The Encyclopedic Reference of Critical Points for $SO(8)$-Gauged $\mathcal{N} = 8$ Supergravity

Part 1: Cosmological Constants in the Range $-\Lambda/g^2 \in [6; 14.7)$

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Abstract

This article is part of a collection that strives to collect and provide in an
unified form data about all the critical points on the scalar manifold of $SO(8)$-
\textit{gauged} $\mathcal{N} = 8$ supergravity in four dimensions known so far. The vast ma-
jority of these were obtained using the enhanced sensitivity backpropagation
method introduced by the author in 2008.

This part of the collection describes 41 critical points, 7 of which have been
known for more than two decades, 8 of which were discovered recently, and 26
are novel. The residual gauge symmetries of these 41 critical points (likely) are
$SO(8)$ with $N=8$ SUSY (1x), $SO(7)$ (2x), $SU(4)$ (1x), $G2$ with $N=1$ SUSY (1x),
$SU(3)\times U(1)$ with $N=2$ SUSY (1x), $SO(3)\times SO(3)$ (2x), $SO(3)\times U(1)\times U(1)$ (1x),
$SO(3)\times U(1)$ (3x), $SU(3)$ (3x), $U(1)\times U(1)$ with $N=1$ SUSY (1x), $U(1)\times U(1)$
without SUSY (4x), $U(1)$ (11x), and None (10x). Analytic conjectures (not
yet proven but overwhelmingly likely correct) are given for the locations and
cosmological constants of some critical points.

0 About this document

Thanks to novel powerful methods introduced by the author in \cite{1, 2}, the symme-
try breaking structure of field theories with large numbers of scalars can now be
investigated to unprecedented depth. A systematic application of these methods to
$SO(8)$-gauged $\mathcal{N} = 8$ supergravity \cite{3, 4} greatly expanded our knowledge of loca-
tions and properties of critical points, increasing the number of known solutions of
the stationarity condition from seven to well over one hundred.

While these new methods manage to produce a far more detailed picture than
earlier approaches, they neither are exhaustive, nor do they give analytical results
straightaway. (It is indeed possible to obtain analytic expressions in a fully auto-
mated way, as has been demonstrated for some novel critical points, but this often
requires considerable computing time.) Hence, it is natural to expect that many
more details will become known about this previously mostly uncharted territory,
the scalar sector of gauged maximal four-dimensional supergravity. These critical
points are interesting for a number of reasons, not least because we think to have
some useful degree of control over the mathematical techniques needed to work with
supergravity backgrounds.

Traditionally, the discovery and detailed discussion of one or a few new solutions
of the stationarity conditions would have warranted a dedicated research article, cf.
e.g. \cite{5, 6, 7, 8}. If the novel methods now available would have led to a mere
doubling of the number of known critical points, it may have been appropriate to
discuss them individually in detail in one article each. Now that the supergravity
scalar potential turned out to have a much more complicated structure than what
one may have anticipated, the question of \textit{organizing} and presenting the data in a
useful way becomes quite relevant. Pondering over the detailed list, one can make a few remarkable observations; often, there are pairs of critical points with quite similar properties, sometimes perhaps related to one another by triality, sometimes not (or not obviously) so. Clearly, such things would be very difficult to spot were all this information scattered over many individual articles, each perhaps using different conventions for the parametrization of some particular scalar sub-manifold under study. Therefore, a compact and unified overview presentation of all results makes sense.

0.1 Purpose
The purpose of this series, initially consisting of four parts, is to:

- Collect information about critical points of maximal $SO(8)$-gauged supergravity in four dimensions that have become known.
- Provide (at least on the arXiv) an up-to-date reference listing solutions, properties, and research articles where they are used or investigated further in a systematic way.

It is planned to provide updates to the master list on arXiv whenever new solutions become known, or relevant new properties of specific solutions are discovered, accompanied by separate overviews over updates after relevant major new discoveries.

In order to make this update process manageable and accessible:

- The complete list has been split into parts of manageable size, each containing about 40 solutions in the version 1 release. Each part can receive independent updates and grow independently from one another, using independent version numbers.
- The discussion of properties of solutions carefully is worded in such a way that global statements which may be invalidated by further discoveries (such as “this is the only known solution with property X”) are avoided. This ensures that each part can evolve without the need for ever deleting statements, or for modifying statements that involve different parts of the collection.

This means, of course, that global statements such as “there is only one known non-supersymmetric solution that satisfies the Breitenlohner-Freedman stability condition” (true at the time of the release of the collection of ‘version 1’ parts) cannot be made. This should be kept in mind when using the list.

A “big picture” overview of the solutions contained in the initial release of the collection will be given in a separate article [9].

Two questions naturally arise here: One may wonder whether the number of solutions actually might be so large that such a classification effort might be futile. Also, the question how the claimed properties of such a large number of solutions can be validated is quite relevant.

Concerning the expected total number of solutions, including yet undiscovered ones, the raw data produced by the deep search that resulted in the greatly enlarged list contained numerous duplicates; while the methods used so far seem to systematically miss certain solutions, it also now appears somewhat unlikely for the total number of critical points to be larger than 1000 or so.

The validation problem is addressed by the article [10]. The arXiv source code of that article, available at http://arxiv.org/e-print/0912.1636, contains a direct Python transcription of the definitions presented in detail in [2] that can be used to numerically check all claims about critical points. Raw numerical data files for each
of the published solutions can be obtained by downloading the source file of the corresponding part of the collection from arXiv. The numerical accuracy provided by these data files considerably goes beyond the number of digits actually listed in the text.

0.2 Structure of the tables

Each table lists, in its headline, a boldface identifier that uniquely names the critical point. As it is expected that new solutions will be discovered that fall between known ones, these identifiers cannot be consecutive numbers. As the cosmological constant turns out to be a useful proxy to discern solutions, these names are based on the value of the potential at the critical point. In most cases, the name of a stationary point consists of the letter ‘S’ followed by a seven-digit code which is the truncated (rather than rounded) cosmological constant multiplied by $-10^5$. The headline furthermore lists the value of the potential to somewhat greater accuracy, and the base-10 logarithm of the Frobenius norm square of the ‘Q-tensor’ (which gives the violation of the stationarity condition) for the numerical solution. This roughly matches the number of valid digits in the cosmological constant; as the potential is quadratic around a stationary point, the number of valid digits in the $\phi$ coordinates is roughly half the number of valid digits in the potential. Furthermore, the headline lists information about the dimension of the residual unbroken gauge group (if any – many solutions do not have any residual symmetry), and possible the amount of supersymmetry.

The headline is followed by a block listing the non-zero coordinates of the 70-dimensional vector $\phi$. These have been converted from the $35_s+35_c$ form used in [2] to the language of self-dual/anti-self-dual 4-forms. A term such as $-0.2011801_{1238s}$ e.g. means that the corresponding contribution to the symmetric traceless matrices over the spinors (index tag ‘$s$’) is given by $-0.201180 \cdot \gamma_{1238}$. Users of these tables may find both the validation code given in [10] as well as data describing previously known critical points useful to match their conventions against the ones used here. As discussed in [1, 2], the minimization of $|Q|^2$ (the violation of the stationarity condition) via sensitivity backpropagation produces a solution in random position on a gauge group orbit. A second optimization step is used to find a gauge group rotation that sets as many coordinates to zero as possible, giving both a nice presentation with substantially fewer than 70 nonzero coordinates, as well as numerically suggesting simplifying relations between coefficients. While this second optimization step usually is much faster than the first, there are some odd cases where it has a tendency to ‘get stuck’, resulting in a presentation with many more nonzero entries than necessary. It is quite possible that some (few) of the solutions listed in the tables suffer from this problem.

Finally, each table contains three blocks listing the (ordered) masses-squared of gravitini ($m^2/m_0^2$), spin-1/2 fermions, and the masses of scalars, with multiplicities. All these masses are given in AdS units. Each unbroken supersymmetry corresponds to a gravitino mode with mass-squared +1, and the Breitenlohner-Freedman [11, 12] stability criterion for scalar masses is $(m/m_0)[\phi] \geq -(d-1)^2/4 = -2.25$.

While these tables give a first numerical overview over properties of solutions, and all numbers shown are expected to be correct to as many digits as are actually listed, it must again be pointed out that the raw source files (obtainable from arXiv via the ‘other formats’ article download format link) contain more accurate data than what is shown in print. These hence may be useful for some investigations.
1 Introduction

Supersymmetry \cite{13} is a highly nontrivial extension of space-time symmetries with far reaching consequences. As supersymmetry transformations mix bosonic and fermionic degrees of freedom, supersymmetric field theories are constrained both in their particle spectra as well as in the form of their interaction terms. These constraints become stronger as one goes from minimally supersymmetric field theories to models with extended supersymmetry. One finds that it is possible to not only construct supersymmetric field theories in flat space, but also to supersymmetrize general relativity, which leads to supergravity \cite{14, 15}. As one can obtain a spacetime translation from two supersymmetry transformations, a supersymmetric theory including general relativity must necessarily feature supersymmetry as a local symmetry, which then gives rise to diffeomorphisms rather than constant translations (which in general do not exist in curved spacetime). Both the attempt to promote supersymmetry to a local (gauge) symmetry as well as the attempt to combine supersymmetry with general relativity hence lead to the same goal, namely supergravity. Therefore, should supersymmetry be proven experimentally to play a role in particle physics, then accepting general relativity as the appropriate classical description of gravity inevitably forces us to accept the idea of supergravity. Just as there are non-gravitational field theories with extended (i.e. more than minimal) supersymmetry, there also are supergravities with extended supersymmetry.

The simplest supergravity features a spin-2 graviton as well as a spin-3/2 gravitino. Each additional supersymmetry enlarges the supermultiplet helicity range by 1/2, up to a maximum of eight times the minimal amount of supersymmetry. In this maximally supersymmetric supergravity, the helicity +2 graviton state is unified with all other particle states down to the helicity -2 graviton state into a single supermultiplet whose structure is completely fixed by supersymmetry. As gravitational theories with more than the minimal number of graviton states do not seem to make sense, and it seems impossible to construct theories of interacting massless particles with helicity larger than 2, the maximal amount of supersymmetry in a 3+1-dimensional field theory with gravity is constrained to be \( \mathcal{N} = 8 \) times the minimal amount of supersymmetry, with 32 supersymmetries in total (eight real four-spinors).

