The Influence of Free Quintessence on Gravitational Frequency Shift and Deflection of Light with 4D momentum

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Based on the 4D momentum, the influence of quintessence on the gravitational frequency shift and the deflection of light are examined in modified Schwarzschild space. We find that the frequency of photon depends on the state parameter of quintessence \( w_q \): the frequency increases for \(-1 < w_q < -1/3\) and decreases for \(-1/3 < w_q < 0\). Meanwhile, we adopt an integral power number \( a \) \((a = 3w_q + 2)\) to solve the orbital equation of photon. The photon’s potentials become higher with the decrease of \( w_q \). The behavior of bending light depends on the state parameter \( w_q \) sensitively. In particular, for the case of \( w_q = -1 \), there is no influence on the deflection of light by quintessence. Else, according to the H-masers of GP-A redshift experiment and the long-baseline interferometry, the constraints on the quintessence field in Solar system are presented here.

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I. INTRODUCTION

It is well known that the universe accelerating is proved by the type Ia Supernovae (SNe Ia) [1] and the cosmic microwave background (CMB) [2] in Wilkinson Microwave Anisotropy Probe (WMAP). The large scale distribution of galaxies [3] shows the existence of cold dark matter (CDM) with the ratio of \( \Omega_{CDM} = 0.27 \pm 0.04 \). The current universe is dominated by the dark energy component which contributes \( \Omega_{de} = 0.67 \pm 0.06 \) to the critical energy density. Else, dark energy should also have a negative pressure to ensure the acceleration of universe.

These astronomical observations constrain current state parameter \( \omega \) to be close to the cosmological constant case [4] which corresponds to a fluid with \( \omega = -1 \). Actually, the observations also indicate a little bit time evolution of \( w \). Thus, many people considered the corresponding situation in which the state equation of dark energy changes with time. From particle physics, the scalar fields is natural candidates of dark energy. A wide variety of scalar field dark energy models have been proposed such as quintessence [5], K-essence [6], tachyon field [7], Phantom (ghost) [8], dilatonic dark energy [9] and so on. The quintessence model refers to a minimally coupled scalar field with a potential which decreases as the field increases. The field is coupled by the gravity through a Lagrangian taking the form

\[
L_q = \sqrt{-g}\left(\frac{1}{2}\partial^\phi \partial^\phi - V(\phi)\right),
\]

where \( \phi \) is the scalar field and \( V(\phi) \) is the potential.

On the other hand, many people have discussed minutely how to combine the quintessence energy momentum tensor with Einstein equations [11] [12] [13]. However, the initial solutions obtained by [11] [12] have no horizon and no ‘hair’. Hence, the black hole can not be formed in those metrics. Later, Kiselev [13] adopted a nonzero off-diagonal energy momentum tensor being proportional to diagonal terms, i.e. \( C(r) \neq 0 \propto B(r) \), where \( C(r) \) and \( B(r) \) are the coefficients of the energy momentum tensors. The energy momentum tensor of the quintessence \( T^\mu_\nu \) are

\[
T^t_t = A(r), \quad T^t_i = 0, \quad T^i_i = C(r)r^i_i + B(r)\delta^i_i.
\]
The constant coefficient $C(r)/B(r)$ satisfies the additivity and linearity conditions. So a exact black hole solution surrounding by quintessence matter is obtained by the static coordinates \[13\]. Since then, many people have performed its quasinormal modes \[14\], entropy \[15\], geodesic precession \[16\] and so on. However, as far as we know, there is no work discussing its influence on gravity test in the view of 4D momentum. In this paper, we start from the 4D momentum and calculate carefully the gravitational shift and the deflection of light in the modified Schwarzschild space encoded the quintessence matter. From the consideration of the supernovae dimming, we should make the dark energy satisfies $\omega \leq -2/3$ to fit the observations \[16\]. But in this paper we evaluate the quintessence matter with the state parameter in the range of $\omega_q \in [-1, 0]$. Meanwhile, we also compare the result of quintessence with that of cosmological constant \[17\] to verify its rationality based on the fact of that $\Lambda$CDM is reduced from a special quintessence field of $\omega_q = -1$ to a certain extent. It should also be noted that the problems of gravitational frequency shift and light deflection are studied in the context of central gravitational field rather than the cosmology here.

This paper is organized as follows: in section II, we present the Kiselev solution, i.e. the Schwarzschild black hole surrounded by quintessence matter. In section III, we calculate the frequency gravitational shift of photons in which the frequency is inverse proportional to local $\sqrt{g_{00}}$. In section IV, we study the deflection of light according to the different state parameter $\omega_q$. In section V, we use the astronomical observations to constrain the quintessence and examine whether the observable effect is big enough to be measured. Section VI is a conclusion. We adopt the signature $(+, −, −, −)$ and put $\hbar, c$, and $G$ equal to unity.

