Origin of the Strong Toroidal Magnetic Field in Magnetars

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We propose a simple model of chiral asymmetry in order to interpret the possible origin of the strong toroidal magnetic field \( \approx 1 \times 10^{12} \text{T} \) suggested by analysis of the hard X-ray detector data. In a typical size magnetar, the electron is considered to be a quantum degenerate system exhibiting non-local feature due to total anti-symmetrization in the entire system. Since Fermi energy is of the order of 100 MeV, the electrons relevant to dynamics forming the strong toroidal magnetic field are ultra-relativistic, which cause behavior as massless particles. In rotating neutron star, the electron system shares the AM with the hadron and forms a circular flux of an orbiting current. During the gravitational collapse, high energy electrons captured by protons converted to neutrons through parity violating weak interaction, leaving positive chirality electrons as remnant resulting in chiral asymmetry. It is demonstrated that the relevant orbiting electrons having positive chirality are predominantly in positive helicity states resulting in spin-like currents of the Gordon decomposition of Dirac current, and that the current gives rise to strong toroidal magnetic field; in other words, spin of orbiting electrons in the toroidal magnetic field are aligned parallel to the field, The interaction between the toroidal field and the magnetic moment sustains the helicity and the chirality.

§1. Introduction

Neutron stars (NS) observed as a pulsar are rotating with extremely regular spin period \( P \) increasing also very regularly. The spinning down phenomenon is attributed to magnetic dipole radiation. The rotating magnetic dipole model estimates surface magnetic field (MF) by \( B_s = 3.2 \times 10^3 \sqrt{\dot{P} P} \text{TT} \) (in this paper, tera Tesla, \( 10^{16} \text{Gauss} \), is used as a unit of the magnetic flux density) with time derivative of the period \( \dot{P} \). In more direct manner, the MF is estimated from the energy spectrum of X-ray, according to the electron cyclotron resonance scattering feature. Neutron stars having strong magnetic field, e.g. higher than the quantum critical field \( B_{\text{QED}} = 4.414 \times 10^{-3} \text{TT} \) is referred as magnetar\textsuperscript{2}.

The periodgram analysis of the hard x-ray detector data suggests distributions of MF in a magnetar having a structure of the toroidal and the strength of \( B_t (\approx 1 \text{TT}) \) much stronger than the poloidal (or dipole) field \( B_d (0.013 \text{TT}) \) as an inner structure. This suggestion is made in terms of deformation caused by the magnetic pressure, and therefore the strong toroidal MF is difficult to observe directly. It is worthwhile to study theoretically on the possible origin of such MF.

The toroidal MF was theoretically discussed under the consideration of the chirality asymmetry of electrons produced through the parity-violating weak process during core collapse of supernovae by A. Ohnishi and N. Yamamoto\textsuperscript{4}. They relied on somewhat mathematical concept of magnetic helicity being proportional to
the Gauss liking number and approximate conserved number to explain the long sustainability of such MF. However they considered MF formed in the plasma or magnetohydrodynamic matter and did not show its structure or mechanism of formation. It was criticized in terms of the quick decay due to chiral mixing through mass term. In the present study following the idea of chiral asymmetry, we solve these difficulties by demonstrating a more practical structure and dynamics in terms of the spin current together with the convection current in Gordon decomposition of Dirac current described by Clifford number.

In order to figure out a general feature of the system, let us take an example of a NS whose mass and radius are $1.5M_\odot$ and 12 km, respectively. The total number of baryon is estimated to be $1800N_\star$, where a big number $N_\star$ stands for $(10^{18})^3$ based on scaling factor $1\text{km} = 10^{18}\text{fm}$. The average baryon density turns out to be $0.25N_\star\text{km}^{-3} = 0.25\text{fm}^{-3}$, which is 1.4 times higher than the one of an atomic nucleus. In this density, Fermi energy of neutron is about 76 MeV. Fermi energy of neutron is higher than that of a proton by $\approx 100\text{MeV}$, including the effect of nuclear interaction, i.e., Lane potential. Therefore in order to convert a proton into a neutron through weak interaction, an electron needs an energy higher than this. Indeed, the quantum degenerate electrons with such a high Fermi energy preserve the neutron system from $\beta$-decay by Pauli blocking. From these considerations, it is seen that the system is a quantum degenerate electrons of ultra-relativistic state. The density $\rho_e$ is $\approx 0.004\text{ fm}^{-3}$, 2 $\times 10^{12}$ times higher than the one in iron metal, that is crucial in realizing the strong MF of order of Tera Tesla.

The NS is usually rotating and has large angular momentum (AM). The electron system shares AM with the baryon system. Electrons carrying AM are located in peripheral region in the vicinity of the equatorial plane, and this current flux forms a toroidal zone. Electrons having negative chirality in the region are selectively captured by protons through the parity violating weak interaction. Eventually, electrons having positive chirality remain. In this density $\rho_e$, the strength of MF (magnetization) constituted from aligned electrons amounts $\mu_0\mu_B\rho_e \approx 50\text{TT}$, where $\mu_B$ denotes the Bohr magneton. This is just the strength desired in the explanation of the strong toroidal MF.

Neutron density depends on the radius and is determined to keep local balance of $\beta$-equilibrium. Since we concentrate on global structure of MF and dynamics producing the field through interaction with the electric current. Therefore, the distribution of hadrons given by a certain theory necessarily determines the one of electrons. The degree of freedom left for the electron system is collective excitation nearby Fermi surface, whose wave functions are spread all over NS. For the sake of simplicity the gravitational effects of general relativity is also ignored, since they are not important at the peripheral region.

It is realized from these analyses, that the electron system is a super many-body quantum system being degenerated with ultra-relativistic Fermi energy. This system has to be treated with Dirac Hartree-Fock like calculation together with Maxwell equation. It is not feasible, however, to handle such a gigantic system of $\approx 3 \times 10^{55}$ particles. Hence we will solve the equation with scaled $\hbar$ method, e.g., $\hbar = 10^{18}\hbar$. This means that Dirac equation gives only number of states for quantum degenerate
assemblies of $N_\ast$ electrons. Wave functions obtained by this equation correspond to amplitude modulation waves for standing waves, and to frequency modulation waves for running waves, respectively. The scaling factor $\gamma$ in $\hbar = \gamma \hbar_\ast = \gamma \times 10^{18} \hbar$ can be chosen in consideration of balance with accuracy of calculations and feasibility of computations (distinguish asterisk $\ast$ from star $\star$ in suffixes). The electromagnetic field as a gauge field for charged particles propagates global information related with phase and determines the significant structure.

In section 2.1 we show basic formulae necessary to describe current of electrons in terms of Dirac equation coupled with electromagnetic field. First: we employ diagonal $4 \times 4$ matrix $\beta$ called Dirac-representation. The spinor in Dirac equation is expressed by $j-j$ coupling scheme using the quantum number $\kappa = \pm (j + \frac{1}{2})$. Second: in order to discuss the chiral asymmetry characterizing the present model, we treat, in section 2.2, with the Weyl (or chiral) representation in which the chiral operator $\gamma_5$ is diagonal, two-component theory being parity mixed state for massless particles. While a free massless spinor has a definite helicity, a particle in bound state is not a good quantum number, so that the helicity representation is employed to describe electrons forming the strong toroidal MF.

In section 2.3 we expand the density and the current density in terms of spherical harmonics and vector spherical harmonics. The matrix elements of coupling between particle and electromagnetic field are calculated by using Racah algebra, to find a simple expression for the coupling matrix elements of fields between an initial $\kappa_i$ and a final $\kappa_f$ states. The MF produced by chiral asymmetric current, i.e., polarized in helicity (or helical current) brings about magnetic helicity, which strengthens and stabilizes the helical currents. The helical current is composed from convection and spin like currents of the Gordon decomposition of the Dirac current.

