Passive tracer in non-Markovian, Gaussian velocity field

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Abstract

We consider the trajectory of a tracer that is the solution of an ordinary differential equation \( \dot{X}(t) = V(t, X(t)) \), \( X(0) = 0 \), with the right hand side, that is a stationary, zero-mean, Gaussian vector field with incompressible realizations. It is known, see [2, 1, 4], that \( X(t)/\sqrt{t} \) converges in law, as \( t \to +\infty \), to a normal, zero mean vector, provided that the field \( V(t, x) \) is Markovian and has the spectral gap property. We wish to extend this result to the case when the field is not Markovian and its covariance matrix is given by a completely monotone Bernstein function.

1 Introduction and some assumptions

In this paper we would like to show the central limit theorem for a passive tracer model, when the velocity field is non-Markovian but Gaussian and exponentially mixing in time.

Passive tracer model is given by the following equation,

\[
\begin{align*}
\frac{dX(t)}{dt} &= V(t, X(t)), \quad t > 0, \\
X(0) &= 0,
\end{align*}
\]

(1.1)

where \( V : \mathbb{R}^{1+d} \times \Omega \to \mathbb{R}^d \) is a real, \( d \)-dimensional, incompressible i.e. \( \sum_{p=1}^d \partial_{x_p} V_p(t, x) \equiv 0 \), zero mean, Gaussian random vector field over a probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

Some basic problems concerning the asymptotic behavior of the tracer are: the law of large numbers (LLN) i.e. whether \( X(t)/t \) converges to a constant vector \( v_* \) (called the Stokes drift), as \( t \to +\infty \) and the central limit theorem (CLT), i.e. whether \( (X(t) - v_* t)/\sqrt{t} \) is convergent in law to a normal vector \( N(0, \kappa) \). The covariance matrix \( \kappa = [\kappa_{ij}]_{i,j=1,...,d} \) is called turbulent diffusivity of the tracer.

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It is expected that both the LLN, with \( v_* = 0 \), and the CLT for the tracer trajectory hold when the velocity field is zero mean, Gaussian, incompressible and its covariance matrix \( R(t, x) = [R_{pq}(t, x)]_{p,q=1,...,d} \), given by

\[
R_{pq}(t, x) := \mathbb{E}[V_p(t, x)V_q(0, 0)], \quad p, q = 1, \ldots, d, \quad (t, x) \in \mathbb{R}^{1+d},
\]

exponentially decays in time, i.e. there exists \( C > 0 \) such, that

\[
\sum_{p,q=1}^{d} |R_{pq}(t, x)| \leq Ce^{-|t|/C}, \quad \text{for all} \quad (t, x) \in \mathbb{R}^{1+d}.
\]  

(1.2)

The CLT has been established in [5], in the case of \( T \)-dependent fields, i.e. those for which exists \( T > 0 \) such that \( R(t, x) = 0, \quad |t| > T, \quad x \in \mathbb{R}^d \).

In case when the vector field \( V(t, x) \) is Markovian (not necessarily Gaussian) and satisfies the spectral gap condition, the CLT has been established in [2], Theorem A, see also [4, 1, 6]. In the Gaussian case when the covariance matrix is of the form

\[
R_{pq}(t, x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi - \gamma(\xi)|t|} \hat{R}_{pq}(d\xi), \quad p, q = 1, \ldots, d, \quad (t, x) \in \mathbb{R}^{1+d},
\]

(1.3)

where both \( \gamma(\cdot) \) and non-negative Hermitian matrix valued measure \( \hat{R}(\cdot) = [\hat{R}_{pq}(\cdot)] \) are even (because the field is real), then the field is Markovian. It can be shown, see Chapter 12 of [4], that the spectral gap condition holds, provided there exists \( \gamma_0 > 0 \) such that

\[
\gamma(\xi) \geq \gamma_0, \quad \xi \in \mathbb{R}^d.
\]  

(1.4)

We will consider fields for which the exponential factor is replaced by a function \( h : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R} \) i.e. fields with the covariance defined as follows

\[
R_{pq}(t, x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi - \gamma(\xi)|t|} \hat{R}_{pq}(d\xi), \quad p, q = 1, \ldots, d, \quad (t, x) \in \mathbb{R}^{1+d},
\]

(1.5)

where \( \hat{R}_{pq}(\xi) \) is a density of \( \hat{R}_{pq}(\cdot) \).

