On the Dividend Strategies with Non-Exponential Discounting

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May 11, 2014

Abstract

In this paper, we study the dividend strategies for a shareholder with non-constant discount rate in a diffusion risk model. We assume that the dividends can only be paid at a bounded rate and restrict ourselves to Markov strategies. This is a time inconsistent control problem. The equilibrium HJB-equation is given and the verification theorem is proved for a general discount function. Considering a mixture of exponential discount functions and a pseudo-exponential discount function, we get equilibrium dividend strategies and the corresponding equilibrium value functions by solving the equilibrium HJB-equations.

Keywords: Dividend strategies; Non-exponential discounting; Time inconsistency; Equilibrium strategies; Equilibrium HJB-equation

1 Introduction

Since it was proposed by De Finetti (1957), the optimization of dividend strategy has been investigated by many researchers under various risk models. This problem is usually phrased as the management’s problem of determining the optimal timing and the size of dividend payments in the presence of bankruptcy risk. For more literature on this problem, we refer the reader to a recent survey paper by Avanzi (2009).

In the very rich literature, a common assumption is that the discount rate is constant over time so the discount function is exponential. However, some empirical studies of human behavior suggest that the assumption of constant discount rate is unrealistic, see, e.g., Thaler (1981), Ainslie (1992) and Loewenstein and Prelec (1992). Indeed, there is experimental evidence that people are impatient about choices in the short term but are more patient when choosing between long-term alternatives. More precisely, events in the near future tend to be discounted at a higher rate than events that occur

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in the long run. Considering such effect, individual behavior is best described by the hyperbolic discounting (see Phelps and Pollak (1968)), which has been extensively studied in the areas of microeconomics, macroeconomics, and behavioral finance, such as Laibson (1997) and Barro (1999) among others.

However, difficulties arise when we try to solve an optimal control problem with a non-constant discount rate by the standard dynamic programming approach. In fact, the standard optimal control techniques give rise to time inconsistent strategies, i.e., a strategy that is optimal for the initial time may not be optimal later. This is the so-called time inconsistent control problem and the classical dynamic programming principle does no longer hold. Strotz (1955) studies the time inconsistent problem within a game theoretic framework by using Nash equilibrium points. They seek the equilibrium policy as the solution of a subgame-perfect equilibrium where the players are the agent and her future selves.

Recently, there is an increasing attention in the time inconsistent control problem due to the practical applications in economics and finance. A modified HJB equation is derived in Marín-Solano and Navas (2010) which solves an optimal consumption and investment problem with the non-constant discount rate for both naive and sophisticated agents. A similar problem is also considered by another approach in Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008), which provide the precise definition of the equilibrium concept in continuous time for the first time. They characterize the equilibrium policies through the solutions of a flow of BSDEs, and they show, for a special form of the discount factor, that this BSDE reduces to a system of two ODEs which has a solution. Considering the hyperbolic discounting, Ekeland et al. (2012) studied the portfolio management problem for an investor who is allowed to consume and take out life insurance, and they characterize the equilibrium strategy by an integral equation. Following this definition of the equilibrium strategy, Björk and Murgoci (2010) studied the time-inconsistent control problem in a general Markov framework, and derived the equilibrium HJB-equation together with the verification theorem. Björk et al. (2012) studied the Markowitz’s problem with state-dependent risk aversion by utilizing the equilibrium HJB-equation obtained in Björk and Murgoci (2010).

In this paper, we study the dividend strategies for the shareholders with non-constant discount rate in a diffusion risk model. We assume that the dividends can only be paid at a bounded rate and restrict ourselves to Markov strategies. We use the equilibrium HJB-equation to solve this problem. In contrast to the papers mentioned above which consider a fixed time horizon or an infinite time horizon, in the dividend problem the ruin risk should be taken into account and the time horizon is a random variable (the time of ruin). Thus, the equilibrium HJB-equation given in this paper looks different with the one obtained in Björk and Murgoci (2010). We first give the equilibrium HJB-equation which is motivated by Yong (2012) and the verification theorem for a general discount function. Then we solve the equilibrium HJB-equation for two special non-exponential discount functions: a mixture of exponential discount function and a pseudo-exponential discount function. For more details about these discount functions, we refer the reader to Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008). Under the mixture of exponential discount function, our results show
that if the bound of the dividend rate is small enough then the equilibrium strategy is to always pay the maximal dividend rate; otherwise, the equilibrium strategy is to pay the maximal dividend rate when the surplus is above a barrier and pay nothing when the surplus is below the barrier. Given some conditions, the results are similar under the pseudo-exponential discount function. These features of the equilibrium dividend strategies are similar to the optimal strategies obtained in Asmussen and Taksar (1997) which considers the exponential discounting in the diffusion risk model.

The remainder of this paper is organized as follows. The dividend problem and the definition of an equilibrium strategy are given in Section 2. The equilibrium HJB-equation and a verification theorem are presented in Section 3. In Section 4, we study two cases with a mixture of exponential discount functions and a pseudo-exponential discount function.

2 The model

In the case of no control, the surplus process is assumed to follow

\[ dX_t = \mu dt + \sigma dW_t, \quad t \geq 0, \]

where \(\mu, \sigma\) are positive constants and \(\{W_t\}_{t \geq 0}\) is a one-dimensional standard Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) satisfying the usual hypotheses. The filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) is completed and generated by \(\{W_t\}_{t \geq 0}\).

A dividend strategy is described by a stochastic process \(\{l_t\}_{t \geq 0}\). Here, \(l_t \geq 0\) is the rate of dividend payout at time \(t\) which is assumed to be bounded by a constant \(M > 0\). We restrict ourselves to the feedback control strategies (Markov strategies), i.e. at time \(t\), the control \(l_t\) is given by

\[ l_t = \pi(t, x), \]

where \(x\) is the surplus level at time \(t\) and the control law \(\pi : [0, \infty) \times [0, \infty) \to [0, M]\) is a Borel measurable function.

When applying the control law \(\pi\), we denote by \(\{X^\pi_t\}_{t \geq 0}\) the controlled risk process. Considering the controlled system starting from the initial time \(t \in [0, \infty)\), \(\{X^\pi_t\}\) evolves according to

\[
\begin{cases}
    dX^\pi_t = \mu ds + \sigma dW_s - \pi(s, X^\pi_s)ds, & s \geq t, \\
    X^\pi_t = x.
\end{cases}
\]

Let

\[ \tau^\pi_t := \inf\{s \geq t : X^\pi_s \leq 0\} \]

be the time of ruin under the control law \(\pi\). Without loss of generality, we assume that \(X^\pi_s \equiv 0\) for \(s \geq \tau^\pi_t\).

Let \(h : [0, \infty) \to [0, \infty)\) be a discount function which satisfies \(h(0) = 1, h(s) \geq 0\) and \(\int_0^\infty h(t)dt < \infty\).
Furthermore, \( h \) is assumed to be continuously differentiable on \([0, \infty)\) and \( h'(x) \leq 0 \).

**Definition 2.1.** A control law \( \pi \) is said to be admissible if it satisfies: \( 0 \leq \pi(t, x) \leq M \) for all \((t, x) \in [0, \infty) \times [0, \infty)\), \( \pi(t, 0) \equiv 0 \) for all \( t \in [0, \infty) \). We denote by \( \Pi \) the set of all admissible control laws.

For a given admissible control law \( \pi \) and an initial state \((t, x) \in [0, \infty) \times [0, \infty)\), we define the return function \( V^\pi \) by

\[
V^\pi(t, x) = \mathbb{E}_{l, x} \left[ \int_t^{\tau^l} h(s-t)\pi(s, X^\pi_s)ds \right],
\]

where \( \mathbb{E}_{l, x} \) is the expectation conditioned on the event \( \{X^\pi_t = x\} \). Note that for any admissible strategy \( \pi \in \Pi \), we have

\[
\mathbb{E}_{l, x} \left[ \int_t^{\tau^l} |h(s-t)\pi(s, X^\pi_s)|ds \right] \leq M \int_0^\infty h(t)dt < \infty, \quad \forall (t, x) \in [0, \infty) \times [0, \infty),
\]

which means the performance functions \( V^\pi(t, x) \) are well-defined for all admissible strategies.

In classical risk theory, the optimal dividend strategy, denoted by \( \pi^* \), is an admissible strategy such that

\[ V^{\pi^*}(t, x) = \sup_{\pi \in \Pi} V^\pi(t, x). \]

However, in our settings, this optimization problem is time-inconsistent in the sense that the Bellman optimality principle fails.

Similar to Ekeland and Pirvu (2008) and Björk and Murgoci (2010), we view the entire problem as a non-cooperative game and look for Nash equilibria for the game. More specifically, we consider a game with one player for each time \( t \), where player \( t \) can be regarded as the future incarnation of the decision maker at time \( t \). Given state \((t, x)\), player \( t \) will choose a control action \( \pi(t, x) \), and she/he wants to maximize the functional \( V^\pi(t, x) \). In the continuous-time model, Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008) give the precise definition of this equilibrium strategy for the first time. Intuitively, equilibrium strategies are the strategies such that, given that they will be implemented in the future, it is optimal to implement them right now.

