ON A CONJECTURE OF MEEKS, PÉREZ AND ROS

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ABSTRACT. Meeks, Pérez and Ros conjectured that a closed Riemannian 3-manifold which does not admit any closed embedded minimal surface whose two-sided covering is stable must be diffeomorphic to a quotient of the 3-sphere. We give a counterexample to this conjecture. Also, we show that if we consider immersed surfaces instead of embedded ones, then the corresponding statement is true.

1. INTRODUCTION

A closed minimal surface $\Sigma$ immersed in a Riemannian 3-manifold $(M, g)$ is called stable if the second variation of area is non-negative for all smooth variations of $\Sigma$. Existence of stable minimal surfaces can be obtained when the ambient space presents nontrivial topology; for instance, one can minimize area in the isotopy class of an incompressible surface (see [10]). On the other hand, quotients of the 3-sphere endowed with metrics of positive Ricci curvature do not admit two-sided, closed, stable, minimal surfaces. These facts suggest that the existence of closed stable minimal surfaces is related to the topology of the ambient space. In [8], Meeks, Pérez and Ros stated the following conjecture.

Conjecture (Meeks, Pérez, Ros). Let $(M, g)$ be a closed Riemannian 3-manifold. If $(M, g)$ does not admit any closed, embedded, minimal surface whose two-sided covering is stable, then $M$ is finitely covered by the 3-sphere.

Remark 1. When the surface is one-sided, we consider the two-sided immersed surface associated to it, otherwise we consider the surface itself.

In this work we present a family of counter-examples to this conjecture. The examples are locally homogeneous Riemannian manifolds modelled on the Thurston geometries $\text{Nil}_3$ and $\tilde{\text{SL}}_2(\mathbb{R})$. The construction uses a topological characterization of these spaces and the classification of its stable minimal surfaces (due to Pitts-Rubinstein [12]). We next state our first main result.

Theorem 1. Let $(\tilde{M}, \tilde{h})$ be $\text{Nil}_3$ or $\tilde{\text{SL}}_2(\mathbb{R})$ endowed with a homogeneous metric $\tilde{h}$ whose isometry group has dimension 4. There are closed Riemannian 3-manifolds $(M, h)$ obtained as quotients of $(\tilde{M}, \tilde{h})$ and which do not contain any closed, embedded, minimal surface whose two-sided cover is stable.

We prove Theorem 1 in Section 3, using a family of examples presented in Section 2.2.

One can wonder about what is the correct statement to the conjecture of Meeks, Pérez and Ros. In this direction, we prove the following.
Theorem 2. Let \((M, g)\) be a closed Riemannian 3-manifold. If \((M, g)\) does not admit any closed, two-sided, immersed, stable minimal surface, then \(M\) is finitely covered by the 3-sphere.

The proof of Theorem 2 uses results of existence of area-minimizing surfaces and some of the developments in the theory of 3-manifolds of the last years, in particular the work of Perelman on the Geometrization conjecture \[11\], and the work of Kahn-Markovic on the Surface subgroup conjecture \[6\].

In relation to the existence of the closed geometric 3-manifolds in Theorem 1 that provide counterexamples to the Meeks, Perez Ros conjecture, we ask the following question, which we highlight is not even known in the special case that the metric \(g\) is hyperbolic; see the end of Section 4 for further discussion and motivation for this question.

Question: Let \(M\) be a closed, non-Haken hyperbolic 3-manifold and let \(g\) be an arbitrary Riemannian metric on \(M\). Does \((M, g)\) admit a closed, embedded, minimal surface whose two-sided covering is stable?

Acknowledgments: It is a pleasure to thank Lucas Ambrozio and Harold Rosenberg for discussions and suggestions on the manuscript. I also thank Laurent Mazet for a discussion about the main results, and Joaquin Pérez for personal correspondence about this work. Finally, I would like to thank the anonymous referees by suggestions ans corrections.

