Numerical Simulation guided Lazy Abstraction Refinement for Nonlinear Hybrid Automata

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Abstract. This draft suggests a new counterexample guided abstraction refinement (CEGAR) framework that uses the combination of numerical simulation for nonlinear differential equations with linear programming for linear hybrid automata (LHA) to perform reachability analysis on nonlinear hybrid automata. A notion of $\epsilon$-structural robustness is also introduced which allows the algorithm to validate counterexamples using numerical simulations.

1 Introduction

The model checking of hybrid automata remains a challenge and the existing tools [6,4] do not scale up to the needs of the industry. Because of the well known fundamental undecidability results [5], the model checking of general hybrid automata often proceeds by building successive tighter approximations to these hybrid automata in a relatively easy-to-analyze fragment of hybrid automata like Linear Hybrid Automata [7]. Theoretical results about the asymptotic completeness of this approximation procedure form the backbone of such a strategy behind the model checking of nonlinear hybrid automata.

There has been considerable interest in applying Counterexample Guided Abstraction Refinement (CEGAR), which works so well with discrete systems, to the problem of hybrid system verification [2]. There has also been some exploration of using fragments instead of counterexamples during abstraction refinement [3] and the application of CEGAR specifically to LHA [8]. However, our ongoing work makes the following new contributions to the abstraction refinement based analysis of hybrid systems.

- We address the problem of abstraction refinement for nonlinear hybrid automata and use CEGAR to construct successively refined LHA approximations. Our refinement is lazy and hence, refines some parts of the state space more finely than others.
- We use the distance between a feasible path in the abstract linear hybrid automata and the numerically simulated trajectory in the nonlinear hybrid automata to refine those locations in the LHA that do not faithfully represent the behavior of the nonlinear hybrid automata.
We define a structural notion of robustness and use it to present a counterexample validation algorithm (for a rich class of nonlinear hybrid automata) using linear programming [9]. Hence, it is possible to detect reachability of a bad state even before the abstraction refinement loop terminates.

2 Background on LP based path feasibility analysis of LHA

Informally, a linear hybrid automaton is a conventional automaton extended with a set of continuous variables. The states of the automaton called *locations* are annotated with a change rate for each continuous variable such as $\dot{x} = [a, b]$ ($x$ is a variable, and $[a, b]$ is a rational interval), and the transitions of the automaton are labeled with constraints on the variables such as $a \leq \sum_{i=0}^{m} c_i x_i \leq b$ and/or with reset actions such as $x := c$ ($x$ and $x$ are variables, $a, b$, and $c_i$ are real numbers). Such linear hybrid automata are essentially equivalent to the definition given in [5]. It is known that this subclass of linear hybrid automata are sufficiently expressive to allow asymptotic completeness of the abstraction process for a general hybrid automata. “A restricted form of linear phase portrait approximations are asymptotically complete, namely, when all automaton constraints are over-approximated using independent, rational lower and upper bounds on the values and derivatives of each variable” [1]. For simplicity, we suppose that in any linear hybrid automaton considered in this paper, there is just one initial location with no initial conditions and no transitions to the initial location (we assume that each variable with an initial value is reset to the initial value by the transitions from the initial location).

**Definition 1.** A linear hybrid automaton is a tuple $H = (X, V, E, v_I, \alpha, \beta)$, where

- $X$ is a finite set of real-valued variables.
- $V$ is a finite set of locations.
- $E$ is transition relation whose elements are of the form $(v, \phi, \psi, v')$ where $v, v'$ are in $V$, $\phi$ is a set of guards or variable constraints of the form $a \leq \sum_{i=0}^{m} c_i x_i \leq b$, and $\psi$ is a set of reset actions of the form $x := c$ where $x_i \in X$ ($0 \leq i \leq m$), $x \in X$, $a, b$ and $c_i$ ($0 \leq i \leq m$) are real numbers, and $a$ and $b$ may be $\infty$.
- $v_I$ is an initial location.
- $\alpha$ is a labeling function which maps each location in $V - \{v_I\}$ to a state invariant which is a set of variable constraints of the form $a \leq \sum_{i=0}^{m} c_i x_i \leq b$ where $x_i \in X$ ($0 \leq i \leq m$), $y \in X$, $a, b$, and $c_i$ ($0 \leq i \leq m$) are real numbers, $a$ and $b$ may be $\infty$.
- $\beta$ is a labeling function which maps each location in $V - \{v_I\}$ to a set of change rates which are of the form $\dot{x} = [a, b]$ where $x \in X$, and $a, b$ are rational numbers ($a \leq b$). For any location $v$, for any $x \in X$, there is one and only one change rate definition $\dot{x} = [a, b] \in \beta(v)$.
For a linear hybrid automaton $H = (X, V, E, v_I, \alpha, \beta)$, a path segment is a sequence of locations

$$v_1 (\phi_1, \psi_1) \xrightarrow{} v_2 (\phi_2, \psi_2) \xrightarrow{} \cdots \xrightarrow{} v_n (\phi_{n-1}, \psi_{n-1})$$

which satisfies $(v_i, \phi_i, \psi_i, v_{i+1}) \in E$ for each $i$ ($1 \leq i \leq n - 1$). A path in $H$ is a path segment starting at $v_I$. The behavior of linear hybrid automata can be represented by timed sequences. Any timed sequence is of the form $(v_1, t_1) \xrightarrow{} (v_2, t_2) \xrightarrow{} \cdots \xrightarrow{} (v_n, t_n)$, where $v_i$ ($1 \leq i \leq n$) is a nonnegative real number, which represents a behavior of an automaton that the system starts at the initial location and changes to the location $v_1$, stays there for $t_1$ time units, then changes to the location $v_2$ and stays in $v_2$ for $t_2$ time units, and so on.

**Definition 2.** For a linear hybrid automaton $H = (X, V, E, v_I, \alpha, \beta)$, a timed sequence $(v_1, t_1) \xrightarrow{} (v_2, t_2) \xrightarrow{} \cdots \xrightarrow{} (v_n, t_n)$ represents a behavior of $H$ if the following condition is satisfied:

- there is a path in $H$ of the form
  
  $$v_0 (\phi_0, \psi_0) \xrightarrow{} v_1 (\phi_1, \psi_1) \xrightarrow{} \cdots \xrightarrow{} v_n$$

- $t_1, t_2, \ldots, t_n$ satisfy all the variable constraints in $\phi_i$ ($1 \leq i \leq n - 1$), i.e. for each variable constraint $a \leq c_0 x_0 + c_1 x_1 + \ldots + c_m x_m \leq b$ in $\phi_i$,
  
  $$\delta_k \leq \gamma_i(x_k) \leq \delta'_k$$
  
- $t_1, t_2, \ldots, t_m, \gamma_i(x_k), \lambda_i(x_k)$ satisfy the state invariant for each location $v_i$ ($1 \leq i \leq n$), where $\gamma_i(x_k)$ ($0 \leq k \leq m$) represents the value of the variable $x_k$ when the automaton stay at $v_i$ with the delay $t_i$; and, similarly,

- $t_1, t_2, \ldots, t_m, \gamma_i(x_k), \lambda_i(x_k)$ satisfy the state invariant for each location $v_i$ ($1 \leq i \leq n$), where $\gamma_i(x_k)$ ($0 \leq k \leq m$) represents the value of the variable $x_k$ when the automaton stay at $v_i$ with the delay $t_i$, and $\lambda_i(x_k)$ ($0 \leq k \leq m$) represents the value of the variable $x_k$ after leaving state $v_i$ and after the reset conditions have been applied.
Now, we use linear programming to test the feasibility of a single path for the reachability analysis of general hybrid automata. Let $H = (X, V, E, v_I, \alpha, \beta)$ be a linear hybrid automaton, and $\rho$ be a path in $H$ of the form

$$v_0 \xrightarrow{\phi_0, \psi_0} v_1 \xrightarrow{\phi_1, \psi_1} \ldots v_n$$

where $v_n = v$. For any timed sequence of the form $(v_1, t_1)^\prec (v_2, t_2)^\prec \ldots (v_n, t_n)$, if $\rho$ is feasible, then the following condition must hold:

