An efficient numerical algorithm for solving range-dependent underwater acoustic waveguides based on a direct global matrix of coupled modes and the Chebyshev-Tau spectral method

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ABSTRACT

Sound propagation in a range-dependent ocean environment has long been a matter of widespread concern in ocean acoustics. Stepwise coupled modes is one of the main methods to solve range-dependent acoustic propagation problems. Underwater sound propagation satisfies a Helmholtz equation, the solution of which represents the core of computational ocean acoustics. Due to its high accuracy in solving differential equations, the spectral method has been introduced into computational ocean acoustics in recent years and has achieved remarkable results. However, the existing underwater acoustic propagation algorithms based on the spectral method can calculate only range-independent ocean acoustic waveguides, which excludes applications in more general range-dependent environments. In this paper, a complete and efficient algorithm is designed using an improved global matrix of coupled modes to solve the range dependence of the ocean environment and using the Chebyshev-Tau spectral method to precisely solve the eigenmodes in a stepped range-independent stair. Based on this algorithm, a complete and efficient numerical program is developed, and the numerical simulation results verify that this algorithm is extremely computationally fast and accurate for various range dependence and seabed environments.
Keywords: Chebyshev-Tau spectral method; coupled modes; range-dependent; underwater acoustic propagation; computational ocean acoustics.

I. INTRODUCTION

Sound propagation in a range-dependent marine environment has always been a popular research topic in underwater acoustics. At present, the theories to solve range-dependent acoustic propagation problems include coupled normal modes, adiabatic normal modes, and the wide-angle parabolic approximation$^{1-3}$. The classic normal mode theory proposed by Pekeris$^4$ is only suitable to solve range-independent acoustic waveguides and is powerless for range-dependent problems. The theory of coupled normal modes proposed by Pierce$^5$ and Milder$^6$ in 1965 and 1967, respectively, asserts that energy is exchanged between normal modes in a horizontally changing waveguide. However, Rutherford et al$^7$ noted that Pierce and Milder’s use of vertical derivative operators to replace normal derivative operators caused the nonconservation of energy in sloping terrain; consequently, they proposed a first-order modification to coupled mode theory to maintain the first-order conservation of energy on the slope. In 1983, Evans$^8$ proposed the idea of using a stair-step geometry to discretize sloping terra, where each step was considered as a range-independent segment. In combination with boundary conditions, the propagator matrix between the coupling coefficients of each segment can be obtained, and the coupling coefficients of the segments can be obtained by considering the radiation conditions. The acoustic field solution of each segment contains both the forward scattering mode, which exponentially decays with the range, and the backward scattering mode, which exponentially grows with the range. Moreover,
when considering leaky modes, the traditional superposition method suffers from numerical instability. Hence, in 1985, Mattheij proposed a decoupling matrix algorithm to solve the two-point boundary value problem, which could separate the exponential growth component from the exponential decay component. Soon after, Evans applied this decoupling algorithm to stepwise coupled modes, successfully resolved the numerical instability caused by leaky modes, and developed the computational program COUPLE. The latest version, COUPLE07, can accurately calculate the full elliptic two-way solution of Helmholtz equation.

Nevertheless, Luo et al. and Yang et al. reported that COUPLE exhibited numerical instability due to unreasonable normalized range solutions and proposed an accurate and stable direct global matrix of coupled normal mode model (DGMCM). DGMCM adopts a reasonable normalized range solution, which eliminates the numerical overflow problem that may occur in previous two-way models. The model uses a global matrix to simultaneously obtain the coupling coefficients of all segments, and this global matrix is unconditionally stable. The work of this article will be improved based on this method, so it will be introduced in more detail later.

In solving the range-independent normal modes, COUPLE employs the Galerkin method; however, finite difference method such as Kraken are traditionally used. In recent years, many studies have begun to solve the acoustic propagation model by applying more accurate spectral methods. In 1993, Dzieciuch first used the Chebyshev-Tau spectral method to solve for the normal modes of the water column. In 2016, Evans devised a Legendre-Galerkin spectral method to solve the problem of acoustic propagation in a two-layer marine
environment that contained bottom sediment. In 2020, Tu et al.\textsuperscript{19,20} used the Chebyshev-Tau spectral method to more efficiently solve this problem. Numerical experiments show that the NM-CT program\textsuperscript{28} based on the Chebyshev-Tau spectral method is faster than the rimLG program\textsuperscript{27} based on Legendre-Galerkin spectral method and more accurate than the classic finite difference method\textsuperscript{18}. Recently, Sabatini et al.\textsuperscript{29} and Tu et al.\textsuperscript{25,30} used Chebyshev collocation method and Legendre collocation method, respectively, to solve the problem of acoustic propagation in an ocean environment with any number of layers.

Existing studies have shown that spectral methods solve underwater acoustic propagation problems with high accuracy. However, the current programs\textsuperscript{26–28,30} based on spectral methods can solve for only the range-independent acoustic waveguides. Therefore, this article combines the global matrix of coupled modes with the Chebyshev-Tau spectral method to develop a new algorithm that can efficiently solve for range-dependent acoustical waveguides.

II. PHYSICAL MODEL

A. Range-independent normal modes

We consider a two-dimensional point source acoustic field in a cylindrical axisymmetric environment, where the angular frequency of the acoustic source is $\omega$, and the simple harmonic point source is located at $r = 0, z = z_s$. Let the acoustic pressure be $p = p(r, z)$ and omit the time factor $\exp(i\omega t)$. The acoustic governing equation (Helmholtz equation) can be written as\textsuperscript{1}:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \rho(z) \frac{\partial}{\partial z} \left( \frac{1}{\rho(z)} \frac{\partial p}{\partial z} \right) + \frac{\omega^2}{c^2(z)} p = -\frac{\delta(r)\delta(z - z_s)}{2\pi r}$$

(1)
where \( \omega = 2\pi f \), \( f \) is the frequency of the sound source, and \( c(z) \) and \( \rho(z) \) are the sound speed and density profiles, respectively.

Using the technique of the separation of variables\(^4\), the acoustic pressure can be decomposed into:

\[
p(r, z) = \psi(z)R(r)
\]

(2)

where \( R(r) \) is related only to the range \( r \) and satisfies:

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dR(r)}{dr} \right) + k^2_r R(r) = -\frac{\delta(r)\psi(z_s)}{2\pi r \rho(z_s)}
\]

(3)

where \( k_r \) is the horizontal wavenumber. We solve the above formula:

\[
R(r) = \frac{i}{4\rho(z_s)} \psi(z_s) \mathcal{H}_0^{(1)}(k_r r)
\]

(4)

where \( \mathcal{H}_0^{(1)}(\cdot) \) is the Hankel function of the first type. \( \psi(z) \) in Eq. (2) satisfies the following modal equation:

\[
\rho(z) \frac{d}{dz} \left( \frac{1}{\rho(z)} \frac{d\psi(z)}{dz} \right) + k^2 \psi(z) = k^2_r \psi(z), \quad k = (1 + i\eta\alpha)\omega/c(z)
\]

(5)

where \( k \) is the complex wavenumber, \( \alpha \) is the attenuation coefficient in dB/\( \lambda \) (\( \lambda \) is the wavelength), and \( \eta = (40\pi \log_{10} e)^{-1} \). This is the essential equation to be solved in this paper.

