General fractional Sobolev space with variable exponent and applications to nonlocal problems

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Abstract
In this paper, we extend the fractional Sobolev spaces with variable exponents $W^{s,p(x,y)}$ to include the general fractional case $W^{s,p(x,y)}_K$, where $p$ is a variable exponent, $s \in (0,1)$ and $K$ is a suitable kernel. We are concerned with some qualitative properties of the space $W^{s,p(x,y)}_K$ (completeness, reflexivity, separability, and density). Moreover, we prove a continuous and a compact embedding theorem of these spaces into variable exponent Lebesgue spaces. As applications, we discuss the existence of a nontrivial solution for a nonlocal $p(x,\cdot)$-Kirchhoff type problem. Further, we establish the existence and uniqueness of a solution for a variational problem involving the integro-differential operator of elliptic type $L^{p(x,\cdot)}_K$.

Keywords  Generalized fractional Sobolev spaces · Nonlocal and integro-differential operators · $p(x,\cdot)$-Kirchhoff type problems · Mountain pass theorem · Minty–Browder theorem

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1 Introduction

Our main goal in this paper is to extend the fractional Sobolev spaces with variable exponent to cover the nonlocal general case with singular kernel. For this, we begin this work by remembering the definition of fractional Sobolev spaces with variable exponent, see for instance [4, 5, 13, 17].

Let $\Omega$ be a smooth bounded open set in $\mathbb{R}^N$. We start by fixing $s \in (0, 1)$ and let $p : \overline{\Omega} \times \overline{\Omega} \rightarrow (1, +\infty)$ be a continuous bounded function such that

$$1 < p^- = \min_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x, y) \leq p(x, y) \leq p^+ = \max_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x, y) < +\infty,$$

and

$p$ is symmetric, that is, $p(x, y) = p(y, x)$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$. 

We denote by :

$$\bar{p}(x) = p(x, x) \quad \text{for all} \quad x \in \overline{\Omega}.$$

We define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows

$$W = W^{s, p(x,y)}(\Omega) = \left\{ u \in L^{\bar{p}}(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{\bar{p}(x,y)}}{\bar{p}(x,y)|x-y|^{\bar{p}(x,y)+N}} \, dx \, dy < +\infty, \text{ for some } \lambda > 0 \right\},$$

where $L^{\bar{p}}(\Omega)$ is the Lebesgue space with variable exponent, (see Sect. 2).

The space $W^{s, p(x,y)}(\Omega)$ is a Banach space (see [13]) if it is endowed with the norm

$$\|u\|_{W^{s, p(x,y)}(\Omega)} = \|u\|_W = \|u\|_{L^{\bar{p}}(\Omega)} + [u]_{x, p(x,y)},$$

where $[.,.]_{x, p(x,y)}$ is a Gagliardo semi-norm with variable exponent which is defined by

$$[u]_{x, p(x,y)} = \inf \left\{ \lambda > 0 : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{p(x,y)+N}} \, dx \, dy \leq 1 \right\}.$$

The space $(W, \|\cdot\|_W)$ is separable and reflexive, see ( [6, Lemma 3.1]).

Let us consider the fractional $p(x, .)$-Laplacian operator given by

$$(-A_{p(x,.)})^s u(x) = p.v. \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x-y|^{n+sp(x,y)}} \, dy \quad \text{for all } x \in \Omega,$$

where $p.v.$ is a commonly used abbreviation in the principal value sense. One typical feature of this operator is the nonlocality, in the sense that the value of $(-A_{p(x,.)})^s u(x)$ at any point $x \in \Omega$ depends not only on the values of $u$ on $\Omega$, but actually on the entire space $\mathbb{R}^N$. 

\[ \text{Birkhäuser} \]
Note that the operator $(-\Delta_{p(x,\cdot)})^s$ is the fractional version of well known $p(x)$-Laplacian operator $A_{p(x)}u(x) = \text{div}\left( |\nabla u(x)|^{p(x)-2} u(x) \right)$. On the other hand, we remark that in the constant exponent case it is known as the fractional $p$-Laplacian operator $(-\Delta)^s$. This nonlinear operator is consistent, up to some normalization constant depending upon $N$ and $s$, with the linear fractional Laplacian $(-\Delta)^s$ in the case $p = 2$. The interest in this last operator and more generally pseudo-differential operators has constantly increased over the last few years, although such operators have been a classical topic of functional analysis since long ago. Nonlocal operators such as $(-\Delta)^s$ and its generalisation $\mathcal{L}_K$ (see for instance [16, 18, 22–24]) naturally arise in continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastical stabilization of Lévy processes, see e.g. [11, 20, 21]. We refer the reader to [8, 14] and to the references included for a self-contained overview of the basic properties of fractional Sobolev spaces and fractional Laplacian operator.

Now, we introduce the nonlocal integro-differential operator of elliptic type $\mathcal{L}_K^{p(x,\cdot)}$ that generalizes the operator $(-\Delta_{p(x,\cdot)})^s$ as follows

$$
\mathcal{L}_K^{p(x,\cdot)}(u(x)) = p.v. \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))K(x,y)dy
$$

$$
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))K(x,y)dy
$$

for all $x \in \mathbb{R}^N$, where $p.v.$ is a commonly used abbreviation in the principal value sense, $p : \mathbb{R}^N \times \mathbb{R}^N \to (1, +\infty)$ is a continuous bounded function satisfies (1.1), (1.2) on $\mathbb{R}^N \times \mathbb{R}^N$ and

$$
p((x,y) - (z,z)) = p(x,y), \quad \text{for all } (x,y), (z,z) \in \mathbb{R}^N \times \mathbb{R}^N. \quad (1.3)
$$

The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ is a measurable function with the following properties:

$$
K(x,y) = K(y,x) \quad \text{for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N, \quad (1.4)
$$

there exists $k_0 > 0$ such that

$$
K(x,y) \geq k_0 |x - y|^{-(N+sp(x,y))} \quad \text{for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } x \neq y, \quad (1.5)
$$

$$
mK \in L^1(\mathbb{R}^N \times \mathbb{R}^N), \quad \text{where } m(x,y) = \min\left\{ 1, |x - y|^{p(x,y)} \right\}. \quad (1.6)
$$

A typical example for $K$ is given by the singular kernel $K(x,y) = |x - y|^{-(N+sp(x,y))}$. In this case $\mathcal{L}_K^{p(x,\cdot)} = (-\Delta_{p(x,\cdot)})^s$. Another example for $K$ is given by the kernel

$$
K_1(x,y) = |x - y|^{-(N+sp(x,y))}a(x-y),
$$
where \( a : \mathbb{R}^N \to [1, +\infty) \) is a function which bounded below. Hence, it is easy to see that \( K_1 \) satisfies the assumptions (1.4)–(1.6).

It is worth to mention that the assumption 1.3 will be used just in Lemma 10.

This paper is organized as follows. In Sect. 2, we give some definitions and fundamental properties of the spaces \( L^{q(x)} \) and \( W^{s,p(x,y)}_K \). In Sect. 3, we compare the space \( W^{s,p(x,y)}_K \) with \( W^{s,p(x,y)}_K \) and we study the completeness, reflexivity, separability, and density of these spaces. Moreover, we prove a continuous and a compact embedding theorem of these spaces into variable exponent Lebesgue spaces. In Sect. 4, we prove some basic properties of the operator \( \mathcal{L}^{p(x,\cdot)}_K \). As applications, in Section 5, we show the existence of a nontrivial solution for a nonlocal \( p(x,\cdot) \)-Kirchhoff type problem by means of the mountain pass theorem. Finally, we apply the Minty–Browder theorem to establish the existence and uniqueness of a solution for a variational problem involving the integro-differential operator \( \mathcal{L}^{p(x,\cdot)}_K \).

2 Some preliminary results

In this section, we recall some useful properties of the variable exponent spaces. For more details we refer the reader to [12, 15, 19], and the references therein.

Consider the set

\[
C_+(\Omega) = \{ q \in C(\Omega) : q(x) > 1, \text{ for all } x \in \Omega \}.
\]

For all \( q \in C_+(\Omega) \), we define \( q^+ = \sup_{x \in \Omega} q(x) \) and \( q^- = \inf_{x \in \Omega} q(x) \) such that

\[
1 < q^- \leq q(x) \leq q^+ < +\infty.
\]

(2.1)

For any \( q \in C_+(\Omega) \), we define the variable exponent Lebesgue space by

\[
L^{q(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_{\Omega} |u(x)|^{q(x)} \, dx < +\infty \right\}.
\]

This vector space endowed with the Luxemburg norm which is defined by

\[
\|u\|_{L^{q(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x)|^{q(x)}}{\lambda} \, dx \leq 1 \right\}
\]

is a separable and reflexive Banach space.

Let \( \tilde{q} \in C_+(\Omega) \) be the conjugate exponent of \( q \), that is, \( \frac{1}{q(x)} + \frac{1}{\tilde{q}(x)} = 1 \). Then we have the following Hölder-type inequality

**Lemma 1** (Hölder’s inequality). If \( u \in L^{q(x)}(\Omega) \) and \( v \in L^{\tilde{q}(x)}(\Omega) \), so

\[
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{q(x)} + \frac{1}{\tilde{q}(x)} \right) \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{\tilde{q}(x)}(\Omega)} \leq 2 \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{\tilde{q}(x)}(\Omega)}.
\]
A very important role in manipulating the generalized Lebesgue spaces with variable exponent is played by the modular \( \rho_{q(\cdot)} : L^{q(\cdot)}(\Omega) \rightarrow \mathbb{R} \) which is defined by

\[
\rho_{q(\cdot)}(u) = \int_{\Omega} |u(x)|^{q(x)} dx.
\]

**Proposition 1** Let \( u \in L^{q(\cdot)}(\Omega) \) and \( \{u_k\} \subset L^{q(\cdot)}(\Omega), k \in \mathbb{N}, \) then we have,

(i) \( \|u\|_{L^{q(\cdot)}(\Omega)} < 1 \) (resp \( 1 \geq \|u\|_{L^{q(\cdot)}(\Omega)} < 1 \)) \( \Leftrightarrow \rho_{q(\cdot)}(u) < 1 \) (resp \( 1 > 1 \)),

(ii) \( \|u\|_{L^{q(\cdot)}(\Omega)} < 1 \) \( \Rightarrow \|u\|_{L^{q(\cdot)}(\Omega)}^{q^+(\cdot)} \leq \rho_{q(\cdot)}(u) \leq \|u\|_{L^{q(\cdot)}(\Omega)}^{q^-(\cdot)}, \)

(iii) \( \|u\|_{L^{q(\cdot)}(\Omega)} > 1 \) \( \Rightarrow \|u\|_{L^{q(\cdot)}(\Omega)}^{q^-(\cdot)} \leq \rho_{q(\cdot)}(u) \leq \|u\|_{L^{q(\cdot)}(\Omega)}^{q^+(\cdot)}. \)

(iv) \( \lim_{k \to +\infty} \|u_k - u\|_{L^{q(\cdot)}(\Omega)} = 0 \iff \lim_{k \to +\infty} \rho_{q(\cdot)}(u_k - u) = 0. \)

From Theorems 1.6, 1.8 and 1.10 in [15], we obtain the following proposition:

**Proposition 2** Suppose that (2.1) is satisfied. If \( \Omega \) is a bounded open domain, then \( (L^{q(\cdot)}(\Omega), \|u\|_{L^{q(\cdot)}(\Omega)}) \) is a uniformly convex (reflexive) and separable Banach space.

