Einstein equations with cosmological constant in Super Space-Time

F. Gholami, F. Darabi, M. Mohammadi, S. Varsaie, M. Roshandel

Abstract We introduce a new kind of super warped product spaces \( \bar{M}_{(I)} = I^{[0]} \times_f M^{m|n}, \bar{M}_{(II)} = I^{[1]} \times_f M^{m|n}, \) and \( \bar{M}_{(III)} = I^{[1]} \times_f M^{m|n}, \) where \( M^{m|n} \) is a supermanifold of dimension \( m|n, I^{[\delta, \delta']} \) is standard superdomain with \( I = (0,1) \) and \( \delta, \delta' \in \{0,1\}, \) subject to the warp functions \( f(t), f(\bar{t}), \) and \( f(t, \bar{t}), \) respectively. In each super warped product space, \( \bar{M}_{(I)}, \bar{M}_{(II)}, \) and \( \bar{M}_{(III)}, \) it is shown that Einstein equations \( \bar{G}_{\alpha\beta} = -\bar{\Lambda}\bar{g}_{\alpha\beta}, \) with cosmological term \( \bar{\Lambda} \) are reducible to the Einstein equations \( G_{\alpha\beta} = -\Lambda g_{\alpha\beta} \) on the super space \( M^{m|n} \) with cosmological term \( \Lambda, \) where \( \bar{\Lambda} \) and \( \Lambda \) are functions of \( f(t), f(\bar{t}), \) and \( f(t, \bar{t}), \) as well as \( (m, n). \) This dependence points to the origin of cosmological terms which turn out to be within the warped structure of the super space-time. By using the Generalized Robertson-Walker space-time, as a super space-time, and demanding for constancy of \( \bar{\Lambda} \) we can determine the warp functions and \( \Lambda \) which result in finding the solutions for Einstein equations \( \bar{G}_{\alpha\beta} = -\bar{\Lambda}\bar{g}_{\alpha\beta} \) and \( G_{\alpha\beta} = -\Lambda g_{\alpha\beta}. \) We have discussed the cosmological solutions, for each kind of super warped product space, in the special case of \( M^{3|0}. \)
1 Introduction

Bishop and O’Neill were the pioneering in introducing the warped product spaces to construct Riemannian manifolds having negative curvature [1]. The warped product spaces were widely studied in the context of general relativity theory to construct new metrics with interesting geometrical and physical properties. In this regard, some solutions of Einstein equations, like generalized Friedmann-Robertson-Walker metric, generalized Schwarzschild black hole metric, generalized Reissner-Nordstrom black hole metric, generalized (2 + 1)-dimensional Banados-Teitelboim-Zanelli (BTZ) black hole metric, generalized (2+1)-dimensional de Sitter black hole metric, generalized standard static metric, 5-dimensional Randall-Sundrum and Dvali-Gabadadze-Porrati metrics, generalized twisted product structure and special base conformal warped product structure were shown by the authors to be expressed in terms of multi warped products in Lorentzian geometry [2]-[5].

By considering the importance of warped products in geometry and physics, we present a meaningful generalization of warped product which is called super warped product. This generalization is novel in its kind and has not been defined before. We study Einstein equations with cosmological term in this super warped product space and show how to reduce them to the Einstein equations with cosmological term in the lower dimensional space. It is shown that the cosmological terms are defined in terms of the warp function in the super warped product space and the dimensions of supermanifold. In explicit words, we consider three types of super warped product metric of the types $\bar{M}_1(I) = I^{10} \times f M^{m|n}$, $\bar{M}_2(I) = I^{01} \times f M^{m|n}$, and $\bar{M}_3(I) = I^{11} \times f M^{m|n}$ where $M^{m|n}$ is a supermanifold of dimension $m|n$, $I^{0\delta}$ is standard superdomain with $I = (0,1)$ and $\delta, \delta' \in \{0,1\}$, subject to the warp functions $f(t)$, $f(t)$, and $f(t, \bar{t})$, respectively. Similar to the previous examples, we show that in each super warped product space, $\bar{M}_1(I)$, $\bar{M}_2(I)$, and $\bar{M}_3(I)$, the Einstein equations $\bar{G}_{AB} = -\bar{\Lambda} \bar{g}_{AB}$, with cosmological term $\bar{\Lambda}$ are reducible to the Einstein equations $\bar{G}_{\alpha\beta} = -\bar{\Lambda} \bar{g}_{\alpha\beta}$ on the super space $\bar{M}$ with cosmological term $\bar{\Lambda}$, where $\bar{\Lambda}$ and $\bar{\Lambda}$ are functions of $f(t)$, $f(t)$, and $f(t, \bar{t})$, respectively. Moreover, by using the Generalized Robertson-Walker space-times and demanding for constancy of $\bar{\Lambda}$ we can determine the warp functions and $\bar{\Lambda}$ which result in finding the solutions for Einstein equations $\bar{G}_{AB} = -\bar{\Lambda} \bar{g}_{AB}$ and $\bar{G}_{\alpha\beta} = -\bar{\Lambda} \bar{g}_{\alpha\beta}$. For each kind of super warped product space, in the special case of $M^{3|0}$, we have discussed the cosmological solutions.

2 Preliminaries

In this section we need to state some definitions of super manifolds and super Riemannian metric.

