On anti-Ramsey numbers for complete bipartite graphs and the \textit{Turán} function

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Abstract

Given two graphs $G$ and $H$ with $H \subseteq G$ we consider the anti-Ramsey function $AR(G, H)$ which is the maximum number of colors in any edge-coloring of $G$ so that every copy of $H$ receives the same color on at least one pair of edges. The classical Turán function for a graph $G$ and family of graphs $\mathcal{F}$, written $ex(G, \mathcal{F})$, is defined as the maximum number of edges of a subgraph of $G$ not containing any member of $\mathcal{F}$. We show that there exists a constant $c > 0$ so that $AR(K_n, K_{s,t}) - ex(K_n, K_{s,t}) < cn^{2-\frac{1}{t}}$, for $s \leq t$ by a result of Kővari, Sós, and Turán.

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1 Introduction

For basic graph theoretic terminology and definitions see Diestel \cite{Di}. For specific definitions, we follow \cite{1}. Given a graph $H$ and an edge-coloring $c$ of $H$, we say that $c$ is \textit{rainbow} if no two edges of $H$ receive the same color. Given a copy $A$ of $K_{s,t}$ for $t \geq s$, let $X(A)$ and $Y(A)$ denote the parts of $A$ of order $s$ and $t$, respectively and call them the \textit{interior} and the \textit{exterior}. For any $l > s$, consider the set of vertices $U = \{u_1, \ldots, u_l\}$, and let $T \subseteq \binom{U}{s}$, the set of $s$-tuples of $U$. Let $T = \{x_1, \ldots, x_k\}$ where $x_i$ is an $s$-tuple for $1 \leq i \leq k$. If $x_1 \cap x_2 \neq \emptyset, \ldots, x_{k-1} \cap x_k \neq \emptyset$, and $S$ is a graph containing $T$ so that $S$ can be partitioned into $k$ edge-disjoint copies $A_1, \ldots, A_k$ of $K_{s,t}$ with $X(A_i) = x_i$ for $1 \leq i \leq k$, then we call $S$ a $K_{s,t}$-\textit{string} of length $k$. Furthermore, if $x_1 \cap x_k \neq \emptyset$, then we call $S$ a $K_{s,t}$-\textit{ring} of length $k$. If $S$ is a $K_{s,t}$-string and there exists a vertex $x \notin \{x_1, \ldots, x_k\}$, adjacent to $s$ vertices of $x_1$ and a vertex of $x_k$ which is not in $x_1$, then we call $S$ a $K_{s,t}$-\textit{string-tie}.

Let $G$ be a graph and $c : E(G) \to \mathbb{Z}$ a coloring of $E(G)$. A \textit{representing graph} of $c$ is a spanning subgraph $L$ of $G$ containing exactly one edge of each color of $c$.

Given a multigraph $G$ we define the \textit{edge-multiplicity} $m(G)$ as the maximum number of edges between two vertices $x$ and $y$.

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The Turán function for a graph \( G \) and family of graphs \( \mathcal{F} \), written \( ex(G, \mathcal{F}) \), is defined as the maximum number of edges of a subgraph of \( G \) not containing any member of \( \mathcal{F} \).

The anti-Ramsey function for graphs \( G \) and \( H \subseteq G \), written \( AR(G, H) \), is the maximum number of colors in any edge-coloring of \( G \) so that every copy of \( H \) receives the same color on at least one pair of edges.

2 Brief History

The anti-Ramsey function and its relation to the Turán function were studied by Erdős, Simonovits, and Sós in [3], where they showed that \( AR(K_n, H) - ex(K_n, \mathcal{H}) = o(n^2) \) with \( \mathcal{H} = \{ H - e : e \in E(H) \} \). Since then, many authors have worked on determining the asymptotic order of \( AR(G, H) \) (see [4] for example). We follow the investigation of Axenovich and Jiang [1] who were able to determine that \( AR(K_n, K_{2,n}) = (\frac{\sqrt{2}}{2})n^\frac{3}{2} + O(n^{\frac{3}{4}}) \) and \( AR(K_{n,n}, K_{2,n}) = \sqrt{7} - 2n^{\frac{3}{2}} + O(n^{\frac{3}{4}}) \). We show that if we exclude all rainbow complete bipartite graphs of fixed order, we can extend the previous technique and produce a general upper bound that follows the result from [5].

