CLASSIFICATION OF OKAMOTO–PAINLEVÉ PAIRS

MASA-HIKO SAI TO AND T ARO TAKE BE

ABSTRACT. In this paper, we introduce the notion of an Okamoto–Painlevé pair $(S, Y)$ which consists of a compact smooth complex surface $S$ and an effective divisor $Y$ on $S$ satisfying certain conditions. Though spaces of initial values of Painlevé equations introduced by K. Okamoto give examples of Okamoto–Painlevé pairs, we find a new example of Okamoto–Painlevé pairs not listed in [Oka]. We will give the complete classification of Okamoto–Painlevé pairs.

0. Introduction

In this paper, we will introduce the notion of an Okamoto–Painlevé pair $(S, Y)$, which is defined as follows:

Definition 0.1. (Cf. Definition [23]). Let $S$ be a compact smooth complex surface and $Y = \sum_{i=1}^{r} a_i Y_i$ an effective divisor on $S$. We say that a pair $(S, Y)$ is an Okamoto–Painlevé pair if it satisfies the following conditions:

(i) There exists a meromorphic 2-form $\omega$ on $S$ such that $(\omega) = -Y$, that is, $\omega$ has the pole divisor $Y$ (counting multiplicities) and has no zero outside $Y$.

(ii) For all $i$ $(1 \leq i \leq r)$, $Y \cdot Y_i = \deg[Y]|_{Y_i} = 0$.

(iii) Let us set $D := Y_{\text{red}} = \sum_{i=1}^{r} Y_i$. Then $S - D$ contains $\mathbb{C}^2$ as a Zariski open set.

(iv) Set $F = S - \mathbb{C}^2$ where $\mathbb{C}^2$ is the same Zariski open set as in (ii). Then $F$ is a (reduced) divisor with normal crossings.

Historically, Okamoto [Oka] introduced the space $M_J(t)$ of initial values for each Painlevé equation of type $P_J$ $(J = I, \ldots, VI)$ with the time parameter $t$, which is a noncompact complex surface. We can obtain a nice compactification $\overline{M}_J(t)$ of $M_J(t)$ so that $(\overline{M}_J(t), D_J(t))$ becomes an Okamoto–Painlevé pair, where $D_J(t) = \overline{M}_J(t) - M_J(t)$ (cf. [ST] and [MMT]). Conversely, for each Okamoto–Painlevé pair one can associate a Hamiltonian system via the deformation theory of pairs, and such a Hamiltonian system is equivalent to a differential equation of Painlevé type (cf. [Sa–T]).

The main purpose of this paper is the classification of Okamoto–Painlevé pairs. We will classify Okamoto–Painlevé pairs $(S, Y)$ into seven types according to the configuration of the divisor $Y \in |-K_S|$. (See Theorem [24].) One can easily see that the configuration of the divisor $Y$ is same as one of singular fibers of elliptic surfaces [Kod]. According to the Kodaira’s classification, the seven configurations can be denoted by $II^*, III^*, IV^*, I^*_5, I^*_2, I^*_1, I^*_0$, or in the notation of the dual graph of $Y$ they can be denoted by $\tilde{E}_8, \tilde{E}_7, \tilde{E}_6, \tilde{D}_7, \tilde{D}_6, \tilde{D}_5, \tilde{D}_4$, respectively.

It is known that there is a one-to-one correspondence between Okamoto–Painlevé pairs of the six types $\tilde{E}_8, \tilde{E}_7, \tilde{E}_6, \tilde{D}_7, \tilde{D}_6, \tilde{D}_5, \tilde{D}_4$ and Painlevé equations of six types $P_I, P_{II}, P_{III}, P_{IV}, P_V, P_{VI}$, respectively (cf. [Oka], [MMT] and [ST]). Though an Okamoto–Painlevé pair of type $\tilde{D}_7$ did

\[1991 \text{ Mathematics Subject Classification.} \quad 14D15, 14J26.\]

\[\text{Key words and phrases.} \quad \text{Okamoto–Painlevé pairs, Rational Surfaces, Painlevé equations, Kodaira degenerate elliptic curves.}\]

Partly supported by Grant-in Aid for Scientific Research (B-09440015), the Ministry of Education, Science and Culture, Japan.
Lemma 1.1. We obtain the following lemma.

Let us assume that there exists a normal crossing effective divisor \( F \). In [14], we will construct an Okamoto–Painlevé pair of type \( D_7 \) by blowing-up of \( F_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) with centers on the anti-canonical divisor \(-K_P\).

Let us discuss main ideas of classifying Okamoto–Painlevé pairs. For an Okamoto–Painlevé pair \((S, Y)\), the surface \( S \) is the compactification of \( \mathbb{C}^2 \), namely \( S \) includes \( \mathbb{C}^2 \) as a Zariski open set. Let \( F = S - \mathbb{C}^2 \). By the definition of Okamoto–Painlevé pairs, \( F \) is a normal crossing divisor. Each component of \( Y_{\text{red}} \) is also a component of \( F \), namely,

\[
Y_{\text{red}} = \sum_{i=1}^r Y_i \subset \sum_{i=1}^l F_i = F,
\]

defines \( Y_{\text{red}} \) to be a normal crossing divisor. One can easily see that the divisor \( Y \) has the same configuration of a singular fiber of an elliptic surface, and the number of irreducible components of \( F \) is 10. Therefore, the number of the irreducible components of \( Y_{\text{red}} \) is less than or equal to 10. Moreover, the configuration of \( F \) must be a tree, so is \( Y_{\text{red}} \). These arguments show that the configuration of \( Y \) is one of the types \( \bar{E}_{r-1} \) for \( r = 9, 8, 7 \) and \( \bar{D}_{r-1} \) for \( r = 5, \ldots, 10 \).

Now we will consider the classification of the configurations of normal crossing divisors \( F \). The divisor \( F \) can be obtained by adding some components to \( Y_{\text{red}} \). We call an irreducible component of \( F - Y_{\text{red}} \) an additional component. We will classify all of configurations of \( F \), and this gives the complete classification of Okamoto–Painlevé pairs and also the configurations of \( F \).

The complete list of all normal crossing divisors \( F \) is given in [14]. Each of those is a divisor with ten components including one of the seven Kodaira types. All configurations of \( F \) in the list can be transformed into the anti-canonical divisors of \( \mathbb{P}^2, F_0 \) and \( F_2 \) by blowings-up and blowings-down.

Let us discuss the relation between our results and results of Sakai in [14]. Sakai defined the notion of generalized Halphen surfaces. By definition, a generalized Halphen surface \( S \) is a compact complex surface satisfying the condition of (i) and (ii) of Definition 1.1. He related generalized Halphen surfaces to the discrete Painlevé equations via Cremona transformations. The Painlevé differential equations can be obtained as limits of discrete Painlevé equations. Most essential extra conditions of an Okamoto–Painlevé pair \((S, Y)\) are that \( S \) is a compactification of \( \mathbb{C}^2 \) and \( F \) is a normal crossing divisor. Our classification gives more direct correspondences between Okamoto–Painlevé pairs and Painlevé differential equations (cf. [Sa–3]).