Considering the particle content of this \( \mathcal{N} = 8 \) supergravity, which is completely fixed by supersymmetry, there are one graviton, eight gravitini \( \psi \), 28 vectors \( A \), 56 spin-1/2 fermions \( \chi \), and finally a total of 70 scalars \( \phi \). Historically, the construction of the Lagrangian for \( \mathcal{N} = 8 \) supergravity did not succeed by using the iterative procedure (the Noether method) employed for models with less supersymmetry, due to the complicated interactions between the scalars. Ultimately, the construction succeeded by a detour employing a Kaluza-Klein reduction \cite{16}: first, a supergravity was constructed in the maximum possible spacetime dimension that permits fermion-boson matching (and has one time direction). This gave rise to eleven-dimensional supergravity \cite{17}, which, when compactified on a 7-torus \( T^7 \) to four spacetime dimensions, gives the desired \( \mathcal{N} = 8 \) supergravity \cite{18}. A surprising discovery was that generalized electric-magnetic duality plays a very prominent role for this theory and involves a non-compact real form of one of the exceptional simple Lie groups in the Cartan-Dynkin classification, the group \( E_{7(7)} \). Specifically, the scalars of the theory parametrize the 70-dimensional riemannian symmetric space \( E_{7(7)}/SU(8) \). While this, like 11-dimensional supergravity, was originally regarded mostly as a mathematical curiosity, the modern perspective considers 11-dimensional supergravity to play a central role as the low energy limit of an at the time of this writing only partially understood ‘mother’ theory (‘\( \mathcal{M}\)-Theory’) which, in different limits, gives rise to the five 10-dimensional superstring theories (and supergravities in various dimensions) \cite{19}. The emergence of the \( E_{7(7)} \) symmetry
group is now considered to be a consequence of duality symmetries, already present in higher dimensions, of \(\mathcal{M}\)-Theory. It is likely that the most promising strategy to obtain a deeper understanding of the structure and properties of \(\mathcal{M}\)-Theory is to focus on these ‘\(U\)-duality’ symmetries \[20, \ 21\].

Closer investigation of \(\mathcal{N} = 8\) supergravity in four-dimensional spacetime has shown that maximal supersymmetry does not fix the construction completely; it is possible to promote the 28 vector bosons to gauge bosons of a non-abelian symmetry, in particular \(SO(8)\) \[4\]. This corresponds to a compactification of 11-dimensional supergravity on the 7-sphere \(S^7\).

Surprisingly, it was later found that it is just as well possible to alternatively introduce certain \emph{non-compact} nonabelian gauge groups. In particular, it is e.g. possible to use non-compact real forms \(SO(p, 8 - p)\) instead of \(SO(8)\). In supergravity, this works (and does not introduce ghosts) because the kinetic term for the vectors is not of the form \(\sim \delta_{AB} F^A F^B\) (\(A, B\) being gauge group adjoint representation indices), but of the form \(\sim S_{AB} (\phi) F^A F^B\) with some positive definite symmetric inner product \(S\) that depends on the scalars \[22, \ 23\].

In the 80s, the construction of a fully unified theory of matter and force particles including gravity that was pretty much uniquely determined by its symmetry properties gave rise to considerable enthusiasm. This made some authors (most notably Stephen Hawking in his inaugural lecture as the Lucasian Professor of Mathematics at Cambridge \[24\]) proclaim that we would, in the not-too-distant future, have a Unified Theory of Everything. Considering attempts to link \(\mathcal{N} = 8\) supergravity with experimentally observed particle physics immediately face a number of problems; perhaps most importantly, it is not possible to accommodate the standard model gauge group \(SU(3) \times SU(2) \times U(1)\) in \(SO(8)\). For this and other reasons, present interest in \(\mathcal{N} = 8\) supergravity is more due to its remarkable mathematical properties, as well as its relation to three-dimensional CFTs via the AdS/CFT correspondence \[25\] than due to it being a viable theory of quantum gravity containing the Standard Model. While initial hopes that supergravity may directly give renormalizable quantum theory of gravity via boson-fermion loop cancellations turned out to be too optimistic, the renormalizability question is not settled yet for \(\mathcal{N} = 8\) supergravity \[26, \ 27, \ 28\]. It is already clear that \(E_7\) symmetry plays a crucial role for the question whether \(\mathcal{N} = 8\) supergravity may make sense in the UV limit.

Whenever some symmetry that rotates supersymmetry generators into one another is gauged, extra terms have to be added to the Lagrangian. In particular, a potential for the scalar arises in second order of the gauge coupling constant. This scalar potential is a quadratic function of the mass matrices (and couplings) of the fermionic fields. In the simplest case, this potential is just a cosmological constant \[29\], but for theories with more complicated scalar sectors, it can be fairly involved. Both the possibility to gauge extended supergravities as well as the scalar potential arising from this procedure are general features of extended supergravities in various dimensions. These potentials have a number of properties that at first seemed rather awkward; they give rise to a (typically negative) Planck-scale cosmological constant, are unbounded from below, and stationary points normally are saddle points (or, in some cases, maxima), rather than minima. However, it has been shown that even saddle points can lead to (at least perturbatively) stable vacua, as one has to carefully analyze the interplay between kinetic and potential energy in a negatively curved background spacetime \[11, \ 12\].

It is easy to see why conventional approaches to analyze the potentials of gauged supergravities with a large number of scalars cannot succeed: one would first have to construct an analytic parametrization of a high-dimensional coset manifold such as \(E_7(7)/SU(8)\), then use this parametrization to express the potential in terms of (hyperbolic) sines and cosines of the coordinates, and finally solve a complicated set of coupled polynomial equations. Even the first step – parametrizing the coset...
manifold—would generate, in the $E_7$ case, a complex $56 \times 56$ matrix with entries that are polynomial in (hyperbolic) sines/cosines of 70 variables. As a substantial fraction of all conceivable combinations of factors does indeed arise, one would guess estimate the order of magnitude of the number of terms in each matrix entry to be around $2^{70} \sim 10^{21}$—certainly well beyond reach. It is nevertheless possible to modify this approach in such a way that it becomes analytically feasible, but the price to be paid is that the analysis is constrained to solutions with a large amount of residual gauge symmetry. Basically, the idea is that if $H$ is a subgroup of the gauge group $G$ and $M_H$ is a $H$-invariant submanifold of the scalar manifold, then any critical point $P_H$ on $M_H$ must also be a critical point on the full scalar manifold, as the first-order term in the potential expanded around $P_H$ is $G$-invariant and there is no way to form a $G$-singlet from a ‘1-element tensor product’ of nontrivial $H$-representations.

The first detailed studies of these potentials were based on such techniques, with which a number of solutions were found in the 80s. The present approach does not have such limitations and indeed produced a large number of solutions that break $SO(8)$ completely and leave no residual gauge symmetry. However, it still has shortcomings. In particular, it randomly ‘fishes’ for solutions and will produce some stationary points with much greater likelihood than others; in particular, relative probabilities are such that it seems to effectively miss some solutions that were obtained by other means, cf. the discussion of critical points S0983994 and S1400000.

## 2 Critical Points and their Properties

The following tables list properties for all critical points that have been found so far (either by conventional means, or by the novel sensitivity backpropagation methods) and have a cosmological constant in the range given in the title of this part of the series.

### Table 1: Critical Points S0600000

| $\phi$ | $m^2/m_0^2$ | $m/m_0$ |
|--------|------------|---------|
| 0      | $+1.000000$ | $-2.000000$ |

### Table 2: Critical Points S0668740

| $\phi$ | $m^2/m_0^2$ | $m/m_0$ |
|--------|------------|---------|
| $-0.201180_{1238}$ | $+2.700000_{+0.075000}$ | $+6.000000_{+0.000000}$ |

### Table 3: Critical Points S0698771

| $\phi$ | $m^2/m_0^2$ | $m/m_0$ |
|--------|------------|---------|
| $-0.201180_{1238}$ | $+2.700000_{+0.075000}$ | $+6.000000_{+0.000000}$ |
| Page 7 |
| $(m^2/m_0^2) [\psi]$ | $+2.326179_{(x;0)}$; $+1.874815_{(x;0)}$ |
| $(m^2/m_0^2) [\chi]$ | $+1.465245_{(x;0)}$; $+3.749630_{(x;0)}$; $+3.745622_{(x;0)}$ |
| $(m^2/m_0^2) [\phi]$ | $+2.723885_{(x;0)}$; $+0.515437_{(x;10)}$; $+0.369264_{(x;10)}$; $+0.278072_{(x;0)}$; $+0.124339_{(x;0)}$; $+0.081783_{(x;2)}$; $+0.003699_{(x;0)}$ |

$\phi_{0}$

$|V/g|^2 = -8.80733895008$, Quality $- \log_{10}(|Q|^2) = 33.55$, $\text{dim}(GG)=6$

| $(m^2/m_0^2) [\psi]$ | $+2.049938_{(x;8)}$ |
| $(m^2/m_0^2) [\chi]$ | $+1.098107_{(x;8)}$; $+3.744343_{(x;8)}$; $+0.390544_{(x;32)}$ |
| $(m^2/m_0^2) [\phi]$ | $+0.006655_{(x;8)}$; $+0.105954_{(x;8)}$; $+0.000000_{(x;22)}$; $+0.535898_{(x;8)}$; $-1.132444$; $-1.901924_{(x;15)}$; $-10.504659_{(x;10)}$ |

$\phi_{0}$

$|V/g|^2 = -9.8994835633$, Quality $- \log_{10}(|Q|^2) = 32.75$, $\text{dim}(GG)=4$

| $(m^2/m_0^2) [\psi]$ | $+2.703129_{(x;2)}$; $+1.953125_{(x;0)}$ |
| $(m^2/m_0^2) [\chi]$ | $+5.402440_{(x;2)}$; $+3.362227_{(x;6)}$; $+3.306250_{(x;6)}$ |
| $(m^2/m_0^2) [\phi]$ | $+2.039063_{(x;2)}$; $+1.549016_{(x;10)}$; $+0.278288_{(x;6)}$; $+0.164063_{(x;6)}$; $+0.164062_{(x;6)}$; $+0.139132_{(x;6)}$ |

