0. Introduction

Due to diagonal arguments, it is known that there are no universal primitive recursive functions. At minimum, such would be a $k$-ary p.r. function that evaluates any $k$-ary p.r. function $f_n$ on $k-1$ inputs and $n$. In what follows, however, an algorithm is presented that appears to compute precisely such a function. This appearance must be in error; nevertheless, it is an error worth investigating, for it is not clear upon strenuous reflection where it lies. I thus offer up the algorithm to readers, with a humble request for assistance in locating the error.

One preliminary note: The following normally does not discuss p.r. functions directly but rather function symbols $f^{n}_m$ within a version of Primitive Recursive Arithmetic (PRA). (From this angle, the algorithm thus purports to show that PRA is inconsistent.) The reason for this focus is that the algorithm relies crucially on numeric subscripts, and it is unproblematic which numeric subscript a function symbol has. In contrast, it is dubious to assume that, e.g., the factorial function itself has some specific index attached to it (in Plato’s heaven, as it were). The discussion thus proceeds primarily by considering function symbols rather than functions per se.

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1 My thanks to [redacted] for discussion of issues relevant to this paper. I also express my gratitude to an audience at the 2022 meeting of the [redacted].
2 That PRA is inconsistent was argued previously by Princeton mathematician Edward Nelson (2011). But Tao’s (2011) response caused Nelson (2011a) to retract his claim. The argument of this paper is not related to Nelson’s: He was focused on the induction axiom; this paper concerns more minimal versions of PRA which do not include the induction axiom.
1. Exposition of PRA–

We will focus on a weakened version of PRA; cf. Skolem (1923); Hilbert & Bernays (1934, ch. 7); Curry (1941). Call it PRA–. The weakening is that the induction axiom is replaced with the axiom that any positive integer has a predecessor (cf. Q, Robinson’s 1950 arithmetic). But for our purposes, the most important feature of PRA– is the way a p.r. function is assigned a subscripted function symbol; this is detailed later in this section.

First, terms of PRA– are defined inductively as follows:

(i) ‘0’ is a term.
(ii) If τ is a term, then so is τ’. (‘0’ followed by zero or more occurrences of ‘′’ are the numerals. Let τ be the numeral co-referring with τ.)
(iii) vτ is a term (a variable).
(iv) If τ₁, …, τₘ are terms, then so is fnmτ₁…τₘ (a non-numeric or nn-term).

For convenience, however, Arabic numerals will be used and I revert to using ‘χ’, ‘γ’, etc., as variables. Also, a nn-term will normally be written as fn(τ₁, ..., τₘ).

The well-formed formulae (wff) of PRA– are defined thus:

(v) If τ₁ and τ₂ are terms, then τ₁=τ₂ is a wff.
(vi) If φ is a wff, then so is ~φ.
(vii) If φ and ψ are wff, then so is (φ ⊃ ψ).

Assume that ‘∈’, ‘~’, and ‘⊃’, have their standard interpretations. Parentheses will be omitted when there is no danger of confusion.

The system has all axioms from the following three schemes:

(L1) φ ⊃ (ψ ⊃ φ)
(L2) (φ ⊃ (ψ ⊃ ρ)) ⊃ ((φ ⊃ ψ) ⊃ (φ ⊃ ρ))
(L3) (~ψ ⊃ ~φ) ⊃ (φ ⊃ ψ)

3 Also, PRA sometimes includes principles for course-of-values recursion. Such things are left aside also.
4 In what follows, I often elide the distinction between use and mention. Quine’s (1951) corner quotes could be used to distinguish an expression τⁿ¹ from what it represents. (Some other notation for Gödel numbers would then be required.) But to avoid clutter, I instead rely on context to disambiguate.
PRA– also has the standard axiomatic analysis of ‘=’, as per the Law of Universal Identity and the Indiscernability of Identicals:

\[
\begin{align*}
(U_=) & \quad x = x \\
(I_=) & \quad \tau_1 = \tau_2 \supset (\phi[x/\tau_1] \supset \phi[x/\tau_2])
\end{align*}
\]

Here and elsewhere, \(\phi[x/\tau]\) is the result of uniformly replacing \('x'\ in \(\phi\) with \(\tau\).

Further, in PRA– each p.r. function is assigned to at least one basic function symbol \(f_\omega\); axioms defining those symbols appropriately are included. Per Gödel (1931, pp. 179-180), the axioms shall be specified inductively.\(^5\)

The inductive definition provided is more involved than is usual, but the index for each axiomatically defined function will effectively code the composition of the function, and this will be useful later. (I assume some standard way of Godel numbering each symbol of the language, each string of symbols (incl. terms and wff), and each sequence of strings; cf. Gödel, pp. 179.) Moreover, it is important that the coding (and thus, the enumeration) of the axioms for the p.r. functions is primitive recursive. Where ‘*’ indicates the concatenation of symbols:

**Axioms for base p.r. functions:**

- \(f_0(x) = 0\) is the 0\(^{th}\) axiom;
- \(f_1(x) = x'\) is the 1\(^{st}\) axiom;
- If \(c\) codes \(f^*k\) where \(1 \leq f \leq k \leq c\) then: \(f_c(x_1, ..., x_k) = x_0\) is the \(c^{th}\) axiom.

**Axioms for composed p.r. functions:**

If \(c\) codes the string \(0^*b^*c_1^* ..., c_n^* k\) (where each of \(b, c_1, ..., c_n\) and \(k\) is less than \(c\), and the \(b^{th}\), \(c_1^{th}\), ..., and \(c_n^{th}\) axioms define, respectively, \(f_b(x_1, ..., x_k), f_{c_1}(x_1, ..., x_k), ...,\) and \(f_{c_n}(x_1, ..., x_k)\), then:

\(f_c(x_1, ..., x_k) = f_b(f_{c_1}(x_1, ..., x_k), ..., f_{c_n}(x_1, ..., x_k))\) is the \(c^{th}\) axiom.

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\(^5\) All references to Gödel will be in relation to his (1931).
Axioms for p.r. recursions:
If \( c \) codes the string ""* \( a \) * \( d \) * \( k \) (where each of \( a \), \( d \), and \( k \) is less than \( c \)), and the \( \alpha \)th and \( \beta \)th axioms define, respectively, \( f_\alpha(x_1, ..., x_k) \) and \( f_\beta(x_1, ..., x_{k+2}) \), then where \( (\phi \land \psi) \) is shorthand for \( \neg(\phi \land \neg\psi) \):
\[
\begin{align*}
  f_c(x_1, ..., x_k, 0) &= f_\alpha(x_1, ..., x_k) \\
  f_c(x_1, ..., x_k, y') &= f_\beta(f_c(x_1, ..., x_k, y'), x_1, ..., x_k, y)
\end{align*}
\]
defines the \( c \)th axiom.

