A STRUCTURAL APPROACH TO THE LOCALITY OF PSEUDOVARIETIES OF THE FORM $\mathbf{LH} \bowtie \mathbf{V}$

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Dedicated to the memory of Bret Tilson

Abstract. We show that if $\mathbf{H}$ is a Fitting pseudovariety of groups and $\mathbf{V}$ is a local pseudovariety of monoids, then $\mathbf{LH} \bowtie \mathbf{V}$ is local if either $\mathbf{V}$ contains the six element Brandt monoid, or $\mathbf{H}$ is a non-trivial pseudovariety of groups closed under extension.

1. Introduction

Since the seminal work of Tilson [28], as well as further work of Tilson and Rhodes [17], Rhodes and Weil [18,19] and work of Pin, Straubing and Therien [13], it has become clear that finite categories play a crucial role in finite monoid and formal language theory. Tilson [28] defined a pseudovariety of monoids $\mathbf{V}$ to be local if the pseudovariety of categories generated by the elements of $\mathbf{V}$, viewed as one-object categories, consists precisely of those categories whose local monoids at each object belong to $\mathbf{V}$. Many important pseudovarieties of monoids are local, but not all; see [28] for more background.

Let $\mathbf{H}$ be a pseudovariety of groups. A pseudovariety of groups $\mathbf{H}$ is said to be Fitting [3] if whenever $G$ is a finite group and $H, K \triangleleft G$ are normal subgroups belonging to $\mathbf{H}$, then $HK \in \mathbf{H}$. For instance, pseudovarieties of groups closed under extension are Fitting; so is the pseudovariety of nilpotent groups. Notice that if $\mathbf{H}$ is Fitting and $G$ is a finite group, then there is a largest normal subgroup $\rho_{\mathbf{H}}(G)$ of $G$ belonging to $\mathbf{H}$, namely the product of all normal subgroups $N \triangleleft G$ with $N \in \mathbf{H}$. This subgroup is called the $\mathbf{H}$-radical of $G$. For example, the $p$-group radical of a finite group $G$ is just the intersection of all its $p$-Sylow subgroups; the nilpotent radical of $G$ is the product of the $p$-group radicals taken over all prime divisors $p$ of the order of $G$. For more on Fitting pseudovarieties of groups and their relationship with semigroup theory, see [3].

We denote by $\mathbf{LH}$ the pseudovariety of all semigroups $S$ whose local monoids $eSe$, with $e$ an idempotent, belong to $\mathbf{H}$. If $\mathbf{V}$ is a pseudovariety of semigroups and $\mathbf{W}$ is a pseudovariety of monoids, then the Mal’cev product

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\(V \oplus W\) is the pseudovariety generated by all monoids \(M\) with a homomorphism \(\varphi : M \rightarrow N \in W\) such that \(e\varphi^{-1} \in V\) for each idempotent \(e \in N\). It in fact consists precisely of all homomorphic images of such monoids \(M\).

In this paper, we shall be interested in Mal’cev products \(LH \oplus V\) where \(H\) is a Fitting pseudovariety of groups. Let \(G_p\) be the pseudovariety of \(p\)-groups, where \(p\) is prime, and let \(G_{\text{sol}}\) be the pseudovariety of solvable groups: both are Fitting. The work of Weil [29] describes the languages in \(LG_p \oplus V\) and \(LG_{\text{sol}} \oplus V\) in terms of the languages of \(V\) using marked products with modular counters. The trivial pseudovariety of groups \(I\) is also Fitting. It is shown in [13] that the languages of \(LI \oplus V\) can be described in terms of unambiguous marked products of \(V\)-languages.

Our main theorem is the following.

**Theorem 1.1.** Let \(H\) be a pseudovariety of groups and \(V\) a local pseudovariety of monoids. Then \(LH \oplus V\) is local under either of the following circumstances:

1. \(H\) is a Fitting pseudovariety of groups and \(V\) contains the 6 element Brandt monoid \(B_2^1\);
2. \(H\) is non-trivial and closed under extension.

The proof follows the scheme of [24, Section 6], where locality is established for a one-sided version of \(LG \oplus V\) (here \(G\) is the pseudovariety of all finite groups). We also use ideas from [9,17,19,25,26].

