Principal factors and lattice minima

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Dedicated to H. C. Williams.

Abstract: Let $k = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$, where $d > 1$ is a cube-free positive integer, $k_0 = \mathbb{Q}(\zeta_3)$ be the cyclotomic field containing a primitive cube root of unity $\zeta_3$, and $G = \text{Gal}(k/k_0)$. The possible prime factorizations of $d$ in our main result [2, Thm. 1.1] give rise to new phenomena concerning the chain $\Theta = (\theta_i)_{i \in \mathbb{Z}}$ of lattice minima in the underlying pure cubic subfield $L = \mathbb{Q}(\sqrt[3]{d})$ of $k$. The aims of the present work are to give criteria for the occurrence of generators of primitive ambiguous principal ideals $(\alpha) \in \mathcal{P}_k^G/\mathcal{P}_{k_0}$ among the lattice minima $\Theta = (\theta_i)_{i \in \mathbb{Z}}$ of the underlying pure cubic field $L = \mathbb{Q}(\sqrt[3]{d})$, and to explain exceptional behavior of the chain $\Theta$ for certain radicands $d$ with impact on determining the principal factorization type of $L$ and $k$ by means of Voronoi’s algorithm.

Keywords: Pure cubic field, 3-rank, primitive ambiguous principal ideals, principal factorization type, chain of lattice minima, Voronoi’s algorithm.

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1 Introduction

Let $k = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$, where $d > 1$ is a cube-free positive integer, $k_0 = \mathbb{Q}(\zeta_3)$, where $\zeta_3$ is a primitive cube root of unity, and $k^*$ be the relative genus field of $k/k_0$.

In our previous work [2], we implemented Gerth’s methods [7] and [6] for determining the rank of the group of ambiguous ideal classes of $k/k_0$ and obtained all integers $d$ and conductors $f$ for which $\text{Gal}(k^*/k) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. In contrast with the radicands $d$ of the shape in [1, Thm. 1.1], the possible prime factorizations of $d$ in our main result [2, Thm. 1.1] are more complicated and give rise to new phenomena concerning the chain $\Theta = (\theta_i)_{i \in \mathbb{Z}}$ of lattice minima in the underlying pure cubic subfield $L = \mathbb{Q}(\sqrt[3]{d})$ of $k$. A lattice minimum $\theta_i$ is an algebraic integer with norm not exceeding the Minkowski bound of the maximal order $\mathcal{O}_L$ of $L$. In particular, all positive units $\eta > 0$ of $L$, which have norm 1, are lattice minima and the original purpose of Voronoi’s algorithm [9] was to find the fundamental unit $0 < \varepsilon < 1$ by constructing the chain $\Theta$ and stopping at the first unit encountered, which must be $\varepsilon$.

More recently, however, it was the idea of Barrucand, Cohn [4] and Williams [12] to use Voronoi’s algorithm for the classification of pure cubic fields into three principal factorization types, which we have rederived with cohomological techniques in [1, § 2.1]. The clue was to keep track of the norms $n_i = N_{L/\mathbb{Q}}(\theta_i)$ of all lattice minima on the way through the chain $\Theta$, starting at the trivial unit $\theta_0 = 1$. When some $n_i$ divides the square of the conductor $f$ of $k/k_0$, then $\theta_i$ is generator of a primitive ambiguous principal ideal in $\mathcal{P}_L^G/\mathcal{P}_{Q} \leq \mathcal{P}_k^G/\mathcal{P}_{k_0}$, more precisely an absolute principal factor, and $L$ is of type $\beta$ [1, Thm. 2.1]. Now, the new phenomenon which arises for numerous radicands of the form in Equation (1) of [2, Thm. 1.1] is the occasional failure of the chain $\Theta$ to lead to an absolute principal factor although $L$ is of type $\beta$. 
After explaining the connection between radicand $d$, conductor $f$, and ramification in $k/k_0$ in section 2, the formalism of canonical divisors in section 3, and the concept of lattice minima in section 4, we prove necessary and sufficient conditions for the occurrence of generators $\alpha$ of primitive ambiguous principal ideals $(\alpha) \in \mathcal{P}_k^G/\mathcal{P}_k$ among the lattice minima in the chain $\Theta = (\theta_i)_{i \in \mathbb{Z}}$ in section 5. We develop a powerful new algorithm which elegantly avoids all mentioned problems by using a non-maximal order $\mathcal{O}_{L,0}$ for $d \equiv \pm 1 \pmod{9}$, and by exploiting the impossibility of type $\gamma$ [1, Thm. 2.1] for $d \equiv \pm 2, \pm 4 \pmod{9}$, in section 6, and we give an explicit criteria for M0-fields in rational integers in section 7.

The new techniques were implemented for an extensive classification of all normalized radicands $2 \leq d < 10^6$ and they detected serious defects in the previous table [12, § 6, p. 272, and Tbl. 2, p. 273]. The usual notations is given as follows:

- $L = \mathbb{Q}(\sqrt[3]{d})$ is a pure cubic field, where $d > 1$ is a cube-free positive integer;
- $k_0 = \mathbb{Q}(\zeta_3)$, where $\zeta_3 = e^{2\pi i/3}$ denotes a primitive third root of unity;
- $k = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$ is the normal closure of $L$;
- $f$ is the conductor of the relative Kummer extension $k/k_0$;
- $\nu_l(x)$ is the $l$-valuation of the integer $x$;
- $\Theta = (\theta_i)_{i \in \mathbb{Z}}$ is the chain of lattice minima in the underlying pure cubic subfield $L$ of $k$.
- $Q$ is the index of the subgroup $E_0$ generated by the units of intermediate fields of the extension $k/\mathbb{Q}$ in the group of units of $k$;
- $\langle \tau \rangle = \text{Gal}(k/L)$ such that $\tau^2 = id$, $\tau(\zeta_3) = \zeta_3^2$ and $\tau(\sqrt[3]{d}) = \sqrt[3]{d}$;
- $\langle \sigma \rangle = \text{Gal}(k/k_0)$ such that $\sigma^3 = id$, $\sigma(\zeta_3) = \zeta_3$, $\sigma(\sqrt[3]{d}) = \zeta_3 \sqrt[3]{d}$ and $\tau\sigma = \sigma^2\tau$;
- For an algebraic number field $F$:
  - $\mathcal{O}_F$, $E_F$ : the ring of integers and the group of units of $F$;
  - $\mathcal{I}_F$, $\mathcal{P}_F$ : the group of ideals and the subgroup of principal ideals of $F$;

2 Conductor and ramification

Let $L = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field with normalized radicand $d = a \cdot b^2$, where $a > b \geq 1$ are square-free coprime integers. The normalization enforces that the co-radicand $d = a^2 \cdot b$ is strictly bigger than $d$. It generates an isomorphic field $\mathbb{Q}(\sqrt[3]{d}) \simeq L$, since $a^2 \cdot b$ differs from the square $a^2 \cdot b^4$ of $d$ by the complete third power $b^3$.

The class field theoretic conductor $f$ of the associated relative Kummer extension $k/k_0$ is

$$f = \begin{cases} 3ab & \text{if } d \not\equiv \pm 1 \pmod{9} \text{ (Dedekind’s species 1)}, \\ ab & \text{if } d \equiv \pm 1 \pmod{9} \text{ (Dedekind’s species 2)}. \end{cases}$$

(1)

This means that all prime divisors of $ab$ are ramified in $k/k_0$. If $L$ is of Dedekind’s second species with $d \equiv \pm 1 \pmod{9}$, then $3 \nmid ab$ and $3$ is unramified in $k/k_0$. However, if $L$ is of Dedekind’s first species with $d \not\equiv \pm 1 \pmod{9}$, then either $3 \mid ab$ (species 1a) or $d \equiv \pm 2, \pm 4 \pmod{9}$ (species 1b), and in both cases $3$ is ramified in $k/k_0$ [1, § 2.2].

