Φ-Entropy Inequality and Invariant Probability Measure for SDEs with Jump

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Abstract

By using the Φ-entropy inequality derived in [14, 3] for Poisson measures, the same type of inequality is established for a class of stochastic differential equations driven by purely jump Lévy processes. The semigroup Φ-entropy inequality for SDEs driven by Poisson point processes as well as a sharp result on the existence of invariant probability measures are also presented.

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1 Introduction

Let Φ ∈ C([0, ∞)) ∩ C²((0, ∞)) be convex such that Φ(0) = 0 and the function

ΨΦ(u, v) := Φ(u) − Φ(v) − Φ′(v)(u − v), u, v ≥ 0

is non-negative and convex. Typical examples of Φ include Φ(u) = u log u and Φ(u) = up for p ∈ [1, 2].

Let (ℰ, ℰ(ℰ)) be a Dirichlet form on L²(µ) for a probability measure µ. The Φ-entropy inequality considered in [3] is of type

\[ \text{Ent}_µ^Φ(f) := µ(Φ(f)) − Φ(µ(f)) ≤ Cℰ(Φ′(f), f), f, Φ′(f) ∈ ℰ(ℰ), f ≥ 0 \]

for some constant C > 0. This inequality is equivalent to (see [3, Corollary 1.1])

\[ \text{Ent}_µ^Φ(P_tf) ≤ e^{-t/C} \text{Ent}_µ^Φ(f), t ≥ 0, f ∈ ℰ_b^+, \]

where P_t is the associated Markov semigroup and ℰ_b^+ is the set of all bounded positive elements in L²(µ). When Φ(u) = u log u, the inequality [14] reduces to the modified log-Sobolev inequality studied in [14, 15].
In this paper, we investigate the \( \Phi \)-entropy inequality for the following stochastic differential equation (SDE) on \( \mathbb{R}^d \):

\[
(1.3) \quad dX_t = b(X_t)dt + \sigma dL_t,
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) is \( C^1 \)-smooth with bounded \( \nabla b \), \( \sigma \) is an invertible \( d \times d \)-matrix, and \( L_t \) is a purely jump Lévy process on \( \mathbb{R}^d \) with Lévy measure \( \nu \), i.e. \( L_t \) is generated by

\[
(1.4) \quad \mathcal{L}_0 f := \int_{\mathbb{R}^d} \left[ f(\cdot + z) - f - \langle \nabla f, z \rangle 1_{\{|z| \leq 1\}} \right] \nu(dz), \quad f \in C^2_b(\mathbb{R}^d).
\]

Since \( b \) is Lipschitz continuous, for any initial data \( x \in \mathbb{R}^d \) the equation (1.3) has a unique solution \( X_t(x) \) for \( t \in [0, \infty) \). Let \( P_t \) be the associated Markov semigroup, i.e.

\[
P_t f(x) := \mathbb{E}f(X_t(x)), \quad t \geq 0, f \in \mathcal{B}_b(\mathbb{R}^d), x \in \mathbb{R}^d,
\]

where \( \mathcal{B}_b(\mathbb{R}^d) \) is the set of all bounded measurable functions on \( \mathbb{R}^d \).

When \( P_t \) has an invariant probability measure \( \mu \), we consider the corresponding (possibly non-sectorial) form

\[
(1.5) \quad \mathcal{E}(f, g) := -\int_{\mathbb{R}^d} f \mathcal{L} g \, d\mu, \quad f, g \in C^2_b(\mathbb{R}^d),
\]

where \( \mathcal{L} \) is the generator of \( P_t \), i.e.

\[
(1.6) \quad \mathcal{L} f = \langle \nabla f, b \rangle + \int_{\mathbb{R}^d} \left[ f(\cdot + \sigma z) - f - \langle \nabla f, \sigma z \rangle 1_{\{|z| \leq 1\}} \right] \nu(dz).
\]

Let

\[
\Gamma_{\Phi,\nu}(f)(x) = \int_{\mathbb{R}^d} \Psi_{\Phi}(f(\cdot + \sigma z), f(x)) \nu(dz), \quad f \in \mathcal{B}_b^+(\mathbb{R}^d),
\]

where \( \mathcal{B}_b^+(\mathbb{R}^d) \) is the set of all positive elements in \( \mathcal{B}_b(\mathbb{R}^d) \). Let \( C^2_{c, +}(\mathbb{R}^d) \) be the set of any \( C^2 \) positive function on \( \mathbb{R}^d \) which is constant outside a compact set. Then for any \( f \in C^2_{c, +}(\mathbb{R}^d) \) we have \( \int_{\mathbb{R}^d} \mathcal{L} \Phi(f) d\mu = 0 \), so that (1.6) yields

\[
\mathcal{E}(\Phi'(f), f) := -\int_{\mathbb{R}^d} \Phi'(f) \mathcal{L} f \, d\mu
\]

\[
= \int_{\mathbb{R}^d} d\mu \int_{\mathbb{R}^d} \Psi_{\Phi}(f(\cdot + \sigma z), f(x)) \nu(dz) - \int_{\mathbb{R}^d} \mathcal{L} \Phi f \, d\mu
\]

\[
= \int_{\mathbb{R}^d} \Gamma_{\Phi,\nu}(f) \, d\mu.
\]

Thus, for the present model, the \( \Phi \)-entropy inequality (1.1) reduces to

\[
(1.8) \quad \text{Ent}_\mu^\Phi(f) \leq C \int_{\mathbb{R}^d} \Gamma_{\Phi,\nu}(f) d\mu, \quad f \in \mathcal{B}_b^+(\mathbb{R}^d).
\]

**Theorem 1.1.** Assume that

\[
\frac{\kappa_1}{|x|^\alpha} \leq \nu(dz) \leq \frac{\kappa_2}{|z|^\alpha} \quad \text{for some constants } \kappa_1, \kappa_2 > 0 \text{ and } \alpha \in (0, 2).
\]

Let \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that

\[
\lambda_1 |v|^2 \leq \langle \sigma^{-1}(\nabla b(x)) \sigma v, v \rangle \leq \lambda_2 |v|^2, \quad x, v \in \mathbb{R}^d.
\]
(1) For any \( T > 0 \) and \( f \in \mathcal{B}_b^+(\mathbb{R}^d) \),

\[
\text{Ent}_\mu^\Phi(f) := P_T \Phi(f) - \Phi(P_T f) \leq \frac{\kappa_2 \left( \exp\left[ \kappa_2 (d + 3) T - \lambda_1 T d \right] - 1 \right)}{\kappa_1 (\lambda_2 (d + 3) - \lambda_1 d)} P_T \Phi_{\epsilon_2}(f).
\]

(2) If \( \lambda_2 (d + 3) < \lambda_1 d \), then \( P_t \) has a unique invariant probability measure \( \mu \) and (1.8) holds for

\[
C := \frac{\kappa_2}{\kappa_1 (\lambda_2 d - \lambda_1 (d + 3))}.
\]

The following result partly extends Theorem 1.1 to the case where the Lévy process \( L_t \) merely has large (e.g. \( \rho = 1_{[1, \infty)} \)) or small (e.g. \( \rho = 1_{(0, 1]} \)) jumps. In particular, (1.10) holds in the situation of Theorem 1.2.

