Abstract

We show that: (1) unimodular simplices in a lattice 3-polytope cover a neighborhood of the boundary of the polytope if and only if the polytope is very ample, (2) the convex hull of lattice points in every ellipsoid in $\mathbb{R}^3$ has a unimodular cover, and (3) for every $d \geq 5$, there are ellipsoids in $\mathbb{R}^d$, such that the convex hulls of the lattice points in these ellipsoids are not even normal. Part (c) answers a question of Bruns, Michałek, and the author.

Mathematics Subject Classifications: 52B20, 11H06

1 Introduction

1.1 Main result

A convex polytope $P \subset \mathbb{R}^d$ is normal if it is lattice, i.e., has vertices in $\mathbb{Z}^d$, and satisfies the condition

$$\forall c \in \mathbb{N} \quad \forall x \in (cP) \cap \mathbb{Z}^d \quad \exists x_1, \ldots, x_c \in P \cap \mathbb{Z}^d \quad x_1 + \cdots + x_c = x.$$ 

A necessary condition for $P$ to be normal is that the subgroup

$$\text{gp}(P) := \sum_{x, y \in P \cap \mathbb{Z}^d} \mathbb{Z}(x - y) \subset \mathbb{Z}^d$$

must be a direct summand. Also, a face of a normal polytope is normal.

Normality is a central notion in toric geometry and combinatorial commutative algebra [7]. A weaker condition for lattice polytopes is very ample; see Section 1.2 for the definition. Normal polytopes define projectively normal embeddings of toric varieties whereas very ample polytopes correspond to normal projective varieties [3, Proposition 2.1].
A sufficient condition for a lattice polytope $P$ to be normal is the existence of a \textit{unimodular cover}, which means that $P$ is a union of unimodular simplices. Unimodular covers play an important role in integer programming through their connection to the \textit{Integral Carathéodory Property} \cite{8, 12, 15}.

There exist normal polytopes in dimensions $\geq 5$ without unimodular cover \cite{6}. It is believed that all normal 3-polytopes have unimodular cover. But progress in this direction is scarce. Recent works \cite{4, 11} show that all lattice 3-dimensional parallelepipeds and centrally symmetric 3-polytopes with unimodular corners have unimodular cover.

The normality of the convex hull of lattice points in an ellipsoid naturally comes up in \cite{9}. We consider general ellipsoids, neither centered at 0 nor aligned with the coordinate axes. According to \cite[Theorem 6.5(c)]{9}, the convex hull of the lattice points in any ellipsoid $E \subset \mathbb{R}^3$ is normal. \cite[Question 7.2(b)]{9} asks whether this result extends to higher dimensional ellipsoids.

Here we prove the following

\textbf{Theorem.} Let $P \subset \mathbb{R}^3$ be a lattice polytope, $E \subset \mathbb{R}^d$ an ellipsoid, and $P(E)$ the convex hull of the lattice points in $E$.

(a) The unimodular simplices in $P$ cover a neighborhood of the boundary $\partial P$ in $P$ if and only if $P$ is very ample.

(b) If $d = 3$ then the polytope $P(E)$ is covered by unimodular simplices.

(c) For every $d \geq 6$, there exists $E$ such that $\text{gp}(P(E)) = \mathbb{Z}^d$ and $P(E)$ is not normal.

If in (c) we drop the condition $\text{gp}(P(E)) = \mathbb{Z}^d$, then ellipsoids $E \subset \mathbb{R}^d$ with $P(E)$ non-normal already exist for $d = 5$; see Remark 7.

1.2 Preliminaries

$\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the sets of non-negative integers and reals, respectively.

The convex hull of a set $X \subset \mathbb{R}^d$ is denoted by $\text{conv}(X)$. The relative interior of a convex set $X \subset \mathbb{R}^d$ is denoted by $\text{int} X$. The boundary of $X$ is denoted by $\partial X = X \setminus \text{int} X$.

Polytopes are assumed to be convex. For a polytope $P \subset \mathbb{R}^d$, its vertex set is denoted by $\text{vert}(P)$.