**Definition 1** By a super ringed space, we mean a pair $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of supercommutative $\mathbb{Z}_2$-graded rings on...
A morphism between \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) is a pair \(\psi := (\bar{\psi}, \psi^*)\) such that \(\bar{\psi} : X \to Y\) is a continuous map and \(\psi^* : \mathcal{O}_Y \to \bar{\psi}_*\mathcal{O}_X\) is a homomorphism between the sheaves of supercommutative \(\mathbb{Z}_2\)-graded rings. The super ring space \(U|m|n := (U, \mathcal{O}_{C^\infty_R m} \otimes \wedge \mathbb{R}^n)\), \(U\) is an open subset of \(\mathbb{R}^m\), is called standard superdomain where \(\mathcal{O}_{C^\infty_R m}\) is the sheaf of smooth functions on \(U\) and \(\wedge \mathbb{R}^n\) is the exterior algebra of \(\mathbb{R}^n\).

**Definition 2** A supermanifold of dimension \(m|n\) is a super ringed space \((\bar{M}, \mathcal{O}_M)\) that is locally isomorphic to \(\mathbb{R}^m|n\) and \(M\) is a second countable and Hausdorff topological space.

It can be shown that the stalks of the structure sheaf of a supermanifold are local rings.

A morphism between two supermanifolds \(\bar{M} = (\bar{M}, \mathcal{O}_M)\) and \(\bar{N} = (\bar{N}, \mathcal{O}_N)\) is just a morphism say \((\bar{\psi}, \psi^*)\) between two super ringed spaces such that for \(y = \bar{\psi}(x)\), \(\psi^* : \mathcal{O}_N \to \bar{\psi}_*\mathcal{O}_M\) is a morphism of local rings. By \(\mathcal{J}_M(U\subset \bar{M})\) for \(U\subset \bar{M}\), we mean the set of all nilpotent elements in \(\mathcal{O}_M(U)\).

It can be seen that the quotient sheaf \(\mathcal{O}_M/\mathcal{J}_M\) is locally isomorphic to the sheaf \(\mathcal{O}_{C^\infty_R m}\). (See [6].) Thus \(\tilde{M} := (\bar{M}, \mathcal{O}_M/\mathcal{J}_M)\) is a classical smooth manifold that is called reduced manifold associated to \(M\). In addition each morphism \(\psi : M \to N\) induces a smooth map \(\tilde{\psi} : \tilde{M} \to \tilde{N}\).

In supergeometry one may define the tangent bundle \(\mathcal{T}_M\) and differential forms \(\Omega^k_M\) for each supermanifold \(M\). see [6] and [7] for more details. The category of supermanifolds admits products. Let \(M_i(1 < i < n)\) be spaces in the category. A ringed space \(M\) together with (“projection”) maps \(P_i : M \to M_i\) is called a product of the \(M_i\), and is denoted by

\[
M = M_1 \times ... \times M_n,
\]

if the following is satisfied: for any ringed space \(N\), the map

\[
f \mapsto (P_1 \circ f,...,P_n \circ f),
\]

from \(\text{Hom}(N,M)\) to \(\Pi_i\text{Hom}(N,M_i)\) is a bijection. In other words, the morphisms \(f\) from \(N\) to \(M\) are identified with \(n\)-tuples \((f_1,...,f_n)\) of morphisms such that \(f_i(N \to M_i)\) for all \(i\).[6]

**Definition 3** Let \(V = V_0 \oplus V_1\) be a super vector space over a field \(\mathbb{K}\). By a scalar superproduct on \(V\), we mean a non-degenerate and supersymmetric even \(\mathbb{K}\)-bilinear form \(\langle ., . \rangle : V \times V \to V\). By the non-degeneracy, we mean that the mapping \(v \mapsto \langle v, . \rangle\) is an isomorphism for all \(v \in V\) and by graded symmetric, we mean that for all homogeneous elements \(v,w \in V\), we have

\[
\langle v, w \rangle = (-1)^{|v||w|}\langle w, v \rangle,
\]

where \(|v|\) is the parity of \(v\).

**Definition 4** A Riemannian supermetric on a supermanifold \(M\) is a graded symmetric even non-degenerate \(\mathcal{O}_M\)-linear morphism of sheaves

\[
g : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M.
\]
where by the non-degeneracy we mean that the mapping $X \mapsto g(X,\cdot)$ is an isomorphism from $T_M$ to $\Omega^1_M$. A supermanifold equipped with a Riemannian supermetric is called a Riemannian supermanifold.

3 Super warped product

In this section, we define super warped product space. To see the analogue definition in classical geometry one may refer to [1].

**Definition 5** Let $(B, g_B)$ and $(M, g_M)$ be semi-Riemannian supermanifolds of dimensions $k|l$ and $m|n$ respectively. Let $f \in O(B)$ be a superfunction with $\tilde{f} > 0$ where $\tilde{f}$ is a smooth function on $\tilde{B}$ such that, for each $x \in \tilde{B}$, $f = \tilde{f}(x)$ is not invertible on every neighbourhood of $x$. For more detailed description see [6]. The super warped product is the product supermanifold $M = B \times M$ together with the supermetric defined by $g = \pi_1^*(g_B) \oplus \pi_2^*(f)\pi_2^*(g_M)$ where $\pi_1$ and $\pi_2$ are the natural projections of $B \times M$ to $B$ and $M$ respectively. In addition $f$ is called super warp function and the supermanifolds $(M, g_M)$ and $(B, g_B)$ are called fiber and base supermanifolds of the super warped product respectively.

4 Generalization of Robertson-Walker Space-times in Super-geometry

Generalized Robertson-Walker space-times were introduced in 1995 by Alías, Romero and Sánchez (see Ref.[8,9]). In this section, we study a generalization of Robertson-Walker space-times in super-geometry and obtain the Einstein equations with cosmological constant.

