Arithmetic Milnor invariants and multiple power residue symbols in number fields

Fumiya Amano and Masanori Morishita

Dedicated to Professor Takayuki Oda

Abstract. We introduce arithmetic Milnor invariants and multiple power residue symbols for primes in number fields, following the analogies between primes and knots. Our symbols generalize the Legendre, power residue symbols and the Rédei triple symbol, and describe the decomposition law of a prime in certain nilpotent extensions of number fields. As a new example, we deal with triple cubic residue symbols by constructing concretely Heisenberg extensions of degree 27 over the cubic cyclotomic field with prescribed ramification. We also give a cohomological interpretation of our multiple power residue symbols by Massey products in étale cohomology.

Introduction

The present paper is concerned with a multiple (higher order) generalization of the power residue symbol in a number field. It is L. Rédei who firstly studied such a generalization of the Legendre symbol in 1939 ([Rd]). Aiming to generalize the arithmetic of quadratic fields such as the theory of genera initiated by Gauss ([G]), Rédei introduced a triple symbol \([p_1, p_2, p_3]\) for certain rational primes \(p_1, p_2, p_3\), which describes the decomposition law of \(p_3\) in a certain dihedral extension \(\mathcal{R}\), determined by \(p_1\) and \(p_2\), of degree 8 over the rationals \(\mathbb{Q}\). Here the Rédei extension \(\mathcal{R}\) is given concretely by

\[
\mathcal{R} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}),
\]

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where \( \alpha = x + y \sqrt{p_1} \) and \( x, y \) are certain integers satisfying \( x^2 - p_1 y^2 - p_2 z^2 = 0 \) with some non-zero integer \( z \) so that the extension \( \mathcal{R}/\mathbb{Q} \) is unramified outside \( p_1, p_2 \) and the infinite prime with ramification index for each \( p_i \) being 2. It might not be clear, however, why such a dihedral extension and triple symbol should be considered as a natural generalization of a quadratic field and the Legendre symbol, and it seemed that his work had been overlooked for a long time (except some related works \([Fö], [Fu]\) etc).

In the late 1990s, M. Kapranov and the second author independently interpreted the Rédei symbol as an arithmetic analogue of a triple linking number in 3-dimensional topology, and further the second author introduced arithmetic analogues for rational primes of the Milnor invariants (higher order linking numbers) for a link in the 3-sphere ([Mi2]), based on the analogies between primes and knots in arithmetic topology ([Ka], [Mo1]∼[Mo4], [Rz]). For example, the mod 2 arithmetic Milnor invariant \( \mu_2(12\cdots n) \in \mathbb{Z}/2\mathbb{Z} \) of length \( n \geq 2 \) for certain rational primes \( p_1, p_2, \ldots, p_n \) describes the decomposition law of \( p_n \) in a certain nilpotent extension \( \mathcal{K}(n) \), determined by \( p_1, \ldots, p_{n-1} \), of degree \( 2^\frac{1}{2}n(n-1) \) over \( \mathbb{Q} \), where the extension \( \mathcal{K}(n)/\mathbb{Q} \) is unramified outside \( p_1, p_2, \ldots, p_{n-1} \) and the infinite prime with ramification index for each \( p_i \) being 2, and the Galois group \( \text{Gal}(\mathcal{K}(n)/\mathbb{Q}) \) is isomorphic to the group of \( n \) by \( n \) upper-triangular unipotent matrices over \( \mathbb{F}_2 \). In particular, when \( n = 2 \) and 3, the arithmetic Milnor invariants \( \mu_2(12) \) and \( \mu_2(123) \) give the Legendre symbol \( \left( \frac{p_1}{p_2} \right) \) and the Rédei symbol \( [p_1, p_2, p_3] \), respectively, in the relations

\[
(0.2) \quad (-1)^{\mu_2(12)} = \left( \frac{p_1}{p_2} \right), \quad (-1)^{\mu_2(123)} = [p_1, p_2, p_3],
\]

and further it is also shown that

\[
(0.3) \quad \mathcal{R}(2) = \mathbb{Q}(\sqrt{p_1}), \quad \mathcal{R}(3) = \mathcal{R}
\]

for \( p_i \equiv 1 \mod 4 \) ([A1], [Mo1]∼[Mo3]). This unified interpretation may tell us that Rédei’s dihedral extension and triple symbol would be a natural generalization of a quadratic field and the Legendre symbol.

However, these results are concerned with only rational primes, because they are based on an analogy between the structure of a certain pro-2 Galois group over \( \mathbb{Q} \) with restricted ramification and that of the group of a link in the 3-sphere and there are obstructions to extend this analogy for number
fields. It was expected that we could introduce the arithmetic Milnor invariants at least for primes in a number field whose $l$-class group ($l$-primary part of the ideal class group) is trivial, where $l$ is a prime number, since such a number field corresponds to an $l$-homology 3-sphere in the analogy of arithmetic topology and the topological Milnor invariants are well defined for a link in an $l$-homology 3-sphere ([Tu]).

In this paper, we introduce the mod $m$ arithmetic Milnor invariants $\mu_m(12 \cdots n) \in \mathbb{Z}/m\mathbb{Z}$ ($m$ being a power of $l$) for finite primes $p_1, p_2, \ldots, p_n$ in a number field $k$ which contains a primitive $m$-th root of unity $\zeta_m$ and whose $l$-class group is trivial. Although there is an obstruction for a pro-$l$ Galois group over such a number field $k$ with restricted ramification to be a link group like, we overcome this difficulty by enlarging the given set of primes $p_1, p_2, \ldots, p_n$ and then by showing that our invariants are independent of the choice of added auxiliary primes. Using the arithmetic Milnor invariant, we introduce the $n$-tuple $m$-th power residue symbol by

$$[p_1, p_2, \ldots, p_n]_m := \zeta_m(12 \cdots n)$$

in the way generalizing (0.2), and show that it describes the decomposition law of $p_n$ in a certain nilpotent extension $\mathfrak{R}_m(n)$, determined by $p_1, \ldots, p_{n-1}$ and $m$, of degree $m^{\frac{1}{n}(n-1)}$ over $k$, where the extension $\mathfrak{R}_m(n)/k$ is unramified outside $p_1, p_2, \ldots, p_{n-1}$ and the infinite primes with ramification index for each $p_i$ being $m$, and the Galois group $\text{Gal}(\mathfrak{R}_m(n)/k)$ is isomorphic to the group of $n$ by $n$ upper-triangular unipotent matrices over $\mathbb{Z}/m\mathbb{Z}$. When $k = \mathbb{Q}$, $\mathfrak{R}_2(n)$ coincides with the above $\mathfrak{R}(n)$.

As a new concrete example, we deal with the triple cubic residue symbol $[p_1, p_2, p_3]_3$ for certain primes $p_i = (\pi_i)$ of the cubic cyclotomic field $\mathbb{Q}(\zeta_3)$. For this, we construct concretely the extension $\mathfrak{R}_3(3)$ over $\mathbb{Q}(\zeta_3)$ as

$$(0.4) \quad \mathfrak{R}_3(3) = \mathbb{Q}(\zeta_3)(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta}),$$

where $\theta = x + y\sqrt[3]{\pi_1} + z(\sqrt[3]{\pi_2})^2$ and $x, y, z$ are certain algebraic integers in $\mathbb{Z}[\zeta_3]$ satisfying $x^3 + \pi_1 y^3 + \pi_1^2 z^3 - 3\pi_1 xyz - \pi_2^3 w^3 = 0$ with some $w \in \mathbb{Z}[\zeta_3]$, so that (0.4) may be regarded as a generalization over $\mathbb{Q}(\zeta_3)$ of the equality (0.3) for the Rédei extension $\mathfrak{R}$ in (0.1).

The power residue symbol is known to be described by the cup product in Galois cohomology ([Ko1; 8.11], [Se; XIV, §2]). We generalize this relation
for our multiple power residue symbols by describing them by Massey products in étale cohomology. It is also an extension of our earlier work ([Mo3]) in the case of the rationals to a number field.

One direction in the developments of algebraic number theory after the study of quadratic fields \( \mathbb{Q}(\sqrt{p}) \) has been the study of \( p \)-th cyclotomic fields \( \mathbb{Q}(\zeta_p) \) and the tower of \( p^n \)-th cyclotomic fields \( \mathbb{Q}(\zeta_{p^n}) \) such as Iwasawa theory ([Iw3]), which was motivated by the analogies with function fields. Our present work may suggest another direction of development of classical algebraic number theory, the study of the tower of nilpotent extensions \( \mathbb{K}_m(n) \) and multiple power residue symbols, which was originated from the works of Gauss and Rédei, and motivated by the analogies with knot and link theory.

Here are contents of this paper. In Section 1, we introduce the notion of a pro-\( l \) Galois group of link type, a class of pro-\( l \) Galois groups over number fields with restricted ramification, and recall a theorem by Koch on the structure of such a Galois group, from the view point of the analogy with the theorem by Milnor and Turaev on a link group. In Section 2, we introduce the mod \( m \) arithmetic Milnor invariants for primes \( p_1, \ldots, p_n \) in a number field \( k \) which contains a primitive \( m \)-th root of unity and whose \( l \)-class number is one, and show some basic properties of them. In Section 3, using the arithmetic Milnor invariants, we introduce the \( n \)-tuple \( m \)-th power residue symbol \([p_1, \ldots, p_n]_m\) which describe the decomposition law of \( p_n \) in a certain nilpotent extension \( \mathbb{K}_m(n) \) over \( k \), and we discuss the uniqueness and construction of \( \mathbb{K}_m(2) \) and \( \mathbb{K}_2(3) \) for the power residue symbol and the Rédei symbol, respectively. In Section 4, as a new example of multiple power residue symbols, we deal with the triple cubic residue symbols by constructing concretely Heisenberg extensions of degree 27 over the cubic cyclotomic field. In Section 4, we give a cohomological interpretation of our multiple power residue symbols by Massey products in étale cohomology.

Throughout this paper, let \( l \) denote a fixed prime number.

§1 Pro-\( l \) Galois groups of link type

In this section, we recall a theorem by Koch ([Ko1], [Ko2]) on the structure of certain pro-\( l \) Galois groups over number fields with restricted ramification, which is analogous to a theorem by Milnor and Turaev ([Mi1], [Mi2],
[Tu]) on the structure of the group of a link in an $l$-homology 3-sphere. We then give a sufficient condition for the vanishing of an obstruction involved in Koch’s theorem.

Let $m$ be a fixed positive power of $l$. Let $k$ be a finite algebraic number field containing a fixed primitive $m$-th root of unity $\zeta_m$. Let $\mathcal{O}_k$ denote the ring of algebraic integers in $k$ and let $N\mathfrak{a}$ denote the norm of an ideal $\mathfrak{a}$ of $\mathcal{O}_k$. Let $S_{k}^{\text{non-l}}$ denote the set of finite primes of $k$ (maximal ideals of $\mathcal{O}_k$) which are not lying over $l$. We start to note the elementary

**Lemma 1.1.** For a finite prime $p$ of $k$, $p$ is in $S_{k}^{\text{non-l}}$ if and only if $Np \equiv 1 \pmod{m}$.

**Proof.** Suppose $p \in S_{k}^{\text{non-l}}$. By the assumption, $k$ contains the $m$-th cyclotomic field $\mathbb{Q}(\zeta_m)$. Let $p'$ and $(p)$ denote the maximal ideals of $\mathbb{Z}[\zeta_m]$ and $\mathbb{Z}$, respectively, which are lying below $p$. Let $f$ denote the degree of the residue field $\mathbb{Z}[\zeta_m]/p'$ over $\mathbb{Z}/(p)$. Then we have $Np' = p^f$ and the Frobenius automorphism $\sigma_p$ over $p$ has the order $f$ in the Galois group $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. On the other hand, we have the isomorphism $(\mathbb{Z}/m\mathbb{Z})^\times \cong \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ where $p$ mod $m$ is sent to $\sigma_p$. Therefore $p$ mod $m$ has the order $f$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ and hence $Np' = p^f \equiv 1 \pmod{m}$. Since $Np$ is a power of $Np'$, we have $Np \equiv 1 \pmod{m}$. Conversely, suppose $Np \equiv 1 \pmod{m}$. Since $Np$ is prime to $l$, $p$ is in $S_{k}^{\text{non-l}}$. $\Box$

Let $S$ be a finite subset of $S_{k}^{\text{non-l}}$ consisting of $s$ distinct finite primes, $S := \{p_1, \ldots, p_s\}$, $s = \#S$. We have $Np_i \equiv 1 \pmod{m}$ ($1 \leq i \leq s$) by Lemma 1.1. Let $k_S(l)$ denote the maximal pro-$l$ extension of $k$, unramified outside $S \cup S_{k}^\infty$, in a fixed algebraic closure $\overline{k}$, where $S_{k}^\infty$ denotes the set of infinite primes of $k$. Let $\mathcal{G}_{k_S}(l)$ denote the Galois group of $k_S(l)$ over $k$, $\mathcal{G}_{k,S}(l) := \text{Gal}(k_S(l)/k)$. We describe the structure of the pro-$l$ group $\mathcal{G}_{k,S}(l)$ in a certain unobstructed case. For this, we first recall a result due to Iwasawa on the local Galois group ([Iw2]). For each $i$ ($1 \leq i \leq s$), let $k_{p_i}$ be the $p_i$-adic field with a fixed prime element $\pi_i$. We fix an algebraic closure $\overline{k}_{p_i}$ of $k_{p_i}$ and an embedding $\overline{k} \hookrightarrow \overline{k}_{p_i}$. Let $k_{p_i}(l)$ denote the maximal pro-$l$ extension of $k_{p_i}$ in $\overline{k}_{p_i}$ and $\mathcal{G}_{k_{p_i}}(l)$ denote the Galois group of $k_{p_i}(l)$ over $k_{p_i}$, $\mathcal{G}_{k_{p_i}}(l) := \text{Gal}(k_{p_i}(l)/k_{p_i})$. Then we have

$$k_{p_i}(l) = k_{p_i}(\zeta_{a_i}, \sqrt[n]{{\pi_i}} | a_i \geq 1),$$
where \( \zeta_l \) denotes a primitive \( l^a \)-th root of unity in \( \bar{k} \) such that \((\zeta_l)^{b} = \zeta_{l^{b-c}} \) for all \( b \geq c \). We also impose the condition that

\[
(1.2) \quad \zeta_l = (\zeta_m)^{m/l^a} \text{ if } m \text{ is divisible by } l^a
\]

with the fixed primitive \( m \)-th root of unity \( \zeta_m \) in \( k \). The local Galois group \( \mathfrak{G}_{k_{p_i}}(l) \) is then topologically generated by the monodromy \( \tau_i \) and (an extension of) the Frobenius automorphism \( \sigma_i \) which are defined by

\[
(1.3) \quad \begin{align*}
\tau_i(\zeta_m) &:= \zeta_m, \\
\tau_i\left(\sqrt[l^a]{\pi_i}\right) &:= \zeta_m \sqrt[l^a]{\pi_i}, \\
\sigma_i(\zeta_m) &:= \zeta_{m^{N_{p_i}}}, \\
\sigma_i\left(\sqrt[l^a]{\pi_i}\right) &:= \sqrt[l^a]{\pi_i}
\end{align*}
\]

and subject to the relation

\[
(1.4) \quad \tau_i^{N_{p_i}-1}[\tau_i, \sigma_i] = 1,
\]

where \([x, y]\) denotes the commutator \( xyx^{-1}y^{-1} \). For each \( i \) (\( 1 \leq i \leq s \)), the fixed embedding \( k \hookrightarrow \bar{k}_{p_i} \) gives an embedding \( k_S(l) \hookrightarrow k_{p_i}(l) \), hence a prime \( \mathfrak{p}_i \) of \( k_S(l) \) lying over \( p_i \). We denote by the same letters \( \tau_i \) and \( \sigma_i \) the images of \( \tau_i \) and \( \sigma_i \) under \( \varphi_{p_i,S} \), respectively, under the homomorphism

\[
\varphi_{p_i,S} : \mathfrak{G}_{k_{p_i}}(l) \longrightarrow \mathfrak{G}_{k,S}(l)
\]

induced by the embedding \( k_S(l) \hookrightarrow k_{p_i}(l) \). Then \( \tau_i \) is a topological generator of the inertia group of the prime \( \mathfrak{p}_i \) and \( \sigma_i \) is an extension of the Frobenius automorphism of the maximal subextension of \( k_S(l)/k \) for which \( \mathfrak{p}_i \) is unramified. We call simply \( \tau_i \) and \( \sigma_i \) a monodromy over \( p_i \) in \( k_S(l)/k \) and a Frobenius automorphism over \( p_i \) in \( k_S(l)/k \), respectively.

Throughout this paper, we always assume that the \( l \)-class number of \( k \) is one, namely, the \( l \)-class group \( H_k(l) \) (the \( l \)-primary part of the ideal class group \( H_k \)) of \( k \) is trivial:

\[
(1.5) \quad H_k(l) = \{1\}.
\]

By this assumption (1.5) and class field theory, the global Galois group \( \mathfrak{G}_{k_S}(l) \) is topologically generated by \( \tau_1, \ldots, \tau_s \). Then the Galois group \( \mathfrak{G}_{k_S}(l) \) is said to be of link type if the relations among \( \tau_i \)'s are given by the local relations (1.4) for all \( 1 \leq i \leq s \), namely, the pro-\( l \) group \( \mathfrak{G}_{k_S}(l) \) has the following presentation

\[
\mathfrak{G}_{k_S}(l) = \langle x_1, \ldots, x_s \mid x_1^{N_{p_i}-1}[x_1, y_1] = \cdots = x_s^{N_{p_i}-1}[x_s, y_s] = 1 \rangle,
\]

where \( N_{p_i} \) denotes the \( l \)-class number of \( k_{p_i} \). We also impose the condition that

\[
(1.6) \quad \zeta_{l^a} = (\zeta_{m^{N_{p_i}}})^{m/l^a} \text{ if } m \text{ is divisible by } l^a
\]

with the fixed primitive \( m \)-th root of unity \( \zeta_m \) in \( k_{p_i} \). The local Galois group \( \mathfrak{G}_{k_{p_i}}(l) \) is then topologically generated by the monodromy \( \tau_i \) and (an extension of) the Frobenius automorphism \( \sigma_i \) which are defined by

\[
(1.7) \quad \begin{align*}
\tau_i(\zeta_m) &:= \zeta_m, \\
\tau_i\left(\sqrt[l^a]{\pi_i}\right) &:= \zeta_m \sqrt[l^a]{\pi_i}, \\
\sigma_i(\zeta_m) &:= \zeta_{m^{N_{p_i}}}, \\
\sigma_i\left(\sqrt[l^a]{\pi_i}\right) &:= \sqrt[l^a]{\pi_i}
\end{align*}
\]

and subject to the relation

\[
(1.8) \quad \tau_i^{N_{p_i}-1}[\tau_i, \sigma_i] = 1,
\]

where \([x, y]\) denotes the commutator \( xyx^{-1}y^{-1} \). For each \( i \) (\( 1 \leq i \leq s \)), the fixed embedding \( k_S(l) \hookrightarrow k_{p_i}(l) \) gives an embedding \( k_S(l) \hookrightarrow k_{p_i}(l) \), hence a prime \( \mathfrak{p}_i \) of \( k_S(l) \) lying over \( p_i \). We denote by the same letters \( \tau_i \) and \( \sigma_i \) the images of \( \tau_i \) and \( \sigma_i \) under \( \varphi_{p_i,S} \), respectively, under the homomorphism

\[
\varphi_{p_i,S} : \mathfrak{G}_{k_{p_i}}(l) \longrightarrow \mathfrak{G}_{k,S}(l)
\]

induced by the embedding \( k_S(l) \hookrightarrow k_{p_i}(l) \). Then \( \tau_i \) is a topological generator of the inertia group of the prime \( \mathfrak{p}_i \) and \( \sigma_i \) is an extension of the Frobenius automorphism of the maximal subextension of \( k_S(l)/k \) for which \( \mathfrak{p}_i \) is unramified. We call simply \( \tau_i \) and \( \sigma_i \) a monodromy over \( p_i \) in \( k_S(l)/k \) and a Frobenius automorphism over \( p_i \) in \( k_S(l)/k \), respectively.

