Solitary waves in mixtures of Bose gases confined in annular traps

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Introduction. Cold atoms provide an ideal system for the study of nonlinear effects including solitary waves. The study of solitary waves in these systems has a number of interesting features that extend the results of the well-known nonlinear Schrödinger equation in homogeneous systems, first studied by Zakharov and Shabat \(^1\) three decades ago.

Among among the many novel aspects associated with the physics of solitary waves in trapped gases \(^2, 3, 4\) we note (i) the presence of an external trapping potential which renders these systems finite, (ii) the fact that they can be constructed as quasi-one or quasi-two dimensional systems, and (iii) the possibility of bound states of solitary waves in two- (or multi-) component gases. We refer the reader to the article by Carretero-Gonzalez et al. \(^5\) for a review of the extensive work that has been performed on all these problems.

Remarkably, recent experimental advances now permit the realization of elongated, quasi-one-dimensional traps. Solitary waves have been observed in elongated traps \(^2\) as well as in more spherical traps \(^3\). More recently, solitary waves have also been observed \(^4\) in a two-component Bose-Einstein condensate.

In the present study we examine the problem of solitary waves in a two-component Bose-Einstein condensate \(^4, 7, 8, 9, 10, 11, 12, 13, 14\) on an infinite line. Making an ansatz for the density of the two species, we derive exact solutions of the two coupled nonlinear equations, which describe the order parameters of the two gases and derive the profiles of grey-grey and grey-bright solitary waves, with the assumption that the density is constant, either zero or nonzero, far from the center of the two waves. We then demonstrate the uniqueness of the derived solutions. Finally, we derive the dispersion relation associated with these solitary waves, imposing periodic boundary conditions, i.e., assuming that the bosons are confined on a ring. Since the derived solutions are exponentially localized, the solutions with periodic boundary conditions can be approximated accurately by the present infinite-line calculations if the ring is sufficiently large. We compare the resulting dispersion relation with that of a single-component Bose gas confined in one dimension under periodic boundary conditions and interacting via a contact potential, that was first studied by Elliott Lieb \(^15\).

Model. A Bose-Einstein condensate consisting of two distinguishable species, \(A\) and \(B\), is described within the mean-field approximation by the two order parameters \(\Psi_A(x, t)\) and \(\Psi_B(x, t)\), which satisfy two Gross-Pitaevskii-like equations, i.e., two coupled nonlinear Schrödinger equations. Assuming for simplicity equal masses for the two species, \(M_A = M_B = M\), and setting \(\hbar = 2M = 1\), these equations have the form

\[
i\partial \Psi_j / \partial t = -\partial^2 \Psi_j / \partial x^2 + \gamma( |\Psi_j|^2 + |\Psi_k|^2 ) \Psi_j, \quad (1)
\]

where we have also assumed equal s-wave scattering lengths \(a_{AA} = a_{BB} = a_{AB} = a\) for all elastic collisions between the atoms. The coefficient \(\gamma\) that multiplies the nonlinear terms is thus equal for all terms and is also \(\propto a\). In the above equations we have also chosen the normalization condition \(\int |\Psi_j|^2 dx = N_j\), where \(N_j\) is the number of atoms in each component.

Solitary waves. We first determine the solitary wave profiles that result from Eqs. (1). Writing \(\Psi_j(x, t) = \sqrt{n_j(x, t)} e^{i\Phi_j(x, t)}\) and separating real and imaginary parts, we get two continuity equations for the two species, as well as two Euler equations, namely

\[
\partial n_j / \partial t + 2(n_j \Phi_j)' = 0, \quad (2)
\]

\[
\partial \Phi_j / \partial t = (\sqrt{n_j})'' / \sqrt{n_j} - (\Phi_j)^2 - \gamma(n_j + n_k), \quad (3)
\]

where the primes denote spatial derivatives. From the asymptotic behavior of Eqs. (1), we find that \(\Psi_j \propto e^{-i\mu t}\), where \(\mu = \gamma(n_0^A + n_0^B)\) is the chemical potential and \(n_j^0\) is the density of species \(j\) at \(|z| = \pm \infty\).