The global structure of electromagnetic field is studied more specifically with axially symmetric ansatz in section 3. First in subsection 3.1 monopole electric polarization is considered for the quantum degenerate electron system with ultra-relativistic Fermi energy to be confined. Quasi-bound states are introduced in the subsection 3.2 to describe the electron in a potential deeper than $2m_e c_0^2$ to circumvent the difficulty of escaping through Klein tunnel, where $c_0$ stands for the light velocity in the vacuum. We will show correspondence between energies and quantum numbers of single particles obtained by using different scaling factors $\gamma = 1$ and 0.5. Next, the cranking model is employed to make the NS rotate together with electron system, in which Coriolis force breaks time reversal symmetry resulting asymmetric population in time reversal partners in the vicinity of Fermi surface. Convection like current is calculated for the orbiting electrons. Finally, we discuss in subsection 3.8 the helical current constituted by convection and spin like currents, in Weyl representation. Population to positive chirality states dominate due to the chiral asymmetric reaction of electron capture. The spin and convection like currents generate toroidal and poloidal fields, respectively. As for the convection like current, it is important to account the proton current being competitive or predominant to electron current, while spin like current contributed from proton is small enough to be ignored due to non-relativistic behavior, or large mass, 938 MeV.
§2. Basic Formulae for Calculations

2.1. Dirac representation and $j$-$j$ coupling scheme

Let us study mathematically representations of stationary states of the electron in electromagnetic field. In the NS the Fermi energy of electron is for example, $\approx 100$ MeV, much higher than $m_e e_0^2$ and therefore the electrons have to be described by Dirac equation. Wave functions are expressed by four component eigenfunction $\psi$ and eigen-vector of Hamiltonian $\hat{H}$

$$\hat{H}\psi = \left[ c_0 \alpha \cdot \left( \hat{p} + \frac{e}{c_0} \mathbf{A} \right) - eA_0 + \beta m_e c_0^2 \right] \psi , \quad (2.1)$$

where $\alpha$ and $\beta$ are $4 \times 4$ matrices and $(A_0(r), \mathbf{A}(r))$ is electromagnetic potential. The wave function $\psi$ may be divided into two spinors $(\psi^u(r\sigma), \psi^l(r\sigma))$, upper and lower components in so called Dirac representation;

$$\begin{pmatrix} m_0 c_0^2 - eA_0 \\ \sigma \cdot (c_0 \hat{p} + e\mathbf{A}) - m_0 c_0^2 - eA_0 \end{pmatrix} \begin{pmatrix} \psi^u(r\sigma) \\ \psi^l(r\sigma) \end{pmatrix} = E \begin{pmatrix} \psi^u(r\sigma) \\ \psi^l(r\sigma) \end{pmatrix} . \quad (2.2)$$

Using the spinors, the probability density and current density are written as

$$\begin{pmatrix} \rho(r) \\ j(r) \end{pmatrix} = \langle \psi | \begin{pmatrix} 1_4 \\ c_0 \alpha \end{pmatrix} | \psi \rangle_\sigma = \begin{pmatrix} \langle \psi^u | \psi^u \rangle_\sigma + \langle \psi^l | \psi^l \rangle_\sigma \\ \langle \psi^u | c_0 \sigma | \psi^l \rangle_\sigma + \langle \psi^l | c_0 \sigma | \psi^u \rangle_\sigma \end{pmatrix} . \quad (2.3)$$

The suffix $\sigma$ attached to the bracket $\langle \cdots | \cdots \rangle_\sigma$ expresses calculating trace in spin matrices. For discussions about symmetries, the two spinors are expanded by $\sigma$-matrices. For orbital AM $j_\kappa$ and total AM $j_\kappa$ defined by Rose$^{[10],[12]}$

$$| \kappa | = k - \frac{1}{2}, \quad \ell_\kappa = j_\kappa \pm \frac{1}{2} \quad \text{for} \quad \kappa = \pm \kappa. \quad (2.4)$$

Actually the quantum number $\kappa$ is eigen-value of the operator

$$\hat{K} = \beta \left( \sigma \cdot \hat{\ell} + \hbar \right) , \quad (2.5)$$

and appears specifically in the four components theory$^{[13]}$ where $\hbar$ stands for scaled $\hbar$, and $k$ takes on natural number. It corresponds to "$j$" in the Dirac’s original$^{[5]}$ and to "$j$'" in Gordon’s papers$^{[17]}$. We introduce notations $\ell_\kappa = \ell_\kappa + \frac{1}{2}$, $\kappa = \pm \kappa$ and the signature $S_\kappa \equiv \kappa/k = \ell_\kappa - \ell_\kappa = \pm 1$. The two spinors are expanded as follows

$$\begin{pmatrix} \psi^u(r\sigma) \\ \psi^l(r\sigma) \end{pmatrix} = \begin{pmatrix} \sum_{\kappa \mu} g_{\kappa \mu}(r) \chi_{\kappa \mu}(\hat{r}\sigma) \\ \sum_{\kappa \mu} f_{\kappa \mu}(r) \chi_{\kappa \mu}(\hat{r}\sigma) \end{pmatrix} . \quad (2.6)$$

The spherical spinors $\chi_{\kappa \mu}(\hat{r}\sigma)$ are defined by

$$\chi_{\kappa \mu}(\hat{r}\sigma) = \sum_{\sigma = \pm \frac{1}{2}} \langle \ell_\kappa m |_{ \sigma} | j_\kappa \mu \rangle \mathcal{V}_{\ell_\kappa m}(\hat{r}) \chi_{\sigma}^{(\frac{1}{2})} , \quad (2.7)$$
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with $\mathcal{Y}_{\ell, m}(\vec{r}) = i^\ell \mathcal{Y}_{\ell, m}(\vec{r})$, differing from the Condon-Shortley convention $\mathcal{Y}_{\ell, m}(\vec{r})$ by phase factor $i^\ell$, and $(\ell, m \frac{1}{2} \sigma | j_\mu \mu)$ stands for Clebsch-Gordan coefficient.

Let us define a pseudo scalar operator

$$\sigma_r \equiv \sigma \cdot n_r, \quad \text{with} \quad n_r = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),$$

(2.8)

having a property of $\sigma_r^2 = 1_2$ and altering the parity without changing $j_{\mu \mu}$. We redefine radial wave function for convenience,

$$G_{\alpha}(r), F_{\alpha}(r) = (rg_{\alpha}(r), rf_{\alpha}(r)).$$

(2.10)

In case of spherically symmetric system with potential $U(r) = -eA_0(r)$, (necessarily $A = 0$), the radial part of the equation turns out to be

$$\frac{d}{dr} \begin{pmatrix} G_{\alpha}(r) \\ F_{\alpha}(r) \end{pmatrix} = \begin{pmatrix} -\frac{\kappa_\alpha}{r} & \frac{E + mc_0^2 - U(r)}{h_sc_0} \\ \frac{E - mc_0^2 - U(r)}{h_sc_0} & \frac{\kappa_\alpha}{r} \end{pmatrix} \begin{pmatrix} G_{\alpha}(r) \\ F_{\alpha}(r) \end{pmatrix},$$

(2.11)

where $\alpha$ expresses combined quantum numbers of $\kappa$ and $\mu$. The wave function is orthonormalized eigenfunction labeled by $\nu$ or $\nu'$ as follows,

$$\sum_\alpha \int_0^\infty dr \left\{ G^{(\nu)}_{\alpha}(r) G^{(\nu')}_{\alpha}(r) + F^{(\nu)}_{\alpha}(r) F^{(\nu')}_{\alpha}(r) \right\} = \delta_{\nu \nu'}.$$

(2.12)

While, in using $\hbar$, this normalization means the probability of single electron, the above normalization in the equation using $h_s$, represents the probability of quantum assembly $N_s = \gamma^3 N_s$ of quantum degenerate electrons.

