Quantizing String Theory in $AdS_5 \times S^5$: Beyond the pp-Wave

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Abstract

In a certain kinematic limit, where the effects of spacetime curvature (and other background fields) greatly simplify, the light-cone gauge world-sheet action for a type IIB superstring on $AdS_5 \times S^5$ reduces to that of a free field theory. It has been conjectured by Berenstein, Maldacena, and Nastase that the energy spectrum of this string theory matches the dimensions of operators in the appropriately defined large $R$-charge large-$N_c$ sector of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in four dimensions. This holographic equivalence is thought to be exact, independent of any simplifying kinematic limits. As a step toward verifying this larger conjecture, we have computed the complete set of first curvature corrections to the spectrum of light-cone gauge string theory that arises in the expansion of $AdS_5 \times S^5$ about the plane-wave limit. The resulting spectrum has the complete dependence on $\lambda = g_{YM}^2 N_c$; corresponding results in the gauge theory are known only to second order in $\lambda$. We find precise agreement to this order, including the $\mathcal{N} = 4$ extended supermultiplet structure. In the process, we demonstrate that the complicated schemes put forward in recent years for defining the Green–Schwarz superstring action in background Ramond-Ramond fields can be reduced to a practical (and correct) method for quantizing the string.

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1 Introduction

A dramatic prediction of the AdS/CFT correspondence [1] is that the excited state energies of the first-quantized type IIB superstring in the $AdS_5 \times S^5$ geometry should match the dimensions of certain operators in the strong coupling limit of $\mathcal{N} = 4$ super Yang–Mills field theory in four dimensions [2]. The obstacles to verifying this conjecture are, on one hand, the difficulty of quantizing superstring theory in the presence of Ramond-Ramond (RR) fields (an essential feature of the $AdS_5 \times S^5$ background) and, on the other, the need to calculate dimensions of non-BPS operators in strongly coupled gauge theory. Substantial progress in both regards has been made recently.

The first step was the realization that the Green–Schwarz (GS) superstring, evaluated in the light-cone gauge, becomes a free (albeit massive) worldsheet theory if the true $AdS_5 \times S^5$ background is replaced by a Penrose limit describing the near neighborhood of an equatorial lightlike geodesic on the $S^5$ subspace [3, 4]. The energy spectrum of this free theory is simply that of a string moving around the equator of the $S^5$ and boosted to large angular momentum $J$. By the AdS/CFT correspondence, these string energies should match the dimensions of operators with large $R$-charge ($R \sim J$) in strongly coupled four-dimensional $\mathcal{N} = 4$ super Yang–Mills theory. This general expectation was precisely realized by Berenstein, Maldacena, and Nastase (BMN) [5] who identified the subspace of gauge theory operators corresponding to specific free string excited states (i.e., states with different numbers and types of string oscillator modes applied to the string ground state) and showed that perturbative calculations of the dimensions of these operators are reliable in the large $R$-charge limit and gave evidence that they agree with the string theory predictions. This work demonstrated an equivalence between energies of string excited states and gauge theory operator dimensions in a kinematic limit where the difficulties of quantizing the string and computing operator dimensions are neatly circumvented.

This equivalence should not be restricted to the Penrose limit of the geometry (or the large $R$-charge limit of operator dimensions). It is not easy to verify this stronger prediction, primarily because the superstring propagating in the general $AdS_5 \times S^5$ geometry is governed by a complicated interacting worldsheet theory. However, these interactions vanish in the limit of large $S^5$ angular momentum $J$ and it should be possible to develop a perturbative expansion (in inverse powers of $J$) of the string energy spectrum. Corrections to the string spectrum should subsequently be compared with an expansion in inverse powers of $R$-charge of the dimensions of BMN-type gauge theory operators. For various reasons, a lot of attention has been paid to the problem of calculating these operator dimensions, and there is an extensive literature ranging over many topics: finding the proper limit to take to see the correspondence [6], careful calculations of operator dimensions at one loop [7], making explicit the extended supersymmetry structure of the problem [8], and, more recently, calculations of higher-loop anomalous dimensions [9] (and this is by no means an all-inclusive list of relevant papers!). The problem of developing the perturbation theory of worldsheet string dynamics in the $AdS_5 \times S^5$ background has, however, received much less attention. There has been one study along these lines of the bosonic string [10] with promising results.
which do not, however, address the crucial supersymmetry issues. The perturbative analysis of the full GS superstring in the $AdS_5 \times S^5$ background remains to be done, and that is the subject of this paper. We present here a condensed summary of our findings, relegating the many cumbersome technical details to a longer paper \[11\].

In brief, we find that string energies organize themselves into supermultiplets that match the dimensions of BMN operators in $\mathcal{N} = 4$ super Yang–Mills theory, thus verifying the AdS/CFT correspondence in a new and challenging context. To achieve this, we have had to perform a completely explicit quantization of the interacting superstring in a RR background. A side benefit of our results is therefore a verification of the practical utility (and correctness) of the rather complicated nonlinear action that has been proposed for the fermionic degrees of freedom of the type IIB GS superstring \[12\] in the $AdS_5 \times S^5$ background \[13\] \[14\].

2 Setup, Notation, Recap of BMN

We begin with a brief review of essential results from recent literature on the AdS/CFT correspondence in the Penrose limit. In convenient global coordinates, the $AdS_5 \times S^5$ metric can be written in the form

$$ds^2 = \hat{R}^2(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_5^2 + \cos^2 \theta \, d\phi^2 + d\theta^2 + \sin^2 \theta \, d\tilde{\Omega}_3^2), \quad (1)$$

where $\hat{R}$ denotes the radius of both the sphere and the AdS space. (The hat is introduced because we reserve the symbol $R$ for $R$-charge.) The coordinate $\phi$ is periodic with period $2\pi$ and, strictly speaking, the time coordinate $t$ exhibits the same periodicity. In order to accommodate string dynamics, it is necessary to pass to the covering space in which time is not taken to be periodic. This geometry is accompanied by an RR field with $N_c$ units of flux on the $S^5$. It is a consistent, maximally supersymmetric type IIB superstring background provided that

$$\hat{R}^4 = g_s N_c (\alpha')^2, \quad (2)$$

where $g_s$ is the string coupling. The AdS/CFT correspondence asserts that this string theory is equivalent to $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions with an $SU(N_c)$ gauge group and coupling constant $g_{YM}^2 = g_s$. To simplify both sides of the correspondence, we study the duality in the simultaneous limits $g_s \to 0$ (the classical limit of the string theory) and $N_c \to \infty$ (the planar diagram limit of the gauge theory) with the 't Hooft coupling $g_{YM}^2 N_c$ held fixed. The holographically dual gauge theory is defined on the conformal boundary of $AdS_5 \times S^5$, which, in this case, is $R \times S^3$. Duality demands that operator dimensions in the conformally invariant gauge theory be equal to the energies of corresponding states of the ‘first-quantized’ string propagating in the $AdS_5 \times S^5$ background. This conjecture is motivated by the fact that both theories are invariant under the same supergroup $PSU(2,2|4)$, but a vast amount of more specific evidence in support of the AdS/CFT correspondence has been accumulated.

As explained above, the quantization problem is simplified by boosting the string to lightlike momentum along some direction or, equivalently, by quantizing the string in the
background obtained by taking a Penrose limit of the original geometry using the lightlike geodesic corresponding to the boosted trajectory. The simplest choice is to boost along an equator of the \(S^5\) or, equivalently, to take a Penrose limit with respect to the lightlike geodesic \(\phi = t, \rho = \theta = 0\). To perform light-cone quantization about this geodesic, it is helpful to make the reparametrizations

\[
\cosh \rho = \frac{1 + z^2/4}{1 - z^2/4} \quad \cos \theta = \frac{1 - y^2/4}{1 + y^2/4},
\]

and work with the metric

\[
ds^2 = \hat{R}^2 \left[ -\left(\frac{1 + \frac{1}{4}z^2}{1 - \frac{1}{4}z^2}\right)^2 dt^2 + \left(\frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2}\right)^2 d\phi^2 + \frac{dz_k dz_k}{(1 - \frac{1}{4}z^2)^2} + \frac{dy_k' dy_k'}{(1 + \frac{1}{4}y^2)^2} \right],
\]

where \(y^2 = y_k' y_k'\) with \(k' = 5, \ldots, 8\) and \(z^2 = z_k z^k\) with \(k = 1, \ldots, 4\) define eight ‘Cartesian’ coordinates transverse to the geodesic. This form of the metric is well-suited to the present calculation; the spin connection, which will be important for the superstring action, turns out to have a simple functional form, and the \(AdS_5\) and \(S^5\) subspaces appear nearly symmetrically. This metric is invariant under the full \(SO(4, 2) \times SO(6)\) symmetry, but only translation invariance in \(t\) and \(\phi\) and the \(SO(4) \times SO(4)\) symmetry of the transverse coordinates remain manifest in this form. The translation symmetries mean that string states have a conserved energy \(\omega\), conjugate to \(t\), and a conserved (integer) angular momentum \(J\), conjugate to \(\phi\). Boosting along the equatorial geodesic is equivalent to studying states with large \(J\) and the lightcone Hamiltonian will give the (finite) allowed values for \(\omega - J\) in that limit.

On the gauge theory side, the \(S^5\) geometry is replaced by an \(SO(6)\) \(R\)-symmetry group, and \(J\) corresponds to the eigenvalue of an \(SO(2)\) \(R\)-symmetry generator. The AdS/CFT correspondence implies that string energies in the large-\(J\) limit should match operator dimensions in the limit of large \(R\)-charge.

On dimensional grounds, taking the \(J \to \infty\) limit on string states is equivalent to taking the \(\hat{R} \to \infty\) limit of the geometry (in properly chosen coordinates). The coordinate redefinitions

\[
t \to x^+ \quad \phi \to x^+ + \frac{x^-}{\hat{R}^2} \quad z_k \to \frac{z_k}{\hat{R}} \quad y_k' \to \frac{y_k'}{\hat{R}}
\]

make it possible to take a smooth \(\hat{R} \to \infty\) limit. (The lightcone coordinates \(x^\pm\) are a bit unusual, but have been chosen for future convenience in quantizing the worldsheet Hamiltonian). Expressing the metric (4) in these new coordinates, we obtain the following expansion in powers of \(1/\hat{R}^2\):

\[
ds^2 \approx 2 dx^+ dx^- + dz^2 + dy^2 - (z^2 + y^2)(dx^+)^2 + \frac{1}{\hat{R}^2} \left[ -2y^2 dx^- dx^+ + \frac{1}{2} (y^4 - z^4) (dx^+)^2 + (dx^-)^2 + \frac{1}{2} z^2 dz^2 - \frac{1}{2} y^2 dy^2 \right] + O(1/\hat{R}^4).
\]
The leading $\hat{R}$-independent part is the Penrose limit, or pp-wave geometry: it describes the geometry seen by the infinitely boosted string. For future reference, we define this limiting metric as

$$ds_{pp}^2 = 2dx^+dx^- + dz^2 + dy^2 - (z^2 + y^2)\,(dx^+)^2. \quad (7)$$

The $x^+$ coordinate is dimensionless, $x^-$ has dimensions of length squared, and the transverse coordinates now have dimensions of length.

In light-cone gauge quantization of the string dynamics, one identifies world-sheet time $\tau$ with the $x^+$ coordinate, so that the world-sheet Hamiltonian corresponds to the conjugate space-time momentum $P_+ = \omega - J$. Additionally, one sets the world-sheet momentum density $p_- = 1$ so that the other conserved quantity carried by the string, $P_- = J/\hat{R}^2$, is encoded in the length of the $\sigma$ interval. Once $x^\pm$ are eliminated, the quadratic dependence of $ds_{pp}^2$ on the remaining eight transverse bosonic coordinates leads to a quadratic (and hence soluble) bosonic lightcone Hamiltonian $P_+$. Things are less simple when $1/\hat{R}^2$ corrections to the metric are taken into account: they add quartic interactions to the lightcone Hamiltonian and lead to non-trivial shifts in the spectrum of the string. This phenomenon, generalized to the superstring, will be the primary subject of the rest of the paper.

While it is clear how the Penrose limit can bring the bosonic dynamics of the string under perturbative control, the RR field strength survives this limit and causes problems for quantizing the superstring. The GS action is the only practical approach to quantizing the superstring in RR backgrounds, and we must construct this action for the IIB superstring in the $AdS_5 \times S^5$ background \cite{13}, pass to lightcone gauge and then take the Penrose limit. The latter step reduces the otherwise extremely complicated action to a worldsheet theory of free, equally massive transverse bosons and fermions \cite{4}. For reference, we give a concise summary of the construction and properties of the lightcone Hamiltonian $H_{pp}^{GS}$ that describes the superstring in this limit. This will be a helpful preliminary to our principal goal of evaluating the corrections to the Penrose limit of the GS action.

Gauge fixing eliminates both lightcone coordinates $x^\pm$, leaving eight transverse coordinates $x^I$ as bosonic dynamical variables. Type IIB supergravity has two ten-dimensional supersymmetries that are described by two sixteen-component Majorana–Weyl spinors of the same ten-dimensional chirality. The GS superstring action contains just such a set of spinors (so that the desired spacetime supersymmetry comes out ‘naturally’). In the course of lightcone gauge fixing, half of these fermi fields are set to zero, leaving behind a complex eight-component worldsheet fermion $\psi$. This field is further subject to the condition that it transform in an $8_s$ representation under $SO(8)$ rotations of the transverse coordinates (while the bosons of course transform as an $8_v$). In a sixteen-component notation the restriction of the world-sheet fermions to the $8_s$ representation is implemented by the condition $\gamma^9 \psi = +\psi$ where $\gamma^9 = \gamma^1 \cdots \gamma^8$ and the $\gamma^A$ are eight real, symmetric gamma matrices satisfying a Clifford algebra $\{\gamma^A, \gamma^B\} = 2\delta^{AB}$. Another quantity, which proves to be important in what follows, is $\Pi \equiv \gamma^1\gamma^2\gamma^3\gamma^4$. One could also define $\Pi = \gamma^5\gamma^6\gamma^7\gamma^8$, but $\Pi \psi = \Pi \psi$ for an $8_s$ spinor.
In the Penrose limit, the lightcone GS superstring action takes the form
\[ S_{pp} = \frac{1}{2\pi\alpha'} \int d\tau \int_{0}^{2\pi\alpha' P_-} d\sigma (\mathcal{L}_B + \mathcal{L}_F) , \] where
\[ \mathcal{L}_B = \frac{1}{2} [(\dot{x}^A)^2 - (x'^A)^2 - (x^A)^2] , \quad \mathcal{L}_F = i\psi^\dagger \dot{\psi} + \psi^\dagger \Pi \psi + \frac{i}{2} (\psi \psi'^\dagger + \psi'^\dagger \psi) . \]

The fermion mass term $\psi^\dagger \Pi \psi$ arises from the coupling to the background RR 5-form field strength, and matches the bosonic mass term (as required by supersymmetry). It is important that the quantization procedure preserve supersymmetry. However, as is typical in lightcone quantization, some of the conserved generators are linearly realized on the $x^A$ and $\psi^\alpha$, and others have a more complicated non-linear realization.

