Path Decomposition of Spectrally Negative Lévy Processes

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Abstract

Path decomposition is performed to analyze the pre-supremum, post-supremum, post-infimum and the intermediate processes of a spectrally negative Lévy process as motivated by the aim of finding the joint distribution of the maximum loss and maximum gain. In addition, the joint distribution of the supremum and the infimum is found before an exponential time. As an application of path decomposition, the distributions of supremum of the post-infimum process and the maximum loss of the post-supremum process are obtained.

Keywords: maximum drawdown, maximum drawup, scale function, expansion of filtration, extreme values

1. Introduction

A spectrally negative Lévy process $X$ is that with no positive jumps and has found place naturally in many applications such as risk theory, mathematical finance and queuing theory, e.g. [1, 3, 7, 8]. It is an advantageous model due to its tractability through the so-called scale functions. In this paper, path decomposition of a spectrally negative Lévy process is performed to analyze the pre-supremum, post-supremum, post-infimum and the inter-
mediate processes as motivated by questions which can be expressed through trajectories and extreme values.

Let $X$ be defined on the filtered probability space $(\Omega, \mathcal{H}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual hypothesis. By definition, the Lévy measure $\Pi$ of a spectrally negative Lévy process is concentrated on $(-\infty, 0)$. We assume that $X$ is not the negative of a subordinator or a deterministic drift. Let the Laplace exponent of $X$ be given by

$$
\psi(\lambda) = -\mu \lambda + \frac{\sigma^2}{2} \lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x 1_{\{x>1\}}) \Pi(dx)
$$

where $\mu \in \mathbb{R}$, and $\sigma > 0$. Then, Brownian motion with drift $\mu$ can be recovered as a special case when $\Pi \equiv 0$. Using a Brownian motion $B$ and a Poisson random measure $N$, for $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$, we can write

$$
dX_t = \mu dt + \sigma dB_t + \int_{-1}^0 y \tilde{N}(dy,dt) + \int_{-\infty}^{-1} y N(dy,dt)
$$

where $\tilde{N} = N - \Pi$, and assume $\int_{-1}^0 y^2 \Pi(dy) < \infty$ and $\int_{-\infty}^{-1} y \Pi(dy) < \infty$. Then, the infinitesimal generator $\Gamma$ of $X$ is given by

$$
\Gamma f(x) = \mu f'(x) + \frac{1}{2} \sigma^2 f''(x) + \int_{-\infty}^0 [f(x+y) - f(x) - f'(x)y 1_{\{y>1\}}] \Pi(dy).
$$

We consider a spectrally negative Lévy process $X$ taken up to an exponential time $T$ with parameter $\gamma$. We denote the first passage times above and below $x$ respectively by

$$
\tau_x^+ = \inf\{t \geq 0 : X_t > x\} \quad \tau_x^- = \inf\{t \geq 0 : X_t < x\}.
$$

Let the supremum and the infimum defined by $S_T := \sup\{X_s : 0 \leq s \leq T\}$ and $I_T := \inf\{X_s : 0 \leq s \leq T\}$. The fluctuation identities that we rely on involve a class of functions known as scale functions. The $\gamma$–scale function of $X$ satisfies

$$
\int_0^\infty e^{-\lambda x} W^{(\gamma)}(x) dx = \frac{1}{\psi(\lambda) - \gamma} \quad \text{for} \quad \lambda > \Phi(\gamma)
$$

2
where $\Phi$ denotes the right inverse of $\psi$ and the second scale function is given by

$$Z^{(\gamma)}(x) = 1 + \gamma \int_0^x W^{(\gamma)}(y)dy$$

(1)

see e.g. [1 Thm. 8.1].

We first find the joint distribution of the infimum and the supremum of $X$ in Proposition [1] as

$$\mathbb{P}_0(a < I_T, S_T < b) = 1 - Z^{(\gamma)}(-a) + \left[Z^{(\gamma)}(b - a) - 1\right] \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b - a)}.$$

Second, we perform path decomposition through the extremes and find the distributions of pre-supremum, post-supremum and post-infimum processes in Theorem [1]. As a corollary, the cumulative distribution function of the supremum of the post-infimum process given $I_t = a$ turns out to be

$$\frac{\Phi(\gamma)(Z^{(\gamma)}(b - a) - 1)}{\gamma W^{(\gamma)}(b - a)}$$

at $b \geq a$. Afterwards, Proposition [2] characterizes the post-infimum process up to a given level. As in [13 Sec.3], we apply the theory of expansion of filtrations to have the last exit from the infimum a stopping time and analyze the post-infimum process.

The result in Proposition [2] concerns the intermediate process between the infimum and the supremum given that the infimum of the Lévy process occurs before its supremum. We state it in Theorem [2] as a part of path decomposition, together with the distribution of the post-supremum process under the same conditions. Such a path decomposition is motivated by an ultimate aim of finding the joint distribution of the maximum loss and the maximum gain defined by

$$M_t^- := \sup_{0 \leq u \leq v \leq t} (X_u - X_v) \quad M_t^+ := \sup_{0 \leq u \leq v \leq t} (X_v - X_u)$$

which are also known as maximum drawdown and maximum drawup. This approach has been originally followed by Salminen and Vallois [11] to obtain the same joint distribution for standard Brownian motion. In our case, the characterization of pre-infimum process remains unsolved. Nevertheless, the cumulative distribution function of the maximum loss of the post-supremum
process at $0 < d < b - a$ under the condition that infimum occurs before the supremum and $I_T = a$, $S_T = b$ is found as

$$1 - \left[ 1 - Z^{(\gamma)}(b - a - d) + (Z^{(\gamma)}(b - a) - 1) \frac{W^{(\gamma)}(b - a - d)}{W^{(\gamma)}(b - a)} \right]$$

$$\cdot \frac{Z^{(\gamma)'}(d) - Z^{(\gamma)}(d) \frac{W^{(\gamma)'}(d)}{W^{(\gamma)}(d)}}{-Z^{(\gamma)'}(b - a) + (Z^{(\gamma)}(b - a) - 1) \frac{W^{(\gamma)'}(b - a)}{W^{(\gamma)}(b - a)}}$$

as given in Corollary 2. Clearly, the joint distribution of the supremum and the infimum given in Proposition 1 could be combined with Theorem 2 in order to remove the conditions.