1. Preliminary

Let \( S \) be a compact complex surface and let \(-K_S\) denote the anti-canonical divisor class of \( S \). For any divisor \( D \) on \( S \), we denote by \(|D|\) linear equivalence class of \( D \), and by \(|D| = |D|\) the linear space of effective divisors \( C \), such that \( C \sim D \). Here “\( \sim \)” means the linear equivalence of divisors. In this paper, we often identify the divisor class \(|D|\) with the isomorphism class of the corresponding line bundle \(|D|\) or the corresponding invertible sheaf \( \mathcal{O}_S(D) \).

Let us assume that there exists a normal crossing effective divisor \( Y \in |-K_S| \). Moreover, assume that every irreducible component of \( Y \) is a smooth rational curve. Set \( Y = \sum_{i=1}^n m_i C_i \).

Consider the blowing-up \( \pi : \tilde{S} \rightarrow S \) with center of \( P \in C_{i_0} \), and let \( C'_i \) be the strict transform of \( C_i \) by \( \pi \), and \( E \) the exceptional curve of \( \pi \). Note that \( m_i \geq 0 \). By a standard calculation, we can obtain the following lemma.

Lemma 1.1. If \( p \in C_{i_0} \setminus \bigcup_{i \neq i_0} (C_{i_0} \cap C_i) \), then

\[
-K_{\tilde{S}} = \left[ \sum_{i=1}^n m_i C'_i + (m_{i_0} - 1)E \right],
\]

\[
(C'_i)^2 = (C_i)^2 \quad (i \neq i_0),
\]
and
\[(C'_0)^2 = (C_0)^2 - 1.\]
If \(p \in C_{i0} \cap C_{i1}\) for some \(i_1 (\neq i_0)\), then
\[-K_S = \left[ \sum_{i=1}^{n} m_i C'_i + (m_{i0} + m_{i1} - 1)E \right],\]
\[(C'_i)^2 = (C_i)^2 \quad (i \neq i_0, i_1),\]
and
\[(C'_i)^2 = (C_i)^2 - 1 \quad (i = i_0, i_1).\]

Let \(F_n\) denote the Hirzebruch surface \(P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(n))\). We denote by \(f\) the class of fiber of \(F_n\), \(s_0\) the class of the minimal section of \(F_n\) respectively. We also set \(s_\infty = s_0 + nf\). Then, the self-intersection numbers are given by
\[f^2 = 0, \quad (s_0)^2 = -n, \quad (s_\infty)^2 = n.\]

It is well-known that \(F_n\) can be obtained by performing several blowings-up and blowings-downs to \(P^2\).

Now we consider the anti-canonical divisor class of \(P^2\) and \(F_n\). Let \(h\) denote the class of lines in \(P^2\). Then, they are given as
\[-K_{P^2} = \quad 3h,\]
\[-K_{F_n} = 2s_0 + (n + 2)f = 2s_\infty - (n - 2)f.\]

In this paper, we often use the formula \(-K_{F_0} = 2s_0 + 2f\) and \(-K_{F_2} = 2s_\infty\). Note that \(-K_{F_0}\) and \(-K_{F_2}\) have complements in \(F_0\) and \(F_2\) respectively which contain \(C^2\) as Zariski open sets.

2. Okamoto–Painlevé Pairs

Now let us give the definition of Okamoto–Painlevé pairs.

**Definition 2.1.** Let \(S\) be a compact smooth complex surface and \(Y = \sum_{i=1}^{r} a_i Y_i\) an effective divisor on \(S\). We say that a pair \((S, Y)\) is an Okamoto–Painlevé pair if it satisfies the following conditions:

(i) There exists a meromorphic 2-form \(\omega\) on \(S\) such that \((\omega) = -Y\), that is, \(\omega\) has the pole divisor \(Y\) (counting multiplicities) and has no zero outside \(Y\).

(ii) For all \(i (1 \leq i \leq r), \quad Y \cdot Y_i = \deg[Y]|_{Y_i} = 0.\)

(iii) Let us set \(D := Y_{\text{red}} = \sum_{i=1}^{r} Y_i\). Then \(S - D\) contains \(C^2\) as a Zariski open set.

(iv) Set \(F = S - C^2\) where \(C^2\) is the same Zariski open set as in (ii). Then \(F\) is a (reduced) divisor with normal crossings.

**Remark 2.1.** (i) The meromorphic 2-form \(\omega\) as above induces a holomorphic symplectic structure on \(S - D\).

(ii) Let \(K_S\) denote the canonical divisor class of \(S\). The condition (i) means that \(K_S = -Y\) or \(-K_S = [Y]\).

(iii) The condition (v) implies that the reduced part \(D = Y_{\text{red}}\) of \(Y\) is also a divisor with normal crossings.

(iv) The spaces of initial values of Painlevé equations can be written as \(S - D\) for some Okamoto–Painlevé pair \((S, Y)\). (See [Oka], [ST] and [MMT]).

The following is the main theorem of this paper.
Theorem 2.1. Let \((S,Y)\) be an Okamoto–Painlevé pair. Then we have the following assertions.

(i) The surface \(S\) is a projective rational surface.

(ii) The configuration of \(Y\) counting multiplicities is in the list of Kodaira’s classification of singular fibers of elliptic surfaces. More precisely, they are given by one of the following lists.

(iii) All Okamoto–Painlevé pairs can be obtained by blowings-up and blowings-down of \((\mathbb{P}^2, 3H)\), where \(H\) is a line in \(\mathbb{P}^2\).

| \(Y\) | \(\tilde{E}_8\) | \(\tilde{E}_7\) | \(\tilde{D}_7\) | \(\tilde{D}_6\) | \(\tilde{E}_6\) | \(\tilde{D}_5\) | \(\tilde{D}_4\) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Kodaira’s notation | \(II^*\) | \(III^*\) | \(I_3^*\) | \(I_2^*\) | \(IV^*\) | \(I_1^*\) | \(I_0^*\) |
| Painlevé equation | \(P_I\) | \(P_{II}\) | \(P_{III}^*\) | \(P_{III}^*\) | \(P_{IV}\) | \(P_{V}\) | \(P_{VI}\) |
In Figure 1, each line denotes $\mathbb{P}^1$ whose self-intersection number is equal to $-2$, and the number next to each line denotes the multiplicity of corresponding component in $-K_S$.

Let us start our proof with the following easy lemmas.

**Lemma 2.1.** Let $(S, Y)$ be an Okamoto–Painlevé pair, then we have:
(i) $Y_i \cong \mathbb{P}^1$ for $i$ ($1 \leq i \leq r$).
(ii) The following conditions are equivalent for $i$ ($1 \leq i \leq r$).
   (a) $Y \cdot Y_i = 0$.
   (b) $(Y_i)^2 = -2$.

**Proof.** (i) It is sufficient to show that the component $F_i$ of $F$ is isomorphic to $\mathbb{P}^1$. We will show that $H^1(F_i, \mathbb{Z}) = 0$.  

