SHUFFLE QUADRI-ALGEBRAS AND CONCATENATION

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ABSTRACT. In this article, we study the shuffle quadri-algebra $\mathcal{H}$. We prove the existence of some relations between quadri-algebra laws which constitute shuffle product, the concatenation product and the deconcatenation coproduct. We also show that $\mathcal{H}$ has two module-algebra structures on $(\mathcal{H}, \Pi)$.

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1. Introduction

A dendriform algebra is a vector space equipped with an associative product which can be written as a sum of two operations $≺$ and $≻$ called left and right respectively, which satisfy the following three operations:

\[
(x ≺ y) ≺ z = x ≺ (y ≺ z) + x ≺ (y ≻ z) \\
(x ≻ y) ≺ z = x ≻ (y ≺ z) \\
(x ≺ y) ≻ z + (x ≻ y) ≻ z = x ≻ (y ≻ z)
\]

They were introduced by Jean-Louis Loday [11, §5] in 1995 with motivation from algebraic K-theory and have been studied by other authors in different domains [1, 5, 6, 7, 12, 13, 16].

In 2004, Marcelo Aguiar and Jean-Louis Loday introduced the notion of quadri-algebra in [2]. A quadri-algebra is an associative algebra the multiplication of which can be decomposed as the sum of four operations $\downarrow, \uparrow, \leftarrow$ and $\rightarrow$ satisfying nine axioms. Two dendriform

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structures are attached to a quadri-algebra: the first dendriform structure is given by the two operations \( \succ \) and \( \prec \) such that:

\[
\begin{align*}
x \succ y & := x \uparrow y + x \downarrow y, \\
x \prec y & := x \downarrow y + x \uparrow y,
\end{align*}
\]

and the second is given by the two operations \( \vee \) and \( \wedge \) where:

\[
\begin{align*} 
x \vee y & := x \downarrow y + x \uparrow y, \\
x \wedge y & := x \uparrow y + x \downarrow y.
\end{align*}
\]

Quadri-algebras were studied by Loïc Foissy together with quadri-coalgebras and quadri-bialgebras [9].

In this article we revisit the canonical example of shuffle quadri-algebra, called also the symmetric functions quadri-algebra, [4, 11, 13, 15] treated by Marcelo Aguiar, Jean-Louis Loday and Loïc Foissy. We prove that there exists relations between the quadri-algebra laws, the concatenation product and the deconcatenation coproduct. We show that, for any \( u, v, w \in \mathcal{H} \), we have:

\[
\begin{align*}
u \uparrow (vw) & = \sum_{u=u^1u^2} (u^1 \downarrow v)(u^2 \wedge w) \\
& = \sum_{u=u^1u^2} (u^1 \succ v)(u^2 \wedge w), \\
u \downarrow (vw) & = \sum_{u=u^1u^2} (u^1 \downarrow v)(u^2 \vee w) \\
& = \sum_{u=u^1u^2} (u^1 \succ v)(u^2 \downarrow w), \\
u \uparrow (vw) & = \sum_{u=u^1u^2} (u^1 \wedge v)(u^2 \vee w) \\
& = \sum_{u=u^1u^2} (u^1 \prec v)(u^2 \downarrow w), \\
u \downarrow (vw) & = \sum_{u=u^1u^2} (u^1 \wedge v)(u^2 \wedge w) \\
& = \sum_{u=u^1u^2} (u^1 \prec v)(u^2 \uparrow w).
\end{align*}
\]
We derive from these results relations between the dendriform laws, the concatenation and the deconcatenation coproduct. We show that, for any $u, v, w \in H$, we have:

$$u \wedge (vw) = u \leftarrow (vw) + u \ll (vw)$$

$$= \sum_{u = u^1 u^2} (u^1 \lor v)(u^2 \land w),$$

$$u \prec (vw) = u \ll (vw) + u \leftarrow (vw)$$

$$= \sum_{u = u^1 u^2} (u^1 \prec v)(u^2 \succ w),$$

$$u \lor (vw) = u \leftarrow (vw) + u \ll (vw)$$

$$= \sum_{u = u^1 u^2} (u^1 \lor v)(u^2 \lor w),$$

$$u \succ (vw) = u \ll (vw) + u \leftarrow (vw)$$

$$= \sum_{u = u^1 u^2} (u^1 \succ v)(u^2 \succ w),$$

and consequently, two relations between the shuffle product, the concatenation and the deconcatenation coproduct. We show that, for any $u, v, w \in H$, we have:

$$u \equiv (vw) = \sum_{u = u^1 u^2} (u^1 \lor v)(u^2 \equiv w)$$

$$= \sum_{u = u^1 u^2} (u^1 \equiv v)(u^2 \equiv w).$$

At the end of this article, we prove the existence of two module-algebra structures on $H$ given by $\lor$ and $\succ$, in other words $\lor$ and $\succ$ verify:

$$m \circ (\lor \otimes \lor) \circ \tau_{23} \circ (\Delta \otimes I \otimes I) = \lor \circ (I \otimes m),$$

$$m \circ (\succ \otimes \succ) \circ \tau_{23} \circ (\Delta \otimes I \otimes I) = \succ \circ (I \otimes m).$$

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2. Dendriform algebras

A dendriform algebra is a vector space $D$ together with two operations $\prec: D \otimes D \to D$ and $\succ: D \otimes D \to D$, called left and right respectively, such that:

$$(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z)$$

$$(x \succ y) \prec z = x \succ (y \prec z)$$

$$(x \prec y) \succ z + (x \succ y) \succ z = x \succ (y \succ z).$$
Dendriform algebras were introduced [11, §5]. See also [1, 5, 6, 7, 12, 13, 16] for additional work on this subject. Defining a new operation by:

\[ x \star y := x \prec y + x \succ y \]  

permits us to rewrite axioms (1) as:

\[(x \prec y) \prec z = x \prec (y \star z)\]

\[(x \succ y) \prec z = x \succ (y \prec z)\]

\[(x \star y) \succ z = x \succ (y \succ z)\].

By adding the three relations we see that the operation \(\star\) is associative. For this reason, a dendriform algebra may be regarded as an associative algebra \((D, \star)\) for which the multiplication \(\star\) can be decomposed as the sum of two coherent operations.

