CURRENT ALGEBRAS, HIGHEST WEIGHT CATEGORIES AND QIERS

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Abstract. We study the category of graded finite-dimensional representations of the polynomial current algebra associated to a simple Lie algebra. We prove that the category has enough injectives and compute the graded character of the injective envelopes of the simple objects as well as extensions between simple objects. The simple objects in the category are parametized by the affine weight lattice. We show that with respect to a suitable refinement of the standard ordering on affine the weight lattice the category is highest weight. We compute the Ext quiver of the algebra of endomorphisms of the injective cogenerator of the subcategory associated to an interval closed finite subset of the weight lattice. Finally, we prove that there is a large number of interesting quivers of finite, affine and tame type that arise from our study. We also prove that the path algebra of star shaped quivers are the Ext-algebra of a suitable subcategory.

Introduction

In this paper we study the category $\mathcal{G}$ of graded finite-dimensional representations of the polynomial current algebra $\mathfrak{g}[t]$ associated to a simple finite dimensional Lie algebra $\mathfrak{g}$. There are numerous interesting and related families of examples of such representations: the Demazure modules arising from the positive level representations of the affine algebra, the fusion products of finite-dimensional representations of $\mathfrak{g}[t]$ defined in [9], the Kirillov-Reshetikhin modules studied in [4, 5] and the Weyl modules introduced in [6] and studied in [3, 10]. All these representations are in general reducible but always indecomposable.

The isomorphism classes of simple objects in $\mathcal{G}$ are indexed by the set $\Lambda = P^+ \times \mathbb{Z}_+$ where $P^+$ is the set of dominant integral weights of $\mathfrak{g}$. The set $\Lambda$ can be identified in a natural way with a subset of the lattice of integral weights $\hat{P}$ of the untwisted affine Lie algebra associated to $\mathfrak{g}$. We define an interval finite partial order $\preceq$ on $\Lambda$ which is a refinement of the usual order on $\hat{P}$ and show (Theorem 1) that $\mathcal{G}$ is a highest weight category, in the sense of [7], with the poset of weights $(\Lambda, \preceq)$. To do this, we study first the category $\hat{\mathcal{G}}$ of graded $\mathfrak{g}[t]$-modules with finite-dimensional graded pieces. This category has enough projectives and the graded character of the projective modules can be described explicitly. Then, using a certain duality, we are able to show that the category $\mathcal{G}$ has enough injectives and we compute the graded character of the injective envelope of any simple object. We then prove that $\mathcal{G}$ is a directed highest weight category by computing the extensions between simple objects.

In Section 3 we study algebraic structures associated with Serre subcategories of $\mathcal{G}$. For an interval closed subset $\Gamma$ of $\Lambda$, let $\mathcal{G}[\Gamma]$ be the full subcategory of $\mathcal{G}$ consisting of objects

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whose simple constituents are parametrized by elements of \( \Gamma \) and let \( I(\Gamma)_R \) be the injective cogenerator of \( \mathcal{G}[\Gamma] \). It is well-known that there is an equivalence of categories between \( \mathcal{G}[\Gamma] \) and the category of finite-dimensional right \( \mathfrak{a}(\Gamma) = \text{End}_{\mathcal{G}[\Gamma]} I(\Gamma)_R \)-modules. Moreover \( \mathfrak{a}(\Gamma) \) is a quotient of the path algebra of its Ext quiver \( Q(\Gamma) \) and has a compatible grading. By using the character formula for the injective envelopes, we show that \( Q(\Gamma) \) can be computed quite explicitly in terms of finite dimensional representations of \( \mathfrak{g} \).

In Sections 4 and 5 we show that there are many interesting quivers arising from our study. Thus, in Section 4 we see that for all \( \mathfrak{g} \) (in some cases one has to exclude \( \mathfrak{sl}_2 \) or \( \mathfrak{g} \) of type \( C_\ell \)), there exists interval closed finite subsets \( \Gamma \) such that the corresponding algebra \( \mathfrak{a}(\Gamma) \) is hereditary and \( Q(\Gamma) \) is (a) a generalized Kronecker quiver; (b) a quiver of type \( A_\ell, D_\ell \); (c) an affine quiver of type \( \tilde{D}_\ell \); (d) any star shaped quiver with three branches. In Section 5 we study an example which arises from the theory of Kirillov-Reshetikhin modules for \( \mathfrak{g}[t] \) where \( \mathfrak{g} \) is of type \( D_n \). In this case the algebra \( \mathfrak{a}(\Gamma) \) is not hereditary, but is still of tame representation type.

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1. The category \( \mathcal{G} \)

1.1. The simple Lie algebras and the associated current algebras. Throughout the paper \( \mathfrak{g} \) denotes a finite-dimensional complex simple Lie algebra and \( \mathfrak{h} \) a fixed Cartan subalgebra of \( \mathfrak{g} \). Set \( I = \{1, \ldots, \dim \mathfrak{h} \} \) and let \( \{\alpha_i : i \in I \} \subset \mathfrak{h}^* \) be a set of simple roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \). Let \( R \subset \mathfrak{h}^* \) (respectively, \( R^+, P^+, Q^+ \)) be the corresponding set of roots (respectively, positive roots, dominant integral weights, the \( \mathbb{Z}_+ \)-span of \( R^+ \)) and let \( \theta \in R^+ \) be the highest root. Let \( W \subset \text{Aut}(\mathfrak{h}^*) \) be the Weyl group of \( \mathfrak{g} \) and \( w_0 \) be the longest element of \( W \). For \( \alpha \in R \) denote by \( \mathfrak{g}_\alpha \) the corresponding root space. The subspaces \( \mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \), \( \mathfrak{n}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha \), are Lie subalgebras of \( \mathfrak{g} \). Fix a Chevalley basis \( x^+_\alpha, \alpha \in R^+, h_i, i \in I \) of \( \mathfrak{g} \) and for \( \alpha \in R^+ \), set \( h_\alpha = [x_\alpha, x_{-\alpha}] \). Note that \( h_{\alpha_i} = h_i, i \in I \). For \( i \in I \), let \( \omega_i \in P^+ \) be defined by \( \omega_i(h_j) = \delta_{ij} \) for all \( j \in I \).

Let \( \mathcal{F}(\mathfrak{g}) \) be the category of finite-dimensional \( \mathfrak{g} \)-modules with the morphisms being maps of \( \mathfrak{g} \)-modules. In particular, we write \( \text{Hom}_\mathfrak{g} \) for \( \text{Hom}_{\mathcal{F}(\mathfrak{g})} \). The set \( P^+ \) parametrizes the isomorphism classes of simple objects in \( \mathcal{F}(\mathfrak{g}) \). For \( \lambda \in P^+ \), let \( V(\lambda) \) be the simple module in the corresponding isomorphism class which is generated by an element \( v_\lambda \in V(\lambda) \) satisfying the defining relations:

\[
n^+v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda, \quad (x^-_{\alpha_i})^{\lambda(h_i)+1}v_\lambda = 0,
\]
for all \( h \in \mathfrak{h}, i \in I \). The module \( V(-w_\circ \lambda) \) is the \( \mathfrak{g} \)-dual of \( V(\lambda) \). If \( V \in \mathcal{F}(\mathfrak{g}) \), write
\[
V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,
\]
where \( V_\lambda = \{ v \in V : hv = \lambda(h)v, \ \forall \ h \in \mathfrak{h} \} \). Set \( \text{wt}(V) = \{ \lambda \in \mathfrak{h}^*: V_\lambda \neq 0 \} \). Finally, recall also that the category \( \mathcal{F}(\mathfrak{g}) \) is semi-simple, i.e. any object in \( \mathcal{F}(\mathfrak{g}) \) is isomorphic to a direct sum of the modules \( V(\lambda), \lambda \in P^+ \). We shall use the following standard results in the course of the paper (cf. [14] for (iv)).

**Lemma.**

(i) Let \( \lambda \in P^+ \). Then \( \text{wt}(V(\lambda)) \subset \lambda - Q^+ \).

(ii) Let \( V \in \mathcal{F}(\mathfrak{g}) \). Then \( w \text{wt}(V) \subset \text{wt}(V) \) for all \( w \in W \) and \( \dim V_\lambda = V_{w\lambda} \).

(iii) Let \( V \in \mathcal{F}(\mathfrak{g}) \). Then
\[
\dim \text{Hom}_\mathfrak{g}(V(\lambda),V) = \dim \{ v \in V_\lambda : n^+v = 0 \}.
\]

(iv) Let \( \lambda, \mu \in P^+ \). Then the module \( V(-w_\circ \lambda) \otimes V(\mu) \) is generated as a \( \mathfrak{U}(\mathfrak{g}) \)-module by the element \( v = v_{-\lambda} \otimes v_\mu \) with defining relations:
\[
(x_{\alpha_i}^+)^{\mu(h_i)+1} v = (x_{\alpha_i}^-)^{\lambda(h_i)+1} v = 0, \quad hv = (\mu - \lambda)(h)v,
\]
for all \( i \in I \) and \( h \in \mathfrak{h} \). \( \square \)

Given any Lie algebra \( \mathfrak{a} \) let \( \mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t] \) be the polynomial current algebra of \( \mathfrak{a} \). Let \( \mathfrak{a}[t]_+ \) be the Lie ideal \( \mathfrak{a} \otimes t\mathbb{C}[t] \). Both \( \mathfrak{a}[t] \) and \( \mathfrak{a}[t]_+ \) are \( \mathbb{Z}_+ \)-graded Lie algebras with the grading given by powers of \( t \). Let \( \mathfrak{U}(\mathfrak{a}) \) denote the universal enveloping algebra of \( \mathfrak{a} \). Then \( \mathfrak{U}(\mathfrak{a}[t]) \) has a natural \( \mathbb{Z}_+ \)-grading as an associative algebra and we let \( \mathfrak{U}(\mathfrak{a})[k] \) be the \( k \)-th graded piece. The algebra \( \mathfrak{U}(\mathfrak{a}) \) is a Hopf algebra, the comultiplication being given by extending the assignment \( x \to x \otimes 1 + 1 \otimes x \) for \( x \in \mathfrak{a} \) to an algebra homomorphism of \( \mathfrak{U}(\mathfrak{a}) \).

In the course of the paper, we shall repeatedly use the fact that \( \mathfrak{U}(\mathfrak{a}[t]) \) is generated as a graded algebra by \( \mathfrak{a} \) and \( \mathfrak{a} \otimes t \) without further comment.

### 1.2. The category \( \widehat{\mathcal{G}} \)

Let \( \widehat{\mathcal{G}} \) be the category whose objects are graded \( \mathfrak{g}[t] \)-modules with finite-dimensional graded pieces and where the morphisms are graded maps of \( \mathfrak{g}[t] \)-modules. More precisely, if \( V \in \text{Ob} \widehat{\mathcal{G}} \) then
\[
V = \bigoplus_{r \in \mathbb{Z}_+} V[r],
\]
where \( V[r] \) is a finite-dimensional subspace of \( V \) such that \( (xt^k)V[r] \subset V[r+k] \) for all \( x \in \mathfrak{g} \) and \( r, k \in \mathbb{Z}_+ \). In particular, \( V[r] \in \text{Ob} \mathcal{F}(\mathfrak{g}) \). Also, if \( V, W \in \text{Ob} \widehat{\mathcal{G}} \), then
\[
\text{Hom}_{\widehat{\mathcal{G}}}(V,W) = \{ f \in \text{Hom}_{\mathfrak{g}[t]}(V,W) : f(V[r]) \subset W(r), r \in \mathbb{Z}_+ \}.
\]
For \( f \in \text{Hom}_{\widehat{\mathcal{G}}}(V,W) \) let \( f[r] \) be the restriction of \( f \) to \( V[r] \). Clearly, \( f[r] \in \text{Hom}_{\mathfrak{g}}(V[r], W[r]) \).

Define a covariant functor \( \text{ev} : \mathcal{F}(\mathfrak{g}) \to \widehat{\mathcal{G}} \) by the requirements:
\[
\text{ev}(V)[0] = V, \quad \text{ev}(V)[r] = 0, \quad r > 0,
\]
and with \( \mathfrak{g}[t] \)-action given by
\[
(xt^k)v = \delta_{k,0}xv, \quad \forall x \in \mathfrak{g}, k \in \mathbb{Z}_+, v \in V
\]
and
\[ \text{Hom}_G(\text{ev}(V), \text{ev}(W)) = \text{Hom}_\mathfrak{g}(V, W). \] (1.1)

For \( s \in \mathbb{Z}_+ \) let \( \tau_s \) be the grading shift given by
\[(\tau_s V)[k] = V[k - s], \quad k \in \mathbb{Z}_+, \quad V \in \text{Ob} \widehat{\mathcal{G}}.\]
Clearly \( \tau_s(V) \in \text{Ob} \widehat{\mathcal{G}}. \)

1.3. Simple objects in \( \widehat{\mathcal{G}} \). For \( \lambda \in P^+ \) and \( r \in \mathbb{Z}_+ \), set
\[ V(\lambda, r) = \tau_r(\text{ev}(V(\lambda))). \] (1.2)

**Proposition.** The isomorphism classes of simple objects in \( \widehat{\mathcal{G}} \) are parametrized by pairs \((\lambda, r)\) and we have
\[ \text{Hom}_G(V(\lambda, r), V(\mu, s)) = 0, \quad (\lambda, r) \neq (\mu, s), \]
\[ \text{Hom}_G(V(\lambda, r), V(\lambda, r)) \cong \mathbb{C}. \]
Moreover if \( V \in \text{Ob} \mathcal{G} \) is such that \( V = V[n] \) for some \( n \in \mathbb{Z}_+ \), then \( V \) is semi-simple.