The equation of motion of the transverse string coordinates is
\[ \ddot{x}^A - x''^A + x^A = 0 . \] The requirement that $x^A$ be periodic in the worldsheet coordinate $\sigma$ (with period $2\pi\alpha' P_-)$ leads to the mode expansion
\[ x^A(\sigma, \tau) = \sum_{n=-\infty}^{\infty} x_n^A(\tau)e^{-ik_n}\sigma , \quad k_n = \frac{n}{\alpha' P_-} = \frac{n\hat{R}^2}{\alpha' J} . \] The canonical momentum $p^A$ also has a mode expansion, related to that of $x^A$ by the free-field equation $p^A = \dot{x}^A$. The coefficient functions are most conveniently expressed in terms of harmonic oscillator raising and lowering operators:
\[ x_n^A(\tau) = \frac{i}{\sqrt{2\omega_n P_-}}(a_n^A e^{-i\omega_n \tau} - a_n^A | e^{i\omega_n \tau}) , \quad p_n^A(\tau) = \sqrt{2\omega_n P_-}(a_n^A e^{-i\omega_n \tau} + a_n^A | e^{i\omega_n \tau}) . \] The harmonic oscillator frequencies are determined by the equation of motion (10) to be
\[ \omega_n = \sqrt{1 + k_n^2} = \sqrt{1 + (n\hat{R}^2/\alpha' J)^2} = \sqrt{1 + (g_{YM}^2 N_c n^2/J^2)} , \] where the mode index $n$ runs from $-\infty$ to $+\infty$. (Because of the mass term, there is no separation into right-movers and left-movers). The canonical commutation relations are satisfied by imposing the usual creation and annihilation operator algebra:
\[ [a_m^A, a_n^B] = \delta_{mn}\delta^{AB} \Rightarrow [x^A(\sigma), p^B(\sigma')] = i2\pi\alpha' \delta(\sigma - \sigma')\delta^{AB} . \] The fermion equation of motion is
\[ i(\dot{\psi} + \psi'^\dagger) + \Pi \psi = 0 . \]
The expansion of $\psi$ in terms of creation and annihilation operators is achieved by expanding the field in worldsheet momentum eigenstates

$$\psi(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \psi_n(\tau)e^{-ik_n\sigma},$$

which are further expanded in terms of convenient positive and negative frequency solutions of the fermion equation of motion:

$$\psi_n(\tau) = \frac{1}{\sqrt{4P_n\omega_n}}(e^{-i\omega_n\tau}(\Pi + \omega_n - k_n)b_n + e^{i\omega_n\tau}(1 - (\omega_n - k_n)\Pi)b_n^\dagger).$$

The frequencies and momenta in this expansion are equivalent to those of the bosonic coordinates. In order to reproduce the anticommutation relations

$$\{\psi(\tau, \sigma), \psi^\dagger(\tau, \sigma')\} = 2\pi\alpha'\delta(\sigma - \sigma'),$$

we impose the standard oscillator algebra

$$\{b_\alpha^\dagger b_\beta^\dagger n, b_\gamma n\} = \frac{1}{2}(1 + \gamma_9)^{\alpha\beta}\delta_{m,n}.$$  

The spinor fields $\psi$ carry sixteen components, but the $8_s$ projection reduces this to eight anticommuting oscillators, exactly matching the eight transverse oscillators in the bosonic sector. The final expression for the light-cone Hamiltonian is

$$H_{pp}^{GS} = \sum_{n=-\infty}^{+\infty} \omega_n \left( \sum_A (a_n^A)^\dagger a_n^A + \sum_\alpha (b_n^\alpha)^\dagger b_n^\alpha \right).$$

The harmonic oscillator zero-point energies nicely cancel between bosons and fermions for each mode $n$. The frequencies $\omega_n$ depend on the single parameter

$$\omega_n = \sqrt{1 + \lambda'n^2},$$

so that one can take $J$ and $g_{YM}^2 N_c$ to be simultaneously large while keeping $\lambda'$ fixed. If $\lambda'$ is kept fixed and small, $\omega_n$ may be expanded in powers of $\lambda'$, suggesting that contact with perturbative Yang–Mills gauge theory is possible.

The spectrum is generated by $8 + 8$ transverse oscillators acting on ground states labeled by an $SO(2)$ angular momentum taking integer values $-\infty < J < \infty$ (note that the oscillators themselves carry zero $SO(2)$ charge). Any combination of oscillators may be applied to a ground state, subject to the constraint that the sum of the oscillator mode numbers must vanish (this is the level-matching constraint, the only constraint not eliminated by lightcone gauge-fixing). The energies of these states are the sum of the individual oscillator energies (13), and the spectrum is very degenerate.\(^1\) For example, the 256 states of the form

\(^1\)Note that the $n = 0$ oscillators raise and lower the string energy by a protected amount $\delta P_+ = 1$, independent of the variable parameters. These oscillators play a special role, enlarging the degeneracy of the string states in a crucial way, and we will call them ‘zero-modes’ for short.
for a given mode number $n$ (where $A^\dagger$ and $B^\dagger$ each can be any of the 8+8 bosonic and fermionic oscillators) all have the energy

$$P_+ = \omega - J = 2\sqrt{1 + (g_{YM}^2 N_c n^2 / J^2)} \sim 2 + (g_{YM}^2 N_c n^2 / J^2) + \ldots. \quad (22)$$

In the weak coupling limit ($\lambda' \to 0$) the degeneracy is even larger because the dependence on the oscillator mode number $n$ goes away! This actually makes sense from the dual gauge theory point of view where $P_+ \to D - R$ ($D$ is the dimension and $R$ is the $R$-charge carried by gauge-invariant operators of large $R$); at zero coupling, operators have integer dimensions and the number of operators with $D - R = 2$, for example, grows with $R$, providing a basis on which string multiplicities are reproduced. Even more remarkably, BMN were able to show [5] that subleading terms in a $\lambda'$ expansion of the string energies match the first perturbative corrections to the gauge theory operator dimensions in the large $R$-charge limit. We will review the details of this agreement in the next section.

More generally, we expect exact string energies in the $AdS_5 \times S^5$ background to have a joint expansion in the parameters $\lambda'$, defined above, and $1/J$. We also expect the degeneracies found in the $J \to \infty$ limit (for fixed $\lambda'$) to be lifted by interaction terms that arise in the worldsheet Hamiltonian describing string physics at large but finite $J$. Large degeneracies must nevertheless remain in order for the spectrum to be consistent with the $PSU(2,2|4)$ global supergroup that should characterize the exact string dynamics. The specific pattern of degeneracies should also match that of operator dimensions in the $\mathcal{N} = 4$ super Yang-Mills theory. Since the dimensions must be organized by the $PSU(2,2|4)$ superconformal symmetry of the gauge theory, consistency is at least possible, if not guaranteed. In the rest of this paper we will explore this question. We will first summarize the information about gauge theory operator anomalous dimensions that is needed to test the predictions of this duality. We will then describe our evaluation of the interaction terms that must be added to $H_{GS}$ to accommodate corrections to superstring worldsheet physics in the large-$J$ limit. Finally, we will report the results of a first-order degenerate perturbation theory treatment of these corrections to the string worldsheet Hamiltonian, and show that they precisely match the relevant gauge theory expectations.

### 3 Gauge Theory, Group Theory, Dimension Expansion

Before tackling the calculation of the energy spectrum of the string, we first present some information on dimensions of gauge theory operators that will be needed to test the predictions of this duality. Most of what we will say in this section can be found in the literature in one form or another, especially in work by Beisert [8]. However, in order to organize things in the most suitable way for our subsequent comparison with string theory results (and to make a few points that don’t seem to have been made elsewhere), we found it useful, at least for ourselves, to rederive and restate mostly known results. We focus on the noninteracting string ($g_s \to 0$) which, as explained earlier, is dual to the gauge theory in the large-$N_c$ limit (the Yang-Mills genus-counting parameter is $g_2 = J^2 / N_c$ [3]). In the large-$N_c$ limit, the
operators of interest are single-gauge-trace monomials of fields of $\mathcal{N} = 4$ SUSY Yang–Mills theory.

In the string theory, we look at states of large $J$ but finite $P_+ = \omega - J$.

In the gauge theory we classify operators by $R$-charge (in some $SO(2)$ subgroup of the $SU(4)$ $R$-symmetry group) and dimension $D$. To match the kinematic limit of the string states, we look for operators with dimension $D$ and $R$ both large but with $\Delta = D - R$ finite. The dimension $D$ approaches the naive engineering dimension, which we will denote by $K$, in the limit $\lambda' \to 0$. Thus, the limit of interest is $K, R \to \infty$ with the integer difference $\Delta_0 = K - R$ held fixed. Thus $\Delta$ will be the sum of the integer $\Delta_0$ and the anomalous dimension $D - K$ of the operator (which we assume to have a finite limit). Our problem is to identify an appropriate basis of gauge operator monomials which are mixed by the dimension operator, calculate the anomalous dimension matrix to some order in perturbation theory, and then diagonalize that matrix to get the allowed values of $\Delta$. The question is whether they match the string theory spectrum of $P_+$. The component fields available to us in $\mathcal{N} = 4$ SYM are a gauge field, a set of gluinos and a set of scalars, all in the adjoint of the gauge group. The theory has an exact global $SU(4)$ $R$-symmetry, under which the gluinos transform as a $4$ and $\tilde{4}$ and the scalars as a $6$. Since the dimension matrix commutes with the full $R$-symmetry group, it is helpful to classify operators according to their $SU(4)$ representation. Irreducible tensor representations of $SU(4)$ are indexed by Young diagrams describing their symmetries under permutations of the tensor indices. Such diagrams contain up to three rows of boxes with non-increasing numbers of boxes per row and are denoted by a set of three integers $(n_1, n_2, n_3)$ giving the differences in length of successive rows. The total number of boxes in the diagram is the total number of $SU(4)$ indices in the tensor. The boxes are filled in with tensor indices in some canonical order and the representations are antisymmetric under the exchange of any pair of indices in the same column. More specifically, the scalars are in the 6-dimensional $\mathbf{(0, 1, 0)}$ representation of $SU(4)$, the gluinos are 2-component Weyl spacetime spinors in the 4-dimensional fundamental $\mathbf{(1, 0, 0)}$ plus an adjoint field in the 4-dimensional anti-fundamental $\mathbf{(0, 0, 1)}$:

\begin{align*}
\text{Scalars : } & \phi \quad \text{Gluinos : } \chi^a \quad \chi_{\dot{a}} \quad \tilde{\chi}_{\dot{a}}.
\end{align*}

The $a$ (resp. $\dot{a}$) indices on the gluinos indicate that they transform in the $\mathbf{(2, 1)}$ (resp. $\mathbf{(1, 2)}$) representations of the $SL(2, C)$ covering group of the spacetime Lorentz group. All of these fields, as well as the $SU(4)$-singlet gauge field, are adjoint matrices in the gauge group algebra. The Young diagram superscript is a shorthand for indicating the $SU(4)$ tensor character of the fields (viz. $\phi$ is a rank-two antisymmetric tensor, $\chi^a$ is a rank-one tensor and so on).

We want to use these fields to construct gauge-singlet composite operators. As mentioned before, we work in the leading large-$N_c$ limit and need only consider monomials involving a single gauge trace. For the moment, we limit our attention to operators that are spacetime scalars. The $SO(2)$ scalar $R$-charge that will eventually be taken to infinity (to match the

\footnote{Multiple trace operators appear when we go beyond the large-$N_c$ limit; they mix with single-trace operators when non-planar diagrams are included.}
$J \to \infty$ limit of the string spectrum) is defined by the decomposition $SU(4) \supset SU(2) \times SU(2) \times U(1)_R$ (the same thing as $SO(6) \supset SO(4) \times SO(2)$). The scalar $R$-charge of the various components of the gauge theory fields is assessed by distributing indices in the boxes of the Young diagram superscripts, subject to the rule of column antisymmetry and assigning $R = \frac{1}{2}(-\frac{1}{2})$ to $SU(4)$ indices $(1, 2) (3, 4)$ respectively. The result is as follows:

\[ R = 1 : \phi^A(Z), \quad R = 0 : \phi^A, \phi^1, \phi^2, (\phi^A), \quad R = -1 : \phi^A(\bar{Z}), \]

\[ R = 1/2 : \chi^\alpha, \chi^\beta, \bar{\chi}^\gamma, \bar{\chi}^\delta, \quad R = -1/2 : \chi^\alpha, \chi^\beta, \bar{\chi}^\gamma, \bar{\chi}^\delta. \tag{24} \]

We have introduced an alternate notation for the scalars (to be used later) $(Z, \bar{Z}, \phi^A, A = 1, \ldots, 4)$ that emphasizes their $SO(4)$ content. As discussed earlier, we need a basis of operators with large naive dimension $K$, large scalar $R$-charge and fixed $\Delta_0 = K - R$. BMN showed that, in this limit, such operators correspond to string states created by a fixed finite number $(\Delta_0)$ of string oscillators acting on the pp-wave ground state of angular momentum $R$. Operators with $\Delta_0 = 0$ are BPS, and their dimensions are protected by supersymmetry. In what follows, we will, for simplicity, restrict the discussion to $\Delta_0 = 2$ operators, corresponding to string states created by two oscillators acting on the vacuum (so-called ‘two-impurity’ states). The list of all single-trace spacetime scalar operators of naive dimension $K$ which can have $\Delta_0 \leq 2$ is as follows:

\[ \text{tr} \left( (\phi^A)^K \right), \quad (R_{\text{max}} = K) \]

\[ \text{tr} \left( (\chi^\alpha \sigma_2 \phi^A)^K \right), \quad \text{tr} \left( (\chi^\alpha \phi^A \sigma_2 \chi^\beta)^K \right), \ldots \quad (R_{\text{max}} = K - 2) \]

\[ \text{tr} \left( (\bar{\chi}^\alpha \sigma_2 \bar{\chi}^\beta)^K \right), \quad \text{tr} \left( (\bar{\chi}^\alpha \phi^A \sigma_2 \bar{\chi}^\beta)^K \right), \ldots \quad (R_{\text{max}} = K - 2) \]

\[ \text{tr} \left( \nabla^\mu \phi^A \nabla_\mu \phi^A \right)^K, \quad (R_{\text{max}} = K - 2). \tag{25} \]

The fields inside the operators are $SU(N_c)$ adjoint matrices and the trace is taken over gauge indices; spacetime spinor indices on the $\chi$ are contracted to produce a spacetime scalar (note that a product of a $\chi^\alpha$ and a $\chi^\beta$ cannot make a scalar because they transform under inequivalent irreps of spacetime $SL(2, C)$); $\nabla$ is the spacetime gauge-covariant derivative. There are multiple versions of operators involving gluinos and spacetime derivatives arising from the different ways that scalars may be distributed among them (and the cyclic symmetry of the gauge trace reduces the number of independent operators one can construct). These operators provide a basis for a reducible representation of the global $SU(4)$ $R$-symmetry group. Since the anomalous dimension operator commutes with this $SU(4)$, it will have no matrix elements between different $SU(4)$ irreps, and our first task is to find linear combinations of the above operators that provide a basis for these irreps (and find the multiplicities of inequivalent occurrences of the same irrep). The group theory analysis helps us obtain precise control of the subleading corrections in $1/K$ to the structure of the operators and their anomalous dimensions.
For the bosonic operators with no derivatives, we have a reducible $SU(4)$ tensor of rank $2K$ which we must decompose into irreducible $SU(4)$ tensors of rank $2K$. These irreps are symbolized by Young diagrams with $2K$ boxes; the main problem is to determine the multiplicity with which each such diagram appears. The standard algorithm for projecting a reducible character onto irreducible characters [15] cannot be implemented because of the cyclic symmetry of single-trace monomials. The algorithm, however, can be adapted with some effort to the case at hand to compute the desired multiplicities. Although the total number of irreducible tensors in the expansion grows rapidly with $K$, only a few can have $\Delta_0 = K - R = 0, 2$ and we report only the multiplicities of that limited set of irreps. The results are slightly different for odd and even $K$, but we will eventually see that this even/odd difference is harmless. For $K$ odd we have

\[
\text{tr} (\phi^K) \rightarrow 1 \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + \left( \frac{K - 1}{2} \right) \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + \left( \frac{K - 1}{2} \right) \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + \ldots,
\]

while for $K$ even we have

\[
\text{tr} (\phi^K) \rightarrow 1 \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + \left( \frac{K - 2}{2} \right) \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + \left( \frac{K - 2}{2} \right) \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + \ldots
\]

These irrep expansions could equally well have been done using the bosonic R-symmetry group $SO(6)$: this is what is done, with the same results, in [8]. The irreps with larger minimal values of $\Delta_0 = K - R$ have multiplicities that grow as higher powers of $K$. This is very significant for the eventual string theory interpretation of the anomalous dimensions, but we will not expand on this point here.

