Asymptotically Efficient Multi-Unit Auctions via Posted Prices

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Abstract

We study the asymptotic average-case efficiency of static and anonymous posted prices for
n agents and m(n) multiple identical items with m(n) = o \left( \frac{n^{\log n}}{n} \right).

When valuations are drawn i.i.d from some fixed continuous distribution (each valuation is
a vector in \( \mathbb{R}^m \) and independence is assumed only across agents) we show: (a) for any “upper
mass” distribution there exist posted prices such that the expected revenue and welfare of the
auction approaches the optimal expected welfare as n goes to infinity; specifically, the ratio
between the expected revenue of our posted prices auction and the expected optimal social
welfare is 1 – O \left( \frac{m(n) \log n}{n} \right), and (b) there do not exist posted prices that asymptotically obtain
full efficiency for most of the distributions that do not satisfy the upper mass condition.

When valuations are complete-information and only the arrival order is adversarial, we pro-
vide a “tiefree” condition that is sufficient and necessary for the existence of posted prices that
obtain the maximal welfare. This condition is generically satisfied, i.e., it is satisfied with
probability 1 if the valuations are i.i.d. from some continuous distribution.

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1 Introduction

Posted prices are a common selling mechanism, with several important advantages. For example, it is simple to grasp, it works well in the presence of budgets, and it fits an online environment where buyers arrive over time. Despite these clear advantages, the current theoretical understanding of how to design posted prices is still partial. In this context, Feldman, Gravin, and Lucier (2014) pose the following general question: “To what extent can approximately efficient outcomes be implemented using anonymous posted prices in settings of incomplete information?”. Their main result shows how to devise posted prices whose expected resulting social welfare is guaranteed to be at least one half of the expected optimal social welfare. Their construction holds for any fixed number of items and agents. Improvements of this one-half guarantee under various assumptions were subsequently made – see a discussion below on related literature. This literature continues to focus on providing guarantees for any fixed number of items and agents and hence the welfare guarantees remain quite far from the optimal expected welfare. One might be tempted to conclude that, with posted prices, significant welfare (and revenue) loss relative to the optimal expected welfare is inevitable. While this may be true for small numbers of agents and/or items, it need not necessarily be true asymptotically.

This paper investigates the possibility of approaching full efficiency as the numbers of items and agents grow. We focus on the most fundamental setting of multiple identical items. There are $m$ identical items and $n$ buyers that arrive one after the other. Each buyer has a monotone valuation $v_i : [m] \to \mathbb{R}_+$ where $v_i(j)$ is i’s private value for receiving $j$ items.1 Before the first buyer arrives, the seller fixes static and anonymous (i.e., fixed throughout the game) prices $p_j$ for every bundle size $j \in [m]$. Every buyer $i = 1, \ldots, n$, here re-numerated upon arrival order, chooses a number of items $0 \leq j_i \leq m - \sum_{k=1}^{i-1} j_k$ that maximizes her utility $u_i(j_i) = v_i(j_i) - p_{j_i}$ ($v_i(0) = p_0 = 0$). An allocation of items to buyers is efficient if it maximizes the resulting social welfare $SW = \sum_{i=1}^{n} v_i(j_i)$. The aim is to obtain social welfare as close as possible to the optimal one which is denoted by OPT. This paper describes two sets of results for this setting, that correspond to two different informational frameworks regarding the seller’s partial information about $v_1, \ldots, v_n$, as described next.

The first framework is incomplete (Bayesian) information: $v_1(\cdot), \ldots, v_n(\cdot)$ are drawn i.i.d from some fixed $m$-dimensional distribution with a bounded support.2 We allow arbitrary distributions that represent arbitrary connections between the different bundle values of a player. All our requirements are expressed only for the marginal distribution of $v_i(1)$. We assume that this marginal distribution belongs to a class $C[0,1]$ of all absolutely continuous distributions with full support on $[0,1]$.3 Our analysis in this setting is asymptotic in $n, m$ and we assume that $m = o\left(\frac{n}{\log(n)}\right)$, i.e.,

1Let $\mathbb{N} = \{1, 2, \ldots\}$ and for $x \in \mathbb{N}$, define $[x] = \{1, \ldots, x\}$.
2The assumption that the distribution is fixed and independent of $n$ is important, since Feldman et al. (2014) show a distribution that depends on $n$ for which any posted price mechanism obtains at most one-half of the optimal social welfare.
3The assumption that the support is in $[0,1]$ is just a normalization. Absolute continuity is convenient since it is
the number of items might grow with the number of buyers but (if it does grow) it grows slower.