When supplemented by boundary conditions, Eq. (5) has a set of solutions \((k_{r,m}, \psi_m), m = 1, 2, \ldots \) where \( \psi_m \) is also called the eigenmode. The eigenmodes of Eq. (5) satisfy the orthogonal normalization:

\[
\int_0^H \frac{\psi_m(z)\psi_n(z)}{\rho(z)} dz = \delta_{mn}, \quad m, n = 1, 2, \ldots
\]

(6)
where $H$ is the depth of the ocean and $\delta$ is Kronecker delta function. Finally, the fundamental solution of Helmholtz equation can be written as:

$$p(r, z) = \frac{i}{4\rho(z_s)} \sum_{m=1}^{\infty} \psi_m(z_s) \psi_m(z) \mathcal{H}_0^{(1)}(kr,mr) \tag{7}$$

To accurately obtain the sound pressure, it is necessary to synthesize an infinite number of eigenmodes, which is obviously impossible in actual calculations. It is usually more practical to take $M$ physically meaningful eigenmodes to synthesize the sound field.

For marine environments containing sediment, $\rho(z)$ and $k(z)$ are usually discontinuous at the interface ($z = h$) between water column and bottom sediment. Considering an intermittent environment, the ocean is divided into two discontinuous layers. The environmental parameters are separately defined in the water column and bottom sediment as:

$$c(z) = \begin{cases} 
  c_w(z), & 0 \leq z \leq h \\
  c_b(z), & h \leq z \leq H \\
  c_\infty, & z \geq H
\end{cases} \quad \tag{8a}$$

$$\rho(z) = \begin{cases} 
  \rho_w(z), & 0 \leq z \leq h \\
  \rho_b(z), & h \leq z \leq H \\
  \rho_\infty, & z \geq H
\end{cases} \quad \tag{8b}$$

$$\alpha(z) = \begin{cases} 
  \alpha_w(z), & 0 \leq z \leq h \\
  \alpha_b(z), & h \leq z \leq H \\
  \alpha_\infty, & z \geq H
\end{cases} \quad \tag{8c}$$
Boundary conditions should be imposed at the sea surface \((z = 0)\) and seabed \((z = H)\), and interface conditions should be imposed at the discontinuous surface \((z = h)\). Taking the pressure release boundary condition as an example, the upper boundary condition is:

\[
\psi(0) = 0 \quad (9)
\]

The bottom boundary is either perfectly free or rigid:

\[
\psi(H) = 0 \quad (10a)
\]
\[
\psi'(H) = 0 \quad (10b)
\]

In addition, the use of an acoustic halfspace is quite common in underwater acoustic modeling:

\[
\psi(H) + \frac{\rho_\infty}{\rho_b(H)\gamma_\infty} \psi'(H) = 0, \quad \gamma_\infty = \sqrt{k_r^2 - \omega^2 c_\infty^2} \quad (11)
\]

At the interface, both acoustic pressure and normal particle velocity must be continuous. Thus, two constraints on continuity are explicitly imposed by:

\[
\psi(h^-) = \psi(h^+) \quad (12)
\]
\[
\frac{1}{\rho(h^-)} \frac{d\psi(h^-)}{dz} = \frac{1}{\rho(h^+)} \frac{d\psi(h^+)}{dz} \quad (13)
\]

where the superscripts \(-\) and \(+\) of \(h\) indicate the limits from above and below, respectively.

**B. Improved global matrix of coupled normal modes**

For range-dependent marine environments, the classic solution is to divide the terrain into many sufficiently narrow segments, e.g., to resemble stair steps, as in Figure 1; we treat each segment as range-independent. After the eigenmodes and horizontal wavenumbers of
each segment are obtained, the segment conditions of \( J \) segments are used to couple the subfields of each segment to obtain the acoustic field of the entire waveguide.

\[
p_j(r, z) \approx M \sum_{m=1}^{M} \left[ a^j_m H^1_m(r) + b^j_m H^2_m(r) \right] \psi^j_m(z), \quad j = 1, 2, \ldots, J
\]

where \( M \) is the total number of normal modes to synthesize the acoustic field, \( \psi^j_m(z) \) is the \( m \)-th eigenmode of the \( j \)-th segment, and \( \{a^j_m\}_{m=1}^M \) and \( \{b^j_m\}_{m=1}^M \) are the coupling coefficients, which denote the amplitudes of the forward and backward propagation modes in the \( j \)-th segment, respectively. \( H^1_m(r) \) and \( H^2_m(r) \) are the normalized Hankel functions of the first
and second types, respectively, and are defined as follows:

\[
H^1_{jm}(r) = \frac{\mathcal{H}^{(1)}_0(k^j_{r,m} r)}{\mathcal{H}^{(1)}_0(k^j_{r,m} r_{j-1})} \approx \sqrt{\frac{r_{j-1}}{r}} e^{ik^j_{r,m}(r-r_{j-1})} \quad (15)
\]

\[
H^2_{jm}(r) = \frac{\mathcal{H}^{(2)}_0(k^j_{r,m} r)}{\mathcal{H}^{(2)}_0(k^j_{r,m} r_{j-1})} \approx \sqrt{\frac{r_{j-1}}{r}} e^{-ik^j_{r,m}(r-r_{j})} \quad (16)
\]

where \(k^j_{r,m}\) is the horizontal wavenumber of the \(m\)-th mode in the \(j\)-th segment. For special cases when \(j = 1\), we let \(r_{j-1} = r_1\). The definition of \(H^1_{jm}(r)\) here is identical to that in COUPLE\textsuperscript{11}, but the definition of \(H^2_{jm}(r)\) is different from COUPLE. In COUPLE, \(H^2_{jm}(r)\) is defined as:

\[
H^2_{jm}(r) = \frac{\mathcal{H}^{(2)}_0(k^j_{r,m} r)}{\mathcal{H}^{(2)}_0(k^j_{r,m} r_{j-1})} \approx \sqrt{\frac{r_{j-1}}{r}} e^{-ik^j_{r,m}(r-r_{j-1})} \quad (17)
\]