![Diagram](image-url)
We have the following isomorphisms of cohomology groups with $\mathbb{Z}$-coefficients by Poincaré duality.

\[
H^i(S, F; \mathbb{Z}) \cong H^i_0(C^2, \mathbb{Z}) \\
\cong H_{4-i}(C^2, \mathbb{Z}) \\
\cong \begin{cases} 
0 & i = 0, 1, 2, 3 \\
\mathbb{Z} & i = 4 
\end{cases}
\]

On the other hand, consider the long exact sequence of cohomology groups for pair $(S, F)$

\[
0 \rightarrow H^0(F, \mathbb{Z}) \rightarrow H^0(S, \mathbb{Z}) \rightarrow H^0(S, F; \mathbb{Z}) = 0 \\
\rightarrow H^1(F, \mathbb{Z}) \rightarrow H^1(S, \mathbb{Z}) \rightarrow H^1(S, F; \mathbb{Z}) = 0 \\
\rightarrow H^2(F, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z}) \rightarrow H^2(S, F; \mathbb{Z}) = 0 \\
\rightarrow H^3(F, \mathbb{Z}) \rightarrow H^3(S, \mathbb{Z}) \rightarrow H^3(S, F; \mathbb{Z}) = 0 \\
\rightarrow H^4(F, \mathbb{Z}) \rightarrow H^4(S, \mathbb{Z}) \rightarrow H^4(S, F; \mathbb{Z}) \cong \mathbb{Z} \\
\rightarrow \ldots
\]

Then we have

\[
H^i(S, \mathbb{Z}) \cong H^i(F, \mathbb{Z}) \quad (i = 0, 1, 2, 3). \tag{1}
\]

Especially,

\[
H^3(S, \mathbb{Z}) \cong H^3(F, \mathbb{Z}) = 0.
\]

By Poincaré duality again, we have

\[
H^1(S, \mathbb{Z}) \cong H^3(S, \mathbb{Z}) = 0. \tag{2}
\]

Now we consider an irreducible component $F_i$ of $F$. Let $F' = \bigcup_{k \neq i} F_k$. Then $F_i \cap F'$ consists of a finite set of points. And we have the Mayer–Vietoris exact sequence

\[
0 \rightarrow H^0(F, \mathbb{Z}) \rightarrow H^0(F_i, \mathbb{Z}) \oplus H^0(F', \mathbb{Z}) \rightarrow H^0(F_i \cap F', \mathbb{Z}) \\
\rightarrow H^1(F, \mathbb{Z}) \rightarrow H^1(F_i, \mathbb{Z}) \oplus H^1(F', \mathbb{Z}) \rightarrow H^1(F_i \cap F', \mathbb{Z}) \\
\rightarrow \ldots \tag{3}
\]

By (1) and (2), we have $H^1(F, \mathbb{Z}) = 0$. Since $H^1(F_i \cap F', \mathbb{Z}) = 0$, we have $H^1(F_i, \mathbb{Z}) \oplus H^1(F', \mathbb{Z}) = 0$. Therefore, $H^1(F_i, \mathbb{Z}) = 0$.

If the irreducible component $F_i$ is a singular nodal curve, then $H^1(F_i, \mathbb{Z}) \neq 0$. So $F_i$ is nonsingular, namely $F_i \cong \mathbb{P}^1$.

(ii) Since $Y_i$ is a nonsingular rational curve, by the adjunction formula, we have $K_S \cdot Y_i + (Y_i)^2 = -2$. Since $K_S = [-Y]$, the conditions (a) and (b) are equivalent to each other.

**Lemma 2.2.**

\[
H^2(F, \mathbb{Z}) \cong H^2(F_i, \mathbb{Z}) \oplus H^2(F', \mathbb{Z}).
\]

**Proof.** Let $F' = \bigcup_{k \neq i} F_k$. We see that $H^1(F_i \cap F', \mathbb{Z}) = H^2(F_i \cap F', \mathbb{Z}) = 0$. From the exact sequence (3), we have

\[
H^2(F, \mathbb{Z}) \cong H^2(F_i, \mathbb{Z}) \oplus H^2(F', \mathbb{Z}).
\]

Using the argument inductively, we obtain the assertion.

**Lemma 2.3.** The configuration of $F$ is a tree, that is, the dual graph of $F$ is connected and contains no cycles.

**Proof.** Since $H^0(S, \mathbb{Z}) \cong H^0(F, \mathbb{Z})$, the connectivity of $S$ implies the connectivity of $F$. To show that $F$ contains no cycles, it is sufficient to show that there are no irreducible components $F_{i_1}, \ldots, F_{i_m} (m \geq 3)$ such that

\[
F_{i_k} \cap F_{i_{k+1}} \neq \emptyset \quad (k = 1, \ldots, m-1), \\
F_{i_1} \cap F_{i_m} \neq \emptyset.
\]

We assume that there exist such $F_{i_1}, \ldots, F_{i_m}$. Let $F^{(1)} = F_{i_1} \cup \cdots \cup F_{i_m}$ and $F^{(2)} = F_{i_1} \cup \cdots \cup F_{i_m-1}$. Since $F^{(1)}$ and $F^{(2)}$ are connected, we have $H^0(F^{(1)}, \mathbb{Z}) \cong H^0(F^{(2)}, \mathbb{Z}) \cong \mathbb{Z}$. $F^{(2)} \cap F_{i_m}$ consists...
of at least two points, and let \( \nu \) be the number of the points of \( F^{(2)} \cap F_{1m} \). Then we have \( H^0(F^{(2)} \cap F_{1m}, \mathbb{Z}) \cong \mathbb{Z}^\nu \) (\( \nu \geq 2 \)). Note that \( H^0(F_{1m}, \mathbb{Z}) \cong H^0(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z} \). We consider the Mayer–Vietoris exact sequence

\[
0 \to H^0(F^{(1)}, \mathbb{Z}) \to H^0(F^{(2)}, \mathbb{Z}) \oplus H^0(F_{1m}, \mathbb{Z}) \to H^0(F^{(2)} \cap F_{1m}, \mathbb{Z}) \cong \mathbb{Z}^\nu.
\]

If we assume \( H^1(F^{(1)}, \mathbb{Z}) = 0 \) then \( \nu \) must be one, which contradicts the fact that \( \nu \geq 2 \). So we have \( H^1(F^{(1)}, \mathbb{Z}) \neq 0 \). Now let \( F^{(3)} \) be the union of irreducible components of \( F \) which do not belong to \( F^{(1)} \). Namely, \( F = F^{(1)} \cup F^{(3)} \). We consider the exact sequence

\[
H^1(F, \mathbb{Z}) \to H^1(F^{(1)}, \mathbb{Z}) \oplus H^1(F^{(3)}, \mathbb{Z}) \to H^1(F^{(1)} \cap F^{(3)}, \mathbb{Z}).
\]

Since \( F^{(1)} \cap F^{(3)} \) consists of a finite set of points, we have \( H^1(F^{(1)} \cap F^{(3)}, \mathbb{Z}) = 0 \), so \( H^1(F^{(1)}, \mathbb{Z}) \oplus H^1(F^{(3)}, \mathbb{Z}) = 0 \). Therefore we have \( H^1(F^{(1)}, \mathbb{Z}) = 0 \), which is a contradiction. This completes the proof.

