Mathematical modeling of a waterjet jet for structural material cutting

Vasiliy Shpilev¹,², Mikhaile Reshetnikov¹, Svetlana Borodulina¹, Mikhail Zakharchenko¹ and Yuri Ivashchenko¹

¹Yuri Gagarin State Technical University of Saratov, Saratov, Russian Federation

²Corresponding author: vasya-shpilev@rambler.ru

Abstract. When modeling a jet during hydroabrasive processing of structural materials, we will represent it as a heterogeneous environment. And accordingly, the hydroabrasive flow will be considered as a heterogeneous flow. The equations describing the formation of a heterogeneous flow are based on the equation of conservation of momentum of the total movement and allow for the interaction of the background flow and particles of the abrasive. A mathematical model of a heterogeneous flow, reveals the mechanism of formation of a waterjet jet. Based on the mathematical model, the hydrodynamic forces and surface stresses acting between the abrasive and the liquid can be taken into account. It also takes into account speed, concentration, mass, density of the fluid and abrasive. Knowing these parameters, it is possible to use the mathematical model of a heterogeneous flow to find the optimal conditions for the formation of a waterjet flow and the processing of structural materials.

1. Introduction

Waterjet cutting is an alternative method to many methods of processing various types of structural materials [1-3], especially those that do not tolerate temperature effects [4]. The process of forming a waterjet flow [5-8] itself has not yet been fully studied, and it is of certain scientific interest.

The greatest influence on the process of forming the surface of the workpiece during waterjet cutting is exerted by the speed and diameter of the jet [9-10]. In early studies on the process of waterjet cutting, a liquid stream was considered as a two-phase or single-phase environment [11-13]. This does not quite correctly reflect its essence. For the most complete reflection of the formation of a waterjet flow, it is proposed to consider it as a heterogeneous environment, consisting of a background stream and abrasive particles.

2. Methods

Relying on work [14-15] we make assumptions allowing the use of equations for a two-phase flow in the case of a heterogeneous environment. As a result, the obtained equations will take into account the speed of the particles of abrasive and liquid, their concentration, mass and stress in the allocated volume.

In mathematical modeling of the processes and movements of heterogeneous mix, two main assumptions will always be valid [16-17]:

1. The sizes of inclusions and heterogeneities in the mix are many times larger than the molecular kinetic distances between the molecules. Thus, these heterogeneities contain a large number of
molecules. This allows the use of classical representations and equations of mechanics of continuous single-phase media to describe processes on the scale of inhomogeneity itself, that is, processes inside or near individual inclusions or heterogeneities. In this case, to describe the physical properties, one can use equations and parameters obtained from experiments with the corresponding substances in a single-phase state [18].

2. The dimensions of the indicated inhomogeneity are many times smaller than the distances at which the averaged or macroscopic parameters of the mix change significantly. Thus, the dimensions of the inhomogeneity are much smaller than the lengths of the waves considered in the mix, the lengths and diameters of the channels in which the heterogeneous mixture flows. This assumption allows one to describe macroscopic processes in a heterogeneous mix by the methods of continuum mechanics using averaged or macroscopic parameters.

Consider a heterogeneous (N=2) dispersed mix of particles, droplets or bubbles with a supporting background flow. The subscript \( i = 1 \) will be assigned to the parameters of the background carrier flow, and \( i = 2 \) – to the parameters of inclusion of the weighted flow. We also assume that there are no processes of crushing and the formation of new dispersed particles [19].

In the process of the movement of such an environment, as a result of the interaction of the background flow and the abrasive particles due to different speeds, the appearance of hydrodynamic forces, for example resistance forces, is predetermined. Therefore, in the conservation equations it is necessary to take into account the indicated interactions of the background flow and moving solid particles.

