Novel conditions for finite-time stability of neural networks with time varying delay

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Abstract. In this paper, the problem of finite-time stability of neural networks with time varying delay is considered by using improved reciprocally convex inequalities and some integral inequalities, combined with Wirtinger’s inequalities and some useful integral inequality. New sufficient conditions for finite-time stability are established to guarantee finite-time stability for the system are given in terms of linear matrix inequalities (LMIs). Numerical examples are given to illustrate the effectiveness of the obtained result.

1. Introduction
These guidelines show how to prepare articles for publication in phenomenon of time delays are often met in many systems such as networked control systems, process control systems. The existence of these delays may be the source of instability and serious deterioration in the performance of the closed-loop systems. Especially, in real systems, the delay should be assumed to be time-varying satisfying \( d_1 \leq d(t) \leq d_2 \) and \( d_1 \) is not necessarily restricted to be 0.

Finite-time stability (FTS) is one of the important role in mathematical control theory, which has been studied by different approaches and for different kind of systems [1, 2, 3, 4, 5]. The concept of finite-time stability, presented by Dorato [1], plays a fundamental concepts in stability theory of dynamical systems. A system is said to be finite-time stable if the state of a system does not exceed some bound during a fixed time interval. Finite-time stability focused its on the boundedness of system during a fixed finite-time interval. In many interesting results of finite-time stability and stabilization of linear time-delay have been obtained [1, 2, 3, 4, 5].

In this paper, we consider problem of the finite-time stability for neural network with time-varying delay. The improved reciprocally convex inequalities [10] and some integral inequalities, which are employed to provide a tight upper bound for the time-derivative of some Lyapunov-Krasovskii functional. New sufficient conditions for finite-time stability are established to guarantee finite-time stability for the system are given in terms of linear matrix inequalities (LMIs). Numerical examples are given to illustrate the effectiveness of the obtained result.

2. Problem statement with preliminaries
The following notations will be used in this paper: \( \mathbb{R}^+ \) denotes the set of all non-negative real numbers; \( \mathbb{R}^n \) denotes the \( n \)-dimensional space with the Euclidean norm \( \| . \| \); \( M^{n \times r} \) denotes the space of all matrices of \( (n \times r) \)-dimensions.

\( A^T \) denotes the transpose of matrix \( A \); \( A \) is symmetric if \( A = A^T \); \( I \) denotes the identity matrix; \( \lambda(A) \) denotes the set of all eigenvalues of \( A \); \( \lambda_{\text{max}}(A) = \sup_{\lambda \in \lambda(A)} \{ \text{Re} \lambda \} \).

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Lemma 2 [6] For a given matrix \( R > 0 \), the following inequality holds for any continuously differentiable function \( w : [a, b] \rightarrow \mathbb{R}^n \)

\[
\int_a^b \dot{w}^T(u)R\dot{w} du \geq \frac{1}{b-a}(\Gamma_1^T R \Gamma_1 + 3\Gamma_2^T R \Gamma_2)
\]

where

\[
\Gamma_1 := \omega(b) - \omega(a), \\
\Gamma_2 := \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u) du.
\]
Lemma 3 [8] Let $R$ and $Z$ be $n \times n$ constant real matrices satisfying $R = R^T > 0$ and $Z = Z^T > 0$, and $\omega : [a, b] \to \mathbb{R}^n$ a vector function, such that the integrations below are well defined, where $a$ and $b$ are two scalars with $b > a$. Then, the following two inequalities hold:

$$\int_a^b \omega^T(s)R\omega ds \geq \frac{1}{b-a} \xi^T(\omega, a, b)\Delta_1^T R\Delta_1 \xi(\omega, a, b),$$  \hspace{1cm} (7)

$$\int_a^b (s-a)\omega^T(s)Z\omega ds \geq \xi^T(\omega, a, b)\Delta_2^T Z\Delta_1 \xi(\omega, a, b)$$  \hspace{1cm} (8)