**Theorem 2.1.** \( ex(K_n, K_{s,t}) \leq cn^{2 - \frac{1}{4}} \) where \( s \leq t \) and \( c \) depends on \( s \) and \( t \).

3 Excluding Rainbow Complete Bipartite Graphs

The following proposition was shown in [3]:

**Proposition 3.1.** \( ex(G, \mathcal{H}) + 1 \leq AR(G, \mathcal{H}) \leq ex(G, \mathcal{H}) \)

**Proof.** For the upper bound, any representing subgraph of an \( H \)-free coloring of \( E(G) \) is a subgraph of \( G \) containing no \( H \) subgraph. The number of colors used in a representing graph is equal to the number of edges of the representing graph = \( AR(G, H) \). However, this number of colors is also the number of edges avoiding \( H \).

For the lower bound, we consider a subgraph \( G' \) in \( G \) that has \( ex(G, \mathcal{H}) \) edges which does not contain any member of \( \mathcal{H} \) as a subgraph. Color the edges of \( G' \) using distinct colors. Color the rest of \( G \) by some other color (all same color). The resulting coloring contains no rainbow copy of \( H \) and uses \( ex(G, \mathcal{H}) + 1 \) colors. \( \square \)

The arguments in the following two lemmas are similar to the case when \( s = 2 \) which can be found in [1].

**Lemma 3.2.** If \( c \) is a coloring of \( E(K_n) \) with no rainbow \( K_{s,t} \), then \( c \) does not contain a rainbow \( K_{s,t-1}\)-string-tie.

**Proof.** Let \( M \) be a rainbow \( K_{s,t-1} \) string-tie in \( c \) that is of minimum length. Let the interior \( X = \{ x_1, \ldots, x_k \} \) where \( x_i \) are \( s \)-tuples and let the copies of \( K_{s,t-1} \) that form \( P \) be labeled \( B_1, \ldots, B_k \) where \( X(B_i) = \{ x_i \} \). Suppose \( M \) is obtained from a string \( P \) of length \( k \) by adding a vertex \( x \) (not in \( P \)) and making it adjacent to a vertex \( u_k \in x_k \backslash x_1 \), and the vertices \( s_1 \subseteq x_1 \) where \( |s_1| = s - 1 \). If \( k = 2 \), then \( M \) is a rainbow \( K_{s,t} \). Let us assume that \( k \geq 3 \). Let \( M_1 = B_1 \cup x_{s_1} \) and
deletion. Notice that the resulting graph is simple with $n$ edges, so it must contain a cycle $C_H$ edges and edge multiplicity $s$. Thus, $xu_2$ completes a rainbow $K_{s,t}$-string-tie with either $M_1$ or $M_2$ which is shorter than $M$ and a contradiction.

**Lemma 3.3.** If $H$ is a graph not containing a $K_{s,t}$-string-tie, then $H$ is not a $K_{s,t}$-ring.

**Proof.** Suppose that $H$ is a $K_{s,t}$-ring of length $k$, $X(H) = \{x_1, \ldots, x_k\}$ where $x_i$ are $s$-tuples and all the copies of $K_{s,t}$ forming $H$ are $B_1, \ldots, B_k$ with $B_i = \{x_i\}$. Suppose first that the $Y(B_i)$ are pairwise disjoint so that $|Y(H)| = kt > k(s-1)+1 \geq |X(H)|$. By the pigeonhole principle, there exists $w \in Y(H) \setminus X(H)$. Without loss of generality, assume $w \in Y(B_1)$. Note that for $s_1 \subseteq x_1$ where $|s_1| = s-1$ and a vertex $u_k \in x_k \setminus x_1$, $\bigcup_{i=2}^k B_i \cup \{ws_1, wu_k\}$ is a $K_{s,t}$ string-tie.

