The pressure of deconfined QCD for all temperatures and quark chemical potentials

Andreas Ipp

ECT*, Villa Tambosi, Strada delle Tabarelle 286, I-38050 Villazzano Trento, Italy

A new method for the evaluation of the perturbative expansion of the QCD pressure is presented which is valid for all temperatures and quark chemical potentials in the deconfined phase, and worked out up to and including order $g^4$. This new approach unifies several distinct perturbative approaches to the equation of state, and agrees with dimensional reduction, HDL and HTL resummation schemes, and the zero-temperature result in their respective ranges of validity.

1. Introduction

In a recent paper [1] Kajantie, Rebhan, Vuorinen, and myself have proposed a new method for calculating the QCD equation of state in a perturbative setup for all temperatures $T$ and quark chemical potentials $\mu$. Previous perturbative approaches have been limited to certain regions only: when the temperature is large compared to the Debye screening mass $T \gtrsim m_D$ (with $m_D^2 = g^2(T^2 + \mu^2/\pi^2)$ for $N_c = 3$ colors and $N_f = 2$ flavors), hard modes can be integrated out to leave an effective theory for the zero mode, and dimensional reduction applies [2]-[4]. In the opposite case $T \lesssim m_D$, a four-dimensional resummation in terms of hard dense loops (HDL) works, giving rise to anomalous contributions in entropy and specific heat of a non-Fermi liquid [5]. The latter approach smoothly connects to the well-known zero-temperature limit by Freedman and McLerran [6]. Our new method covers all of these cases, and provides an independent check on those calculations.

These parametrically distinct regions are suitably visualized in a QCD phase diagram in polar coordinates (Figs. 1-3). Due to asymptotic freedom, weak-coupling techniques apply at high temperatures or chemical potentials $r \approx \sqrt{T^2 + \mu^2}$. We label the abscissa in Figs. 2 and 3 by powers of the coupling $g^2$, $g^3$, ... and the ordinate by parametrically different regimes $T \sim \mu$, $g\mu$, $g^3\mu$, ... which for larger $r$ are displayed as horizontal lines (somewhat deviating from strict polar coordinates). In the following, we will describe the various existing perturbative approaches, and connect them by our new method.

2. Previous methods

If the temperature $T$ is larger than all other dynamical scales, one can apply effective field theory methods [2] and integrate out the degrees of freedom corresponding to non-zero Matsubara modes to obtain a simpler three-dimensional (dimensionally reduced)
Andreas Ipp

\( \mu = 0 \quad 90^\circ \)

\( T = 0 \quad 0^\circ \)

\( r \approx \sqrt{T^2 + \mu^2} \quad 304 \text{ MeV} \)

\( \theta \approx \arctan \frac{T}{\mu} \quad \text{csc} \)

\( \frac{T}{\mu} \approx \sqrt{T^2 + \mu^2} \)

\( \text{vacuum} \quad \text{hadron gas} \quad \text{LHC} \quad \text{RHIC} \)

\( 170 \text{ MeV} \)

\( \text{quark-gluon plasma} \)

Figure 1. QCD phase diagram in polar coordinates.

effective theory, known as electrostatic QCD (EQCD). The effective theory approach can be applied a second time to remove also the massive longitudinal gluon from the theory, thus producing an effective three-dimensional pure Yang-Mills theory, which is called magnetostatic QCD (MQCD). Practically, the effective Lagrangians and effective masses \( m_E \) and couplings \( g_E, \lambda_E, g_M, \ldots \) can be obtained by matching a set of physical quantities between a generic Lagrangian and the full theory. The perturbative pressure can then be written as [2]

\[
p = p_E(T, g) + p_M(m^2_E, g_E, \lambda_E, \ldots) + p_C(g_M, \ldots),
\]

where the first two terms correspond to the coefficients of the unit operator of EQCD and MQCD respectively, and the last term corresponds to the pressure of MQCD, which constitutes a fundamentally non-perturbative contribution to full QCD. Figure 2 shows at which orders the latter two terms start to contribute, by the lines labeled EQCD and MQCD. The plot shows for example that the non-perturbative contribution of MQCD which sets in at order \( g^6 \) for \( T \gtrsim \mu \), only affects the pressure at order \( g^8 \) if the temperature is of the parametric order \( T \sim g\mu \). So far the perturbative pressure has been obtained up to and including order \( g^4 \) and extended to finite chemical potential \( \mu \) [4].

The other result depicted in Fig. 2 is the zero-temperature limit by Freedman and McLerran (FMcL) [6] which is known up to and including order \( g^4 \), with an error of order \( g^6 \). At order \( g^4 \) dimensional reduction diverges as \( T \to 0 \) and does not connect to FMcL, revealing obviously the breakdown of dimensional reduction.

