STABILITY AND ASYMPTOTIC PROPERTIES OF DISSIPATIVE EVOLUTION EQUATIONS COUPLED WITH ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we obtain some stability results of systems corresponding to the coupling between a dissipative evolution equation (set in an infinite dimensional space) and an ordinary differential equation. Many problems from physics enter in this framework, let us mention dispersive medium models, generalized telegraph equations, Volterra integro-differential equations, and cascades of ODE-hyperbolic systems. The goal is to find sufficient (and necessary) conditions on the involved operators that guarantee stability properties of the system, i.e., strong stability, exponential stability or polynomial one. We also illustrate our abstract statements for different concrete examples, where new results are achieved.

1. Introduction. In this paper we analyze the stability of different systems corresponding to the coupling between a dissipative evolution equation (set in an infinite dimensional space) and an ordinary differential equation. Namely we consider $U, P$ solution of the system

$$\begin{cases}
U_t = AU + MP, & \text{in } H, \\
P_t = BP + NU, & \text{in } X,
\end{cases}$$

(1.1)

where $A$ is an unbounded operator that is the generator of a $C_0$ semigroup in the Hilbert space $H$, $B$ is a bounded operator from another Hilbert space $X$, and $M, N$ are supposed to be bounded operators. Many problems from physics enter in this framework, let us mention dispersive medium models \[32, 34, 55, 43, 46, 31\], the generalized telegraph equations $\[28, 29, 44, 26\]$, the heat equation with memory effects $\[20, 50, 35\]$, and cascades of ODE-hyperbolic systems $\[30, 36\]$, see below for more explanations.

Eliminating $P$ by the formula

$$P(t) = e^{Bt}P_0 + \int_0^t e^{B(t-s)}U(s) \, ds,$$

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we see that the first equation of (1.1) becomes

\[ U_t = A u + Me^{Bt} P_0 + \int_0^t Me^{B(t-s)} U(s) \, ds. \]  

(1.2)

Hence, the last term of this right-hand side is of memory type. But it has not the standard form studied in the literature concerning viscoelastic problems [18, 17, 21, 22, 16, 1].

Coming back to the different physical models mentioned above, we may notice similarities in the proofs of stability results made in each situation. Therefore, a unified analysis would have the advantage to obtain general statements that can be applied in different situations (avoiding to repeat many times the same arguments).

The present paper fills this gap and finds sufficient (and necessary) conditions on the involved operators \( A, B, M \) and \( N \) that guarantee stability properties of system (1.1), i.e., strong stability, exponential stability or polynomial one. More precisely, we first concentrate on strong stability results of (1.1) where, under different dissipativeness assumptions, we find sufficient conditions that guarantee the strong stability of the semigroup \( T(t) \) associated with (1.1). In a second step, we analyze the exponential stability of the semigroup: the first main result is that, up to some technical assumptions, \( T(t) \) is exponentially stable in \( H \times X \) if and only if the semigroup generated by \( A \) is exponentially stable in \( H \), the second main result says that, up to some technical assumptions, if the semigroup generated by \( A \) is exponentially stable in an invariant subspace of \( H \), then \( T(t) \) is exponentially stable in an invariant subspace of \( H \times X \). We further obtain a third main result about polynomial decay, namely if there exists a bounded operator \( C \) from \( X \) to \( H \) such that \( A + CN \) generates polynomially decaying semigroup on an invariant subspace of \( H \), then, up to some technical assumptions, \( T(t) \) is also polynomially decaying in an invariant subspace of \( H \times X \). Finally our abstract statements allow to achieve new results for dispersive medium models, the generalized telegraph equations, the heat equation with memory effects, and an ODE-hyperbolic system.

The paper is organized as follows: in section 2 we give the exact general setting and briefly discuss well-posedness of the problem. Section 3 is devoted to the proof of different strong stability results, while section 4 provides some exponential or polynomial decay results. Finally in section 5 different illustrative examples are treated.

Let us finish this introduction with some notation used in the remainder of the paper: The usual norm and semi-norm of \( H^s(\Omega) \) \( (s \geq 0) \) are denoted by \( || \cdot ||_{s,\Omega} \) and \( | \cdot |_{s,\Omega} \), respectively. For \( s = 0 \) we drop the index \( s \). Similarly \( (\cdot,\cdot) \) (resp. \( || \cdot || \)) denotes the Euclidean inner product (resp norm) in \( \mathbb{C}^k \), for some \( k \in \mathbb{N}^* \). For a (unbounded) operator \( A \) in a Hilbert space \( H \), \( \sigma(A) \) will denote its spectrum, while \( \rho(A) = \mathbb{C} \setminus \sigma(A) \) will be its resolvent set. For two non-negative quantities \( \Phi \) and \( \Psi \) (that may vary at different occurences and may depend on a real parameter \( \xi \), the space variable \( x \) and the time variable \( t \)), by \( \Phi \lesssim \Psi \), we mean that there exists a constant \( C > 0 \) independent of \( \Phi, \Psi \), a real parameter \( \xi \), the space variable \( x \) and the time variable \( t \) such that \( \Phi \leq C \Psi \). By \( \Phi \gtrsim \Psi \), we mean that \( \Psi \lesssim \Phi \) and by \( \Phi \sim \Psi \), we mean that both \( \Phi \lesssim \Psi \) and \( \Phi \gtrsim \Psi \) hold.

2. Well-posedness of the system. Well-posedness of problem (1.1) is direct using a standard perturbation argument. Before its statement, we recall some notations and the exact assumptions. We fix two Hilbert spaces \( H \) and \( X \) with the respective inner product denoted by \( (\cdot,\cdot)_H \) and \( (\cdot,\cdot)_X \) and associated norm \( || \cdot ||_H \)
and \( \| \cdot \|_X \). We suppose given an (unbounded) operator \( A \) from \( H \) into itself with domain \( D(A) \) that generate a \( C_0 \) semigroup, and three operators
\[
B : X \to X, \\
M : X \to H, \\
N : H \to X,
\]
that are supposed to be bounded.

With these assumptions, we can introduce the (unbounded) operator \( A \) from \( H \times X \) into itself as follows
\[
D(A) = D(A) \times X,
\]
and
\[
A(U, P)^\top = \begin{pmatrix} AU + MP \\ BP + NU \end{pmatrix}, \forall (U, P)^\top \in D(A).
\]
This allows to recast (1.1) as the Cauchy problem
\[
\begin{cases}
U_t = AU \text{ in } H \times X, \\
U(0) = (U_0, P_0)^\top,
\end{cases}
\]
with \( U = (U, P)^\top \).

Now we can state the following existence result.

**Theorem 2.1.** Under the previous assumptions, the operator \( A \) defined above generates a \( C_0 \)-semigroup \( T(t) \) on \( H \times X \). Therefore, for all \( U_0 \in H \times X \), problem (2.1) has a unique (weak) solution \( U \in C([0, \infty), H \times X) \) given by \( U = T(t)U_0 \).

Moreover, if \( U_0 \in D(A^\ell) \), with \( \ell \in \mathbb{N}^* \), then problem (2.1) has a unique (strong) solution \( U \in C([0, \infty), D(A^\ell)) \cap C^1([0, \infty), D(A^{\ell - 1})) \).

**Proof.** We define the operator \( A_0 : D(A) \to H \) by
\[
A_0(U, P)^\top = \begin{pmatrix} AU \\ 0 \end{pmatrix}, \forall (U, P)^\top \in D(A).
\]
Since \( A \) generates a \( C_0 \)-semigroup \( S(t) \) on \( H \), we directly see that \( A_0 \) generates a \( C_0 \)-semigroup \( T_0(t) \) on \( H \times X \) given by
\[
T_0(t)(U_0, P_0)^\top = (S(t)U_0, P_0)^\top, \forall (U_0, P_0)^\top \in H \times X.
\]

As \( A - A_0 \) is a bounded operator, a standard perturbation argument (see [47, Theorem 3.1.1] for instance) allows to conclude that \( A \) also generates a \( C_0 \)-semigroup on \( H \times X \).

3. **Strong stability.** One simple way to prove the strong stability of (2.1) is to use the following theorem due to Arendt & Batty and Lyubich & Vu (see [4, 39]).

**Theorem 3.1** (Arendt & Batty/Lyubich & Vu). Let \( X \) be a reflexive Banach space and \( (T(t))_{t \geq 0} \) be a \( C_0 \) semigroup generated by \( A \) on \( X \). Assume that \( (T(t))_{t \geq 0} \) is bounded and no eigenvalues of \( A \) lie on the imaginary axis. If \( \sigma(A) \cap i\mathbb{R} \) is countable, then \( (T(t))_{t \geq 0} \) is strongly stable, i.e.,
\[
\lim_{t \to \infty} \| T(t)x \|_X = 0, \forall x \in X.
\]
We now want to take advantage of this Theorem. Hence, the first step is to show that the semigroup $T(t)$ is bounded, namely that there exists a positive real number $M$ such that

$$
\|T(t)(U,P)^\top\|_{H \times X} \leq M\|(U,P)^\top\|_{H \times X}, \forall (U,P)^\top \in H \times X.
$$

(3.1)

In a second step, since the resolvent of our operator is not compact in general, we have to analyze the full spectrum of $\mathbb{A}$ on the imaginary axis.

To prove the boundedness property of the semigroup, we can use a criterion on the resolvent of $\mathbb{A}$ [5, Theorem 5.2.1], which may be a difficult task. A more restrictive condition, but satisfied in many applications, is to assume that $\mathbb{A}$ is dissipative, namely that

$$
\Re(\mathbb{A}(U,P)^\top,(U,P)^\top)_{H \times X} \leq 0, \forall (U,P)^\top \in D(A) \times X.
$$

(3.2)

Indeed in such a case, by Lumer-Phillips’ theorem it generates a $C_0$-semigroup of contractions on $H \times X$ since for $\lambda > 0$ large enough, by Theorem 2.1, $\lambda I - \mathbb{A}$ is an isomorphism. Let us notice that in such a case, the resolvent set $\rho(\mathbb{A})$ contains the open half-plane $\{ \lambda \in \mathbb{C} | \Re \lambda > 0 \}$. Therefore, the use of Theorem 3.1 is reduced to the analysis of $\rho(\mathbb{A}) \cap i\mathbb{R}$.

Let us now give a necessary condition for the dissipativeness of $\mathbb{A}$ and then a necessary and sufficient condition under a condition between $M$ and $N$.

**Lemma 3.2.** If $\mathbb{A}$ is dissipative, then $\mathbb{A}$ and $B$ are dissipative. Furthermore, if $M = -N^*$, then $\mathbb{A}$ is dissipative if and only if $\mathbb{A}$ and $B$ are dissipative.

**Proof.** Notice that we trivially have

$$
\Re(\mathbb{A}(U,P)^\top,(U,P)^\top)_{H \times X} = \Re(AU,U)_H + \Re(BP,P)_X + \Re(MP,U)_H + \Re(NU,P)_X,
$$

(3.3)

for all $(U,P)^\top \in D(A) \times X$.

For the first assertion, it suffices to take pairs $(U,0)^\top$ with $U \in D(A)$ and $(0,P)^\top$ with $P \in X$ to get the conclusion.

The second assertion follows similarly because (3.3) reduces to

$$
\Re(\mathbb{A}(U,P)^\top,(U,P)^\top)_{H \times X} = \Re(AU,U)_H + \Re(BP,P)_X,
$$

if $M = -N^*$.

$\square$

Let us now give other necessary conditions.

**Lemma 3.3.** If $\mathbb{A}$ is dissipative, then the following properties hold

1. $$(M + N^*)P_0 = 0, \forall P_0 \in \ker(B + B^*).$$

(3.4)

In particular, if $B$ is skew-Hermitian (i.e. $B^* = -B$), then $M + N^* = 0$.

2. $$(M^* + N)U_0 = 0, \forall U_0 \in \ker(A + A^*).$$

(3.5)

In particular, if $A$ is skew-Hermitian (i.e. $A^* = -A$), then $M + N^* = 0$.

**Proof.** Let us prove the first assertion, since the second is fully similar. Fix $P_0 \in \ker(B + B^*)$. For an arbitrary real number $\varepsilon$ different from zero, in (3.3), we take $P = \varepsilon P_0$ and $U \in D(A)$ arbitrary. Then due to (3.2), we get

$$
\Re(AU,U)_H + \varepsilon\Re((M + N^*)P_0,U)_H \leq 0.
$$
because $\Re(BP_0, P_0)_X = ((B + B^*)P_0, P_0)_X = 0$. Dividing by $\varepsilon$, we find

$$
\varepsilon^{-1}\Re(AU, U)_H + \Re((M + N^*)P_0, U)_H \leq 0, \forall \varepsilon > 0,
$$

$$
\varepsilon^{-1}\Re(AU, U)_H + \Re((M + N^*)P_0, U)_H \geq 0, \forall \varepsilon < 0.
$$

Letting $\varepsilon$ goes to $+\infty$ (resp. $-\infty$), we then find

$$
\Re((M + N^*)P_0, U)_H = 0, \forall U \in D(A).
$$

As for $U \in D(A)$, $iU$ also belongs to $D(A)$, and because $\Re((M + N^*)P_0, iU)_H = \Im((M + N^*)P_0, U)_H$, we obtain

$$
((M + N^*)P_0, U)_H = 0, \forall U \in D(A).
$$

Since $D(A)$ is dense in $H$, we conclude that (3.4) holds.

**Corollary 3.4.** If $B$ is skew-Hermitian, then $A$ is dissipative if and only if $A + M + N^* = 0$. Conversely if $A$ is skew-Hermitian, $A$ is dissipative if and only if $B$ is dissipative and $M + N^* = 0$.

Let us go on with the analysis of the full spectrum of $A$ on the imaginary axis under some additional assumptions that are satisfied by many examples. Our approach is based on the characterization of the point spectrum of $A$ and its adjoint as well as on the closedness of the range of $i\xi I - A$ for appropriate $\xi \in \mathbb{R}$.

The proof of following characterization of the adjoint of $A$ is an easy exercise, and is then left to the reader.

**Exercise 3.5.** The adjoint $A^*$ of $A$ is defined by

$$
A^*(U, P)^\top = \left(\begin{array}{c}
A^*U + N^*P \\
B^*P + M^*U
\end{array}\right), \forall (U, P)^\top \in D(A^*) = D(A^*) \times X.
$$

Let us now give a first result concerning the point spectrum $\sigma_p(A)$ of $A$ under the assumption that

$$
\Re(A(U, P)^\top, (U, P)^\top)_{H \times X} \preceq -\|U\|_H^2, \forall (U, P)^\top \in D(A) \times X. \quad (3.6)
$$

**Lemma 3.6.** If (3.6) holds, then for all $\xi \in \mathbb{R}$, one has

$$
\ker(i\xi \mathbb{I} - A) = \{(0, P)^\top | P \in \ker M \cap \ker(i\xi I - B)\}. \quad (3.7)
$$

In particular $\sigma_p(A) \cap i\mathbb{R} = \emptyset$ if and only if

$$
\ker M \cap \ker(i\xi \mathbb{I} - B) = \{0\}, \forall \xi \in \mathbb{R}. \quad (3.8)
$$

**Proof.** Let us fix $\xi \in \mathbb{R}$ and look for $(U, P)^\top \in \ker(i\xi \mathbb{I} - A)$ or equivalently such that

$$
\left\{\begin{array}{c}
i\xi U - AU - MP = 0, \\
i\xi P - BP - NU = 0.
\end{array}\right. \quad (3.9)
$$

By the dissipativeness assumption (3.6), we directly deduce that $U = 0$ and (3.9) reduces to

$$
\left\{\begin{array}{c}
MP = 0, \\
i\xi P - BP = 0.
\end{array}\right.
$$

This means that $P \in \ker M \cap \ker(i\xi I - B)$ and the conclusion immediately follows.

A direct consequence of this result concerns the range of $i\xi I - A$ for all $i\xi \in \rho(B)$. 

Corollary 3.7. Let the assumption (3.6) be satisfied. If \( \xi \in \mathbb{R} \) is such that \( i\xi \in \rho(B) \), then \( i\xi \not\in \sigma_p(\mathbb{A}) \) and there exists a positive constant \( c(\xi) \) (depending on \( \xi \)) such that
\[
\| (i\xi I - \mathbb{A})(U, P)^\top \|_{H \times X} \geq c(\xi)\| (U, P)^\top \|_{H \times X}, \forall (U, P)^\top \in D(\mathbb{A}) \times X,
\] (3.10)
in particular the range \( R(i\xi I - \mathbb{A}) \) of \( i\xi I - \mathbb{A} \) is closed.

