New Travelling Wave Solution-Based New Riccati Equation for Solving KdV and Modified KdV Equations

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Abstract
A large family of explicit exact solutions to both Korteweg- de Vries and modified Korteweg- de Vries equations are determined by the implementation of the new extended direct algebraic method. The procedure starts by reducing both equations to related ODEs by compatible travelling wave transforms. The balance between the highest degree nonlinear and highest order derivative terms gives the degree of the finite series. Substitution of the assumed solution and some algebra results in a system of equations are found. The relation between the parameters is determined by solving this system. The solutions of travelling wave forms determined by the application of the approach are represented in explicit functions of some generalized trigonometric and hyperbolic functions and exponential function. Some more solutions with different characteristics are also found.

Keywords: new extended direct algebraic method, KdV equation, modified KdV equation, exact solution, travelling wave solution.

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1 Introduction

The Korteweg-de Vries (KdV) family of nonlinear PDEs attracts many researchers due to having soliton-type solutions that preserve their shapes and heights after interacting with each other. In the related literature, it
is accepted that the story of Korteweg-de Vries equation (KdVE) represented by the following equation

\[ u_t + pu_{xx} + qu_{xxx} = 0, \tag{1} \]

where \( p \) and \( q \) are nonzero constant coefficients and superscripts denoting partial derivatives of \( u(x,t) \), starts in the late 1800s in [1]. The KdVE is an integrable equation having pulse-type travelling wave solutions with positive or negative amplitudes. The KdVE has multiple soliton-type solutions having particle-like behaviours that maintain their velocity, shape, and amplitudes after collision [2, 3]. Another significant property of the KdVE is to have infinitely many conservation laws describing different physical meanings covering energy, momentum, and mass [4]. Besides, it is also useful to model internal waves in waters with multiple density layers, crystal lattice acoustic waves and ion-acoustic wave models in plasma. Due to the above-mentioned properties of the KdV model, researchers search for different ways to provide accurate solutions to this model.

When the quadratic nonlinear term is replaced by a cubic one, the equation is named as the modified Korteweg-de Vries equation (mKdVE)

\[ u_t + pu^2u_x + qu_{xxx} = 0, \tag{2} \]

where and \( q \) are nonzero real constants and subscripts \( x \) and \( t \) denote the partial derivatives of \( u(x,t) \). A plenty of efficient methods have been implemented to derive solutions in various forms to the KdVE- and mKdV-type equations. For example, Ankiewicz et al. in [5] provided three lowest order exact rational solutions to the Kdv-type equations. Jia et al. [6] constructed a Darboux transformation for the nonlocal mKdV-type equations finding exact solutions in the form of soliton, kink and antikink solutions. Inverse scattering transformation is proposed for finding soliton solutions to the modified KdV in [7]. The consistent Riccati expansion method has been utilized for investigating an interacting solution to the mKdV models in [8]. Some soliton-like and periodic solutions were constructed by the assistance of hyperbolic tangent and cotangent functions [9]. Rational function solutions having some trigonometric or/and hyperbolic finite series in both numerator and denominator were set by classical \((G'/G)\)-expansion approach [10]. The extended homoclinic test technique was applied to the KdVE to find solitary wave solutions in periodic function forms [11]. Composite function solutions were suggested under Wronskain expansion in [12]. Periodic wave and hyperbolic function-type solutions have been derived by simple ansatz techniques in trigonometric function forms [13]. Exp function approach is proposed periodic solutions of rational function forms of trigonometric and exponential functions [14].

