras are continuous with respect to this topology.

Conditions \((2.10)\) and \(L = \mathcal{F}_1 L\) imply that the \(\mathfrak{GLie}_\infty\)-algebra \(L\) is pronilpotent, and hence the left hand side of the MC equation:

\[
\partial\alpha + \sum_{m=2}^{\infty} \frac{1}{m!}\{\underbrace{\alpha, \alpha, \ldots, \alpha}_m\} = 0 \tag{2.11}
\]

is well-defined for every \(\alpha \in L\).

A degree zero element \(\alpha \in L\) satisfying equation \((2.11)\) is called a MC element of \(L\). In this paper, the notation \(\text{MC}(L)\) is reserved for the set of MC elements of a filtered \(\mathfrak{GLie}_\infty\)-algebra \(L\).

We next record some well known facts regarding MC elements:

1. Given a filtered \(\mathfrak{GLie}_\infty\) algebra \(L = (L, \partial, \{\cdot, \cdot\}, \{\cdot, \cdot, \cdot\}, \ldots)\) and a MC element \(\alpha \in L\), we can construct a new filtered \(\mathfrak{GLie}_\infty\) structure \(L^\alpha = (L, \partial^\alpha, \{\cdot, \cdot\}^\alpha, \{\cdot, \cdot, \cdot\}^\alpha, \ldots)\) on the cochain complex \((L, \partial^\alpha)\) with the new differential

\[
\partial^\alpha (v) := \partial(v) + \sum_{k=1}^{\infty} \frac{1}{k!}\{\underbrace{\alpha, \ldots, \alpha}_k, v\} \tag{2.12}
\]

and the new multi-brackets

\[
\{v_1, v_2, \ldots, v_m\}^\alpha := \sum_{k=0}^{\infty} \frac{1}{k!}\{\underbrace{\alpha, \ldots, \alpha, v_1, v_2, \ldots, v_m}_k\} \tag{2.13}
\]

2. If \(F : L \to \tilde{L}\) is an \(\infty\)-morphism of \(\mathfrak{GLie}_\infty\) algebras and \(\alpha \in L\) is a MC element then

\[
F_*(\alpha) := \sum_{k=1}^{\infty} \frac{1}{k!} p_{\tilde{L}} \circ F(\alpha^k) \tag{2.14}
\]

is a MC element of \(\tilde{L}\).\(^5\)

3. If \(F : L \to \tilde{L}\) is an \(\infty\)-morphism of \(\mathfrak{GLie}_\infty\) algebras and \(\alpha \in L\) is a MC element then we can construct a new \(\infty\)-morphism

\[
F^\alpha : L^\alpha \to \tilde{L}^{F_*(\alpha)}
\]

with

\[
p_{\tilde{L}} \circ F^\alpha (v_1 v_2 \ldots v_m) = \sum_{k=0}^{\infty} \frac{1}{k!} p_{\tilde{L}} \circ F(\alpha^k v_1 v_2 \ldots v_m). \tag{2.15}
\]

\(^5\)For more details about twisting we refer the reader to [1, Section 2.4], [8], [9, Section 4] \(^6\)Note that \(F_*(\alpha)\) is well defined for any degree 0 element \(\alpha\).
Note that the infinite sums in equations (2.14) and (2.15) are well defined because $F$ is continuous and $L = \mathcal{F}_1 L$.

More generally, we denote by $\text{curv}$ the map of sets $L^0 \mapsto L^1$ given by the formula

$$\text{curv}(\alpha) := \partial \alpha + \sum_{m \geq 1} \frac{1}{m!} \{\alpha, \ldots, \alpha\}_{m_m}.$$  

For example, elements $\alpha \in L^0$ satisfying $\text{curv}(\alpha) = 0$ are precisely MC elements of the $\mathcal{S}\text{Lie}_\infty$-algebra $L$. Various useful properties of the operation $\text{curv}$ are listed in the following proposition.