Proof. Obviously if \( i\xi \in \rho(B) \), then \( \ker(i\xi I - B) = \{0\} \), which yields \( i\xi \not\in \sigma_p(\mathbb{A}) \) by the previous Lemma. Let us fix \( \xi \in \mathbb{R} \) such that \( i\xi \in \rho(B) \), and prove (3.10) by a contradiction argument. Indeed assume that (3.10) does not hold, there then exists a sequence of \( (U_n, P_n)^\top \in D(\mathbb{A}) \times X, n \in \mathbb{N} \) such that
\[
\| (U_n, P_n)^\top \|_{H \times X} = 1, \forall n \in \mathbb{N},
\] (3.11)
and
\[
\| (i\xi I - \mathbb{A})(U_n, P_n)^\top \|_{H \times X} \to 0, \text{ as } n \to \infty.
\] (3.12)
Note that this last property is equivalent to
\[
\begin{align*}
&i\xi U_n - AU_n - MP_n \to 0 \text{ in } H, \text{ as } n \to \infty, \\
i\xi P_n - BP_n - NU_n \to 0 \text{ in } X, \text{ as } n \to \infty.
\end{align*}
\] (3.13)
By the dissipativeness assumption (3.6), we have
\[
\| U_n \|^2 \leq \Re((i\xi I - \mathbb{A})(U_n, P_n)^\top, (U_n, P_n)^\top)_{H \times X} \leq \| (i\xi I - \mathbb{A})(U_n, P_n)^\top \|_{H \times X},
\]
which by (3.12) guarantees that
\[
U_n \to 0 \text{ in } H, \text{ as } n \to \infty.
\] (3.14)
Using (3.13), and the boundedness of \( N \), we find
\[
i\xi P_n - BP_n \to 0 \text{ in } X, \text{ as } n \to \infty.
\]
Since \( i\xi I - B \) is an isomorphism, we deduce that
\[
P_n \to 0 \text{ in } X, \text{ as } n \to \infty.
\]
This property and (3.14) yield a contradiction with (3.11); hence, (3.10) holds. \( \square \)

Corollary 3.8. Let the assumption (3.6) and (3.8) be satisfied and suppose that \( \rho(B) \cap \sigma_p(-\mathbb{A}^*) \cap \mathbb{R} = \emptyset \). Then
\[
\sigma(\mathbb{A}) \cap i\mathbb{R} \subset \sigma(B) \cap i\mathbb{R},
\] (3.15)
and if additionally \( \sigma(B) \cap i\mathbb{R} \) is countable, the semigroup \( T(t) \) generated by \( \mathbb{A} \) is strongly stable.

Proof. First by Lemma 3.6 we have \( \sigma_p(\mathbb{A}) \cap i\mathbb{R} = \emptyset \). So, if we show that (3.15) holds, the strong stability result follows from Theorem 3.1 if we assume that \( \sigma(B) \cap i\mathbb{R} \) is countable.

Let us then fix \( i\xi \in \rho(B) \cap i\mathbb{R} \), then by Corollary 3.7, \( i\xi I - \mathbb{A} \) is injective and has a closed range. As
\[
R((i\xi I - \mathbb{A})^\dagger) = \ker((-i\xi I - \mathbb{A})^\dagger) = \ker((-i\xi I - \mathbb{A})^\dagger),
\]
we conclude that
\[
R((i\xi I - \mathbb{A})^\dagger) = H \times X,
\]
since \( i\xi \not\in \sigma_p(-\mathbb{A}^*) \). In other words, \( i\xi I - \mathbb{A} \) is an isomorphism; therefore, \( i\xi \in \rho(\mathbb{A}). \)
This proves the inclusion
\[
\rho(B) \cap i\mathbb{R} \subset \rho(\mathbb{A}) \cap i\mathbb{R},
\]
which is clearly equivalent to (3.15). \( \square \)
We now prove similar results if
\[ \Re(A(U, P)^\top, (U, P)^\top)_{H \times X} \lesssim -\|P\|_X^2, \forall (U, P)^\top \in D(A) \times X. \] (3.16)

**Lemma 3.9.** If (3.16) holds, then for all \( \xi \in \mathbb{R} \), it holds
\[ \ker(\xi I - A) = \{ (U, 0)^\top \mid U \in \ker N \cap \ker(\xi I - A) \}. \] (3.17)
In particular \( \sigma_p(A) \cap \text{iR} = \emptyset \) if and only if
\[ \ker N \cap \ker(\xi I - A) = \{ 0 \}, \forall \xi \in \mathbb{R}. \] (3.18)

**Proof.** The proof of this Lemma is exactly the same as the one of Lemma 3.6 using the fact that for \( (U, P)^\top \in \ker(\xi I - A) \), \( P = 0 \) due to (3.16).

The consequence of this result to the range of \( (\xi I - A) \) differs from Corollary 3.7 because \( B \) is bounded (and not \( A \)).

**Corollary 3.10.** Let the assumption (3.16) be satisfied and suppose given a bounded operator \( C \) from \( X \) to \( H \). If \( \xi \in \mathbb{R} \) is such that \( \xi I \in \rho(A + CN) \), then \( \xi \notin \sigma_p(A) \) and (3.10) holds, in particular \( R(\xi I - A) \) is closed.

**Proof.** We obviously have the inclusion
\[ \ker N \cap \ker(\xi I - A) \subset \ker(\xi I - A - CN), \]
therefore, by our assumption
\[ \ker N \cap \ker(\xi I - A) = \{ 0 \}, \]
and hence, by Lemma 3.9, \( \xi \notin \sigma_p(A) \).

Let us fix \( \xi \in \mathbb{R} \) such that \( \xi I \in \rho(A + CN) \), and prove (3.10) by a contradiction argument. Indeed assume that (3.10) does not hold, there then exists a sequence of \( (U_n, P_n)^\top \in D(A) \times X, n \in \mathbb{N} \) satisfying (3.11) to (3.13). By the dissipativeness assumption (3.16), we have
\[ P_n \to 0 \text{ in } X, \text{ as } n \to \infty. \] (3.19)
Hence, as \( B \) and \( M \) are bounded, by (3.13), we get
\[ \begin{cases} 
\xi U_n - AU_n \to 0 \text{ in } H, \text{ as } n \to \infty, \\
NU_n \to 0 \text{ in } X, \text{ as } n \to \infty.
\end{cases} \] (3.20)
Since \( C \) is bounded, this obviously leads to
\[ \xi U_n - AU_n - CNU_n \to 0 \text{ in } H, \text{ as } n \to \infty, \]
and since \( \xi I - A - CN \) is an isomorphism, we deduce that
\[ U_n \to 0 \text{ in } H, \text{ as } n \to \infty. \]
This property and (3.19) yields a contradiction with (3.11).

In the previous result, the second property from (3.20) is partially lost, while it is of importance in the case when \( N \) has a left inverse. The next result is dedicated to this particular case.

**Corollary 3.11.** Let the assumption (3.16) be satisfied and suppose that \( N \) has a left inverse. Then for all \( \xi \in \mathbb{R} \), \( \xi \notin \sigma_p(A) \) and \( R(\xi I - A) \) is closed.
Proof. Obviously if $N$ has a left inverse, $N$ is injective and the first assertion follows from (3.17). Further it is well known (see [12, Exercice 5.18] for instance) that $N$ has a left inverse if and only if
\[ \|U\|_H \lesssim \|NU\|_X, \forall U \in H. \]
Consequently by using the contradiction argument from the previous Corollary we see that the second identity of (3.20) combined with this estimate directly shows that
\[ U_n \to 0 \text{ in } H, \quad n \to \infty. \]
Hence, (3.11) holds, which leads to the closedness property.

These two results yield the following stability properties with the same proof than the one of Corollary 3.8.

**Corollary 3.12.** Let the assumption (3.16) and (3.18) be satisfied and suppose that there exists a bounded operator $C$ from $X$ to $H$ such that $\rho(A+CN) \cap \sigma_p(-\mathcal{A}^*) \cap i\mathbb{R} = \emptyset$. Then
\[ \sigma(A) \cap i\mathbb{R} \subset \sigma(A + CN) \cap i\mathbb{R}, \tag{3.21} \]
and if additionally $\sigma(A + CN) \cap i\mathbb{R}$ is countable, the semigroup $T(t)$ generated by $\mathcal{A}$ is strongly stable.

**Corollary 3.13.** Let the assumption (3.16) be satisfied and suppose $N$ has a left inverse. If $\sigma_p(-\mathcal{A}^*) \cap i\mathbb{R} = \emptyset$, then $i\mathbb{R} \subset \rho(\mathcal{A})$ and therefore, the semigroup $T(t)$ generated by $\mathcal{A}$ is strongly stable.

We end up with similar results with the mixed variant of (3.6) and (3.16), namely
\[ \Re(\mathcal{A}(U,P)^T, (U,P)^T)_{H \times X} \lesssim -\|NU - \tilde{M}P\|_X^2, \forall (U,P)^T \in D(A) \times X, \tag{3.22} \]
for some bounded operator $\tilde{M}$ from $X$ into itself. For convenience introduce further the bounded operator
\[ \mathcal{L} : H \times X \to X : (U,P)^T \to NU - \tilde{M}P. \]

**Lemma 3.14.** If (3.22) holds and suppose given a bounded operator $C$ from $X$ to $H$ such that $CM = M$. Then for all $\xi \in \mathbb{R}$, one has
\[ \ker(\mathcal{L} \subset \ker(\xi I - \mathcal{A} \subset (\mathcal{A} + CN) \times \ker(\xi I - B - \tilde{M})). \tag{3.23} \]
In particular $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ if and only if
\[ \ker\mathcal{L} \cap \ker(\xi I - \mathcal{A} \subset (\mathcal{A} + CN) \times \ker(\xi I - B - \tilde{M}) = \{(0,0)^T\}, \forall \xi \in \mathbb{R}. \tag{3.24} \]

**Proof.** Let us fix $\xi \in \mathbb{R}$ and look for $(U,P)^T \in \ker(\xi I - \mathcal{A})$ (or equivalently solution of (3.9)). Then by the dissipativeness assumption (3.22), we directly deduce that
\[ NU - \tilde{M}P = 0, \tag{3.25} \]
and
\[ MP = C\tilde{M}P = CNU, \]
due to our assumption and as $C$ is linear. Consequently (3.9) reduces to
\[ \begin{cases} \xi U - AU - CNU = 0, \\ \xi P - BP - \tilde{M}P = 0. \end{cases} \]
As (3.25) means equivalently that $(U,P)^T \in \ker\mathcal{L}$, we obtain (3.23). \[ \square \]
Corollary 3.15. Let the assumptions of Lemma 3.14 be satisfied. If \( \xi \in \mathbb{R} \) is such that \( \iota \xi \in \rho(A + CN) \cap \rho(B + \tilde{M}) \), then \( \iota \xi \not\in \sigma_p(\mathcal{L}) \) and (3.10) holds, in particular \( R(\iota \xi I - \mathcal{L}) \) is closed.

Proof. The first assertion directly follows from (3.23).

For \( \xi \in \mathbb{R} \) such that \( \iota \xi \in \rho(A + CN) \cap \rho(B + \tilde{M}) \) we again prove (3.10) by a contradiction argument. Indeed if (3.10) does not hold, there then exists a sequence of \((U_n, P_n)^\top \in D(A) \times X, n \in \mathbb{N}\) satisfying (3.11) to (3.13). By the dissipativeness assumption (3.22) we then have

\[
NU_n - \tilde{M}P_n \to 0 \text{ in } X, \quad \text{as } n \to \infty, \quad (3.26)
\]

and

\[
CN U_n - MP_n \to 0 \text{ in } H, \quad \text{as } n \to \infty, \quad \text{as } C \text{ is bounded.}
\]

These properties in (3.13) yield

\[
\begin{cases}
\iota \xi U_n - AU_n - CNU_n \to 0 \text{ in } H, \quad \text{as } n \to \infty, \\
\iota \xi P_n - BP_n - MP_n \to 0 \text{ in } X, \quad \text{as } n \to \infty.
\end{cases} \quad (3.27)
\]

By our assumption on \( \iota \xi \), we deduce that

\((U_n, P_n)^\top \to (0, 0)^\top \text{ in } H \times X, \quad \text{as } n \to \infty, \)

which is in contradiction with (3.11). \(\square\)

Corollary 3.16. In addition to the assumptions of Lemma 3.14, assume that \( \tilde{M} \) is invertible. If \( \xi \in \mathbb{R} \) is such that \( \iota \xi \in \rho(A + CN) \), then \( \iota \xi \not\in \sigma_p(\mathcal{L}) \) and (3.10) holds, in particular \( R(\iota \xi I - \mathcal{L}) \) is closed.

Proof. For the first assertion as \( \ker(\iota \xi I - A - CN) = \{0\} \), if \((U, P)^\top \in \ker(\iota \xi I - \mathcal{L})\), by (3.23), \( U = 0 \) and then \((0, P)\) being in \( \ker \mathcal{L} \), we find

\[
\tilde{M} P = 0,
\]

which yields \( P = 0 \) as \( \tilde{M} \) is invertible.

For \( \xi \in \mathbb{R} \) such that \( \iota \xi \in \rho(A + CN) \) we still prove (3.10) by a contradiction argument. Indeed if (3.10) does not hold, there then exists a sequence of \((U_n, P_n)^\top \in D(A) \times X, n \in \mathbb{N}\) satisfying (3.11) to (3.13). By the dissipativeness assumption, (3.22), (3.26) and (3.27) are still valid. Here our assumption on \( \iota \xi \) only yields

\[
U_n \to 0 \text{ in } H, \quad \text{as } n \to \infty.
\]

This property in (3.26) leads to

\[
\tilde{M} P_n \to 0 \text{ in } X, \quad \text{as } n \to \infty,
\]

and as \( \tilde{M} \) is invertible, we get

\[
P_n \to 0 \text{ in } X, \quad \text{as } n \to \infty,
\]

which is in contradiction with (3.11). \(\square\)

As before we deduce the following stability properties.

Corollary 3.17. Let the assumptions of Lemma 3.14 be satisfied. Suppose that \( \rho(A + CN) \cap \rho(B + \tilde{M}) \cap \sigma_p(-\mathcal{L}^*) \cap i\mathbb{R} = \emptyset \). Then

\[
\sigma(\mathcal{L}) \cap i\mathbb{R} \subset (\sigma(A + CN) \cup \sigma(B + \tilde{M})) \cap i\mathbb{R}, \quad (3.28)
\]

and if additionally \( \rho(A + CN) \cap i\mathbb{R} \) and \( \rho(B + \tilde{M}) \cap i\mathbb{R} \) are countable, the semigroup \( T(t) \) generated by \( \mathcal{L} \) is strongly stable.
Corollary 3.18. In addition to the assumptions of Lemma 3.14, assume that \( \tilde{M} \) is invertible. Suppose that \( \rho(A + CN) \cap \sigma_p(-A^*) \cap i\mathbb{R} = \emptyset \). Then
\[
\sigma(A) \cap i\mathbb{R} \subset \sigma(A + CN) \cap i\mathbb{R},
\tag{3.29}
\]
and if additionally \( \sigma(A + CN) \cap i\mathbb{R} \) is countable, the semigroup \( T(t) \) generated by \( A \) is strongly stable.

4. Stability results. Our stability results are based on a frequency domain approach, namely for the exponential decay of the semigroup we use the following result (see [48] or [27]):

Lemma 4.1. Let \( (e^{t\mathcal{L}})_{t \geq 0} \) be a bounded \( C_0 \) semigroup on a Hilbert space \( \mathcal{H} \). Then it is exponentially stable, i.e., it satisfies
\[
\| e^{t\mathcal{L}} U_0 \|_\mathcal{H} \leq C e^{-\omega t} \| U_0 \|_\mathcal{H}, \quad \forall U_0 \in \mathcal{H}, \quad \forall t \geq 0,
\]
for some positive constants \( C \) and \( \omega \) if and only if
\[
i\mathbb{R} \subset \rho(\mathcal{L}),
\tag{4.1}
\]
and
\[
\sup_{\xi \in \mathbb{R}} \| (\alpha \xi - \mathcal{L})^{-1} \| < \infty.
\tag{4.2}
\]

On the contrary the polynomial decay is based on the following result stated in Theorem 2.4 of [9] (see also [6, 7, 37] for weaker variants).