$|V/g|^2 = -9.98703840034$, Quality $- \log_{10}(|Q|^2) = 34.85$, $U(1)$
\[
\phi = \begin{pmatrix}
+0.004543_{123} & +0.132740_{12377} & +0.106683_{1238} & +0.162666_{12317} \\
-0.121240_{1238} & +0.523185_{12356} & -0.100317_{1258} & +0.179180_{1345} \\
-0.167907_{1367} & +0.132440_{1368} & +0.004542_{13578} & +0.121420_{1465} \\
+0.155365_{1468} & -0.100017_{1567} & +0.172114_{1578} & -0.172114_{2346} \\
+0.100317_{2345} & +0.155365_{2357} & +0.121240_{2358} & -0.004542_{2456} \\
+0.132440_{2457} & +0.167907_{2458} & -0.179180_{2657} & -0.100317_{3467} \\
+0.523185_{3478} & +0.121240_{3456} & +0.162666_{3568} & +0.106683_{34567} \\
-0.132440_{3568} & +0.004542_{3578} & -0.102479_{3265} & +0.004620_{12347} \\
+0.541792_{1238} & +0.005972_{1246} & +0.032183_{1248} & -0.093253_{1257} \\
+0.007467_{1258} & +0.111285_{1345} & +0.078664_{13467} & +0.004620_{1368} \\
-0.102479_{1378} & +0.032183_{1467} & -0.363013_{1478} & -0.007467_{1576} \\
+0.558308_{1568} & -0.583080_{2347} & -0.007467_{2348} & +0.363013_{2356} \\
-0.032183_{2358} & -0.102479_{2346} & -0.004626_{23457} & +0.078664_{2456} \\
+0.111285_{2678} & -0.007467_{2647} & -0.093253_{3468} & +0.032183_{34567} \\
+0.005972_{3578} & -0.541792_{1257} & +0.004626_{12568} & +0.102479_{124567} \\
\end{pmatrix}
\]

\[
\langle m^2/m_0^2 \rangle[\psi] = \begin{pmatrix}
+2.79126_{120} & +2.19717_{130} & +2.080922 & +1.890468_{123} \\
+1.599729 & +3.861081_{120} & +3.780930_{120} & +3.770279 & +3.749695 \\
+3.195858 & +2.563712_{120} & +2.390787 & +1.896312_{120} & +1.451463_{120} & +1.400002_{120} & +1.389043_{120} & +1.375448 \\
+0.794933_{120} & +0.583644 & +0.507523_{120} & +0.224515 & +0.201140_{120} & +0.154827 & +0.152738 & +0.130668_{120} & +0.124411_{120} & +0.103886_{120} & +0.097648_{120} & +0.080484_{120} & +0.076549 & +0.049297 & +0.040171_{120} & +0.038041_{120} & +0.028638_{120} & +0.016556 \\
\end{pmatrix}
\]

\[
\langle m^2/m_0^2 \rangle[\chi] = \begin{pmatrix}
+7.458621_{120} & +7.355300 & +5.576413_{120} & +4.883761_{120} & +4.520104 & +1.732487 & +0.000000_{120} & -0.662286_{120} & -1.006973 & -1.268409_{120} & -1.279149_{120} & -1.330024_{120} & -1.332481_{120} & -1.349294 & -1.397747_{120} & -1.605804 \\
-1.763565_{120} & -1.767763_{120} & -1.799041_{120} & -1.814667_{120} & -1.815003_{120} & -1.967626_{120} & -2.094064 & -2.196647_{120} & -2.433077_{120} & -2.475673 & -3.051457 \\
\end{pmatrix}
\]

S1006758: \[ V/g^2 = -10.0675803596, \text{ Quality } - \log_{10}(|Q|^2) = 30.19, U(1) \]

\[
\phi = \begin{pmatrix}
+0.543353_{1236} & -0.012650_{1245} & +0.185896_{1247} & -0.178390_{1258} & -0.024558_{1278} & -0.167188_{1348} & -0.234276_{1357} & -0.185890_{1456} \\
-0.012650_{1467} & +0.024558_{1568} & -0.178390_{1678} & +0.185890_{1345} & +0.024558_{2347} & +0.126502_{2358} & -0.185890_{2358} & -0.234276_{2346} \\
-0.167188_{2567} & -0.024558_{3456} & +0.178390_{3467} & +0.185890_{3568} & +0.012650_{3457} & +0.543353_{3578} & +0.342403_{2358} & -0.113920_{2346} \\
+0.025164_{1247} & +0.005184_{1236} & +0.139237_{1238} & +0.565212_{12267} & +0.025164_{12456} & +0.025164_{1456} & +0.113920_{1457} & +0.139237_{1568} \\
-0.005184_{1578} & +0.005184_{2346} & +0.139237_{2347} & +0.139237_{2346} & -0.005184_{2346} & +0.565212_{2367} & +0.139237_{3456} & +0.005184_{3457} & +0.025164_{3578} & -0.113920_{2367} & +0.342403_{3456} \\
\end{pmatrix}
\]

\[
\langle m^2/m_0^2 \rangle[\psi] = \begin{pmatrix}
+2.963527 & +2.7470417 & +2.222988 & +2.223988 & +2.223988_{120} & +2.033524 & +1.779289_{120} \\
\end{pmatrix}
\]

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\[ m^2/m_0^2 \{ \chi \} \\
\begin{align*}
+5.926113; & +4.940833; +4.417576; +4.430996(x_{22}); \\
+4.083724; & +4.067048; +3.939762; +3.814870(x_{22}); \\
+3.580705(x_{22}); & +3.558578(x_{22}); +3.207573; +2.650040(x_{22}); \\
+1.942763; & +1.759836; +1.608160(x_{22}); +1.591935; \\
+1.471656(x_{22}); & +1.075863(x_{22}); +0.983415; +0.945554; \\
+0.715150(x_{22}); & +0.595775; +0.222462; +0.188618(x_{22}); \\
+0.178994(x_{22}); & +0.149837; +0.146569(x_{22}); +0.102327(x_{22}); \\
+0.099537; & +0.093973(x_{22}); +0.093234; +0.086951(x_{22}); \\
+0.083796; & +0.047140; +0.043789(x_{22}); +0.034985(x_{22}); \\
+0.014739(x_{22}); & +0.013685(x_{22}); \\
\end{align*}

\[ m/m_0 \{ \phi \} \\
\begin{align*}
+7.101137; & +7.223490; +7.039714; +6.046063(x_{22}); \\
+5.916868; & +4.004818(x_{22}); +2.247246; +0.804249(x_{22}); \\
+0.000000(x_{22}); & -0.898234; -1.007302; -1.209946(x_{22}); \\
-1.242514(x_{22}); & -1.274356; -1.358007; -1.377478; \\
-1.384020; & -1.424337(x_{22}); -1.696104(x_{22}); -1.719716; \\
-1.775265(x_{22}); & -1.794323; -1.801592; -1.844495(x_{22}); \\
-1.886209(x_{22}); & -1.967410; -1.987750(x_{22}); -2.270745(x_{22}); \\
-2.571390; & -2.687770(x_{22}); -3.119747; \\
\end{align*}

**S1039624**: \( V/g^2 = -10.396246071 \), Quality – \( \log_{10}(|Q|^2) = 33.84 \), \( U(1) \)

\[ (m^2/m_0^2)_{\{ \psi \}} \]
\begin{align*}
+2.956723(x_{22}); & +2.280515(x_{22}); +2.239739(x_{22}); +1.656801(x_{22}); \\
+5.915448(x_{22}); & +4.579384(x_{22}); +4.461478(x_{22}); \\
+4.150924(x_{22}); & +3.865095(x_{22}); +3.788446(x_{22}); \\
+3.313602(x_{22}); & +3.139216(x_{22}); +2.821160(x_{22}); \\
+1.993222(x_{22}); & +1.710684(x_{22}); +1.430449(x_{22}); \\
+1.120788(x_{22}); & +0.951514(x_{22}); +0.830383(x_{22}); \\
+0.630034(x_{22}); & +0.005847(x_{22}); +0.370922(x_{22}); \\
+0.128437(x_{22}); & +0.104398(x_{22}); +0.082885(x_{22}); \\
+0.054524(x_{22}); & +0.051525(x_{22}); +0.038441(x_{22}); \\
+0.029004(x_{22}); & +0.021810(x_{22}); +0.006602(x_{22}); +0.006200(x_{22}); \\
\end{align*}

\[ (m^2/m_0^2)_{\{ \chi \}} \]
\begin{align*}
+7.428033; & +7.157433(x_{22}); +6.847443; +5.990270(x_{22}); \\
+4.181315(x_{22}); & +3.785710(x_{22}); +1.160452; +0.000000(x_{22}); \\
-0.364070(x_{22}); & -0.899193(x_{22}); -1.131535; -1.144653(x_{22}); \\
-1.170485; & -1.233837; -1.306560(x_{22}); -1.531413(x_{22}); \\
-1.761344(x_{22}); & -1.788939(x_{22}); -1.814783; -1.871691(x_{22}); \\
-1.902077(x_{22}); & -2.051521(x_{22}); -2.329507(x_{22}); -2.392258; \\
-2.416567(x_{22}); & -2.416694; -2.982023; \\
\end{align*}

**S1043471**: \( V/g^2 = -10.434712595 \), Quality – \( \log_{10}(|Q|^2) = 34.40 \)
$$\phi +0.000744_{1235s} +0.218613_{1236s} +0.014376_{1237s} +0.01376_{1238s} \\
-0.60069_{1239s} +0.014376_{1240s} +0.014376_{1241s} +0.218613_{1242s} \\
+0.000744_{1243s} -0.000744_{1244s} -0.191674_{1245s} +0.191674_{1246s} \\
-0.014376_{1247s} -0.014376_{1248s} +0.014376_{1249s} +0.014376_{1250s} \\
+0.191674_{1251s} +0.191674_{1252s} +0.014376_{1253s} +0.016668_{1254s} \\
+0.014376_{1255s} -0.000744_{1256s} -0.191674_{1257s} \\
+0.010668_{1258s} +0.013762_{1259s} -0.91674_{1260s} -0.191674_{1261s} \\
-0.00744_{1262s} -0.00744_{1263s} +0.218613_{1264s} -0.014376_{1265s} \\
-0.014376_{1266s} -0.60069_{1267s} -0.014376_{1268s} -0.014376_{1269s} \\
+0.218613_{1270s} -0.00744_{1271s} +0.068273_{1272s} -0.278841_{1273s} \\
+0.068273_{1274s} +0.038078_{1275s} +0.068273_{1276s} -0.306271_{1277s} \\
+0.068273_{1278s} +0.65553_{1279s} +0.306271_{1280s} +0.068273_{1281s} \\
-0.068273_{1282s} +0.068273_{1283s} -0.278841_{1284s} \\
-0.068273_{1285s} +0.306271_{1286s} +0.65553_{1287s} +0.068273_{1288s} \\
+0.306271_{1289s} +0.068273_{1290s} +0.038078_{1291s} +0.068273_{1292s} \\
+0.278841_{1293s} +0.068273_{1294s}$$

$$m^2/m_0^2[\phi] +3.023937; +2.620070_{(x; 2)}; +2.330369; +2.241258; +1.950556_{(x; 2)}; +1.650829$$

$$m^2/m_0^2[\psi] +6.046179; +5.240140_{(x; 2)}; +4.600739; +4.382516; +4.056155; +4.004562; +3.939774_{(x; 2)}; +3.901112_{(x; 2)}; +3.781889; +3.628010_{(x; 2)}; +3.301658; +3.034565; +3.014018; +2.576136_{(x; 2)}; +2.073452; +1.644046; +1.497837_{(x; 2)}; +1.394536; +1.142081; +1.051713_{(x; 2)}; +0.945559; +0.896430; +0.789259_{(x; 2)}; +0.647684; +0.612589; +0.463087_{(x; 2)}; +0.411036; +0.140304; +0.117881_{(x; 2)}; +0.107120; +0.106267; +0.080590; +0.077880; +0.071176; +0.057268_{(x; 2)}; +0.049223_{(x; 2)}; +0.031893_{(x; 2)}; +0.031455; +0.022130; +0.005559; +0.001503_{(x; 2)}; +0.001484$$

$$m/m_0 +7.396579; +7.462654; +7.383699; +6.150012_{(x; 2)}; +5.378473; +4.205243_{(x; 2)}; +4.074111; +3.516195; +1.294398; +0.496079; +0.000000_{(x; 28)}; -0.640035; -0.971212; -1.027047; -1.158021; -1.166072; -1.189389_{(x; 2)}; -1.351917; -1.376854; -1.526283_{(x; 2)}; -1.691252; -1.698745; -1.735871_{(x; 2)}; -1.791754_{(x; 2)}; -1.825650; -1.846817_{(x; 2)}; -1.862716; -1.874241; -2.118273; -2.349034; -2.406651; -2.450894; -2.567766; -2.598128; -3.075931$$