Let $\bar{M} = I^{\delta|\delta'} \times_f M^{m|n}$ be a super warped product where $M^{m|n}$ is semi-Riemannian supermanifold and $I^{\delta|\delta'}$ is standard superdomain with $I = (0, 1)$ and $\delta, \delta' \in \{0, 1\}$. The supermanifold $\bar{M}$ is equipped by superlorentzian metric

$$\bar{g} = -\pi_{1}^{*}(g_{I^{\delta|\delta'}}) + f\pi_{2}^{*}(g_{M^{m|n}}).$$

Then we call $(\bar{M}, \bar{g})$ a super semi-Riemannian warped product. We use the Einstein convention, that is, repeated indices with one upper index and one lower index denote summation over their range. If not stated otherwise, throughout the paper we use the following ranges for indices: $i, j, k, \ldots \in \{1, \ldots, m\}$ and $\alpha, \beta, \ldots \in \{m + 1, \ldots, m + n\}$ for even and odd indices of $M^{m|n}$ respectively.

Now, in the following cases, we study Einstein equations with cosmological constant in the Generalized Robertson-Walker space-times.
Case I

Let $M = I^{10} \times f M^{m|n}$, $(x^i, x^\alpha)$ be an even-odd coordinate system on $M$, $t$ be a coordinate on $(0,1)$ and $f \equiv f(t)$. By $|t|, |i|$ and $|\alpha|$, we mean the parities of $t, x^i$ and $x^\alpha$ respectively. So $|t| = 0 = |i|$ and $|\alpha| = 1$.

Then, we have

$$\bar{g}_{tt} = \bar{g}(\partial_t, \partial_t) = -1,$$
$$\bar{g}_{st} = \bar{g}(\partial_s, \partial_t) = \bar{g}_{ts} = \bar{g}_{ss} = 0,$$
$$\bar{g}_{\alpha \beta} = fg_{\alpha \beta},$$
$$\bar{g}_{ij} = fg_{ij},$$
$$\bar{g}_{i\alpha} = fg_{i\alpha},$$

where $\bar{g}_{ij} = \bar{g}(\partial_i, \partial_j)$ and $\bar{g}_{\alpha \beta} = \bar{g}(\partial_\alpha, \partial_\beta)$.

For the next proposition we need the following lemma.

**Lemma 1** Let $\bar{M} = I^{10} \times f M^{m|n}$ be a super semi-Riemannian warped product. The Christoffel symbol of $(\bar{M}, \bar{g})$ admits the following expression

$$\bar{\Gamma}^{L}_{IJ} = \frac{1}{2} \left[ \partial_I \bar{g}_{JK} - \partial_K \bar{g}_{IJ} + (-1)^{|I||J|} \partial_J \bar{g}_{IK} \right] \bar{g}^{KL}$$
$$+ \frac{1}{2} \left[ \partial_J \bar{g}_{IS} - \partial_S \bar{g}_{IJ} + (-1)^{|I||J|} \partial_I \bar{g}_{JS} \right] \bar{g}^{IL}$$
$$+ \frac{1}{2} \left[ \partial_{\alpha} \bar{g}_{IJ} - (-1)^{|I||J|} \partial_I \bar{g}_{\alpha J} + (-1)^{|I||J|+1} \partial_J \bar{g}_{\alpha I} \right] \bar{g}^{\alpha L},$$

where $I, J, K, L$ are arbitrary indices on $\bar{M}$, i.e. they can stand for even and odd indices $t, s, \alpha$.

**Proof** Let $N$ be a supermanifold; for Christoffel symbols we have the following equation

$$\bar{\Gamma}^L_{IJ} = \frac{1}{2} \left[ \partial_I \bar{g}_{JK} - (-1)^{|K||I|+|J|} \partial_K \bar{g}_{IJ} + (-1)^{|J||I|+|K|} \partial_J \bar{g}_{KI} \right] \bar{g}^{KL},$$

where $K \in \{t, s, \alpha\}$.

Simply, one can show that

$$\bar{\Gamma}^I_{tt} = \bar{\Gamma}^I_{tt} = \bar{\Gamma}^L_{tt} = 0, \quad \bar{\Gamma}^I_{tt} = \bar{\Gamma}^L_{tt} = \frac{f'}{2f} \bar{g}_{tt},$$
$$\bar{\Gamma}^I_{ij} = \frac{1}{2} f' \bar{g}_{ij}, \quad \bar{\Gamma}^L_{ij} = \bar{\Gamma}^L_{ij},$$

where $I, J, L \in \{s, \alpha\}$ and $'$ denotes $\partial_t$. 
Lemma 2 Let $\hat{M} = I^{[0]} \times_f M^{m|n}$ be a super semi-Riemannian warped product.

\[
\hat{R}_{IJK}^L = \partial_I \hat{F}_{JK}^L + (-1)^{\left[1\right]+\left[1\right]} \hat{F}_{IJK}^L \partial_t + (-1)^{\left[1\right]} \hat{F}_{JK}^L \partial_I \hat{F}_{t}^L
+ (-1)^{\left[1\right]+\left[1\right]+1} \hat{F}_{JK}^L \partial_\alpha \hat{F}_{I}^L - (-1)^{\left[1\right]} \hat{F}_{JK}^L \partial_I \hat{F}_{t}^L
- (-1)^{\left[1\right]} \hat{F}_{IJK}^L \partial_s \hat{F}_{t}^L - (-1)^{\left[1\right]+\left[1\right]+1} \hat{F}_{IJK}^L \partial_\alpha \hat{F}_{t}^L,
\]

where $\hat{R}$ is the curvature tensor of $\nabla$. In addition $[\nabla_{\partial_t} , \nabla_{\partial_\alpha}] \partial_\kappa = \hat{R}_{IJK}^L \partial_\kappa$ and $I, J, K, L$ stand for even and odd indices $t, i, \alpha$

