Optimal Scalar Linear Index Codes for Three Classes of Two-Sender Unicast Index Coding Problem

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Abstract—Communication problems where a set of messages are distributed among multiple senders, can avail distributed transmissions to reduce the delay in content delivery. One such scenario is the two-sender index coding problem. In this paper, two-sender unicast index coding problem (TUICP) is studied, where the senders possibly have some messages in common, and each receiver requests a unique message. It is analyzed using three independent sub-problems (which are single-sender unicast index coding problems (SUICPs)) and the interactions among them. These sub-problems are described by three disjoint vertex-induced subgraphs of the side-information graph of the TUICP respectively, based on the availability of messages at the senders. The TUICP is classified based on the type of interactions among the sub-problems. Optimal scalar linear index codes for a class of TUICP are obtained using those of the sub-problems. For two classes, we identify a sub-class for which scalar linear codes are obtained using the notion of joint extensions of SUICPs. An SUICP \( I_E \) is said to be a joint extension of \( l \) SUICPs if the fitting matrices of all the \( l \) SUICPs are disjoint submatrices of that of \( I_E \). Joint extensions generalize the notion of rank-invariant extensions. Scalar linear codes and a condition for optimality of the codes are given for a class of joint extensions. Using this result, scalar linear codes and the conditions for their optimality are obtained for two classes of the TUICP.

I. INTRODUCTION

The classical index coding problem (ICP) introduced in [1] is a source coding problem, where a single sender has to broadcast coded messages to a set of receivers. Each receiver demands a different subset of messages and has a different subset of messages as its side-information. The sender takes advantage of the knowledge of side-information at the receivers and encodes the messages optimally (with minimum code length), such that every receiver can decode its demands. In some scenarios, messages are distributed among many senders which may be due to constraints on storage capacity of the senders or erroneous reception of some messages by the senders over noisy channels. For example, video streaming with caching helpers [2] and multiple ground stations being in the coverage of multiple satellites, each of which have some subset of messages [3]. In such cases, distributed transmissions help in reducing the delay of content delivery [4]. Ong et al [5] studied a class of multi-sender ICP, where each receiver knows a unique message and demands a subset of other messages. Thapa et al [3] extended single-sender index coding schemes based on graph theory to two-sender unicast ICP (TUICP), where each receiver demands a unique message. Several works provide inner and outer bounds on the capacity region of variations of multi-sender ICP [6]-[9]. Thapa et al [4] studied the broadcast rates of TUICP using a two-sender graph coloring technique for the confusion graph. They obtain upper bounds based on code constructions of sub-problems which are SUICPs. Optimal codes for some classes of SUICPs are found using those of already solved SUICPs [10]-[12]. Rank invariant extensions of SUICPs were presented in [12], where the extended problems have same optimal code lengths as that of the original SUICPs. Optimal codes are obtained using those of the original SUICPs. It is important to study such extensions, as bigger SUICPs can be solved easily using the solutions of smaller SUICPs. As the SUICP is also related to distributed storage problem in a dual sense [13], rank invariant extensions provide insights into construction of new optimal distributed storage codes from known codes. Such results are of interest due to the relation between SUICP and related problems like topological interference management problem [14].

In this work, we decompose the side-information graph of the TUICP into three disjoint vertex-induced subgraphs (each of which represents a single unicast ICP (SUICP)) along with the interactions [Section II] among the subgraphs, following the approach given in [4]. Optimal scalar linear codes for the TUICPs are obtained from those of the constituent SUICPs, based on their optimal code lengths and the associated interactions. The interaction between any two subgraphs is given by a fitting matrix, which also contains the fitting matrices of the sub-problems as disjoint submatrices. Generalizing this idea, we define joint extensions of any finite number of SUICPs and solve a particular class of joint extensions for optimal length scalar linear index codes. The optimal code is given in terms of those of the sub-problems.

The contributions of this paper are highlighted below.

- Optimal scalar linear index codes for three classes of TUICP are given. According to the authors, this is the first work to provide optimal scalar linear codes when there are partially-participated interactions [Section II] among the sub-problems of the TUICP.
function is said to be linear, if it is a linear transformation.

The paper is organized as follows. Section II introduces the SUICP and TUICP setup and gives the required definitions. Section III provides necessary lemmas and optimal scalar linear codes for a class of TUICPs. Section IV defines joint extensions and provides scalar linear codes for a class of jointly extended SUICPs. It also provides conditions for optimality of the resulting code. Section V provides scalar linear codes for a sub-class of two classes of TUICPs using joint extensions along with the condition for optimality. Section VII concludes the paper.