For a prime number $\ell \in \mathbb{P}$, we denote by $\nu_\ell : \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$ the $\ell$-valuation of non-zero rational numbers.
The species of the field $L$ can be expressed by the 3-valuation of the conductor $f$:

$$v_3(f) = \begin{cases} 
2, & \text{if } L \text{ is of species } 1a, \\
1, & \text{if } L \text{ is of species } 1b, \\
0, & \text{if } L \text{ is of species } 2. 
\end{cases}$$

(2)

Since the conductor is divisible by 9 for fields of species 1a, it is convenient to define a ramification invariant $R$ which is the product of all primes which are ramified in $k/k_0$:

$$R := \begin{cases} 
 f = ab & \text{if } d \equiv \pm 1 \pmod{9} \text{ (and thus } 3 \nmid ab), \\
 f = 3ab & \text{if } d \equiv \pm 2, \pm 4 \pmod{9} \text{ (and thus } 3 \nmid ab), \\
 f/3 = ab & \text{if } 3 \nmid ab.
\end{cases}$$

(3)

### 3 Formalism of canonical divisors

For the investigation of principal factors, that is, generators $\alpha \in \mathcal{O}_L$ of primitive ambiguous principal ideals $(\alpha) = \alpha \mathcal{O}_L \in \mathcal{P}_L^L / \mathcal{P}_L$, which have divisors of the square $R^2$ of the ramification invariant $R$ as norms, $n = |\mathcal{N}_{L/k}(\alpha)|$ with $n \mid R^2$, it is useful to introduce the formalism of canonical divisors of the radicand $d = ab^2$ with respect to the norm $n$ [3, § 7, p. 18]:

$$d_1 := \prod_{\ell \in \mathbb{P}} \{\ell \mid v_\ell(a) = 1, v_\ell(n) = 1\}, \quad d_2 := \prod_{\ell \in \mathbb{P}} \{\ell \mid v_\ell(a) = 1, v_\ell(n) = 2\},$$

$$d_4 := \prod_{\ell \in \mathbb{P}} \{\ell \mid v_\ell(b) = 1, v_\ell(n) = 1\}, \quad d_5 := \prod_{\ell \in \mathbb{P}} \{\ell \mid v_\ell(b) = 1, v_\ell(n) = 2\},$$

and two additional silent divisors for expressing the radicand and its components,

$$d_3 := \prod_{\ell \in \mathbb{P}} \{\ell \mid v_\ell(a) = 1, v_\ell(n) = 0\}, \quad d_6 := \prod_{\ell \in \mathbb{P}} \{\ell \mid v_\ell(b) = 1, v_\ell(n) = 0\}.$$

(4)

Then the norm $n$, the radicands $d, \bar{d}$, and their components $a, b$ have the following shape:

$$n = d_1d_2d_4d_5, \quad a = d_1d_2d_3, \quad b = d_4d_5d_6, \quad d = d_1d_2d_3d_4d_5d_6, \quad \bar{d} = \bar{d}_1\bar{d}_2\bar{d}_3\bar{d}_4\bar{d}_5\bar{d}_6.$$  

(5)

### 4 Lattice minima with principal factor norm

We assume that $\sqrt[3]{d}$ denotes the unique real zero of the pure equation $X^3 - d = 0$ and therefore the pure cubic field $L = \mathbb{Q}(\sqrt[3]{d})$ is a real field with two complex conjugates, $L' = \mathbb{Q}(\zeta_3 \sqrt[3]{d})$ and $L'' = \mathbb{Q}(\zeta_3^2 \sqrt[3]{d})$, that is, with signature $(1, 1)$ and torsion-free Dirichlet unit rank $1 + 1 - 1 = 1$. Thus the total order of the field $\mathbb{R}$ of real numbers restricts to $L$, which we shall need for investigating lattice minima. We point out that the second algebraic conjugate $\alpha'' \in L''$ of an element $\alpha \in L$ is exactly the complex conjugate of the first (algebraic) conjugate $\alpha' \in L'$ of $\alpha$, since $L = \text{Fix}(\tau)$, $\tau^2 = \sigma^2 \tau$, and thus $\alpha'' = \alpha\sigma^2 = (\alpha')^\sigma = \alpha^\sigma\tau = \alpha^\sigma = (\alpha')^\tau$ where $\tau$ with $\tau(\zeta_3) = \zeta_3^2 = \bar{\zeta}_3$ is the complex conjugation restricted to $k$.

The Minkowski mapping $\psi : \mathcal{O}_L \rightarrow \mathbb{R}^3$, $\alpha \mapsto (\text{Re}(\alpha'), \text{Im}(\alpha'), \alpha)$ is an injective embedding of the maximal order $\mathcal{O}_L$ into Euclidean 3-space $\mathbb{R}^3$. The number geometric image $\psi(\mathcal{O}_L)$ is a discrete free $\mathbb{Z}$-module of rank three, i.e., a complete lattice in $\mathbb{R}^3$.

**Definition 4.1.** The norm cylinder of a point $x = (x, y, z)$ in Euclidean 3-space is defined by

$$\mathcal{N}(\mathcal{O}) := \{u = (u, v, w) \in \mathbb{R}^3 \mid 0 \leq u^2 + v^2 < x^2 + y^2, \quad 0 \leq w < |z|\}.$$  

(7)

If $\mathcal{O} \subseteq \mathcal{O}_L$ is an order of the field $L$, not necessarily the maximal order, then an algebraic integer $\alpha \in \mathcal{O}$ with $\alpha > 0$ is called a lattice minimum of $\mathcal{O}$ if

$$\mathcal{N}(\psi(\alpha)) \cap \psi(\mathcal{O}) = \{\mathcal{O}\}, \quad \text{where } \mathcal{O} = (0, 0, 0) \text{ denotes the origin of } \mathbb{R}^3,$$

(8)

or, equivalently, observing that $\text{Re}(\alpha'')^2 + \text{Im}(\alpha'')^2 = |\alpha'|^2 = \alpha'(\alpha')^\tau = \alpha^\tau$, if

$$\forall \beta \in \mathcal{O} \left(0 \leq \beta' \beta'' < \alpha'\alpha'', \quad 0 \leq \beta < \alpha \implies \beta = 0\right).$$

(9)
Note that the volume of the cylinder is given by \( \text{vol}_3(N(\psi(\alpha))) = \pi \cdot \alpha' \cdot \alpha'' \cdot \alpha = \pi \cdot N_{L/Q}(\alpha) \), which justifies the designation **norm cylinder**.

The set of all lattice minima of \( \mathcal{O} \) is denoted by \( \text{Min}(\mathcal{O}) \).

**Lemma 4.1.** All positive units in \( E_L^+ := \{ \eta \in E_L \mid \eta > 0 \} \) are lattice minima of \( \mathcal{O}_L \), but the radical \( \delta := \sqrt[3]{d} \) and the co-radical \( \bar{\delta} := \sqrt[3]{\bar{d}} \) never belong to \( \text{Min}(\mathcal{O}_L) \). More generally, if \( \alpha \in \text{Min}(\mathcal{O}_L) \) then \( \alpha \delta, \alpha \bar{\delta} \notin \text{Min}(\mathcal{O}_L) \).

**Proof.** Let \( \eta > 0 \) be a positive unit in \( E_L = \langle -1, \varepsilon \rangle \), where \( \varepsilon > 1 \) denotes the fundamental unit of \( L \). For an algebraic integer \( \alpha \in \mathcal{O}_L \) with \( \psi(\alpha) \in N(\psi(\eta)) \), we have \( 0 \leq \alpha' \alpha'' < \eta' \eta'' \) and \( 0 \leq \alpha < \eta \) and thus \( 0 \leq N_{L/Q}(\alpha) < N_{L/Q}(\eta) = 1 \). Since \( N_{L/Q}(\alpha) \in \mathbb{Z} \) is an integer, this is only possible for \( \alpha = 0 \). Thus we have \( \eta \in \text{Min}(\mathcal{O}_L) \). In particular, the fundamental unit \( \varepsilon \) and the trivial unit \( 1 \) with \( \psi(1) = (1, 0, 1) \) are lattice minima of \( \mathcal{O}_L \).

Concerning the second claim, which is also valid for any algebraic integer \( \alpha \in \mathcal{O}_L \) with \( \alpha > 0 \) (not necessarily \( \alpha \in \text{Min}(\mathcal{O}_L) \)), we firstly observe that \( \delta, \bar{\delta} \geq \sqrt[3]{2} \approx 1.26 > 1 \) since \( d, \bar{d} \geq 2 \), furthermore \( N_{L/Q}(\delta) = \delta \delta' \delta'' = \delta \cdot \zeta_3 \cdot \zeta_3^2 \delta = \zeta_3^3 \delta^3 = d \) and thus \( \delta' \delta'' = d/\delta = \delta^2 \geq \sqrt[3]{4} \approx 1.59 > 1 \). Consequently \( (\alpha \delta')(\alpha \delta'') = \alpha' \cdot \delta' \delta'' > \alpha' \alpha'' \) and \( \alpha \delta > \alpha \), which means that \( \mathcal{O} \neq \psi(\alpha) \in N(\psi(\alpha \delta)) \) and therefore \( \alpha \delta \notin \text{Min}(\mathcal{O}_L) \). Similarly, the proof for \( \alpha \bar{\delta} \). \( \square \)
Figure 1: Chain $\Theta = (\theta_j)_{j \in \mathbb{Z}}$ of lattice minima in Minkowski signature space

Definition 4.2. A pure cubic field $L = \mathbb{Q}(\sqrt[3]{d})$ of principal factorization type $\beta$ is called an

- **M2-field** if $\text{Min}(\mathcal{O}_L) \cap \Delta_{L/\mathbb{Q}} = E_L^+ \cup E_L^+ \alpha \cup E_L^+ \beta$,
- **M1-field** if $\text{Min}(\mathcal{O}_L) \cap \Delta_{L/\mathbb{Q}} = E_L^+ \cup E_L^+ \alpha$ or $E_L^+ \cup E_L^+ \beta$,
- **M0-field** if $\text{Min}(\mathcal{O}_L) \cap \Delta_{L/\mathbb{Q}} = E_L^+$.

Here, $\beta$ denotes one of $\bar{\alpha}$, $\bar{\alpha}^2$, $\bar{\alpha}^3$.