Throughout this paper, we always assume that the \( l \)-class number of \( k \) is one, namely, the \( l \)-class group \( H_k(l) \) (the \( l \)-primary part of the ideal class group \( H_k \)) of \( k \) is trivial:

\[
(1.9) \quad H_k(l) = \{1\}.
\]

By this assumption (1.9) and class field theory, the global Galois group \( \mathfrak{G}_{k_S}(l) \) is topologically generated by \( \tau_1, \ldots, \tau_s \). Then the Galois group \( \mathfrak{G}_{k_S}(l) \) is said to be of link type if the relations among \( \tau_i \)'s are given by the local relations (1.8) for all \( 1 \leq i \leq s \), namely, the pro-\( l \) group \( \mathfrak{G}_{k_S}(l) \) has the following presentation

\[
\mathfrak{G}_{k_S}(l) = \langle x_1, \ldots, x_s \mid x_1^{N_{p_i}-1}[x_1, y_1] = \cdots = x_s^{N_{p_i}-1}[x_s, y_s] = 1 \rangle,
\]
where $x_i$ denotes the word which represents a monodromy over $p_i$ in $k_S(l)/k$ and $y_i$ denotes the free pro-$l$ word of $x_1, \ldots, x_n$ which represents a Frobenius automorphism over $p_i$ in $k_S(l)/k$. It was shown by Höchsmann ([H]) and by Koch ([Ko1, Satz 6.11]) that the relations among $\tau_i$'s are given by the local relations (1.4) for all $1 \leq i \leq s$ if and only if the localization homomorphism, denoted by $\varphi^*_S$, of the 2nd continuous cohomology groups with coefficients in $\mathbb{F}_l := \mathbb{Z}/l\mathbb{Z}$, which is induced by $\varphi_i$ for $1 \leq i \leq s$,

$$\varphi^*_S : H^2(G_{k,S}(l), \mathbb{F}_l) \longrightarrow \prod_{i=1}^s H^2(G_{k_{p_i}}(l), \mathbb{F}_l)$$

is injective. Koch also gave an estimate of $\text{Ker}(\varphi^*_S)$, namely, an embedding of $\text{Ker}(\varphi^*_S)$ into the Pontryagin dual of a certain Abelian group $B_S$ which is defined as follows ([K1, Satz 11.3]): We firstly define the subgroup $V_S$ of $k^\times$ by

\begin{equation}
V_S := \{ a \in k^\times \mid aO_k = a^l, \ a \in (k_p^\times)^l \ (p \in S \cup S_\infty^\infty) \},
\end{equation}

where $a$ is a fractional ideal of $O_k$, and we define $B_S$ by

\begin{equation}
B_S := V_S/(k^\times)^l.
\end{equation}

Summing up, a theorem by Koch is then stated as follows.

**Theorem 1.8.** Notations being as above, we assume that $H_k(l) = \{ 1 \}$ and $B_S = \{ 1 \}$. Then the pro-$l$ Galois group $G_{k,S}(l)$ is of link type, namely, it has the following presentation

$$G_{k,S}(l) = \langle x_1, \ldots, x_s \mid x_1^{N_{p_1} - 1}[x_1, y_1] = \cdots = x_s^{N_{p_s} - 1}[x_s, y_s] = 1 \rangle,$$

where $x_i$ denotes the word which represents a monodromy over $p_i$ in $k_S(l)/k$ and $y_i$ denotes the free pro-$l$ word of $x_1, \ldots, x_n$ which represents a Frobenius automorphism over $p_i$ in $k_S(l)/k$.

**Remark 1.9.** (1) We note that the Galois group $G_{k,S}(l)$ is the maximal pro-$l$ quotient of the étale fundamental group of the complement of $S$ in $\text{Spec}(O_k)$, $G_{k,S}(l) = \pi_1^{\text{ét}}(\text{Spec}(O_k) \setminus S)(l)$. Following the analogies between knots and primes, 3-manifolds and number rings, Theorem 1.7 may be regarded as an arithmetic analogue of the following theorem by Milnor and Turaev on link
groups ([Mi1], [Mi2], [Tu]). Let $M$ be an $l$-homology 3-sphere, namely, a closed oriented 3-manifold with $H_1(M; \mathbb{F}_l) = \{1\}$. Let $\mathcal{L} := K_1 \cup \cdots \cup K_s$ be a link of $s$ components in $M$. For each $i$ ($1 \leq i \leq s$), let $\partial V_{K_i}$ be the boundary of a tubular neighborhood of $K_i$ and let $G_{K_i}$ denote the fundamental group of $\partial V_{K_i}$, $G_{K_i} := \pi_1(\partial V_{K_i})$. Then $G_{K_i}$ is generated by a meridian $\alpha_i$ and a longitude (parallel) $\beta_i$ with the relation

\[(1.9.1) \quad [\alpha_i, \beta_i] = 1.
\]

Let $G_{\mathcal{L}, M}$ denote the link group of $\mathcal{L}$, namely, the fundamental group of the complement of $\mathcal{L}$ in $M$, $G_{\mathcal{L}, M} := \pi_1(M \setminus \mathcal{L})$. For each $i$ ($1 \leq i \leq s$), the inclusion $\partial V_{K_i} \hookrightarrow M \setminus \mathcal{L}$ induces the homomorphism

$$\varphi_{K_i, \mathcal{L}} : G_{K_i} \longrightarrow G_{\mathcal{L}, M}.$$  

The images of $\alpha_i$ and $\beta_i$ under $\varphi_{K_i, \mathcal{L}}$ is denoted by the same letters $\alpha_i$ and $\beta_i$, respectively, which are called a meridian around $K_i$ and a longitude (parallel) around $K_i$, respectively. Then a theorem of Milnor and Turaev asserts that a nilpotent quotient of $G_{\mathcal{L}}$ is generated by the images of $\alpha_1, \ldots, \alpha_s$ and the relations among those images of $\alpha_i$’s are given by the local relations (1.8.1) for all $1 \leq i \leq n$. In particular, the pro-$l$ completion $\mathfrak{G}_{\mathcal{L}, M}(l)$ of $G_{\mathcal{L}}$ has the following presentation

$$\mathfrak{G}_{\mathcal{L}, M}(l) = \langle x_1, \ldots, x_s \mid [x_1, y_1] = \cdots = [x_s, y_s] = 1 \rangle,$$

where $x_i$ denotes the word which represents a meridian $\alpha_i$ around $K_i$ and $y_i$ denotes the free pro-$l$ word of $x_1, \ldots, x_s$ which represents a longitude $\beta_i$ around $K_i$. Here is a dictionary of the analogous objects:

| finite set of primes | link |
|----------------------|------|
| $\mathcal{L} = K_1 \cup \cdots \cup K_s$ | \[ \mathcal{L} = K_1 \cup \cdots \cup K_s \] in a 3-manifolds $M$ |

| pro-$l$ local Galois group | peripheral group |
|----------------------------|------------------|
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p))(l)$ | $G_{K_i} = \pi_1(\partial V_{K_i}) = \langle x_i, y_i \mid [x_i, y_i] = 1 \rangle$ |
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p))(l)$ | $x_i$ : a meridian around $K_i$ |
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p))(l)$ | $y_i$ : a longitude around $K_i$ |

| pro-$l$ Galois group with restricted ramification | pro-$l$ completion of link group |
|-------------------------------------------------|---------------------------------|
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p) \setminus S)(l)$ | $\mathfrak{G}_{\mathcal{L}, M}(l) = \pi_1(M \setminus \mathcal{L})^\wedge(l)$ |
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p) \setminus S)(l)$ | $\mathfrak{G}_{\mathcal{L}, M}(l) = \pi_1(M \setminus \mathcal{L})^\wedge(l)$ |
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p) \setminus S)(l)$ | $\mathfrak{G}_{\mathcal{L}, M}(l) = \pi_1(M \setminus \mathcal{L})^\wedge(l)$ |
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p) \setminus S)(l)$ | $\mathfrak{G}_{\mathcal{L}, M}(l) = \pi_1(M \setminus \mathcal{L})^\wedge(l)$ |

| pro-$l$ Galois group with restricted ramification | pro-$l$ completion of link group |
|-------------------------------------------------|---------------------------------|
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p) \setminus S)(l)$ | $\mathfrak{G}_{\mathcal{L}, M}(l) = \pi_1(M \setminus \mathcal{L})^\wedge(l)$ |
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p) \setminus S)(l)$ | $\mathfrak{G}_{\mathcal{L}, M}(l) = \pi_1(M \setminus \mathcal{L})^\wedge(l)$ |
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p) \setminus S)(l)$ | $\mathfrak{G}_{\mathcal{L}, M}(l) = \pi_1(M \setminus \mathcal{L})^\wedge(l)$ |
| $\mathfrak{G}_{k, \mathcal{L}}(l) = \pi_1^{et}(\text{Spec}(k_p) \setminus S)(l)$ | $\mathfrak{G}_{\mathcal{L}, M}(l) = \pi_1(M \setminus \mathcal{L})^\wedge(l)$ |
Next, we give a sufficient condition for the vanishing of the obstruction $B_S$. Let $T_k$ be the set of finite primes $p$ of $k$ satisfying the following conditions:

\[
\begin{align*}
\bullet & \ p \in S^{\text{non-l}}_k, \ \text{equivalently}, \ Np \equiv 1 \mod m, \\
\bullet & \ p \text{ is inert in any cyclic extension } k(\sqrt[l]{\varepsilon})/k \text{ of degree } l \text{ for some unit } \varepsilon \in \mathcal{O}^\times_k.
\end{align*}
\]

(1.10)

Lemma 1.11. The set $T_k$ is infinite.

Proof. We let $E$ be the field defined by

\[E := k(\sqrt[l]{\varepsilon} | \varepsilon \in \mathcal{O}^\times_k).\]

Since $\mathcal{O}^\times_k/(\mathcal{O}^\times_k)^l$ is finite and $\zeta_m$ is contained in $k$, $E$ is a finite Abelian extension over $k$ of degree a power of $l$. By the conditions (1.10), $T_k$ contains all finite primes in $S^{\text{non-l}}_k$ which are inert in $E/k$. So the assertion follows from the Chebotareff density theorem applied to the extension $E/k$. □

A sufficient condition for the vanishing of $B_S$ is given by the following

Proposition 1.12. If there is a prime in $S \cap T_k$, then $B_S = \{1\}$.

Proof. Let $a \in V_S$ so that $a \in k^\times$, $(a) = a^l$ and $a \in (k_p^\times)^l$ for all $p \in S \cup S_\infty^\times$. Since $a$ has the order dividing $l$, by our assumption (1.5), we have $a = (b)$ for some $b \in k^\times$. Therefore we can write $a = \varepsilon b^l$ for $\varepsilon \in \mathcal{O}^\times_k$. If $\varepsilon \in (k^\times)^l$, $a \in (k^\times)^l$ and we are done. Suppose $\varepsilon \notin (k^\times)^l$. Then the extension $k(\sqrt[l]{\varepsilon})/k$ is a cyclic extension of degree $l$. Let $q \in S \cap T_k$. By (1.10), $k_q(\sqrt[l]{\varepsilon})/k_q$ is also of degree $l$ and so $u \notin (k_q^\times)^l$. This is a contradiction. Hence $\varepsilon \in (k^\times)^l$ again, and the assertion is proved. □

By Theorem 1.8, Lemma 1.11 and Proposition 1.12, we have the following

Corollary 1.13. For a given finite subset $S$ of $S^{\text{non-l}}_k$, there is a finite subset $R$ of $S^{\text{non-l}}_k$ such that $S \subset R$ and the pro-$l$ Galois group $\mathfrak{G}_{k,R}(l)$ is of link type.

Examples 1.14. (1) Let $k = \mathbb{Q}$ and $l = m = 2$. Let $S$ be a finite subset of $S^{\text{non-2}}_\mathbb{Q}$. Then we have $H_\mathbb{Q} = 1$ and, by the definition (1.6), (1.7) of $B_S$, we have $B_S = \{1\}$. Therefore, by Theorem 1.8, the pro-2 Galois group
$G_{Q,S}(2)$ is always of link type.

(2) Let $k = Q(\zeta_3) = Q(\sqrt{-3})$ and $l = m = 3$. Let $S$ be a finite subset of $S^{\text{non-3}}_{Q(\sqrt{-3})}$. We have $H_{Q(\sqrt{-3})} = 1$ and $O_k^\times = Z[\zeta_3]^\times = \{\pm 1, \pm \zeta_3, \pm \zeta_3^2\}$. We note that $Np \equiv 1, 4$ or 7 mod 9 for any $p \in S$. Since $k(\sqrt{-3})/Q(\zeta_9)$, where $\zeta_9$ is a primitive 9-th root of unity, $p$ is in $T_k$ if and only if $p$ is inert in the cyclic extension $Q(\zeta_9)/Q(\sqrt{-3})$ of degree 3. It is easy to see that the latter condition is equivalent to $Np \equiv 4$ or 7 mod 9. On the other hand, if $S$ consists of only primes $p$ with $Np \equiv 1$ mod 9, then $\zeta_3 \in V_S$ but $\zeta_3 \not\in (Q(\sqrt{-3})^\times)^3$, and hence $B_S \neq \{1\}$. Therefore, by Proposition 1.12, we have

$$B_S = \{1\} \iff S \text{ contains a prime } p \text{ with } Np \equiv 4 \text{ or } 7 \text{ mod } 9.$$ 

By Theorem 1.8, if $S$ satisfies this condition, the Galois group $G_{Q(\sqrt{-3}),S}(3)$ is of link type.

§2 Arithmetic Milnor invariants

In this section, we introduce the mod $m$ arithmetic Milnor invariants for a finite set of primes of a number field $k$ which contains a primitive $m$-th root of unity and whose $l$-class group is trivial. The idea is to pursue the arithmetic analogy of the construction of Milnor invariants for a link ([Mi2], [Tu]), based on the analogy between the structure of a pro-$l$ Galois group with restricted ramification and that of a link group presented in Remark 1.9. We actually carried out this construction in the previous work ([Mo1] ∼ [Mo4]. See also [V]) for the case of $k = Q$. We meet, however, a difficulty to work out it for a general number field $k$, because there is the obstruction $B_S$ discussed in Section 1 such that the Galois group $G_{k,S}(l)$ may not be of link type in general. We overcome this difficulty by enlarging $S$ to $R$ such that the larger Galois group $G_{k,R}(l)$ is of link type and then by proving that the Milnor invariants derived from $G_{k,R}(l)$ are independent of the choice of $R$.

We first recall the Magnus expansion for a free pro-$l$ group. Let $m$ be a power of $l$ and let $\mathfrak{F}_r$ denote the free pro-$l$ group on the words $x_1, \ldots, x_r$. Let $Z/mZ[[\mathfrak{F}_r]]$ denote the completed group algebra of $\mathfrak{F}_r$ over $Z/mZ$ and let $\epsilon_{Z/mZ[[\mathfrak{F}_r]]} : Z/mZ[[\mathfrak{F}_r]] \to Z/mZ$ be the augmentation homomorphism with the augmentation ideal $I_{Z/mZ[[\mathfrak{F}_r]]} := \text{Ker}(\epsilon_{Z/mZ[[\mathfrak{F}_r]]})$. Let $Z/mZ\langle\langle X_1, \ldots, X_r \rangle\rangle$
denote the formal power series algebra over \( \mathbb{Z}/m\mathbb{Z} \) in non-commuting variables \( X_1, \ldots, X_r \). Sending \( x_i \) to \( 1+X_i \) for \( 1 \leq i \leq r \), we have the isomorphism of topological \( \mathbb{Z}/m\mathbb{Z} \)-algebras

\[
\Theta_{r,m} : \mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]] \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}\langle\langle X_1, \ldots, X_r \rangle\rangle,
\]

which we call the (pro-l) Magnus isomorphism over \( \mathbb{Z}/m\mathbb{Z} \). The augmentation ideal \( I_{\mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]]} \) corresponds to the two-side ideal of \( \mathbb{Z}/m\mathbb{Z}\langle\langle X_1, \ldots, X_r \rangle\rangle \) generated by \( X_1, \ldots, X_r \). For \( \alpha \in \mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]] \), \( \Theta_{r,m}(\alpha) \) is called the Magnus expansion of \( \alpha \) over \( \mathbb{Z}/m\mathbb{Z} \). In the following, for a multi-index \( I = (i_1 \cdots i_n) \), \( 1 \leq i_1, \ldots, i_n \leq r \), we set

\[
|I| := n \text{ and } X_I := X_{i_1} \cdots X_{i_n}.
\]

We call the coefficient of \( X_I \) in the Magnus expansion \( \Theta_{r,m}(\alpha) \) the mod \( m \) Magnus coefficient of \( \alpha \) for \( I \) and denote it by \( \mu_m(I; \alpha) \) so that we have

\[
\Theta_{r,m}(\alpha) = \epsilon_{\mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]]}(\alpha) + \sum_{|I| \geq 1} \mu_m(I; \alpha)X_I.
\]

In terms of the pro-l Fox free derivatives over \( \mathbb{Z}/m\mathbb{Z} \)

\[
\frac{\partial}{\partial x_i} : \mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]] \to \mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]]
\]

(cf. [Ih; §2], [Mo4; 8.3], [O]), the mod \( m \) Magnus coefficient \( \mu_m(I; \alpha) \) of \( \alpha \) for \( I = (i_1 \cdots i_n) \) is written as

\[
\mu_m(I; \alpha) = \epsilon_{\mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]]}\left( \frac{\partial^n \alpha}{\partial x_{i_1} \cdots \partial x_{i_n}} \right).
\]

Restricting \( \Theta_{r,m} \) to \( \mathfrak{F}_r \), we have an injective group homomorphism, denoted by the same \( \Theta_{r,m} \),

\[
\Theta_{r,m} : \mathfrak{F}_r \hookrightarrow \mathbb{Z}/m\mathbb{Z}\langle\langle X_1, \ldots, X_r \rangle\rangle^\times,
\]

which we call the Magnus embedding of \( \mathfrak{F}_r \) over \( \mathbb{Z}/m\mathbb{Z} \). The Magnus expansion of \( f \in \mathfrak{F}_r \) over \( \mathbb{Z}/m\mathbb{Z} \) is then of the form

\[
\Theta_{r,m}(f) = 1 + \sum_{|I| \geq 1} \mu_m(I; f)X_I.
\]

Here are some properties of mod \( m \) Magnus coefficients which will be used in the following.

(2.1) For \( \alpha, \beta \in \mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]] \) and a multi-index \( I \), we have

\[
\mu_m(I; \alpha \beta) = \sum_{I=JK} \mu_m(I; \alpha)\mu_m(K; \beta),
\]
where the sum ranges over all pairs \((J, K)\) of multi-indices such that \(JK = I\).

(2.2) (Shuffle relation) For \(f \in \mathfrak{F}_r\) and multi-indices \(I, J\) with \(|I|, |J| \geq 1\), we have

\[
\mu_m(I; f) \mu_m(J; f) = \sum_{H \in \text{Sh}(I, J)} \mu_m(H; f),
\]

where \(\text{Sh}(I, J)\) denotes the set of the results of all shuffles of \(I\) and \(J\) ([CFL]).

(2.3) For \(f \in \mathfrak{F}_r\) and \(d \geq 2\), we have

\[
\mu_m(I; f) = 0 \text{ for } |I| < d \iff f \in \mathfrak{F}_r^{(d)},
\]

where \(\mathfrak{F}_r^{(d)} := \mathfrak{F}_r \cap (\mathbb{Z}/m\mathbb{Z}[[\mathfrak{F}_r]])^d\), the \(d\)-th term of the mod \(m\) Zassenhaus filtration of \(\mathfrak{F}_r\).

Now, let us be back to the arithmetic situation in Section 1 and keep the same notations and assumptions there. So let \(k\) be a finite algebraic number field containing a fixed primitive \(m\)-th root of unity \(\zeta_m\), where \(m\) is a fixed power of \(l\), and the \(l\)-class number of \(k\) is assumed to be one. Let \(S\) be a finite subset of \(S_k^{\text{non-l}}\) consisting of \(s\) distinct primes of \(k\), \(S := \{p_1, \ldots, p_s\}\).

