Metrizability of CHART groups

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Abstract

For compact Hausdorff admissible right topological (CHART) group \( G \), we prove \( w(G) = \pi\chi(G) \). This equality is well known for compact topological groups. This implies the criteria for the metrizability of CHART groups: if \( G \) is first-countable (2013, Moors, Namioka) or \( G \) is Fréchet (2013, Glasner, Megrelishvili), or \( G \) has countable \( \pi \)-character (2022, Reznichenko) then \( G \) is metrizable. Under the continuum hypothesis (CH) assumption, a sequentially compact CHART group is metrizable. Namioka’s theorem that metrizable CHART groups are topological groups extends to CHART groups with small weight.

Keywords: compact right topological groups, admissible groups, CHART groups, metrizable spaces, \( \pi \)-character

1. Introduction

A group \( G \) with a topology is called right topological if all right shifts \( \rho_h : G \to G, g \mapsto gh \) are continuous. The set of \( g \in G \) for which the left shift \( \lambda_g : G \to G, h \mapsto gh \) is continuous is called the topological center and is denoted as \( \Lambda(G) \). A group \( G \) with topology is called semitopological if all right and left shifts are continuous, that is, if \( G \) is right topological and \( \Lambda(G) = G \). A right topological group \( G \) is called admissible if \( \Lambda(G) \) is a dense subset of \( G \). We write “CHART” for “compact Hausdorff admissible right topological”. A paratopological group \( G \) is a group \( G \) with a topology such that the product map of \( G \times G \) into \( G \) is jointly continuous.

The study of groups with topology in which not all operations are continuous and conditions implying the continuity of operations began with the 1936 paper \cite{1} of Montgomery, who, among other things, proved that a Polish (i.e., separable metrizable by a complete metric) semitopological group is a topological group. Interest in such groups was renewed in relation to topological dynamics. The autohomeomorphism group of a locally compact space in the compact open topology is a paratopological group \cite{2}. In the same paper Arens obtained conditions under which an autohomeomorphism group is a topological group. In

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1957 Ellis proved that a locally compact paratopological group is a topological group [3]. Shortly afterwards, he strengthened this theorem to semitopological locally compact groups and proved the celebrated Ellis theorem [4]: any semitopological locally compact group is a topological group. The enveloping semigroup of a dynamical system was introduced by Ellis in 1960 [5]. It has become a fundamental tool in the abstract theory of topological dynamical systems. The study of dynamical systems for which the enveloping semigroup is a CHART group plays a large role in the abstract theory of topological dynamical systems.

Let $G$ be a CHART group. Any of the following conditions implies that $G$ is a topological group.

$(C_1)$ $G$ is metrizable (Theorem 2.1 [6]).

$(C_2)$ $G$ is first-countable (Remark after Proposition 2.7 [7]).

$(C_3)$ $G$ is Fréchet (Corollary 8.8 [8]).

$(G_4)$ $G$ has countable $\pi$-character (for example, $G$ is a compact space with countable tightness) (Corollary 2 (3) [9]).

A CHART group $G$ is tame if for every $g \in G$, the mapping $x \mapsto g \cdot x$ is fragmented. A compact semitopological group is tame.

Note even more conditions that imply that $G$ is a topological group: $(G_5)$ the multiplication of $G$ is separately continuous (Ellis theorem [4]); $(G_6)$ $G$ is tame [8] Theorem 21]; $(G_7)$ the multiplication of $G$ is continuous at $(e, e)$; (follows from [10] and [11] Theorem 5]); $(G_8)$ the multiplication of $G$ is feebly continuous [12] Proposition 3.2] (see also [9] Corollary 2 (2)); $(G_9)$ the inversion $g \mapsto g^{-1}$ is continuous at $e$ [11] Theorem 5]; $(G_{10})$ $\Lambda(G)$ is a topological group (or merely contains a dense topological group) [11] Theorem 5]; the right translations $g \mapsto gh$ form an equicontinuous family of maps from $G$ onto $G$ [11] Theorem 5].

Note that the group $G$ in $(C_1)$–$(C_4)$ is metrizable because compact first-countable, Fréchet topological groups and groups of countable $\pi$-character are metrizable (Corollary 4.2.2 and Corollary 5.7.26 of [13]). Recall that for compact Hausdorff spaces [14]:

\[
\text{metrizable} \Rightarrow \text{first-countable} \Rightarrow \text{Fréchet} \Rightarrow \text{countable tightness} \Rightarrow \text{countable $\pi$-character}.
\]

In this note, we prove that $w(G) = \pi \chi(G)$ for CHART group $G$ (Theorem 2). Whence it follows that $G$ is metrizable if $G$ has countable $\pi$-character. Note that this fact reduces $(C_2)$–$(C_4)$ to $(C_1)$.

Under the continuum hypothesis (CH) assumption, a sequentially compact CHART group is metrizable (Corollary 2). Namioka’s theorem that metrizable CHART groups are topological groups extends to CHART groups with small weight (Theorem 3).
2. Definitions and notation

We will denote by \( G \) a group and by \( e \in G \) the identity of the group.

