Bregman circumcenters: monotonicity and forward weak convergence

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Abstract
Recently, we systematically studied the basic theory of Bregman circumcenters in another paper. In this work, we aim to apply Bregman circumcenters to optimization algorithms. Here, we propose the forward Bregman monotonicity which is a generalization of the powerful Fejér monotonicity and show a weak convergence result of the forward Bregman monotone sequence. We also naturally introduce the Bregman circumcenter mappings associated with a finite set of operators. Then we provide sufficient conditions for the sequence of iterations of the forward Bregman circumcenter mapping to be forward Bregman monotone. Furthermore, we prove that the sequence of iterations of the forward Bregman circumcenter mapping weakly converges to a point in the intersection of the fixed point sets of relevant operators, which reduces to the known weak convergence result of the circumcentered method under the Euclidean distance. In addition, particular examples are provided to illustrate the Bregman isometry and Browder’s demiclosedness principle, and our convergence result.

Keywords  Bregman distance · Legendre function · Backward Bregman (pseudo)-circumcenter · Forward Bregman (pseudo)-circumcenter · Fixed point set · Forward Bregman monotonicity · Convergence · Bregman circumcenter method

1 Introduction
Throughout the work, we assume that \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and \( \{m, n\} \subseteq \mathbb{N} \setminus \{0\} \), and that

\( \mathcal{H} \) is a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and induced norm \( \|\cdot\| \).

Projection methods, including the method of cyclic projections based on Bregman distances, are employed in many applications (see, e.g., [4], [5], [6], and [19] and
the references therein). The convergence rate of certain circumcentered methods is no worse than that of the method of cyclic projections under the Euclidean distance, and, in some cases, some circumcentered methods converge much faster than the method of cyclic projections, for solving the best approximation problem or the feasibility problem (see, [1], [2], [11], [12], [13], [15], [16], [17], [18], and [22] for details).

In our recent work [14], we presented multiple interesting theoretical results on the Bregman circumcenter. Various examples under general Bregman distances were also provided to illustrate our main results.

In this work, our objects are to introduce the circumcenter mappings and methods and investigate the convergence of circumcenter methods under general Bregman distances.

We present the main results in this work below.

R1: Theorem 3.3 characterizes the weak convergence of the forward Bregman monotone sequence.

R2: Theorem 4.12 and 4.14 show that the sequence of iterations of the forward Bregman circumcenter mapping converges weakly to a point in the intersection of the fixed point sets of operators inducing the mapping.

The remainder of the work is organized as follows. In Sect. 2, we collect fundamental definitions and facts. In Sect. 3, we introduce the forward Bregman monotonicity and investigate the weak convergence of the forward Bregman monotone sequence. In Sect. 4, we introduce the forward Bregman circumcenter mapping induced by a finite set of operators and specify sufficient conditions for the weak convergence of the forward Bregman circumcenter method to a point in the intersection of fixed point sets of the related operators. Moreover, we provide particular examples to illuminate our hypotheses and main results under general Bregman distances.

We now turn to the notation used in this paper. \( \Gamma_0(H) \) is the set of proper closed convex functions from \( H \) to \([-\infty, +\infty] \). Let \( f : H \to [-\infty, +\infty] \) be proper. The domain (conjugate function, gradient, subgradient, respectively) of \( f \) is denoted by \( \text{dom} f (f^*, \nabla f, \partial f, \text{respectively}) \). We say \( f \) is coercive if \( \lim_{\|x\| \to +\infty} f(x) = +\infty \). Let \( C \) be a nonempty subset of \( H \). Its interior and boundary are abbreviated by \( \text{int} C \) and \( \text{bd} C \), respectively. \( C \) is an affine subspace of \( H \) if \( C \neq \emptyset \) and \((\forall \rho \in \mathbb{R}) \rho C + (1 - \rho)C = C \). The smallest affine subspace of \( H \) containing \( C \) is denoted by \( \text{aff} C \) and called the affine hull of \( C \). The best approximation operator (or projector) onto \( C \) under the Euclidean distance is denoted by \( P_C \), that is, \( (\forall x \in H) P_C x := \arg\min_{y \in C} \|x - y\| \). \( \iota_C \) is the indicator function of \( C \), that is, \( (\forall x \in C) \iota_C(x) = 0 \) and \( (\forall x \in H \setminus C) \iota_C(x) = +\infty \). \( \iota_C \) stands for the identity mapping. For every \( x \in H \) and \( \delta \in \mathbb{R}_{++} \), \( B[x; \delta] \) is the closed ball with center at \( x \) and with radius \( \delta \). Let \( A : H \to 2^H \) and let \( x \in H \). Then \( A \) is locally bounded at \( x \) if there exists \( \delta \in \mathbb{R}_{++} \) such that \( A(B[x; \delta]) \) is bounded. Denote by \( \text{Fix} A := \{ x \in H : x \in A(x) \} \). For other notation not explicitly defined here, we refer the reader to [7].

2 Bregman distances and projections

In this section, we collect some essential definitions and facts to be used subsequently.
It is clear that if \( f = \frac{1}{2} \| \cdot \|^2 \) in the following Definition 2.1, we recover the Euclidean distance \( D : \mathcal{H} \times \mathcal{H} \to [0, +\infty] : (x, y) \mapsto \frac{1}{2} \| x - y \|^2 \).

**Definition 2.1** ([4, Definitions 7.1 and 7.7]) Suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom} f \neq \emptyset \) and that \( f \) is Gâteaux differentiable on \( \text{int dom} f \). The Bregman distance \( D_f \) associated with \( f \) is defined by

\[
D_f : \mathcal{H} \times \mathcal{H} \to [0, +\infty] : (x, y) \mapsto \begin{cases} 
    f(x) - f(y) - \langle \nabla f(y), x - y \rangle, & \text{if } y \in \text{int dom} f; \\
    +\infty, & \text{otherwise}.
\end{cases}
\]

Moreover, let \( C \) be a nonempty subset of \( \mathcal{H} \). For every \( (x, y) \in \text{dom} f \times \text{int dom} f \), define the **backward Bregman projection** (or simply Bregman projection) of \( y \) onto \( C \) and **forward Bregman projection** of \( x \) onto \( C \), respectively, as

\[
\overleftarrow{P}^f_C(y) := \{ u \in C \cap \text{dom} f : (\forall c \in C) D_f (u, y) \leq D_f (c, y) \}, \quad \text{and} \\
\overrightarrow{P}^f_C(x) := \{ v \in C \cap \text{int dom} f : (\forall c \in C) D_f (x, v) \leq D_f (x, c) \}.
\]

Abusing notation slightly, we shall write \( \overleftarrow{P}^f_C(y) = u \) and \( \overrightarrow{P}^f_C(x) = v \), if \( \overleftarrow{P}^f_C(y) \) and \( \overrightarrow{P}^f_C(x) \) happen to be the singletons \( \overleftarrow{P}^f_C(y) = \{u\} \) and \( \overrightarrow{P}^f_C(x) = \{v\} \), respectively.

The following definitions are compatible with their classical counterparts as [3, Definitions 2.1, 2.3 and 2.8] in the finite-dimensional Euclidean space (see, [4, Theorem 5.11] for more details).

**Definition 2.2** ([4, Definition 5.2 and Theorem 5.6]) Suppose that \( f \in \Gamma_0(\mathcal{H}) \). We say \( f \) is:

(i) **essentially smooth**, if \( \text{dom} \partial f = \text{int dom} f \neq \emptyset \), \( f \) is Gâteaux differentiable on \( \text{int dom} f \), and \( \| \nabla f(x_k) \| \to +\infty \), for every sequence \( (x_k)_{k \in \mathbb{N}} \) in \( \text{int dom} f \) converging to some point in \( \text{bdom} f \).

(ii) **essentially strictly convex**, if \( (\partial f)^{-1} \) is locally bounded on its domain and \( f \) is strictly convex on every convex subset of \( \text{dom} \partial f \).

(iii) **Legendre** (or a **Legendre function** or a convex function of Legendre type), if \( f \) is both essentially smooth and essentially strictly convex.

**Fact 2.3** ([4, Lemma 7.3]) Suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom} f \neq \emptyset \), that \( f \) is Gâteaux differentiable on \( \text{int dom} f \), and that \( f \) is essentially strictly convex. Let \( x \in \mathcal{H} \) and \( y \in \text{int dom} f \). Then \( D_f (\cdot, y) \) is coercive. Moreover, \( D_f (x, y) = 0 \iff x = y \).

The following Fact 2.4, 2.6 on the existence and uniqueness of backward and forward Bregman projections are fundamental to some results in Sect. 4 below.

**Fact 2.4** ([4, Corollary 7.9]) Suppose that \( f \in \Gamma_0(\mathcal{H}) \) is Legendre, that \( C \) is a closed convex subset of \( \mathcal{H} \) with \( C \cap \text{int dom} f \neq \emptyset \), and that \( y \in \text{int dom} f \). Then \( \overleftarrow{P}^f_C(y) \) is a singleton contained in \( C \cap \text{int dom} f \).
Not surprisingly, not every Legendre function allows forward Bregman projections, and the backward and forward Bregman projections are different notions (see, e.g., [3], [6] and [8] for details). In particular, we can find well-known functions satisfying the hypotheses of the following Fact 2.6 in [6, Examples 2.1 and 2.7]. We refer the interested readers to [3], [6] and [8] for details on the backward and forward Bregman projections.

