A General Criterion of Quantum Integrability Accommodating Central Charges and ”Anomalies”

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Abstract

A simple quantum generalisation of the Liouville–Arnold criterion of classical integrability is proposed: a system is quantum-integrable if it has an abelian Lie group of Wigner symmetries of dimension equal to the number of degrees of freedom. The criterion goes significantly beyond the familiar case of involutive conserved operators to cover systems with anomalies, in which involutivity is modified by central charges. ”Anomalous” quantum integrability is shown to have all the expected consequences including exact diagonalsability. The approach throws new light on the origin of Weyl group invariance.
1. Classical and quantum integrability. A constraint-free conservative mechanical system of a finite number $n$ of degrees of freedom is said to be completely integrable classically (abbreviated to LA-integrable, for Liouville-Arnold)[1] if there are $n$ functions $f_i$ on its phase space $P$ having the properties of (i) involutivity: $[f_i, f_j] = 0$; (ii) conservation: $[h, f_i] = 0$ where $h$ is the Hamiltonian function; and (iii) completeness: the differentials $df_i$ are linearly independent. The conserved quantities $f_i$ are action variables and they form one half of a set of canonical coordinates $f_i, q_i$ for $P$, $[f_i, q_j] = \delta_{ij}$. More precisely, $q_i$ are local coordinates for a maximal submanifold $C$ (of dimension $n$) of $P$ on which all $f_i$ take constant values. $C$ is a generalised configuration space adapted to the set of action variables. It is not necessarily compact; despite this, we shall refer to $q_i$ as an angle variable. It is an immediate consequence of LA-integrability that $h$ is a function only of $f_i$ and that the equation of motion for $q_i$ takes the form $dq_i/dt = \phi_i(f)$ for some function $\phi_i$ and so can be trivially integrated.

Our purpose here is to propose an answer to the question: What is a criterion for quantum (q-)integrability that is as concise and general as the above classical criterion? Historically, the study of q-integrability began with certain soluble models[2,3] exhibiting the properties (i) to (iii), with obvious reinterpretations of $f$ and $h$ as selfadjoint operators on the space of states and of $[,]$ as the commutator. (We shall later be using $[,]$ for the Lie brackets also; the context will make the meaning clear). That these properties are a satisfactory general characterisation of q-integrability seems to have general acceptance, though their verification in models remains a case by case exercise. In addition to having diagonalisable Hamiltonians, such models exhibit certain finite(Weyl) group "symmetries"[3,4] whose provenance has remained a matter of some mystery.

What is proposed here is an inherently quantum formulation of q-integrability that relies on the classical theory only for motivation. It provides a framework covering the models mentioned above, namely those obtained by a direct transcription ("naive quantisation") of an integrable classical system to the quantum domain. But it also extends the scope of the notion, beyond such "normal" q-integrable systems, to a class of "anomalous" q-integrable systems for which involutivity is modified by the presence of central charges. (These terms are explained later). In particular, the latter class will be shown to possess the properties of diagonalisability and Weyl group invariance.

2. The criterion. The standard criterion of LA-integrability stated above says that the
system has an \( n \)-dimensional abelian Lie algebra of classical symmetries. We shall assume that this can be exponentiated to a Lie group \( G \) (connected, abelian, \( n \)-dimensional) of symmetries. Classically, this only requires that \( P \) is a manifold\[5\]; in the quantum context, symmetry under \( G \) is a tighter demand than under its Lie algebra. Our criterion for q-integrability is then simply:

*A quantum system of \( n \) degrees of freedom is q-integrable if it has a connected, abelian, \( n \)-dimensional Lie group \( G \) as a group of Wigner symmetries.*

By a Wigner symmetry is meant as usual a one-one onto map of states to themselves preserving transition probabilities. Significantly, \( G \)-invariance of the Hamiltonian is not part of this criterion. Indeed demanding it will prove to be unjustified in general—we will see below that it is the proper handling of this issue that extends the scope of q-integrability to systems ”with anomalies”.