The mass constancy equation for the background flow and abrasive particles can be represented as follows [20]:

\[
\begin{align*}
\frac{1}{\phi_1} \frac{d\phi_1}{dx} + \frac{1}{\overline{v}_1} \frac{d\overline{v}_1}{dx} + \frac{1}{F} \frac{dF}{dx} &= 0 \\
\frac{1}{\phi_2} \frac{d\phi_2}{dx} + \frac{1}{\overline{v}_2} \frac{d\overline{v}_2}{dx} + \frac{1}{F} \frac{dF}{dx} &= 0
\end{align*}
\]

where \( \overline{v}_1 \) - background flow speed; 
\( \overline{v}_2 \) - particle abrasive velocity; 
\( \phi_1, \phi_2 \) - concentration of the first and second components of the weighted stream; 
\( F \) – living cross-sectional area.

In integral form, the equation for the flow rate of a heterogeneous environment has the form:

\[
m = m_1 + m_2 = F(\rho_1\phi_1\overline{v}_1 + \rho_2\phi_2\overline{v}_2).
\]

Equation of momentum for background flow:

\[
\frac{m_1}{F} \frac{d\overline{v}_1}{dx} = \frac{d\phi_1 E_1}{dx} + \phi_1 \rho_1 B - R;
\]

Equation of momentum for abrasive particles:

\[
\frac{m_2}{F} \frac{d\overline{v}_2}{dx} = \frac{d\phi_2 E_2}{dx} + \phi_2 \rho_2 B + R.
\]

Where \( m_1 = \phi_1 \rho_1 \overline{v}_1 F; \ m_2 = \phi_2 \rho_2 \overline{v}_2 F; \ E_1 \) and \( E_2 \) - intensity of energy exchange between the background stream and particles of abrasive; \( \rho_1 \) and \( \rho_2 \) - background flux density and abrasive particles; \( B \) - mass force vector; \( R \) - force of hydrodynamic interaction of flow components, determined by the formula:
\[ \vec{R} = 0.5C_s F_s \rho_f (\vec{v}_1 - \vec{v}_2) \left| \vec{v}_1 - \vec{v}_2 \right|, \]  

(5)

where \( C_s \) - drag coefficient of abrasive particles; \( F_s \) - particle mean sectional area.

The momentum equation for the environment, as a whole is obtained by adding equations (3) and (4):

\[ m d\vec{v} / dx = F d (\phi_1 E_1 + \phi_2 E_2) / dx + \rho \vec{B} \vec{F}, \]  

(6)

where \( \vec{v} = u_1 k_1 + u_2 k_2 \), where \( k_1 \) and \( k_2 \) consumable mass concentration of liquid and solid particles.

The energy conservation equation for a one-dimensional weighted flow has the form [21]:

\[ \rho_1 \phi_1 \vec{v}_1 \frac{d}{dx} \left( \vec{v}_1^2 / 2 \right) = \frac{d\phi_1 E_1}{dx} \vec{v}_1 + A_{i2} + A_{mp} + \rho_1 \phi_1 \vec{B} \vec{v}_1; \]  

(7)

\[ \rho_2 \phi_2 \vec{v}_2 \frac{d}{dx} \left( \vec{v}_2^2 / 2 \right) = \frac{d\phi_2 E_2}{dx} \vec{v}_2 - A_{i2} + A_{mp} + \rho_2 \phi_2 \vec{B} \vec{v}_2, \]  

(8)

where \( A_{i2} \) - inertia forces work; \( A_{mp} \) - friction forces work.

In heterogeneous environments, the description of the laws of the relative motion of components is complicated, since this motion is determined not by diffusion processes associated with the collision and chaotic motion of inclusion particles, and the processes of interaction of macroscopic systems. For example, a stream of particles around inclusions of a carrier fluid. These processes are described using forces and with a more consistent consideration of the energy of the components.

Thus, the problem of heterogeneous motion in the framework of a multi-speed (multi-fluid) model is reduced to setting the conditions for the joint movement of the components and determining the quantities that describe the internal (force, energy, thermal) and external (mass \( M_{ij} \), force \( P_{ij} \), energy \( E_{ij} \) ) interactions.

In some cases, when the inertial effects of the relative motion of the components are insignificant, the diffusion (single-fluid) approximation can also be used to describe heterogeneous mix.

Consider in detail the equation of conservation of momentum.

