A Nearly Tight Lower Bound for the $d$-Dimensional Cow-Path Problem

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Abstract

In the $d$-dimensional cow-path problem, a cow living in $\mathbb{R}^d$ must locate a $(d-1)$-dimensional hyperplane $H$ whose location is unknown. The only way that the cow can find $H$ is to roam $\mathbb{R}^d$ until it intersects $H$. If the cow travels a total distance $s$ to locate a hyperplane $H$ whose distance from the origin was $r \geq 1$, then the cow is said to achieve competitive ratio $s/r$.

It is a classic result that, in $\mathbb{R}^2$, the optimal (deterministic) competitive ratio is 9. In $\mathbb{R}^3$, the optimal competitive ratio is known to be at most $\approx 13.811$. But in higher dimensions, the asymptotic relationship between $d$ and the optimal competitive ratio remains an open question. The best upper and lower bounds, due to Antoniadis et al., are $O(d^{3/2})$ and $\Omega(d)$, leaving a gap of roughly $\sqrt{d}$. In this note, we achieve a stronger lower bound of $\tilde{\Omega}(d^{3/2})$. 

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1 Introduction

The cow-path problem is one of the simplest algorithmic problems taught to undergraduates: A cow begins at the origin on the number line, and must find a hay-stack located at some unknown point \( p \in \mathbb{R} \) satisfying \(|p| \geq 1\). How should the cow go about locating point \( p \), if the cow wishes to optimize its worst-case competitive ratio, which is given by \( s/|p| \) where \( s \) is the total distance traveled by the cow?

The optimal solution is to perform a repeated doubling argument: the cow travels between the points \((-2)^0, (-2)^1, (-2)^2, \ldots\). This path can be shown to have a competitive ratio of 9, which is optimal for any deterministic solution \([5]\) – more generally, the class of paths that achieve competitive ratio 9 has been studied in great detail \([11]\). Randomized solutions can do even better, achieving a competitive ratio of \( \approx 4.591 \). \([5]\).

In the decades since it was first introduced, the cow path problem has been generalized in many natural ways: to consider paths in a multi-lane highway \([10][11]\), paths that search for various type of objects \([3]\), etc. For a (slightly out of date) survey on these types of problems, see \([7]\).

Perhaps surprisingly, however, one of the most natural generalizations still remains quite enigmatic. In the \( d \)-dimensional cow-path problem, the cow begins at the origin in \( \mathbb{R}^d \), and must travel in search of a \((d-1)\)-dimensional hyperplane. Once the cow has intersected the hyperplane, their path is complete. Even in \( d = 2 \) dimensions, the optimal competitive ratio remains an open question – it is conjectured to \( \approx 13.811 \), and to be achieved by a logarithmic spiral \([3][6]\). In higher dimensions, even the asymptotic behavior of the optimal competitive ratio remains open. The best upper and lower bounds, due to Antoniadis et al. \([2]\), are \( O(d^{3/2}) \) and \( \Omega(d) \), leaving a gap of roughly \( \sqrt{d} \).

In this note, we settle the optimal \( d \)-dimensional competitive ratio up to low-order terms, presenting a simple argument for a lower bound of \( \tilde{\Omega}(d^{3/2}) \), or, more precisely, \( \Omega(d^{3/2}/\log d) \).

Concurrent work by Ghomi and Wenk \([9]\), posted on arXiv a few days before this note, establishes a \( d \)-dimensional competitive ratio of \( \Omega(d^{3/2}) \) (communicated by Nazarov). The earlier work of Ghomi and Wenk \([8]\) considered the special case of \( d = 3 \).

