Superradiant quantum phase transition in a circuit QED system: a revisit from a fully microscopic point of view

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In order to examine whether or not the quantum phase transition of Dicke type exists in realistic systems, we revisit the model setup of the superconducting circuit QED from a microscopic many-body perspective based on the BCS theory with pseudo-spin presentation. By deriving the Dicke model with the correct charging terms from the minimum coupling principle, it is shown that the circuit QED system can exhibit superradiant quantum phase transition in the limit $N \to \infty$. The critical point could be reached at easiness by adjusting the extra parameters, the ratio of Josephson capacitance $C_J$ to gate capacitance $C_g$, as well as the conventional one, the ratio of Josephson energy $E_J$ to charging energy $E_C$.

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Introduction.—In the conventional Dicke model [1], which ignores two-photon interaction term $A^2$, the superradiant quantum phase transition (QPT) [2–4] can happen when the atom-radiation field coupling $g$ is strong enough. However, for the realistic systems with the minimum coupling from $U(1)$-gauge theory, the Thomas-Reiche-Kuhn (TRK) sum rule means that the $A^2$ term cannot be neglected when the increase of $A^2$ term follows the increase of $g$ [5, 6]. The $A^2$ term shifting the effective frequencies hence prevents the considered system from reaching the critical point, so that no superradiant QPT happens in the natural atom systems. This fact was stated as a no-go theorem for the realistic cavity QED systems.

On the contrary, it was ad hoc pointed out that this superradiant QPT could be realized in some artificial system [7]. Later on, P. Nataf and C. Ciuti showed that the circuit QED system consisting of a collection of Josephson atoms capacitively coupled to a transmission line resonator (TLR) is capable for such kind of QPT [8]. Viehmann et al questioned this judgment based on an overall microscopic model. They argued [9–11] that the phenomenological Hamiltonian used in Ref. [8] cannot adequately describe the superradiant QPT of the circuit QED system with large atom numbers; if all the degrees of freedom are considered properly, the no-go theorem of superradiant QPT based on the TRK sum rule still works for this artificial system. In this sense, they excluded the existence of the superradiant QPT in the circuit QED system.

Viehmann et al claimed that a fully microscopic approach was utilized by themselves, but it seems difficult to straightforwardly deduce the phenomenological Hamiltonian used in Ref. [8] from their overall microscopic model. In this paper, we try to carry out this necessary task to deduce it from a microscopic model with the minimum coupling form $U(1)$-gauge theory. To this end, we provide a description of the circuit QED system using the pseudo-spin representation [12] of the BCS theory, which explicitly displays the superconducting characteristics of the Josephson atoms. Our microscopic approach correctly gives the additional quadratic quantum voltage term, which is usually ignored in current references, e.g., [13]. Applying this result to the low excited ensemble of artificial atoms, we conclude that the no-go theorem in the cavity QED system could not rule out the superradiant QPT in the circuit QED system for some experimentally accessible parameters.

Microscopic modeling of superconducting circuit QED.—As a key element in the circuit QED system, as illustrated in Fig. I superconducting Josephson junction consists of a thin insulating barrier sandwiched
between two superconductors. Microscopically, we use the collective pseudo-spin operators

\[ S_\alpha = \frac{1}{\sqrt{2N}} \sum_k c_{-k,\alpha} c_{k,\alpha}, \]
\[ S_{z,\alpha} = \frac{1}{2} \sum_k \left( c_{-k,\alpha} c_{k,\alpha} + c_{-k,\alpha}^\dagger c_{k,\alpha} - 1 \right) \]  

(1)

to describe the charging (tunneling) process of Cooper pairs in the junction \([14, 15]\). Here, \(c_{k,\alpha}^\dagger (c_{k,\alpha})\) is the electron creation (annihilation) operator for the superconductor on the \(\alpha\)-hand side with \(\alpha = L, R\). The index \(k (\sim k)\) denotes the momentum of the electron with spin up (down), and it is summed over the energy shell \([\epsilon_F - \hbar \omega_D, \epsilon_F + \hbar \omega_D]\), which is around the Fermi energy \(\epsilon_F\) up to the Debye frequency \(\omega_D\). The normalization factor \(N\) equals to half of the number of momentum states within this energy shell. Obviously, \(S_{z,\alpha}\) is the operator counting the number of the Cooper pairs in excess of the electroneutrality of the superconductor on the \(\alpha\) side.