Allocate an arbitrary volume in the stream \( dV \) (fig. 1) and apply the theorem on the change in momentum to it: the main vector of all external mass and surface forces acting on the flow is equal to the sum of the changes in its momentum convectively given or perceived by the system per unit time.

![Fig. 1 - Heterogeneous flow](image)

For a heterogeneous environment, the change in the momentum vector of the background flow will be:
\[
\frac{d}{d\tau} \int (\rho_1 \mathbf{V}_1 dV) = \int \frac{d}{d\tau} (\rho_1 \phi_1 \mathbf{V}_1 dV).
\]

Let us reveal the derivative of the product of the background flow velocity vector \( \mathbf{V}_1 \) na mass of background flow \( \rho_1 \phi_1 dV \) in elementary capacity:

\[
\frac{d}{d\tau} (\rho_1 \phi_1 \mathbf{V}_1 dV) = \rho_1 \phi_1 dV \frac{d\mathbf{V}_1}{d\tau} + \mathbf{V}_1 \frac{d(\rho_1 \phi_1 dV)}{d\tau}.
\]

The time derivative in the second term is replaced by the background flow continuity equation (1), in the end we get:

\[
\frac{d}{d\tau} \int (\rho_1 \phi_1 \mathbf{V}_1 dV) = \int \rho_1 \phi_1 \mathbf{dV}_1 dV.
\] (9)

We now determine the vectors of external forces acting on the background flow of the allocated capacity \( V \). The main vector of surface forces:

\[
\mathbf{P}_1 = \int S_1 \mathbf{dS}.
\]

where \( S_1 \) - capacity flow limiting surface \( V \); \( \Pi_1 \) – surface force tensor, acting on the background flow. We divide the surface \( S_1 \) into two components: \( S_{11} \) – the surface along which the background flow is in contact with itself, \( S_{12} \) – the surface along which the background flow is in contact with the particles of abrasive. Then

\[
\mathbf{P}_1 = \int S_{11} \mathbf{dS} + \int S_{12} \mathbf{dS}.
\] (10)

We transform the first term using the Gauss-Ostrogradsky theorem:

\[
\int S_{11} \mathbf{dS} = \int \text{div} \Pi_{11} \phi_1 dV.
\] (11)

The second term in (10) formally represented through the volume integral of a certain interaction force between the components of the mix \( \mathbf{R}_{12} \) per unit mass:

\[
\int S_{12} \mathbf{dS} = \int \rho_1 \phi_1 \mathbf{R}_{12} dV.
\] (12)

The vector of mass force referred to the unit mass acting on the background flow is denoted by \( \mathbf{B}_1 \).

Then the total force:

\[
\mathbf{P}_1 = \int \rho_1 \phi_1 \mathbf{B}_1 dV.
\] (13)
Equating the change in momentum (9) and (10) to the impulse of all forces (11)-(13), given the randomness in the choice of capacity, we finally obtain for the background flow:

$$\rho \varphi_1 \frac{d\vec{v}_1}{d\tau} + x(\vec{v}_1 - \vec{v}_1) = \text{div} \varphi_1 \Pi_{11} + \rho \varphi_1 \vec{R}_{12} + \rho \varphi_1 \vec{B}_1;$$  \hspace{1cm} (14)

analogically for abrasive particles:

$$\rho_2 \varphi_2 \frac{d\vec{v}_2}{d\tau} + x(\vec{v}_2 - \vec{v}_3) = \text{div} \varphi_2 \Pi_{22} + \rho_2 \varphi_2 \vec{R}_{21} + \rho_2 \varphi_2 \vec{B}_2.$$  \hspace{1cm} (15)

Add (14) and (15) to get the momentum equation for the whole environment:

$$\rho \varphi_1 \frac{d\vec{v}_1}{d\tau} + \rho_2 \varphi_2 \frac{d\vec{v}_2}{d\tau} + x(\vec{v}_2 - \vec{v}_1) = \text{div} \Pi + \rho \vec{B}.$$  \hspace{1cm} (16)