Lemma 4.2. Let \( (e^{t\mathcal{L}})_{t \geq 0} \) be a bounded \( C_0 \) semigroup on a Hilbert space \( \mathcal{H} \) such that its generator \( \mathcal{L} \) satisfies (4.1) and let \( \ell \) be a fixed positive real number. Then the following properties are equivalent
\[
\| e^{t\mathcal{L}} U_0 \|_\mathcal{H} \leq t^{-\frac{1}{\ell}} \| U_0 \|_{\mathcal{D}(\mathcal{L})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}), \quad \forall t > 1,
\tag{4.3}
\]
\[
\| e^{t\mathcal{L}} U_0 \|_\mathcal{H} \leq t^{-1} \| U_0 \|_{\mathcal{D}(\mathcal{L}^\ell)}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}^\ell), \quad \forall t > 1,
\]
\[
\sup_{\xi \in \mathbb{R}} \frac{1}{1 + |\xi|^\ell} \| (\alpha \xi - \mathcal{L})^{-1} \| < \infty.
\tag{4.4}
\]

If (4.3) holds, we say that the semigroup \( (e^{t\mathcal{L}})_{t \geq 0} \) is polynomially stable with a decay in \( t^{-\frac{1}{\ell}} \).

The results of the previous section allow to manage the assumption (4.1). Therefore, we are reduced to check the assumption (4.2) or (4.4). Let us start with the first one. As (4.2) is based on the bounded behavior of the resolvent on \( i\mathbb{R} \) at infinity, we first formulate a result between the resolvents of \( A \) and \( \mathcal{A} \).

Lemma 4.3. The following assertions are equivalent
1. There exist two positive real numbers \( R_1 \) and \( C_1 \) such that for all \( \xi \in \mathbb{R} \) satisfying \( |\xi| > R_1 \), \( i\mathbb{R} - A \) is an isomorphism and
\[
\| (\alpha \xi I - A)^{-1} \| \leq C_1.
\tag{4.5}
\]
2. There exist two positive real numbers \( R_2 \geq 2\| B \| \) and \( C_2 \) such that for all \( \xi \in \mathbb{R} \) satisfying \( |\xi| > R_2 \), \( i\mathbb{R} - A \) is an isomorphism and
\[
\| (\alpha \xi I - A)^{-1} \| \leq C_2.
\tag{4.6}
\]
Proof. We start with the following remark. As $B$ is supposed to be bounded, for $\xi \in \mathbb{R}$ such that $|\xi| \geq \|B\|$, $\xi I - B$ is invertible and satisfies

$$
\|(\iota \xi I - B)^{-1}\| \leq \frac{2}{|\xi|}. \tag{4.7}
$$

Indeed such an estimate is valid because due to the constraint on $\xi$, we may write

$$(\iota \xi I - B)^{-1} = (\iota \xi)^{-1} \sum_{n=0}^{\infty} ((\iota \xi)^{-1}B)^n,$$

and therefore,

$$
\|(\iota \xi I - B)^{-1}\| \leq |\xi|^{-1} \frac{1}{1 - |\xi|^{-1}\|B\|} \leq \frac{2}{|\xi|}.
$$

For such a $\xi$, we can then define the operator $A_\xi$ by

$$
A_\xi = \iota \xi I - A - M(\iota \xi I - B)^{-1}N. \tag{4.8}
$$

We first prove that point 1 is equivalent to 3. There exist two positive real numbers $R_3 \geq 2\|B\|$ and $C_3$ such that for all $\xi \in \mathbb{R}$ satisfying $|\xi| > R_3$, $A_\xi$ is an isomorphism and

$$
\|A_\xi^{-1}\| \leq C_3. \tag{4.9}
$$

Indeed if point 1 holds, for $|\xi| > \max\{R_1, 2\|B\|\}$ we can write

$$
A_\xi = (\iota \xi I - A)(I - (\iota \xi I - A)^{-1}M(\iota \xi I - B)^{-1}N). \tag{4.10}
$$

By the assumption (4.5), the estimate (4.7) and as $M$ and $N$ are bounded, we have

$$
\|(\iota \xi I - A)^{-1}M(\iota \xi I - B)^{-1}N\| \leq 2|\xi|^{-1}C_1\|M\|\|N\|.
$$

Hence, by assuming that $2|\xi|^{-1}C_1\|M\|\|N\| \leq \frac{1}{2}$ or equivalently $|\xi| \geq 4C_1\|M\|\|N\|$, the Neumann series

$$
\sum_{n=0}^{\infty} ((\iota \xi I - A)^{-1}M(\iota \xi I - B)^{-1}N)^n
$$

converges and coincides with the inverse of $I - (\iota \xi I - A)^{-1}M(\iota \xi I - B)^{-1}N$ with

$$
\|(I - (\iota \xi I - A)^{-1}M(\iota \xi I - B)^{-1}N)^{-1}\| \leq 2.
$$

Therefore, by (4.10) and the assumption from point 1, $A_\xi$ is invertible with

$$
\|A_\xi^{-1}\| \leq 2\|(\iota \xi I - A)^{-1}\|.
$$

This yields point 3 with $C_3 = 2C_1$ and $R_3 = \max\{R_1, 2\|B\|, 4C_1\|M\|\|N\|\}$.

The converse implication is proved in the same manner by writing (see (4.8))

$$
\iota \xi I - A = A_\xi + M(\iota \xi I - B)^{-1}N.
$$

Now we show the equivalence between points 2 and 3. We first prove that point 3 implies point 2. So, for $\xi \in \mathbb{R}$ and $(F,G)^{\top} \in H \times X$, we look for a solution $(U,P)^{\top} \in D(A) \times X$ of

$$
\begin{cases}
\iota \xi U - AU - MP = F, \\
\iota \xi P - BP - NU = G,
\end{cases} \tag{4.11}
$$

satisfying

$$
\|U\|_H + \|P\|_X \leq C_2(\|F\|_H + \|G\|_X), \tag{4.12}
$$

for $|\xi|$ large enough.
Now we fix $\xi$ such that $|\xi| > R_3$. Coming back to (4.11), its second identity is equivalent to
\begin{equation}
P = (i\xi I - B)^{-1} (NU + G),
\end{equation}
and inserting this expression in the first identity we find
\[ i\xi U - AU - M(i\xi I - B)^{-1} NU = F + M(i\xi I - B)^{-1} G, \]
or equivalently
\begin{equation}
A\xi U = F + M(i\xi I - B)^{-1} G.
\end{equation}
By our assumption, we then find
\begin{equation}
\|U\|_H \leq C_3 \left( \|F\|_H + \|M(i\xi I - B)^{-1} G\|_H \right).
\end{equation}
Using (4.7) and the boundedness of $M$, we get
\begin{equation}
\|U\|_H \leq C_3 \left( \|F\|_H + 2\|M\| \|G\|_X \right).
\end{equation}
Hence, assuming that $|\xi| \geq 1$, we obtain
\begin{equation}
\|U\|_H \leq C_3 (\|F\|_H + 2\|M\| \|G\|_X).
\end{equation}
Coming back to (4.13), using (4.7), the boundedness of $N$ and this last estimate (4.15), we arrive at
\begin{equation}
\|P\|_X \leq 2\|N\|C_3 (\|F\|_H + 2\|M\| \|G\|_X) + 2\|G\|_X.
\end{equation}
This estimate and (4.15) show that (4.12) holds with $R_2 = \max\{1, R_3\}$.

Conversely if we assume that point 2 holds, then for any $F \in H$, as datum in (4.11) we take $(F, 0)$ which yields $(U, P)$ that satisfies $A\xi U = F$, and
\[ \|U\|_H + \|P\|_X \leq C_3 \|F\|_H \]
by the assumption (4.6). This proves point 3 with $C_3 = C_2$ and $R_3 = R_2$. \qed

**Remark 4.4.** The previous result shows that the boundedness of the resolvent of $A$ and $A$ at infinity on $i\mathbb{R}$ are equivalent. This essentially means that the exponential decay of system (1.1) cannot be obtained via the equation in $P$ and in general follows from the exponential decay of the evolution equation
\begin{equation}
\begin{cases}
U_t = AU, & \text{in } H, \\
U(0) = U_0.
\end{cases}
\end{equation}

A precise statement emanating from this remark is the following equivalent exponentially decaying result.

**Corollary 4.5.** Assume that $\mathcal{A}$ (resp. $A$) generates a bounded $C_0$ semigroup $T(t)$ (resp. $S(t)$) on $H \times X$ (resp. $H$) satisfying (4.1), namely $i\mathbb{R} \subset \rho(\mathcal{A})$ (resp. $i\mathbb{R} \subset \rho(A)$). Then $T(t)$ is exponentially stable if and only if the semigroup generated by $A$ on $H$ is exponentially stable.

**Proof.** As the mapping $\lambda \to (\mathcal{A} - \lambda I)^{-1}$ (resp. $\lambda \to (A - \lambda I)^{-1}$) is holomorphic on the resolvent set $\rho(\mathcal{A})$ (resp. $\rho(A)$), see [5] for instance, it is continuous on any compact set of $i\mathbb{R}$. Therefore, (4.2) holds for $\mathcal{A}$ (resp. $A$) if and only if there exists $R_2 > 0$ (resp. $R_1 > 0$) large enough such that (4.6) (resp. (4.5)) holds for all $|\xi| > R_2$ (resp. $|\xi| > R_1$). The equivalence then follows from Lemma 4.3. \qed
Before giving another consequence of Lemma 4.3, let us state a result concerning the asymptotic behavior of the resolvent of a bounded perturbation of $A$.

For shortness let us first state the next definition. First let us recall that a family $\{\varphi_j\}_{j=1}^{\infty}$ is a Riesz basis of a Hilbert space $V$ if there is an isomorphism $\Xi : V \to V$ and an orthonormal basis $\{\psi_j\}_{j=1}^{\infty}$ such that $\varphi_j = \Xi \psi_j$, for all $j \in \mathbb{N}^*$. It is well known that $\{\varphi_j\}_{j=1}^{\infty}$ is a Riesz basis of $V$ if and only if $\{\varphi_j\}_{j=1}^{\infty}$ is complete in $V$ and

$$\sum_{j=1}^{\infty} |c_j|^2 \sim \sum_{j=1}^{\infty} \|c_j\varphi_j\|^2_V,$$

for all finite sequences of scalars $(c_j)_{j=1}^{\infty}$. Note that this equivalence remains valid for all sequences of scalars $(c_j)_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty} |c_j|^2 < \infty$.

**Definition 4.6.** A subspace $V$ of $H$ is called a Riesz eigenspace with bounded eigenvalues of $A$ if and only if $V$ is closed and is spanned by linearly independent eigenvectors $\varphi_j$, $j = 1, \ldots, J$ of $A$ of associated eigenvalues $\lambda_j$, with $J$ finite or infinite; furthermore, when $J$ is infinite, we suppose that the set $\{\varphi_j\}_{j=1}^{\infty}$ forms a Riesz basis of $V$ and that the set $\{\lambda_j\}_{j=1}^{\infty}$ is bounded.

**Lemma 4.7.** Let $C$ be a bounded operator from $X$ to $H$ such that $A + CN$ generates a $C_0$ semigroup on $H$ and let a Riesz eigenspace $V$ with bounded eigenvalues of $A + CN$ be fixed. If we assume that there exist a positive real number $R_3$ and a non negative real number $m$ such that for all $\xi \in \mathbb{R}$ with $|\xi| > R_3$, $\xi I - A - CN$ is an isomorphism on $V^\perp$ and

$$\|(\xi I - A - CN)^{-1}F\|_H \lesssim |\xi|^m \|F\|_H, \forall F \in V^\perp, \tag{4.18}$$

then there exists a positive real number $R_4 \geq R_3$ such that for all $\xi \in \mathbb{R}$ with $|\xi| > R_4$, $\xi I - A - CN$ is an isomorphism on $H$ and

$$\|(\xi I - A - CN)^{-1}F\|_H \lesssim |\xi|^m \|F\|_H, \forall F \in H. \tag{4.19}$$

**Proof.** With the notation from Definition 4.6, denote by $R = 1 + \max_{j=1,\ldots,J} |\lambda_j|$, and let us show that for $|\xi| > R$, we have

$$\|(\xi I - A - CN)^{-1}F\|_H \lesssim \|F\|_H, \forall F \in V. \tag{4.20}$$

Indeed writing $F \in V$ in the form $F = \sum_{j=1}^{J} \alpha_j \varphi_j$, with $\alpha_j \in \mathbb{C}$, the equivalence of norms in a finite dimensional space or our Riesz basis assumption in the case $J = \infty$ yields

$$\|F\|_H^2 \sim \sum_{j=1}^{J} |\alpha_j|^2.$$

Then one directly checks that

$$(\xi - A - CN)^{-1}F = \sum_{j=1}^{J} (\xi - \lambda_j)^{-1} \alpha_j \varphi_j,$$

and therefore,

$$\|(\xi I - A - CN)^{-1}F\|_H^2 \sim \sum_{j=1}^{J} |\xi - \lambda_j|^{-2} |\alpha_j|^2.$$
As $|\xi| > 1 + \max_{j=1,\ldots,j}|\lambda_j|$ we have $|i\xi - \lambda_j| \geq 1$, and therefore,

$$
\|(i\xi - A - CN)^{-1}F\|_H \lesssim \sum_{j=1}^{\infty} |\alpha_j|^2.
$$

This directly yields (4.20).

Combining (4.18) and (4.20), we deduce that (4.19) holds for all $|\xi| > \max\{R, R_3\}$. 

\[\square\]

**Corollary 4.8.** Assume that $A$ generates a $C_0$ semigroup $S(t)$ on $H$. Suppose that there exists a Riesz eigenspace $V$ with bounded eigenvalues of $A$ such that the restriction $A_0$ of $A$ to $V^\perp$ generates an exponentially stable semigroup $S_0(t)$. Assume further that there exists a closed subspace $W$ of $H \times X$ such that the restriction $A_0$ of $A$ to $W$ generates a bounded $C_0$ semigroup $T_0(t)$ on $W$ satisfying (4.1), namely $i\mathbb{R} \subset \rho(A_0)$, then $T_0(t)$ is exponentially stable.

**Proof.** By Lemma 4.1, we know that $A_0$ satisfies (4.2), namely

$$
\|(i\xi - A)^{-1}F\|_H \lesssim \|F\|_H, \forall \xi \in \mathbb{R}, F \in V^\perp.
$$

Further by Lemma 4.7 with $C = 0$ and $m = 0$, there exists a positive real number $R_1$ such that

$$
\|(i\xi - A)^{-1}F\|_H \lesssim \|F\|_H, \forall F \in H, |\xi| > R_1,
$$

hence, (4.5) holds for all $|\xi| > R_1$. The conclusion now follows with the help of Lemma 4.3 and the holomorphic property of the resolvent of $\mathcal{A}_0$ on its resolvent set (see above). \[\square\]

**Remark 4.9.** If $W = H \times X$, this last result has two (philosophical) applications:

1. if system (4.17) is exponentially stable up to a finite-dimensional space of eigenvectors (or more generally a Riesz eigenspace with bounded eigenvalues), via the coupling with the equation in $P$ it becomes exponentially stable,
2. the exponential decay of system (4.17) up to a finite-dimensional space of eigenvectors (or more generally a Riesz eigenspace with bounded eigenvalues) is sufficient to stabilize the ODE if the coupling is appropriately chosen.

The case $W \neq H \times X$ has many applications that will be detailed below.

Let us go on with a polynomial decay result.

