A Graph Theoretic Analysis of Leverage Centrality

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Abstract

In 2010, Joyce et. al defined the leverage centrality of vertices in a graph as a means to analyze functional connections within the human brain. In this metric a degree of a vertex is compared to the degrees of all its neighbors. We investigate this property from a mathematical perspective. We first outline some of the basic properties and then compute leverage centralities of vertices in different families of graphs. In particular, we show there is a surprising connection between the number of distinct leverage centralities in the Cartesian product of paths and the triangle numbers.

1 Introduction

In a social network people influence each other and those with lots of friends often have more leverage (or influence) than those with fewer friends. However the true influence of a person not only depends on the number of friends that they have, but also on the number of friends that their friends have. A person that is well connected can pass information to many friends, but if their friends are also receiving information from others, their influence on others is lessened. The extreme cases of influence occurs with a person who has a large number of friends, and for each of the friends, their only source of information is the original person. In this situation, the original person has the highest possible influence and all of the others have the lowest possible influence.

The level of influence can be quantified by a property defined by Joyce et al. [6] known as leverage centrality. We recall that the degree of a vertex $v$ is the number of edges incident to $v$ and is denoted $\deg(v)$.

We next give a formal definition of leverage centrality [6].

Definition 1 (leverage centrality) Leverage centrality is a measure of the relationship between the degree of a given node $v$ and the degree of each of its neighbors $v_i$, averaged over all neighbors $N_v$, and is defined as shown below:

$$l(v) = \frac{1}{\deg(v)} \sum_{v_i \in N_v} \frac{\deg(v) - \deg(v_i)}{\deg(v) + \deg(v_i)}.$$

This property was used by Joyce et al. [6] in the analysis of functional magnetic resonance imaging (fMRI) data [6] and has also been applied to real-world networks including airline connections, electrical power grids, and coauthorship collaborations [8]. However despite these studies leverage centrality has yet to be explored from a mathematical standpoint. The formula gives a measure of the relationship between a vertex and its

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neighbors. A positive leverage centrality means that this vertex has influence over its neighbors, whereas a negative leverage centrality indicates that a vertex is being influenced by its neighbors.

We begin with an elementary result involving the bounds of leverage centrality (Li et al. [8]).

**Lemma 2** Let $G$ be a graph with $n$ vertices. For any vertex $v$, $|l(v)| \leq 1 - \frac{2}{n}$. Furthermore, these bounds are tight in the cases of stars and complete graphs.

We note that the bounds are also tight for regular graphs.

There exist graphs $G$ where the leverage centrality of all vertices is equal and where the leverage centrality of vertices is distinct. It is clear that if $G$ is a regular graph than $l(v) = 0$ for every $v \in G$. We give an example below of a graph that has distinct leverage centralities.

\[
\begin{array}{c|c}
0 & \frac{1}{5} \left( \frac{3-6}{3+3} \right) = \frac{1}{15} \\
1 & \frac{1}{5} \left( \frac{5-6}{5+6} \right) = -\frac{1}{55} \\
2 & 0 \\
3 & \frac{1}{6} \left( 3 \left( \frac{6-5}{6+5} \right) + \left( \frac{4-3}{4+3} \right) \right) = \frac{10}{99} \\
4 & \frac{1}{7} \left( \left( \frac{4-5}{4+5} \right) + 2 \left( \frac{4-3}{4+3} \right) + \left( \frac{3-2}{3+2} \right) \right) = -\frac{42}{495} \\
5 & \frac{1}{8} \left( \frac{3-6}{3+6} \right) = -\frac{2}{9} \\
6 & \frac{1}{9} \left( 2 \left( \frac{6-5}{6+5} \right) + \frac{6-4}{6+4} + \left( \frac{6-3}{6+3} \right) \right) = \frac{59}{495} \\
7 & \frac{1}{10} \left( 2 \left( \frac{5-6}{5+6} \right) + \frac{5-4}{5+4} \right) = -\frac{7}{295} \\
8 & \frac{1}{11} \left( \frac{4-6}{4+6} \right) = -\frac{2}{7} \\
9 & \frac{1}{12} \left( 3 \left( \frac{6-5}{6+5} \right) + \left( \frac{6-1}{6+1} \right) \right) = \frac{38}{231} \\
\end{array}
\]

Figure 1. A graph with distinct leverage centralities

Intuitively one would think that the sum of the leverage centralities over a graph would be zero. This is in fact the case when a graph is regular. However, for non-regular graphs the sum of leverage centralities is negative. This arises since each edge between two vertices of different degrees contributes a negative amount to the sum of the leverage centralities. Let $G$ be the graph $K_3$ with a pendant edge (see Figure 2).
Then \( l(v_1) = \frac{1}{2} \left( \frac{3}{2} - \frac{3}{2} \right) + \frac{1}{2} \left( \frac{3}{2} - \frac{3}{2} \right); \ l(v_2) = \frac{1}{3} \left( \frac{3}{2} - \frac{3}{2} \right) + \frac{1}{3} \left( \frac{3}{2} - \frac{3}{2} \right); \ l(v_3) = \frac{1}{3} \left( \frac{3}{2} - \frac{3}{2} \right); \) and \( l(v_4) = \frac{1}{3} \left( \frac{3}{2} - \frac{3}{2} \right) + \frac{1}{3} \left( \frac{3}{2} - \frac{3}{2} \right). \) We can regroup the sum to be \( \sum_{v \in G} l(v) = \frac{1}{2} \left( \frac{3}{2} - \frac{3}{2} \right) + \frac{1}{2} \left( \frac{3}{2} - \frac{3}{2} \right) + \frac{1}{3} \left( \frac{3}{2} - \frac{3}{2} \right) + \frac{1}{3} \left( \frac{3}{2} - \frac{3}{2} \right). \)

Since the first three parts are negative and the last part is zero, the sum must be negative.

