QUANTUM HAMILTONIAN REDUCTION OF W-ALGEBRAS AND CATEGORY \( \mathcal{O} \)

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Abstract. We define a quantum version of Hamiltonian reduction by stages, producing a construction in type A for a quantum Hamiltonian reduction from the W-algebra \( U(\mathfrak{g}, e_1) \) to an algebra conjecturally isomorphic to \( U(\mathfrak{g}, e_2) \), whenever \( e_2 \geq e_1 \) in the dominance ordering. This isomorphism is shown to hold whenever \( e_1 \) is sub-regular, and in \( \mathfrak{sl}_n \) for all \( n \leq 4 \).

We next define embeddings of various categories \( \mathcal{O} \) for the W-algebras associated to \( e_1 \) and \( e_2 \), amongst them the embeddings \( \mathcal{O}(e_2, p) \hookrightarrow \mathcal{O}(e_1, p) \), where \( p \) is a parabolic subalgebra containing both \( e_1 \) and \( e_2 \) in its Levi subalgebra.

1. Introduction

Let \( \mathfrak{g} \) be a semi-simple complex Lie algebra with universal enveloping algebra \( U(\mathfrak{g}) \), and let \( G \) be the simply-connected algebraic group satisfying \( \text{Lie}(G) = \mathfrak{g} \). To any nilpotent element \( e \in \mathfrak{g} \), one can associate a certain non-commutative algebra \( U(\mathfrak{g}, e) \) known as the W-algebra associated to the nilpotent \( e \). There are several definitions of the W-algebra depending on different choices and parameters, but it is known that they are all equivalent up to isomorphism, and depend only on the nilpotent orbit of \( e \) under the adjoint action of \( G \).

The definition of W-algebras of primary use in this paper is as a quantum Hamiltonian reduction of the universal enveloping algebra \( U(\mathfrak{g}) \) with respect to a choice of nilpotent subalgebra \( \mathfrak{m} \) and character \( \chi \) thereof, both derived from the nilpotent \( e \). In short, given a Lie algebra \( \mathfrak{m} \) coming from a good grading of \( \mathfrak{g} \) and the character \( \chi \in \mathfrak{m}^* \) associated to \( e \) under the identification \( \mathfrak{g} \simeq \mathfrak{g}^* \) given by the Killing form. Considering the shift \( \mathfrak{m}_{\chi} := \{ y - \chi(y) : y \in \mathfrak{m} \} \), the W-algebra can be defined as the algebra of invariants in the quotient \( U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{\chi} \) under the adjoint action of \( \mathfrak{m} \); i.e. \( U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{\chi})^\mathfrak{m} \). Since all objects involved are filtered by the Kazhdan filtration, the W-algebra is itself a filtered algebra. Taking the associated graded algebra yields the ring of functions on the Slodowy slice \( \mathcal{S}_\chi \subseteq \mathfrak{g}^* \), and quantum Hamiltonian reduction reduces to ordinary Hamiltonian reduction of Slodowy slices [GG].

With this framework in mind, one can ask whether this quantum Hamiltonian reduction can be decomposed into steps, analogous to the classical construction of Hamiltonian reduction by stages. In particular, given a pair of W-algebras defined by quantum Hamiltonian reduction, when can an intermediate reduction between the two be found which commutes with the original reductions up to isomorphism, as in fig. 1.

In section 3, we give a partial answer to this question in type A. We first present a construction using the combinatorics of pyramids, which for any pair of nilpotent elements \( e_1, e_2 \in \mathfrak{sl}_n \), where \( e_2 \) covers \( e_1 \) in the dominance ordering, produces an intermediate reduction from the W-algebra \( U(\mathfrak{g}, e_1) \) in type A to a certain algebra

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associated to $e_2$. We conjecture that this algebra is isomorphic to the W-algebra $U(g, e_2)$ (conjecture 3.13), and present a proof in known cases.

In section 4, we turn our attention to the representation theory of W-algebras. This is a subject which has been widely studied, and a number of connections to the representation theory of the Lie algebra $g$ itself have been found. The construction of quantum Hamiltonian reduction by stages produces a $(U(g, e_1), U(g, e_2))$-bimodule for any pair of nilpotents $e_1$ and $e_2$ as above. This in turn provides a pair of adjoint functors $U(g, e_1) \to U(g, e_2)\mod$ for any such pair. A modification of an argument of Loseu [Los3] can be used to produce embeddings of the corresponding categories $O(e_2, p) \hookleftarrow O(e_1, p)$, whenever $e_1 \leq e_2$.

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2. W-algebras and quantum Hamiltonian reduction

We first recall the definition of W-algebras via quantum Hamiltonian reduction. Let $g$ be a semisimple Lie algebra over $\mathbb{C}$, and let $e \in g$ be a chosen nilpotent element. By the Jacobson–Morozov theorem any non-zero $e$ can be completed to an $\mathfrak{sl}_2$-triple $(e, h, f)$. The semisimple element $h$ determines a $\mathbb{Z}$-grading of the Lie algebra $g$ by declaring $g = \bigoplus_{j \in \mathbb{Z}} g_j$, where $g_j = \{x \in g : [h, x] = jx\}$. This grading satisfies the following list of useful properties, where $g_j$ is the centraliser of $e$ in $g$, and $\langle \cdot, \cdot \rangle$ is the Killing form:

(GG1) $e \in g_2$,
(GG2) $ad_e : g_j \to g_{j+2}$ is injective for $j \leq -1$,
(GG3) $ad_e : g_j \to g_{j+2}$ is surjective for $j \geq -1$,
(GG4) $g_j(e) \subseteq \bigoplus_{j \geq 0} g_{j}$,
(GG5) $\langle g_i, g_j \rangle = 0$ unless $i + j = 0$,
(GG6) $\dim g_j(e) = \dim g_0 + \dim g_1$.

It is a well-known result that any grading which satisfies properties GG1 to GG3 will necessarily satisfy all of them (and even more strongly that properties GG2 and GG3 are equivalent for any $\mathbb{Z}$-grading). This motivates the following definition, which provides a generalisation of the gradings coming from $\mathfrak{sl}_2$-triples.

Definition 2.1. A $\mathbb{Z}$-grading of $g$ is called a good grading for the nilpotent $e$ if it satisfies properties GG1 to GG3. A good grading which comes from an $\mathfrak{sl}_2$-triple containing $e$ is called a Dynkin grading. A good grading which vanishes in odd degree is called an even grading.

Note that although all Dynkin gradings are good, there exist good gradings which are not Dynkin: these non-Dynkin good gradings will be important for our work. From this point on, fix a good grading of the Lie algebra $g$. 

![Figure 1. Reduction of W-algebras by stages.](image-url)
The space $\mathfrak{g}_{-1}$ has a natural symplectic form $\omega$ given by $\omega(x, y) := \langle e, [x, y] \rangle$. Choosing a Lagrangian subspace $I \subseteq \mathfrak{g}_{-1}$ with respect to this form, one can define a Premet subalgebra $m := I \oplus \bigoplus_{j < -2} \mathfrak{g}_j$. Premet subalgebras enjoy a number of properties we record for future reference. Let $\mathcal{O}_e$ be the adjoint orbit through $e$.

1. $m$ is an ad-nilpotent subalgebra of $\mathfrak{g}$,
2. $\dim m = \frac{1}{2} \dim \mathcal{O}_e$,
3. $m \cap \mathfrak{g}_0(e) = 0$,
4. $\chi := \langle e, \cdot \rangle$ restricts to a character of $m$.

Given a Lie algebra $m$ with character $\chi$, one can define the shifted Lie algebra $m_\chi := \{ y - \chi(y) : y \in m \}$. With this in hand, we can define the $W$-algebra.

**Definition 2.2.** Let $e \in \mathfrak{g}$ be a nilpotent element with a chosen good grading and Lagrangian subspace $I \subseteq \mathfrak{g}_{-1}$, and let $m$ be the associated Premet subalgebra. The (finite) $W$-algebra $U(\mathfrak{g}, e)$ is the set of invariants in the quotient $U(\mathfrak{g})/U(\mathfrak{g}) m_\chi$, under the adjoint action of $m$:

$$U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g}) m_\chi)^m = \{ u \in U(\mathfrak{g})/U(\mathfrak{g}) m_\chi : [a, u] \in U(\mathfrak{g}) m_\chi \quad \forall a \in m \}.$$

**2.1. Slodowy slices.** For any nilpotent element $e \in \mathfrak{g}$, one can construct an $\mathfrak{sl}_2$-triple $(e, h, f)$ by the Jacobson–Morozov theorem. In fact, any such pair of triples $(e, h, f)$ and $(e', h', f')$ for which $\mathcal{O}_e = \mathcal{O}_{e'}$ are conjugate by the adjoint action, and so in particular one can speak unambiguously of “the $\mathfrak{sl}_2$-triple associated to $e$". If one additionally has a good grading $\Gamma$ for the nilpotent $e$, the $\mathfrak{sl}_2$-triple can be chosen to be $\Gamma$-graded, so that $e, h$ and $f$ lie in graded degrees 2, 0 and -2, respectively.

Associated to an $\mathfrak{sl}_2$-triple $(e, h, f)$ is a certain subvariety $\mathcal{S}_e \subseteq \mathfrak{g}$ known as the Slodowy slice. It is an affine space which forms a transverse slice to the nilpotent orbit $\mathcal{O}_e$, and is defined as a translate of the centraliser of $f$. We shall usually deal with the Slodowy slice in the dual Lie algebra $\mathfrak{g}^*$, transported via the Killing isomorphism $\kappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.

$$\mathcal{S}_e := e + \mathfrak{g}_0(f) \subseteq \mathfrak{g} \quad \mathcal{S}_\chi := \chi + \mathfrak{g}/[\mathfrak{g}, f]^* = \kappa(\mathcal{S}_e) \subseteq \mathfrak{g}^*$$

The Slodowy slice $\mathcal{S}_\chi$, and hence $\mathcal{S}_e$, inherits a natural Poisson structure from the variety $\mathfrak{g}^*$ equipped with the Lie–Poisson bracket.

Slodowy slices are of independent interest, but for our purposes we concentrate on their relation to $W$-algebras. Gan and Ginzburg proved in [GG] that $U(\mathfrak{g}, e)$ has the structure of a filtered algebra, and that the corresponding associated graded algebra $gr U(\mathfrak{g}, e)$ is the ring of functions on the Slodowy slice $\mathbb{C}[\mathcal{S}_e]$. Further, taking $M$ to be the algebraic group with Lie algebra $m$, one can consider the moment map of the co-adjoint action of $M$ on $\mathfrak{g}^*$; this is just restriction map $\mu : \mathfrak{g}^* \rightarrow m^*$. The Slodowy slice $\mathcal{S}_\chi$ can be expressed as a Hamiltonian reduction of $\mathfrak{g}^*$:

$$\mathcal{S}_\chi \simeq \mathfrak{g}^*/\mu (\mathcal{S}_e) := \mu^{-1}(\chi)/M.$$  

Expressed in terms of rings of functions, and taking $I_\chi$ to be the ideal of functions which vanish on $\mu^{-1}(\chi)$, this takes the form

$$\mathbb{C}[\mathcal{S}_\chi] \simeq (\mathbb{C}[\mathfrak{g}^*]/I_\chi)^M.$$  

The Poisson structure which $\mathcal{S}_\chi$ inherits as a Hamiltonian reduction of $\mathfrak{g}^*$ agrees with the Poisson structure it inherits as a subvariety of $\mathfrak{g}^*$ (cf. [GG, Section 3.4]). The similarity between definition 2.2 and eq. (1) has led to definition 2.2 to be referred to as a quantum Hamiltonian reduction, by analogy. This can be formalised in the language of deformation quantisations.
2.2. Deformation quantisations and quantum Hamiltonian reduction.