To model the tunneling process as the Josephson effect [12], the single electron tunneling Hamiltonian \(H_T = T \sum_{k,a} \left( c_{k,a}^\dagger c_{k,L} + h.c. \right)\) is re-expressed in terms of the collective pseudo-spin operators as

\[ H_T = -2\hbar N^2 T \left( S^+ + S^- \right), \]  

(2)

where we have ignored the single electron tunneling terms for the system in superconducting phase. Here, the operator \(S^- = S_L S_R^\dagger \left( S^+ = (S^-)^\dagger \right)\) denotes that a Cooper pair tunnels from the left (right) superconductor to the right (left), where the operator \(S_z = (S_{z,L} - S_{z,R})/2\) is defined and the commutation relations \([S^\pm, S_z] = \mp S^\pm\) and \([S^+, S^-] \approx 2S_z/N^2\) are fulfilled. Note that \(S_{z,L} + S_{z,R} = 0\) since the Josephson junction is electroneutral. \(S_z\) also represents the number of the excess Cooper pairs on the left bulk of superconductor. In the case of small number of the excess tunneling Cooper pairs, i.e., \(N \gg \langle S_z \rangle\), \(S^+/S^-\) is central to the algebra generated by \(S^\pm\) and \(S_z\) because of \([S^+, S^-] \rightarrow 0\) \([16, 17]\). Then the polar decomposition \(S^z = \exp(\pm i\theta)\) defines the macroscopic phase operator \(\theta\), which obeys \([\theta, S_z] = i\). Using the phase operator, we can rewrite the tunneling Hamiltonian in the conventional fashion \(H_T = -E_J \cos \theta\), where \(E_J = \Phi_0 I/2\pi\) is the Josephson energy, \(\Phi_0 = \hbar/2e\) is the quantized flux, and \(I = 8eN^2 T\) is the maximum tunneling current.

According to the reference [12], the microscopic meaning of \(\theta = \phi_L - \phi_R\) could be explained as the difference between the order parameters of the right- and left-hand superconductors in the BCS ground states \(|\phi_\alpha\rangle = \prod_k u_k + v_k \exp(\pm i\phi_\alpha) c_{-k,\alpha}^\dagger c_{k,\alpha} |0\rangle\), where \(\phi_\alpha\) is the common phase of superconductor. Then the tunneling current \(\langle J \rangle = I \sin (\phi_L - \phi_R)\) is obtained by the average of \(J = -2e [N_L, H_T]\) over the product state \(|\phi\rangle = |\phi_L\rangle \otimes |\phi_R\rangle\), where \(N_L\) is the number of the electrons on the left-hand superconductor which equals to 2\(S_z\) plus a constant.

Next we model the charging process for a simple Josephson device, which is a superconducting island [or Cooper pair box (CPB)] connected to a gate capacitor \(C_g\) and a bulk of superconducting electrode through a thin junction with capacitance \(C_J\). The geometry of the superconducting circuit is shown in Fig.1(a), where the voltage \(V_g\) is applied to the gate capacitor by a classical source. This device is also coupled to a quantized electromagnetic field provided by a superconducting TLR in a coplanar-waveguide geometry, which gives an additional quantum voltage \(V_q = V_q \left( a + a^\dagger \right)\) where \(a^\dagger (a)\) is the corresponding creation (annihilation) operator for the single mode of the TLR with eigenfrequency \(\omega_r\).

With the charges \(Q\) distributed on the island, the electrostatic potential \(V\) of the junction is determined by

\[ C_J V - C_g (V_g - V) = Q \]  

(3)

with the total gate voltage \(V_g = V_c + \hat{V}_g\). Initially, we assume no excess electron exists, i.e., \(Q = 0\), hence the potential of the CPB \(V_0 = C_g V_g / (C_g + C_J)\) is formally quantized as an operator. The total electrostatic energy \(U = C_g (V_0 - V_g)^2/2 + C_J V_0^2/2\) for both the gate and the Josephson capacitors connected to the electroneutral CPB is calculated as

\[ U = 4E_c n_g^2 C_J / C_g, \]  

(4)

where \(E_c = e^2/2C_g\) is the charging energy for a single electron, \(C_2 = C_J + C_g\) is the total capacitance and \(n_g = C_g V_g / 2e\).