A lattice $n$-simplex $\Delta = \text{conv}(x_0, \ldots, x_n) \subset \mathbb{R}^d$ is \textit{unimodular} if $\{x_1 - x_0, \ldots, x_n - x_0\}$ is a part of a basis of $\mathbb{Z}^d$.

A \textit{unimodular pyramid} over a lattice polytope $Q$ is a lattice polytope $P = \text{conv}(v, Q)$, where the point $v$ is not in the affine hull of $Q$ and the lattice height of $v$ above $Q$ inside the affine hull of $P$ equals 1.

Cones $C$ are assumed to be \textit{pointed, rational, and finitely generated}, which means $C = \mathbb{R}_+ x_1 + \cdots + \mathbb{R}_+ x_k$, where $x_1, \ldots, x_k \in \mathbb{Z}^d$ and $C$ does not contain a nonzero linear subspace. For a cone $C \subset \mathbb{R}^d$, the smallest generating set of the additive submonoid $C \cap \mathbb{Z}^d \subset \mathbb{Z}^d$ consists of the indecomposable elements of this monoid. This is a finite set, called the \textit{Hilbert basis} of $C$ and denoted by $\text{Hilb}(C)$. See \cite[Chapter 2]{7} for a detailed
discussion on Hilbert bases. For a lattice polytope $P \subset \mathbb{R}^d$, we have the inclusion of finite subsets of $\mathbb{Z}^{d+1}$:

$$ (P \cap \mathbb{Z}^d, 1) \subset \text{Hilb}(\mathbb{R}_+(P, 1)). $$

This inclusion is an equality if and only if $P$ is normal.

A lattice polytope $P$ is very ample if $\text{Hilb}(\mathbb{R}_+(P - v)) \subset P - v$ for every vertex $v \in \text{vert}(P)$. All normal polytopes are very ample, but already in dimension 3 there are very ample non-normal polytopes [7, Exercise 2.24]. For a detailed analysis of the discrepancy between the two properties see [3].

For a cone $C \subset \mathbb{R}^d$, we say that $C$ has a unimodular Hilbert triangulation (cover) if $C$ can be triangulated (resp., covered) by cones of the form $\mathbb{R}_+ x_1 + \cdots + \mathbb{R} x_n$, where $\{x_1, \ldots, x_n\}$ is a part of a basis of $\mathbb{Z}^d$ as well as of $\text{Hilb}(C)$.

An ellipsoid $E \subset \mathbb{R}^d$ is a set of the form

$$ \{x \in \mathbb{R}^d \mid (l_1(x) - a_1)^2 + \cdots + (l_d(x) - a_d)^2 = 1\} \subset \mathbb{R}^d, $$

where $l_1, \ldots, l_d$ is a full-rank system of real linear forms and $a_1, \ldots, a_d \in \mathbb{R}^d$.

For a lattice polytope $P$, the union of unimodular simplices in $P$ will be denoted by $U(P)$.

## 2 Unimodular covers close to the boundary

The following result of Sebő was later rediscovered in [1, 5] in a refined form in the context of toric varieties.

**Theorem 1.** ([16]) Every 3-dimensional cone $C$ has a unimodular Hilbert triangulation.

**Notice.** There exist 4-dimensional cones without unimodular Hilbert triangulation [5] and it is not known whether all 4- and 5-dimensional cones have unimodular Hilbert cover. According to [6], in all dimensions $\geq 6$ there are cones without unimodular Hilbert cover.

If $P \subset \mathbb{R}^3$ is very ample, then by Theorem 1, for every $v \in \text{vert}(P)$, the cone $\mathbb{R}_+(P - v)$ has a unimodular Hilbert triangulation:

$$ \mathbb{R}_+(P - v) = \bigcup_{T(v)} C_t, $$

where $T(v)$ is a finite index set, depending on $v$. In particular, the following unimodular simplices form a neighborhood of $v$ in $P$:

$$ \Delta_{v,t} = \text{conv}(\text{Hilb}(C_t), 0) + v, \quad t \in T(v). $$

Also, lattice polygons have unimodular triangulation [7, Corollary 2.54]. Therefore, the following lemma completes the proof of Theorem (a):
Lemma 2. For a lattice polytope $P$ of an arbitrary dimension, the following conditions are equivalent:

(a) $U(P)$ is a neighborhood of $\partial P$ within $P$;

(b) $U(P)$ is a neighborhood within $P$ of every vertex of $P$ and $\partial P \subset U(P)$.

