GALILEAN THEORIES OF GRAVITATION

by

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Talk given by Roberto De Pietri

Summary: A generalization of Newtonian gravitation theory is obtained by a suitable limiting procedure from the ADM action of general relativity coupled to a mass-point. Three particular theories are discussed and it is found that two of them are invariant under an extended Galilei gauge group.

1. The general framework

Assuming the existence of a global 3+1 splitting of the space-time manifold, we consider the Einstein-Hilbert-De Witt action for the gravitational field plus a matter action corresponding to a single mass-point:

\[ A = A_F + A_M = \frac{c^3}{16\pi G} \int dtd^3z \sqrt{gN} \left[ 3R + 3g^{ik}g^{jl}(K_{ij}K_{kl} - K_{ik}K_{jl}) \right] - mc \int d\lambda \sqrt{-g_{\mu\nu}x'^{\mu}x'^{\nu}}, \]

(1.1)

where we have adopted the standard ADM notations¹ for the field part.

We want to do the non-relativistic limit of (1.1) in its most general fashion. By parametrizing the covariant metric tensor as:

\[ \mathring{g}_{\mu\nu} \equiv \left| -N^2 + 3g^{ij}N_iN_j \frac{N_j}{3g_{ij}} - c^2t_\mu t_\nu + \mathring{h}_{\mu\nu} + O \left( \frac{1}{c^2} \right) \right|, \]

(1.2)

(c = velocity of light), the inverse metric takes the form:

\[ \mathring{g}^{\mu\nu} \equiv \left| -\frac{1}{N^2} - 3g^{ij}N_iN_j \frac{3g_{jk}N_k}{N^2} \right| \equiv h^{\mu\nu} - \frac{1}{c^2}K^{\mu\nu} + O \left( \frac{1}{c^4} \right). \]

\[ \mathring{g}^{\mu\nu} \equiv \left| \frac{3g^{ij}N_i}{N^2} - 3g^{ij}N_iN_j \frac{3g_{jk}N_k}{N^2} \right| \equiv h^{\mu\nu} - \frac{1}{c^2}K^{\mu\nu} + O \left( \frac{1}{c^4} \right). \]

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From the relation $4g^{\mu\nu}g_{\nu\rho} = \delta_\rho^\mu$, we see that, at first order in $1/c^2$, the following identities are fulfilled:

$$h^{\mu\nu}t_\nu = 0 \quad , \quad h^{\mu\nu}\tilde{h}_{\nu\rho} = \delta_\rho^\mu - t_\rho \kappa^{\mu\nu}t_\nu \ .$$

(1.4)

Accordingly, in the limit $c \to +\infty$, we must identify $h^{\mu\nu}$ with so-called Newtonian space metric and $t_\mu t_\nu$ with so-called Newtonian time metric (see for example Refs.[2],[3],[4]).

On the basis of the geometrical assumption made above, we have to admit the existence of an absolute time ($T$) foliation of space-time and of a global coordinate system in which $t_\mu = T_\mu = (\Theta(t), \vec{0})$. In such a system, $g_{\mu\nu}$ takes the form:

$$4g_{\mu\nu} = -c^2 T_\mu T_\nu + \tilde{h}_{\mu\nu}$$

$$\equiv -c^2 \begin{vmatrix} \Theta^2 & 0 & 0 \\ 0 & 0 \end{vmatrix} + \frac{2A_0}{A_i} A_j \begin{vmatrix} \alpha_i & \alpha_j \\ \gamma_{ij} & \beta_{ij} \end{vmatrix} + \frac{1}{c^2} \begin{vmatrix} 0 & \beta_{ij} \\ \alpha_i & \gamma_{ij} \end{vmatrix} + \frac{1}{c^4} \begin{vmatrix} 0 & \beta_{ij} \\ \alpha_i & \gamma_{ij} \end{vmatrix} + O \left( \frac{1}{c^6} \right) \ ,$$

and

$$4g^{\mu\nu} = \begin{vmatrix} 0 & 0 \\ 0 & g^{ij} \end{vmatrix} - \frac{1}{c^2 \Theta^2} \begin{vmatrix} 1 & -g^{jk} A_k \\ -g^{ij} & g^{jl}(\gamma_{kl} + A_k A_l) \end{vmatrix} + O \left( \frac{1}{c^4} \right) .$$

