An application of the catastrophe theory to building the model of elastic-plastic behaviour of materials. Part 2. 3D model

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Abstract

The three-dimensional elastic-plastic deformation is considered. The catastrophe theory underlies the construction of this process model. It was shown that the variety of stable states consists on elastic states and can be depicted as a lattice on $T - \Gamma$ plane, where $T$ is shearing stress intensity and $\Gamma$ is shearing strain intensity.

The uniaxial model of elastic-plastic deformation was built in [1]. Just as in that article we use the catastrophe theory to construct the model. To define the state function we introduce two variables. The first one is the shearing stress intensity $T$:

$$T = \sqrt{\frac{1}{2} \cdot s_{ij} s_{ij}},$$

where $s_{ij}$ are the components of stress deviator. The second one is the shearing strain intensity:

$$\Gamma = \sqrt{2 \cdot e_{ij} e_{ij}},$$

where $e_{ij}$ are the components of strain deviator.

With these variables Hook’s law takes the following form: $T = G \cdot \Gamma$, where $G$ is the shear modulus or in dimensionless units: $T = \Gamma$. We make a substitution here, where $T$ stands for $T/G$.

To construct the model we take into account the assumption of existence of the equilibrium static states of the deforming material. These equilibrium states are elastic states. Plastic states are not static equilibrium. They are implemented in the transforming from one equilibrium state (elastic state) to another. Thus we can construct the variety of equilibrium states that is depicted on Fig. 1a for ideal plastic materials and on Fig. 1b for strengthening material. As it is seen from Fig. 1 the variety of stable states is the lattice which we will call the $\Delta$-lattice, here $T$ is non-dimensional quantity.

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So in 3D model the variety of stable states forms the lattice the same way as in one-dimensional model. The variety of unstable states consists of segments joining the upper end \( (A_i) \) of each rod with lower end \( (O_{i+1}) \) of next rod (see the dotted lines on Fig. 2).

Just as in one-dimensional model we distinguish two types of models: the model with parameter \( T \) and the model with parameter \( \Gamma \). Different types of transitions from one rod of \( \Delta \)-lattice to next rod correspond to different types of models. Transitions in the model with parameter \( T \) are the same as in the model with parameter \( \sigma \). And transitions in the model with parameter \( \Gamma \) are the same as in the model with parameter \( \varepsilon \) (see [1] and Fig. 3). That is why we refer the reader to [1] in order to omit redundant discussions.

Let us summarize the demands that the state function have to satisfy. The state function must be smooth and has minima at each rod of \( \Delta \)-lattice and maxima for rods of additional variety. In accordance with these demands we start to construct the state functions. Firstly we consider the state function with parameter \( T \). The function looks like this:

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\begin{align*}
(1) & \quad \Phi = \Phi_n(\Gamma; T), \\
(2a) & \quad \Phi_n' \equiv \frac{d\Phi_n}{d\Gamma} = -k \prod_{s=-k}^{s} (T - \varphi_{n-s}(\Gamma)), \\
(2b) & \quad \Gamma_i \leq \Gamma < \Gamma_{i+1}, \text{ if } \Gamma_i - \text{ projection of lower end of rod onto } \Gamma \text{ axis,} \\
& \quad \Gamma_{i+1} - \text{ projection of lower end of next rod onto } \Gamma \text{ axis;}
(2c) & \quad \Gamma_i \leq \Gamma \leq \Gamma_{i+1}, \text{ if } \Gamma_i - \text{ projection of lower end of rod onto } \Gamma \text{ axis,} \\
& \quad \Gamma_{i+1} - \text{ projection of upper end of rod onto } \Gamma \text{ axis,} \\
& \quad \Gamma_{i+1} - \text{ projection of upper end of next rod onto } \Gamma \text{ axis,} \\
(2d) & \quad \Gamma_i < \Gamma < \Gamma_{i+1}, \text{ if } \Gamma_i - \text{ projection of upper end of rod onto } \Gamma \text{ axis,} \\
& \quad \Gamma_{i+1} - \text{ projection of lower end of next rod onto } \Gamma \text{ axis;}
(2e) & \quad \Gamma_i < \Gamma \leq \Gamma_{i+1}, \text{ if } \Gamma_i - \text{ projection of upper end of rod onto } \Gamma \text{ axis,} \\
& \quad \Gamma_{i+1} - \text{ projection of upper end of next rod onto } \Gamma \text{ axis;}
(3) & \quad \Phi_n(\Gamma_{n+1}) = \Phi_{n+1}(\Gamma_{n+1}) - \text{ lacing condition, where } n < (N - 1), \\
\end{align*}
\]

