COMPREHENSIVE SUBCLASSES OF ANALYTIC FUNCTIONS AND
COEFFICIENT BOUNDS

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ABSTRACT. In this paper, we introduce two general subclasses of analytic functions by means
of the principle of subordination and investigate the coefficient bounds for functions in these
classes. The well-known results are obtained as a corollary of our main results. Especially, we
improve the results of Altınta¸s and Kılıç [1].

1. DEFINITIONS AND PRELIMINARIES

Let $A$ be the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disk $D = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. For
analytic functions $f$ and $g$ with $f(0) = g(0)$, $f$ is said to be subordinate to $g$ in $D$ if
there exists an analytic function $h$ on $D$ such that

$$h(0) = 0, \quad |h(z)| < 1 \quad \text{and} \quad f(z) = g(h(z)) \quad (z \in D).$$

We denote the subordination by

$$f(z) \prec g(z) \quad (z \in D).$$

Note that if the function $g$ is univalent in $D$, then we have

$$f(z) \prec g(z) \quad (z \in D) \iff f(0) = g(0) \quad \text{and} \quad f(D) \subset g(D).$$

Let $N$ be the class consisting of analytic and univalent functions $\varphi : D \to \mathbb{C}$ such that $\varphi(D)$
is convex with

$$\varphi(0) = 1 \quad \text{and} \quad \Re(\varphi(z)) > 0 \quad (z \in D).$$

By means of functions belong to the class $\mathcal{N}$ and the principle of subordination, we consider
following subclasses of analytic function class $A$:

$$S^*(\varphi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (\varphi \in N; \ z \in D) \right\}, \quad (1.2)$$

$$K(\varphi) = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \quad (\varphi \in N; \ z \in D) \right\}, \quad (1.3)$$

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The classes $S^* (\varphi)$ and $K (\varphi)$ are introduced by Ma and Minda [5], and the class $C (\varphi, \psi)$ is introduced by Kim et al. [3]. Since $f (z) \in K (\varphi) \iff zf' (z) \in S^* (\varphi)$, we also have $f (z) \in C (\varphi, \psi) \iff \exists g \in S^* (\psi) \text{ s.t. } \frac{zf' (z)}{g' (z)} \prec \varphi (z) \quad (z \in \mathbb{D}).$

**Remark 1.** If we choose
\[ \varphi (z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1) \]
in (1.2) and (1.3), then we get the classes of Janowski starlike functions and Janowski convex functions
\[ S^* \left( \frac{1 + Az}{1 + Bz} \right) = S^* (A, B) \quad \text{ and } \quad K \left( \frac{1 + Az}{1 + Bz} \right) = K (A, B), \]
respectively, introduced by Janowski [2].

**Remark 2.** If we choose
\[ \varphi (z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1) \quad \text{ and } \quad \psi (z) = \frac{1 + z}{1 - z} \]
in (1.4) and (1.5), then we obtain the classes
\[ C \left( \frac{1 + Az}{1 + Bz}, \frac{1 + z}{1 - z} \right) = CCV (A, B), \quad C \left( \frac{1 + Az}{1 + Bz}, \frac{1 + z}{1 - z} \right) = CST (A, B) \]
introduced by Reade [8]; and from (1.6), we have the class
\[ QK \left( \frac{1 + Az}{1 + Bz}, \frac{1 + z}{1 - z} \right) = QCV (A, B) \]
introduced by Altıntaş and Kılıç [1].

