THE GEL’FAND-KALININ-FUKS CLASS AND CHARACTERISTIC CLASSES OF TRANSVERSELY SYMPLECTIC FOLIATIONS

D. KOTSCHICK AND S. MORITA

Abstract. In the early 1970’s, Gel’fand, Kalinin and Fuks found an exotic characteristic class of degree 7 in the Gel’fand-Fuks cohomology of the Lie algebra of formal Hamiltonian vector fields on the plane. We prove that this cohomology class can be decomposed as a product $\eta \wedge \omega$ of a certain leaf cohomology class $\eta$ of degree 5 and the transverse symplectic class $\omega$. This is similar to the well known factorization of the Godbillon-Vey class for codimension $n$ foliations[8], [13]. We also interpret the characteristic classes of transversely symplectic foliations introduced by Kontsevich[12] in terms of the known classes and prove non-triviality for some of them.

1. Introduction

Let $a_n$ denote the Lie algebra consisting of all the formal vector fields on $\mathbb{R}^n$ and let $\mathfrak{ham}_{2n} \subset a_{2n}$ denote the subalgebra of Hamiltonian formal vector fields on $\mathbb{R}^{2n}$ with respect to the standard symplectic form $\omega$. Hereafter we denote by $H^{2n}_R$ this standard symplectic vector space, which is the fundamental representation of the symplectic group $\text{Sp}(2n, \mathbb{R})$.

The Lie algebra $\mathfrak{ham}_{2n}$ contains the Lie subalgebra $\mathfrak{sp}(2n, \mathbb{R})$ consisting of linear Hamiltonian vector fields. In [7] Gel’fand, Kalinin and Fuks studied the Gel’fand-Fuks cohomology of $\mathfrak{ham}_{2n}$ and showed that

$$H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{\leq 0} \cong \mathbb{R}[\omega, p_1, \ldots, p_n]/I.$$ 

Throughout this paper, all the cohomology groups of Lie algebras are with trivial coefficients in $\mathbb{R}$, and we omit the coefficients from the
notation. In the formula above $\omega \in H^2_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2n, \mathbb{R}))$ denotes the symplectic class of weight $-2$ (for the definition of weights, see the next section) and $p_i \in H^4_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2n, \mathbb{R}))_0$ $(i = 1, \cdots, n)$ denote the Pontrjagin classes. Further $I$ is the ideal generated by the classes

$$\omega^k p_1^{k_1} \cdots p_n^{k_n} \quad (k + k_1 + 2k_2 \cdots + nk_n > n)$$

that vanish. (In the context of $\mathfrak{a}_n$ this corresponds to the Bott vanishing theorem.) Thus, in the non-positive weight part, the result is similar to the case of $\mathfrak{a}_n$. However, in the positive weight part, Gel’fand, Kalinin and Fuks found an exotic class $GKL \in H^7_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8$ of degree 7 and weight 8, which is now called the Gel’fand-Kalinin-Fuks class. They raised the problem of determining whether their class is non-trivial as a characteristic class of transversely symplectic foliations, or not. Recall that the Godbillon-Vey class, which corresponds to $h_1 c_1 \in H^2_{GF}(\mathfrak{a}_n, \mathbb{O}(n))$, was shown to be non-trivial almost immediately after its discovery. Namely Roussarie first proved the non-triviality and Thurston [21] proved the remarkable result that this class can vary continuously. In sharp contrast with this, non-triviality of the GKF-class has now been an open problem for nearly 40 years.

In the late 1990’s, Kontsevich [12] interpreted the Rozansky-Witten invariants in terms of the Gel’fand-Fuks cohomology and characteristic classes for foliations. As an application, he constructed certain characteristic classes for transversely symplectic foliations. More precisely, he considered the two Lie subalgebras

$$\mathfrak{ham}^1_2 \subset \mathfrak{ham}^0_2 \subset \mathfrak{ham}_2$$

of $\mathfrak{ham}_2$, where $\mathfrak{ham}^\epsilon_2$ denotes the formal Hamiltonian vector fields without constant terms and without constant or linear terms for $\epsilon = 0, 1$ respectively. Kontsevich constructed a natural homomorphism

$$\land^n : H^*_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2n, \mathbb{R})) \cong H^*_{GF}(\mathfrak{ham}^1_2, \mathfrak{sp}(2n, \mathbb{R})) \to H^{*+2n}_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2n, \mathbb{R})).$$

Since the abelianization of the Lie algebra $\mathfrak{ham}^1_2$ can be written as

$$\mathfrak{ham}_2^1 \to S^3 H^2_{\mathbb{R}},$$

where $S^3 H^2_{\mathbb{R}}$ denotes the third symmetric power of $H^2_{\mathbb{R}}$, he obtained a homomorphism

$$\Phi : H^*(S^3 H^2_{\mathbb{R}}) \to H^{*+2n}_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2n, \mathbb{R})).$$

Roughly speaking, Kontsevich first considered the leaf or foliated cohomology classes of transversely symplectic foliations, rather than the de Rham cohomology, and then produced characteristic classes for
such foliations (in de Rham cohomology) by taking the wedge product with the maximal power $\omega^n$ of the transverse symplectic form.

The purpose of this paper is twofold. Firstly we interpret the Gel’fand-Kalinin-Fuks class in this framework of Kontsevich. This interpretation shows that the GKF class can be decomposed as a product $\eta \wedge \omega$ of a certain leaf cohomology class $\eta$ of degree 5 and the transverse symplectic class $\omega$. This is similar to the case of the Godbillon-Vey class $h_1c^n_1$ for codimension $n$ foliations \[8\], which can be expressed as the product of a 1-dimensional leaf cohomology class $h_1$, the Reeb class, and the primary characteristic class $c^n_1$. (Similar factorizations are known for some other characteristic classes of foliations; see e.g. \[13\].) Although the problem of geometric non-triviality of the GKF class remains open, we hope that our result will shed some light on the geometric meaning of this class. Secondly we determine Kontsevich’s homomorphism $\Phi$ in \((1)\) completely up to degree $2n$ and prove that some of these classes are non-trivial as characteristic classes of transversely symplectic foliations.