Proof. The implication (a)$\implies$(b) is obvious.

For the opposite implication, let:

- $x \in \partial P$;
- $F$ be the minimal face of $P$ containing $x$;
- $v \in \text{vert}(F)$;
- $T_F$ be a unimodular cover of $F$ with $\dim(F)$-simplices, contained in $F$;
- $T_v$ be a unimodular cover of a neighbourhood of $v$ in $P$;
- $T_{v,F}$ be the sub-family of $T_v$, consisting of simplices that have a $\dim(F)$-dimensional intersection with $F$;
- $T_v/F$ be the collection of faces of simplices in $T_{v,F}$, opposite to $F$ (that is, from each simplex in $T_{v,F}$ remove the $\dim(F)+1$ vertices that lie in $F$, so that one is left with a $(\dim(P) - \dim(F))$-simplex).

Then, the collection of $\text{conv}(T_v/F, T_F)$ covers a neighbourhood of $x$ in $P$ and consists of unimodular simplices.

3 Unimodular covers inside ellipsoids

3.1 Proof of Theorem (b)

The set of normal polytopes $P \subset \mathbb{R}^d$ carries a poset structure, where the order is generated by the elementary relation

$$P \preceq Q \text{ if } P \subset Q \text{ and } \#(Q \cap \mathbb{Z}^d) = \#(P \cap \mathbb{Z}^d) + 1.$$ 

In [9] this poset is denoted by $NPol(d)$. The trivial minimal elements of $NPol(d)$ are the singletons from $\mathbb{Z}^d$. It is known that $NPol(d)$ has nontrivial minimal elements for $d \geq 4$ [7, Exercise 2.27] and the first maximal elements for $d = 4, 5$ were found in [9]. It is possible that $NPol(d)$ has isolated elements for some $d$.

Computer searches so far have found neither maximal nor nontrivial minimal elements in $NPol(3)$ [9]. The next lemma is yet another evidence that all normal 3-polytopes have unimodular cover.
Lemma 3. Let $P$ be a normal 3-polytope. If $\ast \leq P$ in $\text{NPol}(3)$ for a singleton $\ast \in \mathbb{Z}^3$ then $P = \text{U}(P)$.

Proof. If $Q \leq P$ is an elementary relation in $\text{NPol}(d)$ and $\dim Q < \dim P$ then $P$ is a unimodular pyramid over $Q$. In this case every full-dimensional unimodular simplex $\Delta \subset P$ is the unimodular pyramid over a unimodular simplex in $Q$ and with the same apex as $P$. On the other hand, lattice segments and polygons are unimodularly triangulable. Therefore, it is enough to show that a polytope $P \in \text{NPol}(3)$ has a unimodular cover if there is a 3-polytope $Q \in \text{NPol}(3)$, such that $Q$ has a unimodular cover and $Q \leq P$ is an elementary relation in $\text{NPol}(3)$. Assume $\{v\} = \text{vert}(P) \setminus Q$. By Theorem (a) we have the inclusion $P \setminus \text{U}(P) \subset Q$. Since $Q = \text{U}(Q)$ we have $P = \text{U}(P)$.

Call a subset $E \subset \mathbb{Z}^d$ ellipsoidal and a point $v \in E$ extremal if there is an ellipsoid $E \subset \mathbb{R}^d$, such that $E = \text{conv}(E) \cap \mathbb{Z}^d$ and $v \in E$.

Lemma 4. Let $E \subset \mathbb{R}^d$ be an ellipsoidal set. Then $E$ has an extremal point and $E \setminus \{v\}$ is also ellipsoidal for every extremal point $v \in E$.

