A NEW CHARACTERIZATION OF THE HARDY SPACE AND OF OTHER SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. The Fock space can be characterized (up to a positive multiplicative factor) as the only Hilbert space of entire functions in which the adjoint of derivation is multiplication by the complex variable. Similarly (and still up to a positive multiplicative factor) the Hardy space is the only space of functions analytic in the open unit disk for which the adjoint of the backward shift operator is the multiplication operator. In the present paper we characterize the Hardy space in term of the adjoint of the differentiation operator. We use reproducing kernel methods, which seem to also give a new characterization of the Fock space.

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1. INTRODUCTION

The Fock (or Bargmann-Fock-Segal) space is the unique Hilbert space of entire functions in which
\begin{equation}
\partial^*_z = M_z.
\end{equation}

where $\partial_z$ denote the derivative with respect to $z$ (also denoted simply $\partial$ when with respect to $z$), and will be used throughout the work along with the notation $(\partial f)(z) = f'(z)$. Furthermore, in (1.1) $M_z$ stands for multiplication by the variable $z$, e.g. $(M_z f)(z) = z f(z)$. Formula (1.1) suggests to find similar characterizations for other important spaces of analytic functions, such as the Bergman space, the Dirichlet space, the Hardy spaces and more. In the present work we approach this problem using reproducing kernel Hilbert spaces methods. We prove:

Theorem 1.1. The Hardy space is, up to a positive multiplicative factor, the only reproducing kernel Hilbert space of functions analytic in $\mathbb{D}$, in which the equality
\begin{equation}
\partial^*_z = M_z \partial_z M_z
\end{equation}

makes sense on the linear span of the kernel.
Note that, both in this, and in the next, theorem, one could assume that the functions are analytic only in a neighborhood of the origin, and then use analytic continuation. We also note that the unbounded operator $M_z \partial_z$ is diagonal, and acts on the polynomials as the number operator of quantum mechanics:

$$M_z \partial_z (z^n) = nz^n, \quad n = 0, 1, \ldots,$$

see e.g. [4, p. 548] for the latter.

The Hardy space of the open unit disk $\mathbb{D}$ has reproducing kernel $\frac{1}{1-z\bar{\omega}}$. More generally, for every $\alpha \geq 1$, the function $\frac{1}{(1-z\bar{\omega})^\alpha}$ is positive definite in $\mathbb{D}$, as can be seen from the power series expansion of the function $\frac{1}{(1-z\bar{\omega})^\alpha}$ centered at the origin. We will use a similar notation to Bargmann, and denote by $\mathcal{H}_\alpha$ to be the associated reproducing kernel Hilbert space, characterized by the following result:

**Theorem 1.2.** Let $\alpha \geq 1$. Then the space $\mathcal{H}_\alpha$ is, up to a multiplicative factor, the only reproducing kernel Hilbert space of functions analytic in $\mathbb{D}$, in which the equality

(1.3)$$\partial^*_z = M_z \partial_z M_z - (1 - \alpha)M_z, \quad \alpha \geq 1$$

makes sense on the linear span of the kernel.

The case $\alpha = 1$ corresponds to the Hardy space and Theorem [1.1] and $\alpha = 2$ corresponds to the Bergman space. The case $\alpha = 0$ would “correspond” to the Dirichlet space, in the sense that

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \left( \frac{1}{(1-z\bar{\omega})^\alpha} - 1 \right) = -\ln(1-z\bar{\omega}).$$

Note that $\partial_z$ is not densely defined in the Dirichlet space (since $\partial^2_z k$ is not in the Dirichlet space), and therefore its adjoint is a relation and not an operator. We were not able to get a counterpart of Theorem [1.2] but we can prove:

**Theorem 1.3.** The Dirichlet space is, up to a positive multiplicative factor, the only reproducing kernel Hilbert space of functions analytic in $\mathbb{D}$, for which the equality

(1.4)$$\partial^2_{z\bar{\omega}} k = \bar{\omega}^2 \partial_{\bar{\omega}} \partial_z k$$

holds for the kernel.

Note that (1.4) is not an equality in the Dirichlet space, but is an equality between analytic functions. For a similar characterization of the Fock space, see Proposition [2.5].