After \(l\) excess electrons are added in the CPB, the total energy is the electrostatic energy \(U\) plus the work \(W\) done to tunneling Cooper pairs. \(W\) is actually the work cost by the excess electrons to cross the barrier, which is actually supplied by the voltage source. The corresponding potential \(V_1\) is calculated by substituting the excess charges \(Q = Q_l = -e \sum_{j=0}^l n_j\) into Eq.4, where \(n_j = c_{-j}^\dagger c_j\) is the single electron number operator. As we concern the charge accumulation process, the momentum states indexes \(k\) of the electron operators are of no importance. Instead, we assign to each electron operator a subscript \(j\) indicating the order of accumulating on the island.

According to classical electrodynamics, to add one more electron on the island with \(l\) excess electrons already on it, the work is calculated according to the formula \(W_l = -e n_{l+1} V_l\). This formula seems phenomenological, but now we can derive it from the minimum coupling principle based on \(U\) (1-gauge) theory with a single particle Hamiltonian \(H_c = [p - e A(\mathbf{x})]^2 / 2m - e \phi(\mathbf{x})\). It describes an electron moving in the vector potential \(A(\mathbf{x})\) and scalar potential \(\phi(\mathbf{x})\). Here we use the coulomb
gauge $\nabla \cdot \mathbf{A}(x) = 0$ and the dipole approximation with $\mathbf{A}(x) \approx \mathbf{A}(x_0) = \mathbf{A}_0$ and $\nabla \phi(x) \approx \nabla \phi(x)|_{x=x_0}$, which is consistent with the prerequisite of the discussion about the superradiant phenomenon in this paper. It leads to

$$H_e \approx \frac{1}{2m}(\mathbf{p}^2 + e^2 \mathbf{A}_0^2) + e \mathbf{x} \cdot \mathbf{E}_0,$$

where $\mathbf{E}_0 = -[\partial \mathbf{A}(x)/\partial t + \nabla \phi(x)]|_{x=x_0}$.

In second quantization, the field operators $\psi(x) = \sum_k \psi_k(x) c_k$ is used with $\psi_k(x)$ approximately being the plane wave. Then the energy cost of a single electron crossing from one electrodes to the other one at $d$ apart is calculated as

$$\int d^3x \left[ \hat{\psi}^\dagger(\mathbf{x}) H_e \hat{\psi}(\mathbf{x}) - \hat{\psi}^\dagger(\mathbf{x} + d) H_e \hat{\psi}(\mathbf{x} + d) \right] = e\bar{n}V,$$

where $V = d \cdot \mathbf{E}_0$ and $\bar{n} = \sum_k \psi_k^\dagger(x) c_k$. The momentum term $\mathbf{p}^2$ and quadratic vector potential term $\mathbf{A}_0^2$ are both canceled out, and the remaining term $-e\bar{n}V$ verifies our phenomenological form of $W_j$.

When $N$ excess electrons are added in the CPB, the total work is $W = \sum_{j=0}^{N-1} W_j$ is obtained as

$$W = 4E_C S_z^2 - 8E_C n_g S_z - 2E_C S_z,$$

where we use the fact $S_z = \sum_{j=0}^{N} n_j/2$. The linear term $2E_C S_z$ can be neglected because it will merely shift $n_g$ by 1/4, which can be adjusted by tuning the gate voltage without influence the further discussion.

At last, the charging Hamiltonian $H_C = U + W$ is explicitly written as

$$H_C = 4E_C(S_z - n_g)^2 + \mu n_g^2.$$  

We remark that the last term $\mu n_g^2 = 4E_C(C_1/C_g - 1)n_g^2$ was neglected in some current references [18][19], since it is a constant for the classical voltage and not related with the charges in the CPB. However, in the case of the gate voltage $V_g$ contains a quantized component, this term provides a nonzero quadratic voltage term, which is evidently crucial in determining whether the superradiant QPT exists.

**Superradiant QPT in the Dicke model based on circuit QED.**—Now we further consider the circuit QED system, as shown in Fig.4(b), with a TLR coupled to $N$ small junctions, which are modeled as the artificial atoms of two energy levels. The total Hamiltonian $H_{cir} = \sum_{j=1}^{N} H_j + h\omega_r a^\dagger a$ is defined by

$$H_j = 4E_C (S_{z,j} - n_g)^2 - E_J \cos \theta_j + \mu n_g^2,$$

which correctly includes the tunneling part Eq. (2) and the charging part Eq. (1). Obviously, $H_{cir}$ is very similar to the cavity QED system for the atoms interacting with cavity modes through minimum coupling. Generally, it is very hard in experiments to realize the strong atom-field coupling in the conventional cavity QED systems, but the strong coupling regime is feasible in the current experiments of the superconducting circuit QED. Therefore, the circuit QED system is more ideal to investigate the superradiant QPT.