In Definition 4.2, which presents the mysterious M0-fields as the central objects of our subsequent investigations, because of their unpleasant impact on the classification problem and corresponding serious defects in tables of cubic fields [12], we use the isomorphism

$$\mathcal{P}_L^\mathbb{C}/\mathcal{P}_\mathbb{Q} \simeq \Delta_{L/\mathbb{Q}}/(E_L \cdot \mathbb{Q}^\times),$$

(10)
induced by the principal ideal mapping \( \iota : L^\times \to \mathcal{P}_L \), \( \alpha \mapsto (\alpha) = \alpha \mathcal{O}_L \), with inverse image \( \Delta_{L/Q} := \iota^{-1}(\mathcal{P}_L) \), and we assume that the integral part \( \Delta_{L/Q} \cap \mathcal{O}_L \), which always contains the radical group \( \Delta := \{ 1, \delta, \delta^2 \} \), is generated by the trivial principal factor \( \delta \) and an additional non-trivial principal factor \( \alpha \). For the same reason as for replacing the non-primitive square \( \delta^2 = \sqrt[4]{a^2b^4} = b \cdot \sqrt[4]{a^2b} = b \cdot \delta \) by \( \delta := \frac{b^2}{d_2d_6} \) we also replace \( \alpha^2 \) by \( \bar{\alpha} := \frac{a^2}{d_2d_6} \), as explained below by means of the canonical divisors. Then we have

\[
\Delta_{L/Q} \cap \mathcal{O}_L \simeq \{ 1, \delta, \delta^2; \alpha, \frac{\alpha \delta}{d_2d_4d_6}, \frac{\alpha \delta}{d_2d_4d_6}; \bar{\alpha}, \frac{\bar{\alpha} \delta}{d_2d_4d_6}, \frac{\bar{\alpha} \delta}{d_2d_4d_6} \}, \tag{11}
\]

represented by the norms (with abbreviations \( ab^2 = d_1d_2d_3d_4d_5d_6, a^2b = d_1^2d_2^2d_4^2d_5d_6) \)

\[
\{ 1, ab^2, a^2b; d_2d_4d_5d_6, d_2d_4d_5d_6, d_2d_4d_5d_6, d_2d_4d_5d_6, d_2d_4d_5d_6, d_2d_4d_5d_6 \}, \tag{12}
\]

**Theorem 4.1.** Among the 12220 pure cubic fields \( L = \mathbb{Q}(\sqrt[3]{d}) \) with normalized radicands in the range \( 2 \leq d \leq 15000 \), there occur more M0-fields than the 16 cases listed by H. C. Williams [12, § 6, Tbl. 2, p. 273],

\[
2, 455, 833, 850, 1078, 1235, 1573, 3857, 4901, 6061, 6358, 8294, 8959, 12121, 12818, 14801. \tag{13}
\]

The five missing radicands are:

\[
1430, 6370, 9922, 11284, 12673. \tag{14}
\]

So there are precisely 21 cases of M0-fields in this range.

**Lemma 4.2.** If the fundamental unit \( \varepsilon \) is the \( \ell \)th lattice minimum, counted from the trivial unit 1 in the direction of increasing height, then the norms of lattice minima are periodic with primitive period length \( \ell \), that is,

\[(\forall 0 \leq j \leq \ell - 1)(\forall n \in \mathbb{Z}) N_{L/Q}(\theta_{j+n \cdot \ell}) = N_{L/Q}(\theta_j). \tag{15}\]

**Proof.** Let \( \varepsilon > 1 \) be the normpositive fundamental unit bigger than the trivial unit 1 of \( L \). Then \( 0 < \varepsilon^{-1} < 1 \) is the inverse normpositive fundamental unit of \( L \). Due to the decomposition

\[
\Theta = (\theta_j)_{j \in \mathbb{Z}} = ((\theta_{j+n \cdot \ell})_{n \in \mathbb{Z}})_{0 \leq j < \ell}, \quad \text{respectively} \quad \text{Min}(\mathcal{O}_L) = \bigcup_{j=0}^{\ell-1} E_L^+ \cdot \theta_j \tag{16}
\]

of the chain \( \Theta \), respectively of the set \( \text{Min}(\mathcal{O}_L) \), where \( \theta_{n \cdot \ell} = \varepsilon^n \) for all \( n \in \mathbb{Z} \), into orbits under the action of \( E_L^+ = \{ \varepsilon^n \mid n \in \mathbb{Z} \} \) with representatives \( 1 \leq \theta_j < \varepsilon \), \( 0 \leq j < \ell \), in the first primitive period, visualized impressively in Figure 1, we have

\[(\forall 0 \leq j \leq \ell - 1)(\forall n \in \mathbb{Z}) \theta_{j+n \cdot \ell} = \varepsilon^n \cdot \theta_j, \tag{17}\]

and thus \( N_{L/Q}(\theta_{j+n \cdot \ell}) = N_{L/Q}(\varepsilon^n \cdot \theta_j) = N_{L/Q}(\varepsilon)^n \cdot N_{L/Q}(\theta_j) = 1 \cdot N_{L/Q}(\theta_j) = N_{L/Q}(\theta_j) \). \(\square\)

5 Necessary and sufficient conditions for minimal principal factors

We now state the main theorem on principal factors among the lattice minima.
Theorem 5.1. Let \( L = \mathbb{Q}(\sqrt[3]{d}) \) be a pure cubic field of principal factorization type \( \beta \) with normalized cube-free radicand \( d = ab^2 > 1 \). Suppose that \( \alpha \in \mathcal{O}_L \setminus E_L \) is generator of a primitive ambiguous principal ideal \( (\alpha) \in \mathcal{P}_L^2/\mathcal{I}_L^2 \) of \( L \) with norm \( n = N_{L/\mathbb{Q}}(\alpha) = 3^v \cdot d_1 d_2^2 d_4 d_5^2 \), where \( v \geq 1 \) at most for \( d \equiv \pm 2, \pm 4 \pmod{9} \), and that \( \gamma = \sqrt[3]{ab^2}/d_2 d_4 d_5 > 1 \) and \( \tilde{\gamma} = \sqrt[3]{a^2 b}/d_1 d_2 d_5 > 1 \). Then the criteria for the occurrence of \( \alpha \) among the lattice minima of the chain \( \Theta \) of the maximal order \( \mathcal{O}_L \), respectively \( \Phi \) of the non-maximal order \( \mathcal{O}_{L,0} \) with conductor \( \mathfrak{f} \mathfrak{l} \), where \( 3 \mathcal{O}_L = \mathfrak{f} \mathfrak{f} \mathfrak{l}^2 \) [5], if \( d \equiv \pm 1 \pmod{9} \), can be partitioned in the following way:

- **Unconditional criteria:**
  
  1. If \( L \) is of species 1a, \( 3 \mid d \), then \( \alpha \in \text{Min}(\mathcal{O}_L) \).
  2. If \( L \) is of species 1b, \( d \equiv \pm 2, \pm 4 \pmod{9} \), and \( v = 0 \), then \( \alpha \in \text{Min}(\mathcal{O}_L) \).
  3. If \( L \) is of species 2, \( d \equiv \pm 1 \pmod{9} \), then \( \alpha \in \text{Min}(\mathcal{O}_{L,0}) \).

- **Conditional criteria in dependence on \( u_1 \equiv d_1 d_2 d_4 d_5 \pmod{3} \) and \( u_2 \equiv d_1 d_2 d_4 d_6 \pmod{3} \):**
  
  1. If \( L \) is of species 1b, \( d \equiv \pm 2, \pm 4 \pmod{9} \), and \( v = 1 \), or \( L \) is of species 2, \( d \equiv \pm 1 \pmod{9} \), let two critical bivariate polynomials be defined by
     
     \[
     P_2(X,Y) := X^2 + Y^2 - XY - X - Y + 1 \in \mathbb{Z}[X,Y], \\
     P_4(X,Y) := X^4 - X^3 + X^2 Y - 8X^2 + XY + Y^2 \in \mathbb{Z}[X,Y].
     \] (18)

     Then the following necessary and sufficient criterion holds:
     
     \[
     \alpha \not\in \text{Min}(\mathcal{O}_L) \iff (u_1, u_2) \neq (1, 1) \text{ and } P_2(u_1 L, u_2 \gamma) < 9 \iff (u_1, u_2) \neq (1, 1) \text{ and } P_4(u_1 \gamma, -u_1 u_2 y) < 0 \iff (u_1, u_2) \neq (1, 1) \text{ and } P_4(u_2 \gamma, -u_1 u_2 y) < 0. \] (19)

     For \( (u_1, u_2) \neq (1, 1) \), a coarse sufficient, but not necessary, condition is given by:
     
     \[
     \max \left( \frac{\gamma}{B(u_2)}, \frac{\tilde{\gamma}}{B(u_1)} \right) \geq 1 \implies \alpha \in \text{Min}(\mathcal{O}_L),
     \] (20)

     where the bound is defined by
     
     \[
     B(u) := \begin{cases} 
     \sqrt{6} \approx 2.44948974278318 & \text{if } u = -1, \\
     2 & \text{if } u = 1. 
     \end{cases} 
     \] (21)

  2. If \( L \) is of species 1b, \( d \equiv \pm 2, \pm 4 \pmod{9} \), and \( v = 2 \), let a critical bound be defined by
     
     \[
     C(u) := \begin{cases} 
     \frac{1}{2}(-1 + \sqrt{33}) \approx 2.37228132326901 & \text{if } u = 1, \\
     2 & \text{if } u = -1. 
     \end{cases} 
     \] (22)

     Then the following necessary and sufficient criterion holds:
     
     \[
     \alpha \not\in \text{Min}(\mathcal{O}_L) \iff \min \left( \frac{\gamma}{C(u_1)}, \frac{\tilde{\gamma}}{C(u_2)} \right) < 1. \] (23)

**Proof.** The major part of the proof is due to Williams. However, it is scattered among several papers [10, 11, 12], and some cases have never been formulated as necessary and sufficient criteria. Generally, let \( \alpha \in \mathcal{O}_L \) be a principal factor with norm \( n = N_{L/\mathbb{Q}}(\alpha) = 3^v \cdot d_1 d_2^2 d_4 d_5^2 \), where \( v \in \{0, 1, 2\} \) and \( v \geq 1 \) at most for \( d \equiv \pm 2, \pm 4 \pmod{9} \).

