Decomposition of Exact pfd Persistence Bimodules

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Abstract
We characterize the class of persistence modules indexed over $\mathbb{R}^2$ that are decomposable into summands whose supports have the shape of a block—i.e. a horizontal band, a vertical band, an upper-right quadrant, or a lower-left quadrant. Assuming the modules are pointwise finite dimensional (pfd), we show that they are decomposable into block summands if and only if they satisfy a certain local property called exactness. Our proof follows the same scheme as the proof of decomposition for pfd persistence modules indexed over $\mathbb{R}$, yet it departs from it at key stages due to the product order on $\mathbb{R}^2$ not being a total order, which leaves some important gaps open. These gaps are filled in using more direct arguments. Our work is motivated primarily by the stability theory for zigzags and interlevel-sets persistence modules, in which block-decomposable bimodules play a key part. Our results allow us to drop some of the conditions under which that theory holds, in particular the Morse-type conditions.

Keywords Topological data analysis · Two-parameter persistence · Persistence modules · Decomposition theorems · Functorial filtration method · Level-sets persistence

Mathematics Subject Classification 62-07 · 68U05 · 16G20

1 Introduction

Decomposition theorems are one of the pillars of topological persistence theory, as they provide sufficient conditions under which topological descriptors for data, called barcodes or diagrams, can be defined. We refer the reader to [18] for an introduction to persistence theory and to the role played by decompositions therein.
The 1-dimensional setting is pretty well-understood by now. In this setting, the objects of interest are the so-called 1-d persistence modules, which are functors from the real line $\mathbb{R}$ (equipped with its natural order) to the vector spaces over a fixed field. The category of such functors is abelian, and several theorems [1,5,13,14,21,22] identify conditions under which these functors decompose as direct sums of interval modules—functors that are constantly equal to the field over some interval and trivial elsewhere. Hence the notion of barcode of a persistence module, defined as the collection of such intervals appearing in its (essentially unique) decomposition.

By contrast, the higher-dimensional setting is much less understood. While decomposition theorems exist [6,19], the underlying quivers are wild-representation type, meaning that their sets of indecomposables are hard to classify [11]. Hence the difficulties to extend the concept of barcode to higher dimensions.

In this paper we consider the 2-dimensional setting and we characterize the subclass of the $\mathbb{R}^2$-indexed modules that decompose into block modules, i.e., functors that are constantly equal to the field on 2-d shapes called blocks (upper-right or lower-left quadrants, vertical or horizontal bands) and trivial elsewhere. We prove that these modules are precisely the ones that satisfy a certain local property called exactness (Theorem 2.1). Thus, we provide a necessary and sufficient local condition (exactness) by which a global property (decomposition into block modules) can be checked. We show some applications of this result, most notably to the study of zigzag persistence and the development of a stability theory for interlevel-sets persistence based on earlier work by Bjerkevik, Botnan and Lesnick [4,7].

Related work. To our knowledge, our result was historically the first decomposition theorem proposed for $\mathbb{R}^2$-indexed persistence modules (or a certain class thereof, not restricted to grid-indexed modules). Since its first appearance, a general Krull–Schmidt theorem for pointwise finite-dimensional modules indexed over small categories has been proven by Botnan and Crawley-Boevey [6] following an approach developed by Ringel for locally finite quivers [19], from which another proof of our result can be derived by studying the structure of the indecomposables.

Let us also mention a different and complementary approach to the problem, which consists in defining barcodes via rectangle measures in the plane. This definition does not require any decomposition, thus it allows to relax somewhat the hypotheses on the considered persistence module. The approach was initiated in [12] in the 1-d setting, then it was extended to exact 2-d modules in [8,9]. As it does not yield any decomposition theorem, it cannot be combined with the stability theory developed in [4,7], however it constitutes a serious alternative to our work.

The special case of interlevel-sets persistence yields exact 2-d modules with some specific properties. Most notably, the structure of such modules is fully determined by the interval decomposition of their restriction to some zigzag along the anti-diagonal—by the Mayer–Vietoris theorem, assuming all homology degrees are considered at once. This fact was leveraged in early work on the subject [3,10], which established a decomposition theorem for such modules in the discrete setting via the interval decomposition theorem by Botnan [5] for $\mathbb{Z}$-indexed zigzag modules.

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1 By discrete setting we mean that the indexing set sits inside the integer lattice $\mathbb{Z}^2$. In [3,10] the result is established in the case where $\mathbb{Z}^2$ is replaced with a finite grid, however the analysis extends to the entire lattice via the more recent interval decomposition theorem by Botnan [5] for $\mathbb{Z}$-indexed zigzag modules.
decomposition of zigzag modules. These early contributions were seminal in that they introduced the problem, proved a discrete analogue of our Theorem 2.1 in the special case of interlevel-sets persistence, and provided some of the key ideas that were exploited in later work such as [9].

2 Main Result

Throughout the exposition, a field of coefficients is fixed and denoted by \( k \). The set on which the vector spaces of our modules will be indexed is \( \mathbb{R}^2 \), equipped with the product order:

\[
\text{for all } s, t \in \mathbb{R}^2, \quad s \leq t \iff s_x \leq t_x \text{ and } s_y \leq t_y.
\]

A persistence module indexed over \( \mathbb{R}^2 \) (or persistence bimodule\(^2\) for short) is a functor \( M \) from the poset \((\mathbb{R}^2, \leq)\) to the category of vector spaces over \( k \). By default we will denote by \( M_t, t \in \mathbb{R}^2 \), its constituent vector spaces, and by \( \rho_s^t, s \leq t \in \mathbb{R}^2 \), its constituent linear maps. For clarity, \( \rho_s^t \) will be sometimes renamed \( v_s^t \) when \( s_x = t_x \) (‘\( v \’ \) for ‘vertical’), and \( h_s^t \) when \( s_y = t_y \) (‘\( h \’ \) for ‘horizontal’). For any \( s \leq t \in \mathbb{R}^2 \) we have the following commutative diagram where the spaces and maps are taken from \( M \):

\[
\begin{array}{ccc}
M_{(s_x,t_y)} & \xrightarrow{h^t_{(t_x,t_y)}} & M_t \\
\downarrow v_{(s_x,t_y)} & & \downarrow v_{(t_x,t_y)} \\
M_s & \xrightarrow{h_s^{(t_x,t_y)}} & M_{(t_x,t_y)}
\end{array}
\] (1)

\( M \) is called pointwise finite-dimensional (pfd) if \( M_t \) is finite-dimensional for every \( t \in \mathbb{R}^2 \). It is called exact if, for every \( s \leq t \in \mathbb{R}^2 \), the following sequence induced by diagram (1) is exact (i.e., \( \text{Im} \phi = \text{Ker} \psi \)):

\[
\begin{array}{ccc}
M_s & \xrightarrow{\phi} & M_{(t_x,t_y)} \oplus M_{(s_x,t_y)} \\
\downarrow \psi & & \downarrow \psi = v_s^{t_x,t_y} - h_s^{(t_x,t_y)} \\
M_t & & M_t
\end{array}
\]

There are several ways in which this condition can be interpreted, including:

- At a low level, \( \text{Im} \phi \subseteq \text{Ker} \psi \) means that diagram (1) commutes, while \( \text{Im} \phi \supseteq \text{Ker} \psi \) means that every element of \( M_t \) that has preimages in \( M_{(t_x,t_y)} \) and \( M_{(s_x,t_y)} \) has a preimage common to both in \( M_s \).

\(^2\) We abuse terminology here, bimodules being a different concept in abstract algebra.
At a higher level, exactness of the sequence means that diagram (1) is a weak form of pushout or pullback square, in which surjectivity of $\psi$ or injectivity of $\phi$ are not required.

The exactness condition also implies (and is stronger than) the following equalities, which will be instrumental in our analysis:

$$\begin{align*}
\text{Ker } \rho'^t &= \text{Ker } h'_s(t_x, s_y) + \text{Ker } v'_s(s_x, t_y), \\
\text{Im } \rho'^t &= \text{Im } v'_s(t_x, s_y) \cap \text{Im } h'_s(s_x, t_y).
\end{align*}$$

$$\text{(2)}$$

In this paper we are interested in exact pfd bimodules. In some places our analysis extends to modules satisfying (2), for which the natural shapes to consider are rectangles.

### Rectangles, Blocks, and Their Associated Modules

We use cuts to parametrize the rectangles in the plane. A cut is a partition $c$ of $\mathbb{R}$ into two (possibly empty) sets $c^-, c^+$ such that $x < y$ for all $x \in c^-$ and $y \in c^+$. For instance, $c = (c^-, c^+)$ with $c^- = (-\infty, 1]$ and $c^+ = (1, +\infty)$ is a cut. A cut $c$ with either $c^+ = \emptyset$ or $c^- = \emptyset$ is said to be at infinity or trivial.

A non-empty rectangle $R$ in the plane is then uniquely defined by four cuts: two horizontal (say $c_-$ and $c_+$, standing respectively for left cut and right cut), and two vertical (say $c_-$ and $c_+$, standing respectively for bottom cut and top cut), so that $R = (c_+ \cap c_-) \times (c_- \cap c_+)$. Note that $R$ may not necessarily be open or closed, in fact the nature of each cut determines which boundaries belong to the rectangle.

To any rectangle $R$ we associate a unique rectangle module $k_R$ having a copy of the field $k$ at every point $t \in R$ and zero vector spaces elsewhere, the copies of $k$ being connected by identities and the rest of the maps being zero. It is easily seen that any such bimodule is pfd and satisfies the equalities of (2). However, not every rectangle module is exact, as illustrated in Fig. 1.

Among the rectangles with at least two cuts at infinity, we distinguish the following four types, illustrated in Fig. 2:
Fig. 2  The four block types. From left to right: birth quadrant (bquad), death quadrant (dquad), horizontal band (hband), vertical band (vband)

- **birth quadrants** (shorthand: bquad), for which $c^+ = c^- = \emptyset$;
- **death quadrants** (shorthand: dquad), for which $c^+ = c^- = \emptyset$;
- **horizontal bands** (shorthand: hband), for which $c^- = c^+ = \emptyset$;
- **vertical bands** (shorthand: vband), for which $c^- = c^+ = \emptyset$.

A block is a rectangle of any of these types. Note that these types are not mutually exclusive, for instance $\mathbb{R}^2$ belongs to all four of them. The rectangle module associated to a given block is called a block module. It is immediate that any such bimodule is both pfd and exact, and that, among the rectangles in the plane, only those that are blocks yield exact modules—see the counterexample in Fig. 1. Since being exact is invariant under taking direct sums, any pfd bimodule that is decomposable into block summands (or block-decomposable for short) is also exact. Our main result states that the converse is also true:

**Theorem 2.1** (Decomposition of exact pfd bimodules) Any exact pfd bimodule $M$ decomposes as a direct sum of block modules:

$$M \cong \bigoplus_{B \in \mathcal{B}(M)} k_B,$$

(3)

where $\mathcal{B}(M)$ is some multiset of blocks that depends on $M$. The decomposition is unique up to isomorphism and reordering of the terms.

Thus, among the pfd bimodules, the ones that are block-decomposable are precisely the ones that are exact. Some applications of this result are described in Sect. 9. Among them, the study of the stability of zigzags in the context of interlevel-sets persistence (Sect. 9.3) served as the initial motivation for this work. Exactness in that setting is ensured by the Mayer–Vietoris theorem.

**Proof Outline**

The bulk of the paper (Sects. 3 to 8) is devoted to the proof of Theorem 2.1. The uniqueness of the decomposition is a straightforward consequence of Azumaya’s theorem [2], the endomorphism ring of any block module being clearly isomorphic to $k$ and therefore local. It remains to prove the existence of a decomposition, for which we follow the same scheme as in the 1-d setting [13], with some significant adjustments at each step due to the product order on $\mathbb{R}^2$ not being total:
In Sects. 3 and 4, to each rectangle $R$ in the plane we associate a counting functor $C_R$ (see (9)) that maps pfd persistence bimodules satisfying (2) to $k$-vector spaces. This functor captures the elements whose lifespan is exactly $R$ in such a module $M$. In particular, if we assume $M$ to be decomposable into rectangle summands, then the dimension of the space $C_R(M)$ is the same as the multiplicity of the summand $k_R$ in the decomposition of $M$ (Lemma 4.2). The functor is built using the so-called functorial filtration technique (see e.g. [20]), which consists in filtering each space $M_t$ by the kernels and images of the internal morphisms $\rho^s_t$ (for $s \leq t$) and $\rho^u_t$ (for $u \geq t$). The technique was used in the 1-d setting, where the total order on the real line made it simple to show that the image and kernel subspaces get transported from one index to the other within $R$, which is the key property to define the counting functor. Here the transportation property is preserved (Corollary 3.5) modulo some adjustments to the definitions of the image and kernel subspaces, leveraging the equalities in (2). See (5) for the precise definitions, Remark 3.2 for the underlying intuition, and Example 3.3 for an illustrative example.

From Sect. 5 onward, we assume that the bimodule $M$ is pfd and exact. In order to build an explicit internal decomposition of $M$, to any block $B$ we associate a submodule $M_B$ of $M$ whose structure is that of a direct sum of $\dim C_B(M)$ copies of $k_B$ (see Lemma 5.5). As in the 1-d setting, the construction of $M_B$ leverages that of the counting functor $C_B$, and it boils down to choosing some vector-space complement in a certain inverse limit. However, in contrast to the 1-d case, the vector-space complement cannot be chosen arbitrarily, as care must be taken of the way the image of the complement through the cone maps transitions outside the block $B$ (see Proposition 5.3).

In Sect. 6 we show that the submodules $M_B$ are in direct sum, which it is sufficient to check pointwise at every index $t \in \mathbb{R}^2$. The proof in the 1-d setting uses the concept of disjoint sections of a vector space, from which the direct sum follows. In our setting, while the submodules associated to bands satisfy the disjointness property leveraged in 1-d, the whole family of submodules $\{M_B\}_{B:\text{block}}$ does not (see Example 6.4). We therefore resort to more direct arguments, based on the observation that modules associated to different blocks have different supports, so that showing the direct sum amounts for the most part to showing that, for a finite family of linearly related block modules, there is one whose support extends further than the others to the right or to the top (Propositions 6.6 and 6.7). This fact is not true in all cases, but sufficiently widely so that the special cases can be handled individually using exactness.

In Sects. 7 and 8 we show that the submodules $M_B$ generate the whole module $M$, which it is also sufficient to check pointwise at every index $t \in \mathbb{R}^2$. The proof in the 1-d setting uses the concept of covering sections, from which the result follows. In our setting, the family of submodules $\{M_B\}_{B:\text{block}}$ does not satisfy the covering property (see Example 7.4), unless we remove from $M$ those elements that live since $-\infty$ both horizontally and vertically (Proposition 7.5). We therefore study separately the submodule generated by those elements, showing by a duality argument that it itself decomposes as a direct sum of block modules (Corol-
lary 8.6). This allows us to complete the construction of the internal direct-sum decomposition of \( M \) (Corollary 8.7).