**Definition 2.5** ([6, Definition 2.4]) Suppose that \( f \in \Gamma_0(\mathcal{H}) \) is Legendre such that \( \text{dom} f^* \) is open. We say the function \( f \) allows forward Bregman projections if it satisfies the following properties.

(i) \( \nabla^2 f \) exists and is continuous on \( \text{int dom} f \).
(ii) \( D_f \) is convex on \( \text{int dom} f \times \text{int dom} f \).
(iii) For every \( x \in \text{int dom} f \), \( D_f(x, \cdot) \) is strictly convex on \( \text{int dom} f \).

**Fact 2.6** ([6, Fact 2.6]) Suppose that \( \mathcal{H} = \mathbb{R}^n \) and \( f \in \Gamma_0(\mathcal{H}) \) is Legendre such that \( \text{dom} f^* \) is open, that \( f \) allows forward Bregman projections, and that \( C \) is a closed convex subset of \( \mathcal{H} \) with \( C \cap \text{int dom} f \neq \emptyset \). Then \( (\forall x \in \text{int dom} f) \, \widehat{\mathcal{T}}_{C} f(x) \) is a singleton contained in \( C \cap \text{int dom} f \).

### 3 Forward Bregman monotonicity

In this section, we show the weak convergence of the forward Bregman monotone sequence, which plays a critical role in the proof of our main result in the next section.

The following is a variant of the Bregman monotonicity defined in [5, Definition 1.2]. Note that both [5, Definition 1.2] and the following Definition 3.1 are natural generalizations of the classical Fejér monotonicity. In view of the broad applications of Fejér monotonicity and Bregman monotonicity in the proof of the convergence of iterative algorithms (see, e.g., [7], [5], [21] and the references therein), we assume the forward Bregman monotonicity defined in Definition 3.1 below is interesting on its own and probably can be used in many other iterative algorithms under general Bregman distances.

**Definition 3.1** A sequence \((x_k)_{k \in \mathbb{N}}\) in \( \mathcal{H} \) is **forward Bregman monotone with respect to a set** \( C \subseteq \mathcal{H} \) if \( C \cap \text{int dom} f \neq \emptyset \), \((x_k)_{k \in \mathbb{N}}\) lies in \( \text{int dom} f \), and

\[
(\forall c \in C \cap \text{int dom} f) \, (\forall k \in \mathbb{N}) \quad D_f(x_{k+1}, c) \leq D_f(x_k, c).
\]

**Fact 3.2** Suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom} f \neq \emptyset \), that \( f \) is Gâteaux differentiable on \( \text{int dom} f \), and that \( f \) is essentially strictly convex. Then \( f \) is strictly convex on \( \text{int dom} f \), which is equivalent to

\[
(\forall x \in \text{int dom} f) (\forall y \in \text{int dom} f) \quad x \neq y \Rightarrow (x - y, \nabla f(x) - \nabla f(y)) > 0.
\]

**Proof** Because \( f \) is convex, by [7, Propositions 3.45(ii), 8.2 and 16.27], \( \text{int dom} f \) is a convex subset of \( \text{dom} \partial f \). Then employ Definition 2.2(ii) to see that \( f \) is strictly convex.
on \( \text{int dom } f \). Hence, the last equivalence is obtained by applying [7, Proposition 17.10] to the function \( f + \iota_{\text{int dom } f} \).

**Theorem 3.3** Suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom } f \neq \emptyset \) and that \( f \) is Gâteaux differentiable on \( \text{int dom } f \). Let \( C \) be a closed convex set in \( \mathcal{H} \) with \( C \cap \text{int dom } f \neq \emptyset \), and let \( (x_k)_{k \in \mathbb{N}} \) be in \( \text{int dom } f \) and be forward Bregman monotone with respect to \( C \). Then the following statements hold.

(i) \( (\forall c \in C \cap \text{int dom } f) \left( D_f(x_k, c) \right)_{k \in \mathbb{N}} \) is decreasing, nonnegative and convergent.

(ii) Let \( \{y, z\} \subseteq C \cap \text{int dom } f \). Then the limit \( \lim_{k \to \infty} \langle \nabla f(y) - \nabla f(z), x_k \rangle \) exists.

(iii) Suppose that \( f \) is essentially strictly convex. Then \((x_k)_{k \in \mathbb{N}} \) is bounded.

(iv) Suppose that \( f \) is essentially strictly convex. Then \((x_k)_{k \in \mathbb{N}} \) weakly converges to some point in \( C \cap \text{int dom } f \) if and only if all weak sequential cluster points of \((x_k)_{k \in \mathbb{N}} \) are in \( C \cap \text{int dom } f \).

**Proof**

(i): This is clear from Definition 3.1,2.1.

(ii): According to Definition 2.1, for every \( k \in \mathbb{N} \),

\[
D_f(x_k, y) - D_f(x_k, z) = f(x_k) - f(y) - \langle \nabla f(y), x_k - y \rangle
- \left( f(x_k) - f(z) - \langle \nabla f(z), x_k - z \rangle \right)
\]

\[
\iff \langle \nabla f(z) - \nabla f(y), x_k \rangle = D_f(x_k, y) - D_f(x_k, z) + f(y) - f(z)
- \langle \nabla f(y), y \rangle + \langle \nabla f(z), z \rangle.
\]

which, via (i), yields the desired result.

(iii): Let \( c \in C \cap \text{int dom } f \). Then based on Fact 2.3, the boundedness of \((D_f(x_k, c))_{k \in \mathbb{N}} \) implies the boundedness of \((x_k)_{k \in \mathbb{N}} \).

(iv): \( \Rightarrow \): This is trivial.

\( \Leftarrow \): Suppose that all weak sequential cluster points of \((x_k)_{k \in \mathbb{N}} \) are in \( C \cap \text{int dom } f \). Because the boundedness of \((x_k)_{k \in \mathbb{N}} \) is proved in (iii) above, bearing [7, Lemma 2.46] in mind, we know that it remains to prove that \((x_k)_{k \in \mathbb{N}} \) has at most one weak sequential cluster point in \( C \cap \text{int dom } f \). Assume that \( \bar{x} \) and \( \hat{x} \) are two weak sequential cluster points of \((x_k)_{k \in \mathbb{N}} \) in \( C \cap \text{int dom } f \). Then there exist subsequences \((x_{k_j})_{j \in \mathbb{N}} \) and \((x_{k_t})_{t \in \mathbb{N}} \) of \((x_k)_{k \in \mathbb{N}} \) such that \( x_{k_j} \to \bar{x} \) and \( x_{k_t} \to \hat{x} \). Then

\[
\left\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), x_{k_j} \right\rangle \to \left\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \bar{x} \right\rangle; \quad (3.1a)
\]

\[
\left\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), x_{k_t} \right\rangle \to \left\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \hat{x} \right\rangle. \quad (3.1b)
\]

On the other hand, apply (ii) with \( y = \bar{x} \) and \( z = \hat{x} \) to deduce that \( \lim_{k \to \infty} \langle \nabla f(\bar{x}) - \nabla f(\hat{x}), x_k \rangle \) exists. This combined with (3.1) implies that \( \langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \bar{x} \rangle = \langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \hat{x} \rangle \), that is,

\[
\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \bar{x} - \hat{x} \rangle = 0,
\]

which, combining with Fact 3.2, yields \( \hat{x} = \bar{x} \). Altogether, the proof is complete. \( \square \)
4 Bregman circumcenter methods

As we mentioned in the introduction, projection methods based on Bregman distances have broad applications and some circumcentered methods accelerate the method of cyclic projections for finding the best approximation point onto, or a feasibility point in, the intersection of finitely many closed convex sets. In this section, we introduce backward and forward Bregman (pseudo-)circumcenter mappings and methods. We shall also investigate the convergence of the forward Bregman circumcenter method.

Throughout this section, suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom } f \neq \emptyset \) and that \( f \) is Gâteaux differentiable on \( \text{int dom } f \). Set \( \mathcal{I} := \{1, \ldots, m\} \).

Bregman circumcenter mappings

Henceforth, for every \( D \subseteq \mathcal{H} \), denote by \( \mathcal{P}(D) \) the set of all nonempty subsets of \( D \) containing finitely many elements.

Suppose that \( K \in \mathcal{P}(\text{dom } f) \). Set
\[
\widehat{E}_f(K) := \{q \in \text{int dom } f : (\forall x \in K) D_f(x, q) \text{ is identical}\}.
\]
Suppose additionally \( K \in \mathcal{P}(\text{int dom } f) \). Denote by
\[
\widehat{E}_f(K) := \{p \in \text{dom } f : (\forall y \in K) D_f(p, y) \text{ is identical}\}.
\]

Definition 4.1 ([14, Definition 3.1]) Let \( K \in \mathcal{P}(\text{int dom } f) \).