3. Quadri-algebras

In this section, we use definitions and results on quadri-algebra structures given by Marcelo Aguiar and Jean-Louis Loday in [2] and Loïc Foissy in [9]. A quadri-algebra structure consists in splitting an associative product into four operations, which in turn gives rise to two distinct dendriform structures.

**Definition 1.** A quadri-algebra is a vector space \(Q\) together with four operations:

\[ \downarrow, \nearrow, \nwarrow, \swarrow : Q \otimes Q \rightarrow Q, \]

satisfying the nine axioms below. In order to state them, consider the following operations:

\[(x \succ y) := x \nearrow y + x \nwarrow y\]

\[(x \prec y) := x \swarrow y + x \searrow y\]

\[(x \vee y) := x \downarrow y + x \uparrow y\]

\[(x \wedge y) := x \nearrow y + x \searrow y\]

and:

\[(x \star y) := x \nearrow y + x \nwarrow y + x \swarrow y + x \searrow y\]

\[= x \succ y + x \prec y\]

\[= x \vee y + x \wedge y.\]

The nine axioms, stated by Marcelo Aguiar and Jean-Louis Loday in [2] are:

\[(x \nwarrow y) \nwarrow z = x \nwarrow (y \star z)\]

\[(x \nearrow y) \nwarrow z = x \nearrow (y \star z)\]

\[(x \swarrow y) \nwarrow z = x \swarrow (y \star z)\]

\[(x \prec y) \searrow z = x \searrow (y \star z)\]

\[(x \succ y) \searrow z = x \searrow (y \star z)\]

\[(x \star y) \searrow z = x \searrow (y \star z)\]
We refer to the operations $\nwarrow, \swarrow, \nearrow, \searrow$ as southeast, northeast, northwest, and southwest, respectively. Accordingly, we use north, south, west, and east for $\wedge, \vee, \prec$ and $\succ$. The axioms are displayed in the form of a $3 \times 3$ matrix. We will make use of standard matrix terminology (entries, rows and columns) to refer to them.

Let $Q$ be a quadri-algebra. Following [9], we extend the four products to $\bar{Q} \otimes \bar{Q} := (K \otimes Q) \oplus (Q \otimes Q) \oplus (Q \otimes K)$ in the following way: if $a \in Q$,

$$
a \nwarrow 1 = a \quad a \swarrow 1 = 0 \quad 1 \nwarrow a = 0 \quad 1 \swarrow a = 0
$$

$$
a \nearrow 1 = 0 \quad a \searrow 1 = 0 \quad 1 \nearrow a = 0 \quad 1 \searrow a = a
$$

It follows that we have for any $a \in Q$:

$$
a \wedge 1 = a \quad 1 \wedge a = 0 \quad 1 \vee a = a \quad a \vee 1 = 0
$$

$$
a \succ 1 = 0 \quad 1 \succ a = a \quad 1 \prec a = 0 \quad a \prec 1 = a
$$

4. FROM QUADRI-ALGEBRAS TO DENDRIFORM ALGEBRAS

The three column sums in the matrix of quadri-algebra axioms yield:

$$(x \prec y) \prec z = x \prec (y \star z), \quad (x \succ y) \prec z = x \succ (y \prec z) \quad \text{and} \quad (x \star y) \succ z = x \succ (y \succ z).$$

Thus, endowed with the operations west for left and east for right, $Q$ is a dendriform algebra. We denote it by $Q_h$ and call it the horizontal dendriform algebra associated to $Q$. Considering instead the three row sums in the matrix of quadri algebra axioms yields:

$$(x \wedge y) \wedge z = x \wedge (y \star z), \quad (x \vee y) \wedge z = x \vee (y \wedge z) \quad \text{and} \quad (x \star y) \vee z = x \vee (y \vee z).$$

Thus, endowed with the operations north for left and south for right, $Q$ is a dendriform algebra. We denote it by $Q_v$ and call it the vertical dendriform algebra associated to $Q$. The associative operations corresponding to the dendriform algebras $Q_h$ and $Q_v$ by means of (2) coincide, according to (8).

5. SHUFFLE QUADRI-ALGEBRA

Let $k$ be a field, and let $V$ be a $k$-vector space. Let $\mathcal{H} = T(V) = \bigoplus_{n \geq 0} V^\otimes n$ be the tensor algebra of $V$, where we denote by $\Delta$ the deconcatenation coproduct and by $m$ the concatenation product. For all $u, v \in \mathcal{H}$, we have:

$$m(u \otimes v) = uv, \quad (9)$$

and

$$\Delta(u) = \sum_{u = u^1 \otimes u^2} u^1 \otimes u^2, \quad (10)$$
The shuffle product \( \triangledown \) is defined for any \( u = u_1u_2 \ldots u_p \) and \( v = u_{p+1}u_{p+2} \ldots u_{p+q} \) in \( H \) by:

\[
(11) \quad u \triangledown v = \sum_{\sigma \in \text{Sh}(p,q)} u_{\sigma^{-1}(1)}u_{\sigma^{-1}(2)} \ldots u_{\sigma^{-1}(p+q)},
\]

where \( \text{Sh}(p, q) \) denotes the set of \( \sigma \in S_{p+q} \) verifying \( \sigma(1) < \ldots < \sigma(p) \) and \( \sigma(p+1) < \ldots < \sigma(q) \).

The triple \((H, \triangledown, \Delta)\) becomes a commutative Hopf algebra called the shuffle Hopf algebra. The shuffle algebra of a vector space \( V \) provides an example of a commutative quadri-algebra (see Remark [1]). The quadri-algebra laws on \( H \) are defined by Marcelo Aguiar and Jean-Louis Loday in [2] recursively on the degrees of \( u \) and \( v \). Let \( a, b, c, d \in V \) and \( w, \theta \in V^{\otimes n} \).