2. Seifert Fibered Spaces

In this section, we recall some facts about Seifert Fibered Spaces. The main references used here are \[2, 17\]. Let \(D\) denote the unit disc of the complex plane centered at the origin. Let \(p, q\) be two coprime integers with \(p > 0\) and \(0 \leq q \leq p/2\). A standard fibered solid torus with coefficients \((p, q)\) is an open solid torus

\[
\{ (z, 0) \sim (\psi_{p,q}(z), 1) \}
\]

where \(\psi_{p,q} : D \to D\) is given by \(\psi_{p,q}(z) = e^{(2\pi i q/p)} z\). The fibration by vertical segments \(\{x\} \times [0, 1]\) extends to a fibration by circles of the solid torus. The central fiber obtained by identifying the endpoints of \(\{0\} \times [0, 1]\) is the core of the solid torus, and every non-central fiber winds \(p\) times around the core of \(M\). The positive number \(p\) is the multiplicity of the central fiber. If \(p = 1\) the fibered solid torus is diffeomorphic to the usual product \(D \times S^1\) and the central fiber is called regular. If \(p > 1\), the central fiber is called singular.

A Seifert Fibered Space is a 3-manifold \(M\) foliated by circles, such that every circle has a fibered neighbourhood diffeomorphic to a standard fibered solid torus. Let \(S\) be the topological space obtained from \(M\) by quotienting the circles to points. Then, \(S\) is a connected surface and has a natural orbifold structure: if the preimage of \(x \in S\) is a fiber of multiplicity \(p\), we see \(x\) as a cone point of angle \(\frac{2\pi}{p}\). The quotient map \(\Pi : M \to S\) is called the Seifert fibration (one should think of the total space of this fibration...
as a circle bundle over the orbifold $S$). A Seifert fibration without singular fibers is a circle bundle over a surface, in the usual sense.

Let $\Sigma \subset M$ be a closed, embedded surface in a Seifert Fibered space $M$. We say $\Sigma$ is \textit{vertical} if it is the union of regular fibers. In this case we have two possibilities: $\Sigma$ is either a torus or a Klein bottle, and in both cases the projection $\Pi(\Sigma)$ to the base surface is a simple closed curve contained in the complement of the cone points of $S$. We also say that $\Sigma$ is \textit{horizontal} if it is everywhere transverse to the fibers. On a closed Seifert manifold $M$ there is an invariant associated to it, called the Euler number and denoted by $e(M)$, which detects the presence of horizontal surfaces: $M$ contains a horizontal surface if, and only if, $e(M) = 0$, see [7][Section 10.4.2].

2.1. \textbf{Geometric metrics.} Let $E$ denote one of the fibered \textit{Thurston geometries}, i.e., $\mathbb{R}^3$, $S^3$, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, Nil$_3$, $\tilde{SL}_2(\mathbb{R})$ (we will denote $\tilde{SL}_2(\mathbb{R})$ by $\tilde{SL}_2$). Each of these spaces admits a homogeneous metric with isometry group of dimension at least 4 and these metrics satisfy the following properties: there is a Riemannian submersion $\Pi' : E \to \mathbb{M}^2(\kappa)$, where $\mathbb{M}^2(\kappa)$ is the complete, simply connected Riemannian surface with constant curvature $\kappa$, and the fibers of $\Pi'$ are the integral curves of a unit Killing vector field $\xi$. We have $\kappa = 0$ in the case of $\mathbb{R}^3$ and $\text{Nil}_3$, $\kappa = 1$ in the case of $S^2 \times \mathbb{R}$ and $S^3$, and $\kappa = -1$ in the case of $H^2 \times \mathbb{R}$ and $\tilde{SL}_2$. Moreover each geometry has the structure of a line or circle bundle over $\mathbb{M}^2(\kappa)$, such that the bundle projection coincides with the submersion $\Pi'$, and all the isometries of these metrics preserve this bundle structure.

Every Seifert Fibered Space $M$ is diffeomorphic to a quotient of some $E$ as above, and conversely, every manifold which is a quotient of some $E$ has a Seifert Fibered structure. Hence, we see that each Seifert fibered space has a locally homogeneous metric, and its Seifert fibration is induced by the bundle structure on the universal cover and the projection $\Pi'$. Thus, away from the singular fibers, we have an $SO(2)$-action on the fibers by isometries and the projection $\Pi : M \to S$ is a Riemannian submersion, where the base orbifold is endowed with a constant curvature metric with isolated cone singularities. We call such a metric a \textit{geometric metric}.

\textbf{Remark 2.} Two closed 3-manifolds with different Thurston geometries are not diffeomorphic (see [17] Theorem 5.2). This fact will be crucial to our counterexample in the next Section.