where $R = \text{diag}(3R, 5R), \hat{Z} = \text{diag}(2Z, 4Z)$ and

$\xi(\omega, a, b) := \text{col}(\omega(a), \omega(b)), \frac{1}{b-a} \int_a^b \omega(2)ds, \frac{1}{(b-a)^2} \int_a^b (b-\omega(2)ds$

$$\Delta_1 := \begin{bmatrix} -\frac{1}{4} & I & I & 0 & 0 \\
-I & I & -2I & 12I & 0 \
-I & I & -6I & 0 & 0 \end{bmatrix}, \Delta_2 := \begin{bmatrix} 0 & I & -I & 0 \\
0 & I & -4I & 6I \end{bmatrix}. $$  \hspace{1cm} (9)

Remark 1: It is clear to see that the inequality in lemma 5 provides a tighter lower bound for $\int_a^b \omega^T(u)R\omega(u)du$ than Jensen’s inequality since $\Delta_1^T R \Delta_1 > 0$ for $\Delta_2 \neq 0$. Thus, the inequality (6) is an improvement over Jensen’s inequality.

Lemma 4 Let $R_i \in \mathbb{R}^{m \times m}$ ($R_i = R_i^T > 0$) and $\omega \in \mathbb{R}^m (i = 1, 2)$ and a scalar $\alpha \in (0, 1)$. If there exist real matrices $S \in \mathbb{R}^{m \times m}$ such that

$$\begin{bmatrix} R_1 & S \\
S^T & R_2 \end{bmatrix} \geq 0$$  \hspace{1cm} (10)

the following inequality holds:

$$f(\alpha) := \frac{1}{\alpha} \omega_1^T R_1 \omega_1 - \frac{1}{1-\alpha} \omega_2^T R_2 \omega_2 \geq \omega_1^T R_1 \omega_1 + 2\alpha \omega_1^T S \omega_2 + \omega_2^T R_2 \omega_2 \geq 0$$  \hspace{1cm} (11)

Lemma 5 [7] Let $R_1, R_2 \in \mathbb{R}^{m \times m}$ be real symmetric positive definite matrices and $\sigma_1, \sigma_2 \in \mathbb{R}^m$ and a scalar $\alpha \in (0, 1)$. If there exist real symmetric matrices $X_1, X_2 \in \mathbb{R}^{m \times m}$ and real matrices $Y_1, Y_2 \in \mathbb{R}^{m \times m}$ such that

$$\begin{bmatrix} R_1 - X_1 & Y_1 \\
Y_1^T & R_2 \end{bmatrix} \geq 0, \begin{bmatrix} R_1 & Y_1 \\
Y_1^T & R_2 - X_2 \end{bmatrix} \geq 0,$$

the following inequality holds

$$F(\alpha) := \frac{1}{\alpha} \sigma_1^T R_1 \sigma_1 + \frac{1}{1-\alpha} \sigma_2^T R_2 \sigma_2 \geq S(X_1, X_2)$$

$$S(X_1, X_2) := \sigma_1^T [R_1 + (1-\alpha)X_1] \sigma_1 + 2\sigma_1^T Y_1 \sigma_2 + \sigma_2^T [R_2 + \alpha X_2] \sigma_2.$$  \hspace{1cm} (13)

Remark 2 Lemma 5 present a general lower bound $S(X_1, X_2)$ for the reciprocally convex combination $F(\alpha)$ by introducing four slack matrix variable, which bring additional degree of freedom if compared with the one in Park et al.(2011). However, more slack matrix variables usually lead to high computation complexity. The following Lemma [10] gives a maximum lower bound for $F(\alpha)$ from the set $S(X_1, X_2)/(X_1, X_2)$ satisfies $S(13)$

