Hidden E type structures in dilute A models

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Abstract

The hidden $E_7(E_6)$ structure has been conjectured for the minimal model $\mathcal{M}_{4,5}(\mathcal{M}_{6,7})$ perturbed by $\Phi_{1,2}$ in the context of conformal field theory (CFT). Motivated by this, we examine the dilute $A_{4,6}$ models, which are expected to be corresponding lattice models. Thermodynamics of the equivalent one dimensional quantum systems is analyzed via the quantum transfer matrix approach. Appropriate auxiliary functions, related to kinks in the theory, play a role in constructing functional relations among transfer matrices. We successfully recover the universal $Y-$ systems and thereby Thermodynamic Bethe Ansatz equations for $E_{6,7}$ from the dilute $A_{6,4}$ model, respectively.

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1 Introduction

The impact of perturbed conformal field theory (CFT) has many aspects \cite{1,2}. In this communication we explore one of its predictions; the ”Trinity” among minimal unitary CFT theory $\mathcal{M}_{p,p+1}, p = 3, 4, 6$ perturbed by $\Phi_{1,2}$, lattice models off criticality, (the Ising model in a field, the tricritical Ising model off the critical temperature and the tricritical 3 state Potts model) and the dilute $A_{3,4,6}$ models. Many results have already been accumulated on the equivalence in universality \cite{3}-\cite{18}. The scaling exponents of the dilute $A_3$ model in periodic or open boundary conditions have been evaluated analytically \cite{12}-\cite{17}. They agree with numerical results for the Ising model in a magnetic field. Masses for eight elementary excitations of the dilute $A_3$ model are found to be proportional to components of the (largest) eigenvector of Cartan matrix for $E_8$ \cite{16,18}. Vertex operators of $A_2^{(2)}$, which is the symmetry of the dilute $A_3$ model at criticality, satisfy a set of relations indicating the hidden $E_8$ structure \cite{19}.

Especially, we like to call attention to thermodynamics of a 1D system related to the dilute $A_3$ model in \cite{20}. A set of solutions to the eigenvalue problem of the 1D Hamiltonian has been identified in exquisite ”string” forms \cite{20,21,22}. Nine of them are expected to contribute nontrivially in the thermodynamic limit. This observation leads to a set of integral equations (Thermodynamic Bethe ansatz , TBA ) which determines the free energy. Remarkably, TBA exhibits the underlying $E_8$ structure \cite{23}.

In \cite{24}, we have attacked the same problem in a different setting. By following general frameworks, one represents the free energy by the largest eigenvalue of the ”quantum transfer matrix” (QTM) acting on a virtual space \cite{25} -\cite{31}. We have managed to solve the (single) eigenvalue problem of commuting QTM by introducing auxiliary functions related to fusion of QTM \cite{32,34,35}. We will simply call them fusion QTMs. Eight fusion QTMs are found to satisfy a closed set of functional relations related to $E_8$. A quantum analogue of the Jacobi-Trudi formula \cite{36,39}, as well as combinatorial aspects in terms of ”Yangian analogue” of Young tableaux \cite{36,37,38,39,40} play a fundamental part in the proof of the relations. Nice analytic properties of fusion QTMs allow for the transformation of functional relations into coupled integral equations. The resultant thermodynamic Bethe ansatz (TBA) equation yields a direct evaluation of free energy. Again it coincides with a hypothetical TBA for the $E_8$ theory.

As promised in \cite{24}, we carry out this program for the dilute $A_4$ and $A_6$ models. A novel feature lies in the fact that ”a box” of the ”Young tableaux” is no longer the fundamental constituent. This may be natural in view of the representation theory; the vector representation is no longer minimal. In the language of the S-matrix theory, boxes present breathers, rather than kinks. We conjecture explicit forms of QTMs related to these
kinks. The investigation on these kink QTM{s reveals connections between
$U_q(A_2^{(2)})$ modules of symmetric tensors of the vector representation and
$U_q(\widehat{E}_6), U_q(\widehat{E}_7)$ modules when $q$ equals to proper root of unity. With help
of this observation, closed functional relations among fusion QTM{s are also
found for the dilute $A_{4,6}$ models. Quite parallel to the dilute $A_3$ model,
one recovers TBAs expected for $E_6,7$ \cite{23}.

The paper is organized as follows. As the subject may not be so familiar
to readers, we present a brief survey on the QTM approach, together with
the sketch of the idea of analytic Bethe ansatz in section 2. The dilute
$A_L$ model is briefly described in section 3. The $sl_3$ fusion structure of
the model is presented in view of analytic Bethe ansatz, the ”Yangian
analogue” of Young tableaux and the quantum Jacobi-Trudi formula in
section 4. We concentrate on the dilute $A_4$ model in sections 5 and 6.
The explicit form of QTM related to the ”kink” is proposed in section 5.
Yangian homomorphisms among $Y(E_7)$ modules serve as a useful guide
in search of the form. The rest of QTM{s are defined and their functional
relations are examined in section 6. The result coincides with prediction
in \cite{58}. Similar results for the dilute $A_6$ model are given in section 7. With
piece of information on analyticity of these QTM{s, supported by numerics,
the desired TBAs are recovered in section 8. We conclude the paper with
a short summary in section 9.

2 Survey on the QTM approach

Exact evaluation of physical quantities at finite temperatures poses serious
difficulties even for integrable models. One has to go much beyond mere
diagonalization of a Hamiltonian; summation over eigenspectra must be
performed.

The string hypothesis brought the first breakthrough and success. It
postulates dominant solutions to the Bethe ansatz equation (BAE) in the
thermodynamic limit. In a sense, the method tackles the combinatorial
aspect of the problem directly.

2.1 QTM

The quantum transfer matrix (QTM) method takes a different route. It
utilizes the famous mapping between the Hamiltonian $H_M$ of a 1D quan-
tum system and the row to row transfer matrix $T_{RTR}(u)$ of the correspond-
ing 2D classical model \cite{25} - \cite{31}. In the present context, the latter is given by

$$
(T_{RTR}(u))_{[a]}^{[b]} = \prod_{j=1}^{M} b_j \frac{b_{j+1}}{a_j a_{j+1}}.
$$
Here the box represents the RSOS weight and $M$ is the number of sites. See appendix for explicit weights for the dilute $A_L$ models. The parameter $u$ represents the anisotropy of interactions between horizontal and vertical directions, and is called the spectral parameter. The explicit relation between $\mathcal{H}_M$ and $T_{RTR}(u)$ reads,

$$T_{RTR}(u) \sim T_{RTR}(0)(1 + \frac{u}{\epsilon}\mathcal{H}_M + O(u^2)),$$

where $\epsilon$ is the normalization parameter of the Hamiltonian. The essential idea in the QTM approach is encoded in the following identity,

$$\exp(-\beta\mathcal{H}_M) = \lim_{N \to \infty} (1 + \frac{u}{\epsilon}\mathcal{H}_M)^N|_{u \to -\beta\epsilon/N} = \lim_{N \to \infty} (T_{RTR}'(u = -\beta\epsilon/N))^N$$

$$T_{RTR}'(u) := (T_{RTR}(u)(T_{RTR}(0))^{-1})^N.$$

Namely, the partition function of the original problem is transformed into that of the 2D classical models on $M \times N$ sites. The fictitious dimension $N$ is sometimes referred to as the Trotter number. We can interpret $T_{RTR}'(u)$ as the row to row transfer matrix in the "vertical" direction. Similarly, one can construct a transfer matrix propagating in the "horizontal" direction $T_{QTM}'(u)$, which acts on $N$ sites. Thereby we have

$$\text{Tr} \exp(-\beta\mathcal{H}_M) = \lim_{N \to \infty} \text{Tr} T_{QTM}'(u = -\beta\epsilon/N)^M.$$ 

It may be better to rewrite this in the form,

$$\lim_{M \to \infty} \frac{1}{M} \log \left( \text{Tr} \exp(-\beta\mathcal{H}_M) \right) = \lim_{N \to \infty} \lim_{M \to \infty} \frac{1}{M} \left( \text{Tr} T_{QTM}'(u = -\beta\epsilon/N)^M \right).$$

The exchangeability of two limits are proven in [25].

The gap opens up between the largest and the second largest eigenvalues of $T_{QTM}'(u)$. In the thermodynamic limit $M \to \infty$, we only have to deal with the largest eigenvalue of QTM. This strongly contrasts to the spectra of $T_{RTR}'(u)$. One observes almost degenerate low lying excitations in the latter case as $M \to \infty$. The evaluation of free energy per site of the 1D quantum system is thus reduced to the largest eigenvalue problem of $T_{QTM}'(u)$. We are free from the summation problem.

This is, unfortunately, not the happy end of the story. The Trotter number should be sent infinity at the end. The diagonalization of QTM is accomplished by the application of the Bethe ansatz method. The BAE depends nontrivially on $N$, which originates from the local interaction parameter $u$. Thus we can not retort to the simple-mind application of the usual scheme of converting the transcendental equation into the integral equation. This makes the extrapolation $N \to \infty$ quite nontrivial.
2.2 commuting QTM

Instead of dealing with BAE roots directly, we employ a different idea. The integrable structure of the underlying model allows for the introduction of one parameter family of commuting QTMs which is labeled by a novel complex parameter \( x \). For the explicit demonstration of this, we adopt a more sophisticated approach \([32, 33]\) than the one presented above. One introduces a "staggered manner" QTM to avoid \( (T_{RTR}(0))^{-1} \) factor in the definition of \( T'_{RTR}(u) \).

\[
(T_{QTM}(u, x))_{\{b\}} = \prod_{j=1}^{N/2} \frac{b_{2j-1}}{a_{2j-1}} \frac{b_{2j}}{a_{2j}} \frac{a_{2j+1}}{b_{2j+1}} u + i x \frac{b_{2j}}{a_{2j}} \frac{a_{2j+1}}{b_{2j+1}} u - i x .
\]

The following relation is still valid,

\[
\beta f = - \lim_{M \to \infty} \frac{1}{M} \ln \text{Tr} \exp(-\beta \mathcal{H}_x)
= - \lim_{N \to \infty} \ln \left( \text{the largest eigenvalue of } T_{QTM}(u = -\frac{\beta}{N}, x = 0) \right).
\]

As is emphasized above, we find the intriguing fact, commutativity of QTMs,

\[
[T_{QTM}(u, x), T_{QTM}(u, x')] = 0.
\]

Generally, one can construct a "higher spin" QTM \( T_{\text{fusion}}(u, x) \) by the fusion procedure. The Yang Baxter integrability also assures commutativity among these generalized QTMs,

\[
[T_{\text{fusion}}(u, x), T_{\text{fusion}}'(u, x')] = 0.
\]

Note that the factor \( u \) is in common. Hereafter we will sometimes drop this common factor in commuting QTMs. We also sometimes use same notations for transfer matrices and their eigenvalues as we are considering them on the identical eigenspace.