**Theorem 1.2.** Let \( \frac{\kappa_1 \rho(|z|)}{|z|^{d + \alpha}} \leq \frac{\nu(dz)}{dz} \leq \frac{\kappa_2 \rho(|z|)}{|z|^{d + \alpha}} \) for some constants \( \kappa_1, \kappa_2 > 0 \) and some non-negative measurable function \( \rho \) on \( (0, \infty) \). Assume that (1.9) holds.

(I) If \( \lambda_0 \leq 0 \) and \( \rho \) is decreasing, then assertions (1) and (2) in Theorem 1.1 hold. In particular, if \( \lambda_2 (d + 3) < \lambda_1 d \) then \( P_t \) has a unique invariant probability measure \( \mu \) such that

\[
\text{Ent}_\mu^\Phi(P_t f) \leq \exp \left[ -\frac{\kappa_1 (\lambda_1 d - \lambda_2 (d + 3))}{\kappa_2} t \right] \text{Ent}_\mu^\Phi(f), \quad t \geq 0, \, f \in \mathcal{B}_b^+(\mathbb{R}^d).
\]

(II) If \( \lambda_1 \geq 0 \) and \( \rho \) is increasing, then the assertion (1) in Theorem 1.1 holds.

**Remark 1.1.** (1) We would like to mention a nice entropy inequality derived recently in [13] for non-local Dirichlet forms. Let \( \mu(dx) := e^{-V(x)}dx \) be a probability measure on \( \mathbb{R}^d \) and let \( \rho \) be a positive function on \( (0, \infty) \) such that

\[
(1.11) \quad c := \inf_{x, y \in \mathbb{R}^d} \rho(|x - y|) \{ e^{V(x)} + e^{V(y)} \} > 0,
\]

then

\[
\text{Ent}_\mu(f) := \mu(f \log f) - \mu(f) \log \mu(f)
\]

\[
\leq \frac{1}{c} \int_{\mathbb{R}^d} \mu(dx) \int_{\mathbb{R}^d} \left\{ (f(x + z) - f(z)) \log \frac{f(x + z)}{f(x)} \right\} \rho(|z|) dz, \quad f \in \mathcal{B}_b^+(\mathbb{R}^d).
\]

Since \( \Psi \log(u, v) \leq (u - v) \log \frac{u}{v} \) for \( u, v > 0 \), this inequality follows from the corresponding \( \Phi \)-entropy inequality with \( \Phi(r) = r \log r \). But, in general this result is incomparable with ours for \( \Phi(r) = r \log r \). In our case the invariant probability measure of \( P_t \) (if exists) is not explicitly known, so that the condition (3.1) is hard to verify. Moreover, condition (3.1) implies that \( \nu(dz) := \rho(|z|) dz \) has full support on \( \mathbb{R}^d \) which does not apply to the situations of Theorem 1.2 if \( \rho \) is not strictly positive on \( (0, \infty) \).

(2) When \( b(x) = -x \) and \( \nu(dz) = N(z)dz \) for \( N \geq 0 \) satisfying

\[
\int_1^\infty N(sz)s^{d-s} ds \leq C N(z)
\]

for some constant \( C > 0 \), the \( \Phi \)-entropy inequality (1.8) was proved in [6]. This condition is satisfied for \( \nu \) given in Theorem 1.1 but fails in the situation of Theorem 1.2(1) for e.g. \( \rho(s) = 1_{s \leq 1} \). Moreover, as shown in (d) in the proof of Theorem 1.2, to deduce the exponential convergence from (1.8) an approximation argument should be included in the proof of [6, Theorem 1], since
the formula in [6] Lemma 1] only makes sense for functions \( w_1, w_2 \) such that \( w_1 L w_2 \in L^1(u_\infty dx) \). In general, the form \((\mathcal{E}, C^0_0(\mathbb{R}^d))\) given in (1.5) does not provide a Dirichlet form, so that the equivalence between (1.1) and (1.2) for Dirichlet forms does not apply.

Next, partly for the proof of Theorem (1.1)(2), we consider the existence of invariant probability measures for the following more general SDE:

(1.12) \[
    dX_t = b(X_t)dt + \sigma_1(X_t)dW_t + \sigma_2(X_{t-})dL_t,
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma_1, \sigma_2 : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) are locally Lipschitz continuous, \( L_t \) is the Lévy process in (1.3), and \( W_t \) is a \( d \)-dimensional Brownian motion independent of \( L_t \). Then (1.12) has a unique solution up to the life time.

Although the existence of invariant probability measures for SDEs with jumps has been investigated in the literature, we did not find any existing result which directly applies to the framework in Theorem (1.1). For instance, in [2, Theorem 4.5] it is assumed that \( \int_{\mathbb{R}^d} |z|^2 \nu(dz) < \infty \), while in [1] the Lévy process is assumed to be the \( \alpha \)-stable process and \( b(x) \) is a perturbation by \( -\gamma x \) for some constant \( \gamma > 0 \), see also [4, 5, 8] for the study of semilinear SPDEs with jump. We aim to present a new result which is sharp in terms of the Lévy measure and, in particular, implies the existence of invariant probability measure in the situation of Theorem (1.1)(2).

**Theorem 1.3.** Let \( B \in C^1([0, \infty)) \) be strictly positive. For any \( \varepsilon > 0 \) let

(1.13) \[
    \bar{B}_\varepsilon(x) = \sup \left\{ \frac{B(r) - r B'(r)}{2B(r)^2(1 + r)} : r \geq 0, \ |r - |x|| \leq \varepsilon ||\sigma_2(x)|| \right\}, \quad x \in \mathbb{R}^d.
\]

If there exists \( \varepsilon \in (0, 1) \) such that either

(1.14) \[
    A_\varepsilon := \limsup_{|x| \to \infty} \left\{ \frac{\langle b(x), x \rangle + \text{Tr}(\sigma_1 \sigma_1^*)(x)}{B(|x|)(|x| + 1)} + \frac{||\sigma_2(x)||}{B(|x|)} \int_{\{\varepsilon < |z| \leq 1\}} |z| \nu(dz) \right\} + \frac{|\sigma_1^*(x)|^2 B'(|x|)}{B(|x|)^2(1 + |x|)|x|} + \int_{\{|z| > \varepsilon\}} \nu(dz) \int_{|z| \leq \varepsilon} \frac{1}{|z|} \frac{\nu(dz)}{B(s)} ds + ||\sigma_2(x)|| \int_{\{|z| \leq \varepsilon\}} |z|^2 \nu(dz)
\]

or

(1.14) \[
    A_\varepsilon < 0 \quad \text{and} \quad \int_0^\infty \frac{ds}{B(s)} = \infty,
\]

then the solution to (1.12) is non-explosive and the associated Markov semigroup has an invariant probability.

The following is a consequence of Theorem 1.3 which provides some more explicit sufficient conditions for the existence of invariant probability measures.

