A NOTE ON THE BMO AND CALDERÓN-ZYGMUND ESTIMATE

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Abstract. In this note, we give a simple proof of the pointwise BMO estimate for Poisson’s equation. Then the Calderón-Zygmund estimate follows by the interpolation and duality.

1. Introduction

Consider the following Poisson’s equation

\(\Delta u = f\) in \(B_1\),

where \(B_1 \subset \mathbb{R}^n\) is the unit ball. In 1952, Calderón and Zygmund ([3], see also [7, Chapter 9]) proved the classical \(W^{2,p}\) estimate \((1 < p < \infty)\) for (1.1) by the method of singular integral. That is, for any strong solution \(u \in W^{2,p}(B_1)\) of (1.1),

\[\|u\|_{W^{2,p}(B_1/2)} \leq C(\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)})\],

where \(C\) depends only on \(n\) and \(p\). In 2003, Wang [14] gave a direct and elementary proof of the \(W^{2,p}\) estimate without using the theory of singular integral. Wu, Yin and Wang [16, Chapter 9] presented another elementary proof based on energy estimates.

It is also well known that the \(W^{2,p}\) estimate fails for \(p = 1\) and \(p = \infty\) (see [6, Chapter 7.1.3]). It was found that the Hardy space \(H^1\) and the BMO (bounded mean oscillation) space are appropriate substitutes for \(L^1\) and \(L^\infty\) respectively. The estimates in the \(H^1\) and the BMO space are proved by means of singular integral as well (see [13, Theorem 4, Chapter 3] and [11]).

In this note, we give a simple proof of the \(W^{2,BMO}\) estimate for (1.1) (see Theorem 2.3 below). By combining with the interpolation and duality argument, we have the \(W^{2,p}\) estimate for any \(1 < p < \infty\).

We introduce some notations. The \(B_r(x) \subset \mathbb{R}^n\) denotes the ball with radius \(r\) and center \(x\), and \(B_r := B_r(0)\). Let

\[f_\Omega := \frac{1}{|\Omega|} \int_{\Omega} f\] and \[\|f\|_{L^p(\Omega)} := \left(\frac{1}{|\Omega|} \int_{\Omega} |f|^p\right)^{1/p}, \forall 1 \leq p < \infty\],

where \(\Omega \subset \mathbb{R}^n\) is a bounded domain and \(|\Omega|\) denotes its Lebesgue measure.

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For \( x_0 \in \Omega \) and \( r_0 > 0 \), if
\[
|f|_{*,x_0} := \sup_{0 < r < r_0} \|f - f_{B_r(x_0) \cap \Omega}\|_{L^2(B_r(x_0) \cap \Omega)} < +\infty,
\]
we say that \( f \) is BMO at \( x_0 \) or \( f \in BMO(x_0) \) (with radius \( r_0 \)). If \( f \in BMO(x_0) \) for any \( x_0 \in \Omega \) with the same radius \( r_0 \) and
\[
|f|_{*,\Omega} := \sup_{x \in \Omega} |f|_{*,x} < +\infty,
\]
we say that \( f \) is a BMO function or \( f \in BMO(\Omega) \) (with radius \( r_0 \)). Moreover, we endow \( BMO(\Omega) \) with the following norm:
\[
\|f\|_{BMO(\Omega)} := \|f\|_{L^2(\Omega)} + |f|_{*,\Omega}.
\]
If \( f \) has weak derivatives (denoted by \( Df \)) and \( Df \in BMO(x_0) \) (\( BMO(\Omega) \)), we say that \( u \in W^{1,BMO}(x_0) \) (\( W^{1,BMO}(\Omega) \)). Similarly, we can define \( W^{2,BMO} \) etc.

Note that the usual BMO space is defined based on \( L^1 \) norm rather than \( L^2 \) norm. In fact, these definitions are equivalent (see [13, Corollary on P. 144] and [6, Corollary 6.22]).

Unless stated otherwise, \( C \) always denotes a constant depending only on the dimension \( n \) throughout this note.