2. Gel’fand-Fuks cohomology of formal Hamiltonian vector fields

As is well-known, each element $X \in \mathfrak{ham}_{2n}$ corresponds bijectively to a formal Hamiltonian function

$$H \in \mathbb{R}[[x_1, \cdots, x_n, y_1, \cdots, y_n]]/\mathbb{R}$$

which is defined up to constants, via the correspondence

$$X \leftrightarrow \sum_{i=1}^{n} \left\{ \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} - \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} \right\}.$$ 

Thus, on the one hand, we have an isomorphism of topological Lie algebras

$$\mathfrak{ham}_{2n} \cong \mathbb{R}[[x_1, \cdots, x_n, y_1, \cdots, y_n]]/\mathbb{R},$$

where the Lie bracket on the right hand side is given by the Poisson bracket. On the other hand, this topological Lie algebra is the completion of that of polynomial Hamiltonian functions

$$\mathbb{R}[x_1, \cdots, x_n, y_1, \cdots, y_n]/\mathbb{R} = \bigoplus_{k=1}^{\infty} S^k H^{2n}_\mathbb{R}.$$

Here $S^k H^{2n}_\mathbb{R}$ denotes the $k$-th symmetric power of $H^{2n}_\mathbb{R}$, which is identified with the space of all the homogeneous polynomials of degree $k$.

The Poisson bracket is given by

$$S^k H^{2n}_\mathbb{R} \otimes S^\ell H^{2n}_\mathbb{R} \ni f \otimes g \mapsto \{f, g\} \in S^{k+\ell-2} H^{2n}_\mathbb{R}.$$
Hence the cochain complex $C^\ast_{GF}(\mathfrak{ham}_{2n})$ of $\mathfrak{ham}_{2n}$ splits as a direct sum of finite dimensional subcomplexes

$$C^\ast_{GF}(\mathfrak{ham}_{2n}) \cong \bigoplus_{w=-2n}^\infty C^\ast_{GF}(\mathfrak{ham}_{2n})_w,$$

so that we have

$$H^\ast_{GF}(\mathfrak{ham}_{2n}) \cong \bigoplus_{w=-2n}^\infty H^\ast_{GF}(\mathfrak{ham}_{2n})_w.$$

Here

$$C^\ast_{GF}(\mathfrak{ham}_{2n})_w = \sum_{-k_1+k_2+2k_3+3k_5=\cdots=w} \Lambda^{k_1}(S^1H^2_{\mathbb{R}})^\ast \otimes \Lambda^{k_2}(S^2H^2_{\mathbb{R}})^\ast \otimes \cdots$$

denotes the set of cochains with weight $w$, where we define the weight of each element in $(S^kH^2_{\mathbb{R}})^\ast$ to be $k - 2$, so that the coboundary operator preserves the weights. Similar decompositions hold in the case of the relative cohomology $H^\ast_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))$, and in the cases of the Lie subalgebras $\mathfrak{ham}^0_{2n}$, $\mathfrak{ham}_{2n}$.

Now, as was already mentioned in the introduction, Gel’fand, Kalinin and Fuks proved in [7] that the cohomology $H^\ast_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{\leq 0}$ in the non-positive weight part is described in terms of the usual characteristic classes, namely the Pontrjagin classes and the transverse symplectic class. However, contrary to their initial working hypothesis, they found an exotic class for the case $n = 1$:

**Theorem 1 (Gel’fand-Kalinin-Fuks [7]).** The relative cohomology $H^\ast_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$ for $w \leq 8$ is given by:

$$H^\ast_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{\leq 0} \cong \mathbb{R}[\omega, p_1]/(\omega^2, \omega p_1, p_1^3),$$

$$H^\ast_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w = 0 \quad (w = 1, \cdots, 7),$$

$$H^\ast_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_8 = \begin{cases} \mathbb{R} & (* = 7) \\ 0 & (otherwise) \end{cases}.$$

Perchik [20] gave a formula for the generating function

$$\sum_{w=0}^\infty \chi(H^\ast(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w)t^w$$

of the Euler characteristic of the relative cohomology of $\mathfrak{ham}_{2n}$. By computing it for the case $n = 1$, he showed the existence of many more exotic classes. Later, Metoki [17] found an explicit exotic class in $H^9(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{14}$ which we call the Metoki class.
Similar to the case of $H^*_G(a_n, O(n))$, which provides characteristic classes for foliations of codimension $n$ (see [11]), the relative cohomology $H^*_G(\mathfrak{ham}_{2n}, \text{Sp}(2n, \mathbb{R}))$ provides characteristic classes for transversely symplectic foliations of codimension $2n$. More precisely, let $\mathbb{B} \Gamma_{\omega}^{2n}$ denote the Haefliger classifying space for transversely symplectic foliations of codimension $2n$. Then we have a homomorphism

\[
(2) \quad H^*_G(\mathfrak{ham}_{2n}, \text{Sp}(2n, \mathbb{R})) \longrightarrow H^*_G(\mathfrak{ham}_{2n}, U(n)) \longrightarrow H^*(\mathbb{B} \Gamma_{\omega}^{2n}; \mathbb{R}) ,
\]

where $U(n) \subset \text{Sp}(2n, \mathbb{R})$ is a maximal compact subgroup. In particular, we have the important problem of determining whether the Gel’fand-Kalinin-Fuks class in $H^7(\mathfrak{ham}_{2n}, \text{Sp}(2n, \mathbb{R}))$ defines a non-trivial characteristic class in $H^7(\mathbb{B} \Gamma_{\omega}^{2n}; \mathbb{R})$, or not. We also have this problem for the Metoki class.