1. If \( u = 1 \) and \( v \in H \), we have:

\[
1 \triangledown v = v
\]

and:

\[
1 \uparrow v = 0, \quad 1 \downarrow v = v, \\
1 \downarrow v = 0, \quad 1 \uparrow v = v,
\]

which immediately gives:

\[
1 \rhd v = v \quad 1 \lhd v = 0 \\
1 \uparrow v = 0 \quad 1 \downarrow v = v.
\]

2. If \( u, v \in V \), we have:

\[
u \triangledown v = uv + vu
\]

and:

\[
u \uparrow v = vu, \quad u \downarrow v = 0, \\
u \downarrow v = uv, \quad u \uparrow v = 0,
\]

which immediately gives:

\[
u \rhd v = vu \quad u \lhd v = uv \\
u \uparrow v = 0 \quad u \downarrow v = uv.
\]

3. If \( u \in V \), and \( v = c\theta d \in V^{\otimes n} \) for \( n \geq 2 \), we have:

\[
u \triangledown v = u \triangledown c\theta d
\]

\[
= uc\theta d + c(u \triangledown \theta)d + c\theta du + 0.
\]

The four quadri-algebra laws on \( H \) are given by:

\[
u \uparrow v = c\theta du \quad u \downarrow v = c(u \triangledown \theta)d \\
u \downarrow v = uc\theta d \quad u \uparrow v = 0,
\]

(12)
which immediately gives:

\[
\begin{align*}
  u \prec v &= u c \theta d \\
  u \succ v &= c(u \shuffle \theta) d + c \theta d u \\
  u \wedge v &= c \theta d u \\
  u \vee v &= c(u \shuffle \theta) d + u c \theta d.
\end{align*}
\]

(4) If \( u, v \in \mathcal{H} \), such that \( u, v \) of degree \( \geq 2 \), i.e., \( u = awb \) and \( v = c \theta d \), we have:

\[
\begin{align*}
  u \shuffle v &= a(wb \shuffle c \theta)d + c(awb \shuffle \theta)d + a(w \shuffle c \theta d)b + c(aw \shuffle \theta d)b.
\end{align*}
\]

The four quadri-algebra operations on \( \mathcal{H} \) are defined by:

\[
\begin{align*}
  u \nearrow v &= c(aw \shuffle \theta d)b & u \searrow v &= c(awb \shuffle \theta)d \\
  u \searrow v &= a(wb \shuffle c \theta)d & u \swarrow v &= a(w \shuffle c \theta d)b.
\end{align*}
\]

The dendriform algebra operations on \( \mathcal{H} \) are defined by:

\[
\begin{align*}
  u \succ v &= c(awb \shuffle \theta d) & u \prec v &= a(awb \shuffle \theta d) \\
  u \wedge v &= (aw \shuffle c \theta d)b & u \vee v &= (awb \shuffle c \theta d).
\end{align*}
\]

We verify easily then:

\[
\begin{align*}
  u \shuffle v := u \nearrow v + u \searrow v + u \swarrow v + u \searrow v &= u \succ v + u \prec v \\
  &:= u \vee v + u \wedge v.
\end{align*}
\]

The nine axioms of quadri-algebra laws can now be easily verified.

**Remark 1.** By the commutativity of the shuffle product the quadri-algebra laws verify:

\[
\begin{align*}
  u \nearrow v &= c(aw \shuffle \theta d)b \\
  &= c(\theta d \shuffle aw)b \\
  &= v \swarrow u.
\end{align*}
\]

\[
\begin{align*}
  u \searrow v &= c(awb \shuffle \theta)d \\
  &= c(\theta \shuffle awb)d \\
  &= v \swarrow u.
\end{align*}
\]
Remark 2. The four quadri-algebra operations also admit a non-recursive definition in terms of shuffles:

\[ u \downarrow v = \sum_{\sigma \in Sh(p,q), \sigma^{-1}(1) \geq p+1} \sigma^{-1}(1) \sigma^{-1}(2) \ldots \sigma^{-1}(p+q), \]

\[ u \triangleright v = \sum_{\sigma \in Sh(p,q), \sigma^{-1}(1) \geq p+1} \sigma^{-1}(1) \sigma^{-1}(2) \ldots \sigma^{-1}(p+q), \]

\[ u \wedge v = \sum_{\sigma \in Sh(p,q), \sigma^{-1}(1) \leq p} \sigma^{-1}(1) \sigma^{-1}(2) \ldots \sigma^{-1}(p+q), \]

\[ u \triangledown v = \sum_{\sigma \in Sh(p,q), \sigma^{-1}(1) \leq p} \sigma^{-1}(1) \sigma^{-1}(2) \ldots \sigma^{-1}(p+q). \]

We can now state the main result of this article.

Theorem 1. For any \( u, v, w \in H \), we have:

\[ u \triangleright (vw) = \sum_{u = u^1 u^2} (u^1 \triangleright v)(u^2 \wedge w), \]

\[ u \downarrow (vw) = \sum_{u = u^1 u^2} (u^1 \downarrow v)(u^2 \lor w), \]

\[ u \wedge (vw) = \sum_{u = u^1 u^2} (u^1 \wedge v)(u^2 \triangleright w), \]

\[ u \triangledown (vw) = \sum_{u = u^1 u^2} (u^1 \triangledown v)(u^2 \wedge w). \]

Proof. We will prove this theorem by induction on the length of \( u \). Let us verify that the theorem is true for \( u = 1 \) and for \( u \in V \).
For $u = 1$ and for $v, w \in \mathcal{H}$, we have:

$$u \triangleright (vw) = 1 \triangleright (vw) = 0,$$

and:

\[
\sum_{u=1} (u^1 \triangleright v)(u^2 \vee w) = (1 \triangleright v)(1 \vee w) = 0 = 1 \triangleright (vw).
\]

\[
\sum_{u=1} (u^1 \smalltriangleright v)(u^2 \triangleright w) = (1 \smalltriangleright v)(1 \triangleright w) = 0 = 1 \triangleright (vw).
\]

Similarly:

$$u \smalltriangleright (vw) = 1 \smalltriangleright vw = vw.$$

\[
\sum_{u=1} (u^1 \smalltriangleright v)(u^2 \vee w) = (1 \smalltriangleright v)(1 \vee w) = vw = 1 \smalltriangleright (vw).
\]

\[
\sum_{u=1} (u^1 \triangleright v)(u^2 \smalltriangleright w) = (1 \triangleright v)(1 \smalltriangleright w) = vw = 1 \smalltriangleright (vw),
\]

and by a similar computation, we prove that the two other assertions are true for $u = 1$.