To derive solitonic solutions, we assume a constant and common velocity of propagation \(u\) for the two waves, and therefore we assume that \(\Psi_j(x, t) = \sqrt{n_j(z)} e^{i(\Phi_j(z) - ut)}\), where \(z = x - ut\). We note that while the density \(n_j = |\Psi_j(x, t)|^2\) is indeed a function of \(z = x - ut\) only, the phase has a more general dependence on \(x\) and \(t\) as indicated.

Grey-grey solitary waves. The assumption of travelling-wave solutions allows us to convert the time derivatives into spatial derivatives. The continuity equations may then be integrated to give

\[
\phi_j' = (u/2)(1 - n_j^0 / n_j). \quad (4)
\]
Since the local fluid velocity is \( v_j = h \phi_j'/M \), we see that 
\[ v_j = u(1 - n_0^o/n_j) \]. The Euler equations now take the form

\[
\left( \frac{\sqrt{n_j}}{\sqrt{n_j'}} \right)'' = \frac{u^2}{4} \left( \frac{(n_0^o)^2}{n_j^2} - 1 \right) + \gamma (n_j - n_0^o + n_k - n_0^o),
\]  
(5)

where we have imposed the boundary conditions \( n_j \to n_0^o \) and \( (\sqrt{n_j'})'' \to 0 \) for \( |z| \to \infty \).

To integrate the above equations, we make the following ansatz, which is also consistent with the boundary conditions,

\[ n_B - n_0^B = \kappa(n_A - n_0^A). \]  
(6)

This ansatz allows us to perform the integration and obtain

\[
\begin{align*}
\frac{n_A^o}{2} & = \frac{[\gamma(1 + \kappa)n_A - u^2/2](n_A - n_0^A)^2}{\kappa}, \quad (7) \\
\frac{n_B^o}{2} & = \frac{[\gamma + 1/\kappa]n_B - u^2/2( n_B - n_0^B)^2}{\kappa}. \quad (8)
\end{align*}
\]

The consistency of Eqs. (6), (7), and (8) requires that \( n_B = \kappa n_A \), which establishes the value of \( \kappa \) as \( n_B^o/n_A^o \) independent of the velocity of propagation \( u \). Since the sign of \( \kappa \) is clearly positive, there can be no dark-antidark solitary solutions that are consistent with the ansatz of Eq. (6). In other words, if the \( n_j^o \) are nonzero, the waves must both be density depressions, i.e., both must be grey solitary waves.

The solution of the above equations in the case of grey-grey solitary waves has the form

\[ n_j - n_0^j = -n_j^0 \cos^2 \theta / \cosh^2 (z \cos \theta / \xi_D), \]  
(9)

where \( \sin^2 \theta = u^2/c^2 \) with \( c^2 = 2\gamma n_0^A(1 + \kappa) \) is the (common) speed of sound and where the coherence length \( \xi_D \) is given by \( \xi_D^2 = \gamma n_0^A(1 + \kappa)/2 \).

Grey-bright solitary waves. The above analysis requires some modifications in the case where the density of one of the components vanishes at infinity. Assuming, for example, that \( n_A - n_0^A \neq 0 \) and \( n_B \to 0 \), the phase of \( A \) is still given by Eq. (4). However, the local velocity of species \( B \) is constant and equal to \( u \), since \( \phi_B' = u/2 \).

The equations obtained in this case are

\[
\begin{align*}
\left( \frac{\sqrt{n_A}}{\sqrt{n_A'}} \right)'' = \frac{u^2}{4} \left( \frac{(n_0^o)^2}{n_A^2} - 1 \right) + \gamma (n_A - n_0^A + n_B), \quad (10) \\
\left( \frac{\sqrt{n_B}}{\sqrt{n_B'}} \right)'' = \gamma (n_A - n_0^A + n_B) + s, \quad (11)
\end{align*}
\]

where \( s \) is the limit of \( (\sqrt{n_B'})''/\sqrt{n_B} \) as \( |z| \to \infty \). The parameter \( n_0^B \) in the case of grey-grey solutions is now replaced by \( s \), which can be determined from the number of particles \( N_B \) in the bright component \( B \). The ansatz for the solution assumes the form \( n_B = \kappa(n_A - n_0^A) \) with \( \kappa \) negative. Here, \( \kappa \) depends on the propagation velocity \( u \), in contrast to the result obtained above for grey-grey solitons.