Lemma 6 [10] Let $R_1, R_2 \in \mathbb{R}^{m \times m}$ be real symmetric positive definite matrices and $\sigma_1, \sigma_2 \in \mathbb{R}^m$ and a scalar $\alpha \in (0, 1)$. Then for any $Y_1, Y_2 \in \mathbb{R}^{m \times m}$, the following inequality holds

$$F(\alpha) \geq \sigma_1^T [R_1 + (1-\alpha)(R_1 - Y_1 Y_1^T)] \sigma_1 + \sigma_2^T [R_2 + \alpha(R_2 - Y_2 Y_2^T)] \sigma_2 + 2\sigma_1^T [(1-\alpha)Y_1] \sigma_2$$

$$+ 2\sigma_1^T [1+(1-\alpha)Y_2] \sigma_2.$$  \hspace{1cm} (12)

Remark 3 Lemma 6 provides a maximum lower bound $S(R_1 - Y_1 R_2^{-1} Y_1^T, R_2 - Y_2 R_2^{-1} Y_2)$ for $F(\alpha)$. It is worth pointing out that the slack matrix variable $X_1$ and $X_2$ in (13) are removed from Theorem 1
3. Main Results

In this section, we give a design of finite-time stability for the system (1). First, we present a delay-dependent asymptotical stabilizability analysis conditions for the neural network system with time-varying delay (1). For our goal, we choose the following Lyapunov-Krasovskii functional

\[V(t, x_t) = V_1(t, x_t) + V_2(t, x_t) + V_3(t, x_t)\]  \hspace{1cm} (13)

where

\[V_1(t, x_t) := \zeta_1^T(t)P\zeta_1, 0\]  \hspace{1cm} (14)

\[V_2(t, x_t) := \int_{t-h}^{t} \zeta_2^T(t)Q_1\zeta_2ds + \int_{t-h}^{t-\tau(t)} \zeta_2^T(t)Q_2\zeta_2ds,\]  \hspace{1cm} (15)

\[V_3(t, x_t) := \int_{t-h}^{t} \dot{x}^T(s)[h\sigma(s)R + \sigma^2(s)Z\dot{x}(s)]ds,\]  \hspace{1cm} (16)

where

\[\zeta_1(t) := \text{col}[x(t), x(t-\tau(t)), x(t-h), (h-\tau(t))\rho_1(t), d(t)\rho_2(t), (h-\tau(t))\rho_3(t), \tau(t)\rho_4(t)]\]

\[\zeta_2(t) := \text{col}[\dot{x}(s), x(s)],\]

\[\rho_1(t) := \frac{1}{h-\tau(t)} \int_{t-h}^{t-\tau(t)} x(s)ds,\]

\[\rho_2(t) := \frac{1}{\tau(t)} \int_{t-h}^{t} x(s)ds,\]

\[\rho_3(t) := \frac{1}{(h-\tau(t))^2} \int_{t-h}^{t-\tau(t)} (t-\tau(t)-s)x(s)ds,\]

\[\rho_4(t) := \frac{1}{\tau^2(t)} \int_{t-h}^{t} (t-s)x(s)ds,\]

and \(\sigma(s) := h-t+s.\)

**Theorem 1** The system (1) is finite-time stable with respect to \((c_1, c_2, T, R)\) if there exist positive definite matrices \(P, Q_j > 0, (j = 1, 2), R > 0, Z > 0, T = \text{diag}[t_1, t_2, \ldots, t_n] > 0, \Pi_1, \Pi_2, \Pi_3 \text{ and } Y \) such that the following LMIs holds

\[
\begin{bmatrix}
\Gamma(h, d)_{|\delta_{\mu_1, \mu_2}} \
\Theta_1^T & \Theta_2^T & \zeta_7^T & \zeta_5^T \Delta_1^T S^T \\
\Theta_1 & -2T & \Theta_3 & 0 & 0 \\
\Theta_2^T & \Theta_3^T & I & 0 & 0 \\
\zeta_6^T & 0 & 0 & -I & 0 \\
S \Delta_4 \zeta_5 & 0 & 0 & 0 & -R - 2\dot{Z}
\end{bmatrix} < 0, \hspace{1cm} (17)
\]