By Corollary 1.13, we choose a finite subset \(R\) of \(S_k^{\text{non-l}}\) such that \(S \subset R\) and \(\mathfrak{G}_{k, R}(l)\) is a pro-\(l\) Galois group of link type. We let \(R := \{p_1, \ldots, p_s, \ldots, p_r\}\), \(r = \#R \geq s\). Let \(F_r\) denote the free group on the words \(x_1, \ldots, x_r\), where each \(x_i\) represents a monodromy \(\tau_i\) over \(p_i\) in \(k_R(l)/k\), and let \(\mathfrak{F}_r\) denote the pro-\(l\) completion of \(F_r\). We have

\[
(2.4) \quad \mathfrak{G}_{k, R}(l) = \langle x_1, \ldots, x_r \mid x_1^{Np_1-1}[x_1, y_1] = \cdots = x_r^{Np_r-1}[x_r, y_r] = 1 \rangle = \mathfrak{F}_r/\mathfrak{N},
\]

where \(y_i\) is the free pro-\(l\) word of \(x_1, \ldots, x_r\) representing a Frobenius automorphism \(\sigma_i\) over \(p_i\) in \(k_R(l)/k\) and \(\mathfrak{N}\) is the closed subgroup of \(\mathfrak{F}_r\) generated normally by \(x_1^{Np_1-1}[x_1, y_1], \ldots, x_r^{Np_r-1}[x_r, y_r]\).

Let \(I = (i_1 \cdots i_n)\) be a multi-index such that \(1 \leq i_1, \ldots, i_n \leq s\). Let \(e_I\) be the maximal integer \(e\) satisfying \(Np_i \equiv 1 \text{ mod } l^e\) for all \(i = i_1, \ldots, i_n\) and we set \(m_I := l^{e_I}\). We note by Lemma 1.1 that \(m_I\) is divisible by \(m\). Let \(I' := (i_1 \cdots i_{n-1})\) and we define the mod \(m\) arithmetic Milnor number \(\mu_m(I)\) for \(I\) by the mod \(m\) Magnus coefficient of \(y_{i_n}\) for \(I'\):

\[
(2.5) \quad \mu_m(I) := \mu_m(I'; y_{i_n}).
\]

Here we set \(\mu_m(I) := 0\) if \(|I| = 1\). Let \(P(I)\) denote the set of all cyclic permutations of proper sub-sequences of \(I\) and we define the indeterminacy
$\Delta_m(I)$ for $I$ by the ideal of $\mathbb{Z}/m\mathbb{Z}$ generated by the binomial coefficients $\binom{m}{a}$ for all $1 \leq a < |I|$ and $\mu_m(J)$ for all $J \in P(I)$:

(2.6) $\Delta_m(I) := \left( \binom{m}{a} \right) (1 \leq a < |I|), \mu_m(J) (J \in P(I))$.

Then we set

(2.7) $\overline{\mu}_m(I) := \mu_m(I) \mod \Delta_m(I)$.

Theorem 2.8. Notations being as above, the following assertions hold: For a multi-index $I = (i_1 \cdots i_n)$ satisfying $1 \leq i_1, \ldots, i_n \leq s$ and $2 \leq |I| \leq m_I$,

1. $\overline{\mu}_m(I)$ is independent of a choice of a monodromy $\tau_i$ and a Frobenius $\sigma_i$ over $p_i$ for $1 \leq i \leq r$, namely, a choice of a prime $\mathfrak{p}_i$ in $k_{R(l)}$ lying over $p_i$ (equivalently an embedding $k_{p_i(l)} \hookrightarrow k_{k,R(l)}$), so an invariant of $G_{k,R(l)}$.
2. $\overline{\mu}_m(I)$ is independent of a choice of $R$.

Hence $\overline{\mu}_m(I)$ is an invariant of $S$ and determined by $I$ and $m$.

Proof. (1) Since $G_{k,R(l)}$ has a presentation (2.5), we need to verify

(i) $\overline{\mu}_m(I)$ is unchanged when $x_i$ is replaced by its conjugate for $1 \leq i \leq r$.

(ii) $\overline{\mu}_m(I)$ is unchanged when $y_{i_1} \cdots y_{i_n}$ is replaced by its conjugate in $\mathfrak{G}_r$.

(iii) $\overline{\mu}_m(I)$ is unchanged when $y_{i_1} \cdots y_{i_n}$ is multiplied by a product of conjugates of $(x_i^{N_{p_i}} - 1[x_i, y_i])^e$ for $1 \leq i \leq r, e = \pm 1$.

We set $I' := (i_1 \cdots i_{n-1})$.

Proof of (i). Suppose that $x_i$ is replaced by $x_i^* = x_j x_i x_j^{-1}$ ($1 \leq i, j \leq r$) with $\Theta_{r,m}(x_i^*) = 1 + X_i^*$. Since $x_i = x_i^{-1} x_i^* x_i$, we have $X_i = (1 - X_j + X_j^2 - \cdots) X_i^*(1 + X_j)$ and hence

(2.8.1) $X_i = X_i^* + (\text{terms involving } X_j X_i^* \text{ or } X_i^* X_j)$.

Each time $X_i$ appears in the Magnus expansion $\Theta_{r,m}(y_{i_1}) = 1 + \sum J \mu_m(J_i y_{i_n}) X_J$, it is to be replaced by the above expansion (2.8.1) and we finally reach the new expansion of $\Theta_{r,m}(y_{i_1})$ in $X_i^*, \ldots, X_r^*$, by which we denote $\Theta_{r,m}(y_{i_1})$. Then we can easily see that the coefficient of $X_{i_1}^* \cdots X_{i_{n-1}}^*$ in $\Theta_{r,m}(y_{i_n})$, denoted by $\mu_m^*(I)$, is of the form

$$\mu_m^*(I) = \mu_m(I) + \sum_j \mu_m(J_i y_{i_n}).$$
where \( J \) runs over some proper subsequences of \( I' \). Therefore we have

\[
\mu_m^*(I) \equiv \mu_m(I) \mod \Delta_m(I).
\]

Similarly, \( \overline{\mu}_m(I) \) is proved to be unchanged when \( x_i \) is replaced by \( x_j^{-1}x_ix_j \) (\( 1 \leq i,j \leq n \)). So \( \overline{\mu}_m(I) \) is unchanged when \( x_i \) is replaced by its conjugate in \( F_r \). Since \( F_r \) is dense in \( \overline{F_r} \) and \( \overline{\mu}_m(I) \) takes discrete values, \( \overline{\mu}_m(I) \) is unchanged when \( x_i \) is replaced by its conjugate in \( \overline{F_r} \).

Proof of (ii). By comparing the coefficients of \( X_I' \) in the both sides of the equality

\[
\Theta_{r,m}(x_iy_i,x_i^{-1}) = (1 + X_i)\Theta_{r,m}(y_i,1 - X_i + X_i^2 - \cdots)
\]

for \( 1 \leq i \leq r \), we have

\[
\mu_m(I'; x_iy_i,x_i^{-1}) \equiv \mu_m(I) \mod a(I'),
\]

where \( a(I') \) is the ideal of \( \mathbb{Z}/m\mathbb{Z} \) generated by \( \mu_m(J_i) \) for all proper subsequences of \( I' \). Since \( a(I') \subset \Delta_m(I) \), we have

\[
\mu_m(I'; x_iy_i,x_i^{-1}) \equiv \mu_m(I) \mod \Delta_m(I).
\]

Similarly, we have \( \mu_m(I'; x_i^{-1}y_ix_i) \equiv \mu_m(I) \mod \Delta_m(I) \). By the same argument as in the proof of (i), the assertion (ii) is proved.

Proof of (iii). Let \( J \) be any subsequence of \( I' \), \( 1 \leq i \leq r \) and \( e = \pm 1 \). We will prove that

(2.8.2) \[ \mu_m(J; \{x_i^{Np_i-1}[x_i,y_i]\}^e) \equiv 0 \mod \Delta_m(I). \]

First we prove that

(2.8.3) \[ \mu_m(J; [x_i,y_i]^e) \equiv 0 \mod \Delta_m(I). \]

Comparing the coefficients of \( X_J \) in the equality

\[
\Theta_{r,m}([x_i,y_i]^e) = \begin{cases} 
1 + (\Theta_{r,m}(x_iy_i) - \Theta_{r,m}(y_ix_i))\Theta(x_i^{-1})\Theta_{r,m}(y_i^{-1}) & (e = 1) \\
1 + (\Theta_{r,m}(y_ix_i) - \Theta_{r,m}(x_iy_i))\Theta(y_i^{-1})\Theta_{r,m}(x_i^{-1}) & (e = -1),
\end{cases}
\]

we have

\[
\mu_m(J; [x_i,y_i]^e) = e(\mu_m(J; x_iy_i) - \mu_m(J; y_ix_i)) + \sum_A (\mu_m(A; x_iy_i) - \mu(A; y_ix_i))c_A,
\]

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where $A$ runs over some proper subsequences of $J$ and $c_A \in \mathbb{Z}/m\mathbb{Z}$. So, in order to prove (2.8.3), it is enough to show that for any subsequence $J$ of $I'$ and $1 \leq i \leq r$,

(2.8.4) \[ \mu_m(J; x_i y_i) - \mu_m(J; y_i x_i) \equiv 0 \mod \Delta_m(I). \]

Let $J = (j_1 \cdots j_a)$. By the straightforward computation, we have

\[ \mu_m(J; x_i y_i) = \begin{cases} \mu_m(Ji) & (i \neq j_1), \\ \mu_m(Jj_1) + \mu_m(j_2 \cdots j_a j_1) & (i = j_1), \end{cases} \]

and

\[ \mu_m(J; y_i x_i) = \begin{cases} \mu_m(Ji) & (i \neq j_a), \\ \mu_m(j_2 \cdots j_a j_1) + \mu_m(J) & (i = j_a), \end{cases} \]

Hence we have

(2.8.5) \[ \mu_m(J; x_i y_i) - \mu_m(J; y_i x_i) = \begin{cases} \mu_m(j_2 \cdots j_a j_1) - \delta_{j_1, j_a} \mu_m(J) & (i = j_1), \\ \mu_m(j_2 \cdots j_a j_1) \delta_{j_1, j_a} - \mu_m(J) & (i = j_a), \\ 0 & \text{(otherwise)}, \end{cases} \]

where $\delta_{a,b}$ is the Kronecker delta. By the definition (2.6) of $\Delta_m(I)$, (2.8.5) yields (2.8.4) and hence (2.8.3). (It is here that cyclic permutations are needed.)

Next, we prove that for any subsequence $J$ of $I'$, $1 \leq i \leq r$ and $e = \pm 1$,

(2.8.6) \[ \mu_m(J; (x_i^{Np_i-1})^e) \equiv 0 \mod \Delta_m(I). \]

Suppose $i \in I$. Then $Np_i \equiv 1 \mod m_I$ and so we write $Np_i - 1 = m_I q$. By the definition (2.5) of $\Delta_m(I)$, we have

\[ \Theta_{r,m}((x_i^{Np_i-1})^e) = (1 + X_i)^{em_I q} = \left( \sum_{j=0}^{m_I} \binom{m_I}{j} X_i^j \right)^{eq} \equiv 1 + \text{(term of degree} \geq |I| \text{) mod} \Delta_m(I), \]

from which (2.8.6) follows.

Suppose $i \notin I$. Then $\Theta_{r,m}((x_i^{Np_i-1})^e)$ does not contain the term of $X_J$, and hence $\mu_m(J; (x_i^{Np_i-1})^e) = 0$. So (2.8.6) is proved.

By (2.1), (2.8.3) and (2.8.6), we have

\[ \mu_m(J; (x_i^{Np_i-1}[x_i, y_i]^e) = \sum_{J', J''} \mu_m(J'; (x_i^{Np_i-1})^e) \mu_m(J''; [x_i, y_i]^e) \equiv 0 \mod \Delta_m(I), \]

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where $J = J'J''$ if $e = 1$ and $J = J''J'$ if $e = -1$. Thus (2.8.2) is proved. By the same argument as in the proof of (ii), we have, for any $f \in \mathfrak{F}_r$,

\[(2.8.7)\]

\[
\mu_m(J; f(x_i^{N_{p_i}^{-1}}[x_i, y_i])^e f^{-1}) \equiv \mu(J; (x_i^{N_{p_i}^{-1}}[x_i, y_i])^e) \equiv 0 \mod \Delta_m(I).
\]

Finally, by (2.1) and (2.8.7), we have

\[
\mu_m(I'; f(x_i^{N_{p_i}^{-1}}[x_i, y_i])^e f^{-1} y_{in}) = \sum_{p=JK} \mu_m(J; f(x_i^{N_{p_i}^{-1}}[x_i, y_i])^e f^{-1}) \mu_m(K; y_{in}) \equiv \mu_m(I) \mod \Delta_m(I).
\]

By the argument similar to the above, we can prove that

\[
\mu_m(I'; y_{in} f(x_i^{N_{p_i}^{-1}}[x_i, y_i])^e f^{-1}) \equiv \mu_m(I) \mod \Delta_m(I).
\]

Hence the assertion (iii) is proved.

(2) Let $T$ be another choice of a finite subset of $S_k^{non-l}$ such that $S \subset T$ and the Galois group $\mathfrak{G}_{k,T}(l)$ is of link type. We let $T := \{p_1, \ldots, p_s, \ldots, p_t\}$, $t = \#T \geq s$. In order to make clear the dependence of our choice of a finite subset of $S_k^{non-l}$ containing $S$, we use the following notations. Let the letter $z$ (resp. $Z$) stand for $r$ or $t$ (resp. $R$ or $T$). Let $\tau_i^Z$ denote a monodromy over $p_i$ in $k_z(l)/k$ for $1 \leq i \leq z$, and let $\mathfrak{F}_z^{Z}$ denote the free pro-$l$ group on the words $x_i^Z, \ldots, x_z^Z$, where each $x_i^Z$ represents $\tau_i^Z$. We then have

\[
\mathfrak{G}_{k,z}(l) = \langle x_1^Z, \ldots, x_z^Z \mid (x_i^{N_{p_i}^{-1}}[x_i^Z, y_i^Z]) = \cdots = (x_z^{N_{p_z}^{-1}}[x_z^Z, y_z^Z]) = 1 \rangle,
\]

where $y_i^Z$ denotes the free pro-$l$ word of $x_1^Z, \ldots, x_z^Z$ which represents a Frobenius automorphism over $p_i$ in $k_z(l)/k$. We let $\mu_m^Z(I)$ and $\Delta_m^Z(I)$ denote the mod $m$ Magnus coefficient and the indeterminacy, respectively, obtained by using the Magnus embedding $\Theta_{z,m}^Z: \mathfrak{F}_z^Z \to \mathbb{Z}/m\mathbb{Z}\langle(X_1^Z, \ldots, X_z^Z)\rangle$ ($\Theta_{z,m}^Z(x_i^Z) := 1 + X_i^Z$). We set $\overline{\mu}_m^Z(I) := \mu_m^Z(I) \mod \Delta_m^Z(I)$. We claim

(iv) $\Delta_m^R(I) = \Delta_m^T(I)$ and $\overline{\mu}_m^R(I) = \overline{\mu}_m^T(I)$.

Proof of (iv). By the definition of $\Delta_m^Z(I)$ and $\overline{\mu}_m^Z(I)$, it suffices to show that $\mu_m^R(J) = \mu_m^T(J)$ for any subsequence $J$ of $I$. Let $U := R \cup T$ so that we may write $U := \{p_1, \ldots, p_s, \ldots, p_u\}, u \geq r, t$. Let $\tau_i^U$ and $\sigma_i^U$ denote a monodromy and a Frobenius automorphism, respectively, over $p_i$ in $k_U(l)/k$ such that

\[(2.8.8)\]

\[
\tau_i^U|_{k_z(l)} = \tau_i^Z, \quad \sigma_i^U|_{k_z(l)} = \sigma_i^Z \quad (Z = R \text{ or } T),
\]

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if \( p_i \in Z \). Let \( F_U \) denote the free pro-\( l \) group on the words \( x_1^U \), ..., \( x_u^U \), where each \( x_i^U \) represents \( \tau_i^U \), and we have

\[
\mathfrak{G}_{k,U}(l) = \langle x_1^U, \ldots, x_u^U \mid (x_1^U)^{Np_1-1}[x_1^U, y_1^U] = \cdots = (x_u^U)^{Np_u-1}[x_u^U, y_u^U] = 1 \rangle,
\]

where \( y_i^U \) denotes the free pro-\( l \) word of \( x_1^U \), ..., \( x_u^U \) which represents a Frobenius automorphism over \( p_i \) in \( k^U(l)/k \). Let \( \mu_m^U(J) \) denote the mod \( m \) Magnus coefficient obtained by using the Magnus embedding \( \Theta_{U,m} : \mathfrak{F}_U \to \mathbb{Z}/m\mathbb{Z}\langle\langle X_1^U, \ldots, X_u^U \rangle\rangle \) \((\Theta_{U,m}(x_i^U) := 1 + X_i^U)\).

Now let \( \psi_U^Z : \mathfrak{F}_U \to \mathfrak{F}_Z \) be the group homomorphism defined by

\[
\psi_U^Z(x_i^U) := \begin{cases} x_i^Z & (p_i \in Z), \\ 1 & (p_i \notin Z), \end{cases}
\]

and let \( \lambda_Z^U : \mathbb{Z}/m\mathbb{Z}\langle\langle X_1^Z, \ldots, X_z^Z \rangle\rangle \to \mathbb{Z}/m\mathbb{Z}\langle\langle X_1^Z, \ldots, X_z^Z \rangle\rangle \) be the \( \mathbb{Z}/m\mathbb{Z} \)-algebra homomorphism defined by

\[
\lambda_Z^U(X_i^U) := \begin{cases} X_i^Z & (p_i \in Z), \\ 0 & (p_i \notin Z). \end{cases}
\]

By (2.8.8), we have the following commutative diagram

\[
\begin{array}{ccc}
\mathfrak{F}_U & \xrightarrow{\Theta_{U,m}} & \mathbb{Z}/m\mathbb{Z}\langle\langle X_1^U, \ldots, X_u^U \rangle\rangle \\
\psi_U^Z \downarrow & & \downarrow \lambda_Z^U \\
\mathfrak{F}_Z & \xrightarrow{\Theta_{Z,m}} & \mathbb{Z}/m\mathbb{Z}\langle\langle X_1^Z, \ldots, X_z^Z \rangle\rangle.
\end{array}
\]

It follows that, for \( J = (j_1 \cdots j_a) \subset I \) \((a \geq 2)\) with \( J' = (j_1 \cdots j_{a-1}) \),

\[
\mu_m^Z(J) = \mu_m^Z(J', y_{j_a}^U) = \mu_m^Z(J; \psi_U^Z(y_{j_a}^U)) = \mu_m^U(J', y_{j_a}) \quad \text{(by 1 \leq j_1, \ldots, j_b \leq z)} = \mu_m^U(J),
\]

and hence (iv) is proved. \( \square \)

By Theorem 2.8, we call \( \overline{\mu}_m(I) \) the mod \( m \) arithmetic Milnor invariant of \( S \) for the multi-index \( I \).

**Remark and Example 2.9.** (1) Let \( k = \mathbb{Q} \) and \( l = m = 2 \). Then the
Galois group $\mathfrak{G}_{k,S}(l)$ is of link type for any finite set of odd rational primes (Example 1.14 (1)) and so we can take $R$ to be $S$ (no need to add auxiliary primes) to define the mod 2 arithmetic Milnor invariants. In this case, we defined the Milnor invariants in the previous work [Mo3], [Mo4; 8.4], where the indeterminacy depended on the set $S$ and is larger than our present one (2.6) in general. So our present definition (2.7) is more refined than the previous one even in this case.