**Proof.** The modules \( V(\lambda, r), (\lambda, r) \in \Lambda \) are obviously simple and non-isomorphic. Moreover, if \( V \in \text{Ob} \widehat{G} \) is such that \( V \neq V[r] \) for some \( r \), there exists \( m, m' \in \mathbb{Z}_+ \) with \( m' > m \) such that \( V[j] \neq 0 \), for \( j \in \{m, m'\} \). Hence the subspace \( \bigoplus_{k \geq m} V[k] \) is a nontrivial proper graded \( \mathfrak{g}[t] \)-submodule of \( V \) and it follows that \( V \) is not simple. Assume now that \( V \) is simple so that \( V = V[r] \) for some \( r \). This implies that \( V \) is finite-dimensional and also that \( \mathfrak{g}[t]_+ V = 0. \) It follows that \( V \) must be isomorphic to \( V(\lambda) \) for some \( \lambda \in P^+ \) as a \( \mathfrak{g} \)-module and hence \( V \cong V(\lambda, r) \) as \( \mathfrak{g}[t] \)-modules. The other statements are now obvious. \( \Box \)

1.4. Tensor structure of the category \( \widehat{\mathcal{G}} \). Let \( V, W \in \text{Ob} \widehat{G} \). Then \( V \otimes W \) is a \( \mathfrak{g}[t] \)-module with the action being given by the comultiplication. Given \( k \in \mathbb{Z}_+ \), set
\[ (V \otimes W)[k] = \bigoplus_{i \in \mathbb{Z}_+} V[i] \otimes W[k - i], \]
with the usual convention that \( W[j] = 0 \) if \( j < 0 \). The following is trivially checked.

**Lemma.** (i) \( V \otimes W = \bigoplus_{k \in \mathbb{Z}_+} (V \otimes W)[k] \) and for all \( r \in \mathbb{Z}_+ \), we have
\[ (xt^r)((V \otimes W)[k]) \subset (V \otimes W)[k + r]. \]
In particular, \( \widehat{\mathcal{G}} \) is a tensor category.

(ii) For all \( r, s \in \mathbb{Z}_+ \)
\[ \tau_s V \cong V \otimes V(0, s), \quad \tau_{r+s}(V \otimes W) \cong \tau_r V \otimes \tau_s W. \] (1.3)
1.5. The subcategories \( \mathcal{G} \) and \( \mathcal{G}_{\leq s} \). Let \( \mathcal{G}_{\leq s} \) be the full subcategory of \( \widehat{\mathcal{G}} \) whose objects \( V \) satisfy

\[
V[r] = 0, \quad \forall r > s,
\]

and let \( \mathcal{G} \) be the full subcategory of \( \mathcal{G} \) consisting of \( V \in \text{Ob} \widehat{\mathcal{G}} \) such that \( V \in \text{Ob} \mathcal{G}_{\leq s} \) for some \( s \in \mathbb{Z}_+ \). It follows from the definition that \( \mathcal{G}_{\leq s} \) is a full subcategory of \( \mathcal{G}_{\leq r} \) for all \( s < r \in \mathbb{Z}_+ \).

Given \( V \in \text{Ob} \mathcal{G} \), let \( \text{soc}(V) \in \text{Ob} \mathcal{G} \) be the maximal semi-simple subobject of \( V \). Similarly, given \( V \in \text{Ob} \widehat{\mathcal{G}} \), let \( \text{head}(V) \) be the maximal semi-simple quotient of \( V \).

Given \( s \in \mathbb{Z}_+ \) and \( V \in \text{Ob} \widehat{\mathcal{G}} \), define

\[
V_{>s} = \bigoplus_{r>s} V[r], \quad V_{\leq s} = V/V_{>s}
\]

Then \( V_{\leq s} \in \text{Ob} \mathcal{G}_{\leq s} \). Furthermore, if \( f \in \text{Hom}_\mathcal{G}(V,W) \), then \( V_{>s} \) is contained in the kernel of the canonical morphism \( \bar{f} : V \to W_{\leq s} \) and hence we have a natural morphism \( f_{\leq s} \in \text{Hom}_{\mathcal{G}_{\leq s}}(V_{\leq s}, W_{\leq s}) \).

**Lemma.** (i) For all \( r, s \in \mathbb{Z}_+ \), and \( V \in \text{Ob} \mathcal{G}_{\leq r}, W \in \text{Ob} \mathcal{G}_{\leq s} \) we have

\[
V \otimes W \in \text{Ob} \mathcal{G}_{\leq r+s}.
\]

In particular \( \mathcal{G} \) is a tensor subcategory of \( \widehat{\mathcal{G}} \).

(ii) The assignments \( V \mapsto V_{\leq r} \) for all \( V \in \text{Ob} \widehat{\mathcal{G}} \) and \( f \mapsto f_{\leq r} \) for all \( f \in \text{Hom}_\mathcal{G}(V,W) \),

\[
V, W \in \text{Ob} \widehat{\mathcal{G}} \text{ define a full, exact and essentially surjective functor from } \widehat{\mathcal{G}} \text{ to } \mathcal{G}_{\leq r}.
\]

(iii) For any \( V \in \text{Ob} \widehat{\mathcal{G}}, \lambda \in P^+ \) and \( r, s \in \mathbb{Z}_+ \) with \( s \geq r \), we have

\[
(V \otimes V(\lambda, r))_{\leq s} \cong V_{\leq s-r} \otimes V(\lambda, r).
\]

**Proof.** Parts (i) and (ii) are obvious. For the last part, consider the natural map of graded \( \mathfrak{g}[t] \)-modules \( V \otimes V(\lambda, r) \to V_{\leq s-r} \otimes V(\lambda, r) \). The assertion follows by noting that \( (V \otimes V(\lambda, r))[k] = V[k-r] \otimes V(\lambda, r) \) for all \( k \in \mathbb{Z}_+ \). \( \square \)

From now on, given \( V \in \text{Ob} \mathcal{G} \) we denote by \( [V : V(\lambda, r)] \) the multiplicity of \( V(\lambda, r) \) in a composition series for \( V \). Furthermore, given \( W \in \text{Ob} \widehat{\mathcal{G}} \), we set \( [W : V(\lambda, r)] := [W_{\leq r} : V(\lambda, r)] \). Observe that \( [V : V(\lambda, r)] \) equals the \( \mathfrak{g} \)-module multiplicity of \( V(\lambda) \) in \( V[r] \). For any \( V \in \text{Ob} \widehat{\mathcal{G}} \), define

\[
\Lambda(V) = \{ (\lambda, r) \in \Lambda : [V : V(\lambda, r)] \neq 0 \}.
\]

1.6. We recall the following definition (which motivated much of this paper) of a directed category following [7,13], in the context of interest to us. Thus let \( \mathcal{C} \) be an abelian category over \( \mathcal{C} \) whose objects are complex vector spaces, have finite length and such that \( \text{Hom}_\mathcal{C}(M,N) \) is finite-dimensional for all \( M, N \in \text{Ob} \mathcal{C} \).

**Definition.** We say that \( \mathcal{C} \) is a directed category if

1°. The simple objects in \( \mathcal{C} \) are parametrized by a poset \((\Pi, \leq)\) \textit{(the poset of weights)} such that for all \( \tau \in \Pi \), the set \( \{ \xi \in \Pi : \xi < \tau \} \) is finite.

2°. Given \( \sigma \in \Pi \), let \( S(\sigma) \) be a simple object in the corresponding isomorphism class. Then \( \text{Ext}^1_\mathcal{C}(S(\sigma), S(\tau)) \neq 0 \implies \sigma < \tau \).
It is immediate from [8, 16, 17] that a directed category has enough injectives. Given \( \Xi \subset \Pi \), let \( \mathcal{C}[\Xi] \) be the full subcategory of \( \mathcal{C} \) whose objects satisfy

\[
M \in \text{Ob} \mathcal{C}[\Xi], \quad [M : S(\tau)] \neq 0 \implies \tau \in \Xi.
\]

It is clear that if \( \mathcal{C} \) is a directed category with poset of weights \( \Pi \), then for any subset \( \Xi \subset \Pi \), the category \( \mathcal{C}[\Xi] \) is also directed with poset of weights \( \Xi \). Given \( \sigma, \tau \in \Xi \) with \( \sigma < \tau \), we denote by \( [\sigma, \tau] \) the interval \( \{ \xi : \sigma \leq \xi \leq \tau \} \). A subset \( \Xi \) of \( \Pi \) is said to be interval closed if \( \sigma < \tau \in \Xi \) implies that \( [\sigma, \tau] \subset \Xi \).

### 1.7. We can now state the main result of this section. Set

\[
\Lambda = P^+ \times \mathbb{Z}^+, \quad \Lambda_{\leq r} = \{(\nu, l) \in \Lambda : l \leq r\}, \quad r \in \mathbb{Z}_+.
\]

Define a strict partial order on \( \Lambda \) in the following way. Given \( (\lambda, r), (\mu, s) \in \Lambda \), say that \( (\mu, s) \) covers \( (\lambda, r) \) if and only if \( s = r + 1 \) and \( \mu - \lambda \in R \cup \{0\} \). It follows immediately that for any \( (\mu, s) \in \Lambda \) the set of \( (\lambda, r) \in \Lambda \) such that \( (\mu, s) \) covers \( (\lambda, r) \) is finite. Let \( \preceq \) be the unique partial order on \( \Lambda \) generated by this cover relation. Then \( \{(\mu, s) : (\mu, s) \prec (\lambda, r)\} \) is finite for all \( (\lambda, r) \in \Lambda \). Note that if \( (\lambda, r) \prec (\mu, s) \in \Lambda_{\leq k} \) then \( (-w_0 \mu, k - s) \prec (-w_0 \lambda, k - r) \).

**Theorem 1.** For all \( \Gamma \subset \Lambda \), the category \( \mathcal{G}[\Gamma] \) is a directed category with poset of weights \( (\Gamma, \preceq) \).

We prove this theorem in the next section (Proposition 2.5).

### 2. Injective and Projective objects

#### 2.1. Projectives in \( \widehat{\mathcal{G}} \) and \( \mathcal{G}_{\leq r} \).

Given \( (\lambda, r) \in \Lambda \), set

\[
P(\lambda, r) = U(g[t]) \otimes U(g) V(\lambda, r).
\]

Clearly, \( P(\lambda, r) \) is an infinite dimensional graded \( g[t] \)-module. Using the PBW theorem we have an isomorphism of graded vector spaces

\[
U(g[t]) \cong U(g[t]_+) \otimes U(g),
\]

and hence we get

\[
P(\lambda, r)[k] = U(g[t]_+)[k - r] \otimes V(\lambda, r), \quad (2.1)
\]

where we understand that \( U(g[t]_+)[k - r] = 0 \) if \( k < r \). This shows that \( P(\lambda, r) \in \text{Ob} \widehat{\mathcal{G}} \) and also that

\[
P(\lambda, r)[r] = 1 \otimes V(\lambda, r).
\]

Set \( p_{\lambda, r} = 1 \otimes v_{\lambda, r} \).

**Proposition.** Let \( (\lambda, r) \in \Lambda \), \( s \in \mathbb{Z}_+ \) and \( s \geq r \).

1. \( P(\lambda, r) \) is generated as a \( g[t] \)-module by \( p_{\lambda, r} \) with defining relations:

\[
(n^+)p_{\lambda, r} = 0, \quad hp_{\lambda, r} = \lambda(h)p_{\lambda, r}, \quad (x_{\alpha_i}^-)^{\lambda(h_i) + 1}p_{\lambda, r} = 0,
\]

for all \( h \in \mathfrak{h}, i \in I \). Hence, \( P(\lambda, r) \) is the projective cover in the category \( \widehat{\mathcal{G}} \) of its unique simple quotient \( V(\lambda, r) \). Moreover the kernel \( K(\lambda, r) \) of the canonical projection \( P(\lambda, r) \twoheadrightarrow V(\lambda, r) \) is generated as a \( g[t] \) module by \( P(\lambda, r)[r + 1] = g \otimes V(\lambda, r) \).

2. \( P(\lambda, r) \cong P(0, 0) \otimes V(\lambda, r) \) as objects in \( \widehat{\mathcal{G}} \).
(iii) The modules \( P(\lambda, r)_{\leq s} \) are projective in \( \mathcal{G}_{\leq s} \) and
\[
P(\lambda, r)_{\leq s} \cong P(0, 0)_{\leq s-r} \otimes V(\lambda, r)
\]

(iv) As \( \mathfrak{g} \)-modules, we have
\[
P(0, 0)[k] \cong U(\mathfrak{g}[t^+]_+)[k] \cong \bigoplus_{(r_1, \ldots, r_k) \in \mathbb{Z}^+ : \sum j_r = k} S^{r_1}(\mathfrak{g}) \otimes \cdots \otimes S^{r_k}(\mathfrak{g}),
\]
where \( S^p(\mathfrak{g}) \) denotes the \( p \)-th symmetric power of \( \mathfrak{g} \).