**Corollary 1.4.** Assume that for some \( \theta \in \mathbb{R} \)

(1.15) \[
    D := \limsup_{|x| \to \infty} \left\{ \frac{\langle b(x), x \rangle + \text{Tr}(\sigma_1 \sigma_1^*)(x)}{(1 + |x|)^{1+\theta}} - \frac{\theta |\sigma_1^*(x)|^2}{|x|(1 + |x|)^{\theta+2}} \right\} < 0,
\]
and that
\[ \Theta := \limsup_{|x| \to \infty} \frac{\|\sigma_2(x)\|}{|x|} < \infty. \]

Then the solution to (1.12) is non-explosive and the associated Markov semigroup has an invariant probability measure in each of the following three situations:

1. \( \theta > 1 \).
2. \( \theta = 1, \int_{\{|z| \geq 1\}} \log(1 + |z|) \nu(dz) < \infty \), and there exists \( \varepsilon \in (0, \Theta^{-1}) \cap (0, 1] \) such that
\[ (1.16) \quad \varepsilon^2 \Theta^2 \frac{\int_{\{|z| \leq \varepsilon\}} |z|^2 \nu(dz)}{(2(1 - \varepsilon \Theta)^2)} + \Theta \int_{\{z < |z| \leq 1\}} |z| \nu(dz) + \int_{\{|z| > \varepsilon\}} \log(1 + \varepsilon \Theta |z|) \nu(dz) < -D. \]
3. \( \theta \in (0, 1), \|\sigma_2\| \) is bounded, and \( \int_{\{|z| \geq 1\}} |z|^{1 - \theta} \nu(dz) < \infty \).
4. \( \theta \in (-\infty, 1), \int_{\{|z| \geq 1\}} |z|^{1 + \theta} \nu(dz) < \infty \), and
\[ (1.17) \quad \limsup_{|x| \to \infty} \frac{\|\sigma_2(x)\|}{|x|^\theta} \int_{\{|z| > 1\}} |z| \nu(dz) < -D. \]

Note that when (1.16) holds with \( \theta = 1 \) and \( \lim_{|x| \to \infty} \frac{\|\sigma_2(x)\|}{|x|} = 0 \), Corollary 1.4 implies the existence of the invariant probability measure provided
\[ (1.18) \quad \int_{\{|z| \geq 1\}} \log(1 + |z|) \nu(dz) < \infty. \]

According to [10, Theorems 17.5 and 17.11], (1.18) is sharp (i.e. sufficient and necessary) for the purely jump Ornstein-Uhlenbeck process (i.e. \( \sigma_1 = 0, \sigma_2 = I, b(x) = -x \)) to have invariant probability measure. When \( \theta \in (0, 1), \sigma_1 = 0, \sigma_2 = I \) and \( b(x) = -x|x|^{\theta-1} \), we would believe that the condition \( \int_{\{|z| \geq 1\}} |z|^{1 - \theta} \nu(dz) < \infty \) in case (3) is also sharp for the existence of the invariant probability measure. However, in this case the distribution of the solution is no longer infinitely divisible, so that the proof of [10, Theorem 17.11] does not apply.

The remainder of the paper is organized as follows. In Section 2, by using the \( \Phi \)-entropy inequality derived in [14] and [3] for Poisson measures, we prove a result on the semigroup \( \Phi \)-entropy inequality for SDEs driven by Poisson point processes. In Section 3 we prove Theorem 1.3 and Corollary 1.4. Finally, proofs of Theorems 1.1 and 1.2 are presented in Section 4.

2 The semigroup \( \Phi \)-entropy inequality

Let \( N(dt, dz) \) be a Poisson point process on \( \mathbb{R}^d \) with compensator \( dt \nu(dz) \), where \( \nu \) is a \( \sigma \)-finite measure on \( \mathbb{R}^d \). Then for any \( T > 0, 1_{[0,T]}(t)N(dt, dz) \) is a random variable on the configuration space
\[ \Gamma_T := \left\{ \gamma = \sum_{i=1}^n \delta_{(s_i, z_i)} : n \in \mathbb{Z}_+, \cup \{ \infty \}, (s_i, z_i) \in [0, T] \times \mathbb{R}^d \right\} \]
equipped with the σ-field induced by \( \{\gamma \mapsto \gamma(A) : A \in \mathcal{B}([0, T] \times \mathbb{R}^d)\} \), where \( \mathcal{B}([0, T] \times \mathbb{R}^d) \) is the Borel σ-field on \([0, T] \times \mathbb{R}^d \) and \( \delta_{(s_i, x_i)} \) stands for the Dirac measure at point \((s_i, x_i)\). The distribution of \(1_{[0,T]}(t)N(dt, dz)\) is the Poisson measure with intensity \(dt \nu(dz)\) on \([0, T] \times \mathbb{R}^d \).

Let
\[
\nu : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d
\]
be measurable such that for every \(s \geq 0\), \(a_s\) is invertible and
\[
\int_{[0,t] \times \mathbb{R}^d} (1 \wedge |a_s(z)|^2) ds \nu(dz) < \infty, \quad t \geq 0.
\]
Let
\[
\bar{N}_a(dt, dz) = N(dt, dz) - 1_{\{|a_t(z)| \leq 1\}} dt \nu(dz).
\]
Then the stochastic integral
\[
\int_{[0,t] \times \mathbb{R}^d} a_s(z)1_{\{|a_s(z)| \leq 1\}} \bar{N}(ds, dz), \quad t \geq 0
\]
is well defined (see e.g. [12, page 36-37]). Moreover, since (2.1) implies that \(1_{\{|s| > 1\}} N(ds, dz) \in \Gamma^0_t := \{\gamma \in \Gamma_t : \gamma([0, t] \times \mathbb{R}^d) < \infty\}\),
the stochastic integral
\[
\int_{[0,t] \times \mathbb{R}^d} a_s(z)\bar{N}_a(ds, dz), \quad t \geq 0
\]
is well defined as well.

Now, consider the following equation on \(\mathbb{R}^d\):

\[
dX_t = b_t(X_t)dt + \int_{\mathbb{R}^d} a_t(z)N_a(dt, dz), \quad t \geq 0,
\]
where \(b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d\) is measurable such that \(b_t\) is Lipschitz continuous for every \(t \geq 0\) and the Lipschitz constant is locally bounded in \(t\). It is standard that for any \(x \in \mathbb{R}^d\), this equation has a unique solution \(X_t(x)\) with \(X_0 = x\), see e.g. [12, Theorem 17].

Let
\[
P_t f(x) = \mathbb{E} f(X_t(x)), \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad f \in \mathcal{B}(\mathbb{R}^d).
\]
We aim to establish the \(\Phi\)-entropy inequality for \(P_T\). To state our main result, we introduce the following equation driven by \(N_a + \delta_{(s,z)}\) for \((s, z) \in (0, \infty) \times \mathbb{R}^d\):

\[
X^{s,x}_t(z) = x + \int_0^t b_r(X^{s,x}_r(z))dr + \int_{[0,t] \times \mathbb{R}^d} a_r(y)\{N_a + \delta_{(s,z)}\}(dr, dy), \quad t \geq 0.
\]

**Theorem 2.1.** For fixed \(T > 0\) and \(x \in \mathbb{R}^d\), let
\[
\psi_s(z) = a_s^{-1}(X^{s,x}_T(z) - X_T(x)), \quad s \in (0, T], \quad z \in \mathbb{R}^d.
\]
If \(\nu \circ \psi_s^{-1}\) is absolutely continuous w.r.t. \(\nu\) such that
\[
\xi_s := \text{ess}_{\mathbb{P} \times \nu} \frac{d\nu \circ \psi_s^{-1}}{d\nu} < \infty, \quad s \in (0, T],
\]
then
\[
\text{Ent}^\Phi_{P_T}(f)(x) \leq \mathbb{E} \int_0^T \xi_t dt \int_{\mathbb{R}^d} \Psi_\Phi(f(X_T(x) + a_t(z)), f(X_T(x))) \nu(dz), \quad x \in \mathbb{R}^d, \quad f \in \mathcal{B}^+_b(\mathbb{R}^d).
\]
Throughout this section, we fix $T > 0$ and $x \in \mathbb{R}^d$, and simply denote

$$N_T := 1_{[0,T]}(t)N(ds, dz).$$

To prove Theorem 2.1, we shall use the following $\Phi$-entropy inequality for the Poisson point process on $[0, T] \times \mathbb{R}^d$:

$$
\mathbb{E}(\Phi \circ F(N_T)) - \Phi(\mathbb{E} F(N_T)) \\
\leq \mathbb{E} \int_{[0,t] \times \mathbb{R}^d} \Psi_\Phi(F(N_T + \delta_{(s,z)}), F(N_T))ds \nu(dz), \quad F \in \mathcal{B}^+_{\mathbb{P}}(\Gamma_T).
$$

(2.5)

This inequality was first proved by Wu [14] for $\Phi(u) = u \log u$, and as explained in [11] §5.1 that Wu’s proof also applies to general $\Phi$ considered in the paper.