**Definition 1** Let \( p : \overline{\Omega} \times \overline{\Omega} \rightarrow (1, +\infty) \) be a continuous variable exponent and \( s \in (0, 1) \). For any \( u \in W \), we define the modular \( \rho_{p^{(\cdot)}} : W \rightarrow \mathbb{R} \) by

\[
\rho_{p^{(\cdot)}}(u) = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx dy + \int_{\Omega} |u(x)|^{\bar{p}(x)} \, dx,
\]

and

\[
\|u\|_{\rho_{p^{(\cdot)}}} = \inf\left\{ \lambda > 0 : \rho_{p^{(\cdot)}}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.
\]

**Remark 1**

(i) It is easy to see that \( \|\cdot\|_{\rho_{p^{(\cdot)}}} \) is a norm on \( W \) which is equivalent to the norm \( \|\cdot\|_W \).

(ii) \( \rho_{p^{(\cdot)}} \) also checks the results of Proposition 1.

**Lemma 2** ([5]) Let \( p : \overline{\Omega} \times \overline{\Omega} \rightarrow (1, +\infty) \) be a continuous variable exponent and \( s \in (0, 1) \). For any \( u \in W_0 \), we have

(i) \( 1 \leq \|u\|_{p^{(\cdot)}, p(x,y)} \Rightarrow \|u\|_{p^{(\cdot)}, p(x,y)}^{p^-(\cdot)} \leq \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx dy \leq \|u\|_{p^{(\cdot)}, p(x,y)}^{p^+(\cdot)}, \)

(ii) \( \|u\|_{p^{(\cdot)}, p(x,y)} \leq 1 \Rightarrow \|u\|_{p^{(\cdot)}, p(x,y)}^{p^+(\cdot)} \leq \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx dy \leq \|u\|_{p^{(\cdot)}, p(x,y)}^{p^-(\cdot)}. \)

In [17], the authors introduced the variable exponent Sobolev fractional space as follows.

\[ \varepsilon \text{ Birkhäuser} \]
\[ E = W^{s,q(x),p(x,y)}(\Omega) \]
\[ = \left\{ u \in L^{q(x)}(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{sp(x,y)+N}} \, dx dy < +\infty \text{ for some } \lambda > 0 \right\}, \]

where \( q : \overline{\Omega} \rightarrow (1, +\infty) \) be a continuous function satisfying (2.1). We would like to mention that the continuous and compact embedding theorem has been proved in [17] under the assumption \( q(x) > \bar{p}(x) = p(x,x) \). The authors in [5] gave a slightly different version of the continuous compact embedding theorem assuming that \( q(x) = \bar{p}(x) = p(x,x) \).

**Theorem 1** ([5]). Let \( \Omega \) be a Lipschitz bounded domain in \( \mathbb{R}^N \) and let \( s \in (0, 1) \). Let \( p : \overline{\Omega} \times \overline{\Omega} \rightarrow (1, +\infty) \) be a continuous function satisfying (1.1) and (1.2) with \( sp^+ < N \). Let \( r : \overline{\Omega} \rightarrow (1, +\infty) \) be a continuous variable exponent such that

\[ 1 < r^- = \min_{x \in \overline{\Omega}} r(x) < r^+ = \frac{N\bar{p}(x)}{N - sp(x)} \text{ for all } x \in \overline{\Omega}. \]

Then, there exists a constant \( C = C(N,s,p,r,\Omega) > 0 \) such that for any \( u \in W \)

\[ \|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_W. \]

That is, the space \( W \) is continuously embedded in \( L^{r(x)}(\Omega) \). Moreover, this embedding is compact.

**Remark 2** Let \( W_0 \) denotes the closure of \( C_0^\infty(\Omega) \) in \( W \), that is,

\[ W_0 = C_0^\infty(\Omega) \|\cdot\|_W. \]

(i) Theorem 1 remains true if we replace \( W \) by \( W_0 \).

(ii) Since \( 1 < p^- \leq \bar{p}(x) < p^+_s(x) \), for any \( x \in \overline{\Omega} \), then Theorem 1 implies that \( \|\cdot\|_{L^{p(x,y)}} \) is a norm on \( W_0 \) which is equivalent to the norm \( \|\cdot\|_W \). So \( (W_0, \|\cdot\|_{L^{p(x,y)}}) \) is a Banach space.

Let us denoted by the \( \mathcal{L} \) the operator associated to \((-A_{p(x,y)})^s\) defined as

\[ \mathcal{L} : W_0 \rightarrow W_0^* \]
\[ u \rightarrow \mathcal{L}(u) : W_0 \rightarrow \mathbb{R} \]
\[ \varphi \rightarrow \langle \mathcal{L}(u), \varphi \rangle \]

such that

\[ \langle \mathcal{L}(u), \varphi \rangle = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp(x,y)}} \, dx dy, \]

where \( \langle \cdot, \cdot \rangle \) denotes the usual duality between \( W_0 \) and its dual space \( W_0^* \).
Lemma 3  ([6]). Assume that hypothesis (1.1) and (1.2) are satisfied and $s \in (0, 1)$. Then, the following assertions hold:

- $\mathcal{L}$ is a bounded and strictly monotone operator.
- $\mathcal{L}$ is a mapping of type $(S_+)$, that is, if $u_k \rightarrow u$ in $W_0$ and $\limsup_{k \rightarrow \infty} (\mathcal{L}(u_k) - \mathcal{L}(u), u_k - u) \leq 0$, then $u_k \rightarrow u$ in $W_0$.
- $\mathcal{L}$ is a homeomorphism.

3 Functional framework

One of the aims of this paper is to study nonlocal problems driven by $\mathcal{L}^{p(x, \cdot)}_K$ and $(-\Delta_{p(x, \cdot)})^s$ with Dirichlet boundary data via variational methods. For this purpose, we need to work in a suitable fractional Sobolev space. For this, we consider a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces in order to correctly encode the Dirichlet boundary datum in the variational formulation.

This section is devoted to the definition of this space as well as to its properties. Further, we will prove a continuous compact embedding theorem of these spaces into variable exponent Lebesgue spaces. Finally, we establish a convergence property for a bounded sequence in $W^{s, p(x, y)}_{K, 0}(\Omega)$.

Let $\Omega$ be a Lipschitz open bounded subset of $\mathbb{R}^N$, $s \in (0, 1)$ be fixed such that $sp^+ < N$. Denote by $Q$ the set

$$Q := \mathbb{R}^N \times \mathbb{R}^N \setminus (C\Omega \times C\Omega), \quad \text{where} \quad C\Omega = \mathbb{R}^N \setminus \Omega.$$ 

Now, due to the nonlocality of the operator $\mathcal{L}^{p(x, \cdot)}_K$ we introduce the general fractional Sobolev space with variable exponent as follows

$$W^{s, p(x, y)}_K(\Omega) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable such that } u|_\Omega \in L^{p(x)}(\Omega) \text{ with } \int_Q \frac{|u(x) - u(y)|^{p(x, y)}}{J^{p(x, y)}(x, y)} K(x, y) \, dx \, dy < +\infty, \text{ for some } \lambda > 0 \right\}.$$ 

The norm in $W^{s, p(x, y)}_K(\Omega)$ can be defined as follows:

$$\|u\|_{W^{s, p(x, y)}_K(\Omega)} = \|u\|_{K, p(x, y)} = \|u\|_{L^{p(x)}(\Omega)} + [u]_{K, p(x, y)},$$

where $[u]_{K, p(x, y)} = \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x, y)}}{J^{p(x, y)}(x, y)} K(x, y) \, dx \, dy \leq 1 \right\}$, (see Lemma 5).

For any $u \in W^{s, p(x, y)}_K(\Omega)$, we define the functional
\[ \rho_{K,p(\cdot)}^o(u) = \int_Q |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy + \int_{\Omega} |u(x)|^{\rho(x)} \, dx. \]

It is easy to see that \( \rho_{K,p(\cdot)}^o \) is a convex modular on \( W^{s,p(x,y)}_K(\Omega) \). The norm associated with \( \rho_{K,p(\cdot)}^o \) is given by

\[ \|u\|_{\rho_{K,p(\cdot)}^o} = \inf \left\{ \lambda > 0 : \rho_{K,p(\cdot)}^o \left( \frac{u}{\lambda} \right) \leq 1 \right\}. \]

Using the same argument as in [12, Theorem 2.17], we prove that \( \|u\|_{\rho_{K,p(\cdot)}^o} \) is a norm on \( W^{s,p(x,y)}_K(\Omega) \) which is equivalent to the norm \( \|u\|_{K,p(x,y)} \). We also define the closed linear subspace of \( W^{s,p(x,y)}_K(\Omega) \) by

\[ W^{s,p(x,y)}_{K,0}(\Omega) = \left\{ u \in W^{s,p(x,y)}_K(\Omega) : u(x) = 0 \text{ a.e. } x \in \mathbb{R}^N \setminus \Omega \right\}. \]

On the other hand, for any \( u \in W^{K,p(x,y)}_0(\Omega) \), we define the functional

\[ \rho_{K,p(\cdot)}^o(u) = \int_Q |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy. \]

\( \rho_{K,p(\cdot)}^o \) is a convex modular on \( W^{K,p(x,y)}_0(\Omega) \). The norm associated with \( \rho_{K,p(\cdot)}^o \) is given by

\[ \|u\|_{\rho_{K,p(\cdot)}^o} = [u]_{K,p(x,y)} = \inf \left\{ \lambda > 0 : \rho_{K,p(\cdot)}^o \left( \frac{u}{\lambda} \right) \leq 1 \right\}. \]

**Remark 3** The modular \( \rho_{K,p(\cdot)}^o \) does not satisfy the triangle inequality, that is,

\[ \rho_{K,p(\cdot)}^o(u + v) \leq \rho_{K,p(\cdot)}^o(u) + \rho_{K,p(\cdot)}^o(v). \]

However, there is a substitute that is sometimes useful.

\[ \rho_{K,p(\cdot)}^o(u + v) \leq 2^{p^* - 1} \left( \rho_{K,p(\cdot)}^o(u) + \rho_{K,p(\cdot)}^o(v) \right). \]

We will refer to this as the modular triangle inequality.

