A KAM Theorem for the Hamiltonian with Finite Zero Normal Frequencies and Its Applications (In Memory of Professor Walter Craig)

Yuan Wu · Xiaoping Yuan

Received: 2 September 2020 / Revised: 4 February 2021 / Accepted: 13 February 2021 / Published online: 20 March 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract

we investigate the existences of KAM tori for the infinite dimensional Hamiltonian system with finite number of zeros among normal frequencies. By constructing a constant vector we show that, for “most” tangent frequencies in the sense of Lebesgue measure, either if the vector is zero, there is a KAM torus or if the vector is not zero, there is no KAM torus in some domain. As an application, we show that the nonlinear Schrödinger equation with a zero among normal frequencies possesses many quasi-periodic solutions.

Keywords Zero normal frequency · KAM tori · Quasi-periodic solution · Schrödinger equation

1 Introduction and the Main Results

Consider a Hamiltonian function

$$H = \langle \omega, y \rangle + \sum_{|j| \leq \iota} \Omega_j z_j \bar{z}_j + \varepsilon R(x, y, z, \bar{z}, \omega),$$

where \((x, y, z, \bar{z}) \in \mathbb{T}^n \times \mathbb{C}^n \times \mathcal{H} \times \mathcal{H}\) and \(\mathcal{H}\) is a finite dimensional Euclidean space \((\iota < \infty)\) or a Hilbert space \((\iota = \infty)\). Endow \(H\) with symplectic structure \(dy \wedge dx + i d\bar{z} \wedge dz\) where \(i^2 = -1\).

Clearly, when \(\varepsilon = 0\), \(\mathbb{T}_y^n = \mathbb{T}^n \times \{y = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}\) is a \(n\)-dimensional invariant torus with rotational frequency \(\omega\) for the Hamiltonian system defined by \(H\). Assume

$$\langle k, \omega \rangle \neq 0, \ k \in \mathbb{Z}^n \setminus \{0\},$$

1 School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, Hubei, People’s Republic of China

2 School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China
Melnikov [31,32] announced that the invariant torus $T^n_0$ is preserved under a small analytic perturbation $\varepsilon R$ for the finite dimensional Hamiltonian $H$ (i.e. $\nu < \infty$). Eliasson [16] gave out the detail proof. Also see Pöschel [33]. The above result is called Melnikov’s preservation theorem of KAM tori. The conditions (1.3) and (1.4) are so-called the first Melnikov conditions and the second Melnikov conditions, respectively. Kuksin [23,24], Wayne [35] initiated the study of the Melnikov’s preservation theorem for infinite dimensional Hamiltonian systems (i.e. $\nu = \infty$) and thus proved the existence of time-quasi-periodic solutions (KAM tori) for the nonlinear Schrödinger equations and wave equations of spatial dimension 1.

Clearly, the first and the second Melnikov conditions imply that $\Omega_j \neq 0$ and multiplicity $\Omega_j^\# = 1$, respectively. For 1-dimensional nonlinear Schrödinger equation of 0-mass

$$iu_t + u_{xx} + |u|^4u = 0, \quad x \in \mathbb{T}^1,$$

the normal frequency $\Omega_0 = 0$ is the first eigenvalue of the differential operator $-\partial_{xx}$ with periodic boundary condition $x \in \mathbb{T}^1$. Consider $d > 1$ dimensional nonlinear Schrödinger equation of mass $m > 0$

$$iu_t + \Delta u + mu + |u|^4u = 0, \quad x \in \mathbb{T}^d,$$

for which $\Omega_j = |j|^2 + m$ for $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$. At this time, one has

$$\Omega_j^\# \approx |j|^{d-1} \rightarrow \infty.$$

Naturally one proposes the following two problems:

**Problem 1.** Is there Melnikov preservation theorem of KAM tori for some $\Omega_j = 0$?

**Problem 2.** Is there Melnikov preservation theorem of KAM tori for some $\Omega_j^\# > 1$?

They were originally proposed by Kuksin [24]. At present time, Problem 2 has been deeply investigated when the perturbation is bounded. Bourgain [8–14] developed a new method initiated by Craig–Wayne [15] to deal with the KAM tori for the PDEs in high spatial dimension, based on the Newton iteration, Fröhlich–Spencer techniques, harmonic analysis, and semialgebraic geometry theory (see [14]). This is called the Craig–Wayne–Bourgain (C–W–B) method. We also mention the work by Eliasson–Kuksin [18] and Eliasson-Grébert–Kuksin [17], where the classical KAM theorem is extended in the direction of [4,16,20,23,26,34,35] to deal with high spatial dimensional nonlinear Schrödinger equations and beam equations by introducing an elegant analysis of the Töplitz–Lipschitz operator. The obtained KAM tori by [18] are linear stable. In previous KAM theorems, the normal frequencies $\Omega_j$’s cluster at infinity for the infinite dimensional Hamiltonian system. Recently, in [38], a new KAM theorem was constructed where $\Omega_j^\# \geq 1$ and $\Omega_j$’s are bounded with application to Benjamin–Bona–Mahony equation and Pochhammer–Chree equation. Incidentally, the KAM theory is also developed to deal some 1-dimensional PDEs of unbounded perturbation. See, for example, [1–3,5,7,19,22,25,28,29,39].

However, there are fewer results about Problem 1. In the present paper, we will investigate this problem.

In order to state our theorem, we need some notations. Let $\mathbb{N}_+ = \{1, 2, \ldots \}$. Fix an integer $b > 0$ and take a subset $\mathcal{J}$ of $\mathbb{N}_+$

$$\mathcal{J} = \{j_1 < j_2 < \cdots < j_b\} \subseteq \mathbb{N}_+.$$
Denote
\[ z^* = (z_0, z), \quad \bar{z}^* = (\bar{z}_0, \bar{z}), \]
where
\[ z_0 = (z_{jm} \in \mathbb{C}, m = 1, \ldots, b), \quad z = (z_j \in \mathbb{C}, j \in \mathbb{N}_+ \setminus \mathcal{J}), \]
and
\[ \bar{z}_0 = (\bar{z}_{jm} \in \mathbb{C}, m = 1, \ldots, b), \quad \bar{z} = (\bar{z}_j \in \mathbb{C}, j \in \mathbb{N}_+ \setminus \mathcal{J}). \]
Usually the variables \( z^* \) and \( \bar{z}^* \) are regarded as being independent unless we point out that \( \bar{z}^* \) is the complex conjugate of \( z^* \). Introduce the phase space
\[ (x, y, z^*, \bar{z}^*) \in \mathcal{P}^a,p = \mathbb{T}^n \times \mathbb{C}^n \times l^a,p \times l^a,p, \]
where \( n \in \mathbb{N}_+, \ a \geq 0 \) and \( p \geq 1 \) are given and \( l^a,p \) is the Hilbert space of all complex sequences \( z^* = (z_0, z) \) with
\[ \|z^*\|_{a,p}^2 = \sum_{jm \in \mathcal{J}} |z_{jm}|^2 \|2 p e^{2aj}\| + \sum_{j \in \mathbb{N}_+ \setminus \mathcal{J}} |z_j|^2 |j|^2 p e^{2aj} < \infty. \]
Endow \( \mathcal{P}^a,p \) with the symplectic structure \( dy \wedge dx + i d\bar{z}^* \wedge dz^* \). Let \( N(y, z^*, \bar{z}^*, \xi) \) be an integrable Hamiltonian depending on parameters \( \xi \in \Pi, \Pi \) a parameter set of positive Lebesgue measure in \( \mathbb{R}^n \), precisely,
\[ N(y, z^*, \bar{z}^*, \xi) = \langle \omega(\xi), y \rangle + \langle \Omega^*(\xi)z^*, \bar{z}^* \rangle = \langle \omega(\xi), y \rangle + \langle \Omega_0(\xi)z_0, \bar{z}_0 \rangle + \langle \Omega(\xi)z, \bar{z} \rangle, \]
where \( \omega = (\omega^1, \ldots, \omega^n), \Omega^*(\xi) = \text{diag} (\Omega_0(\xi), \Omega(\xi)) \) with
\[ \Omega_0(\xi) = \text{diag} (\Omega_{0j}^i(\xi), \ldots, \Omega_{0j}^b(\xi)) \in \mathbb{R}^{b \times b}, \quad \Omega(\xi) = \text{diag} (\Omega^i(\xi) \in \mathbb{R} : j \in \mathbb{N}_+ \setminus \mathcal{J}), \]
and
\[ \langle \Omega_0(\xi)z_0, \bar{z}_0 \rangle = \sum_{1 \leq m \leq b} \Omega_{0jm}^i(\xi) z_{jm} \bar{z}_{jm}, \quad \langle \Omega(\xi)z, \bar{z} \rangle = \sum_{j \in \mathbb{N}_+ \setminus \mathcal{J}} \Omega^i(\xi) z_j \bar{z}_j. \]
We always assume \( \Omega_0(\xi) \equiv 0, \ m = 1, \ldots, b. \)
Quite evidently, \( T_0^n = \mathbb{T}^n \times \{ y = 0 \} \times \{ z^* = 0 \} \times \{ \bar{z}^* = 0 \} \) is a \( n \)-dimensional invariant rotational torus for the Hamiltonian system defined by \( N \). Our aim is to give out a criterion for preservation or non-preservation of the torus \( T_0^n \) under small perturbation \( \epsilon R \) for “most” parameters \( \xi \in \Pi \) in the sense of Lebesgue measure. To this end, introduce complex neighborhoods of \( T_0^n \)
\[ D(s, r) = \{(x, y, z^*, \bar{z}^*) \in \mathcal{P}^a,p : |y| \leq s, |y| \leq r^2, \|z^*\|_{a,p} + \|\bar{z}^*\|_{a,p} \leq r\}, \]
\[ D_{\tilde{R}}(s, r) = \{(x, y, z^*, \bar{z}^*) \in D(s, r) : x, y \in \mathbb{R}^n, \bar{z}^* \text{ is the complex conjugate of } z^* \}, \]
where \( s, r > 0 \) are constants and \( |\cdot| \) denotes the sup-norm for complex vectors.
For \( r > 0 \) and \( \tilde{p} \geq p \), we define the weighted phase norm
\[ \|W\| = |X| + \frac{1}{r^2} |Y| + \frac{1}{r} \|U\|_{a,\tilde{p}} + \frac{1}{r} \|V\|_{a,\tilde{p}}, \ W = (X, Y, U, V) \in \mathcal{P}^{a,\tilde{p}}. \]
Furthermore, for a map $W : D(s, r) \times \Pi \to \mathcal{P}^{a, \tilde{p}}$, (for example, $W$ is the Hamiltonian vector field $X_R$,) we define the norms
\[
\|W\|_{D(s, r) \times \Pi} := \sup_{D(s, r) \times \Pi} \|W\|,
\]
\[
\|W\|_{\mathcal{L}}_{D(s, r) \times \Pi} := \sup_{D(s, r) \times \Pi} \|\partial_\xi W\|,
\]
where $\partial_\xi$ is the derivative with respect to $\xi$ in Whitney’s sense.

Denote by $\mathcal{A}(l^{a, p}, l^{a, \tilde{p}})$ the set of all bounded linear operators from $l^{a, p}$ to $l^{a, \tilde{p}}$ and by $||| \cdot |||$ the operator norm. For any subset $S \subset \mathbb{N}_+$, let us consider a vector $u = (u_j \in \mathbb{C} : j \in S)$ and a matrix $U = (U_{ij} \in \mathbb{C} : i, j \in S)$. We expand $u$ into
\[
\tilde{u} = (\tilde{u}_j : j \in \mathbb{N}_+), \text{ here } \tilde{u}_j = \begin{cases} u_j, & j \in S, \\ 0, & j \in \mathbb{N}_+ \setminus S \end{cases}
\]
and also expand $U$ into
\[
\tilde{U} = (\tilde{U}_{ij} : i, j \in \mathbb{N}_+), \text{ here } \tilde{U}_{ij} = \begin{cases} U_{ij}, & i, j \in S, \\ 0, & i \text{ or } j \in \mathbb{N}_+ \setminus S \end{cases}
\]
Define $\|u\|_{a, p} = ||\tilde{u}||_{a, p}$ and $\|U\| = ||\tilde{U}||$.
We also denote $| \cdot |_2$ the Euclidean norm and $|| \cdot ||$ the operator norm induced by $| \cdot |_2$.

Now, we state our main theorem.

**Theorem 1.1** Consider a perturbation of the integrable Hamiltonian $N$
\[
H(x, y, z^*, \bar{z}^*, \xi) = N(y, z^*, \bar{z}^*, \xi) + R(x, y, z^*, \bar{z}^*, \xi)
\]  \hspace{1cm} (1.5)
defined on the domain $D(s, r) \times \Pi$. Suppose the following assumptions hold.

**Assumption A:** (Nondegeneracy). There exist positive constants $E_1$ and $L_0$ such that $|\omega|_\Pi, |\omega|_\Pi^{\tilde{p}} \leq E_1$ and $|\text{det}(\partial_\xi \omega(\xi))|_\Pi \geq L_0$.

**Assumption B:** (Spectral Asymptotics). There exists $d \geq 1$ such that
\[
\Omega_j^i(\xi) = j^d + \cdots + O(\xi^p), \quad j \in \mathbb{N}_+ \setminus J,
\]  \hspace{1cm} (1.6)
where the dots stands for fixed lower order terms in $j$ and $p \geq 1$. More precisely, there exists positive constant $E_2$ such that $|\Omega|_\Pi^{\tilde{p}} \leq E_2$.

**Assumption C:** (Regularity). The perturbation $R$ is analytic in the space coordinates $(x, y, z^*, \bar{z}^*) \in D(s, r)$ and $1$ order Whitney smooth in the parameter $\xi \in \Pi$, and for each $\xi \in \Pi$, its Hamiltonian vector field $X_R = (R_y, -R_x, iR_{z^*}, -iR_{\bar{z}^*})^T$ ($T = \text{transpose}$) defines near $T^0_0$ a real analytic map $X_R : \mathcal{P}^{a, p} \to \mathcal{P}^{a, \tilde{p}}$, $\tilde{p} \geq p$ for $d > 1$, $\tilde{p} > p$ for $d = 1$.

**Assumption D:** (Reality). For any $(x, y, z^*, \bar{z}^*, \xi) \in D(s, r) \times \Pi$, the perturbation $R$ is real, that is,
\[
\overline{R}(x, y, z^*, \bar{z}^*, \xi) = R(x, y, z^*, \bar{z}^*, \xi),
\]  \hspace{1cm} (1.7)
where the bar means complex conjugate.
Set $E = E_1 + E_2$. There exists a positive constant $C_0$, depending on $n, s, r$, and $E$, such that, for every perturbation $R$ described above with

$$
\varepsilon := \|X_R\|_{D(s,r)}, \Pi + \frac{\gamma}{E} \|X_R\|^2_{D(s,r), \Pi} \leq C_0 \gamma^2,
$$

for given $r > 0, s > 0$ and sufficiently small $0 < \gamma \ll 1$, there exist a sequence of constants $s_m \approx 2^{-(m+1)} s, \epsilon_m \approx \epsilon_s^{(4/3)m}, r_m \approx \epsilon_s^{1/3} r, \gamma_m = \frac{\gamma}{\epsilon_s}(1 + 2^{-m+1}), E_m = E + 2(\epsilon_0 + \cdots + \epsilon_{m-1})$, a sequence of domains $D(s_m, r_m), \Pi_m$ with $\cap_{m=0}^{\infty} \Pi_m = \Pi \subset \Pi$ and

$$
|\Pi \backslash \Pi'_m| = O(\gamma),
$$
as well as symplectic transformations

$$
\Phi^{m-1} = \Phi_0 \circ \cdots \circ \Phi_{m-1} : D(s_m, r_m) \times \Pi_m \rightarrow D(s, r),
$$
such that the Hamiltonian $H$ is changed by $\Phi^{m-1}$ into

$$
H_m = H \circ \Phi^{m-1} = N_m + R_m,
$$

where the sequences of the new normal form $N_m$

$$
N_m = J_m^N(\xi) + \langle \omega_m(\xi), y \rangle + \langle \Omega_m(\xi)z, \bar{z} \rangle + \langle J_m^{\zeta_0}(\xi), z_0 \rangle
$$

$$
+ \langle J_m^{\bar{z}_0}(\xi), \bar{z}_0 \rangle + \langle J_m^{\bar{z}0}(\xi)z_0, \bar{z}_0 \rangle + \langle J_m^{\bar{z}0}(\xi)\bar{z}_0, \bar{z}_0 \rangle,
$$

and for each $\xi \in \Pi_m$, the perturbation $R_m(x, y, z^*, \bar{z}^*, \xi)$ is analytic on $D(s_m, r_m)$. Moreover, the following estimates hold:

1. for any $x \in \mathbb{T}^n$, the symplectic map $\Phi^{m-1}$ obeys

$$
\|\Phi^{m-1} - id\|_{D(s_m, r_m), \Pi_m} + \frac{\gamma_m}{E_m} \|\Phi^{m-1} - id\|^2_{D(s_m, r_m), \Pi_m} \leq \varepsilon_{m-1},
$$

$$
\|D\Phi^{m-1} - Id\|_{D(s_m, r_m), \Pi_m} + \frac{\gamma_m}{E_m} \|\Phi^{m-1} - id\|^2_{D(s_m, r_m), \Pi_m} \leq \varepsilon_{m-1},
$$

where $D\Phi^{m-1}$ denotes the tangent map of $\Phi^{m-1}$,

2. the frequencies $\omega_m(\xi)$ and $\Omega_m(\xi)$ satisfy

$$
|\omega_m(\xi)|_{\Pi_m}^2 + |\Omega_m(\xi)|^2_{\Pi_m} \leq E_m;
$$

3. the perturbation $R_m$ satisfies

$$
\|X_{R_m}\|_{D(s_m, r_m), \Pi_m} + \frac{\gamma_m}{E_m} \|X_{R_m}\|^2_{D(s_m, r_m), \Pi_m} \leq \varepsilon_m,
$$

where $u \ll v$ means there exists a constant $c > 0$ depending on $n, b$ such that $u \leq cv$.

Furthermore, denote

$$
\bar{N}^{\zeta_0}(\xi) = (\bar{N}^{\zeta_{j1}}(\xi), \cdots, \bar{N}^{\zeta_{j\hat{b}}}(\xi)) = \left(\lim_{m \to \infty} J_m^{\zeta_{j1}}(\xi), \cdots, \lim_{m \to \infty} J_m^{\zeta_{j\hat{b}}}(\xi)\right),
$$

$$
\bar{N}^{\bar{z}_0}(\xi) = (\bar{N}^{\bar{z}_{j1}}(\xi), \cdots, \bar{N}^{\bar{z}_{j\hat{b}}}(\xi)) = \left(\lim_{m \to \infty} J_m^{\bar{z}_{j1}}(\xi), \cdots, \lim_{m \to \infty} J_m^{\bar{z}_{j\hat{b}}}(\xi)\right),
$$

$$
\omega_m = \lim_{m \to \infty} \omega_m(\xi), \Phi = \lim_{m \to \infty} \Phi^{m-1}.
$$

We have
Corollary 1.2 If \( \vec{N} := (\vec{N}^0(\xi), \vec{N}^0(\xi)) = 0 \), then there exist a Cantor set \( \Pi_{\gamma} \subset \Pi \), a Whitney smooth family of torus embedding
\[
\Phi : T^n \times \{ y = 0 \} \times \{ z^* = 0 \} \times \{ \bar{z}^* = 0 \} \times \Pi_{\gamma} \rightarrow P^a,p,
\]
and a Whitney smooth map \( \omega_* : \Pi_{\gamma} \rightarrow \mathbb{R}^n \), such that for each \( \xi \in \Pi_{\gamma} \) the map \( \Phi \) restricted to \( T^n \times \{ y = 0 \} \times \{ z^* = 0 \} \times \{ \bar{z}^* = 0 \} \times \{ \xi \} \) is a real analytic embedding of a rotational torus with the frequencies \( \omega_* \) for the Hamilton system defined by (1.5) at \( \{ \xi \} \).

Corollary 1.3 If \( \vec{N} := (\vec{N}^0(\xi), \vec{N}^0(\xi)) \neq 0 \), that is, \( \sqrt{|\vec{N}^0(\xi)|^2 + |\vec{N}^0(\xi)|^2} := \delta_0 > 0 \), then there exist an integer \( m_0 > 100 \log(\frac{\log \delta_0}{\log \epsilon^2}) \), a domain
\[
\Xi_{m_0} = \{(x, y, z^*, \bar{z}^*) : |\Im x| \leq s_{m_0}, |y| \leq r_{m_0}^2, \|z^*\|_{a,p} + \|\bar{z}^*\|_{a,p} \leq e_{m_0-1}^2\},
\]
a Cantor set \( \Pi_{m_0-1} \subset \Pi \), a Whitney smooth family of embedding \( \Phi_{m_0-1} : \Xi_{m_0} \times \Pi_{m_0-1} \rightarrow D((s, r), \Pi) \), and a Whitney smooth map \( \omega_{m_0} : \Pi_{m_0-1} \rightarrow \mathbb{R}^n \), such that for each \( \xi \in \Pi_{m_0} \), there is no torus in the domain \( \Phi_{m_0-1}(\Xi_{m_0} \times \{ \xi \}) \subset D((s, r)) \) for the Hamilton system defined by (1.5).

Some remarks and a “guide to the proof” of Theorem 1.1.

1.1. The Linearized Equation (Sect. 2)

This is the heart of the proof. The idea consists in a quadratic-convergent iterative procedure apt to reduce at each step of the scheme, which is done in order to beat the divergence introduced by small divisors arising in the inversion of non-elliptic differential operators. Since there are finite zero normal frequencies, the main difficulties we encounter are
\[
\langle k, \omega(\xi) \rangle \pm \Omega^{jm} = 0, \quad 1 \leq m \leq b,
\]
and
\[
\langle k, \omega(\xi) \rangle \pm \Omega^{jm} \pm \Omega^{jm'} = 0, \quad 1 \leq m, m' \leq b,
\]
when \( k = 0 \).

To overcome these difficulties, a basic idea is that we preserve the terms related to (1.8) and (1.9).