**Proof of Theorem 2.1.** (i) Since \( S \) is a compactification of \( \mathbb{C}^2 \), we see that \( S \) is a rational surface by Theorem 5 of Kodaira [Kod2].

(ii) Let \((S, Y)\) be an Okamoto–Painlevé pair. We know that \( S \) is a projective rational surface and the configuration of the dual graph of \( F = S - \mathbb{C}^2 \) is a tree. Let \( F = \sum_{i=1}^l F_i \) be the irreducible decomposition of \( F \), where \( l \) denotes the number of the irreducible components of \( F \). Let \([F_i]\) denote the class in \( H^2(S, \mathbb{Z}) \) dual to the class of \( F_i \subset S \). From Lemma 2.2 and (1), we obtain the isomorphism

\[
\bigoplus_{i=1}^l \mathbb{Z}[F_i] \cong H^2(F, \mathbb{Z}) \cong H^2(S, \mathbb{Z}).
\]

This implies that \( \{[F_i] \mid 1 \leq i \leq l\} \) forms an integral basis of \( H^*(S, \mathbb{Z}) \).

From the condition (iii) of Definition 1.3, we have

\[
Y^2 = \sum_{i=1}^r a_i(Y \cdot Y_i) = 0.
\]

Since \( K_S = -Y \), we know that \((K_S)^2 = (-Y)^2 = Y^2 = 0 \). By Noether’s formula, we have

\[
e(S) + (K_S)^2 = 12\chi(S, \mathcal{O}_S),
\]

and \( \chi(S, \mathcal{O}_S) = 1 \). Then we obtain

\[
e(S) = 2 - 2b_1(S) + b_2(S) = 12.
\]

Since \( b_1(S) = 0 \), we have \( b_2(S) = 10 \), and hence

\[
l = \text{(the number of the components of } F) = 10.
\]

Since \( D = \sum_{i=1}^r Y_i \subset F = \sum_{i=1}^{10} F_i \), we have

\[
r \leq 10.
\]

Since \( F = \sum_{i=1}^{10} F_i \) is a connected divisor with normal crossings, the divisor \( D = Y_{\text{red}} = \sum_{i=1}^r Y_i \) is also a connected divisor with normal crossings. To see that \( D = Y_{\text{red}} \) is connected, it is sufficient to show that \( Y \) is connected. By Serre duality, we have \( H^1(S, \mathcal{O}_S) \cong H^1(S, \mathcal{O}(K_S)) \). Hence we have \( H^1(S, \mathcal{O}(K_S)) = 0 \). By the exact sequence

\[
0 \to \mathcal{O}(K_S) \to \mathcal{O}_S \to \mathcal{O}_Y \to 0,
\]
we obtain the long exact sequence
\[
0 \to H^0(S, \mathcal{O}(K_S)) \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_Y) = 0
\]
\[
\to H^1(S, \mathcal{O}(K_S)) \to H^1(S, \mathcal{O}_S) \to \cdots
\]
Therefore we have \( H^0(S, \mathcal{O}_Y) \cong H^0(S, \mathcal{O}_S) \cong \mathbb{C} \). This shows that \( Y \) is connected.

Now we prove that the divisor \( Y = \sum_{i=1}^r a_i Y_i \) has one of the configurations in the list of singular fibers of elliptic surfaces (cf. Theorem 6.2, [Kod1]). First let us show that the greatest common divisor of \( \{a_i\}_{i=1}^r \) is equal to one. Since \( S \) is a rational surface with \( b_2(S) = 10 \), \( Y \) is not relatively minimal. Hence \( S \) contains an exceptional curve \( E \) of the first kind, that is, \( E \cong \mathbb{P}^1 \) and \( E^2 = -1 \). Note that \( K_S = [-Y] \). By the adjunction formula, we have \( K_S \cdot E + E^2 = -2 \) and this implies that \( K_S \cdot E = -1 \) and equivalently \( Y \cdot E = 1 \). Since \( K_S = [-Y] \) and \( Y \cdot Y_i = 0 \) for every irreducible component \( Y_i \) of \( Y \), we know that \( E \) is not a component of \( Y \) hence \( E \cdot Y_i \geq 0 \). On the other hand, since we have
\[
1 = Y \cdot E = \sum_{i=1}^r a_i(Y_i \cdot E),
\]
there exists an irreducible component \( Y_i \) with \( a_i = 1 \). Under the condition, in order to see that the configuration of \( Y \) is one in the list of the singular fibers of elliptic surfaces in [Kod1], we can follow the proof of Theorem 6.2 of [Kod1]. In fact, only the following conditions are needed to determine all of the configuration of singular fibers of elliptic surfaces:

1. \( D = Y_{\text{red}} \) is connected.
2. \( K_S \cdot Y = 0 \).
3. \( Y \cdot Y_i = 0 \), for \( i \leq 1 \leq i \leq r \).

We have proved the first assertion, and the third assertion follows from Definition 0.1. The second assertion follows from the third assertion because \( K_S = [-Y] \). Hence the configuration of \( Y \) is in the list of Kodaira. Next, we note that \( D = Y_{\text{red}} \) must be a divisor with normal crossings and the dual graph of \( D \) must be a tree. Then \( Y \) must be one of the types of \( \overline{E}_r \) for \( r = 9, 8, 7 \) and \( \overline{D}_{r-1} \) for \( r = 5, \ldots, 10 \). At this moment, we see that each irreducible component \( Y_i \) of \( D = Y_{\text{red}} \) is a smooth rational curve with \( (Y_i)^2 = -2 \).

We will show that the configurations of types \( \overline{D}_9 \) and \( \overline{D}_8 \) cannot occur. Let us set \( \Lambda_Y = \sum_{i=1}^r \mathbb{Z}[Y_i] \) and \( \Lambda_F = \sum_{j=1}^{10} \mathbb{Z}[F_j] = \text{Pic}(S) \cong H^2(S, \mathbb{Z}) \). Then we have the inclusion map
\[
i : \Lambda_Y \hookrightarrow \Lambda_F \cong H^2(S, \mathbb{Z}).
\]
By this inclusion map \( \iota \), \( \Lambda_Y \) can be considered as a sublattice of \( H^2(S, \mathbb{Z}) \) and \( H^2(S, \mathbb{Z}) \) is generated by \( [F_j] \) (1 \leq j \leq 10). The intersection matrix \( I_F := ([F_i \cdot F_j])_{1 \leq i, j \leq 10} \) is the unimodular matrix with the signature \( (b_+, b_-) = (1, 9) \). On the other hand, the intersection matrix
\[
I_Y = ([Y_i \cdot Y_j])_{1 \leq i, j \leq r}
\]
has a null eigenvalue corresponding to \( Y \) because \( Y^2 = 0 \). This means that the rank of \( \Lambda_Y \) is strictly less than ten, that is,
\[
r < 10.
\]
This proves that the configuration of type \( \overline{D}_9 \) does not occur. If \( Y \) is of type \( \overline{D}_8 \), then \( r = 9 \) and hence we can write \( F = \sum_{i=1}^9 Y_i + F_{10} \). Since \( F \) is connected and the dual graph of \( F \) is a tree, we see that \( F_{10} \) intersects only one irreducible component \( Y_i \) for some \( i \) (1 \leq i \leq 9). In this case, \( \Lambda_F = H^2(S, \mathbb{Z}) \) is generated by \( Y_1, \ldots, Y_9 \) and \( F_{10} \) where the dual graph of \( Y_1, \ldots, Y_9 \) is of type \( \overline{D}_8 \). Now by a direct calculation we see that the intersection matrix of \( \{Y_1, \ldots, Y_9, F_{10}\} \) cannot have the determinant \(-1 \). Hence \( \overline{D}_8 \) does not occur. Then we see that the dual graph of \( Y \) must be one of the following types:
\[
\overline{E}_8, \overline{E}_7, \overline{E}_6, \overline{D}_7, \overline{D}_6, \overline{D}_5, \overline{D}_4.
\]
Conversely, if $Y$ is one of the types as above, we can construct Okamoto–Painlevé pair $(S, Y)$ by blowing up and blowing down of $(P^2, Y = 3H)$ where $H$ denotes a line in $P^2$. For detail, see [8]. This proves our theorem.

