Quasi-periodic solutions for p-Laplacian equations with jumping nonlinearity and unbounded potential terms

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Abstract
In this paper, we are concerned with the boundedness of all the solutions for a kind of second order differential equations with p-Laplacian term \((\phi_p(x')') + a\phi_p(x^+) - b\phi_p(x^-) + f(x) = e(t)\), where \(x^+ = \max(x, 0)\), \(x^- = \max(-x, 0)\), \(\phi_p(s) = |s|^{p-2}s\), \(p \geq 2\), \(a\) and \(b\) are positive constants \((a \neq b)\), and satisfy \(\frac{1}{a_p} + \frac{1}{b_p} = 2\omega^{-1}\), where \(\omega \in \mathbb{R}^+ \setminus \mathbb{Q}\), the perturbation \(f\) is unbounded, \(e(t) \in C^6\) is is a smooth \(2\pi_p\)-periodic function on \(t\), where \(\pi_p = \frac{2\pi(p-1)}{p\sin \frac{\pi}{p}}\).

Keywords: Quasi-periodic solutions; Boundedness of solutions; p-Laplacian equations; Canonical transformation; Moser’s small twist theorem.

1 Introduction
Due to the relevance with applied mechanics, for example modelling some kind of suspension bridge(see[12]), the following semilinear Duffing’s equations is widely studied
\[
x'' + ax^+ - bx^- = f(x, t),
\]
where \(x^+ = \max(x, 0)\), \(x^- = \max(-x, 0)\), \(f(x, t)\) is a smooth \(2\pi\)-periodic function on \(t\), \(a\) and \(b\) are positive constants \((a \neq b)\).

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If \( f(x, t) \) depends only on \( t \), the equation (1.1) becomes
\[
x'' + ax^+ - bx^- = f(t), \quad f(t + 2\pi) = f(t),
\]
which had been studied by Fucik [6] and Dancer [3] in their investigations of boundary value problems associated to equations with “jumping nonlinearities”. For recent developments, we refer to [7, 8, 11] and references therein.

In 1996, Ortega [21] proved the Lagrangian stability for the equation
\[
x'' + ax^+ - bx^- = 1 + \gamma h(t)
\]
if \( |\gamma| \) is sufficiently small and \( h \in C^4(S^1) \).

On the other hand, when \( \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q} \), Alonso and Ortega [2] proved that there is a \( 2\pi \)-periodic function \( f(t) \) such that all the solutions of Eq. (1.2) with large initial conditions are unbounded. Moreover for such a \( f(t) \), Eq. (1.2) has periodic solutions.

In 1999, Liu [17] removed the smallness assumption on \( |\gamma| \) in Eq. (1.3) when \( \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q} \) and obtained the same result.

Liu [15] studied the boundedness and Aubry-Mather sets for the semilinear equation
\[
x'' + \lambda^2 x + \phi(x) = e(t),
\]
where \( \phi(x) = o(x) \) as \( |x| \to +\infty \). The methods and formulations developed in [15] are benefit to treat the more general equation
\[
x'' + ax^+ - bx^- + \phi(x) = e(t). \tag{1.4}
\]

For the equation, Wang [24] and Wang [25] considered the Lagrangian stability when the perturbation \( \phi(x) \) is bounded. While Yuan [26] considered the existence of quasiperiodic solutions and Lagrangian stability when \( \phi(x) \) is unbounded.

Fabry and Mawhin [5] investigated the equation
\[
x'' + ax^+ - bx^- = f(x) + g(x) + e(t) \tag{1.5}
\]
under some appropriate conditions, they get the boundedness of all solutions.

Yang [28] considered more complicated nonlinear equation with p-Laplacian operator
\[
((\phi_p(x'))' + (p - 1)[a\phi_p(x^+) - b\phi_p(x^-)] + f(x) + g(x) = e(t). \tag{1.6}
\]

Using Moser’s small twist theorem, he proved that all the solutions are bounded, when \( \frac{1}{a^p} + \frac{1}{b^p} = \frac{2m}{n} \), \( m, n \in \mathbb{N} \), the perturbation \( f(x) \) and the oscillating term \( g \) are bounded. For the case when \( \frac{1}{a^p} + \frac{1}{b^p} = 2\omega^{-1} \), where \( \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \), the perturbation \( f(x) \) is bounded, Yang [27] studied the following equation
\[
(\phi_p(x'))' + a\phi_p(x^+) - b\phi_p(x^-) + f(x) = e(t). \tag{1.7}
\]
and came to the conclusion that every solution of the equation is bounded.

In 2004, Liu [18] studied equation

\[(\phi_p(x'))' + a\phi_p(x^+) - b\phi_p(x^-) = f(x, t), f(x, t + 2\pi) = f(x, t)\]  \hspace{1cm} (1.8)

where \(p > 1\), for the cases when \(\frac{\pi}{p} + \frac{\pi}{b} = \frac{2\pi}{n}\) and \(f \in C^{(7,6)}(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})\) and satisfies that

(i) the following limits exists uniformly in \(t\)

\[\lim_{x \to \infty} f(x, t) = f_{\pm}(t)\]

(ii) the following limits exists uniformly in \(t\)

\[\lim_{x \to \infty} x^m \frac{\partial^{m+n}}{\partial x^m \partial t^n} f(x, t) = f_{\pm,m,n}(t)\]

for \((n, m) = (0, 6), (7, 0)\) and \((7, 6)\). Moreover, \(f_{\pm,m,n}(t) \equiv 0\) for \(m = 6, n = 0, 7\). He comes to the conclusion that all solutions are bounded and the existence of quasi-periodic solutions.