II. SCHWARZSCHILD SPACE SURROUNDED BY STATIC SPHERICALLY SYMMETRIC QUINTESSENCE MATTER

Before performing the gravitational test of modified general relativity (GR), we would like to introduce Kiselev’s black hole solution \[13\]. The derivation starts from a spherically symmetric static gravitational field,

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (3)

It assumes that the quintessence matter distributes evenly outside black hole and the energy momentum tensor takes the forms

$$T_{i}^{t} = \rho_q(r),$$ \hspace{1cm} (4)

$$T_{j}^{j} = \rho_q(r)\gamma \left[ -(1 + 3B) \frac{r_i r_j}{r_i r^n} + B \delta_{i}^{j} \right],$$  \hspace{1cm} (5)

where $\rho_q$ is the density of quintessence matter. The parameter $B$ depends on the internal structure of quintessence. The isotropic average over the angle components is

$$< T_{i}^{j} >= -\rho_q(r)\frac{\gamma}{3} \delta_{i}^{j} = -\rho_q(r)\delta_{i}^{j},$$ \hspace{1cm} (6)

where the relationship $< r_i r_j > = \frac{1}{4} \delta_{i}^{j} r_i r^n$ is used and the state equation is $P_q = \omega_q \rho_q$ where $\omega_q = \gamma/3$.

According to the additivity and linearity principle, i.e. $T_i^t = T_c^t$, the free parameter $B(\omega_q)$ can be fixed as

$$B = \frac{3\omega_q + 1}{6\omega_q}.$$ \hspace{1cm} (7)

Meanwhile, this assumption directly leads to a relationship $\lambda + \nu = 0$. Hence the energy momentum tensor can be written as follows,

$$T_{i}^{t} = T_{c}^{t} = \rho_q,$$ \hspace{1cm} (8)

$$T_{\theta}^{\theta} = T_{\phi}^{\phi} = -\frac{1}{2} \rho_q(3\omega_q + 1).$$ \hspace{1cm} (9)

If we set $\lambda = -\ln(1 + f)$, a differential equation of $f$ can be obtained

$$r^2 \frac{d^2 f}{dr^2} + 3(1 + \omega_q) r \frac{df}{dr} + (3\omega_q + 1) f = 0,$$ \hspace{1cm} (10)
which has two exact solutions

\[ f_g = \frac{\alpha}{r^{3\omega_q+1}}, \]  
\[ f_{BH} = -\frac{r_g}{r}, \]  

where \( \alpha \) and \( r_g \) are the normalization factors. \( f_{BH} \) is the ordinary point-like black hole solution such as the Schwarzschild black hole. The quintessence density is

\[ \rho_q = \frac{\alpha}{2} \frac{3\omega_q}{r^{3(\omega_q+1)}}. \]  

(13)

So if we require the density of energy positive, \( \rho_q > 0 \), we deduce that \( \alpha \) is negative for \( \omega_q \) negative. The curvature has the form of

\[ R = 2T_\mu^\mu = 3\alpha \omega_q \frac{1-3\omega_q}{r^{3(\omega_q+1)}}. \]  

(14)

Apparently, \( r = 0 \) is the singularity for \( \omega_q \neq \{ -1, 0, 1/3 \} \). Combining linearly solutions (11) and (12), we can get a general solution.

\[ e^{-\lambda} = 1 - \frac{r_g}{r} - \sum_n \left( \frac{r_n}{r} \right)^{3\omega_n+1}, \]  

(15)

where \( n \) indicates several fields and \( r_q \) is a normalization factor which has the dimension of length. The free quintessence creates an outer horizon of de Sitter universe at \( r = r_q \) for

\[ -1 < \omega_q < -\frac{1}{3}, \]  

(16)

and also generates an inner horizon of black hole at \( r = r_q \) for

\[ -\frac{1}{3} < \omega_q < 0. \]  

(17)

Here we only consider the influence of quintessence matter on Schwarzschild space. The metric of Schwarzschild space is modified by a new form,

\[ ds^2 = \left( 1 - \frac{2M}{r} - \frac{\alpha}{r^{3\omega_q+1}} \right) dt^2 - \left( 1 - \frac{2M}{r} - \frac{\alpha}{r^{3\omega_q+1}} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \]  

(18)

where \( M \) is the black hole mass. When \( \alpha = 0 \) this metric reduces to a pure Schwarzschild case. Otherwise, when \( \omega_q = -1 \) this metric reduces to a Schwarzschild-de Sitter one. It should be noticed that comparing with the large total mass of dark energy, the relatively small mass of black hole can be treated as a constant.