We utilize the existence of the complex \( x \) plane in which QTMs are simultaneously diagonalizable. There exist functional relations among these fusion QTMs in the \( x \) plane. Our idea is to utilize these functional relations in place of BAE. See \([32, 33]\) for the discussion in the case of usual row to row transfer matrices. Our motivation is simple. The number of roots is of order \( N \), the Trotter number. All these locations are changing with \( N \), while functional relations depend on \( N \) weakly. The dependence can be summarized in the known scalar factors in functional relations. One may expect the tractable limit \( N \to \infty \) for functional relations. The problem of combinatorics (summation over eigenspectra) is then reduced to the
study of functional relations \footnote{One can adopt different auxiliary functions from the fusion hierarchy. Results on other choices of auxiliary functions for several models, see \cite{33,45-51}.} and analytic structures of fusion QTMs, as will be discussed soon below.

### 2.3 Analytic Bethe ansatz and functional relations

The QTMs should not possess singularities in the $x$ plane as Boltzmann weights are regular functions of $x$. BAE can be interpreted as the pole-free condition of QTMs in the complex $x$ plane. Conversely, the analyticity requirement imposes restrictions on explicit eigenvalues of QTMs.

The analytic Bethe ansatz was proposed in \cite{55}, as a tool in deriving expressions of eigenvalues of transfer matrices. It starts from a simple observation; the eigenvalue at the ”vacuum sector” is determined by diagonal elements of $R$ matrix, which are referred to as vacuum expectation values. In general sectors, eigenvalues should be modified such that each vacuum expectation value is ”dressed” by appropriate combination of Baxter’s $Q$ operators. See eq.\cite{4} for a typical example. The combination is determined by requiring analyticity of the transfer matrix. We call the resultant expression, Dressed Vacuum Form (DVF).

A ”universal” BAE has been proposed in \cite{55,56}; one can write down BAE for the model based on $U_q(\hat{g})$ using only algebraic data of $U_q(\hat{g})$. Starting from a properly chosen ”highest weight term”, we can construct a pole-free set of functions under BAE. In \cite{38} a similarity has been pointed out between the above procedure and the construction of the highest weight module of Lie algebra. It leads to an assumption that there exists a set of functions, pole-free under BAE, corresponding to an irreducible module of quantum affine Lie algebra. The set is naturally identified with the eigenvalue of the transfer matrix of which trace is taken over the irreducible module. This has been promoted as an axiom in \cite{38} and subsequent papers \cite{39,40} producing fruitful results. We take $sl_2$ as the simplest example. By $V_m(x)$ ($T_m(x)$), we mean the $m+1$-dimensional $sl_2$ module, and the associated transfer matrix. The DVF of $T_1(x)$ consists of two terms. We do not specify their forms and represent them by boxes with letters 1 and 2,

$$T_1(x) = \begin{array}{c} 1 \\ 2 \end{array}.$$  

Each box carries spurious poles, which are actually canceled due to BAE. We represent this situation graphically as,

$$\begin{array}{c} 1 \\ 2 \end{array} \rightarrow \begin{array}{c} 2 \\ \end{array}.$$  

The eigenvalue of a fusion QTM can be apparently represented by sum over products of ”boxes” with various letters and spectral parameters.
For example, one can construct a transfer matrix of which auxiliary space acts on the symmetric subspace of $V_1 \times V_1$. We associate to this the set of glued boxes, $[i_1]_{x-i_1},[i_2]_{x+i_2}$ ($i_1 \leq i_2$). The difference in spectral parameters is fixed so as to match the singularity of $R$ matrix. The cancellation of spurious singularities is again depicted as

$$[1\ 1] \rightarrow [1\ 2] \rightarrow [2\ 2]$$

where we omit spectral parameters. The eigenvalue of $T_2(x)$ is given by the sum of three diagrams in the above. Extension to general $T_m$ is now obvious. Starting from the "highest weight" term, $[1\ 1 \cdots 1]$, we have a cancellation diagram,

$$[1\ 1 \cdots 1] \rightarrow [1\ 1 \cdots 2] \rightarrow \cdots \rightarrow [2\ 2 \cdots 2]$$

The sum over them, regarded as expressions, yields the eigenvalue of $T_m(x)$. One can easily identified the diagram with the crystal graph of the $m$-fold tensor of $U_q(sl_2)$ representing the irreducible module.

Suppose that we have a short exact sequence among tensor products of irreducible modules of quantum affine Lie algebra,

$$0 \rightarrow W_0 \otimes W_1 \rightarrow W_2 \otimes W_3 \rightarrow W_4 \otimes W_5 \rightarrow 0,$$

then a functional relation follows,

$$0 = T_{W_0}T_{W_1} - T_{W_2}T_{W_3} + T_{W_4}T_{W_5}.$$ 

Remark that spectral parameter dependencies are implicit in $W_i$s. The desired functional relations are derived as a consequence of relations among affine modules. Even if exact sequences are not available, one can still check the validity of hypothetical functional relations using explicit forms of transfer matrices, which can be derived by applying the analytic Bethe ansatz. Indeed, functional relations for $sl_2$ are easily derived without knowledge on exact sequences. By using the above box-representation, one can derive graphically,

$$T_m(x-i)T_m(x+i) = g_m(x) + T_{m-1}(x)T_{m+1}(x) \quad m \geq 1,$$  \hspace{1cm} (1)

where $g_m(x)$ is a known scalar function which depends on $N$.

Such functional relations are sometimes referred to as the $T-$ system.

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2To be precise, we first consider a vertex model of which quantum space (auxiliary space) is given by $W_i$ and denote the transfer matrix by $T_{W_i}$. Later section we use the same notation for the transfer matrix of the corresponding RSOS model.
2.4 Functional relations and Thermodynamic Bethe Ansatz

Unfortunately, functional relations alone do not provide enough information on the explicit eigenvalues. This can be easily seen from the fact that excited states’ eigenvalues satisfy same algebraic relations. One needs additional information on analyticities of fusion QTMs. Let us again demonstrate this for the $sl_2$ case. One conveniently rewrites the $T-$ system (1) in terms of $Y_m(x) = T_{m-1}(x)T_{m+1}(x)/g_m(x),$

$$Y_m(x - i)Y_m(x + i) = (1 + Y_{m-1}(x))(1 + Y_{m+1}(x)), \quad (m \geq 2)$$

where a known property of the $g-$ function, $g_m(x + i)g_m(x - i) = g_{m-1}(x)g_{m+1}(x)$ is used.

Consider functions in the largest eigenvalue sector of $T_1(x).$ We have convincing numerical evidences for the conjecture that zeros of $T_m(x)$ approximately lie on the curve $\Im x \sim \pm (m + 1).$ Then both sides of (2) are analytic, nonzero and asymptotically constant (ANZC) within the strip $\Im x \in [-1, 1].$ Strictly speaking, we must modify the lhs for $m = 1.$ We will not go into such detail in this introductory part.

This piece of information is now sufficient to transform the algebraic relations to integral equations which enable the explicit evaluation of $Y_m$ and then $T_m.$

Take the logarithmic derivatives of both sides of (2) and perform Fourier transformations. Let $\tilde{d}l Y_m[k]$ be the Fourier transformation of the logarithmic derivative of $Y_m(x).$ Thanks to the ANZC property, the Cauchy theorem applies. The resultant equation is simply given by

$$2 \cosh k \tilde{d}l Y_m[k] = \tilde{d}l(1 + Y_{m-1})[k] + \tilde{d}l(1 + Y_{m+1})[k]. \quad (m \geq 2)$$

Remarkably, both sides only contain functions with the same Fourier mode. Dividing both sides by $2 \cosh k,$ performing inverse Fourier transformation and integrating once over $x,$ we reach the integral equation, which is identical to TBA,

$$\log Y_m(x) = \int_{-\infty}^{\infty} K(x - y) \log(1 + Y_{m-1})(1 + Y_{m+1})(y)dy. \quad (m \geq 2)$$

(3)

$K(x)$ denotes the Fourier transformation of $1/2 \cosh k.$ Though we have omitted above, the rhs of the equation (3) for $m = 1$ has a nontrivial scalar factor originated from $g$ function. It thus brings a $N-$ dependency, however, by the combination $uN$ in the $N \to \infty$ limit. Remembering that $u$ is inversely proportional to $N,$ we can send $N \to \infty$ analytically! The resultant drive term depends only on $\beta.$
In this way, we take a completely different route from the string hypothesis but reach the same conclusion. In the absence of appropriate conjectures on dominant patterns of roots, our method has an explicit advantage in attacking the problem. This is the case with the dilute $A_{4,6}$ models.\footnote{Although yet unpublished, there is also progress in view of string hypothesis for these cases. The author thanks V.V. Bazhanov and O. Warnaar for information.}

In the rest of this paper, we shall extensively apply the above ideas to these cases. Before that, we repeat lessons from the above. To find functional relations is not enough. One must find them having ANZC property in appropriate domains of the complex $x$ plane. This is the most crucial step in the present approach.

3 The dilute $A_L$ model

The dilute $A_L$ model is proposed in \cite{12, 13} as an elliptic extension of the Izergin-Korepin model \cite{11}. (See \cite{12} for an elliptic extension of the different type.) The model is of the restricted SOS type with local variables $\in \{1, 2, \cdots, L\}$. The variables $\{a, b\}$ on neighboring sites should satisfy adjacency condition, $|a - b| \leq 1$. The solvable weights contain parameters $u, q$ and $\lambda$. We supplement their explicit forms in appendix. The model exhibits four different physical regimes depending on parameters,

- regime 1. $0 < u < 3$, $\lambda = \frac{\pi L}{4(L+1)}$, $L \geq 2$
- regime 2. $0 < u < 3$, $\lambda = \frac{\pi(L+2)}{4(L+1)}$, $L \geq 3$
- regime 3. $3 - \pi < u < 0$, $\lambda = \frac{\pi(L+2)}{4(L+1)}$, $L \geq 3$
- regime 4. $3 - \pi < u < 0$, $\lambda = \frac{\pi L}{4(L+1)}$, $L \geq 2$

We are interested in regimes 2 and 3. As in section 2, one defines the Hamiltonian of the associated 1D quantum chain by

$$H_\epsilon = \epsilon \frac{\partial}{\partial u} \ln T_{\text{RTR}}(u)|_{u=0}$$

as in \cite{20}, $\epsilon = -1$, (1) corresponds to regimes 2 (3), respectively.

