Symmetries at stationary Killing horizons

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Abstract

It has often been suggested (especially by Carlip) that spacetime symmetries in the neighborhood of a black hole horizon may be relevant to a statistical understanding of the Bekenstein–Hawking entropy. A prime candidate for this type of symmetry is that which is exhibited by the Einstein tensor. More precisely, it is now known that this tensor takes on a strongly constrained (block-diagonal) form as it approaches any stationary, non-extremal Killing horizon. Presently, exploiting the geometrical properties of such horizons, we provide a particularly elegant argument that substantiates this highly symmetric form for the Einstein tensor. It is, however, duly noted that, on account of a “loophole”, the argument does fall just short of attaining the status of a rigorous proof.

I. THE MOTIVATION

No one seriously disputes the notion of black holes as thermodynamic entities; nevertheless, the Bekenstein–Hawking entropy [1,2] remains as enigmatic as ever from a statistical viewpoint. The “company line” has been, more often than not, to hope that quantum gravity will eventually provide the resolution; but this could require, cynically speaking, a rather long wait. A more pragmatic expectation might be to hope for a statistical explanation that interpolates between the quantum-gravitational and semi-classical regimes, and that is not particularly sensitive to the fundamental micro-constituents. Such a perspective appears to be in compliance with the general stance of S. Carlip — who has long advocated for horizon boundary conditions as a means of altering the physical content of the theory, thereby inducing new degrees freedom that can account for the black hole entropy [3].

One might then query as to what physical principle determines the correct choice of boundary conditions. On this point, Carlip has stressed the importance of asymptotic symmetries \(^1\) in the neighborhood of the horizon [4]. Ideally, these symmetries should be based on semi-classical concepts that can be enhanced into a quantum environment.

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\(^1\)Asymptotic in the sense that such symmetries need only be exact at the horizon itself.
and that are strong enough to physically constrain the theory so that it describes only black holes. Indeed, the merit of this philosophy has been substantiated by the various statistical calculations of Carlip [5,6] and others (e.g., [7–14]). Remarkably, these studies have successfully replicated the Bekenstein–Hawking entropy (although sometimes up to a constant factor) while following, more or less, along the stated lines. ²

One can find a viable candidate for Carlip’s notion of an asymptotic symmetry by looking directly at the near-horizon geometry of a stationary (but otherwise arbitrary) black hole spacetime. To elaborate, the current author, M. Visser and D. Martin have demonstrated a highly constrained and symmetric form for the Einstein tensor near the horizon of any stationary (non-extremal) Killing horizon. Irregardless of whether the horizon is static [17] or rotating (but still stationary) [18], we have shown that the on-horizon Einstein tensor block-diagonalizes into transverse (⊥) and parallel (∥) blocks. ³ It was also demonstrated that the transverse block must be directly proportional to the (induced) transverse metric. ⁴

To better appreciate these findings (especially in the context of Carlip’s ideas), let us make a couple of pertinent comments. Firstly, the block-diagonalization indicates that, at the horizon, the “r–t”-plane dissociates from the in-horizon coordinates, which is a necessary ingredient in most statistical entropy calculations [4]. Secondly, given the Einstein field equations, it immediately follows that the on-horizon stress tensor is subject to an identical set of constraints. As a consequence of the form of the transverse block, one finds that (as the horizon is approached) the transverse pressure goes as the negative of the energy density. This, in turn, suggests that the matter near a horizon can be effectively described as a two-dimensional conformal field theory (see [17,18] for an explanation); a theory that necessarily underlies any entropy calculation that calls upon the Cardy formula [20] (as most happen to do [4]).

A brief discussion on the methodology used in [17,18] is also in order: After establishing a suitable coordinate system, we wrote out each of the metric components as a Taylor expansion with respect to a Gaussian normal coordinate (measuring distance from the horizon). We then constrained the expansion coefficients by enforcing regularity on geometric invariants as the horizon is approached. Once all possible constraints have been exhausted, it is straightforward to calculate the Einstein tensor to any order in the perturbative expansion.