- Concerning the unconditional criteria:
1. The claim that generally $\alpha \in \text{Min}(O_L)$ for $d \equiv 0, \pm 3 \pmod{9}$ (whence $v = 0$) is proved in [10, § 4, Thm. 2, p. 1427] and again in [11, § 5, Thm. 5.1(i), p. 643].

2. $\alpha \in \text{Min}(O_L)$ for $d \equiv \pm 2, \pm 4 \pmod{9}$ with $v = 0$ is also proven in [10, Thm. 2].

3. The statement that $\alpha \in \text{Min}(O_{L,0})$ for $d \equiv \pm 1 \pmod{9}$ (and hence $v = 0$) is due to ourselves, and provides considerable computational simplification, as Theorem 6.1 will show. For fields of the second species, $(1, \delta, \bar{\delta})$ is not an integral basis of the maximal order $O_L$, but it is a basis of the non-maximal order $O_{L,0} = \mathbb{Z} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\bar{\delta}$ with conductor $l_\sigma l$, where $3O_L = l_\sigma l^2$. The proof in [10, § 4, Thm. 2, p. 1427] is generally valid for the order $\mathbb{Z} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\bar{\delta}$ and does not use the congruence $d \not\equiv \pm 1 \pmod{9}$. Thus it also holds for $d \equiv \pm 1 \pmod{9}$.

- Concerning the conditional criteria for either $d \equiv \pm 2, \pm 4 \pmod{9}$ with $v \geq 1$ or the maximal order in the case $d \equiv \pm 1 \pmod{9}$, [11, § 3, Thm. 3.4, p. 638] establishes a diophantine criterion for the existence of a non-trivial lattice point within the norm cylinder of an algebraic integer with principal factor norm. In [11, § 4, Lem. 4.1, p. 639], the possible solutions of this critical system of diophantine ternary quadratic inequalities are narrowed down generally.

1. For either $d \equiv \pm 2, \pm 4 \pmod{9}$ with $v = 1$ or the maximal order in the case $d \equiv \pm 1 \pmod{9}$, it is shown in [11, § 4, Lem. 4.2, p. 640] that the diophantine criterion has a unique solution in dependence on $(u_1, u_2)$, except for $(u_1, u_2) = (1, 1)$, where $\alpha \in \text{Min}(O_L)$ turns out generally. The final conclusion is given in the later paper [12, § 4, Thm. 4.1, p. 268] in terms of our quadratic polynomial $P_2(X,Y)$. Our transformation in terms of the fourth degree polynomial $P_4(X,Y)$ is new and permits the deduction of a coarse sufficient condition for the converse statement in formulas (20) and (21) by investigating the zero locus of $P_4(X,Y)$ in the $XY$-plane. An even coarser sufficient condition is given in [11, § 5, Thm. 5.1(ii)–(iii), p. 643] by generally taking the bigger bound $\sqrt{6} > 2$.

2. Finally, for $d \equiv \pm 2, \pm 4 \pmod{9}$ with $v = 2$, a few solutions of the diophantine criterion are found in [11, § 4, Lem. 4.3, p. 642] in dependence on $(u_1, u_2)$, but no concluding theorem is stated. We proved that the solution in dependence on $(u_1, u_2)$ is in fact unique for each of the normalized radicals $\gamma$ and $\bar{\gamma}$, which leads to the necessary and sufficient criterion in formulas (22) and (23). A coarse sufficient condition for the converse statement is given in [11, § 5, Thm. 5.1(vi), p. 643] by generally taking the bigger bound $\frac{1}{4}(-1 + \sqrt{33}) > 2$
In Figure 2, the upper part $Y \geq 4$ of the zero locus of the bivariate polynomial $P_4(X,Y) \in \mathbb{Z}[X,Y]$ is plotted. This is the part which is relevant for deciding whether a principal factor whose norm is not divisible by 9 is a lattice minimum or not, because in Equation (19) of Theorem 5.1 the conditions $P_4(u_1\gamma, -u_1u_2y) < 0$ and $P_4(u_2\bar{\gamma}, -u_1u_2y) < 0$ must be checked, both for $(u_1, u_2) \neq (1, 1), \gamma > 1, \bar{\gamma} > 1$ and $y = \gamma \bar{\gamma} \geq \max(\gamma, \bar{\gamma})$. Consequently, the quadrant $X > 0, Y < 0$, where the zero locus reaches down to $Y = -16$, does not concern the decision. In the green triangles $Y \leq \sqrt{6}$, respectively $Y \geq 2$, the condition holds automatically, in the blue regions, only the left, and in the red regions the left and right inequality must be tested.

**Corollary 5.1.1.** Under the assumptions and notations of Theorem 5.1, a further coarse sufficient, but not necessary condition, is given by:

$$(u_1, u_2) \neq (1, 1) \text{ and } y \leq B(-u_1u_2) \implies \alpha \notin \text{Min}(O_L),$$

for either $d \equiv \pm 2, \pm 4 \pmod{9}$ with $v = 1$ or $d \equiv \pm 1 \pmod{9}$.

**Proof.** This also follows from investigating the zero locus of $P_4(X,Y)$ in the $XY$-plane. □
6 Classification algorithm

We continue with another main theorem on the classification of pure cubic fields into principal factorization types \cite[§ 2.1]{1} with the aid of Voronoï’s algorithm. The decisive innovation in contrast to previous classification algorithms is the use of a non-maximal order for species 2.

**Theorem 6.1.** Let \( L = \mathbb{Q}(\sqrt[3]{d}) \) be a pure cubic field with normalized cube-free radicand \( d \geq 2 \), ramification invariant \( R \), according to equation (3), and subfield unit index \( Q \), according to \cite[§ 2.1]{1}. Denote the chain of lattice minima of the maximal order \( \mathcal{O}_L \) by \( \Theta = (\theta_j)_{j \in \mathbb{Z}} \) and its primitive period length by \( \ell \geq 1 \). Then the following necessary and sufficient criteria determine the principal factorization type of \( L \) in dependence on the Dedekind species of the radicand \( d \).

1. If \( L \) belongs to species 1a, \( d \equiv 0, \pm 3 (\text{mod } 9) \), then \( L \) is of
   
   (a) type \( \alpha \) \iff \( Q = 1 \),
   
   (b) type \( \beta \) \iff \((\exists 1 \leq j \leq \ell - 1) N_{L/Q}(\theta_j) \mid R^2 \),
   
   (c) type \( \gamma \) \iff \((\forall 1 \leq j \leq \ell - 1) N_{L/Q}(\theta_j) \not\mid R^2 \) and \( Q = 3 \).

2. If \( L \) belongs to species 1b, \( d \equiv \pm 2, \pm 4 (\text{mod } 9) \), then \( L \) is of
   
   (a) type \( \alpha \) \iff \( Q = 1 \),
   
   (b) type \( \beta \) \iff either \((\exists 1 \leq j \leq \ell - 1) N_{L/Q}(\theta_j) \mid R^2 \) or \( Q = 3 \).

   (c) For this species, \( L \) can never be of type \( \gamma \).

3. If \( L \) belongs to species 2, \( d \equiv \pm 1 (\text{mod } 9) \), let \( \Phi = (\phi_j)_{j \in \mathbb{Z}} \) be the chain of lattice minima of the non-maximal order \( \mathcal{O}_{L,0} \) with conductor \( \Gamma^2 \), where \( 3\mathcal{O}_L = \Gamma^2 \) \cite{5}, and \( \ell_0 \geq 1 \) its primitive period length. Then \( L \) is of
   
   (a) type \( \alpha \) \iff \( Q = 1 \),
   
   (b) type \( \beta \) \iff \((\exists 1 \leq j \leq \ell_0 - 1) N_{L/Q}(\phi_j) \mid R^2 \),
   
   (c) type \( \gamma \) \iff \((\forall 1 \leq j \leq \ell_0 - 1) N_{L/Q}(\phi_j) \not\mid R^2 \) and \( Q = 3 \).

