AN ARITHMETIC GROUP ASSOCIATED WITH A PISOT UNIT AND ITS SYMBOLIC-DYNAMICAL REPRESENTATION

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Abstract. To a given Pisot unit \( \beta \) we associate a finite abelian group whose size appears to be equal to the discriminant of \( \beta \). We call it the Pisot group and find its representation in the two-sided \( \beta \)-compactum in the case of \( \beta \) satisfying the relation \( \text{Fin}(\beta) = \mathbb{Z}[\beta] \cap [0,1) \). As a motivation for the definition, we show that the Pisot group is the kernel of some important arithmetic coding of the toral automorphism given by the companion matrix naturally associated with \( \beta \).

1. The definition of the Pisot group and its basic properties

Let \( \beta > 1 \) be a Pisot number, i.e. an algebraic integer whose conjugates have the moduli strictly less than 1. Let the characteristic polynomial of \( \beta \) be \( g(x) = x^m - k_1 x^{m-1} - k_2 x^{m-2} - \cdots - k_m \). We assume \( \beta \) to be a unit, i.e. \( k_m = \pm 1 \).

We recall that since \( \beta \) is a Pisot number, \( \| \beta^n \| \to 0 \) as \( n \to +\infty \), where \( \| y \| = \min \{ |y - k|, k \in \mathbb{Z} \} \). Let

\[ P_\beta = \{ \xi : \| \xi \beta^n \| \to 0, \ n \to +\infty \}. \]

Let us first establish some auxiliary facts. It is well-known that \( \xi \in \mathbb{Q}(\beta) \) (see, e.g., [Cas]). Let \( \text{Tr}(\xi) \) denotes the trace of \( \xi \), i.e. the sum of \( \xi \) and all its conjugates.

Lemma 1.1. The set \( P_\beta \) is a commutative group under addition containing \( \mathbb{Z}[\beta] \). It can be characterized as follows:

\[ P_\beta = \{ \xi \in \mathbb{Q}(\beta) : \text{Tr}(a \xi) \in \mathbb{Z} \text{ for any } a \in \mathbb{Z}[\beta] \}. \]

Proof. The first claim is a consequence of the inequalities \( \| \xi_1 \pm \xi_2 \| \leq \| \xi_1 \| + \| \xi_2 \| \) and of the fact that \( \| \beta^n \| \to 0 \). To prove (1.1), we observe that the classical theorem due to Pisot and Vijayaraghavan says that for any Pisot \( \beta \), there exists \( k_0 \in \mathbb{N} \) such that \( P_\beta = \{ \xi \in \mathbb{Q}(\beta) : \text{Tr}(\beta^k \xi) \in \mathbb{Z}, \ k \geq k_0 \} \) (see, e.g., [Cas]), whence follows (1.1), as \( \beta \) is a unit, and \( \text{Tr}(\xi) \in \mathbb{Z} \) implies \( \text{Tr}(\beta^{-1} \xi) \in \mathbb{Z} \). □

Thus, if we regard \( \mathbb{Z}[\beta] \) as a lattice over \( \mathbb{Z} \), then by (1.1) and the definition, \( P_\beta \) is the dual lattice for \( \mathbb{Z}[\beta] \) (notation: \( P_\beta = (\mathbb{Z}[\beta])^* \)). The following claim follows from [FrolTa, Chapter III, (2.20)].

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Proposition 1.2. There exists $\xi_0 \in \mathbb{Q}(\beta) \setminus \mathbb{Z}[\beta]$ such that

$$P_\beta = \xi_0 \cdot \mathbb{Z}[\beta].$$

It can be given by the formula $\xi_0 = (g'(\beta))^{-1}$.

**Definition.** Let the Pisot group $A_\beta$ be defined as the quotient group $P_\beta / \mathbb{Z}[\beta]$.

Let $\beta_1 = \beta$, and $\beta_2, \beta_3, \ldots, \beta_m$ denote its conjugates. Finally, let $D = D(\beta) = \prod_{i \neq j} (\beta_i - \beta_j)^2$, i.e. the discriminant of $\beta$.