### III. FREQUENCY GRAVITATIONAL SHIFT

When photon propagates in a stable gravitational field, the stationary observers at different positions obtain different frequencies. This phenomenology is the so called gravitational redshift which is one of the special effects purely due to curved space. Certainly, it is also a natural result from the equivalence principle in general relativity. For the static metric \( g_{\mu\nu} \), the photon’s energy can be written as

\[ E = p_\mu U^\mu, \]  

(19)

where \( p_\mu \) is the 4D momentum of photon, \( U^\mu \) is the 4D velocity of observer which is in the form

\[ U^\mu = \frac{1}{\sqrt{g_{00}}} (1, 0, 0, 0). \]  

(20)
Submitting Eq. (20) into Eq. (19), the energy \( E \) can be rewritten as

\[
E = \frac{p_0}{\sqrt{g_{00}}}.\tag{21}
\]

Using the Planck relation \( E = h\nu \), the frequency and metric component \( g_{00} \) yield

\[
\sqrt{g_{00} \nu} = p_0 / h.\tag{22}
\]

The right hand side of Eq. (22) is a constant since \( p_0 \) is a conserved parameter when photon moves in this stable space.

When Schwarzschild space is surrounded by the free static spherically symmetric quintessence matter, the time component of metric (18) is

\[
g_{00} = 1 - \frac{2M}{r} - \frac{\alpha}{r^{3\omega_q+1}}.\tag{23}
\]

Meanwhile, we assume that the emitter is at \( x_0 = (r_0, 0, 0, 0) \) and the receiver is at \( x = (r, 0, 0, 0) \). The photon signal has fixed spatial coordinates and their 4-velocity is tangent to the static Killing field \( \xi^a \). Furthermore, the photon frequency is inverse proportion to the local \( \sqrt{g_{00}} \) and the frequencies of emitter and receiver must satisfy

\[
\nu / \nu_0 = \sqrt{\frac{g_{00}(r_0)}{g_{00}(r)}}.\tag{24}
\]

Submitting the lapse function Eq. (23) into above ratio \( \nu / \nu_0 \), we can obtain

\[
\nu / \nu_0 = \sqrt{\frac{g_{00}^{(0)}(r_0)}{g_{00}^{(0)}(r)}} \left( 1 + \frac{\alpha}{2} \left( \frac{1}{g_{00}^{(0)}(r)r^{3\omega_q+1}} - \frac{1}{g_{00}^{(0)}(r_0)r_0^{3\omega_q+1}} \right) \right),\tag{25}
\]

where \( g_{00}^{(0)} = 1 - 2M/r \) and we assume \( 2M/r \gg \alpha/r^{3\omega_q+1} \). In the weak field limit

\[
M/r \ll 1,\tag{26}
\]

Eq. (25) can be simplified to

\[
\nu / \nu_0 = 1 + \frac{M}{r} - \frac{M}{r_0} + \Delta_\nu,\tag{27}
\]

where

\[
\Delta_\nu = \frac{\alpha}{2} \left( \frac{1}{r^{3\omega_q+1}} - \frac{1}{r_0^{3\omega_q+1}} \right).\tag{28}
\]

Hence, the relative gravitational redshift (or redshift parameter) \( z \) is

\[
z = \frac{\nu - \nu_0}{\nu_0} = \frac{M}{r} - \frac{M}{r_0} + \Delta_\nu,\tag{29}
\]

where the first and second terms are the result of GR and the last one \( \Delta_\nu \) describes the amendment to Schwarzschild space by quintessence matter. Furthermore, the modified term \( \Delta_\nu \) satisfies the following relation

\[
\Delta_\nu \rightarrow \begin{cases} 
> 0 & \text{for } -1 < \omega_q < -1/3, \\
< 0 & \text{for } -1/3 < \omega_q < 0.
\end{cases}\tag{30}
\]

Comparing with the usual pure Schwarzschild result \[18\], we can find the redshift increases for \(-1 < \omega_q < -1/3\) and decreases for \(-1/3 < \omega_q < 0\).
IV. DEFLECTION OF LIGHT

Because that the movement of photon is described by the null line element $ds = 0$, $s$ can not be treated as the affine parameter any more. The original definition of 4D momentum $p^\mu = m \frac{dx^\mu}{d\tau}$ is not suitable to describe massless photons. Considering the above reasons, we choose an arbitrary scalar parameter $\zeta$ as a new affine parameter. Therefore the 4D momentum of photon is redefined as

$$p^\mu = \frac{dx^\mu}{d\zeta}. \quad (31)$$

When photon goes in the static spherically symmetric space [18], $p_0$ and $p_3$ are still conserved parameters. The first and second motion equations about $t$ and $\varphi$ are obtained naturally in the following equations:

$$r^2 \frac{d\varphi}{d\zeta} = L, \quad (32)$$

$$\left(1 - \frac{2M}{r} - \frac{\alpha}{r^{3\omega_q+1}}\right) \frac{dt}{d\zeta} = E. \quad (33)$$

The third motion equation can be given by the normalization relation of photons $g_{\mu\nu}p^\mu p^\nu = 0$,

$$\left(\frac{dr}{d\zeta}\right)^2 = E^2 - \frac{L^2}{r^2} \left(1 - \frac{2M}{r} - \frac{\alpha}{r^{3\omega_q+1}}\right). \quad (34)$$

Meanwhile, we define two new parameters, one is the impact parameter $b = L/E$, which means the effective sighting range, the other is the photon’s effective potential $1/B^2(r)$ where

$$B(r) = r \left(1 - \frac{2M}{r} - \frac{\alpha}{r^{3\omega_q+1}}\right)^{-1/2}. \quad (36)$$

The effective potential, which is illustrated by Fig. 1 becomes higher with smaller state parameter $\omega_q$.

