Differential Forms on Hyperelliptic Curves with Semistable Reduction

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ABSTRACT. Let \( C \) be a hyperelliptic curve over a local field \( K \) with odd residue characteristic, defined by some affine Weierstraß equation \( y^2 = f(x) \). We assume that \( C \) has semistable reduction and denote by \( \mathcal{X} \to \text{Spec} \mathcal{O}_K \) its minimal regular model with relative dualizing sheaf \( \omega_{\mathcal{X}/\mathcal{O}_K} \). We show how to directly read off a basis for \( H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{O}_K}) \) from the cluster picture of the roots of \( f \). Furthermore we give a formula for the valuation of \( \lambda \) such that \( \lambda \cdot \frac{dx}{y} \wedge \cdots \wedge x^{g-1} \frac{dx}{y} \) is a generator for \( \det H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{O}_K}) \).

1 Introduction

Let \((R, v)\) be a discrete valuation ring with maximal ideal \( p = (\pi) \) and field of fractions \( K \). For an element \( r \in R \), we write \( \bar{r} \) for the reduction modulo \( p \). The separable closure of \( K \) is denoted by \( K^{\text{sep}} \) and the algebraic closure by \( \bar{K} \). We assume that the residue field \( k = R/p \) has characteristic \( p \neq 2 \).

Let \( C/K \) be a hyperelliptic curve given by a Weierstraß equation

\[ C : y^2 = f(x). \]

Throughout this paper, we will always assume that the curve \( C \) has semistable reduction and genus \( g > 1 \). Since \( p \neq 2 \), it is easy to read off from the polynomial \( f \) whether \( C \) is semistable and determine the semistable reduction. This is described in \([1]\) and \([2]\).

To a Weierstraß equation, we associate the differential forms

\[ \omega_0 = \frac{dx}{2y}, \quad \omega_1 = \frac{x dx}{2y}, \ldots, \quad \omega_{g-1} = \frac{x^{g-1} dx}{2y}. \]

These form a basis of \( H^0(C, \Omega_{C/K}) \). Let \( \mathcal{X} \to \text{Spec} R \) be the minimal regular model of \( C \) and \( \omega_{\mathcal{X}/R} \) the relative dualizing sheaf \([3]\) Definition 6.4.18]. In our situation, the relative dualizing sheaf is isomorphic to the canonical sheaf \([3]\) Theorem 6.4.32]. We
have that $H^0(C, \Omega_{C/K}) = H^0(X, \omega_{X/R}) \otimes_R K$ and $H^0(X, \omega_{X/R})$ is a free $R$-module of rank $g$. So $\det H^0(X, \omega_{X/R}) = \Lambda^g H^0(X, \omega_{X/R})$ is free of rank one over $R$.

In this paper, we are going to study the following two problems:

1. Let $\omega := \omega_0 \wedge \cdots \wedge \omega_{g-1} \in \det H^0(C, \Omega_{C/K})$. Determine $\lambda_C \in K$, such that $\lambda_C \cdot \omega$ generates $\det H^0(X, \omega_{X/R})$ as an $R$-module.

2. Explicitly determine a basis for the global sections of $\omega_{X/R}$.

Our approach is based on results of [4]. Under simplified hypotheses a formula for $\lambda_C$ and a description of a basis for the global sections of $\omega_{X/R}$ is given in Proposition 5.5. of that paper.

1.1 Motivation Our motivation for studying the differential forms on the minimal regular model comes from the Birch and Swinnerton-Dyer conjectures. Originally formulated for elliptic curves, the conjectures were later generalized to abelian varieties over number fields by Tate [6].

Let $C$ be a hyperelliptic curve defined over the rational numbers $\mathbb{Q}$ and let $J$ denote its Jacobian. The second BSD conjecture in this situation is

$$\lim_{s \to 1} (s-1)^{-r} L(J, s) = \Omega \cdot \text{Reg} \cdot \prod_p c_p \cdot \# \text{III}(J, \mathbb{Q}) \cdot (\# J(\mathbb{Q})_{\text{tors}})^{-2}.$$ 

Here $L(J, s)$ is the $L$-series of $J$ and $r$ its analytic rank. Reg denotes the regulator of $J(\mathbb{Q})$. For a prime $p$, the Tamagawa number is denoted by $c_p$. The Shafarevich-Tate group is represented by $\text{III}(J, \mathbb{Q})$ and $J(\mathbb{Q})_{\text{tors}}$ is the torsion subgroup of $J(\mathbb{Q})$. The results of the present paper can be applied to calculate the sixth quantity, that is the period $\Omega$. For the description of this quantity we follow the outline in [7, Section 3] and [3, Section 3.5]. Both papers provide numerical evidence for the BSD conjectures for hyperelliptic curves.

Let $J \to \text{Spec} \mathbb{Z}$ denote the Néron model of $J$. We have that $H^0(J, \Omega_{J/\mathbb{Z}})$ is a free $\mathbb{Z}$-module of rank $g$. Let $(\mu_0, \ldots, \mu_{g-1})$ be a basis for this module. Then $\mu := \mu_0 \wedge \cdots \wedge \mu_{g-1}$ is a generator for $\Lambda^g H^0(J, \Omega_{J/\mathbb{Z}})$ and $\Omega$ is defined as

$$\Omega := \int_{J(\mathbb{R})} |\mu|.$$ 

Finding a basis for $H^0(J, \Omega_{J/\mathbb{Z}})$ can be done locally. Let $p \in \mathbb{Z}$ be a prime. We write $R = \mathbb{Z}_p$ and use the notation introduced in the beginning. We have that $\Omega_{J/R}$ is isomorphic to the canonical sheaf $\omega_{X/R}$, where $X$ is the minimal regular model of $C$. So it is enough to find a basis for the global sections of $\omega_{X/R}$.

For computational purposes one does not necessarily need to know the basis of regular differentials. In [7] and [3] the authors first evaluate $\int_{J(\mathbb{R})} |\omega|$, where $\omega$ is the exterior product of the differentials $\omega_0, \ldots, \omega_{g-1}$ associated to the Weierstraß equation for $C$ and then compute a correction term in order to get the value $\Omega := \int_{J(\mathbb{R})} |\mu|$.

1.2 Results We give a brief overview of our main results and illustrate them by means of an example. The results are stated in terms of cluster pictures. A cluster
picture is a combinatorial object associated to the equation of a curve. It encodes different invariants of the curve. The concept has been studied in [2]. For definitions we refer to [2] or Section 2 of the present paper. Here we only recall the necessary notation.