The group \( G \) is the product of a vector group and a torus group: \( G = \mathbb{R}^k \times \mathbb{T}^{n-k} \), for some \( k, 0 \leq k \leq n \). By Wigner’s theorem, the state space \( \mathcal{H} \) must carry (continuous) projective unitary representations (PURs in short) of this group. We summarise the facts concerning them, relevant to the present work, briefly[6,7].

PURs of the group \( \mathbb{R}^n \) are in one-one correspondence with, and can be obtained \textit{via} the exponential map from, the URs of a Lie algebra \( g_\alpha \) with basis \( X_i \) and brackets

\[
[X_i, X_j] = i\alpha_{ij}, \quad 1 \leq i, j \leq n,
\]

obtained by adjoining a set of real central charges \( \alpha_{ij} = -\alpha_{ji} \) to the abelian Lie algebra of \( \mathbb{R}^n \).

These brackets define a central extension of \( \mathbb{R}^n \), as a Lie algebra, by \( \mathbb{R} \) and corresponding to each distinct set \( \alpha = \{\alpha_{ij}\} \) we get a distinct central extension by \( U(1) \), and a distinct equivalence class of PURs, of the group \( \mathbb{R}^n \). The representation space \( \mathcal{H}_\alpha \) is a superselection sector in the total state space \( \mathcal{H} \) for a fixed \( \alpha \) and the latter is therefore a collection of sectors with no inter-sector transitions allowed, rather than their direct sum. The trivial sector \( \mathcal{H}_0 \) is special; it carries a UR (or a trivial PUR) of \( \mathbb{R}^n \).

In sharp contrast, a torus group has only trivial central extensions and trivial PURs[8] even though \( \mathbb{T}^n \) and \( \mathbb{R}^n \) have the same Lie algebra; the state space consists of just the trivial sector. Thus the possible central charges are determined by the compactness properties of \( G \).

3.\textit{Normal q-integrability; the trivial sector.} We consider first the case \( G = \mathbb{T}^n \). Then
the total state space $\mathcal{H}$ itself is the trivial sector; it carries a UR $U$ of $\mathbb{T}^n$ and hence of its Lie algebra ($\cong \mathbb{R}^n$) with basis $X_i$. We can choose the $n$ independent selfadjoint operators $F_i = U(X_i)$ as action operators on $\mathcal{H}$ and they are trivially involutive. Next, we note that the conservation of $F_i$ is equivalent to the $\mathbb{T}^n$-invariance of the Hamiltonian operator $H$ of the system, by virtue of the equation of motion

$$i\dot{F}_i = [H, F_i] = 0.$$ 

But the invariance of $H$ follows from the fact that $\mathbb{T}^n$ acts unitarily on all states in $\mathcal{H}$ at all times; thus conservation is ensured by our criterion. Consequently, as in the classical case, $H$ is a function of $F_i$ alone. To show this, it is best to choose the concrete realisation of $\mathcal{H}$ as $L^2(C)$, the Hilbert space of square-integrable functions $\psi$ (generalised wave functions) on the generalised configuration space $C$ of angle variables $q_i$. The action

$$(F_i \psi)(q) = -i\frac{\partial \psi}{\partial q_i}$$

makes $L^2(C)$ a UR space of $\mathbb{T}^n$ (imposing suitable boundary conditions if needed). We have also the "angle operators"

$$(Q_i \psi)(q) = q_i \psi(q),$$

so that on $L^2(C)$ there is an irreducible (by the Stone–von Neumann theorem) UR of the Heisenberg algebra defined by the canonical commutators $[F_i, Q_j] = -i\delta_{ij}$. Every vector of $\mathcal{H}$ can then be approximated by polynomials in the creation operators $F_i + iQ_i$ operating on the unique state annihilated by $F_i - iQ_i$ for all $i$. Hence every densely defined operator, in particular $H$, can be approximated by polynomials in ("is a function of") $F_i, Q_i$, implying $[H, F_i] = i\partial H/\partial Q_i = 0$ from the conservation of $F_i$ (invariance of $H$).

