Chiral $f$-wave Topological Superfluid in Triangular Optical Lattices

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We demonstrate that an exotically chiral $f$-wave topological superfluid can be induced in cold-fermionic-atom triangular optical lattices through the laser-field-generated effective non-Abelian gauge field, controllable Zeeman fields and $s$-wave Feshbach resonance. We find that the chiral $f$-wave topological superfluid is characterized by three gapless Majorana edge states located on the boundary of the system. More interestingly, these Majorana edge states degenerate into one Majorana fermion bound to each vortex in the superfluid. Our proposal enlarges topological superfluid family and specifies a unique experimentally controllable system to study the Majorana fermion physics.

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I. INTRODUCTION

Topological superconductors (TSCs) and superfluids (TSFs)$^{[1,2]}$ have attracted considerable interest in condensed matter physics because of their potential applications on the fault-tolerant topological quantum computation (TQC)$^{[3]}$. One of the remarkable features in TSCs/TSFs is the helicity or chirality of the unconventional pairings. Unfortunately, there are very limited natural materials$^{[4,5]}$ exhibiting these kinds of unconventional pairings. Starting with the pioneer work by Fu et al.$^{[6]}$, recently, some new classes of hybridized systems$^{[7–10]}$ have been proposed as possible candidates for TSCs, where the unconventional pairings are induced from proximity effects of conventional $s$-wave superconductor films. However, the impurities or disorders in the materials hosting the electron gas increase the difficulties to investigate the topological properties in the hybridized systems from experiments$^{[11]}$. Therefore, it should be not only interesting but also necessary to design other systems that present TSC/TSF phases.

The ultra-cold atom gas associated with optical lattice technology provide an ideal platform to realize and investigate the topological phases$^{[12–10]}$ due to the controllability and cleanly. In particular, some chiral $p$-wave TSFs$^{[13–15]}$ have been proposed based on the laser-induced artificial gauge fields$^{[16–18]}$ in cold atom systems. The effect of the artificial gauge fields is equivalent to the spin-orbit coupling, a key factor to induce topological phase. More recently, some experimental groups reported the realization of strong spin-orbit coupling in ultra-cold fermionic atoms gas$^{[19,20]}$ and $^6$Li$^{[19,21]}$. This new technique brings the huge hope to realize many exotic states related with spin-orbit coupling.

Recently, triangular optical lattices (TOLs) have been widely investigated in experiments and theory, and the external fields and the different types of interactions among the filled ultra-cold atoms can induce rich quantum phases in the TOLs$^{[22–23]}$. In this paper, we propose that an exotically chiral $f$-wave TSF can be realized through the effective $k^3$ Rashba spin-orbit coupling (RSOC)$^{[24]}$, Zeeman field (ZF) and $s$-wave Feshbach resonance in triangular optical lattices (TOLs). The effective $k^3$ Rashba SOC and ZF are produced by the laser-atom interactions through modulating applied laser beams. The $s$-wave Feshbach resonance is utilized to induce the SF states$^{[25]}$ of the trapped atoms. We find that there exists a phase transition separating the TSF and normal superfluid (NSF), which is determined by the bulk gap closing mechanism$^{[26]}$. The TSF resembles the SF with $f$-wave pairing symmetry$^{[27]}$, which is consistent with the geometrical symmetry of the TOLs. The chiral $f$-wave TSF is fully gapped in bulk and has three chiral gapless edge states located on the boundary. More interestingly, the TSF can be modulated through initializing the lasers. Furthermore, there is one stable Majorana fermion bound to each vortex in the TSF, and the commensurability between the SF vortex lattice structures and the TOLs shows advantages to investigate the properties of Majorana fermions. Hence, these properties make the system a potential candidate to perform QTC.

The paper is organized as follows. In Sec. II, we propose a scheme to simulate the RSOC and ZF through the laser-atom interaction in triangular lattices and an effective tight-binding Hamilton describing the fermionic atom in dark states is deduced. In Sec. III, through applying the $s$-wave Feshbach resonance, we discuss the properties of the TSF and NSF with mean-field approximation. Furthermore, we discuss the Majorana zero mode in the vortex structure in the TSF states. In Sec. IV, we summarize our results.