Let \(DS\) be the pseudovariety of monoids whose regular \(J\)-classes are subsemigroups. If \(H\) is a pseudovariety of groups, then \(\overline{H}\) denotes the pseudovariety of monoids whose subgroups belong to \(H\). Consider the pseudovariety \(DS \cap \overline{H}\). Independent work of Putcha and Schützenberger [14,21] (see also [1]) shows that

\[
DS \cap \overline{H} = LH \oplus Sl.
\]

Since \(Sl\) is local by a theorem of Simon [4,5], Theorem 1.1(2) shows \(DS \cap \overline{H}\) is local when \(H\) is extension-closed and non-trivial, a result first proved in [9]. The case \(H\) is trivial is the pseudovariety \(DA\), which was shown to be local by Almeida [2]. This was first announced by Therien [26], but this paper contains an unproven lemma, which was left as an exercise and which is generally believed to be an open problem. If this lemma were true, our techniques would show that Theorem 1.1 applies in the case \(H\) is trivial without the assumptions of (1).

If \(DO\) is the pseudovariety of monoids from \(DS\) whose idempotents in each \(J\)-class form a subsemigroup, then each subpseudovariety of \(DO\) whose associated variety of languages is closed under unambiguous product was shown to be local in [2] using profinite techniques. The results of [13] imply that any such pseudovariety is of the form \(LI \oplus V\) where \(V\) is a pseudovariety of semilattices of groups; see [2, Corollary 3.3]. Such pseudovarieties \(V\) are local by work of Jones and Szendrei [10], so Theorem 1.1 is true in these cases even when \(H\) is trivial (without the \(B_2^1\) assumption). We leave it
as an open question as to whether the assumption that \( H \) is non-trivial is needed in Theorem 1.1(2). More generally, does Theorem 1.1(2) hold for any Fitting pseudovariety of groups (i.e. is the hypothesis on \( V \) in Theorem 1.1(1) necessary)?

Our paper is organized as follows. We first consider \( LH \)-morphisms of categories. We show that any category admitting an \( LH \)-morphism to an element of \( gV \) belongs to \( g(LH \circledast mV) \) under the assumptions of Theorem 1.1. Afterwards, we show that if \( V \) is local, then any category that is locally in \( LH \circledast V \) admits an \( LH \)-morphism to an element of \( gV \). This is our technically most difficult theorem and requires the use of radical congruences from [3,7,11,16,27] and generalized group mapping monoids [11,20]. In order to keep this paper as short as possible, we prove only what is necessary. In particular, we don’t fully develop the theory of \( LH \)-morphisms of categories, where most results from the monoid setting can be shown to hold.

In this paper all semigroups, monoids, categories and semigroupoids are finite.

2. \( LH \)-morphisms of categories

We begin by defining \( LH \)-morphisms. The reader is referred to [12,28] for the notion of a quotient morphism of categories. If \( C \) is a category, \( C(c,c') \) denotes the hom set of arrows from \( c \) to \( c' \). We use \( C_c \) for the local monoid \( C(c,c) \). The set of arrows of \( C \) is denoted \( Arr(C) \); the set of objects of \( C \) is denoted \( Obj(C) \). If \( m \in Arr(C) \), then \( m\iota \) is the initial vertex of \( m \) and \( m\tau \) is the terminal vertex. We write \( m : m\iota \rightarrow m\tau \).

**Definition 2.1 (LH-morphism).** A morphism of categories \( \varphi : C \rightarrow D \) is called an \( LH \)-morphism if it is a quotient morphism and if, for each idempotent \( e \in Arr(D) \), \( e\varphi^{-1} \in LH \).

Since the congruence class of an idempotent is contained in a local monoid, the notion of an \( LH \)-morphism is local in the following sense.

**Proposition 2.2.** Let \( \varphi : C \rightarrow D \) be a quotient morphism of categories. Then \( \varphi \) is an \( LH \)-morphism if and only if, for all \( c \in Obj(C) \), \( \varphi|_{C_c} : C_c \rightarrow D_c \) is an \( LH \)-morphism (of monoids).

