CODING OF GEODESICS ON SOME MODULAR SURFACES AND APPLICATIONS TO ODD AND EVEN CONTINUED FRACTIONS

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Abstract. The connection between geodesics on the modular surface $\text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}$ and regular continued fractions, established by Series, is extended to a connection between geodesics on $\Gamma \backslash \mathbb{H}$ and odd and grotesque continued fractions, where $\Gamma \cong \mathbb{Z}_3 \ast \mathbb{Z}_3$ is the index two subgroup of $\text{PSL}(2, \mathbb{Z})$ generated by the order three elements $(\frac{0}{1}, -1)$ and $(\frac{0}{1}, 1)$, and having an ideal quadrilateral as fundamental domain. A similar connection between geodesics on $\Theta \backslash \mathbb{H}$ and even continued fractions is discussed in our framework, where $\Theta$ denotes the Theta subgroup of $\text{PSL}(2, \mathbb{Z})$ generated by $(\frac{0}{1}, -1)$ and $(\frac{1}{2}, 0)$.

1. Introduction

The connection between geodesics on the modular surface $\mathcal{M} = \text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ and regular continued fractions (RCF), originating in the seminal work of Artin [6], has generated a significant amount of interest. Inspired by earlier work of Moeckel [21], Series [29] established explicit connections between the geodesic flow on $\mathcal{M}$, geodesic coding, and RCF dynamics. The way ideal triangles of the Farey tessellation $\mathbb{F}$ cut oriented geodesics $\gamma$ on the upper half-plane $\mathbb{H}$ play a central role in this approach. The geodesic crosses two sides of a triangle, and the geodesic arc is labeled $L$ or $R$ according to whether the vertex shared by the sides is to the left or right, respectively, of the geodesic. This labeling is invariant under $\text{SL}(2, \mathbb{Z})$, and hence under any of its subgroups. These geodesics $\gamma$ are lifts of geodesics $\bar{\gamma}$ on $\mathcal{M}$, which are uniquely determined by infinite two-sided cutting sequences $\ldots L_{n-1}^1 R_{n-1}^0 L_n^1 \ldots$. The sequence of positive integers $(n_i)_{i=-\infty}^{\infty}$ is intimately related with regular continued fractions, since

$$\gamma_{-\infty} = \frac{-1}{n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \ldots}}} = \left[\frac{-1}{n_0} \right]\quad \text{and} \quad \gamma_{\infty} = \frac{1}{n_2 + \frac{1}{n_3 + \ldots}} = \left[\frac{1}{n_2} \right]$$

are the negative and positive endpoints of some lift $\gamma$ of $\bar{\gamma}$ to $\mathbb{H}$. Shifting along the cutting sequences is related to the Gauss map $T$ on $[0, 1)$ and its natural extension $\bar{T}$ on $[0, 1)^2$, defined by

$$T([n_1, n_2, n_3, \ldots]) = [n_2, n_3, n_4, \ldots], \quad \bar{T}([n_1, n_2, n_3, \ldots], [n_0, n_{-1}, n_{-2}, \ldots]) = ([n_2, n_3, n_4, \ldots], [n_1, n_0, n_{-1}, \ldots]).$$

Some different approaches for coding the geodesic flow on $\mathcal{M}$ were considered by Arnoux in [4] and by Katok and Ugarcovici [14].

A large class of continued fractions has been studied in the context of the geodesic flow and symbolic dynamics. A non-exhaustive list includes backward continued fractions [2] [3], even continued fractions [1] [7] [9] [13], Rosen continued fractions [20], $(a, b)$-continued fractions [15], Nakada $\alpha$-continued fractions and $\alpha$-Rosen continued fractions [5], or other classes of complex or Heisenberg continued fractions [18]. In different directions, the symbolic dynamics associated with the billiard flow on modular surfaces of uniform triangle groups, and respectively with the geodesic
flow on two-dimensional hyperbolic good orbifolds, have been thoroughly investigated in \[12\] and respectively \[24\].

This note describes codings of geodesics on the modular surfaces \(\mathcal{M}_o\) and \(\mathcal{M}_e\), associated with subgroups of index two and respectively three in \(\text{PSL}(2, \mathbb{Z})\). The coding of \(\mathcal{M}_o\) is hereby connected, in the spirit of \[29\], to the dynamics of odd and grotesque continued fractions (OCF, respectively GCF), and the coding of \(\mathcal{M}_e\) is connected to the even (ECF) and extended even continued fractions. These continued fractions were first investigated by Rieger \[25\] and by Schweiger \[26, 27\].

We consider the modular surface \(\mathcal{M}_o = \Gamma \backslash \mathbb{H} = \pi_o(\mathbb{H})\), where \(\Gamma\) is the index two subgroup of \(\Gamma(1) = \text{PSL}(2, \mathbb{Z})\) generated by the Möbius transformations \(S(z) = \frac{z - 1}{z + 1}\) and \(T(z) = z + 2\) acting on \(\mathbb{H}\). Equivalently, \(\Gamma\) is generated by the order three matrices \(S = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\) and \(ST^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\). The corresponding Dirichlet fundamental domain with respect to \(i\) is the quadrilateral \(\mathcal{F}\) bounded by the geodesic arcs \([0, \omega], [\omega, \infty], [0, \omega^2]\) and \([\omega^2, \infty]\), where \(\omega = \frac{1}{2}(1 + i\sqrt{3})\) (see Fig. 1). The transformation \(S\) fixes \(\omega^2 = \omega - 1\) and cyclically permutes the points \(\infty, 0, -1\), and respectively \(i, \frac{1+i}{2}, -1+i\). On the other hand \(ST^{-1}\) fixes \(\omega\) and permutes \(\infty, 0, 1\), and respectively \(1, \frac{1+i}{2}, 1+i\) (see Figs. 1 and 3). As shown by Lemma 7, the point \(\pi_o(\infty)\) is the only cusp of \(\Gamma\), while \(\pi_o(i)\) is a regular point for \(\mathcal{M}_o\) since we deal with a two-fold cover ramified at \(i\) which makes the singularity disappear.

The parts of the fundamental region \(\mathcal{F}\) on either side of the imaginary axis are considered separately. First, we consider the triangle \(\omega^2, 0, \infty\). The union of the images of this triangle under \(I, S,\) and \(S^2\) gives the ideal triangle \(-1, 0, \infty\). Similarly, the triangle with vertices \(\omega, 0, \infty\), under \(I, ST^{-1}\) and \((ST^{-1})^2\) is the ideal triangle \(1, 0, \infty\). Together, these regions form the ideal quadrilateral \(\Delta\) with vertices \(-1, 0, 1\) and \(\infty\). The images of \(\Delta\) under \(\Gamma\) form the Farey tessellation. That is, two rational numbers \(\frac{p}{q}, \frac{p'}{q'}\) are joined by a side of the Farey tessellation precisely when \(pq' - p'q = \pm 1\).
This is the same tessellation considered by Series [29], but we add a checkerboard coloring, as shown in Fig. 2. The triangle $-1, 0, \infty$ is light, while $1, 0, \infty$ is dark, then continue in a checkerboard pattern, so that each of the three neighboring Farey cells of a light cell are dark, and vice versa. We code oriented geodesics by including the shade of the Farey cell. Concretely, a light $L$ is denoted by $L$, a dark $R$ by $R$, and a light $L$ by $L$. This way, every geodesic in $\mathbb{H}$ with irrational endpoints is assigned an infinite two-sided sequence of symbols $\mathbb{L}, \mathbb{L}, \mathbb{R}$ and $\mathbb{R}$. We also require that a light letter $L$ or $R$ can only be succeeded and preceded by a dark one, and vice versa.

The coding of geodesics on $M_0$ is described by concatenating words of the following type:

$$((\mathbb{L})^k)^{-1} L R, \quad ((\mathbb{L})^k)^{-1} R L, \quad ((\mathbb{R})^k)^{-1} L R, \quad ((\mathbb{R})^k)^{-1} R L, \quad k \geq 1$$

(1.1)

Single letter strings are allowed. We require colors of individual letters to alternate. Thus, strings of the first and third type must be succeeded by those of the first or second.

A powerful tool in the study of endomorphisms in ergodic theory is provided by the natural extension, an invertible transformation that dilates the given endomorphism and preserves many of its ergodic properties, such as ergodicity or mixing [10]. Here, we describe the natural extension of the Gauss type OCF map (or, by reversing order, of the Gauss type GCF map) as a factor of its ergodic properties, such as ergodicity or mixing [10].

In this section we review some properties and dynamics of odd, grotesque and even continued fractions. Other applications include a characterization of quadratic surds in terms of their OCF expansion and their conjugate GCF expansion, as well as a tail-equivalence type description of the orbits of the action of $\Gamma$ on the real line.

We also observe that similar results can be obtained for even continued fractions, using the Farey tessellation without coloring and the modular surface $M_e = \Theta \setminus \mathbb{H}$, where $\Theta$ denotes the index three Theta subgroup in $\Gamma(1)$ generated by the transformations $S(z) = -\frac{1}{z}$ and $T(z) = z + 2$.

2. Odd, grotesque, and even continued fractions

In this section we review some properties and dynamics of odd, grotesque and even continued fractions. Odd and even continued fractions are part of the broader class of $D$-continued fractions introduced in [16] (see also [13, 10, 11]), while the GCF is the dual algorithm of the odd continued fractions. Instead of giving a compact presentation, we chose to be more repetitive, in order to clarify notation and the difference between these three classes of continued fractions.