**Proposition 2.2** Let $L$ and $\tilde{L}$ be filtered $\mathcal{S}\text{Lie}_\infty$-algebras and $F$ be a continuous $\infty$-morphism from $L$ to $\tilde{L}$. Then for every $\alpha, \beta \in L^0, v \in L$ we have

$$\partial(\text{curv}(\alpha)) + \sum_{m=1}^{\infty} \frac{1}{m!} \{\alpha, \ldots, \alpha, \text{curv}(\alpha)\} = 0,$$  

$$\text{curv}(F_*(\alpha)) = \sum_{m \geq 0} \frac{1}{m!} F'(\alpha^m \text{curv}(\alpha)),$$  

$$\partial^\alpha \circ \partial^\alpha(v) = -\{\text{curv}(\alpha), v\}_2,$$  

$$\text{curv}(\alpha + \beta) = \text{curv}(\alpha) + \partial^\alpha(\beta) + \sum_{m=2}^{\infty} \frac{1}{m!} \{\beta, \ldots, \beta\}_{m_m}.$$  

Proof. Equation (2.17) is proved in [9, Lemma 4.5]. Equations (2.19) and (2.20) follow from relations (2.6).

To prove (2.18) we use equation (2.7) which implies that

$$F(\exp(\alpha) - 1) = \exp(F_*(\alpha)) - 1,$$  

where $\exp(\alpha) - 1$ (resp. $\exp(F_*(\alpha)) - 1$) is considered as the element of the completion of $S(L)$ (resp. $S(\tilde{L})$) defined by the corresponding Taylor series.

A similar computation shows that, for every $\beta \in \tilde{L}$, we have

$$\tilde{Q}(\exp(\beta) - 1) = \exp(\beta)\text{curv}(\beta),$$  

where $\tilde{Q}$ is the coderivation of $S(\tilde{L})$ corresponding to the $\mathcal{S}\text{Lie}_\infty$-algebra structure on $\tilde{L}$.

On the other hand,

$$\tilde{Q} \circ F = F \circ Q.$$  

Hence (2.21) and (2.22) imply that

$$\tilde{Q}(\exp(F_*(\alpha)) - 1) = F(\exp(\alpha)\text{curv}(\alpha)).$$  

Applying the canonical projection $p_* L$ to both sides of (2.23) we get desired identity (2.18). \[\square\]

7Here we tacitly assume that $Q$ and $\tilde{Q}$ are extended in the natural way to the completions of $S(L)$ and $S(\tilde{L})$, respectively.
Remark 2.3 It is often convenient to “absorb” the differential $\partial$ on a $\mathcal{S}\text{Lie}_\infty$-algebra $L$ into the collection of multi-brackets (2.4) treating it as the unary operation:
\[
\{\cdot\} := \partial : L \to L.
\] (2.24)

Then relations (2.6) can be rewritten in the more concise form
\[
\sum_{k=1}^{m} \sum_{\sigma \in \text{Sh}_{k,m-k}} (-1)^{\varepsilon(\sigma,v_1,\ldots,v_m)} \{v_{\sigma(1)}, v_{\sigma(k)}, v_{\sigma(k+1)}, \ldots, v_{\sigma(m)}\} = 0,
\] (2.25)
where $(-1)^{\varepsilon(\sigma,v_1,\ldots,v_m)}$ is, as above, the sign coming from the Koszul rule.

Thus one can say that a $\mathcal{S}\text{Lie}_\infty$-structure on a cochain complex $(L, \partial)$ is a collection of degree 1 multi-brackets
\[
\{\ldots, \cdot, \ldots\} : S^m(L) \to L
\] (2.26)
for $m \geq 1$ satisfying (2.25) and the condition
\[
\{v\} = \partial v, \quad \forall \ v \in L.
\] (2.27)

3 The symmetric monoidal category $\mathcal{S}\text{Lie}^{\text{MC}}_\infty$

In this section we introduce the main hero of this note: an “enhanced” version $\mathcal{S}\text{Lie}^{\text{MC}}_\infty$ of the category of $\mathcal{S}\text{Lie}_\infty$-algebras. Then, we will show that $\mathcal{S}\text{Lie}^{\text{MC}}_\infty$ is a symmetric monoidal category.