In 2012, Jiao, Piao and Wang [9] considered the boundedness of equations

\[x'' + \omega^2 x + \phi(x) = G_x(x, t) + p(t),\] \hspace{1cm} (1.9)

and

\[x'' + ax^+ - bx^- = G_x(x, t) + p(t).\] \hspace{1cm} (1.10)

Inspired by the above references, especially by Liu [15] and Yuan [26], we are going to study the boundedness of all solutions for the following equation

\[(\phi_p(x'))' + a\phi_p(x^+) - b\phi_p(x^-) + f(x) = e(t)\] \hspace{1cm} (1.11)

where \(f(x)\) is unbounded, which generalized the equation in [26] for \(p \geq 2\). Our main results are as follows:

**Theorem 1** Assume

(H1) \(f \in C^6(\mathbb{R} \setminus \{0\}) \cap C^0(\mathbb{R})\), and there are constants \(c\) and \(0 < \gamma < \frac{1}{p-1}\) such that

\[|x^k f^{(k)}(x)| \leq c|x|^\gamma\]

for all \(x \in \mathbb{R} \setminus \{0\}\), where \(0 \leq k \leq 6\);

(H2) There are two constants \(\beta_1, \beta_2 > 0\), and \(\beta_1, \beta_2\) satisfy that \(p\beta_1 > q\beta_2 > 0\), where \(\frac{1}{p} + \frac{1}{q} = 1\) such that for all \(x \in \mathbb{R} \setminus \{0\}\), we have

\[xf(x) \geq \beta_1 |x|^\gamma + 1, x^2 f'(x) \leq \beta_2 |x|^\gamma + 1;\]
(H3) \( e(t) \in C^6(S_p) \), where \( S_p \triangleq \mathbb{R}/2\pi p \mathbb{Z} \).

Then there exists \( \delta_0 > 0 \) such that for a given \( 0 < \rho < \frac{1}{2} \) and any \( 0 < \delta < \delta_0 \) and every irrational number \( \omega = \omega(\delta) \) satisfying

\[
1 + \rho \leq \frac{\omega - \omega_0}{\delta} \leq 2 - \rho,
\]

where \( \omega_0 = -\pi_p\left(\frac{1}{a^p} + \frac{1}{b^p}\right) \), and

\[
|m\omega - 2\pi_p n| \geq \rho \delta |m|^{-\frac{3}{2}}
\]

for all integers \( n \) and \( m \neq 0 \), the time \( 2\pi_p \) mapping \( P: (x, x')_{t=0} \to (x, x')_{t=2\pi_p} \) of the flow of Eq. (1.11) possesses an invariant curve \( \Lambda_\delta \) with rotation number \( \omega = \omega(\delta) \) and the curve surrounds the origin \( (x, x') = (0, 0) \) and goes arbitrarily far from the origin as \( \delta \to 0 \). Moreover, the curve is an intersection of the invariant torus in the \( (x, x', t(mod 2\pi_p)) \)-space with the \( t = 0 \)-plane and any motion starting from the torus is quasiperiodic, of basic frequencies \( 2\pi_p \) and \( \omega \).

**Corollary 1.1** The equation (1.11) possesses Lagrange stability, i.e. if \( x(t) \) is any solution of equation (1.11), then it exists for all \( t \in \mathbb{R} \) and \( \sup_{t \in \mathbb{R}}(|x(t)| + |x'(t)|) < \infty \).

**Corollary 1.2** Most motions with large amplitude are quasiperiodic, i.e. most initial conditions (in the sense of Lebesgue measure) with large \( |x(0)| + |x'(0)| \) give rise to quasiperiodic motions: \( x(t) = f(2\pi_p t, \omega t) \), where \( f \) is a function on a 2-torus.

**Remark 1.1** For the equation in (1.11), We believe that for the following two cases:(i) \( \omega \in \mathbb{Q} \);(ii) \( 1 < p < 2 \) similar results still hold true. We will study those problems in our future work.

The main idea is as follows: By means of transformation theory the original system outside of a large disc \( D = \{(x, x') \in \mathbb{R}^2 : x^2 + x'^2 \leq r^2\} \) in \((x, x')\)-plane is transformed into a perturbation of an integrable Hamiltonian system. The Poincaré map of the transformed system is closed to a so-called twist map in \( \mathbb{R}^2 \setminus D \). Then Moser’s twist theorem guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the \((x, x')\)-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space \((x, x', t) \in \mathbb{R}^2 \times \mathbb{R}\), which confines the solutions in the interior and which leads to a bound of these solutions.

The remain part of this paper is organized as follows. In section 2, we introduce action-angle variables and exchange the role of time and angle variables. And we construct canonical transformations such that the new Hamiltonian system is closed to an integrable one. In section
3, we give some estimates. In section 4, we will prove the Theorem 1 and the Corollaries by Moser’s twist theorem.
Throughout this paper, \( F(x) = \int_0^x f(s)ds, F(0) = 0, \) and \( C \) are some positive constants without concerning their quantity.

## 2 Action-Angle Variables

In this section, we will introduce action-angle variables \((r, \theta)\) via symplectic transformations.
We introduce a new variables \( y \) as \( y = -\varphi_p(\omega^{-1}x) \), let \( q \) be the conjugate exponent of \( p : p^{-1} + q^{-1} = 1 \). Then (1.11) is changed into the form

\[
x' = -\omega \varphi_q(y), \quad y' = \omega [a_1 \varphi_p(x^+) - b_1 \varphi_p(x^-)] + \omega^{1-p}[f(x) - e(t)]
\]

where \( a = \omega^p a_1, b = \omega^p b_1 \) and \( a_1, b_1 \) satisfy

\[
a_1^{-\frac{1}{p}} + b_1^{-\frac{1}{p}} = 2,
\]

which is a planar non-autonomous Hamiltonian system

\[
x' = -\frac{\partial H}{\partial y}(x, y, t), \quad y' = \frac{\partial H}{\partial y}(x, y, t)
\]

where

\[
H(x, y, t) = \frac{\omega}{q} |y|^q + \frac{\omega}{p} (a_1 |x^+|^p + b_1 |x^-|^p) + \omega^{1-p}[F(x) - e(t)x].
\]

Let \( C(t) = \sin_p t \) be the solution of the following initial value problem

\[
(\varphi_p(C'(t)))' + \varphi_p(C(t)) = 0, \quad C(0) = 0, C'(0) = 1.
\]