**Definition 2.2.** Choose a control law \( \hat{\pi} \in \Pi \), a fixed \( l \in [0, M] \) and a fixed real number \( \epsilon > 0 \). For any fixed initial point \((t, x) \in [0, \infty) \times [0, \infty)\), we define the control law \( \pi^\epsilon \) by

\[
\pi^\epsilon(s, y) = \begin{cases} 0, & \text{for } s \in [t, \infty), \ y = 0; \\ l, & \text{for } s \in [t, t + \epsilon], \ y \in (0, \infty); \\ \hat{\pi}(s, y), & \text{for } s \in [t + \epsilon, \infty), \ y \in (0, \infty). \end{cases}
\]

If

\[
\liminf_{\epsilon \to 0} \frac{V^{\hat{\pi}}(t, x) - V^{\pi^\epsilon}(t, x)}{\epsilon} \geq 0,
\]

for all \( l \in [0, M] \), we say that \( \hat{\pi} \) is an equilibrium control law. And the equilibrium value function \( V \)
is defined by
\[ V(t, x) = V^x(t, x). \] (2.3)

In the following section, we will first give the equilibrium HJB-equation for the equilibrium value function \( V \), and then prove a verification theorem.

### 3 The equilibrium Hamilton-Jacobi-Bellman Equation

In this section, we consider the objective function having the form

\[ V^\pi(t, x) = E_{t,x} \left[ \int_t^\tau C(t, s, \pi(s, X_s^x)) \, ds \right], \] (3.1)

where \( C(t, s, \pi(s, X_s^x)) = h(s-t)\pi(s, X_s^x) \), for \( s \geq t \).

For all \( \pi \in \Pi \) and any real valued function \( f(t, x) \in C^{1,2}([0, \infty) \times (0, \infty)) \), which means that the partial derivatives \( \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2} \) exist and are continuous on \([0, \infty) \times (0, \infty)\), we define the infinitesimal generator \( \mathcal{L}^\pi \) by

\[ \mathcal{L}^\pi f(t, x) = \frac{\partial f}{\partial t}(t, x) + (\mu - \pi(t, x)) \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, x). \]

Let \( \mathcal{D}[0, \infty) := \{(s, t) \mid 0 \leq s \leq t < \infty\} \). The following equilibrium HJB-equation is motivated by Equation (4.77) of Yong (2012) and the proof of Theorem 3.2.

**Definition 3.1.** For a smooth function \( c(s, t, x) \) defined on \( \mathcal{D}[0, \infty) \times [0, \infty) \), the equilibrium HJB-equation is given by

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\partial c}{\partial t}(s, t, x) + H\left(s, t, \phi(t, t), \frac{\partial c}{\partial t}(t, t, x), \frac{\partial^2 c}{\partial x^2}(t, t, x)\right) = 0, \\
\frac{\partial c}{\partial t}(s, t, x) = 0, \quad \forall (s, t, x) \in \mathcal{D}[0, \infty) \times (0, \infty), \\
\end{array} \right.
\end{aligned}
\] (3.2)

where
\[
\begin{aligned}
&H(s, t, l, p, P) = \frac{1}{2} \sigma^2 P + (\mu - l) p + C(s, t, l), \\
&\phi(s, t, P, P) = \text{arg max } H(s, t, \cdot, p, P),
\end{aligned}
\] (3.3)

for \((s, t, l, p, P) \in \mathcal{D}[0, \infty) \times [0, M] \times \mathbb{R}^2\).

Since the equilibrium HJB-equation given in Definition 3.1 is informal, we are now giving a strict verification theorem.

**Theorem 3.2.** (Verification Theorem) Assume that there exists a bounded function \( c(s, t, x) \), which is smooth enough, solves the equilibrium HJB-equation in Definition 3.1. Let

\[
\hat{\pi}(t, x) := \phi\left(t, t, \frac{\partial c}{\partial x}(t, t, x), \frac{\partial^2 c}{\partial x^2}(t, t, x)\right)
\] (3.4)
and

\[ V(t,x) := c(t,t,x). \]  

(3.5)

If for any \((s,t,x) \in \mathcal{D}[0,\infty) \times [0,\infty)\) it holds that

\[ \lim_{n \to \infty} c(s,\tau_n, X_{\tau_n}^\pi) = 0, \quad \text{a.s.,} \]  

(3.6)

where \(\tau_n = n \land \tau_1^\pi, n \geq t, \ n = 1, 2, \cdots\), and \(X_{\tau_1}^\pi\) is the unique solution to the SDE (2.1) with \(\pi\) replaced by \(\hat{\pi}\) and initial state \((t,x)\), then \(\hat{\pi}\) given by (3.4) is an equilibrium control law, and \(V\) given by (3.5) is the corresponding equilibrium value function.

**Proof.** We give the proof in two steps: 1. We show that \(V\) is the value function corresponding to \(\hat{\pi}\), i.e., \(V(t,x) = V_{\hat{\pi}}(t,x)\); 2. We prove that \(\hat{\pi}\) is indeed the equilibrium control law which is defined by Definition 2.2.

Step 1.

With (3.4), we rewrite (3.2) as

\[
\begin{align*}
\mathcal{L}_{\hat{\pi}}^c(s,t,x) + C(s,t,\hat{\pi}(t,x)) &= 0, \quad (s,t,x) \in \mathcal{D}[0,\infty) \times (0,\infty), \\
c(s,t,0) &= 0, \quad \forall (s,t) \in \mathcal{D}[0,\infty),
\end{align*}
\]

(3.7)

where the operator \(\mathcal{L}_{\hat{\pi}}^c\) applies to the function \(c(s, \cdot, \cdot)\).

By (3.7), applying Dynkin’s formula to the function \(c(s, \cdot, \cdot)\) yields that

\[
\begin{align*}
\mathbb{E}_{t,x}\left[ C\left(s, z, \hat{\pi}(z, X_{\tau_1}^\pi)\right) \right] - \mathbb{E}_{t,x}\left[ \int_t^{\tau_1} \mathcal{L}_{\hat{\pi}}^c\left(s, z, X_{\tau_1}^\pi\right) \right] dz
\end{align*}
\]

Recalling Definition 2.1 of admissible strategies (see also (2.2)), for given \(s \leq t\), we have

\[
\mathbb{E}_{t,x}\left[ \int_t^{\tau_1} \left| C\left(s, z, \hat{\pi}(z, X_{\tau_1}^\pi)\right) \right| dz \right] < \infty, \quad \forall (t,x) \in [0,\infty) \times [0,\infty).
\]

Since \(c(\cdot, \cdot, \cdot)\) is bounded, by (3.6), letting \(n \to \infty\) and applying dominated convergence theorem yield that

\[
\mathbb{E}_{t,x}\left[ \int_t^{\tau_1} \left| C\left(s, z, \hat{\pi}(z, X_{\tau_1}^\pi)\right) \right| dz \right] \rightarrow 0,
\]

(3.8)

Thus, we have

\[
V(t,x) := c(t,t,x) = V_{\hat{\pi}}(t,x).
\]

Step 2. For a given \(l \in [0,M]\), and a fixed real number \(\epsilon > 0\), we define \(\pi^\epsilon\) by Definition 2.2. For simplicity, we denote by \(X^\epsilon\) the path under the control law \(\pi^\epsilon\). Without loss of generality, we consider the case where \(\epsilon\) is sufficient small such that \(t + \epsilon < \tau_1^\pi \land \tau_1^\hat{\pi}\). By the definition of \(V_{\pi}\), we
Here \( \hat{\pi}(s,X^e_s) \) and \( \hat{\pi}(s,X^\hat{\pi}_s) \) are the equilibrium control processes associated with the paths of \( X^e \) and \( X^\hat{\pi} \), respectively.

According to the equation (3.9), we now consider the limitation \( \lim_{\epsilon \to 0} \frac{V^e(t,x) - V^\hat{\pi}(t,x)}{\epsilon} \) in three parts separately:

1. Noting that \( \int_0^\infty h(t)dt < \infty \), \( l \) and \( \hat{\pi} \) are bounded and applying the dominated convergence theorem, we get

\[
\lim_{\epsilon \to 0} \frac{E_{t,x} \left[ \int_t^{t+\epsilon} h(s-t) \left( \hat{\pi}(s,X^\hat{\pi}_s) - \pi^e(s,X^e_s) \right) ds \right]}{\epsilon} = \hat{\pi}(t,x) - \pi^e(t,x).
\]

2. We rewrite the second part in the right-side of the equation (3.9) by

\[
E_{t,x} \left[ V^\hat{\pi}(t+\epsilon,X^\hat{\pi}_{t+\epsilon}) - V^\hat{\pi}(t,X^\hat{\pi}_t) \right] = E_{t,x} \left[ V^\hat{\pi}(t+\epsilon,X^\pi_{t+\epsilon}) - V^\hat{\pi}(t,X^\pi_t) \right] - E_{t,x} \left[ V^\hat{\pi}(t+\epsilon,X^e_{t+\epsilon}) - V^\hat{\pi}(t,X^\pi_t) \right] = E_{t,x} \left[ \int_t^{t+\epsilon} dV^\hat{\pi}(u,X^\hat{\pi}_u) \right] - E_{t,x} \left[ \int_t^{t+\epsilon} dV^\pi(u,X^\pi_u) \right].
\]