(2) Let $k = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ and $l = m = 3$. By Example 1.14 (2), $B_S = \{0\}$ if and only if $S$ contains a prime $p$ with $Np \equiv 4$ or $7 \pmod{9}$ and so $\mathfrak{G}_\mathbb{Q}(\sqrt{-3},S)(3)$ may not be of link type for any $S$. For example, if $S$ consists of primes $p$ with $Np \equiv 1 \pmod{9}$, the mod 3 arithmetic Milnor invariants of $S$ can be defined by enlarging $S$. This case will be studied in Section 4.

(3) Let $I = (ij), 1 \leq i \neq j \leq s$. Then $\Delta_m(I) = 0$ and $\overline{\mu}_m(I) = \mu_m(ij)$. By the definition of $\mu_m(ij)$, we have

$$\sigma_j \equiv \prod_{i \neq j} t_i^\mu_{m(ij)} \mod \mathfrak{G}_{k,S}(l)(2),$$

where $\mathfrak{G}_{k,S}(l)(2) = (\mathfrak{G}_{k,S}(l))^m[\mathfrak{G}_{k,S}(l), \mathfrak{G}_{k,S}(l)]$. The invariant $\mu_m(ij)$ may be regarded as an arithmetic analogue of the mod $m$ linking number for $p_i$ and $p_j$.

By the proof of Theorem 2.8 (2), we can easily see the following

**Corollary 2.10.** Let $S$ and $S'$ be given finite subsets of $S_k^{\text{non-l}}$ such that there are $p_{i_1}, \ldots, p_{i_n} \in S \cap S'$. Then the mod $m$ arithmetic Milnor invariants of $S$ and $S'$ for the multi-index $(i_1 \cdots i_n)$ coincide.

A meaning of the arithmetic Milnor invariants is given as follows. We choose a finite subset $R$ of $S_k^{\text{non-l}}$ such that $S \subset R$ and $\mathfrak{G}_{k,R}(l)$ is a pro-$l$ Galois group of link type, and we use the same notations as in the paragraph before Theorem 2.8. Let $N_n(A)$ denote the group of $n$ by $n$ upper-triangular unipotent matrices over a commutative ring $A$. Let $I = (i_1 \cdots i_n)$ be a multi-index such that $1 \leq i_1, \ldots, i_n \leq s$ and $\Delta_m(I) \neq \mathbb{Z}/m\mathbb{Z}$. We define the map

$$\rho_m(I) : \mathfrak{F}_r \longrightarrow N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I))$$
by
(2.11)

\[
\rho_m(I)(f) := \begin{pmatrix}
1 & \mu_m(I_{1,1}; f) & \mu_m(I_{1,2}; f) & \cdots & \mu_m(I_{1,n-1}; f) \\
0 & 1 & \mu_m(I_{2,1}; f) & \cdots & \mu_m(I_{2,n-1}; f) \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & \mu_m(I_{n-1,1}; f) \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix} \mod \Delta_m(I)
\]

for \( f \in \mathcal{F} \), where \( I_{a,b} = (i_a \cdots i_b) \). It is, in fact, a homomorphism by the
property (2.1). The following theorem may be seen as an arithmetic analog
of Murasugi’s theorem for a link ([Mu]).

**Theorem 2.12.** (1) The homomorphism \( \rho_m(I) \) factors through the Galois

\[
\rho_m(I) : \mathfrak{S}_{k,R}(l) \longrightarrow N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I)),
\]

d group \( \mathfrak{S}_{k,R}(l) \)

and it is surjective if \( i_1, \ldots, i_{n-1} \) are all distinct.

Let \( \mathfrak{K}_m(I) \) denote the subextension of \( k_R(l)/k \) corresponding to the subgroup

\( \text{Ker}(\rho_m(I)) \) of \( \mathfrak{S}_{k,R}(l) \).

(2) The field \( \mathfrak{K}_m(I) \) is independent of a choice of \( R \) and depends only on \( I \)

and \( m \).

(3) The extension \( \mathfrak{K}_m(I)/k \) is a finite Galois extension whose Galois group is

isomorphic to \( N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I)) \), and it is unramified outside \( p_{i_1}, \ldots, p_{i_{n-1}} \)

and \( S_{k}^\infty \) and the ramification index for each \( p_{i_a} \) is \( \#(\mathbb{Z}/m\mathbb{Z})/\Delta_m(I) \). Further,

for a Frobenius automorphism \( \sigma_{i_n} \) over \( p_i \) in \( k_R(l)/k \), we have

\[
\rho_m(I)(\sigma_{i_n}) = \begin{pmatrix}
1 & 0 & 0 & \cdots & \overline{\mu}_m(I) \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}.
\]

In particular, we have

\[
\overline{\mu}_m(I) = 0 \Longleftrightarrow p_{i_n} \text{ is completely decomposed in } \mathfrak{K}_m(I)/k.
\]
Proof. (1) The former assertion follows from $\rho_m(I)(x_i^{Np_i-1}[x_i, y_i])$ is the identity matrix, which can be shown by the same manner as in the proof of (2.4.2). The latter one follows from that $N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I))$ is generated by

$$\rho_m(I)(x_{ia}) = \begin{pmatrix} 1 & & & & \\
& 1 & 1 & & \\
& & 1 & & \\
& & & \ddots & \\
& & & & 1 \end{pmatrix}$$

for all $1 \leq a \leq n$.

(2) Let $T$ be another choice of a finite subset of $S_k^{\text{non-}l}$ such that $S \subset T$ and the Galois group $\mathfrak{G}_{k,T}(l)$ is of link type. In order to make clear the dependence of our choice of a finite subset of $S_k^{\text{non-}l}$ containing $S$, as in the proof of Theorem 2.8 (2), let $Z$ stand for $R$ or $T$, and we denote $\rho_m(I)$ and $\mathfrak{R}_m(I)$ by $\rho^Z_m(I) : \mathfrak{G}_{k,Z}(l) \to N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I))$ and $\mathfrak{R}^Z_m(I) := \text{Ker}(\rho^Z_m(I))$, respectively, according as we consider $Z = R$ or $T$. Then we need to show $\mathfrak{R}^R_m(I) = \mathfrak{R}^T_m(I)$.

Let $U := R \cup T$. Let $\psi^U_Z : \mathfrak{G}_{k,U}(l) \to \mathfrak{G}_{k,Z}(l)$ be the natural surjective homomorphism defined by $\psi^U_Z(g) := g|_{kZ(l)}$, and let $\rho^U_m(I) : \mathfrak{G}_{k,U}(l) \to N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I))$ be defined in the same manner as for $\rho^Z_m$. Then, by the definition of $\rho^Z_m$ and $\rho^U_m(I)$ (see (2.11)), we have

$$\rho^U_m(I) = \rho^Z_m(I) \circ \psi^U_Z$$

and the commutative diagram

$$\begin{array}{ccc}
\mathfrak{G}_{k,R}(l) & \xrightarrow{\rho^R_m(I)} & N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I)) \\
\psi^U_Z & & \\
\mathfrak{G}_{k,U}(l) & \xrightarrow{\rho^U_m(I)} & \\
\psi^U_T & & \\
\mathfrak{G}_{k,T}(l) & \xrightarrow{\rho^T_m(I)} & \\
\end{array}$$

Therefore we have

$$(\psi^U_T)^{-1}(\text{Ker}(\rho^R_m(I))) = \text{Ker}(\rho^U_m(I)) = (\psi^U_T)^{-1}(\text{Ker}(\rho^R_m(I))).$$
Since $\mathcal{R}_m^Z(I)$ is the subfield of $k_U(I)/k$ corresponding to $(\psi_Z^{U})^{-1}(\text{Ker}(\rho_m^Z(I)))$, we have

$$\mathcal{R}_m^R(I) = \mathcal{R}_m^T(I).$$

(3) By (1), $\mathcal{R}_m(I)$ is a Galois extension of $k$ whose Galois group $\text{Gal}(\mathcal{R}_m(I)/k)$ is isomorphic to $N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I))$. Since $\rho_m(I)(x_j)$ is the identity matrix if $j \neq i_1, \ldots, i_{n-1}$, $\mathcal{R}_m(I)/k$ is unramified outside $p_{i_1}, \ldots, p_{i_{n-1}}$ and $S_k^\infty$. The ramification index for $p_{i_n}$ is the order of $\rho_m(I)(x_{i_n})$, which is $\#((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I))$ as we see in the proof of (1). Since $\mu_m(I_{a,b}; y_{i_n}) = \mu(i_a \cdots i_b i_{i_n}) \equiv 0 \mod \Delta_m(I)$ if $(a, b) \neq (1, n)$ and $\mu_m(I_{1,n-1}; y_{i_n}) = \mu(I)$, we obtain the assertion for $\rho_m(I)(\sigma_{i_n})$. Finally, since $\sigma_{i_n}|_{\mathcal{R}_m(I)} = \rho_m(I)(\sigma_{i_n})$, we have

$$\overline{\mu}_m(I) = 0 \iff \rho_m(I)(\sigma_{i_n}) = \text{identity in } N_n((\mathbb{Z}/m\mathbb{Z})/\Delta_m(I))$$

$$\iff \sigma_{i_n}|_{\mathcal{R}_m(I)} = \text{identity in } \text{Gal}(\mathcal{R}_m(I)/k)$$

$$\iff p_{i_n} \text{ is completely decomposed in } \mathcal{R}_m(I)/k. \quad \Box$$

Finally, as a property the arithmetic Milnor invariants enjoy, we note the following shuffle relation which follows from the property (2.2).

**Proposition 2.13.** Suppose that multi-indices $I = (i_1 \cdots i_a)$ and $J = (j_1 \cdots j_b)$ satisfy $1 \leq i_1, \ldots, i_a, j_1, \ldots, j_b \leq s$ $(a, b \geq 1)$ and $a + b \leq m_{IJ} - 1$ with $1 \leq i \leq s$. Then we have

$$\sum_{H \in \text{PSh}(I,J)} \overline{\mu}_m(Hi) \equiv 0 \mod \text{g.c.d}\{\Delta_m(Hi) \mid H \in \text{PSh}(I,J)\},$$

where $\text{PSh}(I,J)$ denotes the set of results of all proper shuffles of $I$ and $J$ ([CFL]).

**Proof.** By (2.2) with $f = y_i$, we have

$$\mu_m(Ii)\mu_m(Ji) = \sum_{H \in \text{Sh}(I,J)} \mu_m(Hi).$$

Here we take mod g.c.d$\{\Delta_m(Hi) \mid H \in \text{PSh}(I,J)\}$ in the both sides. Then the left hand side $\equiv 0$ and $\mu_m(Hi) \equiv 0$ in the right hand side if $H$ is not the result of a proper shuffle. Hence the assertion follows. $\Box$
Remark 2.14. We take this opportunity to point out a mistake in [Mo3; Theorem 1.2.5]. The proof of the cyclic symmetry of $\mu_m(I)$ therein is wrong. At present, we do not know a proof of the cyclic symmetry of $\mu_m(I)$, except for the cases that $|I| = 2$, $k = \mathbb{Q}$ with $p_i \equiv 1 \mod 4$ (quadratic reciprocity law) or that $|I| = 3$, $k = \mathbb{Q}$ with $p_i \equiv 1 \mod 4$ (cyclic symmetry for the Rédei symbol ([Rd], [A1]). For the Rédei symbol, see Example 3.8).

§3 Multiple power residue symbols

In this section, using the mod $m$ arithmetic Milnor invariants in Section 2, we introduce the $n$-tuple $m$-th power residue symbols $[p_1, p_2, \ldots, p_n]_m$ for finite primes $p_1, p_2, \ldots, p_n$, prime to $l$, of a number fields which contains a primitive $m$-th root of unity and whose $l$-class group is trivial. We then show the relations with the Legendre, power residue symbols and the Rédei triple symbol together with some results obtained by the first author ([A1], [A2]).

We keep the same notations as in Sections 1 and 2. Let $k$ be a finite algebraic number field containing a fixed primitive $m$-th root of unity $\zeta_m$, where $m$ is a fixed power of $l$, and we assume that the $l$-class group $H_k(l)$ of $k$ is trivial. Let $p_1, \ldots, p_n$ be a set of $n(\geq 2)$ distinct primes in $S_k^{\text{non-l}}$. Let $m_{(12\cdots n)}$ be the maximal power $l^e$ of $l$ such that $Np_i \equiv 1 \mod l^e$ for $1 \leq i \leq n$. Let $\mu_m(i_1 \cdots i_a)$ be the mod $m$ Milnor number in (2.5), let $\Delta_m(i_1 \cdots i_a)$ be the indeterminacy in (2.6), and let $\mu_m(i_1 \cdots i_a)$ be the mod $m$ Milnor invariant in (2.7) for $1 \leq i_1, \ldots, i_a \leq n$. In this section, we assume that

\begin{align}
(3.1) \quad \begin{cases} \\
\quad \begin{array}{l}
\left( \frac{m_{(12\cdots n)}}{j} \right) \equiv 0 \mod m \text{ for } 1 \leq j < n, \\
\mu_m(J) = 0 \text{ for any cyclic permutation } J \text{ of any proper subsequence of } 12\cdots n.
\end{array}
\end{cases}
\end{align}

By the assumption (3.1) and the definition (2.6) of the indeterminacy, we have

\begin{align}
(3.2) \quad \Delta_m(12\cdots n) = 0
\end{align}

and

\begin{align}
(3.3) \quad \overline{\mu}_m(12\cdots n) = \mu_m(12\cdots n) \in \mathbb{Z}/m\mathbb{Z}.
\end{align}
**Definition 3.4.** We define the \( n \)-tuple \( m \)-th power residue symbol by

\[
[p_1, p_2, \ldots, p_n]_m := \zeta_m^{(12 \cdots n)}.
\]

We denote simply by \( \mathcal{R}_m(n) \) the field \( \mathcal{R}_m(12 \cdots n) \) in Theorem 2.12. By the definition of \( \rho_n(I) \) in (2.11) and (3.2), the field \( \mathcal{R}_m(n) \) depends only on \( p_1, \ldots, p_{n-1} \). Here is a re-statement of Theorem 2.1.12.

**Theorem 3.5.** Let the notations and assumptions be as above. The extension \( \mathcal{R}_m(n) \) over \( k \) has the following properties:

\[
\begin{align*}
\bullet \quad \mathcal{R}_m(n)/k \text{ is a Galois extension whose Galois group is} \\
\text{to isomorphic to } N_n(\mathbb{Z}/m\mathbb{Z}), \text{ in particular, the degree} \\
of \mathcal{R}_m(n) \text{ over } k \text{ is } m^{\frac{1}{2}(n-1)}, \\
\bullet \quad \mathcal{R}_m(n)/k \text{ is unramified outside } p_1, \ldots, p_{n-1} \text{ and } S_k^\infty, \\
\text{and the ramification index of each } p_i \text{ is } m,
\end{align*}
\]

and further we have

\[
[p_1, p_2, \ldots, p_n]_m = 1 \iff p_n \text{ is completely decomposed in } \mathcal{R}_m(n)/k.
\]

In view of Theorem 3.5, the following problems are important in the arithmetic of multiple power residue symbols.

\[
\begin{align*}
(i) \text{ Do the properties (3.5.1) characterize } \mathcal{R}_m(n)? \\
& \text{If not the case, can one characterize the extension } \mathcal{R}_m(n)/k \\
& \text{in an arithmetic manner?} \\
(ii) \text{ Can one construct the field } \mathcal{R}_m(n) \text{ in a concrete manner?}
\end{align*}
\]

**Example 3.7 (Power residue symbol).** Let \( n = 2 \). We have \( N_2(\mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z} \) and so \( \mathcal{R}_m(2) \) is a cyclic Kummer extension of degree \( m \) over \( k \). The following theorem answers the problem (3.6) (i) affirmatively when \( m > 2 \) or when \( m = 2 \) and \( \mathbb{Q} \).

**Theorem 3.7.1.** Let \( k \) be as above and let \( p_1 \in S_k^\nonl \). We assume that
\(m > 2\) or that \(m = 2\) and \(k = \mathbb{Q}\). Then there is a unique cyclic extension over \(k\) of degree \(m\) which is unramified outside \(p_1\) and \(S_k^\infty\) and totally ramified over \(p_1\). Hence the properties (3.5.1) determine \(\mathfrak{K}_m(2)\) uniquely under the assumption.

For the proof of Theorem 3.7.1, let us recall a known fact on the ramification in a Kummer extension and a classical theorem by Iwasawa. In the following, for a finite prime \(p\) of a finite algebraic number field \(F\), let \(v_p\) denote the normalized additive valuation on \(F_p\).

**Lemma 3.7.2 ([B]).** Let \(F\) be a finite algebraic number field containing a primitive \(l\)-th root of unity, and let \(L\) be a cyclic extension of degree \(l\) over \(F\) so that \(L = F(\sqrt{a})\) for some \(a \in F^\times\). Let \(\mathfrak{p}\) be a finite prime of \(F\) which is not lying over \(l\). Then we have the followings.

1. If \(v_{\mathfrak{p}}(a)\) is prime to \(l\), then \(\mathfrak{p}\) is totally ramified in \(L/F\).
2. If \(v_{\mathfrak{p}}(a)\) is divisible by \(l\), then \(\mathfrak{p}\) is unramified in \(L/F\).

**Lemma 3.7.3 ([Iw1]).** Let \(F\) be a finite algebraic number field whose \(l\)-class number is one, and let \(L\) be a Galois extension over \(F\) of degree a power of \(l\). We assume that there is at most one prime in \(F\) which is ramified in \(L/F\). Then the \(l\)-class number of \(L\) is also one.

**Proof of Theorem 3.7.1.** Suppose that \(m = 2\) and \(k = \mathbb{Q}\) and \(p_1\) is an odd prime. Then it is easy to see that \(\mathbb{Q}(\sqrt{p_1^\ast})\) \((p_1^\ast := (-1)^{(p_1 - 1)/2}p_1)\) is the unique quadratic extension over \(\mathbb{Q}\) which is unramified outside \(p_1\) and the infinite prime \(\infty\).

Suppose \(m > 2\), so that \(S_k^\infty\) is unramified in any extension over \(k\). Since \(\mathfrak{K}_m(2)\) satisfies (3.5.1), it suffices to show the uniqueness. Suppose that \(K_1\) and \(K_2\) are distinct cyclic extensions over \(k\) of degree \(m\) such that they are unramified outside \(p_1\) and \(p_1\) is totally ramified in \(K_i/k\) for \(i = 1, 2\). Let \(F := K_1 \cap K_2\). We note by Lemma 3.7.3 that the \(l\)-class number of \(F\) is 1. Let \(\mathfrak{p}\) be the prime in \(F\) lying over \(p_1\). Since \(K_1 \neq K_2\), there is a (unique) cyclic extension \(L_i\) over \(F\) of degree \(l\) in \(K_i\) for \(i = 1, 2\), where only \(\mathfrak{p}\) is (totally) ramified in \(L_i/F\). By Lemma 3.7.2, there is an integer \(c \in \mathbb{Z}\) such that \(e_i := v_{\mathfrak{p}}(a_i)\) such that

\[L_i = F(\sqrt{a_i}),\quad (e_i, l) = 1\]

for \(i = 1, 2\). Choose an integer \(c \in \mathbb{Z}\) such that \(e_1 + e_2c \equiv 0 \mod l\), and
consider the field $K_3 := F(\sqrt[3]{a_1a_2^5})$. Since we have $v_F(a_1a_2^5) = e_1 + ce_2 \equiv 0 \mod l$, $K_3$ is an unramified cyclic extension over $F$ of degree $l$ by Lemma 3.7.2. By class field theory, this contradicts to that the $l$-class number of $F$ is one. Hence $K_1 = K_2$.

As for the problem (3.6) (ii), we have only the following conditional statement, which follows immediately from the ramification theory of a Kummer extension and Theorem 3.7.1.

**Proposition 3.7.4.** Let $h_k$ be the class number of $k$ and $p_1^{h_k} = (a)$. Then the field $K_m(2)$ is given by

$$K_m(2) = k(\sqrt[2]{a}),$$

if for any prime $l$ of $k$ lying over $l$, there is $\alpha_l \in O_k$ such that $\alpha_l^{d_l} \equiv a \mod l^{d_l}$, where $d_l := v_l(1 - \zeta_m)$.