We show in Proposition 2.1 that the function \( h \) is non-negative definite iff \( R(t, x) = [R_{pq}(t, x)] \) is non-negative definite. Therefore, the largest (in the sense of inclusion) set of functions \( h \) in (1.5) which can be examined is the class of non-negative definite functions. We study a smaller family of functions, namely we assume that \( h \) is completely monotone in the sense of Bernstein, see (1.7).

Let us denote by \( \hat{r} \) the power-energy spectrum. It is a scalar non-negative, integrable function given by formula

\[
\hat{r}(\xi) := \text{tr} \hat{R}(\xi), \quad \xi \in \mathbb{R}^d,
\]  

(1.6)

where \( \text{tr} \) is the trace. Let \( \gamma(d\xi) := \hat{r}(\xi)d\xi \).
The main result of the paper, see Theorem 2.1, is the CLT for the trajectory of a tracer moving in a field whose covariance matrices are given by (1.5), where the function $h$ is completely monotone i.e. $h \in C^\infty(0, +\infty) \cap C[0, +\infty), \ (-1)^n h^{(n)}(t, \xi) \geq 0$, $t > 0$, $\forall$ a.e. $\xi$, $n = 0, 1, \ldots$, and satisfies (1.8). From the Bernstein Theorem ([7] Theorem 3., p. 138) we know that

$$h(t, \xi) = \int_0^{+\infty} e^{-\lambda t} \mu(\xi, d\lambda), \quad t \in [0, +\infty), \ \forall \ a.e. \ \xi,$$

where $\mu(\xi, \cdot)$ is a non-negative, finite Borel measure on $[0, +\infty)$ for $\forall \ a.e. \ \xi$. For example from [11], Lemma 4.5, we know, that all completely monotone functions are non-negative definite. We assume that there exists $\lambda_0 > 0$ such that

$$\text{supp} \ \mu(\xi, \cdot) \subset [\lambda_0, +\infty), \ \forall \ a.e. \ \xi.$$  

Observe that this assumption implies (1.2).

In Section 3.1 we show (see (3.7)) an example of a covariance matrix which is of the form (1.5) but not of the form (1.3).

We prove Theorem 2.1 in Section 3 by embedding the field $V(t, x)$ into a larger space where we add one dimension and one argument i.e a space of $d + 1$ dimensional fields $\tilde{V}(t, x, y)$. We define a field $\tilde{V}(t, x, y)$ in such a way that the field $(\tilde{V}(t, x, 0))$ has the same distribution as the field $(V(t, x), 0)$. The process $\tilde{V}(t, \cdot, \cdot)$ has the Markov property in appropriate function space. This process also has the spectral gap property. At the end we use Theorem 12.13 from [1].

In Section 4 we show the proof of Proposition 2.1.

### 2 Preliminaries and the statement of the main result

First, we present some assumptions on matrix valued function $\hat{R}(\xi) = [\hat{R}_{pq}(\xi)]_{p,q=1,\ldots,d}$, which guarantee that function $R(t, x)$ (defined in (1.5)) is non-negative definite:

$$\hat{R}_{pq}(\xi) = \hat{R}^*_{qp}(\xi), \quad p, q = 1, \ldots, d, \ \forall \ a.e. \ \xi,$$  

$$\hat{R}(\xi) \eta \cdot \eta \geq 0, \quad \eta \in \mathbb{C}^d, \ \forall \ a.e. \ \xi.$$  

Now we present the result which explains why we need to deal with functions $h$, which are non-negative definite in $t$ for $\forall \ a.e. \ \xi$.