In [12], Kontsevich proposed a new approach to the theory of characteristic classes for transversely symplectic foliations. Here we briefly summarize his construction. Kontsevich considered the two Lie subalgebras

\[
\mathfrak{ham}_{2n}^0 \supset \mathfrak{ham}_{2n}^1
\]

of $\mathfrak{ham}_{2n}$ consisting of formal Hamiltonian vector fields without constant terms and without constant or linear terms. In terms of Hamiltonian functions, we can write

\[
\mathfrak{ham}_{2n}^0 = \bigoplus_{k=2}^{\infty} S^k H^2_{\mathbb{R}} \quad \text{and} \quad \mathfrak{ham}_{2n}^1 = \bigoplus_{k=3}^{\infty} S^k H^2_{\mathbb{R}}
\]

and we have an isomorphism

\[
H^*_G(\mathfrak{ham}_{2n}^0, \text{Sp}(2n, \mathbb{R})) \cong H^*_G(\mathfrak{ham}_{2n}^1)_{\text{Sp}(2n, \mathbb{R})} .
\]

Let $\mathcal{F}$ be a foliation on a smooth manifold $N$ and let $T \mathcal{F} \subset TN$ be the tangent bundle of $\mathcal{F}$. The leaf cohomology or foliated cohomology of $\mathcal{F}$, denoted by $H^*_\mathcal{F}(N; \mathbb{R})$, is defined to be the cohomology of $\Omega^*_\mathcal{F}(N) = \bigoplus_k \Gamma(\Lambda^k T^* \mathcal{F})$, which is the quotient of the de Rham complex $\Omega^* N$ of $N$ by the ideal $I^*(\mathcal{F})$ of $\mathcal{F}$. If $\mathcal{F}$ is a transversely symplectic foliation of codimension $2n$, then there is a transverse symplectic form $\omega$, and the homomorphism

\[
\wedge \omega^n : I^*(\mathcal{F}) \longrightarrow \Omega^* N
\]

vanishes identically, so that there is a well-defined homomorphism

\[
\wedge \omega^n : H^*_\mathcal{F}(N; \mathbb{R}) \longrightarrow H^{*+2n}(N; \mathbb{R}) .
\]
Now Kontsevich pointed out that the relative cohomology
\[ H^*_GF(\mathfrak{ham}_0^{2n}, \text{Sp}(2n, \mathbb{R})) \cong H^*_GF(\mathfrak{ham}_1^{2n}; \mathbb{R})^{\text{Sp}(2n, \mathbb{R})} \]
serves as the universal model for \( H^*_F(N; \mathbb{R}) \), so that one has the following commutative diagram:
\[
\begin{array}{ccc}
H^*_GF(\mathfrak{ham}_1^{2n})^{\text{Sp}(2n, \mathbb{R})} & \longrightarrow & H^*_F(N; \mathbb{R}) \\
\wedge \omega^n & & \wedge \omega^n \\
H^{*-2n}(\mathfrak{ham}_2^{2n}, \text{Sp}(2n, \mathbb{R})) & \longrightarrow & H^{*-2n}(N; \mathbb{R}) \\
\end{array}
\]

It is easy to show that the natural projection
\[ \mathfrak{ham}_1^{2n} \longrightarrow S^3 H_{\mathbb{R}}^{2n} \]
on the lowest weight part gives the abelianization of the Lie algebra \( \mathfrak{ham}_2^{2n} \) because the Poisson bracket
\[ S^3 H_{\mathbb{R}}^{2n} \otimes S^k H_{\mathbb{R}}^{2n} \longrightarrow S^{k+1} H_{\mathbb{R}}^{2n} \]
is easily seen to be surjective for any \( k \geq 3 \). It follows that, as mentioned in [1], there is a homomorphism \( \Phi \) defined by the following composition:

\[ H^*(S^3 H_{\mathbb{R}}^{2n})^{\text{Sp}(2n, \mathbb{R})} \longrightarrow H^*_GF(\mathfrak{ham}_1^{2n})^{\text{Sp}(2n, \mathbb{R})} \wedge \omega^n \longrightarrow H^*GF(\mathfrak{ham}_2^{2n}, \text{Sp}(2n, \mathbb{R})) \]

Further composing \( \Phi \) with the homomorphism in (2), we obtain

\[ \tilde{\Phi} : H^*(S^3 H_{\mathbb{R}}^{2n})^{\text{Sp}(2n, \mathbb{R})} \longrightarrow H^{*-2n}(\text{ESymp}_\delta^\omega(\mathbb{R}) \mathbb{R}) \]

For any symplectic manifold \((M, \omega)\) of dimension \(2n\), let \( \text{Symp}^\delta(M) \) denote the symplectomorphism group of \( M \) equipped with the discrete topology. Then the above construction gives rise to a homomorphism

\[ H^*(S^3 H_{\mathbb{R}}^{2n})^{\text{Sp}(2n, \mathbb{R})} \longrightarrow H^{*-2n}(\text{ESymp}_\delta^\omega(M)) \overset{\int_M}{\longrightarrow} H^*(\text{BSymp}_\delta^\omega(M); \mathbb{R}) \]

where \( \text{ESymp}_\delta^\omega(M) \) denotes the total space of the universal foliated \( M \)-bundle over the classifying space \( \text{BSymp}_\delta^\omega(M) \) of \( \text{Symp}^\delta(M) \), and \( \int_M \) is the integration over the fiber in this universal bundle.

One of the merits of the above construction of Kontsevich is that the stable cohomology of \( \mathfrak{ham}_2^{2n} \) is not interesting because by [9] it is just the polynomial algebra on the symplectic class while that of \( \mathfrak{ham}_0^{2n} \) is one of the three versions of Kontsevich’s graph cohomology (see [10], [11]), more precisely the commutative version, which is very rich and still mysterious.
3. Statements of the main results

In this section, we state the main results of this paper.