Today, there are many different techniques used to derive exact solutions to nonlinear PDEs of both integer and fractional order due to their importance in modelling physical phenomenon. Time fractional nonlinear dispersive PDE is introduced and examined in [15], and a solitary wave solution to the problem is obtained. Numerical treatment of a fractional model of the Newell weighted Segel equation of arbitrary order is investigated using a residual power series method that provides excellent results for the problem [26]. Also, a biological model that describes the immune system and tumour cells in the immunogenic tumour model is presented and discussed in [17] with a new definition of the Atangana fractional definition and studied using the Adam Bashforth Moulton method. Some new fractional definitions are introduced in [18], and those definitions are then used to simulate a new Yang-Abdel-Aty-Cattani fractional diffusion equation. Also, time-fractional wave equations are presented in the sense of Yang-Abdel-Aty-Cattani fractional definition and then solved with the aid of the homotopy perturbation transform method [19], and the existence and uniqueness of the fractional Cauchy reaction-diffusion equations are also solved in [20] with the same technique. Several other methods including simple trigonometric ansatz methods [21, 22], modified auxiliary equation technique [23, 24], unified method [25,27], Jacobi elliptic function expansion method [28,29], Sine-Gordon expansion method [30,31], Exp \((-\varphi(\xi))\)-Expansion method [32, 33], and modified simple equation method [34, 35] are some other important and efficient methods to set exact solutions to nonlinear PDEs. For more details regarding models of nonlinear PDE and their solutions, one may refer to [36–39] and references therein. These methods

In this article, we are concerned with implementing a new extended direct algebraic approach to derive a large family of exact solutions to both the KdV equation represented in Eq. (1) and the mKdV in Eq. (2).
This article is organised as follows: Section 2 gives brief properties of the new extended direct algebraic approach and summarizes the steps of the implementation of the approach. Section 3 and Section 4 are reserved to solve the KdV and the mKdV by the proposed approach given in Section 2. Section 5 provides a conclusion to the study.

2 Fundamentals of the New Extended Direct Algebraic Method

In this section, we introduce the main steps for developing the new extended direct algebraic method \cite{35,40}. The main steps for the method are as follows:

**Step 1.** Consider a nonlinear partial differential equation of the form

\[ F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0 \]  \hspace{1cm} (3)

which can be converted to an ODE as the following form

\[ G(u, u', u'', \ldots) = 0, \]  \hspace{1cm} (4)

using the wave transformation

\[ u(x, t) = u(\xi), \quad \xi = x - \mu t. \]

**Step 2.** Suppose that the solution of Eq. (4) can be presented as follows

\[ u(\xi) = \sum_{j=0}^{m} b_j Q^j(\xi), \quad b_m \neq 0, \]  \hspace{1cm} (5)

where \( b_k \) (\( 0 \leq k \leq m \)) are constant coefficients to be determined later and \( Q(\xi) \) satisfies the following ODE

\[ Q'(\xi) = \ln(A)(\alpha + \beta Q(\xi)) + \sigma Q^2(\xi), \quad A \neq 0, 1. \]  \hspace{1cm} (6)

The above ODE has a wide range of exact solutions. Here just some of them are listed.

**Family 1:** If \( \beta^2 - 4\alpha \sigma < 0 \) and \( \sigma \neq 0 \), then

\[ Q_1(\xi) = -\frac{\beta}{2\sigma} + \sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2\sigma}} \tan A \left( \frac{\sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}}{2} (\xi + \xi_0) \right), \]

\[ Q_2(\xi) = -\frac{\beta}{2\sigma} - \sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2\sigma}} \cot A \left( \frac{\sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}}{2} (\xi + \xi_0) \right), \]

\[ Q_3(\xi) = -\frac{\beta}{2\sigma} + \frac{-\sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}}{2\sigma} \tan A \left( \sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}(\xi + \xi_0) \right) \pm \sqrt{pq} \sec A \left( \sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}(\xi + \xi_0) \right), \]

\[ Q_4(\xi) = -\frac{\beta}{2\sigma} - \frac{-\sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}}{2\sigma} \cot A \left( \sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}(\xi + \xi_0) \right) \pm \sqrt{pq} \csc A \left( \sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}(\xi + \xi_0) \right), \]

\[ Q_5(\xi) = -\frac{\beta}{2\sigma} + \frac{-\sqrt{-\frac{(\beta^2 - 4\alpha \sigma)}{2}}}{4\sigma} \tan A \left( \frac{-\sqrt{\frac{(\beta^2 - 4\alpha \sigma)}{2}}}{4} (\xi + \xi_0) \right) - \cot A \left( \frac{-\sqrt{\frac{(\beta^2 - 4\alpha \sigma)}{2}}}{4} (\xi + \xi_0) \right). \]