Let
\[
R = R^x(\xi) + \langle R^y(x, \xi), y \rangle + \langle R^{z0}(x, \xi)z_0, z \rangle + \langle R^{\bar{z}0}(x, \xi)\bar{z}_0, \bar{z} \rangle
\]
\[
+ \langle R^{\bar{z}0}(x, \xi)\bar{z}_0, z \rangle + \langle R^{z0}(x, \xi)\bar{z}_0, \bar{z} \rangle + \langle R^z(x, \xi), z \rangle + \langle R^{\bar{z}}(x, \xi), \bar{z} \rangle
\]
\[
+ \langle R^{\bar{z}}(x, \xi)z, z \rangle + \langle R^z(x, \xi)z, \bar{z} \rangle + \langle R^{\bar{z}2}(x, \xi)\bar{z}, \bar{z} \rangle
\]
\[
+ \langle R^{z2}(x, \xi)z, \bar{z} \rangle + \langle R^{z0}(x, \xi)z_0, \bar{z}_0 \rangle + \langle R^{\bar{z}0}(x, \xi)\bar{z}_0, z_0 \rangle
\]
Thus, the terms we will preserve are $\hat{R}^{(0)}(0, \xi, \xi), \hat{R}^{(0)}(0, \xi), \hat{R}^{(20)}(0, \xi), \hat{R}^{(20)}(0, \xi), \hat{R}^{(20)}(0, \xi), \hat{R}^{(20)}(0, \xi), N_0 = N, R_0 = R$ and $H_0 = 0$. Suppose the Hamiltonian

$$H_0 = N_0 + R_0 = (\omega_0(\xi), y) + (\Omega_0^*(\xi)z^*, \bar{z}^*) + R_0$$

$$= (\omega_0(\xi), y) + (\Omega_0(\xi)z_0, \bar{z}_0) + (\Omega_0(\xi)\bar{z}, \bar{z}) + R_0, \quad \Omega_0 = 0, \quad (1.10)$$

described above satisfies assumptions A, B, C, D and

$$\|X_{R_0}\|_{D(s_0, r_0)}, \pi_0 + \frac{\gamma_0}{E_0} \|X_{R_0}\|_{D(s_0, r_0), \pi_0} = \varepsilon_0,$$

there exists a symplectic transformation $\Phi_0$, such that

$$H_1 = H_0 \circ \Phi_0 = (N_0 + R_0) \circ \Phi_0 = N_1 + R_1, \quad (1.11)$$

where the new normal form

$$N_1 = \hat{N}_0^*(\xi) + (\omega_1(\xi), y) + (\Omega_1(\xi)z, \bar{z}) + (J_1^{(0)}(\xi), z_0) + (J_1^{(b)}(\xi), \bar{z}_0)$$

$$+ (J_1^{(0)}(\xi)z_0, 0) + (J_1^{(b)}(\xi)z_0, \bar{z}_0) + (J_1^{(0)}(\xi)z_0, \bar{z}_0), \quad (1.12)$$

while the new perturbation is of smaller size

$$\|X_{R_1}\|_{D(s_1, r_1)}, \pi_1 + \frac{\gamma_1}{E_1} \|X_{R_1}\|_{D(s_1, r_1), \pi_1} < \varepsilon_1^2 := \varepsilon_1.$$  

The parameter $\xi$ appearing in (1.12) will vary in small compact set $\Pi_1$ (of relatively large Lebesgue measure).

Obviously, after 1-th iteration, we obtain a new normal form $N_1$, which has more terms than the usual KAM normal form. Thus, our iterative scheme from $H_1$ is non-standard and, from a technical point of view, represents the most novel part of the proof.

Similarly, for $H_1$ in (1.11), there exists a symplectic transformation $\Phi_1$, such that

$$H_2 = H_1 \circ \Phi_1 = (N_1 + R_1) \circ \Phi_1 = N_2 + R_2, \quad (1.13)$$

where the new normal form

$$N_2 = \sum_{j=0}^{1} \hat{N}_j^*(\xi) + (\omega_2(\xi), y) + (\Omega_2(\xi)z, \bar{z}) + (J_2^{(0)}(\xi), z_0) + (J_2^{(b)}(\xi), \bar{z}_0)$$

$$+ (J_2^{(0)}(\xi)z_0, 0) + (J_2^{(b)}(\xi)z_0, \bar{z}_0) + (J_2^{(0)}(\xi)z_0, \bar{z}_0),$$

while the new perturbation is of smaller size

$$\|X_{R_2}\|_{D(s_2, r_2)}, \pi_2 + \frac{\gamma_2}{E_2} \|X_{R_2}\|_{D(s_2, r_2), \pi_2} < \varepsilon_1^2,$$

where the parameter $\xi$ will vary in small compact set $\Pi_2$ (of relatively large Lebesgue measure).

Since the preserved terms are put into the normal form $N_1$, the homological equations in this iteration are of the following forms

$$\omega \cdot \partial_x F_1 + A_1 F_1 + F_1 B_1 = R_1, \quad (1.14)$$

$$\omega \cdot \partial_x F_2 + A_2 F_2 = R_2, \quad (1.15)$$

$\$
\[ \omega \cdot \partial_\xi F_3 + \Lambda F_3 + F_3 \Lambda = R_3, \tag{1.16} \]

where \( A_1, A_2, B_1 \) depending only on \( \xi \) are not diagonal while \( \Lambda \) is diagonal (Instead of 1-st iteration with normal form \( N_0 \), (1.16) is the only homological equations we have to solve). More narrowly, Eq. (1.14) is derived from the homological equation of the coefficients \( F^{\gamma_0, \delta}, F^{\gamma_0 \delta}, F^{\gamma_0, \delta} \), whose Fourier coefficient matrixes related to the preserved terms are finite dimension (less than \( 4b^2 \times 4b^2 \)). Thus, by introducing Kronecker product and column straightening, the coefficient equation can be solved provided that its Fourier coefficient matrixes are non-degenerate. Furthermore, observing that the \( k \)-th Fourier coefficient matrix is self-adjoint after making small changes, the essential non-resonant conditions imposed on the coefficient matrixes can be converted to its eigenvalues. For Eqs. (1.15) and (1.16), they are also solvable as long as any \( k \)-th Fourier coefficient matrixes are non-degenerate and satisfy some non-resonant conditions. These are the places where small divisors arise. Such divisors are

1. \( \mathcal{K}_d(1) = \{ \xi \in \Pi_1 : | \langle k, \omega_1(\xi) \rangle + \langle l, \Omega_1(\xi) \rangle | < \frac{\gamma_1}{(k+1)^\tau}, (k, l) \in \mathbb{Z} \}, \)
   where \( \gamma_1 = 3/4\gamma_0, (l)_d = \max(1, \sum j^d l_j), \tau > n + 1 \) and \( \mathbb{Z} = \{(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^{\infty} : |l| \leq 2 \}; \)
2. \( \mathcal{K}_d(1) = \{ \xi \in \Pi_1 : \max |\lambda_{A_{11}}(\xi)| < \frac{\gamma_1}{\lambda^{1}} \}, \)
   where \( A_{11}(\xi) = (k, \omega_1)I_{3b^2} + B_{11}(\xi), \tau_1 = \tau - 1, \gamma_1 = \frac{\gamma_1}{3b^2} \) and the \( 3b^2 \times 3b^2 \) order matrix \( B_{11}(\xi) \)'s norm is small enough;
3. \( \mathcal{K}_d(1) = \{ \xi \in \Pi_1 : \max |\lambda_{A_{31}}(\xi)| < \frac{\gamma_1}{\lambda^{1}} \}, \)
   where \( A_{31}(\xi) = (k, \omega_1)I_{3b^2} + B_{11}(\xi), \tau_3 = \tau, \gamma_3 = \frac{\gamma_1}{4b^2} \) and the \( 4b^2 \times 4b^2 \) order matrix \( B_{31}(\xi) \)'s norm is small enough;
4. \( \mathcal{K}_d(1) = \{ \xi \in \Pi_1 : \max |\lambda_{A_{41}}(\xi)| < \frac{\gamma_1}{\lambda^{1}} \}, \)
   where \( A_{41}(\xi) = (k, \omega_1)I_{2b^2} + B_{41}(\xi), \tau_4 = \tau - 1, \gamma_4 = \frac{\gamma_1}{2b^2} \) and the \( 2b^2 \times 2b^2 \) order matrix \( B_{41}(\xi) \)'s norm is small enough;

where \( I_n \) is the \( n \times n \) identity matrix and \( \lambda_A \) denotes the eigenvalue of matrix \( A \).

Therefore, the homological equation associated to (1.13) can be solved and the KAM machinery still works well.

### 1.1.2 The Iterative Lemma (Sect. 4)

We want to construct, inductively, symplectic transformations \( \Phi_m, m \geq 0 \), such that

\[ H_{m+1} = H_m \circ \Phi_m = (N_m + R_m) \circ \Phi_m = N_{m+1} + R_{m+1}, \]

where the sequences of the new normal form \( N_{m+1} \)

\[ N_{m+1} = \sum_{j=0}^{\infty} N_j^{\infty}(\xi) + \langle \omega_{m+1}(\xi), y \rangle + \langle \Omega_{m+1}(\xi), z \rangle + \langle J_{m+1}^{\xi}(\xi), z_0 \rangle + \langle J_{m+1}^{\xi}(\xi), z_0 \rangle \]

\[ + \langle J_{m+1}^{\xi}(\xi), z_0 \rangle + \langle J_{m+1}^{\xi}(\xi), z_0 \rangle + \langle J_{m+1}^{\xi}(\xi), z_0 \rangle, \]

while the sequences of perturbations \( R_{m+1} \) are of smaller and smaller size

\[ \| X R_{m+1} \|_{D(s_{m+1}, r_{m+1}), \Pi_{m+1}} + \frac{\gamma_{m+1}}{E_{m+1}} \| X R_{m+1} \|_{D(s_{m+1}, r_{m+1}), \Pi_{m+1}} \leq \varepsilon_{m+1}. \]
The parameter \( \xi \) will vary in smaller and smaller compact sets \( \Pi_m \) (of relatively large Lebesgue measure)

\[
\Pi_0 \supset \Pi_1 \supset \cdots \Pi_m \supset \cdots \supset \Pi_\infty \supset \bigcap_{m=0}^{\infty} \Pi_m.
\]

The smallness assumption on \( \varepsilon \) will allow to turn on the iteration procedure.

The symplectic map \( \Phi^m \) will be sought of the form

\[
\Phi^m = \Phi^{m-1} \circ \Phi_m = \Phi_0 \circ \cdots \circ \Phi_m.
\]

In order to work for the approach, one has to show that

\[
\Phi_m : D(s_{m+1}, r_{m+1}) \times \Pi_{m+1} \to D(s_m, r_m), (\forall m \geq 0), \quad (1.17)
\]

\[
\Phi^m : D(s_{m+1}, r_{m+1}) \times \Pi_{m+1} \to D(s_0, r_0), (\forall m \geq 0). \quad (1.18)
\]

Relations (1.17) and (1.18) are checked in Sect. 4.

**1.1.3 Proof of Theorem 1.1 and Corollaries 1.2, 1.3 (Sect. 5)**

Once the iterative step is set up, it has to be equipped with estimates. This technique part follows the corresponding part in [34]. Particularly, the key results of theorem 1.1 concerning the new Hamiltonian \( H_m \) and the measure of \( \Pi_m \) follow easily.

Moreover, when \( \tilde{N} = (\tilde{N}^z_0(\xi), \tilde{N}^\bar{z}_0(\xi)) = 0 \), from the fast convergence of \( H_m \), there exists a family of torus embedding \( \Phi \) such that \( \Phi(T^0_n \times \{\xi\}) \) is an invariant torus for the Hamiltonian \( H \) in (1.5).

When \( \tilde{N} = (\tilde{N}^z_0(\xi), \tilde{N}^\bar{z}_0(\xi)) \neq 0 \), that is, \( \sqrt{|\tilde{N}^z_0(\xi)|^2 + |\tilde{N}^\bar{z}_0(\xi)|^2} = \delta_0 > 0 \). Since

\[
\lim_{m \to \infty} \tilde{J}_m^z(\xi) = \tilde{N}^z_0(\xi) \quad \text{and} \quad \lim_{m \to \infty} \tilde{J}_m^\bar{z}(\xi) = \tilde{N}^\bar{z}_0(\xi),
\]

there exists a sufficiently large \( M_0 \) such that for any \( m \geq M_0 \),

\[
\sqrt{|J_m^z(\xi)|^2 + |J_m^\bar{z}(\xi)|^2} \geq \frac{\delta_0}{2}. \quad (1.19)
\]

More exactly, we will choose \( m_0 > M_0 \) such that

\[
\delta_0 > 28\varepsilon_{m_0-1}^7. \quad (1.20)
\]

Consider the Hamiltonian equation defined by \( H_{m_0} = N_{m_0} + R_{m_0} \) and fix an initial value \( \|z^*(0)\|_{a,p} + \|\tilde{z}^*(0)\|_{a,p} \leq \varepsilon_{m_0-1}^7 \). From (1.19), (1.20) and using some ordinary differential equation tools, we will obtain

\[
\|z^*(1)\|_{a,p} + \|\tilde{z}^*(1)\|_{a,p} > \varepsilon_{m_0-1}^7.
\]

That is, there exists no torus in the domain \( \Phi^{m_0-1}(\Xi_{m_0} \times \{\xi\}) \) for the Hamiltonian \( H \) in (1.5) when we denote \( \Xi_{m_0} = \{(x, y, z^*, \tilde{z}^*) : |x| \leq s_{m_0}, |y| \leq r_{m_0}^2, \|z^*\|_{a,p} + \|\tilde{z}^*\|_{a,p} \leq \varepsilon_{m_0-1}^7\} \). Theorem 1.1 and Corollaries 1.2, 1.3, at this point, are completely proven.
1.2 Application to Nonlinear Schrödinger Equation (NLS) (Sect. 6)

Consider a specific nonlinear Schrödinger equation
\[ iu_t - u_{xx} + |u|^4u = 0 \]  
(1.21)
on the finite \( x \)-interval \([0, 2\pi]\) with even periodic boundary conditions
\[ u(t, x) = u(t, x + 2\pi), \quad u(x, t) = u(-x, t). \]

From the assumption \( u(x, t) = u(-x, t) \) which simplifies the proof, it follows \( \Omega_j^x = 1 \). We see that \( \tilde{N} = (\tilde{N}^0(\xi), \tilde{N}^{\Omega_1}(\xi)) = 0 \). We will arrive at this end by proving \( R_{m0}^0(0, \xi) = 0 \), \( R_{m0}^0(0, \xi) = 0 \) in any \( m \)-th iteration. The detailed, quantitative results are collected in Sect. 6.

2 The Linearized Equation

Assume that all the assumptions of Theorem 1.1 are satisfied. Recall that
\[ H_0 = H_0(x, y, z^*, \bar{z}^*, \xi) = N_0(y, z^*, \bar{z}^*, \xi) + R_0(x, y, z^*, \bar{z}^*, \xi), \]
where
\[ N_0(y, z^*, \bar{z}^*, \xi) = \langle \omega_0(\xi), y \rangle + \langle \Omega_{00}(\xi)z_0, \bar{z} \rangle + \langle \Omega_0(\xi)z, \bar{z} \rangle \]
\[ = \sum_{1 \leq j \leq N} \omega_{0j} y_j + \sum_{J_m \in J} \Omega_{00}^{m} z_{jm} \bar{z}_{jm} + \sum_{j \in N_+ \setminus J} \Omega_{0j}^{m} z_j \bar{z}_j. \]

In the following, we abbreviate \( \Omega_{00}^{m} \) and \( z_{jm} \), respectively, as \( \Omega_{00}^{m} \) and \( z_{0m} \) (\( m = 1, \ldots, b \)), for convenience.

Denote \( R_0(x, y, z^*, \bar{z}^*, \xi) = R_0^{low}(x, y, z^*, \bar{z}^*, \xi) + R_0^{high}(x, y, z^*, \bar{z}^*, \xi) \). Then we have
\[ R_0^{low} = \sum_{\alpha \in \mathbb{N}^p, \beta, \gamma \in \mathbb{N}^l, 2|\alpha| + |\beta| + |\gamma| \leq 2} R_0^{\alpha\beta\gamma}(x, \xi) y^\alpha \{ z^* \}^\beta \{ \bar{z}^* \}^\gamma, \]
\[ R_0^{high} = \sum_{\alpha \in \mathbb{N}^p, \beta, \gamma \in \mathbb{N}^l, 2|\alpha| + |\beta| + |\gamma| \geq 3} R_0^{\alpha\beta\gamma}(x, \xi) y^\alpha \{ z^* \}^\beta \{ \bar{z}^* \}^\gamma. \]

We desire to eliminate the terms \( R_0^{low} \) by the coordinate transformation \( \Phi_0 \), which is obtained as the time-1-map \( X_{F_0}^t \) of a Hamiltonian vector field \( X_{F_0} \), where \( F_0(x, y, z^*, \bar{z}^*, \xi) \) is of the form
\[ F_0(x, y, z^*, \bar{z}^*, \xi) = F_0^{low}(x, y, z^*, \bar{z}^*, \xi) \]
\[ = \sum_{\alpha \in \mathbb{N}^p, \beta, \gamma \in \mathbb{N}^l, 2|\alpha| + |\beta| + |\gamma| \leq 2} F_0^{\alpha\beta\gamma}(x, \xi) y^\alpha \{ z^* \}^\beta \{ \bar{z}^* \}^\gamma. \]

Using Taylor formula, we have
\[ H_1 = H_0 \circ X_{F_0}^t |_{t=1} = N_0 + \{ N_0, F_0 \} + \int_0^1 (1 - t) \{ N_0, F_0 \} \circ X_{F_0}^t dt \]
For some fixed constant $\tau > 0$, then the modified homological equation writes

$$
N_1 + R_1.
$$

where

$$
N_1 = N_0 + \hat{N}_0
$$

\begin{align*}
&= N_0 + \hat{R}_0^0 (0, \xi) + \langle \hat{R}_0^0 (0, \xi), y \rangle + \langle \hat{R}_0^{00} (0, \xi), z_0 \rangle \\
&\quad + \langle \hat{R}_0^{00} (0, \xi), \hat{z}_0 \rangle + \langle \hat{R}_0^{000} (0, \xi)z_0, z_0 \rangle \\
&\quad + \langle \hat{R}_0^{000} (0, \xi)z_0, \hat{z}_0 \rangle + \langle R_0^{000} (0, \xi)\hat{z}_0, \hat{z}_0 \rangle + \sum_{j \in \mathbb{N} \setminus \mathcal{J}} R_0^{j} (0, \xi)\hat{z}_j \hat{z}_j,
\end{align*}

and

$$
R_1 = \int_0^1 \{ (1 - t) \hat{N}_0 + t R_0^{low} (0, \xi), y \} + R_0^{high} \circ X_{F_0}^t |_{t=1}.
$$

Then the modified homological equation writes

$$
\{ N_0, F_0 \} + R_0^{low} = \hat{N}_0,
$$

(2.1)

For any $m \geq 0$, we also denote

$$
\hat{N}_m^x (\xi) = \hat{R}_m^x (0, \xi), \hat{N}_m^y (\xi) = \hat{R}_m^y (0, \xi), \hat{N}_m^{0x} (\xi) = \hat{R}_m^{0x} (0, \xi), \hat{N}_m^{0y} (\xi) = \hat{R}_m^{0y} (0, \xi),
$$

$$
\hat{N}_m^{00x} (\xi) = \hat{R}_m^{00x} (0, \xi), \hat{N}_m^{00y} (\xi) = \hat{R}_m^{00y} (0, \xi), \hat{N}_m^{000} (\xi) = \hat{R}_m^{000} (0, \xi).
$$

2.1 The Solution of Homological Equation (2.1)

To solve the equation, we need:

Lemma 2.1 Suppose $H_0 = N_0 + R_0$ described above satisfies assumptions A, B, C and D, and

$$
\varepsilon_0 := \| X_{R_0} \|_{D(s_0, r_0), \Pi_0} + \frac{\gamma_0}{E_0} \| X_{R_0} \|_{D(s_0, r_0), \Pi_0} \leq C_0 \gamma_0^2.
$$

For some fixed constant $\tau > n + 1$, let

$$
\mathcal{R}_{kl}(0) = \left\{ \xi \in \Pi_0 : |(k, \omega_0 (\xi)) + (l, \Omega_0 (\xi))| < \frac{\gamma_0 (l)_d}{(1 + |k|)^\tau} \right\},
$$

where $(l)_d = \max (1, | \sum j^d l_j |)$, $\mathcal{Z} = \{(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty : |l| \leq 2 \}$, and let

$$
\Pi_1 = \Pi_0 \setminus \bigcup_{|k| > K_0, |l| \leq 2} \mathcal{R}_{kl}(0),
$$

where $K_0$ will be given later. Then for each $\xi \in \Pi_1$, the homological equation (2.1) has a solution $F_0 (x, y, z^*, \hat{z}^*, \xi)$ satisfying

$$
\overline{F}_0 (x, y, z^*, \hat{z}^*, \xi) = F_0 (x, y, z^*, \hat{z}^*, \xi), \ \forall (x, y, z^*, \hat{z}^*, \xi) \in D_{kl}(s_0 - \sigma_0, r_0) \times \Pi_1,
$$

$$
\| X_{F_0} \|_{D(s_0 - \sigma_0, r_0), \Pi_1} + \frac{\gamma_0}{E_0} \| X_{F_0} \|_{D(s_0 - \sigma_0, r_0), \Pi_1} \leq \frac{\varepsilon_0}{\gamma_0}.
$$
where $0 < \sigma_0 = s_0/40 \leq \frac{1}{4}$ and $u < v$ means there exists a constant $c > 0$ depending on
$n, b$ such that $u \leq cv$.