\[
\begin{bmatrix}
\Gamma(0, d)_{|\delta_{\mu_1, \mu_2}} \
\Theta_1^T & \Theta_2^T & \zeta_7^T & \zeta_4^T \Delta_1^T S \\
\Theta_1 & -2T & \Theta_3 & 0 & 0 \\
\Theta_2^T & \Theta_3^T & I & 0 & 0 \\
\zeta_6^T & 0 & 0 & -I & 0 \\
S \Delta_4 \zeta_4 & 0 & 0 & 0 & -R
\end{bmatrix} < 0, \hspace{1cm} (18)
\]
where $\mathcal{R} := \text{diag}(R, 3R, 5R)$, $\hat{Z} := \text{diag}(Z, 3Z, 5Z)$, $e_i$ is the $i$thn $10n \times 10n$ row-block vector of the $10n \times 10n$ identity matrix ($i = 1, 2, \ldots, 10$) and

\[
\begin{align*}
\Theta_1 & := \hat{\Pi} + T \Lambda \tilde{e}_6, \\
\Theta_2 & := \hat{\Pi}^T (B_1 - K_1 B_2), \\
\Theta_3 & := -T \Lambda_2 K_2 B_2, \\
\hat{\Pi} & := \Pi_1 e_{10} + \Pi_2 e_1 + \Pi_3 e_2, \\
\Gamma(\tau(t), \dot{\tau}(t)) & := \Gamma_1(\tau(t), \dot{\tau}(t)) - \Gamma_2(\tau(t)), \\
\Gamma_1(\tau(t), \dot{\tau}(t)) & := He_1(2 \mathcal{P} \mathcal{D}_1 + \hat{X} \mathcal{G}_0) + h^2 e_{10}^T (\mathcal{R} + Z) e_{10} + \mathcal{G}_1^T Q_1 \mathcal{G}_1 \\
& \quad + (1 - \hat{\Pi} \tau) \mathcal{G}_4^T (Q_2 - Q_1) \mathcal{G}_3 - \mathcal{H}_2^T \mathcal{H}_2, \\
\mathcal{G}_0 & := -(A - BK)e_1 + W_1 e_2 + W_2 e_{10}, \\
\mathcal{G}_1 & := \text{col}(e_{10}, e_1), \\
\mathcal{G}_2 & := \text{col}(e_9, e_3), \\
\mathcal{G}_3 & := \text{col}(e_8, e_2), \\
\mathcal{G}_4 & := \text{col}(e_2, e_1, e_5, e_7), \\
\mathcal{G}_5 & := \text{col}(e_3, e_2, e_4, e_6), \\
\mathcal{D}_1 & := \text{col}(e_1, e_2, e_3, (h - \tau(t)) e_4, \tau(t) e_5, (h - \tau(t)) e_6, \tau(t) e_7), \\
\mathcal{D}_2 & := \text{col}(e_{10}, (1 - \hat{\tau}) e_8, e_9, (1 - \hat{\tau}) e_2 - e_3, e_1 - (1 - \hat{\tau}) e_2 \\
& \quad + (1 - \hat{\tau}) e_4 + \tau e_6 - e_3, e_5 - (1 - \hat{\tau}) e_2 - \tau e_6, \\
\Gamma_2(\tau(t)) & := \mathcal{G}_4^T \Lambda_1^T \mathcal{R} + (1 - \beta)(\mathcal{R} + 2Z) \Delta_1 + 2\Delta_2^T \hat{Z} \Delta_2 \mathcal{G}_4 \\
& \quad + \mathcal{G}_5^T \Lambda_2^T \Delta_2(1 + \beta) \Lambda_1^T \mathcal{R} \Delta_1 \mathcal{G}_5 + He_1^T \Lambda_1^T S \Delta_1 \mathcal{G}_5), \\
\end{align*}
\]

with $\hat{Z} = \text{diag}(2Z, 4Z)$ $\Delta_1, \Delta_2$ defined in (9), $\beta = \frac{\tau(t)}{h}$ and

\[
\frac{\alpha_2 c_1}{\alpha_1} \leq c_2.
\]