**Proposition 3** For any graph \( G, \sum_{v \in G} l(v) \leq 0. \)

**Proof.** If \( G \) is a regular graph, then \( l(v) = 0 \) for all \( v \), and hence \( \sum_{v \in G} l(v) = 0. \) If \( G \) is not regular, there must exist an edge \( e \) with end vertices \( u \) and \( v \) where \( d(u) > d(v). \) We note that the contribution of each edge \( uv \) to the sum of the leverage centralities is 
\[
\frac{1}{d(u)} \left( \frac{d(u) - d(v)}{d(u) + d(v)} \right) - \frac{1}{d(v)} \left( \frac{d(u) - d(v)}{d(u) + d(v)} \right) < 0.
\]
Hence for a non-regular graph, the sum of the leverage centralities is 
\[
\sum_{v \in G} l(v) = \sum_{(u,v) \in G} \frac{1}{d(u)} \left( \frac{d(u) - d(v)}{d(u) + d(v)} \right) - \frac{1}{d(v)} \left( \frac{d(u) - d(v)}{d(u) + d(v)} \right) < 0.
\]

# 2 Vertices with positive / negative leverage centrality

A vertex of lowest degree cannot have a positive leverage centrality and a vertex of highest degree cannot have a negative leverage centrality. However it is possible to have all the vertices in a graph except for one to have negative leverage centrality, or all but one have positive leverage centrality. The star graph \( K_{1,n-1} \) has \( n-1 \) vertices with negative leverage centrality. We show in the next theorem there exist graphs where there are \( n-1 \) vertices with positive leverage centrality.

**Theorem 4** The maximum number of vertices with positive leverage centrality is \( n - 1. \)

**Proof.** Since the sum of leverage centralities over all vertices in a graph is less than or equal to zero, it is impossible for a graph to have \( n \) vertices with positive leverage centrality. Let \( G \) be a graph with vertices \( v_1, ..., v_n \), where \( n \geq 11 \), and edges \( \{v_iv_j | 1 \leq i < j \leq n-4\} \cup \{v_i | 1 \leq i \leq n-4 \} \cup \{v_{n-3}v_{n-1} \mid n-3 \leq i \leq n-1\}. \) We note that \( \text{deg}(v_i) = n-2 \) for \( 1 \leq i \leq n-4 \), \( \text{deg}(v_i) = n-3 \), for \( n-3 \leq i \leq n-1 \), and \( \text{deg}(v_n) = 3. \) Then \( l(v_i) > 0 \) for all \( 1 \leq i \leq n-4 \) since these vertices have the largest degree in \( G. \) Then for \( n-3 \leq i \leq n-1, l(v_i) = \frac{1}{n-3} \left( (n-4) - \frac{n-3}{n-3} - \frac{n-3}{n-3} \right) = \frac{1}{n(n-3)} (n-10). \) Here \( l(v_i) > 0 \) whenever \( n \geq 11. \) Hence we have \( n-1 \) vertices with positive leverage centrality.

We present a second example. Let \( G \) be a graph with \( n \geq 12 \) vertices \( v_1, v_2, ..., v_n \) and edges: \( \{v_iv_j | 1 \leq i < j \leq n-5\} \cup \{v_iv_j | 1 \leq i \leq n-5 \} \cup \{v_{n-4}v_{n-2} \} \cup \{v_{n-3}v_{n-1} \} \cup \{v_iv_n | n-3 \leq i \leq n-1\}. \) It is clear that \( l(v_i) > 0 \) for all \( 1 \leq i \leq n-5 \) since these vertices have the maximum degree. Then for \( n-4 \leq i \leq n-1, l(v_i) = \frac{1}{n-4} \left( (n-5) - \frac{n-3}{n-3} - \frac{n-3}{n-3} \right) = \frac{n^2 - 15n + 40}{2n^2 - 9n + 4n + 15}, \) which is positive when \( n > 11.531. \)

## 2.1 Leverage Centrality vs. Degree Centrality

Degree centrality weights a vertex based on its degree. A vertex with higher (lower) degree is deemed more (less) central. This property has been well-studied (for early works see Czepiel [1], Fauchaux and Moscovici [2], Freeman [3], Garrison, [4], Hanneman and Newman [5], Kajitani and Maruyama [7], Mackenzie [9], Nieminen [10], Pitts [12], Rogers [13], and Shaw [14]). For some families of graphs the leverage centrality and degree centralities of vertices are closely related. For example, in scale-free networks where the distribution of degrees follows the power law, vertices with large degree will be adjacent to many vertices with much lower degrees. Hence the leverage centrality of these vertices will also be high.
However, for other families of graphs leverage centrality and degree centrality are not closely related. We show in the following example it is possible to construct infinite families of graphs where the vertex of largest degree does not have the highest leverage centrality. We do this by connecting nearly complete graphs as shown in Figure 3.

\[ \text{Figure 3. A family of connected nearly complete graphs} \]

For all \( n \geq 5 \), we have \( \text{deg}(u) > \text{deg}(v) \), however \( l(u) < l(v) \).

Let \( u \) be a vertex in \( K_{n+1} \) that has a neighbor vertex on the \( K_n \) graph. Then, \( \text{deg}(u) = n \) and as \( n \to \infty \), it follows that \( \text{deg}(u) \to \infty \). Let \( v \) be the vertex that is the base of the claw graph found on the right side of the graph shown in Figure 2. Thus, the degree of \( v \) will always equal 4 and therefore, for all \( n \geq 5 \), \( \text{deg}(u) > \text{deg}(v) \).