Set
\[
\gamma_1 = \frac{3}{4} \gamma_0, \quad E_1 = E_0 + 2\varepsilon_0, \quad s_1 = s_0 - 5\sigma_0,
\]
\[
r_1 = \eta_0 r_0, \quad \eta_0^3 = \frac{\varepsilon_0}{\gamma_0^3} \frac{2^\tau+n+2}{\gamma_0}, \quad K_0^{r+1} = \frac{1}{\gamma_0}, \quad \varepsilon_1 = \frac{\varepsilon_0^4}{\gamma_0^2 \gamma_0^2 2^{\tau+n+2}}.
\]

Then we get the new Hamiltonian $H_1$
\[
H_1 = N_1 + R_1,
\]
where
\[
N_1 = \hat{N}_0^3(\xi) + \omega_1(\xi, y) + \Omega_1(\xi) z, \quad \hat{N}_1(\xi, z, \tilde{z}) + \langle \hat{N}_0^{\tau_0}(\xi), z_0 \rangle + \langle \hat{N}_0^{\tau_0}(\xi), \tilde{z}_0 \rangle
\]
\[
+ \langle \hat{N}_0^{\tau_0}(\xi) z_0, \tilde{z}_0 \rangle + \langle \hat{N}_0^{\tau_0}(\xi) \tilde{z}_0, z_0 \rangle + \langle \hat{N}_0^{\tau_0}(\xi) \tilde{z}_0, \tilde{z}_0 \rangle,
\]
with
\[
\omega_1(\xi) = \omega_0(\xi) + \hat{N}_0^3(\xi), \quad 1 \leq j \leq n.
\]
\[
\Omega_1(\xi) = \text{diag} \left( \Omega_0^j(\xi) + R_0^{\tau_0}(0, \xi) : j \in \mathbb{N}_+ \right).
\]

and
\[
R_1 = \int_0^1 \left\{ (1 - t) \hat{N}_0 + t R_1^{low}, F_0 \right\} \circ X_{F_0}^t \, dt + R_0^{high} \circ X_{F_0}^t |_{t=1}.
\]

Moreover, the following estimates hold:

(a) the matrices $\hat{N}_0^{\tau_0}(\xi), \hat{N}_0^{\tau_0}(\xi)$ and $\hat{N}_0^{\tau_0}(\xi)$ are symmetric and satisfy
\[
\hat{N}_0^{\tau_0}(\xi) = \hat{N}_0^{\tau_0}(\xi), \quad \hat{N}_0^{\tau_0}(\xi) = \hat{N}_0^{\tau_0}(\xi),
\]
and
\[
\| \partial_\xi \hat{N}_0^{\tau_0}(\xi) \|, \quad \| \partial_\xi \hat{N}_0^{\tau_0}(\xi) \|, \quad \| \partial_\xi \hat{N}_0^{\tau_0}(\xi) \| \leq \frac{\varepsilon_0}{\gamma_0};
\]
(b) the frequencies $\omega_1(\xi)$ and $\Omega_1(\xi)$ satisfy
\[
|\omega_1(\xi)| \leq E_1;
\]
(c) the symplectic map $\Phi_0 = X_{F_0}^t |_{t=1}$ satisfies
\[
\| \Phi_0 - i \|_{D(0-3\sigma_0, \rho/4, \Pi_1)} + \frac{\gamma_1}{E_1} \| \Phi_0 - i \|_{D(0-3\sigma_0, \rho/4, \Pi_1)} \leq \varepsilon_0^{5/6};
\]
(d) the perturbation $R_1(x, y, z, \tilde{z}, \xi)$ satisfies
\[
R_1(x, y, z, \tilde{z}, \xi) = R_1(x, y, z, \tilde{z}, \xi), \quad \forall (x, y, z, \tilde{z}, \xi) \in D_0(\sigma_1, r_1) \times \Pi_1,
\]
\[
\| X_{R_1} \|_{D(\sigma_1, r_1, \Pi_1)} + \frac{\gamma_1}{E_1} \| X_{R_1} \|_{D(\sigma_1, r_1, \Pi_1)} \leq \varepsilon_1;
\]
(e) the measure of the $\Pi_1$ satisfies
\[
| \Pi_1 \setminus \Pi_0 | = O(\gamma_0).\]
Proof See Lemma 1 in [34] and Lemma 4.5 in [37]. □

Thus we have

\[ H_1 = H_1(x, y, z^*, \tilde{z}^*, \xi) = N_1(y, z^*, \tilde{z}^*, \xi) + R_1(x, y, z^*, \tilde{z}^*, \xi). \]

Denote

\[ R_1(x, y, z^*, \tilde{z}^*, \xi) = R_1^{low}(x, y, z^*, \tilde{z}^*, \xi) + T_1(x, y, z^*, \tilde{z}^*, \xi) + R_1^{high}(x, y, z^*, \tilde{z}^*, \xi), \]

where

\[ R_1^{low}(x, y, z^*, \tilde{z}^*, \xi) = \sum_{|k| \leq K_1, \alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, 2|\alpha| + |\beta| + |\gamma| \leq 2} R_1^{\alpha\beta\gamma}(k, \xi) y^\alpha \{ z^* \}^\beta \{ \tilde{z}^* \}^\gamma e^{i(k, x)}, \]

\[ T_1(x, y, z^*, \tilde{z}^*, \xi) = \sum_{|k| > K_1, \alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, 2|\alpha| + |\beta| + |\gamma| \leq 2} R_1^{\alpha\beta\gamma}(k, \xi) y^\alpha \{ z^* \}^\beta \{ \tilde{z}^* \}^\gamma e^{i(k, x)}, \]

\[ R_1^{high}(x, y, z^*, \tilde{z}^*, \xi) = \sum_{k \in \mathbb{Z}^n, \alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^n, 2|\alpha| + |\beta| + |\gamma| \geq 3} R_1^{\alpha\beta\gamma}(k, \xi) y^\alpha \{ z^* \}^\beta \{ \tilde{z}^* \}^\gamma e^{i(k, x)}. \]

It follows that

\[
\| X_{T_1} \|_{D(s_1 - \sigma, r_1), \Pi_1} \leq \sum_{|k| > K_1} e^{|k|(s_1 - \sigma)} \| X_{R_1} \|_{D(s_1, r_1), \Pi_1} \leq K_1 e^{-K_1 \sigma} \| X_{R_1} \|_{D(s_1, r_1), \Pi_1},
\]

\[
\| X_{T_1} \|_{\dot{D}(s_1 - \sigma, r_1), \Pi_1} \leq K_1 e^{-K_1 \sigma} \| X_{R_1} \|_{\dot{D}(s_1, r_1), \Pi_1},
\]

(2.2)

where \( K_1 \) determines later.

We desire to eliminate the term \( R_1^{low} \) by the coordinate transformation \( \Phi_1 \) which is obtained as the time-1-map \( X_{F_1}^t |_{t=1} \). One then has

\[ H_2 = H_1 \circ \Phi_1 = N_2 + R_2, \]

(2.3)

where

\[ N_2 = N_2 + \hat{N}_1 \]

\[ = \sum_{j=0}^1 \hat{N}_j^1(\xi) + \langle \omega_2(\xi), y \rangle + \langle \Omega_2(\xi) z, \tilde{z} \rangle + \left( \sum_{j=0}^1 \hat{N}_j^{z_0}(\xi), z_0 \right) + \left( \sum_{j=0}^1 \hat{N}_j^{\tilde{z}_0}(\xi), \tilde{z}_0 \right) + \left( \sum_{j=0}^1 \hat{N}_j^{\tilde{z}_0}(\xi), \tilde{z}_0 \right), \]

(2.4)

with

\[ \omega_2(\xi) = \omega_1(\xi) + \hat{N}_1^1(\xi), 1 \leq j \leq n, \]

\[ \Omega_2(\xi) = diag \left( \Omega_1^j(\xi) + R_1^{\alpha\beta\gamma}(0, \xi) : j \in \mathbb{N}_+ \setminus \mathcal{J} \right), \]

and

\[ R_2 = \int_0^1 \{(1-t)\hat{N}_1 + tR_1^{low}, F_1\} \circ X_{F_1}^t dt + (T_1 + R_1^{high}) \circ X_{F_1}^t |_{t=1}. \]

(2.5)
Similarly, we need to solve the homological equation

\[ \{N_1, F_1\} + R_1^{low} = \dot{N}_1. \]  

(2.6)

Let \( \partial_\omega = \omega \cdot \partial_x \). Then the homological equation (2.6) decomposes into

\[ \partial_\omega, F_1^{\omega_0 \omega_j} + i \sum_{l=1}^{b} \left( N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} + F_1^{\omega_l \omega_j} N_0^{\omega_0 \omega_l}(\xi) \right) \]

\[ -i \sum_{l=1}^{b} \left( N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} + F_1^{\omega_l \omega_j} N_0^{\omega_0 \omega_l}(\xi) \right) = R_1^{\omega_0 \omega_j}, 1 \leq i, j \leq b; \]  

(2.7)

\[ \partial_\omega, F_1^{\omega_0 \omega_j} + 2i \sum_{l=1}^{b} \left( N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} + F_1^{\omega_l \omega_j} N_0^{\omega_0 \omega_l}(\xi) \right) \]

\[ -i \sum_{l=1}^{b} \left( N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} - F_1^{\omega_l \omega_j} N_0^{\omega_0 \omega_l}(\xi) \right) \]

\[ -2i \sum_{l=1}^{b} \left( N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} + F_1^{\omega_l \omega_j} N_0^{\omega_0 \omega_l}(\xi) \right) = R_1^{\omega_0 \omega_j}, 1 \leq i, j \leq b; \]  

(2.8)

\[ \partial_\omega, F_1^{\omega_0 \omega_j} - i \sum_{l=1}^{b} \left( N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} + F_1^{\omega_l \omega_j} N_0^{\omega_0 \omega_l}(\xi) \right) \]

\[ + i \sum_{l=1}^{b} \left( N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} + F_1^{\omega_l \omega_j} N_0^{\omega_0 \omega_l}(\xi) \right) = R_1^{\omega_0 \omega_j}, 1 \leq i, j \leq b; \]  

(2.9)

\[ \left( \partial_\omega + i \Omega_1^j \right) F_1^{\omega_0 \omega_j} + i \sum_{l=1}^{b} N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} \]

\[ -2i \sum_{l=1}^{b} N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} = R_1^{\omega_0 \omega_j}, 1 \leq i \leq b, j \in \mathbb{N}_+ \setminus \mathcal{J}; \]  

(2.10)

\[ \left( \partial_\omega - i \Omega_1^j \right) F_1^{\omega_0 \omega_j} - i \sum_{l=1}^{b} N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} \]

\[ -2i \sum_{l=1}^{b} N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} = R_1^{\omega_0 \omega_j}, 1 \leq i \leq b, j \in \mathbb{N}_+ \setminus \mathcal{J}; \]  

(2.11)

\[ \left( \partial_\omega + i \Omega_1^j \right) F_1^{\omega_0 \omega_j} + i \sum_{l=1}^{b} N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} \]

\[ + 2i \sum_{l=1}^{b} N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} = R_1^{\omega_0 \omega_j}, 1 \leq i \leq b, j \in \mathbb{N}_+ \setminus \mathcal{J}; \]  

(2.12)

\[ \left( \partial_\omega - i \Omega_1^j \right) F_1^{\omega_0 \omega_j} - i \sum_{l=1}^{b} N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} \]

\[ + 2i \sum_{l=1}^{b} N_0^{\omega_0 \omega_l}(\xi) F_1^{\omega_l \omega_j} = R_1^{\omega_0 \omega_j}, 1 \leq i \leq b, j \in \mathbb{N}_+ \setminus \mathcal{J}; \]  

(2.13)
where \(\kappa_{ij} = 1, i = j; \kappa_{ij} = 0, i \neq j\).

Before solving the homological equation (2.6), we introduce Kronecker Product of matrices.

**Definition 2.2** Let \(A = (a_{ij}) \in C^{m \times n}\) and \(B = (b_{ij}) \in C^{p \times q}\). Then the following partial matrix

\[
A \otimes B = \begin{pmatrix}
    a_{11}B & a_{12}B & \cdots & a_{1n}B \\
    a_{21}B & a_{22}B & \cdots & a_{2n}B \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{pmatrix} \in C^{mp \times nq}
\]

is called Kronecker product.

**Definition 2.3** Let \(A = (a_{ij}) \in C^{m \times n}\) and note \(a_{i} = (a_{1i}, a_{2i}, \cdots, a_{mi})^T (i = 1, 2, \cdots, n)\). Denote

\[
vec(A) = \begin{pmatrix}
    a_{1} \\
    a_{2} \\
    \vdots \\
    a_{n}
\end{pmatrix}.
\]
Then \( \text{vec}(A) \) is called column straightening of \( A \).

**Lemma 2.4** Let \( A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times p} \) and \( C \in \mathbb{C}^{p \times q} \). Then
\[
\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B).
\]

**Lemma 2.5** Let \( A \in \mathbb{C}^{m \times m}, B \in \mathbb{C}^{n \times n} \) and \( X \in \mathbb{C}^{m \times n} \). Then

1. \( \text{vec}(AX) = (I_n \otimes A)\text{vec}(X) \);
2. \( \text{vec}(XB) = (B^T \otimes I_m)\text{vec}(X) \);
3. \( \text{vec}(AX + XB) = (I_n \otimes A + B^T \otimes I_m)\text{vec}(X) \).

Now we begin to solve the homological equation (2.6). After a brief observation, the equation (2.6) can be divided into following four types.

(†1) The coefficient equations of \( F_{1 \hat{b} \hat{b} 0}^{20} \), \( F_{1 \hat{b} 0}^{20} \), and \( F_{1 \hat{b} 0}^{20} \).

\[
\begin{align*}
\frac{\partial}{\partial \omega_1} F_{1 \hat{b} \hat{b} 0}^{20} &+ i \left( \tilde{N}_0^{20} (\xi) F_{1 \hat{b} \hat{b} 0}^{20} + F_{1 \hat{b} \hat{b} 0}^{20} \tilde{N}_0^{20} (\xi) \right) \\
&- i \left( \tilde{N}_0^{20} (\xi) F_{1 \hat{b} \hat{b} 0}^{20} + F_{1 \hat{b} \hat{b} 0}^{20} \tilde{N}_0^{20} (\xi) \right) = R_{1 \hat{b} \hat{b} 0}^{20}, \\
\frac{\partial}{\partial \omega_1} F_{1 \hat{b} \hat{b} 0}^{20} &+ 2i \left( \tilde{N}_0^{20} (\xi) F_{1 \hat{b} \hat{b} 0}^{20} + F_{1 \hat{b} \hat{b} 0}^{20} \tilde{N}_0^{20} (\xi) \right) \\
&- i \left( \tilde{N}_0^{20} (\xi) F_{1 \hat{b} \hat{b} 0}^{20} - F_{1 \hat{b} \hat{b} 0}^{20} \tilde{N}_0^{20} (\xi) \right) = R_{1 \hat{b} \hat{b} 0}^{20}, \\
\frac{\partial}{\partial \omega_1} F_{1 \hat{b} \hat{b} 0}^{20} &- i \left( \tilde{N}_0^{20} (\xi) F_{1 \hat{b} \hat{b} 0}^{20} + F_{1 \hat{b} \hat{b} 0}^{20} \tilde{N}_0^{20} (\xi) \right) \\
&+ i \left( \tilde{N}_0^{20} (\xi) F_{1 \hat{b} \hat{b} 0}^{20} + F_{1 \hat{b} \hat{b} 0}^{20} \tilde{N}_0^{20} (\xi) \right) = R_{1 \hat{b} \hat{b} 0}^{20}.
\end{align*}
\]

Introducing column straightening, we have

\[
\begin{align*}
\frac{\partial}{\partial \omega_1} \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) &+ i (J_b \otimes \tilde{N}_0^{20} (\xi) + \tilde{N}_0^{20} (\xi) \otimes J_b) \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) \\
&- i (J_b \otimes \tilde{N}_0^{20} (\xi) + \tilde{N}_0^{20} (\xi) \otimes J_b) \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) = \text{vec}(R_{1 \hat{b} \hat{b} 0}^{20}), \\
\frac{\partial}{\partial \omega_1} \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) &+ 2i (J_b \otimes \tilde{N}_0^{20} (\xi) + \tilde{N}_0^{20} (\xi) \otimes J_b) \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) \\
&- i (J_b \otimes \tilde{N}_0^{20} (\xi) - \tilde{N}_0^{20} (\xi) \otimes J_b) \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) = \text{vec}(R_{1 \hat{b} \hat{b} 0}^{20}), \\
\frac{\partial}{\partial \omega_1} \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) &- i (J_b \otimes \tilde{N}_0^{20} (\xi) + \tilde{N}_0^{20} (\xi) \otimes J_b) \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) \\
&+ i (J_b \otimes \tilde{N}_0^{20} (\xi) + \tilde{N}_0^{20} (\xi) \otimes J_b) \text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) = \text{vec}(R_{1 \hat{b} \hat{b} 0}^{20}).
\end{align*}
\]

It follows that
\[
\mathcal{A} \begin{pmatrix}
\text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) \\
\text{vec}(F_{1 \hat{b} \hat{b} 0}^{20}) \\
\text{vec}(F_{1 \hat{b} \hat{b} 0}^{20})
\end{pmatrix} = \begin{pmatrix}
\text{vec}(R_{1 \hat{b} \hat{b} 0}^{20}) \\
\text{vec}(R_{1 \hat{b} \hat{b} 0}^{20}) \\
\text{vec}(R_{1 \hat{b} \hat{b} 0}^{20})
\end{pmatrix}, \quad (2.23)
\]
where the operator
\[
\mathcal{A} = \begin{pmatrix}
\partial_{\omega_1} + i(A_3 + A_4) & -i(A_1 + A_2) & 0 \\
2i(A_5 + A_6) & \partial_{\omega_1} - i(A_3 - A_4) & -2i(A_1 + A_2) \\
0 & i(A_5 + A_6) & \partial_{\omega_1} - 2i(A_3 + A_4)
\end{pmatrix}_{3b^2 \times 3b^2}
\]
with
\[
A_1 = I_b \otimes N_0^{\circ\circ\circ}(\xi), \quad A_2 = N_0^{\circ\circ\circ}(\xi) \otimes I_b,
\]
\[
A_3 = I_b \otimes N_0^{\circ\circ\circ}(\xi), \quad A_4 = N_0^{\circ\circ\circ}(\xi) \otimes I_b,
\]
\[
A_5 = I_b \otimes N_0^{\circ\circ\circ}(\xi), \quad A_6 = N_0^{\circ\circ\circ}(\xi) \otimes I_b.
\]
(\dagger 2). The coefficient equations of $F^{20\bar{z}_j}, F^{20\bar{z}_j}, F^{20z_j}$ and $F^{20\bar{z}_j}$ for $j \in \mathbb{N}_+ \setminus \mathcal{J}$.

\[
\begin{pmatrix}
\partial_{\omega_1} + i\Omega_1^j \\
\partial_{\omega_1} - i\Omega_1^j \\
\partial_{\omega_1} + i\Omega_1^j \\
\partial_{\omega_1} - i\Omega_1^j
\end{pmatrix}
\begin{pmatrix}
F_1^{20z_j} \\
F_1^{20\bar{z}_j} \\
F_1^{20\bar{z}_j} \\
F_1^{20z_j}
\end{pmatrix}
= \begin{pmatrix}
R_1^{20z_j} \\
R_1^{20\bar{z}_j} \\
R_1^{20\bar{z}_j} \\
R_1^{20z_j}
\end{pmatrix},
\]
(2.24)

where the operator
\[
\mathcal{B} = i \begin{pmatrix}
N_0^{\circ\circ\circ}(\xi) & 0 & -2N_0^{\circ\circ\circ}(\xi) & 0 \\
0 & N_0^{\circ\circ\circ}(\xi) & 0 & -2N_0^{\circ\circ\circ}(\xi) \\
2N_0^{\circ\circ\circ}(\xi) & 0 & -N_0^{\circ\circ\circ}(\xi) & 0 \\
0 & 2N_0^{\circ\circ\circ}(\xi) & 0 & -N_0^{\circ\circ\circ}(\xi)
\end{pmatrix}_{4b^2 \times 4b^2}
\]

\[
+ \begin{pmatrix}
\partial_{\omega_1} + i\Omega_1^j & 0 & 0 & 0 \\
0 & \partial_{\omega_1} - i\Omega_1^j & 0 & 0 \\
0 & 0 & \partial_{\omega_1} + i\Omega_1^j & 0 \\
0 & 0 & 0 & \partial_{\omega_1} - i\Omega_1^j
\end{pmatrix}_{4b^2 \times 4b^2}.
\]

(\dagger 3). The coefficient equations of $F^{z_0}$ and $F^{\bar{z}_0}$.

\[
\mathcal{C} \begin{pmatrix}
F_1^{z_0} \\
F_1^{\bar{z}_0}
\end{pmatrix}
= \begin{pmatrix}
R_1^{z_0} \\
R_1^{\bar{z}_0}
\end{pmatrix}
+ i \begin{pmatrix}
N_0^{\circ\circ\circ}(\xi) & -2N_0^{\circ\circ\circ}(\xi) \\
-2N_0^{\circ\circ\circ}(\xi) & N_0^{\circ\circ\circ}(\xi)
\end{pmatrix}_{2b^2 \times 2b^2}
\]

\[
\frac{1}{2b^2 \times 2b^2}
\]

where the operator
\[
\mathcal{C} = i \begin{pmatrix}
N_0^{\circ\circ\circ}(\xi) & -2N_0^{\circ\circ\circ}(\xi) \\
-2N_0^{\circ\circ\circ}(\xi) & N_0^{\circ\circ\circ}(\xi)
\end{pmatrix}_{2b^2 \times 2b^2}
\]

\[
\partial_{\omega_1} + 0 \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}_{2b^2 \times 2b^2}.
\]
(†4). The coefficient equations of \( F^{zi,j} \), \( F^{zi,\bar{z}j} \), \( F^{\bar{z}i,\bar{z}j} \) for \( i, j \in \mathbb{N}_+ \) \( J \), \( F_1^{zi,j} \), \( F_1^{zi,\bar{z}j} \) for \( j \in \mathbb{N}_+ \) \( J \) and \( F_1^y, F_1^x \).

\[
\partial_{\omega_1} F_1^{zi,j} + i(\Omega_1^i + \Omega_1^j) F_1^{zi,j} = R_1^{zi,j},
\]

(2.26) \( \partial_{\omega_1} F_1^{zi,\bar{z}j} + i(\Omega_1^i - \Omega_1^j) F_1^{zi,\bar{z}j} = R_1^{zi,\bar{z}j} - \kappa_{ij} \tilde{N}_1(\xi), \)

(2.27) \( \partial_{\omega_1} F_1^{\bar{z}i,j} - i(\Omega_1^i + \Omega_1^j) F_1^{\bar{z}i,j} = R_1^{\bar{z}i,j}, \)

(2.28) \( (\partial_{\omega_1} + i\Omega_1^i) F_1^y = R_1^y - \tilde{N}_1(\xi), \)

(2.29) \( (\partial_{\omega_1} - i\Omega_1^i) F_1^x = R_1^x - \tilde{N}_1(\xi) + i \sum_{l=1}^{b} \left( \tilde{N}_0(\xi) F_1^{zi,\bar{z}j} - \tilde{N}_0(\xi) F_1^{zi,\bar{z}j} \right), \)

(2.30) \( \partial_{\omega_1} F_1^y = R_1^y - \tilde{N}_1(\xi), \)

(2.31) \( \partial_{\omega_1} F_1^x = R_1^x - \tilde{N}_1(\xi) + i \sum_{l=1}^{b} \left( \tilde{N}_0(\xi) F_1^{zi,\bar{z}j} - \tilde{N}_0(\xi) F_1^{zi,\bar{z}j} \right). \)

(2.32)

**Remark 2.6** Compared to the results given in [27], one can easily see that (†4) is the standard equations which are essential in their analysis. In this paper, we have to prove another three type’s equations except (†4). Moreover, to solve the (†1), (†2) and (†3) homological equations, we must find the inverses of operators \( A, B \) and \( C \) and calculate the measures newly.