Proof Let $\nabla$ be the Levi-Civita connection of the metric $\hat{g}$. Then one has $\nabla_{\partial_\alpha} \partial_j = \hat{F}_{IJK}^L \partial_\kappa$. Since the connection $\nabla$ is torsion free we get

\[
\hat{F}_{IJK}^L \hat{g}_{JK} + \hat{F}_{IJK}^L \hat{g}_{LK} = \frac{1}{2} \partial_j \hat{g}_{JK} - (-1)^{\left[1\right]} \partial_\kappa \hat{g}_{JK} + (-1)^{\left[1\right]} \partial_j \hat{g}_{KJ},
\]

\[
\hat{F}_{IJK}^L \hat{g}_{JK} + \hat{F}_{IJK}^L \hat{g}_{LK} = \frac{1}{2} \partial_j \hat{g}_{JK} - (-1)^{\left[1\right]} \partial_\kappa \hat{g}_{JK} + (-1)^{\left[1\right] + 1} \partial_j \hat{g}_{KJ}.
\]

By the equation (3), and a straightforward computation according the argument in [10] we will have the desired result.

Proposition 1 Let $\hat{M} = I^{[0]} \times_f M^{m|n}$ be a super semi-Riemannian warped product then one has

\[
\hat{Ric}_{tt} = (n - m) \left( f'' - \frac{f'^2}{4f^2} \right),
\]

\[
\hat{Ric}_{t\alpha} = \hat{Ric}_{tt} = 0,
\]

\[
\hat{Ric}_{\alpha\beta} = \hat{Ric}_{\alpha\beta} + \left( m - n - 2 \right) \left( \frac{f'^2}{4f^2} + \frac{f''}{2f} \right) g_{\alpha\beta},
\]

\[
\hat{Ric}_{ij} = \hat{Ric}_{ij} + \left( m - n - 2 \right) \left( \frac{f'^2}{4f^2} + \frac{f''}{2f} \right) g_{ij},
\]

\[
\hat{Ric}_{ai} = \hat{Ric}_{ai} + \left( m - n - 2 \right) \left( \frac{f'^2}{4f^2} + \frac{f''}{2f} \right) g_{ai},
\]

where $\hat{Ric}_{\alpha\beta}, \hat{Ric}_{ai}, \hat{Ric}_{ij}$ are the Ricci tensors of semi-Riemannian supermanifold $(M, g)$.

Proof Let $N$ be an arbitrary supermanifold then we have

\[
\hat{Ric}(\partial_\alpha, \partial_j) = R_{sIJ}^t \left[ (\frac{1}{2} + \left[1\right] + \left[1\right] + \left[1\right] + \left[1\right]) \right] R_{\alpha sIJ}^t,
\]

where $s, \alpha$ are even and odd indices over $N$. If $N := \hat{M} = I^{[0]} \times_f M^{m|n}$ we have

\[
\hat{Ric}(\partial_\alpha, \partial_j) = R_{sIJK}^t + R_{sIJ}^t \left[ (\frac{1}{2} + \left[1\right] + \left[1\right]) \right] R_{\alpha sIJ}^t.
\]

where $\hat{R}_{IJK}^t$ is curvature tensor as above and $I, J \in \{t, s, \alpha\}.$
Proposition 2 Let $\bar{M} = I^{[0]} \times_f M^{[m]}$ be a super semi-Riemannian warped product. Then the scalar curvature $\bar{S}$ of $(\bar{M}, \bar{g})$ admits the following expressions

$$\bar{S} = \frac{S_M}{f} + (m - n)\frac{f''}{f} + (m - n)(m - n - 3)\frac{f'^2}{4f^2}. \quad (9)$$

Proof We use the following equation

$$\bar{S} = \bar{Ric}_{tt} \bar{g}_{tt} + \bar{Ric}_{ij} \bar{g}^{ji} - \bar{Ric}_{\alpha\beta} \bar{g}_{\beta\alpha}. \quad (10)$$

Then, with a straightforward computation we will have the desired result.

Proposition 3 Let $\bar{G}$ be the Einstein gravitational tensor field of $(\bar{M}, \bar{g})$, then we have the following equations

$$\bar{G}_{tt} = \frac{S_M}{2f} + (m - n)(m - n - 1)\frac{f'^2}{8f^2},$$

$$\bar{G}_{ij} = G_{ij} + (1 - m + n)\frac{f''}{2}g_{ij} + (m - n - 2)\frac{f'^2}{4f}g_{ij} - (m - n)(m - n - 3)\frac{f'^2}{8f}g_{ij},$$

$$\bar{G}_{\alpha\alpha} = G_{\alpha\alpha} + (1 - m + n)\frac{f''}{2}g_{\alpha\alpha} + (m - n - 2)\frac{f'^2}{4f}g_{\alpha\alpha} - (m - n)(m - n - 3)\frac{f'^2}{8f}g_{\alpha\alpha},$$

$$\bar{G}_{\alpha\beta} = G_{\alpha\beta} + (1 - m + n)\frac{f''}{2}g_{\alpha\beta} + (m - n - 2)\frac{f'^2}{4f}g_{\alpha\beta} - (m - n)(m - n - 3)\frac{f'^2}{8f}g_{\alpha\beta},$$

$$\bar{G}_{tt} = \bar{G}_{tt} = 0. \quad (11)$$

Proof The Einstein gravitational tensor field of $(\bar{M}, \bar{g})$ is

$$\bar{G} = \bar{Ric} - \frac{1}{2} \bar{S}_{\bar{g}}. \quad (12)$$

By using (12), (7), (9) and (2), we obtain (11).

