Families of 2D superintegrable anisotropic Dunkl oscillators and algebraic derivation of their spectrum

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Abstract
We generalize the construction of integrals of motion for quantum superintegrable models and the deformed oscillator algebra approach. This is presented in the context of 1D systems admitting ladder operators satisfying a parabosonic algebra involving reflection operators and more generally \( c_3 \) extended oscillator algebras with grading. We apply the construction on two-dimensional \( c_3 \) oscillators. We also introduce two new superintegrable Hamiltonians that are the anisotropic Dunkl and the singular Dunkl oscillators. Integrals are constructed by extending the approach of Daskaloyannis to include grading. An algebraic derivation of the energy spectra of the two models is presented, making use of finite dimensional unitary representations. We show how the spectra divide into sectors, and make comparisons with the physical case.

Keywords: superintegrable systems, polynomial algebras, deformed oscillator, Dunkl oscillator

1. Introduction

It is well known that the deformed oscillator algebra approach (also referred to as the Daskaloyannis method) may be used to obtain finite dimensional unitary representations of finitely generated polynomial algebras involving three generators, and to obtain algebraic derivations of the energy spectrum of various superintegrable systems [1–5]. Some examples of higher rank quadratic algebras have also been considered [6–8]. Some of these algebras were direct sums of Lie algebras and quadratic algebras with structure constants involving Casimir operators of some higher rank Lie algebras, that are central elements of the quadratic
algebra. Other cases of quadratic algebras have been shown to have a more complicated embedded structure. In addition, methods to generate polynomial algebras and integrals of motion of systems from ladder operators satisfying polynomial Heisenberg algebras have been used [9]. These methods have been extended to the case of integrals constructed with combinations of various types of supercharges, ladder and shift operators [10–14].

One-dimensional quantum systems involving the Dunkl operator or reflection operators were studied in the context of Calogero models and supersymmetric quantum mechanics [15–22]. In recent years, a 2D superintegrable system involving reflection operators has been discovered and studied [23]. A series of papers then introduced isotropic Dunkl oscillators in 2D and 3D, singular Dunkl oscillators, and a particular anisotropic case with frequency 2:1 [23–27, 29]. These works made interesting connections between the resulting quadratic algebras and orthogonal polynomials of various types such as Jacobi–Dunkl polynomials and dual-1 Hahn polynomials. Some works on angular momentum involving such operators and the related algebra were also undertaken recently [30]. The 1D Dunkl and singular oscillators also admit ladder operators involving reflection operators and from an algebraic perspective they comprise a particular case among the class of $c_3$ extended oscillators that involve generators of the cyclic group [31–33].

The purpose of this paper is to extend the construction of integrals for 2D Hamiltonians from ladder operators satisfying a $c_3$ extended oscillator algebra for the 1D components of the Hamiltonian, obtain the corresponding polynomial algebras and demonstrate how the deformed oscillator algebra approach can be applied to obtain the spectrum. The other main objectives are to introduce two new families of 2D superintegrable systems that are anisotropic generalizations of the 2D Dunkl and singular Dunkl oscillators, present their integrals, polynomial algebras, the realizations as deformed oscillator algebras and obtain the spectrum.

In section 2, we present a construction of integrals of motion using ladder operators satisfying a $c_3$ extended oscillator algebra and present the finitely generated polynomial algebras. In section 3, we study an anisotropic generalization of Dunkl and singular Dunkl oscillator and present algebraic derivation of the energy spectrum from the finite-dimensional unitary representation. In section 4, we make a comparison with the physical spectrum obtained via separation of variables in Cartesian coordinates.