Notations: Matrices and vectors are denoted by bold uppercase and bold lowercase letters respectively. For any positive integer \( m \), \( [m] \) denotes \{1, ..., m\}. \( \mathbb{F}_q \) denotes the finite field of order \( q \). \( \mathbb{F}_q^{n \times d} \) denotes the vector space of all \( n \times d \) matrices over \( \mathbb{F}_q \). For a matrix \( M \in \mathbb{F}_q^{n \times d} \), \( M_{j\cdot} \) denotes the \( j \)th row of \( M \), and \( M_{\cdot j} \) denotes the \( j \)th column of \( M \) and \( (M) \) denotes the row space of \( M \). \( 0_{a \times b} \) denotes a matrix of size \( a \times b \), whose all entries are zero. Rank of \( M \) is denoted as \( rk(M) \). The transpose of \( M \) is denoted by \( M^T \). A matrix obtained by deleting some of the rows and/or columns of \( M \) is said to be a submatrix of \( M \). A set of submatrices of the given matrix are said to be disjoint, if no two of the submatrices have elements with same indices in the given matrix. A matrix (a vector) with elements from a given field along with unknowns denoted by \( x \)'s, is denoted with the subscript \( x \) as in \( \mathbb{F}_q(x) \). A matrix with only zeros and unknowns is denoted with the subscript \( T \) as in \( B_T \). The matrix \( X \) denotes a matrix consisting of all \( x \)'s. If there is ambiguity in the size of \( X \), it is explicitly mentioned, otherwise it is clear from the context.

II. INDEX CODING PROBLEM SETUP

In this section, we formulate single-sender unicast ICP (SUICP) and two-sender unicast ICP. There are \( m \) independent message symbols given by \( M = \{x_1, x_2, ..., x_m\} \), where \( x_i \in \mathbb{F}_q^{d \times 1}, \forall i \in [m] \) and \( d \geq 1 \). There are \( m \) receivers. The \( i \)th receiver demands \( x_i \) and has \( \mathcal{H}_i \subseteq M \setminus \{x_i\} \) as its side-information. The \( j \)th sender is denoted by \( S_j, j \in \{1, 2\} \) and has the set of messages \( M_j \), such that \( M_j \subseteq M \) and \( M_1 \cup M_2 = M \). In the case of SUICP, we assume that \( S_1 \) is the sender and \( M_1 = M \). The senders know the identity of messages of each other. The transmission is through a noiseless broadcast channel, which carries symbols from \( \mathbb{F}_q \). For an instance of TUICP, every codeword of the index code consists of two sub-codewords broadcasted by the two senders respectively, one after the other. The encoding function for \( S_j \) is given by \( \mathbb{E}_j : \mathbb{F}_q^{\{M_j|d \times 1\}} \rightarrow \mathbb{F}_q^{p_j \times 1} \), where \( p_j \) is the length of the sub-codeword transmitted by \( S_j \). The encoding function is said to be linear, if it is a linear transformation.

An index code for a TUICP is said to be linear, if both the encoding functions are linear. The \( i \)th receiver has a decoding function given by \( \mathbb{D}_i : \mathbb{F}_q^{(|\mathcal{H}_i|d + p_1 + p_2) \times 1} \rightarrow \mathbb{F}_q^{d \times 1} \), such that it can decode \( x_j \) using its side-information and the received codeword. For SUICP, there is no \( S_2 \) and hence \( p_2 = 0 \). If \( d = 1 \), the code is said to be a scalar code, otherwise it is said to be a vector code. In this paper, we assume \( d = 1 \) and linear encoding functions at the senders. Next, we provide some definitions from graph theory [15].

A directed graph (also called digraph) given by \( D = (\mathcal{V}(D), \mathcal{E}(D)) \) consists of a set of vertices \( \mathcal{V}(D) \) and a set of edges \( \mathcal{E}(D) \), which is a set of ordered pairs of vertices. A sub-digraph \( G \) of a digraph \( D \) is a digraph whose vertex set \( \mathcal{V}(G) \subseteq \mathcal{V}(D) \) and whose edge set \( \mathcal{E}(G) \subseteq \mathcal{E}(D) \) is restricted to the vertices in \( \mathcal{V}(G) \). The sub-digraph of \( D \) induced by the vertex set \( \mathcal{V}(G) \) is a digraph whose vertex set is \( \mathcal{V}(G) \), and edge set is \( \{(u, v) : u, v \in \mathcal{V}(G), (u, v) \in \mathcal{E}(D)\} \). A path in a digraph \( D \) is a sequence of distinct vertices \( \{v_1, \ldots, v_r\} \) such that \( (v_s, v_{s+1}) \in \mathcal{E}(D) \) for all \( s \in [r - 1] \). A digraph is called strongly connected if there is a directed path from each vertex in the digraph to every other vertex. The strongly connected components (SCCs) of a digraph are its maximal strongly connected sub-digraphs. A cycle in a digraph \( D \) is a sequence of distinct vertices \( \{v_1, \ldots, v_m\} \) such that \( (v_s, v_{s+1}) \in \mathcal{E}(G) \) for all \( s \in [c - 1] \) and \( (v_0, v_1) \in \mathcal{E}(D) \). A digraph is called acyclic if it contains no cycles.