Proposition 4 Let $(\bar{G}, \bar{A})$ be the Einstein gravitational tensor field and cosmological constant of $(\bar{M}, \bar{g})$ for the case I, then we have the following equations

$$\bar{A}(t) = \frac{1}{4}(m - n - 1) \left( (m + n)\frac{f''}{f} - (m + 3n)\frac{f'^2}{2f^2} \right), \quad (13)$$

$$G_{\alpha\beta} = \frac{1}{4}(1 - m + n)(m + n - 2) \left( \frac{f''}{f} - \frac{f'^2}{f^2} \right) g_{\alpha\beta}. \quad (14)$$

Proof The Einstein equations with cosmological constant $\bar{A}$ is

$$\bar{G} = -\bar{A}\bar{g}. \quad (15)$$

By using (15), (2) and (11), we obtain (13) and (14).
The Einstein equations on \((\bar{M}, \bar{g})\) with cosmological constant \(\bar{\Lambda}\) induces the Einstein equations on \((M, g)\), where the cosmological constant \(\Lambda\) is given by

\[
\Lambda(t) = \frac{1}{4}(m - n - 1)(m + n - 2) \left( \frac{f''}{f} - \frac{f'^2}{f^2} \right) f(t). \tag{16}
\]

We call \(\bar{\Lambda}\) and \(\Lambda\) here as \(\bar{\Lambda}_I\) and \(\Lambda_I\), respectively. The requirement \(\bar{\Lambda}_I = \text{Const}\) can determine the warp function \(f(t)\) and \(\Lambda_I(t)\), using which the Einstein equations \(\bar{G}_{AB} = -\bar{\Lambda}_I \bar{g}_{AB}\) and \(G_{\alpha\beta} = -\Lambda_I g_{\alpha\beta}\) are solved.

Let us consider \(\bar{\Lambda}_I = L^{-2} = \text{Const}\), where \(L\) has dimension of length. Then, we obtain

\[
f(t) = \frac{L^2}{4k} \exp \frac{t}{\sqrt{kL}}, \quad k = \frac{1}{8}(m + n)(m - n - 1). \tag{17}
\]

Assuming a supermanifold \(M^{3|0}\), the warp function in this warp product space is identified with the squared scale factor \(a^2(t)\) in \((1+3)\)-dimensional Robertson-Walker cosmology which describes the time evolution of the 3-dimensional spatial hypersurface \(M^{3|0}\) as

\[
a(t) = \frac{L}{\sqrt{3}} \exp \frac{t}{\sqrt{3L}}, \tag{18}
\]

which shows a de Sitter expansion of the universe in agreement with the solution of FRW cosmological model with a cosmological constant \(L^{-2}\).

Using the warp function \(17\), we obtain \(\Lambda_I = 0\). This result is in agreement with the fact that the cosmological constant is an energy density corresponding to the \(tt\) component of Einstein equations and has no counterpart on the supermanifold \(M^{3|0}\), namely we have a Ricci flat Einstein equation

\[
G_{\alpha\beta} = 0. \tag{19}
\]

Case II

Let \(\bar{M} = I^{0|1} \times_f M^{m|n}\) be a super semi-Riemannian warped product. Then \(O_{I^{0|1}} = \mathbb{R} \oplus \mathbb{R} = a + bt\) such that \(a, b \in \mathbb{R}\) and \((M, g)\) is a super semi-Riemannian manifold an, \((x^t, x^\alpha)\) be an even - odd coordinate system on \(M^{m|n}\), \(\bar{t}\) be a coordinate on \(I^{0|1}\) and \(f \equiv f(\bar{t})\). By \(|\bar{t}|, |i|, |\alpha|\) we mean the parities of \(\bar{t}, x^t, x^\alpha\) respectively. So \(|\bar{t}| = |\alpha| = 1\) and \(|i| = 0\). we have

\[
\begin{align*}
g_{\bar{tt}} &= g(\partial_{\bar{t}}, \partial_{\bar{t}}) = 0, \\
g_{\alpha\bar{t}} &= g_{\alpha\alpha} = g_{\bar{t}\bar{t}} = 0, \\
g_{\alpha\beta} &= f g_{\alpha\beta}, \\
g_{ij} &= f g_{ij}, \\
g_{i\alpha} &= f g_{i\alpha}. \tag{20}
\end{align*}
\]
The Christoffel symbols $\Gamma_{IJ}^L$ satisfies the following equation

$$\bar{\Gamma}_{IJ}^L \bar{g}_{LK} = \frac{1}{2} \left[ \partial_I \bar{g}_{JK} - (-1)^{|J|(|I|+|J|)} \partial_J \bar{g}_{IK} + (-1)^{|I|(|J|+|K|)} \partial_J \bar{g}_{KI} \right],$$

(21)

where $I, J, K, L$ stand for $t, i$ and $\alpha$. If at least one of indices $K, I$ or $J$ is equal to $t$ then the both sides of equality are zero. So $\Gamma_{tI}^i, \Gamma_{t\alpha}^i, \Gamma_{\alpha t}^i, \Gamma_{t\alpha}^\alpha$ are arbitrary elements. Therefore from now on for computation we consider them to be zeros.

Similar to case I, we can obtain the following equations for Ricci tensor

$$\bar{\text{Ric}}_{tt} = (n - m) \frac{f'^2}{f^2} + \frac{f'^2}{2f^2},$$

$$\bar{\text{Ric}}_{ti} = \bar{\text{Ric}}_{t\alpha} = 0,$$

$$\bar{\text{Ric}}_{\alpha \beta} = \text{Ric}_{\alpha \beta},$$

$$\bar{\text{Ric}}_{ij} = \text{Ric}_{ij},$$

$$\bar{\text{Ric}}_{\alpha i} = \text{Ric}_{\alpha i},$$

(22)

where $'$ denotes $\partial_\bar{t}$, and the scalar curvature

$$\bar{S} = \frac{S}{f}.$$  

(23)

**Proposition 5** Let $(\bar{M}, \bar{g})$ be super semi-Riemannian warped product as above. If $(\bar{M}, \bar{g})$ is an Einstein space, then $\text{Ric}(\bar{t}, \partial_\bar{t}) = 0$ and $m - n = -2$.