Then it follows from [16] that \( C(t) = \sin_p t \) is a 2\( \pi_p \)-period \( C^2 \) odd function with \( \sin_p(\pi_p - t) = -\sin_p(t) \), for \( t \in [0, \frac{\pi_p}{2}] \) and \( \sin_p(2\pi_p - t) = -\sin_p(t) \), for \( t \in [\pi_p, 2\pi_p] \). Moreover for \( t \in (0, \frac{\pi_p}{2}) \), \( C(t) > 0, C'(t) > 0, \) and \( C : [0, \frac{\pi_p}{2}] \to [0, (p-1)^{\frac{1}{p}}] \) can be implicitly given by

\[
\int_0^{\sin_p t} \frac{ds}{(1 - \frac{s^p}{p-1})^{\frac{1}{p}}} = t.
\]

**Lemma 2.1** For \( p \geq 2 \) and for any \((x_0, y_0) \in \mathbb{R}^2, t_0 \in \mathbb{R}, \) the solution

\[
z(t) = (x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))
\]

of (2.1) satisfying the initial condition \( z(t_0) = (x_0, y_0) \) is unique and exists on the whole \( t \)-axis.
The proof of uniqueness can be obtained similarly as the proof of Proposition 2 in [17], the global existence result can be proved similarly as Lemma 3.1 in [10]. Consider an auxiliary equation

\[(\phi_p(x'))' + a_1\phi_p(x^+) - b_1\phi_p(x^-) = 0\]

Let \(v(t)\) be the solution with initial condition: \((v(0), v'(0)) = ((p - 1)^{\frac{1}{p}}, 0)\). Setting \(\phi_p(v') = u\), then \((v, u)\) is a solution of the following planar system:

\[x' = \phi_q(y), \quad y' = -a_1\phi_p(x^+) + b_1\phi_p(x^-)\]

where \(q = p/(p - 1) > 1\). It is not difficult to prove that:

(i) \(q^{-1}|u|^q + p^{-1}(a_1|v|^p + b_1|v|^{-p}) \equiv \frac{a_1}{q}\);

(ii) \(v(t)\) and \(u(t)\) are \(2\pi p\)-periodic functions.

(iii) \(v(t)\) can be given by

\[
v(t) = \begin{cases} 
\sin_p(a_1^{\frac{1}{p}}t + \frac{\pi}{2}), & 0 \leq t \leq \frac{\pi}{2a_1^{\frac{1}{p}}}, \\
-(\frac{a_1}{b_1})^{\frac{1}{p}} \sin_p b_1^{\frac{1}{p}}(t - \frac{\pi}{2a_1^{\frac{1}{p}}}), & \frac{\pi}{2a_1^{\frac{1}{p}}} < t \leq \pi.
\end{cases} \quad (2.5)
\]

\(v(2\pi_p - t) = v(t), t \in [\pi_p, 2\pi_p]. \quad (2.6)\)

**Lemma 2.2** Let \(I_p = \int_0^{\frac{\pi}{2}} \sin_p t \, dt\). Then

\[I_p = \frac{(p - 1)^{\frac{2}{p}}}{p} B\left(\frac{2}{p}, 1 - \frac{1}{p}\right),\]

where \(B(r, s) = \int_0^1 r^{r-1}(1 - t)^{s-1} \, dt\) for \(r > 0, s > 0\).

From the expression of \(v(t)\) in (2.5), we obtain

\[
\int_0^{\frac{\pi}{2a_1^{\frac{1}{p}}}} v(t) \, dt = \frac{I_p}{a_1^{\frac{1}{p}}} ; \quad (2.7)
\]

\[
\int_{\frac{\pi}{2a_1^{\frac{1}{p}}}}^{\frac{\pi}{2a_1^{\frac{1}{p}}}} v(t) \, dt = -\frac{a_1^{\frac{1}{p}} I_p}{b_1^{\frac{1}{p}}} ; \quad (2.8)
\]

This method has been used in [27].

We introduce the action and angle variables via the solution \((v(t), u(t))\) as follows.

\[x = d^{\frac{1}{p}} \tau^r v(\theta), \quad y = d^{\frac{1}{q}} \tau^s u(\theta) \quad (2.9)\]
where \( d = qa_1^{-1} \). This transformation is called a generalized symplectic transformation as its Jacobian is 1. Under this transformation, the system (2.1) is changed to

\[
\theta' = \frac{\partial h}{\partial r}(r, \theta, t), r' = -\frac{\partial h}{\partial \theta}(r, \theta, t)
\]  

(2.10)

with the Hamiltonian function

\[
h(r, \theta, t) = \omega r + \omega\frac{1}{1} r^{\frac{1}{p}} v(\theta) - \omega_1 - \frac{1}{p} d_1\frac{1}{r^{\frac{1}{p}}} v(\theta)e(t)
\]  

(2.11)

Observing the facts \( F(\cdot) \in C^7(\mathbb{R}\setminus\{0\}) \cap C^1(\mathbb{R}) \) by (H1) and \( c(t) \in C^6(S_p) \) by (A3) and \( v(\theta) \in C^1(S_p) \), we have that \( h(r, \theta, t) \in C^{1,6}(\mathbb{R} \times S_p \times S_p) \). Let \( \Xi = \{ \theta \in S_p : v(\theta) = 0 \} \), clearly, the Lebesgue measure of the set \( \Xi \) vanishes. From the above, we have that \( h(r, \theta, t) \) is of class \( C^6 \) in \( r \) when \( \theta \in S_p \setminus \Xi \) and \( t \in S_p \) are regarded as parameters. Note that

\[
rd\theta - hdt = -(hdt - rd\theta)
\]

This means that if we can solve \( r = r(h, t, \theta) \) from (2.11) as a function of \( h, t \) and \( \theta \), then \( r = r(h, t, \theta) \) is the Hamiltonian function of the following system:

\[
\frac{dh}{d\theta} = -\frac{\partial r}{\partial t}(h, t, \theta), \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h, t, \theta)
\]  

(2.12)

i.e. (2.12) is a Hamiltonian system with \( r = r(h, t, \theta) \) as Hamiltonian function. Now the new action, angle and time variables are \( h, t, \theta \). This method have been used in Levi [13].