(1.5)

(1.6)

Thus, we are lead to identify the ADM variables as:

$$\begin{cases}
3g_{ij} \equiv g_{ij} + \frac{1}{c^2} \gamma_{ij} + \frac{1}{c^4} \beta_{ij} + O \left( \frac{1}{c^6} \right) \\
3R \equiv R + \frac{1}{c^2} R_1 (g_{ij}, \gamma_{ij}) + \frac{1}{c^4} R_2 (g_{ij}, \gamma_{ij}, \beta_{ij}) + O \left( \frac{1}{c^6} \right) \\
N_i \equiv A_i + \frac{1}{c^2} \alpha_i + \frac{1}{c^4} \beta_i + O \left( \frac{1}{c^6} \right) \\
N^2 \equiv c^2 \Theta^2 - 2A_0 + \frac{2}{c^2} \left[ \alpha_0 - g^{ij} \alpha_i A_j - \frac{1}{2} \gamma_{rs} g^{ij} g^{s} j A_i A_j \right] + O \left( \frac{1}{c^4} \right) \\
NK_{ij} \equiv 3B_{ij} = B_{ij} + \frac{1}{c^2} B^{(1)}_{ij} + O \left( \frac{1}{c^4} \right) ,
\end{cases}$$

(1.7)

where we have defined:

$$A = A_0 - \frac{1}{2} g^{ij} A_i A_j$$

$$B_{ij} = \frac{1}{2} [\nabla_i A_j + \nabla_j A_i - \frac{\partial g_{ij}}{\partial t}]$$

$$B^{(1)}_{ij} = \frac{1}{2} [\nabla_i \alpha_j + \nabla_j \alpha_i - \frac{\partial \gamma_{ij}}{\partial t} - A_k g^{kl}(\nabla_j \gamma_{il} + \nabla_i \gamma_{lj} - \nabla l \gamma_{ij})] .$$

(1.8)
We shall assume the zeroth-order term in the $1/c^2$ expansion of Eq.(1.1) as our Galilean Action. Precisely, we have:

$$\tilde{A} \equiv \frac{1}{16\pi G} \int dt d^3z \sqrt{g} \left[ \Theta R_2 + \frac{1}{2} g^{ij} \gamma_{ij} \Theta R_1 + \frac{1}{2} g^{ij} \beta_{ij} \Theta R \right.$$

$$- \frac{1}{2} \Theta \left( g^{ik} g^{jl} - \frac{1}{2} g^{ij} g^{kl} \right) \gamma_{ij} \gamma_{kl} R - \frac{A}{\Theta} R_1$$

$$- \frac{1}{\Theta} \left( \alpha_0 - g^{ij} A_i \alpha_j + \frac{1}{2} A g^{ij} \gamma_{ij} + \frac{1}{2} \gamma_{ij} g^{im} A_i A_m \right) R - \frac{A^2}{2\Theta^3} R$$

$$+ \frac{2}{\Theta} g^{ik} g^{jl} \left( B_{ij} B_{kl}^{(1)} - B_{ik} B_{jl}^{(1)} \right) + \frac{2}{\Theta} g^{ik} g^{jl} \gamma_{rs} g^{kl} \left( B_{ij} B_{kl} - B_{ik} B_{jl} \right)$$

$$+ \frac{A}{\Theta^3} g^{ik} g^{jl} \left( B_{ij} B_{kl} - B_{ik} B_{jl} \right)$$

$$+ m \int d\lambda \frac{m}{\Theta^t} \left[ \frac{1}{2} g_{ij} (x^i + g^{ik} A_k t') (x^j + g^{jl} A_l t') + A t' \right].$$

We see that 27 fields survive the limiting procedure, namely $\Theta$, $A$, $A_i$, $g_{ij}$, $\alpha_0$, $\alpha_i$, $\gamma_{ij}$, $\beta_{ij}$, where we have used $A$ as independent variable instead of $A_0$.