Since a p.r. operation must be total, assume that \( n \) maps to 0 when \( n \) corresponds to no axiom.

In both occurrences, the parenthetical “where each of....is less than \( c \)” is eliminable.

For the index \( c \) is independently guaranteed to exceed the arity \( k \), as well as the index for any function used in defining \( f_c \). At least, on a standard Gödel numbering scheme, the code for \( n \) is greater than \( n \); hence, since \( c \) codes a string consisting of \( k \) and the subscripts of the component function symbols, \textit{inter alia}, \( c \) will be greater than any of those numbers.\(^6\) This fact about \( c \) will be important to the arguments later.

Moving on, PRA– also has the following arithmetical axioms:

\[
\begin{align*}
  (A1) \quad & \neg0 = x' \\
  (A2) \quad & x' = y' \Rightarrow x = y \\
  (A3) \quad & \neg x = 0 \Rightarrow (\mu y \leq x) x = y'
\end{align*}
\]

As usual, \( (\mu y \leq x) \) expresses bounded minimization (a p.r. function).

The rules of inference in PRA– are \textit{modus ponens} and variable substitution:

\[
\begin{align*}
  (\text{MP}) \quad & \text{From } \phi \text{ and } (\phi \Rightarrow \psi), \psi \text{ is derivable.} \\
  (\text{VS}) \quad & \text{From } \phi(x_1, ..., x_0), \phi[x/\tau_1], ..., [x_0/\tau_0] \text{ is derivable.}
\end{align*}
\]

A finite sequence of wff counts as a \textit{derivation} of \( \phi \) in PRA– from a (possibly empty) set of wff \( \Gamma \) iff: The first members are the members of \( \Gamma \) (if any), the last member is \( \phi \), and any member of the sequence is either a member of \( \Gamma \), an axiom, or is derivable from previous

\(^6\) Relatedly, we know that with the composed p.r. functions, if \( f_\beta \) has arity \( n \), it must be that \( n < c \). This is because \( c \) codes \( n+2 \) symbols, and the code for \( n+2 \) symbols is always greater than \( n \) itself.
members via some inference rule in the system. When $\Gamma$ is empty, we say that the sequence is a *proof* of $\phi$ in PRA–.

2. Towards a universal p.r. function?

This section begins an attempt to show that the following function $u$ is p.r:

$$u(i, n) = m \text{ iff } f(i, n) = m \text{ is a theorem of PRA–.}$$

The tactics for this will require a p.r. predicate in PRA– which strongly represents relations between theorems and proofs. However, like PRA, PRA– has no predicate that represents *in toto* the relations between theorems and their proofs, given that it has no unbounded quantifiers. Nonetheless, if $\phi$ has Gödel number $\lceil \phi \rceil$, we can define an “$i$-bounded” p.r. proof predicate $B(i, n, \lceil \phi \rceil)$ in PRA–, which is satisfied iff $\phi$ has a proof coded by $n$ where any axiom may occur except those beyond the $i$th axiom defining a p.r. function symbol.

In defining the $i$-bounded proof predicate, we will not bother with many of the details since they are somewhat tedious and are identical to the details for Gödel’s predicate $B(x, y)$ (p. 186, #45 in Gödel’s list of p.r. functions and relations). Or in some cases, they just require straightforward revisions. (For instance, unlike Gödel’s system, PRA– is first-order only and has no quantifiers; many of Gödel’s definitions are thus more complex than they need to be for our purposes.) The only revisions which may not be straightforward concern the definition of an $i$-bounded “axiom predicate” $Ax(i, n)$ for PRA– (cf. #42 in Godel’s list). This is a predicate which represents the axioms for PRA– except those beyond the $i$th axiomatic definition of a p.r. function symbol.

In defining $Ax(i, n)$, we first construct a p.r. predicate $AxPR(i, j)$ which is satisfied iff $j$ codes the $j$th or earlier axiom for a p.r. function. Recall that we gave an enumeration of (the
codes of) the axioms for p.r. functions using only p.r. functions. Hence, we know that there is a p.r. enumeration $e$ such that if $e(i) \neq 0$, $e(i)$ codes the axiom defining the $i^{th}$ p.r. function symbol. This allows us to interpret $AxPR(i,j)$ as follows:

(D1) $AxPR(i,j)$ iff $(\exists x \leq i)(e(x) = j \& j \neq 0)$

The predicates representing the logical axioms, arithmetical axioms, and axioms for ‘$=$’ are presumed given. So we have what we need to define $Ax(i,n)$, and thus, to define an $i$-bounded p.r. proof predicate $B(i,n,\mathsf{⌜φ⌝})$ in the manner of Gödel.

This much should be uncontroversial. But given $B(i,n,\mathsf{⌜φ⌝})$, we now consider an attempt to define $U(i,n,m)$ as a p.r. predicate that strongly represents the function $u$. Briefly, where $s(i,n)$ codes the $i^{th}$ “$n$-sequence” in a series to be specified, the proposal is to understand $U(i,n,m)$ as follows:

(D2) $U(i,n,m)$ iff $B(i,s(i,n),\mathsf{⌜f_i(i,n)=m⌝})$

This tells us that $U(i,n,m)$ holds iff the relevant sequence counts as an $i$-bounded proof in PRA– of $f_i(i,n)=m$. The suggestion is that the $s(i,n)$ codes an $i$-bounded proof of $f_i(i,n)=m$ iff that formula is indeed a theorem. It purportedly codes the sort of sequence where, if this formula has a proof at all, that sequence will count as one (and otherwise it will not).

Roughly, the proposal is to define $s(i,n)$ as follows, where $e(i)$ is the earlier function enumerating (the codes of) the axioms for the function symbols:

$$s(i,n) = \begin{cases} cp(\mathsf{⌜f_i(i,n)=m⌝}) & \text{if the axiom coded by } e(i) \text{ defines the symbol } f_i^2 \\ 0 & \text{otherwise} \end{cases}$$

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7 This does not imply the existence of a p.r. function $g(i)$ that outputs the $i^{th}$ p.r. function. (That alone would allow lead to a contradictory diagonal function.) Our function does not have functions in its range, but rather codes for various syntactic strings. These happen to be definitions of p.r. functions. But a malignant diagonal function would need to compute the $i^{th}$ p.r. function on input $i$. And that would require an unbounded search of, e.g., the proofs of PRA–, to find one ending with an equality that identifies what $f_i(i)$ is.
When the defining condition is not satisfied, the function outputs 0. Otherwise, \( e(i) \) codes an axiom that defines a binary symbol \( f_i \). Intuitively, in that case, \( s \) purportedly outputs the code for the “canonical proof” of \( f(i,n) = m \).