3 Some Estimates

In this section, we will give some estimates which will be used in the proof of Theorem. Throughout this section, we assume the hypotheses of Theorem 1 hold.

Lemma 3.1 For \( r \) large enough and \( t \in S_p \), it holds that:

\[
\left| \frac{\partial^k}{\partial r^k} F(d^{\frac{1}{1}} r^{\frac{1}{p}} v(\theta)) \right| \leq c \cdot r^{-k + \frac{\gamma + 1}{p}}, \quad 0 \leq k \leq 6
\]  

(3.1)

\[
\left| \frac{\partial^k}{\partial r^k} f(d^{\frac{1}{1}} r^{\frac{1}{p}} v(\theta)) \right| \leq c \cdot r^{-k + \frac{\gamma + 1}{p}}, \quad 0 \leq k \leq 6
\]  

(3.2)

where \( \theta \in S_p \) if \( k = 1 \) or \( \theta \in S_p \setminus \Xi \) if \( k \geq 2 \).

Proof. Using the assumption (H1) and letting \( x = d^{\frac{1}{1}} r^{\frac{1}{p}} v(\theta) \), we have

\[
|F(d^{\frac{1}{1}} r^{\frac{1}{p}} v(\theta))| = \int_0^x f(s) ds \leq \int_0^{|x|} |f(s)| ds \leq c|x|^{\gamma + 1} \leq cr^{\frac{\gamma + 1}{p}}
\]
This means that (3.1) holds for \( k = 0 \). Observe that
\[
\frac{\partial^k}{\partial r^k} F\left(\frac{1}{r} \frac{1}{p} v(\theta)\right) = \sum_{s=1}^{k} c_s F^{(s)}\left(\frac{1}{r} \frac{1}{p} v(\theta)\right) \partial_r^{j_1} \left(\frac{1}{r} \frac{1}{p} v(\theta)\right) \cdots \partial_r^{j_s} \left(\frac{1}{r} \frac{1}{p} v(\theta)\right)
\]
where \( 1 \leq j_1, \cdots, j_s \leq k \), \( j_1 + \cdots + j_s = k \). Hence the terms are bounded by
\[
c |F^{(s)}\left(\frac{1}{r} \frac{1}{p} v(\theta)\right)||\partial_r^{j_1} \left(\frac{1}{r} \frac{1}{p} v(\theta)\right)|| \cdots ||\partial_r^{j_s} \left(\frac{1}{r} \frac{1}{p} v(\theta)\right)||
\leq c |f^{(s-1)}\left(\frac{1}{r} \frac{1}{p} v(\theta)\right)| |\frac{1}{r} \frac{1}{p} v(\theta)|^{s-1} |\frac{1}{r} \frac{1}{p} v(\theta)|^{-j_1-\cdots-j_s}
\leq c |\frac{1}{r} \frac{1}{p} v(\theta)|^{\gamma} |\frac{1}{r} \frac{1}{p} v(\theta)|^{r-\gamma k} \leq c r^{-k+\frac{2+1}{p}},
\]
where \( \theta \in \mathbb{S}_p \) if \( k = 1 \) or \( \theta \in \mathbb{S}_p \setminus \Xi \) if \( k \geq 2 \). This ends the proof of (3.1). The proof of (3.2) is similar to the one above.

**Lemma 3.2** Let
\[
h_1(r, \theta, t) = \omega^{1-p} F\left(\frac{1}{r} \frac{1}{p} v(\theta)\right) - \omega^{1-p} \frac{1}{r} \frac{1}{p} v(\theta) e(t).
\]
Then, for \( r \) large enough and \( t \in \mathbb{S}_p \), we have
\[
|\partial_r^k \partial_t^l h_1(r, \theta, t)| \leq c r^{-k+l+\frac{2+1}{p}}, \quad 0 \leq k \leq 6,
\]
where \( \theta \in \mathbb{S}_p \) if \( k = 1 \) or \( \theta \in \mathbb{S}_p \setminus \Xi \) if \( k \geq 2 \).

**Proof.** The proof is finished by using Lemma 3.1 and (A3).

It follows from (2.11) and (3.3) that
\[
h(r, \theta, t) = \omega t + h_1(r, \theta, t).
\]
Using Lemma 3.2 and noting \( 0 < \gamma < \frac{1}{p-1} \leq 1 \), we have that, for \( r \) large enough and \( t \in \mathbb{S}_p \), the following inequalities hold:
\[
0 \leq c_1 r \leq h(r, \theta, t) \leq c_2 r, \quad \theta \in \mathbb{S}_p,
\]
\[
|\partial_r h(r, \theta, t)| \geq \frac{1}{2} \omega > 0, \quad \theta \in \mathbb{S}_p,
\]
\[
|\partial_r^k \partial_t^l h(r, \theta, t)| \leq c r^{-k+l+1} \leq c r^{-k} h(r, \theta, t), \quad 0 \leq k + l \leq 6, \quad \theta \in \mathbb{S}_p \setminus \Xi.
\]
Using the implicit theorem and (3.7) we can, indeed, solve (2.11) for \( r = r(h, t, \theta) \) as a function of \( h, t \) and \( \theta \).
In the following, we will give Lemma 3.3, which is Lemma A1.1 in M. Levi [13], for convenience, here we just give the Lemma and omit the proof.