(i) Define the backward Bregman circumcenter operator \( \overleftarrow{CC} \) w.r.t. \( f \) as
\[
\overleftarrow{CC} : \mathcal{P}(\text{int dom } f) \to 2^\mathcal{H} : K \mapsto \text{aff}(K) \cap \widehat{E}_f(K).
\]

(ii) Define the backward Bregman pseudo-circumcenter operator \( \overleftarrow{CC}^{ps} \) w.r.t. \( f \) as
\[
\overleftarrow{CC}^{ps} : \mathcal{P}(\text{int dom } f) \to 2^\mathcal{H} : K \mapsto \text{aff}(\nabla f(K)) \cap \widehat{E}_f(K).
\]

In particular, for every \( K \in \mathcal{P}(\text{int dom } f) \), we call the element in \( \overleftarrow{CC}(K) \) and \( \overleftarrow{CC}^{ps}(K) \) Backward Bregman circumcenter and Backward Bregman pseudo-circumcenter of \( K \), respectively.

Definition 4.2 ([14, Definition 4.1]) Let \( K \in \mathcal{P}(\text{dom } f) \).

(i) Define the forward Bregman circumcenter operator w.r.t. \( f \) as
\[
\overrightarrow{CC} : \mathcal{P}(\text{dom } f) \to 2^\mathcal{H} : K \mapsto \text{aff}(K) \cap \widehat{E}_f(K).
\]
(ii) Define the forward Bregman pseudo-circumcenter operator w.r.t. $f$ as

$$\overrightarrow{CC}^{ps} : \mathcal{P}(\text{dom } f) \rightarrow 2^\mathcal{H} : K \mapsto (\nabla f^* (\text{aff } K)) \cap \overrightarrow{E} f(K).$$

In particular, for every $K \in \mathcal{P}(\text{dom } f)$, we call the element in $\overrightarrow{CC}(K)$ and $\overrightarrow{CC}^{ps}(K)$ forward Bregman circumcenter and forward Bregman pseudo-circumcenter of $K$, respectively.

Suppose that $D_f$ is symmetric, that is, $\text{dom } f = \text{int dom } f$ and $(\forall \{x, y\} \subseteq \text{dom } f) \ D_f(x, y) = D_f(y, x)$. Then the backward Bregman circumcenter and forward Bregman circumcenter are consistent. In particular, if $f := \frac{1}{2} \parallel \cdot \parallel^2$, then $\nabla f = \text{Id}$ and hence, the notions backward Bregman circumcenter, forward Bregman pseudo-circumcenter, forward Bregman circumcenter and forward Bregman pseudo-circumcenter are all the same and reduce to the circumcenter defined in [9, Definition 3.4] under the Euclidean distance.

From now on, for every set-valued operator $A : \mathcal{H} \rightarrow 2^\mathcal{H}$, if $A(x)$ is a singleton for some $x \in \mathcal{H}$, we sometimes by slight abuse of notation allow $A(x)$ to stand for its unique element. The intended meaning should be clear from the context.

**Fact 4.3** ([14, Corollary 6.1]) Suppose that $K := \{q_0, q_1, \ldots, q_m\} \subseteq \text{dom } f$ is nonempty. Then the following assertions hold.

(i) Suppose that $\mathcal{H} = \mathbb{R}^n$, that $f$ is Legendre such that $\text{dom } f^*$ is open, that $\overrightarrow{E} f(K) \neq \emptyset$, that $f$ allows forward Bregman projections, that $\text{aff } K \subseteq \text{int dom } f$, and that $\nabla f(\text{aff } K)$ is a closed affine subspace. Then $\left(\forall z \in \overrightarrow{E} f(K)\right)$

$$\overrightarrow{P}_{\text{aff } (K)}(z) \in \overrightarrow{CC}(K).$$

(ii) Suppose that $\overrightarrow{E} f(K) \neq \emptyset$ and that $K \subseteq \text{int dom } f$ and $\text{aff } (\nabla f(K)) \subseteq \text{dom } f$. Then $\left(\forall z \in \overrightarrow{E} f(K)\right)$ $\overrightarrow{CC}^{ps}(K) = \overrightarrow{P}_{\text{aff } (f(K))}(z)$.

(iii) Suppose that $K \subseteq \text{int dom } f$ and $\text{aff } (\nabla f(K)) \subseteq \text{dom } f$, and that $\nabla f(q_0), \nabla f(q_1), \ldots, \nabla f(q_m)$ are affinely independent. Then $\overrightarrow{CC}^{ps}(K)$ uniquely exists and has an explicit formula.

(iv) Suppose that $f$ is Legendre, that $\text{aff } (K) \cap \text{int dom } f \neq \emptyset$, and that $\overrightarrow{E} f(K) \neq \emptyset$. Then $\left(\forall z \in \overrightarrow{E} f(K)\right)$ $\overrightarrow{P}_{\text{aff } (K)}(z) \in \overrightarrow{CC}(K)$.

(v) Suppose that $f$ is Legendre, and that $\text{aff } (K) \subseteq \text{int dom } f^*$ and $\overrightarrow{E} f(K) \neq \emptyset$. Then $\left(\forall z \in \overrightarrow{E} f(K)\right)$ $\overrightarrow{CC}^{ps}(K) = \nabla f^* (P_{\text{aff } (K)}(\nabla f(z)))$.

(vi) Suppose that $f$ is Legendre, that $\text{aff } (K) \subseteq \text{int dom } f^*$, and that $q_0, q_1, \ldots, q_m$ are affinely independent. Then $\overrightarrow{CC}^{ps}(K)$ uniquely exists and has an explicit formula.

Note that we have particular examples in [14] with functions $f \neq \frac{1}{2} \parallel \cdot \parallel^2$ and sets $K$ such that the hypotheses of each item of Fact 4.3 above hold.

**Definition 4.4** Let $t \in \mathbb{N} \setminus \{0\}$. Suppose that $G_1, \ldots, G_t$ are operators from $\mathcal{H}$ to $\mathcal{H}$. Set

$$S := \{G_1, \ldots, G_t\} \quad \text{and} \quad (\forall x \in \mathcal{H}) \ S(x) := \{G_1 x, \ldots, G_t x\}.$$
The forward Bregman circumcenter mapping induced by $S$ is

$$\overrightarrow{CC}_S : \mathcal{H} \to 2^{\mathcal{H}} : y \mapsto \overrightarrow{CC}(S(y)), \quad (4.1)$$

that is, for every $y \in \mathcal{H}$, if the forward Bregman circumcenter of the set $S(y)$ defined in Definition 4.2(i) does not exist, then $\overrightarrow{CC}_S y = \emptyset$. Otherwise, $\overrightarrow{CC}_S y$ is the set of points $v$ satisfying the two conditions below:

(i) $v \in \text{aff}(S(x)) \cap \text{int dom } f$, and

(ii) $D_f(G_1 x, v) = \cdots = D_f(G_t x, v)$.

Let $\mathcal{C}$ be a subset of $\mathcal{H}$. If $(\forall y \in \mathcal{C}) \overrightarrow{CC}_S y$ contains at most one element, we say that $\overrightarrow{CC}_S$ is at most single-valued on $\mathcal{C}$. Naturally, if $\mathcal{C} = \mathcal{H}$, then we omit the phrase “on $\mathcal{C}$”.

Analogously, we define the forward Bregman pseudo-circumcenter mapping $\overrightarrow{CC}^{ps}_S$ induced by $S$, backward Bregman circumcenter mapping $\overleftarrow{CC}_S$ induced by $S$, and backward Bregman pseudo-circumcenter mapping $\overleftarrow{CC}^{ps}_S$ induced by $S$ with replacing the forward Bregman circumcenter operator $\overrightarrow{CC}$ in (4.1) by the corresponding operator, $\overrightarrow{CC}^{ps}$, $\overleftarrow{CC}$, and $\overleftarrow{CC}^{ps}$, respectively.

We can also directly deduce the following result by [11, Theorem 3.3(ii)].

**Corollary 4.5** Suppose that $f := \frac{1}{2} \| \cdot \|^2$. Let $(\forall i \in I) T_i : \mathcal{H} \to \mathcal{H}$ be a linear isometry. Set $S := \{T_1, \ldots, T_m\}$ and $(\forall x \in \mathcal{H}) S(x) = \{T_1 x, \ldots, T_m x\}$. Then $\overrightarrow{CC}_S(x) = \overrightarrow{CC}^{ps}_S(x) = \overrightarrow{CC}_S(x) = \overrightarrow{CC}^{ps}_S(x) = P_{\text{aff}(S(x))}(0)$.

**Proof** Notice that $\nabla f = \text{Id}$ and $\text{dom } f = \mathcal{H}$ in this case. Then $0 \in \cap_{i=1}^m \text{Fix } T_j \subseteq \overleftarrow{E}_f(K) = \overleftarrow{E}_f(K)$. Therefore, the required result follows easily from Fact 4.3(ii) or Fact 4.3(v).