The one particle excitations for the dilute $A_L$ case have been examined in \cite{43, 44} for $L = 3, 4, 6$. (See also \cite{18} for another derivation for $L = 3$).

Eight, seven and six particles are identified respectively, and their masses are summarized by a single formula in the trigonometric limit,

$$m_j \sim \sum_a \sin \left( \frac{a\pi}{q^j} \right),$$

The one particle excitations for the dilute $A_L$ case have been examined in \cite{43, 14} for $L = 3, 4$ and 6. (See also \cite{18} for another derivation for $L = 3$)
where $g^\vee$ is 30, 18, 12 for $L = 3, 4, 6$ and is nothing but the dual Coxeter number for $E_8$, $E_7$ and $E_6$, respectively. We present sets of allowed $a$'s in table 1,2 for $L = 4, 6$, which are of our current interest.

| $j$  | set of allowed $a$’s for $A_4$       |
|------|--------------------------------------|
| 1 (2)| {1, 7}                               |
| 2 (5)| {2, 6, 8}                             |
| 3 (7)| {3, 5, 7, 9}                          |
| 4 (6)| {4, 6, 8}                             |
| 5 (4)| {5, 7}                                |
| 6 (1)| {6}                                  |
| 7 (3)| {4, 8}                                |

Table 1

| $j$  | set of allowed $a$’s for $A_6$       |
|------|--------------------------------------|
| 1, 5 (1)| {4}                           |
| 2, 4 (4)| {3, 5}                           |
| 6 (3)  | {1, 5}                           |
| 3 (6)  | {2, 4, 6}                         |

Table 2

A number $k$ in the bracket means that it corresponds to the $k$-th light particle. Note that vectors of the form $(m_1, m_2, \cdots)$, coincide with the eigenvectors of Cartan matrices for $E_7$ and $E_6$, respectively. The exponents $\{a\}$ will re-appear in a novel context later. The leftmost numbers, which are just indices up to the present, will be connected to indices for nodes in the Dynkin diagrams of $E_7$ and $E_6$.

![Figure 1: Dynkin diagrams for $E_7$ and $E_6$.](image)

4. $sl_3$ fusion structure and Quantum Jacobi Trudi formula

The $sl_3$ type fusion structure in the dilute $A_L$ model has been discussed in [54]. This comes from the singularity of RSOS weights at $u = \pm 3$; the face
operator becomes a projector related to $sl_3$ at these points. One picks up a desired subspace from tensor products of spaces using these projectors. The adjacency conditions of local states are described by combinatorics of tableaux.

We are interested in eigenvalues of fusion QTMs. Then the most relevant is the fact that these eigenvalues are again expressible in terms of "Young tableaux" depending on spectral parameters, as exemplified in section 2.3 for the $sl_2$ case.

Explicitly, the eigenvalue $T_1(u, x)$ of $T_{QTM}(u, x)$ is given by

$$T_1(u, x) = w\phi(x + \frac{3}{2}i)\phi(x + \frac{1}{2}i)Q(x - 5/2i)Q(x - 1/2i)$$

$$+ \phi(x + \frac{3}{2}i)\phi(x - \frac{3}{2}i)Q(x - 3/2i)Q(x + 3/2i)Q(x - 1/2i)Q(x + 1/2i)$$

$$+ w^{-1}\phi(x - \frac{3}{2}i)\phi(x - \frac{1}{2}i)Q(x + 5/2i)Q(x + 1/2i),$$

$$Q(x) := \prod_{j=1}^{N} h[x - x_j]$$

$$\phi(x) := \left(\frac{h[x + (3/2 - u)i]h[x - (3/2 - u)i]}{h[2i]h[3i]}\right)^{N/2}, \quad h[x] := \theta_1(ix),$$

and $w = \exp(i\pi \frac{\ell}{L+1})$ where $\ell = 1$ for the largest eigenvalue sector. $\theta_1(ix)$ is defined in appendix. The parameters, $\{x_j\}$ are solutions to BAE,

$$w \frac{\phi(x_j + i)}{\phi(x_j - i)} = \frac{Q(x_j - i)Q(x_j + 2i)}{Q(x_j + i)Q(x_j - 2i)}, \quad j = 1, \cdots, N. \quad (5)$$

As in section 2, we represent these three terms by three boxes with letters 1,2 and 3,

$$T_1(u, x) = 1_1 + 2_1 + 3_1. \quad (6)$$

One infers from the $sl_2$ example that a combinatorial aspect may also appear. This turns out to be true. The eigenvalues of fusion QTMs are given by the sum of combinations of boxes, which can be identified with semi-standard Young tableaux for $sl_3$. On each diagram, the spectral parameter changes $+2i$ from the left to the right and $-2i$ from the top to the bottom. (Fig 2).

We restrict ourselves to diagrams of the rectangular shape for a while. Firstly we note that QTM associated to $2 \times m$ $(3 \times m)$ Young diagram can be reduced to the one associated to $1 \times m$ (or just scalars). This is
The spectral parameters are assigned from the left to the right. The renormalization factor, common factor to all expressions of length \( m \) tableaux, is given by

\[
f_m(x) := \prod_{j=1}^{m-1} \phi(x \pm i(\frac{2m-1}{2} - j)).
\]

Hereafter we sometimes denote \( f(x \pm iy) := f(x + iy)f(x - iy) \).

The resultant \( T_m \)'s are all degree 2N w.r.t. \( h[x + \text{shift}] \). Obviously, we have a periodicity due to Boltzmann weights;

\[
T_m(x + \frac{10}{3}i) = T_m(x), \quad (T_m(x + \frac{14}{4}i) = T_m(x)) \quad (8)
\]

for the dilute \( A_4 (A_6) \) model. From the \( sl_3 \) structure, together with the above property, one can prove the following functional relations,

\[
T_m(x - i)T_m(x + i) = g_m(x)T_m(x) + T_{m+1}(x)T_{m-1}(x), \quad m \geq 1
\]

\[
g_m(x) = \phi(x \pm i(m + 3/2)),
\]

\[
T_{-1}(x) := 0
\]

\[
T_0(x) := f_2(x).
\]
The periodicity for the dilute $A_4$ model, $\phi(x + 10/3i) = \phi(x)$, leads to $g_{m+10}(x) = g_m(x) \quad m \geq 0$ and $g_{7-m}(x) = g_m(x), (0 \leq m \leq 7)$. From the adjacency matrix, one concludes $T_{8,9}(x) = 0$.

Thus functional relations are invariant under the transformation, $T_m(x) \rightarrow T_{7-m}(x) \quad (m = 0, \cdots, 7)$ or $T_m(x) \rightarrow T_{m+10}(x) \quad (m \geq -1)$. The symmetry of functional relations is not necessarily inherited to its solution in general. We verify, however, the "duality",

$$T_m(x) = T_{7-m}(x), \quad m = 0, \cdots, 7$$

(10)

and $T_{m+10}(x) = T_m(x), m \geq -1$ numerically for the largest eigenvalue sector, only which we are interested in. These duality relations are also observed and numerically verified for $A_6$ models with change in the period. See section 7.

Functional relations among them, however, do not possess the desired analytical property, as discussed in [24]. We thus introduce other class of QTMs related to skew Young diagrams.

Let $\mu$ and $\lambda$ be a pair of Young diagrams satisfying $\mu_i \geq \lambda_i, \forall i$. We subtract a diagram $\lambda$ from $\mu$. We call the result a skew Young diagram $\mu - \lambda$, consisted of $(\mu_1 - \lambda_1, \mu_2 - \lambda_2, \cdots)$ boxes. In the theory of symmetric polynomials, the Jacobi-Trudi formula tells that a complex Schur function associated to a skew Young diagram can be expressed by a determinant of a matrix, of which elements are given by elementary Schur functions associate to "one-row" diagrams or "one-column" ones. The quite parallel formula holds for the present situation, which we call the "quantum" Jacobi-Trudi formula [36, 38, 39].

Consider a set of semi-standard skew Young tableaux of the shape $\mu - \lambda$. We assign an expression to each table. The spectral parameter of the "top-left" box is fixed to $x + i(\mu'_1 - \mu_1)$ \footnote{Using this opportunity we remark misprints in the 4th row in the second paragraph of section 5 and the caption of Fig2 in [24].} where $\mu'_1$ denotes the depth of the tableaux.

One identifies each box in a table with an expression under the rule (6) with the shift of the spectral parameter. Then the product over all constituting boxes yields the desired expression for the table.
Figure 4: The spectral parameter $x + i(\mu'_1 - \mu_1)$ is assigned to the hatched place.

**Theorem 1** Let $T_{\mu/\lambda}(x)$ be the sum over the resultant expressions divided by a common factor, $\prod_{j=1}^{\mu'_1} f_{\mu_j - \lambda_j}(x + i(\mu'_1 - \mu_1 + \mu_j + \lambda_j - 2j + 1))$. Then the following equality holds,

$$T_{\mu/\lambda}(x) = \det_{1 \leq j, k \leq \mu'_1} (T_{\mu_j - \lambda_k - j + k}(x + i(\mu'_1 - \mu_1 + \mu_j + \lambda_k - j - k + 1)))$$

(11)

where $T_{m<0} := 0$.

The proof is quite similar to the one for Young tableaux[38, 39]. One must only keep in mind that the allowed position of a box is restricted by its spectral parameter.

The crucial fact is that the so defined $T_{\mu/\lambda}(x)$ is an analytic function of $x$ due to BAE, and contains $T_1(x)$ as a special case. The former is not so obvious from the original definition by the tableaux, but it follows trivially from the quantum Jacobi-Trudi formula.

In the same spirit, we introduce $\Lambda_{\mu/\lambda}(x)$, which is analytic under BAE, by putting $T_{m\geq8}(x) = 0$ in $T_{\mu/\lambda}(x)$,

$$\Lambda_{\mu/\lambda}(x) := T_{\mu/\lambda}(x)/\{T_{m\geq8}(x) = 0\},$$

(12)

for the dilute $A_4$ model. The pole-free property of $\Lambda_{\mu/\lambda}(x)$ is obvious from (11). For the dilute $A_6$ model, we define $\Lambda_{\mu/\lambda}(x)$ by setting $T_{m\geq11}(x) = 0$.