Since we know the degree of the neighbors of \( u \), we can calculate the leverage centrality of \( u \) as shown:

\[
\begin{align*}
l(u) &= \frac{1}{n} \left( \frac{n - (n - 1)}{n + (n - 1)} + (n - 1) \left( \frac{n - n}{n + n} \right) \right) = \frac{1}{2n^2 - n}.
\end{align*}
\]

Thus, if we take the limit of the leverage centrality of \( u \) as \( n \to \infty \) we get:

\[
\lim_{n \to \infty} \frac{1}{2n^2 - n} = 0.
\]

We can also calculate the leverage centrality of \( v \):

\[
\begin{align*}
l(v) &= \frac{1}{4} \left( \frac{4 - 2}{4 + 2} + (3) \left( \frac{4 - 1}{4 + 1} \right) \right) = \frac{8}{15}.
\end{align*}
\]

Since the leverage centrality of \( u \) converges to 0 as \( n \to \infty \), and the leverage centrality of \( v \) is equal to \( \frac{8}{15} \), then \( l(u) > l(v) \forall n \geq 5 \).

### 2.2 Leverage Centrality Zero

We note that bounds given in Lemma 2 are tight for regular graphs, where the leverage centrality of all vertices is zero. In fact, it is straightforward to show that \( l(v) = 0 \) for every vertex \( v \) if and only if \( G \) is a regular graph. It is also clear that for a vertex \( v \) with degree \( k \) that if all of the neighbors of \( v \) have degree \( k \), then \( l(v) = 0 \). However, it is possible for a vertex to have a leverage centrality of zero without all of its neighbors having the same degree as the original vertex. We investigate this property below.

**Example 5** Let \( G \) be a graph containing a vertex \( v \) of degree \( k \) where \( k - 1 \) of \( v \)'s neighbors have degree \( k = 2 \) and the remaining neighbor has degree 1. Then \( l(v) = \frac{1}{k} \left( \frac{k-1}{k+1} + (k - 1) \left( \frac{k-(k+2)}{k+(k+2)} \right) \right) = 0 \).

We also give an example of a graph with a vertex \( v \) whose neighbors all have distinct degrees and \( l(v) = 0 \).

**Example 6** Let \( G \) be a graph containing a vertex \( v \) of degree 3 and the neighbors of \( v \) have degrees 1, 2, and 17. The leverage centrality of \( v \) is \( l(v) = \frac{1}{3} \left( \frac{3-1}{3+1} + \frac{3-2}{3+2} + \frac{3-17}{3+17} \right) = 0 \).
It would be an interesting problem indeed to determine necessary and sufficient conditions for a vertex \( v \) to have leverage centrality zero, particularly when the neighbors of \( v \) all have distinct degrees. A computer search gives several examples for vertices with small degree.

| \( d(v) \) | degrees of the neighbors of \( v \) |
|-------|------------------|
| 7     | 1, 2, 3, 7, 11, 33, 77 |
| 7     | 1, 2, 3, 9, 11, 33, 41 |
| 3     | 1, 2, 17           |
| 3     | 1, 3, 9           |
| 4     | 1, 2, 5, 41       |
| 5     | 1, 2, 4, 13, 37   |
| 5     | 1, 2, 5, 10, 37   |
| 5     | 1, 3, 5, 7, 35    |
| 6     | 1, 2, 3, 6, 36, 66 |

| degrees of the neighbors of \( v \) |
|------------------|
| 7, 1, 2, 3, 7, 11, 33, 77 |
| 7, 1, 2, 3, 9, 11, 33, 41 |
| 1, 2, 17           |
| 1, 3, 9           |
| 1, 2, 5, 41       |
| 1, 2, 4, 13, 37   |
| 1, 2, 5, 10, 37   |
| 1, 3, 5, 7, 35    |
| 1, 2, 3, 6, 36, 66 |

3 Complete Multipartite Graphs

We use \( K_{t_1, t_2, \ldots, t_r} \) to denote the complete multipartite graph with parts of sizes \( t_1, t_2, \ldots, t_r \) and each vertex in a part is adjacent to every vertex in each of the other parts. As noted in [8] for vertices in the star graph \( K_{1, n-1} \) the leverage centrality meets the two extremes. The vertex in a part by itself has leverage centrality 

\[
\frac{1}{n-1} (n-1) \left( \frac{n-1}{n-1} \right) = 1 - \frac{2}{n}
\]

and all other vertices have a leverage centrality of 

\[
\frac{1}{r} \left( \frac{1-(n-1)}{1+(n-1)} \right) = -1 + \frac{2}{n}.
\]

We can extend the same idea to the general case of complete multipartite graphs. We will use \( G = K_{t_1, t_2, \ldots, t_r} \) to denote a complete multipartite graph with \( r \) parts \( n_1, n_2, \ldots, n_r \) where each part \( n_i \) has order \( t_i \) for all \( 1 \leq i \leq r \).