Note well: Since \( f_i(i,n) \) is a p.r. term, there is a unique \( m \) such that \( f_i(i,n) = m \) has a proof in PRA– (assuming PRA– is sound). So, looking at the definition of \( s \), the function basically purports to compute what \( m \) is, given only \( i \) and \( n \). It does so by checking if \( e(i) \) codes an axiom for \( f_i^2 \), and then generating a proof wherein the term \( f(i,n) \) is computed. So the crux of the matter is whether there is a p.r. algorithm which can generate a proof for any \( f_i(i,n) \). Again, diagonal arguments suggest that there could not be such an algorithm.

The peculiarity, however, is that it seems one can describe such an algorithm. Here is a rough, intuitive gloss. (The detailed description starts in section 4). First, where \( e(i) \) codes an axiom defining \( f_i^2 \), the algorithm starts a proof with that axiom. It then instantiates the axiom on \( i \) and \( n \), so that the proof has a line that is roughly of the form \( f_i(i,n) = \phi_i(i,n) \) (unless \( f_i \) expresses a basic binary projection, which the algorithm can handle separately). Now, since the righthand term is a p.r. term, it seems we can generate a proof computing the term by cycling through the recursive procedures that define the term in a designated order.

If this is right, then since a p.r. function symbol is here defined only by function symbols with lower subscripts, a proof of \( f_i(i,n) = m \) yielded by the algorithm would count as an “\( i \)-bounded” proof. In which case, \( s(i,n) \) would render true the sentence \( B(i,s(i,n),\forall f(i,n) = m) \), as intended.

But at first blush, our algorithm seems to require an unbounded search, for it is unknown how many p.r. operations are necessary to compute a given term. Our algorithm
will avoid this, however, by utilizing a dynamically updated counter which estimates the number of operations needed. The counter is set at a given number at the beginning of the process and is adjusted as the algorithm proceeds. The counter will very often be incorrect in its estimation, but it has one redeeming feature: It estimates the number of remaining computations to be 0 if, and only if, the number of remaining computations is 0. With such a counter, the process will halt exactly when it should halt. Add to this that the subscripts on function symbols will guide the algorithm on which p.r. operations should be applied at which stages. The result seems to be an algorithm that seems able to compute any binary p.r. term at the requisite (primitive) recursive depth.

3. Preparatory remarks, re: function symbols within proofs

Before detailing the algorithm, we should appreciate the distinctive feature of the axiomatically defined function symbols, namely, that the subscripts indicate the composition of the definition for the symbol. For again, our algorithm will exploit the information encoded in these subscripts in a way that reduces a p.r. term to a single numeral, within the context of a proof.

It may be objected that a well-defined algorithm should not be able to “access” the subscript on a function symbol. Officially, however, a nn-term has no subscript; it is of the form \( f^{n}m_{1} \ldots m_{m} \). More importantly, the objection is at odds with the use of \( AxPR(i, n) \) or Kleene’s (1952) T-predicate, which are defined with reference to indices on functions.

As an illustration of the information coded by the indices, consider the term \( f_{c}(1, 2) \), where \( c \) codes the string “”*q*alted*1. The 1st member of the string tells us that \( f_{c} \) expresses a
p.r. recursion, whereas the 2nd and 3rd members indicate that \( f_c \) is defined by \( f_d \) and \( f_d \). So we can recover that the axiom for \( f_c \) is:

\[
f_c(x, 0) = f_d(x) \quad \text{and} \quad f_c(x, y') = f_d(f_c(x, y), x, y)
\]

Suppose, moreover, that \( q \) codes \( 1^*1 \). We can then discern that \( f_d \) expresses the unary 1st projection function, i.e., simple equality. As for \( d \) suppose it codes \( 0^*p^*1^*q^*q^*3 \). Since the string here begins with 0, we know that \( d \) indexes a composed p.r. function—and we can discern further that \( f_d \) is defined in the system by \( f_d, f_3 \), and two occurrences of \( f_d \). So we can recover the axiom for \( f_d \):

\[
f_d(z, x, y) = f_d(f_1(z), f_d(x), f_d(y))
\]

(It may seem unnecessary for the axiom to invoke \( f_d \), but this is so that it has the right form for the definition of a composed p.r. function.) Suppose now that \( p \) codes \( 1^*3 \); then, \( f_e \) must express the ternary 1st-projection function. Then, \( f_c \) expresses the addition function and that \( f_c(1,2) = 3 \). Again, it is important that \( c \) codes information about how \( f_c \) is defined; this will make possible the algorithm for building canonical proofs of the relevant p.r. formulae.

For expediency’s sake, the proofs shall make use of inference rules corresponding to the standard evaluation rules for p.r. terms. Where \( n_1, n_2, n_3... \) are any numerals:

Rules for evaluating base p.r. terms

- \((Z)\) From \( \phi(f_0(n)) \), \( \phi(0) \) is derivable.
- \((S)\) From \( \phi(f_1(n)) \), \( \phi(n') \) is derivable.
- \((P)\) If \( c \) codes \( f^*k \) where \( 1 \leq j \leq k \leq c \), then:
  - From \( \phi(f_c(n_1, ..., n_k)) \), \( \phi(n_j) \) is derivable.

Rule for evaluating composed p.r. terms

- \((C)\) If \( c \) codes the string \( 0^*b^*c_1^*..., ^*c_r^*k \), and the \( b^h, c_1^h, ..., \) and \( c_r^h \) axioms define \( f_b(x_1, ..., x_b), f_{c_1}(x_1, ..., x_{c_1}), ..., \) and \( f_{c_r}(x_1, ..., x_{c_r}) \), respectively, then:
  - From \( \phi(f_c(n_1, ..., n_k)) \), \( \phi(f_b(f_{c_1}(n_1, ..., n_{c_1}), ..., f_{c_r}(n_1, ..., n_{c_r}))) \) is derivable.
Rules for evaluating p.r. recursion terms

If \( c \) codes the string \("* a* d* k\) and the \( d^\text{th} \) and \( d^\text{th} \) axioms define \( f_d(x_1, \ldots, x_k) \) and \( f_d(x_1, \ldots, x_{k+2}) \), respectively, then:

(R1) From \( \phi(f_c(n_1, \ldots, n_i, 0)) \), \( \phi(f_a(n_1, \ldots, x_k)) \) is derivable, and

(R2) From \( \phi(f_c(n_1, \ldots, n_k, n')) \), \( \phi(f_d(f_c(n_1, \ldots, n_k), n_1, \ldots, n_k, n')) \) is derivable.

