Scalar induced gravitational waves from primordial black hole Poisson fluctuations in \( f(R) \) gravity

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Abstract. The gravitational potential of a gas of initially randomly distributed primordial black holes (PBH) can induce a stochastic gravitational-wave (GW) background through second-order gravitational effects. This GW background can be abundantly generated in a cosmic era dominated by ultralight primordial black holes, with masses \( m_{\text{PBH}} < 10^9 \text{g} \). In this work, we consider \( f(R) \) gravity as the underlying gravitational theory and we study its effect at the level of the gravitational potential of Poisson distributed primordial black holes. After a general analysis, we focus on the \( R^2 \) gravity model. In particular, by requiring that the scalar induced GWs (SIGWs) are not overproduced, we find an upper bound on the abundance of PBHs at formation time \( \Omega_{\text{PBH},f} \) as a function of their mass, namely that
\[
\Omega_{\text{PBH},f} < 5.5 \times 10^{-5} \left( \frac{10^9 \text{g}}{m_{\text{PBH}}} \right)^{1/4},
\]
which is 45% tighter than the respective upper bound in...
general relativity. Afterwards, by considering $R^2$ gravity as an illustrative case study of an $f(R)$ gravity model, we also set upper bound constraints on its mass parameter $M$. These mass parameter constraints, however, should not be regarded as physical given the fact that the Cosmic Microwave Background (CMB) constraints on $R^2$ gravity are quite tight. Finally, we conclude that the portal of SIGWs associated to PBH Poisson fluctuations can act as a novel complementary probe to constrain alternative gravity theories.

**Keywords:** gravitational waves / theory, modified gravity, primordial black holes

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1 Introduction

Primordial black holes (PBHs), firstly proposed in the early ‘70s [1–4], are formed in the early universe before the birth of stars, out of the collapse of overdensity regions whose energy density perturbations are higher than a critical threshold [5–7]. They are currently attracting an increasing attention since they can address a number of issues of modern cosmology. According to recent arguments, they can potentially account for a part or all of the dark matter content of the Universe [8], and additionally they can offer an explanation for the large-scale structure formation through Poisson fluctuations [9, 10]. Furthermore, they can provide seeds for the supermassive black holes residing in the centre of galaxies [11, 12], as well as constitute viable candidates for the progenitors of the black-hole merging events recently...
detected by the LIGO/VIRGO collaboration [13] through the emission of gravitational waves (GWs). Other evidence in favor of the PBH scenario can be found in [14].

Due to the significance of PBHs and the huge progress achieved in the field of gravitational-wave astronomy, there have been many attempts connecting PBHs and GWs [15]. Firstly, a large amount of research has been devoted to the GW background signals associated to PBH merging events [16–21]. Moreover, extensive research has been also performed regarding the PBH Hawking radiated-graviton background [22, 23] as well as concerning the scalar induced GWs (SIGWs) connected to the primordial high curvature perturbations which gave rise to PBHs [24–30] (for a recent review see [31]). However, apart from the aforementioned GW signals, it has been recently noted in [32], and further studied in [33, 34], that the Poisson fluctuations of a gas of randomly distributed PBHs can induce second-order GWs at distances much larger than the PBH mean separation scale. These GWs are not induced by the primordial curvature perturbations, which gave rise to PBHs, but instead by the PBH density fluctuations themselves and can be abundantly produced during an early PBH dominated era naturally driven by ultralight PBHs, which evaporate before BBN time [20, 35–37].

At the same time, there are many reasons indicating that one should construct modified gravitational theories. At the theoretical level, gravitational modifications are known to be able to improve the renormalizability issues of general relativity [38, 39]. At the phenomenological level, modified gravity can offer an alternative way to explain the two phases of the Universe’s accelerated expansion, namely the early-time, inflationary one [40, 41], and/or the late-time, dark-energy one [42–44]. In all cases, these modified gravitational theories possess general relativity as a particular limit, but in general they have a richer structure and extra degrees of freedom that can describe the Universe’s evolution.

One of the simplest classes of modified gravity is $f(R)$ gravity, which is obtained through the extension of the Einstein-Hilbert Lagrangian to an arbitrary function of the Ricci scalar [45]. Apart from its general cosmological application, in the inflationary framework the particular subclass of the theory known as Starobinsky, or $R^2$ gravity [46], proves to be one of the best-fitted models to the cosmological data [47]. Hence, due to its success, $f(R)$ gravity has been extensively studied in the literature. In particular, in such investigations one is in general interested in extracting the corrections on various observational signals, induced by the $f(R)$ modifications on top of the corresponding general-relativity predictions (see [48–71] and references therein).

Therefore, in the present work we are interested in investigating the GW signal induced by PBH Poisson fluctuations, in the framework of $f(R)$ gravity. In particular, since all the relevant studies up to now have been performed in the framework of general relativity, apart from [72–74] where the authors study the primordial SIGWs in modified gravity constructions, in the following we calculate the effect of $f(R)$ corrections on the PBH gravitational potential power spectrum and subsequently on the associated SIGW background. In this way, one may use it as an extra and novel method to constrain on the one hand the PBH abundances and on the other hand possible $f(R)$ modifications, constituting in this way an independent test of general relativity.

The plan of the work is as follows: in section 2, we review the PBH gravitational potential in general relativity and in section 3 we perform the extended analysis, extracting the PBH gravitational potential in the framework of $f(R)$ gravity. In section 4, we make a case study within $f(R)$ gravity theories and extract the relevant SIGW signal focusing on the simplest $f(R)$ gravity model, namely the $R^2$ gravity, treating in this way its mass parameter $M$ as a free parameter. Then, in section 5 by demanding that SIGWs are not overproduced at
PBH evaporation time, we obtain, on the one hand, upper bound constraints on the PBH abundance at formation time $\Omega_{\text{PBH},f}$ as a function of the PBH mass $m_{\text{PBH}}$ and, on the other hand, upper bounds on the mass parameter $M$ of $R^2$ gravity as a function of $m_{\text{PBH}}$ and $\Omega_{\text{PBH},f}$. Finally, section 6 is devoted to the conclusions.

2 The primordial black hole gravitational potential in general relativity

In the context of general relativity (GR), the action is written as follows:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda \right) + \int d^4x \sqrt{-g} \mathcal{L}_m,$$

(2.1)

with $G$ being the gravitational Newton constant (throughout this paper we work in units where $c = 1$), $R$ the Ricci scalar, $\Lambda$ the cosmological constant, $\mathcal{L}_m$ the total matter Lagrangian density (radiation, baryonic and dark matter) of the Universe and $T_{\mu\nu}^m = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}}$ the corresponding total matter energy-momentum tensor. Varying the action (2.1) with respect to the metric $g^{\mu\nu}$ we obtain the usual Einstein field equations, namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}^m.$$

(2.2)

Note that the Bianchi identity $\nabla_\mu G^{\mu}_\nu = 0$ implies the conservation of the total energy-momentum tensor.

2.1 Background evolution

Proceeding to a cosmological setup, we consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background metric of the form

$$ds^{2}_b = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,$$

(2.3)

where $a(t)$ is the scale factor. By adopting this background metric and assuming that the total matter content of the Universe is described by the perfect fluid energy-momentum tensor $T^{m}_{\mu\nu} = \text{diag}(\bar{\rho}, \bar{p}, \bar{p}, \bar{p})$, where $\bar{\rho}$ and $\bar{p}$ are the total matter (i.e. including radiation, baryonic and dark matter) energy density and pressure respectively, the GR field equations give rise to the two Friedmann equations:

$$H^2 = \frac{8\pi G}{3} \bar{\rho} + \frac{\Lambda}{3} \equiv \frac{8\pi G}{3} \bar{\rho}_{\text{tot}},$$

(2.4)

$$\dot{H} + H^2 = -\frac{4\pi G}{3} (\bar{\rho} + 3\bar{p}) + \frac{\Lambda}{3} \equiv -\frac{4\pi G}{3} (\bar{\rho}_{\text{tot}} + 3\bar{p}_{\text{tot}}),$$

(2.5)

where $H = \dot{a}/a$ is the Hubble parameter, with dots denoting derivatives with respect to the cosmic time $t$. In the above expressions $\bar{\rho}_{\text{tot}}$ and $\bar{p}_{\text{tot}}$ correspond to the total background energy density and pressure of the Universe, i.e the total matter sector as well as the cosmological constant term, which is interpreted as a dark energy fluid with $\rho_{\text{de}} = -p_{\text{de}} \equiv \frac{\Lambda}{8\pi G}$ whose energy-momentum tensor is $T^{\text{de}}_{\mu\nu} = \text{diag}(-\rho_{\text{de}}, -p_{\text{de}}, -p_{\text{de}}, -p_{\text{de}})$. Nevertheless, since in this work we focus on the early-time matter (i.e. PBH) dominated era, the contribution of the cosmological constant or effective dark energy at the background level can be neglected.
Lastly, it proves convenient to introduce the conformal time \( \eta \) defined through 
\[ \text{d}t \equiv a \text{d}\eta, \]
and similarly the conformal Hubble parameter defined as 
\[ \mathcal{H} \equiv \frac{a'}{a} = aH, \]
where primes denote derivatives with respect to \( \eta \). Hence, the above two Friedmann equations become simply
\[
\mathcal{H}^2 = \frac{8\pi G a^2}{3} \bar{\rho}_{\text{tot}} \quad (2.6)
\]
\[
\mathcal{H}' = -\frac{4\pi G a^2}{3} \left( \bar{\rho}_{\text{tot}} + 3\bar{p}_{\text{tot}} \right). \quad (2.7)
\]

### 2.2 Scalar perturbations

Let us now refer to the perturbation evolution. Focusing on scalar perturbations, the perturbed FLRW metric in the Newtonian gauge reads as
\[
ds^2 = a^2(\eta) \left\{ -(1 + 2\Psi) \text{d}\eta^2 + [(1 - 2\Phi)\delta_{ij}] \text{d}x^i \text{d}x^j \right\},
\]
where for convenience we perform the calculations using the conformal time \( \eta \). In the above ansatz, \( \Psi \) and \( \Phi \) stand for the Bardeen potentials [75], which are first order quantities in cosmological perturbation theory.

Further, we allow perturbations around the background stress-energy tensor of the total matter content of the Universe (matter and radiation) which we write as follows:
\[
T_{0}^{0} = - \left( \bar{\rho} + \delta \rho \right)
\]
\[
T_{i}^{0} = \left( \bar{\rho} + \bar{p} \right) v_i, \quad v_i \equiv a \delta u_i
\]
\[
T_{j}^{i} = \bar{p} \left( \delta_{j}^{i} + \Pi_{j}^{i} \right), \quad (2.9)
\]
where \( \delta \equiv \delta \rho / \bar{\rho} \) is the relative energy density perturbation, \( \delta u_i \equiv v_i / a \) is the velocity perturbation and \( \Pi_{j}^{i} \) is the (dimensionless) anisotropic stress. The evolution of \( \Phi \) and \( \Psi \) is governed by the perturbed Einstein equations, which are [76]:
\[
3\mathcal{H}(\Phi' + \mathcal{H}\Psi) - \nabla^2 \Phi = -4\pi G a^2 \delta \rho \quad (2.10)
\]
\[
(\Phi' + \mathcal{H}\Psi)_i = 4\pi G a^2 \left( \bar{\rho} + \bar{p} \right) v_i \quad (2.11)
\]
\[
\Phi'' + \mathcal{H}(\Phi' + 2\Psi') + (\mathcal{H}' + 2\mathcal{H})\Phi + \nabla^2(\Phi - \Psi)/3 = 4\pi G a^2 \bar{p} \quad (2.12)
\]
\[
\Phi - \Psi = 8\pi G a^2 \bar{p} \Pi. \quad (2.13)
\]

During the time period we are concerned with, namely before BBN, the anisotropic stress of the Universe is negligible since we do not have the presence of free-streaming particles. Thus, from (2.13) we see that \( \Phi \approx \Psi \), which we will adopt from now on. This potential can actually be identified with the PBH gravitational potential, whose behavior will be derived in the following analysis.\(^1\)

We proceed by defining the total entropy perturbation as
\[
S \equiv \mathcal{H} \left( \frac{\delta p}{\bar{p}} - \frac{\delta \rho}{\bar{\rho}} \right). \quad (2.14)
\]