Taking the derivative of $V$ along the solution of system (1), we obtain

\[
\begin{align*}
\dot{V}_1 & = 2\mathcal{G}_4^T(t) P_2 \tilde{z}(t), \\
\dot{V}_2 & = \mathcal{G}_2^T(t) Q_1 \mathcal{G}_2(t) - \mathcal{G}_2^T(t - h) Q_2 \mathcal{G}_2(t - h) + (1 - \hat{\tau}) \mathcal{G}_2^T(t - \tau(t))(Q_2 - Q_1) \mathcal{G}_2(t - \tau(t)), \\
\dot{V}_3 & = h^2 \dot{x}^T(t) (R + Z) \dot{x}(t) - h \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds - 2 \int_{t-h}^{t} (h - t + s) \dot{x}^T(s) Z \dot{x}(s) ds.
\end{align*}
\]

Let

\[
\eta(t) := \text{col}(x(t), x(t - \tau(t)), x(t - h), \rho_1(t), \rho_2(t), \rho_3(t), \rho_4(t), \dot{x}(t - \tau(t)), \dot{x}(t - h), \dot{x}(t)).
\]

Then, the equation (1) can be rewritten as

\[
0 = \mathcal{G}_0 \eta(t) + W_0 f(x(t))
\]
where \( C_0 \) is defined in (25). For any real matrices \( \Pi_i \in \mathcal{R}^{n \times n} (i = 1, 2, 3) \). So we obtain
\[
2\eta^T(t)\hat{\Pi}^T[C_0\eta(t) + W_0 f(x(t))] = 0
\]
where \( \hat{\Pi} \) defined in (20). \( f(x(t)) = [f_1(x_1(t), f_2(x_2(t), \ldots, f_n(x_n(t))]^T \) is the activation function. \( f_i(x_i(t)) (i = 1, 2, \ldots, n) \) satisfy the following sector-bound conditions:
\[
f_i(x_i(t))[f_i(x_i(t)) - l_i x_i] \leq 0.
\]
Thus for any \( T = \text{diag}[t_1, t_1, \ldots, t_n] \) with \( t_i \geq 0 (i = 1, 2, \ldots, n) \), we have the following
\[
0 \leq -2f^T(x(t))T[f(x(t)) - \Lambda x(t)] \leq 0,
\]
where \( \Lambda = \text{diag}[l_1, l_2, \ldots, l_n] \). From (28), (29), (30) and (32), we obtain
\[
\dot{V}(t, x_i) = \eta^T(t)\Gamma_1(\tau(t), \dot{\tau}(t))\eta(t) - h \int_{t-h}^t \dot{x}^T(s)R\dot{x}(s)ds - 2 \int_{t-h}^t (h - t + s)\dot{x}^T(s)Z\dot{x}(s)ds + W_0 w(t),
\]
where \( \Gamma_1(\tau(t), \dot{\tau}(t)) \) defined in (24). Now, we estimate the integral term as the following
\[
\int_{t-h}^t (h - t + s)\dot{x}^T(s)Z\dot{x}(s)ds = \int_{t-h}^t (\tau(t) - t + s)\dot{x}^T(s)Z\dot{x}(s)ds + (h - \tau(t)) \int_{t-h}^t \dot{x}^T(s)Z\dot{x}(s)ds + \int_{t-h}^t (h - t + s)\dot{x}^T(s)Z\dot{x}(s)ds = \ell_1(t) + \ell_2(t) + \ell_3(t).
\]
Apply Lemma5, we obtain
\[
\int_{t-h}^t \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{\beta} \varphi_1^T(t)\Delta_1^T R\Delta_1 \varphi_1(t),
\]
\[
\int_{t-h}^t (\tau(t) - t + s)\dot{x}^T(s)Z\dot{x}(s)ds \geq \varphi_1^T(t)\Delta_1^T Z\Delta_1 \varphi_1(t),
\]
\[
(h - \tau(t)) \int_{t-h}^t \dot{x}^T(s)Z\dot{x}(s)ds \geq (\frac{1}{\beta} - 1) \varphi_1^T(t)\Delta_1^T Z\Delta_1 \varphi_1(t),
\]
\[
\int_{t-h}^t (h - t + s)\dot{x}^T(s)Z\dot{x}(s)ds \geq \varphi_2^T(t)\Delta_2^T Z\Delta_2 \varphi_2(t)
\]
where \( \Delta_1 \) and \( \Delta_2 \) are given in (9) and
\[ \varphi_2(t) := \varphi(x, t - \tau(t), t), \varphi_2(t) := \varphi(x, t - h, t - \tau(t)) \]