### 2.2. Chiral and Helicity Representations

As discussed in section[1], electrons nearby Fermi surface are ultra-relativistic and therefore behave like a massless particle. To describe them, we choose the Weyl[15] or chiral representation of the Dirac equation (2.2) obtained by the transformation

$$\begin{pmatrix} \psi^{(\pm)}(r \sigma) \\ \psi^{(\mp)}(r \sigma) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi^u(r \sigma) + \psi^e(r \sigma) \\ -\psi^u(r \sigma) + \psi^e(r \sigma) \end{pmatrix},$$

(2.13)

where the superscript $(\pm \frac{1}{2})$ specifies chirality. Then, Dirac equation is rewritten as

$$\begin{pmatrix} \sigma \cdot (c_0 \hat{p} + eA) - eA_0 & -mc_0^2 \\ -mc_0^2 & -\sigma \cdot (c_0 \hat{p} + eA) - eA_0 \end{pmatrix} \begin{pmatrix} \psi^{(\pm)}(r \sigma) \\ \psi^{(\mp)}(r \sigma) \end{pmatrix} = E \begin{pmatrix} \psi^{(\pm)}(r \sigma) \\ \psi^{(\mp)}(r \sigma) \end{pmatrix}.$$

(2.14)

In case of $m_0 = 0$, the four component equation is split into a decoupled pair of two-component equations. While, in the absence of an electromagnetic field, a positive chirality state is eigenvector of helicity $\hbar \equiv \sigma \cdot \hat{p}/p$ with positive helicity for the particle confined in potential $U(r)$ the helicity $\hbar$ is mixed.
It will be found in subsection 3.4 that the convection like current forming the torus is constituted mainly from high $k$ and high $\mu$ states whose quantum fluctuation of spin is small. Hence the plus (minus) sign of $\kappa$, i.e., $S_\kappa = 1$ ($= -1$), corresponds to down (up) spin. The following operation is applied to the states orbiting a circle, which rotates the spin by $\pi/2$ about an axis $\mathbf{n}_\rho = (\cos \phi, \sin \phi, 0)$ perpendicular to polar axis,

$$\exp \left( -i \frac{\pi}{4} S_\kappa \sigma_\rho \right) = \cos \frac{\pi}{4} - i S_\kappa \sigma_\rho \sin \frac{\pi}{4}, \quad \text{with} \quad \sigma_\rho = \mathbf{\sigma} \cdot \mathbf{n}_\rho, \quad (2.15)$$

then the states in which the spin points to tangential direction of the circle, considered as positive helicity state are obtained. The negative helicity state is obtained by changing the rotation angle from $\pi/2$ to $-\pi/2$. Since in the relevant current region, $\sin \theta$ and $\cos \theta$ are actually $\approx 1$ and $\approx 0$, respectively, then one makes approximation like $\sigma_\rho \approx \sigma_r$, the pseud scalar operator defined in eq.(2.8). This operation projecting out the helicity plus state from the state labelled by $\kappa$ turns out to be

$$P^{(\epsilon)}_h \chi_{\kappa\mu}(\hat{r}, \sigma) = \frac{1}{\sqrt{2}} (\chi_k - (-1)^{h-\frac{1}{2}} \chi_{\kappa\mu}^{(c)}(\hat{r}, \sigma)), \quad (2.16)$$

From eq.(2.9), this equation gives a unitary transformation between complementary states of helicity representation, introduced for the analysis of collisions of relativistic particles with spin\cite{14} and $j-j$ coupling states (canonical representation). We chose negative value of $\kappa$ in eq. (2.16) to fit the phase convention\cite{16}, e.g.,

$$\chi_{hjm}(\hat{r}, \sigma) = \frac{1}{\sqrt{2}} \left( \chi_{-\kappa\mu}(\hat{r}, \sigma) - (-1)^{h-\frac{1}{2}} \chi_{\kappa\mu}(\hat{r}, \sigma) \right), \quad (2.17)$$

where $h = 1/2$ is taken on for positive helicity and $h = -1/2$ for negative helicity. To distinguish the helicity $\hat{h}$ from that of helicity representation in eq.(2.17), we will call the former as linear helicity and the latter as circular helicity. To solve each equation, the spinors of helicity representation are employed,

$$\psi^{(c)}(r \sigma) = \sum_{h=\pm \frac{1}{2}} \sum j_\mu u^{(c)}_{hj\mu}(r) \chi_{hj\mu}(\hat{r} \sigma), \quad (2.18)$$

Now, we redefine $H^{(c)}_{hj\mu}(r) = ru^{(c)}_{hj\mu}(r)$ and operate $\mathbf{\sigma} \cdot \hat{\mathbf{p}}$ to $\psi^{(c)}(r \sigma)$ to obtain

$$\mathbf{\sigma} \cdot \hat{\mathbf{p}} \left( \frac{H^{(c)}_{hj\mu}(r)}{r} \right) \chi_{hj\mu} = (-1)^{h-\frac{1}{2}} \frac{\hbar_s}{r} \left( \frac{dH^{(c)}_{hj\mu}}{dr} \chi_{-h\mu} + \frac{k}{r} H^{(c)}_{hj\mu}(r) \chi_{h\mu} \right). \quad (2.19)$$

From the calculation, it is seen that only radial motion mixes the circular helicity. In other words, the higher $j$ and the larger $|m_\mu|$ states have the purer circular helicity.

It is characteristic in the equation (2.19) that the square root of centrifugal potential, $\pm k/r$ like Coulomb potential, comes to the same sector as central potential,
i.e., diagonal, of which charge $k$ is positive for the same sign and negative for opposite sign of helicity and chirality, respectively. As shown in subsection 3.5 this fact results that the plus chirality states dominate plus circular helicity and vice versa.

The radial functions $k$ that the plus chirality states dominate plus circular helicity and vice versa.

The probability density and current density are expressed as

$$
\begin{pmatrix}
\rho(r) \\
j(r)
\end{pmatrix} = \begin{pmatrix}
\rho^{(+)}(r) + \rho^{(-)}(r) \\
n^{(+)}(r) + n^{(-)}(r)
\end{pmatrix},
$$

with

$$
\begin{pmatrix}
\rho^{(+)}(r) \\
n^{(+)}(r)
\end{pmatrix} = \begin{pmatrix}
\langle \psi^{(\pm)}(r, \sigma) | \psi^{(\pm)}(r, \sigma) \rangle_{\sigma} \\
\langle \psi^{(\pm)}(r, \sigma) | \pm c_{0}\sigma | \psi^{(\pm)}(r, \sigma) \rangle_{\sigma}
\end{pmatrix}.
$$

In the Weyl representation, the matrix $\alpha'$ is diagonal in chirality, and therefore the current density is a separable sum of each component, rather than transitions between upper and lower components in the Dirac representation.