The integration of Eqs. (10) and (11) yields

\[
\begin{align*}
n_A^o/2 &= \frac{[\gamma(1 + \kappa)n_A - u^2/2](n_A - n_0^A)^2}{\kappa}, \quad (12) \\
n_B^o/2 &= \frac{[\gamma + 1/\kappa]n_B - u^2/2( n_B - n_0^B)^2}{\kappa}. \quad (13)
\end{align*}
\]

Consistency of these equations with the ansatz demands that

\[ u^2/2 = \gamma(1 + \kappa)n_A^o - 2s, \]  
(14)

and therefore \( \kappa \geq -1 \). Since \( \kappa \) must also be negative, \( -1 \leq \kappa \leq 0 \).

The solution of Eq. (12) for the grey component is the same as in Eq. (9). The solution of Eq. (13) for the bright component is

\[ n_B = \frac{N_B}{2\xi_B \cosh^2(z/\xi_B)} = -\frac{\kappa n_0^A \cos^2 \theta / \cosh^2 (z \cos \theta / \xi_D)}{\xi_B}. \]  
(15)

where \( f \) \( n_B \) \( dz = N_B \), in accordance with the normalization condition, and \( s = 1/\xi_B^2 = \cos^2 \theta / \xi_D^2 \).

For completeness we mention that on a ring of finite radius, our equations also support bright solitary waves in both components provided that \( \gamma \) is negative. We will examine this problem in a future study.

To get some insight into the above results, it is instructive to write the initial equations taking into account the ansatz,

\[ i \frac{\partial \Psi_A}{\partial t} = -\frac{\partial^2 \Psi_A}{\partial x^2} + \gamma|n_A(1 + \kappa) + n_B - \kappa n_0^A|\Psi_A, \]  
(16)

\[ i \frac{\partial \Psi_B}{\partial t} = -\frac{\partial^2 \Psi_B}{\partial x^2} + \gamma|n_B(1 + \kappa) + n_A - n_0^A|\Psi_B. \]  
(17)

In a sense, the ansatz decouples the two equations, although consistency forces additional conditions as shown above. The final terms on the right of Eqs. (16) and (17) either vanish for the grey-grey case, or are constant for the grey-bright case and are not important. The important terms are those involving the combinations \( n_A(1 + \kappa) \) and \( n_B(1 + 1/\kappa) \) or, equivalently, the “effective” couplings \( U_A = \gamma(1 + \kappa) \) and \( U_B = \gamma(1 + 1/\kappa) \). Clearly \( U_A = \kappa U_B \).

In the case of grey-grey solitary waves, \( U_A, U_B \) and \( \kappa \) are all positive, which is consistent with the fact that the solitary waves are grey in both components. On the other hand, in the case of grey-bright solitary waves, \( U_B \) is negative and \( U_A \) is positive since \( -1 \leq \kappa \leq 0 \). This is consistent with a bright wave in the \( B \) component and a grey wave in the \( A \) component.