Applying the Itô formula, we get

\[
\lim_{\epsilon \to 0} \frac{E_{t,x} \left[ \int_t^{t+\epsilon} dV^\hat{\pi}(u,X^\hat{\pi}_u) \right]}{\epsilon} = \frac{\partial V^\hat{\pi}(t,x)}{\partial t} + (\mu - \hat{\pi}(t,x)) \frac{\partial V^\hat{\pi}(t,x)}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V^\hat{\pi}(t,x)}{\partial x^2} = \left( L^V \hat{V} \right)(t,x) = \left( L^\pi V \right)(t,x),
\]
and

\[
\lim_{\epsilon \to 0} E_{t,x} \left[ \int_{t}^{t+\epsilon} dV^\hat{\pi}_\epsilon (u, X^\epsilon_u) \right] = \frac{\partial V^\hat{\pi}(t,x)}{\partial t} + (\mu - l) \frac{\partial V^\hat{\pi}(t,x)}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V^\hat{\pi}(t,x)}{\partial x^2} = (\mathcal{L}^\pi V^\hat{\pi})(t,x) = (\mathcal{L}^\pi V)(t,x).
\]

3. Considering the cases with \( \tau^\hat{\pi}_t \geq \tau^\pi_t \) and \( \tau^\hat{\pi}_t \leq \tau^\pi_t \) and noting that \( \hat{\pi}(s, X^\pi_s) \equiv 0 \) for \( s \geq \tau^\pi_t \), we have

\[
E_{t,x} \left[ \int_{t}^{\tau^\hat{\pi}_t} (h(s-t) - h(s-t-\epsilon)) \hat{\pi}(s, X^\hat{\pi}_s) ds \right] - E_{t,x} \left[ \int_{t+\epsilon}^{\tau^\pi_t} (h(s-t) - h(s-t-\epsilon)) \hat{\pi}(s, X^\pi_s) ds \right] \geq E_{t,x} \left[ \int_{t+\epsilon}^{\tau^\hat{\pi}_t} (h(s-t) - h(s-t-\epsilon)) \left[ \hat{\pi}(s, X^\hat{\pi}_s) - \hat{\pi}(s, X^\pi_s) \right] ds \right].
\]

Noting that \( \hat{\pi} \) is bounded and \( \int_0^\infty h(s) ds < \infty \), by the dominated convergence theorem, we get

\[
\lim_{\epsilon \to 0} \frac{E_{t,x} \left[ \int_{t+\epsilon}^{\tau^\hat{\pi}_t} [h(s-t) - h(s-t-\epsilon)] (\hat{\pi}(s, X^\hat{\pi}_s) - \hat{\pi}(s, X^\pi_s)) ds \right]}{\epsilon} = 0.
\]

Therefore, we obtain

\[
\lim_{\epsilon \to 0} \frac{V^\hat{\pi}(t,x) - V^\pi(t,x)}{\epsilon} \geq \left[ \mathcal{L}^\pi V(t,x) + C(t,t,\hat{\pi}(t,x)) \right] - \left[ \mathcal{L}^\pi V(t,x) + C(t,t,\pi(t,x)) \right]. \tag{3.10}
\]

It follows from (3.3) and (3.4) that

\[
(\mathcal{L}^\pi V)(t,x) + C(t,t,\hat{\pi}(t,x)) = \sup_{\pi \in \Pi} \{(\mathcal{L}^\pi V)(t,x) + C(t,t,\pi(t,x))\}. \tag{3.11}
\]

Therefore, (3.10) and (3.11) imply that

\[
\lim_{\epsilon \to 0} \frac{V^\hat{\pi}(t,x) - V^\pi(t,x)}{\epsilon} \geq 0.
\]

This completes the proof. \(\square\)
4 Solutions to Two Special Cases

In this section, we try to find a solution of the equilibrium HJB-equation in Definition 3.1 for specific discount functions. First of all, we make a conjecture of equilibrium strategy for a general discount function. Since

\[ H(s,t,l,p,P) = \frac{1}{2} \sigma^2 P + (\mu - l)p + C(s,t,l) \]
\[ = \frac{1}{2} \sigma^2 P + \mu p + [h(t-s) - p]l, \]

we have

\[ \phi(s,t,p,P) = \begin{cases} 0, & \text{if } p \geq h(t-s), \\ M, & \text{if } p < h(t-s). \end{cases} \]

We assume that there exists a constant \( b \geq 0 \) such that \( \frac{\partial c}{\partial x}(t,t,x) \geq 1 \), if \( 0 \leq x < b \), and \( \frac{\partial c}{\partial x}(t,t,x) < 1 \), if \( x \geq b \). Thus, the equilibrium strategy is given by

\[ \hat{\pi}(t,x) = \phi \left( t, t, \frac{\partial c}{\partial x}(t,t,x), \frac{\partial^2 c}{\partial x^2}(t,t,x) \right) = \begin{cases} 0, & \text{if } 0 \leq x < b, \\ M, & \text{if } x \geq b. \end{cases} \]

Then the equilibrium HJB-equation (3.2) becomes

\[
\begin{cases}
\frac{\partial c}{\partial t}(s,t,x) + \frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial x^2}(s,t,x) + \mu \frac{\partial c}{\partial x}(s,t,x) = 0, & (s,t,x) \in D[0,\infty) \times (0,b), \\
\frac{\partial c}{\partial t}(s,t,x) + \frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial x^2}(s,t,x) + (\mu - M) \frac{\partial c}{\partial x}(s,t,x) + h(t-s)M = 0, & (s,t,x) \in D[0,\infty) \times [b,\infty), \\
c(s,t,0) = 0, & \forall (s,t) \in D[0,\infty). 
\end{cases}
\] (4.2)

4.1 A Mixture of Exponential Discount Functions

Let us consider a case where the dividends are proportionally paid to \( N \) inhomogenous shareholders. In terms of inhomogenous, we mean that the shareholders have different discount rates. Then given a control law \( \pi \), the return function is

\[ V^\pi(t,x) = \sum_{i=1}^N E_{t,x} \left[ \int_t^{\tau^\pi_i} \omega_i e^{-\delta_i(z-t)} \pi(z,X^\pi_z)dz \right], \]

where \( \omega_i > 0 \) satisfying \( \sum_{i=1}^N \omega_i = 1 \) is the proportion at which the dividends are paid to the shareholders, \( \delta_i > 0, i = 1,2,\cdots,N, \) are the constant discount rates of the shareholders, respectively.

In fact, a mixture of exponential discount functions is used in the above example. We consider a discount function defined by

\[ h(t) = \sum_{i=1}^N \omega_i e^{-\delta_i t}, \quad t \geq 0, \]

(4.3)
where \( \delta_i > 0 \), and \( \omega_i > 0 \) satisfies \( \sum_{i=1}^{N} \omega_i = 1 \).

We consider the following ansatz:

\[
c(s,t,x) = \sum_{i=1}^{N} \omega_i e^{-\delta_i(t-x)} V_i(x), \quad (s,t,x) \in D[0,\infty) \times [0,\infty),
\]

(4.4)

where the functions \( V_i(x) \), \( i = 1,2,\cdots,N \), are given by the system of ODEs

\[
\begin{align*}
\frac{1}{2} \sigma^2 \frac{\partial^2 V_i}{\partial x^2}(x) + \mu \frac{\partial V_i}{\partial x}(x) - \delta_i V_i(x) &= 0, \quad x \in [0,b), \\
\frac{1}{2} \sigma^2 \frac{\partial^2 V_i}{\partial x^2}(x) + (\mu - M) \frac{\partial V_i}{\partial x}(x) - \delta_i V_i(x) + M &= 0, \quad x \in [b,\infty), \\
V_i(0) &= 0.
\end{align*}
\]

(4.5)

Denote by \( \theta_1(\eta,c) \) and \(-\theta_2(\eta,c)\) the positive and negative roots of the equation \( \frac{1}{2} \sigma^2 y^2 + \eta y - c = 0 \), respectively. Then

\[
\begin{align*}
\theta_1(\eta,c) &= -\frac{\eta + \sqrt{\eta^2 + 2\sigma^2 c}}{\sigma^2}, \\
\theta_2(\eta,c) &= \frac{\eta + \sqrt{\eta^2 + 2\sigma^2 c}}{\sigma^2}.
\end{align*}
\]

Thus a general solution of the equation (4.5) has the form

\[
V_i(x) = \begin{cases} 
C_{i1} e^{\theta_1(\mu,\delta_i)x} + C_{i2} e^{-\theta_2(\mu,\delta_i)x}, & x \in [0,b), \\
\frac{M}{\delta_i} + C_{i3} e^{\theta_1(\mu-M,\delta_i)x} + C_{i4} e^{-\theta_2(\mu-M,\delta_i)x}, & x \in [b,\infty),
\end{cases}
\]

(4.6)

for \( i = 1,2,\cdots,N \).

Since \( V_i(0) = 0 \), and \( V_i(x) > 0 \), for all \( x > 0 \), we have \( C_{i1} = -C_{i2} := C_i > 0 \), \( i = 1,2,\cdots,N \). Since we are looking for a bounded function \( c(\cdot,\cdot) \) (see Theorem 3.2), we have \( C_{i3} = 0 \), \( i = 1,2,\cdots,N \). To simplify the notation, let \( C_{i4} := -d_i, i = 1,2,\cdots,N \).