**Proposition 2.1.** For any $N \geq 1$, $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ and $t_1, \ldots, t_N \in \mathbb{R}$ we have

$$\sum_{p,q=1}^{N} h(|t_p - t_q|, \xi) \alpha_p \bar{\alpha}_q \geq 0, \quad \forall \ a.e. \ \xi,$$

iff the matrix valued function $R$ given by (1.5) is non-negative definite.
The proof of this Proposition is presented in Section 4.

Since $E|V(0,0)|^2 < +\infty$, we have

$$\int_{\mathbb{R}^d} h(0, \xi) \hat{r}(\xi) d\xi < +\infty. \tag{2.4}$$

Recall that we have assumed that $h$ is of the form (1.7). To fulfill the assumption (2.4) we require that

$$\text{esssup}_{\xi} \mu(\xi, [0, +\infty)) < +\infty. \tag{2.5}$$

It implies that $|h|$ is bounded. Above esssup is the essential supremum with respect to $r$. Then (2.4) is implied by

$$\int_{\mathbb{R}^d} \hat{r}(\xi) d\xi < +\infty. \tag{2.6}$$

Assumptions (2.1)–(2.6) imply that the matrix valued function $R(\cdot, \cdot)$ is non-negative definite. From [3], Section I.3, we know that there exits a unique (in the sense of law) stationary Gaussian random vector field $V(t, x)$ such that $R(t, x)$ is its covariance matrix.

We want to deal with a real field, so we need to assume, see [10] Theorem 4.2., p. 18, that

- For $r$ a.e. $\xi$ we have
  $$\hat{R}_{pq}(\xi) = \hat{R}_{qp}(-\xi), \quad p, q = 1, \ldots, d. \tag{2.7}$$

- For any $t \in \mathbb{R}$
  $$h(|t|, \xi) = h(|t|, -\xi), \quad r \text{ a.e. } \xi. \tag{2.8}$$

Assumption (2.8) implies (see (1.7))

$$\mu(\xi, \cdot) = \mu(-\xi, \cdot), \quad r \text{ a.e. } \xi. \tag{2.9}$$

To sum up, in this paper we consider the model in which a $d$-dimensional random vector field $V(t, x)$ has the covariance matrix given by (1.5), where $\hat{R}(\xi)$ and $h(|t|, \xi)$ satisfy assumptions (2.1)–(2.9).

To be able to solve the equation (1.1) we need to assume an appropriate regularity of the field. Namely, we assume that there exists the second derivative in $x$ of the field $V(t, x)$ and it is continuous. This implies (see (1.5))

$$\int_{\mathbb{R}^d} (1 + |\xi|^4) \hat{r}(\xi) d\xi < +\infty. \tag{2.10}$$

Moreover, we assume that the field $V(t, x)$ is incompressible, i.e.

$$\nabla_x \cdot V(t, x) = \sum_{p=1}^d \partial_{x_p} V_p(t, x) \equiv 0, \quad (t, x) \in \mathbb{R}^{1+d}.$$
We can rewrite this using (1.5) as follows
\[
\sum_{p=1}^{d} \xi_p \hat{R}_{pq}(\xi) = 0, \quad q = 1, \ldots, d, \text{ a.e. } \xi. \quad (2.11)
\]
From the above equality we obtain
\[
\sum_{p=1}^{d} \partial_{x_p} R_{pq}(t, x) = 0, \quad (t, x) \in \mathbb{R}^{1+d}, \quad q = 1, \ldots, d. \quad (2.12)
\]
Formula (1.5) and condition (2.10) imply in particular (see Theorem 3.1 from [8]) that there exists a version of \( V(t, x) \) with the realizations which grows slower than linearly in \( (t, x) \) a.s., therefore the equation (1.1) can be solved globally in \( t \), and the process \( X(t) \) is defined for all \( t \in [0, +\infty) \).

Now we can present our main result of this paper.