### 3 Images and Kernels

As in the 1-dimensional setting [13], the basic ingredients in our analysis are certain limits of images and kernels. Given a rectangle \( R = (c^+ \cap c^-) \times (\bar{c}^+ \cap \bar{c}^-) \), and a point \( t \in R \), we first construct these limits along the 1-dimensional restrictions of the module \( M \) to the horizontal and vertical lines passing through \( t \) (with the convention that \( \text{Im}_{c,t}^- = 0 \) when \( c^- = \emptyset \) and \( \text{Ker}_{c,t}^- = M_t \) when \( c^+ = \emptyset \)):

\[
\begin{align*}
\text{Im}_{c,t}^+(M) &= \bigcap_{x \in c^+} \text{Im} h^t_{(x,t_y)}, & \text{Im}_{c,t}^-(M) &= \sum_{x \in c^-} \text{Im} h^t_{(x,t_y)}, \\
\text{Ker}_{c,t}^+(M) &= \bigcap_{x \in c^+} \text{Ker} h^t_{(x,t_y)}, & \text{Ker}_{c,t}^-(M) &= \sum_{x \in c^-} \text{Ker} h^t_{(x,t_y)}, \\
\text{Im}_{c,t}^+(M) &= \bigcap_{y \in c^+} \text{Im} h^t_{(t_x,y)}, & \text{Im}_{c,t}^-(M) &= \sum_{y \in c^-} \text{Im} h^t_{(t_x,y)}, \\
\text{Ker}_{c,t}^+(M) &= \bigcap_{y \in c^+} \text{Ker} h^t_{(t_x,y)}, & \text{Ker}_{c,t}^-(M) &= \sum_{y \in c^-} \text{Ker} h^t_{(t_x,y)}.
\end{align*}
\]

See Fig. 3 for an illustration. In the following we omit \( M \) from our notations whenever the considered module is obvious.
Lemma 3.1 (Realization) Assume $M$ is pfd, and extend it to a representation of the extended plane $[-\infty, +\infty]^2$ by letting $M_{(-\infty, \cdot)} = M_{(\cdot, \pm\infty)} = 0$. Then

$$\text{Im}_{c, t}^+ = \text{Im} h'_{(x, t_x)} \text{ for some } x \in \mathcal{E}^+ \cap (-\infty, t_x] \text{ and any lower } x \in \mathcal{E}^+, \text{ }$$

$$\text{Im}_{c, t}^- = \text{Im} h'_{(x, t_x)} \text{ for some } x \in \mathcal{E}^- \cup \{-\infty\} \text{ and any greater } x \in \mathcal{E}^-,$$

$$\text{Ker}_{c, t}^+ = \text{Ker} h_{(x, t_x)} \text{ for some } x \in \mathcal{E}^+ \cup \{+\infty\} \text{ and any lower } x \in \mathcal{E}^+,$$

$$\text{Ker}_{c, t}^- = \text{Ker} h_{(x, t_x)} \text{ for some } x \in \mathcal{E}^- \cap [t, +\infty) \text{ and any greater } x \in \mathcal{E}^-.$$

And similarly for the vertical cuts $c, \overline{c}$. Note that the spaces $\text{Im}_{c, t}^\pm$ and $\text{Ker}_{c, t}^\pm$ mentioned here, which are those of the extension of $M$, are the same as those of $M$ since $t \in \mathbb{R}^2$ and the cuts considered are cuts of $\mathbb{R}$.

Proof This is [13, Lem. 2.1], and a direct consequence of the finite dimensionality of $M_t$. \qed

We now combine the contributions from the horizontal and vertical restrictions of $M$ at point $t$ as follows (where equalities between formulas come from the inclusions $\text{Im}_{c, t}^- \subseteq \text{Im}_{c, t}^+$ and $\text{Ker}_{c, t}^- \subseteq \text{Ker}_{c, t}^+$):

$$\text{Im}_{R, t}^+ = \text{Im}_{c, t}^+ \cap \text{Im}_{\overline{c}, t}^+,$$

$$\text{Im}_{R, t}^- = (\text{Im}_{c, t}^- + \text{Im}_{\overline{c}, t}^-) \cap \text{Im}_{R, t}^+$$

$$= \text{Im}_{c, t}^- \cap \text{Im}_{\overline{c}, t}^+ + \text{Im}_{\overline{c}, t}^- \cap \text{Im}_{c, t}^+,$$

$$\text{Ker}_{R, t}^+ = (\text{Ker}_{c, t}^+ + \text{Ker}_{\overline{c}, t}^-) \cap (\text{Ker}_{c, t}^- + \text{Ker}_{\overline{c}, t}^+),$$

$$\text{Ker}_{R, t}^- = \text{Ker}_{c, t}^- + \text{Ker}_{\overline{c}, t}^-.$$ (5)

It is immediate from the definition that $\text{Im}_{R, t}^- \subseteq \text{Im}_{R, t}^+$ and $\text{Ker}_{R, t}^- \subseteq \text{Ker}_{R, t}^+$.

Remark 3.2 The above definitions are motivated as follows. The straightforward generalization of the definitions from the 1-d setting [13] gives the following spaces:

$$I_{R, t}^+ = \bigcap_{s \in R} \text{Im} \rho_s^t,$$

$$I_{R, t}^- = \sum_{s \in R} \text{Im} \rho_s^t;$$

$$K_{R, t}^+ = \bigcap_{u \in R} \text{Ker} \rho_u^t,$$

$$K_{R, t}^- = \sum_{u \in R} \text{Ker} \rho_u^t.$$ (6)

Unfortunately, due to the order $\leq$ not being total on $\mathbb{R}^2$, we may not always have $I_{R, t}^- \subseteq I_{R, t}^+$ nor $K_{R, t}^- \subseteq K_{R, t}^+$. The fix is to consider sums and intersections as follows:

$$\text{Im}_{R, t}^+ = I_{R, t}^+,$$

$$\text{Im}_{R, t}^- = I_{R, t}^- \cap I_{R, t}^+,$$

$$\text{Ker}_{R, t}^+ = K_{R, t}^+ + K_{R, t}^-,$$

$$\text{Ker}_{R, t}^- = K_{R, t}^-.$$ (7)
Other combinations of sums and intersections could be considered, e.g. letting $\text{Im}^+_{t, R} = I^+_{R,t} + I^-_{R,t}$ and $\text{Im}^-_{t, R} = I^-_{R,t}$, however the ones above are the only ones ensuring that the spaces can be transported from one index $t$ to another $t' \geq t$ (see the Transportation Corollary 3.5 below). Furthermore, they induce a duality between image and kernel spaces, through vector-space duality, as will be emphasized and exploited in Sect. 8 (see Lemma 8.5). Finally, assuming the module $M$ is exact or satisfies the equalities of (2), the horizontal and vertical contributions to the images and kernels in (6) can be decoupled so that the definitions in (7) are equivalent to those in (5) (see Appendix A). In the following proof of Theorem 2.1 we only use the definitions from (5), not the ones from (7), for consistency.

**Example 3.3** Take for $M$ the direct sum of the modules associated with a birth quadrant $A$ and a death quadrant $B$, such that the intersection $A \cap B$ is non-empty (see Fig. 4 for an illustration). Given any $t \in A \cap B$, call $\alpha$ a generator of the 1-dimensional subspace of $M_t$ spanned by $A$, and $\beta$ a counterpart for $B$. Then, applying the formulas in (5) (or equivalently the ones in (7)), we get:

$$\begin{align*}
\text{Im}^+_{A,t} &= M_t, & \text{Im}^-_{A,t} &= \langle \beta \rangle, & \text{Ker}^+_{A,t} &= M_t, & \text{Ker}^-_{A,t} &= \langle \beta \rangle, \\
\text{Im}^+_{B,t} &= \langle \beta \rangle, & \text{Im}^-_{B,t} &= 0, & \text{Ker}^+_{B,t} &= \langle \beta \rangle, & \text{Ker}^-_{B,t} &= 0.
\end{align*}$$

Then, we see that

$$\begin{align*}
\text{Im}^+_{A,t} / \text{Im}^-_{A,t} &\simeq \langle \alpha \rangle \simeq \text{Ker}^+_{A,t} / \text{Ker}^-_{A,t}, \\
\text{Im}^+_{B,t} / \text{Im}^-_{B,t} &\simeq \langle \beta \rangle \simeq \text{Ker}^+_{B,t} / \text{Ker}^-_{B,t}.
\end{align*}$$

Thus, for each block $A, B$ the quotient $\text{Im}^+ / \text{Im}^-$ captures those elements that are born on the bottom-left boundary of the block, while $\text{Ker}^+ / \text{Ker}^-$ captures those that die on its top-right boundary. This fact holds generally for blocks, and (beyond that) also for rectangles, as we shall see in Sect. 4. Moreover, it is independent of the choice of index $t$ within the block or rectangle, as a consequence of the following transportation property.

It turns out that images get transported to images, and kernels to kernels, through the internal morphisms of $M$—see Corollary 3.5 below. The proof says something slightly more precise, namely:
Lemma 3.4 Assume $M$ is pfd and satisfies (2). Let $R = (c^+ \cap c^-) \times (\bar{c}^+ \cap \bar{c}^-)$ be a rectangle, $s \leq t \in R$, and $\bullet, \bigtriangleup \in \{+, -\}$. Then

$$\rho_t^s(\text{Im}_{\bullet,s} \cap \text{Im}_{\bigtriangleup,t}) = \text{Im}_{\bullet,t} \cap \text{Im}_{\bigtriangleup,t},$$

$$(\rho_t^s)^{-1}(\text{Ker}_{\bullet,t} + \text{Ker}_{\bigtriangleup,t}) = \text{Ker}_{\bullet,t} + \text{Ker}_{\bigtriangleup,t}.$$  

Proof For convenience, we extend $M$ to a representation of the extended plane $[-\infty, +\infty]^2$ by letting $M(\pm\infty, \cdot) = M(\cdot, \pm\infty) = 0$. Note that this extension may not be exact when $M$ itself is exact, however it still satisfies the equalities of (2) when $M$ does so.

We first consider images. The Realization Lemma 3.1 tells us that there exist $x \leq s_x \leq t_x$ and $y \leq s_y \leq t_y$ (possibly equal to $-\infty$) such that

$$\text{Im}_{\bullet,s} = \text{Im} h_s^{(x, s_y)} \quad \text{and} \quad \text{Im}_{\bullet,t} = \text{Im} h_t^{(x, t_y)},$$

$$\text{Im}_{\bigtriangleup,s} = \text{Im} v_s^{(s_x, y)} \quad \text{and} \quad \text{Im}_{\bigtriangleup,t} = \text{Im} v_t^{(t_x, y)}.$$  

We then have the following commutative diagram:

Chasing through this diagram gives

$$\text{Im} h_t^{(x, t_y)} \cap \text{Im} v_t^{(t_x, y)} \quad \overset{\text{(2)}}{=} \quad \text{Im} \rho_t^s(\text{Im} \rho_t^s) \quad \overset{\text{(2)}}{=} \quad \rho_t^s(\text{Im} h_t^{(x, t_y)} \cap \text{Im} v_t^{(t_x, y)}),$$

thus proving the first part of the lemma.

We now consider kernels. The Realization Lemma 3.1 tells us that there exist $x \geq t_x \geq s_x$ and $y \geq t_y \geq s_y$ (possibly equal to $+\infty$) such that

$$\text{Ker}_{\bullet,s} = \text{Ker} h_s^{(x, s_y)} \quad \text{and} \quad \text{Ker}_{\bullet,t} = \text{Ker} h_t^{(x, t_y)},$$

$$\text{Ker}_{\bigtriangleup,s} = \text{Ker} v_s^{(s_x, y)} \quad \text{and} \quad \text{Ker}_{\bigtriangleup,t} = \text{Ker} v_t^{(t_x, y)}.$$  

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We then have the following commutative diagram:

\[
\begin{array}{ccc}
(s, y) & \xrightarrow{(t, y)} & (x, y) \\
| & (t_x, y) \downarrow & (x, t_y) \\
| & (x, s_y) \uparrow \\
S & \xrightarrow{t} & (x, t_y) \\
\end{array}
\]

Chasing through this diagram gives

\[
(\rho \circ t \circ s)^{-1} (\operatorname{Ker} h_t^{(x, t_y)} + \operatorname{Ker} v_t^{(t_x, y)}) \overset{\text{Eq. (2)}}{=} (\rho \circ s)^{-1} (\operatorname{Ker} \rho s^{(x, y)}) = \operatorname{Ker} \rho s^{(x, y)},
\]

\[
\overset{\text{Eq. (2)}}{=} \operatorname{Ker} h_s^{(x, t_y)} + \operatorname{Ker} v_s^{(s_x, y)}
\]

thus proving the second part of the lemma.

Proof All four cases are proven by the same argument, which we detail here in the first case. The Realization Lemma 3.1 tells us that there exist finite values \(x \geq t_x\) and \(y \leq t_y\) such that \(\operatorname{Ker} c_{t,t} \subseteq \operatorname{Ker} h_t^{(x, t_y)}\) and \(\operatorname{Im}^+_{x,t} = \operatorname{Im}^+_{v(t_x, y)}\). We then have the

\[
\begin{array}{c}
\operatorname{Ker} c_{t,t} \subseteq \operatorname{Ker} h_t^{(x, t_y)} \\
\operatorname{Im}^+_{x,t} \subseteq \operatorname{Im}^+_{v(t_x, y)} \\
\operatorname{Im}^-_{x,t} \subseteq \operatorname{Im}^-_{v(t_x, y)}
\end{array}
\]

Corollary 3.5 (Transportation) Assume \(M\) is pfd and satisfies (2). Let \(R\) be a rectangle and let \(s \leq t \in R\). Then (using \(\pm\) as a shorthand for either \(+\) or \(\cdot\), with the same sign on both sides of the equality)

\[
\rho (\operatorname{Im}^+_{R, s}) = \operatorname{Im}^+_{R, t},
\]

\[
(\rho)^{-1} (\operatorname{Ker} \pm_{R, t}) = \operatorname{Ker} \pm_{R, s}.
\]

Proof Follows from Lemma 3.4 and from the facts that \(f(U + V) = f(U) + f(V)\) and \(f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)\) — recall the definitions in (5).
following exact square:

\[
\begin{array}{ccc}
M_t & \to & M_{(x,y)} \\
\uparrow & & \uparrow \\
M_{(x,y)} & \to & M_{(x,y)}
\end{array}
\]

The exactness of this square implies that every \( \alpha \in \text{Ker } h_t^{(x,y)} \) has a common antecedent \( \beta \in M_{(x,y)} \) with \( 0 \in M_{(x,y)} \), meaning that \( \alpha = v_t^{(x,y)}(\beta) \in \text{Im } v_t^{(x,y)} \). Therefore, \( \text{Ker}_{c,t} \subseteq \text{Im}_{c,t} \). The inclusion \( \text{Ker}_{c,t} \subseteq \text{Im}_{c,t} \) is obtained symmetrically, with the Realization Lemma 3.1 giving some finite \( x \leq t_s \) and \( y \geq t_y \) such that \( \text{Ker}_{c,t} = v_t^{(t_s,y)} \) and \( \text{Im}_{c,t} = h_t^{(x,t_y)} \). \( \square \)

4 The Counting Functor

Our exposition in this section follows [13] closely. To define our functor we consider certain combinations of images and kernels that, intuitively, capture the elements appearing at the bottom and left boundaries of a rectangle and that die at its top and right boundaries. Specifically, given a rectangle \( R \) and a point \( t \in R \), we define as in [13]:

\[
\begin{align*}
V_{R,t}^+ &= \text{Im}_{R,t}^+ \cap \text{Ker}_{R,t}^+ , \\
V_{R,t}^- &= \text{Im}_{R,t}^+ \cap \text{Ker}_{R,t}^- + \text{Im}_{R,t}^- \cap \text{Ker}_{R,t}^+ .
\end{align*}
\] (8)

Since \( \text{Im}_{R,t}^- \subseteq \text{Im}_{R,t}^+ \) and \( \text{Ker}_{R,t}^- \subseteq \text{Ker}_{R,t}^+ \), we have \( V_{R,t}^- \subseteq V_{R,t}^+ \). Note that these spaces depend a priori on the location of \( t \) in the rectangle. In fact, it turns out not to be the case, as the following result shows:

**Lemma 4.1** Assume \( M \) is pfd and satisfies (2). Then, for all \( s \leq t \in R \) we have \( \rho_s^i(V_{R,s}^{\pm}) = V_{R,t}^{\pm} \). Furthermore, the induced map \( \rho_s^i : V_{R,s}/V_{R,s} \to V_{R,t}/V_{R,t} \) is an isomorphism.