- $t_1, t_2, \ldots, t_n$ satisfy all the variable constraints in $\phi_i (0 \leq i \leq n)$, and
- $t_1, t_2, \ldots, t_n$ satisfy all the variable constraints in $\alpha(v_i) (1 \leq i \leq n)$,

which form a group of linear inequalities on $t_1, t_2, \ldots, t_n, \gamma_i(x_k), \lambda_i(x_k)$ (see Definition 2), denoted by $\Theta(\rho)$ or $LP_\rho(t_1, \gamma_i(x_k), \lambda_i(x_k))$. It follows that we can check if $\rho$ is a feasible path by checking if the group $\Theta(\rho)$ (or $LP_\rho(t_1, \gamma_i(x_k), \lambda_i(x_k))$) of linear inequalities has a solution, which can be solved by linear programming [9].

## 3 Background on Abstraction of Affine dynamics by LHA

Given a general hybrid system $H = (X, V, E, v_I, \alpha, \beta)$ where $X$ is a finite set of real-valued variables, $V$ is a finite set of locations, $E$ is transition relation whose elements are of the form $(v, \phi, \psi, v')$ where $v, v'$ are in $V$, $\phi$ is a set of guards or variable constraints of the form $a \leq \sum_{i=0}^{m} c_i x_i \leq b$, and $\psi$ is a set of reset actions of the form $x := c$ where $x_i \in X \ (0 \leq i \leq m)$, $x \in X$, $a, b$ and $c_i (0 \leq i \leq m)$ are real numbers, and $a$ and $b$ may be $\infty$, $v_I$ is an initial location, $\alpha$ is a labeling function which maps each location in $V - \{v_I\}$ to a state invariant which is a set of variable constraints of the form $a \leq \sum_{i=0}^{m} c_i x_i \leq b$ where $x_i \in X \ (0 \leq i \leq m)$, $y \in X$, $a, b$, and $c_i (0 \leq i \leq m)$ are real numbers, $a$ and $b$ may be $\infty$, $\beta$ is a labeling function which maps each location in $V - \{v_I\}$ to a set of change rates which are of the form $x_i := f(x_0, x_1, \ldots, x_m, \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_m, a_0, a_1, \ldots a_m)$ where $x_0, x_1, \ldots x_m \in X$, and $a_0, a_1, \ldots, a_n$ are real numbers. For any location $v$, for any $x \in X$, there is one and only one change rate definition.

We construct (in a fashion similar to [44]) a linear hybrid automata $H_a$ which is an over-approximate abstraction of the general hybrid automaton $H$. We define an operator $\text{split}$ which divides a location into smaller locations and use this to divide each location $v_i$ of the original hybrid automaton.

**Definition 3.** Let $v$ be a location and $x_i \in X$ be a variable in a hybrid automata $H = (X, V, E, v_i, \alpha, \beta)$. Suppose $In_v = \{v_i^{in}, v_i^{in} \ldots v_n^{in}\}$ be the locations from which there is a transition into the location $v$ and $Out_v = \{v_i^{out}, v_i^{out} \ldots v_n^{out}\}$ be the locations to which there is a transition from $v$. Also, $C$ be a set of linear constraints on $X$. Then, the operator $\text{split} (H, v, C)$ constructs a new hybrid automata $H' = (X, V', E', v_I, \alpha', \beta')$, where

- $V' = V \cup \{v', v''\} \setminus v$
\[ \alpha' = \alpha \cup \{ \alpha(v'), \alpha(v'') \} \setminus \{ \alpha(v) \}, \text{ where} \]

- \[ \alpha(v') = \alpha(v) \cup \{ C \} \]
- \[ \alpha(v'') = \alpha(v) \cup \{ \neg C \} \]

\[ E' = E \setminus \left\{ \{(v_{in}, \phi_{v_{in},v}, \psi_{v_{in},v}, v)|v_{in} \in I_{v_{in}}\} \cup \{(v, \phi_{v,v_{out}}, \psi_{v,v_{out}}, v_{out})|v_{out} \in O_{v_{out}}\} \cup \{(v_{in}, \phi_{v_{in},v}, \psi_{v_{in},v}, v')|v_{in} \in I_{v_{in}}\} \cup \{(v', \phi_{v_{in},v}, \psi_{v_{in},v}, v_{out})|v_{out} \in O_{v_{out}}\} \cup \{(v'', \phi_{v_{in},v}, \psi_{v_{in},v}, v''')|v_{in} \in I_{v_{in}}\} \cup \{(v'', \phi_{v_{in},v}, \psi_{v_{in},v}, v_{out})|v_{out} \in O_{v_{out}}\} \cup \{(v', \alpha(v'), v'') = C, \emptyset \}, (v''', \alpha(v'''), v') = \neg C, \emptyset \}} \]

- \[ E'' = E \setminus \{ \{ v_{in}, \phi_{v_{in},v}, \psi_{v_{in},v}, v \}| v_{in} \in I_{v_{in}} \} \cup \{ (v', \phi_{v,v_{out}}, \psi_{v,v_{out}}, v_{out})| v_{out} \in O_{v_{out}} \} \cup \{ (v'', \phi_{v,v_{out}}, \psi_{v,v_{out}}, v_{out})| v_{out} \in O_{v_{out}} \} \cup \{ (v', \alpha(v'), v'') = C, \emptyset \}, (v''', \alpha(v'''), v') = \neg C, \emptyset \} \}

\[ \beta' = \beta \cup \{ \beta(v'), \beta(v'') \} \setminus \{ \beta(v) \}, \text{ where} \]

- \[ \beta(v') = \beta(v) \]
- \[ \beta(v'') = \beta(v) \]

We call the new locations \( v' \) and \( v'' \) as the children of \( v \). In particular, \( v' = \text{child}(v, C) \) and \( v'' = \text{child}(v, \neg C) \), where \( C \) is the set of linear constraints used to split \( v \). We also call \( v \) as the parent of \( v' \) and \( v'' \). Thus, the \textit{split} operator naturally defines a tree of locations that we call the \textit{location tree}, where the children location are formed by splitting the parent location.

**Definition 4.** \textit{LHA-approximation to a general hybrid automata:} Given a general hybrid automata \( H = (X, V, E_v, v_i, \alpha, \beta) \), \( H_a = (X, V_a, E_a, v_{i_a}, \alpha_a, \beta_a) \) is a LHA-approximation iff

- There exists a hybrid automata \( H' = (X, V', E', v_i', \alpha', \beta') \), where \( H' = \text{split}^n(H) \).
- \( V_a = V', E_a = E', v_{i_a} = V_{i_a}, \alpha' = \alpha_a \)
- \( \forall v \in V_a, \beta_a(v) \supset \beta'_v \) and if \( c \in \beta_a(v) \), then \( c \) is of the form \( x := [a, b] \), where \( a, b \in \mathbb{R} \).
4 Definitions

The linear hybrid automaton \( H_a \) is an over-approximate approximation of the general hybrid automata \( H \). A path \( \rho = \{v_0, v_1, \ldots, v_m\} \), where \( v_i \in V \) is said to exist in \( H_a \) if \( (v_i, v_{i+1}) \in E, 0 \leq i < m \).