**Proof** By Einstein equation with cosmological constant $\bar{G} = -\bar{\Lambda} \bar{g}$.

(24)

one has $\bar{G}_{tt} = -\bar{\Lambda} \bar{g}_{tt}$. Since $\bar{g}_{tt} = 0$, thus $\bar{G}_{tt} = 0$ and $\text{Ric}_{tt} = 0$. So the first equality in (22) shows $m - n = -2$.

One can easily show that the Einstein gravitational tensor fields of $(\bar{M}, \bar{g})$ and $(M, g)$ are equal i.e. $\bar{G} = G$.

**Proposition 6** Let $(\bar{G}, \bar{\Lambda})$ be the Einstein gravitational tensor field and cosmological constant of $(\bar{M}, \bar{g})$, then we have the following equations

$$\bar{\Lambda} = a \in \mathbb{R},$$

(25)

$$\bar{G}_{\alpha \beta} = G_{\alpha \beta}.$$  

(26)

The Einstein equations on $(\bar{M}, \bar{g})$ with cosmological constant $\bar{\Lambda}$ induces the Einstein equations on $(M, g)$, where the cosmological term $\Lambda$ is given by

$$\Lambda(\bar{t}) = a f(\bar{t}).$$

(27)

We call $\bar{\Lambda}$ and $\Lambda$ here as $\Lambda_{(tt)}$ and $\Lambda_{(t\bar{t})}$, respectively. It is seen that, unlike the case I, $\Lambda_{(tt)}$ itself is constant independent of $f(\bar{t})$ and $(m, n)$. Therefore,
f(t) is no longer determined by the equation \( \dot{A} = a \) and this case describes a super warped product spacetime with an arbitrary warp function \( f(t) \) which is a result of identities \( \tilde{g}_{tt} = \text{Ric}(\partial_t, \partial_t) = \tilde{G}_{tt} = 0 \). In other words, since there is no information in the \( \tilde{t} \) component of Einstein equations, the warp function is not determined. Using (27), the cosmological term \( A_{(tt)} \) is not determined too, and so the Einstein equation \( G_{\alpha\beta} = -A_{(tt)}g_{\alpha\beta} \) is free of physical information.

Moreover, assuming a supermanifold \( M^{3|0} \) and identifying the warp function with the squared scale factor \( a^2(t) \) in a \((1+3)\)-dimensional Robertson-Walker cosmology, it turns out that the \( t \) evolution of the supermanifold \( M^{3|0} \) is completely arbitrary and so this case does not describe a physically viable cosmology.

Case III

Let \((\tilde{M}, \tilde{g})\) be a super semi-Riemannian warped product where \( \tilde{M} = I^{1|1} \times_f M^{m|n} \) and \( \tilde{g} = -dt^2 + dt \times g_M \). In addition \( I^{1|1} = I^{1|0} \times I^{0|1} \) and define \( f = h + t k \) where \( h, k \) are smooth functions on \( I \) and \( h > 0 \).

Let \((t, \tilde{t})\) be an even - odd coordinate system on \( I^{1|1}, (x^i, x^n) \) be a coordinate system on \( M^{m|n} \) and \( f \equiv f(t, \tilde{t}) \). Then, all equalities (2) and (20) are satisfied simultaneously. Similar to case I and case II we can obtain the following equations for Ricci tensor and scalar curvature

\[
\tilde{\text{Ric}}_{tt} = \frac{(m^2 - n^2)}{2}(\frac{m + n}{f^2}f_t^2 - f_{tt}),
\]

\[
\tilde{\text{Ric}}_{tt} = \frac{(m^2 - n^2)}{2}\left(\frac{f_{tt}}{f} - \frac{(m + n - 2)f_t^2}{f^2}\right),
\]

\[
\tilde{\text{Ric}}_{\tilde{t}t} = \frac{(m^2 - n^2)}{2}\left(\frac{f_{\tilde{t}t}}{f} - \frac{(m + n - 2)f_{\tilde{t}}f_t}{f^2}\right),
\]

\[
\tilde{\text{Ric}}_{\tilde{t}\tilde{t}} = \frac{(m^2 - n^2)}{2}\left(\frac{f_{\tilde{t}\tilde{t}}}{f} - \frac{(m + n - 2)f_{\tilde{t}}f_{\tilde{t}}}{f^2}\right),
\]

\[
\tilde{\text{Ric}}_{\alpha\beta} = \tilde{\text{Ric}}_{\beta\alpha} = \tilde{\text{Ric}}_{\alpha\tilde{t}} = \tilde{\text{Ric}}_{\tilde{t}\alpha} = 0,
\]

\[
\tilde{\text{Ric}}_{\alpha\beta} = \tilde{\text{Ric}}_{\alpha\beta} - \frac{(m - n - 1)(m + n)}{2}\left(\frac{f_t f_t}{f}(g_{\alpha\beta}) - \frac{(m^2 - n^2)}{4}\left(\frac{f^2 g_{\alpha\beta}}{f}\right)\right),
\]

\[
\tilde{\text{Ric}}_{\alpha\beta} = \tilde{\text{Ric}}_{\alpha\beta} + \frac{(m^2 - n^2)}{4}\left(\frac{f^2 g_{\alpha\beta}}{f}\right),
\]

\[
\tilde{\text{Ric}}_{\alpha\tilde{t}} = \tilde{\text{Ric}}_{\beta\tilde{t}} = \tilde{\text{Ric}}_{\alpha\beta} - \frac{(m - n - 1)(m + n)}{2}\left(\frac{f_t f_t}{f}(g_{\beta\alpha}) - \frac{(m + n)^2}{4}\left(\frac{f^2 g_{\alpha\beta}}{f}\right)\right), \quad (28)
\]