II. SIMULATION AND MODEL HAMILTONIAN
Firstly, we apply three blue detuned laser beams to create two-dimensional TOLs, which can trap atoms at lattice sites. The three laser beams have same wave-vector length but different polarizations and are applied along three different directions: \( \pm \frac{\sqrt{3}}{2} \hat{x} \) and \( \pm \frac{1}{2} \hat{y} \), respectively. The total potential is given by \( V(\mathbf{r}) = V_L \sum_{i=1}^{3} \cos(\mathbf{k}_i \cdot \mathbf{r}) \) with wave vectors \( \mathbf{k}_{1,2} = \left( \pm \frac{\sqrt{3}}{2}, \frac{1}{2} \right) \) and \( \mathbf{k}_3 = k (0, 1) \). The pattern of the potential is shown in Fig.1(a), where the maxima form perfect TOLs.

In order to simulate the RSOC, we consider the ultra-cold fermionic atoms trapped in the TOLs and having tripod-type level configuration (e.g., the lowest three Zeeman levels of \(^6\)Li atoms near the broad s-wave Feshbach resonance)\(^{12, 18, 28}\) shown in Fig. 2(a).

\( \sum_{i,\alpha} | a_{i,\alpha} \rangle = - \sum_{i,\alpha} \mu a_{i,\alpha}^\dagger a_{i,\alpha} - \sum_{i,j,\alpha,\beta} t_{i,j} a_{i,\alpha}^\dagger a_{j,\beta} \) \hspace{1cm} (2)

\( H_{1-a} = \sum_{i,v} \delta_v a_{i,4}^\dagger a_{i,4} - \sum_{i,v,\alpha} \hbar \Omega_{v,\alpha} a_{i,\alpha}^\dagger a_{i,\alpha} + H.c \) \hspace{1cm} (3)

Here, \( H_{1-a} \) is the tight-binding Hamiltonian describing the atom hopping between different sites, and \( H_{1-a} \) describes the laser-atom coupling. \( \mu \) is the chemical potential. \( a_{i,\alpha}^\dagger \) is the creation operator of atom on site \( i \) and in state \( \alpha \) with \( \alpha = 1, 2, 3 \). \( t_{i,j} \) is the hopping integral between site \( i \) and \( j \). \( \delta_v \) is the detuning to the excited state 4.

Since the energy scale of \( \delta_v \) is much larger than that of \( \mu \) and \( t_{i,j} \) (See the below discussions about the parameters parts.), we firstly consider \( H_{1-a} \). The eigenvalues of \( H_{1-a} \) can be obtained from the diagonalization. Namely, \( E_{i,v,n=1,2,3,4} = 0, 0, \frac{1}{2} \delta_v \pm \sqrt{\delta_v^2 + \hbar^2 \Omega_v^2} \). The corresponding eigenstates (dressed states) are

\[
\begin{align*}
|D_{i,v,1}\rangle &= \sin \phi_v e^{iS_{v,3.1}} |1\rangle - \cos \phi_v e^{iS_{v,3.2}} |2\rangle \\
|D_{i,v,2}\rangle &= \cos \theta_v \cos \phi_v e^{iS_{v,3.1}} |1\rangle + \cos \theta_v \sin \phi_v e^{iS_{v,3.2}} |2\rangle - \sin \theta_v |3\rangle \\
|B_{i,v,1}\rangle &= \frac{1}{\sqrt{\Omega_v^2 + (2E_{i,v,3})^2}} \left[ \Omega_v \left( \sin \theta_v \cos \phi_v e^{iS_{v,3.1}} |1\rangle + \sin \theta_v \sin \phi_v e^{iS_{v,3.2}} |1\rangle + \cos \theta_v |3\rangle \right) - 2E_{i,v,3} e^{iS_{v,3}} |4\rangle \right] \\
|B_{i,v,2}\rangle &= \frac{1}{\sqrt{\Omega_v^2 + (2E_{i,v,4})^2}} \left[ \Omega_v \left( \sin \theta_v \cos \phi_v e^{iS_{v,3.1}} |1\rangle + \sin \theta_v \sin \phi_v e^{iS_{v,3.2}} |1\rangle + \cos \theta_v |3\rangle \right) - 2E_{i,v,4} e^{iS_{v,3}} |4\rangle \right].
\end{align*}
\]