2. MAIN RESULT

Before starting to prove the main result, we describe the idea briefly. We adopt perturbation argument, i.e., we use harmonic functions to approximate the solution of (1.1). If \( f \) is small enough, then \( u \) can be approximated by a polynomial of degree 2 at some scale (called key step, see Lemma 2.1). This is proved by the method of compactness. Next, by the standard scaling argument, we have a sequence of polynomials approximating \( u \) at different scales (see Lemma 2.2). Then this essentially implies that \( u \in BMO(0) \) (see Theorem 2.3). This technique has been used widely to prove the \( C^{k,\alpha} \) regularity since the seminal work of Caffarelli (see [1, 2, 8, 10, 12, 15] etc.). It turns out that the BMO regularity can be regarded as a kind of pointwise regularity as the \( C^{k,\alpha} \) regularity.

First, we prove the key step.

**Lemma 2.1.** Let \( u \in W^{1,2}_{loc}(B_1) \) be a weak solution of
\[
\Delta u = f \text{ in } B_1.
\]
Suppose that \( \|u\|_{L^2(B_1)} \leq 1 \) and \( \|f\|_{L^2(B_1)} \leq \delta \), where \( 0 < \delta < 1 \) depends only on \( n \).

Then there exists a polynomial \( P \) of degree 2 such that
\[
\|u - P\|_{L^2(B_n)} \leq \eta^2,
\]
\[
\Delta P = f_{B_n},
\]
\[
|P(0)| + |DP(0)| + |D^2P(0)| \leq \bar{C},
\]
where \( 0 < \eta < 1 \) and \( \bar{C} \) depend only on \( n \).

**Proof.** We prove the lemma by contradiction. Suppose that the lemma is false. Then there exist sequences of \( u_m \) and \( f_m \) such that
\[
\Delta u_m = f_m \text{ in } B_1
\]
with $\|u_m\|_{L^2(B_1)}^* \leq 1$ and $\|f_m\|_{L^2(B_1)}^* \leq 1/m$. But for any polynomial $P$ of degree 2 with $\Delta P = f_{m,B_q}$ and $|P(0)| + |D_P(0)| + |D^2P(0)| \leq \bar{C}$, we have

$$\tag{2.3} \|u_m - P\|_{L^2(B_q)}^* > \eta^2,$$

where $0 < \eta < 1$ and $\bar{C}$ are to be specified later.

Then there exist $u \in W^{1,2}_{loc}(B_1)$ and subsequences of $u_m$ (denoted by $u_m$ again) such that $u_m \to u$ in $W^{1,2}_{loc}(B_1)$ weakly and in $L^2_{loc}(B_1)$ strongly. Moreover,

$$\Delta u = 0 \text{ in } B_1.$$

Since $u$ is a harmonic function, there exists a polynomial $\bar{P}$ of degree 2 such that

$$\|u - \bar{P}\|_{L^2(B_r)}^* \leq Cr^3, \quad \forall \ 0 < r < 1/2,$$

$$\Delta \bar{P} = 0,$$

$$|\bar{P}(0)| + |D\bar{P}(0)| + |D^2\bar{P}(0)| \leq C.$$

Take $\bar{C} = C + 1$ and $\eta$ small enough such that

$$\eta \bar{C} \leq 1/2.$$

Thus,

$$\tag{2.4} \|u - \bar{P}\|_{L^2(B_r)}^* \leq \frac{1}{2} \eta^2.$$

Set

$$P_m(x) = \bar{P}(x) + \frac{f_{m,B_q}}{2m}|x|^2.$$

Then $\Delta P_m = f_{m,B_q}$ and $|P_m(0)| + |D_P(0)| + |D^2P(0)| \leq \bar{C}$ (for $m$ large enough). By (2.3),

$$\|u_m - P_m\|_{L^2(B_q)}^* > \eta^2.$$

Let $m \to \infty$ (noting $f_{m,B_q} \to 0$) and we have

$$\|u - \bar{P}\|_{L^2(B_q)}^* \geq \eta^2,$$

which contradicts with (2.4).

Let $0 < \delta < 1$ be as in Lemma 2.1 and $u \in W^{1,2}_{loc}(B_1)$ be a weak solution of

$$\Delta u = f \text{ in } B_1.$$

Suppose that $\|u\|_{L^2(B_1)}^* \leq 1$, $\|f\|_{L^2(B_1)}^* \leq \delta$ and $|f|_{*,0} \leq \delta$ (i.e., $\|f - f_{B_r}\|_{L^2(B_r)}^* \leq \delta$ for any $0 < r < 1$).