The modular \( \rho_{K,p(\cdot)}^o \) checks the following result which is similar to Proposition 1 and Lemma 2.

**Lemma 4** Let \( p : \mathbb{R}^N \times \mathbb{R}^N \to (1, +\infty) \) be a continuous variable exponent and \( K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty) \) is a measurable function satisfying (1.4) and (1.6). Then

For any \( u \in W^{s,p(x,y)}_{K,0} \), we have

(i) \( 1 \leq [u]_{K,p(x,y)} \Rightarrow [u]_{K,p(x,y)}^{p^*} \leq \rho_{K,p(\cdot)}^o(u) \leq [u]_{K,p(x,y)}^{p^*} \),

(ii) \( [u]_{K,p(x,y)} \leq 1 \Rightarrow [u]_{K,p(x,y)}^{p^*} \leq \rho_{K,p(\cdot)}^o(u) \leq [u]_{K,p(x,y)}^{p^*} \),

(iii) \( \lim_{k \to +\infty} [u_k - u]_{K,p(x,y)} = 0 \iff \lim_{k \to +\infty} \rho_{K,p(\cdot)}^o(u_k - u) = 0. \)
Proof. We prove the first pair of inequalities; the proof of the second is essentially the same. Indeed, it is easy to see that for all $\lambda \in (0, 1)$, we obtain
\[
\lambda^p \rho_{K,p,(\cdot)}^p(u) \leq \rho_{K,p,(\cdot)}^p(\lambda u) \leq \lambda^p \rho_{K,p,(\cdot)}^p(u).
\]
Now, if $[u]_{K,p,(x,y)} > 1$, then $0 < \frac{1}{[u]_{K,p,(x,y)}} < 1$, so we have
\[
\frac{\rho_{K,p,(\cdot)}^p(u)}{[u]_{K,p,(x,y)}^p} \leq \rho_{K,p,(\cdot)}^p \left( \frac{u}{[u]_{K,p,(x,y)}} \right) \leq \frac{\rho_{K,p,(\cdot)}^p(u)}{[u]_{K,p,(x,y)}^p}.
\]
Since $\rho_{K,p,(\cdot)}^p \left( \frac{u}{[u]_{K,p,(x,y)}} \right) = 1$. Then the desired result follows.

The assertion (iii) can be easily obtained from the first and the second assertions. \qed

Lemma 5. $\| \cdot \|_{K,p,(x,y)}$ is a norm on $W_{K,p,(x,y)}^{\alpha}(\Omega)$.

Proof. Since $\| \cdot \|_{L^p(\Omega)}$ is a norm on $L^p(\Omega)$. So we need to prove that:

(i) $\| u \|_{K,p,(x,y)} = 0$ if and only if $u = 0$, and $[\cdot]_{K,p,(x,y)}$ is a semi-norm on $W_{K,p,(x,y)}^{\alpha}(\Omega)$, that is,
(ii) for all $z \in \mathbb{R}$, $[zu]_{K,p,(x,y)} = |z|[u]_{K,p,(x,y)}$,
(iii) $[u + v]_{K,p,(x,y)} \leq [u]_{K,p,(x,y)} + [v]_{K,p,(x,y)}$.

Indeed, For (i), if $u = 0$, then
\[
\int_Q |u(x) - u(y)|^p K(x, y) \, dx \, dy = 0 \quad \text{and} \quad \| u \|_{L^p(\Omega)} = 0.
\]

So, by Lemma 4 we get
\[
[u]_{K,p,(x,y)} = 0 \quad \text{and} \quad \| u \|_{L^p(\Omega)} = 0 \implies \| u \|_{K,p,(x,y)} = 0.
\]

Conversely, if $\| u \|_{K,p,(x,y)} = 0$, then
\[
\| u \|_{L^p(\Omega)} = 0, \quad (3.1)
\]

and
\[
[u]_{K,p,(x,y)} = 0. \quad (3.2)
\]

By (3.1), we have
\[
u = 0 \quad \text{a.e. in } \Omega. \quad (3.3)
\]

Using (3.2), we get
\[
\lambda^* = \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^p K(x,y)}{\lambda^p} \, dx \, dy \leq 1 \right\} = 0.
\]

Let $\lambda_n > 0$, $n \in \mathbb{N}$, such that $\lambda_n$ decreases to $\lambda^*$ and
\[
\int_Q \left| u(x) - u(y) \right|^{p(x,y)} N(x,y) \, dx \, dy \leq 1.
\]

If \( \lambda_n < 1 \), then we infer
\[
\int_Q \left| u(x) - u(y) \right|^{\lambda_n} N(x,y) \, dx \, dy \leq 1.
\]

Hence
\[
\int_Q \left| u(x) - u(y) \right|^{p(x,y)} K(x,y) \, dx \, dy \leq \lambda_n^{-p^*}.
\]

When \( n \to +\infty \), we obtain \( \lambda_n^{p^*} \to (\lambda^*)^{p^*} = 0 \). Thus
\[
0 \leq \int_Q \left| u(x) - u(y) \right|^{p(x,y)} K(x,y) \, dx \, dy \leq 0 \quad \Rightarrow \quad \int_Q \left| u(x) - u(y) \right|^{p(x,y)} K(x,y) \, dx \, dy = 0.
\]

We conclude that \( u(x) = u(y) \) a.e. \((x,y) \in Q\), then \( u = c \in \mathbb{R} \) a.e. in \( \mathbb{R}^N \). Finally, by (3.3) it easily follows that \( c = 0 \), so \( u = 0 \) a.e. in \( \mathbb{R}^N \).

To prove (ii), note that if \( \lambda = 0 \), this follows from (i). Fix \( \lambda \neq 0 \), then by a change of variable, we have
\[
[z u]_{K, p(x,y)} = \inf \left\{ \lambda > 0 : \int_Q \left| \frac{z u(x) - z u(y)}{\lambda} \right|^{p(x,y)} K(x,y) \, dx \, dy \leq 1 \right\}
\]
\[
= \inf \left\{ \lambda > 0 : \int_Q \left| \frac{u(x) - u(y)}{(\frac{\lambda}{|z|})^{p(x,y)}} \right|^{p(x,y)} K(x,y) \, dx \, dy \leq 1 \right\}
\]
\[
= |z| \inf \left\{ \frac{\lambda}{|z|} > 0 : \int_Q \left| \frac{u(x) - u(y)}{(\frac{\lambda}{|z|})^{p(x,y)}} \right|^{p(x,y)} K(x,y) \, dx \, dy \leq 1 \right\}
\]
\[
= |z| \inf \left\{ \mu > 0 : \int_Q \left| \frac{u(x) - u(y)}{\mu^{p(x,y)}} \right|^{p(x,y)} K(x,y) \, dx \, dy \leq 1 \right\}
\]
\[
= |z|[u]_{K, p(x,y)}.
\]

Finally, to prove (iii), fix \( \lambda_u > [u]_{K, p(x,y)} \) and \( \lambda_v > [v]_{K, p(x,y)} \). Then
Now, Let \( \lambda = \lambda_u + \lambda_v \), then by the convexity of \( \rho_{K,p}(\cdot) \), we have

\[
\rho_{K,p}(\cdot)\left(\frac{u + v}{\lambda}\right) \leq \frac{\lambda_u}{\lambda} \rho_{K,p}(\cdot)\left(\frac{u}{\lambda_u}\right) + \frac{\lambda_v}{\lambda} \rho_{K,p}(\cdot)\left(\frac{v}{\lambda_v}\right).
\]

Since, \( \frac{\lambda_u}{\lambda} + \frac{\lambda_v}{\lambda} = 1 \), then

\[
\rho_{K,p}(\cdot)\left(\frac{u + v}{\lambda}\right) \leq 1.
\]

Hence,

\[
[u + v]_{K,p(x,y)} \leq \lambda = \lambda_u + \lambda_v,
\]

we take the infimum over all such \( \lambda_u \) and \( \lambda_v \), we get the desired inequality.