## 5 Kink Transfer Matrix

In following few sections, we restrict our discussion to the dilute $A_4$ model. We will summarize results for the dilute $A_6$ model in section 6.

For the $E_8$ case, the vector representation is minimal and other representations are constructed by fusion of it. Eigenvalues of associated transfer matrices of the dilute $A_3$ model are thus derived from products of $T_1(x + \text{some shift})$. 
This is no longer true for the dilute $A_4$ model. Let $W_a(x)$ be the Yangian highest weight module associated to the node $a$ in Figure 1. Through several evidences, we identify $T_1(x)$ of the model with the transfer matrix connected to $W_1(x)$, which is not minimal. The most fundamental object in $E_7$ is $W_6(x)$ rather than $W_1(x)$. Any other object may be constructed from $T^{(6)}(x)$, the transfer matrix of $W_6(x)$, of which explicit form is not known from the dilute $A_4$ model. The determination of explicit form of $T^{(6)}(x)$ is thus vital in the present approach. For this purpose, homomorphisms of $U_q(\hat{g})$ modules deserve attentions. They lead to non-trivial algebraic relations among $T^{(6)}(x)$ and other QTMs. Although such information is not available for $U_q(\hat{E}_7)$, there exists a list of homomorphisms of $Y(E_7)$ modules in [57],

\[
W_{6-a}(x) \hookrightarrow W_{7-a}(x - \frac{1}{6}i) \otimes W_6(x + \frac{a}{6}i), \quad a = 1, 2, 3 \quad (13)
\]

\[
W_1(x) \hookrightarrow W_6(x + \frac{5}{6}i) \otimes W_6(x - \frac{5}{6}i) \quad (14)
\]

\[
C \hookrightarrow W_6(x + \frac{3}{2}i) \otimes W_6(x - \frac{3}{2}i) \quad (15)
\]

\[
W_2(x) \hookrightarrow W_1(x + \frac{1}{6}i) \otimes W_1(x - \frac{1}{6}i) \quad (16)
\]

\[
W_7(x) \hookrightarrow W_1(x + \frac{1}{2}i) \otimes W_6(x - \frac{2}{3}i). \quad (17)
\]

Note that the normalization of spectral parameter is different from [57].

For example, the second relation implies,

\[
T^{(6)}(x + \frac{5}{6}i)T^{(6)}(x - \frac{5}{6}i) = T^{(1)}(x) + \cdots.
\]

After trials and errors, we find the following ansatz compatible with the above homomorphisms,

\[
T^{(6)}(x) = \frac{1}{\sqrt{2}} \left( w^2 \phi(x + \frac{2}{3}i) \frac{Q(x + \frac{2}{3}i)Q(x - \frac{2}{3}i)}{Q(x - \frac{1}{3}i)Q(x + \frac{1}{3}i)} + w \phi(x - \frac{4}{3}i) \frac{Q(x - \frac{4}{3}i)Q(x + \frac{2}{3}i)}{Q(x - \frac{1}{3}i)Q(x + \frac{1}{3}i)} \right.
\]

\[
\left. + \phi(x) \frac{Q(x + \frac{5}{3}i)Q(x + \frac{5}{3}i)Q(x + \frac{5}{3}i)}{Q(x + i)Q(x - i)} + \frac{1}{w} \phi(x + \frac{4}{3}i) \frac{Q(x + \frac{4}{3}i)Q(x - \frac{5}{3}i)}{Q(x + \frac{1}{3}i)Q(x - i)} + \phi(x - \frac{2}{3}i) \frac{Q(x - \frac{2}{3}i)}{Q(x + \frac{1}{3}i)} \right).
\quad (18)
\]

As explicit RSOS weights are not yet derived, it may be inappropriate to call $T^{(6)}(x)$ as the eigenvalue of transfer matrix. In the following discussion, however, we do not use the assumption that it coincides with the actual eigenvalue of transfer matrix of $W_6$. Rather, we simply use facts
(1) it is pole free under BAE (2) it satisfies the desired relations expected from eqs. (14) and (15). Readers should understand this terminology just as a "nickname".

The following functional relations between $T(x)$ and $T_m(x)$ will facilitate discussions in later sections.

**Lemma 1**

\[
\begin{align*}
T^6(x + \frac{1}{6}i)T^6(x - \frac{1}{6}i) &= T_2(x + \frac{10}{6}i) + T_0(x) \quad (19) \\
T^6(x + \frac{3}{6}i)T^6(x - \frac{3}{6}i) &= T_3(x) \quad (20) \\
T^6(x + \frac{5}{6}i)T^6(x - \frac{5}{6}i) &= T_1(x) + T_1(x + \frac{10}{6}i) \quad (21) \\
T^6(x + \frac{7}{6}i)T^6(x - \frac{7}{6}i) &= T_3(x + \frac{10}{6}i) \quad (22) \\
T^6(x + \frac{9}{6}i)T^6(x - \frac{9}{6}i) &= T_2(x) + T_0(x + \frac{10}{6}i). \quad (23)
\end{align*}
\]

Proof: The duality relation (10) plays a fundamental role in the proof. The direct substitutions of eq.(18) in the lhs of eqs. (20), (21) and (23) yield $\frac{1}{2}(T_3(x) + T_4(x))$, $\frac{1}{2}(T_1(x) + T_1(x + \frac{10}{6}i)) + T_6(x + \frac{10}{6}i)$ and $\frac{1}{2}(T_2(x) + T_5(x)) + T_0(x + \frac{10}{6}i)$, respectively. We remark that non-trivial cancellation of terms occurs due to $w^5 = -1$.

These results coincide with the rhs due to dualities $T_3 = T_4$, $T_1 = T_6$ and $T_2 = T_5$.

Rests follow from these by $x \to x + \frac{10}{6}i$. $\square$

These relations suggest the underlying homomorphisms between $U_q(\tilde{E}_7)$ and $U_q'(A^{(2)}_2)$ modules at $q = \exp(i\pi/20)$, $q' = \exp(i3\pi/10)$. This may be an interesting but an independent subject from the present problem thus we will not go into detail here.

6 **Fusion Quantum transfer matrices and the $T-$ system for the dilute $A_4$ model**

Having defined the "kink" QTM $T^{(6)}$, we are in position to introduce other QTMs and explore functional relations among them.

First we present QTMs defined by skew Young tableaux.
Definition 1

\[
T(1)(x) = \Lambda_{(1)}(x) \quad (= T_1(x)) \quad (24)
\]

\[
T(2)(x) = \frac{1}{\phi(x - \frac{5}{3}i)} \Lambda_{(6,1)}(x - \frac{5}{6}) \quad (25)
\]

\[
T(3)(x) = \frac{1}{\phi(x + \frac{5}{2}i)} \Lambda_{(11,6)/(5,5)}(x) \quad (26)
\]

\[
T(5)(x) = \Lambda_{(2)}(x + \frac{5}{3}i) \quad (= T_2(x + \frac{5}{3}i)) \quad (27)
\]

We have two comments. First, one can equivalently rewrite (25) and (26) in terms of \(\Lambda_{(6,6)/(5)}\) or \(\Lambda_{(6,6,1)/(5)}(x)\) using relations,

\[
\Lambda_{(6,6)/(5)}(x) = \Lambda_{(6,1)}(x + 5i) \quad (28)
\]

\[
\Lambda_{(6,6,1)/(5)}(x) = \Lambda_{(11,6)/(5,5)}(x). \quad (29)
\]

These are outcome of the quantum Jacobi-Trudi formula and duality (10). The second comment concerns the complex conjugate property. In the largest eigenvalue sector of QTM, we confirm numerically \(Q(x) = \overline{Q(x)}\). The explicit forms of DVFs in eq. (4) and eq. (18) then conclude \(T(1)(x) = T(1)(\bar{x})\), \(T(6)(x) = T(6)(\bar{x})\). By remembering \(5/3i\) equals to the half period of our elliptic function \(\theta_1(x)\), one also verifies \(T(5)(x) = T(5)(\bar{x})\). The conjugate property of \(T(2)(x)\) and \(T(3)(x)\) is less obvious. One can nevertheless show this by the use of the quantum Jacobi-Trudi formula. \(T(6)\) comes into expressions of remaining QTM-s. The Yangian homomorphisms turn out to be useful in deriving their explicit forms. We take \(T(4)(x)\), related to \(W_4(x)\), for instance. As argued in section 2.3, we identify an analytic set with an affine irreducible module. Thus eq. (13) for \(a = 2\) implies,

\[
T(5)(x - i/6)T(6)(x + i/3) \sim T(4)(x) + T'(x).
\]

That is, the product of DVFs \(T(5)(x - I/6)T(6)(x + i/3)\) decomposes into two (or more) subsets and each subset is analytic within itself. We look at the explicit DVF of the lhs and find that it contains analytic subset given by \(\phi(x + \frac{5}{3}i)\phi(x - i)T(6)(x - \frac{1}{3}i)\). The sum of remaining terms must be analytic under BAE. We identify them as \(T(4)(x)\). In a similar way, we deduce \(T(7)(x)\).
Definition 2

\[
T^{(4)}(x) = (T^{(5)}(x - \frac{1}{6}i)T^{(6)}(x + \frac{1}{3}i) - \phi(x + \frac{2}{3}i)\phi(x - \frac{1}{3}i)T^{(6)}(x - \frac{1}{3}i))
\]

\[
T^{(7)}(x) = \frac{1}{\phi(x - \frac{1}{3}i)}(T^{(1)}(x + \frac{1}{2}i)T^{(6)}(x - \frac{2}{3}i) - \phi(x)\phi(x - i)T^{(6)}(x + \frac{4}{3}i)).
\]

(30)

(31)

The common factor \(\frac{1}{\phi(x - \frac{1}{3}i)}\) is divided out for \(T^{(7)}(x)\). We note that the conjugate property, discussed for other \(T^{(a)}s\), is also verified for \(T^{(7)}(x)\) when it is written in terms of its explicit DVF. On the other hand, the property is not so apparent for \(T^{(4)}(x)\), although one can prove it by a different route. See the discussion after the proof of eq. (36).

We are now ready to describe the statement as to functional relations among QTMs.