This improved definition (16) was presented by Luo\textsuperscript{12–15} and Yang\textsuperscript{16,17}. The existence of leaky modes may cause the value of \(H^2_{jm}(r)\) defined in COUPLE to overflow. Specifically, for leaky mode \(k^j_{r,m} = \mathcal{R} + \mathcal{I}i\), where \(\mathcal{I} > 0\), \(\mathcal{R}\) and \(\mathcal{I}\) are the real and imaginary parts of \(k^j_{r,m}\), respectively; in Eq. (17), if \(r - r_{j-1} > 0\), then Eq. (17) contains \(\exp[\mathcal{I}(r - r_{j})]\). When \(\mathcal{I}\) is large, using Eq. (17) may cause numerical instability. In contrast, in Eq. (16), since the exponential part contains \(\exp[\mathcal{I}(r - r_{j})]\) at time \((r - r_{j} < 0)\), regardless of the value of \(\mathcal{I}\) at this time, the value of \(H^2_{jm}(r)\) is limited, and no numerical overflow phenomenon occurs. In other words, in this improved global matrix of coupled modes, the left boundary is used to normalize the forward acoustic field, and the right boundary is used to normalize the backward acoustic field, which ensures the numerical stability of the calculation.

The method of coupling segments explicitly imposes two segment continuity conditions on the sides of the segments. The first segment condition is that the acoustic pressure must
be continuous at the $j$-th side:

$$p^{j+1}(r_j, z) = p^j(r_j, z)$$

(18)

The second segment condition is that the radial velocity at acoustic pressure is continuous at the $j$-th side:

$$\frac{1}{\rho_{j+1}(z)} \frac{\partial p^{j+1}(r_j, z)}{\partial r} = \frac{1}{\rho_j(z)} \frac{\partial p^j(r_j, z)}{\partial r}$$

(19)

For the first segment condition, we have:

$$\sum_{m=1}^M \left[ a_m^{j+1} H_1^{j+1}(r_j) + b_m^{j+1} H_2^{j+1}(r_j) \right] \psi^{j+1}_m(z) = \sum_{m=1}^M \left[ a_m^j H_1^j(r_j) + b_m^j H_2^j(r_j) \right] \psi^j_m(z)$$

(20)

where $H_1^{j+1}(r_j) = H_2^j(r_j) = 1$. We apply the following operator to both sides of the above equation:

$$\int_0^H \frac{\psi^{j+1}_\ell(z)}{\rho_{j+1}(z)} \, dz$$

Then, we use the orthogonal normalization relationship Eq. (6) of the eigenmodes in the $(j+1)$-th segment. Accordingly, Eq. (20) is equivalent to:

$$a^{j+1}_\ell + b^{j+1}_\ell H_2^{j+1}(r_j) = \sum_{m=1}^M \left[ a_m^j H_1^j(r_j) + b_m^j \right] \tilde{c}_\ell m, \quad \ell = 1, \ldots, M$$

(21)

where

$$\tilde{c}_\ell m = \int_0^H \frac{\psi^{j+1}_\ell(z) \psi^j_m(z)}{\rho_{j+1}(z)} \, dz$$

The above formula can be easily written in the following matrix-vector form:

$$a^{j+1} + H_2^{j+1} b^{j+1} = \tilde{C}^j (H_1^j a^j + b^j)$$

(22)
To similarly address the segment condition (19), we first write the derivative expression of \( p \) with respect to \( r \), which can be known from Eq. (14):

\[
\frac{1}{\rho_j} \frac{\partial p^j(r,z)}{\partial r} \simeq \frac{1}{\rho_j} \sum_{m=1}^{M} k_{r,m}^j \left[ a_m^j H^1_m(r) - b_m^j H^2_m(r) \right] \psi_m^j(z) \tag{23}
\]

Then, the second segment condition is equivalent to:

\[
\frac{1}{\rho_{j+1}} \sum_{m=1}^{M} k_{r,m}^{j+1} \left[ a_m^{j+1} - b_m^{j+1} H^2_m^{j+1}(r_j) \right] \psi_m^{j+1}(z) = \frac{1}{\rho_j} \sum_{m=1}^{M} k_{r,m}^j \left[ a_m^j H^1_m(r_j) - b_m^j \right] \psi_m^j(z) \tag{24}
\]

Likewise, we apply the following operator to the above equation:

\[
\int_0^H (\cdot) \psi_{\ell}^{j+1}(z) dz
\]

Next, we utilize the orthogonal normalization relationship Eq. (6) of the eigenmodes in the \((j+1)\)-th segment to obtain:

\[
a_{\ell}^{j+1} - b_{\ell}^{j+1} H^2_{\ell}^{j+1} = \sum_{m=1}^{M} \left[ a_m^j H^1_m(r_j) - b_m^j (r_j) \right] \hat{c}_{tm}, \quad \ell = 1, \ldots, M \tag{25}
\]

where

\[
\hat{c}_{tm} = \frac{k_{r,m}^j}{k_{r,\ell}^{j+1}} \int \frac{\psi_{\ell}^{j+1}(z) \psi_m^j(z)}{\rho_j(z)} dz
\]

The above formula can be easily written in the following matrix-vector form:

\[
a^{j+1} - H^{j+1} b^{j+1} = \hat{C}^j \left( H^1_j a^j - b^j \right) \tag{26}
\]

Eqs. (22) and (26) can be combined into the following form:

\[
\begin{bmatrix}
a^{j+1} \\
b^{j+1}
\end{bmatrix} =
\begin{bmatrix}
R^j_1 & R^j_2 \\
R^j_3 & R^j_4
\end{bmatrix}
\begin{bmatrix}
a^j \\
b^j
\end{bmatrix} \tag{27}
\]
where

\[ R_j^1 = \frac{1}{2} \left( \tilde{C}^j + \hat{C}^j \right) H_j^1 \]  

(28a)

\[ R_j^2 = \frac{1}{2} \left( \tilde{C}^j - \hat{C}^j \right) \]  

(28b)

\[ R_j^3 = \frac{1}{2} \left( H_{2j}^{j+1} \right)^{-1} \left( \tilde{C}^j - \hat{C}^j \right) H_j^1 \]  

(28c)

\[ R_j^4 = \frac{1}{2} \left( H_{2j}^{j+1} \right)^{-1} \left( \tilde{C}^j + \hat{C}^j \right) \]  

(28d)

Finally, the segment condition and radiation condition should be imposed at the acoustic source \( r = 0 \) and \( r \to \infty \). The segment condition at the acoustic source \( r = 0 \) is:

\[ a_{m} = i \frac{\rho_1(z_s)}{4} \psi_{m}^{(1)}(z_s) H_{0}^{(1)}(k_{r,m}^{1} r_1) + b_{m} \frac{\rho_1(z_s)}{4} \psi_{m}^{(2)}(z_s) H_{0}^{(2)}(k_{r,m}^{1} r_1), \quad m = 1, \ldots, M \]  

(29)

This condition can be written in a matrix-vector form:

\[ \mathbf{a}^1 - D \mathbf{b}^1 = \mathbf{s} \]  

(30a)

\[ D_{mm} = \frac{H_{0}^{(1)}(k_{r,m}^{1} r_1)}{H_{0}^{(2)}(k_{r,m}^{1} r_1)} \], \quad s_{m} = i \frac{\rho_1(z_s)}{4} \psi_{m}^{(1)}(z_s) H_{0}^{(1)}(k_{r,m}^{1} r_1) \]  

(30b)

For the radiation condition at \( r \to \infty \), \( \mathbf{b}^{J} = 0 \) is sufficient.