Henceforth, suppose $T_1, \ldots, T_m$ are operators from $\mathcal{H}$ to $\mathcal{H}$ with $\text{int dom } f \cap (\cap_{i \in I} \text{Fix } T_i) \neq \emptyset$. Unless stated otherwise, we always set

$$S := \{\text{Id}, T_1, \ldots, T_m\} \quad \text{and} \quad (\forall x \in \mathcal{H}) S(x) := \{x, T_1 x, \ldots, T_m x\}. \quad (4.2)$$

**Lemma 4.6** Suppose that $f$ is essentially strictly convex. Then the following statements hold.

(i) $\text{Fix } \overrightarrow{CC} \subseteq \text{int dom } f \cap (\cap_{i \in I} \text{Fix } T_i)$ and $\text{Fix } \overrightarrow{CC}^{ps} \subseteq \text{int dom } f \cap (\cap_{i \in I} \text{Fix } T_i) \cap \text{Fix } \nabla f^*$.

(ii) $\text{Fix } \overrightarrow{CC} \subseteq \text{int dom } f \cap (\cap_{i \in I} \text{Fix } T_i)$ and $\text{Fix } \overrightarrow{CC}^{ps} \subseteq (\cap_{i \in I} \text{Fix } T_i) \cap \text{Fix } \nabla f$.

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1 A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be isometric or an isometry if $(\forall x \in \mathcal{H}) (\forall y \in \mathcal{H}) \|Tx - Ty\| = \|x - y\|$. Springer
\textbf{Proof} (i): Let \( z \in \mathcal{H} \).

\[ z \in \text{Fix} \overrightarrow{CC}_S \]
\[ \iff z \in \overrightarrow{CC}_S z \]
\[ \iff z \in \text{aff}(S(z)) \cap \text{int dom} f \text{ and } D_f(z, z) = D_f(T_1z, z) = \cdots = D_f(T_mz, z) \]
\[ \iff z \in \text{int dom} f \text{ and } (\forall i \in I) z \in \text{Fix} T_i \quad \text{(by Fact 2.3)} \]
\[ \iff z \in \text{int dom} f \cap (\cap_{i \in I} \text{Fix} T_i), \]

which guarantees that \( \text{Fix} \overrightarrow{CC}_S = \text{int dom} f \cap (\cap_{i \in I} \text{Fix} T_i) \).

Similarly,

\[ z \in \text{Fix} \overrightarrow{CC}^ps_S \]
\[ \iff z = \overrightarrow{CC}^ps_S z \]
\[ \iff z \in \nabla f^*(\text{aff}(S(z))) \cap \text{int dom} f \text{ and } \]
\[ D_f(z, z) = D_f(T_1z, z) = \cdots = D_f(T_mz, z) \]
\[ \iff z \in \text{Fix} (\nabla f^*(\text{aff}(S))) \cap \text{int dom} f \text{ and } (\forall i \in I) z \in \text{Fix} T_i \quad \text{(by Fact 2.3)} \]
\[ \iff z \in \text{int dom} f \cap (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} (\nabla f^*(\text{aff}(S))), \]

which deduces that \( \text{Fix} \overrightarrow{CC}^ps_S = \text{int dom} f \cap (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} (\nabla f^*(\text{aff}(S))). \)

In addition, clearly, \( \text{Id} \in S \) entails that \( \text{Fix} \nabla f^* \subseteq \text{Fix} (\nabla f^*(\text{aff}(S))). \) Hence, \( (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} \nabla f^* \subseteq (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} (\nabla f^*(\text{aff}(S))). \) On the other hand, \( (\forall y \in (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} (\nabla f^*(\text{aff}(S)))) y \in \nabla f^*(\text{aff}(S(y))) = \nabla f^*(y), \) which necessitates that \( (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} (\nabla f^*(\text{aff}(S))) \subseteq (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} \nabla f^*. \) Therefore, \( (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} (\nabla f^*(\text{aff}(S))) = (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} \nabla f^*. \)

Altogether, \( \text{Fix} \overrightarrow{CC}^ps_S = \text{int dom} f \cap (\cap_{i \in I} \text{Fix} T_i) \cap \text{Fix} \nabla f^*. \)

(ii): The proof is similar to the proof of (i) above. \( \square \)

\textbf{Bregman isometry and Browder’s Demiclosedness Principle}

In this subsection, we generalize the traditional isometry and Browder’s Demiclosedness Principle from the Euclidean distance to general Bregman distances and investigate the Bregman isometry and Browder’s Demiclosedness Principle, which play critical roles in the main result in this section. The Bregman isometry and Browder’s Demiclosedness Principle are interesting in their own right.

\textbf{Definition 4.7} Let \( \mathcal{C} \) be a nonempty subset of \( \mathcal{H} \) and let \( T : \mathcal{C} \to \mathcal{H} \). We say \( T \) is \textit{Bregman isometric} (or a \textit{Bregman isometry}) w.r.t. \( f \), if \( (\forall (x, y) \in \text{dom} f \times \text{int dom} f) (Tx, Ty) \in \text{dom} f \times \text{int dom} f \), and

\[ (\forall [x, y] \subseteq \text{dom} f \times \text{int dom} f) D_f(Tx, Ty) = D_f(x, y). \quad (4.3) \]
In view of the definition, it is trivial that given an arbitrary function $g$ satisfying the statements of Definition 2.1, the identity operator $\text{Id}$ is Bregman isometric w.r.t. $g$.

Notice that if $f = \frac{1}{2} \| \cdot \|^2$, then the Bregman isometry deduces the traditional isometry (see, e.g., [11, Lemma 2.23] for examples of isometries under the Euclidean distance).

The following definition is a generalization of the well-known Browder’s Demi-closedness Principle [20, Theorem 3(a)]. In view of [7, Corollary 4.25], when $f = \frac{1}{2} \| \cdot \|^2$, the Bregman Browder’s Demiclosedness Principle holds for all non-expansive operators.

**Definition 4.8** Let $C$ be a nonempty subset of $H$, let $T : C \to H$ and let $x \in \text{int dom } f$. We say the Bregman Browder’s demiclosedness principle associated with $f$ holds at $x \in C \cap \text{int dom } f$ for $T$ if for every sequence $(x_k)_{k \in \mathbb{N}}$ in $\text{int dom } f$, $(Tx_k)_{k \in \mathbb{N}}$ in $\text{int dom } f$ and

$$\begin{cases}
    x_k \rightharpoonup x \\
    D_f(x_k, Tx_k) \to 0
\end{cases} \Rightarrow x \in \text{Fix } T. \quad (4.4)$$

In addition, we say $T$ is $f$-demiclosed, if the Bregman Browder’s demiclosedness principle associated with $f$ holds for every $x \in C \cap \text{int dom } f$ for $T$.

Because $\text{Fix Id} = H$, given an arbitrary function $g$ satisfying the statements of Definition 2.1, $\text{Id}$ is $g$-demiclosed.

The following examples are used to illustrate the two new concepts above.

**Example 4.9** Suppose that $H = \mathbb{R}^n$. Denote by $(\forall x \in \mathbb{R}^n) \ x = (x_i)_{i=1}^n$. Define $f : x \mapsto -\sum_{i=1}^n \ln(x_i)$, with $\text{dom } f = ]0, +\infty[^n$, which is the Burg entropy.

Then the following statements hold.

(i) $f$ is Legendre such that $\text{dom } f$ and $\text{dom } f^*$ are open.

(ii) Set $J := \{1, \ldots, n\}$. Let $(c_i)_{i \in J} \in \mathbb{R}^n$ with $(\forall i \in J) \ c_i \in \mathbb{R}_{++}$. Define $T : \mathbb{R}^n \to \mathbb{R}^n : (x_i)_{i \in J} \mapsto (c_i x_i)_{i \in J}$. Then:

(a) $T$ is Bregman isometric w.r.t. $f$.

(b) Suppose that there exists $j \in J$ such that $c_j \neq 1$. Then $T$ is $f$-demiclosed.

(iii) Suppose that $H = \mathbb{R}$. Let $c \in \mathbb{R}_{++}$ and $\mu \in \mathbb{R}_{++} \setminus \{1\}$. Define $T : \mathbb{R} \to \mathbb{R} : t \mapsto ct^\mu$. Then:

(a) $\text{int dom } f \cap \text{Fix } T = \left\{ c^{-\frac{1}{\mu-1}} \right\}$, which is nonempty closed and convex.

(b) $T$ is $f$-demiclosed.

**Proof** (i): This is immediately from [6, Examples 2.1].

(ii)(a): This is clear from Definition 2.1 and the definitions of $f$ and $T$.

(ii)(b): Let $(x^{(k)})_{k \in \mathbb{N}}$ be in $\text{dom } f$. Because $(\forall x \in \text{dom } f) \ Tx \in \text{dom } f$, we have $(Tx^{(k)})_{k \in \mathbb{N}}$ is a sequence in $\text{dom } f$. 