According to the inequality (2.5), to prove Theorem 2.1 we need to formulate $X_T(x) + a_t(z)$ using $N_T + \delta_{(\tau, \xi)}$ for some $\xi \in \mathbb{R}^d$ and $\tau \in [0, T]$. To this end, we let $F : [0, T] \times \Gamma_T \to \mathbb{R}^d$ be measurable such that $X_t(x) = F_t(N_T), t \in [0, T]$. Then we would suggest that $Y_t := F_t(N_T + \delta_{(\tau, \xi)})$ solves the equation

$$Y_t = x + \int_0^t b_s(Y_s)ds + \int_{[0,t] \times \mathbb{R}^d} a_s(z)\{N_a + \delta_{(\tau, \xi)}\}(ds, dz), \quad t \in [0, T].$$

Thus, taking $\xi$ and $\tau$ such that $a_t(z) = Y_T - X_T(x)$, we obtain

$$X_T(x) + a_t(z) = F_T(N_T + \delta_{(\tau, \xi)}).$$

However, since $X_T(x) = F_T(N_T)$ holds on $[0, T]$ merely $\mathbb{P}$-a.s., to make this argument rigorous we need to verify the quasi-invariance for the transform $N_T \to N_T + \delta_{(\tau, \xi)}$, which is ensured by the following Girsanov type theorem, see [11] for a similar result for Lévy processes.

**Lemma 2.2.** Let $g$ be a strictly positive function on $[0, T] \times \mathbb{R}^d$ such that $\nu^g(ds, dz) := g(s, z)ds \nu(dz)$ is a probability measure on $[0, T] \times \mathbb{R}^d$. Let

$$N_T(g) = \int_{[0,T] \times \mathbb{R}^d} g(s, z)N(ds, dz).$$

Moreover, let $(\tau, \xi)$ be a random variable independent of $N_T$ and with distribution $\nu^g$. Then

$$R := \frac{1}{g(\tau, \xi) + N_T(g)}$$

is a strictly positive probability density w.r.t. $\mathbb{P}$ such that the distribution of $N_T + \delta_{(\tau, \xi)}$ under $d\mathbb{Q} := Rd\mathbb{P}$ coincides with that of $N_T$ under $\mathbb{P}$.

**Proof.** Let $\pi$ be the Poisson measure with intensity $d\nu^g(dz)$ on $[0, T] \times \mathbb{R}^d$. Then $\pi \times \nu^g$ is the distribution of $(N_T, \tau, \xi)$. By the Mecke formula for the Poisson measure (see (3.1) in [9]), for any $F \in \mathcal{B}^+_{\mathbb{P}}(\Gamma_T)$ we have

$$
\mathbb{E}\{RF(N_T + \delta_{(\tau, \xi)})\} = \int_{\Gamma \times [0,T] \times \mathbb{R}^d} \frac{F(\gamma + \delta_{(s,z)})g(s, z)}{(\gamma + \delta_{(s,z)})(g)} \pi(d\gamma)d\nu^g(dz)
$$

$$
= \int_{\Gamma} \frac{F(\gamma)\gamma(g)}{\gamma(g)} \pi(d\gamma) = \pi(F).
$$

Therefore, $\mathbb{Q} := Rd\mathbb{P}$ is a probability measure, and the distribution of $N_T + \delta_{(\tau, \xi)}$ under $\mathbb{Q}$ coincides with that of $N_T$ under $\mathbb{P}$. \hfill \Box
Proof of Theorem 2.1 Let $F : \Gamma_T \to \mathbb{R}^d$ be measurable such that $X_T(x) = F(N_T)$. We intend to prove
\begin{equation}
X_T^{\tau,x}(z) = F(N_T + \delta_{(s,z)}), \quad \mathbb{P} \times ds \times \nu(dz)-a.e.
\end{equation}

To this end, for $g \in \mathcal{B}^+([0,T] \times \mathbb{R}^d)$ in Lemma 2.2 consider the product probability space:
\[
\bar{\Omega} = \Omega \times [0,T] \times \mathbb{R}^d, \quad \bar{\mathbb{P}}(d\omega, ds, dz) = g(s, z)\mathbb{P}(d\omega)ds\nu(dz).
\]

Let $\bar{N} = (N, \tau, \xi)$ be defined by
\[
N(\omega, s, z) = N(\omega), \quad \xi(\omega, s, z) = z, \quad \tau(\omega, s, z) = s, \quad (\omega, s, z) \in \bar{\Omega}.
\]

Then under $\bar{\mathbb{P}}$ the random variable $(\tau, \xi)$ is independent of $N_T$ and has distribution $
u^\theta(ds, dz) := g(s, z)ds\nu(dz)$. Let $R$ be in Lemma 2.2 Then the distribution of $N_T + \delta_{(\tau, \xi)}$ under $\mathbb{Q}$ coincides with that of $N_T$ under $\bar{\mathbb{P}}$ (equivalently, under $\mathbb{P}$). Thus, by the weak uniqueness of solutions to (2.2), the distribution of $(N_T + \delta_{(\tau, \xi)}, Y_T)$ under $\mathbb{Q}$ coincides with that of $(N_T, X_T(x))$ under $\mathbb{P}$. In particular, the distribution of $Y_T - F(N_T + \delta_{(\tau, \xi)})$ under $\mathbb{Q}$ coincides with that of $X_T(x) - F(N_T)$ under $\mathbb{P}$. Since $X_T(x) = F(N_T)$ $\mathbb{P}$-a.s., this implies that
\[
Y_T = F(N_T + \delta_{(\tau, \xi)}), \quad \mathbb{Q}$-a.s.
\]

As $\mathbb{Q}$ is equivalent to $\bar{\mathbb{P}}$, it also holds $\bar{\mathbb{P}}$-a.s. Then (2.6) follows by noting that $Y_T = X_T^{\tau,x}(\xi)$ and $g > 0$ such that $\bar{\mathbb{P}}$ is equivalent to $\mathbb{P} \times ds \times \nu(dz)$.