**Theorem 1.3.** The order of the Pisot group is $|D(\beta)|$.

**Proof.** Again, we will use the facts from classical Number Theory. By Dedekind’s Ramification Theorem, for any lattice $M$ over $\mathbb{Z}$, the order of $M^* : M$ is $|D|$, where $D = \det M_\beta$,

$$M_\beta = \begin{pmatrix} \text{Tr}(1) & \text{Tr}(\beta) & \ldots & \text{Tr}(\beta^{m-1}) \\ \text{Tr}(\beta) & \text{Tr}(\beta^2) & \ldots & \text{Tr}(\beta^m) \\ \vdots & \vdots & \ldots & \vdots \\ \text{Tr}(\beta^{m-1}) & \text{Tr}(\beta^m) & \ldots & \text{Tr}(\beta^{2m-1}) \end{pmatrix}$$

(see [FrolTa, Chapter III, (2.8)]). It suffices to show that $\det M_\beta = D$. Sometimes, this relation is taken as a definition for the discriminant. Otherwise, observe that $M_\beta = V_\beta \cdot V_\beta^T$, where

$$V_\beta = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \beta_1 & \beta_2 & \ldots & \beta_m \\ \beta_1^2 & \beta_2^2 & \ldots & \beta_m^2 \\ \vdots & \vdots & \ldots & \vdots \\ \beta_1^{m-1} & \beta_2^{m-1} & \ldots & \beta_m^{m-1} \end{pmatrix},$$

whence $\det M_\beta = (\det V_\beta)^2 = \prod_{i \neq j} (\beta_i - \beta_j)^2 = D$. $\square$

The following simple fact answers the question about the finer structure of the Pisot group. Let $d = \min \{l \geq 1 : lM_\beta^{-1} \in \mathbb{Z}^{m \times m}\}$.

**Lemma 1.4.** The Pisot group $A_\beta$ contains $\mathbb{Z}/d\mathbb{Z}$ as a subgroup. In particular, if $d = |D|$, then $A_\beta$ is cyclic, and if $|D|/d$ is prime, then $A_\beta$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z}) \times (\mathbb{Z}/(|D|/d)\mathbb{Z})$.

**Corollary 1.5.** Assume the entries of $DM_\beta^{-1}$ to be coprime. Then $A_\beta$ is cyclic.

**Remark.** All the above results are valid for any algebraic unit, if one takes (1.1) as the definition of $P_\beta$.

**Examples.** 1. Quadratic units. Let $\beta^2 = k\beta \pm 1$ with $k \geq 1$ for $+1$ and $k \geq 3$ for $-1$. Then by direct inspection,

$$M_\beta = \begin{pmatrix} 2 & k \\ k & k^2 + 2 \end{pmatrix},$$

and $\xi_0 = \frac{1}{\sqrt{D}}$. Then by Lemma 1.4, $A_\beta$ is isomorphic to $\mathbb{Z}/D\mathbb{Z}$ if $k$ is odd, and to $(\mathbb{Z}/(D/2)\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ otherwise. The Pisot group for this case was considered in [SV2].
2. Let $\beta$ be the “tribonacci number”, i.e. the positive root of $x^3 = x^2 + x + 1$. Here $D = -44$, 

$$M_\beta = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 3 & 7 \\ 3 & 7 & 11 \end{pmatrix}, \quad M_\beta^{-1} = \frac{1}{22} \begin{pmatrix} 8 & -5 & 1 \\ -5 & -12 & 9 \\ 1 & 9 & -4 \end{pmatrix},$$

$\xi^{-1}_0 = g'(\beta) = -1 - 2\beta + 3\beta^2$, and $\xi_0 = \frac{1}{22}(1 + 9\beta - 4\beta^2)$. By Lemma 1.4, $A_\beta \cong (\mathbb{Z}/22\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

3. Finally, let $\beta$ be the smallest Pisot number, i.e. the root of $\beta^3 = \beta + 1$ (see, e.g., [DuPi]). Here $D = -23$, 

$$M_\beta = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, \quad M_\beta^{-1} = \frac{1}{23} \begin{pmatrix} 5 & -6 & 4 \\ -6 & -2 & 9 \\ 4 & 9 & -6 \end{pmatrix},$$

and since 23 is a prime, $A_\beta \cong \mathbb{Z}/23\mathbb{Z}$.

At the end of the section, we would like to establish a link between the groups $P_\beta$ and $A_\beta$ and certain groups of recurrent sequences, which will be used in the next section. Let

$$R_\beta = \{\{T_n\}_{n=1}^\infty \in \mathbb{Z}^\infty \mid \exists j : T_{n+m} = k_1T_{n+m-1} + k_2T_{n+m-2} + \cdots + k_mT_n, \ n \geq j\}.$$ 

Obviously, $R_\beta$ is a group under addition.

**Proposition 1.8.** The groups $P_\beta$ and $R_\beta$ are isomorphic.

**Proof.** Let $\{T_n\}_{n=1}^\infty \in R_\beta$. Put

$$\xi := \lim_{n \to +\infty} \beta^{-n}T_n.$$ 

Then $\xi \in P_\beta$, because $\|\xi\beta^n\| = |\xi\beta^n - T_n| \to 0$ (in view of $\beta$ being a Pisot number, whence $|\xi - \beta^nT_n| = o(\beta^{-n})$).