**Theorem 7** Let \( G = K_{t_1, t_2, \ldots, t_r} \) where \( t_i \) is the order of part \( n_i \). Then

\[
l(v_i) = \frac{1}{\sum_{j \neq i} t_j} \left( \sum_{k \neq i} t_k \left( \frac{t_k - t_i}{\sum_{j \neq i} t_j + \sum_{j \neq k} t_j} \right) \right)
\]

**Proof.** Let \( v_i \) be a vertex in part \( n_i \) with degree \( \sum_{j \neq i} t_j \). Due to the nature of a complete multipartite graph, it follows that \( v_i \) will have \( t_1 \) neighbors in part \( n_1 \), \( t_2 \) neighbors in part \( n_2 \), \( t_3 \) neighbors in part \( n_3 \), and the pattern continues for all \( 1 \leq i \leq r \) groups. Note that every vertex \( v_k \in n_k \) will have degree \( \sum_{j \neq k} t_k \). Thus the leverage centrality of \( v_i \) can be calculated as follows:

\[
l(v_i) = \frac{1}{\sum_{j \neq i} t_j} \left( t_1 \left( \frac{\sum_{j \neq i} t_j - \sum_{j \neq 1} t_j}{\sum_{j \neq i} t_j + \sum_{j \neq 1} t_j} \right) + t_2 \left( \frac{\sum_{j \neq i} t_j - \sum_{j \neq 2} t_j}{\sum_{j \neq i} t_j + \sum_{j \neq 2} t_j} \right) + \cdots + t_r \left( \frac{\sum_{j \neq i} t_j - \sum_{j \neq r} t_j}{\sum_{j \neq i} t_j + \sum_{j \neq r} t_j} \right) \right)
\]

\[
= \frac{1}{\sum_{j \neq i} t_j} \left( \sum_{k \neq i} t_k \left( \frac{\sum_{j \neq i} t_j - \sum_{j \neq k} t_j}{\sum_{j \neq i} t_j + \sum_{j \neq k} t_j} \right) \right)
\]

\[
= \frac{1}{\sum_{j \neq i} t_j} \left( \sum_{k \neq i} t_k \left( \frac{t_k - t_i}{\sum_{j \neq i} t_j + \sum_{j \neq k} t_j} \right) \right)
\]

This completes the proof. ■
4 Cartesian Product of Graphs

Definition 8 Given a graph $F$ with vertex set $V(F)$ and edge set $E(F)$, and a graph $H$ with vertex set $V(H)$ and edge set $E(H)$ we let $G$ define the Cartesian Product of $F$ and $H$ to be the graph $G = F \times H$ which is defined as follows: $V(G) = \{(u,v)| u \in V(F) \text{ and } v \in V(H)\}$ and $E(G) = \{(u_1,v_1),(u_2,v_2) \text{ where } u_1 = u_2 \text{ and } (v_1,v_2) \in E(H) \text{ or } v_1 = v_2 \text{ and } (u_1,u_2) \in E(F)\}$. We use $\times G_i$ to denote the Cartesian product of $m$ copies of a graph $G_i$.

We next present an elementary result from graph theory.

Lemma 9 If $G = F \times H$, then the degree of a vertex $(u,v)$ in $G$ is the sum of the degrees of vertices $u$ and $v$, where $u \in V(F)$ and $v \in V(H)$.

Theorem 10 Let $G$ be a graph and let $G_r$ be a regular graph where each vertex has degree $r$. Let $u \in V(G_r)$ and let $v_i$ and $v_j$ be vertices in $G$ with degrees $k_i$ and $k_j$ respectively. For each vertex $(u,v_i) \in V(G \times G)$ we have

$$l(u,v_i) = \frac{1}{r + k_i} \sum_{j \neq i} \frac{k_i - k_j}{2r + k_i + k_j}.$$ 

Proof. Consider a vertex $(u,v_i) \in V(G \times G)$. We note that $\deg((u,v_i)) = \deg(u) + \deg(v_i) = r + k_i$.

Then

$$l(u,v_i) = \frac{1}{r + k_i} \sum_{j \neq i} \frac{(r+k_i)-(r+k_j)}{2r+k_i+k_j} = \frac{1}{r + k_i} \sum_{j \neq i} \frac{k_i-k_j}{2r+k_i+k_j}. \quad \blacksquare$$

Corollary 11 Let $(u,v_i)$ be a vertex in $K_m \times G$ where $u \in V(K_m)$ and $v_i \in V(G)$. Then for all $(v_i,v_j) \in E(G)$

$$l(u,v_i) = \frac{1}{(m-1)+\deg(v_i)} \sum_j \frac{(m-1)+\deg(v_i)-((m-1)+\deg(v_j))}{(m-1)+\deg(v_i)+((m-1)+\deg(v_j))}$$

$$= \frac{1}{(m-1)+\deg(v_i)} \sum_j \frac{\deg(v_i)-\deg(v_j)}{2(m-2)+\deg(v_i)+\deg(v_j)}.$$ 

Proof. By Lemma 9 we have that $\deg((u,v_i)) = m - 1 + \deg(v_i)$ and for all neighbors $v_j$ of vertex $v_i$ we have that $\deg((u,v_j)) = m - 1 + \deg(v_j)$. The result then follows. \hfill \blacksquare

4.1 Cartesian Products of $P_n$

In this section we will consider the lattice, $\times P_n$. As the calculation of the degrees of vertices in a lattice is straightforward we will present results only involving the degrees without proof. We continue with some definitions.

Definition 12 Any vertex of $\times P_n$ can be defined by an $m$-tuple:

$$v = (v_1,v_2,\ldots,v_m) \text{ such that } v_i \in \{1,\ldots,n\} \text{ } \forall i \in \{1,\ldots,m\}.$$ 

Definition 13 We define a corner vertex of $\times P_n$ to be

$$v_c = (v_1,v_2,\ldots,v_m) \text{ such that } v_i \in \{1,n\} \text{ } \forall i \in \{1,\ldots,m\}.$$ 

A non-corner vertex is a vertex $v = (v_1,v_2,\ldots,v_m)$ of $\times P_n$ such that at least one $v_i \in \{2,\ldots,n-1\}$.
An inner corner vertex of $\times P_n$ is defined as follows.
\[
v_{ic} = (v_1, v_2, \ldots, v_m)
\text{ such that } v_i \in \{2, n-1\} \quad \forall i \in \{1, \ldots, m\}.
\]
It follows by definition that all vertices that are inner corner vertices are also non-corner vertices.