\[
S = \frac{1}{f}S_M + \frac{(m + n)}{2}\left(\frac{m(m - n - 1) - m(m - 1)}{2}\frac{f_t f_t}{f^2} - \frac{(m + n)}{2}\left(\frac{4n - m^2 + n^2 - 2m}{f^2}\right)\right)\left(\frac{m^2 - n^2 - 2}{2}\right)\frac{f_t}{f}. \quad (29)
\]
Let \((\bar{G}, \bar{A})\) be the Einstein gravitational tensor field and cosmological constant of \((M, \bar{g})\) respectively, then we have the following equations

\[
\bar{G}_{tt} = \left(\frac{m^2 - n^2}{2}\right) \left(\frac{m + n f_t^2}{f^2} - \frac{f_t}{f}\right),
\]

\[
\bar{G}_{\alpha\beta} = \left(\frac{m^2 - n^2}{2}\right) \left(\frac{f_{tt}}{f} - \frac{1}{4} \frac{(m + n - 2)f_t^2}{f^2}\right) + \frac{S_M}{2f} + \frac{(m + n)(m - n - 1) - m(m - 1)}{4} \frac{f_t f_t}{f^2} + \frac{(m + n)(m - n - 2)}{4} \frac{f_{tt}}{f^2},
\]

\[
\bar{G}_{ij} = \frac{1}{2} \left(\frac{m + n}{2} \right) \left(\frac{m - n - 1}{2} \right) \frac{f_{ij}}{f} - \frac{1}{4} \frac{(m + n)(2n - n^2 - 2)}{2f^2} \frac{f_t f_t}{f^2} + \frac{(m + n)(2n^2 - mn + m - m)}{4} \frac{f_t f_t}{f^2}.
\]

**Proof** We use relation \(\bar{G} = \bar{Ric} - 1/2 \bar{S} \bar{g}\) and similar to Case I, II we obtain gravitational tensor field \(\bar{G}\).

**Proposition 7** Let \((\bar{G}, \bar{A})\) be the Einstein gravitational tensor field and cosmological constant of \((M, \bar{g})\), then we have the following equations

\[
\bar{A}(t, i) = \left(1 - \frac{n}{2}\right) \left(\frac{m^2 - n^2}{2}\right) \left(\frac{f_{tt}}{f} - \frac{1}{4} \frac{(m + n - 2)f_t^2}{f^2}\right) + \frac{(m + n)(2n - n^2 - 1) - m(m - 1)}{4} \frac{f_t f_t}{f^2} + \frac{m^2(n - 2) + n^2(m - 8) - (m - n + 2) - m(n - 1)}{2f^2} \frac{f_t f_t}{f^2} + \frac{(m^2 - n^2 - 2)f_{tt}}{4f},
\]

\[
\bar{G}_{\alpha\beta} = f g_{\alpha\beta} \left(1 - \frac{n}{2}\right) \left(\frac{m + n}{2} \right) \left(\frac{m - n - 1}{2} \right) \frac{f_{ij}}{f} - \frac{1}{4} \frac{(m + n)(6n + m^2 - n^2 + 4m)}{8} \frac{f_t f_t}{f^2} - \frac{(m^2 - n^2)(-f_{tt})}{2} \left(\frac{m + n - 2}{2} \right) \frac{f_t^2}{f^2}.
\]

**Proof** The Einstein equations with cosmological constant \(\bar{A}\) is

\[
\bar{G} = -\bar{A} \bar{g}.
\]

Similar to case I, II we obtain \(\bar{A}\) and \(\bar{G}_{\alpha\beta}\).
The Einstein equations on \((\bar{M}, \bar{g})\) with cosmological constant \(\bar{\Lambda}\) induces the Einstein equations on \((M, g)\), where the cosmological constant \(\Lambda\) is given by

\[
\Lambda(t, \bar{t}) = -f(1 - \frac{n}{2}) \times \left(\frac{(m + n)(m - n - 1)}{2} f_{\bar{t}} f_t + \frac{(m + n)(-6n + m^2 - n^2 + 4m)}{8} f_t^2 - \frac{(m^2 - n^2)}{2} (-\frac{f_{\bar{t}}}{f_t} - \frac{m + n - 2}{2} f_t^2)\right).
\]

We call \(\bar{\Lambda}\) and \(\Lambda\) here as \(\bar{\Lambda}_{(tt)}, \Lambda_{(tt)}\), respectively. The requirement \(\Lambda_{(tt)} = \text{Const}\) can determine the warp function \(f(t, \bar{t})\) and \(\Lambda_{(tt)}(t, \bar{t})\), using which the Einstein equations \(\bar{G}_{AB} = -\bar{\Lambda}_{(tt)} \bar{g}_{AB}\) and \(\bar{G}_{\alpha\beta} = -\Lambda_{(tt)} g_{\alpha\beta}\) are solved. Solving \(\bar{\Lambda}_{(tt)} = \text{Const}\) as a nonlinear quadratic partial differential equation for \(f(t, \bar{t})\) is a hard task. However, for instance, one can propose an ansatz

\[
f(t, \bar{t}) = e^{\alpha t + \beta \bar{t}},
\]

for which the equation \(\dot{\bar{\Lambda}}_{(tt)} = \text{Const}\) reads as the following algebraic equation

\[
\alpha^2 (1 - \frac{n}{2}) \left(\frac{(m^2 - n^2)}{2} \right) (\frac{m + n}{2}) + \alpha \beta \frac{(m + n)(2n(m - n - 1) - m(m - 1))}{4} + \beta^2 \frac{m^2(n - 2) + n^2(m - 8)}{8} - \frac{(m^3 + n^3)}{2} + 2(m^2 - n^2 - 2) + 10mn = \text{Const}.
\]