**Family 2:** If \( \beta^2 - 4\alpha \sigma > 0 \) and \( \sigma \neq 0 \), then

\[ Q_6(\xi) = -\frac{\beta}{2\sigma} - \frac{-\sqrt{\frac{(\beta^2 - 4\alpha \sigma)}{2}}}{2\sigma} \tan A \left( \frac{\sqrt{\beta^2 - 4\alpha \sigma}}{2} (\xi + \xi_0) \right), \]
Remark 1. Where generalized triangular and hyperbolic functions are defined as

\[ Q_7(\xi) = -\frac{\beta}{2\sigma} - \sqrt{\frac{\beta^2 - 4\alpha\sigma}{2\sigma}} \coth_4\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}(\xi + \xi_0)\right), \]

\[ Q_8(\xi) = -\frac{\beta}{2\sigma} - \sqrt{\frac{\beta^2 - 4\alpha\sigma}{2\sigma}} \left(\tanh_4\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}(\xi + \xi_0)\right) \pm i\sqrt{pq}\sech_4\left(\sqrt{\beta^2 - 4\alpha\sigma}(\xi + \xi_0)\right)\right), \]

\[ Q_9(\xi) = -\frac{\beta}{2\sigma} - \sqrt{\frac{\beta^2 - 4\alpha\sigma}{2\sigma}} \left(\coth_4\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}(\xi + \xi_0)\right) \pm \sqrt{pq}\csch_4\left(\sqrt{\beta^2 - 4\alpha\sigma}(\xi + \xi_0)\right)\right), \]

\[ Q_{10}(\xi) = -\frac{\beta}{2\sigma} - \sqrt{\frac{\beta^2 - 4\alpha\sigma}{2\sigma}} \left(\tanh_4\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4}(\xi + \xi_0)\right) + \coth_4\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4}(\xi + \xi_0)\right)\right). \]

**Family 3:** If \( \beta = \lambda, \alpha = m\lambda (m \neq 0) \) and \( \sigma = 0 \), then

\[ Q_{11}(\xi) = A^\lambda(\xi + \xi_0) - m. \]

**Family 4:** If \( \beta = \sigma = 0 \), then

\[ Q_{12}(\xi) = \alpha(\xi + \xi_0)\ln A. \]

**Family 5:** If \( \beta = \alpha = 0 \), then

\[ Q_{13}(\xi) = -\frac{1}{\sigma(\xi + \xi_0)\ln A}. \]

**Family 6:** If \( \alpha = 0 \) and \( \beta \neq 0 \), then

\[ Q_{14}(\xi) = \frac{p\beta}{(\cosh_4(\beta(\xi + \xi_0)) - \sinh_4(\beta(\xi + \xi_0)) + p)}, \]

\[ Q_{15}(\xi) = -\frac{\beta}{\sigma(\sinh_4(\beta(\xi + \xi_0)) + \cosh_4(\beta(\xi + \xi_0))) + \cosh_4(\beta(\xi + \xi_0)) + q}. \]

**Family 7:** If \( \beta = \lambda, \sigma = m\lambda (m \neq 0) \) and \( \alpha = 0 \), then

\[ Q_{16}(\xi) = \frac{pA^\lambda(\xi + \xi_0)}{p - mqA^\lambda(\xi + \xi_0)}, \]

where \( \xi_0 \) is an arbitrary constant.