If \( V \) is a pseudovariety of monoids, the pseudovariety \( LH \circledast V \) consists of all quotients of monoids admitting an \( LH \)-morphism to an element of \( V \). If \( H \) is a Fitting pseudovariety of groups, then \( LH \circledast V \) consists of all monoids admitting an \( LH \)-morphism to a monoid in \( V \); see Section 3.

The following proposition generalizes a result from the monoid case. The converse is true if \( H \) is extension-closed, although we do not prove it as we shall not use it.

**Proposition 2.3.** Suppose \( \varphi : C \rightarrow D \) is an \( LH \)-morphism and \( \varphi = \psi\gamma \) where \( \psi : C \rightarrow C' \) and \( \gamma : C' \rightarrow D \) are quotient morphisms. Then \( \psi \) and \( \gamma \) are \( LH \)-morphisms.
Proof. By Proposition 2.2 it suffices to show that for each $c \in \text{Obj}(C)$, $\psi|_c$ and $\gamma|_c$ are LH-morphisms. Since $\varphi_c$ is an LH-morphism by Proposition 2.2, we have that $\psi|_c$ and $\gamma|_c$ are LH-morphisms by the well-known analogue of Proposition 2.3 for monoids [17, 19]. □

A key idea in this paper is to translate things from categories to monoids via the consolidation operator. If $C$ is a category, we denote by $C^{cd}$ the consolidation of $C$. It is the monoid $C \cup 0 \cup 1$ obtained by adjoining a zero 0 and an identity 1 to $C$ and making all previously undefined products zero.

It was observed in [25] that if $\varphi : C \rightarrow D$ is a quotient morphism of categories, then there is an induced monoid morphism $\varphi^{cd} : C^{cd} \rightarrow D^{cd}$ given by $\varphi$ on $\text{Arr}(C)$ and sending 0 to 0 and 1 to 1. The next proposition shows that the notion of LH-morphism carries over nicely to the consolidation.

Proposition 2.4. Let $\varphi : C \rightarrow D$ be a quotient morphism. Then $\varphi$ is an LH-morphism if and only if $\varphi^{cd} : C^{cd} \rightarrow D^{cd}$ is an LH-morphism.

Proof. First note that the idempotents of $D^{cd}$ are the idempotents of $D$ along with 0 and 1. Moreover, if $e \in \text{Arr}(D)$ is an idempotent, then $e\varphi^{-1} = e\varphi^{-1}$. Since $1\varphi^{-1} = 1$, $0\varphi^{-1} = 0$ it now is clear that $\varphi^{cd}$ is an LH-morphism if and only if $e\varphi^{-1} \in \text{LH}$ for each idempotent $e \in \text{Arr}(D)$ if and only if $\varphi$ is an LH-morphism. □

To prove Theorem 1.1 we need the following technical result, whose proof we defer to Section 3.

Theorem 2.5. Let $V$ be a pseudovariety of monoids and $H$ be a Fitting pseudovariety of groups. Let $C$ be a category. Then $C \in \ell(LH \circledast V)$ if and only if $C$ admits an LH-morphism to a member of $\ell V$.

2.1. Proof of Theorem 1.1(1). Suppose that $C \in \ell(LH \circledast V)$. Then there is, by Theorem 2.5, an LH-morphism $\varphi : C \rightarrow D$ with $D \in \ell V = gV$. Hence, by Proposition 2.4, $\varphi^{cd} : C^{cd} \rightarrow D^{cd}$ is an LH-morphism. But since $D \in gV$, we have $D^{cd} \in V$, as $V$ contains $B_1$ [28]. Thus $C^{cd} \in LH \circledast V$, whence $C \in g(LH \circledast V)$, since $C$ divides $C^{cd}$. □

2.2. Proof of Theorem 1.1(2). We now recall the notion of a maximal proper quotient (MPQ) [25], generalizing Rhodes' notion of a maximal proper surmorphism (MPS) [11, 15, 18]. This notion will play a key role in the proof of Theorem 1.1(2).

Definition 2.6 (Maximal proper quotient). A quotient morphism $\varphi : C \rightarrow D$ of categories is called a maximal proper quotient (MPQ), if the associated congruence ($\varphi$) is minimal amongst non-trivial congruences on $C$.