Consider a path \( \rho = \{v_0, v_1, \ldots, v_m\} \) that exists in the abstract linear hybrid automata model \( H_a \) and let \( LP_\rho(t_i, \gamma_i(x_k), \lambda_i(x_k)) \) be the linear program corresponding to the path. If the linear program \( LP_\rho \) has a feasible solution, then the path is said to be feasible in the abstract model i.e. the linear hybrid automaton; otherwise it is said to be infeasible.

**Definition 5.** Trace: Given a feasible path \( \rho \) in the abstract linear hybrid automaton model \( H_a \), the feasible solution to the linear program \( LP_\rho(t_i, \gamma_i(x_k), \lambda_i(x_k)) \) is called a trace of \( H_a \).

We write \( \text{trace}(H_a) = \{v_0, \lambda_0(x_0), \lambda_0(x_1), \ldots, \lambda_0(x_n), \gamma_0(x_0), \gamma_0(x_1), \ldots, \gamma_0(x_n), t_0, (v_1, \lambda_1(x_0), \lambda_1(x_1), \ldots, \lambda_1(x_n), \gamma_1(x_0), \gamma_1(x_1), \ldots, \gamma_1(x_n), t_1), \ldots, (v_m, \lambda_m(x_0), \lambda_m(x_1), \ldots, \lambda_m(x_n), \gamma_m(x_0), \gamma_m(x_1), \ldots, \gamma_m(x_n), t_m)\} \).

It is known [9] that the trace obtained by the linear program is a real execution trace of the over-approximate linear hybrid automata.

**Definition 6.** Concretization of a path: Consider a path \( \rho = \{v_0, v_1, \ldots, v_m\} \) that is feasible in the abstract linear hybrid automata model \( H_a \). Then, the concretization of this path in the original hybrid automaton \( H \) is the trace \( \rho_{\text{concrete}} = \{v^\text{root}_0, v^\text{root}_1, \ldots, v^\text{root}_m\} \), where \( v^\text{root}_i \) is the root of the location tree in which \( v_i \) is a leaf.

As the split operator forms a tree of locations in the abstract linear hybrid automata, the root of the location tree is known and the concretization of an abstract path is well defined.

**Definition 7.** Concretization of a trace: Consider a trace \( tr = \{v_0, \lambda_0(x_0), \ldots, \lambda_0(x_n), \gamma_0(x_0), \gamma_0(x_1), \ldots, \gamma_0(x_n), t_0, (v_1, \lambda_1(x_0), \lambda_1(x_1), \ldots, \lambda_1(x_n), \gamma_1(x_0), \gamma_1(x_1), \ldots, \gamma_1(x_n), t_1), \ldots, (v_m, \lambda_m(x_0), \lambda_m(x_1), \ldots, \lambda_m(x_n), \gamma_m(x_0), \gamma_m(x_1), \ldots, \gamma_m(x_n), t_m)\} \) corresponding to the path \( \rho = \{v_0, v_1, \ldots, v_m\} \) that is feasible in the abstract linear hybrid automata model \( H_a \). Then, the concretization of this trace in the original hybrid automaton \( H \) is the trace \( tr_{\text{concrete}} = \{v^\text{root}_0, \lambda_0(x_0), \lambda_0(x_1), \ldots, \lambda_0(x_n), \gamma_0(x_0), \gamma_0(x_1), \ldots, \gamma_0(x_n), t_0, (v^\text{root}_1, \lambda_1(x_0), \lambda_1(x_1), \ldots, \lambda_1(x_n), \gamma_1(x_0), \gamma_1(x_1), \ldots, \gamma_1(x_n), t_1), \ldots, (v^\text{root}_m, \lambda_m(x_0), \lambda_m(x_1), \ldots, \lambda_m(x_n), \gamma_m(x_0), \gamma_m(x_1), \ldots, \gamma_m(x_n), t_m)\} \), where \( v^\text{root}_i \) is the root of the location tree of which \( v_i \) is a leaf.

**Definition 8.** \( \epsilon \)-Simulation Trajectory: Given a valuation of \( X \) i.e. \( Val_0(X) = (x_0 = x_0, x_1 = x_1, \ldots, x_n = x_n) \) in a location \( v \) of a general hybrid system \( H = (X, V, E, v, 0, 1) \), then \( \tau(x_0, x_1, \ldots, x_n, v, t) \) is said to be an \( \epsilon \)-simulation trajectory for location \( v \) with initial valuation \( Val_0(X) \) iff

\[-\tau(t = 0) = (x_0, x_1, \ldots, x_n)\]
if \( f(x_0, x_1, \ldots, x_n, t) \) is the solution to the initial value problem \((\beta(v), Val_0(X))\), then \( f(t) - \epsilon \leq \tau(t) \leq f(t) + \epsilon \)

It is known that numerical techniques can solve the initial value problem for ODEs (including non-linear ODEs) quite efficiently.

**Definition 9.** \( \epsilon \)-Hybrid Simulation Trajectory: Given an initial valuation of \( X \) i.e. \( Val_0(X) = (x_0 = x_0, x_1 = x_1, \ldots, x_n = x_n) \) and a path \( \rho = \{v_0, v_1 \ldots v_m\} \) in a general hybrid system \( H = (X, V, E, v_i, \alpha, \beta) \), then \( \tau(x_0, x_1, \ldots x_n) = f(t) \) is said to be an \( \epsilon \)-hybrid simulation trajectory iff

- \( \tau(0) = Val_0(X) \) and \( Val_0(X) \in \alpha(v_0) \)
- if \( x_k := e \in \psi(v_i, v_{i+1}) \) then \( \tau(x_k, \sum_0^{i}(t_i) + = e) \) else \( \tau(x_k, \sum_0^i(t_i)+) = \tau(x_k, \sum_0^i(t_i)) \)
- Before executing the jump \((v_i, v_{i+1})\), \( \tau(x_k, \sum_0^i(t_i)-) \) satisfies every precondition in \( \phi(v_i, v_{i+1}) \)
- Within each location \( v_i \) where the timed path has spent time \( t_i \),
  \[ \forall t, t_i < t < t_{i+1}, \tau(v_i, \sum_0^i(t_i) + t) \]
  is an \( \epsilon \)-simulation trajectory for location \( v_i \) with initial valuation \( Val_0 = \tau(\sum_0^i(t_i)) \).

**Definition 10.** Guided Simulation Trajectory of the concretization of a trace: An \( \epsilon \)-hybrid simulation trajectory \( \tau \) is said to be a Guided Simulation Trajectory of a concretized trace \( tr_{\text{concrete}} \) iff

- The initial valuation of \( X \) i.e. \( Val_0(X) = (x_0 = x_0, x_1 = x_1, \ldots, x_n = x_n) \) for the trajectory \( \tau \) is the initial point in the concretized trace \( tr_{\text{concrete}} \).
- The \( \epsilon \)-hybrid simulation trajectory \( \tau \) corresponds to the path \( \rho = \{v_0, v_1 \ldots v_m\} \) corresponding to \( tr_{\text{concrete}} \)

5 CEGAR based Refinement of the abstract linear hybrid automata

The CEGAR algorithm repeatedly constructs LHA over-approximations to the given (possibly nonlinear) hybrid system and then asks a LHA analysis engine if the over-approximate LHA admits any counterexample. If it does not, we are done and we report that the original hybrid system has no counterexample either. Otherwise, we take the reported counterexample of the over-approximate LHA and attempt to validate it using numerical simulation. If we succeed in validating the counterexample, we report an error that the bad state is reachable and STOP. Otherwise, we find a location where we need to split the nonlinear hybrid automata and then rebuild a more precise over-approximate abstraction.
Algorithm for CEGAR

(Input: Nonlinear Hybrid Automata A. Output: Error No error)

1. $A_0 = A$; $i := 0$; $L = \text{Universe}$.
2. $LHA_i = \text{LHA-approximation } (A_i)$
3. $L(LHA_i) := \text{Language of } LHA_i$. $L$ represents the set of potential counterexamples in $A_i$ (finitely expressible as a regular expression) [8].
4. $L = L \cap L(LHA_i)$
5. If $L$ is empty, report "BAD STATES NOT REACHABLE" and stop.
6. Pick a counterexample $ce$ in $L$.
7. Validate the counterexample $ce$ in the original hybrid automata $A$.
8. If $ce$ is validated in $A$, stop and report that ERROR STATE IS REACHABLE.
9. Compute a refinement operator $split$, and $A_{i+1} = split(A_i)$.
   Also, compute $L = split(L)$.
10. $i := i + 1$
11. Loop to Step 2.