Combining the continuity conditions at the boundaries of the \( J \) segments with the boundary condition at the acoustic source and the radiation condition at infinity, the following
system of linear algebraic equations is obtained:

\[
\begin{bmatrix}
I & -D & 0 \\
R_1^1 & R_2^1 & -I & 0 \\
R_3^1 & R_4^1 & 0 & -I \\
\vdots & \ddots & \ddots & \ddots \\
R_1^{J-2} & R_2^{J-2} & -I & 0 \\
R_3^{J-2} & R_4^{J-2} & 0 & -I \\
R_1^{J-1} & R_2^{J-1} & -I & 0 \\
R_3^{J-1} & R_4^{J-1} & 0 & \vdots
\end{bmatrix}
\begin{bmatrix}
a^1 \\
b^1 \\
a^2 \\
\vdots \\
b^{J-2} \\
b^{J-1} \\
a^J \\
0
\end{bmatrix}
= 
\begin{bmatrix}
s \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

This system of linear algebraic equations can be solved to obtain the coupling coefficients \(\{a^j\}_{j=1}^J, \{b^j\}_{j=1}^J\); then Eq. (14) is used to synthesize the acoustic pressure field.

Since \(r_{j-1} = r_1\) when \(j = 1\) is defined above, in the first segment, \(H_{1m}^j(r)\) is normalized to the right side. For leaky modes, when \(\exp[I(r_1 - r)]\) is large, calculating \(H_{1m}^j(r)\) causes numerical instability. To avoid this problem, the superposition principle is used to solve the acoustic field. Substituting \(a^1\) in Eq. (30) into Eq. (14) reveals the following:

\[
p^1(r, z) \approx \frac{i}{4\rho(z_s)} \sum_{m=1}^M \psi_m^1(z_s) \psi_m^1(z) H_0^{(1)}(k_{r,m}^1 r) + 2 \sum_{m=1}^M b_m^1 \frac{J_0(k_{r,m}^1 r)}{H_0^{(2)}(k_{r,m}^1 r_1)} \psi_m^1(z)
\]

where the first term on the right side represents the range-independent acoustic field, the second term represents the scattered acoustic field caused by range dependency, and \(J_0(\cdot)\) is the Bessel function.\(^{12}\)
III. METHODOLOGY AND ALGORITHM

A. Chebyshev-Tau spectral method

The classic spectral method is the Galerkin-type spectral method, which is derived from the Galerkin method of weighted residual method. A special feature of the Galerkin-type spectral method is that the basis/weight functions are selected as the same set of orthogonal polynomials. Since the classic Galerkin-type spectral method requires the basis function to satisfy the boundary conditions (generally a linear combination of orthogonal polynomials of a certain kind), it is not easy to apply to problems with complex boundary conditions. To resolve this problem, Lanczos proposed the Tau method in 1938\textsuperscript{31}. The Tau method also uses the same set of orthogonal polynomials as the basis/weight functions but does not require the basis function to satisfy the boundary conditions and only imposes boundary constraints on the coefficients of the spectral expansion. In other words, force coefficients are used to satisfy the boundary conditions in spectral space. The Chebyshev-Tau spectral method is a type of spectral method that uses Chebyshev polynomials as the basis/weight functions. In our previous research\textsuperscript{19,20}, we concisely introduced the Chebyshev-Tau spectral method and its application to range-independent normal modes, and we developed the related NM-CT program, which is included in the open-source code and available at Ocean Acoustics Library (OALIB)\textsuperscript{28}. Similarly, here, we employ the Chebyshev-Tau spectral method to solve the horizontal wavenumbers and eigenmodes of modal equation (Eq. (5)) in the range-independent segments and refine the method.
When using the Chebyshev-Tau spectral method to solve the modal equation, the modal equation must be scaled to the domain of Chebyshev polynomials $T_i(x)$:

$$
\frac{4}{|\Delta h|^2} \rho(x) \frac{d}{dx} \left( \frac{1}{\rho(x)} \frac{d\psi(x)}{dx} \right) + k^2 \psi(x) = k_r^2 \psi(x), \quad x \in [-1, 1]
$$

(33)

Moreover, the modal function must be transformed into the spectral space formed by Chebyshev orthogonal polynomial $T_i(x)$:

$$
\psi(x) \approx \sum_{i=0}^{N} \hat{\psi}_i T_i(x)
$$

(34)

where $\{\hat{\psi}_i\}_{i=0}^N$ is the spectral expansion coefficients of $\psi(x)$. Due to the good properties of Chebyshev polynomial basis function, the following relations are easily derived:

$$
\hat{\psi}'_i \approx \frac{2}{c_i} \sum_{j=i+1, \ j+i=\text{odd}}^N j \hat{\psi}_j, \quad c_0 = 2, c_{i>1} = 1 \iff \hat{\psi}' \approx D_N \hat{\psi}
$$

(35a)

$$
\left(\nabla \psi\right)_i \approx \frac{1}{2} \sum_{m+n=i}^N \hat{\psi}_m \hat{\psi}_n + \frac{1}{2} \sum_{|m-n|=i}^N \hat{\psi}_m \hat{\psi}_n \iff \left(\nabla \psi\right) \approx C_N \hat{\psi}
$$

(35b)

Eq. (35a) gives the relationship between the spectral expansion coefficients of a function and those of its derivative function. Likewise, Eq. (35b) describes the relationship between the spectral expansion coefficients of a product of two functions and the spectral expansion coefficients of one of the functions; the right-hand side is the matrix-vector representation of this relationship.

When using the Chebyshev-Tau method to solve differential equations, we should start from the variational form of differential equations, which is:

$$
\int_{-1}^{1} \left[ \frac{4}{|\Delta h|^2} \rho(x) \frac{d}{dx} \left( \frac{1}{\rho(x)} \frac{d\psi(x)}{dx} \right) + k^2 \psi(x) - k_r^2 \psi(x) \right] \frac{T_i(x)}{\sqrt{1-x^2}} \, dx = 0
$$

$$
x \in (-1, 1), \quad i = 0, 1, \ldots, N - 2
$$

(36)
By substituting Eq. (34) into Eq. (36) and considering Eq. (35), the modal equation can be directly discretized into the following matrix form:

\[
\left( \frac{4}{|\Delta h|^2} C_p D_N C_{1/\rho} D_N + C_{k^2} \right) \hat{\Psi} = k_r^2 \hat{\Psi}
\]  

(37)

From a formal viewpoint, this is an ordinary matrix eigenvalue problem, and boundary constraints must be added to the actual solution. For details regarding the discretization process, please see Eq. (29) in reference\(^\text{20}\).