At the degenerate point the Josephson junction behaves as a two-level system, and the total Hamiltonian reads

$$H_{cir} = \sum_{j=1}^{N} \left[ E_C - \frac{E_J}{2} \sigma_j^z + \frac{2}{e} E_C C_g V_g (a + a^\dagger) \sigma_j^x \right] + h\omega_r a^\dagger a + hD (a^\dagger + a)^2 + hF (a^\dagger + a),$$

where the correct two-photon term is included with $hF = \mu NC_g V_g/2e$ and $hD = NE_C C_g V_g^2/e^2$, $\sigma_j^x = |0\rangle_j \langle 1|_j$ and $\sigma_j^z = |0\rangle_j \langle 0|_j + |1\rangle_j \langle 1|_j$. Here, we neglect a constant $N(C_1/C_g - 1)/4$ as it is a pure $c$ number.

To study the superradiant phenomenon in this circuit QED system, it is necessary to explore the circumstance that the atom number $N$ is large and the total excitation number is low. In this case, the collective excitation operator

$$b^\dagger = \frac{1}{\sqrt{N}} \sum_j |e\rangle_j \langle g|$$

behaves as bosonic operator in the atomic quasi-spin wave [20], which is defined by the eigenstates $|e\rangle_j = i(|0\rangle_j - |1\rangle_j)/\sqrt{2}$ and $|g\rangle_j = (|0\rangle_j + |1\rangle_j)/\sqrt{2}$ of the $j$-th CPB. Then, $H_{cir}$ is rewritten as

$$H_{cir} = h\omega_r a^\dagger a + h\omega_r b^\dagger b - i\hbar \Omega (a^\dagger + a) (b^\dagger - b) + hD (a^\dagger + a)^2 + hF (a^\dagger + a).$$

Apparently, our circuit QED system is reduced into an equivalent system of two coupled harmonic oscillators (CHO) with frequencies $\omega_J = E_J/h$ and $\omega_r$, and the coupling strength is

$$\Omega = \frac{2\sqrt{N} E_C C_g V_g}{\hbar e}.$$  

Generally, we consider a CHO system with canonical coordinates $x_1$ and $x_2$, eigenfrequencies $\omega_1$ and $\omega_2$, and masses $m_1$ and $m_2$, respectively. If the coupling term were inappropriately chosen as $-gx_1 x_2$, the eigenvalues of the coupled system would be imaginary when the coupling strength $g$ is strong enough, specifically, when $g^2 > m_1 m_2 \omega_1^2 \omega_2^2$. Somebody describes this phenomenon as a kind of QPT, but the natural coupling in the conventional coupled CHO should be $g(x_1 - x_2)^2/2 = gx_1^2/2 + gx_2^2/2 - gx_1 x_2$, so the two quadratic coordinate terms renormalize the eigenfrequencies as $\tilde{\omega}_i = \sqrt{\omega_i^2 + g^2/m_i}$, $i = 1, 2$. Therefore, there would not be QPT, since $g^2 < m_1 m_2 \omega_1^2 \omega_2^2$ is always valid [21]. This is the very
reason that the correct non-linear term is particularly important in the discussion of QPT. However, in our present circuit QED system, there is no such intrinsic relation between the renormalized eigenfrequency and the effective coupling, thus it is possible to observe such kind of QPT phenomenon therein. We would like to point out our model Hamiltonian contains an additional linear term $hF(a + a^\dagger)$, which is introduced accompanying the correct two-photon term. It can be eliminated by displaced transformations $a = a + \eta$ and $\beta = b + \xi$ ($\eta$ and $\xi$ are c numbers). Then we diagonalize $H_{\text{circuit}}$ in a conventional way and obtain two eigenfrequencies as

$$\omega_{\pm}^2 = \frac{1}{2}\left[\Omega_{\text{c}}^2 \pm \sqrt{\Omega_{\text{c}}^4 + 16\Omega_{\text{r}}^2\omega_r\omega_j}\right],$$

(13)

where $\Omega_{\text{c}} = \sqrt{\omega_{\text{c}}^2 + 4D\omega_r}\pm\omega_j$. It is obvious that $\omega_+^2$ is always positive, while the $\omega_-^2$ can be negative when

$$1 - \kappa\gamma > \frac{\omega_j^2\omega_{\text{r}}}{4\Omega_{\text{c}}^2}.$$  

(14)