The paper is organized as follows. In Section 2, the joint distribution of the supremum and the infimum of the spectrally negative Lévy process is found before an exponential time. The path is decomposed through its infimum and supremum separately in Section 3. The post-infimum process is characterized in Section 4. Finally, Section 5 provides a more detailed path decomposition including the intermediate process between the infimum and the supremum, and displays the distribution of the maximum loss of the post-supremum process as an application.

# 2. Joint distribution of the supremum and the infimum

In this part, we find the joint distribution of the supremum and the infimum of a spectrally negative Lévy process $X$, which is taken up to an independent exponential time $T$ with parameter $\gamma$. The distribution is characterized in view of the formulas for the Laplace transform of the first passage times.

**Proposition 1.** Let, $X$, be a spectrally negative Lévy process and let $T$ be an exponentially distributed random variable with parameter $\gamma$, independent of $X$. For $a < 0 < b$, the joint distribution of $I_T$ and $S_T$ is given by

$$\mathbb{P}(a < I_T, S_T < b) = 1 - Z^{(\gamma)}(-a) + \left[ Z^{(\gamma)}(b - a) - 1 \right] \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b - a)}.$$
Proof: For $a < 0 < b$, we have
\[
\mathbb{P}_0\{a < I_T, S_T < b\} = \mathbb{P}_0\{T < \tau_a^- \land \tau_b\}
\]
\[
= \mathbb{P}_0\{T < \tau_a^-, \tau_a^- < \tau_b\} + \mathbb{P}_0\{T < \tau_b, \tau_b < \tau_a^-\}
\]
\[
= 1 - (\mathbb{P}_0\{T > \tau_a^-, \tau_a^- < \tau_b\}) + \mathbb{P}_0\{T > \tau_b, \tau_b < \tau_a^-\})
\]
\[
= 1 - (\mathbb{E}_0[e^{-\gamma \tau_a^-} 1_{\{\tau_a^- < \tau_b\}}] + \mathbb{E}_0[e^{-\gamma \tau_a^-} 1_{\{\tau_b < \tau_a^-\}}])
\]
\[
= 1 - (\mathbb{E}_0[e^{-\gamma \tau_0^+} 1_{\{\tau_0^+ < \tau_b-a\}}] + \mathbb{E}_0[e^{-\gamma \tau_0^+} 1_{\{\tau_b-a < \tau_0^+\}}]).
\]
The expectations in the above equation are known as two-sided exit from below and two-sided exit from above a level, respectively. For a spectrally negative Lévy process, two sided exit problems are well defined and the expressions for them are well known. After substituting these expressions, we find the joint distribution as
\[
\mathbb{P}_0\{a < I_T, S_T < b\}
\]
\[
= 1 - \left[Z^{(\gamma)}(-a) - Z^{(\gamma)}(b-a) \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b-a)} + \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b-a)}\right]
\]
\[
= 1 - Z^{(\gamma)}(-a) + \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b-a)}[Z^{(\gamma)}(b-a) - 1].
\]
The joint distribution above is a stand alone formula for the extremes of a spectrally negative Lévy process and generalizes that for a Brownian motion with drift. In particular, it is useful towards our analysis in Section about the path decomposition under the condition that the infimum and the supremum occur at given levels.

3. Decomposition through infimum or supremum

In this section, we perform a path decomposition of the Lévy process to characterize the distributions of pre-supremum, post-supremum, and post-infimum processes before an independent exponential time $T$ with parameter $\gamma > 0$. Let
\[
H_S := \sup\{t < T : X_t = S_t\} \quad \text{and} \quad H_I := \sup\{t < T : X_t = I_t\}.
\]
Then, we have the following theorem for pre-$H_S$ process $\{X_u : 0 \leq u \leq H_S\}$, post-$H_S$ process $\{X_{H_S+u} : u \leq T - H_S\}$, and post-$H_I$ process $\{X_{H_I+u} : u \leq T - H_I\}$.
Theorem 1. i. The pre-$H_I$ process and the post-$H_I$ process are independent. Given $I_T = a$, the law for the post-$H_I$ process is given by $h$-transform of the law $I^\uparrow$ of the original spectrally negative Lévy process conditioned to stay positive and killed at time $T$ with

$$h(z) = \frac{\gamma W^{(\gamma)}(z)}{\Phi(\gamma) W(z)} - \frac{\gamma}{W(z)} \int_0^z W^{(\gamma)}(r) \, dr,$$

that is, the transition semigroup of the post-$H_I$ process is given by

$$P_t(x, dy) = \frac{h(y - a)}{h(x - a)} I^\uparrow_{x-a} \{ X_t \in dy - a, t < T \}$$

for $x, y > a$, and its entrance law is obtained by letting $x \to a^+$.  

ii. The pre-$H_S$ process and the post-$H_S$ process are independent. Given $S_T = b$, the pre-$H_S$ process is a spectrally negative Lévy process with Laplace exponent

$$\tilde{\psi}(\lambda) = \psi(\lambda + \Phi(\gamma)) - \gamma$$

for $\lambda \geq -\Phi(\lambda)$, killed at the first passage time above $b$. On the other hand, the transition semigroup of the post-$H_S$ process is given by

$$P_t(x, dy) = \frac{h(b - y)}{h(b - x)} I_{b-x} \{ \hat{X}_t \in b - dy, t < T \land \hat{\tau}_0^- \}$$

for $x, y < b$, where

$$h(z) = 1 - e^{-\Phi(\gamma)z},$$

$\hat{X}$ is a spectrally positive Lévy process distributed like $-X$, and the entrance law is obtained as $x \to b^-$.  