Now to find the value of \( C_i,d_i, i = 1,2,\cdots,N \) and \( b \), we use “the principle of smooth fit” to get

\[
\begin{align*}
V_i(b+) &= V_i(b-), \quad i = 1,2,\cdots,N, \\
V'_i(b+) &= V'_i(b-), \quad i = 1,2,\cdots,N, \\
\frac{\partial c}{\partial x}(t,t,b+) &= 1 \quad \text{(or equivalently, } \frac{\partial c}{\partial x}(t,t,b-) = 1) \}.
\end{align*}
\]

(4.7)

Therefore by denoting

\[
\theta_1 = \theta_1(\mu,\delta_i), \quad \theta_2 = \theta_2(\mu,\delta_i), \quad \theta_3 = \theta_2(\mu-M,\delta_i), \quad i = 1,2,\cdots,N
\]

we can rewrite (4.7) as for \( i = 1,2,\cdots,N \),

\[
C_{i1} \left( e^{\theta_1 b} - e^{-\theta_2 b} \right) = \frac{M}{\delta_i} - d_i e^{-\theta_3 b},
\]

(4.8)

\[
C_{i1} \left( \theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b} \right) = d_i \theta_3 e^{-\theta_3 b},
\]

(4.9)
and
\[ \sum_{i=1}^{N} \omega_i C_i \left( \theta_{i1} e^{\theta_{i1}b} + \theta_{i2} e^{-\theta_{i2}b} \right) = 1. \]  
(4.10)

From (4.8) - (4.9) we can get \( C_i \) and \( d_i \) in the expression of \( b \):

\[ C_i = \frac{M \theta_{i3}}{\delta_i} \left[ (\theta_{i1} + \theta_{i3}) e^{\theta_{i1}b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2}b} \right]^{-1}, \]  
(4.11)

\[ d_i = \frac{M \theta_{i3}}{\delta_i} \frac{\theta_{i1} e^{\theta_{i1}b} + \theta_{i2} e^{-\theta_{i2}b}}{(\theta_{i1} + \theta_{i3}) e^{\theta_{i1}b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2}b}}, \]  
(4.12)

for \( i = 1, 2, \cdots, N \).

Substituting \( C_i \) into (4.10), we obtain

\[ \sum_{i=1}^{N} \omega_i \frac{M \theta_{i3}}{\delta_i} \frac{\theta_{i1} e^{\theta_{i1}b} + \theta_{i2} e^{-\theta_{i2}b}}{(\theta_{i1} + \theta_{i3}) e^{\theta_{i1}b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2}b}} = 1. \]

Let

\[ F(b) := \sum_{i=1}^{N} \omega_i \frac{M \theta_{i3}}{\delta_i} \frac{\theta_{i1} e^{\theta_{i1}b} + \theta_{i2} e^{-\theta_{i2}b}}{(\theta_{i1} + \theta_{i3}) e^{\theta_{i1}b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2}b}} - 1. \]

**Lemma 4.1.** If \( \sum_{i=1}^{N} \omega_i \frac{M \theta_{i3}}{\delta_i} > 1 \), then \( F(b) = 0 \) has a unique positive solution.

**Proof.** The condition \( \sum_{i=1}^{N} \omega_i \frac{M \theta_{i3}}{\delta_i} > 1 \) implies that \( F(0) > 0 \). From Lemma 2.1 of Asmussen and Taksar (1997), we know that

\[ \frac{M}{\delta_i} - \frac{1}{\theta_{i3}} - \frac{1}{\theta_{i1}} < 0, \quad i = 1, 2, \cdots, N. \]

Thus,

\[ F(+\infty) = \sum_{i=1}^{N} \omega_i \frac{M \theta_{i3}}{\delta_i} \frac{\theta_{i1}}{\theta_{i1} + \theta_{i3}} - 1 \]

\[ = \sum_{i=1}^{N} \omega_i \left( \frac{M \theta_{i3}}{\delta_i} \frac{\theta_{i1}}{\theta_{i1} + \theta_{i3}} - 1 \right) \]

\[ = \sum_{i=1}^{N} \omega_i \frac{\theta_{i1} \theta_{i3}}{\theta_{i1} + \theta_{i3}} \left( \frac{M}{\delta_i} - 1 \frac{1}{\theta_{i3}} - \frac{1}{\theta_{i1}} \right) \]

\[ < 0. \]

Furthermore, we have

\[ F'(b) = \sum_{i=1}^{N} \omega_i \frac{M \theta_{i3}}{\delta_i} \frac{\Delta_i}{\left[ (\theta_{i1} + \theta_{i3}) e^{\theta_{i1}b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2}b} \right]^2}, \]
where
\[
\Delta_i = \left(\theta_{12}^2 e^{\theta_{12} b} - \theta_{12}^2 e^{-\theta_{12} b}\right) \left[\left(\theta_{11} + \theta_{12}\right) e^{\theta_{11} b} + \left(\theta_{12} - \theta_{13}\right) e^{-\theta_{12} b}\right] \\
- \left(\theta_{12} e^{\theta_{12} b} + \theta_{13} e^{-\theta_{12} b}\right) \left[\theta_{11} \left(\theta_{11} + \theta_{13}\right) e^{\theta_{11} b} - \theta_{12} \theta_{13}\right) e^{-\theta_{12} b}\right]
\]
\[
= \left[\theta_{11} \left(\theta_{12} - \theta_{13}\right) - \theta_{12}^2 \left(\theta_{11} + \theta_{13}\right) + \theta_{11} \theta_{12} \left(\theta_{12} - \theta_{13}\right) - \theta_{12} \theta_{11} \left(\theta_{11} + \theta_{13}\right)\right] e^{\theta_{11} b - \theta_{12} b}
\]
\[
= \left[\theta_{11} \left(\theta_{12} - \theta_{13}\right) - \theta_{12} \theta_{13}\right) \left(\theta_{11} + \theta_{13}\right)\right] e^{\theta_{11} b - \theta_{12} b}
\]
\[
= -\theta_{12} \theta_{13} \left(\theta_{11} + \theta_{13}\right) e^{\theta_{11} b - \theta_{12} b}
\]
\[
< 0.
\]

Therefore, the equation \( F(b) = 0 \) admits a unique solution on \((0, \infty)\). \( \square \)

**Theorem 4.2.** Given the discount function (4.3), there exists a smooth function \( c(\cdot, \cdot, \cdot) \) satisfying the equilibrium HJB-equation (3.2).

(i) If \( \sum_{i=1}^{N} \omega_i \frac{M \theta_{13}}{\delta_i} \leq 1 \), then \( b = 0 \) and the function \( c(\cdot, \cdot, \cdot) \) is given by

\[
c(s, t, x) = \sum_{i=1}^{N} \omega_i e^{-\delta_i (t-s)} \frac{M}{\delta_i} \left(1 - e^{-\theta_{13} x}\right), \quad x \in [0, \infty).
\]

(ii) If \( \sum_{i=1}^{N} \omega_i \frac{M \theta_{13}}{\delta_i} > 1 \), then

\[
c(s, t, x) = \begin{cases} 
\sum_{i=1}^{N} \omega_i e^{-\delta_i (t-s)} C_i \left(e^{\theta_{13} x} - e^{-\theta_{13} x}\right), & x \in [0, b), \\
\sum_{i=1}^{N} \omega_i e^{-\delta_i (t-s)} \left(\frac{M}{\delta_i} - d_i e^{-\theta_{13} x}\right), & x \in [b, \infty),
\end{cases}
\]

where \( C_i, d_i, i = 1, 2, \cdots, N \), and \( b \) is the unique solution to the system (4.8)-(4.10).

**Proof.** (i) It is easy to check the function \( c(s, t, x) \) given by (4.13) and \( b = 0 \) satisfy the system of ODEs (4.2). Obviously, we have

\[
\frac{\partial c}{\partial x}(s, t, 0) = \sum_{i=1}^{N} \omega_i e^{-\delta_i (t-s)} \frac{M}{\delta_i} \left(1 - e^{-\theta_{13} x}\right), \quad (s, t) \in D[0, \infty),
\]
\[
\frac{\partial^2 c}{\partial x^2}(s, t, x) = -\sum_{i=1}^{N} \omega_i e^{-\delta_i (t-s)} \frac{M^2}{\delta_i} \theta_{13}^2 e^{-\theta_{13} x} < 0, \quad (s, t, x) \in D[0, \infty) \times [0, \infty).
\]

Thus, \( \frac{\partial c}{\partial x}(t, t, x) < 1 \), for \( x \geq 0 \), which implies \( c(\cdot, \cdot, \cdot) \) satisfies the equilibrium HJB-equation (3.2).