### 2. Daskaloyannis approach with grading

We consider Hamiltonians built from two commuting parts
\[ H = H_1 + H_2, \]
i.e. such that $[H_1, H_2] = 0$, with each part $H_k$ associated to ladder operators $a_k, a_k^\dagger$ satisfying deformed $c_3$-extended oscillator algebra commutation relations [32]. Specifically, for $k = 1, 2$, we let $T_k$ denote generators of the cyclic group of order $\lambda_k$, satisfying $T_k^{\lambda_k} = I$, with $C_{\lambda_k} = \text{span} \{I, T_k, T_k^2, \ldots, T_k^{\lambda_k-1}\}$ being the corresponding group algebra such that
\[ a_k^\dagger T_k^\ell = q_k^{-\ell} T_k a_k^\dagger, \]
where we have set $q_k = e^{2\pi i/\lambda_k}$. An immediate consequence of this relation is the identity
\[ a_k^\dagger T_k^\ell = q_k^{-\ell} T_k a_k^\dagger, \quad \ell = 1, 2, \ldots, \lambda_k - 1. \]
We furthermore define $T_k^\ell = T_k^{\lambda_k-\ell}$, from which it easily follows that
\[ (T_k^\ell)^\dagger = T_k^{\lambda_k-\ell}, \quad \ell = 1, 2, \ldots, \lambda_k - 1. \]
Taking the hermitian conjugate of equation (2.2) then leads to
\[ T_k^\ell a_k = q_k^{-\ell} q_k T_k^\ell, \quad \ell = 1, 2, \ldots, \lambda_k - 1. \]
In fact it is straightforward to establish the more general relation
\[ a_k^N T_k^m = q_k^{mN} T_k^m a_k^N. \quad (2.3) \]
We now impose the relations
\[ [H_k, a_k] = -a_k \left( \alpha_{k,0} I + \sum_{m=1}^{\lambda_k-1} \alpha_{k,m} T_k^m \right). \quad (2.4) \]
Taking
\[ \beta_{k,m} = \bar{\alpha}_{k,\lambda_k-m} q_k^m, \quad m = 1, 2, \ldots, \lambda_k - 1, \quad \beta_{k,0} = \bar{\alpha}_{k,0} \quad (2.5) \]
(using the overbar to denote complex conjugation), the hermitian conjugate of (2.4) gives
\[ [H_k, a_k^\dagger] = a_k \left( \beta_{k,0} I + \sum_{m=1}^{\lambda_k-1} \beta_{k,m} T_k^m \right). \quad (2.6) \]
If we further impose
\[ \alpha_{k,0} = \bar{\alpha}_{k,0} \text{ and } \alpha_{k,m} = \bar{\alpha}_{k,\lambda_k-m}, \quad (2.7) \]
then we may deduce
\[ [H_k, a_k^\dagger a_k] = 0. \quad (2.8) \]
By contrast to our approach, we note that in the work of Quesne and Vansteenkiste [32], the authors set
\[ [a, a^\dagger] = 1 + \sum_{\mu=1}^{\lambda_k-1} \kappa_{\mu} T^\mu, \]
and impose the condition \( \bar{\kappa}_{\mu} = \kappa_{\lambda_k-\mu} \). The condition (2.7) could be considered an analogue of this condition.