**Remark 2.2.** We will classify not only the configurations of $Y$ but also $F$ of an Okamoto–Painlevé pair $(S, Y)$. By using the result, one can show that $S - D$ is covered by a finite number of Zariski open sets $\{U_i\}$ each of which is isomorphic to $C^2$ if $Y$ is not of type $\tilde{E}_8$. (See [ST] and [MMT]).

**Example 2.1.** We consider the case where the configuration of $Y$ is of type $\tilde{D}_4$. Namely, the corresponding Painlevé equation is type $P_{VI}$. For more details, see [ST]. At first, we take the Hirzebruch surface $\Sigma^{(2)}_y$ which is obtained by gluing four copies of $C^2$ via following identification. Note that $\Sigma^{(2)}_{(0)} = F_2$.

\[
U_i = \text{Spec } C[x_i, y_i] \cong C^2 \quad (i = 0, 1, 2, 3)
\]
\[
\begin{align*}
x_0 &= x_1, & y_0 &= 1/y_1, \\
x_0 &= 1/x_2, & y_0 &= x_2(\varepsilon - x_2y_2), \\
x_2 &= x_3, & y_2 &= 1/y_3
\end{align*}
\] (4)

We consider a fiber space $(\Sigma^{(2)}_y \times B_{VI}, \pi, B_{VI})$, where $B_{VI} = C \setminus \{0, 1\}$. Let us take

\[
\varepsilon = (\kappa_0 + \kappa_1 + \kappa_t - 1 + \kappa_{\infty})/2,
\]
where $\kappa_{\nu}$ ($\nu = 0, 1, t, \infty$) are complex constants in the Hamiltonian function $H_{VI}$ (cf. [ST]). For any parameter $t \in B_{VI}$, we define a divisor $D^{(0)}(t)$ on $\Sigma^{(2)}_y \times t$:

\[
D^{(0)}(t) = \{(x_1, y_1, t) \in U_1 \times t \mid y_1 = 0\} \cup \{(x_3, y_3, t) \in U_3 \times t \mid y_3 = 0\},
\]

Note that $(D^{(0)}(t))^2 = 2$, and $2D^{(0)}(t) \in \{-K_{\Sigma^{(2)}_y}\}$. And we take four points $a^{(0)}_{(0)}(t) \in D^{(0)}(t)$ ($\nu = 0, 1, t, \infty$):

\[
\begin{align*}
a^{(0)}_{(0)}(t) &= \{(x_1, y_1, t) = (\nu, 0, t)\} \in U_1 \cap D^{(0)}(t) \quad (\nu = 0, 1, t), \\
a^{(0)}_{(0)}(t) &= \{(x_3, y_3, t) = (0, 0, t)\} \in U_3 \cap D^{(0)}(t).
\end{align*}
\]

\[\text{Figure 2.}\]

We perform blowings-up to $\Sigma^{(2)}_y \times t$ at $a^{(0)}_{(0)}(t)$ for all $t \in B_{VI}$, and let $D^{(1)}_{(0)}(t)$ be the exceptional curves of the blowings-up at $a^{(0)}_{(0)}(t)$ for $\nu = 0, 1, t, \infty$. We can take four coordinate systems $(z_{\nu}, w_{\nu})$ around the points at infinity of the exceptional curves $D^{(1)}_{(0)}(t)$ ($\nu = 0, 1, t, \infty$), where

\[
\begin{align*}
(z_{\nu}, w_{\nu}) &= ((x_1 - \nu)y_1^{-1}, y_1) \quad (\nu = 0, 1, t), \\
(z_{\infty}, w_{\infty}) &= (x_3y_3^{-1}, y_3).
\end{align*}
\]
Note that we have $w_\nu = 0$ on $D^{(1)}_\nu(t)$ for $\nu = 0, 1, t, \infty$. In order to perform the second blowings-up, let us take four points $a^{(1)}_\nu(t)$ for $\nu = 0, 1, t, \infty$.

$$a^{(1)}_\nu(t) = \{(z_\nu, w_\nu, t) = (\kappa_\nu, 0, t)\} \in D^{(1)}_\nu(t) (\nu = 0, 1, t, \infty),$$

Let us perform blowings-up at $a^{(1)}_\nu(t)$, and denote $D^{(2)}_\nu(t)$ for the exceptional curves, respectively. We take four coordinate systems $(Z_\nu, W_\nu)$ around the points at infinity of $D^{(2)}_\nu(t)$ for $\nu = 0, 1, t, \infty$, where

$$(Z_\nu, W_\nu) = ((x_1 y_1^{-1} - \kappa_0) y_1^{-1}, y_1), \quad \nu = 0, 1, t,$$

$$(Z_\infty, W_\infty) = ((x_3 y_3^{-1} - \kappa_\infty) y_3^{-1}, y_3).$$
For the strict transform of $D^{(i)}_\nu(t)$ by the blowing-up, we also denote by $D^{(i)}_\nu(t)$, respectively. Let $S(t) \rightarrow \Sigma^{(2)} \times t$ be the composition of above eight blowings-up for the parameter $t$. Then, we see that the configuration of the divisor

$$D(t) := 2D^{(0)}(t) + \sum_{\nu=0,1,t,\infty} D^{(1)}_\nu(t)$$

on $S(t)$ is of type $\tilde{D}_4$. And we see that the complements of $D(t)$ in $S(t)$ is covered by six Zariski open sets

$$\text{Spec } \mathbb{C}[Z_\nu, W_\nu] \quad (\nu = 0, 1, t, \infty),$$

$$\text{Spec } \mathbb{C}[x_0, y_0],$$

$$\text{Spec } \mathbb{C}[x_2, y_2].$$

Note that this example corresponds to the Okamoto–Painlevé pair of type $\tilde{D}_4–(2)$ in our classification. (See §5).