\(^1\)The first-order gravitational potential due to the primordial energy density perturbations is ignored here as we concentrate on the induced GW signal due to the PBH energy density perturbations. This contribution can be added to the contribution calculated in our work, if we desire to include the primordial SIGWs [31] too.
Since the (total) energy-momentum tensor is conserved, the background continuity equation holds, namely
\[ \bar{\rho}' = -3H(\bar{\rho} + \bar{p}). \]
Therefore, from (2.14) we acquire:
\[ \delta p = c_s^2[\delta \rho - 3(\bar{\rho} + \bar{p})S], \tag{2.15} \]
where \( w \equiv \bar{p}/\bar{\rho} \) is the equation-of-state parameter and \( c_s^2 \equiv \bar{p}'/\bar{\rho}' \) is the sound speed square of the total matter content of the Universe. Finally, one can combine (2.10) with (2.12) and (2.15) to get the following equation governing the behavior of the gravitational potential \( \Phi \):
\[ \Phi'' + 3H(1 + c_s^2)\Phi' - c_s^2\nabla^2 \Phi + 3(c_s^2 - w)H^2\Phi = -\frac{9}{2}c_s^2(1 + w)H^2S. \tag{2.16} \]

2.3 The power spectrum of the PBH gravitational potential

Having extracted above the background and the perturbation equations for the PBH gravitational potential, we derive here the corresponding power spectrum following closely [32]. As it is standardly adopted in the literature, we assume that PBHs are formed in the radiation-dominated (RD) era. Hence, considering PBHs as a matter fluid, their formation process can be regarded as a transition of a fraction of the radiation energy density into PBHs. Thus, assuming that PBHs are randomly distributed in space at formation time, their energy density is inhomogeneous while the total energy density of the background is homogeneous. Consequently, the PBH energy density perturbation can be viewed as an isocurvature Poisson fluctuation. As it was found in [32], the Poissonian power spectrum for the PBH density contrast, assuming monochromatic PBH mass function [77], reads as
\[ P_\delta(k) = \frac{k^3}{2\pi^2}P_\delta(k) = \frac{2}{3\pi} \left( \frac{k}{k_{UV}} \right)^3 \Theta(k_{UV} - k), \tag{2.17} \]
where \( k_{UV} \equiv a/\bar{r} \) is a UV cut-off scale related to the mean PBH separation scale. This UV cut-off scale is introduced here since at scales smaller than the mean PBH separation scale the PBH fluid description is not valid. In particular, at these scales one probes the granularity of the PBH energy density field entering the non-linear regime where \( P_\delta(k) > 1 \). Straightforwardly, one can show that the UV cut-off scale reads as [32]
\[ k_{UV} = H_f\Omega_{PBH,f}^{1/3}, \tag{2.18} \]
where \( H_f \) and \( \Omega_{PBH,f} \) are respectively the conformal Hubble parameter and the PBH abundance at PBH formation time.

Then the next step is to relate the above power spectrum of the PBH energy density perturbations to the power spectrum for the PBH gravitational potential \( \Phi \). In order to achieve this we should have in mind that since in the RD era, \( \Omega_{PBH} \equiv \frac{\rho_{PBH}}{\rho_{tot}} \propto a \), hence if the initial abundance of PBHs is large enough, then PBHs can potentially dominate the Universe energy budget. Consequently, the isocurvature PBH energy density perturbation in the RD era will be converted to an adiabatic curvature perturbation in the subsequent PBH dominated era [78, 79], which will be related to a gravitational potential \( \Phi \).

To derive now \( \Phi \) from \( \delta_{PBH} \), we use as an intermediate variable the uniform-energy density curvature perturbation of a fluid, \( \zeta \), which is related with the Bardeen potential \( \Phi \) and the respective energy density perturbation by the following definition [80]:
\[ \zeta = -\Phi - \frac{H\delta \rho}{\rho}. \tag{2.19} \]
If the total energy-momentum tensor is conserved, the (background) continuity equation
\[ \dot{\rho} = -3H(\rho + p) \]
holds, and thus \( \zeta \) is expressed as
\[ \zeta \equiv -\Phi + \frac{\delta}{3(1 + w)}, \] (2.20)
where \( w \equiv p/\rho \) is the equation-of-state parameter of the total matter content of the Universe.

In our case, since the energy-momentum tensors of radiation and PBH-matter are separately conserved, we can use (2.20) for \( \zeta_r \) and \( \zeta_{PBH} \) and acquire:
\[ \zeta_r = -\Phi + \frac{1}{4}\delta_r, \] (2.21)
\[ \zeta_{PBH} = -\Phi + \frac{1}{3}\delta_{PBH}. \] (2.22)

Finally, we introduce the isocurvature perturbation defined as:
\[ S = 3(\zeta_{PBH} - \zeta_r) = \delta_{PBH} - \frac{3}{4}\delta_r. \] (2.23)

On superhorizon scales, \( \zeta \) and \( \zeta_{PBH} \) are conserved separately [80], like the isocurvature perturbation \( S \). Thus, in the PBH-dominated era, \( \zeta \simeq \zeta_{PBH} = \zeta_r + S/3 \simeq S/3 \). Since \( S \) is conserved, it can be calculated at formation time \( t_f \). Therefore, neglecting the adiabatic contribution associated to the radiation fluid at the PBH formation time, since it is negligible for the scales considered here, from eq. (2.23) we obtain that \( S = \delta_{PBH}(t_f) \). Hence, we finally find
\[ \zeta \simeq \frac{1}{3}\delta_{PBH}(t_f) \text{ if } k \ll H. \] (2.24)

Using now the fact that \( \zeta \simeq -R \) on superhorizon scales (see e.g. [80]), where \( R \) is the comoving curvature perturbation defined by
\[ R = \frac{2}{3}\frac{\Phi'}{H} + \Phi, \] (2.25)
one gets straightforwardly that in the PBH-matter dominated era, where \( w = 0 \) and \( \Phi \) is constant in time [80],
\[ \Phi \simeq -\frac{1}{3}\delta_{PBH}(t_f) \text{ if } k \ll H. \] (2.26)

On sub-Hubble scales, one can determine the evolution of \( \delta_{PBH} \) by solving the evolution equation for the matter density perturbations, namely the Mészáros growth equation [81], which, in the case of a Universe with radiation and PBH-matter, takes the form:
\[ \frac{d^2 \delta_{PBH}}{ds^2} + \frac{2 + 3s}{2s(s+1)} \frac{d\delta_{PBH}}{ds} - \frac{3}{2s(s+1)}\delta_{PBH} = 0. \] (2.27)

By solving the above equation one can find that the dominant solution deep in the PBH-dominated era can be written as \( \delta_{PBH} \simeq 3s\delta_{PBH}(t_f)/2 \). Now, the relation between the Bardeen potential and the density contrast is dictated by the Poisson equation, and in a matter-dominated era takes the form
\[ \delta_{PBH} = -\frac{2}{3} \left( \frac{k}{H} \right)^2 \Phi. \] (2.28)

– 6 –
Therefore, plugging the solution for $\delta_{\text{PBH}}$ into the aforementioned formula, one obtains

$$\Phi \simeq - \frac{9}{4} \left( \frac{H_d}{k} \right)^2 \delta_{\text{PBH}}(t_f) \quad \text{if} \quad k \gg H_d,$$  \hspace{0.5cm} (2.29)

where $H_d$ is the conformal Hubble function at PBH domination time. Finally, making an interpolation between (2.29) and (2.26), and using (2.17) one obtains that

$$P_{\Phi}(k) \equiv k^3 \frac{2}{2\pi^2} P_{\Phi}(k) = \frac{2}{3\pi} \left( \frac{k}{k_{\text{UV}}} \right)^3 \left( 5 + \frac{4k^2}{9 k_d^2} \right)^{-2},$$  \hspace{0.5cm} (2.30)

where $k_d \equiv H_d$ is the comoving scale exiting the Hubble radius at PBH domination time. From eq. (2.30), one can see that $P_{\Phi}$ has a broken power-law behavior: when $k \ll k_d$ we have that $P_{\Phi} \propto k^3$, while when $k \gg k_d$ we acquire $P_{\Phi} \propto 1/k$. We mention that it reaches its maximum when $k \sim k_d$, where $P_{\Phi}$ is of order $(k_d/k_{\text{UV}})^3$.

3 The primordial black hole gravitational potential in $f(R)$ gravity

In the previous section we presented the calculation of the PBH gravitational potential power spectrum in the framework of general relativity. In this section we proceed to the bulk of our analysis, which is to perform the same calculation but in the case of $f(R)$ modified gravity, extracting the corresponding corrections.

We consider a modified action of the form [45]

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + \int d^4x \sqrt{-g} L_m,$$  \hspace{0.5cm} (3.1)

where $f(R)$ is a general function of the Ricci scalar $R$. Variation of the action (3.1) with respect to the metric $g^{\mu\nu}$ yields the following field equations:

$$FR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) F = 8\pi G T^m_{\mu\nu}, \hspace{0.5cm} (3.2)$$

where we have set $F \equiv df(R)/dR$. One characteristic feature of the richer structure of $f(R)$ gravity is the existence of an additional propagating degree of freedom, the so-called scalaron field [46]. Its equation can be obtained by taking the trace of (3.2), which yields:

$$\Box F(R) = \frac{1}{3} \left[ 2f(R) - F(R)R + 8\pi G T^m \right] \equiv \frac{dV}{dF}, \hspace{0.5cm} (3.3)$$

where $T^m$ is the trace of the energy-momentum tensor of the (total) matter content of the Universe. As we observe, equation (3.3) is a wave equation for $\phi_{\text{sc}} \equiv F(R)$ whose mass is given by $m_{\text{sc}}^2 \equiv d^2V/dF^2$, which reads:

$$m_{\text{sc}}^2 = \frac{1}{3} \left( \frac{F}{F_R} - R \right), \hspace{0.5cm} (3.4)$$

where $F_R \equiv dF/dR = d^2f/dR^2$. An alternative way to see this is by performing a conformal transformation to the Einstein frame [45]. Amongst others, the presence of this additional degree of freedom induces an extra polarization mode for the gravitational waves [45], as we will see in the next section.
For our purposes, we shall formulate $f(R)$ gravity in terms of an effective curvature-induced fluid. Specifically, we shall express the equations (3.2) as the corresponding ones in GR (2.2), with the addition of the following energy-momentum tensor \[ T^{(f(R))}_{\mu \nu} \equiv (1 - F) R^{\mu}_{\nu} + \frac{1}{2} \delta^{\mu}_{\nu} (f - R) - (\delta^{\mu}_{\nu} \Box - \nabla^{\mu} \nabla^{\nu}) F. \] (3.5)

Similarly to the GR case, we will first examine the evolution at the background and perturbation levels, and then we will calculate the power spectrum of the PBH gravitational potential.

3.1 Background evolution

Applying $f(R)$ gravity to a cosmological framework, namely using the FLRW metric (2.3), we extract the Friedmann equations, which in terms of the conformal time are written as

\[
\mathcal{H}^2 = \frac{8 \pi G a^2}{3} \bar{\rho}_{\text{tot}} \tag{3.6}
\]

\[
\mathcal{H}' = -\frac{4 \pi G a^2}{3} (\bar{\rho}_{\text{tot}} + 3 \bar{p}_{\text{tot}}). \tag{3.7}
\]

We mention that these equations acquire the same form as in the GR case, with the only difference being that in the total content of the Universe we need to take into account the contribution of the $f(R)$ curvature-induced effective fluid, whose energy density and pressure are given by [45]:

\[
\bar{\rho}_{(f(R))} \equiv -T^{(f(R))}_{00} = \frac{1}{8 \pi G a^2} \left( 3 \mathcal{H}^2 - \frac{3}{2} a^2 f + 3 F \mathcal{H}' - 3 \mathcal{H} F' \right) \tag{3.8}
\]

\[
\bar{p}_{(f(R))} \equiv T^{(f(R))}_{i i} = \frac{1}{8 \pi G a^2} \left( -2 \mathcal{H}' - \mathcal{H}^2 + \frac{3}{2} a^2 f - F \mathcal{H}' - 2 F \mathcal{H}^2 + F'' + \mathcal{H} F' \right). \tag{3.9}
\]

3.2 Scalar perturbations

In order to describe the evolution of scalar perturbations, we shall use again the metric (2.8) and the perturbed form of the (total) energy-momentum tensor (2.9). The perturbed field equations are similar in form with the corresponding ones of GR, with the addition of $\delta \rho_{(f(R))}, \delta p_{(f(R))}, v_{(f(R))}$ and $\Pi_{(f(R))}$. They are provided explicitly in appendix A. Therefore, we need to take into account the contribution of the $f(R)$ curvature-induced effective fluid to the expressions introduced in subsection 2.2.