Proof. Let $E = \text{conv}(E) \cap \mathbb{Z}^d$ for an ellipsoid $E \subset \mathbb{R}^d$. Applying an appropriate homothetic contraction, centered at the center of $E$, we can always achieve $E \cap E \neq \emptyset$. In particular, $E$ has an extremal point. For $v \in E \cap E$, after changing $E$ to its homothetic image with factor $(1 + \varepsilon)$ and centered at $v$, where $\varepsilon$ is a sufficiently small positive real number, we can further assume $E \cap E = \{v\}$. Finally, applying a parallel translation to $E$ by $\delta(z - v)$, where $z$ is the center of $E$ and $\delta > 0$ is a sufficiently small real number, we achieve $\text{conv}(E) \cap \mathbb{Z}^d = E \setminus \{v\}$.

Next we complete the proof of Theorem (b). It follows from Lemma 3.2 that, for any natural number $d$ and an ellipsoidal set $E \subset \mathbb{Z}^d$, there is a descending sequence of ellipsoidal sets of the form

$$E = E_k \supset E_{k-1} \supset \cdots \supset E_1,$$

with $\#E_i = i$ for $i = 1, \ldots, k$.

By [9, Theorem 6.5(c)], for $d = 3$, the $\text{conv}(E_i)$ are normal polytopes. Therefore, $\ast \leq \text{conv}(E)$ in $\text{NPol}(3)$ for some $\ast \in \mathbb{Z}^3$. Thus Lemma 3 applies.

3.2 Alternative algorithmic proof in symmetric case

For the ellipsoids $E$ with center in $\frac{1}{2}\mathbb{Z}^3$, there is a different proof of Theorem (b). It yields a simple algorithm for constructing a unimodular cover of $P(E)$.

Instead of Theorem 1 and [9, Theorem 6.5] this approach uses Johnson’s 1916 Circle Theorem [13, 14]. We only need Johnson’s theorem to derive the following fact, which does not extend to higher dimensions: for any lattice $\Lambda \subset \mathbb{R}^2$ and any ellipse $E' \subset \mathbb{R}^2$, such that $\text{conv}(E')$ contains a triangle with vertices in $\Lambda$, every parallel translate $\text{conv}(E') + v$, where $v \in \mathbb{R}^2$, meets $\Lambda$.

Assume an ellipsoid $E \subset \mathbb{R}^3$ has center in $\frac{1}{2}\mathbb{Z}^3$ and $\dim(P(E)) = 3$ (notation as in the theorem). Assume $U(P(E)) \not\subseteq P(E)$. Because $\partial P(E)$ is triangulated by unimodular
triangles, there is a unimodular triangle \( T \subset P(E) \), not necessarily in \( \partial P(E) \), and a point \( x \in \text{int} \, T \), such that the points in \([0, x]\), sufficiently close to \( x \), are not in \( U(P(E)) \). For the plane, parallel to \( T \) on lattice height 1 above \( T \) and on the same side as 0, the intersection \( E' = \text{conv}(E) \cap H \) is at least as large as the intersection of \( \text{conv}(E) \) with the affine hull of \( T \): a consequence of the fact that \( P(E) \cap \mathbb{Z}^3 \) is symmetric relative to the center of \( E \). The mentioned consequence of Johnson’s theorem implies that \( \text{conv}(E') \) contains a point \( z \in \mathbb{Z}^3 \). In particular, all points in \([x, 0]\), sufficiently close to \( x \) are in the unimodular simplex \( \text{conv}(T, z) \subset P(E) \), a contradiction.

### 4 High dimensional ellipsoids

For a lattice \( \Lambda \subset \mathbb{R}^d \), define a \( \Lambda \)-polytope as a polytope \( P \subset \mathbb{R}^d \) with \( \text{vert}(P) \subset \Lambda \). Using \( \Lambda \) as the lattice of reference instead of \( \mathbb{Z}^d \), one similarly defines \( \Lambda \)-normal polytopes and \( \Lambda \)-ellipsoidal sets.

Consider the lattice \( \Lambda(d) = \mathbb{Z}^d + \mathbb{Z} \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \subset \mathbb{R}^d \). We have \([\mathbb{Z}^d : \Lambda(d)] = 2\). Consider the \( \Lambda(d) \)-polytope \( P(d) = \text{conv} \left( B(d) \cap \Lambda(d) \right) \), where \( B(d) = \{ (\xi_1, \ldots, \xi_d) \mid \sum_{i=1}^d (\xi_i - \frac{1}{2})^2 \leq \frac{d}{4} \} \subset \mathbb{R}^d \), i.e., \( \partial(B(d)) \) is the circumscribed sphere for the cube \([0, 1]^d\).