For a unicast ICP (SUICP or TUICP), the information about the identity of side-information and demands of all the receivers is represented by a side-information digraph given by \( D = (\mathcal{V}(D), \mathcal{E}(D)) \), where the vertex set is given by \( \mathcal{V}(D) = \{v_1, \ldots, v_m\} \). The vertex \( v_i \) represents \( i \)th receiver which demands \( x_i \). The edge set is given by \( \mathcal{E}(D) = \{(v_i, v_j) : x_j \in \mathcal{H}_i, i \in [m]\} \). The same information given by side-information digraph is given by the fitting matrix, where each row represents a receiver and each column represents a message [12]. In the general case of a single-sender ICP, where more than one receiver may demand a message, the fitting matrix is defined as follows. Let \( n \) be the number of receivers.

Definition 1 (Fitting Matrix, [12]). An \( n \times m \) matrix \( \mathbf{F}_x \) is called the fitting matrix of a single sender ICP, where the \((i, j)\)th entry is given by,

\[
[F_x]_{i,j} = \begin{cases} 
  x & \text{if } x_j \in \mathcal{H}_i, \\
  1 & \text{if } i \text{th receiver demands } x_j, \\
  0 & \text{otherwise.}
\end{cases}
\]

\( \forall \ i \in [n], j \in [m]. \)

The minimum rank of \( \mathbf{F}_x \), obtained by replacing the \( x \)'s in \( \mathbf{F}_x \) with arbitrary values from \( \mathbb{F}_q \), is called the minrank of \( \mathbf{F}_x \). For SUICP, the optimal code length of a scalar linear index code is equal to the minrank of \( \mathbf{F}_x \), denoted as \( mrk(\mathbf{F}_x) \) or \( mrk(D) \) [16].

Consider the messages given by \( \mathcal{M}_1 = M_1 \setminus M_2 = \cdots \).
$\{x_1^{(1)}, \ldots, x_m^{(1)}\}$ and $\overline{M}_2 = M_2 \setminus M_1 = \{x_1^{(2)}, \ldots, x_m^{(2)}\}$ which are present only with $S_1$ and $S_2$ respectively, and the common messages $M_3 = M_1 \cap M_2 = \{x_1^{(3)}, \ldots, x_m^{(3)}\}$, where $m_s = |M_s|$, $\forall s \in \{1, 2, 3\}$. Let $\overline{M} = (M_1, M_2, M_3)$.

Any TUICP $I$ can now be described in terms of the two tuple $(D, \overline{M})$ as $I(D, \overline{M})$. The optimal (minimum) length of a scalar linear index code for the TUICP described by $(D, \overline{M})$ is denoted as $l(D, \overline{M})$. As described in (4), we define the following vertex-induced sub-diagraphs of $D$ and related interactions. Let $D_s$ be the sub-diagraph of $D$, induced by vertices $\{v_j : v_j \in M_s\}$. Let the SUICP represented by $D_s$ be denoted as $I_s$ and equivalently described by the fitting matrix $F_s^{(i)}$. If there exists an edge from a vertex in $\nu(D_i)$ to some vertex in $\nu(D_j)$ in $D$, then we say that there is an interaction from $D_i$ to $D_j$ and denote it as $D_i \rightarrow D_j$. We say that the interaction $D_i \rightarrow D_j$ is fully-participated if there are edges from every vertex in $D_i$ to every vertex in $D_j$ in $D$. Otherwise, it is called partially-participated interaction. Consider the graph $J$ with $\nu(J) = \{i, j\}$ and $E(J) = \{(i, j) : D_i \rightarrow D_j, i, j \in \{1, 2, 3\}\}$. Consider the function $f : D \rightarrow J$ where $f(v) = j$ if $v \in \nu(D_j)$. There are totally 64 possibilities for the graph $J$ as shown in Figure 2.

Note that a pair of interactions given by $D_i \rightarrow D_j$ and $D_j \rightarrow D_i$ is denoted by lines having two-sided arrows.