**Remark 4** We remark that in the model case \( K(x,y) = |x - y|^{-(N + sp(x,y))} \), the norms \( \|\cdot\|_{K,p(x,y)} \) and \( \|\cdot\|_{s,p(x,y)} \) are not the same, because \( \Omega \times \Omega \) is strictly contained in \( Q \). Moreover, in the variational formulation, we integrate on \( Q \) to correctly encode the Dirichlet boundary datum (see for example Definition 2). Consequently, we are able to handle the Dirichlet nonlocal problems

**Lemma 6** Let \( K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty) \) be a measurable function satisfying (1.4) and (1.6), let \( p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty) \) be a continuous bounded function satisfying (1.1) and (1.2). Then

\[
C_0^\infty(\Omega) \subset W_0^{K,p(x,y)}(\Omega).
\]

**Proof** Using the same argument as in [24], this lemma can be proved. For completeness, we give its proof. For \( u \in C_0^\infty(\Omega) \), we only need to check that \( [u]_{K,p(x,y)} < +\infty \). From Lemma 4, it is enough to prove that

\[
\int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx dy < +\infty.
\]

Indeed, since \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \), we have that
\[
\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy = \int_{\Omega \times \Omega} |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy \\
+ 2 \int_{\Omega \times (\mathbb{R}^N \setminus \Omega)} |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy \\
\leq 2 \int_{\Omega \times \mathbb{R}^N} |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy.
\]

(3.4)

For all \((x, y) \in \mathbb{R}^{2N}\), we notice that
\[
|u(x) - u(y)| \leq \|\nabla u\|_{L^\infty(\mathbb{R}^N)} |x - y| \quad \text{and} \quad |u(x) - u(y)| \leq 2\|u\|_{L^\infty(\mathbb{R}^N)},
\]

Hence,
\[
|u(x) - u(y)|^{p(x,y)} \leq \|\nabla u\|_{L^\infty(\mathbb{R}^N)}^{p(x,y)} |x - y|^{p(x,y)} \quad \text{for all} \ (x, y) \in \mathbb{R}^{2N},
\]

and
\[
|u(x) - u(y)|^{p(x,y)} \leq 2^{p(x,y)}\|u\|_{L^\infty(\mathbb{R}^N)}^{p(x,y)} \quad \text{for all} \ (x, y) \in \mathbb{R}^{2N}.
\]

Thus, for all \((x, y) \in \mathbb{R}^{2N}\), we get
\[
|u(x) - u(y)|^{p(x,y)} \leq \left( \|\nabla u\|_{L^\infty(\mathbb{R}^N)}^{p_+} + \|\nabla u\|_{L^\infty(\mathbb{R}^N)}^{p_-} \right) |x - y|^{p(x,y)}
\]

and
\[
|u(x) - u(y)|^{p(x,y)} \leq 2^{p_+} \left( \|u\|_{L^\infty(\mathbb{R}^N)}^{p_+} + \|u\|_{L^\infty(\mathbb{R}^N)}^{p_-} \right).
\]

So, we have
\[
|u(x) - u(y)|^{p(x,y)} \leq 2^{p_+ + 1} \left( \|u\|_{C^1(\mathbb{R}^N)}^{p_+} + \|u\|_{C^1(\mathbb{R}^N)}^{p_-} \right) \min\left\{ 1, |x - y|^{p(x,y)} \right\}.
\]

From (3.4), we deduce that
\[
\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy = 2^{p_+ + 1} \left( \|u\|_{C^1(\mathbb{R}^N)}^{p_+} + \|u\|_{C^1(\mathbb{R}^N)}^{p_-} \right) \\
\times \int_{\Omega \times \mathbb{R}^N} m(x,y)K(x,y) \, dx \, dy \\
\leq C_1 \int_{\mathbb{R}^N \times \mathbb{R}^N} m(x,y)K(x,y) \, dx \, dy.
\]

From assumption (1.6), it follows that \(u \in W_0^{K,p(x,y)}(\Omega)\). \(\square\)

**Remark 5** We observe that \(C_0^\infty(\Omega) \subset C_0^2(\Omega) \subset W_0^{K,p(x,y)}(\Omega)\). Then a trivial consequence of Lemma 6, \(W_0^{K,p(x,y)}(\Omega)\) and \(W_0^{K,p(x,y)}(\Omega)\) are non-empty.
In the following lemma, we compare the spaces $W^{s,p(x,y)}_K$ and $W^{x,p(x,y)}$. This result is crucial in the proof of the continuous and compact embedding theorem.

**Lemma 7** Let $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ be a measurable function satisfying (1.4)–(1.6). Let $p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be a continuous bounded function satisfying (1.1) and (1.2). Then the following assertions hold:

(i) If $u \in W^{s,p(x,y)}_K(\Omega)$, then $u \in W^{x,p(x,y)}(\Omega)$. Moreover,

$$\|u\|_{x,p(x,y)} \leq \max\{1, \tilde{k}_0\} \|u\|_{K,p(x,y)},$$

where $\tilde{k}_0 = \tilde{k}_0(k_0, p^-, p^+)$ is a positive constant, i.e., the space $W^{s,p(x,y)}_K(\Omega)$ is continuously embedded in $W^{x,p(x,y)}(\Omega)$.

(ii) If $u \in W^{K,p(x,y)}(\Omega)$, then $u \in W^{x,p(x,y)}(\mathbb{R}^N)$. Moreover,

$$\|u\|_{x,p(x,y)} \leq \|u\|_{W^{K,p(x,y)}(\Omega)} \leq \max\{1, \tilde{k}_0\} \|u\|_{K,p(x,y)}$$

**Proof** (i)- Let $\lambda > 0$, for $u \in W^{s,p(x,y)}_K(\Omega)$ and by (1.5), we have

$$\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{q(x,y)}} \frac{K(x,y)}{k_0} \, dx\, dy \leq \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{q(x,y)}} \frac{K(x,y)}{k_0} \, dx\, dy. \quad (3.5)$$

We define

$$A^{s}_{\lambda,\Omega} = \left\{ \lambda > 0 : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{q(x,y)} + N} \, dx\, dy \leq 1 \right\},$$

and

$$A^{K,k_0}_{\lambda,\Omega} = \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{q(x,y)}} \frac{K(x,y)}{k_0} \, dx\, dy \leq 1 \right\}.$$

By (3.5), it is easy to see that $A^{K,k_0}_{\lambda,\Omega} \subseteq A^{s}_{\lambda,\Omega}$. Hence $\inf_{\lambda > 0} A^{s}_{\lambda,\Omega} \leq \inf_{\lambda > 0} A^{K,k_0}_{\lambda,\Omega}$. Then, we have

$$[u]_{x,p(x,y)} = \inf_{\lambda > 0} A^{s}_{\lambda,\Omega} \leq \inf_{\lambda > 0} A^{K,k_0}_{\lambda,\Omega}. \quad (3.6)$$

Now, let

$$A^{K}_{\lambda,\Omega} = \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{q(x,y)}} \frac{K(x,y)}{k_0} \, dx\, dy \leq 1 \right\}.$$

And we set

$$\tilde{k}_0 = \max\left\{k_0^{-}, k_0^{+}\right\} \quad \text{and} \quad \lambda = \bar{\lambda}\tilde{k}_0.$$
So, we obtain

\[
\tilde{k}_0 \inf_{\lambda > 0} \mathcal{A}^K_{\lambda, Q} = \inf_{\lambda > 0} \left\{ \lambda \tilde{k}_0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^\lambda} K(x,y) \, dx \, dy \leq 1 \right\}
\]

\[
= \inf_{\lambda > 0} \left\{ \lambda : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^\lambda} \tilde{k}_0 \, K(x,y) \, dx \, dy \leq 1 \right\}.
\]

(3.7)

Since \( \tilde{k}_0 \geq \frac{1}{\tilde{k}_0} \), then \( \tilde{K}^{p(x,y)}_{\lambda, Q} \geq \frac{1}{\tilde{k}_0} \). Then, we get

\[
\int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^\lambda} \frac{K(x,y)}{\tilde{k}_0} \, dx \, dy \leq \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^\lambda} \frac{\tilde{k}_0}{\tilde{k}_0} K(x,y) \, dx \, dy
\]

Let

\[
\mathcal{B}^{\tilde{k}_0}_{\lambda, Q} = \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^\lambda} \frac{K(x,y)}{\tilde{k}_0} \, dx \, dy \leq 1 \right\}.
\]

We remark that \( \mathcal{B}^{\tilde{k}_0}_{\lambda, Q} \subset \mathcal{A}^{\tilde{k}_0}_{\lambda, Q} \). This implies that \( \inf_{\lambda > 0} \mathcal{A}^{\tilde{k}_0}_{\lambda, Q} \leq \inf_{\lambda > 0} \mathcal{B}^{\tilde{k}_0}_{\lambda, Q} \).

Using (3.7), we get

\[
\inf_{\lambda > 0} \mathcal{A}^{\tilde{k}_0}_{\lambda, Q} \leq \inf_{\lambda > 0} \mathcal{B}^{\tilde{k}_0}_{\lambda, Q} = \tilde{k}_0 \inf_{\lambda > 0} \mathcal{A}^{\tilde{k}_0}_{\lambda, Q}.
\]

Hence, by (3.6), we have

\[
[u]_{s,p(x,y)} \leq \tilde{k}_0 [u]_{K,p(x,y)} < + \infty.
\]

In fact, from the definition of norms \( \|u\|_{s,p(x,y)} \) and \( \|u\|_{K,p(x,y)} \), we infer

\[
\|u\|_{s,p(x,y)} \leq \max\{1, \tilde{k}_0\} \|u\|_{K,p(x,y)}.
\]

The first assertion is proved.

(ii) For \( u \in W^{s,p(x,y)}_0(\Omega) \), we have \( u = 0 \, a.e. \, \text{in} \, \mathbb{R}^N \setminus \Omega \). Then

\[
\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\mathbb{R}^N)} < + \infty.
\]

By the same argument in the assertion (i), we get \( u \in W^{s,p(x,y)}(\mathbb{R}^N) \) and

\[
[u]_{W^{s,p(x,y)}(\Omega)} \leq [u]_{W^{s,p(x,y)}(\mathbb{R}^N)} \leq \tilde{k}_0 [u]_{K,p(x,y)}.
\]

So the estimate on the norm is easily follows.

\[\square\]

**Theorem 2** Let \( \Omega \) be a Lipschitz bounded domain in \( \mathbb{R}^N \) and \( s \in (0, 1) \). Let \( p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty) \) be a continuous variable exponent satisfying (1.1) and (1.2) with \( sp^+ < N \). Let \( r : \overline{\Omega} \rightarrow (1, +\infty) \) be a continuous bounded variable exponent such that
1 < r^- \leq r(x) < p^*_s(x) \quad \text{for all } x \in \Omega.

Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ is a measurable function satisfying (1.4)–(1.6). Then

\begin{enumerate}[(i)]
\item There exists a positive constant $C = C(N, r, s, \Omega) > 0$ such that for any $u \in W^{s,p(x)}_K(\Omega)$, we have
$$
\|u\|_{L^{r}(\Omega)} \leq C \|u\|_{W^{s,p(x)}_K(\Omega)} \leq C \max\{1, \tilde{k}_0\} \|u\|_{K,p(x,y)},
$$
that is, the space $W^{s,p(x)}_K(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$.
Moreover, this embedding is compact.