S1046017: \(V/q^2 = -10.4601767468\). Quality - \(log_0(||Q||^2) = 21.53\)
\begin{align*}
(m/m_0)_\phi & = +8.334901; +8.129854; +7.321861; +5.639686 (\times 2) ; \\
& +4.717397; +4.561460 (\times 2); +4.205247; +3.192269; \\
& +1.509513; +0.463247 (\times 2); +0.000000 (\times 26); -0.881641; \\
& -0.943848; -1.152030; -1.192406 (\times 2); -1.271731 (\times 2); \\
& -1.281890; -1.460578 (\times 2); -1.532622; -1.692522; \\
& -1.714226 (\times 2); -1.764738 (\times 2); -1.771229 (\times 2); -1.824905; \\
& -1.843184 (\times 2); -2.135287; -2.451489; -2.531327 (\times 2); \\
& -2.541931 (\times 2); -2.776048; -3.185045.
\end{align*}

S1067475: \( V/g^2 = -10.674754183, \) Quality \(- \log_{10}(|Q|^2) = 22.69, U(1) \times U(1) \)

\begin{align*}
\phi & = -0.593541_{12385} + 0.038131_{12465} - 0.199941_{12488} - 0.199941_{12671}; \\
& + 0.031708_{13278} - 0.199941_{13465} + 0.031708_{13488} + 0.033831_{13671}; \\
& - 0.199941_{13785} + 0.296371_{14578} + 0.195765_{15687} - 0.195765_{23478}; \\
& - 0.329895_{15378} + 0.031708_{15465} - 0.033831_{15488} - 0.031708_{25678}; \\
& - 0.199941_{25785} + 0.031708_{25465} - 0.199941_{25488} + 0.199941_{35674}; \\
& + 0.331315_{34678} + 0.593541_{34678} + 0.328895_{12368} - 0.328895_{12456}; \\
& + 0.328895_{25678} + 0.328895_{25468} + 0.328895_{15678} - 0.328895_{34578}; \\
\end{align*}

\begin{align*}
(m^2/m_0^2)_\psi & = +2.655750 (\times 4); +2.136690 (\times 4); \\
\end{align*}

\begin{align*}
(m^2/m_0^2)_\chi & = +5.311510 (\times 4); +4.273380 (\times 4); +3.994898 (\times 4); \\
& +3.672547 (\times 4); +3.462577 (\times 4); +1.344334 (\times 4); \\
& +1.194729 (\times 4); +0.802447 (\times 4); +0.637759 (\times 4); \\
& +0.177085 (\times 4); +0.090020 (\times 4); +0.085628 (\times 4); \\
& +0.084559 (\times 4); +0.000532 (\times 4); \\
\end{align*}

\begin{align*}
(m/m_0)_\phi & = +9.689014; +7.898348 (\times 2); +7.145826; +5.761924 (\times 2); \\
& +4.328875 (\times 2); +3.813643 (\times 2); +2.131649; +0.987027 (\times 2); \\
& +0.000000 (\times 20); -0.040019 (\times 2); -0.748284; -0.780190; \\
& -1.021059 (\times 2); -1.240824 (\times 2); -1.282105 (\times 2); \\
& -1.634993 (\times 4); -1.696293 (\times 2); -1.923737 (\times 2); \\
& -2.085550 (\times 4); -2.131594 (\times 2); -2.472441; -2.638383; \\
& -3.291881 (\times 2); -3.366751.
\end{align*}

S1068971: \( V/g^2 = -10.6897177038, \) Quality \(- \log_{10}(|Q|^2) = 21.93, U(1) \times U(1) \)

\begin{align*}
\phi & = -0.679593_{13458} + 0.679593_{26785} + 0.129174_{12355} - 0.639602_{12466}; \\
& + 0.129174_{12278} + 0.129174_{13486} + 0.390696_{13675} - 0.129174_{14157}; \\
& + 0.639602_{15268} - 0.639602_{23478} - 0.129174_{26665} + 0.390696_{23458}; \\
& - 0.129174_{34576} - 0.129174_{34658} - 0.639602_{35756} + 0.129174_{34678}; \\
\end{align*}

\begin{align*}
(m^2/m_0^2)_\psi & = +2.597091 (\times 4); +2.039597 (\times 4); \\
\end{align*}

\begin{align*}
(m^2/m_0^2)_\chi & = +5.194182 (\times 4); +4.079194 (\times 4); +4.026394 (\times 4); \\
& +3.908826 (\times 4); +2.484867 (\times 4); +1.127723 (\times 4); \\
& +0.910972 (\times 4); +0.786650 (\times 4); +0.107702 (\times 4); \\
& +0.052989 (\times 4); +0.040961 (\times 4); +0.044933 (\times 4); \\
\end{align*}

\begin{align*}
(m/m_0)_\phi & = +9.851669; +9.527399; +7.238720; +4.688077; \\
& +4.120364 (\times 4); +2.824418 (\times 4); +2.634825; +0.000000 (\times 20); \\
& -0.410361; -1.033164 (\times 4); -1.167795 (\times 4); -1.357886 (\times 2); \\
& -1.49336; -1.603920 (\times 4); -1.649304; -1.915972 (\times 4); \\
& -2.169029; -2.183838 (\times 4); -2.724879 (\times 4); -3.109369.
\end{align*}

S1075828: \( V/g^2 = -10.7582870728, \) Quality \(- \log_{10}(|Q|^2) = 33.23, \text{dim(GG)}=4 \)
$$\phi$$

$$\begin{align*}
\phi & = +0.117884_{13578} + 0.654358_{1468} + 0.382994_{1468} + 0.693190_{1468} + 0.104591_{1468} + 0.44670_{1468} + 0.249519_{1468} + 0.702362_{1468} + 0.882406_{1468} + 0.124368_{1468} \\
(m^2/m_0^2) & (\psi) = +2.562132_{1468} + 2.137868_{1468} \\
(m^2/m_0^2) & (\chi) = +5.121623_{1468} + 4.273730_{1468} + 3.987684_{1468} + 2.787320_{1468} + 1.105330_{1468} + 0.693190_{1468} + 0.107828_{1468} + 0.056802_{1468} + 0.044670_{1468} \\
(m/m_0) & (\phi) = +10.94811; +9.396152; +7.145562; +5.246738; +2.98975; +2.946052; +0.439230; +0.000000; -0.800000; -0.98975; -0.996152; -1.272303; -1.639230; -1.646738; -2.000000; -2.400000; -3.200000; -3.313671 \\
\end{align*}$$

$$S1165685: V/g^2 = -11.6568542495, \text{ Quality } - \log_{10}(|Q|^2) = 31.35 , U(1) \times U(1)$$

$$\begin{align*}
\phi & = +0.764285_{1468} + 0.764285_{1468} ; +0.764285_{1468} + 0.764285_{1468} \\
(m^2/m_0^2) & (\psi) = +2.560669_{1468} \\
(m^2/m_0^2) & (\chi) = +5.121623_{1468} + 3.987684_{1468} + 3.644106_{1468} \\
(m/m_0) & (\phi) = +8.485281_{1468} ; +7.029137; +6.363961_{1468} + 5.634160_{1468} + 4.801195; +3.266128_{1468}; +2.485281; +0.000000; -0.514719_{1468} + 0.782846_{1468} - 1.091883_{1468} - 1.286476; -1.408975_{1468}; -2.123204_{1468} - 2.485281; -3.514179_{1468} \\
\end{align*}$$

$$S1176725: V/g^2 = -11.7672503174, \text{ Quality } - \log_{10}(|Q|^2) = 33.22$$

$$\begin{align*}
\phi & = +0.246799_{13578} ; +0.246799_{13578} ; +0.246799_{13578} ; +0.246799_{13578} + 0.162463_{13578} ; +0.162463_{13578} ; +0.162463_{13578} ; +0.162463_{13578} \\
(m^2/m_0^2) & (\psi) = +0.291285_{13578} ; +0.219283_{13578} ; +0.117840_{13578} ; +0.001772_{13578} \\
(m^2/m_0^2) & (\chi) = +8.485281_{1468} ; +7.029137; +6.363961_{1468} + 5.634160_{1468} + 4.801195; +3.266128_{1468}; +2.485281; +0.000000; -0.514719_{1468} + 0.782846_{1468} - 1.091883_{1468} - 1.286476; -1.408975_{1468}; -2.123204_{1468} - 2.485281; -3.514179_{1468} \\
(m/m_0) & (\phi) = +0.100173_{13578} ; +0.027710_{1468} \\
\end{align*}$$

$$S1195898: V/g^2 = -11.958930843, \text{ Quality } - \log_{10}(|Q|^2) = 21.12 , \dim(GG)=3$$
\[ \phi = -0.231863_{12348} + 0.119571_{112386} - 0.246369_{123584} - 0.246360_{123679} \\
+0.246360_{123584} + 0.246360_{123784} + 0.246351_{145784} + 0.246360_{14688} \\
+0.119571_{156784} - 0.119571_{234848} - 0.246360_{235784} - 0.724351_{23688} \\
-0.246360_{24568} - 0.246360_{24788} - 0.246360_{24858} - 0.246360_{24968} \\
+0.119571_{45784} - 0.231631_{67846} + 0.240574_{12386} + 0.132260_{12456} \\
+0.240574_{124784} + 0.220384_{12568} - 0.240574_{13456} - 0.132260_{13478} \\
+0.703180_{138676} + 0.240574_{14686} - 0.240574_{15786} - 0.240574_{23466} \\
+0.240574_{235784} + 0.703180_{24568} - 0.132260_{25686} - 0.240574_{26786} \\
-0.220384_{245784} - 0.240574_{24686} - 0.132263_{67868} - 0.240574_{14678}.\]

\[ (m^2/m_0^2)_{\psi} = +3.073009_{(x,y)} + 2.518511_{(x,y)} \]

\[ (m^2/m_0^2)_{\chi} = +6.146019_{(x,y)} + 3.891993_{(x,y)} + 5.037922_{(x,y)} + 4.805271_{(x,y)} \]

\[ (m^2/m_0^2)_{\phi} = +3.998704_{(x,y)} + 3.906094_{(x,y)} + 3.931958_{(x,y)} \]

\[ (m^2/m_0^2)_{\alpha} = +0.415199_{(x,y)} + 0.391172_{(x,y)} + 0.375233_{(x,y)} + 0.263746_{(x,y)} + 0.051804_{(x,y)} \]

\[ (m/m_0)_{\phi} = +9.852022_{(x,y)} + 7.449141_{(x,y)} + 5.391412_{(x,y)} + 0.851174_{(x,y)} \]

