Algebraic decoupling of variables for systems of ODEs of quasipolynomial form

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Abstract

A generalization of the reduction technique for ODEs recently introduced by Gao and Liu is given. It is shown that the use of algebraic methods allows the extension of the procedure to much more general flows, as well as the derivation of simple criteria for the identification of reducible systems.

Keywords: Ordinary differential equations, integrability, algebraic methods, reduction techniques.

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1. Introduction

The problem of finding first integrals and identifying integrability conditions of dynamical systems has deserved a continued effort along many decades. The relevance and interest of the problem is reflected in the number of different procedures developed for these purposes, such as the Lie symmetries method [1], Carleman embedding [2], Prelle-Singer procedure [3] or Painlevé test [4], to cite a sample. However, none of the presently known methods can account for the problem in its full generality, and only partial answers have been developed.

Among the different analytic tools available, the Quasipolynomial (QP) formalism for ODEs has received increasing attention in the last years. This interest was initially centered in integrability properties [5]–[11] and canonical forms [12, 13], but applications are also starting to reach different fields such as chemical kinetics [14], theoretical biochemistry [15], normal forms [16] and Hamiltonian systems [17].

In a recent article [18], Gao and Liu have applied a changing variables method (CVM from now on) in order to find first integrals of 3D quadratic systems of Lotka-Volterra form. This is done by decoupling one of the variables of the initial 3D flow, thus reducing the effective dimension of the system in one unit. Analysis of integrability conditions and identification of first integrals is thus a much simpler task in the reduced 2D system.

It is worth noting that most transformations employed in [18] find their place in a natural way within the QP formalism. In this work we explore the consequences arising from this fact. As Gao and Liu, we shall be primarily concerned with the possibility of reducing a flow into a two-dimensional one. In addition to the possibility of finding first integrals and integrability conditions already mentioned in [18], it should be added that knowledge that a system of dimension three or higher can be reduced into a two-dimensional one is interesting in itself because it excludes the possibility of chaotic behavior—a problem to which a considerable effort has been devoted recently [19] in the case of 3D systems.

We shall demonstrate that the CVM can be completely reformulated in terms of the QP formalism. This has four major implications:

1) The first is that the procedure can be made more systematic and simpler, since all manipulations can be carried out easily in terms of matrix algebra.

2) The second is that the use of the QP formalism allows generalizing the
scope of the procedure. Generically, the method allows a reduction of one unit in the effective dimension of an \( n \)-dimensional system, with arbitrary \( n \). This makes the technique particularly interesting for the reduction of 3D flows into two-dimensional ones, thus precluding the existence of irregular motion. Most examples will accordingly be on standard 3D systems. However, such reduction into a two-dimensional flow may also be possible for some \( n \)-dimensional systems (an example is given in Subsection 3.3). Moreover, we shall demonstrate that the procedure is not limited to Lotka-Volterra quadratic systems, but is equally valid for flows with much more general nonlinearities.

\( iii) \) The third is that the use of matrix algebra leads to simple criteria for the identification of reducible systems.

\( iv) \) The fourth is that some of the CVM transformations are just particular cases of wider transformation families that we characterize. We shall see in the examples that, in some cases, different members of those families are preferable to the CVM choice.

Before describing the reduction technique, it is convenient to give a short account of some relevant features of the QP formalism.

2. Transformations on QP systems

We shall begin by briefly recalling those basic properties of QP equations that will be necessary in what is to come. We refer the reader to the cited literature for further details.

The starting point of the formalism are QP systems of ODE’s of the form:

\[
\dot{x}_i = x_i \left( \lambda_i + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_{B_{jk}} \right), \quad i = 1 \ldots n, \quad m \geq n
\] (1)

where \( n \) and \( m \) are positive integers, and \( A, B \) and \( \lambda \) are \( n \times m \), \( m \times n \) and \( n \times 1 \) real matrices, respectively. In what follows, \( n \) will always denote the number of variables of a QP system, and \( m \) the number of quasimonomials:

\[
\prod_{k=1}^{n} x_{B_{jk}}^{B_{jk}}, \quad j = 1 \ldots m
\] (2)
We will assume that \( m \geq n \) and that \( B \) is of maximal rank, i.e. \( \text{rank}(B) = n \). If \( m < n \) and/or \( \text{rank}(B) \) is not maximal, then it can be shown [13] that the system is redundant and can always be reduced to the standard situation \( m \geq n \) and \( \text{rank}(B) = n \), which is our starting assumption.