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Suppose that \( x^{(k)} \rightarrow \bar{x} \) with \( \bar{x} := (\bar{x}_i)_{i \in J} \in ]0, +\infty[^n \). Denote by \((\forall k \in \mathbb{N})\) \( x^{(k)} := (x_i^{(k)})_{i \in J} \). Now,

\[
D_f(x^{(k)}, Tx^{(k)}) = f(x^{(k)}) - f(Tx^{(k)}) - \left( \nabla f(Tx^{(k)}) \cdot x^{(k)} - Tx^{(k)} \right)
\]

\[
= - \sum_{i=1}^{n} \ln(x_i^{(k)}) + \sum_{i=1}^{n} \ln(c_i x_i^{(k)}) + \sum_{i=1}^{n} \frac{x_i^{(k)} - c_i x_i^{(k)}}{c_i x_i^{(k)}}
\]

\[
= \sum_{i=1}^{n} \ln(c_i x_i^{(k)}) + \sum_{i=1}^{n} \frac{x_i^{(k)} - c_i x_i^{(k)}}{c_i x_i^{(k)}},
\]

which implies that

\[
D_f(x^{(k)}, Tx^{(k)}) \rightarrow \sum_{i=1}^{n} \left( \ln(c_i) + \frac{1}{c_i} - 1 \right).
\]

On the other hand, consider the function \( g : \mathbb{R}^+ \rightarrow \mathbb{R} : t \mapsto \ln(t) + \frac{1}{t} - 1 \). By some easy calculus, \((\forall t \in \mathbb{R}^+ \setminus \{1\})\) \( g(t) > g(1) = 0 \). Because there exists \( j \in J \) such that \( c_j \neq 1 \), we know that \( \sum_{i=1}^{n} \left( \ln(c_i) + \frac{1}{c_i} - 1 \right) > 0 \). Thus it never holds that \( D_f(x^{(k)}, Tx^{(k)}) \rightarrow 0 \), and so, vacuously, (4.4) holds.

(iii): This is trivial.

(iv): Let \( a \in ]0, +\infty[ \). Let \((a_k)_{k \in \mathbb{N}}\) be in \( \text{int dom } f \) such that \( a_k \rightarrow a \). Then clearly \((T a_k)_{k \in \mathbb{N}}\) is a sequence in \( ]0, +\infty[ \). Suppose that \( D_f(a_k, T a_k) \rightarrow 0 \). Note that

\[
D_f(a_k, T a_k) = - \ln(a_k) + \ln(c a_k^\mu) + \frac{a_k - c a_k^\mu}{c a_k^\mu} \rightarrow \ln(c a^\mu) - \ln(a) + \frac{a^{1-\mu}}{c} - 1.
\]

Denote by \( h : \mathbb{R}^+ \rightarrow \mathbb{R} : t \mapsto \ln(c t^\mu) - \ln(t) + \frac{1-\mu}{c t^\mu} - 1 \). Then \((\forall t \in \mathbb{R}^+)\) \( h'(t) = (\mu - 1)\left( \frac{1}{t} - \frac{1}{c t^\mu} \right) \). By considering the two cases \( \mu > 1 \) and \( \mu < 1 \) separately, we easily get that \( \ln(c a^\mu) - \ln(a) + \frac{a^{1-\mu}}{c} - 1 = 0 \) implies that \( a = c a^\mu \), that is, \( a = c^{-1/\mu-1} \in \text{Fix} T \). Altogether, by Definition 4.8, \( T \) is \( f \)-demiclosed. \( \Box \)

Consider our examples \( f \) and \( T \) in Example 4.9(ii). Notice that if there exists \( j \in J \) such that \( c_j \neq 1 \), then \( \text{int dom } f \cap (\bigcap_{i \in 1} \text{Fix} T_i) = \emptyset \).

The following example of \( f \) and \( T \) satisfies all requirements in Theorem 4.14 below.

**Example 4.10** Suppose that \( \mathcal{H} = \mathbb{R}^n \). Denote by \((\forall x \in \mathbb{R}^n)\) \( x = (x_i)_{i=1}^n \). Define \( f : x \mapsto \sum_{i=1}^{n} x_i \ln(x_i) + (1 - x_i) \ln(1 - x_i) \), with \( \text{dom } f = [0, 1]^n \), which is the Fermi-Dirac entropy. \( ^2 \) Set \( J := \{1, \ldots, n\} \). Then the following statements hold.

(i) \( f \) is Legendre with \( \text{dom } f^* \) open, and allows forward Bregman projections.

\( ^2 \) Here and elsewhere, we use the convention that \( 0 \ln(0) = 0 \).
(ii) \( \text{Id is } f\)-demiclosed.

(iii) Define \( T : \mathbb{R}^n \to \mathbb{R}^n : (x_i)_{i=1}^n \mapsto (1 - x_i)_{i=1}^n \). Then:

(a) \( T \) is Bregman isometric w.r.t. \( f \).
(b) \( T \) is \( f \)-demiclosed.
(c) \( \text{Fix } T \cap \text{int dom } f = \{(\frac{1}{2})_{i=1}\} \) is nonempty, closed and convex.

(iv) Let \( \Lambda \) be a subset of \( J \). Define \( T : \mathbb{R}^n \to \mathbb{R}^n \) by \((\forall x \in \mathbb{R}^n) \ T x := (y_i)_{i=1}^n \) where \((\forall i \in J \setminus \Lambda) \ y_i = x_i \) and \((\forall i \in \Lambda) \ y_i = (1 - x_i) \).

(a) \( \text{Fix } T = \{x \in \mathbb{R}^n : (\forall i \in \Lambda)x_i = \frac{1}{2}\} \).
(b) \( T \) is Bregman isometric w.r.t. \( f \).
(c) \( T \) is \( f \)-demiclosed.

**Proof**

(i): This follows immediately from [6, Examples 2.1 and 2.7].

(ii): This is trivial.

(iii)(a): This follows easily from Definition 2.1 and the definitions of \( f \) and \( T \).

(iii)(b): Let \( x \in ]0,1[^n \) and let \((x^{(k)})_{k \in \mathbb{N}} \) be a sequence in \( ]0,1[^n \) such that \( x^{(k)} \to \bar{x} \). Denote by \( 
\bar{x} := (\bar{x}_i)_{i=1}^n \) and \((\forall k \in \mathbb{N}) \ x^{(k)} = (x^{(k)}_i)_{i \in J} \). In view of the definition of \( T \), \((T x^{(k)})_{k \in \mathbb{N}} \) is also a sequence in \( ]0,1[^n \). Suppose that \( D_f(x_k, Tx_k) \to 0 \). Notice that

\[
D_f(x^{(k)}, T x^{(k)}) = \sum_{i=1}^n \left[ x^{(k)}_i \ln(x^{(k)}_i) + (1 - x^{(k)}_i) \ln(1 - x^{(k)}_i) \right]
\]

\[
- \sum_{i=1}^n \left[ (1 - x^{(k)}_i) \ln(1 - x^{(k)}_i) + x^{(k)}_i \ln(x^{(k)}_i) \right]
\]

\[
- \sum_{i=1}^n \left( x^{(k)}_i - (1 - x^{(k)}_i) \right) \ln \left( \frac{1 - x^{(k)}_i}{x^{(k)}_i} \right)
\]

\[
= - \sum_{i=1}^n \left( x^{(k)}_i - (1 - x^{(k)}_i) \right) \ln \left( \frac{1 - x^{(k)}_i}{x^{(k)}_i} \right) \to
\]

\[
- \sum_{i=1}^n (\bar{x}_i - (1 - \bar{x}_i)) \ln \left( \frac{1 - \bar{x}_i}{\bar{x}_i} \right).
\]

Consider the function \( g : ]0,1[ \to \mathbb{R} : t \mapsto - (t - (1 - t)) \ln \left( \frac{1-t}{t} \right) \). Since \((\forall t \in ]0,1[) \ g''(t) = \frac{1}{(t-1)^2} > 0 \) and \( g'(\frac{1}{2}) = 0 \), thus \((\forall t \in ]0,1[ \setminus \{\frac{1}{2}\}) \ g(t) > 0 \). Hence, \(- \sum_{i=1}^n (\bar{x}_i - (1 - \bar{x}_i)) \ln \left( \frac{1-\bar{x}_i}{\bar{x}_i} \right) = 0 \) implies that \( \bar{x} = (\frac{1}{2})_{i \in J} \in \text{Fix } T \). Therefore, the assertion is true.

(iii)(c): The required results are trivial.
(iv): According to the definitions of $T$ and $f$, it is easy to see that
\[
(\forall x \in \text{dom } f)(\forall y \in \text{dom } f) \quad f(x) - f(Tx) = 0 \quad \text{and} \quad \left\langle \nabla f(Ty), Tx - Ty \right\rangle
\]
\[
= \left\langle \nabla f(y), x - y \right\rangle.
\]
(4.5)

(iv)(a): This is clear.
(iv)(b): This follows immediately from Definition 4.7 and Definition 2.1.
(iv)(c): Applying a proof similar to that of (iii)(b) and invoking (iv)(a), we obtain the required result. \(\square\)

**Convergence of forward Bregman circumcenter methods**

Motivated by the Lemma 4.6, to find a point in the intersection $\cap_{i \in \mathbb{I}} \text{Fix } T_i$, we prefer the backward and forward Bregman circumcenter mappings induced by $S$ rather than the backward and forward Bregman pseudo-circumcenter mappings induced by $S$ with smaller fixed point sets. Moreover, notice that comparing with the hypothesis of Fact 4.3(i) on the existence of the backward Bregman circumcenter, we don’t have a strong requirement for $f$ in the Fact 4.3(iv) on the existence of the forward Bregman circumcenter. Therefore, we consider only the convergence of sequences of iterations generated by forward Bregman circumcenter mappings in this work.