**Definition 2.3.** Let $A$ be a Poisson algebra with Poisson bracket $\{\cdot, \cdot\}$. A deformation quantisation of $A$ is an associative unital product $*: A \otimes A \rightarrow A[[\hbar]]$ such that, when extended $\mathbb{C}[[\hbar]]$-bilinearly, satisfies the following conditions:

1. $*$ is an associative binary product on $A[[\hbar]]$, continuous in the $\hbar$-adic topology;
2. $f * g = fg + O(\hbar)$ for all $f, g \in A$;
3. $f * g - g * f = (f, g)\hbar + O(\hbar^2)$ for all $f, g \in A$.

Writing $f * g = \sum_{k \geq 0} D_k(f, g)\hbar^k$, we shall further require that $*$ be a differential deformation quantisation, that is one satisfying the additional condition:

4. For each $k$, $D_k(\cdot, \cdot)$ is a bidifferential operator of order at most $k$ in each variable.

The vector space $A[[\hbar]]$ equipped with the multiplication $*$ shall be denoted $A_\hbar$, and is often referred to as a deformation quantisation itself. If $X$ is a Poisson variety and $A$ is its ring of functions, we often say that $A_\hbar$ is a deformation quantisation of $X$.

The product $*$ can also be used to introduce a new associative product on $A$ through the projection $A[[\hbar]] \rightarrow A$, given by sending $\hbar$ to 1. More concretely, define the product $:\odot: A \otimes A \rightarrow A$ by $f \odot g := \sum_{k \geq 0} D_k(f, g)$. Let the vector space $A$ equipped with this new algebra structure be denoted $A$. By abuse of terminology, the algebra $A$ is often referred to as a deformation quantisation of $A$.

By results of Gan and Ginzburg [GC] and Loseu [Los1], the Rees algebra of the $W$-algebra considered with the Kazhdan filtration, denoted $U_h(\mathfrak{g}, e)$, is a deformation quantisation of the ring of functions of the Slodowy slice $X = \mathbb{C}[\mathcal{Z}_\lambda]$. The $W$-algebra $U(g, e)$ itself is then $A$.

As a special case, we consider the $\mathbb{C}[[\hbar]]$-extended universal enveloping algebra $U_h(\mathfrak{g})$, which is a deformation quantisation of $\mathbb{C}[\mathfrak{g}]$. Consider the vector space $g_h := \mathfrak{g} \otimes \mathbb{C}[[\hbar]]$ equipped with the Lie bracket $[,]_h$ defined as $[x, y]_h := [x, y]_\hbar$ for $x, y \in \mathfrak{g}$ and extended $\mathbb{C}[[\hbar]]$-bilinearly. The algebra $U_h(\mathfrak{g})$ is then the universal enveloping algebra of $g_h$, and can be concretely presented as the tensor algebra $\mathcal{T}(g_h)$ modulo the relation $xy - yx = [x, y]_\hbar$. This algebra $U_h(\mathfrak{g})$ is just the Rees algebra of $U(\mathfrak{g})$ considered with the $\mathbb{C}[[\hbar]]$-filtration.

Assume now that $G$ is an algebraic group which acts on $A_h$ by $\mathbb{C}[[\hbar]]$-algebra automorphisms, and preserves the grading. This induces an action of $\mathfrak{g}$ on $A_h$ by derivations, and we denote the derivation corresponding to $\xi \in \mathfrak{g}$ by $\xi_A$. Let there furthermore exist a quantum comoment map for the action of $G$ on $A_h$, i.e. a linear map $\Phi: \mathfrak{g} \rightarrow A_h$, which is $G$-equivariant and satisfies $\frac{1}{\hbar}[\Phi(\xi), \cdot] = \xi_A$. It shall be useful to extend this $\mathbb{C}[[\hbar]]$-linearly to a map $\Phi: U_h(\mathfrak{g}) \rightarrow A_h$.

**Definition 2.4.** Let $A_h$ be a deformation quantisation on which $G$ acts with quantum comoment map $\Phi$. Let $\gamma \in \mathfrak{g}^*$ be fixed under the co-adjoint action of $G$, and define $\mathcal{I}_\gamma$ as the two-sided ideal in $U_h(\mathfrak{g})$ generated by $\mathfrak{g}_{h, \gamma} := \{x - \gamma(x)\hbar : x \in \mathfrak{g}\}$. The quantum Hamiltonian reduction of $A_h$ at $\gamma$ under the action of $G$ is

$$A_h/\mathcal{I}_\gamma^G := (A_h/\mathfrak{a}_h\Phi(\mathcal{I}_\gamma))^G.$$

This has a natural algebra structure with multiplication given by

$$(a + A_h\Phi(\mathcal{I}_\gamma))(b + A_h\Phi(\mathcal{I}_\gamma)) = ab + A_h\Phi(\mathcal{I}_\gamma).$$

**Remark 2.5.** Let $A_h$ be a deformation quantisation of the Poisson variety $X$, and let $G$ act on $A_h$ with quantum comoment map $\Phi$. Assume further that the action of $G$ on $A_h$ is induced by an action of $G$ on $X$. Then the quantum comoment map $\Phi$ induces a classical moment map $\mu: X \rightarrow \mathfrak{g}^*$, and for any $\gamma \in \mathfrak{g}^*$ fixed under the
co-adjoint action of $G$, the quantum Hamiltonian reduction $A_\hbar \sslash / \gamma G$ is a deformation quantisation of the classical Hamiltonian reduction $X \sslash / \gamma G$.

We shall now give a justification for calling definition 2.2 the definition of the W-algebra by quantum Hamiltonian reduction. Choosing a Premet subalgebra $m$ for $e$ naturally produces an algebraic group $M \subseteq G$ by exponentiation, since $m$ is an ad-nilpotent subalgebra ($\chi_1$). This acts on $\mathfrak{g}^*$ by the restriction of the co-adjoint action, and on $U_h(\mathfrak{g})$ by extending the adjoint action. Furthermore, this action has a quantum comoment map $\Phi: m \to U_h(\mathfrak{g})$ which comes from the natural inclusion of $m$ into $\mathfrak{g}$, and extends to the natural inclusion of $U_h(m)$ into $U_h(\mathfrak{g})$.

Since $\chi \in m^*$ is a character of $m$ ($\chi 4$) it is fixed under the co-adjoint action of $M$, and so we can consider the quantum Hamiltonian reduction $U_h(\mathfrak{g}) \sslash / \chi M$. Since $M$ is a unipotent algebraic group, invariants under adjoint action of $M$ are completely equivalent to invariants under the adjoint action of $m$. As a result,

$$U_h(\mathfrak{g}) \sslash / \chi M := (U_h(\mathfrak{g})/U_h(\mathfrak{g})\Phi(\mathcal{I}_\chi))^M = (U_h(\mathfrak{g})/U_h(\mathfrak{g})\mathcal{I}_\chi)^m,$$

and passing through the projection $h \mapsto 1$ results in the definition of the W-algebra given in definition 2.2. We can therefore without ambiguity denote the above by $U(\mathfrak{g}) \sslash / \chi m$.

2.3. Quantum Hamiltonian reduction by stages. Consider the Slodowy slice associated to the zero nilpotent, $\mathcal{S}_0 = \mathfrak{g}^*$. Equation (1) can then be restated in the following way: the Slodowy slice $\mathcal{S}_\chi$ can be expressed as a Hamiltonian reduction of $\mathcal{S}_0$. In order to answer for which other pairs of nilpotent elements $e_1$ and $e_2$ this can be done, we need to introduce the machinery of Hamiltonian reduction by stages.

Reduction by stages is a technique for decomposing a Hamiltonian reduction into a sequence of smaller reductions. The general theory is quite highly developed (cf. [MMO$^+$]), but we shall be interested in the specific case of reduction by a semidirect product.

Let $G \simeq H \rtimes K$ be an algebraic group which is a semidirect product of the closed subgroups $H$ and $K$, with $H$ normal in $G$. Let $X$ be a Poisson variety with a Hamiltonian action of $G$ with equivariant moment map $\mu: X \to \mathfrak{g}^*$. Let $\gamma \in \mathfrak{g}^*$ be a regular value of $\mu$, which is identified with $(\eta, \kappa)$ under the decomposition $\mathfrak{g}^* = \mathfrak{h}^* \times \mathfrak{t}^*$. Under certain mild conditions on the subgroup $K$ and the values $\eta$ and $\kappa$, there exists an isomorphism of Poisson varieties

$$X \sslash / \gamma G \simeq (X \sslash / \eta H) \sslash / \kappa K,$$

where all the induced actions are well-defined and Hamiltonian.

With this in mind, we seek to define an analogous notion of quantum Hamiltonian reduction by stages.

**Theorem 2.6.** Let $A_\hbar$ be a deformation quantisation, and let $G \simeq H \rtimes K$ be an algebraic group which acts on it with quantum comoment map $\Phi: \mathfrak{g} \to A_\hbar$. Let $\gamma \in \mathfrak{g}^*$ be an invariant under the co-adjoint action of $G$, which decomposes as $\gamma = (\eta, \kappa)$ under the identification
$g^* \cong \mathfrak{h}^* \times \mathfrak{t}^*$. Then there exists a natural action of $K$ on $\mathcal{A}_h/\mathcal{I}_h$ with an induced quantum comoment map $\Phi_K : \mathfrak{t} \to \mathcal{A}_h/\mathcal{I}_h$, and there is a natural homomorphism of algebras

$$(\mathcal{A}_h/\mathcal{I}_h) \mathcal{K} \to \mathcal{A}_h/\mathcal{I}_h G.$$  

Proof. We first show that the reduced spaces are properly defined and there exists an action of $K$ on $\mathcal{A}_h/\mathcal{I}_h$ with a quantum comoment map denoted $\Phi_K$. First note that restricting the action of $G$ yields an action of $H$ on $\mathcal{A}_h$, and the restriction $\Phi_H := \Phi|_h : h \to \mathcal{A}_h$ is $H$-equivariant due to $G$-equivariance and the normality of $H$. Further, $\eta$ is fixed by $G$ and hence by $H$, so the reduction $\mathcal{A}_h/\mathcal{I}_h$ is well-defined.