Let $M$ be a closed Seifert Fibered Space with base orbifold $S$. The underlying geometry for $M$ is determined in terms of its Euler number $e(M)$ and the orbifold Euler characteristic of $S$ (denoted $\chi(S)$), as given by the following table.

| $\chi$ | $e$ | $S^2 \times \mathbb{R}$ | $\mathbb{R}^3$ | $\tilde{SL}_2$ | Nil$_3$ | $S^3$ |
|--------|-----|-------------------------|----------------|--------------|--------|------|
| $< 0$  | 0   | $\mathbb{R}^3$         | $S^2 \times \mathbb{R}$ | $\tilde{SL}_2$ | Nil$_3$ | $S^3$ |
| $= 0$  | 0   | $\mathbb{R}^3$         | $S^2 \times \mathbb{R}$ | $\tilde{SL}_2$ | Nil$_3$ | $S^3$ |
| $> 0$  | 0   | $\mathbb{R}^3$         | $S^2 \times \mathbb{R}$ | $\tilde{SL}_2$ | Nil$_3$ | $S^3$ |
Lemma 1 (Pitts, Rubinstein, [12]). Let $\Sigma$ be a closed two-sided embedded stable minimal surface in a Seifert Fibered Space $M$ endowed with a geometric metric $h$. Then $\Sigma$ is vertical or horizontal.

Proof. If $M$ is a Lens space, then the metric $h$ has positive Ricci curvature, so in this case there is no closed, two-sided, stable minimal surface. So, suppose $M$ is not a Lens space, and let $\Pi : M \to S$ be the Seifert Fibration. In this case, there exist a finite-degree covering $\tilde{F} : \tilde{M} \to M$, an orbifold covering $f : \tilde{S} \to S$ and a Seifert Fibration $\tilde{\Pi} : \tilde{M} \to \tilde{S}$, such that $\tilde{S}$ is

2.2. Main example. Let $p_1, p_2, p_3 \geq 2$ be natural numbers and let $\Delta$ be a geodesic triangle either in $\mathbb{R}^2$ or in $\mathbb{H}^2$, with inner angles $\frac{\pi}{p_1}, \frac{\pi}{p_2}, \frac{\pi}{p_3}$. By reflecting iteratively $\Delta$ along its sides we get a tessellation $T$. The triangle group $\Gamma(p_1, p_2, p_3)$ is the group of isometries of $\mathbb{R}^2$ or of $\mathbb{H}^2$, generated by reflections along the three sides of $\Delta$. This group acts freely and transitively on the triangles of the tessellation $T$, hence it is discrete and $\Delta$ is a fundamental domain for $\Gamma(p_1, p_2, p_3)$. Consider the index-two subgroup $\Gamma = \Gamma^o(p_1, p_2, p_3) \lt \Gamma(p_1, p_2, p_3)$ of orientation-preserving isometries. Taking the quotient of the model space by $\Gamma$ we obtain an orbifold $(S, g)$, which is a sphere with exactly 3 cone points of orders $p_1, p_2$ and $p_3$. $(S, g)$ is hyperbolic, or flat, according to whether $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$ is smaller than 1, or equal to 1, respectively.

Assume that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$. Recall that the geometry of $\tilde{\text{SL}}_2$ can be constructed in the following way: consider the unit tangent bundle $U\mathbb{H}^2$ of $\mathbb{H}^2$ endowed with the Sasaki metric, and then take the universal cover of $U\mathbb{H}^2$ endowed with the pullback metric.

For each isometry $f$ of $\mathbb{H}^2$, the differential $f_*$ is an isometry of $U\mathbb{H}^2$, and therefore using the covering $\tilde{\text{SL}}_2 \to U\mathbb{H}^2$, $f_*$ lifts to an isometry of $\tilde{\text{SL}}_2$. So, there is a group of isometries of $\tilde{\text{SL}}_2$, denoted by $\tilde{\Gamma}$, which is an extension of $\Gamma$ by the central $\mathbb{Z}$ subgroup of $\text{Isom}(\tilde{\text{SL}}_2)$: we have the exact sequence

$$0 \to \mathbb{Z} \to \tilde{\Gamma} \to \Gamma \to 0.$$ 

Note that the action of $\tilde{\Gamma}$ has no fixed points. In order to prove this, it suffices to show that for any $f \in \Gamma, f_*$ has no fixed points, unless $f$ is the identity. But the construction of $\Gamma$ gives that either $f$ is conjugate to a rotation or to a hyperbolic translation, from where it follows the claim.