**Proof** This follows from the Transportation Corollary 3.5. First, we have

\[
\begin{align*}
\rho_s^i(V_{R,s}^+) &= \rho_s^i(\text{Im}_{R,s}^+ \cap \text{Ker}_{R,s}^+) \\
&\subseteq \rho_s^i(\text{Im}_{R,s}^+ \cap \rho_s^i(\text{Ker}_{R,s}^+)) = \text{Im}_{R,t}^+ \cap \text{Ker}_{R,t}^+ = V_{R,t}^+ , \\
\rho_s^i(V_{R,s}^-) &= \rho_s^i(\text{Im}_{R,s}^+ \cap \text{Ker}_{R,s}^- + \text{Im}_{R,s}^- \cap \text{Ker}_{R,s}^+) \\
&\subseteq \rho_s^i(\text{Im}_{R,s}^+ \cap \rho_s^i(\text{Ker}_{R,s}^-) + \rho_s^i(\text{Im}_{R,s}^-) \cap \rho_s^i(\text{Ker}_{R,s}^+)) \\
&= \text{Im}_{R,t}^+ \cap \text{Ker}_{R,t}^- + \text{Im}_{R,t}^- \cap \text{Ker}_{R,t}^+ = V_{R,t}^- .
\end{align*}
\]

Thus, \( \rho_s^i(V_{R,s}^{\pm}) \subseteq V_{R,t}^{\pm} \) and the induced map \( \rho_s^i \) is well-defined. We will now show that \( \rho_s^i \) is both injective and surjective, proving that \( \rho_s^i(V_{R,s}^{\pm}) = V_{R,t}^{\pm} \) along the way.
Surjectivity: Take $\beta \in V^+_{R,t} = \text{Im}^+_{R,t} \cap \text{Ker}^+_{R,t}$. Then $\beta = \rho^i_s(\alpha)$ for some $\alpha \in \text{Im}^+_{R,s}$. Now, $\alpha \in (\rho^i_s)^{-1}(\beta) \subseteq (\rho^i)^{-1}(\text{Ker}^+_{R,t}) = \text{Ker}^+_{R,s}$, so $\alpha \in V^+_{R,s}$. Thus, $\rho^i_s(V^+_{R,s}) = V^+_{R,t}$, which implies that the induced map $\overline{\rho^i_s}$ is surjective.

Injectivity: Take $\alpha \in V^+_{R,s}$ such that $\beta = \rho^i_s(\alpha) \in V^+_{R,t}$. Then, $\beta = \beta_1 + \beta_2$ with $\beta_1 \in \text{Im}^+_{R,t} \cap \text{Ker}^+_{R,t}$ and $\beta_2 \in \text{Im}^−_{R,t} \cap \text{Ker}^−_{R,t}$. By the same argument as before, $\beta_1 = \rho^i_s(\alpha_1)$ for some $\alpha_1 \in \text{Im}^−_{R,s} \cap \text{Ker}^−_{R,s}$. Now, $\rho^i_s(\alpha - \alpha_1) = \beta_2 \in \text{Ker}^−_{R,s}$, so $\alpha - \alpha_1 \in \text{Ker}^−_{R,s}$. Moreover, $\alpha - \alpha_1 \in \text{Im}^+_{R,s}$, so $\alpha \in V^−_{R,s}$. Thus, $(\rho^i_s)^{-1}(V^−_{R,s}) \subseteq V^−_{R,s}$, which implies that the induced map $\overline{\rho^i_s}$ is injective. It also implies that $\rho^i_s(V^−_{R,s}) = V^−_{R,t}$ since we already know that $\rho^i_s(V^−_{R,s}) \subseteq V^−_{R,t} \subseteq V^+_{R,t} = \rho^i_s(V^+_{R,s})$.

We can now define an intrinsic quotient, independent of the location of $t \in R$, by considering the inverse system $\mathcal{V}$ of vector spaces $V^+_{R,t}/V^−_{R,t}$ with transition maps $\overline{\rho^i_s}$, and by taking its inverse limit:

$$C_R(M) = \lim_{t \in R} V^+_{R,t}/V^−_{R,t}. \tag{9}$$

By Lemma 4.1, this limit is isomorphic to $V^+_{R,t}/V^−_{R,t}$ for all $t \in R$. Moreover, its construction is entirely functorial, since for any morphism of modules $\phi: M \to N$ there are canonically induced maps $\text{Im}^\pm_{R,t}(M) \to \text{Im}^\pm_{R,t}(N)$ and $\text{Ker}^\pm_{R,t}(M) \to \text{Ker}^\pm_{R,t}(N)$, then $V^\pm_{R,t}(M) \to V^\pm_{R,t}(N)$, then $V^+_{R,t}/V^−_{R,t}(M) \to V^+_{R,t}/V^−_{R,t}(N)$, and finally $C_R(\phi): C_R(M) \to C_R(N)$ by universality of the limit. Thus, $C_R$ is a functor from the category of pfd bimodules satisfying the equalities of (2) to the category of finite-dimensional vector spaces. This functor is additive because the inverse limit commutes with direct products, and direct products coincide with direct sums in the category of pfd bimodules.

We refer to $C_R$ as the counting functor associated to the rectangle $R$ because, as we shall see in the following sections, what it does is, literally, to count the multiplicity of the summand $k_R$ in the direct-sum decomposition of $M$. In particular, we can already prove the following fact:

**Lemma 4.2** Assume $M$ is pfd and decomposes as a direct sum of rectangle modules. Then, for any rectangle $R$, the multiplicity of the summand $k_R$ in the direct-sum decomposition of $M$ is given by $\dim C_R(M)$.

**Proof** Since $C_R$ is an additive functor, it is enough to prove the result on a single summand $k_{R'}$. Let us write $R = (c^+ \cap c^-) \times (c'^+ \cap c'^-) \times (c^+ \cap c^-)$. Suppose first that $R' \neq R$. Then, there is a cut that differs between $R$ and $R'$, i.e., there is some $c \in \{c, c', c\} \cap \{c', c\}$ such that $c \neq c'$. For all $t \in R \cap R'$, we then have $\bullet_{c,t}(k_{R'}) = \bullet_{c,t}(k_{R})$, where $\bullet$ stands for either $\text{Im}$ or $\text{Ker}$ depending on whether $c \in \{c, c\}$ or $c \in \{c', c\}$. Then, by (5) we have $\bullet_{R',t}(k_{R'}) = \bullet_{R,t}(k_{R'})$, which by (8) implies that $V^+_{R,t}(k_{R'}) = V^−_{R,t}(k_{R'})$ and so $V^+_{R,t}(k_{R'})/V^−_{R,t}(k_{R'}) = 0$. Meanwhile, for

---

3 Strictly speaking, we use the opposite ordering on $R$, to fit with the convention of [15, Chap. 0, §13.1]. This switch in the ordering is implicit in the rest of the paper.
all \( t \in R \setminus R' \), we have \((k_{R'})_t = 0\) and so \(V_{R,t}^{+}(k_{R'})/V_{R,t}^{-}(k_{R'}) = 0\). Taking the inverse limit as in (9), we obtain that \(C_R(k_{R'}) = 0\).

Suppose now that \( R' = R \). For any \( t \in R \) and any \( c \in \{c, \underline{c}, c_{\partial}, \overline{c}\} \), we have \( \bullet_{c,t}(k_R) = (k_R)_t \simeq k \) and \( \bullet_{c_{\partial},t}(k_R) = 0 \), where \( \bullet \) stands for either \( \text{Im} \) or \( \text{Ker} \) depending on whether \( c \in \{c, \underline{c}\} \) or \( c \in \{c_{\partial}, \overline{c}\} \). Then, by (5) we have \( \text{Im}_{R,t}^{+}(k_R) = \text{Ker}_{R,t}^{+}(k_R) = (k_R)_t \simeq k \) while \( \text{Im}_{R,t}^{-}(k_R) = \text{Ker}_{R,t}^{-}(k_R) = 0 \), which by (8) implies that \( V_{R,t}^{+}(k_R) = (k_R)_t \simeq k \) while \( V_{R,t}^{-}(k_R) = 0 \), and so \( V_{R,t}^{+}(k_R)/V_{R,t}^{-}(k_R) \simeq k \), which, taking the inverse limit as in (9), gives \( C_R(k_R) \simeq k \) and so \( \dim C_R(k_R) = 1 \) as desired. □

5 Submodules

While the counting functor allows us to retrieve the multiplicity of a summand given a decomposition of \( M \), it is not enough to prove the existence of such a decomposition. For this we need to further assume that \( M \) is exact, and to specify a submodule \( M_B \) of \( M \) for each block \( B \), so that in the following sections we can exhibit an internal direct-sum decomposition. To define \( M_B \) we use the spaces \( V_{B,t}^{\pm} \) from (8) and consider their inverse limits:

\[
V_B^{\pm}(M) = \lim_{\leftarrow t \in B} V_{B,t}^{\pm}.
\]

Lemma 5.1 Assume \( M \) is pfd and satisfies (2). Then, the quotient of limits \( V_B^{+}(M)/V_B^{-}(M) \) is isomorphic to the limit of quotients \( C_B(M) \).

Proof Recall from Lemma 4.1 that \( \rho_s^{t}(V_{B,s}^{-}) \subseteq V_{B,t}^{-} \) for any \( s \leq t \in B \). Thus, we have an inverse system of vector spaces \( V_{B,t}^{-} \) for \( t \in B \), with transition maps \( \rho_s^{t} \) for \( s \leq t \in B \). This system satisfies the Mittag-Leffler condition [15, Chap. 0, (13.1.2)] because every space \( V_{B,t}^{-} \) is finite-dimensional. Now, since \( B \) contains a countable subset\(^4\) that is coinitial for the product order \( \leq \), the hypotheses of Proposition 13.2.2 of [15, Chap. 0] hold for the system of exact sequences

\[
0 \rightarrow V_{B,t}^{-} \rightarrow V_{B,t}^{+} \rightarrow V_{B,t}^{+}/V_{B,t}^{-} \rightarrow 0
\]

and hence the limit sequence

\[
0 \rightarrow V_{B}^{-}(M) \rightarrow V_{B}^{+}(M) \rightarrow C_B(M) \rightarrow 0
\]

is also exact, giving the result. □

\(^4\) Take either \([\min B], \) or \((\min\{t_s \mid t \in B]\} \times \mathbb{Q}) \cap B, \) or \((\mathbb{Q} \times \{\min\{t_s \mid t \in B]\}) \cap B, \) or \(\mathbb{Q}^2 \cap B, \) depending on whether (respectively) \( B \) contains both its left and bottom boundaries, or only the left one, or only the bottom one, or none.
Letting $\pi_t : V_B^+(M) \to V_{B,t}^+$ denote the natural map given by the limit, we can make the following identification:

$$V_B^-(M) = \bigcap_{t \in B} \pi_t^{-1}(V_{B,t}^-) \subseteq V_B^+(M).$$

**Lemma 5.2** Assume $M$ is pfd and satisfies (2). Then, for all $t \in B$, the induced map $\pi_t^*: V_B^+(M)/V_B^-(M) \to V_{B,t}^+/V_{B,t}^-$ is an isomorphism.

**Proof** Following-up the proof of Lemma 5.1, for every $t \in B$ we have the commutative diagram below, where the vertical arrows are the natural maps given by the limit, and where each row is known to be exact:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & V_B^-(M) & \longrightarrow & V_B^+(M) & \longrightarrow & C_B(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_{B,t}^- & \longrightarrow & V_{B,t}^+ & \longrightarrow & V_{B,t}^+/V_{B,t}^- & \longrightarrow & 0.
\end{array}
\]

Hence the (commutative) diagram:

\[
\begin{array}{ccccccccc}
V_B^+(M)/V_B^-(M) & \longrightarrow & C_B(M) \\
\downarrow & & \downarrow & & \\
V_{B,t}^+/V_{B,t}^- & \longrightarrow & V_{B,t}^+/V_{B,t}^-.
\end{array}
\]

Now, by Lemma 4.1 the maps $\rho_{us}^t$ are isomorphisms for all $s \leq u \in B$, therefore $\nu_t$ itself is an isomorphism. The conclusion follows.

Now we can identify, for each block $B$, a submodule $M_B$ of $M$ that is isomorphic to a direct sum of $\dim C_B(M)$ copies of the interval module $k_B$ (see Lemma 5.5 below). Note that, unlike in the 1-d setting [13], not every vector space complement of $V_B^-(M)$ in $V_B^+(M)$ will work to get a submodule whose support is precisely $B$. We need to choose that complement with care so that the submodule does vanish outside $B$, and for this we use exactness. Here are the details:

**Proposition 5.3** Assume $M$ is pfd and exact. Then, for each block $B$ there is a vector space complement $M_B^0$ of $V_B^-(M)$ in $V_B^+(M)$ such that the family of subspaces defined by

$$(M_B)_t = \begin{cases} 
\pi_t(M_B^0), & t \in B, \\
0, & t \notin B,
\end{cases}$$

defines a submodule $M_B$ of $M$. 

$\square$
Proof Given a fixed block $B$, let us first observe that, whatever choice of subspace $M^0_B$ we make such that $V^+_B(M) = M^0_B \oplus V^-_B(M)$, the following properties will be satisfied:

- For any $s \leq t$ sitting both in $B$, $\rho'_s((M^B_B)_s) \subseteq (M^B_B)_t$. This is because $\rho'_s \circ \pi_s = \pi_t$ by definition of $\pi$.
- For any $s \leq t$ such that $s \notin B$ and $t \in B$, $\rho'_s((M^B_B)_s) = \rho'_s(0) = 0 \subseteq (M^B_B)_t$.

There only remains to show that, for a suitable choice of subspace $M^0_B$, we also have $\rho'_s(\pi_s(M^B_B)) = 0$ for all $s \leq t$ with $s \in B$ and $t \notin B$. For this we let $B = (c^+ \cap c^-) \times (c^+ \cap c^-)$ and we distinguish between the various block types:

**Case $B$ is a birth quadrant (possibly with $c^- = \emptyset$ or $c^- = \emptyset$).** Then any choice of vector space complement $M^0_B$ works trivially, because there are no indices $s \leq t$ with $s \in B$ and $t \notin B$.