2.3. Moments of Density and Current Density

The density and current density are expanded like

$$
\begin{align*}
\rho(r) &= \sum_{LM} \rho_{LM}(r) \mathcal{Y}_{LM}(\hat{r}) \\
n(r) &= \sum_{LJM} n_{LJM}(r) \mathcal{V}_{LJM}(\hat{r}),
\end{align*}
$$

in terms of spherical harmonics and vector spherical harmonics as follows, where $\mathcal{V}_{LJM}(\hat{r})$ is defined as

$$
\mathcal{V}_{LJM}(\hat{r}) = \sum_{M_{L}S} \langle LM_{L}1S | JM \rangle \mathcal{Y}_{LML}(\hat{r}) \chi^{(1)}_{S}.
$$

The notation $\chi^{(1)}_{S}$ is tensor representation of rank 1. For the same $J$, there are three functions with different $L = J \pm 1, J$. In time independent states, the current density is divergent free, i.e., $\nabla \cdot J = 0$. In case of $L = J$, a vector field

$$
\mathcal{C}_{JM}(r, \mathcal{M}) = c_{J}(r) \mathcal{V}_{J,J,M}(\hat{r}),
$$

is shown to be divergent free and to have parity $(-1)^{J+1}$. It may be called the magnetic $2^{J}$-pole field in analogy with the multipole radiation field. A linear combination of two others, ($L = J \pm 1$) being solenoidal, is defined

$$
\mathcal{C}_{JM}(r, \mathcal{E}) = C^{(+)}_{J} k_{J-1}(r) \mathcal{V}_{(J-1),J,M}(\hat{r}) + C^{(-)}_{J} k_{J+1}(r) \mathcal{V}_{(J+1),J,M}(\hat{r}),
$$

with

$$
(C^{(+)}_{J}, C^{(-)}_{J}) \equiv \left( \sqrt{\frac{J}{2J+1}}, \sqrt{\frac{J+1}{2J+1}} \right).
$$

The radial functions $k_{J-1}(r)$ and $k_{J+1}(r)$ are not independent due to divergent free. This field has parity $(-1)^{J}$ and may be called as electric $2^{J}$-pole field denoted as $\mathcal{C}_{JM}(r, \mathcal{E})$. The last field orthogonal to $\mathcal{C}_{JM}(r, \mathcal{E})$ is given by

$$
\mathcal{C}_{JM}(r, \mathcal{L}) = C^{(+)}_{J} k_{J-1}(r) \mathcal{V}_{(J-1),J,M}(\hat{r}) - C^{(-)}_{J} k_{J+1}(r) \mathcal{V}_{(J+1),J,M}(\hat{r}).
$$
together with harmonics, the moments of density and current density are calculated as by the electric- and magnetic- multipole fields. The label $L_N$. Onishi and T. Maruyama

\[
\begin{align*}
\rho_{LM}(r) &= \sum_{\alpha\beta} C_{\alpha\beta}^{\ell J} \left( \rho_{\alpha\beta}(r) \right), \\
\eta_{LM}(r) &= \sum_{\alpha\beta} C_{\alpha\beta}^{\ell J} \left( \eta_{\alpha\beta}(r) \right),
\end{align*}
\]

(2.28)

together with

\[
\begin{align*}
\left( \begin{array}{c}
\rho_{\alpha\beta}(r) \\
\eta_{\alpha\beta}(r)
\end{array} \right) &= \frac{1}{r^2} \sum_{\nu \in \text{occ}} \left( \begin{array}{c}
G^{(\nu)}(r) G^{(\nu)}_{\beta}(r) + F^{(\nu)}_{\alpha}(r) F^{(\nu)}_{\beta}(r) \\
G^{(\nu)}(r) F^{(\nu)}_{\beta}(r) + F^{(\nu)}_{\alpha}(r) G^{(\nu)}_{\beta}(r)
\end{array} \right),
\end{align*}
\]

(2.29)

where $\nu$ specifies occupied quantum states. The kinematical factor $C_{\alpha\beta}^{\ell J}$ is given by

\[C_{\alpha\beta}^{\ell J} = D_{\alpha\beta}^{\ell J}(-1)^{\ell - \mu_{\beta}}(\rho_{\alpha\beta} - \mu_{\beta}JM),\]

(2.30)

and $D_{\alpha\beta}^{\ell J}$ represents

\[D_{\alpha\beta}^{\ell J} = \frac{(\ell + \frac{3}{2}J + 1)}{2\ell + 1} = S_{\alpha\beta} \cos\left(\frac{\pi}{2}(\ell + L - \ell_{\alpha\beta})\right) d_{\ell k'},\]

(2.31)

and

\[d_{\ell k'}^{J} = \frac{1}{1/2(2J + 1)} (-1)^{J' - \ell_{\ell k'}} \frac{1}{2} (j_{k}^1 - \frac{1}{2} | J0) ,\]

(2.32)

with

\[
\xi_{\ell k}^{\ell' k'} = -j^{-1} \left\{ \begin{array}{cc}
C_{J}^{(x)}(\xi_{\ell k}) \mp C_{J}^{(x)}(\xi_{\ell k'}) & \text{for } \left\{ \begin{array}{c}
J = L + 1 \\
J = L
\end{array} \right. , \text{and } \hat{j} = \sqrt{J(J + 1)}
\end{array} \right. \]

(2.33)

is independent of magnetic quantum numbers $\mu_{\alpha}, \mu_{\beta}$ and $M$ due to Wigner-Eckart theorem, where $\langle \ell_{\alpha\beta} j_{\alpha\beta} || Y_{\lambda} || \ell_{\alpha\beta} j_{\alpha\beta} \rangle$ is called as reduced matrix element.

2.4. **Coupling Matrices of Electromagnetic Field**

Next, let us expand electromagnetic potentials $A_{0}(r)$ and $A(r)$ in terms of spherical and vector spherical harmonics as follows,

\[
\begin{align*}
A_{0}(r) &= \sum_{LM} R_{LM}(r) Y_{LM}(\hat{r}) \\
A(r) &= \sum_{LM} H_{LM}(r) Y_{LM}(\hat{r}) ,
\end{align*}
\]

(2.34)

From Maxwell equation, the potentials are given by the densities through the Poisson equation

\[
\sum_{LM} \left( \begin{array}{c}
R_{LM}(\hat{r}) \\
H_{LM}(\hat{r})
\end{array} \right) = -(-\varepsilon \nabla) \left( \begin{array}{c}
\varepsilon_{0}^{-1} \rho_{LM}(r) \\
\mu_{0} \eta_{LM}(r)
\end{array} \right) ,
\]

(2.35)
and are solved in making use of Green’s function as
\[
\begin{pmatrix}
R_{LM}(r) \\
H_{LJM}(r)
\end{pmatrix} = -\frac{eN_s}{2L + 1} \int_0^\infty r'^2 dr' (r' < r) \frac{1}{r^{L+1}} \left( \varepsilon_0^{-1} \rho_{LM}(r') \right),
\]

with
\[
(r_-, r_+) = \{(r', r) \text{ or } (r, r')\} \quad \text{for } \{r' < r \text{ or } r < r'\},
\]
where \(\varepsilon_0\) and \(\mu_0\) express dielectric constant and magnetic permeability of the vacuum, respectively. It is noted again that \(H_{LJM}(r)\) is real and depends on only \(\eta_{LJM}(r)\), because the Laplacian included in Poisson equation is a scalar operator.

Finally we obtain Dirac equation in the coupled channel form,
\[
\begin{pmatrix}
\frac{dG_\alpha(r)}{dr} + \frac{\kappa_\alpha}{r} G_\alpha(r) - \sum_\beta V_{\alpha\beta}(r) G_\beta(r) \\
\frac{dF_\alpha(r)}{dr} - \frac{\kappa_\alpha}{r} F_\alpha(r) - \sum_\beta V_{\alpha\beta}(r) F_\beta(r)
\end{pmatrix}
= \frac{1}{\hbar c_0} \begin{pmatrix}
(E + m_e c_0^2) F_\alpha(r) + \sum_\beta U_{\alpha\beta}(r) F_\beta(r), \\
-(E - m_e c_0^2) G_\alpha(r) - \sum_\beta U_{\alpha\beta}(r) G_\beta(r).
\end{pmatrix}
\]
The suffixes \(\tilde{\alpha}\) indicate the dual state of \(\alpha\), i.e., \((\ell_{\tilde{\alpha}} = \tilde{\ell}_\alpha)\). The matrix elements are given by
\[
\begin{pmatrix}
U_{\alpha\beta}(r) \\
V_{\alpha\beta}(r)
\end{pmatrix} = \frac{e}{\hbar c_0} \sum_{LJM} c_{\alpha\beta}^{LJ} \begin{pmatrix}
R_{LM}(r) \\
\xi_{LJ}^{LJM} H_{LJM}(r)
\end{pmatrix}.
\]

Mixing channel of different AM breaks spherical symmetry.