Let $M = \tilde{\text{SL}}_2/\tilde{\Gamma}$. Then, it is easy to prove that $M$ is a closed Seifert Fibered Space projecting on $(S, g)$. Moreover, $e(M) \neq 0$ and $M$ has 3 singular fibers whose respective fibered solid tori have coefficients $(p_i, 1)$, $i = 1, 2, 3$. Geometrically we can see $M$ as the unit tangent bundle of the orbifold $(S, g)$. In general, it is possible to construct other circle bundles over $(S, g)$ where $e \neq 0$ and at least one of the singular fibers has coefficients $(p_i, q_i)$ satisfying $q_i \neq 1$. Similar constructions in the case $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ produces quotients of $\text{Nil}_3$. For more details on these examples see [17].

3. The counterexample

The next two results together with the main Example 2.2 are the keys steps in the construction of our counterexample in Theorem 3.
a smooth surface, \( \tilde{M} \) is a \( S^1 \)-bundle over \( \tilde{S} \) and \( \Pi \circ F = f \circ \tilde{\Pi} \) (see [7, Section 10.3.7]). Also, \( \tilde{h} = F^*h \) is a geometric metric, and \( \tilde{S} \) and \( S \) are endowed with respective metrics \( \tilde{g} \) and \( g \) of (the same) constant curvature.

Consider a lift \( \Sigma \) of \( \Sigma \) to \( \tilde{M} \). So, \( \tilde{\Sigma} \) is a closed, two-sided, embedded, stable minimal surface in \( (\tilde{M}, \tilde{h}) \). Let \( \xi \) be the unit Killing vector field associated to \( \tilde{\Pi} \) and let \( N \) be a unit normal to \( \tilde{\Sigma} \). Denote \( \phi = \langle N, \xi \rangle \). Suppose \( \tilde{\Sigma} \) is neither horizontal nor a union of fibers. Then \( \phi \) is not identically zero, but \( \phi(x) = 0 \) for some \( x \in \tilde{\Sigma} \). Since \( \xi \) is a Killing vector field we have

\[
L(\phi) = 0,
\]

where \( L \) is the Jacobi operator of \( \tilde{\Sigma} \). Since \( \tilde{\Sigma} \) is stable and equation \( (\Pi) \) holds, by standard elliptic theory we have that \( \phi \) does not vanish anywhere, or it is identically zero, which contradicts what we established before. Thus, \( \tilde{\Sigma} \) is horizontal or it is a union of fibers. Since these two properties are local, the same conclusion holds for \( \Sigma \).

Let \( \Sigma \) be a union of fibers, and suppose it contains a singular fiber. Thus \( \gamma = \Pi(\Sigma) \) is a simple closed curve containing a cone point \( x \in S \) of angle \( 2\pi/p \), for some integer \( p > 1 \). Consider a neighbourhood \( D \) of \( x \) such that \( f^{-1}(D) \) is a collection of disjoint geodesic disks \( D_1, \ldots, D_m \) of \( (\tilde{S}, \tilde{g}) \). Each \( D_i \) is decomposed into \( 2p \) circular sectors of angle \( \pi/p \), and \( \tilde{\gamma} = f^{-1}(\gamma \cap D) \) can be described as follows: on each sector there are two arcs meeting (uniquely) at the center of \( D_i \). Hence, \( \tilde{\Pi}^{-1}(\tilde{\gamma}) \) is not embedded. Since \( \Pi \circ F = f \circ \tilde{\Pi} \), we conclude that \( \Sigma \) is not embedded, which is a contradiction. Therefore, \( \Sigma \) cannot contain a singular fiber.

**Proposition 1.** Let \( (S, g) \) be a two-sphere endowed with a flat or hyperbolic metric \( g \) with three cone points. Then, \( (S, g) \) does not contain a simple closed geodesic inside its regular part.

**Proof.** We argue by contradiction and suppose there is such a geodesic \( \gamma \). Consider disks \( D_i \) around the cone points \( x_i, i = 1, 2, 3 \), such that \( \gamma \subset \tilde{S} = S \setminus (\cup_i D_i) \). We can choose \( D_i \) as the quotient of a geodesic disc \( \tilde{D}_i \) in \( \mathbb{R}^2 \) or \( \mathbb{H}^2 \) by a rotation of angle \( 2\pi/p_i \), for some integer \( p_i > 1 \), where the center of this disc projects on the cone point. The boundary \( \partial D_i \) is the projection of \( \partial \tilde{D}_i \), thus the boundary components of \( \tilde{S} \) have negative geodesic curvature with respect to the unit normal vector pointing towards \( \tilde{S} \).