2 Preliminaries

In the \( d \)-dimensional cow-path problem, a cow starts at 0 in \( \mathbb{R}^d \), and wishes to find a \((d-1)\)-dimensional hyperplane \( H \) whose distance \( r \geq 1 \) from 0 is unknown. The cow travels along a path until the cow intersects \( H \). At this point, if the cow has traveled a total distance of \( s \), then the cow is said to have achieved a competitive ratio of \( \frac{s}{r} \). In general, a cow path is said to achieve competitive ratio \( \alpha \) if its competitive ratio is bounded above by \( \alpha \) for all \((d-1)\)-dimensional hyperplanes \( H \) that are of distance at least 1 from the origin.

Up to a constant factor in the optimal competitive ratio, using the standard doubling argument, one can assume without loss of generality that \( H \) has distance 1 from 0 \([2]\). Defining \( \mathcal{H} \) to be the set of such hyperplanes, the competitive ratio of the cow path is simply the total distance traveled until the cow path has hit every \( H \in \mathcal{H} \).

As shown in \([2]\), using the polar duality between points and hyperplanes, this latter problem becomes equivalent to the following sphere-inspection problem. Let \( U \) be the unit sphere in \( \mathbb{R}^d \) centered at the origin. We say that a point \( p \in \mathbb{R}^d \), satisfying \(|p| \geq 1\) where \(| \cdot | \) denotes the standard Euclidean norm, views a point \( q \in U \) if the segment \( \overline{pq} \) intersects \( U \) only at \( q \). (If \(|p| < 1\), it does not view any points in \( U \).) One can show that a path \( P \) intersects every hyperplane \( H \in \mathcal{H} \) if and only if each point \( q \in U \) is visible from some point \( p \in P \). This is because in order for the path \( P \) (starting from the origin) to view a point \( q \in U \), the path must intersect the hyperplane \( H \in \mathcal{H} \) that is orthogonal to \( q \).

Thus, the main result of this note can be reformulated as follows: any path \( P \) that starts at the origin and views every point in \( U \) must have length at least \( \Omega(d^{3/2}/\sqrt{\log d}) \). This then implies a \( \Omega(d^{3/2}/\sqrt{\log d}) \)
lower bound for the optimal competitive ratio of the $d$-dimensional cow-path problem.

It is worth commenting on several other ways to think about the same problem \cite{2}: A path $P$ views all of $U$ if and only if $U$ is contained in the convex hull of $P$. Also, a point $p \in P$ views a point $q \in U$ if and only if the angle $\angle(0, q, p)$ is greater than or equal to $90^\circ$—this, in turn, is equivalent to $\langle p - q, 0 - q \rangle \leq 0$, which can be rewritten as $\langle p, q \rangle \geq |q|^2 = 1$. Both the interpretations of sphere visibility (the convex-hull interpretation and the $\langle p, q \rangle \geq 1$ interpretation) will be useful throughout our proofs.

We remark that, although the sphere-inspection problem and the $d$-dimensional cow path problem are asymptotically equivalent, their optimal competitive ratios differ within constant factors. Indeed, the exact optimal competitive ratio for the sphere-inspection problem is known for $d \leq 3$ \cite{8}, while the optimal bound for the cow path problem remains open even for $d = 2$ \cite{2,3,6}.

### 3 The Lower Bound

We will use $d$ as a global variable, indicating that we are in $\mathbb{R}^d$. And we will use $U = \{x \in \mathbb{R}^d \mid |x| = 1\}$ to denote the unit sphere. We begin by recalling a standard result on the distribution of $U$’s mass.

**Lemma 1** (Lemma 2.2 of \cite{4}). Let $v$ be any point satisfying $|v| = 1$. The fraction of points $u \in U$ satisfying $\langle u, v \rangle \geq \varepsilon$ is at most $e^{-d\varepsilon^2/2}$.

Lemma 1 directly gives the following bound on the fraction of $U$ that is visible from any given point $p$.

**Lemma 2.** Let $p$ be any point of size $|p| \geq 1$. The fraction of points on the surface of $U$ that are visible from point $p$ is at most $e^{-d/(2|p|^2)}$.