**Remark 3.** If we choose
\[ \varphi (z) = \frac{1 + (1 - 2\alpha) z}{1 - z} \quad (0 \leq \alpha < 1) \quad \text{ and } \quad \psi (z) = \frac{1 + (1 - 2\beta) z}{1 - z} \quad (0 \leq \beta < 1) \]
in (1.4), then we obtain the class of close-to-convex functions of order $\alpha$ and type $\beta$,
\[ C \left( \frac{1 + (1 - 2\alpha) z}{1 - z}, \frac{1 + (1 - 2\beta) z}{1 - z} \right) = C (\alpha, \beta), \]
introduced by Libera [4].
Remark 4. If we choose
\[ \varphi(z) = \frac{1 + z}{1 - z} = \psi(z) \]
in \((1.2)-(1.4)\), then we get the familiar class \(S^*\) consists of starlike functions in \(D\), \(K\) consists of convex functions in \(D\) and \(C\) consists of close-to-convex function in \(D\), respectively. Also, from \((1.5)\) and \((1.6)\), we get the class \(CS\) of close-to-starlike functions in \(D\) introduced by Reade [8], and the class \(Q\) of quasi-convex functions in \(D\) introduced by Noor and Thomas [7], respectively.

Throughout this paper
\[ 0 \leq \delta \leq \lambda \leq 1 \quad \text{and} \quad \varphi, \psi \in \mathcal{N}. \]

Now we define new comprehensive subclasses of analytic function class \(A\), as follows:

**Definition 1.** A function \(f \in A\) is said to be in the class \(K_{\lambda,\delta}(\varphi, \psi)\) if
\[
\frac{f'(z) + (\lambda - \delta + 2\lambda \delta) zf''(z) + \lambda \delta z^2 f'''(z)}{g'(z)} < \varphi(z) \quad (z \in D),
\]
where \(g \in K_{\psi}(\psi)\).

**Definition 2.** A function \(f \in A\) is said to be in the class \(S_{\lambda,\delta}(\varphi, \psi)\) if
\[
\frac{(1 - \lambda + \delta) f(z) + (\lambda - \delta) zf'(z) + \lambda \delta z^2 f''(z)}{g(z)} < \varphi(z) \quad (z \in D),
\]
where \(g \in S^*(\psi)\).

**Remark 5.** If we set \(\delta = 0\) and \(\lambda = 1\) in Definition 1 and Definition 2, then we have the classes
\[ K_{1,0}(\varphi, \psi) = QK(\varphi, \psi) \quad \text{and} \quad S_{1,0}(\varphi, \psi) = C(\varphi, \psi). \]
Also when \(\delta = 0\) and \(\lambda = 0\), we get the classes
\[ K_{0,0}(\varphi, \psi) = C(\varphi, \psi) \quad \text{and} \quad S_{0,0}(\varphi, \psi) = CS(\varphi, \psi). \]

**Remark 6.** If we set \(\delta = 0\) and
\[ \varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1) \quad \text{and} \quad \psi(z) = \frac{1 + z}{1 - z} \]
in Definition 1 and Definition 2, then we obtain the classes \(Q_{CV}(\lambda, A, B)\) and \(Q_{ST}(\lambda, A, B)\), respectively, introduced very recently by Altıntaş and Kılıç [1]. These classes consist of functions \(f \in A\) satisfying
\[
\frac{f'(z) + \lambda zf''(z)}{g'(z)} < \frac{1 + Az}{1 + Bz} \quad (g \in K, \ z \in D)
\]
and
\[
\frac{(1 - \lambda) f(z) + \lambda zf'(z)}{g(z)} < \frac{1 + Az}{1 + Bz} \quad (g \in S^*, \ z \in D),
\]
respectively.

Altıntaş and Kılıç [1] obtained following coefficient bounds for functions belong to the classes \(Q_{CV}(\lambda, A, B)\) and \(Q_{ST}(\lambda, A, B)\), as follows:
Theorem 1. If $f \in Q_{CV}(\lambda, A, B)$, then
\[ |a_n| \leq \frac{1}{1 + (n - 1) \lambda} \left( 1 + \frac{(n - 1)(A - B)}{1 - B} \right) \quad (n = 2, 3, \ldots). \]

Theorem 2. If $f \in Q_{ST}(\lambda, A, B)$, then
\[ |a_n| \leq \frac{n}{1 + (n - 1) \lambda} \left( 1 + \frac{(n - 1)(A - B)}{1 - B} \right) \quad (n = 2, 3, \ldots). \]

In this work, we obtain coefficient bounds for functions in the comprehensive subclasses $K_{\lambda,\delta}(\varphi, \psi)$ and $S_{\lambda,\delta}(\varphi, \psi)$ of analytic functions. Our results improve the results of Altıntaş and Kılıç [1] (Theorem 1 and Theorem 2).