Now, by (2.5) and (2.6), for any $f \in \mathcal{B}^+_b(\mathbb{R}^d)$ we have
\begin{equation}
\text{Ent}_{\mathbb{P}}^\Phi(f) \leq \mathbb{E} \int_{[0,T] \times \mathbb{R}^d} \Phi(f \circ F(N_T + \delta_{(s,z)}), f \circ F(N_T))ds\nu(dz)
\end{equation}
\begin{equation}
= \mathbb{E} \int_{[0,T] \times \mathbb{R}^d} \Phi(f(X_T^{\tau,x}(z)), f(X_T(x)))ds\nu(dz).
\end{equation}

Noting that $X_T^{\tau,x}(z) = X_T(x) + a_s \circ \psi_s(z)$, it follows from (2.4) that
\begin{align*}
\mathbb{E} \int_{\mathbb{R}^d} \Phi(f(X_T^{\tau,x}(z)), f(X_T(x)))\nu(dz)
&= \mathbb{E} \int_{\mathbb{R}^d} \Phi(f(X_T(x) + a_s(z)), f(X_T(x)))\nu(\psi_s^{-1})(dz)
&\leq \mathbb{E} \int_0^T \xi_s ds \int_{\mathbb{R}^d} \Phi(f(X_T(x) + a_s(z)), f(X_T(x)))\nu(dz), \quad s \in (0,T).
\end{align*}

Combining this with (2.7) we finish the proof. $\square$

3 Proofs of Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3. Take $W(x) = \varphi(|x|)$, where $\varphi(r) := \int_0^r \frac{s}{(1+s)^2}ds, \quad r \geq 0$. Then $W \in C^2(\mathbb{R}^d)$. Let $\mathcal{L}$ be the generator of the solution $X_t$ to (1.12). By the Itô formula we have
\begin{equation}
\mathcal{L}W(x) = \langle b(x), \nabla W(x) \rangle + \text{Tr}[\sigma_1^1 \nabla^2 W](x) + \int_{\mathbb{R}^d} \left[ W(x + \sigma_2(x)z) - W(x) - \langle \nabla W(x), \sigma_2(x)z \rangle 1_{|z| \leq 1} \right] \nu(dz)
\end{equation}
if the integral in the right hand side exists. We observe that it suffices to prove that \( \mathcal{L}W \) is a well defined locally bounded function with

\[
\mathcal{L}W(x) \leq \frac{\langle b(x), x \rangle + \text{Tr}(\sigma_1 \sigma_1^T)(x)}{B(|x|)(|x| + 1)} + \frac{\|\sigma_2(x)\| \int_{\{\varepsilon < |z| \leq 1\}} |z| \nu(dz)}{B(|x|)}
\]

\[
- \frac{|\sigma_1^*(x)|^2 B'(|x|)}{B(|x|)^2(1 + |x|)|x|} + \int_{\{\varepsilon > |z| \geq |\sigma_2(x)||z|\}} \nu(dz) \int_{|x|} \frac{d s}{B(s)}
\]

\[
+ \|\sigma_2(x)\|^2 \mathcal{B}_\varepsilon(x) \int_{(|z| - \varepsilon)} |z|^2 \nu(dz).
\]

(3.2)

In fact, by this and \( A_{\varepsilon} = -\infty \) we see that \( -\mathcal{L}W \) is a compact function (i.e. \( \{r > 0\} \) is relatively compact for \( r > 0 \)). Therefore, by the Itô formula we see that the solution is non-explosive with

\[
\lim_{t \to \infty} \mathbb{E}(\mathcal{L}W)(X_t)ds < \infty,
\]

which implies the existence of the invariant probability measure by a standard tightness argument. Moreover, if \( \int_0^\infty \frac{ds}{B(s)} = \infty \) and \( A < 0 \), then \( W \) is a compact function and by the Itô formula the solution is non-explosive with \( \mathbb{E}W(X_t) < \infty, t \geq 0 \). Thus, according to [4, Theorem 4.1], \( A < 0 \) also implies that the associated Markov semigroup has an invariant probability measure. Below we prove that \( \mathcal{L}W \) is locally bounded such that (3.2) holds.

(a) It is easy to see that

\[
\left\{ \langle b, \nabla W \rangle + \text{Tr}(\sigma_1 \sigma_1^T \nabla^2 W) \right\}(x) - \int_{\{\varepsilon < |z| \leq 1\}} \langle \nabla W(x), \sigma_2(x)z \rangle \nu(dz)
\]

\[
\leq \varphi'(|x|) \left( \frac{\langle b(x), x \rangle + \text{Tr}(\sigma_1 \sigma_1^T)(x)}{|x|} - \frac{|\sigma_1^*(x)|^2 |x|^2}{|x|^3} + \|\sigma_2(x)\| \int_{\{\varepsilon < |z| \leq 1\}} |z| \nu(dz) \right)
\]

\[
+ \varphi''(|x|) \langle b(x), x \rangle \frac{|\sigma_1^*(x)|^2 |x|^2}{|x|^2}
\]

\[
\leq \frac{\langle b(x), x \rangle + \text{Tr}(\sigma_1 \sigma_1^T)(x)}{B(|x|)(|x| + 1)} + \frac{\|\sigma_2(x)\| \int_{\{\varepsilon < |z| \leq 1\}} |z| \nu(dz)}{B(|x|)}
\]

\[
- \frac{|\sigma_1^*(x)|^2 B'(|x|)}{B(|x|)^2(1 + |x|)|x|}, \quad x \in \mathbb{R}^d.
\]

(b) Since \( \int_{\{|z| \leq \varepsilon\}} |z|^2 \nu(dz) < \infty \),

\[
\int_{\{|z| \leq \varepsilon\}} |W(x + \sigma_2(x)z) - W(x) - \langle \nabla W(x), \sigma_2(x)z \rangle| \nu(dz)
\]

\[
\leq \frac{1}{2} \sup_{|y| \leq |x| + \|\sigma_2(x)\|} \|\nabla^2 W(y)\| \int_{\{|z| \leq \varepsilon\}} |\sigma_2(x)z|^2 \nu(dz)
\]

is locally bounded in \( x \in \mathbb{R}^d \). Noting that

\[
\nabla^2 W(x)(v, v) = \varphi'(|x|) \left( \frac{|v|^2}{|x|} - \frac{\langle x, v \rangle^2}{|x|^3} \right) + \varphi''(|x|) \frac{\langle x, v \rangle^2 |x|^2}{|x|^2}
\]

\[
\leq \frac{|v|^2 (B(|x|) - |x| B'(|x|))}{B(|x|)^2(1 + |x|)}.
\]
Therefore, (1.15) implies (1.13).

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By Theorem 1.3, for each situations it suffices to choose

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is locally bounded and

Since ν(\{ |z| > \varepsilon \}) < \infty and A_\varepsilon < 0, this implies that

\[
\int_{\{ |z| > \varepsilon \}} [W(x + \sigma_2(x)z) - W(x)] \nu(dz)
\]

is locally bounded and

\[
\int_{\{ |z| > \varepsilon \}} [W(x + \sigma_2(x)z) - W(x)] \nu(dz) \leq \int_{\{ |z| > \varepsilon \}} \nu(dz) \int_{|x|}^{\|x + \sigma_2(x)\||x|} \frac{ds}{B(s)}.
\]

By combining (3.1) with (a)-(c), we conclude that \( \mathcal{L}W \) is locally bounded satisfying (3.2).

Proof of Corollary 1.4. By Theorem 1.3, for each situations it suffices to choose \( B \) such that one of (1.13) and (1.14) holds for some \( \varepsilon \in (0, \Theta^{-1}) \).