Other spacetime scalar operators that can have $\Delta_0 = K - R = 2$ are the ‘bifermions’, or products of two gluinos and $K - 3$ scalars. Including only the irreps that can actually have $\Delta_0 = 2$, their expansions are as follows:

\[
\text{tr} \left( \chi^\square \sigma_2 \chi^\square (\phi)^{K-3} \right) \rightarrow 1 \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + 1 \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + \ldots
\]

\[
\text{tr} \left( \chi^\square \sigma_2 \chi^\square (\phi)^{K-3} \right) \rightarrow 1 \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + 1 \times \begin{array}{c}
\vspace{0.5cm}
\end{array} + \ldots
\]

\(^3\)We report here only the pertinent results and leave the exposition of the group theory particulars to a longer publication.
There are identical expansions for operators arising from different placements of the fermions with respect to the bosons. Because of cyclicity of the gauge trace and the fermi statistics of the gluino fields, these operators are not all independent. The counting of independent operators depends, once again, on whether $K$ is even or odd. Using an obvious shorthand notation, the multiplicities of bifermion irreps are as follows for $K$ odd:

$$\text{tr} \left( \chi \sigma_2 \chi (\phi \bar{B})^{K-3} \right) \rightarrow \left( \frac{K-3}{2} \right) \times \begin{array}{c}
\hline
\hline
\hline
\end{array} \oplus \left( \frac{K-1}{2} \right) \times \begin{array}{c}
\hline
\hline
\hline
\end{array} \oplus \ldots \quad (30)$$

$$\text{tr} \left( \chi \sigma_2 \chi (\phi \bar{B})^{K-3} \right) \rightarrow \left( \frac{K-3}{2} \right) \times \begin{array}{c}
\hline
\hline
\hline
\end{array} \oplus \left( \frac{K-1}{2} \right) \times \begin{array}{c}
\hline
\hline
\hline
\end{array} \oplus \ldots \quad (31)$$

The results for $K$ even are, once again, slightly different:

$$\text{tr} \left( \chi \sigma_2 \chi (\phi \bar{B})^{K-3} \right) \rightarrow \left( \frac{K-2}{2} \right) \times \begin{array}{c}
\hline
\hline
\hline
\end{array} \oplus \left( \frac{K-2}{2} \right) \times \begin{array}{c}
\hline
\hline
\hline
\end{array} \oplus \ldots \quad (32)$$

$$\text{tr} \left( \chi \sigma_2 \chi (\phi \bar{B})^{K-3} \right) \rightarrow \left( \frac{K-2}{2} \right) \times \begin{array}{c}
\hline
\hline
\hline
\end{array} \oplus \left( \frac{K-2}{2} \right) \times \begin{array}{c}
\hline
\hline
\hline
\end{array} \oplus \ldots \quad (33)$$

The point of all this is that the dimension operator can only have matrix elements between operators belonging to the same $SU(4)$ irrep. There is a unique irrep, $(0, K, 0)$ (i.e., two rows of $K$ boxes), which contains ‘top’ states with dimension equal to $R$-charge (or $\Delta_0 = \Delta = 0$). The latter are known to be BPS states and get no correction to their dimension. Thus the dimension of the whole irrep, including all its components with $\Delta_0 > 0$, is unmodified by interactions. The other irreps displayed above have multiplicities that grow roughly as $K/2$ for large $K$. The irreps we have not displayed have higher values of $\Delta_0$ and multiplicities that grow as higher powers of $K$. The dimension operator will, in general, have matrix elements between all the operators belonging to a given representation. We therefore have to diagonalize a matrix of size roughly $K/2 \times K/2$ and will find $O(K/2)$ eigenvalues. The key question will then be the evolution of the spectrum as $K \to \infty$. From the work of BMN, we expect to find a spectrum that can be interpreted, at large $K = R + 2$ and fixed $\Delta_0 = K - R = 2$, as due to the action of two string modes on a string ground state of angular momentum $J = R$. Our goal is to evaluate and compare the $1/R$ corrections on both sides of this correspondence. One benefit of the group theory analysis is immediately apparent: the irrep $(2, K-4, 2)$ appears only in the reduction of the purely bosonic operator. For this irrep, the anomalous dimension matrix must act purely within the space of bosonic operators, a welcome simplification. By contrast, the irrep $(0, K-3, 2)$ appears both in the purely bosonic operators and in one of the two-fermion operators (with the same multiplicity in both cases). Thus, there can be matrix elements of the dimension operator between boson and fermion states and the diagonalization problem will be more complicated. In fact, the results of the diagonalization will test the fermionic structure of the string Hamiltonian, which makes this a particularly important test to carry out.
Having calculated the multiplicity of specific irreps, we turn to the perturbative diagonalization of the dimension operator. A simple approach begins with the two-point function between elements of the operator basis \( \{ O_a(x) \} \), calculated to first non-trivial order in perturbation theory. The typical result is

\[
\langle O_a(x)O_b(0) \rangle \sim (x)^{-2d_0}(\delta_{ab} + \ln(x^2)d_{1ab}^b),
\]

where \( d_0 \) is the naive dimension. The leading Kronecker \( \delta_{ab} \) implies that the operator basis is orthonormal in the free theory (in the large-\(N_c\) limit, this is enforced by multiplying the operator basis by a common overall normalization constant). The anomalous dimensions are then the eigenvalues of the mixing matrix \( d_{1ab}^b \), and the eigenoperators of definite dimension are linear combinations of basis operators defined by the eigenvectors. One should be careful to pick out conformal primary operators, but this subtlety is not too troublesome for one-loop perturbative calculations.

Group theory tells us that the dimension operator \( D \) block-diagonalizes under the different \( SU(4) \) irreps, and it is not too hard to show in concrete detail how it works in the purely bosonic sector. Consider a basis of \( K - 1 \) bosonic operator monomials of dimension \( K \) and \( \Delta_0 = K - R = 2 \):

\[
\{ O_{K,1}^{AB}, \ldots, O_{K,K-1}^{AB} \} = \{ \text{tr}(ABZ^{K-2}), \text{tr}(AZBZ^{K-3}), \ldots, \text{tr}(AZ^{K-3}BZ), \text{tr}(AZ^{K-2}B) \},
\]

where \( Z \) stands for \( \phi^A \) and has \( R = 1 \), while \( A, B \) stand for any of the four \( \phi^A \) \((A = 1, \ldots, 4)\) and have \( R = 0 \). The overall constant needed to orthonormalize this basis (in the large-\(N_c\) limit) is easy to compute, but not needed for present purposes. In the \( SO(2) \times SO(4) \) decomposition of \( SU(4) \), \( A, B \) are \( SO(4) \) vectors so that the operators of this basis are rank-two \( SO(4) \) tensors. We won’t give the detailed argument here, but it can be shown that the members of this basis can be assigned to \( SU(4) \) irreps by splitting them into irreducible rank-two \( SO(4) \) tensors. In particular, the symmetric traceless tensor belongs to the \((2, K-4, 2)\) irrep of \( SU(4) \), the antisymmetric tensor belongs to the pair \((0, K-3, 2) + (2, K-3, 0)\), and the \( SO(4) \) trace (when completed to a full \( SO(6) \) trace) belongs to the \((0, K-2, 0)\) irrep. In what follows, we refer to these three classes of operator as \( T_K^{(\pm)}, T_K^{(-)} \) and \( T_K^{(0)} \), respectively. If we take \( A \neq B \), the trace part drops out and the \( T_K^{(\pm)} \) operators are isolated by symmetrizing and antisymmetrizing on \( A, B \).

A simple extension of the BMN argument can be used to give the \( O(g_{YM}^2N_c) \) action of the anomalous dimension operator on the basis \( \{ 35 \} \), correct to all orders in \( 1/K \). In the leading large-\(N_c\) limit and leading order in \( g_{YM}^2 \), the gauge theory interaction term \( \text{tr}(\phi^a \phi^b)[\phi^a, \phi^b] \) has a very simple action on single-trace monomials in the \( \phi^a \): it produces a sum of interchanges of all nearest-neighbors in the trace. Diagrams that lead to exchanges at greater distances are non-planar and suppressed by powers of \( 1/N_c \). For the restricted case \( A \neq B \), the leading action of the anomalous dimension on the \( K - 1 \) bosonic monomials
of (35) has the following detailed structure:

\[
(ABZ^K) \rightarrow (BAZ^K) + 2(AZBZ^K) + (K - 3)(ABZ^K),
\]

\[
(AZBZ^K) \rightarrow 2(ABZ^K) + 2(AZBZ)^K + (K - 4)(AZBZ^K).
\]

\[
(ABZ^K) \rightarrow 2(AZBZ^K) + (K - 3)(BAZ^K) + (ABZ^K),
\]

\[
(36)
\]

(omitting the overall factor coming from the details of the Feynman diagram). The action on the trace parts when \(A = B\) is more complicated, and we will omit the detailed argument for that case. In an obvious matrix notation, we have

\[
\begin{pmatrix}
(K - 3) & 2 & 0 & \ldots & 1 \\
2 & K - 4 & 2 & \ldots & 0 \\
0 & \ldots & 2 & K - 4 & 2 \\
1 & \ldots & 0 & 2 & K - 3
\end{pmatrix}
\]

\[
(37)
\]

The logic of renormalization theory allows for a subtraction on the diagonal of this matrix, and in fact one is needed. The vector \(\vec{X}_0 = (1, \ldots, 1)\), corresponding to the operator in which all operators in (35) are summed over with equal weight, is an eigenvector with eigenvalue \(K\). This particular operator actually belongs to the special representation \((0, K, 0)\), whose anomalous dimensions must vanish because it contains the chiral primary BPS operator \(\text{tr}(Z^K)\) (whose dimension is equal to the \(R\)-charge). To properly normalize (37) and ensure that this eigenvector has eigenvalue zero, we subtract \(K\) times the unit matrix and drop the zero eigenvector of the anomalous dimension matrix on the grounds that it belongs to the ‘uninteresting’ \((0, K, 0)\) representation. The anomalous dimensions we seek are therefore the non-zero eigenvalues of the matrix

\[
\begin{pmatrix}
-3 & +2 & 0 & \ldots & 1 \\
+2 & -4 & +2 & \ldots & 0 \\
0 & \ldots & +2 & -4 & +2 \\
+1 & \ldots & 0 & +2 & -3
\end{pmatrix}
\]

\[
(38)
\]

This looks very much like the lattice Laplacian for a particle hopping from site to site on a periodic lattice. The special structure of the first and last rows assigns an extra energy to the particle when it hops past the origin. This breaks strict lattice translation invariance but makes sense as a picture of the dynamics involving two-impurity states: the impurities propagate freely when they are on different sites and have a contact interaction when they collide. This picture has lead people to map the problem of finding operator dimensions onto the technically much simpler one of finding the spectrum of an equivalent
quantum-mechanical Hamiltonian [16]. In one version, the map is to a spin-chain system with integrable dynamics [17], suggesting that exact results for many quantities of interest may be possible. This is an important topic, but we will not pursue it further in this paper.

Before diagonalizing (38), we note a useful symmetry of the problem: the operator monomials in the basis (35) go into each other pairwise under $A \leftrightarrow B$ and, at the same time, the vector $\vec{C} = (C_1, \ldots, C_{K-1})$ representing a linear combination of monomials transforms as $C_i \rightarrow C_{K-i}$. Since (38) is invariant under this transformation, its eigenvectors will be either even ($C_i = C_{K-i}$) or odd ($C_i = -C_{K-i}$) under it. Since the two options (even or odd under $A \leftrightarrow B$) correspond to different $SU(4)$ irreps, assessing the $SU(4)$ assignment of the different eigenvalues will be easy. The two classes of eigenvalues and normalized eigenvectors are as follows:

$$\lambda_n^{(K^+)} = 8 \sin^2 \left( \frac{n \pi}{K-1} \right) \quad n = 0, 1, 2, \ldots, n_{\text{max}} = \begin{cases} (K-3)/2 & K \text{ odd} \\ (K-2)/2 & K \text{ even} \end{cases},$$

$$C_{n,i}^{(K^+)} = \frac{2}{\sqrt{K-1}} \cos \left[ \frac{2 \pi n}{K-1} \left( i - \frac{1}{2} \right) \right] \quad i = 1, \ldots, K-1 , \quad (39)$$

$$\lambda_n^{(K^-)} = 8 \sin^2 \left( \frac{n \pi}{K} \right) \quad n = 1, 2, \ldots, n_{\text{max}} = \begin{cases} (K-1)/2 & K \text{ odd} \\ (K-2)/2 & K \text{ even} \end{cases},$$

$$C_{n,i}^{(K^-)} = \frac{2}{\sqrt{K}} \sin \left[ \frac{2 \pi n}{K} i \right] \quad i = 1, \ldots, K-1 . \quad (40)$$

For the case of $\lambda_n^{(K^+)}$, we indicate that $n = 0$ is a possible eigenvalue, but we must remember that it belongs to the $(0, K, 0)$ irrep when we count irrep multiplicities. The eigenoperators corresponding to the various dimensions are constructed from the eigenvectors according to

$$\mathcal{T}_{K,n}^{(\pm)}(x) = \sum_{i=1}^{K-1} C_{n,i}^{(K^{\pm})} O_{K,i}^{4AB}(x) . \quad (41)$$

The subscript $n$ will not be displayed in the following.

To get $\Delta = D - R$, we multiply these eigenvalues by the appropriate overall normalization factor and add the zeroth order value $\Delta_0 = 2$. The results for $\mathcal{T}_{K}^{(+)}}$ (symmetric traceless, belonging to the $(2, K-4, 2)$ irrep), $\mathcal{T}_{K}^{(-)}$ (antisymmetric, belonging to the $(0, K-3, 2) +$
(2, K − 3, 0) irreps) and \( T_K^{(0)} \) (trace, belonging to the (0, K − 2, 0) irrep) are

\[
\Delta(T_K^{(+)}) = 2 + \frac{g_{YM}^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{K-1} \right) \quad n = 1, 2, \ldots, n_{\text{max}} = \begin{cases} (K-3)/2 & K \text{ odd} \\ (K-2)/2 & K \text{ even} \end{cases}
\]

\[
\Delta(T_K^{(-)}) = 2 + \frac{g_{YM}^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{K} \right) \quad n = 1, 2, \ldots, n_{\text{max}} = \begin{cases} (K-1)/2 & K \text{ odd} \\ (K-2)/2 & K \text{ even} \end{cases}
\]

\[
\Delta(T_K^{(0)}) = 2 + \frac{g_{YM}^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{K+1} \right) \quad n = 1, 2, \ldots, n_{\text{max}} = \begin{cases} (K-1)/2 & K \text{ odd} \\ (K)/2 & K \text{ even} \end{cases}
\]

(42)

Note that the counting of eigenvalues corresponds exactly to the multiplicities of these irreps as reported in (26) and (27). The above results on dimensions and eigenoperators can all be found in [8] and, piecemeal, in earlier discussions of the one-loop operator dimension problem.

The expressions in (42) are the first terms in a perturbative expansion. Since we must work in the limit of large \( g_{YM}^2 N_c \), this expansion is not guaranteed to be reliable. The string theory discussion will show that the eigenvalue index \( n \) is to be interpreted as the mode number of an excited string oscillator. This implies a limiting procedure in which \( n \) is held fixed while \( R \) and \( g_{YM}^2 N_c \) are taken to infinity such that there are two controlled, small parameters, \( g_{YM}^2 N_c / R^2 \) and \( 1/R \). We will assume, as proposed by BMN, that the smallness of \( g_{YM}^2 N_c / R^2 \) makes perturbation theory reliable, at least for fixed-\( n \) eigenvalues (without this assumption, there is little one can calculate on the gauge theory side). At the same time, the smallness of \( 1/R \) controls the size of interaction corrections to the Penrose limit string worldsheet Hamiltonian. If we express the dimension formulae (42) in terms of \( R \)-charge \( R \), rather than naive dimension \( K \) (using \( K = R + 2 \)) and take the limit in this way, we find

\[
\Delta(T_{R+2}^{(+)}) \to 2 + \frac{g_{YM}^2 N_c}{R^2} n^2 \left( 1 - \frac{2}{R} + O(R^{-2}) \right),
\]

\[
\Delta(T_{R+2}^{(-)}) \to 2 + \frac{g_{YM}^2 N_c}{R^2} n^2 \left( 1 - \frac{4}{R} + O(R^{-2}) \right),
\]

\[
\Delta(T_{R+2}^{(0)}) \to 2 + \frac{g_{YM}^2 N_c}{R^2} n^2 \left( 1 - \frac{6}{R} + O(R^{-2}) \right).
\]

(43)

To leading order in \( 1/R \), the dimensions of these operator multiplets are degenerate and agree with the corresponding expression in the Penrose limit (22). The degeneracy is lifted at subleading order in \( 1/R \), just as the Penrose limit degeneracy of string worldsheet energies should be lifted by string worldsheet interactions. Our goal is show that the two approaches to the lifting of operator dimension (string energy) degeneracy give equivalent results on each side of the duality.