Uniqueness of the solutions. It is useful to regard the present problem as that of the motion of a particle in a two-dimensional potential, with \( \sqrt{n_j} \) playing the role of spatial coordinates and \( z \) that of time. For example, in the case of two grey solitary waves,

\[ (\partial \sqrt{n_A}/\partial z)^2/2 + (\partial \sqrt{n_B}/\partial z)^2/2 + V(n_A, n_B) = 0, \]  
(18)

where

\[ V = \frac{u^2}{8} \left[ \frac{(n_A - n_0^A)^2}{n_A} + \frac{(n_B - n_0^B)^2}{n_B} \right] - \frac{\gamma}{4}(n_A - n_0^A + n_B - n_0^B)^2. \]  
(19)
In the case of grey-bright solutions the potential becomes,
\[ V = \frac{n^2}{8} \left( \frac{n_A - n_A^0}{n_A} \right)^2 - \frac{\gamma}{4} (n_A - n_A^0 + n_B)^2 - \frac{1}{2} sn_B. \] (20)

In both expressions for \( V \) above, we have imposed the boundary condition that \( V \) vanishes when \( n_A \) and \( n_B \) have their asymptotic values.

In the grey-grey case, the directional derivative of the potential perpendicular to the straight line defined by the ansatz vanishes,
\[ \left( \sqrt{n_B^0} - \sqrt{n_A^0} \right) \left( \frac{\partial V}{\partial n_A} - \frac{\partial V}{\partial n_B} \right) = 0. \] (21)

Starting at the asymptotic field values at “time” \( z = -\infty \), the system will move along this minimum in \( V \) and return to its starting point at \( z = +\infty \).

The solutions found above for the specific boundary conditions are unique. The only physically interesting trajectory must start at rest on the \( V = 0 \) contour with the initial values \( \sqrt{n_A^0} \) and \( \sqrt{n_B^0} \) and must end at the same point. Linearization of the two coupled nonlinear equations for \( |z| \to \infty \) sets the asymptotic form of the two solutions. Given this asymptotic form, the solutions follow a well-defined path for all \( z \) determined by the potential of Eqs. (19) and (20). In other words, there is one and only one trajectory which will result from starting the system at the point with \( (\sqrt{n_A^0}, \sqrt{n_B^0}) = (\sqrt{n_A^0}, \sqrt{n_B^0}) \).

Dispersion relation. Having calculated the profiles of the solitary waves, it is instructive to evaluate the corresponding dispersion relation. In the case of a single-component Bose gas in one spatial dimension, Elliott Lieb \[15\] found that when bosons interact via a contact potential, the excitation spectrum consists of two branches. The one corresponds to the usual Bogoliubov mode, while the other was later shown by Kulish et al. and by Ishikawa and Takayama to correspond to solitary waves \[16, 17, 18, 19\].

The present calculation generalizes these results to the case of a two-component system with some similarities and some differences. Starting with grey-grey solitary waves, the kinetic energy is given by
\[ KE = \int \left[ \left( \partial \sqrt{n_A} / \partial z \right)^2 + n_i (\partial \phi / \partial z)^2 \right] dz. \] (22)

From Eqs. (21) and (22) the total kinetic energy is
\[ KE = KE_A + KE_B = (1 + \kappa)^2 \gamma \int (n_A - n_A^0)^2 \, dz. \] (23)

The interaction (free) energy less the infinite energy of the background density of the two components is
\[ PE - \mu_A N_A - \mu_B N_B = \frac{\gamma}{2} \int (n_A - n_A^0 + n_B - n_B^0)^2 \, dz, \] (24)
which is equal to \( KE \). The total energy is thus
\[ E/E_{GG} = (4/3)(1 + \kappa)^{1/2} \cos^3 \theta, \] (25)
where \( E_{GG} = \gamma n_A^0(1 + \kappa)(n_A^0 \xi_0) \) with \( 1/\xi_0^2 = n_A^0 \gamma/2 \), or \( E_{GG} = (1 + \kappa)[2\gamma(n_A^0 \xi_0^3)^{1/2}] \). Turning to the momentum,
\[ P/n_A^0 = (1 + \kappa)(\pi - 2\theta - \sin 2\theta), \] (26)

for which we have imposed periodic boundary conditions; for a detailed analysis of the derivation of Eq. (26), see Ref. \[19\]. When Eq. (26) is combined with the result of Eq. (25), we get the dispersion relation \( E = E(P) \) plotted in the upper graph of Fig. 1 for the case \( n_A^0 = n_B^0 \).