**Example 2.2.** We will construct the space of initial values of the Painlevé equation of type $P_I$ for an example of the Okamoto–Painlevé pair of type $E_8$. As we remarked in Remark 2.2, the spaces of initial values of $P_I$ can not be covered by some Zariski open sets each of which is isomorphic to $\mathbb{C}^2$. For any $t \in B_I$, we consider the Hirzebruch surface $\Sigma(t) := \Sigma^{(2)}(0) = F_2$, and take two curves $D_0(t) = \{y_0 = 0\} \cup \{y_2 = 0\}$ and $D'_0(t) = \{x_2 = 0\} \cup \{x_3 = 0\}$ on $\Sigma(t)$. Furthermore, we have to consider the multiplicities of each component of the anti-canonical divisor. For the surface $\Sigma(t)$, we can take the anti-canonical divisor $2D_0(t) + 4D'_0(t) \in | - K_{\Sigma(t)}|$. Let us perform blowing-up at the point $a_0(t) = \{(x_3, y_3) = (0, 0)\}$.

![Figure 5](image-url)

Let $D_1$ denote the exceptional curve, and take two coordinate systems $(z_1, W_1)$ and $(Z_1, w_1)$ on $D_1$ which satisfies $D_1 = \{z_1 = 0\} \cup \{w_1 = 0\}$ and $Z_1 = W_1^{-1}$. Note that

$$\begin{align*}
(z_1, W_1) &= (x_3, x_3^{-1}y_3), \\
(Z_1, w_1) &= (x_3y_3^{-1}, y_3).
\end{align*}$$
Next, let us perform the blowing-up at the point $a_1(t) = \{(Z_1, w_1) = (0,0)\}$, and denote the exceptional curve by $D_2(t)$. Take two coordinate systems $(z_2, W_2)$ and $(Z_2, w_2)$ on $D_2$ such that $D_2 = \{z_2 = 0\} \cup \{w_2 = 0\}$ and $Z_2 = W_2^{-1}$. Note that
\[
\begin{align*}
(z_2, W_2) &= (x_3y_3^{-1}, x_3^{-1}y_3^2), \\
(Z_2, w_2) &= (x_3y_3^2, y_3).
\end{align*}
\]

Let $P = P(t) = \{(Z_2, w_2) = (0,0)\}$ and $Q = Q(t) = \{(z_2, W_2) = (0, 0)\}$. Let us perform blowing-up at the point $a_2(t) = \{(Z_2, w_2) = (1/4, 0)\} = \{(z_2, W_2) = (0, 4)\}$, and denote the exceptional curve by $D_3(t)$. Note that we can consider this blowing-up by using the coordinate system either $(Z_2, w_2)$ or $(z_2, W_2)$.

At first, we consider the blowing-up at $P$ with the coordinate system $(Z_2, w_2)$. Take two coordinate systems $(z_3, W_3)$ and $(Z_3, w_3)$ on $D_3(t)$ such that $D_3(t) = \{z_3 = 0\} \cup \{w_3 = 0\}$ and $Z_3 = W_3^{-1}$. Then we have
\[
\begin{align*}
\{ & z_3 = x_3y_3^{-2} - 1/4, \\
& W_3 = (x_3y_3^{-2} - 1/4)^{-1}y_3, \\
& Z_3 = (x_3y_3^{-2} - 1/4)y_3^{-1}, \\
& w_3 = y_3.
\end{align*}
\]
In the same way, we have to perform five more blowings-up, and take the coordinate systems $(z_i, W_i)$ and $(Z_i, w_i)$ which satisfies that $D_i(t) = \{ z_i = 0 \} \cup \{ w_i = 0 \}$ and $Z_i = W_i^{-1}$, where $D_i(t)$ is the exceptional curve of each blowing-up ($i = 4, 5, 6, 7$). Similarly, let $a_i(t)$ be the center of each blowing-up. Then we have to take $a_i(t)$ as follows. For more details, see [Oka].

\[
\begin{align*}
    a_3(t) &= \{(Z_3, w_3) = (0, 0)\}, \\
    a_4(t) &= \{(Z_4, w_4) = (0, 0)\}, \\
    a_5(t) &= \{(Z_5, w_5) = (0, 0)\}, \\
    a_6(t) &= \{(Z_6, w_6) = (t/2, 0)\}, \\
    a_7(t) &= \{(Z_7, w_7) = (1/2, 0)\}.
\end{align*}
\]
Let $Y = 2D_0 + 4D'_0 + 3D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6 + D_7$, and $D = Y_{red}$. For simplicity, we rewrite $(x_1, y_1)$ as $(x, y)$, and set $U = \text{Spec} \mathbb{C}[x, y]$. Note that $(U; (x, y)) \subset S - D$.

Now we consider the curve $D_8$. Let us take the coordinate system $(u, v)$ around the point at infinity of $D_8$. Let $U'$ be the coordinate system $(U''; (u, v)) \cong \mathbb{C}^2$. Note that $D_8 \cap U' = \{ u = 0 \}$. By considering the coordinate transformations via above eight blowings-up, we have the following coordinate transformation between $(U; (x, y))$ and $(U'; (u, v))$:

\begin{align*}
u &= x^{15} y^{-8} - \frac{1}{4} x^{12} y^{-6} - \frac{t}{2} x^4 y^{-2} + \frac{1}{2} x^2 y^{-1}, \\
v &= -x^{-2} y. \tag{5}
\end{align*}

By calculating exterior derivations of $u$ and $v$, we have

$$du \wedge dv = x^{12} y^{-8} dx \wedge dy.$$ 

On the other hand, by solving the system of equations (5) and (6), we have

\begin{align*}
x &= A^{-1} v^{-2}, \\
y &= -A^{-2} v^{-3},
\end{align*}

where

$$A = uv^6 + \frac{1}{2} v^5 + \frac{t}{2} v^4 + \frac{1}{4}.$$ 

So,

$$x^{12} y^{-8} = A^4 = (uv^6 + \frac{1}{2} v^5 + \frac{t}{2} v^4 + \frac{1}{4})^4.$$
Therefore, we have
\[ dx \wedge dy = \frac{du \wedge dv}{(uv^6 + \frac{1}{2}v^5 + \frac{1}{2}v^4 + \frac{1}{4})^4} \]

On the other hand, note that we are denoting \((x_1, y_1)\) by \((x, y)\). By using \(x_1 = x_3^{-1}\) and \(y_1 = -x_3^{-2}y_3\) (see (1)), from (3) and (4), we see that
\[
\begin{align*}
u &= x_3y_3^{-8} - \frac{1}{4}y_3^{-6} - t\frac{2}{2}y_3^{-2} - \frac{1}{2}y_3^{-1}, \\
v &= y_3.
\end{align*}
\]

Then we have
\[
uv^6 + \frac{1}{2}v^5 + \frac{t}{2}v^4 + \frac{1}{4} = x_3y_3^{-2}.
\]

From Figure 9, the curve \(C = \{uv^6 + (1/2)v^5 + (t/2)v^4 + (1/4) = 0\}\) coincides with the component \(D_0'(t)\) of the divisor \(Y\). Note that the order of the pole at \(C\) of the 2-form \(dx_0 \wedge dy_0\) and the multiplicity of \(D_0'(t)\) in \(Y\) are both equal to four. Namely the affine chart \((U'; (u, v)) \cong \mathbb{C}^2\) intersects \(Y\), and satisfies that \(U' \cap Y = D_0'(t)\).