**Case $B$ is a death quadrant and not a band ($c^+ \neq \emptyset \neq c^-)$.** Then we will enforce $\pi_s(M^B_B) \subseteq \ker^+_{c^+_s} \cap \ker^+_{c^-_s}$ for every $s \in B$, which will imply that $\rho'_s(\pi_s(M^B_B)) \subseteq \rho'_s(\ker^+_{c^+_s} \cap \ker^+_{c^-_s}) = 0$ for every $t \geq s$ with $t \notin B$. Let then $K^+_{B,s} = \ker^+_{c^+_s} \cap \ker^+_{c^-_s}$ for each $s \in B$, and consider the inverse system formed by these vector spaces with the transition maps $\rho^u_s$ for $s \leq u \in B$. Since $K^+_{B,s} \subseteq \text{im}^+_{B,s}$ by Lemma 3.6, we have $K^+_{B,s} \subseteq V^+_{B,s}$ and so the inverse limit $K^+_{B}(M)$ of the system can be identified as follows:

$$K^+_{B}(M) = \lim_{s \in B} K^+_{B,s} = \bigcap_{s \in B} \pi^{-1}_s(K^+_{B,s}) \subseteq V^+_{B}(M).$$

We claim that $V^-_{B}(M) + K^+_{B}(M) = V^+_{B}(M)$. Indeed, this equality holds at every index $s \in B$ because $K^+_{B,s} \subseteq \text{im}^+_{B,s}$:

$$V^+_{B,s} = \text{im}^+_{B,s} \cap (\ker^-_{B,s} + K^+_{B,s}) = \text{im}^+_{B,s} \cap \ker^-_{B,s} + K^+_{B,s} = V^-_{B,s} + K^+_{B,s}.$$  

In other words, at every index $s \in B$ we have the following exact sequence:

$$0 \longrightarrow V^-_{B,s} \cap K^+_{B,s} \overset{\alpha \mapsto (\alpha, -\alpha)}{\longrightarrow} V^-_{B,s} \oplus K^+_{B,s} \overset{(\alpha, \beta) \mapsto \alpha + \beta}{\longrightarrow} V^+_{B,s} \longrightarrow 0.$$  

Since every space $V^-_{B,s} \cap K^+_{B,s}$ is finite-dimensional, the Mittag-Leffler condition is satisfied by this system of exact sequences, and so, by Proposition 13.2.2 of [15, Chap. 0], the limit sequence is exact. After noticing that $\lim V^-_{B,s} \cap K^+_{B,s} = V^-_{B}(M) \cap K^+_{B}(M)$ inside $V^+_{B}(M)$, and that the canonical morphism $V^-_{B}(M) \oplus K^+_{B}(M) \rightarrow \lim V^-_{B,s} \oplus K^+_{B,s}$ is an isomorphism, we obtain the following exact sequence:

$$0 \longrightarrow V^-_{B}(M) \cap K^+_{B}(M) \overset{\alpha \mapsto (\alpha, -\alpha)}{\longrightarrow} V^-_{B}(M) \oplus K^+_{B}(M) \overset{(\alpha, \beta) \mapsto \alpha + \beta}{\longrightarrow} V^+_{B}(M) \longrightarrow 0.$$
which implies that $V_B^-(M) + K_B^+(M) = V_B^+(M)$, as claimed earlier. We can then choose\footnote{This is done via a finite induction since $V_B^+(M)/V_B^-(M)$ is finite-dimensional by Lemma 5.2.} our vector space complement $M_0^B(M)$ inside $K_B^+(M)$, which ensures that $\pi_s(M_0^B(M)) \subseteq K^+_{B,s}$ for every $s \in B$.

Case $B$ is a horizontal band and not a birth quadrant ($c^+ \neq \emptyset$). Then we will enforce $\pi_s(M_0^B(M)) \subseteq \Im c^+, s \cap \Ker c^+, s$ for every $s \in B$, which will imply that $\rho^t_s(\pi_s(M_0^B(M))) \subseteq \rho^t_s(\Ker c^+, s) = 0$ for every $t \geq s$ with $t \notin B$. Let then $K^+_{B,s} = \Im c^+, s \cap \Ker c^+, s$ for each $s \in B$. We have

$$V_B^+ = \Im c^+, s \cap \Ker c^+, s = \Im c^+, s \cap \Im c^+, s \cap (\Ker c^+, s + \Ker c^+, s).$$

Since $\Ker c^+, s \subseteq \Im c^+, s$ and $\Ker c^+, s \subseteq \Ker c^+, s \subseteq \Im c^+, s$ by Lemma 3.6, we get

$$V_B^+ = \Im c^+, s \cap \Ker c^+, s + \Im c^+, s \cap \Ker c^+, s = \Im c^+, s \cap \Ker c^+, s + K^+_{B,s}.$$ 

Meanwhile, we have

$$V_B^- = \Im c^+, s \cap \Ker c^+, s + \Im c^-, s \cap \Ker c^+, s$$

$$= \Im c^+, s \cap \Im c^+, s \cap (\Ker c^+, s + \Ker c^+, s) + \Im c^-, s \cap \Ker c^+, s$$

$$= \Im c^+, s \cap \Ker c^+, s + \Im c^+, s \cap \Ker c^+, s + \Im c^-, s \cap \Ker c^+, s \supseteq \Im c^+, s \cap \Ker c^+, s.$$

Hence, $V_B^+ + K^+_{B,s} = V_B^+$. By the same argument as in the previous case, we deduce that the limits satisfy $V_B^+(M) + K_B^+(M) = V_B^+(M)$. We can then choose our vector space complement $M_0^B(M)$ inside $K_B^+(M)$, which ensures that $\pi_s(M_0^B(M)) \subseteq K^+_{B,s}$ for every $s \in B$.

Case $B$ is a vertical band and not a birth quadrant ($c^+ \neq \emptyset$). This case is symmetric to the previous one.\hfill \Box

\begin{remark}
Since we have chosen $M_0^B(M)$ such that $V_B^+(M) = V_B^-(M) \oplus M_0^B(M)$, for every $t \in B$ we have $M_0^B(M) \cong (M_B)_t$ and $V_B^+ = V_B^- \oplus (M_B)_t$, by Lemma 5.2.

Assuming that $M$ is pfd and decomposes as a direct sum of block modules (hence exact), we have by construction and Lemma 4.2 that $\dim M_0^B(M) = \dim C_B(M)$ is equal to the multiplicity of the summand $k_B$ in the decomposition of $M$. More generally:

\begin{lemma}
Assume that $M$ is pfd and exact. Then, for every block $B$, $M_B$ is isomorphic to the direct sum of $\dim C_B(M)$ copies of the block module $k_B$.
\end{lemma}

\begin{proof}
For every $t \in B$, the restriction of $\pi_t$ to $M_0^B(M)$ is an isomorphism onto $(M_B)_t$, by Lemma 5.2. Take a (finite) basis $\Gamma$ of $M_0^B(M)$. For each $\gamma \in \Gamma$, the elements $\pi_t(\gamma)$ for $t \in B$ are non-zero and they satisfy $\rho^t_s(\pi_t(\gamma)) = \pi_t(\gamma)$ for all $s \leq t \in B$, so they span a submodule $N(\gamma)$ of $M_B$ that is isomorphic to $k_B$. Now, for all $t \in B$ the family $\{\pi_t(\gamma)\}_{\gamma \in \Gamma}$ is a basis of $(M_B)_t$, so $M_B = \bigoplus_{\gamma \in \Gamma} N(\gamma)$. Finally, the size of the basis $\Gamma$ is $\dim M_0^B(M) = \dim C_B(M)$.\hfill \Box
\end{proof}
6 Sections and Direct Sum

In this section we assume that $M$ is pfd and exact, and we show that the submodules $M_B$ introduced in Proposition 5.3 are in direct sum. For this we introduce the so-called sections associated to these submodules. A section in a vector space $U$ is a pair $(F^-, F^+)$ of subspaces such that $F^- \subseteq F^+ \subseteq U$. We say that a family of sections $\{(F^\lambda_-, F^\lambda_+)\}_{\lambda \in \Lambda}$ in $U$ is disjoint if for all $\lambda \neq \mu$, either $F^\lambda_+ \subseteq F^-\mu$ or $F^\mu_+ \subseteq F^-\lambda$. Disjointness is a useful concept for proving direct sums thanks to the following result:

**Lemma 6.1** [13] Suppose that $\{(F^\lambda_-, F^\lambda_+)\}_{\lambda \in \Lambda}$ is a disjoint family of sections in $U$. For each $\lambda \in \Lambda$, let $M_\lambda$ be a subspace with $F^+_\lambda = M_\lambda \oplus F^-\lambda$. Then, the family of spaces $\{M_\lambda\}_{\lambda \in \Lambda}$ is in direct sum.

In our context, it turns out that the individual families of image and kernel subspaces induced by horizontal or vertical cuts are disjoint:

**Lemma 6.2** [13] Given a fixed $t \in \mathbb{R}^2$, each of the families $\{(\text{Im}^-_c, \text{Im}^+_c, t), \text{Ker}^-_c, \text{Ker}^+_c, t)\}_{c \ni tx}$, $\{(\text{Im}^-_c, \text{Im}^+_c, t), \text{Ker}^-_c, \text{Ker}^+_c, t)\}_{c \ni ty}$, and $\{(\text{Ker}^-_c, \text{Ker}^+_c, t), \text{Im}^-_c, \text{Im}^+_c, t)\}_{c \ni ty}$ is disjoint in $M_t$.

Moreover, disjoint families can be combined in certain ways to get new disjoint families, for instance:

**Lemma 6.3** [13] If $\mathcal{F} = \{(F^\lambda_-, F^\lambda_+)\}_{\lambda \in \Lambda}$ and $\mathcal{G} = \{G^-_\sigma, G^+_\sigma\}_{\sigma \in \Sigma}$ are two families of sections in $U$ such that $\mathcal{F}$ is disjoint, then the family

$$\{(F^\lambda_- + G^-_\sigma \cap F^\lambda_+ + G^+_\sigma \cap F^\lambda_+)\}_{(\lambda, \sigma) \in \Lambda \times \Sigma}$$

is disjoint.

However, unlike in the 1-d setting [13], these results do not suffice to conclude on the direct sum of the submodules $M_B$ in our 2-d setting, because the full family of sections $\mathcal{V}_t = \{(V^-_{B,t}, V^+_{B,t})\}_{B:\text{block} \ni t}$ is not disjoint.

**Example 6.4** Take for $M$ the direct sum of the modules associated with two birth quadrants $B_1$ and $B_2$ whose lower-left corners are not comparable in the product order.
on \( \mathbb{R}^2 \) (see Fig. 5 for an illustration). Take \( t \) in the intersection of the two quadrants. Call \( \beta_1 \) a generator of the 1-dimensional subspace of \( M_B \) spanned by \( B_1 \), and call \( \beta_2 \) a counterpart for \( B_2 \). Then

\[
V_{B_1}^+ = \langle \beta_1 \rangle, \quad V_{B_2}^+ = \langle \beta_2 \rangle, \quad V_{B_1}^- = V_{B_2}^- = 0.
\]

As a result, we have neither \( V_{B_1}^+ \subseteq V_{B_2}^- \) nor \( V_{B_2}^+ \subseteq V_{B_1}^- \), which means that the family \( \{(V_{B_1}^-, V_{B_2}^+)\}_{B: \text{block} \ni t} \) is not disjoint in this example.

This lack of global disjointness calls for a special treatment to establish the direct sum in our 2-d setting. For this we will combine the above results with some new (more direct) arguments. To start with, we consider a slightly different family, called \( \mathcal{F}_t = \{(F_{B,t}^-, F_{B,t}^+)\}_{B: \text{block} \ni t} \), where the spaces \( F_{B,t}^\pm \) are defined as follows:

\[
F_{B,t}^+ = \operatorname{Im} - B_{t} + V_{B,t}^+, \quad F_{B,t}^- = \operatorname{Ker} + B_{t} \cap \operatorname{Im} + B_{t}.
\] (10)

As the following lemma shows, we can work indifferently with \( \mathcal{V}_t \) or \( \mathcal{F}_t \) to study the spaces \((M_B)_t\):

**Lemma 6.5** \( F_{B,t}^+ = F_{B,t}^- \oplus (M_B)_t \) for any block \( B \) and any \( t \in B \).

**Proof** From (10) and Remark 5.4 we deduce

\[
F_{B,t}^+ = V_{B,t}^+ + \operatorname{Im} - B_{t} = V_{B,t}^- + (M_B)_t + \operatorname{Im} - B_{t} = F_{B,t}^- + (M_B)_t.
\]

Meanwhile, we have \((M_B)_t \subseteq V_{B,t}^+\) therefore

\[
F_{B,t}^- \cap (M_B)_t = F_{B,t}^- \cap V_{B,t}^+ \cap (M_B)_t = V_{B,t}^- \cap (M_B)_t = 0.
\]

The result follows. \( \square \)

The reason for choosing \( \mathcal{F}_t \) over \( \mathcal{V}_t \) in our context (as in the 1-d setting) is that it is somewhat easier to work with.

The proof that the family of submodules \( \{M_B\}_{B: \text{block}} \) is in direct sum is divided into two parts: first, we show that, for each individual block type, the associated subfamily is in direct sum (Proposition 6.6); second, we show that the sum is also direct across block types (Proposition 6.7).

**Proposition 6.6** For any fixed block type, the submodules \( M_B \), where \( B \) ranges over the blocks of this type, are in direct sum.

**Proof** Submodules are in direct sum if and only if they are such pointwise. Let then \( t \in \mathbb{R}^2 \) be fixed. We focus on each block type individually:

**Horizontal bands** (including ones that extend to infinity vertically, either upwards or downwards or both). Let \( \omega \) denote the trivial horizontal cut with \( \omega^- = \emptyset \). By
Lemma 6.2, the family \( \{(\text{Im}_{c,t}^-, \text{Im}_{c,t}^+)\}_{c \in \mathcal{F}_t} \) is disjoint. It follows, by intersecting all the spaces of this family with \( \text{Im}_{c,t}^+ \), that \( \{(\text{Im}_{c,t}^- \cap \text{Im}_{c,t}^+, \text{Im}_{c,t}^+ \cap \text{Im}_{c,t}^-)\}_{c \in \mathcal{F}_t} \) is also disjoint. By definition this is the same family as \( \{(\text{Im}_{B,i}^-, \text{Im}_{B,i}^+)\}_{B:\text{hand} \in \mathcal{F}_t} \). Then, by Lemma 6.3 the family \( \{(F_{B,i}^-, F_{B,i}^+)\}_{B:\text{hand} \in \mathcal{F}_t} \) itself is disjoint. Hence, by Lemma 6.1 the family of subspaces \( \{(M_B)_t\}_{B:\text{hand} \in \mathcal{F}_t} \) is in direct sum.

**Vertical bands** (including ones that extend to infinity horizontally, either to the left or to the right or both). The treatment is symmetric.

**Death quadrants** (including ones that extend to infinity upwards or to the right or both). Take any finite family of distinct death quadrants \( B_1, \ldots, B_n \) that contain \( t \). Because they are all distinct, there must be one of them (say \( B_1 \)) that is not included in the union of the others. Hence, there is some \( u \geq t \) such that \( u \in B_1 \setminus \bigcup_{i \geq 1} B_i \). Now, suppose there is some relation \( \sum_{i=1}^n \alpha_i = 0 \) with \( \alpha_i \in (M_B)_t \), non-zero for all \( i \). Then, by linearity of \( \rho_i^u \) we have \( \sum_{i=1}^n \rho_i^u(\alpha_i) = 0 \). But each \( \alpha_i \) with \( i > 1 \) is sent to zero through \( \rho_i^u \) because \( u \) lies outside \( B_i \). Hence, \( \rho_i^u(\alpha_1) = -\sum_{i=2}^n \rho_i^u(\alpha_i) = 0 \).