(ii) Similarly, it is easy to check that \( b \) and \( c(\cdot, \cdot, \cdot) \) given by (4.8)-(4.10) and (4.14) satisfy the system of ODEs (4.2). It is sufficient to show

\[
\begin{cases} 
\frac{\partial c}{\partial x}(t, t, x) \geq 1, & x \in [0, b), \\
\frac{\partial c}{\partial x}(t, t, x) < 1, & x \in [b, \infty).
\end{cases}
\]

\[ (4.15) \]
The first and second derivatives of \( c(s, t, x) \) given by (4.14) with respective to \( x \) are

\[
\frac{\partial c}{\partial x}(t, t, x) = \begin{cases} 
\sum_{i=1}^{N} \omega_i C_i \left( \theta_{1i} e^{\theta_{1i} x} + \theta_{2i} e^{-\theta_{2i} x} \right), & (t, x) \in [0, \infty) \times [0, b), \\
\sum_{i=1}^{N} \omega_i d_i \theta_{i3} e^{-\theta_{i3} x}, & (t, x) \in [0, \infty) \times [b, \infty),
\end{cases}
\]

and

\[
\frac{\partial^2 c}{\partial x^2}(t, t, x) = \begin{cases} 
\sum_{i=1}^{N} \omega_i C_i \left( \theta_{1i}^2 e^{\theta_{1i} x} - \theta_{2i}^2 e^{-\theta_{2i} x} \right), & (t, x) \in [0, \infty) \times [0, b), \\
- \sum_{i=1}^{N} \omega_i d_i \theta_{i3}^2 e^{-\theta_{i3} x}, & (t, x) \in [0, \infty) \times [b, \infty),
\end{cases}
\]

respectively.

It is easy to check that \( \frac{\partial c}{\partial x}(t, t, x) > 0 \), for all \((t, x) \in [0, \infty) \times [0, \infty)\), which implies that \( c(t, t, \cdot) \) is strictly increasing. Next we show that \( c(t, t, \cdot) \) is a concave function on \([0, \infty)\), i.e. \( \frac{\partial^2 c}{\partial x^2}(t, t, x) < 0 \), for all \((t, x) \in [0, \infty) \times [0, \infty)\). First we show that \( \frac{\partial^2 c}{\partial x^2}(t, t, x) \) is continuous at \( x = b \). Apparently, \( \frac{\partial^2 c}{\partial x^2}(t, t, x) < 0 \), for all \((t, x) \in [0, \infty) \times [b, \infty)\). Recalling (4.4), (4.5) and (4.7), we have

\[
\frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial x^2}(t, t, b-) = - \mu \frac{\partial c}{\partial x}(t, t, b) + \sum_{i=1}^{N} \omega_i \delta_i V_i(b), \\
\frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial x^2}(t, t, b+) = - (\mu - M) \frac{\partial c}{\partial x}(t, t, b) + \sum_{i=1}^{N} \omega_i \delta_i V_i(b) - M.
\]

Since \( \frac{\partial c}{\partial x}(t, t, b) = 1 \), we get \( \frac{\partial^2 c}{\partial x^2}(t, t, b-) = \frac{\partial^2 c}{\partial x^2}(t, t, b+) = \frac{\partial^2 c}{\partial x^2}(t, t, b) \).

Obviously, for all \( 0 \leq x \leq b \), we have

\[
\frac{\partial^3 c}{\partial x^3}(t, t, x) = \sum_{i=1}^{N} \omega_i C_i \left( \theta_{1i}^3 e^{\theta_{1i} x} + \theta_{2i}^3 e^{-\theta_{2i} x} \right) > 0,
\]

which means that \( \frac{\partial^2 c}{\partial x^2}(t, t, x) \leq \frac{\partial^2 c}{\partial x^2}(t, t, b) < 0 \), for all \( 0 \leq x \leq b \). Thus, we proved (4.15). \( \square \)

**Corollary 4.3.** Consider the discount function (4.3).

(i) If \( \sum_{i=1}^{N} \omega_i \frac{M_0 \theta_{1i}}{\sigma_i} \leq 1 \), then for \( t \in [0, \infty) \)

\[
\hat{\pi}(t, x) = \phi \left( t, t, \frac{\partial c}{\partial x}(t, t, x), \frac{\partial^2 c}{\partial x^2}(t, t, x) \right) = M, \quad x \in [0, \infty),
\]

is an equilibrium dividend strategy, and

\[
V(t, x) = c(t, t, x) = \sum_{i=1}^{N} \omega_i \frac{M}{\delta_i} \left( 1 - e^{-\theta_{1i} x} \right), \quad x \in [0, \infty),
\]

is the corresponding equilibrium value function.
Figure 4.1: Equilibrium value functions with a mixture of exponential discount functions

(ii) If \( \sum_{i=1}^{N} \omega_i \frac{M_{\theta_i}}{\delta_i} > 1 \), then for \( t \in [0, \infty) \)

\[
\hat{r}(t, x) = \phi \left( t, t, \frac{\partial c}{\partial x}(t, t, x), \frac{\partial^2 c}{\partial x^2}(t, t, x) \right) = \begin{cases} 
0, & x \in [0, b), \\
M, & x \in [b, \infty), 
\end{cases}
\]

is an equilibrium dividend strategy, and

\[
V(t, x) = c(t, t, x) = \begin{cases} 
\sum_{i=1}^{N} \omega_i C_i \left( e^{\theta_1 x} - e^{-\theta_2 x} \right), & x \in [0, b), \\
\sum_{i=1}^{N} \omega_i \left( \frac{M_{\theta_i}}{\delta_i} - d_i e^{-\theta_3 x} \right), & x \in [b, \infty), 
\end{cases}
\]

is the corresponding equilibrium value function. Here \( C_i, d_i, i = 1, 2, \ldots, N \), and \( b \) is the unique solution to the system (4.8)-(4.10).

Proof. By Theorem 3.2 and Theorem 4.2, it is sufficient to verify (3.6). If \( M \geq \mu \), in both cases (i) and (ii), it is well known that \( P\left( \tau_t^x < \infty \right) = 1 \) (see, e.g. Gerber and Shiu (2006)). Since \( c(s, t, 0) = 0 \) for all \( (s, t) \in D[0, \infty), \) we get (3.6). If \( M < \mu \), in both cases (i) and (ii), we have \( P\left( \tau_t^x = \infty \right) > 0 \) and \( x_{\tau_t^x}^\pm = +\infty \) on \( \{ \tau_t^x = \infty \} \). However, for any \( s \in [0, \infty) \) we have \( \lim_{t \to \infty, x \to \infty} c(s, t, x) = 0 \). Thus, we still have (3.6).

Example 4.4. Let \( N = 2, \mu = 1, \sigma = 1, M = 0.8, \delta_1 = 0.2, \delta_2 = 0.4 \). Figure 4.1 illustrates the equilibrium value functions for the mixture of exponential discount functions with \( \omega = 0, 0.4, 0.7 \) and 1. The barriers are 0.6525, 0.8781, 1.0207 and 1.1452, respectively. The cases with \( \omega = 0 \) and 1 are time consistent and the equilibrium strategies are optimal.
4.2 A Pseudo-Exponential Discount Function

We now consider a pseudo-exponential discount function defined as

\[ h(t) = (1 + \lambda t)e^{-\delta t}, \quad t \geq 0, \]  

(4.16)

where \( \lambda > 0, \delta > 0 \) are parameters. We refer the reader to Ekeland and Pirvu (2008) for explanations of this discount function. To ensure \( h \) is decreasing, we assume that \( \lambda < \delta \). To simplify the calculations, we shall impose more conditions on \( \lambda \) in the following.

We consider the following ansatz:

\[ c(s, t, x) = e^{-\delta(t-s)} \{ \lambda(t-s)V_3(x) + V_4(x) \}, \quad (s, t, x) \in D[0, \infty) \times [0, \infty), \]  

(4.17)

where \( V_3(\cdot) \) and \( V_4(\cdot) \) are given by

\[
\begin{align*}
\frac{1}{2} \sigma^2 \frac{\partial^2 V_3}{\partial x^2}(x) + \mu \frac{\partial V_3}{\partial x}(x) - \delta V_3(x) &= 0, \quad x \in [0, b), \\
\frac{1}{2} \sigma^2 \frac{\partial^2 V_3}{\partial x^2}(x) + (\mu - M) \frac{\partial V_3}{\partial x}(x) - \delta V_3(x) + M &= 0, \quad x \in [b, \infty), \\
V_3(0) &= 0, 
\end{align*}
\]

(4.18)

and

\[
\begin{align*}
\frac{1}{2} \sigma^2 \frac{\partial^2 V_4}{\partial x^2}(x) + \mu \frac{\partial V_4}{\partial x}(x) - \delta V_4(x) + \lambda V_3(x) &= 0, \quad x \in [0, b), \\
\frac{1}{2} \sigma^2 \frac{\partial^2 V_4}{\partial x^2}(x) + (\mu - M) \frac{\partial V_4}{\partial x}(x) - \delta V_4(x) + \lambda V_3(x) + M &= 0, \quad x \in [b, \infty), \\
V_4(0) &= 0, 
\end{align*}
\]

(4.19)

respectively. It is easy to check that the function \( c(\cdot, \cdot, \cdot) \) given by (4.17)-(4.19) satisfies the system (4.2).

Recalling the situation we discussed in Subsection 4.1, the equation (4.18) has a general solution

\[ V_3(x) = \begin{cases} 
C \left( e^{\theta_1 x} - e^{-\theta_2 x} \right), & x \in [0, b), \\
\frac{M}{\delta} - d e^{-\theta_2 (\mu-M) x}, & x \in [b, \infty),
\end{cases} \]

(4.20)

where \( C > 0, d > 0 \) are two unknown constants to be determined, \( \theta_1(\eta) \) and \( -\theta_2(\eta) \) are the positive and negative roots of the equation \( \frac{1}{2} \sigma^2 y^2 + \eta y - \delta = 0 \), respectively.