(v) Let \((\mu, s) \in \Lambda\). Then \([K(\lambda, r) : V(\mu, s)] \neq 0 \) only if \((\lambda, r) \prec (\mu, s)\).

(vi) Let \( V \in \text{Ob} \widehat{\mathcal{G}}\). Then \( \dim \text{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V) = [V : V(\lambda, r)]\).

**Proof.** The fact that \( P(\lambda, r) \) is projective in the category \( \widehat{\mathcal{G}} \) is standard in relative homological algebra (cf. \( \square \)). The other statements in \( \square \) are immediate from the discussion preceding the proposition. For part \( \square \), note that the element \( p_{0,0} \otimes v_{\lambda,r} \) satisfies the defining relations of \( P(\lambda, r) \). Moreover it is easily seen that
\[
U(\mathfrak{g}[t])(p_{0,0} \otimes v_{\lambda,r}) = P(0, 0) \otimes V(\lambda, r).
\]
Hence we have a surjective morphism \( P(\lambda, r) \to P(0, 0) \otimes V(\lambda, r) \) in \( \widehat{\mathcal{G}} \). On the other hand, \( \square \) implies that \( P(\lambda, r) \cong P(0, 0) \otimes V(\lambda, r) \) as vector spaces and \( \square \) is proved. Part \( \square \) is immediate from Lemma \( \square \). The first isomorphism in \( \square \) is obvious. To prove the second, we may assume that \( k > 0 \) since \( U(\mathfrak{g}[t^+]_+[0] = \mathbb{C}\). For \( s \geq 0 \), let \( U(\mathfrak{g}[t^+]_{\leq s}, s \geq 0 \) be the subspace of \( U(\mathfrak{g}[t^+]_+) \) spanned by the set
\[
\{(y_1t^{r_1}) \cdots (y_kt^{r_k}) : y_jt^{r_j} \in \mathfrak{g}[t^+], 1 \leq j \leq k \leq s\},
\]
and set
\[
U(\mathfrak{g}[t^+]_+[k]_{ \leq s} = U(\mathfrak{g}[t^+]_+[k] \cap U(\mathfrak{g}[t^+]_+[s].
\]
Then \( U(\mathfrak{g}[t^+]_+[k] = U(\mathfrak{g}[t^+]_+[k]_{ \leq k} \) and for \( 0 \leq s \leq k \) the subspaces \( U(\mathfrak{g}[t^+]_+[k]_{ \leq s} \) define an increasing filtration on \( U(\mathfrak{g}[t^+]_+[k]. \) Moreover, regarding \( \mathfrak{g}[t^+]_+ \) as a \( \mathfrak{g} \)-module via the adjoint action on \( \mathfrak{g}[t^+]_+ \), we see that this filtration is in fact \( \mathfrak{g} \)-equivariant. Since \( U(\mathfrak{g}[t^+]_+[k]_{ \leq r} \) is finite dimensional we get an isomorphism of \( \mathfrak{g} \)-modules,
\[
U(\mathfrak{g}[t^+]_+[k]_{ \leq r} \cong \bigoplus_{r=1}^k S^r(\mathfrak{g}[t^+]_+[k] = S(\mathfrak{g}[t^+]_+[k].
\]
the second isomorphism being a consequence of the PBW theorem. It follows by a downward induction on \( r \) that
\[
U(\mathfrak{g}[t^+]_+[k] \cong \bigoplus_{r=1}^k S^r(\mathfrak{g}[t^+]_+[k] = S(\mathfrak{g}[t^+]_+[k].
\]
Given a partition \( \mathbf{n} = (n_s \geq n_{s-1} \geq \cdots \geq n_1 > 0) \) of \( k \), let \( V(\mathbf{n}) \) be the subspace of \( S(\mathfrak{g}[t^+]_+[k] \) spanned by the elements
\[
\{(x_{j_1}t^{n_1}) \cdots (x_{j_r}t^{n_r}) : x_j \in \mathfrak{g}\}.
\]
Clearly \( V(\mathbf{n}) \) is a \( \mathfrak{g} \)-submodule of \( S(\mathfrak{g}[t^+]_+[k] \) and we have
\[
S(\mathfrak{g}[t^+]_+[k] = \bigoplus_{\mathbf{n},k} V(\mathbf{n}).
\]
The result follows since
\[ V(\mathfrak{u}) \cong \mathfrak{g} S^{r_1}(\mathfrak{g}) \otimes \cdots \otimes S^{r_k}(\mathfrak{g}), \]
where \( r_j = \{|1 \leq r \leq s : n_r = j\} \).

Part (vi) is now obvious. To establish (vii), it is enough to observe that by (vi) the natural map
\[ \text{Hom}_G(P(\lambda, r), V) \to \text{Hom}_{G_{\leq r}}(P(\lambda, r)_{\leq r}, V_{\leq r}) \]
is injective and hence is an isomorphism (Proposition 1.5(i)). The statement follows since \( P(\lambda, r)_{\leq r} \cong V(\lambda, r) \) is projective in \( G_{\leq r} \) and every object in \( G_{\leq r} \) has finite length. \( \square \)

In what follows, we shall write
\[ S^{(k)}(\mathfrak{g}) = \bigoplus_{(r_1, \ldots, r_k) \in \mathbb{Z}_+: \sum r_j = k} S^{r_1}(\mathfrak{g}) \otimes \cdots \otimes S^{r_k}(\mathfrak{g}). \]
Observe that \( S^{(k)}(\mathfrak{g}) \) is a \( \mathfrak{g} \)-module quotient of \( \mathfrak{g}^{\otimes k} \). Indeed, the map \( \mathfrak{g}^{\otimes k} \to \text{U}(\mathfrak{g}[t])_+[k] \cong \mathfrak{g} S^{(k)}(\mathfrak{g}) \) given by extending \( x_1 \otimes \cdots \otimes x_k \mapsto (x_1t) \cdots (x_kt) \), \( x_j \in \mathfrak{g}, 1 \leq j \leq k \), is a surjective \( \mathfrak{g} \)-module homomorphism.

### 2.2. Morphisms between projectives

The next proposition is trivial, but we include it here explicitly since it is used repeatedly in the later sections where we construct examples of interesting subcategories of \( G \).

Let \( s \leq r \leq \ell \in \mathbb{Z}_+ \). It is immediate from the definitions and Proposition 2.1 that
\[ \text{Hom}_\mathfrak{g}(P(\lambda, r)_{\leq \ell}, P(\mu, s)_{\leq \ell}) = \text{Hom}_\mathfrak{g}(P(\lambda, r), P(\mu, s)) \cong \text{Hom}_\mathfrak{g}(\text{U}(\lambda), \text{U}(\mathfrak{g}[t]_+)[r - s] \otimes \text{V}(\mu)). \] (2.2)

In this section we make the last isomorphism explicit. Let \( s \leq r \in \mathbb{Z}_+ \) and \( \lambda, \mu \in P^+ \). Given \( f \in \text{Hom}_\mathfrak{g}(\text{U}(\lambda), \text{U}(\mathfrak{g}[t]_+)[r - s] \otimes \text{V}(\mu)) \), define \( f \in \text{Hom}_\mathfrak{g}(P(\lambda, r), P(\mu, s)) \) by
\[ f(u \otimes v) = \sum_p u_p \otimes v_p, \]
where \( u \in \text{U}(\mathfrak{g}[t]_+), v \in \text{V}(\lambda, r) \) and \( \sum_p u_p \otimes v_p = f(v) \). Let \( m : \text{U}(\mathfrak{g}[t]) \otimes \text{U}(\mathfrak{g}[t]) \to \text{U}(\mathfrak{g}[t]) \) be the multiplication map. The next proposition is a straightforward consequence of Proposition 2.1 and we omit the details of the calculations.

**Proposition.** Let \((\lambda, r), (\mu, s) \in \Lambda \).

(i) The assignment \( f \mapsto f[r] \) defines an isomorphism of vector spaces
\[ \phi^{(\lambda, r)}_{(\mu, s)} : \text{Hom}_\mathfrak{g}(P(\lambda, r), P(\mu, s)) \to \text{Hom}_\mathfrak{g}(\text{U}(\lambda), \text{U}(\mathfrak{g}[t]_+)[r - s] \otimes \text{V}(\mu)). \]
Moreover if \( f \in \text{Hom}_\mathfrak{g}(P(\lambda, r), P(\mu, s)), g \in \text{Hom}_\mathfrak{g}(P(\mu, s), P(\nu, k)) \), then
\[ (g \circ f)[r] = (m \otimes 1) \circ (1 \otimes g[s]) \circ f[r]. \]

(ii) The assignment \( f \mapsto f \) is an isomorphism
\[ \psi^{(\lambda, r)}_{(\mu, s)} : \text{Hom}_\mathfrak{g}(\text{U}(\lambda), \text{U}(\mathfrak{g}[t]_+)[r - s] \otimes \text{V}(\mu)) \to \text{Hom}_\mathfrak{g}(P(\lambda, r), P(\mu, s)) \]
which is the inverse of $\phi_{\mu,s}^{(\lambda)}$. Moreover, if $f \in \text{Hom}_g(V(\lambda), U(g[t])_{-1}[r-s] \otimes V(\mu))$, $g \in \text{Hom}_g(V(\mu), U(g[t])_{-1}[s-k] \otimes V(\nu))$ then
\[
\psi_{(\nu,k)}(m \otimes 1) \circ (1 \otimes g) \circ f = \psi_{(\mu,s)}(g) \circ \psi_{(\lambda)}(f).
\]

### 2.3. Duality in $G_{\leq s}$. Given $V, W \in \text{Ob} G$, the vector space $\text{Hom}_C(V, W)$ is a $g[t]$-module with respect to the usual action
\[
((xt^r) \cdot f)(v) = (xt^r)f(v) - f((xt^r)v),
\]
for all $x \in g$, $r \in \mathbb{Z}_+$, $f \in \text{Hom}_C(V, W)$ and $v \in V$. For $k \in \mathbb{Z}_+$, set
\[
\text{Hom}_C(V, W)[k] = \bigoplus_{i \in \mathbb{Z}_+} \text{Hom}_C(V[i], W[i + k])
\]
and define
\[
\text{Hom}_C^+(V, W) = \bigoplus_{k \in \mathbb{Z}_+} \text{Hom}_C(V, W)[k].
\]
Since $V[i] = 0$ for all but finitely many $i \in \mathbb{Z}_+$, $\dim \text{Hom}_C(V, W)[k] < \infty$. Notice that $\text{Hom}_C^+(V, W) = \text{Hom}_C(V, W)$ provided that $V \in \text{Ob} G_{\leq r}$ and $W[i] = 0$ for all $i < r$. The proof of the following proposition is quite standard and is omitted.

**Proposition.** Let $V, W \in \text{Ob} G$, $r, s \in \mathbb{Z}_+$.

(i) For all $x \in g$, $i, r, k \in \mathbb{Z}_+$ and $f \in \text{Hom}_C(V[i], W[i + k]),$
\[
(xt^r) \cdot f \in \text{Hom}_C(V[i], W[i + k + r]) \oplus \text{Hom}_C(V[i - r], W[i + k]).
\]
In particular, $\text{Hom}_C^+(V, W) \in \text{Ob} G$ and
\[
W \in \text{Ob} G_{\leq s} \implies \text{Hom}_C^+(V, W) \in \text{Ob} G_{\leq s}.
\]
(ii) Let $\#^s : G_{\leq s} \to G_{\leq s}$ be the contravariant functor defined by $\text{Hom}_C(-, V(0, s))$. Then $\#^s$ is exact and for all $\lambda \in P^+$, $r \leq s$,
\[
(V^\#)^s \cong \tau_{s-r}V, \quad V(\lambda, r)^\# \cong V(-w_\lambda s - r).
\]
In particular, $\#^s$ defines an involutive auto-duality on the category $G_{\leq s}$.

(iii) Suppose that $V \in \text{Ob} G_{\leq r}, W \in \text{Ob} G_{\leq s}$. As objects in $G$,
\[
(V \otimes W)^{\#}_{r+s} \cong V^\# \otimes W^\#, \quad V \otimes W^\# \cong \text{Hom}_C(W, V(0, s) \otimes V).
\]
(iv) For all $V, W' \in \text{Ob} G$, $W \in \text{Ob} G_{\leq s}$, we have an isomorphism of vector spaces,
\[
\text{Hom}_G(V, W \otimes W') \cong \text{Hom}_G(V \otimes W^\#, V(0, s) \otimes W').
\]

### 2.4. Injective objects in $G$ and $\hat{G}$. We begin with the following remark: any injective object of $G$ is also injective in $\hat{G}$. To prove this, let $I \in \text{Ob} G$ be injective and assume that $I[s] = 0$ for all $s \geq r$. Suppose that $i \in \text{Hom}_{\hat{G}}(V, W)$ is injective and let $f \in \text{Hom}_G(V, I)$. Since $f_{\leq r} \in \text{Hom}_G(V_{\leq r}, I)$ and $i_{\leq r} \in \text{Hom}_{\hat{G}}(V_{\leq r}, W_{\leq r})$ is injective there exists $\tilde{f} \in \text{Hom}_{\hat{G}}(W_{\leq r}, I)$ such that $\tilde{f} \circ i_{\leq r} = f_{\leq r}$. Let $f = \tilde{f} \circ p_r(W)$ where $p_r(W) : W \to W_{\leq r}$ be the canonical projection. It is now easily checked that $f \circ i = f$. 
For \((\lambda, r) \in \Lambda\), set
\[ I(\lambda, r) \cong P(-w_0, 0)_{\leq r}^{\# r}. \]

It follows from Proposition 2.5 and Proposition 2.3 that \( I(\lambda, r) \cong I(\lambda, 0) \otimes V(0, r) \).