### 2.2 The Solvability of Homological Equation (2.6)

Different from the homological equations in [34] which is diagonal and in view of the four type’s homological equations above, the usual non-resonant conditions are not adequate for us to solve the homological equation (2.6). In order to solve (2.6), we have to introduce some new non-resonant conditions (2), (3) and (4) which are related to (†2), (†3) and (†4)’s homological equations. The non-resonant conditions are

1. \( R_{kl}(1) = \{ \xi \in \Pi_1 : |\langle k, \omega_1(\xi) \rangle + l, \Omega_1(\xi) \rangle| < \frac{\gamma_1}{|k| + \tau} \}, \)
   where \( \gamma_1 = 3/4 \gamma_0, \langle l \rangle_d = \max(1, |\sum j^d |l_j|), \tau > n + 1 \) and \( \mathbb{Z} = \{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^{\infty} : |l| \leq 2 \}; \)
2. \( R_{kl}(1) = \{ \xi \in \Pi_1 : \max |\lambda_{A_{11}}(\xi)| < \frac{\gamma_1}{|k| \tau} \}, \)
   where \( A_{11}(\xi) = (k, \omega_1)I_{3b^2} + B_{11}(\xi), \tau_1 = \tau - 1, \gamma_{11} = \frac{\gamma_1}{3b^2} \) and the \( 3b^2 \times 3b^2 \) order matrix \( B_{11}(\xi) \)’s norm is small enough;
3. \( R_{3jk}(1) = \{ \xi \in \Pi_1 : \max |\lambda_{A_{31}}(\xi)| < \frac{\gamma_1}{|k| \tau} \}, \)
   where \( A_{31}(\xi) = (k, \omega_1)I_{2b^2} + B_{31}(\xi), \tau_3 = \tau, \gamma_{31} = \frac{\gamma_1}{4b^2} \) and the \( 4b^2 \times 4b^2 \) order matrix \( B_{31}(\xi) \)’s norm is small enough;
4. \( R_{4k}(1) = \{ \xi \in \Pi_1 : \max |\lambda_{A_{41}}(\xi)| < \frac{\gamma_1}{|k| \tau} \}, \)
   where \( A_{41}(\xi) = (k, \omega_1)I_{2b^2} + B_{41}(\xi), \tau_4 = \tau - 1, \gamma_{41} = \frac{\gamma_1}{2b^2} \) and the \( 2b^2 \times 2b^2 \) order matrix \( B_{41}(\xi) \)’s norm is small enough.
Let
\[ \Pi_2 = \Pi_1 \setminus \bigcup_{k_0 < |k| \leq k_1, (k, l) \in \mathbb{Z}} \mathcal{R}_{kl}(1) \setminus \bigcup_{0 < |k_1| \leq k_1, i=1,4} \mathcal{R}_{ik}(1) \setminus \bigcup_{j \leq 2|k|, |k| \leq K_1} \mathcal{R}_{jk}(1). \] (2.33)

For \( s_2, \eta_1 > 0 \), denote
\[ D_i^2 = D \left( s_2 + \frac{i}{4} (s_1 - s_2), \frac{i}{4} \eta_1 r_1 \right), \quad 0 < i \leq 4. \]

For \( \varsigma > 0 \), define
\[ \|X_F\|_{D(s_r), \Pi}^\varsigma = \|X_F\|_{D(s_r), \Pi} + \varsigma \|X_F\|_{D(s_r), \Pi}. \]

**Lemma 2.7** If the parameter \( \xi \in \Pi_2 \), then the homological equation (2.6) has a solution \( F_1(x, y, z, \bar{z}, \bar{x}, \bar{\xi}) \) with the estimate
\[ \|X_{F_1}\|_{D_2, \Pi_2}^\varsigma \leq c_1^{-6} K_1^{(10b^2 + 2)\tau + 10b^2} (s_1 - s_2)^{-n-1-\tau \varepsilon_1}, \] (2.34)
where \( 0 \leq \varsigma_1 \leq \frac{\varepsilon_1}{\varepsilon_1}. \)

**Proof** First of all, we expand \( F_1 \) and \( R_1 \) into Fourier series. Then we solve the homological equation (2.6) through the following order.

**(P1)** (2.23) yields
\[ A_{11}(\xi) \begin{pmatrix} \text{vec}(\tilde{P}_{12}^{20}(k, \xi)) \\ \text{vec}(\tilde{P}_{11}^{20}(k, \xi)) \\ \text{vec}(\tilde{P}_{11}^{20}(k, \xi)) \end{pmatrix} = -i \begin{pmatrix} \text{vec}(\tilde{R}_{12}^{20}(k, \xi)) \\ \text{vec}(-\frac{1}{2}\tilde{R}_{12}^{20}(k, \xi)) \\ \text{vec}(\tilde{R}_{12}^{20}(k, \xi)) \end{pmatrix}, \]
where
\[ A_{11}(\xi) = \begin{pmatrix} (k, \omega_1) + (A_3 + A_4) & -(A_1 + A_2) & 0 \\ -(A_5 + A_6) & -1/2(k, \omega_1) + 1/2(A_3 - A_4) & (A_1 + A_2) \\ 0 & (A_5 + A_6) & (k, \omega_1) - (A_3 + A_4) \end{pmatrix}_{3b^2 \times 3b^2}. \]

Denote
\[ (k, \omega_1) I_{3b^2} = \begin{pmatrix} (k, \omega_1) & 0 & 0 \\ 0 & -1/2(k, \omega_1) & 0 \\ 0 & 0 & (k, \omega_1) \end{pmatrix}_{3b^2 \times 3b^2} \]
and
\[ B_{11}(\xi) = \begin{pmatrix} (A_3 + A_4) & -(A_1 + A_2) & 0 \\ -(A_5 + A_6) & 1/2(A_3 - A_4) & (A_1 + A_2) \\ 0 & (A_5 + A_6) & -(A_3 + A_4) \end{pmatrix}_{3b^2 \times 3b^2}. \]

One then has \( A_{11}(\xi) = (k, \omega_1) I_{3b^2} + B_{11}(\xi) \). Since \( \tilde{N}_0^{2\xi} = N_0^{2\xi} \) and \( \tilde{N}_0^{2\xi} = N_0^{2\xi} \), it is easy to prove that \( A_{11}(\xi) \) is self-adjoint. Moreover, we obtain the estimate
\[ \|\|\partial_\xi B_{11}(\xi)\|\| \leq \frac{c_0}{\varepsilon_0}. \]

Let
\[ \mathcal{R}_{ik}(1) = \{ \xi \in \Pi_1 : \max \{ |\lambda_{A_{11}}(\xi)| < \gamma_{11}/|k|^\tau_1 \}, 0 < |k| \leq K_1, \]
for \( \tau_1 = \tau, \gamma_{11} = \frac{\gamma_1}{3b^2} \), where the \( \lambda_{A_{11}} \) is the eigenvalue of \( A_{11} \).
When \( \xi \in \Pi_1 \setminus \bigcup_{0 < |k| \leq K_1} \mathcal{R}_{1k}(1) \), one gets
\[
\|A_{11}^{-1}(\xi)\| \leq |\min\{\lambda_{A_{11}}(\xi)\}|^{-1} \leq \frac{1}{\gamma_{11}/|k|^{\tau_1}} < \gamma_{11}^{-1}|k|^{\tau_1}.
\]
Noting that the order of \( A_{11} \) is \( 3b^2 \), we have
\[
\|A_{11}^{-1}(\xi)\| \leq (3b^2)^{p_0} e^{b^2} \gamma_{11}^{-1}|k|^{\tau_1} < \gamma_{11}^{-1}|k|^{\tau_1}.
\]
Since \( A_{11} = A_{11}(\xi) \) is continuously differentiable on every component \( \xi_j \) of the parameter in \( \Pi_1 \), it follows from Whitney extension theorem that there exists \( A_{11}(\xi) \) which is continuously differentiable on every component \( \xi_j \) of the parameter in \( \mathbb{R} \) such that \( A_{11}(\xi) = A_{11}(\xi) \) for any \( \xi \in \Pi_1 \). Using Lemma 7.1, there are continuously differentiable functions \( \lambda^1(\xi), \ldots, \lambda^{3b^2}(\xi) \) on every component \( \xi_j \) of the parameter \( \xi \) which represents the eigenvalues of \( A_{11}(\xi) \) for \( \xi \in \mathbb{R} \). In particular, they also represent the eigenvalues of \( A_{11}(\xi) \) for \( \xi \in \Pi_1 \). Moreover, there exists a matrix-valued function \( U(\xi) \) of order \( 3b^2 \), which depends on \( \xi \), such that for \( \xi \in \Pi_1 \), the following equality hold:
\[
A_{11}(\xi) = U(\xi) diag(\lambda^1(\xi), \ldots, \lambda^{3b^2}(\xi))U^*(\xi),
\]
and
\[
U(\xi)U^*(\xi) = U^*(\xi)U(\xi) = I_{3b^2},
\]
where \( U^* \) is the conjugate transpose of \( U \). It follows that
\[
\|U(\xi)\| = \|U^*(\xi)\| = 1,
\]
where \( \| \cdot \| \) is the operator norm defined above.

Denote \( \lambda = \lambda(\xi) \in \{\lambda^1(\xi), \ldots, \lambda^{3b^2}(\xi)\} \). Let \( \phi \) be the normalized eigenvector corresponding \( \lambda \). Since \( |\det(\partial_\xi \omega_0(\xi))|_{\Pi} \geq L_0 \), it follows easily that
\[
|\det(\partial_\xi \omega_1(\xi))|_{\Pi} = |\det(\partial_\xi (\omega_0(\xi) + \widehat{N}^{-1}_0(\xi)))|_{\Pi} \geq \frac{1}{2} L_0.
\]
Let \( \eta = (\eta_1, \ldots, \eta_n) = \omega_0 \). Note \( B_{11}(\xi) \) is self-adjoint and \( ||\partial_\xi B_{11}(\xi)||_{\Pi} \leq \frac{\epsilon_0}{\gamma_0} \). Thus the map \( \omega_1 : \xi \in \Pi \mapsto \omega_1(\xi) \) is homeomorphism in both direction. Denote \( \omega_1^{-1} \) the inverse of the map. From Lemmas 7.1 and 7.2 , we get that the eigenvalue \( \lambda(\eta) = \lambda(\xi) = \lambda(\omega_1^{-1}(\eta)) \) is \( C^1 \) in each entry of \( \eta \), say \( \eta_1 \) and
\[
|\partial_\eta_1 \lambda(\xi(\eta))| = |(\partial_\eta_1 A_{11}(\omega_1^{-1}(\eta))|_{\Pi} \phi, \phi)|
\]
\[
= |\langle diag(\phi, \partial_\eta_1 \eta) \rangle \phi, \phi| + |\langle \partial_\eta_1 B_{11}(\omega_1^{-1}(\eta)) \phi, \phi| |
\]
\[
\geq |k_1| + \frac{2\epsilon_0}{\gamma_0} E_0 \quad \text{(in view of } |\partial_\xi \omega_1(\xi)|_{\Pi} \leq |\partial_\xi (\omega_1(\xi) + \widehat{N}_0^{-1}(\xi)))|_{\Pi} \leq 2E_0)\]
\[
\geq \frac{1}{2} |k_1|, \quad (\epsilon_0 \text{ is small enough})
\]
Thus, we have
\[
|\mathcal{R}_{1k}(1)| \leq \frac{2\gamma_1}{|k|^{\tau_1}} \cdot \frac{1}{|\det(\partial_\xi \omega_1(\xi))|_{\Pi}} \leq \frac{4\gamma_1}{L_0 |k|^{\tau_1}}.
\]
For any \( \xi \in \Pi_1 \setminus \bigcup_{0 < |k| \leq K_1} \mathcal{R}_{1k}(1) \), one obtains
\[
|\partial_\xi A_{11}^{-1}(\xi)| = |A_{11}^{-1}(\xi)(\partial_\xi A_{11}(\xi))A_{11}^{-1}(\xi)| \leq \gamma_{11}^{-1} |k|^{2\tau_1+1}.
\]
It follows that

\[ \| \partial_\xi A_{11}^{-1}(\xi) \| \leq (3b^2)^{p_0} e^{b^2 a_{11}^{-2}} |k|^{2\tau_1+1} < \gamma_{11}^{-2} |k|^{2\tau_1+1}. \]

Thus, for any \( \xi \in \Pi_1 \setminus \bigcup_{0 < |k| \leq K_1} \mathcal{R}_{kk}(1) \), one has

\[
\begin{pmatrix}
\| F_{1}^{0,0} (k, \xi) \| \\
\| F_{1}^{2,0} (k, \xi) \| \\
\| F_{1}^{0,2} (k, \xi) \|
\end{pmatrix}
\leq \gamma_{11}^{-1} |k|^{\tau-1}
\begin{pmatrix}
\| R_{1}^{0,0} (k, \xi) \| \\
\| R_{1}^{2,0} (k, \xi) \| \\
\| R_{1}^{0,2} (k, \xi) \|
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\| \partial_\xi F_{1}^{0,0} (k, \xi) \| \\
\| \partial_\xi F_{1}^{2,0} (k, \xi) \| \\
\| \partial_\xi F_{1}^{0,2} (k, \xi) \|
\end{pmatrix}
\leq \gamma_{11}^{-2} |k|^{2\tau-1}
\begin{pmatrix}
\| R_{1}^{0,0} (k, \xi) \| + \gamma_{11} \| \partial_\xi R_{1}^{0,0} (k, \xi) \| \\
\| R_{1}^{2,0} (k, \xi) \| + \gamma_{11} \| \partial_\xi R_{1}^{2,0} (k, \xi) \| \\
\| R_{1}^{0,2} (k, \xi) \| + \gamma_{11} \| \partial_\xi R_{1}^{0,2} (k, \xi) \|
\end{pmatrix}.
\]

\((P2)\) : For \( i \neq j \in \mathbb{N}_+ \setminus \mathcal{J} \), (2.26)–(2.28) yield

\[
\begin{pmatrix}
\partial_\xi F_{1}^{i,j} (k, \xi) \\
\partial_\xi F_{1}^{i,j} (k, \xi) \\
\partial_\xi F_{1}^{i,j} (k, \xi)
\end{pmatrix}
= \begin{pmatrix}
R_{1}^{i,j} (k, \xi) \\
R_{1}^{i,j} (k, \xi) \\
R_{1}^{i,j} (k, \xi)
\end{pmatrix}
\]

where

\[ \gamma_1 = 3/4 \gamma_0, \quad (l)_d = \max(1, | \sum j^d l_j |), \quad \tau > n + 1 \quad \text{and} \quad \mathcal{Z} = \{ (k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty : |l| \leq 2 \}. \]

When \( \xi \in \Pi_1 \setminus \bigcup_{|l|=1, K_0 < |k| \leq K_1} \mathcal{R}_{kk}(1) \), we have

\[
\begin{pmatrix}
\| F_{1}^{i,j} (k, \xi) \| \\
\| F_{1}^{i,j} (k, \xi) \| \\
\| F_{1}^{i,j} (k, \xi) \|
\end{pmatrix}
\leq \gamma_{11}^{-1} |k|^{\tau}
\begin{pmatrix}
\| R_{1}^{i,j} (k, \xi) \| \\
\| R_{1}^{i,j} (k, \xi) \| \\
\| R_{1}^{i,j} (k, \xi) \|
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\| \partial_\xi F_{1}^{i,j} (k, \xi) \| \\
\| \partial_\xi F_{1}^{i,j} (k, \xi) \| \\
\| \partial_\xi F_{1}^{i,j} (k, \xi) \|
\end{pmatrix}
\leq \gamma_{11}^{-2} |k|^{2\tau-1}
\begin{pmatrix}
\| R_{1}^{i,j} (k, \xi) \| + \gamma_{11} \| \partial_\xi R_{1}^{i,j} (k, \xi) \| \\
\| R_{1}^{i,j} (k, \xi) \| + \gamma_{11} \| \partial_\xi R_{1}^{i,j} (k, \xi) \| \\
\| R_{1}^{i,j} (k, \xi) \| + \gamma_{11} \| \partial_\xi R_{1}^{i,j} (k, \xi) \|
\end{pmatrix}.
\]
(P3) : (2.24) yields

\[
A_j^{31}(\xi) \begin{pmatrix}
\tilde{F}_{z0}^{20j}(k, \xi) \\
\tilde{F}_{z0}^{20j}(k, \xi) \\
\tilde{F}_{z0}^{20j}(k, \xi) \\
\tilde{F}_{z0}^{20j}(k, \xi)
\end{pmatrix} = i \begin{pmatrix}
\tilde{R}_{z0}^{20j}(k, \xi) \\
\tilde{R}_{z0}^{20j}(k, \xi) \\
-R_{z0}^{20j}(k, \xi) \\
-R_{z0}^{20j}(k, \xi)
\end{pmatrix},
\]

where

\[
A_j^{31}(\xi) = \begin{pmatrix}
-\langle k, \omega_1 \rangle - \Omega^j_1 & 0 & 0 & 0 \\
0 & -\langle k, \omega_1 \rangle + \Omega^j_1 & 0 & 0 \\
0 & 0 & \langle k, \omega_1 \rangle + \Omega^j_1 & 0 \\
0 & 0 & 0 & \langle k, \omega_1 \rangle - \Omega^j_1
\end{pmatrix}
\]

and

\[
B_{31}(\xi) = \begin{pmatrix}
-\tilde{N}_{z0}^{020}(\xi) & 0 & 2\tilde{N}_{z0}^{020}(\xi) & 0 \\
0 & -\tilde{N}_{z0}^{020}(\xi) & 0 & 2\tilde{N}_{z0}^{020}(\xi) \\
0 & 2\tilde{N}_{z0}^{020}(\xi) & 0 & -\tilde{N}_{z0}^{020}(\xi) \\
0 & 0 & -\tilde{N}_{z0}^{020}(\xi) & 0
\end{pmatrix}.
\]

Here, \(A_j^{31}(\xi)\) is also self-adjoint.

Let \(\mu_1 = \frac{1}{\omega_1}, \mu_2 = \frac{\omega_2}{\omega_1}, \ldots, \mu_n = \frac{\omega_n}{\omega_1}\). It is easy to get

\[
\left| \det \left( \frac{\partial (\mu_1, \ldots, \mu_n)}{\partial (\omega_1, \ldots, \omega_1^m)} \right) \right| = (\omega_1^1)^{-n-1} > 1.
\]

Combining (2.36) and \(|\partial_\xi \Omega_j^j| \leq E_1\), we can regard \(\mu = (\mu_1, \ldots, \mu_n)\) as parameter instead of \(\xi\), and we can show that

\[
\| |\partial_\mu B_{31}(\mu)\| \leq \frac{E_0}{\gamma_0}
\]

and

\[
|\partial_\mu \Omega_j^j| \leq 2E_1
\]

Denote \(C(\mu) = k_1 + \sum_{l=2}^n k_l \mu_l\) and

\[
\hat{A}_j^{31}(\mu) = \begin{pmatrix}
-C(\mu) - \mu_1 \Omega_1^j & 0 & 0 & 0 \\
0 & -C(\mu) + \mu_1 \Omega_1^j & 0 & 0 \\
0 & 0 & C(\mu) + \mu_1 \Omega_1^j & 0 \\
0 & 0 & 0 & C(\mu) - \mu_1 \Omega_1^j
\end{pmatrix} + B_{31}(\mu)
\]

Let

\[
\hat{R}_{3jk}(1) = \{ \mu \in \Pi_1 : \max |\lambda_{\hat{A}_j^{31}}(\mu)| < \gamma_31 / |k|^\tau_3 \},
\]

for \(\tau_3 = \tau, \gamma_31 = \frac{\gamma_1}{4b^2}\), where the \(\lambda_{\hat{A}_j^{31}}\) is the eigenvalue of \(\hat{A}_j^{31}\).
Following (P1)'s methods, we then have

\[ |\partial_{\mu_1} \lambda_{A^j_{31}}| \geq \min_j \Omega^j_{1} + 2E_1 \]

Thus, we have

\[ \partial_{\mu_1} \lambda_{A^j_{31}} \geq c > 0. \]

It follows that

\[ |\tilde{R}_{3jk}(1)| \leq \frac{\gamma_1}{c|k|^3}. \]

Let

\[ R_{3jk}(1) = \{ \xi \in \Pi_1 : \max |\lambda_{A^j_{31}}(\xi)| < \gamma_1/|k|^3 \}. \]

We obtain

\[ |R_{3jk}(1)| \leq \frac{\gamma_1}{c|k|^3}. \]

When \( \xi \in \Pi_1 \setminus \bigcup_{j < 2|k|_1, |k|_1 \leq K_1} R_{3jk}(1) \), following (P1)'s methods, also have

\[ \|(A^j_{31})^{-1}(\xi)\|, \|(A^j_{31})^{-1}(\xi)\| \leq \gamma_{31}^{-1}|k|^3, \]

and

\[ \|\partial_\xi (A^j_{31})^{-1}(\xi)\|, \|\partial_\xi (A^j_{31})^{-1}(\xi)\| \leq \gamma_{31}^{-1}|k|^{3r_3+1}. \]

Consequently, for any \( \xi \in \Pi_1 \setminus \bigcup_{j < 2|k|_1, |k|_1 \leq K_1} R_{3jk}(1) \), we have

\[
\begin{pmatrix}
\|\hat{F}_{1}^0(k, \xi)\| \\
\|\hat{F}_{1}^{20}(k, \xi)\| \\
\|\hat{F}_{1}^{20}(k, \xi)\| \\
\|\hat{F}_{1}^{20}(k, \xi)\|
\end{pmatrix}
\leq \gamma_{31}^{-1}|k|^r 
\begin{pmatrix}
\|\hat{R}_{1}^{0}(k, \xi)\| \\
\|\hat{R}_{1}^{20}(k, \xi)\| \\
\|\hat{R}_{1}^{20}(k, \xi)\| \\
\|\hat{R}_{1}^{20}(k, \xi)\|
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\|\partial_\xi \hat{F}_{1}^{0}(k, \xi)\| \\
\|\partial_\xi \hat{F}_{1}^{20}(k, \xi)\| \\
\|\partial_\xi \hat{F}_{1}^{20}(k, \xi)\| \\
\|\partial_\xi \hat{F}_{1}^{20}(k, \xi)\|
\end{pmatrix}
\leq \gamma_{31}^{-2}|k|^{2r_3+1} 
\begin{pmatrix}
\|\hat{R}_{1}^{0}(k, \xi)\| + \gamma_{31}\|\partial_\xi \hat{R}_{1}^{20}(k, \xi)\| \\
\|\hat{R}_{1}^{20}(k, \xi)\| + \gamma_{31}\|\partial_\xi \hat{R}_{1}^{20}(k, \xi)\| \\
\|\hat{R}_{1}^{20}(k, \xi)\| + \gamma_{31}\|\partial_\xi \hat{R}_{1}^{20}(k, \xi)\| \\
\|\hat{R}_{1}^{20}(k, \xi)\| + \gamma_{31}\|\partial_\xi \hat{R}_{1}^{20}(k, \xi)\|
\end{pmatrix}.
\]