Lemma 3.3 If a real function \( f \) of two real variables \( x, t \) (\( t \) viewed as a parameter) satisfies for some \( c > 0 \) and \( n \in \mathbb{N} \):
\[
|\partial_x^k \partial_t^i f(x, t)| \leq cx^{-k} f(x, t)
\]
for all \( x > 0 \) large enough and for all \( k, i : k + i \leq N \) and if, moreover,
\[
\partial_x f(x, t) \geq cx^{-1} f(x, t) > 0
\]
for all \( x > 0 \) large enough, then the inverse function \( g(y, t) \) of \( f \) in \( x \) satisfies
\[
|\partial_y^k \partial_t^i g(y, t)| \leq cy^{-k} g(y, t),
\]
for all \( k + i \leq N \) and for all \( y \) large enough.

Lemma 3.4 For \( h \) large enough and \( t \in S_p \), it holds that:
\[
|\partial_h^k \partial_t^l r(h, t, \theta)| \leq ch^{-k} r(h, t, \theta) \leq ch^{-k+1}, \quad 0 \leq k \leq 6, \quad \theta \in S_p \setminus \Xi.
\]
(3.9)

Proof. Regard \( \theta \) as a parameter. The proof is finished by using Lemma 3.3 together with (3.6) to (3.8).

Let
\[
g(r, \theta, t) := r^{-\frac{1}{p}} h_1(r, \theta, t) = r^{-\frac{1}{p}} \omega^{1-p} [F(d^{\frac{1}{p}} r^{\frac{1}{p}} v(\theta)) - d^{\frac{1}{p}} r^{\frac{1}{p}} v(\theta)e(t)].
\]
(3.10)
By Lemma 3.2 we have that for \( r \gg 1 \) and \( \theta, t \in S_p \),
\[
|\partial_r^k \partial_t^l g(r, \theta, t)| \leq cr^{-k+s}, \quad 0 \leq k + l \leq 6.
\]
(3.11)

Now (3.5) can be rewritten in the form of
\[
h = \omega r + r^{\frac{1}{p}} g(r, \theta, t), \quad \text{where} \quad r = r(h, t, \theta).
\]
(3.12)
Hence, by Taylor’s formula,
\[
g(r, \theta, t) = g(\omega^{-1} h - \omega^{-1} r^{\frac{1}{p}} g(r, \theta, t), \theta, t)
\]
\[
= g(\omega^{-1} h, \theta, t) + \int_0^1 g_r(\omega^{-1} h - s \omega^{-1} r^{\frac{1}{p}} g(r, \theta, t), \theta, t) \omega^{-1} r^{\frac{1}{p}} g(r, \theta, t) ds
\]
\[
= g(\omega^{-1} h, \theta, t) + R_0(h, t, \theta)
\]
\[
= (\omega^{-1} h)^{-\frac{1}{p}} \omega^{1-p} [F(d^{\frac{1}{p}} (\omega^{-1} h)^{\frac{1}{p}} v(\omega^{-1} \theta)) - d^{\frac{1}{p}} (\omega^{-1} h)^{\frac{1}{p}} v(\omega^{-1} \theta)e(t)] + R_0(h, t, \theta)
\]
(3.13)
where
\[ R_0(h, t, \theta) = -\int_0^1 g_r'((\omega^{-1}h - s\omega^{-1}r^\frac{1}{p}g(r, \theta, t), \theta, t)\omega^{-1}r^\frac{1}{p}g(r, \theta, t))ds \]  \tag{3.14}

By (3.12) we have
\[ r = \omega^{-1}h - \omega^{-1}r^\frac{1}{p}g(r, \theta, t) \]
\[ = \omega^{-1}h - \omega^{-1}(\omega^{-1}h - \omega^{-1}r^\frac{1}{p}g(r, \theta, t))\frac{1}{p}g \]
\[ = \omega^{-1}h - \omega^{-1}(\omega^{-1}h)\frac{1}{p}(1 - h^{-1}r^\frac{1}{p}g(r, \theta, t))\frac{1}{p}g \]
\[ = \omega^{-1}h - \omega^{-1}(\omega^{-1}h)\frac{1}{p}g + \frac{1}{p}\omega^{-1}(\omega^{-1}h)\frac{1}{p}\int_0^1 (1 - sh^{-1}r^\frac{1}{p}g(r, \theta, t))\frac{1}{p}h^{-1}r^\frac{1}{p}gds \]  \tag{3.15}

where we have expressed \((1 - h^{-1}r^\frac{1}{p}g(r, \theta, t))\frac{1}{p}\) by Taylor’s formula. Putting (3.13) into the second term in the last line of (3.15) we have
\[ r(h, t, \theta) = \omega^{-1}h - \omega^{-p}F(d^\frac{1}{p}(\omega^{-1}h)\frac{1}{p}v(\omega^{-1}h)) + R_1 + R_2 + R_3, \]  \tag{3.16}

where
\[ R_1 = \omega^{-(2+\gamma)}h^\frac{1}{p}\int_0^1 g_r'(\omega^{-1}h - s\omega^{-1}r^\frac{1}{p}g(r, \theta, t), \theta, t)r^\frac{1}{p}g(r, \theta, t))ds \]  \tag{3.17}
\[ R_2 = \frac{1}{p}\omega^{-(1+\gamma)}h^\frac{1}{p}-1\int_0^1 (1 - sh^{-1}r^\frac{1}{p}g(r, \theta, t))\frac{1}{p}h^{-1}r^\frac{1}{p}g^2ds \]  \tag{3.18}
\[ R_3 = d^\frac{1}{p}\omega^{-(p+\gamma)}h^\frac{1}{p}v(\theta)e(t) \]  \tag{3.19}

where \(r = r(h, t, \theta), g = g(r(h, t, \theta), \theta, t)\).