- $c_f$ leading coefficient of $f$
- $\mathcal{R}$ set of roots of $f$ in $K^{\text{sep}}$
- $\mathfrak{s}$ a cluster
- $\mathfrak{s} \wedge \mathfrak{s}'$ smallest cluster containing $\mathfrak{s}$ and $\mathfrak{s}'$
- $z_s$ a centre of $\mathfrak{s}$
- $d_s$ depth of $\mathfrak{s}$
- $\delta_s$ relative depth of $\mathfrak{s}$

The following theorem shows how to read off a basis for $H^0(X, \omega_{X/R})$ from the cluster picture of a curve.

**Theorem 1.1 (Theorem 4.1).** Let $C/K$ be a hyperelliptic curve defined by an integral Weierstrass equation $C : y^2 = f(x)$ and $\Sigma$ the associated cluster picture.

Let $X/R$ be the minimal regular model. Assume that the residue field $k$ is algebraically closed. Then an $R$-basis for the global sections of the relative dualizing sheaf $\omega_{X/R}$ is given by $(\mu_0, \ldots, \mu_{g-1})$, where

$$
\mu_i = \pi^e_i \prod_{j=0}^{i-1} (x - z_{s_j}) \frac{dx}{y}
$$

with

$$
e_i = \frac{\nu_{s_i}}{2} - \sum_{j=0}^{i} d_{s_j \wedge s_i}.
$$

The clusters $\mathfrak{s}_0, \ldots, \mathfrak{s}_{g-1}$ are chosen inductively such that

$$
\frac{\nu_{s_i}}{2} - \sum_{j=0}^{i} d_{s_j \wedge s_i} = \max_{\mathfrak{s} \in \Sigma} \left( \frac{\nu_{s_i}}{2} - \sum_{j=0}^{i-1} d_{s_j \wedge s} - d_s \right).
$$

If the maximal value is obtained by two different clusters $\mathfrak{s}$ and $\mathfrak{s}'$ such that $\mathfrak{s}' \subset \mathfrak{s}$, we choose $\mathfrak{s}_i = \mathfrak{s}$.

We are going to illustrate the procedure described in this theorem by an example.

**Example 1.2.** Let $p > 3$ and $C/\mathbb{Q}_p$ the hyperelliptic curve of genus $g = 5$ defined by

$$
y^2 = x(x-p^6)(x-2p^6)(x-p^4)(x-2p^4)(x-3p^4)(x-1)(x-1-p^8)(x-1-2p^8)(x-3p^8)(x-2)(x-3).
$$

The proper clusters are

$$
\mathcal{R}, \ t_1 = \{0, p^6, 2p^6, p^4, 2p^4, 3p^4\}, \ t_2 = \{0, p^6, 2p^6\}, \ t_3 = \{1, 1 + p^8, 1 + 2p^8, 1 + 3p^8\}.
$$

These clusters have depths $d_{\mathcal{R}} = 0$, $d_{t_1} = 4$, $d_{t_2} = 6$, $d_{t_3} = 8$ and relative depths $\delta_{t_1} = 4$, $\delta_{t_2} = 2$, $\delta_{t_3} = 8$. This information is contained in the cluster picture $\Sigma$:
The subscript of the top cluster is its depth. The subscripts of the other clusters are their relative depths.

We construct a basis for $H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{R}})$ as described in Theorem 1.2. First we choose $s_0$ to be the cluster that maximises $\frac{d}{2} - d_i$. The evaluation of this term for each cluster can be found in the first column (after the double line) of the table below. Next we choose $s_1$ to be the cluster that maximises $\frac{d}{2} - d_i - d_{v_id}$, the cluster $s_2$ is the one that maximises $\frac{d}{2} - d_i - d_{v_is_0} - d_{v_2s_0}$ and so on.

| $\nu$ | $d_i$ | $\frac{d}{2} - d_i$ | $\frac{d}{2} - d_i - d_{v_1s_0}$ | $\ldots - d_{v_1s_0}$ | $\ldots - d_{v_2s_0}$ |
|-------|-------|-------------------|---------------------|-----------------|------------------|
| $\mathcal{R}$ | 0 | 0 | 0 | 0 | 0 |
| $t_1$ | 24 | 4 | 8 | 4 | [4] | 0 | 0 |
| $t_2$ | 30 | 6 | 9 | 3 | 3 | -1 | -1 |
| $t_3$ | 32 | 8 | 8 | 8 | 0 | 0 | 0 |

In each column the maximal value is circled to indicate which cluster is chosen in the respective step. Three dots always represent the entire expression in the previous column. We can read off from the table

$s_0 = t_2, \quad s_1 = t_3, \quad s_2 = t_1, \quad s_3 = \mathcal{R}, \quad s_4 = \mathcal{R}$,

and $e_0 = 9, \quad e_1 = 8, \quad e_2 = 4, \quad e_3 = 0, \quad e_4 = 0$.

If we choose $z_0 = z_{t_1} = z_{t_2} = 0$ and $z_{t_3} = 1$ as centres for the clusters, we get the following basis for the global sections of $\omega_{\mathcal{X}/\mathcal{R}}$:

$$
\left(\mu_0 = p^0 \frac{dx}{y}, \quad \mu_1 = p^1 x \frac{dx}{y}, \quad \mu_2 = p^2 x(x - 1) \frac{dx}{y}, \quad \mu_3 = x^2(x - 1) \frac{dx}{y}, \quad \mu_4 = x^3(x - 1) \frac{dx}{y}\right)
$$

As mentioned before, it is often not necessary to determine the basis for $H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{R}})$ explicitly, but it suffices to know a generator for $\det H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{R}})$. Theorem 1.3 gives a convenient formula for the determination of this generator.