Proof: i) The independence of pre-$H_I$ and post-$H_I$ process follows from [2, Lem.VI.6] in view of duality between $X - I$ and $\hat{S} - \hat{X}$ and with the analysis as in (4) below, but with the excursion measure $\hat{n}$ for $\hat{S} - \hat{X}$. Then, the post-$H_I$ process evolves as a spectrally negative Lévy process which is conditioned to stay above $I_T$ and killed at an exponential time $T$. The law of this process can be found by studying

$$I_x \{ X_t \in dy, t < T \mid T < \tau_0^- \}$$

(2)
which is equal to

\[ \mathbb{P}_x \{ X_t \in dy, t < T \wedge \tau_0^- \} \]

\[ \frac{\mathbb{P}_x \{ T < \tau_0^- \} }{ \mathbb{P}_x \{ T < \tau_0^- \} } \]

\[ = \frac{\mathbb{P}_x \{ T < \tau_0^- | X_t \in dy, t < T \wedge \tau_0^- \} }{ \mathbb{P}_x \{ T < \tau_0^- \} } \]

\[ = \frac{\mathbb{P}_y \{ T \circ \theta_t < \tau_0^- \circ \theta_t \} \mathbb{P}_x \{ X_t \in dy, t < T \wedge \tau_0^- \} }{ \mathbb{P}_x \{ T < \tau_0^- \} } \]

by Markov property and the fact that \( T \) is memoryless, where \( \theta \) is the shift operator. Let

\[ \tilde{h}(x) := \mathbb{P}_x \{ T < \tau_0^- \} . \]

The function \( \tilde{h} \) can be evaluated as

\[ \tilde{h}(x) = \mathbb{E}_x[\mathbb{P}_x \{ T < \tau_0^- \} | \tau_0^-] = \mathbb{E}_x[1 - e^{-\gamma \tau_0^-} 1_{\{\tau_0^- < \infty\}}] \]

\[ = 1 - \mathbb{E}_x[e^{-\gamma \tau_0^-} 1_{\{\tau_0^- < \infty\}}] \]

\[ = 1 - Z(\gamma)(x) + \frac{\gamma}{\Phi(\gamma)} W(\gamma)(x) \]

\[ = \frac{\gamma W(\gamma)(x)}{\Phi(\gamma)} - \gamma \int_0^x W(\gamma)(z) \, dz \quad (3) \]

by \([7, \text{Thm.8.1}]\) and \([1] \). Note that the post-\( H_I \) process can now be viewed as a transform of the law of the Lévy process conditioned to stay positive \([2, \text{pg.198}]\) by rearranging \((2)\) as

\[ \mathbb{P}_x \{ X_t \in dy, t < T \mid T < \tau_0^- \} \]

\[ = \frac{\tilde{h}(y)}{\tilde{h}(x)} \mathbb{P}_x \{ X_t \in dy, t < \tau_0^-, t < T \} \]

\[ = \frac{\tilde{h}(y)}{\tilde{h}(x)} W(y) e^{-\gamma t} \frac{W(y)}{W(x)} \mathbb{P}_x \{ X_t \in dy, t < \tau_0^- \} \]

\[ = \frac{\tilde{h}(y)}{\tilde{h}(x)} e^{-\gamma t} \mathbb{P}_x^\dagger \{ X_t \in dy \} = \frac{\tilde{h}(y)}{\tilde{h}(x)} \mathbb{P}_x^\dagger \{ X_t \in dy, t < T \} \]
where $\mathbb{P}^x_+$ denotes the law of the Lévy process started at $x$ and conditioned to stay positive, and $W(x) = W^{(0)}(x)$. Explicitly, $h = \dot{h}/W$ is given by

$$h(x) = \frac{\gamma W^{(\gamma)}(x)}{\Phi(\gamma)W(x)} - \frac{\gamma}{W(x)} \int_0^x W^{(\gamma)}(z) \, dz.$$  

As a result, the entrance law of the post-$H_T$ process is $h$-transform of the law of the Lévy process conditioned to stay positive and killed at an exponential time [2, Prop.VII.14].

ii) [2, Lem.VI.6] gives the independence by noting that $X_{H_S-} = S_T$ if 0 is a regular point for $S - X$, equivalently, 0 is regular for $(-\infty, 0)$, which in turn is equivalent to $X$ being of unbounded variation since $X$ is spectrally negative, and $X_{H_S} = S_T$ if 0 is irregular for $S - X$. Using similar arguments, also as in [11, Thm.3.2], and the excursions of $X$ from its maximum, we get

$$\mathbb{E} [F_1(X_u : u \leq H_S) F_2(S_T - X_{H_S+u} : u \leq T - H_S)] = \mathbb{E} \left[ \int_0^{\gamma b} db F_1(X_u : u \leq \tau_b) e^{-\gamma \tau_b} \right] \mathbb{E} \left[ \int_\varepsilon n(d\varepsilon) F_2(\varepsilon(u) : u \leq T) 1_{\{T<\zeta\}}(\varepsilon) \right]$$

for bounded measurable functionals $F_1$ and $F_2$. Note that we have used the fact that $T$ is an exponential random variable with parameter $\gamma > 0$.