**Remark 6.1.** This remarkable algorithm deserves several remarks.

1. Our progressive innovation to use the non-maximal order for the guaranteed detection of an absolute principal factor is an incredibly powerful and easily implementable technique which circumvents the error prone method of Williams in \cite[§ 4, pp. 268–271]{12].

2. Actually, we have used this algorithm to achieve the extensive classification of all 827 600 fields with \( d < 10^6 \), as described in \cite[Exm. 2.1]{1}. For more detailed statistics see Table 1, where column \( B = 15 \) 000 is included with corrected results for \cite[§ 6, p. 272]{12}.

3. For item 2.(b) of Theorem 6.1, \( Q = 3 \) alone would be sufficient, but the determination of \( Q \) requires the fundamental unit \( \varepsilon = \theta_\ell \) at the end of the full period, whereas usually a \( \theta_j \) with \( N_{L/Q}(\theta_j) \mid R^2 \) has a subscript \( 1 \leq j < \ell \) of approximate magnitude \( \ell/3 \) or \( 2\ell/3 \), and thus admits an earlier termination of the algorithm at a third or two thirds of the period.

**Proof.** The equivalence of type \( \alpha \) with a subfield unit index \( Q = 1 \) is true independently of the Dedekind species, according to \cite[Eqn. (5) in Rmk. 2.1]{1}. For the other two types \( \beta \) and \( \gamma \), where \( Q = 3 \) for both, we distinguish the species.

1. For species 1a, \( d \equiv 0, \pm 3 (\text{mod } 9) \), the unconditional criterion 1 in Theorem 5.1 proves that a non-unit \( \alpha \) with norm \( n = N_{L/Q}(\alpha) \) dividing \( R^2 \) must occur as a lattice minimum \( \alpha = \theta_j \) in the chain \( \Theta \) of the maximal order \( \mathcal{O}_L \). Thus the occurrence of such a \( \theta_j \) is equivalent with type \( \beta \). The lack of such a \( \theta_j \) implies type \( \alpha \) or \( \gamma \) and type \( \alpha \) must be discouraged by \( Q = 3 \).
2. A necessary condition for type $\gamma$, that is, the occurrence of a unit $Z \in E_k$ such that $N_{k/k_0}(Z) = \zeta_3$, is that the conductor $f$ of $k/k_0$ is divisible only by 3 or primes $\ell \equiv \pm 1 \pmod{9}$. For species 1b, $d \equiv \pm 2, \pm 4 \pmod{9}$, there must exist a prime divisor $\ell \equiv \pm 2, \pm 4 \pmod{9}$ of $f$ and type $\gamma$ is impossible. Therefore, type $\beta$ is equivalent with $Q = 3$, and only for accelerating the algorithm it is worth while to check the possible occurrence of a lattice minimum with norm dividing $R^2$.

3. For species 2, $d \equiv \pm 1 \pmod{9}$, the unconditional criterion 3 in Theorem 5.1 shows that a non-unit $\alpha$ with norm $n = N_{L/Q}(\alpha)$ dividing $R^2$ must occur as a lattice minimum $\alpha = \phi_j$ in the chain $\Phi$ of the non-maximal order $O_{L,0}$. Therefore the occurrence of such a $\phi_j$ is equivalent with type $\beta$. The lack of such a $\phi_j$ enforces type $\alpha$ or $\gamma$ and type $\alpha$ must be eliminated by $Q = 3$. (Note that $\alpha$ is coprime to the conductor $t^2\ell$ [5].) \hfill \Box

7 Explicit criteria for M0-fields in rational integers

It is useful to specialize the general Theorem 5.1 to situations, where the occurrence of a principal factor among the lattice minima can be characterized in terms of the canonical divisors $d_1, \ldots, d_6$. The most convenient situation appears for a squarefree radicand $d = d_1d_2d_3$, where $d_4 = d_5 = d_6 = 1$, a priori.

**Theorem 7.1.** Let the squarefree radicand $d = d_1d_2d_3$ be of second species, $d \equiv \pm 1 \pmod{9}$, and assume there exists a principal factor $\alpha \in O_L$ with norm $n = d_1d_2^2$, minimal in the first coset $\{d_1d_2^2, d_1^2d_3, d_2d_3^2\}$, that is

$$d_2^2 < d_1d_3, \quad d_1d_2 < d_3^2. \quad (25)$$

- If $d_1^2 < d_2d_3$, then $\bar{n} = d_2d_3$ is minimal in the second coset $\{d_1^2d_2, d_2^2d_3, d_1d_3^2\}$, and $L$ is an M0-field (neither $\alpha \in Min(O_L)$ nor $\bar{\alpha} \in Min(O_L)$), if

  either $d_1 \equiv d_2 \equiv -d_3 \pmod{3}, \quad d_3 \leq 2\min(d_1, d_2)$

  or $d_1 \equiv -d_2 \equiv d_3 \pmod{3}, \quad d_3 \leq \min(\sqrt{6}d_1, 2d_2) \quad (26)$

  or $-d_1 \equiv d_3 \equiv d_2 \pmod{3}, \quad d_3 \leq \min(2d_1, \sqrt{6}d_2)$.

- If $d_2d_3 < d_1^2$, then $\bar{n} = d_2d_3$ is minimal in the second coset $\{d_2^2d_3, d_1d_3^2, d_1^2d_3\}$, and $L$ is an M0-field (neither $\alpha \in Min(O_L)$ nor $\bar{\alpha}d/\delta \in Min(O_L)$), if

  either $d_1 \equiv d_2 \equiv -d_3 \pmod{3}, \quad d_1 \leq \sqrt{6}d_2, \quad d_3 \leq 2d_2$

  or $d_1 \equiv -d_2 \equiv d_3 \pmod{3}, \quad \max(d_1, d_3) \leq 2d_2 \quad (27)$

  or $-d_1 \equiv d_3 \equiv d_2 \pmod{3}, \quad d_1 \leq 2d_2, \quad d_3 \leq \sqrt{6}d_2$.

**Proof.** The claim concerns both non-trivial cosets of principal factors, the first coset of $\alpha$ with norm $n = d_1d_2^2$ and the second coset of $\bar{\alpha} = \alpha^2/d_2$, respectively $\bar{\alpha}d/\delta_1$, with norm $\bar{n}$.

First, we consider the coset of $\alpha$. Here, we have the congruence invariants $u_1 \equiv d_1d_3 \pmod{3}$, $u_2 \equiv d_1d_2 \pmod{3}$, the normalized radicals $\gamma = \sqrt{d_1d_2d_3}d_2 > 1$, $\bar{\gamma} = \sqrt{d_1^2d_2^2d_3^2}d_1d_2 > 1$, and

| bound $B$ | 10  | 100 | 1 000 | 10 000 | 15 000 | 100 000 | 1 000 000 |
|-----------|-----|-----|-------|--------|--------|---------|-----------|
| $\# \text{type } \alpha$ | 1   | 19  | 182   | 1 755  | 2 606  | 16 935  | 163 527   |
| $\# \text{type } \beta$   | 4   | 49  | 556   | 5 988  | 9 058  | 62 068  | 635 463   |
| $\# \text{type } \gamma$  | 1   | 6   | 50    | 381    | 556    | 3 261   | 28 610    |
| $\# \text{total}$          | 6   | 74  | 788   | 8 122  | 12 220 | 82 264  | 827 600   |

Table 1: Distribution of principal factorization types for $2 \leq d \leq B$
their product \( y = \gamma = (\sqrt[3]{d_1 d_2 d_3}/d_2)(\sqrt[3]{d_1^2 d_2^2 d_3^2}/d_1 d_2) = d_3/d_2 \). The minimality of \( n \) in its coset yields relations between the magnitude of the canonical divisors, \( d_2^2 < d_1 d_3 \) and \( d_1 d_2 < d_3^2 \), that is, formula (25).

We exploit the coarse sufficient condition in Corollary 5.1.1: \( y \leq B(-u_1 u_2) \Rightarrow \alpha \notin \text{Min}(\mathcal{O}_L) \), that is, \( d_3 \leq \sqrt{6} \cdot d_2 \) if \( u_1 = u_2 = -1 \), and \( d_3 \leq 2 \cdot d_2 \) otherwise. The connection between the congruence invariants and the residue class of the canonical divisors is given by the forbidden case \( d_1 \equiv d_2 \equiv d_3 \mod 3 \) \( \iff \) \((u_1, u_2) = (1, 1)\), and the admissible cases \( d_1 \equiv d_2 \equiv -d_3 \mod 3 \) \( \iff \) \((u_1, u_2) = (-1, 1)\), \( d_1 \equiv -d_2 \equiv d_3 \mod 3 \) \( \iff \) \((u_1, u_2) = (1, -1)\), \( -d_1 \equiv d_2 \equiv d_3 \mod 3 \) \( \iff \) \((u_1, u_2) = (1, 1)\).

For the second coset, we have to split the investigation.