A monoid morphism that is an MPQ is called a maximal proper surmorphism or MPS.

Clearly any quotient morphism $\varphi : C \rightarrow D$ of categories factors $\varphi = \varphi_1 \cdots \varphi_n$ with each $\varphi_i$ an MPQ. This combined with Proposition 2.3 proves.
Proposition 2.7. Let $H$ be a pseudovariety of groups. Let $\varphi : C \to D$ be a $\text{LH}$-morphism. Then $\varphi = \varphi_1 \cdots \varphi_n$ where each $\varphi_i$ is both an MPQ and a $\text{LH}$-morphism.

To prove Theorem 1.1(2), we shall essentially use an induction argument on the length of the above decomposition. In [25], we made an observation that allows us reduce many results about MPQs to MPSs. This is done via the consolidation operator. The following is [25, Proposition 2.1].

Proposition 2.8. Let $\varphi : C \to D$ be a quotient morphism. Then $\varphi$ is a maximal proper quotient if and only if $\varphi_{cd} : C_{cd} \to D_{cd}$ is a maximal proper surmorphism.

In order to use some decomposition results from semigroup theory, we shall need to consider the kernel category of a quotient morphism of categories [8, 22]. Let $\varphi : C \to D$ be a quotient morphism; we view $\varphi$ as being the identity on $\text{Obj}(C)$. First we define a category $W_\varphi$. Its object set consists of all pairs $(n_L, n_R)$ of arrows of $D$ such that $n_L \tau = n_R \iota$ (that is, $n_L n_R$ is defined). The arrows are of the form
$$(n_L, m, n_R) : (n_L, m \varphi n_R) \to (n_L m \varphi, n_R),$$
where $n_L \tau = m \iota$ and $m \tau = n_R \iota$ (so $n_L m \varphi n_R$ is defined). Multiplication is given by
$$(n_L, m, m' \varphi n_R)(n_L m \varphi, m', n_R) = (n_L, mm', n_R).$$
The identity at $(n_L, n_R)$ is $(n_L, 1_{n_L \tau}, n_R)$. The kernel category [8, 22] $K_\varphi$ is the quotient of $W_\varphi$ by the congruence that identifies two coterminal arrows $(n_L, m, n_R)$ and $(n_L, m', n_R)$ if and only if, for all $m_L \in n_L \varphi^{-1}$, $m_R \in n_R \varphi^{-1}$,
$$m_L mm_R = m_L m'm_R \quad (2.1)$$
When $C$ and $D$ are monoids, this is the kernel category of [17].

Our next theorem is a reformulation of a result that was first announced in [8] (see also [26]), but was first correctly proved by the author [22] (see [23] for the one-sided analogue). This result plays a key role in [9]. Recall [17] that $V \ast\ast W$ denotes the two-sided semidirect product of pseudovarieties.

Theorem 2.9. Let $V$ and $W$ be pseudovarieties of monoids and let $\varphi : C \to D$ be a quotient morphism with $D \in \text{gW}$ and $K_\varphi \in \text{gV}$. Then $C \in \text{g}(V \ast\ast W)$.

To use the powerful decomposition results of Rhodes et al. [17, 19], we need to relate the $K_\varphi$ and $K_{\varphi_{cd}}$.

Proposition 2.10. Let $\varphi : C \to D$ be a quotient morphism of categories. Then $K_\varphi$ is the full subcategory of $K_{\varphi_{cd}}$ with objects $(n_L, n_R)$ with $n_L, n_R \in \text{Arr}(D)$, $n_L \tau = n_R \iota$. Hence, $K_\varphi$ belongs to any pseudovariety of categories containing $K_{\varphi_{cd}}$. 
Proof. Clearly $W_{\varphi}$ embeds in $W_{\varphi_{cd}}$ as a subsemigroupoid, however the local identities are different. The only arrows of $W_{\varphi_{cd}}$ between objects of $W_{\varphi}$ that do not belong to $W_{\varphi}$ are the arrows of the form $(n_L, 1, n_R)$, where $n_{LT} = n_{RT}$. But (2.1) shows that if $(n_L, n_R)$ is in $\text{Obj}(K_{\varphi})$, then $(n_L, 1, n_R)$ is identified with $(n_L, 1, n_R)$ in $K_{\varphi_{cd}}$. Also it is clear from (2.1) that two arrows of $W_{\varphi_{cd}}$ become identified in $\text{K}_{\varphi}$ if and only if they are identified in $K_{\varphi}$. Putting this all together we see $K_{\varphi}$ is a full subcategory of $K_{\varphi_{cd}}$. □