**Theorem 4.10.** Assume that there exist a bounded operator $C$ from $X$ to $H$, a Riesz eigenspace $V$ with bounded eigenvalues of $A + CN$ such that the restriction $(A + CN)_0$ of $A + CN$ to $V^\perp$ generates a bounded $C_0$ semigroup on $V^\perp$ satisfying (4.1), namely $i\mathbb{R} \subset \rho((A + CN)_0)$, and

$$
\sup_{\xi \in \mathbb{R}} \frac{1}{1 + |\xi|^m} \|(i\xi - (A + CN)_0)^{-1}\| < \infty,
$$

for some non negative real number $m$. Assume that (3.16) holds and that there exists a closed subspace $W$ of $H \times X$ such that the restriction $A_0$ of $A$ to $W$ generates a bounded $C_0$ semigroup $T_0(t)$ on $W$ satisfying (4.1), namely $i\mathbb{R} \subset \rho(A_0)$. Then $T_0(t)$ is polynomially stable with a decay in $t^{-\ell}$, more precisely it satisfies

$$
\|T(t)(U_0, P_0)^\top\|_{H \times X} \leq t^{-\ell} \|(U_0, P_0)^\top\|_{D(A)_X \times X}, \forall (U_0, P_0)^\top \in W \cap (D(A) \times X), \forall t > 1,
$$

with $\ell = \max\{m, 2(m + 1)\}$. 


Proof. As already mentioned it suffices to show that

$$|\xi|^\ell \| (i\xi - \mathcal{A}_0)^{-1} \| \lesssim 1,$$  \hspace{1cm} (4.23)

for $|\xi|$ large enough and then apply Lemma 4.2 (because $\ell \geq 2$). For that purpose, we use a contradiction argument, i.e., we suppose that (4.23) is false. Then there exist a sequence of real numbers $\xi_n \to +\infty$ and a sequence of vectors $(U_n, P_n)^T$ in $W \cap \mathcal{D}(\mathcal{A}) \times X$ with $\|(U_n, P_n)^T\|_{H \times X} = 1$ such that

$$\xi_n^\ell \|(i\xi_n - \mathcal{A})(U_n, P_n)^T\|_{H \times X} \to 0 \text{ as } n \to \infty,$$  \hspace{1cm} (4.24)

or equivalently

$$\xi_n^\ell (i\xi_n U_n - A U_n - MP_n) = F_n \to 0 \text{ in } H,$$  \hspace{1cm} (4.25)

$$\xi_n^\ell (i\xi_n P_n - BP_n - NU_n) = G_n \to 0 \text{ in } X.$$  \hspace{1cm} (4.26)

First using the dissipativeness property (3.16), one finds

$$\xi_n^\ell \|P_n\|_X^2 \lesssim \Re((F_n, G_n)^T, (U_n, P_n)^T)_{H \times X} \leq \|(F_n, G_n)^T\|_{H \times X} \to 0,$$

and therefore,

$$\xi_n^{\ell/2} P_n \to 0, \text{ in } X.$$  \hspace{1cm} (4.27)

Now (4.26) yields

$$NU_n = i\xi_n P_n - BP_n - \xi_n^{-\ell} G_n,$$

hence, multiplying this identity by $\xi_n^m$, we find

$$\xi_n^m NU_n = i\xi_n^{m+1} P_n - \xi_n^m BP_n - \xi_n^{m-\ell} G_n.$$  \hspace{1cm} (4.28)

With our choice of $m$, we see that $m + 1 \leq \ell/2$ and $m \leq \ell$ and therefore,

$$\xi_n^m NU_n \to 0, \text{ in } X,$$  \hspace{1cm} (4.29)

and since $C$ is bounded we further get

$$\xi_n^m CNU_n \to 0, \text{ in } H.$$  \hspace{1cm} (4.29)

Now we come back to (4.25) that yields

$$\xi_n^m (i\xi_n U_n - A U_n) = \xi_n^{m-\ell} F_n + \xi_n^m MP_n.$$  \hspace{1cm} (4.25)

By the boundedness of $M$, (4.27), and the assumption $m \leq \ell$, we conclude that this right-hand side goes to zero in $H$ as $n$ goes to infinity. Added to (4.29), we see that

$$\xi_n^m (i\xi_n - A - CN) U_n = \tilde{F}_n \to 0 \text{ in } H.$$  \hspace{1cm} (4.21)

By our assumption (4.21) and Lemma 4.7, we conclude that

$$\|U_n\|_H \lesssim \|\tilde{F}_n\|_H \to 0.$$  \hspace{1cm} (4.27)

This property and (4.27) is in contradiction with $\|(U_n, P_n)^T\|_{H \times X} = 1$; hence, the result.

Remark 4.11. In the previous Theorem, the same conclusion holds if we replace (3.16) by (3.6) (resp. (3.22) with a bounded operator $C$ from $X$ to $H$ such that $CM = M$), with $\ell = 2m$. \hfill $\square$
5. Some illustrative examples.

5.1. Dispersive medium models. In this subsection \( \Omega \) is a bounded and connected domain of \( \mathbb{R}^3 \) with a Lipschitz boundary \( \Gamma \). The unit outward normal vector along \( \Gamma \) is denoted by \( \mathbf{n} \).

All dispersive medium models from \([32, 34, 55, 43, 15]\) or the cold magnetized plasma model \([31]\) enters in the following setting:

\[
\begin{align*}
\epsilon E_t - \text{curl} \mathbf{H} &= r_{ee}E + r_{eh}H + M_e P \quad \text{in } Q = \Omega \times ]0, +\infty[, \\
\mu H_t + \text{curl} \mathbf{E} &= r_{he}E + r_{hh}H + M_h P \quad \text{in } Q, \\
\rho P_t &= B_p P + N_e E + N_h H \quad \text{in } Q, & \quad (5.1) \\
\mathbf{E} \times \mathbf{n} &= 0 \quad \text{on } \Sigma := \Gamma \times ]0, +\infty[, \\
\mathbf{E}(x, 0) &= \mathbf{E}_0(x), \quad \mathbf{H}(x, 0) = \mathbf{H}_0(x), \quad P(x, 0) = P_0(x) \quad \text{in } \Omega,
\end{align*}
\]

where \( \mathbf{E} \) (resp. \( \mathbf{H} \)) is the electric (resp. magnetic) field and \( P \) is a vectorial unknow with \( 3k \) components, \( k \) being a positive integer (corresponding to the polarization vector or density of particles for instance). The different parameters and their basic properties are the following ones:

- **(H1)** the permittivity \( \epsilon \) and the permeability \( \mu \) of the medium (resp. the density \( \rho \)) are assumed to be in \( L^\infty(\Omega; \mathbb{R}^{3\times3}) \) (resp. \( \rho \in L^\infty(\Omega; \mathbb{R}^{3k\times3k}) \)), symmetric and uniformly positive definite, in the sense that
  \[
  (\epsilon(x)\mathbf{\Xi}, \mathbf{\Xi}) \gtrsim ||\mathbf{\Xi}||^2, \quad (\mu(x)\mathbf{\Xi}, \mathbf{\Xi}) \gtrsim ||\mathbf{\Xi}||^2, \quad (\rho(x)\mathbf{\Upsilon}, \mathbf{\Upsilon}) \gtrsim ||\mathbf{\Upsilon}||^2, \quad \forall x \in \Omega,
  \]

- **(H2)** the matrix valued functions \( r_{ee}, r_{eh}, r_{he} \) and \( r_{hh} \) are supposed to be in \( L^\infty(\Omega; \mathbb{R}^{3\times3}) \),

- **(H3)** the matrix valued function \( B_p \) is supposed to be in \( L^\infty(\Omega; \mathbb{R}^{3k\times3k}) \),

- **(H4)** the matrix valued functions \( M_e \) and \( M_h \) are supposed to be in \( L^\infty(\Omega; \mathbb{R}^{3\times3}) \),

- **(H5)** the matrix valued functions \( N_e \) and \( N_h \) are supposed to be in \( L^\infty(\Omega; \mathbb{R}^{3k\times3}) \).

With these assumptions, system (5.1) enters in the abstract framework (1.1) by defining \( H, X, A, B, M \) and \( N \) as follows: The Hilbert spaces \( H \) and \( X \) are defined by

\[
H = L^2(\Omega)^6, \quad X = L^2(\Omega)^{3k},
\]

with the inner product

\[
((\mathbf{E}, \mathbf{H})^\top, (\mathbf{E}', \mathbf{H}')^\top)_H = \int_\Omega (\varepsilon \mathbf{E} \cdot \mathbf{E}' + \mu \mathbf{H} \cdot \mathbf{H}') \, dx, \quad (\mathbf{E}, \mathbf{H})^\top \in L^2(\Omega)^6,
\]

\[
(P, P')_X = \int_\Omega \rho P \cdot P' \, dx, \quad P, P' \in L^2(\Omega)^{3k}.
\]

The operators \( B, M \) and \( N \) are given by

\[
B : X \to X : P \to \rho^{-1}B_p P, \\
M : X \to H : P \to (\varepsilon^{-1}M_e P, \mu^{-1}M_h P)^\top, \\
N : H \to X : (\mathbf{E}, \mathbf{H})^\top \to \rho^{-1}(N_e \mathbf{E} + N_h \mathbf{H}),
\]

and we directly check that they are indeed bounded. Finally the operator \( A = A_u + R \), where \( A_u \) and \( R \) are defined as follows: the domain \( D(A_u) \) of \( A_u \) is given by

\[
D(A_u) = \{ (\mathbf{E}, \mathbf{H})^\top \in L^2(\Omega)^6 | \text{curl} \mathbf{E}, \text{curl} \mathbf{H} \in L^2(\Omega)^3, \quad \text{and } \mathbf{E} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \},
\]
and
\[ A_u(E, H)^\top = (\varepsilon^{-1} \text{curl} H, -\mu^{-1} \text{curl} E)^\top, \forall (E, H)^\top \in D(A_u), \]
while
\[ R(E, H)^\top = (\varepsilon^{-1}(r_{ee}E + r_{eh}H), \mu^{-1}(r_{he}E + r_{hh}H)) \]
clearly defines a bounded operator from \( H \) into itself. Therefore, \( A \) will generate a \( C_0 \) semigroup in \( H \) if and only if its principal part \( A_u \) generates a \( C_0 \) semigroup as well.

Since \( D(\Omega)^3 \) is dense in \( H_0(\text{curl}, \Omega) \), Green’s formula (I.2.22) from [24] directly implies that
\[ \int_{\Omega} (E \cdot \text{curl} F - \text{curl} E \cdot \overline{F}) \, dx = 0, \forall E \in H(\text{curl}, \Omega), F \in H_0(\text{curl}, \Omega). \tag{5.2} \]
As a direct consequence, we have
\[ (A_u(E, H)^\top, (E, H)^\top)_{H} = 0, \forall (E, H)^\top \in D(A_u), \tag{5.3} \]
hence, \( A_u \) and \( -A_u \) are dissipative on \( H \). Further it is not difficult to show that \( \lambda I \pm A_u \) is surjective for all \( \lambda > 0 \) (see [42, Lemma 3.1]) and by Lumer-Phillips’ theorem \( A_u \) and \( -A_u \) generate a \( C_0 \)-semigroup of contractions on \( H \). This actually means that \( A_u \) generates a \( C_0 \)-group on \( H \).

Using Theorem 2.1, problem (5.1) is then well-posed in \( H \times X \). We now analyze some stability properties of particular but illustrative examples. Before going on, note that \( D(A) \) is not compactly embedded into \( H \times X \); therefore, the verification of (4.1) is not reduced to the proof of the injectivity of \( \iota \xi I - A \), for all \( \xi \in \mathbb{R} \).

Remark that the dissipativity property (5.3) implies that
\[ R(A(E, H, P)^\top, (E, H, P)^\top)_{H \times X} = R(R(E, H)^\top, (E, H)^\top)_{H} \]
\[ + (BP, P)_X + (MP, (E, H)^\top)_{H} + (N(E, H)^\top, P)_X \], \( \forall (E, H, P)^\top \in D(A) \times X. \tag{5.4} \]

5.1.1. Plasma models. The Lorentz two poles model, the cold plasma model [32, 34, 55, 43] and the cold magnetized plasma model [31] have the following common properties:
\[ R = 0, \tag{5.5} \]
\[ N_h = 0, M_h = 0, M_c = -N_c^*, \tag{5.6} \]
where here \( N_c^* \) means the adjoint of \( N_c \) with respect to the euclidean inner products of \( \mathbb{R}^3 \) and \( \mathbb{R}^{3k} \). Furthermore, \( N_c \) has a uniform left inverse, in the sense that there exists \( L_c \in L^\infty(\Omega; \mathbb{R}^{3 \times 3k}) \) such that
\[ L_c(x)N_c(x) = I, \text{ a.a. } x \in \Omega, \tag{5.7} \]
with
\[ \|L_c(x)Y\| \lesssim \|Y\|, \forall Y \in \mathbb{C}^{3k}, \text{ a. a. } x \in \Omega. \tag{5.8} \]
Finally \( B_p \) is supposed to be definite negative, namely
\[ R(B_p(x)Y, Y) \lesssim -\|Y\|^2, \forall Y \in \mathbb{C}^{3k}, \text{ a. a. } x \in \Omega. \tag{5.9} \]
We shall now give some stability properties of (5.1) under these sole additional assumptions; hence, recovering and extending some results from [43, 31].

As the properties (5.5) and (5.6) imply that
\[ M^* = -N, \tag{5.10} \]
by (5.9) and (5.4), we find
\[ R(\mathcal{A}(E, H, P)\top, (E, H, P)\top)_{H\times X} \lesssim -\|P\|^2_X, \forall (E, H, P)\top \in D(\mathcal{A}) \times X, \] (5.11)
which means that (3.16) holds. Hence, Lemma 3.9 allows us to characterize \( \sigma_p(\mathcal{A}) \cap i\mathbb{R} \).

**Lemma 5.1.** Under the assumptions (5.5) to (5.9), we have
\[ \sigma_p(\mathcal{A}) \cap i\mathbb{R} = \{0\} \]
and
\[ \ker \mathcal{A} = \{(0, H, 0)\top | H \in H(\text{curl} = 0, \Omega)\}, \] (5.12)
where
\[ H(\text{curl} = 0, \Omega) := \{H \in L^2(\Omega)^3 | \text{curl} H = 0\}. \]

**Proof.** We first notice that (5.7) and (5.8) imply that
\[ \|\Xi\| \lesssim \|N_e(\cdot)\Xi\|, \forall \Xi \in \mathbb{C}^3, \text{ a. a. } x \in \Omega, \]
consequently
\[ \ker N = \{(0, H) | H \in L^2(\Omega)^3\}. \]
Therefore, by (3.17), for any \( \xi \in \mathbb{R} \),
\[ \ker(\xi I - \mathcal{A}) = \{(0, H, 0)\top | (0, H)\top \in \ker(\xi I - \mathcal{A}_u)\}. \]
On one hand, for \( \xi \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \), we readily check that \((0, H)\top \in \ker(\xi I - \mathcal{A}_u)\) if and only if \( H = 0 \), and consequently \( \xi \notin \sigma_p(\mathcal{A}) \). On the other hand, \((0, H)\top \in \ker \mathcal{A}_u\) if and only if \( \text{curl} H = 0 \), which proves that \( 0 \notin \sigma_p(\mathcal{A}) \) and (5.12). \( \square \)

As \( \sigma_p(\mathcal{A}) \cap i\mathbb{R} \) is not empty, we cannot hope to have strong stability on the whole space \( H \times X \), but we will reach it for its restriction to \( H_0 \times X \), where
\[ H_0 = L^2(\Omega)^3 \times \{H \in L^2(\Omega)^3 | \int_{\Omega} \rho H \cdot \dot{H'} dx = 0, \forall \dot{H'} \in H(\text{curl} = 0, \Omega)\}, \]
which is also a Hilbert space equipped with the inner product of \( H \). Let us further remark that the restriction \( \mathcal{A}_0 \) of \( \mathcal{A} \) defined by \( D(\mathcal{A}_0) = D(\mathcal{A}) \cap (H_0 \times X) \) and
\[ \mathcal{A}_0(E, H, P)\top = \mathcal{A}(E, H, P)\top, \forall (E, H, P)\top \in D(\mathcal{A}) \cap (H_0 \times X), \]
is well defined since our assumptions guarantee that
\[ R(\mathcal{A}) \subset H_0 \times X, \]
oweing to (5.2). This obviously implies that problem (5.1) is well-posed in \( H_0 \times X \). Furthermore, as the restriction \( \mathcal{A}_0 \) of \( \mathcal{A} \) to \( H_0 \) is also well defined, we can use Lemma 3.9 and Corollaries 3.10 and 3.12 in \( H_0 \times X \).