The inverse mapping from $P_\beta$ to $R_\beta$ assigns to each $\xi \in P_\beta$ the sequence $\{T_n\}$, where $T_n$ is defined as the closest integer for $\xi\beta^n$. By the fact that $\beta$ is a Pisot number, $T_n$ will eventually satisfy the relation in question. □

Let now the equivalence relation $\sim$ on $R_\beta$ be defined as follows: $\{T_n\} \sim \{T'_n\}$ iff $\lim_n \beta^{-n}(T_n - T'_n) \in \mathbb{Z}[\beta]$. Then obviously, the quotient group $R_\beta/\sim$ is isomorphic to $A_\beta$.

2. **Symbolic representation of the Pisot group**

**Definition.** A representation of an $x \in [0, 1)$ of the form

$$x = \sum_{k=1}^\infty \varepsilon_k\beta^{-k}$$

is called the $\beta$-expansion of $x$ if the “digits” $\{\varepsilon_k\}_{k=1}^\infty$ are obtained by means of the greedy algorithm (similarly to the decimal expansions), i.e. $\varepsilon_k = \varepsilon(x) = [\beta x]$, $\varepsilon = \{\varepsilon_k\}$. 


\( \varepsilon_2(x) = [\beta(\beta x)] \), etc. The set of all possible sequences \( \{\{\varepsilon_k(x)\}_{k=1}^{\infty} : x \in [0, 1) \} \) is called the (one-sided) \( \beta \)-compactum and denoted by \( X^+_\beta \).

The \( \beta \)-compactum can be described more explicitly. Let \( 1 = \sum_{k=1}^{\infty} d_k \beta^{-k} \) be the expansion of 1 defined by the greedy algorithm, i.e., \( d_1 = [\beta] \), \( d_2 = \{\beta[\beta]\} \), etc. If the sequence \( \{d_n\} \) is not finite, we put \( d_n = d_1 \cdot \cdot \cdot d_{n-1} \). Otherwise let \( k = \max \{j : d_j \neq 0\} \), and \( (d_1, d_2, \ldots ) := (d_1, \ldots, d_{k-1}, d_k - 1) \), where the bar denotes a period.

We will write \( \{x_n\}_{n=1}^{\infty} \prec \{y_n\}_{n=1}^{\infty} \) if \( x_n < y_n \) for the least \( n \geq 1 \) such that \( x_n \neq y_n \). Then by definition,

\[
X^+_\beta = \{\{\varepsilon_n\}_{n=1}^{\infty} : (\varepsilon_n, \varepsilon_{n+1}, \ldots ) \prec (d_1, d_2, \ldots ) \text{ for all } n \in \mathbb{N}\}
\]

(see [Pa]). Similarly, we define the two-sided \( \beta \)-compactum as

\[
X_\beta = \{\{\varepsilon_n\}_{n=-\infty}^{\infty} : (\varepsilon_n, \varepsilon_{n+1}, \ldots ) \prec (d_1, d_2, \ldots ) \text{ for all } n \in \mathbb{Z}\}.
\]

Both compacta are naturally endowed with the weak topology, i.e. with the topology of coordinate-wise convergence. For \( \beta \) Pisot the properties of the \( \beta \)-compactum are well-studied. Its main property is that it is sofic (see, e.g., the review [Bl]). Let \( \text{Fin}(\beta) \) denote the set of \( x \)'s whose \( \beta \)-expansions are finite, i.e. have the tail \( 0^{\infty} \). Obviously, \( \text{Fin}(\beta) \subseteq \mathbb{Z}[\beta] \cap [0, 1) \), but the inverse inclusion does not hold for an arbitrary Pisot unit.

**Definition.** A Pisot unit is called finitary, if

\[
\text{Fin}(\beta) = \mathbb{Z}[\beta] \cap [0, 1).
\]

**Examples.**

1. The Pisot number being the principal root of the equation \( x^m = k_1 x^{m-1} + \cdots + k_m \), \( k_1 \geq k_2 \geq \cdots \geq k_m \geq 1 \), is known to be finitary [FrSo].

2. A quadratic Pisot unit is finitary if and only if its norm is +1. Indeed, if \( \beta^2 = k\beta + 1 \), \( k \geq 1 \), then the claim follows from the previous one. If \( \beta^2 = k\beta - 1 \), \( k \geq 3 \), then in view of \( (\beta - 1)^2 = \beta^2 - 2\beta + 1 = (k - 2)\beta \), we have the \( \beta \)-expansion \( \beta - 1 = (k - 2)\beta^{-1} + (k - 2)\beta^{-2} + \ldots \), whence \( \mathbb{Z}[\beta] \cap [0, 1) \not\subseteq \text{Fin}(\beta) \).