Although technically sound, our previously used method does have — arguably — one possible shortcoming; namely, the static case and the rotating (but stationary) case have to be treated separately. By this, it is meant that, before proceeding with the calculation, one must first specify a particular coordinate system; with such a choice being unavoidably different for the two stated cases. Meanwhile, it is a common school of thought that any

²Interestingly, the seminal calculation of this nature [15] was based rather on asymptotic symmetries at spatial infinity [16].

³Orientations are always defined relative to an arbitrary spacelike cross-section of the horizon. Also note that a four-dimensional (black hole) spacetime is assumed throughout.

⁴This form had actually been known for quite some time, but only under the stringent conditions of a static and spherically symmetric horizon geometry [19].
profound statements about gravity should be expressible (at least in principle) in a background or coordinate independent framework [21]. With this philosophy in mind, it has been our contention that there should be some geometrical argument which would confirm these on-horizon symmetries, but without resorting to case or coordinate specifics. In fact, the beginnings of such an argument were laid out in [18], although we were unable to proceed past an intuitive level at that time. In the current paper we are, however, able to alleviate this situation and establish the validity of the argument — up to a “loophole”, which is elaborated upon in the final section.

The rest of this paper proceeds as follows. In the next section, we clarify our objective while introducing important notation. The main calculations in support of the argument are then presented in Section III. Finally, there is a summary and some further discussion in the concluding section.

II. SOME PRELIMINARIES

To reiterate, it is our intention to show that, for any stationary (non-extremal) black hole spacetime, the following will occur at the horizon: the Einstein tensor takes on a block-diagonal form (separating into transverse and parallel blocks) and the transverse components of this tensor are directly proportional to the (induced) transverse metric. This is to be accomplished by a purely geometrical argument that, unlike the prior method [17,18], does not have to be reformulated in going from the static to the rotating case (or vice versa).

Our argument will essentially depend on only three inputs: (i) the well-established properties of a stationary and non-extremal Killing horizon, 5 (ii) time-reversal symmetry, and (iii) the conservation of the Einstein tensor. We will also “assume” a bifurcate Killing horizon; that is, one that contains a bifurcation surface or spacelike 2-surface where the Killing vector vanishes. However, thanks to the work of Racz and Wald [24], the presence of such a bifurcation surface can be viewed as a consequence of the first input rather than an additional assumption. To elaborate, these authors demonstrated that, if the surface gravity is constant and non-vanishing over any spacelike cross-section of a Killing horizon, there will exist a stationary extension of the spacetime that does include a well-defined bifurcation surface. Which is to say, even a physically relevant black hole (such as one formed by stellar collapse) will asymptotically approach this type of spacetime.

Before discussing the Einstein tensor, we will first require a suitable basis for the on-horizon coordinate system. 6 For this purpose, let us consider an arbitrarily chosen spacelike section of the Killing horizon. We will use \([m_1]^a\) and \([m_2]^a\) to denote any orthonormal pair

5See [22] for a text-book discussion on Killing horizons. It should be noted, in particular, that stationarity ensures that the horizon surface gravity is a constant [23] (i.e., the “zeroth law” of black hole mechanics), while non-extremality necessitates that this constant is non-vanishing (essentially, the “third law”).

6This step may appear to be contradictory with our earlier comment about coordinate independence; however, see Section IV for why this is not really so.
of spacelike vectors that are tangent to this 2-surface. Then a convenient choice of basis turns out to be [18] the orthonormal set \( \{ \chi^a, N^a, [m_1]^a, [m_2]^a \} \); where \( \chi^a \) denotes the Killing vector (which is null on the horizon), and \( N^a \) represents a vector that is both null on the horizon and orthogonal to \( \chi^a \). Note that the vector \( N^a \) is guaranteed to exist [22], although it can only be fixed up to a constant factor. Here, this constant will be specified (without loss of generality) by the normalization condition \( \chi^a N_a = -1 \). Further note that — except for \([m_1]^a [m_1]_a = 1\) and \([m_2]^a [m_2]_a = 1\) — any other contraction of basis vectors must vanish on the horizon.