The AdS/CFT interpretation of the operator dimensions displayed in (43) is that they are dual to the energies of string states built out of two bosonic mode creation operators:
\((a_n^A)\dagger (a_{-n}^B)\dagger |R\rangle\). It is important to note that these anomalous dimensions are valid for all operators in the representations in question, not just those for which \(\Delta_0 = K - R = 2\); this is a simple consequence of the global \(SU(4)\) \(R\)-symmetry. We believe that this translates on the string theory side into the existence of exact zero-mode oscillators \(a_0^A\) which augment the \(P_+\) eigenvalue of a state by unity, independent of \(g_Y^2 N_c / J^2\) and \(1 / J\). This is true in the Penrose limit, as we can infer from (13), and we expect it to continue to be true to all orders in \(1 / J\). If so, the string states

\[(a_n^A)\dagger (a_{-n}^B)\dagger (a_0^{C_1})\dagger \ldots (a_0^{C_s})\dagger |J - 2 - s\rangle \tag{44}\]

should all have the same energy and correspond to the \(\Delta_0 > 2\) components of the \((2, J - 4, 2)\) irrep (if we project onto operators symmetric and traceless on \(A, B\), for example). This suggests that the interaction terms in the string worldsheet Hamiltonian should not involve zero-mode oscillators at all. We will eventually see that this is the case, at least to the order we are able to study.

We have given a rather detailed treatment of the calculation of the anomalous dimensions of two specific operator multiplets. To fully address the issues that will arise in string theory, we need expressions like (43) for all operator multiplets (not just spacetime scalars) that contain components with \(\Delta_0 = 2\). It is possible to carry out some version of the above lattice Laplacian argument for all the relevant operator classes, but we can use supersymmetry to circumvent this tedious task. The extended superconformal symmetry of the gauge theory means that conformal primary operators are organized into multiplets obtained from a lowest-dimension primary \(\mathcal{O}_D\) of dimension \(D\) by anticommutation with the supercharges \(Q^\alpha_{i}\) (\(i\) is an \(SL(2, C)\) Lorentz spinor index and \(\alpha\) is an \(SU(4)\) index). We need only concern ourselves here with the case in which \(\mathcal{O}_D\) is a spacetime scalar (of dimension \(D\) and \(R\)-charge \(R\)). There are sixteen supercharges and we can choose eight of them to be raising operators; there are \(2^8 = 256\) operators we can reach by ‘raising’ the lowest one. Since the raising operators increase the dimension and \(R\)-charge by \(1 / 2\) each time they act, the operators at level \(L\), obtained by acting with \(L\) supercharges, all have the same dimension and \(R\)-charge. The corresponding decomposition of the 256-dimensional multiplet is shown in table 1.

| Level | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|---|---|
| Multiplicity | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 |
| Dimension | \(D\) | \(D + 1 / 2\) | \(D + 1\) | \(D + 3 / 2\) | \(D + 2\) | \(D + 5 / 2\) | \(D + 3\) | \(D + 7 / 2\) | \(D + 4\) |
| \(R\) - charge | \(R\) | \(R + 1 / 2\) | \(R + 1\) | \(R + 3 / 2\) | \(R + 2\) | \(R + 5 / 2\) | \(R + 3\) | \(R + 7 / 2\) | \(R + 4\) |

Table 1: \(R\)-charge content of a supermultiplet

The states at each level can be classified under the Lorentz group and the \(SO(4) \sim SU(2) \times SU(2)\) subgroup of the \(R\)-symmetry group, which is unbroken after we have fixed the \(SO(2)\) \(R\)-charge. For instance, the 28 states at level 2 decompose under \(SO(4)_{\text{Lor}} \times SO(4)_R\) as
(6, 1) + (1, 6) + (4, 4). For the present, the most important point is that, given the dimension of one operator at one level, we can infer the dimensions of all other operators in the supermultiplet.

We can use this logic to get a complete accounting of the dimensions of the $\Delta_0 = 2$ BMN operators. Here we summarize work by Beisert [8], recasting his results to fit our needs (and adding some further useful information that emerges from our own $SU(4)$ analysis). The supermultiplet of interest is based on the set of scalars $\Sigma_A \operatorname{tr} (\phi^A Z^p \phi^A Z^{R_0-p})$, the operator class we have denoted by $\mathcal{T}^{(0)}_{R+2}$. According to (42), the spectrum of $\Delta = D - R$ eigenvalues associated with this operator basis is

$$\Delta(\mathcal{T}^{(0)}_{R+2}) = 2 + \frac{g_M^2 N_c}{\pi^2} \sin^2 \left(\frac{n\pi}{R+3}\right) \to 2 + \frac{g_M^2 N_c}{R^2} n^2 \left(1 - \frac{6}{R} + O(R^{-2})\right).$$

The other spacetime scalar operators $\mathcal{T}^{(\pm)}_{R+2}$ displayed in (42) have dimension formulae which appear to differ from this. However, when they are put into the context of a supermultiplet and the dimension formulae are expressed in terms of the $R$-charge of the lowest-dimension member of the supermultiplet, it turns out that (45) governs all the operators at all levels in the supermultiplet. We summarize the situation for the spacetime scalar members of the multiplet in table 2. The last column displays the allowed range of the eigenvalue index $n$

| $L$ | $R$ | $SU(4)$ Irreps | Operator | $\Delta - 2$ | Multiplicity |
|-----|-----|----------------|----------|--------------|--------------|
| 0   | $R_0$ | (0, $R_0$, 0) | $\Sigma_A \operatorname{tr} (\phi^A Z^p \phi^A Z^{R_0-p})$ | $\frac{g_M^2 N_c}{\pi^2} \sin^2 \left(\frac{n\pi}{(R_0+1)+3}\right)$ | $n = 1, \ldots, \frac{R_0+1}{2}$ |
| 2   | $R_0 + 1$ | (0, $R_0$, 2) + c.c. | $\operatorname{tr} (\phi^A Z^p \phi^A Z^{R_0+1-p})$ | $\frac{g_M^2 N_c}{\pi^2} \sin^2 \left(\frac{n\pi}{(R_0+1)+2}\right)$ | $n = 1, \ldots, \frac{R_0+1}{2}$ |
| 4   | $R_0 + 2$ | (2, $R_0$, 2) | $\operatorname{tr} (\phi^A Z^p \phi^A Z^{R_0+2-p})$ | $\frac{g_M^2 N_c}{\pi^2} \sin^2 \left(\frac{n\pi}{(R_0+2)+1}\right)$ | $n = 1, \ldots, \frac{R_0+1}{2}$ |
| 4   | $R_0 + 2$ | (0, $R_0 + 2$, 0) × 2 | $\operatorname{tr} (\phi^A Z^p \phi^A Z^{R_0+1-p})$ | $\frac{g_M^2 N_c}{\pi^2} \sin^2 \left(\frac{n\pi}{(R_0+3)+1}\right)$ | $n = 1, \ldots, \frac{R_0+1}{2}$ |
| 6   | $R_0 + 3$ | (0, $R_0 + 2$, 2) + c.c. | $\operatorname{tr} (\phi^A Z^p \phi^A Z^{R_0+2-p})$ | $\frac{g_M^2 N_c}{\pi^2} \sin^2 \left(\frac{n\pi}{(R_0+4)+1}\right)$ | $n = 1, \ldots, \frac{R_0+1}{2}$ |
| 8   | $R_0 + 4$ | (0, $R_0$, 0) | $\operatorname{tr} (\nabla^A Z^p \nabla^A Z^{R_0+2-p})$ | $\frac{g_M^2 N_c}{\pi^2} \sin^2 \left(\frac{n\pi}{(R_0+5)+1}\right)$ | $n = 1, \ldots, \frac{R_0+1}{2}$ |

Table 2: Dimensions and multiplicities of spacetime scalar operators

at each level (for $R_0$ odd only, just to save space) computed from our results for $SU(4)$ irrep multiplicities. It is non-trivial that the result is the same at each level; were it not so, the levels could not be assembled into a single supermultiplet. The universal dimension formula is written at each level in such a way as to emphasize the dependence on the $R$-charge of the particular level. This shows how the different results (42) and (45) are reconciled in the supermultiplet.

The supermultiplet contains operators that are not spacetime scalars (i.e., that transform non-trivially under the $SU(2, 2)$ conformal group) and group theory determines at what levels in the supermultiplet they must lie. A representative sampling of data on such operators
(extracted from Beisert’s paper) is collected in table 3. We have worked out neither the $SU(4)$ representations to which these lowest-$\Delta$ operators belong nor their precise multiplicities. The ellipses indicate that the operators in question contain further monomials involving fermion fields (so that they are not uniquely specified by their bosonic content). This information will be useful in consistency checks to be carried out below.

| $L$ | $R$ | Operator | $\Delta - 2$ | $\Delta - 2 \rightarrow$ |
|-----|-----|----------|---------------|-------------------|
| 2 | $R_0 + 1$ | $\text{tr} \left( \phi^i Z^p \nabla_{\mu} ZZ^{R_0-p} \right) + \ldots$ | $\frac{g^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{R_0+1+2} \right) \frac{g^2 N_c}{R_0} n^2 \left( 1 - \frac{4}{R_0} \right)$ | |
| 4 | $R_0 + 2$ | $\text{tr} \left( \phi^i Z^p \nabla_{\mu} ZZ^{R_0+1-p} \right)$ | $\frac{g^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{R_0+2+1} \right) \frac{g^2 N_c}{R_0} n^2 \left( 1 - \frac{2}{R_0} \right)$ | |
| 4 | $R_0 + 2$ | $\text{tr} \left( \nabla_{(\mu} Z^p \nabla_{\nu)} ZZ^{R_0-p} \right)$ | $\frac{g^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{R_0+2+1} \right) \frac{g^2 N_c}{R_0} n^2 \left( 1 - \frac{2}{R_0} \right)$ | |
| 6 | $R_0 + 3$ | $\text{tr} \left( \phi^i Z^p \nabla_{\mu} ZZ^{R_0+2-p} \right) + \ldots$ | $\frac{g^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{R_0+3} \right) \frac{g^2 N_c}{R_0} n^2 \left( 1 - \frac{0}{R_0} \right)$ | |
| 6 | $R_0 + 3$ | $\text{tr} \left( \nabla_{(\mu} Z^p \nabla_{\nu)} ZZ^{R_0+1-p} \right)$ | $\frac{g^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{R_0+3} \right) \frac{g^2 N_c}{R_0} n^2 \left( 1 - \frac{0}{R_0} \right)$ | |

Table 3: Anomalous dimensions of some operators that are not scalars

As far as dimensions are concerned, all of the above can be summarized by saying that the dimensions of the operators of $R$-charge $R$ at level $L$ in the supermultiplet are given by the general formula (valid for large $R$ and fixed $n$):

$$\Delta_n^{R,L} = 2 + \frac{g^2 N_c}{\pi^2} \sin^2 \left( \frac{n\pi}{R + 3 - L/2} \right) \rightarrow 2 + \frac{g^2 N_c}{R^2} n^2 \left( \frac{1}{2} - \frac{6 - L}{R} + O(R^{-2}) \right) .$$  

(46)

This amounts to a gauge theory prediction for the way in which worldsheet interactions lift the degeneracy of the two-impurity string multiplet. The 256 states of the form $A_n^L \langle B^\dagger_n | R \rangle$, for a given mode number $n$, (where $A^\dagger$ and $B^\dagger$ each can be any of the 8+8 bosonic and fermionic oscillators) should break up as shown in table 4. It should be emphasized that, for fixed $R$, the operators associated with different levels are actually coming from different supermultiplets; this is why they have different dimensions! As mentioned before, we can also precisely identify transformation properties under the Lorentz group and under the rest of the $R$-symmetry group of the degenerate states at each level. This again leads to useful consistency checks, and we will elaborate on this when we analyze the eigenstates of the string worldsheet Hamiltonian.

| Level | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|---|---|
| Multiplicity | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 |
| $\delta E \times (R^2/g^2_{YM} N_c n^2)$ | $-6/R$ | $-5/R$ | $-4/R$ | $-3/R$ | $-2/R$ | $-1/R$ | 0 | $1/R$ | $2/R$ |

Table 4: Predicted energy shifts of two-impurity string states
We now turn to the construction of the classical GS superstring action in the $AdS_5 \times S^5$ target space. We would like to construct a worldsheet action that has the full $SO(4,2) \times SO(6)$ symmetry of this space (we speak only about the bosonic symmetries, but similar considerations apply to their fermionic partners as well). However, it is not possible to make the full symmetry manifest: the fact that one has to expand about a classical trajectory of the string ‘spontaneously breaks’ the symmetry down to $SO(4,1) \times SO(5)$ (the analog of the Poincaré group for this background). The solution to this problem lies in the fact that this target space can be realized as the coset space $SO(4,2) \times SO(6)/SO(4,1) \times SO(5)$: there is a general strategy for writing a nonlinear sigma model action on a target space $G/H$ such that only symmetry under the stabilizer group $H$ is manifest (i.e. linearly realized) while the remaining generators of the full symmetry group $G$ are realized nonlinearly. This coset construction can be generalized to handle supersymmetries as well, provided that the target superspace can be realized as a supercoset manifold. Fortunately, this is true for the superstring on $AdS_5 \times S^5$, as was shown by Metsaev and Tseytlin [13] who constructed an action possessing the full $PSU(2,2|4)$ supersymmetry. Their action consists of a kinetic term and a Wess–Zumino term built as follows out of Cartan (super)one-forms on the supercoset manifold:

$$2\pi \alpha' S_{GS} = \int d^2\sigma (L_{Kin} + L_{WZ}) = \int d^2\sigma \left(-\frac{1}{2} h^{ab} L^\mu_a L^\nu_b \eta_{\mu\nu} - 2i \epsilon^{ab} \int_0^1 dt L^\mu_{a\tau} s^{IJ} \bar{\theta}^I \Gamma^\nu L^J_b \eta_{\mu\nu}\right).$$

(47)

The $\Gamma^\mu$ are $SO(9,1)$ gamma matrices, $\eta_{\mu\nu}$ is the $SO(9,1)$ Minkowski metric and $s^{IJ} = \text{diag}(1,-1)$. The world-sheet fermi fields $\theta^I$ $(I,J = 1,2)$ of the type IIB theory are two $SO(9,1)$ Majorana–Weyl spinors of the same chirality satisfying $\Gamma^1_1 \theta^I = \theta^I$. The gauge-fixing condition $\Gamma^0_0 \Gamma^I_0 \theta^I = \theta^I$ allows us to set half the components of $\theta^I$ to zero. This condition will be kept exact throughout the gauge-fixing and curvature expansion procedure (as will the bosonic gauge condition $x^+ = \tau$). It will also be convenient to define a complex spinor $\psi = \sqrt{2}(\theta^1 + i\theta^2)$. Upon restricting to the $8_s$ representation, the condition $\gamma^0 \psi = +\psi$ selects the upper eight components of $\psi$, since $\gamma^0 = \text{diag}(1,-1)_{16\times16}$. The fermion $\psi_\alpha$ can therefore be thought of as an eight-component complex spinor constructed from the 16 components of $\theta^I$ that survive the above gauge fixing.