**Proof.** Recall that a point $q$ is visible from $p$ iff $\langle p, q \rangle \geq 1$. Thus $\langle q, \hat{p} \rangle \geq 1/|p|$ where $\hat{p}$ is the unit vector along $p$. Applying Lemma 1 with $v = \hat{p}$ and $\varepsilon = 1/|p|$ gives the result. \hfill $\square$

Next we consider the fraction of $U$ that is visible from a given sphere $V$ of radius $1/2$. An important observation here, which has also appeared implicitly in past work \cite{2}, is that if $V$ has center $p$, then the portion of $U$ that is visible from $V$ is also visible from the point $2p$. For an accompanying illustration, see Figure 1.

**Lemma 3.** Let $p$ be a point with $|p| \geq 1$. Let $V$ be a sphere of radius $1/2$ centered around point $p$. Let $Q_1$ denote the set of points $q \in U$ such that $q$ is visible from some point $r \in V$, and let $Q_2$ denote the set of points visible from point $2p$. Then $Q_1 \subseteq Q_2$.

**Proof.** Consider any point $q \in Q_1$. As $q$ is visible from $r$, we have $\langle q, r \rangle \geq 1$. As $r \in V$, we can write $r = p + v$ for some $v$ with $|v| \leq 1/2$. Thus we have,

$$1 \leq \langle q, r \rangle = \langle q, p + v \rangle = \langle q, p \rangle + \langle q, v \rangle \leq \langle q, p \rangle + |q||v| \leq \langle q, p \rangle + 1/2,$$

Figure 1: A visual illustration of how the points $q \in U$ visible from $V$ are also visible from point $2p$. 

where the last step uses that $|q| = 1$ and $|v| \leq 1/2$. Thus $\langle q, p \rangle \geq 1/2$, or equivalently, $\langle 2p, q \rangle \geq 1$, which implies that $q$ is visible from the point $2p$. This completes the proof that $Q_1 \subseteq Q_2$. □

Combining the previous lemmas, we conclude that the only way for a path to see the entire sphere $U$ is if either (1) the path is very long, or (2) the path is, at some point, quite far away from the origin.

**Lemma 4.** Consider a path $P$ that never surpasses distance $r$ from the origin, but that views all of $U$. We must have

$$|P| \geq e^{d/(8r^2)} - 1.$$  

**Proof.** Break $P$ into $|P|$ sub-paths $P_1, P_2, \ldots$, all but one of which have length 1. Each $P_i$ is contained in a sphere $S_i$ of radius $1/2$ centered at the midpoint $p_i$ of $P_i$. By Lemma 3, every point in $U$ visible from the subpath $P_i$ is also visible from the point $2p_i$. Since $2p_i$ is distance at most $2r$ from the origin, it follows by Lemma 2 that the fraction of $U$ visible from $P_i$ is at most $e^{-d/(8r^2)}$. Since the $P_i$s collectively view all of $U$, we can conclude that $|P| \geq e^{d/(8r^2)}$, and thus that $|P| \geq e^{d/(8r^2)} - 1$. □

**Corollary 5.** Consider any path $P$ that views all of $U$. The path $P$ must either reach distance $\sqrt{d/(16 \log d)}$ from the origin or have length $|P| \geq e^{2 \log d - 1} \geq \Omega(d^2)$.

We can now prove our main result.

**Theorem 6.** Any path $P$ that start at the origin and views all of $U$ must have length $|P| \geq \Omega(d^{3/2}/\sqrt{\log d})$.