Within this context, we define the total entropy perturbation as:

\[
S_{\text{tot}} \equiv \mathcal{H} \left( \frac{\delta \rho_{\text{tot}}}{\bar{\rho}_{\text{tot}}} - \frac{\delta p_{\text{tot}}}{\bar{p}_{\text{tot}}} \right). \tag{3.10}
\]

Again the total energy-momentum tensor is conserved, so the background continuity equation holds, namely $\bar{\rho}_{\text{tot}}' = -3 \mathcal{H} (\bar{\rho}_{\text{tot}} + \bar{p}_{\text{tot}})$, so from (3.10) we acquire:

\[
\delta \rho_{\text{tot}} = c_{\text{tot}}^2 [\delta \rho_{\text{tot}} - 3(\bar{\rho}_{\text{tot}} + \bar{p}_{\text{tot}}) S_{\text{tot}}], \tag{3.11}
\]

where $w_{\text{tot}} \equiv \bar{p}_{\text{tot}}/\bar{\rho}_{\text{tot}}$ is the total equation-of-state parameter and $c_{\text{tot}}^2 \equiv \bar{p}_{\text{tot}}/\bar{\rho}_{\text{tot}}$ is the sound speed square of the total content of the Universe. By combining (A.1) with (A.3) and (3.11) we get the following equation governing the behavior of the gravitational potential $\Phi$:

\[
\Phi'' + 3 \mathcal{H} \left( 1 + c_{\text{tot}}^2 \right) \Phi' - c_{\text{tot}}^2 \nabla^2 \Phi + 3 \left( c_{\text{tot}}^2 - w_{\text{tot}} \right) \mathcal{H}^2 \Phi = -\frac{9}{2} c_{\text{tot}}^2 (1 + w_{\text{tot}}) \mathcal{H}^2 S_{\text{tot}}. \tag{3.12}
\]
3.3 The power spectrum of the PBH gravitational potential in $f(R)$ gravity

We can now repeat the procedure of subsection 2.3 and extract the power spectrum of the PBH gravitational potential $P_{\Phi}$ within the context of $f(R)$ gravity. At this point, we should emphasise that we are agnostic about the production mechanism of PBHs within $f(R)$ gravity. We merely assume that they are formed during RD era after the end of inflation\textsuperscript{2} and that they are Poisson distributed at formation time, an assumption which is rather reasonable. For this reason, the methodology adopted in this section for the derivation of $P_{\Phi}$ is model independent. Then, the setup described in section 4 and section 5 for the calculation of the SIGW signal and the derivation of constraints on the parameters of our $f(R)$ model at hand, namely the $R^2$ gravity, can be easily generalised to alternative gravitational theory [See e.g in [85] the generalisation to teleparallel gravity].

In the following, we will make use of cosmological perturbation theory working with perturbations in the Jordan frame in order to extract the power spectrum of the PBH gravitational potential.\textsuperscript{3}

To begin with, we need to take into account the presence of the $f(R)$ curvature-induced effective fluid. Therefore, on top of the usual $\zeta_r$ and $\zeta_{PBH}$, we have $\zeta_{f(R)}$, too. Since by construction its energy-momentum tensor (3.5) is conserved, we can use (2.20) to get:

$$\zeta_{f(R)} = -\Phi + \frac{1}{3(1 + w_{f(R)})} \delta_{f(R)},$$

(3.13)

where $w_{f(R)} \equiv \bar{\rho}_{f(R)}/\bar{\rho}(R) = -a^2 F' + 6(1+2F)H' - 2F'H + F''$ is the equation-of-state parameter of the effective fluid. Thus, we can study now how these curvature perturbations evolve on super-Hubble ($k \ll H$) and sub-Hubble ($k \gg H$) scales.

On super-Hubble scales, $\zeta_r$ and $\zeta_{PBH}$ are separately conserved [80], as is the isocurvature perturbation between them, which is defined by

$$S = 3(\zeta_{PBH} - \zeta_r).$$

(3.14)

However, the total curvature perturbation is not conserved and is equal to

$$\zeta = -\Phi + \frac{\delta_{tot}}{3(1 + w_{tot})} = \frac{4}{3} \bar{\rho} \zeta_r + \bar{\rho}_{PBH} \zeta_{PBH} + \frac{1}{3} \bar{\rho}(R) \zeta_{f(R)}.$$

(3.15)

At this point, we should stress that given the fact that we consider that PBHs are formed during the RD era after the end of inflation, it is reasonable to assume that at the background level, the energy contribution from the $f(R)$ fluid will be negligible compared to the contribution of radiation and matter in form of PBHs, i.e. $\bar{\rho}_r/\bar{\rho}(R) \gg 1$ and $\bar{\rho}_{PBH}/\bar{\rho}(R) \gg 1$. In addition, on the scales we are interested in, namely the PBH scales, the dominant contribution to the curvature perturbation during the PBH dominated era, will be due to the PBH curvature perturbation. Thus one can safely neglect $(1 + w_{f(R)}) \bar{\rho}_{f(R)} \zeta_{f(R)}$ and $(1 + w_{f(R)}) \bar{\rho}(R)$ from the

\textsuperscript{2}Ultralight PBHs may arise as well from the growth of metric perturbations during the matter dominated stage after the end of inflation and before the scalaron decay [83, 84]. However, we do not consider such scenarios in the present work. We focus on PBHs during an RD era as it is standardly assumed in the literature.

\textsuperscript{3}As it was found in [86, 87], it should be emphasized that physics is frame independent and the Jordan and Einstein frame are equivalent giving the same physical observables. One can always define perturbations in both the Einstein and Jordan frames. See e.g. [45] regarding the definition of the curvature perturbation within $f(R)$ theories of gravity.
numerator and the denominator of eq. (3.15) respectively. Consequently, $\zeta$ can be recast in the following form:

$$
\zeta = \frac{4}{4 + 3s} \zeta_r + \frac{3s}{4 + 3s} \zeta_{\text{PBH}},
$$

with $s \equiv \frac{a}{a_d}$, where $a_d$ denotes the value of the scale factor $a$ at the time PBHs start to dominate. From this expression, we can see that $\zeta$ evolves from its initial value $\zeta_r$, deep in the radiation era, to $\zeta_{\text{PBH}}$, deep in the PBH era. As a consequence, in the PBH-dominated era, $\zeta \simeq \zeta_{\text{PBH}} = \zeta_r + S/3$. Since $S$ is conserved on super-Hubble scales, it can be evaluated at formation time $t_f$. Furthermore, the isocurvature perturbation can be identified with $\delta_{\text{PBH}}(t_f)$, which will be calculated in the following subsection, assuming implicitly a uniform radiation energy density in the background. Indeed, in the following we will focus on the PBH contribution and we will ignore the usual adiabatic contribution (associated to the radiation fluid), which is negligible at the scales we are interested in, hence we simply have

$$
\zeta \simeq \frac{1}{3} \delta_{\text{PBH}}(t_f) \quad \text{if} \quad k \ll \mathcal{H}.
$$

Concerning the super-Hubble scales, as we show in appendix C, in $f(R)$ gravity one can also use the property $\zeta \simeq -\mathcal{R}$ (see e.g. [80]), where $\mathcal{R}$ is the comoving curvature perturbation defined in (2.25), by requiring that $\delta F \approx 0$,4 for $k \ll \mathcal{H}$ which ensures (in addition to the usual assumption that the anisotropic stress of the total matter content is negligible at these scales) that $\Psi \approx \Phi$. During a matter-dominated era, such as the one driven by PBHs, $\Phi'$ can be neglected since it is proportional to the decaying mode, thus we obtain $\mathcal{R} = -\zeta = (5/3)\Phi$. Finally, combining with (3.17), this implies that

$$
\Phi \simeq -\frac{1}{5} \delta_{\text{PBH}}(t_f) \quad \text{if} \quad k \ll \mathcal{H}.
$$

Let us now focus on sub-Hubble scales. One can determine the evolution of $\delta_{\text{PBH}}$ by solving the evolution equation of the matter density perturbations, namely the Meszaros equation [81], in a Universe where we have radiation, matter in form of PBHs, and an effective dark energy fluid due to the $f(R)$ gravity modulations, which should however be negligible before Big Bang Nucleosynthesis (BBN) time where one expects a subdominant energetic contribution from the dark energy sector.

At the background level, the Friedman equation (3.6) can be expressed as

$$
\mathcal{H}^2 = \frac{8\pi G a^2}{3} \left[ \bar{\rho}_{\text{PBH}} + \bar{\rho}_r + \bar{\rho}^{(R)} \right]
$$

where $\bar{\rho}^{(R)}$ is given by (3.8), where every sector obeys the conservation equation separately [82]. Neglecting then the effective fluid contribution, as justified above, the Friedmann equation can be recast as

$$
\mathcal{H}^2 \simeq H^2_{f} \Omega_{\text{PBH},f} \left( \frac{1}{s} + \frac{1}{s^2} \right),
$$

where $s \equiv a/a_d$ and $a_d$ denotes the time at the transition from the radiation to the PBH domination era, and where we have assumed that $\Omega_{r,f} \simeq 1$ since PBHs are considered to be formed in the radiation era [32]. Note that the scale factor is normalised at one at formation time, i.e. $a_t = 1$.

4The assumption $\delta F \approx 0$ is a reasonable one since, as it was checked numerically, $(\Phi - \Psi)/\Phi = \delta F/(F\Phi)$ is very small during the time period considered here and can vary between $10^{-30}$ up to $10^{-9}$ depending on the choice of the PBH mass $m_{\text{PBH}}$, the initial PBH abundance $\Omega_{r,\text{PBH},f}$ and the parameter of the underlying gravity theory. See appendix B for more details.
At the perturbation level, we can use the standard cosmological perturbation theory at subhorizon scales, where the matter perturbations obey the growth equation [88, 89]:

$$\dddot{\delta}_m + \mathcal{H} \ddot{\delta}_m - 4\pi G a^2 \dot{\rho}_m \delta_m = 0. \quad (3.20)$$

Treating the gas of PBHs as a matter fluid and accounting for the screening of the gravitational constant due to $f(R)$ gravity modification, one should replace in the above equation $\delta_m$ with $\delta_{PBH}$ and $G$ with $G_{\text{eff}}$ defined as [90]

$$G_{\text{eff}} = \frac{G}{F \left(1 + \frac{4k^2 F_{\mathcal{R}}}{a^2 F_{\mathcal{R}}} \right)^{\frac{1}{2}}}. \quad (3.21)$$

Hence, assembling everything, and using $s$ as the time variable, the growth equation (3.20) can be recast in the following form:

$$\frac{d^2 \delta_{PBH}}{ds^2} + \frac{2 + 3s}{2s(s+1)} \frac{d\delta_{PBH}}{ds} - \frac{3}{2s(s+1)} \frac{1 + \frac{4k^2 F_{\mathcal{R}}}{a^2 F_{\mathcal{R}}}}{1 + \frac{3k^2 F_{\mathcal{R}}}{a^2 F_{\mathcal{R}}}} \delta_{PBH} = 0. \quad (3.22)$$

We proceed by relating our solution for $\delta_{PBH}$ from (3.22) with $\Phi$, via the sub-Hubble scale approximation of the time-time field equation in $f(R)$ gravity for the PBH dominated era (equations (A.1) and (A.7) of appendix A). At the end, one gets the modified Poisson equation which reads as follows:

$$\delta_{PBH} = -\frac{2}{3} \left(\frac{k}{\mathcal{H}}\right)^2 \frac{F}{F_{\mathcal{R}}} \left(1 + \frac{3k^2 F_{\mathcal{R}}}{a^2 F_{\mathcal{R}}} \right) \Phi. \quad (3.23)$$

Hence, making an interpolation between eq. (3.18) and eq. (3.23) as in the case of GR, and using the expression for the PBH matter power spectrum in eq. (2.17) we straightforwardly extract the following PBH gravitational potential power spectrum:

$$P_\Phi(k) = \frac{k^3}{2\pi^2} P_\Phi(k) = \frac{2}{3\pi} \left(\frac{k}{k_{UV}}\right)^3 \left[5 + \frac{2}{3} \left(\frac{k}{\mathcal{H}}\right)^2 \frac{F}{F_{\mathcal{R}}} \left(1 + \frac{3k^2 F_{\mathcal{R}}}{a^2 F_{\mathcal{R}}} \right) \right]^{-2}. \quad (3.24)$$

In the above expression, $\xi(a)$ is defined as

$$\xi(a) = \frac{\delta_{PBH}(a)}{\delta_{PBH}(a_f)}, \quad (3.25)$$

where $\delta_{PBH}(a)$ is the solution of eq. (3.22). As checked numerically, $\xi(a)$ has a mild dependence on the comoving scale $k$, and thus for practical reasons we will consider $\xi(a)$ as $k$ independent. Lastly, note that in the case of GR we have $F = 1$ and $\xi(a) \simeq \frac{3}{2} \frac{a}{a_f}$, and thus recovering the result of (2.30).