The Cartan one-forms satisfy constraint equations, known as the Maurer–Cartan equations, which can be thought of as generalized Bianchi identities. The approach in [13] was to solve these equations order-by-order in powers of the coordinate fields $(x, \theta)$, and the first few terms (to quartic order in fields) of (47) were written explicitly therein. Following [13], it was shown by Kallosh, Rahmfeld and Rajaraman that these equations can be solved exactly.
for the $AdS_5 \times S^5$ geometry with the following results [14]:

\[
L_b^J = \frac{\sinh tM}{M} D_b \theta^J \\
L_b^J = L_b^J \mid_{t=1} \\
L_a^\mu = L_a^\mu \mid_{t=1},
\]

where

\[
(D_a \theta)^I = \left( \partial_a \theta + \frac{1}{4} (\omega^{\mu \nu}_\rho \partial_a x^\rho) \Gamma_{\mu \nu}^I \right)^I - \frac{i}{2} \epsilon_{IJ} \epsilon^\mu_\rho \partial_a x^\rho \Gamma_{\mu \theta}^J,
\]

\[
(M^2)^{IL} = -\epsilon^{IJ} (\Gamma_a \Gamma^\mu \theta^I \theta^L \Gamma^\mu) + \frac{1}{2} \epsilon^{KL} (-\Gamma^{jk} \theta^I \theta^K \Gamma_{jk} \Gamma_a + \Gamma_{jk} \theta^I \theta^K \Gamma_{jk} \Gamma_a) \equiv \Gamma_a.
\]

In the above, $\epsilon^\mu_\rho (\omega^{\mu \nu}_\rho)$ is the vielbein (spin connection) in the $AdS_5 \times S^5$ geometry, and $\Gamma_a \equiv \epsilon^{KL} \Gamma_{1234}$. Factors of the Minkowski metric, needed to contract Lorentz indices, have been suppressed.

The action obtained by substituting these formulas into (47) can now be systematically expanded in powers of the inverse curvature scale $\hat{R}$ by inserting the expansions of the vielbeins and spin connections that follow from the metric expansion (46). The expansion of $L_0^\mu L_0^\mu$, from which one constructs the kinetic term of the worldsheet action, is

\[
L_0^\mu L_0^\mu \approx \left\{ 2 \dot{x}^r - (x^A)^2 + (\dot{x}^A)^2 - 2i \beta^I \Gamma^{-} (\partial_0 \theta^I - \epsilon^{IJ} \Pi \theta^J) \right\}
+ \frac{1}{\hat{R}^2} \left\{ (\dot{x}^r)^2 - 2y^2 \dot{x}^r + \frac{1}{2} (\dot{z}^2 z^2 - y^2 y^2) + \frac{1}{2} (y^2 - z^4) + (\theta \text{ terms}) \right\}
\]

\[
L_1^\mu L_1^\mu \approx \left\{ (x^A)^2 + \frac{1}{\hat{R}^2} \left\{ \frac{1}{2} (\dot{z}^2 z^2 - y^2 y^2) + (x^r)^2 + (\theta \text{ terms}) \right\} \right\}
\]

\[
L_0^\mu L_1^\mu \approx \left\{ x^r \dot{x}^r - \dot{x}^A x^A - i \dot{\theta}^I \Gamma^{-} \partial_0 \theta^I \right\}
+ \frac{1}{\hat{R}^2} \left\{ x^r \dot{x}^r - y^2 \dot{x}^r + \frac{1}{2} (\dot{z}^2 z_k z^k - y^2 \dot{y}_{k'} y_{k'}^r) + (\theta \text{ terms}) \right\}.
\]

The bosonic coordinate $x^A = (y_{k'}, z_k)$ has eight components. (In the previous section the indices $A, B$ took four values, but in this section they take eight values.) As usual, a dot is the same thing as $\partial_0$ and a prime is the same thing as $\partial_1$. The expression for the full worldsheet action expanded in this fashion is not very illuminating, and we will not present it here.

Our ultimate goal is to construct a Hamiltonian for the physical transverse coordinates $x^A$, $\psi^a$ and their associated canonical momenta. The first step is to impose the bosonic gauge condition $x^+ = \tau$ along with the $\kappa$-symmetry gauge-fixing condition $\Gamma^+ \theta^I = 0$. At leading order in $1/\hat{R}$, the gauge $x^+ = \tau$ is consistent with a flat worldsheet metric $h^{ab} = (-1, 1)$. However, to maintain the gauge choice $x^+ = \tau$ beyond leading order, it turns out that we must allow $h^{ab}$ to acquire curvature corrections.\footnote{This is to be contrasted with ref. [10], which imposed a flat world sheet metric and introduced curvature corrections to the gauge choice.} We therefore need to eliminate both $x^-$ and
the worldsheet metric in favor of physical variables. Taken together, the equations of motion for \( x^- \) and the conformal gauge constraints (vanishing of the worldsheet energy-momentum tensor) provide exactly the information needed to do this.

Consider first the conformal constraints obtained by varying the action with respect to the worldsheet metric:

\[
T_{ab} = L_a^\mu L_b^\mu - \frac{1}{2} h_{ab} h^{cd} L_c^\mu L_d^\mu = 0 .
\]  

(51)

Because \( T_{ab} \) is symmetric and traceless, there are only two independent constraints associated with conformal invariance on the worldsheet. Using (50) and (6), and keeping only terms of leading order in \( 1/\tilde{R} \), they read

\[
T_{00} \sim \frac{1}{2} \left( 2\dot{x}^- - (x^A)^2 + (\dot{x}^A)^2 + (x'^A)^2 - 2i\bar{\theta}J \Gamma^- (\partial_0 \theta^I - \epsilon^{IJ} \Pi \theta^J) \right) = 0
\]

\[
T_{01} \sim x^J - \dot{x}^A x^A - i\bar{\theta}J \Gamma^- \partial_1 \theta^J = 0.
\]  

(52)

These constraints can be recast as equations to determine \( x^- \) to leading order in \( 1/\tilde{R} \):

\[
(x^-)_0 = \frac{1}{2} (x^A)^2 - \frac{1}{2} \left[ (\dot{x}^A)^2 + (x'^A)^2 \right] + i\bar{\theta}J \Gamma^- (\partial_0 \theta^I - \epsilon^{IJ} \Pi \theta^J)
\]

\[
(x^J)_0 = -\dot{x}^A x^A + i\bar{\theta}J \Gamma^- \partial_1 \theta^J.
\]  

(53)

We will eventually show how to evaluate the \( O(1/\tilde{R}^2) \) corrections to \( x^- \).

The building blocks of the \( x^- \) equation of motion, to \( O(1/\tilde{R}^2) \), are as follows:

\[
\frac{\delta \mathcal{L}}{\delta x^-} = \frac{1}{2} \frac{h^{00}}{R^2} \left\{ 2 + \frac{1}{R^2} \left[ (2\dot{x}^- - 2y^2) - i\bar{\theta}J \Gamma^- \partial_0 \theta^I + 2i\bar{\theta}J \Gamma^- \epsilon^{IJ} \Pi \theta^J \right] \right\} + \frac{i}{2R^2} s^{IJ} \bar{\theta}J \Gamma^- \partial_1 \theta^J
\]

\[
= \frac{1}{2} \frac{h^{00}}{R^2} \left\{ 2 + \frac{1}{R^2} \left[ (x^2 - y^2) - \left[ (\dot{x}^A)^2 + (x'^A)^2 \right] + i\bar{\theta}J \Gamma^- \partial_0 \theta^I \right] \right\} + \frac{i}{2R^2} s^{IJ} \bar{\theta}J \Gamma^- \partial_1 \theta^J
\]

\[
\frac{\delta \mathcal{L}}{\delta x^J} = h^{01} \frac{1}{R^2} \left\{ -\dot{x}^A x^A + \frac{i}{2} \bar{\theta}J \Gamma^- \partial_1 \theta^I \right\} - \frac{i}{2R^2} s^{IJ} \bar{\theta}J \Gamma^- \partial_1 \theta^J
\]  

(54)

In the first equation, \( x^- \) was eliminated by using the \( T_{00} \) constraint evaluated to leading order. It is obvious from (54) that the choice of a flat Minkowski worldsheet metric (\( h^{00} = -h^{11} = 1, h^{01} = 0 \)) is inconsistent with the \( x^- \) equations of motion. To allow for corrections to the metric, we therefore write

\[
h^{00} = -1 + \frac{\tilde{h}^{00}}{R^2} + \ldots \quad h^{11} = 1 + \frac{\tilde{h}^{11}}{R^2} + \ldots \quad h^{01} = \frac{\tilde{h}^{01}}{R^2} + \ldots .
\]  

(55)

By choosing the metric corrections

\[
\tilde{h}^{00} = \frac{1}{2} (z^2 - y^2) - \frac{1}{2} \left[ (\dot{x}^A)^2 + (x'^A)^2 \right] + \frac{i}{2} \bar{\theta}J \Gamma^- \partial_0 \theta^I - \frac{i}{2} s^{IJ} \bar{\theta}J \Gamma^- \partial_1 \theta^J
\]  

(56)
\[ \tilde{h}^{01} = \dot{x}^A \dot{x}^A - \frac{i}{2} \bar{\theta}^I \Gamma^- \partial_I \theta^J + \frac{i}{2} s^{IJ} \bar{\theta}^I \Gamma^- \partial_0 \theta^J, \]  

The \( x^- \) equation of motion is vastly simplified:

\[ \frac{\delta \mathcal{L}}{\delta \dot{x}^-} = 1 + O(1/R^4) \quad \frac{\delta \mathcal{L}}{\delta x^-} = O(1/R^4). \]  

This choice of worldsheet metric is what is needed to enforce the lightcone gauge condition \( x^+ = \tau \) to \( O(1/R^2) \). With the corrected metric in hand, we can revisit the conformal gauge constraints to determine \( x^- \) to \( O(1/R^2) \).

Upon evaluating \( x^- \) to the order of interest, we are equipped to express the Hamiltonian density for the generator of translations of lightcone time \( x^+ \), \( \delta \mathcal{L}/\delta \dot{x}^+ = p_+ \). The variation is done before any gauge fixing, holding the remaining coordinates and the worldsheet metric fixed. The replacement of \( x^\pm \) and \( h^{ab} \) by fixing conformal gauge is understood to be completed after the variation. The result to \( O(1/R^2) \) is

\[ H_{lc} = \frac{\delta \mathcal{L}_{GS}}{\delta \dot{x}^+} = H_{pp} + H_{int} \]

\[ H_{pp} = \frac{1}{2} (x^A)^2 + \frac{1}{2} [\dot{x}^A]^2 + \frac{1}{8} \left[ (\dot{x}^A)^2 + (x^A)^2 \right] - i \bar{\theta}^I \Gamma^- \epsilon^{IJK} \theta^J + is^{IJ} \bar{\theta}^I \Gamma^- \partial_0 \theta^J \]

\[ \hat{R}^2 H_{int} = \frac{1}{4} \left[ y^2 (\dot{z}^2 - z^2 - 2y^2) + z^2 (-\dot{y}^2 + y^2 + 2z^2) \right] \]

\[ + \frac{1}{8} \left[ 3(\dot{x}^A)^2 - (x^A)^2 \right] \left[ (\dot{x}^A)^2 + (x^A)^2 \right] + \frac{1}{8} [(-\dot{x}^A)^2 - \frac{1}{2} (\dot{x}^A)^2] \]

\[ - \frac{i}{4} \sum_{a=0}^1 \bar{\theta}^I (\partial_a x^A \Gamma^A) \epsilon^{IJK} \Gamma^- \Pi (\partial_a x^B \Gamma^B) \theta^J - \frac{i}{2} (x^A)^2 \bar{\theta}^I \Gamma^- \epsilon^{IJK} \Pi \theta^J \]

\[ - \frac{i}{2} (\dot{x}^A)^2 \bar{\theta}^I \Gamma^- \partial_0 \theta^J - \frac{i}{12} \bar{\theta}^I \Gamma^- (\mathcal{M}^2) \epsilon^{IJK} \Pi \theta^L - \frac{1}{2} (\bar{\theta}^I \Gamma^- \epsilon^{IJK} \Pi \theta^J)^2 \]

\[ + \frac{i}{2} (\dot{x}^A^I \bar{\theta}^I \Gamma^- \partial_0 \theta^J - \frac{i}{4} (y^2 - z^2) s^{IJ} \bar{\theta}^I \Gamma^- \partial_0 \theta^J \]

\[ + \frac{i}{4} s^{IJ} \bar{\theta}^I \Gamma^- (y_j \Gamma^- - z_j \Gamma^-) \theta^J + \frac{i}{4} s^{IJ} \bar{\theta}^I \Gamma^- (z_j, \bar{\theta}^I \Gamma^- - y_j, \Gamma^0 \Gamma^0) \theta^J \]

\[ + \frac{1}{4} \left[ (\dot{x}^A)^2 - (x^A)^2 \right] s^{IJ} \bar{\theta}^I \Gamma^- \partial_0 \theta^J + \frac{i}{12} s^{IJ} \bar{\theta}^I \Gamma^- (\mathcal{M}^2) \epsilon^{IJK} \Pi \theta^L \]

\[ + \frac{1}{2} s^{IJ} \bar{\theta}^I \Gamma^- \partial_0 \theta^J + \frac{i}{2} (x^A)^2 s^{IJ} \bar{\theta}^I \Gamma^- \partial_0 \theta^J \].

To quantize this system, the Hamiltonian must be expressed in terms of canonical coordinates and momenta. The recipe for computing the bosonic momenta \( p^A \) is, once again, to vary \( \mathcal{L} \) with respect to \( \dot{x}^A \), holding the other coordinates and the worldsheet metric fixed, and replacing \( x^\pm \) and \( h^{ab} \) according to the appropriate constraints only after the variation is
done. For example, the result for the momenta in the $SO(4)$ descending from $AdS_5$, correct to $O(1/R^2)$, is

$$p_k = \dot{z}_k + \frac{1}{R^2} \left\{ \frac{1}{2} y'^2 p_k + \frac{1}{2} \left[ (p_A)^2 + (x'^A)^2 \right] p_k - (p_A x'^A) z'_k - \frac{i}{2} p_k \theta^I \Gamma^I \partial_0 \theta^I \right. \\
+ \frac{i}{2} p_k S^{IJ} \theta^I \Gamma^J \partial_0 \theta^J - \frac{i}{4} \tilde{\theta}^I \Gamma^I \partial_0 \theta^I + \frac{i}{4} \bar{\theta}^I \Gamma^I \partial_0 \theta^I + \frac{i}{2} \bar{\theta}_k \theta^I \Gamma^I \partial_0 \theta^J \\
+ \frac{i}{4} p_A \epsilon^{IJ} \theta^I \Gamma^J \Gamma^J (\Gamma^A - \Gamma^A \Pi \Gamma_k) \theta^J - \frac{i}{2} \bar{\theta}_k \bar{\theta}^{IJ} \Gamma^J \Gamma^J \bar{\theta}^I \partial_0 \theta^J \\
+ \frac{i}{4} x^{IJ} \epsilon^{IJ} \Gamma^J \Gamma^J (\Gamma^A - \Gamma^A \Pi \Gamma_k) \theta^J \right\} . \tag{60}$$

To calculate the Hamiltonian to $O(1/R^2)$, we use this relation to eliminate $\dot{z}_k$ from $H_{pp}$ (and use $p_k = \dot{z}_k$ in $H_{int}$).

Performing the analogous operation in the fermionic regime is more complicated. At this point it is convenient to change notation by replacing fermionic coordinates $\theta^I$ with the single complex spinor $\psi = \sqrt{2}(\theta_1 + i \theta_2)$; $\psi$ and $\psi^\dagger$ appear as independent coordinates and there are two fermionic canonical momenta, $\rho_\psi = \delta L/\delta \dot{\psi}$ and $\rho_{\psi^\dagger} = \delta L/\delta \dot{\psi}^\dagger$. In all the standard field theory examples, one can manipulate $L$ (using integration by parts in time) so that the action is independent of $\dot{\psi}^\dagger$. The momentum equations, known as the primary constraints, then read $\rho_\psi - \rho_{\psi^\dagger} = 0$ and $\rho_{\psi^\dagger} = 0$. They are, in effect, constraints that eliminate $\dot{\psi}^\dagger$ as dynamical variables in the system, leaving the standard canonical Poisson brackets for $\psi, \rho_\psi$ unchanged. Things do not work quite so simply in the present problem.