- If \( d_2^2 < d_2 d_3 \), then the minimal norm is \( \bar{n} = \sqrt[3]{d_2^2} \). With new canonical invariants \( \bar{n} = c_1 c_2 \), where \( c_1 := d_2 \) and \( c_2 := d_1 \) are twisted, whereas \( c_3 = d_3 \) remains fixed.

The connection between the congruence invariants and the residue class of the canonical divisors is given by
\[
\begin{align*}
d_1 &\equiv d_2 \equiv -d_3 \mod 3 \iff c_1 \equiv c_2 \equiv -c_3 \mod 3 \iff (u_1, u_2) = (-1, 1), \\
d_1 &\equiv -d_2 \equiv d_3 \mod 3 \iff -c_1 \equiv c_2 \equiv c_3 \mod 3 \iff (u_1, u_2) = (-1, -1), \\
-d_1 &\equiv d_2 \equiv d_3 \mod 3 \iff c_1 \equiv -c_2 \equiv c_3 \mod 3 \iff (u_1, u_2) = (1, -1).
\end{align*}
\]

Again, we employ the coarse sufficient condition in Corollary 5.1.1: \( y \leq B(-u_1 u_2) \Rightarrow \alpha \notin \text{Min}(\mathcal{O}_L) \), that is, \( d_3 = c_3 \leq \sqrt{6} \cdot c_2 = \sqrt{6} \cdot d_1 \) if \( u_1 = u_2 = -1 \), and \( d_3 = c_3 \leq 2 \cdot c_2 = 2 \cdot d_1 \) otherwise.

- If \( d_2 d_3 < d_3^2 \), then the minimal norm is \( \bar{n} = \sqrt[3]{d_2 d_3} \). With new canonical invariants \( \bar{n} = c_1 c_2 \), where \( c_1 := d_3 \) and \( c_2 := d_1 \) are twisted, whereas \( c_2 = d_2 \) remains fixed.

The connection between the congruence invariants and the residue class of the canonical divisors is given by
\[
\begin{align*}
d_1 &\equiv d_2 \equiv -d_3 \mod 3 \iff -c_1 \equiv c_2 \equiv c_3 \mod 3 \iff (u_1, u_2) = (-1, 1), \\
d_1 &\equiv -d_2 \equiv d_3 \mod 3 \iff c_1 \equiv -c_2 \equiv c_3 \mod 3 \iff (u_1, u_2) = (1, -1), \\
-d_1 &\equiv d_2 \equiv d_3 \mod 3 \iff c_1 \equiv c_2 \equiv -c_3 \mod 3 \iff (u_1, u_2) = (1, -1).
\end{align*}
\]

Again, we employ the coarse sufficient condition in Corollary 5.1.1: \( y \leq B(-u_1 u_2) \Rightarrow \alpha \delta/d_1 \notin \text{Min}(\mathcal{O}_L) \), that is, \( d_1 = c_3 \leq \sqrt{6} \cdot c_2 = \sqrt{6} \cdot d_1 \) if \( u_1 = u_2 = -1 \), and \( d_1 = c_3 \leq 2 \cdot c_2 = 2 \cdot d_1 \) otherwise.

Finally we collect all required inequalities for the first and second non-trivial coset, and we must make sure that not \( u_1 = u_2 = 1 \), which is the case if not \( d_1 \equiv d_2 \equiv d_3 \mod 3 \).

Theorem 7.1 gives rise to the following hypothesis, since the assumptions for the three positive integers \( d_1, d_2, d_3 \) in form of simple inequalities and simple congruences modulo 3 seem to be satisfiable even by infinitely many triples \((d_1, d_2, d_3) \in \mathbb{P}^3 \) of prime numbers.

**Conjecture 7.1.** There exist infinitely many squarefree radicands \( d \) of second species such that \( L = \mathbb{Q}(\sqrt[3]{d}) \) is an M0-field.

**Example 7.1.** We prove two defects in [12, § 6, Tbl. 2, p. 273], as claimed in Theorem 4.1, both of species 2, \( d \equiv \pm 1 \mod 9 \). They can be treated by the first variant of Theorem 7.1.

- Let \( d = 1430 = 2 \cdot 5 \cdot 11 \cdot 13 \) and \( n = 1100 = 2^2 \cdot 5^2 \cdot 11 \). Then \( d_1 = 11, d_2 = 2 \cdot 5 = 10, d_3 = 13, and \) (25) is satisfied with \( d_1 d_3 = 11 \cdot 13 = 143 > 100 = 10^2 = d_2^2, d_2 d_3 = 13 \cdot 11 = 143 > 121 = 11^2 = d_1^2 \). Furthermore, (26) is satisfied with \( -d_1 = -11 \equiv d_2 = 10 \equiv d_3 = 13 \mod(3), d_3 = 13 < 22 = 2 \cdot 11 = 2d_1, d_3 = 13 < 24.49 \approx 2.449 \cdot 10 = \sqrt{6}d_2 \). Therefore, \( L = \mathbb{Q}(\sqrt[3]{1430}) \) is an M0-field.

- Let \( d = 12673 = 19 \cdot 23 \cdot 29 \) and \( n = 10051 = 19 \cdot 23^2 \). Then \( d_1 = 19, d_2 = 23, d_3 = 29, \) and (25) is satisfied with \( d_1 d_3 = 19 \cdot 29 = 551 > 529 = 23^2 = d_2^2, d_3^2 = 29^2 = 841 > \)
437 = 19 \cdot 23 = d_1 d_2, \ d_2 d_3 = 23 \cdot 29 = 667 > 361 = 19^2 = d_1^2. \ \text{Furthermore, (26)} \text{ is satisfied with } -d_1 = -19 \equiv d_2 = 23 \equiv d_3 = 29 \pmod{3}, \ d_3 = 29 < 38 = 2 \cdot 19 = 2d_1, \ d_2 = 23 < 36.34 \approx 2.449 \cdot 23 \approx \sqrt{6}d_2. \ \text{Consequently, } L = \mathbb{Q}(\sqrt{12673}) \text{ is an M0-field. We point out that this radicand is of the fifth form in the Main Theorem [2, Thm. 1.1].}

\textbf{Example 7.2.} \ Up to now, no examples of M0-fields of species 2 were known. Since } d = 1430 \text{ was the first discovered radicand of such an exotic field } L = \mathbb{Q}(\sqrt{d}), \text{ we present some details of the actual execution of Voronoi’s algorithm. The procedure starts at the trivial unit } \theta_0 = 1, \text{ respectively } \varphi_0 = 1, \text{ and constructs the chain of lattice minima, } \Theta \text{ of the maximal order } \mathcal{O}_L, \text{ respectively } \Phi \text{ of the non-maximal order } \mathcal{O}_{L,0}, \text{ in direction of decreasing height } h = z \text{ and increasing radius } r = \sqrt{x^2 + y^2} \text{ in Minkowski signature space } \mathbb{R}^3, \text{ and stops at the inverse fundamental unit } 0 < \theta_{-\ell} = \varepsilon^{-1} < 1, \text{ respectively } 0 < \varphi_{-\ell} = \varepsilon^{-1} < 1, \text{ as illustrated in Figure 1. In this particular example the unit groups of maximal order and suborder coincide and } \varepsilon_0 = \varepsilon. \text{ Before the period ended at length } \ell_0 = 48 \text{ we found two principal factors at characteristic locations } j = -16 = \frac{1}{3} \cdot (-48) \text{ exactly and } j = -34 \approx \frac{2}{3} \cdot (-48) \text{ approximately:}

\begin{align*}
\beta &:= \phi_{-16} = -28 490 - 13 120\bar{d} + 1 389\bar{\delta}, \\
\alpha &:= \phi_{-34} = -5 130 804 470 + 350 650 663\delta + 9 298 918\bar{\delta}, \\
\varepsilon_0 &:= \phi_{-48} = -6 074 553 925 441 - 689 057 082 849\delta + 109 019 548 011\bar{\delta}. \\
\end{align*}

\text{For instance the norm of } \beta = x + y\delta + z\bar{\delta} \text{ can be computed with the homogeneous pure cubic norm form } N(\beta) = x^3 + d \cdot y^3 + d^2 \cdot z^3 - 3d \cdot xyz

\begin{align*}
&= -28 490^3 + 1 430 \cdot (-13 120^3) + 2044 900 \cdot 1 389^3 - 3 \cdot 1 430 \cdot (-28 490) \cdot (-13 120) \cdot 1 389 \\
&= -23 124 766 049 000 - 3 229 516 759 040 000 + 5 479 977 964 418 100 - 2 227 336 439 328 000 \\
&= 1 100 = 2^2 \cdot 5^2 \cdot 11.
\end{align*}