**Theorem 5.2.** Let the assumptions (5.5) to (5.9) be satisfied. Then \( i\mathbb{R} \subset \rho(\mathcal{A}_0) \) and therefore, the semigroup \( T_0(t) \) generated by \( \mathcal{A}_0 \) is strongly stable.

**Proof.** By Lemma 5.1 and the definition of \( \mathcal{A}_0 \), we have
\[ \sigma_p(\mathcal{A}_0) \cap i\mathbb{R} = \emptyset. \]
Now to apply Corollaries 3.10 and 3.12, we chose the operator \( C \) as follows
\[ CP = -(L\rho P, 0)\top, \forall P \in X, \] (5.13)
so that \( C \) is indeed bounded from \( X \) to \( H_0 \) and
\[ CN(E, H)\top = -(E, 0)\top. \] (5.14)
This yields
\[(A + CN)(E, H)^\top = A_n(E, H)^\top - (E, 0)^\top, \forall (E, H)^\top \in D(A_n) \cap H_0.\]

Since it is well known that this operator generates an exponentially stable semigroup, by Lemma 4.1, we have
\[
\{\lambda \in \mathbb{C} \mid \mathfrak{R}\lambda \geq 0\} \subset \rho(A + CN),
\]
and
\[
\sup_{\xi \in \mathbb{R}} \| (\iota \xi I - (A + CN))^{-1} \| < \infty.
\]
The first property and Corollary 3.10 allow to conclude that \(R(\iota \xi I - A_0)\) is closed, for all \(\xi \in \mathbb{R}\).

We further notice that \(A_0^* = -A_0\); therefore, by Exercise 3.5 and (5.10) we then have \(D(\mathbb{A}^*) = D(\mathbb{A}) = D(A) \times X\) and
\[
\mathbb{A}^*(E, H, P)^\top = \left(\begin{array}{c}
-A_n(E, H)^\top - MP \\
B^*P - N_e E
\end{array}\right), \forall (E, H, P)^\top \in D(\mathbb{A}^*). \tag{5.17}
\]
As a consequence of (5.2), we see that
\[
R(\mathbb{A}^*(E, H, P)^\top, (E, H, P)^\top)_{H \times X} = R(BP, P)_{X} \lesssim -\|P\|_{X}^2, \forall (E, H, P)^\top \in D(A) \times X. \tag{5.18}
\]
With the help of the same arguments than in Lemmas 3.9 and 5.1, we can show that
\[
\sigma_p(\mathbb{A}^*) \cap i\mathbb{R} = \{0\}
\]
and
\[
\ker \mathbb{A}^* = \ker \mathbb{A}.
\]
As before, we can show that the restriction \((\mathbb{A}^*)_0\) of \(\mathbb{A}^*\) to \(H_0 \times X\) is well defined. Furthermore, it turns out that \((\mathbb{A}^*)_0 = (\mathbb{A}_0)^*\). Therefore, one has
\[
\sigma_p((\mathbb{A}_0)^*) \cap i\mathbb{R} = \emptyset.
\]
The stability property of \(T_0\) then follows from Corollary 3.12. \hfill \Box

The previous properties allow us to state the following polynomial stability.

**Theorem 5.3.** Let the assumptions (5.5) to (5.9) be satisfied. Then the semigroup \(T_0(t)\) generated by \(\mathbb{A}_0\) is polynomially stable with a decay in \(t^{-\frac{1}{2}}\), namely
\[
\|T(t)(E, H, P)^\top\|_{H \times X} \lesssim t^{-\frac{1}{2}} \|D(A)_{\times X}, \forall (E, H, P)^\top \in (D(A) \cap H_0) \times X.
\]

**Proof.** We simply apply Theorem 4.10 with \(C\) defined by (5.13). Since (5.15) guarantees that (4.21) holds with \(m = 0\), and since the previous Theorem yields \(i\mathbb{R} \subset \rho(\mathbb{A}_0)\), we obtain that \(T_0\) satisfies (4.22) with \(\ell = 2\). \hfill \Box

5.1.2. *The double-negative (DNG) materials.* The wave propagation in a double-negative (DNG) medium [33, 57, 56] satisfies the properties (5.5) and (5.10), with \(N_e\) and \(N_h\) so that the \(3k \times 6\) matrix-valued function \((N_e, N_h)\) has a uniform left inverse, namely there exists \(L \in L^\infty(\Omega; \mathbb{R}^{6 \times 3k})\) such that
\[
L(x)(N_e(x), N_h(x)) = I, \text{ a.a. } x \in \Omega, \tag{5.19}
\]
with
\[
\|L(x)\| \lesssim \|\cdot\|, \forall \Phi \in C^6, \text{ a. a. } x \in \Omega. \tag{5.20}
\]
Finally \(B_p\) is supposed to be definite negative, namely it satisfies (5.9).
As before we give some stability properties of (5.1) under these additional assumptions; hence, recovering and extending Theorem 4.12 from [43] with a much simpler proof.

Clearly by (5.10), (5.9) and (5.4), the estimate (3.16) remains valid; hence, Lemma 3.9 allows us to characterize $\sigma_p(\mathcal{A}) \cap i\mathbb{R}$ and Corollary 3.13 to deduce a strong stability result.

**Theorem 5.4.** Under the assumptions (5.5), (5.9), (5.10), (5.19), and (5.20), we have
\[ \sigma_p(\mathcal{A}) \cap i\mathbb{R} = \sigma_p(\mathcal{A}^*) \cap i\mathbb{R} = \emptyset. \]

Furthermore, the semigroup $T(t)$ generated by $\mathcal{A}$ is strongly stable.

**Proof.** As the assumptions (5.19) and (5.20) lead to
\[ \Vert (\mathcal{E}, \mathcal{Z})^\top \Vert \lesssim \Vert \mathcal{N}_e \mathcal{E} + \mathcal{N}_h \mathcal{Z} \Vert, \quad \forall (\mathcal{E}, \mathcal{Z}) \in \mathbb{C}^3, \]
hence, $\ker \mathcal{N} = \{(0, 0)^\top\}$ and we conclude that
\[ \sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset, \]
owing to (3.17).

As before, the adjoint of $\mathcal{A}$ is given by (5.17) and satisfies (5.18). This easily implies that
\[ \sigma_p(\mathcal{A}^*) \cap i\mathbb{R} = \emptyset, \]
and we conclude with the help of Corollary 3.13.

We directly deduce a polynomial stability by using Theorem 4.10 with $C$ defined by (5.13).

\[ \mathcal{C}P = -\Pi_e \mathcal{L}P, \forall P \in \mathbb{C}^{3k}, \]
where $\Pi_e(\mathcal{E}, \mathcal{Z})^\top = (\mathcal{E}, 0)^\top$, for all $\mathcal{E}, \mathcal{Z} \in \mathbb{C}^3$, so that $C$ is indeed bounded from $X$ to $H$ and
\[ CN(\mathcal{E}, \mathcal{H})^\top = -(\mathcal{E}, 0)^\top. \]

**Theorem 5.5.** Under the assumptions (5.5), (5.9), (5.10), (5.19), and (5.20), the semigroup $T(t)$ generated by $\mathcal{A}$ is polynomially stable with a decay in $t^{-\frac{1}{2}}$, namely
\[ \| T(t)(\mathcal{E}, \mathcal{H}, P)_d \|_{H \times X} \lesssim t^{-\frac{1}{2}} \| (\mathcal{E}, \mathcal{H}, P)_d \|_{D(\mathcal{A}) \times X}, \forall (\mathcal{E}, \mathcal{H}, P)_d \in D(\mathcal{A}) \times X. \]

5.1.3. The generalized Debye models. We consider the system (5.1) under the assumptions that
\[ r_{eh} = r_{he} = r_{hh} = 0, \quad (5.21) \]
\[ N_h = 0, M_h = 0, M_e = N_e^*, \quad (5.22) \]
that $N_e$ has a uniform left inverse, that $B_p$ is symmetric and definite negative, namely satisfying (5.9) and that
\[ M_eB_p^{-1}N_e = r_{ee}. \quad (5.23) \]

The single pole Debye medium model described in [32, 34, 55, 43] corresponds to the case $k = 1$ and
\[ r_{ee} = -\alpha^2, M_e = N_e = \alpha \beta, B_p = -\beta^2, \]
where the parameters $\varepsilon, \mu, \rho, \alpha$ and $\beta$ are positive constants.

Under the assumptions (5.21), (5.22), we see that (5.4) reduces to
\[ \Re(\mathcal{A}(\mathcal{E}, \mathcal{H}, P)^\top, (\mathcal{E}, \mathcal{H}, P)^\top)_H \times X = \int_\Omega \Re (r_{ee} \mathcal{E} \cdot \bar{\mathcal{E}} + M_e \mathcal{P} \cdot \bar{\mathcal{E}} + N_e \mathcal{E} \cdot \bar{\mathcal{E}} + B_p \mathcal{P} \cdot \bar{\mathcal{P}}) \, dx, \]
for all \((\mathbf{E}, \mathbf{H}, P)^\top \in D(A) \times X\). Using (5.23), we obtain equivalently
\[
\Re(\mathfrak{A}(\mathbf{E}, \mathbf{H}, P)^\top, (\mathbf{E}, \mathbf{H}, P)^\top)_{H \times X} = - \int_{\Omega} \|(B_p) - \frac{1}{2} (N_e \mathbf{E} + B_p P)\|^2 \, dx,
\]
for all \((\mathbf{E}, \mathbf{H}, P)^\top \in D(A) \times X\). Since
\[
\|(B_p) - \frac{1}{2} \Upsilon\| \gtrsim \|\Upsilon\|, \forall \Upsilon \in C^3_k,
\]
we deduce that
\[
\Re(\mathfrak{A}(\mathbf{E}, \mathbf{H}, P)^\top, (\mathbf{E}, \mathbf{H}, P)^\top)_{H \times X} \lhd - \int_{\Omega} \|N_e \mathbf{E} + B_p P\|^2 \, dx, \forall (\mathbf{E}, \mathbf{H}, P)^\top \in D(A) \times X,
\]
which proves that (3.22) holds with
\[
\tilde{M} = -B.
\] (5.24)

Lemma 3.14 then allows to characterize \(\sigma_p(\mathfrak{A}) \cap i\mathbb{R}\).

**Lemma 5.6.** Assume that (5.7) and (5.8) hold, that \(B_p\) is symmetric and satisfies (5.9), and that (5.21) to (5.23) hold. Then
\[
\sigma_p(\mathfrak{A}) \cap i\mathbb{R} = \{0\}
\]
and
\[
\ker \mathfrak{A} = \{ (\mathbf{E}, \mathbf{H}, -B_p^{-1} N_e \mathbf{E})^\top \mid \mathbf{E} \in H_0(\text{curl} = 0, \Omega), \mathbf{H} \in H(\text{curl} = 0, \Omega) \}, \quad (5.25)
\]
where
\[
H_0(\text{curl} = 0, \Omega) := \{ \mathbf{E} \in H(\text{curl} = 0, \Omega) \mid \mathbf{E} \times \mathbf{n} = 0 \text{ on } \Gamma \}.
\]

**Proof.** First we look for a bounded operator \(C\) such that \(C \tilde{M} = M\). This is equivalent to
\[
C \tilde{M} = (M_e, 0)^\top,
\]
and since \(\tilde{M} = -B\) is invertible, the sole choice is
\[
CP = (M_e, 0)^\top (-B_p)^{-1} P, \forall P \in X.
\]
This implies that for all \((\mathbf{E}, \mathbf{H})^\top \in H\), one has
\[
CN(\mathbf{E}, \mathbf{H})^\top = (M_e, 0)^\top (-B_p)^{-1} N_e \mathbf{E} = (M_e (-B_p)^{-1} N_e \mathbf{E}, 0)^\top,
\]
and by (5.23), we get
\[
CN(\mathbf{E}, \mathbf{H})^\top = -(r_{ee} \mathbf{E}, 0)^\top.
\]
In conclusion we deduce that
\[
\mathcal{A} + CN = \mathcal{A}_u, \text{ and } B + \tilde{M} = 0.
\]
We can now apply Lemma 3.14.

First if \(\xi \in \mathbb{R}^*, \) then for \((\mathbf{E}, \mathbf{H}, P)^\top \in \ker(\xi \mathbb{I} - \mathfrak{A})\), by (3.23), we get \(P = 0\) and since \(\tilde{M}\) is invertible, we deduce that \(N_e \mathbf{E} = 0\). As \(N_e\) has a left inverse, \(\mathbf{E} = 0\). So, we are reduced to a pair \((0, \mathbf{H}) \in \ker(\xi \mathbb{I} - \mathcal{A}_u)\), which yields \(\mathbf{H} = 0\).

Secondly if \(\xi = 0\), (3.24) here reduces to
\[
\ker \mathfrak{A} = \ker \mathcal{L} \cap (\ker \mathcal{A}_u \times X).
\]
As \(\tilde{M}\) is invertible, this directly leads to (5.25). \(\square\)
As \( \sigma_p(\mathbb{A}) \cap t \mathbb{R} \) is not empty, a strong stability is not available on the whole space \( H \times X \), but we will reach it for its restriction to \( W = (\ker \mathbb{A})^\perp \). In other words,

\[
W = \{ (E, H, P)^\top \in H \times X \mid \int_{\Omega} (\epsilon E \cdot E' + \rho H \cdot H' - \rho P \cdot B_p^{-1} N_v E') \, dx = 0, \\
\forall E' \in H_0(\text{curl} = 0, \Omega), H' \in H(\text{curl} = 0, \Omega) \},
\]

which is also a Hilbert space equipped with the inner product of \( (\ker \mathbb{A})^\perp \).

Owing to (5.2). This obviously implies that problem (5.1) is well-posed in \( W \). Furthermore, as the restriction \( \mathbb{A}_0 \) of \( \mathbb{A} \) to \( H_0 \) is also well defined, we can use Lemma 3.14 and Corollary 3.18 in \( W \) to find the next strong stability result, whose proof is left to the reader.

**Theorem 5.7.** Let the assumption of Lemma 5.6 be satisfied. Then \( t \mathbb{R} \subset \rho(\mathbb{A}_0) \) and therefore, the semigroup \( T_0(t) \) generated by \( \mathbb{A}_0 \) is strongly stable.

This result combined with Corollary 4.8 leads to

**Theorem 5.8.** In addition to the assumption of Lemma 5.6, assume that \( r_{ee} \) has a uniform left-inverse (as \( r_{ee} \) is symmetric, this is equivalent to have a uniform inverse). Then \( T_0(t) \) is exponentially stable.

This result extends the exponential decay result obtained in Theorem 4.12 of [43] to a larger class of parameters but also to a larger energy space \( W \).

### 5.2. The generalized telegraph equation on networks.

In the present subsection, motivated by [28, 29, 44, 45, 26], we consider the generalized telegraph equation on a \( C^2 \)-network \( \mathcal{N} \). Let us then shortly recall the notion of \( C^2 \)-networks, which is simply those of [53] (we also refer to [2, 3, 52, 54, 8, 38, 41] for more details).

All graphs considered here are non-empty, finite, and simple. Let \( G \) be a connected topological graph imbedded in \( \mathbb{R}^n \), \( n \in \mathbb{N}^+ \), with \( n \) vertices \( V = \{ v_i : 1 \leq i \leq n \} \) and \( n \) edges \( E = \{ e_j : 1 \leq j \leq N \} \). Each edge \( e_j \) is a Jordan curve in \( \mathbb{R}^n \) and is assumed to be parametrized by its arc length parameter \( x_j \), such that the parametrization \( \pi_j : [0, l_j] \to e_j : x_j \mapsto \pi_j(x_j) \) is twice differentiable, i.e., \( \pi_j \in C^2([0, l_j], \mathbb{R}^n) \) for all \( 1 \leq j \leq N \).

We now define the \( C^2 \)-network \( \mathcal{N} \) associated with \( G \) as the union \( \mathcal{N} = E \cup V \).