The **forward Bregman circumcenter method** generates the sequence of iterations of the forward Bregman circumcenter mapping.

According to [10, Theorem 4.3(ii)] and [11, Theorem 3.3(ii)], the circumcenter mappings induced by finite sets of isometries (see [11, Definition 2.27] for the exact definition), are special operators $G$ satisfying conditions in Theorem 4.11 below.

**Theorem 4.11** Suppose that $f$ is Legendre. Let $C$ be a nonempty subset of $\text{int dom } f$. Suppose that $G : C \rightarrow C$ satisfies that $\text{Fix } G \cap \text{int dom } f \neq \emptyset$ and $\text{Fix } G$ is a closed convex subset of $H$, and that $(\forall x \in C \cap \text{int dom } f) \ (\forall z \in \text{Fix } G \cap \text{int dom } f) \ G(x) = \overrightarrow{P}_{aff(S(x))} z$. Let $x \in C \cap \text{int dom } f$. Then the following statements hold.

(i) $(G^k x)_{k \in \mathbb{N}}$ is a well-defined sequence in $C \cap \text{int dom } f$.

(ii) $(G^k x)_{k \in \mathbb{N}}$ is forward Bregman monotone with respect to $\text{Fix } G$. Consequently, $(\forall z \in \text{Fix } G \cap \text{int dom } f) \ D_f (G^k x, z)_{k \in \mathbb{N}}$ converges.

(iii) $\lim_{k \rightarrow \infty} D_f (G^k x, G^{k+1} x) = 0$.

(iv) Suppose that $G$ is $f$-demiclosed and that all weak sequential cluster points of $(G^k x)_{k \in \mathbb{N}}$ lie in $\text{int dom } f$. Then $(G^k x)_{k \in \mathbb{N}}$ weakly converges to a point in $\text{int dom } f \cap \text{Fix } G$.

**Proof** Invoking the definition of the operator $G : C \rightarrow C$ and Fact 2.4, we observe that
\[
(\forall y \in C \cap \text{int dom } f) \ (\forall z \in \text{Fix } G \cap \text{int dom } f) \quad G(y) = \overrightarrow{P}_{aff(S(y))} z \in C \cap \text{int dom } f.
\]
(4.6)
Hence, use \( \{ y, G(y) \} \subseteq \text{aff}(S(y)) \) and apply [14, Theorem 2.1(i)] with \( U = \text{aff}(S(x)) \) to yield that

\[
(\forall y \in \mathcal{C} \cap \text{int dom } f) \ (\forall z \in \text{Fix} G \cap \text{int dom } f) \quad D_f(y, z) = D_f(y, G(y)) + D_f(G(y), z).
\] (4.7)

(i): This is clear from (4.6) by induction.

(ii): Taking (4.7) and Definition 2.1 into account, we deduce that

\[
(\forall y \in \mathcal{C} \cap \text{int dom } f) \ (\forall z \in \text{int dom } f \cap \text{Fix} G) \quad D_f(y, z) \geq D_f(Gy, z).
\] (4.8)

Employing (i) above and Definition 3.1, we know that \( (G^k x)_{k \in \mathbb{N}} \) is forward Bregman monotone with respect to \( \text{Fix} G \), which, combining with Theorem 3.3(i), implies the convergence assertion.

(iii): Let \( z \in \text{Fix} G \cap \text{int dom } f \). For every \( k \in \mathbb{N} \), substitute \( y \) in (4.7) by \( G^k x \) to deduce that

\[
D_f(G^k x, z) = D_f(G^k x, G^{k+1} x) + D_f(G^{k+1} x, z),
\]

which implies that

\[
\sum_{k \in \mathbb{N}} D_f(G^k x, G^{k+1} x) = D_f(x, z) - \lim_{t \to \infty} D_f(G^t x, z) < \infty,
\]

where the existence of the limit \( \lim_{t \to \infty} D_f(G^t x, z) \) is from (ii). Therefore, (iii) holds.

(iv): Let \( \bar{z} \) be a weak sequential cluster point of \( (G^k x)_{k \in \mathbb{N}} \). Then there exists a subsequence \( (G^{k_j} x)_{j \in \mathbb{N}} \) of \( (G^k x)_{k \in \mathbb{N}} \) such that \( G^{k_j} x \to \bar{z} \). Notice that, in view of (iii) above, \( D_f(G^k x, G^{k+1} x) \to 0 \). Due to the assumption, \( \bar{z} \in \text{int dom } f \). Hence, utilizing the assumption that \( G \) is \( f \)-demiclosed, we derive that

\[
\begin{cases}
G^{k_j} x \to \bar{z} \\
D_f(G^{k_j} x, G(G^{k_j} x)) \to 0
\end{cases} \Rightarrow \bar{z} \in \text{Fix} G,
\]

which implies that all weak sequential cluster points of \( (G^k x)_{k \in \mathbb{N}} \) lie in \( \text{Fix} G \cap \text{int dom } f \), since \( \bar{z} \) is an arbitrary cluster point of \( (G^k x)_{k \in \mathbb{N}} \). Therefore, via Theorem 3.3(iv), \( (G^k x)_{k \in \mathbb{N}} \) weakly converges to some point in \( \text{int dom } f \cap \text{Fix} G \). \( \Box \)

It is clear that the characterization (4.7) of the backward Bregman projection \( \overrightarrow{P}_{\text{aff}(S(y))} z \) plays an essential role in the proof of Theorem 4.11 which is critical to prove our main result Theorem 4.12 below. This combined with [14, Theorem 2.1(ii)] suggests that the idea of the proof of Theorem 4.11 does not work if we redefine \( G : \mathcal{C} \to \mathcal{C} \) in Theorem 4.11 by \( (\forall x \in \mathcal{C} \cap \text{int dom } f) \ (\forall z \in \text{Fix} G \cap \text{int dom } f) \)

\[
G(x) = \overrightarrow{P}_{\text{aff}(S(y))} z.
\]

On the other hand, in view of Fact 4.3(i) & (iv), under some conditions, certain backward Bregman projections (resp. forward Bregman projections) are forward Bregman circumcenters (resp. backward Bregman circumcenters).
Therefore, it is easier to work on the forward Bregman circumcenter method than the backward Bregman circumcenter method.

The following Theorem 4.12(vi)(a) and Theorem 4.12(vi)(b) reduce to [10, Theorem 3.17] and [11, Theorem 4.7], respectively, when \( f = \frac{1}{2} \| \cdot \|^2 \). Notice that the circumcenter mappings studied in [11], [12], [13], and [22] under the Euclidean distance are all single-valued operators, and that all of the examples of backward and forward Bregman (pseudo)-circumcenters presented in [14] are singletons. So, our assumption “\( \overrightarrow{C_{CS}} \) is at most single-valued on \( \text{int dom } f \)” in the following Theorem 4.12 is not too restrictive. In addition, it is clear that if \( f = \frac{1}{2} \| \cdot \|^2 \), then the following condition (4.9) is a direct result from the triangle inequality.

**Theorem 4.12** Suppose that \( f \) is Legendre, that \( \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \) is nonempty, that \( (\bigcap_{i \in I} \text{Fix } T_i) \) is closed and convex, that \( (\forall i \in I) T_i \) is Bregman isometric w.r.t. \( f \), and that \( \overrightarrow{C_{CS}} \) is at most single-valued on \( \text{int dom } f \). Let \( y \in \text{int dom } f \). Then the following statements hold.

\[
\begin{align*}
(i) & \quad (\forall x \in \text{dom } f) \text{ dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \subseteq \overrightarrow{E}_f (S(x)). \\
(ii) & \quad (\forall z \in \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i)) \overrightarrow{C_{CS}} y = \overrightarrow{P}_f (S(y))(z) \in \text{int dom } f. \\
(iii) & \quad (\overrightarrow{C_{CS}^k y})_{k \in \mathbb{N}} \text{ is a well-defined sequence in } \text{int dom } f. \\
(iv) & \quad (\overrightarrow{C_{CS}^k y})_{k \in \mathbb{N}} \text{ is forward Bregman monotone with respect to } \bigcap_{i \in I} \text{Fix } T_i. \text{ Consequently, } (\forall z \in \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i)) \left( D_f \left( \overrightarrow{C_{CS}^k y}, z \right) \right)_{k \in \mathbb{N}} \text{ converges.} \\
(v) & \quad \lim_{k \to \infty} D_f \left( \overrightarrow{C_{CS}^k y}, \overrightarrow{C_{CS}^{k+1} y} \right) = 0. \\
(vi) & \quad \text{Suppose that } (\forall i \in I) T_i \text{ is } f \text{-demiclosed, that for every sequence } (u_k)_{k \in \mathbb{N}} \text{ in } \text{int dom } f, \text{ and that} \\
\begin{align*}
& \left\{ \begin{array}{ll}
D_f (u_k, \overrightarrow{C_{CS} u_k}) \to 0 \\
(\forall i \in I) D_f (T_i u_k, \overrightarrow{C_{CS} u_k}) \to 0
\end{array} \right. \\
& \Rightarrow (\forall i \in I) D_f (u_k, T_i u_k) \to 0. \tag{4.9}
\end{align*}
\]

Then the following hold.