2.1. Odd continued fractions. The OCF expansion of a number $x \in [0, 1] \setminus \mathbb{Q}$ is given by

$$x = \lfloor (a_1, \epsilon_1), (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots \rfloor_o = \frac{1}{a_1 + \frac{\epsilon_1}{a_2 + \frac{\epsilon_2}{a_3 + \cdots}}}$$

(2.1)

where $\epsilon_i = \epsilon_i(x) \in \{\pm 1\}$, $a_i = a_i(x) \in 2\mathbb{N} - 1$ and $a_i + \epsilon_i \geq 2$. Such an expansion is unique. We also consider

$$x = \lfloor (a_0, \epsilon_0); (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \rfloor_o := a_0 + \epsilon_0 \lfloor (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \rfloor_o \in [1, \infty),$$

with $\epsilon_0 \in \{\pm 1\}$, $a_0 \in 2\mathbb{N} - 1$ and $a_0 + \epsilon_0 \geq 2$, so that $x \in (a_0, a_0 + 1)$ if $\epsilon_0 = 1$ and $x \in (a_0 - 1, a_0)$ if $\epsilon_0 = -1$. 


The odd Gauss map \( T_o \) acts on \([0, 1]\) by
\[
T_o(x) = \epsilon \left( \frac{1}{x} - 2k + 1 \right) \quad \text{if} \quad x \in B(\epsilon, k) := \begin{cases} 
\left( \frac{1}{2k}, \frac{1}{2k-1} \right) & \text{if} \quad \epsilon = 1, \ k \geq 1 \\
\left( \frac{1}{2k}, \frac{1}{2k-2} \right) & \text{if} \quad \epsilon = -1, \ k \geq 2.
\end{cases}
\]
Symbolically, \( T_o \) acts on the OCF representation by removing the leading digit \((a_1, \epsilon_1)\) of \( x \), i.e.
\[
T_o([[(a_1, \epsilon_1), (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots])_o) = [(a_2, \epsilon_2), (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots]_o.
\]
The probability measure \( d\mu_o(x) = \frac{1}{3\log G} \left( \frac{1}{G-1+x} + \frac{1}{G+1-x} \right) dx \) is \( T_o \)-invariant (see [25, 26]), where we denote
\[
G = \frac{1}{2}(\sqrt{5} + 1).
\]

### 2.2. Grotesque continued fractions
Rieger’s GCF representation of an irrational \( y \in I_G := (G - 2, G) \) is given by
\[
y = \langle \langle (b_0, \epsilon_0), (b_1, \epsilon_1), (b_2, \epsilon_2), \ldots \rangle \rangle_o = \frac{\epsilon_0}{b_0 + \frac{\epsilon_1}{b_1 + \frac{\epsilon_2}{b_2 + \cdots}}},
\]
where \( \epsilon_i \in \{\pm 1\} \), \( b_i = b_i(y) \in 2\mathbb{N} - 1 \) and \( b_i + \epsilon_i \geq 2 \). Every irrational number \( y \in I_G \), \( y \neq 0 \), can be uniquely represented as above, by taking \( \epsilon_0 = \epsilon_0(y) = \text{sign}(y) \) and \( b_0 = b_0(y) \) the unique odd positive integer with \( \frac{1}{|y|} - G \leq b_0(y) \leq \frac{1}{|y|} - G + 2 \). The corresponding Gauss map \( \tau_o \) acts on \( I_G \) by
\[
\tau_o(y) = \frac{\epsilon_0(y)}{y} - b_0(y) = \frac{1}{|y|} - b_0(y),
\]
or on the symbolic representation \((2.2)\) by
\[
\tau_o(\langle \langle (b_0, \epsilon_0), (b_1, \epsilon_1), (b_2, \epsilon_2), \ldots \rangle \rangle_o) = \langle \langle (b_1, \epsilon_1), (b_2, \epsilon_2), (b_3, \epsilon_3), \ldots \rangle \rangle_o,
\]
and \( d\nu_o(y) = \frac{1}{3\log G} \cdot \frac{dy}{y+1} \) provides a \( \tau_o \)-invariant probability measure (see [25, 26]).

Consider \( \Omega_o = (0, 1) \times I_G \). The natural extension of \( T_o \) can be realized as the invertible map \([26, 27]\)
\[
\tilde{T}_0 : \Omega_o \to \Omega_o, \quad \tilde{T}_0(x, y) = \left( T_o(x), \frac{\epsilon_1(x)}{a_1(x) + y} \right),
\]
and it acts on \( \Omega_o \cap (\mathbb{R} \setminus \mathbb{Q})^2 \) as a two-sided shift as follows:
\[
\tilde{T}_0([[(a_1, \epsilon_1), (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots]]_o, \langle \langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots \rangle \rangle_o)
\]
\[
= ([a_2], (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots]_o, \langle \langle (a_1, \epsilon_1), (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots \rangle \rangle_o).
\]

It is also convenient to consider the extension \( \tilde{T}_o \) of \( T_o \) to \( \tilde{\Omega}_o := \Omega_o \times \{-1, 1\} \) defined by
\[
\tilde{T}_o(x, y, \epsilon) := \left( T_o(x, y), -\epsilon_1(x)\epsilon \right),
\]
with inverse
\[
\tilde{T}_o^{-1}([[(a_1, \epsilon_1), (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots]]_o, \langle \langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots \rangle \rangle_o, \epsilon)
\]
\[
= ([a_0, \epsilon_0], (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots]_o, \langle \langle (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), (a_{-3}, \epsilon_{-3}), \ldots \rangle \rangle_o, -\epsilon_0 \epsilon).
\]
2.3. Even continued fractions. The ECF expansion of \( x \in [0, 1] \setminus \mathbb{Q} \) is given by

\[
x = \left[ (a_1, \epsilon_1), (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots \right]_e = \frac{1}{a_1 + \frac{\epsilon_1}{a_2 + \frac{\epsilon_2}{a_3 + \cdots}}},
\]

where \( \epsilon_i = \epsilon_i(x) \in \{\pm 1\} \) and \( a_i = a_i(x) \in 2\mathbb{N} \). Such an expansion is unique. Every number \( x \in [1, \infty) \setminus \mathbb{Q} \) has a unique infinite ECF expansion

\[
x = \left[ (a_0, \epsilon_0); (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \right]_e = a_0 + \epsilon_0 \left[ (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \right]_e \in [1, \infty),
\]

with \( \epsilon_0 \in \{\pm 1\} \) and \( a_0 \in 2\mathbb{N} \), so that \( x \in (a_0, a_0 + 1) \) if \( \epsilon_0 = 1 \) and \( x \in (a_0 - 1, a_0) \) if \( \epsilon_0 = -1 \).

The even Gauss map \( T_e \) acts on \([0, 1]\) by

\[
T_e(x) = \epsilon \left( \frac{1}{x} - 2k \right) \quad \text{if} \quad x \in B(\epsilon, k) = \left( \frac{1}{2k+1}, \frac{1}{2k} \right) \quad \text{if} \quad \epsilon = 1, \ k \geq 1
\]

\[
\frac{1}{2k}, \frac{1}{2k-1} \quad \text{if} \quad \epsilon = -1, \ k \geq 1.
\]

Symbolically, \( T_e \) acts on the ECF representation by removing the leading digit \((a_1, \epsilon_1)\) of \( x \), i.e.

\[
T_e \left[ (a_1, \epsilon_1), (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots \right]_e = \left[ (a_2, \epsilon_2), (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots \right]_e.
\]

Here the infinite measure \( d\mu_e(x) = \left( \frac{1}{1+x} + \frac{1}{1-x} \right) dx \) is \( T_e \)-invariant (see [7, 26, 27]).

The even continued fraction equivalent of the grotesque continued fractions are the extended even continued fractions. Given \( \epsilon_i \in \{\pm 1\} \) and even positive integers \( b_i \), we denote

\[
y = \left[ (b_0, \epsilon_0), (b_1, \epsilon_1), (b_2, \epsilon_2), \ldots \right]_e := \epsilon_0 \left[ (b_1, \epsilon_1), (b_2, \epsilon_2), (b_3, \epsilon_3), \ldots \right]_e = \frac{\epsilon_0}{\epsilon_1} \in (-1, 1).
\]

The corresponding shift \( \tau_e \) acts on \((-1, 1)\) by

\[
\tau_e(y) = \frac{\epsilon_0(y)}{y} - b_0(y),
\]

or in the symbolic representation by

\[
\tau_e \left( \left[ (b_0, \epsilon_0), (b_1, \epsilon_1), (b_2, \epsilon_2), \ldots \right]_e \right) = \left[ (b_1, \epsilon_1), (b_2, \epsilon_2), (b_3, \epsilon_3), \ldots \right]_e.
\]

Consider \( \Omega_e = (0, 1) \times (-1, 1) \) and the natural extension of \( T_e \), realized as the map [27]

\[
\tilde{T}_e : \Omega_e \to \Omega_e, \quad \tilde{T}_e(x, y) = \left( T_e(x), \frac{\epsilon_1(x)}{a_1(x) + y} \right).
\]

Equivalently, \( \tilde{T}_e \) is acting on \( \Omega_e \cap (\mathbb{R} \setminus \mathbb{Q})^2 \) as a two-sided shift:

\[
\tilde{T}_e \left[ (a_1, \epsilon_1), (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots \right]_e, \left[ (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots \right]_e
\]

\[
= \left[ (a_2, \epsilon_2), (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots \right]_e, \left[ (a_1, \epsilon_1), (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots \right]_e.
\]

It is again convenient to consider the extension \( \tilde{T}_e \) of \( \tilde{T}_e \) to \( \tilde{\Omega}_e = \Omega_e \times \{-1, 1\} \) defined by

\[
\tilde{T}_e(x, y, \epsilon) = \left( \tilde{T}_e(x, y), -\epsilon_1(x) \epsilon \right),
\]

with inverse

\[
\tilde{T}_e^{-1}(x, y, \epsilon) = \left( \tilde{T}_e^{-1}(x, y), -\epsilon_0(y) \epsilon \right).
\]
3. Cutting sequences for geodesics on $\mathcal{M}_o$

3.1. The group $\Gamma$ and the modular surface $\mathcal{M}_o = \Gamma \backslash \mathbb{H}$. The fundamental Dirichlet region corresponding to the subgroup $\Gamma$ of $\Gamma(1)$ generated by the Möbius transformations $S(z) = \frac{-1}{z+1}$ and $T(z) = z + 2$, with order three generators $S$ and $ST^{-1}$, with respect to the point $i$ is the quadrilateral

$$\mathfrak{F} = \{z \in \mathbb{H} : |\operatorname{Re} z| \leq \frac{1}{2}, |z - 1| \geq 1, |z + 1| \geq 1\},$$

with edges $[\omega, \infty]$, $[0, \omega]$ identified by $ST^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, and with $[0, \omega^2]$, $[\omega^2, \infty]$ identified by $S$. The resulting quotient space $\mathcal{M}_o = \pi_o(\mathbb{H})$ is homeomorphic to the union of two cones glued along their basis $[0, \infty]$, with vertices corresponding to $\omega$ and $\omega^2$, cuts along the geodesic arcs $[\omega, \infty]$ and $[\omega^2, \infty]$, and a cusp at $\pi_o(\infty)$ (see Fig. 1).