Since the prescribed set forms, by construction, an orthonormal basis for the tangent space, we can express the on-horizon Einstein tensor (indeed, any symmetric on-horizon tensor) in the following pedantic manner:

\[
G_{ab} = G_{++} \chi_a \chi_b + G_{--} N_a N_b + G_{+-} \{ \chi_a N_b + N_a \chi_b \}
+ G_{+1} \{ \chi_a [m_1]_b + [m_1]_a \chi_b \} + G_{+2} \{ \chi_a [m_2]_b + [m_2]_a \chi_b \}
+ G_{-1} \{ N_a [m_1]_b + [m_1]_a N_b \} + G_{-2} \{ N_a [m_2]_b + [m_2]_a N_b \}
+ G_{11} [m_1]_a [m_1]_b + G_{22} [m_2]_a [m_2]_b
+ G_{12} \{ [m_1]_a [m_2]_b + [m_2]_a [m_1]_b \} ,
\]

where the coefficients are, at this point, arbitrary functions of the spacetime geometry. 7 By an inspection of equation (7.1.15) from Wald’s textbook [22], it is not difficult to verify that \( G^b_a \chi_b \propto \chi_a \) must be true on the horizon [18]. (Meaning that the on-horizon Einstein tensor possesses a null eigenvector.) With knowledge of this fact (along with the orthogonality properties of the basis set), we can first contract equation (1) with \( \chi_b \) and then set the coefficient to zero of any term which is not proportional to \( \chi_a \). By way of this procedure, we have \( G_{--} = G_{-1} = G_{-2} = 0 \), so that the on-horizon form of the Einstein tensor reduces to

\[
G_{ab} = H_{ab} + J_{ab} ,
\]

where

\[
H_{ab} \equiv G_{+-} \{ \chi_a N_b + N_a \chi_b \}
+ G_{11} [m_1]_a [m_1]_b + G_{22} [m_2]_a [m_2]_b
+ G_{12} \{ [m_1]_a [m_2]_b + [m_2]_a [m_1]_b \}
\]

and

\[
J_{ab} \equiv G_{++} \chi_a \chi_b
+ G_{+1} \{ \chi_a [m_1]_b + [m_1]_a \chi_b \} + G_{+2} \{ \chi_a [m_2]_b + [m_2]_a \chi_b \} .
\]

We have separated \( G_{ab} \) into two distinct tensors so as to readily distinguish between the part that survives on the bifurcation surface — where \( \chi^a \rightarrow 0 \) — and the remainder. That

7However, even at this stage, we do know the following since \( \chi^a \) is a Killing vector: any of these coefficients vanishes when operated on by \( \chi^a \nabla_a \).

8Note, however, that \( \chi_a N_b \) has a well-defined limit on the bifurcation surface, even though \( \chi^a \rightarrow 0 \). This is because the other null normal limits as \( N^a \rightarrow \infty \), as a consequence of the normalization condition \( \chi^a N_a = -1 \).
is, by definition, $J_{ab} = 0$ and $G_{ab} = H_{ab}$ on the bifurcation surface. It is a further point of interest that $H_{ab}$ can be also written in the following compact form:

$$H_{ab} = G_{+} \{g_{\perp}\}_{ab} + [H_{\parallel}]_{ab},$$

(5)

where

$$[g_{\perp}]_{ab} = \chi_a N_b + N_a \chi_b$$

(6)

is the transverse part of the induced metric \[22\] and we have defined the following “in-horizon” tensor:

$$[H_{\parallel}]_{ab} \equiv G_{11} \{m_1\}_{a}[m_1]_{b} + G_{22} \{m_2\}_{a}[m_2]_{b} + G_{12} \{\{m_1\}_{a}[m_2]_{b} + [m_2]_{a}[m_1]_{b}\}.$$  
(7)

It is clear from the above formulation that $H_{ab}$ is a block-diagonal tensor for which the transverse components are proportional to the transverse metric. Hence, it will be sufficient, for our purposes, to show that $J_{ab} = 0$ everywhere on the horizon. We will now set out to argue that this is indeed the case.