**Example 3.7.5.** (1) Let $k = \mathbb{Q}$, $l = m = 2$ and $p_1 = (p_1)$. Then we have

$$K_2(2) = \mathbb{Q}(\sqrt{p^*}), \quad p^* := (-1)^{\frac{p-1}{2}} p_1.$$

(2) Let $k = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$, $l = m = 3$ and let $p_1$ satisfy $Np_1 \equiv 1 \mod 9$.

It will be shown in Section 4 (Lemma 4.1 and Proposition 4.2) that there is $\pi_1 \in O_k$ such that

$$K_3(2) = k(\sqrt[3]{\pi_1}), \quad p_1 = (\pi_1), \quad \pi_1 \equiv 1 \mod (3\sqrt{-3}).$$

Recall that for a finite prime $p$ and $a \in k_p^\times$ with $v_p(a) \equiv 0 \mod m$, the $m$-th power residue symbol is defined by

$$\left(\frac{a}{p}\right)_m = \frac{\sigma(\sqrt[m]{a})}{\sqrt[m]{a}},$$

where $\sigma$ is the Frobenius automorphism of the unramified extension $k_p(\sqrt[m]{a})/k_p$.

We note that

$$\left(\frac{a}{p}\right)_m = 1 \iff a \in (k_p^\times)^m.$$

Let $p_1$ and $p_2$ be distinct primes in $S_k^{\text{non-l}}$. Since the conditions (3.1) are obviously satisfied for $n = 2$, the symbol $[p_1, p_2]_m$ is defined.
Theorem 3.7.6. Notations being as above, suppose that $p_1$ is a principal ideal $(\pi_1)$ for $\pi_1 \in \mathcal{O}_k$ such that $\pi_1$ coincides with a fixed prime element of $k_{p_1}$ in Section 1. Then we have

$$[p_1, p_2]_m = \left(\frac{\pi_1}{p_2}\right)_m.$$ 

Proof. Let $S := \{p_1, p_2\}$ and let $\tau_i$ and $\sigma_i$ be a monodromy over $p_i$ and a Frobenius automorphism over $p_i$, respectively, in $k_S(l)/k$ for $i = 1, 2$. Let $k_S(l)^{(2)}$ be the subextension of $k_S(l)/k$ corresponding to the subgroup $\mathfrak{G}_{k,S}(l)^m[\mathfrak{G}_{k,S}(l), \mathfrak{G}_{k,S}(l)]$ of $\mathfrak{G}_{k,S}(l)$. Then $k(\sqrt[2]{\pi_1}) \subset k_S(l)^{(2)}$ and $\tau_1|_{k(\sqrt[2]{\pi_1})}$ is a generator of $\text{Gal}(k(\sqrt[2]{\pi_1})/k)$. By (2.9.1), we have

$$\sigma_2|_{k(\sqrt[2]{\pi_1})} = (\tau_1|_{k(\sqrt[2]{\pi_1})})^{\mu_m(12)},$$

and hence

$$\left(\frac{\pi_1}{p_2}\right)_m = \left(\frac{\sqrt[2]{\pi_1}}{p_2}\right)^{\mu_m(12)} = \left(\frac{\pi_1}{p_2}\right)^{\mu_m(12)} = \zeta_m^{\mu_m(12)} \quad \text{(by (1.2), (1.3))} \quad = [p_1, p_2]_m.$$ 

Example 3.8 (Rédei symbol). Let $k = \mathbb{Q}$ and $l = m = 2$. Let $p_1$ and $p_2$ be distinct rational primes satisfying the conditions

(3.8.1) \quad p_i \equiv 1 \mod 4 \quad (i = 1, 2), \quad \left(\frac{p_1}{p_2}\right) = \left(\frac{p_2}{p_1}\right) = 1.

Let us recall the construction of the Rédei extension ([Rd]). By (3.8.1), there are rational integers $x, y, z \in \mathbb{Z}$ satisfying

(3.8.2) \quad x^2 - p_1y^2 - p_2z^2 = 0, \quad (x, y, z) = 1, \quad y \equiv 0 \mod 2, \quad x - y \equiv 1 \mod 4.

Let $\alpha := x + \sqrt{p_1}y$. We define the extension $\mathfrak{A}_{\{p_1, p_2\}}$ over $\mathbb{Q}$ by

(3.8.3) \quad \mathfrak{A}_{\{p_1, p_2\}} := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha})$
and call it the Rédei extension over \( \mathbb{Q} \) associated to \( \{p_1, p_2\} \). In fact, it can be shown that \( R_{\{p_1, p_2\}} \) is independent of a choice of \( \alpha \) and that \( R_{\{p_1, p_2\}}/\mathbb{Q} \) is a Galois extension whose Galois group is isomorphic to the dihedral group \( D_8 \) of order 8 and it is unramified outside \( p_1, p_2 \) and the infinite prime with ramification index for each \( p_i \) being 2 ([A1], [R]). Moreover the first author gave the following arithmetic characterization of the Rédei extension.

**Theorem 3.8.4 ([A1; Theorem 2.1]).** Let \( p_1 \) and \( p_2 \) be rational primes satisfying (3.8.1). For a finite algebraic number field \( K \), the following conditions are equivalent:

1. \( K/\mathbb{Q} \) is a Galois extension whose Galois group is isomorphic to the dihedral group \( D_8 \) of order 8 and it is unramified outside \( p_1, p_2 \) and the infinite prime with ramification index for each \( p_i \) being 2.
2. \( K \) is the Rédei extension \( R_{\{p_1, p_2\}} \).

Noting \( N_3(\mathbb{F}_2) \simeq D_8 \), Theorem 3.8.4 answers the problems (3.6) (i) and (ii) affirmatively.

**Corollary 3.8.5.** Let \( p_1 \) and \( p_2 \) be as above. Then the properties (3.5.1) determines uniquely the extension \( R_2(3) \) over \( \mathbb{Q} \) and moreover we have

\[
R_2(3) = R_{\{p_1, p_2\}}.
\]

Let \( p_1, p_2 \) and \( p_3 \) be distinct rational primes satisfying the conditions

\[
p_i \equiv 1 \mod 4 \ (1 \leq i \leq 3), \quad \left( \frac{p_i}{p_j} \right) = 1 \ (1 \leq i \neq j \leq 3).
\]

Then the Rédei symbol is defined by

\[
[p_1, p_2, p_3]_{\text{Rédei}} := \begin{cases} 
1 & \text{if } p_3 \text{ is completely decomposed} \\
-1 & \text{otherwise}.
\end{cases}
\]

On the other hand, by (3.8.6), the conditions (3.1) are satisfied so that the mod 2 arithmetic invariant \( \mu_2(123) \) in (3.3) and hence the triple quadratic residue symbol \( [p_1, p_2, p_3]_2 \) in Definition 3.4 is defined. By Theorem 3.5, Corollary 3.8.5 and the definition (3.8.7), we have the following

\[
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\]
Theorem 3.8.8. We have

\[ [p_1, p_2, p_3] = [p_1, p_2, p_3]_{\text{Rédei}}. \]

Example 3.9. Let \( k = \mathbb{Q} \) and \( l = m = 2 \). Let \( p_1, p_2, p_3 \) and \( p_4 \) be distinct rational primes satisfying the conditions

\[
\begin{align*}
& p_i \equiv 1 \pmod{4} \ (1 \leq i \leq 4), \\
& \left( \frac{p_1}{p_2} \right) = 1 \ (1 \leq i \neq j \leq 4), \\
& [p_i, p_j, p_k]_2 = 1 \ (i, j, k \text{ are all distinct}).
\end{align*}
\]

Then the first author ([A2]) constructed concretely a Galois extension \( \mathfrak{K} \) of degree 64 over \( \mathbb{Q} \) such that the Galois group \( \text{Gal}(\mathfrak{K}/\mathbb{Q}) \) is isomorphic to \( N_4(\mathbb{F}_2) \) and it is unramified outside \( p_1, p_2, p_3 \) and the infinite prime, and the ramification index for each \( p_i \) is 2, and further showed that

\[ [p_1, p_2, p_3, p_4]_2 = 1 \iff p_4 \text{ is completely decomposed in } \mathfrak{K}/\mathbb{Q}. \]

However, we do not know whether \( \mathfrak{K}(2) = \mathfrak{K} \) or not. So, we are not able to answer neither the problem (3.6) (i) nor (ii).

§4 Triple cubic residue symbols

In this section, as a new example of multiple power residue symbols, we study a triple cubic residue symbol \([p_1, p_2, p_3]_3\) for certain primes \( p_1, p_2 \) and \( p_3 \) of the cubic cyclotomic field \( k = \mathbb{Q}(\zeta_3) \). For this, we construct a Heisenberg extension \( \mathfrak{K}_3(3) \) of degree 27 over \( k \) in Section 3 concretely, which may be regarded as a cubic generalization of the Rédei extension in (3.8.3), and describe the symbol \([p_1, p_2, p_3]_3\) arithmetically in the extension \( \mathfrak{K}_3(3)/k \).

Let \( \zeta_3 \) be the primitive cubic root of unity, \( \zeta_3 := \frac{-1 + \sqrt{-3}}{2} \), and let \( k \) be the cubic cyclotomic field \( \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3}) \). We note that the ring of integers of \( k \) is \( \mathcal{O}_k = \mathbb{Z}[\zeta_3] \), the unit group is \( \mathcal{O}_k^\times = \{ \pm \zeta_3^e \mid e = 0, 1, 2 \} \) and the class number of \( k \) is one. We denote by \( \mathfrak{l} \) the (unique) maximal ideal of \( \mathcal{O}_k \) lying over 3, \( \mathfrak{l} = (1 - \zeta_3) \). Note that \( \mathfrak{l}^3 = (3\sqrt{-3}) \). As in the previous sections, let \( S_k^{\text{non-3}} \) be the set of finite primes of \( k \) which are not lying over 3.
Lemma 4.1. Let $p$ be a finite prime in $S_{k}^{\text{non-3}}$ satisfying $Np \equiv 1 \mod 9$. Then there exists uniquely $\pi \in \mathcal{O}_k$ which satisfies

$$p = (\pi), \quad \pi \equiv 1 \mod (3\sqrt{3}).$$

Proof. Since the class number of $k$ is one, there is $\pi' \in \mathcal{O}_k$ such that $p = (\pi')$. So $N_{k/Q}(\pi') = \pm Np \equiv \pm 1 \mod 9$. Since it is easy to see that $N_{k/Q}(\alpha) \equiv 0$ or $1 \mod 3$ for any $\alpha \in \mathcal{O}_k$, we see that

(4.1.1) \hspace{1cm} N_{k/Q}(\pi') \equiv 1 \mod 9.

Let $\mathfrak{U} := (\mathcal{O}_k/(3\sqrt{3}))^\times$. Since $(3\sqrt{3}) = \mathfrak{l}^3$ and $\mathcal{O}_k/\mathfrak{l} = \mathbb{Z}/3\mathbb{Z}$, we have

(4.1.2) \hspace{1cm} \mathfrak{U} = \{a_0 + a_1\sqrt{-3} + a_2(\sqrt{-3})^2 \mod (3\sqrt{3}) \mid a_0 = 1, 2, a_1, a_2 = 0, 1, 2\}

$$= \{a + b\sqrt{-3} \mod (3\sqrt{3}) \mid a = 1, 2, 4, 5, 7, 8, \quad b = 0, 1, 2\}.$$  

Let $\mathfrak{U}^1$ be the subgroup of $(\mathcal{O}_k/(3\sqrt{3}))^\times$ consisting of $\alpha \mod (3\sqrt{3})$ with $N_{k/Q}(\alpha) = 1$. Here we note that $N_{k/Q}(\alpha) \equiv N_{k/Q}(\alpha') \mod 9$ if $\alpha \equiv \alpha' \mod (3\sqrt{3})$ and hence $\mathfrak{U}^1$ is well defined. By the straightforward calculation using (4.1.2), we have

(4.1.3) \hspace{1cm} \mathfrak{U}^1 = \{a + b\sqrt{-3} \mod (3\sqrt{3}) \mid (a, b) = (1, 0), (8, 0), (4, 1), (4, 2), (5, 1), (5, 2)\}

$$= \langle -1 \mod (3\sqrt{3}) \rangle \times \langle \zeta_3 \mod (3\sqrt{3}) \rangle,$$

where $-1 \equiv 8, \zeta_3 \equiv 4 + 2\sqrt{-3} \mod (3\sqrt{3})$. By (4.1.1) and (4.1.3), there is a unit $\varepsilon \in \mathcal{O}_k^\times$ such that $\pi := \varepsilon\pi' \equiv 1 \mod (3\sqrt{3})$ and $(\pi) = (\pi')$.

Next, suppose that $p = (\pi) = (\varpi)$ and $\pi \equiv \varpi \equiv 1 \mod (3\sqrt{3})$. We can write $\varpi = \varepsilon\pi$ for some $\varepsilon \in \mathcal{O}_k^\times$. So $\varepsilon \equiv \varepsilon\pi = \varpi \equiv 1 \mod (3\sqrt{3})$. By (4.1.3), we must have $\varepsilon = 1$ and hence $\varpi = \pi$. \hfill \Box

Let $\pi$ be as in Lemma 4.1. Let $K := k(\sqrt[3]{\pi})$ so that $K$ is a cyclic Kummer extension of degree 3 over $k$.

Proposition 4.2. Notations being as above, we have the followings.

(1) The extension $K/k$ is unramified outside $p$ and $p$ is totally ramified.
(2) The $l$-class number of $k(\sqrt[3]{\pi})$ is one.

Proof. (1) Since $(\pi) = p$, $p$ is totally ramified by Lemma 3.7.2 (1). Let $\lambda := \frac{\sqrt[3]{\pi} - 1}{\sqrt{-3}}$. Since $\lambda$ satisfies $\lambda^3 - \sqrt{-3}\lambda^2 - \lambda - \frac{1 - \sqrt{-3}}{3 - \sqrt{-3}} = 0$ and $\frac{1 - \sqrt{-3}}{3 - \sqrt{-3}} \in O_k$ by Lemma 4.1, we find $\lambda \in O_k$. The relative discriminant of $\lambda$ in $k(\sqrt[3]{\pi})/k$ is computed as

$$d(\lambda, K/k) = \left| \begin{array}{c} 1 \\ 1 \\ 1 \\ \end{array} \right| \left( \begin{array}{c} \lambda^{(1)} \\ \lambda^{(2)} \\ \lambda^{(3)} \\ \end{array} \right)^2 = -\frac{\pi^2}{27} \left| \begin{array}{ccc} 1 & 1 & 1 \\ \zeta_3 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3^2 & \zeta_3 \\ \end{array} \right|^2 = \pi^2,$$

where $\lambda^{(1)} := \lambda$, $\lambda^{(2)} := (\zeta_3 \sqrt[3]{\pi} - 1)/\sqrt{-3}$ and $\lambda^{(2)} := (\zeta_3^2 \sqrt[3]{\pi} - 1)/\sqrt{-3}$. So $K/k$ is unramified outside $p$.

(2) This follows from Lemma 3.7.3. $\Box$

**Proposition 4.3.** For any $\varepsilon \in O_K^\times$, there is $u \in O_K^\times$ such that $N_{K/k}(u) = \varepsilon$.

For the proof, we prepare a Lemma. For a prime $q$ of $k$, let $(\frac{a}{q})$ the cubic Hilbert symbol in the local field $k_q$ defined by

$$(a, k_q(\sqrt[3]{b})/k_q) \sqrt[3]{b} = \left( \frac{a}{q} \right)_3 \sqrt[3]{b} \quad (a, b \in k_q^\times),$$

where $( , k_q(\sqrt[3]{b})/k_q) : k_q^\times \to Gal(k_q(\sqrt[3]{b})/k_q)$ is the norm residue symbol of local class field theory.

**Lemma 4.4.** For any prime $q$ of $k$, we have

$$\left( \frac{\pi, \zeta_3}{q} \right)_3 = 1.$$

Proof. We verify the assertion according to the four cases.

Case 1. $q$ is prime to $p$, $3$, $\infty$: Then $\pi$ and $\zeta_3$ are in the local unit group of $k_q$, and hence $\left( \frac{\pi, \zeta_3}{q} \right)_3 = 1$ by local class field theory.

Case 2. $q = p$: Since $Np \equiv 1 \pmod{9}$, $\sqrt[3]{\zeta_3} \in k_p$ and hence $k_p(\sqrt[3]{\zeta_3}) = k_p$. Therefore the assertion immediately follows.

Case 3. $q = \infty$: Since $k = \mathbb{C}$, the assertion is obvious.
Case 4. \( q | 3: q = (\sqrt{-3}) \), the above cases and the product formula for the Hilbert symbol yield \( (\frac{\pi \cdot q}{\sqrt{-3}})_3 = 1 \). \( \square \)

Proof of Proposition 4.3. Since \( \mathcal{O}_K^\times = \{ \pm \zeta_3^e \mid e = 0, 1, 2 \} \) and \( N_{K/k}(-1) = -1 \), it suffices to show that there is \( u \in \mathcal{O}_K^\times \) such that \( N_{K/k}(u) = \zeta_3 \). As we easily see, it follows from the weaker assertion \( \zeta_3 \in N_{K/k}(K^\times) \). By the Hasse norm principle for the cyclic extension \( K/k \) ([Ta]), we need to show that for any prime \( q \) of \( k \), \( \zeta_3 \in N_{k_q}(\sqrt{\pi})_{k_q}/k_q(k_q(\sqrt{\pi})^\times) \), which follows from Lemma 4.4. \( \square \)

Now, let \( p_1 \) and \( p_2 \) be distinct finite primes of \( k \) satisfying \( Np \equiv 1 \mod 9 \). By Lemma 4.1, there exists uniquely \( \pi_i \in \mathcal{O}_k \) such that \( p_i = (\pi_i), \pi_i \equiv 1 \mod (3\sqrt{-3}) \) \( (i = 1, 2) \).

We set \( K_i := k(\sqrt[3]{\pi_i}) \) \( (i = 1, 2) \).

In the following, we assume

\[
\left( \frac{\pi_1}{\pi_2} \right)_3 = \left( \frac{\pi_2}{\pi_1} \right)_3 = 1,
\]

which is equivalent to that \( p_1 \) (resp. \( p_2 \)) is completely decomposed in \( K_2/k \) (resp. \( K_1/k \)). Let \( \mathfrak{P} \) be a fixed prime of \( K_1 \) lying over \( p_2 \).

Proposition 4.6. There is an algebraic integer \( \alpha \) in \( K_1 \) which satisfies the following properties.
(1) \( N_{K_1/k}(\alpha) = \pi_2^{-3} \beta^3 \) for some \( \beta \in k \).
(2) The principal ideal \( (\alpha) \) has the decomposition of the form

\[
(\alpha) = \mathfrak{P}^a \mathfrak{B}^b, \quad (a, 3) = 1, \quad (\mathfrak{B}, 3) = 1, \quad b \equiv 0 \mod 3.
\]

Proof. (1) Let \( h \) be the class number of \( K_1 \) and \( \mathfrak{P}^h = (\alpha') \) for some \( \alpha' \in \mathcal{O}_{K_1} \). Since \( p_2 \) is completely ramified in \( K_1/k \), we have \( N_{K_1/k}((\alpha')) = N_{K_1/k}\mathfrak{P}^h = (\pi_2^h) \) and so \( N_{K_1/k}(\alpha') = \varepsilon \pi_2^h \) for some \( \varepsilon \in \mathcal{O}_k^\times \). By Proposition 4.3, there is \( u \in \mathcal{O}_{K_1}^\times \) such that \( N_{K_1/k}(u) = \varepsilon \). Letting \( \alpha := u\alpha' \), we have \( N_{K_1/k}(\alpha) = \pi_2^h \).