2. Particular cases

2.1. Post-newtonian parametrization

The maximum of similarity to Newton’s theory is achieved confining to a post-Newtonian-like parametrization defined by $\Theta = 1$, $g_{ij} = \delta_{ij}$, $A_i = 0$ and $A = -\varphi$, so that:

$$\bar{g}_{\mu\nu} = \left| \begin{array}{cc}
-\epsilon^2 & \frac{2\alpha_0}{\epsilon^3} \\
\frac{\alpha}{\epsilon^2} & \delta_{ij} + \frac{\gamma_{ij}}{\epsilon^2} \end{array} \right|. \quad (2.1)$$

The action (1.9) becomes:

$$\tilde{A} = \frac{1}{16\pi G} \int dt d^3z \left[ \varphi R_1 + R_2 - \frac{1}{2} \delta_{ij} \gamma_{ij} R_1 + m \left( \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j - \varphi \right) \delta^3 [z - x(t)] \right]. \quad (2.2)$$

The constraint analysis shows that the secondary constraints are just the Euler-Lagrange field equations, which determine the behaviour of the fields $\varphi$, $\gamma_{ij}$:

$$\chi_\varphi(z, T) \equiv \frac{1}{16\pi G} R_1 - m \delta^3 [z - x(t)] \simeq 0$$

$$\chi_\gamma^{ij}(z, T) \equiv \frac{1}{16\pi G} \left\{ \left[ \delta_{ir} \delta_{js} - \delta_{ij} \delta_{rs} \right] \partial_r \partial_s \varphi 
+ \frac{1}{2} \left[ \delta_{ir} \delta_{js} - \frac{1}{2} \delta_{ij} \delta_{rs} \right] \delta_{ab} \left[ \gamma_{ab,rs} + \gamma_{rs,ab} - \gamma_{ar,sb} - \gamma_{as,rb} \right] \right\} \simeq 0, \quad (2.3)$$
while the mass-point equations of motion are the standard Newton equations with potential $\varphi$, i.e., $m\ddot{x}^i = -\delta^{ik}\partial_k\varphi$. By performing the contraction $\delta_{ij}\chi^{ij}$ and using the constraint $\chi\varphi \simeq 0$, we obtain the Poisson equation for the potential $\varphi(z, t)$, i.e.:

$$\delta^{ij}\partial_i\partial_j\varphi(z, t) = 4\pi Gm\delta^3[z - x(t)].$$  \hfill (2.4)

### 2.2. A ten-fields theory

If we keep only the fields that explicitly interact with the mass-point, i.e., if we set $\alpha_0 = \alpha_i = \gamma_{ij} = \beta_{ij} = 0$, the action (1.9) becomes:

$$\tilde{A} = \frac{1}{16\pi G} \int dtd^3z \sqrt{g} \left[ -\frac{A^2}{2\Theta^2} R + \frac{A}{\Theta^2} g^{ik}g^{jl}(B_{ij}B_{kl} - B_{ik}B_{jl}) \right]$$

$$+ m \int dtd^3z \frac{m}{\Theta} \left[ \frac{1}{2} g_{ij}(\dot{x}^i + g^{lk}A_k)(\dot{x}^j + g^{jk}A_l) + \tilde{A} \right] \delta[z - x(T)].$$  \hfill (2.5)