Consider the \( d \)-dimensional \( \Lambda(d) \)-polytope and the \((d - 1)\)-dimensional \( \Lambda(d) \)-simplex:

\[
Q(d) = \text{conv} \left( \left( P(d) \cap \Lambda(d) \right) \setminus \{ e_1 + \cdots + e_d \} \right),
\]

\[
\Delta(d-1) = \text{conv} \left( e_1 + \cdots + e_{i-1} + e_{i+1} + \cdots + e_d \mid i = 1, \ldots, d \right),
\]

where \( e_1, \ldots, e_d \in \mathbb{R}^d \) are the standard basic vectors.

**Notice.** Although \( P(d) \cap \mathbb{Z}^d = \{0, 1\}^d \) for all \( d \), yet \([0, 1]^d \subset \not\subset P(d) \) for all \( d \geq 4 \). In fact, \((\frac{1}{2}, \ldots, \frac{1}{2}) + k e_i \in P(d) \cap \Lambda(d) \) for \( 1 \leq i \leq d \) and \(-\left\lfloor \frac{\sqrt{2}}{2} \right\rfloor \leq k \leq \left\lceil \frac{\sqrt{2}}{2} \right\rceil \).

**Lemma 5.** If \( d \geq 5 \) then \( \Delta(d-1) \) is a facet of \( Q(d) \) and \( \Delta(d-1) \cap \Lambda(d) = \text{vert}(\Delta(d-1)) \).

**Proof.** Assume \( x = (\xi_1, \ldots, \xi_d) \in P(d) \cap \Lambda(d) \) satisfies \( \xi_1 + \cdots + \xi_d \geq d-1 \). We claim that there are only two possibilities: either \( x = e_1 + \cdots + e_d \) or \( x = e_1 + \cdots + e_{i-1} + e_{i+1} + \cdots + e_d \) for some index \( i \). Since \( P(d) \cap \mathbb{Z}^d = \{0, 1\}^d \), only the case \( x \in \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) + \mathbb{Z}^d \) needs to be ruled out. Assume \( \xi_i = \frac{1}{2} + a_i \) for some integers \( a_i \), where \( i = 1, \ldots, d \). Then we have the inequalities

\[
\sum_{i=1}^d a_i^2 \leq \frac{d}{4} \quad \text{and} \quad \sum_{i=1}^d a_i \geq \frac{d}{2} - 1.
\]

Since the \( a_i \) are integers we have \( \frac{d}{4} \geq \frac{d}{2} - 1 \), a contradiction because \( d \geq 5 \). \( \square \)

**Lemma 6.** For every even natural number \( d \geq 6 \), there exists a point in \( \left( \frac{d}{2} : Q(d) \right) \cap \Lambda(d) \) which does not have a representation of the form \( x_1 + \cdots + x_2 \) with \( x_1, \ldots, x_2 \in Q(d) \cap \Lambda(d) \). In particular, \( Q(d) \) is not \( \Lambda(d) \)-normal.
Assume the baricenter \( \beta(d) = \frac{d-1}{d} \cdot (e_1 + \cdots + e_d) \) of \( \Delta(d-1) \). The point \( \frac{d}{2} \cdot \beta(d) \) is the baricenter of the dilated simplex \( \frac{d}{2} \cdot \Delta(d-1) \) and, simultaneously, a point in \( \Lambda(d) \). Assume \( \frac{d}{2} \cdot \beta = x_1 + \cdots + x_d \) for some \( x_1, \ldots, x_d \in Q(d) \cap \Lambda(d) \). Lemma 5 implies \( x_1, \ldots, x_d \in \Delta(d-1) \cap \Lambda(d) = \text{vert}(\Delta(d-1)) \). But this is not possible because the dilated \( (d-1) \)-simplex \( c\Delta(d-1) \) has an interior point of the form \( z_1 + \cdots + z_c \) with \( z_1, \ldots, z_c \in \text{vert}(\Delta(d-1)) \) only if \( c \geq d \).