Since \( h \) is prime to 3 by Proposition 4.2 (2), the assertion follows.
(2) This follows immediately from (1). \( \square \)
Let $\alpha$ be an element on $O_{K_1}$ satisfying the properties (1), (2) of Lemma 4.6. Let $\tau$ be the element of $\text{Gal}(K_1/k)$ defined by

$$\tau(\sqrt[3]{\pi_1}) := \zeta_3 \sqrt[3]{\pi_1}. $$

Then we have $\text{Gal}(K_1/k) = \langle \tau \mid \tau^3 = 1 \rangle$. We set

$$\begin{cases}
\alpha^{(1)} := \alpha, \\
\alpha^{(2)} := \tau(\alpha), \\
\alpha^{(3)} := \tau^2(\alpha),
\end{cases} \quad \begin{cases}
\mathfrak{P}^{(1)} := \mathfrak{p}, \\
\mathfrak{P}^{(2)} := \tau(\mathfrak{p}), \\
\mathfrak{P}^{(3)} := \tau^2(\mathfrak{p}),
\end{cases}
$$

where $\mathfrak{P}^{(1)}, \mathfrak{P}^{(2)}$ and $\mathfrak{P}^{(3)}$ are distinct all prime ideals of $O_{K_1}$ lying over $p_2$. By Proposition 4.6, we easily see the following.

**Theorem 4.7.** Let $\theta := \zeta_3^e \alpha^{(1)}^2 \alpha^{(2)} \in O_{K_1}$ ($e = 0, 1, 2$). Then $\theta$ satisfies the following property:

(1) $N_{K_1/k}(\theta) = \pi_2^3 w^3$ for some $w \in k$. Writing $\theta = x + y\sqrt[3]{\pi} + z(\sqrt[3]{\pi})^2$ ($x, y, z \in k$), it is written as

$$x^3 + \pi_1 y^3 + \pi_1^2 z^3 - 3\pi_1 x y z = \pi_2^3 w^3.$$

(2) $(\theta) = (\mathfrak{P}^{(1)})^{2e}(\mathfrak{P}^{(2)})^e \mathfrak{A}^a$, $(e, 3) = 1$, $(\mathfrak{A}, 3) = 1$ ($i = 1, 2$), $a \equiv 0 \mod 3$.

In the following, we assume that $\theta$ satisfies the properties (1), (2) of Theorem 4.7 and the following condition: There is $\eta \in O_{K_1}$ such that

$$(4.7.3) \quad \eta^3 \equiv \theta \mod (3\sqrt{-3}).$$

A sufficient condition for (4.7.3) to hold is given as follows. Let $\mathfrak{U}_{K_1} := (O_{K_1}/(3\sqrt{-3}))^\times$ and let $\mathfrak{U}_{K_1}(3)$ denote the 3-Sylow subgroup of $\mathfrak{U}_{K_1}$.

**Proposition 4.8.** (1) The group $\mathfrak{U}_{K_1}(3)$ is given by

$$\mathfrak{U}_{K_1}(3) = \langle \mathfrak{a}_1 \rangle \times \langle \mathfrak{a}_2 \rangle \times \langle \mathfrak{a}_3 \rangle \times \langle \mathfrak{a}_4 \rangle \times \langle \mathfrak{a}_5 \rangle \times \langle \mathfrak{a}_6 \rangle, \quad \langle \mathfrak{a}_i \rangle \simeq \mathbb{Z}/3\mathbb{Z},$$

where $a_1 := 4, a_2 := \zeta_3, a_3 := \sqrt[3]{\pi_1}, a_4 := 1 + \sqrt{-3} \sqrt[3]{\pi_1} = 1 + \sqrt{-3} + 3\lambda, a_5 := -1 + 3\sqrt[3]{\pi_1} - (\sqrt[3]{\pi_1})^2 = 1 + \sqrt{-3}\lambda + 3\lambda^2, a_6 := \frac{1}{\sqrt[3]{3}}(1 + (\sqrt{-3} - 2) \sqrt[3]{\pi_1} + (\sqrt[3]{\pi_1})^2) = 1 + \sqrt{-3}\lambda + \sqrt{-3}\lambda^2$ with $\lambda := \frac{1}{\sqrt[3]{3}}(\sqrt[3]{\pi_1} - 1)$, and $\mathfrak{a}_i := a_i \mod (3\sqrt{-3})$. 

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(2) Let \( \alpha \) be as in Lemma 4.6 and let \( b \) be an integer prime to 3. Then we have \( \alpha^b \in \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \langle \alpha_3 \rangle \times \langle \alpha_4 \rangle \). If we have \( \alpha^b \in \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \langle \alpha_3 \rangle \), then the condition (4.7.3) is satisfied.

Proof. (1) Since \((\sqrt{-3})\) is unramified in \( K/k \), we can write \((\sqrt{-3}) = q_1 \cdots q_r\) with prime ideals \( q_i \)’s of \( O_K \). Then we have

\[
U \simeq (O_{K_1}/q_1^3 \cdots q_r^3)^\times \simeq (O_{K_1}/q_1^3)^\times \times \cdots \times (O_{K_1}/q_r^3)^\times.
\]

We note that \#\( U = \prod_{i=1}^r Nq_i^3(Nq_i - 1) = N((\sqrt{-3})^2) \prod_{i=1}^r (Nq_i - 1) = 729b \), where \( b = \prod_{i=1}^r (Nq_i - 1) \) is an integer prime to 3, and that \( a_i^3 \equiv 1 \mod (3\sqrt{-3}) \). Define \( \tau \in \text{Gal}(K_1/k) \), \( \tau(\sqrt{3}) = \zeta_3 \sqrt{3} \). Then we have

\[
\tau(a_3) \equiv a_2 \cdot a_3, \quad \tau(a_4) \equiv a_1 \cdot a_4 \mod (3\sqrt{-3})
\]

and hence \( \overline{a_i} \not\in (\overline{a_i}) \). We note that \( N_{K_1/k}(a_i) \equiv 1 \mod (3\sqrt{-3}) \) and

\[
N_{K_1/k}(a_5) = -1 + 27\pi - \pi^2 \mod (3\sqrt{-3}) = 7 \mod (3\sqrt{-3})
\]

\[
N_{K_1/k}(a_6) = \sqrt{-3}(1 + (\sqrt{-3} - 2)^2\pi + \pi^2 - 3 \cdot (\sqrt{-3} - 2) \cdot 1 \cdot \pi)
\]

\[
\equiv \sqrt{-3}(1 + (16 + 6\sqrt{-3})\pi + \pi^2) = \frac{\sqrt{-3}}{9}((1 - \pi_1)^2 + (18 + 6\sqrt{-3})\pi_1)
\]

\[
= \frac{\sqrt{-3}}{9}(1 - \pi_1)^2 + (2\sqrt{-3} - 2)\pi_1
\]

\[
\equiv (2\sqrt{-3} - 2)\pi_1 \mod (3\sqrt{-3}) = 7 + 2\sqrt{-3} \mod (3\sqrt{-3})
\]

and hence \( \overline{a_5} \not\in \prod_{i=1}^4(\overline{a_i}) \), \( \overline{a_6} \not\in \prod_{i=1}^5(\overline{a_i}) \). Since the order of the group \( \prod_{i=1}^6(\overline{a_i}) \) is \( 3^6 = 729 \), it must be \( \mathcal{U}_{K_1}(3) \).

(2) Since \((\alpha)\) is prime to 3, we have \( \alpha^b \in \mathcal{U}_{K_1} \). Since \( N_{K_1/k}(\alpha) \equiv 1 \mod (3\sqrt{-3}) \), by (1), we have \( \overline{\alpha^b} \in \prod_{i=1}^4(\overline{a_i}) \). Suppose \( \overline{\alpha^b} \in \prod_{i=1}^4(\overline{a_i}) \). Then we can write \( \overline{\alpha^b} = \overline{a_1^{b_1}} \cdot \overline{a_2^{b_2}} \cdot \overline{a_3^{b_3}} \) and \( \tau(\alpha) = \overline{a_1^{b_1}} \cdot \overline{a_2^{b_2+b_3}} \cdot \overline{a_3^{b_3}} \) by (1). Therefore we have \( \theta \equiv 1 \) with \( \theta = \zeta_3 a_1^{b_1} a_2^{b_2} \). \( \square \)

Let \( \theta \) be an element of \( O_{K_1} \) satisfying (1), (2) of Theorem 4.7 and (4.7.3). We then set

\[
\mathfrak{R}_\theta := k(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta}).
\]

Remark 4.9. The field \( \mathfrak{R}_\theta \) will be proved to be independent of the choice
of $\theta$ and depends only on the set $\{p_1, p_2\}$ (see Theorem 4.12, Theorem 4.13 below), just like the classical Rédei extension $\mathcal{R}_{\{p_1, p_2\}}$ over $\mathbb{Q}$ in (3.8.3). More precisely, the properties (1), (2) of Theorem 4.7 and (4.7.1) correspond to the conditions in (3.8.2) as follows: the equation $x^3 + \pi_1 y^3 + \pi_1^2 z^3 - 3\pi_1 xyz = \pi_3^2 w^3$ corresponds the equation $x^2 - p_1 y^2 = p_2 z^2$, the property (2) corresponds to $(x, y, z) = 1$ and the condition (4.7.3) corresponds to $y \equiv 0 \mod 2$ and $x - y \equiv 1 \mod 4$.

We let

$$
\begin{align*}
\theta^{(1)} := \theta &= \zeta_3 \alpha_1^2 \alpha_2, \\
\theta^{(2)} := \tau(\theta) &= \zeta_3 \alpha_2^2 \alpha_3, \\
\theta^{(3)} := \tau^2(\theta) &= \zeta_3 \alpha_3^2 \alpha_1.
\end{align*}
$$

Theorem 4.10. (1) We have $\mathcal{R}_\theta = k(\sqrt[3]{\theta}^{(1)}, \sqrt[3]{\theta}^{(2)}, \sqrt[3]{\theta}^{(3)})$.
(2) The extension $\mathcal{R}_\theta/k$ is a Galois extension such that it is unramified outside $p_1$ and $p_2$ and the ramification index of each $p_i$ is 3.
(3) The Galois group $\text{Gal}(\mathcal{R}_\theta/k)$ is isomorphic to $N_3(\mathbb{F}_3)$.

Proof. (1) $k(\sqrt[3]{\theta}^{(1)}, \sqrt[3]{\theta}^{(2)}, \sqrt[3]{\theta}^{(3)}) \subseteq \mathcal{R}_\theta$: Since $\sqrt[3]{\pi_2}, \sqrt[3]{\pi_1} \in \mathcal{R}_\theta$ and we see

$$
\theta^{(1)} \theta^{(2)2} = \pi_2^2 (\alpha^{(2)} \beta)^3
$$

we have $\sqrt[3]{\theta^{(2)}} \in \mathcal{R}_\theta$ and hence the assertion is proved.

$\mathcal{R}_\theta \subseteq k(\sqrt[3]{\theta}^{(1)}, \sqrt[3]{\theta}^{(2)}, \sqrt[3]{\theta}^{(3)})$: Writing $\theta = x + y \sqrt[3]{\pi_1} + z \sqrt[3]{\pi_2}$, we have

$$
\zeta_3 \theta^{(1)} + \theta^{(2)} + \zeta_3^2 \theta^{(3)} = 3 \zeta_3 y \sqrt[3]{\pi_1}.
$$

So, $\sqrt[3]{\pi_1} \in k(\sqrt[3]{\theta^{(1)}}, \sqrt[3]{\theta^{(2)}}, \sqrt[3]{\theta^{(3)}})$ if $y \neq 0$. If $y = 0$, then $z \neq 0$ and $\zeta_3 \theta^{(1)} + \theta^{(2)} = (1 + \zeta_3) x + z \zeta_3 (\sqrt[3]{\pi_1})^2$. Hence $\sqrt[3]{\pi_1} \in k(\sqrt[3]{\theta^{(1)}}, \sqrt[3]{\theta^{(2)}}, \sqrt[3]{\theta^{(3)}})$. Using $\theta^{(1)} \theta^{(2)} = \pi_3^2 (\alpha^{(2)} \beta)^3$ again, we have $\sqrt[3]{\pi_2} \in k(\sqrt[3]{\theta^{(1)}}, \sqrt[3]{\theta^{(2)}}, \sqrt[3]{\theta^{(3)}})$. Thus the assertion is proved.

(2) Since $\mathcal{R}_\theta$ is the splitting field of $\prod_{i=1}^3 (T^3 - \theta_i) \in \mathcal{O}_k[T]$, $\mathcal{R}_\theta$ is a Galois extension over $k$. We shall show that only $p_1$ and $p_2$ are ramified in $\mathcal{R}_\theta/k$ with ramification index 3. Since $\sqrt[3]{\theta} \notin K_1$ by Theorem 4.7 (2), we have $[K_1(\sqrt[3]{\theta}) : K_1] = 3$. Let $\xi := \sqrt[3]{-3(\eta - \sqrt[3]{\theta})}/3$. Then $\xi \in \mathcal{O}_{K_1(\sqrt[3]{\theta})}$; since $\xi$ satisfies $\xi^3 - \sqrt[3]{-3} \eta \xi^2 - \eta^2 \xi + \sqrt[3]{-3}(1 - \theta)/9 = 0$ and $\sqrt[3]{-3}(\eta^3 - \theta)/9 \in \mathcal{O}_{K_1}$ by
\( \eta^3 \equiv \theta \mod 3\sqrt{-3} \) (4.7.3). The relative discriminant of \( \xi \) in \( K_1(\sqrt[3]{\bar{\theta}})/K_1 \) is computed as

\[
d(\xi, K_1(\sqrt[3]{\bar{\theta}})/K_1) = \begin{vmatrix}
1 & \xi^{(1)} & (\xi^{(1)})^2 \\
1 & \xi^{(2)} & (\xi^{(2)})^2 \\
1 & \xi^{(3)} & (\xi^{(3)})^2
\end{vmatrix}^2 = -\frac{\theta^2}{27} \begin{vmatrix}
1 & 1 & 1 \\
1 & \zeta_3 & \zeta_2^2 \\
1 & \zeta_2 & \zeta_3
\end{vmatrix} = \theta^2,
\]

where \( \xi^{(1)} := \xi, \xi^{(2)} := \sqrt[3]{-3(\eta - \zeta_3 \sqrt[3]{\bar{\theta}})/3} \) and \( \xi^{(3)} := \sqrt[3]{-3(\eta - \zeta_2 \sqrt[3]{\bar{\theta}})/3} \). So, only \( \mathfrak{P}^{(1)} \) and \( \mathfrak{P}^{(2)} \) are ramified in \( K_1(\sqrt[3]{\bar{\theta}})/K_1 \). Similarly, only \( \mathfrak{P}^{(2)} \) and \( \mathfrak{P}^{(3)} \) are ramified in \( K_1(\sqrt[3]{\bar{\theta}^{(2)}})/K_1 \). Since \( \mathfrak{R}_\theta = K_1(\sqrt[3]{\bar{\theta}^{(1)}}) \cdot K_1(\sqrt[3]{\bar{\theta}^{(2)}}) \) and only \( p_1 \) is ramified in \( K_1/k \), we conclude that only \( p_1 \) and \( p_2 \) are ramified in \( \mathfrak{R}_\theta/k \) and their ramification indices are 3.

(3) First, we show \([\mathfrak{R}_\theta : k] = 27\). By (2), \( K_1(\sqrt[3]{\bar{\theta}})/K_1 \) is a cyclic extension of degree 3 where only \( \mathfrak{P}^{(1)} \) and \( \mathfrak{P}^{(2)} \) are ramified. By Proposition 4.2 (1), \( K_1(\sqrt[3]{\bar{\theta}^2})/K_1 \) is a cyclic extension of degree 3 where only \( \mathfrak{P}_2 \) is ramified. So, \( K_1(\sqrt[3]{\bar{\theta}}) \cap K_1(\sqrt[3]{\bar{\theta}^2}) = K_1 \). Since \( \mathfrak{R}_\theta = K_1(\sqrt[3]{\bar{\theta}}) \cdot K_1(\sqrt[3]{\bar{\theta}^2}) \), \( [\mathfrak{R}_\theta : K_1] = [K_1(\sqrt[3]{\bar{\theta}}) : K_1][K_1(\sqrt[3]{\bar{\theta}^2}) : K_1] = 9 \). Hence \( [\mathfrak{R}_\theta : k] = [\mathfrak{R}_\theta : K_1][K_1 : k] = 27 \).

By the computer calculation using GAP, we have the following presentation of the group \( N_3(\mathbb{F}_3) \):

\[
N_3(\mathbb{F}_3) = \langle g_1, g_2, g_3 \mid g_1^3 = g_2^3 = g_3^3 = 1, g_2g_1g_3g_2g_1g_3 = g_1g_3g_2g_1g_3 = g_1g_3g_2g_1g_3, g_3g_2 = g_2g_3 \rangle.
\]

where \( g_1, g_2 \) and \( g_3 \) are words representing the following matrices respectively:

\[
g_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad g_2 = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad g_3 = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} (= [g_2, g_1]).
\]

On the other hand, we define \( \gamma_1, \gamma_2, \gamma_3 \in \text{Gal}(\mathfrak{R}_\theta/k) \) by

\[
\begin{align*}
\gamma_1 &: (\sqrt[3]{\bar{\pi}_1}, \sqrt[3]{\bar{\pi}_2}, \sqrt[3]{\bar{\theta}_1}, \sqrt[3]{\bar{\theta}_2}, \sqrt[3]{\bar{\theta}_3}) \\
&\mapsto (\zeta_3 \sqrt[3]{\bar{\pi}_1}, \sqrt[3]{\bar{\pi}_2}, \sqrt[3]{\bar{\theta}_2}, \sqrt[3]{\bar{\theta}_3}) \\
\gamma_2 &: (\sqrt[3]{\bar{\pi}_1}, \sqrt[3]{\bar{\pi}_2}, \sqrt[3]{\bar{\theta}_1}, \sqrt[3]{\bar{\theta}_2}, \sqrt[3]{\bar{\theta}_3}) \\
&\mapsto (\sqrt[3]{\bar{\pi}_1}, \zeta_3 \sqrt[3]{\bar{\pi}_2}, \sqrt[3]{\bar{\theta}_1}, \sqrt[3]{\bar{\theta}_2}, \sqrt[3]{\bar{\theta}_3}) \\
\gamma_3 &: (\sqrt[3]{\bar{\pi}_1}, \sqrt[3]{\bar{\pi}_2}, \sqrt[3]{\bar{\theta}_1}, \sqrt[3]{\bar{\theta}_2}, \sqrt[3]{\bar{\theta}_3}) \\
&\mapsto (\sqrt[3]{\bar{\pi}_1}, \sqrt[3]{\bar{\pi}_2}, \zeta_3^2 \sqrt[3]{\bar{\theta}_1}, \zeta_3^2 \sqrt[3]{\bar{\theta}_2}, \zeta_3^2 \sqrt[3]{\bar{\theta}_3}).
\end{align*}
\]
Then we have
\[ \gamma_1^3 = \gamma_2^3 = \gamma_3^3 = \text{id}, \gamma_2 \gamma_1 = \gamma_3 \gamma_1, \gamma_3 \gamma_1 = \gamma_1 \gamma_3 \text{ and } \gamma_3 \gamma_2 = \gamma_2 \gamma_3, \]
equivalently,
\[ (4.10.2) \quad \gamma_3 = [\gamma_2, \gamma_1], \; \gamma_1^3 = \gamma_2^3 = (\gamma_1 \gamma_2^2)^3 = (\gamma_1^2 \gamma_2^3)^3 = 1. \]
Thus the correspondence \( g_i \mapsto \gamma_i \) (\( i = 1, 2 \)) gives a homomorphism \( \kappa : N_3(F_3) \to \text{Gal}(R_\theta/k) \). Since we easily see that the fixed subfields of \( R_\theta \) by \( \langle \gamma_2 \rangle \) and \( \langle \gamma_3 \rangle \) are \( K_1(\sqrt[3]{\theta}) \) and \( K_1(\sqrt[3]{\pi}) \), respectively, \( \text{Gal}(R_\theta/k) \) is generated by \( \gamma_1 \) and \( \gamma_2 \), and so \( \kappa \) is surjective. Since \( N_3(F_3) \) and \( \text{Gal}(R_\theta/k) \) have the same order 27, \( \kappa \) is an isomorphism. \( \Box \)

All subgroups of \( \text{Gal}(R_\theta/k) \) and the corresponding subfields of \( R_\theta \) are illustrated as follows.