It can be easily shown that the field $\Theta(t)$ lacks real dynamical content. Actually, its role amounts only to a redefinition of the evolution parameter $t$ in the expression $T(t) = \int_0^t d\tau \Theta(\tau)$, which, in turn, has to be identified to Newtonian absolute time. Indeed, if we redefine the fields $A, A_0, A_i$ and $B_{ij}$ as:

$$\left\{ \begin{array}{l} \tilde{A}_0 \equiv \frac{A_0}{\Theta^2} ; \quad \tilde{A} \equiv \frac{A}{\Theta^2} ; \quad \tilde{A}_i \equiv \frac{A_i}{\Theta} \\ \tilde{B}_{ij} \equiv \frac{B_{ij}}{\Theta} = \frac{1}{2}[\nabla_i \tilde{A}_j + \nabla_j \tilde{A}_i - \partial g_{ij}/\partial T] \end{array} \right.$$  \hfill (2.6)

the total action (2.5) can be rewritten in the form:

$$\tilde{A} = \frac{1}{16\pi G} \int dT d^3z \sqrt{g} \left[ -\frac{\tilde{A}^2}{2} R + \tilde{A}g^{ik}g^{jl}(\tilde{B}_{ij}\tilde{B}_{kl} - \tilde{B}_{ik}\tilde{B}_{jl}) \right]$$

$$+ m \int dT d^3z \left[ \frac{1}{2} g_{ij}(\frac{dx^i}{dT} + g^{lk}\tilde{A}_k)(\frac{dx^j}{dT} + g^{jk}\tilde{A}_l) + \tilde{A} \right] \delta[z - x(T)],$$  \hfill (2.7)

where $\Theta$ has disappeared. Then, the Dirac Hamiltonian is:

$$H_D = \int d^3z \left\{ \frac{\tilde{A}^2}{32\pi G} \mathcal{H}_I + \frac{16\pi G}{A} \mathcal{H}_E + \left[ \frac{1}{2m} g^{ij}p_ip_j - m\tilde{A} \right] \delta[z - x(T)] \right\}$$

$$+ \int d^3z \left[ \tilde{A}_i g^{ij}\phi_j + \pi^i\lambda_i + \pi_A\lambda_A \right],$$  \hfill (2.8)
where $\pi^{ij}, \pi^i, \pi_A$ are the conjugate fields of $g_{ij}, \tilde{A}_i, \tilde{A}$, respectively, and $\mathcal{H}_I = \sqrt{g} R$, $\mathcal{H}_E = \frac{1}{\sqrt{g}} [\text{Tr} \pi^2 - \frac{1}{3} (\text{Tr} \pi)^2]$, $\phi_J = -2g_{jk} \nabla_l \pi^{kl} - p \delta^j_z \delta^i [z - x(T)]$. The constraints’ chains are:

\[
\begin{align*}
\frac{\partial S}{\partial g_{ij}} & = 0, \\
\frac{\partial S}{\partial g_{ij}} & = 0, \\
\frac{\partial S}{\partial g_{ij}} & = 0,
\end{align*}
\]

where the arrows denote requirement of time-conservation of the constraints*. Thus, the fields $A_i$ and three independent functionals of $g_{ij}$ are gauge variables, while $\tilde{A}$ is determined by the constraint $\chi_A \simeq 0$. The surviving physical degrees of freedom correspond to three independent functionals of $g_{ij}$.

It is worth considering the particular solution of the variational problem corresponding to the static sector, defined by $A_k = 0, \frac{dg_{ij}}{dT} = 0, \frac{d\pi^i}{dT} = 0$. The consistency of the constraints’ chains, in this case, implies that the field $\tilde{A}$ satisfies the following modified Poisson equation:

\[
\sqrt{g} g^{ij} \nabla_i \nabla_j \tilde{A}^2 = 4\pi G m \tilde{A} \delta[z - x(T)].
\]

It is remarkable that, under the assumption that $\tilde{A} = \tilde{A}(r)$ and $g_{ij} \to \delta_{ij}$ for $r = |\vec{z} - \vec{x}(T)| \to \infty$, the field $\tilde{A}$, at fixed $T$, has the asymptotic expansion

\[
\tilde{A} = \frac{k_1}{r} + \frac{k_2}{r^2} + . . .
\]

2.3. An eleven-fields theory

The action for this case is given by the expression (2.5) where now $\Theta$ is allowed to be a function depending also on the space coordinate $\vec{z}$: $\Theta = \Theta(\vec{z}, t)$. This has

* The explicit expression of the constraints can be found in Ref.[5]
the effect that the resulting theory has the same number of degrees of freedom of Einstein’s theory. Although the function $\Theta(\vec{z}, t)$ cannot be rescaled in this case, the corresponding additional degree of freedom, which is a classical analogue of the \textit{dilaton}, does not have any propagation properties.