For the law of the pre-$H_S$ process, we observe that

$$\mathbb{E}[F_1(X_u : u \leq \tau_b) e^{-\gamma \tau_b}]$$

$$= \int \mathbb{P}(d\omega) F_1(X_u(\omega) : u \leq \tau_b(\omega)) e^{-\gamma \tau_b(\omega)}$$

$$= \int \mathbb{P}^{\Phi(\gamma)}(d\omega) e^{-\Phi(\gamma)X_{\tau_b}(\omega)+\psi(\Phi(\gamma))\tau_b(\omega)} F_1(X_u(\omega) : u \leq \tau_b(\omega)) e^{-\gamma \tau_b(\omega)}$$

$$= \mathbb{E}^{\Phi(\gamma)}[F_1(X_u : u \leq \tau_b)] e^{-\Phi(\gamma)b}$$

by change of measure with $c = \Phi(\gamma)$ as given in [7, pg.232], where $\mathbb{P}^c$ is the law of another spectrally negative Lévy process with Laplace exponent $\tilde{\psi}(\lambda) = \psi(\lambda + c) - \psi(c)$, and we put $X_{\tau_b} = b$ since the spectrally negative Lévy process $X$ creeps upwards almost surely. From (1), we get

$$\mathbb{E} [F_1(X_u : u \leq H_S) F_2(S_T - X_{H_S+u} : u \leq T - H_S)]$$

$$= \int_0^{\gamma b} db \mathbb{E}[F_1(X_u : u \leq \tau_b)] e^{-\Phi(\gamma)b} \int_\varepsilon n(d\varepsilon) \mathbb{E}[F_2(\varepsilon(u) : u \leq T) 1_{\{T<\zeta\}}(\varepsilon)]$$

$$= \frac{1}{\Phi(\gamma)} \int_0^{\gamma b} db \mathbb{E}[F_1(X_u : u \leq \tau_b)] f_{S_T}(b) \int_\varepsilon n(d\varepsilon) \mathbb{E}[F_2(\varepsilon(u) : u \leq T) 1_{\{T<\zeta\}}(\varepsilon)]$$
where we wrote $f_{S_T}$ for the probability density function of $S_T$, which has exponential distribution with parameter $\Phi(\gamma)$ [2, Cor. VII.2]. The assertion about the conditional distribution of the pre-$H_S$ process given $S_T = b$ follows from (5).

The law of the post-$H_S$ process is characterized in view of the dual process $\hat{X} := -X$, which is spectrally positive. Similar to the analysis for post-$H_I$ process, we get

$$\mathbb{P}_{b-x}\{\hat{X}_t \in b - dy, t < T \mid T < \hat{\tau}_0^-\}$$

by Markov property and the fact that $T$ is memoryless. This is the $h$-transform of the law of a spectrally positive Lévy process killed at the minimum of an exponential time and passage time below 0, with

$$h(x) = \mathbb{P}_x\{T < \hat{\tau}_0^-\} = \mathbb{P}_x\{T < \tau_0^+\} = \mathbb{P}\{T < \tau_x^+\}$$

$$= 1 - \mathbb{E}[e^{-\gamma\tau_x^+}I_{\{\tau_x^+ < \infty\}}]$$

$$= 1 - e^{-\Phi(\gamma)x}$$

for $x > 0$ [7, pg.233].

As a special case, the auxiliary function $\tilde{h}(x)$ as well as $h(x)$ corresponding to post-$H_S$ in the proof above are equal to $1 - e^{-x\sqrt{2\gamma}}$ for standard Brownian motion due to symmetry and we recover the result in [11, Thm.3.2.1]. Moreover, we obtain the following corollary, as a generalization of [11, Eq.3.23].

**Corollary 1.** The distribution of the supremum of the post-infimum process, $S_{H_I,T}$, when $I_T = a$ is given by

$$\mathbb{P}(S_{H_I,T} \leq b\mid I_T = a) = \frac{\Phi(\gamma)(Z^{(\gamma)}(b - a) - 1)}{\gamma W^{(\gamma)}(b - a)}$$

for $b > a$.

**Proof:** Note that the transition density of the post-infimum process can be written as

$$P_t(x, dy) = \frac{\tilde{h}(b - a)}{\tilde{h}(x - a)} \mathbb{P}_{x-a}\{X_t \in dy - a, t < T \wedge \tau_0^-\}$$
for $x, y > a$ where $\tilde{h}$ is given in (3). We have

$$
\mathbb{P}(S_{H_{t}, T} > b \mid I_T = a) = \lim_{x \to a} \frac{\tilde{h}(b - a)}{\tilde{h}(x - a)} \mathbb{P}_{x-a} \{ \tau_{\tilde{h}(b-a)} < \tau_0^- \}, \quad \tau_{b-a}^+ < T \}
$$

which is simplified by (1) and we have used the fact that $W(\gamma)(0) = 0$ because $X$ is assumed to be of unbounded variation.

4. Post-infimum process up to a given level

In this section, we decompose the process according to its infimum until the first passage time above a given level which occurs before an independent, exponentially distributed time $T$. The last exit time from the infimum is not a stopping time under the natural filtration generated by the spectrally negative Lévy process, denoted by $\mathcal{F}$ below. On the other hand, after applying the theory of expansion of filtrations described in [10, Sec.VI.3], the last time the process exits from its infimum becomes a stopping time. As in [13, Sec.3], we expand the filtration and characterize the distribution of the post-infimum process up to a given level before the exponential time.