According to “the principle of smooth fit”, we have

\[
\begin{align*}
V_3(b+) &= V_3(b-), \\
V'_3(b+) &= V'_3(b-),
\end{align*}
\]

(4.21)

which yields that

\[ C = \frac{M \theta_3}{\delta} \left[ \left( \theta_1 + \theta_3 \right) e^{\theta_3 b} + \left( \theta_2 - \theta_3 \right) e^{-\theta_3 b} \right]^{-1}, \]

(4.22)
\[ d = \frac{M}{\delta} e^{\theta_1 b} \frac{\theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b}}{\theta_1 + \theta_3} e^{\theta_1 b} + \frac{(\theta_2 - \theta_3) e^{-\theta_2 b}}{\theta_1 + \theta_3} e^{-\theta_2 b}, \]

where

\[ \theta_1 = \theta_1(\mu), \quad \theta_2 = \theta_2(\mu), \quad \theta_3 = \theta_2(\mu - M). \]

After obtaining \( V_3 \), solving ODE (4.19) yields that

\[
V_4(x) = \begin{cases} 
(D_1 - B_1 x) e^{\theta_1 x} + (D_2 + B_2 x) e^{-\theta_2 x}, & 0 \leq x < b, \\
\frac{M}{\delta} \left( 1 + \frac{1}{\delta} \right) + (D_3 + B_3 x) e^{-\theta_3 x}, & x \geq b, 
\end{cases}
\]

where

\[ B_1 = \frac{\lambda \sigma}{\mu + \sigma^2 \theta_1} > 0, \quad B_2 = \frac{\lambda C}{\mu - \sigma^2 \theta_2} < 0, \quad B_3 = \frac{\lambda d}{\mu - M - \sigma^2 \theta_3} < 0. \]

Since \( V_4(0) = 0 \), we have \( D_1 = -D_2 = \hat{C} \). Also noting that \( B_1 + B_2 = 0 \), we rewrite (4.24) as

\[
V_4(x) = \begin{cases} 
(D_1 - B_1 x) e^{\theta_1 x} - (\hat{C} + B_1 x) e^{-\theta_2 x}, & 0 \leq x < b, \\
\frac{M}{\delta} \left( 1 + \frac{1}{\delta} \right) + (D_3 + B_3 x) e^{-\theta_3 x}, & x \geq b.
\end{cases}
\]

Applying "the principle of smooth fit", we obtain

\[
\begin{cases}
V_4(b+) = V_4(b-), \\
V'_4(b+) = V'_4(b-), \\
\frac{\partial c}{\partial x}(t, t, b+) = 1 \quad (\text{or equivalently}, \frac{\partial c}{\partial x}(t, t, b-) = 1).
\end{cases}
\]

From the first two equations in (4.27), we obtain

\[
\hat{C} = \frac{[(\theta_1 + \theta_3) b + 1] B_1 e^{\theta_1 b} - [(\theta_2 - \theta_3) b - 1] B_1 e^{-\theta_2 b} + B_3 e^{-\theta_3 b} + \theta_3 \left( 1 + \frac{1}{\delta} \right) \frac{M}{\delta}}{(\theta_1 + \theta_3) e^{\theta_1 b} + (\theta_2 - \theta_3) e^{-\theta_2 b}},
\]

\[ D_3 = e^{\theta_3 b} \left[ (\hat{C} - B_1 b) e^{\theta_1 b} - (\hat{C} + B_1 b) e^{-\theta_2 b} - \left( 1 + \frac{1}{\delta} \right) \frac{M}{\delta} \right] - B_3 b. \]

Furthermore, using \( \frac{\partial c}{\partial x}(t, t, b+) = \frac{\partial c}{\partial x}(t, t, b-) = 1 \), we have

\[ \left[ (\hat{C} - B_1 b) \theta_1 - B_1 \right] e^{\theta_1 b} + \left[ (\hat{C} + B_1 b) \theta_2 - B_1 \right] e^{-\theta_2 b} - 1 = 0, \]

i.e.,

\[ \hat{C} = \frac{1 + (\theta_1 b + 1) B_1 e^{\theta_1 b} - (\theta_2 b - 1) B_1 e^{-\theta_2 b}}{\theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b}}, \]

and

\[ (-\theta_3 D_3 - \theta_3 B_3 b + B_3) e^{-\theta_3 b} - 1 = 0, \]
\[ D_3 = \frac{1}{\theta_3} \left( B_3 - e^{\theta_3 b} \right) - B_3 b. \]  

Putting (4.28) and (4.29) into the left-hand-side of (4.30), it can be rewritten as

\[ \left[ (\theta_1 + \theta_3) e^{\theta_1 b} + (\theta_2 - \theta_3) e^{-\theta_2 b} \right]^{-1} G(b), \]

where

\[ G(b) := -\theta_3 B_1 e^{2\theta_1 b} + \theta_3 B_1 e^{-2\theta_2 b} + \theta_1 B_3 e^{(\theta_1 - \theta_3) b} + \theta_2 B_3 e^{-(\theta_2 + \theta_3) b} + 2 (\theta_1 + \theta_2) \theta_3 B_1 e^{(\theta_1 - \theta_2) b} + \left[ \theta_1 \theta_3 \left( 1 + \frac{\lambda}{\delta} \right) \frac{M}{\delta} - (\theta_1 + \theta_3) \right] e^{\theta_1 b} + \left[ \theta_2 \theta_3 \left( 1 + \frac{\lambda}{\delta} \right) \frac{M}{\delta} - (\theta_2 - \theta_3) \right] e^{-\theta_2 b}, \]

and

\[ G(0) = (\theta_1 + \theta_2) \left\{ \frac{\lambda}{\mu - M - \sigma^2 \theta_3} + \theta_3 \left( 1 + \frac{\lambda}{\delta} \right) \frac{M}{\delta} - 1 \right\}. \]

**Lemma 4.5.** If

\[ \frac{\delta}{M} - \theta_3 \frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta} < \lambda < \left( \frac{\delta}{M} \right) \left( \frac{1}{\mu - M - \sigma^2 \theta_3} \right) + \frac{\theta_3}{\delta} \left( \frac{\theta_1}{\delta} - \frac{\lambda}{\mu + \sigma^2 \theta_1} \right), \]

then \( G(b) = 0 \) has a positive solution.

The proof of Lemma 4.5 is shown in Appendix A. Now we show the main result of this subsection in the following theorem.

**Theorem 4.6.** Assume that \( 0 < \lambda < \delta \). Given the discount function (4.16), there exists a smooth function \( c(\cdot, \cdot, \cdot) \) satisfying the equilibrium HJB-equation (3.2).

(i) If \( \frac{\delta}{M} > \theta_3 \) and \( \lambda \leq \left( \frac{\delta}{M} - \theta_3 \right) \left[ \frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta} \right]^{-1} \), then \( b = 0 \) and \( c(\cdot, \cdot, \cdot) \) is given by (4.17) with

\[
V_3(x) = \frac{M}{\delta} \left( 1 - e^{-\theta_3 x} \right), \quad x \in [0, \infty), \]

\[
V_4(x) = \left( 1 + \frac{\lambda}{\delta} \right) \frac{M}{\delta} + \frac{M}{\mu - M - \sigma^2 \theta_3} x - \left( 1 + \frac{\lambda}{\delta} \right) e^{-\theta_3 x}, \quad x \in [0, \infty). \]

(ii) If (4.33) and (B.1) hold, then \( c(\cdot, \cdot, \cdot) \) is given by (4.17) with

\[
V_3(x) = \begin{cases} 
C \left( e^{\theta_1 x} - e^{-\theta_2 x} \right), & x \in [0, b), \\
\frac{M}{\delta} - de^{-\theta_3 x}, & x \in [b, \infty), 
\end{cases}
\]

\[
V_4(x) = \begin{cases} 
\left( \hat{C} - B_1 x \right) e^{\theta_1 x} - \left( \hat{C} + B_1 x \right) e^{-\theta_2 x}, & 0 \leq x < b, \\
\left( 1 + \frac{\lambda}{\delta} \right) \frac{M}{\delta} + (D_3 + B_3 x) e^{-\theta_3 x}, & x \in [b, \infty), 
\end{cases}
\]

(4.35)
where \((b, C, d, \hat{C}, B_1, B_3, D_3)\) is a solution to \((4.21)\) and \((4.27)\).

**Proof.** It is easy to check that the function \(c(\cdot, \cdot, \cdot)\) given by \((4.17)-(4.19)\) satisfies the system \((4.2)\).