5.1 Counterexample Validation and Structural Robustness

Let $v_0, \gamma_0(x_0), \gamma_0(x_1) \ldots \gamma_0(x_n), t_0$ be the initial point in a concretized trace $tr_{\text{concrete}}$ for the abstraction i.e. the linear hybrid system $H_a$ corresponding to the general hybrid system $H$. Let $\tau(v_0, x_0, x_1, \ldots x_n)$ be the $\epsilon-$ Hybrid Simulation Trajectory starting from this initial point.

Definition 11. $\epsilon-$ Structurally Robust Hybrid System: A hybrid system $H = (X, V, E, \nu_i, \alpha, \beta)$ is said to be structurally robust iff

- $\forall i, (v_i, \phi, \psi, v_{i+1}) \in E$, every constraint $c$ in $\phi$ is satisfied by at least a dense set of size $\epsilon$ i.e. If $S = \{Val = (x_0, x_1, \ldots x_n) | Val \text{ satisfies } c\}$, then $\max_{a \in S} \min_{b \in S} d(a, b) > \epsilon$.
  In particular, we allow only sampled comparisons $x := c$, which is a shorthand for $[\frac{x}{\epsilon}] \times \epsilon < x < [\frac{x}{\epsilon}] \times \epsilon$.

Definition 12. $\epsilon$ Robust Hybrid Simulation Trajectory: Given an initial valuation of $X$ i.e. $Val_0(X) = (x_0 = x0, x_1 = x1, \ldots x_n = xn)$ and a path $\rho = \{v_0, v_1 \ldots v_m\}$ in a general hybrid system $H = (X, V, E, \nu_i, \alpha, \beta)$, then $\tau(x_0, x_1, \ldots x_n) = f(t)$ is said to be an $\epsilon-$ robust hybrid simulation trajectory iff

- $\tau(0) = Val_0(X)$ and $Val_0(X) \in \alpha(v_0)$
- if $x_k := e \in \psi(v_i, v_{i+1})$ then $\tau(x_k, \sum^i t_i +) = e$ else $\tau(x_k, \sum^i t_i +) = \tau(x_k, \sum^i t_i -)$
- Before executing the jump $(v_i, v_{i+1})$, $\tau(x_k, \sum^i t_i -)$ satisfies every precondition in $\phi(v_i, v_{i+1})$ $\epsilon-$ robustly.
• A linear constraint $c$ is $\epsilon$– robustly satisfied by $X = (x_0, x_1, \ldots)$ iff for every $X'$ such that $d(X, X') \leq \epsilon$, $c(X')$ is true.

– Within each location $v_i$ where the timed path has spent time $t_i$,
  • $\forall t_i < t < t_{i+1}, \tau(v, \sum_i^t (t_i) + t)$ is an $\epsilon$– simulation trajectory for location $v_i$ with initial valuation $Val_0 = \tau(\sum_i^t (t_i))$.

**Theorem 1.** If $\tau(v_0, x_0, x_1, \ldots x_n)$ be the $\epsilon$– Robust Hybrid Simulation Trajectory starting from the initial valuation $Val_0 = \gamma_0(x_0), \gamma_0(x_1), \ldots \gamma_0(x_n)$, and $H$ be a $\epsilon$– structurally robust hybrid system, then $tr_{\text{concrete}}$ corresponds to a real counterexample for the hybrid system $H$.

**Proof.** The proof follows from the definition of $\epsilon$– robust hybrid automata and the notion of $\epsilon$– hybrid simulation trajectory.

### 5.2 Simulation Based Abstraction Refinement

Consider the concretization of a trace $tr_{\text{concrete}}$ with respect to the general hybrid automata $H$ obtained from a trace $tr$ of the abstract linear hybrid automata $H_a$. Also, consider the guided hybrid simulation trajectory $\tau_{tr_{\text{concrete}}}$ corresponding to the concretization of the trace $tr$ with respect to the general hybrid automata $H$.

**Metrics for distance between trace and trajectory** We define two distance metrics between a trace and the corresponding guided hybrid simulation trajectory.

– $D(t) = d(\tau_{tr_{\text{concrete}}}(t), tr_{\text{concrete}}(t))$.
  This is simply a distance metric between corresponding points on the trace and the trajectory. The metric $d$ may be the Euclidean distance metric or the Manhattan distance metric (linear function).

– $D'(t) = d'(d(\tau_{tr_{\text{concrete}}}(t), tr_{\text{concrete}}(t)), d(\tau_{tr_{\text{concrete}}}(t-), tr_{\text{concrete}}(t-))$.
  This metric measures how rapidly the guided hybrid simulation trajectory is moving away from the trace. The metric $d'$ may be the Euclidean distance metric or the Manhattan distance metric (linear function), while the metric $d'$ may be the real difference. $t-$ represents the last instant of time for which the value of the concretized trace is known.

**Strategies to choose the location to be refined** Let $t_i$ be the discrete point on the concretization of a trace i.e. on $tr_{\text{concrete}}$ for which $tr_{\text{concrete}}(t)$ is known from the solution of the LP problem. There are few different strategies to choose the location in the approximate linear hybrid automata, where one needs to refine the abstract hybrid automata $H_a$.

– $min_i |D(t_i)| > \epsilon$, where $\epsilon$ is an empirically determined constant.
– $min_i |D'(t_i) - D'(t_{i-1})| > \epsilon$, where $\epsilon$ is an empirically determined constant.
– $\min_i |D'(t_i)/D'(t_{i-1})| > \epsilon$, where $\epsilon$ is an empirically determined constant.

After finding out the point $t_i$ where one needs to refine the location, the location $v_i$ at the time $t_i$ which needs to be split is easily known from the concretized trace.

Choosing the variable to split the location When a simulation trajectory differs substantially from the trace obtained by the LP solution, we need to split the location at which the difference is substantial along a hyperplane such that the abstract hybrid automata formed by the linear hybrid automata has a trace that is close to the simulation trajectory. Let $D$ be the metric used to decide if a given location should be refined; then we split those variables into half-spaces which have contributed beyond a threshold to $D$.

6 Conclusion and Future Work

This early draft discusses the core issues involved in building a CEGAR framework for analyzing nonlinear hybrid systems. The central idea is to use linear programming as a mechanism for obtaining feasible traces of the over-approximate linear hybrid automata (LHA) abstractions and numerical simulation for obtaining a corresponding trace of the original (possibly nonlinear) hybrid system. The distance between these two traces is then used to guide the refinement step in our CEGAR loop.