and \( \varphi(., .) \) denote in Lemma 5. Thus, we get

\[ h \int_{t-h}^{t} \chi^T(s) R \dot{s} ds + 2 \int_{t-h}^{t} (h - t + s) \chi^T(s) Z \dot{s} ds \geq 1 - (1 - \beta) \psi_1^T \Delta^T_\beta (R + 2 \hat{Z}) \Delta_1 \psi_1(t) \]

\[ + 2 \psi_1^T(t) (\Delta^T \hat{Z} \Delta_2 - \Delta^T_1 Z \Delta_1) \psi_1(t) + 2 \psi_2^T(t) \Delta^T_2 \hat{Z} \Delta_2 \psi_2(t) \]

\[ \mathcal{F}(\beta) = \frac{1}{\beta} \psi_1^T(t) \Delta^T_\beta (R + 2 \hat{Z}) \Delta_1 \psi_1(t) + \frac{1}{1 - \beta} \psi_2^T(t) \Delta^T_2 \psi_2(t). \]

Applying (12), let

\[ \sigma_1 = \Delta_1 \psi_1(t), \sigma_2 = \Delta_2 \psi_2(t), R_1 = R + 2 \hat{Z} R_2 = R, \alpha = \beta. \]

We get

\[ \mathcal{F}(\beta) \geq \psi_1^T(t) \Delta^T_\beta [R + 2 \hat{Z} + (1 - \beta)(R - 2 \hat{Z})] \Delta_1 \psi_1(t) \]

\[ + \psi_2^T(t) \Delta^T_\beta [R + \beta(\hat{R} - R + 2 \hat{Z})] \Delta_1 \psi_2(t) + 2 \psi_2^T(t) \Delta^T_2 \psi_2(t) \]

(38)

From (37) we have

\[ h \int_{t-h}^{t} \chi^T(s) R \dot{s} ds + 2 \int_{t-h}^{t} (h - t + s) \chi^T(s) Z \dot{s} ds \geq \eta^T \Gamma_2(\tau(t) - \Gamma_21) \eta(t) \]

(39)

where

\[ \Gamma_2(\tau(t)) := \mathcal{C}_4^T [\Delta^T_\beta (R + 2 \hat{Z})] \Delta_1 + 2 \Delta^T_2 \hat{Z} \Delta_2 \mathcal{C}_4 \]

\[ + \mathcal{C}_5^T [\Delta^T_2 \hat{Z} \Delta_2 (1 + \beta) \Delta^T_3 \mathcal{R} \Delta_1] \mathcal{C}_5 + \mathcal{H} e(\mathcal{C}_4^T \Delta^T_1 \Sigma) \]

(41)

and

\[ \Gamma_{21} := (1 - \beta) \mathcal{C}_4^T \Delta^T_\beta \mathcal{S} \mathcal{R}^{-1} \mathcal{S} \Delta_1 \mathcal{C}_4 + \beta \mathcal{C}_5^T \Delta^T_3 \mathcal{S} \Delta_1 \mathcal{C}_5. \]

(42)

Let \( \mathcal{E}_1 = \mathcal{C}_5^T \Delta^T_\beta \mathcal{S} \mathcal{R}^{-1} \mathcal{S} \Delta_1 \mathcal{C}_4 \) and \( \mathcal{E}_2 = \mathcal{C}_4^T \Delta^T_\beta \mathcal{S} \mathcal{R}^{-1} \mathcal{S} \Delta_1 \mathcal{C}_4 \), so \( \Gamma_{21} := (1 - \beta) \mathcal{E}_1 + \beta \mathcal{E}_2. \)