According to Eq. (36), photons’ orbital equation (35) can be rewritten as

$$\left(\frac{1}{r^2} \frac{dr}{d\varphi}\right)^2 = \left(\frac{E}{L}\right)^2 - \frac{1}{r^2} \left(1 - \frac{2M}{r} - \frac{\alpha}{r^{3\omega_q+1}}\right). \quad (37)$$

Calculating the first order derivation of the trajectory equation (35) with respect to $\varphi$, the photon Binet equation can be obtained as follows,

$$\frac{d^2 u}{d\varphi^2} = -u + 3a^2 + \frac{3\alpha(\omega_q + 1)}{2M^{3\omega_q+1}}u^a, \quad (38)$$

where $u = M/r$ and $a = 3\omega_q + 2$. In the right hand side of the above equation, the second term is the general relativity correction and the third one is the quintessence contribution. Because $u$ can be treated as a small quantity in the weak field regime, we can solve Eq. (38) by the successive approximation method.

The negative pressure of quintessence matter $(p_q = \omega_q \rho_q)$ supplies the state parameter $-1 \leq \omega_q \leq 0$. The exponential $a$ in the additional term of Eq. (38) is in the scope $-1 \leq a \leq 2$. It is more difficult to solve Eq. (38) just by the undetermined state parameter $\omega_q$ and normalization $\alpha$. In order to solve this problem, we select the solvable integers $(a = -1, 0, 1, 2)$ in the range of $\omega_q \in [-1, 0]$. The four cases’ results, which are $(a = 2, \omega_q = 0)$, $(a = 1, \omega_q = -1/3)$, $(a = 0, \omega_q = -2/3)$ and $(a = -1, \omega_q = -1)$ respectively, are listed in the following TABLE I.
Here, the parameters \( u_0, y \) and \( y_0 \) are \( u_0 = \frac{GM}{R}, \ y = u + \frac{\sqrt{1-6\alpha M}}{6} \) and \( y_0 = u_0 + \frac{\sqrt{1-6\alpha M}}{6} \), where \( R \) is the solar radius. The calculation detail are shown in Appendix A, B, C and D.

The forms of influence on light deflection by quintessence are different according to various state parameters \( \omega_q \). The deflection angles depend sensitively on quintessence’s normalization parameter \( \alpha \). The deflection angles are illustrated in Fig. 2. For the case \( \omega_q = -2/3 \), the deflection angle increases monotonically as \( |\alpha| \) increases. On the contrary, the deflection angles decrease with increasing \( |\alpha| \) in the cases of \( \omega_q = -1/3 \) and \( \omega_q = -2/3 \). Meanwhile, comparing to the pure Schwarzschild case, the deflection becomes larger in the case of \( \alpha = 0 \) and becomes smaller in the cases \( \alpha = 1 \) and 2. These variations are caused by the fact that the modified term \( \frac{3\alpha(\omega_q+1)}{2M^{2\omega_q+1}}u^2 \), in the photon’s orbital equation (38), gives the different solutions. The behaviors of deflection heavily depend on the quantities of state parameter \( \omega_q \). It should be notice that in order to easily analyse the results, we recover the international system of units from the natural unit.

**FIG. 1:** The effective potentials \( 1/B^2(r) \) of photons with \( \alpha = -0.05 \) and unit mass \( M = 1 \). The corresponding Schwarzschild case \( \alpha = 0 \) is drawn by solid line.

**FIG. 2:** Deflection angle \( \delta \beta \) versus quintessence’s normalization parameter \( \alpha \) with \( M = 1, R = 10 \). The solid line denotes the pure Schwarzschild case 18 whose deflection angle \( \delta \beta = 4GM/R \) corresponds to the case of \( \alpha = -1 \).
TABLE I: The influence of quintessence on deflection of light