(P4) : (2.25) reduces to

\[ A_{41}(\xi) \begin{pmatrix} \hat{F}_{1}^{0}(k, \xi) \\ \hat{F}_{1}^{20}(k, \xi) \end{pmatrix} = i \begin{pmatrix} \hat{R}_{1}^{00}(k, \xi) \\ -\hat{R}_{1}^{20}(k, \xi) \end{pmatrix}, \]

where

\[ A_{41}(\xi) = \begin{pmatrix} -\langle k, \omega_1 \rangle & 0 \\
0 & \langle k, \omega_1 \rangle \end{pmatrix}_{2b^2 \times 2b^2} + B_{41}(\xi) \]
with
\[ B_{41}(\xi) = \begin{pmatrix} -N_{0}^{-2,0}(\xi) & 2N_{0}^{-2,0}(\xi) \\ 2N_{0}^{0,2}(\xi) & -N_{0}^{0,2}(\xi) \end{pmatrix} \],
and
\[ \left( R_{0}^{0}(k, \xi), R_{1}^{0}(k, \xi) \right) = \begin{pmatrix} -R_{1}^{0}(k, \xi) - i(F_{1}^{0,2}(k, \xi)(N_{0}^{0,0}(\xi) - F_{1}^{0,2}(k, \xi)N_{0}^{0,0}(\xi)) \\ R_{1}^{0}(k, \xi) - i(F_{1}^{0,2}(k, \xi)N_{0}^{0,0}(\xi) - F_{1}^{0,2}(k, \xi)N_{0}^{0,0}(\xi)) \end{pmatrix}. \]
Moreover, \( A_{41}(\xi) \) is self-adjoint, which is easy to check we omit it here.
Let
\[ R_{4k}(1) = \{ \xi \in \Pi_{1} : \max |\lambda_{A_{41}}(\xi)| < \gamma_{41}/|k|^{\tau_{4}} \}, \]
for \( \tau_{4} = \tau - 1, \gamma_{41} = \frac{\pi^{1}}{2b^{2}} \), where the \( \lambda_{A_{41}}(\xi) \) is the eigenvalue of \( A_{41}(\xi) \).
When \( \xi \in \Pi_{1} \setminus \bigcup_{0<|k| \leq K_{1}} R_{4k}(1) \), we have
\[ \|A_{41}^{-1}(\xi)\|, \|A_{41}^{-1}(\xi)\| < \gamma_{41}^{-1}|k|^{\tau_{4}}, \]
\[ \|\partial_{\xi} A_{41}^{-1}(\xi)\|, \|\partial_{\xi} A_{41}^{-1}(\xi)\| < \gamma_{41}^{-2}|k|^{2\tau_{4}+1}, \]
together with
\[ |R_{4k}(1)| \leq \gamma_{1}/|k|^{\tau_{4}}. \]
We also have
\[ \|R_{1}^{0}(k, \xi)\|_{a,p} < \|R_{1}^{0}(k, \xi)\| + \left( \|F_{1}^{0,2}(k, \xi)\| + \|F_{1}^{0,2}(k, \xi)\| \right) \varepsilon_{0} \]
and
\[ \|\partial_{\xi} R_{1}(k, \xi)\|_{a,p} < \|\partial_{\xi} R_{1}^{0}(k, \xi)\|_{a,p} + \left( \|F_{1}^{0,2}(k, \xi)\| + \|F_{1}^{0,2}(k, \xi)\| \right) \varepsilon_{0} \]
\[ + \left( \|\partial_{\xi} F_{1}^{0,2}(k, \xi)\| + \|\partial_{\xi} F_{1}^{0,2}(k, \xi)\| \right) \varepsilon_{0}, \]
\[ \|\partial_{\xi} R_{1}^{0}(k, \xi)\|_{a,p} < \|\partial_{\xi} R_{1}^{0}(k, \xi)\|_{a,p} + \left( \|F_{1}^{0,2}(k, \xi)\| + \|F_{1}^{0,2}(k, \xi)\| \right) \varepsilon_{0} \]
\[ + \left( \|\partial_{\xi} F_{1}^{0,2}(k, \xi)\| + \|\partial_{\xi} F_{1}^{0,2}(k, \xi)\| \right) \varepsilon_{0}. \]
Therefore, we get
\[ \|F_{1}^{0,2}(k, \xi)\|_{a,p} < \gamma_{41}^{-1}|k|^{\tau_{1}-1}\|R_{1}^{0}(k, \xi)\|_{a,p} \]
\[ + \gamma_{41}^{-1}\gamma_{11}^{-1}|k|^{2\tau_{1}-1}\left( \|R_{1}^{0,2}(k, \xi)\| + \|R_{1}^{0,2}(k, \xi)\| \right) \varepsilon_{0}, \]
\[ \|F_{1}^{0,2}(k, \xi)\|_{a,p} < \gamma_{41}^{-1}|k|^{\tau_{1}-1}\|R_{1}^{0}(k, \xi)\|_{a,p} \]
\[ + \gamma_{41}^{-1}\gamma_{11}^{-1}|k|^{2\tau_{1}-1}\left( \|R_{1}^{0,2}(k, \xi)\| + \|R_{1}^{0,2}(k, \xi)\| \right) \varepsilon_{0}. \]
and

\[
||\partial_\xi \tilde{F}_1^{z_0}(k, \xi)||_{a,p} \leq \gamma_{41}^{-2}|k|^{2\tau-1}||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)||_{a,p} \\
+ \gamma_{11}^{-2} \gamma_{41}^{-2}| k |^{4\tau-2} \left( ||\tilde{R}_1^{z_0}(k, \xi)|| + ||\tilde{R}_1^{z_0}(k, \xi)|| \right) \varepsilon_0 \\
+ \gamma_{11}^{-2} \gamma_{41}^{-2}| k |^{4\tau-2} \left( ||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)|| + ||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)|| \right) \varepsilon_0,
\]

\[
||\partial_\xi \tilde{F}_1^{z_0}(k, \xi)||_{a,p} \leq \gamma_{41}^{-2}|k|^{2\tau-1}||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)||_{a,p} \\
+ \gamma_{11}^{-2} \gamma_{41}^{-2}| k |^{4\tau-2} \left( ||\tilde{R}_1^{z_0}(k, \xi)|| + ||\tilde{R}_1^{z_0}(k, \xi)|| \right) \varepsilon_0 \\
+ \gamma_{11}^{-2} \gamma_{41}^{-2}| k |^{4\tau-2} \left( ||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)|| + ||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)|| \right) \varepsilon_0.
\]

(P5) : For \( j \in \mathbb{N}_+ \setminus \mathcal{J} \), (2.29)–(2.30) yield

\[
A_{S1}(\xi) \left( \frac{\tilde{F}_1^{z_1}(k, \xi)}{\tilde{F}_1^{z_1}(k, \xi)} \right) = \left( \frac{\tilde{R}_1^{z_1}(k, \xi) + \mathbf{i}(N_{1}^{z_1}(\xi))^T \tilde{R}_1^{z_0}(k, \xi) - (N_{1}^{z_0}(\xi))^T \tilde{F}_1^{z_0}(k, \xi)}{\tilde{R}_1^{z_1}(k, \xi) + \mathbf{i}(N_{1}^{z_1}(\xi))^T \tilde{F}_1^{z_1}(k, \xi) - (N_{1}^{z_0}(\xi))^T \tilde{F}_1^{z_0}(k, \xi)} \right),
\]

where \{ \cdot \}^T means transposition of vector and

\[
A_{S1}(\xi) = \mathbf{i} \begin{pmatrix} (k, \omega_1) + \Omega_1^j & 0 \\ 0 & (k, \omega_1) - \Omega_1^j \end{pmatrix}.
\]

Let

\[
\mathcal{R}_{kl}(1) = \{ \xi \in \Pi_1 : \langle k, \omega_1 \rangle + \langle l, \Omega_1 \rangle |< - \frac{\gamma_{11}(l)_d}{||k|| + 1}^2 \}, \quad (k, l) \in \mathbb{Z},
\]

where \( \gamma_{11} = 3/4 \gamma_0 \), \( (l)_d = \max(1, | \sum j^d l_j |), \quad \tau > n + 1 \) and \( \mathcal{Z} = \{ (k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty : ||l|| \leq 2 \} \).

When \( \xi \in \Pi_1 \setminus \bigcup_{K_0 < |k|, |l|=1} \mathcal{R}_{kl}(1) \), we have

\[
||\widetilde{F}_1(k, \xi)||_{a,p} \leq \gamma_{11}^{-1}|k|^\tau ||\tilde{R}_1^{z_0}(k, \xi)||_{a,p} \\
+ \gamma_{11}^{-1} \gamma_{31}^{-1}|k|^{(4b^2+1)\tau + 4b^2 - 1} \left( ||\tilde{R}_1^{z_0}(k, \xi)|| + ||\tilde{R}_1^{z_0}(k, \xi)|| \right) \varepsilon_0,
\]

\[
||\widetilde{F}_1(k, \xi)||_{a,p} \leq \gamma_{11}^{-1}|k|^\tau ||\tilde{R}_1^{z_0}(k, \xi)||_{a,p} \\
+ \gamma_{11}^{-1} \gamma_{31}^{-1}|k|^{(4b^2+1)\tau + 4b^2 - 1} \left( ||\tilde{R}_1^{z_0}(k, \xi)|| + ||\tilde{R}_1^{z_0}(k, \xi)|| \right) \varepsilon_0,
\]

and

\[
||\partial_\xi \tilde{F}_1(\xi)||_{a,p} \leq \gamma_{11}^{-2}|k|^{2\tau+1}||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)||_{a,p} \\
+ \gamma_{11}^{-2} \gamma_{31}^{-1}|k|^{4\tau+2} \left( ||\tilde{R}_1^{z_0}(k, \xi)|| + ||\tilde{R}_1^{z_0}(k, \xi)|| \right) \varepsilon_0 \\
+ \gamma_{11}^{-2} \gamma_{31}^{-1}|k|^{4\tau+2} \left( ||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)|| + ||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)|| \right) \varepsilon_0,
\]

\[
||\partial_\xi \tilde{F}_1(\xi)||_{a,p} \leq \gamma_{11}^{-2}|k|^{2\tau+1}||\partial_\xi \tilde{R}_1^{z_0}(k, \xi)||_{a,p}
\]

\[
\text{ Springer}
\]
\( + \gamma_1^{-2} \gamma_3^{-1} |k|^2 \left( \| R_1^{\alpha_2} (k, \xi) \| + \| \partial_k R_1^{\alpha_2} (k, \xi) \| \right) \varepsilon_0 \)

\( + \gamma_1^{-2} \gamma_3^{-1} |k|^2 \left( \| \partial_k R_1^{\alpha_2} (k, \xi) \| + \| \partial_k R_1^{\alpha_2} (k, \xi) \| \right) \varepsilon_0. \)

(P6) : (2.31)–(2.32) reduce to

\[
\begin{align*}
\mathbf{i} \hat{F}_1^\gamma (k, \xi) &= \frac{\hat{R}_1^\gamma (k, \xi)}{\langle k, \omega \rangle}, \\
\mathbf{i} \hat{F}_1^\bar{\gamma} (k, \xi) &= \frac{\hat{R}_1^\bar{\gamma} (k, \xi) + \mathbf{i} \left( \{ N_0^\gamma (\xi) \}^T \hat{F}_1^{\bar{\gamma}} (k, \xi) - \{ N_0^{\bar{\gamma}} (\xi) \}^T \hat{F}_1^\gamma (k, \xi) \right)}{\langle k, \omega \rangle},
\end{align*}
\]

where \( \{ \cdot \}^T \) means transposition of vector.

Let

\[
\mathcal{R}_{kl} (1) = \{ \xi \in \mathbb{P} : |\langle k, \omega \rangle + \langle l, \Omega \rangle | < \frac{\gamma_1 (l)_d}{(|k| + 1)^\tau}, (k, l) \in \mathcal{Z}, \}
\]

where \( \gamma_1 = 3/4 \gamma_0, (l)_d = \max(1, |\sum j^2 j^j|), \tau > n + 1 \) and \( \mathcal{Z} = \{ (k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty : |l| \leq 2 \} \).

When \( \xi \in \mathbb{P} \setminus \bigcup_{K_0 < |k| \leq K_1, |l| = 0} \mathcal{R}_{kl} (1) \), one obtains

\[
\begin{align*}
|\hat{F}_1^\gamma (k, \xi)| &\leq \gamma_1^{-1} |k| \| \hat{R}_1^\gamma (k, \xi) \| , \\
|\hat{F}_1^\bar{\gamma} (k, \xi)| &\leq \gamma_1^{-1} |k| \| \hat{R}_1^{\bar{\gamma}} (k, \xi) \| + \gamma_1^{-1} \gamma_4^{-1} |k| (2b^2 + 1)^\tau + 2b^2 - 1 \varepsilon_1 \\
&+ \gamma_1^{-1} \gamma_4^{-1} \gamma_1^{-1} |k| (5b^2 + 1)^\tau + 5b^2 - 2 \varepsilon_1,
\end{align*}
\]

and

\[
\begin{align*}
|\partial_k \hat{F}_1^\gamma (k, \xi)| &\leq \gamma_1^{-2} |k|^{2 \tau + 1} \| \partial_k \hat{R}_1^\gamma (k, \xi) \| , \\
|\partial_k \hat{F}_1^{\bar{\gamma}} (k, \xi)| &\leq \gamma_1^{-2} |k|^{2 \tau + 1} \| \partial_k \hat{R}_1^{\bar{\gamma}} (k, \xi) \| + \gamma_1^{-2} \gamma_4^{-2} |k|^{2 \tau + 1} \varepsilon_1 + \gamma_1^{-4} \gamma_4^{-2} |k|^{4 \tau + 2} \varepsilon_1.
\end{align*}
\]

Set \( 0 \leq \varepsilon_1 \). In view of (P1) – (P6), we have

\[
\| X_{R_1} \|_{D_2} \leq \left( \gamma_1^{-2} K_1^{8b^2 + 8b^2 - 1} + \gamma_1^{-4} K_1^{2(10b^2 + 2) + 10b^2 - 2} + \gamma_1^{-6} K_1^{(10b^2 + 2) + 10b^2 - 1} \right) \\
\times \sum_{|k| \leq K_1} |k| e^{-\frac{1}{2} |k| (s_1 - s_2)} \\
\ll (s_1 - s_2)^{-\frac{1}{2} - \tau} \left( \gamma_1^{-2} K_1^{8b^2 + 2 + 8b^2} + \gamma_1^{-4} K_1^{(10b^2 + 2) + 10b^2 - 1} + \gamma_1^{-6} K_1^{(10b^2 + 2) + 10b^2 - 1} \right) \varepsilon_1 \\
\ll \gamma_1^{-6} (s_1 - s_2)^{-\frac{1}{2} - \tau} K_1^{(10b^2 + 2) + 10b^2 - 1} \varepsilon_1.
\]

3 The New Hamiltonian

In view of (2.3), we obtain the new Hamiltonian

\[ H_2 = N_2 + R_2, \]
where $N_2$ and $R_2$ are given in (2.4) and (2.5) respectively.

### 3.1 The New Normal Form $N_2$

The new normal form is given in (2.4). Denote

$$
\tilde{J}_{m+1}^j (\xi) = \sum_{j=0}^{m} \tilde{N}^j_m (\xi), \quad \tilde{J}_{m+1}^0 (\xi) = \sum_{j=0}^{m} \tilde{N}^0_m (\xi),
$$

$$
\tilde{J}_{m+1}^{20} (\xi) = \sum_{j=0}^{m} \tilde{N}^{20}_m (\xi), \quad \tilde{J}_{m+1}^{200} (\xi) = \sum_{j=0}^{m} \tilde{N}^{200}_m (\xi),
$$

We then have

$$
N_2 = \tilde{J}_2^j (\xi) + \langle \omega_2 (\xi), y \rangle + \langle \Omega_2 (\xi) z, \bar{z} \rangle + \langle \tilde{J}_2^0 (\xi), z_0 \rangle + \langle \tilde{J}_2^{20} (\xi), \bar{z}_0 \rangle + \langle \tilde{J}_2^{200} (\xi), \bar{z}_0 \rangle.
$$

### 3.2 The New Perturbation $R_2$

The new perturbation is given in (2.5), i.e.

$$
R_2 = \int_0^1 \{ (1-t) \tilde{N}_1 + t R_1^{low} (F_1) \} \circ X_1^t F_1 dt + T_1 \circ X_1^t F_1 |_{t=1} + R_1^{high} \circ X_1^t F_1 |_{t=1}.
$$

where $R_1 (t) = (1-t) \tilde{N}_1 + t R_1^{low}$, then

$$
X_{R_2} = \int_0^1 (\Phi_1^t)^* X_1 R_1 (t), F_1) dt + (\Phi_1^t)^* X_1 R_1^{high}.
$$

By the generalized Cauchy’s inequality and the estimates (2.2) and (2.34), one has

$$
\| X_{R_2} \|_{D_1^1, \Pi_2} \leq \| X_1 \|_{R_1 (t), F_1} \| \Phi_1^t \|_{D_2^1, \Pi_2} + \| X_1 \|_{D_2^1, \Pi_2} + \| X_1 R_1^{high} \|_{D_2^1, \Pi_2} < \eta_1^{-2} \gamma_1^{-6} K_1^{(10b^2 + 2) \tau + 10b^2} (s_1 - s_2)^{-n-1-\tau} \varepsilon_1^2 + \eta_1 \varepsilon_1 + K_1^n e^{-K_1(s_1-s_2)} \varepsilon_1.
$$

### 4 Iteration Lemma

To iterate the KAM step infinitely, we should choose sequences for the pertinent parameters. The guiding principle is to choose geometric sequences for $s_m$, $r_m$ and the error estimate $\eta_m$, $\gamma_m$, $E_m$. We define for all $m \geq 0$ the following sequences

$$
s_m = \frac{7 \gamma_0}{8} (1 - \sum_{i=1}^{m+1} 2^{-i}),
$$

$$
\gamma_m = \frac{3 \gamma_0}{8} (1 - 2^{-m+1}),
$$

$$
\varepsilon_1 = \frac{\sigma_0}{(\gamma_0 r_0^{2 \alpha + \tau + 2})}, \quad \sigma_0 = 1/40 \gamma_0, \quad \varepsilon_m = \frac{\gamma_m^{-6} (m - 1)^{64 b^4} (s_m - 1)^{-n-1-\tau} \varepsilon_m^2 n, m \geq 2.
$$
Then for each \( \xi \in \Pi_m \), there exist a symplectic coordinate transformation \( \Phi_m : D_{m+1} \to D_m \) satisfying

\[
\| \Phi_m - id \|_{D_{m+1}, \Pi_{m+1}} \leq \varepsilon_m^6,
\]

and a closed set

\[
\Pi_{m+1} = \Pi_m \setminus \bigcup_{\substack{K_{m-1} \leq |k| \leq K_m \setminus |l| \leq 2 \atop 0 < |k| \leq K_m, |l| \leq 1 \leq 4, \atop j \leq 2k, 0 < |k| \leq K_m}} \mathcal{R}_{kl}(m) \setminus \bigcup_{j \leq 2k} \mathcal{R}_{ik}(m) \setminus \bigcup_{0 < |k| \leq K_m} \mathcal{R}_{3jk}(m),
\]

and the new non-resonant conditions

(1) \( \mathcal{R}_{kl}(m) = \{ \xi \in \Pi_m : (k, \omega_m(\xi)) + (l, \Omega_m(\xi)) \leq \frac{\gamma_m^{(l)}}{(k|l|+1)} , (k, l) \in \mathbb{Z} \}, \)

where \( (l) \) is \( \text{max}(1, |\sum k l j I j|), \tau > n + 1 \) and \( \mathbb{Z} = \{(k, l) : 2 \in \mathbb{Z} \times \mathbb{Z}^2 : |l| \leq 2 \}, \)

(2) \( \mathcal{R}_{ik}(m) = \{ \xi \in \Pi_m : \max |\lambda A_{ikm}(\xi)| < \frac{\gamma_m^{(i)}}{|k|+1}, \}

where \( A_{ikm}(\xi) = (k, \omega_m) I_{3b^2} + B_{1m}(\xi), \tau_1 = \tau - 1, \gamma_{1m} = \frac{\gamma_m}{3b^2 m^2} \) and the \( 3b^2 \times 3b^2 \) order matrix \( B_{1m}(\xi) \)'s norm is small enough;

(3) \( \mathcal{R}_{3jk}(m) = \{ \xi \in \Pi_m : \min |\lambda A_{3jm}(\xi)| < \frac{\gamma_m^{(3j)}}{|k|^2}, \}

where \( A_{3jm}(\xi) = (k, \omega_m) I_{3b^2} + B_{3m}(\xi), \tau_3 = \tau, \gamma_{3m} = \frac{\gamma_m}{4b^2 m^2} \) and the \( 4b^2 \times 4b^2 \) order matrix \( B_{3m}(\xi) \)'s norm is small enough;

(4) \( \mathcal{R}_{4k}(m) = \{ \xi \in \Pi_m : \max |\lambda A_{4m}(\xi)| < \frac{\gamma_m^{(4)}}{|k|+1}, \}

where \( A_{4m} = (k, \omega_m) I_{2b^2} + B_{4m}(\xi), \tau_4 = \tau - 1, \gamma_{4m} = \frac{\gamma_m}{2b^2 m^2} \) and the \( 2b^2 \times 2b^2 \) order matrix \( B_{4m}(\xi) \)'s norm is small enough.
such that for \( H_{m+1} = H_m \circ \Phi_m = N_{m+1} + R_{m+1} \), the same estimates as above are satisfied with \( m + 1 \) in place of \( m \), that is,

\[
|\omega_{m+1}(\xi)|_{\Pi_{m+1}}^c + |\Omega_{m+1}(\xi)|_{\Pi_{m+1}}^c \leq E_{m+1},
\]

and

\[
\|X_{R_{m+1}}\|_{D_{m+1}^1, \Pi_{m+1}} \leq \varepsilon_{m+1}.
\]

Moreover, we have

\[
|\Pi_{m+1}\setminus \Pi_m| \leq \gamma_1 \cdot \frac{1}{1 + K_{m-1}} + \gamma_1 \cdot \frac{1}{m^2}.
\]

**Proof** Take \( \varepsilon_0 \) satisfying

\[
\varepsilon_0 \leq \gamma_0 \alpha_0^2 n^{2n+4\tau+2}, \quad \alpha_0 \leq \frac{1}{2} \left( \frac{2n+4\tau+2}{\varepsilon_0} \right)^{n+1} \theta C_0 + 2^{2n+\tau+5}.
\]

In the \( m \)-th step, we take \( K_m = \min \{ K : \int K^n \tau e^{-K(s_m - s_{m+1})} \} \leq \varepsilon_m \) and \( \eta_m = \varepsilon_m \). Thus we have

\[
\eta_m^{-2} \varepsilon_m^{-6} K_m^{10b^2} (s_m - s_{m+1})^{-n-1-\tau} \varepsilon_m \leq \varepsilon_m^{1/6}.
\]

This verified the smallness of \( \|X_{R_{m+1}}\|_{D_{m+1}^1, \Pi_{m+1}} \). Similarly, in the \( m + 1 \)-th step, assumptions \( A, B, C \) and \( D \) are fulfilled. We omit the details.