**Lemma 1.** The group $\Gamma$ is an index two subgroup of $\Gamma(1)$ and it coincides with

$$\Gamma_0 := \left\{ M \in \Gamma(1) : M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2} \right\}.$$

**Proof.** The generators $S$ and $T$ are contained in the group $\Gamma_0$. Since $|\operatorname{SL}(2, \mathbb{Z}_2)| = 6$, we have $[\Gamma(1) : \Gamma_0] = 2$. Finally, we notice that $\mathfrak{F} = \mathcal{F} \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{F}$, where $\mathcal{F} = \{z \in \mathbb{H} : |z| \geq 1, |\operatorname{Re} z| \leq \frac{1}{2}\}$ is the standard fundamental domain for $\Gamma(1) \backslash \mathbb{H}$. This yields $[\Gamma(1) : \Gamma] = 2$ and $\Gamma = \Gamma_0$. \hfill $\square$

A lift of the group $\Gamma$ to $\text{GL}(2, \mathbb{Z})$ played an important role in the analysis of the renewal time for odd continued fractions, pursued by Vandehey and one of the authors in [8].

We denote by $\mathcal{A}_o$ the set of geodesics $\gamma$ in $\mathbb{H}$ with endpoints satisfying

$$(\gamma_\infty, \gamma_{-\infty}) \in \mathcal{S}_o := ((1, \infty) \times (-I_G)) \cup ((-\infty, -1) \times I_G).$$

**Lemma 2.** Every geodesic $\tilde{\gamma}$ on $\mathcal{M}_o$ lifts to $\mathbb{H}$ to a geodesic $\gamma \in \mathcal{A}_o$. 

![Figure 3. The Farey tessellation in the disk model and the rotations $S$ and $ST^{-1}$](image)
Proof. Without loss of generality, we can take \( \bar{\gamma} \) to be a positively oriented geodesic arc in \( \mathcal{F} \) connecting \([\omega^2, \infty)\) to \([\omega, \infty)\), \([0, \omega^2)\) to \([\omega, \infty)\), \([\omega^2, \infty)\) to \([0, \omega)\), or \([0, \omega^2)\) to \([0, \omega)\). Note that \([0, 1] \subset I_o = (G - 2, G)\). There are four cases to consider. First, \( \gamma_{-\infty} < -1 < 1 < \gamma_{\infty} \), and an appropriate \( 2\mathbb{Z} \)-translation gives \( \gamma_{-\infty} \in -I_G \) and \( \gamma_{\infty} > 1 \). Second, \( \gamma \) connects the arc \([-1, 0] \) to \([\omega, \infty)\), hence \(-1 < \gamma_{-\infty} < 0 < 1 < \gamma_{\infty}\). Third, we have \( \gamma_{-\infty} < -1 < 0 < \gamma_{\infty} < 1 \) in \( A_o \).

Finally, in the fourth case \( TS^{-1} \) maps \( \bar{\gamma} \) to a geodesic arc connecting \([\omega + 1, \infty)\) to \([\omega, \infty)\), hence \( \gamma_{-\infty} > 2 > 0 > \gamma_{\infty} \). An appropriate \( 2\mathbb{Z} \)-translation ensures that \( \gamma_{-\infty} \in I_G \) and \( \gamma_{\infty} < -1 \). □

3.2. Cutting sequences and odd/grotesque continued fraction expansions. As described in the introduction, our coding of geodesics on \( \mathcal{M}_o \) refines the Series coding. An oriented geodesic \( \gamma \) in \( \mathbb{H} \) is cut into segments as it crosses triangles in the Farey tessellation \( \mathcal{F} \). Each segment of the geodesic crosses two sides of a triangle in the tessellation. If the vertex where the two sides meet is on the left, we label the segment \( L \), if it is on the right we label it \( R \). We use \( L \) and \( R \) when the geodesic is in a light cell and \( \mathbb{L} \) or \( \mathbb{R} \) for a dark cell. This way, we assign to every geodesic in \( \mathbb{H} \) with irrational endpoints an infinite two-sided sequence of symbols \( \mathbb{L}, \mathbb{L}, \mathbb{R} \) and \( \mathbb{R} \), with alternating shades.

Next, we analyze in detail the connection between the GCF expansion of \( \gamma_{-\infty} \), the OCF expansion of \( \gamma_{\infty} \), and the strings in (1.1).

For every Möbius transformation \( \bar{\rho} \) leaving \( \mathcal{S}_o \) invariant, we still denote by \( \bar{\rho} \) the product map \( \bar{\rho} \times \bar{\rho} \), viewed as a transformation of \( \mathcal{S}_o \). To every geodesic \( \gamma \in \mathcal{A}_o \) we associate the positively oriented geodesic arc \([\xi_{\gamma}, \eta_{\gamma}]\) where

\[
\xi_{\gamma} := \begin{cases} 
\gamma \cap [1, \infty) & \text{if } \gamma_{\infty} > 1 \\
\gamma \cap [-1, \infty) & \text{if } \gamma_{\infty} < -1
\end{cases}
\quad \text{and} \quad
\eta_{\gamma} := \begin{cases} 
\gamma \cap [a_1, a_1 + \epsilon_1] & \text{if } \gamma_{\infty} > 1 \\
\gamma \cap [-a_1, -a_1 - \epsilon_1] & \text{if } \gamma_{\infty} < -1.
\end{cases}
\]

When \( \gamma_{\infty} > 1 \), we write \( \gamma_{\infty} = \langle (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \rangle_o \), \( \gamma_{-\infty} = -\langle (a_0, -\epsilon_0), (a_-1, \epsilon_1), \ldots \rangle_o \in -I_G \). When \( \gamma_{\infty} < -1 \), \( \gamma_{-\infty} = -\langle (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \rangle_o \), \( \gamma_{\infty} = \langle (a_0, \epsilon_0), (a_-1, \epsilon_1), \ldots \rangle_o \in I_G \). Four cases will occur:

(A) \( \gamma_{\infty} \in (2k, 2k + 1) \), \( \epsilon_1 = -1, a_1 = 2k + 1 \). The Möbius transformation \( \rho_o(x) = \frac{1}{a_1 - x} \) belongs to \( \Gamma \) and

\[
\rho_o(\langle (a_1, \epsilon_1), (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots \rangle_o, -\langle (a_0, \epsilon_0), (a_-1, \epsilon_1), (a_-2, \epsilon_2), \ldots \rangle_o) = \langle (a_2, \epsilon_2), (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots \rangle_o, -\langle (a_1, \epsilon_1), (a_0, \epsilon_0), (a_-1, \epsilon_1), \ldots \rangle_o \rangle_o.
\]

(3.1)

In this situation \( \rho_o \) transforms the arc \([\xi_{\gamma}, \eta_{\gamma}]\) of \( \gamma \) connecting the geodesics \([1, \infty)\) and \([a_1 - 1, a_1]\) into an arc connecting \([0, \frac{1}{a_1 - 1}]\) with \([1, \infty] \). Following the orientation of \( \gamma \), we assign the string \( \xi_{\gamma} (\mathbb{L})^{k-1} \mathbb{L} \mathbb{R} \eta_{\gamma} \) to the arc \([\xi_{\gamma}, \eta_{\gamma}]\) (see Fig. 4).
and $-\rho$ with inverse $A$ and $C$ are followed by $A$ or $B$, and cases $B$ and $D$ are followed by cases $C$ or $D$. In this situation

$$\rho$$

assign the string $(D)$ and the arc $\xi$ (see Fig. 4). Following the orientation of $\gamma$, to the arc $[\xi, \eta]$ we assign the string $\xi (L \mathbb{R})^{k-1} \rho \eta$ (see Fig. 5).

(B) $\gamma_\infty \in (2k - 1, 2k), \epsilon_1 = +1, a_1 = 2k - 1$. The transformation $\rho_o(x) = \frac{1}{a_1 - x}$ belongs to $\Gamma$ and

$$\rho_o([a_1, \epsilon_1];(a_2, \epsilon_2), (a_3, \epsilon_3), \ldots]_o, -\langle\langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots\rangle\rangle_o)$$

$$= (\ - \langle\langle (a_2, \epsilon_2), (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots\rangle\rangle_o, \langle\langle (a_1, \epsilon_1), (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots\rangle\rangle_o).$$

(3.2)

In this situation $\rho_o$ transforms the arc $[\xi, \eta]$ of $\gamma$ connecting the geodesics $[1, \infty]$ and $[a_1, a_1 + 1]$ into an arc connecting $[0, \frac{1}{a_1 - 1}]$ with $[-1, \infty]$. Following the orientation of $\gamma$, to the arc $[\xi, \eta]$ we assign the string $\xi (L \mathbb{R})^{k-1} \rho \eta$ (see Fig. 5).