4 Scalar induced gravitational waves in Starobinsky $R^2$ gravity

In the previous section we derived the power spectrum of the gravitational potential of initially Poisson-distributed PBHs, therefore in this section we are able to extract the stochastic gravitational wave background induced at second order from the PBH Poisson
fluctuations. Since we will perform specific calculations, we have to specify our \( f(R) \) form. As we mentioned in the Introduction, one of the most studied cases, which can also give rise to an inflationary scenario with a very efficient agreement with observations, is the Starobinsky or \( R^2 \) gravity \cite{Starobinsky}, in which

\[
f(R) = R + \frac{R^2}{6M^2},
\]

with \( M \) being the model parameter with dimensions of mass. This mass parameter is well fixed by the amplitude of the curvature power spectrum on CMB scales and it is equal to \( M = 10^{-5} M_{pl} \) \cite{M}. However, given the simplicity of the Starobinsky gravity model — it constitutes the simplest realisation beyond GR within \( f(R) \) gravity — we will use it in the following as our case study \( f(R) \) gravity model in order to see how one can constrain an \( f(R) \) gravity theory using the portal of the SIGWs associated to PBH Poisson fluctuations. Consequently, in the following sections the mass parameter \( M \) of Starobinsky gravity will be considered as a free parameter of the underlying gravity theory.

Before deriving the GW spectrum induced from a gas of PBHs, it is important to highlight here a major issue emerging from the study of induced GWs at second order. In particular, while the tensor modes are gauge invariant at first order, this is not valid at second order \cite{91–95}. This implies that, a priori, one needs to specify in which gauge the gravitational waves are observed. However, in this work we explore a GW backreaction problem without paying attention to observational predictions. In particular, if the energy density associated to the induced gravitational waves overcomes the one of the background, one expects perturbation theory to break down in any gauge. Hence, it is legitimate to assume that our findings bear little dependence on the gauge choice.

4.1 Tensor perturbations

Having clarified the gauge choice issue, we continue by studying the tensor perturbations \( h_{ij} \) induced by the gravitational potential \( \Phi \). In particular, the perturbed metric in the Newtonian gauge, assuming as usual zero anisotropic stress and \( \delta F/F \approx 0 \) [See appendix B], is written as

\[
ds^2 = a^2(\eta) \left\{ -(1 + 2\Phi)dr^2 + \left(1 - 2\Phi \right) \delta_{ij} \frac{dr^i dr^j}{2} \right\}, \tag{4.2}
\]

where we have multiplied by a factor \( 1/2 \) the second order tensor perturbation as is standard in the literature.\(^5\) Then, by Fourier transforming the tensor perturbations and taking into account the three polarization modes of the GWs in \( f(R) \) gravity, namely the \( \times \) and the + as in GR and the scalaron one, denoted with \( sc \), the equation of motion for the tensor modes \( h_k \) reads as

\[
h_k^{s''} + 2\mathcal{H} h_k^{s'} + (k^2 - \lambda m_{sc}^2) h_k^s = 4S_k^s, \tag{4.3}
\]

where \( \lambda = 0 \) when \( s = (+), (\times) \) and \( \lambda = 1 \) when \( s = (sc) \). The scalaron mass term, \( m_{sc}^2 \), is given by equation (3.4), and thus in the case of the Starobinsky model it becomes simply \( m_{sc}^2 = M^2 \). The source function \( S_k^s \) is given by

\[
S_k^s = \int \frac{d^3 q}{(2\pi)^{3/2}} e_i^s(k) \eta_i q_j \left[ 2\Phi_q \Phi_{q-k-q} + \frac{4}{3(1+w_{tot})} (\mathcal{H}^{-1} \Phi'_q + \Phi_q) (\mathcal{H}^{-1} \Phi'_{q-k-q} + \Phi_{q-k-q}) \right], \tag{4.4}
\]

\(^5\)The contribution from the first-order tensor perturbations is not considered here since we concentrate on gravitational waves induced by scalar perturbations at second order.
where $s = (+), (\times), (sc)$. The polarization tensors $e_{ij}^s(k)$ are defined as [43]

\[
e_{ij}^{(+)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{ij}^{(\times)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{ij}^{(sc)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.5)
\]

Regarding the time evolution of the potential $\Phi$ considering $\epsilon_{\text{tot}}^2 \approx w_{\text{tot}}$ and neglecting entropic perturbations eq. (3.12), can be recast as

\[
\Phi'' + \frac{6(1 + w_{\text{tot}})}{1 + 3w_{\text{tot}}} \frac{\Phi'}{\eta} + w_{\text{tot}}k^2 \Phi_k = 0. \quad (4.6)
\]

The above equation accepts a solution with one constant and one decaying mode on super sound-horizon scales. In the late-time limit, one can neglect the decaying mode, and write the solution for the Fourier transform of $\Phi$ as $\Phi_k(\eta) = T_\Phi(\eta) \phi_k$, where $\phi_k$ is the value of the gravitational potential at some initial time (which here we consider it to be the time at which PBHs dominate the energy content of the Universe, $x_d$) and $T_\Phi(\eta)$ is a transfer function, defined as the ratio of the dominant mode between the times $x$ and $x_d$. Consequently, eq. (4.4) can be written in a more compact form as

\[
S_k^2 = \int \frac{d^3q}{(2\pi)^{3/2}} e^s(k, q) F(q, k - q, \eta) \phi_q \phi_{k - q}, \quad (4.7)
\]

where

\[
F(q, k - q, \eta) \equiv 2T_\Phi(q\eta)T_\Phi(|k - q|\eta) + \frac{4}{3(1 + w)} \left[ H^{-1}qT_\Phi'(q\eta) + T_\Phi(q\eta) \right]
\]

\[
= \left[ H^{-1}|k - q|T_\Phi'(|k - q|\eta) + T_\Phi(|k - q|\eta) \right], \quad (4.8)
\]

and the contraction $e_{ij}^s(q)q_iq_j \equiv e^s(k, q)$ can be expressed in terms of the spherical coordinates $(q, \theta, \varphi)$ of the vector $q$ as

\[
e^s(k, q) = \begin{cases} 
\frac{1}{\sqrt{2}} q^2 \sin^2 \theta \cos 2\varphi & \text{for } s = (+) \\
\frac{1}{\sqrt{2}} q^2 \sin^2 \theta \sin 2\varphi & \text{for } s = (\times) \\
\frac{1}{\sqrt{2}} q^2 \cos^2 \theta & \text{for } s = (sc)
\end{cases}. \quad (4.9)
\]

Finally, the solution of eq. (4.3) for the tensor modes $h_k^s$ can be obtained using the Green’s function formalism where one can write for $h_k^s$ that

\[
a(\eta)h_k^s(\eta) = 4 \int_{d4} d\bar{\eta} G_k^s(\eta, \bar{\eta}) a(\bar{\eta}) S_k^2(\bar{\eta}), \quad (4.10)
\]

where the Green’s function $G_k^s(\eta, \bar{\eta})$ is the solution of the homogeneous equation

\[
G_k^{s''}(\eta, \bar{\eta}) + \left( k^2 - \lambda m_{\text{sc}}^2 - \frac{a''}{a} \right) G_k^s(\eta, \bar{\eta}) = \delta(\eta - \bar{\eta}), \quad (4.11)
\]

with the boundary conditions $\lim_{\eta \to -\bar{\eta}} G_k^s(\eta, \bar{\eta}) = 0$ and $\lim_{\eta \to -\bar{\eta}} G_k^{s'}(\eta, \bar{\eta}) = 1$. 

\[\text{---}\]
Having extracted above the tensor perturbations, the next step is to derive the tensor power spectrum, \( P_h(\eta, k) \) for the different polarization modes, which is defined as the equal time correlator of the tensor perturbations through the following relation:

\[
(h^s_k(\eta)h^{s'*}(\eta)) \equiv \delta^{(3)}(k - k')\delta^3 \frac{2\pi^2}{k^3} P_h^s(\eta, k),
\]

where \( s = (\times) \) or \((+)\) or \((sc)\). At the end, after a straightforward but rather long calculation one acquires that \( P_h(\eta, k) \) for the \((\times)\) and \((+)\) polarization states can be recast as [96–99]

\[
P_h^{(\times)}(\eta, k) = 4 \int_{0}^{\infty} dv \int_{|1-v|}^{1+v} du \left[ \frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right]^2 I^2(u, v, x) P_\Phi(kv)P_\Phi(\eta k),
\]

whereas for the scalaron polarization one obtains that

\[
P_h^{(sc)}(\eta, k) = 8 \int_{0}^{\infty} dv \int_{|1-v|}^{1+v} du \left[ \frac{(1 + v^2 - u^2)^2}{4uv} \right]^2 I^2(u, v, x) P_\Phi(kv)P_\Phi(\eta k). \tag{4.14}
\]

The two auxiliary variables \( u \) and \( v \) are defined as \( u \equiv |k - q|/k \) and \( v \equiv q/k \), and the kernel function \( I(u, v, x) \) is given by

\[
I(u, v, x) = \int_{x A}^{x} d\bar{x} \frac{a(\bar{x})}{a(x)} k G^s_k(x, \bar{x}) F_k(u, v, \bar{x}). \tag{4.15}
\]

In the above expressions, \( x = k\eta \) and we use the notation \( F_k(u, v, \eta) \equiv F(k, |k - q|, \eta) \) since the function \( F(q, k - q, \eta) \) depends only on the modulus of its first two arguments. In the following, since we focus on second-order effects, we assume that the background evolution is close to that of \( \Lambda \)CDM scenario, and since in the time period we are investigating the Universe is matter (i.e. PBH) dominated, we have \( \omega_{\text{tot}} \approx \omega_{\text{PBH}} = 0 \). Under these considerations, in a matter era, the Bardeen potential is, up to a decaying mode, constant in time, hence \( T_\Phi = 1 \) and from eq. (4.8), one gets that \( F = 10/3 \).

Finally, note also that the power spectrum of the PBH gravitational potential should be calculated at a reference initial time, which here is considered to be the PBH domination time.

### 4.2 The gravitational wave energy density spectrum

Since we have extracted the power spectrum of the tensor perturbations, we can now calculate the energy density associated to the SIGWs. We focus only on subhorizon scales, in which one does not feel the curvature of spacetime and hence he can use a flat spacetime approximation. Consequently, after a straightforward but lengthy calculation the GW energy density can be recast as [100]

\[
\rho_{\text{GW}}(\eta, \mathbf{x}) = \frac{M_{\text{Pl}}^2}{32\pi^2} (\partial_\eta h_{\alpha\beta}(\partial_\eta h^{\alpha\beta} + \partial_\eta h_{\alpha\beta} \partial^\eta h^{\alpha\beta}), \tag{4.16}
\]

which is simply the sum of a kinetic term and a gradient term. The overall bar stands for an oscillation averaging on sub-horizon scales, which is performed to deduce only the envelope of the gravitational-wave spectrum. The GW spectral abundance is just the GW energy density per logarithmic comoving scale, i.e.

\[
\Omega_{\text{GW}}(\eta, k) = \frac{1}{\rho_{\text{tot}}} \frac{d\rho_{\text{GW}}(\eta, k)}{d\ln k}. \tag{4.17}
\]
Let us now focus on a matter-dominated era driven by PBHs, where \( w = 0 \). Under these conditions, the transfer function \( T_\Phi \) is constant in time, and we normalise it to one at PBH domination time, namely \( T_\Phi(x_d) = 1 \). This forces the source term \( S^s_k \) to be constant in time and as a consequence at sub-horizon scales, where \( k \gg H \), from eq. (4.3) we acquire that \( h^s_k \simeq \frac{4S^s_k}{x_d} \). Consequently, the tensor modes have a mild dependence on time and therefore the kinetic term in relation (4.16) gives a negligible contribution to the GW energy density. Therefore, we straightforwardly obtain that

\[
\langle \rho_{GW}(\eta, x) \rangle \simeq \langle \rho_{GW, \text{grad}}(\eta, x) \rangle = \sum_{s=+, \times, \text{sc}} \frac{M^2_{\text{Pl}}}{32a^2} \left\langle \left( \nabla h^s_{\alpha \beta} \right)^2 \right\rangle \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} k_1 k_2 \langle h^s_{k_1}(\eta) h^{s\ast}_{k_2}(\eta) \rangle e^{i(k_1-k_2) \cdot x},
\]

where the brackets stand for an ensemble average. At the end, by combining eq. (4.18), eq. (4.17) and eq. (4.12) and taking into account from eq. (4.13) that the (\( \times \)) and (\( + \)) polarization modes give an equal contribution, we find that

\[
\Omega_{GW}(\eta, k) \simeq \frac{1}{\rho_{tot}} \frac{d\rho_{GW, \text{grad}}(\eta, k)}{d \ln k} = \frac{1}{96} \left( \frac{k}{H(\eta)} \right)^2 \left[ 2P^{(\times)}_{h}(\eta, k) + P^{(\text{sc})}_{h}(\eta, k) \right].
\]

5 The case of Starobinsky \( R^2 \) gravity

In this section, by demanding that SIGWs are not overproduced at PBH evaporation time, firstly we derive constraints on the PBH abundances in the context of Starobinsky \( R^2 \) modified gravity with the mass scale \( M \) taking its fiducial value \( M = 10^{-5}M_{\text{Pl}} \). Afterwards, by treating \( R^2 \) as an illustrative \( f(R) \) gravity case study theory, we treat its mass parameter \( M \) as a free parameter and by avoiding again GW overproduction, we set constraints this time on \( M \). These constraints on \( M \) however should not be viewed as physical ones since \( M \) is well fixed by the amplitude of the scalar perturbations through CMB probes. They are derived only to demonstrate that with the case study example of \( R^2 \) gravity, the portal of SIGWs associated to PBH Poisson fluctuations can be used as a novel probe to constrain the underlying gravity theory.