Our analysis suggests the following question: for which polynomial with two variables $p(x, y)$ the equation

$$\partial^* = p(M_z, \partial)$$

characterized a reproducing kernel Hilbert space.

The paper consists of four sections besides the introduction. In section 2 we review a number of definitions and results on reproducing kernel Hilbert spaces of analytic functions. Sections 3, 4, and 5 contain proofs of Theorems [1.1] [1.2] and [1.3] respectively.
2. Reproducing kernel Hilbert spaces

In this section we will briefly review the properties of reproducing kernel Hilbert spaces needed in the following sections. We first recall the definition:

**Definition 2.1.** A reproducing kernel Hilbert space is a Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) of functions defined in a non-empty set \(\Omega\), such that there exists a complex-valued function \(k(z, \omega)\) defined on \(\Omega \times \Omega\) and with the following properties:

1. \(\forall \omega, \quad k_\omega : z \mapsto k(z, \omega) \in \mathcal{H}\)
2. \(\forall f \in \mathcal{H}, \quad \langle f, k_\omega \rangle = f(\omega)\)

The function \(k(z, \omega)\) is uniquely defined by the Riesz representation theorem, and is called the reproducing kernel of the space. The reproducing kernel (kernel, for short) has a very important property: it is positive definite, that is for all \(N \in \mathbb{N}, \omega_1, \ldots, \omega_N \in \Omega\), and \(c_1, \ldots, c_N \in \mathbb{C}\); we have

\[
\sum_{i,j} c_i \bar{c}_j k(\omega_i, \omega_j) \geq 0.
\]

We refer to the book [6] for more information on reproducing kernel Hilbert spaces, and we recall that there is a one-to-one correspondence between positive definite functions on a given set and reproducing kernel Hilbert spaces of functions defined on that set. In the present work we are interested in the case where \(\Omega\) is an open neighborhood of the origin, and where the kernels are analytic in \(z\) and \(\omega\). The following result is a direct consequence of Hartog’s theorem, and will be used in the sequel. For a different proof, see [3, p. 92].

**Proposition 2.2.** Let \(\mathcal{H}\) be a reproducing kernel Hilbert space of functions analytic in \(\Omega \subset \mathbb{C}\), with reproducing kernel \(k(z, \omega)\). Then the reproducing kernel is jointly analytic in \(z\) and \(\omega\).

**Proof.** Since the kernels belong to the space, we have that for every \(w \in \Omega\) the function \(z \mapsto k(z, \omega)\) is analytic in \(\Omega\). Since \(k(z, \omega) = \overline{k(\omega, z)}\), the kernel is also analytic in \(\omega\). Hartog’s theorem (see [2, p. 39]) allows us to conclude that \(k(z, \omega)\) is jointly analytic in \(z\) and \(\omega\). \(\square\)

One then has:

**Proposition 2.3.** Under the hypothesis of the above discussion, the elements of the associated reproducing kernel Hilbert space are analytic in \(\Omega\) and the following hold:

\[
(2.1) \quad \partial f(z) = \langle f, \partial_{\omega}k_\omega \rangle
\]

and

\[
(2.2) \quad \partial_z k(z_0, \omega_0) = \overline{\partial_\omega k(\omega_0, z_0)}, \quad z_0, \omega_0 \in \Omega.
\]

**Proof.** The proof of (2.1) can be found in [7, theorem 9, p. 41]. We give the proof of (2.2), where (and in the rest of the work) we use the notation: \(k_\beta : z \mapsto k(z, \beta)\) where \(\beta \in \Omega\).

Using \((\partial f)(z_0) = \langle f, \partial_{\omega}k_{z_0} \rangle\), we calculate the right hand side: \(\partial_z k(z_0, \omega_0) = \langle k_{\omega_0}, \partial_\omega k_{z_0} \rangle\). Therefore

\[
\partial_z k(z_0, \omega_0) = \langle \partial_z k_{z_0}, k_{\omega_0} \rangle = \partial_\omega k_{z_0}|_{z=\omega_0} = \partial_\omega k(\omega_0, z_0),
\]

and hence the result. \(\square\)
For a special case, the reader could also check (2.2) for \( k(z, \omega) = f(z\overline{\omega}) \).