(C) $\gamma_\infty \in (-2k - 1, -2k), \epsilon_1 = -1, a_1 = 2k + 1$. The transformation $\rho_o(x) = \frac{1}{-a_1 - x}$ belongs to $\Gamma$ and

$$\rho_o([a_1, \epsilon_1];(a_2, \epsilon_2), (a_3, \epsilon_3), \ldots]_o, \langle\langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots\rangle\rangle_o)$$

$$= (\ - \langle\langle (a_2, \epsilon_2), (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots\rangle\rangle_o, \langle\langle (a_1, \epsilon_1), (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots\rangle\rangle_o).$$

(3.3)

In this situation $\rho_o$ transforms the arc $[\xi, \eta]$ of the geodesic $\gamma$ connecting the geodesics $[-1, \infty]$ and $[-a_1, -a_1 + 1]$ into an arc connecting $[\frac{1}{a_1 - 1}, 0]$ with $[-1, \infty]$. Following the orientation of $\gamma$, to the arc $[\xi, \eta]$ we assign the string $\xi (L \mathbb{R})^{k-1} \rho \eta$ (see Fig. 5).

(D) $\gamma_\infty \in (-2k - 1, -2k), \epsilon_1 = +1, a_1 = 2k - 1$. The transformation $\rho_o(x) = \frac{1}{a_1 - x}$ belongs to $\Gamma$ and

$$\rho_o([a_1, \epsilon_1];(a_2, \epsilon_2), (a_3, \epsilon_3), \ldots]_o, \langle\langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots\rangle\rangle_o)$$

$$= (\langle\langle (a_2, \epsilon_2), (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots\rangle\rangle_o, -\langle\langle (a_1, \epsilon_1), (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots\rangle\rangle_o).$$

(3.4)

In this situation $\rho_o$ transforms the arc $[\xi, \eta]$ of the geodesic $\gamma$ connecting the geodesics $[-1, \infty]$ and $[-a_1 - 1, -a_1]$ into an arc connecting $[\frac{1}{a_1 - 1}, 0]$ with $[1, \infty]$. Following the orientation of $\gamma$, to the arc $[\xi, \eta]$ we assign the string $\xi (L \mathbb{R})^{k-1} \rho \eta$ (see Fig. 5).

Summarizing (3.1) - (3.4), we obtain the following general formula for the action of $\rho_o$ on $S_o$:

$$\rho_o(\epsilon([a_1, \epsilon_1];(a_2, \epsilon_2), (a_3, \epsilon_3), \ldots)_o, -\epsilon\langle\langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots\rangle\rangle_o)$$

$$= (\ - \epsilon\langle\langle (a_2, \epsilon_2), (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots\rangle\rangle_o, -\epsilon\langle\langle (a_1, \epsilon_1), (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots\rangle\rangle_o),$$

(3.5)

with inverse

$$\rho_o^{-1}(\epsilon([a_1, \epsilon_1];(a_2, \epsilon_2), (a_3, \epsilon_3), \ldots)_o, -\epsilon\langle\langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots\rangle\rangle_o)$$

$$= (\ -\epsilon_o(\epsilon\langle\langle (a_0, \epsilon_0); (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots\rangle\rangle_o, -\epsilon\langle\langle (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots\rangle\rangle_o) if \epsilon \in \{\pm 1\}.$$

This also gives that $\rho_o$ reflects across the imaginary axis exactly when $\epsilon = +1$, so that $\rho_o$ agrees with the transformation $\rho$ from [29] exactly where the RCF and OCF agree. We also get that cases A and C are followed by A or B, and cases B and D are followed by cases C or D.
Proposition 3. The map $\rho_o : S_o \to S_o$ is invertible, and the diagram

$$
\begin{array}{ccc}
S_o & \overset{\rho_o}{\longrightarrow} & S_o \\
\downarrow J_o & & \downarrow J_o \\
\tilde{\Omega}_o & \overset{T_o}{\longrightarrow} & \tilde{\Omega}_o
\end{array}
$$

commutes, where $J_o : S_o \to \tilde{\Omega}_o$ is the invertible map defined by

$$
J_o(x, y) := \text{sign}(x)(1/x, -y, 1) = \begin{cases} 
(1/x, -y, 1) & \text{if } x > 1, y \in -I_G; \\
(-1/x, y, -1) & \text{if } x < -1, y \in I_G.
\end{cases}
$$

Proof. Set $x = \epsilon[[a_1, \epsilon_1]; (a_2, \epsilon_2), \ldots, \epsilon_o]$, $y = -\epsilon\langle ((a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots) \rangle_o$ with $\epsilon \in \{\pm 1\}$, so that $(x, y) \in S_o$. Using formulas (3.5) and (2.4), we get that

$$
J_o \rho_o(x, y) = J_o(\epsilon\langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots \rangle_o) = \langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots \rangle_o, \pm \epsilon_o
$$

Corollary 4. If $\alpha = [[(a_1, \epsilon_1); (a_2, \epsilon_2), \ldots, (a_r, \epsilon_r)]_o > 1$ and $\beta = -\langle ((a_r, \epsilon_r), (a_1, \epsilon_1)) \rangle_o \in -I_G$, then

(i) $\rho_o^\alpha(\alpha, \beta) = (-\epsilon_1) \cdots (-\epsilon_r)(\alpha, \beta)$.

(ii) $\rho_o^{\alpha \beta}(\alpha, \beta) = (\alpha, \beta)$.

Proof. To check (i), we use Proposition 3 and compute

$$
J_o^{-1} T_o^\alpha J_o(\alpha, \beta) = J_o^{-1} T_o^\alpha (1/\alpha, -\beta, 1) = J_o^{-1} (1/\alpha, -\beta, (-\epsilon_1) \cdots (-\epsilon_r)) = (-\epsilon_1) \cdots (-\epsilon_r)(\alpha, \beta).
$$

(ii) We consider the case $(-\epsilon_1) \cdots (-\epsilon_r) = -1$, when we use Proposition 3 and

$$
J_o^{-1} T_o^\alpha J_o(\alpha, -\beta) = J_o^{-1} T_o^\alpha (1/\alpha, -\beta, -1) = J_o^{-1} (1/\alpha, -\beta, (-\epsilon_1) \cdots (-\epsilon_r)) = (\alpha, \beta).
$$

Finally, we notice the equality

$$
T_o^{-1}(u, v) = \left( -\frac{\text{sign}(v)}{\rho_o(1/u)}, \text{sign}(v) \rho_o(-v) \right), \quad \forall (u, v) \in \tilde{\Omega}_o \cap (\mathbb{R} \setminus \mathbb{Q})^2. 
$$

4. Connection with cutting sequence and RCF

4.1. Odd continued fractions. We now explore the connection between the cutting sequences of the regular continued fractions and the odd continued fractions. Here, we use $x$ to mark the imaginary axis, as in [29], and $\xi = \gamma \cap [1, \infty]$ as defined above.

In cases $A$ and $B$, we get the cutting sequence $\ldots xL^{n_1}R^{n_2}L^{n_3} \ldots$ with regular continued fraction expansion $[n_1; n_2, n_3, \ldots]$. Without coloring, this corresponds to $\ldots L \xi L^{n_1-1}R^{n_2}L^{n_3} \eta, \ldots$. We have two cases to consider for the first digit of the odd continued fraction expansion.

(A) $n_1 = 2k$ is even, and $\gamma \in (2k, 2k+1)$. This gives the cutting sequence $\ldots \xi(\mathbb{L})^{k-1}L \mathbb{R} \eta$, and we get $(2k, 1, -1) = (a_1, \epsilon_1)$. The next digit is represented by $L^n R$ for $n \geq 0$. When
\[ n_2 > 1, \text{ the next digit is } L^0 \mathbb{R}, \text{ corresponding to } (1,+1). \] This gives the cutting sequence \[ \ldots \xi_\gamma (L L)^{k-1} L R \eta_1 \mathbb{R} \ldots . \] This corresponds to
\[ 2k + \frac{1}{n_2 + z} = 2k + 1 - \frac{1}{1 + \frac{1}{n_2 - 1 + z}}. \]

When \( n_2 = 1 \), we proceed with \( \ldots \xi_\gamma (L L)^{k-1} L \mathbb{R} \eta_1 L \ldots . \) This corresponds to
\[ 2k + \frac{1}{1 + \frac{1}{n_3 + z}} = 2k + 1 + \frac{-1}{n_3 + 1 + z}. \]

These equalities correspond to the singularization and insertion algorithm introduced by Kraaikamp in [16] and explicitly computed for the OCF in Masarotto’s master’s thesis [19] (see also [13]). This algorithm is based on the identity
\[ a + \frac{\epsilon}{1 + \frac{1}{b + z}} = a + \epsilon + \frac{-\epsilon}{b + 1 + z} \quad \text{where } \epsilon \in \{ \pm 1 \}. \]

(B) \( n_1 = 2k - 1 \) is odd, and \( \gamma_\alpha \in (2k - 1, 2k) \). This gives the cutting sequence \( \ldots \xi_\gamma (L L)^{k-1} \mathbb{R} \eta_1 \), and we get \( (2k - 1, +1) = (a_1, \epsilon_1) \). The next digit is represented by \( R^n L \), where \( n \geq 0 \). If \( n_2 = 1 \), we have \( \ldots \xi_\gamma (L L)^{k-1} \mathbb{R} \eta_1, L \ldots \), and the next digit corresponds to \( R^k L \), which gives \((1,+1)\).

Strings starting with \( R \) are treating similarly, with \((R R)^{k-1} L \) corresponding to \((2k - 1, +1)\) and \((R R)^{k-1} R L \) to \((2k + 1, -1)\).

Geodesics of type C and D can be classified similarly. In this case, we get the cutting sequence \( \ldots x R^{a_1} L^{n_2} R^{n_3} \ldots \), where \( \gamma_\alpha = -\{n_1, n_2, n_3, \ldots \} \). This gives, without coloring, the cutting sequence \( \ldots R \xi_\gamma R^{n_1-1} L^{n_2} R^{n_3} \ldots \), and we interpret the strings in the same way as above.