**Theorem 2.** In the range of weights \( w \leq 10 \) the relative cohomology groups \( H^*_{GF}(\mathfrak{ham}_0^m, \mathfrak{sp}(2, \mathbb{R})) \) are non-trivial only for the following three combinations of degree and weight:

\[
\begin{align*}
H^0_{GF}(\mathfrak{ham}_0, \mathfrak{sp}(2, \mathbb{R}))_0 & \cong \mathbb{R} \\
H^2_{GF}(\mathfrak{ham}_0^2, \mathfrak{sp}(2, \mathbb{R}))_2 & \cong \mathbb{R} \\
H^5_{GF}(\mathfrak{ham}_0^2, \mathfrak{sp}(2, \mathbb{R}))_{10} & \cong \mathbb{R}.
\end{align*}
\]

Furthermore, the following homomorphisms are all isomorphisms:

\[
\begin{align*}
\wedge \omega : H^0_{GF}(\mathfrak{ham}_0^0, \mathfrak{sp}(2, \mathbb{R}))_0 & \longrightarrow H^2_{GF}(\mathfrak{ham}_0^2, \mathfrak{sp}(2, \mathbb{R}))_{-2} \cong \mathbb{R} < \omega >, \\
\wedge \omega : H^2_{GF}(\mathfrak{ham}_0^2, \mathfrak{sp}(2, \mathbb{R}))_2 & \longrightarrow H^4_{GF}(\mathfrak{ham}_0^2, \mathfrak{sp}(2, \mathbb{R}))_0 \cong \mathbb{R} < p_1 >, \\
\wedge \omega : H^5_{GF}(\mathfrak{ham}_0^2, \mathfrak{sp}(2, \mathbb{R}))_{10} & \longrightarrow H^7_{GF}(\mathfrak{ham}_0^2, \mathfrak{sp}(2, \mathbb{R}))_8 \cong \mathbb{R} < GKF >.
\end{align*}
\]

It follows that both the first Pontrjagin class \( p_1 \) and the Gel’fand-Kalinin-Fuks class \( GKF \) can be decomposed as wedge products of certain leaf cohomology classes and the transverse symplectic class \( \omega \).

Combining Theorem 2 with our earlier result in [14] (see also [15]) we obtain the following non-triviality result for the characteristic classes defined by Kontsevich [12].

**Corollary 3.** Under the homomorphisms

\[
\begin{align*}
H^2(S^3 H^2_\mathbb{R}; \mathbb{R})^{\mathfrak{sp}(2, \mathbb{R})} & \longrightarrow H^2(\text{BSymp}^\delta(\Sigma_g); \mathbb{R}), \\
H^2(S^3 H^2_\mathbb{R}; \mathbb{R})^{\mathfrak{sp}(2, \mathbb{R})} & \longrightarrow H^4(\text{Bi}\Gamma_2^\omega; \mathbb{R}),
\end{align*}
\]

the generator of \( H^2(S^3 H^2_\mathbb{R}; \mathbb{R})^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R} \) is mapped to

\[
e_1 \in H^2(\text{BSymp}^\delta(\Sigma_g); \mathbb{R}), \quad p_1 \in H^4(\text{Bi}\Gamma_2^\omega; \mathbb{R})
\]

respectively (up to non-zero constants), where \( \text{Symp}(\Sigma_g) \) denotes the symplectomorphism group of \( \Sigma_g \) with respect to a fixed symplectic form. It follows that both homomorphisms are non-trivial.

We can further generalize the above result to higher dimensions as follows.

**Theorem 4.** In the range \( * \leq 2n \), the image of the homomorphism

\[
\Phi : H^*(S^3 H^{2n}_\mathbb{R}; \mathbb{R})^{\mathfrak{sp}(2n, \mathbb{R})} \longrightarrow H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))
\]

introduced by Kontsevich is precisely the subspace spanned by the classes

\[
\omega^k p_1^{k_1} \cdots p_n^{k_n} \quad (k + k_1 + 2k_2 \cdots + nk_n = n)
\]
that are borderline with respect to Bott vanishing in this context. Furthermore, the elements

\[ \omega^n, \omega^{n-1} p_1, \ldots, p_n \]

are mapped non-trivially under the homomorphism

\[ \tilde{\Phi}: H^*(S^3 H_{2n}^2, \mathbb{R})^{Sp(2n, \mathbb{R})} \longrightarrow H^{*+2n}(B\Gamma_2^w, \mathbb{R}) \]

so that

\[ \dim \text{Im} \tilde{\Phi} \geq n + 1. \]

**Remark 5.** It seems reasonable to conjecture that the above homomorphism \( \Phi \) is trivial in the range \( \ast > 2n \). This is true for the case \( n = 1 \) by Theorem 2.

It is an important problem to determine whether the classes involving the higher Pontrjagin classes \( p_i \) (\( i \geq 2 \)) are non-trivial, or not.

### 4. Proofs of the main results

In this section we write \( H \) as a shorthand for \( H_{2n}^{2n} \).

We begin with the proof of Theorem 2. For this, notice that

\[
C^*_{GF}(\text{ham}_0^0, \text{sp}(2, \mathbb{R}))_w = \sum_{k_3 + 2k_4 + 3k_5 \cdots = w} (\Lambda^{k_3} S^3 H^* \otimes \Lambda^{k_4} S^4 H^* \otimes \Lambda^{k_5} S^5 H^* \otimes \cdots)^{Sp(2, \mathbb{R})},
\]

\[
C^*_{GF}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_w = \sum_{-k_1 + k_3 + 2k_4 \cdots = w} (\Lambda^{k_1} H^* \otimes \Lambda^{k_3} S^3 H^* \otimes \Lambda^{k_4} S^4 H^* \otimes \cdots)^{Sp(2, \mathbb{R})}.
\]

It is easy to see that both \( C^*_{GF}(\text{ham}_0^0, \text{sp}(2, \mathbb{R}))_w \) and \( C^*_{GF}(\text{ham}_2^0, \text{sp}(2, \mathbb{R}))_w \) vanish if \( w \) is odd. Moreover, for \( C^*_{GF}(\text{ham}_0^0, \text{sp}(2, \mathbb{R}))_w \) with \( w = 2, 4, 6, 8 \) we have the following Table 1, where \( \chi \) denotes the Euler characteristic.