Using the relations (2.4) and (2.3) we may easily deduce by induction that
\[ [H_k, a_k^N] = -N \alpha_{k,0} I + \sum_{m=1}^{\lambda_k-1} \sum_{j=1}^{N} q_k^{jm} \alpha_{k,m} T_k^m a_k^N. \quad (2.9) \]
Furthermore, for the primitive \( \lambda_k \)th root of unity \( q_k \) and for any \( m = 1, 2, \ldots, \lambda_k - 1 \), we have
\[ \sum_{j=1}^{\lambda_k} q_k^{jm} = 0. \quad (2.10) \]
Setting \( N = \lambda_k \) in equation (2.9) and applying (2.10) leads to the commutation relation
\[ [H_k, a_k^{\lambda_k}] = -\lambda_k \alpha_{k,0} a_k^{\lambda_k}. \]
In a similar way, we may also deduce the relation
\[ [H_k, (a_k^{\dagger})^{\lambda_k}] = \lambda_k \alpha_{k,0} (a_k^{\dagger})^{\lambda_k}. \]
We also note, from (2.3), that
\[ a_k^{\lambda_k} T_k = T_k a_k^{\lambda_k}. \]
We may express equation (2.6) in the form
\[
H_k a_k^\dagger = a_k^\dagger \left( H_k + \sum_{m=0}^{\lambda_k-1} \beta_{k,m} T_k^m \right),
\]
and similarly write equation (2.4) as
\[
a_k^\dagger H_k = \left( H_k + \sum_{m=0}^{\lambda_k-1} \beta_{k,m} T_k^m \right) a_k,
\]
where we have made use of (2.5) and (2.7) above. We then deduce that for any analytic function \( f \), we have the relations
\[
f(H_k) a_k^\dagger = a_k^\dagger f \left( H_k + \sum_{m=0}^{\lambda_k-1} \beta_{k,m} T_k^m \right),
\]
\[
a_k^\dagger f(H_k) = f \left( H_k + \sum_{m=0}^{\lambda_k-1} \beta_{k,m} T_k^m \right) a_k.
\]
This indicates that if we set the analytic function \( f \) as
\[
a_k^\dagger a_k = f(H_k),
\]
then we may consistently set
\[
a_k^\dagger a_k^\dagger = f \left( H_k + \sum_{m=0}^{\lambda_k-1} \beta_{k,m} T_k^m \right).
\]
Even though this is justified by equation (2.8), it turns out that we may consistently impose the condition that the \( T_k \) also commute with \( H_k \). Consider the following argument.

We start by investigating the commutator of both sides of the relation (2.1). Using the standard derivation rule, we have
\[
[H_k, a_k^\dagger T_k] = a_k^\dagger [H_k, T_k] + a_k^\dagger \sum_{m=0}^{\lambda_k-1} \beta_{k,m} T_k^{m+1},
\]
and
\[
[H_k, q_k^{-1} T_k a_k^\dagger] = a_k^\dagger \sum_{m=0}^{\lambda_k-1} \beta_{k,m} T_k^{m+1} + q_k^{-1} [H_k, T_k] a_k^\dagger.
\]
Equating both expressions gives
\[
a_k^\dagger [H_k, T_k] = q_k^{-1} [H_k, T_k] a_k^\dagger.
\]
Based on the form of (2.1), we make the assumption that
\[
[H_k, T_k] = \phi_k T_k,
\]
where \( \phi_k \) is some constant of proportionality. This then implies that
\[
H_k T_k = T_k (H_k + \phi_k),
\]
which may easily be extended to powers of \( T_k \), namely
\[
H_k T_k^m = T_k^m (H_k + m \phi_k).
\]
Setting \( m = \lambda_k \), however, leads to the conclusion that \( \phi_k = 0 \), and therefore we have
\[
[H_k, T_k] = 0. \tag{2.11}
\]

The outcome of this discussion is that we can impose the relations
\[
a_k^\dagger a_k = \sum_{m=0}^{\lambda_k-1} Q_{k,m}(H_k)T_k^m \tag{2.12}
\]
and
\[
a_k^\dagger a_k^\dagger = \sum_{m=0}^{\lambda_k-1} Q_{k,m}\left(H_k + \sum_{n=0}^{\lambda_k-1} \beta_kT_k^n\right)T_k^m, \tag{2.13}
\]
where the \( Q_{k,m} \) are some specified analytic functions.

Now introduce the new operators
\[
B_k = a_k^\lambda, \quad B_k^\dagger = (a_k^\dagger)^\lambda. \tag{2.14}
\]

We immediately have
\[
[H_k, B_k] = -\gamma_k B_k, \quad [H_k, B_k^\dagger] = \gamma_k B_k^\dagger, \tag{2.15}
\]
where we have set \( \gamma_k = \lambda_k \phi_k,0 \), which must be a real number (from (2.7)). This then leads to
\[
B_k^\dagger B_k = G_k(H_k), \quad B_k B_k^\dagger = G_k(H_k + \gamma_k),
\]
where
\[
G_k(H_k) = \prod_{t=1}^{\lambda_k} Z_t^{(k)}(H_k, T_k).
\]