\item There exists a positive constant $C_0 = C_0(N, p, s, \tilde{k}_0, \Omega) > 0$ such that
$$
[u]_{K,p(x,y)} \leq \|u\|_{K,p(x,y)} \leq C_0[u]_{K,p(x,y)}.
$$
\end{enumerate}

\textbf{Proof} (i)- Let $u \in W^{s,p(x)}_K(\Omega)$, by Lemma 7, we have $u \in W^{s,p(x)}_K(\Omega)$ and
$$
\|u\|_{W^{s,p(y)}_K(\Omega)} = \|u\|_{K,p(x,y)} \leq \max\{1, \tilde{k}_0\} \|u\|_{K,p(x,y)}.
$$
Combining (3.8) with Theorem 1, we obtain
$$
\|u\|_{L^{r(y)}(\Omega)} \leq C \|u\|_{W^{s,p(x)}_K(\Omega)} \leq C \max\{1, \tilde{k}_0\} \|u\|_{K,p(x,y)}.
$$
Hence, we deduce that

\begin{align*}
W^{s,p(x)}_K(\Omega) &\hookrightarrow W^{s,p(x)}_K(\Omega) \hookrightarrow L^{r(x)}(\Omega).
\end{align*}

Since the latter embedding is compact, then the embedding $W^{s,p(x)}_K(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ is also compact.

(ii) This assertion is easily follows by combining the definition of $\|\cdot\|_{K,p(x,y)}$ with assertion (i) and assumptions (1.4)–(1.6).

\textbf{Remark 6}

\begin{enumerate}[(i)]
\item The assertion (i) implies also that $W^{s,p(x)}_{K,0}(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$, where $1 < r^- \leq r(x) < p^*_s(x)$ for any $x \in \Omega$. Moreover, this embedding is compact.

\item As a consequence of assertion (ii), $[\cdot]_{K,p(x,y)}$ is an equivalent norm of $\|u\|_{K,p(x,y)}$ on $W^{s,p(x)}_{0,K}(\Omega)$.
\end{enumerate}

\textbf{Lemma 8} \( W^{s,p(x)}_{K,0}(\Omega), [\cdot]_{K,p(x,y)} \) is a separable, and uniformly convex (reflexive) Banach space.
**Proof** We first prove that $W^{s,p(x,y)}_{K,0}(\Omega)$ is complete with respect to the norm $\|\cdot\|_{K,p(x,y)}$. Let $\{u_n\}$ be a Cauchy sequence in $W^{s,p(x,y)}_{K,0}(\Omega)$. Since $\bar{p}(x) < p_*^+(x)$, so, combining (i) and (ii) of Theorem 2, for any $\varepsilon > 0$, there exists $n_\varepsilon^*$ such that if $n, m \geq n_\varepsilon^*$, we get

\[
\frac{1}{C} \|u_n - u_m\|_{L^{\bar{p}(x)}(\Omega)} \leq \|u_n - u_m\|_{K,p(x,y)} \leq \varepsilon,
\]

where $C = C_0 C \max\{1, \bar{K}_0\}$. By the completeness of $L^{\bar{p}(x)}(\Omega)$, there exists $u \in L^{\bar{p}(x)}(\Omega)$ such that $u_n \rightharpoonup u$ strongly in $L^{\bar{p}(x)}(\Omega)$ as $n \to +\infty$. Since $u_n = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. Then, we define $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. Hence, $u_n \rightharpoonup u$ strongly in $L^{\bar{p}(x)}(\mathbb{R}^N)$ as $n \to +\infty$. So there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ in $W^{s,p(x,y)}_{K,0}(\Omega)$, such that $u_{n_k} \rightharpoonup u$ a.e. in $\mathbb{R}^N$.

Now, we need to prove that $\|u\|_{K,p(x,y)} < +\infty$. By Lemma 4, it is enough to show that

\[
\rho^o_{K,p(\cdot)\cdot}(u) < +\infty.
\]

Indeed, by the Fatou Lemma, with $\varepsilon = 1$, we have

\[
\rho^o_{K,p(\cdot)\cdot}(u) = \int_Q |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy
\]

\[
\leq \liminf_{j \to +\infty} \int_Q |u_{n_j}(x) - u_{n_j}(y)|^{p(x,y)} K(x,y) \, dx \, dy
\]

\[
\leq \liminf_{j \to +\infty} \int_Q |u_{n_j}(x) - u_{n_j}(y) - (u_{n_j}^*(x) - u_{n_j}^*(y)) + (u_{n_j}^*(x) - u_{n_j}^*(y))|^{p(x,y)}
\]

\[
\times K(x,y) \, dx \, dy
\]

\[
\leq \liminf_{j \to +\infty} \int_Q 2^{p^+-1} \left( |u_{n_j}(x) - u_{n_j}(y) - (u_{n_j}^*(x) - u_{n_j}^*(y))|^{p(x,y)}
\right.
\]

\[
+ \left. |(u_{n_j}^*(x) - u_{n_j}^*(y))|^{p(x,y)} \right) K(x,y) \, dx \, dy
\]

\[
\leq 2^{p^+-1} \liminf_{j \to +\infty} \left\{ \int_Q |(u_{n_j} - u_{n_j}^*)(x) - (u_{n_j} - u_{n_j}^*)(y)|^{p(x,y)} K(x,y) \, dx \, dy
\right.
\]

\[
+ \int_Q |u_{n_j}^*(x) - u_{n_j}^*(y)|^{p(x,y)} K(x,y) \, dx \, dy \right\}
\]

\[
\leq 2^{p^+-1} \left\{ \liminf_{j \to +\infty} \rho^o_{K,p(\cdot)\cdot}(u_{n_j} - u_{n_j}^*) + \rho^o_{K,p(\cdot)\cdot}(u_{n_j}^*) \right\}.
\]

Using Lemma 4, we obtain

\[
\rho^o_{K,p(\cdot)\cdot}(u) \leq 2^{p^+-1} \left\{ \liminf_{j \to +\infty} \left( |u_{n_j} - u_{n_j}^*|^{p^+}_{K,p(x,y)} + |u_{n_j} - u_{n_j}^*|^{p^-}_{K,p(x,y)} \right)
\right.
\]

\[
+ \left. \left( |u_{n_j}^*|^{p^+}_{K,p(x,y)} + |u_{n_j}^*|^{p^-}_{K,p(x,y)} \right) \right\}.
\]
By the inequality (3.9) with $\varepsilon = 1$, we get
\[
p^+_K(u) \leq 2^{p^+-1} \left( 2 + \left[ u_{n_1^+} \right]_K + \left[ u_{n_2^+} \right]_K \right) < +\infty.
\]
Thus $u \in W^{s,p(\Omega)}_{K,0}$. 

On the other hand, let $n \geq n_0^*$, combining (3.9), Lemma 4, and the Fatou Lemma, we infer
\[
p^+_K(u_n - u) \leq \liminf_{j \to +\infty} p^+_K(u_n - u_j) \leq \frac{\varepsilon^+ + \varepsilon^-}{2} = \varepsilon'.
\]
Hence,
\[
\lim_{n \to +\infty} p^+_K(u_n - u) = 0.
\]
Using Lemma 4-(iii), we conclude that
\[
\lim_{n \to +\infty} [u_n - u]_K = 0.
\]
That is, $u_n \to u$ strongly in $W^{s,p(\Omega)}_{K,0}$, as $n \to +\infty$.

Let us now prove that the space $W^{s,p(\Omega)}_{K,0}$ is a separable and uniformly convex (reflexive) space. For this, we define the operator
\[
\mathcal{P} : W^{s,p(\Omega)}_{K,0} \to L^{p(\Omega)}(Q, dx dy)
\]
\[
u \mapsto (u(x) - u(y))K(x, y)^{-\frac{1}{p(x)}},
\]
Clearly $\mathcal{P}$ is an isometry from $W^{s,p(\Omega)}_{K,0}$ into $L^{p(\Omega)}(Q)$. Since $W^{s,p(\Omega)}_{K,0}$ is a Banach space, then $\mathcal{P}(W^{s,p(\Omega)}_{K,0})$ is a closed subset of $L^{p(\Omega)}(Q)$ (which is a separable and uniformly convex (reflexive) space, see Proposition 2). It follows that $\mathcal{P}(W^{s,p(\Omega)}_{K,0})$ is separable and uniformly convex (reflexive) space. Consequently, $W^{s,p(\Omega)}_{K,0}$ is also a separable and uniformly convex (reflexive) space.

This concludes the proof. □

**Corollary 1**

(i) \( \left( W^{s,p(\Omega)}_{K}(\Omega), \| \cdot \|_{K,p(\Omega)} \right) \) is a separable and reflexive uniformly convex space.

(ii) If $\Omega \subset \mathbb{R}^N$ is a domain of class $C^{0,1}$, then \( \left( W^{s,p(\Omega)}_{K}(\Omega), \| \cdot \|_{K,p(\Omega)} \right) \) is a Banach space.

**Proof**

(i) We consider the operator

\[
\mathcal{P} : W^{s,p}(\Omega) \to L^{p(\Omega)}(Q, dx dy)
\]
\[
u \mapsto (u(x) - u(y))K(x, y)^{-\frac{1}{p(x)}},
\]
\[ \tilde{P} : W^{s,p(x,y)}_K(\Omega) \longrightarrow L^p(\Omega) \times L^p(x,y)(Q, dxdy) = E \]

\[ u \longrightarrow \left( u(x), (u(x) - u(y))K(x,y)^{\frac{1}{p-1}} \right), \]

which is an isometry from \( W^{s,p(x,y)}_K(\Omega) \) to \( E \). The rest of proof is similar to Lemma 8.

(ii) Since \( \Omega \) is an open domain of class \( C^{0,1} \), then, by the same way in [7, Theorem 2.1], we can prove that \( \Omega \) is a \( W^{s,p(x,y)}_K \)-extension domain. So, for any \( u \in W^{s,p(x,y)}_K(\Omega) \) we define the extension function \( \tilde{u} \) by

\[ \tilde{u}(x) = \begin{cases} 
  u(x) & \text{if } x \in \Omega, \\
  0 & \text{if } x \in \mathbb{R}^N \setminus \Omega.
\end{cases} \]

The rest of proof is similar to Lemma 8.

In the following lemma, we prove a convergence property for a bounded sequence in \( W^{s,p(x,y)}_{K,0}(\Omega) \).