The valency of each vertex \( v \) is denoted by \( \gamma(v) \). For shortness, we later on denote by \( V_{\text{ext}} = \{ v \in V : \gamma(v) = 1 \} \) the set of boundary (or exterior) vertices and \( V_{\text{int}} = V \setminus V_{\text{ext}} \), corresponding to the set of interior vertices. For each vertex \( v \), we also denote by \( J_v = \{ j \in \{1, \ldots, N \} : v \in e_j \} \) the set of edges adjacent to \( v \) and let \( N_v \) be the cardinal of \( J_v \). Note that if \( v \in I_{\text{ext}} \) then \( N_v \) is a singleton that we write \( \{ j_v \} \). For each vertex \( v \) and \( j \in N_v \), we further denote by

\[
\nu_j(v) = \begin{cases} 1 & \text{if } \pi_j(l_j) = v, \\ -1 & \text{if } \pi_j(0) = v, \end{cases}
\]

the normal vector in \( e_j \) at \( v \).
For a function $u : \mathcal{N} \to \mathbb{C}$, we set $u_j = u \circ \pi_j : [0, l_j] \to \mathbb{C}$, its “restriction” to the edge $e_j$ and use the abbreviations:

$$u_j(v) = u_j(\pi_j^{-1}(v)), \quad u'_j(v) = \frac{du_j}{dx_j}(\pi_j^{-1}(v)), \quad \partial_x u_j(v) = u_j(v)u'_j(v),$$

for a vertex $v \in e_j$. Finally, differentiations are carried out on each edge $e_j$ with respect to the arc length parameter $x_j$. Further for $u \in L^1(\mathcal{N})$, we write

$$\int_\mathcal{N} u \, dx = \sum_{j=1}^N \int_{l_j}^{l_j} u_j(x) \, dx.$$  

Finally, we denote by $PH^1(\mathcal{N})$ the set of piecewise $H^1$ functions on $\mathcal{N}$, in other words $u \in PH^1(\mathcal{N})$ if and only if $u_j \in H^1(0, l_j)$, for all $j = 1, \ldots, N$. This space is clearly a Hilbert space with its natural inner product. In the same spirit, let us set $PP_0(\mathcal{N})$ the set of piecewise constant functions on $\mathcal{N}$, in other words $u \in PP_0(\mathcal{N})$ if and only if $u_j \in P_0(0, l_j)$, for all $j = 1, \ldots, N$, where $P_0(0, l_j)$ is the space of constant functions from $(0, l_j)$ to $\mathbb{C}$.

Let us now fix a $C^2$-network $\mathcal{N}$. For each edge $e_j$, we also fix different real valued and non-negative functions $a_j, b_j, c_j, k_j, r_j$ and $g_j$ in $L^\infty(0, l_j)$ satisfying the following assumption

$$a_j \geq 1, \quad b_j \geq 1, \quad c_j \geq 1, \quad k_j + g_j \geq 1 \quad \text{a.e. in} \ (0, l_j), \quad \forall j = 1, \ldots, N. \quad (5.26)$$

These assumptions are in agreement with the physical setting from [28, 29]. We finally fix a decomposition of $V^\text{ext} = V^\text{Dir} \cup V^\text{Diss} \cup V^\text{ext}$ with two disjoint subsets $V^\text{Dir}$ and $V^\text{Diss}$.

With these assumptions, we consider the problem

$$\begin{cases}
V_{j,t} + g_j V_j + a_j I_{j,x} + k_j W_j &= 0, \quad \text{in} \ Q_j, \\
I_{j,t} + r_j I_j + b_j V_{j,x} &= 0, \quad \text{in} \ Q_j, \\
W_{j,t} + c_j W_j - V_j &= 0, \quad \text{in} \ Q_j, \\
\sum_{j \in J_s} \nu_j(v) I_j(v, t) &= 0, \quad \forall v \in V^\text{int}, t > 0, \\
V_j(v, t) - V_k(v, t) &= 0, \quad \forall j, k \in J_s, \forall v \in V^\text{int}, t > 0, \\
V_{j,v}(v, t) &= 0, \quad \forall v \in V^\text{Diss}, t > 0, \\
V_{j,v}(v, t) - \alpha_v \nu_j(v) I_j(v, t) &= 0, \quad \forall v \in V^\text{Diss}, t > 0, \\
V(\cdot, 0) = V_0, I(\cdot, 0) = I_0, W(\cdot, 0) = W_0 & \quad \text{in} \ N,
\end{cases}$$

(5.27)

where $Q_j := (0, l_j) \times (0, \infty)$ and $\alpha_v > 0$, for all $v \in V^\text{Diss}_v$. On each edge $e_j$, the generalized telegraph equation is a coupling between the usual telegraph equation where the electric unknowns are $V_j$ and $I_j$ representing the electric potential and the electric current respectively with a first order differential equation of parabolic type involving an auxiliary variable $W_j$ representing the non-local effects, see [28, 29].

In the above system, the transmission conditions at interior nodes are the so-called Kirchhoff conditions [51, 10, 11, 13, 14], where the first condition describes the charge conservation, while the second condition means that the voltages agree at the junction.

We recognize in the boundary condition in $v \in V^\text{Diss}$ a standard dissipative boundary condition used for $2 \times 2$ hyperbolic systems (like the condition (3) in [19] or the condition (52) in [25]) once the system is transformed into a diagonal one.
System \((5.27)\) enters in the abstract framework \((1.1)\) by defining \(H\), \(X\), \(A\), \(B\), \(M\) and \(N\) as follows: The Hilbert spaces \(H\) and \(X\) are defined by
\[
H = L^2(\mathcal{N})^2, \quad X = L^2(\mathcal{N}),
\]
with the inner product
\[
((V, I)^\top, (V', I')^\top)_H = \int_{\mathcal{N}} (\alpha^{-1}VV' + b^{-1}II')^2 \, dx, \forall (V, I)^\top, (V', I')^\top \in L^2(\mathcal{N})^2,
\]
\[
(W, W')_X = \int_{\mathcal{N}} \gamma W\tilde{W}' \, dx, \forall W, W' \in L^2(\mathcal{N}),
\]
where \(\gamma \in L^\infty(\mathcal{N})\) is defined by
\[
\gamma = a^{-1}(2k + gc + \sqrt{4kgc + g^2c^2})). \tag{5.28}
\]
Under the assumption \((5.26)\), we directly see that
\[
\gamma_j \gtrsim 1, \text{ a.e. in } (0, l_j), \quad \forall j = 1, \ldots, N.
\]
The operators \(B\), \(M\) and \(N\) are given by
\[
B : X \to X : W \mapsto -cW,
\]
\[
M : X \to H : W \mapsto (-kW, 0)^\top,
\]
\[
N : H \to X : (V, I)^\top \mapsto V,
\]
and are indeed bounded. Finally the operator \(A\) is defined as follows: the domain \(D(A)\) of \(A\) is given by
\[
D(A) = \{(V, I)^\top \in PH^1(\mathcal{N})^2 \text{ satisfying } (5.29) - (5.32) \text{ below}\},
\]
\[
\sum_{j \in J_v} \nu_j(v) I_j(v) = 0, \quad \forall v \in V_{\text{int}}, \tag{5.29}
\]
\[
V_j(v) - V_k(v) = 0, \quad \forall j, k \in J_v, \forall v \in V_{\text{int}}, \tag{5.30}
\]
\[
V_{j_v}(v) = 0, \quad \forall v \in V_{\text{ext}}^{\text{Dir}}, \tag{5.31}
\]
\[
V_{j_v}(v) - \alpha_v \nu_{j_v}(v) I_{j_v}(v) = 0, \quad \forall v \in V_{\text{ext}}^{\text{Diss}}, \tag{5.32}
\]
and
\[
A(V, I)^\top = -(aI_x + gV, bV_x + rI)^\top, \forall (V, I)^\top \in D(A).
\]

It is not difficult to check (see \cite[Lemma 3.10]{23}) that the adjoint of \(A\) is given by
\[
D(A^*) = \{(V, I)^\top \in PH^1(\mathcal{N})^2 \text{ satisfying } (5.29) - (5.31) \text{ and } (5.33) \text{ below}\},
\]
\[
V_{j_v}(v) + \alpha_v \nu_{j_v}(v) I_{j_v}(v) = 0, \quad \forall v \in V_{\text{ext}}^{\text{Diss}}, \tag{5.33}
\]
and
\[
A^*(V, I)^\top = (aI_x - gV, bV_x - rI)^\top, \forall (V, I)^\top \in D(A^*). \tag{5.34}
\]
By simple integrations by parts, one can show that

\[
\Re(A(V,I)^\top,(V,I)^\top)_H = -\int_{\mathcal{N}} (a^{-1}g|V|^2 + b^{-1}r|I|^2) \, dx
\]
\[
- \sum_{v \in V_{\text{ext}}} \alpha_v |I_{j_v}(v)|^2, \forall (V,I)^\top \in D(A),
\]

\[
\Re(A^*(V,I)^\top,(V,I)^\top)_H = -\int_{\mathcal{N}} (a^{-1}g|V|^2 + b^{-1}r|I|^2) \, dx
\]
\[
- \sum_{v \in V_{\text{ext}}} \alpha_v |I_{j_v}(v)|^2, \forall (V,I)^\top \in D(A^*),
\]

hence, \( A \) and \( A^* \) are dissipative. As \( A \) is clearly densely defined and closed in \( H \), by [47, Corollary 1.4.4], \( A \) generates a \( C_0 \) semigroup of contractions. Hence, problem (5.27) is well-posed in \( H \times X \).

Let us now show that \( A \) is dissipative. Using (5.35), (3.3) reduces to

\[
\Re(A(V,I,W)^\top,(V,I,W)^\top)_{H \times X} = -\int_{\mathcal{N}} (a^{-1}g|V|^2 + b^{-1}r|I|^2 + \gamma c|W|^2) \, dx
\]
\[
+ \int_{\mathcal{N}} (\gamma - \frac{k}{a}) \Re(WV) \, dx
\]
\[
- \sum_{v \in V_{\text{ext}}} \alpha_v |I_{j_v}(v)|^2,
\]

for all \((V,I,W)^\top \in D(A) \times X\). But the choice (5.28) of \( \gamma \) allows to show that

\[
2(\gamma_j - \frac{k_j}{\alpha_j}) \Re(wv) \leq \gamma_j c_j |w|^2 + \frac{q_j}{\alpha_j} |v|^2 \quad \text{a.e. in } (0,l_j), \forall v,w \in \mathbb{C},
\]

and therefore, the previous identity can be transformed into

\[
\Re(A(V,I,W)^\top,(V,I,W)^\top)_{H \times X} \leq -\int_{\mathcal{N}} (\frac{q}{2a} |V|^2 + b^{-1}r|I|^2 + \frac{2c}{2} |W|^2) \, dx
\]
\[
- \sum_{v \in V_{\text{ext}}} \alpha_v |I_{j_v}(v)|^2,
\]

for all \((V,I,W)^\top \in D(A) \times X\). This implies in particular that (3.16) holds. Hence, by Lemma 3.9, we obtain the next result.

Lemma 5.9. One has

\[
\ker(i\xi I - A) = \{0\}, \forall \xi \in \mathbb{R}^*,
\]

while

\[
\ker A = \{0\} \times K_0 \times \{0\},
\]

where

\[
K_0 = \{I \in PP_0(\mathcal{N}) \text{ satisfying (5.29) and (5.39)-(5.40) below}\}.
\]

Proof. Since \( \ker N = \{0\} \times L^2(\mathcal{N}) \), Lemma 3.9 yields

\[
\ker(i\xi I - A) = \{(0,I,0)^\top | (0,I)^\top \in \ker(i\xi I - A)\}, \forall \xi \in \mathbb{R}.
\]

Now by the definition of \( A \), if \((0,I)^\top \in \ker(i\xi I - A)\), one gets

\[
I_x = (i\xi + r)I = 0.
\]

This directly implies that \( I = 0 \) if \( \xi \neq 0 \). On the contrary if \( \xi = 0 \), then \( I \) belongs to \( PP_0(\mathcal{N}) \) and satisfies (5.39). But as \((0,I)^\top \) belongs to \( D(A) \), it satisfies (5.29)-(5.32), which here reduces to (5.39) and (5.40). \( \square \)
 Remark 5.10. The fact that \( \ker A \) may be reduced to \( \{0\} \) depends on the number of constraints generated by (5.29), (5.39), and (5.40), that correspond to a homogeneous linear system in \( N \) unknowns. This system has a trivial solution if and only if the rank of its associated matrix is \( N \). This rank actually depends on the geometry of the network, the property of \( r \) and the set \( V^\text{ext} \). Let us mention some cases for which \( \ker A = \{0\} \).

1. If \( \int_0^j r_j \, dx > 0 \), for all \( j = 1, \ldots, N \), then (5.39) suffices to deduce that \( \ker A = \{0\} \).

2. If \( N \) is a tree and \( V^\text{ext} \) contains the set \( V^\text{ext} \) of all other exterior vertices except one (called the root), then \( \ker A = \{0\} \). Indeed in such a case, we may classify the edges by generation, the first generation being the edge having the root as exterior vertex and the last generation the edges having one vertex in \( V^\text{ext} \). Then by (5.40), \( I_j = 0 \) for all edges \( e_j \) of the last generation. But since one edge \( e_{j^*} \) of the next to last generation has a vertex in common with some edges of the last generation, by (5.29) and the previous property, we find \( I_{j^*} = 0 \). By iteration, from one generation to the previous one, we arrive at \( I = 0 \).

3. If \( N \) is a tree, \( V^\text{ext} = \emptyset \) but \( \int_0^j r_j \, dx > 0 \), for all edge \( e_j \) of the last generation, then clearly \( \ker A = \{0\} \).

4. The situation may be more complex if \( N \) has some cycles, but it is easy to find situation for which \( \ker A = \{0\} \). For instance if \( N \) is reduced to one cycle, and if \( \int_0^j r_j \, dx > 0 \), for one edge \( e_j \), then \( \ker A = \{0\} \).

If \( \sigma_p(A) \cap i\mathbb{R} = \{0\} \), we cannot obtain a strong stability on the whole space \( H \times X \). Hence, as before we will restrict ourselves to \( H_0 \times L^2(N) \), where

\[
H_0 = L^2(N) \times \{ I \in L^2(N) \mid \int_N b^{-1} I' \, dx = 0, \forall I' \in K_0 \},
\]

which is also a Hilbert space equipped with the inner product of \( H \). Let us notice that the restriction \( A_0 \) of \( A \) defined by \( D(A_0) = D(A) \cap (H_0 \times X) \) and

\[
A_0(V, I, W)^\top = A(V, I, W)^\top, \forall (V, I, W)^\top \in D(A) \cap (H_0 \times X),
\]

is well defined since for all \( (V, I, W)^\top \in D(A) \) and \( I' \in K_0 \), we have

\[
\int_N b^{-1}(bV_x + rI)' \, dx = \int_N V_x I' \, dx,
\]

as \( rI' = 0 \). By integration by parts in each edge and using the fact that \( V \) satisfies (5.30) and (5.31) and \( I' \) is piecewise constant and satisfies (5.29) and (5.31), we have

\[
\int_N V_x I' \, dx = 0,
\]

which shows that

\[
R(A_0) \subset H_0 \times X.
\]

This obviously implies that problem (5.27) is well-posed in \( H_0 \times X \). Furthermore, as the restriction \( A_0 \) of \( A \) to \( H_0 \) is also well defined, we can use Lemma 3.9 and Corollaries 3.10 and 3.12 in \( H_0 \times X \).

**Theorem 5.11.** \( i\mathbb{R} \subset \rho(A_0) \); therefore, the semigroup \( T_0(t) \) generated by \( A_0 \) is strongly stable.

**Proof.** By Lemma 5.9 and the definition of \( A_0 \), we have

\[
\sigma_p(A_0) \cap i\mathbb{R} = \emptyset.
\]
Now to apply Corollaries 3.10 and 3.12, we chose the operator $C$ as follows

$$CW = -(kW,0)^T, \forall W \in X,$$

so that $C$ is bounded from $X$ to $H_0$ and

$$CN(V,I)^T = -(kV,0)^T.$$ 

This yields

$$(A + CN)(V,I)^T = A(V,I)^T - (kV,0)^T, \forall (V,I)^T \in D(A) \cap H_0.$$ 

Due to the assumption $g + k \geq 1$, it is not difficult to show that this operator generates an exponentially stable semigroup, by Lemma 4.1, (5.15) and (5.16) hold. The first property and Corollary 3.10 allow to conclude that $R(\xi \mathbb{I} - A_0)$ is closed, for all $\xi \in \mathbb{R}$.