§3. Electromagnetic Structure of NS

3.1. Electric Monopole Polarization

The electron system in the NS is quantum degenerate with ultra-relativistically high Fermi energy is confined in the star. The atomic nucleus fails to hold electrons inside of body size, e.g. 6 fm, is much smaller than Bohr radius. The size of NS seems large enough to accommodate electrons, but still some force is necessary. It is only Coulomb force, which is passed over behind the local charge neutrality anzats.

In practice the density of neutron is strongly dependent on radius together with those of proton and electron. Assuming Thomas-Fermi approximation, the potential \(U(r)\) is determined by Fermi energy depending on local density \(\rho_e(r)\) through
\[
U(r) = \sqrt{\left(\hbar c_0 (3\pi^2 \rho_e(r))^{1/3}\right)^2 + m_e^2 c_0^4}.
\]
Therefore proton density is mostly same as electron’s due to local charge neutralization. A slightly deferent charge distribution of proton from electron gives rise to monopole charge polarization \(\rho^{(pol)}(r)\), which is the source term of Poisson equation
\[
\varepsilon_0 \frac{1}{e} \frac{d}{dr} \frac{1}{r^2} \frac{d}{dr} U(r) = -\rho^{(pol)}(r).
\]
For the convenience of analysis of electric monopole field, let us introduce the following type of potential

$$U(r) = U_0 \left\{ 1 + \left( \frac{r}{R_*} \right)^\lambda \right\}^{-1}. \quad (3.2)$$

If $r$ is much larger than radius $R_*$, the potential decreases as $r^{-\lambda}$. For large $\lambda$, $U(r)$ is flat in the region of $r < R_*$ and decreases sharply in $r > R_*$. The radial component of electric field $E_r(r)$ is expressed as

$$E_r(r) = -\frac{U_0 \lambda}{er} \left( \frac{r}{R_*} \right)^\lambda \left\{ 1 + \left( \frac{r}{R_*} \right)^\lambda \right\}^{-2}, \quad (3.3)$$

the polarization charge density turns out to be

$$\rho^{(pol)}(r) = \frac{\varepsilon_0}{r} E_r(r) \left\{ (\lambda+1)-(\lambda-1) \left( \frac{r}{R_*} \right)^\lambda \right\} \left( 1+\left( \frac{r}{R_*} \right)^\lambda \right)^{-1}. \quad (3.4)$$

The electric field is the strongest at $r_{mx} = \lambda \sqrt{(\lambda - 1)/(\lambda + 1)} R_*$. In case of $\lambda = 4 (= 6)$, $r_{mx} = 0.88 R_*$ ($= 0.95 R_*$). The electric field and charge density at $r_{mx}$ are given by

$$E_r(r_{mx}) = \frac{U_0 \lambda^2}{4\varepsilon_0} \frac{1}{r_{mx}}, \quad \text{and} \quad \rho^{(pol)}(r_{mx}) = \frac{2\varepsilon_0}{r_{mx}} E_r(r_{mx}), \quad (3.5)$$

respectively. Let us take the values of $U_0$, $R_*$ and $\lambda$ as -160MeV, 12km and 4, respectively. Then, we obtain $r_{mx} = 10.6$ [km], $E_r^{(max)} = 1.4 \times 10^4$ [V/m], $\rho^{(pol)}(r_{mx}) = 2.4 \times 10^{-11}$ [C/m$^3$], corresponding to the electron number $1.5 \times 10^8$ [m$^{-3}$] = $1.5 \times 10^{-37}$ [fm$^3$]. This number is negligibly small compared to $\rho_e(r_{mx}) = 0.0044$ [fm$^{-3}$], namely $\rho^{(pol)}(r_{mx})/\rho_e(r_{mx}) = 3.4 \times 10^{-35}$.

This small number is found in gravitational force compared to electrostatic force between two protons, i.e., $Gm_p^2/(\varepsilon_0^2/4\pi\varepsilon_0) = 8.093 \times 10^{-35}$. So it is interesting to consider the Newtonian gravitational force and electrostatic force acting on a proton at $r_{mx}$. The former and the latter are estimated as 2.4 [f N] and 2.3 [f N], respectively, assuming mass inside of sphere of radius $r_{mx}$ to be 80%. This similarity in the strength seems more than a coincidence.

According to above considerations, it is seen that only a tiny fraction, $10^{-35}$, of electrons contributes to forming electrostatic field to confine the major part of the other high energy electrons. It would be thought the local neutrality satisfied. At the same time, the electric field induced by this polarization affects proton pushed out against the gravitational force. The proton density pushed out outer part of sphere produces a force pushing out the neutrons through the gradient of Lane potential i.e., the isovector potential, resulting a reduction of pressure inside of NS. The reduction of pressure may be important in the discussions on the equation of state for nuclear matter, which is intensively argued in the context of stability in the massive NS such as $2M_\odot$. It is necessary to study more systematically including general relativity together with electromagnetic force as well as hadronic matter.
3.2. Quasi Bound States

The electron bounded in the potential deeper than $2m_e c_0^2$ has always a probability to escape through Klein tunnel, similarly to alpha decay in nuclei heavier than mass number $A \approx 150$, where alpha particle decays through, let’s say, Gamow tunnel. To study the behavior of wave function in the Klein tunnel, we deal with Dirac equation (2.11) together with the central potential in eq. (3.2). The escaping probability may be reduced by tuning the energy $E$ to make the amplitude at a certain point $r_1$ small compared to the probability inside of it, that is

$$P_\kappa(r_1) = \frac{r_1 A^2_\kappa(r_1)}{\int_0^{r_1} A^2_\kappa(r) dr}.$$  \hspace{1cm} (3.6)

Here we describe a set of amplitudes of upper and lower component shown in the left-hand side (l.h.s.) of Fig.[1] as

\[
\begin{pmatrix}
G_\kappa(r) \\
F_\kappa(r)
\end{pmatrix} = A_\kappa(r) \begin{pmatrix}
\cos \theta_\kappa(r) \\
\sin \theta_\kappa(r)
\end{pmatrix}, \quad \text{with} \quad A_\kappa(r) = \sqrt{G_\kappa(r)^2 + F_\kappa(r)^2}, \hspace{1cm} (3.7)
\]

illustrated in the right-hand side (r.h.s.) of Fig. 1 for $\kappa = 5$ and $\nu = 2$. We call this representation as Angle-Amplitude representation, abbreviated as AAr. The orbit starts from the origin and, the radious $A(r)$ increases until the point between two points indicated by P1 and P2. From the Dirac equation (2.11), we obtain

$$\frac{1}{A_\kappa(r)} \frac{dA_\kappa(r)}{dr} = - \frac{\kappa}{r} \cos 2\theta_\kappa(r) + m \sin 2\theta_\kappa(r),$$  \hspace{1cm} (3.8)

and

$$\frac{d\theta_\kappa(r)}{dr} = -(\varepsilon - u(r)) + \frac{\kappa}{r} \sin 2\theta_\kappa(r) + m \cos 2\theta_\kappa(r),$$  \hspace{1cm} (3.9)

with $\varepsilon = E/h_\kappa c_0$, $u(r) = U(r)/h_\kappa c_0$ and $m = m_e c_0/h_\kappa$ respectively. The angle variable $\theta_\kappa(r)$, as a function of $r$ for a given $\varepsilon$, starts from $\pi/2$ at the origin $r = 0$, and

![Fig. 1. Wave functions of upper and lower component for $\kappa = 5, \nu = 2$ are depicted in two different ways in l.h.s. and their AAr. is illustrated in r.h.s.](image-url)
is decreasing, i.e., going around clockwise, until the radial coordinate $r$ gets closer to the point $r_c$ satisfying $\varepsilon = u(r_c)$. Then we set up the end point $r_t$ as $d\theta/dr|_{r=r_t} = 0$. The angle $\theta_{\kappa}(r_t)$ is monotonously decrease function of $\varepsilon$. The probability ratio $P_{\kappa}(r_t)$ oscillates as a function of $\theta(r_t)$ and reaches zero line when $dA/dr$ crosses the zero line with falling right shoulder as shown in Fig. 2 with thick arrows. We determine the energy of quasi bound states by tuning $\varepsilon$ to make $dA/dr|_{r=r_t} = 0$ hold.