Let \( X \) be a space, \( x \in X \), \( \mathcal{P} \) be a family of open subsets of \( X \). A family \( \mathcal{P} \) is called a base in \( x \) if for any neighborhood \( U \) of \( x \) there exists \( V \in \mathcal{P} \) so that \( x \in V \subset U \). A family \( \mathcal{P} \) is called a \( \pi \)-base in \( x \) if for any neighborhood \( U \) of \( x \) there exists \( V \in \mathcal{P} \) so that \( V \subset U \). A family \( \mathcal{P} \) is called a base of \( X \) if \( \mathcal{P} \) is a base at every point of \( X \). A family \( \mathcal{P} \) is called a \( \pi \)-base of \( X \) if \( \mathcal{P} \) is a \( \pi \)-base at every point of \( X \). Denote the diagonal \( \Delta_X = \{(x, x) : x \in X\} \) in \( X^2 \).

Recall the necessary definitions of cardinal functions from [14].

weight

\[ w(X) = \min \{|B| : B \text{ a base for } X\}; \]

diagonal degree

\[ \Delta(X) = \min \{|\mathcal{P}| : \mathcal{P} \text{ a family of open neighborhoods} \]

\[ \text{of the diagonal } \Delta_X \text{ and } \bigcap \mathcal{P} = \Delta_X \}; \]

character

\[ \chi(x, X) = \min\{|\mathcal{P}| : \mathcal{P} \text{ a base for } x\}, \]

\[ \chi(X) = \sup\{\chi(x, X) : x \in X\}; \]

\( \pi \)-character

\[ \pi\chi(x, X) = \min\{|\mathcal{P}| : \mathcal{P} \text{ a } \pi \text{-base for } x\}, \]

\[ \pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}; \]

tightness

\[ t(x, X) = \min \{\tau : \text{ for all } A \subset X \text{ with } x \in \overline{A} \]

\[ \text{there is } M \subset A \text{ with } |M| \leq \tau \text{ and } x \in \overline{M}\}, \]

\[ t(X) = \sup\{t(x, X) : x \in X\}. \]

A space \( X \) is called sequentially compact provided that every sequence in \( X \) has a convergent subsequence.

A topological space \( X \) is said to satisfy the countable chain condition, or to be ccc, if the partially ordered set of non-empty open subsets of \( X \) satisfies the countable chain condition, i.e. every pairwise disjoint collection of non-empty open subsets of \( X \) is countable.

In what follows, it is assumed that the spaces are Hausdorff.
3. Diagonal degree of groups

**Theorem 1.** Let $G$ be a right topological group and $\Lambda(G)^{-1}$ dense in $G$. Then $\Delta(G) \leq \pi\chi(G)$.

**Proof.** Let $\tau = \pi\chi(G)$ and $\{U_\alpha : \alpha < \tau\}$ be a $\pi$-base for $e$. We set $C = \Lambda(G)$ and

$$W_\alpha = \bigcup_{g \in C} gU_\alpha \times gU_\alpha$$

for $\alpha < \tau$. Let us show that $\bigcap_{\alpha < \tau} W_\alpha = \Delta_X$.

Let us show that $\Delta_X \subset \bigcap_{\alpha < \tau} W_\alpha$. Let $x \in G$ and $\alpha < \tau$. Since $C^{-1}$ is dense in $G$, then $g^{-1} \in U_\alpha x$ for some $g \in C$. Then

$$(x, x) \in gU_\alpha \times gU_\alpha \subset W_\alpha.$$ 

Let us show that $\bigcap_{\alpha < \tau} W_\alpha \subset \Delta_X$. Assume the opposite, that is, there is

$$(x, y) \in \bigcap_{\alpha < \tau} W_\alpha \setminus \Delta_X.$$ 

Since $x \neq y$ the group $G$ is Hausdorff and right topological, there exists a neighborhood $U$ of the identity for which $Ux^{-1} \cap Uy^{-1} = \emptyset$. Since $\{U_\alpha : \alpha < \tau\}$ is a $\pi$-base for $e$, then $U_\alpha \subset U$ for some $\alpha < \tau$. There is $g \in C$ so $(x, y) \in gU_\alpha \times gU_\alpha$. Then $x, y \in gU_\alpha$ and $g \in U_\alpha x^{-1} \cap U_\alpha y^{-1}$. Hence $g \in Ux^{-1} \cap Uy^{-1}$, a contradiction. \qed

**Proposition 1.** If $G$ is a CHART group, then $\Lambda(G) = \Lambda(G)^{-1}$ is a subgroup.

**Proof.** Clearly, $\Lambda(G)$ is a subsemigroup. Let $g \in \Lambda(G)$. Since $\lambda_g$ is a continuous bijection of a compact space, then $\lambda_g$ is a homeomorphism and the mapping $\lambda_g^{-1} = \lambda_{g^{-1}}$ is continuous. \qed

**Theorem 2.** Let $G$ be a CHART group. Then $w(G) = \pi\chi(G)$.

**Proof.** Always $w(G) \leq \pi\chi(G)$. Proposition \[1\] and Theorem \[1\] imply $\Delta(G) \leq \pi\chi(G)$. For compact Hausdorff spaces $w(G) = \Delta(G)$ \[14\] Corollary 7.6]. Hence $w(G) = \pi\chi(G)$.