**Theorem 1.3** (Theorem 3.1). Let $C/K$ be a semistable hyperelliptic curve defined by $C : y^2 = f(x)$ with $f(x) = c_f \prod_{r \in \mathcal{R}} (x - r)$. We write $\omega_0 = \frac{dx}{y}, \ldots, \omega_{g-1} = \frac{x^{g-1} dx}{y}$ for the differentials associated to this equation and $\omega := \omega_0 \wedge \cdots \wedge \omega_{g-1} \in \det H^0(C, \Omega^1_{C/K})$. Let $\mathcal{X} \to \text{Spec} \mathcal{R}$ be the minimal regular model of $C$. Suppose that $\lambda_C \cdot \omega$ is a basis for $\det H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{R}})$. Then

$$
8 \nu(\lambda_C) = 4g \cdot v(c_f) + \sum_{|s| \text{ even}} \delta_s(|s| - 2)|s| + \sum_{|s| \text{ odd}} \delta_s(|s| - 1)^2
$$

$$
+ d_{\mathcal{R}} \begin{cases} (|\mathcal{R}| - 2)|\mathcal{R}|, & \text{if } |\mathcal{R}| = 2g + 2 \\ (|\mathcal{R}| - 1)^2, & \text{if } |\mathcal{R}| = 2g + 1 \end{cases}
$$

Let us revisit the above example.
Example 1.4. Let \( p > 3 \) and \( C/\mathbb{Q}_p \) be the hyperelliptic curve Example 1.2. Recall the cluster picture \( \Sigma \):

Then the formula of Theorem 1.3 yields

\[
8v(\lambda_C) = 4 \cdot g \cdot v(c_f) + \delta_{t_1}(|t_1| - 2)|t_1| + \delta_{t_2}((|t_2| - 1)^2 + \delta_{t_3}((|t_3| - 2)|t_3| + d_\mathfrak{R} (|\mathfrak{R}| - 2)|\mathfrak{R}|
\]

\[
= 4 \cdot 5 \cdot 0 + 4 \cdot 4 \cdot 6 + 2 \cdot 2^2 + 8 \cdot 2 \cdot 4 + 0 \cdot 10 \cdot 12
\]

\[
= 8 \cdot 21.
\]

So

\[
\mu := p^{21} \frac{dx}{y} \wedge \frac{xdx}{y} \wedge \frac{x^2dx}{y} \wedge \frac{x^3dx}{y} \wedge \frac{x^4dx}{y}
\]

generates \( \det H^0(X, \omega_X \otimes R) \). Note that the value \( v(\lambda_C) = 21 \) is equal to the sum over the \( e_i \) determined by Theorem 1.3.

1.3 Outline In Section 2 we review some definitions and facts about cluster pictures. In Section 3 we translate Proposition 5.5. of [4] into the language of cluster pictures and generalise it in order to prove Theorem 1.3. The last section is dedicated to the proof of Theorem 1.1.

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2 Cluster Pictures

In this section, we describe the cluster picture associated to an equation defining a hyperelliptic curve and briefly introduce the notation used in the subsequent sections. All information is taken from [2].

Let \( C/\mathbb{K} \) be a hyperelliptic curve defined by a Weierstrass equation \( C : y^2 = f(x) \). We write \( \mathfrak{R} \) for the set of roots of \( f(x) \) in \( K^{sep} \) and \( c_f \) for its leading coefficient, so that

\[
f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r).
\]

Definitions 2.1. (i) ([2] Definition 1.1.) A cluster is a non-empty subset \( s \subset \mathfrak{R} \) of the form \( s = D \cap \mathfrak{R} \) for some disc \( D = \{ x \in \mathring{K} \mid v(x - z) \geq d \} \) for some \( d \in \mathbb{Q} \) and \( z \in \mathring{K} \). We say that \( z \) is a center of the cluster and write \( z = z_s \).

(ii) ([2] Definition 1.1.) If \( |s| > 1 \), then \( s \) is called a proper cluster and its depth is defined to be \( d_s = \min_{r, r' \in s} v(r - r') \).
(iii) ([2] Definition 1.4.) A cluster \( s \) is principal if \( |s| \geq 3 \), except if either \( s = R \) is even and has exactly two children, or if \( s \) has a child of size \( 2g \).

(iv) ([2] Definition 1.3.) If \( s' \subsetneq s \) is a maximal subcluster, we write \( s' < s \). For two clusters (or roots) \( s, s' \) we write \( s \wedge s' \) for the smallest cluster that contains them.

(v) ([2] Definition 1.5.) If \( s \neq R \), the relative depth of \( s \) is defined as the difference between the depth of \( s \) and the depth of the smallest cluster strictly containing \( s \). It is denoted by \( \delta_s \).

(vi) ([2] Definition 1.6.) For a cluster \( s \), set \( \nu_s = v(c_f) + \sum_{r \in R} d_{r \wedge s} \).

See Example 1.2 in the Introduction for an illustration of these definitions.

Remark 2.2. Note that for a root \( r \in R \) and a cluster \( s \), we have

\[
d_{r \wedge s} = d_R + \sum_{s' \neq R, s \wedge r \subseteq s'} \delta_{s'}.
\]

So \( \nu_s \) can also be calculated via

\[
\nu_s = v(c_f) + d_R[R] + \sum_{s' \neq R, s \subseteq s'} \delta_{s'}|s'|.
\]

There is a notion of equivalence for cluster pictures that respects isomorphisms of hyperelliptic curves. For a complete discussion of the topic we refer to [2, Section 14].

The following proposition is important for the proof of Theorem 3.1. So we state it here for the convenience of the reader.

Proposition 2.3 ([2], Proposition 14.6.). Let \( f(x) \in K[x] \) be a separable polynomial with roots \( R \subset K \), such that \( G_K \) acts tamely on \( R \), and let \( \Sigma \) be the associated cluster picture. Suppose \( \Sigma' \) is a cluster picture obtained from \( \Sigma \) by one of the following constructions:

1. Increasing the depth of all clusters by some \( n \in \mathbb{Z} \);
2. Adding a root to \( R \), provided \( |R| \) is odd, \( d_R \in \mathbb{Z} \) and \( |k| > \# \{ s < R : s \text{ is } G_K\text{-stable} \} \);
3. Redistributing the depth between \( s \) and \( R \setminus s \) by decreasing the depth of \( s \) by 1, provided \( |R| \) is even, \( s < R \) is \( G_K \)-stable with \( d_R, d_s \in \mathbb{Z} \) and \( |k| > \# \{ t < s : t \text{ is } G_K\text{-stable} \} \).