Case (1). We take \( B(r) = (1 + r)^\delta \) for some \( \delta \in (1, \theta) \cap (0,1) \). Then \( \int_0^\infty \frac{ds}{B(s)} < \infty \) such that

\[
\limsup_{|x| \to \infty} \left( \frac{\|\sigma_2(x)\|}{B(|x|)} + \int_{\{ |z| > \varepsilon \}} \nu(dz) \int_{|x|}^{\|x + \sigma_2(x)\||x|} \frac{ds}{B(s)} \right) = 0.
\]

Next, since \( \delta > 1 \), for any \( \varepsilon \in (0, \Theta^{-1}) \) we have

\[
\limsup_{|x| \to \infty} \frac{\|\sigma_2(x)\|^2}{|x| \to \infty} \frac{|\|\sigma_2(x)\|^2}{(1 - \varepsilon \Theta)^{1 + \delta} |x|^{1 + \delta}} = 0.
\]

Therefore, (1.15) implies (1.13).

Case (2). We take \( B(r) = 1 + r \). Then \( \int_0^\infty \frac{ds}{B(s)} = \infty \) and by (1.16),

\[
\limsup_{|x| \to \infty} \left\{ \frac{\|\sigma_2(x)\|^2}{B(|x|)} \int_{\{ |z| \leq \varepsilon \}} |z|^2 \nu(dz) + \frac{\|\sigma_2(x)\|}{\|\sigma_2(x)\| |z|} \nu(dz) \right\}
\]

\[
\leq \frac{\varepsilon^2 \Theta^2 \int_{\{ |z| \leq \varepsilon \}} |z|^2 \nu(dz)}{2(1 - \varepsilon \Theta)^2} + \Theta \int_{\{ \varepsilon < |z| \leq 1 \}} |z| \nu(dz) + \int_{\{ |z| > \varepsilon \}} \log(1 + \varepsilon \Theta |z|) \nu(dz) < -D.
\]
Thus, (1.14) follows from (1.15).

**Case (3).** We take $B(r) = (1 + r)\theta$. Since $\|\sigma_2\|$ is bounded and $\theta \in (0, 1)$, we have $\Theta = 0$, $\int_0^\infty \frac{ds}{B(s)} = \infty$ and

$$
\lim_{|x| \to \infty} \|\sigma_2(x)\|^2 \tilde{B}_c(x) = 0.
$$

Moreover,

$$
\int_{|x|}^{\left|\sigma_2(x)\right| |z|} \frac{ds}{B(s)} \leq \min \left\{ \frac{\|\sigma_2(x)\| \cdot |z|}{(1 + |x|)^\theta}, \frac{\|\sigma_2(x)\|^{1-\theta} |z|^{1-\theta}}{1 - \theta} \right\}.
$$

Since $\|\sigma_2\|$ is bounded and $\int_{\{|z| \geq 1\}} |z|^{1-\theta} \nu(dz) < \infty$, by (3.4) and the dominated convergence theorem we obtain

$$
\lim_{|x| \to \infty} \sup_{\{|z| \geq 1\}} \int_{|x|}^{\left|\sigma_2(x)\right| |z|} \frac{ds}{B(s)} = 0.
$$

Then (1.14) with $\varepsilon = 1$ follows from (1.15).

**Case (4).** We first observe that (1.17) implies

$$
\lim_{|x| \to \infty} \sup_{\{|z| > 1\}} \frac{|z| \cdot \|\sigma_2(x)\|}{|x|^\theta \wedge \left(|x| + \|\sigma_2(x)\| \cdot |z|\right)^\theta} \nu(dz) < -D.
$$

Since when $\theta \geq 0$ (1.17) is equivalent to (3.5), we only consider $\theta < 0$. In this case, for any $s > 1$ there exists a constant $C(s) > 0$ such that

$$
(|x| + \|\sigma_2(x)\| \cdot |z|)^{-\theta} \leq s|x|^{-\theta} + C(s)(\|\sigma_2(x)\| \cdot |z|)^{-\theta}.
$$

Moreover, by (1.17) and $\int_{\{|z| > 1\}} |z|^{1-\theta} \nu(dz) < \infty$, we have $\|\sigma_2(x)\|^{1-\theta} \to 0$ as $|x| \to \infty$ so that

$$
\lim_{|x| \to \infty} \sup_{\{|z| > 1\}} \frac{|z| \cdot \|\sigma_2(x)\|}{|x|^\theta \wedge \left(|x| + \|\sigma_2(x)\| \cdot |z|\right)^\theta} \nu(dz)
\leq \lim_{|x| \to \infty} \sup_{\{|z| > 1\}} \left( \frac{s\|\sigma_2(x)\|}{|x|^\theta} \int_{\{|z| > 1\}} \nu(dz) + C(s)\|\sigma_2(x)\|^{1-\theta} \int_{\{|z| > 1\}} |z|^{1-\theta} \nu(dz) \right)
\leq \lim_{|x| \to \infty} \sup_{\{|z| > 1\}} \frac{s\|\sigma_2(x)\|}{|x|^\theta} \int_{\{|z| > 1\}} \nu(dz), \ s > 1.
$$

Since $s > 1$ is arbitrary, we conclude that (1.17) implies (3.5).

Now, we take $B(r) = (1 + r)^\theta$. Then $\int_0^\infty \frac{ds}{B(s)} = \infty$, $\tilde{B}_c(x) = \Theta((1 + |x|)^{-1 + \theta})$ for large $|x|$, and by (1.17) we have $\Theta = 0$. So, combining (3.5) with (3.3) and (3.4), we obtain

$$
\lim_{|x| \to \infty} \left\{ \|\sigma_2(x)\|^2 \tilde{B}_c(x) \int_{\{|z| \leq 1\}} |z|^2 \nu(dz) + \int_{\{|z| > 1\}} \nu(dz) \int_{|x|}^{\left|\sigma_2(x)\right| |z|} \frac{ds}{B(s)} \right\}
\leq \lim_{|x| \to \infty} \sup_{\{|z| > 1\}} \frac{|z| \cdot \|\sigma_2(x)\|}{|x|^\theta \wedge \left(|x| + \|\sigma_2(x)\| \cdot |z|\right)^\theta} \nu(dz) < -D.
$$

Then (1.15) implies (1.14) with $\varepsilon = 1$. 

\[\square\]
4 Proofs of Theorems 1.1 and 1.2

To apply Theorem 2.1 we take \( b_t = b \) and \( a_t(z) = \sigma z \) such that (2.2) reduces back to (1.3). In this case we have

\[
\psi_s(z) = \sigma^{-1}(X^s_t(z) - X_T(x))
\]

and for \( s \in (0, T] \)

\[
d\nabla X_t = \{\nabla b(X^s_t)\} \nabla X_t dt + \sigma \delta_x dt, \quad \nabla X_0 = 0.
\]

Thus, \( \nabla \psi_s(z) = \sigma^{-1} \nabla X_t \) and for \( t \geq s \),

\[
d(\sigma^{-1} \nabla X_t) = \{\sigma^{-1} \nabla b(X^s_t)\} \sigma^{-1} \nabla X_t dt, \quad \sigma^{-1} \nabla X_t = I.
\]

Combining this with (1.9) we obtain

\[
(4.1) \quad |\det(\nabla \psi_s)^{-1}| = \frac{1}{|\det(\nabla \psi_s)|} \leq e^{-\lambda_1(T-s)d},
\]

and

\[
(4.2) \quad \sup_{z \in \mathbb{R}^{d}\{0\}} \frac{|z|}{|\psi^{-1}_s(z)|} = \sup_{z \in \mathbb{R}^{d}\{0\}} \frac{|\psi_s(z)|}{|z|} \leq \sup_{z \in \mathbb{R}^{d}\{0\}} \frac{1}{|z|} \leq e^{\lambda_2(T-s)}.
\]