QP equations (1) are form-invariant under quasimonomial transformations (QMTs):

\[
x_i = \prod_{k=1}^{n} y^C_{ik}, \quad i = 1, \ldots, n
\]

for any invertible real matrix \( C \). After (3), matrices \( B, A, \) and \( \lambda \) change to

\[
B' = B \cdot C, \quad A' = C^{-1} \cdot A, \quad \lambda' = C^{-1} \cdot \lambda,
\]

respectively, but the QP format is preserved.

Quasimonomial transformations are complemented by the new-time transformations (NTTs) of the form [20]:

\[
d\tau = \xi(x_1, \ldots, x_n) \, dt
\]

where \( \xi(x_1, \ldots, x_n) \) is a smooth function. The most important choice for \( \xi \) in the QP formalism is [10]:

\[
\xi(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\beta_i}
\]

where the \( \beta_i \) are real constants. With \( \xi \) given by (6), transformation (5) also preserves the QP format.

Although we will need in certain specific steps of the procedure some additional sets of transformations, it is not necessary to elaborate on them now. Therefore, we can proceed to describe the reduction method. For this, we shall distinguish three cases of increasing complexity.

3. Criteria and algorithms for the reduction of systems

3.1 Case I: \( \lambda = 0 \) and \( m = n \)

In this case we shall see that the reduction in one dimension of the system is always possible. The set of ODEs takes the form:

\[
\dot{x}_i = x_i \left( \sum_{j=1}^{n} A_{ij} \prod_{k=1}^{n} x_k^B_{jk} \right), \quad i = 1 \ldots n
\]
We look for a QMT such that for the new QP flow we have:

\[ B' = B \cdot C = \begin{pmatrix} 1 & B'_{12} & \cdots & B'_{1n} \\ 1 & B'_{22} & \cdots & B'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & B'_{n2} & \cdots & B'_{nn} \end{pmatrix} \]  

(8)

where the columns 2 to \( n \) can be chosen arbitrarily (with the obvious restriction \( \text{rank}(B') = n \)). Note that the CVM choice [18] is a particular case of (8) with \( B'_{ij} = \delta_{ij} \), with \( 2 \leq j \leq n \) and symbol \( \delta \) standing for Kronecker’s delta. Given \( B' \), we immediately find from (8) that \( C \) exists and is unique: \( C = B^{-1} \cdot B' \). Let \( y_i \) denote the variables obtained after the QMT of matrix \( C \). Then we arrive at the following system:

\[ \dot{y}_i = y_1 y_i \left( \sum_{j=1}^{n} A'_{ij} \prod_{k=2}^{n} y_{B'_{jk}} \right), \quad i = 1 \ldots n \]  

(9)

We now rescale the time variable by means of the NTT:

\[ d\tau = y_1 dt \]  

(10)

where \( t \) is the old time variable and \( \tau \) the new one. Let us denote from now on the derivative of any function \( \chi(\tau) \) of a new time \( \tau \) as \( d\chi/d\tau \equiv \dot{\chi} \). Then after (10) we are led to:

\[ \dot{y}_i = y_i \left( \sum_{j=1}^{n} A'_{ij} \prod_{k=2}^{n} y_{B'_{jk}} \right), \quad i = 1 \ldots n \]  

(11)

Now notice that the only variables appearing in the r.h.s. of equations 2 to \( n \) of system (11) are precisely \( \{y_2, \ldots, y_n\} \), i.e. \( y_1 \) has been decoupled. Thus the equation for \( y_1 \) in (11) is a quadrature, and system (7) has been reduced to an \( (n-1) \)-dimensional one in the variables \( \{y_2, \ldots, y_n\} \) and the new time \( \tau \).

**Example of Case I: Euler equations for the free rigid body**

As an example of Case I we can choose Euler’s equations for the free rigid body [21] which are given by:

\[ \begin{align*}
\dot{x}_1 &= a_1 x_2 x_3 \\
\dot{x}_2 &= a_2 x_1 x_3 \\
\dot{x}_3 &= a_3 x_1 x_2
\end{align*} \]  