**Table 1.**

| \( k \) | 1 | 2 | 3 | 4 | 5 | \( \chi \) |
|--------|---|---|---|---|---|--------|
| \( \dim C^k_{GF}(\text{ham}_0^0, \text{sp}(2, \mathbb{R}))_2 \) | 0 | 1 | 0 | 0 | 0 | 1 |
| \( \dim C^k_{GF}(\text{ham}_0^0, \text{sp}(2, \mathbb{R}))_4 \) | 0 | 0 | 1 | 1 | 0 | 0 |
| \( \dim C^k_{GF}(\text{ham}_0^0, \text{sp}(2, \mathbb{R}))_6 \) | 0 | 1 | 1 | 0 | 0 | 0 |
| \( \dim C^k_{GF}(\text{ham}_0^0, \text{sp}(2, \mathbb{R}))_8 \) | 0 | 0 | 4 | 5 | 1 | 0 |
Here we have used well-known facts about the representations of $\text{Sp}(2, \mathbb{R})$ such as $S^k H^* \cong S^k H$ and
\[
S^k H \otimes S^\ell H \cong S^{k+\ell} H \oplus S^{k+\ell-2} H \oplus \cdots \oplus S^{k-\ell} H \quad (k \geq \ell),
\]
as well as various formulae for the irreducible decomposition of $\Lambda^m S^k H$; see e.g. [3].

In the weight 2 part, we find that the homomorphism
\[
H^2_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_2 = (\Lambda^2 S^3 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R} \wedge \omega
\]
\[
H^4_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_0 = (\Lambda^2 H^* \otimes \Lambda^2 S^3 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R} < p_1>
\]
is an isomorphism because $\omega$ is a generator of $(\Lambda^2 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R}$. It follows that the first Pontrjagin class $p_1$ can be decomposed as a wedge product
\[
p_1 = \gamma_1 \wedge \omega
\]
of a class $\gamma_1 \in H^2_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_2 \cong \mathbb{R}$ in the first leaf cohomology with the transverse symplectic class $\omega$. In the weight 4 part, we find that the coboundary operator
\[
C^3_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_4 = (\Lambda^2 S^3 H^* \otimes S^4 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R}
\]
\[
\delta \eta C^4_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_4 = (\Lambda^4 S^3 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R}
\]
is an isomorphism. This, together with the computation shown in Table [1] shows that $H^*(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_4$ is trivial. Similarly the weight 6 part $H^6(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_6$ is trivial because the coboundary operator
\[
C^2_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_6 = (\Lambda^2 S^5 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R}
\]
\[
\delta \eta C^3_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_6 = (S^3 H^* \otimes S^4 H^* \otimes S^5 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R}
\]
can be seen to be an isomorphism.

The cochain complex $C^*_GF(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_8$ for the weight 8 part is given in Table [2] where the symbols $(347)$, $(4^2 6)$, for example, stand for
\[
(S^3 H^* \otimes S^4 H^* \otimes S^7 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R}
\]
\[
(\Lambda^2 S^4 H^* \otimes S^6 H^*)^{\mathfrak{sp}(2, \mathbb{R})} \cong \mathbb{R}
\]
respectively, and similarly for the other ones.

**Table 2.** generators for $C^*_GF(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_8$

| $C^*_GF(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_8$ | dim | generators |
|-----------------|-----|-------------|
| $C^3_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_8$ | 4 | $(347)(356)(4^2 6)(45^2)$ |
| $C^4_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_8$ | 5 | $(3^2 46)(3^2 5^2)(34^2 5)_2$ |
| $C^*_GF(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_8$ | 1 | $(3^4 45)$ |
The subscript 2 in the symbol $\left(3^2\overline{5}^2\right)_2$ means that its dimension is 2, namely we have
\[
(\Lambda^2 S^3 H^* \otimes \Lambda^2 S^5 H^*)^{\text{Sp}(2,\mathbb{R})} \cong \mathbb{R}^2.
\]
A direct computation of the coboundary operators shows that this cochain complex is acyclic.

The dimensions of the cochain complex for the weight 10 part are given in the first line of Table 3. In the second line, the dimensions of the cochain complex for the weight 8 part of $H^*_G(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_8$ are given. These were first computed by Gel’fand-Kalinin-Fuks [7] and were later re-computed by Metoki [17].

| k   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | \chi |
|-----|---|---|---|---|---|---|---|---|-----|
| \dim C^k_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{10} | 0 | 0 | 0 | 1 | 3 | 9 | 12 | 4 | -1  |
| \dim C^k_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_8  | 0 | 0 | 5 | 13 | 17 | 18 | 14 | 4 | -1  |

As was already mentioned, Gel’fand, Kalinin and Fuks determined $H^*_G(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_8$ by a computer calculation and found that it is 1-dimensional, generated by their class of degree 7. Metoki re-computed this cohomology group, again with the aid of a computer program, and constructed an explicit (but complicated) cocycle for the $GKF$-class. His cocycle is not divisible by $\omega$, and, so far, no cocycle divisible by $\omega$ has been known.

Now it can be checked that the homomorphism
\[
\wedge \omega : C^*_G(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{10} \longrightarrow C^{*+2}_G(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_8
\]
induces an embedding of cochain complexes which shifts the degree by 2 and the weight by $-2$. Our purpose is to prove that it induces an isomorphism in cohomology. By an explicit computation, we determined a system of generators for the first chain complex $C^*_G(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{10}$ as shown in Table 4.

In general, there are three equivalent ways of expressing elements in $C^*_G(\mathfrak{ham}_{2n}, \text{Sp}(2n,\mathbb{R}))$. The first one is in terms of (duals of) Sp(2n,\mathbb{R})-invariant tensors of Hamiltonian functions. The second one is by means of vertex oriented graphs which encode ways of contraction of tensors of Hamiltonian functions by applying the symplectic pairing $H^2_{\mathbb{R}} \otimes H^2_{\mathbb{R}} \to \mathbb{R}$ along the edges. The third one is in terms of tautological 1-forms
\[
\delta^i_{j_1 \ldots j_k} \in C^1_{GF}(\mathfrak{a}_{2n})
\]
Table 4. Generators for $C^*_G(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$

| dim | Generators |
|-----|------------|
| $C^2_G(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$ | 1 | $(7^2)$ |
| $C^3_G(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$ | 3 | $(358)(367)(457)$ |
| $C^4_G(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$ | 9 | $(3^248)(3^257)(3^47^2)(3456)(35^3)(4^6)$ |
| $C^5_G(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$ | 12 | $(3^247)(3^25^6)(3^24^26)(3^24^25)(34^25_2)(4^5)$ |
| $C^6_G(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$ | 4 | $(3^45^2)(3^24^25_2)(3^24^2)$ |

restricted to the Lie subalgebra $\mathfrak{ham}_{2n}$, defined by

$$\delta_{j_1\ldots j_k}^i (X) = (-1)^k \frac{\partial f_i}{\partial x_{j_1}\ldots \partial x_{j_k}}(0, \ldots, 0) ,$$

where

$$X = \sum_{i=1}^{2n} f_i \frac{\partial}{\partial x_i} \in \mathfrak{a}_{2n} ;$$

see e.g. [2]. For example, a generator of $(\Lambda^2 S^3 H^*)^{\mathfrak{sp}(2, \mathbb{R})}$ in the first line of Table 4 can be given in either of the following three ways:

1. $(x^3 \wedge y^3 - 3x^2 y \wedge xy^2)^*$ ,
2. a graph with 2 vertices and 3 edges joining them ,
3. $- \delta_2^1 \wedge \delta_1^2 - 3 \delta_1^1 \wedge \delta_2^2$.

Metoki [17] gave an explicit basis for $C^*_G(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_8$ and computed the coboundary operators in terms of this basis. Although he did not use the representation theory of $\mathfrak{sp}(2, \mathbb{R})$ explicitly, it turns out that our basis given in Table 4 appears as a subbasis of his. Therefore we can use his computation (which we checked by our method). In particular, the coboundary maps

$$C^4(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10} \xrightarrow{A} C^5(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10} \xrightarrow{B} C^6(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10}$$
are represented by the following $(12, 9)$-matrix

\[
A = \begin{pmatrix}
45 & 18 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -9 & 0 & 0 & 5 & 6 & 6 & 0 & 0 \\
-10 & 0 & -2 & 10 & -10 & -19 & -33 & 0 & 1 \\
-120 & 0 & 72 & 10 & -16 & -3 & 16 & 0 & 6 \\
30 & 0 & -30 & 0 & -2 & -12 & -21 & 0 & -3 \\
-8 & 50 & 0 & 10 & -32 & -48 & -60 & 13 & 0 \\
-1 & -2 & 0 & 10 & -4 & -15 & -25 & 11 & 0 \\
-15 & 18 & 0 & 20 & -34 & -57 & -73 & 0 & 0 \\
-70 & 16 & 0 & 0 & 0 & 6 & -14 & 20 & 0 \\
0 & 0 & 0 & -9 & -20 & 20 & 39 & 52 & 0 & -1 \\
0 & 0 & 57 & 10 & 4 & 12 & 16 & 0 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -140
\end{pmatrix}
\]

and the $(4, 12)$-matrix

\[
B = \begin{pmatrix}
0 & 30 & -39 & 0 \\
-56 & -8 & -12 & 0 \\
0 & 6 & 9 & 0 \\
0 & 28 & -28 & -14 \\
0 & 68 & -66 & -56 \\
-10 & -6 & 12 & 0 \\
10 & -2 & 4 & 0 \\
0 & -22 & 9 & 0 \\
1 & 5 & -10 & 0 \\
0 & -12 & 12 & -14 \\
0 & -6 & 9 & -14 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Of course we have $BA = O$.

The corresponding coboundary maps in

\[
C^6(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8 \xrightarrow{\tilde{A}} C^7(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8 \xrightarrow{\tilde{B}} C^8(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8
\]

are given by the following $(14, 18)$-matrix $\tilde{A}$ and $(4, 14)$-matrix $\tilde{B}$:

\[
\tilde{A} = \begin{pmatrix} A & A_1 \\ O & A_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B \\ B_1 \end{pmatrix}
\]

Here $O$ denotes the zero matrix of size $(2, 9)$ and this checks the fact that

\[
C^*_G(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_{10} \subset C^*_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R}))_8
\]

is indeed a subcomplex. Now an explicit computation shows that the above inclusion induces an isomorphism in cohomology. This completes the proof of Theorem 2. □
Remark 6. The unique leaf cohomology class
\[ \eta \in H^5_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_{10} \]
such that \( \eta \wedge \omega = GKF \) can be represented by an explicit cocycle in \( C^5_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R}))_{10} \) which is a linear combination of the cochains of the forms \((3^347), (3^24^26), (3^245^2)\) in Table 4. We omit the precise formula.

Proof of Corollary 3. We proved in [14] (see also [15]) that both the first Pontrjagin class \( p_1 \in H^4(\mathrm{ESymp}^\delta(\Sigma_g); \mathbb{R}) \) and its fiber integral \( e_1 \in H^2(\mathrm{BSymp}^\delta(\Sigma_g); \mathbb{R}) \), which is the first Mumford-Morita-Miller class, are non-trivial. More precisely, we proved the existence of foliated \( \Sigma_g \)-bundles over closed oriented surfaces such that the signatures of their total spaces are non-zero, while their total holonomy groups are contained in the group \( \mathrm{Symp}(\Sigma_g) \) of area-preserving diffeomorphisms of \( \Sigma_g \) (with respect to some area form). By Theorem 2 the homomorphism
\[ \wedge \omega : H^2_{GF}(\mathfrak{ham}_0^0, \mathfrak{sp}(2, \mathbb{R})) \cong \mathbb{R} \longrightarrow H^4_{GF}(\mathfrak{ham}_2^0, \mathfrak{sp}(2, \mathbb{R})) \cong \mathbb{R} \]
is an isomorphism, where the target is generated by the first Pontrjagin class \( p_1 \). The result follows. \( \square \)

Proof of Theorem 4. We begin with the proof of the first statement. On the one hand, the weight of the elements in \( S^* H^2_{\mathbb{R}} \) is 1 while that of \( \omega^n \) is \(-2n\). Hence the weights of elements of \( \text{Im}\Phi \) restricted to the range \(* \leq 2n\) are non-positive. By the result of Gel’fand-Kalinin-Fuks [7] mentioned in the Introduction, we can conclude that \( \text{Im}\Phi \) is contained in the span of the classes
\[ \omega^kp_1^{k_1} \cdots p_n^{k_n} \in H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R})) \]
with \( k + k_1 + 2k_2 + \cdots + nk_n \leq n \). On the other hand, any element in \( \text{Im}\Phi \) is annihilated by taking the wedge product with a single \( \omega \) because \( \omega^{n+1} \) vanishes identically. Therefore \( \text{Im}\Phi \) is contained in the span of the above classes with the condition that \( k + k_1 + 2k_2 + \cdots + nk_n \) is precisely equal to \( n \).