The terms in $L_{GS}$ that depend on $\dot{\psi}^\dagger$ (and are therefore relevant for the fermionic momentum constraints) are

$$L \sim \frac{i}{R^2} \left\{ \frac{1}{4} \left[ \dot{x} - \frac{1}{2} (z^2 - y^2) \right] (\dot{\psi}^\dagger \dot{\psi} + \psi \dot{\psi}^\dagger) - \frac{\dot{h}_{00}}{2} \left( \psi \dot{\psi}^\dagger + \psi^\dagger \dot{\psi} \right) \right. \\
+ \frac{1}{96} \left( \psi \gamma^{jk} \psi^\dagger \right) \left( \psi \gamma^{jk} \Pi \dot{\psi}^\dagger - \psi \gamma^{jk} \Pi \dot{\psi} + \frac{x'^{jk}}{4} \left( \psi \dot{\psi} + \psi^\dagger \dot{\psi}^\dagger \right) - (j, k \neq j', k') \right\} . \tag{61}$$

Some of the curvature correction terms in (61) contain $\dot{\psi}^\dagger$ in such a way that it cannot be eliminated from the action. The fermionic constraints, or primary constraints, therefore take
on a more complicated form than usual:

\[
\rho = i\psi^\dagger + \frac{1}{R^2} \left\{ \frac{1}{4} y^2 \rho + \frac{1}{8} \left[ (p^2_A) + (x'^A)^2 \right] \right. \\
\left. - \frac{i}{8} (\bar{\psi} \rho' + \rho \bar{\psi}') \psi + \frac{i}{8} (\bar{\psi} \psi' - \rho \rho') \rho \\
+ \frac{i}{48} \left[ (\psi \gamma^{jk} \rho) (\rho \gamma^{jk} \Pi) - (j, k, \leftrightarrow j', k') \right] \right\} \\
\rho_{\psi^\dagger} = \frac{1}{R^2} \left\{ \frac{i}{4} y^2 \psi + \frac{i}{8} \left[ (p^2_A) + (x'^A)^2 \right] \right. \\
\left. \psi + \frac{i}{4} (p_A x'^A) \rho - \frac{1}{4} (\rho \Pi \psi) \psi \\
- \frac{1}{8} (\psi \rho' + \rho \bar{\psi}') \rho - \frac{1}{8} (\bar{\psi} \psi' - \rho \rho') \psi \right\}.
\]

(62)

(63)

Again, this is the result of varying the Lagrangian first and then substituting in solutions for the gauge-fixed coordinates and worldsheet metric. For clarity, we express the primary constraints as follows:

\[
\chi_\psi \equiv \rho - i\psi^\dagger - \left( 1/R^2 \text{ corrections} \right) = 0 \\
\chi_{\psi^\dagger} \equiv \rho_{\psi^\dagger} - \left( 1/R^2 \text{ corrections} \right) = 0.
\]

(64)

We are now ready to proceed with rewriting the Hamiltonian as a function of canonical coordinates and momenta. The elimination of the bosonic velocities via the equations that define the bosonic canonical momenta (60) is straightforward, at least to first non-leading order in $1/R^2$. Although it requires a messy calculation to show it, this step also eliminates all terms involving $\dot{\psi}$ or $\dot{\psi}^\dagger$. As a result, all terms involving fermi fields are built out of $\psi$, $\psi^\dagger$ and their $\sigma$ derivatives. It is again straightforward, at least to first subleading order in $1/R^2$, to use the $\rho_{\psi}$ constraint to eliminate $\psi^\dagger$ in favor of $\rho_{\psi}$. The result is exactly what we want: a Hamiltonian expressed as a function of $x^A, p^A$ and $\psi^\alpha, \rho_{\psi}^\alpha$. We are not quite done, however. In general, a set of primary constraints $\chi = 0$ (64) can be categorized as either first or second-class constraints. The so-called second-class constraints arise when canonical momenta (defined by the primary constraints) do not have vanishing Poisson brackets with the primary constraints themselves: $\{\rho_{\psi}, \chi_\psi\} \neq 0, \{\rho_{\psi^\dagger}, \chi_{\psi^\dagger}\} \neq 0$. (First-class constraints are characterized by the more typical condition $\{\rho_{\psi^\dagger}, \chi_{\psi^\dagger}\} = \{\rho_{\psi}, \chi_\psi\} = 0.$) In the presence of second-class constraints, consistent quantization requires that the quantum anticommutator of two fields be identified with their Dirac bracket (which depends on the Poisson bracket algebra of the constraints) rather than with their classical Poisson bracket. To the order of interest to us, the net effect of all this can be implemented by saying that the following nonlinear field redefinition restores the conventional anticommutation relations
and Fourier mode expansion (17):

\[ \tilde{\rho}_\alpha = \rho_\alpha \]
\[ \tilde{\psi}_\beta = \psi_\beta + \frac{i}{8R^2} \left\{ (\psi'\psi)\psi_\beta - 2(\rho\Pi\psi)\psi_\beta - (\rho'\rho)\psi_\beta + 2(p_Ax^A)\rho_\beta \right. \\
\left. + [(\rho'\psi)\rho_\beta - (\rho\psi')\rho_\beta] + 2i \left[ y^2\psi_\beta + \frac{1}{2} ((p_A)^2 + (x^A)^2) \psi_\beta \right] \right\}. \] (65)

To be precise, we have removed the second-class constraints on the fermionic variables by imposing, via field redefinition, the proper quantization condition. In turn, the Hamiltonian has been recast in terms of fermionic variables (\(\rho_\psi\) and \(\psi\)) that exhibit the usual anticommutation relations. The field redefinition has no effect on \(H_{\text{int}}\) to the order of interest, but, when applied to \(H_{pp}\), it generates new interaction terms of \(O(1/R^2)\). This procedure gives rise to non-trivial cancellations among terms that would potentially renormalize the spectrum of supergravity modes. Without these cancellations, the resulting spectrum of energy corrections would be nonsensical.

Since we are treating these curvature corrections to the pp-wave background in first-order perturbation theory, we are only interested in physical string states that are eigenstates of the pp-wave theory. The level matching condition on these states is met by fixing \(x'^-\) such that \(T_{01}\) vanishes at leading order. \((T_{01})\) is the current associated with translation symmetry on the closed-string worldsheet. Fixing \(T_{01} = 0\) gives the usual level-matching condition for physical pp-wave eigenstates.

Conformal invariance demands that \(T_{01}\) vanish order by order in the expansion, and this is satisfied by fixing higher-order corrections to \(x'^-\). For some set of physical eigenstates of the \(1/R^2\) corrected geometry, the vanishing of \(T_{01}\) to this order would translate to an exact level matching condition on these states. Since we are not trying to solve this theory exactly, this has no bearing on the present calculation, and an explicit expression for \(x'^-\) at \(O(1/R^2)\) is not needed.

The final result for the Hamiltonian density in the perturbed theory is

\[ \mathcal{H} = \mathcal{H}_{pp} + \mathcal{H}_{\text{int}}, \quad \mathcal{H}_{\text{int}} = \mathcal{H}_{BB} + \mathcal{H}_{FF} + \mathcal{H}_{BF}, \] (66)

where

\[ \mathcal{H}_{pp} = \frac{1}{2} \left[ (x^A)^2 + (p_A)^2 + (x'^A)^2 \right] + \frac{i}{2} \left[ \psi'\psi - \rho\rho' + 2\rho\Pi\psi \right], \] (67)

\[ \mathcal{H}_{BB} = \frac{1}{R^2} \left\{ \frac{1}{4} \left[ z^2 \left( p_y^2 + y^2 + 2z'^2 \right) - y^2 \left( p_z^2 + z'^2 + 2y'^2 \right) \right] + \frac{1}{8} \left[ (x^A)^2 \right]^2 \right. \\
\left. - \frac{1}{8} \left\{ \left[ (p_A)^2 \right]^2 + 2(p_A)^2(x'^A)^2 + \left[ (x'^A)^2 \right]^2 \right\} + \frac{1}{2} \left( x'^A p_A \right)^2 \right\}, \] (68)
\[ H_{FF} = -\frac{1}{4R^2} \left\{ \left[ (\psi'\psi) + (\rho\rho') \right] (\rho\Pi\psi) - \frac{1}{2} (\psi'\psi)^2 - \frac{1}{2} (\rho'\rho)^2 + (\psi'\psi)(\rho'\rho) \\
+ (\rho\psi')(\rho'\psi) - \frac{1}{2} \left[ (\psi\rho')(\psi\rho') + (\psi'\rho)^2 \right] + \left[ \frac{1}{12} (\psi\gamma^j k \rho)(\rho\gamma^j k \Pi\rho') \right] \\
- \frac{1}{48} (\psi\gamma^j k \psi - \rho\gamma^j k \rho) \left( \rho'\gamma^j k \Pi\psi - \rho\gamma^j k \Pi\psi' \right) - (j,k \leftrightarrow j',k') \right\}, \] (69)

\[ H_{BF} = \frac{1}{R^2} \left\{ -\frac{i}{4} \left[ (p_A)^2 + (x^A)^2 + (y^2 - z'^2) \right] (\psi\psi' - \rho\rho') \\
- \frac{1}{2} \left[ (p_A x^A)(\rho\psi' + \psi\rho') - \frac{i}{2} \left( p_k^2 + y^2 - z^2 \right) \rho\Pi\psi \right] \\
+ \frac{i}{4} (z_j' z_k) \left( \psi\gamma^j k \psi - \rho\gamma^j k \rho \right) - \frac{i}{4} (y_j' y_k') \left( \psi\gamma^j k' \psi - \rho\gamma^j k' \rho \right) \\
- \frac{i}{8} (z_k y_{k'} + z_k y_{k'}) \left( \psi\gamma^{k' k} \psi - \rho\gamma^{k' k} \rho \right) + \frac{1}{4} (p_k y_{k'} + z_k p_{k'}) \psi\gamma^{k' k} \rho \\
+ \frac{1}{4} (p_j y_k) \left( \psi\gamma^j k \Pi\psi + \rho\gamma^j k \Pi\rho \right) - \frac{1}{4} (p_j' y_{k'}) \left( \psi\gamma^{j' k'} \Pi\psi + \rho\gamma^{j' k'} \Pi\rho \right) \\
- \frac{i}{2} \left( p_k p_{k'} - z_k' y_{k'} \right) \psi\gamma^{k' k} \Pi\rho \right\}. \] (70)

Repeated indices are summed over; indices \( j,k \) run over \( 1,\ldots,4 \) while indices \( i',j' \) run over \( 5,\ldots,8 \). The need for this notation arises because the coordinates that descend from AdS\(_5\) are treated differently from those that descend from S\(_5\). Put another way, the residual symmetry of the problem is \( SO(4) \times SO(4) \), not \( SO(8) \).

## 5 Quantization and Diagonalization of the Perturbation Hamiltonian

The Hamiltonian (66) is written using the same conventions as in our discussion of the pp-wave limit in the first section. To quantize it, we replace the canonical fields and momenta by their expansion in string mode creation and annihilation operators. The canonical commutation relations are unchanged by the interactions, and we may therefore use the mode expansions (11,12) and mode (anti)commutators of the pp-wave limit without modification. The terms quadratic in mode operators of course reproduce the pp-wave Hamiltonian (20). Terms quartic in mode oscillators constitute the interaction Hamiltonian, the operator we must diagonalize to find the perturbed spectrum. In this paper, we will implement perturbation theory on the degenerate multiplets of states created by acting on the ground state with two creation operators. For these purposes, we only need the terms in \( H_{int} \) containing two creation and two annihilation operators. As an example of the outcome of this procedure,
we display the purely bosonic part of the expansion of $H_{\text{int}}$:

$$H_{BB} = -\frac{1}{32J} \sum \frac{\delta(n + m + l + p)}{\xi} \times$$

$$\left\{ 2 \left[ \xi^2 - (1 - k_l k_p k_n k_m) + \omega_n \omega_m k_l k_p + \omega_i \omega_j k_n k_m + 2\omega_n \omega_l k_m k_p \right.$$

$$+ 2\omega_m \omega_p k_n k_l + \omega_n \omega_p k_m k_l \left[ a^{+A}_{-n} a^{-B}_{m} a^{A}_{-m} a^{B}_{n} + 4 \left( \xi^2 - (1 - k_l k_p k_n k_m) - 2\omega_n \omega_m k_l k_p + \omega_l \omega_m k_n k_p \right.$$

$$- \omega_n \omega_l k_m k_p - \omega_m \omega_p k_n k_l + \omega_n \omega_p k_m k_l \right) a^{+A}_{-n} a^{+B}_{-l} a^{-A}_{m} a^{-B}_{n} + 4 \left[ 8k_l k_p a^{+i}_{-n} a^{+j}_{l} a^{-i}_{m} a^{-j}_{n} \right.$$

$$+ 2(k_l k_p + k_n k_m) a^{+i}_{-n} a^{+j}_{n} a^{j}_{l} a^{i}_{p} + (\omega_l \omega_m + k_l k_p - \omega_n \omega_m - k_n k_m) a^{+i}_{-n} a^{+j}_{-l} a^{-i}_{m} a^{-j}_{p}$$

$$\left. - 4(\omega_l \omega_m - k_l k_p) a^{+i}_{-n} a^{+j}_{-l} a^{-i}_{m} a^{-j}_{n} - (i, j \leftrightarrow i', j') \right]\right\}, \quad (71)$$

where $\xi \equiv \sqrt{\omega_n \omega_m \omega_i \omega_j}$, $\omega_n = \sqrt{1 + k_n^2}$ and $k_n^2 = \lambda n^2$. The indices $l, m, n, p$ run from $-\infty$ to $+\infty$. The notation distinguishes sums over indices in the first $SO(4)$ $(i, j, ..)$, the second $SO(4)$ $(i', j', ..)$ and over the full $SO(8)$ $(A, B, ..)$. Note that the powers of $P$ that come from expanding the fields in terms of creation operators and doing the integral over $\sigma$ combine to convert the small parameter governing the strength of the perturbation from $1/R^2$ to $1/J$, where $J$ is the integrally quantized (and large) angular momentum of the string in the $S^5$ subspace. We have written operator monomials in normal-ordered form. Since $\mathcal{H}_{\text{int}}$ was derived as a classical object, the ‘correct’ ordering of the operators is not defined; we will allow for this ambiguity via an appropriate normal-ordering constant. The corresponding oscillator expressions for $H_{FF}$ and $H_{BF}$ are too complicated to display here.

To compute the string energy shifts induced by $H_{\text{int}}$ in first-order degenerate perturbation theory, we must find its matrix elements between the states of a multiplet degenerate under $H_{pp}$, and then diagonalize the resulting finite-dimensional matrix. We will execute this program on the 256-dimensional space of string excited states created by acting on the ground state of angular momentum $J$ with two creation operators of equal and opposite moding (the latter condition is needed to satisfy the level-matching constraint). The bosonic creation operators are denoted $a^{+A}_{n}$, where $n$ is the integer mode index and $A = 1, .., 8$ is an $SO(8)$ vector index which decomposes as $(4, 1) + (1, 4)$ under the manifest $SO(4) \times SO(4)$ symmetry. Passing to a $SU(2)^2 \times SU(2)^2$ notation, we rewrite these representations as $(2, 2; 1, 1) + (1, 1; 2, 2)$. The fermi creation operators are $b^{+\alpha}_{n}$ where $\alpha$ is an $SO(8)$ spinor index which decomposes as $(2, 1; 2, 1) + (1, 2; 1, 2)$ under the spinor version of the manifest $SO(4) \times SO(4)$ symmetry. The two four-dimensional irreps of $SO(4) \times SO(4)$ are distinguished by their $\Pi$ eigenvalues (+1 or −1), and we will sometimes use $\Pi$ to categorize the fermi creation operators accordingly. The eight bosonic and eight fermionic creation operators allow us to create 256 ‘two-impurity’ states as follows:

$$a^{+A}_{n} a^{-B}_{-n} |J\rangle \quad b^{+\alpha}_{n} b^{-\alpha}_{-n} |J\rangle \quad a^{+A}_{n} b^{-\alpha}_{-n} |J\rangle \quad a^{+A}_{-n} b^{+\alpha}_{n} |J\rangle \quad (72)$$
Half of these states are bosons and half are fermions. They all have the same lightcone energy

\[ \Delta = 2\sqrt{1 + k^2_n} \sim 2 + \lambda' n^2 + \ldots \]  

(73)

under \( H_{pp} \). We expect to find non-zero matrix elements of \( H_{int} \) between these states according to the scheme shown in table 5. The matrix is block diagonal because half the states in the multiplet are bosons and half are fermions, and there are of course no matrix elements between the two. Because of the complicated form of \( H_{int} \) itself, especially in its dependence on fermi fields, we found it necessary to use symbolic manipulation programs to organize the calculation of explicit forms for the matrices according to the various blocks in the table. The results, as we will now show, turn out to be surprisingly simple.