2. Main results

Lemma 1. [9] Let the function $\Phi$ given by
\[ \Phi(z) = \sum_{n=1}^{\infty} A_n z^n \quad (z \in \mathbb{D}) \]
be convex in $\mathbb{D}$. Also let the function $\Psi$ given by
\[ \Psi(z) = \sum_{n=1}^{\infty} B_n z^n \quad (z \in \mathbb{D}) \]
be holomorphic in $\mathbb{D}$. If $\Psi(z) \prec \Phi(z)$ $(z \in \mathbb{D})$,
then
\[ |B_n| \leq |A_1| \quad (n = 1, 2, \ldots). \]

Lemma 2. [10] Let $f \in K(\psi)$ and be of the form (1.1), then
\[ |a_n| \leq \frac{\prod_{j=0}^{n-2} (j + |\psi'(0)|)}{n!} \quad (n = 2, 3, \ldots). \]

Lemma 3. [10] Let $f \in S^*(\psi)$ and be of the form (1.1), then
\[ |a_n| \leq \frac{\prod_{j=0}^{n-2} (j + |\psi'(0)|)}{(n - 1)!} \quad (n = 2, 3, \ldots). \]

Theorem 3. Let $f \in K_{\lambda,\delta}(\varphi, \psi)$ and be of the form (1.1), then
\[ |a_n| \leq \frac{\prod_{j=0}^{n-2} (j + |\psi'(0)|)}{n!} + \frac{|\varphi'(0)|}{n} \left( 1 + \sum_{k=1}^{n-2} \frac{\prod_{j=0}^{n-k-2} (j + |\psi'(0)|)}{(n - k - 1)!} \right) \quad (n = 2, 3, \ldots). \quad (2.1) \]
Proof. Let the function \( f \in K_{\lambda, \delta}(\varphi, \psi) \) be defined by (1.1). Therefore, by Definition 1, there exists a function
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K(\psi), \psi \in \mathcal{M}
\] (2.2)
so that
\[
f'(z) + (\lambda - \delta + 2\lambda \delta)zf''(z) + \lambda \delta z^2 f'''(z) < \varphi(z) \quad (z \in \mathbb{D}).
\] (2.3)
Note that by (2.2) and Lemma 2, we have
\[
|b_n| \leq \frac{n^{n-2}}{n!} \prod_{j=0}^{n-2} (j + |\psi'(0)|) \quad (n = 2, 3, \ldots).
\] (2.4)
Let us define the function \( p(z) \) by
\[
p(z) = \frac{f'(z) + (\lambda - \delta + 2\lambda \delta)zf''(z) + \lambda \delta z^2 f'''(z)}{g'(z)} \quad (z \in \mathbb{D}).
\] (2.5)
Then according to (2.3) and (2.5), we get
\[
p(z) < \varphi(z) \quad (z \in \mathbb{D}).
\] (2.6)
Hence, using Lemma 1, we obtain
\[
|p^{(m)}(0)| = |c_m| \leq |\varphi'(0)| \quad (m = 1, 2, \ldots),
\] (2.7)
where
\[
p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{D}).
\] (2.8)
Also from (2.5), we find
\[
f'(z) + (\lambda - \delta + 2\lambda \delta)zf''(z) + \lambda \delta z^2 f'''(z) = p(z)g'(z).
\] (2.9)
Since \( a_1 = b_1 = 1 \), in view of (2.9), we obtain
\[
n \left[ 1 + (n-1) (\lambda - \delta + 2\lambda \delta) + (n-1) (n-2) \lambda \delta \right] a_n = nb_n
\]
\[
= c_{n-1} + 2c_{n-2}b_2 + \cdots + (n-1) c_1 b_{n-1}
\]
\[
= \sum_{k=1}^{n-1} (n-k) c_k b_{n-k} \quad (n = 2, 3, \ldots).
\] (2.10)
Now we get the desired result given in (2.1) by using (2.4), (2.7) and (2.10).
\[\square\]