**Remark 1.** Where generalized triangular and hyperbolic functions are defined as

\[ \sin_4(\xi) = \frac{pA^\xi - qA^{-\xi}}{2i}, \quad \cos_4(\xi) = \frac{pA^\xi + qA^{-\xi}}{2}, \]

\[ \tan_4(\xi) = -i\frac{pA^\xi - qA^{-\xi}}{pA^\xi + qA^{-\xi}}, \quad \cot_4(\xi) = i\frac{pA^\xi + qA^{-\xi}}{pA^\xi - qA^{-\xi}}, \]

\[ \sec_4(\xi) = \frac{2}{pA^\xi + qA^{-\xi}}, \quad \csc_4(\xi) = \frac{2i}{pA^\xi - qA^{-\xi}}, \]

\[ \sinh_4(\xi) = \frac{pA^\xi - qA^{-\xi}}{2}, \quad \cosh_4(\xi) = \frac{pA^\xi + qA^{-\xi}}{2}, \]

\[ \tanh_4(\xi) = \frac{pA^\xi - qA^{-\xi}}{pA^\xi + qA^{-\xi}}, \quad \csch_4(\xi) = \frac{2}{pA^\xi - qA^{-\xi}}, \]

\[ \sech_4(\xi) = \frac{2}{pA^\xi + qA^{-\xi}}, \quad \quadsch_4(\xi) = \frac{2}{pA^\xi - qA^{-\xi}}. \]
where $\xi$ is an independent variable, $p$ and $q$ are arbitrary constants greater than zero and called deformation parameters.

**Step 3.** Determine the positive integer $m$ in Eq. (5). It can be done by balancing the highest-order derivative term and the highest-order nonlinear term in (4). If the degree of $u(\xi)$ is $D[u(\xi)] = n$, then the degree of the other expressions will be given by

$$D \left[ \frac{d^p u(\xi)}{d\xi^p} \right] = n + \rho, \quad \text{and} \quad D \left[ u^\rho \left( \frac{d^nu(\xi)}{d\xi^n} \right)^s \right] = np + s(n + \nu).$$

Therefore, we can find the value of $m$ in Eq. (7), using Eq. (5).

**Step 4.** Substitute Eq. (5) along with its required derivatives into Eq. (4) and compare the coefficients of powers of $f(\xi)$ in resultant equation for obtaining the set of algebraic equations.

**Step 5.** Solve the set of algebraic equations using the Maple package and put the results generated in Eq. (5) to extract the exact solutions of Eq. (3).

Next, we use the above-mentioned steps for solving the KdV problem represented in Eq. (1).

### 3 Solutions to the KdV Equation

First, consider the wave transformation in the following form

$$u(x,t) = u(\xi), \quad \xi = x - \mu t,$$

then by using the transform in Eq. (8) to the KdV problem, Eq. (1) is reduced as follows:

$$-\mu u' + puu' + qu''' = 0.$$  \hspace{1cm} (9)

Now, by balancing the highest-order derivative term and the highest-order nonlinear term in Eq. (9), we find that $m = 2$. So, Eq. (9) will have a formal solution of the form

$$u(\xi) = b_0 + b_1 Q(\xi) + b_2 Q^2(\xi),$$  \hspace{1cm} (10)

Next, by substituting Eq. (10) into Eq. (9) and collecting all the terms with the same order of $Q(\xi)$ together, the left-hand side of Eq. (9) is then converted into polynomial in $Q(\xi)$. Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for $b_0, b_1, b_2$ and $\mu$ for the coefficient of $Q'(\xi)$ in the following form

$$Q^0(\xi) : \alpha LnA(p_b v_0 + q(\ln^2 A)) \beta^3 v_1 + 6 q(\ln^2 A) \alpha \beta b_2 + 2 q a b_1 \sigma (\ln^2 A) = 0,$$

$$Q^2(\xi) : LnA(8q a b_1 \sigma (\ln^2 A) - 2 \mu b_2 \alpha + pb_1^2 \alpha + pb_0 b_1 \beta + 2 pb_0 b_2 \alpha + q b^3 b_1 (\ln^2 A) + 14 q a \beta^3 b_2 (\ln^2 A) + 16 q a \beta^2 \sigma (\ln^2 A) = 0,$$