To prove \(c(\cdot, \cdot, \cdot)\) satisfies the equilibrium HJB-equation \((3.2)\), it is sufficient to show

\[
\begin{cases}
\frac{\partial c}{\partial x}(t, t, x) \geq 1, & x \in [0, b), \\
\frac{\partial c}{\partial x}(t, t, x) < 1, & x \in [b, \infty).
\end{cases}
\]

(i) Firstly, we show that the function \(V_4\) defined by \((4.34)\) is a concave function. Recalling Lemma A.1 and \(\lambda > 0\), we obtain

\[
V_4'(x) = \frac{M}{\delta} \left( \frac{\lambda}{\mu - M - \sigma^2 \theta_3} + \theta_3 \left( 1 + \frac{\lambda}{\delta} \right) - \theta_3 \frac{\lambda}{\mu - M - \sigma^2 \theta_3} x \right) e^{-\theta_3 x}
\]

\[
\geq \frac{M}{\delta} \left( \frac{\lambda}{\mu - M - \sigma^2 \theta_3} + \theta_3 \left( 1 + \frac{\lambda}{\delta} \right) \right) e^{-\theta_3 x}
\]

\[
= \frac{M}{\delta} \left[ \left( \frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta} \right) \lambda + \theta_3 \right] e^{-\theta_3 x} > 0.
\]

Also note that \(V_3(0) = V_4(0) = 0\) and \(V_4'(0) = \left[ \frac{\lambda}{\mu - M - \sigma^2 \theta_3} + \theta_3 \left( 1 + \frac{\lambda}{\delta} \right) \right] \frac{M}{\delta} \in (0, 1]\). Recalling the second equation of \((4.19)\), we have

\[
\frac{1}{2} \sigma^2 V_4''(0) = -(\mu - M) V_4'(0) + \delta V_4(0) - \lambda V_3(0) - M
\]

\[
= -(\mu - M) V_4'(0) - M
\]

\[
= -\mu V_4'(0) + M \left( V_4'(0) - 1 \right)
\]

\[
< 0.
\]

Thus,

\[
V_4''(x) = -\theta_3 \frac{M}{\delta} \left( \frac{2 \lambda}{\mu - M - \sigma^2 \theta_3} + \theta_3 \left( 1 + \frac{\lambda}{\delta} \right) - \theta_3 \frac{\lambda}{\mu - M - \sigma^2 \theta_3} x \right) e^{-\theta_3 x}
\]

\[
= \left[ V_4''(0) + \theta_3^2 \frac{M}{\delta} \frac{\lambda}{\mu - M - \sigma^2 \theta_3} x \right] e^{-\theta_3 x} < 0.
\]

Therefore \(\frac{\partial c}{\partial x}(t, t, x) = V_4'(x) \leq 1\) for all \(x > 0\).

(ii) For \(x \geq b\), recalling \((4.31)\),

\[
V_4'(x) = (\theta_3 D_3 + B_3 - \theta_3 B_3 x) e^{-\theta_3 x}
\]

\[
\geq (\theta_3 D_3 + B_3 - \theta_3 B_3 b) e^{-\theta_3 x}
\]

\[
= -[\theta_3 (D_3 + B_3 b) - B_3] e^{-\theta_3 x}
\]

\[
= e^{\theta_3 (b-x)} > 0,
\]
It follows from Lemma B.2 that if (B.1) holds, then
\[
V(x) = \theta_3 (\theta_3 D_3 - 2B_3 + \theta_3 B_3 x) e^{-\theta_3 x}
\]
\[
\leq \theta_3 [\theta_3 (D_3 + B_3 b) - 2B_3] e^{-\theta_3 x}
\]
\[
= \theta_3 (-B_3 - e^{\theta_3 b}) e^{-\theta_3 x}
\]
\[
< \theta_3 \left( -\frac{\lambda}{\mu - M - \sigma^2 \theta_3 \frac{M}{\delta} - 1} \right) e^{\theta_1 (b-x)}
\]
\[
\leq \theta_3 \left( \theta_3 \frac{M \lambda}{\delta} - 1 \right) e^{\theta_3 (b-x)}.
\]
The last inequality follows from Lemma A.1. Furthermore, by (B.1), we have \( \theta_3 \frac{M \lambda}{\delta} - 1 \leq 0 \). Therefore, \( V''(x) < 0 \), for \( x \geq b \).

Now we see the case when \( 0 \leq x < b \). It follows from (4.19) and (4.27) that
\[
\frac{1}{2} \sigma^2 V''(b-) = -\mu V'_4(b) + \delta V_4(b) - \lambda V_3(b),
\]
\[
\frac{1}{2} \sigma^2 V''(b+) = -(\mu - M) V'_4(b) + \delta V_4(b) - \lambda V_3(b) - M
\]
\[
= -\mu V'_4(b) + \delta V_4(b) - \lambda V_3(b),
\]
which yields that \( V''(b+) = V''(b-) = V''(b) \). Furthermore, for \( 0 \leq x < b \),
\[
V''''(x) = \theta_1^2 \left[ \theta_1 \hat{C} - 3B_1 - \theta_1 B_1 x \right] e^{\theta_1 x} + \theta_2^2 \left[ \theta_2 \hat{C} - 3B_1 + \theta_2 B_1 x \right] e^{-\theta_2 x}
\]
\[
> \theta_1^2 \left[ \theta_1 \hat{C} - 3B_1 - \theta_1 B_1 b \right] e^{\theta_1 x} + \theta_2^2 \left[ \theta_2 \hat{C} - 3B_1 \right] e^{-\theta_2 x}.
\]
It follows from Lemma B.2 that if (B.1) holds, then \( V''''(x) > 0 \) for \( 0 \leq x < b \). Since \( V''''(x) \) is continuous at \( x = b \) and \( V''''(b) < 0 \), we get that \( V''(x) < 0 \), for \( 0 \leq x < b \). Therefore, \( c(t,t,x) = V_4(x) \) is a concave function on \((0,\infty)\), which together with (4.27) implies (4.36). \(\Box\)

Similar to the proof of Corollary 4.3, it is easy to verify (3.6). We have the following corollary immediately by Theorem 3.2 and Theorem 4.6.

**Corollary 4.7.** Assume that \( 0 < \lambda < \delta \). Consider the discount function (4.16).

1. If \( \frac{\delta}{\bar{M}} > \theta_3 \) and \( \lambda \leq \left( \frac{\delta}{\bar{M}} - \theta_3 \right) \left[ \frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta} \right]^{-1} \), then for \((t,x) \in (0,\infty) \times (0,\infty)\),
\[
\hat{c}(t,x) = \phi \left( t, \frac{\partial c}{\partial x}(t,t,x), \frac{\partial^2 c}{\partial^2 x^2}(t,t,x) \right) = M,
\]
is an equilibrium dividend strategy, and
\[
V(t,x) = c(t,t,x) = \left( 1 + \frac{\lambda}{\delta} \right) M = M \left[ \frac{\lambda}{\mu - M - \sigma^2 \theta_3} x - \left( 1 + \frac{\lambda}{\delta} \right) \right] e^{-\theta_3 x},
\]
is the corresponding equilibrium value function.
(ii) If \((4.33)\) and \((B.1)\) hold, then for \(t \in [0, \infty)\),

\[
\hat{\pi}(t, x) = \phi\left( t, t, \frac{\partial c}{\partial x}(t, t, x), \frac{\partial^2 c}{\partial x^2}(t, t, x) \right) = \begin{cases} 
0, & x \in [0, b), \\
M, & x \in [b, \infty), 
\end{cases}
\]

is an equilibrium dividend strategy, and

\[
V(t, x) = c(t, t, x) = \begin{cases} 
(\hat{C} - B_1 x) e^{\theta_1 x} - (\hat{C} + B_1 x) e^{-\theta_2 x}, & x \in [0, b), \\
(1 + \frac{\lambda}{\delta}) \frac{M}{\delta} + (D_3 + B_3 x) e^{-\theta_3 x}, & x \in [b, \infty), 
\end{cases}
\]

is the corresponding equilibrium value function. Here \((b, \hat{C}, B_1, B_3, D_3)\) is the solution to \((4.27)\).

Example 4.8. Let \(\mu = 1, \sigma = 1, M = 1, \delta = 0.8\). Figure 4.2 shows the equilibrium value functions for pseudo-exponential discount functions with \(\lambda = 0, 0.1\) and 0.2. The barriers \(b\) are 0.3470, 0.4141 and 0.4796, respectively. The case with \(\lambda = 0\) is time consistent and the equilibrium strategy is optimal.

Appendix A

Lemma A.1. \(\frac{\theta_1}{\delta} - \frac{1}{\mu + \sigma^2 \theta_1} > 0\) and \(\frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta} > 0\).

Proof. Recall that \(\theta_1\) and \(\theta_3\) are given by

\[
\theta_1 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 \delta}}{\sigma^2}, \quad \theta_3 = \frac{\mu - M + \sqrt{(\mu - M)^2 + 2\sigma^2 \delta}}{\sigma^2}.
\]
Then it follows that

\[
\frac{\theta_1}{\delta} - \frac{1}{\mu + \sigma^2 \theta_1} = -\frac{1}{\sqrt{\mu^2 + 2\sigma^2 \delta}} + \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 \delta}}{\sigma^2 \delta} \\
= \left(-\mu + \sqrt{\mu^2 + 2\sigma^2 \delta}\right) \frac{1}{\sqrt{\mu^2 + 2\sigma^2 \delta} - \sigma^2 \delta} \\
= -2 \frac{\sqrt{\theta_1}}{\pi} \mu \frac{\sqrt{\frac{1}{2} \mu^2 + \sigma^2 \delta + \left[ \frac{1}{2} \mu^2 + \sigma^2 \delta \right] + \frac{1}{2} \mu^2}}{\sigma^2 \delta \sqrt{\mu^2 + 2\sigma^2 \delta}} \\
= \left[ \frac{\sqrt{\theta_1}}{\pi} \mu - \sqrt{\frac{1}{2} \mu^2 + \sigma^2 \delta} \right]^2 \sigma^2 \delta \sqrt{\mu^2 + 2\sigma^2 \delta} > 0
\]