Objects of the category $\mathcal{S}\text{Lie}^{\text{MC}}_\infty$ are filtered $\mathcal{S}\text{Lie}_\infty$ algebras and morphisms are defined in the following way:

Definition 3.1 An enhanced morphism

\[
L_1 \overset{(\alpha,F)}{\longrightarrow} L_2
\]

between filtered $\mathcal{S}\text{Lie}_\infty$ algebras is a pair consisting of a MC element $\alpha \in L_2$ and a continuous $\infty$-morphism $F : L_1 \to L_2^\alpha$.

Note that every $\infty$-morphism $F$ of filtered $\mathcal{S}\text{Lie}_\infty$ algebras is canonically the enhanced morphism $(0,F)$.

Let $(\alpha,F)$ be an enhanced morphism from $L_1$ to $L_2$. Then $F$ is a homomorphism of dg coCom-coalgebras
\[
F : (\mathcal{S}(L_1), Q_1) \to (\mathcal{S}(L_2), Q_2^\alpha),
\] (3.1)
where $Q_1$ and $Q_2^\alpha$ are the codifferentials defining the $\mathcal{S}\text{Lie}_\infty$-structures on $L_1$ and $L_2^\alpha$, respectively.

To define the composition of enhanced morphisms, we need to extend the codifferentials $Q_1$ and $Q_2$ to the completions
\[
\mathcal{S}(L_1)^\wedge \quad \text{and} \quad \mathcal{S}(L_2)^\wedge
\] (3.2)
with respect to the natural descending filtrations coming from those on $L_1$ and $L_2$, respectively. We do this via extending $Q_1$ and $Q_2$ by continuity and setting

$$Q_1(1) = Q_2(1) = 0.$$  \hfill (3.3)

**Remark 3.2** Note that the completion $S(L)^\wedge$ of $S(L)$ with respect to the filtration coming from some $L$ is, strictly speaking, not a coalgebra because the natural analog of the comultiplication $\Delta$ on $S(L)^\wedge$ lands in the completed tensor product

$$S(L)^\wedge \hat{\otimes} S(L)^\wedge.$$  

This subtlety should be kept in mind when writing equations like

$$\Delta Q(X) = (Q \otimes 1 + 1 \otimes Q) \Delta(X)$$ \hfill (3.4)

or claiming that $Q$ is a coderivation of the “coalgebra” $S(L)^\wedge$.

Passing to the completions allows us to consider such elements as $e^\alpha$, where $\alpha \in L_2^0$, and rewrite the MC equation (2.11) for $\alpha$ in the form

$$Q(e^\alpha) = 0.$$ \hfill (3.5)

Let us also extend the coalgebra morphism (3.1) to the morphism

$$F : (S(L_1)^\wedge, Q_1) \rightarrow (S(L_2)^\wedge, Q_2^\alpha)$$ \hfill (3.6)

by continuity and declaring that

$$F(1) := 1.$$ \hfill (3.7)

It is easy to see that this extension is compatible with the comultiplications on $S(L_1)^\wedge$ and $S(L_2)^\wedge$, respectively:

$$\Delta \circ F = (F \otimes F) \circ \Delta.$$ \hfill (3.8)

Since the codifferential $Q_2^\alpha$ can be written in the form

$$Q_2^\alpha = e^{-\alpha} Q_2 e^\alpha$$ \hfill (3.9)

the compatibility of $F$ with $Q_1$ and $Q_2^\alpha$ is equivalent to the equation

$$Q_2(e^\alpha F) = (e^\alpha F) Q_1.$$ \hfill (3.10)

On the other hand, $e^\alpha F$ is obviously compatible with comultiplications on $S(L_1)^\wedge$ and $S(L_2)^\wedge$. Thus we arrive at the following statement:

**Claim 3.3** Every enhanced morphism $(\alpha, F)$ from $L_1$ to $L_2$ gives rise to the following homomorphism of cocommutative “coalgebras”

$$U_{\alpha,F} : (S(L_1)^\wedge, Q_1) \rightarrow (S(L_2)^\wedge, Q_2)$$

$$U_{\alpha,F}(X) := e^\alpha F(X),$$ \hfill (3.11)

where

$$F(1) := 1,$$ \hfill (3.12)

and $F$ is extended to the completion $S(L_1)^\wedge$ by continuity.
Let \( L_1, L_2, L_3 \) be filtered \( \mathfrak{S} \text{Lie}_\infty \)-algebras and let \((\alpha_2,F)\) (resp. \((\alpha_3,G)\)) be an enhanced morphism from \( L_1 \) to \( L_2 \) (resp. from \( L_2 \) to \( L_3 \)). According to the above claim, \((\alpha_2,F)\) and \((\alpha_3,G)\) give us morphisms of cocommutative “coalgebras”:

\[
U_{\alpha_2,F} : (S(L_1);Q_1) \to (S(L_2);Q_2)
\]  

(3.13)

and

\[
U_{\alpha_3,G} : (S(L_2);Q_2) \to (S(L_3);Q_3).
\]  

(3.14)

We observe that the composition \( U_{\alpha_3,G} \circ U_{\alpha_2,F} \) acts on \( X \in S(L_1) \) as

\[
U_{\alpha_3,G} \circ U_{\alpha_2,F}(X) = e^{\alpha_2}G(e^{\alpha_2}F(X)) = e^{\alpha_3}e^{G_e(\alpha_2)}(e^{-G_e(\alpha_2)}Ge^{\alpha_2}) \circ F(X) = e^{\alpha_3+G_e(\alpha_2)}G^{\alpha_2} \circ F(X).
\]  

(3.15)

Therefore the composition \( U_{\alpha_3,G} \circ U_{\alpha_2,F} \) is a morphism from \( S(L_1) \) to \( S(L_3) \) of the form

\[
(\alpha_3 + G_e(\alpha_2), G^{\alpha_2} \circ F).
\]  

(3.16)

Thus we proved the following statement:

**Proposition 3.4** Equation (3.13) defines a composition of enhanced morphisms for \( \mathfrak{S} \text{Lie}_\infty \)-algebras. Moreover, this composition is associative. \( \square \)

### 3.1 \( \mathfrak{S} \text{Lie}_\infty^{MC} \) is a symmetric monoidal category

Given two filtered \( \mathfrak{S} \text{Lie}_\infty \) algebras \((L, \{\cdot,\cdot\}, \{\cdot,\cdot,\cdot\}, \ldots)\) and \((\tilde{L}, \{\cdot,\cdot\}, \{\cdot,\cdot,\cdot\}, \ldots)\), one obtains a filtered \( \mathfrak{S} \text{Lie}_\infty \) structure on the direct sum \( L \oplus \tilde{L} \) by setting

\[
\{x_1 + x'_1, x_2 + x'_2, \ldots, x_k + x'_k\} := \{x_1, x_2, \ldots, x_k\} + \{x'_1, x'_2, \ldots, x'_k\},
\]

and

\[
F_k(L \oplus \tilde{L}) := (F_kL) \oplus (F_k\tilde{L}).
\]

If \( \alpha \) and \( \tilde{\alpha} \) are MC elements of \( L \) and \( \tilde{L} \), respectively, then \( \alpha + \tilde{\alpha} \in L \oplus \tilde{L} \) is clearly a MC element of the \( \mathfrak{S} \text{Lie}_\infty \)-algebra \( L \oplus \tilde{L} \). Furthermore, the operation of twisting (by a MC element) is compatible with \( \oplus \), i.e. the \( \mathfrak{S} \text{Lie}_\infty \)-algebra \( L^\alpha \oplus \tilde{L}^{\tilde{\alpha}} \) is canonically isomorphic to the \( \mathfrak{S} \text{Lie}_\infty \)-algebra to \((L \oplus \tilde{L})^{\alpha + \tilde{\alpha}} \).