Substituting (39) into (35), we get

\[ \bar{V}(t, x_h) \leq \eta^T(t) \Gamma(\tau(t), \hat{\tau}(t)) + \Gamma_{21} \eta(t) + 2 \eta^T(t) \Pi^T \eta(t) + 2 \eta^T(t) \Pi^T \eta(t) \]

(43)

where \( \Gamma(\tau(t), \hat{\tau}(t)) \) is defined in (20), so we obtain

\[ \bar{V}(t, x_h) \leq \eta^T(t) \left[ \begin{array}{c} \Gamma(\tau(t), \hat{\tau}(t)) + \Gamma_{21} \\ \Theta_1^T \\ \Theta_2^T \\ \Theta_3^T \end{array} \right] \eta(t) \]

(44)

where \( \eta(t) = \text{col} [\eta(t), f(x(t)), 0] \) and \( \Theta_1, \Theta_2 \) and \( \Theta_3 \) are given in (20), (21) and (22). If the LMI's in (17) are satisfies, by the convex combination property we have \( \Gamma(\tau(t), \hat{\tau}(t)) + \beta \mathcal{E}_2 + (1 - \beta) \mathcal{E}_1 < 0 \) for \( \beta \in [0, 1]. \)

By utilizing the Schur complement lemma it follow from (17) and (44). As a result, there exists a scalar \( \varepsilon > 0 \) such that \( V(t, x_h) \leq -\varepsilon \| \eta(t) \|^2 \| \eta(t) \| \leq -\varepsilon \| x(t) \|^2 \), therefore, we can conclude that the system (1) is
asymptotically stable if the LMIs in (17) are satisfied. To complete the proof of theorem, next we consider note that

\[
\alpha_1 x^T(t)Rx(t) \leq V(t,x(t)), \forall t : 0 \leq T.
\]  

(45)

Since, we have

\[
V_1(t,x(t)) = x^T(t)P^{-1}x(t) \\
\quad = x^T(t)R^2_2 \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix} R^2_2 x(t) \\
\quad = x^T(t)R^2_2 \hat{P}^{-1}R^2_2 x(t) \\
\quad \geq \lambda_{\min}(\hat{P})^{-1}x^T(t)Rx(t)
\]

(46)

and \(\alpha_1 = \lambda_{\min}(\hat{P})^{-1}\).

Consider

\[
V(0,x_0) := \zeta_1^T(0)P\zeta_1(0)0 \\
\quad + \int_{0}^{\mu(t)} \zeta_2^T(0)Q_1\zeta_2 ds + \int_{0-h}^{0} \zeta_2^T(0)Q_2\zeta_2 ds0 \\
\quad + \int_{0-h}^{0} \dot{x}^T(0)[h\sigma(0)R + \sigma^2(0)Z\dot{x}(0)] ds \\
\quad \leq [\lambda_{\max}(P) + h\lambda_{\max}(Q_1) + h^2 \lambda_{\max}(Q_2) + h^2 \lambda_{\max}(R) \\
\quad + h\lambda_{\max}(Z)][\|\phi\|^2]
\]

(50)

so we obtain

\[
V(0,x_0) \leq [\lambda_{\max}(P) + h\lambda_{\max}(Q_1) + h^2 \lambda_{\max}(Q_2) + h^2 \lambda_{\max}(R) \\
\quad + h\lambda_{\max}(Z)]c_1 \\
\quad = \alpha_2 c_1
\]

(51)

where

\[
\alpha_2 = (\lambda_{\max}(P) + h\lambda_{\max}(Q_1) + h^2 \lambda_{\max}(Q_2) + h^2 \lambda_{\max}(R) \\
\quad + h\lambda_{\max}(Z)).
\]

(52)

Therefore, from (46),(51), it follows that

\[
\alpha_1 x^T(t)Rx(t) < V(t,x(t)) \leq \alpha_2 c_1, \forall t \in [0, T].
\]

Hence from (45), we have

\[
x^T(t)Rx(t) < \left(\frac{\alpha_2 c_1}{\alpha_1}\right) \leq c_2, \forall t \in [0, T],
\]

which implies that the closed-loop system is finite-time stable w.r.t \((c_1, c_2, T, R)\).
4. Numerical examples.
In this section, we provide a numerical example with their simulations to demonstrate the effectiveness of our results.