Our first main result for this setting is for a class of distributions which we term “Upper Mass” (UM): \( F \in C[0,1] \) has upper mass if there exists \( x_0 < 1 \) such that \( F(x) \leq x \) for any \( x > x_0 \). For example, the uniform distribution has UM. We construct a simple and asymptotically efficient posted prices mechanism for any such distribution. More specifically, for any \( n,m \), the ratio between the expected social welfare of this mechanism and the expected optimal social welfare is \( 1 - O\left( \frac{m \log(n)}{n} \right) \). The same result holds for the revenue of our mechanism. Interestingly, the posted prices that we construct do not depend on the distribution. I.e., we show specific static and anonymous posted prices (that depend on \( n,m \) but not on the distribution) that exhibit the above bounds for any UM distribution.

Given the above result, one might be tempted to believe that there always exist posted prices that provide asymptotic efficiency, even for non upper mass distributions. A-priori this seems particularly reasonable for \( m = 1 \), i.e., when we have just one item and the number of buyers goes to infinity. However, we show that this is not true. Specifically, our second main result shows that even if \( m = 1 \) any distribution which is not UM and in addition satisfies \( f(1) < 1 \) does not have a fixed and anonymous posted prices mechanism whose expected efficiency approaches the expected optimal efficiency as the number of agents goes to infinity.

The second framework that we consider is complete information with unknown arrival order: the \( n \) valuations are known to the mechanism designer but the arrival order is determined by an adversary. In this part we assume submodular valuations. It is not a-priori clear whether one can obtain the efficient outcome with static and anonymous posted prices (this is not immediately clear even if the arrival order is known). Ezra, Feldman, Roughgarden, and Suksompong (2017) (also in Ezra, Feldman, Roughgarden, and Suksompong (2018)) study this issue in length, and they obtain several upper and lower bounds. In particular (Ezra et al., 2017, Proposition 4.3) shows that there exists a tuple of valuations (two items and two buyers) such that no static and anonymous posted prices mechanism can obtain more than \( \frac{2}{3} \) of the optimal social welfare.

Our main positive result for this setting is a posted prices mechanism that obtains the efficient outcome whenever the tuple of valuations satisfies a “tiefree” condition that we define. This condition is generically satisfied, i.e., if we draw \( n \) valuations i.i.d as previously defined, then with probability 1 this condition will be satisfied (even if \( n = m = 2 \)).

The above results obviously reveal just part of the picture and many open questions remain. Three important ones, in our opinion, are: (1) What are the best approximation guarantees for natural classes of lower mass distributions? (2) What are the best approximation guarantees for the case of \( m = O(n) \)? (3) Can similar results be obtained for the case of non-identical items?

**Paper Organization.** The rest of the paper is organized as follows. Section 2 studies the setting equivalent to assuming \( F(x) = \int_0^x F'(t) dt \) for any \( F \in C[0,1] \), where \( dt \) is the Lebesgue measure.
of Incomplete (Bayesian) Information, with the positive result for upper mass distributions in Section 2.1, and the negative result for most other distributions in Section 2.2. In Section 3, we study the setting of complete information with an adversarial arrival order.

**Related Literature.** Feldman et al. (2014) study the case of combinatorial auctions (different items) with XOS valuations. They construct posted prices that obtain in expectation at least one half of the optimal social welfare, for any fixed number of agents. This bound holds even if the arrival order is adversarial, i.e., an adversary determines the arrival order after the valuations are drawn. If the arrival order is random as well, Ehsani, Hajiaghayi, Kesselheim, and Singla (2018) improve the one-half bound to $1 - \frac{1}{e}$. The case of complete information and adversarial arrival order is extensively studied by Ezra et al. (2017, 2018). They provide various upper and lower bounds on approximation guarantees with various types of posted prices. Dürring, Feldman, Kesselheim, and Lucier (2017) give explicit formal connections between the two frameworks of complete information and incomplete (Bayesian) information, showing how to transform posted prices for the former into posted prices for the latter.