$$Q^3(\xi) : LnA(-\mu b_1 \sigma - 2 \mu b_2 \beta + pb_1^3 \beta + pb_0 b_1 \sigma + 2 pb_0 b_2 \beta + 3 pb_1 b_2 \alpha + 8 q b^3 b_2 (\ln^2 A) + 8 q a \sigma \sigma^3 b_1 (\ln^2 A) + 7 q \beta^2 b_1 \sigma (\ln^2 A) + 52 q a \beta b_2 \sigma (\ln^2 A) + 3 pb_0 b_2 \sigma + 3 pb_1 b_2 \beta) = 0,$$

$$Q^4(\xi) : LnA(54 q b \beta \sigma^2 b_2 (\ln^2 A) + 3 pb_1 b_2 \sigma + 6 q a \sigma^3 b_1 (\ln^2 A) + 2 pb_2 \beta) = 0.$$
where \( \xi = x - (q(Ln^2 A) (8\alpha\sigma + \beta^2) + pb_0)t \).

**Family 2:** If \( \beta^2 - 4\alpha\sigma > 0 \) and \( \sigma \neq 0 \), then

\[
\begin{align*}
    u_6(x,t) &= b_0 + \frac{3q Ln^2 A}{p} \left( \beta^2 - \Lambda \tanh_A \left( \frac{\sqrt{-\Lambda}}{4} (\xi + \xi_0) \right) \right), \\
    u_7(x,t) &= b_0 + \frac{3q Ln^2 A}{p} \left( \beta^2 - \Lambda \coth_A \left( \frac{\sqrt{-\Lambda}}{4} (\xi + \xi_0) \right) \right), \\
    u_8^+(x,t) &= b_0 - \frac{12q\sigma(Ln^2 A)}{p} \left[ -\frac{\beta}{2\sigma} - \frac{\sqrt{-\Lambda}}{2\sigma} \left( \tanh_A \left( \frac{\sqrt{-\Lambda}}{2} (\xi + \xi_0) \right) \pm i\sqrt{pq}\sech_A \left( \sqrt{-\Lambda}(\xi + \xi_0) \right) \right) + \sigma \left( -\frac{\beta}{2\sigma} - \frac{\sqrt{-\Lambda}}{2\sigma} \left( \tanh_A \left( \frac{\sqrt{-\Lambda}}{2} (\xi + \xi_0) \right) \pm i\sqrt{pq}\sech_A \left( \sqrt{-\Lambda}(\xi + \xi_0) \right) \right) \right]^2 \right], \\
    u_9^-(x,t) &= b_0 - \frac{12q\sigma(Ln^2 A)}{p} \left[ -\frac{\beta}{2\sigma} - \frac{\sqrt{-\Lambda}}{2\sigma} \left( \coth_A \left( \frac{\sqrt{-\Lambda}}{2} (\xi + \xi_0) \right) \pm i\sqrt{pq}\csch_A \left( \sqrt{-\Lambda}(\xi + \xi_0) \right) \right) + \sigma \left( -\frac{\beta}{2\sigma} - \frac{\sqrt{-\Lambda}}{2\sigma} \left( \coth_A \left( \frac{\sqrt{-\Lambda}}{2} (\xi + \xi_0) \right) \pm i\sqrt{pq}\csch_A \left( \sqrt{-\Lambda}(\xi + \xi_0) \right) \right) \right]^2 \right],
\end{align*}
\]
Now, by balancing the highest-order derivative term and the highest-order nonlinear term in Eq. (27), we find which reduce Eq. (2) into the following

\[ u_{10}(x,t) = b_0 - \frac{12q\Lambda(Ln^2A)}{p} \left[ \frac{-\beta - \frac{\sqrt{\Lambda}}{2\sigma}}{4\sigma} \left( \tanh_A \left( \frac{\sqrt{\Lambda}}{4}(\xi + \xi_0) \right) + \coth_A \left( \frac{\sqrt{\Lambda}}{4}(\xi + \xi_0) \right) \right) \right], \]

where \( \Lambda = \beta^2 - 4\alpha\sigma \) and \( \xi = x - (q(Ln^2A)/(8\alpha\sigma + \beta^2) + pb_0)t \).