![Diagram of subfields and subgroups](image-url)
where $h_i$ are words representing the following matrices respectively:

$$
\begin{align*}
    h_1 &= g_1 g_3, \\
    h_2 &= g_1 g_2, \\
    h_4 &= g_2 g_1^2, \\
    h_5 &= g_2 g_1 g_3, \\
    h_6 &= g_2 g_2^2, \\
    h_7 &= g_2 g_1, \\
    h_8 &= g_2 g_1 g_3, \\
    h_9 &= g_2 g_1 g_2^2, \\
    h_{10} &= g_2 g_3, \\
    h_{11} &= g_2 g_2^2 g_3.
\end{align*}
$$

We shall show that the field $R_\theta$ is independent of the choice of $\theta$ satisfying the conditions (1), (2) of Theorem 4.7 and (4.7.3), and hence depends only on the ordered pair $(p_1, p_2)$.

**Theorem 4.11** The field $R_\theta$ is independent of the choice of $\theta \in O_{K_1}$ satisfying the conditions (1), (2) of Theorem 4.7 and (4.7.3).

**Proof.** Let $\theta'$ be another integers satisfying the conditions (1), (2) of Theorem 4.7 and (4.7.3). Then, we have

$$(\theta') = (\mathfrak{p}^{(1)} e_{\theta'}^2) (\mathfrak{p}^{(2)} e_{\theta'}^3) \mathfrak{a}^3 \quad (3 \not| e', \mathfrak{a} \text{ is an ideal of } O_{K_1}), \quad (\theta', 3) = 1.$$ 

We shall show that $K_1(\sqrt[3]{\theta}) \cong K_1(\sqrt[3]{\theta'})$. By $3 \not| e$ and $3 \not| e'$, we have $e + e' \equiv 0 \pmod{3}$ or we have $e + 2e' \equiv 0 \pmod{3}$. If $e + e' \equiv 0 \pmod{3}$, we have the following decomposition in $K_1$

$$(\theta \theta') = (\mathfrak{p}^{(1)} e_{\theta'}^2) (\mathfrak{p}^{(2)} e_{\theta'}^3) \mathfrak{a}^3 \mathfrak{a}^3.$$ 

By Lemma 3.7.2 and Theorem 4.10, $K_1(\sqrt[3]{\theta'}^2)/K_1$ is an unramified extension. By class field theory and Proposition 4.2 (2), $K_1(\sqrt[3]{\theta'}^2) = K_1$. Hence $K_1(\sqrt[3]{\theta}) = K_1(\sqrt[3]{\theta'})$. If $e + 2e' \equiv 0 \pmod{3}$, we have the following decomposition in $K_1$

$$(\theta \theta^2) = (\mathfrak{p}^{(1)} e_{\theta'}^2) (\mathfrak{p}^{(2)} e_{\theta'}^3) \mathfrak{a}^3 \mathfrak{a}^6.$$ 

By Lemma 1.1.13, $K_1(\sqrt[3]{\theta}^2)/K_1$ is an unramified extension. By class field theory and Proposition 4.2 (2), $K_1(\sqrt[3]{\theta}^2) = K_1$. Hence $K_1(\sqrt[3]{\theta}) = K_1(\sqrt[3]{\theta'})$.

The following theorem gives an arithmetic characterization of the field $R_\theta$.

**Theorem 4.12.** Let $F$ be a finite extension of $k(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2})$. Then the following conditions are equivalent.

1. $F = R_\theta$ for some $\theta \in O_{K_1}$ satisfying the conditions (1), (2) of Theorem 4.7 and (4.7.3).
(2) $F/k$ is a Galois extension such that the Galois group $\text{Gal}(F/k)$ is isomorphic to $N_3(\mathbb{F}_3)$ and only $p_1$ and $p_2$ are ramified in $F/k$ with ramification index 3.

**Proof.** (1) $\Rightarrow$ (2) is nothing but Theorem 4.10. Therefore it suffices to show (2) $\Rightarrow$ (1). By the structure of the group $N_3(\mathbb{F}_3)$, we have four distinct quadratic subextensions of $F/K$. Let $L$ be one of these three fields which is different from $K_1(\sqrt[3]{\pi})$. Then there is $\eta \in K_1$ such that $L = K_1(\sqrt[3]{\pi})$ and $F = K_1(\sqrt[3]{\pi_2}, \sqrt[3]{\pi})$. Then, by Lemma 3.7.2 and the assumption that all primes ramified in $F/k$ are only $p_1$ and $p_2$ with ramification index 3, we have the following decomposition in $K_1$:

$$\eta = (\mathfrak{P}^{(1)})^{b_1} (\mathfrak{P}^{(2)})^{b_2} (\mathfrak{P}^{(3)})^{b_3} \theta a^3,$$

where $b_1, b_2, b_3$ are non-negative integers and $a$ is an integral ideal of $K_1$ prime to $\mathfrak{P}^{(1)}$, $\mathfrak{P}^{(2)}$ and $\mathfrak{P}^{(3)}$. We may assume that $b_3 = 0$. In fact, let $b = \min\{b_1, b_2, b_3\}$. Then \(\eta/\pi_2^B \in K\) and we can take $\eta$ to be a suitable conjugate of $\eta/\pi_2^B$ over $k$ so that $(\eta) = (\mathfrak{P}^{(1)})^{b_1} (\mathfrak{P}^{(2)})^{b_2} a^3$. We shall show that integers $b_1, b_2$ satisfies the following condition.

$$b_1 \equiv 0 \pmod{3}, \ b_2 \equiv 0 \pmod{3}, \ b_1 \not\equiv b_2 \pmod{3}.$$

If $b_1 \equiv 0 \pmod{3}$, by Lemma 3.7.2 and Proposition 4.2 (2), only $\mathfrak{P}^{(2)}$ are ramified in $L/K_1$ and only $\mathfrak{P}^{(3)}$ are ramified in $K_1(\sqrt[3]{\sigma(\eta)})/K_1$. By Lemma 3.7.2, $\mathfrak{P}^{(1)}$ and $\mathfrak{P}^{(3)}$ are ramified in $K_1(\sqrt[3]{\pi_2^B})/K_1$ and $K_1(\sqrt[3]{\pi_2^B})/K_1$. By the structure of the group $N_3(\mathbb{F}_3)$, $K_1(\sqrt[3]{\pi})$ is $K_1(\sqrt[3]{\pi_2^B})$ or $K_1(\sqrt[3]{\pi_2^B})/K_1$, which is a contradiction. Therefore $b_1 \not\equiv 0 \pmod{3}$. Similarly we can show that $b_2 \not\equiv 0 \pmod{3}$, $b_1 \not\equiv b_2 \pmod{3}$.

We let $\mathfrak{R}_\theta = k(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\pi}) \langle (\theta) = (\mathfrak{P}^{(1)})^{2e}(\mathfrak{P}^{(2)})^{e} a^3, \ e \nmid e \rangle$. We shall show that $K_1(\sqrt[3]{\theta}) = L$. By $b_1 \not\equiv 0 \pmod{3}$, $b_2 \not\equiv 0 \pmod{3}$, $b_1 \not\equiv b_2 \pmod{3}$, we have $b_1 + 2e \equiv 0 \pmod{3}, b_2 + e \equiv 0 \pmod{3}$ or we have $2b_1 + 2e \equiv 0 \pmod{3}, 2b_2 + m \equiv 0 \pmod{3}$. If $b_1 + 2e \equiv 0 \pmod{3}$, $b_2 + e \equiv 0 \pmod{3}$, we have the following decomposition in $K_1$:

$$(\theta \eta) = (\mathfrak{P}^{(1)})^{b_1 + 2e}(\mathfrak{P}^{(2)})^{b_2 + e} a^3.$$
$E = F$. If $2b_1 + 2e \equiv 0 \pmod{3}$, $2b_2 + e \equiv 0 \pmod{3}$, we have the following decomposition in $K_1$:

$$(\theta \eta^2) = (\mathfrak{P}^{(1)})^{2b_1+2e}(\mathfrak{P}^{(2)})^{2b_2+e}\mathfrak{a}^6\mathfrak{A}^3.$$ 

By Lemma 3.7.2, $K_1(\sqrt[3]{\theta \eta^2})/K_1$ is an unramified extension. By class field theory and Proposition 4.2 (2), $K_1(\sqrt[3]{\eta^2}) = K_1$. Hence $K_1(\sqrt[3]{\theta}) = L$ and so $\mathfrak{R}_\theta = F$. 

By Theorem 4.12, the field $\mathfrak{R}_\theta$ is determined by a given set $\{p_1, p_2\}$. So we denote it by

$$\mathfrak{R}_{\{p_1, p_2\}} := \mathfrak{R}_\theta$$

and call it the Rédei extension over $k$ associated to $\{p_1, p_2\}$.

In particular, we obtain an affirmative answer to the problems (3.6) (i), (ii).

**Corollary 4.13.** For the cubic cyclotomic field $k = \mathbb{Q}(\zeta_3)$, the properties (3.5.1) determine $\mathfrak{R}_3(3)$ uniquely and it is given by

$$\mathfrak{R}_3(3) = \mathfrak{R}_{\{p_1, p_2\}}.$$ 

Let $S := \{p_1, p_2, p_3\}$ be a set of distinct primes in $S_k^{\text{non-3}}$ such that

$$p_i = (\pi_i), \quad \pi_i \equiv 1 \pmod{3\sqrt{-3}} \quad (i = 1, 2, 3).$$

We assume that

$$(4.14) \quad \left(\frac{\pi_i}{\pi_j}\right) = 1 \quad (1 \leq i \neq j \leq 3).$$

Then the condition (3.1) is satisfied so that the triple cubic residue symbol

$$(4.15) \quad [p_1, p_2, p_3] = \zeta_3^{\mu_3(123)}$$

is defined (Definition 3.4). On the other hand, let $\mathfrak{R}_{\{p_1, p_2\}}$ be the Rédei extension over $k$ associated to $\{p_1, p_2\}$ so that $\mathfrak{R}_{\{p_1, p_2\}} = \mathfrak{R}_\theta$ for some $\theta \in \mathcal{O}_{K_1}$ satisfying (1), (2) of Theorem 4.7 and (4.7.3). Let $\mathfrak{P}_3$ be a prime of
\( \mathcal{R}_{\{p_1,p_2\}} \) lying over \( p_3 \). By Theorem 4.12, \( \widetilde{\mathcal{R}}_3 \) is unramified in \( \mathcal{R}_{\{p_1,p_2\}}/k \), we have the Frobenius automorphism \( \left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) \in \text{Gal}(\mathcal{R}_{\{p_1,p_2\}}/k) \). Since \( p_3 \) is completely decomposed in \( k(\sqrt[p_3]{\pi_1}, \sqrt[p_3]{\pi_2}) \) by the assumption (4.14), we have \( \left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) \in (\gamma_3) \). Since \( \gamma_3 \) is in the center of \( \text{Gal}(\mathcal{R}_{\{p_1,p_2\}}/k) \), the Frobenius automorphism \( \left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) \) is independent of the choice of \( \widetilde{\mathcal{R}}_3 \) lying over \( p_3 \) and so we denote it by \( \left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) \).

**Theorem 4.16.** Notations being as above, we have

\[
[p_1,p_2,p_3] = \left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) \frac{\sqrt[p_3]{\theta}}{\sqrt[p_3]{\theta}}.
\]

**Proof.** Let \( \gamma_1, \gamma_2, \gamma_3 \in \text{Gal}(\mathcal{R}_{\{p_1,p_2\}}/k) \) be as in (4.10.1). Then we have

\[
\frac{\left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) \sqrt[p_3]{\theta}}{\sqrt[p_3]{\theta}} = \begin{cases} 1 & \left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) = \text{id}_{\mathcal{R}_{\{p_1,p_2\}}} \big/ \zeta_3 & \left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) = \gamma_2^2 = [\gamma_1, \gamma_2], \\
\zeta_3 & \left( \frac{\mathcal{R}_{\{p_1,p_2\}}/k}{p_3} \right) = \gamma_3 = [\gamma_2, \gamma_1].
\end{cases}
\]

By Theorem 1.8 and Example 1.14, we choose a prime \( p_4 \) of \( k \) with \( Np_4 \equiv 4, 7 \) mod 9 so that the Galois group \( \mathfrak{S}_{k,R}(3) \) is of link type for \( R := S \cup \{p_4\} = \{p_1, p_2, p_3, p_4\} \):

\[
\mathfrak{S}_{k,R}(3) = \text{Gal}(k_R(3)/k) = \langle x_1, x_2, x_3, x_4 \mid x_1^{N_{p_4}-1}[x_1, y_1] = x_2^{N_{p_2}-1}[x_2, y_2] = x_3^{N_{p_3}-1}[x_3, y_3] = x_4^{N_{p_4}-1}[x_4, y_4] = 1 \rangle,
\]

where \( x_i \) is the word representing a monodromy \( \tau_i \) over \( p_i \) in \( k_R(3)/k \) and \( y_i \) is the free pro-3 word of \( x_1, x_2, x_3, x_4 \) representing a Frobenius automorphism \( \sigma_i \) over \( p_i \) in \( k_R(3)/k \). Let \( \mathfrak{F}_4 \) be the free pro-3 group on \( x_1, x_2, x_3, x_4 \) and let \( \psi : \mathfrak{F}_4 \to \mathfrak{S}_{k,R}(3) \) be the natural homomorphism. Since \( \mathcal{R}_{\{p_1,p_2\}} \subset k_R(3) \) by Theorem 4.10, we have the natural homomorphism \( \rho : \mathfrak{S}_{k,R}(3) \to \text{Gal}(\mathcal{R}_{\{p_1,p_2\}}/k) \). Let \( \varphi := \rho \circ \psi : \mathfrak{F}_4 \to \text{Gal}(\mathcal{R}_{\{p_1,p_2\}}/k) \). Since \( \tau_1|_{\mathcal{R}_{\{p_1,p_2\}}} = \gamma_1, \tau_2|_{\mathcal{R}_{\{p_1,p_2\}}} = \gamma_2 \), we have

\[
\varphi(x_1) = \gamma_1, \quad \varphi(x_2) = \gamma_2, \quad \varphi(x_3) = 1, \quad \varphi(x_4) = 1,
\]
and we have

\[ \varphi(y_3) = \left( \frac{\mathfrak{N}_{(p_1, p_2)} / k}{p_3} \right). \]

By (4.10.2), the relations among \( \gamma_1 \) and \( \gamma_2 \) are equivalent to the following relations:

\[ \varphi(x_1^3) = \varphi(x_2^3) = 1, \quad \varphi(x_3) = \varphi(x_4) = 1, \quad \varphi(x_1x_2^2)^3 = \varphi(x_1^2x_2^3)^3 = 1, \]

and so \( \text{Ker}(\varphi) \) is generated as a normal subgroup of \( \mathfrak{g} \) by

\[ x_1^3, x_2^3, x_3, x_4, (x_1x_2^2)^3 \text{ and } (x_1^2x_2^3)^3. \]

Let \( \Theta_{4,3} : \mathfrak{g} \rightarrow \mathbb{F}_3 \langle \langle X_1, X_2, X_3, X_4 \rangle \rangle^\times \) be the Magnus embedding of \( \mathfrak{g} \) over \( \mathbb{F}_3 \). Then we have

\[
\begin{align*}
\Theta_{4,3}(x_1^3) &= (1 + X_1)^3 = 1 + X_1^3, \\
\Theta_{4,3}(x_2^3) &= (1 + X_2)^2 = 1 + X_2^2, \\
\Theta_{4,3}(x_3) &= 1 + X_3, \\
\Theta_{4,3}(x_4) &= 1 + X_4, \\
\Theta_{4,3}(x_1x_2^2)^3 &= ((1 + X_1)(1 + X_2)^2)^3 \equiv 1 \text{ mod deg } \geq 3, \\
\Theta_{4,3}(x_1^2x_2^3)^3 &= ((1 + X_1)^2(1 + X_2)^2)^3 \equiv 1 \text{ mod deg } \geq 3.
\end{align*}
\]

Therefore \( \mu_3((1); *) \), \( \mu_3((2); *) \), \( \mu_3((12); *) \) take their values 0 on \( \text{Ker}(\varphi) \).

Case \( \left( \frac{\mathfrak{N}_{(p_1, p_2)} / k}{p_3} \right) = \text{id} \): Then \( \varphi(y_3) = 1 \) and \( \mu_3(123) = \mu_3((12); y_3) = 0 \) by \( y_3 \in \text{Ker}(\varphi) \).

Case \( \left( \frac{\mathfrak{N}_{(p_1, p_2)} / k}{p_3} \right) = [\gamma_1, \gamma_2] \): Then \( \varphi(y_3) = [\gamma_1, \gamma_2] = \varphi([x_1, x_2]) \) and so we can write \( y_3 = [x_1, x_2]f \) for some \( f \in \text{Ker}(\varphi) \). Comparing the coefficients of \( X_1X_2 \) in the equality \( \Theta_{4,3}(y_3) = \Theta_{4,3}([x_1, x_2])\Theta_{4,3}(f) \), we have

\[
\begin{align*}
\mu_3(123) &= \mu_3((12); y_3) \\
&= \mu_3((12); [x_1, x_2]) + \mu_3((1); [x_1, x_2])\mu_3((2); f) + \mu_3((12); f) \\
&= \mu_3((12); [x_1, x_2]) \\
&= 1.
\end{align*}
\]

Case \( \left( \frac{\mathfrak{N}_{(p_1, p_2)} / k}{p_3} \right) = [\gamma_2, \gamma_1] \): Then \( \varphi(y_3) = [\gamma_2, \gamma_1] = \varphi([x_2, x_1]) \) and so we can write \( y_3 = [x_2, x_1]f' \) for some \( f' \in \text{Ker}(\varphi) \). Then comparing the coefficients of \( X_1X_2 \) in the equality \( \Theta_{4,3}(y_3) = \Theta_{4,3}([x_2, x_1])\Theta_{4,3}(f') \), we have

\[
\begin{align*}
\mu_3(123) &= \mu_3((12); y_3) \\
&= \mu_3((12); [x_2, x_1]) + \mu_3((1); [x_2, x_1])\mu_3((2); f') + \mu_3((12); f') \\
&= \mu_3((12); [x_2, x_1]) \\
&= -1 = 2.
\end{align*}
\]
By (4.15) and (4.16.1), the assertion is proved. □

**Example 4.17.** Let \((\pi_1, \pi_2) := (17, 53)\). Then we have
\[
\begin{align*}
\alpha^{(1)} &= \alpha = 8 - 3\sqrt[3]{17}, \\
\alpha^{(2)} &= 8 - 3\zeta_3\sqrt[3]{17}, \\
\alpha^{(3)} &= 8 - 3\zeta_2\sqrt[3]{17}, \\
\end{align*}
\]
and
\[
\begin{align*}
\theta^{(1)} &= \theta = (\alpha^{(1)})^2\alpha^{(2)}, \\
\theta^{(2)} &= (\alpha^{(2)})^2\alpha^{(3)}, \\
\theta^{(3)} &= (\alpha^{(3)})^2\alpha^{(1)}. \\
\end{align*}
\]
Then we easily see that \(\theta\) satisfies (1), (2) in Theorem 4.7. Since \(\alpha^{(1)} \equiv \alpha^{(2)} \mod (3\sqrt[3]{-3})\), \(\alpha^3 \equiv \theta \mod (3\sqrt[3]{-3})\) and so (4.7.3) is also satisfied. Hence
\[
K_3(3) = k(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta}).
\]

Suppose \(\pi_3 = 71, 89, 107, 179, 197\). Then we have
\[
[(17), (53), (71)]_3 = \zeta_3^2, \\
[(17), (53), (89)]_3 = \zeta_3, \\
[(17), (53), (107)]_3 = \zeta_3^2, \\
[(17), (53), (179)]_3 = \zeta_3, \\
[(17), (53), (197)]_3 = \zeta_3.
\]
The right hand side of the equality in Theorem 4.16 depends on the choice of \(\theta\) and the order of \(p_1\) and \(p_2\). In fact, the shuffle relation in Proposition 2.13 yields the following

**Proposition 4.18.** Notations and assumptions being as above, we have
\[
[p_1, p_2, p_3]_3 = [p_2, p_1, p_3]_3^{-1}.
\]

§5 A cohomological interpretation by Massey products

In this section, we give a cohomological interpretation of the multiple power residue symbols by Massey products in étale cohomology. Our theorem is seen as a generalization of the known relation between the cup product and the power residue symbol to the higher order operations, and also a generalization of the previous result by the second author ([Mo3], [V]) in the case of the rationals to a number field.