Here, for convenience, we have set
\[
Z_t^{(k)}(H_k, T_k) = \sum_{m=0}^{\lambda_k-1} q_k^{tm} Q_{k,m}\left(H_k - \sum_{n=0}^{\lambda_k-1} \sum_{j=0}^{\lambda_k-1-t} \beta_k q_k^{-jn}T_k^n\right)T_k^m,
\]
and in particular
\[
Z_{\lambda_k,\lambda_k}(H_k, T_k) = \sum_{m=0}^{\lambda_k-1} Q_{k,m}(H_k)T_k^m.
\]

The generators of the cyclic group are now central elements of the finitely generated polynomial algebra. We introduce the following generators
\[
I_\pm = (B_1)^{n_1}(B_2)^{n_2}, \quad I_\mp = (B_1)^{n_1}(B_2)^{n_2}, \quad K = \frac{1}{2\gamma}(H_1 - H_2), \tag{2.16}
\]
with constraints \( n_1 \gamma_1 = n_2 \gamma_2 = \gamma \). These generate the following polynomial algebra
\[
[K, I_\pm] = \pm I_\pm,
\]
\[
[I_-, I_+] = S_{n_1,n_2}(K + 1, H) - S_{n_1,n_2}(K, H),
\]
where
\[
S_{n_1,n_2}(K, H) = \prod_{i=1}^{n_1} G_1 \left( \frac{1}{2}H + \gamma K - (n_1 - i)\gamma_1 \right) \prod_{j=1}^{n_2} G_2 \left( \frac{1}{2}H - \gamma K + j\gamma_2 \right). \tag{2.17}
\]

The polynomial algebra can be written in the form of a deformed oscillator algebra by taking \( b^\dagger = I_+, \quad b = I_-, \quad N = K - u \), and structure function
\[ \Phi(N, u, H) = S_{n,n_2}(N + u, H). \]  
(2.18)

The constraints for finite dimensional unitary representations are known and take the form \( \Phi(0, u, E) = 0, \Phi(p + 1, u, E) = 0 \) and \( \Phi(x, u, E) > 0 \) for \( x = 0, \ldots, p \).

### 3. Dunkl oscillators

#### 3.1. 1D Dunkl oscillator

The Hamiltonian of the Dunkl oscillator with one parameter is given by

\[ H = -\frac{1}{2} (D^m_x)^2 + \frac{1}{2} m^2 x^2, \]

where \( m \) is some integer and the Dunkl operator given by \( D^m_x = \partial_x + \frac{\mu}{x} (1 - R) \). We remark that the operator \( R \), defined by the action \( R(f(x)) = f(-x) \), satisfies \( R^2 = I \) and so, in the context of our presentation so far, can be viewed as the generator of the cyclic group of order 2. The square of the Dunkl operator evaluates to

\[ (D^m_x)^2 = \partial_x^2 + \frac{2\mu}{x} \partial_x - \frac{\mu}{x^2} (1 - R). \]

We can introduce the following operators

\[ a^\dagger = \frac{1}{\sqrt{2}} (mx + D^m_x), \quad \text{ (3.19)} \]
\[ a = \frac{1}{\sqrt{2}} (mx - D^m_x) \quad \text{ (3.20)} \]

that generate the following algebraic relations [24, 25]

\[ a^\dagger a = H + \frac{m}{2} (-1 - 2\mu R), \quad aa^\dagger = H + \frac{m}{2} (1 + 2\mu R), \]

\[ [H, a^\dagger] = ma^\dagger, \quad [H, a] = -ma. \]

Defining

\[ B = \frac{a^2}{2}, \quad B^\dagger = \frac{(a^\dagger)^2}{2}, \]

these can be shown to satisfy the following commutation relations

\[ [H, B] = -2mB, \quad [H, B^\dagger] = 2mB^\dagger, \]

\[ B^\dagger B = G(H), \quad BB^\dagger = G(H + 2m), \]

where

\[ G(H) = \frac{1}{4} \left[ H - \frac{3}{2} m + m\mu R \right] \left[ H - \frac{1}{2} m - m\mu R \right]. \]