Since

\[
\Pi_{m+1} = \Pi_m \setminus \bigcup_{K_{m-1} < |k| \leq K_m, |l| \leq 2} \mathcal{R}_{kl}(m) \setminus \bigcup_{0 < |k| \leq K_m, i = 1, 4} \mathcal{R}_{ik}(m) \setminus \bigcup_{j \geq 2} \mathcal{R}_{3jk}(m),
\]

we have

\[
|\| \Pi_{m+1}| \leq \sum_{K_{m-1} < |k| \leq K_m} |\mathcal{R}_{kl}(m)| + \sum_{0 < |k| \leq K_m} |\mathcal{R}_{ik}(m)| + \sum_{j \geq 2} |\mathcal{R}_{3jk}(m)|
\]

\[
\leq \sum_{K_{m-1} < |k| \leq K_m} \frac{\gamma_1}{(|k| + 1)^\tau} + \sum_{0 < |k| \leq K_m} \frac{2\gamma_1}{|k|} + \frac{1}{m^2} + \sum_{0 < |k| \leq K_m} \frac{2\gamma_1}{|k|} \cdot \frac{1}{m^2}
\]

\[
\leq \gamma_1 \cdot \frac{1}{1 + K_{m-1}} + \gamma_1 \cdot \frac{1}{m^2}.
\]

The detailed proof can be seen in [34,36]. \( \square \)

### 5 Proof of Theorem 1.1 and Corollaries 1.2,1.3

**Proof** In view of Lemma 4.1, we get

\[
H_m = N_m + R_m,
\]

where

\[
N_m = \tilde{J}_m^c(\xi) + \omega_m(\xi), y) + (\Omega_m(\xi) z, \bar{z}) + (\tilde{J}_m^{z_0}(\xi), z_0) + (\tilde{J}_m^{\bar{z}_0}(\xi), \bar{z}_0) + (\tilde{J}_m^{z_0\bar{z}_0}(\xi) z_0, \bar{z}_0) + (\tilde{J}_m^{\bar{z}_0z_0}(\xi) \bar{z}_0, z_0),
\]

and the perturbation \( R_m(x, y, \bar{z}, \bar{z}, \xi) \) is defined on \( D(s_m, r_m) \times \Pi_m \).
If $\hat{N} = (\hat{N}^{\alpha_0}(\xi), \hat{N}^{\alpha_0}(\xi)) = 0$, from the fast convergence of $H_m$, there exists a family of torus embedding $\Phi$ such that $\Phi(T^0 \times \{\xi\})$ is an invariant torus for the Hamiltonian $H$ defined in (1.5). We finish the proof of the existence of KAM torus in this case.

If $\hat{N} = (\hat{N}^{\alpha_0}(\xi), \hat{N}^{\alpha_0}(\xi)) \neq 0$, that is, $\sqrt{|\hat{N}^{\alpha_0}(\xi)|^2 + |\hat{N}^{\alpha_0}(\xi)|^2} = \delta_0 > 0$. Since

$$\lim_{m \to \infty} J_m^{\alpha_0}(\xi) = \hat{N}^{\alpha_0}(\xi) \quad \text{and} \quad \lim_{m \to \infty} J_m^{\alpha_0}(\xi) = \hat{N}^{\alpha_0}(\xi),$$

there exists sufficiently large $M_0$ such that for any $m \geq M_0$,

$$\sqrt{|J_m^{\alpha_0}(\xi)|^2 + |J_m^{\alpha_0}(\xi)|^2} \geq \frac{\delta_0}{2}. \quad (5.1)$$

More exactly, we will choose $m_0 > M_0$ such that

$$\delta_0 > 28\epsilon^7_{m_0-1}. \quad (5.2)$$

Thus, considering $H_{m_0}(x, y, z^\ast, \hat{z}^\ast, \xi) = N_{m_0}(y, z^\ast, \hat{z}^\ast, \xi) + R_{m_0}(x, y, z^\ast, \hat{z}^\ast, \xi)$ on $D(s_{m_0}, r_{m_0}) \times \Pi_{m_0-1}$, one obtains

$$\begin{align*}
\dot{x} &= \frac{\partial H_{m_0}}{\partial y} - \frac{\partial R_{m_0}}{\partial x}, \\
\dot{y} &= -\frac{\partial H_{m_0}}{\partial x} = -\frac{\partial R_{m_0}}{\partial x}, \\
\dot{z}_0 &= i\frac{\partial H_{m_0}}{\partial z_0} = i\left(\hat{J}_{m_0}^{\alpha_0}(\xi) + J_{m_0}^{\alpha_0}(\xi)z_0 + 2\hat{J}_{m_0}^{\alpha_0}(\xi)\hat{z}_0 + \frac{\partial R_{m_0}}{\partial z_0}\right), \\
\dot{z}_0 &= -i\frac{\partial H_{m_0}}{\partial z_0} = -i\left(\hat{J}_{m_0}^{\alpha_0}(\xi)z_0 - 2J_{m_0}^{\alpha_0}(\xi)z_0 + J_{m_0}^{\alpha_0}(\xi)\hat{z}_0 + \frac{\partial R_{m_0}}{\partial z_0}\right), \\
\dot{z}_j &= i\frac{\partial H_{m_0}}{\partial z_j} = i\left(\Omega_{m_0}(\xi)z_j + \frac{\partial R_{m_0}}{\partial z_j}\right), \\
\dot{z}_j &= -i\frac{\partial H_{m_0}}{\partial z_j} = -i\left(\Omega_{m_0}(\xi)\hat{z}_j + \frac{\partial R_{m_0}}{\partial z_j}\right),
\end{align*} \quad (5.3)$$

Let

$$X_0 = \left(\begin{array}{c} z_0 \\ \hat{z}_0 \end{array}\right), \quad X_j = \left(\begin{array}{c} z_j \\ \hat{z}_j \end{array}\right), \quad j \in \mathbb{N}_+ \setminus J,$$

and

$$\alpha_0 = \left(\begin{array}{c} i\hat{J}_{m_0}^{\alpha_0}(\xi) \\ -i\hat{J}_{m_0}^{\alpha_0}(\xi) \end{array}\right).$$

We also let

$$A_0 = \left(\begin{array}{cc} i\hat{J}_{m_0}^{\alpha_0}(\xi) & 2iJ_{m_0}^{\alpha_0}(\xi) \\ -2iJ_{m_0}^{\alpha_0}(\xi) & -i\hat{J}_{m_0}^{\alpha_0}(\xi) \end{array}\right), \quad A_j = \left(\begin{array}{cc} i\Omega_j(\xi) & 0 \\ 0 & -i\Omega_j(\xi) \end{array}\right),$$

and

$$g_0(t) = \left(\begin{array}{c} \frac{\partial R_{m_0}}{\partial z_0} \\ -i\frac{\partial R_{m_0}}{\partial z_0} \end{array}\right), \quad g_j(t) = \left(\begin{array}{c} \frac{\partial R_{m_0}}{\partial z_j} \\ -i\frac{\partial R_{m_0}}{\partial z_j} \end{array}\right), \quad j \in \mathbb{N}_+ \setminus J.$$

Then the last four equations of (5.3) can be written into the form

$$\dot{X}_0 = \alpha_0 + A_0 X_0 + g_0, \quad \dot{X}_j = A_j X_j + g_j, \quad j \in \mathbb{N}_+ \setminus J. \quad (5.4)$$
Let us pass to the new variables
\[ \tilde{X}_0(t) = e^{A_0 t} X_0, \quad \tilde{X}_j(t) = e^{A_j t} X_j, \quad j \in \mathbb{N}_+ \setminus \mathcal{J}, \] (5.5)
and rewritten (5.4) as
\[ \dot{\tilde{X}}_0(t) = e^{A_0 t} \alpha_0 + \tilde{g}_0, \quad \dot{\tilde{X}}_j(t) = \tilde{g}_j, \quad j \in \mathbb{N}_+ \setminus \mathcal{J}, \] (5.6)
where
\[ \tilde{g}_0(t) = e^{-A_0 t} g_0, \quad \tilde{g}_j(t) = e^{-A_j t} g_j, \quad j \in \mathbb{N}_+ \setminus \mathcal{J}. \]
Since
\[ \|A_0\| = \sup_{|x|_2 \neq 0} \frac{|A_0 x|_2}{|x|_2} \leq 2b \varepsilon_0 \ll 1, \quad x \in \mathbb{C}^{2b}, \]
then for any \(0 \leq t \leq 1\), one has
\[ \|e^{-A_0 t}\| = \left\| E + \sum_{k \geq 1} \frac{(-A_0 t)^k}{k!} \right\| \leq 1 + \sum_{k \geq 1} \frac{\|A_0\|^k}{k!} < 2, \] (5.7)
and
\[ \|e^{-A_0 t}\| = \left\| E + \sum_{k \geq 1} \frac{(-A_0 t)^k}{k!} \right\| \geq 1 - \sum_{k \geq 1} \frac{\|A_0\|^k}{k!} > \frac{1}{2}. \] (5.8)

Fix an initial value \(\|z^*(0)\|_{a,p} + \|\tilde{z}^*(0)\|_{a,p} \leq \varepsilon_{m-1}^\frac{7}{6}\). Since
\[ \|z^*(0)\|_{a,p}^2 = \sum_{j \in \mathcal{J}} |z_{jm}(0)|^2 j^{2p} e^{2\alpha_{jm}} + \sum_{j \in \mathbb{N}_+ \setminus \mathcal{J}} |z_j(0)|^2 j^{2p} e^{2\alpha_j}, \]
and
\[ \|\tilde{z}^*(0)\|_{a,p}^2 = \sum_{j \in \mathcal{J}} |\tilde{z}_{jm}(0)|^2 j^{2p} e^{2\alpha_{jm}} + \sum_{j \in \mathbb{N}_+ \setminus \mathcal{J}} |\tilde{z}_j(0)|^2 j^{2p} e^{2\alpha_j}, \]
one then has
\[ |\tilde{X}_0(0)|_2 = |X_0|_2 \leq \varepsilon_{m-1}^\frac{7}{6}. \] (5.9)
By integrating \(t\) from 0 to 1 of (5.6), one thus obtains
\[ \left| \tilde{X}_0(1) - \int_0^1 e^{-A_0 t} \alpha_0 dt \right|_2 \leq \sqrt{2} |\tilde{X}_0(0)|_2 + \sqrt{2} \int_0^1 |\tilde{g}_0|_2 dt \]
\[ \leq \sqrt{2} |X_0|_2 + 2\sqrt{2} \int_0^1 |g_0|_2 dt \text{ (by (5.7))} \]
\[ \leq 2\varepsilon_{m-1}^\frac{7}{6} + 4r_m \varepsilon_m \text{ (by (5.9))} \]
\[ \leq 3\varepsilon_{m-1}^\frac{7}{6}. \]

Consequently, we obtain
\[ |\tilde{X}_0(1)|_2 \geq \left| \int_0^1 e^{-A_0 t} \alpha_0 dt \right|_2 - 3\varepsilon_{m-1}^\frac{7}{6}. \]
\[
\begin{align*}
\Leftrightarrow \frac{\delta_0}{4} - 3\epsilon_{m-1}^7 & > 4\epsilon_{m-1}^7,
\end{align*}
\]
which implies
\[
|X_0(1)|_2 = |e^{-A_0 \tilde{X}_0(1)}|_2 > 2\epsilon_{m-1}^7 \text{ (by (5.8))}. \tag{5.10}
\]
Consequently, one has
\[
\|z^*(1)\|_{a,p} + \|\tilde{z}^*(1)\|_{a,p} \geq |z_0(1)|_2 + |\tilde{z}_0(1)|_2
\geq \sqrt{|z_0(1)|_2^2 + |\tilde{z}_0(1)|_2^2}
\geq 2\epsilon_{m-1}^7 \text{ (by (5.10))}. \tag{5.11}
\]
Let \(\Xi_{m_0} = (x, y, z^*, \tilde{z}^*) : |\Im x| \leq s_{m_0}, |y| \leq r_{m_0}^2, \|z^*\|_{a,p} + \|\tilde{z}^*\|_{a,p} \leq \epsilon_{m_0}^7 \) and \(\Phi_{m_0} = \prod_{j=0}^{m_0-1} \Phi_j\). It follows from (5.11) that there exists no invariant torus for the Hamiltonian system defined by (1.5) on \(\Phi_{m_0}^{-1}(\Xi_{m_0} \times \{\xi\})\).

6 Application to NLS

We discuss the nonlinear Schrödinger equation
\[
iu_t - u_{xx} + |u|^4u = 0 \tag{6.1}
\]
on the finite \(x\)-interval \([0, 2\pi]\) with even boundary conditions
\[
u(t, x) = u(t, x + 2\pi), \quad u(x, t) = u(-x, t).
\]
Denote the Sobolev space of complex valued \(L^2\)-functions \([0, 2\pi]\) with an \(L^2\)-derivative and vanishing boundary values by \(P = W^1([0, 2\pi])\). With the inner product
\[
\langle u, v \rangle = Re \int_0^{2\pi} u\overline{v} \, dx,
\]
the Hamiltonian
\[
H = \frac{1}{2} \langle Au, u \rangle + \frac{1}{6} \int_0^{2\pi} |u|^6 \, dx
\]
where \(A = -\frac{d^2}{dx^2}\), the system can be written in the Hamiltonian form
\[
\dot{u} = i\nabla H(u)
\]
the gradient of \(H\) is defined with respect to \(\langle \cdot, \cdot \rangle\), and the dot indicates differentiation with respect to time.

Denote \(\mathbb{N} = \{0, 1, \ldots, n, \ldots\}\) and \(\mathbb{N}_+ = \{1, \ldots, n, \ldots\}\). Let
\[
\begin{align*}
\phi_0(x) &= \sqrt{\frac{1}{2\pi}}, \quad \lambda_0 = 0, \\
\phi_j(x) &= \sqrt{\frac{1}{\pi}} \cos jx, \quad \lambda_j = j^2, \quad j \in \mathbb{N}_+,
\end{align*}
\]
be the basic modes and their frequencies for the linear Schrödinger equation $i u_t - u_{xx} = 0$ with even boundary conditions. We rewrite $H$ as a Hamiltonian in infinitely many coordinates by making the ansatz

$$u(t, x) = \sum_{j \in \mathbb{N}} q_j(t) \phi_j(x).$$

The coordinates are taken from the Hilbert space $l^{a,p}$ of all complex-valued sequences $q = (q_0, q_1, \ldots)$ with

$$||q||_{a,p}^2 = |q_0|^2 + \sum_{j \in \mathbb{N}_+} |q_j|^2 j^2 e^{2a_j} < \infty,$$

where $a > 0$ and $p > \frac{1}{2}$ will be fixed later.

We then obtain the Hamiltonian

$$H = \Lambda + G = \frac{1}{2} \sum_{j \in \mathbb{N}} \lambda_j |q_j|^2 + \frac{1}{6} \int_0^{2\pi} |u|^6 \mathrm{d}x$$

on the phase space $l^{a,p}$ with the symplectic structure $\frac{1}{2} i \sum_{j \in \mathbb{N}} dq_j \wedge d\bar{q}_j$. The corresponding equation is

$$\dot{q}_j = 2i \frac{\partial H}{\partial \bar{q}_j}, \quad j \in \mathbb{N}.$$ (6.3)

**Lemma 6.1** Let $a > 0$ and $p > \frac{1}{2}$. If a curve $I \to l^{a,p}, t \mapsto q(t)$ is an analytic solution of (6.3), then

$$u(t, x) = \sum_{j \in \mathbb{N}} q_j(t) \phi_j(x),$$

is a solution of (6.3) which is analytic on $I \times [0, 2\pi]$.

**Proof** More details can be found in [27]. $\square$

For the nonlinearity $|u|^4 u$, we find

$$G(q, \bar{q}) = \frac{1}{6} \int_0^{2\pi} |u|^6 \mathrm{d}x = \frac{1}{6} \sum_{i,j,k,l,m,n} G_{ijklmn} q_i q_j q_k q_l \bar{q}_i \bar{q}_j \bar{q}_k \bar{q}_l$$

with

$$G_{ijklmn} = \int_0^{2\pi} \phi_i \phi_j \phi_k \phi_l \phi_m \phi_n \mathrm{d}x.$$ (6.4)

It is not difficult to verify that $G_{ijklmn} = 0$ unless $i \pm j \pm k \pm l \pm m \pm n = 0$, for some combination of plus and minus signs. For simplicity, we denote $G_{ijk} = G_{iijjkk}, G_i = G_{iiiiii}$. If we choose $n_1, n_2, \ldots, n_b \in \mathbb{N}_+$ satisfying

$$n_i \neq n_j + n_k, \forall i, j, k \in 1, 2, \ldots, b,$$ (6.5)

one can get

$$G_{n_1} = \cdots = G_{n_b} = \frac{5}{8\pi^2}, \quad G_{n_1 n_j n_j} = \frac{3}{8\pi^2}, \quad G_{n_i n_j n_k} = \frac{1}{4\pi^2}.$$
and

$$G_{n,n/l} = \frac{1}{16\pi^2} (4 - \delta_{n_i}^l - \delta_{n_j}^l - \delta_{n_k}^l), \quad G_{n,n/l} = \frac{1}{16\pi^2} (6 - \delta_{i}^{2n_i}),$$

where \(i \neq j, j \neq k, k \neq i, i, j, k \in \{1, 2, \ldots, b\}, l \notin \{n_1, n_2, \ldots, n_b\}\), and for \(v \in \mathbb{N}\)

$$\delta_i^v = \begin{cases} 1, & i = v, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (6.4) that

$$\overline{G(q, \bar{q})} = G(q, \bar{q}), \forall q \in l^a_p. \quad (6.6)$$

For fixed \(\{n_1, n_2, \ldots, n_b\}\), we define the index sets \(\Delta_*, \# = 0, 1, 2\) and \(\Delta_3\) in the following way:

$$\Delta_* = \{(i, j, k, l, m, n) \in \mathbb{N}^6 : \left(\{i, j, k, l, m, n\} \cap \{n_1, n_2, \ldots, n_b\}\right)^* = 6 - \#\}$$

and

$$\Delta_3 = \{(i, j, k, l, m, n) \in \mathbb{N}^6 : \left(\{i, j, k, l, m, n\} \cap \{n_1, n_2, \ldots, n_b\}\right)^z \leq 3\}.$$

Define the resonance sets \(\mathcal{N} = \{(i, j, k, i, j, k)\} \cap \Delta_0\) and \(\mathcal{M} = \{(i, j, k, i, j, k)\} \cap \Delta_2\).

For our convenience, rewrite \(G = G^0 + G^1 + G^2 + \hat{G}\), where

$$G^* = \frac{1}{6} \sum_{i \pm j \pm k \pm l \pm m \pm n = 0, (i, j, k, l, m, n) \in \Delta_*} G_{ijklmn} q_i q_j q_k q_l q_m q_n \quad (6.7)$$

**Definition 6.2** The index set \(\mathcal{I} = \{n_1 < n_2 < \cdots < n_b\}\) is said to be admissible if and only if \(n_1, n_2, \ldots, n_b\) satisfy the following Assumptions \(A, B, C\) and (6.5).

\(A\). If \(i \pm j \pm k \pm l \pm m \pm n = 0, (i, j, k, l, m, n) \in \Delta_0 \setminus \mathcal{N}\), then

$$\lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m - \lambda_n \neq 0.$$

\(B\). If \(i \pm j \pm k \pm l \pm m \pm n = 0, (i, j, k, l, m, n) \in \Delta_1\), then

$$\lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m - \lambda_n \neq 0.$$

\(C\). If \(i \pm j \pm k \pm l \pm m \pm n = 0, (i, j, k, l, m, n) \in \Delta_2 \setminus \mathcal{M}\), then

$$\lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m - \lambda_n \neq 0.$$

**Proposition 6.3** There exist infinite many admissible index sets.

**Proof** As in [30], we choose \(\mathcal{I} = \{n_i : n_1 < n_2 < \cdots < n_b\} \text{ and } n_i \geq 11n_{i-1}^2\}.\) Clearly, there are infinite many \(\mathcal{I}\)'s of such kind. In order to prove that \(\mathcal{I}\) is admissible, it is enough to prove that the conditions of Assumptions \(A, B, C\) are satisfied. First of all, consider \(A\). At this time, each of \(i, j, k, l, m, n\) is non-zero because of \((i, j, k, l, m, n) \in \Delta_0 \setminus \mathcal{N}\). So Assumptions \(A\) hold true by provoking the proof in [30].

We then check Assumption \(B\) and Assumption \(C\). Since \((i, j, k, l, m, n) \in \Delta_1\), we assume \(i, j, k, l, m \in \mathcal{I}\) and \(i < j < k < l < m\).

If \(n \neq 0\), then \(\lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m - \lambda_n \neq 0\).
If \( n = 0 \), then
\[
-\lambda_i + \lambda_j + \lambda_k + \lambda_l - \lambda_m - \lambda_n = j^2 + k^2 + l^2 - m^2 - i^2 \leq \frac{3}{11} m - m^2 - i^2 < 0.
\]
Thus Assumption \( B \) is fulfilled.

Since \( (i, j, k, l, m, n) \in \Delta \setminus M \), we assume \( i, j, k, l \in \mathcal{J} \) and \( i < j < k < l \).
If \( m \neq 0 \) and \( n \neq 0 \), then \( \lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m - \lambda_n \neq 0 \).
If \( m \neq 0 \) and \( n = 0 \), then
\[
\lambda_i - \lambda_j - \lambda_k + \lambda_l - \lambda_m + \lambda_n = -j^2 - k^2 + l^2 - m^2 + i^2.
\]
We next consider different cases of \( m \):

1. \( 1 \leq m \leq l - 1 \).
   \[
   \lambda_i - \lambda_j - \lambda_k + \lambda_l - \lambda_m + \lambda_n = -j^2 - k^2 + l^2 - m^2 + i^2 \\
   = (l - m)(l + m) + i^2 - j^2 - k^2 \\
   \geq l + 1 + i^2 - \frac{2}{11} l \\
   > 0.
   \]

2. \( m \geq l + 1 \).
   \[
   -\lambda_i + \lambda_j + \lambda_k + \lambda_l - \lambda_m - \lambda_n = j^2 + k^2 + l^2 - m^2 - i^2 \\
   = (l - m)(l + m) + j^2 + k^2 - i^2 \\
   \leq -(2l + 1) + \frac{2}{11} l - i^2 \\
   < 0.
   \]

If \( m = 0 \) and \( n = 0 \), then
\[
\lambda_i + \lambda_j + \lambda_k - \lambda_l - \lambda_m + \lambda_n = i^2 + j^2 + k^2 - l^2 \leq \frac{3}{11} l - l^2 < 0.
\]
These lead Assumption \( C \).