For the ocean acoustic waveguide in Eqs. (8) through (13), the modal equation (Eq. (5)) must be established in both water column and bottom sediment. As shown in Figure 1, in a range-independent segment, a single set of basis functions cannot span two layers since the interface \( h \) is not continuously differentiable. Thus, we use the domain decomposition strategy\(^\text{32}\) in Eq. (5) and split the domain interval into two subintervals. For every splitting event, the discontinuous point is the endpoint of one subinterval:

\[
\psi(z) = \begin{cases} 
\psi_w(z) = \psi_w(x) \approx \sum_{i=0}^{N_w} \hat{\psi}_{w,i} T_i(x_w), & x_w = -\frac{2}{h} z + 1, \quad 0 \leq z \leq h \\
\psi_b(z) = \psi_b(x) \approx \sum_{i=0}^{N_b} \hat{\psi}_{b,i} T_i(x_b), & x_b = -\frac{2}{H-h} z + \frac{H+h}{H-h}, \quad h \leq z \leq H 
\end{cases}
\]

(38)

where \( N_w \) and \( N_b \) are the spectral truncated orders in the water column and bottom sediment, respectively; \( \{\hat{\psi}_{w,i}\}_{i=0}^{N_w} \) and \( \{\hat{\psi}_{b,i}\}_{i=0}^{N_b} \) are the modal spectral expansion coefficients in these two layers. Similar to Eq. (37), the modal equations in the water column and bottom sediment can be directly discretized into the matrix-vector form:

\[
A \hat{\Psi}_w = k_r^2 \hat{\Psi}_w, \quad A = \frac{4}{k^2} C_{\rho_w} D_{N_w} C_{1/\rho_w} D_{N_w} + C_{k^2_w}
\]

(39a)

\[
B \hat{\Psi}_b = k_r^2 \hat{\Psi}_b, \quad B = \frac{4}{(H-h)^2} C_{\rho_b} D_{N_b} C_{1/\rho_b} D_{N_b} + C_{k^2_b}
\]

(39b)
where $A$ and $B$ are square matrices of order $(N_w + 1)$ and $(N_b + 1)$, respectively, and $\hat{\Psi}_w$ and $\hat{\Psi}_b$ are column vectors composed of $\{\hat{\psi}_{w,i}\}^{N_w}_{i=0}$ and $\{\hat{\psi}_{b,i}\}^{N_b}_{i=0}$, respectively. Since the interface conditions are related to both water column and bottom sediment, Eqs. (39a) and (39b) should be simultaneously solved as follows:

$$
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\begin{bmatrix}
\hat{\Psi}_w \\
\hat{\Psi}_b
\end{bmatrix} = k_r^2
\begin{bmatrix}
\hat{\Psi}_w \\
\hat{\Psi}_b
\end{bmatrix}
$$

(40)

The boundary conditions and interface conditions in Eqs. (9)–(13) must also be expanded to the Chebyshev spectral space and expressed as row vectors. Let the $(N_w + N_b + 2)$-order square matrix on the left side of Eq. (40) be $L$, and replace the $N_w$-th, $(N_w + 1)$-th, $(N_w + N_b + 1)$-th and $(N_w + N_b + 2)$-th rows of the $L$ matrix with the boundary conditions in the spectral space. By rearranging the modified rows together by elementary row transformation, Eq. (40) can be rewritten into the form of the following block matrix:

$$
\begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{\Psi}_1 \\
\hat{\Psi}_2
\end{bmatrix} = k_r^2
\begin{bmatrix}
\hat{\Psi}_1 \\
0
\end{bmatrix}
$$

(41)

where $L_{11}$ is a square matrix of order $(N_w + N_b - 2)$, and $L_{22}$ is a square matrix of order 4. Solving this mixed linear algebraic system can yield the horizontal wavenumbers and spectral expansion coefficients of the eigenmodes $(k_r, \hat{\Psi}^j_w, \hat{\Psi}^j_b)$. For details on the treatment of the boundary conditions in Eqs. (9), (10), (12) and (13), please see Eq. (38) in reference$^{20}$. We must emphasize that for the acoustic halfspace boundary condition in Eq. (11), since $\gamma_\infty$ contains the eigenvalue $k_r$ to be solved, Eq. (41) is no longer a general matrix eigenvalue problem and can be solved iteratively only by a root-finding algorithm. The biggest shortcoming of root-finding algorithms is that they must make a reasonable initial guess about
eigenvalue $k_r$ being sought\textsuperscript{29}. Since the prior estimate of $k_r$ is usually not available, many of the existing numerical programs following similar principles fail to converge to a specific root in some cases. To avoid the same problem when using the Chebyshev-Tau spectral method to solve the waveguides with an acoustic halfspace boundary condition, we consider a cleverer approach here: we use $k_{z,\infty} = \sqrt{k_{\infty}^2 - k_r^2}$ to transform the modal equation and Eq. (11) as follows:

$$
\rho(z) \frac{d}{dz} \left( \frac{1}{\rho(z)} \frac{d\psi}{dz} \right) + \left( k^2(z) - k_{\infty}^2 + k_{z,\infty}^2 \right) \psi = 0 \quad (42a)
$$

$$
\left. \frac{i\rho_{\infty}}{\rho_b(H)} \frac{d\psi(z)}{dz} \right|_{z=H} + k_{z,\infty} \psi(H) = 0 \quad (42b)
$$

where $k_{\infty} = \omega/c_{\infty}$ is a constant. Eq. (42a) can naturally be discretized into the following form:

$$
[U + k_{z,\infty} I] \hat{\Psi} = 0, \quad U = L - k_{\infty}^2 I \quad (43)
$$

However, due to the addition of Eq. (42b) including $k_{z,\infty}$, Eq. (43) finally takes the following form:

$$
[U + k_{z,\infty} V + k_{z,\infty}^2 W] \hat{\Psi} = 0 \quad (44)
$$

The $U$ term in Eq. (44) is not exactly identical to that in Eq. (43), as it has been modified by boundary conditions and interface conditions; nevertheless, we denote it as $U$. $V$ is a zero matrix of order $(N_w + N_b + 2)$ with only the last row corresponding to the boundary condition in Eq. (42b), and $W$ is simply the identity matrix that has changed by modifying the boundary conditions. This polynomial eigenvalue problem can be efficiently solved by the $QZ$ algorithm; alternatively, it can be transformed into a general matrix eigenvalue
problem using the following formula, although the scale of the matrices is doubled.