### 4.2. Grotesque continued fractions.

The grotesque continued fractions are the dual continued fraction expansion of the odd continued fractions, which changes the restriction on the digits. That is, for \( \epsilon_1/(a_1 + \epsilon_2/\ldots) \), the odd continued fractions require \( a_i + \epsilon_{i+1} \geq 2 \), and the grotesque require \( a_i + \epsilon_i \geq 2 \). This means we need a different insertion and singularization algorithm to convert regular continued fractions to grotesque continued fractions that the one used for odd continued fractions. As with Series’ description of the regular continued fractions [29], the forward endpoints are read from left to right, but the backwards endpoints are read from right to left. Thus, for the odd continued fractions, we can read the strings one at a time, but for grotesque continued fractions, we must also consider whether the preceding string would be valid.

We consider cases A, B, C, and D similar to those above. To stay consistent with how we normally read, we say that the string ends in the letter on the right. The preceding string is the string to the left of the one we are considering.

(A) \( \gamma_\alpha > 1, \gamma_{-\alpha} \in (0, 2 - G) \), and \( \epsilon_0 = -1 \). We get the cutting sequence \( \ldots (L L)^{k-1} L R \xi_\gamma \ldots \)
and \((a_0, \epsilon_0) = (2k + 1, -1)\) as in Fig. 3. The preceding string must end in \( R \) or \( L \), as in case A or B.

(B) \( \gamma_\alpha > 1, \gamma_{-\alpha} \in (-G, 0) \), and \( \epsilon_0 = +1 \). We get the cutting sequence \( \ldots (R R)^{k-1} L \xi_\gamma \ldots \)
and \((a_0, \epsilon_0) = (2k - 1, +1)\). Note that we cannot have \( a_0 = 1 \). The preceding string must end in \( R \) or \( L \), as in case C or D.
(C) \( \gamma_\infty < -1, \gamma_{-\infty} \in (G-2,0) \), and \( \epsilon_0 = -1 \). We get the cutting sequence \( \ldots (R\mathbb{R})^{k-1}R\mathbb{R}\ldots \) and \((a_0, \epsilon_0) = (2k + 1, -1)\) as in the image on the left of Fig. 5. The preceding string must end in \( \mathbb{R} \) or \( \mathbb{L} \), as in case C or D.

(D) \( \gamma_\infty < -1, \gamma_{-\infty} \in (0,G) \), and \( \epsilon_0 = +1 \). We get the cutting sequence \( \ldots (L\mathbb{L})^{k-1}R\mathbb{L}\ldots \) and \((a_0, \epsilon_0) = (2k - 1, +1)\) as in the image on the right of Fig. 5. The preceding string must end in \( \mathbb{R} \) or \( \mathbb{L} \), as in case A or B.

To illustrate the difference between odd and grotesque continued fraction expansions, we consider two example strings. First, note that for case C, we have the RCF cutting sequence \( \ldots \) if we ignore coloring. We compare the GCF expansions for \( \ldots \) for the RCF \( \ldots \)

We need to change the grouping for the first digit to make the second digit an allowable string, as in \([2;2,2,\ldots]\).

In the first case, we have the string \( \ldots \mathbb{R}\mathbb{L}\mathbb{R}\mathbb{L}\mathbb{R}\mathbb{L}\ldots \). Following the above rules, we get the grouping \( \ldots \mathbb{R}(L\mathbb{R})(L\mathbb{R})\mathbb{R}\ldots \) and \([2,1,2,\ldots] = \langle (3, +1), (3, -1), \ldots \rangle\).

In the second case, we have \( \ldots \mathbb{R}\mathbb{L}\mathbb{L}\mathbb{R}\mathbb{L}\ldots \) and \([2,2,2,\ldots] = \langle (1, +1), (1, +1), (3, -1), \ldots \rangle\).

We need to change the grouping for the first digit to make the second digit an allowable string, since \( \mathbb{R} \) must be preceded by \( (L\mathbb{L})^{k-1}L \). Thus, we get \( \ldots \mathbb{R}\mathbb{L}\mathbb{L}(\mathbb{R})(\mathbb{L})(\mathbb{R})\mathbb{L}\ldots \), where the final grouping is determined based on whether the previous letter is \( \mathbb{L} \) or \( \mathbb{R} \).

Finally, we contrast this with the regular continued fraction \( \frac{1}{\gamma_\infty} = [2;2,2,\ldots] \). The corresponding cutting sequence \( \ldots \mathbb{L}\mathbb{R}(\mathbb{L})(\mathbb{R})(\mathbb{L})\mathbb{L}\mathbb{R}\ldots \) gives the OCF. We can see the connection between the RCF, GCF, and OCF by noting that

\[
2 + \frac{1}{2 + z} = 1 + \frac{1}{\frac{1}{1 - \frac{1}{3 + z}}} = 3 - \frac{1}{\frac{1}{1 + \frac{1}{1 + z}}}
\]

5. Some applications

5.1. Invariant measures. Section 4 provides a cross-section \( \mathcal{X} \) for the geodesic flow on \( T_1(M_o) \), consisting of unit tangent vectors that point along geodesics with base point \( \xi \) on the closed line of \( M_o = \Gamma \setminus \mathbb{H} = \pi_o(\mathbb{H}) \) going through \( \pi_o(\infty) \) and \( \pi_o(i) \) (see the right-hand side of Fig. 4), and whose cutting sequence changes type at \( \xi \). Proposition 4 shows that the first return map of the geodesic flow to \( \mathcal{X} \) corresponds to the transformation \( \tilde{T}_o \) on \( \tilde{\Omega}_o \).

As in [29], it is convenient to express every \( u \in T_1(\mathbb{H}) \) in coordinates \( (\alpha, \beta, t) \) given by the endpoints \( \alpha, \beta \) of the geodesic \( \gamma(u) \) through \( u \) and the distance \( t \) from the midpoint of \( \gamma(u) \) to \( u \). It is possible to transform the usual measure Haar measure \( y^{-2}dx dy d\theta \) on \( \mathbb{H} \times S^1 \cong T_1\mathbb{H} \cong \text{PSL}(2,\mathbb{R}) \) to \((\alpha - \beta)^{-2}d\alpha d\beta dt \). Section 4 shows that \( S_o \) and \( \mathcal{X} \) are naturally identified, up to a null-set, by mapping \( \gamma \in S_o \) to \( \pi_o(u_\gamma) \in \mathcal{X} \), where \( u_\gamma \in T_1(\mathbb{H}) \) corresponds to \( \gamma \) at position \( \xi_\gamma \). The first return map to \( \mathcal{X} \) is subsequently identified with \( \rho_o \), acting on \( S_o \) with corresponding invariant measure \( \mu = (\alpha - \beta)^{-2}d\alpha d\beta \). We define \( \pi, \pi_1, \pi_2 : \Omega_o \rightarrow (0,1) \) by \( \pi(x,y,\epsilon) = (x,y), \pi_1(x,y,\epsilon) = x, \) and \( \pi_2(x,y,\epsilon) = y \). The push-forward of \( \mu \) under \( \pi \circ J_o \) provides a \( \tilde{T}_o \)-invariant measure \( \tilde{\mu}_o \) on \( \Omega_o \).

The push-forward of \( \mu \) under \( \pi_1 \circ J_o \) provides a \( \tau_o \)-invariant measure \( \nu_o \) on \( I_G \). For every rectangle \( E = [a,b] \times [c,d] \subset \Omega_o \), we have

\[
(\pi \circ J_o)^{-1}(E) = [b^{-1},a^{-1}] \times [-d,-c] \cup [-a^{-1},-b^{-1}] \cup [c,d] \quad \text{and}
\]

\[
\tilde{\mu}_o(E) = \int_{1/b}^{1/a} \int_{-c}^{-d} \frac{d\alpha d\beta}{(\alpha - \beta)^2} + \int_{-1/a}^{-1/b} \int_{c}^{d} \frac{d\alpha d\beta}{(\alpha - \beta)^2} = 2 \int \int_{E} \frac{dx dy}{(1 + xy)^2},
\]
showing that $\bar{\mu}_0 = (1+xy)^{-2}dxdy$ is a finite $T_o$-invariant measure. Using $\frac{1}{G} = G-1$ and $\frac{1}{2-G} = G+1$, we see that for every interval $[a, b] \subset (0, 1)$,

$$\mu_o([a, b]) = \frac{1}{a} \int_a^b 2^{G-2} \frac{d\alpha d\beta}{(\alpha - \beta)^2} + \int_{-1/a}^{-1/b} G \frac{d\alpha d\beta}{G-2 (\alpha - \beta)^2} = \frac{1}{a} \int_a^b 2^{G-2} \frac{du dv}{(1/u-v)^2} + \int_{-1/a}^{-1/b} G \frac{d\alpha d\beta}{G-2 (\alpha - \beta)^2} = 2 \int_a^b \left( \frac{1}{u+G-1} - \frac{1}{u-1} \right) du.$$

This gives that $\mu_o = \left( \frac{1}{u+G-1} - \frac{1}{u-G-1} \right) du$ is a finite $T_o$-invariant measure. Finally, for every interval $[c, d] \subset I_G$ we have

$$\nu_o([c, d]) = \mu_o\left( [1, \infty) \times [-d, -c] \cup (-\infty, -1] \times [c, d] \right) = \int_{-\infty}^{\infty} \int_{-d}^{c} \frac{d\alpha d\beta}{(\alpha - \beta)^2} = 2 \int_{-1}^{1} \frac{1}{u^2} \left( \frac{1}{u+1} + \frac{1}{u-1} \right) du,$$

showing that $\nu_o = \frac{du}{1+u}$ is a finite $T_o$-invariant measure.