**Theorem 2.1.** Assume that \( V(t, x) \) is a zero-mean Gaussian random vector field whose covariance matrix \( R(t, x) = [R_{pq}(t, x)]_{p,q=1,\ldots,d} \) is given by (1.5). In addition suppose that (2.1)–(2.11) hold. Then the random variables \( X(t)/\sqrt{t} \), where \( X(t) \) is defined in (1.1), converge weakly to the normal vector \( N(0, \kappa) \), where the limit covariance matrix \( \kappa = [\kappa_{pq}] \) satisfies:
\[
\kappa_{pq} = \lim_{t \to +\infty} \frac{1}{t} \mathbb{E}\left[ X_p(t)X_q(t) \right], \quad p, q = 1, \ldots, d.
\]

### 3 Proof of Theorem 2.1

Consider a \( d+1 \)-dimensional field \( \tilde{V} : \mathbb{R}^{2+d} \times \Omega \to \mathbb{R}^{1+d} \) with the covariance matrix
\[
\tilde{R}_{pq}(t, x, y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{-|\lambda|} e^{i\xi \cdot x + i\lambda y} \hat{R}_{pq}(\xi)m(d\xi, d\lambda), \quad t, y \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad (3.1)
\]
for \( p, q = 1, \ldots, d, \) \( \tilde{R}_{pq} = 0 \), if \( p \) or \( q = d+1 \). Above \( m(d\xi, d\lambda) := \tilde{\mu}(\xi, d\lambda)d\xi \) would be the measure given by
\[
m(A \times B) = \int_{A} d\xi \left\{ \int_{B} \tilde{\mu}(\xi, d\lambda) \right\}, \quad A \in \mathcal{B}(\mathbb{R}^d), \quad B \in \mathcal{B}(\mathbb{R}), \quad (3.2)
\]
where \( \tilde{\mu} \) is defined as follows
\[
\tilde{\mu}(\xi, d\lambda) := \frac{1}{2} 1_{(0, +\infty)}(\lambda)\mu(\xi, d\lambda) + \frac{1}{2} 1_{(-\infty, 0]}(\lambda)\mu(\xi, -d\lambda),
\]
recall that \( \mu \) is given by (1.7). The covariance matrix in full Fourier transform form is given by
\[
\tilde{R}_{pq}(t, x, y) = \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\lambda}{\lambda^2 + \tau^2} e^{i\tau + ix \cdot \xi + i\lambda y} \hat{R}_{pq}(\xi)\tilde{\mu}(\xi, d\lambda)d\tau d\xi.
\]
This equality together with the assumptions (2.1)-(2.11) imply that \( \tilde{R}_{pq}(t, x, y) \) is the covariance matrix of some zero mean, Gaussian, stationary in time and space, random, real valued vector field \( \tilde{V}(t, x, y) \) (3, Section I.3). Observe that for \( y = 0 \) we have

\[
\tilde{R}_{pq}(t, x, 0) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\lambda |t|} e^{i \xi \cdot x} \hat{\mu}(\xi, d\lambda) \tilde{R}_{pq}(\xi) d\xi = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{0}^{+\infty} e^{-\lambda |t|} e^{i \xi \cdot x} \mu(\xi, d\lambda) \tilde{R}_{pq}(\xi) d\xi + \frac{1}{2} \int_{\mathbb{R}^d} \int_{-\infty}^{0} e^{\lambda |t|} e^{i \xi \cdot x} \mu(\xi, -d\lambda) \tilde{R}_{pq}(\xi) d\xi = R_{pq}(t, x), \quad p, q = 1, \ldots, d, \ (t, x) \in \mathbb{R}^{1+d},
\]

(3.3)

where matrix \( R_{pq}(t, x) \) is given by (1.3). Observe that \( \tilde{V}_{d+1}(t, x, y) \equiv 0, \) because \( \tilde{R}_{d+1, d+1} \equiv 0. \) So the field satisfies

\[
\{ \tilde{V}(t, x, 0), \ (t, x) \in \mathbb{R}^{1+d} \} \overset{d}{=} \{ (V(t, x), 0), \ (t, x) \in \mathbb{R}^{1+d} \},
\]