The fact that there is a quadratic expression of the Hamiltonian is related to the existence of two sectors unlike the case of the standard harmonic oscillator.
3.2. 2D anisotropic Dunkl oscillator

We now introduce the anisotropic Dunkl oscillator in two-dimensions:

\[
H = H_x + H_y = -\frac{1}{2}(D^x)^2 - \frac{1}{2}(D^y)^2 + \frac{m}{2}x^2 + \frac{n}{2}y^2.
\]

For both coordinate axes, we have two polynomial Heisenberg algebras involving the generators \(R_x\) and \(R_y\) (respectively) that are central elements of the algebra with \(n_1\gamma_x = n_2\gamma_y = \gamma\) (using the notation of equations (2.15)). In particular, we have \(\gamma_x = 2m,\ \gamma_y = 2n,\ m_1 = n,\ m_2 = m,\ \gamma = 2mn\). However, this differs from the usual anisotropic case as the analysis of finite dimensional unitary representations will involve different sectors due to the presence of the reflection operators \(R_x\) and \(R_y\) in the structure function.

By contrast to the 1D case, we set

\[
a_x = \frac{1}{\sqrt{2}}(mx + D^x), \quad a_x^\dagger = \frac{1}{\sqrt{2}}(ny + D^y),
\]

and also

\[
B_k = \frac{a_k^2}{2}, \quad B_k^\dagger = \frac{(a_k^\dagger)^2}{2}, \quad k = x, y
\]

so that

\[
[H_x, B_k^\dagger] = -2mB_k, \quad [H_x, B_k^\dagger] = 2mB_k^\dagger,
\]

\[
[H_y, B_k] = -2nB_k, \quad [H_y, B_k] = 2nB_k^\dagger.
\]

Making use of the generators \(L_x\) defined in (2.16), namely

\[
L_x = B_x^n(B_y)^m, \quad L_y = (B_x^\dagger)^nB_y^m,
\]

the structure function of the 2D Dunkl oscillator and algebraic derivation of the energy spectrum is determined by

\[
\Phi_{1h} = \prod_{i=1}^{n} \left[ \frac{1}{4} \left( \frac{E}{2} + 2mn(x + u) - (n - i)2m - \frac{3}{2}m + m\mu_x s_x \right) \right. \\
\left. \times \left( \frac{E}{2} + 2mn(x + u) - (n - i)2m - \frac{1}{2}m + m\mu_x s_x \right) \right]
\]

\[
\times \prod_{j=1}^{m} \left[ \frac{1}{4} \left( \frac{E}{2} - 2mn(x + u) + 2jn - \frac{3}{2}n + n\mu_y s_y \right) \right. \\
\left. \times \left( \frac{E}{2} - 2mn(x + u) + 2jn - \frac{1}{2}n - n\mu_y s_y \right) \right]
\]

where equations (2.17) and (2.18) have been used, \(s_x\) and \(s_y\) are the eigenvalues of \(R_x\) and \(R_y\) respectively, taking values 1 or \(-1\). We have two types of solutions when we impose \(\Phi(0, u, E) = 0\):

\[
u = -\frac{1}{2mn} \left( \frac{E}{2} - (n - k_1)2m - m - \frac{\epsilon_1}{2}m + \epsilon_1m\mu_x s_x \right).
\]
with the two cases corresponding to the choice $\epsilon_1 = \{1, -1\}$ and where $k_1 = 1, \ldots, n$. Substituting these expressions we obtain the energies using $\Phi(p + 1, u, E) = 0$:

$$
E = 2mn(p + 1) + m + n + \frac{\epsilon_1}{2}m + \frac{\epsilon_2}{2}n - (\epsilon_1 m\mu_s s_x + \epsilon_2 n\mu_s s_y) + 2(mn - k_2 n - k_1 m),
$$