**Proposition.** Let \((\lambda, r) \in \Lambda\).

(i) The object \(I(\lambda, r)\) is the injective envelope of \(V(\lambda, r)\) in \(G\).

(ii) For \(k \in \mathbb{Z}_+\) we have
\[ I(\lambda, r)[r - k] \cong_\mathfrak{g} S^{(k)}(\mathfrak{g}) \otimes V(\lambda). \]

(iii) Let \((\mu, s) \in \Lambda\). Then \([I(\lambda, r)/V(\lambda, r) : V(\mu, s)] \neq 0\) only if \((\mu, s) \prec (\lambda, r)\).

**Proof.** It is immediate from Proposition 2.5 that \(I(\lambda, r)\) is injective in \(G_{\leq r}\). To prove that \(I(\lambda, r)\) is injective in \(G\), it suffices now to show that \(\text{Ext}^1_G(V(\mu, s), I(\lambda, r)) = 0\) if \(s > r\), in other words that every short exact sequence of the form
\[ 0 \to I(\lambda, r) \to V \to V(\mu, s) \to 0 \]
splits if \(s > r\). Writing \(V = \bigoplus_{k \in \mathbb{Z}_+} V[k]\), we see that \(V[k] = 0\) if \(k > s\). Hence \(\mathfrak{g}[t]V[s] \subset V[s]\). Moreover, since \(\bigoplus_{k \leq r} V[k] \cong_\mathfrak{g} I(\lambda, r)\) it follows now that we have a decomposition of \(\mathfrak{g}[t]\)-modules
\[ V \cong I(\lambda, r) \oplus V(\mu, s) \]
and hence the short exact sequence splits. To prove that it is the injective envelope of \(V(\lambda, r)\) it suffices to use Proposition 2.1 and Lemma 1.1 to notice that \(V(\lambda, r)\) is the unique irreducible subobject of \(I(\lambda, r)\).

The proof of (ii) and (iii) are immediate from Lemma 1.1 and Proposition 2.1.

**Corollary.** Let \(V, W \in \text{Ob } G\).

(i) For all \(j \geq 0\), we have
\[ \text{Ext}^j_G(V, W) \cong \text{Ext}^j_G(V, W). \]

(ii) Let \(I\) be the injective envelope of \(V\). If \((\lambda, r) \in \Lambda(I)\) then \((\lambda, r) \preceq (\mu, s)\) for some \((\mu, s) \in \Lambda(\text{soc}(V))\).

**2.5. Extensions between simple objects.** The following proposition proves Theorem 1.

**Proposition.** For \((\lambda, r), (\mu, s) \in \Lambda\), we have
\[ \text{Ext}^1_G(V(\lambda, r), V(\mu, s)) = 0, \quad s \neq r + 1, \]
\[ \text{Ext}^1_G(V(\lambda, r), V(\mu, r + 1)) \cong \text{Hom}_\mathfrak{g}(V(\lambda), \mathfrak{g} \otimes V(\mu)). \]

In other words, \(\text{Ext}^1_G(V(\lambda, r), V(\mu, s)) = 0\) unless \((\mu, s)\) covers \((\lambda, r)\).

**Proof.** Applying \(\text{Hom}_G(V(\lambda, r), -)\) to the short exact sequence
\[ 0 \to V(\mu, s) \to I(\mu, s) \to J(\mu, s) \to 0 \]
gives
\[ \text{Hom}_G(V(\lambda, r), J(\mu, s)) \cong \text{Ext}^1_G(V(\lambda, r), V(\mu, s)). \]
The proposition obviously follows if we prove that

\[ \text{Hom}_G(V(\lambda, r), J(\mu, s)) \cong \begin{cases} \text{Hom}_G(g \otimes V(\lambda), V(\mu)), & \text{if } s = r + 1, \\ 0, & \text{otherwise}. \end{cases} \]

Let \( \psi : V(\lambda, r) \rightarrow J(\mu, s) \) be a non-zero element of \( \text{Hom}_G(V(\lambda, r), J(\mu, s)) \). It follows from Proposition 2.4 that \((\lambda, r) \prec (\mu, s)\) and hence in particular that \( r < s \). Suppose that \( r < s - 1 \). Since \( V(\mu, s) \) is essential in \( I(\mu, s) \), there exists \( V \subset I(\mu, s) \) such that \( V/V(\mu, s) \cong \psi(V(\lambda, r)) \). Then \( V = V[s] \oplus V[r] \) and since \( r < s - 1 \), it follows that \( g[t]V[r] \subset V \). Hence \( V[r] \) is in \( \text{soc}(I(\mu, s)) \) which is impossible. Thus, \( s = r - 1 \).

The following isomorphisms which are consequences of Proposition 2.4 establish the proposition.

\[ \text{Hom}_g(V(\lambda), g \otimes V(\mu)) \cong \text{Hom}_G(V(\lambda, r), \tau_r \text{ev}(g \otimes V(\mu))), \]

\[ \tau_r \text{ev}(J(\mu, r + 1)[r]) \cong_G \text{Hom}_g(V(\lambda), \tau_r \text{ev}(g \otimes V(\mu))), \]

\[ \text{Hom}_G(V(\lambda, r), J(\mu, r + 1)) \cong \text{Hom}_G(V(\lambda, r), \tau_r \text{ev}(J(\mu, r + 1)[r])). \]

\[ \square \]

2.6. Given \( \Gamma \subset \Lambda \), set

\[ V_{\Gamma}^+ = \{ v \in V[s] : n^+v = 0, (\mu, s) \in \Gamma \}, \]

\[ V_{\Gamma} = U(g) V_{\Gamma}^+, \quad V^G = V/V_{\Lambda \setminus \Gamma}. \]

Furthermore, given \( f \in \text{Hom}_G(V, W) \), let \( f_{\Gamma} := f|_{V_{\Gamma}} \) and let \( f^G \) be the induced map \( V^G \rightarrow W^G \). It follows from the definitions that \( f_{\Gamma} \) and \( f^G \) are morphisms of graded vector spaces and \( g \)-modules.

**Proposition.** Let \( V \in \text{Ob} \mathcal{G} \) and let \( \Gamma \) be an interval closed subset of \( \Lambda \).

(i) Suppose that for each \( (\lambda, r) \in \Lambda(V) \setminus \Gamma \) there exists \( (\mu, s) \in \Gamma \) with \( (\lambda, r) \prec (\mu, s) \). Then \( V_{\Gamma} \in \text{Ob} \mathcal{G}[\Gamma] \) and \( V/V_{\Gamma} \in \text{Ob} \mathcal{G}[\Lambda \setminus \Gamma] \).

(ii) Suppose that for each \( (\lambda, r) \in \Lambda(V) \setminus \Gamma \) there exists \( (\mu, s) \in \Gamma \) with \( (\mu, s) \prec (\lambda, r) \). Then \( V_{\Lambda \setminus \Gamma} \in \text{Ob} \mathcal{G}[\Lambda \setminus \Gamma] \) and \( V^G \in \text{Ob} \mathcal{G}[\Gamma] \).

(iii) Let

\[ 0 \rightarrow V \xrightarrow{f} U \xrightarrow{g} W \rightarrow 0 \]

be a short exact sequence in \( \mathcal{G} \). Then \( U \) satisfies (ii) (respectively, (iii)) if and only if \( V \) and \( W \) satisfy (i) (respectively, (iii)) and

\[ 0 \rightarrow V_{\Gamma} \xrightarrow{f_{\Gamma}} U_{\Gamma} \xrightarrow{g_{\Gamma}} W_{\Gamma} \rightarrow 0 \]

(respectively,

\[ 0 \rightarrow V^G \xrightarrow{f^G} U^G \xrightarrow{g^G} W^G \rightarrow 0 \])

is an exact sequence of objects in \( \mathcal{G}[\Gamma] \).

**Proof.** Consider the map \( g \otimes V_{\Gamma} \rightarrow V \) given by \( x \otimes v \mapsto (xt)v \). This is clearly a map of \( g \)-modules. Let \( U \) be a \( g \)-module complement to \( V_{\Gamma} \) in \( V \). Suppose that \( V_{\Gamma} \) is not a subobject of \( V \) in \( \mathcal{G} \), i.e. there exists \( x \in g \) and \( v \in V_{\Gamma} \) such that \( (xt)v \notin V_{\Gamma} \). Since \( (g \otimes t)U(g) \subset U(g)(g \otimes t) \), we may assume without loss of generality that \( v \in V_{\Gamma}^+ \cap V[r]_\lambda \) for some \( (\lambda, r) \in \Lambda \) and hence
\( U(\mathfrak{g})v \cong V(\lambda, r) \). In other words, the induced map of \( \mathfrak{g} \)-modules \( \mathfrak{g} \otimes V(\lambda, r) \to U \) is non-zero and so there exists \( (\nu, r + 1) \not\in \Gamma \) such that the composite map \( \mathfrak{g} \otimes V(\lambda, r) \to U \to V(\nu, r + 1) \) is non-zero. This implies immediately that \( (\lambda, r) \prec (\nu, r + 1) \). Choose \( (\mu, s) \in \Gamma \) such that \( (\nu, r + 1) \prec (\mu, s) \) for some \( (\mu, s) \in \Gamma \). This gives \( (\nu, r + 1) \in [(\lambda, r), (\mu, s)] \) which is impossible since \( \Gamma \) is interval closed. Hence

\[(xt)v \in V_\Gamma, \quad \forall x \in \mathfrak{g}, \: v \in V_\Gamma^+,\]

and (ii) is proved.

In order to prove the second part, assume that \( V_{\Lambda \setminus \Gamma} \) is not a \( \hat{G} \)-subobject of \( V \). Then \( (xt)v \not\in V_{\Lambda \setminus \Gamma} \) for some \( x \in \mathfrak{g}, \: v \in V_{\Lambda \setminus \Gamma} \) and as before we may assume, without loss of generality that \( v \in V_{\Lambda \setminus \Gamma} \cap V[\lambda] \) for some \( (\lambda, r) \in \Lambda \setminus \Gamma \). Let \( U' \) be a \( \mathfrak{g} \)-module complement of \( V_{\Lambda \setminus \Gamma} \). Then we have a non-zero \( \mathfrak{g} \)-module map \( \mathfrak{g} \otimes V(\lambda, r) \to U' \), given by extending \( x \otimes v \mapsto (xt)v \), and hence a non-zero \( \mathfrak{g} \)-module map \( \mathfrak{g} \otimes V(\lambda, r) \to V(\nu, r + 1) \) for some \( (\nu, r + 1) \in \Gamma \). By assumption, there exists \( (\mu, s) \in \Gamma \) such that \( (\mu, s) \prec (\lambda, r) \). Thus, \( (\lambda, r) \in [(\mu, s), (\nu, r + 1)] \cap \Gamma \) since \( \Gamma \) is interval closed, which is a contradiction.

The first statement of (iii) is obvious since \( \Lambda(U) = \Lambda(V) \cup \Lambda(W) \). For the second, note that we have

\[ U \cong V \oplus W, \]

as graded \( \mathfrak{g} \)-modules and hence as \( \mathfrak{g} \)-module we have

\[ U_\Gamma \cong V_\Gamma \oplus W_\Gamma. \]

Since \( V_\Gamma, W_\Gamma \) and \( U_\Gamma \) are objects in \( \mathcal{G}[\Gamma] \) the result follows. \( \square \)

**Remark.** Part (i) of the preceding proposition holds for all \( V \in \text{Ob} \mathcal{G} \) and \( \Gamma \subset \Lambda \) such that \( \Lambda(\text{soc}(V)) \subset \Gamma \) while part (ii) holds if \( \Lambda(\text{head}(V)) \subset \Gamma \). In fact, for any \( (\lambda, r) \in \Lambda(V) \), there exists \( (\mu, s) \in \Lambda(\text{soc}(V)) \) and \( (\nu, p) \in \Lambda(\text{head}(V)) \) such that

\[ (\nu, p) \preceq (\lambda, r) \preceq (\mu, s). \]

This is an immediate consequence of the fact that \( V \) embeds in the injective envelope of \( \text{soc}(V) \) and is a quotient of \( P(\text{head}(V)) \) together with Proposition 2.4(iii) and Proposition 2.4(iii). Moreover, in this case if we let \( \overline{V_\Gamma} \) be the maximal subobject of \( V \) that is in \( \mathcal{G}[\Gamma] \), then the Proposition implies that

\[ \overline{V_\Gamma} \cong V_\Gamma \]

if for each \( (\lambda, r) \in \Lambda(V) \setminus \Gamma \) there exists \( (\mu, s) \in \Gamma \) with \( (\lambda, r) \preceq (\mu, s) \) and similarly for \( \overline{V_\Gamma^+} \).