Table 5: Structure of the matrix of first-order energy perturbations in the space of two-impurity string states

| \( (H)_{int} \) | \( a_n^A a_{-n}^B \langle J \rangle \) | \( b_n^\alpha b_{-n}^{\alpha'} \langle J \rangle \) | \( a_n^A b_{-n}^{\alpha'} \langle J \rangle \) | \( a_n^{A'} b_{-n}^{\alpha} \langle J \rangle \) |
|---|---|---|---|---|
| \( \langle J | a_n^A a_{-n}^B \rangle \) | \( H_{BB} \) | \( H_{BF} \) | 0 | 0 |
| \( \langle J | b_n^\alpha b_{-n}^{\alpha'} \rangle \) | \( H_{BF} \) | \( H_{FF} \) | 0 | 0 |
| \( \langle J | a_n^A b_{-n}^{\alpha'} \rangle \) | 0 | 0 | \( H_{BF} \) | \( H_{BF} \) |
| \( \langle J | a_n^{A'} b_{-n}^{\alpha} \rangle \) | 0 | 0 | \( H_{BF} \) | \( H_{BF} \) |

The matrix elements of \( H_{int} \) between spacetime bosons built out of bosonic string oscillators only turn out to have the following explicit form:

\[
\langle J | a_n^A a_{-n}^B (H_{BB}) a_{-n}^{C_1} a_{n}^{D_1} | J \rangle = \left( N_{BB}(k^2_n) - 2n^2\lambda' \right) \frac{\delta^{AD} \delta^{BC}}{J} + \frac{n^2\lambda'}{J(1 + n^2\lambda')} \left[ \delta^{ab} \delta^{cd} + \delta^{ad} \delta^{bc} - \delta^{ac} \delta^{bd} \right] \\
- \frac{n^2\lambda'}{J(1 + n^2\lambda')} \left[ \delta^{d'b'} \delta^{c'd'} + \delta^{d'd'} \delta^{b'c'} - \delta^{a'c'} \delta^{b'd'} \right] \\
\approx (n_{BB} - 2) \frac{n^2\lambda'}{J} \delta^{AD} \delta^{BC} + \frac{n^2\lambda'}{J} \left[ \delta^{ab} \delta^{cd} + \delta^{ad} \delta^{bc} - \delta^{ac} \delta^{bd} \right] \\
- \frac{n^2\lambda'}{J} \left[ \delta^{d'b'} \delta^{c'd'} + \delta^{d'd'} \delta^{b'c'} - \delta^{a'c'} \delta^{b'd'} \right] + O(\lambda'^2), \quad (74)
\]

where lower-case \( SO(4) \) indices \( a, b, c, d \in 1, \ldots, 4 \) mean that the corresponding \( SO(8) \) labels \( A, B, C, D \) all lie in the first \( SO(4) \), while the indices \( a', b', c', d' \in 5, \ldots, 8 \) mean that the \( SO(8) \) labels lie in the second \( SO(4) \) \( (A, B, C, D \in 5, \ldots, 8) \). Note that we have written both the exact matrix element and its expansion in powers of \( \lambda' \). Since \( \lambda' \) is related to the gauge coupling constant by the AdS/CFT correspondence, the \( \lambda' \) expansion of the energy eigenvalues is what is needed to make comparisons with perturbative gauge theory calculations of operator dimensions.
A further important point is that we have included a function \( N_{BB}(k_n^2) \) to account for operator ordering ambiguities. In a matrix element of \( H_{\text{int}} \) between states of this type, different operator orderings differ by terms proportional to \( \delta^A D \delta^B C \) (the ambiguity arises from commuting creation and annihilation operators of the same mode number past each other: \( A, D \) go with mode \( n \) and \( B, C \) go with mode \(-n\)). The coefficient of this term is an a priori arbitrary function of \( k_n^2 = n^2 \lambda' \) which we will define by its power series. In order for the energy shift to be perturbative in the gauge coupling (i.e. to vanish as \( \lambda' \to 0 \)) the \( k_0^0 \) term in the power series must vanish. Therefore we can write \( N_{BB}(k_n^2) = n_{BB} k_n^2 + O(k_n^4) \), which says that the one-loop \( (O(\lambda') \) or \( O(k_n^2) \)) normal-ordering ambiguity is contained in the single constant \( n_{BB} \). Although we are careful to include such additions at this level, it will be shown that these extra normal-ordering constants must vanish to all orders; the standard operator-ordering prescription used to define (74) is correct as it stands.

The matrix elements between bosonic states created by pairs of fermionic creation operators have a remarkably similar form:

\[
\langle J \big| \hat{b}^\dagger_{-n} a^\dagger_{-n} (H_{FF}) \hat{b}^\dagger_n b_n | J \rangle = \left( N_{FF}(k_n^2) - 2n^2 \lambda' \right) \frac{\delta^{\alpha \beta} \delta^{\gamma \delta}}{J} + \frac{n^2 \lambda'}{24 J (1 + n^2 \lambda')} \left[ (\gamma^{ij})^{\alpha \beta} (\gamma^{ij})^{\gamma \delta} + (\gamma^{ij})^{\alpha \beta} (\gamma^{ij})^{\gamma \delta} - (\gamma^{ij})^{\alpha \gamma} (\gamma^{ij})^{\beta \delta} \right]
\]

\[
- \frac{n^2 \lambda'}{24 J (1 + n^2 \lambda')} \left[ (\gamma^{i'j'})^{\alpha \beta} (\gamma^{i'j'})^{\gamma \delta} + (\gamma^{i'j'})^{\alpha \beta} (\gamma^{i'j'})^{\gamma \delta} - (\gamma^{i'j'})^{\alpha \gamma} (\gamma^{i'j'})^{\beta \delta} \right] \approx \left( n_{FF} - 2 \right) \frac{n^2 \lambda'}{J} \delta^{\alpha \beta} \delta^{\gamma \delta} + \frac{n^2 \lambda'}{24 J} \left[ (\gamma^{ij})^{\alpha \beta} (\gamma^{ij})^{\gamma \delta} + (\gamma^{ij})^{\alpha \beta} (\gamma^{ij})^{\gamma \delta} - (\gamma^{ij})^{\alpha \gamma} (\gamma^{ij})^{\beta \delta} \right]
\]

\[
- \frac{n^2 \lambda'}{24 J} \left[ (\gamma^{i'j'})^{\alpha \beta} (\gamma^{i'j'})^{\gamma \delta} + (\gamma^{i'j'})^{\alpha \beta} (\gamma^{i'j'})^{\gamma \delta} - (\gamma^{i'j'})^{\alpha \gamma} (\gamma^{i'j'})^{\beta \delta} \right] + O(\lambda^2) .
\]

The discussion of the normal-ordering function \( N_{FF} \) follows exactly the same lines as the discussion of \( N_{BB} \) in (74). The gamma matrices are \( SO(8) \) generators, lower-case Roman characters are \( SO(8) \) indices with the prime/unprime notation distinguishing \( i \in 1, 2, 3, 4 \) from \( i' \in 5, 6, 7, 8 \). Repeated indices are summed over. Note that the generators \( \gamma^{ij} \) and \( \gamma^{i'j'} \) all act within one or the other \( SO(4) \), and therefore commute with \( \Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \). A careful analysis shows that \( H_{FF} \) is non-zero only for transitions of the types ++ \( \to \) ++ and \(-\) \(\to\) \(-\) (using \pm to denote the \( \Pi \) eigenvalues of the two fermionic mode operators).

The matrix that mixes bi-fermionic bosons with ordinary bosons has the interesting structure

\[
\langle J \big| \hat{b}^\dagger_{-n} a^\dagger_{-n} (H_{BF}) a_n \hat{b}_n | J \rangle = \frac{n^2 \lambda'}{2 J (1 + n^2 \lambda')} \left\{ \sqrt{1 + n^2 \lambda'} \left[ (\gamma^{ab'})^{\alpha \beta} - (\gamma^{a'b})^{\alpha \beta} \right] + \frac{n \sqrt{\lambda'}}{(1 + n^2 \lambda')} \left[ (\gamma^{ab'})^{\alpha \beta} - (\gamma^{a'b})^{\alpha \beta} + \left( \delta^{ab} - \delta^{a'b'} \right) \delta^{\alpha \beta} \right] \right\} \approx \frac{n^2 \lambda'}{2 J} \left[ (\gamma^{ab'})^{\alpha \beta} - (\gamma^{a'b})^{\alpha \beta} \right] + O(\lambda^{3/2}) .
\]
The complex conjugate of this gives the other off-diagonal component of the bosonic block of the perturbation matrix of $H_{int}$. Note that there is no operator-ordering ambiguity in this matrix element and thus no need for an adjustable normal-ordering constant. The appearance of factors of $\sqrt{\lambda'}$ in the matrix elements is alarming, since it could lead to string energies that are not analytic at the origin and which could therefore not be matched to gauge perturbation theory. Fortunately, to the order we have explored, the half-integer powers of $\lambda'$ cancel out of the string energies so that the spectrum is in fact analytic in $\lambda'$. We do not have a general proof of this important property at the moment. For future use, we note that the leading order in $\lambda'$ limit of the matrix element (the last line in (76)) vanishes unless the two bosonic mode operators are in different $SO(4)$’s. The limiting matrix element also vanishes unless the two fermionic mode operators have opposite $\Pi$ values (this is because a $\gamma^{ab'}$ has one gamma matrix from each $SO(4)$ and anticommutes with the matrix $\Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4$).

Finally, we record the matrix elements of $H_{BF}$ between fermionic states created by acting on the string ground state with one bosonic and one fermionic creation operator (the lower block of the perturbation matrix):

\[
\langle J | b^\alpha_n a^A_{-n} (H_{BF}) b^{\beta\dagger}_n a^{B\dagger}_{-n} | J \rangle = N_{BF} (k^2_n) \frac{\delta^{AB} \delta^{\alpha\beta}}{J} + \frac{n^2 \lambda'}{2J(1 + n^2 \lambda')} \left\{ (\gamma^{ab})^{\alpha\beta} - (\gamma^{a'b'})^{\alpha\beta} - (3 + 4n^2 \lambda') \delta^{ab} \delta^{\alpha\beta} - (5 + 4n^2 \lambda') \delta^{a'b'} \delta^{\alpha\beta} \right\} \\
\approx \frac{n^2 \lambda'}{2J} \left\{ (\gamma^{ab})^{\alpha\beta} - (\gamma^{a'b'})^{\alpha\beta} + \left[ (2n_{BF} - 3) \delta^{ab} + (2n_{BF} - 5) \delta^{a'b'} \right] \delta^{\alpha\beta} \right\} + O(\lambda'^2) ,
\]

(77)

\[
\langle J | b^\alpha_n a^A_{-n} (H_{BF}) b^{\beta\dagger}_n a^{B\dagger}_{-n} | J \rangle = -\frac{n^2 \lambda'}{2J \sqrt{1 + n^2 \lambda'}} \left\{ (\gamma^{ab})^{\alpha\beta} - (\gamma^{a'b'})^{\alpha\beta} \right. \\
- \frac{n \lambda'^{1/2}}{\sqrt{1 + n^2 \lambda'}} \left[ (\gamma^{ab})^{\alpha\beta} - (\gamma^{a'b'})^{\alpha\beta} \right] - \delta^{\alpha\beta} \left( \delta^{ab} - \delta^{a'b'} \right) \right\} \\
\approx \frac{n^2 \lambda'}{2J} \left\{ (\gamma^{ab})^{\alpha\beta} - (\gamma^{a'b'})^{\alpha\beta} - \left( \delta^{ab} - \delta^{a'b'} \right) \delta^{\alpha\beta} \right\} + O(\lambda'^{3/2}) .
\]

(78)

The matrix elements have terms that are non-analytic in $\lambda'$, but it once again turns out that the energy eigenvalues are analytic in $\lambda'$ (as must be the case to make contact with perturbative gauge theory).

Equation (77) has its own normal-ordering function, $N_{BF}$, but the structure of the perturbing Hamiltonian implies that it is related to the other normal-ordering functions by $N_{BF} = N_{BB} + N_{FF} \Pi$. It turns out that $N_{BB}$ alone shifts the energies of string states that correspond to the dimensions of operators at supermultiplet levels $L = 0, 8$. For finite $\lambda'$, these levels are shifted by the function $N_{BB}$ itself; in the small-$\lambda'$ expansion they are shifted by some constant coefficient at each order in the series. In the same way, $N_{FF}$ and
$N_{BB}$ provide energy shifts to levels $L = 2, 4, 6$, and $N_{BF}$ controls $L = 1, 3, 5, 7$. In the gauge theory, supersymmetry dictates that the level spacing must be uniform throughout the supermultiplet, i.e. the spectrum of anomalous dimensions is a linear function of $L$ \[^{10}\].

To meet this condition in the string theory, we require $N_{BB} = N_{BF}$. Furthermore, levels $L = 2, 4, 6$ are populated in such a way that $N_{BB}$ must also be equal to $N_{FF}$. Combined with the above constraint $N_{BF} = N_{BB} + N_{FF}$, however, the normal-ordering functions must vanish to all orders in $\lambda'$. We therefore set $N_{BB} = N_{FF} = 0$, which eliminates all normal-ordering ambiguity from the string theory.

An additional observation about all of the above matrix elements is that they vanish for $n = 0$. This means that states made from two zero-mode oscillators receive no interaction corrections (because of the level-matching constraint, these are the only two-impurity states that involve zero modes). As has been argued elsewhere in the paper, non-renormalization of the zero-mode oscillators is the simplest way to understand how the string states manage to reproduce the large degeneracies implicit in the $PSU(2,2|4)$ superconformal symmetry. To put this conjecture to a more stringent test, we would have to look at higher-impurity states, an exercise we will defer to a subsequent paper. The calculations done here just scratch the surface of this subject, but are at least consistent with the larger conjecture.

We now turn to the problem of finding the eigenvalues of $H_{int}$ and comparing the results with gauge theory predictions. Given the functional form of the matrix elements of the string theory perturbing Hamiltonian \[^{74, 78}\], the energy eigenvalues will in general be fairly complicated functions of $\lambda'$. However, since we want to compare them to perturbative gauge theory anomalous dimensions (which are found as power expansions in $\lambda'$), we can simplify the analysis by expanding the string Hamiltonian matrix elements to the appropriate order in $\lambda'$ before diagonalizing. Since it is instructive, we will first do the leading-order calculation of the string spectrum and its comparison with one-loop anomalous dimensions in some detail. We will then quote the results of the calculation of the string spectrum to higher orders in $\lambda'$ and discuss their comparisons with recently-determined higher-loop gauge theory anomalous dimensions \[^{9}\].

If we expand the string Hamiltonian $H = H_{pp} + H_{int}$ to first order in $\lambda'$, we know that the energy eigenvalues will have the general form

$$E(n, J) = 2 + \lambda' n^2 \left( 1 + \frac{\Lambda}{J} + O(J^{-2}) \right), \quad (79)$$

where $\Lambda$ is a dimensionless quantity that distinguishes the different eigenvalues of $H_{int}$. This is to be compared with the generic formula for one-loop anomalous dimensions of gauge theory operators of large $R$-charge \[^{13}\]:

$$\Delta(n, R) = 2 + \frac{g_{YM}^2 N_c}{R^2} n^2 \left( 1 + \frac{\bar{\Lambda}}{R} + O(R^{-2}) \right), \quad (80)$$

where, in this case, $\bar{\Lambda}$ is a dimensionless quantity that depends on the operator multiplet. The AdS/CFT correspondence asserts that, with the identifications $R \simeq J$ and $g_{YM}^2 N_c / R^2 \simeq \lambda'$,
the two expressions should match. This will indeed be true, provided that the two ways of calculating \( \Lambda \) give the same result.