We next define the action of $K$ on $\mathcal{A}_h/\mathcal{I}_h$. Recall that

$$(2) \quad \mathcal{A}_h/\mathcal{I}_h = \{a + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta) : \text{Ad}_h(a) \in a + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta) \quad \forall \ h \in H\}.$$  

Define the action of $K$ by $\text{Ad}_k(a + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta)) := \text{Ad}_k(a) + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta)$. This is independent of the choice of representative, as can be seen by the following calculation, taking $k \in K$ and $x \in \mathfrak{k}$:

$$\text{Ad}_k \Phi(x - \eta(x)h) = \Phi(\text{Ad}_k x - \eta(x)h) \quad \Phi \text{ is } G \text{- and so } K \text{-equivariant}$$

$$\begin{align*}
= & \Phi(\text{Ad}_k x - \text{Ad}^*_k(\eta)(x)h) \\
= & \Phi(\text{Ad}_k x - \eta(\text{Ad}_k x)h) \in \Phi(\mathcal{I}_\eta) \\
\text{H } & \subseteq G \text{ and so } \text{Ad}_k x \in \mathfrak{h}
\end{align*}$$

That $\text{Ad}_k(a + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta))$ remains $H$-invariant again follows from the normality of $H$, as $\text{Ad}_h \text{Ad}_k(a + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta)) = \text{Ad}_h(a + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta)) = \text{Ad}_h \text{Ad}_k(a + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta)) = \text{Ad}_h(a + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta))$.

Lastly, we need to exhibit a quantum comoment map $\Phi_K : \mathfrak{t} \to \mathcal{A}_h/\mathcal{I}_h$. We first define an $\eta$-twisted quantum comoment map, extending $\eta$ by zero on $\mathfrak{t}$ and defining

$$\Phi_\eta : g = \mathfrak{h} \times \mathfrak{t} \to \mathcal{A}_h$$

$$\Phi_\eta(x) := \Phi(x) - \eta(x)h$$

To define the quantum comoment map $\Phi_K$, consider $\mathfrak{t}$ as the quotient $g/\mathfrak{h}$, choose an arbitrary lift from $\mathfrak{t}$ to $g$, and apply the function $\pi_H \circ \Phi_\eta$, where $\pi_H$ is the projection $\pi_H : \mathcal{A}_h \to \mathcal{A}_h/\mathcal{A}_h \Phi_H (\mathcal{I}_\eta)$. To see this is well-defined, note that for any $y \in \mathfrak{t}$ and $x \in \mathfrak{h}$,

$$\Phi_K(y) = \Phi_\eta(y + x) = \Phi(y) + \Phi(x - \eta(x)h) \in \Phi(y) + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta).$$

To see that the image of $\Phi_K$ lies within $H$-invariants, note that the semidirect product structure of $g$ guarantees that for $y \in \mathfrak{t}$ and $h \in H$ there exists an $x \in \mathfrak{h}$ such that $\text{Ad}_h y = y + x$. From this we can see that

$$\text{Ad}_h \Phi_K(y) = \Phi_\eta(\text{Ad}_h y) = \Phi(y) + \Phi(x - \eta(x)h) \in \Phi_K(y) + \mathcal{A}_h \Phi_H (\mathcal{I}_\eta).$$

That this map is $K$-equivariant follows directly from the $G$-equivariance of $\Phi$, and the quantum comoment condition $\xi | [\Phi_K (\xi)] = \xi | [\mathcal{A}_h \mathcal{I}_h G]$ follows from the corresponding condition for $\Phi$ and the above calculations showing the action of $K$ is well-defined on $\mathcal{A}_h/\mathcal{I}_h$. Since $\kappa \in \mathfrak{t}^*$ is fixed by the action of $G$, and hence by $K$, we can therefore talk sensibly about the two-stage reduction $(\mathcal{A}_h/\mathcal{I}_h)\mathcal{K}$. Furthermore, the action of $K$ on $\mathcal{A}_h/\mathcal{I}_h$ descends to an action on $(\mathcal{A}_h/\mathcal{I}_h)/(\mathcal{A}_h/\mathcal{I}_h)\Phi_K (\mathcal{I}_\eta)$.

It remains to show that there is a map from the two-stage reduction $(\mathcal{A}_h/\mathcal{I}_h)\mathcal{K}$ to the one-shot reduction $\mathcal{A}_h/\mathcal{I}_h G$. To this end we shall construct the map $\varphi$ from fig. 3. Consider the maps

$$\tilde{\varphi} : (\mathcal{A}_h/\mathcal{I}_h)\Phi_K (\mathcal{I}_\eta) \
\varphi : (\mathcal{A}_h/\mathcal{I}_h)\mathcal{K} \to \mathcal{A}_h/\mathcal{I}_h G,$$
where \( \varphi \) is defined by first lifting to \( A_h/\iota_H \) and then applying \( \pi^1_G \circ \iota_H \), and \( \varphi \) is defined as the composition \( \varphi \circ \iota_K \). To show these are well-defined requires checking that \( \varphi \) doesn’t depend on the lift chosen, and that the image of \( \varphi \) lies in \( G \)-invariants. These can both be checked by careful but straightforward calculations. We have therefore constructed a homomorphism \( \varphi \) from the two-stage to the one-shot reduction; it is merely the identity map suitably interpreted in the appropriate cosets:

\[
\varphi((a + A_h \Phi_H(I_{\eta})) + (A_h/\iota_H) \Phi_K(I_{\kappa})) = a + A_h \Phi(I_{\gamma}).
\]

\[\square\]

**Corollary 2.7.** If this homomorphism \( \varphi \) induces an isomorphism \( \varphi \colon (A_h/\iota_H) / \iota_K \cong A_h/\iota_K \) of the corresponding Poisson algebras, then it is itself an isomorphism. In particular, this holds if \( A_h \) is a deformation quantisation of a Poisson manifold for which the classical reduction by stages hypotheses hold.

**Proof.** The first statement follows from the fact that \( \varphi \) induces a homomorphism of Poisson algebras, and from the fact that \( A \) and \( \mathcal{A} \) have identical underlying vector spaces. That this induces an isomorphism if \( A_h \) is a deformation quantisation of a Poisson manifold is just the classical Hamiltonian reduction by stages construction, which can be found in e.g. [MMO^*], §5.3.

\[\square\]

**Example 2.8.** Consider the algebra \( U_h(sl_3) \), which is a deformation quantisation of the Poisson variety \( sl_3^* \). By restriction of the adjoint action of \( SL_3, U_h(sl_3) \) is acted on by the following groups:

\[
N = \left\{ \left( \begin{array}{ccc} 1 & 0 & 0 \\ r & 1 & 0 \\ s & t & 1 \end{array} \right) : r, s, t \in \mathbb{C} \right\}, \quad M = \left\{ \left( \begin{array}{ccc} 1 & 0 & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) : r \in \mathbb{C} \right\}, \quad K = \left\{ \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) : s \in \mathbb{C} \right\}.
\]

Note that \( N = M \times K \), and the quantum comoment maps associated to the actions are given by the inclusions of their respective Lie algebras into \( U_h(sl_3) \). Let \( \chi \in \mathfrak{n}^* \) be the character corresponding to the regular nilpotent element \( e = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right) \) under the Killing isomorphism, and let \( \eta \in \mathfrak{m}^* \) and \( \kappa \in \mathfrak{s}^* \) be the restrictions of \( \chi \).

The quantum Hamiltonian reduction \( U_h(sl_3) / \chi N \) is the polynomial ring \( \mathbb{C}[z_1, z_2] \), where

\[
z_1 := h_1^2 + h_1h_2 + h_2^2 + 3h(e_1 + e_2)
\]

\[
z_2 := 2h_1^3 + 3h_1^2h_2 - 3h_1h_2^2 - 2h_2^3 + 9he_1(h_1 + 2h_2) - 9he_2(2h_1 + h_2) + 27h^2(e_3 + e_2).
\]
These can be lifted to invariants under the action of $M$ in $U_h(\mathfrak{sl}_3)/U_h(\mathfrak{sl}_3)\Phi_M(\mathcal{I}_\eta)$ as
\[
z_1 \mapsto z_1 + 3e_2(f_2 - h) \quad \quad \quad z_2 \mapsto z_2 + 9(e_2b_2 + he_3 - he_2)(f_2 - h).
\]
Passing to the quotient $(U_h(\mathfrak{sl}_3)/\mathfrak{sl}_3)M)/(U_h(\mathfrak{sl}_3)/\mathfrak{sl}_3)\Phi_K(\mathcal{I}_\eta)$ yields the well-defined map $\psi$ of fig. 3, and its image lies in $K$-invariants.

3. Reduction by stages for $W$-algebras in type $A$

Given the framework of quantum Hamiltonian reduction by stages, we can now try to find an explicit realisation in the case of $W$-algebras, as in fig. 1. From this point we shall work over $\mathbb{C}$ and in type A, assuming that $\mathfrak{g} = \mathfrak{sl}_n$. In this case, we have a simple classification of both the conjugacy classes of nilpotent elements and of their good gradings.

Recall the set of nilpotent orbits in $\mathfrak{sl}_n$ is parameterised by partitions of $n$, corresponding to the sizes of the Jordan blocks for the nilpotent. The set of nilpotent orbits in a Lie algebra always has a natural partial ordering, where $\mathcal{O}_{e_1} \subseteq \mathcal{O}_{e_2}$ is defined to mean $\mathcal{O}_{e_1} \subseteq \mathcal{O}_{e_2}$. In type A, this coincides with the dominance ordering on partitions: take $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$, where $\lambda$ and $\mu$ are padded on the right with zeros if necessary, and define $\lambda \leq \mu$ to mean that $\sum_{i=1}^{\ell} \lambda_i \leq \sum_{i=1}^{\ell} \mu_i$ for every $\ell = 1, \ldots, k$. A classical theorem of Gerstenhaber classifies the covering relations in the dominance ordering, and roughly corresponds to ‘sliding a box up’ in the corresponding Young diagram.

**Lemma 3.1.** [Gerstenhaber] The partition $\lambda$ covers $\mu$ if and only if there exist indices $j < k$ with $\mu_j = \lambda_j - 1$, $\mu_k = \lambda_k + 1$ and $\lambda_i = \mu_i$ otherwise, where $j$ is the smallest index such that $0 \leq \lambda_k < \lambda_j - 1$ and either $k = j + 1$ or $\lambda_k = \lambda_j - 2$.

**Example 3.2.** The partitions $\lambda = (3)$, $\mu = (2, 1)$ and $\nu = (1, 1, 1)$ cover one another in turn.

\[
\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \end{array} \quad \quad \mu = \begin{array}{|c|c|} \hline 1 & 2 \end{array} \quad \quad \nu = \begin{array}{|c|} \hline 3 \end{array}
\]

3.1. Pyramids. The problem of classifying all good gradings has been solved by Elashvili and Kac [EK]; in the classical types, this is accomplished using a combinatorial structure known as a pyramid. In type A, pyramids are an enriched version of Young diagrams, allowing horizontal shifts of the rows according to certain conditions. In this paper we shall use the French convention for Young diagrams.