**Example 1** Consider the neural network with time-varying delays (1) which was considered in [5] where

\[
A = \begin{bmatrix} 3.99 & 0 \\ 0 & 2.99 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1.188 & 0.09 \\ 0.09 & 1.188 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.09 & 0.14 \\ 0.05 & 0.09 \end{bmatrix}.
\]

\(\tau_1 = 0.3, \tau_2 = 0.5, \mu_1 = -0.1, \mu_2 = 0.1.\) And the condition (27) is satisfied with \(T = 10, c_1 = 1, c_1 = 50.\) By using LMI Toolbox in Matlab, the LMI in Theorem (1) is feasible. A set of solution are

\[
P = \begin{bmatrix} 2.7248 & 0.1632 \\ 0.1632 & 3.1304 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.5182 & -0.0562 \\ -0.0562 & 0.8325 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 23.8276 & 2.1214 \\ 2.1214 & 20.5712 \end{bmatrix},
\]

\[
R = \begin{bmatrix} 23.3478 & 2.1985 \\ 2.1985 & 20.3251 \end{bmatrix}, \quad Z = \begin{bmatrix} 3.8276 & 2.1214 \\ 2.1214 & 0.5712 \end{bmatrix}.
\]

**Table 1.** show that the smallest value of \(c_2\) with different \(T = 2, 4, 6, 8, 10.\)

| \(T\) | 2 | 4 | 6 | 8 | 10 |
|------|---|---|---|---|----|
|      | 5.5527 | 6.2606 | 7.0589 | 7.9589 | 8.9736 |

**Example 2** Consider the neural networks with interval time-varying delays and control input with the following parameter:

\[
\dot{x}(t) = -Ax(t) + W_0f(x(t)) + W_1g(x(t - h(t)))
\]

where

\[
A = \begin{bmatrix} -0.2 & 0 \\ 1 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.2 \end{bmatrix}.
\]

It is worth noting that, the delay function \(h(t) = 0.3 + 0.3\sin t, h_1 = 0.1, h_2 = 0.3.\) Given initial function \(\phi(t) = C^1([-0.4, 0], R^2).\) Using the Matlab LMI toolbox, we obtain

\[
P = \begin{bmatrix} 0.0071 & 0.0002 \\ 0.002 & 0.0328 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.0003 & 0.0001 \\ 0.0001 & 0.0005 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.0026 & 0.0004 \\ 0.0004 & 0.0362 \end{bmatrix},
\]

\[
R = \begin{bmatrix} 0.0068 & 0 \\ 0 & 0.0409 \end{bmatrix}, Z = \begin{bmatrix} 0.0315 & 0 \\ 0 & 0.6053 \end{bmatrix}.
\]

**Table 2.** show that the smallest value of \(c_2\) with different \(T = 2, 4, 6, 8, 10.\)

| \(T\) | 2 | 4 | 6 | 8 | 10 |
|------|---|---|---|---|----|
|      | 4.9621 | 5.2206 | 6.0219 | 6.95489 | 7.9136 |
5. Conclusions
In this paper, finite-time stability for neural networks with time-varying delay is studied. using improved reciprocally convex inequalities and some integral inequalities, which are employed to provide a tight upper bound on the time-derivative of some Lyapunov-Krasovskii functional. New sufficient conditions for finite-time stability are established to guarantee finite-time stability for the system are given in terms of linear matrix inequalities(LMIs). Numerical examples is given to illustrate the effectiveness of the obtained result.

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