**Lemma 9** Under the same assumptions of Theorem 2. Let \( \{u_j\} \) be a bounded sequence in \( W^{s,p(x,y)}_{K,0}(\Omega) \). Then there exists \( u \in L^{r(x)}(\mathbb{R}^N) \) with \( u = 0 \) a.e in \( \mathbb{R}^N \setminus \Omega \) such that up to a subsequence

\[ u_n \longrightarrow u \text{ strongly in } L^{r(x)}(\Omega), \text{ as } n \longrightarrow +\infty. \]

**Proof** Since \( u_j \in W^{s,p(x,y)}_{K,0}(\Omega) \), so by Lemma 7-(ii), we have \( u_j \in W^{s,p(x,y)}(\mathbb{R}^N) \), hence \( u_j \in W^{s,p(x,y)}(\Omega) \). Moreover, by Lemma 7-(ii), Theorem 2-(ii), and the definition of \( W^{s,p(x,y)}_{K,0}(\Omega) \), we get

\[ \|u_j\|_{W^{s,p(x,y)}(\Omega)} \leq \|u_j\|_{W^{s,p(x,y)}(\mathbb{R}^N)} \leq C_0[u_j]_{K,p(x,y)}. \]

Using this fact and since \( \{u_j\} \) is bounded in \( W^{s,p(x,y)}_{K,0}(\Omega) \), we have that \( \{u_j\} \) is bounded in \( W^{s,p(x,y)}(\Omega) \). By Theorem 1, there exists \( u \in L^{r(x)}(\Omega) \) such that up to a subsequence \( u_n \longrightarrow u \) strongly in \( L^{r(x)}(\Omega) \). Since \( u_j = 0 \) a.e. in \( \mathbb{R}^N \setminus \Omega \), we can define \( u = 0 \) a.e. in \( \mathbb{R}^N \setminus \Omega \).

As in the classic case with \( s \) being an integer, any function in the fractional Sobolev space \( W^{s,p(x,y)}_K(\Omega) \) can be approximated by a sequence of smooth functions with compact support.

**Lemma 10** Let (1.1)–(1.3) be satisfied. Then the space \( C^\infty_0(\mathbb{R}^N) \) of smooth functions with compact support is dense in \( W^{s,p(x,y)}_K(\Omega) \).
Proof The proof is similar to the model case $K(x, y) = |x - y|^{-(N + sp(x, y))}$ in [6, Lemma 2.3].

4 Properties of the nonlocal fractional operator $L^{p(x, \cdot)}_K$

In this section, we give some basic properties of the nonlocal integro-differential operator of elliptic type $L^{p(x, \cdot)}_K$.

Let (1.1) and (1.2) be satisfied and $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ is a measurable function satisfying (1.4)–(1.6). Then

$$L^{p(x, \cdot)}_K : W^{s, p(x, \cdot)}_{K, 0}(\Omega) \rightarrow \left(W^{s, p(x, \cdot)}_{K, 0}(\Omega)\right)^*,$$

$$u \mapsto \langle L^{p(x, \cdot)}_K(u), \varphi \rangle,$$

such that

$$\langle L^{p(x, \cdot)}_K(u), \varphi \rangle = \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p(x, y)} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x, y) \, dx \, dy,$$

where $\langle ., . \rangle$ denotes the usual duality between $W^{s, p(x, \cdot)}_{K, 0}(\Omega)$ and its dual space $\left(W^{s, p(x, \cdot)}_{K, 0}(\Omega)\right)^*$.

In the following Lemma, we show some fundamental properties of the operator $L^{p(x, \cdot)}_K$.

Lemma 11 Suppose that (1.1) and (1.2) be satisfied and let $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ be a measurable function satisfying (1.4)–(1.6). Then, the following assertions hold:

(i) $L^{p(x, \cdot)}_K$ is well defined and bounded,

(ii) $L^{p(x, \cdot)}_K$ is a strictly monotone operator,

(iii) $L^{p(x, \cdot)}_K$ is a mapping of type $(S_+)$, that is, if $u_k \rightharpoonup u$ in $W^{s, p(x, \cdot)}_{K, 0}$ and

$$\limsup_{k \rightarrow +\infty} \langle L^{p(x, \cdot)}_K(u_k) - L^{p(x, \cdot)}_K(u), u_k - u \rangle \leq 0,$$

then $u_k \rightarrow u$ in $W^{s, p(x, \cdot)}_{K, 0}$,

(iv) $L^{p(x, \cdot)}_K : W^{s, p(x, \cdot)}_{K, 0}(\Omega) \rightarrow \left(W^{s, p(x, \cdot)}_{K, 0}(\Omega)\right)^*$ is a homeomorphism,

(v) $L^{p(x, \cdot)}_K$ is coercive.

Proof (i)- Let $u, \varphi \in W^{s, p(x, \cdot)}_{K, 0}(\Omega)$. Then,
\[ \langle L^p_{K}(x), \varphi \rangle \leq \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)}-2(u(x) - u(y))(\varphi(x) - \varphi(y))K(x,y) \, dx \, dy \]

\[ \leq \int_{\mathbb{R}^N} \left( |u(x) - u(y)|^{p(x,y)} - 1 \right) K(x,y)^{\frac{1}{p(x,y)}} \right) \, dx \, dy, \]

where \( \tilde{p} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty) \) is the conjugate exponent of \( p \), that is, \( \frac{1}{p(x,y)} + \frac{1}{\tilde{p}(x,y)} = 1 \). If we set

- \( \Psi(x,y) = |u(x) - u(y)|^{p(x,y)} - 1 \in L^{\tilde{p}(x,y)}(Q, dx \, dy) \),
- \( \Phi(x,y) = |\varphi(x) - \varphi(y)|K(x,y)^{\frac{1}{p(x,y)}} \in L^{\tilde{p}(x,y)}(Q, dx \, dy) \).

Then, by the Hölder’s inequality, we obtain

\[ \left| \langle L^p_{K}(x), \varphi \rangle \right| \leq 2 \left\| \Psi \right\|_{L^{\tilde{p}(x,y)}(Q, dx \, dy)} \left\| \Phi \right\|_{L^{\tilde{p}(x,y)}(Q, dx \, dy)} \]

\[ \leq C \left\| \Phi \right\|_{L^{\tilde{p}(x,y)}(Q, dx \, dy)}. \]

It follows that

\[ \left\| L^p_{K}(x) \right\|_{(W^p_{K,0}(\Omega))'} \leq C < +\infty. \]

For the proof of the properties (ii), (iii) and (iv), we follow the same argument in [6, Lemma 4.2-(i), (ii), and (iii)].

(v) Let \( u \in W^{p_{K,0}}_{p(x,y)}(\Omega) \). Then, we have

\[ \langle L^p_{K}(x), u \rangle = \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx \, dy = \rho_{K,p(x,y)}(u). \]

If \( u_{K,p(x,y)} > 0 \). From Lemma 4-(i), we get

\[ \langle L^p_{K}(x), u \rangle = \rho_{K,p(x,y)}(u) \geq \left[ u \right]_{K,p(x,y)}^{\frac{p}{p(x,y)}} = \left[ u \right]_{W^p_{K,0}}^{\frac{p}{p(x,y)}}. \quad (4.1) \]

If \( u_{K,p(x,y)} < 1 \). By Lemma 4-(ii), we obtain

\[ \langle L^p_{K}(x), u \rangle = \rho_{K,p(x,y)}(u) \geq \left[ u \right]_{K,p(x,y)}^{\frac{p}{p(x,y)}} = \left[ u \right]_{W^p_{K,0}}^{\frac{p}{p(x,y)}}. \quad (4.2) \]

Combining (4.1) and (4.2), we deduce that

\[ \lim_{\|u\|_{W^{p_{K,0}}^{p(x,y)}} \to +\infty} \frac{\langle L^p_{K}(x), u \rangle}{\|u\|_{W^{p_{K,0}}^{p(x,y)}}} = +\infty. \]

This concludes the proof. \( \Box \)
5 Applications to nonlocal fractional problems with variable exponent

In this section, we work under the hypotheses of Theorem 2. We aim to study two problems driven by the nonlocal operator $L^{p(x, \cdot)}$ and its particular case $(-A_{p(x, \cdot)})^s$.

5.1 Application to Kirchhoff type problems

At first, we discuss the existence of a nontrivial solution for a nonlocal $p(x, \cdot)$-Kirchhoff type problem of the following form

\[
(P^M_M) \quad \left\{ \begin{array}{ll}
M(\sigma_{p(x,y)}(u))L^{p(x, \cdot)}_K u + |u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\
\quad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{array} \right.
\]

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded open domain, $N \geq 3$, and

\[
\sigma_{p(x,y)}(u) = \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)}K(x,y) \, dx \, dy.
\]

$M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function which satisfies the following polynomial growth condition

\[ (M_1) : (1 - \mu)t^{r(x)-1} \leq M(t) \leq (1 + \mu)t^{r(x)-1} \quad \text{for all } t > 0, \quad \mu \in [0, 1) \text{ with } \]

and $M : \mathbb{R}^+ \rightarrow (1, +\infty)$ is a bounded function such that $1 < x^- \leq \alpha(x) \leq x^+ < +\infty$. $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfies the following conditions:

\[ (f_0) : |f(x,t)| \leq c_1(1 + |t|^\beta(x)-1) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}, \]

where $\beta \in C_+(\bar{\Omega})$ such that $\beta(x) < p^*_s(x)$ for all $x \in \bar{\Omega}$, and $\frac{p^-}{\alpha} > p^+$. (f1) : \lim_{t \to 0} \frac{f(x,t)}{|t|^{p^- - 1}} = 0 \quad \text{uniformly for } x \in \Omega.$

\[ (AR) : \text{There exist } A > 0 \text{ and } \theta > \left( \frac{1+\mu}{1-\mu} \right) \frac{x^+(p^*_s)^+}{(p^-)^+} \text{ such that } \]

\[ 0 < \theta F(x,t) = \theta \int_0^t f(x,\tau) \, d\tau \leq f(x,t) t \quad \text{for all } |t| > A \text{ and } \text{a.e. } x \in \Omega. \]

Actually, Ambrosetti-Rabinowitz condition (AR) is quite natural and important not only to ensure that the Euler-Lagrange functional has a mountain pass geometry, but also to guarantee the boundedness of the Palais-Smale (PS) sequences.