Proof of Theorem 1.1. Let \( \frac{\kappa_1}{\|\psi\|_{\infty}} \leq \frac{\nu(dz)}{dz} \leq \frac{\kappa_2}{\|\psi\|_{\infty}} \). Then by (4.1) and (4.2) we obtain

\[
(\nu \circ \psi^{-1})(dz) \leq \frac{\kappa_2}{\|\psi\|_{\infty}} \frac{|\det(\nabla \psi^{-1}_s(z))|}{|\psi^{-1}_s(z)|^d \alpha} dz = \frac{\kappa_2}{\|\psi\|_{\infty}} \frac{|\det(\nabla \psi_s)^{-1}(\psi^{-1}_s(z))|}{|\psi^{-1}_s(z)|^d \alpha} dz \\
\leq \frac{\kappa_2 e^{\lambda_2(T-s)(d+\alpha)-\lambda_1(T-s)d} \nu(dz)}{\kappa_1}.
\]

By Theorem 2.1 this proves Theorem 1.1 (1).

Next, to prove the existence of invariant probability measure using Corollary 1.4, we take \( \sigma_1(x) = 0 \) and \( \sigma_2(x) = \sigma \). It is easy to see that \( \Theta = 0 \) and \( \int_{\{|z| \geq 1\}} \log(1 + |z|) \nu(dz) < \infty \). Since \( \lambda_2(d + \alpha) - \lambda_1 d < 0 \) implies \( \lambda_2 < 0 \), (1.15) and (1.16) hold for \( \theta = 1 \). Then according to Corollary 1.4 for \( \theta = 1 \), \( P_t \) has an invariant probability measure \( \mu \). Moreover, since \( \lambda_2 < 0 \) implies

\[
\lim_{t \to \infty} |X_t(x) - X_t(y)| \leq \lim_{t \to \infty} |x - y| e^{\lambda_2 t} = 0, \quad x, y \in \mathbb{R}^d,
\]

we conclude that \( P_t f \to \mu(f) \) as \( t \to \infty \) holds for all \( f \in C_b(\mathbb{R}^d) \). Thus, \( \mu \) is the unique invariant probability measure of \( P_t \). Since \( \Phi \in C^2((0, \infty)) \), for \( f \in C^2_b(\mathbb{R}^d) \) with \( \inf f > 0 \) we have \( \Gamma_{\Phi, \nu}(f) \in C^2_b(\mathbb{R}^d) \). By letting \( t \to \infty \) in the semigroup \( \Phi \)-entropy inequality in Theorem 1.1 (1), we prove (1.8) for the desired constant \( C \) and positive \( f \in C^2_b(\mathbb{R}^d) \) with \( \inf f > 0 \). By a simple approximation argument, (1.8) holds for all \( f \in \mathcal{B}_0^+(\mathbb{R}^d) \). \(\square\)

Proof of Theorem 1.2. (a) Let \( \lambda_2 \leq 0 \) and \( \rho \) be decreasing. By (4.2) we have \( |z| \leq |\psi^{-1}_s(z)| \), so that

\[
(4.3) \quad \rho(|\psi^{-1}_s(z)|) \leq \rho(|z|).
\]
Combining this with (4.1) and (4.2) we obtain
\[
(\nu \circ \psi^{-1})(dz) \leq \frac{\kappa_2 |\det(\nabla \psi)|^{-1} |(z)\rho(|\psi^{-1}(z)|)}{|\psi^-1(z)|^{d+\alpha}} dz \leq e^{\lambda_2(T-s)(d+\alpha)} - \lambda_1(T-s)^d \nu(dz).
\]
According to the proof of Theorem 1.1 this proves the first assertion in (I).

(b) By an approximation argument, for (1.10) we may assume that \( f \in C^2_{c,+}(\mathbb{R}^d) \). We first consider the case that \( b \in C^2(\mathbb{R}^d; \mathbb{R}^d) \) with bounded \( \nabla^2 b \) and \( \nu(1_{\{|\cdot| > 1\}} \cdot |) < \infty \). Then by the boundedness of \( \nabla b \) (due to (4.1)) and \( \nabla^2 b \), we see that \( \|\nabla X_t\|_\infty \) and \( \|\nabla^2 X_t\|_\infty \) are locally bounded in \( t \geq 0 \), since for any \( u, v \in \mathbb{R}^d \),
\[
d\nabla_u X_t = \nabla b(X_t)\nabla_u X_t dt, \quad \nabla_u X_0 = u,
\]
\[
d\nabla_u \nabla_v X_t = \{\nabla^2 b(X_t)(\nabla_u X_t, \nabla_v X_t) + \nabla b(X_t)\nabla_u \nabla_v X_t\} dt, \quad \nabla_u \nabla_v X_0 = 0.
\]
This implies that \( P_t C^2_b(\mathbb{R}^d) \subseteq C^2_b(\mathbb{R}^d) \) for any \( t \geq 0 \) with \( \|\nabla P_t f\|_\infty \) and \( \|\nabla^2 P_t f\|_\infty \) locally bounded in \( t \). Next, by (1.9) and \( \nu(\cdot |1_{\{|\cdot| > 1\}} < \infty \), it is easy to see that \( W(x) := \sqrt{|x|^2 + 1} \)
\[
\mathcal{L} W(x) \leq C_1 - C_2 |x|, \quad x \in \mathbb{R}^d
\]
for some constants \( C_1, C_2 > 0 \). Thus, the invariant probability measure \( \mu \) satisfies \( \mu(|\cdot|) < \infty \). Moreover, by the boundedness of \( \nabla b \), for any \( f \in C^2_b(\mathbb{R}^d) \) there exists a constant \( C_3 > 0 \) such that \( \mathcal{L} f \leq C_3(1 + |x|) \). So, by (1.6) and \( |\cdot| \leq W \),
\[
\frac{|P_t f - f|}{t} \leq \frac{C_3}{t} \int_0^t (1 + E|X_s|)ds \leq C_3(1 + W + C_1), \quad t \in (0, 1].
\]
Since the upper bound is integrable with respect to \( \mu \), by the dominated convergence theorem we obtain
\[
\int_{\mathbb{R}^d} \mathcal{L} f d\mu = \int_{\mathbb{R}^d} \lim_{t \to 0} \frac{P_t f - f}{t} d\mu = \lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^d} (P_t f - f) d\mu = 0, \quad f \in C^2_b(\mathbb{R}^d).
\]
Since \( P_t C^2_b(\mathbb{R}^d) \subseteq C^2_b(\mathbb{R}^d) \), for any \( f \in C^2_{c,+}(\mathbb{R}^d) \) we have \( \Phi(P_t f) \in C^2_b(\mathbb{R}^d) \), so that
\[
\int_{\mathbb{R}^d} \mathcal{L} \Phi(P_t f) d\mu = 0.
\]
Hence, (1.7) holds for \( P_t f \) in place of \( f \). Therefore, it follows from (1.8) that
\[
\frac{d}{dt} \text{Ent}_\mu^\Phi(P_t f) = -\mathcal{E}(\Phi'(P_t f), P_t f) = -\int_{\mathbb{R}^d} \Gamma_{\Phi,\nu}(P_t f) d\mu \leq -\frac{1}{C} \text{Ent}_\mu^\Phi(P_t f), \quad t \geq 0.
\]
This implies (1.10) for \( f \in C^2_{c,+}(\mathbb{R}^d) \) since according to the first assertion (1.8) holds for \( C = \frac{\kappa_2}{\kappa_2} (\lambda_1 d - \lambda_2 (d + \alpha)) \).