**Family 3:** If \( \beta = \alpha = -\), then

\[ u_{11}(x,t) = b_0 - \frac{12q(Ln^2A)}{p} \left[ \frac{-1}{(x-pb_0t + \xi_0)LnA} + \left( \frac{1}{(x-pb_0t + \xi_0)LnA} \right)^2 \right]. \]

**Family 4:** If \( \alpha = 0 \) and \( \beta \neq 0 \), then

\[ u_{12}(x,t) = b_0 - \frac{12q(Ln^2A)}{p} \left[ \frac{-\beta}{(\cosh_A(\beta(\xi + \xi_0)) - \sinh_A(\beta(\xi + \xi_0)) + p)} \right], \]

\[ u_{13}(x,t) = b_0 - \frac{12q(Ln^2A)}{p} \left[ \frac{-\beta}{(\sinh_A(\beta(\xi + \xi_0)) + \cosh_A(\beta(\xi + \xi_0)) + q)} \right], \]

where \( \xi = x - (qB^2(Ln^2A) + pb_0)t \).

**Family 5:** If \( \beta = \lambda, \sigma = m\lambda (m \neq 0) \) and \( \alpha = 0 \), then

\[ u_{14}(x,t) = b_0 - \frac{12qm\lambda(Ln^2A)}{p} \left[ \frac{pA^\lambda(x-(q\lambda^2(Ln^2A) + pb_0)t + \xi_0)}{p - mQA^\lambda(x-(q\lambda^2(Ln^2A) + pb_0)t + \xi_0)} + m\lambda \right], \]

where \( \xi_0 \) is an arbitrary constant.

Next, we will apply the same newly developed method for the mKdV problem represented in Eq. (2).

### 4 Solutions to the Modified KdV Equation

Consider the wave transformation in the following form

\[ u(x,t) = u(\xi), \quad \xi = x - \mu t, \]

which reduce Eq. (2) into the following

\[ -\mu u' + pu^2u' + qu''' = 0. \]

Now, by balancing the highest-order derivative term and the highest-order nonlinear term in Eq. (27), we find that \( m = 1 \). Then, Eq. (27) will have a formal solution of the form

\[ u(\xi) = b_0 + b_1 Q(\xi). \]
Fig. 1 Graphics of the solution equation (14) \((u_{5}(x,t))\) corresponding to the values \(\beta = 3, \alpha = 4, \sigma = 2, A = 2.65, p = 1, q = 0.9, b_{0} = 1\) and \(\xi_{0} = -0.5\) (a) 3D plot and (b) contour plot.

Fig. 2 Graphics of the solution equation (28) \((u_{14}(x,t))\) corresponding to the values \(\lambda = 1.5, m = 3, A = 2.7, p = 1.2, q = 0.9, b_{0} = 3\) and \(\xi_{0} = 1.5\) (a) 3D plot and (b) contour plot.

By substituting Eq. (28) into Eq. (27) and collecting all terms with the same order of \(Q(\xi)\) together, the left-hand side of Eq. (27) is converted into a polynomial in \(Q(\xi)\). Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for \(b_{0}, b_{1}, \) and \(\mu\) for the coefficients of \(Q^{i}(\xi)\) in the following form

\[Q^{0}(\xi): \quad b_{1}\alpha LnA(-\mu + pb_{0}^{2} + 2q\sigma\alpha(Ln^{2}A) + q\beta^{2}(Ln^{2}A)) = 0,\]