(12)
The QP matrices of system (12) are:
\[
B = \begin{pmatrix}
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{pmatrix},
A = \begin{pmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{pmatrix},
\lambda = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\] (13)
We now apply a QMT of matrix:
\[
C = \begin{pmatrix}
1 & 1/2 & 1/2 \\
1 & 0 & 1/2 \\
1 & 1/2 & 0
\end{pmatrix}
\] (14)
The resulting QP system is characterized by matrices:
\[
B' = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix},
A' = \begin{pmatrix}
-a_1 & a_2 & a_3 \\
2a_1 & -2a_2 & 0 \\
2a_1 & 0 & -2a_3
\end{pmatrix},
\lambda' = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\] (15)
In this case we have chosen \( C \) so as to arrive to a matrix \( B' \) of the form used in [18]. Let \{\( y_1, y_2, y_3 \)\} be the variables of the transformed equations corresponding to matrices (15). If we finally perform the NTT \( d\tau = y_1 dt \) we arrive to:
\[
\begin{align*}
\dot{y}_1 &= y_1(-a_1 + a_2 y_2 + a_3 y_3) \\
\dot{y}_2 &= y_2(2a_1 - 2a_2 y_2) \\
\dot{y}_3 &= y_3(2a_1 - 2a_3 y_3)
\end{align*}
\] (16)
This completes the decoupling procedure. In (16) the equation of \( y_1 \) has been reduced to a quadrature, while the equations for \( y_2 \) and \( y_3 \) do not depend on \( y_1 \). Moreover, the equations for \( y_2 \) and \( y_3 \) are also decoupled from each other, and can be integrated straightforwardly. The integrability of the system is thus made manifest.

3.2 Case II: \( \lambda = 0 \) and \( m > n \)

We now write the system as:
\[
\dot{x}_i = x_i \left( \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_{B_{jk}} \right), \quad i = 1 \ldots n, \quad m > n
\] (17)
We again look for a QMT of matrix \( C \) such that:
\[
B' = B \cdot C = \begin{pmatrix}
1 & B'_{12} & \ldots & B'_{1n} \\
1 & B'_{22} & \ldots & B'_{2n} \\
& \vdots & \ddots & \vdots \\
1 & B'_{m2} & \ldots & B'_{mn}
\end{pmatrix}
\] (18)
As in Case I, columns 2 to \( n \) of \( B' \) in (18) can be chosen freely in such a way that \( \text{Rank}(B') = n \). Now note that matrices \( B \) and \( B' \) are not square but \( m \times n \). This implies that a suitable \( C \) may or may not exist, depending on the form of \( B \). Let us denote by \( \tilde{B} \) the following \( m \times (n + 1) \) matrix:

\[
\tilde{B} \equiv \begin{pmatrix}
B_{11} & B_{12} & \ldots & B_{1n} & 1 \\
B_{21} & B_{22} & \ldots & B_{2n} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
B_{m1} & B_{m2} & \ldots & B_{mn} & 1
\end{pmatrix}
\tag{19}
\]

It is not difficult to prove that \( C \) exists if and only if \( \text{Rank}(\tilde{B}) = n \), and in this case it is unique.

If, according to the matrix criterion \( \text{Rank}(\tilde{B}) = n \), a suitable \( C \) exists, then the rest of the procedure is completely similar to Case I: We first perform the QMT of matrix \( C \). Let \( \{y_1, \ldots, y_n\} \) be the new variables of the transformed system. Then we can factor out \( y_1 \) in each of the equations, and eliminate it later by means of the NTT \( d\tau = y_1 dt \). The result is the decoupling of \( y_1 \) and the reduction in one unit of the effective dimension of the vector field.

**Example of Case II: Halphen system**

We now consider the Halphen equations [22, 23] which describe the two-monopole system. We shall write them in the form given in [24]:

\[
\begin{align*}
\dot{x}_1 &= x_2 x_3 - x_1 x_2 - x_1 x_3 \\
\dot{x}_2 &= x_1 x_3 - x_1 x_2 - x_2 x_3 \\
\dot{x}_3 &= x_1 x_2 - x_1 x_3 - x_2 x_3
\end{align*}
\tag{20}
\]

In QP terms we have:

\[
B = \begin{pmatrix}
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad A = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & -1 & 0
\end{pmatrix}, \quad \lambda = 0 \tag{21}
\]

It is a simple task to check that \( \text{rank}(\tilde{B}) = 3 \) when \( B \) is given by (21). Consequently, the system can be reduced. For this purpose we may choose a QMT of
matrix:

\[
C = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix}
\]  

(22)

After the QMT, the first variable of the transformed system, say \( y_1 \), can be factored out. If we then eliminate it by means of the NTT (10) the result is:

\[
\begin{align*}
\hat{y}_1 &= y_1(y_2y_3 - y_2 - y_3) \\
\hat{y}_2 &= -y_2 + y_2^2 - y_2y_3 + y_3 \\
\hat{y}_3 &= -y_3 + y_3^2 - y_2y_3^2 + y_2
\end{align*}
\]

(23)

As expected, the first variable is reduced to quadrature and the study of system (20) is reduced to a 2D flow corresponding to the equations for \( y_2 \) and \( y_3 \) in (23).