Then there is a Möbius transformation \( \phi(z) = \frac{az + b}{cz + d} \) with \( a, b, c, d \in K \), such that \( \Sigma' \) is the cluster picture of \( \mathcal{R}' = \{ \phi(r) : r \in \mathcal{R} \} \setminus \{ \infty \} \) if \( |R| \) is even and of \( \mathcal{R}' = \{ \phi(r) : r \in \mathcal{R} \cup \{ \infty \} \} \setminus \{ \infty \} \) if \( |R| \) is odd. Moreover, if \( y^2 = f(x) \) is a hyperelliptic curve, then there is a \( K \)-isomorphic curve given by a Weierstraß model whose cluster picture is \( \Sigma' \).
3 A Basis for $\det H^0(\mathcal{X}, \omega_{\mathcal{X}/R})$

Let $C/K$ be a semistable hyperelliptic curve of genus $g$ defined by $C : y^2 = f(x)$ with $f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r)$. We write $\omega_0 = \frac{dx}{y}, \ldots, \omega_{g-1} = \frac{d^{g-1}x}{y}$ for the differentials associated to this equation and $\omega := \omega_0 \wedge \cdots \wedge \omega_{g-1} \in \det H^0(C, \Omega^1_{C/K})$. Let $\mathcal{X} \to \text{Spec} R$ be the minimal regular model of $C$. The main result of this section is Theorem 3.1, where we determine $\lambda_C \in K$, such that $\lambda_C \cdot \omega$ generates $\det H^0(\mathcal{X}, \omega_{\mathcal{X}/R})$ as an $R$-module. Note that $\lambda_C$ is only well-defined up to a unit. Moreover it is not a curve invariant, but depends on the equation.

**Theorem 3.1.** Let $C/K$ be a semistable hyperelliptic curve of genus $g$ defined by $C : y^2 = f(x)$ with $f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r)$. We write $\omega_0 = \frac{dx}{y}, \ldots, \omega_{g-1} = \frac{d^{g-1}x}{y}$ for the differentials associated to this equation and $\omega := \omega_0 \wedge \cdots \wedge \omega_{g-1} \in \det H^0(C, \Omega^1_{C/K})$. Let $\mathcal{X} \to \text{Spec} R$ be the minimal regular model of $C$. Suppose that $\lambda_C \cdot \omega$ is a basis for $\det H^0(\mathcal{X}, \omega_{\mathcal{X}/R})$. Then

$$8 \nu(\lambda_C) = 4g \cdot v(c_f) + \sum_{|s| \text{ even } s \in \mathfrak{R}} \delta_2(|s| - 2)|s| + \sum_{|s| \text{ odd } s \in \mathfrak{R}} \delta_2(|s| - 1)^2$$

$$+ d_{\mathfrak{R}} \begin{cases} (|\mathfrak{R}| - 2)|\mathfrak{R}|, & \text{if } |\mathfrak{R}| = 2g + 2 \\ (|\mathfrak{R}| - 1)^2, & \text{if } |\mathfrak{R}| = 2g + 1 \end{cases} \quad (1)$$

A result in [4] shows that this formula is true under some additional assumptions (Lemma 3.2). We show how one can reduce to this case and work out the necessary correction terms.

**Lemma 3.2.** Let $C/K$ be a hyperelliptic curve defined by $C : y^2 = f(x)$ with $f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r)$. Assume that

(i) $\mathfrak{R} \subset R$,

(ii) $v(r - s) \in 2\mathbb{Z}$ for all $r, s \in \mathfrak{R}$,

(iii) $c_f$ is a unit in $R$,

(iv) $|\mathfrak{R}| = 2g + 2$,

(v) $\# \{ \bar{r} \mid r \in \mathfrak{R} \} \geq 3$.

Then $\nu(\lambda_C)$ can be computed using Equation (1).

**Proof.** Under the conditions in the lemma, Equation (1) reduces to

$$8 \nu(\lambda_C) = \sum_{|s| \text{ even } s \in \mathfrak{R}} \delta_2(|s| - 2)|s| + \sum_{|s| \text{ odd } s \in \mathfrak{R}} \delta_2(|s| - 1)^2. \quad (2)$$

This formula is due to Kausz, see [4, Proposition 5.5.2.]. He does not use cluster pictures to express the formula, but it is easy to see that his formula translates into Formula (2) and the conditions of Proposition 5.5.2. coincide with Conditions (i)-(v).

Note that the conditions in the above Lemma imply semistability, see for example [2, Theorem 7.1.]. Conversely, after a tamely ramified extension of the base field, there always exists an equation for $C$ satisfying the conditions listed in the above Lemma if $C$ is semistable.
Proposition 3.3. Let $C/K$ be a semistable hyperelliptic curve with minimal regular model $\mathcal{X} \to \text{Spec} R$. Let $K'/K$ be a finite field extension. Write $C'$ for the base-change of $C$ to $K'$, $R' = \mathcal{O}_{K'}$ and $\mathcal{X}' \to \text{Spec} R'$ for the minimal regular model of $C'$. Then

$$H^0(\mathcal{X}', \omega_{\mathcal{X}'/R'}) = H^0(\mathcal{X}, \omega_{\mathcal{X}/R}) \otimes_R R'$$

inside $H^0(C', \Omega_{C'/K'}) = H^0(C, \Omega_{C/K}) \otimes_K K'$.

Proof. Let $\mathcal{Y} \to \text{Spec} R$ and $\mathcal{Y}' \to \text{Spec} R'$ be the stable models of $C/K$ and $C'/K'$ respectively. The stable model $\mathcal{Y}$ is obtained from $\mathcal{X}$ by contraction of all components $\Gamma$ of the special fibre for which $K_{\mathcal{X}/R}^\Delta, \Gamma = 0$. Write $f : \mathcal{X} \to \mathcal{Y}$ for the contraction morphism. Since the intersection matrix of the contracted components is negative definite, it follows from $[5, \text{Corollary 9.4.18}.$] that $\omega_{\mathcal{X}/R} = f_* \omega_{\mathcal{Y}/R}$. Therefore $H^0(\mathcal{X}, \omega_{\mathcal{X}/R}) = H^0(\mathcal{Y}, \omega_{\mathcal{Y}/R})$ inside $H^0(C, \Omega_{C/K})$. By the same reasoning $H^0(\mathcal{X}', \omega_{\mathcal{X}'/R'}) = H^0(\mathcal{Y}', \omega_{\mathcal{Y}'/R'})$. So it suffices to show that $H^0(\mathcal{Y}', \omega_{\mathcal{Y}/R}) = H^0(\mathcal{Y}, \omega_{\mathcal{Y}/R}) \otimes_R R'$.