(3.4)

where \( \overset{d}{=} \) denotes equality in the law. Let \( \hat{\tau} \) be a measure \( \hat{\tau}(d\xi, d\lambda) := \tilde{\mu}(\xi, d\lambda) \hat{r}(\xi) d\xi. \) From (1.8) we know that \( |\lambda| \geq \lambda_0, \ \hat{r} \ a.e \ \xi. \)

Let \( \rho > (d + 1)/2. \) Denote by \( \mathcal{E} \) the Hilbert space that is the completion of the space of functions \( v = (v_1, \ldots, v_{d+1}) : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) with components in \( C^\infty_c(\mathbb{R}^{d+1}) \) (the space of infinitely differentiable compactly supported functions), under the norm

\[
\|v\|^2_\mathcal{E} := \int_{\mathbb{R}^{d+1}} (|v(z)|^2 + |\nabla v(z)|^2) (1 + |z|^2)^{-\rho}dz.
\]

Here

\[
|v(z)|^2 := \sum_{i=1}^{d+1} v_i^2(z) \quad \text{and} \quad |\nabla v(z)|^2 := \sum_{i,j=1}^{d+1} \left( \partial_{z_j} v_i \right)^2 (z), \quad z \in \mathbb{R}^{d+1}.
\]

Consider the process

\[
\tilde{V}_t := \tilde{V}(t, \cdot, \cdot), \ t \geq 0.
\]

From (3) Chapter 12, we know that the process takes values in space \( \mathcal{E}. \) This space contains the realization of the field \( \tilde{V}(0, \cdot, \cdot). \) This is a separable Hilbert space which turns out to be a subspace of the space of \( C^1 \) maps \( v : \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}. \) It can be shown that process \( \{\tilde{V}_t, t \geq 0\} \) is stationary. On the space \( \mathcal{E} \) we introduce a measure \( \pi \) as a distribution of \( \tilde{V}(0, \cdot, \cdot). \) Process \( \{\tilde{V}_t, t \geq 0\} \) has the Markov property with the corresponding transition semigroup \( (P_t)_{t \geq 0} \) i.e.

\[
\mathbb{E} \left[ F(\tilde{V}_{t+h}) | \mathcal{V}_t \right] = P_h F(\tilde{V}_t), \quad h \geq 0, t \geq 0, \ F \in \mathcal{B}_b(\mathcal{E}),
\]

(3.5)

where \( \mathcal{V}_t := \sigma(\tilde{V}_s, \ s \leq t) \) is a natural filtration of the process \( \{\tilde{V}_t, t \geq 0\}. \) \( (P_t)_{t \geq 0} \) is a semigroup of Markov contractions on \( L^2(\pi). \) Its generator \( L : D(L) \to L^2(\pi) \) has the spectral gap property (see (4), Corollary 12.15) i.e.

\[
-\langle LF, F \rangle_{L^2(\pi)} \geq \gamma_0 \|F\|^2_{L^2(\pi)}, \quad \text{for} \ F \in D(L), \ \text{such that} \ \int_{\mathcal{E}} Fd\pi = 0,
\]
where $\gamma_0$ was introduced in (1.4).

We consider the following equation

\[
\begin{aligned}
\frac{d\tilde{X}_p(t)}{dt} &= \tilde{V}_p(t, \tilde{X}(t), Y(t)), \quad p = 1, \ldots, d, \ t \geq 0, \\
\frac{dY(t)}{dt} &= \tilde{V}_{d+1}(t, \tilde{X}(t), Y(t)), \\
\tilde{X}(0) &= 0, \\
Y(0) &= 0.
\end{aligned}
\]  \tag{3.6}

The existence and uniqueness of the solution (3.6) comes from the form of the covariance of the field $\tilde{V}(\cdot)$. Because $\tilde{V}_{d+1}(t, x, y) \equiv 0$ we have $Y(t) = 0$ for $t \geq 0$. This and (3.4) imply $\{\tilde{X}(t), t \geq 0\} \overset{d}{=} \{X(t), t \geq 0\}$, where $X(t)$ was defined in (1.1). So we will omit writing the symbol tilde over $X(t)$.