Revenue issues with posted prices are studied in Dürring, Fischer, and Klimm (2016). They compare the revenue loss with static pricing compared to a dynamic (optimal) one. In an earlier paper Chawla, Hartline, Malec, and Sivan (2010) approximate the optimal revenue (as well as the optimal welfare) with posted prices. Several papers compare posted prices mechanisms with other types of auctions, for example, Kultti (1999), Wang (1995) and Wang (1993). A comparison of the best welfare guarantees of dynamic auctions versus posted prices, for the case of one item, is done in Blumrosen and Holenstein (2008). They show that on the one hand discriminatory (personalized) prices can be asymptotically equivalent to full revelation auctions, but on the other hand those auctions with symmetric prices are inferior. The case of prior-independent posted prices is analyzed in Babaioff, Blumrosen, Dughmi, and Singer (2011), where the seller does not know the distributions. They find that in general they cannot even guarantee a constant fraction of the optimal social welfare, but in the special case of MHR distributions, a constant fraction can be guaranteed. Posted prices for procurement settings are studied in Badanidiyuru, Kleinberg, and Singer (2012).

## 2 Incomplete (Bayesian) information

Throughout, the number of items $m : \mathbb{N} \to \mathbb{N}$ is a given function of the number of agents, although for brevity we write $m$ instead of $m(n)$. As discussed in Section 1, our results are given for the following class of distributions:

**Definition 1.** The distribution $F \in \mathcal{C}[0,1]$ has upper mass (UM) if there exists an $x_0 < 1$ such that $F(x) \leq x$, for all $x > x_0$. It has lower mass (LM) if there exists an $x_0 < 1$ such that $F(x) > x$, for all $x > x_0$. 

The uniform distribution provides a (non-unique) lower bound for upper mass distributions. Here is one more UM example.

**Example 2 (Upper Mass).**
\[
f_4(x) = 2x
\]
\[
F_4(x) = x^2
\]

Here is a generic example for lower mass, in the sense of Lemma 11, to come.

**Example 3 (Lower Mass).**
\[
f_3(x) = 2 - 2x
\]
\[
F_3(x) = 2x - x^2
\]

**Definition 4.** In the setting of incomplete information, prices are efficient if \( \frac{\mathbb{E}[\text{SW}]}{\mathbb{E}[\text{OPT}]} \to 1 \), as the number of buyers tend to infinity.

### 2.1 Upper mass distributions

**Notation 5.** For a given sequence \( \tau = (\tau_n)_{n \in \mathbb{N}} \), with, for all \( n \), \( \tau_n \in (0,1) \), and a given distribution \( F \in \mathcal{C}[0,1] \), let \( F_n = F(\tau_n) = \Pr(X \leq \tau_n) \).

**Definition 6** (A Posted Price for UM Distributions). For any fixed number of agents \( n \), and a number of items \( m = o \left( \frac{n \log(n)}{n} \right) \), the price for a single item is \( p_1 = p_1(n,m) = n \frac{m+1}{n} \), and for larger bundle sizes \( j > 1 \), the price is \( p_j = \infty \).

**Theorem 7.** Consider an upper mass distribution \( F \in \mathcal{C}[0,1] \). With a Posted Price for UM Distributions (as in Definition 6),
\[
\frac{\mathbb{E}[\text{SW}]}{\mathbb{E}[\text{OPT}]} = 1 - \mathcal{O} \left( \frac{m \log(n)}{n} \right)
\]
Moreover, the prices are efficient, and in addition
\[
\mathbb{E}[\text{revenue}] \to \mathbb{E}[\text{OPT}]
\]
as the number of buyers \( n \to \infty \).
Proof. Let \( q_j(n) = \binom{n}{j} \) denote the binomial coefficient. For any fixed \( n \), the probability that all items get sold is

\[
1 - \sum_{j=0}^{m-1} q_j(n)(F_n)^{n-j}(1 - F_n)^j
\]

Namely, conditioned on that there is still an item available when agent \( i \) arrives at the market, they will not buy any item with probability \( F_n \). The number of combinations of \( 1 \leq j \leq m \) unsold items is a polynomial in \( n \) of degree \( j \), namely \( q_j(n) = \binom{n}{j} \). Therefore \( q_j(n)(F_n)^{n-j}(1 - F_n)^j \) is the probability that exactly \( n-j \) agents will not buy, and so

\[
\sum_{j=0}^{m-1} q_j(n)(F_n)^{n-j}(1 - F_n)^j
\]

is the probability that not all items will get sold.