### 2.7

We isolate some consequences of the preceding Proposition since we use them repeatedly in the following sections.

**Proposition.** Let \( \Gamma \) be finite and interval closed and assume that \( (\lambda, r), (\mu, s) \in \Gamma \).

(i) The object \( I(\lambda, r)_\Gamma \) is the injective envelope of \( V(\lambda, r) \) in \( \mathcal{G}[\Gamma] \) while \( P(\lambda, r)^\Gamma \) is the projective cover of \( V(\lambda, r) \) in \( \mathcal{G}[\Gamma] \). In particular, \( \mathcal{G}[\Gamma] \) has enough projectives.

(ii) We have

\[ [P(\lambda, r)^\Gamma : V(\mu, s)] = [P(\lambda, r) : V(\mu, s)] = [I(\mu, s) : V(\lambda, r)] = [I(\mu, s)^\Gamma : V(\lambda, r)]. \]
(iii) For all $j \geq 0$, we have
\[ \text{Ext}_j^G(V(\lambda, r), V(\mu, s)) \cong \text{Ext}_j^{G[\Gamma]}(V(\lambda, r), V(\mu, s)). \]
(iv) Let $p^\Gamma(\lambda, r) : P(\lambda, r) \rightarrow P(\lambda, r)^\Gamma$ (respectively, $i_\Gamma(\lambda, r) : I(\lambda, r) \hookrightarrow I(\lambda, r)$) be the canonical projection (respectively, the canonical embedding). There exists an isomorphism
\[ \text{Hom}_G(P(\lambda, r), P(\mu, s)) \rightarrow \text{Hom}_{G[\Gamma]}(P(\lambda, r)^\Gamma, P(\mu, s)^\Gamma) \]
given by $f \rightarrow f^\Gamma$ such that
\[ p^\Gamma(\mu, s) \circ f = f^\Gamma \circ p^\Gamma(\lambda, r), \]
and similarly an isomorphism $\text{Hom}_G(I(\mu, s), I(\lambda, r)) \rightarrow \text{Hom}_{G[\Gamma]}(I(\mu, s)^\Gamma, I(\lambda, r)^\Gamma)$ given by $g \rightarrow g^\Gamma$ such that
\[ g \circ i_\Gamma(\mu, s) = i_\Gamma(\lambda, r) \circ g^\Gamma. \]

Proof. It follows from Proposition 2.4(iii) that
\[(\nu, k) \prec (\lambda, r) \quad \forall (\nu, k) \in \Lambda(I(\lambda, r)) \setminus \{(\lambda, r)\}.
Proposition 2.6 now gives,
\[ I(\lambda, r)^\Gamma \in \text{Ob} G[\Gamma], \quad \text{soc}(I(\lambda, r)^\Gamma) = V(\lambda, r) \]
and
\[ \text{Hom}_G(V(\mu, s), I(\lambda, r)^\Gamma) = 0, \quad \forall (\mu, s) \in \Gamma. \]

It follows immediately that $\text{Ext}_1^{G[\Gamma]}(V(\mu, s), I(\lambda, r)^\Gamma) = 0$ for all $(\mu, s) \in \Gamma$ which implies the first statement in (i). The proof of the second statement is similar. The first and the last equality in (ii) follow immediately from Proposition 2.6 while the second equality is an obvious consequence of Proposition 2.1 and Proposition 2.4.

Part (iii) is obvious if $j = 0$. Set $Q_{-1}(\mu, s) = V(\mu, s)$. For $j \geq 0$ define inductively the objects $I_j(\mu, s)$ as the injective envelope of $Q_{j-1}(\mu, s)$ in $G$ and $Q_j(\mu, s) = \text{coker}(Q_{j-1}(\mu, s) \hookrightarrow I_j(\mu, s))$. Then
\[ 0 \rightarrow V(\mu, s) \rightarrow I_0(\mu, s) \rightarrow I_1(\mu, s) \rightarrow \cdots \rightarrow I_k(\mu, s) \rightarrow 0 \]
is an injective resolution for $V(\mu, s)$ in $G$ and
\[ \text{Ext}_j^G(V(\lambda, r), V(\mu, s)) \cong \text{Hom}_G(V(\lambda, r), I_j(\mu, s)), \quad j > 0. \tag{2.6} \]
It follows from Corollary 2.4(iii) by a straightforward induction on $j$ that
\[ (\nu, k) \neq (\mu, s) \in \Lambda(I_j(\mu, s)) \cup \Lambda(Q_j(\mu, s)) \implies (\nu, k) \prec (\mu, s). \]
Hence by Proposition 2.6
\[ Q_j(\mu, s)^\Gamma, I_j(\mu, s)^\Gamma \in \text{Ob} G[\Gamma], \quad I_j(\mu, s)/I_{j-1}(\mu, s)^\Gamma \in G[\Lambda \setminus \Gamma] \]
and the sequence
\[ 0 \rightarrow V(\mu, s) \rightarrow I_0(\mu, s)^\Gamma \rightarrow I_1(\mu, s)^\Gamma \rightarrow \cdots \rightarrow I_k(\mu, s)^\Gamma \rightarrow 0 \tag{2.7} \]
is exact in $G[\Gamma]$. It follows from part (i) that (2.7) is an injective resolution of $V(\mu, s)$ in $G[\Gamma]$. Furthermore, for all $(\lambda, r) \in \Gamma$
\[ \text{Hom}_G(V(\lambda, r), I_j(\mu, s)) \cong \text{Hom}_G(V(\lambda, r), I_j(\mu, s)^\Gamma) \tag{2.8} \]
and similarly
\[ \text{Hom}_G(V(\lambda, r), Q_j(\mu, s)) \cong \text{Hom}_G(V(\lambda, r), Q_j(\mu, s)^\Gamma). \]
In particular, this implies that
\[ \text{soc}(I_j(\mu, s)_\Gamma) \cong \text{soc}(Q_{j-1}(\mu, s)_\Gamma) \]
and so \( I_j(\mu, s)_\Gamma \) is the injective envelope of \( Q_{j-1}(\mu, s)_\Gamma \) in \( \mathcal{G}[\Gamma] \). By Proposition 2.6(iii), \( I_j(\mu, s)_\Gamma/Q_{j-1}(\mu, s)_\Gamma \cong Q_j(\mu, s)_\Gamma \). Then
\[ \text{Ext}^j_{\mathcal{G}[\Gamma]}(V(\lambda, r), V(\mu, s)) \cong \text{Hom}_{\mathcal{G}[\Gamma]}(V(\lambda, r), I_j(\mu, s)_\Gamma) \]
and (iii) follows from (2.6) and (2.8).

To prove (iv), let \( f \in \text{Hom}_\mathcal{G}(P(\lambda, r), P(\mu, s)) \), \( f \neq 0 \). Then \( f^\Gamma \neq 0 \) since \( f^\Gamma(1 \otimes V(\lambda, r)) = f(1 \otimes V(\lambda, r)) \mod P(\lambda, r)_{\Lambda[\Gamma]} \neq 0 \). Thus, we have an injective map
\[ \text{Hom}_\mathcal{G}(P(\lambda, r), P(\mu, s)) \to \text{Hom}_{\mathcal{G}[\Gamma]}(P(\lambda, r)^\Gamma, P(\mu, s)^\Gamma). \]
Since both spaces have the same dimension by (iii) and Proposition 2.2, the isomorphism follows. To prove the statement for injectives, observe that the natural map
\[ \text{Hom}_\mathcal{G}(I(\mu, s), I(\lambda, r)) \to \text{Hom}_\mathcal{G}(I(\mu, s)_\Gamma, I(\lambda, r)_\Gamma) \]
is surjective, while \( \text{Hom}_\mathcal{G}(I(\mu, s)_\Gamma, I(\lambda, r)_\Gamma) \cong \text{Hom}_\mathcal{G}(I(\mu, s)_\Gamma, I(\lambda, r)_\Gamma) \). The assertion follows from (iii).

\section{3. Algebras associated with the category \( \mathcal{G} \)}

In this section, we let \( \Gamma \) be a finite interval closed subset of \( \Lambda \). Given a finite dimensional algebra \( A \), let \( A - \text{mod}_f \) (respectively, \( \text{mod}_f - A \)) be the category of finite dimensional left (respectively, right) \( A \)-modules.

\subsection*{3.1. The algebra \( \mathfrak{A}(\Gamma) \) and an equivalence of categories.}

Set
\[ I(\Gamma) = \bigoplus_{(\lambda, r) \in \Gamma} I(\lambda, r), \quad \mathfrak{A}(\Gamma) = \text{End}_\mathcal{G} I(\Gamma). \]
Then \( \mathfrak{A}(\Gamma) \) is an associative algebra. Moreover, it is immediate from Proposition 2.7 that
\[ \mathfrak{A}(\Gamma) \cong \mathfrak{A}_f(\Gamma)_\Gamma := \text{End}_\mathcal{G} I(\Gamma)_\Gamma. \tag{3.1} \]
In particular, \( \mathfrak{A}(\Gamma) - \text{mod}_f \) is equivalent to \( \mathfrak{A}_f(\Gamma) - \text{mod}_f \) and similarly for the categories of right modules. Since \( I(\Gamma)_\Gamma \) is the injective cogenerator of \( \mathcal{G}[\Gamma] \), a standard argument now shows that the contravariant functor \( \text{Hom}_\mathcal{G}(-, I(\Gamma)_\Gamma) \) from \( \mathcal{G}[\Gamma] \) to the category \( \mathfrak{A}_f(\Gamma) - \text{mod}_f \) is exact and provides a duality of categories. Similarly, the functor \( \text{Hom}_\mathcal{G}(-, I(\Gamma)_\Gamma)^* \) from \( \mathcal{G}[\Gamma] \) to the category \( \text{mod}_f - \mathfrak{A}_f(\Gamma) \) is exact and provides an equivalence of categories. Thus, \( \mathcal{G}[\Gamma] \) is equivalent to the category \( \mathfrak{A}_f(\Gamma) - \text{mod}_f \).

It is clear from the definition that the simple objects in \( \mathfrak{A}(\Gamma) - \text{mod}_f \) are one-dimensional, that is to say \( \mathfrak{A}(\Gamma) \) is basic, and their isomorphism classes are parametrized by elements of \( \Gamma \). Given \( (\lambda, r) \in \Gamma \), let \( S_{\lambda, r} \) be the corresponding simple left \( \mathfrak{A}(\Gamma) \)-module.

\textbf{Proposition.} Let \( \Gamma \subset \Lambda \) be finite and interval closed. For all \( (\lambda, r), (\mu, s) \in \Gamma \), we have
\[ \dim \text{Ext}_{\mathfrak{A}[\Gamma]}(S_{\lambda, r}, S_{\mu, s}) = \delta_{r,s+1} \dim \text{Hom}_\mathcal{G}(V(\mu), g \otimes V(\lambda)). \tag{3.2} \]
In particular, the algebra \( \mathfrak{A}(\Gamma) \) is quasi-hereditary.
Proposition. Let paths from $(\lambda, r) \in \Gamma$, then
\[
\text{Hom}_{G[\Gamma]}(V(\mu, s), I(\lambda, r)_{\Gamma}/V(\lambda, r)) \cong \text{Hom}_{G}(V(\mu, s), I(\lambda, r)/V(\lambda, r)).
\]
Hence, we have,
\[
\text{Ext}^1_{\mathcal{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, s}) \cong \text{Ext}^1_{G[\Gamma]}(V(\mu, s), V(\lambda, r)) \cong \text{Ext}^1_{G}(V(\mu, s), V(\lambda, r)).
\]
Equation (3.2) follows from Proposition 2.5 which also proves that \(\mathcal{A}(\Gamma) - \text{mod}_f\) is a directed highest weight category with poset of weights \((\Gamma, \preceq)\).

Now [7, Theorem 3.6] implies that the algebra \(\mathcal{A}(\Gamma)\) is quasi-hereditary.