One typical feature of problem $(P^M_M)$ is the nonlocality, in the sense that the value of $L^{p(x, \cdot)}_K$ at any point $x \in \Omega$ depends not only on the values of $u$ on $\Omega$, but actually on the entire space $\mathbb{R}^N$. Moreover, the presence of the function $M$, which implies that the first equation in $(P^M_M)$ is no longer a pointwise equation, it is no longer a pointwise identity, therefore it is often called nonlocal problem. Furthermore, the Dirichlet datum is given in $\mathbb{R}^N \setminus \Omega$ (which is different from the classical case of the
$p(x)$-Laplacian) and not simply on $\partial \Omega$. This causes some mathematical difficulties which make the study of such a problem particularly interesting. Motivated by the results in [1–5, 9], we will prove that the problem $(\mathcal{P}^{K}_{M})$ has at least one nontrivial weak solution, by means of the mountain pass theorem of Ambrosetti and Rabinowitz [25].

Throughout this part, for simplicity, we use $c_{i}$ to denote the general nonnegative or positive constant (the exact value may change from line to line), we set also $X_{0} = W_{0}^{K,p(x,y)}(\Omega)$.

**Definition 2** We say that $u \in X_{0}$ is a weak solution of problem $(\mathcal{P}^{K}_{M})$ if

$$M\left( \sigma_{p(x,y)}(u) \right) \int_{Q} |u(x) - u(y)|^{p(x,y) - 2} (u(x) - u(y)) (\varphi(x) - \varphi(y)) K(x,y) \, dx \, dy$$

$$+ \int_{Q} |u|^{p(x) - 2} u \varphi dx - \int_{Q} f(x,u) \varphi dx = 0,$$

(5.1)

for all $\varphi \in X_{0}$.

Let us consider the Euler–Lagrange functional $J : X_{0} \rightarrow \mathbb{R}$ which associated to $(\mathcal{P}^{K}_{M})$ and defined by

$$J(u) = \hat{M}\left( \sigma_{p(x,y)}(u) \right) + \frac{1}{p(x)} \int_{Q} |u|^{p(x)} dx - \int_{Q} F(x,u) dx,$$

where $\hat{M}(t) = \int_{0}^{t} M(\tau) d\tau$.

Standard arguments (see, for instance [5, Lemma 3.1]) and the continuity of $M$ imply that $J$ is well defined and $J \in C^{1}(X_{0}, \mathbb{R})$. Moreover, for all $u, \varphi \in X_{0}$, its Gateaux derivative is given by

$$\langle J'(u), \varphi \rangle = M\left( \sigma_{p(x,y)}(u) \right) \int_{Q} |u(x) - u(y)|^{p(x,y) - 2} (u(x) - u(y)) (\varphi(x) - \varphi(y)) K(x,y) \, dx \, dy$$

$$+ \int_{Q} |u|^{p(x) - 2} u \varphi dx - \int_{Q} f(x,u) \varphi dx.$$

Thus, the weak solutions of $(\mathcal{P}^{K}_{M})$ coincide with the critical points of $J$.

Now, we are in a position to state our existence result as follows

**Theorem 3** Let $\Omega$ be a Lipschitz bounded domain in $\mathbb{R}^{N}$ and let $s \in (0,1)$, let $p : \overline{Q} \rightarrow (1, +\infty)$ be a continuous function satisfying (1.1) and (1.2) with $sp^{+} < N$. Assume that the assumptions $(M_{1}), (f_{0}), (f_{1})$ and $(AR)$ hold. Then, problem $(\mathcal{P}^{K}_{M})$ has at least one nontrivial weak solution.

The proof of Theorem 3 based on mountain pass theorem of Ambrosetti and Rabinowitz, and it follows from the following Lemmas.

**Lemma 12** Suppose that the assumptions $(M_{1}), (f_{0}),$ and $(AR)$ hold. Then, $J$ satisfies the $(PS)$ condition.
Proof Let us assume that there exists a sequence \( \{u_n\} \subset X_0 \) such that
\[
|J(u_n)| \leq c_2 \quad \text{and} \quad J'(u_n) \rightharpoonup 0. \quad (5.2)
\]
Using \((M_1), (AR),\) Proposition 1, Lemma 4 and Remark 6-(i), for \( n \) large enough, we get
\[
c_2 + \|u_n\|_{W_0^n} \]
\[
\geq J(u_n) - \frac{1}{\theta} \langle J'(u_n), u_n \rangle
\]
\[
= M(\sigma_{p(x,y)}(u_n)) + \int_{\Omega} \frac{1}{p(x)}|u_n|^{p(x)} \, dx - \int_{\Omega} F(x, u_n) \, dx
\]
\[
- \frac{1}{\theta} M(\sigma_{p(x,y)}(u_n)) \rho_{0,\rho}^{\theta}(u_n) - \frac{1}{\theta} \int_{\Omega} |u_n|^{p(x)} \, dx + \frac{1}{\theta} \int_{\Omega} f(x, u_n) u_n \, dx
\]
\[
\geq \frac{1 - \mu}{\alpha^+} (\sigma_{p(x,y)}(u_n))^{2(x)} - \frac{1 + \mu}{\theta} (\sigma_{p(x,y)}(u_n))^{2(x) - 1} \rho_{0,\rho}^{\theta}(u_n)
\]
\[
+ \left( \frac{1}{p^+} - \frac{1}{\theta} \right) \int_{\Omega} |u_n|^{p(x)} \, dx + \int_{\Omega} \left[ \frac{1}{\theta} f(x, u_n) u_n - F(x, u_n) \right] \, dx
\]
\[
\geq \left( \frac{1 - \mu}{\alpha^+ (p^+)^{x^+}} - \frac{1 + \mu}{\theta (p^-)^{x^- - 1}} \right) \rho_{0,\rho}^{\theta}(u_n)^{x^+} + \left( \frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|_{L^{p(x)}(\Omega)}^{p^{-}}
\]
\[
\geq \left( \frac{1 - \mu}{\alpha^+ (p^+)^{x^+}} - \frac{1 + \mu}{\theta (p^-)^{x^- - 1}} \right) \|u_n\|_{X_0}^{2^{-}} + \left( \frac{1}{p^+} - \frac{1}{\theta} \right) c_3 \|u_n\|_{X_0}^{p^-}.
\]
Since \( 1 < p^- < \alpha^+ p^- \) and \( \theta > \left( \frac{1 + \mu}{1 - \mu} \right) \frac{x^+ (p^+)^{x^+}}{(p^-)^{x^-}} > p^+ \), we obtain that \( \{u_n\} \) is bounded in \( X_0 \). This information, combined with the fact that \( X_0 \) is reflexive, implies that there exists a subsequence, still denoted by \( \{u_n\} \), and \( u \in X_0 \) such that \( \{u_n\} \) converges weakly to \( u \) in \( X_0 \). Next, as \( \beta(x) < \rho_{0}^+(x) \) for all \( x \in \Omega \), then by Remark 6-(1), \( X_0 \) is compactly embedded in \( L^{p(x)}(\Omega) \), it follows that
\[
u_n \rightharpoonup u \quad \text{(strongly) in} \quad L^{p(x)}(\Omega) \quad \text{and} \quad u_n(x) \rightarrow u(x) \quad \text{a.e.} \quad x \in \Omega. \quad (5.3)
\]
Using (5.2), we have
\[
\langle J'(u_n), u_n - u \rangle \rightarrow 0, \quad \text{as} \quad n \rightarrow +\infty,
\]
that is,
\[
M(\sigma_{p(x,y)}(u_n)) \langle \mathcal{L}_K^{p(x)}(u_n), u_n - u \rangle
\]
\[
+ \int_{\Omega} |u_n|^{p(x) - 2} u_n (u_n - u) \, dx - \int_{\Omega} f(x, u_n) (u_n - u) \, dx \rightharpoonup 0. \quad (f_0)
\]
From \((f_0)\) and Lemma 1, it follows that
\[
\left| \int_{\Omega} f(x, u_n)(u_n - u) \, dx \right| \leq c \left\| u_n^{\frac{1}{\beta(x)} - 1} \right\|_{L^{\beta(x)}(\Omega)} \left\| u_n - u \right\|_{L^{p(x)}(\Omega)} + c_1 \int_{\Omega} |u_n - u| \, dx,
\]
where \( \frac{1}{p(x)} + \frac{1}{\beta(x)} = 1 \). So, by (5.3), we have
\[
\int_{\Omega} f(x, u_n)(u_n - u) \, dx \longrightarrow 0 \quad \text{as} \quad n \to +\infty. \tag{5.4}
\]
Using again the Hölder’s inequality, we obtain
\[
\int_{\Omega} |u|^{p(x)-2} u(u_n - u) \, dx \leq 2 \left\| u_n \right\|_{L^{p(x)}(\Omega)} \left\| u_n - u \right\|_{L^{p(x)}(\Omega)}.
\]
Since \( u_n \) converges weakly to \( u \) in \( X_0 \), then the compact embedding of \( X_0 \) into \( L^{p(x)}(\Omega) \) helps us to get
\[
\lim_{n \to +\infty} \int_{\Omega} |u|^{p(x)-2} u(u_n - u) \, dx = 0. \tag{5.5}
\]
Hence, by (5.4) and (5.5), we get
\[
M\left( \sigma_{p(x,y)}(u_n) \right) \left\langle L^p_x(u_n), u_n - u \right\rangle \longrightarrow 0. \tag{5.6}
\]
Now, since \( \{u_n\} \) is bounded in \( X_0 \), we may assume that
\[
\sigma_{p(x,y)}(u_n) \longrightarrow t_1 \geq 0.
\]
If \( t_1 = 0 \), then \( \{u_n\} \) converge strongly to \( u = 0 \) in \( X_0 \) and the proof is finished.
If \( t_1 > 0 \), since the function \( M \) is continuous, we have
\[
M(\sigma_{p(x,y)}(u_n)) \longrightarrow M(t_1) \geq 0.
\]
Hence, by (\( M_1 \)), for \( n \) large enough, we have that
\[
0 < c_4 \leq M(\sigma_{p(x,y)}(u_n)) \leq c_5. \tag{5.7}
\]
Combining (5.6) and (5.7), we deduce that
\[
\left\langle L^p_x(u_n), u_n - u \right\rangle \longrightarrow 0. \tag{5.8}
\]
On the other hand, since \( \{u_n\} \) converge weakly to \( u \) in \( X_0 \), we have that
\[
\left\langle J'(u), u_n - u \right\rangle \longrightarrow 0,
\]
that is,
\[
M\left( \sigma_{p(x,y)}(u) \right) \left\langle L^p_x(u), u_n - u \right\rangle + \int_{\Omega} |u|^{p(x)-2} u(u_n - u) \, dx - \int_{\Omega} f(x, u)(u_n - u) \, dx \longrightarrow 0, \tag{5.9}
\]
which implies by using the same argument as before that
\[
\langle L_K^{p(x,y)}(u), u_n - u \rangle \xrightarrow{n \to +\infty} 0. \tag{5.9}
\]
Combining (5.8) and (5.9), we deduce that
\[
\limsup_{n \to +\infty} \langle L_K^{p(x,y)}(u_n) - L_K^{p(x,y)}(u), u_n - u \rangle \leq 0.
\]
By Lemma 11-(iii) \(L_K^{p(x,y)}\) is a mapping of type \((S_+)\), thus
\[
\limsup_{n \to +\infty} \langle L_K^{p(x,y)}(u_n) - L_K^{p(x,y)}(u), u_n - u \rangle \leq 0, \quad u_k \to u \text{ in } X_0,
\]
\[
\Rightarrow u_n \to u \text{ (strongly) in } X_0.
\]
Consequently, \(J\) satisfies the (PS) condition.