For $b > 0$, we consider the trace of the original probability space on the event $D = \{ \tau_{b}^+ \leq T \}$ [4, Exer.I.1.15, I.3.12]. On this probability space, define the random variable

$$
\rho = \sup \{ t : 0 \leq t \leq \tau_{b}^+ \leq T, X_t = I_{\tau_{b}^+} \}
$$

as the last exit time from the infimum before the first passage time above $b$, which occurs before an exponential time $T$. Here, $\rho$ is the end of the random set

$$
\Lambda = \{ (t, \omega) \subset \mathbb{R}_+ \times \Omega : 0 \leq t \leq \tau_{b}^+(\omega) \leq T(\omega), X_t(\omega) = I_{\tau_{b}^+(\omega)} \},
$$
that is, \( \rho = \sup\{t : (t, \omega) \in \Lambda\} \), and we can apply the progressive expansions described in [10, Sec.VI.3]. The trace of the original filtration \( F \) is given by 
\( \mathcal{F}_t \cap \{\tau_b^+ \leq T\} := \{A \cap D : A \in \mathcal{F}_t\} \). Let \( \mathcal{F}^\rho \) denote the smallest expanded filtration making \( \rho \) a stopping time of the trace of \( F \), satisfying the usual hypotheses. Let 
\[
Y_t = \mathbb{P}\{\rho \leq t | \mathcal{F}_t \cap \{\tau_b^+ \leq T\}\} \quad t \geq 0.
\tag{7}
\]

**Lemma 1.** The process \( Y \) is a submartingale with respect to \( \mathcal{F}_t \cap \{\tau_b^+ \leq T\} \) given by 
\[
Y_t = \frac{W^{(\gamma)}(X_{t \wedge \tau_b^+} - I_{t \wedge \tau_b^+})}{W^{(\gamma)}(b - I_{t \wedge \tau_b^+})}
\]
for \( t \geq 0 \) with Doob-Meyer decomposition \( Y = M + A \) where 
\[
M_t = \int_0^{t \wedge \tau_b^+} \frac{W^{(\gamma)}(X_s - I_s)}{W^{(\gamma)}(b - I_s)} \sigma dB_s \\
+ \int_0^{t \wedge \tau_b^+} \int_{-\infty}^0 \left[ \frac{W^{(\gamma)}(X_s + y - I_s) - W^{(\gamma)}(X_s - I_s)}{W^{(\gamma)}(b - I_s)} \right] \tilde{N}(dy, ds)
\]
is a martingale and 
\[
A_t = \int_0^{t \wedge \tau_b^+} \left[ \frac{W^{(\gamma)}(X_s - I_s)}{W^{(\gamma)}(b - I_s)} \mu + \frac{\sigma^2}{2} \frac{W^{(\gamma)}(X_s - I_s)}{W^{(\gamma)}(b - I_s)} \right] ds \\
+ \int_0^{t \wedge \tau_b^+} \int_{-\infty}^0 \left[ \frac{W^{(\gamma)}(X_s + y - I_s) - W^{(\gamma)}(X_s - I_s) - W^{(\gamma)}(X_s - I_s) y 1_{y > 1}}{W^{(\gamma)}(b - I_s)} \right] \Pi(dy) ds \\
+ \int_0^{t \wedge \tau_b^+} \frac{W^{(\gamma)}(X_s - I_s) W^{(\gamma)}(b - I_s) - W^{(\gamma)}(X_s - I_s) W^{(\gamma)}(b - I_s)}{[W^{(\gamma)}(b - I_s)]^2} d\mathcal{I}^c_s \\
+ \sum_{s \leq t} \left[ \frac{W^{(\gamma)}(X_s - I_s)}{W^{(\gamma)}(b - I_s)} - \frac{W^{(\gamma)}(X_s - I_s)}{W^{(\gamma)}(b - I_s)} \right]
\]
is of bounded variation.

**Proof:** Let \( t \geq 0 \) be fixed. When \( \rho \leq t \), after time \( t \), the process passes above the level \( b \) before it passes below the level \( I_{\tau_b^+} \), which is equal to \( I_t \) in
this case. Therefore, we get
\[
\mathbb{P}\{\rho \leq t|\mathcal{F}_t \cap \{\tau^+_b \leq T\}\} = \mathbb{P}_{X_t}\{\tau^+_b < \tau^+_I, \ \tau^+_b \leq T\} 1_{\{\tau^+_b \}}
\]
\[
= \mathbb{E}_{X_t}[e^{-\gamma \tau^+_b} 1_{\{\tau^+_b < \tau^+_I\}}] 1_{\{\tau^+_b \}}
\]
\[
= \mathbb{E}_{X_t-I_t}[e^{-\gamma \tau^+_b-I_t} 1_{\{\tau^+_b-I_t < \tau^+_I\}}] 1_{\{\tau^+_b \}}
\]
\[
= \frac{W^{(\gamma)}(X_t - I_t)}{W^{(\gamma)}(b - I_t)} 1_{\{\tau^+_b \}}
\]
where $W^{(\gamma)}$ is the $\gamma$-scale function and the last equality follows from \cite{[7], Thm.8.1}. Therefore, we get
\[
Y_t = \frac{W^{(\gamma)}(X_t - I_t)}{W^{(\gamma)}(b - I_t)} \quad t \leq \tau^+_b .
\]
Note that for $t \geq \tau^+_b$ $Y_t = 1$ by definitions \cite{(4), (7)}, and the expression for $Y_t$ follows. Since $Y_t$ is increasing in $t$, it is a $(\mathcal{F}_t \cap \{\tau^+_b \leq T\})$-submartingale.