Meanwhile, we have \( \rho_i^u(\alpha_1) \neq 0 \) because the restriction of \( \rho_i^u \) to \( (M_B)_t \) is injective by Lemma 5.5. This raises a contradiction.

**Birth quadrants** (including ones that extend to infinity downwards or to the left or both). All we need to prove is that, for any finite family of distinct birth quadrants \( B_1, \ldots, B_n \), there is at least one of them (say \( B_1 \)) whose corresponding subspace \( (M_B)_t \subset M_t \) is in direct sum with the ones of the other quadrants in the family. The result follows then from a simple induction on the size \( n \) of the family.

Let then \( B_1, \ldots, B_n \) be such a family. Each quadrant \( B_i \) is bounded to the left by a horizontal cut \( \xi_i \) and to the bottom by a vertical cut \( \xi_i \). Up to reordering, we can assume that \( B_1 \) has the rightmost horizontal cut and, in case of ties, it also has the uppermost vertical cut among the quadrants with the same horizontal cut. Formally

\[
\xi_1^+ \subseteq \bigcap_{i=2}^n \xi_i^+, \quad \xi_1^- \subseteq \bigcap_{i>1} \xi_i^-.
\]

It follows that \( B_1 \) contains none of the other quadrants. Those can be partitioned into two subfamilies: the ones (say \( B_2, \ldots, B_k \)) contain \( B_1 \) strictly, while the others \( (B_{k+1}, \ldots, B_n) \) neither contain \( B_1 \) nor are contained in \( B_1 \). See Fig. 6 for an illustration. We analyze the two subfamilies separately.

For every \( i \in (1, k) \), we have both \( \xi_i^+ \supseteq \xi_1^+ \) and \( \xi_i^+ \supseteq \xi_1^+ \), moreover we have either \( \xi_i^+ \supseteq \xi_1^+ \) or \( \xi_i^+ \supseteq \xi_1^- \) or both. It follows that \( \text{Im}_{\xi_i,t}^+ \subseteq \text{Im}_{\xi_1,t}^+ \) and \( \text{Im}_{\xi_i,t}^- \subseteq \text{Im}_{\xi_1,t}^- \), moreover either \( \text{Im}_{\xi_i,t}^+ \subseteq \text{Im}_{\xi_1,t}^+ \) or \( \text{Im}_{\xi_i,t}^- \subseteq \text{Im}_{\xi_1,t}^- \) or both. Hence,

\[
\text{Im}_{B_1,t}^+ = \text{Im}_{B_1,t}^+ \cap \text{Im}_{\xi_i,t}^+ \subseteq \text{Im}_{\xi_1,t}^+ \cap \text{Im}_{\xi_i,t}^+ + \text{Im}_{\xi_1,t}^- \cap \text{Im}_{\xi_i,t}^- = \text{Im}_{B_1,t}^-.
\]
Summing over \(i = 2, \ldots, k\) we obtain
\[
\sum_{i=2}^{k} \text{Im}^+_{B_i, t} \subseteq \text{Im}^-_{B_1, t}.
\] (11)

For every \(i \in (k, n]\), we have \(\xi_i^+ \supseteq \xi_1^+\) and \(\xi_i^+ \subsetneq \xi_1^+\). Let \(\vec{B} = \bigcap_{i=k+1}^{n} B_i\) — this birth quadrant neither contains \(B_1\) nor is contained in it. Let now \(B\) be the smallest quadrant containing both \(B_1\) and \(\vec{B}\) — this quadrant strictly contains them both. It follows, using the same argument as in the case \(i \in (1, k]\):
\[
\text{Im}^+_{B_1, t} \cap \left( \sum_{i=k+1}^{n} \text{Im}^+_{B_i, t} \right) \subseteq \text{Im}^+_{B_1, t} \cap \text{Im}^+_{B, t} = \text{Im}^+_{B, t} \subseteq \text{Im}^-_{B_1, t}.
\] (12)

Combining (11) and (12), we obtain
\[
(M_{B_1})_t \cap \left( \sum_{i=2}^{k} (M_{B_i})_t + \sum_{i=k+1}^{n} (M_{B_i})_t \right) \subseteq \text{Im}^+_{B_1, t} \cap \left( \sum_{i=2}^{k} \text{Im}^+_{B_i, t} + \sum_{i=k+1}^{n} \text{Im}^+_{B_i, t} \right)
\]
\[
= \sum_{i=2}^{k} \text{Im}^+_{B_i, t} + \text{Im}^+_{B_1, t} \cap \left( \sum_{i=k+1}^{n} \text{Im}^+_{B_i, t} \right)
\]
\[
\subseteq \text{Im}^-_{B_1, t} \subseteq F_{B_1, t},
\]
which by Lemma 6.5 is itself in direct sum with \((M_{B_1})_t\). Hence the result.

To establish the direct sum across block types, we adopt the following convention regarding blocks that belong to more than one type:

- All the blocks whose support extends to infinity both upwards and to the right are assigned to the birth quadrants.
- Among the remaining blocks, the ones whose support extends to infinity upwards are assigned to the vertical bands, while the ones whose support extends to infinity to the right are assigned to the horizontal bands.
Proposition 6.7 Under the previous convention, the submodules \( \bigoplus_{B:bquad} M_B \), \( \bigoplus_{B:vband} M_B \) and \( \bigoplus_{B:hband} M_B \) are in direct sum.

Proof Again, we only need to prove the direct sum pointwise. Let then \( t \in \mathbb{R}^2 \) be fixed. We order the block types as follows: birth quadrants, vertical bands, horizontal bands, death quadrants. We will prove that the summands of each block type are in direct sum with the summands of the following block types in the sequence.

Birth quadrants. Suppose that

\[
\left( \bigoplus_{B:bquad \ni t} (M_B)_t \right) \cap \left( \bigoplus_{B:vband \ni t} (M_B)_t + \bigoplus_{B:hband \ni t} (M_B)_t + \bigoplus_{B:dquad \ni t} (M_B)_t \right) \neq 0,
\]

where by our convention we treat all the blocks extending to infinity both upwards and to the right as birth quadrants. Take then a non-zero vector \( \alpha \) in the intersection. It can be written as a linear combination of non-zero vectors \( \alpha_1, \ldots, \alpha_n \) taken from the summands of finitely many birth quadrants \( B_1, \ldots, B_n \), but also as a linear combination of non-zero vectors \( \beta_1, \ldots, \beta_m \) taken from the summands of finitely many blocks \( B'_1, \ldots, B'_m \) of other types:

\[
\sum_{i=1}^n \alpha_i = \alpha = \sum_{j=1}^m \beta_j.
\]

Pick a point \( u \geq t \in \mathbb{R}^2 \) that is large enough so that it lies outside the blocks \( B'_1, \ldots, B'_m \). Such a point \( u \) exists because, by our convention, none of the blocks \( B'_1, \ldots, B'_m \) extends to infinity both upwards and to the right. Meanwhile, \( u \) still lies in the birth quadrants \( B_1, \ldots, B_n \). Let us then consider the image of \( \alpha \) in \( M_u \) through the map \( \rho^u_t \). On the one hand it is zero since \( \rho^u_t (\beta_j) = 0 \) for all \( j \). On the other hand it is non-zero since the restriction of \( \rho^u_t \) to \( \bigoplus_{i=1}^n (M_{B_i})_t \) is injective by Lemma 5.5 and Proposition 6.6. This raises a contradiction.

Vertical bands. Suppose that

\[
\left( \bigoplus_{B:vband \ni t} (M_B)_t \right) \cap \left( \bigoplus_{B:hband \ni t} (M_B)_t + \bigoplus_{B:dquad \ni t} (M_B)_t \right) \neq 0,
\]

where by our convention we treat the blocks extending to infinity upwards as vertical bands. Then we can reproduce the previous reasoning: take a non-zero vector \( \alpha \) in the intersection, and decompose it as a sum of finitely many non-zero vectors taken from the summands of vertical bands on the one hand, as a sum of finitely many non-zero vectors taken from the summands of horizontal bands or death quadrants on the other hand. Pick then a point \( u \geq t \) with \( u_x = t_x \) and with \( u_y \) large enough so that \( u \) lies outside all the horizontal bands and death quadrants involved in the decomposition of \( \alpha \). By looking at the image \( \rho^u_t (\alpha) \in M_u \) we can raise the same contradiction as before.

Horizontal bands. They are treated symmetrically to the vertical bands. \(\square\)

---

6 The horizontal bands extending to infinity upwards have already been treated as birth quadrants.
7 Sections and Covering

In this section and the next we still assume that $M$ is pfd and exact, and we want to show that the submodules $M_B$ introduced in Proposition 5.3, for $B$ ranging over all blocks, cover the whole module $M$, which will conclude the proof of Theorem 2.1. For this we work again with the family of sections $\mathcal{F}_t = \{(F^-_{B,t}, F^+_{B,t})\}_{B: \text{block}\not\exists t}$ defined in (10), and we use the concept of a covering family, borrowed from [13]. Given a vector space $U$, $\{(F^-_\lambda, F^+_{\lambda})\}_{\lambda \in \Lambda}$ covers $U$ if for every proper subspace $X \subsetneq U$ there is a $\lambda \in \Lambda$ such that

$$X + F^-_{\lambda} \neq X + F^+_{\lambda}.$$ 

We say this family strongly covers $U$ if for all subspaces $X \subsetneq U$ and $Z \subsetneq X$ there is a $\lambda \in \Lambda$ such that

$$X + (F^-_{\lambda} \cap Z) \neq X + (F^+_{\lambda} \cap Z).$$

The use of covering sections is justified by the following result from [13]:

**Lemma 7.1** Suppose that $\{(F^-_{\lambda}, F^+_{\lambda})\}_{\lambda \in \Lambda}$ is a family of sections that covers $U$. For each $\lambda \in \Lambda$, let $M_\lambda$ be a subspace with $F^+_{\lambda} = M_\lambda \oplus F^-_{\lambda}$. Then, $U = \sum_{\lambda \in \Lambda} M_\lambda$.

Given a fixed index $t$, the proof in the 1-d case [13] proceeds by showing that the families of kernels and of images, namely $\{(\text{Ker}^-_{I,t}, \text{Ker}^+_{I,t})\}_{I: \text{interval}\not\exists t}$ and $\{(\text{Im}^−_{I,t}, \text{Im}^+_{I,t})\}_{I: \text{interval}\not\exists t}$, both strongly cover $M_t$. It then applies the following result to deduce that the family $\{(F^-_{I,t}, F^+_{I,t})\}_{I: \text{interval}\not\exists t}$ covers $M_t$:

**Lemma 7.2** If $\{(F^-_{\lambda}, F^+_{\lambda})\}_{\lambda \in \Lambda}$ is a family of sections that covers $U$, and $\{G_{\sigma}, G^+_{\sigma}\}_{\sigma \in \Sigma}$ is a family of sections that covers $U$, then the following family covers $U$:

$$\{(F^-_{\lambda} + G^-_{\sigma} \cap F^+_{\lambda}, F^+_{\lambda} + G^+_{\sigma} \cap F^-_{\lambda})\}_{(\lambda, \sigma) \in \Lambda \times \Sigma}.$$ 

From there the proof concludes using Lemma 7.1. Unfortunately, this strategy does not work in our 2-d setting, where the family of images $\{(\text{Im}^-_{B,t}, \text{Im}^+_{B,t})\}_{B: \text{block}\not\exists t}$ may not strongly cover $M_t$, while the family of kernels $\{(\text{Ker}^-_{B,t}, \text{Ker}^+_{B,t})\}_{B: \text{block}\not\exists t}$ may not even cover it, as shown in the following two examples:

**Example 7.3** Consider the module $M$ from Example 6.4. Take $X = 0$, and for $Z$ take the linear span of $\beta_1 + \beta_2$ in $M_t$. Since $Z$ is 1-dimensional, for any block $B$ such that the space $\text{Im}^+_{B,t} \cap Z \neq 0$, $\text{Im}^+_{B,t}$ must contain at least $Z$, therefore $B$ must be included in $B_1 \cap B_2$. But then $\text{Im}^-_{B,t} = \text{Im}^+_{B,t} = M_t$, which means that $X + \text{Im}^-_{B,t} \cap Z = Z = X + \text{Im}^+_{B,t} \cap Z$. Hence, the family $\{(\text{Im}^-_{B,t}, \text{Im}^+_{B,t})\}_{B: \text{block}\not\exists t}$ does not strongly cover $M_t$ in this example.

**Example 7.4** Take for $M$ the direct sum of the modules associated with two death quadrants $B_1$ and $B_2$ whose upper-right corners are not comparable in the product...
order on $\mathbb{R}^2$ (see Fig. 7 for an illustration). Let $t \in B_1 \cap B_2$, and call $\beta_1$ (resp. $\beta_2$) a generator of the 1-dimensional subspace of $M_t$ spanned by $B_1$ (resp. by $B_2$). Take $X$ to be the linear span of $\beta_1 + \beta_2$. Then, for any choice of block $B = (c^+ \cap c^-) \times (c^+ \cap c^-)$ containing $t$, we have $\text{Ker}_{c,t}^+ \in \{0, \langle \beta_1 \rangle, M_t \}$ while $\text{Ker}_{c,t}^+ \in \{0, \langle \beta_2 \rangle, M_t \}$. This implies that $\text{Ker}_{c,t}^+ \cup \text{Ker}_{c,t}^- \neq 0$ (and thus that $\text{Ker}_{B,t}^+$ is potentially different from $\text{Ker}_{B,t}^-$) only when $\text{Ker}_{c,t}^+ = M_t$ or $\text{Ker}_{c,t}^- = M_t$. But then we have $\text{Ker}_{B,t}^+ \in \{\langle \beta_1 \rangle, \langle \beta_2 \rangle, M_t \}$, which implies that $X + \text{Ker}_{B,t}^- = M_t = X + \text{Ker}_{B,t}^+$. Hence, the family $\{\langle \text{Ker}_{B,t}^-, \text{Ker}_{B,t}^+ \rangle\}_{B: \text{block} \ni t}$ does not cover $M_t$ in this example.

The lack of covering by kernels is particularly troublesome in our setting. Since it impacts essentially the way we analyze death quadrants, for now we will leave the contribution of the death quadrants aside and focus on the rest of the blocks. For this we introduce the following submodules $L_t, N_t$ of $M$, defined by

$$L_t = F_{\mathbb{R}^2,t}^+ = V_{\mathbb{R}^2,t}^+ = \text{Im}_{\mathbb{R}^2,t}^+, \quad N_t = F_{\mathbb{R}^2,t}^- = V_{\mathbb{R}^2,t}^- = \text{Im}_{\mathbb{R}^2,t}^- \cap \text{Ker}_{\mathbb{R}^2,t}^- \subseteq L_t.$$ 

Intuitively, the submodule $N_t$ should be the one spanned by the block modules associated to death quadrants that are not $\mathbb{R}^2$ itself. Here we prove the following result, deferring the analysis of $N$ to the next section.