Theorem 4. Let \( f \in S_{\lambda, \delta}(\varphi, \psi) \) and be of the form (1.1), then
\[
\left[ 1 + (n-1) (\lambda - \delta + 2\lambda \delta) + (n-1) (n-2) \lambda \delta \right] |a_n|
\]
\[
\leq \frac{n^{n-2}}{(n-1)!} \prod_{j=0}^{n-2} (j + |\psi'(0)|) \left( 1 + \sum_{k=1}^{n-2} \frac{\prod_{j=0}^{n-k-2} (j + |\psi'(0)|)}{(n-k-1)!} \right) \quad (n = 2, 3, \ldots).
\] (2.11)
Proof. Let the function $f \in \mathcal{S}_{\lambda, \delta}(\varphi, \psi)$ be defined by (1.1). Therefore, by Definition 2 there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*_\psi, \psi \in \mathcal{M} \tag{2.12}$$

so that

$$
\frac{(1 - \lambda + \delta) f(z) + (\lambda - \delta) z f'(z) + \lambda \delta z^2 f''(z)}{g(z)} \prec \varphi(z) \quad (z \in \mathbb{D}). \tag{2.13}
$$

Note that by (2.12) and Lemma 3, we have

$$
|b_n| \leq \frac{\prod_{j=0}^{n-2} (j + |\psi'(0)|)}{(n - 1)!} \quad (n = 2, 3, \ldots). \tag{2.14}
$$

Let us define the function $q(z)$ by

$$q(z) = \frac{(1 - \lambda + \delta) f(z) + (\lambda - \delta) z f'(z) + \lambda \delta z^2 f''(z)}{g(z)} \quad (z \in \mathbb{D}). \tag{2.15}
$$

Then according to (2.13) and (2.15), we get

$$q(z) \prec \varphi(z) \quad (z \in \mathbb{D}). \tag{2.16}
$$

Hence, using Lemma 1, we obtain

$$
\left| \frac{q^{(m)}(0)}{m!} \right| = |d_m| \leq |\varphi'(0)| \quad (m = 1, 2, \ldots), \tag{2.17}
$$

where

$$q(z) = 1 + d_1 z + d_2 z^2 + \cdots \quad (z \in \mathbb{D}). \tag{2.18}
$$

Also from (2.15), we find

$$(1 - \lambda + \delta) f(z) + (\lambda - \delta) z f'(z) + \lambda \delta z^2 f''(z) = q(z) g(z). \tag{2.19}
$$

Since $a_1 = b_1 = 1$, in view of (2.19), we obtain

$$
[1 - \lambda + \delta + n (\lambda - \delta) + n (n - 1) \lambda \delta] a_n - b_n
= c_{n-1} + c_{n-2} b_2 + \cdots + c_1 b_{n-1}
= \sum_{k=1}^{n-1} c_k b_{n-k} \quad (n = 2, 3, \ldots). \tag{2.20}
$$

Now we get the desired result given in (2.11) by using (2.14), (2.17) and (2.20). \qed
3. Corollaries and Consequences

Letting \( \delta = 0 \) and \( \lambda = 1 \) in Theorem 3 and Theorem 4, we obtain the following consequences, respectively.