3. For the cubic Pisot units also exists a full criterion due to Sh. Akiyama [Ak]. Namely, the norm of a finitary cubic \( \beta \) must be +1, i.e. \( \beta^3 = k_1\beta^2 + k_2\beta + 1 \). Then \( \beta \) is finitary if and only if \( k_1 > 0 \) and \( -1 \leq k_2 \leq k_1 + 1 \). Thus, the tribonacci number and the smallest Pisot number are both finitary, whereas, for example, the principal root of \( x^3 = 3x^2 - 2x + 1 \) is not.

4. Finally, the second in order Pisot number, i.e. the positive root of \( x^4 = x^3 + 1 \), is non-finitary, as \( \beta^{-2} + \beta^{-3} = \beta^{-1} + \beta^{-6} + \beta^{-11} + \ldots \).

From here on we will assume \( \beta \) to be finitary.

**Remark.** The \( \beta \)-expansions can be easily extended from \([0, 1)\) to \( \mathbb{R}_+ \) in the way analogous to the decimal expansions. This allows to add in \( X_\beta \) two sequences finite to the left. Namely, if \( \overline{x} \) and \( \overline{y} \) are both finite to the left, put \( x := \sum_{k=1}^{\infty} (\varepsilon_k + \varepsilon'_k) \beta^{-k} \); then by definition, \( \overline{x} + \overline{y} \) is the \( \beta \)-expansion of \( x \). The same is true for subtraction, though \( \overline{x} - \overline{y} \) is well defined only for the sequences, for which \( \sum (\varepsilon_k - \varepsilon'_k) \beta^{-k} \geq 0 \).
Theorem. (A. Bertrand, K. Schmidt) For a Pisot $\beta$, any $x \in \mathbb{Q}(\beta) \cap \mathbb{R}_+$ has an ultimately periodic $\beta$-expansion.

Therefore, all elements of $\mathcal{P}_\beta \cap \mathbb{R}_+$ have ultimately periodic $\beta$-expansions. Since their denominators in the standard basis of the field are bounded (by $|D|$), these periods cannot be too long, whence it follows that there is a finite set of such periods.

Our goal is to represent the Pisot group $A_\beta$ in $X_\beta$. We perform the following operation: to a $[\xi] \in A_\beta$ the mapping $g$ assigns all purely periodic two-sided sequences in $X_\beta$ which are extensions to the left of the periodic tails that occur in the $\beta$-expansions of the nonnegative $\xi' \sim \xi$. By the above argument, $\#g([\xi]) < \infty$ for any $\xi \in \mathcal{P}_\beta \cap \mathbb{R}_+$. Now we identify all the images $\{g(\xi') : \xi' \in [\xi]\}$. This leads to the mapping $g$ acting from $A_\beta$ onto the set that we will denote by $\mathfrak{A}_\beta$.

Lemma 2.1. The mapping $g : A_\beta \to \mathfrak{A}_\beta$ is 1-to-1.

Proof. We need to prove the injectivity of $g$ only. Let $g([\xi]) = g([\xi'])$. This means that there exist $\xi_1 \in [\xi]$, $\xi_2 \in [\xi']$ whose tails of the $\beta$-expansions coincide. But this implies $\xi_1 - \xi_2 \in \mathbb{Z}[\beta]$, whence $[\xi] = [\xi']$. □

Thus, we have defined a certain finite arithmetic group which may be regarded as a representation of the Pisot group in $X_\beta$. Although its elements are not sequences but certain classes of equivalence, arithmetically they represent one and the same sequence, because besides arithmetic operations inherited from $A_\beta$, one may define some internal arithmetic in $X_\beta$. More precisely, addition in $\mathfrak{A}_\beta$ can be carried out directly through the whole compactum in the following way: let $\overline{\varepsilon}, \overline{\varepsilon}' \in X_\beta$, then $\overline{\varepsilon} + \overline{\varepsilon}' = \lim_{N \to +\infty} (\varepsilon(N) + \varepsilon'(N))$, where $\varepsilon(N) = (\ldots, 0, 0, \ldots, 0, \varepsilon_{-N}, \varepsilon_{-N+1}, \ldots)$, addition of the sequences finite to the left being executed as explained in the remark above.

Let us prove accurately that $\mathfrak{A}_\beta$ is a group under addition in this sense as well. Let $Fin_k(\beta)$ denote the set of sequences $\overline{\varepsilon}$, for which $\varepsilon_j = 0$, $j \geq k$. The theorem due to Ch. Frougny and B. Solomyak says that if $\beta$ is finitary, then there exists $L = L(\beta)$ such that $\overline{\varepsilon}, \overline{\varepsilon}' \in Fin_k(\beta)$ implies $\overline{\varepsilon} + \overline{\varepsilon}' \in Fin_{k+L}(\beta)$, i.e. for the finitary Pisot numbers the “waiting time” is bounded [FrSo].