Proposition 1 The above QTMs enjoy the following \(T-\) system,

\[
T^{(1)}(x - \frac{1}{6}i)T^{(1)}(x + \frac{1}{6}i) = \phi(x - \frac{5}{3}i)T^{(2)}(x) + T_0(x \pm \frac{5}{6}i),
\]

(32)

\[
T^{(2)}(x - \frac{1}{6}i)T^{(2)}(x + \frac{1}{6}i) = T_0(x)T_0(x \pm \frac{4}{6}i) + T^{(1)}(x)T^{(3)}(x),
\]

(33)

\[
T^{(3)}(x - \frac{1}{6}i)T^{(3)}(x + \frac{1}{6}i) = T_0(x \pm \frac{1}{2}i)T_0(x \pm \frac{1}{6}i) + T^{(2)}(x)T^{(4)}(x)T^{(7)}(x),
\]

(34)

\[
T^{(4)}(x - \frac{1}{6}i)T^{(4)}(x + \frac{1}{6}i) = T_0(x)T_0(x \pm \frac{1}{3}i) + T^{(3)}(x)T^{(5)}(x),
\]

(35)

\[
T^{(5)}(x - \frac{1}{6}i)T^{(5)}(x + \frac{1}{6}i) = T_0(x \pm \frac{1}{6}i) + T^{(4)}(x)T^{(6)}(x),
\]

(36)

\[
T^{(6)}(x - \frac{1}{6}i)T^{(6)}(x + \frac{1}{6}i) = T_0(x) + T^{(5)}(x),
\]

(37)

\[
T^{(7)}(x - \frac{1}{6}i)T^{(7)}(x + \frac{1}{6}i) = T_0(x \pm \frac{1}{3}i) + T^{(3)}(x).
\]

(38)

These coincide with the \(T-\) system for \(E_7\) proposed in [58] in a different context.

Note that eq. (37) has already been proven in eq. (19) by the definition (27).

Proof of eqs. (32) and (33).

Take the simpler case (32) first. We consider the decomposition of the product \(T_1(x-6i)T_6(x+i)\). Quite similar to combinatorics of semi-standard tableaux, we have

\[
T_1(x-6i)T_6(x+i) = \Lambda_{(6,1)}(x) + T_7(x)T_0(x-5i),
\]

18
which can also be verified from the quantum Jacobi-Trudi formula. By utilizing dualities \( T_6(x) = T_1(x) \), \( T_7(x) = T_0(x) \) and the periodicity, one recovers eq. (32) after the shift \( x \to x - \frac{5}{6}i \) in both sides. 

Eq. (33) also follows by considering the decomposition of \( \Lambda_{(6,6)/(5)}(x)\Lambda_{(6,1)}(x + 12i) \). See Fig. 5. The lhs of the arrow contains two Young diagrams corresponding to \( \Lambda_{(6,6)/(5)}(x) \) and \( \Lambda_{(6,1)}(x + 12i) \) (hatched) in the relative position, compatible with the shift in the spectral parameter. We then employ the "recombination" of boxes, as in usual Young diagrams. Two rules need particular attention. First, we must put box so as to match the spectral parameter. Second, due to the condition \( T_m \geq 8 \) in the definition of \( \Lambda_{\mu/\lambda}(x) \), the width of a column in the resultant Young diagram must be less equal to 7. Then two terms result from the "recombination" as depicted in the rhs of the arrow, \( \Lambda_{(11,6,6)/(5,5)}(x - 8i)T_1(x + 6i) \) and \( T_7(x)T_7(x + 12i) \). We still leave hatch to boxes which once belong to \( \Lambda_{(6,1)}(x + 12i) \). Note that the figure represents the equality of the DVFs in terms of boxes. Thus it needs proper normalization factors, viewed as the relation between \( \Lambda_{\mu/\lambda}(x) \) or \( T_m(x) \), due to scalar factors in the definitions (7) of \( T_m(x) \) and of \( \Lambda_{\mu/\lambda}(x) \) in Theorem 1. Then one finds, with the property (28),

\[
\Lambda_{(6,1)}(x - 5i)\Lambda_{(6,1)}(x + 12i) = \Lambda_{(11,6,6)/(5,5)}(x - 8i)T_1(x + 6i) + \phi(x - \frac{9}{2}i)\phi(x - \frac{11}{2}i)\phi(x + \frac{13}{2}i)\phi(x + \frac{15}{2}i)T_7(x)T_7(x + 12i)
\]

where the factor in front of \( T_7(x)T_7(x + 12i) \) comes from \( f_7(x)f_7(x + 12i)/f_6(x + i)f_6(x + 13i) \). Finally let us shift the spectral parameter by \( +i/6 \) and use the periodicity (8). Then eq. (33) follows from definitions (25), (26) and \( T_7 = T_0 \).

\[\text{Figure 5: The decomposition of } \Lambda_{(6,6)/(5)}(x)\Lambda_{(6,1)}(x + 12i).\]

\[\Box\]

Proof of eq. (36).

The proof utilizes (19) as follows. Consider the product \( T^{(4)}(x)T^{(6)}(x) \).
Substituting eq. (30), we have

\[
T^{(4)}(x)T^{(6)}(x) = T^{(5)}(x - \frac{1}{6}i)T^{(6)}(x)T^{(6)}(x + \frac{1}{3}i) - T_0(x + \frac{1}{6}i)T^{(6)}(x)T^{(6)}(x - \frac{1}{3}i).
\]
One applies the rule (19) to the product of two $T^{(6)}$s by shifting $x \pm i/6$. The result leads to

$$T^{(4)}(x)T^{(6)}(x) = T^{(5)}(x - \frac{1}{6}i)T^{(5)}(x + \frac{1}{6}i) - T_0(x \pm \frac{1}{6}i),$$

which coincides with eq. (39).

As mentioned previously, the complex conjugate property of $T^{(4)}(x)$ is not obvious from its DVF. Instead, we can now shown it from the established relation (36) with the help of the conjugate property of $T^{(5)}(x)$ and $T^{(6)}(x)$ commented below (27).

To prove remaining relations, we need to prepare further lemmas.

**Lemma 2** The following decompositions are valid,

$$T^{(5)}(x + \frac{1}{3}i)T^{(5)}(x - \frac{1}{3}i) = T_0(x)T_4(x) + T^{(3)}(x) \tag{39}$$

$$T^{(5)}(x + \frac{1}{3}i)T^{(5)}(x - \frac{1}{3}i) = T_4(x)(T_0(x) + T^{(5)}(x)) - T_0(x \pm \frac{1}{3}i)$$

$$- (T_0(x + \frac{1}{3}i)T^{(5)}(x - \frac{1}{3}i) + T_0(x - \frac{1}{3}i)T^{(5)}(x + \frac{1}{3}i)). \tag{40}$$

Proof of Lemma 2. The relation (39) is checked by comparing DVF's of both sides directly. To prove eq. (40), we rewrite $T^{(5)}$ in the lhs in terms of $T^{(6)}$ by using eq. (19),

$$T^{(5)}(x + \frac{1}{3}i)T^{(5)}(x - \frac{1}{3}i) = T_0(x \pm \frac{1}{3}i)$$

$$+ T^{(6)}(x - \frac{1}{6}i)T^{(6)}(x + \frac{1}{6}i)T^{(6)}(x - \frac{1}{2}i)T^{(6)}(x + \frac{1}{2}i)$$

$$- (T_0(x + \frac{1}{3}i)T^{(6)}(x - \frac{1}{6}i)T^{(6)}(x - \frac{1}{2}i) + T_0(x - \frac{1}{3}i)T^{(6)}(x + \frac{1}{6}i)T^{(6)}(x + \frac{1}{2}i))$$

By applying eqs. (19) and (24), one reaches the rhs of (40). □

For later use, we rearrange the sum of eqs. (39) and (40),

$$\left(2T_0(x)T_4(x) + T^{(3)}(x) + T_4(x)T^{(5)}(x) - T_0(x + \frac{1}{3}i)T^{(5)}(x - \frac{1}{3}i)\right)$$

$$- T_0(x - \frac{1}{3}i)T^{(5)}(x + \frac{1}{3}i) - 2T^{(5)}(x - \frac{1}{3}i)T^{(5)}(x + \frac{1}{3}i)) = T_0(x \pm \frac{1}{3}i). \tag{41}$$

**Lemma 3**

$$T_0(x \pm i)T_3(x) + T_1(x - 2i)T_1(x + 2i)$$

$$- T_0(x - i)T_1(x + 2i)T_2(x - i) - T_0(x + i)T_1(x - 2i)T_2(x + i)$$

$$= \phi(x \pm \frac{7}{2}i)\phi(x \pm \frac{5}{2}i)\phi(x \pm \frac{3}{2}i).$$

20
This is the analogue of the relation,
\[
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\]
and it can be shown in a similar manner.

In proving eq. (34), one needs decomposition of a huge QTM.

**Lemma 4**

\[
\Lambda_{(11,11,6,6)/(10,5,5)}(x) = \phi(x + \frac{1}{2}i)\phi(x - \frac{5}{2}i) \times
\left( T_1(x + 3i)T_2(x + i)T_2(x - 3i) - T_0(x + 2i)T_1(x - 2i)T_2(x + i)
- T_0(x - i)T_1(x + 3i)T_3(x - i) + T_0(x - i)T_0(x + 2i)T_4(x) \right).
\]

Proof of Lemma 4. For convenience, we go back to the original definition of \( T_{(11,11,6,6)/(10,5,5)}(x) \). Thanks to the semi-standard condition, it decomposed into pieces. We consequently have

\[
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\]

Figure 6: Decomposition of the skew diagram \((11,11,6,6)/(10,5,5)\). The height 3 piece in the middle of the rhs reduces to a known scalar factor.

\[
f_6(x + 6i)f_6(x - 8i)T_{(11,11,6,6)/(10,5,5)}(x) = \frac{f_5(x - i)f_5(x - 9i)f_5(x + 7i)}{f_2(x - i)}
\times T_5(x - 9i)\left( T_1(x + 13i)T_5(x + 5i) - T_0(x + 12i)T_6(x + 8i) \right).
\]

(42)

On the other hand, the quantum Jacobi-Trudi formula relates the same quantity to

\[
f_6(x + 6i)f_6(x - 8i)\Lambda_{(11,11,6,6)/(10,5,5)}(x)
+ f_{13}(x - i)T_1(x + 13i)T_{13}(x - i) - f_{14}(x)T_{14}(x).
\]

(43)

Then the equality (42)=(43) with properties \( T_{m+10}(x) = T_m(x) \) and \( T_{7-m}(x) = T_m(x) \) leads to Lemma 4 after renormalization. □