**Proof.** Let $\tau = \sqrt{\frac{d}{8}}/(16 \log \frac{d}{8})$. Suppose that $|P|$ has length $O(d^{3/2})$. By Corollary 5 $P$ must reach distance $\tau$ from the origin – let $P(t_1)$ be the first point at which this happens. Let $P_2$ be the path $P$ projected onto the $(d-1)$-dimensional hyperplane orthogonal to $P(t_1)$ (and going through the origin). Since $|P_2| \leq |P|$, we know that $|P_2| \leq O(d^{3/2})$, so by Corollary 2 $P_2$ must reach distance $\tau$ from the origin – let $P_2(t_2)$ be the first point at which this happens. Let $P_3$ be the path $P_2$ projected onto the $(d-2)$-dimensional hyperplane orthogonal to $P(t_2)$ (and going through the origin). Since $|P_3| \leq |P_2|$, we know that $|P_3| \leq O(d^{3/2})$, so by Corollary 2 $P_3$ must reach distance $\tau$ from the origin – let $P_3(t_3)$ be the first point at which this happens. Continuing like this, we can define $t_1, \ldots, t_{d/2}$ and $P_1, \ldots, P_{d/2}$ so that each $P_i(t_i)$ is distance at least $\tau$ from the origin and so that each $P_i+1$ is the path $P$ projected onto the $(d-i+1)$-dimensional hyperplane (going through the origin) that is orthogonal to each of $P(t_1), P_2(t_2), \ldots, P_i(t_i)$.

An important observation is that $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_{d/2}$. Indeed, if $t_{i+1} < t_i$, then since $|P_i(t_{i+1})| \geq |P_{i+1}(t_{i+1})| \geq \tau$, we would have that $t_i$ was not the first time at which $P_i$ reached distance $\tau$ from the origin, a contradiction.

Also observe that the distance traveled by $P$ between time $t_i$ and time $t_{i+1}$ is at least the distance traveled by $P_{i+1}$ during that time interval. Since $P_{i+1}(t_i) = 0$ and $|P_{i+1}(t_{i+1})| = \tau$, it follows that $P$ travels distance at least $\tau$ between $t_i$ and $t_{i+1}$. The total length of $P$ is therefore at least

$$\sum_{i=1}^{d/2} |P_i| \geq \frac{d}{2} \cdot \tau \geq \Omega(d^{3/2}/\sqrt{\log d}).$$ □

**Corollary 7.** The optimal competitive ratio for the $d$-dimensional cow-path problem is $\Omega(d^{3/2}/\sqrt{\log d})$.

**References**

[1] Spyros Angelopoulos, Christoph Dürr, and Shendan Jin. Best-of-two-worlds analysis of online search. In *36th International Symposium on Theoretical Aspects of Computer Science*, 2019.
[2] Antonios Antoniadis, Ruben Hoeksma, Sándor Kisfaludi-Bak, and Kevin Schewior. Online search for a hyperplane in high-dimensional euclidean space. *Information processing letters*, 177:106262, 2022.

[3] Ricardo Baeza-Yates and René Schott. Parallel searching in the plane. *Computational Geometry*, 5(3):143–154, 1995.

[4] Keith Ball et al. An elementary introduction to modern convex geometry. *Flavors of geometry*, 31(1-58):26, 1997.

[5] Anatole Beck and Donald J Newman. Yet more on the linear search problem. *Israel journal of mathematics*, 8(4):419–429, 1970.

[6] Steven R Finch and Li-Yan Zhu. Searching for a shoreline. *arXiv preprint math/0501123*, 2005.

[7] Shmuel Gal. Search games. *Wiley encyclopedia of operations research and management science*, 2010.

[8] Mohammad Ghomi and James Wenk. Shortest closed curve to inspect a sphere. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2021(781):57–84, 2021.

[9] Mohammad Ghomi and James Wenk. Shortest closed curve to contain a sphere in its convex hull. *arXiv preprint arXiv:2209.05988*, 2022.

[10] Ming-Yang Kao, Yuan Ma, Michael Sipser, and Yiqun Yin. Optimal constructions of hybrid algorithms. *Journal of Algorithms*, 29(1):142–164, 1998.

[11] Ming-Yang Kao, John H Reif, and Stephen R Tate. Searching in an unknown environment: An optimal randomized algorithm for the cow-path problem. *Information and Computation*, 131(1):63–79, 1996.