We now pull out a big hammer, namely the following deep result of Rhodes et al. [17, 19].

Theorem 2.11. Let $H$ be a pseudovariety of groups. Let $\varphi : M \to N$ be a maximal proper surmorphism of monoids and an LH-morphism. Then $K_{\varphi} \in \ell H$.

Putting everything together we obtain:

Corollary 2.12. Let $H$ be a pseudovariety of groups and let $\varphi : C \to D$ be a maximal proper quotient and an LH-morphism of categories. Then $K_{\varphi} \in \ell H$.

Proof. Since $\varphi$ is MPQ, $\varphi_{cd}$ is MPS by Proposition 2.8. Since $\varphi$ is an LH-morphism, so is $\varphi_{cd}$ by Proposition 2.1. By Theorem 2.11, $K_{\varphi_{cd}} \in \ell H$. Hence by Proposition 2.10, $K_{\varphi} \in \ell H$. □

We proceed to prove Theorem 1.1(2) assuming Theorem 2.5.

2.2.1. Proof of Theorem 1.1(2). Fix a non-trivial extension-closed pseudovariety of group $H$. We first use the following theorem [17, 19] (see also [29]) whose difficult direction relies on Theorem 2.11 and the fact that $\ell H = g H$ [28] if $H$ is non-trivial.

Theorem 2.13. Let $V$ be a pseudovariety of monoids and $H$ be a non-trivial extension closed pseudovariety of groups. Then $\text{LH} \otimes V$ is the smallest pseudovariety $W$ containing $V$ such that $H ** W = W$.

Suppose now that $V$ is a local pseudovariety of monoids. Let $C \in \ell (\text{LH} \otimes V)$. By Theorem 2.5, $C$ admits an LH-morphism $\varphi : C \to D$ with $D \in \ell V = \text{gV}$. Hence by Lemma 2.7, there is a sequence $\varphi_1, \ldots, \varphi_n$ of LH-morphisms that are MPQ such that $C$ is the domain $\varphi_1$, the codomain of $\varphi_n$ belongs to $\text{gV}$ and $\varphi_1 \varphi_2 \cdots \varphi_n$ is defined. We proceed by induction on $n$. If $n = 0$, then $\varphi$ is the identity map and so $C \in \text{gV}$ and we are done. Else, suppose $\varphi_1 : C \to C'$. Then $C' \in \ell (\text{LH} \otimes V)$ and by considering $\varphi_2, \ldots, \varphi_n$ and the induction hypothesis, $C' \in g(\text{LH} \otimes V)$.

By Corollary 2.12, $K_{\varphi_1} \in \ell H = g H$, the equality following from [28]. So by Theorem 2.9 and Theorem 2.13,

$$C \in g(H ** (\text{LH} \otimes V)) = g(\text{LH} \otimes V).$$

This establishes Theorem 1.1(2). □
3. Radical Congruences

The goal of this section is to prove Theorem 2.5. In particular, we show how certain radical congruences on $C^{cd}$ restrict nicely to local monoids $C_c$. We shall need to use the description of the maximal congruence associated to an $\mathbf{LH}$-morphism of monoids for Fitting pseudovarieties of groups.

A semigroup $S$ is called generalized group mapping [11, 20] (GGM) if it has a 0-minimal ideal $I$ on which it acts faithfully on both the left and right. GGM semigroups play a crucial role in finite semigroup theory, both in relation to complexity theory [11, 16, 20, 27] and to Malcev products of the form $\mathbf{LH} \otimes \mathbf{V}$ [3, 20].