**Proposition 7.5** For $B$ ranging over the birth quadrants (including $\mathbb{R}^2$) and the strict bands (i.e., bands that are not quadrants, noted sband for short), we have

$$M = N + \bigoplus_{B: \text{bquad or sband}} M_B.$$ 

To prove this result we use the strong covering property of image and kernel sections in the 1-d setting:

**Lemma 7.6** [13] Given a fixed $t \in \mathbb{R}^2$, for any subsets $X \subseteq M_t$ and $Z \not\subseteq X$, there is a horizontal cut $c$ with $t_x \in c^+$ such that $\text{Im}_{c,t}^- \cap Z \subseteq X \not\supseteq \text{Im}_{c,t}^- \cap Z$. Similarly, there is a vertical cut $c$ with $t_y \in c^+$ such that $\text{Im}_{c,t}^- \cap Z \subseteq X \not\supseteq \text{Im}_{c,t}^- \cap Z$. Same for kernels.
The proof of the 1-d analogue of Proposition 7.5 (see Lemma 6.1 in [13]) proceeds by contradiction: assuming there is an index $t$ such that $N_t + \sum_B(M_B)_t \subsetneq M_t$, it applies Lemma 7.6 to exhibit some interval $B'$ (the 1-d analogue of a block) such that $F_{B',t}^- \subsetneq N_t + \sum_B(M_B)_t \not\supsetneq F_{B',t}^+$, then it deduces by Lemma 6.5 that $(M_{B'})_t \not\subsetneq N_t + \sum_B(M_B)_t$, a contradiction. Here we follow the same approach, applying Lemma 7.6 in each dimension sequentially, then combining the results using the exactness of $M$ (via Lemma 3.6) in order to exhibit a block $B'$ that yields the same contradiction.

Proof of Proposition 7.5 Given a fixed $t \in \mathbb{R}^2$, let $X = N_t + \sum_B(M_B)_t$, where $B$ ranges over those birth quadrants (including $\mathbb{R}^2$) and strict bands that contain $t$. Suppose for a contradiction that $X \subsetneq M_t$. Then, apply Lemma 7.6 with $Z = M_t$ to get a horizontal cut $\mathcal{c}$ such that $t_x \in \mathcal{c}^+$ and $\text{Im}_{\mathcal{c},t}^- \subsetneq X \not\supsetneq \text{Im}_{\mathcal{c},t}^+$. Apply then Lemma 7.6 again, this time with $Z = \text{Im}_{\mathcal{c},t}^+$, to get a vertical cut $\mathcal{z}$ such that $t_y \in \mathcal{z}^+$ and $\text{Im}_{\mathcal{z},t}^- \cap \text{Im}_{\mathcal{c},t}^+ \subsetneq X \not\supsetneq \text{Im}_{\mathcal{z},t}^+ \cap \text{Im}_{\mathcal{c},t}^+$. Note that the cuts $\mathcal{c}, \mathcal{z}$ cannot be both trivial, as otherwise (for $\mathcal{c}^- = \emptyset = \mathcal{z}^-$) we would have

$$\text{Im}_{\mathcal{z},t}^+ \cap \text{Im}_{\mathcal{c},t}^+ = \text{Im}_{\mathbb{R}^2,t}^+ = F_{\mathcal{z},t}^+ = F_{\mathcal{c},t}^- + (M_{\mathbb{R}^2})_t = N_t + (M_{\mathbb{R}^2})_t \subsetneq X,$$

which contradicts what precedes. Hence, either $\mathcal{c}^- \neq \emptyset$, or $\mathcal{z}^- \neq \emptyset$, or both. We distinguish these three cases below:

**Case $\mathcal{c}^- \neq \emptyset$ and $\mathcal{z}^- \neq \emptyset$.** Let $B'$ be the birth quadrant $\mathcal{c}^+ \cap \mathcal{z}^+$. Then, we have $\text{Im}_{B',t}^+ = \text{Im}_{\mathcal{c},t}^+ \cap \text{Im}_{\mathcal{z},t}^+$, and by Lemma 3.6 we also have $\text{Ker}_{B',t}^- \subsetneq \text{Im}_{\mathcal{c},t}^- + \text{Im}_{\mathcal{z},t}^-$, which gives

$$F_{B',t}^+ = \text{Im}_{\mathcal{c},t}^- + \text{Im}_{B',t}^+ \cap M_t = \text{Im}_{B',t}^+ = \text{Im}_{\mathcal{c},t}^+ \cap \text{Im}_{\mathcal{z},t}^+ \not\subsetneq X,$$

$$F_{B',t}^- \subsetneq \text{Im}_{\mathcal{c},t}^- + \text{Im}_{B',t}^+ \cap (\text{Im}_{\mathcal{c},t}^+ + \text{Im}_{\mathcal{z},t}^-) = \text{Im}_{B',t}^- \subsetneq \text{Im}_{\mathcal{c},t}^- \cap \text{Im}_{\mathcal{z},t}^- + \text{Im}_{\mathcal{z},t}^+ \subsetneq X.$$

Hence, by Lemma 6.5 we have $(M_{B'})_t \not\subsetneq X$, which contradicts the definition of $X$.

**Case $\mathcal{c}^- = \emptyset$ and $\mathcal{z}^- \neq \emptyset$.** By Lemma 7.6, applied with $Z = \text{Im}_{\mathcal{c},t}^+ \cap \text{Im}_{\mathcal{z},t}^+$, there is a vertical cut $\mathcal{z}$ such that $t \in \mathcal{z}^-$ and

$$\text{Im}_{\mathcal{c},t}^- \cap \text{Im}_{\mathcal{z},t}^- \cap \text{Ker}_{\mathcal{z},t}^- \subsetneq X \not\supsetneq \text{Im}_{\mathcal{c},t}^+ \cap \text{Im}_{\mathcal{z},t}^- \cap \text{Ker}_{\mathcal{z},t}^+.$$

Let $B'$ be the horizontal band $\mathbb{R} \times (\mathcal{z}^+ \cap \mathcal{z}^-)$. Lemma 3.6 tells us that $\text{Ker}_{\mathcal{z},t}^- \subsetneq \text{Im}_{\mathcal{c},t}^-$ and $\text{Ker}_{\mathcal{z},t}^- \subsetneq \text{Im}_{\mathcal{z},t}^-$, where $c_1$ is the trivial cut with $c_1^+ = \emptyset$. Then

$$\text{Im}_{B',t}^- = \text{Im}_{\mathcal{z},t}^- \cap \text{Im}_{\mathcal{z},t}^- \subsetneq X,$$

$$\text{Im}_{B',t}^+ \cap \text{Ker}_{B',t}^- = \text{Im}_{\mathcal{c},t}^+ \cap \text{Im}_{\mathcal{z},t}^- \cap (\text{Ker}_{\mathcal{z},t}^- + \text{Ker}_{\mathcal{z},t}^-) \subsetneq \text{Im}_{\mathcal{c},t}^+ \cap \text{Im}_{\mathcal{z},t}^- \cap (\text{Ker}_{\mathcal{z},t}^- + \text{Im}_{\mathcal{z},t}^-)$$

$$= \text{Im}_{\mathcal{c},t}^+ \cap \text{Im}_{\mathcal{z},t}^- \cap \text{Ker}_{\mathcal{z},t}^- + \text{Im}_{\mathcal{z},t}^- \cap \text{Im}_{\mathcal{z},t}^- \subsetneq X,$$

$$\text{Im}_{B',t}^+ \cap \text{Ker}_{B',t}^+ \supsetneq \text{Im}_{\mathcal{c},t}^+ \cap \text{Im}_{\mathcal{z},t}^- \cap \text{Ker}_{\mathcal{z},t}^+ \not\subsetneq X.$$
Hence, $F_{B^t}^- \subseteq X \neq F_{B^t}^+$, which by Lemma 6.5 implies that $(M_{B^t})_t \nsubseteq X$, thus contradicting the definition of $X$.

Case $c^- \neq \emptyset$ and $c^- = \emptyset$. This case is symmetric to the previous one and therefore raises the same contradiction. \hfill \Box

### 8 Completion of the Proof

We will now show that the sum in (13) is direct (Corollary 8.3) and that $N$ itself decomposes as a direct sum of block modules (Corollary 8.6), which will conclude the proof of Theorem 2.1. For completeness, we will also show that the submodules $M_B$ provide an internal direct-sum decomposition of $M$ (Corollary 8.7).

The proof that the sum in (13) is direct proceeds in two steps, given by the following two lemmas.

**Lemma 8.1** The submodules $\left( N + \bigoplus_{B: \text{band}} M_B \right)$ and $\bigoplus_{B: \text{bquad}} M_B$ are in direct sum.

**Proof** Suppose the contrary, and let $t \in \mathbb{R}^2$ such that the intersection between the two submodules is non-trivial. Then there are $\alpha \in N_t$, $\alpha_1 \in (M_{B_1})_t$, $\ldots$, $\alpha_r \in (M_{B_r})_t$ and $\alpha_{r+1} \in (M_{B_{r+1}})_t$, $\ldots$, $\alpha_n \in (M_{B_n})_t$ such that $B_1, \ldots, B_r$ are strict bands, $B_{r+1}, \ldots, B_n$ are birth quadrants, and we have

$$\alpha + \sum_{i=1}^r \alpha_i = \sum_{i=r+1}^n \alpha_i \neq 0.$$ 

The strict bands being not birth quadrants, there is some $u \geq t$ such that $u \notin \bigcup_{i=1}^r B_i$. Moreover, since $\alpha \in N_t \subseteq \text{Ker}_{\mathbb{R}^2,t}$, we have $\alpha = \alpha_h + \alpha_v$ for some $\alpha_h \in \text{Ker}_{c_{\text{tl}}}$ and $\alpha_v \in \text{Ker}_{c_{\text{tl}}}$, where $c_{\text{tl}}, c_{\text{tl}}$ are the trivial cuts with $c_{\text{tl}} = \overline{c_{\text{tl}}} = \emptyset$. Then, by the Realization Lemma 3.1, there exist finite coordinates $x \geq t_x$ and $y \geq t_y$ such that $\alpha_h \in \text{Ker}_{\rho_{\nu}^{(x,t_y)}}$ and $\alpha_v \in \text{Ker}_{\rho_{\nu}^{(t_x,y)}}$. Letting $v$ be the point with coordinates $(\max\{u_x, x\}, \max\{u_y, y\})$, we obtain

$$\rho_{\nu}^v (\alpha + \sum_{i=1}^r \alpha_i) = 0.$$ 

But since $\rho_{\nu}^v$ restricted to $\bigoplus_{i=r+1}^n (M_{B_i})_t$ is injective (Lemma 5.5 and Proposition 6.6), we have $\rho_{\nu}^v (\sum_{i=r+1}^n \alpha_i) \neq 0$. This raises a contradiction. \hfill \Box

**Lemma 8.2** The submodules $N$ and $\bigoplus_{B: \text{band}} M_B$ are in direct sum.

**Proof** Suppose once again the contrary, and let $t \in \mathbb{R}^2$ be such that the two submodules have a non-trivial intersection. Then there are $\alpha \in N_t$, $\alpha_1 \in (M_{B_1})_t$, $\ldots$, $\alpha_n \in (M_{B_n})_t$.
such that $B_1, \ldots, B_n$ are strict bands and

$$\alpha = \sum_{i=1}^{n} \alpha_i \neq 0.$$ 

Assume without loss of generality that $B_1, \ldots, B_r$ are horizontal bands while $B_{r+1}, \ldots, B_n$ are vertical bands. Assume also (still without loss of generality) that none of the $\alpha_i$‘s are zero. Since the bands $B_{r+1}, \ldots, B_n$ are not birth quadrants, there is some point $u$ with $u_x \geq t_x$ and $u_y = t_y$ such that $u \notin \bigcup_{i=r+1}^{n} B_i$. Then, we have $\rho_i^u(\sum_{i=r+1}^{n} \alpha_i) = 0$, while $\rho_i^u(\sum_{i=1}^{r} \alpha_i) \neq 0$ since the restriction of $\rho_i^u$ to $\bigoplus_{i=1}^{r} (M_{B_i})_u$ is injective by Lemma 5.5 and Proposition 6.6. Thus,

$$\beta = \sum_{i=1}^{r} \beta_i \neq 0,$$  \hspace{1cm} (14)

where $\beta = \rho_i^u(\alpha) \in N_u$ and each $\beta_i = \rho_i^u(\alpha_i) \in (M_{B_i})_u$. Now, we have $N_u \subseteq \text{Im}_{\mathbb{R}_1^{op}, u}^+ = F_{\mathbb{R}_1^{op}, u}^+$. As we saw in the proof of Proposition 6.6 (case of horizontal bands), the family of sections $\{(F_{\mathbb{R}_1^{op}, u}^-, F_{\mathbb{R}_1^{op}, u}^+), (F_{B_1}^-, F_{B_1}^+), \ldots, (F_{B_n}^-, F_{B_n}^+)\}$ is disjoint. Since $(F_{\mathbb{R}_1^{op}, u}^-, F_{\mathbb{R}_1^{op}, u}^+)$ is the minimum element in this family, meaning that $F_{\mathbb{R}_1^{op}, u}^+ \subseteq F_{B_i}^-$ for every $i$, the family $\{(0, F_{\mathbb{R}_1^{op}, u}^+), (F_{B_1}^-, F_{B_1}^+), \ldots, (F_{B_n}^-, F_{B_n}^+)\}$ is also disjoint, which by Lemma 6.1 implies that $F_{\mathbb{R}_1^{op}, u}^+$ is in direct sum with $\bigoplus_{i=1}^{r} (M_{B_i})_u$. As a subspace of $F_{\mathbb{R}_1^{op}, u}^+$, $N_u$ itself is also in direct sum with $\bigoplus_{i=1}^{r} (M_{B_i})_u$, thereby contradicting (14). \hfill $\square$

It follows from Lemmas 8.1 and 8.2 that the sum in (13) is direct, that is:

**Corollary 8.3** $M = N \oplus \bigoplus_{B: \text{qbirth or sband}} M_B$.

Now we show that $N$ decomposes as a direct sum of block modules. For this we use vector-space duality. Let $N^*$ be the pointwise dual of $N$. Since duality is a contravariant functor, $N^*$ is a persistence module indexed over $(\mathbb{R}^{op})^2$, where $\mathbb{R}^{op}$ denotes the poset $\mathbb{R}$ with the opposite order $\geq$.

**Lemma 8.4** $N^*$ is pfd and exact.

**Proof** We first need to show that $N$ itself is pfd and exact. That it is pfd follows immediately from the fact that it is a submodule of the pfd module $M$. For exactness, let $s \leq t \in \mathbb{R}_1^+$ and take an element $\delta \in N_t$ that has preimages $\beta \in N_{(t_x, t_y)}$ and $\gamma \in N_{(s_x, t_y)}$. Then, by exactness of $M$, $\beta$ and $\gamma$ have a common preimage $\alpha \in M_s$. Let us show that $\alpha \in N_s$.