with the corresponding structure functions

$$
\Phi_{HO}(x, p) = \prod_{i=1}^{n} \frac{1}{4} (2mn + (i - k_1)2m)(2mn + (i - k_1)2m + m + (1 - \epsilon_1)\mu_s s_x)
$$

$$
\times \prod_{j=1}^{m} \frac{1}{4} (2mn(p + 1) - 2mnx + 2n(j - k_2))(2mn(p + 1) - 2mnx + 2n(j - k_2) + n - 2\epsilon_2 n\mu_s s_y),
$$

where $k_1 = 1, \ldots, n$ and $k_2 = 1, \ldots, m$ and $\epsilon_1, \epsilon_2 \in \{1, -1\}$. It can be verified that $\Phi_{HO}(x, p) > 0$ for $x = 1, \ldots, p$, which ensures that the finite dimensional representations are unitary.

4. Singular Dunkl oscillators

4.1. 1D singular Dunkl oscillator

The Hamiltonian of the singular Dunkl oscillator with three parameters is given by

$$
H = -\frac{1}{2}(D^\mu)^2 + \frac{1}{2}m^2x^2 + \frac{\alpha + \beta R}{2x^2}.
$$

with the Dunkl operator $D^\mu$ defined as before. In this case we introduce the following operators which can be considered deformations of those defined in (2.14), and make use of the operators $a$ and $a^\dagger$ defined in (3.19) and (3.20) [26]:

$$
B^\dagger = (a^\dagger)^2 = \frac{\alpha + \beta R}{2x^2}, \quad B = a^2 = \frac{\alpha + \beta R}{2x^2}.
$$

These operators satisfy $[B^\dagger, R] = [B, R] = 0$, and moreover

$$
[H, B] = -2mB, \quad [H, B^\dagger] = 2mB^\dagger
$$

$$
= B^\dagger B = H^2 - 2mH + m^2\left(\frac{3}{4} + \mu R - \mu^2 - (\alpha + \beta R)\right) = G(H), \quad (4.23)
$$

$$
BB^\dagger = G(H + 2m).
$$

The expression for $G(H)$ given in (4.23) can be factorized to the form

$$
G(H) = (H - \Lambda_+)(H - \Lambda_-),
$$

with

$$
\Lambda_{\pm} = m \left(1 \pm \frac{1}{2}\sqrt{1 + 4\alpha + 4\beta s - 4\mu s + 4\mu^2 s^2}\right), \quad (4.24)
$$

where $s = -1, 1$ is the eigenvalue of the operator $R$.

We see that this case corresponds to a nontrivial deformation (provided $\alpha, \beta \neq 0$) of the regular 1D Dunkl oscillator introduced earlier, so the situation is genuinely different and
warrants further study. To this end, we now look at the singular Dunkl oscillator in two-dimensions.

4.2. 2D singular anisotropic Dunkl oscillator

The Hamiltonian we consider in this section can be expressed as

$$H = H_x + H_y = -\frac{1}{2} D_i D_i^\mu + \frac{1}{2} m^2 x^2 + \frac{\alpha_x + \beta_x R_x}{2x^2} + \frac{\alpha_y + \beta_y R_y}{2y^2}.$$  

We have the following two-polynomial Heisenberg algebras that contain the grading elements $R_x$ and $R_y$. We have similar integrals and a similar form of the deformed oscillator with $n_1 = n$, $n_2 = m$, $\gamma_x = 2m$, $\gamma_y = 2n$, $\gamma = mn$. Specifically, making use of the operators $a_x, a_x^\dagger, a_y, a_y^\dagger$ given in (3.21) and (3.22), we set

$$B_x = a_x^2 - \frac{\alpha_x + \beta_x R_x}{2x^2}, \quad B_x^\dagger = (a_x^\dagger)^2 - \frac{\alpha_x + \beta_x R_x}{2x^2},$$