For $u \in V$ and for any $v, w \in \mathcal{H}$, we have:

$$u \triangleright (vw) = vwu,$$

and:

\[
\sum_{u=1} (u^1 \triangleright v)(u^2 \vee w) = (1 \triangleright v)(u \vee w) + (u \triangleright v)(1 \vee w)
\]

\[= vw + u \triangleright (vw).
\]
\[ \sum_{u=\theta \delta \theta} (u^1 \triangleright v)(u^2 \triangleright w) = (1 \triangleright v)(u \triangleright w) + (u \triangleright v)(0) \]
\[ = uvw \]
\[ = u \triangleright (vw). \]

Similarly:
\[ u \triangleright (vw) = uvw. \]

\[ \sum_{u=\theta \delta \theta} (u^1 \triangleright v)(u^2 \triangleright w) = (1 \triangleright v)(u \triangleright w) + (u \triangleright v)(0) \]
\[ = uvw \]
\[ = u \triangleright (vw). \]

\[ \sum_{u=\theta \delta \theta} (u^1 \triangleright v)(u^2 \triangleright w) = (1 \triangleright v)(u \triangleright w) + (u \triangleright v)(0) \]
\[ = uvw \]
\[ = u \triangleright (vw). \]

By a similar computation, we prove that the two other assertions are true for \( u \in V \).

We will now use the induction hypothesis to prove the theorem. Let \( u = \theta \delta \theta \), \( v \) and \( w \) be three elements of \( \mathcal{H} \), we have:

**Proof of (1):**
\[ \sum_{u=\theta \delta \theta} (u^1 \triangleright v)(u^2 \triangleright w) = (1 \triangleright v)(u \triangleright w) + \sum_{u=\theta \delta \theta} (u^1 \triangleright v)(u^2 \triangleright w). \]

The condition \( u^2 \neq 1 \) gives: \( u^2 = u^{12} \delta \theta \) where \( u^1 u^{12} = \theta \delta \theta \), hence:
\[ \sum_{u=\theta \delta \theta} (u^1 \triangleright v)(u^2 \triangleright w) = \sum_{\theta \delta \theta = u^1 u^{12}} (u^1 \triangleright v)(u^{12} \delta \theta \triangleright w). \]

We distinguish here two cases, the first case where \( v \) is a single-letter word and the second case where \( v \) is a word of length \( \geq 2 \), i.e \( v = c \xi \delta \theta d \), where \( c, d \in V \) and \( \xi \in V^{\otimes n} \).

If \( v \in V \), by Remark \( \mathbf{1} \) we have \( u^1 \triangleright v = v \triangleright u^1 = 0 \) for all \( u^1 \neq 1 \) (see equation \( \mathbf{12} \)). Hence the sum \( \sum_{\theta \delta \theta = u^1 u^{12}} (u^1 \triangleright v)(u^{12} \delta \theta \triangleright w) \) gives one term where \( u^1 = 1 \), the other terms all vanish, we have:
\[
\sum_{u=u^1u^2}(u^1 \triangledown v)(u^2 \land w) = \sum_{a\theta=u^1u^{12}}(u^1 \triangledown v)(u^{12}b \land w) \\
= (1 \triangledown v)(u \land w) \\
= v(u \land w) \\
= u \mapsto (vw).
\]

Now if \( v = c\xi d \) we obtain the same result:

\[
\sum_{u=u^1u^2}(u^1 \triangledown v)(u^2 \land w) = \sum_{a\theta=u^1u^{12}}(u^1 \triangledown v)(u^{12}b \land w) \\
= \sum_{a\theta=u^1u^{12}}(u^1 \triangledown c\xi)(u^{12} \triangledown w)b \\
= \sum_{a\theta=u^1u^{12}}[(u^1 \triangledown c\xi)(u^{12} \triangledown dw)b + (u^1 \triangledown c\xi)(u^{12} \triangledown dw)b] \\
= (a\theta \triangledown vw)b + (a\theta \triangledown vw)b \quad \text{(induction hypothesis)} \\
= (a\theta \triangledown vw)b \\
= (a\theta b) \triangledown (c\xi dw) \\
= u \triangledown (vw).
\]

Similarly, we have:

\[
\sum_{u=u^1u^2}(u^1 \triangledown v)(u^2 \triangledown w) = (u \triangledown v)(1 \triangledown w) + \sum_{u^1 \neq u_1, u^2 \neq 1}(u^1 \triangledown v)(u^2 \triangledown w).
\]

The condition \( u^1 \neq u \) and \( u^2 \neq 1 \) gives: \( u^2 = u^{12}b \) where \( u^1u^{12} = a\theta \), hence:

\[
\sum_{u=u^1u^2}(u^1 \triangledown v)(u^2 \triangledown w) = \sum_{a\theta=u^1u^{12}}(u^1 \triangledown v)(u^{12}b \triangledown w) \\
= \sum_{a\theta=u^1u^{12}}(u^1 \triangledown v)(u^{12} \triangledown w)b \\
= \sum_{a\theta=u^1u^{12}}(u^1 \triangledown v)(u^{12} \triangledown w)b + \sum_{a\theta=u^1u^{12}}(u^1 \triangledown v)(u^{12} \triangledown w)b \\
= (a\theta \triangledown vw)b + (a\theta \triangledown vw)b \quad \text{(induction hypothesis)} \\
= (a\theta \triangledown vw)b \\
= c(a\theta \triangledown \xi dw)b \\
= (a\theta b) \triangledown (c\xi dw) \\
= u \triangledown (vw).
\]

**Proof of (2):** By a similar method we prove the second assertion:
We distinguish here two cases, the first case where $v$ is a single-letter word and the second case where $v$ is a word of length $\geq 2$, i.e. $v = c\xi d$, where $c, d \in V$ and $\xi \in V^\otimes n$.