It remains to prove that all these classes are indeed contained in \( \text{Im}\Phi \). For this, we use the well-known formulae which express \( \omega \) and Pontrjagin classes in terms of the tautological 1-forms. With respect to the standard basis \( x_1, \cdots, x_n, y_1, \cdots, y_n \) of the symplectic vector space \( H^2_{\mathbb{R}} \) and the tautological 1-forms
\[ \delta^1_{j_1 \cdots j_k} \in C^1_{GF}(\mathfrak{ham}_n), \]
we have
\[ \omega = \delta^1 \wedge \delta^{n+1} + \cdots + \delta^n \wedge \delta^{2n}. \]
The universal curvature form $\Omega = (\Omega^i_j)$ can be written as
\[ \Omega^i_j = \sum_{k=1}^{n} \delta^i \wedge \delta^k_{jk}, \]
see e.g. [2], and the Pontrjagin classes $p_i$ ($i = 1, 2, \cdots$) are certain homogeneous polynomials on $\Omega^i_j$ of degree $2i$. In terms of the duals of Hamiltonian functions, the tautological forms $\delta^i$ and $\delta^k_{jk}$ correspond to elements of $H^*$ and $S^3H^*$, respectively. We can now conclude that $p_i \in (\Lambda^{2i}H^* \otimes \Lambda^{2i}S^3H^*)^{Sp(2n, \mathbb{R})}$ ($i = 1, 2, \cdots$).

It follows that any element $\omega^k p_1^{k_1} \cdots p_n^{k_n}$ with $k + k_1 + 2k_2 \cdots + nk_n = n$ is contained in $(\Lambda^{2n}H^* \otimes \Lambda^{2n-2k}S^3H^*)^{Sp(2n, \mathbb{R})}$ because $\Lambda^{2n}H^* \cong \mathbb{R}$ generated by $\omega^n$. Hence such elements are contained in $\text{Im}\Phi$ proving the first part of the Theorem.

Next we prove the second part. Let $\pi: E \to X$ be a foliated $\Sigma_g$-bundle over a closed oriented surface with non-vanishing signature such that the total holonomy group is contained in the group $\text{Symp}(\Sigma_g)$. The existence of such bundles was proved in our paper [14]. The classifying map $f: E \to B\Gamma_2$ of the transversely symplectic foliation on $E$ of codimension 2 has the property that $f^*(p_1) \neq 0$. Now consider the manifold $\mathbb{C}^P^{n-k} \times E^k$ equipped with transversely symplectic foliation of codimension $2n$ which is induced from the point foliation on $\mathbb{C}^P^{n-k}$ and the above foliation on $E$. Then it is easy to see that the characteristic class $\omega^{n-k} p_1^k$ of this foliation is non-trivial. This completes the proof. \[ \square \]

**Remark 7.** By the calculation in the proof $p_1$ is divisible by $\omega$ only if $n = 1$, although in general $p_1^n$ is divisible by $\omega$. The dimension of $(\Lambda^{2i}H^* \otimes \Lambda^{2i}S^3H^*)^{Sp(2n, \mathbb{R})}$ is 1 for $n = 1$ and is 2 for $n \geq 2$.

5. **The Euler characteristic of $H^*(\mathfrak{ham}^0_{2n}, \text{Sp}(2n, \mathbb{R}))$**

As we mentioned already, Perchik [20] gave a formula for the generating function
\[ \sum_{w=0}^{\infty} \chi(H^*(\mathfrak{ham}^0_{2n}, \text{Sp}(2n, \mathbb{R})))_w t^w \]
for the Euler characteristic of the relative cohomology of $\mathfrak{ham}_{2n}$. In this section, we prove a similar formula for $H^*_{GF}(\mathfrak{ham}^0_{2n}, \text{Sp}(2n, \mathbb{R}))$. 
Following [20], let us define rational functions \( p_i(n) \) \((i = 0, 1, \cdots)\) on \( n + 1 \) variables \( x_1, \cdots, x_n, t \) (polynomials with respect to \( t \)) as follows. First consider
\[
a = (a_1, \cdots, a_n), \quad b = (b_1, \cdots, b_n) \quad (a_i, b_i \geq 0)
\]
and put
\[
|a + b| = \sum (a_i + b_i), \quad x^{a - b} = x_1^{a_1 - b_1} \cdots x_n^{a_n - b_n}.
\]
Then define
\[
p_0(n) = \prod_{|a + b| = 2, \, a \neq b} (1 - x^{a - b}),
\]
\[
p_k(n) = \prod_{|a + b| = 2 + k} (1 - t^k x^{a - b}).
\]

**Theorem 8.** The constant term with respect to \( x_i \) \((i = 1, \cdots, n)\) of the infinite product \( \prod_{i=0}^\infty p_i(n) \) is equal to
\[
n!2^n \sum_{w=0}^{\infty} \chi(H^*(\mathfrak{ham}_{2n}^0, \text{Sp}(2n, \mathbb{R})))_w t^w,
\]
where the subscript \( w \) denotes the weight \( w \) part of the cohomology.