Next, let us perform the third blowing-up (the blowing-up with center of \(a_2(t)\)), by using the coordinate system whose origin is \(Q(t)\), namely \((z_2, W_2)\). For simplicity, let us use the same notations \((Z_i, w_i), (z_i, W_i)\), or \(D_i\) as above. But we denote \((u', v')\) for the coordinate system around the point at infinity of \(D_8\). And let \(U''\) be the coordinate system \((U''; (u', v')) \cong \mathbb{C}^2\). Now we have the following results:
\[
\begin{align*}
u' &= x^{-9}y^8 - 4x^{-6}y^6 - \frac{t}{2}x^{-2}y^2 + \frac{1}{2}x^{-1}y \\
&= x_3^{-7}y_3^{-8} - 4x_3^{-6}y_3^{-6} - \frac{t}{2}x_3^{-2}y_3^{-2} - \frac{1}{2}x_3^{-1}y_3 \\
v' &= -xy^{-1} \\
&= -x_3y_3^{-1}
\end{align*}
\]

By the similar calculation, we have
\[
\begin{align*}
&dx \wedge dy = dx_1 \wedge dy_1 = \frac{du' \wedge dv'}{(u'(v'))^6 + \frac{1}{2}(v')^5 + \frac{t}{2}(v')^4 + 4)^3}, \\
&\text{and} \\
&x^{-1}y_3^2 = \frac{1}{2}(v')^5 + \frac{t}{2}(v')^4 + 4 = x_3^{-1}y_3^2.
\end{align*}
\]

From Figure 9, the curve \(C' = \{uv'(v')^6 + (1/2)(v')^5 + (t/2)(v')^4 + 4 = 0\}\) coincides with the component \(D_1(t)\) of the divisor \(Y\). The order of the pole \(C'\) of the 2-form \(dx_1 \wedge dy_1\) and the multiplicity of \(D_1(t)\) in \(Y\) are both equal to three. We see that \(U'' \cap Y = D_0'(t)\).

Consequently, for an Okamoto–Painlevé pair of type \(E_8\), \(S - Y\) is covered by \((U; (x_1, y_1)) \cong \mathbb{C}^2\) and \((U'; (u, v)) - C \cong \mathbb{C}^2 - C\) (or \((U''; (u', v')) - C'\).

3. The Okamoto–Painlevé pairs of non-elliptic type

**Definition 3.1.** An Okamoto–Painlevé pair \((S, Y)\) is of **elliptic type** if there exists a fibration \(f : S \to \mathbb{P}^1\) of elliptic curves such that a scheme theoretic fiber \(f^*(\infty)\) at \(\infty\) is \(Y\), that is, \(f^*(\infty) = Y\). If \((S, Y)\) is not of elliptic type, we call \((S, Y)\) is of **non-elliptic type**.

A rational elliptic surface can be obtained by blowings-up of 9 base points of a cubic pencil on \(\mathbb{P}^2\). Actually, one can obtain an Okamoto–Painlevé pairs \((S, Y)\) of type \(E_6\) by blowings-up of infinitely near base points of a pencil of \(3 \times \text{line}\) and a non-singular cubic curve. This example gives an elliptic fibration \(f : S \to \mathbb{P}^1\) with \(f^*(\infty) = Y\), and hence \((S, Y)\) is an Okamoto-Painlevé pair of elliptic type. On the other hand, if one blows up 8 (infinitely near) base points of the pencil and blow up a point on anti-canonical divisor which is not a base point of the pencil, one can not obtain an elliptic fibration, so \((S, Y)\) becomes an Okamoto-Painlevé pair of non-elliptic type.
For each type of Okamoto-Painlevé pair, one can obtain both elliptic type and non-elliptic type depending on the position of the points of blowings-up and blowings-down. Note that non-elliptic type is general in the moduli space of Okamoto-Painlevé pairs of each type.

The following proposition is shown in [Sa–T].

**Proposition 3.1.** Let \((S, Y)\) be an Okamoto–Painlevé pair. Then \((S, Y)\) is of non-elliptic type if and only if \(H^0(S - Y, \mathcal{O}_{alg}) \cong \mathbb{C}\). (Here \(H^0(S - Y, \mathcal{O}_{alg}) \cong \mathbb{C}\) means that all of algebraic regular functions on \(S - Y\) are constant functions.)

4. **Construction of type \(\tilde{D}_7\)**

Now we will construct the Okamoto–Painlevé pair of type \(\tilde{D}_7\) by blowing-up of \(F_0 = \mathbb{P}^1 \times \mathbb{P}^1\) on \(-K_{F_0} = 2s_0 + 2f\). In the following figure two numbers near each solid line denotes the multiplicity and the self-intersection number in \(-K_S\). The broken lines denote \((-1)\)-curves, whose multiplicities in \(F\) are zero. Namely, they are additional components of \(F\).

![Diagram of construction of \(\tilde{D}_7\)](image)

**Figure 10.**
5. Additional Components

By definition, each Okamoto–Painlevé pair \((S, Y)\) contains the affine plane \(\mathbb{C}^2\) as a Zariski open set and we set

\[ F = S - \mathbb{C}^2. \]

We can see \(F\) as a divisor with ten components which is obtained by adding some irreducible smooth rational curves to \(Y_{\text{red}}\). We call such curves additional components.

Now we will consider \(F\) counting the multiplicity \(m_j\) for the component \(F_j\) \((j = 1, \ldots, 10)\). We set

\[ F = \sum_{j=1}^{10} m_j F_j \]

and

\[ Y = \sum_{i=1}^r a_i Y_i. \]

Note that \(r < 10\). If \(F_j = Y_i\), then we set \(m_j := a_i\). And if \(F_j\) is an additional component, then we set \(m_j = 0\).

Lemma 5.1. Let \(C\) be an additional component of \(F\) which intersects \(F_j\). Then we have \(C^2 = m_j - 2\).

Proof. Note that \(F_j\) which satisfies \(m_j > 0\) and which \(C\) intersects exists uniquely. By the adjunction formula, we have \(C^2 = C \cdot Y - 2\). Since \(C \cdot Y = m_j\), our assertion is proved.

Now we assume that \(F = \sum_{i=1}^{10} F_i\) satisfies the following condition.

\((\ast)\) Both \((F_1)^2\) and \((F_2)^2\) are even numbers, and both \(F_1\) and \(F_2\) intersect only \(F_3\).

\[ F_1 \]

\[ \cdots \]

\[ F_2 \]

\[ F_3 \]

Figure 11.

Lemma 5.2. There exists no Okamoto–Painlevé pairs which satisfy \((\ast)\).

Proof. We consider the determinant of the intersection matrix of \(F\).

\[
\det I_F = \begin{vmatrix}
(F_1)^2 & 0 & 1 & 0 & \cdots & 0 \\
0 & (F_2)^2 & 1 & 0 & \cdots & 0 \\
1 & 1 & (F_3)^2 & * & \cdots & * \\
0 & 0 & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & * & * & \cdots & * \\
\end{vmatrix}
\]

We see that \(\det I_F\) must be an even number, if we expand the determinant. Namely \(\det I_F \neq -1\). This contradicts the fact that \(I_F\) is a unimodular matrix with the signature \((b_+, b_-) = (1, 9)\). This completes the proof.
Lemma 5.3. There is not more than one additional component which intersects the components of $F$ whose multiplicities are $\geq 2$.