If $v \in V$, by Remark 1 we have $u^1 \downarrow v = v \downarrow u^1 = 0$ for all $u^1 \neq 1$ (see equation (12)). Hence the sum $\sum_{u=u^1u^2}(u^1 \downarrow v)(u^2 \vee w)$ gives one term where $u^1 = 1$ and $u^2 = u$, the other terms all vanish, we have:

$$\sum_{u=u^1u^2}(u^1 \downarrow v)(u^2 \vee w) = (1 \downarrow v)(u \vee w) = v(u \vee w) = u \downarrow (vw),$$

and if $v = c\xi d$, we have:

$$\sum_{u=u^1u^2}(u^1 \downarrow v)(u^2 \vee w) = \sum_{u=u^1u^2}(u^1 \downarrow c\xi d)(u^2 \vee w)$$

$$= \sum_{u=u^1u^2} \sum_{u^1=1}^{u_{11}} (u^1 \downarrow c)(u^2 \vee \xi d)(u^2 \vee w) \quad \text{(induction hypothesis)}$$

$$= \sum_{u=u_{11}u_{12}}(u^1 \downarrow c)(u^2 \vee \xi d)(u^2 \vee w)$$

$$= \sum_{u=u_{11}u'}(u^1 \downarrow c)(u' \vee \xi dw) \quad \text{(induction hypothesis)}$$

$$= \sum_{u=u_{11}u'}(c \downarrow u^1)(u' \vee \xi dw).$$

The last sum contain one term because $c \downarrow u^1 = 0$ if $u^1 \neq 1$, then we have:

$$\sum_{u=u^1u^2}(u^1 \downarrow v)(u^2 \vee w) = c(u \vee \xi dw)$$

$$= c(u \vee \xi dw)$$

$$= u \downarrow (c\xi dw)$$

$$= u \downarrow (vw).$$

Similarly, we distinguish here two cases, the first case where $w$ is a single-letter word and the second case where $w$ is a word of length $\geq 2$, i.e. $w = e\eta f$, where $e, f \in V$ and $\eta \in V^\otimes n$. 
If \( w \in V \), the sum \( \sum_{u=u^1u^2}(u^1 \succ v)(u^2 \searrow w) \) gives one term where \( u^1 = u \) and \( u^2 = 1 \), the other terms vanish, which gives:

\[
\sum_{u=u^1u^2} (u^1 \succ v)(u^2 \searrow w) = (u \succ v)(1 \searrow w) = (u \succ v)w = u \searrow (vw),
\]

and if \( w = e\eta f \), we have:

\[
\sum_{u=u^1u^2} (u^1 \succ v)(u^2 \searrow w) = \sum_{u=u^1u^2} (u^1 \succ v)(u^2 \searrow e\eta f) \quad \text{(induction hypothesis)}
\]

\[
= \sum_{u=u^1u^2} \sum_{u^1u^2 = u^2u^2} (u^1 \succ v)(u^2 \succ e\eta)(u^2 \searrow f)
\]

\[
= \sum_{u=u^1u^2} (u^1 \succ v)(u^2 \succ e\eta)(u^2 \searrow f)
\]

\[
= \sum_{u=u' u^22} \sum_{u'=u^1u^21} (u^1 \succ v)(u^2 \succ e\eta)(u^2 \searrow f) \quad \text{(induction hypothesis)}
\]

\[
= \sum_{u'=u' u^22} (u' \succ v e\eta)(u^2 \searrow f).
\]

The last sum contain one term because \( f \searrow u^{22} = 0 \) if \( u^{22} \neq 1 \), then we obtain:

\[
\sum_{u=u^1u^2} (u^1 \succ v)(u^2 \searrow w) = (u \succ v e\eta f)
\]

\[
= (u \succ c\xi d e\eta f)
\]

\[
= c(u \mathbin{n} \xi d e\eta f)
\]

\[
= u \searrow (c\xi d e\eta f)
\]

\[
= u \searrow (vw).
\]

Proof of (3):

\[
\sum_{u=u^1u^2} (u^1 \not\succ v)(u^2 \triangledown w) = \underbrace{(1 \not\succ v)(u \triangledown w)}_{0} + \sum_{u^1 \neq 1, u^2 \neq u} (u^1 \not\succ v)(u^2 \triangledown w).
\]
The condition $u^1 \neq 1$ gives: $u^1 = au^{11}$ where $u^{11}u^2 = \theta b$, hence:

$$
\sum_{u=u^1u^2} (u^1 \prec v)(u^2 \lor w) = \sum_{u=au^{11}u^2} (au^{11} \prec v)(u^2 \lor w) = \sum_{u=au^{11}u^2} a(u^{11} \lor v)(u^2 \lor w)
$$

$$
= \sum_{u=au^{11}u^2} a(u^{11} \prec v)(u^2 \lor w) + \sum_{u=au^{11}u^2} a(u^{11} \lor v)(u^2 \lor w)
$$

$$
= a(\theta b \prec vw) + a(\theta b \lor vw) \quad \text{(induction hypothesis)}
$$

$$
= a(\theta b \lor vw)
$$

$$
= (a\theta b) \lor (vw)
$$

$$
= u \lor (vw).
$$

Similarly, we have:

$$
\sum_{u=u^1u^2} (u^1 \prec v)(u^2 \land w) = (1 \prec v)(u \land w) + \sum_{u^1 \neq 1, u^2 \neq u} (u^1 \prec v)(u^2 \land w).
$$

The condition $u^1 \neq 1$ gives: $u^1 = au^{11}$ where $u^{11}u^2 = \theta b$, hence:

$$
\sum_{u=u^1u^2} (u^1 \prec v)(u^2 \land w) = \sum_{u=au^{11}u^2} (au^{11} \prec v)(u^2 \land w)
$$

we distinguish here two cases, the first case where $w$ is a single-letter word and the second case where $w$ is a word of length $\geq 2$, i.e $w = e\eta f$, where $e, f \in V$ and $\eta \in V^\otimes n$.