The Dirac Hamiltonian is:

$$H_D = \int d^3\vec{z} \Theta \left\{ \frac{\vec{A}^2}{32\pi G} \mathcal{H}_I + \frac{16\pi G}{A} \mathcal{H}_E + \left[ \frac{1}{2m} g^{ij} p_i p_j - m \vec{A} \right] \delta[\vec{z} - \vec{x}(T)] \right\} + \int d^3\vec{z} \left[ \vec{A}_i g^{ij} \phi_j + \pi^i \lambda_i + \pi_A \lambda_A + \pi_{\Theta} \lambda_{\Theta} \right],$$

(2.11)

where $\pi^{ij}, \pi^i, \pi_A, \pi_{\Theta}$ are the conjugate fields of $g_{ij}, \vec{A}_i, \vec{A}, \Theta$, respectively, and $\mathcal{H}_I, \mathcal{H}_E, \phi_j$ have the same functional form as before. The constraints’ chains are now:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig.png}
\caption{Fields \textit{A} and \textit{three} independent functionals of $g_{ij}$ are still \textit{gauge} variables. On the other hand, despite the additional degree of freedom introduced by $\Theta$, we have now two more pairs of \textit{second-class} constraints. Therefore only two dynamical \textit{graviton-like} degrees of freedom survive in the three-metric $g_{ij}$.}
\end{figure}

3. Gauging the Extended Galilei group

Within the re-parametrization invariant formulation of the non-relativistic free mass-point system with coordinates $t(\lambda), x^i(\lambda)$, the action and the Lagrangian are:

$$\vec{A}_M = \int_{\lambda_1}^{\lambda_2} d\lambda \ L_M(\lambda) = \int_{\lambda_1}^{\lambda_2} d\lambda \ \frac{1}{2} \frac{\delta_{ij} x'^i(\lambda) x'^j(\lambda)}{t'(\lambda)} ,$$

(3.1)
where \( f'(\lambda) \equiv \frac{d}{d\lambda} f(\lambda) \). The infinitesimal Galilei transformations, defined as “equal-\( \lambda \)” ones (i.e., \( \delta \lambda = 0 \)), are:

\[
\begin{align*}
\delta t &= -\varepsilon \\
\delta x^i &= \varepsilon^i + c_{jk}^i \omega^j x^k - t v^i \equiv \eta^i .
\end{align*}
\]  

(3.2)

Under the transformations (3.2), it follows:

\[
\delta L_M = \frac{d}{d\lambda} \left[ -m \delta_{ij} v^i x^j \right] ,
\]  

(3.3)

which shows that the quasi-invariance of the Lagrangian is an effect of the central-charge term.

The infinitesimal gauge Galilei transformations are defined by allowing the following space-time dependence for the infinitesimal parameters of (3.2): \( \varepsilon(t) \), \( \varepsilon^i(x, t) \), \( \omega^j(x, t) \), \( v^i(x, t) \). Accordingly, the mass-point coordinates transform as:

\[
\begin{align*}
\delta t &= -\varepsilon(t) \\
\delta x^i &= \eta^i(x, t) \\
\delta t' &= -t' \frac{d\varepsilon(t)}{dt} \\
\delta x'^i &= x'^k \frac{\partial \eta^i(x, t)}{\partial x^k} + t' \frac{\partial \eta^i(x, t)}{\partial t} .
\end{align*}
\]  

(3.4)