(c) Assume that \( \nu(1_{\{|\cdot| > 1\}} \cdot |) < \infty \). To apply the assertion proved in (b), we make a standard regularization of \( b \) as follows:
\[
b_\varepsilon(x) = \frac{1}{(\pi \varepsilon)^{d/2}} \int_{\mathbb{R}^d} b(y) e^{-|x-y|^2/\varepsilon} dy, \quad x \in \mathbb{R}^d, \varepsilon \in (0, 1).
\]
Since (1.9) is equivalent to the dissipative property of \( b(x) - \lambda_3 x \) and \( \lambda_1 x - b(x) \), according to [5 Theorem 9.1] we conclude that for every \( \varepsilon \in (0, 1) \), \( b_\varepsilon \in C^2(\mathbb{R}^d; \mathbb{R}^d) \) with bounded \( \nabla^2 b_\varepsilon \) and (1.9) holds for \( b_\varepsilon \) in place of \( b \). Then by (b), we have
\[
(4.5) \quad \text{Ent}_{\mu_\varepsilon}^{\Phi}(P_t f) \leq e^{-t/C} \text{Ent}_{\mu_\varepsilon}^{\Phi}(f), \quad f \in \mathcal{A}^+_b(\mathbb{R}^d), t \geq 0,
\]
where $C = \frac{\kappa_2}{\kappa_2\lambda_1 - \lambda_2(d + \alpha)}$, $P_t^\varepsilon$ and $\mu_\varepsilon$ are the semigroup and invariant probability measure for the equation
\[ dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sigma dL_t, \quad X_0^\varepsilon = X_0. \]

Moreover, by the boundedness of $\nabla b$,
\[ |b_\varepsilon(x) - b(x)| \leq \frac{\|\nabla b\|_\infty}{(\pi\varepsilon)^{d/2}} \int_{\mathbb{R}^d} |x - y|e^{-|x-y|^2/\varepsilon}dy \leq c\sqrt{\varepsilon}, \quad x \in \mathbb{R}^d, \varepsilon \in (0, 1) \]
holds for some constant $c > 0$. Combining this with (1.9) we obtain
\[ d|X_t - X_t^f| = \frac{\langle X_t - X_t^f, b(X_t) - b(X_t^f) \rangle + \langle X_t - X_t^f, b(X_t^f) - b_\varepsilon(X_t^f) \rangle}{|X_t - X_t^f|} dt \leq \left\{ \lambda_2 |X_t - X_t^f| + c\sqrt{\varepsilon} \right\} dt. \]

Since $\lambda_2 < 0$, this implies
\[ |X_t - X_t^f| \leq \frac{c\sqrt{\varepsilon}}{-\lambda_2} =: c'\sqrt{\varepsilon}, \quad t \geq 0, \varepsilon \in (0, 1). \]

Then for any $f \in C_0^1(\mathbb{R}^d)$,
\[ \|P_t^\varepsilon f - P_tf\|_\infty \leq \|\nabla f\|_\infty c'\sqrt{\varepsilon}, \quad t \geq 0, \varepsilon \in (0, 1). \]  
(4.6)

Hence,
\[ |\mu_\varepsilon(f) - \mu(f)| = \lim_{t \to \infty} |P_t^\varepsilon f(0) - P_tf(0)| \leq \|\nabla f\|_\infty c'\sqrt{\varepsilon}, \quad f \in C_0^1(\mathbb{R}^d), \varepsilon \in (0, 1). \]  
(4.7)

Combining (4.6) and (4.7), for any $f \in C_0^1(\mathbb{R}^d)$ with $\inf f > 0$ we obtain
\[ \lim_{t \to \infty} \sup \left| \mu_\varepsilon(\Phi(P_t^\varepsilon f)) - \mu(\Phi(P_tf)) \right| \leq \lim_{t \to \infty} \sup \left| \mu_\varepsilon(\Phi(P_t f)) - \mu(\Phi(P_tf)) \right| = 0, \quad t \geq 0. \]

Therefore, letting $\varepsilon \downarrow 0$ in (1.5), we prove (1.10) for $f \in C_0^1(\mathbb{R}^d)$ with $\inf f > 0$, and thus also for $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ by an approximation argument.

(d) In general, for any $n \geq 1$ let $\nu_n(dz) = 1_{\{|z| \leq n\}}\nu(dz)$ and $\tilde{\nu}_n = \nu - \nu_n$. Write $L_t = L_t^n + \tilde{L}_t^n$, where $L_t^n$ and $\tilde{L}_t^n$ are Lévy processes with Lévy measures $\nu_n$ and $\tilde{\nu}_n$ respectively. Consider the equation
\[ dX_t^n = b(X_t^n)dt + \sigma dL_t^n, \quad X_0^n = X_0, \]
and let $P_t^n$ and $\mu_n$ be the associate semigroup and invariant probability measure respectively. Then by (c), we have
\[ \text{Ent}_{\mu_n}^\Phi(P_t^n f) \leq e^{-t/C} \text{Ent}_{\mu_n}^\Phi(f), \quad t \geq 0, n \geq 1, f \in \mathcal{B}_b^+(\mathbb{R}^d). \]  
(4.8)

Noting that
\[ d(X_t - X_t^n) = (b(X_t) - b(X_t^n))dt + \sigma d\tilde{L}_t^n, \quad X_0 - X_0^n = 0, \]
by (1.9) with $\lambda_2 < 0$ we may find a constant $\lambda > 0$ such that such that
\[ d|X_t - X_t^n|^c \leq \left( \int_{\{|z| > n\}} (|X_t - X_t^n + \sigma z|^c - |X_t - X_t^n|^c)\nu(dz) - \lambda |X_t - X_t^n|^c \right) dt + M_t \]
holds for some local martingale $M_t$. Since
$$\int_{\{|z|>n\}} \left(|X_t - X^n_t + \sigma z|^\varepsilon - |X_t - X^n_t|^\varepsilon\right)\nu(dz) \leq \int_{\{|z|>n\}} |\sigma z|^\varepsilon\nu(dz) =: \delta_n \downarrow 0 \text{ as } n \uparrow \infty,$$
we obtain
$$\mathbb{E}|X_t - X^n_t|^\varepsilon \leq \frac{\delta_n}{\lambda}, \quad t \geq 0.$$  
Thus,
$$|P^n_tf - P_t f| \leq (\|\nabla f\|_{\infty} + 2\|f\|_{\infty})\mathbb{E}(1 \wedge |X_t - X^n_t|) \leq \frac{\delta_n}{\lambda}(\|\nabla f\|_{\infty} + 2\|f\|_{\infty}), \quad f \in C^1_b(\mathbb{R}^d).$$
Then, according to the argument in the end of (c) using this estimate to replace (4.6), we prove (1.10) by letting $n \to \infty$ in (4.8).

(e) Let $\lambda_1 \geq 0, \theta_1 \geq 1$ and $\rho$ be increasing. Then $|z| \geq |\psi^{-1}_s(z)|$, so that (4.3) holds and the remainder of the proof is similar to (a).

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