Equation (16) takes into account that \( \rho \varphi_1 \vec{R}_{12} + \rho_2 \varphi_2 \vec{R}_{21} = 0 \) and the symbol:

$$\varphi_1 \Pi_{11} + \varphi_2 \Pi_{22} = \Pi; \quad \rho \varphi_1 \vec{B}_1 + \rho_2 \varphi_2 \vec{B}_2 = \rho \vec{B} \text{ at } \vec{B}_1 = \vec{B}_2 = \vec{B}; \quad \Pi - \text{tensor of surface stresses acting in a heterogeneous environment.}$$

The equation of momentum for the whole medium as a whole can be written as:

$$\frac{d(\vec{v}_1 \rho \varphi_1 V)}{V d\tau} + \frac{d(\vec{v}_2 \rho_2 \varphi_2 V)}{V d\tau} = \text{div} \Pi + \rho \vec{B},$$  \hspace{1cm} (17)

Since for the environment as a whole \( \frac{d(\rho V)}{d\tau} = 0 \), we take out on the left side of equation (17) from under the sign of the derivative \( \rho V \). Given the equalities \( \varphi_1 V = V_1; \ \varphi_2 V = V_2; \ \varphi_1 = \frac{\rho_1 V_1}{\rho V} \) and \( \varphi_2 = \frac{\rho_2 V_2}{\rho V} \), finally get:

$$\frac{d\vec{v}}{d\tau} = \frac{1}{\rho} \text{grad} \Pi + \vec{B},$$  \hspace{1cm} (18)

where \( \vec{v} = x_1 \vec{v}_1 + x_2 \vec{v}_2; \ \rho = \rho x_1 + \rho x_2 = \varphi_1 + \varphi_2. \)

This equation in structure coincides with the equation of motion for a single-phase fluid.

The tensor of surface forces or stresses through the projections of the corresponding forces is denoted as follows:

$$\Pi = \begin{pmatrix}
\Pi_{xx} & \Pi_{xy} & \Pi_{xz} \\
\Pi_{yx} & \Pi_{yy} & \Pi_{yz} \\
\Pi_{zx} & \Pi_{zy} & \Pi_{zz}
\end{pmatrix}.$$  \hspace{1cm} (19)

In the accepted symbols, the first index with the components indicates which axis the area on which the voltage acts is perpendicular, and the second indicates the axis on which this voltage is projected. So, \( \Pi_{xx} \) is the normal tension acting on a pad perpendicular to the axis \( x \), and \( \Pi_{xy} \Pi_{zz} \) - tangential tension, acting on other platforms perpendicular to the axes \( y \) and \( z \). The relationship between the components of the tensions tensor and the strain rate tensor for isotropic media is established by the generalized Newton law:
\( \Pi_{xx} = -p + \lambda \text{div} \vec{v} + 2\mu \frac{\partial \vec{v}}{\partial x}; \)

\( \Pi_{yy} = -p + \lambda \text{div} \vec{v} + 2\mu \frac{\partial \vec{v}}{\partial y}; \)

\( \Pi_{zz} = -p + \lambda \text{div} \vec{v} + 2\mu \frac{\partial \vec{v}}{\partial z}; \)

\( \Pi_{xy} = \Pi_{yx} = \mu \left\{ \frac{\partial \vec{v}}{\partial y} + \frac{\partial \vec{v}}{\partial x} \right\}; \)

\( \Pi_{xz} = \Pi_{zx} = \mu \left\{ \frac{\partial \vec{v}}{\partial z} + \frac{\partial \vec{v}}{\partial x} \right\}; \)

\( \Pi_{yz} = \Pi_{zy} = \mu \left\{ \frac{\partial \vec{v}}{\partial z} + \frac{\partial \vec{v}}{\partial y} \right\}; \)

\[ (20) \]

where \( \mu \) - dynamic viscosity coefficient, and \( \lambda \) - second viscosity coefficient.

The value \( p \) does not depend on the strain rate tensor and is determined by the equation of state of the fluid. The average value of normal stresses at any point \( p \) (pressure) is defined as:

\[ -\bar{p} = \frac{\Pi_{xx} + \Pi_{yy} + \Pi_{zz}}{3} = -p + \left( \frac{2}{3} \mu + \frac{\lambda}{\mu} \right) \text{div} \vec{v}. \]

\[ (21) \]

In case of incompressible fluid \( \text{div} \vec{v} = 0 \), value is equal \( \bar{p} \) to pressure \( p \).