First of all, by the Transportation Corollary 3.5, we know that $\alpha \in (\rho_s^1)^{-1}(N_t) \subseteq (\rho_s^1)^{-1}(\text{Ker}_{\mathbb{R}_1^{op}, t}) = \text{Ker}_{\mathbb{R}_1^{op}, s}^-$). Meanwhile, since $\beta \in N_{(t_x, t_y)} \subseteq \text{Im}_{\mathbb{R}_1^{op}, (t_x, t_y)}^+$, for any $u \leq s$ with $u_x = s_x$, there is some preimage $\beta_u$ of $\beta$ in $M_{(t_x, u_y)}$, which, by exactness of $M$, implies that there is some common preimage $\alpha_u$ of $\alpha$ and $\beta_u$ in $M_u$. Thus,
\( \alpha \in \text{Im}_{\mathbb{C},s}^+ \), where \( \mathbb{C} \) is the trivial vertical cut with \( \mathbb{C}^- = \emptyset \). Symmetrically, and using \( \gamma \), we have \( \alpha \in \text{Im}_{\mathbb{C},s}^+ \) where \( \mathbb{C} \) is the trivial horizontal cut with \( \mathbb{C}^- = \emptyset \). Hence \( \alpha \) belongs also to \( \text{Im}_{\mathbb{R}^2,s}^+ \) and therefore to \( N_s \). This means that \( N \) is exact.

Now, duality sends finite-dimensional vector spaces to finite-dimensional vector spaces, hence \( N^* \) is pfd just as \( N \). And since duality is an additive and exact functor, for every \( s \leq t \) the exact sequence

\[
N_s \xrightarrow{\phi = (h_s(t_x, s_y), v_s(t_x, s_y))} N(t_x, s_y) \oplus N(s_x, t_y) \xrightarrow{\psi = v'(t_x, s_y) - h'(t_x, s_y)} N_t
\]

associated with the commutative diagram

![Diagram](https://via.placeholder.com/150)

turns into the exact sequence

\[
N_s^* \xleftarrow{\phi^* = h_s^*(t_x, s_y) + v_s^*(t_x, s_y)} N^*(t_x, s_y) \oplus N^*(s_x, t_y) \xrightarrow{\psi^* = v'^*(t_x, s_y) - h'^*(t_x, s_y)} N_t^*
\]

which means that the sequence

\[
N_s^* \xleftarrow{h_s^*(t_x, s_y) - v_s^*(t_x, s_y)} N^*(t_x, s_y) \oplus N^*(s_x, t_y) \xrightarrow{(v'^*(t_x, s_y) - h'^*(t_x, s_y))} N_t^*
\]

associated with the diagram

![Diagram](https://via.placeholder.com/150)

is also exact. Hence the result. \( \square \)
We can now apply Corollary 8.3 to $N^*$, and add the following observation:

**Lemma 8.5** For all $t \in (\mathbb{R}^{op})^2$, $\text{Im}^+_{(\mathbb{R}^{op})^2, t}(N^*) = 0$.

**Proof** Denote by $X^\perp$ the annihilator of any subspace $X \subseteq N_t$:

$$X^\perp = \{ \phi \in N_t^* \mid \phi(\alpha) = 0 \text{ for all } \alpha \in X \}.$$

Since taking the annihilator turns sums into intersections and kernels into images, we have

$$\text{Im}^+_{(\mathbb{R}^{op})^2, t}(N^*) = (\text{Ker}_{\mathbb{R}^2, t}(N))^\perp = N_t^\perp = 0.$$

\qed

It follows that $N^*$ decomposes as a direct sum of block modules indexed over $(\mathbb{R}^{op})^2$, and by duality.

**Corollary 8.6** $N$ decomposes as a direct sum of block modules indexed over $\mathbb{R}^2$.

Corollaries 8.3 and 8.6 conclude the proof of Theorem 2.1. Note that one can further show that $N$ has surjective internal morphisms, so that all the blocks involved in its decomposition are death quadrants. This is not mandatory for the proof of the main theorem, however it justifies our previous intuition that $N$ is the submodule of $M$ spanned by the block modules associated to death quadrants (except $\mathbb{R}^2$ itself since $N_t \subseteq \text{Ker}_{\mathbb{R}^2, t}$).

For completeness, we also show (a posteriori, using the Decomposition Theorem 2.1) that the submodules $M_B$, for $B$ ranging over all blocks, do give an internal direct-sum decomposition of $M$.

**Corollary 8.7** $M = \bigoplus_{B: \text{block}} M_B$.

**Proof** By Theorem 2.1 we know that $M$ decomposes as follows:

$$M \cong \bigoplus_{B: \text{block}} \mathbf{k}^{n_B}_B,$$

where $n_B \in \mathbb{N}$ denotes the multiplicity of the summand $\mathbf{k}_B$ in the decomposition. Then, for each $B$, Lemmas 4.2 and 5.5 ensure that $M_B \cong \mathbf{k}^{n_B}_B$, while Propositions 6.6 and 6.7 ensure that the $M_B$’s are in direct sum in $M$. Hence, at every index $t \in \mathbb{R}^2$ we have $M_t \cong \bigoplus_{B: \text{block}} (M_B)_t$, and therefore $M_t = \bigoplus_{B: \text{block}} (M_B)_t$, since $M_t$ is finite-dimensional.

\qed

### 9 Applications

To conclude the paper, we briefly discuss some of the implications of our decomposition theorem.
9.1 Barcodes and Stability for Exact pfd Bimodules

By Theorem 2.1, to any exact pfd persistence bimodule $M$ we can associate the multiset of blocks involved in its decomposition (3). This multiset is called the barcode of $M$ and denoted by $B(M)$. The following isometry result follows\(^\text{7}\) then from [4,7]:

**Corollary 9.1** For any exact pfd persistence bimodules $M$ and $N$ we have

$$d_I(M, N) = d_b(B(M), B(N)),$$

where $d_I$ denotes the interleaving distance as defined in [16], and where $d_b$ denotes the bottleneck distance as defined in [4,7].

9.2 Bimodules Indexed over a Rectangle or the Open Half-Plane Above the Anti-diagonal

We can easily adapt our main theorem to decompose bimodules that are indexed only over a rectangle in the plane, possibly extending to infinity. The proof is given in Appendix B.

**Theorem 9.2** Let $M$ be a bimodule indexed over some (axis-aligned) rectangle $R \subseteq \mathbb{R}^2$. If $M$ is pfd and exact, then it is block-decomposable, more precisely

$$M \simeq \bigoplus_{B \in \mathcal{B}(M)} k_{B \cap R}.$$

Moreover, the decomposition is unique up to isomorphism and reordering of the terms (note that two blocks $B$, $B'$ having the same intersection with $R$ yield the same indecomposable $k_{B \cap R} = k_{B' \cap R}$).

We can also adapt the decomposition theorem to decompose bimodules that are indexed over the open half-plane $\mathbb{U}$ above the anti-diagonal: $\mathbb{U} = \{ t \in \mathbb{R}^2 \mid t_x + t_y > 0 \}$. The proof is given in Appendix C; it follows the scheme of [6, Sect. 5] but does not make use of the general decomposition result in [6].

**Theorem 9.3** Let $M$ be a bimodule indexed over $\mathbb{U}$. If $M$ is pfd and exact, then it is block-decomposable, more precisely

$$M \simeq \bigoplus_{B \in \mathcal{B}(M)} k_{B \cap \mathbb{U}}.$$

Moreover, the decomposition is unique up to isomorphism and reordering of the terms (again, two blocks $B$, $B'$ having the same intersection with $\mathbb{U}$ yield the same indecomposable $k_{B \cap \mathbb{U}} = k_{B' \cap \mathbb{U}}$).

---

\(^7\) Strictly speaking, the result in [4,7] is stated for bimodules indexed over the open half-plane above the anti-diagonal $x + y = 0$. However, a careful look at the proof reveals that the result extends easily to bimodules indexed over $\mathbb{R}^2$. 

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9.3 Interlevel-Sets Persistence

We now consider the bimodules that arise in the study of interlevel-sets persistence, which served as the initial motivation for this work. Let \( U = \{ t \in \mathbb{R}^2 \mid t_x + t_y > 0 \} \) as in the previous section. We observe that \( U \) is naturally identified with the set of non-empty bounded open intervals of \( \mathbb{R} \) via the following bijection:

\[
\mathbb{R} \supset (x, y) \mapsto (-x, y) \in U.
\]

Moreover, if we equip \( U \) with the product order inherited from \( \mathbb{R}^2 \), and the set of bounded open intervals with the inclusion order, then the above bijection is an isomorphism of posets.

Given now a topological space \( X \) and an \( \mathbb{R} \)-valued function \( f : X \to \mathbb{R} \), let \( S_f \) denote the interlevel-sets filtration of \( f \), which assigns the space \( (S_f)_t = f^{-1}((-t_x, t_y)) \) to any point \( t \in U \). \( S_f \) can be viewed as a functor from the poset \( U \) to the category of topological spaces. The composition \( H \circ S_f \) (where \( H \) is a shorthand for singular homology with coefficients in a fixed field \( k \)) is then a functor from \( U \) to the category of \( k \)-vector spaces. The map \( f \) is called pfd whenever \( H \circ S_f \) is a pfd module. Note that this module is always exact because, for any \( s < t \in U \), the following diagram:

\[
\begin{array}{ccc}
  f^{-1}((-s_x, t_y)) & \subset & f^{-1}((-t_x, t_y)) \\
  \uparrow & & \uparrow \\
  f^{-1}((-s_x, s_y)) & \subset & f^{-1}((-t_x, s_y))
\end{array}
\]

induces an exact diagram in homology, by the Mayer–Vietoris theorem. We then have the following decomposition result as a byproduct of Theorem 9.3:

**Corollary 9.4** For any topological space \( X \) and any pfd function \( f : X \to \mathbb{R} \), the bimodule \( H \circ S_f \) is block decomposable, that is

\[
H \circ S_f \simeq \bigoplus_{B \in \mathcal{B}(H \circ S_f)} k_{B \cap U}.
\]

Moreover, the decomposition is unique up to isomorphism and reordering of the terms (again, two blocks \( B, B' \) having the same intersection with \( U \) yield the same indecomposable \( k_{B \cap U} = k_{B' \cap U} \)).

This result answers a conjecture of Botnan and Lesnick [7, Conjecture 8.3]. Combined with Corollary 9.1, it induces a general stability result for interlevel-sets persistence, in which the functions considered do not have to be of Morse type [7,9]:

**Corollary 9.5** For any pfd functions \( f, f' : X \to \mathbb{R} \), the barcodes \( B(H \circ S_f) \) and \( B(H \circ S_{f'}) \) are well-defined and we have

\[
d_b(B(H \circ S_f), B(H \circ S_{f'})) \leq \| f - f' \|_{\infty}.
\]
Alternatively, one may consider interlevel-sets filtrations obtained by taking preimages of bounded \textit{closed} intervals—excluding singletons. The corresponding bimodules are also indexed over \( U \), moreover they are \( pfd \) and exact by the Mayer–Vietoris theorem, therefore they also decompose as direct sums of block summands.

### 9.4 \( \mathbb{Z}^2 \)-Indexed Modules

Assume now that \( M \) is indexed over \( \mathbb{Z}^2 \) only, while being still \( pfd \) and exact. Then we can extend \( M \) to \( \mathbb{R}^2 \) by taking its left or right Kan extension (say left) \( \overline{M} \). Since the inclusion \( \mathbb{Z}^2 \rightarrow \mathbb{R}^2 \) is a fully faithful functor between posets, \( \overline{M} \) restricts to a module isomorphic to \( M \) on \( \mathbb{Z}^2 \)—see e.g. [17, Corollary X.3.3]. In fact, since every down-set in \( \mathbb{R}^2 \) has a greatest element in \( \mathbb{Z}^2 \), \( M \) is piecewise constant:

\[
\text{for all } t = (x, y) \in \mathbb{R}^2, \quad \overline{M}_t = \lim_{\xrightarrow{\longrightarrow}} M_{\{x \in \mathbb{Z}^2 \mid x \leq t\}} \simeq M_{\lfloor x \rfloor, \lfloor y \rfloor}.
\]

Therefore, \( \overline{M} \) is exact and \( pfd \) like \( M \), which implies that it decomposes into block summands, by Theorem 2.1. Then, by restriction, \( M \) itself also decomposes into block summands, which yields the following result:

**Theorem 9.6** For any exact \( pfd \) bimodule \( M \) indexed over \( \mathbb{Z}^2 \), we have

\[
M \simeq \bigoplus_{B \in \mathcal{B}(M)} k_{B \cap \mathbb{Z}^2}.
\]

Note that the above construction works as well when \( \mathbb{Z}^2 \) is replaced by \( \mathbb{N}^2 \) or by any finite grid, leading to a similar conclusion.

### 9.5 \( \mathbb{Z} \)-Indexed Zigzag Modules

It is well-known that \( pfd \) zigzag modules with a countable index set decompose as direct sums of interval modules — see e.g. [5] for a recent treatment. This result can be obtained as a byproduct of our main theorem. The construction is similar to the one in [7]: given such a zigzag module \( M \), assume without loss of generality that its index set is \( \mathbb{Z} \), and furthermore that the arrow orientations in the zigzag are alternating (which can be ensured by inserting in isomorphisms at the right places):

\[
\cdots \leftarrow M_{2i-2} \rightarrow M_{2i-1} \leftarrow M_{2i} \rightarrow M_{2i+1} \leftarrow M_{2i+2} \rightarrow \cdots
\]

Embed then the zigzag as an infinite staircase \( S \) in the plane:

\[
\cdots \leftarrow (i-1, -i+1) \rightarrow (i, -i+1) \leftarrow (i, -i) \rightarrow (i+1, -i) \leftarrow (i+1, -i-1) \rightarrow \cdots
\]

and realize \( M \) as a representation \( \overline{M} \) of that staircase by letting \( \overline{M}_{(i, -i)} = M_{2i} \) and \( \overline{M}_{(i+1, -i)} = M_{2i+1} \) for all \( i \in \mathbb{Z} \). Then, take left and right Kan extensions of \( \overline{M} \) to
get a bimodule $\overline{M}$ indexed over $\mathbb{Z}^2$. As in the previous section, this extension restricts
to a zigzag module isomorphic to $M$ on the staircase. Moreover, since every up-set or
down-set in $\mathbb{Z}^2$ intersects only a finite portion of the staircase, $\overline{M}$ is pfd. Finally, since
left (resp. right) Kan extensions of diagrams of the form $\bullet \leftarrow \bullet \rightarrow \bullet$ (resp. $\bullet \rightarrow \bullet \leftarrow \bullet$) are pushouts (resp. pullbacks), and since iterated Kan extensions
are also Kan extensions, $\overline{M}$ is isomorphic to a bimodule obtained by taking pushouts
and pullbacks iteratively, which implies that $\overline{M}$ is exact. Then, by Theorem 9.6 we
have
\[
\overline{M} \simeq \bigoplus_{B \in B(\overline{M})} k_{B \cap \mathbb{Z}^2},
\]
and by restriction we deduce
\[
M \simeq \bigoplus_{B \in B(\overline{M})} k_{B \cap S},
\]
where each $B \cap S$ is an interval. Hence the result.