Now we introduce the so-called environment process

\[U_t := \tilde{V}(t, X(t) + \cdot, Y(t) + \cdot) \in \mathcal{E}, \quad \mathbb{P} \text{ a.e. } t \geq 0.\]

Observe that

\[X(t) = \int_0^t F(U_s)ds, \quad t \geq 0,
\]

where $F : \mathcal{E} \to \mathbb{R}^d$ is given by $F(\omega) = (\omega_1(0, 0), \ldots, \omega_d(0, 0))$, $\omega \in \mathcal{E}$. Process $\{U_t, t \geq 0\}$ is also Markovian, see [4] Proposition 12.19 (i). Observe that

\[\nabla \cdot \tilde{V}(t, x, y) = \sum_{i=1}^d \partial_{x_i} \tilde{V}_i(t, x, y) + \partial_y \tilde{V}_{d+1}(t, x, y) = \sum_{i=1}^d \partial_{x_i} \tilde{V}_i(t, x, y), \quad t, y \in \mathbb{R}, x \in \mathbb{R}^d.
\]

Let $Z(t, x, y) := \sum_{i=1}^d \partial_{x_i} \tilde{V}_i(t, x, y)$. Observe that

\[\mathbb{E}Z^2(t, x, y) = \mathbb{E}Z^2(0, 0, 0) = \sum_{i,j=1}^d \partial_{x_i x_j} \tilde{R}_{ij}(0, 0, 0) = \sum_{j=1}^d \partial_{x_j} \left\{ \sum_{i=1}^d \partial_{x_i} \tilde{R}_{ij}(0, 0) \right\} = 0,
\]

where the last two equalities above come respectively from (3.3) and (2.12). Since $\nabla \cdot \tilde{V} = 0$ we know that $\pi$ is an invariant measure for the process $\{U_t, t \geq 0\}$; see [4] Proposition 12.19 (ii). Moreover, the transition semigroup $(R_t)_{t \geq 0}$ for $\{U_t, t \geq 0\}$, on $L^2(\pi)$, has the spectral gap property ([4] Corollary 12.22) i.e.

\[-\langle LF, F \rangle_{L^2(\pi)} \geq \gamma_0 \|F\|_{L^2(\pi)}^2, \quad \text{for } F \in D(\mathcal{L}), \text{ such that } \int_{\mathcal{E}} Fd\pi = 0.
\]

The CLT for the process $\{X(t), t \geq 0\}$ (which is defined in (1.1)) comes directly from Theorem 12.13 of [4].
3.1 Example of non-Markovian field which satisfies the assumption of Theorem 2.1

Observe that Theorem 2.1 can be applied to a field whose covariance matrix is given by

\[ R_{pq}(t,x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi - \gamma(|t|\xi)} \hat{R}_{pq}(d\xi), \quad p, q = 1, \ldots, d, \quad 0 < \alpha \leq 1, \quad (t, x) \in \mathbb{R}^{1+d}, \quad (3.7) \]

where functions \( \tilde{\gamma}, \gamma : \mathbb{R}^d \to \mathbb{R} \) satisfy (1.4). Observe that the function

\[ h(t,\xi) = \int_0^{+\infty} e^{-t^\alpha \lambda} \mu(\xi, d\lambda), \quad t \geq 0, \quad \text{r. a. e.} \xi, \quad (3.8) \]

where \( \alpha \in (0, 1] \) is completely monotone in \( t \). Indeed, we know that \( \hat{H}(t,\xi) = \int_0^{+\infty} e^{-t\lambda} \mu(\xi, d\lambda), \quad t \geq 0, \quad \text{r. a. e.} \xi \)

is completely monotone. However, \( \phi(t) := t^\alpha \) can be represented by an integral from \( \psi(t) := \alpha t^{\alpha-1} \), which is completely monotone on \( (0, +\infty) \), if \( \alpha \in (0, 1] \). Therefore, from Theorem 8 of [12] we have that \( h(t,\xi) = H(\phi(t),\xi) \) is completely monotone on \([0, +\infty)\).