(a) \( \overrightarrow{C_{CS}} \) is \( f \)-demiclosed.

(b) Suppose that all weak sequential cluster points of \( (\overrightarrow{C_{CS}^k y})_{k \in \mathbb{N}} \) lie in \( \text{int dom } f \). Then \( (\overrightarrow{C_{CS}^k y})_{k \in \mathbb{N}} \) weakly converges to some point in \( \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \).

**Proof** Because \( (\forall i \in I) T_i \) is Bregman isometric, we know that \( (\forall i \in I) (\forall (x, y) \in \text{dom } f \times \text{int dom } f) (T_i x, T_i y) \in \text{dom } f \times \text{int dom } f \), and that

\[
(\forall i \in I) (\forall (x, y) \subseteq \text{dom } f \times \text{int dom } f) D_f (T_i x, T_i y) = D_f (x, y). \tag{4.10}
\]

(i): Let \( x \in \text{dom } f \) and \( z \in \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \). Then

\[
(\forall i \in I) D_f (x, z) = D_f (T_i x, z) \Leftrightarrow (\forall i \in I) D_f (x, z) = D_f (T_i x, T_i z),
\]
where the second equality holds by (4.10) and \( z \in \cap_{j \in \mathbb{N}} \text{Fix}T_j \). Hence, \( z \in \overrightarrow{E}_f(S(x)) \) by the definition of \( \overrightarrow{E}_f(S(x)) \) presented in Definition 4.2.

(ii): Let \( z \in \text{int dom } f \cap (\cap_{i \in \mathbb{N}} \text{Fix}T_i) \). Because \( y \in \text{int dom } f \), we see that \( y \in \text{aff}(S(y)) \cap \text{int dom } f \neq \emptyset \), and that, by (i) above, \( z \in \overrightarrow{E}_f(S(y)) \). Applying Fact 4.3(iv) with \( K \) replaced by \( S(y) \), we deduce that \( \overrightarrow{P}_{\text{aff}(S(y))}z \in \overrightarrow{CC}(S(y)) \).

Combine this with Definition 4.4, Fact 2.4, and the assumption that \( \overrightarrow{CC}S \) is at most single-valued, to yield that \( \overrightarrow{CC}S = \overrightarrow{CC}(S(y)) = \overrightarrow{P}_{\text{aff}(S(y))}z \in \text{int dom } f \).

(iii)&(iv)&(v): By Lemma 4.6(i), \( \text{Fix}\overrightarrow{CC}S = \text{int dom } f \cap (\cap_{i \in \mathbb{N}} \text{Fix}T_i) \). Hence, (ii) implies that

\[
(\forall x \in \text{int dom } f) \ (\forall z \in \text{Fix}\overrightarrow{CC}S \cap \text{int dom } f) \quad \overrightarrow{CC}Sx = \overrightarrow{P}_{\text{aff}(S(x))}z \in \text{int dom } f.
\]

Therefore, the required results follow directly from Theorem 4.11(i) & (ii) & (iii), respectively, with \( C = \text{int dom } f \) and \( G = \overrightarrow{CC}S \).

(vi)(a): Suppose that \( x \in \text{int dom } f \) and that \( (x_k)_{k \in \mathbb{N}} \) is a sequence in \( \text{int dom } f \) such that \( x_k \xrightarrow{f} x \). Notice that, due to (ii) above, \( (\overrightarrow{CC}Sx_k)_{k \in \mathbb{N}} \) is a sequence in \( \text{int dom } f \).

Assume that \( D_f(x_k, \overrightarrow{CC}Sx_k) \to 0 \). In view of Definition 4.4,

\[
(\forall i \in I) \ (\forall k \in \mathbb{N}) \quad D_f\left(x_k, \overrightarrow{CC}Sx_k\right) = D_f\left(T_i x_k, \overrightarrow{CC}Sx_k\right),
\]

which, connecting with \( D_f(x_k, \overrightarrow{CC}Sx_k) \to 0 \) and applying (4.9) with \( (\forall k \in \mathbb{N}) u_k = x_k \), implies that \( (\forall i \in I) \ D_f(x_k, T_i x_k) \to 0 \). Let \( i \in I \). Note that \( T_i \) is \( f \)-demiclosed. So the results \( x_k \xrightarrow{f} x \) and \( D_f(x_k, T_i x_k) \to 0 \) imply that \( x \in \text{Fix}T_i \). Because \( i \in I \) is chosen arbitrarily, by Lemma 4.6(i), \( x \in \text{int dom } f \cap (\cap_{i \in \mathbb{N}} \text{Fix}T_i) = \text{Fix}\overrightarrow{CC}S \).

(vi)(b): Bearing (vi)(a) in mind, we observe that the desired convergence follows from Theorem 4.11(iv) with \( C = \text{int dom } f \) and \( G = \overrightarrow{CC}S \). \( \square \)

**Remark 4.13** We uphold assumptions in Theorem 4.12 and have a closer look at the condition (4.9). As we mentioned before, in the classical Euclidean distance, the condition (4.9) is immediate from the triangle inequality of the Euclidean distance. Because generally Bregman distances do not obey the triangle inequality, (4.9) is no longer trivial if \( f \neq \frac{1}{2} \| \cdot \| \). We explain below that, even if the classical triangle inequality may not hold in a general Bregman distance, there is no need to be pessimistic.

Let \( (u_k)_{k \in \mathbb{N}} \) be in \( \text{int dom } f \). According to Definition 4.4 and 4.2,

\[
(\forall k \in \mathbb{N})(\forall i \in I) \quad D_f(T_i u_k, \overrightarrow{CC}Su_k) = D_f(u_k, \overrightarrow{CC}Su_k),
\]

which yields that (4.9) is equivalent to the following statement

\[
D_f(u_k, \overrightarrow{CC}Su_k) \to 0 \Rightarrow (\forall i \in I) D_f(u_k, T_i u_k) \to 0.
\]
Combine this with Definition 2.1 to ensure that (4.9) holds if and only if
\[ f(u_k) - f(\overrightarrow{CC}_S u_k) = \nabla f(\overrightarrow{CC}_S u_k), u_k - \overrightarrow{CC}_S u_k \rightarrow 0 \Rightarrow (\forall i \in I) f(u_k) - f(T_i u_k) - \nabla f(T_i u_k), u_k - T_i u_k \rightarrow 0. \]

Therefore, we observe that (4.9) depends not only on the function $f$ but also the set $S := [\text{Id}, T_1, \ldots, T_m]$. For example,

(i) if $S = [\text{Id}]$, then (4.9) holds for an arbitrary function $f$ satisfying the statements of Definition 2.1;

(ii) if there exists a particular constant $\rho \in \mathbb{R}_{++}$ such that $(\forall i \in I) (\forall k \in \mathbb{N})$
\[ D_f(u_k, T_i u_k) \leq \rho D_f(u_k, \overrightarrow{CC}_S u_k), \text{ then (4.9) is also satisfied.} \]

Notice that if we can find a function $f \neq \frac{1}{2} \|\cdot\|^2$ such that $(\forall [x, y, z] \subseteq \mathcal{H})$
\[ D_f(x, y) \leq D_f(x, z) + D_f(z, y), \text{ then we obtain (4.9) immediately. (We believe such functions actually have a wide range of applications in various areas.) Based on our explanations in Remark 4.13, even if we cannot find such beautiful functions, there is a large probability that there exist some special functions $f$ together with appropriate sets $S := [\text{Id}, T_1, \ldots, T_m]$ (the set might be dependent on the corresponding $f$) such that (4.9) is satisfied.} \]

Given a function $f$ and a set $S$, there may be infinitely many sequences $(u_k)_{k \in \mathbb{N}}$ in $\text{int dom } f$ satisfying $D_f(u_k, \overrightarrow{CC}_S u_k) \rightarrow 0$ and $(\forall i \in I) D_f(T_i u_k, \overrightarrow{CC}_S u_k) \rightarrow 0$, so the workload of verifying (4.9) might be really large if we don’t use tricks like Remark 4.13(ii) or some better ones. The following result demonstrates that to show the weak convergence of forward Bregman circumcenter methods, the condition (4.9) might be unnecessary.

**Theorem 4.14** Suppose that $f$ is Legendre, that $\text{int dom } f \cap (\cap_{i \in [1, m]} \text{Fix } T_i)$ is nonempty, that $(\cap_{i \in [1, m]} \text{Fix } T_i)$ is closed and convex, that $(\forall i \in I) T_i$ is Bregman isometric w.r.t. $f$, and that $\overrightarrow{CC}_S$ is at most single-valued on $\text{int dom } f$. Let $x_0 \in \text{int dom } f$. Set $(\forall k \in \mathbb{N})$
\[ x_{k+1} = \overrightarrow{CC}_S x_k. \]
Then the following assertions hold.

(i) $(x_k)_{k \in \mathbb{N}}$ is a well-defined sequence in $\text{int dom } f$.

(ii) $(x_k)_{k \in \mathbb{N}}$ is forward Bregman monotone with respect to $\cap_{i \in [1, m]} \text{Fix } T_i$. Consequently, $(\forall z \in \text{int dom } f \cap (\cap_{i \in [1, m]} \text{Fix } T_i))$
\[ (\forall z \in \text{int dom } f \cap (\cap_{i \in [1, m]} \text{Fix } T_i)) \rightarrow \lim_{k \rightarrow \infty} f(x_k, \overrightarrow{CC}_S z) = 0. \]

(iii) $(x_k)_{k \in \mathbb{N}}$ converges.