Thus normal q-integrability, which is a special case of our general criterion, is a direct quantum transcription of LA-integrability in all respects. The models of [2,3] are examples of this class. Their LA-integrability guarantees their normal q-integrability and vice versa; stated differently, every LA-integrable system has a normal q-integrable quantisation whose classical limit it is. It is also possible, from our quantum point of view, to determine the circumstances under which such systems possess finite (Weyl) group symmetries; this will be done elsewhere.

4. Nontrivial sectors. The fundamental difference between the trivial and a nontrivial sector is best seen in the maximally nontrivial case, namely, when the central charges form
a nonsingular matrix $A$, $\det A = 0$. This means that $G = \mathbb{R}^n$ with no torus factor and that $n = 2l$ is even. $\mathcal{H}_\alpha$ can still be taken as the space $L^2(C)$ of generalised wave functions (many-valued wave functions may have to be considered, but this is easily done) with the angle operators acting, as before, by multiplication by $q_i$. As for the action operators, it is quickly checked that

$$F_{ai} = U_\alpha(X_i) = -i \frac{\partial}{\partial q_i} + \frac{1}{2} \alpha_{ij} q_j$$

is a UR of the Lie algebra $g_\alpha$:

$$[F_{ai}, F_{aj}] = i \alpha_{ij},$$

while preserving the action–angle CCR. This UR is not irreducible. The operators

$$F'_{ai} = -i \frac{\partial}{\partial q_i} - \frac{1}{2} \alpha_{ij} q_j$$

satisfy

$$[F'_{ai}, F_{aj}] = 0, \quad [F'_{ai}, F'_{aj}] = -i \alpha_{ij},$$

showing that $\mathcal{H}_\alpha$ carries a UR of the direct sum $g_\alpha \oplus g_{-\alpha}$. Hence $\mathcal{H}_\alpha$ has the tensor factorisation

$$\mathcal{H}_\alpha = \mathcal{V}_\alpha \otimes \mathcal{V}_{-\alpha},$$

where $\mathcal{V}_\alpha$ is the unique (upto equivalence, by Stone–von Neumann) irreducible (by Schur) UR of the Heisenberg Lie algebra $g_\alpha$.

This factorisation[9] is a key result. It suggests that the operators best adapted to the study of q-integrability in the presence of central charges are not the action and angle operators but rather the action operators for both signs of the central charges. Note also that $g_{-\alpha}$ is not part of the integrability Lie algebra; it just describes concisely the (infinite) multiplicity in $\mathcal{H}_\alpha$ of the unique irreducible UR of $g_\alpha$.

5. Conservation, anomalies. The superselection structure requires the Hamiltonian, like all observables, to be block-diagonal with respect to the sectors—it is actually a family of operators $H_\alpha$, one for each sector. Now the group unitarily represented in $\mathcal{H}_\alpha$ is not $G$ but rather its central extension $G_\alpha$ (of which $G$ is not a subgroup except for $\alpha = 0$). Thus for each $\alpha$, $H_\alpha$ must be invariant under the unitary action of $G_\alpha$ on $\mathcal{H}_\alpha$. Infinitesimally, this demand generalises the trivial sector equation of motion (Sec.3) to

$$i \dot{F}_{ai} = [H_\alpha, F_{ai}] = 0,$$
valid in an arbitrary sector. Thus all $F_{a_i}$, $i = 1, .., 2l$, are necessarily conserved simply as a consequence of the criterion of $q$-integrability; conservation in $H_\alpha$ and the $g_\alpha$- invariance of $H_\alpha$ are, once again, the same. Contrarily, $F'_{a_i}$ are not conserved as they do not represent the symmetries of integrability.