Lemma 2.2. Let $[\overline{\varepsilon}], [\overline{\varepsilon}'] \in \mathfrak{A}_\beta$. Then regardless of the choice of representatives in the equivalence classes, $[\overline{\varepsilon} \pm \overline{\varepsilon}']$ is well defined.

Proof. Let $\delta(N) := \varepsilon(N) + \varepsilon'(N)$. Then $\delta(N+k) - \delta(N) \in Fin_{-N+L}(\beta)$ by the theorem cited above, whence the limit does exist. To obtain $-\overline{\varepsilon}$, we subtract $\varepsilon(N)$ from the sequence $(\ldots, 0, 0, \ldots, 1, 0, 0, \ldots)$, where 1 stands at the $(-N-1)$‘th place and then pass to the limit. Even if it did not exist, all partial limits would belong to one and the same equivalence class, since the difference of their periodic parts would belong to $\mathbb{Z}[\beta]$. □

Thus, $\mathfrak{A}_\beta$ is an additive group in $X_\beta$ in the sense of its natural arithmetics. Its obvious property is that it is shift-invariant, i.e. contains any sequence together will all its shifts. Pursuing the idea from Section 1, we will give another way of obtaining the sequences from $\mathfrak{A}_\beta$, which in a sense looks more traditional (see [FrSa1] for the case of the golden mean).
Lemma 2.3. For any element \([\alpha] \in \mathfrak{A}_\beta\) there exists an eventually nonnegative sequence \(\{T_n\}_1^\infty \in R_\beta\) such that the set of partial limits for the sequence of its \(\beta\)-expansions \(\{\alpha_n\}_1^\infty\) is just the class \([\alpha]\).

Proof. Take \([\xi] = g^{-1}(\bar{\alpha})\), and, as above, define \(T_n\) as the closest integer to \(\xi'\beta^n\) for some nonnegative \(\xi' \sim \xi\). Then \(T_n = \xi'\beta^n + o(1)\), and \(T_n\) are eventually nonnegative, as \(\xi' > 0\). Hence it follows that to take the set of partial limits of its \(\beta\)-expansions is the same operation as “pulling out” the periodic tail for a \(\xi \in \mathcal{P}_\beta \cap \mathbb{R}_+\). \(\square\)

Examples. 1. For the quadratic Pisot units \(\beta^2 = k\beta \pm 1\) we proved in Section 1 that \(\mathcal{P}_\beta = \frac{1}{\sqrt{D}} \cdot \mathbb{Z}[\beta]\). Since \(\frac{1}{\sqrt{D}} = \frac{k\beta^2}{\beta^2-1}\), all sequences in \(\mathfrak{A}_\beta\) are of period 4. Let us consider some subcases. We will use the convention to write just the period instead of the whole periodic sequence.

1.1. \(\beta^2 = \beta + 1\). Here \(\mathfrak{A}_\beta = \{0000, 1000, 0100, 0010, 0001\}\). Recall that by definition, \(X_\beta\) does not contain sequences ending by 101010\ldots, that is why there is no need to identify 0000 with 1010 and 0101. Thus, every class in \(\mathfrak{A}_\beta\) consists of just one element. If \(F_1 = 1, F_2 = 2, \ldots\) is the Fibonacci sequence, then by Lemma 2.3, the set of partial limits for the \(\beta\)-expansions of \((F_n)_{k=1}^\infty\) in \(X_\beta\) will give us just the four nonzero sequences in \(\mathfrak{A}_\beta\). However, this may be checked directly as well:

\[
F_k = \beta^{k-1} + \beta^{k-5} + \beta^{k-9} + \ldots + \beta^{-k+3} + \beta^{-k}, \quad k \text{ even},
\]

\[
F_k = \beta^{k-1} + \beta^{k-5} + \beta^{k-9} + \ldots + \beta^{-k+5} + \beta^{-k+1}, \quad k \text{ odd},
\]

whence the limits of \((F_{4k+j})_{k=1}^\infty\) in \(X_\beta\) do exist for \(j = 0, 1, 2, 3\) and yield the four sequences in \(\mathfrak{A}_\beta\). For more details see [FrSa1], [SV1].

1.2. \(\beta^2 = 2\beta + 1\). Here \#\(\mathfrak{A}_\beta = 8\), and \(\mathfrak{A}_\beta = \{0000, 1010, 0101, 2000, 0200, 0020, 1000, 1111\}\). The elements 1010, 0101, 1111 are of order 2, whereas all other nonzero elements are of order 4. A similar pattern takes place for \(\beta^2 = k\beta + 1\), \(k\) even.