In Table 2, we compare the crucial locations in the chains of both orders. By the general theory of principal factors, we have the characteristic relations \( \varepsilon^{-1}_0 = \frac{\theta}{N(\theta)} \), \( \varepsilon^{-2}_0 = \frac{\alpha}{N(\alpha)} \), which shows that Voronoi’s algorithm can be terminated at \( \beta \) already, only a third of the period, to get the fundamental unit. Of course, by Example 7.1, we cannot find principal factors in the chain \( \Theta \). However, instead we encounter the \textit{shadows} of \( \beta \) and \( \alpha \) in the maximal order, that is, the actual lattice minima within the norm cylinders of \( \beta \) and \( \alpha \):

\begin{align*}
\theta_{-17} &= \frac{1}{3}(56 557 + 28 328\delta - 2 960\bar{\delta}) = \frac{1}{3}(-1 + \frac{1}{10}\delta + \frac{1}{110}\bar{\delta}) \cdot \phi_{-16}, \\
\theta_{-28} &= \frac{1}{3}(-112 505 639 + 13 815 812\delta + 339 929\bar{\delta}), \\
\theta_{-35} &= \frac{1}{3}(-8 480 403 749 - 236 672 041\delta + 87 819 928\bar{\delta}) = \frac{1}{3}(1 + \frac{1}{11}\delta - \frac{1}{110}\bar{\delta}) \cdot \phi_{-34}.
\end{align*}

The shadow norms \( N(\theta_{-17}) = 239 \) and \( N(\theta_{-35}) = 183 \) can be computed with the results in [12, § 4, pp. 268–271]. As opposed to the principal factor norms, the shadow norms are not unique, and this fact causes complications, since for instance \( \theta_{-28} \) with norm 183 has nothing to do with principal factors, indicated by the symbol \( \smash{\frac{1}{3}} \).

Hence \( L = \mathbb{Q}(\sqrt{1430}) \) is the first M0-field of species 2 and type \( \beta \). It has inadvertently been overlooked for some reason by H. C. Williams in [12, Tbl. 2, p. 273].

\textbf{Example 7.3.} \ Outside of the range \( d < 15 \times 1000 \) of radicands in the computations of [12, § 6, Tbl. 2, p. 273] there also occur examples of the second variant of Theorem 7.1.

- \text{Let } d = 33 337 = 17 \cdot 37 \cdot 53 \text{ and } n = 15 317 = 17^2 \cdot 53. \ \text{Then } d_1 = 53, \ d_2 = 17, \ d_3 = 37, \ \text{and (25)} \text{ is satisfied with } d_1 d_3 = 53 \cdot 37 = 1961 > 289 = 17^2 = d_2^2, \ d_2 d_3 = 17 \cdot 37 = 629 < 2909 = 53^2 = d_1^2. \ \text{Unfortunately, (27)} \text{ with } d_1 = 53 \equiv d_2 = 17 \equiv -d_3 = -37 \pmod{3} \text{ is not satisfied, since both inequalities}
Let the square-part radicand \( \sqrt{2} \) be an element of second species. Their cubes are \( \theta_3 = 239 \approx 37228132326901 \), \( \theta_4 = 183 \approx 40080587094953 \), that is \( \theta_3 = 239 \approx 37228132326901 \), \( \theta_4 = 183 \approx 40080587094953 \), and (25) is satisfied with \( d = 37 \approx 2 \cdot 17 = 2d_2 \) are in the false direction so that the fine criteria of Theorem 5.1 must be applied. However, the field is interesting for another reason, since all prime factors are \( \equiv \pm 1 \pmod{9} \) and thus the multiplicity of the conductor \( f = d \) is given by \( m(f) = 2^3 \cdot X_{-1} = 2^3 \cdot \frac{1}{2} = 4 \) giving rise to one of the rare quartets of second species.

- Let \( d = 52417 = 23 \cdot 43 \cdot 53 \) and \( n = 22747 = 23^2 \cdot 43 \). Then \( d_1 = 43, d_2 = 23, d_3 = 53 \), and (25) is satisfied with \( d_1d_3 = 43 \cdot 53 = 2279 > 529 = 23^2 = d_2^2, d_2^3 = 53^2 = 2809 > 989 = 43 \cdot 23 = d_1d_2, d_2d_3 = 23 \cdot 53 = 1219 < 1849 = 43^2 = d_1^2 \). Furthermore, (27) is satisfied with \( -d_1 = -43 \equiv d_2 = 23 \equiv d_3 = 53 \pmod{3} \), \( d_1 = 43 < 46 = 2 \cdot 23 = 2d_2 \), \( d_3 = 53 < 56.34 \approx 2.449 \cdot 23 \approx \sqrt{6d_2} \). Consequently, \( L = \mathbb{Q}(\sqrt[3]{52417}) \) is an M0-field. We point out that this radicand is of the seventh form in [2, Thm. 1.1].

**Theorem 7.2.** Let the square-part radicand \( d = d_3d_4^2 \) be of species 1a, \( d \equiv \pm 2, \pm 4 \pmod{9} \), and assume there exists a principal factor \( \alpha \in \mathcal{O}_L \) with norm \( n = 9d_4 \), minimal in the first coset \( \{9d_4, 9d_3, 9d_3d_4^2\} \), that is

\[
d_4 < d_3,
\]

then \( \bar{\alpha} = 3d_4^2 \) is minimal in the second coset \( \{3d_4^2, 3d_3d_4, 3d_3^2\} \) and \( \bar{\alpha} = \alpha^2/3 \). Denote by \( Z_+ \) the unique positive zero of the univariate polynomial \( Q_4(X) := X^4 + X^3 + X - 8 \in \mathbb{Z}[X] \), that is, \( Z_+ \approx 1.40080587094953 \) with cube \( Z_+^3 \approx 2.74874124930414 \). Further, put \( C_1 := (-1 + \sqrt{33})/2 \approx 2.37228132326901 \) with cube \( C_1^3 \approx 13.350531904211 \). Then

- \( L \) is an M0-field (neither \( \alpha \in \text{Min}(\mathcal{O}_L) \) nor \( \bar{\alpha} \in \text{Min}(\mathcal{O}_L) \)) \( \iff \)

\[
d_3 \equiv -d_4 \pmod{3}, \quad d_3 < Z_+^3 \cdot d_4.
\]

- \( L \) is an M1-field (\( \alpha \notin \text{Min}(\mathcal{O}_L) \) but \( \bar{\alpha} \in \text{Min}(\mathcal{O}_L) \)) \( \iff \)

\[
either \quad d_3 \equiv -d_4 \pmod{3}, \quad Z_+^3 \cdot d_4 \leq d_3 < 8 \cdot d_4
\]

or \( d_3 \equiv d_4 \pmod{3}, \quad d_3 < C_1^3 \cdot d_4 \).

- \( L \) is an M2-field (both, \( \alpha \in \text{Min}(\mathcal{O}_L) \) and \( \bar{\alpha} \in \text{Min}(\mathcal{O}_L) \)) \( \iff \)

\[
either \quad d_3 \equiv -d_4 \pmod{3}, \quad 8 \cdot d_4 \leq d_3
\]

or \( d_3 \equiv d_4 \pmod{3}, \quad C_1^3 \cdot d_4 \leq d_3 \).