We recall some notions and results of Krohn and Rhodes. The reader is referred to [11, 20] for details. Fix a finite semigroup $S$. Let us set some notation. Choose, for each regular $J$-class $J$, a fixed maximal subgroup $G_J$. Let $J$ be a regular $J$-class of $S$. Let $F(J) = \{ s \in S \mid s \not\geq J \}$. Then $F(J) \cup J$ and $F(J)$ are ideals of $S$. We identify $(F(J) \cup J)/F(J)$ with $G_{J,0}$. Choose an isomorphism of $J,0$ with a Rees matrix semigroup $\mathcal{M}(G_J,A,B,C)$. We do not distinguish between $J,0$ and this Rees matrix semigroup group. Now let $N \trianglelefteq G_J$ be a normal subgroup. Then we let $\eta : F(J) \cup J \rightarrow J,0$ and $\psi : \mathcal{M}(G_J,A,B,C) \rightarrow \mathcal{M}(G_J/N,A,B,C)$ be the natural projections, where $C$ is the matrix $C$ reduced modulo $N$.

We define a congruence $\equiv(J,G_J,N)$ as follows [11]. Let $s,t \in S$ and $x,y \in J$. Then $xsy, xty \in F(J) \cup J$. Define, for $s,t \in S$,

$$s \equiv(J,G_J,N) t \iff xsy\eta\psi = xty\eta\psi \text{ for all } x,y \in J. \tag{3.1}$$

The quotient $S/\equiv(J,G_J,N)$ is denoted GGM$(J,G_J,N)$ [11]. It can be shown that GGM$(J,G_J,N)$ is GGM, its definition does not depend on the choice of the Rees matrix representation of $J,0$ and that all generalized group mapping images of $S$ are of this form for some regular $\mathfrak{F}$-class $J$ and normal subgroup $N$ of $G_J$; see [11, Proposition 8.3.28, Remark 8.3.29].

Let us make an observation about the definition of $\equiv(J,G_J,N)$. Namely, let $x, y \in J$. Let $x', y'$ be inverses of $x, y$ respectively. Let $s, t \in S$. Then it is clear that (3.1) holds if and only if

$$x'xsy'\eta\psi = x'xtyy'\eta\psi.$$

Thus in (3.1), we may always assume that $x$ and $y$ are idempotent. We shall use this observation without further remark.

Proofs of the following results are implicit in [11, 16, 27], where they are proved for every Fitting pseudovariety of groups that had been considered in semigroup theory up until the time they were written; notice these works were written before Eilenberg and Schützenberger introduced the notion of pseudovarieties to semigroup theory [5, 6] and before anybody had considered Mal’cev products in the theory. The general case, whose proof is no different than the case of $p$-groups already considered by Rhodes and Tilson [16, 27],
can be found in Hall and Weil [7]; see also [3]. We state the results in the form that we shall use them.

**Theorem 3.1** (Rhodes, Tilson). Let $H$ be a Fitting pseudovariety of groups and $S$ a semigroup. Then $\varphi : S \to \prod GGM(J,G,J,\rho_{H}(G))$, the product of the canonical morphisms (where the product runs over all regular $J$-classes), is an $LH$-morphism and the congruence associated to $\varphi$ contains the congruence associated to any other $LH$-morphism.

**Theorem 3.2** (Rhodes, Tilson). Let $H$ be a Fitting pseudovariety of groups, $V$ a pseudovariety of monoids and $M$ a finite monoid. Then $M$ belongs to $LH \boxtimes V$ if and only if one has $GGM(J,G,J,\rho_{H}(G)) \in V$ for all regular $J$-classes $J$ of $M$.

We are primarily interested in the following corollary.

**Corollary 3.3.** Let $H$ be a Fitting pseudovariety of groups and let $V$ be a pseudovariety of monoids. Then $M$ belongs to $LH \boxtimes V$ if and only if it has an $LH$-morphism to an element of $V$. In particular if $H$ and $V$ are decidable, then so is $LH \boxtimes V$.

The next two results are analogous to [24, Lemma 6.2]. Let $C$ be a category and $c \in \text{Obj}(C)$. To prove Theorem 3.5 we must understand how the $J$-relation on $C^{cd}$ and $C_c$ relate.