III. THE MAIN ARGUMENT

Let us begin this section by focusing on the tensor $H_{ab}$; that is, the bifurcation-surface form of the Einstein tensor. Our initial objective is to demonstrate that, when propagated away from the bifurcation surface but along the horizon, this tensor does not change in relation to the metric (and, hence, in relation to the spacetime geometry). Such a propagation is clearly a Killing translation, and so it is significant that $H_{ab}$ is a Killing invariant or $L^{\chi} H_{ab} = 0$ (which follows from each individual constituent of $H_{ab}$ being a Killing invariant \[9\]). Consequently, this tensor will be formally unaltered under any such translation along the horizon (as, incidentally, so will $J_{ab}$). It is then sufficient for our purposes to establish the on-horizon validity of $\chi^c \nabla_c H_{ab} = 0$. The point being that, if this is indeed correct, $H_{ab}$ will be parallel transported as it moves away from the bifurcation surface; meaning that its relationship with the metric (which is certainly parallel transported) will remain intact.

First of all, let us consider the transverse part of $H_{ab}$ or $G_{+} \{g_{\perp}\}_{ab}$. This is the product of a scalar coefficient and a block-diagonal part of the “on-horizon metric” (i.e., the projection of the spacetime metric onto the horizon). As long as we are staying on the horizon — which is implied by the Killing translation — this induced metric (and, hence, any block-diagonal constituent thereof) will also be invariant under the action of $\chi^a \nabla_a$. Therefore, on the horizon,

$$\chi^c \nabla_c \{G_{+} \{g_{\perp}\}_{ab}\} = 0.$$  
(8)

We are now left with the task of verifying the on-horizon validity of $\chi^c \nabla_c [H_{\parallel}]_{ab} = 0$. To demonstrate this result, let us consider, in turn, the following two possibilities for the state

\[9\]Note that $L^{\chi}$ represents a Lie differentiation with respect to the Killing vector. See \[22\] for some pertinent background.
of a stationary black hole spacetime: either the black hole is static or it is axially symmetric (and possibly rotating) [22]. The essential point here is that, if a rotating black hole is embedded in a spacetime that is not axially symmetric, tidal forces will act to both slow down the rotation and smooth out the asymmetry. Hence, the spacetime will continue to evolve until such time as either staticity or axial symmetry has been achieved.

Firstly, if the black hole is static, there must be a timelike Killing vector (say, $[\partial_t]^a$) for the spacetime. Moreover, the on-horizon action of the operator $\chi^a \nabla_a$ is equivalent to that of $[\partial_t]^a \nabla_a$ and, therefore, $\chi^c \nabla_c [H_{\parallel}]^{ab} = \nabla_t [H_{\parallel}]^{ab}$. Since there can be no timelike components for the in-horizon 2-metric (say, $[g_{\parallel}]_{ab}$) in the static case, one can readily verify that this covariant derivative reduces to

$$\nabla_t [H_{\parallel}]^{ab} = \partial_t [H_{\parallel}]^{ab} + \frac{1}{2} [H_{\parallel}]^{ac}[g_{\parallel}]^{db} \partial_t [g_{\parallel}]_{cd} + \frac{1}{2} [H_{\parallel}]^{db}[g_{\parallel}]^{ad} \partial_t [g_{\parallel}]_{cd},$$

which is, of course, trivially vanishing.

Secondly, if the black hole is axially symmetric, then the axis of rotation picks out a particular spacelike direction (say, $\phi$), which is significant for the following reason: Given our "input" of time-reversal symmetry, it follows that every physically relevant quantity should be invariant under the simultaneous change of $t \rightarrow -t$ and $\phi \rightarrow -\phi$. In view of this fact and $[\partial_\phi]^a$ being a Killing vector [22], if we choose (without loss of generality) to orientate $[m_1]$ in the $\phi$ direction, there can be no off-diagonal elements in $[H_{\parallel}]^{ab}$. By similar reasoning, the in-horizon 2-metric ($[g_{\parallel}]^{ab}$) will also have no off-diagonal elements. And so we can, after diagonalizing both tensors, arrive at the form