**Proof.** We begin by seeking conditions for \( \alpha \in \text{Min}(\mathcal{O}_L) \). The normalized radicals are \( 1 < \gamma = \delta/d_4, 1 < \tilde{\gamma} = \tilde{\delta} \). Their cubes are \( 1 < \gamma^3 = \frac{d_3^2}{d_4} = \frac{9}{4} < d_3^2d_4 = \tilde{\gamma}^3 \), whence \( \min(\gamma, \tilde{\gamma}) = \gamma \). Their product is \( y = \gamma \tilde{\gamma} = d_3 \). The congruence invariants are \( u_1 \equiv d_3d_4 \pmod{3} \) and \( u_2 \equiv d_4 \pmod{3} \). Thus, we have four cases according to Formula (23) in Theorem 5.1:
If \( d_3 \equiv d_4 \equiv 1 \pmod{3} \), then \((u_1, u_2) = (1, 1)\) and 
\[ \alpha \in \operatorname{Min}(O_L) \iff \gamma < C_1 \iff d_3 < C_3^2 \cdot d_4. \]
If \( d_3 \equiv d_4 \equiv -1 \pmod{3} \), then \((u_1, u_2) = (1, -1)\) and 
\[ \alpha \in \operatorname{Min}(O_L) \iff \gamma < C_1 \iff d_3 < C_2^2 \cdot d_4, \]
since \( \tilde{\gamma} < 2 \implies \gamma < \tilde{\gamma} < 2 < C_1. \)
If \(-d_3 \equiv d_4 \equiv 1 \pmod{3} \), then \((u_1, u_2) = (-1, 1)\) and 
\[ \alpha \in \operatorname{Min}(O_L) \iff \gamma < 2 \iff d_3 < 2^3 \cdot d_4. \]
(Note that the smallest possible square-part radicand is \( 12 = 2^2 \cdot 3 \), whence \( \tilde{\gamma} = \delta = \frac{d_3}{d_4} \geq 12 > C_1 > 2 \))
If \( d_3 \equiv -d_4 \equiv 1 \pmod{3} \), then \((u_1, u_2) = (-1, -1)\) and 
\[ \alpha \in \operatorname{Min}(O_L) \iff \gamma < 2 \iff d_3 < 8 \cdot d_4. \] Herewith, the first coset is done.
We turn to the second coset. The basic assumption \( d_4 < d_3 \) in Formula (30) is equivalent with minimality of \( n = 9d_4 \) in the first coset and minimality of \( \bar{n} = 3d_4^2 \) in the second coset. However, \( \alpha^2 \) has norm \( 81d^2_4 \) and thus \( \bar{\alpha} = \alpha^2 \equiv 3 \pmod{3} \) has norm \( \bar{n} \). The new non-trivial canonical divisors of \( \bar{n} = 3d_4^2 \equiv 3 \pmod{3} \) are \( c_3 = d_3 \) (fixed) and \( c_4 = d_4 \) (twisted). Therefore, the new congruence invariants are \( u_1 \equiv c_3c_5 = d_3d_4 \pmod{3} \) as before, but \( u_2 \equiv 1 \pmod{3} \) is constant. Consequently, we have only two cases according to Formula (19) in Theorem 5.1, since \((u_1, u_2) = (1, -1)\) and \((u_1, u_2) = (-1, 1)\) cannot occur:
If \( d_3 \equiv d_4 \pmod{3} \), then \((u_1, u_2) = (1, 1)\) and \( \alpha \in \operatorname{Min}(O_L) \).
If \( d_3 \equiv -d_4 \pmod{3} \), then \((u_1, u_2) = (-1, 1)\) and 
\[ \alpha \in \operatorname{Min}(O_L) \iff P_4(-\gamma, y) < 0 \iff P_4(\gamma, y) < 0. \]
Now we come to a phenomenon which is very peculiar for the present situation. The new normalized radicals are \( \gamma = \delta/c_5 = \delta/d_4 = \sqrt[3]{\frac{d_3d_4}{d_4^2}} = \sqrt[3]{\frac{d_3}{d_4}} \), and their product is \( y = \gamma \tilde{\gamma} = \frac{d_3}{d_4} = \gamma^2 \). Actual substitution into \( P_4(X, Y) = X^4 - \delta \cdot 2X^2 \cdot \delta \cdot 2X^2 \cdot \delta \cdot 2X^2 + X^4 + Y^2 \) yields \( P_4(-\gamma, y) = P_4(-\gamma, \gamma^2) = \gamma^4 + \gamma^2 + \gamma^2 \cdot 2^3 - 8\gamma^2 - \gamma^3 + \gamma^2 \cdot 2^2 \cdot \gamma \cdot 2 - 8\gamma + \gamma^3 + \gamma^2 \).
Since \( \gamma \geq 1 \), we obtain \( P_4(-\gamma, y) < 0 \iff Q_4(\gamma) = \gamma^4 + \gamma^3 + \gamma - 8 < 0 \iff \gamma < Z_+ \iff y = \frac{d_3}{d_4} = \gamma^3 < Z_+^2 \iff d_3 < Z_+^2 \cdot d_4 \), because the negative zero \( Z_- \) of \( Q_4(X) \) is irrelevant.

**Example 7.4.** We confirm six results in [12, § 6, Tbl. 2, p. 273], as reproduced in Theorem 4.1, all of species 1b, \( d = \pm 2, \pm 4 \pmod{9} \). They can be treated by Theorem 7.2.

- Let \( d = 833 = 7^2 \cdot 17 \) and \( n = 63 = 3^2 \cdot 7 \). Then \( d_3 = 17, d_4 = 7 \), and (30) is satisfied with \( d_4 = 7 < 17 = d_3 \). Further, (31) is satisfied with \( d_3 = 17 \equiv -d_4 = -7 \equiv -1 \pmod{3} \), \( d_3 = 17 < 19.24 \approx 2.7487 \cdot 7 \approx Z_+^2 \cdot d_4 \). Therefore, \( L = \mathbb{Q}(\sqrt{833}) \) is an M0-field.
- Let \( d = 1573 = 11^2 \cdot 13 \) and \( n = 99 = 3^2 \cdot 11 \). Then \( d_3 = 13, d_4 = 11 \), and (30) is satisfied with \( d_4 = 11 < 13 = d_3 \). Also, (31) is satisfied with \( d_3 = 13 \equiv -d_4 = -11 \equiv 1 \pmod{3} \), \( d_3 = 13 < 30.2 \approx 2.7487 \cdot 11 \approx Z_+^2 \cdot d_4 \), and \( L = \mathbb{Q}(\sqrt{1573}) \) is an M0-field.
- Let \( d = 4901 = 13^2 \cdot 29 \) and \( n = 117 = 3^2 \cdot 13 \). Then \( d_3 = 29, d_4 = 13 \), and (30) is satisfied with \( d_4 = 13 < 29 = d_3 \). Also, (31) is satisfied with \( d_3 = 29 \equiv -d_4 = -13 \equiv -1 \pmod{3} \), \( d_3 = 29 < 35.73 \approx 2.7487 \cdot 13 \approx Z_+^2 \cdot d_4 \), and \( L = \mathbb{Q}(\sqrt{4901}) \) is an M0-field.
- Let \( d = 6358 = 2^2 \cdot 11 \cdot 17^2 \) and \( n = 153 = 3^2 \cdot 17 \). Then \( d_3 = 22, d_4 = 17 \), and (30) is satisfied with \( d_4 = 17 < 22 = d_3 \). Also, (31) is satisfied with \( d_3 = 22 \equiv -d_4 = -17 \equiv 1 \pmod{3} \), \( d_3 = 22 < 46.7 \approx 2.7487 \cdot 17 \approx Z_+^2 \cdot d_4 \), and \( L = \mathbb{Q}(\sqrt{6358}) \) is an M0-field.
- Let \( d = 8959 = 17^2 \cdot 31 \) and \( n = 153 = 3^2 \cdot 17 \). Then \( d_3 = 31, d_4 = 17 \), and (30) is satisfied with \( d_4 = 17 < 31 = d_3 \). Also, (31) is satisfied with \( d_3 = 31 \equiv -d_4 = -17 \equiv 1 \pmod{3} \), \( d_3 = 31 < 46.7 \approx 2.7487 \cdot 17 \approx Z_+^2 \cdot d_4 \), and \( L = \mathbb{Q}(\sqrt{8959}) \) is an M0-field.
- Let \( d = 14801 = 19^2 \cdot 41 \) and \( n = 171 = 3^2 \cdot 19 \). Then \( d_3 = 41, d_4 = 19 \), and (30) is satisfied with \( d_4 = 19 < 41 = d_3 \). Also, (31) is satisfied with \( d_3 = 41 \equiv -d_4 = -19 \equiv -1 \pmod{3} \), \( d_3 = 41 < 52.2 \approx 2.7487 \cdot 19 \approx Z_+^2 \cdot d_4 \), and \( L = \mathbb{Q}(\sqrt{14801}) \) is an M0-field.

Note that all these radicands, except 6358, are of the third form in [2, Thm. 1.1].
In Table 3, we show for some radicands $d$ of $M_0$-fields whether the proof is possible either by coarse rational integer criteria $y = \gamma \bar{\gamma} < C (\checkmark)$ or only by fine multiprecision criteria $P_2(u_1 \gamma, u_2 \bar{\gamma}) < B$ involving irrationalities, when $y \geq C (\checkmark)$.

Table 3: Justifications for $M_0$-fields of species 2 with coarse and fine criteria

| $d$  | first coset of $\alpha$ | $y$ | $C$ | $P_2$ | $B$ | $y$ | $C$ | $P_2$ | $B$ |
|------|-------------------------|-----|-----|-------|-----|-----|-----|-------|-----|
| 1430 | 1.3000  2.4494  | ✓   | 4.5812 | 9.0000 | | 1.1818 | 2.0000 | ✓ | 4.6919 | 9.0000 |
| 12673| 1.2608  2.4494  | ✓   | 4.5713 | 9.0000 | | 1.5263 | 2.0000 | ✓ | 5.5960 | 9.0000 |
| 20539| 2.0434  2.4494  | ✓   | 6.2265 | 9.0000 | | 2.4736 | 2.0000 | ✓ | 8.7714 | 9.0000 |
| 33337| 2.1764  2.0000  | ✓   | 8.8258 | 9.0000 | | 3.1176 | 2.4494 | ✓ | 7.7183 | 9.0000 |
| 52417| 2.3043  2.4494  | ✓   | 6.3921 | 9.0000 | | 1.8695 | 2.0000 | ✓ | 7.3155 | 9.0000 |

8 Conclusion

In our previous work [2], we have characterized in all Kummer extensions $k/k_0$, which possess a relative 3-genus field $k^*$ with elementary bicyclic Galois group $\text{Gal}(k^*/k)$. The underlying pure cubic subfields $L = \mathbb{Q}(\sqrt[3]{d})$ partially reveal the rare behavior that none of the generators of primitive ambiguous principal ideals occurs among the lattice minima of the maximal order $O_L$. We have given necessary and sufficient conditions for these exotic fields. Since their existence has an unpleasant impact on the classification of pure cubic fields $L$ by means of Voronoi’s algorithm, we have developed and implemented a marvellous technique for unambiguously determining the principal factorization type of $L$, thereby correcting serious defects in earlier tables.

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