Several practical issues like the choice of the distance metrics, the choice of picking up a particular solution to the linear program and a characterization of the nonlinear functions which can be handled using this paradigm have been left to a more complete version of this draft. The techniques presented here are also being implemented into a tool which will be a successor to the IRA meta-tool for analyzing LHAs.

7 Acknowledgement

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Numerical Simulation guided Lazy Abstraction Refinement for Nonlinear Hybrid Automata

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Abstract. This draft suggests a new counterexample guided abstraction refinement (CEGAR) framework that uses the combination of numerical simulation for nonlinear differential equations with linear programming for linear hybrid automata (LHA) to perform reachability analysis on nonlinear hybrid automata. A notion of \( \epsilon \)-structural robustness is also introduced which allows the algorithm to validate counterexamples using numerical simulations.

1 Introduction

The model checking of hybrid automata remains a challenge and the existing tools [?,?] do not scale up to the needs of the industry. Because of the well known fundamental undecidability results [?], the model checking of general hybrid automata often proceeds by building successive tighter approximations to these hybrid automata in a relatively easy-to-analyze fragment of hybrid automata like Linear Hybrid Automata [?]. Theoretical results about the asymptotic completeness of this approximation procedure form the backbone of such a strategy behind the model checking of nonlinear hybrid automata.

There has been considerable interest in applying Counterexample Guided Abstraction Refinement (CEGAR), which works so well with discrete systems, to the problem of hybrid system verification [?]. There has also been some exploration of using fragments instead of counterexamples during abstraction refinement [?] and the application of CEGAR specifically to LHA [?]. However, our ongoing work makes the following new contributions to the abstraction refinement based analysis of hybrid systems.

- We address the problem of abstraction refinement for nonlinear hybrid automata and use CEGAR to construct successively refined LHA approximations. Our refinement is lazy and hence, refines some parts of the state space more finely than others.
- We use the distance between a feasible path in the abstract linear hybrid automata and the numerically simulated trajectory in the nonlinear hybrid automata to refine those locations in the LHA that do not faithfully represent the behavior of the nonlinear hybrid automata.
We define a structural notion of robustness and use it to present a counterexample validation algorithm (for a rich class of nonlinear hybrid automata) using linear programming [4]. Hence, it is possible to detect reachability of a bad state even before the abstraction refinement loop terminates.

2 Background on LP based path feasibility analysis of LHA

Informally, a linear hybrid automaton is a conventional automaton extended with a set of continuous variables. The states of the automaton called locations are annotated with a change rate for each continuous variable such as \( \dot{x} = [a, b] \) (\( x \) is a variable, and \([a, b]\) is a rational interval), and the transitions of the automaton are labeled with constraints on the variables such as \( a \leq \sum_{i=0}^{m} c_i x_i \leq b \) and/or with reset actions such as \( x := c \) (\( x \) and \( x \) are variables, \( a, b \), and \( c_i \) are real numbers). Such linear hybrid automata are essentially equivalent to the definition given in [4]. It is known that this subclass of linear hybrid automata are sufficiently expressive to allow asymptotic completeness of the abstraction process for a general hybrid automata. A restricted form of linear phase portrait approximations are asymptotically complete, namely, when all automaton constraints are over-approximated using independent, rational lower and upper bounds on the values and derivatives of each variable” [4]. For simplicity, we suppose that in any linear hybrid automaton considered in this paper, there is just one initial location with no initial conditions and no transitions to the initial location (we assume that each variable with an initial value is reset to the initial value by the transitions from the initial location).

Definition 1. A linear hybrid automaton is a tuple \( H = (X, V, E, v_I, \alpha, \beta) \), where

- \( X \) is a finite set of real-valued variables.
- \( V \) is a finite set of locations.
- \( E \) is transition relation whose elements are of the form \( (v, \phi, \psi, v') \) where \( v, v' \) are in \( V \), \( \phi \) is a set of guards or variable constraints of the form \( a \leq \sum_{i=0}^{m} c_i x_i \leq b \), and \( \psi \) is a set of reset actions of the form \( x := c \) where \( x_i \in X \) \((0 \leq i \leq m)\), \( x \in X \), \( a, b \) and \( c_i \) \((0 \leq i \leq m)\) are real numbers, and \( a \) and \( b \) may be \( \infty \).
- \( v_I \) is an initial location.
- \( \alpha \) is a labeling function which maps each location in \( V \) \(- \{v_I\} \) to a state invariant which is a set of variable constraints of the form \( a \leq \sum_{i=0}^{m} c_i x_i \leq b \) where \( x_i \in X \) \((0 \leq i \leq m)\), \( y \in X \), \( a, b \) and \( c_i \) \((0 \leq i \leq m)\) are real numbers, \( a \) and \( b \) may be \( \infty \).
- \( \beta \) is a labeling function which maps each location in \( V \) \(- \{v_I\} \) to a set of change rates which are of the form \( \dot{x} = [a, b] \) where \( x \in X \), and \( a, b \) are rational numbers \( (a \leq b) \). For any location \( v \), for any \( x \in X \), there is one and only one change rate definition \( \dot{x} = [a, b] \in \beta(v) \).
For a linear hybrid automaton $H = (X, V, E, v_I, \alpha, \beta)$, a path segment is a sequence of locations

$$v_1 (\phi_1, \psi_1) \xrightarrow{} v_2 (\phi_2, \psi_2) \xrightarrow{} \cdots \xrightarrow{} (\phi_{n-1}, \psi_{n-1}) v_n$$

which satisfies $(v_i, \phi_i, \psi_i, v_{i+1}) \in E$ for each $i$ ($1 \leq i \leq n - 1$). A path in $H$ is a path segment starting at $v_I$. The behavior of linear hybrid automata can be represented by timed sequences. Any timed sequence is of the form $(v_1, t_1) \rightarrow (v_2, t_2) \rightarrow \cdots \rightarrow (v_n, t_n)$, where $v_i$ ($1 \leq i \leq n$) is a location and $t_i$ ($1 \leq i \leq n$) is a nonnegative real number, which represents a behavior of an automaton that the system starts at the initial location and changes to the location $v_1$, stays there for $t_1$ time units, then changes to the location $v_2$ and stays in $v_2$ for $t_2$ time units, and so on.

**Definition 2.** For a linear hybrid automaton $H = (X, V, E, v_I, \alpha, \beta)$, a timed sequence $(v_1, t_1) \rightarrow (v_2, t_2) \rightarrow \cdots \rightarrow (v_n, t_n)$ represents a behavior of $H$ if the following condition is satisfied:

- there is a path in $H$ of the form

$$v_0 \xrightarrow{} v_1 \xrightarrow{} \cdots \xrightarrow{} v_n$$

- $t_1, t_2, \ldots, t_n$ satisfy all the variable constraints in $\phi_i$ ($1 \leq i \leq n - 1$), i.e. for each variable constraint $a \leq c_0 x_0 + c_1 x_1 + \ldots + c_m x_m \leq b$ in $\phi_i$,

$$\delta_k \leq \gamma_i(x_k) \leq \delta_k'$$

for any $k$ ($0 \leq k \leq m$), and

$$a \leq c_0 \gamma_i(x_0) + c_1 \gamma_i(x_1) + \ldots + c_m \gamma_i(x_m) \leq b$$

where $\gamma_i(x_k)$ ($0 \leq k \leq m$) represents the value of the variable $x_k$ when the automaton stay at $v_i$ with the delay $t_i$; and, similarly,