3.2. Let \(Q(\Gamma)\) be the Ext-quiver of \(\mathcal{A}(\Gamma)\), that is, the quiver whose set of vertices is \(\Gamma\) and the number of arrows from \((\lambda, r)\) to \((\mu, s)\) in \(Q(\Gamma)\) is \(\dim \text{Ext}^1_{\mathcal{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, s})\). Note that the number of paths from \((\lambda, r)\) to \((\mu, s)\) is non-zero only if \((\mu, s) \prec (\lambda, r)\). In particular, \(Q(\Gamma)\) has no oriented loops. Let \(C Q(\Gamma)\) be the path algebra of \(Q(\Gamma)\) and \(C Q(\Gamma)[k]\) be the subspace spanned by all paths of length \(k\). Then
\[
C Q(\Gamma) = \bigoplus_{k \in \mathbb{Z}_+} C Q(\Gamma)[k],
\]
is a tightly graded associative algebra. Since \(\mathcal{A}(\Gamma)\) is basic, a classical result of Gabriel’s (cf. for example [15, 2.1(2)]) proves that \(\mathcal{A}(\Gamma)\) is isomorphic to a quotient of the path algebra \(C Q(\Gamma)\) of \(Q(\Gamma)\) by an ideal \(R(\Gamma)\) which is contained in the ideal of paths of length at least two. In particular this means that an arrow between \((\lambda, r)\) and \((\mu, r-1)\) maps to a non-zero element of \(\text{Hom}_{G}(I(\lambda, r), I(\mu, r-1))\). Given \((\lambda, r) \in \Gamma\), let \(1_{\lambda, r}\) be the corresponding primitive idempotent in \(C Q(\Gamma)\). Note that \(1_{\lambda, r}\) maps to the element \(1_{\lambda, r} \in \text{End}_{G} I(\Gamma)\) defined by \(1_{\lambda, r}(I(\mu, s)) = \delta_{(\lambda, r), (\mu, s)} 1_{\Gamma}\) for \((\mu, s) \in \Gamma\). In particular, \(1_{\mu, s} \mathcal{A}(\Gamma) 1_{\lambda, r} \cong \text{Hom}_{G}(I(\lambda, r), I(\mu, s))\) as a vector space.

3.3. A grading on \(\mathcal{A}(\Gamma)\). Given \(k \leq r \in \mathbb{Z}_+\) define
\[
\mathcal{A}(\Gamma)[k] = \bigoplus_{(\lambda, r), (\mu, r-k) \in \Gamma} \text{Hom}_{G}(I(\lambda, r), I(\mu, r-k)).
\]
Since \([I(\lambda, r) : V(\mu, s)] = 0\) unless \((\mu, s) \preceq (\lambda, r)\) (cf. Proposition 2.4[(ii)\]), it follows immediately that
\[
\mathcal{A}(\Gamma) = \bigoplus_{k \in \mathbb{Z}_+} \mathcal{A}(\Gamma)[k], \quad \mathcal{A}(\Gamma)[j] \mathcal{A}(\Gamma)[k] \subset \mathcal{A}(\Gamma)[j+k], \quad \forall j, k \in \mathbb{Z}_+.
\]
Thus, \(\mathcal{A}(\Gamma)\) is a graded associative algebra and \(\mathcal{A}(\Gamma)[0]\) is a commutative semi-simple subalgebra of \(\mathcal{A}(\Gamma)\). It is trivial to observe that with this grading the algebra \(\mathcal{A}(\Gamma)\) is in fact a graded quotient of \(C Q(\Gamma)\) and hence the ideal \(R(\Gamma)\) is graded. In particular, \(\mathcal{A}(\Gamma)\) is tightly graded.

3.4. The dimension of \(R(\Gamma)\).

Proposition. Let \(\Gamma\) be interval closed and finite and \((\lambda, r), (\mu, s) \in \Gamma\). Then the number of paths from \((\lambda, r)\) to \((\mu, s)\) is \(\dim \text{Hom}_{G}(V(\mu), g^{\otimes (r-s)} \otimes V(\lambda))\).
Proof. Let \( N(\lambda, r), (\mu, r - s) \) be the number of paths in \( CQ(\Gamma) \) from \( (\lambda, r) \) to \( (\mu, r - s) \) and set \( N'(\lambda, \mu, s) = \dim \text{Hom}_g(V(\mu), g^{\otimes s} \otimes V(\lambda)). \) It is easy to see that

\[
N'(\lambda, \mu, 0) = \delta_{\lambda, \mu} = N((\lambda, r), (\mu, r)),
\]

while

\[
N'(\lambda, \mu, 1) = \dim \text{Hom}_g(V(\mu), g \otimes V(\lambda)) = \dim \text{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, r - 1}) = N((\lambda, r), (\mu, r - 1)),
\]

where we used (3.2). We now prove that \( N \) and \( N' \) satisfy the same recurrence relation which establishes the proposition. It is clear that

\[
N((\lambda, r), (\mu, r - s)) = \sum_{\nu \in P^+} N((\nu, r - s + 1), (\mu, r - s))N((\lambda, r), (\nu, r - s + 1)).
\]

On the other hand, note that we can write

\[
g^{\otimes (s-1)} \otimes V(\lambda) \cong \bigoplus_{\nu \in P^+} V(\nu)^{N'(\lambda, \nu, s-1)}.
\]

Tensoring with \( g \) gives,

\[
N'(\lambda, \mu, s) = \dim \bigoplus_{\nu \in P^+} \text{Hom}_g(V(\mu), g \otimes V(\nu))^{N'(\lambda, \nu, s-1)} = \sum_{\nu \in P^+} N'(\lambda, \nu, s-1)N'(\nu, 1),
\]

and the proof is complete. \( \square \)

Corollary. Given \((\mu, r - s), (\lambda, r) \in \Gamma\), we have

\[
\dim 1_{\mu, r - s} R(\Gamma)1_{\lambda, r} = \dim \text{Hom}_g(V(\mu), g^{\otimes s} \otimes V(\lambda)) - \dim \text{Hom}_g(V(\mu), S^{(s)}(g) \otimes V(\lambda)).
\]

In particular, the algebra \( \mathfrak{A}(\Gamma) \) is hereditary if and only if

\[
\dim \text{Hom}_g(V(\mu), g^{\otimes s} \otimes V(\lambda)) = \dim \text{Hom}_g(V(\mu), S^{(s)}(g) \otimes V(\lambda))
\]

for all \((\mu, r - s), (\lambda, r) \in \Gamma\).

Proof. Observe that

\[
\dim 1_{\mu, r - s} R(\Gamma)1_{\lambda, r} = N((\lambda, r), (\mu, r - s)) - \dim id_{\mu, r - s} \mathfrak{A}(\Gamma) id_{\lambda, r} = N((\lambda, r), (\mu, r - s)) - \dim \text{Hom}_g(I(\lambda, r), I(\mu, r - s)).
\]

The first assertion is now immediate from the above Proposition and Proposition 2.4. For the second, it is enough to observe that \( \mathfrak{A}(\Gamma) \) is hereditary if and only if \( R(\Gamma) = 0 \) and that

\[
R(\Gamma) = \bigoplus_{(\lambda, r), (\mu, r - s) \in \Gamma} 1_{\mu, r - s} R(\Gamma)1_{\lambda, r}.
\]

\( \square \)

3.5. Given \( \Gamma \subset \Lambda_{\leq r} \), let \( \Gamma_{\#} = \{(-w_0 \mu, r - s) : (\mu, s) \in \Gamma\} \). It is easy to see that \( \Gamma_{\#} \) is interval closed if and only if \( \Gamma \) is interval closed.

Proposition. Suppose that \( \Gamma \subset \Lambda_{\leq r} \) is finite and interval closed. Then \( \mathfrak{A}(\Gamma_{\#}) \cong \mathfrak{A}(\Gamma)^{op} \).

Proof. Let \((\lambda, r) \in \Gamma\). Then \( V(\lambda, r)^{\#} \) is an object in \( \mathcal{G}[\Gamma_{\#}] \) and it follows that \( (\mathcal{G}[\Gamma])^{\#} = \mathcal{G}[\Gamma_{\#}] \). Thus, \( \mathcal{G}[\Gamma] \) is dual to \( \mathcal{G}[\Gamma_{\#}] \). It follows from 3.1 that \( \mathfrak{A}(\Gamma)^{op} \) and \( \mathfrak{A}(\Gamma_{\#}) \) are Morita equivalent. Since they are both basic, the assertion follows. \( \square \)
4. Examples of $\Gamma$ with $\mathfrak{A}(\Gamma)$ hereditary

Throughout this section we use the notations of [15] for the various types of quivers. This should eliminate confusion with the notation for the types of simple Lie algebras. For instance, $T_{n_1,\ldots,n_r}$ denotes a quiver whose underlying graph is a star with $r$ branches where the $i$th branch contains $n_i$ vertices, while $\tilde{X}_k$ denotes the quiver whose underlying graph is of affine Dynkin type $X_k^{(1)}$. Notice that if $\Gamma \subset \Lambda \leq r$, then Proposition 3.5 implies that $Q(\Gamma^\#_r)$ is the opposite quiver of $Q(\Gamma)$.

4.1. The generalized Kronecker quivers. Let $\lambda \in \mathbb{P}^+$ be non-zero and let

$$k_\lambda = |\{i \in I : \lambda(h_i) > 0\}|.$$ 

It is easily checked, by using Lemma 1.1 that

$$\dim \text{Hom}_g(V(\lambda), g \otimes V(\lambda)) = k_\lambda.$$ 

For $r \in \mathbb{Z}_+$, set $\Gamma_{\lambda,r} = \{(\lambda, r), (\lambda, r + 1)\}$. It follows that $Q(\Gamma_{\lambda,r})$ is the quiver with $k_\lambda$ arrows from $(\lambda, r + 1)$ to $(\lambda, r)$. If $k_\lambda = 1$, the quiver is of type $\tilde{A}_2$, if $k_\lambda = 2$ it is the Kronecker quiver $\tilde{A}_1$, while for $k_\lambda > 2$ we get the generalized Kronecker quiver. Since there are no paths of length two in these quivers, it follows that $A(\Gamma_{\lambda,r}) \cong CQ(\Gamma_{\lambda,r})$.

4.2. Quivers of type $\tilde{D}_4$. Suppose that $g$ is not of type $\mathfrak{sl}_2$. Let $I_* = \{i \in I : \theta - \alpha_i \in R^+\}$. Note that $|I_*| = 1$ if $g$ is not of type $\mathfrak{sl}_{n+1}$ and let $i_*$ be the unique element of $I_*$. If $g \cong \mathfrak{sl}_{n+1}$, $n > 1$, $I_* = \{1, n\}$.

Let $r \in \mathbb{Z}_+$. If $g$ is not of type $\mathfrak{sl}_{n+1}$ set

$$\Gamma = \{(\theta, r), (0, r + 1), (2\theta, r + 1), (2\theta - \alpha_{i_*}, r + 1), (\theta, r + 1)\}.$$ 

Otherwise, set

$$\Gamma = \{(\theta, r), (0, r + 1), (2\theta, r + 1), (2\theta - \alpha_1, r + 1), (2\theta - \alpha_n, r + 1)\}.$$ 

In the first case, we find that $Q(\Gamma)$ is

$$\begin{array}{c}
(2\theta - \alpha_{i_*}, r + 1) \\
(0, r + 1) \\
(\theta, r) \\
(\theta, r + 1) \\
(\theta, r + 1)
\end{array}$$

while in the second case $Q(\Gamma)$ is

$$\begin{array}{c}
(2\theta - \alpha_1, r + 1) \\
(0, r + 1) \\
(\theta, r) \\
(2\theta, r + 1) \\
(2\theta - \alpha_n, r + 1)
\end{array}$$
The algebra $\mathfrak{A}(\Gamma)$ is hereditary since any path in $Q(\Gamma)$ has length at most one. Note that $\Gamma$ can be shifted by any $\lambda \in P^+$ such that $\dim \text{Hom}_g(V(\lambda + \theta), g \otimes V(\lambda + \theta)) = 1$.

4.3. Quivers of type $\mathbb{A}_k$. If $g$ is not of type $\mathfrak{sl}_2$ choose $i_\bullet \in I_\bullet$.

**Proposition.** Fix $\lambda \in P^+$ with $\lambda(h_{i_\bullet}) \neq 0$, $\ell \in \mathbb{Z}_+$. Let $r_j \in \mathbb{Z}_+$, $0 \leq j \leq \ell$ be such that $|r_k - r_{k+1}| = 1$ for $0 \leq k \leq \ell - 1$. Let $\alpha \in \{\theta, \theta - \alpha_\bullet\}$. The set

$$\Gamma = \{(\lambda + j\alpha, r_j) : 0 \leq j \leq \ell\}$$

is interval closed, the quiver $Q(\Gamma)$ is of type $\mathbb{A}_{k+1}$ and the algebra $\mathfrak{A}(\Gamma)$ is hereditary.

**Proof.** We prove the proposition in the case when $\alpha = \theta - \alpha_\bullet$; the proof in the other case being similar and in fact simpler. Suppose that for some $0 \leq j, j' \leq \ell$ and $(\mu, s) \in \Lambda$, we have,

$$(\lambda + j\alpha, r_j) < (\mu, s) < (\lambda + j'\alpha, r_{j'})$$

then we have

$$\mu = \lambda + j'\alpha - \sum_{p=1}^{r_{j'} - s} \beta_p = \lambda + j\alpha + \sum_{q=1}^{s - r_j} \gamma_q,$$

and

$$r_j < s < r_{j'},$$

for some $\beta_p, \gamma_q \in R \cup \{0\}$. Assuming without loss of generality that $j \leq j'$, we find in particular, $0 < r_{j'} - r_j \leq j' - j$ and

$$(j' - j)(\theta - \alpha_\bullet) = \sum_{p=1}^{r_{j'} - s} \beta_p + \sum_{q=1}^{s - r_j} \gamma_q.$$

Equating the coefficients of $\alpha_\bullet$, $i \neq i_\bullet$ on both sides of the above expression we conclude that

$$\beta_p, \gamma_q \in \{\theta, \theta - \alpha_\bullet\}, \quad r_{j'} - r_j = j' - j,$$

for $1 \leq p \leq r_{j'} - s$ and $1 \leq q \leq s - r_j$, which gives

$$r_p = r_j + (p - j), \quad j \leq p \leq j'.$$

Next, equating the coefficient of $\alpha_\bullet$ on both sides now gives $\beta_p = \gamma_q = \theta - \alpha_\bullet$ for all $1 \leq p \leq r_{j'} - s$, $1 \leq q \leq s - r_j$. This proves that

$$(\mu, s) = (\lambda + s\alpha, s) = (\lambda + s\alpha, r_s),$$

and hence $(\mu, s) \in \Gamma$.