**Definition 3.3.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $n$. A pyramid of shape $\lambda$ is a Young diagram of shape $\lambda$ consisting of boxes of size 2, along with integer horizontal row shifts such that the co-ordinates of the first (resp. last) boxes in each row form an increasing (resp. decreasing) sequence.

A filling of a pyramid is a labelling of each of the boxes with a number between 1 and $n$, such that there are no repeated labels. Given a filled pyramid, the column and row of the box labelled $k$ are denoted $\text{col}(k)$ and $\text{row}(k)$, respectively. We say $\ell$ is right-adjacent to $k$, denoted $k \rightarrow \ell$, if $\text{row}(k) = \text{row}(\ell)$ and $\text{col}(k) + 2 = \text{col}(\ell)$.

**Note.** The row and column of a box are only well-defined up to an integer shift. However, since we’ll only ever be concerned with differences of row and column numbers, this will not cause a problem.

When filling pyramids, we shall most often choose the labelling so that it increases first up columns and then left to right.
Example 3.4. The three pyramids of shape \((4, 3)\) follow, each with a sample filling.

\[
\begin{array}{c|c|c|c}
2 & 4 & 6 \\
1 & 3 & 5 & 7 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
2 & 4 & 6 \\
1 & 3 & 5 & 7 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
3 & 5 & 7 \\
1 & 2 & 4 & 6 \\
\end{array}
\]

Theorem 3.5. [EK, Theorem 4.2] There is a bijection between the pyramids of size \(n\) and the set of good gradings in \(\mathfrak{sl}_n\) up to conjugacy. The same holds in \(\mathfrak{gl}_n\).

Consider a filled pyramid \(P\). The nilpotent element \(e_P\) associated to \(P\) is just the nilpotent element associated to \(P\) considered as a Young tableau, namely \(\sum_{i \to j} E_{ij}\). The grading \(\Gamma_P\) associated to \(P\) is defined by declaring \(E_{ij}\) to be of graded degree \(\text{col}(j) - \text{col}(i)\). It can be checked that this grading is good for \(e_P\).

3.2. Reduction by stages for \(W\)-algebras. In this section we shall use the machinery of pyramids to produce a quantum Hamiltonian reduction by stages for \(W\)-algebras in type \(A\). Since our reductions are by nilpotent groups, it will suffice to work with the Lie algebras, which completely determine the actions of the corresponding algebraic groups.

Objective 3.6. Let \(g = \mathfrak{sl}_n\) and \(e_1, e_2 \in g\) be two nilpotent elements such that \(\Theta_{e_1} < \Theta_{e_2}\). We would like to construct an algebraic group \(K\) with a quantum Hamiltonian action on \(U(g, e_1)\), along with a character \(\kappa \in \mathfrak{t}^*\), such that \(U(g, e_2) \simeq U(g, e_1) \bowtie \kappa K\).

It will suffice to produce such a construction for every pair such that \(\Theta_2\) covers \(\Theta_1\). For any such pair we will chose nilpotent elements \(e_i \in \Theta_i\) with respective duals \(\chi_i \in g^*\) for \(i = 1, 2\), a good grading \(\Gamma_1\) for \(e_1\) with a Premet subalgebra \(m_1\), and a subalgebra \(m_2 \supset m_1\) satisfying:

- SR1. \(m_2\) decomposes as a semidirect product \(m_2 = m_1 \rtimes \mathfrak{t}\).
- SR2. \(\chi_2\) restricts to a character of \(m_2\), and \(\chi_2 = (\chi_1, \kappa)\) in the above decomposition.
- SR3. the subalgebra \(\mathfrak{t}\) annihilates \(\chi_1\).

Since \(m_1\) is a Premet subalgebra for \(e_1\), the corresponding quantum Hamiltonian reduction by stages will ensure that

\[
U(g) \bowtie \chi_2 m_2 \simeq (U(g) \bowtie \chi_1 m_1) \bowtie \mathfrak{t} = U(g, e_1) \bowtie \kappa \mathfrak{t}.
\]

We will therefore provide a construction satisfying these conditions, and conjecture that the algebra \(U(g) \bowtie \chi_2 m_2\) is isomorphic to the \(W\)-algebra \(U(g, e_2)\). Provided this conjecture holds, this will accomplish objective 3.6.

3.2.1. The general construction. Let \(\mu\) be the partition corresponding to the nilpotent \(e_1\). We will construct a right-aligned pyramid for \(\mu\), i.e. a pyramid for which the rightmost boxes in each row all lie in the same column, and number the boxes from bottom to top and left to right. This determines an even good grading \(\Gamma_1\) and Premet subalgebra \(m_1\) for \(e_1\). By lemma 3.1, for every partition \(\lambda\) which covers \(\mu\) in the dominance ordering, there is a pair of integers \(i < j\) for which \(\lambda\) is obtained from \(\mu\) by ‘sliding a box down’ from row \(j\) to row \(i\). Define \(e_2\) as

\[
e_2 := e_1 + \sum_{\text{row}(k) = i, \text{col}(\ell) = j, \text{col}(k) = \text{col}(\ell)} E_{k\ell},
\]

and define the Lie algebras \(m_2\) and \(\mathfrak{t}\) by

\[
\mathfrak{t} := \langle E_{m} \rangle_{m=1}^{j-i} \quad \text{and} \quad m_2 := m_1 + \mathfrak{t}, \quad \text{where} \quad E_m := \sum_{\text{row}(k) < \text{row}(\ell) \leq j, \text{row}(\ell) - \text{row}(k) = m, \text{col}(k) = \text{col}(\ell)} E_{k\ell}.
\]
Let us further define a semisimple element $h'_2$:

$$(6) \quad h'_2 := \sum_{\text{row}(\ell) = s, s \neq i, j \atop \ell \text{ is } t\text{-th from the left}} (\lambda_s - 1 - 2t)E_{\ell \ell} + \sum_{\text{col}(m) = \text{col}(\ell) - 2 \atop \ell \text{ is } t\text{-th from the left}} (\lambda_i + K - 2t)(E_{\ell \ell} + E_{mm}).$$

In the second term, the $E_{mm}$ term is omitted if there is no $m$ satisfying the conditions for the given $t$, and $E_{\ell \ell}$ is omitted for $t = \lambda_i$. Here, $K$ is the unique constant so that $h'_2$ has trace zero. Note that $h'_2$ is a semisimple element for which $[h'_2, e_2] = 2e_2$; we shall show that $h'_2$ determines a good grading for $e_2$ in lemma 3.12, however $m_2$ is not in general a Premet subalgebra for this grading, nor does there necessarily exist an $s_{l_2}$-triple containing $e_2$ and $h'_2$.

Remark 3.7. Note that $\mathfrak{t}$ is an abelian Lie algebra, and that

$$[m_2, m_2] = [m_2, m_1] \subseteq \bigoplus_{k \leq 0} g_k, \bigoplus_{k \leq -2} g_k \subseteq \bigoplus_{k \leq -2} g_k = m_1 \subseteq m_2.$$ 

This confirms that $m_2$ is closed under the Lie bracket, and further that $m_1$ is an ideal in $m_2$; hence, $m_2$ is a semi-direct product $m_1 \rtimes \mathfrak{t}$.

Example 3.8. Let $\mathfrak{g} = sl_6$ and consider $\mu = (2, 2, 2)$. The right-aligned pyramid $P_1$, nilpotent element $e_1$ and Premet subalgebra $m_1$ are as follows:

$$P_1 = \begin{array}{ccc} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{array} \quad e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The unique covering partition is $\lambda = (3, 2, 1)$, which is obtained by ‘sliding a box from row 3 to row 1’. Applying the above procedure with $i = 1$ and $j = 3$ results in

$$e_2 = e_1 + E_{13} + E_{46} \quad m_2 = m_1 + \langle E_1, E_2 \rangle$$

$$e_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \end{pmatrix}$$

where $E_1 = E_{21} + E_{32} + E_{45} + E_{65}$ and $E_2 = E_{31} + E_{64}$. Further,

$$h'_2 = (E_{22} - E_{55}) + (2E_{11} + 0E_{14} + 0E_{33} - 2E_{66})$$

3.2.2. Properties of the construction.

Theorem 3.9. Under the above circumstances, $e_2$ is a nilpotent element of type $\lambda$, $m_2$ is a Lie algebra and conditions SR1 to SR3 hold. Consequently, theorem 2.6 holds, and so there is homomorphism from the quantum Hamiltonian reduction by stages to the one-shot reduction:

$$\left( U(\mathfrak{g})/\mathfrak{\mathfrak{g}}_1, m_1 \right)_{/\mathfrak{t}} = U(\mathfrak{g}, e_1)_{/\mathfrak{t}} \rightarrow U(\mathfrak{g})/\mathfrak{\mathfrak{g}}_2, m_2.$$ 

We shall prove in theorem 4.4 that this homomorphism is, in fact, an isomorphism, but will leave this discussion until the necessary framework has been developed. Before proving the theorem, we should introduce a result of [EK]: given any filled pyramid $P$ with corresponding nilpotent element $e$, the centraliser $Z_P(e)$ can be read
off the pyramid $P$. Let $\mu = (\mu_1, \ldots, \mu_k)$, so the $i$th row of the pyramid has $\mu_i$ boxes, and let $b_{i,j}$ be the standard basis vector corresponding to the index of the box in the $i$th row, $j$th from the right in the filled pyramid. We can represent an endomorphism of $\mathbb{C}^n$ by specifying where each of the basis vectors $b_{i,j}$ is sent in an arrow diagram.

Elashvili and Kac define a collection of endomorphisms in $\mathfrak{gl}_n$, denoted $E^i_j[r]$, where $i$ and $j$ range over the rows of the pyramid and $r \in \mathbb{N}$ varies over a range depending on $\mu_i$ and $\mu_j$. These endomorphisms are defined in fig. 4, where any basis vector not specified is sent to zero.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Endomorphisms of $\mathbb{C}^n$ commuting with $e$.}
\end{figure}

Note. The nilpotent $e$ is shown in grey in each row for reference.

**Lemma 3.10.** [EK] Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of $n$, and consider a filled pyramid of shape $\mu$ with associated nilpotent $e$. Then the collection $\{E^i_j[r]\}$, where

- $i, j \in \{1, \ldots, k\}$
- $0 \leq r < \mu_j$ if $i \leq j$
- $0 \leq r < \mu_j$ if $i > j$
- $\mu_j - \mu_i \leq r < \mu_j$ if $i > j$

forms a basis of the centraliser $\mathfrak{z}_{\mathfrak{gl}_n}(e)$, and those which lie in $\mathfrak{sl}_n$ form a basis of $\mathfrak{z}_{\mathfrak{sl}_n}(e)$.