Let us now consider a pair of enhanced morphisms

\[
(\alpha, F) : L_1 \to L_2, \quad \text{and} \quad (\tilde{\alpha}, \tilde{F}) : \tilde{L}_1 \to \tilde{L}_2.
\]  

(3.17)

Extending the corresponding \( \infty \)-morphisms

\[
F : S(L_1) \to S(L_2^\alpha), \quad \tilde{F} : S(\tilde{L}_1) \to S(\tilde{L}_2^{\tilde{\alpha}}),
\]

to \( S(L_1) \) and \( S(\tilde{L}_1) \) respectively by declaring that

\[
F(1) := 1, \quad \text{and} \quad \tilde{F}(1) := 1
\]

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and tensoring the resulting homomorphism of cocommutative coalgebras (with counits) we get
\[ F \otimes \tilde{F} : S(L_1) \otimes S(\tilde{L}_1) \cong S(L_1 \oplus \tilde{L}_1) \to S(L_2^o) \otimes S(\tilde{L}_2^\alpha) \cong S(L_2^o \oplus \tilde{L}_2^\alpha). \quad (3.18) \]

Since the operation of twisting by a MC element commutes with \( \oplus \), the restriction of (3.18) to \( S(L_1 \oplus \tilde{L}_1) \) gives us a desired \( \infty \)-morphism
\[ F \otimes \tilde{F} \bigg|_{S(L_1 \oplus \tilde{L}_1)} : S(L_1 \oplus \tilde{L}_1) \to S((L_2 \oplus \tilde{L}_2)^{\alpha + \tilde{\alpha})} \quad (3.19) \]
from the \( \mathcal{GLie}_\infty \)-algebra \( L_1 \oplus \tilde{L}_1 \) to the \( \mathcal{GLie}_\infty \)-algebra \( (L_2 \oplus \tilde{L}_2)^{\alpha + \tilde{\alpha}}. \)
Thus every pair (3.17) gives us the enhanced morphism
\[ (\alpha, F) \otimes (\tilde{\alpha}, \tilde{F}) := \bigg( \alpha + \tilde{\alpha}, F \otimes \tilde{F} \bigg|_{S(L_1 \oplus \tilde{L}_1)} \bigg) \quad (3.20) \]
from \( L_1 \oplus \tilde{L}_1 \) to \( L_2 \oplus \tilde{L}_2. \)

It is easy to verify that the operations \( (L, \tilde{L}) \mapsto L \oplus \tilde{L} \) and (3.20) give \( \mathcal{GLie}_\infty^{MC} \) the structure of a symmetric monoidal category whose unit object is \( 0. \)

4 Integration of a \( \mathcal{GLie}_\infty^{MC} \)-enriched category to a simplicial category

In this section, we show that every \( \mathcal{GLie}_\infty^{MC} \)-enriched category \( \mathcal{C} \) can be integrated to a simplicial category. We use this construction in subsequent paper [3] to find a higher categorical structure formed by homotopy algebras of a fixed type.

For this purpose, we need to recall the construction [9] of the Deligne-Getzler-Hinich (DGH) \( \infty \)-groupoid \( \mathcal{MC}_\bullet(L) \) of a nilpotent \( \mathcal{GLie}_\infty \)-algebra \( L. \)

Let \( \Omega_n = \Omega^\bullet(\Delta^n) \) denote the polynomial de Rham complex on the \( n \)-simplex with coefficients in \( \mathbb{k} \), and \( \{ \Omega_n \}_{n \geq 0} \) the associated simplicial dg commutative \( \mathbb{k} \)-algebra.

Since, for every \( n \), \( \Omega_n \) is a dg commutative algebra, the tensor product
\[ L \otimes \Omega_n \]
is naturally a nilpotent \( \mathcal{GLie}_\infty \)-algebra. Furthermore, the simplicial structure on the collection \( \{ \Omega_n \}_{n \geq 0} \) gives us the structure of a simplicial set on the collection
\[ \mathcal{MC}_n(L) := \text{MC}(L \otimes \Omega_n), \quad (4.1) \]
where, as above, \( \text{MC}(L) \) denotes the set of MC elements of a \( \mathcal{GLie}_\infty \)-algebra \( L \).

Due to [9] Lemma 4.6, the simplicial set \( \mathcal{MC}_\bullet(L) \) (4.1) is a Kan complex (a.k.a. an \( \infty \)-groupoid).