(33)

For a given constant and given values of \(m\) and \(n\), the coefficients \(\alpha\) and \(\beta\) satisfy the equation (33). Moreover, \(\Lambda_{(tt)}(t, \bar{t})\) is also obtained as follows

\[
\Lambda_{(tt)}(t, \bar{t}) = -e^{\alpha t + \beta \bar{t}} \left(1 - \frac{n}{2}\right) \times \left(\alpha^2 \frac{(m^2 - n^2)}{2} \left(\frac{m + n}{2}\right) + \alpha \beta \frac{(m + n)(m - n - 1)}{2} + \beta^2 \frac{(m + n)(-6n + m^2 - n^2 + 4m)}{8}\right).
\]

(34)

Assuming a supermanifold \(M^{3|0}\), the equation \(\dot{\bar{\Lambda}}_{(tt)} = \text{Const}\) and \(\Lambda_{(tt)}(t, \bar{t})\) read as

\[
-\left(\frac{27}{4} \alpha^2 + \frac{9}{2} \alpha \beta + \frac{31}{8} \beta^2\right) = \text{Const},
\]

(35)

\[
\Lambda_{(tt)}(t, \bar{t}) = -e^{\alpha t + \beta \bar{t}} \left(\frac{27}{4} \alpha^2 + 3 \alpha \beta + \frac{63}{8} \beta^2\right).
\]

(36)

By choosing one of the coefficients \(\alpha\) or \(\beta\), the other coefficient is fixed through the equation (35) and the \(\bar{t}\) evolution of supermanifold \(M^{3|0}\), namely \(e^{\beta \bar{t}}\), determines its \(t\) evolution \(e^{\alpha t}\), and vice versa. A wide variety of positive, negative, vanishing and imaginary values for \(\alpha\) and \(\beta\) is available. This is an interesting result in that we obtain a two parameter \((\alpha, \beta)\) class of two-times cosmological models for which the \(t\) and \(\bar{t}\) evolutions of the cosmological scale factor \(a(t, \bar{t})\) correspond to the same cosmological constant \(\bar{\Lambda}_{(tt)}\). Positive and/or negative
values of $\alpha$ and $\beta$ may result in inflationary expanding or contracting (depending on the signs and values of $\alpha$ and $\beta$) supermanifold $M^{3|0}$. Vanishing values of both $\alpha$ and $\beta$ are possible if $\bar{\Lambda}^{(III)} = 0$ and result in a static state of supermanifold $M^{3|0}$, which corresponds to Einstein static universe [11]. Imaginary values of $\alpha$ and/or $\beta$ may result in a full or partial wave-like evolution of the supermanifold $M^{3|0}$ [12].

Conclusion

In this paper, we have studied three types of super warped product space-times $\bar{M}^{(I)} = I^{1|0} \times_{f} M^{m|n}$, $\bar{M}^{(II)} = I^{0|1} \times_{f} M^{m|n}$, and $\bar{M}^{(III)} = I^{1|1} \times_{f} M^{m|n}$, where $M^{m|n}$ is a supermanifold of dimension $m|n$, $I^{(\delta, \delta')}_{\delta, \delta' \in \{0, 1\}}$ is standard superdomain with $I = (0, 1)$ and $\delta, \delta' \in \{0, 1\}$, subject to the warp functions $f(t)$, $f(\bar{t})$, and $f(t, \bar{t})$, respectively. By using the Generalized Robertson-Walker space-time, as super warped product space, we have shown that the Einstein equations $\bar{G}_{AB} = -\bar{\Lambda} \bar{g}_{AB}$ are reducible to the Einstein equations $G_{\alpha\beta} = -\Lambda g_{\alpha\beta}$ on the supermanifold $M$ with cosmological term $\Lambda$, where $\bar{\Lambda}$ and $\Lambda$ are functions of $f(t)$, $f(\bar{t})$, and $f(t, \bar{t})$, as well as $(m, n)$. Then, by demanding for constancy of $\bar{\Lambda}$ we have determined the warp functions and $\Lambda$ which result in finding the solutions for Einstein equations $\bar{G}_{AB} = -\bar{\Lambda} \bar{g}_{AB}$ and $G_{\alpha\beta} = -\Lambda g_{\alpha\beta}$.

Interested in the cosmological solutions of Einstein equations with cosmological constant in the special case of supermanifold $M^{3|0}$ (3-dimensional space) we have found, in each case of super warp product space, the following results:

- In the first case, namely $\bar{M}^{(I)} = I^{1|0} \times_{f} M^{3|0}$, the $t$ evolution of the supermanifold $M^{3|0}$ is described by the $t$ evolution of the cosmological scale factor as $a(t) = \frac{\Lambda}{3} \exp \left( \frac{\Lambda}{3} t \right)$ which describes a de Sitter expansion of the Universe with initial scale factor $a(0) = \frac{\Lambda}{3}$.
- In the second case, namely $\bar{M}^{(II)} = I^{0|1} \times_{f} M^{3|0}$, the $\bar{t}$ evolution of the supermanifold $M$ is completely arbitrary and so this case does not describe a physically viable cosmology.
- In the third case, namely $\bar{M}^{(III)} = I^{1|1} \times_{f} M^{3|0}$, the two-times evolution of supermanifold $M^{3|0}$ is described by $(t, \bar{t})$ evolution of the cosmological scale factor $a(t, \bar{t}) = e^{\frac{\Lambda}{3 \sqrt{3}}} t$ which defines a two parameter $(\alpha, \beta)$ class of two-times cosmological models corresponding to the same cosmological constant $\bar{\Lambda}^{(III)}$.

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