Now we will check condition (1.8). Observe that we can write for \( \mu(\xi, d\lambda) := \delta_{\gamma(\xi)}(d\lambda), \quad t \geq 0, \)

\[ \int_0^{+\infty} e^{-t^\alpha \lambda} \mu(\xi, d\lambda) = e^{-\gamma(\xi)t^\alpha} = \int_0^{+\infty} e^{-\tilde{\gamma}(\xi)\lambda^t} g(\lambda)d\lambda, \quad t \geq 0, \quad \text{r. a. e.} \xi, \]

where \( g \) is a smooth density of \( T(1) \), where \( \{ T(t), t \geq 0 \} \) is a \( \alpha \)-stable subordinator ([13], p.201, formula (2.10.7)). Observe that for \( t \geq 0 \)

\[ e^{-\gamma(\xi)t-\tilde{\gamma}(\xi)t^\alpha} = \int_0^{+\infty} e^{-(\tilde{\gamma}(\xi)+\gamma(\xi))t} g(\lambda)d\lambda = \int_0^{+\infty} e^{-\lambda^t} \tilde{g}(\xi, \lambda)d\lambda, \quad \text{r.a.e.} \xi, \quad (3.9) \]

where

\[ \tilde{g}(\xi, \lambda) = \begin{cases} \gamma(\xi)^{-1} \frac{(\lambda-\gamma(\xi))}{\tilde{\gamma}(\xi)}, & \lambda > \gamma(\xi), \\ 0, & \lambda \leq \gamma(\xi). \end{cases} \]

We can see that the last expression in (3.9) is of the form (1.7) and the measure \( \tilde{\mu}(\xi, d\lambda) := \tilde{g}(\xi, \lambda)d\lambda \) fulfills (1.8).

4 Proof of the Proposition 2.1

Denote by \( A(\xi) \) some non-negative Hermitian, matrix valued function such that

\[ \hat{R}(\xi) = A(\xi)\hat{r}(\xi), \quad \xi \in \mathbb{R}^d, \]
where
\[
\text{tr} A(\xi) \equiv 1 \quad (4.1)
\]
f\text{or } r \text{ a.e. } \xi. \text{ Choose a function } \hat{\phi} \in \mathcal{S}(\mathbb{R}^d), \text{ where } \mathcal{S}(\mathbb{R}^d) \text{ is the class of } \mathbb{C}^d \text{ valued Schwartz functions. Then }
\[
0 \leq \mathbb{E} \left| \sum_{p=1}^{N} \alpha_p \int_{\mathbb{R}^d} V(t_p, x) \cdot \hat{\phi}(x) dx \right|^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left( \sum_{p,q=1}^{N} \alpha_p \tilde{\alpha}_q h(t_p - t_q, \xi) \right) A(\xi) \phi(\xi) \cdot \phi(\xi) \hat{r}(\xi) d\xi,
\]
where \( \phi(\xi) \) is the inverse Fourier transform of a function \( \hat{\phi}(x) \). If we choose \( \hat{\phi}_j(x) \), such that \( \phi_j(\xi) = e_j \psi(\xi) \), where \( \psi(\xi) \) is a scalar function, \( e_j = (0, \ldots, 1, \ldots, 0) \) and summing by \( j = 1, \ldots, d \), we obtain
\[
0 \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left( \sum_{p,q=1}^{N} \alpha_p \tilde{\alpha}_q h(t_p - t_q, \xi) \right) |\psi(\xi)|^2 \sum_{j=1}^{d} A(\xi) e_j \cdot e_j \hat{r}(\xi) d\xi.
\]
Considering (4.1) and the fact that \( \psi \) was arbitrary, we obtain (2.3).

\[\square\]

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