It follows from Proposition 4.1 that

$$\dim \text{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda + j\alpha, r_j}, S_{\lambda + k\alpha, r_k}) = \delta_{r_j - r_k, 1} \dim \text{Hom}_g(V(\lambda + k\alpha), g \otimes V(\lambda + j\alpha)),$$

and applying Lemma 1.3 now gives

$$\dim \text{Ext}^1_{\mathfrak{A}(\Gamma)}(S_{\lambda + j\alpha, r_j}, S_{\lambda + k\alpha, r_k}) = \delta_{r_j - r_k, 1} \delta_{|j - k|, 1}.$$

This shows that there is precisely one arrow between $(\lambda + j\alpha, r_j)$ and $(\lambda + (j \pm 1)\alpha, r_{j \pm 1})$ and no other arrow which has $(\lambda + j\alpha, r_j)$ as its head or tail. Therefore, $Q(\Gamma)$ is of type $\mathbb{A}_{k+1}$.

To prove that the algebra $\mathfrak{A}(\Gamma)$ is hereditary, let $(\lambda + k\alpha, r_k), (\lambda + k'\alpha, r_{k'}) \in \Gamma$. The number of paths in $Q(\Gamma)$ between these vertices is zero unless $(\lambda + k\alpha, r_k), (\lambda + k'\alpha, r_{k'})$ are strictly
comparable. Assume without loss of generality that \((\lambda + k\alpha, r_k) < (\lambda + k'\alpha, r_{k'})\) and also that \(k \leq k'\). But in this case, we have proved that \(r_k = r_{k'} - k' + k\) and that there is exactly one path from \((\lambda + k\alpha, r_k)\) to \((\lambda + k\alpha, r_{k'} - k' + k)\). The result now follows from Corollary 3.4 if we prove that
\[
\dim \text{Hom}_g(V(\lambda + k\alpha), S^{(k'-k)}(g) \otimes V(\lambda + k'\alpha)) = 1.
\]
Since \(\text{Hom}_g(V((k-k')\theta)), S^{(k'-k)}(g)) \neq 0\) it suffices to prove that
\[
\dim \text{Hom}_g(V(\lambda + k\alpha), V((k-k')\theta)) \otimes V(\lambda + k'\alpha) = 1.
\]
But this again follows from Lemma 1.1. \(\square\)

**Remark.** The restriction \(\lambda(h_{i\alpha}) \neq 0\) is not necessary if \(\alpha = \theta\).

4.4. **Quivers of type** \(\mathbb{D}_{\ell+1}, \ell \geq 4\). The arguments given in the previous section can be used with obvious modifications to prove the following. Let \(\alpha = \theta - \alpha_i, \ell \in \mathbb{Z}_+\) with \(\ell \geq 4\) and \(\lambda \in P^+\). Let \(r_j \in \mathbb{Z}_+, 2 \leq j \leq \ell - 1\) be such that \(|r_{k} - r_{k+1}| = 1\) for \(2 \leq k \leq \ell - 2\) and let
\[
\Gamma = \Gamma_1 \cup \Gamma_2
\]
where \(\Gamma_1 = \{(\lambda + j\theta, r_j) : 2 \leq j \leq \ell - 1\}\) and
\[
\Gamma_2 = \{(\lambda + \theta, r_2 + 1), (\lambda + \theta + \alpha, r_2 + 1), (\lambda + \ell\theta, r_{\ell-1} - 1), (\lambda + (\ell - 1)\theta + \alpha, r_{\ell-1} - 1)\}.
\]
Then \(\Gamma\) is interval closed and the quiver \(Q(\Gamma)\) is of type \(\mathbb{D}_{\ell+1}\), where the set \(\Gamma_2\) consists of precisely those vertices which are either the head or the tail of precisely one arrow. Moreover, the algebra \(\mathcal{A}(\Gamma)\) is hereditary.

4.5. **Star-shaped quivers.** Suppose that \(g\) is not of type \(C_n\). Fix \(\lambda \in P^+\). For \(1 \leq p \leq 3\) and \(0 \leq j \leq \ell_p\), let \(r_{p,j} \in \mathbb{Z}_+\) be such that \(|r_{p,j} - r_{p,j+1}| = 1\). Let
\[
\Gamma_1 = \{(\lambda + (\ell_1 - j)\theta, r_{1,j}) : 0 \leq j \leq \ell_1\},
\]
\[
\Gamma_2 = \{(\lambda + (\ell_1 + j)\theta, r_{2,j}) : 0 \leq j \leq \ell_2\},
\]
\[
\Gamma_3 = \{(\lambda + (\ell_1 + j)\theta - j\alpha_i, r_{3,j}) : 0 \leq j \leq \ell_3 - 1\}.
\]
Set \(\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\).

**Proposition.** Suppose that for all \(1 \leq p \leq 3\) and \(0 \leq j_p < \ell_p\) we have
\[
|r_{3,j_3} - r_{2,j_2}| \leq |j_3 - j_2|, \quad |r_{3,j_3} - r_{1,j_1}| \leq |j_3 - j_1|.
\]
Then \(\Gamma\) is interval closed, \(Q(\Gamma)\) is of type \(T_{\ell_1,\ell_2,\ell_3}\) and \(\mathcal{A}(\Gamma)\) is hereditary.

**Proof.** Note that the conditions imply that
\[
r_{1,j} = r_{2,j} = r_{3,j}, \quad 0 \leq j \leq \ell_3, \quad \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 = \{(\lambda + \ell_1\theta, r_{1,0})\}.
\]
Using Proposition 4.3 we see that \(\Gamma_1 \cup \Gamma_2\) and \(\Gamma_3\) are interval closed and that \(\mathcal{A}(\Gamma_1 \cup \Gamma_2)\) and \(\mathcal{A}(\Gamma_3)\) are hereditary of type \(A_{\ell_1+\ell_2-1}\) and \(A_{\ell_3}\). The proposition follows at once if we prove that an element in \(\Gamma_3\), is not comparable in the partial order \(<\) to any element in \((\Gamma_1 \cup \Gamma_2)\) except possibly to \((\lambda + \ell_1\theta, r_{1,0})\).
Suppose first that $(\lambda + (\ell_1 + j_3)\theta - j_3\alpha_i, r_{3,j_3})$ is strictly comparable with $(\lambda + (\ell_1 + j_2)\theta, r_{2,j_2})$. Then $0 < |r_{2,j_2} - r_{3,j_3}| \leq |j_3 - j_2|$ and

\[
(j_2 - j_3)\theta + j_3\alpha_i = \sum_{p=1}^{\frac{|r_{2,j_2} - r_{3,j_3}|}{j_3 - j_2}} \beta_p, \quad \beta_p \in R \cup \{0\}. \tag{4.1}
\]

If $j_3 > j_2$, then we see by comparing the coefficients of $\alpha_i$ with $i \neq i_\bullet$ on both sides, that $\beta_p \in \{-\theta, -\theta + \alpha_i\}$ and that $|r_{2,j_2} - r_{3,j_3}| = j_3 - j_2$. Suppose that $-\theta$ occurs $s$ times in the set $\{\beta_p : 1 \leq p \leq j_3 - j_2\}$. Then $0 \leq s \leq j_3 - j_2$ and $-\theta + \alpha_i$ occurs $j_3 - j_2 - s$ times. It follows that $-(j_3 - j_2)\theta + j_3\alpha_i = \sum_p \beta_p = -s\theta + (j_3 - j_2 - s)(-\theta + \alpha_i) = -(j_3 - j_2)\theta + (j_3 - j_2 - s)\alpha_i$, which implies $j_2 = s = 0$.

Furthermore, suppose that $j_2 > j_3$. By comparing the coefficients of $\alpha_i$ with $i \neq i_\bullet$ in both sides of (4.1), we conclude that $\beta_p \in \{\theta, \alpha_i\}$ and $|r_{2,j_2} - r_{3,j_3}| = j_2 - j_3$. Suppose that $\alpha_i$ occurs $s'$ times. Then we must have $s' = j_3$ and $j_2 - j_3 - s' = (j_2 - j_3)$ which is only possible if $s' = j_3 = 0$.

Suppose now that $(\lambda + (k + j_3)\theta - j_3\alpha_i, r_{3,j_3})$ is strictly comparable with $(\lambda + (k - j_1)\theta, r_{1,j_1})$. Then we must have $0 \leq r = |r_{3,j_3} - r_{1,j_1}| \leq |j_3 - j_1|$ and

\[
(j_1 + j_3)\theta - j_3\alpha_i = \sum_{p=1}^{r} \gamma_p, \quad \gamma_p \in R \cup \{0\}.
\]

Comparing the coefficients of $\alpha_i$, $i \neq i_\bullet$ in both sides shows that $\beta_p \in \{\theta, \theta - \alpha_i\}$. If $\theta$ appears $s$ times, then we have

\[
r\theta - (r - s)\alpha_i = (j_1 + j_3)\theta - j_3\alpha_i,
\]

which implies $r = j_1 + j_3$ and $r - s = j_3$. Since $r \leq |j_1 - j_3|$ the first equality implies that either $j_1 = 0$ or $j_3 = 0$. \qed

5. QUIVERS WITH RELATIONS

In this section we give an example of $\Gamma$ for which $\mathfrak{A}(\Gamma)$ is not hereditary. The example is motivated by a family of $\mathfrak{g}[t]$-modules called the Kirillov-Reshetikhin modules (cf. [4]). We assume in this section that $\mathfrak{g}$ is of type $D_n$, $n \geq 6$. Recall that for $i \in I$ with $i \neq n - 1, n$ we have

\[
\omega_i = \sum_{j=1}^{i} j\alpha_j + i \sum_{j=i+1}^{n-2} \alpha_j + \frac{i}{2}(\alpha_{n-1} + \alpha_n).
\]

Let $\Gamma$ be the interval $[(2\omega_4, 0), (0, 4)]$. It is easily checked that

\[
\Gamma = \{(2\omega_4, 0), (\omega_2 + \omega_4, 1), (\omega_4, 2), (2\omega_2, 2), (\omega_1 + \omega_3, 2), (\omega_3, 3), (0, 4)\}.
\]
Since $g \cong V(\omega_2)$, it is now not hard to see by using Proposition 3.2 that the quiver $Q(\Gamma)$ is as follows:

\[
\begin{array}{c}
(0,4) \\
\downarrow \\
(\omega_2,3) \\
\downarrow \\
\omega_1 \\
\downarrow \\
(\omega_2,2) \\
\downarrow \\
(\omega_2 + \omega_4,1) \\
\downarrow \\
(2\omega_4,0)
\end{array}
\]

The path algebra $CQ(\Gamma)$ has a basis consisting of the paths of length at most four which we list below for the readers convenience:

\[
\{1_{\lambda,r} : (\lambda,r) \in \Gamma\}, \quad \{a,b_1,c_1,d : 1 \leq i \leq 3\}, \quad \{b_1a,dc_1,c_ib_1 : 1 \leq i \leq 3\}, \quad \{c_ib_1a,dc_ib_1 : 1 \leq i \leq 3\}, \quad \{dc_ib_1a : 1 \leq i \leq 3\}
\]

We now compute the dimension of $\dim 1_{\mu,s}R(\Gamma)1_{\lambda,r}$ for $(\mu,s),(\lambda,r) \in \Gamma$ with $r - s \geq 2$. By Corollary 3.4 it suffices to calculate $\dim \text{Hom}_G(I(\lambda,r),I(\mu,s))$. Using Proposition 2.7(ii) and the graded characters of injective envelopes of simples in $G[\Gamma]$ listed in Appendix A.1 we find that

\[
\dim 1_{\mu,s}R(\Gamma)1_{\lambda,r} = 1,
\]

if

\[
((\lambda,r),(\mu,s)) \in \{(0,4),(\omega_1 + \omega_3,2)\}, \quad ((\omega_2,3),(\omega_2 + \omega_4,1)), \quad ((\omega_1 + \omega_3,2),(2\omega_4,0))
\]

and

\[
\dim 1_{\mu,s}R(\Gamma)1_{\lambda,r} = 2,
\]

if

\[
((\lambda,r),(\mu,s)) \in \{(0,4),(\omega_2 + \omega_4,1)\}, \quad ((0,4),(2\omega_4,0))
\]

while $\dim 1_{\mu,s}R(\Gamma)1_{\lambda,r} = 0$ otherwise. This implies that there exists a unique (up to multiplication by non-zero constants) choice of complex numbers $x_i, 1 \leq i \leq 3$ and $\xi_j, \zeta_j, \eta_j, j = 1,2$ such that $A(\Gamma)$ is the quotient of $CQ(\Gamma)$ by the following relations

\[
b_3a = 0 = dc_3, \quad x_1c_1b_1 + x_2c_2b_2 + x_3c_3b_3 = 0, \tag{5.1}
\]

\[
c_3b_3a = 0, \quad \xi_1c_1b_1a + \xi_2c_2b_2a = 0, \quad \zeta_1dc_1b_1 + \zeta_2dc_2b_2 = 0, \tag{5.2}
\]

and

\[
\eta_1dc_1b_1a + \eta_2dc_2b_2a = 0, \quad dc_3b_3a = 0. \tag{5.3}
\]

**Proposition.** The algebra $A(\Gamma)$ is the quotient of $CQ(\Gamma)$ by the relations:

\[
b_3a = 0, \quad dc_3 = 0, \quad c_1b_1 + c_2b_2 + c_3b_3 = 0. \tag{5.4}
\]

In particular $A(\Gamma)$ is quadratic, of global dimension 2 and of tame representation type.
Remark. It is not hard to see by using the results of [4] and the equivalence of categories between $\mathfrak{A}(\Gamma)$-modules and $G[\Gamma]$, that the projective cover in $\mathfrak{A}(\Gamma) - \text{mod}_\mathfrak{f}$ of $S_{0,4}$ or the injective envelope of $S_{2\omega_1,0}$ corresponds to the Kirillov-Reshetikhin module $KR(2\omega_4)$, which is thus injective and projective in $G[\Gamma]$. This connection will be pursued elsewhere.