In the examples we keep in mind [1], [2], [3], [4], [5], [6], [7], [8], [13], [14] the \( \mathcal{GLie}_\infty \)-algebras are rarely nilpotent. However, the above construction can be easily extended to the case when the \( \mathcal{GLie}_\infty \)-algebra \( L \) is filtered in the sense of Definition [2,1]
In this case we have to replace $L \otimes \Omega_n$ by the completed tensor product

$$L \hat{\otimes} \Omega_n$$

of the topological space $L$ (with the topology coming from the filtration) and the discrete topological space $\Omega_n$.

We claim that

**Proposition 4.1** For every filtered $\mathcal{G}\text{Lie}_\infty$-algebra $L$ the simplicial set with

$$\mathcal{MC}_n(L) := \text{MC}(L \hat{\otimes} \Omega_n) \quad (4.2)$$

is a Kan complex.

**Proof.** The canonical maps $L/F_k L \to L/F_{k-1} L$ are surjective strict morphisms between nilpotent $\mathcal{G}\text{Lie}_\infty$-algebras. Hence, Proposition 4.7 in [9] implies that the induced maps $\mathcal{MC}_*(L/F_k L) \to \mathcal{MC}_*(L/F_{k-1} L)$ are fibrations between Kan complexes. Therefore, the inverse limit of this tower of fibrations is a Kan complex (cf. [10][Sec. VI.1]). Our proposition then follows since

$$\mathcal{MC}_*(L) = \lim_{\leftarrow k} \mathcal{MC}_*(L/F_k L).$$

□

Let us next observe that any continuous $\infty$-morphism $F : L \to \tilde{L}$ of filtered $\mathcal{G}\text{Lie}_\infty$-algebras gives a collection of $\infty$-morphisms of $\mathcal{G}\text{Lie}_\infty$-algebras

$$F^{(n)} : L \otimes \Omega_n \to \tilde{L} \otimes \Omega_n,$$

$$F^{(n)}(v_1 \otimes \omega_1, v_2 \otimes \omega_2, \ldots, v_m \otimes \omega_m) = \pm F(v_1, v_2, \ldots, v_m) \otimes \omega_1 \omega_2 \cdots \omega_m, \quad (4.3)$$

where $v_i \in L, \omega_i \in \Omega_n$, and $\pm$ is the usual Koszul sign. This collection is obviously compatible with all the faces and all the degeneracies. Hence, $F$ induces a morphism of simplicial sets

$$\mathcal{MC}_*(F) : \mathcal{MC}_*(L) \to \mathcal{MC}_*(\tilde{L}) \quad (4.4)$$

given by the formula

$$\mathcal{MC}_n(F)(\alpha) := F^{(n)}(\alpha). \quad (4.5)$$

**Proposition 4.2** The assignment

$$L \mapsto \mathcal{MC}_*(L)$$

extends naturally to a monoidal functor from the category $\mathcal{G}\text{Lie}_\infty^{MC}$ to the category of simplicial sets.

For the proof of Prop. 4.2 we need the following lemma.

**Lemma 4.3** Let $\alpha$ be a MC element in $L$ and $L^\alpha$ be the filtered $\mathcal{G}\text{Lie}_\infty$ algebra which is obtained from $L$ via twisting by $\alpha$. Then the following assignment

$$\beta \in \text{MC}(L^\alpha \hat{\otimes} \Omega_n) \mapsto \alpha + \beta \in \text{MC}(L \hat{\otimes} \Omega_n) \quad (4.6)$$

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which sends the zero MC element of $L^\alpha$ to the MC element $\alpha$ in $L$. For every $\infty$-morphism $F$ of filtered $\mathcal{S}$Lie$\infty$-algebras $L \to \tilde{L}$ the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{MC}_\bullet(L^\alpha) & \xrightarrow{\text{Shift}_\alpha} & \mathcal{MC}_\bullet(L) \\
\mathcal{MC}_\bullet(F^\alpha) \downarrow & & \downarrow \mathcal{MC}_\bullet(F) \\
\mathcal{MC}_\bullet(\tilde{L}^{F_*(\alpha)}) & \xrightarrow{\text{Shift}_{F_*(\alpha)}} & \mathcal{MC}_\bullet(\tilde{L}),
\end{array}
\]

where $F^\alpha$ denotes the $\infty$-morphism $L^\alpha \to \tilde{L}^{F_*(\alpha)}$ which is obtained from $F$ via twisting by the MC element $\alpha$.