If $w \in V$, the sum $\sum_{u=au^{11}u^2}(au^{11} \prec v)(u^2 \land w)$ gives one term where $u^2 = 1$, the other terms all vanish, we have:

$$
\sum_{u=u^1u^2} (u^1 \prec v)(u^2 \land w) = \sum_{u=au^{11}u^2} (au^{11} \prec v)(u^2 \land w)
$$

$$
= (u \prec v)(1 \land w)
$$

$$
= (u \prec v)w
$$

$$
= u \lor (vw).
$$
Now if \( w = e\eta f \), we have:

\[
\sum_{u = u^1u^2} (u^1 \prec v)(u^2 \searrow w) = \sum_{u = au^{11}u^2} (au^{11} \times v)(u^2 \searrow w)
\]

\[
= \sum_{u = au^{11}u^2} (au^{11} \times v)(u^2 \searrow e\eta f)
\]

\[
= \sum_{u = au^{11}u^2} a(u^{11} \vDash v)e(u^2 \vDash \eta f)
\]

\[
= \sum_{u = au^{11}u^2} a(u^{11} \lor ve)(u^2 \lor \eta f)
\]

\[
= \sum_{u = au^{11}u^2} a(u^{11} \searrow ve)(u^2 \searrow \eta f) + \sum_{u = au^{11}u^2} a(u^{11} \lor ve)(u^2 \lor \eta f)
\]

\[
= a(\theta b \searrow ve\eta f) + a(\theta b \lor ve\eta f) \quad \text{(induction hypothesis)}
\]

\[
= a(\theta b \lor vw)
\]

\[
= u \lor (vw),
\]

which proves the third assertion.

**Proof of (4):**

\[
\sum_{u = u^1u^2} (u^1 \lor v)(u^2 \land w) = \underbrace{(1 \lor v)(u \land w) + (u \lor v)(1 \land w)}_{0} + \sum_{u = u^1u^2, u \neq 1} (u^1 \lor v)(u^2 \land w)
\]

the condition \( u^1, u^2 \neq 1, u \) gives: \( u^1 = au^{11} \) and \( u^2 = u^{12}b \) where \( u^{11}u^{12} = \theta \), hence:

\[
\sum_{u = u^1u^2} (u^1 \lor v)(u^2 \land w) = \sum_{\theta = u^{11}u^{12}} (au^{11} \lor v)(u^{12}b \vDash w)b
\]

\[
= \sum_{\theta = u^{11}u^{12}} (au^{11} \lor v)(u^{12} \lor w)b + \sum_{\theta = u^{11}u^{12}} (au^{11} \lor v)(u^{12} \land w)b
\]

\[
= (a\theta \lor vw)b + (a\theta \land vw)b \quad \text{(induction hypothesis)}
\]

\[
= (a\theta \lor vw)b
\]

\[
= a(\theta \vDash vw)b
\]

\[
= (a\theta b) \vDash (vw)
\]

\[
= u \vDash (vw).
\]

Similarly we have:

\[
\sum_{u = u^1u^2} (u^1 \prec v)(u^2 \nearrow w) = \underbrace{(1 \prec v)(u \nearrow w) + (u \prec v)(1 \nearrow w)}_{0} + \sum_{u = u^1u^2, u \neq 1} (u^1 \prec v)(u^2 \nearrow w).
\]
The condition $u^1, u^2 \neq 1, u$ gives: $u^1 = au^{11}$ and $u^2 = u^{12}b$ where $u^{11}u^{12} = \theta$, hence:

$$\sum_{u=u^1u^2} (u^1 \prec v)(u^2 \succ w) = \sum_{\theta=u^{11}u^{12}} (au^{11} \prec v)(u^{12}b \succ w)$$

$$= \sum_{\theta=u^{11}u^{12}} (au^{11} \prec v)(u^{12} \succ w)b$$

$$= \sum_{\theta=u^{11}u^{12}} (au^{11} \prec v)(u^{12} \succ w)b + \sum_{\theta=u^{11}u^{12}} (au^{11} \prec v)(u^{12} \preceq w)b$$

$$= (a\theta \prec vw)b + (a\theta \preceq vw)b \text{ (induction hypothesis)}$$

$$= (a\theta \prec vw)b$$

$$= \theta \in \text{id} vw)b$$

$$= (a\theta)b \preceq (vw)$$

$$= u \preceq (vw),$$

which proves the fourth assertion. □

**Remark 3.** A non-recursive proof of Theorem 1 is available, at least when $u, v$ and $w$ are non-empty. Indeed, to prove the first assertion of (1) we note that $u \preceq (vw)$ is obtained by summing the shuffle of $u$ with $vw$ so that the first letter belongs to $v$ and the last letter belongs to $w$. We cut each of these terms just after the last letter of $v$. The left part is obtained by shuffling a prefix of $u$ with $v$ such that the first and last letters are in $v$. The right part is obtained by shuffling a suffix of $u$ with $w$ such that the last letter is in $w$. We proceed similarly for the second assertion, cutting just before the first letter of $w$. Items (2), (3) and (4) can be handled similarly.

**Corollary 1.** Given three elements $u, v$ and $w$ of $\mathcal{H}$, we have:

(1) $$u \preceq (vw) = u \succ (vw) + u \preceq (vw)$$

$$= \sum_{u=u^1u^2} (u^1 \lor v)(u^2 \land w)$$

(2) $$u \prec (vw) = u \preceq (vw) + u \succeq (vw)$$

$$= \sum_{u=u^1u^2} (u^1 \prec v)(u^2 \succ w).$$

(3) $$u \lor (vw) = u \succeq (vw) + u \preceq (vw)$$

$$= \sum_{u=u^1u^2} (u^1 \lor v)(u^2 \lor w).$$
\[(4)\]

\[u \succ (vw) = u \triangleleft (vw) + u \triangleright (vw)\]
\[= \sum_{u = u^1 u^2} (u^1 \succ v)(u^2 \succ w).\]

**Proof.** We prove these results by summing the operations obtained in the previous theorem:

\[(1)\]

\[u \wedge (vw) = u \triangleright (vw) + u \triangleleft (vw)\]
\[= \sum_{u = u^1 u^2} (u^1 \wedge v)(u^2 \wedge w) + \sum_{u = u^1 u^2} (u^1 \wedge v)(u^2 \wedge w)\]
\[= \sum_{u = u^1 u^2} (u^1 \wedge v)(u^2 \wedge w)\]
\[= \sum_{u = u^1 u^2} (u^1 \wedge v)(u^2 \wedge w).\]

\[(2)\]

\[u \prec (vw) = u \triangledown (vw) + u \triangleright (vw)\]
\[= \sum_{u = u^1 u^2} (u^1 \prec v)(u^2 \prec w) + \sum_{u = u^1 u^2} (u^1 \prec v)(u^2 \prec w)\]
\[= \sum_{u = u^1 u^2} (u^1 \prec v)(u^2 \prec w)\]
\[= \sum_{u = u^1 u^2} (u^1 \prec v)(u^2 \prec w).\]

\[(3)\]