Here, \( |B_{i,v,1/2}\rangle \) represent zero-energy dark states, while \( |D_{i,v,1/2}\rangle \) represent the nonzero-energy bright states. That means the energy of dark states is not adjusted by the laser fields. Moreover, the dark states \( |D_{i,v,1/2}\rangle \) have no coupling with the initial excited state 4. Therefore, the dark states are stable under atomic spontaneous emission. With the adiabatic approximation\(^{12, 28}\), we can neglect all the couplings that simultaneously involve the dark states and bright states and reduce the Hamiltonian \( H_0 \) into the subspace spanned by the dark states. In general, the dark states \( |D_{i,v,1/2}\rangle \) produced by laser set \( a \) (See the red lines in Fig. 2(a)) are different from the dark states \( |D_{i,b,1/2}\rangle \) produced by laser set \( b \) (See the green lines in Fig. 2(a)). However, the two sets of dark
subjected into the subspace spanned by fine levels of ultra-cold atom and three sets of laser beams effective Zeeman fields. (a) The configuration of the hyperfine energy levels modulated by the atom-laser couplings. For Ω

For the unitary operator:

Here, c_i is the creation operator of atom on site i in eigenstate |D_i⟩. The Peierls phase factor is,

Where \tilde{A}_{v,\sigma,\sigma'} is the laser-field-induced gauge vector potential, and \tilde{A}_{v,\sigma,\sigma'} = i\hbar\langle D_{v,\sigma'}|\nabla|D_{v,\sigma'}\rangle. We list the forms of \tilde{A}_{v,\sigma,\sigma'} for completeness:

Note that since the two sets of lasers a and b have different detunings \delta_a and \delta_b to the excited state |4⟩, there are no interference effects between the two sets, and they interact with the atoms independently. The total gauge vector potentials are the simple sum of \tilde{A}_{a,\sigma,\sigma'} + \tilde{A}_{b,\sigma,\sigma'}.

Now, we evaluate that the effective RSOC can be simulated by the aforementioned two sets of lasers a and b. For convenience, we define the (next) nearest neighbor lattice vectors: s_9 = |B_9⟩ shown in Fig. 1 (a) with n=1...6, and set the lattice constant 1. For the two sets of lasers configuration illustrated in Fig. 2 (b) and (c), we can find that U_{s_9} = e^{i4\pi \cos \varphi_x} and U_{s_9} = e^{i4\pi \cos \varphi_y} are the only nontrivial phase factors and other U_{s_n/d_s} are trivial and equal 1. Since the TOLs have the rotation symmetry of point group C_{3v}, rotating the laser beams or the lattice systems with \pm \frac{\pi}{3} gives another two groups of nontrivial phase factors:

For a two-component spin system, we have the following relation for the unitary operator:

With Eq. (8), we can find that H_{eff} in Eq. (5) has the form as follows,
The corresponding Rabi frequencies are parameterized as $H_{i\sigma} = \frac{\mu}{2} c_{i\sigma}^\dagger c_{i\sigma} - t \cos \alpha \sum_{n} c_{i\sigma}^\dagger c_{i+n\sigma} - t' \cos \alpha \sum_{n} c_{i\sigma}^\dagger c_{i+n\sigma}$. Hence, we can only consider the effect of hopping and RSOC. That is nearly impossible in the condensed matter system. The $k^3$ type of RSOC can be easily found in the momentum space form of Eq. (9), which we will discuss in the next section.