Since the underlying surface is a sphere, the curve \( \gamma \) separates \( S \). Suppose \( \gamma \) is the boundary of a disc \( U \) on \( \tilde{S} \). Then, by the Gauss-Bonnet formula and the hypothesis on the curvature of the orbifold we have

\[
2\pi = \int_U K_S \, dA + \int_{\gamma} \kappa_g \, dL = \begin{cases} 0, & \text{if } S \text{ is flat;} \\ -|U|, & \text{if } S \text{ is hyperbolic;} \end{cases}
\]

which is a contradiction.

Thus, \( \gamma \) must be homotopic to a the boundary components \( \tilde{\gamma} \) of \( \tilde{S} \). Let \( A \) be the annulus in \( \tilde{S} \) bounded by \( \gamma \) and \( \tilde{\gamma} \). Since \( \tilde{\gamma} \) has negative geodesic curvature we obtain

\[
0 = \int_A K_S \, dA + \int_{\gamma} \kappa_g + \int_{\tilde{\gamma}} \kappa_g \, dL < 0,
\]
which is again a contradiction.

Let us recall some definitions concerning 3-manifolds.

**Definition.** 1) We say that a 3-manifold $M$ is irreducible if every 2-sphere embedded in $M$ bounds an embedded 3-ball in $M$. $M$ is $P^2$-irreducible if it is irreducible and contains no embedded two-sided projective plane.

2) We say that a closed oriented 3-manifold $M$ is Haken if it is irreducible and contains an embedded, two-sided, incompressible surface $\Sigma$ (i.e., the map $\pi_1(\Sigma) \to \pi_1(M)$ induced by the embedding is one-to-one) of genus $n \geq 1$. Otherwise, we say $M$ is non-Haken.

**Theorem 3.** Let $M$ be a closed, irreducible, non-Haken Seifert Fibered Space with infinite fundamental group, and let $h$ be a geometric metric on $M$. Then $(M,h)$ does not contain any closed, embedded, minimal surface whose two-sided cover is stable.

**Proof.** It is well-known that a closed, irreducible, non-Haken Seifert Fibered Space $M$ with infinite fundamental group has $\epsilon(M) \neq 0$ and the base orbifold $S$ is a sphere with three cone points (see for example Proposition 2 in [3]). The possible geometric structures for $M$ are $\text{Nil}_3$ and $\tilde{\text{SL}}_2$, and for the geometric metrics on $M$, $S$ has a flat or hyperbolic metric. So, $M$ belong to the class of manifolds discussed in the Main example 3.

We argue by contradiction and suppose that there is $\Sigma \subset M$, a closed embedded minimal surface whose two-sided covering is stable. In the case $\Sigma$ is two-sided, by Lemma 1, $\Sigma$ is either vertical or horizontal. If $\Sigma$ is vertical then its projection $\gamma$ on the base surface is a simple closed curve in the regular part of $S$. We next prove that $\gamma$ is a geodesic.

Let $T$ and $\eta$ be respectively a unit tangent field and a unit normal field to $\gamma$. Given a point $x \in \Sigma$, consider a small neighborhood $U$ of $\Pi(x)$ in $S$ such that the projection $\Pi : \Pi^{-1}(U) \to U$ is a Riemannian submersion. Let $\tilde{T}$ and $\tilde{\eta}$ be the corresponding horizontal lifts to $\Pi^{-1}(U)$ of $T$ and $\eta$. Then, $\{\tilde{T},\xi\}$ is an orthonormal basis on $T_x\Sigma$ and $\tilde{\eta}$ a unit normal to $\Sigma$, where $\xi$ is the unit Killing vertical vector field of $\Pi^{-1}(U)$. Also, let $\nabla$ and $\tilde{\nabla}$ denote the connections on $M$ and $S$ respectively. Then, the mean curvature of $\Sigma$ at $x$ is

$$0 = -\langle \nabla_{\tilde{T}}\tilde{\eta},T \rangle - \langle \nabla_{\xi}\tilde{\eta},\xi \rangle = \langle \nabla_{\tilde{T}}\tilde{T},\tilde{\eta} \rangle = \langle \nabla_{T}T,\eta \rangle$$

the geodesic curvature of $\gamma$ at the point $\Pi(x)$.