Firstly, we recall some general materials on Massey products. Let \((C^\bullet, d)\) be a differential graded algebra over a commutative ring \(A\) with identity,
namely, \( C^\bullet = \bigoplus_{j \geq 0} C^j \) is an associative graded algebra over \( A \) with differential \( d \) of degree 1 such that \( C^0 = A \), \( d^2 = 0 \) and \( d(ab) = da \cdot b + (-1)^i a \cdot db \) for \( a \in C^i \) and \( b \in C^j \). Then we can introduce the Massey product in the cohomology ring \( H^*(C^\bullet) \) of \( C^\bullet \) according to the general procedure ([Ma]). However, we deal with only one or two dimensional cohomology groups in the following, and so we introduce the Massey product only on \( H^1(C^\bullet) \). For the sign convention, we follow [Dw].

Let \( \chi_1, \ldots, \chi_n \in H^1(C^\bullet) \) \((n \geq 2)\). An \( n \)-th Massey product \( \langle \chi_1, \ldots, \chi_n \rangle \) is said to be defined if there is an array

\[
\Omega = \{ \omega_{ij} \in C^1 \mid 1 \leq i < j \leq n + 1, (i, j) \neq (1, n + 1) \}
\]

such that

\[
\begin{cases}
[\omega_{i,i+1}] = \chi_i \ (1 \leq i \leq n), \\
d\omega_{ij} = \sum_{a=i+1}^{j-1} \omega_{ia} \cdot \omega_{aj} \ (j \neq i + 1).
\end{cases}
\]

Such an array \( \Omega \) is called a defining system for \( \langle \chi_1, \ldots, \chi_n \rangle \). The value of \( \langle \chi_1, \ldots, \chi_n \rangle \) relative to \( \Omega \) is defined by the cohomology class represented by the 2-cocycle

\[
\sum_{a=2}^{n} \omega_{1a} \cdot \omega_{a,n+1},
\]

and denoted by \( \langle \chi_1, \ldots, \chi_n \rangle_\Omega \). A Massey product \( \langle \chi_1, \ldots, \chi_n \rangle \) itself is usually taken to be the subset of \( H^2(C^\bullet) \) consisting of elements \( \langle \chi_1, \ldots, \chi_n \rangle_\Omega \) for some defining system \( \Omega \). By convention, \( \langle \chi \rangle = 0 \).

We recall a basic fact on Massey products which will be used later (cf. [Kr]).

**Lemma 5.1.** We have \( \langle \chi_1, \chi_2 \rangle = \chi_1 \cdot \chi_2 \). For \( n \geq 3 \), \( \langle \chi_1, \ldots, \chi_n \rangle \) is defined and consists of a single element if \( \langle \chi_{j_1}, \ldots, \chi_{j_a} \rangle = 0 \) for all proper subsets \( \{j_1, \ldots, j_a\} \ (a \geq 2) \) of \( \{1, \ldots, n\} \).

Next, we recall a relation between Massey products and the Magnus coefficients. Let \( \mathfrak{G} \) be a pro-\( l \) group acting trivially on a commutative ring \( A \) with identity. Let \( C^j(\mathfrak{G}, A) \) be the \( A \)-module of inhomogeneous \( j \)-cochains \((j \geq 0)\) of \( \mathfrak{G} \) with coefficients in \( A \) and we consider the differential graded algebra \( (C^\bullet(\mathfrak{G}, A), d) \), where the product structure on \( C^\bullet(\mathfrak{G}, A) = \bigoplus_{j \geq 0} C^j(\mathfrak{G}, A) \) is given by the cup product and the differential \( d \) is the coboundary operator.
Then we have the cohomology $H^\ast(G, A) = H^\ast(C^\ast(G, A))$ of the pro-$l$ group $G$ with coefficients in $A$.

Suppose that $G$ is a finitely generated pro-$l$ group with a minimal presentation

$$1 \longrightarrow \mathcal{N} \longrightarrow \mathfrak{f}_r \xrightarrow{\psi} G \longrightarrow 1,$$

where $\mathfrak{f}_r$ is a free pro-$l$ group on $x_1, \ldots, x_r$ with $r = \dim_\mathbb{F}_l H^1(G, \mathbb{F}_l)$. We set $\tau_i := \psi(x_i)$ ($1 \leq i \leq r$). Let $m$ be a positive power of $l$ and we shall consider the case that the ring $A$ of coefficients is $\mathbb{Z}/m\mathbb{Z}$. We suppose that $\psi$ induces the isomorphism $(\mathbb{Z}/m\mathbb{Z})^r = \mathfrak{f}_r/\mathfrak{f}_r^m[\mathfrak{f}_r, \mathfrak{f}_r] \simeq G/[G, G]$, so that $\psi$ induces the isomorphism $H^1(G, \mathbb{Z}/m\mathbb{Z}) \simeq H^1(\mathfrak{f}_r, \mathbb{Z}/m\mathbb{Z})$.

We let $\text{tg} : H^1(\mathcal{N}, \mathbb{Z}/m\mathbb{Z}) \rightarrow H^2(G, \mathbb{Z}/m\mathbb{Z})$ be the transgression defined as follows. For $a \in H^1(\mathcal{N}, \mathbb{Z}/m\mathbb{Z})$, choose a 1-cochain $b \in C^1(\mathfrak{f}_r, \mathbb{Z}/m\mathbb{Z})$ such that $b|_{\mathcal{N}} = a$. Since the value $db(f_1, f_2)$ depends only on the cosets $f_i$ mod $\mathcal{N}$, there is a 2-cocyle $c \in Z^2(G, \mathbb{Z}/m\mathbb{Z})$ such that $\psi^*(c) = db$. We then define $\text{tg}(a)$ by the class of $c$.

By the Hochschild-Serre spectral sequence, $\text{tg}$ is an isomorphism and so we have the dual isomorphism, called the Hopf isomorphism,

$$\text{tg}^\vee : H_2(G, \mathbb{Z}/m\mathbb{Z}) \rightarrow H^1(\mathcal{N}, \mathbb{Z}/m\mathbb{Z}) = \mathcal{N}/[\mathcal{N}, \mathfrak{f}_r].$$

Then we have the following Proposition (cf. [St; Lemma 1.5]). The proof goes in the same manner as in [Mo3, Theorem 2.2.2].

**Proposition 5.2.** Notations being as above, let $\chi_1, \ldots, \chi_n \in H^1(G, \mathbb{Z}/m\mathbb{Z})$ and $\Omega = (\omega_{ij})$ a defining system for the Massey product $\langle \chi_1, \ldots, \chi_n \rangle$. Let $f \in \mathcal{N}$ and set $\delta := (\text{tg}^\vee)^{-1}(f \mod \mathcal{N}/[\mathcal{N}, \mathfrak{f}_r])$. Then we have

$$\langle \chi_1, \ldots, \chi_n \rangle \Omega(\delta) = \sum_{j=1}^n (-1)^{j+1} \sum_{c_1 + \cdots + c_j = n} \sum_{I = (i_1 \cdots i_j)} \omega_{1,1+c_1}(\tau_{i_1}) \cdots \omega_{r+1-c_j,n+1}(\tau_{i_j}) \mu_m(I; f),$$

where $c_1, \ldots, c_j$ run over positive integers satisfying $c_1 + \cdots + c_j = n$ and $\mu_m(I; f)$ denotes the mod $m$ the Magnus coefficient.

Let us go back to the arithmetic situation in Section 3 and keep the same notations and assumptions there. So let $k$ be a finite algebraic number field.
containing a fixed primitive $m$-th root of unity $\zeta_m$, where $m$ is a fixed power of $l$, and the $l$-class group of $k$ is assumed to be trivial. Let $p_1, \ldots, p_n$ be $n(\geq 2)$ distinct primes in $S_k^{\text{non-l}}$. Let $m_{(12\ldots n)}$ be the maximal power $l^e$ of $l$ such that $Np_i \equiv 1 \mod l^e$ for $1 \leq i \leq n$. Let $\mu_m(i_1 \ldots i_a)$ (resp. $\overline{\mu}_m(i_1 \ldots i_a)$) be the mod $m$ arithmetic Milnor number (resp. arithmetic Milnor invariant) for $1 \leq i_1, \ldots, i_a \leq n$. We assume the following condition, slightly stronger than (3.1),

\begin{equation}
\begin{array}{l}
\bullet \left( m_{(12\ldots n)} \right) \equiv 0 \mod m \text{ for } 1 \leq j < n, \\
\bullet \mu_m(j_1 \ldots j_a) = 0 \text{ for any proper subset } \{j_1, \ldots, j_a\} \text{ of } \{1, 2, \ldots, n\}.
\end{array}
\end{equation}

(5.3)

So we have $\overline{\mu}_m(12 \ldots n) = \mu_m(12 \ldots n)$.

By Corollary 1.13, we choose a finite subset $R := \{p_1, \ldots, p_n, \ldots, p_r\}$ of $S_k^{\text{non-l}}$ containing $S$ so that the pro-$l$ Galois group $G_{k,R}(l) = \text{Gal}(k_R(l)/k)$ is of link type

\[ G_{k,R}(l) = \langle x_1, \ldots, x_r \mid x_1^{N_{p_1}} = \cdots = x_r^{N_{p_r}} = 1 \rangle = \mathfrak{G}/\mathfrak{H}. \]

Here $x_i$ is the word representing a monodromy $\tau_i$ over $p_i$ in $k_R(l)/k$ and $y_i$ is the free pro-$l$ word of $x_1, \ldots, x_r$ representing a Frobenius automorphism $\sigma_i$ over $p_i$ in $k_R(l)/k$, and $\mathfrak{G}$ is the free pro-$l$ group on $x_1, \ldots, x_r$ and $\mathfrak{H}$ is the closed subgroup of $\mathfrak{G}$ generated normally by $x_1^{N_{p_1}} = \cdots = x_r^{N_{p_r}} = 1$.

Let $\mathfrak{X} := \text{Spec}(O_k) \setminus R$. We recall Verdier’s construction of the étale cohomology $H^*(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ (cf. [AGV; Exposé V], [AM; §8], [Fi; 3]) and introduce the Massey product on it. Let $U_{\bullet}$ be a hypercovering of the étale site on $\mathfrak{X}$ and let $C^j(U_{\bullet}, \mathbb{Z}/m\mathbb{Z})$ be the $\mathbb{Z}/m\mathbb{Z}$-module of Čech $j$-cochains ($j \geq 0$) associated to $U_{\bullet}$ with coefficients in the constant sheaf $\mathbb{Z}/m\mathbb{Z}$. We consider the differential graded algebra $(C^\bullet(U_{\bullet}, \mathbb{Z}/m\mathbb{Z}), d)$, where the product structure on $C^\bullet(U_{\bullet}, \mathbb{Z}/m\mathbb{Z}) = \bigoplus_{j \geq 0} C^j(U_{\bullet}, \mathbb{Z}/m\mathbb{Z})$ is given by the Alexander-Whitney cup product and the differential $d$ is the coboundary operator. Since the étale cohomology is given by

\[ H^*(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}) = \lim_{\rightarrow} H^*(C^\bullet(U_{\bullet}, \mathbb{Z}/m\mathbb{Z})), \]

where the colimit is taken over the homotopy category of hypercoverings $U_{\bullet}$, we have the Massey product structure on $H^*(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$. 

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Let $\chi_1, \ldots, \chi_r \in H^1(\mathcal{X}_\text{ét}, \mathbb{Z}/m\mathbb{Z}) = H^1(\mathfrak{G}_{k,R}, \mathbb{Z}/m\mathbb{Z})$ be the Kronecker dual to the monodromies $\tau_1, \ldots, \tau_r$. Let $\delta_i$ be the canonical generator of $H_2(\text{Spec}(k_{p_i})\text{ét}, \mathbb{Z}/m\mathbb{Z})$, which is the dual to that of $H^2(\text{Spec}(k_{p_i})\text{ét}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$ given by local class field theory ([Se; XIII, §3]). We write the same letter $\delta_i$ for the image of $\delta_i$ under the homomorphism $H^2(\text{Spec}(k_{p_i})\text{ét}, \mathbb{Z}/m\mathbb{Z}) \rightarrow H_2(\mathcal{X}_\text{ét}, \mathbb{Z}/m\mathbb{Z})$ induced by the natural morphism $\text{Spec}(k_{p_i}) \rightarrow \mathcal{X}$. Geometrically, the homology class $\delta_i \in H_2(\mathcal{X}_\text{ét}, \mathbb{Z}/m\mathbb{Z})$ represents the boundary of a tubular neighborhood of $\text{Spec}(\mathcal{O}_k/p_i)$. Group-theoretically, $\delta_i$ corresponds to the relator $x_i^{Np_i-1}[x_i, y_i]$ of the pro-$l$ étale fundamental group $\mathfrak{G}_{k_{p_i}}(l) = \pi_1^{\text{ét}}(\text{Spec}(k_{p_i}))(l)$ in (1.4) as follows. By the Hochschild-Serre spectral sequence, we have $H_*(\text{Spec}(k_{p_i})\text{ét}, \mathbb{Z}/m\mathbb{Z}) = H_*(\mathfrak{G}_{k_{p_i}}(l), \mathbb{Z}/m\mathbb{Z})$. As in Section 1, the pro-$l$ group $\mathfrak{G}_{k_{p_i}}(l)$ has the presentation

$$\mathfrak{G}_{k_{p_i}}(l) = \langle x_i, y_i \mid x_i^{Np_i-1}[x_i, y_i] = 1 \rangle = \mathfrak{F}_i/\mathfrak{N}_i,$$

where $x_i$ and $y_i$ are the words representing a monodromy $\tau_i$ and a Frobenius automorphism $\sigma_i$ in $k_{p_i}(l)/k_{p_i}$, respectively ((1.3), (1.4)), and $\mathfrak{F}_i$ is the free pro-$l$ group on $x_i, y_i$ and $\mathfrak{N}_i$ is the closed subgroup of $\mathfrak{F}_i$ generated normally by $x_i^{Np_i-1}[x_i, y_i]$. So we have the Hopf isomorphism

$$t_{\mathfrak{G}_{k_{p_i}}^\vee}: H_2(\mathfrak{G}_{k_{p_i}}(l), \mathbb{Z}/m\mathbb{Z}) \sim H_1(\mathfrak{N}_i, \mathbb{Z}/m\mathbb{Z}) \mathfrak{G}_{k_{p_i}}(l) = \mathfrak{N}_i/\mathfrak{N}_i^m[\mathfrak{N}_i, \mathfrak{F}_i],$$

where $\delta_i$ corresponds to $x_i^{Np_i-1}[x_i, y_i]$. Thus the image of $\delta_i$ under the natural homomorphism $H_2(\mathcal{X}_\text{ét}, \mathbb{Z}/m\mathbb{Z}) \rightarrow H_2(\mathfrak{G}_{k,R}(l), \mathbb{Z}/m\mathbb{Z})$ coincides with the image of $x_i^{Np_i-1}[x_i, y_i]$ under the natural homomorphism $\mathfrak{N}_i/\mathfrak{N}_i^m[\mathfrak{N}_i, \mathfrak{F}_i] \rightarrow \mathfrak{N}/\mathfrak{N}_i^m[\mathfrak{N}, \mathfrak{F}_r] = H_2(\mathfrak{G}_{k,R}(l), \mathbb{Z}/m\mathbb{Z})$.

Now we present our theorem, which is analogous to the case for a link in an $\mathbb{F}_l$-homology 3-sphere ([Tu]).

**Theorem 5.4.** Notations and assumptions being as above, the Massey product $\langle \chi_1, \ldots, \chi_n \rangle$ in $H^2(\mathcal{X}_\text{ét}, \mathbb{Z}/m\mathbb{Z})$ is uniquely defined and we have

$$\langle \chi_1, \chi_2, \ldots, \chi_n \rangle(\delta_n) = (-1)^n \mu_m(12 \cdots n)$$

and

$$[p_1, p_2, \ldots, p_n]_m = \zeta_m(-1)^n \langle \chi_1, \chi_2, \ldots, \chi_n \rangle(\delta_n).$$

**Proof.** When $n = 2$, we have $\langle \chi_1, \chi_2 \rangle = \chi_1 \cup \chi_2$ and, by Proposition 5.2, we
have $\langle \chi_1, \chi_2 \rangle (\delta_2) = \mu_m(12)$. Suppose $n \geq 3$ and let \( \{ j_1, \ldots, j_a \} \) \((a \geq 2)\) be any proper subset of \( \{1, \ldots, n\} \). By our assumption (5.3) and the proof of (2.8.2), we have

$$\mu_m(J; x_i^{N_{p_i}-1}[x_i, y_i]) = 0$$

for \( J = (j_1 \cdots j_a) \) and \( 1 \leq i \leq r \). So, by Proposition 5.2, we have

$$\langle \chi_{j_1}, \ldots, \chi_{j_a} \rangle (\delta_i) = 0$$

for \( 1 \leq i \leq r \). Since \( H_2(\delta_{k,R}(l), \mathbb{Z}/m\mathbb{Z}) \) is generated by \( x_i^{N_{p_i}-1}[x_i, y_i], 1 \leq i \leq r \), we have

$$\langle \chi_{j_1}, \ldots, \chi_{j_a} \rangle = 0.$$  

By Definition 3.4, the last assertion follows. \( \Box \)

**Example 5.5.** (1) (Power residue symbol) Let \( n = 2 \). By Theorem 5.4, we have

$$[p_1, p_2]_m = \zeta_m^{\langle \chi_1, \chi_2 \rangle (\delta_2)}.$$  

When \( p_i \) is a principal ideal \((\pi_i)\) for \( i = 1, 2 \), we have, by Theorem 3.7.6,

$$[p_1, p_2]_m = \left( \frac{\pi_1}{\pi_2} \right)_m.$$  

So (5.5.1) is nothing but the known description of the \( m \)-th norm residue symbol by the cup product ([Ko; 8.11], [Se; XIV, §2]), since \( \chi_i \) is the Kummer character associated to \( \pi_i \).

(2) (The Rédei symbol) Let \( l = m = 2 \) and \( n = 3 \). Let \( p_1, p_2 \) and \( p_3 \) be distinct rational primes satisfying the conditions

$$p_i \equiv 1 \mod 4 \ (1 \leq i \leq 3), \quad \left( \frac{p_i}{p_j} \right)_m = 1 \ (1 \leq i \neq j \leq 3).$$

Then, by Theorem 5.4, the Rédei symbol is expressed by

$$[p_1, p_2, p_3]_2 = (-1)^{\langle \chi_1, \chi_2, \chi_3 \rangle (\delta_3)}.$$
(3) (Triple cubic residue symbol) Let \( l = m = 3 \) and \( n = 3 \). Let \( p_1 = (\pi_1), p_2 = (\pi_2) \) and \( p_3 = (\pi_3) \) be distinct primes of \( \mathbb{Q}(\zeta_3) \) satisfying the conditions

\[
N_{p_i} \equiv 1 \mod 9, \quad \pi_i \equiv 1 \mod (3\sqrt{-3}) \quad (1 \leq i \leq 3), \quad \left( \frac{\pi_i}{\pi_j} \right) = 1 \quad (1 \leq i \neq j \leq 3). 
\]

Then, by Theorem 5.4, the triple cubic symbol is expressed by

\[
[p_1, p_2, p_3]_3 = \zeta_3^{-\langle \chi_1, \chi_2, \chi_3 \rangle (\delta_3)}. 
\]

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Fumiya Amano  
2077-1, Jyuni-cho, Ibusuki, Kagoshima, 891-0403, JAPAN  
e-mail: ca-solitudeam.p@ezweb.ne.jp

Masanori Morishita  
Faculty of Mathematics, Kyushu University  
744, Motooka, Nishi-ku, Fukuoka, 819-0395, JAPAN  
e-mail: morisita@math.kyushu-u.ac.jp