Our final lemma concerns the decomposition of \( T^{(4)}(x)T^{(7)}(x) \).
Lemma 5

$$\phi(x - \frac{4}{3}i)T^{(4)}(x)T^{(7)}(x) =$$

$$T_2(x + \frac{11}{6}i)\left\{T_1(x + \frac{1}{2}i)T_2(x + \frac{7}{6}i) - T_0(x + \frac{17}{6}i)T_1(x - \frac{7}{6}i)\right\}$$

$$-T_0(x - \frac{1}{6}i)T_1(x + \frac{1}{2}i)T_3(x - \frac{1}{6}i) + T_0(x - \frac{1}{2}i)T_0(x - \frac{1}{6}i)T_3(x + \frac{5}{6}i).$$

Proof of Lemma 5. We use a trick; the complex conjugate property, established soon below the proof of eq. (36), allows us to replace $T^{(4)}(x)$ by $\overline{T^{(4)}(x)}$ in the lhs. After this, we substitute definitions (30) and (31) into the lhs,

$$\phi(x - \frac{4}{3}i)T^{(4)}(x)T^{(7)}(x) =$$

$$T_2(x + \frac{11}{6}i)T_1(x + \frac{1}{2}i)T^{(6)}(x - \frac{1}{2}i + \frac{1}{6}i)T^{(6)}(x - \frac{1}{2}i - \frac{1}{6}i)$$

$$-\phi(x - i)T_2(x + \frac{11}{6}i)T^{(6)}(x + \frac{1}{2}i + \frac{5}{6}i)T^{(6)}(x + \frac{1}{2}i - \frac{5}{6}i)$$

$$-\phi(x + \frac{1}{3}i)\phi(x - \frac{2}{3}i)T_1(x + \frac{1}{2}i)T^{(6)}(x - \frac{1}{6}i + \frac{1}{2}i)T^{(6)}(x - \frac{1}{6}i - \frac{1}{2}i)$$

$$+\phi(x - i)\phi(x - \frac{2}{3}i)\phi(x + \frac{1}{3}i)T^{(6)}(x + \frac{5}{6}i + \frac{1}{2}i)T^{(6)}(x + \frac{5}{6}i - \frac{1}{2}i).$$

where we have used definitions (24) and (27) to rewrite $T^{(1)}$ and $T^{(5)}$ by $T_1$ and $T_2$. Thanks to the trick, the differences in arguments of products of $T^{(6)}$ are such that one can apply eqs. (19), (20) and (21). The result of the application then agrees with the rhs of the equality in Lemma 5. \[\square\]

By comparing the rhs of Lemma 4 and Lemma 5 with use of the duality $T_3 = T_4$, one notices the following equality.

Lemma 6

$$\Lambda_{(11,11,6,6)/(10,5,5)}(x + \frac{5}{6}i) = \phi(x + \frac{8}{6}i)\phi(x - \frac{10}{6}i)T^{(4)}(x)T^{(7)}(x).$$

With these preparations, we prove remaining relations.

Proof of (35).

Let us rewrite the lhs by substituting the definition (30). In doing so, we employ a similar trick to the above; we replace $T^{(4)}(x + \frac{1}{6}i)$ in the lhs
by $T^{(4)}(x - \frac{1}{6}i)$. Then it follows,

$$T^{(4)}(x + \frac{1}{6}i)T^{(4)}(x - \frac{1}{6}i) =$$

$$T^{(5)}(x - \frac{1}{3}i)T^{(5)}(x + \frac{1}{3}i)T^{(6)}(x - \frac{1}{6}i)T^{(6)}(x + \frac{1}{6}i)$$

$$-T_0(x)T^{(5)}(x - \frac{1}{3}i)T^{(6)}(x + \frac{1}{3}i - \frac{1}{6}i)T^{(6)}(x + \frac{1}{3}i + \frac{1}{6}i)$$

$$-T_0(x)T^{(5)}(x + \frac{1}{3}i)T^{(6)}(x - \frac{1}{3}i - \frac{1}{6}i)T^{(6)}(x - \frac{1}{3}i + \frac{1}{6}i)$$

$$+(T_0(x))^2T^{(6)}(x - \frac{1}{2}i)T^{(6)}(x + \frac{1}{2}i).$$

In this form, eq. (13) applies to the first three terms of the rhs, and eq. (20) can be used in the fourth. Also we use eq. (39) the first term of the rhs. The result reads,

$$T^{(4)}(x + \frac{1}{6}i)T^{(4)}(x - \frac{1}{6}i) = T^{(3)}(x)T^{(5)}(x)$$

$$+T_0(x)\left(2T_0(x)T_4(x) + T^{(3)}(x) + T_4(x)T^{(5)}(x)\right)$$

$$-T_0(x + \frac{1}{3}i)T^{(5)}(x - \frac{1}{3}i) - T_0(x - \frac{1}{3}i)T^{(5)}(x + \frac{1}{3}i)$$

$$-2T^{(5)}(x - \frac{1}{3}i)T^{(5)}(x + \frac{1}{3}i),$$

where we represent $T_2(x)$ by $T^{(5)}(x + 5/3i)$ and use the duality $T_3 = T_4$. One notices that the content of bracket in the rhs reduces to $T_0(x \pm \frac{1}{3}i)$ due to eq. (14), which completes the proof.

Proof of (39). In the same manner, we start from the equivalent expression $T^{(7)}(x - \frac{1}{6}i)T^{(7)}(x - \frac{1}{6}i)$ and substitute (31). After rearrangement using eqs. (21), (22) and (23), we find this expression equal to

$$T^{(7)}(x + \frac{1}{6}i)T^{(7)}(x - \frac{1}{6}i) = \frac{1}{\phi(x + \frac{3}{2}i)}\left(T_0(x + \frac{3}{2}i)T_3(x + \frac{10}{6}i)ight)$$

$$+T_1(x - \frac{1}{3}i)T_1(x + \frac{1}{3}i)T_1(x + \frac{10}{6}i) + T_1(x - \frac{1}{3}i)T_1(x + \frac{1}{3}i)T_1(x)$$

$$-T_0(x + \frac{2}{3}i)T_0(x + \frac{1}{3}i)T_1(x + \frac{1}{3}i) - T_0(x + \frac{2}{3}i)T_1(x + \frac{1}{3}i)T_2(x + \frac{2}{3}i)$$

$$-T_0(x - \frac{2}{3}i)T_0(x - \frac{1}{3}i)T_1(x - \frac{1}{3}i) - T_0(x - \frac{2}{3}i)T_1(x - \frac{1}{3}i)T_2(x - \frac{2}{3}i).$$

Let us subtract $T^{(3)}(x)$ from the above. Note that $T^{(3)}(x)$ should be understood as the result of the application of (12) to (24) and of the duality
$T_7 \rightarrow T_0$. Then we have

$$T^{(7)}(x + \frac{1}{6}i)T^{(7)}(x - \frac{1}{6}i) - T^{(3)}(x) =$$

$$\frac{1}{\phi(x \pm \frac{3}{2}i)} \left( T_0(x \pm \frac{2}{3}i)T_3(x + \frac{10}{6}i) + T_1(x - \frac{1}{3}i)T_3(x + \frac{10}{6}i) - T_1(x - \frac{1}{3}iT_2(x + \frac{2}{3}i) - T_0(x - \frac{2}{3}i)T_1(x - \frac{1}{3}i)T_2(x - \frac{2}{3}i) \right).$$

The content of the bracket is identical to the lhs of Lemma \[ with the shift $x \rightarrow x + 10/6i$ by noticing the periodicity \[. Immediately, one verifies that the rhs reduces to $T_0(x \pm 1/3i)$, and \[ is proven. □

Proof of \[(34)\]. The decomposition of $\Lambda_{(11,6,6)/(5,5)}(x)\Lambda_{(6,6,1)/(5)}(x + 7i)$ can be done by the formula \[(12)\]. Equivalently, we can argue it in a graphic manner as shown in Fig. 7, just as in Fig. 5.

$$\Lambda_{(11,6,6)/(5,5)}(x)\Lambda_{(6,6,1)/(5)}(x + 7i) = \Lambda_{(6,1)}(x + 6i)\Lambda_{(11,11,6,6)/(10,5,5)}(x + i) + T_0(x \pm i)T_0(x \pm 6i)T_0(x + 8i)T_0(x + 13i),$$

where $T_7$ is replaced by $T_0$.

Figure 7: Graphical rule for decomposition of two diagram $(11,6,6)/(5,5)$ and $(6,6,1)/(5)$.

By using eqs. \[(26)\] and \[(29)\] and taking account of normalization factors, one finds

$$T^{(3)}(x - \frac{1}{6}i)T^{(3)}(x + \frac{1}{6}i) = T_0(x \pm \frac{1}{2}i)T_0(x \pm \frac{1}{6}i)$$

$$+ \frac{\phi(x - \frac{10}{6}i)}{\phi(x \pm \frac{5}{6}i)\phi(x \pm \frac{10}{6}i)}T^{(2)}(x)\Lambda_{(11,11,6,6)/(10,5,5)}(x + \frac{5}{6}i).$$

Then the equivalent statement to eq.\[(38)\] is,

$$\Lambda_{(11,11,6,6)/(10,5,5)}(x + \frac{5}{6}i) = \phi(x \pm \frac{8}{6}i)\phi(x - \frac{10}{6}i)T^{(4)}(x)T^{(7)}(x).$$

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This is nothing but Lemma 6. One thus completes the proof of proposition. 

In next section, we summarize similar results for the dilute $A_6$ model without proof.

7 The $T-$ system for the dilute $A_6$ model

We firstly comment on the "duality property" in the dilute $A_6$ model,

$$T_m(x) = T_{11-m}(x), \quad m = 0, \ldots, 11$$

and $T_{12}(x) = T_{13}(x) = 0$. This is again compatible with the symmetry of functional relations but is still a conjecture supported by numerics.