**Proposition 3.4.** Let $x, y \in C_c$. Then $x \not J y$ in $C^{cd}$ if and only if $x \not J y$ in $C_c$.

*Proof.* Clearly $x \not J y$ in $C_c$ implies $x \not J y$ in $C^{cd}$. For the converse, observe that if $uxv = y$ with $u, v \in C^{cd}$, then if $u$ is not 1, it must start at $y_\tau$ and end at $x_\tau$. Since $c = y_\tau = x_\tau$, we see $u \in C_c$. Similarly, $v = 1$ or $v \in C_c$. Hence $x \not J y$ in $C_c$. □

**Proposition 3.5.** Let $x \in C_c$. Then the $H$-class of $x$ in $C^{cd}$ and $C_c$ coincide. In particular, if $e \in C^{cd}$ is an idempotent other than 0 and 1, then $e \in C_c$ for some $c \in \text{Obj}(C)$ and the maximal subgroup at $e$ in $C^{cd}$ and $C_c$ coincide.

*Proof.* The second statement is an immediate consequence of the first. For the first, observe that in a category $\mathcal{R}$-equivalent elements must start at the same vertex and $L$-equivalent elements must end at the same vertex. Hence $\mathcal{H}$-equivalent elements are coterminial. Suppose now that $x \in C_c$. Then its $\mathcal{H}$-class is contained in $C_c$. The fact that the $H$-class of $x$ in $C^{cd}$ and $C_c$ coincide now follows from [24, Lemma 6.2]. □

These two propositions can be summarized as follows:

**Corollary 3.6.** Let $C$ be a category and $c \in \text{Obj}(C)$. Let $J$ be a $J$-class of $C^{cd}$. If $J_c = J \cap C_c$ is not empty, then it is a $J$-class of $C_c$ and is a union of $H$-classes of $J$. Moreover, $J$ is regular if and only if $J_c$ is regular.
Proof. Only the last statement has not been proved. Clearly if $J_c$ is regular, so is $J$. Suppose $J$ is regular. Let $x \in J_c$. Let $x'$ be an inverse of $x$. Then $xx'x = x$ implies that $x' \in C_c \cap J = J_c$. Thus $J_c$ is regular. \hfill \Box

Let $C$ be a category and fix a maximal subgroup $G_J$ for each regular $\mathcal{J}$-class $J$ of $C^{cd}$. If $J$ intersects $C_c$, we continue to use $J_c$ for $J \cap C_c$. Corollary 3.4 shows that $J_c$ is a regular $\mathcal{J}$-class of $C_c$ and that $G_J$ is a maximal subgroup of $J_c$, that is we may take $G_J = G_{J_c}$.

Let $s, t \in C_c$. Suppose $J$ is a regular $\mathcal{J}$-class of $C^{cd}$ that doesn’t intersect $C_c$. Clearly if $x, y \in J$ are idempotents, then $x sy \notin J$ and $x ty \notin J$. Thus

\[ s \equiv_{(J_G \cup J, \rho_{H(G_J)})} t \text{ if } J \cap C_c = \emptyset \tag{3.2} \]

The $\mathcal{J}$-classes of 0 and 1, of course, don’t intersect $C_c$.

Lemma 3.7. Let $J$ be a regular $\mathcal{J}$-class of $C^{cd}$ intersecting $C$. Then

\[ \equiv_{(J_G \cup J, \rho_{H(G_J)})} \cap (C_c \times C_c) = \equiv_{(J, G_{J_c} \cup J, \rho_{H(G_J)})} \cap (C_c \times C_c) \]