$$B_y = a_y^2 - \frac{\alpha_y + \beta_y R_y}{2y^2}, \quad B_y^\dagger = (a_y^\dagger)^2 - \frac{\alpha_y + \beta_y R_y}{2y^2},$$

with

$$[H_x, B_x] = -2mB_x, \quad [H_x, B_x^\dagger] = 2mB_x^\dagger,$$

$$[H_y, B_y] = -2nB_y, \quad [H_y, B_y^\dagger] = 2nB_y^\dagger.$$  

The structure function (again making use of the generators of the form (2.16)) for the singular oscillator can be written as

$$\Phi_{SIO} = \prod_{i=1}^\mu \left[ \left( \frac{E}{2} + 2mn(x + u) - (n - i)2m - m\left(1 - \frac{s_x}{2} + \nu_{x,s_x} \right) \right) \times \left( \frac{E}{2} + 2mn(x + u) - (n - i)2m - m\left(1 + \frac{s_x}{2} - \nu_{x,s_x} \right) \right) \right] \times \prod_{j=1}^\mu \left[ \left( \frac{E}{2} - 2mn(x + u) + 2jn - n\left(1 - \frac{s_y}{2} + \nu_{y,s_y} \right) \right) \times \left( \frac{E}{2} - 2mn(x + u) + 2jn - n\left(1 + \frac{s_y}{2} - \nu_{y,s_y} \right) \right) \right].$$

As before, equations (2.17) and (2.18) have been used, $s_x$ and $s_y$ represent eigenvalues of the operators $R_x$ and $R_y$ respectively, and so take on values 1 or -1. To avoid overly complicated expressions, particularly arising from the form of $G(H)$ as given in (4.23), we have also introduced the notation

$$\nu_{x,s_x} = \frac{s_x}{2} - \frac{1}{2}\sqrt{1 + 4\alpha_x + 4\beta_x s_x - 4\mu_x s_x + 4\mu_x^2 s_x}$$

(and a similar expression for $\nu_{y,s_y}$), the form of which has been chosen for its connection to the physical spectrum. This last point shall be discussed in section 5 below.
Using $\Phi(0, u, E) = 0$, we obtain

$$u = \frac{1}{2mn} \left( -\frac{E}{2} + \frac{(n - k_1)2m + m\left(1 - \epsilon_1 \frac{s_2}{2} + \epsilon_1 \nu_{x,y}\right)}{2} \right).$$

Applying the condition $\Phi(p + 1, u, E) = 0$ determines the energy spectrum as

$$E = 2mn(p + 1) + 2(nm - k_2n - k_3m) + m\left(1 - \epsilon_1 \frac{s_2}{2} + \epsilon_1 \nu_{x,y}\right) + n\left(1 - \epsilon_2 \frac{s_2}{2} + \epsilon_2 \nu_{x,y}\right).$$

Once again, we have made use of the notation $\epsilon_1, \epsilon_2 = 1, -1$ to characterize the various cases that arise in this Daskaloyannis approach. The associated structure functions reduce to the form

$$\Phi_{SHO}(x, p)$$

$$= \prod_{i=1}^{n} (2mnx + (i - k_1)2m)(2mnx + (i - k_2)2m + n(-\epsilon_1 s_x + \epsilon_1 \nu_{x,y}))$$

$$\times \prod_{j=1}^{m} (2mn(p + 1) - 2mnx + 2n(j - k_2))(2mn(p + 1) - 2mnx + 2n(j - k_3))$$

$$- 2mnx + 2n(j - k_2) + n(-\epsilon_2 s_y + \epsilon_2 \nu_{x,y}).$$

Where $k_1 = 1, \ldots, n$ and $k_2 = 1, \ldots, m$. It can be verified also for this structure function that $\Phi_{SHO}(x, p) > 0$ for $x = 1, \ldots, p$.