These are in fact more restrictive than the usual evaluation rules, for they apply to terms loaded with numerals only. (This is to simplify things later.) Regardless, we shall give a proof in section 5 that the rules are sound. Soundness means, moreover, that these rules are mere shortcuts for what could be proven otherwise in PRA– (assuming PRA– is complete with respect to p.r. formulae). Our remarks will therefore bear on PRA–, even though the shortcut rules are not officially part of that system.

For now, it is best to focus on how the subscripts on function symbols enable the application of these rules. An example will help. To save time, I make use of the standard elimination rule for &; also, some occurrences of ‘\( f_q(x) \)’ are omitted as trivial (but a few are included since they make clearer how (R1) is applied). Apart from these omissions, note that what follows is in conformity with the algorithm given later. Where \( \langle c^{-1} \rangle \) is the string coded by \( c \):

1. \( f_d(x, 0) = f_d(x) \& f_d(x, y') = f_d(f_c(x, y), x, y) \) \hspace{1cm} [Axiom for \( f_d \)]
2. \( f_d(0', 0) = f_d(0') \& f_d(0', 0'') = f_d(f_c(0', 0'), 0', 0') \) \hspace{1cm} [(VS), 1: \( x/0', y/0' \)]
3. \( f_d(0', 0'') = f_d(f_d(0', 0'), 0', 0') \) \hspace{1cm} [(&E), 2]
4. \( f_d(0', 0'') = f_d(f_d(f_c(0', 0'), 0', 0'), 0', 0') \) \hspace{1cm} [(R2), 3: \( d = 3^\text{rd in } \langle c^{-1} \rangle \)]
5. \( f_d(0', 0'') = f_d(f_d(f_d(0', 0'), 0', 0'), 0', 0') \) \hspace{1cm} [(R1), 4: \( g = 2^\text{nd in } \langle c^{-1} \rangle \)]
6. \( f_d(0', 0'') = f_d(f_d(0', 0', 0'), 0', 0') \) \hspace{1cm} [(P), 5: \( q \) codes \( 1*1 \)]
7. \( f_d(0', 0'') = f_d(f_d(f_c(0', 0'), 0'), 0', 0') \) \hspace{1cm} [(C), 6: \( p, q \) are \( 2^\text{nd, 3^rd in } \langle d^{-1} \rangle \)]
8. \( f_d(0', 0'') = f_d(f_d(f_d(0', 0', 0'), 0', 0'), 0', 0') \) \hspace{1cm} [(S), 7: subscript is 1]
9. \( f_d(0', 0'') = f_d(f_d(0'', 0', 0') \hspace{1cm} [(P), 8: \( p \) codes \( 1*3 \)]
10. \( f_d(0', 0'') = f_d(f_d(0', 0'), 0', 0') \) \hspace{1cm} [(C), 9: \( p, q \) are \( 2^nd, 3^rd \) in \( \langle d^{-1} \rangle \)]
11. \( f_d(0', 0'') = f_d(0'', 0', 0') \) \hspace{1cm} [(S), 10: subscript is 1]
12. \( f_d(0', 0'') = 0'' \) \hspace{1cm} [(P), 11: \( p \) codes \( 1^*3 \)]
This example elucidates how the subscripts on function symbols enable the construction of the proofs. Specifically, the subscripts allow applications of the above shortcut rules for evaluating p.r. terms in a p.r. setting, for it is p.r. decidable what the indices are or what they encode.

4. An algorithm for canonical proofs?

We now present the algorithm for canonical proofs of \( f(i, n) = m \) (when the formula has a proof at all). As will be detailed later, the algorithm appears to be free of unbounded searches, in which case, it would be computable by a p.r. operation.

Terminology: The indices for some functions will code a string \( 0^* b^* c_1^*, ..., c_n^* k \), where the \( b^h \), \( c_1^h \), \( c_n^h \) axioms define \( f_b(x_1, ..., x_k), f_{c_1}(x_1, ..., x_k), ..., f_{c_n}(x_1, ..., x_k) \), respectively. Call such an index a “composition index.” Other indices will code a string \( a^* d^* \), where the \( a^h \) and \( d^h \) axioms define \( f_a(x_1, ..., x_k) \) and \( f_d(x_1, ..., x_{k+2}) \), respectively. Call such an index a “recursion index.” Also, let us jointly refer to composition indices and recursion indices as “non-basic indices.” The other indices we shall call “basic indices.”

Since the coding process is p.r., it suffices to indicate the wff that will be members of \( s(i, n) \). Suppose, then, \( i \) enumerates the axiom for \( f_i^2 \). (If it does not, \( s(i, n) = 0 \).) Then, \( s(i, n) \) is the code of the sequence that is determined as follows:

Step 1. [Basic Binary Projections] If \( i \) does not code \( j^2 \), where \( j \) equals 1 or 2, then go to Step 2. Otherwise:

1A. If \( j = 1 \), output the following sequence:
1. \( f(x, y) = x \) [Axiom for \( f_1 \)]
2. \( f(i, n) = i \) [(VS), 1: \( x/i, y/n \)]
Then, halt.
1B. If \( j = 2 \), output the following sequence:

1. \( f(x, y) = y \)  
2. \( f(i, n) = n \)  

Then, halt.

Step 2. [Getting the Baseline Term for P.R. Recursions] If \( i \) is not a recursion index, then go to Step 3. Otherwise:

2A. If \( n = 0 \), start the sequence as follows:

1. \( f(x, 0) = f_s(x) \)  
2. \( f(i, 0) = f(s) \)  
3. \( f(i, 0) = f_s(i) \)  

Then, go to Step 4.

2B. If \( n \neq 0 \), start the sequence as follows:

1. \( f(x, 0) = f_s(x) \)  
2. \( f(i, 0) = f(s) \)  
3. \( f(i, 0) = f_s(i) \)  

Then, go to Step 4.

Step 3. [Getting the Baseline Term for Other P.R. Functions] Start the sequence as follows:

1. \( f(x, y) = \phi(x, y) \)  
2. \( f(i, n) = \phi(i, n) \)  

Then, go to Step 4.