3.3 Case III: \( \lambda \neq 0 \)

We now focus on QP systems of the general form:

\[
\dot{x}_i = x_i \left( \lambda_i + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \right), \quad i = 1 \ldots n, \quad m \geq n
\]

(24)

The line of action now consists in reducing the problem to either Case I or II, i.e. in suppressing the \( \lambda \) terms. For this we first introduce the following new variables:

\[
y_i = e^{-\lambda_i t} x_i, \quad i = 1 \ldots n
\]

(25)

Transformation (25) is similar to the one employed in [18], and was also used in [6] in connection with the construction of canonical forms for ODEs. We find:

\[
\dot{y}_i = y_i \left( \sum_{j=1}^{m} A_{ij} e^{F_{ij}t} \prod_{k=1}^{n} y_k^{B_{jk}} \right), \quad i = 1 \ldots n
\]

(26)

where \( \Gamma = B \cdot \lambda \). In order to reduce (26) to Cases I or II, the following condition is sufficient:

\[
\Gamma_1 = \Gamma_2 = \ldots = \Gamma_m = \gamma
\]

(27)

Provided (27) holds, we can perform the transformation:

\[
d\tau = e^{\gamma t} dt
\]

(28)
We finally arrive to the system:

\[ \dot{y}_i = y_i \left( \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} y_{B_{jk}^i} \right), \quad i = 1 \ldots n \]  

(29)

Thus, if condition (27) is satisfied, equations (24) can be reduced to system (29), which corresponds to Case I if \( m = n \) and to Case II if \( m > n \).

This completes our enumeration of criteria and reduction algorithms. We now give two examples corresponding to this last situation.

A first example of Case III: Maxwell-Bloch system

As an example we may consider the Maxwell-Bloch equations for laser systems. In the case of periodic boundary conditions, the equations are [25]:

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 \\
\dot{x}_2 &= -a_3 x_2 + a_2 x_1 x_3 \\
\dot{x}_3 &= -a_4 (x_3 - x_{30}) - 4a_2 x_1 x_2
\end{align*}
\]

(30)

The QP matrices are:

\[
B = \begin{pmatrix}
-1 & 1 & 0 \\
1 & -1 & 1 \\
1 & 1 & -1 \\
0 & 0 & -1
\end{pmatrix}, \quad A = \begin{pmatrix}
a_2 & 0 & 0 & 0 \\
0 & a_2 & 0 & 0 \\
0 & 0 & -4a_2 & a_4 x_{30}
\end{pmatrix}, \quad \lambda = \begin{pmatrix}
-a_1 \\
-a_3 \\
-a_4
\end{pmatrix}
\]

(31)

We first compute \( \Gamma \):

\[
\Gamma = \begin{pmatrix}
a_1 - a_3 \\
-a_1 + a_3 - a_4 \\
-a_1 - a_3 + a_4 \\
a_4
\end{pmatrix}
\]

(32)

Let us look at the compatibility condition (27) for \( \Gamma \) in (32). If we impose \( \Gamma_1 = \Gamma_2 = \Gamma_3 \) we immediately find:

\[ 2a_1 = a_3 = a_4 \]  

(33)

However, it is not possible to simultaneously verify the last requirement \( \Gamma_4 = \Gamma_i \), for \( i = 1, 2, 3 \). Then, system (30) cannot be reduced in general. However, we can follow Gümral and Nutku [24] and consider the case in which \( x_{30} = 0 \) in
equations (30). The resulting system is given by the following QP matrices:

\[
B = \begin{pmatrix}
-1 & 1 & 0 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{pmatrix}, \quad A = \begin{pmatrix}
a_2 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & -4a_2
\end{pmatrix}, \quad \lambda = \begin{pmatrix}
-a_1 \\
-a_3 \\
-a_4
\end{pmatrix}
\] (34)

Now condition (27) is satisfied iff (33) holds. The parameter values that we have obtained in order to verify equation (27)

\[
2a_1 = a_3 = a_4, \quad x_{30} = 0,
\] (35)

are precisely those found after some ad hoc transformations by Gümral and Nutku in [24] when characterizing the values of the parameters for which the Maxwell-Bloch system (30) is bi-Hamiltonian. Notice that such parameter values arise here in a natural way and allow the identification of these integrable cases.