and

\[
\frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta} = -\frac{1}{\sqrt{(\mu - M)^2 + 2\sigma^2 \delta}} + \frac{\mu - M + \sqrt{(\mu - M)^2 + 2\sigma^2 \delta}}{\sigma^2 \delta} \\
= \frac{\left(\mu - M + \sqrt{(\mu - M)^2 + 2\sigma^2 \delta}\right) \sqrt{(\mu - M)^2 + 2\sigma^2 \delta} - \sigma^2 \delta}{\sigma^2 \delta \sqrt{(\mu - M)^2 + 2\sigma^2 \delta} \sqrt{(\mu - M)^2 + 2\sigma^2 \delta}} \\
= \frac{2 \frac{\sqrt{\theta_1}}{\pi} (\mu - M) \left( \sqrt{\frac{1}{2} (\mu - M)^2 + \sigma^2 \delta + \left[ \frac{1}{2} (\mu - M)^2 + \sigma^2 \delta \right] + \frac{1}{2} (\mu - M)^2} \right)}{\sigma^2 \delta \sqrt{(\mu - M)^2 + 2\sigma^2 \delta}} \\
= \left[ \frac{\sqrt{\theta_1}}{\pi} (\mu - M) + \sqrt{\frac{1}{2} (\mu - M)^2 + \sigma^2 \delta} \right]^2 \sigma^2 \delta \sqrt{(\mu - M)^2 + 2\sigma^2 \delta} > 0
\]

\[\square\]

**Proof of Lemma 4.5.** It is easy to check that

\[
G(0) = (\theta_1 + \theta_2) \left\{ \frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta} \left[ \frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta} \right] \right\} M \frac{\theta_3}{\delta} + \theta_3 \frac{M}{\delta} - 1
\]

By Lemma A.1 and (4.33) we have \(G(0) > 0\). Now by (4.22), (4.23) and (4.25), we rewrite \(G(b)\) as

\[
G(b) = \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M \theta_3}{\delta} (\theta_1 + \theta_3) e^{\theta_1 b} + (\theta_2 - \theta_3) e^{-\theta_2 b} \left( e^{2\theta_1 b} + e^{-2\theta_2 b} \right) \\
+ \frac{\lambda}{\mu - M - \sigma^2 \theta_3} \frac{M}{\delta} (\theta_1 + \theta_3) e^{\theta_1 b} + (\theta_2 - \theta_3) e^{-\theta_2 b} \left( \theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b} \right)^2 \\
+ 2 (\theta_1 + \theta_2) \theta_3 \frac{M \theta_3}{\delta} (\theta_1 + \theta_3) e^{\theta_1 b} + (\theta_2 - \theta_3) e^{-\theta_2 b} \theta e^{(\theta_1 - \theta_2) b}
\]

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where

\[
g(b) = 2(\theta_1 + \theta_2) \frac{\lambda}{\mu + \sigma^2 \theta_1} \left( M \theta_3 + b e^{(\theta_1 - \theta_2) b} + e^{2\theta_1 b} \left\{ \frac{\lambda}{\mu + \sigma^2 \theta_1} \left( \frac{1 - \frac{1}{\mu + \sigma^2 \theta_1}}{\mu - M - \sigma^2 \theta_3} \right) + \theta_1^2 \left( \frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{1}{\delta} \right) \right\} + \theta_1 \theta_3 (\theta_1 + \theta_3) \left( \frac{M}{\delta} - \frac{1}{\theta_3} - \frac{1}{\theta_1} \right) \right) + e^{(\theta_1 - \theta_2) b} \left\{ \frac{M}{\delta} \left( \theta_1^2 \frac{1}{\mu - M - \sigma^2 \theta_3} + \theta_3^2 \left( \frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{1}{\delta} \right) \right) + \theta_2 - \theta_3 \right\} \right\}.
\]

From Lemma 2.1 of Asmussen and Taksar (1997), we have

\[
\frac{M}{\delta} - \frac{1}{\theta_3} - \frac{1}{\theta_1} < 0.
\]

By Lemma A.1 and (4.33), it is easy to see that \(G(\infty) < 0\). Thus, the equation \(G(b) = 0\) admits a positive solution.

\[\square\]

**Appendix B**

**Lemma B.2.** If

\[
\lambda \leq \frac{\theta_1 + \theta_2}{\theta_1 + 3\theta_2 M \theta_3} \frac{\delta^2}{2 \theta_1 (\theta_1 + 2 \theta_2)} \frac{(\theta_1 + \theta_3)(\theta_1 + \theta_2)}{M \theta_3}, \tag{B.1}
\]

then

\[
\theta_1 \hat{C} - 3B_1 - \theta_1 B_1 b > 0.
\]
Proof. It follows that

\[
\theta_1 \mathcal{C} - 3B_1 - \theta_1 B_1 b = \theta_1 - 2\theta_1 \theta_2 B_1 e^{-\theta_2 b} - 2B_1 \theta_1 e^{\theta_1 b} + B_1 (\theta_1 - 3\theta_2) e^{-\theta_2 b} - \theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b} = \frac{\theta_1 + B_1 \left[-2\theta_1 e^{\theta_1 b} + (-2\theta_1 \theta_2 b + \theta_1 - 3\theta_2) e^{-\theta_2 b}\right]}{\theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b}} = \theta_1 \left[1 + \theta_3 - 2 \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta}\right] e^{(\theta_1 + \theta_2)b} + \left[\theta_1 (\theta_2 - \theta_3) + \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta} (-2\theta_1 \theta_2 b + \theta_1 - 3\theta_2)\right] e^{\theta_1 b} (\theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b}) \left[(\theta_1 + \theta_3) e^{\theta_1 b} + (\theta_2 - \theta_3) e^{-\theta_2 b}\right].
\]

(B.2)

Let

\[
q(b) := \theta_1 \left[1 + \theta_3 - 2 \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta}\right] e^{(\theta_1 + \theta_2)b} + \left[\theta_1 (\theta_2 - \theta_3) + \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta} (-2\theta_1 \theta_2 b + \theta_1 - 3\theta_2)\right].
\]

Then

\[
q'(b) = \theta_1 (\theta_1 + \theta_2) \left[1 + \theta_3 - 2 \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta}\right] e^{(\theta_1 + \theta_2)b} + \left[\theta_1 (\theta_2 - \theta_3) + \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta} (-2\theta_1 \theta_2 b + \theta_1 - 3\theta_2)\right],
\]

\[
q''(b) = \theta_1 (\theta_1 + \theta_2)^2 \left[1 + \theta_3 - 2 \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta}\right] e^{(\theta_1 + \theta_2)b},
\]

and

\[
q(0) = \theta_1 \left[1 + \theta_3 - 2 \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta}\right] + \theta_1 (\theta_2 - \theta_3) + \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta} (\theta_1 - 3\theta_2).
\]

\[
q'(0) = \theta_1 (\theta_1 + \theta_2) \left[1 + \theta_3 - 2 \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta}\right] + 2 \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta} \theta_1 \theta_2.
\]

If \(\lambda \leq \frac{\theta_1 + \theta_2}{\theta_1 + 3\theta_2} \frac{\delta^2}{M\theta_3} \wedge \frac{(\theta_1 + \theta_2)(\theta_1 + \theta_3)}{2\theta_1 (\theta_1 + 2\theta_2)} \frac{\delta^2}{M\theta_3}\) holds, then it follows from Lemma A.1 that

\[
q(0) = (\theta_1 + \theta_2) \left[1 - \frac{\theta_1 + 3\theta_2}{\theta_1 + \theta_2} \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta}\right] \geq \delta (\theta_1 + \theta_2) \left(\frac{\theta_1}{\delta} - \frac{1}{\mu + \sigma^2 \delta}\right) > 0.
\]

and similarly,

\[
q'(0) = (\theta_1 + \theta_2) (\theta_1 + \theta_3) \left[1 - \frac{2\theta_1 (\theta_1 + 2\theta_2)}{(\theta_1 + \theta_2)(\theta_1 + \theta_3)} \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta}\right] \geq \delta (\theta_1 + \theta_2) (\theta_1 + \theta_3) \left(\frac{\theta_1}{\delta} - \frac{1}{\mu + \sigma^2 \delta}\right) > 0.
\]
Thus, it follows from $q'(0) \geq 0$ that

$$
\theta_1 (\theta_1 + \theta_2) \left[ \theta_1 + \theta_3 - 2 \frac{\lambda}{\mu + \sigma^2} \frac{M\theta_3}{\delta} \right] \geq 2 \frac{\lambda}{\mu + \sigma^2} \frac{M\theta_3}{\delta} \theta_1 \theta_2 > 0.
$$

Therefore, $q''(b) > 0$, $q'(b) > 0$, and then $q(b) > 0$. Finally, it follows from (B.2) that

$$
\theta_1 \hat{C} - 3B_1 - \theta_1 B_1 b > 0.
$$

\[\square\]

**Acknowledgments**

We would like to thank the referee(s) for valuable comments and suggestions. This work was supported by National Natural Science Foundation of China (10971068, 11231005), Doctoral Program Foundation of the Ministry of Education of China (20110076110004), Program for New Century Excellent Talents in University (NCET-09-0356) and “the Fundamental Research Funds for the Central Universities”.

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