$$[H_{\parallel}]_{ab} = \tilde{G}_{11} [g_{m_1}]_{ab} + \tilde{G}_{22} [g_{m_2}]_{ab},$$

where the tilde on a coefficient signifies a suitable redefinition and the metric "blocks" have been labeled accordingly. Since each term in $[H_{\parallel}]_{ab}$ is the product of a scalar and a block-diagonal constituent of the metric, it can be deduced [recalling the discussion leading up to equation (8)] that

$$\chi^c \nabla_c [H_{\parallel}]^{ab} = 0$$

is, once again, true on the horizon. (Note that, even though a distinction was made, the end result is a coordinate-independent statement.)

Let us now bring the conservation of the (on-horizon) Einstein tensor into the game. More specifically, we will use

\[1\] For the sceptical reader, there is another way of seeing the very same thing. Let us begin with $\mathcal{L}_\chi [m_1]^a [m_2]^b = [m_1]^a \chi^c \nabla_c [m_1]^b + [m_1]^b \chi^c \nabla_c [m_1]^a - [m_1]^a [m_1]^c \nabla_c \chi^b - [m_1]^b [m_1]^c \nabla_c \chi^a = 0$. Next, we will contract this Lie derivative with the (diagonalized) on-horizon metric or $g_{ab} = [g_{\parallel}]_{ab} + g_{11} [m_1]_a [m_1]^b + g_{22} [m_2]_a [m_2]^b$. This process yields $2g_{11} [m_1]_a \chi^b \nabla_b [m_1]^a - 2g_{11} [m_1]_a [m_1]^b \nabla_b \chi^a = 0$. The antisymmetry property $\nabla_a \chi_b = -\nabla_b \chi_a$ [22] means that the second term vanishes, which then implies that $\chi^c \nabla_c [m_1]_a [m_1]^a = g_{ab} \chi^c \nabla_c [m_1]^a [m_1]^b = 0$. This outcome (and the $[m_2]$ analogue) and the diagonality of $[H_{\parallel}]^{ab}$ is sufficient to reproduce equation (11). Moreover, the same argument substantiates equation (8), since the transverse metric can always be recast in a diagonal form.
\[ \nabla_a G^{ab} = \nabla_a H^{ab} + \nabla_a J^{ab} \to 0 \; ; \tag{12} \]

where the arrow signifies that the resulting derivatives have been suitably “pulled back” to the horizon. As implied by this last remark, we must now deal with an important caveat: In general, the action of these covariant derivatives will move the tensors off of the horizon (albeit, infinitesimally), and it is no longer obvious how to handle computations involving the Killing vector. One way around this dilemma is to “replace” the Killing vector \( \chi^a \) with a vector \( \rho^a \) which is defined by 

\[ \nabla_a |\chi|^2 = -2\kappa \rho_a \, . \tag{13} \]

(Here, \( \kappa \) is the surface gravity — which can itself be defined by the relation \( \chi^a \nabla_a \chi^b = \kappa \chi^b \) on the horizon [22].) The relevant point being that \( \rho^a \) has been precisely defined so that it always satisfies \( \rho^a \chi_a = 0 \) and becomes null on the horizon; that is, \( \rho^a \) limits to \( \chi^a \) at the horizon [6]. Hence, in obtaining any on-horizon result, we can first do the calculation in terms of \( \rho^a \) and then take the limit to the horizon afterwards.

Let us begin here with the observation that, on the bifurcation surface in particular, \( \nabla_a J^{ab} \to 0 \). To see this, it is useful to call upon a few on-horizon limits: \( \nabla_a \rho^a \to 2\kappa \) [cf, [6]; equations (A.7) and (A.10)], \( \rho^a \nabla_a \rho^b \to \kappa \chi^b \) [cf, [6]; equations (A.8) and (A.10)], and \( [m_1]^a \nabla_a \rho^b \to [m_1]^a \nabla_a \chi^b \) (and similarly for \( [m_2]^a \)). By application of these limiting relations, it is not difficult to determine from equation (4) what the “residue” of \( \nabla_a J^{ab} \) turns out to be (after pulling back to the bifurcation surface, where \( J^{ab} \) is itself identically vanishing). Given that \( \nabla_a \rho^a \to 2\kappa \) whereas (on the horizon) \( \nabla_a \chi^a = 0 \), the residue in question is just \( 2\kappa \{G_{++} \chi^b + G_{+1} [m_1]^b + G_{+2} [m_2]^b \} \). Now, from this last expression, we can see that the residue is, in fact, directly proportional to the on-horizon contraction of \( N_a \) with \( J^{ab} \). Since this contraction must certainly vanish on the bifurcation surface, \( \nabla_a J^{ab} \to 0 \) then follows.