**Proof of theorem 3.9.** To prove that $e_2$ has the correct Jordan type, it suffices to exhibit a Jordan basis. Note that a Jordan basis can be read off the rows of the pyramid, proceeding from right to left.
The Jordan basis for $e_1$ in row $i$ of this pyramid is therefore
\[ b_{i,\mu_i} \leftrightarrow \cdots \leftrightarrow b_{i,\mu_j+1} \leftrightarrow b_{i,j} \leftrightarrow b_{i,\mu_j-1} \leftrightarrow \cdots \leftrightarrow b_{i,2} \leftrightarrow b_{i,1}. \]

The Jordan basis for $e_2$ is identical to that of $e_1$ except for those strings corresponding to rows $i$ and $j$. The Jordan basis in those rows is
\[ \mu_j b_{i,\mu_i} \leftrightarrow \cdots \leftrightarrow \mu_j b_{i,j} \leftrightarrow ((\mu_j - 1)b_{i,\mu_j+1} + b_{i,j}) \leftrightarrow \cdots \leftrightarrow (kb_{i,k} + b_{j,k+1}) \leftrightarrow \cdots \leftrightarrow b_{j,1}, \]

of lengths $\mu_i + 1 = \lambda_i$ and $\mu_j - 1 = \lambda_j$, respectively.

Condition SR1 is shown in remark 3.7. To check SR2, note that $\chi_2|_{m_1} = \chi_1$ by construction, and so $\chi_2 = (\chi_1, \kappa)$ for some $\kappa \in \mathfrak{t}^*$. To confirm that $\chi_2$ is a character of $m_2$, recall from remark 3.7 that $[m_2, m_2] = [m_1 + \mathfrak{t}, m_1]$. However, since $\chi_2([m_1, m_1]) = \chi_1([m_1, m_1]) = 0$, it remains only to check that $\chi_2([\mathfrak{t}, m_1]) = 0$. We shall check this on the generating set $\{[E_m, E_{\ell k}] : 1 \leq m \leq j - i, \text{col}(k) < \text{col}(\ell)\}$.

Using the language of fig. 4, note that
\begin{equation}
[\chi_2(E_m, E_{\ell k})] = (e_2, [E_m, E_{\ell k}]) = (e_2, [E_m, E_{\ell k}]) \in \mathfrak{g}_0.
\end{equation}

Here the second equality follows from the fact that $E^{s+m}_s[0]$ commutes with $e_2$; that $\chi_2$ annihilates eq. (7) now follows from property GG5, and the fact that $E_{\ell k} \in \bigoplus_{\ell < 0} \mathfrak{g}_1$. This further establishes the claim that $\mathfrak{t}$ annihilates both $\chi_2$ and $\chi_1$: hence condition SR3 also holds. This completes the proof of theorem 3.9. \hfill \Box

**Theorem 3.11.** The pair $e_2$ and $m_2$ satisfy properties $\chi_1$ to $\chi_4$.

**Proof.** Property $\chi_1$ is manifest from the construction, and property $\chi_4$ is a subclaim of condition SR2. Property $\chi_2$ follows from the fact that $e_1$ itself satisfies it, along with an application of the orbit–stabiliser theorem and lemma 3.10.

We prove property $\chi_3$ by directly calculating $m_2 \cap \mathfrak{g}_0(e_2)$. In the coming calculation, we use the following conventions:

- Recall that $k \to \ell$ means that $\ell$ is right-adjacent to $k$.
- If $\text{row}(k) = i$, then $i^p$ indicates that $p$ is the box such that $\text{row}(p) = j$ and $\text{col}(p) = \text{col}(k)$, if such exists. Similarly, if $\text{row}(s) = j$, then $j^s$ indicates that $q$ is such that $\text{row}(q) = i$ and $\text{col}(q) = \text{col}(s)$, if such exists.
- $A_{k \ell} = 0$ if there do not exist $k$ and $\ell$ which satisfy the adjacency relations specified below, and $B_m = 0$ if $m < 1$ or $m > j - i$.

Taking the commutator of $e_2$ with a generic element of $m_2$ results in the following:

\[ [e_2, \sum_{\text{col}(u) < \text{col}(u)} A_{uv} E_{uv} + \sum_{m=1}^{j-i} B_m E_m] = \]

\[ \sum_{\text{col}(s) < \text{col}(k)} \left(A_{k\ell}^{p} - A_{kr}^{p} \begin{cases} \text{row}(k) = i, k^p & \text{row}(s) = j, j^{s} \\ 0 & \text{otherwise} \end{cases} \right) E_{ks} \]

\[ + \sum_{\text{col}(s) = \text{col}(k)} \left( A_{k\ell} - A_{kr} \begin{cases} B_m & \text{row}(k) = i, \text{row}(s) = j - m \\ -B_m & \text{row}(s) = j, \text{row}(k) = i + m \end{cases} \right) E_{ks}. \]

For eq. (8) to vanish, we will prove that all of $A_{uv}$ and $B_m$ must vanish as well.
Case 1 ($A_{uv} = 0$ for $\text{col}(v) < \text{col}(u)$, $\text{row}(u) \neq i$ and $\text{row}(v) \neq j$).

Examining the coefficient of $E_{uv}$ for $v \to w$ yields $A_{tw} - A_{uv}$, where $u \to t$. We can prove the claim by induction on the distance of $u$ from the right of the pyramid. The base case is when $u$ is rightmost in its row; then $A_{tw} = 0$, and so $A_{uv} = 0$. The same argument assuming the inductive hypothesis for all $u$ within $n$ boxes of the right proves the claim for all $u$ within $n+1$ boxes of the right, completing the induction.

Case 2 ($A_{uv} = 0$ for $\text{col}(v) < \text{col}(u) - 1$, $\text{row}(u) = i$ and $\text{row}(v) \neq j$).

Examining the coefficient of $E_{uv}$ for $v \to w$ yields $A_{tw} - A_{uv} + A_{pw}$, where $u \to t$ and $u \uparrow'$. But $A_{pw} = 0$ by case 1, and so the same argument as above completes the case.

Case 3 ($A_{uv} = 0$ for $\text{col}(v) < \text{col}(u) - 1$, $\text{row}(u) \neq i$ and $\text{row}(v) = j$).

Apply the argument of case 2 mutatis mutandis.

Case 4 ($A_{uv} = 0$ for $\text{col}(v) < \text{col}(u) - 1$, $\text{row}(u) = i$ and $\text{row}(v) = j$).

The conclusions of cases 2 and 3 allow the same argument to again be applied mutatis mutandis.

Case 5 ($A_{uv} = 0$ for $\text{col}(v) = \text{col}(u) - 1$).

If neither $\text{row}(u) = i$ or $\text{row}(v) = j$ this is dealt with by case 1, while if both of these hold there is no contribution from any $B_m$. Since the arguments are symmetric, we’ll assume that $\text{row}(u) = i$.

Assume that $\text{row}(v) = j - m$ for $1 \leq m \leq j - i$; otherwise there is no contribution from any $B_m$ and we’re done. Since $e_2$ covers $e_1$ there are exactly the same number of boxes in the two rows; let the boxes of row $i$ be labelled from the left $u_1, \ldots, u_k$ and the boxes of row $j$ be labelled $v_1, \ldots, v_k$. The sum of the co-efficients of $E_{u_1v_1}$ up to $E_{u_kv_k}$ is $kB_m$, which proves that $B_m = 0$. The argument of case 1 proves that the remaining $A_{uv}$ must vanish, which completes this last case and the proof of the theorem.

Finally, recalling the element $h'_2$ from eq. (6), we establish the following result for future reference.

**Lemma 3.12.** The grading coming from the semisimple element $h'_2$ is good for $e_2$. Further, $\text{ad } h'_2$ preserves $m_1$.

**Proof.** By construction we have that $[h'_2, e_2] = 2e_2$, so all that remains is to show that $\bigoplus_{i<0} g_i \cap \mathfrak{z}(e_2) = \{0\}$, which is equivalent to property GG2. Considering the basis of the centraliser $\mathfrak{z}(e_1)$ given in lemma 3.10, note that the basis of the centraliser $\mathfrak{z}(e_2)$ is closely related: it differs only in that endomorphisms which involve basis vectors in rows $i$ and $j$ have these replaced by an appropriate linear combination of basis vectors as in the proof of lemma 3.10. However, these linear combinations lie in the zero weight space of $h'_2$ by construction; this proves that any element of the centraliser cannot lie in strictly negative degree. That $\text{ad } h'_2$ preserves $m_1$ follows immediately from the fact that $h'_2$ is diagonal. \hfill \Box

3.3. Relation to W-algebras.

**Conjecture 3.13.** For nilpotents $e_1, e_2 \in \mathfrak{g}$ and subalgebras $m_1, m_2 \subseteq \mathfrak{g}$ as defined in eqs. (4) and (5), the reduction by stages $U(\mathfrak{g}, e_1) \parallel \mathfrak{k} \cong U(\mathfrak{g}) \parallel_{X_2} m_2$ is isomorphic to the W-algebra $U(\mathfrak{g}, e_2)$.

**Remark 3.14.** This conjecture is a special case of a more general conjecture due to Premet, based on his work in [Pre]. Specifically, Premet conjectures that for any
pair of subalgebra \( m \) and nilpotent \( e \) which satisfy properties \( \chi 1 \) to \( \chi 4 \), the quantum Hamiltonian reduction \( U(\mathfrak{g})/\mathfrak{m} \) is isomorphic to the W-algebra \( U(\mathfrak{g}, e) \). In fact, Premet has proven this conjecture in the case that the base field is of non-zero characteristic \( p \).

**Proposition 3.15.** Conjecture 3.13 holds for \( e_1 \) a subregular and \( e_2 \) a regular nilpotent.

*Proof.* The subalgebra \( m_2 \) constructed is simply the maximal nilpotent subalgebra of lower-triangular matrices \( n^− \). This is a Premet subalgebra for \( e_2 \). \( \square \)

**Remark 3.16.** The construction detailed in this section can be modified slightly to give a stronger version of proposition 3.15. Instead of choosing a right-aligned pyramid of shape \( \mu \), one can choose a pyramid which is right-aligned but for a leftward shift of 1 at row \( i \) and another leftward shift of 1 at row \( j + 1 \). This necessitates a choice of Lagrangian \( I \subseteq \mathfrak{g}^{−1} \); this choice can be made so that the resulting Premet subalgebra can be extended to a Premet subalgebra for a pyramid of shape \( \lambda \), which is right-aligned but for a leftward shift of 1 at row \( i + 1 \) and another leftward shift of 2 at row \( j \).

\[
\begin{array}{cccc}
6 & 4 & 9 & \\
3 & 8 & \\
2 & 7 & \\
1 & 5 & 10 & \\
\end{array} < \begin{array}{cccc}
6 & 4 & \\
3 & 8 & \\
2 & 7 & 9 & \\
1 & 5 & 10 & \\
\end{array}
\]

\( \mu = (3, 2, 2, 2, 1) \)

\( \lambda = (3, 3, 2, 1, 1) \)

For this new pyramid and compatible choice of Lagrangian, theorem 3.9 remains true. Furthermore, proposition 3.15 and its proof hold not only for \( e_1 \) a subregular and \( e_2 \) a regular nilpotent, but more generally for any pair of nilpotent elements \( e_1 \) and \( e_2 \) of types \( \mu = (\mu_1, \ldots, \mu_k, 1) \) and \( \lambda = (\mu_1, \ldots, \mu_k + 1) \), respectively.