\[ (m/m_0)_{\alpha} = -0.000000_{(x,y)} + 0.489000_{(x,y)} - 0.489000_{(x,y)} - 0.879698_{(x,y)} - 0.198118_{(x,y)} - 2.244850_{(x,y)} - 3.432790_{(x,y)} \]

\[ S1200000: \ V/g^2 = -12.0, \ Quality - \log_{10}(|Q|^2) = 26.75, \ U(1) \times U(1) \quad N = 1 \]

\[ \phi = +0.573108_{123587} - 0.573108_{235876} + 0.573108_{236875} - 0.573108_{245876} \]

\[ (m^2/m_0^2)_{\psi} = +4.000000_{(x,y)} + 3.000000_{(x,y)} + 2.250000_{(x,y)} + 1.000000_{(x,y)} \]

\[ (m^2/m_0^2)_{\chi} = +8.000000_{(x,y)} + 6.000000_{(x,y)} + 4.746708_{(x,y)} + 4.500000_{(x,y)} \]

\[ (m^2/m_0^2)_{\phi} = +3.936492_{(x,y)} + 3.732051_{(x,y)} + 2.165560_{(x,y)} + 2.000000_{(x,y)} \]

\[ (m^2/m_0^2)_{\alpha} = +1.500000_{(x,y)} + 1.125000_{(x,y)} + 0.584310_{(x,y)} + 0.003292_{(x,y)} + 0.000000_{(x,y)} \]

\[ (m/m_0)_{\phi} = +8.678807_{(x,y)} + 8.190152_{(x,y)} + 4.412278_{(x,y)} + 3.067100_{(x,y)} \]

\[ (m/m_0)_{\alpha} = +2.732051_{(x,y)} + 0.000000_{(x,y)} - 0.732051_{(x,y)} - 1.250000_{(x,y)} \]

\[ -1.316589_{(x,y)} - 1.912878_{(x,y)} - 2.000000_{(x,y)} - 2.196152_{(x,y)} - 2.229377_{(x,y)} - 2.250000_{(x,y)} \]

\[ S1212986: \ V/g^2 = -12.1298657377, \ Quality - \log_{10}(|Q|^2) = 27.73, \ U(1) \]

\[ \phi = +0.235254_{12348} - 0.254267_{12358} + 0.254267_{12678} + 0.306918_{13578} \]

\[ +0.306918_{13688} - 0.714110_{14568} - 0.205576_{14788} - 0.205570_{23568} \]

\[ -0.254267_{34678} + 0.306918_{24787} + 0.306918_{24868} + 0.254267_{3458} \]

\[ -0.254267_{34678} + 0.235254_{26788} - 0.271941_{2568} + 0.271941_{24178} \]

\[ +0.816939_{1358} + 0.273947_{13678} + 0.273947_{1258} + 0.816939_{24678} \]

\[ +0.271941_{24568} + 0.271941_{14786} \]

\[ (m^2/m_0^2)_{\psi} = +3.535597_{(x,y)} + 2.920604_{(x,y)} + 2.398064_{(x,y)} + 2.276432_{(x,y)} \]

\[ (m^2/m_0^2)_{\chi} = +7.071193_{(x,y)} + 5.841328_{(x,y)} + 5.395954_{(x,y)} + 5.388200_{(x,y)} \]

\[ +5.388200_{(x,y)} + 4.796128_{(x,y)} + 4.526841_{(x,y)} + 4.339888_{(x,y)} \]

\[ +4.069349_{(x,y)} + 4.023181_{(x,y)} + 3.998405_{(x,y)} + 3.833180_{(x,y)} \]

\[ +2.250937_{(x,y)} + 1.140589_{(x,y)} + 1.133068_{(x,y)} + 1.124823_{(x,y)} \]

\[ +1.061286_{(x,y)} + 0.960255_{(x,y)} + 0.842616_{(x,y)} + 0.594763_{(x,y)} \]

\[ +0.594763_{(x,y)} + 0.376813_{(x,y)} + 0.203637_{(x,y)} + 0.124094_{(x,y)} \]

\[ +0.050622_{(x,y)} + 0.013642_{(x,y)} + 0.006222_{(x,y)} + 0.003630_{(x,y)} \]
\[
\begin{align*}
(m/m_0)[\phi] &= +12.641025; \quad +11.237761_{(\pm 2)}; \quad +8.601317_{(\pm 2)}; \quad +7.520357; \\
&\quad +4.968644; \quad +4.647099_{(\pm 2)}; \quad +4.050277_{(\pm 2)}; \quad +3.993562; \\
&\quad +3.646665_{(\pm 2)}; \quad +3.463951_{(\pm 2)}; \quad +0.828099_{(\pm 2)}; \quad +0.295051; \\
&\quad +0.000000_{(\pm 2)}; \quad -0.873003_{(\pm 2)}; \quad -0.883431_{(\pm 2)}; \\
&\quad -1.093132; \quad -1.178440_{(\pm 2)}; \quad -1.407864_{(\pm 2)}; \quad -1.644078_{(\pm 2)}; \\
&\quad -1.834720_{(\pm 2)}; \quad -1.920523_{(\pm 2)}; \quad -1.947453_{(\pm 2)}; \quad -1.952274; \\
&\quad -2.336639_{(\pm 2)}; \quad -2.414964; \quad -2.910854_{(\pm 2)}; \quad -3.375215
\end{align*}
\]

| S1271622: \(V/q^2 = -12.7162248888\), Quality = \(-\log_{10}(|Q|^2) = 29.31\), dim(GG)=3 |
|-----------------------------------------------|---------------|
| \(\phi\) | +3.948474_{(\pm 2)}; \quad +2.012973_{(\pm 2)} |
| \(m^2/m_0^2)[\psi]\) | +7.189999_{(\pm 4)}; \quad +5.142313_{(\pm 4)}; \quad +4.633511_{(\pm 4)} |
| \(m^2/m_0^2)[\chi]\] | +4.025947_{(\pm 4)}; \quad +3.596011_{(\pm 4)}; \quad +2.797159_{(\pm 4)} |
| \(m/m_0)[\phi]\] | +1.676916_{(\pm 4)}; \quad +1.471722_{(\pm 4)}; \quad +0.350381_{(\pm 4)} |
| \(m^2/m_0^2)[\psi]\] | +0.064058_{(\pm 4)}; \quad +0.019667_{(\pm 4)}; \quad +0.013615_{(\pm 4)} |
| \(m^2/m_0^2)[\chi]\] | +9.373520; \quad +8.326452_{(\pm 2)}; \quad +8.107084; \quad +7.297178_{(\pm 3)} |
| \(m/m_0)[\phi]\] | +7.247748; \quad +6.291101_{(\pm 1)}; \quad +2.383617; \quad +1.288334 |
| \(m^2/m_0^2)[\psi]\] | +0.000000_{(\pm 25)}; \quad -1.055457_{(\pm 6)}; \quad -1.172410_{(\pm 9)} |
| \(m^2/m_0^2)[\chi]\] | -1.362824_{(\pm 6)}; \quad -1.483922; \quad -1.714334; \quad -1.864898_{(\pm 9)} |
| \(m/m_0)[\phi]\] | -1.931500; \quad -2.625043_{(\pm 3)}; \quad -3.353494 |

| S1301601: \(V/q^2 = -13.0160180374\), Quality = \(-\log_{10}(|Q|^2) = 21.08\), \((U(1) \times U(1))\) |
|-----------------------------------------------|---------------|
| \(\phi\) | +3.005710_{(\pm 8)} |
| \(m^2/m_0^2)[\psi]\] | +6.011240_{(\pm 8)}; \quad +5.321842_{(\pm 8)}; \quad +4.126058_{(\pm 8)} |
| \(m^2/m_0^2)[\chi]\] | +0.855344_{(\pm 10)}; \quad +0.784661_{(\pm 8)}; \quad +0.096209_{(\pm 8)} |
| \(m/m_0)[\phi]\] | +14.515899; \quad +12.935358_{(\pm 22)}; \quad +11.056222; \quad +10.008967 |
| \(m^2/m_0^2)[\psi]\] | +7.458673; \quad +4.469179_{(\pm 4)}; \quad +3.198206_{(\pm 4)}; \quad +3.026592_{(\pm 4)} |
| \(m^2/m_0^2)[\chi]\] | +2.913149_{(\pm 4)}; \quad +0.000000_{(\pm 26)}; \quad -0.214642; \quad -0.514678 |
| \(m/m_0)[\phi]\] | -0.647720; \quad -1.083233_{(\pm 4)}; \quad -1.478523_{(\pm 4)}; \quad -1.716653 |
| \(m^2/m_0^2)[\psi]\] | -1.909115_{(\pm 4)}; \quad -2.590041_{(\pm 4)}; \quad -3.590693; \quad -3.806792 |

| S1362365: \(V/q^2 = -13.6236525917\), Quality = \(-\log_{10}(|Q|^2) = 18.70\), \((U(1))\) |
|-----------------------------------------------|---------------|
| \(\phi\) | +3.424801; \quad +3.391334; \quad +3.377415; \quad +3.080922_{(\pm 2)} |
| \(m^2/m_0^2)[\psi]\] | +2.306966_{(\pm 2)}; \quad +1.439073 |
\[
(m^2/m_0^2)[\chi]
\]
\[
\begin{array}{c}
+6.849601; +6.782667; +6.754830; +6.161841; \\
+5.62127; +5.388167; +4.864558; +4.861528; \\
+4.613933; +4.366510; +4.279957; +4.045993; \\
+3.402663; +2.861945; +2.755513; +2.753857; \\
+2.284207; +2.004265; +1.936863; +1.901416; \\
+1.552619; +1.504583; +1.349929; +1.346186; \\
+1.271529; +0.956640; +0.540350; +0.304967; \\
+0.241586; +0.172498; +0.088053; +0.069907; \\
+0.067235; +0.042941; +0.027701; +0.018851; \\
+0.007926; +0.005053
\end{array}
\]

\[
(m/m_0)[\phi]
\]
\[
\begin{array}{c}
+12.258386; +11.973987; +8.228326; +7.072909; \\
+7.006771; +5.860918; +5.735844; +5.203470; \\
+4.667988; +4.023814; +3.994495; +1.789900; \\
+1.321663; +0.000000; +0.300034; +0.480035; \\
-0.526508; -1.118467; -1.158780; -1.193964; \\
-1.231666; -1.339218; -1.373506; -1.409821; \\
-1.488227; -1.754013; -2.056017; -2.174017; \\
-2.595156; -2.714364; -2.812468
\end{array}
\]

\[
S1363782: V/g^2 = -13.6378280148, \text{ Quality} - \log_{10}(|Q|^2) = 32.39
\]