In the following part of this section, we simulate how to generate an effective ZF to split two pseudo-spin states $|D_{i\uparrow}\rangle$ and $|D_{i\downarrow}\rangle$. We apply two additional laser beams that couple the states $|1\rangle$ and $|2\rangle$ to the excited state $|3\rangle$ (See the blue lines in Fig. 2 (a)). The laser-atom interaction is

$$H_{\text{int}} = -\sum_i \delta \sigma_{i} a_{i,a}^\dagger a_{i,a} + \sum_{i,a=1}^{a=2} \hbar \Omega_{c,a} a_{i,a}^\dagger a_{i,a} + H.c.$$ (10)

The corresponding Rabi frequencies are parameterized as $\Omega_{c,1} = \frac{\sqrt{2}}{4} \Omega c e^{i4\pi x}$ and $\Omega_{c,2} = \frac{\sqrt{2}}{4} \Omega c e^{-i4\pi x}$ with $\sqrt{\Omega_{c,1}^2 + \Omega_{c,2}^2} = \frac{\Omega c}{2}$ and $\hbar \Omega c \ll \delta_c$. The lasers configuration is illustrated in Fig. 2 (d).

The eigenvalues of $H_{\text{int}}$ can be obtained from the diagonalization. Namely, $E_{i,n=1,2,3} = 0, -\frac{1}{2}(\delta_c + \sqrt{\delta_c^2 + \hbar^2 \Omega_c^2})$. The corresponding eigenstates are:

$$|1\rangle = \frac{\sqrt{2}}{2} e^{-i4\pi x} |1\rangle - \frac{\sqrt{2}}{2} e^{i4\pi x} |2\rangle$$

$$|2\rangle = \frac{\sqrt{2}}{2} \cos \beta e^{-i4\pi x} |1\rangle + \frac{\sqrt{2}}{2} \cos \beta e^{i4\pi x} |2\rangle - \sin \beta |3\rangle$$

$$|3\rangle = \frac{\sqrt{2}}{2} \sin \beta e^{-i4\pi x} |1\rangle + \frac{\sqrt{2}}{2} \sin \beta e^{i4\pi x} |2\rangle + \cos \beta |4\rangle$$ (11)

Here, tan $\beta = (\delta_c^2 - \hbar^2 \Omega_c^2) / \delta_c$. Due to $\hbar \Omega_c \ll \delta_c$, we can get $\tan \beta \sim \hbar \Omega_c / \delta_c \sim 0$, and $E_{i,2}^{\prime} \sim \hbar^2 \Omega_c^2 / 4 \delta_c$.

$$|1\rangle \sim \frac{\sqrt{2}}{2} e^{-i4\pi x} |1\rangle + \frac{\sqrt{2}}{2} e^{i4\pi x} |2\rangle, |3\rangle \sim |4\rangle.$$ (12)

Since $E_{1}^{\prime} = 0$, there is no effect of $|1\rangle$ to the ground states $|1\rangle$ and $|2\rangle$, and $|3\rangle$ also has no effect to $|1\rangle$ and $|2\rangle$. Hence, we can only consider the effect of $|2\rangle$ to $|1\rangle$ and $|2\rangle$. If we define that $d_{i}^\dagger$ is an operator to create an atom on site $i$ in eigenstate $|\chi_2\rangle$. A perturbation Hamiltonian can be written as: $H_p = \hbar \Omega_p \sum_i d_{i}^\dagger d_i$. With Eq. (12), $H_p$ has the form:

$$H_p = H_{ac} + \hbar \Omega_p \sum_i e^{-i8\pi x} a_{i,1}^\dagger a_{i,2} + H.c.$$ (13)

Where $\Omega_p = \hbar \Omega_c^2 / 8 \delta_c$. $H_{ac} = \hbar \Omega_p \sum_i (a_{i,1}^\dagger a_{i,1} + a_{i,2}^\dagger a_{i,2})$ is a constant ac-Stark shift, whose effect can be canceled with a frequency offset of the laser beams $\Omega_{c,3}$ applied to the level $|3\rangle$ [29]. Therefore, we can only consider the effect of the second term in Eq. (13). From Eq. (14), we can get the following relations

$$|1\rangle = \frac{\sqrt{2}}{2} (|D_{i,\uparrow}\rangle + \cos \varphi |D_{i,\downarrow}\rangle)$$

$$|2\rangle = \frac{\sqrt{2}}{2} (-|D_{i,\uparrow}\rangle + \cos \varphi |D_{i,\downarrow}\rangle)$$ (14)