We see that the interactions between $D_1$ and $D_2$ (both $D_1 \rightarrow D_2$ and $D_2 \rightarrow D_1$) can be represented by a $(m_i + m_j) \times (m_i + m_j)$ fitting matrix given by

$$F_x^{(ij)} = \begin{pmatrix} F_x^{(ij)} & \mathbb{C}^{(ij)}_x \\ \mathbb{C}^{(ij)}_x & F_x^{(ji)} \end{pmatrix},$$

where $F_x^{(ij)}$ is the fitting matrix of the SUICP represented by $D_i$, the matrices $\mathbb{A}^{(ij)}_x$ and $\mathbb{C}^{(ij)}_x$ are matrices with 0’s and $x$’s which indicate side-information of receivers in $D_i$ and $D_j$ present in $D_j$ and $D_i$, respectively. We illustrate the above definitions with an example.

**Example 1.** Consider the TUICP with $m = 5$ messages, where the $i$th receiver demands $i$th message $x_i$. The side-information of all receivers are given as follows: $H_1 = \{x_2, x_3, x_3\}$, $H_2 = \{x_1, x_3\}$, $H_3 = \{x_4, x_5\}$, $H_4 = \{x_3, x_5\}$, $H_5 = \{x_1, x_2\}$. The side-information graph $D$ and the corresponding $J$ are shown in the Figure 1. Note that the interactions $D_2 \rightarrow D_3$ and $D_3 \rightarrow D_1$ are fully-participated. Others are partially-participated interactions. Consider $x_1^{(1)} = x_2, x_1^{(2)} = x_3, x_1^{(3)} = x_4, x_1^{(4)} = x_5$. The same information is given in terms of fitting matrices as shown below.

$$F_x^{(13)} = \begin{pmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_x^{(23)} = \begin{pmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Hence, receivers in $V(D_1)$ can not take advantage of the side-information present in $V(D_2)$. Hence, an optimal codeword of TUICP consists of optimal codewords of the SUICPs described by $D_1$ and $D_2$ respectively. Hence the result.

**Lemma 5.** For any TUICP $I(D, \overline{M})$, $l(D, \overline{M}) \geq \text{mrk}(D_1) + \text{mrk}(D_2)$

**Proof.** We know that the optimal code length of any ICP is greater than that of any sub-problem, which is obtained by deleting any set of receivers. Deleting the vertex set $V(D_3)$ and using Lemma 4 we obtain the result.

Next, we state the main result of this section.

**Theorem 1.** For any TUICP having any type of interaction (fully-participated or partially-participated) among its induced sub-digraphs $D_1$, $D_2$ and $D_3$, if $f(D) \in \{J_1, ..., J_{29}\}$, then $l(D, \overline{M}) = \text{mrk}(D_1) + \text{mrk}(D_2) + \text{mrk}(D_3)$

**Proof.** (First Case : $f(D) \in \{J_1, ..., J_{25}\}$) Note that $f(D)$ is acyclic. Let $D_1, ..., D_{25}^{s_2}$ be the SCCs of $D_s$, where $s \in \{1, 2, 3\}$. Due to acyclicity of $f(D)$, the SCCs of $D$ is the union of those of $D_s$ for $s \in \{1, 2, 3\}$. Hence using Lemma 1 \( \text{mrk}(D) = \sum_{s=1}^{3} \sum_{k=1}^{b_k} \text{mrk}(D_k^{s}) = \text{mrk}(D_1) + \text{mrk}(D_2) + \text{mrk}(D_3) \). Using Lemma 2 and 4 we get the required result.

(Second Case : $f(D) \in \{J_{26}, ..., J_{29}\}$) In this case, no receiver represented by vertices in $V(D_3)$ has side-information in $\overline{M}_1$ and $\overline{M}_2$. Hence, only a sub-codeword obtained by encoding messages in $\overline{M}_3$ is required for these receivers to decode their demands. This is satisfied by an optimal scalar linear index code for the SUICP represented by the side-information graph $D_3$, whose length is given by $\text{mrk}(D_3)$, and broadcasted by either $S_1$ or $S_2$. After all the receivers in $V(D_3)$ are satisfied, the remaining TUICP is same as having $V(D_3) = \phi$ in the original TUICP. Hence messages in $\overline{M}_3$ do not contribute in reducing the optimal code length for the sub-codewords used to decode the demands of receivers in $V(D_1)$ and $V(D_2)$. Using Lemma 4 we see that the optimal code length for this sub-problem is $\text{mrk}(D_1) + \text{mrk}(D_2)$. This gives the lower bound. Using Lemma 5 we see that both the bounds match. Hence, the result.

**IV. JOINTLY EXTENDED INDEX CODING PROBLEMS**

In this section, we define joint extensions of any finite number of single-sender ICPs (SICPs), which need not be SUICPs necessarily, and investigate a special class of joint extensions. Encoding matrices for this class of extended problems are obtained from those of the constituent problems. A necessary condition for the optimality of the encoding matrices of the extended problems is given. The following two definitions from [12] are used in this paper.