The main result we will need is as follows:

**Proposition 2.4.** Let \( k(z, \omega) \) be positive definite and jointly analytic in \( z \) and \( \overline{\omega} \) for \( z, \omega \) in the open subset \( \Omega \) of the complex plane. Assume that the operator \( \partial \) is densely defined in the associated reproducing kernel Hilbert space \( \mathcal{H}(k) \). Then \( \partial \) is closed and in particular has a densely defined adjoint \( \partial^* \) which satisfies \( \partial^{**} = \partial \).

**Proof.** Let \((f_n)\) be a sequence of elements in \( \text{Dom} \partial \) and let \( f, g \in \mathcal{H} \) be such that

\[
\begin{align*}
    f_n & \to f \\
    \partial f_n & \to g
\end{align*}
\]

where the convergence is in the norm. Since weak convergence follows from strong convergence, and using (2.1), we have for every \( \omega \in \Omega \):

\[
\langle f_n, \partial \overline{\omega} k_{\omega} \rangle \to \langle f, \partial \overline{\omega} k_{\omega} \rangle \quad \text{and} \quad \langle \partial f_n, k_{\omega} \rangle \to \langle g, k_{\omega} \rangle ,
\]

where the brackets denote the inner product in \( \mathcal{H}(k) \). Hence it follows that

\[
\lim_{n \to \infty} f'_n(\omega) = f'(\omega) \quad \text{and} \quad \lim_{n \to \infty} f'_n(\omega) = g(\omega)
\]

Thus \( g = f' \), and hence \( \partial \) is closed. Hence, \( \partial \) has a densely defined adjoint and \( \partial^{**} = \partial \); see e.g. [5] for the latter. \( \square \)

As an application we prove the following characterization of the Fock space, different from the one given by Bargmann ([1, footnote 4, p. 188]). In the statement, one could assume the functions analytic only in a neighborhood of the origin, and then use analytic continuation.

**Proposition 2.5.** The Fock space is the unique (up to a positive multiplicative factor) reproducing kernel Hilbert space of entire functions where the equation

\[
\partial^* = M_z
\]

makes sense on the linear span of the kernels (in particular the kernel functions are in the domain of \( \partial^* \) and of \( M_z \)).

**Proof.** From Proposition 2.2, the kernel is jointly analytic in \( \mathbb{D} \). Since \( \partial^* = M_z \), it follows that

\[
\langle \partial^* k(z, \omega), k(z, \nu) \rangle = \langle M_z k(z, \omega), k(z, \nu) \rangle.
\]

Evaluating each side yields the following: For the right hand side we get:

\[
\langle M_z k(z, \omega), k(z, \nu) \rangle = \nu k(\nu, \omega)
\]

since \( M_z k(z, \omega) = \nu k(z, \omega) \). The left hand side yields:

\[
\langle \partial^* k(z, \omega), k(z, \nu) \rangle = \langle k(z, \omega), \partial_z k(z, \nu) \rangle
\]

\[
= \langle \partial_z k(z, \nu), k(z, \omega) \rangle
\]

\[
= \partial_z k(z, \nu) \big|_{z=\omega}
\]

\[
= \partial_\omega k(\omega, \nu)
\]

\[
= \partial_\omega k(\nu, \omega),
\]
where we have used (2.2) to go form the penultimate line to the last one. Thus we obtain that
\[ \partial \overline{\omega} k(\nu, \omega) = \nu k(\nu, \omega), \]
which is a differential equation with the solution:
\[ k(\nu, \omega) = c(\nu) e^{\nu \overline{\omega}}. \]

But \( k(\nu, \omega) = \overline{k(\omega, \nu)} \). Hence \( c(\nu) = \overline{c(\nu)} = c \), where \( c \) is a positive constant since the kernel is positive. \( \square \)