\[
\tilde{U} \tilde{\Psi} = k_{z,\infty}^2 \tilde{V} \tilde{\Psi},
\]

\[
\tilde{U} = \begin{bmatrix}
-V & -U \\
0 & 0 \\
I & 0
\end{bmatrix}, \quad
\tilde{V} = \begin{bmatrix}
W & 0 \\
0 & I
\end{bmatrix}, \quad
\tilde{\Psi} = \begin{bmatrix}
k_{z,\infty} \hat{\Psi}
\end{bmatrix}
\]

(45a)

(45b)

It is necessary to take the inverse transform of the eigenvectors \( \hat{\Psi}_w \) and \( \hat{\Psi}_b \) to \([0, h]\) and \([h, H]\). The vectors \( \Psi_w \) and \( \Psi_b \) are stacked into a single column vector to form \( \Psi \); then, Eq. (6) is used to normalize \( \Psi \); finally, a set of modes \((k_r, \Psi)\) is obtained. We must emphasize that the spectrally expanded coefficients of the eigenmodes obtained from the \( J \) range-independent segments must be transformed using the same resolution in the vertical direction. Otherwise, \( \hat{c}_{\ell m} \) in Eq. (21) and \( \hat{c}_{\ell m} \) in Eq. (25) cannot be calculated.

B. Numerical algorithm

Summarizing the above derivation, we provide a complete description of the algorithm below:

1. Set up environmental data.

The data include the frequency \( f \) and depth \( z_s \) of the sound source, total depth of the ocean \( H \), topography of the seabed, number of acoustic profiles, and specific information of each group of acoustic profiles. In addition, the data should include the spectral truncated order \( (N_w, N_b) \), horizontal and vertical resolutions, the phase velocity range to screen the modes, and type of bottom boundary condition. If the
bottom boundary is the upper interface of an acoustic halfspace, the speed \( c_\infty \), density \( \rho_\infty \) and attenuation \( \alpha_\infty \) of the halfspace must also be specified.

2. Segment the marine environment based on the seabed topography and sound speed profiles.

Stair-step discretization criteria have been established to accurately represent smoothly varying bathymetry in numerical models. Jensen\(^{33}\) showed that the strictest segmentation criterion was \( \Delta r \leq \lambda/4 \), where \( \lambda = \min\{c_w, c_b\}/f \). Thus, we suppose that the entire waveguide is divided into \( J \) segments.

3. Apply the Chebyshev-Tau spectral method to form the mixed linear systems and solve for the horizontal wavenumbers and eigenmodes \((k_j^r, \Psi_j)\) of \( J \) range-independent segments.

The modal spectral expansion coefficients \( \hat{\Psi}_j \) obtained for \( J \) segments should be transformed to a uniform vertical resolution. This process can be computed in parallel because the range-independent segments are irrelevant.

4. According to the phase velocity range, select the appropriate modes and denote the number of modes as \( M \).

If the number of modes that satisfy the phase velocity range in the \( J \) segments are not equal, the number of modes \( M \) is set to the maximum value, and segments with fewer than \( M \) modes are allowed to retain a small number of evanescent modes to make all segments have equal number of modes.
5. Calculate the coupling submatrices \( \{ R^j_1 \}_{j=1}^{J-1}, \{ R^j_2 \}_{j=1}^{J-1}, \{ R^j_3 \}_{j=1}^{J-1}, \text{ and } \{ R^j_4 \}_{j=1}^{J-1} \) according to Eqs. (15), (16), (21), (25) and (28). This step is also naturally parallel.

6. Calculate \( D \) and \( s \) in the boundary conditions, construct the global matrix according to Eq. (31), and solve Eq. (31) to obtain the coupling coefficients \( \{ a^j \}_{j=1}^{J}, \{ b^j \}_{j=1}^{J} \) of \( J \) segments. The global matrix is a band matrix of order \((2J - 1) \times M\), and its bandwidth is \((3M - 1)\). The inverse of a band matrix can be efficiently solved using mature numerical algorithms and libraries.

7. Calculate the synthetic sound field.

   The sound pressure field of each segment is calculated according to Eq. (14), and the sound pressure field of the first segment is corrected according to Eq. (32). The sound fields of \( J \) segments are individually embedded into the entire waveguide to obtain the final global sound pressure field.

IV. NUMERICAL SIMULATION

To validate the performance of the numerical algorithm in solving range-dependent waveguides, the following tests and analyses are performed through five experiments. For comparison, we take the widely used Kraken program based on the finite difference discretization\(^1\), COUPLE program based on Galerkin method\(^1\), and RAM program\(^3\) based on the parabolic approximation as comparison. In this article, the developed program based on the above numerical algorithm is named SPEC. The above codes are implemented in FORTARN language.
To present the acoustic field results, the transmission loss (TL) of the acoustic pressure is defined as
\[
\text{TL} = -20 \log_{10} \left( \frac{|p|}{|p_0|} \right)
\]
in units of decibels (dB), where \( p_0 = \exp(ik_0)/(4\pi) \) is the acoustic pressure at a range of 1 m from the point source. In actual displays, TL fields are often used to compare and analyze the sound fields.

### A. Special case: Range-independent waveguide

Example 1 involves a range-independent waveguide as a simple special case that the above numerical algorithm can handle. This special case is also included in case 1 of the COUPLE manual. The specific configuration of this example is depicted in Figure 2. Figure 3 shows the sound fields calculated by COUPLE, KRAKEN, and SPEC and the TL curves at a depth of 90 m. All three programs use 28 modes. For COUPLE and SPEC, we segment the field at \( r = 1000 \) m and set the acoustic parameters and configuration to be identical in both segments. The results of the three programs are very consistent; in the shaded area, the results of COUPLE and SPEC almost overlap.

**FIG. 2.** Schematic diagram of a range-independent waveguide.
FIG. 3. Sound fields of example 1 and the TL curves at a depth of 90 m calculated by COUPLE, KRAKEN and SPEC.

B. Simple sloping terrain

Sloping terrain is one of the most common and classic range dependencies in underwater acoustic propagation. The specific configuration of this example is displayed in Figure 4. We applied COUPLE, RAM and SPEC to calculate the sound fields of this example and set the receiver at a depth of 36 m. The COUPLE and SPEC programs use 5 modes, and both take the truncated order of 10 and 225 segments. The horizontal and vertical resolutions used by RAM are 2 m and 0.2 m, respectively. Overall, the sound fields calculated by the
three programs are highly consistent. Beyond a range of 1500 m (in the far field), the sound fields of SPEC and RAM remain consistent, while the result of COUPLE is slightly less reliable.