**Definition 2** (Extension of an SICP, [12]). An SICP $I_c$ with fitting matrix $F_x^c$ is called an extended SICP of $I$ (or an extension of $I$) with fitting matrix $F_x$, if $F_x^c$ contains $F_x$ as a submatrix.

**Definition 3** (Rank-Invariant Extension, [12]). $I_c$ is called a rank-invariant extension of $I$, if the optimal scalar code lengths of both the SICPs are equal.

We say $F \approx F_x$ (F completes $F_x$), if $F$ is obtained from $F_x$ by replacing all the unknown elements by arbitrary elements from the field. We now define jointly-extended SICPs.

**Definition 4** (Joint Extension). Consider $l$ SICPs, where the $i$th SICP $I_i$ is described using the fitting matrix $F_x^{(i)}$, $i \in [l]$. An SICP $I_E$ whose fitting matrix is given by $F_x$ is called a jointly extended SICP (or simply a joint extension) of $l$ SICPs $I_1, ..., I_l$, if $F_x$ consists of all $F_x^{(i)}$’s, $i \in [l]$, as disjoint submatrices.

Note that $I_E$ is also an extension of $I_i$, for every $i \in [l]$. Throughout this section, we assume the following setting, unless otherwise stated explicitly. Let the $i$th SICP $I_i$ have $n_i \times m_i$ fitting matrix $F_x^{(i)}$, $i \in [l]$. We assume that $F_x^{(i)} \approx F_x^{(i)}$ (that is, $F_x^{(i)}$ completes $F_x^{(i)}$), with $r_i = \text{rk}(F_x^{(i)})$. Note that $F_x^{(i)}$ need not be an optimal completion (that is, $r_i \neq \text{mrk}(F_x^{(i)})$). This implies that any $r_i$ independent vectors which span $F_x^{(i)}$, form the rows of an encoding matrix for $I_i$. Without loss of generality, we can assume that the first $r_i$ rows of $F_x^{(i)}$ span $F_x^{(i)}$. This would require a permutation of rows of $F_x^{(i)}$ and the same permutation being applied to rows of $F_x^{(i)}$, which is equivalent to renaming the receivers. We denote the first $r_i$ rows of $F_x^{(i)}$ by $F_x^{(i)}_{r_i \times m_i}$. Let $P_x^{(i)}$ be a $(n_i - r_i) \times r_i$ matrix such that the last $n_i - r_i$ rows of $F_x^{(i)}$ is given by $P_x^{(i)}F_x^{(i)}_{r_i \times m_i}$. Such a $P_x^{(i)}$ must exist as we have assumed that $F_x^{(i)}_{r_i \times m_i}$ spans $F_x^{(i)}$. Let $r_{max} = \max\{r_i, i \in [l]\}$. If the encoding function is linear, the encoded message is given by $G_x$, where $G \in \mathbb{F}_q^{r_{max} \times m}$. The encoding matrix, $x$ is the message vector whose $i$th component is $x_i$. Lemma 1 and Lemma 2 in [12] are used to derive the results in this paper and are given below.

**Lemma 6** (Vamsi et. al. [12]). For an SICP $I$ with $n \times m$ fitting matrix $F_x$, a matrix $G \in \mathbb{F}_q^{r \times m}$ is an encoding matrix if there exists a matrix $D \in \mathbb{F}_q^{n \times r}$ such that $DG \approx F_x$, i.e. $DG \approx F_x$.

**Lemma 7** (Vamsi et. al. [12]). If the fitting matrix $F_x$ of an SICP $I$ is a submatrix of the fitting matrix $F_x$ of an SICP $I_c$, then $\text{mrk}(F_x) \geq \text{mrk}(F_x)$.

The following lemma which is Theorem 2 in [18] is required to prove the main results of this section.

**Lemma 8** (Theorem 2, [18]). Let $M_x$ be a $(\sum_{i=1}^{l} n_i) \times (\sum_{i=1}^{l} m_i)$ matrix constructed as shown in (1). The diagonal elements in $M_x$ are matrices, the $i$th matrix being $F_x^{(i)}$. Other elements in $M_x$ are matrices, all of whose elements are unknowns, denoted as $X$. Then, $\text{mrk}(M_x) =$
max\{mrk(F^{(1)}_x), i \in [l]\}.

\[
M_x = \begin{pmatrix}
F^{(1)}_x & X & \ldots & \ldots & X \\
X & F^{(2)}_x & X & \ldots & \vdots \\
\vdots & X & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & F^{(l-1)}_x & X \\
X & \ldots & \ldots & X & F^{(l)}_x
\end{pmatrix}.
\tag{1}
\]