By Ito’s formula \cite{[5], Thm.I.4.57}, we have
\[
dY_t = \frac{W^{(\gamma)}(X_t - I_t)}{W^{(\gamma)}(b - I_t)} (\mu dt + \sigma dB_t) + \frac{1}{2} \frac{W^{(\gamma)''}(X_t - I_t)}{W^{(\gamma)}(b - I_t)} \sigma^2 dt
\]
\[
+ \int_{-\infty}^0 \left[ \frac{W^{(\gamma)}(X_t + y - I_t) - W^{(\gamma)}(X_t - I_t)}{W^{(\gamma)}(b - I_t)} \] \Pi(dy) dt
\]
\[
+ \int_{-\infty}^0 \left[ \frac{W^{(\gamma)}(X_t - y - I_t) - W^{(\gamma)}(X_t - I_t)}{W^{(\gamma)}(b - I_t)} \right] \tilde{N}(dy, dt)
\]
\[
+ \frac{W^{(\gamma)}(X_t - I_t)W^{(\gamma)''}(b - I_t) - W^{(\gamma)''}(X_t - I_t)W^{(\gamma)}(b - I_t) - W^{(\gamma)''}(X_t - I_t)W^{(\gamma)}(b - I_t)}{[W^{(\gamma)}(b - I_t)]^2} dI^c
\]
\[
+ \frac{W^{(\gamma)}(X_t - I_t) - W^{(\gamma)}(X_t - I_{-})}{W^{(\gamma)}(b - I_t)} - \frac{W^{(\gamma)}(X_t - I_{-})}{W^{(\gamma)}(b - I_{-})}
\]
where $I^c$ denotes the continuous part of the decreasing process $I$.

Now, $M_t$ is a martingale because both summands in its expression are martingales. First, the integrand of the Brownian integral is continuous and bounded as the process $X_t$ is bounded when $t \in [0, \tau^+_b]$. Then, the numerator of the integrand in the Poisson integral can be bounded by a multiple of “$y$” by mean value theorem as $W^{(\gamma)'}(X_t - I_t)$ is continuous and bounded over $t \in [0, \tau^+_b]$. As a result, the second integral is a stochastic integral with
respect to the compensated Poisson random measure under the conditions that the spectrally negative Lévy process \( X \) exists. Its construction is similar to that of \( X \), see e.g. [7, Thm.2.10]. Therefore, the second term in \( A_t \) is well-defined. Clearly, the other terms are well-defined as well. \( \square \)

We can now characterize the post-\( \rho \) process \( X_{t+\rho} - X_\rho \) in the following proposition.

**Proposition 2.** The process \( X_{t+\rho} - X_\rho \) satisfies the stochastic integral equation

\[
Z_t = \left( \mu + \int_{-\infty}^{-1} y \Pi(dy) \right) t + \sigma Q_t + L_t \\
+ \sigma \int_0^t \frac{W^{(\gamma)}(Z_s)}{W^{(\gamma)}(Z_s)} \, ds + \int_0^t \int_{-\infty}^{0} \frac{W^{(\gamma)}(Z_s + y) - W^{(\gamma)}(Z_s)}{W^{(\gamma)}(Z_s)} \Pi(dy) \, ds
\]

as a weak solution, which is unique in distribution.

**Proof:** By [10, Thm.VI.18], if \( Z \) is a martingale for the original filtration then \( Z - \Upsilon \) is a martingale for the enlarged filtration where

\[
\Upsilon_t := -\int_0^{t_{\wedge}^\rho} \frac{1}{1 - Y_s} d\langle Z, M \rangle_s + 1_{\{t \geq \rho\}} \int_{\rho}^t \frac{1}{Y_s} d\langle Z, M \rangle_s
\]

and \( Y \) and \( M \) are as in Lemma [11]. For \( t < \tau_b^+ \), let us define

\[
\Lambda_t = -\int_0^{t_{\wedge}^\rho} \frac{1}{1 - Y_s} d\langle B, M \rangle_s + 1_{\{t \geq \rho\}} \int_{\rho}^t \frac{1}{Y_s} d\langle B, M \rangle_s
\]

\[
= -\int_0^{t_{\wedge}^\rho} \frac{W^{(\gamma)}(X_s - I_s)}{W^{(\gamma)}(b - I_s) - W^{(\gamma)}(X_s - I_s)} \sigma \, ds + 1_{\{t \geq \rho\}} \int_{\rho}^t \frac{W^{(\gamma)}(X_s - I_s)}{W^{(\gamma)}(X_s - I_s)} \sigma \, ds
\]

for Brownian motion \( B \). Also, consider the martingale

\[
K_t := \int_0^t \int_{-\infty}^{0} y \tilde{N}(dy, ds)
\]

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for the original filtration and let

\[
\Gamma_t = - \int_0^{t \wedge \rho} \frac{1}{1 - Y_s} d\langle K, M \rangle_s + 1_{\{t \geq \rho \}} \int_\rho^t \frac{1}{Y_s} d\langle K, M \rangle_s
\]

\[
= - \int_0^{t \wedge \rho} \frac{W'(\gamma)(b - I_s)}{W'(\gamma)(b - I_s) - W'(\gamma)(X_s - I_s)} d\langle K, M \rangle_s
\]

\[
+ 1_{\{t \geq \rho \}} \int_\rho^t \frac{W'(\gamma)(b - I_s)}{W'(\gamma)(X_s - I_s)} d\langle K, M \rangle_s
\]

\[
= - \int_0^{t \wedge \rho} \int_0^t \int_{-\infty}^0 \frac{W'(\gamma)(X_s + y - I_s) - W'(\gamma)(X_s - I_s)}{W'(\gamma)(b - I_s) - W'(\gamma)(X_s - I_s)} \Pi(dy) ds
\]

\[
+ 1_{\{t \geq \rho \}} \int_\rho^t \int_0^t \int_{-\infty}^0 \frac{W'(\gamma)(X_s + y - I_s) - W'(\gamma)(X_s - I_s)}{W'(\gamma)(X_s - I_s)} \Pi(dy) ds
\]

for \( t < \tau_b^+ \). It follows that \( B - \Lambda \) and \( K - \Gamma \) are martingales for the enlarged filtration \( \mathcal{F}_\rho \).