| state parameter $\omega_q$ reduced Binet equation | the first order approximate solution | deflection angle |
|--------------------------------------------------|--------------------------------------|------------------|
| $a = 2, \omega_q = 0$  
$\frac{d^2 u}{d\varphi^2} + u = (3 + \frac{\omega_q}{2}) u^2$  
$u = u_0 \cos \varphi + (1 + \frac{\omega_q}{2}) u_0^2 (1 + \sin^2 \varphi)$ | $\delta \varphi = 2\beta = 4 \left(1 + \frac{\omega_q}{2}\right) \frac{GM}{R^2}$ | |
| $a = 1, \omega_q = -1/3$  
$\frac{d^2 u}{d\varphi^2} + (1 - \alpha)u = 3u^2$  
$u = u_0 \cos \varphi \sqrt{1 - \alpha} + \frac{u_0}{1} (1 + \sin^2 \varphi \sqrt{1 - \alpha})$ | $\delta \varphi = \frac{4}{(1 - \alpha)^{3/2}} \frac{GM}{R}$ | |
| $a = 0, \omega_q = -2/3$  
$\frac{d^2 u}{d\varphi^2} + u - \frac{\omega_q}{2} = 3u^2$  
$y = y_0 \cos[\varphi(1 - 6\alpha)M^{1/4}] + \frac{3}{2}\frac{\alpha M}{\sqrt{1 - \varphi}} (1 + \sin^2 \varphi (1 - 6\alpha M)^{1/4})$ | $\delta \varphi = \frac{4\omega_q}{(1 - 6\alpha M)^{7/4}}$ | |
| $a = -1, \omega_q = -1$  
$\frac{d^2 u}{d\varphi^2} + u = 3u^2$  
$u = u_0 \cos \varphi + u_0^2 (1 + \sin^2 \varphi)$ | $\delta \varphi = 4GM/R$ | |

V. EXPERIMENTAL CONSTRAINTS

We know that astronomical observations, such as the gravitational lensing measurements, become important in determining cosmological parameters. Here we will do this work in the view of black hole. We compare the theoretical predictions of gravitational frequency shift and deflection of light with the typical experiments and give the experimental constraints on the cosmological parameters.

Firstly, we consider the case of gravitational frequency shift. The results of frequency comparison/clock comparison are given by the hydrogen masers frequency standard in the GP-A redshift experiment [19]. The continuous microwave signals are generated from H-masers located in the spacecraft and at an Earth station where the spacecraft was launched nearly vertically upward to 10000 km. This experiment reached a $10^{-14}$ accuracy. So, if we consider the quintessence field’s effect, the parameter $\alpha$ has to satisfy the constraint

$$|\alpha| \lesssim 2 \times 10^{-14} \times \frac{(rr_0)^{3\omega_q+1}}{|r^{3\omega_q+1} - r_0^{3\omega_q+1}|}.$$  \hspace{1cm} (39)

The constraints of field parameter $\alpha$ are presented in TABLE [II] Generally speaking, the upper limit of field parameter $|\alpha|$ increases with bigger state parameter $\omega_q$. When $\omega_q$ crosses over $-1$ the maximum $|\alpha|$ is below the value of $10^{-28}$, which is consistent with the cosmological constant case $|\alpha| \lesssim 10^{-28}$ [17]. However, there is a singular point located at $\omega = -1/3$ since the definition of $\omega_q = -1/3$ does not exist in Eq.(39). Else, when $\omega_q \to -1/3$, i.e. $\Delta_\nu \to 0$, the redshift parameter $z$ [20] reduces to the result of pure Schwarzschild case.

TABLE II: The constraint on field parameter $\alpha$ by H-masers of GP-A redshift experiment.

| $\omega_q$  | 0  | -0.2  | -0.4  | -2/3  | -0.8  | -1  |
|-------------|----|-------|-------|-------|-------|-----|
| Estimate on $|\alpha|$  | $\lesssim 3 \times 10^{-7}$  | $\lesssim 6 \times 10^{-11}$  | $\lesssim 8 \times 10^{-15}$  | $\lesssim 5 \times 10^{-21}$  | $\lesssim 6 \times 10^{-24}$  | $\lesssim 3 \times 10^{-28}$  |

Secondly, we consider the other case of deflection of light. In order to obtain experimental constraints from the light deflection result, the final light deflection angle $\delta \varphi$ should be expressed in terms of the deviation $\Delta_{LD}$ from the general relativity prediction $\delta \varphi_{GR}$ for the Sun.

$$\delta \varphi = \delta \varphi_{GR}(1 + \Delta_{LD}),$$  \hspace{1cm} (40)

where $\delta \varphi_{GR} = 4GM/R$. The best available constraints on $\delta \varphi_{GR}$ come from long-baseline radio interferometry which shows that $|\Delta_{LD}| \lesssim 0.0017$ [20]. Submitting the deflection angles of $\omega_q = -1, -2/3, -1/3, 0$ shown in TABLE II into Eq.(40), we can obtain the corresponding constraints on the field parameter $\alpha$, which are present in TABLE III. The constraint on $\alpha$ becomes stronger with the decreasing state parameter $\omega_q$.