**Lemma 3.5** For \(h \gg 1\), \(t \in S_p, \theta \in S_p \setminus \Xi\), we have
\[ |\partial^k_h \partial^l_t R_1(h, t, \theta)| \leq ch^{-k+\gamma}, \quad 0 \leq k + l \leq 5, \]  \tag{3.20}
\[ |\partial^k_h \partial^l_t R_2(h, t, \theta)| \leq ch^{-k+\gamma}, \quad 0 \leq k + l \leq 5, \]  \tag{3.21}
\[ |\partial^k_h \partial^l_t R_3(h, t, \theta)| \leq ch^{-k+\gamma}, \quad 0 \leq k + l \leq 5. \]  \tag{3.22}

**Proof.** By (3.9) and (3.6), it is easy to verify that for \(s \in [0, 1], h \gg 1, t \in S_p,\) and \(\theta \in S_p \setminus \Xi,\)
\[ |\partial^k_h \partial^l_t r^\frac{1}{p}(h, t, \theta)| \leq ch^{-k+\frac{1}{p}}, \quad 0 \leq k + l \leq 6 \]  \tag{3.23}

Noting (3.11) and (3.23) we have that for any \(s \in [0, 1],\)
\[ |\partial^k_h \partial^l_t (s\omega^{-1}r^\frac{1}{p}g)| \leq ch^{-k+\frac{1+k}{p}}, \quad 0 \leq k + l \leq 6 \]  \tag{3.24}
Applying $\partial^k_h \partial^l_t$ to $g_r'(\omega^{-1} h - s\omega^{-1} r^p g(r, \theta, t), \theta, t)$ with $r = r(h, t, \theta)$ and using (3.11), (3.24) together with $0 < \gamma < \frac{1}{p-1} \leq 1$, we have

$$|\partial^k_h \partial^l_t g_r'(\omega^{-1} h - s\omega^{-1} r^p g(r, \theta, t), \theta, t)| \leq c h^{-k-\frac{1}{p} \gamma}, \quad 0 \leq k + l \leq 5. \tag{3.25}$$

By applying $\partial^k_h \partial^l_t$ to $gh^{\frac{1}{p-1}}$ and noting (3.23) and (3.11), we obtain

$$|\partial^k_h \partial^l_t (gh^{\frac{1}{p-1}})| \leq c h^{-k+\frac{2+\gamma}{p}}, \quad 0 \leq k + l \leq 6. \tag{3.26}$$

By combining (3.17), (3.25), (3.26) and $p \geq 2$, the proof of (3.20) is completed. The proof of (3.21) is similarly completed by using (3.9) and (3.11). The proof of (3.22) is obvious.

**Lemma 3.6** Let

$$\bar{F}(h) = \int_0^{2\pi \rho} F(d^\frac{1}{p} (\omega^{-1} h)^\frac{1}{p} v(\theta)) d\theta. \tag{3.27}$$

For $h \gg 1$, we have the estimates:

$$|\bar{F}^{(k)}(h)| \leq c h^{-k+\frac{2+\gamma}{p}}, \quad 0 \leq k \leq 6, \tag{3.28}$$

$$\bar{F}'(h) \geq c h^{-1+\frac{2+\gamma}{p}}, \tag{3.29}$$

$$\bar{F}''(h) \leq -c h^{-2+\frac{2+\gamma}{p}}. \tag{3.30}$$

**Proof.** It follows from (3.1) that (3.28) holds true. For simplicity we write $x = d^\frac{1}{p} (\omega^{-1} h)^\frac{1}{p} v(\theta)$. Using (3.27) and the fact that the set $\Xi \cap [0, 2\pi \rho]$ is finite, we have

$$\bar{F}'(h) = \frac{1}{ph} \int_{[0,2\pi \rho] \setminus \Xi} xf(x) d\theta. \tag{3.31}$$

In view of assumption (H2), we have

$$\bar{F}'(h) = \frac{1}{ph} \int_{[0,2\pi \rho] \setminus \Xi} xf(x) d\theta \geq \frac{\beta_1}{ph} \int_{[0,2\pi \rho] \setminus \Xi} |x|^\gamma+1 d\theta = c h^{-1+\frac{2+\gamma}{p}}. \tag{3.32}$$

This ends the proof of (3.29).

Differentiating (3.31) with respect to $h$ we have

$$\bar{F}''(h) = \frac{1}{p^2 h^2} \int_{[0,2\pi \rho] \setminus \Xi} x^2 f(x) d\theta - \frac{1}{qh} \bar{F}'(h).$$
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\[ \leq \frac{\beta_2}{p^2 h^2} \int_{[0,2\pi_p] \setminus \Xi} |x|^{\gamma+1} d\theta - \frac{1}{q h} \bar{F}'(h) \]

\[ \leq \frac{\beta_1 \beta_2}{p h \beta_1} F'(h) - \frac{1}{q h} \bar{F}'(h) \]

\[ \leq \left( \frac{\beta_2}{p} - \frac{1}{q} \right) \frac{1}{h} \bar{F}'(h) \]

\[ \leq \frac{1}{p} \left( \frac{\beta_2}{\beta_1} - \frac{p}{q} \right) ch^{-2 + \frac{1}{p} + \gamma} + \frac{1}{p}. \]

The proof of (3.30) is now completed by assumption (H2): \( \beta_1 > \beta_2 > 0, \ p\beta_1 > q\beta_2 > 0 \) and \( p \geq 2. \)

Let

\[ S(h, \theta) = \omega^{-p} \int_0^\theta (F(d^{1/p}(\omega^{-1}h)^{1/p} v(\theta))) - \bar{F}(h) d\theta. \]  

(3.33)

In view of Lemma 3.1 and 3.6, we have

\[ |\partial^{k+1} h S(h, \theta)| \leq ch^{-k-1 + \frac{p+1}{p}}. \]  

(3.34)

Define a map \( \Psi_1 : (h, t) \to (h, \tau) \) by formula:

\[ \Psi_1 : \quad h = h, \quad t = \tau + \frac{\partial S}{\partial h}, \]

where the time variable \( \theta \) is regarded as a parameter. Clearly the map is of period \( 2\pi_p \) in \( \theta. \)

Since \( dh \wedge dt = dh \wedge d\tau, \) the map is symplectic. The Hamiltonian function \( r = r(h, t, \theta) \) defined in (3.16) is transformed into a new Hamiltonian function

\[ \hat{r} = \omega^{-1} h - \omega^{-p} F(d^{1/p}(\omega^{-1}h)^{1/p} v(\theta)) + R_1 + R_2 + R_3 + \frac{\partial S}{\partial \theta}, \]

i.e.