**Proof.** Perchik’s formula was obtained by multiplying the above infinite product \( \prod_{i=0}^\infty p_i(n) \) with one more rational function
\[
p_{-1}(n) = \prod_{|a + b| = 1, \, a \neq b} (1 - t^{-1} x^{a - b}).
\]
This part corresponds to the constant term of \( \mathfrak{ham}_{2n} \) which is isomorphic to \( H^2_{\mathbb{R}} \) as a representation of \( \text{Sp}(2n, \mathbb{R}) \) and whose weight is \(-1\). Since the relative cohomology \( H^*_{GF}((\mathfrak{ham}_{2n}^0, \text{Sp}(2n, \mathbb{R}))) \) is defined by ignoring this part, the proof follows by eliminating \( p_{-1}(n) \) from the original formula of Perchik. \( \square \)

**Remark 9.** In the case of \( n = 1 \), a computer computation carried out with the help of M. Suzuki shows that \( \frac{1}{2} \) times the above constant term in low degrees in \( t \) is given by
\[
1 + t^2 - t^{10} + t^{12} - t^{14} - t^{16} + t^{18} - 3t^{20} + 2t^{26} + \cdots,
\]
while the corresponding series for \( H^*_{GF}((\mathfrak{ham}_{2}^0, \mathfrak{sp}(2, \mathbb{R}))) \) due to Perchik is
\[
t^{-2} + 2 - t^8 - t^{14} - t^{22} - t^{28} + t^{30} - t^{32} + \cdots.
\]
The coefficient \(-1\) of \( t^{10} \) in the former series corresponds to our leaf cohomology class \( \eta \). Observe also that the coefficient of \( t^{16} \) is \(-1\). The corresponding coefficient of \( t^{14} \) in the latter series is also \(-1\) which represents the Metoki class in \( H^9_{GF}((\mathfrak{ham}_{2}, \mathfrak{sp}(2, \mathbb{R})))_14 \) (see [17]). Although the cocycle given by Metoki himself is not divisible by \( \omega \), it seems
highly likely that his class can also be decomposed as \( \eta' \wedge \omega \) for some leaf cohomology class \( \eta' \in H^7_{GF}(\mathfrak{ham}^0_{2n}, \text{Sp}(2, \mathbb{R}); \mathbb{R})_16 \).

6. Concluding remarks

It is easy to see that the relative cohomology

\[
H^*_G(\mathfrak{ham}^0_{2n}, \text{Sp}(2n, \mathbb{R})),_w \cong H^*_G(\mathfrak{ham}^1_{2n}, \text{Sp}(2n, \mathbb{R}))
\]

stabilizes as \( n \) goes to infinity. In fact, the limit cohomology is nothing but one of Kontsevich’s theories of graph cohomologies developed in [10], [11], more precisely the commutative case (see [12]). As before, the abelianization homomorphism

\[
\mathfrak{ham}^1_{2n} \longrightarrow \mathcal{S}_3 H^{2n}
\]

induces a homomorphism

\[
\Phi_n : H^*(\mathcal{S}_3 H^{2n})_{\text{Sp}(2n, \mathbb{R})} \longrightarrow H^*_G(\mathfrak{ham}^1_{2n}, \text{Sp}(2n, \mathbb{R})).
\]

However, the stable cohomology

\[
\lim_{n \to \infty} H^*(\mathcal{S}_3 H^{2n})_{\text{Sp}(2n, \mathbb{R})}
\]

is isomorphic to

\[
\mathbb{R}[\text{vertex oriented connected trivalent graph}]/(\text{AS}),
\]

where \( \text{AS} \) denotes the anti-symmetric relation. If we add another relation, called the \( \text{IHX} \) relation, to the above, we obtain the algebra

\[
\mathcal{A}(\phi) = \mathbb{R}[\text{vertex oriented connected trivalent graph}]/(\text{AS}, \text{IHX}).
\]

This algebra plays a fundamental role in the theory of finite type invariants for homology 3-spheres due to Ohtsuki [18], who extended the foundational theory of Vassiliev for knots, and developed by Le, Murakami and Ohtsuki (see [16]).

Garoufalidis and Nakamura proved the following result:

**Theorem 10 (Garoufalidis and Nakamura [6])**. The ideal \( (S^1 H^{2n}) \) of \( \Lambda^* \mathcal{S}_3 H^{2n} \) generated by \( S^1 H^{2n} \subset \Lambda^2 \mathcal{S}_3 H^{2n} \) corresponds exactly to the \( \text{IHX} \)-relation so that there is an isomorphism

\[
\mathcal{A}(\phi) \cong (\Lambda^* \mathcal{S}_3 H^{2n} / (S^1 H^{2n}))^{\text{Sp}(2n, \mathbb{R})}.
\]

Since it can be seen that \( \ker \Phi_{\infty} \) coincides with the \( \text{Sp} \)-invariant part \( ((S^1 H^{2n}))^{\text{Sp}(2n, \mathbb{R})} \) of the above ideal, we conclude that

\[
\text{Image } \Phi_{\infty} \cong \mathcal{A}(\phi).
\]
Thus it is a very important problem to determine $\text{Coker } \Phi_{\infty}$. We have tried to determine whether our leaf cohomology class $\eta \in H^5_{GF}(\mathfrak{ham}^0, \mathfrak{sp}(2, \mathbb{R}))_{10}$ survives in the stable cohomology 

$$\lim_{n \to \infty} H^5_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{10},$$

or not. We have the same problem for other unstable leaf cohomology classes. So far this attempt has remained unsuccessful. One method of attacking this problem would be to compute the generating function $c(t)$ for the commutative graph cohomology by making use of Theorem 8. More precisely, there is important problem of computing

$$c(t) = \lim_{n \to \infty} \sum_{w=0}^{\infty} \chi(H^*(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w)t^w,$$

which is the limit as $n \to \infty$ of the formula given in Theorem 8.

Our computations so far imply

$$c(t) = 1 + t^2 + 2t^4 + 3t^6 + 6t^8 + \cdots.$$

Recall here that the algebra $\mathcal{A}(\phi)$ is known (see [19]) to be a polynomial algebra whose numbers of generators are $1, 1, 1, 2, 2, 3, \cdots$ in degrees $2, 4, 6, 8, 10, 12, \cdots$ so that the generating function for this algebra is

$$1 + t^2 + 2t^4 + 3t^6 + 6t^8 + 9t^{10} + 16t^{12} + \cdots.$$

It should be nice to know how these two generating functions differ from each other.

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Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany
E-mail address: dieter@member.ams.org

Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan
E-mail address: morita@ms.u-tokyo.ac.jp