The Yangian representation theory asserts that irreducible modules of $Y(E_6)$ are made by tensoring minimal objects, $W_1(x)$ and $W_5(x')$. On the other hand, we assume

$$T_1(x) = T^{(6)}(x)$$

as QTM for $W_6(x)$. Thus the situation is similar to the $E_7$ case; one must figure out eigenvalues of QTMs associated to $W_1(x)$ and $W_5(x')$ independently from the knowledge of the dilute $A_6$ model. We conjecture that eigenvalues of QTMs for these are same and its explicit form reads,

$$T^{(1)}(x) = T^{(5)}(x)$$

and

$$= \frac{1}{2} \left( \omega^3 \phi(x + \frac{3}{4}i) Q(x + \frac{3}{4}i) + \omega^2 \phi(x - \frac{5}{4}i) Q(x - \frac{5}{4}i) + \omega \phi(x + \frac{5}{4}i) Q(x + \frac{5}{4}i) + \phi(x - \frac{7}{4}i) Q(x - \frac{7}{4}i) \right)$$

The following relations hold in parallel to eqs. (19)-(23), which are key ingredients in the proof of the $T-$ system.

$$T^{(1)}(x)^2 = T_2(x + \frac{7}{4})$$

$$T^{(1)}(x + \frac{1}{4})T^{(1)}(x - \frac{1}{4}) = \frac{1}{2} \left( T_1(x) + T_0(x) \right)$$

$$T^{(1)}(x + \frac{1}{2})T^{(1)}(x - \frac{1}{2}) = \frac{1}{2} T_5(x + \frac{7}{4})$$

$$T^{(1)}(x + \frac{3}{4})T^{(1)}(x - \frac{3}{4}) = T_1(x) + \phi(x) T^{(1)}(x + \frac{7}{4}).$$
(We omit relations obtained by $x \to x + \frac{7}{4}i$.)

Let other QTMs be

$$
T^{(2)}(x) = T^{(4)}(x) = T^{(1)}(x - \frac{1}{4}i)T^{(1)}(x + \frac{1}{4}i) - T_0(x)
$$

$$
T^{(3)}(x) = \frac{1}{\phi(x + i\frac{7}{4})} \left( T_1(x - \frac{1}{4}i)T_1(x + \frac{1}{4}i) - T_0(x \pm \frac{3i}{4} \right),
$$

then the following $T-$ system is valid.

**Proposition 2**

$$
T^{(1)}(x - \frac{1}{4}i)T^{(1)}(x + \frac{1}{4}i) = T_0(x) + T^{(2)}(x),
$$

$$
T^{(5)}(x - \frac{1}{4}i)T^{(5)}(x + \frac{1}{4}i) = T_0(x) + T^{(4)}(x),
$$

$$
T^{(6)}(x - \frac{1}{4}i)T^{(6)}(x + \frac{1}{4}i) = T_0(x \pm \frac{3i}{4} + \phi(x + i\frac{7}{4})T^{(3)}(x),
$$

$$
T^{(2)}(x - \frac{1}{4}i)T^{(2)}(x + \frac{1}{4}i) = T_0(x \pm \frac{1}{4}i) + T^{(1)}(x)T^{(3)}(x),
$$

$$
T^{(3)}(x - \frac{1}{4}i)T^{(3)}(x + \frac{1}{4}i) = T_0(x)T_0(x \pm \frac{1}{2}i) + (T^{(2)}(x)\delta T^{(6)}(x),
$$

$$
T^{(4)}(x - \frac{1}{4}i)T^{(4)}(x + \frac{1}{4}i) = T_0(x \pm \frac{1}{4}i) + T^{(3)}(x)T^{(5)}(x).
$$

First three relations are trivial re-writings of definitions. Last three equations need nontrivial proof which we omit for brevity.

### 8 The $Y-$systems and $E$ type Thermodynamic Bethe Ansatz

We define $Y-$ functions by combinations of $T^{(n)}$s and transform the $T-$ systems into equivalent but desired forms. In the case of the dilute $A_4$ case, they explicitly read,

**Definition 3**

$$
Y^{(1)}(x) := \frac{\phi(x + \frac{10}{6}i)T^{(2)}(x)}{T_0(x \pm \frac{3i}{6})}, \quad Y^{(2)}(x) := \frac{T^{(1)}(x)T^{(3)}(x)}{T_0(x)T_0(x \pm \frac{3i}{6})}
$$

$$
Y^{(3)}(x) := \frac{T^{(2)}(x)T^{(4)}(x)}{T_0(x \pm \frac{1}{6}i)T_0(x \pm \frac{1}{2}i)}, \quad Y^{(4)}(x) := \frac{T^{(3)}(x)T^{(5)}(x)}{T_0(x)T_0(x \pm \frac{3i}{6})}
$$

$$
Y^{(5)}(x) := \frac{T^{(4)}(x)T^{(6)}(x)}{T_0(x \pm \frac{3i}{6}), \quad Y^{(6)}(x) := \frac{T^{(5)}(x)}{T_0(x)}
$$

$$
Y^{(7)}(x) := \frac{T^{(3)}(x)}{T_0(x \pm \frac{3i}{6})}.
$$
Similarly for the dilute $A_6$ model,

**Definition 4**

\[
Y^{(1)}(x) = Y^{(5)}(x) := \frac{T^{(2)}(\frac{x}{4})}{T_0(\frac{x}{4})} \\
Y^{(2)}(x) = Y^{(4)}(x) := \frac{T^{(1)}(\frac{x}{4})T^{(3)}(\frac{x}{4})}{T_0(\frac{x}{4} \pm \frac{1}{4}i)} \\
Y^{(3)}(x) := \frac{(T^{(2)}(\frac{x}{4}))^2T^{(6)}(\frac{x}{4})}{T_0(\frac{x}{4})T_0(\frac{x}{4} \pm \frac{1}{2}i)} \\
Y^{(6)}(x) := \frac{\phi(\frac{x}{4} + i\frac{x}{4})T^{(3)}(\frac{x}{4})}{T_0(\frac{x}{4} \pm \frac{3i}{4})}.
\]

Then new sets of functional relations (the $Y-$ system) follow from the $T-$systems.

**Theorem 2** Functional relations among $Y-$ functions exhibit the $E_{6,7}$ structure in the following form,

\[
Y^{(a)}(x - i)Y^{(a)}(x + i) = \prod_{b \sim a} (1 + Y^{(b)}(x)), \quad a = 1, \ldots, a_{\text{max}}.
\]

Here $a_{\text{max}} = 6(7)$ for the dilute $A_6$ ($A_4$) model, respectively. We denote $a \sim b$ if $a$ and $b$ are adjacent nodes in the $E_6(E_7)$ Dynkin diagram.

This coincides with the $E_{6,7}$ case of the universal $Y-$ system in [59].

The derivation of TBA from the $Y-$ system needs some information on the analytic structures of $Y^{(a)}(x), 1 + Y^{(a)}(x)$. As stressed in the survey section, only the $Y-$ system with nice analytic properties (ANZC) yields an explicit algorithm in the evaluation of free energy.

We employ numerical calculations for some fixed values of $N$ and $\beta$, this purpose. This is relatively facile as one has only to deal with the largest eigenvalue sector. Though we have performed the numerical calculation for small values of $N$, it already reveals intriguing patterns for zeros of $T^{(a)}(x)$, which are also observed for the dilute $A_3$ model. Namely, imaginary parts of coordinates of zeros show the remarkable coincidence with exponents related to mass spectra in Table 1. We state it as a conjecture for arbitrary $N$.

**Conjecture 1** Zeros of $T^{(a)}$ distribute along approximately on lines, $\Im x \sim \pm \frac{1}{6}(a_j + 1)$ for the dilute $A_4$ model and $\pm \frac{1}{4}(a_j + 1)$ for the dilute $A_6$ model. The set $\{a_j\}$ agrees with $\{a\}$ for the particle $j$ in Table 1 (Table 2).
Therefore, we have a lemma parallel to the dilute $A_3$ case.

**Lemma 7** Assume that the above conjecture is valid. Then $\tilde{Y}^{(a)}(x)$ and $1 + Y^{(a)}(x)$ are Analytic, NonZero and have Constant asymptotic behavior (ANZC) in strips $\Im x \in [-1, 1], [-0^+, 0^+]$, respectively.

$\tilde{Y}^{(a)}(x)$ is defined by

\[
\tilde{Y}^{(a)}(x) = \begin{cases} 
Y^{(a)}(x)/\{\kappa(x + i(1 + \tilde{u}))\kappa(x - i(1 + \tilde{u}))\}, & \text{for } a = 1(6), u < 0 \\
Y^{(a)}(x)\kappa(x + i(1 - \tilde{u}))\kappa(x - i(1 - \tilde{u})), & \text{for } a = 1(6), u > 0 \\
Y^{(a)}(x), & \text{otherwise}
\end{cases}
\]

and $\tilde{u} = 6u(4u)$ for the dilute $A_4$ ($A_6$) model. The renormalization factor is given by

\[
\kappa(x) = \left(\frac{v_1(i\pi v/4, \tau')}{v_2(i\pi v/4, \tau')}\right)^{N/2},
\]

where $\tau' = 5/2(\pi/2)\tau$ for the dilute $A_4$ ($A_6$) model.

The significance of the above property is clear when one considers these relations (to be precise, logarithmic derivatives of them) in Fourier space, or "$k$" space. Cauchy’s theorem assures that all quantities satisfy algebraic equations at same $k$, i.e., without mixing of modes. Thus they can be solved in an elementary way. We omit the explicit procedure, for it has been given for other models \[34, 35, 24\]. The resultant coupled integral equations read

\[
\ln Y^{(a)}(x) = -\epsilon\delta_{a,t}\tilde{\beta}s(x) + C_{a,b}^* \ln(1 + Y^{(b)})(x),
\]

\[
s(x) = \frac{\delta}{2\pi} \sum_n e^{ik_n x} \frac{1}{2 \cosh k_n},
\]

\[
C_{a,b}(x) = s(x)(2I - C^g)_{a,b},
\]

where $t = 1(6)$, $\tilde{\beta} = 12\pi\beta(8\pi\beta)$, and $C^g$ denotes the Cartan matrix for $E_7(E_6)$. $\delta$ also depends on whether we are dealing with the dilute $A_4$ model or the dilute $A_6$ model through $\tilde{\delta} = \pi^2/(2\tau')$ and $k_n = n\delta$. We also adopt the abbreviation, $A*B(x) := \int_{-2\tau'/\pi}^{2\tau'/\pi} A(x - x')B(x')dx'$.

This is nothing but the conjectured TBA for the $E_{6,7}$ RSOS model at level 2 \[23\].

The free energy is expressed via $Y-$ functions with the aid of eq.(32). We shall only give the result for $\epsilon = 1$.

\[
-\beta f = -\beta e_0 - \tilde{b}_1 * s(0) + s * \ln(1 + Y^{(1)})(0)
\]

\[
e_0 := \begin{cases} 
\lambda[\ln(\vartheta_1(\pi/10)\vartheta_1(4\pi/10))]' & \text{for } A_4 \\
\lambda[\ln(\vartheta_1(\pi/7)\vartheta_1(3\pi/7))]' & \text{for } A_3
\end{cases}
\]

\[
\tilde{b}_1(x) := \begin{cases} 
\frac{\sinh 3x + \sinh 9x}{\sinh 10x}, & \text{for } A_4 \\
\frac{\sinh 6x}{\sinh 7x}, & \text{for } A_3
\end{cases}
\]

(55)
9 Conclusion

We have seen that the $E_{6,7,8}$ structure appears in the dilute $A_{6,4,3}$ model; exponents of mass scale, zeros of QTM and TBAs. These results strongly support the underlying $E$ type symmetry in the dilute $A_L$ model.