![Diagram of the simple sloping terrain problem](image)

**FIG. 4.** Schematic diagram of the simple sloping terrain problem.

**C. Seamount waveguide**

The topography of a seamount represents a typical range-dependent marine environment. This example considers a seamount configuration, as shown in Figure 6. Figure 7 illustrates the sound fields calculated by COUPLE, RAM and SPEC and the TL curves at a depth of 200 m. The COUPLE and SPEC programs use 8 modes, both of which take the same truncated order of 16 and 126 segments. The horizontal and vertical resolutions used by RAM are 2 m and 0.2 m, respectively. From the perspective of the entire sound field, the results calculated by the three programs are very similar, with only slight differences after
FIG. 5. Sound fields of example 2 and the TL curves at a depth of 36 m calculated by COUPLE, RAM and SPEC.

crossing the seamount. This conclusion can also be drawn from the TL curves at a depth of 200 m.

D. Plane-parallel waveguide

Range dependence can occur due to both bathymetric variations (seamounts and continental slopes) and variations in material properties (oceanographic features such as fronts and eddies or changes in the bottom type)\(^1\). In this example, we consider a plane-parallel waveguide, and the actual sound speed profiles are shown in Figure 8. The plane-parallel
waveguide problem was originally developed by Jensen et al\textsuperscript{34} and Thomson et al\textsuperscript{35}. The depth of the waveguide is $H = 500$ m, and the sound speed $c(r, z)$ varies in both range and depth as follows:

\[
\frac{c_0^2}{c^2(r, z)} = 1 + \frac{\pi^2 f_1^2}{H^2} \exp \left( -\frac{2\pi r}{H} \right) + \frac{4\pi^2 f_2^2}{H^2} \exp \left( -\frac{4\pi r}{H} \right) - \frac{2\pi f_1}{H} \left[ 1 - \frac{2\pi f_2}{H} \exp \left( -\frac{2\pi r}{H} \right) \right] \cos \left( \frac{\pi z}{H} \right) \exp \left( -\frac{\pi r}{H} \right) - \frac{4\pi f_2}{H} \cos \left( \frac{2\pi z}{H} \right) \exp \left( -\frac{2\pi r}{H} \right)
\]

where $c_0 = 1500$ m/s, $f_1/H = 0.032$, and $f_2/H = 0.016$. The source frequency is $f = 25$ Hz, the density of the water is $\rho = 1$ g/cm$^3$, the attenuation coefficient is $\alpha = 0$ dB/$\lambda$, and the maximum range to calculate the sound field is 4 km. This example constitutes a benchmark problem presented by the Acoustic Society of America (ASA), and Desanto applied conformal mapping theory to give its analytical solution\textsuperscript{35}. Figure 9 shows the sound
FIG. 7. Sound fields of example 3 and TL curves at a depth of 200 m calculated by COUPLE, RAM and SPEC.

field of the plane-parallel waveguide calculated by Desanto’s analytical solution, COUPLE and SPEC. Since the three programs can specify only discrete sound speed profiles in the environmental file, we adopt 851 discrete sound speed profiles in the horizontal direction to simulate continuously varying sound speed profiles. We take sound speed profiles at intervals of 1 m from 0 to 500 m in the horizontal direction and at intervals of 10 m from 500 m to 4000 m. SPEC uses 16 modes in total, and COUPLE uses 30 modes in total, both of which take the truncated order of 30. The seafloor is designated as a perfectly rigid boundary. Since COUPLE cannot directly handle a perfectly rigid bottom boundary, the rigid bottom
is simulated as a homogeneous bottom with a sound speed of $10^7 \text{ m/s}$ and a density of $10^5 \text{ g/cm}^3$, and an artificial bottom boundary is introduced at a depth of 1000 m. As shown in Figure 9, the sound fields calculated by the three programs are approximately identical, especially those of SPEC and the analytical solution, and good consistency is observed in both near field and far field.

E. Warm-core eddy

Eddies are common hydrological phenomena in the ocean that change the temperature and salinity of seawater, which change in the acoustic properties of the ocean. Hence, the propagation of sound through an eddy is different from that through seawater without an eddy. Here, we consider the warm-core eddy in Figure 10, which is a classic example
FIG. 9. Sound fields of example 4 and the TL curves at a depth of 200 m calculated by the analytical solution, COUPLE and SPEC.

for range-dependent waveguides, as mentioned by Jensen et al\textsuperscript{1} and Porter\textsuperscript{18}. An acoustic halfspace is adopted below a depth of 5100 m.

Figure 11 plots the sound fields through the warm-core eddy calculated using KRAKEN and SPEC. The number of discrete points used by KRAKEN is automatically selected by the program, while the spectral truncated orders used by SPEC in the water column and bottom sediment are 300 and 30, respectively. The sound fields calculated by SPEC and KRAKEN are approximately identical, although there are subtle differences that may arise
mainly because the range dependence in KRAKEN is handled by the theory of one-way coupled modes.

These numerical simulations strongly confirm the correctness of the proposed algorithm and its implementation in this article and fully demonstrate that SPEC can handle the three types of seabed conditions with ease.
Table I shows the running times of these examples. The tests were performed on a Dell XPS8930 personal computer, and each program was run ten times. The running time listed in the table is the average result. The compiler used is gfortran 7.5.0; all programs used for comparison are also compiled with this compiler. For the same experiments, when using identical numbers of segments, SPEC has a much shorter running time than COUPLE, which directly demonstrates efficiency of the proposed algorithm.

**TABLE I.** Comparison among the running times of the numerical examples (unit: seconds).

| No. | SPEC  | COUPLE | KRAKEN | RAM  |
|-----|-------|--------|--------|------|
| 1   | 0.158 | 17.996 | 0.327  | /    |
| 2   | 1.198 | 4.985  | /      | 0.358|
| 3   | 0.814 | 6.724  | /      | 3.963|
| 4   | 21.541| 767.888| /      | 2.545|
| 5   | 99.718| /      | 0.988  | /    |

V. ANALYSIS AND PARALLELIZATION

A. Analysis

From a computational cost perspective, the bulk of the calculations performed by this algorithm is divided into two parts: one part is to solve $J$ matrix eigenvalue problems for the horizontal wavenumbers and eigenmodes in the range-independent segments (see Eqs. (41)
and (45)); the other is to solve the global matrix of linear equations (see Eq. (31)). The amount of calculations in the first part depends on the number of segments $J$ when solving for the eigenvalues and eigenvectors of $J$ square matrices of order $(N_w + N_b + 2)$. The amount of calculations in the second part depends on the number of segments $J$ and the number of modes $M$ to couple when solving $(2J-1) \times M$-order banded sparse linear equations. In other words, the main computational load of the algorithm is concentrated in the third and sixth steps. The test results in Table II also support this analysis. Likewise, the computational load of the COUPLE program is concentrated in these two steps. COUPLE uses the Galerkin method to solve the modal information of the range-independent segments, which forms in each segment a generalized eigenvalue problem of symmetric matrices $A$ and $B^{11}$:

$$Au = \lambda Bu$$

(47a)

$$A_{n,m} = \int_0^H \frac{k^2(z)\phi_n(z)\phi_m(z)}{\rho(z)}dz - \int_0^H \frac{1}{\rho(z)} \frac{d\phi_n(z)}{dz} \frac{d\phi_m(z)}{dz}dz$$