We have $\mathcal{Y}' = \mathcal{Y} \times_R R'$ and by $[5, \text{Theorem 6.4.9.b}.$] $\omega_{\mathcal{Y}'/R'} = p^* \omega_{\mathcal{Y}/R}$, where $p : \mathcal{Y} \times_R R' \to \mathcal{Y}$ is the first projection. So the result follows from $[5, \text{Corollary 5.2.27}].$

□

Lemma 3.4. Let $C/K$ be a hyperelliptic curve with semistable reduction defined by $C : y^2 = f(x)$. Let $K'/K$ be a tamely ramified extension and write $C'$ for the base-change of $C$ to $K'$. Then Equation [1] holds for $C/K$ if and only if it holds for $C'/K'$.

Proof. Let $e_{K'/K}$ denote the ramification degree of the extension $K'/K$. We write $\Sigma$ for the cluster picture associated to $C : y^2 = f(x)$ and $\Sigma'$ for the cluster picture associated to the equation $C' : y^2 = f(x)$ over $K'$. The clusters themselves do not change under a ramified extension, but their depths do. More precisely we have $\delta'_s = e_{K'/K} \cdot \delta_s$ for each cluster $s$ and $d'_s = e_{K'/K} \cdot d_s$. We write $R' = \mathcal{O}_{K'}$ and $v'$ for the normalized valuation. That is $v'(r) = e_{K'/K} \cdot v(r)$ for all $r \in R$.

For $\lambda_{C'}$, Equation [1] yields

$$8 v'(\lambda_{C'}) = e_{K'/K} \cdot \left( g \cdot v(c_f) + \sum_{|s| \text{ even}} \delta_s (|s| - 2)|s| + \sum_{|s| \text{ odd}} \delta_s (|s| - 1)^2 \right) + d_R \begin{cases} (|\mathcal{R}| - 2)|\mathcal{R}|, & \text{if } |\mathcal{R}| = 2g + 2 \\ (|\mathcal{R}| - 1)^2, & \text{if } |\mathcal{R}| = 2g + 1 \end{cases}.$$

From Proposition [3.3] it follows that $v'(\lambda_{C'}) = e_{K'/K} \cdot v(\lambda_C)$. So the above calculation shows that Formula [1] is true for $C/K$ if and only if it is true for $C'/K'$.

□

Definition 3.5. Let $C/K$ be a hyperelliptic curve of genus $g$, defined by some Weierstrass equation $y^2 = f(x)$. We denote by $c_f$ the leading coefficient of $f$. Then the discriminant $\Delta$ of the equation is defined as

$$\Delta := 2^{4g} c_f^{4g+2} \text{disc} \left( \frac{1}{c_f} f(x) \right).$$

8
While the discriminant defined above is not a curve invariant but depends on the equation, there exists a more natural definition of discriminant. See also the paragraph before Proposition 2.2. in [4].

**Definition 3.6.** Let \( C/K \) be a hyperelliptic curve of genus \( g \), defined by some Weierstraß equation \( y^2 = f(x) \) with discriminant \( \Delta \). We associate to this equation the differential forms \( \omega_0, \ldots, \omega_{g-1} \) and write \( \omega = \omega_0 \wedge \cdots \wedge \omega_{g-1} \in H^0(C, \Omega^1_{C/K}) \).

Then the element
\[
\Lambda := \Delta^g \cdot \omega \wedge 8g+4 \in (\det H^0(C, \Omega^1_{C/K}))^{\otimes 8g+4}
\]
is called **hyperelliptic discriminant** of \( C \).

The following proposition shows that \( \Lambda \) is well defined.

**Proposition 3.7.** Let \( C/K \) be a hyperelliptic curve with hyperelliptic discriminant \( \Lambda \). Let \( y^2 = f(x) \) be some Weierstraß equation for \( C \). We associate to this equation the elements \( \Delta \) and \( \lambda \). Then the following statements are true.

1. The element \( \Lambda \) is independent of the choice of equation.
2. Viewed as a rational section of \( (\det H^0(X, \omega_X \otimes K))^\otimes 8g+4 \), the order of vanishing in \( p \) is given by
   \[
   \text{ord}_p(\Lambda) = g \cdot v(\Delta) - (8g + 4) \cdot v(\lambda_C).
   \]
3. Let \( y^2 = g(x') \) be another equation defining the same curve with \( \Delta' \) and \( \lambda_{C'} \) the corresponding quantities. Then
   \[
   \frac{v(\Delta') - v(\Delta)}{8g + 4} = \frac{v(\lambda_{C'}) - v(\lambda_C)}{g}.
   \]

**Proof.**

1. This is Proposition 2.2.1. in [4].
2. This follows from the definition of \( \Lambda \). See also [4, Section 5, Formula 1].
3. This is a direct consequence of the first two statements.

**Lemma 3.8.** Let \( C/K \) be a hyperelliptic curve defined by \( C : y^2 = f(x) \) with \( f(x) = c_f \prod_{r \in R} (x - r) \) and \( v(c_f) \) in \( 2\mathbb{Z} \). Then the curve \( C'/K \) defined by \( y^2 = \prod_{r \in R} (x - r) \) is isomorphic to \( C \) over \( K^{nr} \) and

\[
v(\lambda_C) = v(\lambda_{C'}) + g \cdot \frac{v(c_f)}{2}.
\]

**Proof.** Since \( v(c_f) \) is even, \( c_f \) is a square in \( K^{nr} \) and the two equations define isomorphic curves over \( K^{nr} \).

By definition, the discriminant of the equation \( y^2 = c_f \prod_{r \in R} (x - r) \) is
\[
\Delta = 2^{4g} c_f^{4g+2} \text{disc}(\prod_{r \in R} (x - r)).
\]
So
\[ v(\Delta) = v(\Delta') + (4g + 2) v(c_f). \]

Using part 3 of Proposition 3.7 we get
\[ v(\lambda_C) = v(\lambda_{C'}) + g \cdot \frac{v(c_f)}{2}. \]

\[ \square \]

Lemma 3.9. Let \( C/K \) be a hyperelliptic curve and \( y^2 = f(x) \) a Weierstraß equation defining this curve. Let \( \Sigma \) be its cluster picture and \( \lambda_C \) the quantity associated to this equation.

Let \( y'^2 = g(x') \) be a different equation for \( C \). Denote by \( \Sigma' \) and \( \lambda_{C'} \) the corresponding elements.