\[
\phi
\]
\[
\begin{array}{c}
+0.049498; -0.075759; -0.005984; +0.054500; \\
+0.072193; -0.000963; -0.001279; +0.144635; \\
+0.005984; +0.000963; -0.001279; +0.060130; \\
-0.049498; +0.028514; +0.920210; -0.920210; \\
-0.028514; +0.049498; +0.060130; +0.001279; \\
+0.000963; +0.005984; -0.144635; +0.001279; \\
-0.000963; +0.072193; -0.054500; -0.005984; \\
+0.075759; +0.049498; +0.011261; +0.228668; \\
+0.049498; +0.108395; +0.212712; +0.003275; \\
+0.000745; -0.378219; -0.049153; +0.003275; \\
-0.000745; +0.920224; +0.011261; +0.234756; \\
+0.204611; -0.204611; +0.234756; +0.011261; \\
+0.920224; +0.000745; +0.003275; +0.049153; \\
-0.378219; +0.000745; -0.003275; -0.011261; \\
+1.018395; -0.018395; -0.228668; +0.011261; \\
+1.018395; +0.342808; +0.342808; +0.342808; \\
+2.478208; +2.118747; +1.500004
\end{array}
\]

\[
(m^2/m_0^2)[\psi]
\]
\[
\begin{array}{c}
+7.017426; +6.850096; +6.795540; +6.122906; +5.951174; \\
+5.564370; +5.370984; +4.956416; +4.893853; +4.882039; \\
+4.833766; +4.831103; +4.405594; +4.251284; +4.237493; \\
+4.068340; +3.748283; +3.134138; +3.000008; +2.885676; \\
+2.838759; +2.723049; +2.444570; +2.126322; +2.098829; \\
+1.939040; +1.983231; +1.785699; +1.678400; +1.499595; \\
+1.421309; +1.394087; +1.311186; +1.267888; +1.187442; \\
+1.076417; +0.854942; +0.643456; +0.438856; +0.304580; \\
+0.240693; +0.237791; +0.227807; +0.134889; +0.132434; \\
+0.104710; +0.091965; +0.067234; +0.053249; +0.048322; \\
+0.048643; +0.018905; +0.019074; +0.008691; +0.0004854; \\
+0.0004421
\end{array}
\]
\begin{align*}
(\phi) & = \phi \\
(\phi^2) & = \phi^2 \\
S136684: \quad V/g^2 = -13.6686490616, \quad \text{Quality} - \log_{10}(|Q|^2) = 32.28
\end{align*}

\begin{align*}
(m/m_0)[\phi] & = +12.015895; +11.694678; +8.205019; +7.998997; \\
& +7.347287; +6.559691; +6.411703; +5.669881; \\
& +5.610098; +5.358082; +5.079754; +4.788509; +4.471018; \\
& +3.776346; +3.296343; +1.617945; +1.212060; +0.442827; \\
& +0.000000_{(2.28)}; -0.463482; -0.683535; -0.850415; \\
& -1.005595; -1.106717; -1.153270; -1.161027; -1.188733; \\
& -1.285827; -1.303540; -1.332068; -1.386838; -1.418999; \\
& -1.507445; -1.664690; -1.706247; -1.824473; -2.062555; \\
& -2.166120; -2.334913; -2.699591; -2.849194; \\
& -2.955394
\end{align*}

\begin{align*}
S1367611: \quad V/g^2 = -13.6761142184, \quad \text{Quality} - \log_{10}(|Q|^2) = 31.31
\end{align*}

\begin{align*}
(\phi) & = \phi \\
(\phi^2) & = \phi^2 \\
\end{align*}

\begin{align*}
(m/m_0)[\phi] & = +0.102432_{(2.88)}; +0.056950_{(2.47)}; -0.052291_{(2.28)}; +0.082782_{(2.96)}; \\
& +0.082782_{(2.34)}; +0.921226_{(1.367)}; +0.052291_{(1.388)}; -0.102432_{(1.468)}; \\
& -0.086344_{(1.578)}; +0.086344_{(1.234)}; +0.102432_{(1.257)}; +0.052291_{(1.307)}; \\
& -0.921226_{(2.458)}; -0.082782_{(2.678)}; +0.082782_{(2.478)}; +0.052291_{(2.307)}; \\
& +0.056950_{(3.0568)}; +0.102432_{(2.567)}; -0.327140_{(2.235)}; +0.225595_{(1.245)}; \\
& -0.210554_{(1.267)}; -0.053658_{(1.260)}; -0.196874_{(1.347)}; +0.053658_{(1.384)}; \\
& +0.225595_{(1.356)}; +0.327140_{(1.456)}; -0.327140_{(2.287)}; +0.225595_{(2.247)}; \\
& -0.053658_{(2.256)}; -0.916874_{(2.568)}; -0.053658_{(1.457)}; +0.210554_{(3.458)}; \\
& -0.225595_{(3.678)}; +0.327140_{(4.676)}
\end{align*}

\begin{align*}
(m^2/m_0^2)[\phi] & = +3.019733; +3.338898_{(2.21)}; +3.719778; +2.973868; \\
& +2.612441; +1.797021_{(2.1)}
\end{align*}
| $\phi / (m_e/\sqrt{\alpha})$ | \(\phi / (m_e/\sqrt{\alpha})\) |
|-------------------------|-------------------------|
| \(\phi / (m_e/\sqrt{\alpha})\) | \(\phi / (m_e/\sqrt{\alpha})\) |
### S1384135: $V/g^2 = -13.841358302$, Quality – $\log_{10}([|Q|^2]) = 20.96\ , \ U(1)$

| $m^2/m_0^2$ | $|\psi|$ | $|\chi|$ | $|\phi|$ |
|-------------|----------|----------|----------|
| $\phi$      | +0.161449, 1257s | -0.010602, 268s | -0.185149, 158s |
|             | +0.752922, 136s | +0.351922, 165s | +0.161446, 178s |
|             | +0.351922, 257s | -0.752924, 240s | +0.185149, 257s |
|             | -0.185149, 268s | +0.161446, 158s | +0.185149, 158s |
|             | +0.161446, 158s | +0.351922, 165s | +0.161446, 178s |
|             | +0.155065, 235s | -0.581767, 237s | +0.161446, 178s |
|             | -0.161446, 178s | +0.351922, 165s | +0.161446, 178s |
|             | -0.318508, 246s | +0.155065, 235s | +0.161446, 178s |

| $m^2/m_0^2$ | $|\psi|$ | $|\chi|$ | $|\phi|$ |
|-------------|----------|----------|----------|
| $\phi$      | +3.846097, 110s | +3.184549, 110s | +2.033699, 110s |
|             | +2.887794, 121s | +1.006714, 121s |  |

### S1400000: $V/g^2 = -14.00$, Quality – $\log_{10}([|Q|^2]) = 106.43\ , \ dim(GG)=6$

| $m^2/m_0^2$ | $|\psi|$ | $|\chi|$ | $|\phi|$ |
|-------------|----------|----------|----------|
| $\phi$      | +1.020804, 123s | -1.020804, 467s | +1.020804, 123s | -1.020804, 467s |
|             | +3.857143, 145s | +2.142857, 145s |  |

### S1400056: $V/g^2 = -14.00005638418$, Quality – $\log_{10}([|Q|^2]) = -29.79$

| $m^2/m_0^2$ | $|\psi|$ | $|\chi|$ | $|\phi|$ |
|-------------|----------|----------|----------|
| $\phi$      | +0.210306, 123s | +0.127154, 123s | +0.210306, 123s | -0.127154, 123s |
|             | -0.043212, 125s | +0.019964, 125s | +0.093485, 347s | -0.208103, 134s |
|             | -0.631440, 135s | -0.053430, 165s | -0.358927, 145s | -0.093485, 145s |
|             | -0.091964, 147s | -0.053430, 165s | -0.263432, 158s | -0.263432, 158s |
|             | +0.053430, 234s | -0.091964, 235s | -0.093485, 236s | +0.358927, 236s |
|             | +0.053430, 245s | +0.634140, 247s | -0.208103, 256s | +0.093485, 256s |
|             | -0.091964, 267s | -0.043212, 140s | +0.127154, 140s | -0.210306, 140s |
|             | -0.127154, 140s | -0.053430, 165s | -0.093485, 236s | -0.093485, 236s |
|             | +0.064107, 134s | -0.019321, 125s | +0.105175, 127s | +0.040642, 124s |
|             | +0.007433, 134s | -0.046308, 134s | -0.940395, 136s | +0.005357, 136s |
|             | +0.703297, 145s | -0.007433, 134s | +0.040642, 124s | -0.005357, 136s |
|             | -0.077777, 134s | -0.007433, 134s | -0.005357, 136s | +0.040642, 124s |
|             | +0.007433, 236s | +0.703297, 236s | -0.005357, 247s | -0.940395, 247s |
|             | +0.046308, 256s | +0.007433, 256s | +0.040642, 267s | -0.150175, 346s |
|             | -0.019321, 140s | +0.064107, 134s | +0.019321, 140s | -0.058355, 140s |
\[
\begin{array}{|c|}
\hline
(m^2/m_0^2)[\psi] & +3.599967; +4.619372; +3.401156; +3.390972; +2.857991; \\
+2.717967; +2.675207; +1.314345 \\
\hline
(m^2/m_0^2)[\chi] & +7.919934; +7.238744; +7.017672; +6.927079; +6.802300; \\
+6.781945; +5.715982; +5.559961; +5.435935; +5.404235; \\
+5.350415; +5.110000; +5.031542; +4.839588; +4.729669; \\
+3.769438; +3.712840; +3.553037; +3.268525; +2.947269; \\
+2.818573; +2.800407; +2.758125; +2.728802; +2.628691; \\
+2.254191; +2.212630; +2.148602; +2.089141; +1.982196; \\
+1.557265; +1.539720; +1.428459; +1.413870; +0.854259; \\
+0.842935; +0.815777; +0.808692; +0.745839; +0.317567; \\
+0.298295; +0.193023; +0.151709; +0.143706; +0.130319; \\
+0.129703; +0.118747; +0.093219; +0.091178; +0.067937; \\
+0.043026; +0.041270; +0.025538; +0.022028; +0.015303; \\
+0.000930; \\
\hline
(m/m_0)[\phi] & +14.782906; +14.617867; +9.29603; +9.052751; \\
+7.804477; +7.790402; +7.604398; +6.302453; +6.099548; \\
+5.980469; +5.178642; +5.063846; +4.763895; +4.457394; \\
+3.862797; +3.641781; +2.039083; +1.502449; +1.260835; \\
+0.545977; +0.000000; +9.35619; -0.952700; \\
-0.955505; -1.034421; -1.045971; -1.068879; -1.108013; \\
-1.170975; -1.214071; -1.657788; -1.763508; -1.801483; \\
-1.942550; -2.059556; -2.122654; -2.146817; -2.439615; \\
-2.450999; -2.466806; -2.506403; -2.801326; -2.870221; \\
\end{array}
\]