Claim 1. For all sufficiently large \( n \), \( \sum_{j=0}^{m-1} q_j(n)(F_n)^{n-j}(1 - F_n)^j < n^m(F_n)^n \).

Proof of Claim 1. Since \( \tau_n \to 1 \) and since \( F \) has full support, for all sufficiently large \( n \), \( F_n > 1/2 \). Therefore \( \sum_{j=0}^{m-1} q_j(F_n)^{n-j}(1 - F_n)^j < \sum_{j=0}^{m-1} q_j(n)(F_n)^n = (F_n)^n \sum_{j=0}^{m-1} q_j(n) < n^m(F_n)^n \), which suffices because the largest degree of the \( q_j(n) \) is \( m-1 \).

By Claim 1 and (5), if \( n^m(F_n)^n \to 0 \) then all items get sold, and this implies \( \mathbb{E}[SW] \geq m\tau_n \), for all sufficiently large \( n \). (If all items get sold, the social welfare is at least \( m \) times the price for one item.) Hence, if \( \tau_n \to 1 \), then intuitively \( \mathbb{E}[SW] \to m = \mathbb{E}[OPT] \), for reasonably small \( m \) (for example \( m \) a constant), but we must prove a little more.

Now, for all \( n \), with \( m = o \left( \frac{n}{\log(n)} \right) \), set the prices \( p_1(n,m) \) as \( \tau_n = n^{-m+1} \).

Claim 2. For prices \( \tau_n = n^{-m+1} \), then \( \tau_n \to 1 \) and \( n^m(F_n)^n = o(1) \).

Proof of Claim 2. For the first part, note that

\[
\log \left( n^{-m/n} \right) = \frac{m}{n} \log(n)
\]

\[
= -o \left( \frac{n}{\log(n)} \right) \frac{\log(n)}{n}
\]

\[
= o(1)
\]

That is \( \log \left( n^{-m/n} \right) \to 0 \), as \( n \to \infty \), and hence \( \tau_n \to 1 \). For the second part, by upper mass, \( \exists x_0: \)
\( F(x) \leq x \), for all \( x > x_0 \). Therefore, for all sufficiently large \( n \),

\[
(F_n)^n \leq (\tau_n)^n = o(n^{-m}),
\]

which suffices to prove Claim 2.

Assume that \( \tau_n = n^{-m+1} \) is as in Claim 2. We proceed to show that, if \( 1 \leq m = o\left(\frac{n}{\log(n)}\right) \), then

\[
\mathbb{E}[\text{OPT}] - \mathbb{E}[\text{SW}] = O\left(\frac{m^2 \log(n)}{n}\right),
\]

and hence

\[
\mathbb{E}[\text{SW}] / \mathbb{E}[\text{OPT}] = 1 - O\left(\frac{m \log(n)}{n}\right),
\]

because \( \mathbb{E}[\text{OPT}] \to m \) if \( m = o(n) \). (See also Lemma 9 in Section 2.2.)

For all sufficiently large \( n \), we have the following sequence of inequalities, where (12) is by Claim 1 and (13) is by the upper mass assumption

\[
\mathbb{E}[\text{SW}] \geq m \tau_n \left( 1 - n^m (F_n)^n \right) \geq m \tau_n \left( 1 - n^m (\tau_n)^n \right) = mn^{-\frac{m+1}{n}} (1 - n^{-1}),
\]

and where (14) follows by \( \tau_n = n^{-\frac{m+1}{n}} \). Therefore, for all sufficiently large \( n \),

\[
\mathbb{E}[\text{OPT}] - \mathbb{E}[\text{SW}] \leq m - mn^{-\frac{m+1}{n}} (1 - n^{-1}) = O\left( m \left( 1 - n^{-\frac{m+1}{n}} \right) \right) = O\left( m \left( 1 - \left( 1 - \frac{m+1}{n} \log(n) + o\left( \frac{m+1}{n} \log(n) \right) \right) \right) \right) = O\left( m \left( \frac{m+1}{n} \log(n) \right) \right).
\]

For (17), we have used that \( \tau_n = e^{-\frac{m+1}{n} \log(n)} \), together with Maclaurin series expansion of \( e^x \), which we can do since \( m = o\left(\frac{n}{\log(n)}\right) \). This proves (11) and hence the result. The reason that we get the same limiting behavior for the expected revenue is that the inequality in (12) holds also for the expected revenue.