Then there exist a sequence of polynomials $P_m$ of degree 2 such that for any $m \geq 1$,

$$\tag{2.5} \|u - P_m\|_{L^2(B_{2m})}^* \leq \eta^{2m},$$

$$\Delta P_m = f_{B_{2m}},$$

$$|(P_m - P_{m-1})(0)| + \eta^{m-1}|D(P_m - P_{m-1})(0)| + \eta^{2(m-1)}|D^2(P_m - P_{m-1})(0)| \leq \bar{C}\eta^{2(m-1)},$$

where $0 < \eta < 1$ and $\bar{C}$ are as in Lemma 2.1.
Proof. We prove the lemma by induction. For $m = 1$, by setting $P_0 \equiv 0$ and Lemma 2.1, the conclusion holds. Suppose that the conclusion holds for $m$ and we need to prove that it holds for $m + 1$.

Let $r = \eta^m$, $y = x/r$ and

$$v(y) = \frac{u(x) - P_m(x)}{r^2}.$$ 

Then

$$\Delta v = \tilde{f} \text{ in } B_1,$$

where $\tilde{f}(y) = f(x) - \Delta P_m = f(x) - f_{B_r}$. Clearly, $\|v\|_{L^2(B_{1})} \leq 1$. In addition,

$$\|\tilde{f}\|_{L^2(B_{1})} = \|f - f_{B_r}\|_{L^2(B_{1})} \leq \delta.$$ 

By Lemma 2.1, there exists a polynomial $P$ of degree 2 such that

$$\|v - P\|_{L^2(B_{r})} \leq \eta^2,$$

$$\Delta P = \tilde{f}_{B_r},$$

$$|P(0)| + |DP(0)| + |D^2P(0)| \leq C.$$ 

By rescaling back to $u$ with $P_{m+1}(x) = P_m(x) + \eta^2kP(y)$, (2.5) holds for $m + 1$. By induction, the proof is complete. \hfill \Box

Now, we show that Lemma 2.2 implies the $W^{2, BMO}$ regularity of the solution.

**Theorem 2.3.** Let $u \in W^{1, 2}_{\text{loc}}(B_1)$ be a weak solution of

$$\Delta u = f \text{ in } B_1.$$

Suppose that $f \in BMO(0)$ with radius 1. Then $u \in W^{2, BMO}(0)$ with radius $\eta^2$ and

$$|D^2 u|_{*,0} \leq C \left(\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)} + |f|_{*,0}\right),$$

where $\eta$ is as in Lemma 2.1 and $C$ depends only on $n$.

If $f \in BMO(B_1)$, then $u \in W^{2, BMO}(B_{1/2})$ and

$$\|u\|_{W^{2, BMO}(B_{1/2})} \leq C \left(\|u\|_{L^2(B_1)} + \|f\|_{BMO(B_1)}\right).$$

Proof. Without loss of generality, we may assume that $\|u\|_{L^2(B_1)} \leq 1$, $\|f\|_{L^2(B_1)} \leq \delta$ and $|f|_{*,0} \leq \delta$, where $\delta$ is as in Lemma 2.1. Otherwise, we may consider

$$v = \frac{\delta u}{\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)} + |f|_{*,0}}.$$ 

For any $0 < r < \eta^2$, there exists $m \geq 2$ such that $\eta^{m+1} \leq r < \eta^m$. By Lemma 2.2, there exists a polynomial $P_{m-1}$ such that

$$\|u - P_{m-1}\|_{L^2(B_{\eta^m})} \leq \eta^{2(m-1)}.$$ 

Let $y = x/\eta^m$ and $v(y) = (u - P_{m-1})/\eta^{2(m-1)}$. Then

$$\Delta v = \tilde{f} \text{ in } B_1,$$

where $\tilde{f}(y) = f(x) - f_{B_{\eta^m}}$. By the standard $W^{2,2}$ regularity for $v,$

$$\|D^2u - D^2P_{m-1}\|_{L^2(B_{\eta^m})} \leq C \left(\|v\|_{L^2(B_1)} + \|	ilde{f}\|_{L^2(B_1)}\right) \leq C.$$
Hence,\(^{(2.7)}\)
\[\|D^{2}u-(D^{2}u)_{B_{r}}\|_{L^{2}(B_{r})}^{2}\leq\|D^{2}u-D^{2}P_{m-1}\|_{L^{2}(B_{r})}^{2}\leq\frac{1}{\eta}\|D^{2}u-D^{2}P_{m-1}\|_{L^{2}(B_{m})}^{2}\leq C.\]
That is, \(u \in W^{2, BMO}(0)\).