Proof. We consider additional components $C_1$ and $C_2$. We assume that $C_1$ and $C_2$ intersect $F_1$ and $F_2$ respectively, and that $m_1 \geq 2$ and $m_2 \geq 2$. Note that $C_1 \cdot C_2 = 0$. If $C_1 \cdot C_2 \neq 0$, then $F$ includes a cycle. From Lemma 5.3, it follows that $(C_1)^2 = m_1 - 2$ and $(C_2)^2 = m_2 - 2$. Note that $(C_1)^2 \geq 0$ and $(C_2)^2 \geq 0$. We will prove by separating into some cases.

[i] The case where $m_1 \geq 3$ or $m_2 \geq 3$.

For example, we assume $m_1 \geq 3$. We have $(C_1)^2 > 0$ and $C_1 \cdot C_2 = 0$. Therefore, the Hodge index theorem shows that $(C_2)^2 < 0$ or $C_2 \equiv 0$. This is a contradiction.

[ii] The case where another additional component intersects $C_1$ or $C_2$.

In this case, we suppose that $C_3$ intersects $C_1$ as an example. Then we have $(C_3)^2 = -2$, $C_1 \cdot C_3 = 1$ and $C_2 \cdot C_3 = 0$. Now we consider a divisor $A = nC_1 + C_3$ for $n \in \mathbb{Z}_{\geq 2}$. We have $A^2 = 2n - 2 > 0$ and $A \cdot C_2 = 0$. By applying the Hodge index theorem again, it follows that $(C_2)^2 < 0$ or $C_2 \equiv 0$. Thus we obtained a contradiction.

(1) The case where another additional component intersects neither $C_1$ nor $C_2$.

In this case, $C_1$ and $C_2$ can not intersect the same component of $F$ by Lemma 5.2. Hence we see that $C_1$ and $C_2$ intersects the different components. Let us denote them by $F_1$ and $F_2$ respectively. Note that $(C_1)^2 = (C_2)^2 = 0$, $C_1 \cdot F_1 = 1$, $C_2 \cdot F_1 = 0$ and $(F_1)^2 = -2$. Now we consider a divisor $B = nC_1 + F_1$ for $n \in \mathbb{Z}_{\geq 2}$. Then we have $B^2 = 2n - 2 > 0$ and $B \cdot C_2 = 0$. This contradicts the Hodge index theorem. This completes the proof.

By using the above lemmas, we will be able to give the complete list of configurations of $F$ and $Y \subset F$.

Moreover, one can also check that each pattern in the following list can be transformed into $(\mathbb{P}^2, -K_{\mathbb{P}^2} = 3h)$, $(F_0, -K_{F_0} = 2s_0 + 2f)$ or $(F_2, -K_{F_2} = 2s_\infty)$ by performing blowing-up and blowing-down to $S$ on $-K_S$. Since $(F_0, -K_{F_0} = 2s_0 + 2f)$ or $(F_2, -K_{F_2} = 2s_\infty)$ can be transformed into $(\mathbb{P}^2, -K_{\mathbb{P}^2} = 3h)$ by birational transformations, as a consequence, each pattern can be transformed into $(\mathbb{P}^2, -K_{\mathbb{P}^2} = 3h)$ by performing blowing-up and blowing-down of $(S, Y)$. (This gives a proof of the assertion (iii) of Theorem 2.1) This also shows that all of the configuration of $F$ in the following list do exist. In fact, one can perform the inverse birational transformation from $(\mathbb{P}^2, -3f)$ to $(S, Y)$ or $(S, F)$.

**COMPLETE LIST OF CONFIGURATIONS OF $F$.**

We will give the list of configurations of $F$ in the form of dual graphs. In each figure, we denote by each vertex a component with the positive multiplicity in $F$, and by each circle an additional component. The number near each vertex means the multiplicity of the component in $F$, and the number inside each circle means the self-intersection number of the corresponding additional component. Note that the multiplicity of an additional component in $F$ is zero.
Type $\tilde{D}_4$

$\tilde{D}_4^{(1)}$

$\tilde{D}_4^{(2)}$

$\tilde{D}_4^{(3)}$

$\tilde{D}_4^{(4)}$

$\tilde{D}_4^{(5)}$

$\tilde{D}_4^{(6)}$
\[ \tilde{D}_4-(7) \]

\[ \text{Type } \tilde{D}_5 \]

\[ \tilde{D}_5-(1) \]

\[ \tilde{D}_5-(2) \]

\[ \tilde{D}_5-(3) \]

\[ \tilde{D}_5-(4) \]
\[ \tilde{D}_5^{-}(5) \]

\[ \tilde{D}_5^{-}(6) \]

\[ \tilde{D}_5^{-}(7) \]

\[ \tilde{D}_5^{-}(8) \]

**Type \( \tilde{D}_6 \)**

\[ \tilde{D}_6^{-}(1) \]

\[ \tilde{D}_6^{-}(2) \]
Type $\tilde{D}_7$

Type $\tilde{E}_6$

$\tilde{E}_6^{-}(1)$

$\tilde{E}_6^{-}(2)$

$\tilde{E}_6^{-}(3)$

$\tilde{E}_6^{-}(4)$
\[
\tilde{E}_7 \quad (1)
\]
\[
\tilde{E}_7 \quad (2)
\]
\[
\tilde{E}_7 \quad (3)
\]

Type $\tilde{E}_7$
Type $\tilde{E}_8$

-1 1 2 3 4 5 6 3

References

[Kod1] K. Kodaira, On compact analytic surfaces II, Annals of Math., 77 (1963), 563–626.
[Kod2] —, Holomorphic mappings of polydiscs into compact complex manifolds, J. Differential Geometry, 6 (1971), 33–46.
[MMT] T. Matano, A. Matsumiya, and K. Takano, On Some Hamiltonian Structures of Painlevé Systems, II, J. Math. Soc. Japan., 51, (1999), no. 4, 843–866.
[Oka] K. Okamoto, Sur les feuilletages associés aux équation du second ordre à points critiques fixes de P. Painlevé, Japan. J. Math. 5 (1979), 1–79.
[Ram] C. P. Ramanujam, A topological characterization of the affine plane as an algebraic variety, Ann. of Math., 94 (1971), 69–88.
[Sa–T] M.-H. Saito and T. Takebe, Deformation of Okamoto–Painlevé pairs and the Painlevé equations, in preparation.
[Sak] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Kyoto-Math 99-10, preprint(1999).
[ST] T. Shioda and K. Takano, On Some Hamiltonian Structures of Painlevé Systems, I, Funkcial. Ekvac. 40 (1997), 271–291.
[Tak] T. Takebe, Classification of Okamoto–Painlevé pairs, Master thesis, Kobe University (1999), February.

Department of Mathematics, Faculty of Science, Kobe University, Kobe, Rokko, 657-8501, Japan.
E-mail address: mhsaito@math.kobe-u.ac.jp
E-mail address: takebe@math.kobe-u.ac.jp