Step 4. [Initializing the Counter] The latest line of the sequence is an equality; call the term to the right of ‘=’ the “baseline term” for the sequence. Identify how many times ‘\( f \)’ occurs in the baseline term and initialize the counter at that number (not greater than the total number of symbols in the wff\(^8\)). Go to Step 5.

Step 5. [Entering the Main Loop.] If the counter is at zero, halt. Otherwise, (re)start the Main Loop: For the present iteration of the Loop, let \( \varepsilon \) be the equation on the latest line. To the right of ‘=’, find the nn-term embedded in the most parentheses (a.k.a., the “innermost” nn-term). When there is a tie, choose the rightmost one. Let \( \tau \) be the rightmost, innermost nn-term (to the right of ‘=’) for the current iteration of the Main Loop. (N.B., \( \tau \) will be loaded with numerals only; we prove this later.) Go to Step 6.

Step 6. [Computing \( \tau \)] Check the index \( c \) for \( \tau \). It is either 0, 1, or codes \( f^k_p \), where \( 1 \leq j \leq k \leq c \)—alternatively, it is a composition or recursion index.

6A. If \( c = 0 \), apply (Z): Add a line where \( \tau \) is replaced in \( \varepsilon \) with 0. Subtract 1 from the counter and go back to Step 5.

6B. If \( c = 1 \), then \( \tau \) is loaded with some \( m \). Apply (S): Add a line where \( \tau \) is replaced in \( \varepsilon \) with \( m' \). Subtract 1 from the counter and go back to Step 5.

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\(^8\) Cf. Godel’s function \( \ell(x) \); #7 in his list of p.r. functions and relations, p. 182.
6C. If $c$ codes $f^k_i$ where $1 \leq j \leq k \leq c$, then apply (P): Add a line where $\tau$ is replaced in $\varepsilon$ with the numeral in the $j$th position of $\tau$. Subtract 1 from the counter and go back to Step 5.

6D. If $c$ is a composition index, then apply (C): Add a line where $\tau$ is replaced in $\varepsilon$ with $f \delta(f_\beta(m, ..., n_k), ..., f_\delta(m, ..., n_3))$, where $n_1, ..., n_k$ are the same as in $\tau$, and $b$ is the 2nd member of the string coded by $c$, $c_1$ is 3rd member of the coded string coded by $c$, ..., and $c_l$ is $l+2$th member of the string coded by $c$. Add $l$ to the counter and go back to Step 5.

6E. If $c$ is a recursion index, then:

6Ei. If $\tau$ is loaded with $n_1, ..., n_k 0$, then apply (R1): Add a line where $\tau$ is replaced in $\varepsilon$ with $f_\alpha(m, ..., n_k)$, where $\alpha$ is the 2nd member of the string coded by $c$. Leave the counter unchanged and go back to Step 5.

6Eii. If $\tau$ is loaded with $n_1, ..., n_k m$ where $m \neq 0$, then apply (R2): Add a line where $\tau$ is replaced in $\varepsilon$ with $f_\delta(f_\beta(m, ..., n_k m-1), ..., n_k m-1)$, where $c$ is the same as in $\tau$ and $d$ is the 3rd member of the string coded by $c$. Add 1 to the counter and go back to Step 5.

Again, the algorithm purports to generate a proof where $f_i(j, n)$ is computed for any $n$, provided that the $j$th axiom for the function symbols defines a binary symbol. We now consider an apparent proof of this claim. In the section after that, we consider an apparent proof that the algorithm is p.r.

5. Alleged proof that the algorithm is correct

The claim in exact terms is:

Claim: If $\varepsilon(i)$ codes the axiom defining $f_i^2$, then for any $n$, there is a $m$ such that the algorithm generates a unique proof of $f_i^2(j, n) = m$, at which point the algorithm halts.

Note that here and elsewhere in this section, our statements employ unbounded quantification; however, this alone does not undermine that the algorithm is p.r. Claim is thesis about the algorithm, not part of the algorithm itself.

Establishing Claim is best approached by considering p.r. terms with basic indices first, and then considering separately those with non-basic indices.
Basic Indices: Vacuously, **Claim** holds if the index \( i = 0 \); after all, the 0th function symbol is not binary, and a binary function-symbol with subscript 0 has no axiom in the system. Similar remarks apply when \( i = 1, 2, 3, \ldots, j - 1 \), where \( j = \) the code for \( 1 \times 2 \) or the code for \( 2 \times 2 \). Moreover, when \( i = j \), then \( f_i \) is a basic binary projection, and the algorithm clearly handles these under Step 1 in a way that satisfies **Claim**.

Non-Basic Indices. We want to show that **Claim** holds when \( i \) is a non-basic index. This part of the proof will require four lemmas. The first is as follows:

**Lemma 1**: If \( f_i^2 \) has a non-basic index, then for any \( n \), the algorithm starts with a proof of \( f_i(x, 0) = f_i(x, y) \), where \( f_i(x, y) \) is a baseline term and has no free variables.

Proof: If \( i \) is non-basic, there are three kinds of case to consider. One is where \( i \) is a recursion index and \( n = 0 \); the second is where \( i \) is a recursion index and \( n \neq 0 \); the third is where \( i \) is a composition index. In the first case, the algorithm starts thusly:

1. \( f_i(x, 0) = f_a(x) \) & \( f_i(x, y) = f_b(f_i(x, y), x, y) \)  \[Axiom for \( f_i \)\]
2. \( f_i(i, 0) = f_a(i) \) & \( f_i(i, n) = f_b(f_i(i, n), i, n) \)  \[(VS), 1: x/i, y/n\]
3. \( f_i(i, 0) = f_a(i) \)  \[(&E), 2\]

The rules used are obviously sound; also, \( f_a(i) \) has no free variables and is by definition a baseline term. So the third line verifies **Lemma 1** in this first case. In the second case, the algorithm begins with the lines:

1. \( f_i(x, 0) = f_a(x) \) & \( f_i(x, y) = f_b(f_i(x, y), x, y) \)  \[Axiom for \( f_i \)\]
2. \( f_i(i, 0) = f_a(i) \) & \( f_i(i, n) = f_b(f_i(i, n-1), i, n-1) \)  \[(VS), 1: x/i, y/n\]
3. \( f_i(i, 0) = f_a(i) \)  \[(&E), 2\]

The same rules are used as before, and again, \( f_b(f_i(i, n-1), i, n-1) \) has no free variables and is by definition a baseline term. So the third line verifies **Lemma 1** in the second case. In the third case, the algorithm starts as follows:

1. \( f_i(x, y) = \phi(x, y) \)  \[Axiom for \( f_i \)\]
2. \( f_i(i, n) = \phi(i, n) \)  \[(VS), 1: x/i, y/n\]
The rule is sound; $\phi_i(i, n)$ has no free variables and is by definition a baseline term. So the second line verifies Lemma 1 in the third case, which completes the proof of Lemma 1.