**Appendix A: Equivalence of definitions**

Let $R = (c^+ \cap c^-) \times (c^+ \cap \overline{c}^-)$ be a rectangle, and let $t \in R$. To prove that
the definitions in (7) are equivalent to those in (5), it is sufficient to show the following
equalities:

\[
I_{R,t}^+ = \text{Im}_{c,t}^+ \cap \text{Im}_{c,t}^+,
\]
\[
I_{R,t}^- = \text{Im}_{c,t}^- + \text{Im}_{c,t}^-,
\]
\[
K_{R,t}^+ = \text{Ker}_{c,t}^+ \cap \text{Ker}_{c,t}^+,
\]
\[
K_{R,t}^- = \text{Ker}_{c,t}^- + \text{Ker}_{c,t}^-.
\]

**Proof of (15a):**

\[
I_{R,t}^+ = \bigcap_{s \in R} \text{Im} \rho_s^t = \bigcap_{l^+ \exists x \leq t_x} \text{Im} \rho_{(x,y)}^t \text{ Eq. (2)} = \bigcap_{l^+ \exists x \leq t_x} \text{Im} \rho_{(x,y)}^t \cap \text{Im} \rho_{(tx,y)}^t
\]
\[
= \bigcap_{l^+ \exists x \leq t_x} \bigcap_{c^+ \exists y \leq t_y} \text{Im} \rho_{(x,y)}^t \cap \text{Im} \rho_{(tx,y)}^t
\]
\[
= \bigcap_{l^+ \exists x \leq t_x} \left( \text{Im} \rho_{(x,y)}^t \cap \bigcap_{c^+ \exists y \leq t_y} \text{Im} \rho_{(tx,y)}^t \right)
\]
\[
= \left( \bigcap_{x \in \mathbb{I}^+ \exists x \leq t_x} \text{Im } \rho_t^{l(x,ty)} \right) \cap \left( \bigcap_{y \in \mathbb{I}^- \exists y \leq t_y} \text{Im } \rho_t^{l(tx,y)} \right) = \text{Im}^+_{i_c,t} \cap \text{Im}^+_{i,c,t}.
\]

**Proof of (15b):**
\[
I_{i,R,t} = \sum_{x \in \mathbb{I}^-} \text{Im } \rho_t^l = \sum_{x \in \mathbb{I}^+ \exists x \leq t_x} \text{Im } \rho_t^{l(x,ty)} \cap \text{Im } \rho_t^{l(tx,y)} + \sum_{y \in \mathbb{I}^- \exists y \leq t_y} \text{Im } \rho_t^{l(x,ty)} \cap \text{Im } \rho_t^{l(tx,y)}
\]
\[
= \sum_{x \in \mathbb{I}^- \exists y \leq t_y} \text{Im } \rho_t^{l(x,ty)} \cap \text{Im } \rho_t^{l(tx,y)} + \sum_{x \in \mathbb{I}^- \exists y \leq t_y} \text{Im } \rho_t^{l(x,ty)} \cap \text{Im } \rho_t^{l(tx,y)}
\]
\[
= \sum_{x \in \mathbb{I}^- \exists y = t_y} \text{Im } \rho_t^{l(x,ty)} \cap \text{Im } \rho_t^{l(tx,y)} + \sum_{y \in \mathbb{I}^- \exists x = t_x} \text{Im } \rho_t^{l(x,ty)} \cap \text{Im } \rho_t^{l(tx,y)}
\]
\[
= \sum_{x \in \mathbb{I}^-} \text{Im } \rho_t^{l(x,ty)} + \sum_{y \in \mathbb{I}^-} \text{Im } \rho_t^{l(tx,y)} = \text{Im}^-_{i,c,t} + \text{Im}^-_{i,c,t}.
\]

**Proof of (15c):** it is symmetric to that of (15b):
\[
K_{i,R,t}^+ = \bigcap_{u \in \mathbb{I}^+ \exists u \geq t_u} \text{Ker } \rho_t^u = \bigcap_{u \in \mathbb{I}^+ \exists u \geq t_u} \text{Ker } \rho_t^{l(x,ty)} = \bigcap_{x \in \mathbb{I}^+ \exists x \geq t_x \land y \geq t_y} \text{Ker } \rho_t^{l(x,ty)} + \text{Ker } \rho_t^{l(tx,y)}
\]
\[
= \bigcap_{x \in \mathbb{I}^+ \exists y = t_y} \text{Ker } \rho_t^{l(x,ty)} + \text{Ker } \rho_t^{l(tx,y)} \cap \bigcap_{y \in \mathbb{I}^+ \exists x = t_x} \text{Ker } \rho_t^{l(x,ty)} + \text{Ker } \rho_t^{l(tx,y)}
\]
\[
= \bigcap_{x \in \mathbb{I}^+ \exists y = t_y} \text{Ker } \rho_t^{l(x,ty)} + \text{Ker } \rho_t^{l(tx,y)} \cap \bigcap_{y \in \mathbb{I}^+ \exists x = t_x} \text{Ker } \rho_t^{l(x,ty)} + \text{Ker } \rho_t^{l(tx,y)}
\]
\[
= \bigcap_{x \in \mathbb{I}^+ \exists y = t_y} \text{Ker } \rho_t^{l(x,ty)} \cap \bigcap_{y \in \mathbb{I}^+ \exists x = t_x} \text{Ker } \rho_t^{l(x,ty)} + \text{Ker } \rho_t^{l(tx,y)}\]

**Proof of (15d):** it is symmetric to that of (15a):
\[
K_{i,R,t}^- = \sum_{u \in \mathbb{I}^- \exists u \geq t_u} \text{Ker } \rho_t^u = \sum_{u \in \mathbb{I}^- \exists u \geq t_u} \text{Ker } \rho_t^{l(x,ty)} \cap \text{Ker } \rho_t^{l(tx,y)}
\]
\[
= \sum_{c \mathbb{I}^- \exists x \geq t_x \land y \geq t_y} \sum_{c \mathbb{I}^- \exists y \geq t_y} \text{Ker } \rho_t^{l(x,ty)} + \text{Ker } \rho_t^{l(tx,y)}
\]
\[
\sum_{c} \bigg( \operatorname{Ker} \rho_t^{(x,t_y)} + \sum_{c} \operatorname{Ker} \rho_t^{(t_x,y)} \bigg) = \left( \sum_{c} \operatorname{Ker} \rho_t^{(x,t_y)} \right) + \left( \sum_{c} \operatorname{Ker} \rho_t^{(t_x,y)} \right) = \operatorname{Ker} e_{t,t} + \operatorname{Ker} \bar{e}_{t,t}.
\]

**Appendix B: Proof of Theorem 9.2**

Write \( R = I \times J \), where \( I, J \) are two intervals of the real line. The easiest situation is when both \( I \) and \( J \) are open. Then we can reparametrize each of them monotonously over the real line. The induced functor \( F \), going from the representation category of \( R \) to the representation category of \( R^2 \), is clearly an isomorphism of abelian categories. Moreover, the reparametrization preserves rectangles, therefore \( F \) preserves the exactness property in addition to pointwise finite-dimensionality on objects. It follows that the \( R^2 \)-indexed module \( F(M) \) is both pfd and exact, and therefore block-decomposable by Theorem 2.1. The result for \( M \) follows.

Now, in the general case, each of \( I, J \) can be either open, right-open, left-open, or closed. In each case, monotonous reparametrizations as above allow us to reindex the module over some closed rectangle, possibly extending to infinity to the right, top, left, or bottom. From now on we assume without loss of generality that \( R \) is that closed rectangle. Define \( R^+ \) to be the set \( \{ t \in R^2 \mid \exists s \in R \text{ with } t \leq s \} \). Note that \( R^+ \) is the closed birth quadrant having the same lower-left corner as \( R \). Extend \( M \) to a module \( M^+ \) indexed over \( R^+ \) by taking its left Kan extension along the inclusion \( R \hookrightarrow R^+ \). Clearly, we have \( M^+|_R \simeq M \) since the inclusion is a fully faithful functor between poset categories. Moreover, using the pointwise definition of the left Kan extension via colimits, at each point \( t \in R^+ \) we have \( M^+_t = \lim_{\longrightarrow} M|_{R \cap t^-} \), where \( t^- \) denotes the set of points \( s \in R^2 \) such that \( s \leq t \) (i.e., the closed death quadrant having \( t \) as upper-right corner). Since \( R \cap t^- \) is a closed rectangle, it is a directed poset with a maximum element \( s \), so we have \( M^+_t \simeq M_s \). As a result, \( M^+ \) is clearly pfd, and it is easily seen to be exact as well. Now we do a similar operation to extend \( M^+ \) to a module \( M^{++} \) indexed over \( R^2 \), that is, we take the right Kan extension of \( M^+ \) along the inclusion \( R^+ \hookrightarrow R^2 \). Again, by the fully faithful nature of the inclusion we have \( M^{++}|_{R^+} \simeq M^+ \), and it follows from the pointwise definition of the right Kan extension via limits that \( M^{++} \) is both pfd and exact. Then, Theorem 2.1 implies that \( M^{++} \) is block-decomposable, from which we deduce by restriction that \( M \) itself is block-decomposable.

**Appendix C: Proof of Theorem 9.3**

We follow the scheme of [6, Sect. 5] but do not make use of the general decomposition theorem in [6]. For any \( s \leq t \in U \), let us call \((s,t)-square\) the commutative diagram (1), and say that it is injective if the first morphism in the exact sequence is
injective, and surjective if the second morphism in the sequence is surjective. Assume from now on that $M$ is an exact pfd bimodule indexed over $\mathbb{U}$.

**Lemma C.1** Suppose there exists some $(s, t)$-square that is not surjective. Then, we have $M \cong M' \oplus k_{B \cap U}$ where $B$ is a birth quadrant such that $s \notin B$ and $t \in B$. Similarly, if the $(s, t)$-square is not injective, then we have $M \cong M'' \oplus k_{B \cap U}$ where $B$ is a death quadrant such that $s \in B$ and $t \notin B$.

**Proof** The proof is the same as the ones of [6, Lems. 5.12 and 5.13], with the invocations of Lemmas 5.5 and 5.6 replaced by the one of our Theorem 9.2, observing that the only blocks $B$ that can make the $(s, t)$-square non-surjective (resp. non-injective) are the birth quadrants $B$ such that $s \notin B \ni t$ (resp. death quadrants $B$ such that $s \in B \not\ni t$). Note that the conclusion of the proof of Lemma 5.13 is that there are morphisms $k_{B \cap U} \hookrightarrow M \twoheadrightarrow k_{B \cap U}$ composing to the identity of $k_{B \cap U}$, which implies that $M \cong M' \oplus k_{B \cap U}$. Meanwhile, the conclusion of the proof of Lemma 5.12 is that there is an injective morphism $k_{B \cap U} \hookrightarrow M$, which implies that $M \cong M'' \oplus k_{B \cap U}$ since $k_{B \cap U}$ is an injective module when $B$ is a death quadrant.

**Lemma C.2** If an $(s, t)$-square is injective (resp. surjective) in $M$, then so are all the $(s', t')$-squares for $s \leq s' \leq t'$.

**Proof** We prove the injectivity part, the surjectivity part following by duality. First, observe that for any $t'$ such that $s \leq t' \leq t$ we have $\ker h_s^{(t', s_y)} \cap \ker v_s^{(s_x, t')}$ \subseteq $\ker h_s^{(t, s_y)} \cap \ker v_s^{(s_x, t_y)} = 0$, so we may assume without loss of generality that $t = t'$. Now, for any $s'$ such that $s \leq s' \leq t$ we have the following commutative diagram, where $a = (s_x, t_y), b = (s_x', t_y), c = (s_x, s_y'), d = (t_x, s_y'), e = (s_x', s_y), f = (t_x, s_y)$, and where every square is exact:

$$
\begin{array}{ccc}
M_a & \longrightarrow & M_b \\
\downarrow & & \downarrow \\
M_c & \longrightarrow & M_d \\
\downarrow & & \downarrow \\
M_e & \longrightarrow & M_f \\
\end{array}
$$

Assume for a contradiction that the $(s', t)$-square is not injective, which means that $\ker h_s^{d} \cap \ker v_s^{b} \neq 0$. Take $\alpha \in (\ker h_s^{d} \cap \ker v_s^{b}) \setminus \{0\}$. Then, $0 \in M_d$ has preimages $\alpha \in M_{c'}$ and $0 \in M_f$, therefore the latter have a common preimage $\beta \in M_e$. Similarly, $\alpha \in M_{c'}$ and $0 \in M_a$ have a common preimage $\gamma \in M_c$. Since $v_s^{d'}(\beta) = \alpha = h_s^{d'}(\gamma)$, $\beta$ and $\gamma$ have a common preimage $\delta \in M_e$. This preimage cannot be zero since it is sent to $\alpha \neq 0$ through $h_s^{d'}$. Meanwhile, it is sent to $h_s^{d'}(\beta) = 0 \in M_f$ through $h_s^{d'}$, and to $u_s^{a'}(\gamma) = 0 \in M_a$ through $u_s^{a'}$. Hence, it belongs to $\ker h_s^{d'} \cap \ker v_s^{a'}$, which implies that $\ker h_s^{d'} \cap \ker v_s^{a'} \neq 0$ and thus contradicts the hypothesis that the $(s, t)$-square is injective.
Lemma C.3 The module $M$ decomposes as follows:

$$M \simeq N \oplus \bigoplus_{B \in \mathcal{B}} k_{B \cap \mathbb{U}},$$

where $\mathcal{B}$ is a set of birth and death quadrants, and where every $(s, t)$-square in $N$ is both injective and surjective.

Proof Consider the sequence of points $(t_n)_{n \in \mathbb{N}}$ defined by $t_n = (n+1, n+1) \in \mathbb{U}$. For each $n$, take a finite zigzag $s^0_n \leftarrow s^1_n \rightarrow s^2_n \leftarrow \cdots \leftarrow s^{r_n}_n$ as illustrated in Fig. 8, and call $T_n$ the region bounded by $t_n$ and this zigzag (in gray in the figure). Furthermore, make sure that the zigzag converges to the anti-diagonal $x + y = 0$ as $n$ goes to infinity, in such a way that $T_n \subseteq T_{n+1}$ for all $n \in \mathbb{N}$.

Then, consider the sequence of submodules of $M$ obtained as follows:

- $M_0 = M$;
- For $n \geq 1$, while there exist internal summands of $M_{n-1}$ that are isomorphic to $k_{B \cap \mathbb{U}}$ for $B$ a birth (resp. death) quadrant with $s^1_n \notin B \ni t_n$ (resp. $s^1_n \in B \ni t_n$), peel off such a summand from $M_{n-1}$; then do the same for the $(s^3_n, t_n)$-square, and so on, until the $(s^{r_n-1}_n, t_n)$-square. This procedure terminates because each square considered is finite-dimensional (hence finitely many internal summands are peeled off for that square) and there are finitely many such squares. Call $M_n$ the resulting submodule of $M_{n-1}$. Lemma C.1 ensures that each square that has been considered is both injective and surjective in $M_n$, and therefore so is every $(s', t')$-square with $s' \leq t' \in T_n$ by Lemma C.2.

We thus get a decreasing family $(M_n)_{n \in \mathbb{N}}$ of submodules of $M$. Since $M$ is pfd, this family converges pointwise to some submodule $N$. Every $(s, t)$-square in $N$ is both injective and surjective, because it belongs to $T_n$ for some finite index $n$ and for all the indices beyond.

To conclude the proof of Theorem 9.3, we must show that the submodule $N$ defined above is block-decomposable. The argument is in fact the same as in the conclusion of
the proof of [6, Thm. 5.11]: using the prior that every square is now both injective and surjective, we apply [6, Cor. 5.9] (whose proof uses our main Theorem 2.1 and not the general decomposition theorem in [6]) to get a direct-sum decomposition of \( N \) into summands that are obtained from Kan extensions of interval summands of a certain zigzag in the plane, each such extension giving rise to a particular block module.

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