**Proof of Theorem (c).** Since \( e_1, \ldots, e_d, (\frac{1}{2}, \ldots, \frac{1}{2}) \in Q(d) \) we have the equality \( \text{gp}(Q(d)) = \Lambda(d) \). By Lemmas 4 and 5, the set \( Q(d) \cap \Lambda(d) \) is \( \Lambda(d) \)-ellipsoidal for \( d \geq 5 \). By applying a linear transformation, mapping \( \Lambda(d) \) isomorphically to \( \mathbb{Z}^d \), Lemma 6 already implies Theorem (c) for \( d \) even.

One involves all dimensions \( d \geq 6 \) by observing that (i) if \( \mathcal{E} \subset \mathbb{R}^d \) is an ellipsoidal set then \( \mathcal{E} \times \{0,1\} \subset \mathbb{R}^{d+1} \) is also ellipsoidal and (ii) the normality of \( \text{conv}(\mathcal{E} \times \{0,1\}) \) implies that of \( \text{conv}(\mathcal{E}) \). While (ii) is straightforward, for (i) one applies an appropriate affine transformation to achieve \( \mathcal{E} = \text{conv}(S^{d-1}) \cap \Lambda \), where \( S^{d-1} \subset \mathbb{R}^d \) is the unit sphere, and \( \Lambda \subset \mathbb{R}^d \) is a shifted lattice. In this case the ellipsoid \( E = \{ (\xi_1, \ldots, \xi_d) \mid \frac{\xi_1^2}{a^2} + \cdots + \frac{\xi_{d-1}^2}{a^2} + \frac{(\xi_d+1)^2}{b^2} = 1 \} \subset \mathbb{R}^{d+1} \) with \( b > \frac{1}{2} \) and \( a = \frac{2b}{\sqrt{d^2-1}} \), is within the \( (b - \frac{1}{2}) \)-neighborhood of the region of \( \mathbb{R}^{d+1} \) between the hyperplanes \( \mathbb{R}^d, 0 \) and \( \mathbb{R}^d, 1 \) and satisfies the following conditions: \( E \cap \mathbb{R}^d = (S^{d-1}, 0) \) and \( E \cap (\mathbb{R}^d, 1) = (S^{d-1}, 1) \). In particular, when \( \frac{1}{2} < b < \frac{3}{2} \) we have \( \text{conv}(E) \cap (\Lambda \times \mathbb{Z}) = \mathcal{E} \times \{0,1\} \).

**Remark 7.** The definition of a normal polytope in the introduction is stronger than the one in [7, Definition 2.59]: the former is equivalent to the notion of an *integrally closed* polytope, whereas ‘normal’ in the sense of [7] is equivalent to \( \text{gp}(P) \)-normal. Examples of \( \text{gp}(P) \)-normal polytopes, which are not normal, are lattice non-unimodular simplices, whose only lattice points are the vertices. Lemma 5 and the proof of Lemma 6 show that the 5-simplex \( \Delta(5) \) is not \( \Lambda(6) \)-unimodular. Applying an appropriate affine transformation we obtain a lattice non-unimodular simplices \( \Delta' \subset \mathbb{R}^5 \) with \( \text{vert}(\Delta') \) ellipsoidal. Such examples in \( \mathbb{R}^5 \) have been known since the 1970s: a construction of Voronoi [2] yields a lattice \( \Lambda \subset \mathbb{R}^5 \) and a 5-simplex \( \Delta \subset \mathbb{R}^5 \) of \( \Lambda \)-multiplicity 2, whose circumscribed sphere does not contain points of \( \Lambda \) inside except \( \text{vert}(\Delta) \).

We do not know whether there are ellipsoidal subsets \( \mathcal{E} \subset \mathbb{R}^5 \) with \( \text{conv}(\mathcal{E}) \) non-normal and \( \text{gp}(\text{conv}(\mathcal{E})) = \mathbb{Z}^5 \). For instance, \( Q(5) \) is \( \Lambda(5) \)-normal, as checked by Normaliz [10].

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