In our setup, we investigate and extract the SIGW spectrum produced during a cosmic era driven by PBHs. In order to achieve this we treat the PBHs as a matter fluid, and thus with zero equation-of-state parameter, an approximation which is justifiable for scales larger than the PBH mean PBH separation scale where \( k < k_{UV} \) (see the discussion in subsection 2.3).

5.1 The theoretical parameters involved

Before going into the investigation of the GW signal let us discuss the relevant theoretical parameters involved in the problem at hand. These parameters are actually the mass of the PBH \( m_{\text{PBH}} \), the initial PBH abundance at formation time \( \Omega_{\text{PBH}, f} \), and the dimensionless parameter \( \alpha \) defined as the ratio of the Hubble parameter at PBH formation time over the energy scale parameter of Starobinsky (or \( R^2 \)) gravity \( M \)

\[
\alpha \equiv H_f/M. \tag{5.1}
\]

Regarding the PBH mass range we assume that the PBHs considered here are formed after the end of inflation and evaporate before BBN time. In particular, we extract a lower
and an upper bound on the PBH mass, $m_{\text{PBH}}$ by accounting for the current Planck upper bound on the tensor-to-scalar ratio for single-field slow-roll models of inflation, which gives $\rho_{\text{inf}}^{1/4} < 10^{16}\text{GeV}$ [47] as well conservative a lower bound on the reheating energy scale, i.e. $\rho_{\text{reh}}^{1/4} > 4\text{MeV}$ [101–104]. Consequently, by requiring that $\rho_{\text{reh}} \geq \rho_{\text{BBN}}$ and considering the fact that the mass of a PBH is roughly equal to the mass inside the Hubble volume at PBH formation time, $m_{\text{PBH}} = 4\pi \rho_f H_f^{-3}/3$, we can straightforwardly show that the relevant PBH mass range is given by

$$10^g < m_{\text{PBH}} < 10^9\text{g}, \quad (5.2)$$

where moreover we have used the fact that the Hawking evaporation time of a black hole scales with the mass $m_{\text{PBH}}$ as $t_{\text{evap}} = \frac{160}{\pi g_{\text{eff}}} \frac{m_{\text{PBH}}^3}{M_{\text{Pl}}^4}$ [105], where $g_{\text{eff}}$ is the effective number of relativistic degrees of freedom. In our numerical applications, we take $g_{\text{eff}} = 100$ since it is the order of magnitude predicted by the Standard Model before the electroweak phase transition [106].

Concerning now the range of $\Omega_{\text{PBH}},f$ in order to have a transient PBH domination era, this can be set by demanding that the PBH evaporation time $t_{\text{evap}}$, is larger than the PBH domination time $t_d$. In particular, knowing that during a radiation domination era $\Omega_{\text{PBH}} = \frac{\rho_{\text{PBH}}}{\rho_d} \propto a^{-3}/a^{-4} \propto a$, then the PBHs dominate the energy budget of the Universe when $\Omega_{\text{PBH}} = 1$, from which we find that $a_d = a_f/\Omega_{\text{PBH},f}$. Thus, knowing that during radiation domination era $H \simeq 1/(2t)$, and demanding that $t_{\text{evap}} > t_d$, we obtain that

$$\Omega_{\text{PBH},f} > 10^{-15} \sqrt{\frac{g_{\text{eff}}}{100}} \frac{10^g}{m_{\text{PBH}}}. \quad (5.3)$$

Finally, regarding the dimensionless parameter $\alpha$, knowing that in Starobinsky gravity $M \sim H_{\text{inf}}$, and assuming as mentioned above that PBHs are formed after inflation, i.e. $H_{\text{inf}} \geq H_f$, we get that $M \geq H_f$. In addition, we know that in Starobinsky-like inflationary models $M < M_{\text{max}} \equiv 10^{-5}M_{\text{Pl}}$ in order to be compatible with the amplitude of curvature power spectrum from CMB observations. Consequently, for our considerations we have that $H_f \leq M \leq M_{\text{max}}$ and the relevant range for $\alpha$ can be recast as

$$\frac{H_f}{M_{\text{max}}} \leq \alpha \leq 1. \quad (5.4)$$

In the limit $\alpha \to 0 \iff M \to \infty$ one recovers GR. However, given the fact that we constrain our analysis to regimes where $M \leq 10^{-5}M_{\text{Pl}}$, the GR limit $\alpha \to 0$ is not included here.

At this point, we need to stress that given the fact that the energy scale $M$ is more or less the energy scale at the end of inflation, the regime where $\alpha \sim 1$, or equivalently $M \sim H_f$, corresponds to a regime where PBHs are created right after the end of inflation considering instantaneous reheating. This assumption of instantaneous reheating is a simplistic one but rather reasonable given the fact that in the following we aim to illustrate with the case study of $R^2$ gravity how one can set constraints on the theoretical parameters of the underlying $f(R)$ gravity theory using the novel probe of the SIGWs associated to PBH Poisson fluctuations rather than extract precise constraints on $M$.

### 5.2 Gravitational waves from an era driven by primordial black holes

Having introduced in the previous subsection the relevant parameters involved we derive here the GW spectrum during an era of PBH domination. To do so, the first step is to
calculate the kernel function $I(u, v, x)$ defined in eq. (4.15). Since we are in a matter (i.e. PBHs) dominated era, namely with $w = 0$, in the subhorizon limit, i.e. $x \gg 1$, $I(u, v, x)$ reads as (see appendix D)

$$I^2(x) = \frac{100}{9} \frac{1}{k^4} \begin{cases} 1 & \text{if } s = (\times), (+) \\ \frac{M}{k^4} & \text{if } s = (sc) \end{cases}. \quad (5.5)$$

As one may notice from the expression (5.5), we have a suppression factor of the order $k^4/M^4$, which suppresses the scalaron contribution. One expects the highest contribution of this factor in the region close to the UV cut-off scale and the regimes where $M$ takes its minimum value, namely $M = M_{\text{min}} = H_f$. In particular, when $k = k_{UV}$ and $M = H_f$ one gets that

$$I^2_{(sc)}(x) = \frac{100}{9} \Omega_{\text{PBH},f}^{4/3} < 1 < I^2_{(+)or(\times)}(x) = \frac{100}{9}, \quad (5.6)$$

since $\Omega_{\text{PBH},f} < 1$. As a result, one anticipates that the scalaron contribution should be negligible with respect to the contributions from the $(+) \text{ and } (\times)$ polarisations. This can be seen from the right panel of figure 2 where we see that for $m_{\text{PBH}} = 10^3 \text{g}$, $\Omega_{\text{PBH},f} = 10^{-3}$ and $M = H_f$ the scalaron contribution to the GW signal is indeed the subdominant one.

Under this approximation, neglecting the scalaron contribution, the GW spectrum (4.19) can be recast in the following form:

$$\Omega_{\text{GW}}(\eta, k) = \frac{4}{75\pi^2} \left( \frac{k}{aH} \right)^2 \left( \frac{k}{k_{UV}} \right)^6 \mathcal{F}(y, \Omega_{\text{PBH},f}, \alpha), \quad (5.7)$$

where

$$\mathcal{F}(y, \Omega_{\text{PBH},f}, \alpha) = \int_0^{\Lambda_{UV}} dv \int_{[1-v]}^{\min \Lambda_{UV}, 1+v} du \left[ \frac{4v^2 - (1 + v^2 - u^2)^2}{4 \left( 3 + \frac{2\Xi(\alpha, \Omega_{\text{PBH},f})}{\xi(\alpha, \Omega_{\text{PBH},f})} y^2u^2 \right) \left( 3 + \frac{2\Xi(\alpha, \Omega_{\text{PBH},f})}{\xi(\alpha, \Omega_{\text{PBH},f})} y^2u^2 \right)} \right]^2 uv, \quad (5.8)$$

with $y = k/(a_d H_d)$. $\Lambda_{UV}$ is the upper bound of the integral in $v$ due to the UV cut-off scale, discussed in subsection 2.3, and is defined as [32]

$$\Lambda_{UV} = \frac{k_{UV}}{k}. \quad (5.9)$$

Finally, the function $\Xi(\alpha, \Omega_{\text{PBH},f})$ is defined as

$$\Xi(\alpha, \Omega_{\text{PBH},f}) = \frac{F(a_d)}{\xi(\alpha, \Omega_{\text{PBH},f})} \left[ 1 + \frac{3k^2 F_R(a_d)}{a_d^2 F(a_d)} \right], \quad (5.10)$$

where $\Xi(\alpha, \Omega_{\text{PBH},f}) \approx 1/\xi(\alpha, \Omega_{\text{PBH},f})$ since as we have verified numerically $F(a_d) = 1 + \alpha^2 \Omega_{\text{PBH},f}^2 \sim 1$ and $\left( 1 + 3k^2 F_R(a_d) \right) / \left( 1 + 2k^2 F_R(a_d) \right) \sim 1$. Note that we have dropped the argument $a_d$ from $\xi$ in order not to have a heavy notation and we will keep this convention throughout the paper.

In figure 1 we depict the function $\xi(\alpha, \Omega_{\text{PBH},f})$ as a function of $\alpha$, taking different values of $\Omega_{\text{PBH},f}$. As we observe, $\xi(\alpha, \Omega_{\text{PBH},f})$ is a decreasing function of $\alpha$, with a plateau behaviour.

---

\[6\] For the derivation of the expression of $I^2_{(sc)}(x)$ and $I^2_{(+)or(\times)}(x)$ see appendix D.
\( \alpha \equiv \frac{H_f}{M} \).

\[ m_{\text{PBH}} = 10^5 \text{g} \]

\( \xi(\alpha, \Omega_{\text{PBH}}, f) \) given in (3.25), as a function of \( \alpha \), for fixed \( m_{\text{PBH}} = 10^5 \text{g} \) and for various values of \( \Omega_{\text{PBH}}, f \).

Figure 1. The ratio of the PBH density contrast computed at PBH domination time over the PBH density contrast at PBH formation \( \xi(\alpha, \Omega_{\text{PBH}}, f) \) given in (3.25), as a function of \( \alpha \), for fixed \( m_{\text{PBH}} = 10^5 \text{g} \) and for various values of \( \Omega_{\text{PBH}}, f \).

for small values of \( \alpha \). For relatively small \( \Omega_{\text{PBH}} \) values we can also infer that \( \xi(\alpha, \Omega_{\text{PBH}}, f) \) depends slightly on \( \Omega_{\text{PBH}}, f \).

Consequently, having calculated \( \xi(\alpha, \Omega_{\text{PBH}}, f) \), we can insert it in expression (5.6) and extract the GW spectrum. In the left panel of figure 2 we show the GW spectral abundance at PBH evaporation time, \( \Omega_{\text{GW}}(\eta_{\text{evap}}, k) \), namely at the end of the PBH-dominated era, for different values of the parameter \( \alpha = \frac{H_f}{M} \). As one may see, as \( \alpha \) increases we have a departure from the GR limit which can be clearly observed in the regime where \( \alpha \sim 1 \) or equivalently when \( M \sim H_f \). In particular, the increase of \( \alpha \) decreases the GW signal, due to the fact that for fixed \( \Omega_{\text{PBH}}, f \), \( \xi(\alpha, \Omega_{\text{PBH}}, f) \) is a decreasing function of \( \alpha \), as it can be seen from figure 1. In terms now of the mass parameter \( M \), an increase in \( M \) is equivalent with an increase in the amplitude of the GW signal. Concerning now the contribution of the different polarisation states to the amplitude of GWs we find a negligible contribution of the scalaron polarisation, a fact which leaves the shape of the GW spectrum the same as that of GR. This behavior can be confirmed by right panel of figure 2.