**Proof.** We give a lower bound for \(mrk(M_x)\) and construct a completion of \(M_x\) whose rank meets this lower bound. Let \(r_i = mrk(F^{(1)}_x), \forall i \in [l]\). Without loss of generality, we can assume that \(r_1 \geq r_2 \geq \ldots \geq r_{l-1} \geq r_l\). Otherwise, we can permute the sets of columns and rows corresponding to \(F^{(i)}_x\)'s in \(M_x\), to obtain the above order. This requires renaming of \(F^{(i)}_x\)'s and the corresponding messages and receivers. From Lemma 7 we know that \(mrk(M_x) \geq r_i\). Hence, \(mrk(M_x) \geq \max\{r_i, i \in [l]\}\). Now, we give a completion of \(M_x\) with rank equal to \(\max\{r_i, i \in [l]\}\). Consider any \(k\)th row of block matrices in \(M_x\) of size \(n_k \times \sum_{j=1}^{n_k} m_j, k \in [l]\), given by \(M^{(k)}_x = (X) \ldots (F^{(k)}_x) \ldots (X)\). Complete the first \(r_k\) rows of \(M^{(k)}_x\) with \((\hat{F}^{(k1)}_x) \ldots (\hat{F}^{(kk)}_x) \ldots (\hat{F}^{(kl)}_x)\), where the \(r_k \times m_j, j \in [l]\) is given in (2).

\[
\hat{F}^{(kj)}_x = \begin{cases} 
F^{(j)}_{r_k \times m_j} & \text{if } r_j \geq r_k, \\
\frac{F^{(j)}_{r_k \times m_j}}{0_{(r_k-r_j) \times m_j}} & \text{otherwise}.
\end{cases}
\tag{2}
\]

Complete the remaining \(n_k - r_k\) rows of \(M^{(k)}_x\) with \(P^{(k)}_x(\hat{F}^{(k1)}_x) \ldots (\hat{F}^{(kk)}_x) \ldots (\hat{F}^{(kl)}_x)\). Hence, they are in the row space of the first \(r_k\) rows of the completion of \(M^{(k)}_x\). It can be easily verified that this is a valid completion of \(M^{(k)}_x\). Now consider the completion of \(M^{(1)}_x\) as given by the above procedure. It is easy to verify that the first \(r_k\) rows of the completion of \(M^{(k)}_x\), for any \(k \in [l], k \neq 1\) is same as that of \(M^{(3)}_x\). Hence, the rank of the given completion of \(M_x\) is \(r_1\). Hence, the lemma is proved.

We illustrate the completion given above for \(l = 2\). As before, we assume that \(r_1 \geq r_2\). We complete the matrix \(M_x\) as shown in (3).

\[
\begin{pmatrix}
P^{(1)}_x & P^{(2)}_x \\
0_{(r_1-r_2) \times m_2} & 0_{(r_2-r_3) \times m_2}
\end{pmatrix} \approx M_x.
\tag{3}
\]

Let \(B^{(ij)}_x = \left(\frac{\hat{F}^{(ij)}_x}{0_{(r_1-r_2) \times m_2}}\right), i, j \in [l]\), be the completion of the \((i, j)\)th block matrix in \(M_x\). For any positive integer \(l \geq 2\), a completion of \(M_x\) as given in Lemma 8 is shown in (4).

From the proof of Lemma 8 we see that any \((i, j)\)th block matrix \(B^{(ij)}_x\), \(i \neq j\), in the completion of \(M_x\) can have some of it’s entries as 0’s. Hence, all the messages in the has-set of any receiver need not be used while decoding the messages in its want-set. From the completion given in the proof, we see that the matrices with all \(x\)'s (\(X\)) in \(M_x\) can be replaced by other matrices, which contain 0’s and \(x\)’s only. In such cases too, we can obtain the same optimal code lengths, as stated and proved in Theorem 2. We assume that \(r_k = rk(F^{(i)}_x)\) and \(r_j\) is not necessarily equal to \(mrk(F^{(l)}_x), i \in [l]\). Without loss of generality, we can assume that \(r_1 \geq r_2 \geq \ldots \geq r_{l-1} \geq r_l\).