**Observation 8.** Consider a related setting, where each marginal valuation \( v_i - v_{i-1} \) is upper mass i.i.d. Then, by adjusting Definition 6 to uniform prices such that \( p_i - p_{i-1} = n^{-\frac{m+1}{n}} \), for all \( i > 0 \), then Theorem 7 still holds.
2.2 Lower mass distributions

In this section, we prove most results for the case \( m = 1 \), since if we cannot even sell one item efficiently, we could not hope to sell more items efficiently. We prove that, in the case of lower mass distributions \( F \in C[0,1] \), there is no asymptotically efficient fixed and anonymous posted price to sell a single item, if \( f(1) < 1 \). This is Theorem 15, but the engine to the proof is Lemma 11. We use \( \tau = (\tau_n)_{n \in \mathbb{N}} \) as a generic price vector for the price of a single item, depending on the number of buyers, \( n \).

**Lemma 9.** Consider a distribution \( F \in C[0,1] \). Then, for \( m = 1 \), \( \lim_{n \to \infty} E[OPT] = 1 \).

**Proof.** Since \( F \) is continuous, then for all \( \epsilon > 0 \), for all agents \( i \), \( \Pr_F(v_i(1) > 1 - \epsilon) > 0 \). So for sufficiently large \( n \), w.h.p. some buyer valuation will be in the interval \((1 - \epsilon, 1] \). Since \( \epsilon > 0 \) is arbitrary, the lemma holds. \( \square \)

**Lemma 10.** Given a distribution \( F \in C[0,1] \), the following are equivalent.

(i) \( \tau \) is efficient;

(ii) \( \tau_n \to 1 \) as \( n \to \infty \), and \( \Pr(\text{the item gets sold}) \to 1 \) as \( n \to \infty \).

**Proof.** Item (ii) implies item (i) since for all \( n \), \( E[SW] \geq \tau_n \Pr(\text{an item gets sold}) \) and by item (2) this goes to 1 as \( n \) goes to infinity. For the other direction, suppose first that \( \limsup \tau_n = 1 - \delta \), for some \( \delta > 0 \). Thus there exists an \( 0 < \epsilon < \delta \) such that, for any \( n \),

\[
\Pr[\text{the item is sold to a buyer with value in}(1 - \delta, 1 - \delta/2)] \geq \epsilon
\]

Therefore \( E[SW] \leq (1 - \epsilon) \cdot 1 + \epsilon \cdot (1 - \frac{\delta}{2}) = 1 - \frac{\epsilon \delta}{2} \). Since \( \epsilon \) is independent of \( n \), it follows that \( \lim_{n \to \infty} E[SW] \leq 1 - \frac{\epsilon \delta}{2} \). Now suppose that \( \tau_n \to 1 \) as \( n \to \infty \) but \( \Pr(\text{the item gets sold}) < 1 - \epsilon \) for all \( n \). In this case, because the prices and buyer valuations are no larger than 1, \( \lim_{n \to \infty} E[SW] < 1 - \epsilon \). By Lemma 9 the expected optimal social welfare is 1 and therefore in both cases \( \tau \) is not efficient. \( \square \)

The following Lemma generalizes Example 3.

**Lemma 11.** Define a distribution \( F \) by a pdf of the form \( f(x) = c + 1 - 2cx \), for some given constant \( c \in (0,1] \). Then \( F \in C[0,1] \) has lower mass, and it does not have efficient prices.

**Proof.** By continuity, we get \( F(x) = (c + 1)x - cx^2 \), so for any \( c \in (0,1] \), \( F(0) = 0 \), \( F(1) = 1 \), and \( 0 \leq F(x) \leq 1 \) for all \( x \in [0,1] \) since \( F \) is non-decreasing. Hence \( F \in C[0,1] \). Moreover, \( F \) is lower mass, since for all \( x \in [0,1] \), \( c(x - x^2) > 0 \) implies for all \( x \), \( F(x) > x \).