**Example 3.17.** Consider \( sl_4 \) and \( e_1 \) a nilpotent of type \( (2, 2) \). This is covered by the subregular nilpotent \( e_2 \), and so the construction will produce an algebra \( U(\mathfrak{g})/\chi_2 \mathfrak{m}_2 \). The associated graded algebra of \( U(\mathfrak{g}, e_2) \) is the ring of functions on the Slodowy slice \( \mathcal{S}_{\chi_2} \), and the associated graded algebra of the reduced space is \( \mathbb{C}[\mathfrak{g}^*/\chi_2 \mathcal{M}_2] \).

\[
\mathcal{S}_{\chi_2} = \left\{ \begin{pmatrix} a & 1 & 0 & 0 \\ b - 3a^2 & a & 1 & 0 \\ c + 20a^3 & b - 3a^2 & a & d \\ f & 0 & 0 & -3a \end{pmatrix} : a, b, c, d, f \in \mathbb{C} \right\}
\]

\[
\mathfrak{g}^*/\chi_2 \mathcal{M}_2 \cong \left\{ \begin{pmatrix} x + \frac{u+v}{3} & 1 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{x + u+v}{3} & 0 & 1 \\ z + \frac{u+v}{2} & \frac{x + u+v}{3} & 0 \end{pmatrix} : u, v, x, y, z \in \mathbb{C} \right\}
\]

\[
\mathbb{C}[\mathcal{S}_{\chi_2}] = \mathbb{C}[a, b, c, d, f], \quad \mathbb{C}[\mathfrak{g}^*/\chi_2 \mathcal{M}_2] = \mathbb{C}[u, v, x, y, z]
\]

These are isomorphic as Poisson algebras, as shall be shown below.

Consider the ring homomorphism \( \varphi : \mathbb{C}[\mathcal{S}_{\chi_2}] \to \mathbb{C}[\mathfrak{g}^*/\chi_2 \mathcal{M}_2] \) defined by

\[
\varphi(a) = \frac{-1}{3} y \quad \varphi(b) = x \quad \varphi(c) = 2z - \frac{8}{3} xy \quad \varphi(d) = v + x + y^2 \quad \varphi(f) = -u - x - y^2.
\]
The non-zero Poisson brackets are given by the formulae:
\[
\begin{align*}
\{a, d\} &= \frac{1}{2}d^2 & \{c, d\} &= \frac{1}{6}bd & \{u, y\} &= \frac{1}{8}(u + x + y^2) & \{u, z\} &= \frac{1}{4}x(u + x + y^2) \\
\{a, f\} &= \frac{1}{12}f & \{c, f\} &= \frac{1}{6}bf & \{v, y\} &= \frac{1}{8}(v + x + y^2) & \{v, z\} &= \frac{1}{4}x(v + x + y^2) \\
\{d, f\} &= -\frac{27}{2}a^3 + ab - \frac{1}{8}c & \{u, v\} &= \frac{1}{41}(z + xy + 2(u + v)y)
\end{align*}
\]

It can be checked that this map is a ring isomorphism and also preserves the Poisson bracket; it hence induces an isomorphism of the Poisson varieties \(S_{X_2} \simeq g^*/X_2 M_2\). Furthermore, this map preserves the characteristic polynomial.

Thus, we know that our algebra \(U(g)/X_2 M_2\) is a deformation quantisation of the Slodowy slice \(S_{X_2}\). Since there is, up to isomorphism, a unique such deformation quantisation which is \(G\)-equivariant, and the W-algebra \(U(g, e_2)\) is such a deformation quantisation, it follows that \(U(g)/X_2 M_2 \simeq U(g, e_1)/M_2\) is isomorphic to \(U(g, e_2)\).

### 4. The representation theory of W-algebras

The construction of quantum Hamiltonian reduction by stages has a number of applications to the representation theory of W-algebras. Theorem 3.9 has an immediate corollary relating the categories of modules over \(U(g, e_1)\) and \(U(g, e_2)\).

**Corollary 4.1.** Let \(e_1\) and \(e_2\) be two nilpotent elements of \(sl_\mathfrak{n}\) such that \(e_2\) covers \(e_1\) in the dominance ordering; then the quantum Hamiltonian reduction by stages construction produces an adjoint pair of functors \(U(g, e_1)\mod \simeq (U(g)/X_2 M_2)\mod\). If conjecture 3.13 holds, then there exists an adjunction \(U(g, e_1)\mod \simeq U(g, e_2)\mod\) for any pair of nilpotents \(e_2 \geq e_1\).

**Proof.** Note that the quotient \(U(g, e_1)/U(g, e_1)\mathfrak{t}_\mathfrak{s}\) is a \((U(g, e_1), U(g)/X_2 M_2)\)-bimodule, where the left module structure comes from left multiplication by \(U(g, e_1)\) and the right module structure comes from the fact that \(U(g)/X_2 M_2 \simeq (U(g, e_1)/\mathfrak{t}_\mathfrak{s})^\mathfrak{g}\). This proves the existence of the first adjunction.

If conjecture 3.13 holds, then the latter algebra is isomorphic to \(U(g, e_2)\). Since adjunctions can be composed, and their composition is itself an adjunction, we can form such an adjunction for any pair of nilpotents \(e_2 \geq e_1\) by composing along a sequence of covering relations. \(\square\)

There are also applications to the W-algebraic analogue of the \(s\mathfrak{c}g\mathfrak{g}\) category \(\mathcal{C}\): a full subcategory of \(U(g)\mod\) whose definition we shall recall in the next section, and which has been studied in, e.g. [BGK, Web]. In [Los3], Loseu investigates its structure and constructs an equivalence between it and a certain subcategory of Whittaker modules in \(U(g)\mod\). The objective of this section is to prove a similar equivalence in type \(A\), relating the W-algebraic categories \(\mathcal{C}\) for different nilpotents to one another. In what follows we shall always assume conjecture 3.13 holds.

### 4.1. Categories \(\mathcal{C}\) and other related categories for W-algebras

To discuss the categories \(\mathcal{C}\) for the W-algebra \(U(g, e)\), we need to fix a choice of parabolic subalgebra \(p \subseteq g\) such that \((e, h, f)\) is contained in the Levi subalgebra \(l \subseteq p\). Further, we shall fix a maximal torus \(t\) of the centraliser \(Z_g(e)\), noting that \(t \subseteq l\).

In place of the choice of parabolic, Loseu instead chooses a cocharacter \(\theta\) of \(T\) viewed as an element of \(t\); this uniquely determines a parabolic as the positive eigenspaces of \(\theta\). Different choices of \(\theta\) will only matter inasmuch as they determine different parabolics, so as far as we are concerned the two points of view are equivalent.

This choice of parabolic and maximal torus allows us to define a pre-order on the weights of \(t\): \(\lambda \geq \mu\) if and only if \(\lambda - \mu\) is a linear combination of the weights of \(t\) acting on \(p\). The existence of an embedding \(U(t) \hookrightarrow U(g, e)\) (cf. [BGK, Theorem 3.3])
allows any \( U(\mathfrak{g}, e) \)-module to be decomposed into generalised weight spaces with respect to \( t \).

Note also that, as proven by Premet, \( Z(\mathfrak{g}, e) \equiv Z(U(\mathfrak{g}, e)) \) is isomorphic to the ordinary centre of the universal enveloping algebra \( U(\mathfrak{g}) \), and that the natural map \( Z(\mathfrak{g}) \to U(\mathfrak{g}, e) \) is an isomorphism onto the centre. Thus, central characters of \( U(\mathfrak{g}) \) can be translated to central characters of \( U(\mathfrak{g}, e) \).

This allows for a number of different full subcategories of \( U(\mathfrak{g}, e)\)-mod to be defined, the objects of which satisfy various subsets of the following conditions.

- (O1) The \( t \)-weights are contained in a finite union of sets of the form \( \{ \mu : \mu \leq \lambda \} \).
- (O2) The generalised weight spaces with respect to \( t \) are finite-dimensional.
- (O3) The action of \( t \) on the module is semisimple.
- (O4) The action of \( Z(\mathfrak{g}, e) \equiv Z(U(\mathfrak{g}, e)) \) on the module is semisimple.

The notation used for these categories differs amongst different papers. We will mostly keep to a pared-down version of the notation used in Webster’s papers \cite{Web}, but since the machinery and proof of Loseu’s work \cite{Los3} is extremely important here, we shall present it as well. Loseu’s notation leaves the nilpotent \( e \) implicit, which would render it ambiguous in the context of this paper.

| Conditions | 1 | 1,2 | 1,3 | 1,2,3 | 1,2,4 |
|------------|---|-----|-----|-------|-------|
| Notation   | \( \mathcal{O}(e, p) \) | \( \mathcal{O}(e, p) \) | \( \mathcal{O}(e, p) \) | \( \mathcal{O}(e, p) \) | \( \mathcal{O}(e, p) \) |
| Webster    | \( \mathcal{O}(\mathcal{W}_e, p) \) | \( \mathcal{O}(\mathcal{W}_e, p) \) | \( \mathcal{O}(\mathcal{W}_e, p) \) | \( \mathcal{O}(\mathcal{W}_e, p) \) |
| Loseu      | \( \mathcal{O}(\theta) \) | \( \mathcal{O}(\theta) \) | \( \mathcal{O}(\theta) \) | \( \mathcal{O}(\theta) \) | \( \mathcal{O}(\theta) \) |

Table 1. Definitions of the \( W \)-algebraic categories \( \mathcal{O} \)

**Note.** The full subcategories on which the centre \( Z(\mathfrak{g}, e) \) acts by a given generalised central character \( \xi \) are denoted \( \mathcal{O}(\xi, e, p) \), and so on.

**Remark 4.2.** There are a number of equivalent ways of phrasing the above conditions, some of which are used in Loseu’s original paper.

Condition (O1) is equivalent to:

- (O1’) \( U(\mathfrak{g}, e)_{\geq 0} \) acts by locally nilpotent endomorphisms.

Further, conditions (O1) and (O2) together are equivalent to condition (O1) and:

- (O2’) The \( U(\mathfrak{g}, e)^0 \)-module obtained after taking \( U(\mathfrak{g}, e)_{\geq 0} \)-invariants is of finite dimension, where \( U(\mathfrak{g}, e)^0 := U(\mathfrak{g}, e)_{\geq 0}/(U(\mathfrak{g}, e)U(\mathfrak{g}, e)_{\geq 0} \cap U(\mathfrak{g}, e)_{\geq 0}) \).

In addition, there are a number of full subcategories of \( U(\mathfrak{g}) \)-mod of interest to us; these are variations on subcategories of modules known as **generalised Whittaker modules**. To define these categories, we need to fix a choice of maximal nilpotent subalgebra \( \mathfrak{n} \) along with a character \( \chi : \mathfrak{n} \to \mathbb{C} \); we can then define the shifted Lie algebra \( \mathfrak{n}_\chi := \{ \xi - \chi(\xi) : \xi \in \mathfrak{n} \} \). We will again consider full subcategories whose objects satisfy various subsets of the following conditions.