5.2. Quadratic surds and their conjugates.

**Proposition 5.** A real number $\alpha > 1$ has a purely periodic OCF expansion if and only if $\alpha$ is a quadratic surd with $-G < \bar{\alpha} < 2 - G$. Furthermore, if

$$\alpha = \ll ([a_1, \epsilon_1]; (a_2, \epsilon_2), \ldots, (a_r, \epsilon_r)]_o, \ (5.1)$$

then

$$\bar{\alpha} = -\ll ([a_r, \epsilon_r], \ldots, (a_1, \epsilon_1)]_o. \ (5.2)$$

**Proof.** In one direction, suppose that $\alpha$ is given by (5.1). Consider the geodesic $\gamma \in A_o$ with endpoints at $\gamma_{-\infty} = \alpha$ and $\gamma_{-\infty} = \beta = -\ll ([a_r, \epsilon_r], \ldots, (a_1, \epsilon_1)]_o \in -I_G$. Corollary 4 shows that the geodesic $\gamma$ is fixed by $\rho^2_{\bar{\alpha}}$, so it is fixed by some $M \in \Gamma$, $M \neq I$. Hence both $\alpha$ and $\beta$ are fixed by $M$; in particular, $\beta = \bar{\alpha}$.

In the opposite direction, suppose that $A\alpha^2 + B\alpha + C = 0$ with $\gcd(A, B, C) = 1$, $A > 1$, and $\bar{\alpha} \in -I_G$. The quadratic surds $\alpha, \bar{\alpha}, -\alpha, -\bar{\alpha} = -\bar{\alpha}$, and $M\alpha = \frac{ao+b}{co+d}$ with $M \in \Gamma$ have the same discriminant. From $\alpha - \bar{\alpha} = \sqrt{\Delta} > G - 1$ and $-2AG < 2A\bar{\alpha} = -B - \sqrt{\Delta} < 2A(2-G)$, we infer that the number of quadratic surds $\alpha$ with fixed discriminant $\Delta = B^2 - 4AC$ and satisfying these restrictions must be finite. Employing equality (3.6), it follows that both components of $\bar{T}_o^k\left( \frac{1}{\alpha}, -\bar{\alpha} \right)$ are quadratic surds with discriminant $\Delta$ for every $k \geq 0$. Since they satisfy the same kind of restrictions as $\alpha$ above, there exist $k, k' > 0, k \neq k'$ such that $\bar{T}_o^k\left( \frac{1}{\alpha}, -\bar{\alpha} \right) = \bar{T}_o^{k'}\left( \frac{1}{\alpha}, -\bar{\alpha} \right)$. The map $\bar{T}_o$ is invertible, hence there exists $r \geq 1$ such that $\bar{T}_o^r\left( \frac{1}{\alpha}, -\bar{\alpha} \right) = \left( \frac{1}{\alpha}, -\bar{\alpha} \right)$, showing that $\alpha$ must be of the form (5.1) and $\bar{\alpha}$ of the form (5.2).

5.3. Action of $\Gamma$ on the real line and continued fractions. Define the $m$-tail of an irrational number $\alpha = \ll ([a_1, \epsilon_1]; (a_2, \epsilon_2), \ldots]_o > 1$ by

$$t_m(\alpha) := (-\epsilon_1) \cdots (-\epsilon_m) \ll ([a_{m+1}, \epsilon_{m+1}]; (a_{m+2}, \epsilon_{m+2}), \ldots]_o.$$

**Proposition 6.** Two irrationals $\alpha, \beta > 1$ are $\Gamma$-equivalent if and only if there exist $r, s \geq 0$ such that

$$t_r(\alpha) = t_s(\beta). \ (5.3)$$
Proof. The proof follows closely the outline of statement 3.3.3 in [29]. In one direction, if (5.3) holds, then α and β are Γ-equivalent because \(-a_1 - \frac{1}{\alpha} = t_1(\alpha)\).

Conversely, suppose that \(g\alpha = \beta\) for some \(g \in \Gamma\). Fix \(\delta \in -I_G\) and consider the geodesics \(\gamma, \gamma' \in A_\infty\) with \(\gamma_\infty = \gamma'_\infty = \delta\), \(\gamma_\infty = \alpha\) and \(\gamma'_\infty = \beta\). Their cutting sequences are \(\ldots \xi_\gamma A_1 A_2 \ldots\) and respectively \(\ldots \xi_\gamma B_1 B_2 \ldots\) with \(A_i, B_i\) strings of type \(A, B, C\) or \(D\). The geodesics \(\gamma'' = g\gamma\) and \(\gamma'\) have the same endpoint \(\beta\). Since their \(\Gamma(1)\)-cutting sequences in \(L\) and \(R\) coincide (cf. [29], Lemma 3.3.1), their cutting sequences also coincide, implying that the cutting sequence of \(\gamma''\) is of the form \(\xi_{\gamma''} \ldots B_k B_{k+1} \ldots\) for some \(k \geq 1\). As \(\gamma\) and \(\gamma''\) are Γ-equivalent geodesics, their cutting sequences (after equivalent initial points) will coincide, implying that the cutting sequences of \(\gamma\) and \(\gamma'\) are of the form \(\ldots \xi_\gamma A_1 \ldots A_r D_1 D_2 \ldots\) and respectively \(\ldots \xi_\gamma B_1 \ldots B_s D_1 D_2 \ldots\). Upon (5.5) this implies (5.3). □

Two rational numbers are Γ-equivalent, as shown by the following elementary

Lemma 7. \(\Gamma_\infty = \mathbb{Q}\).

The problem of characterizing Γ-equivalence classes for a broad class of subgroups of \(\Gamma(1)\) has been recently investigated, with a different approach, in [23].

Proposition 8. The OCF expansion of an irrational \(\alpha\) is eventually periodic if and only if \(\alpha\) is a quadratic surd.

Proof. If the OCF tail of \(\alpha\) is eventually periodic, then \(g\alpha = \epsilon[(a_1, e_1), \ldots, (a_r, e_r)]_o\) for some \(g \in \Gamma\) and \(\epsilon \in \{\pm 1\}\). Proposition 6 gives that \(g\alpha\) is a quadratic surd, hence \(\alpha\) is a quadratic surd.

Conversely, assume that \(\alpha\) is a quadratic surd and \(\gamma\) is the geodesic connecting \(\alpha\) to its conjugate root \(\overline{\alpha}\). Let \(g \in \Gamma\) such that the geodesic \(g\gamma\) is a lift of \(\pi(\gamma)\) to \(\mathbb{H}\) with \(g\gamma \in S_o\). We can assume that \(g > 1\), reversing \(\alpha\) and \(\overline{\alpha}\) if necessary. Proposition 8 shows that \(g\alpha\) has purely periodic OCF expansion, and Proposition 7 shows that the OCF expansion of \(\alpha\) is eventually periodic. □

Proposition 8 is known in more general situations, for instance it holds for all \(D\)-continued fractions (see, e.g., [19], [22]).

5.4. Closed geodesics on \(M_o\). Employing Proposition 6 and standard arguments, one can prove

Proposition 9. A geodesic \(\tilde{\gamma}\) on \(M_o\) is closed if and only if it has a lift \(\gamma \in A_o\) with purely periodic endpoints

\[
\gamma_\infty = \epsilon[(a_1, e_1), (a_2, e_2), \ldots, (a_r, e_r)]_o \quad \text{and} \quad \gamma_\infty = -\epsilon[(a_r, e_r), (a_2, e_2), (a_1, e_1)]_o
\]

for some \(\epsilon \in \{\pm 1\}\) and \((-\epsilon_1) \cdots (-\epsilon_r) = 1\).

5.5. The roof function and length of closed geodesics on \(M_o\). Our construction describes the geodesic flow on \(T_1 M_o\) as a suspension flow over the measure preserving transformation \((\widehat{\Omega}_o, \widehat{T}_o, \widehat{\mu}_o)\) identified with \((S_o, \rho_o, (\alpha - \beta)^{-2}d\alpha d\beta)\). The roof function is given by the hyperbolic distance between two consecutive return points to \(S_o\):

\[
r_o(\xi_\gamma) = d(\xi_\gamma, \eta_\gamma),
\]

with \(\xi_\gamma, \eta_\gamma \in \mathbb{H}\) as in Section 3.2. The points \(\xi_\gamma\) and \(\eta_\gamma\) can also be identified with elements in \(T_1 M_o\) and represent two consecutive changes in type for the cutting sequence of the geodesic \(\gamma\). It is convenient to replace the geodesic arc \([\xi_\gamma, \eta_\gamma]\) with \([\xi, \eta]\), where \(\xi = \rho_o(\xi_\gamma), \eta = \rho_o(\eta_\gamma)\). Considering \(\gamma_\pm = \rho_o(\gamma_\pm\infty)\), we employ as in [29] the formula

\[
d(\xi_\gamma, \eta_\gamma) = d(\xi, \eta) = \log \left| \frac{\gamma_- - \eta_-}{\gamma_- - \xi_-} \cdot \frac{\gamma_+ - \xi}{\gamma_+ - \eta} \right|.
\]
Figure 6. The geodesic arc $[\xi, \eta]$ between consecutive return points in $S_0$

Assume first that $\gamma$ is in case A, which means $\rho_o(x) = \frac{1}{a_1-x}$ and $\xi = x + iy = [0, \beta] \cap [\gamma_-, \gamma_+]$, where $\beta = \frac{1}{a_1-1}$ and $\eta = [1, \infty] \cap [\gamma_-, \gamma_+]$. Trigonometry of right triangles with vertices $\gamma_-, \eta, \gamma_+$ and respectively $\gamma_-, \xi, \gamma_+$ provides

$$\frac{|\gamma_+ - \eta|}{|\gamma_+ - \eta|} = \sqrt{\frac{1 - \gamma_-}{\gamma_+ - 1}} \text{ and } \frac{|\gamma_+ - \xi|}{|\gamma_+ - \xi|} = \frac{\sqrt{\gamma_+ - \eta}}{\gamma_+ - \gamma_-}.$$  

The equalities $|x + iy - \frac{\beta}{2}| = \frac{\beta}{2}$ and $|x + iy - \frac{1}{2}(\gamma_+ + \gamma_-)| = \frac{1}{2}(\gamma_+ - \gamma_-)$ lead to

$$x = \text{Re} \xi = \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_- - \beta},$$

and thus

$$d_A(\xi, \eta) = \frac{1}{2} \log \left( \frac{\gamma_+ - x}{\gamma_- - x} \right) = \frac{1}{2} \log \left( \frac{\gamma_+ - 1}{\gamma_- - x} \right) = \frac{1}{2} \log \left( \frac{\gamma_+ - 1}{\gamma_- - x} \right).$$