- $t_1, t_2, \ldots, t_m, \gamma_i(x_k), \lambda_i(x_k)$ satisfy the state invariant for each location $v_i$ ($1 \leq i \leq n$), where $\gamma_i(x_k)$ ($0 \leq k \leq m$) represents the value of the variable $x_k$ when the automaton stay at $v_i$ with the delay $t_i$, and $\lambda_i(x_k)$ ($0 \leq k \leq m$) represents the value of the variable $x_k$ after leaving state $v_i$ and after the reset conditions have been applied.
Now, we use linear programming to test the feasibility of a single path for the reachability analysis of linear hybrid automata. Let $H = (X, V, E, v_I, \alpha, \beta)$ be a linear hybrid automaton, and $\rho$ be a path in $H$ of the form

$$v_0 \xrightarrow{\phi_0, \psi_0} v_1 \xrightarrow{\phi_1, \psi_1} \cdots \xrightarrow{\phi_{n-1}, \psi_{n-1}} v_n$$

where $v_n = v$. For any timed sequence of the form $(v_1, t_1)^\ast(v_2, t_2)^\ast \cdots (v_n, t_n)$, if $\rho$ is feasible, then the following condition must hold:

- $t_1, t_2, \ldots, t_n$ satisfy all the variable constraints in $\phi_i (0 \leq i \leq n)$, and
- $t_1, t_2, \ldots, t_n$ satisfy all the variable constraints in $\alpha(v_i) (1 \leq i \leq n)$,

which form a group of linear inequalities on $t_1, t_2, \ldots, t_n, \gamma_i(x_k), \lambda_i(x_k)$ (see Definition 2), denoted by $\Theta(\rho)$ or $LP(\rho)(t_1, \gamma_i(x_k), \lambda_i(x_k))$. It follows that we can check if $\rho$ is a feasible path by checking if the group $\Theta(\rho)$ (or $LP(\rho)(t_1, \gamma_i(x_k), \lambda_i(x_k))$) of linear inequalities has a solution, which can be solved by linear programming [?].

### 3 Background on Abstraction of Affine dynamics by LHA

Given a general hybrid system $H = (X, V, E, v_I, \alpha, \beta)$ where $X$ is a finite set of real-valued variables, $V$ is a finite set of locations, $E$ is transition relation whose elements are of the form $(v, \phi, \psi, v')$ where $v, v'$ are in $V$, $\phi$ is a set of guards or variable constraints of the form $a \leq \sum_{i=0}^m c_i x_i \leq b$, and $\psi$ is a set of resets actions of the form $x := c$ where $x_i \in X (0 \leq i \leq m)$, $x \in X$, $a, b$ and $c_i (0 \leq i \leq m)$ are real numbers, and $a$ and $b$ may be $\infty$, $v_I$ is an initial location, $\alpha$ is a labeling function which maps each location in $V - \{v_I\}$ to a state invariant which is a set of variable constraints of the form $a \leq \sum_{i=0}^m c_i x_i \leq b$ where $x_i \in X (0 \leq i \leq m)$, $y \in X$, $a, b$, and $c_i (0 \leq i \leq m)$ are real numbers, $a$ and $b$ may be $\infty$, $\beta$ is a labeling function which maps each location in $V - \{v_I\}$ to a set of change rates which are of the form $x_i := f(x_0, x_1, \ldots, x_m, \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m, a_0, a_1 \ldots a_m)$ where $x_0, x_1 \ldots x_m \in X$, and $a_0, a_1 \ldots a_m$ are real numbers). For any location $v$, for any $x \in X$, there is one and only one change rate definition.

We construct (in a fashion similar to [?, ?]) a linear hybrid automata $H_a$ which is an over-approximate abstraction of the general hybrid automaton $H$. We define an operator split which divides a location into smaller locations and use this to divide each location $v_i$ of the original hybrid automaton.

**Definition 3.** Let $v$ be a location and $x_i \in X$ be a variable in a hybrid automata $H = (X, V, E, v_i, \alpha, \beta)$. Suppose $In_v = \{v_i^n, v_{i1}^n, \ldots, v_{in}^n\}$ be the locations from which there is a transition into the location $v$ and $Out_v = \{v_0^o, v_1^o, \ldots, v_m^o\}$ be the locations to which there is a transition from $v$. Also, $C$ be a set of linear constraints on $X$. Then, the operator split $(H, v, C)$ constructs a new hybrid automata $H' = (X, V', E', v_i, \alpha', \beta')$, where

- $V' = V \cup \{v', v''\} \setminus v$
\(- \alpha' = \alpha \cup \{\alpha(v'), \alpha(v'')\} \setminus \{\alpha(v)\}, \) where
- \(\alpha(v') = \alpha(v) \cup \{C\}\)
- \(\alpha(v'') = \alpha(v) \cup \{-C\}\)
\(- E' = E \setminus \{(v_{in}, \phi_{v_{in}}, v_{out}, v), (v_{in} \in In_v) \cup (v, \phi_{v_{out}}, \psi_{v_{out}}, v_{out}) \mid v_{out} \in Out_v\} \cup \{(v_{in}, \phi_{v_{in}}, v, v') \mid v_{in} \in In_v \cup (v', \phi_{v_{out}}, \psi_{v_{out}}, v_{out}) \mid v_{out} \in Out_v\} \cup \{(v_{in}, \phi_{v_{in}}, \psi_{v_{in}}, v', v'') \mid v_{in} \in In_v \cup (v'', \phi_{v_{out}}, \psi_{v_{out}}, v_{out}) \mid v_{out} \in Out_v\} \cup \{(v', \alpha(v', v'') = C, \{} \cup (v'') \mid v' 	ext{ is of the form } x := [a, b] \text{, where } a, b \in \mathbb{R}\}\)
\(- \beta' = \beta \cup \{\beta(v'), \beta(v'')\} \setminus \{\beta(v)\}, \) where
- \(\beta(v') = \beta(v)\)
- \(\beta(v'') = \beta(v)\)

We call the new locations \(v'\) and \(v''\) as the children of \(v\). In particular, \(v' = \text{child}(v, C)\) and \(v'' = \text{child}(v, -C)\), where \(C\) is the set of linear constraints used to split \(v\). We also call \(v\) as the parent of \(v'\) and \(v''\). Thus, the split operator naturally defines a tree of locations that we call the location tree, where the children location are formed by splitting the parent location.

**Definition 4.** LHA-approximation to a general hybrid automata: Given a general hybrid automata \(H = (X, V, E, v_i, \alpha, \beta)\), \(H_a = (X, V_a, E_a, v_{i_a}, \alpha_a, \beta_a)\) is a LHA-approximation iff

\(- \text{ There exists a hybrid automata } H' = (X, V', E', V'_i, \alpha', \beta'), \text{ where } H' = \text{split}''(H)\).
\(- V_a = V', E_a = E', V'_a = V_i, \alpha' = \alpha_a\)
\(- \forall v \in V_a, \beta_a(v) \supset \beta'_v \text{ and if } c \in \beta_a(v), \text{ then } c \text{ is of the form } x := [a, b] \text{, where } a, b \in \mathbb{R}.\)
4 Definitions

The linear hybrid automaton $H_a$ is an over-approximate approximation of the general hybrid automata $H$. A path $\rho = \{v_0, v_1, \ldots, v_m\}$, where $v_i \in V$, is said to exist in $H_a$ if $(v_i, v_{i+1}) \in E$, $0 \leq i < m$.

Consider a path $\rho = \{v_0, v_1, \ldots, v_m\}$ that exists in the abstract linear hybrid automata model $H_a$ and let $LP_\rho(t_i, \gamma_i(x_k), \lambda_i(x_k))$ be the linear program corresponding to the path. If the linear program $LP_\rho$ has a feasible solution, then the path is said to be feasible in the abstract model i.e. the linear hybrid automaton; otherwise it is said to be infeasible.