\[Q^{1}(\xi): \quad b_{1}\alpha LnA(2pb_{0}b_{1} + 6q\beta\sigma(Ln^{2}A)) + b_{1}\beta LnA(-\mu + pb_{0}^{2} + 2q\sigma\alpha(Ln^{2}A) + q\beta^{2}(Ln^{2}A)) = 0,\]

\[Q^{2}(\xi): \quad b_{1}\alpha LnA(6q\sigma^{2} + pb_{1}^{2}) + b_{1}\beta LnA(2pb_{0}b_{1} + 6q\beta\sigma(Ln^{2}A)) + b_{1}\sigma LnA(-\mu + pb_{0}^{2} + 2q\sigma\alpha(Ln^{2}A) + q\beta^{2}(Ln^{2}A)) = 0,\]

\[Q^{3}(\xi): \quad b_{1}\beta LnA(6q\sigma^{2}(Ln^{2}A) + pb_{1}^{2}) + b_{1}\sigma LnA(2pb_{0}b_{1} + 6q\beta\sigma(Ln^{2}A)) = 0,\]

\[Q^{4}(\xi): \quad b_{1}\sigma LnA(6q\sigma^{2}(Ln^{2}A) + pb_{1}^{2}) = 0.\]
Last, by solving the above system of equations for \( b_0, b_1 \) and \( \mu \) we obtain the following values

\[
b_0 = \pm \sqrt{\frac{3q}{2p} \beta \ln A}, \quad b_1 = \pm \sqrt{-\frac{6q}{p} \sigma \ln A}, \quad \mu = -\frac{1}{2} q (\ln^2 A) (\beta^2 - 4\sigma \alpha)
\]  

The solutions family of Eq. (2) that corresponds to Eq. (26) and Eq. (29) is as follows:  

**Family 1:** If \( \beta^2 - 4\alpha \sigma < 0 \) and \( \sigma \neq 0 \), then

\[
u^+_1(x,t) = \pm \ln A \sqrt{\frac{3q\Lambda}{2p}} \tan \left(\frac{\sqrt{-\Lambda}}{2} (\xi + x_0)\right)
\]  

\[
u^+_2(x,t) = \pm \ln A \sqrt{\frac{3q\Lambda}{2p}} \tan \left(\frac{\sqrt{-\Lambda}}{2} (\xi + \xi_0)\right)
\]  

\[
u^+_3(x,t) = \pm \ln A \sqrt{\frac{3q\Lambda}{2p}} \left(\tan \left(\frac{\sqrt{-\Lambda}}{4} (\xi + \xi_0)\right) \pm \sqrt{pq} \sec A \left(\sqrt{-\Lambda}(\xi + \xi_0)\right)\right)
\]  

\[
u^+_4(x,t) = \pm \ln A \sqrt{\frac{3q\Lambda}{2p}} \left(\cot \left(\frac{\sqrt{-\Lambda}}{4} (\xi + \xi_0)\right) \pm \sqrt{pq} \csc A \left(\sqrt{-\Lambda}(\xi + \xi_0)\right)\right)
\]  

where \( \Lambda = \beta^2 - 4\alpha \sigma \) and \( \xi = x + \frac{q}{4} (\ln A) \Delta t. \)

**Family 2:** If \( \beta^2 - 4\alpha \sigma > 0 \) and, then

\[
u^{-}_6(x,t) = \pm \ln A \sqrt{-\frac{3q\Lambda}{2p}} \tanh \left(\frac{\sqrt{\Lambda}}{2} (\xi + x_0)\right)
\]  

\[
u^{-}_7(x,t) = \pm \ln A \sqrt{-\frac{3q\Lambda}{2p}} \coth \left(\frac{\sqrt{\Lambda}}{2} (\xi + \xi_0)\right)
\]  

\[
u^{-}_8(x,t) = \pm \ln A \sqrt{-\frac{3q\Lambda}{2p}} \left(\tanh \left(\frac{\sqrt{\Lambda}}{4} (\xi + \xi_0)\right) \pm i \sqrt{pq} \text{sech} A \left(\sqrt{\Lambda}(\xi + \xi_0)\right)\right)
\]  