We can now use the factorisation property of $H_\alpha$ to show that $H_\alpha$ is not a function of $F_{a_i}$, but only of $F'_{a_i}$: briefly, from the irreducibility of $H_\alpha$ under $g_\alpha \oplus g_{-\alpha}$, $H_\alpha$ is a priori a polynomial in $F_{a_i}$ and $F'_{a_i}$ but, since it commutes with $F_{a_i}$, it cannot depend on them. The argument, based on properties of the Heisenberg Lie algebra, is the same as used for the trivial sector Hamiltonian $H$, but the conclusion is diametrically different. (Of course, for $\alpha = 0$, $F_i$ and $F'_i$ coincide). We also note the implication that $H_\alpha$ cannot be chosen as one of the conserved charges $F_{a_i}$, as in normal $q$- (and LA-) integrable systems.

Given the generality of scope of our discussion, invariance is about the only guiding principle available in choosing a Hamiltonian—starting with a system of particles in a real configuration space with a specified Hamiltonian and transforming to action–angle operators to exhibit integrability is not part of our present aim. We can only assert that when the Hamiltonian is so transformed, it should not depend on the action operators if the system is integrable. In the rest of this paper we make a specific choice for $H_\alpha$ as a polynomial in $F'_{a_i}$, namely,

$$H_\alpha = \sum_{i=1}^{2l} (F'_{a_i})^2.$$  

This is the Hamiltonian operator in $H_\alpha$ resulting from the appropriate quantisation of the classical Hamiltonian function $h = \Sigma_i(f_i)^2$. The details below are for this particularly nice Hamiltonian, but similar results hold and can be worked out for other acceptable choices.

On the other hand, a ”naive” quantisation (see Sec. 1) of the same system will lead to the choice $H = H_0 = \Sigma_i(F_i)^2$ for commuting $F_i$, which is correct only for the trivial sector. Using this $H_0$ to compute time evolution in $H_\alpha$ results in an apparent lack of conservation of $F_{a_i}$:

$$[H_0, F_{a_i}] = -\alpha_{ij} \frac{\partial}{\partial q_j} = i\alpha_{ij} F_j.$$  

This is a typical instance of ”anomalous conservation” arising from the use of a classically indicated Hamiltonian in a quantum context where it is inappropriate—the anomaly gets cancelled on adding the anomalous piece $H_\alpha - H_0$ to the ”naive Hamiltonian” $H_0$. Nontrivial sectors are anomalous in this sense. (For more on the link between PURs and anomalies,
The Heisenberg equations for the angle operators $Q_i$ (and $F'_{\alpha i}$),

$$
\dot{Q}_i = i[H_\alpha, Q_i] = 2 F'_{\alpha i},
$$

$$
\ddot{Q}_i = 2 \dot{F}'_{\alpha i} = 2i[H_\alpha, F'_{\alpha i}] = -4 \alpha_{ij} F'_{\alpha j},
$$

are easily integrated. Thus the term 'integrable' is justified despite the presence of central charges and anomalies.

6. Spectral properties, Weyl group invariance. The Hamiltonian $H_\alpha$ can be explicitly diagonalised as follows.

As an operator on $H_\alpha$, $H_\alpha$ is invariant under $F'_{\alpha i} \rightarrow M_{ij} F'_{\alpha j} =: I_{\alpha i}$ for $M$ a real orthogonal matrix:

$$
H_\alpha = \sum_{i=1}^{2l} (I_{\alpha i})^2, \quad [I_{\alpha i}, I_{\alpha j}] = i (MAM^T)_{ij}.
$$