### 3. PROOF OF THEOREM 1.1

The kernel \( \frac{1}{1 - z \overline{\omega}} \) is a solution of (1.2). Indeed (1.2) applied to the kernel gives
\[ \frac{\nu^2 \overline{\omega}}{(1 - \nu \overline{\omega})^2} + \frac{\nu}{1 - \nu \overline{\omega}} = \frac{\nu}{(1 - \nu \overline{\omega})^2} \Rightarrow \nu^2 \overline{\omega} + \nu(1 - \nu \overline{\omega}) = \nu \]
To prove the converse we apply (1.2) to kernels, and find a partial differential equation satisfied by the reproducing kernel. Then we use analyticity to find the kernel via its Taylor expansion at the origin. Let \( \omega, \nu \in \mathbb{D} \). From (1.2) we get
\[ (3.1) \quad \langle \partial k_{\omega}, k_{\nu} \rangle = \langle k_{\omega}, \partial^* k_{\nu} \rangle = \langle k_{\omega}, M_z \partial M_z k_{\nu} \rangle \]
We rewrite (1.2) as
\[ \partial^* f = z(\partial z f) = z(z f' + f) = z^2 f' + zf. \]
We have \( \langle \partial k_{\omega}, k_{\nu} \rangle = \partial k(\nu, \omega) \), and by using the two end sides of (3.1), we get:
\[ \langle \partial k_{\omega}, k_{\nu} \rangle = \langle k_{\omega}, M_z \partial M_z k_{\nu} \rangle = \langle k_{\omega}, M_z \partial M_z k_{\nu} \rangle = \langle \nu^2 \partial k_{\nu} + \nu k_{\nu}, k_{\omega} \rangle = \overline{\omega} \partial k(\nu, \omega) + \overline{\omega} k(\nu, \omega) \]
Thus we get the partial differential equation \( \partial k - \overline{\omega} \partial \overline{\omega} k = \overline{\omega} k \), which can be written as
\[ (3.2) \quad \partial k = \overline{\omega} \partial \overline{\omega} k + \overline{\omega} k \]
where \( k = k(\nu, \omega) \).

The kernel is analytic in \( z \) and \( \overline{\omega} \) near the origin, and hence can be written as
\[ k(\nu, \omega) = \sum_{n,m=0}^{\infty} c_{n,m} \nu^n \overline{\omega}^m. \]
So we can rewrite (3.2) as
\[ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n c_{n,m} \nu^{n-1} \overline{\omega}^m = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m c_{n,m} \nu^n \overline{\omega}^{m+1} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \nu^n \overline{\omega}^{m+1} \]
which can also be written as:
\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n + 1) c_{n+1,m} \nu^n \overline{\omega}^m = \sum_{n=0}^{\infty} \sum_{m=2}^{\infty} (m - 1) c_{n,m-1} \nu^{n-1} \overline{\omega}^m + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{n,m-1} \nu^n \overline{\omega}^{m-1}. \]
Now we can consider the following cases:
Let \( m = 0 \); then we have:
\[
\sum_{n=0}^{\infty} (n + 1)c_{n+1,0} \nu^n = 0.
\]
Hence
\[
c_{n+1,0} = 0
\]
for all \( n \).

Let \( m = 1 \); then we have:
\[
\sum_{n=0}^{\infty} (n + 1)c_{n+1,1} \nu^n \bar{\omega}^m = \sum_{n=0}^{\infty} c_{n,0} \nu^n.
\]
Hence
\[
(n + 1)c_{n+1,1} = c_{n,0}
\]
for all \( n \). Note that for \( n = 0 \) we get \( c_{0,0} = c_{1,1} \).

Let \( m \geq 2 \); then we have:
\[
(n + 1)c_{n+1,m} = (m - 1)c_{n,m-1} + c_{n,m-1}.
\]
Hence
\[
(3.3)
\]
for all \( n \). Note that if \( m = n + 1 \), then \( (n + 1)c_{n+1,n+1} = (n + 1)c_{n,n} \). So
\[
(3.4)
\]
We now check that \( c_{n,m} = 0 \) when \( n \neq m \). From (3.3) it follows that
\[
c_{n+1,m} = \alpha_{n,m}c_{n+1,m-0}
\]
for \( \alpha_{n,m} \neq 0 \).