Remark: Like the previous two cases, the baseline term in the third case will have at least one occurrence of $'f'$. After all, $f_i$ in the third case expresses a composed p.r. function, and the axiom for $f_i$ therefore will contain at least one $'f$ on the righthand side (and the baseline term is an instantiation of the righthand side).

The second lemma required is the following:

**Lemma 2:** [Soundness] If the algorithm applies (Z), (S), (P), (C), (R1), or (R2) to a line in order to produce a new line of a sequence, then in the standard model, if the former line is true, so is the latter.

The lemma is claiming that that algorithm uses the shortcut rules in a way that is sound in the standard model. We prove this by considering its use of each of the rules.

Preliminary observation: The algorithm is designed to apply the shortcut rules only to the rightmost, innermost nn-term $\tau$ on a line. So, $\tau$ will be loaded with numerals only. For if $\tau$ were loaded with a nn-term, it would not be the innermost. (Also, we also know from Lemma 1 that a baseline term has no free variables, and none of the inference rules introduce free variables into the picture.)

**Applying (Z):** Suppose the algorithm applies (Z) so to produce $f_i(i, n)=\phi(\tau_2)$ from $f_i(i, n)=\phi(\tau_1)$. Then, per the instructions on Step 6A, $\tau_1$ must be of the form $f_0(m)$, for some numeral $m$. Also, per those instructions, $\tau_2$ must be 0. Since $f_0$ expresses the constantly-zero function, Lemma 2 is verified in the case of (Z).

**Applying (S):** Suppose the algorithm applies (S) so to produce $f_i(i, n)=\phi(\tau_2)$ from $f_i(i, n)=\phi(\tau_1)$. Then, per the instructions on Step 6B, $\tau_1$ must be of the form $f_1(m)$, for some
numeral \( m \). Also, per those instructions, \( \tau_2 \) must be \( m' \). Thus, since \( f_1 \) expresses the successor function, Lemma 2 is verified in the case of (S).

Applying (P): Suppose the algorithm applies (P) so to produce \( f(i, n) = \phi(\tau_2) \) from \( f(i, n) = \phi(\tau_1) \). Then, per the instructions on Step 6C, \( \tau_1 \) must be of the form \( f_c(m, \ldots, n) \), where \( c \) codes \( *^k \) and \( 1 \leq j \leq k \leq c \). Also, per those instructions, \( \tau_2 \) must be \( n_j \). Thus, since \( f_c \) expresses the \( k \)-ary \( j \)-th projection function, Lemma 2 is verified in the case of (P).

Applying (C): Suppose the algorithm applies (C) so to produce \( f(i, n) = \phi(\tau_2) \) from \( f(i, n) = \phi(\tau_1) \). Then, per the instructions on Step 6D, \( \tau_1 \) must be of the form \( f_c(m, \ldots, n) \), where \( c \) codes \( 0^* b^* c^* \ldots c^* k \). Also, per those instructions, \( \tau_2 \) must be \( f_b(f_c(m, \ldots, n), \ldots, f_c(m, \ldots, n)) \). Observe that, given how the indices for composed functions are assigned, \( f_c \) expresses a function where the \( l \)-ary function expressed by \( f_b \) is composed with the \( k \)-ary functions expressed by \( f_c, \ldots, f_c \). The consequence is that \( f_c(m, \ldots, n) \) co-refers with \( f_b(f_c(m, \ldots, n), \ldots, f_c(m, \ldots, n)) \). Thus, the transition from \( f(i, n) = \phi(\tau_1) \) to \( f(i, n) = \phi(\tau_2) \) is sound, and Lemma 2 is verified in the case of (C).

Applying (R1): Suppose the algorithm applies (R1) so to produce \( f(i, n) = \phi(\tau_2) \) from \( f(i, n) = \phi(\tau_1) \). Then, per the instructions on Step 6Ei, \( \tau_1 \) must be of the form \( f_c(m, \ldots, n, 0) \) where \( c \) codes \( "\"^* d^* e^* k \). Also, per those instructions, \( \tau_2 \) must be \( f_d(n, \ldots, n) \). Observe that, given how the indices for composed functions are assigned, \( f_c \) expresses a function which is recursively defined by \( f_d \) and \( f_b \). The consequence is that \( f_c(m, \ldots, n, 0) \) co-refers with \( f_d(n, \ldots, n) \). So the transition from \( f(i, n) = \phi(\tau_1) \) to \( f(i, n) = \phi(\tau_2) \) is sound, and Lemma 2 is verified in the case of (R1).
**Applying (R2):** Suppose the algorithm applies (R2) so to produce $f_i(i, n) = \phi(\tau_2)$ from $f_i(i, n) = \phi(\tau_1)$. Then, per the instructions on Step 6Eii, $\tau_1$ must be of the form $f_c(n_1, ..., n_k, m)$ for $m > 0$, where $c$ codes "*a*d* k. Also, per those instructions, $\tau_2$ must be $f_d(f_c(n_1, ..., n_k, m-1), n_1, ..., n_k, m-1)$. Observe that, given how the indices for composed functions are assigned, $f_c$ expresses a function which is recursively defined by $f_a$ and $f_d$. The consequence is that $f_c(n_1, ..., n_k, m)$ co-refers with $f_d(f_c(n_1, ..., n_k, m-1), n_1, ..., n_k, m-1)$. So the transition from $f_i(i, n) = \phi(\tau_1)$ to $f_i(i, n) = \phi(\tau_2)$ is sound, and Lemma 2 is verified in the case of (R1).

This completes the proof of Lemma 2.

The third lemma will be needed for proving the fourth lemma, later:

**Lemma 3:** If $\beta$ is the baseline term for a sequence and the algorithm produces a derivation from $f_i(i, n) = \beta$ to $f_i(i, n) = \chi$, then at that point, the counter is set to the number of times 'f' occurs in $\chi$.

We argue this by induction on the number of times that the Main Loop iterates in deriving $f_i(i, n) = \chi$ from $f_i(i, n) = \beta$.