\[u \lor (vw) = u \triangleleft (vw) + u \triangledown (vw)\]
\[= \sum_{u = u^1 u^2} (u^1 \lor v)(u^2 \lor w) + \sum_{u = u^1 u^2} (u^1 \lor v)(u^2 \lor w)\]
\[= \sum_{u = u^1 u^2} (u^1 \lor v)(u^2 \lor w)\]
\[= \sum_{u = u^1 u^2} (u^1 \lor v)(u^2 \lor w).\]

\[(4)\]

\[u \triangleright (vw) = u \triangleright (vw) + u \triangleleft (vw)\]
\[= \sum_{u = u^1 u^2} (u^1 \triangleright v)(u^2 \triangleright w) + \sum_{u = u^1 u^2} (u^1 \triangleright v)(u^2 \triangleright w)\]
\[= \sum_{u = u^1 u^2} (u^1 \triangleright v)(u^2 \triangleright w)\]
\[= \sum_{u = u^1 u^2} (u^1 \triangleright v)(u^2 \triangleright w).\]
Corollary 2. For any $u, v, w \in \mathcal{H}$, we have:

\begin{align*}
(18) \quad u \ll (vw) &= \sum_{u=u'v^2} (u^1 \lor v)(u^2 \ll w) \\
(19) \quad u \gg (vw) &= \sum_{u=u'v^2} (u^1 \gg v)(u^2 \gg w)
\end{align*}

Proof. We use in this proof the property given by the Corollary

\begin{align*}
u \ll (vw) &= u \lor (vw) + u \land (vw) \\
&= \sum_{u=u'v^2} (u^1 \lor v)(u^2 \lor w) + \sum_{u=u'v^2} (u^1 \lor v)(u^2 \land w) \\
&= \sum_{u=u'v^2} (u^1 \lor v)(u^2 \lor w + u^2 \land w) \\
&= \sum_{u=u'v^2} (u^1 \lor v)(u^2 \ll w).
\end{align*}

Similarly, we prove the second assertion:

\begin{align*}
u \gg (vw) &= u \gg (vw) + u \ll (vw) \\
&= \sum_{u=u'v^2} (u^1 \gg v)(u^2 \gg w) + \sum_{u=u'v^2} (u^1 \ll v)(u^2 \gg w) \\
&= \sum_{u=u'v^2} (u^1 \gg v + u^1 \ll v)(u^2 \gg w) \\
&= \sum_{u=u'v^2} (u^1 \gg v)(u^2 \gg w).
\end{align*}

Example 1. An example of computation for $u = u_1u_2 \in V^\otimes 2$, $v = v_1v_2 \in V^\otimes 2$ and $w \in V$.

\[
 u \ll (vw) = (u_1u_2) \ll (v_1v_2w) = u_1u_2v_1v_2w + u_1v_1u_2v_2w + u_1v_1v_2u_2w + u_1u_2v_1v_2w + u_1v_1v_2wu_2 + v_1v_2u_1u_2w + v_1u_1v_2u_2w + v_1u_1v_2wu_2 + v_1v_2wu_1u_2 \\
\text{Also we have:} \\
\sum_{u=u'v^2} (u^1 \lor v)(u^2 \ll w) &= (1 \lor v)(u \ll w) + (u_1 \lor v)(u_2 \ll w) + (u_1u_2 \lor v)(1 \ll w) \\
&= v(u_1u_2w + u_1wu_2 + wu_1u_2) + (u_1v_1v_2 + v_1u_1v_2)(u_2w + wu_2) \\
&\quad + (u_1u_2v_1v_2 + u_1v_1u_2v_2 + v_1u_1u_2v_2)w \\
&= v_1v_2u_1u_2w + v_1v_2u_1wu_2 + v_1v_2wu_1u_2 + u_1v_1v_2u_2w + v_1u_1v_2u_2w + u_1v_1v_2wu_2 + v_1u_1v_2w + v_1u_1v_2w.
\]
and,
\[ \sum_{u=u^1u^2} (u^1 \triangledown v)(u^2 \triangleright w) = (1 \triangledown v)(u \triangleright w) + (u_1 \triangledown v)(u_2 \triangleright w) + (u_1u_2 \triangledown v)(1 \triangleright w) \]
\[ = v_1v_2wu_1u_2 + (u_1v_1v_2 + v_1u_1v_2 + v_1v_2u_1)wu_2 \]
\[ + (u_1u_2v_1v_2 + u_1v_1u_2v_2 + v_1u_1u_2v_2 + v_1v_1v_2u_2 + v_1v_2u_2u_2)w \]
\[ = v_1v_2wu_1u_2 + u_1v_1v_2wu_2 + v_1u_1v_2wu_2 + v_1v_1u_2wu_2 + u_1u_2v_1v_2w \]
\[ + u_1u_2v_2w + u_1u_2v_2w + u_1v_1v_2u_2w + v_1u_1u_2w + v_1v_2v_2u_2w. \]

Then we have:
\[ u \triangledown (vw) = \sum_{u=u^1u^2} (u^1 \triangledown v)(u^2 \triangledown w) \]
\[ = \sum_{u=u^1u^2} (u^1 \triangledown v)(u^2 \triangleright w). \]

6. Module-algebra structures on Shuffle quadri-algebra

We consider the bialgebra \((\mathcal{H}, \triangledown, \Delta)\) and the infinitesimal bialgebra \((\mathcal{H}, m, \Delta)\).