If \(f \in BMO(B_{1})\) with radius \(r_{0}\), \(f \in BMO(x_{0})\) for any \(x_{0} \in B_{1/2}\) with the same radius \(r_{0}\). Then by similar arguments to the above, we have \(u \in BMO(x_{0})\) for any \(x_{0} \in B_{1/2}\) with the same radius \(\eta^{2}r_{0}/2\). Hence, \(u \in W^{2, BMO}(B_{1/2})\) and the estimate \((2.8)\) holds. \(\square\)

Remark 2.4. The techniques in Lemma 2.1 and Lemma 2.2 are adopted (with minor modifications) from [9] (see Section 11 there). The observation that \((2.5)\) implies the \(BMO\) regularity is from [5] (see Remark 6.3 there).

The \(W^{2,2}\) estimate for \((1.1)\) is easy to establish (see [7, Theorem 8.8, Theorem 9.9]). Since we have derived \(W^{2, BMO}\) estimate, by the interpolation between \(L^{2}\) and \(BMO\) (see [4, Appendix 4] and [6, Chapter 6.3.3]), the \(W^{2,p}\) estimate follows for any \(2 < p < \infty\). Furthermore, we can also obtain the \(W^{2,p}\) estimate for any \(1 < p < 2\) due to the duality (see [7, Theorem 9.9]). In conclusion, we have the following Calderón-Zygmund estimate:

**Corollary 2.5.** Let \(u \in W^{1,1}_{loc}(B_{1})\) be a weak solution of\n\[
\Delta u = f \text{ in } B_{1}.
\]
Suppose that \(f \in L^{p}(B_{1})\) with \(1 < p < \infty\). Then \(u \in W^{2,p}_{loc}(B_{1})\) and
\[
\|u\|_{W^{2,p}(B_{1/2})} \leq C \left(\|u\|_{L^{1}(B_{1})} + \|f\|_{L^{p}(B_{1})}\right),
\]
where \(C\) depends only on \(n\) and \(p\).

Based on the notion of the sharp maximal function \(f^{\sharp}(x) := |f|_{*,x}\), we have another proof of the \(W^{2,p}\) estimate without interpolation. Recall a property of the sharp maximal function due to Fefferman and Stein (see [6, Theorem 6.30]):

**Lemma 2.6.** Suppose that \(g \in L^{1}(B_{1})\) and \(g^{\sharp} \in L^{p}(B_{1})\) for some \(1 < p \leq \infty\). Then \(g \in L^{p}(B_{1})\) and
\[
\|g\|_{L^{p}(B_{1})} \leq C \left(\|g^{\sharp}\|_{L^{p}(B_{1})} + \|g\|_{L^{1}(B_{1})}\right),
\]
where \(C\) depends only on \(n\) and \(p\).

In fact, we infer from the proof of Theorem 2.3 (see \((2.7)\)) that
\[
(D^{2}u)^{\sharp}(x) \leq C \left(\|u\|_{L^{2}(B_{1})} + \|f\|_{L^{2}(B_{1})} + f^{\sharp}(x)\right), \quad \forall \ x \in B_{1/2}.
\]
Then for \(2 \leq p < \infty\),
\[
\|D^{2}u\|_{L^{p}(B_{1/2})} \leq C \left(\|(D^{2}u)^{\sharp}\|_{L^{p}(B_{1/2})} + \|D^{2}u\|_{L^{2}(B_{1/2})}\right)
\leq C \left(\|u\|_{L^{2}(B_{1})} + \|f\|_{L^{2}(B_{1})} + \|f^{\sharp}\|_{L^{p}(B_{1/2})}\right)
\leq C \left(\|u\|_{L^{2}(B_{1})} + \|f\|_{L^{p}(B_{1})}\right),
\]
where \(C\) depends only on \(n\) and \(p\).

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