5.3 Gravitational wave backreaction constraints

Interestingly enough, according to the above analysis we deduce that for some values of the involved parameters one is met with an overproduction of gravitational waves at PBH evaporation time, which is something unphysical. This GW overproduction issue seems rather intriguing since one would expect that the energy density of gravitational waves generated
Following the aforementioned discussion we extract below analytical constraints on the initial abundance of PBHs $\Omega_{\text{PBH},f}$ as a function of the PBH mass $m_{\text{PBH}}$ and the mass parameter $M$ of $R^2$ gravity. In order to achieve this, one can expand $\mathcal{F}$ in the regimes $y \ll 1$ and $y \gg 1$. Following the procedure described in appendix B of [32] one obtains that\footnote{The full expression for $\mathcal{F}(y, \Omega_{\text{PBH},f})$ independently of $\Omega_{\text{PBH},f}$ is given by}

$$
\mathcal{F}(y, \Omega_{\text{PBH},f}) \simeq \begin{cases} 
\frac{125}{48} \sqrt{\frac{5}{6}} \pi \xi^{7/2} (\alpha, \Omega_{\text{PBH},f}) \frac{y^2}{625 \pi^2 \xi^4 (\alpha, \Omega_{\text{PBH},f})} & \text{for } y \ll 1 \text{ and } \Omega_{\text{PBH},f} \ll 1 \\
\frac{6}{128 y^8} & \text{for } y \gg 1
\end{cases} \quad (5.10)
$$

by PBHs inhomogeneities decays like radiation as $a^{-4}$, i.e. faster than the energy density of PBHs themselves which decays like matter as $a^{-3}$. This is true in the case where GWs decay as free waves. However, in our case we study the SIGW production during an early PBH domination era with the source of the induced GWs, namely eq. (4.4) not being zero. Under these conditions, the tensor perturbations are not decoupled from the scalar ones and the GWs are not freely propagating. One then expects a continuous production of GWs up to the time where the source term (4.4) has sufficiently decayed, namely after the PBH evaporation time. After this time, GWs evolve as radiation with $\rho_{\text{GW}} \sim a^{-4}$. Therefore, in order to avoid this GW backreaction issue we demand that $\Omega_{\text{GW},\text{tot}}(\eta_{\text{evap}}) < 1$. Hence, this condition will lead to bounds for the relevant parameters of the problem at hand.

### 5.3.1 Constraints on the primordial black hole abundance

Following the aforementioned discussion we extract below analytical constraints on the initial abundance of PBHs $\Omega_{\text{PBH},f}$ as a function of the PBH mass $m_{\text{PBH}}$ and the mass parameter $M$ of $R^2$ gravity. In order to achieve this, one can expand $\mathcal{F}$ in the regimes $y \ll 1$ and $y \gg 1$. Following the procedure described in appendix B of [32] one obtains that\footnote{The full expression for $\mathcal{F}(y, \Omega_{\text{PBH},f})$ independently of $\Omega_{\text{PBH},f}$ is given by}

$$
\mathcal{F}(y, \Omega_{\text{PBH},f}) = \begin{cases} 
\frac{500 \xi^{7/2} (\alpha, \Omega_{\text{PBH},f})}{576} \left[ \sqrt{30} \text{Arctan} \left( \sqrt{\frac{2}{15 \xi (\alpha, \Omega_{\text{PBH},f})}} \frac{1}{\Omega_{\text{PBH},f}} \right) \right] & \text{for } y \ll 1 \text{ and } \Omega_{\text{PBH},f} \ll 1 \\
\frac{6}{\sqrt{\xi (\alpha, \Omega_{\text{PBH},f})}} \left[ 1125 \Omega_{\text{PBH},f}^{10/3} + 44 \xi^{-2} (\alpha, \Omega_{\text{PBH},f}) \Omega_{\text{PBH},f}^{2/3} + 400 \Omega_{\text{PBH},f}^{2/3} / \xi (\alpha, \Omega_{\text{PBH},f}) \right] & \text{for } y \gg 1
\end{cases} \quad (5.10)
$$

The full expression for $\mathcal{F}(y, \Omega_{\text{PBH},f})$ independently of $\Omega_{\text{PBH},f}$ is given by

$$
\mathcal{F}(y \ll 1, \Omega_{\text{PBH},f}) = \frac{500 \xi^{7/2} (\alpha, \Omega_{\text{PBH},f})}{576} \left[ \sqrt{30} \text{Arctan} \left( \sqrt{\frac{2}{15 \xi (\alpha, \Omega_{\text{PBH},f})}} \frac{1}{\Omega_{\text{PBH},f}} \right) \right] 
$$

\begin{align*}
\text{for } y \ll 1 \text{ and } \Omega_{\text{PBH},f} \ll 1 \\
\frac{6}{\sqrt{\xi (\alpha, \Omega_{\text{PBH},f})}} \left[ 1125 \Omega_{\text{PBH},f}^{10/3} + 44 \xi^{-2} (\alpha, \Omega_{\text{PBH},f}) \Omega_{\text{PBH},f}^{2/3} + 400 \Omega_{\text{PBH},f}^{2/3} / \xi (\alpha, \Omega_{\text{PBH},f}) \right] & \text{for } y \gg 1
\end{align*}

\begin{align*}
\text{for } y \ll 1 \text{ and } \Omega_{\text{PBH},f} \ll 1 \\
\frac{6}{\sqrt{\xi (\alpha, \Omega_{\text{PBH},f})}} \left[ 1125 \Omega_{\text{PBH},f}^{10/3} + 44 \xi^{-2} (\alpha, \Omega_{\text{PBH},f}) \Omega_{\text{PBH},f}^{2/3} + 400 \Omega_{\text{PBH},f}^{2/3} / \xi (\alpha, \Omega_{\text{PBH},f}) \right] & \text{for } y \gg 1
\end{align*}

\begin{align*}
\text{for } y \ll 1 \text{ and } \Omega_{\text{PBH},f} \ll 1 \\
\frac{6}{\sqrt{\xi (\alpha, \Omega_{\text{PBH},f})}} \left[ 1125 \Omega_{\text{PBH},f}^{10/3} + 44 \xi^{-2} (\alpha, \Omega_{\text{PBH},f}) \Omega_{\text{PBH},f}^{2/3} + 400 \Omega_{\text{PBH},f}^{2/3} / \xi (\alpha, \Omega_{\text{PBH},f}) \right] & \text{for } y \gg 1
\end{align*}
Then, inserting the above expression into (5.6) we acquire
\[ \Omega_{GW}(\eta_{\text{evap}}, k \ll H_d) \simeq \frac{8\sqrt{2}}{3} \frac{\xi^{7/2}(\alpha, \Omega_{PBH,f})}{\pi} \left( \frac{g_{\text{eff}}}{100} \right)^{-2/3} \frac{k}{H_d} \left( \frac{m_{PBH}}{M_{\text{Pl}}} \right)^{4/3} \Omega_{PBH,f}^{16/3}, \] (5.11)

\[ \Omega_{GW}(\eta_{\text{evap}}, k \gg H_d) \simeq 50 \left( \frac{3}{5} \right)^{3/2} \xi^4(\alpha, \Omega_{PBH,f}) \left( \frac{g_{\text{eff}}}{100} \right)^{-2/3} \left( \frac{m_{PBH}}{M_{\text{Pl}}} \right)^{4/3} \Omega_{PBH,f}^{16/3}. \] (5.12)

Finally, by integrating over \( \ln k \) we obtain the total amount of GWs produced during the PBH domination era, namely
\[ \Omega_{GW,\text{tot}}(\eta_{\text{evap}}) = \int d \ln k \, \Omega_{GW}(\eta_{\text{evap}}, k). \] (5.13)

Specifically, by replacing (5.11) and (5.12) into (5.13), \( \Omega_{GW,\text{tot}}(\eta_{\text{evap}}) \) is written as
\[ \Omega_{GW,\text{tot}}(\eta_{\text{evap}}) = \mu [\kappa - \ln(\Omega_{PBH,f})] \Omega_{PBH,f}^{16/3}, \] (5.14)

with
\[ \mu = 20 \xi^{7/2}(\alpha, \Omega_{PBH,f}) \left( \frac{3}{5} \right)^{1/2} \left( \frac{g_{\text{eff}}}{100} \right)^{-2/3} \left( \frac{m_{PBH}}{M_{\text{Pl}}} \right)^{4/3} \] (5.15)

and
\[ \kappa = \frac{2\sqrt{2}}{9} \frac{1}{\pi \sqrt{\xi(\alpha, \Omega_{PBH,f})}} + \frac{3}{2} \ln 2. \] (5.16)

As a last step, let us extract the bounds for the parameters \( m_{PBH}, \Omega_{PBH,f} \) and \( \alpha \). To do so, we need to solve the equation \( \Omega_{GW,\text{tot}}(\eta_{\text{evap}}) = 1 \). This equation can be solved in terms of the Lambert function [107], obtaining
\[ \Omega_{PBH,f}^{\text{max}} = \left[ \frac{3\mu}{16} W_{-1} \left( -16 \frac{e^{-\frac{16\mu}{3\pi}}}{3\pi} \right) \right]^{-3/16}, \] (5.17)

where \( W_{-1} \) is the “\(-1\)”-branch of the Lambert function. Given the fact that \( m_{PBH} > 10^g \) [see eq. (5.2)], we find that \( \mu \gg 1 \), while \( \kappa \) is of order one. Consequently, the argument of the Lambert function is close to zero, and in this regime it can be approximated by a logarithmic function, i.e. \( W_{-1} \left( -16 \frac{e^{-\frac{16\mu}{3\pi}}}{3\pi} \right) \simeq \ln \left( -16 \frac{e^{-\frac{16\mu}{3\pi}}}{3\pi} \right) \). Now taking into account the mild dependence of the logarithmic function on its argument, for our numerical purposes we will choose a central value in PBH mass range, namely \( m_{PBH} = 10^5 g \), and we will consider the logarithm as constant. Concerning the value of \( \xi(\alpha) \), given the fact that for \( \Omega_{PBH,f} \leq 0.01 \) it varies between 2.2 and 2.5 (see figure 1) we will take it equal to 2.4.

At the end, we straightforwardly obtain that
\[ \Omega_{PBH,f} \leq 10^{-4} \left( \frac{10^9 g}{m_{PBH}} \right)^{1/4} \frac{1}{\xi^{21/32}(M, \Omega_{PBH,f})}, \] (5.18)

where \( \xi(M, \Omega_{PBH,f}) \) is expressed in terms of the mass parameter of Starobinsky gravity. Choosing now the fiducial value of \( M = 10^{-5} M_{\text{Pl}} \) as dictated by the CMB observations on the amplitude of the curvature power spectrum and exploiting the fact that for \( M = 10^{-5} M_{\text{Pl}} \), \( \xi(M, \Omega_{PBH,f}) \) has a very mild dependence on \( \Omega_{PBH,f} - \xi(M = 10^{-5} M_{\text{Pl}}, \Omega_{PBH,f}) \sim 2.5 \) for every value of \( \Omega_{PBH,f} \) — we find that the upper constraint on \( \Omega_{PBH,f} \) reads as
\[ \Omega_{PBH,f} \leq 5.5 \times 10^{-5} \left( \frac{10^9 g}{m_{PBH}} \right)^{1/4}. \] (5.19)
This upper bound on $\Omega_{\text{PBH},f}$ is depicted with the dotted blue line in figure 3. As it was checked, it is 45% reduced compared to the respective upper bound within GR which reads as [32] $\Omega_{\text{PBH},f} \leq 10^{-4} \left( \frac{10^9}{m_{\text{PBH}}} \right)^{1/4}$. Thus, one finds that despite the fact that the corrections from the $R^2$ term are very small at the level of the background and perturbations, as it can be seen already from the left panel of figure 2, we find almost an order of magnitude tighter constraints on $\Omega_{\text{PBH},f}$ compared to GR. This result has important consequences at the level of the detectability of the SIGW signal associated to PBH Poisson fluctuations since, as we can see from eq. (5.11) and eq. (5.12), given the mild dependence of $\xi(\Omega_{\text{PBH},f}, M)$ on $\Omega_{\text{PBH},f}$, the amplitude of the signal scales as $\Omega_{\text{PBH},f}^{16/3}$ a scaling which also holds in GR.