Hence from these cases and by symmetry, all off-diagonal entries of the kernel will be zero, and it follows from (3.4) that \( k(z, \omega) = \frac{c_{n,0}}{1 - \bar{z}\omega} \). This ends the proof of the theorem.

If we assume that the powers of \( z \) are in the domain of \( \partial^* \) and of \( M_z \) one has a simpler proof, close in spirit to Bargmann’s arguments.

**Proposition 3.1.** Let \( \mathcal{H} \) be a reproducing kernel Hilbert space of functions analytic in a neighborhood of the origin and such that:

1. \( M_z \) bounded.
2. \( \{z^n\} \subset \text{Dom } \partial \).
3. \( \text{Dom } \partial \subset \text{Dom } \partial^* \).
4. \( \partial^* = M_z \partial M_z \).

Then \( \mathcal{H} = \mathcal{H}^2 \) up to a scalar.

**Proof.** From Proposition 2.2, the kernel is jointly analytic in \( \mathbb{D} \). Take \( f = z^n \) and \( g = f^m \), then:
\[
\langle f, \partial g \rangle = \langle z^n, mz^m-1 \rangle = m \langle z^n, z^{m-1} \rangle
\]
and
\[
\langle \partial^* f, g \rangle = \langle z^2f' + zf, g \rangle = \langle nz^{n+1} + z^{n+1}, z^m \rangle = (n + 1)\langle z^{n+1}, z^m \rangle.
\]

Since \( \langle f, \partial g \rangle = \langle \partial^* f, g \rangle \), we obtain:
\[
(3.5) \quad (n + 1)\langle z^{n+1}, z^m \rangle = m\langle z^n, z^{m-1} \rangle
\]
For \( m = n + 1 \), we have
\[
(n + 1)\langle z^n, z^n \rangle = (n + 1)\langle z^{n+1}, z^{n+1} \rangle \implies \langle z^n, z^n \rangle = \langle z^{n+1}, z^{n+1} \rangle
\]
thus the diagonal entries are nonzero. Now we are left to show that if \( n \neq m \), \( \langle z^n, z^m \rangle = 0 \). From (3.5) we get

\[
\langle z^{n+1}, z^m \rangle = \frac{m}{n+1} \langle z^n, z^{m-1} \rangle
\]

Take \( f = z^n, n \neq 0, g = 1 \), then:

\[
\langle f, \partial g \rangle = \langle \partial^* f, g \rangle = \langle z^2 f' + zf, g \rangle = (nz^{n+1} + z^{n+1}, 1) = (n+1)\langle z^{n+1}, 1 \rangle
\]

However \( \langle f, \partial g \rangle = 0 \), hence \( (n+1)\langle z^{n+1}, 1 \rangle = 0 \), and from (3.6) all the off-diagonal coefficients \( c_{n,m} \) are equal to 0. □

More generally, with the same hypothesis as in Proposition 3.1, one could replace \( M_z \partial_z \) by a (possibly unbounded) diagonal operator defined as follows:

\[
D(z^n) = \alpha_n z^n, \quad n = 0, 1, 2, \ldots
\]

with \( \alpha_n > 0 \) for \( n \geq 1 \) and \( \alpha_0 \) arbitrary. Then we get

\[
\langle z^n, z^m \rangle = \delta_{n,m} \frac{n!}{\alpha_n \cdots \alpha_1} \langle 1, 1 \rangle,
\]

and the reproducing kernel will then be:

\[
k(z, \omega) = \sum_{n=0}^{\infty} \frac{\alpha_n \cdots \alpha_1}{n!} z^n \omega^n,
\]

provided the radius of convergence of the above series is strictly positive.