**Definition 5.** Trace: Given a feasible path $\rho$ in the abstract linear hybrid automaton model $H_a$, the feasible solution to the LP program $LP_\rho(t_i, \gamma_i(x_k), \lambda_i(x_k))$ is called a trace of $H_a$.

We write $\text{trace}(H_a) = (v_0, \lambda_0(x_0), \lambda_0(x_1), \ldots, \lambda_0(x_n), \gamma_0(x_0), \gamma_0(x_1), \ldots, \gamma_0(x_n), t_0), (v_1, \lambda_1(x_0), \lambda_1(x_1), \ldots, \lambda_1(x_n), \gamma_1(x_0), \gamma_1(x_1), \ldots, \gamma_1(x_n), t_1), \ldots, (v_n, \lambda_n(x_0), \lambda_n(x_1), \ldots, \lambda_n(x_n), \gamma_n(x_0), \gamma_m(x_1), \ldots, \gamma_m(x_n), t_m)$.

It is known [?] that the trace obtained by the linear program is a real execution trace of the over-approximate linear hybrid automata.

**Definition 6.** Concretization of a path: Consider a path $\rho = \{v_0, v_1, \ldots, v_m\}$ that is feasible in the abstract linear hybrid automata model $H_a$. Then, the concretization of this path in the original hybrid automata $H$ is the trace $\rho_{\text{concrete}} = \{v_{i, \text{root}}, v_{i, \text{root}}^{\text{root}} \ldots v_{m, \text{root}}^{\text{root}}\}$, where $v_{i, \text{root}}^{\text{root}}$ is the root of the location tree in which $v_i$ is a leaf.

As the split operator forms a tree of locations in the abstract linear hybrid automata, the root of the location tree is known and the concretization of an abstract path is well defined.

**Definition 7.** Concretization of a trace: Consider a trace $tr = (v_0, \lambda_0(x_0), \lambda_0(x_1), \ldots, \lambda_0(x_n), \gamma_0(x_0), \gamma_0(x_1), \ldots, \gamma_0(x_n), t_0), (v_1, \lambda_1(x_0), \lambda_1(x_1), \ldots, \lambda_1(x_n), \gamma_1(x_0), \gamma_1(x_1), \ldots, \gamma_1(x_n), t_1), \ldots, (v_n, \lambda_n(x_0), \lambda_n(x_1), \ldots, \lambda_n(x_n), \gamma_n(x_0), \gamma_m(x_1), \ldots, \gamma_m(x_n), t_m)$ corresponding to the path $\rho = \{v_0, v_1, \ldots, v_m\}$ that is feasible in the abstract linear hybrid automata model $H_a$. Then, the concretization of this trace in the original hybrid automata $H$ is the trace $tr_{\text{concrete}} = (v_{0, \text{root}}, \lambda_0(x_0), \lambda_0(x_1), \ldots, \lambda_0(x_n), \gamma_0(x_0), \gamma_0(x_1), \ldots, \gamma_0(x_n), t_0), (v_{1, \text{root}}, \lambda_1(x_0), \lambda_1(x_1), \ldots, \lambda_1(x_n), \gamma_1(x_0), \gamma_1(x_1), \ldots, \gamma_1(x_n), t_1), \ldots, (v_{m, \text{root}}, \lambda_m(x_0), \lambda_m(x_1), \ldots, \lambda_m(x_n), \gamma_m(x_0), \gamma_m(x_1), \ldots, \gamma_m(x_n), t_m)$, where $v_{i, \text{root}}^{\text{root}}$ is the root of the location tree of which $v_i$ is a leaf.

**Definition 8.** $\epsilon$-Simulation Trajectory: Given a valuation of $X$ i.e. $Val_0(X) = (x_0 = x_0, x_1 = x_1, \ldots, x_n = x_n)$ in a location $v$ of a general hybrid system $H = (X, V, E, v_0, \alpha, \beta)$, then $\tau(x_0, x_1, \ldots, x_n, v, t)$ is said to be an $\epsilon$-simulation trajectory for location $v$ with initial valuation $Val_0(X)$ iff

\[\tau(t = 0) = (x_0, x_1, \ldots, x_n)\]
if \( f(x_0, x_1, \ldots, x_n, t) \) is the solution to the initial value problem \((\beta(v), Val_0(X))\), then \( f(t) - \epsilon \leq \tau(t) \leq f(t) + \epsilon \).

It is known that numerical techniques can solve the initial value problem for ODEs (including non-linear ODEs) quite efficiently.

**Definition 9.** \( \epsilon \)-Hybrid Simulation Trajectory: Given an initial valuation of \( X \) i.e. \( Val_0(X) = (x_0 = x_0, x_1 = x_1, \ldots, x_n = x_n) \) and a path \( \rho = \{v_0, v_1 \ldots v_m\} \) in a general hybrid system \( H = (X, V, E, v_i, \alpha, \beta) \), then \( \tau(x_0, x_1, \ldots, x_n) = f(t) \) is said to be an \( \epsilon \)-hybrid simulation trajectory iff:

- \( \tau(0) = Val_0(X) \) and \( Val_0(X) \in \alpha(v_0) \)
- if \( x_k := e \in \psi(v_i, v_{i+1}) \) then \( \tau(x_k, \sum_0^i(t_i)\) + = \( e \) else \( \tau(x_k, \sum_0^i(t_i)\) − = \( \tau(x_k, \sum_0^i(t_i)\) −)
- Before executing the jump \((v_i, v_{i+1})\), \( \tau(x_k, \sum_0^i(t_i)\) −) satisfies every precondition in \( \phi(v_i, v_{i+1}) \)
- Within each location \( v_i \) where the timed path has spent time \( t_i \),
  \[ \forall t, t_i < t < t_{i+1}, \tau(v_i, \sum_0^i(t_i) + t) \text{ is an } \epsilon \text{-simulation trajectory for location } v_i \text{ with initial valuation } Val_0 = \tau(\sum_0^i(t_i)) \].

**Definition 10.** Guided Simulation Trajectory of the concretization of a trace: An \( \epsilon \)-hybrid simulation trajectory \( \tau \) is said to be a Guided Simulation Trajectory of a concretized trace \( tr_{\text{concrete}} \) iff:

- The initial valuation of \( X \) i.e. \( Val_0(X) = (x_0 = x_0, x_1 = x_1, \ldots, x_n = x_n) \) for the trajectory \( \tau \) is the initial point in the concretized trace \( tr_{\text{concrete}} \).
- The \( \epsilon \)-hybrid simulation trajectory \( \tau \) corresponds to the path \( \rho = \{v_0, v_1 \ldots v_m\} \) corresponding to \( tr_{\text{concrete}} \).