By Lemma 10, if prices are efficient, two conditions must be satisfied: both \( \tau_n \to 1 \), and the probability that an item does not get sold must tend to zero. We show that we cannot have both properties simultaneously.
For a fixed number of agents $n$, the probability that no item gets sold, with prices $\tau$, is $(F_n)^n$. Now, by inserting prices $\tau_n \to 1$, into our CDF $F(x) = (c + 1)x - cx^2$, and by expanding terms, we get the following inequality, for all $n > 1$,

\[
(F_n)^n = (c + 1)^n(\tau_n)^n - (c + 1)^{n-1}(\tau_n)^{n-1}c(\tau_n)^2 + \cdots \geq (c + 1)^{n-1}(\tau_n)^{n-1}((c + 1)\tau_n - c(\tau_n)^2)
\]  

(19)

Namely, to prove this inequality (where we for large $n$ have ignored most terms), it suffices to note that for all $i < n/2$, then

\[
((c + 1)\tau_n)^{n-2i}(c\tau_n^2)^{2i} - ((c + 1)\tau_n)^{n-2i-1}(c\tau_n^2)^{2i+1} > 0
\]

if and only if

\[
(c + 1)\tau_n - c\tau_n^2 > 0
\]

which holds, since $c + 1 > 1$ (is a constant) and $\tau_n < 1$.

Now, return to inequality (20). We have $(c + 1)\tau_n - c(\tau_n)^2 \geq \tau_n$ and $\tau_n > 1/2$ for sufficiently large $n$ if $\tau_n \to 1$.

Therefore, assuming that $(F_n)^n = \Pr(\text{no item gets sold}) \to 0$, the product $(c + 1)^{n-1}(\tau_n)^{n-1}$ must tend to zero for large $n$. This forces $\tau_n < \frac{1}{c+1}$, for all sufficiently large $n$.

But this contradicts that $\tau_n \to 1$. Therefore, we cannot have both the desired properties simultaneously.

To obtain our main result for this section, we define an order over distributions in $\mathcal{C}[0,1]$.

**Definition 12.** Let $F, G \in \mathcal{C}[0,1]$. If there exists an $x_0 \in [0,1]$ such that for all $x > x_0$, $\bar{F}(x) \leq \bar{G}(x)$, then $F \leq G$.

Note that if $F$ has lower mass and $G$ has upper mass, then $F \leq G$, and if $f(1) < 1$ then lower mass is equivalent with not upper mass. In general Definition 12 is a partial order; for example $f(x) = 1 + C(1 - x)\sin(2\pi/(1 - x))$ is continuous, and it can be normalized as a pdf, by setting the constant $C$ such that $F(1) = 1$, so $F \in \mathcal{C}[0,1]$, but it has neither lower- nor upper mass, and in particular it is incomparable with the uniform distribution.

**Notation 13.** We say that the price function $\tau$ is $F$-efficient if $\tau$ is efficient with respect to $F \in \mathcal{C}[0,1]$.

**Lemma 14.** If $F \leq G$ and $\tau$ is $F$-efficient, then $\tau$ is $G$-efficient. Equivalently, if $F \leq G$ and there are no efficient prices for $G$, then there are no efficient prices for $F$.
Proof. For all agents $i$, the probability that they buy the item is no smaller given the distribution $G$ than given the distribution $F$. Hence, for a fixed price function, and for all $n$,

$$\Pr_F(\text{an item gets sold}) \leq \Pr_G(\text{an item gets sold})$$

Therefore $E_F[\text{SW}] \to 1$ implies $E_G[\text{SW}] \to 1$.

\[ \square \]

**Theorem 15.** Suppose that $G \in \mathcal{C}[0,1]$ satisfies $g(1) < 1$. Then $G$ does not have efficient prices.

Proof. Let $f(x) = c + 1 - 2cx$ be as in Lemma 11, where the constant $c$ will depend on the given pdf $g$. By combining that result with Lemma 14, we get: if $F \geq G$, with $f = F'$, then no efficient prices exist for $G$. Hence, it suffices to find an $x_0 \in (0,1]$, such that, for all $x \in (x_0,1]$, $g(x) \leq f(x)$.

By $g(1) < 1$ we can take $c = (1 - g(1))/2 > 0$. Then $f(1) = 1 - c > 1 - 2c = g(1)$, and so, by continuity, there is an $x_0 < 1$ such $g(x) \leq f(x)$, for all $x \in (x_0,1]$.

We finish this section with an example that not yet understood in the sense of Open problem 1 below. This is because it has lower mass but $f(1) = 1$, and so Theorem 15 does not apply.