1. If \( \Sigma' \) is obtained from \( \Sigma \) by increasing the depths of all clusters by some \( t \in \mathbb{Z} \), then
\[ v(\lambda_C) = v(\lambda_{C'}) - \frac{t}{8} \cdot \begin{cases} (|R| - 2)|R|, & \text{if } |R| = 2g + 2 \\ (|R| - 1)^2, & \text{if } |R| = 2g + 1 \end{cases} \]

2. If \( \Sigma' \) is obtained from \( \Sigma \) by adding a root to \( R \), then
\[ v(\lambda_C) = v(\lambda_{C'}) - \frac{d_R (|R| - 1)}{4} \cdot \frac{(|R| - 2)(|R| - 2|s|)}{8}. \]

3. If \( \Sigma \) has even size and \( \Sigma' \) is obtained from \( \Sigma \) by redistributing the depth between \( s < R \) and \( R \setminus s \) to \( d'_s = d_s - t \) and \( d'_{R \setminus s} = d_{R \setminus s} + t \), then
\[ v(\lambda_C) = v(\lambda_{C'}) - t \cdot \frac{(|R| - 2)(|R| - 2|s|)}{8}. \]

Proof. From [2, Lemma 16.6.], we know how the discriminant changes under the above modifications of the cluster picture. We will combine these results with Proposition 3.7, part 3.

1. By [2, Lemma 16.6.(i)]
\[ v(\Delta') - v(\Delta) = t|R|(|R| - 1). \]

Now Proposition 3.7 implies
\[ v(\lambda_{C'}) - v(\lambda_C) = g \cdot \frac{t|R|(|R| - 1)}{8g + 4} \]
\[ = \frac{t}{8} \cdot \begin{cases} (|R| - 2)|R|, & \text{if } |R| = 2g + 2 \\ (|R| - 1)^2, & \text{if } |R| = 2g + 1 \end{cases} \]

2. By [2, Lemma 16.6.(ii)]
\[ v(\Delta') - v(\Delta) = 2d_R |R|. \]
This follows from Lemma 3.8.

defined by

Proof of Theorem 3.1. Let \( R \) may assume that it suffices to prove that the formula holds after a tamely ramified extension. So we if \( /\text{divides.}\) alt0 \( \lambda \) formula for \( d \) Lemma 3.9 this corresponds to subtracting \( d \) sponding to the new equation has outer depth \( d \) can apply Part 4 of Proposition 2.3 that is redistribute dept h between a (\( \sigma \)). Without loss of generality we may always assume \( G \) with Conditions (i)-(iv) of the lemma being satisfied and \( d \) implies Condition (v) of Lemma 3.2. That is \( 8 \) with Conditions (i)-(iv) of the lemma being satisfied and \( d \) 0. Without loss of generality we may always assume \( G_K \)-stability for clusters because the formula for \( \lambda \) behaves well under tamely ramified extension (see Lemma 3.4). So we can apply Part 4 of Proposition 2.3 that is redistribute depth between a (\( G_K \)-stable) cluster \( \mathfrak{s} \) and \( \mathfrak{R}\backslash\mathfrak{s} \). With this method we can manipulate the cluster picture such that \( \mathfrak{R} \) has at least three maximal subclusters. Together with the fact that \( d_\mathfrak{R} = 0 \) this implies Condition (v) of Lemma 3.2 Assuming that the formula behaves well when redistributing depths, this proves the theorem. So we only have to show the latter.
Let $s^* < \mathfrak{R}$ be a cluster in $\Sigma$. Let $\Sigma'$ be the cluster picture obtained after redistributing depth between $s^*$ and $\mathfrak{R}\setminus s^*$. That is, $d'_{s^*} = d_{s^*} - t$ and $d'_{\mathfrak{R}\setminus s^*} = d_{\mathfrak{R}\setminus s^*} + t$ for some $t \in \mathbb{Z}$. We have already seen in Lemma 3.9, part 3 that

$$8 \left( v(\lambda_{C'}) - v(\lambda_C) \right) = t \cdot (|\mathfrak{R}| - 2)(|\mathfrak{R}| - 2|s|).$$

The calculation below shows that this equals the change on the right hand side of the equation.

$$\sum_{|s| \text{ even}} \delta'_s(|s| - 2|s|) + \sum_{|s| \text{ odd}} \delta'_s(|s| - 1)^2$$

$$= \sum_{|s| \text{ even}} \delta_s(|s| - 2|s|) + \sum_{|s| \text{ odd}} \delta_s(|s| - 1)^2$$

$$+ \begin{cases} 
(\delta_{s^*} - t)(|s^*| - 2)s^* + (\delta_{\mathfrak{R}\setminus s^*} + t)(|\mathfrak{R}\setminus s^*| - 2)|\mathfrak{R}\setminus s^*|, & |s^*| \text{ even} \\
(\delta_{s^*} - t)(|s^*| - 1)^2 + (\delta_{\mathfrak{R}\setminus s^*} + t)(|\mathfrak{R}\setminus s^*| - 1)^2, & |s^*| \text{ odd}
\end{cases}$$

$$= \sum_{|s| \text{ even}} \delta_s(|s| - 2|s|) + \sum_{|s| \text{ odd}} \delta_s(|s| - 1)^2$$

$$+ \begin{cases} 
-t(|s^*| - 2)s^* + t(2g - |s^*|)(2g + 2 - |s^*|), & |s^*| \text{ even} \\
-t(|s^*| - 1)^2 + t(2g + 1 - |s^*|)^2, & |s^*| \text{ odd}
\end{cases}$$

$$= \sum_{|s| \text{ even}} \delta_s(|s| - 2|s|) + \sum_{|s| \text{ odd}} \delta_s(|s| - 1)^2 + t(|\mathfrak{R}| - 2)(|\mathfrak{R}| - 2|s|) \quad \square$$

### 4 A Basis for $H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathfrak{R}})$

Let $C/K$ be a semistable hyperelliptic curve defined by a Weierstraf equation $C : y^2 = f(x)$ with $f(x) = cf \prod_{r \in \mathfrak{R}} (x - r)$. To this equation we associate the cluster picture $\Sigma$. Let $\mathcal{X} \to \text{Spec} \mathfrak{R}$ be the minimal regular model of $C$.

In this section we show how to read off a basis for the global sections of the canonical sheaf $\omega_{\mathcal{X}/\mathfrak{R}}$ from the cluster picture $\Sigma$.