The dispersion relation for grey-bright solitary waves is calculated in a similar way. The kinetic energy of the two components is
\[ KE_A = (\gamma/2)(1 + \kappa) \int (n_A - n_A^0)^2 \, dz, \] (27)
\[ KE_B = \frac{1}{2} \int (n_A^0 \gamma) N_B (1 + \kappa) + \frac{\gamma}{2} \kappa (1 + \kappa) \int (n_A - n_A^0)^2 \, dz, \] (28)
while the potential energy minus the infinite energy due to the background density of the \( A \) component is
\[ PE - \mu_A N_A = \frac{\gamma}{2} (1 + \kappa)^2 \int (n_A - n_A^0)^2 \, dz. \] (29)

![FIG. 1: The dispersion relation \( E = E(P) \) (solid line), \( dE/dP \) (dashed line), and \( u \) (dashed line) as functions of \( P \) for the case of grey-grey (upper graph) and grey-bright (lower graph) solitary waves. The energy is measured in units of \( E_{GG} \) in the grey-grey case, and in \( E_{GB} \) in the bright case. The derivative \( dE/dP \) and \( u \) are measured in units of \( \sqrt{2n_A \gamma} \). The figures show the case \( n_A^0 = n_B^0 \) in the grey-grey case, and \( N_B \) is set equal to \( n_A^0 \xi_0 \) in the grey-bright case. The equality \( dE/dP = u \) is satisfied in both cases.](image-url)
Collecting terms yields
\[ \frac{E}{E_{GB}} = \frac{1}{2} (1 + \kappa) \left( 1 - \frac{4}{3} \kappa \cos^2 \theta \right), \]  
(30)
where \( E_{GB} = N_B n_0^0 \gamma_n \). The momentum of the two components is given as \[19\]
\[ P_A / n_0^0 = \pi - 2\theta - \sin 2\theta, \]  
(31)
\[ P_B = N_B \sin \theta \sqrt{1 + \kappa} n_0^0 \gamma_n / 2. \]  
(32)

The parameter \( \kappa \) is now a function of \( u \) (and therefore of \( \theta \)). Their relationship is established by the normalization condition,
\[ N_B / (n_0^0 A_0) = -2\kappa \cos \theta / \sqrt{1 + \kappa}. \]  
(33)

From Eqs. (31), (32) and (33) we find that \( P = P_A + P_B \) is given by
\[ P / n_0^0 = \pi - 2\theta - (1 + \kappa) \sin 2\theta. \]  
(34)

Combining Eqs. (30), (33) and (34), we obtain the dispersion relation for a grey-bright solitary wave, which is plotted in the lower graph of Fig. 1 for the case \( n_0^0 A_0 = N_B \).

Specifically, for the case of grey-grey solitary waves when \( P \to 0 \), we find \( E = c P \), with the speed of sound \( c = \sqrt{2\gamma(n_0^0 + n_B^0)} \) identical to the value obtained by linearizing the original equations \[20\]. For the case of grey-bright solitary waves when \( P \to 0 \), the dispersion relation vanishes quadratically, \( E \to P^2 / N_B \). Then, \( \partial E / \partial P \to 2n_0^0 \gamma_n (1 + \kappa) \), which agrees with the value of \( u \) obtained from Eq. (14) for \( s \to 0 \).

Conclusions. Solitary waves in a two-component Bose-Einstein condensate confined in a ring potential show interesting physics. Considering repulsive effective interactions, this system supports bound states of grey-grey and grey-bright solitary-wave solutions, which are unique. The dispersion relation resembles that found for single-component solitary waves for large density depressions for both grey-grey and grey-bright solitary waves.

Experimental verification of the derived dispersion relation shown in Fig. 1 would confirm the expected spectrum for the first time, as this has never been confirmed in any nonlinear system so far. Such an experiment should be possible with use of the method of Bragg spectroscopy \[21\] \[22\] \[23\], which has been developed to probe the excitation spectrum of cold atomic systems.