S1402217: \( V/g^2 = -14.0221724366 \), Quality - \( \log_{10}(|Q|^2) = 21.64 \), \( U(1) \)

\[
\phi
\]

\[
(m^2/m_0^2)[\psi] & +4.287422; +3.478015; +2.554300; +1.939201; \\
+8.574844; +8.956029; +5.805348; \\
+5.642960; +5.363076; +5.342117; \\
+5.108600; +4.051767; +3.878403; \\
+3.826227; +3.602032; +3.132572; \\
+3.015534; +1.768447; +1.727864; \\
+1.724713; +1.694192; +0.931396; \\
+0.795854; +0.544649; +0.324214; \\
+0.271714; +0.141445; +0.104996; \\
+0.020344; +0.008360; +0.000397; \\
\hline
\]

\[
(m^2/m_0^2)[\chi] & +11.118916; +10.519630; +8.704129; +8.165741; \\
+7.630408; +7.433008; +4.914640; +4.277371; \\
+3.923000; +3.669504; +2.311649; +0.788136; \\
+0.000000; -0.281385; -0.747762; -0.754799; \\
-0.787340; -0.919348; -1.154049; -1.186690; \\
-2.294017; -2.403310; -2.767474; -2.803941; \\
-3.022824; -3.034867; -3.209646; \\
\hline
\]

S1424025: \( V/g^2 = -14.2402599725 \), Quality - \( \log_{10}(|Q|^2) = 22.48 \), \( \dim(GG)=3 \)
| $\phi_{0}$ | $-0.427976_{1344}$ | $-0.518819_{1356}$ | $-0.045421_{2784}$ | $-0.066192_{3584}$ |
|-----------|-----------------|-----------------|-----------------|-----------------|
| $\phi_{0}$ | $0.539590_{2367}$ | $+0.539590_{2457}$ | $+0.066192_{2367}$ | $-0.045421_{3456}$ |
| $\phi_{0}$ | $-0.518819_{3478}$ | $-0.427976_{3678}$ | $+0.445727_{2356}$ | $+0.445727_{2346}$ |
| $\phi_{0}$ | $+0.136284_{3486}$ | $-0.582012_{1457}$ | $-0.582012_{1457}$ | $-0.582012_{2456}$ |
| $\phi_{0}$ | $-0.136284_{3478}$ | $-0.582012_{2968}$ | $-0.582012_{2456}$ | $-0.582012_{2567}$ |

\[(m^2/m_0^2)_{\psi}\] +3.861312_{(x; 2)} +2.331953_{(x; 2)}

\[(m^2/m_0^2)_{\chi}\] +7.722628_{(x; 2)} +4.066390_{(x; 2)} +4.243506_{(x; 2)}

\[(m^2/m_0^2)_{\phi}\] +8.390313_{(x; 3)} +7.954234_{(x; 3)} +5.442560_{(x; 3)}

\[(m/m_0)_{\phi}\] +0.000000_{(x; 23)} +0.386162_{(x; 4)} +0.663247_{(x; 4)} -1.121077_{(x; 4)}

\[(m/m_0)_{\phi}\] -1.251253_{(x; 4)} -1.461729_{(x; 4)} -1.739652_{(x; 4)} -1.858801

\[(m^2/m_0^2)_{\gamma}\] +3.987299_{(x; 2)} +3.742063_{(x; 2)} +2.874011_{(x; 2)} +2.327290_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +7.974599_{(x; 2)} +7.674781_{(x; 2)} +7.484120_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +5.808191_{(x; 2)} +5.748023_{(x; 2)} +5.125001_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +4.911162_{(x; 2)} +4.654580_{(x; 2)} +4.286759_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +3.987223_{(x; 2)} +3.680813_{(x; 2)} +3.536477_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +2.953605_{(x; 2)} +2.905381_{(x; 2)} +2.866629_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +1.950762_{(x; 2)} +1.359332_{(x; 2)} +1.333032_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +0.678343_{(x; 2)} +0.490432_{(x; 2)} +0.204120_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +0.188891_{(x; 2)} +0.179534_{(x; 2)} +0.161957_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +0.110696_{(x; 3)} +0.106758_{(x; 3)} +0.011528_{(x; 3)} +0.003240_{(x; 3)}

\[(m^2/m_0^2)_{\gamma}\] +14.07674_{(x; 2)} +13.723956_{(x; 2)} +12.618363_{(x; 2)} +10.083687_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +10.001246_{(x; 2)} +8.425351_{(x; 2)} +7.540331_{(x; 2)} +7.481800_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +7.104544_{(x; 2)} +6.758381_{(x; 2)} +5.285155_{(x; 2)} +4.153573_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +2.801419_{(x; 2)} +2.726072_{(x; 2)} +1.689121_{(x; 2)} +1.378209_{(x; 2)}

\[(m^2/m_0^2)_{\gamma}\] +0.000000_{(x; 27)} +0.221648_{(x; 27)} -0.443510_{(x; 27)} -0.488338_{(x; 27)}

\[(m^2/m_0^2)_{\gamma}\] -0.840532_{(x; 28)} +0.900212_{(x; 28)} -1.163517_{(x; 28)} -1.262184_{(x; 28)}

\[(m^2/m_0^2)_{\gamma}\] -1.343510_{(x; 29)} -1.494277_{(x; 29)} -1.924022_{(x; 29)} -1.980403_{(x; 29)}

\[(m^2/m_0^2)_{\gamma}\] -2.151596_{(x; 29)} -2.352526_{(x; 29)} -3.026056_{(x; 29)} -3.402853_{(x; 29)}

\[(m^2/m_0^2)_{\gamma}\] -3.683294

**S1441574:** $V/g^2 = -14.4157405138$, Quality $= \log_{10}(|Q|^2) = 26.21$, $U(1)$
S1442018: $V/g^2 = -14.4201873779$, Quality $-\log_{10}(|Q|^2) = 25.03$ , $U(1)$

$\phi$

$\frac{m^2}{m_0^2} [\psi]$

$\frac{m^2}{m_0^2} [\chi]$

S1443834: $V/g^2 = -14.4383474481$, Quality $-\log_{10}(|Q|^2) = 27.02$ , $U(1)$

$\phi$

$\frac{m^2}{m_0^2} [\psi]$

$\frac{m^2}{m_0^2} [\chi]$
\[
(m/m_0) \phi = \begin{pmatrix}
+18.000637_{(x;2)} & +15.679089_{(x;2)} & +9.632280_{(x;2)} & +9.450695_{(x;2)} \\
+8.110325_{(x;2)} & +7.722708_{(x;2)} & +5.660438_{(x;2)} & +5.198748_{(x;2)} \\
+5.036314_{(x;2)} & +4.581299_{(x;2)} & +3.028791_{(x;2)} & +2.838237_{(x;2)} \\
+0.355790_{(x;2)} & +0.000000_{(x;2)} & -0.549039_{(x;2)} & -0.631015_{(x;2)} \\
-0.631015_{(x;2)} & -0.910739_{(x;2)} & -1.133950_{(x;2)} & -1.142968_{(x;2)} \\
-1.564321_{(x;2)} & -1.635876_{(x;2)} & -1.688063_{(x;2)} & -1.840754_{(x;2)} \\
-1.876506 & -2.115054 & -2.654684 & -2.793380_{(x;2)} \\
-2.894339 & & & \\
\end{pmatrix}
\]

\[\phi \]

\[S1464498: \quad V/g^2 = -14.6449847504, \quad \text{Quality} = \log_{10}(|Q|^2) = 21.04\]

\[\psi \]

\[m^2/m_0^2 \psi = \begin{pmatrix}
+7.576784_{(x;2)} & +6.259250_{(x;2)} & +4.875050_{(x;2)} \\
+4.796329_{(x;2)} & +4.790684_{(x;2)} & +4.527843_{(x;2)} \\
+2.940722_{(x;2)} & +2.940722_{(x;2)} & +4.090825_{(x;2)} \\
+3.585904_{(x;2)} & +2.983410_{(x;2)} & +2.972991_{(x;2)} \\
+2.964264_{(x;2)} & +2.336620_{(x;2)} & +1.544922_{(x;2)} \\
+1.539806_{(x;2)} & +1.128006_{(x;2)} & +1.066355_{(x;2)} \\
+0.705875_{(x;2)} & +0.570854_{(x;2)} & +0.556030_{(x;2)} \\
+0.468260_{(x;2)} & +0.232695_{(x;2)} & +0.119834_{(x;2)} \\
+0.074663_{(x;2)} & +0.049568_{(x;2)} & +0.036994_{(x;2)} & +0.010656_{(x;2)} \\
\end{pmatrix}
\]

\[\chi \]

\[m^2/m_0 \chi = \begin{pmatrix}
+9.935311_{(x;2)} & +9.518337_{(x;2)} & +8.349777_{(x;2)} & +7.980431_{(x;2)} \\
+7.644513_{(x;2)} & +7.518213_{(x;2)} & +6.547322_{(x;2)} & +5.548293_{(x;2)} \\
+5.167956 & +4.168627 & +3.259355_{(x;2)} & +3.038787_{(x;2)} \\
+2.676089_{(x;2)} & +2.509770 & +2.453734 & +2.175704_{(x;2)} \\
+0.000000_{(x;2)} & -0.614032 & -0.844275 & -0.926226_{(x;2)} \\
-0.941913 & -1.222260_{(x;2)} & -1.220572 & -1.250165_{(x;2)} \\
-1.327939 & -1.419113 & -1.627333_{(x;2)} & -1.975641_{(x;2)} \\
-2.287093 & -2.287555 & -2.382066 & -2.438224 & -2.500817 & -2.522582 & -2.543940 \\
\end{pmatrix}
\]

\[S1465354: \quad V/g^2 = -14.6535427384, \quad \text{Quality} = \log_{10}(|Q|^2) = 31.89\]

\[\phi \]

\[\phi = \begin{pmatrix}
-0.127715_{(x;2)} & -0.297003_{(x;2)} & -0.062670_{(x;2)} & +0.013046_{(x;2)} \\
+0.127715_{(x;2)} & +0.088381_{(x;2)} & -0.297003_{(x;2)} & -0.479849_{(x;2)} \\
-0.023651_{(x;2)} & -0.297003_{(x;2)} & +0.127715_{(x;2)} & +0.001162_{(x;2)} \\
+0.036850_{(x;2)} & -0.297003_{(x;2)} & +0.033906_{(x;2)} & +0.002391_{(x;2)} \\
-0.297003_{(x;2)} & +0.368507_{(x;2)} & +0.127715_{(x;2)} & +0.001162_{(x;2)} \\
+0.297003_{(x;2)} & -0.023651_{(x;2)} & +0.127715_{(x;2)} & +0.001162_{(x;2)} \\
+0.088381_{(x;2)} & +0.127715_{(x;2)} & +0.001162_{(x;2)} & +0.002391_{(x;2)} \\
+0.297003_{(x;2)} & -0.023651_{(x;2)} & +0.127715_{(x;2)} & +0.001162_{(x;2)} \\
-0.011260_{(x;2)} & +0.011260_{(x;2)} & +0.002391_{(x;2)} & +0.002391_{(x;2)} \\
+0.046542_{(x;2)} & +0.046542_{(x;2)} & +0.001162_{(x;2)} & +0.002391_{(x;2)} \\
\end{pmatrix}
\]
2.1 Version 1

Preamble This subsection discusses special properties of solutions known at the release of version 1 of this part of the collection. Subsequent incremental updates will be added in subsections titled “Version 2” etc. Version numbers of the four parts of the encyclopedic reference evolve independently from one another. As discussed in the explanation at the beginning of part 1 of the series, the discussion deliberately avoids global statements along the lines of “this is the only known solution with property X” that may be invalidated by subsequent updates of this, or any other, part of the collection.