Note that in [FrSa2] the authors study two similar groups for arbitrary quadratic Pisot units. One of them, \(H_\beta\), is the group of all sequences of period 4; its order is \(k^2D\), and it is in fact isomorphic to \(\frac{\mathbb{Z}[\beta]}{k\sqrt{D}} [\mathbb{Z}[\beta]]\). Another one, \(G_\beta\), is, on the contrary, smaller than \(\mathfrak{A}_\beta\); it is related to a certain finite automaton. For example, in the case \(\beta^2 = 2\beta + 1\), \(G_\beta = \{0000, 1010, 1010, 1111\} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})\). In the general quadratic case one can show that \(G_\beta \subset \mathfrak{A}_\beta \subset H_\beta\), both inclusions being, generally speaking, proper. However, in the case of the golden mean (and only in this case), \(G_\beta = \mathfrak{A}_\beta = H_\beta\).

2. For the tribonacci number the situation with the periods of the sequences in \(\mathfrak{A}_\beta\) proves to be more complicated. Expanding the elements of \(\mathcal{P}_\beta\), we see that these periods are 1, 2, 3 and 10. More precisely, besides 0, \(\mathfrak{A}_\beta\) consists of 40 sequences of period 10, namely, of \(\{1000110000, 1010000110, 1001011000, 1001101100\}\) together with all their shifts, two sequences of period two: \(\{01, 10\}\) and one sequence of period 3: 100. One may ask: where are all its shifts, mustn’t they belong to \(\mathfrak{A}_\beta\) as well? The answer is simple: \(\text{arithmetically}\) they are all equivalent, because \((\beta + \beta^{-2} + \beta^{-5} + \ldots) - (1 + \beta^{-3} + \beta^{-6} + \ldots) = \frac{\beta^{-1}}{1-\beta^{-1}} = 1 \in \mathbb{Z}[\beta]\). This example shows that even for a finitary \(\beta\) not necessarily any \(\xi \in \mathcal{P}_\beta\) and \(\xi + l\), \(l \in \mathbb{Z}[\beta]\)

\footnote{\text{It is easy to see that the Fibonacci sequence is the basis of the module } R_\beta, \text{i.e. for any sequence } \{t_n\} \subseteq R_\beta, T = \sum_{n=1}^{\infty} t_n \mathfrak{A}_\beta, \text{the } \mathfrak{A}_\beta^\infty \text{ together with } \{0000, 1000, 0100, 0010, 0001\} \text{ comprise a basis of the module } R_\beta \text{ in } \mathbb{Z}[\beta].}
will have one and the same tail. Therefore the definition of $\mathfrak{A}_\beta$ as a quotient set is essential.

4. $\beta^3 = \beta + 1$. Here $D = -23$. Let us prove a simple claim about the structure of the group $\mathfrak{A}_\beta$ in the case of prime $|D|$.

**Lemma 2.4.** If $|D|$ is a prime number, then $|D|$ consists of $|D| - 1$ sequences of period $|D| - 1$ being the shifts of each other, and $0$.

**Proof.** Since $\mathfrak{A}_\beta$ is shift-invariant and cyclic, any $\tau \in \mathfrak{A}_\beta \setminus \{0\}$ belongs to it together with all its shifts, and they are arithmetically non-equivalent. □

Return to the example. Here the period of $\xi_0$ is 10000100000000100000000, and $\mathfrak{A}_\beta$ consists of this sequence together with its 21 shifts and 0.

### 3. Symbolic dynamics associated with the Pisot group

Our goal is to show that a certain natural “arithmetic” coding of the “companion” toral automorphism, is not one-to-one a.e. and that its kernel is in fact coincides with the group $\mathfrak{A}_\beta$. In this section we still assume $\beta$ to be a finitary Pisot unit.

We will need some facts from hyperbolic dynamics. Let $T$ be a hyperbolic automorphism of the torus $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$, $L_s$ and $L_u$ denote respectively the leaves of the stable and unstable foliations passing through $0$. Recall that a point homoclinic to $0$ or simply a homoclinic point is a point belonging to $L_s \cap L_u$. In other words, $t$ is homoclinic iff $T^n t \to 0$ as $n \to \pm \infty$. The homoclinic points are a group under addition isomorphic to $\mathbb{Z}^m$, and we will denote it by $\mathcal{H}(T)$. Each homoclinic point $t$ can be obtained as follows: take some $n \in \mathbb{Z}^m$ and project it onto $L_u$ along $L_v$ and then onto $\mathbb{T}^m$ by taking the fractional parts of all coordinates of the vector (see [Ver]).