5.3.2 Constraints on the $f(R)$ gravity model at hand

In our previous analysis we showed how one can constrain the PBH abundances by avoiding a GW overproduction issue. Conversely if one fixes the initial PBH abundance $\Omega_{\text{PBH},f}$ and their mass $m_{\text{PBH}}$, one can translate the GW backreaction constraints to constraints on the
underlying $f(R)$ gravity theory. Here we consider as an illustrative example, the case of Starobinsky $R^2$ gravity given that it is the simplest monoparametric extension of GR within the class of $f(R)$ gravity theories. In this sense, we will ignore the fact that the mass parameter $M$ is fixed by CMB observations to the value $M = 10^{-5}M_{\text{Pl}}$ and we will treat it as a free parameter.

Under these considerations, as we can see from the left panel of figure 2, if one fixes $\Omega_{\text{PBH}}$ and $m_{\text{PBH}}$ by increasing the mass parameter $M$ or equivalently by decreasing the parameter $\alpha = H_{\text{f}}/M$, the amplitude of SIGWs is increasing as well signalling that one can set an upper bound constraint on the mass parameter $M$. To do so, we solved numerically the equation $\Omega_{\text{GW, tot}}(\eta_{\text{evap}}) = 1$ and found the upper bound $M_{\text{max}}$ on $M$ as a function of $m_{\text{PBH}}$ and $\Omega_{\text{PBH}}$. In figure 3 we show this upper bound constraint on $M$ in the color bar axis. The lower left triangular region in “olive” stands for the region in the parameter space $(m_{\text{PBH}}, \Omega_{\text{PBH}})$ where PBHs dominate the Universe energy content after their evaporation, hence it not of special interest. The upper grey region with large values of $\Omega_{\text{PBH}}$ corresponds to regimes where GWs are overproduced during the PBH dominated era, so it is excluded.

The interesting region which permits an early PBH dominated era not presenting a GW overproduction issue is the intermediate one, where we show in the lateral color bar axis the upper bound on the mass scale $M$. As expected, for the majority of the parameter space $(m_{\text{PBH}}, \Omega_{\text{PBH}})$ $M_{\text{max}}$ is found to be equal to $10^{-5}M_{\text{Pl}}$ which is the fiducial value of $M$ as obtained by CMB observations. However, there is an interesting region between the “bordeaux” region of $M_{\text{max}} = 10^{-5}M_{\text{Pl}}$ and the grey region of GW overproduction where the upper bound $M_{\text{max}}$ becomes smaller than $10^{-5}M_{\text{Pl}}$ reaching very small values up to $10^{-14}M_{\text{Pl}}$. This behavior can be explained from the fact that as in this region whose location is described by eq. (5.19) $\Omega_{\text{PBH}}$ takes its greatest value more or less between $10^{-4}$ up to $10^{-2}$ where one expects a high amplitude of GWs. To re-compensate therefore for this increased amplitude of GWs one should lower the mass scale $M$ since as we show in figure 2 the amplitude of GWs is an increasing function of $M$.

At this point, we should highlight that the upper bounds in $M$ should not be interpreted as physical since $M$ is very well fixed by the amplitude of the scalar perturbations as measured by CMB probes. As we already mentioned above, the choice of $R^2$ should rather be regarded as an illustrative example which demonstrates the fact that the SIGW portal associated to PBH Poisson fluctuations can serve as a novel probe to constrain alternative gravity theories.

6 Conclusions

Primordial black holes are of great significance, since they may constitute a part or all of the dark matter sector, they may provide an explanation for the large-scale structure formation through Poisson fluctuations, and moreover they can offer the seeds for the progenitors of the black-hole merging events as well as for the supermassive black holes formation. Their effect on the GW background signals, and in particular the second-order GWs induced by the gravitational potential of Poisson-distributed PBHs, has been studied only in the framework of general relativity. Hence, in this work we extended the analysis of the literature in the case of $f(R)$ gravity. In order to illustrate the effect of $f(R)$ gravity theory, we worked with the Starobinsky $R^2$ gravity, which constitutes the simplest monoparametric generalisation beyond GR within $f(R)$ theories as well as one of the most favored inflationary models from the observational side. However, our formalism is applicable for every model in the context of $f(R)$ gravity.

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8In particular, the very low $M_{\text{max}}$ regions are rather questionable since it is not easy to achieve scalaron decay quite early and thus one should account for very strong restrictions of $M$ in these regimes.
Firstly, we calculated the effect of $f(R)$ modification on the PBH gravitational potential power spectrum and we extracted the associated SIGW spectrum during an era driven by ultralight PBHs ($m_{\text{PBH}} < 10^{9}g$), which evaporate before BBN. In particular, we found its dependence on the relevant parameters involved, namely the PBH mass $m_{\text{PBH}}$, the initial PBH abundance at formation time $\Omega_{\text{PBH},f}$, and the mass parameter of the $R^2$ gravity $M$ by accounting as well for the three polarization states of GWs in $f(R)$ gravity, namely the $(\times)$, the $(\pm)$ and the scalaron one.

Concerning the contribution of the different polarisation states to the amplitude of GWs, we found a negligible contribution of the scalaron polarisation, a fact which left the shape of the GW spectrum the same as that of GR [See the right panel of figure 2]. The only difference with respect to GR was observed at the level of the amplitude of GWs. In particular, a decrease of the mass parameter $M$ leads to a decrease of the GW amplitude which becomes distinguishable from the GW amplitude within GR in the region where $M \sim H_{f}$, where $H_{f}$ is the Hubble parameter at the PBH formation time [See the left panel of figure 2].

Interestingly, in some region of our parameter space ($m_{\text{PBH}}$, $\Omega_{\text{PBH},f}$, $M$) we found regimes where the overall energy density of the induced GWs at PBH evaporation time becomes greater than the total energy density of the Universe, which is unphysical and thus needs to be avoided. Thus, in order to avoid this GW backreaction problem we demanded that the overall energy density contribution of the GWs at evaporation time is less than one, $\Omega_{\text{GW,tot}}(\eta_{\text{evap}}) < 1$. This condition allowed us to extract an upper bound on $\Omega_{\text{PBH,f}}$ as a function of the PBH mass and the mass parameter $M$. Intriguingly, this upper bound is the respective GR bound screened by a function of $M$ and $\Omega_{\text{PBH,f}}$, namely

$$\Omega_{\text{PBH,f}} \leq 10^{-4} \left(\frac{10^{9}g}{m_{\text{PBH}}}\right)^{1/4} \frac{1}{\xi^{21/32}(M,\Omega_{\text{PBH}})},$$

(6.1)

Given the above inequality condition, on the one hand, by fixing the mass parameter $M$ of $R^2$ gravity to its fiducial value $M = 10^{-5}M_{\text{Pl}}$ as imposed by CMB observations and exploiting the mild dependence of $\xi(M = 10^{-5}M_{\text{Pl}},\Omega_{\text{PBH,f}})$ on $\Omega_{\text{PBH,f}}$, we found that

$$\Omega_{\text{PBH,f}} \leq 5.5 \times 10^{-5} \left(\frac{10^{9}g}{m_{\text{PBH}}}\right)^{1/4},$$

(6.2)

which gives an upper bound $\Omega_{\text{PBH,f}}$ 45% tighter than that in GR [32].

On the other hand, by fixing $m_{\text{PBH}}$ and $\Omega_{\text{PBH,f}}$ we saturated the above inequality in order to find an upper bound on the mass scale $M$ given the fact that the GW amplitude is an increasing function of $M$ as it can be seen by the left panel of figure 2. These upper bounds can be seen in the color bar axis of figure 3 but they should not be considered as physical ones given the fact that the value of $M$ is very well fixed by CMB observations. As mentioned before already, given the simplicity of the $R^2$ gravity model we use it as a case study in order not to set precise constraints on $M$ but rather to illustrate that one can use the SIGW portal associated to PBH Poisson fluctuations as a novel probe to constrain alternative gravitational theories.

One should comment here on the observational prospects of the aforementioned SIGW signal within the class of $f(R)$ gravity theories. Regarding the frequency of the GW signal

\footnote{As it was found numerically $\xi(M = 10^{-5}M_{\text{Pl}},\Omega_{\text{PBH,f}})$ varies between the values 2.5 and 2.47 within the range for $\Omega_{\text{PBH,f}} \in [10^{-15}, 10^{-1}]$. Thus, for our numerical purposes we take $\xi(M = 10^{-5}M_{\text{Pl}},\Omega_{\text{PBH,f}}) = 2.5$.}
these are given by \( f \equiv k/(2\pi a_0) \), where \( a_0 \) is the scale factor today and \( k \) is the comoving wavenumber lying within the range \([k_{\text{evap}}, k_{\text{UV}}]\) with \( k_{\text{evap}} \) being the comoving number crossing the Hubble radius at the PBH evaporation time and \( k_{\text{UV}} \) the UV cut-off scale introduced to avoid entering to the non-linear regime where \( P_\delta(k) > 1 \). These two comoving wavenumbers depend on the details of the gas of PBHs, namely on their mass \( m_{\text{PBH}} \) and their initial abundance \( \Omega_{\text{PBH},f} \). Therefore, the only effect of the underlying gravitational theory will be at the level of the upper bound on \( \Omega_{\text{PBH},f} \) in order to avoid GW overproduction. Consequently, the peak GW frequency at \( k_d \) as a function of \( m_{\text{PBH}} \) and \( \Omega_{\text{PBH},f} \) can be recast after a straightforward calculation as \([32]\)

\[
\frac{f}{\text{Hz}} \simeq \frac{1}{(1+z_{\text{eq}})^{1/4}} \left( \frac{H_0}{70\text{km s}^{-1}\text{Mpc}^{-1}} \right)^{1/2} \left( \Omega_{\text{PBH},f}^2/\beta \right)^{1/6} \left( \frac{m_{\text{PBH}}}{10^9\text{g}} \right)^{-5/6}, \tag{6.3}\]

where \( H_0 \) is the value of the Hubble parameter today and \( z_{\text{eq}} \) is the redshift at matter-radiation equality. We show in figure 4 how the SIGW frequency varies with \( m_{\text{PBH}} \) and \( \Omega_{\text{PBH},f} \). Interestingly, depending on the choice of the PBH mass and the initial PBH abundance, the SIGW frequency can lie within the frequency detection bands of the Einstein Telescope (ET) \([108]\), the Laser Interferometer Space Antenna (LISA) \([109]\) and the Square Kilometre Array (SKA) facility \([110]\) pointing out the ability of these GW experiments to potentially detect such a signal and measure deviations from GR.

At this point, we should stress that the contribution to the GW spectrum coming from the transition from the PBH dominated era to the radiation dominated one, which in the case of GR and in the regimes of a monochromatic PBH mass function enhances considerably the GW signal as pointed out in \([111, 112]\), was not considered in this work. This aspect should be considered in future works in order to check which values of GW amplitudes have the potential to be observed by GW experiments.

We close this work by making a comment on the aforementioned procedure. By making use of the cosmological perturbation theory we extracted the power spectrum of the gravitational potential, and by imposing the UV cut-off scale we ensured that we are well within the perturbative regime. This is very important since it is \( \Phi \) that induces the second-order gravitational waves. Nevertheless, from the point of view of the energy density perturbation \( \delta \), as it is well established in the context of GR, during matter domination \( \delta \) grows linearly with the scale factor. A similar picture was found here too, namely \( \delta \) grows with the scale factor although non linearly. Thus, there will be scales where \( \delta \) can acquire values larger than one, entering into the non-linear regime although \( \Phi \) remains much smaller than one. Therefore, in order to clarify the status of these scales one should follow the full virialisation dynamics \([32, 34]\) something which is beyond the scope of this work. However, we may speculate that a growth of \( \delta \) will enhance the power spectrum above the Poissonian value, which in turn will lead to an even larger signal than that extracted above. In that sense, the bounds obtained here in particular regarding the initial abundances of PBHs, \([\text{see eq. (6.2)}]\) can be considered as conservative ones.

In summary, through the above analysis we showed that the condition to avoid an overproduction of scalar induced gravitational waves associated to PBH Poisson fluctuations at PBH evaporation time can act as a novel method to extract constraints on PBH parameters as well as on gravitational theories, independent from other methods such as the BBN or cosmological confrontations. Hence, by the combined application of all these approaches we can have an improved tool to constrain proposed scenarios and test possible deviations from general relativity.
Figure 4. The peak frequency of the SIGW signal within $R^2$ gravity produced during an early PBH-dominated era as a function of the initial PBH abundance at formation $\Omega_{\text{PBH},f}$ (horizontal axis) and the PBH mass $m_{\text{PBH}}$ (colour coding). The region of parameter space that is shown corresponds to values of $m_{\text{PBH}}$ and $\Omega_{\text{PBH},f}$ such as that the black holes dominate the energy budget of the Universe for a transient period, see eq. (5.3), that they form after inflation and Hawking evaporate before big-bang nucleosynthesis, see eq. (5.2), and that the induced gravitational waves do not lead to a backreaction problem, see eq. (6.2). For our numerical applications, $g_{\text{eff}} = 100$, $z_{\text{eq}} = 3387$ and $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$. For comparison, the frequency detection bands of ET, LISA and SKA are also shown.