**Corollary 1.** Let \( f \in QK(\varphi, \psi) \) and be of the form (1.1), then

\[
|a_n| \leq \frac{1}{n^2} \prod_{j=0}^{n-2} (j + |\psi'(0)|) \left( 1 + \sum_{k=1}^{n-2} \frac{\prod_{j=0}^{n-k-2} (j + |\psi'(0)|)}{(n-k-1)!} \right) \quad (n = 2, 3, \ldots). 
\]

**Corollary 2.** Let \( f \in C(\varphi, \psi) \) and be of the form (1.1), then

\[
|a_n| \leq \frac{1}{n!} \prod_{j=0}^{n-2} (j + |\psi'(0)|) \left( 1 + \sum_{k=1}^{n-2} \frac{\prod_{j=0}^{n-k-2} (j + |\psi'(0)|)}{(n-k-1)!} \right) \quad (n = 2, 3, \ldots). 
\]

Letting \( \delta = 0 \) and \( \lambda = 0 \) in Theorem 4, we obtain the following consequence.

**Corollary 3.** Let \( f \in CS(\varphi, \psi) \) and be of the form (1.1), then

\[
|a_n| \leq \frac{1}{(n-1)!} \prod_{j=0}^{n-2} (j + |\psi'(0)|) \left( 1 + \sum_{k=1}^{n-2} \frac{\prod_{j=0}^{n-k-2} (j + |\psi'(0)|)}{(n-k-1)!} \right) \quad (n = 2, 3, \ldots). 
\]

If we choose \( \varphi(z) = \frac{1 + (1 - 2\alpha) z}{1 - z} \) (\( 0 \leq \alpha < 1 \)) and \( \psi(z) = \frac{1 + (1 - 2\beta) z}{1 - z} \) (\( 0 \leq \beta < 1 \)) in Corollary 2, then we get following consequence.

**Corollary 4.** Let \( f \in C(\alpha, \beta) \) (\( 0 \leq \alpha, \beta < 1 \)) and be of the form (1.1), then

\[
|a_n| \leq \frac{2(3 - 2\beta)(4 - 2\beta) \cdots (n - 2\beta)}{n!} \left[ n(1 - \alpha) + (\alpha - \beta) \right] \quad (n = 2, 3, \ldots). 
\]

Letting

\[
\delta = 0, \quad \varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1), \quad \psi(z) = \frac{1 + z}{1 - z} 
\]

in Theorem 3 and Theorem 4, we obtain the following consequences, respectively.

**Corollary 5.** Let \( f \in QC_{CV}(\lambda, A, B) \) and be of the form (1.1), then

\[
|a_n| \leq \frac{1}{1 + (n - 1) \lambda} \left( 1 + \frac{(n-1)(A-B)}{2} \right) \quad (n = 2, 3, \ldots). 
\]

**Corollary 6.** Let \( f \in QC_{ST}(\lambda, A, B) \) and be of the form (1.1), then

\[
|a_n| \leq \frac{n}{1 + (n - 1) \lambda} \left( 1 + \frac{(n-1)(A-B)}{2} \right) \quad (n = 2, 3, \ldots). 
\]
Remark 7. It is clear that
\[ 1 + \frac{(n - 1)(A - B)}{2} \leq 1 + \frac{(n - 1)(A - B)}{1 - B} \quad (-1 \leq B < A \leq 1, \ n = 2, 3, \ldots), \]
which would obviously yield significant improvements of Theorem 1 and Theorem 2.

Letting
\[ \lambda = 0, \quad A = 1, \quad B = -1 \]
in Corollary 5 and Corollary 6 we have following consequences, respectively.

**Corollary 7.** [8] Let \( f \in \mathcal{C} \) and be of the form \((1.1)\), then
\[ |a_n| \leq n^2 \quad (n = 2, 3, \ldots). \]

**Corollary 8.** [8] Let \( f \in \mathcal{CS} \) and be of the form \((1.1)\), then
\[ |a_n| \leq n^2 \quad (n = 2, 3, \ldots). \]

Letting
\[ \lambda = 1, \quad A = 1, \quad B = -1 \]
in Corollary 5 we have following consequence.

**Corollary 9.** [6] Let \( f \in \mathcal{Q} \) and be of the form \((1.1)\), then
\[ |a_n| \leq 1 \quad (n = 2, 3, \ldots). \]

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