Now, consider the post-\( \rho \) process \( X_{\rho + t} - X_\rho \) given by

\[
X_{\rho + t} - X_\rho = \mu t + \sigma(B_{\rho + t} - B_\rho) + \int_\rho^{\rho + t} \int_{-\infty}^0 y \tilde{N}(dy, ds) + t \int_{-\infty}^{-1} y \Pi(dy)
\]

(8)

Also, for \( \rho + t < \tau_b^+ \)

\[
\Lambda_{\rho + t} - \Lambda_\rho = \int_\rho^{\rho + t} \frac{W'(\gamma)(X_s - I_s)}{W'(\gamma)(X_s - I_s)} \sigma ds
\]

and

\[
\Gamma_{\rho + t} - \Gamma_\rho = \int_\rho^{\rho + t} \int_{-\infty}^0 \frac{W'(\gamma)(X_s + y - I_s) - W'(\gamma)(X_s - I_s)}{W'(\gamma)(X_s - I_s)} \Pi(dy) ds
\]

Putting \( X_\rho = I_\rho \), and \( \bar{B} = B - \Lambda \) and \( \bar{K} = K - \Gamma \), we can write (8) as

\[
X_{\rho + t} - I_\rho = \left( \mu + \int_{-\infty}^{-1} y \Pi(dy) \right) t + \sigma(\bar{B}_{\rho + t} - \bar{B}_\rho) + \sigma(\Lambda_{\rho + t} - \Lambda_\rho) + \bar{K}_{\rho + t} - \bar{K}_\rho + \rho_{\rho + t} - \rho_\rho
\]

Letting

\[
Q_t := \bar{B}_{\rho + t} - \bar{B}_\rho \quad L_t := \bar{K}_{\rho + t} - \bar{K}_\rho
\]

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we see that $Q$ is a continuous martingale and $L$ is a martingale with jumps independent from $\mathcal{F}_\rho$. Therefore, the post-process $Z_t := X_{\rho+t} - X_{\rho} = X_{\rho+t} - I_\rho$, $0 < t < \tau^+_b - \rho$, satisfies

$$Z_t = \left( \mu + \int_{-\infty}^{-1} y \Pi(dy) \right) t + \sigma Q_t + L_t$$

$$+ \sigma \int_0^t \frac{W^{(\gamma)}(Z_s)}{W(\gamma)} \, ds + \int_0^t \int_{-\infty}^0 \frac{W(\gamma)(Z_s + y) - W(\gamma)(Z_s)}{W(\gamma)(Z_s)} \Pi(dy) \, ds$$

due to the fact that $X_s - I_s = X_s - I_\rho = Z_{s-\rho}$ for $\rho < s < \tau^+_b$. The martingale problem associated with the generator of this equation is well-posed in view of [12, Thm.4.3] for Lévy generators. The only point to verify is that the drift function is bounded. Clearly, $W^{(\gamma)}/W(\gamma)$ is continuous, hence bounded over the interval $[0, b - I_\rho]$ where $Z_s$ can take values. On the other hand, the function

$$\int_{-\infty}^0 \frac{W^{(\gamma)}(\cdot + y) - W^{(\gamma)}(\cdot)}{W(\gamma)(\cdot)} \Pi(dy)$$

is also continuous by bounded convergence theorem, and hence bounded over compact intervals. As a result, Equation (9) uniquely characterizes the distribution of the post-$\rho$ process. □

5. Path decomposition conditioned on extremes

In this section, we display the distribution of various parts of the path conditioned on the supremum and the infimum based on the analysis of the previous sections.

**Theorem 2.** Given $H_I < H_S$, $I_T = a$, $I_S = b$, it follows that

i. the intermediate process $\{X_{H_I+t} : 0 \leq t \leq H_S - H_I\}$ satisfies integral equation (8),

ii. the transition semigroup of the post-$H_S$ process is the $h$-transform of the law of a spectrally positive Lévy process killed at $T \wedge \hat{t}^+_{b-a} \wedge \hat{t}^-_0$, with

$$h(z) = 1 - Z^{(\gamma)}(-z+b-a) + Z^{(\gamma)}(b-a) \frac{W^{(\gamma)}(-z + b - a)}{W^{(\gamma)}(b - a)} - \frac{W^{(\gamma)}(-z + b - a)}{W^{(\gamma)}(b - a)},$$

that is,

$$P_t(x, dy) = \frac{h(b-y)}{h(b-x)} \mathbb{P}_{b-x} \{ \hat{X}_t \in b - dy, t < T \wedge \hat{t}^+_{b-a} \wedge \hat{t}^-_0 \}$$
for $a < x, y < b$, and the entrance law is obtained as $x \rightarrow b$.

**Proof:**

i) The result follows from Section [14].