**Example 16.**

$$f_5(x) = \begin{cases} -6x + 3, & \text{if } x < 1/2. \\ 2x - 1, & \text{otherwise}. \end{cases}$$ (21)

$$F_5(x) = \begin{cases} -3x^2 + 3x, & \text{if } x < 1/2. \\ x^2 - x + 1, & \text{otherwise}. \end{cases}$$ (22)

**Open problem 1.** Do there exist efficient prices in those special cases of lower mass when $f(1) = 1$?

### 3 Complete information with adversarial arrival order

In this section we assume submodular valuations. Define the marginal value of item $j > 0$ to agent $i$ as $\delta_{i,j} = v_i(j) - v_i(j - 1)$. Submodularity means that marginal valuations are non-increasing. That is, $0 \leq \delta_{i,j+1} \leq \delta_{i,j}$, for all $0 \leq j < m$. Throughout this section, the tuple of valuations $v_1, \ldots, v_n$ is fixed and known to the seller. Let $\Delta^j$ (derived from $v_1, \ldots, v_n$) be the multiset of the $j$ largest marginal valuations in the matrix $\Delta = (\delta_{i,j})_{i \in [n], j \in [m]}$. For example, the multiset $\Delta^{mn}$ contains all marginal values, and $\Delta^m$ contains the $m$ largest values.

**Observation 17.** Given a matrix of submodular marginal valuations $\Delta$, $\text{OPT} = \sum_{\delta \in \Delta^m} \delta$.

By this we sometimes call the multiset $\Delta^m$, the OPT-set.
Definition 18. If $\min \Delta^m > \min \Delta^{m+1}$, then $\Delta$ is tiefree, and otherwise, i.e. if $\min \Delta^m = \min \Delta^{m+1}$, then $\Delta$ is non-tiefree.

Our main result for complete information (Theorem 23) is that if the matrix of marginal valuations is tiefree, then there exist posted prices such that, independent of arrival order, $SW = OPT$, and we describe an algorithm which finds such prices (Definition 21).

If the marginal valuations in a submodular matrix $\Delta$ are pairwise distinct, then, for all $j \in [m]$, $\Delta^j$ reduces to a set of size $j$ and notation simplifies.\footnote{Note that, if the agents’ marginal valuations are independently drawn from some continuous distribution, then they will be pairwise distinct with probability 1.} In particular, it is convenient (but not necessary) to have a unique bijection between the multiset $\Delta^{mn}$ and the matrix $\Delta$. In general the matrix $\Delta^m$ need not be unique. However if $\Delta$ is tiefree, $\Delta^m$ is unique.

Definition 19. Consider a submodular matrix $\Delta$ and a fixed $\Delta^m$. Let $\delta(j) = \{\delta_{i,j} | \delta_{i,j} \in \Delta^m, i \in [n]\}$ denote the (multi)set of marginal values from the (multi)set $\Delta^m$ that originate in the agents’ marginal valuations of bundle size $j \in [m]$.

Notation 20. Bundle size $j$ is tagged in $\Delta^m$ if $\delta(j) \neq \emptyset$.

Note that, if bundle size $j > 1$ is tagged, then $j - 1$ is tagged, since $\Delta^m$ is constructed from the optimal allocation.

We assign anonymous static prices $p_j$ for each bundle size $j$. It is convenient to state the algorithm in full generality, for multisets, the emphasis being that the origins of $\Delta^m$ are fixed, in which case, we say that $\Delta^m$ is fixed. For the reader’s convenience, we advice to think about the simpler setting with pairwise distinct elements in $\Delta$.

Definition 21 (Complete Information Posted Prices, CIPP). Consider a submodular matrix $\Delta$ and a fixed $\Delta^m$. For each bundle size $j$, tagged in $\Delta^m$, let $\mu_j = \min \delta(j)$. If bundle size $k$ is tagged, then its price is

$$p_k = \sum_{j \leq k} \mu_j,$$

and otherwise $p_k = \infty$.

Note that bundle sizes that are not tagged in $\Delta^m$ will never be sold, because the utility is $-\infty$. All agents have the same indifference action, and unless otherwise stated, we adapt the following convention:

Definition 22 (Indifference action). If an agent is indifferent between two or more bundles, she selects the largest bundle that maximizes her utility.