We note that
\[
\deg(v) = \sum_{i=1}^m x_i \quad \text{such that } x_i = \begin{cases} 
1 & \text{if } v_i \in \{1, n\} \\
2 & \text{if } v_i \in \{2, \ldots, n-1\}
\end{cases}
\]
We also observe that neighbor $v'$ of vertex $v = (v_1, v_2, \ldots, v_m)$ is defined as $v' = (v'_1, v'_2, \ldots, v'_m)$ such that $v'_i = v_i$ for $m-1$ elements of $(v'_1, v'_2, \ldots, v'_m)$ and
\[
|v'_i - v_i| = 1
\]
for the remaining element of the $m$-tuple $(v'_1, v'_2, \ldots, v'_m)$. Notice, there are two special cases for this remaining element. If the remaining element $v_i = 1$, then $v'_i = 2$ and if the remaining element $v_i = n$, then $v'_i = n-1$.

4.1.1 General Lemmas

We begin with a basic result involving the degrees of vertices and its neighbors in a lattice.

Lemma 14 Let $G$ be a lattice $\times P_n$. Any vertex adjacent to a vertex with degree $k$ must have degree $k-1$, $k$, or $k+1$.

4.1.2 Extreme Leverage Centralities

We next identify vertices with the minimum and maximum leverage centralities. We will show that the vertices with the minimum leverage centrality are the corners and the vertices with the maximum leverage centrality are the inner corners. Furthermore, we will show that for any vertex $v$ in the lattice $G = \times P_n$,
\[
\frac{1}{2m+1} \leq l(v) \leq \frac{1}{8m-2}.
\]

Minimum Leverage Centrality We first characterize the vertices with the minimum leverage centrality. We begin by stating two elementary lemmas involving degrees of vertices in a lattice.

Lemma 15 Any corner vertex $v_c$ in $G = \times P_n$ will have a degree of $m$. Furthermore, each neighbor of $v_c$ will have degree of $m + 1$.

Lemma 16 Let $G$ be the lattice $\times P_n$. A vertex $v$ that is non-corner vertex of $G$ must have at least one neighbor $u$ such that:
\[
\deg(u) \leq \deg(v).
\]

Theorem 17 Let $v_c$ be a corner vertex of $G = \times P_n$. Then,
\[
l(v_c) = -\frac{1}{2m+1}.
\]

Proof. By Lemma 15 we have that for $G = \times P_n$, $\deg(v_c) = m$ and that for a neighbor $u$ of $v_c$, $\deg(u) = m + 1$. We can compute the leverage centrality of $v_c$ with Definition 4.
\[
l(v_c) = \frac{1}{m} \sum_{i=1}^m \frac{m - (m + 1)}{m + (m + 1)} = \frac{1}{m} \sum_{i=1}^m \frac{-1}{2m+1} = \frac{1}{m} \cdot m \left( -\frac{1}{2m+1} \right) = -\frac{1}{2m+1}.
\]
Theorem 18 (Minimum Leverage Centrality) Let \( u \) be any vertex in \( G = \times P_n \) that is not a corner vertex and let \( v_c \) be a corner vertex in \( G \). Then, \( l(v_c) < l(u) \).

Proof. Let \( v \) be a non-corner vertex in \( G \) with degree \( k \). We know from Lemma [16] that at least one adjacent node has degree at most \( k \). We know from Lemma [14] that the remaining adjacent nodes can have degree at most \( k + 1 \).

Let \( v \) have one adjacent node with degree \( k \) and \( k - 1 \) adjacent nodes with degree \( k + 1 \). We now calculate the leverage centrality of \( v \).

\[
l(v) = \frac{1}{k} \left( \frac{k - (k + 1)}{k + (k + 1)} \cdot (k - 1) + \frac{k - k}{k + k} \right) = \left( \frac{1 - k}{k(2k + 1)} \right).
\]

From Theorem [17] we have that for a corner vertex \( v_c \) of degree \( k \), the leverage centrality is:

\[
l(v_c) = \left( \frac{1}{2k + 1} \right).
\]

Given that the degree of any adjacent node must be greater than 0, we know that \( 0 < \frac{k - 1}{k(2k + 1)} < 1 \). It follows that \( \frac{1 - k}{k(2k + 1)} < \frac{1 - k}{k(k + 1)} \) and hence \( l(v_c) < l(v) \).

If the neighbors of any non-corner vertex \( u \) differ from that of \( v \), then it follows from our construction of \( v \) and Lemma [14] that for any corresponding neighbors \( u_i \) from \( u \) and \( v_i \) from \( v \), that \( \deg(u_i) \leq \deg(v_i) \) and hence, \( l(v_i) \leq l(u_i) \). So we have that \( l(v_c) < l(v) \leq l(u) \).

This implies that \( l(v_c) < l(u) \) which completes the proof. ■

Maximum Leverage Centrality. We next characterize the vertices with the largest leverage centrality, beginning with two elementary results involving degrees of vertices in a lattice.

Lemma 19 Let \( v_{ic} \) be an inner vertex of \( G = \times P_n \). Then \( v_{ic} \) has \( 2m \) neighbors, such that \( m \) neighbors have degree \( 2m \) and the remaining \( m \) neighbors have degree \( 2m - 1 \).