VI. CONCLUSION

In this paper, we have studied the influence of free quintessence on gravitational frequency shift and deflection of light based on the 4D momentum. We summarize what have been achieved.
TABLE III: Estimates on α from Solar system observation.

| ωq | ΔLD | Estimate on α |
|-----|------|---------------|
| 0   | \( \frac{\alpha}{2M} \) | \(|\alpha| \lesssim 10^{27}\) |
| −1/3| \( 1 - (1 - \alpha)^{-3/2} \) | \(|\alpha| \lesssim 10^{-3}\) |
| −2/3| \( (1 + \frac{2}{3}M)(1 - \frac{\alpha M}{2}) - 1 \) | \(|\alpha| \lesssim 10^{-34}\) |
| −1  | 0    | —             |

1. The influence of quintessence matter on our space is expressed mathematically as a new metric form. Taking advantage of the symmetries of Kiselev solution, we can avoid to solve directly the original geodesic equation. The inner product \( u^a \xi_a \) is constant along the geodesic in a Killing field \( \xi^a \) with a geodesic tangent \( u^a \). In this paper, starting from the 4D momentum the geodesics problem is reduced to the problem of one-dimensional motion of a particle in an effective potential. The null geodesic of the modified Schwarzschild geometry containing the quintessence matter is solved by the usual method \[18\]. We analyze the behavior of light ray in the weak field regime \( r \gg M \).

Also, this result can be applied to the exterior field of ordinary body such as the Sun or more large astronomical scale. Meanwhile, the quintessence matter is also treated as a very small component in whole space, i.e. \( M/r \gg \alpha/r^{3\omega + 1} \). These appropriate approximation can help us to simplify the calculation of the gravitational redshift and the bending of light.

2. The small \( g_{00} \) in a metric is corresponding to the high frequency because the frequency of photon is inverse proportional to local \( \sqrt{g_{00}} \), i.e. \( \nu \propto 1/\sqrt{g_{00}} \). The gravitational redshift expression (29) is obtained naturally in the weak field limit. It is clearly that the frequency is larger (or smaller) for \( -1 < \omega_q < -1/3 \) (or \( -1/3 < \omega_q < 0 \)) than that of the pure Schwarzschild case from the result of redshift (27).

3. Comparing with the gravitational redshift, the deflection of light is more or less complex with the uncertain power \( a \) in the orbital equation \[38\]. We choose the solvable integral numbers \( a \) in the range of \( \omega_q \in [-1,0] \) to obtain the deflected angle. The solutions of the orbital equation \[38\] depend sensitively on the value of \( \omega_q \). Furthermore, when \( a = -1 \), i.e. \( \omega_q = -1 \), there is no influence on bending of light by quintessence matter, which is in accordance with the theoretical prediction of SdS case \[13\]. This point also illustrates well that the border case of the extraordinary quintessence \( \omega_q = -1 \) covers the cosmological constant term \[4\].

4. The parameter α is of course a crucial parameter, which indicates the influence of quintessence matter over the space, it is therefore necessary to give its realistic order of magnitude. Based on the the relation \[13\], we evaluate roughly the order of α in TABLE IV in the four different astronomical scales, i.e. Solar, Galaxy, Cluster of Galaxies and Supercluster. In the range of Solar system, the measuring unit of the distance between two planets is Astronomical Unit (AU). It is well known that the nearest Mercury is about 0.4 AU and the most remote Pluto is about 40 AU far from the sun. Here the average orbital radius of Pluto is chosen as the value of the parameter \( r \) in Solar system. The other three large astronomical scales are estimated in the order of magnitude. If quintessence matter can be treated as the dark energy model, its density \( \rho_q \) can be assumed as the critical density of the universe, namely,

\[
\rho_q = \begin{cases} 
10^{-46} \text{GeV}^4 & \text{(natural unit),} \\
10^{-4} \text{eV/cm}^3 & \text{(energy density unit),} \\
10^{-29} \text{g/cm}^3 & \text{(mass density unit).}
\end{cases}
\]

Hence, substituting the density \( \rho_q \) and the different scales \( r \) into relationship \[13\], we can get the order of magnitude of α with different state parameter \( \omega_q \), which is shown in TABLE IV. Apparently, the quintessence matter makes more influence in the large scale. Comparing the long-baseline radio interferometry experiment results with the theoretical expectation values TABLE IV, we can find that the case of \( \omega_q = -2/3 \) violates many orders of magnitude of \( |\alpha| \) in solar. So the case of \( \omega_q = -2/3 \) should be abandoned in solar system.
There are two easily confused points should be clarified in the end. Firstly, the parameter $\alpha$ in Eq. (11) and Eq. (18) are the same as a matter of fact. The original Kiselev’s black hole solution is the general form of exact spherically-symmetric solutions for the Einstein equations describing black holes surrounded by the quintessential matter with the energy momentum tensor yielding the additive and linear conditions (8) and (9), i.e. Eqs. (13) — (14) in original paper [13]. If we only consider the simplest singular Schwarzschild case, the general solution to Eq. (10) should be in the form of

$$e^{-\lambda} = 1 - \frac{r_g}{r} - \frac{\alpha}{r^{3\omega_q+1}}.$$  \hspace{1cm} (42)