**Theorem 2.** Consider \(F^{(i)}_x\) such that there exists an \(F^{(i)}_x \approx F^{(i)}_x\), where the first \(r_i\) rows of \(F^{(i)}_x\) span \((F^{(l)}_x), i \in [l]\). Let \(B^{(ij)}_x\) be any matrix such that \(B^{(ij)}_x \approx B^{(ij)}_x, \forall i, j \in [l], i \neq j.\)
\[
\begin{pmatrix}
\hat{F}^{(1)} & \hat{F}^{(12)} & \cdots & \cdots & \cdots & \hat{F}^{(11)} \\
P^{(1)}F^{(1)} & P^{(1)}F^{(12)} & \cdots & \cdots & \cdots & P^{(1)}F^{(11)} \\
\vdots & \hat{F}^{(22)} & \cdots & \cdots & \cdots & \vdots \\
\vdots & P^{(2)}F^{(22)} & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\
\hat{F}^{(l)} & \vdots & \cdots & \cdots & \cdots & \vdots \\
P^{(l)}F^{(l)} & \vdots & \cdots & \cdots & \cdots & P^{(l)}F^{(l)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
B^{(1)} & \cdots & \cdots & \cdots & B^{(l)} \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
B^{(1)} & \vdots & \cdots & \cdots & \vdots \\
\end{pmatrix}
\approx M_x.
\]
The following examples illustrate that optimal encoding matrix for $I_E$ can be obtained even if non optimal encoding matrices are used for some of the constituent SUICPs, as long as the conditions given in Theorem 2 are satisfied.

**Example 3.** Considering the SUICPs $I_1$ and $I_2$ in Example 2 we show that we can obtain an optimal code for a jointly extended problem using a non optimal code for $I_2$, as long as $mrk(F_x^{(1)}) \geq r_2$. $F^{(1)}$ is same as in Example 2 We take a completion of $F_x^{(2)}$ as shown below.

$$F^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$P^{(1)}$ is as given in the previous example. The minrank of $I_2$ is 2. Let $F_x^E$ be the fitting matrix for $I_E$, partitioned as shown below.

$$F_x^E = \begin{pmatrix} 1 & x & x & 0 & 0 & x & 0 & 0 \\ 0 & 1 & x & 0 & 0 & 0 & x \\ 0 & 0 & 1 & x & x & 0 & 0 \\ x & x & 0 & 0 & 1 & x & x \\ x & x & 0 & x & 0 & 0 & 1 \\ 0 & x & 0 & x & 0 & 1 & 0 \\ 0 & 0 & x & x & 0 & x & 1 \\ 0 & 0 & x & x & 0 & 0 & x \end{pmatrix}.$$

The optimal $G_E$ as obtained from Theorem 2 is shown below.

$$G_E = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

**Example 4.** Let the fitting matrices of SUICPs $I_1$ and $I_2$ be given as shown below along with their completions.

$$F_x^{(1)} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, F_x^{(2)} = \begin{pmatrix} 1 & 0 & x \\ 0 & x & 1 \\ 0 & x & 1 \end{pmatrix}.$$

$$F^{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, F^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that $P^{(1)} = (1 \ 1)$. We know that $r_2 = 3$, $\text{minrk}(F_x^{(2)}) = 2$ and $r_1 = \text{minrk}(F_x^{(1)}) = 2$. Let $F_x^E$ be the fitting matrix partitioned as shown below.

$$F_x^E = \begin{pmatrix} 1 & x & 0 & x & 0 & 0 \\ 0 & 1 & x & 0 & x \\ 0 & x & 0 & x & 0 \\ x & x & 0 & 1 & 0 \\ x & 0 & x & 1 & 0 \\ x & 0 & x & 0 & x \end{pmatrix}.$$

An encoding matrix $G_E$ as obtained using Theorem 2 is shown below. However, no claim on the optimality of $G_E$ can be made.

$$G_E = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

**V. Application of Joint Extension to Class D and Class E of TUICP**

In this section, we apply the results presented in section IV on joint extensions to obtain upper bounds on optimal scalar linear code lengths for a sub-class of TUICPs belonging to class D and class E as shown in Figure 2 Then, we provide conditions for optimality of the codes.

**Theorem 3.** Let the TUICP given by $I(D, \overline{M})$ be such that $f(D) \in \{J_{91}, ..., J_{92}\}$. Let the interaction between $D_1$ and $D_3$ be given by $F_x^{(13)}$. Let the fitting matrices of the SUICPs described by $D_2$ be $F_x^{(2)}$. Let the submatrices $A_T^{(13)}$ and $C_T^{(13)}$ of $F_x^{(13)}$ be given by $B_T^{(13)}$ and $D_T^{(31)}$ as given in section IV. Then, $l(D, \overline{M}) \leq \text{mrk}(D_2) + \text{max}(\text{mrk}(D_1), \text{mrk}(D_3))$. If $\text{max}(\text{mrk}(D_1), \text{mrk}(D_3)) = \text{mrk}(D_1)$, then $l(D, \overline{M}) = \text{mrk}(D_1) + \text{mrk}(D_2)$.