Then, in what follows we will write in (34):

\[
a_1 \equiv \alpha, \quad a_3 = a_4 \equiv 2\alpha
\] (36)

Thus we can perform transformation (25):

\[
y_1 = e^{\alpha t}x_1, \quad y_2 = e^{2\alpha t}x_2, \quad y_3 = e^{2\alpha t}x_3,
\] (37)

and then the change \(d\tau = e^{-\alpha t}dt\). The result is:

\[
\dot{y}_1 = a_2 y_2 \\
\dot{y}_2 = a_2 y_1 y_3 \\
\dot{y}_3 = -4a_2 y_1 y_2
\] (38)

The QP matrices \(A'\) and \(B'\) of system (38) coincide, respectively, with \(A\) and \(B\) in (34), while now \(\lambda' = 0\). Since we have \(m = n = 3\) in (38), equations (30) have been reduced to Case I of the algorithm. As we know from Subsection 3.1, the reduction to a 2D flow is always possible in this case.

According to the procedure for Case I, we first apply to system (38) a QMT of matrix:

\[
C = \begin{pmatrix}
1 & 1/2 & 1/2 \\
2 & 1/2 & 1/2 \\
2 & 1 & 0
\end{pmatrix}
\] (39)

Let \(\{z_1, z_2, z_3\}\) be the variables of the transformed system. The last step is a NTT \(d\tau = a_2 z_1 dt\). The final result is:
\[ \dot{z}_1 = z_1(z_2 - 1) \]
\[ \dot{z}_2 = 2z_2(1 - z_2 - 2z_3) \]  \hspace{1cm} (40)
\[ \dot{z}_3 = 2z_3(1 + 2z_3) \]

Then the first variable is decoupled and we obtain a reduced 2D system. Note also that the equation for \( z_3 \) is directly integrable, so the whole system is, in fact, reduced to a one-dimensional problem.

**An n-dimensional example: Riccati projective systems**

We conclude the examples with the Riccati projective equations which have recently deserved some attention in different areas, such as selection dynamics [26] or normal forms [16]. These systems are given by:

\[ \dot{x}_i = \lambda_i x_i + x_i \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1 \ldots n \]  \hspace{1cm} (41)

We can follow the steps given in Case III and evaluate \( \Gamma \). Thus we could simplify the system provided condition (27) is satisfied, i.e. \( \lambda_1 = \ldots = \lambda_n \).

We shall not proceed according to this line of action, however. Instead, we shall demonstrate that the techniques described above allow solving equations (41) in general. Since \( \lambda \neq 0 \) in (41) we start, as usual, by applying transformation (25):

\[ y_i = e^{-\lambda_it}x_i, \quad i = 1 \ldots n \]  \hspace{1cm} (42)

The result is:

\[ \dot{y}_i = \sum_{j=1}^{n} a_{ij}e^{\lambda_j t}y_j, \quad i = 1 \ldots n \]  \hspace{1cm} (43)

In general, we cannot factor out the exponentials in equation (43). In other words, relations (27) will not be usually satisfied. However, this is not an unavoidable difficulty in the case of system (43): We can anyhow perform a QMT of the form (8) described in Subsection 3.1. The best possibility can be easily seen to be:

\[ C = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & -1 & 0 & \ldots & 0 \\
1 & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & -1
\end{pmatrix} \]  \hspace{1cm} (44)
Notice that this QMT corresponds to a different choice to the one considered in [18]. When we perform the QMT of matrix (44) on equations (43) the result is:

\[
\dot{z}_1 = z_1^2 \left( a_1 e^{\lambda_1 t} + \sum_{j=2}^{n} \frac{a_j e^{\lambda_j t}}{z_j} \right)
\]  
(45)

and

\[
\dot{z}_i = 0 , \quad i = 2 \ldots n
\]  
(46)

The outcome is that, in its final form (45)–(46), the integrability of system (41) is made completely explicit —in fact, equations (45)–(46) can be integrated trivially.

4. Final remarks

The QP formalism provides the natural operational framework for the changing variables method as given in [18]: Not only allows its reformulation in simpler matrix terms, but also leads naturally to extensions, for example to the case of nonquadratic flows. The use of matrix algebra has also made possible the derivation of some simple criteria for the identification of reducible systems —criteria which are quite convenient for practical purposes.

It is worth insisting that this kind of approach should be especially appropriate in the context of 3D sets of ODEs. However, our treatment has been completely general in what concerns to the dimension of the system, since the possibility of finding higher-dimensional applications cannot be excluded, as our last example illustrates. In any case, the final goal has always been the reduction into a 2D flow: When this is possible, chaotic dynamics is precluded, and further analysis (on parameter space, for instance) can be carried out in much simpler terms.

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