So the modified Schwarzschild black hole solution (18) can be obtained without changing parameter $\alpha$. Mathematically, if we adopt a singular field, not the multiple fields, the first term of Kiselev’s solution should be

$$\tilde{g}_{00} = 1 - \frac{r_g}{r} - \left(\frac{r_0}{r}\right)^{3\omega_q+1}.$$  \hspace{1cm} (43)

Comparing with the exact solutions (11) and (12), we can get following equation

$$\left(\frac{r_0}{r}\right)^{3\omega_q+1} = \frac{\alpha}{r^{3\omega_q+1}}.$$  \hspace{1cm} (44)

So it is clear that two relationships $r_0^{3\omega_q+1} = \alpha$ and $\omega_0 = \omega_q$ exist here. Hence, the parameters $\alpha$ in Eqs. (11) and (18) are the same.

Secondly, the parameter $M$ is actually the mass of black hole (or the center mass of gravitational field), not the mass of outside quintessence matter. The form of quintessence field is denoted by the energy momentum tensor yielding the conditions (8) and (9). Since the dark energy almost has the ratio $\Omega_{de} \approx 0.67 \pm 0.06$ in universe, the mass of quintessence $M_q$ is far more than the mass of a stellar black hole $M$, i.e. $M_q \gg M$. So the quintessence density $\rho$ (13) does not depend largely on the mass of black hole and $M$ can be assumed as a constant. This assumption also can be found in the corresponding extensions of this quintessence black hole solution [14], [15], [16].

| Table IV: The theoretical magnitude order of parameter $\alpha$ in several astronomical scales |
|---------------------------------|---------------------------------|------------------|------------------|------------------|
| astronomical scales           | Solar (40AU)                     | Milky Way (10⁴Pc) | Cluster of Galaxies (1MPc) | Supercluster (10MPc) |
| $\omega_q = -1/3$              | $7.16 \times 10^{-13}$ $Kg \cdot m^{-1}$ | $1.90 \times 10^{9}$ $Kg \cdot m^{-1}$ | $1.90 \times 10^{13}$ $Kg \cdot m^{-1}$ | $1.90 \times 10^{14}$ $Kg \cdot m^{-1}$ |
| $\omega_q = -2/3$              | $5.98 \times 10^{-23}$ $Kg \cdot m^{-2}$ | $3.08 \times 10^{-12}$ $Kg \cdot m^{-2}$ | $3.08 \times 10^{-10}$ $Kg \cdot m^{-2}$ | $3.08 \times 10^{-11}$ $Kg \cdot m^{-2}$ |
| $\omega_q = -1$               | $6.67 \times 10^{-33}$ $Kg \cdot m^{-3}$ |                               |                               |                               |

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APPENDIX A: THE CASE OF $a = 2, \omega_q = 0$

When the state parameter $\omega_q = 0$, Eq. (38) reduces to

$$\frac{d^2 u}{d\varphi^2} + u = \left(3 + \frac{3\alpha}{2M}\right) u^2.$$  \hspace{1cm} (A1)

Because of the small $u$, we first ignore the 2th order small quantity $3u^2$. The zeroth order approximate solution is

$$u = u_0 \cos \varphi,$$  \hspace{1cm} (A2)
where \( u_0 = GM/R \) and \( R \) is the solar radius. This is a straight line normal to pole axis. Substituting Eq. (A2) into the right hand side of Eq. (A1), we can get
\[
\frac{d^2 u}{d\varphi^2} + u = \left(3 + \frac{3\alpha}{2M}\right) u_0^2 \cos^2 \varphi.
\] (A3)

This equation has a particular solution,
\[
u = \left(1 + \frac{\alpha}{2M}\right) u_0^2 (1 + \sin^2 \varphi).
\] (A4)

So the first order approximate solution to Eq. (A1) is
\[
u = u_0 \cos \varphi + \left(1 + \frac{\alpha}{2M}\right) u_0^2 (1 + \sin^2 \varphi).
\] (A5)

The azimuth angle of null order approximate solutions (A2) are \( \pm \pi/2 \) in very far place \( u = 0 \). However, the azimuth angles of the first order one (A5) are \( \pm (\pi/2 + \beta) \) in \( u = 0 \). The deflection angle \( \beta \) is a small quantity which satisfies
\[-u_0 \sin \beta + \left(1 + \frac{\alpha}{2M}\right) u_0^2 (1 + \cos^2 \beta) = 0.
\] (A6)

We use the Taylor expansions of sine and cosine functions and keep the basic term. The deflection angle is written formally as
\[
\delta \varphi = 2\beta = 4 \left(1 + \frac{\alpha}{2M}\right) \frac{GM}{R}.
\] (A7)

Comparing with the result of pure Schwarzschild space [18], the additional term containing parameter \( \alpha \) is the contribution of quintessence matter. The case of \( \omega_q = 0 \) affects the light deflection only through the particular solution (A4). But the zeroth order approximate (A2) is unchanged.