**Theorem 4.1.** Let $C/K$ be a hyperelliptic curve defined by an integral Weierstraf equation $C : y^2 = f(x)$ and $\Sigma$ the associated cluster picture.

Let $\mathcal{X}/\mathfrak{R}$ be the minimal regular model. Assume that the residue field $k$ is algebraically closed. Then an $\mathfrak{R}$-basis for the global sections of the relative dualizing sheaf $\omega_{\mathcal{X}/\mathfrak{R}}$ is given by $(\mu_0, \ldots, \mu_{g-1})$, where

$$\mu_i = \pi^{\epsilon_i} \prod_{j=0}^{i-1} \frac{dx}{y} \int_{z_j}^{x} (x - z_j) \frac{dx}{y}$$

with

$$\epsilon_i = \frac{\nu_i}{2} - \sum_{j=0}^{i} d_{s_j \setminus s_i}.$$
The clusters \( s_0, \ldots, s_{g-1} \) are chosen inductively such that
\[
\frac{\nu_{s_j}}{2} - \sum_{j=0}^{i-1} d_{s_j \land s_i} = \max_{s_i \in \Sigma} \left( \frac{\nu_{s_i}}{2} - \sum_{j=0}^{i-1} d_{s_j \land s_i} - d_s \right).
\]

If the maximal value is obtained by two different clusters \( s \) and \( s' \) such that \( s' \subset s \), we choose \( s_i = s \).

Note that the same cluster can appear multiple times in the tuple \((s_0, \ldots, s_{g-1})\). Since \( k \) is algebraically closed, one can find a centre \( z_s \in K \) for every cluster \( s \). This follows from [2, Lemma 4.2.]. For an illustration of the theorem, we refer to Example 1.2 in the Introduction.

The following lemmas will be used to prove Theorem 4.1.

**Lemma 4.2.** Let \( C/K \) be a hyperelliptic curve defined by an integral Weierstraß equation \( C : y^2 = f(x) \) and \( \Sigma \) the associated cluster picture. Let the clusters \( s_0, \ldots, s_{g-1} \in \Sigma \) be chosen according to Theorem 4.1. Then for every cluster \( s \in \Sigma \):
\[
\gamma(s) := \#\{s_i \mid s_i \subset s\} = \left\lfloor \frac{|s| - 1}{2} \right\rfloor
\]

**Proof.** Since \( \gamma(\mathcal{R}) = g \), the statement is true for \( s = \mathcal{R} \). Let \( s \neq \mathcal{R} \) and let \( s' \) be the parent of \( s \), that is \( s < s' \). Then
\[
\begin{align*}
d_s &= d_{s'} + \delta_s, \\
\nu_s &= \nu_{s'} + \delta_s|s|, \\
d_{s \land s'} &= \begin{cases} 
    d_{s \land s'} + \delta_s & \text{if } s_j \subset s \\
    d_{s \land s'} & \text{otherwise}
\end{cases}
\end{align*}
\]

for \( j \in \{0, \ldots, i - 1\} \).

So for any \( i \in \{0, \ldots, g-1\} \)
\[
\frac{\nu_s}{2} - \sum_{j=0}^{i-1} d_{s \land s'} - d_s = \frac{\nu_{s'}}{2} - \sum_{j=0}^{i-1} d_{s \land s'} - d_{s'} + \delta_s \left( \frac{|s|}{2} - 1 - \#\{s_j \mid s_j \subset s, \ 0 \leq j \leq i - 1\} \right)
\]

This means that \( \frac{\nu_s}{2} - \sum_{j=0}^{i-1} d_{s \land s'} - d_s > \frac{\nu_{s'}}{2} - \sum_{j=0}^{i-1} d_{s \land s'} - d_{s'} \) if and only if \( |s| - 1 > \#\{s_j \mid s_j \subset s, \ 0 \leq j \leq i - 1\} \). So \( \gamma(s) \leq \left\lfloor \frac{|s|-1}{2} \right\rfloor \) for all \( s \in \Sigma \).

To show equality, we proceed by induction. Again let \( s \neq \mathcal{R} \) and assume that the statement holds for every cluster strictly containing \( s \). If \( \gamma(s) < \left\lfloor \frac{|s|-1}{2} \right\rfloor \), then at any step \( i \), we have \( \frac{\nu_s}{2} - \sum_{j=0}^{i-1} d_{s \land s'} - d_s > \frac{\nu_{s'}}{2} - \sum_{j=0}^{i-1} d_{s \land s'} - d_{s'} \), hence \( s_i \neq s' \). This implies \( \gamma(s') = \sum_{t \subset s'} \gamma(t) \). Using that \( |s'| = \sum_{t \subset s'} |t| \), \( \gamma(t) \leq \left\lfloor \frac{|s|-1}{2} \right\rfloor \) for every \( t < s' \) and \( \gamma(s) < \left\lfloor \frac{|s|-1}{2} \right\rfloor \), we get \( \gamma(s') < \left\lfloor \frac{|s|-1}{2} \right\rfloor \). Contradiction. So \( \gamma(s) = \left\lfloor \frac{|s|-1}{2} \right\rfloor \).

**Lemma 4.3.** Let \( C/K \) be a hyperelliptic curve defined by an integral Weierstraß equation \( C : y^2 = f(x) \) and \( \Sigma \) the associated cluster picture. Let the clusters \( s_0, \ldots, s_{g-1} \in \Sigma \) be chosen according to Theorem 4.1. Let \( \mathcal{X}/\mathcal{R} \) be the minimal regular model.
Then for any principal cluster $s \in \Sigma$, we have that $-(\frac{1}{\pi} - \sum_{j=0}^{i-1} d_{s_j, s} - d_s)$ is a lower bound for the order of the element $\prod_{j=0}^{i-1} (x - z_{s_j})^{d_s} y_s$ on the component of the special fibre of $X$ that corresponds to $s$.

Proof. Let $s$ be a principal cluster in $\Sigma$ and denote by $\Gamma_s$ the component (or possibly the two components) of the special fibre of $X/R$ corresponding to this cluster. The connection between the cluster picture of a curve and the special fibre of its minimal regular model is explained in [2, Theorem 8.5.].