**Proposition 6.4** For any given admissible index set \( \mathcal{J} = \{n_1 < n_2 < \cdots < n_b\} \), there exists a real analytic, symplectic change coordinates \( \Gamma \) in some neighborhood of the origin that takes the Hamiltonian \( H \) in (6.2) into
\[
H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K,
\]
where Hamiltonian vector fields \( X_{\bar{G}} \) and \( X_K \) are real analytic vector fields in a neighborhood of the origin in \( l^{\mu, \nu} \),

\[
\bar{G} = \frac{5}{48 \pi^2} (|q_{n_1}|^6 + \cdots + |q_{n_b}|^6) \\
+ \frac{3}{16 \pi^2} (|q_{n_1}|^4 |q_{n_2}|^2 + \cdots + |q_{n_1}|^4 |q_{n_b}|^2) \\
+ |q_{n_2}|^4 |q_{n_1}|^2 + \cdots + |q_{n_2}|^4 |q_{n_b}|^2 + \cdots + |q_{n_b}|^4 |q_{n_b}|^2) \\
+ \frac{3}{2 \pi^2} (|q_{n_1}|^2 |q_{n_2}|^2 |q_{n_1}|^2 + \cdots + |q_{n_{b-1}}|^4 |q_{n_{b-1}}|^2 |q_{n_b}|^2) \\
+ \frac{3}{2} \left( \sum_{i \notin \mathcal{J}} G_{n_{1}, n_{1}} |q_{n_1}|^4 |q_i|^2 + \cdots + \sum_{i \notin \mathcal{J}} G_{n_{b}, n_{b}} |q_{n_b}|^4 |q_i|^2 \right)
\]
for any monomial 

\[ +6 \left( \sum_{i \neq j} G_{ni,nj}|q_{ni}|^2|q_{nj}|^2|q_i|^2 + \cdots + \sum_{i \neq 1} G_{n_{b-1}n_{b-1}}|q_{n_{b-1}}|^2|q_{n_{b-1}}|^2|q_i|^2 \right), \]  

(6.9)

\[ \hat{G} = \frac{1}{6} \sum_{i \pm j \pm k \pm l \pm m = \pm 0 \,(i,j,k,l,m,n) \in \Delta_3} G_{ijklmn} q_i q_j q_k q_l q_m \]  

(6.10)

and

\[ K(q, \bar{q}) = \sum_{a,a' \in \mathbb{N}} K_{aa'} \prod_{n=0}^{\infty} q_n^{a_n} \bar{q}_n^{a'_n} = \mathcal{O}(\|q\|^8), \]  

(6.11)

with properties :

\[ K_{aa'} = 0 \text{ if } \sum_n (a_n - a'_n)n \neq 0, \text{ and } \sum_n a_n + a'_n \in 2\mathbb{Z} \bigcap \{x \in \mathbb{R} : x \geq 8\} \]  

(6.12)

for any monomial \( \prod_{n=0}^{\infty} q_n^{a_n} \bar{q}_n^{a'_n} \).

**Proof** Let \( \Gamma = X^F_F \big|_{t=1} \) be the time-1-map of the flow of the Hamiltonian vector field \( X_F \) given by the Hamiltonian

\[ F = F^0 + F^1 + F^2 \]

\[ = \frac{1}{6} \left\{ \sum_{i,j,k,l,m,n} F^0_{ijklmn} q_i q_j q_k q_l q_m q_n + \sum_{i,j,k,l,m,n} F^1_{ijklmn} q_i q_j q_k q_l q_m q_n + \sum_{i,j,k,l,m,n} F^2_{ijklmn} q_i q_j q_k q_l q_m q_n \right\} \]

with the coefficients

\[ iF^0_{ijklmn} = \begin{cases} G_{ijklmn} \lambda_i + \lambda_j + \lambda_k - \lambda_i + \lambda_j - \lambda_m - \lambda_n & \text{for } i \pm j \pm k \pm l = 0 \text{ and } (i,j,k,i,j,k) \in \Delta_0 \setminus \mathcal{N}, \\ 0 & \text{otherwise}, \end{cases} \]

\[ iF^1_{ijklmn} = \begin{cases} G_{ijklmn} \lambda_i + \lambda_j + \lambda_k - \lambda_i + \lambda_j - \lambda_m - \lambda_n & \text{for } i \pm j \pm k \pm l = 0 \text{ and } (i,j,k,i,j,k) \in \Delta_1, \\ 0 & \text{otherwise}, \end{cases} \]

\[ iF^2_{ijklmn} = \begin{cases} G_{ijklmn} \lambda_i + \lambda_j + \lambda_k - \lambda_i + \lambda_j - \lambda_m - \lambda_n & \text{for } i \pm j \pm k \pm l = 0 \text{ and } (i,j,k,i,j,k) \in \Delta_2 \setminus \mathcal{M}, \\ 0 & \text{otherwise}. \end{cases} \]

Note our Assumptions \( \mathcal{A}, \mathcal{B}, \mathcal{C} \), the remained proof is just a copy of Lemma 4 in [27].

Next we prove \( K(q, \bar{q}) = \sum_{a,a' \in \mathbb{N}} K_{aa'} \prod_{n=0}^{\infty} q_n^{a_n} \bar{q}_n^{a'_n} \) (\( a,a' \in \mathbb{N} \)) which has the properties that \( K_{aa'} = 0 \) if \( \sum_n (a_n - a'_n)n \neq 0 \) and \( \sum_n a_n + a'_n \) is even (\( \geq 8 \)) for any monomial \( \prod_{n=0}^{\infty} q_n^{a_n} \bar{q}_n^{a'_n} \).

Since

\[ K = \{G, F\} + \frac{1}{2!} \{\{\Lambda, F\}, F\} + \frac{1}{2!} \{\{G, F\}, F\} \]

\[ + \cdots + \frac{1}{n!} \{\cdots\{\Lambda, F\} \cdots, F\} + \frac{1}{n!} \{\cdots\{G, F\} \cdots, F\} + \cdots \]

\( \copyright \) Springer
We first consider \{G, F\}, due to

\[
G(q, \tilde{q}) = \sum_{a, a', \sum_n (a_n - a'_n) = 0} G_{aa'} \prod_{n=0}^{\infty} q_n^{a_n} \tilde{q}_n^{a'_n},
\]

\[
F(q, \tilde{q}) = \sum_{b, b', \sum_n (b_n - b'_n) = 0} F_{bb'} \prod_{n=0}^{\infty} q_n^{b_n} \tilde{q}_n^{b'_n},
\]

then

\[
\{G, F\} = i \sum_n \left( \frac{\partial G}{\partial q_n} \frac{\partial F}{\partial \tilde{q}_n} - \frac{\partial G}{\partial \tilde{q}_n} \frac{\partial F}{\partial q_n} \right)
\]

\[
= i \sum_{a, a', \sum_n (a_n - a'_n) = 0, \ b, b', \sum_n (b_n - b'_n) = 0} G_{aa'} F_{bb'} \sum_j \left( \prod_{n \neq j} q_n^{a_n+b_n} \tilde{q}_n^{a'_n+b'_n} \right) \left( \binom{\binom{a_n+b_n}{a'_n+b'_n} a_{j+b}+b_{j-1}}{a'_j+b'_j-1} q_j \right)
\]

where

\[
\sum_n = \sum_{a, a', b, b', \text{when } n \neq j, a_n + b_n = a'_n + b'_n = c_n, \text{when } n = j, a_n + b_n - 1 = c_n, a'_n + b'_n - 1 = c'_n}
\]

It follows easily that for every monomial, one has

\[
\sum_n n(c_n - c'_n) = \sum_{n \neq j} n(a_n + b_n - a'_n - b'_n) + j(a_j + b_j - 1 - a'_j - b'_j + 1) = 0,
\]

and

\[
\sum_n c_n + c'_n = \sum_{n \neq j} (a_n + b_n + a'_n + b'_n) + (a_j + b_j - 1 + a'_j + b'_j - 1)
\]
\[= \sum_n (a_n + a'_n) + \sum_n (b_n + b'_n) - 2.
\]

Analogously, \(1/n! \{\cdots \{A, F\} \cdots, F\}\) and \(1/n! \{\cdots \{G, F\} \cdots, F\}\) have also this properties. Therefore, \(K\) has also this properties.

Now our Hamiltonian is \(H = \Lambda + \tilde{G} + \tilde{G} + K\). Fix a positive \(n\). Pick a set

\[
J_+ = \{ j_1 < j_2 < \cdots < j_n \} \subseteq \mathbb{N}_+.
\]

and take \(\xi = (\xi_1, \ldots, \xi_n) \in \Pi = [1, 2]^n \subseteq \mathbb{R}^n\) as parameters. Introduce symplectic polar and real coordinates \((x, y, z, \bar{z}, \bar{\bar{z}})\) by setting

\[
\left\{ \begin{array}{l}
q_{jb} = \sqrt{2(\xi_b + \lambda b)} e^{-ixb}, \ b = 1, \ldots, n, \\
q_j = \sqrt{2} z_j, \ j \notin J_+,
\end{array} \right.
\]

\(\text{ Springer}\)
where \( z^* = (z_0, z) \). Then we have, up to a constant term, Hamiltonian (6.8) can be rewritten as

\[
H = \langle \omega(\xi), y \rangle + \langle \Omega(\xi)z^*, \tilde{z}^* \rangle + \tilde{G}(x, y, z^*, \tilde{z}^*, \xi) + \tilde{G}(x, y, z^*, \tilde{z}^*, \xi) + K(x, y, z^*, \tilde{z}^*, \xi)
+ \sum_{1 \leq b \leq n} o^b(\xi)y_b + \sum_{j \notin J^+} \Omega_j z_j \tilde{z}_j + R(x, y, z^*, \tilde{z}^*, \xi),
\]

(6.13)

where \( \omega(\xi) = \alpha + A(\xi) \) and \( \Omega(\xi) = \beta^* + B^*(\xi) = \text{diag}(0, \beta) + \text{diag}(B^0(\xi), B(\xi)) \) with

\[
\alpha = (j_1^2, \ldots, j_n^2), \quad \beta = \text{diag}(j^2)_{j \in \mathbb{N}+\setminus J^+}
\]

and

\[
A(\xi) = \left( \frac{1}{16\pi^2} (18\xi_1^2 + \cdots + 10\xi_b^2 + \cdots + 18\xi_n^2
+ 36\xi_b(\cdots + \xi_{b-1} + \xi_{b+1} + \cdots) + 48(\cdots + \xi_{b-1}\xi_b + \xi_{b+1}\xi_b+2 + \cdots)) \right)_{b=1,2,\ldots,n},
\]

(6.14)

\[
B(\xi) = (12(G_{n_1,n_1}\xi_1^2 + \cdots + G_{n_b,n_b}\xi_b^2 + \cdots)
+ 48(G_{n_1,n_2}\xi_1\xi_2 + \cdots + G_{n_{b-1},n_b}\xi_{b-1}\xi_b + \cdots))_{j \in \mathbb{N}+\setminus J^+},
\]

(6.15)

\[
B^0(\xi) = (\cdots + G_{n_b,n_0}0\xi_b^2 + \cdots) + 48(\cdots + G_{n_{b-1},n_0}0\xi_{b-1}\xi_b + \cdots),
\]

(6.16)

and

\[
\tilde{G} = O(|y|^3) + O(|\xi||y|^2) + O(|\xi||y|)|z|_{a,p}^2 + O(|y|^2||z|_{a,p}^2),
\]

\[
\tilde{G} = O(|\xi|^3|z|_{a,p}^3), \quad K = O(|\xi|^4).
\]

with the variables \( q_b, \tilde{q}_b, b = 1, \ldots, n \) expressed in terms of \( y \) and \( x \). Rescaling \( \xi \) by \( e^{\xi}, z, \tilde{z} \) by \( e^{\xi}z, e^{\xi} \tilde{z} \), and \( y \) by \( e^y \), one obtains a Hamiltonian given by the rescaled Hamiltonian

\[
\bar{H}(x, y, z^*, \tilde{z}^*, \xi) = e^{-20}H(x, e^y, e^{\xi}z, e^{\xi} \tilde{z}, e^\xi, \epsilon)
= \langle \bar{\omega}(\xi), y \rangle + \langle \bar{\Omega}(\xi)z, \tilde{z} \rangle + \epsilon \bar{R}(x, y, z^*, \tilde{z}^*, \xi, \epsilon),
\]

where \( \bar{\omega}(\xi) = e^{-12}\alpha + A(\xi), \bar{\Omega}(\xi) = e^{-12}\beta^* + B^*(\xi), \xi \in [1, 2]^n \). For simplicity, we rewrite \( \bar{H} \) by \( \bar{H}, \bar{\omega} \) by \( \omega, \bar{\Omega} \) by \( \Omega, \) and \( \bar{R} \) by \( R \). It follows that \( X_R \) is analytic in \( (x, y, z^*, \tilde{z}^*) \) in a sufficiently small neighborhood of the origin, \( C^1 \) in component \( \xi_j \) of \( \xi \) lying on the closed bounded set \( \Pi \) in the sense of Whitney, and for the argument \( (x, y, z^*, \tilde{z}^*) \in D_{\Pi}(s, r) \) with given \( s, \ r > 0 \),

\[
\bar{R}(x, y, z^*, \tilde{z}^*, \xi) = R(x, y, z^*, \tilde{z}^*, \xi).
\]

(6.17)

We are now in position to estimate the measure of the resonant set

\[
\mathcal{R}_{1k} = \left\{ \xi \in \Pi : \max |\lambda_{Q(\xi)}| < \frac{\epsilon^k}{|k|^r}, \ k > 0 \right\},
\]

where \( Q(\xi) = (k, \omega(\xi))I_{3b^2} + \tilde{Q}(\xi), (k \in 2\mathbb{Z}\setminus \{0\}) \) is a matrix of order \( 3b^2 \times 3b^2, || \tilde{Q}(\xi) || \) is small enough. To this end, we need to verify that \( \omega(\xi) : \xi \in \Pi \rightarrow \omega(\Pi) \) is homeomorphism in both direction. Recall

\[
\omega^b(\xi) = j_b^2 + \frac{1}{16\pi^2} (18\xi_1^2 + \cdots + 10\xi_b^2 + \cdots + 18\xi_n^2 + 36\xi_b(\cdots + \xi_{b-1} + \xi_{b+1} + \cdots)
+ 48(\cdots + \xi_{b-1}\xi_b + \xi_{b+1}\xi_b+2 + \cdots)), \ b = 1, 2, \ldots, n.
\]
Thus, we have
\[
\left| \det \left( \frac{\partial (\omega^1, \ldots, \omega^n)}{\partial (\xi_1, \ldots, \xi_n)} \right) \right|_{\xi=(1, \ldots, 1)} = \frac{1}{4^n \pi^{2n}} \left| \det \begin{pmatrix} 9n - 4 & 12n - 6 & \ldots & 12n - 6 \\ 12n - 6 & 9n - 4 & \ldots & 12n - 6 \\ \vdots & \vdots & \ddots & \vdots \\ 12n - 6 & 12n - 6 & \ldots & 9n - 4 \end{pmatrix} \right|
\]
\[
= \frac{1}{16^n \pi^{2n}} \left( 12n^2 - 9n + 2 \right) (3n - 2)^{n-1} \neq 0,
\]
which implies that there exist a domain \( \Pi_0 \subseteq \Pi \) with \( |\Pi_0| > 0 \) and positive constants \( c_1, c_2 \), such that
\[
c_1 \leq \left| \det \left( \frac{\partial (\omega^1, \ldots, \omega^n)}{\partial (\xi_1, \ldots, \xi_n)} \right) \right| \leq c_2, \quad \forall \xi \in \Pi_0.
\] (6.18)

So we can regard \( \omega = (\omega^1, \ldots, \omega^n) \in \omega(\Pi_0) \) as parameter vectors rather than \( \xi \). Note that \( Q \) is self-adjoint. By Lemmas 7.1 and 7.2, we have that \( \lambda(\omega) \) is \( C^1 \) in \( \omega_1 \) and
\[
|\partial_{\omega^j} \lambda(\xi(\omega))| = |\langle \partial_{\omega^j} Q(\xi(\omega)) \phi, \phi \rangle| = |\text{diag} (\langle k, \partial_{\omega^j} \omega \phi, \phi \rangle)| + |\langle \partial_{\omega^j} \widetilde{Q}(\xi(\omega)) \phi, \phi \rangle|
\geq \frac{1}{2} |k_1|,
\]
Thus, we have
\[
|R_{1k}| \leq \frac{2e^k}{|k|^\tau} \cdot \sup_{\xi} \left| \det \left( \frac{\partial (\omega^1, \ldots, \omega^n)}{\partial (\xi_1, \ldots, \xi_n)} \right) \right| \leq \frac{2e^k}{c_1 |k|^\tau}.
\]
The remained measure estimates are similar as before. We omit them here.

On the other hand, (6.14)–(6.14) and (6.18) imply that the following equivalent assumptions hold.

**Assumption A**: (Nondegeneracy). The map \( \xi \mapsto \omega(\xi) \) in \( \Pi_0 \) is Whitney smooth, i.e. there exist positive constants \( E_1 \) and \( c_1, c_2 \) such that \( |\omega|_1^E \leq E_1 \) and \( c_1 \leq \sup_{\xi \in \Pi_0} \left| \det \left( \frac{\partial (\omega^1, \ldots, \omega^n)}{\partial (\xi_1, \ldots, \xi_n)} \right) \right| \leq c_2 \).

**Assumption B**: (Spectral Asymptotics). For any \( j \in \mathbb{N}_+ \setminus \),
\[
\Omega^j(\xi) = \frac{j^2}{\epsilon^12} + \cdots + O(\xi^2), \quad j \in \mathbb{N}_+ \setminus \mathcal{J}_+,
\] (6.19)
where the dots stands for fixed lower order terms in \( j \). More precisely, there exists positive constant \( E_2 \) such that \( |\Omega|_1^E \leq E_2 \).

Using the Theorem 1.1 and the Corollary 1.2, we have

**Theorem 6.5** There is a subset \( \Pi_\epsilon \subseteq \Pi_0 \) with
\[
|\Pi_0 \setminus \Pi_\epsilon| = O(\epsilon^{\frac{1}{\mathcal{F}}}), \quad 0 < \epsilon \ll 1,
\]
such that for any parameter \( \xi = (\xi_1, \ldots, \xi_n) \in \Pi_\epsilon \), the nonlinear Schrödinger equation with small \( \epsilon \) has a rotational invariant torus of frequency vector \( \omega(\xi) = (\omega^1(\xi), \ldots, \omega^n(\xi)) \), where
\[ \omega^b(\xi) = \frac{j_b^2}{64\pi^2} (18\xi_1^2 + \cdots + 10\xi_b^2 + \cdots + 18\xi_n^2 + 36\xi_b(\cdots + \xi_{b-1} + \xi_{b+1} + \cdots) + 48(\cdots + \xi_{b-1}\xi_b + \xi_{b+1}\xi_{b+2} + \cdots)), \]
\[ b = 1, 2, \ldots n. \]

The motion on the torus can be expressed by \( u(t, x) \) which is quasi-periodic (in time) with frequency \( \omega \) and \( u(t, \cdot) : \mathbb{R} \to I^{a:p}(\mathbb{T}^n) \) is an analytic map.

**Proof** To prove this, we only need to verify conditions in Corollary 1.2.

Let
\[
N_0(y, z^*, \bar{z}^*, \xi) = \langle \omega_0, y \rangle + \langle \Omega_0, z, \bar{z} \rangle = \sum_{j_b \in J_+} \omega^b_0 y_b + \sum_{j \in \mathbb{N}_+} \Omega^j_0 z_j \bar{z}_j,
\]
with
\[
\omega^b(\xi) = \frac{j_b^2}{16\pi^2} (18\xi_1^2 + \cdots + 10\xi_b^2 + \cdots + 18\xi_n^2) + 36\xi_b(\cdots + \xi_{b-1} + \xi_{b+1} + \cdots) + 48(\cdots + \xi_{b-1}\xi_b + \xi_{b+1}\xi_{b+2} + \cdots),
\]
\[ b = 1, 2, \ldots n, \]
\[
\Omega^j(\xi) = j^2 + 12(\cdots + G_{nbnb_j}^2 + \cdots) + 48(\cdots + G_{nb-1nb}\xi_b + \cdots), j \in \mathbb{N}_+ \setminus J_+,
\]
\[
\Omega^0_j(\xi) = 12(\cdots + G_{nb0}\xi_b + \cdots) + 48(\cdots + G_{nb-1nb0}\xi_b + \cdots),
\]
and
\[
R_0(x, y, z^*, \bar{z}^*, \xi) = R_0^{\text{low}}(x, y, z^*, \bar{z}^*, \xi) + R_0^{\text{high}}(x, y, z^*, \bar{z}^*, \xi),
\]
with
\[
R_0^{\text{low}}(x, y, z^*, \bar{z}^*, \xi) = \sum_{\alpha \in \mathbb{N}_+, \beta, \gamma \in \mathbb{N}_+, 2|\alpha|+|\beta|+|\gamma| \leq 2} R_0^{\alpha\beta\gamma}(x, \xi) y^\alpha (z^*)^\beta (\bar{z}^*)^\gamma
\]
\[ = R_0^5(x, \xi) + \langle R_0^y(x, \xi), y \rangle + \langle R_0^z(x, \xi), z \rangle + \langle R_0^{\bar{z}^2}(x, \xi), \bar{z} \rangle
\]
\[ + \langle R_0^{\bar{z}z}(x, \xi)z, z \rangle + \langle R_0^{\bar{z}z}(x, \xi)\bar{z}, \bar{z} \rangle + \langle R_0^{z\bar{z}}(x, \xi, \bar{z}, z) \rangle
\]
\[ + \langle R_0^{z0z}(x, \xi, z) z, \bar{z} \rangle + \langle R_0^{z0z}(x, \xi, \bar{z}, z) \rangle + \langle R_0^{z0\bar{z}}(x, \xi)z, z \rangle + \langle R_0^{z0\bar{z}}(x, \xi)\bar{z}, \bar{z} \rangle
\]
and
\[
R_0^{\text{high}}(x, y, z^*, \bar{z}^*, \xi) = \sum_{\alpha \in \mathbb{N}_+, \beta, \gamma \in \mathbb{N}_+, 2|\alpha|+|\beta|+|\gamma| \geq 3} R_0^{\alpha\beta\gamma}(x, \xi) y^\alpha (z^*)^\beta (\bar{z}^*)^\gamma.
\]

After introducing action-angle coordinates for tangential variables, the monomials of \( R_0 \) take the form
\[
e^{i\theta_1 + \cdots + i\theta_n} \prod_{j \in \mathbb{N}_+ \setminus J_+} z_j^l z_j^\prime \]
with
\[
- \sum_{1 \leq b \leq n} k_b j_b + \sum_{j \in \mathbb{N}_+ \setminus J_+} (l_j - l'_j) j = 0.
\]
Lemma 6.6  If $|k|$ is even, the corresponding Fourier coefficients of $R_0$ satisfy:

\[ \hat{R}_0^{\pm 0}(k) = \hat{R}_0^{\pm 0}(k) = 0, \quad (6.20) \]

\[ \hat{R}_0^{\pm j}(k) = \hat{R}_0^{\pm j}(k) = 0, \quad j \in \mathbb{N}_+ \setminus J_+, \quad (6.21) \]

\[ \hat{R}_0^{y_b \pm 0}(k) = \hat{R}_0^{y_b \pm 0}(k) = \hat{R}_0^{y_b \pm j}(k) = \hat{R}_0^{y_b \pm j}(k) = 0, \quad 1 \leq b \leq n, \quad j \in \mathbb{N}_+ \setminus J_+, \quad (6.22) \]

\[ \hat{R}_0^{\pm 0 \cdot \pm 0 \cdot \pm j \cdot \pm j}(k) = 0, \quad l_0 + l_j + l_j' = 3 \quad \text{and} \quad j \in \mathbb{N}_+ \setminus J_. \quad (6.23) \]

Proof Since

\[ q_{jb} = \sqrt{2(\xi_b + y_b)} e^{-i \xi_b} = \sqrt{2(\xi_b + y_b)} \frac{1}{2} e^{-i \xi_b} = \sqrt{2 \xi_b} \left( 1 + \frac{y_b}{\xi_b} \right)^{\frac{1}{2}} e^{-i \xi_b} \]

\[ = \sqrt{2 \xi_b} \left( 1 + \frac{y_b}{2 \xi_b} - \frac{1}{8} \left( \frac{y_b}{\xi_b} \right)^2 \right) e^{-i \xi_b} \quad b = 1, \ldots, n, \]

and

\[ q_j = \sqrt{2} z_j, \quad j \notin J_+, \]

then the monomials of the Hamiltonian $P = P(q, \bar{q}) = \sum_{a,a'} P_{aa'} \prod_{n=0}^{\infty} q_{an} a_n^a$ with $\sum_{n=0}^{\infty} (a_n + a_n') = 6$ take the form

\[ e^{i \xi_1 y_1 \cdots + i \xi_n y_n} \cdot \prod_{j=0}^{l_j} \prod_{l' j=0}^{l'_j} \xi_j \left( \prod_{j \in \mathbb{N}_+ \setminus J_+} \xi_j \right), \]

where

\[ \begin{cases} 
q_{jb} = \sqrt{\xi_b + y_b} e^{i \xi_b \xi_b}, & \delta_b = -1, \\
\bar{q}_{jb} = \sqrt{\xi_b + y_b} e^{i \xi_b \xi_b}, & \delta_b = 1. 
\end{cases} \]

The corresponding coefficients of $P$ are as follows:

\[ P^{y_b}(x, \xi) = \sum_{a,a'} P_{aa'} \sum_{1 \leq b \leq n} \hat{q}_{jb} e^{i (\xi_1 x_1 + \cdots + \xi_n x_n + \xi_1 \xi_1 + \cdots + \xi_n \xi_n - x_b)} \]

\[ P^{\bar{z}_0}(x, \xi) = \sum_{a,a'} P_{aa'} \sum_{1 \leq b \leq n} \hat{q}_{jb} e^{i (\xi_1 x_1 + \cdots + \xi_n x_n + \xi_1 \xi_1 + \cdots + \xi_n \xi_n - x_b)}, \quad P^{z_0}(0, \xi) = 0, \]

\[ P^{\bar{z}_0}(x, \xi) = \sum_{a,a'} P_{aa'} \sum_{1 \leq b \leq n} \hat{q}_{jb} e^{i (\xi_1 x_1 + \cdots + \xi_n x_n + \xi_1 \xi_1 + \cdots + \xi_n \xi_n - x_b)}, \quad P^{z_0}(0, \xi) = 0. \]
Since R
Lemma 6.7 The solution \( F_0 \) has the same structure with \( R_0 \)(See the properties in Lemma 6.6).