In the region \( T \lesssim g\mu \) perturbation theory requires a different reorganization which can be most efficiently performed via HDL resummation [5] (Fig. 3). A small modification to HTL resummation increases the range of validity of this approach to larger temperatures. The interaction pressure \( \delta p \equiv p - p_{SB} - (p - p_{SB})|_{T=0} \) (\( p_{SB} \) is the interaction-free Stefan-Boltzmann contribution) as well as entropy and specific heat show anomalous logarithmic enhancement of the leading contribution (a signature for non-Fermi-liquid behavior) and
The pressure of deconfined QCD for all temperatures and quark chemical potentials

The pressure of deconfined QCD for all temperatures and quark chemical potentials

\[ \mu = 0 \]
\[ g^0 g^2 g^3 g^4 g^5 g^6 \cdots \]
\[ T \sim \mu \]
\[ T \sim g \mu \]
\[ T \sim g^2 \mu \]
\[ T = 0 \]

\[ T \sim g \mu \]
\[ T \sim g^2 \mu \]
\[ \mu = 0 \]
\[ g^0 g^2 g^3 g^4 g^5 g^6 \cdots \]
\[ T \sim \mu \]
\[ T \sim g \mu \]
\[ T \sim g^2 \mu \]
\[ T = 0 \]

\[ \delta p = \frac{g^2 \mu T^2}{9 \pi^2} \left\{ \ln \frac{g \mu}{\pi T} - 0.7906 - 2.9401 \left( \frac{\pi T}{g \mu} \right)^{2/3} + 2.2312 \left( \frac{\pi T}{g \mu} \right)^{4/3} + \cdots \right\} \] (2)

The fractional powers of the temperature in the regime \( T \lesssim g \mu \) (for \( N_f = 2, N_c = 3 \))

The fractional powers are indicated by lines starting from \( T \sim g \mu \) and \( g^4 \) in Fig. 3 and form the correct continuation of the plasmon term \( \propto T m_D^3 \) (EQCD line in Fig. 2).

3. The new approach

Our new approach [1] is motivated by the observation that only a subset of all diagrams that contribute to the pressure exhibit infrared divergent behavior in the limit \( T \to 0 \). The problematic set consists of the two-gluon reducible diagrams (2GR; vacuum bubble diagrams that can be separated by cutting two gluon lines). Up to order \( g^4 \), the following classes of 2GR diagrams require resummation:

\[ \sum_{n=2}^{\infty} \quad \sum_{n_1, n_2 = 1}^{\infty} \quad \sum_{n_1, n_2, n_3 = 1}^{\infty} \quad \sum_{n_1, n_2 = 1}^{\infty} \]

where the black dots represent the full one-loop gluon polarization tensor. This is only known numerically, but the contribution of diagram \( b \) to the pressure up to \( \mathcal{O}(g^4) \) can be expressed analytically, and diagrams \( c \) and \( d \) would only contribute beyond that order. Only the contribution from diagram \( a \) has to be obtained by means of numerical integration (using the methods developed at large \( N_f \) [7]). The pressure through order \( g^4 \) (labeled by the roman numeral IV) valid for all \( T \) and \( \mu \) is finally given by the sum of analytically calculable pieces and the ring diagrams \( a \) which have to be integrated numerically: \( p_{IV} = p_{anl} + p_{safe} + \mathcal{O}(g^5 T m_D^3) + \mathcal{O}(g^6 \mu^4) \).

Figure 4 shows that dimensional reduction (curves labeled \( g^2, g^3, \) and \( g^4 \)) ceases to be applicable when \( T \lesssim 0.2 m_D \). At this point, non-Fermi-liquid effects take over that can
Figure 4. Thermal contribution to the interaction pressure $\delta p$ as a function of $T/m_D^{T=0}$ for fixed chemical potential $\mu$ and coupling $g = 0.5$. When two lines of the same type run close to each other, they differ by changing the renormalization scale $\Lambda_{\overline{MS}} = \mu ... 4\mu$.

Figure 5. The dividing line between the regime of dimensional reduction (DR) and that of non-Fermi-liquid behavior (NFL) for $N_f = 3$ (full lines) and $N_f = 2$ (dashed lines), in comparison with the weak-coupling result for the critical temperature of color superconductivity (CSC) when extrapolated to large coupling.

only be described by HDL/HTL resummation or the new $p_{IV}$. In Fig. 5 this separation line $T \approx 0.2 m_D$ is plotted along with the weak-coupling result for the critical temperature of color superconductivity [8]. While at larger couplings significant non-perturbative modifications can be expected, at weaker couplings one observes a clear separation of regimes with qualitatively different weak-coupling descriptions.

REFERENCES

1. A. Ipp, K. Kajantie, A. Rebhan and A. Vuorinen, Phys. Rev. D 74 (2006) 045016 [hep-ph/0604060].
2. E. Braaten and A. Nieto, Phys. Rev. D 53 (1996) 3421 [hep-ph/9510408].
3. K. Kajantie, M. Laine, K. Rummukainen and Y. Schröder, Phys. Rev. D 67 (2003) 105008 [hep-ph/0211321].
4. A. Vuorinen, Phys. Rev. D 68 (2003) 054017 [hep-ph/0305183].
5. A. Ipp, A. Gerhold and A. Rebhan, Phys. Rev. D 69 (2004) 011901 [hep-ph/0309019]; A. Gerhold, A. Ipp and A. Rebhan, Phys. Rev. D 70 (2004) 105015 [hep-ph/0406087].
6. B. A. Freedman and L. D. McLerran, Phys. Rev. D 16 (1977) 1169; V. Baluni, Phys. Rev. D 17 (1978) 2092.
7. G. D. Moore, JHEP 0210 (2002) 055 [hep-ph/0209190]; A. Ipp, G. D. Moore and A. Rebhan, JHEP 0301 (2003) 037 [hep-ph/0301057]; A. Ipp and A. Rebhan, JHEP 0306 (2003) 032 [hep-ph/0305030].
8. W. E. Brown, J. T. Liu and H. c. Ren, Phys. Rev. D 61, 114012 (2000) [arXiv:hep-ph/9908248]; A. Schmitt, Q. Wang and D. H. Rischke, Phys. Rev. D 66, 114010 (2002) [arXiv:nucl-th/0209050].