- (Wh1) The shifted Lie algebra \( \mathfrak{n}_\chi \) acts by locally nilpotent endomorphisms.
- (Wh2) The action of the centre \( Z(\mathfrak{g}) \) is locally finite.
- (Wh3) The action of \( t \) on the module is semisimple.
- (Wh4) The action of \( Z(\mathfrak{g}) \) on the module is semisimple.

**Theorem 4.3** ([\cite{Los3}, Theorem 4.1], [\cite{Los1}, Theorem 1.2.2(iii)], [\cite{Web}, Proposition 7]). There are equivalences between each of columns of table 1 and the corresponding columns of table 2. These equivalences still hold if one restricts to a given generalised character of \( Z(\mathfrak{g}) \) and the corresponding character of \( Z(\mathfrak{g}, e) \).
We shall provide a version of this theorem for which the Whittaker categories lie not in \( U(\mathfrak{g})\)-mod, but rather in \( U(\mathfrak{g}, e')\)-mod for another nilpotent \( e' \leq e \).

### 4.2. Equivariant Slodowy slices

The remaining results of this paper follow the techniques and methodology presented in Loseu’s papers [Los1, Los2, Los3], but translated to the context of this paper. We present them here for clarity and to highlight the changes necessary in our situation.

Given a nilpotent element \( e \) with a good grading \( \Gamma \) given by the semisimple element \( h' \), construct a \( \Gamma \)-graded \( \mathfrak{sl}_2 \)-triple \((e, h, f)\). Based on this data, Loseu defines the **equivariant Slodowy slice**:

\[
\widetilde{\mathcal{S}}_\chi := G \times \mathcal{S}_\chi \subseteq G \times \mathfrak{g}^* \simeq T^* G.
\]

This is a symplectic subvariety of the cotangent bundle \( T^* G \), and is stable under a number of group actions. The group \( G \) acts on itself both on the left and right by multiplication, which induces corresponding Hamiltonian actions on the cotangent bundle. Loseu defines the following group actions:

- \( G \) acts by \( g \cdot (g_1, \alpha) = (gg_1, \alpha) \).
- \( \mathbb{C}^\times \) acts by \( t \cdot (g_1, \alpha) = (g_1 \gamma(t)^{-1}, t^{-\gamma(t)} \alpha) \).

Here, \( \gamma : \mathbb{C}^\times \to G \) is the cocharacter determined by exponentiation of the semisimple element \( h' \). Recall also that \( G \) acts on \( \mathfrak{g}^* \) by the coadjoint action \((g \alpha)(\xi) = \alpha(\text{Ad}_g^{-1} \xi)\).

Choosing a Premet subgroup \( M \) for \( e \), we further define an action of \( M \) on \( T^* G \):

- \( M \) acts by \( m \cdot (g_1, \alpha) = (g_1 m^{-1}, ma) \).

This has moment map \( \mu : G \times \mathfrak{g}^* \to \mathfrak{m}^* \) given by \( \mu(g, \alpha) = \alpha |_{m} \). It is therefore clear that \( T^* G/\mathcal{H} M \simeq \widetilde{\mathcal{S}}_\chi \), where this is the usual symplectic Hamiltonian reduction.

Having translated the problem of Hamiltonian reduction of Slodowy slices as Poisson varieties to that of Hamiltonian reduction of equivariant Slodowy slices as symplectic varieties, we can now state the following theorem.

**Theorem 4.4.** The homomorphism \( U(\mathfrak{g}, e_1) /\mathcal{H} \mathfrak{t} \to U(\mathfrak{g}) /\mathcal{H} \mathfrak{m}_2 \) of theorem 3.9 is an isomorphism.

**Proof.** After taking \( G \)-invariants of this symplectic reduction, we obtain the previous Poisson reduction of Slodowy slices. By corollary 2.7, it therefore suffices to prove that the quantum Hamiltonian reduction induces an isomorphism of the classical Hamiltonian symplectic reductions. However this follows by classical symplectic Hamiltonian reduction by stages for semidirect products, which can be found in, e.g. [MMO+, Theorem 4.2.2].

Recall now the constructions of section 3. For any \( e_1 \in \mathfrak{sl}_n \) and any nilpotent orbit \( \mathcal{O}_2 \) which covers the orbit of \( e_1 \), we produce a Premet subalgebra \( m_1, \) a nilpotent \( e_2 \in \mathcal{O}_2, \) a subalgebra \( m_2, \) and a semisimple element \( h'_2 \) which gives a grading \( \Gamma \) which is good for \( e_2 \). Choosing \( (e_2, h_2, f_2) \) to be a \( \Gamma \)-graded \( \mathfrak{sl}_2 \)-triple, we can define
the Slodowy slice $\mathcal{S}_{\chi_2}$. The reduction by stages construction of section 3 therefore produces the following commutative diagram:

\[
\begin{array}{ccc}
T^*G & \rightarrow & T^*G \\
\downarrow & & \downarrow \\
G \times (\chi_2 + m_2^\perp) & \hookrightarrow & G \times (\chi_1 + m_1^\perp) \\
\downarrow & & \downarrow \\
\mathcal{S}_{\chi_2} & \varphi \hookrightarrow & \mathcal{S}_{\chi_1}
\end{array}
\] (10)

Here, the vertical maps $G \times (\chi_i + m_i^\perp) \hookrightarrow T^*G$ and $G \times (\chi_i + m_i^\perp) \rightarrow \mathcal{S}_{\chi_i}$ are the natural maps coming from Hamiltonian reduction, while the inclusions of $\mathcal{S}_{\chi_i}$ come from the natural presentation $\mathcal{S}_{\chi_i} = \chi_i + (g/[g, f_i])^* \subseteq \chi_i + m_i^\perp$. The map $\varphi$ is the natural extension of the inclusion $\chi_2 + m_2^\perp \hookrightarrow \chi_1 + m_1^\perp$, and $\varphi$ is defined as the obvious composition of maps.

In this context, we shall consider the following additional actions on $T^*G$, which preserve each of $\mathcal{S}_{\chi_i}$:

- $Q := Z_G(e_1, h_1, f_1) \cap Z_G(e_2, h_2, f_2) \cap Z_G(h'_2)$ acts by $g_0 \cdot (g_1, \alpha) = (g_1 g_0^{-1}, g_0 \alpha)$.

- $\tilde{G} := G \times \mathbb{C}^\times \times Q$, which acts component-wise.

**Lemma 4.5.** The map $\varphi$ is a $\tilde{G}$-equivariant embedding of symplectic manifolds.

**Proof.** That $\varphi$ is $(G \times Q)$-equivariant is manifest from the construction. To see that $\varphi$ is injective, note that $\varphi(x) = \varphi(y)$ if and only if $i(x)$ and $i(y)$ lie in the same $M_1$-orbit. But then $x$ and $y$ lie in the same $M_2$-orbit, as $M_1 \subseteq M_2$, which would imply that $x = y$.

To see that $\varphi$ is symplectic, note that the $M_i$-orbits form a nilfoliation of $\mathcal{S}_{\chi_i}$ in $G \times (\chi_i + m_i^\perp)$. Hence, lifting along the $M_2$-orbits can equally well be accomplished by restricting to lifting along $M_1$-orbits, and so the symplectic forms will agree.

As in the classical case, the cocharacter $\gamma$ associated to the semisimple element $h'_2$ gives an action of $\mathbb{C}^\times$ on $\mathcal{S}_{\chi_2}$. By lemma 3.12, the adjoint action $\text{Ad}_{\gamma(t)}$ stabilises $m_1$, and hence this also gives a well-defined action on $\mathcal{S}_{\chi_1}$. The map $\varphi$ intertwines these two actions, and in both cases scales the symplectic form: $t \cdot \omega = t^2 \omega$. \hfill $\square$

Putting these facts together yields the following theorem.

**Theorem 4.6.** For any pair of nilpotent elements $e_1 \leq e_2$ in $\mathfrak{sl}_n$ in the dominance ordering, there is a $(G \times Q)$-equivariant embedding of symplectic manifolds $\mathcal{S}_{\chi_2} \hookrightarrow \mathcal{S}_{\chi_1}$. Taking $G$-invariants yields an embedding of Poisson manifolds $\mathcal{S}_{\chi_2} \hookrightarrow \mathcal{S}_{\chi_1}$. Furthermore, there exist $\mathbb{C}^\times$-actions on both sides, intertwined by the embedding, which scale the symplectic forms (resp. Poisson bivectors) by a factor of $t^2$. This $\mathbb{C}^\times$-action is a contracting action on $\mathcal{S}_{\chi_2}$.

### 4.3 The decomposition lemma

Consider the point $x = (1, \chi_2)$; the embedding of theorem 4.6 induces an inclusion of symplectic vector spaces $\varphi_* : T_x \mathcal{S}_{\chi_2} \hookrightarrow T_x \mathcal{S}_{\chi_1}$. This, in turn, induces an inclusion $(g/[g, f_2])^* \hookrightarrow (g/[g, f_1])^*$, and we let $W$ denote its image. Finally, we define the subspace $V \subseteq g_{\mathfrak{h}}(e_1)$ as follows:

\[
V := W_\perp \cap g_{\mathfrak{h}}(e_1) = \{ \xi \in g_{\mathfrak{h}}(e_1) : \alpha(\xi) = 0 \text{ for all } \alpha \in W \}.
\]

This vector space has a symplectic form expressed by $\omega(x, y) = \chi_2([x, y])$. Note that if $e_1 = 0$, then $V = [g, f_2]$.

The symplectic form on $T_x \mathcal{S}_{\chi_1}$ is given by the expression:

$$
\omega(\xi + \alpha, \eta + \beta) = \chi_2([\xi, \eta]) - \langle \xi, \beta \rangle + \langle \eta, \alpha \rangle,
$$
so the symplectic complement \((\varphi_\ast T_x \mathcal{F}_\chi)''\) is \(\{(\xi, \operatorname{ad}_x^* \chi) : \xi \in V\}\). Projecting onto the first component identifies this with \(V\), but we’ll instead identify the it with \(V^\ast\); they are isomorphic as symplectic \(Q\)-modules. This gives a \((\mathbb{C}^\times \times Q)\)-equivariant symplectic isomorphism \(\psi : T_x \mathcal{F}_\chi \to T_x \mathcal{F}_\chi\), given by \(\psi(v, w) = \varphi_\ast(v) + w\).

The standard considerations of Fedosov quantisation (cf. [Los1, Section 2.2]) yield \((G \times Q)\)-invariant, homogeneous, degree 2 star products on each of \(\mathbb{C}[\mathcal{F}_\chi][\hbar]\), the Moyal-Weyl star product on \(\mathbb{C}[V^\ast][\hbar]\), and finally on the product \(\mathbb{C}[\mathcal{F}_\chi \times V^\ast][\hbar]\). Since the star products are differential, they induce star products on the completions \(\mathbb{C}[\mathcal{F}_\chi][\hbar]_G\) and \(\mathbb{C}[\mathcal{F}_\chi \times V^\ast][\hbar]_G\). Applying the argument of [Los1, Theorem 3.3.1] yields a \(\tilde{G}\)-equivariant \(\mathbb{C}^\hbar\)-algebra isomorphism

\[
\Phi_h : \mathbb{C}[\mathcal{F}_\chi][\hbar]_G \to \mathbb{C}[\mathcal{F}_\chi \times V^\ast][\hbar]_G.
\]

Taking \(G\)-invariants produces the following analogue of [Los3, Proposition 2.1].