\[ \hat{r} = r(h, \tau + \partial_h S(h, \theta), \theta) = \omega^{-1} h - \omega^{-p} \bar{F}(h) + R(h, \tau, \theta), \]  

(3.35)

where

\[ R(h, \tau, \theta) := (R_1 + R_2 + R_3)(h, \tau + \partial_h S(h, \theta), \theta). \]  

(3.36)

Applying \( \partial^k_h \partial^l_\tau \) to (3.36) and using Lemma 3.4 together with (3.34), we have that for \( h \gg 1, \ \tau \in S_p, \ \theta \in S_p \setminus \Xi, \)

\[ |\partial^k_h \partial^l_\tau R(h, \tau, \theta)| \leq ch^{-k + \max \{\gamma, \frac{1}{p}\}}, \quad 0 \leq k + l \leq 5. \]  

(3.37)

This method have been used in [13].
4 The proof of theorem

For given $0 < \delta < 1$, define a map $\Psi_2 : (h, \tau) \to (\lambda, \tau)$ by $\Psi_2 : \delta \lambda = \omega^{-p} \tilde{F}'(h)$, $\tau = \tau$, $1 \leq \lambda \leq 4$. This trick was used in [13]. Observe that small $\delta$ corresponds to large $h$ since $\tilde{F}'(h) \to 0$ as $h \to +\infty$ by (3.28) with $k = 1$. It follows from (3.30) that $\Psi_2$ is a diffeomorphism. The equations corresponding to $\hat{\gamma}(h, \tau, -\theta)$ are transformed into

$$\frac{d\lambda}{d\theta} = l_1(\lambda, \tau, \theta, \delta), \quad \frac{d\tau}{d\theta} = -\omega^{-1} + \delta \lambda + l_2(\lambda, \tau, \theta, \delta),$$

(4.1)

where

$$l_1(\lambda, \tau, \theta, \delta) = \omega^{-p} \delta^{-1} \tilde{F}''(h) \partial_{R}(h, \tau, -\theta),$$

(4.2)

$$l_2(\lambda, \tau, \theta, \delta) = -\partial_{h}R(h, \tau, -\theta),$$

(4.3)

with $h = h(\delta \lambda)$ being defined by $\delta \lambda = \omega^{-p} \tilde{F}'(h)$.

**Lemma 4.1** For $h(\delta \lambda)$ we have

$$c_1 \delta^{-1 - \frac{2+1}{p}} \leq h(\delta \lambda) \leq c_2 \delta^{-1 - \frac{2+1}{p}},$$

(4.4)

$$|\partial_{\lambda}^k h(\delta \lambda)| \leq c h(\delta \lambda), \quad 0 \leq k \leq 4.$$  

(4.5)

**Proof.** The abbreviation $h(\delta \lambda) = h$ is used in what follows. By (3.28) and (3.29) we have

$$c_3 h^{-1+\frac{2+1}{p}} \leq \tilde{F}'(h) \leq c_4 h^{-1+\frac{2+1}{p}}.$$  

(4.6)

Putting $h(\delta \lambda)$ into (4.6) and observing $\tilde{F}'(h) = \omega^p \delta \lambda$, we have

$$c_3 h^{-1+\frac{2+1}{p}} \leq \omega^p \delta \lambda \leq c_4 h^{-1+\frac{2+1}{p}},$$

by direct computing we have (4.4). In the following, we give the proof of (4.5), which is similar to the reference [13]. For convenience, here we just give a brief proof.

Differentiating $\tilde{F}'(h) = \omega^p \delta \lambda$ with respect to $\lambda$ we have

$$\partial_{\lambda} h \cdot \tilde{F}''(h) = \omega^p \delta, \quad (\partial_{\lambda} h)^2 \cdot \tilde{F}''(h) + \partial_{\lambda}^2 h \cdot \tilde{F}''(h) = 0 \quad (\partial_{\lambda} h)^3 \cdot \tilde{F}'''(h) + 3 \partial_{\lambda} h \cdot \partial_{\lambda}^2 h \cdot \tilde{F}'''(h) + \partial_{\lambda}^3 h \cdot \tilde{F}''(h) = 0,$$

(4.7)

(4.8)

(4.9)

From (4.7) and (3.30) we have

$$|\partial_{\lambda} h| = \left| \frac{\omega^p \delta}{\tilde{F}''(h)} \right| = \left| \frac{\omega^p \delta h}{h \tilde{F}''(h)} \right| = c \left| \frac{\delta h}{\delta \lambda} \right| \leq c h,$$
By (4.8) and (3.30) we have
\[ |\partial^2_{\lambda}l| = \left| -\frac{F''(h)}{F''(h)}(\partial_\lambda h)^2 \right| \leq ch, \]
similarly we can get
\[ |\partial^k h| \leq ch. \]
Now we finish the proof of the Lemma.

**Lemma 4.2** For $0 < \delta \ll 1$, $\tau \in S_p$, $\theta \in \mathbb{R}$, $\lambda \in [1, 4]$, we have
\[ |\partial^k l_1(\lambda, \tau, \delta)|, \quad |\partial^k l_2(\lambda, \tau, \delta)| \leq c\delta^\sigma, \quad 0 \leq k + l \leq 4, \]  \hspace{1cm} (4.10)
where $\sigma = (-1 + \max\{\gamma, \frac{1}{p}\})\frac{p}{\gamma + 1 - p} > 1$.