**Proposition 1.** The two maps \(\triangledown\) and \(\triangleright\) are two actions of \((\mathcal{H}, \triangledown)\) on \(\mathcal{H}\). In other words, the two following diagrams are commutative:

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & \overset{\triangledown \otimes I}{\longrightarrow} & \mathcal{H} \otimes \mathcal{H} \\
I \otimes \triangledown & \downarrow & \downarrow \triangledown \\
\mathcal{H} \otimes \mathcal{H} & \rightarrow & \mathcal{H}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & \overset{id \otimes \triangleright}{\longrightarrow} & \mathcal{H} \otimes \mathcal{H} \\
\triangledown \otimes I & \downarrow & \downarrow \triangleright \\
\mathcal{H} \otimes \mathcal{H} & \rightarrow & \mathcal{H}
\end{array}
\]

That is to say:

\[(20)\quad \triangledown \circ (I \otimes \triangledown) = \triangledown \circ (\triangledown \otimes I),\]
\[(21)\quad \triangleright \circ (I \otimes \triangleright) = \triangleright \circ (\triangledown \otimes I).\]

**Proof.** Let \(a, u\) and \(v\) be three elements of \(\mathcal{H}\), we have:
\[ \triangledown \circ (I \otimes \triangledown)(a \otimes u \otimes v) = \triangledown(a \otimes u \triangledown v) = a \triangledown (u \triangledown v). \]
\( \vee \circ (\underline{\mu} \otimes I)(a \otimes u \otimes v) = \vee (a \underline{\mu} b \otimes v) = (a \underline{\mu} u) \vee v = a \vee (u \vee v). \)

Then, we have:
\[ \vee \circ (I \otimes \vee) = \vee \circ (\underline{\mu} \otimes I). \]
\[ \succ \circ (I \otimes \succ)(a \otimes u \otimes v) = \succ (a \otimes u \succ v) = a \succ (u \succ v). \]
\[ \succ \circ (\underline{\mu} \otimes I)(a \otimes u \otimes v) = \vee (a \underline{\mu} b \otimes v) = (a \underline{\mu} u) \succ v = a \succ (u \succ v). \]

Then, we have:
\[ \succ \circ (I \otimes \succ) = \succ \circ (\underline{\mu} \otimes I). \]

\[ \square \]

**Theorem 2.** \( \mathcal{H} \) admits two module-algebra structures on \( \mathcal{H} \). In other words, the two following diagrams are commutative:

That is to say:

(22) \[ m \circ (\vee \otimes \vee) \circ \tau_{23} \circ (\Delta \otimes I \otimes I) = \vee \circ (I \otimes m), \]

(23) \[ m \circ (\succ \otimes \succ) \circ \tau_{23} \circ (\Delta \otimes I \otimes I) = \succ \circ (I \otimes m). \]
Proof. To prove the commutativity of these diagrams, we will use the results of Corollary 2. Let $u, v$ and $w$ be three elements of $\mathcal{H}$, we have:

$$m \circ (\vee \otimes \vee) \circ \tau_{23} \circ (\Delta \otimes I \otimes I)(u \otimes v \otimes w) = m \circ (\vee \otimes \vee) \circ \tau_{23}(\sum_{u=u^1 u^2} u^1 \otimes u^2 \otimes v \otimes w)$$

$$= m \circ (\vee \otimes \vee) \circ (\sum_{u=u^1 u^2} u^1 \otimes v \otimes u^2 \otimes w)$$

$$= m(\sum_{u=u^1 u^2} (u^1 \vee v) \otimes (u^2 \vee w))$$

$$= \sum_{u=u^1 u^2} (u^1 \vee v)(u^2 \vee w),$$

whereas:

$$\vee \circ (I \otimes m)(u \otimes v \otimes w) = \vee(u \otimes vw)$$

$$= u \vee (vw)$$

$$= \sum_{u=u^1 u^2} (u^1 \vee v)(u^2 \vee w).$$

We also have:

$$m \circ (\succ \otimes \succ) \circ \tau_{23} \circ (\Delta \otimes I \otimes I)(u \otimes v \otimes w) = m \circ (\succ \otimes \succ) \circ \tau_{23}(\sum_{u=u^1 u^2} u^1 \otimes u^2 \otimes v \otimes w)$$

$$= m \circ (\succ \otimes \succ) \circ (\sum_{u=u^1 u^2} u^1 \otimes v \otimes u^2 \otimes w)$$

$$= m(\sum_{u=u^1 u^2} (u^1 \succ v) \otimes (u^2 \succ w))$$

$$= \sum_{u=u^1 u^2} (u^1 \succ v)(u^2 \succ w).$$

whereas:

$$\succ \circ (I \otimes m)(u \otimes v \otimes w) = \succ(u \otimes vw)$$

$$= u \succ (vw)$$

$$= \sum_{u=u^1 u^2} (u^1 \succ v)(u^2 \succ w).$$

\[\square\]

Remark 4. $\mathcal{H}$ does not admit any module-bialgebra structure [14, 3] on $\mathcal{H}$ given by $\vee$ or $\succ$, because neither $(\mathcal{H}, \vee)$ nor $(\mathcal{H}, \succ)$ is a module-coalgebra on $\mathcal{H}$. In other words the two diagrams below are not commutative:
Proposition 2. The action $\lor$ verifies the following diagram:

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{I \otimes \Delta} & \mathcal{H} \otimes \mathcal{H} \\
\lor & \downarrow & \Delta \otimes I \\
\mathcal{H} & \xrightarrow{\lor \otimes \lor} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}
\end{array}
\]

which means that $\Delta$ is a morphism of left $\mathcal{H}$-modules, making $\mathcal{H}$ a right $\mathcal{H}$-comodule in the category of left $\mathcal{H}$-modules.

Remark 5. The previous diagram again commutes by replacing $\lor$ by $\triangleright$ and $\lor \otimes m$ by $m \otimes \triangleright$.

Remark 6. It is well-known that $(\mathcal{H}, m, \Delta)$ carries an infinitesimal bialgebra structure (in the category of vector spaces). This does not give rise to an infinitesimal bialgebra structure in the category of $(\mathcal{H}, m)$-modules, because $\mathcal{H}$ is not a module-algebra on $\mathcal{H}$, in other words the diagram below is not commutative:

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\Delta \otimes I \otimes I} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \\
I \otimes m & \downarrow & \tau_{23} \\
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{\lor \otimes \lor} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}
\end{array}
\]

Proposition 3. The two following diagrams are commutative:

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{\Delta \otimes I \otimes I} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \\
I \otimes m & \downarrow & \tau_{23} \\
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{\lor \otimes \lor} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{\Delta \otimes I \otimes I} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \\
I \otimes m & \downarrow & \tau_{23} \\
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{\lor \otimes \lor} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}
\end{array}
\]
which means that $\mathcal{H}$ acts on the left on $\mathcal{H}$ by $m$, also $\mathcal{H}$ is an $\mathcal{H}$-module in the category of left-hand $\mathcal{H}$-module and $m$ is a left $\mathcal{H}$-module morphism.

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