The non-singular points on $\Gamma_s$ are all visible on the chart $U \setminus P$, where $P$ is a finite set of points and

$$U := \text{Spec } (R[x_s, y_s]/(y_s^2 - f_s(x_s)))$$

with local coordinates

$$x_s = \frac{x - z_s}{\pi d_s}, \ y_s = \frac{y}{\pi^{\nu_s/2}}.$$ 

This follows from [2, Proposition 5.5.]. Because the set $P$ has codimension 2 in $X$, it suffices to prove the statement for $U \setminus P$ (see for example [5, Theorem 4.1.14.]).

We can write

$$\prod_{j=0}^{i-1} (x - z_{s_j}) dx = \prod_{j=0}^{i-1} (\pi d_s x_s + z_s - z_{s_j}) \cdot \frac{d(\pi d_s x_s + z_s)}{\pi^{\nu_s/2} y_s} = \prod_{j=0}^{i-1} (\pi d_s x_s + z_s - z_{s_j}) \cdot \frac{\pi d_s}{\pi^{\nu_s/2} y_s} \cdot \frac{dx}{y_s} = \pi^{\sum_{j=0}^{i-1} d_{s_j, s} + d_s - \nu_s} \cdot \prod_{j=0}^{i-1} (\pi d_s x_s - \frac{z_s - z_{s_j}}{\pi d_{s_j, s}}) \cdot \frac{dx}{y_s}.$$ 

Since $d_s \geq d_{s_j, s}$ and $\frac{z_s - z_{s_j}}{\pi d_{s_j, s}} \in R$ for all clusters $s, s_j \in \Sigma$, the element $\prod_{j=0}^{i-1} (\pi d_s x_s - \frac{z_s - z_{s_j}}{\pi d_{s_j, s}}) y_s$ is an integral section on $\omega_{X/R} \otimes_{\mathcal{O}_X} U$. The statement of the lemma follows.

Proof of Theorem 4.1. For $i \in \{0, \ldots, g - 1\}$, let $s_i$, $e_i$ and $\mu_i$ be as described in the theorem.

**Claim 1**: The differentials $\mu_0, \ldots, \mu_{g-1}$ are global sections of $\omega_{X/R}$.

Here, we are going to make use of the fact that inside $H^0(C, \Omega_{C/K})$ the global sections of the minimal regular model are equal to those of the stable model (cf. the proof of Proposition 3.3). So by [2, Theorem 5.24.], it suffices to check that the differentials $\mu_0, \ldots, \mu_{g-1}$ are regular on the components of the special fibre corresponding to principal clusters. Let $\Gamma_s$ be a component of the special fibre corresponding to a principal cluster $s \in \Sigma$. From Lemma 3.3 we know that $-(\frac{1}{\pi} - \sum_{j=0}^{i-1} d_{s_j, s} - d_s)$ is a lower bound for the order of the element $\prod_{j=0}^{i-1} (x - z_{s_j})^{d_s} y_s$ on $\Gamma_s$. Since $e_i = \max_{s \in \Sigma} (\frac{1}{\pi} - \sum_{j=0}^{i-1} d_{s_j, s} - d_s)$, the order of $\mu_i = \pi^{e_i} \cdot \prod_{j=0}^{i-1} (x - z_{s_j})^{d_s} y_s$ is non-negative on every such component.

For the horizontal part, we have to consider the restriction of $\mu_i$ to the generic fibre. Clearly $\mu_i \otimes_R K \in H^0(X, \omega_{X/R}) \otimes_R K = H^0(C, \Omega_{C/K})$. This proves the first claim.

**Claim 2**: Let $\lambda_C$ be the quantity defined in the previous section, then $\sum_{i=0}^{g-1} e_i = \nu(\lambda_C)$. 


\[\sum_{i=0}^{g-1} e_i = \sum_{i=0}^{g-1} \left( \frac{\nu_{g, i}}{2} - \sum_{j=0}^{i} d_{g, j} \right)\]

\[= \sum_{i=0}^{g-1} \left( \frac{1}{2} \left( v(c_f) + \sum_{j \leq i} \delta_{j} \right) - \sum_{j=0}^{i} d_{g, j} \right)\]

\[= g \cdot \frac{v(c_f)}{2} + \sum_{s \in \mathbb{N}} \delta_s \cdot g \cdot \frac{\nu_{g, s}}{2} - \sum_{s \in \mathbb{N}} \delta_s \sum_{i=0}^{g-1} \# \{ s_j \subseteq s \mid j \leq i \} - d_{g, i} \sum_{i=0}^{g-1} \# \{ s_j \subseteq \mathfrak{N} \mid j \leq i \}\]

In the last line we used that \(\sum_{i=0}^{g-1} \# \{ s_j \subseteq s \mid j \leq i \} = \frac{2(s)(\gamma(s)+1)}{2}\) for every cluster \(s\). The value of \(\gamma(s)\) is given in Lemma 4.2. Now the claim follows from Theorem 3.1.

**Claim 3:** \(\mu := \mu_{0} \wedge \cdots \wedge \mu_{g-1}\) is a basis for \(\det H^0(\mathcal{X}, \omega_{\mathcal{X}/R})\).

Let \(\omega_{0}, \ldots, \omega_{g-1}\) denote the differentials associated to the Weierstraß equation \(C : y^2 = f(x)\). By construction, we have that

\[(\mu_{0}, \ldots, \mu_{g-1}) = (\omega_{0}, \ldots, \omega_{g-1}) \cdot A,\]

where \(A\) is a matrix of the form \(A = \begin{pmatrix} \pi^{e_0} & \star \\ \vdots & \vdots \\ 0 & \pi^{e_{g-1}} \end{pmatrix}\).

Therefore

\[\mu = \det(A) \cdot \omega_{0} \wedge \cdots \wedge \omega_{g-1} = \pi^{v(c_f)} \cdot \omega_{0} \wedge \cdots \wedge \omega_{g-1}.\]

So by Theorem 3.1, \(\mu\) is a basis for \(\det H^0(\mathcal{X}, \omega_{\mathcal{X}/R})\).

We have seen that \(\mu_{0}, \ldots, \mu_{g-1} \in H^0(\mathcal{X}, \omega_{\mathcal{X}/R})\) and that \(\mu := \mu_{0} \wedge \cdots \wedge \mu_{g-1}\) is a basis for \(\det H^0(\mathcal{X}, \omega_{\mathcal{X}/R})\). Therefore \((\mu_{0}, \ldots, \mu_{g-1})\) is a basis for \(H^0(\mathcal{X}, \omega_{\mathcal{X}/R})\).

\[\square\]

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