5 CEGAR based Refinement of the abstract linear hybrid automata

The CEGAR algorithm repeatedly constructs LHA over-approximations to the given (possibly nonlinear) hybrid system and then asks a LHA analysis engine if the over-approximate LHA admits any counterexample. If it does not, we are done and we report that the original hybrid system has no counterexample either. Otherwise, we take the reported counterexample of the over-approximate LHA and attempt to validate it using numerical simulation. If we succeed in validating the counterexample, we report an error that the bad state is reachable and STOP. Otherwise, we find a location where we need to split the nonlinear hybrid automata and then rebuild a more precise over-approximate abstraction.
Algorithm for CEGAR

(Input: Nonlinear Hybrid Automata A. Output: Error No error)
1. $A_0 = A$; $i := 0$; $L = \text{Universe}$. 
2. $LHA_i = \text{LHA-approximation (A)}$ 
3. $L(LHA_i) := \text{Language of LHA}_i$. $L$ represents the set of potential counterexamples in $A_i$ (finitely expressible as a regular expression) [?]. 
4. $L = L \cap L(LHA_i)$ 
5. If $L$ is empty, report "BAD STATES NOT REACHABLE" and stop. 
6. Pick a counterexample $ce$ in $L$. 
7. Validate the counterexample $ce$ in the original hybrid automata $A$. 
8. If $ce$ is validated in $A$, stop and report that ERROR STATE IS REACHABLE. 
9. Compute a refinement operator $\text{split}$, and $A_{i+1} = \text{split}(A_i)$.
   Also, compute $L = \text{split}(L)$. 
10. $i := i + 1$ 
11. Loop to Step 2.

5.1 Counterexample Validation and Structural Robustness

Let $v_0, \gamma_0(x_0), \gamma_0(x_1) \ldots \gamma_0(x_n), t_0$ be the initial point in a concretized trace $tr_{\text{concrete}}$ for the abstraction i.e. the linear hybrid system $H_a$ corresponding to the general hybrid system $H$. Let $\tau(v_0, x_0, x_1, ... x_n)$ be the $\epsilon-$ Hybrid Simulation Trajectory starting from this initial point.

Definition 11. $\epsilon-$ Structurally Robust Hybrid System: A hybrid system $H = (X, V, E, v_0, \alpha, \beta)$ is said to be structurally robust iff

- $\forall i, (v_i, \phi, \psi, v_{i+1}) \in E$, every constraint $e$ in $\phi$ is satisfied by at least a dense set of size $\epsilon$ i.e. If $S = \{Val = (x_0, x_1, ... x_n) | Val \text{ satisfies } e\}$, then $\max_{a \in S} \min_{b \in S} d(a, b) > \epsilon$.

In particular, we allow only sampled comparisons $x := e$, which is a shorthand for $[x] \times \epsilon < x < [x] \times \epsilon$.

Definition 12. $\epsilon$ Robust Hybrid Simulation Trajectory: Given an initial valuation of $X$ i.e. $Val_0(X) = (x_0 = x_0, x_1 = x_1, ... x_n = x_n)$ and a path $\rho = \{v_0, v_1 \ldots v_m\}$ in a general hybrid system $H = (X, V, E, v_0, \alpha, \beta)$, then $\tau(x_0, x_1, \ldots x_n) = f(t)$ is said to be an $\epsilon-$ robust hybrid simulation trajectory iff

- $\tau(0) = Val_0(X)$ and $Val_0(X) \in \alpha(v_0)$
- if $x_k := e \in \psi(v_i, v_{i+1})$ then $\tau(x_k, \sum t_i) + = e$ else $\tau(x_k, \sum t_i) + = \tau(x_k, \sum t_i) -$
- Before executing the jump $(v_i, v_{i+1})$, $\tau(x_k, \sum t_i) -$ satisfies every precondition in $\phi(v_i, v_{i+1})$ $\epsilon-$ robustly.
• A linear constraint \( c \) is \( \epsilon \)-robustly satisfied by \( X = (x_0, x_1, \ldots) \) iff for every \( X' \) such that \( d(X, X') \leq \epsilon \), \( c(X') \) is true.

- Within each location \( v_i \) where the timed path has spent time \( t_i \),
  - \( \forall t_i < t < t_i + 1, \tau(v, \sum_{0}^{i}(t_i) + t) \) is an \( \epsilon \)-simulation trajectory for location \( v_i \) with initial valuation \( Val_0 = \tau(\sum_{0}^{i}(t_i)) \).

**Theorem 1.** If \( \tau(\gamma_0(x_0), \gamma_0(x_1), \ldots \gamma_0(x_n)) \) be the \( \epsilon \)-Robust Hybrid Simulation Trajectory starting from the initial valuation \( Val_0 = \gamma_0(x_0), \gamma_0(x_1) \ldots \gamma_0(x_n) \), and \( H \) be a \( \epsilon \)-structurally robust hybrid system, then \( tr_{concrete} \) corresponds to a real counterexample for the hybrid system \( H \).

**Proof.** The proof follows from the definition of \( \epsilon \)-robust hybrid automata and the notion of \( \epsilon \)-hybrid simulation trajectory.

### 5.2 Simulation Based Abstraction Refinement

Consider the concretization of a trace \( tr_{concrete} \) with respect to the general hybrid automata \( H \) obtained from a trace \( tr \) of the abstract linear hybrid automata \( H_a \). Also, consider the guided hybrid simulation trajectory \( \tau_{tr_{concrete}} \) corresponding to the concretization of the trace \( tr \) with respect to the general hybrid automata \( H \).

**Metrics for distance between trace and trajectory** We define two distance metrics between a trace and the corresponding guided hybrid simulation trajectory.

- \( D(t) = d(\tau_{tr_{concrete}}(t), tr_{concrete}(t)) \).
  This is simply a distance metric between corresponding points on the trace and the trajectory. The metric \( d \) may be the Euclidean distance metric or the Manhattan distance metric (linear function).

- \( D'(t) = d' (d(\tau_{tr_{concrete}}(t), tr_{concrete}(t)), d(\tau_{tr_{concrete}}(t-1), tr_{concrete}(t-1)) \)
  This metric measures how rapidly the guided hybrid simulation trajectory is moving away from the trace. The metric \( d' \) may be the Euclidean distance metric or the Manhattan distance metric (linear function), while the metric \( d' \) may be the real difference. \( t-1 \) represents the last instant of time for which the value of the concretized trace is known.

**Strategies to choose the location to be refined** Let \( t_i \) be the discrete point on the concretization of a trace i.e. on \( tr_{concrete} \) for which \( tr_{concrete}(t) \) is known from the solution of the LP problem. There are few different strategies to choose the location in the approximate linear hybrid automata, where one needs to refine the abstract hybrid automata \( H_a \).

- \( min_{i} |D(t_i)| > \epsilon \), where \( \epsilon \) is an empirically determined constant.
- \( min_{i} |D'(t_i) - D'(t_{i-1})| > \epsilon \), where \( \epsilon \) is an empirically determined constant.
− \(\min_i |D'(t_i)/D'(t_{i-1})| > \epsilon\), where \(\epsilon\) is an empirically determined constant.

After finding out the point \(t_i\) where one needs to refines the location, the location \(v_i\) at the time \(t_i\) which needs to be split is easily known from the concretized trace.

Choosing the variable to split the location When a simulation trajectory differs substantially from the trace obtained by the LP solution, we need to split the location at which the difference is substantial along a hyperplane such that the abstract hybrid automata formed by the linear hybrid automata has a trace that is close to the simulation trajectory. Let \(D\) be the metric used to decide if a given location should be refined; then we split those variables into half-spaces which have contributed beyond a threshold to \(D\).

6 Conclusion and Future Work

This early draft discusses the core issues involved in building a CEGAR framework for analyzing nonlinear hybrid systems. The central idea is to use linear programming as a mechanism for obtaining feasible traces of the over-approximate linear hybrid automata (LHA) abstractions and numerical simulation for obtaining a corresponding trace of the original (possibly nonlinear) hybrid system. The distance between these two traces is then used to guide the refinement step in our CEGAR loop.

Several practical issues like the choice of the distance metrics, the choice of picking up a particular solution to the linear program and a characterization of the nonlinear functions which can be handled using this paradigm have been left to a more complete version of this draft. The techniques presented here are also being implemented into a tool which will be a successor to the IRA meta-tool for analyzing LHAs.

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