ii) At an intermediate time $t < T$, the law of the post-$H_S$ process given $I_T = a, S_T = b$, $H_I < H_S$ is determined by transition semigroups $P_t(x, dy)$, $x < b$, $a < y < b$. This is equal to transition probability density at $b - y$ of a spectrally positive Lévy process $\hat{X}$ killed at an exponential time $T$ and conditioned to stay positive and not go above $b - a$. Similar to post-$H_I$ process, the probability law of this process is written as

$$
P_{b-x} \{ \hat{X}_t \in b - dy, t < T | T < \hat{\tau}_{b-a}^- \wedge \hat{\tau}_0^- \}
= \frac{\mathbb{P}_{b-x} \{ \hat{X}_t \in b - dy, t < T, t < \hat{\tau}_{b-a}^+ \wedge \hat{\tau}_0^- | T < \hat{\tau}_{b-a}^+ \wedge \hat{\tau}_0^- \}}{\mathbb{P}_{b-x} (T < \hat{\tau}_{b-a}^+ \wedge \hat{\tau}_0^-)}
= \frac{\mathbb{P}_{b-y} \{ T < \hat{\tau}_{b-a}^+ \wedge \hat{\tau}_0^-, \text{ and } \mathbb{P}_{b-x} \{ \hat{X}_t \in b - dy, t < T \wedge \hat{\tau}_{b-a}^+ \wedge \hat{\tau}_0^- \}}}{\mathbb{P}_{b-x} (T < \hat{\tau}_{b-a}^+ \wedge \hat{\tau}_0^-)}
$$

by Markov property and the fact that $T$ is memoryless, and $\hat{\tau}_{a}^- := \{ t \geq 0 : \hat{X}_t \leq a \}$. It follows that the law of the post-$H_S$ process is the $h$-transform of a spectrally positive Lévy process killed at time $T \wedge \hat{\tau}_{b-a}^+ \wedge \hat{\tau}_0^-$ with

$$
h(z) = \mathbb{P}_z \{ T < \hat{\tau}_{b-a}^+ \wedge \hat{\tau}_0^- \} = \mathbb{P}_{-z} \{ T < \tau_{a-b}^- \wedge \tau_0^+ \}
= \mathbb{P}_{-z} \{ T < \tau_{a-b}^- \wedge \tau_0^+ \} + \mathbb{P}_{-z} \{ T < \tau_0^+ \wedge \tau_{a-b}^- \}
= \mathbb{P}_{-z} \{ \tau_{a-b}^- < \tau_0^+ \} - \mathbb{E}_{-z} [e^{-\gamma \tau_{a-b}^-} 1_{\{ \tau_{a-b}^- < \tau_0^+ \}}] + \mathbb{P}_{-z} \{ \tau_0^+ < \tau_{a-b}^- \} - \mathbb{E}_{-z} [e^{-\gamma \tau_0^+} 1_{\{ \tau_0^+ < \tau_{a-b}^- \}}]
= 1 - Z^{(\gamma)}(-z + b - a)
+ Z^{(\gamma)}(b - a) \frac{W^{(\gamma)}(-z + b - a)}{W^{(\gamma)}(b - a)} - \frac{W^{(\gamma)}(-z + b - a)}{W^{(\gamma)}(b - a)}
$$

The $h$-function in [11] has been obtained using the expressions of the two sided exit formulas given in [6, pgs. 17, 24].

Recall that the maximum loss at time $t > 0$ is defined by

$$
M_t^- := \sup_{0 \leq u \leq v \leq t} (X_u - X_v),
$$
Corollary 2. The distribution of the maximum loss of the post-supremum process, $M_{H_S,T}$, when $H_I < H_S$, $I_T = a$, $S_T = b$ is given by

$$
\mathbb{P}(M_{H_S,T} < d | H_I < H_S, I_T = a, S_T = b) = 1 - \left[ 1 - Z^{(\gamma)}(b - a - d) + (Z^{(\gamma)}(b - a) - 1) \frac{W^{(\gamma)}(b - a - d)}{W^{(\gamma)}(b - a)} \right] 
$$

$$
= 1 - \frac{Z^{(\gamma)}(d) - Z^{(\gamma)}(d) \frac{W^{(\gamma)}(d)}{W^{(\gamma)}(d)}}{-Z^{(\gamma)}(b - a) + (Z^{(\gamma)}(b - a) - 1) \frac{W^{(\gamma)}(b - a)}{W^{(\gamma)}(b - a)}}
$$

for $0 < d < b - a$.

Proof: We have

$$
\mathbb{P}(M_{H_S,T} < d | H_I < H_S, I_T = a, S_T = b)
$$

$$
= 1 - \lim_{x \to b} \frac{h(d)}{h(b - x)} \mathbb{P}_{b - x} \left\{ \hat{\tau}_d^+ < T, \hat{\tau}_d^+ < \hat{T}_0 \right\}
$$

$$
= 1 - \lim_{x \to b} \frac{h(d)}{h(b - x)} \mathbb{P}_{b - x} \left\{ \tau_d^- < T, \tau_d^- < \tau_0^- \right\}
$$

$$
= 1 - h(d) \lim_{x \to b} \frac{Z^{(\gamma)}(x - b + d) - Z^{(\gamma)}(d) \frac{W^{(\gamma)}(x - b + d)}{W^{(\gamma)}(d)}}{1 - Z^{(\gamma)}(x - a) + Z^{(\gamma)}(b - a) \frac{W^{(\gamma)}(x - a)}{W^{(\gamma)}(b - a)}} - \frac{W^{(\gamma)}(x - a)}{W^{(\gamma)}(b - a)}
$$

$$
= 1 - \frac{Z^{(\gamma)}(d) - Z^{(\gamma)}(d) \frac{W^{(\gamma)}(d)}{W^{(\gamma)}(d)}}{-Z^{(\gamma)}(b - a) + (Z^{(\gamma)}(b - a) - 1) \frac{W^{(\gamma)}(b - a)}{W^{(\gamma)}(b - a)}}
$$

which yields the result. \(\square\)

As a result, we have characterized the distributions of the intermediate process and the post-supremum process when the infimum and the supremum before an exponential time are known and the infimum occurs before the supremum. Our motivation was to find the joint distribution of the maximum loss and the maximum gain as an application of path decomposition. However, we have a partial answer for the conditional distribution of the maximum loss since the law of the pre-infimum process is still to be found.

Acknowledgement. We are grateful to Andreas E. Kyprianou for his helpful discussions on the post-infimum process.
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