Here, we define two dimensionless parameters, the ratio of Josephson energy to charging energy $\kappa \equiv E_J/4E_C$ and the ratio of Josephson capacitance to gate capacitance $\gamma \equiv C_J/C_g$. It was proven in Ref.\cite{23} that $\omega_- = 0$ is the critical point of the superradiant QPT and the superradiant phase lies in the region that eigenfrequency $\omega_-$ of the system is imaginary, hence Eq.\cite{14} is actually the condition for the appearance of superradiant phase. As the coupling $\Omega$ increases with $\sqrt{N}$, the right hand side of Eq.\cite{14} approaches to positive infinitesimal in the limit $N \to \infty$. Thus the occurrence of the superradiant QPT depends on the condition $\kappa\gamma < 1$. The corresponding critical point of $\Omega$ is

$$\Omega_{\text{c}} = \frac{1}{2}\sqrt{-\frac{4\Omega_{\text{r}}^2\omega_j}{1 - \kappa\gamma}}.$$  

(15)

According to the above arguments, the superradiant QPT indeed can occur in principle, but we need to examine this conclusion for the realistic systems. In order to achieve a good charge qubit with small fluctuation of the Cooper pair number and low classical noise, it is usually chosen $\kappa \ll 1$ and $\gamma \gg 1$ in experiments. These two key factors, $\kappa$ and $\gamma$, compete in determining whether the superradiant QPT can take place. If the Josephson capacitance is large enough, such as $\gamma \approx 10^3$, and the ratio $\kappa$ is set around 0.4, the condition Eq.\cite{14} is violated thus the superradiant QPT cannot happen. In contrast, we can also choose $\kappa \ll 1$ and $\gamma \approx 1$ as in the experiment, which clearly allows the superradiant QPT.

Remarks and conclusion.—Pedantically, we need to understand why the Dicke-type superradiant QPT is allowed in the circuit QED system, while it is forbidden in the cavity QED system, since these two systems possess very similar Hamiltonians with the correspondences between canonical variables as listed in the table. This analogy apparently implies the superradiant QPT can happen neither in the circuit QED system nor the conventional cavity QED system. However, this argument obviously contradicts with the conclusion made above, as well with the analysis by Nataf et al.\cite{10}.

To solve this puzzle, we would like to consider whether or not there exists the correspondence between the basis vectors used for defining the collective operators in two systems. In cavity QED system, we use the two lowest eigenstates $|e\rangle$ and $|g\rangle$ of the total Hamiltonian $H_{\text{na}}$ of a natural atom to define a qubit subspace. Evidently, they are not the eigenstates of the momentum operator $p$ due to the existence of trapping potential. In the artificial atoms, however, though the electron pair number operator $S_z$ corresponds to momentum $p$, the two discrete eigenstates $|0\rangle$ and $|1\rangle$ of $S_z$ does not correspond to $|e\rangle$ and $|g\rangle$ respectively. It follows this observation that the collective operators of the natural atom ensembles and artificial atoms are of different types, and describe different types of quasi-excitations. Thus it is not surprising that the circuit QED system exhibits superradiant QPT while the cavity QED system does not.

In summary, we have theoretically explored the superradiant QPT in the circuit QED system, where $N$ Cooper pair boxes behaves as artificial atoms coupled to a single resonator mode. With the microscopic Hamiltonian based on the pseudo-spin representation of the BCS theory and the minimum coupling principle, we deduce the correct quadratic term $V_{\text{g}}^2$ of the gate voltage from a fully quantum perspective. Then we showed that the circuit QED system is capable for the superradiant QPT, and the critical point is determined by more parameters, the ratios $\kappa$ and $\gamma$. The QPT is morefeasible to be realized when these two ratios are small. We also explained the cavity and circuit QED systems show different collective behaviors is due to the superradiant phenomena in these two systems are based on different types of quasi-excitations.

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| natural atom | artificial atom |
|-------------|----------------|
| $H_{\text{na}} = p^2/2m + U(x)$ | $H_{\text{ar}} = 4E_C (S_z - 1/2)^2 - E_J \cos \theta$ |
| $x$ | $\theta$ |
| $p$ | $S_z - 1/2$ |
| $[x, p] = i\hbar$ | $[0, S_z - 1/2] = i$ |

Table I: The correspondence between the natural atom and the Josephson-type artificial atom.
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