5. A comment on the physical spectrum

The physical spectrum of these two new 2D anisotropic Dunkl and singular Dunkl oscillators can be obtained using separation of variables and results obtained in the 1D cases. Explicit expressions for the wavefunctions in the 1D cases [24–26] have been obtained by solving the corresponding differential equations and written in terms of Hermite and Laguerre polynomials, namely

$$\psi^+_n = \sqrt{\frac{n!}{\Gamma(n + \mu + \frac{1}{2})}} e^{-\frac{x^2}{2}} L_n^{\mu - \frac{1}{2}}(x^2)$$

$$\psi^-_m = \sqrt{\frac{m!}{\Gamma(m + \mu + \frac{1}{2})}} e^{-\frac{x^2}{2}} L_m^{\nu + \frac{1}{2}}(x^2)$$

and

$$\psi^+_n = (-1)^n \sqrt{\frac{n!}{\Gamma(n + \nu^+ + \frac{1}{2})}} e^{-\frac{x^2}{2}} x^{2k^+} L_n^{\nu^- - \frac{1}{2}}(x^2)$$

$$\psi^-_n = (-1)^n \sqrt{\frac{n!}{\Gamma(n + \nu^- + \frac{1}{2})}} e^{-\frac{x^2}{2}} x^{2k^-} L_n^{\nu^+ + \frac{1}{2}}(x^2).$$

In the Dunkl case there are four different sectors, characterized by the eigenvalues of $R_x$ and $R_y$.
In the case of the 2D singular oscillator spectrum we also have four sectors
\[ E = \left( 2mn_{1x} + 2mn_{1y} + m(\nu_{s,x} + 1 - \frac{1}{2}s_x) + n(\nu_{s,y} + 1 - \frac{1}{2}s_y) \right). \]

with
\[ \nu_{s,\pm} = 2k_i^\pm + \mu_i, \quad i = x, y \]
and
\[ \nu_{s,+} + \frac{1}{2} > 0, \quad \nu_{s,-} + \frac{3}{2} > 0. \]

The parameters \( k_i^\pm \) of the 2D singular oscillator must satisfy
\[ \alpha_i = 2k_i^+\left(k_i^+ + \mu_i - \frac{1}{2}\right) + 2k_i^-\left(k_i^- + \mu_i + \frac{1}{2}\right), \]
\[ \beta_i = 2k_i^+\left(k_i^+ + \mu_i - \frac{1}{2}\right) - 2k_i^-\left(k_i^- + \mu_i + \frac{1}{2}\right). \]

It turns out that not all eigenvalues arising from the algebraic approach are physical. In order to compare the physical spectrum and those eigenvalues obtained by the algebraic derivation via deformed oscillator algebra, we define
\[ n_{l_x}' = mn_{1x} + l_x, \quad n_{l_x}' = mn_{2x} + l_x, \quad n_{l_y}' = mn_{1y} + l_x, \quad n_{l_y}' = mn_{2y} + l_x \]
with \( l_x = 0, \ldots, n - 1 \) and \( l_y = 0, \ldots, m - 1 \) and \( n_{l_x}, n_{2x}, n_{l_y}, n_{2y} = 0, 1, 2, \ldots \). We take respectively for the four physical solutions \( p = n_{l_x}' + n_{l_y}' \) for \( i, j = 1, 2 \) and also consider \( l_x = -k_1 + n \) and \( l_y = -k_2 + m \).

6. Conclusion

In this paper, we have extended the construction of integrals to introduce several new families of superintegrable systems with ladder operators involving generator of the cyclic group. We generalize the Daskaloyannis approach for the corresponding class of polynomial algebras. We applied this construction for a sum of two \( c_3 \) extended oscillators.

As an application, we introduced two new families of superintegrable systems that are the anisotropic Dunkl and singular Dunkl oscillators for which we demonstrated how to apply the general construction. We compare the spectrum that divides into four sector to the physical spectrum derived by separation of variables.

Superintegrable systems with higher order integrals of motion expressed in terms of Dunkl operators (even with more general grading) are as yet unexplored, and no systematic study has been undertaken. Thus, the algebraic results of the current article which have been obtained in a general setting for two new families of models—both of which have integrals of motion of arbitrary order—open the door to future work.
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