**Corollary 1.** Let $G$ be a CHART group. Then the following conditions are equivalent.

1. $G$ is metrizable;
2. $G$ is first-countable;
3. $G$ is Fréchet;
4. $G$ has countable tightness;
5. $G$ has countable $\pi$-character.

**Proof.** Always (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Since for compact spaces $\pi\chi(G) \leq t(G)$ \[14\] Theorem 7.13], (4) $\Rightarrow$ (5) is true. Theorem \[2\] implies (5) $\Rightarrow$ (1). \qed
4. Sequentially compact CHART groups

Denote $I = [0, 1]$, 

$$s = \min\{\tau : I^\tau \text{ is not sequentially compact}\}.$$ 

The cardinal $s$ is called splitting number, $\omega < s \leq 2^{\omega}$ [16] [17].

**Proposition 2.** Let $X$ be a compact sequentially compact space. Then 

$$\pi\chi(x, X) < s$$ 

for some $x \in X$.

**Proof.** Let us assume the opposite. Then $\pi\chi(x, X) \geq s$ for all $x \in X$. Then [18] Theorem 1 implies that $X$ maps continuously onto $I^s$. Since sequential compactness is preserved by continuous mappings, $I^s$ is a sequentially compact space. Contradiction.

If $G$ is a compact Hausdorff space and $w(G) = \tau < s$ then $G$ can be embedded in $I^\tau$ and $G$ is sequentially compact. In a homogeneous space $G$, if at some point the $\pi$-character is equal to $\tau$, then the $\pi$-character of the whole space is equal to $\tau$. Therefore, the Proposition 2 and Theorem 2 imply the following proposition.

**Theorem 3.** Let $G$ be a CHART group. Then $G$ is sequentially compact if and only if $w(G) < s$. 

Assuming the continuum hypothesis (CH), $\omega < s \leq 2^{\omega} = \omega_1$, that is, $s = \omega_1$. Theorem 3 implies the following proposition.

**Corollary 2.** (CH) Let $G$ be a sequentially compact CHART group. Then $G$ is metrizable.

5. Martin’s axiom and continuity of operations in CHART groups

Recall the topological characterization of the statement $MA(\tau)$:

$MA(\tau)$ if $X$ is a compact Hausdorff topological space that satisfies the ccc then $X$ is not the union of $\tau$ or fewer nowhere dense subsets.

**Martin’s axiom (MA):** For every $\tau < 2^{\omega}$, $MA(\tau)$ holds.

A topological space $(X, \mathcal{T})$ is called $\Delta_s$-nonmeager space [19] [20] if for any mapping $\Omega : X \to \mathcal{T}$ such that $x \in \Omega(x)$ for $x \in X$, there exists a nonempty $W \in \mathcal{T}$, such that 

$$W \subset \{x \in W : W \subset \Omega(x)\}. \quad (1)$$

**Proposition 3.** $MA(\tau)$. Let $X$ be a ccc compact space and $w(G) \leq \tau$. Then $X$ is $\Delta_s$-nonmeager.
Proof. Let $\mathcal{T}$ be the topology of $X$, $\{U_\alpha : \alpha < \tau\} \subset \mathcal{T}$ is the base of $X$ and $\Omega : X \to \mathcal{T}$ is a mapping such that $x \in \Omega(x)$ for $x \in X$. For $\alpha < \tau$, put $M_\alpha = \{x \in U_\alpha : U_\alpha \subset \Omega(x)\}$. Then $G = \bigcup_{\alpha < \tau} M_\alpha$. $MA(\tau)$ implies that $M_\alpha$ has nonempty interior for some $\alpha < \tau$. Take a nonempty open $W \subset U_\alpha$ such that $M_\alpha \cap W$ dense in $W$. Then (1) holds. □

Proposition 1 and [9, Theorem 13 and Theorem 17(2)] imply the following assertion.

**Theorem 4.** If $G$ is a $\Delta_\tau$-nonmeager CHART group then $G$ is topological group.

**Theorem 5.** $MA(\tau)$. Let $G$ be a CHART group. If $w(G) \leq \tau$ then $G$ is a topological group.

Proof. CHART groups have a right-invariant Haar measure [20] [21]. Hence $G$ is a ccc space. It follows from Proposition 3 that $G$ is $\Delta_\tau$-nonmeager. Theorem 4 implies that $G$ is a topological group. □

**Corollary 3** (Corollary 3 [9]). (MA) Let $G$ be a CHART group. If $w(G) < 2^\omega$ then $G$ is a topological group.

Since $MA(\omega)$ is true in ZFC, Theorem 5 implies the following assertion follows.

**Corollary 4** (Theorem 2.1 [9]). Let $G$ be a metrizable CHART group. Then $G$ is a topological group.

From Theorem 3 and 5 the following assertion follows.

**Corollary 5.** Suppose that $MA(\tau)$ is satisfied for each $\tau < s$. Let $G$ be a sequentially compact CHART group. Then $G$ is a topological group.

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