A Yangian analogue of Young tableaux arises in proof of the $T-$ system. The combinatorial aspects provide interesting problems on their own. We thus believe that the subject is worth of further research.

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Appendix

The RSOS weights for the dilute $A_L$ model is given by

$$
\begin{align*}
\frac{a \quad u \quad a}{a \quad a} &= \frac{\theta_1(6 - u)\theta_1(3 + u)}{\theta_1(6)\theta_1(3)} - \frac{\theta_1(u)\theta_1(3 - u)}{\theta_1(6)\theta_1(3)} \\
&\times \left( \frac{S_{a+1} \theta_4(2a - 5)}{S_a \theta_4(2a + 1)} + \frac{S_{a-1} \theta_4(2a + 5)}{S_a \theta_4(2a - 1)} \right),
\end{align*}
$$

$$
\begin{align*}
\frac{a \pm 1 \quad u \quad a}{a \quad a} &= \frac{\theta_1(3 - u)\theta_4(\pm 2a + 1 - u)}{\theta_1(3)\theta_4(\pm 2a + 1)},
\end{align*}
$$

$$
\begin{align*}
\frac{a \quad u \quad a \pm 1}{a \quad a} &= \frac{\theta_1(2 - u)\theta_1(3 - u)}{\theta_1(2)\theta_1(3)},
\end{align*}
$$

$$
\begin{align*}
\frac{a \quad u \quad a \pm 1}{a \quad a} &= -\left( \frac{S_{a-1}S_{a+1}}{S_a^2} \right)^{1/2} \frac{\theta_1(u)\theta_1(1 - u)}{\theta_1(2)\theta_1(3)},
\end{align*}
$$

$$
\begin{align*}
\frac{a \pm 1 \quad u \quad a}{a \quad a} &= \frac{\theta_1(3 - u)\theta_1(\pm 4a + 2 + u)}{\theta_1(3)\theta_1(\pm 4a + 2)} \\
&\quad + \frac{S_{a\pm 1} \theta_1(u)\theta_1(\pm 4a - 1 + u)}{S_a \theta_1(3)\theta_1(\pm 4a + 2)}, \quad \text{for } 4a + 2 \neq 0,
\end{align*}
$$

$$
\begin{align*}
\frac{a \quad u \quad a \pm 1}{a \quad a} &= \frac{\theta_1(3 + u)\theta_1(\pm 4a - 4 + u)}{\theta_1(3)\theta_1(\pm 4a - 4)} \\
&\quad + \left( \frac{S_{a\pm 1}\theta_1(4)}{S_a\theta_1(2)} - \frac{\theta_1(\pm 2a - 5)}{\theta_1(\pm 2a + 1)} \right) \frac{\theta_1(u)\theta_1(\pm 4a - 1 + u)}{\theta_1(3)\theta_1(\pm 4a - 4)}, \quad \text{otherwise}.
\end{align*}
$$

(57)

Here $\theta_{1,4}(x) = \vartheta_{1,4}(\lambda x, \tau)$,

$$
\begin{align*}
\vartheta_1(x, \tau) &= 2q^{1/4} \sin x \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2x + q^{4n})(1 - q^{2n}),
\end{align*}
$$

$$
\begin{align*}
\vartheta_4(x, \tau) &= \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2x + q^{4n-2})(1 - q^{2n}),
\end{align*}
$$

and $q = \exp(-\tau)$. $\lambda$ is a parameter of the model specified in section 3 and $S_a$ denotes

$$
S_a = (-1)^a \frac{\theta_1(4a)}{\theta_4(2a)}.
$$

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References

[1] A.B. Zamolodchikov, Adv. Stud. Pure. Math. 19 (1989) 641.
[2] A.B. Zamolodchikov, Int. J. Mod. Phys. A 4 (1989) 4235.
[3] M. Henkel and H. Saleur, J. Phys. A22 (1989) L513.
[4] P.G. Lauwers and V. Rittenberg, Phys. Lett. B233 (1989) 197
[5] I.R. Sagdeev and A. B. Zamolodchikov, Mod. Phys. Lett B3 (1989) 1375.
[6] P. Christe and G. Mussardo, Nucl. Phys. B330 (1990) 465.
[7] G. von Gehlen, Nucl. Phys. B330 (1990) 741.
[8] V.P. Yurov and Al. B. Zamolodchikov, Int.J. Mod. Phys. A6 (1991) 4556.
[9] M. Lassig, G. Mussardo and J. L. Cardy Nucl. Phys. B348 (1990) 591.
[10] G. Delfino and G. Musssardo, Nucl Phys. B455 (1995) 724.
[11] C. Acerbi and G. Mussardo, Int. J. Mod. Phys. A11 (1996) 5327.
[12] S.O.Warnaar, B. Nienhuis and K. A. Seaton, Phys. Rev. Lett. 69 (1992) 710.
[13] S.O.Warnaar, B. Nienhuis and K. A. Seaton, Int. J. Nod. Phys. B7 (1993) 3727.
[14] S.O.Warnaar, P.A. Pearce, B. Nienhuis and K. A. Seaton, J. Stat. Phys. 74 (1994) 469.
[15] M.T. Batchelor, V. Fridkin and Y.K. Zhou, J. Phys. A, J. Phys. A29 (1996) L61.
[16] M.T. Batchelor and K.A. Seaton, J. Phys. A30 (1997) L479.
[17] M.T. Batchelor and K.A. Seaton, Nucl. Phys. B520 (1998) 697
[18] B. McCoy and P. Orrick, Phys. Lett. A230 (1997) 24.
[19] Y. Hara, M. Jimbo, H. Konno, S. Odake and J. Shiraishi, "Free Field Approach to the Dilute $A_L$ Models" [math/9902150]
[20] V.V. Bazhanov, O. Warnaar and B. Nienhuis, Phys. Lett. B322 (1994) 198.
[21] U. Grimm and B. Nienhuis, “Scaling Properties of the Ising Model in a Field”, in: Symmetry, Statistical Mechanical Models and Applications, Proceedings of the Seventh Nankai Workshop (Tianjin 1995), edited by M.L. Ge and F.Y. Wu, World Scientific, Singapore (1996) p.384.

[22] U. Grimm and B. Nienhuis, Phys. Rev. E55 (1997) 5011.

[23] V V Bazhanov and N Yu Reshetikhin, Prog. Theoret. Phys. Supplement 102 (1990) 301.

[24] J. Suzuki, Nucl Phys B528 (1998) 683.

[25] M. Suzuki, Phys. Rev. B31 (1985) 2957.

[26] M. Suzuki and M. Inoue, Prog. Theor. Phys. 78 (1987) 787.

[27] T. Koma, Prog. Theor. Phys. 78 (1987) 1213.

[28] J. Suzuki, Y. Akutsu and M. Wadati, J. Phys. Soc. Japan 59 (1990) 2667.

[29] M. Takahashi, Phys. Rev. B43 (1991) 5788, see also vol. 44 p. 12382.

[30] J. Suzuki, T. Nagao and M. Wadati, Int. J. Mod. Phys. B6 (1992) 1119.

[31] C. Destri and H.J. de Vega, Phys. Rev. Lett. 69 (1992) 2313.

[32] A. Klümper, Ann. Physik 1 (1992) 540.

[33] A. Klümper, Z. Phys. B91 (1993) 507.

[34] G. Jüttner, A. Klümper and J. Suzuki, Nucl Phys B512 (1998) 581.

[35] A. Kuniba, K. Sakai and J. Suzuki, Nucl Phys B525 (1998) 597.

[36] V.V. Bazhanov and N. Yu Reshetikhin, J.Phys. A23 (1990) 1477.

[37] J. Suzuki, Phys. Lett. A195 (1994) 190.

[38] A. Kuniba and J. Suzuki, Comm. Math. Phys. 173 (1995) 225.

[39] A. Kuniba, Y.Ohta and J. Suzuki, J. Phys. A28 (1995) 6211.

[40] Z Tsuboi, Physica A252 (1998) 565.

[41] A.G. Izergin and V. E. Korepin, Comm. Math. Phys. 79 (1981) 303.

[42] A. Kuniba, Nucl. Phys. B355 (1991) 801.
[43] M. T. Batchelor and K. A. Seaton, Eur. Phys. J. B5 (1998) 719.

[44] K. A. Seaton and M. T. Batchelor, "E8, E7 and E6 symmetries in the dilute Al lattice model" in 'Group22: Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics', Eds. S. P. Corney, R. Delbourgo and P. D. Jarvis (Cambridge, MA: International Press) (1998).

[45] A. Klümper and R.Z. Bariev, Nucl. Phys. B458 (1996) 623.

[46] G. Jüttner and A. Klümper, Euro. Phys. Lett. 37 (1997) 335.

[47] G. Jüttner, A. Klümper and J. Suzuki, Nucl. Phys. B486 (1997) 650.

[48] G. Jüttner, A. Klümper and J. Suzuki, J. Phys. A30 (1997) 1181.

[49] G. Jüttner, A. Klümper and J. Suzuki, Nucl Phys B522 (1998) 471.

[50] J. Suzuki, J Phys A32 (1999) 2341.

[51] K. Sakai, M. Shiroishi, J. Suzuki and Y. Umeno, Phys. Rev. B60 (1999) 5186.

[52] R.J. Baxter and P.A. Pearce, J. Phys. A15 (1982) 897.

[53] A. Klümper and P.A. Pearce, Physica A183 (1992) 304.

[54] U. Grimm, P.A. Pearce and Y.K. Zhou, Physica A 222 (1995) 261.

[55] N. Yu Reshetikhin, Theoret. Math. Phys. 63 (1985) 555.

[56] N. Yu Reshetikhin, Lett. Math. Phys. 14 (1987) 235.

[57] T. Nakanishi, Nucl. Phys. B439 (1995) 441.

[58] A.Kumiba, T. Nakanishi and J. Suzuki, Int. J. Mod. Phys. A9 (1994) 5215, ibid 5267.

[59] Al.B. Zamolodchikov, Phys. Lett B253 (1991) 391.