Where we have applied the conditions that at all the lattice sites, $e^{\pm i8\pi x} = 1$, $e^{\pm i8\pi \sigma_{x}} = 1$, $\theta = \varphi$ and $\phi = \pi/4$. With Eq. (11), we find the second term in Eq. (13) induce a splitting between $|D_{i,\uparrow}\rangle$ and $|D_{i,\downarrow}\rangle$ as:

$$H_s = -\hbar \Omega_p \sum_i (c_{i,1}^\dagger c_{i,1} - \cos^2 \varphi c_{i,4}^\dagger c_{i,4})$$

$$= -\hbar \Omega_p \sum_{i,\sigma} (c_{i,\sigma}^\dagger c_{i,\sigma} - \hbar z \sum_{i,\sigma} (c_{i,\sigma}^\dagger c_{i,\sigma} - c_{i,\sigma}^\dagger c_{i,\sigma}))$$

with $\hbar_0 = \hbar \Omega_p (1 - \cos^2 \varphi) / 2$ and $\hbar z = \hbar \Omega_p (1 + \cos^2 \varphi) / 2$. $\hbar_0$ can be renormalized into the chemical potential term in $H_{s,eff}$ (Eq. (12)), and $\hbar z$ describes the effective ZF. In order to guarantee that $H_s$ cannot pump the atoms outside of the dark-state subspace, the conditions: $h \Omega_p \ll |E_{b,3}| < |E_{a,3}|$ must be fulfilled (See Fig. 2 (e)). Now, the new Hamiltonian including the effective RSOC and ZF is:

$$H_0 = H_{s,eff} + H_s.$$ (15)

### III. TOPOLOGICAL SF AND MAJORANA FERMION

The SF states can be induced by atomic interaction from the s-wave scattering. The interaction term is described by the Hamiltonian:
where $\alpha$ and $\beta$ label three ground states of atoms and $V_{\alpha\beta}$ are proportional to s-wave scattering lengths between $\alpha$, $\beta$ channel. In s-wave SF state, $H_{int}$ can be decoupled on the mean-field level:

$$H_{mf} = \sum_i \sum_{\alpha < \beta} \Delta_{\alpha\beta} a^\dagger_{i,\alpha} a^\dagger_{i,\beta} a_{i,\beta} a_{i,\alpha} + H.c.$$  \hspace{1cm} (17)

with $\Delta_{\alpha\beta} = V_{\alpha\beta} \langle a_{i,\beta} | a_{i,\alpha} \rangle$, the SF order parameter. Under the condition of $\Delta_{\alpha\beta} \ll \Omega_{a/b}$, it is safe to consider the SF in the dark-state subspace, because $H_{mf}$ cannot pump the atoms outside of the dark-state subspace. Then, we project $H_{mf}$ to the dark-state subspace and have

$$H'_{mf} = \sum_i \Delta_0 c^\dagger_{i,k} c_{i,k} + H.c.$$  \hspace{1cm} (18)

with $\Delta_0$ the linear combinations of $\Delta_{\alpha\beta}$.

In the following parts of the paper, we focus on the total Hamiltonian which describes the SF states:

$$H_t = H_0' + H_{mf}.$$  \hspace{1cm} (19)

After the Fourier transformation, in momentum Nambu bases: $[\epsilon_{k\uparrow}, c^\dagger_{k\uparrow}, c^\dagger_{-k\downarrow}, c_{-k\downarrow}]^T$, $H_t$ can be expressed as:

$$H_t(k) = \begin{bmatrix} \epsilon_k - h_z \sigma_z + g_k \cdot \sigma & -i \Delta_0 \sigma_y \\ i \Delta_0 \sigma_y & -(\epsilon_k - h_z \sigma_z) + g_k \cdot \sigma^* \end{bmatrix}.$$  \hspace{1cm} (20)