4. PROOF OF THEOREM 1.2

To prove Theorem 1.2 we use the same strategy as in the previous section. The kernel \( \frac{1}{(1-z\omega)^\alpha} \) is a solution of (1.3). Indeed, (1.3) applied to the kernel gives us

\[
\frac{z}{(1-z\omega)^\alpha} + \alpha \frac{z^2\omega}{(1-z\omega)^{\alpha+1}} - (1-\alpha) \frac{z}{(1-z\omega)^\alpha} = \frac{z(1-z\omega) + \alpha z^2\omega - (1-\alpha)z(1-z\omega)}{(1-z\omega)^{\alpha+1}}
\]

\[
= \frac{z - z^2\omega + \alpha z^2\omega - z + z^2\omega + \alpha z - \alpha z^2\omega}{(1-z\omega)^{\alpha+1}}
\]

\[
= \frac{\alpha z}{(1-z\omega)^{\alpha+1}}
\]

so

\[
z(1-z\omega) + \alpha z^2\omega - (1-\alpha)z(1-z\omega) = \alpha z
\]

As we see again, indeed for \( \alpha = 1 \) we have the Hardy case. To prove the converse we apply (1.3) to kernels, and find a partial differential equation satisfied by the reproducing kernel. Then we use analyticity to find the kernel via its Taylor expansion at the origin. Let \( \omega, \nu \in \mathbb{D} \), then from (1.3) we get:

\[
\langle \partial k_\omega, k_\nu \rangle = \langle k_\omega, \partial^* k_\nu \rangle = \langle k_\omega, M_z \partial M_\nu k_\nu - (\alpha - 1)M_z k_\nu \rangle
\]
We rewrite (1.3) as

\[ \partial^* f = z(\partial z f) - (\alpha - 1)zf = z^2 f' + zf - \alpha zf + zf = z^2 f' + (2 - \alpha)zf. \] (4.2)

We know \( \langle \partial k_\omega, k_\nu \rangle = \partial k(\nu, \omega) \), thus from (4.2) and the two end sides of (4.1), we get:

\[ \partial k(\nu, \omega) = \langle \partial k_\omega, k_\nu \rangle = \langle k_\omega, M_\omega \partial M_\nu - (\alpha - 1)M_\omega k_\nu \rangle = \langle k_\omega, \nu^2 \partial k_\nu + (2 - \alpha)\nu k_\nu, k_\omega \rangle = \omega^2 \partial k(\nu, \omega) + (2 - \alpha)\bar{\omega}k(\nu, \omega) \]

Thus we get the partial differential equation:

\[ \partial k = \bar{\omega}^2 \partial \bar{\omega}k + (2 - \alpha)\bar{\omega}k \] (4.3)

where \( k = k(\nu, \omega) \).

The kernel is analytic in \( z \) and \( \bar{\omega} \) near the origin, and hence can be written as

\[ k(\nu, w) = \sum_{n,m=0}^{\infty} c_{n,m} \nu^n \bar{\omega}^m. \]

So we can rewrite (4.3) as

\[ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n c_{n,0} \nu^{n-1} \bar{\omega}^m = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m c_{n,m} \nu^n \bar{\omega}^{m+1} + (2 - \alpha) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \nu^n \bar{\omega}^{m+1} \]

which can also be written as:

\[ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (n+1) c_{n+1,m} \nu^n \bar{\omega}^m = \sum_{n=0}^{\infty} \sum_{m=2}^{\infty} (m-1) c_{n,m-1} \nu^n \bar{\omega}^{m} + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (2 - \alpha) c_{n,m-1} \nu^n \bar{\omega}^{m}. \]

Now we can consider the following cases:

Let \( m = 0 \); then we have: \( \sum_{n=0}^{\infty} (n+1) c_{n+1,0} \nu^n = 0 \). Hence

\[ c_{n+1,0} = 0 \]

for all \( n \).

Let \( m = 1 \); then we have: \( \sum_{n=0}^{\infty} (n+1) c_{n+1,1} \nu^n \bar{\omega}^m = \sum_{n=0}^{\infty} (2 - \alpha) c_{n,0} \nu^n. \) Hence

\[ (n+1)c_{n+1,1} = (2 - \alpha)c_{n,0} \]

for all \( n \). Note that for \( n = 0 \) we get \( c_{0,0} = (2 - \alpha)c_{1,1} \).