Proof. The relations in (5.4) are clearly independent of each other. To see that all relations in $\mathfrak{A}(\Gamma)$ are consequences of those in (5.4) it is enough to prove that the space spanned by $b_ic_j, b_jc_i$ with $1 \leq i < j \leq 3$ is always of dimension two. Using the equivalence of categories, Proposition 3.5, Proposition 2.4 and Proposition 2.3(ii) this can be reformulated into the following question on morphisms in $G$. Thus, for $\mu \in \{2\omega_2, \omega_1, \omega_1 + \omega_3\}$ fix non-zero elements $f_\mu \in \text{Hom}_G(P(\omega_2 + \omega_2, 2), P(\mu, 1))$ and $g_\mu \in \text{Hom}_G(P(\mu, 1), P(\omega_2, 0))$. We have to prove that the elements $g_\mu f_\mu$ and $g_\mu f_\lambda$ are linearly independent in $\text{Hom}_G(P(\omega_2 + \omega_2, 2), P(\omega_2, 0))$. In turn, using Proposition 2.2 this question translates into the following question in the category $\mathcal{F}(g)$. Let $f_\mu, g_\mu$ be the restrictions of $f_\mu$ and $g_\mu$ to $V(\omega_2 + \omega_1)$ and $V(\mu)$ respectively and $p : T^3(g) \to S^2(g) \otimes g$ be the canonical projection. The elements $p \circ (1 \otimes g_\mu) \circ f_\mu$ and $p \circ (1 \otimes g_\lambda) \circ f_\lambda$ are linearly independent elements of $\text{Hom}_G(V(\omega_2 + \omega_2), S^2(g) \otimes g)$. This is done by an explicit computation of the maps, and the details can be found in the Appendix A.2.

Since $\mathfrak{A}(\Gamma)$ is quadratic, it follows from [2] Theorem 1.1 that $\text{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, s}) = 0$ unless $r = s + 2$. We have

$$\dim \text{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{0,4}, S_{\omega_1 + \omega_2, 2}) = \dim \text{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{\omega_2, 3}, S_{\omega_2 + \omega_4, 1}) = \dim \text{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{\omega_1 + \omega_3, 2}, S_{2\omega_3, 0}) = 1$$

and $\text{Ext}^2_{\mathfrak{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, r+2}) = 0$ in all other cases. We claim that $\text{Ext}^3_{\mathfrak{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, s}) = 0$ for all $(\lambda, r), (\mu, s) \in \Gamma$. Indeed, by [2] Theorem 1.1

$$\dim \text{Ext}^3_{\mathfrak{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, s}) = \dim \text{Ext}^3_{\mathfrak{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, s})$$

where $\text{CQ}(\Gamma)_+$ is the radical of $\text{CQ}(\Gamma)$. If $r - s < 4$, it is clear that $\dim \text{Ext}^3_{\mathfrak{A}(\Gamma)}(S_{\lambda, r}, S_{\mu, s}) = 0$. For $r - s = 4$, we have a unique pair $(\lambda, r), (\mu, r+4) \in \Gamma$, namely $(0, 4)$ and $(2\omega_4, 0)$, and two linearly independent elements in $1_{2\omega_4, 0}(\text{CQ}(\Gamma)_+ R(\Gamma)) \cap \text{CQ}(\Gamma)_+ 1_{0,4}$, namely $dc_3b_3a$ and $dc_2b_2a + dc_1b_1a$. The first is contained in $R(\Gamma)^2$, since it can be written as $(dc_3)(b_3a)$ and $dc_3, b_3a \in R(\Gamma)$, while the second is contained in $\text{CQ}(\Gamma)_+ R(\Gamma)\text{CQ}(\Gamma)_+$ since it can be written as $d(c_1b_1 + c_2b_2 + c_3b_3)a$. Thus, $\dim \text{Ext}^3_{\mathfrak{A}(\Gamma)}(S_{0,4}, S_{2\omega_4, 0}) = 0$.

It remains to prove that the algebra is tame. Let $\Gamma_0 = \Gamma \setminus \{(2\omega_4, 0), (0, 4)\}$. Note that $\Gamma_0$ is interval closed and consider the subalgebra $\mathfrak{A}(\Gamma_0)$. This algebra is canonical (cf. [16] 3.7) and of type $(2, 2, 2)$, hence tame concealed ([16] 4.3(5)). Let $K$ be the subspace of $\mathfrak{A}(\Gamma_0)$ spanned by $b_3, c_3b_3$. Clearly, $K$ is a $\mathfrak{A}(\Gamma_0)$-submodule of $\mathfrak{A}(\Gamma_0)1_{\omega_2, 3}$. Let $M$ be the quotient of $\mathfrak{A}(\Gamma_0)1_{\omega_2, 3}$ by $K$. This $\mathfrak{A}(\Gamma_0)$-module has dimension vector

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
0 \\
1
\end{pmatrix}
\]
and hence belongs to the tubular family of type $(2, 2, 2)$. Then it is easy to check that $\mathfrak{A}(\Gamma)$ is obtained as one-point extension and one-point coextension of $\mathfrak{A}(\Gamma_0)$ at $M$ and hence is tame (even domestic). We refer the reader to [15, 4.7] for details.  

**Appendix A.**

**A.1.** The graded characters of injectives envelopes of simple objects in $\mathcal{C}[\Gamma]$ given by Proposition 2.4 and Proposition 2.7 can be calculated explicitly by using the LiE computer program ([12]) and we list them below for the reader’s convenience.

\[
\begin{align*}
I(0, 4)_{\Gamma} &= V(0, 4) \oplus V(\omega_2, 3) \oplus \begin{pmatrix} V(\omega_4, 2) \\ + \end{pmatrix} \oplus V(\omega_2 + \omega_4, 1) \oplus V(2\omega_4, 0) \\
I(\omega_2, 3)_{\Gamma} &= V(\omega_2, 3) \oplus \begin{pmatrix} V(\omega_4, 2) \\ + \end{pmatrix} \oplus 2V(\omega_2 + \omega_4, 1) \oplus V(2\omega_4, 0) \\
I(\omega_1 + \omega_3, 2)_{\Gamma} &= V(\omega_1 + \omega_3, 2) \oplus V(2\omega_4, 0) \\
I(\omega_2 + \omega_4, 1)_{\Gamma} &= V(\omega_2 + \omega_4, 1) \oplus V(2\omega_4, 0) \\
I(2\omega_4, 0)_{\Gamma} &= V(2\omega_4, 0).
\end{align*}
\]

**A.2.** We now establish the following result which was used in the proof of Proposition 5. We use the notation of the proof freely.

**Lemma.** Let $\lambda, \mu \in \{2\omega_2, \omega_4, \omega_1 + \omega_4\}$. The elements $\mathbf{p} \circ (1 \otimes \bar{g}_\mu) \circ \bar{f}_\mu$ and $\mathbf{p} \circ (1 \otimes \bar{g}_\lambda) \circ \bar{f}_\lambda$ are linearly independent elements of $\text{Hom}_g(V(\omega_2 + \omega_4), S^2(g) \otimes g)$.

**Proof.** Write $x_i^{-}$ for $x_i^{-}$. Let $w_{\omega_4} = \bar{g}_{\omega_4}(v_{\omega_4})$. Since $g \cong g V(\omega_2)$, we have

\[
\begin{align*}
w_{\omega_4} &= v_{\omega_2} \otimes x_2^{-} x_1^{-} x_3^{-} x_2 v_{\omega_2} + x_2^{-} x_1^{-} x_3^{-} x_2 v_{\omega_2} \otimes v_{\omega_2} - x_2^{-} v_{\omega_2} \otimes x_1^{-} x_3^{-} x_2 v_{\omega_2} \\
&\quad - x_1^{-} x_2^{-} v_{\omega_2} \otimes x_2 v_{\omega_2} + x_1^{-} x_2^{-} v_{\omega_2} \otimes x_3^{-} x_2 v_{\omega_2} + x_1^{-} x_2^{-} v_{\omega_2} \otimes x_1^{-} x_2^{-} v_{\omega_2} \in S^2(g).
\end{align*}
\]

Write $w_{\omega_4} = w_{\omega_4}^{(1)} \otimes w_{\omega_4}^{(2)}$ in Sweezer notation. Then

\[
\begin{align*}
v_1 &= w_{\omega_4}^{(1)} \otimes v_{\omega_2}, \quad v_2 &= v_{\omega_2} \otimes w_{\omega_4}, \quad v_3 &= w_{\omega_4}^{(1)} \otimes v_{\omega_2} \otimes w_{\omega_4}^{(2)}
\end{align*}
\]

is a basis of $U = \{v \in g \otimes g_{\omega_2 + \omega_4} : n^+ v = 0\}$. Let $s_1 = v_1, s_2 = v_2 + v_3, s_3 = v_2 - v_3$. Then $s_1, s_2$ form a basis of $U \cap S^2(g) \otimes g$, while $s_3$ spans $U \cap \wedge^2 g \otimes g$.

We have

\[
\bar{f}_{2\omega_4}(v_{\omega_2 + \omega_4}) = -v_{\omega_2} \otimes x_1^{-} x_2^{-} x_3^{-} x_2 v_{\omega_2} + 2v_{\omega_3} \otimes x_2^{-} x_1^{-} x_3^{-} x_2 v_{\omega_2} - 3x_2^{-} v_{\omega_2} \otimes x_1^{-} x_3^{-} x_2 v_{\omega_2} + 3x_1^{-} x_2^{-} v_{\omega_2} \otimes x_3^{-} v_{\omega_2} + 3x_3^{-} x_2^{-} v_{\omega_2} \otimes x_1^{-} x_2^{-} v_{\omega_2}.
\]
Furthermore, the image of $v_{\omega_2 + \omega_3}$ under the composite map
\[ V(\omega_2 + \omega_3) \stackrel{f_{\omega_2 + \omega_3}}{\longrightarrow} g \otimes V(\omega_1 + \omega_3) \rightarrow g^\otimes 3 \]
is
\[ m_2 = -2v_{\omega_2} \otimes v_{\omega_2} \otimes x_1 x_3 x_2 v_{\omega_2} + 2v_{\omega_2} \otimes x_1 x_3 x_2 v_{\omega_2} \otimes v_{\omega_2} + 2x_2 v_{\omega_2} \otimes v_{\omega_2} \otimes x_1 x_3 x_2 v_{\omega_2} - 2x_1 x_2 v_{\omega_2} \otimes x_2 v_{\omega_2} \otimes v_{\omega_2} + 2x_1 x_2 v_{\omega_2} \otimes x_2 v_{\omega_2} \otimes v_{\omega_2} - 2x_1 x_3 v_{\omega_2} \otimes x_2 v_{\omega_2} \otimes v_{\omega_2} - 2x_3 x_2 v_{\omega_2} \otimes v_{\omega_2} \otimes v_{\omega_2} \]

Finally, the composite map
\[ V(\omega_2 + \omega_4) \stackrel{f_{\omega_4}}{\longrightarrow} g \otimes V(\omega_4) \rightarrow g^\otimes 3 \]
maps $v_{\omega_2 + \omega_4}$ to $m_3 = v_{\omega_2} \otimes v_{\omega_4}$. In particular, this implies that all composite maps are non-zero. Furthermore, it can be shown that
\[ m_1 = 3s_1 + s_2 - 2s_3 \]
\[ m_2 = 2s_1 - s_2 + s_3 \]
\[ 2m_3 = s_2 + s_3. \]
In particular, $m_1$, $m_2$ and $m_3$ are linearly independent. Suppose that $x_1 m_1 + x_2 m_2 + x_3 m_3 \neq 0$ has zero projection onto $S^2(g) \otimes g$. Using the above equations we conclude that $x_i \neq 0$, $i = 1, 2, 3$. □

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