Let $T = T(\beta)$ be the group automorphism of $\mathbb{T}^m$ given by the companion matrix, i.e. by

$$M = M(\beta) = \begin{pmatrix} k_1 & k_2 & k_3 & \cdots & k_{m-1} & k_m \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$ 

Then $(1, \beta^{-1}, \ldots, \beta^{-m+1})$ is an eigenvector corresponding to the eigenvalue $\beta$. In our case $L_u$ is one-dimensional, whence $t \in \mathcal{H}(T)$ iff $t = (\xi, \xi \beta^{-1}, \ldots, \xi \beta^{-m+1})$ mod $\mathbb{Z}^m$ for some $\xi \in \mathcal{P}_\beta$.

Consider two special homoclinic points. Let $\xi_0$ denote the generator of the Pisot group defined in Proposition 1.4, and $t_0$ denote the fundamental homoclinic point given by the formula

$$t_0 = (\xi_0, \xi_0 \beta^{-1}, \ldots, \xi_0 \beta^{-m+1})$$

(we omit the natural projection of $\mathbb{R}^m$ onto the torus, i.e. write in the coordinates of $\mathbb{R}^m$), and

$$t := (1, \beta^{-1}, \ldots, \beta^{-m+1}).$$
Consider the two mappings acting from $X_\beta$ onto $\mathbb{T}^m$, namely:

$$\varphi_0(\varepsilon) = \sum_{k \in \mathbb{Z}} \varepsilon_k T^{-k} t_0 = \lim_{N \to +\infty} \left( \sum_{k=-N}^{\infty} \varepsilon_k \beta^{-k} \right) \left( \begin{array}{c} \xi_0 \\ \xi_0 \beta^{-1} \\ \vdots \\ \xi_0 \beta^{-m+1} \end{array} \right) \mod \mathbb{Z}^m,$$

$$\varphi(\varepsilon) = \sum_{k \in \mathbb{Z}} \varepsilon_k T^{-k} t = \lim_{N \to +\infty} \left( \sum_{k=-N}^{\infty} \varepsilon_k \beta^{-k} \right) \left( \begin{array}{c} 1 \\ \beta^{-1} \\ \vdots \\ \beta^{-m+1} \end{array} \right) \mod \mathbb{Z}^m.$$

Both mappings are well defined, as $\|\beta^N\| \to 0$, $\|\xi_0 \beta^N\| \to 0$ exponentially. They are obviously continuous, and their important property is that they are bounded-to-one (see [Sch] for $\varphi_0$; in the case of $\varphi$ is essentially the same). Their main value is that they both semiconjugate the shift $\tau$ on the compactum $X_\beta$ and the automorphism $T$, i.e., $\varphi_0 \tau = T \varphi_0$, $\varphi \tau = T \varphi$. Thus, both mapping may be regarded as arithmetic codings of $T$. The mapping $\varphi_0$ was introduced in [SV2] for $m = 2$, and in [Sch] for higher dimensions. At the same time, the mapping $\varphi$ had been historically first attempt of arithmetic coding of an automorphism of the torus [Ber].

**Theorem.** (K. Schmidt [Sch]) The mapping $\varphi_0$ is one-to-one on the set of doubly transitive sequences of $X_\beta$, i.e., on the sequences $\varepsilon$ such that the sets $\{\tau^n \varepsilon, n \geq k\}$ and $\{\tau^n \varepsilon, n \leq -k\}$ are both dense in $X_\beta$ for every $k \geq 0$. Therefore, $\varphi_0$ is bijective almost everywhere with respect to the measure of maximal entropy for $\tau$.

Our goal in this section is to show that $\varphi$ is $|D|$-to-1 a.e. and that $\varphi^{-1}(0)$, after the identification of arithmetically equivalent sequences, will coincide with $\mathcal{A}_\beta$.

**Definition.** We will say that $\varepsilon$ is equivalent to $\varepsilon'$ if $\varphi_0(\varepsilon) = \varphi_0(\varepsilon')$.

We will denote this equivalence relation by $\sim$. Let $X'_\beta = X_\beta/\sim$; by Schmidt’s theorem, $\#[\varepsilon] = 1$ for a.e. sequence $\varepsilon$. Let addition (subtraction) in $X'_\beta$ be defined through the torus, i.e., $[\varepsilon] + [\varepsilon'] := \varphi_0^{-1}(\varphi_0(\varepsilon) + \varphi_0(\varepsilon'))$. Obviously, $X'_\beta$ is a group under addition isomorphic to $\mathbb{T}^m$.