Proof We omit the details here.

Now we begin to verify \( \hat{R}_1^c(0, \xi) = 0 \). For the convenience of notations, we note \( z = (z_1, \ldots, z_j, \ldots) \) for any \( j \in \mathbb{N}_+ \setminus J_+ \) while \( z_0 \) will be represented alone.
Denote
\[ P_0 = \{ R_{00}^\text{low}, F_0 \}, \]

Lemma 6.8 \( \hat{P}_0^z(0, \xi) = 0. \)

Proof Denote
\[ P_0 = P_0^x(x, \xi) + \langle P_0^y(x, \xi), y \rangle + \langle P_0^z(x, \xi), z \rangle + \langle P_0^\bar{z}(x, \xi), \bar{z} \rangle \]
\[ + \langle P_0^{\bar{z}}(x, \xi), z \rangle + \langle P_0^{\bar{z}}(x, \xi), \bar{z} \rangle + \langle P_0^{\bar{z}}(x, \xi), \bar{z} \rangle \]
\[ + \langle P_0^z(x, \xi), z_0 \rangle + \langle P_0^{\bar{z}}(x, \xi), \bar{z}_0 \rangle + \langle P_0^{\bar{z}}(x, \xi), z_0 \rangle \]
\[ + \langle P_0^z(x, \xi), z_0 \rangle + \langle P_0^{\bar{z}}(x, \xi), \bar{z}_0 \rangle + \langle P_0^{\bar{z}}(x, \xi), z_0 \rangle \]
\[ + \langle P_0^{\bar{z}}(x, \xi), z_0 \rangle + \langle P_0^z(x, \xi), \bar{z}_0 \rangle + \langle P_0^z(x, \xi), \bar{z}_0 \rangle. \]

Since
\[ P_0 = \{ R_{00}^\text{low}, F_0 \} \]
\[ = \frac{\partial R_{00}^\text{low}}{\partial x} \cdot \frac{\partial F_0}{\partial y} - \frac{\partial R_{00}^\text{low}}{\partial y} \cdot \frac{\partial F_0}{\partial x} + i \frac{\partial R_{00}^\text{low}}{\partial z_0} \cdot \frac{\partial F_0}{\partial \bar{z}} - \frac{\partial R_{00}^\text{low}}{\partial \bar{z}} \cdot \frac{\partial F_0}{\partial z_0} \]
\[ - i \left( \frac{\partial R_{00}^\text{low}}{\partial \bar{z}_0} \cdot \frac{\partial F_0}{\partial \bar{z}} - \frac{\partial R_{00}^\text{low}}{\partial \bar{z}} \cdot \frac{\partial F_0}{\partial \bar{z}_0} + \frac{\partial R_{00}^\text{low}}{\partial \bar{z}} \cdot \frac{\partial F_0}{\partial \bar{z}} + \frac{\partial R_{00}^\text{low}}{\partial \bar{z}_0} \cdot \frac{\partial F_0}{\partial \bar{z}} \right), \]
we can divide \( P_0 \) into two categories:

Case 1. The 0-th Fourier coefficient of the following functions are zero.
\[ P_0^z(x, \xi) = \partial_x R_0^z F_0^y - \partial_x F_0^z R_0^y \]
\[ + i \left( 2 R_0^{z0} F_0^z - 2 R_0^{z0} F_0^{z0} + R_0^z F_0^{z0} - R_0^{z0} F_0^z \right) \]
\[ + R_0^{z0} F_0^z - R_0^{z0} F_0^{z0} + R_0^z F_0^{z0} - R_0^{z0} F_0^z. \]

In fact, the 0-th coefficient of \( P_0^z(x, \xi), P_0^z(x, \xi), P_0^z(x, \xi) \) are zero.

Case 2. The 0-th Fourier coefficients of the following functions are uncertain.
\[ P_0^y(x, \xi) = \partial_x R_0^y F_0^y - \partial_x F_0^y R_0^y, \]
\[ P_0^{z0}(x, \xi) = \partial_x R_0^{z0} F_0^y - \partial_x F_0^{z0} R_0^y \]
\[ + i \left( 2 R_0^{z0} F_0^{z0} - 2 R_0^{z0} F_0^{z0} + R_0^z F_0^{z0} - R_0^{z0} F_0^z \right). \]

In fact, the 0-th coefficient of \( P_0^{z0}(x, \xi), P_0^{z0}(x, \xi), P_0^{z0}(x, \xi), P_0^{z0}(x, \xi), P_0^{z0}(x, \xi) \)
are uncertain.

We firstly consider the term
\[ \partial_x R_0^z F_0^y - \partial_x F_0^z R_0^y(x, \xi) = \sum_{j=1}^n \partial_x R_0^z F_0^{y_j} - \sum_{j=1}^n \partial_x F_0^z R_0^{y_j} \]
\[ = \sum_{j=1}^n \left( \sum_{k \neq 0} i k_j R_0^z(k, \xi) e^{i(k, x)} \right) \left( \sum_{l \neq 0} F_0^y(l, \xi) e^{i(l, x)} \right)' \]
\[
- \sum_{j=1}^n \left( \sum_{k \neq 0} ik_j F^0_j (k, \xi) e^{i(k, x)} \right) \left( \sum_{l \neq 0} R^y_0 (l, \xi) e^{i(l, x)} \right) \\
= \sum_{j=1}^n \left( \sum_{k \neq 0} ik_j R^0_j (k, \xi) e^{i(k, x)} \right) \left( \sum_{l \neq 0} R^y_0 (l, \xi) e^{i(l, x)} \right) \\
- \sum_{j=1}^n \left( \sum_{k \neq 0} R^0_j (k, \xi) e^{i(k, x)} \right) \left( \sum_{l \neq 0} R^y_0 (l, \xi) e^{i(l, x)} \right) \\
= \sum_{j=1}^n \left( \sum_{k, l \neq 0} k_j R^0_j (k, \xi) R^y_0 (l, \xi) e^{i(k+l, x)} \left( \frac{1}{\langle l, \alpha_0 \rangle} - \frac{1}{\langle k, \alpha_0 \rangle} \right) \right).
\]

It follows that
\[
(\partial_x R^0_0 F^y_0 - \partial_x F^0_0 R^y_0)(0, \xi) = 0.
\]

Let
\[
V_1 = \left\{ x = (x_i)_{1 \leq i \leq n} : |x_i| = 1 \text{ and } \sum_{i=1}^n |x_i| = 1, 3 \text{ or } 5 \right\},
\]
\[
V_2 = \left\{ y = (y_i)_{1 \leq i \leq n} : |y_i| = 1 \text{ and } \sum_{i=1}^n |y_i| \text{ is even} \right\},
\]
where the nonzero elements in the above n-dimension vectors are at any place. Observing the structure of $R_0$, it is easy to verify that if and only if $k \in V_1$,
\[
\hat{R}^0_j (k, \xi) \neq 0,
\]
and if and only if $l \in V_2$,
\[
\hat{R}^y_0 (l, \xi) \neq 0.
\]

In order to estimate (6.24), the equation
\[
k + l = 0, k \in V_1, l \in V_2
\]
should be solved. If (6.25) has a solution, let $v_0 = (1, 1, \ldots 1)^T$, then $(k + l)^T v_0 = 0$. But
\[
k \cdot v_0 = \pm 1, \pm 3 \text{ or } \pm 5
\]
for any $k \in V_1$, and
\[
l \cdot v_0 = \pm 2m, \forall m,
\]
for any $l \in V_2$. Clearly, (6.25) is unsolved.

Thus
\[
(\partial_x R^0_0 F^y_0 - \partial_x F^0_0 R^y_0)(0, \xi) = 0.
\]
Similarly, one has
\[
(R_0^{\zeta_0} F_0^{\zeta_0} - R_0^{\zeta_0} F_0^{\zeta_0})(0, \xi) = 0,
\]
\[
(R_0^{\zeta_0} F_0^{\zeta_0} - R_0^{\zeta_0} F_0^{\zeta_0})(0, \xi) = 0,
\]
\[
(R_0^{\zeta_0} F_0^{\zeta_0} - R_0^{\zeta_0} F_0^{\zeta_0})(0, \xi) = 0,
\]
\[
(R_0^{\zeta_0} F_0^{\zeta_0} - R_0^{\zeta_0} F_0^{\zeta_0})(0, \xi) = 0,
\]
which ends the proof.

Similarly, one also has
\[
\hat{P}_0^0(0, \xi) = 0, \hat{P}_0^{-j}(0, \xi) = 0, \hat{P}_0^{-j}(0, \xi) = 0.
\]

Analogously, the coefficients of \( P_0 \) can also be written into the form as follow.

**Case 1.**
\[
\hat{P}_0^0(x, \xi) = \sum_{k \neq 0} b_1(\xi) \hat{R}_0^0(k, \xi) e^{i(k, x)} + \sum_{k \neq 0, l \neq 0} b_2(\xi) \hat{R}_0^{\zeta_0}(k, \xi) \hat{R}_0^0(l, \xi) e^{i(k+l, x)}
\]
\[
+ \sum_{k \neq 0, l \neq 0} b_3(\xi) \hat{R}_0^{\zeta_0}(k, \xi) \hat{R}_0^{\zeta_0}(l, \xi) e^{i(k+l, x)}
\]
\[
+ \sum_{k,l} b_4(\xi) \hat{R}_0^0(k, \xi) \hat{R}_0^{\zeta_0}(l, \xi) e^{i(k+l, x)}.
\]

Similarly, \( \hat{P}_0^0(x, \xi) \), \( \hat{P}_0^{-j}(x, \xi) \), \( \hat{P}_0^{-j}(x, \xi) \) have the same structure as \( \hat{P}_0^0(x, \xi) \).

**Case 2.**
\[
\hat{P}_0^{-j}(x, \xi) = \sum_{k \neq 0} a_1(\xi) \hat{R}_0^{-j}(k, \xi) e^{i(k, x)} + \sum_{k \neq 0, l \neq 0} a_2(\xi) \hat{R}_0^{-j}(k, \xi) \hat{R}_0^{-j}(l, \xi) e^{i(k+l, x)}.
\]
\[
\hat{P}_0^{\zeta_0}(x, \xi) = \sum_{k \neq 0} f_1(\xi) \hat{R}_0^{\zeta_0}(k, \xi) e^{i(k, x)} + \sum_{k \neq 0, l \neq 0} f_2(\xi) \hat{R}_0^{\zeta_0}(k, \xi) \hat{R}_0^{-j}(l, \xi) e^{i(k+l, x)}
\]
\[
+ \sum_{k \neq 0, l \neq 0} f_3(\xi) \hat{R}_0^{\zeta_0}(k, \xi) \hat{R}_0^{\zeta_0}(l, \xi) e^{i(k+l, x)}
\]
\[
+ \sum_{k \neq 0, l \neq 0} f_4(\xi) \hat{R}_0^{\zeta_0}(k, \xi) \hat{R}_0^{\zeta_0}(l, \xi) e^{i(k+l, x)}.
\]

Similarly, \( \hat{P}_0^{-j}(x, \xi) \), \( \hat{P}_0^{-j}(x, \xi) \), \( i, j \in \mathbb{N}_0 \) have the same structure as \( \hat{P}_0^{\zeta_0}(x, \xi) \).

**Lemma 6.9** \( R_1^{\zeta_0}(0, \xi) = 0. \)

**Proof** We omit the details here.

For the iterative process, since
\[
N_2 = N_1 + \hat{N}_1, R_2 = \int_0^1 \{(1-t)\hat{N}_1 + t R_1, F_1\} \circ X_t^F dt,
\]
and the Possion bracket keep the form, it is easy to check that $R_2$ has the same form with $R_1$. Thus we can also obtain

$$\widehat{R_2}^0(0, \xi) = 0, \quad \widehat{\bar{R}_2}^0(0, \xi) = 0.$$  

Inductively, for any $m$, one has

$$\widehat{R_z}^0(0, \xi) = 0, \quad \widehat{\bar{R}_z}^0(0, \xi) = 0.$$  

It follows that

$$\dot{\mathcal{N}} = (\dot{\mathcal{N}}^0(\xi), \dot{\mathcal{N}}^{\bar{0}}(\xi)) = \left( \sum_{m=0}^{\infty} \widehat{R_z}^0(0, \xi) , \sum_{m=0}^{\infty} \widehat{\bar{R}_z}^0(0, \xi) \right) = 0.$$  

Thus, we finish the proof of the existence of KAM torus.  

Remark 6.10 We are glad to mention an elegant paper [6]. The considered model there is

$$i \partial_t u = -\partial_x^2 u + 2|u|^2 u + \epsilon f(x, u).$$

Observing that the equation is integrable when $\epsilon = 0$, the authors regard the integrable equation $i \partial_t u = -\partial_x^2 u + 2|u|^2$ as unperturbed hamiltonian in KAM iteration. So

$$\lambda_j = j^2 + \xi_j \neq 0, \quad j = 0, 1, 2, \cdots.$$  

That implies that the case $\lambda_0 = 0$ does not appear in their paper. In the present paper, we consider

$$i \partial_t u = -u_{xx} + |u|^4 u + \epsilon f(x, u).$$

Then equation with $\epsilon = 0$ is not integrable. So we have to take the linear equation $i \partial_t u = -u_{xx}$ as unperturbed Hamiltonian in implementing KAM iteration. We also would like to mention [30] where the considered equation is

$$iu_t - u_{xx} + mu + |u|^4 u = 0 \quad (6.26)$$

subject to Dirichlet boundary conditions where $\lambda_j = j^2 + m \neq 0$.

Acknowledgements The authors are grateful to anonymous referee for his/her valuable comments, which greatly improve the original manuscript of this paper. The work was supported in part by National Natural Science Foundation of China (11790272 and 11771093).

7 Appendix

Lemma 7.1 Consider an $n \times n$ complex matrix function $Q(\omega)$ which depends on the real parameter $\omega \in \mathbb{R}$. Let $Q(\omega)$ be a matrix function satisfying conditions:

(1) $Q(\omega)$ is self-adjoint for every $\omega \in \mathbb{R}$; i.e. $Q(\omega) = (Q(\omega))^*$, where the $*$ denotes the conjugate transpose matrix;

(2) $Q(\omega)$ is continuous differentiable in an interval $I$ of the real variable $\omega$.

Then there exist $n$ continuously differentiable functions $\mu_1(\omega), \ldots, \mu_n(\omega)$ on $I$ that represent the repeated eigenvalues of $Q(\omega)$.

Proof See [21].
Lemma 7.2 Assume $Q = Q(\omega)$ satisfies the conditions in Lemma 6.1. Let $\mu = \mu(\omega)$ be any eigenvalue of $Q$ and $\phi$ be the normalized eigenfunction corresponding to $\mu$. Then

$$\partial_\omega \mu = \langle (\partial_\omega Q) \phi, \phi \rangle.$$ 

Proof See [21].

References

1. Baldi, P., Berti, M., Haus, E., Montalto, R.: Time quasi-periodic gravity water waves in finite depth. Invent. Math. 214(2), 739–911 (2018)
2. Baldi, P., Berti, M., Montalto, R.: KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation. Math. Ann. 359(1–2), 471–536 (2014)
3. Baldi, P., Berti, M., Montalto, R.: KAM for autonomous quasi-linear perturbations of KdV. Ann. l’Inst. Henri Poincare (C) Non Linear Anal. 33(6), 1589–1638 (2016)
4. Berti, M., Biasco, L.: Branching of Cantor manifolds of elliptic tori and applications to PDEs. Commun. Math. Phys. 305(3), 741–796 (2011)
5. Berti, M., Biasco, L., Procesi, M.: KAM theory for the Hamiltonian derivative wave equation. Ann. Sci. l’Ecole Norm. Super. Soc. Math. France 46(2), 301–373 (2013)
6. Berti, M., Kappeler, T., Montalto, R.: Large KAM tori for perturbations of the defocusing NLS equation. Asterisque 403, 1–160 (2018)
7. Berti, M., Montalto, R.: Quasi-periodic standing wave solutions of gravity-capillary water waves. Mem. Am. Math. Soc. 263(1273), 1–126 (2020)
8. Bourgain, J.: Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to non-linear PDE. Int. Math. Res. Not. 11, 475–497 (1994)
9. Bourgain, J.: Construction of periodic solutions of nonlinear wave equations in higher dimension. Geom. Funct. Anal. GAFA 5(4), 629–639 (1995)
10. Bourgain, J.: On Melnikovs persistency problem. Math. Res. Lett. 4(4), 445–458 (1997)
11. Bourgain, J.: Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. Ann. Math. 148(2), 363–439 (1998)
12. Bourgain, J.: Periodic solutions of nonlinear wave equations. Harmonic Anal. Partial Differ. Equ. 148(2), 69–97 (1998)
13. Bourgain, J.: Recent progress in quasi-periodic lattice Schrödinger operators and Hamiltonian partial differential equations. Uspekhi Mat. Nauk. 59(2), 37–52 (2004)
14. Bourgain, J.: Green’s function estimates for lattice Schrödinger operators and applications (am-158). Ann. Math. Stud. 158, 200 (2005)
15. Craig, W., Wayne, C.E.: Newton’s method and periodic solutions of nonlinear wave equations. Commun. Pure Appl. Math. 46(11), 1409–1498 (1993)
16. Eliasson, L.H.: Perturbations of stable invariant tori for Hamiltonian systems. Ann. Della Sc. Norm. Super. Pisa Classe Di Sci. 15(1), 115–147 (1998)
17. Eliasson, L.H., Grébert, B., Kuksin, S.B.: KAM for the nonlinear beam equation. Geom. Funct. Anal. 26(6), 1588–1715 (2016)
18. Eliasson, L.H., Kuksin, S.B.: KAM for the nonlinear Schrödinger equation. Ann. Math. 172(1), 371–435 (2010)
19. Feola, R., Procesi, M.: Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations. J. Differ. Equ. 259(7), 3389–3447 (2015)
20. Geng, J., You, J.: KAM tori of Hamiltonian perturbations of 1D linear beam equations. J. Math. Anal. Appl. 277(1), 104–121 (2003)
21. Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1980)
22. Kappeler, T., Pöschel, J.: KdV & KAM. A Series of Modern Surveys in Mathematics, vol. 45. Springer, Berlin (2003)
23. Kuksin, S.B.: Perturbation of conditionally periodic solutions of infinite-dimensional Hamiltonian systems. Engl. Transl. Math. USSR Izv. 32(1), 41–63 (1989)
24. Kuksin, S.B.: Nearly Integrable Infinite-dimensional Hamiltonian Systems. Springer, Berlin (1993)
25. Kuksin, S.B.: Analysis of Hamiltonian PDEs. Oxford Lecture Series in Mathematics and Its Applications, vol. 19. Oxford University Press, Oxford (2000)
26. Kuksin, S.B.: Fifteen years of KAM for PDE. Transl. Am. Math. Soc. 212(2), 237–258 (2004)
27. Kuksin, S.B., Pöschel, J.: Invariant cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. Ann. Math. 143(1), 149–179 (1996)
28. Liu, J., Yuan, X.: Spectrum for quantum duffing oscillator and small-divisor equation with large-variable coefficient. Commun. Pure. Appl. Math. 63(9), 1145–1172 (2010)
29. Liu, J., Yuan, X.: A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations. Commun. Math. Phys. 307(3), 629–673 (2011)
30. Liang, Z., You, J.: quasi-periodic solutions for 1D Schrödinger equations with higher order nonlinearity. SIAM J. Math. Anal. 36(6), 1965–1990 (2005)
31. Melnikov, V.K.: On some cases of conservation of almost periodic motions under a small change of Hamiltonian function. Soviet Math. Dokl. 6(3), 1592–1596 (1965)
32. Melnikov, V.K.: A certain family of conditionally periodic solutions of a Hamiltonian systems. Soviet Math. Dokl. 9(3), 882–886 (1968)
33. Pöschel, J.: On elliptic lower dimensional tori in Hamiltonian systems. Math. Z. 202(4), 559–608 (1989)
34. Pöschel, J.: A KAM theorem for some nonlinear partial differential equations. Ann. Della Sc. Norm. Super. Pisa Classe Di Sci. 23(1), 119–148 (1996)
35. Wayne, C.E.: Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. Commun. Math. Phys. 127(3), 479–528 (1990)
36. Yuan, X., Nunes, A.: A note on the reducibility of linear differential equations with quasiperiodic coefficients. Int. J. Math. Math. Sci. 64, 4071–4083 (2003)
37. Yuan, X.: A KAM theorem with applications to partial differential equations of higher dimensions. Commun. Math. Phys. 275, 97–137 (2007)
38. Yuan, X.: KAM theorem with normal frequencies of finite limit-points for some shallow water equations. Commun. Pure Appl. Math. (2021). https://doi.org/10.1002/cpa.21931
39. Zhang, J., Gao, M., Yuan, X.: KAM tori for reversible partial differential equations. Nonlinearity 24(4), 1189–1228 (2011)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.