Here $g_k = (a_k, b_k)$, $\sigma = (\sigma_x, \sigma_y)$, and the explicit forms of $\epsilon_k$, $a_k$ and $b_k$ are listed:

$$\epsilon_k = -2t_1(\cos k_x + 2 \cos \frac{k_x}{2} \cos \frac{\sqrt{3}k_y}{2}) - 2t_3(\cos \frac{\sqrt{3}k_y}{2} + 2 \cos \frac{3k_x}{2} \cos \frac{\sqrt{3}k_y}{2}) - \mu'$$  \hspace{1cm} (21)

$$a_k = 2t_2(\sin k_x - 2 \sin \frac{k_x}{2} \cos \frac{\sqrt{3}k_y}{2})$$

$$b_k = 2t_4(- \sin k_y + 2 \sin \frac{3k_x}{2} \cos \frac{\sqrt{3}k_y}{2})$$  \hspace{1cm} (22)

in which $t_1 = t \cos \alpha$, $t_2 = t \sin \alpha$, $t_3 = t' \cos \alpha$, $t_4 = t' \sin \alpha$ with $\alpha = 4\pi \cos \varphi$ and $\mu' = \mu + \eta_0$.

Before discussing the properties of the SF states described by $H_t$ in Eq. (19), we give the estimations about the parameters related to the aforementioned simulations to ensure the experimental feasibility and rationality. For trapped atoms: $^6$Li, the wave length of laser beams utilized to produce the TOL is $\lambda_L \sim 1 \mu m$, and the lattice constant is $a_L = \frac{2\pi}{\sqrt{3}}$. The recoil energy is $E_r = \frac{\hbar^2 k^2}{2m} \sim h \times 2\pi \times 30 \text{ kHz} \sim 1 \mu K$ with $k_1 = \frac{\pi}{\lambda_L}$ and $t \approx \frac{\hbar}{\sqrt{3}}(\frac{1}{E_r})^2 + e^{-\sqrt{V_0/E_r}}$ when $V_0 >> E_r$ with $V_0 \sim 4V_L$, the depth of the TOL. $t'/t = e^{-\eta(\sqrt{3}-1)\sqrt{(V_0-E_{kin})/E_{kin} a_L}}$ with $E_{kin}$ and $\eta$ the kinetic energy of atom and the renormalized factor. The typical atomic velocity is about several centimeters per second, and $E_{kin}$ has the same order of $E_r$. $\eta$ depends on geometry of the lattice. According to the 1D lattice results and $\eta < 1$, we estimate that $V_0 \sim 3E_r$ is enough to get $t'/t = \frac{1}{\sqrt{3}}$. Actually, the TSF is robust even when $t'/t \sim 10^{-3}$. Here, without lost of generality, we set $t'/t \equiv \frac{1}{\sqrt{3}}$. Then $\sim 0.1E_r$ with the aforementioned formula. From the harmonic-potential approximation, the energies of the atoms tightly confined at a single lattice site are quantized to levels separated by $2\pi \hbar \omega_0 = 2E_r \sqrt{\frac{\eta}{\lambda_L}}$. On the other hand, the maximal band width for the TOL is $W_m \sim 10^5 \sim E_r$. Therefore, it is safe to describe the system with single-band approximation because of $2\pi \hbar \omega_0 > W_m$. The Rabi frequencies $\Omega_{a/b, c}$ are $10^5 E_r/\hbar$, and the $\Omega_{a/b}$ can be tuned from 0 to $E_r/\hbar$ which is enough for $\hbar \Omega_{a/b} \ll |E_{a,b}| < |E_{a,c}|$. Then, adiabatic approximation is reasonable. The typical s-wave pairing potential $\Delta_{a\beta}$ in experiments is about $0.1E_r/\hbar$, which is much smaller than $\Omega_{a/b}$. Hence, our proposal is experimentally feasible when the parameters lie in the estimated region.

For convenience to discuss the properties of SF, we rewrite $H_t(k)$ in the new bases $[\hat{\psi}_{k, +}, \hat{\psi}^\dagger_{k, -}, \hat{\psi}^\dagger_{k', +}, \hat{\psi}^\dagger_{k', -}]^T$ with $\hat{\psi}_{k, \pm} = \frac{1}{\sqrt{2}} [\epsilon_{k\uparrow} \pm c^\dagger_{-k\downarrow}]$, $H_t$ has the following form:

$$H'_t(k) = \begin{bmatrix} H_+(k) - i \epsilon_k \sigma_y \\ \tau \epsilon_k \sigma_y \end{bmatrix}.$$  \hspace{1cm} (23)