Proof. Eq. (2.20) from Prop. 2.2 implies that map (4.6) is well defined, and it is clearly injective. If $\gamma \in \mathcal{MC}(L \hat{\otimes} \Omega_n)$, then $\text{curv}(\gamma) = \text{curv}(\alpha + (\gamma - \alpha)) = 0$. Hence, (2.20) implies that

$$\gamma - \alpha \in \mathcal{MC}(L^\alpha \hat{\otimes} \Omega_n),$$

so map (4.6) is also surjective. Since $\alpha$ is constant as an element of $L \hat{\otimes} \Omega_\bullet$, (4.6) induces the isomorphism of simplicial sets (4.7).

To show that diagram (4.8) commutes, suppose that $\beta \in \mathcal{MC}_\bullet(L^\alpha)$. Then we have

\[
(\mathcal{MC}_\bullet(F) \circ \text{Shift}_\alpha)(\beta) = F_*(\alpha + \beta) = \sum_{k \geq 1} \frac{1}{k!} F'(k + \beta) = \sum_{k \geq 1} \sum_{l=0}^k \frac{1}{l!(k-l)!} F'(\alpha^l \beta^{k-l}).
\]

On the other hand,

\[
(\text{Shift}_{F_*(\alpha)} \circ \mathcal{MC}_\bullet(F^\alpha))(\beta) = F_*(\alpha) + F^\alpha_*(\beta) = \sum_{l \geq 1} \frac{1}{l!} F'(\alpha^l) + \sum_{k \geq 1} \frac{1}{k!} (F^\alpha)'(\beta^k)
\]

\[
= \sum_{l \geq 1} \frac{1}{l!} F'(\alpha^l) + \sum_{k \geq 1} \sum_{l \geq 0} \frac{1}{l!k!} F'(\alpha^l \beta^k).
\]

After rearranging terms, we see that the two compositions above are equal. \qed

Proof of Prop. 4.2. To a morphism

$$L_1 \xrightarrow{(\alpha,F)} L_2$$

in $\mathcal{S}$Lie$^\infty_{\mathcal{MC}}$, we assign the morphism of simplicial sets

\[
\mathcal{MC}_\bullet(\alpha,F) : \mathcal{MC}_\bullet(L_1) \to \mathcal{MC}_\bullet(L_2)
\]

\[
\mathcal{MC}_\bullet(\alpha,F) := \text{Shift}_\alpha \circ F_*.
\]
Given another morphism in $\mathcal{S}\text{Lie}_{\infty}^{\MC}$

\[ L_2 \xrightarrow{(\beta, G)} L_3, \]
we need to show that this assignment respects composition, which, because of Eq. (3.16), is equivalent to verifying the equality

\[ \MC_\bullet((\beta + G_\ast(\alpha), G^\ast \circ F)) = \MC_\bullet(\beta, G) \circ \MC_\bullet(\alpha, F). \quad (4.12) \]

Observe that the $\mathcal{S}\text{Lie}_{\infty}$ algebras $(L_3^\beta G_\ast(\alpha))$ and $L_3^{\beta + G_\ast(\alpha)}$ are equal (not just isomorphic). Therefore, we have the following diagram

\[
\begin{array}{c}
\MC_\bullet(L_1) \xrightarrow{F} \MC_\bullet(L_2) \\
\downarrow G^\ast \quad \quad \quad \downarrow G_\ast \\
\MC_\bullet((L_3^\beta G_\ast(\alpha)) \xrightarrow{\text{Shift}_{\beta + G_\ast(\alpha)}} \MC_\bullet(L_3) \\
\downarrow \equiv \quad \quad \downarrow \text{Shift}_\beta \\
\MC_\bullet(L_3^{\beta + G_\ast(\alpha)}) \xrightarrow{\text{Shift}_{\beta + G_\ast(\alpha)}} \MC_\bullet(L_3) \\
\end{array}
\]

Lemma 4.3 implies that the top rectangle of the diagram commutes, and the definition of Shift$_{\beta + G_\ast(\alpha)}$ implies that the lower rectangle commutes. Hence, Eq. (4.12) holds.