Let us illustrate with a simple example, displayed in Table 1.
Table 1: A simple example with 2 buyers and 3 items

| $j$ | 1 | 2 | 3 |
|-----|---|---|---|
| $\delta_{1,j}$ | 6 | 5 | 3 |
| $\delta_{2,j}$ | 4 | 2 | 1 |

In Table 1, the OPT-set is $\Delta^3 = \{4, 5, 6\}$, and the tagged subsets are $\delta^{(1)} = \{4, 6\}, \delta^{(2)} = \{5\}$, so the price vector is $p = (4, 9, \infty)$. One can see that our indifference rule achieves OPT in this example, for any arrival order. Note that, if agent 1 were not to choose the larger bundle, then OPT would not be achieved.

**Theorem 23.** Consider a tiefree matrix (or one with pairwise disjoint entries) of marginal valuations $\Delta$. Then, independently of arrival order, the complete information posted prices (Definition 21) achieve the optimal social welfare $SW = OPT$.

**Proof.** Define $k(i) = \arg\max_j \{\delta_{i,j} \in \Delta^m\}$. Note that $\sum_i v_i(k(i)) = \sum_i \sum_{j=1}^{k(i)} \delta_{i,j} = OPT$. Thus, by Observation 17, to prove the theorem it suffices to show that with the complete information posted prices and adversarial arrival order, the number of items that each agent $i$ buys is $k(i)$. To show this, suppose towards a contradiction that agent $i$ is the first to buy a quantity different than $k = k(i)$. Then, for agent $i$, with $k'' < k < k'$, either:

(i) $u(k) \leq u(k')$,

(ii) $u(k'') > u(k)$, or

(iii) the remaining number of items is smaller than what would have maximized agent $i$'s utility.

For (iii), note that this cannot happen since $i$ is the first such agent.

Assume first that (i) applies. Thus, $v(k) - p_k \leq v(k') - p_{k'}$, which means that

$$\sum_{j=k+1}^{k'} \min \delta^{(j)} = \sum_{j=k+1}^{k'} \mu_j \quad (24)$$

$$= p_{k'} - p_k \quad (25)$$

$$\leq v(k') - v(k) \quad (26)$$

$$= \sum_{j=k+1}^{k'} \delta_{i,j} \quad (27)$$

Let us justify the inequalities: (24) is by definition of the $\mu_j$s; (25) is by definition of the posted prices; (26) is by the assumption that agent $i$ buys $k'$ items, and (27) is by the definition of marginal values.
This cannot happen, because, by assumption $\delta_{i,j} \notin \Delta^m$, if $j > k$. Namely, $\Delta^m$ contains the $m$ largest marginal values, so by the tiefree assumption (Definition 18), even the smallest member in this set is strictly larger than the ones that were not chosen.

Suppose next that (ii) applies. So, for agent $i$, $v(k) - p_k < v(k'') - p_{k''}$, which means that

$$\sum_{j=k''+1}^{k} \min \delta^{(j)} = \sum_{j=k''+1}^{k} \mu_j$$

$$= p_k - p_{k''}$$

$$> v(k) - v(k'')$$

$$= \sum_{j=k''+1}^{k} \delta_{i,j},$$

This cannot happen, because, by assumption $\delta_{i,j} \in \Delta^m$, for all $k'' < j \leq k$. ∎

The following theorem demonstrates the importance of the tiefree condition. It is given only for the completeness of the presentation, since, as mentioned in the introduction, Proposition 4.3 in Ezra et al. (2017) shows that no static and anonymous posted prices can obtain more than $\frac{2}{3}$ of the optimal social welfare for complete information with adversarial order (obviously, the proof of this statement relies on a tuple of valuations which is non-tiefree.)

**Theorem 24.** For the valuations in Table 2, there do not exist static and anonymous posted prices that obtain $SW=OPT$ for all arrival orders.

**Proof.** Suppose towards a contradiction that there exist such posted prices $p(j)$ for $j = 1, 2, 3$. We achieve $SW=OPT$ if and only if agent 2 buys two items, either agent 1 or agent 3 buys exactly one item and the other one buys nothing. However, if the arrival order is 1, 3, 2, then either both agent 1 and agent 3 buy one item or both buy no item at all. ∎

| $j$ | 1 | 2 | 3 |
|-----|---|---|---|
| $\delta_{1,j}$ | 1 | 0 | 0 |
| $\delta_{2,j}$ | 2 | 2 | 0 |
| $\delta_{3,j}$ | 1 | 0 | 0 |

**Open problem 2.** What is the approximation ratio of our posted prices for non-tiefree valuation tuples?
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