The above outcome and equation (12) allows us to deduce that \( \nabla_a H^{ab} \to 0 \) on the bifurcation surface. But actually, since \( H^{ab} \) is now known to maintain its formal relationship with the geometry when propagated in the Killing direction (cf, the beginning of the section), this tensor should be similarly conserved at any point along the horizon. It immediately follows that, anywhere on the horizon,

\[ \nabla_a J^{ab} \to 0 \tag{14} \]

should be a true statement.

11Technically speaking, a similar replacement should be made for \( N^a \). This is rather tricky, given our lack of knowledge about this vector under general circumstances. Fortunately, the matter never explicitly comes up. However, see Section IV for some discussion on a related subtlety; that is, the “loophole”.

12These limits can be obtained by long but straightforward calculations [6] that incorporate the defining relation (13) for \( \rho^a \), and the definition of the surface gravity as the horizon limit of the Killing orbit acceleration or \( \sqrt{\nabla_a |\chi| \nabla_a |\chi|} \) [22].
Now substituting equation (4) into equation (14), we find that

$$0 = 3\kappa G++ \chi^b$$
$$+ G_{++} \{ 2\kappa [m_1]^b + \chi^a \nabla_a [m_1]^b + [m_1]^a \nabla_a \chi^b + \chi^b \nabla_a [m_1]^a \}$$
$$+ G_{++} \{ 2\kappa [m_2]^b + \chi^a \nabla_a [m_2]^b + [m_2]^a \nabla_a \chi^b + \chi^b \nabla_a [m_2]^a \}$$
$$+ \{ [m_1]^a \nabla_a G_{++} + [m_2]^a \nabla_a G_{++} \} \chi^b \right) \chi^b ,$$

where — once again — the calculation is done initially in terms of $\rho^a$ and then the on-horizon limit is taken.

Next, let us contract the right-hand side with $[m_1]^b$, which then yields

$$0 = 2\kappa G_{++} + G_{++} \{ [m_1]^b \chi^a \nabla_a [m_2]^b + [m_1]^b [m_2]^a \nabla_a \chi^b \} \right) \chi^b \right) \chi^b \right) \chi^b \right) \chi^b \right) \chi^b \right) \chi^b \right) \chi^b \right) \chi^b \right) \chi^b .$$

Here, we have used the various orthogonality properties, as well as $L_\chi [m_1]^b = \chi^a \nabla_a [m_1]^b - [m_1]^a \nabla_a \chi^b = 0$. Doing the same with $[m_2]^b$, we similarly obtain

$$0 = 2\kappa G_{++} + G_{++} \{ [m_2]^b \chi^a \nabla_a [m_1]^b + [m_2]^b [m_1]^a \nabla_a \chi^b \} \right) \chi^b \right) \chi^b \right) \chi^b \right) \chi^b .$$

But actually, if we apply $[m_2]^b [m_1]^b = 0$ to the first term in the curly brackets and $\nabla_a \chi^b = -\nabla_b \chi^a$ to the second, this last equation can be recast as

$$0 = 2\kappa G_{++} - G_{++} \{ [m_1]^b \chi^a \nabla_a [m_2]^b + [m_1]^b [m_2]^a \nabla_a \chi^b \} \right) \chi^b \right) \chi^b .$$