Here,

$$H_\pm(k) = \frac{1}{\sqrt{2}} [(-h_z \mp \Delta_0) \sigma_z + a_k \sigma_x \pm b_k \sigma_y].$$  \hspace{1cm} (24)

The spectrums of Hamiltonian (23) are $\pm E_\pm(k)$, and

$$E_\pm(k) = \sqrt{\epsilon_k^2 + \Delta_0^2 + \Theta_k^2} \pm 2 \sqrt{\frac{\Delta_0^2 + \epsilon_k^2}{2} + \frac{\Theta_k^2}{2}},$$  \hspace{1cm} (25)

in which $\Theta_k = \sqrt{h_z^2 + |g_k|^2}$. The topological transition point is determined by bulk energy gap closing condition: $E_-(k) = 0$ (Fig. 3(d)). The spectrums are fully gapped off this point, and the SF is topologically nontrivial when $h_z > \sqrt{\Delta_0^2 + \epsilon_k^2} |_{k=(0,0)}$ (Fig. 3(c)) and trivial when $h_z < \sqrt{\Delta_0^2 + \epsilon_k^2} |_{k=(0,0)}$ (Fig. 3(c)). Around the $\Gamma$ point in BZ, $H_\pm(k) = \tau(\Delta_0 \pm h_z) \sigma_z + \lambda_{so}(k^3 \sigma_x \pm k^3 \sigma_y)$, (26)
where $\sigma_{x} = \frac{1}{2}(\sigma_{x} + \sigma_{y})$, $k_{x} = \frac{1}{2}(k_{x} + k_{y})$. The term $\lambda_{m}(k \Gamma_{x} + k \Gamma_{y})$ is the $k^{2}$ RSOC with amplitude: $\lambda_{m} = \frac{t_{z}}{2} \sim -0.48t$ for cos $\varphi = -0.101$. It is explicit that $\mathcal{H}_{v}(k)$ has the well-defined $f$-wave chirality. That means the topological Chern number can be calculated

$$C = \frac{1}{2\pi} \int_{BZ} d^{2}k 2\Re \left( \frac{\partial u_{n}(k)}{\partial k_{x}} \frac{\partial u_{n}(k)}{\partial k_{y}} \right),$$

where $u_{n}(k)$ is the ground state wave-function for the $n$th occupied band ($n=1,2$). The straightforward calculation gives $C_{1}=0$ and $C_{2}=3$ when $h_{z} > \sqrt{\Delta_{0}^{2} + \varepsilon_{k}^{2}} \mid k=0,0 \rangle$, and $C_{1}=0$ and $C_{2}=0$ when $h_{z} < \sqrt{\Delta_{0}^{2} + \varepsilon_{k}^{2}} \mid k=0,0 \rangle$. The non-zero Chern number means the same numbers of gapless edge states from the bulk-edge correspondence. From the energy spectrum of $H(k, x)$ shown in Fig. 4, we find that three gapless chiral edge states transport on one edge of TSF. (see Fig.4 (a) and (d)). The effective Hamiltonian describing the chiral edges is

$$\mathcal{H}_{\text{edge}} = \sum_{k_{x} \geq 0} u_{0} k_{y} \hat{\Psi}_{k_{y}}^{\dagger}(x) \hat{\Psi}_{k_{y}}(x),$$

where $u_{0}$ is the effective velocity at $\Gamma$, and $\hat{\Psi}_{k_{y}}^{\dagger}(x) = \sum_{k_{x}} f \, dx \left[ u_{k_{y}, \sigma}(x) c_{\sigma}^{\dagger}(x) + v_{k_{y}, \sigma}(x) c_{\sigma}(x) \right]$. The Majorana condition requires $\hat{\Psi}_{-k_{y}} = \hat{\Psi}_{k_{y}}^{\dagger}$, i.e., $u_{-k_{y}, \sigma} = v_{k_{y}, \sigma}^{*}$. We check that it’s indeed the case in our model. That means three edge states are chiral Majorana edge states.