Let \( m \geq 2 \); then we have \( (n+1)c_{n+1,m} = (m-1)c_{n,m-1} + (2 - \alpha)c_{n,m-1}. \) Hence

\[ (n+1)c_{n+1,m} = (m+1 - \alpha)c_{n,m-1} \]

for all \( n \). Note that if \( m = n + 1 \), then \( (n+1)c_{n+1,n+1} = (n + 2 - \alpha)c_{n,n}. \) So

\[ c_{n,n} = \left( \frac{n + 1}{n + 2 - \alpha} \right) c_{n+1,n+1}. \]
we see that the diagonal entries are same up to a constant.
We are reminded to check that \( c_{n,m} = 0 \) when \( n \neq m \). From (4.4) it follows that
\[
c_{n+1,m} = \phi_{\alpha,n,m}c_{n+1-m,0}
\]
for \( \phi_{\alpha,n,m} \neq 0 \).
Hence from these cases and by symmetry, all off-diagonal entries of the kernel would be zero, and thus completing the proof.

5. PROOF OF THEOREM 1.3

Let \( k(\nu, \omega) \) be a solution of (1.4), with power series expansion
\[
k(\nu, \omega) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \nu^n \bar{\omega}^m.
\]
We have \( c_{0,0} = 0 \) since \( k(0, 0) = 0 \). We have
\[
\partial_\nu^2 k = \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} c_{n,m} n(n - 1)\nu^{n-2} \bar{\omega}^m
\]
\[
\bar{\omega}^2 \partial_\nu \partial_\bar{\omega} k = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n,m} nm\nu^{n-1} \bar{\omega}^{m+1}.
\]
So we can rewrite (1.4) in terms of the kernel as follows:
\[
\sum_{n=2}^{\infty} \sum_{m=0}^{\infty} c_{n,m} n(n - 1)\nu^{n-2} \bar{\omega}^m = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n,m} nm\nu^{n-1} \bar{\omega}^{m+1}.
\]
Let \( n = 2 \); then:
\[
\sum_{m=0}^{\infty} c_{2,m} 2\bar{\omega}^m = \sum_{m=1}^{\infty} c_{2,m} 2m\nu \bar{\omega}^{m+1}.
\]
Let \( n = 1 \); then:
\[
\sum_{m=1}^{\infty} c_{1,m} m\bar{\omega}^{m+1} = 0.
\]
We make the change of index \( M = m + 1 \) to the equation above, and obtain
\[
\sum_{M=2}^{\infty} c_{1,m-1} (M - 1)\bar{\omega}^M = 0.
\]
Hence from equations (5.3) and (5.2), it follows that
\[
c_{2,0} = c_{2,1} = 0 \quad \text{and} \quad 2c_{2,M} = (M - 1)c_{1,M-1} \quad \text{for} \quad M > 2.
\]
Considering equation (5.1) and making the change of index \( N = n - 2, M = m \) to the right side, and \( N = n - 1, M = m + 1 \) to the left side, we get
\[
\sum_{N=0}^{\infty} \sum_{M=0}^{\infty} c_{N+2,M}(N + 2)(N + 1)\nu^N \bar{\omega}^M = \sum_{N=0}^{\infty} \sum_{M=2}^{\infty} c_{N+1,M-1}(N + 1)(M - 1)\nu^N \bar{\omega}^M.
\]
From (5.4) for \( N \in \mathbb{N}_0 \) and \( M \geq 2 \), we have
\[
c_{N+2,M}(N + 2) = (M - 1)c_{N+1,M-1}.
\]
Taking $M = N + 2$ gives us

$$c_{N+2, N+2}(N + 2) = (N + 1)c_{N+1, N+1}.$$ 

Now we want check that all off diagonal entries of the kernel are indeed zero.

Let $M = 0$; then from (5.1) with the change of variable $N = n - 2$ gives us

$$\sum_{N=0}^{\infty} c_{N+2,0}(N + 2)(N + 1)\nu^N = 0$$

so we have

$$c_{N+2,0} = 0 \text{ for } N \geq 0.$$ 

Let $M = 1$; then from (5.4) we get

$$c_{n+2,1} = 0 \text{ for } N \geq 0.$$ 

Since $k(0, 0) = 0$ we get that $c_{0,0} = 0$. Hence all off diagonal entries and $c_{0,0}$ are zero, and hence completing the proof. □

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