Fig. 2. Probability ratio of the boundary point to that of inside of the point, $P_{\kappa}(r_t)$ is plotted as a function of $\theta_{\kappa}(r_t)$. Arrows are pointing the $\theta(r_t)$ at $dA/dr = 0$ for each radial node number $\nu$.

The radial node number $\nu$ is determined by counting points where the upper component $G_{\kappa}^{(\nu)}(r)$ becomes zero, the points indicated as N2 and N4 in Fig. 1.

3.3. Energy Level Density with different Scaling Factors

In order to examine the validity of the scaled $\hbar$ method, we solved Dirac equation with taking on different values of scaling factor $\gamma$ and obtained the quasi-bound states. Their energy spectra are plotted as a function of $\kappa$ for the scaling factors $\gamma = 1$ and $\gamma = 0.5$, in case of $U(0) = -160\text{MeV}$, in l.h.s. Fig. 3. The points of energy spectra are connected by lines for the same radial node number $\nu$, separately.

Fig. 3. Energy spectra versus $\kappa$ are plotted in l.h.s. for different scaling factors $\gamma = 1$ and $\gamma = 0.5$. The comparison of density radial distributions of two scaling factors to that of Thomas-Fermi approximation are shown in r.h.s.
for different two scaling factors. It is found in l.h.s. of Fig. 3 that the slopes are almost same, indicating that the scaled \( h \) method works well. In other words, angularly node number, i.e., AM and radial node number are proportional. As for the density distributions for different scaling factor, the smaller \( \gamma \) is more accurate than the large number, and the density profiles converge to Thomas-Fermi distribution shown in r.h.s. of Fig. 3.

3.4. Convection like Current in Rotating NS

The observed NS is usually rotating and therefore has the total AM. It is slightly decreasing in time due to electromagnetic radiation, but within a characteristic period of motion of materials in NS, it is conserved and is shared by the electron system. In a weak coupling limit, the total energy and the AM of materials, i.e., hadrons and electron are expressed as \( E_{\text{tot}} = E_h + E_e \) and \( I_{\text{tot}} = I_h + I_e \), respectively. Therefore to make the total energy of material minimum, angular velocity is considered as the Lagrange multiplier, a common variable for the total AM conservation including hadrons. The partial AM are determined so as to minimized the total energy like, e.g., \( \omega = \partial E_h / \partial I_h = \partial E_e / \partial I_e \). The moments of inertia may be given as \( J_i = I_i / \omega \).

For simplicity, the electron system is axially symmetry along the rotating axis. The effects of rotation with angular velocity \( \omega \), are taken into account by Coriolis force in the cranking model utilized in nuclear physics\(^\text{18}\) where the single particle energy \( E \) is just replaced by \( E - \hbar_c \omega_{\mu} \alpha \). A typical rotation period of magnetar may be 8 sec and therefore \( \omega \approx 1 \text{sec}^{-1} \) and \( |\hbar_c \omega_{\mu}| \approx 4 \text{kev} \). But this is too low and may be much higher when the NS and MF were created.

This Coriolis term, being time reversal odd, splits \( 2j_\alpha + 1 \) degenerate states in the \( j_\alpha \) level, namely higher \( \mu \) (positive) levels are pushed down and lower \( \mu \) (negative) levels are pushed up. Eventually, the lower \( \mu \) levels of occupied \( j_\alpha \) level in the vicinity of Fermi energy get out to unoccupied levels, while the higher \( \mu \) levels of unoccupied levels get into occupied levels. This rearrangement in occupations breaks spherical and time reversal symmetries of the entire system, and gives rise to circular current density in the region peripheral due to high-\( j \) and near the equatorial plane due to high \( \mu \) states involved in rearrangement through Coriolis forces.

When both of time reversal partners are occupied, these quantum states do not contribute to AM and current. Hence the highest \( j \) and the highest \( \mu = j \) state is relevant to the mechanism forming the MF. It is instructive to see the current produced by the configuration change in magnetic quantum number with leaving the radial wave function unchanged and without mixing orbital AM resulting deformation. From the dependence of \( \kappa \) and \( \kappa' \) in the factor of reduced matrix elements, \( \xi_{\kappa\kappa'}^{LJ} \) in eq. \( 2.33 \), it is seen that only magnetic odd multipolarity having even parity, i.e., \( J = L \), contributes to the current. In a axially symmetric system \( \mu \) is eigenvalue and therefore \( M = 0 \), so that the vector spherical harmonic under the condition has a following property,

\[
\mathbf{V}_{LL0}(\hat{r}) = -i \mathcal{Y}_{L1}(\theta \phi) n_\phi \quad \text{with} \quad n_\phi = (-\sin \phi, \cos \phi, 0).
\]

The probability current of orbital (\( \kappa = j + \frac{1}{2} \; \mu = j \)) has only a component of tangential
direction of circles and forms toroidal zone having density of cross section \( q_{\kappa \mu}(r\theta) \)

\[
j_{\kappa \mu}(r) = q_{\kappa \mu}(r\theta)n_\phi, \tag{3.11}
\]
is a function of \( r \) and \( \theta \) having the following separable form

\[
q_{\kappa \mu}(r\theta) = R_{\kappa \mu}(r)S_{\kappa \mu}(\theta), \tag{3.12}
\]
which is calculated as

\[
(r^2 R_{\kappa \mu}(r), S_{\kappa \mu}(\theta)) = \left( 2G^{(0)}_{\kappa \mu}(r)F^{(0)}_{\kappa \mu}(r), -\sum_{L=\text{odd}} \xi_{\kappa \kappa \mu} C^{LL}_{\kappa \kappa \mu} iY_L(\theta \theta) \right), \tag{3.13}
\]
and the radial wave functions \( G^{(0)}_{\kappa \mu}(r) \) and \( F^{(0)}_{\kappa \mu}(r) \) are depicted in l.h.s. of Fig. 4. The cross-section of convection like current \( q_{\kappa \mu}(r\theta) \), the radial form factor \( R_{\kappa \mu}(r) \) and angular form factor \( S_{\kappa \mu}(\theta) \) are shown in r.h.s. of Fig. 4.*

### 3.5. Circular Helicity in Weyl Equation

The symmetry breaking of chirality is brought about by the electron capture through parity violating weak interaction, the strong toroidal MF is intended to be explained by alignment of electron along the tangent of circular orbit, i.e., convection like current, in other words circular helicity. We will demonstrate the predominance of positive helicity states in the positive chiral states by means of a simple numerical calculation in this subsection. With neglecting mass term, Weyl equation for the two components \( H^{(c)}_{\kappa j\mu} \) is given

\[
\begin{pmatrix}
(-1)^c - \frac{h}{r} k_
u \frac{1}{r} + u(r) & \frac{d}{dr} \\
-\frac{d}{dr} & -(-1)^c - \frac{h}{r} k_
u \frac{1}{r} + u(r)
\end{pmatrix}
\begin{pmatrix}
H^{(c)}_{\frac{1}{2} j\mu}(r) \\
H^{(c)}_{\frac{1}{2} j\mu}(r)
\end{pmatrix} = \varepsilon
\begin{pmatrix}
H^{(c)}_{\frac{1}{2} j\mu}(r) \\
H^{(c)}_{\frac{1}{2} j\mu}(r)
\end{pmatrix}, \tag{3.14}
\]
where \( c = \pm \frac{1}{2} \) expresses positive and negative chiralities. The differentiate with respect to radial coordinate is off-diagonal and therefore mixing of circular helicity