\[
u^{-}_9(x,t) = \pm \ln A \sqrt{-\frac{3q\Lambda}{2p}} \left(\coth \left(\frac{\sqrt{\Lambda}}{4} (\xi + \xi_0)\right) \pm \sqrt{pq} \text{csch} A \left(\sqrt{\Lambda}(\xi + \xi_0)\right)\right)
\]  

\[
u^{-}_{10}(x,t) = \pm \ln A \sqrt{-\frac{3q\Lambda}{8p}} \left(\tanh \left(\frac{\sqrt{\Lambda}}{4} (\xi + \xi_0)\right) + \coth \left(\frac{\sqrt{\Lambda}}{4} (\xi + \xi_0)\right)\right)
\]
where $\Lambda = \beta^2 - 4\alpha \sigma$ and $\xi_0 = x + \frac{1}{2}q(Ln^2A)t$.

**Family 3:** If $\alpha = 0$ and $\beta \neq 0$, then

$$u_{11}^\pm(x,t) = \pm \sqrt{-\frac{q}{p}}(\beta \ln A) \left[ \sqrt{-\frac{3}{2}} \mp \frac{\sqrt{-6p}}{\cosh_A(\beta(\xi + \xi_0)) - \sinh_A(\beta(\xi + \xi_0)) + \sqrt{-3}} \right]$$

$$u_{12}^\pm(x,t) = \pm \sqrt{-\frac{q}{p}}\beta \ln A \left[ \sqrt{-\frac{3}{2}} \mp \frac{\sqrt{-6p}}{\sinh_A(\beta(\xi + \xi_0)) + \cosh_A(\beta(\xi + \xi_0)) + \sqrt{-3}} \right],$$

where $\xi = x - \frac{1}{2}q\beta^2(Ln^2A)t$.

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**Fig. 3** Graphics of the solution equation (28) ($u_{11}^\pm(x,t)$) corresponding to the values $\beta = 4, \alpha = 2, \sigma = 3, A = 2.6, p = 1, q = 0.9$ and $\xi_0 = 0$ (a) 3D plot and (b) contour plot.

**Fig. 4** Graphics of the solution equation (34) ($u_{12}^\pm(x,t)$) corresponding to the values $\beta = 3, \alpha = 1, \sigma = 2, A = e, p = 1, q = 1$ and $\xi_0 = 0$ (a) 3D plot and (b) contour plot.
Family 4: If $\beta = \lambda$, $\sigma = m\lambda (m \neq 0)$ and $\alpha = 0$, then

$$u^\pm_{13}(x,t) \pm \sqrt{\frac{q}{p}pA}\lambdaLnA\left[\sqrt{\frac{3}{2}} + \sqrt{-6m}\left(\frac{pA^{\lambda(x+\frac{q\lambda^2Ln^2At+\xi_0)}{2}}}{p-mqA^{\lambda(x+\frac{q\lambda^2Ln^2At+\xi_0)}}}\right)\right],$$

(40)

where $\xi_0$ is an arbitrary constant.

5 Conclusions

Many classical methods to be used to determine exact solutions of nonlinear PDEs are particular cases of the new extended direct algebraic approach. Thus, one can easily deduce that the new extended direct algebraic approach should derive more exact solutions. Following this idea, we implemented the new extended direct algebraic method to the KdVE and the mKdVE. Various types of explicit exact solution families covering finite trigonometric, exponential, and hyperbolic function series to both the KdVE and the mKdVE were constructed by the approach. The solutions modelling travelling waves in different forms were represented explicitly. The implementation of the new extended direct algebraic approach to the other nonlinear PDEs appears as a future study for many researchers.

Compliance with ethical standards

Conflict of interest The authors involved in this manuscript declare that they have no conflict of interest.

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