Employing $\frac{1}{2} - \rho_o(x) = x - 1$, we gather

$$d_A(\xi, \eta) = \frac{1}{2} \log \left( \frac{F_\infty(\gamma)}{F_\infty(\gamma_-)} \right), \text{ where } F_A(x) = \rho_o(x)^2 \frac{x - 1}{\rho_o(x) - 1}.$$  

Similar computations in each of the cases B, C, D provide

$$F_B(x) = \rho_o(x)^2 \frac{x - 1}{\rho_o(x) + 1}, \quad F_C(x) = \rho_o(x)^2 \frac{x + 1}{\rho_o(x) + 1}, \quad F_D(x) = \rho_o(x)^2 \frac{x + 1}{\rho_o(x) - 1},$$

and actually we get the general formula

$$d(\xi, \eta) = \frac{1}{2} \log \left( \frac{F_\infty(\xi)}{F_\infty(\xi)} \right) \text{ with } F(x) = F_\gamma(x) = \rho_o(x)^2 \frac{x - \text{sign}(\gamma_\infty)}{\rho_o(x) - \text{sign}(\rho_0(\gamma_\infty))}. \quad (5.4)$$

If $(\alpha, \beta) \in S_0$, then there exists $\epsilon \in \{\pm 1\}$ such that $(\epsilon \alpha, \epsilon \beta) = (\alpha', \beta')$ with

$$\alpha' = [(a_1, \epsilon_1); (a_2, \epsilon_2), \ldots]_o > 1, \quad \beta' = -\langle(a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots\rangle_o \in -I_G.$$

Equations (5.4) and (3.5) then provide

$$F(\alpha) = \rho_o(\alpha)^2 (-\epsilon_1) \left[ (a_1, \epsilon_1); (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots \right]_o - 1 \left[ (a_2, \epsilon_2); (a_3, \epsilon_3), (a_4, \epsilon_4), \ldots \right]_o - 1 = \rho_o(\alpha)^2 F_+(\alpha),$$

$$F(\beta) = \rho_o(\beta)^2 (-\epsilon_1) \left\langle (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), (a_{-2}, \epsilon_{-2}), \ldots \right\rangle_o + 1 \left\langle (a_1, \epsilon_1), (a_0, \epsilon_0), (a_{-1}, \epsilon_{-1}), \ldots \right\rangle_o + 1 = \rho_o(\beta)^2 F_-(\beta). \quad (5.5)$$
For C and D, we write $\gamma$

Every geodesic $\bar{\gamma}$ and

while $H$ with endpoints

4. However, in this case we do not use a checkerboard coloring. Here

of the factors

$\rho_{\gamma,0}(\bar{\gamma}) = \frac{\xi_{\gamma}(n)}{2} \rho_{\gamma,0}(\gamma)$

When $\bar{\gamma}$ is a closed geodesic on $\mathcal{M}_o$ with endpoints as in Proposition 1, the contribution of each of the factors $F_+$ and $F_-$ to the length of $\bar{\gamma}$ is one. Using $(a_{n+r}, \epsilon_n) = (a_n, \epsilon_n)$, we find

$$\text{length}(\bar{\gamma}) = \log \left( \prod_{k=1}^{r} \frac{(a_{k+1}, \epsilon_{k+1}); (a_{k+2}, \epsilon_{k+2}), \ldots, (a_{k+r}, \epsilon_{k+r})}{(a_{k}, \epsilon_{k}), (a_{k-1}, \epsilon_{k-1}), \ldots, (a_{k+r-1}, \epsilon_{k+r-1})} \right)^{1/2} = \log \left( \frac{\rho_{\gamma,0}(\gamma)}{\rho_{\gamma,0}(\gamma^{-1})} \right).$$

(5.6)

6. Geodesic coding and even continued fractions

6.1. The group $\Theta$ and the modular surface $\mathcal{M}_e = \Theta \backslash \mathbb{H}$. The Theta group

$$\Theta := \left\{ M \in \Gamma(1) : M \equiv I_2 \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ (mod 2)} \right\},$$

is the index three subgroup of $\Gamma(1)$ generated by $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The image of the left half of the standard Dirichlet region $\{ |\text{Re } z| < 1, |z| > 1 \}$ of $\Theta \backslash \mathbb{H}$ under the transformation $S$ coincides with the quarter-disc $\{ \text{Re } z > 0, |z| < 1, |z - \frac{1}{2}| > \frac{1}{2} \}$. As a result, the standard Farey cell $\{ 0 < \text{Re } z < 1, |z - \frac{1}{2}| > \frac{1}{2} \}$ provides a fundamental domain for $\mathcal{M}_e = \Theta \backslash \mathbb{H} = \pi_\gamma(\mathbb{H})$.

Bauer and Lopes [7] realized the ECF natural extension $\tilde{T}_e$ as a section of the billiard flow on $\mathcal{M}_e$. Here, we describe the extension $\tilde{T}_e$ of $\tilde{T}_e$ as a section of the geodesic flow on $\mathcal{M}_e$.

The coding of geodesics on $\Theta \backslash \mathbb{H}$ is analogous to the coding for $\Gamma \backslash \mathbb{H}$ described in Sections 3 and 4. However, in this case we do not use a checkerboard coloring. Here $\mathcal{A}_e$ is the set of geodesics in $\mathbb{H}$ with endpoints

$$\{(\gamma_\infty, \gamma_{-\infty}) \in \mathcal{S}_e := ((-\infty, -1) \cup (1, \infty)) \times (-1, 1),$$

while $\xi_\gamma$ and $\eta_\gamma$ are defined by

$$\xi_\gamma = \begin{cases} \gamma \cap [1, \infty] & \text{if } \gamma_\infty > 1 \\
\gamma \cap [-1, \infty] & \text{if } \gamma_\infty < -1 \end{cases}$$

$$\eta_\gamma = \begin{cases} \gamma \cap [a_1, a_1 + \epsilon_1] & \text{if } \gamma_\infty > 1 \\
\gamma \cap [-a_1, -a_1 - \epsilon_1] & \text{if } \gamma_\infty < -1 \end{cases}$$

Every geodesic $\bar{\gamma}$ on $\mathcal{M}_e$ lifts to a geodesic $\gamma \in \mathcal{A}_e$. As in Section 5.2, four cases can occur, depicted in Figs. 7 and 8. Again, we consider cases A, B, C, and D. Here, case A corresponds to $\gamma_\infty \in (2k-1, 2k)$ and $\epsilon_1 = -1$, case B to $\gamma_\infty \in (2k, 2k+1)$ and $\epsilon_1 = +1$, case C to $\gamma_\infty \in (-2k, -2k+1)$ and $\epsilon_1 = -1$, and case D to $\gamma_\infty \in (-2k-1, -2k)$ and $\epsilon_1 = +1$.

In cases A and B, we write $\gamma_\infty = \ll (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \rr \epsilon$ and $\gamma_{-\infty} = -\ll (a_0, \epsilon_0), (a_0, \epsilon_0), \ldots \rr \epsilon$. For C and D, we write $\gamma_\infty = -\ll (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \rr \epsilon$ and $\gamma_{-\infty} = \ll (a_0, \epsilon_0), (a_0, \epsilon_0), \ldots \rr \epsilon$.

When $\gamma_\infty > 1$, the Möbius transformation $\rho_e(z) = \frac{1}{a_1 - z}$ belongs to $\Theta$ and

$$\rho_e(\gamma_\infty) = e_1 \ll (a_1, \epsilon_1), (a_0, \epsilon_0), \ldots \rr \epsilon \in (0, 1) \quad \text{and} \quad \rho_e(\gamma_\infty) = (-1) \ll (a_2, \epsilon_2), (a_3, \epsilon_3), \ldots \rr \epsilon.$$
When $\gamma_\infty < -1$ and the transformation $\rho_e(z) = \frac{1}{a_1 - z}$ belongs to $\Theta$ and

$$\rho_e(\gamma_{-\infty}) = -\epsilon_1 \langle (a_1, \epsilon_1), (a_0, \epsilon_0), \ldots \rangle_e \in (-1, 0) \quad \text{and} \quad \rho_e(\gamma_\infty) = \epsilon_1 \langle (a_2, \epsilon_2); (a_3, \epsilon_3), \ldots \rangle_e.$$  

In all cases we have

$$\rho_e(\epsilon \langle (a_1, \epsilon_1); (a_2, \epsilon_2), \ldots \rangle_e, -\epsilon \langle (a_0, \epsilon_0), (a_1, \epsilon_1), (a_2, \epsilon_2), \ldots \rangle_e)$$

$$= -\epsilon_1 \epsilon \langle (a_2, \epsilon_2); (a_3, \epsilon_3), \ldots \rangle_e - \langle (a_1, \epsilon_1), (a_0, \epsilon_0), \ldots \rangle_e \quad \text{if} \ \epsilon \in \{\pm 1\}. \quad (6.1)$$

Again, we get that when $\epsilon_1 = +1$, $\rho_e(\gamma_\infty)$ and $\gamma_\infty$ are on opposite sides of the imaginary axis, and $\rho_e$ agrees with the transformation $\rho$ for the RCF in [29]. As in the OCF case, this is exactly where the RCF and ECF agree, cases A and C are followed by A or B, and cases B and D are followed by cases C or D.