The matrix $A$ of central charges is real antisymmetric and hence belongs to the Lie algebra of $SO(2l)$. If $T$ is the maximal torus of $SO(2l)$, we can choose $M$ such that $MAM^T =: C$ is in the Lie algebra of $T$ (the maximal commuting or Cartan subalgebra)[11]; i.e., $C$ has the form

$$
C = \begin{pmatrix}
0 & B \\
-B & 0
\end{pmatrix}
$$

with $B$ a real $l \times l$ diagonal matrix, $B = \text{diag}(\beta_1, ..., \beta_l)$. Redefining $I_{\alpha l+i} =: J_{\alpha i}$, we have

$$
H_\alpha = \sum_{k=1}^{l} ((I_{\alpha k})^2 + (J_{\alpha k})^2), \quad [I_{\alpha k}, J_{\alpha m}] = i \delta_{km} \beta_k,
$$

with other commutators zero. Without losing generality, we may therefore assume $H_\alpha$ to have the above canonical form which is that for an $l$-dimensional oscillator or $l$ Landau ‘electrons’ with charges proportional to $\{\beta_k\}$. The energy spectrum consists of eigenvalues

$$
E_\nu := E_{\{\nu\}} = \sum_{k=1}^{l} (\nu_k + 1/2) | \beta_k |^2
$$

for arbitrary nonnegative integers $\nu_k$. The degeneracy of $E_\nu$ in $V_{-\alpha}$ is the number of solutions $\{\nu\}$ of this equation for fixed $\{\beta\}$—for this reason, it is fair to term it arithmetic, an especially suitable name when all $\beta$ are rational numbers.

The arithmetic degeneracy has an alternative description by means of an invariance property of $H_\alpha$ under the Weyl group of $SO(2l)[11]$. $SO(2l)$ has a subgroup transforming $I_{\alpha k}, J_{\alpha k}$
linearly among themselves while preserving the canonical structure of their commutators, \(i.e.,\) taking the set \(\{\beta_k\}\) to some \(\{\beta'_k\}\). This group is the normaliser \(N\) of \(T\) in \(SO(2l)\), consisting of \(M\) such that \(MCMT\) is also in the Cartan subalgebra. Obviously, \(H_\alpha\) is invariant under \(N\). But \(N\) itself has a normal subgroup of matrices which take each \(C\) to itself, namely the centraliser \(Z(\cong T)\) of \(T\), and \(Z\) essentially (upto unitary equivalence) fixes each \(I_{\alpha k}, J_{\alpha m}\). Therefore, \(H_\alpha\) is invariant under the quotient group \(N/Z\) which by definition is the Weyl group \(W\) of \(SO(2l)\), alternatively described as the group generated by reflections along \(l\) basic roots in \(R^l[11]\). The Weyl group invariance reflects the fact that \(H_\alpha\) is associated uniquely not to \(\alpha\), but to the adjoint orbit of \(SO(2l)\) through the matrix \(A\) in its Lie algebra. The relevance of Weyl groups and root systems to integrability thus has a transparent explanation in the quantum context.

It follows that the arithmetic degeneracy of every energy level is the dimension of some representation of \(W\). It depends critically on the relative values of \(\beta_k\); for instance, if \(\beta_k/\beta_m\) is irrational for all \(k\) and \(m\), only the identity representation of \(W\) occurs since, in that case, for any eigenvalue \(E_\nu\) there can be only one solution for \(\{\nu_k\}\) in integers. In \(H_\alpha\) itself, there is an additional common infinite (= \(\dim V_\alpha\)) degeneracy since \(H_\alpha\) is independent of \(F_{\alpha i}\); this reflects the symmetry of the system under the integrability group \(G\).

7. Conclusion. In summary, this paper has delineated a general framework for quantum integrability as the most natural quantum generalisation of the classical Liouville-Arnold theory. While incorporating normal q-integrable systems with involutive action operators, its scope extends to systems which, though involutivity is 'broken' by central charges, remain integrable, in particular exactly diagonalisable. The representation-theoretic approach to counting the arithmetic degeneracy appears to be novel and should prove useful in other similar physical problems of a Diophantine nature [12]. A fuller treatment, including concrete examples, will be taken up elsewhere.

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