To save quasi-invariance of a new putative matter Lagrangian \( L_M^g \) under the gauge transformation (3.4), compensating fields must be introduced. If, in addition, a global flat limit of fields’ expressions and transformation properties (including the central charge term) is required, the whole set of conditions are satisfied by the following choices:

\[
L_M^g(\lambda) \equiv \frac{1}{\Theta t'} \frac{m}{2} [g_{ij} x'^i x'^j + 2A_i x'^i t' + 2A_0 t' t' ] ,
\]  

(3.5)

and

\[
\begin{align*}
\delta \Theta &= \dot{\varepsilon}(t) \Theta(t) \\
\delta g_{ij} &= -\frac{\partial \eta^k(x, t)}{\partial x^i} g_{kj} - \frac{\partial \eta^k(x, t)}{\partial x^i} g_{kj} \\
\delta A_i &= \dot{\varepsilon} A_i - A_j \frac{\partial \eta^j(x, t)}{\partial t} - g_{ij} \frac{\partial \eta^j(x, t)}{\partial t} + \Theta \frac{\partial F}{\partial x^i} \\
\delta A_0 &= 2\dot{\varepsilon} A_0 - A_i \frac{\partial \eta^i(x, t)}{\partial t} + \Theta \frac{\partial F}{\partial t} ,
\end{align*}
\]  

(3.6)

where \( F = -g_{ij} v^i x^j \). Indeed

\[
\delta L_M^g = m \frac{dF}{d\lambda} .
\]  

(3.7)
4. Local Galilei invariance

Unlike the theory corresponding to the post-Newtonian parametrization, both the ten-fields and eleven-fields theories admit the just defined gauge Galilei transformations as a local symmetry. Indeed, for both theories, the variation of the total action under the transformations of the mass-point coordinates and gauge fields (3.4), (3.6), is given by \( \delta \tilde{A} = \int dt d^3z \mathcal{L} \) and \( \Gamma = \text{Tr} B^2 - (\text{Tr} B)^2 \) :

\[
\delta \tilde{A} = \int dt d^3z \left\{ \dot{\mathcal{L}} + \left[ \frac{1}{16\pi G} \sqrt{g} (\sqrt{g} \mathcal{L} - A \Gamma) \right] \left[ \frac{\partial F}{\partial t} - A_r g^{rs} \frac{\partial F}{\partial z^s} \right] - \frac{1}{8\pi G} \frac{\partial}{\partial z^i} \left[ \sqrt{g} A \left( B^{ij} - (\text{Tr} B) g^{ij} \right) \right] \right.
\]

\[
- \frac{1}{8\pi G} \frac{\partial}{\partial z^i} \left[ \sqrt{g} A \left( B^{ij} - (Tr B) g^{ij} \right) \right] \frac{\partial F}{\partial z^i} + \frac{1}{8\pi G} \frac{\partial}{\partial z^i} \left( \sqrt{g} A \left( B^{ij} - (Tr B) g^{ij} \right) \frac{\partial F}{\partial z^i} \right) \right\}
\]

\[
= \int dt d^3z \left\{ \dot{\mathcal{L}} + \Theta \mathcal{E} \mathcal{L}_A \left( \frac{\partial F}{\partial t} - A_r g^{rs} \frac{\partial F}{\partial z^s} \right) \right\} + \Theta \mathcal{E} \mathcal{L}_{A_i} \frac{\partial F}{\partial z^i}
\]

\[
+ \frac{1}{8\pi G} \frac{\partial}{\partial z^i} \left( \sqrt{g} A \left( B^{ij} - (Tr B) g^{ij} \right) \frac{\partial F}{\partial z^i} \right)
\]

(4.1)

where \( \mathcal{E} \mathcal{L}_A \) and \( \mathcal{E} \mathcal{L}_{A_i} \) are the Euler-Lagrange derivatives that are zero on the extremals. In conclusion, the total action is quasi-invariant under the considered transformations in force of the equations of motion. This feature is precisely what it should be expected in the case of a variational principle corresponding to a singular Lagrangian.

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