**Remark.** In fact, to add $[\varepsilon]$ to $[\varepsilon']$, one may avoid using the torus. Indeed, following the definition from the previous section, consider the sequence $\{\varepsilon(1) + \varepsilon'(1)\}_{1}^{\infty}$. It has got some set of partial limits (possibly more than one), but they all belong to one and the same equivalence class by the continuity of $\varphi_0$. Thus, briefly, $[[\varepsilon]] + [[\varepsilon']] = [\lim_N(\{\varepsilon(1)\} + \{\varepsilon'(1)\})]$.

**Theorem 3.1.**

1. The mapping $\varphi$ is well defined on the quotient compactum $X'_\beta$.
2. $\varphi : X'_\beta \to \mathbb{T}^m$ is a group homomorphism, its kernel being equal to $\mathcal{A}_\beta$.

**Proof.** (1) We need to show that if $\varphi_0(\varepsilon) = \varphi_0(\varepsilon')$, then $\varphi(\varepsilon) = \varphi(\varepsilon')$. By the definition of $\varphi_0$, we have $\|\xi_0 u_N\| \to 0$, where $u_N = \sum_{k=-N}^{\infty} (\varepsilon_k - \varepsilon'_k) \beta^{-k}$. By Proposition 1.2, $\xi_0^{-1} \in \mathbb{Z}[\beta]$, whence for any $\{u_N\}$ such that $\|\xi_0 u_N\| \to 0$, $\|u_N\| \to 0$.

\footnote{The nature of this measure is unimportant; in fact, one may take any shift-invariant measure that is strictly positive on all cylinders $\{u \in \mathbb{Z}^m : u_i \in \mathcal{A}_{\beta} \} \subset \mathbb{T}^m$.}
as well. Hence by the definition of \( \varphi \), \( \| \varphi(\varepsilon^{(N)}) - \varphi(\varepsilon'^{(N)}) \| \leq \text{const} \cdot \| u_N \| \to 0 \).

(2) It is shown in [Sch] that a mapping \( \varphi \) would be bounded-to-one for any choice of homoclinic point \( t \). Since the set \( \mathcal{O} = \varphi^{-1}(0) \) is finite and shift-invariant, it consists of purely periodic sequences only. Let \( z \in \mathcal{O} \), its period equal to \( d \) and \( \alpha = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k} \). Then by the definition of \( \varphi \), \( \| \alpha \beta^dN \| \to 0 \) as \( N \to +\infty \).

Considering the sequences \( \tau z, \tau^2 z, \ldots, \tau^{d-1} z \), we see that \( \| \alpha \beta^dN+j \| \to 0 \) for \( j = 0, 1, \ldots, d-1 \), whence \( \| \alpha \beta^N \| \to 0 \) as \( N \to +\infty \), i.e., \( \alpha \in \mathcal{P}_\beta \), and the claim follows from Lemma 2.1. Conversely, if \( \alpha \in \mathcal{P}_\beta \), then \( \| \alpha \beta^N \| \to 0 \), whence \( \varphi(z) = 0 \). □

**Corollary 3.2.** The mapping \( \varphi \) is \( |D| \)-to-1 a.e.

Let \( A = \varphi \varphi^{-1} \). This mapping is well defined a.e. on the torus, and we extend it to the whole \( \mathbb{T}^m \) by continuity. The following claim is straightforward.

**Lemma 3.3.** The mapping \( A : \mathbb{T}^m \to \mathbb{T}^m \) is an endomorphism of the torus. Its determinant equals \( \pm D \).

In practice, to find \( A \), one might expand \( \xi_0^{-1} \) into the sum \( \sum_{k=0}^{m-1} a_k \beta^k \), \( a_k \in \mathbb{Z} \). Then \( A = \sum_{k=0}^{m-1} a_k T^k \).

**Examples.** 1. \( \beta^2 = k \beta \pm 1 \). Here, as we know, \( \xi_0 = \frac{1}{\sqrt{D}} \), \( \xi_0^{-1} = \sqrt{D} = 2\beta - k \). Hence

\[
A = 2T - kI = \begin{pmatrix} 2k & 2 & 0 \\ 2 & 0 & k \\ 0 & k & 0 \end{pmatrix} = \begin{pmatrix} k & 2 & \pm 2 \\ 2 & \pm 2 & k \end{pmatrix} = \begin{pmatrix} k & 2 & \pm 2 \\ 2 & \pm 2 & k \end{pmatrix},
\]

and \( \det A = -D(\beta) \).

2. \( \beta^3 = \beta^2 + \beta + 1 \). Here \( \xi_0^{-1} = -1 - 2\beta + 3\beta^2 \), whence

\[
A = -I - 2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 3 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{pmatrix},
\]

and \( \det A = 44 = -D(\beta) \).

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