Lemma 20 Let \( v \) be a vertex in \( G = \times P_n \). Then, \( \deg(v) \leq 2m \).

Theorem 21 (Maximum leverage centrality) Let \( u \) be a vertex in \( G = \times P_n \) that is not an inner corner vertex of \( G \), and let \( v_{ic} \) be an inner corner vertex in \( G \). Then, \( l(u) < l(v_{ic}) \). Furthermore, \( l(v_{ic}) = \frac{1}{8m^2 - 2} \).

Proof. Let \( v_{ic} \) be an inner corner vertex of \( G \). We have that \( v_{ic} = (v_1, v_2, \ldots, v_m) \) such that \( v_i \in \{2, n - 1\} \) \( \forall i \in \{1, \ldots, m\} \).

By Lemma [19] we know that \( \deg(v_{ic}) = 2m \). We are also given that \( m \) neighbors of \( v_{ic} \) have degree \( 2m \) and that \( m \) neighbors of \( v_{ic} \) have degree \( 2m - 1 \). The leverage centrality of \( v_{ic} \) is

\[
l(v_{ic}) = \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - 2m}{2m + 2m} \right] + \left[ \frac{2m - (2m - 1)}{2m + (2m - 1)} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] \cdot m \text{ terms}
\]

By rearranging terms we get:

\[
l(v_{ic}) = \frac{1}{2m} \left[ \frac{(2m - 2m)}{2m + 2m} + \frac{(2m - (2m - 1))}{2m + (2m - 1)} \right] + \cdots + \left[ \frac{(2m - 2m)}{2m + 2m} + \frac{(2m - (2m - 1))}{2m + (2m - 1)} \right] \cdot m \text{ terms}
\]
By distributing $\frac{1}{2m}$ we get that each term of the sum for $l(v_{ic})$ can be expressed as:

$$
\frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right]
$$

and since there are $m$ terms in the sum, we can express $l(v_{ic})$ as:

$$
l(v_{ic}) = m \cdot \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right]
$$

(2)

We simplify this to get:

$$
l(v_{ic}) = m \cdot \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] = \frac{1}{2} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] = \frac{1}{8m - 2},
$$

which proves the second part of the theorem.

Let $u$ be a vertex in $G$ that is not an inner corner vertex of $G$. We have that $\exists u^*_i \in u = (u_1, u_2, \ldots, u_m)$ such that $u^*_i \in \{1, 3, \ldots, n - 2, n\}$. Without loss of generality, we can assume that $u_i = v_i$ when $u_i \neq u^*_i$ and thus $u$ and $v_i$ differ only in one element, $u^*_i \in u$ and $v^*_i \in v_{ic}$ where $u^*_i \neq v^*_i$.

We see that two cases arise in calculating the leverage centrality of $u$.

(i) Let $u^*_i \in \{1, n\}$ and $v^*_i \in \{2, n - 1\}$

By Lemma [19] we have that $\deg(u) = 2m - 1$. In calculating the leverage centrality of $u$, we see that $l(u)$ and $l(v_{ic})$ can differ only in one term of Equation [1] such that:

$$
l(u) = (m - 1) \cdot \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] + \frac{1}{2m - 1} \left[ \frac{(2m - 1) - 2m}{(2m - 1) + 2m} \right]
$$

$$
l(v_{ic}) = (m - 1) \cdot \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] + \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right]
$$

Let $q = (m - 1) \cdot \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right]$.

Then $l(u) = q + \frac{1}{2m - 1} \left[ \frac{(2m - 1) - 2m}{(2m - 1) + 2m} \right] = q - \frac{1}{2m - 1} \left[ \frac{1}{4 - m} \right] = \frac{2 - m}{2m(1 - 4m)}$.

and $l(v_{ic}) = q + \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] = \frac{m - 1}{2m(4 - m)}$.

For the differing terms for the expressions for leverage centrality of $u$ and $v_{ic}$ we see that

$$
- \frac{1}{2m - 1} \left[ \frac{1}{4 - m} \right] < \frac{1}{2m} \left[ \frac{1}{4 - m} \right]
$$

and it follows that

$$
l(u) < l(v_{ic}).
$$

(ii) If $u^*_i \in \{3, n - 2\}$ and $v^*_i \in \{2, n - 1\}$

By Lemma [19] we know that $\deg(u) = 2m$. In calculating the leverage centrality of $u$, we see that $l(u)$ and $l(v_{ic})$ can differ only in one term of Equation [1] such that:

$$
l(u) = (m - 1) \cdot \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] + \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} \right]
$$

$$
l(v_{ic}) = (m - 1) \cdot \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] + \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right]
$$

Let $q = (m - 1) \cdot \frac{1}{2m} \left[ \frac{2m - 2m}{2m + 2m} + \frac{2m - (2m - 1)}{2m + (2m - 1)} \right] = \frac{m - 1}{2m(4m - 1)}$.
From the proof of Case (i), we already have \( l(v_{ic}) \)

\[
l(u) = q + \frac{1}{2m} \left( \frac{2m - 2m}{2m + 2m} \right) = q, \text{ and}
\]

\[
l(v_{ic}) = q + \frac{1}{2m} \left( \frac{1}{4m - 1} \right).
\]

For the differing terms for the expressions for leverage centrality of \( u \) and \( v_{ic} \) we see that

\[
0 < \frac{1}{2m} \left( \frac{1}{4m - 1} \right)
\]

and it follows that

\[
l(u) < l(v_{ic}).
\]

In both Cases (i) and (ii), we find that \( l(u) < l(v_{ic}) \) which proves that first part of the theorem and completes the proof. ■

4.1.3 Convergence of Leverage Centrality as \( m \to \infty \)

We next consider the leverage centrality of different vertices as the number of dimensions is increased.