**Proof.** Irrespective of the interaction between the $D_i$’s, let $S_2$ broadcast optimal scalar linear index code for the SUICP represented by $D_2$, which is of length given by $\text{mrk}(D_2)$. From the result on jointly extended problems, we see that considering the sub-problem given by $F_x^{(13)}$, we see that $S_1$ can transmit an optimal scalar linear code of length $\text{max}(\text{mrk}(D_1), \text{mrk}(D_3))$. From Lemma 5 which gives lower bound, and the given code construction which gives upper bound, we see that they are equal when $\text{max}(\text{mrk}(D_1), \text{mrk}(D_3)) = \text{mrk}(D_1)$. Hence, the result.

We see that TUICPs with interactions given by class E are obtained from those of class D by interchanging the roles of $D_1$ and $D_2$. Hence, we state a corresponding result. The proof is similar to that of Theorem 3.

**Theorem 4.** Let the TUICP given by $I(D, \overline{M})$ be such that $f(D) \in \{J_{93}, ..., J_{94}\}$. Let the interaction between $D_2$ and $D_3$ be given by $F_x^{(23)}$. Let the fitting matrices of the SUICPs described by $D_1$ be $F_x^{(1)}$. Let the submatrices $A_T^{(23)}$ and $C_T^{(23)}$ of $F_x^{(23)}$ be given by $B_T^{(23)}$ and $B_T^{(32)}$ as given by section IV. Then $l(D, \overline{M}) \leq \text{mrk}(D_1) + \text{max}(\text{mrk}(D_2), \text{mrk}(D_3))$. If $\text{max}(\text{mrk}(D_2), \text{mrk}(D_3)) = \text{mrk}(D_2)$, then $l(D, \overline{M}) = \text{mrk}(D_1) + \text{mrk}(D_2)$.

We illustrate Theorem 2 with an example.

**Example 5.** Let $F_x^{(13)}$ be given by $F_x^E$ as in example 2 with $F_x^{(1)}$ and $F_x^{(3)}$ given by top left submatrix and bottom right submatrix shown by the partition. Let $F_x^{(12)}$ and $F_x^{(32)}$ be given
as shown below. From the given interactions $f(D)$ is $J_{52}$.

$$F^{(13)}_x = \begin{pmatrix}
1 & x & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & x & 0 & x & 0 & 0 \\
0 & 0 & 1 & x & 0 & 0 & 0 \\
x & 0 & 0 & 1 & x & 0 & x \\
x & x & 0 & 0 & 1 & 0 & 0 \\
x & 0 & x & 0 & x & 0 & 0 \\
x & x & x & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

$$F^{(12)}_x = \begin{pmatrix}
1 & x & 0 & 0 & x & x & x \\
0 & 1 & x & 0 & x & 0 & 0 \\
0 & 0 & 1 & x & 0 & 0 & x \\
x & 0 & 0 & 1 & x & 0 & x \\
x & x & 0 & 0 & 1 & 0 & 0 \\
x & 0 & x & 0 & x & 0 & x \\
x & x & 0 & 0 & x & 0 & 0 \\
\end{pmatrix}$$

$$F^{(23)}_x = \begin{pmatrix}
1 & 0 & 0 & x & 0 & 0 & 0 \\
x & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & x & 1 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 1 & 0 & x \\
x & 0 & 0 & x & 0 & 1 & 0 \\
x & 0 & 0 & x & 0 & x & 1 \\
\end{pmatrix}$$

From example 2 we know that the encoding matrix for the problem described by $F^{(13)}_x$ is as given below.

$$G_E = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix}$$

Hence, a solution to this TUICP according to Theorem 3 is as follows. $S_1$ transmits

$x_1^{(1)} + x_3^{(1)} + x_4^{(1)} + x_5^{(1)} + x_1^{(2)} + x_2^{(2)} + x_2^{(3)} + x_4^{(2)} + x_1^{(3)} + x_3^{(3)}$, $S_2$ transmits $x_1^{(2)} + x_2^{(2)} + x_1^{(3)} + x_2^{(3)} + x_3^{(3)}$. It can be easily verified that all the receivers can decode their demands from the given sub-codewords. Since $mrk(D_1) = mrk(D_2) = 3$ and $mrk(D_3) = 2$, this is an optimal scalar linear index code for the TUICP.

VI. CONCLUSION

In the current work, the TUICP is studied using its three disjoint sub-problems and the interactions among them. Optimal scalar linear index codes for some classes of TUICP are provided using those of the sub-problems. Jointly extended SUICPs are defined and a class of jointly extended problems are solved for optimal scalar linear index codes. This is used to obtain optimal scalar linear codes for some classes of TUICP. More classes of jointly extended problems can be explored, which not only solve bigger SUICPs (using smaller solved SUICPs), but can be used to obtain results in multi-sender ICP. Obtaining optimal scalar linear codes for the remaining classes of TUICP is left for future work. The presented results can be generalized to get results in multi-sender ICP.

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