**APPENDIX B: THE CASE OF \( a = 1, \omega_q = -1/3 \)**

When \( \omega_q = -1/3 \), Eq. (38) reduces to
\[
\frac{d^2 u}{d\varphi^2} + (1 - \alpha) u = 3u^2.
\] (B1)

The process of solving this equation is similar to the formal one. The zeroth order approximate solution is
\[
u = u_0 \cos(\varphi \sqrt{1 - \alpha}).
\] (B2)

The particular solution is
\[
u = \frac{u_0^2}{1 - \alpha} \left(1 + \sin^2(\varphi \sqrt{1 - \alpha})\right).
\] (B3)

The first order approximate solution is
\[
u = u_0 \cos(\varphi \sqrt{1 - \alpha}) + \frac{u_0^2}{1 - \alpha} \left(1 + \sin^2(\varphi \sqrt{1 - \alpha})\right).
\] (B4)

So the deflection angle is
\[
\delta \varphi = \frac{4}{(1 - \alpha)^{3/2}} \frac{GM}{R}.
\] (B5)

The deflection behavior of the case \( \omega_q = -1/3 \) is different from the case of \( \omega_q = 0 \). On the contrary, the quintessence matter effects the photons deflection through the zero order solution (B2). So the later particular solution and first order approximate solution vary accordingly.
APPENDIX C: THE CASE OF $a = 0, \omega_q = -2/3$

When $\omega_q = -2/3$, Eq. (38) reduces to

$$
\frac{d^2 u}{d\varphi^2} + u - \frac{\alpha M}{2} = 3u^2. \quad (C1)
$$

In this case, the additional term $\alpha M/2$ changes the form of the zeroth order approximate solution. In order to simplify calculation, we use a transformation

$$
u = y + \frac{1 - \sqrt{1 - 6\alpha M}}{6}. \quad (C2)
$$

So the Eq. (C1) can be rewritten as a usual orbital equation

$$
\frac{d^2 y}{d\varphi^2} + \sqrt{1 - 6\alpha M} y = 3y^2. \quad (C3)
$$

Its first order approximate solution is

$$
y = y_0 \cos \left[ \varphi (1 - 6\alpha M)^{1/4} \right] + \frac{y_0^2}{\sqrt{1 - 6\alpha M}} \left\{ 1 + \sin^2 \left[ \varphi (1 - 6\alpha M)^{1/4} \right] \right\}. \quad (C4)
$$

Here we only consider the small contribution of quintessence matter in the large space background. Hence, when $u \rightarrow 0$, the limit of $y$ is also small. So the deflection angle is

$$
\delta \varphi = \frac{4y_0}{(1 - 6\alpha M)^{3/4}}, \quad (C5)
$$

where

$$
y_0 = u_0 + \frac{\sqrt{1 - 6\alpha M} - 1}{6}. \quad (C6)
$$

APPENDIX D: THE CASE OF $a = -1, \omega_q = -1$

The additional term vanishes for the factor $\omega_q + 1 = 0$. Hence, the influence of quintessence can be ignored. This point is also justified by potential analysis. Since the exponential of additional term is negative $3\omega_q + 2 < 0$, $u^a$ should not be considered as a small quantity any more. The successive approximation is failing here. However, the photon’s effective potential can be rewritten as

$$
\frac{1}{B^2(r)} = r^{-2} \left( 1 - \frac{2M}{r} - r^2 \alpha \right). \quad (D1)
$$

The photons’ trajectory equation (35) of $\omega_q = -1$ is

$$
\frac{dr}{d\varphi} = \pm r^2 \left[ \frac{1}{b^2} - \frac{1}{B^2(r)} \right]^{1/2}, \quad (D2)
$$

where $b$ is the impact parameter and the $\pm$ denotes to increasing or decreasing $r$. According to the condition of perihelion position $r_0$, we can get

$$
\left. \frac{dr}{d\varphi} \right|_{r=r_0} = \frac{1}{b^2} - \frac{1}{B^2(r_0)} = 0. \quad (D3)
$$

So the impact parameter $b$ can be expressed by perihelion position $r_0$. Submitting $b$ into Eq. (D2), the quintessence matter can be dropped out,

$$
\frac{dr}{d\varphi} = \pm r^2 \left[ \frac{1}{r_0} \left( 1 - \frac{2M}{r_0} \right) - \frac{1}{r} \left( 1 - \frac{2M}{r} \right) \right]^{1/2}. \quad (D4)$$
Therefore, the quintessence matter has no influence on the light deflection in this case, which is also justified by the articles [17].

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