S0600000 is the solution at the origin with unbroken $N = 8$ supersymmetry. The existence of this solution follows directly from the expressions given in the
article by de Wit and Nicolai that first constructed $SO(8)$-gauged supergravity \[4\]. It first is discussed in detail in \[5\]. It is by no means true that all gauged maximal supergravities have such a trivial critical point. For example, $\mathcal{N} = 8$ supergravity also permits the gauging of other compact forms of $SO(8)$ and contractions thereof (cf. \[23\]), and the $SO(7,1)$-gauged model does not have a $\phi = 0$ stationary point \[8\].

$S0668740$ and $S0698771$ are unstable critical points with remarkable properties. They can be obtained by picking a single $SO(8)$ spinor (respectively co-spinor), forming a symmetric traceless matrix from it and the identity, and setting the co-spinorial (respectively spinorial) part of the 70-vector to zero. These solutions have matching mass spectra, but different cosmological constants: exchanging $35\phi \leftrightarrow 35\phi$ is not a symmetry of $SO(8)$-gauged $\mathcal{N} = 8$ supergravity. Lifting these to 11-dimensional supergravity, these solutions correspond to compactifications found by Francois Englert in \[31\]. Details of these solutions are discussed in \[32, 33, 34, 35, 30\].

$S0719157$ is a stable $\mathcal{N} = 1$ supersymmetric vacuum with residual gauge symmetry $G_2$. This has been discussed in \[32, 33\]. Renormalization group flows that connect this solution with the $SO(8)$-symmetric $\mathcal{N} = 8$ solution $S0600000$ and the $SU(3) \times U(1)$-symmetric $\mathcal{N} = 2$ solution $S779422$ have been discussed in the context of $M2$-brane field theory in \[36\].

$S0779422$ is a stable $\mathcal{N} = 2$ supersymmetric vacuum with residual gauge symmetry $SU(3) \times U(1)$. This solution has first been described in \[30\] and was the object of numerous investigations, in part due to speculations of its potential phenomenological significance \[37\] (see also \[38\]), as well as its relevance to $M2$-brane physics and ABJM/BLG theory (cf. \[39, 40, 41, 42, 36, 43, 44\]).

$S0880733$ has residual $SO(3) \times SO(3)$ gauge symmetry. This unstable solution was first described in some detail in \[48\]; the cosmological constant is $-6\sqrt{1 + \frac{2}{3}\sqrt{3}}$.

$S0983994$ is another novel (unstable) solution with 4-dimensional residual gauge symmetry which again is likely to be $SO(3) \times U(1)$. Mass spectra suggest a strong similarity between this solution and $S0983994$.
similarity between this solution and S0869596. Curiously, a deep scan over the scalar potentials using the method described in [2] with more than 4000 successful trial runs never managed to produce this critical point, despite the search being most intensive around $-V/g \sim 10$. This can be seen as a strong indication that the search strategy to use standard numerical minimizers on the Frobenius norm of the misalignment tensor $|Q|^2$ (cf. [2]) seems to systematically miss some solutions. Eventually, this critical point was obtained using a modification of the search strategy described in the discussion of S1400000. The cosmological constant very likely is $-5 \cdot 15^{1/4}$.

S0998708 is an unstable critical point with residual $U(1)$ symmetry that was first found in [2], where it is given as solution #7.

S? A solution with cosmological constant $V/g^2 = -10$ might exist, considering that the scalar potential is now known to have critical points with integer cosmological constants $V/g^2 \in \{-6, -8, -12, -14, -16, -18\}$. As the discussion of S0983994 and S1400000 shows, the search method employed to find the first 100+ solutions that became known does seem to systematically miss some critical points.

S1006758 is a novel unstable solution with residual gauge symmetry $U(1)$.

S1039624 is a novel unstable solution with residual gauge symmetry $U(1)$.

S1043471 is an unstable solution that breaks all gauge symmetry. This was first described in [2] as solution #8. Numerical information about properties of this critical point is given to 150-digit accuracy in [18].

S1046017 is a novel unstable solution without any residual gauge symmetry.

S1067475 is an unstable solution with residual $U(1) \times U(1)$ gauge symmetry. This was first described in [2] as solution #9. Numerical information about properties of this critical point is given to 150-digit accuracy in [18], where an analytic expression for the cosmological constant is also given that was obtained by inverse symbolic calculation. It is:

$$\frac{-V}{g^2} = \left(\frac{QW^2 + RW + S}{QW}\right)^{1/4}$$

where

\begin{align*}
Q &= 6561 \\
R &= 28482192 \\
S &= 122545537024 \\
W &= (128692865796145152 + 20596696547328\sqrt{2343}i)/1594323)^{1/3}
\end{align*}

S1068971 is a novel unstable solution with residual $U(1) \times U(1)$ gauge symmetry.

S1075828 is a novel unstable solution with residual four-dimensional gauge symmetry; considering mass degeneracies, this is presumably $SO(3) \times U(1)$.

S1165665 is an unstable solution with residual $U(1) \times U(1)$ gauge symmetry. This was first described in [2] as solution #10.

S1176725 is a novel unstable solution without any residual gauge symmetry.

S1195898 is a novel unstable solution with three-dimensional residual gauge symmetry. Mass degeneracies strongly suggest this to be $SO(3)$. This solution seems to share a number of properties with another newly discovered solution, S2503105.

S1200000 is the $\mathcal{N} = 1$ supersymmetric vacuum with residual gauge symmetry $U(1) \times U(1)$ that saturates the Breitenlohner-Freedman bound [11, 12]. This was first discovered in [2] and discussed in detail in [18].

S1212986 is a novel unstable solution with residual gauge symmetry $U(1)$.

S1271622 is a novel unstable solution with residual gauge symmetry (likely) $SO(3)$.

S1301601 is a novel unstable solution with residual gauge symmetry $U(1) \times U(1)$ and a remarkable gravitino mass spectrum.
S1362365 is an unstable solution with residual gauge symmetry $U(1)$ that was first described in [2] as solution #12. This also is one of the solutions that only could be obtained with a modified optimization strategy (minimizing not $|Q|^2$ but instead $|\nabla P|^2$ via double algorithmic differentiation).

S1363782 is a novel unstable solution without any residual gauge symmetry.

S1366864 is a novel unstable solution without any residual gauge symmetry.

S1367611 is an unstable solution without any residual gauge symmetry that was first described in [2] as solution #13.

S1379439 is a novel unstable solution without any residual gauge symmetry.

S1384135 is a fairly remarkable novel solution with residual gauge symmetry $U(1)$, that just violates both the conditions for supersymmetry and stability.

S1400000 is the $SO(3) \times SO(3)$ stable critical point that was first discovered and described in [6], and indeed the first critical point with broken symmetry to be described for four-dimensional $N = 8$ supergravity. The cosmological constant is exactly $V/g^2 = -14$, and the location is given by $\phi_{1235s} = -\phi_{4678s} = \phi_{1234c} = -\phi_{5678c} = \frac{1}{\sqrt{2}} \text{atanh} \left( \frac{\sqrt{5}}{5} \right)$.

Remarkably, all the other non-supersymmetric critical points in maximal supergravities that have ever been found in the pre-2000 era violate the Breitenlohner-Freedman bound [11, 12] and are hence perturbatively unstable. This gave rise to the belief that stability and supersymmetry were closely linked. It was only in 2010 that the full scalar mass spectrum of this solution was computed [48], which demonstrated that one can have stability without supersymmetry. (It may be interesting to note that the corresponding critical point in $N = 5$ supergravity in four dimensions that breaks $SO(5)$ to $SO(3)$ actually was found to be BF-stable already in the initial article by Breitenlohner and Freedman on stability!)

Somewhat surprisingly, it turns out to be very difficult to obtain this solution via the sensitivity backpropagation technique that discovered most of the solutions found in the post-2000 era – it indeed had to be added to the list by hand. The reasons why the basin of attraction of this particular solution in the numerical search is so small is unclear at the time of this writing. Given that sensitivity backpropagation using $|Q|^2$ as the objective function to be minimized [2] almost always seems to produce either unstable non-supersymmetric or stable supersymmetric solutions and at the time of this writing (version 1) never produced this solution, this leaves a somewhat uneasy feeling about the possible existence of (potentially a large number of) other stable non-supersymmetric solutions that are not easily accessible by these methods.

Still, while repeated numerical minimization seems to have a strong chance of missing this solution, it indeed has been re-discovered in 2002 using hybrid group-theoretic/numerical techniques on a certain sub-manifold of ten $SO(3)$ invariant scalars described in the present author’s PhD thesis [49].

This critical point has the curious property that one can, at least in the presentation given, triality-swap the $35_s$ entries with the $35_c$ entries and again obtain the same solution. An attempt to modify the numerical search in such a way that it looks for solutions that also have this specific property was partially successful: the modified search strategy did indeed manage to produce this solution (in about 3% of all runs), as well as another solution that was overlooked by the unmodified search, S0983994. However, the minima of the modified potential $|Q|^2 + |\tilde{Q}|^2$ (where $\tilde{Q}$ denotes the misalignment tensor for the triality-swapped solution) typically are not also zeroes, and indeed S0983994 does not have this triality-swapping property.

S1400056 is a novel unstable solution without any residual gauge symmetry. It is noteworthy that, despite the very similar cosmological constant, this is a different solution than S1400000, with very different properties.

S1402217 is a novel unstable solution with residual gauge symmetry $U(1)$. 

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S1424025 is a novel unstable solution with residual gauge symmetry (likely) \( SO(3) \).

S1441574 is a novel unstable solution with residual gauge symmetry \( U(1) \).

S1442018 is a novel unstable solution with residual gauge symmetry \( U(1) \).

S1443834 is a novel unstable solution with residual gauge symmetry \( U(1) \).

S1446498 is a novel unstable solution without any residual gauge symmetry.

S1465354 is a novel unstable solution without any residual gauge symmetry.

S1469693 is a novel unstable solution with residual gauge symmetry \( U(1) \).

3 Conclusion

Considering the effort that is normally required to obtain a new critical point, the modified sensitivity backpropagation method, like dynamite fishing, has a certain aura of being unsportsmanlike. However, it is the result that counts, and quite obviously, the data obtained through this powerful technique allows us to develop a much better picture of spontaneous symmetry breaking in \( SO(8) \) supergravity than before.

It is amusing to note that obtaining new stationary points in the scalar potential of \( SO(8) \)-gauged supergravity for a long time was considered a hard problem. Now, it once again is.

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