The following lemma shows that the functional \(J\) satisfies the first geometrical condition of the mountain pass theorem.

**Lemma 13** Suppose that the assumptions \((M_1)\), \((f_0)\), and \((f_1)\) hold. Then there exist two positive real numbers \(R\) and \(a > 0\) for all \(u \in X_0\) with \(\|u\|_{X_0} = R\).

**Proof** Let \(u \in X_0\) with \(\|u\|_{X_0} < 1\), then by \((M_1)\), we have
\[
J(u) \geq \frac{1 - \mu}{\alpha^+ (p^+)^p} \|u\|_{X_0}^{p^+} + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} F(x,u) dx. \tag{5.10}
\]
Since \(\beta(x) < p_+^\ast(x)\) and \(p_+ < p_+^\ast(x)\) for all \(x \in \overline{\Omega}\), then by Remark 6-(1), we have that \(X_0\) is continuously embedded in \(L^{\beta(x)}(\Omega)\), \(L^{p(x)}(\Omega)\) and \(L^{p_+}(\Omega)\), that is, there exist three positive constants \(c_6, c_7\) and \(c_8\) such that
\[
\|u\|_{L^{\beta(x)}(\Omega)} \leq c_6 \|u\|_{X_0}, \quad \|u\|_{L^{p_+}(\Omega)} \leq c_7 \|u\|_{X_0} \quad \text{and} \quad \|u\|_{L^{p(x)}(\Omega)} \leq c_8 \|u\|_{X_0}. \tag{5.11}
\]
Now, we assume that \(\|u\|_{X_0} < \min\{1, \frac{1}{c_6}, \frac{1}{c_7}\}\), where \(c_6\) and \(c_8\) are the positive constant given in (5.11), then we get
\[
\|u\|_{L^{\beta(x)}(\Omega)} < 1 \quad \text{and} \quad \|u\|_{L^{p_+}(\Omega)} < 1 \text{ for all } u \in X_0 \text{ with } \|u\|_{X_0} = R \in (0, 1).
\]
Combining \((f_0)\) and \((f_1)\), we have
\[
F(x,t) \leq e[t]^{p_+} + c_6 |t|^\beta(x) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.
\]
Therefore, by \((M_1)\), (5.10), (5.11) and Proposition 1-(i), we obtain
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Hence, by Lemma 14

Assume that condition of mountain pass theorem.

From the assumption \( \|u\|_{X_0} \leq h \), we find that \( \|u\|_{X_0} \leq R \in (0, 1) \).

The following result shows that the functional \( J \) satisfies the second geometrical condition of mountain pass theorem.

**Lemma 14** Assume that \( (M_1) \) and \( (AR) \) hold. Then there exists \( u_0 \in X_0 \) such that \( \|u\|_{X_0} > R, J(u) < 0 \).

**Proof** From the assumption \( (AR) \), we obtain

\[
F(x, tu) > t^\theta F(x, u) \quad \text{for all } t \geq 1 \text{ and a.e. } x \in \Omega.
\]

Hence, by \( (M_1) \), for \( v \in X_0, v \neq 0 \) and \( t > 1 \), we have

\[
J(tv) = \tilde{M}(\sigma_{p(x)}(tv)) + \int_{\Omega} \frac{|tv|^{p(x)}}{p(x)} dx - \int_{\Omega} F(x, tv) dx
\]

\[
\leq \frac{1 + \mu}{\alpha^+(p^+)^{\frac{1}{\alpha^+}}} t^{\alpha^+} \tilde{P}(v) + \frac{1}{p^+} \int_{\Omega} |v|^{p(x)} dx - \int_{\Omega} t^\theta F(x, v) dx.
\]

From \( \theta > \left( \frac{\alpha^+}{1 - \mu} \right) \frac{1}{(p^+)^{\frac{1}{\alpha^+}}} = \frac{\alpha^+}{1 - \mu} \frac{1}{(p^+)^{\frac{1}{\alpha^+}}} \), we find that \( \theta > \alpha^+p^+ > p^+ \). Therefore,

\[
J(tv) \to -\infty \quad \text{as } t \to +\infty.
\]

Now, we are ready to prove Theorem 3.
Proof of Theorem 3 Combining Lemmas 12–14 and the fact that \( J(0) = 0 \), we have that \( J \) satisfies the assumptions of mountain pass theorem (see [25]). Therefore, \( J \) has at least one nontrivial critical point, that is, problem \((P_M^s)\) has at least one nontrivial weak solution. □

Example 1 As a particular case, we can take

- \( M(t) = a + b t^{\alpha(x)-1}, \ a, b > 0, \) with \( \alpha : \overline{\Omega} \rightarrow (1, +\infty) \) is a bounded function such that \( 1 < \alpha^- \leq \alpha(x) \leq \alpha^+ < \infty \).
- \( K(x, y) = |x - y|^{-(N+sp(x,y))} \).
- \( f(x, t) = |t|^{\gamma(x)-2} t, \) where \( \gamma \in C_+(\overline{\Omega}) \) such that \( \gamma(x) < p^*_s(x) \) for all \( x \in \overline{\Omega} \) and \( \gamma^- > \alpha^+ p^+ \).

In this case, the problem \((P_M^s)\) becomes

\[
\begin{cases}
(a + b \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)^{\alpha^-} |x - y|^{N+sp(x,y)}} \, dx dy)^{\frac{\alpha(x)-1}{\alpha^+}} \bigl( -A p(x, \cdot) \bigr)^\gamma u(x) \\
\quad + |u|^{p(x)-2} u = |u|^{\gamma(x)-2} u \quad \text{in } \Omega, \\
\quad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

It is easy to see that the function \( M \) satisfies \((M_1)\) and the function \( f \) verifies the assumptions \((f_0), (f_1)\) and the \((AR)\) conditions. Consequently, problem \((P_{a,b})\) has at least one nontrivial weak solution.

5.2 Existence and uniqueness result for a nonlocal problem

Now, we investigate the existence of a unique weak solution for a variational problem driven by the nonlocal integro-differential operator of elliptic type \( \mathcal{L}_K^{p(x, -)} \).

\[
\begin{cases}
\mathcal{L}_K^{p(x, -)}(u(x)) = f \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^N \) and \( f \in X_0^s \).

Definition 3 We say that \( u \in X_0 \) is a weak solution of problem \((P_K)\), if

\[
\int_Q |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) K(x, y) \, dx dy = \int_\Omega f \varphi \, dx, \quad (5.12)
\]

for any \( \varphi \in X_0 \).

Applying the Minty–Browder theorem, we get the following existence result.
Theorem 4  Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, and $p : \mathbb{R}^N \times \mathbb{R}^N \to (1, +\infty)$ be a continuous variable exponent satisfying (1.1) and (1.2) with $sp^+ < N$. Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ is a measurable function satisfying (1.4)-(1.6) and $f \in X_0$. Then the problem $(\mathcal{P}_K)$ has a unique weak solution $u \in X_0$.

A typical example for $K$ is given by the singular kernel $K(x, y) = \frac{1}{|x-y|^{N+sp(x,y)}}$. In this case, problem $(\mathcal{P}_K)$ becomes
\[
\begin{aligned}
\begin{cases}
\mathcal{A}_{p(x,\cdot)} u &= f & \text{in } \Omega, \\
u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\end{aligned}
\]

As a particular case, we derive an existence result for problem $(\mathcal{P}_K)$, which is given by the following corollary.

Corollary 2  Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ and $p : \mathbb{R}^N \times \mathbb{R}^N \to (1, +\infty)$ be a continuous variable exponent satisfying (1.1) and (1.2) with $sp^+ < N$. Let $s \in (0, 1)$ and $f \in X_0$. Then the following equation
\[
\int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp(x,y)}} \, dx dy = \int_{\Omega} f v dx,
\]

has a unique solution $u \in X_0$.

Remark 7  We observe that (5.13) represents the weak formulation of problem $(\mathcal{P}_s)$.

Proof of Theorem 4  By Lemma 11 the operator $\mathcal{L}^{p(x,\cdot)}_K$ satisfies the conditions of Minty–Browder Theorem, that is,

- From Lemma 11-(i), $\mathcal{L}^{p(x,\cdot)}_K$ is bounded, from $X_0$ into $X_0^*$.
- From Lemma 11-(ii), $\mathcal{L}^{p(x,\cdot)}_K$ is a strictly monotone operator.
- From Lemma 11-(iv), $\mathcal{L}^{p(x,\cdot)}_K$ is a homeomorphism. Hence, $\mathcal{L}^{p(x,\cdot)}_K$ is continuous.
- From Lemma 11-(v), $\mathcal{L}^{p(x,\cdot)}_K$ is coercive.

Consequently, in the light of Minty-Browder theorem [10, Theorem V.15], then there exists a unique weak solution $u \in X_0$ of problem $(\mathcal{P}_K)$.

Proof of Corollary 2  It is a consequence of Theorem 4, by choosing $K(x, y) = |x-y|^{-(N+sp(x,y))}$, and by recalling that $X_0 \subset W_0^{s,p(x,y)}(\Omega)$.

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