Now due to Exercise 3.5 and (5.34), one has

$$A^*(V,I,W)^T = \begin{pmatrix} aI_x - gV + a\gamma W \\ bV_x - rI \\ -cW - a^{-1}\gamma^{-1}kV \end{pmatrix}, \forall (V,I,W)^T \in D(A^*) := D(A^*) \times X.$$ 

By (5.36), we see that

$$\Re(A^*(V,I,W)^T, (V,I,W)^T)_{H \times X} = -\int_N (\alpha^{-1}|g|^2 + b^{-1}|r|^2 + \gamma c|W|^2) \, dx$$

$$+ \int_N (\gamma - \frac{k}{\alpha}) \Re(W \bar{V}) \, dx - \sum_{v \in V^{\text{Dis}}_{\text{ext}}} \alpha_v |I_{j_v}(v)|^2,$$

for all $(V,I,W)^T \in D(A^*) \times X$. By the choice (5.28) of $\gamma$, we obtain

$$\Re(A^*(V,I,W)^T, (V,I,W)^T)_{H \times X} \leq -\int_N \left(\frac{\alpha}{2a} |V|^2 + b^{-1}|r|^2 + \frac{\gamma}{2} |W|^2 \right) \, dx - \sum_{v \in V^{\text{Dis}}_{\text{ext}}} \alpha_v |I_{j_v}(v)|^2,$$

for all $(V,I,W)^T \in D(A^*) \times X$. This allows to show that

$$\ker(\xi \mathbb{I} - A^*) = \{0\}, \forall \xi \in \mathbb{R},$$

and that

$$\ker A^* = \ker A.$$ 

Consequently

$$\sigma_p(\mathbb{A}_0^*) \cap \mathbb{R} = \emptyset,$$

and the stability property of $T_0$ follows from Corollary 3.12.

The previous properties allow us to state the following polynomial stability.

**Theorem 5.12.** The semigroup $T_0(t)$ generated by $\mathbb{A}_0$ is polynomially stable with a decay in $t^{-\frac{1}{2}}$, namely

$$\|T(t)(V,I,P)^T\|_{H \times X} \lesssim t^{-\frac{1}{2}} \|V,I,P)^T\|_{D(A) \times X}, \forall (V,I,P)^T \in (D(A) \cap H_0) \times X.$$ 

**Proof.** We simply apply Theorem 4.10 with $C$ defined by (5.41), see the proof of Theorem 5.3.

This result extends to arbitrary networks Theorem 4.5 from [44] obtained for one interval and $V_{\text{ext}}^{\text{Dis}} = \emptyset$.

By Corollary 4.8 an exponential stability result for $T_0$ is also available as soon as $A$ generates an exponentially stable semigroup on $H_0$. This is in particular true in the following cases:

1. If $\mathcal{N}$ is a tree, $V_{\text{ext}}^{\text{Dis}}$ contains the set of all other exterior vertices except one,
and all $a_j$ and $b_j$ are constant, then by Corollary 5.8 and Theorem 4.2 of [45], $A$ generates an exponentially stable semigroup on $H$.

2. If $r$ is continuous on $\mathcal{N}$, if $r_j \in W^{1,\infty}(0,t_j)$ for all $j = 1, \cdots, N$ and if
\[
 r + g \gtrsim 1,
\]
then $A$ generates an exponentially stable semigroup on $H_0$. Indeed by Lemma 4.1, it suffices to show that (4.2) holds. Assuming that this is wrong, we can find a sequence of real numbers $\xi_n \to +\infty$ and a sequence of vectors $(V_n, I_n)^\top$ in $D(A) \cap H_0$ with \(\|(V_n, I_n)^\top\|_H = 1\) such that
\[
 \begin{align*}
 \xi_n V_n + a I_{n,x} + g V_n &\to 0 \text{ in } L^2(\mathcal{N}), \\
 \xi_n I_n + b V_{n,x} + r I_n &\to 0 \text{ in } L^2(\mathcal{N}).
\end{align*}
\]
By the dissipativeness property (5.35) of $A$, we directly deduce that
\[
 \sqrt{r} I_n \to 0 \text{ in } L^2(\mathcal{N}) \quad \text{and} \quad \sqrt{g} V_n \to 0 \text{ in } L^2(\mathcal{N}),
\]
\[
 I_n(v) \to 0, \forall v \in V_{\text{ext}}^{\text{Diss}}.
\]

Multiplying (5.44) by $\frac{\xi_n}{a} \tilde{V}_n$ and integrating in $\mathcal{N}$, we find
\[
 \int_{\mathcal{N}} \left( \frac{\xi_n}{a} |V_n|^2 + r I_{n,x} \tilde{V}_n \right) dx \to 0.
\]
Integrating by parts, using the boundary conditions (5.29)-(5.32) satisfied by the pair $(V_n, I_n)$, and (5.46)-(5.47), we get
\[
 \int_{\mathcal{N}} \left( \frac{\xi_n}{a} |V_n|^2 - I_n(r_x \tilde{V}_n + r_n \tilde{V}_{n,x}) \right) dx \to 0.
\]
Hence, (5.45) yields
\[
 \int_{\mathcal{N}} \left( \frac{\xi_n}{a} |V_n|^2 - I_n(r_x \tilde{V}_n + r_n b^{-1} \xi_n \tilde{I}_n) \right) dx \to 0.
\]
Dividing by $\xi_n$ and using $\|(V_n, I_n)^\top\|_H = 1$, we arrive at
\[
 \int_{\mathcal{N}} \left( \frac{r}{a} |V_n|^2 - \frac{r}{b} |I_n|^2 \right) dx \to 0.
\]
Using (5.46), one deduces that
\[
 \sqrt{r} V_n \to 0 \text{ in } L^2(\mathcal{N}),
\]
and by (5.46) and the assumption (5.43), we obtain that
\[
 V_n \to 0 \text{ in } L^2(\mathcal{N}).
\]

Now by multiplying (5.44) by $\frac{\xi_n}{a} \tilde{V}_n$, integrating in $\mathcal{N}$, we find as before
\[
 \int_{\mathcal{N}} \left( \frac{r}{a} |V_n|^2 - \frac{r}{b} |I_n|^2 \right) dx \to 0,
\]
which leads to
\[
 I_n \to 0 \text{ in } L^2(\mathcal{N}),
\]
and to a contradiction.
5.3. The heat equation with memory effects. In this subsection $\Omega$ is a bounded domain of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, with a Lipschitz boundary $\Gamma$.

The heat equation with a memory effect corresponds to the problem (see [20, 50, 35])

$$
\begin{aligned}
&u_t - \Delta u + \int_0^t k(x, t - s)u(s) \, ds = 0 \text{ in } \Omega \times [0, +\infty[, \\
u = 0 \text{ on } \Gamma \times [0, +\infty[, \\
u(x, 0) = u_0(x), \text{ in } \Omega,
\end{aligned}
$$

(5.48)

where $u$ is the temperature and $k$ is a memory kernel. Here we assume that $k$ is in the form

$$
k(x, t) = a(x)e^{-b(x)t}, \forall x \in \Omega, t > 0,
$$

where $a, b \in L^\infty(\Omega, \mathbb{R})$ are non negative functions such that

$$\Omega_a = \{x \in \Omega \mid a(x) > 0\}
$$

has a positive Lebesgue measure. This last assumption is meaningful since otherwise (5.48) reduces to the sole heat equation. As in [35], the case $b = 0$ is of particular interest since in that case $k(x, t)$ is not integrable in $(0, \infty)$.

Let us show how problem (5.48) enters in our framework. Indeed introduce the auxiliary variable

$$
p(x, t) = \int_0^t e^{-b(x)(t-s)}u(s) \, ds \text{ in } \Omega_a \times [0, +\infty),
$$

we see that it satisfies

$$
p_t = u_t - b(x)p(x, t).
$$

This property implies that the pair $(u, p)^\top$ satisfies

$$
\begin{aligned}
&u_t - \Delta u + a\tilde{p} = 0 \text{ in } \Omega \times [0, +\infty[, \\
p_t = u - bp \text{ in } \Omega_a \times [0, +\infty[, \\
u = 0 \text{ on } \Gamma \times [0, +\infty[, \\
u(x, 0) = u_0(x), p(x, 0) = p_0 \text{ in } \Omega,
\end{aligned}
$$

(5.49)

where $p_0 = 0$ and $\tilde{p}$ is the extension of $p$ by zero outside $\Omega_a$. Note that conversely if $(u, p)^\top$ is solution of this system with $p_0 = 0$, then $u$ is a solution of (5.48).

Now (5.49) enters in the framework (1.1) by defining $H, X, A, B, M$ and $N$ as follows: $H = L^2(\Omega)$ with the usual inner product of $L^2(\Omega)$, while $X = L_a^2(\Omega_a)$ is the space of square integrable functions in $\Omega_a$ with respect to the measure $a(x)dx$, which is a Hilbert space with the inner product

$$
(p, p')_a = \int_{\Omega_a} p(x)p'(x)a(x) \, dx, \forall p, p' \in L_a^2(\Omega_a),
$$

the operators $B, M$ and $N$ are given by

$$
B : X \rightarrow X : p \rightarrow -bp,
M : X \rightarrow H : p \rightarrow -a\tilde{p},
N : H \rightarrow X : u \rightarrow au,
$$

are indeed bounded as $a$ and $b$ are in $L^\infty(\Omega)$. Finally the operator $A$ is given by

$$
D(A) = D(\Delta_{\text{Dir}}, \Omega)
$$

with

$$
D(\Delta_{\text{Dir}}, \Omega) := \{ u \in H^1_0(\Omega) \mid \Delta u \in L^2(\Omega) \},
$$

(5.50)
and
\[ A u = \Delta u, \quad \forall u \in D(A). \]
Since it is well-known that \( A \) generates an exponentially stable semigroup on \( L^2(\Omega) \), problem (5.49) is well posed in \( L^2(\Omega) \times L^2_a(\Omega_a) \). Now by a standard Green’s formula, we have
\[ \Re(\mathbb{A}(u,p)^\top, (u,p)^\top)_{H \times X} = -\int_\Omega (|\nabla u|^2 + ab|p|^2) \, dx, \forall (u,p)^\top \in D(A) \times X. \quad (5.51) \]
Hence, \( \mathbb{A} \) is dissipative and in particular (3.6) holds because Poincaré’s inequality yields
\[ \|u\|_\Omega \lesssim |u|_{1,\Omega}, \forall u \in H^1_0(\Omega). \]
By Lemma 3.6, we directly deduce the next result.

**Lemma 5.13.** One always has
\[ \sigma_p(\mathbb{A}) \cap i\mathbb{R} = \emptyset. \]

Now we restrict ourselves to two illustrative examples, namely the case \( b \gtrsim 1 \) on \( \Omega_a \) and the case \( b(x) = 0 \) on an open subset \( M_b \) of \( \Omega_a \) such that \( \text{meas } M_b > 0 \).

**Theorem 5.14.** If \( b \gtrsim 1 \) on \( \Omega_a \), then the semigroup \( T(t) \) generated by \( \mathbb{A} \) is exponentially stable.

**Proof.** It is readily checked that the adjoint of \( \mathbb{A} \) is given by
\[ \mathbb{A}^*(u,p)^\top = (\Delta u + ap, -bp-u)^\top, \forall (u,p)^\top \in D(A^*) = D(A), \]
and that
\[ \Re(\mathbb{A}^*(u,p)^\top, (u,p)^\top)_{H \times X} = -\int_\Omega (|\nabla u|^2 + ab|p|^2) \, dx, \forall (u,p)^\top \in D(A) \times X. \quad (5.52) \]
This allows to show that \( \sigma_p(\mathbb{A}^*) \cap i\mathbb{R} = \emptyset \) and by Corollary 3.8 we deduce that \( i\mathbb{R} \subset \rho(\mathbb{A}) \). Hence, the semigroup \( T(t) \) generated by \( \mathbb{A} \) is strongly stable. As \( A \) generates an exponentially stable semigroup on \( L^2(\Omega) \) we conclude by Corollary 4.5.

The second case if more complex since we have the next result.

**Lemma 5.15.** If \( b = 0 \) on an open subset \( M_b \) of \( \Omega_a \) such that \( \text{meas } M_b > 0 \), then
\[ i\mathbb{R} \cap \sigma(\mathbb{A}) = \{0\}. \]

**Proof.** The fact that \( i\mathbb{R}^* \subset \rho(\mathbb{A}) \) is a consequence of Lemma 5.13 and using the properties of \( \mathbb{A}^* \) mentioned before.

Now assume that \( 0 \in \rho(\mathbb{A}) \), then for any \( (f,g)^\top \in L^2(\Omega) \times L^2_a(\Omega_a) \), there exists a unique solution \( (u,p) \in D(A) \times L^2_a(\Omega_a) \) of
\[ \Delta u - ap = f, \]
\[ u - bp = g. \]
Restricting this second identity to \( M_b \), we get
\[ u = g \text{ in } M_b, \]
which is impossible as soon as \( g \) is chosen such that \( g \) does not belong to \( H^1(M_b) \).

This result forbids the use of Lemmas 4.1 and 4.2, but a recent variant allows to manage such a case (see [40, Proposition 3.1] for a weaker variant).
Theorem 5.16 (Theorem 3.6 of [49]). Let \( (T(t))_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on a Hilbert space \( \mathcal{H} \) generated by \( A \). Assume that \( \sigma(A) \cap \mathbb{R} \subset \{0\} \), that \( \sup_{|\tau|} \| (i\tau \mathbb{1} - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty \) and that there exists a continuous non-decreasing function of positive increase \( m : [1, \infty) \to (0, \infty) \) (see [49, (2.3)]/ such that \( \| (i\tau \mathbb{1} - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} \leq m(\frac{1}{|\tau|}) \), for all \( \tau \in \mathbb{R} \) such that \( 0 < |\tau| \leq 1 \). Then there exists a positive constant \( T_1 \) such that
\[
\| T(t)A(I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} \lesssim \frac{1}{m^{-1}(t)}, \forall t > T_1.
\]

With the help of this Theorem, one can prove the next result.

Theorem 5.17. Under the assumption of Lemma 5.15, then the semigroup \( T(t) \) generated by \( \mathcal{A} \) satisfies
\[
\| T(t)\mathcal{A}(I - \mathcal{A})^{-1}(u, p)^\top \|_{H \times X} \lesssim t^{-\frac{1}{2}} \| (u, p)^\top \|_{H \times X}, \forall t > T_1, (u, p)^\top \in H \times X,
\]
for some \( T_1 > 0 \).