The above computations (16,18) may be reinterpreted as a system of two equations with two unknowns ($G_{++}, G_{++}$); that is,

$$0 = 2\kappa G_{++} + Z G_{++}$$
$$0 = 2\kappa G_{++} - Z G_{++} ,$$

where $Z$ is just the contents of the curly brackets. The requirement for a non-trivial solution turns out to be $4\kappa^2 = -Z^2$. But, since $\kappa > 0$ by hypothesis (and $Z$ is clearly real), this must mean that $G_{++} = G_{++} = 0$. Consequently [cf, equation (15)], $G_{++} = 0$. The end result of all this is that $J_{ab} = 0$ [cf, equation (4)], and so $G_{ab} = H_{ab}$. Which is to say, the Einstein tensor takes on the highly symmetric (block-diagonal) form

$$G_{ab} = G_{++} [g_\perp]_{ab} + [H_\parallel]_{ab} ,$$

anywhere on the horizon.

**IV. A DISCUSSION**

In summary, we have utilized a particularly elegant method to demonstrate the highly symmetric nature of the Einstein tensor near a Killing horizon. More specifically, we have shown that, for any stationary (non-extremal) Killing horizon, the on-horizon Einstein tensor block-diagonalizes into transverse and parallel blocks. Additionally, the transverse block is constrained to be directly proportional to the transverse metric.
It is noteworthy that (along with M. Visser and D. Martin) we were able to deduce the very same near-horizon symmetries by a substantially different approach [17,18]. These earlier treatments were based on first expanding out the metric components (in terms of normal distance from the horizon) and then enforcing regularity on the near-horizon geometry. In spite of the apparent duplicity, the current work has a distinct advantage over the former: it has allowed us to handle both relevant cases (static and rotating) with a single iteration. This simplifying feature can be attributed to the use of an essentially coordinate-independent approach. To see the validity of this statement, consider that the Killing vector, $\chi^a$, can be interpreted as a purely geometric entity. Given this interpretation, the other three vectors in our basis ($N^a$, $[m_1]^a$, $[m_2]^a$) can be suitably defined and then viewed as relational constructs with respect to $\chi^a$. Hence, the analysis really required no formal coordinate system in the usual sense.

The attentive reader will notice that the word “proof” was never used in the main text. This is because our analysis cannot be regarded as rigorous in the following sense: By our use of the conservation equation (12), it was necessary (at least implicitly) to extrapolate the tensors $H_{ab}$ and $J_{ab}$ away from the horizon. It was, however, never actually confirmed that these tensors maintain the distinction of being separate entities under such an extrapolation. [Note, though, that the total Einstein tensor or $G_{ab}$ will always be a well-defined object — both on and outside of the horizon — irrespective of the status of $H_{ab}$ and $J_{ab}$ individually.] Hence, it can not be said with absolute certainty that $H^{ab}$ and $J^{ab}$ will be separately conserved, in spite of the compelling arguments of Section III.

Unfortunately, our ability to resolve the above issue is hindered by a lack of knowledge — under general circumstances — about the off-horizon behavior of the vector $N^a$. On a more favorable note, the results of the current derivation do happen to agree with those of our prior and completely independent work [17,18]. Nonetheless, this “loophole” is rather bothersome, and we hope to (somehow) rigorously address this issue at a later time.

Let us re-emphasize that this highly symmetric form of Einstein tensor implies (via the field equations) that the stress tensor near a black hole horizon will be similarly constrained. This can be expected to have severe repercussions on the matter and energy — including any quantum fluctuations — that can exist near a horizon. As discussed and elaborated on elsewhere [17,18], we suspect that the symmetries at hand may be relevant to some recent (statistically based) calculations of the black hole entropy; most notably, those of Carlip [5,6]. It remains an ongoing challenge, however, to place this connection on firmer, more rigorous ground. (But, for some first steps in this direction, see [25].)

Another interesting challenge will be to generalize our findings to extremal Killing horizons, as well as to dynamical spacetimes. [Yet, if the spacetime is evolving slow enough (or is “quasi-stationary”), then our results should still be applicable in an approximate sense.] Alas, the extremal case is plagued by conceptual issues, whereas truly dynamical scenarios present difficulties of a more technical nature. Suffice it to say, such matters (like the previous ones) are currently under investigation.
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