(iv) Suppose that $(\forall i \in I) T_i$ is $f$-demiclosed, that all weak sequential cluster points of $(x_k)_{k \in \mathbb{N}}$ lie in $\text{int dom } f$, and that
\[ D_f(x_k, \overrightarrow{CC}_S x_k) \rightarrow 0 \Rightarrow (\forall i \in I) D_f(x_k, T_i x_k) \rightarrow 0. \]

Then $(x_k)_{k \in \mathbb{N}}$ weakly converges to some point in $\text{int dom } f \cap (\cap_{i \in [1, m]} \text{Fix } T_i)$.

**Proof** (i)&(i)&(i)&(iii): These results are clear from Theorem 4.12(iii)&(iv)&(v), respectively.
(iv): According to (ii) above and Theorem 3.3(iv), it suffices to prove that all weak sequential cluster points of \((x_k)_{k \in \mathbb{N}}\) are in \(\cap_{i \in I} \text{Fix} T_i\).

Suppose that a subsequence \((x_{k_j})_{j \in \mathbb{N}}\) of \((x_k)_{k \in \mathbb{N}}\) weakly converges to \(\tilde{x} \in \text{int dom} \, f\). Taking (iii) above and (4.11) into account, we observe that \((\forall i \in I) \, D_f(x_{k_j}, T_i x_{k_j}) \to 0\), which ensures that \((\forall i \in I) \, D_f(x_{k_j}, T_i x_{k_j}) \to 0\). Now combine the results, \(x_{k_j} \to \tilde{x}\) and \((\forall i \in I) \, D_f(x_{k_j}, T_i x_{k_j}) \to 0\), with the assumption that \((\forall i \in I) \, T_i \text{ is } f\)-demiclosed to necessitate that \(\tilde{x} \in \cap_{i \in I} \text{Fix} T_i\).

Altogether, the proof is done. \(\square\)

**Remark 4.15**

(i) Note that (4.9) is clearly stronger than (4.11) and that (4.9) is implied by the triangle inequality of the Euclidean distance. So (4.11) is trivial in the Euclidean distance as well.

(ii) Suppose that there exists \(K \in \mathbb{N}\) such that \(x_K \in \cap_{i \in I} \text{Fix} T_i\). Then, via Lemma 4.6(i),

\[
x_K \in \text{int dom} \, f \cap (\cap_{i \in I} \text{Fix} T_i) = \text{Fix} \hat{C} \hat{C} S.
\]

Hence,

\[
(\forall i \in I)(\forall k \geq K) \quad D_f(x_k, \hat{C} \hat{C} S x_k) = D_f(T_i x_k, \hat{C} \hat{C} S x_k) = D_f(x_k, T_i x_k) = 0,
\]

which guarantees that if the forward Bregman circumcenter method converges to a point in \(\text{int dom} \, f \cap (\cap_{i \in I} \text{Fix} T_i)\) in finitely many steps, then (4.9) holds.

Note that the finite convergence of the circumcenter method under the Euclidean distance is not a big surprise (for details, see e.g., [18], [2] and [22]). Moreover, the following Example 4.16 also shows the one-step convergence of a forward Bregman circumcenter method under a general Bregman distance.

We leave a systematic study on the conditions (4.9) and (4.11) as future work.

To end this work, we revisit the operator \(T\) and the function \(f\) in Example 4.10, which satisfy all requirements in Theorem 4.14. In view of Example 4.10, the following example illustrates Theorem 4.12,4.14 and demonstrates that the forward Bregman circumcenter method finds the desired fixed point with one iterate.

**Example 4.16** Suppose that \(\mathcal{H} = \mathbb{R}^n\). Define \(f : x \mapsto \sum_{i=1}^n x_i \ln(x_i) + (1 - x_i) \ln(1 - x_i)\), with \(\text{dom} \, f = [0, 1]^n\), which is the Fermi-Dirac entropy. Let \(\Lambda\) be a subset of \(J := \{1, \ldots, n\}\). Define \(T : \mathbb{R}^n \to \mathbb{R}^n\) by \((\forall x \in \mathbb{R}^n) \, Tx := (y_i)_{i=1}^n\) where \((\forall i \in J \setminus \Lambda) \, y_i = x_i\) and \((\forall i \in \Lambda) \, y_i = 1 - x_i\). Set \(S := \{\text{Id}, T\}\). Let \(x := (x_i)_{i \in J} \in [0, 1]^n\).

Denote by \(\Phi := \{i \in J : x_i \neq \frac{1}{2}\}\). Then the following statements hold.

(i) \(\hat{E}_f(S(x)) = \left\{ (p_i)_{i \in J} \in [0, 1]^n : 0 = \sum_{i \in \Lambda} (2x_i - 1) \ln \left( \frac{p_i}{1 - p_i} \right) \right\}\).

(ii) \(\hat{C} \hat{C} S(x) = \left\{ (p_i)_{i \in J} \in \left[0, 1^n \cap \text{aff}[x, 1 - x]\right] : 0 = \sum_{i \in \Lambda} (2x_i - 1) \ln \left( \frac{p_i}{1 - p_i} \right) \right\}\).

Moreover,
\[
\{(p_i)_{i \in J} \in \left( [0, 1]^n \cap \text{aff} \{x, 1 - x\} \right) : (\forall i \in \Lambda \setminus \Phi) p_i = \frac{1}{2}\} \subseteq \overrightarrow{CC} \mathcal{S}(x);
\]
\[
\emptyset \neq \{(p_i)_{i \in J} \in \left( [0, 1]^n \cap \text{aff} \{x, 1 - x\} \right) : (\forall i \in \Lambda) p_i = \frac{1}{2}\}
\subseteq \text{Fix} T \cap \text{int dom } f \cap \overrightarrow{CC} \mathcal{S}(x).
\]

(iii) Suppose that \((\forall i \in \Lambda \setminus \Phi) x_i > \frac{1}{2}\) or \((\forall i \in \Lambda \setminus \Phi) x_i < \frac{1}{2}\). Then
\[
\overrightarrow{CC} \mathcal{S}(x) = \{(p_i)_{i \in J} \in \left( [0, 1]^n \cap \text{aff} \{x, 1 - x\} \right) : (\forall i \in \Lambda \setminus \Phi) p_i = \frac{1}{2}\}.
\]

(iv) Suppose that \((\forall i \in \Lambda \setminus \Phi) x_i > \frac{1}{2}\) or \((\forall i \in \Lambda \setminus \Phi) x_i < \frac{1}{2}\), and that \(\Lambda \cap \Phi = \emptyset\). Then
\[
\overrightarrow{CC} \mathcal{S}(x) = \{(p_i)_{i \in J} \in \left( [0, 1]^n \cap \text{aff} \{x, 1 - x\} \right) : (\forall i \in \Lambda) p_i = \frac{1}{2}\}
\subseteq \text{Fix} T \cap \text{int dom } f.
\]

(v) Suppose that \(\Lambda\) is a singleton, say \(\Lambda := \{i_0\}\). Then if \(x_{i_0} = \frac{1}{2}\), then \(\overrightarrow{CC} \mathcal{S}(x) = [0, 1]^n \cap \text{aff} \{x, 1 - x\}\); if \(x_{i_0} \neq \frac{1}{2}\), then
\[
\overrightarrow{CC} \mathcal{S}(x) = \{(p_i)_{i \in J} \in [0, 1]^n \cap \text{aff} \{x, 1 - x\} : p_{i_0} = \frac{1}{2}\} \subseteq \text{Fix} T \cap \text{int dom } f.
\]

(vi) Suppose that \(\mathcal{H} = \mathbb{R}\). Then \((\forall x \in [0, 1]) \overrightarrow{CC} \mathcal{S}(x) = \{\frac{1}{2}\} = \text{int dom } f \cap \text{Fix} T\).

(vii) Suppose that \(\mathcal{H} = \mathbb{R}^2\). Let \(x = (\frac{1}{4}, \frac{5}{6})\). Then \(\overrightarrow{CC} \mathcal{S}(x) = \{\frac{1}{2}\} = \text{int dom } f \cap \text{Fix} T\).

**Proof** (i): This is easy from the related definitions and (4.5) in the proof of Example 4.10.

(ii): Based on Example 4.10(iv),
\[
\text{Fix} T \cap \text{int dom } f = \left\{ x \in [0, 1]^n : (\forall i \in \Lambda) x_i = \frac{1}{2} \right\}.
\]

Hence, this is clear from Definition 4.4.

(iii): This follows immediately from (ii).

(iv) & (v) & (vi): The required results follows easily from (iii).

(vii): This is from (ii) and some basic calculus. \(\square\)

Searching more particular examples of sets \(\mathcal{S} = \{\text{Id}, T_1, \ldots, T_m\}\) satisfying the requirements in Theorem 4.12, 4.14 under general Bregman distances associated with \(f \neq \frac{1}{2} \| \cdot \|_2^2\) is left as our future work.
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