Proof. By Lemma 5.15, the first condition of Theorem 5.16 is satisfied. Now as \( \mathcal{A} \) generates an exponentially stable semigroup on \( L^2(\Omega) \) by Lemma 4.3 (and Lemma 4.1), we immediately conclude that
\[
\| (i\tau \mathbb{1} - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} \lesssim 1, \forall \tau \in \mathbb{R} : |\tau| \geq 1,
\]
which proves the second condition of Theorem 5.16. Let us now concentrate on the last condition. Fix \( \tau \in \mathbb{R} \) such that \( 0 < |\tau| \leq 1 \), and \( (f, g)^\top \in H \times X \) and let \( (u, p)^\top = (i\tau \mathbb{1} - A)^{-1}(f, g)^\top \) that satisfies equivalently
\[
(\tau \mathbb{1} - A)u + a\bar{p} = f,
\]
\[
(\tau \mathbb{1} - A)p - u = g.
\]
As \( \tau \mathbb{1} - A \) is always different from zero, we may write
\[
p = (i\tau + b)^{-1}(u + g) \tag{5.54}
\]
and eliminate \( p \) in the first identity to obtain
\[
\tau u - \Delta u + a\bar{p} = f + a(i\tau + b)^{-1}\bar{g}.
\]
Multiplying this identity by \( \bar{u} \), integrating in \( \Omega \) and using Green’s formula, we arrive at
\[
\int_{\Omega} (i\tau + a(i\tau + b)^{-1})|u|^2 + |\nabla u|^2 \, dx = \int_{\Omega} (f + a(i\tau + b)^{-1}\bar{g}) \bar{u} \, dx.
\]
As \( \Re(i\tau + a(i\tau + b)^{-1}) = \frac{ab}{\tau^2 + b^2} \) is non negative, taking the real part of the previous identity one finds
\[
\int_{\Omega} |\nabla u|^2 \, dx \leq \Re \int_{\Omega} (f + a(i\tau + b)^{-1}\bar{g}) \bar{u} \, dx.
\]
Using Cauchy-Schwarz’s inequality and Poincaré’s inequality we obtain
\[
\| u \|^2_{L^2(\Omega)} \lesssim \int_{\Omega} |\nabla u|^2 \, dx \leq \left( \| f \|_{L^2(\Omega)} + \sup_{\Omega_a} \frac{\sqrt{a}}{|\tau + b|^2} \| \sqrt{a}g \|_{L^\infty(\Omega_a)} \right) \| u \|_{\Omega}.
\]
As \( |\tau + b| \geq |\tau| \), we find
\[
\| u \|_{\Omega} \lesssim \| f \|_{\Omega} + |\tau|^{-1}\| \sqrt{a}g \|_{L^\infty(\Omega_a)}.
\]
As \( |\sqrt{a}p| \lesssim |\tau|^{-1}\| f \|_{\Omega} + |\tau|^{-2}\| \sqrt{a}g \|_{L^\infty(\Omega_a)} \), this last estimate yields
\[
\| \sqrt{a}p \|_{L^\infty(\Omega_a)} \lesssim |\tau|^{-1}\| f \|_{\Omega} + |\tau|^{-2}\| \sqrt{a}g \|_{L^\infty(\Omega_a)}.
\]
In conclusion, we have
\[ \|(u, p)\|_{H \times X} \lesssim |\tau|^{-2} \|(f, g)\|_{H \times X}, \]
which proves that
\[ \|(rI - A)^{-1}\|_{\mathcal{L}(H \times X)} \lesssim m(|\tau|^{-1}), \forall 0 < |\tau| \leq 1, \]
with \( m(s) = s^2 \), that is clearly a continuous non-decreasing function of positive increase. This property leads to (5.53) using Theorem 5.16 as \( m^{-1}(t) = \sqrt{t} \).

Let us end up this subsection by performing a link with the polynomial stability result from [35].

**Lemma 5.18.** In addition to the assumption of Lemma 5.15, suppose that \( b = 0 \) and
\[ a \gtrsim 1 \text{ in } \Omega, \]
then the semigroup \( T(t) \) generated by \( \mathcal{A} \) satisfies
\[ \| T(t)(u_0, 0)\|_{H \times X} \lesssim t^{-\frac{1}{2}} \| u_0 \|_\Omega, \forall t > T_1, u_0 \in L^2(\Omega), \] (5.55)
for some \( T_1 > 0 \).

**Proof.** We readily check that \((u, p)^\top\) defined by
\[ u = -u_0 \]
\[ p = -\frac{u_0}{a}, \]
satisfies \((u, p)^\top \in H \times X \) and
\[ \mathcal{A}(I - \mathcal{A})^{-1}(u, p)^\top = (u_0, 0)^\top. \]
Furthermore, by our additional assumption, we have
\[ \|(u, p)^\top\|_{H \times X} \lesssim \| u_0 \|_\Omega. \]
The conclusion then follows from (5.53).

The estimate (5.55) yields the estimate
\[ \| u(t, \cdot)\|_H \lesssim \left( \frac{\log t}{t} \right)^{\frac{1}{2}} \| u_0 \|_{D(A)}, \]
for the solution of \( u \) of (5.48) and should be compared with point (iii) of Theorem 1.2 of [35], whose correct formulation is
\[ \| u(t, \cdot)\|_H \lesssim t^{-2} \| u_0 \|_\Omega, \forall u_0 \in L^2(\Omega), \]
where we recall that their assumptions are \( b = 0 \) and \( a \) is a positive constant.

### 5.4. An ODE-hyperbolic system.

In this subsection \( \Omega \) is still a bounded domain of \( \mathbb{R}^d \), \( d \in \mathbb{N}^* \), with a Lipschitz boundary \( \Gamma \).

The stabilization of an unstable linear ODE via a damped wave equation retains the attention of different authors, see for instance [30, 36]. Here we consider the following cascade ODE-hyperbolic system
\[
\begin{cases}
  P_t - BP - \varepsilon Nu = 0 & \text{in } ]0, +\infty[, \\
  u_{tt} - \Delta u - D = 0 & \text{in } \Omega \times ]0, +\infty[, \\
  u = 0 & \text{on } \Gamma \times ]0, +\infty[, \\
  u(x, 0) = u_0(x), \text{ in } \Omega, u_t(x, 0) = u_1(x), \text{ in } \Omega, P(0) = P_0, \text{ in } \Omega,
\end{cases}
\] (5.56)
where $P \in \mathbb{C}^n$ is the ODE state (that depends only on the time variable), $u \in \mathbb{C}$ is the state of the wave equation (that depends on the space and time variables), and $D$ is the scalar input to the system and is designed below. The cascade system is depicted in Figure 1.

The data are the matrices $B \in \mathbb{C}^{n \times n}$ and $N_0 \in \mathbb{C}^n$ so that the pair $(B, N_0)$ is stabilizable, in order words there exists a matrix $K \in \mathbb{C}^{1 \times n}$ such that $B := B + N_0 K$ is Hurwitz, namely all eigenvalues $\lambda$ of $B + N_0 K$ satisfies $\Re \lambda < 0$. With the help of $N_0$, we define $N$ by

$$Nv = (v, h)_{1, \Omega} N_{0}, \forall v \in H^1_0(\Omega),$$

where $h$ is a fixed function that is supposed to belong to $D(\Delta_{\text{Dir}}, \Omega)$, see (5.50) and to satisfy

$$\int_{\Omega} |\nabla h|^2 \, dx = 1,$$

and we recall that

$$(v, h)_{1, \Omega} = \int_{\Omega} \nabla v \cdot \nabla h \, dx, \forall v, h \in H^1_0(\Omega),$$

is an inner product in $H^1_0(\Omega)$. In that way, the operator $N$ is a bounded operator from $H^1_0(\Omega)$ into $\mathbb{C}^n$, whose adjoint $N^*$ is given by

$$N^* X = (N^*_0 X) h, \forall X \in \mathbb{C}^n.$$

Finally $\varepsilon$ is a positive real parameter fixed below.

Inspired from [30, 36], the input $D$ is designed by using the following transformation

$$\tilde{u}(x, t) = u(x, t) - \frac{h(x)}{\varepsilon} K P(t),$$

where $h$ is the function introduced above. In that way, we see that the ODE in (5.56) becomes

$$P_t = BP + \varepsilon N \tilde{u}.$$  

(5.58)

Now we chose $D$ so that $\tilde{u}$ satisfies a wave type equation with damping

$$\ddot{u} + (\Delta - \varepsilon^2 N^* N) \tilde{u} - \alpha \ddot{u} - \varepsilon N^* (\alpha I + B) P,$$  

(5.59)

where $\alpha$ is a fixed positive constant. Such a choice is always possible, indeed first using (5.57) and the ODE from (5.56), we see that

$$\ddot{u}_t = u_t - \frac{h}{\varepsilon} K (BP + \varepsilon Nu),$$

(5.60)

and hence,

$$\ddot{u}_t = u_{tt} - \frac{h}{\varepsilon} K (B^2 P + \varepsilon Bu + \varepsilon Nu_t).$$

(5.57)
Therefore, by the wave type equation from (5.56), we get
\[ \ddot{u} = \Delta u + D - \frac{h}{\epsilon} K(B^2 P + \epsilon B N u + \epsilon N u_t). \]

Similarly using (5.57) and (5.60), we notice that (5.59) is equivalent to
\[ \ddot{u} = \Delta u - \Delta h \epsilon K P - \epsilon^2 N^* N \left(u - \frac{h}{\epsilon} K P\right) - \alpha \left(u_t - \frac{h}{\epsilon} K(BP + \epsilon N u)\right) - \epsilon N^*(\alpha I + B)P. \]

Comparing these two identities, the unique possibility for \( D \) is
\[ D = \frac{h}{\epsilon} K(B^2 P + \epsilon B N u + \epsilon N u_t) - \Delta h \epsilon K P - \epsilon^2 N^* N \left(u - \frac{h}{\epsilon} K P\right) - \alpha \left(u_t - \frac{h}{\epsilon} K(BP + \epsilon N u)\right) - \epsilon N^*(\alpha I + B)P. \]

With this choice of \( D \), we see that the pair \((\bar{u}, P)\) is solution of (5.58) and (5.59). Now we make the link with system (1.1). By setting \( U = (\bar{u}, \bar{v})^\top \) with
\[ \bar{v} = \bar{u}_t - N^* X, \]
we directly see that \((U, P)^\top\) satisfies (formally) (1.1) with
\[ N(u, v)^\top = \epsilon N u, \quad MP = -N^* P = -\epsilon(N^* P, 0)^\top, \]
and
\[ A(u, v)^\top = (v, \Delta u - \alpha v)^\top. \]

Clearly the Hilbert setting for this system is \( H = H^1_0(\Omega) \times L^2(\Omega) \) with its natural inner product
\[ \langle (u, v)^\top, (u_1, v_1)^\top \rangle_H = \int_\Omega (\nabla u \cdot \nabla u_1 + v \overline{v}_1) dx, \forall (u, v)^\top, (u_1, v_1)^\top \in H, \]
while \( X = \mathbb{C}^n \) is equipped with the Euclidean inner product.

Since it is well known that \( A \) generates an exponentially decaying semigroup in \( H \), problem (5.58) and (5.59) is wellposed in \( H \times X \) (hence, problem (5.56) is also wellposed).

We now analyze the strong stability of the associated semigroup. Here contrary to the other examples, \( A \) is not necessarily dissipative because we do not assume that \( B \) is dissipative.

**Lemma 5.19.** For \( \epsilon > 0 \) small enough, the semigroup \( T(t) \) generated by \( \lambda \) is bounded and strongly stable.

**Proof.** As \( A \) (resp. \( B \)) generates an exponentially decaying semigroup in \( H \) (resp. \( X \)), we know that
\[ \mathbb{C}_+ := \{ \lambda \in \mathbb{C} | \Re \lambda \geq 0 \} \subset \rho(A) \cap \rho(B), \]
and that
\[ \| (\lambda I - A)^{-1} \| + \| (\lambda I - B)^{-1} \| \lesssim 1, \forall \lambda \in \mathbb{C}_+. \]  
(5.63)

Now fix \( \lambda \in \mathbb{C}_+ \) and \((F, G)^\top \in H \times X \) and let us look for a solution \((U, P)^\top \in D(A) \times X \) of
\[ (\lambda I - A)(U, P)^\top = (F, G)^\top. \]  
(5.64)
By the definition of $A$, we get equivalently
\[(\lambda I - A)U = \varepsilon(N^*P, 0)^\top + F,\]
\[(\lambda I - B)P - \varepsilon(N, 0)U = G.\]

Since $(\lambda I - A)$ is invertible, we find
\[U = (\lambda I - A)^{-1}(\varepsilon(N^*P, 0)^\top + F),\]
and inserting this expression in the second identity we find
\[(\lambda I - B)P - \varepsilon^2(N, 0)(\lambda I - A)^{-1}(N^*P, 0)^\top = G - (\lambda I - A)^{-1}F.\]
Again as $\lambda I - B$ is invertible, we get equivalently
\[(I - \varepsilon^2 R_\lambda)P = (\lambda I - B)^{-1}
\left(G - (\lambda I - A)^{-1}F \right),\]
where
\[R_\lambda := (\lambda I - B)^{-1}(N, 0)(\lambda I - A)^{-1}(N^*, 0)^\top.\]

Now using (5.63) and the fact that $N$ is bounded operator from $H^1_0(\Omega)$ into $\mathbb{C}^n$, we deduce that
\[\|R_\lambda\|_{L(\mathbb{C}^n)} \leq C,\]
for some positive constant $C$ (independent of $\lambda \in \bar{C}_+$. Consequently choosing $\varepsilon$ such that $\varepsilon^2 C \leq \frac{1}{2}$, we deduce that $I - \varepsilon^2 R_\lambda$ is invertible, its inverse being given by the Neumann series and that
\[\|(I - \varepsilon^2 R_\lambda)^{-1}\|_{L(\mathbb{C}^n)} \leq 2.\]

Hence, for such a $\varepsilon$, there exists a unique solution $P \in \mathbb{C}^n$ of (5.66) that satisfies
\[\|P\| \leq 2\|(\lambda I - B)^{-1}
\left(G - (\lambda I - A)^{-1}F \right)\| \lesssim \|G\| + \|F\|_H.\]

The identity (5.65) then yields $U \in D(A)$ such that
\[\|U\|_H \lesssim \|G\| + \|F\|_H.\]

In summary, we have found a unique solution $(U, P)^\top \in D(A) \times X$ of (5.64) that satisfies
\[\|(U, P)^\top\|_{H \times X} \lesssim \|(F, G)^\top\|_{H \times X}.\]
This first shows that
\[\bar{C}_+ \subset \rho(A),\]
and secondly that
\[s_0(A) = \inf\{\omega > s(A) | \exists C_\omega \in (0, \infty) : \|(\lambda I - A)^{-1}\| \leq C_\omega, \forall \lambda > \omega\} \leq 0.\]

Hence, by Theorem 5.2.1 of [5], the semigroup $T(t)$ generated by $A$ is bounded. Finally by Theorem 3.1, we conclude that the semigroup $T(t)$ is strongly stable. \(\square\)

We directly deduce the

**Theorem 5.20.** For $\varepsilon > 0$ small enough, the semigroup $T(t)$ generated by $A$ is exponentially stable in $H \times \mathbb{C}^n$; therefore, the solution $(u, P)$ of (5.56) exponentially tends to zero, namely there exists a positive constant $\omega$ such that
\[\|u(\cdot, t)\|_{L^1 \Omega} + \|u_t(\cdot, t)\|_{L^1 \Omega} + \|P(t)\| \lesssim e^{-\omega t}, \forall t > 0.\]
Proof. The previous Lemma and Corollary 4.5 allow to deduce that $T(t)$ is exponentially stable in $H \times C^n$, which means that there exists a positive constant $\omega$ such that

$$\|\hat{u}(\cdot, t)\|_{1, \Omega} + \|\hat{v}(\cdot, t)\|_{\Omega} + \|P(t)\| \lesssim e^{-\omega t}, \forall t > 0.$$ 

Since the mapping

$$(u, u_t, P)^\top \rightarrow (\hat{u}, \hat{v}, P)^\top$$

is clearly bijective and since (5.57), (5.60) and (5.62) yield

$$\|u(\cdot, t)\|_{1, \Omega} + \|u_t(\cdot, t)\|_{\Omega} + \|P(t)\| \lesssim \|\hat{u}(\cdot, t)\|_{1, \Omega} + \|\hat{v}(\cdot, t)\|_{\Omega} + \|P(t)\|, \forall t > 0,$$

we conclude that (5.68) holds. \qed

**Remark 5.21.** If we assume that $B$ satisfies

$$\Re(BP, P) \lesssim \|P\|, \forall P \in C^n,$$

then (3.16) holds, in particular $\mathcal{A}$ is dissipative and hence, generates a contraction semigroup. Moreover, as $\mathcal{A}$ generates an exponentially decaying semigroup, by Lemma 3.9 one has $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Furthermore, as $\mathcal{A}^*$ is given by $D(\mathcal{A}^*) = D(\mathcal{A})$ with

$$\mathcal{A}^*(u, v)^\top = -(v, \Delta u + \alpha v)^\top, \forall (u, v)^\top \in D(\mathcal{A}),$$

we get

$$\Re(\mathcal{A}^*(U, P)^\top, (U, P)^\top)_{H \times X} \lesssim -(\|v\|^2_\Omega + \|P\|^2_X), \forall (U, P)^\top \in D(\mathcal{A}) \times X.$$ 

This allows to show that $\sigma_p(\mathcal{A}^*) \cap i\mathbb{R} = \emptyset$ and by Corollary 3.12, one deduces that $i\mathbb{R} \subset \rho(\mathcal{A})$, for all $\varepsilon > 0$, which means that Lemma 5.19 holds without any restriction of smallness on $\varepsilon$. Consequently the conclusions of Theorem 5.20 remain valid for all $\varepsilon > 0$.

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