**Theorem 22** As the number of paths in the Cartesian product increases \( (m \to \infty) \), the leverage centralities of all of the vertices of \( G = \times_{m} P_n \) converge to 0.

**Proof.** Let \( G = \times_{m} P_n \). From Theorem 21 we know that for any \( m \), the maximum leverage centrality of any vertex \( v \) of \( G \) is:

\[
\max(l(v)) = \frac{1}{8m - 2}
\]

From Theorem 18 we know that for any \( m \), the minimum leverage centrality of any vertex \( v \) of \( G \) is:

\[
\min(l(v)) = -\frac{1}{2m + 1}.
\]

Therefore, for any vertex \( v \) in \( G \) the leverage centrality is bounded as follows:

\[
-\frac{1}{2m + 1} \leq l(v) \leq \frac{1}{8m - 2}.
\]

We see that

\[
\lim_{m \to \infty} \left( -\frac{1}{2m + 1} \right) = \lim_{m \to \infty} \left( \frac{1}{8m - 2} \right) = 0.
\]

It follows that

\[
\lim_{m \to \infty} l(v) = 0.
\]

which completes the proof. ■

5 Leverage Centralities in Lattices and Triangle Numbers

In this section we investigate the number of distinct leverage centralities for lattices and show there is a surprising connection to the triangle numbers \( \binom{m+2}{2} \) where \( m \geq 1 \). We can label the vertices of \( \times_{m} P_n \) with using \( m \)-tuples where \( v = (v_1, v_2, \ldots, v_m) \) such that \( v_i \in \{1, \ldots, n\} \ \forall i \in \{1, \ldots, m\} \). For simplicity we will denote \( v_{r,s,t} \) by \( (r, s, t) \).
There are three distinct leverage centralities for $P_n$ where $n \geq 5$. Let $V(P_n) = v_1, v_2, \ldots, v_n$ where $n \geq 5$. Then $l(v_1) = l(v_n) = -\frac{1}{5}$; $l(v_2) = l(v_{n-1}) = \frac{1}{5}$; and $l(v_i) = 0$ for all other $v_i$.

For $P_n \times P_n$ where $n \geq 5$, we have six different leverage centralities:

$$l(1,1) = \frac{-1}{5}, \ l(1,2) = \frac{-1}{5}, \ l(1,3) = \frac{-1}{5}, \ l(2,2) = \frac{-1}{5}, \ l(2,3) = \frac{-1}{5}, \text{ and } l(3,3) = 0.$$  

For $P_n \times P_n \times P_n$ where $n \geq 5$, we have ten different leverage centralities:

$$l(1,1,1) = \frac{1}{4}, \ l(1,1,2) = \frac{1}{252}, \ l(1,1,3) = \frac{1}{4}, \ l(1,2,2) = \frac{1}{35}, \ l(1,2,3) = \frac{1}{35}, \ l(2,2,2) = \frac{1}{14}, \ l(2,3,3) = \frac{1}{14}, \text{ and } l(3,3,3) = 0.$$  

By symmetry, we need only consider vertices with coordinates $1 \leq v_i \leq 3$ and $v_i \leq v_{i+1}$ for all $1 \leq i \leq m - 1$. It is straightforward to count the number of different combinations of a degree of a vertex and the degrees of its neighbors. We need only count the number of solutions to the equation $x_1 + x_2 + x_3 = m$ where $x_i$ is the number of times $i$ appears in the coordinate. This can be done using the following lemma.

We next restate a well-known combinatorial formula.

**Lemma 23** The number of solutions to $x_1 + x_2 + \cdots + x_n = m$ where each $x_i \in \mathbb{N}$ is $\binom{n+m-1}{m-1}$.

Using Lemma 23, the number of solutions to this equation is the $(m+1)$-st triangle number, $\binom{m+2}{2}$.

Hence we have the following upper bound.

**Theorem 24** If $n \geq 5$ the number of distinct leverage centralities in $G = \times_{m} P_n$ is less than or equal to $\binom{m+2}{2}$.

For small cases of $m$ this bound is in fact tight. The first three cases have been shown above. In the next theorem we show that this holds for $m < 7$.

**Theorem 25** Let $k = \binom{m+2}{2}$ and $G = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_m}$ where $k_1 = k_2 = \cdots = k_m \geq 5$ with vertices $V = \{v_0, v_2, \ldots, v_{k-1}\}$.

1. If $t_j$ is the $j$th triangular number for $0 \leq j \leq m$ and $r = t_{j} + i$ where $0 \leq i \leq j$, then leverage centrality of $v_r$ is given by
   $$l(v_r) = \frac{1}{m+j} \left[ \frac{j-i}{2(m+j)-1} - \frac{(m-j)}{2(m+j)+1} \right].$$

2. The number of distinct leverage centralities in $G$ is less than or equal to $\binom{m+2}{2}$. Moreover, if $m < 7$ the equality holds.