Here for rods of \( \Delta \)-lattice

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\( i = 2m, \quad \varphi_i(\Gamma) = \Gamma - \sum_{l=0}^{i-1} \Delta_l, \)

and for rods of additional variety

\[
(5) \quad i = 2 \cdot m + 1, \quad \varphi_i(\Gamma) = k_i \cdot \left( \frac{\Gamma - \sum_{l=0}^{i-1} \Delta_l}{\Gamma - \sum_{l=0}^{m} \Delta_l} \right),
\]

where \( k_i = \frac{\left( \frac{\Gamma - \sum_{l=0}^{m} \Delta_l}{\Gamma - \sum_{l=0}^{m} \Delta_l} \right)}{\left( \frac{\Gamma - \sum_{l=0}^{m} \Delta_l}{\Gamma - \sum_{l=0}^{m} \Delta_l} \right)} = 1 + \frac{\Delta_m}{\sum_{l=0}^{m} \Delta_l}, \) and \( \Gamma \) is equal to the strain in point \( A_m. \)

As it is seen from equations (1) - (3) the state function is constructed of all \( \varphi_i(\Gamma) \) existing on the examining part of the strain axis \( \Gamma. \) It is not difficult to test (using (1) - (5)) that the state function \( \Phi_n \) is minimum on all rods of \( \Delta \) -lattice (i. e. these states are stable) and maximum on all rods of additional variety (unstable states). It is necessary to note that the end points of rods are degenerated critical points.

The second state function type is the state function with parameter \( \Gamma. \) We define this function as:

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(6) \quad \Phi_n(T; \Gamma), \quad \Phi_n(T_{n+1}) = \Phi_{n+1}(T_{n+1}),
\]

\[
(7) \quad \Phi_{n}' = \frac{d\Phi_{n}}{dT} = \frac{2(N-1)}{\prod_{i=2n}^{2(N-1)} (\Gamma - \psi_i(T))} = \prod_{i=2n}^{2(N-1)} (\psi_i(T) - \Gamma),
\]

defined for region \( T_n < T \leq T_{n+1}, \) if \( n \neq 0, \) and for \( T_n \leq T \leq T_{n+1}, \) if \( n = 0, \) here \( N \) stands for number of rods of \( \Delta \)-lattice.

Here \( \psi_i(T) = T + \sum_{l=0}^{i-1} \Delta_l, \) for the rods of \( \Delta \)-lattice, where \( i = 2m, \)

and for the rods of additional variety:

\[
\psi_i(T) = \frac{1}{k_i} \cdot T + \sum_{l=0}^{i-1} \Delta_l, \text{ where } i = 2 \cdot m + 1.
\]

Here \( k_i \) is the same as in the case of a model with parameter \( T. \)

Like the previous state function (with parameter \( T \)) we use all of functions \( \psi_i(T) \) existing on the examining part of the stress axis \( T \) to construct this function, i. e. it is formed by means of all rods of \( \Delta \)-lattice and additional variety existing on the examining part of the \( T \)-axis. It is easy to verify that the state function \( \Phi_n(T; \Gamma) \) (see (6) - (7)) also satisfies the above demands.

So the 3D model of elastic-plastic deformation was built both with parameter \( \Gamma \) and parameter \( T. \)
References

[1] An application of the catastrophe theory to building the model of elastic-plastic behaviour of materials. Part 1. Uniaxial deformation (stress) // cond-mat/0111304 (http://xxx.lanl.gov/abs/cond-mat/0111304)

Fig. 1 a, b. The Δ-lattice of ideal plastic (a) and strengthening (b) materials.

Fig. 2 a, b. The Δ-lattice and additional variety of ideal plastic (a) and strengthening (b) materials.

Fig. 3 a, b. The possible transitions in the model with parameter Τ (a) and in the model with parameter Γ (b).
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