The map $J_e : S_e \to \bar{\Omega}_e$, $J_e(x,y) = \text{sign}(x)(1/x, -y, 1)$ is invertible. Direct verification reveals

$$J_e \rho_e J_e^{-1} = \bar{T}_e. \quad (6.2)$$

As in Section 5.1 the push-forwards of the measure $(\alpha - \beta)^{-2} d\alpha d\beta$ on $S_e$ under the maps $\pi \circ J_e$, $\pi_1 \circ J_e$, and $\pi_2 \circ J_e$ are $\bar{T}_e$-invariant, $T_e$-invariant, and respectively $\tau_e$-invariant. For intervals $[a, b] \subset (0, 1)$, $[c, d] \subset (-1, 1)$ and $E = [a, b] \times [c, d]$, we find

$$\mu_e(E) = 2 \int_E \frac{dxdy}{(1 + xy)^2}; \quad \nu_e([c, d]) = 2 \int_{c+} \frac{dv}{1 + v};$$

$$\mu_e([a, b]) = \int_{1/a}^{1/b} \int_{-1}^{1} \frac{d\alpha d\beta}{(\alpha - \beta)^2} + \int_{-1/a}^{-1/b} \int_{-1}^{1} \frac{d\alpha d\beta}{(\alpha - \beta)^2} = \int_{a}^{b} \left( \frac{1}{1 + u} + \frac{1}{1 - u} \right) du,$$

which coincide with the invariant measures from [27].

The analogues of Propositions 5, 6, 8, and 9 come from changing the subscript $o$ to $e$, since the same equalities hold.

**Proposition 10.** (i) A real number $\alpha > 1$ has a purely periodic ECF expansion if and only if $\alpha$ is a quadratic surd with $-1 < \tilde{\alpha} < 1$. Furthermore, if

$$\alpha = \langle (a_1, \epsilon_1); (a_2, \epsilon_2), \ldots, (a_r, \epsilon_r) \rangle_e,$$

then

$$\tilde{\alpha} = -\langle (a_r, \epsilon_r), \ldots, (a_1, \epsilon_1) \rangle_e.$$

(ii) Two irrational numbers $\alpha, \beta > 1$ are $\Theta$-equivalent if and only if there are $r, s \geq 0$ such that

$$t_r(\alpha) = t_s(\beta), \quad (6.3)$$

where $t_m(\langle (a_1, \epsilon_1); (a_2, \epsilon_2), \ldots \rangle_e) := (-\epsilon_1) \cdots (-\epsilon_m) \langle (a_{m+1}, \epsilon_{m+1}) \ldots \rangle_e$. 

*Figure 8.* ECF cases C ($\gamma_\infty < -1, \epsilon_1 = -1$) and D ($\gamma_\infty < -1, \epsilon_1 = +1$)
(iii) The ECF expansion of an irrational \( \alpha \) is eventually periodic if and only if \( \alpha \) is a quadratic surd.

(iv) A geodesic \( \tilde{\gamma} \) on \( M_\varepsilon \) is closed if and only if it has a lift \( \gamma \in A_\varepsilon \) with purely periodic endpoints

\[
\gamma_{\infty} = \varepsilon \langle (a_1, e_1); (a_2, e_2), \ldots, (a_r, e_r) \rangle_e \quad \text{and} \quad \gamma_{-\infty} = -\varepsilon \langle (a_r, e_r), \ldots, (a_2, e_2), (a_1, e_1) \rangle_e
\]

for some \( \varepsilon \in \{ \pm 1 \} \) and \((-\varepsilon_1) \cdots (-\varepsilon_r) = 1\).

The second point should be compared with the statement of Theorem 1 in [17]. It seems that the definition of the tail \( t_n(x) \) in [17], Eq. (1.3) should be changed to

\[
t_n(x) = (-\varepsilon_1) \cdots (-\varepsilon_n)[0; \varepsilon_{n+1}/a_{n+1}, \varepsilon_{n+2}/a_{n+2}, \ldots]
\]

for that statement to hold.

Analogous formulas for the roof function and for the length of a closed geodesic also hold, as in (5.4) and (5.5). Since \( \langle (b_0, \varepsilon_0), (b_1, \varepsilon_1), \ldots \rangle_e^{-1} = \varepsilon_0 \langle (b_0, \varepsilon_1), (b_1, \varepsilon_2), (b_2, \varepsilon_3), \ldots \rangle_e \), the analogue of (5.6) shows that the length of a closed geodesic \( \tilde{\gamma} \) on \( M_\varepsilon \) with endpoints as in (6.4) is given by

\[
\log \left( \prod_{k=1}^{r} \left( \frac{[a_{k+1}, e_{k+1}]; (a_{k+2}, e_{k+2}), \ldots, (a_{k+r}, e_{k+r})]}{[a_k, e_k]; (a_{k-1}, e_{k-2}), \ldots, (a_{k-r+1}, e_{k-r})]_e^2} \right)^{2}\prod_{k=1}^{r} \left( \frac{[a_{k+1}, e_{k+1}]; (a_{k+2}, e_{k+2}), \ldots, (a_{k+r}, e_{k+r})]}{[a_k, e_k]; (a_{k-1}, e_{k-2}), \ldots, (a_{k-r+1}, e_{k-r})]_e^2} \right)^{2}
\]

(6.5)

In this case \( M_\varepsilon \) has two cusps, \( \pi_{\varepsilon}(\infty) \) and \( \pi_{\varepsilon}(1) \), and the group \( \Theta \) splits the rationals in two equivalence classes, as shown by the following elementary

**Lemma 11.** \( \Theta_{\infty} = \left\{ \frac{m}{n} \in \mathbb{Q} : m \text{ or } n \text{ is even} \right\} \) and \( \Theta 1 = \left\{ \frac{m}{n} \in \mathbb{Q} : m \text{ and } n \text{ are odd} \right\} \).

6.2. **Connection with cutting sequence.** As in Section 4 we connect the action of \( \rho_{\varepsilon} \) with the cutting sequence associated to \( \gamma \). We use the same coding as in [29] and in Section 3.2 but do not use the checkerboard coding.

As before, cases A and B give the cutting sequence \( \ldots x L^{n_1} R^{n_2} L^{n_3} \ldots \), where \( x \) indicates \( \gamma \in [0, \infty] \) and \( [n_1; n_2, n_3 \ldots] \) is the RCF expansion of \( \gamma_{\infty} \). This again corresponds to \( \ldots L^2 R_{\varepsilon} L^{n_1-1} R^{n_2} L^{n_3} \ldots \).

We have two cases to consider for the first digit of the ECF expansion.

(A) \( n_1 = 2k - 1 \) is odd, and \( \gamma_{\infty} \in (2k - 1, 2k) \). This gives the cutting sequence \( \ldots \xi_{\varepsilon} L^{2k-2} R_{\varepsilon} \gamma \) and we get \( (2k, -1) = (a_1, \varepsilon_1) \). The next digit is represented by \( L^n R \). In the case \( n_2 > 1 \), that is the string \( \ldots \xi_{\varepsilon} L^{n_1-1} R_{\varepsilon} R^{n_2-1} \ldots \), the next digit is \( L^n R \), corresponding to \( (2, -1) \). This corresponds to the insertion and singularization algorithm for even continued fractions described by Kraaikamp and Lopes in [17].

\[
2k - 1 + \frac{1}{n_2 + \ldots} = [(2k, -1); (2, -1)^{n_2-1}, (n_2, +1), \ldots]_e
\]

\[
2k - 1 + \frac{1}{1 + \frac{1}{n_3 + \ldots}} = [(2k, -1); (n_3 + 1, +1), \ldots]_e
\]

where \( (2, -1)^t \) means the digit \( (2, -1) \) \( t \)-times.

(B) \( n_1 = 2k \) is even, and \( \gamma_{\infty} \in (2k, 2k + 1) \). This gives the cutting sequence \( \ldots \xi_{\varepsilon} L^{2k-1} R_{\varepsilon} \gamma \) and we get \( (2k, +1) = (a_1, \varepsilon_1) \). The next digit is represented by \( R^n L \). In the first case, we needed to consider what happened when \( n_2 > 1 \), here we need to look at \( n_2 = 1 \). Then, we have \( \ldots L^{n_1-1} R_{\varepsilon} L^{n_3} \ldots \), and the next digit corresponds to \( R^{n_1} L \), which is again \( (2, -1) \).

Strings starting with \( R \) are treating similarly, with the roles of \( L \) and \( R \) switched. In this case, we get the cutting sequence \( \ldots \xi_{\varepsilon} R^{n_1-1} L_{\varepsilon} L^{n_2-1} R^{n_3} \ldots \), where \( \gamma_{\infty} = -[n_1; n_2, n_3, \ldots] \).
6.3. Extended even continued fractions. For the end point $\gamma_{-\infty}$, we must consider the extended even continued fractions. While odd and grotesque continued fraction expansions differ from each other, the extended even continued fractions give the same expansion as the even continued fractions.

We consider cases A, B, C, and D similar to the grotesque continued fraction case.

(A) $\gamma_{\infty} > 1, \gamma_{-\infty} \in (0, 1)$, and $\epsilon_0 = -1$ gives the cutting sequence $\ldots L^{2k-2}R\xi_\gamma \ldots$ and $(a_0, \epsilon_0) = (2k, +1)$.

(B) $\gamma_{\infty} > 1, \gamma_{-\infty} \in (-1, 0)$, and $\epsilon_0 = +1$ gives the cutting sequence $\ldots R^{2k-1}L\xi_\gamma \ldots$ and $(a_0, \epsilon_0) = (2k, -1)$.

(C) $\gamma_{\infty} < -1, \gamma_{-\infty} \in (0, 1)$, and $\epsilon_0 = +1$ gives the cutting sequence $\ldots L^{2k-1}R\xi_\gamma \ldots$ and $(a_0, \epsilon_0) = (2k, +1)$.

(D) $\gamma_{\infty} < -1, \gamma_{-\infty} \in (-1, 0)$, and $\epsilon_0 = -1$ gives the cutting sequence $\ldots R^{2k-2}L\xi_\gamma \ldots$ and $(a_0, \epsilon_0) = (2k, -1)$.

That is, we interpret the strings the same way as in the even continued fractions. However, strings of type A and C must be preceded by those of type A or B, and strings of type B and D must be preceded by C or D. This is similar to the restrictions on the grotesque continued fraction from Section 4.2.

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