**Proof.** We first prove Property 1. Let $v_r$ be $r$-th $n$-tuple that appears in the lexicographical ordering where each term is between 1 and 3 inclusive, i.e.,

$$v_1 = (1, 1, 1, \ldots, 1), v_2 = (1, 1, 1, \ldots, 2), v_3 = (1, 1, 1, \ldots, 3), \ldots, v_k = (3, 3, 3, \ldots, 3).$$

From this set of vertices $V = \{v_1, v_2, \ldots, v_k\}$ we can see that the degree of each vertex $v_r$ is $m+j$ where $r = t_j + i$ and $t_j$ is the $j$th triangular number. The degrees of the vertices adjacent to $v_r$ are as follows: $m-j$ vertices of degree $m+j+1$, there are $j-i$ vertices of degree $m+j$ and there are $j+i$ vertices of degree $m+j$. Therefore, for $0 \leq j \leq m$ and $0 \leq i \leq j$ the leverage centrality for each vertex $v_r$ is:

$$l(v_r) = \frac{1}{m+j} \left[ \frac{j-i}{2m+(2j)-1} - \frac{(m-j)}{2m+(2j)+1} \right].$$

In our proof of Property 2, we show that the leverage centralities of all vertices $v_r$ are distinct if $m < 7$. From a direct calculation on the formula found above the leverage centrality satisfies the following orders. The first three cases were covered at the beginning of Section 5.
1. If \( m = 4 \), then
\[
\begin{align*}
l(v_{m-1}) &< l(v_{m-3}) < l(v_{m-2}) < l(v_{m-2}+m-2) < l(v_{m-1}+m-1) < l(v_{m-1}+m) < l(v_{m-1}+2) < l(v_{m-2}) < l(v_{m-2}+l(v_{m-3}) < l(v_{m-1}+1) < l(v_{m-1}) < l(v_{m}).
\end{align*}
\]

2. If \( m = 5 \), then
\[
\begin{align*}
l(v_{m-1}) &< l(v_{m-4}) < l(v_{m-3}) < l(v_{m-2}) < l(v_{m-2}+m-1) < l(v_{m-1}) < l(v_{m-1}+m) < l(v_{m-1}+m-3) < l(v_{m-1}+m-2) < l(v_{m-1}+m-1) < l(v_{m-1}+m) < l(v_{m-1}+m-2) < l(v_{m-1}+m-1) < l(v_{m-1}) < l(v_{m-1}+1) < l(v_{m-1}+2) < l(v_{m-3}) < l(v_{m-1}) < l(v_{m-2}) < l(v_{m-2}+l(v_{m-3}) < l(v_{m-1}+1) < l(v_{m-1}+2) < l(v_{m-2}) < l(v_{m-2}+l(v_{m-3}) < l(v_{m-1}+1) < l(v_{m-1}+2) < l(v_{m-3}) < l(v_{m-1}+1) < l(v_{m-1}) < l(v_{m-1}+1) < l(v_{m}).
\end{align*}
\]

3. If \( m = 6 \), then
\[
\begin{align*}
l(v_{m-1}) &< l(v_{m-5}) < l(v_{m-4}) < l(v_{m-4}+2) < l(v_{m-4}+1) < l(v_{m-3}) < l(v_{m-3}+3) < l(v_{m-3}) < l(v_{m-3}+2) < l(v_{m-3}+1) < l(v_{m-3}) < l(v_{m-3}+2) < l(v_{m-3}) < l(v_{m-3}+1) < l(v_{m-3}+2) < l(v_{m-3}) < l(v_{m-2}+l(v_{m-3}) < l(v_{m-1}) < l(v_{m-1}+1) < l(v_{m-1}+2) < l(v_{m-1}+1) < l(v_{m-1}) < l(v_{m-1}+1) < l(v_{m}).
\end{align*}
\]

This completes the proof. ■

We have checked this computationally for all graphs \( \times P_n \) for the first \( m \leq 10 \) (10 dimensions) and have verified that there are exactly \( \binom{m+2}{2} \) distinct leverage centralities in each case. We state the general problem for all \( m \) as part of Conjecture 26.

In Theorem 24 we showed that the number of distinct leverage centralities in \( \times P_n \) is bounded above by \( \binom{m+2}{2} \). To show this bound is tight one would need to show that the \( \binom{m+2}{2} \) leverage centralities are all distinct. By [14] given a vertex \( v \) with degree \( k \), its neighbors must have degrees \( k-1 \), \( k \), or \( k+1 \). Suppose \( x \) of \( v \)'s neighbors have degree \( k-1 \) and \( y \) of \( v \)'s neighbors have degree \( k+1 \). Then the number of \( v \)'s neighbors with degree \( k \) is \( k - x - y \). Hence \( l(v) = \frac{1}{k} \left( \frac{x}{2k-1} - \frac{y}{2k+1} \right) \). One approach would be to show that
\[
\frac{1}{k_1} \left( \frac{x_1}{2k_1-1} - \frac{y_1}{2k_1+1} \right) = \frac{1}{k_2} \left( \frac{x_2}{2k_2-1} - \frac{y_2}{2k_2+1} \right) \Rightarrow k_1 = k_2, x_1 = x_2, y_1 = y_2.
\]
However this appears to be a complex problem.

We have also found that the number of distinct leverage centralities for graphs of the form \( \times P_n^k \) is linked to the polygonal numbers, which are numbers that can be represented by a regular geometrical arrangement of equally spaced points. For the first few cases, the triangle numbers are given by \( P_2(m) = \binom{m+1}{2} \), the tetrahedral numbers are given by \( P_3(m) = \binom{m+2}{3} \) and the pentalope numbers are given by \( P_4(m) = \binom{m+3}{4} \). In general, \( P_{k+1}(m) = \binom{m+k}{k+1} \). Since we do not consider a case of a single vertex, we start all our leverage centrality calculations with the second polygonal numbers. Hence the general formula translates to \( \binom{m+k+1}{k+1} \).

Based on our findings for small values of \( k \) we pose the following conjecture.

**Conjecture 26** Let \( n \geq 4k+1 \) and \( G = \times P_n^k \). Then the number of distinct leverage centralities in \( G \) is \( \binom{m+k+1}{k+1} \).

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