(47b)

$$B_{n,m} = \int_0^H \frac{\phi_n(z)\phi_m(z)}{\rho(z)}dz$$

(47c)
where $\phi(x)$ is the basis/weight functions in Galerkin method. Although both matrices $A$ and $B$ are symmetrical in form, the generalized eigenvalue problem is efficient to solve, but the elements in matrices $A$ and $B$ must be individually obtained through numerical integration, which requires many calculations. In addition, Galerkin method must construct basis functions that satisfy the boundary conditions in each segment, which imposes an additional computational cost compared to the Chebyshev-Tau spectral method. Some previous studies have compared the computation time of the Galerkin-type spectral method (the Galerkin-type spectral method is a special case when the basis/weight functions of the Galerkin method take a certain type of orthogonal polynomial and are completely consistent with the principles of the Galerkin method) with those of the Chebyshev-Tau spectral method and spectral collocation method in solving the range-independent normal modes. The conclusions of these comparisons further illustrate our finding $^{20,25}$.

In terms of solving the coupling coefficients, COUPLE uses the propagator matrix of Eq. (27) to recursively perform the solution. This method requires solving $(J-1) (2M \times 2M)$-order dense matrix linear equations, and there are many matrix transformation and matrix multiplication operations; this is another aspect of COUPLE that makes it more time-consuming than SPEC. In addition, a recursive solution obviously gradually accumulates a certain numerical error.

B. Parallelization

The third and fifth steps of the algorithm are naturally parallel. Therefore, we adopt the idea of multithreaded parallel acceleration and use OpenMP to accelerate the algorithm.
Table III shows the effect of multithreaded acceleration. Generally, when 2–4 threads are used, SPEC can achieve a speedup of 2. This considerable acceleration effect further reflects the advantages of SPEC in the simulation of large-scale underwater acoustic propagation problems, which is particularly salient because multicore processors have become immensely popular. Thus, it is not expensive to purchase hardware with 4–8 threads for personal computers.

### Table III. Acceleration effect of SPEC using OpenMP multithreaded parallel computing technology (unit: seconds; the number in brackets is the speedup based on the running time of a single thread).

| No. | serial  | 1  | 2  | 4  | 8  |
|-----|---------|----|----|----|----|
| 2   | 1.198   | 1.344 (1) | 0.876 (1.53) | 0.682 (1.97) | 0.595 (2.26) |
| 3   | 0.819   | 0.846 (1) | 0.608 (1.39) | 0.490 (1.73) | 0.403 (2.1)  |
| 4   | 21.541  | 22.209 (1) | 15.441 (1.44) | 12.403 (1.79) | 9.921 (2.24) |
| 5   | 99.718  | 102.306 (1) | 57.178 (1.79) | 37.067 (2.75) | 31.058 (3.29) |

### VI. DISCUSSION

In general, the main contributions and highlights of the devised SPEC program developed based on the algorithm devised are as follows:
1. An improved global matrix of coupled modes is completely designed as a robust program for computational ocean acoustics. The improved global matrix of coupled modes is unconditionally stable in normalizing backward acoustic field; thus, SPEC does not suffer the problem of numerical overflow. Simultaneously, since the global matrix can calculate the coupling coefficients of all segments, SPEC does not accumulate numerical errors in traditional algorithms that employ a propagator matrix for recursive solutions.

2. The improved global matrix of coupled modes is completely designed as an efficient software suite for computational ocean acoustics. The global matrix formed during the calculation exhibits good sparsity and a banded shape, so SPEC can efficiently solve for the coupling coefficients. In addition, due to the existence of natural parallelism, SPEC can be easily run in parallel and achieve superior acceleration effects.

3. A robust, accurate and capable normal mode solver based on the Chebyshev-Tau spectral method is developed and applied to SPEC. Accordingly, SPEC can accurately process perfectly free and rigid seafloor boundaries instead of setting $\rho \to 0$ and $\rho \to \infty$ in the input file to simulate a perfectly free and rigid seabed.

4. SPEC can accurately solve waveguides whose seabed is a halfspace boundary without using an iterative root-finding algorithm. Therefore, SPEC does not exhibit the iterative divergence problem caused by poor initial guesses in root-finding algorithms. At present, almost no mature normal mode programs have this advantages.
In terms of accuracy, SPEC is mainly controlled by the spectral truncated orders ($N_w$ and $N_b$) and the number of horizontal segments ($J$). The former determines the accuracy of modal information in the segments, and the latter determines the accuracy of coupling coefficients. Finally, in the third step of the algorithm, the eigenmodes of all segments must be transformed to the same vertical resolution. Thus, the Tau-type spectral method is used to solve for the eigenvalues and eigenmodes instead of the more convenient spectral collocation method\textsuperscript{25}. The eigenmodes obtained by the spectral collocation method are the values at the collocation points, and the eigenmodes of $J$ segments must be interpolated to the same vertical resolution. However, the error caused by this interpolation is carried over into the solution of the coupling coefficients, which distort the results.

VII. SUMMARY

In this article, we proposed a new numerical algorithm for range-dependent waveguides in ocean acoustics. An improved global matrix of coupled normal modes is used to solve the range dependence of the ocean environment, and the Chebyshev-Tau spectral method is used to solve the normal modes in stepwise range-independent segments. Numerical simulations involving various range dependencies in deep and shallow ocean environments verified that our devised algorithm is reliable, practical, and efficient for range-dependent waveguides. Due to the natural parallelism of the main steps of the algorithm, we also leverage parallel computing technology to further accelerate the algorithm. At present, the algorithm is both comprehensive and efficient.
Due to the inherent problems to normal modes, this algorithm still has a larger computational cost than both parabolic approximation and ray model in ocean acoustics. Therefore, for high-frequency deep-sea long-range waveguides, it is valuable to further extend and optimize the SPEC program.

ACKNOWLEDGMENTS

The authors are very grateful to Michael B. Porter for providing the warm-core eddy sound speed profile data and his comments on the normal modes.

This work was supported by the National Natural Science Foundation of China [grant numbers 61972406]; the National Key Research and Development Program of China [grant number 2016YFC1401800].

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