**Theorem 4.7.** There is a \((\mathbb{C}^\times \times Q)\)-equivariant \(\mathbb{C}^\hbar\)-algebra isomorphism

\[
\Phi_h : \mathbb{C}[\mathcal{F}_\chi][\hbar]_G \to \mathbb{C}[\mathcal{F}_\chi \times V^\ast][\hbar]_G\]

satisfying:

1. \(\Phi_h(\sum_{i=0}^\infty f_i \hbar^{2i})\) contains only even powers of \(\hbar\).
2. The cotangent map \(d_0(\Phi_h)^* : \mathbb{C}(e_2) \to V \to \mathbb{C}(e_1)\) coincides with \(\psi\).
3. For \(t_1, t_2\) the respective embeddings of \(q\) into the domain and codomain of \(\Phi_h\), then \(\Phi_h \circ t_1 = t_2\).

4.4. *Loseu’s machinery.* In order to continue, we recall the machinery Loseu has developed for proving Theorem 4.3. Let the following be given.

- \(v = \bigoplus_{i \in \mathbb{Z}} v(i)\) is a graded finite-dimensional vector space on which a torus \(T\) acting by preserving the grading.
- \(A := \operatorname{Sym}(v)\), with the induced grading \(A = \bigoplus_{i \in \mathbb{Z}} A_i\) and induced \(T\)-action.
- \((A, \circ)\) is an algebra with the same underlying vector space as \(A\), where the algebra structure comes from a \(T\)-invariant deformation quantisation.
- \(\omega_1\) is a symplectic form on \(v(1)\), where \(\omega_1(u, v)\) is the constant term of the commutator in \(A\), and \(\eta\) is a lagrangian subspace of \(v(1)\).
- \(m := \eta \oplus \bigoplus_{i < 0} v(i)\).
- \(v_1, v_2, \ldots, v_n\) is a homogeneous basis of \(v\) such that \(v_1, v_2, \ldots, v_m\) form a basis of \(m\). Further, let \(d_i\) be the degree of \(v_i\) and assume that \(d_1, d_2, \ldots, d_m\) are increasing and that all \(v_i\) are \(T\)-semi-invariant.
- \(A^\circ\) is the subalgebra of \(\mathbb{C}[v^\ast]\) consisting of elements of the form \(\sum_{i \leq c} f_i\) for some \(c\), where \(f_i\) is a homogeneous power series of degree \(i\).
- \(A^\omega\) is the algebra \(A^\circ\) with multiplication as in \(A\). Any element of \(A^\omega\) can be written as an infinite linear combination of monomials \(v_{i_1} \circ \cdots \circ v_{i_\ell}\), where \(i_1 \geq \cdots \geq i_\ell\) and \(\sum_{j=1}^\ell d_{i_j} \leq c\) for some \(c\). Hence there is a filtration \(F_c A^\omega\).
- \(\theta\) is a co-character of \(T\), and \(v_{\geq 0}\) and \(v_{> 0}\) are, respectively, the sums of the positive and strictly-positive \(\operatorname{ad} \theta\)-eigenspaces of \(v\). We shall further require that \(v_{> 0} \subseteq m \subseteq v_{\geq 0}\).
- \(A_{\geq 0}, A_{> 0}, A^\omega_{\geq 0}, A^\omega_{> 0}\) are all defined analogously.
- \(A^\wedge := \lim A/Am^k\). Note that there is an injective algebra homomorphism \(A^\circ \to A^\wedge\).

**Proposition 4.8.** [Los3, Proposition 5.1] Let \((A, \circ)\) and \((A', \circ')\) be two different algebras coming from \(A\) and \(v\) as above. Suppose there is a subspace \(\eta \subseteq v(1)\) which is Lagrangian for both symplectic forms, and every element of \(A\) can be written as a finite sum of monomials in both \(A\) and \(A'\). Then any homogeneous \(T\)-equivariant isomorphism \(\Phi : A^\circ \to A'^\circ\)
satisfying $\Phi(u_i) - v_i \in F_{d_i-2}A + (F_dA \cap v^2A)$ extends uniquely to a topological algebra isomorphism $\Phi : A^\wedge \to A'^\wedge$ with $\Phi(A^\wedge m) = A'^\wedge m$.

**Corollary 4.9.** [Los3, Corollary 5.2] The isomorphism $\Phi : A^\wedge \to A'^\wedge$ induces an equivalence of categories $Φₗ : \text{Wh}(A, m) \to \text{Wh}(A', m)$, where $\text{Wh}(A, m)$ is the category of $A^\wedge$-modules which are annihilated by some $A^\wedge m^k$. This equivalence preserves the subcategories on which $t$ acts semisimply, and commutes with the functor of taking $m$-invariants; i.e. $Φₗ(M^m) = Φₗ(M)^m$.

Note. $\text{Wh}(A, m)$ can naturally be viewed as the category of $A$-modules on which $m$ acts by locally nilpotent endomorphisms. This justifies the choice of notation in table 2.

With the constructions as before, we seek to make a set of choices which satisfy the hypotheses of proposition 4.8. First, we’ll fix a maximal torus $T \subseteq Q$, and pick an arbitrary cocharacter $θ$, viewed as an element of $t$. Denote by $p$ the parabolic subalgebra of $g$ consisting of the positive eigenspaces of $\text{ad} \; θ$. The zero eigenspace is a Levi subalgebra $l \subseteq p$, which contains each of $e_i, h_i$ and $f_i$ for $i = 1, 2$. The good grading $Γ$ of $g$ induces a good grading of $l$, and so one can pick a Premet subalgebra $m \subseteq l$ as usual, with corresponding shift $m_\chi$. Let $\bar{m} := m \oplus g_{>0}$, where $g_{>0}$ consists of the strictly positive eigenspaces of the action of $\text{ad} \; θ$; the corresponding shift is $\bar{m}_\chi := \bar{m}_\chi \oplus g_{>0}$.

1. Define $v := \{ξ - \chi(ξ) : ξ \in θ(e_1)\}$. Note that $v \simeq θ(1_2) \oplus V$, as shown in section 4.3.
2. Choosing $m > 2 + 2d$, where $d$ is the maximum eigenvalue of $\text{ad} \; h_2'$ on $g$, define the grading on $v$ to be given by
   $$v(i) = \{ξ \in v : (\text{ad} \; h_2' - m \text{ad} \; θ)ξ = (i - 2)ξ\}.$$
3. Set $m$ to be $\bar{m}_\chi \cap v$, which satisfies $v_{>0} \subseteq m \subseteq v_{\geq 0}$ by our choice of $m$.
4. Define $A := U(g, e_1)$ and $A' = A_V \otimes U(g, e_2)$.

These choices satisfy the conditions beginning this section. It remains to prove the following lemma.

**Lemma 4.10.** There is an isomorphism $\Phi : U(g, e_1) \to (A_V \otimes U(g, e_2))^\wedge$ which satisfies the hypotheses of proposition 4.8.

**Proof.** It follows from the considerations of section 4.3 that $A$ and $A'$ are both deformation quantisations of $A = \text{Sym}(v)$. Further, since $A^\wedge$ and $A'^\wedge$ can be identified with the respective quotients by $h - 1$ of the $C^\times$-finite parts of

$$C[\mathcal{Q}_1]^\wedge x_2[h] \quad \text{and} \quad C[\mathcal{Q}_2]^\wedge x_2[h] \otimes C[h]C[V^\wedge]^\wedge,$$

the isomorphism $Φₗ$ of theorem 4.7 provides the necessary map $Φ$. This satisfies the hypotheses of proposition 4.8 by the same considerations as before (cf. [Los1, Corollary 3.3.2]).

4.5. **Category equivalences.** Finally, we shall use this machinery to develop the category equivalences we need. First, there is an equivalence $K' : \text{Wh}(A', m) \to \widehat{O}(e_2, p), \quad K'(M) := (M)^\bar{m}_\chi \overline{\text{V}}$.

To see that the image lies in $\widehat{O}(e_2, p)$, it suffices to note that $U(g, e_2)_{>0}$ is generated by the strictly positive eigenspaces of $\text{ad} \; θ$, all of which lie in $m$ by construction. That this functor is an equivalence follows from results on representations of Heisenberg algebras, and the fact that $\bar{m} \cap \overline{\text{V}}$ is a Lagrangian subspace of $\overline{\text{V}}$.

Combining this with the equivalence of corollary 4.9 yields the following theorem, which is the main result of this section and generalises theorem 4.3.
Theorem 4.11. There exists an equivalence of categories

\[ K : \text{Wh}(U(g, e_1), m) \to \tilde{O}(e_2, p). \]

Furthermore, \( K \) induces the following embeddings of categories \( O \), along with their block decompositions:

\[ \tilde{O}(e_2, p) \hookrightarrow \tilde{O}(e_1, p) \]
\[ O(e_2, p) \hookrightarrow O(e_1, p) \]

Proof. The equivalence \( K \) is defined to be \( K' \circ \Phi_* \). Let \( m_1 \) and \( m_2 \) be as in section 3.2, \( V_1 := [g, f_1] \), and \( V \) and \( m \) be as above. These functors can then be arranged into a commutative diagram.

\[
\begin{array}{ccc}
\text{Wh}(U(g), (\tilde{m}_1)_{\chi_1}) & \sim & \text{Wh}(A_{V_1} \otimes U(g, e_1), (\tilde{m}_1)_{\chi_1}) \sim \tilde{O}(e_1, p) \\
\text{Wh}(A_{V_1} \otimes U(g, e_1), (\tilde{m}_2)_{\chi_2}) \sim \text{Wh}(A_{V_2} \otimes U(g, e_2), (\tilde{m}_2)_{\chi_2}) \sim \tilde{O}(e_2, p) \\
\text{Wh}(U(g, e_1), m) & \Phi_* & \text{Wh}(A_{V_1} \otimes U(g, e_2), m) \sim K' \tilde{O}(e_2, p)
\end{array}
\]

Here, the equivalences between the first two columns of each row are the functors \( \Phi_* \) of corollary 4.9 in the appropriate settings. The functors between the second and third columns are the appropriate analogues of the functor \( K' \), taking invariants with respect to \( m_1 \cap V_1, m_2 \cap (V_1 \oplus V) \) and \( m \cap V \), respectively. Since these are all respective Lagrangian subspaces of \( V_1, V_1 \oplus V \) and \( V \), it follows that they are equivalences.

From the diagram, it can be seen that there is an embedding of categories \( O \),

\[ \tilde{O}(e_2, p) \hookrightarrow \tilde{O}(e_1, p). \]

Since the functor \( K \) intertwines the actions of \( t \) and \( Z(g, e) \), it induces an embedding of each of the above subcategories, and also their block decompositions with respect to generalised central characters. \( \square \)

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