**Base Case:** The Main Loop iterates zero times. Then, the derivation just consists in the single line $f_i(i, n) = \beta$, meaning that $\chi = \beta$. Moreover, the counter is initially set to the number of 'f's in $\beta$ by stipulation. So the base case verifies Lemma 3.

**Inductive Cases:** Suppose that if the algorithm applies the Main Loop $m$ times to produce a derivation from $f_i(i, n) = \beta$, from $f_i(i, n) = \xi$, then and at that point, the counter matches the number of 'f's in $\xi$. We want to show that if the algorithm applies the Main Loop one more time to derive $f_i(i, n) = \chi$, the counter then matches the number of 'f's in $\chi$. There are six cases to consider.
Case 1: The $m+1$th application of the Main Loop is an application of (Z). Then, the algorithm adds a line which is the same as the previous one, except that the rightmost nn-term in $\xi$ is replaced with $0$. The replacement means that the right side of the newest line has one less occurrence of $'f'$. Similarly, whereas the counter was set to the number of $'f's$ in $\xi$, the algorithm now reduces it by $1$. Thus, the counter matches the number of $'f's$ to the right of $'=' in the newest line. So, Case 1 verifies Lemma 3.

Case 2: The $m+1$th application of the Main Loop is an application of (S). Then, the algorithm adds a line which is the same as the previous one, except that the rightmost nn-term in $\xi$, which is of the form $f_i(m)$, is replaced with $m'$. The replacement means that the newest line has one less occurrence of $'f'$. Similarly, whereas the counter was set to the number of $'f's$ in $\xi$, the algorithm now reduces it by $1$. Thus, the counter matches the number of $'f's$ to the right of $'=' in the newest line. So, Case 2 verifies Lemma 3.

Case 3: The $m+1$th application of the Main Loop is an application of (P). Then, the algorithm adds a line which is the same as the previous one, except that the rightmost nn-term in $\xi$, which is of the form $f_c(n_1, \ldots, n_k)$, has been replaced by a numeral $n_j$ where $j \leq k$. The replacement means that the right side of the newest line has one less occurrence of $'f'$. Similarly, whereas the counter was set to the number of $'f's$ in $\xi$, the algorithm now reduces it by $1$. Thus, the counter matches the number of $'f's$ to the right of $'=' in the newest line. So, Case 3 verifies Lemma 3.

Case 4: The $m+1$th application of the Main Loop is an application of (C). Then, the algorithm adds a line which is the same as the previous one, except that the rightmost nn-term in $\xi$, of the form $f_c(n_1, \ldots, n_k)$, is replaced with $f_{\ell}(f_{\ell_1}(n_1, \ldots, n_k), \ldots, f_{\ell_k}(n_1, \ldots, n_k))$. The term that is replaced in $\xi$ has $1$ occurrence of $'f', whereas the term replacing it has $1+l$
occurrences of ‘f’. So the replacement means that the right side of the newest line has \( l \) more occurrences of ‘f’ than in \( \xi \). Similarly, whereas the counter was set to the number of ‘f’s in \( \xi \), the algorithm now increases it by \( l \). Thus, the counter matches the number of ‘f’s to the right of ‘=’ in the newest line. So, Case 4 verifies Lemma 3.

Case 5: The \( m+1 \)th application of the Main Loop is an application of (R1). Then, the algorithm adds a line which is the same as the previous one, except that the rightmost \( n \)-content in \( \xi \), which is of the form \( f_c(n_1, \ldots, n_k, 0) \), is replaced with \( f_d(n_1, \ldots, n_k, m) \). The term that is replaced in \( \xi \) has 1 occurrence of ‘f’, and so does the term replacing it. Similarly, the number on the counter is unchanged. Thus, the counter matches the number of ‘f’s to the right of ‘=’ on the newest line. So, Case 5 verifies Lemma 3.

Case 6: The \( m+1 \)th application of the Main Loop is an application of (R2). Then, the algorithm adds a line which is the same as the previous one, except that the rightmost \( n \)-content in \( \xi \), which is of the form \( f_c(n_1, \ldots, n_k, m) \) where \( m > 0 \), is replaced with the term \( f_d(f_c(n_1, \ldots, n_k, m-1), n_1, \ldots, n_k, m-1) \). Now the term that is replaced in \( \xi \) has 1 occurrence of ‘f’, whereas the term replacing it has 2. So the replacement means that the right side of the newest line has one additional occurrence of ‘f’. Similarly, whereas the counter was set to the number of ‘f’s in \( \xi \), the algorithm now increases it by 1. Thus, the counter matches the number of ‘f’s to the right of ‘=’ in the newest line. So, Case 6 verifies Lemma 3.

By induction, this suffices to establish Lemma 3. We pause to note a consequence:

Corollary: The algorithm halts if and only if it produces a line \( f_i(i, n) = m \), for some \( m \).

Proof: The ‘if’ part is true by stipulation, and if \( i \) is basic, then the ‘only if’ part is obvious. If \( i \) is non-basic, the ‘only if’ is shown as follows: It can be readily verified that the algorithm halts with non-basic indices iff the counter reaches 0. From Lemma 1, if \( i \) is non-basic, then
the algorithm generates a line \( f(i, n) = \beta \). And from **Lemma 3**, if the algorithm derives

\( f(i, n) = \beta \), the counter reaches 0 only if the number of \( f \)'s to the right of \( '=' \) in the latest line of the proof is 0, i.e., only if the term to the right of \( '=' \) in that line is a numeral. So with non-basic indices as well, **Corollary** holds.

The fourth lemma is now stated as follows:

**Lemma 4:** If \( i \) is a non-basic index, then for any \( n \), the algorithm will produce a derivation from \( f(i, n) = \beta \) to \( f(i, n) = m \), for some \( m \), where \( \beta \) is the baseline term, after which the algorithm halts.

Proof: We saw that the baseline term has at least one occurrence of \( 'f' \). Thus, the algorithm will iterate the Main Loop at least once. Now the Main Loop is defined in such a way that, on a given iteration, it looks at the equation \( \varepsilon \) (on the latest line) and to the right of \( '=' \), it finds the rightmost \( nn \)-term \( \tau \), if any. If it does not find such a \( \tau \), then the righthand term of the equation is a numeral and the algorithm halts—in which case, **Lemma 4** is verified. Otherwise, if finds such a \( \tau \), it adds a line, whereby \( \tau \) is replaced in \( \varepsilon \) with a term that is a p.r. reduction of \( \tau \). The counter is then adjusted, and the algorithm then checks whether the counter is at zero. If so, then the algorithm halts and, by **Corollary**, we know that the term right of \( '=' \) on the latest line is a numeral. In which case, **Lemma 4** is verified. Otherwise, **Corollary** tells us that the Main Loop will be restarted to further reduce the righthand term on the latest line.