In general, the topological defects bound Majorana zero modes in TSC/TSF. Here, we consider the topological excitations of vortex structures in our system. The quasi-particle excitations usually are described by the Bogoliubov-de Gennes (BdG) equation. From Fig. 4(a), we can find that the wave function of low energy excitations can be constructed from the contributions of quasi-particle around $\Gamma$. For simplicity, only the non-trivial part $\mathcal{H}_{\text{v}}(k)$ of Eq. (23) is taken into account. Define $z = x + iy$, then $\partial_{x}/\partial_{z} = \partial_{z} + \partial_{y}$. The BdG equation for the quasi-particle has the form: $\mathcal{H}_{-}(z, z^{*}) \Psi_{0} = E \Psi_{0}$ with $\Psi_{0} = [u_{0}^{*}, v_{0}^{*}]^{T}$ and the corresponding quasi-particle creation operator: $\hat{\Psi}_{0}^{\dagger} = \int dz \, dz^{*} \left[ u_{0} \psi_{z}^{\dagger} + v_{0} \psi_{z}^{\dagger} \right]$. In the uniform TSF states, we can assume a trivial wave function: $u_{0} = e^{-i\pi/4} z^{-1/2} e^{-i(\frac{z_{x} - z_{y}}{z_{x} z_{y}})^{1/2}} (z^{*})^{3/2}$ and $v_{0} = u_{0}^{*}$ as a test wave function from the BdG equation. The SF order parameter with a vortex structure can be approximately
expressed as $\Delta_0(r) = 0$ for $r < r_c$ and $\Delta_0(r) = \Delta_0 e^{i\theta}$ for $r > r_c$ with vorticity 1. We imagine that the vortex is created adiabatically by changing the wave function slowly enough so that it always remains an eigenstate. The wave function of the vortex state can be obtained from a singular gauge transformation: $\Psi_0 \rightarrow \Psi_0 e^{iq\theta}$ with $q = \pm 1$ identifying the quasi-particle and quasi-antiparticle. In reverse, the vortex can be gauged away by the inverse singular gauge transformation: $k \rightarrow k - \nabla \theta/2$ and $\Delta_0 e^{i\theta} \rightarrow \Delta_0$. After the inverse transformation, the state is one of eigenstates, namely, the vortex excited state. In analogy to the Laughlin’s argument about the vortex excitation in quantum Hall state, we cite state. In analogy to the Laughlin’s argument [37], the inverse singular gauge transformation: $k \rightarrow k - \nabla \theta/2$ and $\Delta_0 e^{i\theta} \rightarrow \Delta_0$. After the inverse transformation, the state is one of eigenstates, namely, the vortex excited state. In analogy to the Laughlin’s argument [37] about the vortex excitation in quantum Hall state, we get the wave function describing vortex zero mode as $u_0' \sim e^{-i(\frac{\theta}{2} - \frac{i}{2}) r - \frac{2}{15}\left(\frac{\pi}{2}\right) r^3}$ and $v'_0 = (u'_0)^*$. The unique one zero mode for $f$-wave case is proven by the numerical calculation [8].

The stability of Majorana zero mode is measured by the mini-gap $E_g \sim \Delta_0^2/E_f$ with $E_f$ the Fermi energy. In our case, the $E_f$ is measured by $t$ not $E_{kin}$. Hence, The ratio $E_g/\Delta_0$ can be large enough to protect Majorana zero mode. Take half filling as an example, we assume the optimized $\Delta_0 \sim t$ and roughly estimate $E_f \sim 3t$. Then $E_g/\Delta_0 \sim 1/3$. Comparing with the hybridized systems [4], [5], the energy scale of $E_f$ has the order of electrons’ kinetic energy $E_{kin}$ and $\Delta_0$ from the proximity effect is much smaller compared to $E_{kin}$. So, the mini-gap in hybridized systems may be relative small compare to superconductive gap $\Delta_0$.

IV. CONCLUSIONS

In summary, we have proposed a scheme to produce $k^3$ RSOC and ZF through the laser-atom interaction in TOLs, and a novelty chiral $f$-wave TSF is realized thanks to the $s$-wave Feshbach resonance. We find that there exists three Majorana edge states locating on the boundary of the system and one Majorana fermion binding to each vortex in the TSF state. The TSF can be controlled by modulating the parameters of the laser. The controllability provides convenience to investigate the properties of the TSF. Our proposal enlarges TSF family and presents some advantages to study the Majorana fermions.

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