Proof. Choose a Rees matrix representation $\mathcal{M}^0(G_J, A, B, C)$ for $J^0$. Since, by Corollary 3.6, $J_c$ is a union of $\mathcal{J}$-classes of $J$, there are subsets $A_c \subseteq A$ and $B_c \subseteq B$ such that $\mathcal{M}^0(G_J, A_c, B_c, C_c)$ is a Rees matrix representation of $J^0_c$, where $C_c$ is the restriction of $C$ to $B_c \times A_c$. Let us denote by $\eta : F(J) \cup J \to \mathcal{M}^0(G_J, A, B, C)$ the projection. Clearly the restriction to $F(J_c) \cup J_c$ is the projection $\eta_c : F(J_c) \cup J_c \to \mathcal{M}^0(G_J, A_c, B_c, C_c)$. Similarly, if $\psi : \mathcal{M}^0(G_J, A, B, C) \to \mathcal{M}^0(G_J, A_c, B_c, C_c)$ is the natural projection, then its restriction to $\mathcal{M}^0(G_J, A_c, B_c, C_c)$ is the natural projection $\psi_c : \mathcal{M}^0(G_J, A_c, B_c, C_c) \to \mathcal{M}^0(G_J, A_c, B_c, C_c)$.

Let $s, t \in C_c$. Suppose $x, y \in J$ are idempotents. Since $x, y \notin 0, 1$, they must each belong to some local monoid of $C$. Suppose at least one of these local monoids is not $C_c$. Then $x sy = 0 = x ty$ and so

\[ x sy \eta \psi = 0 = x ty \eta \psi \text{ if } x \text{ or } y \text{ is not in } C_c. \tag{3.3} \]

If $x, y \in C_c$, then $x, y \in J_c$. So $x sy, x ty \in F(J_c) \cup J_c$. Then the previous paragraph shows

\[ x sy \eta \psi = x ty \eta \psi \iff x sy \eta \psi_c = x ty \eta \psi_c. \tag{3.4} \]

Putting together (3.3), (3.4) and (3.1) (for both $J$ and $J_c$), plus the observation that in (3.1) we may assume that $x, y$ are idempotents, we obtain

\[ s \equiv_{(J_G \cup J, \rho_{H(G_J)})} t \iff s \equiv_{(J, G_{J_c} \cup J, \rho_{H(G_J)})} t, \]

thereby completing the proof of the lemma. \hfill \Box

We may now deduce Theorem 2.5.
3.1. Proof of Theorem 2.5 Suppose that $C$ admits an LH-morphism $\varphi : C \to D$ with $D \in \ell V$. Then, for $c \in \text{Obj}(C)$, $\varphi|_{C_c} : C_c \to D_c$ is an LH-morphism by Proposition 2.2. It follows $C_c \in \text{LH} \otimes V$ (since $D_c \in V$ by assumption on $D$).

Conversely, suppose that $C \in \ell (\text{LH} \otimes V)$. The inclusion map on arrows induces a faithful semigroupoid morphism $\varphi_0 : C \to C_{cd}$ (it doesn’t send local identities to the identity). Let $\varphi_1 : C_{cd} \to \prod GGM(J,G_J,\rho_H(G_J))$ be the product of the canonical morphisms, where the product runs over all regular $J$-classes of $C_{cd}$. Let $\equiv$ be the congruence on $C$ associated to $\varphi_0 \varphi_1$, let $D = C/\equiv$ and let $\psi : C \to D$ be the quotient map.

Lemma 3.8. The natural projection $\pi : C \to D$ is an LH-morphism and $D \in \ell V$.

Proof. First of all the local monoids of $D$ are of the form $C_c/(\psi)$ with $c \in \text{Obj}(C)$. But the restriction of $\varphi_0 \varphi_1$ to $C_c$ is $\varphi_1|_{C_c}$. However, (3.2) and Lemma 3.7 show that the congruence $\sim$ on $C_c$ associated to $\varphi_1|_{C_c}$ is the intersection of the congruences $\equiv(J_c,G_J,\rho_H(G_J))$ associated to the regular $J_c$-classes $J_c$ of $C_c$. By Theorem 3.2 and the assumption that $C_c \in \text{LH} \otimes V$, we may conclude that each $C_c/\equiv(J_c,G_J,\rho_H(G_J)) \in V$ and hence $D_c = C_c/\sim \in V$. This shows that $D \in \ell V$. Also, by Theorem 3.1, $\varphi_1|_{C_c}$ is an LH-morphism. Hence, by Proposition 2.2, $\psi : C \to D$ is an LH-morphism, as desired. □

Lemma 3.8 completes the proof of Theorem 2.5 and hence establishes Theorem 1.1. □

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