Finally, given a pair of $\mathcal{S}\text{Lie}_{\infty}$-algebras $L$, $\tilde{L}$, there is a natural isomorphism

\[ \MC_\bullet(L \oplus \tilde{L}) \cong \MC_\bullet(L) \times \MC_\bullet(\tilde{L}). \]

Indeed, for each $n \geq 0$, we have the following natural isomorphisms:

\[
\begin{align*}
\lim_{\leftarrow k}(L \oplus \tilde{L}/F_k(L \oplus \tilde{L}) \otimes \Omega_n) &\cong \lim_{\leftarrow k}\left((L/F_k L \oplus \tilde{L}/F_k \tilde{L}) \otimes \Omega_n\right) \\
&\cong \lim_{\leftarrow k}(L/F_k L \otimes \Omega_n \oplus \tilde{L}/F_k \tilde{L} \otimes \Omega_n) \\
&\cong (\lim_{\leftarrow k} L/F_k L \otimes \Omega_n) \oplus (\lim_{\leftarrow k} \tilde{L}/F_k \tilde{L} \otimes \Omega_n),
\end{align*}
\]

where the last line above follows from the fact that both the projective limit and direct sum are limits, and hence commute. So we have exhibited a natural isomorphism of $\mathcal{S}\text{Lie}_{\infty}$-algebras

\[ (L \oplus \tilde{L}) \hat{\otimes} \Omega_n \cong L \hat{\otimes} \Omega_n \oplus \tilde{L} \hat{\otimes} \Omega_n. \]

By combining this with the obvious natural isomorphism of sets:

\[ \MC(L \hat{\otimes} \Omega_n \oplus \tilde{L} \hat{\otimes} \Omega_n) \cong \MC(L \hat{\otimes} \Omega_n) \times \MC(\tilde{L} \hat{\otimes} \Omega_n), \]

it follows that $\MC_\bullet(-)$ is indeed a strong monoidal functor. \qed
Remark 4.4 As alluded to in the introduction, the functor constructed in Prop. 4.2 demonstrates the utility of enhanced morphisms. Simplicial morphisms which lie in the image of $\mathcal{MC}_\bullet(-)$ need not preserve the base point 0. For example,

$$\text{Hom}_{\mathcal{MC}_\infty}(0, L) \cong \text{Hom}_{\mathcal{S}Set}(\Delta^0, \mathcal{MC}_\bullet(L)) \cong \text{MC}(L).$$

Finally, let $\mathcal{C}$ be a $\mathcal{GLie}_\infty$-enriched category \cite{12}. Then, to every pair of objects $A, B$ of $\mathcal{C}$, we may assign the Kan complex

$$\mathcal{MC}_\bullet(\text{map}(A, B)), \quad (4.13)$$

where map$(A, B)$ denotes the mapping space (i.e. a filtered $\mathcal{GLie}_\infty$-algebra) corresponding to the pair $A, B$ in $\mathcal{C}$. Since the functor described in Prop. 4.2 is a monoidal functor from the category $\mathcal{GLie}_\infty$ to the category of simplicial sets, the Kan complexes (4.13) assemble into a category enriched over $\infty$-groupoids and we conclude that

**Theorem 4.5** For every $\mathcal{GLie}_\infty$-enriched category $\mathcal{C}$ the assignment

$$(A, B) \in \text{Objects}(\mathcal{C}) \times \text{Objects}(\mathcal{C}) \mapsto \mathcal{MC}_\bullet(\text{map}(A, B))$$

gives us a category enriched over $\infty$-groupoids (a.k.a. Kan complexes). \hfill \Box

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