The manifest \( \SO(4)_{\text{AdS}} \times \SO(4)_{S^5} \) transverse space symmetry of the problem can be used to classify eigenvectors and greatly simplify the diagonalization problem in the one-loop limit. We begin with the 128-dimensional block of table 5 that acts on bosonic two-impurity states. Recall that the leading-order-in-\( \lambda' \) matrix elements of the mixing Hamiltonian \( H_{BF} \) vanish on states created by two bosonic oscillators from the same \( \SO(4) \), and also on states created by two fermionic oscillators with the same \( \Pi \) eigenvalue. Thus, \( H_{BB} \) and \( H_{FF} \) may be independently diagonalized (without worrying about boson-fermion mixing) on these two separate 32-dimensional subspaces. \( H_{BF} \) has non-vanishing matrix elements on the orthogonal 64-dimensional subspace spanned by two bosonic creation operators from different \( \SO(4) \)'s and two fermionic creation operators of opposite \( \Pi \) eigenvalue: it poses a separate diagonalization problem which mixes boson-boson with fermion-fermion states.

\[
\begin{array}{|c|c|}
\hline
\SO(4)_{\text{AdS}} \times \SO(4)_{S^5} & \Lambda_{BB} \\
\hline
(1,1;1,1) & -6 \\
(1,1;3,3) & -2 \\
(1,1;3,3) + (1,1;1,3) & -4 \\
(1,1;1,1) & 2 \\
(3,3;1,1) & -2 \\
(3,1;1,1) + (1,3;1,1) & 0 \\
\hline
\end{array}
\]

Table 6: \( O(1/J) \) energy shifts for various bosonic modes

We start with the diagonalization of \( H_{BB} \) on states created by two bosonic mode oscillators in the same \( \SO(4) \approx SU(2) \times SU(2) \). In \( SU(2) \) notation, a four-vector of \( \SO(4) \) is represented as \( (2,2) \). Using this notation, the bosonic modes are in the \( \SO(4) \times \SO(4) \) representations \( (2,2;1,1)+(1,1;2,2) \), and the representation content of the states created by two such oscillators is given by the \( \SO(4) \) formula \( (2,2) \times (2,2) = (3,3) + (3,1) + (1,3) + (1,1) \). \( H_{BB} \) is diagonalized by simply projecting it onto the different invariant subspaces. We find the values for \( \Lambda_{BB} \) shown in table 6. The total number of states in the table is 32. The remaining 32 states of the form \( a_n^+ a_{-n}^\dagger J \) and \( a_{n}^\dagger a_{-n}^+ J \) are subject to bose-fermi mixing and will be dealt with shortly.

We now want to ask whether these energy shifts match gauge theory predictions. Tables 2 and 3 in section 3 list predictions for the one-loop anomalous dimension of various operators in a supermultiplet. Among them are a number of purely bosonic operators, distinguished mainly by their different transformation properties under an \( \SO(4) \) \( R \)-charge. Information provided by the group theory analysis allows us to uniquely match these operators to the string energy eigenstates listed in the previous table, with the result shown in table 6. In the column labeled \( \Delta \), we list the expansion of the anomalous dimension as determined from the general formula (46) for the relevant supermultiplet. In the last column we list the inferred
value of \( \bar{\Lambda} \). The results agree with the corresponding sector of the string spectrum.

Table 8: Energy shifts of states created by two fermionic oscillators of the same \( \Pi \) eigenvalue

Now we turn to the diagonalization of \( H_{FF} \) on the 32-dimensional subspace spanned by states created by two fermionic oscillators of the same \( \Pi \) eigenvalue. To classify the fermions, we use that the \( \Pi = + \) oscillators transform as \((2, 1; 2, 1)\) and the \( \Pi = - \) oscillators transform as \((1, 2; 1, 2)\). The matrix elements \([72]\) only connect ++ states with ++ and --- with --- (and the two submatrices are identical), so the problem of diagonalizing \( H_{FF} \) is further simplified. The results for \( \Lambda_{FF} \), obtained by projecting onto the different invariant subspaces, are shown in table \([8]\). There is another copy of this spectrum in which the roles of the two \( SU(2) \) factors inside each \( SO(4) \) are interchanged. These states can be unambiguously matched to the \( \Delta_0 = 2 \) gauge theory operators built out of pairs of gluino fields. The one-loop anomalous dimensions of some of the operators of this type, along with the predicted values of \( \bar{\Lambda} \) are shown in table \([2]\) (the notation for the operator monomials is rather compressed, but self-explanatory we hope). The match with the corresponding string energy eigenvalues and multiplicities is perfect.

To complete our discussion of the bosonic spectrum, we have to diagonalize \( H_{int} \) on the remaining 64-dimensional subspace of states on which \( H_{BF} \) has non-zero matrix elements. As explained earlier in the discussion following \([76]\), the states on which \( H_{BF} \) acts nontrivially at leading order in \( \lambda' \), are created by two bosonic oscillators taken from different \( SO(4)s \), or by two fermionic oscillators with different \( \Pi \) eigenvalues. For two such fermions, the representation content is \((2, 1; 2, 1) \times (1, 2; 1, 2)\), while for two bosons, it is \((2, 2; 1, 1) \times
Table 9: Two gluino operators corresponding to some of the states in table 8

There are two distinct ways of distributing mode indices on the creation operators, and therefore two realizations of this representation for each of the bosonic and fermionic cases. The eigenvalue problem therefore reduces to that of diagonalizing a $4 \times 4$ numerical matrix. Note that $H_{BB}$ and $H_{FF}$ also have matrix elements between these states. The results of the diagonalization, which are extremely simple, are given in table 10.

Table 10: String eigenstates in the subspace for which $H_{BF}$ has non-zero matrix elements

The eigenvectors are simple linear combinations of the bosonic and fermionic basis states. The corresponding operators and their one-loop dimensions, given in table 11, can be read off from Beisert’s listing of the elements of the gauge theory superconformal multiplet. The ellipses indicate that the operator in question contains further terms involving fermi fields. (See App. B of Beisert [8] for details.) The gauge theory analysis says that there is operator mixing between bosonic and fermionic operators of certain restricted symmetry properties and these are precisely the operators that mix according to the string theory analysis. The gauge theory analysis of boson-fermion mixing is, on the face of it, very complicated, but a simple outcome is legislated by the supersymmetry analysis. This outcome matches perfectly
the first-order perturbation theory of string energy levels.

Carrying the above line of argument through to the end, one can find the first order energy shifts of the 128 spacetime bosons of angular momentum $J$ and count the degeneracies of the shifted levels. The results are given in table 12.

| Level | 0 | 2 | 4 | 6 | 8 |
|-------|---|---|---|---|---|
| Mult. | 1 | 28 | 70 | 28 | 1 |
| $\Lambda_{\text{bose}}$ | $-6$ | $-4$ | $-2$ | $0$ | $2$ |

Table 12: First-order energy shifts of the 128 spacetime bosons of angular momentum $J$

We can also carry out the same exercise for the 128 spacetime fermions. The results for energy shifts and multiplicities are given in table 13. These two energy/multiplicity tables, taken together, perfectly reproduce the table of one-loop gauge theory anomalous dimension predictions summarized in table 4. Although we have not presented the complete analysis, the break-up of the various degenerate submultiplets into $SO(4) \times SO(4)$ irreps matches the gauge theory predictions as well.

Now, as mentioned earlier, a certain amount of higher-order information about dimensions of the relevant operators has become available, and this should allow us to carry out a new and independent set of cross-checks between gauge theory and string theory. To explore this issue, it is useful to have the exact, all-orders in $\lambda'$ expression for the $O(1/J)$ shift in string energies. In other words, we now want to diagonalize the matrix defined by (74-78) without expanding in $\lambda'$. This is slightly more painful than the one-loop diagonalization: when we go beyond leading order in $\lambda'$, the bose-fermi interaction term $H_{BF}$ (76) mixes bosonic indices in both of the $SO(4)$ subgroups and this enlarges the effective size of the matrices which must be diagonalized. Nevertheless, with the help of symbolic manipulation programs, one can get a formula for the string theory energy corrections for all the states in the supermultiplet, exact to all orders in $\lambda'$. The final result is the following concise formula for the energy levels including shifts of $O(1/J)$:

$$E_L(n, J) = 2\sqrt{1 + \lambda'_n^2} - \frac{n^2 \lambda'}{J} \left[ 2 + \frac{(4 - L)}{\sqrt{1 + n^2 \lambda'}} \right] + O(1/J^2) .$$ (81)
For comparison with the gauge theory, we expand in small $\lambda'$:

$$E_L(n, J) \approx \left[ 2 + \lambda' n^2 - \frac{1}{4} (\lambda' n^2)^2 + \frac{1}{8} (\lambda' n^2)^3 + \ldots \right]$$

$$+ \frac{1}{J} \left[ n^2 \lambda' (L - 6) + (n^2 \lambda')^2 \left( \frac{4 - L}{2} \right) + (n^2 \lambda')^3 \left( \frac{3L - 12}{8} \right) + \ldots \right].$$  \quad (82)

The exact energy is organized into degenerate sub-levels $L = 0, \ldots, 8$ with the same multiplicities and $SO(4) \times SO(4)$ content as we found for the one-loop energy shift. The expansion in powers of $\lambda'$ shows that the $\lambda' / J$ term matches, as it should, the one-loop results summarized in tables 12 and 13. The higher-order terms amount to predictions for two-loop and three-loop (and higher-loop, if one wanted) gauge theory anomalous dimensions. Note that the shift is predicted to vanish beyond first order for the $L = 4$ level, the level which includes the bosonic $SU(4)$ irrep that is forbidden to mix with fermionic operators.

Parts of this prediction can in fact be checked: Beisert, Kristjansen and Staudacher [9] have computed the two-loop correction to anomalous dimensions of certain BMN operators in the gauge theory planar limit. The calculation is done for operators at level four in the supermultiplet, for which they find the following expression:

$$\delta \Delta_{n}^R = -\frac{g_{YM}^4 N_c^2}{\pi^4} \frac{n \pi}{R + 1} \left( 1 + \frac{\cos^2 \frac{n \pi}{R + 1}}{\pi^2} \right),$$  \quad (83)

where $R$ is the $R$-charge of the operator. (The authors of [9] use the symbol $J$ instead of $R$.) The discussion leading up to (46) implies that $\mathcal{N} = 4$ supersymmetry allows us to infer the dimensions of all operators at all levels of this supermultiplet by making the substitution $R \rightarrow R + 2 - L/2$ in the formula for the dimension of the $L = 4$ operator ($R$ is the $R$-charge of the operator, whatever the level):

$$\delta \Delta_{n, L}^R = -\frac{g_{YM}^4 N_c^2}{\pi^4} \frac{n \pi}{R + 3 - L/2} \left( 1 + \frac{\cos^2 \frac{n \pi}{R + 3 - L/2}}{\pi^2} \right),$$

$$\approx -\frac{1}{4} (\lambda' n^2)^2 + \frac{1}{2} (\lambda' n^2)^3 \frac{4 - L}{R} + O(1/R^2),$$  \quad (84)

where the expansion is done by taking $R \rightarrow \infty$, keeping $L, n$ fixed. With the standard identification $R = J$, this expression matches the $O(\lambda'^2)$ terms in (82), both at $O(1)$ and at $O(1/J)$. This is an impressive confirmation of both the two-loop gauge theory calculation and the string quantization procedure we have developed.

One would of course like to take this sort of consideration to even higher orders. The authors of [9] have used integrability considerations to conjecture an expression for the planar three-loop version of the two-loop anomalous dimension (83). When we generalize that formula to arbitrary level $L$ and take the large $J$ limit, we get something which almost matches the $O(\lambda'^3)$ terms in the string spectrum (82); the difference is that the factor $3L - 12$ multiplying the $O(\lambda'^3 / J)$ term in the string formula is replaced by a factor $3L - 8$ in the
expansion of the three-loop gauge theory formula. In short, the two expressions differ by a common $O(\lambda'^3/J)$ shift of all the levels in the supermultiplet (while the spacings within the multiplet are the same). One would be tempted to absorb this shift into a normal-ordering constant but, as explained in the discussion following eqn. (77), it appears that any further normal-ordering freedom is eliminated by supersymmetry. The near-agreement between the two $O(\lambda'^3)$ expressions is tantalizing, though, at the moment, we do not see how to make them agree perfectly.

It would be instructive to write down a formula that interpolates between the string theory energy eqn. (81), valid for large $J$ and finite $\lambda' = \tilde{R}^4/J^2$, and the gauge theory anomalous dimension eqns. (46,84), valid for small $\lambda = g^2 Y M N c$ and finite $R$-charge. An additional constraint on the interpolation is that anomalous dimensions are predicted to scale like $\Delta^K_n \sim 2(n^2\lambda)^{1/4}$ at large values of $\lambda$ [2]. A simple formula that connects this $\lambda \gg 1$ limit with the large-$J$ limit of the string theory is

$$(\Delta^K_n)^2 - K^2 = 4\left(1 + \sqrt{K^2 + n^2\lambda}\right) ,$$

where $K = J + 2 - \frac{L}{2}$ is assumed to be large compared to $n$, and $\lambda$ is unrestricted. For large $J$ (at $L = 4$) and fixed $\lambda'$ this gives

$$\Delta^J_n - J \approx 2\sqrt{1 + n^2\lambda'} - \frac{2n^2\lambda'}{J} + \frac{4n^2\lambda'}{J^2}\sqrt{1 + n^2\lambda'} + O(1/J^3) .$$

If the conjecture in eqn. (85) is correct, a string theory calculation to $O(1/J^2)$ should yield the coefficient $4n^2\lambda'\sqrt{1 + n^2\lambda'}$; it may be possible to check this by extending the present calculation to higher orders in perturbation theory. While it is possible to generalize (85) to make contact with the finite-$R$ one and two-loop results in the gauge theory, the results are neither simple nor unique.

### 6 Discussion and Conclusions

The objective of this study was to compute the leading finite-radius curvature correction to the Penrose limit of type IIB string theory in $AdS_5 \times S^5$, and thereby verify that the AdS/CFT correspondence continues to work beyond leading order in this expansion. The next-to-leading order perturbation to the plane-wave geometry induces a complicated non-linear interacting theory on the worldsheet, and carrying out its light-cone quantization is a rather elaborate enterprise. The satisfying end result is that the degeneracy of the BMN spectrum of string states is lifted in a way that precisely reproduces the gauge theory operator dimensions at large but finite $R$-charge and correctly accounts for the extended $\mathcal{N} = 4$ supermultiplet structure dictated by supersymmetry. The success of this explicit quantization of string theory in a curved RR background provides, as a side benefit, rather strong evidence for the correctness of the supercoset manifold construction of the Green-Schwarz superstring action.
In this paper, we have restricted attention to the set of physical string states with two worldsheet impurities, or $\Delta_0 = 2$. It would be a straightforward exercise to extend the string theory analysis to include higher-impurity states in this curved background. In some respects, the corresponding calculation in the gauge theory should not be difficult; the lattice Laplacian techniques outlined in section 3 could be a promising starting point, and one would be able to probe the correspondence on a different level.

In an orthogonal direction, it would be interesting to try to extend this explicit comparison to higher orders in the finite-radius expansion. One would be faced with a much more complicated interacting theory, along with the need to perform sums over physical string states in order to do degenerate perturbation theory beyond leading order. This is certainly a worthwhile problem to attack, especially since the relevant results on the gauge theory side are, by and large, known. While the complexity of the methods we were compelled to use in this paper makes the idea of attempting a brute-force extension to higher orders unappealing, there are several simplifications that could make such a calculation possible. What is ultimately needed is some insight that would enable an exact solution of the worldsheet sigma model, along the lines of the WZW solution of the $SU(2)$ nonlinear sigma model.

It has been suggested in [18] that the complete light-cone gauge world-sheet action for the type IIB superstring in the $AdS_5 \times S^5$ background might be integrable. This is an appealing suggestion that seems very worthwhile to pursue. If such a program were to be successfully carried out, it would represent a major advance. The results presented in this paper could be used as a check by expanding the exact answer in powers of $1/J$ (or $1/R$) about the Penrose limit and comparing the first order correction.

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