So in brief, when the algorithm locates a rightmost \( nn \)-term (to the right of \( '=' \)), it adds a line where that term is removed and replaced with a p.r. reduction of the term. The replacement may not be a numeral, but no matter: By **Corollary**, the algorithm will continue iterating and the replacements are made in a unique order, until a line is reached where the righthand term is a numeral. And there will be such a line, given that each p.r. term is
ultimately reducible to a numeral in finitely many steps, as per the definitional rules for p.r. functions. So the algorithm will produce a derivation from $f(i, n) = \beta$ to $f(i, n) = m$, for some $m$ by iterating the Main Loop a sufficient number of times, after which it will stop—which is what Lemma 4 says.

From Lemma 1, Lemma 2, and Lemma 4, it follows that if $i$ is a non-basic index, the algorithm generates a unique proof of $f(i, n) = m$, for some $m$. That suffices for Claim in the case of non-basic indices, which completes the alleged proof of Claim.

6. Alleged proof that the algorithm is p.r.

We have just argued that the algorithm behaves as advertised, and we now consider an argument that the algorithm is p.r. For simplicity’s sake, the algorithm is described as operating on linguistic strings rather than on the codes for these strings—but since coding and decoding is p.r., it is of no import.

The algorithm is patently p.r. apart from steps 4 – 6 (a.k.a. “the Main Loop”). When it comes to step 4, the algorithm first applies Gödel’s “length” operation (cf. note 8 above) to determine the number $w$ of symbols in the latest line of the proof. This number $w$ is set as a bound for a count of the number of ‘‘f’s occurring to the right of ‘=’ in the latest line, thus making the count of ‘‘f’s p.r. Moreover, the count of ‘‘f’s just is the initial value for the counter, meaning that the counter is initialized in a primitive recursive manner.

As for step 5, the algorithm first evaluates if the counter’s value is 0, in which case it halts. Otherwise, the algorithm can find the rightmost, innermost nn-term in the latest line of the proof as follows. First, it counts the parenthesis-pairs (not greater than $w$) which enclose each ‘‘f’ to the right of ‘=’. An occurrence of ‘‘f’ that yields the greatest count begins
an innermost nn-term.\textsuperscript{9} If there is more than one, the rightmost one is the one that is last in the wff. (If the terms in the wff are enumerated left-to-right, it will be the innermost term enumerated by the greatest number, not greater than \( \nu \).) And the relevant maneuvers are p.r.; they amount to bounded arithmetic operations on finite sequences.\textsuperscript{10}

At step 6, the algorithm adds a new line to the proof, which is identical to the previous line, except that the rightmost, innermost nn-term \( \tau \) has been replaced. Importantly, replacement is p.r.; see Godel's \( Sb \) operation, which is \#31 in his list of p.r. functions and relations. In more detail, the algorithm first identifies the index for \( \tau \) as 0, 1, or as coding a specific sort of string of numerals. (The index \( c \) will always code a finite string, if any, and the string can be checked in a p.r. manner to see if \( c \) is a composition index or a recursion index.) The algorithm then replaces \( \tau \) as per the following:

- (Z): If \( \tau \) is \( f_0(m) \), replace it with 0.
- (S): If \( \tau \) is \( f_1(m) \), replace it with \( m' \).
- (P): If \( \tau \) is \( f_i(n_1, \ldots, n_k) \) where \( c \) codes \( j \times k \) and \( 1 \leq j \leq k \leq c \), replace it with \( n_i \).
- (C): If \( \tau \) is \( f_i(n_1, \ldots, n_k) \), and \( c \) codes a composition index, replace \( \tau \) with \( f_b(f_c(n_1, \ldots, n_k)) \).
- (R1): If \( \tau \) is \( f_i(n_1, \ldots, n_k, 0) \), and \( c \) codes a recursion index, replace it with \( f_a(n_1, \ldots, n_k) \).
- (R2): If \( \tau \) is \( f_i(n_1, \ldots, n_k, m') \), and \( c \) codes a recursion index, replace it with \( f_d(f_c(n_1, \ldots, n_k, m-1), n_1, \ldots, n_k, m-1) \).

Given \( c \), the algorithm extracts \( j, k, b, c_1, \ldots, c_l, a, b \) or \( d \), as needed, and such extraction is p.r.

The replacement for \( \tau \) is then constructed in a p.r. manner since this at most requires replacement into finite term-schemes (which are found in the axiom-schemes for the function-symbols).

\textsuperscript{9} Note that codes for terms in PRA– can be identified by Godel's “first-order term” operation; \#18 in Gödel's list of p.r. functions and relations.

\textsuperscript{10} For assistance with this paragraph, I thank [redacted].
After replacing \( \tau \), the counter is unchanged or adjusted, based on the rule applied. For (Z), (S), and (P), it is decreased by 1, and for (R1) and (R2), respectively, it is unchanged or increased by 1. As for (C), it is increased by \( l \), the number of sub-functions in the composition. These updates are p.r.; at most they require addition or subtraction on a pair of numbers. And the counter eventually reaches 0; cf. Lemma 3.

So too, the other operations in the algorithm are patently p.r. (e.g., halting when the counter reaches zero). Apparently, then, the algorithm itself is p.r., as intended.

7. Closing remarks

To repeat, there must be an error. If \( u \) is p.r., then there is also a p.r. function that outputs \( f(j, n) + 1 \), for any \( i \) and \( n \). Since the function would be expressed by some symbol \( f_j \), it would then follow that \( f(j, n) = f(j, n) + 1 \); contradiction. So PRA– would be inconsistent. More, PRA– is a conservative extension of \( |Q| \), where \( |Q| \) is Robinson arithmetic with bounded quantifiers only. So, \( |Q| \) would be inconsistent. And since the theorems of \( |Q| \) are a subset of the theorems of Q, it would follow that Q is inconsistent.

It might be thought that the diagonalization argument becomes dubious in this instance, rather than the function \( u \). In fact, it seems there is a second argument for inconsistency where a Liar-like sentence is explicitly constructed, but this argument shall be reserved for another time. A universal p.r. function would be quite unexpected regardless, since diagonalization arguments are typically militated against such a thing.

I myself remain puzzled. My hope is that others might look into the matter and make sense of it.
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