Next, assume that there exists $l_1 < l_2$ such that $Y(B_{l_1}) \cap Y(B_{l_2}) \neq \emptyset$. Without loss of generality, suppose $l_1 = 1$ and $l_2$ is as small as possible. Let $v \in Y(B_{l_1}) \cap Y(B_{l_2})$. Let $l_3 = \max\{i \in [k]: v \in Y(B_i)\}$. Notice that $l_3 \geq l_2$. By the above observation, we have $l_2 - 1 \geq 2$ and $l_3 \leq k - 1$. Since the vertices in $X(H)$ are all distinct, $v$ is not a member of at least one of $\{x_2, \ldots, (x_{l_2}, x_{l_2+1})\}$ or $\{x_{l_3+1}, \ldots, x_{k}, (x_1, x_2)\}$. Without loss of generality, suppose the statement holds for the first set. By our choice of $l_2$, we have $v \notin Y(\bigcup_{i=2}^{l_2-1} B_i)$ and hence $v \notin \bigcup_{i=2}^{l_2-1} B_i \cup \{vx_2, v(x_{l_2}, x_{l_2+1})\}$ contains a $K_{s,t}$-string-tie.

The next lemma is the key step to generalizing beyond the exclusion of $K_{2,t}$.

**Lemma 3.4.** If $G' \subseteq G$, $|G| = n$, and $G'$ does not contain a $K_{s,t}$-ring where $t \geq s$, then $\|G\| \leq ex(G, K_{s,t}) + st(n-1)$.

**Proof.** We argue by contradiction and assume that $H$ is a maximal collection of pairwise edge-disjoint $K_{s,t}$ in $G'$ and that $H$ contains $k$ copies of $K_{s,t}$. Note that $H$ contains $kst$ edges and that removing the edges of $H$ from $G'$ leaves no copies of $K_{s,t}$. Combining this observation with our assumption produces

\[
ex(G, K_{s,t}) + st(n-1) < e(G') \leq ex(G, K_{s,t}) + kst
\]

from which we see that $k > n - 1$.

Next we construct a graph $F$ so that $V(F) = V(G')$ and for every member $A$ of $H$ where $X(A) = \{u_i, \ldots, u_s\}$, we create the path $u_{i_1} \ldots u_{i_s}$ in $F$. We note that two such paths may intersect on at most $s-1$ vertices and produce no more than $s-2$ multiple edges of multiplicity 2. Thus, $F$ is a loopless multigraph with $k(s-1)$ edges and edge multiplicity $m(F) \leq k$. For every pair of vertices of $F$ with at least two edges, we delete all but one edge between those vertices. Since the maximum number of multiple edges is $k(s-2)$, we are left with at least $k$ edges after the deletion. Notice that the resulting graph is simple with $n$ vertices and $k > n - 1$ edges, so it must contain a cycle $C$. The edges of $C$ are incident to vertices $X(A)$ where $A$ are members of $H$, and therefore a subgraph containing vertices of $C$ forms a $K_{s,t}$-ring in $G'$ which is a contradiction.

**Theorem 3.5.** For $s \leq t$ there exists a constant $c$, so that

$$AR(K_n, K_{s,t}) - ex(K_n, K_{s,t-1}) \leq cn$$
Proof. Given a $K_{s,t}$-free coloring $c$ of $E(K_n)$ with $AR(K_n, K_{s,t})$ colors and a representing graph $H$ of $c$, we apply lemmas 3.2-3.4 in sequence, to obtain the result.

The above theorem together with Theorem 2.1 immediately gives the following

**Corollary 3.6.** $AR(K_n, K_{s,t}) \leq cn^{2-\frac{1}{s}}$ where $c$ depends on $s$ and $t$.

Note: We can extend our result to $K_{n,n}$ by repeating the above argument as in [1], to produce $AR(K_{n,n}, K_{s,t}) \leq cn^{2-\frac{1}{s}}$ where $c$ depends on $s$ and $t$.

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