**Proof.** Since $l_2(\lambda, \tau, \theta, \delta) = -\partial h R(h, \tau, -\theta)$, by the estimate (3.37) we have
\[ |\partial^1 l_2| = |\partial_\lambda \partial^1 R| \leq c\delta^{-1 + \max\{\gamma, \frac{1}{p}\}} \leq c\delta^{-1 + \max\{\gamma, \frac{1}{p}\}} = c\delta^\sigma. \]
This shows that (4.10) holds for $l_2$ and $k = 0$ and $0 \leq l \leq 4$. When $k > 0$ and $\theta \in S_p \setminus \Xi$, by application of $\partial^k \partial^l_\lambda$ to both sides of $l_2(\lambda, \tau, \theta, \delta) = -\partial h R(h, \tau, -\theta)$ we have
\[ \frac{\partial^{k+l}}{\partial^k \lambda \partial^l \tau} l_2 = \sum_{s=1}^{k} \partial^{s+1+l} R(h, \tau, \theta) \cdot (\frac{\partial}{\partial \lambda})^{j_1} h(\frac{\partial}{\partial \lambda})^{j_2} h \cdots (\frac{\partial}{\partial \lambda})^{j_s} h, \]
where $1 \leq j_1, j_2, \ldots, j_s \leq k$, $j_1 + j_2 + \cdots + j_s = k$. Consequently, using (3.37) and Lemma 4.1 we have
\[ \left| \frac{\partial^{k+l}}{\partial^k \lambda \partial^l \tau} l_2 \right| \leq \sum_{s=1}^{k} ch^{-s-1 + \max\{\gamma, \frac{1}{p}\}} \cdot h^s \leq ch^{-1 + \max\{\gamma, \frac{1}{p}\}} \leq c\delta^\sigma. \]
This completes the proof of (4.10) for $l_2$.

Using the same trick above and using Lemma 4.1 together with (3.28), (3.37) and (4.2), the remaining proof can be finished.

**Lemma 4.3** The solutions $(\lambda(\theta), \tau(\theta))$ of Eq.(4.1) with the initial conditions $\lambda(0) = \lambda_0 \in [2, 3]$, $\tau(0) = \tau_0$ do exist for $0 \leq \theta \leq 2\pi_p$ if the $\delta$ is sufficiently small. The Poncaré mapping of Eq.(4.1) is of the form
\[ \mathcal{P}^{2\pi_p} : \left\{ \begin{array}{l} \lambda(2\pi_p) = \lambda_0 + \delta L_1(\lambda_0, \tau_0, \delta), \\ \tau(2\pi_p) = \tau_0 - 2\pi_p\omega^{-1} + \delta(\lambda_0 + L_2(\lambda_0, \tau_0, \delta)), \end{array} \right. \]  \hspace{1cm} (4.11)
where $L_1$, $L_2$ satisfy
\[ |\partial^k l_1(\lambda_0, \tau_0, \delta)|, \quad |\partial^k l_2(\lambda_0, \tau_0, \delta)| \leq c\delta^{\sigma-1}, \quad 0 \leq k + l \leq 4. \]  \hspace{1cm} (4.12)
Proof. By integrating Eq. (4.1) from \( \theta = 0 \) to \( \theta = 2\pi_p \), and using Lemma 4.2 and the contraction principle, the proof can be easily finished. The argument is similar to the one of Lemma 4 in [4]. We omit the details.

Proof of Theorem 1 and the Corollaries. Since \( l_1, l_2 \) satisfy (4.10), it is easy to verify that the solutions \((\lambda(\theta), \tau(\theta))\) of Eq. (4.1) with the initial conditions \( \lambda(0) = \lambda_0 \in [2, 3], \tau(0) = \tau_0 \) do exist for \( 0 \leq \theta \leq 2\pi_p \) if the \( \delta \) is sufficiently small, that is the conditions of Lemma 4.3 come true. Hence the poincaré map in the form of (4.11) does exist, and the map has the intersection property in the domain \([2, 3] \times S_p\), i.e. if \( \Gamma \) is an embedded circle in \([2, 3] \times S_p\) homotopic to a circle \( \lambda = \text{const} \), then \( P(\Gamma) \cap \Gamma \neq \emptyset \). This is a well-known fact. See [4], for instance.

Until now we have verified that the mapping \( P \) satisfy all the conditions of Moser’s small twist theorem [20] in the domain \([2, 3] \times S_p\), if \( \delta \) is small enough. We come to the conclusion that for any \( 0 < \delta < \delta_0 \) with some constants \( \delta_0 \) small enough, the mapping \( P \) has an invariant curve \( \Upsilon_\delta \) in the annulus \([2, 3] \times S_p\) with rotation number \( \omega = \omega(\delta) \) satisfying (4.12) and (4.13).

Retracting the sequence of the transformations back to the original Eq. (1.11), we conclude that the time \( 2\pi_p \) mapping \( P: (x, x')_{t=0} \rightarrow (x, x')_{t=2\pi_p} \) of the flow of Eq. (1.11) possesses an invariant curve \( \Gamma_\delta \) which surrounds the origin in the \((x, y)\)-plane. Going back to (2.11) and using the fact that small \( \delta \) corresponds to large \( h \) and inequality (3.6) we know that \( r = r(h, t, \theta) \rightarrow +\infty \) as \( \delta \rightarrow 0 \). Returning to (2.9) and using the formula (i) in the second section, we have \( x^2 + y^2 \rightarrow \infty \) as \( \delta \rightarrow 0 \). Thus the invariant curve \( \Gamma_\delta \) goes arbitrarily far from the origin in the \((x, y)\)-plane when \( \delta \rightarrow 0 \). Since Eq. (1.11) is of period \( 2\pi_p \) in time \( t \), these curves \( \Lambda_\delta \) are the intersections of the invariant tori in the \((x, x', t(\mod 2\pi_p))\)-space with the \( t = 0 \)-plane and any motion stating from the torus is quasiperiodic of basic frequencies \((2\pi_p, \omega)\). This ends the proof of the Theorem 1. As there exist invariant curves of the poncaré mapping of the system (1.11), which surround the origin \((x, y) = (0, 0)\) and are arbitrarily far from the origin. According to the Moser’s small twist theorem, we can know that the system (1.11) possesses Lagrange stability. The statement of the first Corollary has been proved. And according to [13] and [4], we can get the second Corollary.

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