However, by Proposition 1, $S$ does not admit simple closed geodesics in its regular part. On the other hand, as we mentioned in the Section 2, closed manifolds with the geometries of $\text{Nil}_3$ and $\tilde{\text{SL}}_2$ do not contain horizontal surfaces. So, in any case we obtain a contradiction.

Finally, if $\Sigma$ is one-sided we can pass to a double covering $F : \tilde{M} \to M$ so that the lift $\tilde{\Sigma}$ of $\Sigma$ to $\tilde{M}$ is a connected closed embedded two-sided minimal surface (see [13, Proposition 3.7]), which is stable by hypothesis. The manifold $\tilde{M}$ is also a Seifert Fibered space, endowed with the geometric metric $\tilde{h} = F^*h$, so again by Lemma 1, $\tilde{\Sigma}$ is either vertical or horizontal, thus this also holds for $\Sigma$, and the contradiction follows as before. 

\[\Box\]
4. Further discussion

In this section, we prove Theorem 2 and motivate the question we presented in the Introduction.

Proof of Theorem 2. Suppose $M$ is not a quotient of the 3-sphere. We will prove that $(M, g)$ contains a two-sided, immersed, stable minimal surface. In this case, it follows from the work of Perelman that $M$ has infinite fundamental group, thus there are four possibilities:

1) $M$ is not irreducible: Then, there is an embedded sphere $S \subset M$, which represents a non zero element of $\pi_2(M)$. So, combining the results in [14] and [9], there is an immersed stable minimal sphere $\Sigma$ in $(M, g)$, and either $\Sigma$ is embedded or it double covers an embedded projective plane.

2) $M$ is orientable, irreducible and Haken: Combining the results in [15, 16] with that of [2], $(M, g)$ admits an incompressible, orientable, stable minimal surface of genus $n \geq 1$. Moreover, this surface is either embedded or it double covers an embedded non-orientable surface.

3) $M$ is orientable, irreducible and non-Haken: We have two sub-cases. If $\pi_1(M)$ contains $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup, then there is an immersed incompressible torus in $M$. Thus it follows from the results in [15, 16] that $(M, g)$ contains an immersed stable minimal torus; if $\pi_1(M)$ does not contain any subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, then by the work of Perelman, $M$ admits a hyperbolic metric (which can be different from the metric $g$). It follows from the work of Kahn-Markovic [6] that $M$ admits an orientable, incompressible immersed surface of genus $n \geq 2$. So using again [15, 16] we conclude that $(M, g)$ contains an immersed stable minimal surface of genus $n \geq 2$.

4) $M$ is non-orientable and irreducible: If $M$ contains an embedded 2-sided projective plane $P$, then by [1] Proposition 2.3, $(M, g)$ contains a embedded stable minimal surface $\Sigma$ homeomorphic to $P$. Moreover $\Sigma$ is 2-sided (otherwise, by [4] Lemma 2) $M$ is $\mathbb{R}P^3$, which contradicts the fact that $M$ is non-orientable). If $M$ is $P^2$-irreducible, by Lemmas 6.6 and 6.7 in [4], $M$ contains an embedded, 2-sided nonseparating incompressible surface $S$. Using Theorems 3.1 and 5.1 of [2] we obtain a stable minimal surface $\Sigma$ homotopic to $S$ in $(M, g)$, moreover, $\Sigma$ is embedded or double covers an embedded one-sided surface. □

Analysing the previous proof, we see that if $M$ is either non-orientable, or not irreducible, or irreducible and Haken, then $(M, g)$ contains an embedded minimal surface whose two-sided covering is stable. If $M$ is irreducible, non-Haken and $\pi_1(M)$ contains $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup, it follows from the Seifert Fibered Space theorem (see Section 2.3.2 in [13]) that $M$ is a non-Haken Seifert Fibered Space with infinite $\pi_1$, and as we saw in the Theorem 3 these spaces admit a metric which does not contain closed, embedded minimal surfaces whose two-sided cover is stable. So the only case remaining to study is that when $M$ is a non-Haken hyperbolic manifold. There are examples of non-Haken hyperbolic manifolds which do not contain any non-orientable embedded surface, so in this case the surfaces produced by the last theorem are not embedded and do not cover any non-orientable
Question: Let $M$ be a closed, non-Haken hyperbolic 3-manifold and let $g$ be an arbitrary Riemannian metric on $M$. Does $(M, g)$ admit a closed, embedded, minimal surface whose two-sided covering is stable?

It is important to highlight that this question is open even when the metric $g$ is the hyperbolic metric.

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