---

* The decomposition of currents into convection and spin currents was defined by Gordon only for free particle. Since the terms used here is not exact, we say convection or spin “like” currents.
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is taken place through radial motion. It results reduction of purity of circular helicity for large radial node numbers \( \nu \). By using same potential \( u(r) \) as previous subsections and same technique with boundary condition in obtaining quasi-bound states, we solve eq. 3.14 to obtain the stationary states. The eigen values and the purity of helicity defined by

\[
h_{\kappa \nu} = \frac{\int_0^\infty dr \left[ H^{(2)}_{1/2}(r)^2 - H^{(1)}_{1/2}(r)^2 \right]}{\int_0^\infty dr \left[ H^{(2)}_{1/2}(r)^2 + H^{(1)}_{1/2}(r)^2 \right]}. \tag{3.15}
\]

Fig. 3.5 clearly shows the predominance of the same sign of circular helicity as chirality in the higher \( k \) with the less radial node number, particularly in the relevant orbit of the highest \( k = 10 \), the highest \( \mu = \frac{19}{2} \) and \( \nu = 0 \) forming the toroidal MF.

3.6. Helical Current

In Weyl representation, current density is calculated separately plus and minus chiralities like in eq. (2.20). The remnant of the reaction of electron capture is composed of positive chirality, so that neglecting mixing of chirality, we calculate the current using the wave function obtained in the former subsection.

A helicity representation state includes two states of different sign \( \kappa \). Hence, the probability current between different sign of \( \kappa \) states gives rise to magnetic odd moments, i.e. convection like circular current as shown in subsection 3.4.

The diagonal current of eigenstate of chirality are decomposed into two terms, namely, spin and convection like currents, such as

\[
j_k(r, \theta) = j_k^{(sp)}(r, \theta) + j_k^{(cv)}(r, \theta), \tag{3.16}
\]

which are expressed by

\[
\begin{align*}
  j_k^{(sp)}(r, \theta) &= c_0 \sum_{i=1}^2 R_i(r) \left( S^{(i)}_{\rho}(\theta) \hat{n}_\rho + S^{(i)}_{z}(\theta) \hat{n}_z \right), \\
  j_k^{(cv)}(r, \theta) &= c_0 R_3(r) S^{(3)}_{\phi}(\theta) \hat{n}_\phi
\end{align*}
\]
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Fig. 6. Density distribution of cross section of convection like current is shown l.h.s. and density distribution of spin like current is depicted in r.h.s.

where three radial form factors $R_i(r)$ are calculated by using two helicity component of wave functions $(H_{\frac{j}{2}}(r)H_{\frac{\mu}{2}}(r))$, omitting the superscript $(c) = (\frac{1}{2})$ and suffixes $j = \frac{19}{2}$ and $\mu = \frac{19}{2}$, as follows

\[
\begin{pmatrix}
R_1(r), R_2(r) \\
R_3(r)
\end{pmatrix} = \frac{1}{r^2} \begin{pmatrix}
H_{\frac{j}{2}}(r) + H_{\frac{\mu}{2}}(r), 2H_{\frac{j}{2}}(r)H_{\frac{\mu}{2}}(r), \\
H_{\frac{j}{2}}(r) - H_{\frac{\mu}{2}}(r)
\end{pmatrix},
\]

\[\text{(3.18)}\]

and angular dependent factors are calculated like

\[
\begin{pmatrix}
S_{\rho}^{(i)}(\theta) \\
S_{z}^{(i)}(\theta)
\end{pmatrix} = \sum_L \cos \left(\frac{L\pi}{2}\right) \begin{pmatrix}
S_{\rho L}^{(i)}Y_{L1}(\theta) \\
S_{z L}^{(i)}Y_{L0}(\theta)
\end{pmatrix},
\]

\[\text{(3.19)}\]

from following four coefficients

\[
\begin{pmatrix}
S_{\rho L}^{(1)} S_{\rho L}^{(2)} \\
S_{z L}^{(1)} S_{z L}^{(2)}
\end{pmatrix} = \sum_{J=L\pm 1} C_k^J \begin{pmatrix}
\pm \hat{J}C_j^{(\pm)}C_L^{(\pm)} & 2kC_j^{(\mp)}C_L^{(\mp)} \\
\pm \hat{J}C_j^{(\pm)}C_L^{(\mp)} & 2kC_j^{(\mp)}C_L^{(\mp)}
\end{pmatrix},
\]

\[\text{(3.20)}\]

with

\[
C_k^J = \langle jjj - j | J0 \rangle \delta_{kk}^J \hat{J}^{-1}.
\]

\[\text{(3.21)}\]

The spin like current is parallel to meridian plane having vector components $\mathbf{n}_\rho$ and $\mathbf{n}_z$ only, because of vector spherical harmonics for $L = J \pm 1$ and $M = 0$. The angular dependent factor of convection like current is given as

\[
S_{\phi}^{(3)}(\theta) = 2k \sum_L \cos \left(\frac{(L-1)\pi}{2}\right) C_k^L (-i)^{L1}(\theta).
\]

\[\text{(3.22)}\]

Eventually, the convection like current is perpendicular to meridional plane, while the spin like current is parallel to the plane. So that, two kinds of current are orthogonal to each other and form helical current. This helical current is very characteristics resulted from the Dirac current in asymmetric chirality states given rise
to by parity violating weak interaction. In the previous section, it is shown that the selective capture through parity violating weak interaction brings about a chiral asymmetric configuration in electrons. In the case of axial symmetry, the MF, therefore is perpendicular to the plane, having only component of $\phi$ which is function of $(r, \theta)$, i.e., $B_\phi(r, \theta)$ and is calculated, conveniently to use of cylindrical coordinates $(\rho = r \sin \theta, \phi, z = r \cos \theta)$ as

$$B_\phi(\rho, z) = \frac{e \mu_0}{2\pi \rho} \int_0^\rho j_z(\rho', z)(2\pi \rho')d\rho', \tag{3.23}$$

by Ampere’s law. The maximum strength occurs at circle of radius 8.7 km on the equatorial plane, and the strength turns out to be about $10^8$ TT. It is too strong compared to the analysis $\approx 1$ TT. This value is the contribution from the single orbit for the highest $k$ and the highest $\mu = j = k - \frac{1}{2}$ for a given scaling factor $\gamma = 0.5$. Therefore the final value of calculated strength is directly proportional to $N_\ast = \gamma^3 N_\ast$. In this paper, we take $0.5$ as the scaling factor and $k = 10$ because of a feasibility of available computation code. For the realistic calculation the factor should be taken in careful consideration, especially the angular velocity $\omega$ in Coriolis force.