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A Scalar perturbation equations in $f(R)$ gravity

In the case of $f(R)$ gravity, in the Newtonian gauge one extracts the following scalar perturbed field equations [45]:

\[ 3\mathcal{H}(\Phi' + \mathcal{H}\Psi) + k^2\Phi = -4\pi G a^2 \delta \rho_{\text{tot}}, \quad (A.1) \]
\[ \Phi' + \mathcal{H}\Psi = 4\pi G a^2 (\bar{\rho}_{\text{tot}} + \bar{p}_{\text{tot}}) v_{\text{tot}}, \quad (A.2) \]
\[ \Phi'' + \mathcal{H}(\Phi' + 2\Psi') + (\mathcal{H}^2 + 2\mathcal{H}')\Phi - k^2(\Phi - \Psi)/3 = -4\pi G a^2 \delta \rho_{\text{tot}}, \quad (A.3) \]
\[ \Phi - \Psi = 8\pi G a^2 \bar{p}_{\text{tot}} \Pi_{\text{tot}}, \quad (A.4) \]

where

\[ (\bar{\rho}_{\text{tot}} + \bar{p}_{\text{tot}}) v_{\text{tot}} \equiv \sum_{l=m,r,f} (\bar{\rho}^l + \bar{p}^l) v^l, \quad (A.5) \]

and

\[ \bar{p}_{\text{tot}} \Pi_{\text{tot}} \equiv \sum_{l=m,r,f} \bar{p}^l \Pi^l. \quad (A.6) \]

Additionally, the perturbed energy density and pressure of the effective fluid arising from $f(R)$ mortification, are written respectively as

\[ \delta \rho_{f(R)} \equiv -\delta T_{00}^{f(R)} = \frac{1}{8\pi G a^2} \left\{ (1 - F) \left[ -6\mathcal{H}'\Psi + k^2\Psi - 3\mathcal{H}(\Phi' + \Psi') - 3\Phi'' \right] 
- 3\mathcal{H}'\delta F + a^2\delta f/2 - k^2\Psi + 2k^2\Phi + 6(\mathcal{H}' + \mathcal{H}^2)\Psi + 3\Phi' + 3\mathcal{H}(\Psi' + 3\Psi') 
+ k^2\delta F + 3\mathcal{H}\delta F' - 3\Phi'(2\Phi' + 2\mathcal{H} + \Psi) \right\}, \quad (A.7) \]
\[ \delta p_{f(R)} \equiv \frac{\delta T^{f(R)}_i}{3} = \frac{1}{8\pi G a^2} \left\{ -(\mathcal{H}' + 2\mathcal{H}^2)\delta F + a^2\delta f/2 + k^2(2\Phi - \Psi) + 3\mathcal{H}(\Psi' + 3\Psi') 
+ 3\Phi'' + 6(\mathcal{H}' + \mathcal{H})\Psi + \delta F'' + 2k^2\delta F'/3 + \mathcal{H}\delta F' - F'(2\Phi + 2\mathcal{H} + \Psi') - 3\Psi F'' 
+ (1 - F) \left[ -k^2\Phi - \Phi'' - 3\mathcal{H}(5\Phi' + \Psi') - (2\mathcal{H}' + 4\mathcal{H}^2)\Psi - k^2(\Phi - \Psi)/3 \right] \right\}. \quad (A.8) \]

Finally, we have

\[ (\bar{\rho}_{f(R)} + \bar{p}_{f(R)}) v^l_{\text{tot}} \equiv -\delta T_0^{f(R)} = \frac{1}{8\pi G} \left\{ 2(1 - F)(\Phi' + \mathcal{H}\Psi)_{,i} + \delta F_{,i} + 2\Psi_{,i} - \mathcal{H}\delta F_{,i} \right\}, \quad (A.9) \]

and

\[ \Pi^{f(R)}_{ij} \equiv \delta T_j^{f(R)} = \frac{1}{8\pi G a^2} \left\{ (1 - F)(\Phi - \Psi)_{,ij} + \delta F_{,ij} \right\}, \quad i \neq j. \quad (A.10) \]

In this context, we can define the (total) comoving curvature perturbation in the usual manner, namely

\[ \mathcal{R} \equiv -\Phi - \mathcal{H} v_{\text{tot}}. \quad (A.11) \]

B The anisotropic stress

Before BBN, which is the period we are interested in, there are no free streaming particles, namely neutrinos or photons, and the dominant matter species is in form of PBHs. Thus, one can safely assume that $\Pi_r = \Pi_m = 0$. One then is left with the anisotropic stress of the $f(R)$
gravity effective fluid which has a pure geometrical origin. Combining eq. (A.4), eq. (A.6) and eq. (A.10) with $\Pi_r = \Pi_m = 0$ one can show that

$$\Phi - \Psi = \frac{\delta F}{F}, \quad (B.1)$$

with $\delta F = F_{,R}\delta R$ and $\delta R$, being the first order perturbation of the Ricci scalar, given by [113]

$$\delta R = -2\frac{k^2}{a^2} \frac{\Phi}{1 + 4\frac{k^2}{a^2} F_{,R}/F}, \quad (B.2)$$

At this point, one can naturally define a dimensionless quantity denoted here with $\lambda$ as

$$\lambda \equiv \frac{\Phi - \Psi}{\Phi}, \quad (B.3)$$

which actually quantifies the anisotropic stress of geometrical origin. In the case of $f(R)$ gravity, plugging eq. (B.2) into $\delta F = F_{,R}\delta R$ and inserting then $\delta F$ into eq. (B.1) one can find that

$$\lambda = -2\frac{k^2 F_{,R}}{a^2} \frac{F_{,R}}{1 + 4\frac{k^2}{a^2} F_{,R}/F}, \quad (B.4)$$

Below we plot this quantity for different values of the wave number $k$ within the range $[k_{\text{evap}}, k_{\text{UV}}]$, for different values of masses $m_{\text{PBH}}$ within the range $[10g, 10^9g]$ as well as for different values of the mass parameter of $R^2$ gravity within the range $[H_f, 10^{-5}M_{\text{Pl}}]$. As one can see from figure 5, figure 6 and figure 7, $\lambda$ is extremely small signalling that one can safely consider a vanishing anisotropic stress and as a consequence that $\Phi = \Psi$. Very tiny values of $\lambda$ we also get by varying $\Omega_{\text{PBH},f}$

### C Super-Hubble scales in $f(R)$ gravity

On super-Hubble scales, eq. (A.1) becomes $3\mathcal{H}(\Phi' + \mathcal{H}\Psi) = -4\pi G a^2 \delta \rho_{\text{tot}}$, and thus together with eq. (A.2) yields:

$$\mathcal{R} = -\Phi + \frac{\delta_{\text{tot}}}{3(1 + w_{\text{tot}})} \xrightarrow{(3.15)} \zeta, \quad k \ll \mathcal{H}. \quad (C.1)$$

Furthermore, from eq. (A.11) and eq. (A.2) we can write:

$$\mathcal{R} = \Phi + \frac{\mathcal{H}(\Phi' + \mathcal{H}\Psi)}{4\pi G a^2 \rho_{\text{tot}} (1 + w_{\text{tot}})} \xrightarrow{\mathcal{H}^2 = 8\pi G a^2 \rho_{\text{tot}}/3} \Phi + \frac{2}{3} \frac{\Phi' / \mathcal{H} + \Psi}{1 + w_{\text{tot}}}. \quad (C.2)$$

Moreover, from eq. (A.4) and eq. (A.6) we see that $\Phi - \Psi = 8\pi G a^2 \bar{\rho} \bar{\Pi} + \delta F/F$ and hence by assuming that at super-Hubble modes $\bar{\Pi}' \approx 0$ and $\delta F \approx 0$, we deduce that $\Phi \approx \Psi$. Therefore, under these assumptions and for $k \ll \mathcal{H}$ we can write for $\mathcal{R}$:

$$\mathcal{R} = \frac{2}{3} \frac{\Phi' / \mathcal{H} + \Phi}{1 + w_{\text{tot}}} + \Phi. \quad (C.3)$$
Figure 5. The dimensionless parameter \( \lambda \equiv \Phi - \Psi \) for \( k = k_{\text{evap}} \) and for different values of the parameter space \( (m_{\text{PBH}}, \Omega_{\text{PBH}} f, M) \). The blue line corresponds to \( M = H_1 \) and the orange one to \( M = 10^{-5} M_{\odot} \).

D The kernel function \( I(u, v, x) \)

In this appendix we derive the kernel function \( I(u, v, x) \) defined in eq. (4.15) for all the three polarization modes, namely the \((\times)\), the \((+)\) and the scalaron one. In order to achieve this we firstly extract the Green function \( G_k(\eta, \bar{\eta}) \) by solving eq. (4.11). In particular, eq. (4.11) accepts an analytic solution in the case where \( w = 0 \), which depending on the GW polarization reads as

\[
kG^((\times))_{k}(\eta, \bar{\eta}) = \frac{1}{x \bar{x}} \left[ (1 + x \bar{x}) \sin(x - \bar{x}) - (x - \bar{x}) \cos(x - \bar{x}) \right], \tag{D.1}
\]

\[
kG^((+) or (+))_{k}(\eta, \bar{\eta}) = \frac{k^2}{x \bar{x} (M^2 - k^2)^{3/2}} \left\{ \frac{\sqrt{M^2 - k^2}}{k} (x - \bar{x}) \cosh \left[ \frac{\sqrt{M^2 - k^2}}{k} (x - \bar{x}) \right] \right. \\
+ \left. \frac{M^2 - k^2}{k^2} x \bar{x} - \frac{k^2}{k^2} \sinh \left[ \frac{\sqrt{M^2 - k^2}}{k} (x - \bar{x}) \right] \right\}. \tag{D.2}
\]

The associated \( I(u, v, x) \) function for the \((\times)\) and \((+)\) polarization modes can be recast as

\[
I^2(x) = \frac{100}{9} \left[ 1 + \cos(x - x_d) \left( \frac{3}{x^2} \frac{3 x_d}{x^3} - \frac{x_d^2}{x^4} \right) \right]^2 - \sin(x - x_d) \left( \frac{3}{x^2} + \frac{3 x_d}{x^3} - \frac{x_d^2}{x^4} \right), \tag{D.3}
\]
Figure 6. The dimensionless parameter $\lambda \equiv \Phi - \Psi$ for $k = k_d$ and for different values of the parameter space $(m_{\text{PBH}}, \Omega_{\text{PBH,f}}, M)$. The blue line corresponds to $M = H_f$ and the orange one to $M = 10^{-5} M_{\text{Pl}}$. 

and as we can see it does not depend on $u$ and $v$. Similarly, for the scalaron polarization we have

$$
P^2(x) = \frac{100k^4}{9(M^2 - k^2)^6x^6} \left\{ (M^2 - k^2)^2 \left[ x^3 + M^2 xx_d^2 - k^2 (3x_d + x(x_d^2 - 3)) \right] \right. \right.
$$

$$
\times \cosh \left[ \sqrt{M^2 - k^2} \frac{k^2}{k} (x - x_d) \right] + k \sqrt{M^2 - k^2} [M^2 xx_d (x_d - 3x)] + k^2 (3 + 3x_d - x_d^2) \sinh \left[ \sqrt{M^2 - k^2} \frac{k^2}{k} (x - x_d) \right] \bigg\}^2, \quad (D.4)
$$

which is independent of $u$ and $v$ too. Taking now into account the fact that $k_{\text{UV}} = H_f \Omega_{\text{PBH,f}}^{1/3}$ and that roughly $k < k_{\text{UV}}$ as well as that $H_f \leq M$, then one can easily see that $k/M < \Omega_{\text{PBH,f}}^{1/3} < 1$. Consequently, the above functions in a PBH dominated era and in the subhorizon limit, i.e. $x \gg 1$, become

$$
P^2(x) = \frac{100}{9} \times \begin{cases} 
1 & \text{if } s = (\times), (+) \\
\frac{k^4}{M^4} & \text{if } s = (\text{sc}) 
\end{cases}. \quad (D.5)
$$
Figure 7. The dimensionless parameter $\lambda \equiv \frac{\Phi - \Psi}{k}$ for $k = k_{\text{UV}}$ and for different values of the parameter space ($m_{\text{PBH}}, \Omega_{\text{PBH}, f}, M$). The blue line corresponds to $M = M_1$ and the orange one to $M = 10^{-5} M_{\text{Pl}}$.

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