3.7. Magnetic Helicity

The magnetic helicity is expressed by the volume integral of

$$h_m = \int_V A(r) \cdot B(r) dr, \quad \text{with} \quad B(r) = \nabla \times A(r). \tag{3.24}$$

The operator nabla, $\nabla$, changes the parity of the operand and therefore only the integral is nonzero for the cross terms of even and odd parity fields.

$$\int_V \left( A^{(\text{odd})}(r) \cdot B^{(\text{even})}(r) + A^{(\text{even})}(r) \cdot B^{(\text{odd})}(r) \right) dr \tag{3.25}$$

In the previous case of helical currents, the vector potential of the convection like current being magnetic odd multipolarity has even parity and spin like current electric odd multipolarity has odd parity. It was shown in subsection 3.6 that in the
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axially symmetric configuration considered hitherto, convection like current has only the component of the direction of azimuthal angle \((0, 0, A_{\phi}^{(cv)}(r, \theta))\), i.e., circular orbiting and the MFs parallel to the meridian plane \((B_{\theta}^{(cv)}(r, \theta), 0, 0)\), while the spin like current is parallel to the meridian plane \((A_{\phi}^{(sp)}(r, \theta), (0, B_{\phi}^{(sp)}(r, \theta))\) and the MF has only the component of \(\phi\) direction \((0, 0, B_{\phi}^{(sp)}(r, \theta))\).

Eventually, the magnetic helicity is contributed from the cross terms of the convection like current and the spin like current as follows

\[
\int_{V} \left( A^{(sp)}(r) \cdot B^{(cv)}(r) + A^{(cv)}(r) \cdot B^{(sp)}(r) \right) d^3r = 2\pi \int_{0}^{\infty} r^2 dr \times \int_{0}^{\pi} \sin \theta d\theta \left( A^{(sp)}_{\theta}(r \theta) \cdot B^{(cv)}_{\phi}(r \theta) + A^{(sp)}_{\phi}(r \theta) \cdot B^{(cv)}_{\theta}(r \theta) + A^{(cv)}_{\phi}(r \theta) \cdot B^{(sp)}_{\phi}(r \theta) \right)
\]

The both currents are composed by helicity positive electrons and therefore \(A^{(sp)}(r) \cdot B^{(cv)}(r)\) and \(A^{(cv)}(r) \cdot B^{(sp)}(r)\) are positive values. This leads to positive volume integral i.e., positive magnetic helicity and is resulted from the fact that neutrino has negative helicity in nature.

3.8. Selective Capture and Chiral Asymmetry

Now, we come to the final stage of the scenario for the mechanism forming the strong toroidal MF. The helical current composed from convection like current caused by orbiting electrons and spin like current generated by alignment of electron spin pointing to tangential of the circular orbit, in other words the magnetization current, plays an important role.

The orbiting electrons are captured by protons through parity violating weak interaction. While only electrons in negative chiral state selectively involve in the reaction, electrons having positive chirality remain in the orbiting zone. This results the chiral asymmetry, and therefor remaining electrons possess dominantly positive helicity. At high energy, electrons behave as massless particles so that the effect of mixing the chirality through the mass term becomes weak. The positive helical current gives rise to positive magnetic helicity, because of the similarity relation between the moments of current and vector potential.

Eventually MF with the positive magnetic helicity makes the electrons in positive chiral state stabilized, since the coupling matrices include the same geometrical factors \(\xi_{\kappa \kappa'}\) in \(2.33\) and \(2.39\). In the spinor representation \(\chi_{\kappa \mu}(\hat{r}, \sigma)\), the spherical harmonics includes standing wave along the polar angle \(\theta\), of which the node number is \(\ell_{\kappa} = |m_{\mu}|\). Eventually, large \(\mu\) states, i.e., small node number states is more likely to be pure helicity state, that we have demonstrated in the subsection 3.6 the relevant states, the highest \(k = |\kappa|\) and the highest positive \(\mu = j\), to forming the toroidal MF the sign of the chirality corresponds to that of helicity, and therefore asymmetric population brought about by electron capture which is major process in gravitation collapse results helical structure in MF. Especially, the spin like current, part of Dirac current, parallel to meridian plane generates a strong toroidal MF, that is the subject of present paper.
§4. Discussions

In this work, we have theoretically studied the global structure of magnetic field and electric current for a typical size of magnetar of $M = 1.5M_{\odot}$ and $R = 12$km, and found that the electrons as an entire system can be considered to be quantum degenerate with ultra-relativistic Fermi energy. In case of rotating NS with angular velocity $\omega$ working as Lagrange multiplier in energy minimization with conservation of the total AM cranking term $-\hbar \omega \mu$ breaking time reversal, splits energy of single quantum state proportional to the magnetic quantum number $\mu$. In the vicinity of Fermi surface, the migration of a set of electrons in occupation from lower $\mu$ (negative) of occupied states to higher $\mu$ (positive) of unoccupied states, gives rise to current density in peripheral region near equatorial plane. This current is convection like current producing poloidal MF consisting of the magnetic odd components of vector potential.

The selective electron capture through parity violating weak interaction leaves positive chirality electrons having dominantly positive circular helicity as remnant. This process takes place spin alignment along tangential direction of the circular orbit. The MF produced in this way is toroidal due to the spin like current at the boundary of torus within the meridian plane, constituting of the electric odd components of vector potential.

The magnetic helicity is determined by the relative phase between convection like and spin like currents, which depends on the circular helicity of electrons. Therefore, the self-interaction between currents by aligned spin electron and toroidal MF stabilizes chiral asymmetry against the chiral mixing effect from mass term, because a spin flip to change helicity costs energy to jump up to the higher Landau levels. Once chiral asymmetric configuration dominate, the selective capture of helicity negative electron is enhanced because of raising the single particle energy by the strong toroidal field. According to the present theory, the magnetic helicity is positive and therefore the relative directions of the poloidal and the toroidal MF are determined.

One limitation in this paper is that we did not consider about the hadronic system in detail. This shall be studied in different papers. The proton circular current acts as repulsive to the convection like current of electrons. As the result, two polar axes slightly split in the balance with the increasing of additional rotational energies caused by the splitting. This splitting gives rise to precession motion, which may explain the phase modulation of hard X-ray from soft X-ray period observed by Makishima et. al\(^{10}\). A time dependent theoretical calculation may be necessary in the process of electron capture increasing chiral asymmetry and helical current to stabilize the toroidal MF.

Another limitation is that we completely ignored the general relativity which may play an important role. In order to tackle this limitation, the metric tensor has to be obtained possibly by using TOV’s equation, and Maxwell equation has to be
solved in the curved space.

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References

1) A. Harding and D. Lai, Rep. Prog. Phys, 69 (2006) 2631
2) R.C. Duncan and C. Thompson, Astrophys. J. 392 (1992) 9
3) K. Makishima et al., Phys. Rev. Lett. 112 (2014) 171102
4) A. Ohnishi and N. Yamamoto, (2014), arXiv: 14024769[astro-ph.HE]
5) D. Grabowska, D.B. Kaplan and S. Reddy, (2014), arXiv: 1409.3602v1 [hep-ph]
6) W. Gordon, Z. Physik 50 (1928) 630
7) P.A.M. Dirac, Proc. Roy. Soc (London) 117 (1928) 610
8) A.M. Lane, Phys. Rev. Lett. 8 (1962) 171
9) G. Bertsch, N. Onishi, and K. Yabana, Z. Phys. D34 (1995)
10) M. E. Rose, Phys. Rev. 51 (1937) 484
11) M. Jacob and G. C. Wick, Ann. Phys. (NY) 7 (1959) 404
12) M. E. Rose, Relativistic Electron Theory John Wiley & Sons, INC (1961)
13) K.T. Hecht, Quantum Mechanics Springer-Verlag, New York (2000)
14) W. Gordon, Z. Physik 48 (1928) 11
15) H. Weyl, Zeit. f. Phys. 56 (1929) 330
16) A. Bohr and B.R. Mottelson, Nuclear Structure Vol I, W.A. Benjamin, INC (1969)
17) J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics Wiley, New York (1952)
Appendix B
18) D. R. Inglis, Phys. Rev 103 (1956) 1786