If we restrict ourselves to the case when the tangential tensions inside each component of the flow can be neglected, then the tensor of surface forces of the \( i \)-th component will take the form \( \Pi_i = -\epsilon_{ii} p \), where \( \epsilon \) - unit tensor (\( p_i \) - \( i \)-component pressure).

Then equation (16) and (17) can be written as:

\[ \rho \frac{d\vec{v}}{dt} = -\text{grad} p + \rho B; \]

\[ (22) \]

\[ \rho_i \varphi_1 \frac{d\vec{v}_1}{dt} + \rho_i \varphi_2 \frac{d\vec{v}_2}{dt} = -\text{grad} p + \rho B. \]

\[ (23) \]

The momentum equations for the background flow and abrasive particles will take the form:

\[ \rho_i \varphi_1 \frac{d\vec{v}_1}{dt} + \rho_i \varphi_2 \frac{d\vec{v}_2}{dt} = -\text{grad} p_i + \rho_i \varphi_1 \vec{R} + \rho_i \varphi_2 \vec{B}_i; \]

\[ (24) \]

where \( \nu_3 \) - total flow rate.

formation of a waterjet flow and the processing of structural materials.
3. Results and Discussion

The obtained mathematical model reveals the mechanism of formation of a heterogeneous hydroabrasive jet, takes into account the interaction of the background flow and the abrasive particles, as well as the hydrodynamic forces, surface stresses, their speed, concentration, mass and density between them, knowing which, it is possible to predict the optimal conditions of the processing process.

The obtained equations based on one of the basic principles of mechanics - momentum equation of the total momentum, which take into account the interaction of the background flow and particles of abrasive having different speeds, which determines the appearance of hydrodynamic forces. Based on the equations obtained, it is possible to derive calculated dependencies that determine the main parameters of the waterjet flow, its speed and diameter, knowing which, it is possible to predict the optimal conditions of the processing process, which will increase its efficiency.

4. Conclusions

1. A mathematical model of the formation of a heterogeneous hydroabrasive jet is presented, which is based on the classical equations of hydromechanics of continuous environments.

2. The equations of momentum are obtained for the background flow and particles of the abrasive, as well as the heterogeneous environment as a whole, which allows one to take into account the interaction of the background flow and particles of the abrasive having different speeds, resulting hydrodynamic forces and surface stresses.

References

[1] Molchanova Y S, Bychkov N A and Chernyayev S I 2015 Producing orifices in structural materials by plasma, waterjet and laser cutting and piercing Welding International 29(2) 161-164 DOI: 10.1080/09507116.2014.897812
[2] Averin E 2017 Universal method for the prediction of abrasive waterjet performance in mining Engineering 3(6) 888-891
[3] Hashish M 1999 Characteristics of surfaces machined with abrasive waterjet MD 16 23-32
[4] Zhao J, Zhang G, Xu Y, Wang R, Zhou W, Zhou Y and Han L 2017 Mechanism and effect of jet parameters on particle waterjet rock breaking Powder Technology 29(2) 161-164 DOI: 10.1080/09507116.2014.897812
[5] Znamenskaya I A, Koroteeva E Y, Shirshov Y N, Novinskaya A M and Sysoev N N 2017 High speed imaging of a supersonic waterjet flow Quantitative InfraRed Thermography 14(2) 185-192 DOI: 10.1080/17686733.2016.1243749
[6] Znamenskaya I A, Naumov D S, Nersesyan D A, Shirshov Y N and Sysoev N N 2015 Dynamic characteristics of high-speed water jets in waterjet cutting machines Journal of Flow Visualization and Image Processing 22(4) 165-173 DOI: 10.1080/17686733.2016.1243749
[7] Li D, Kang Y, Ding X, Wang X and Liu W 2017 Effects of feeding pipe diameter on the performance of a jet-driven helmholtz oscillator generating pulsed waterjets Journal of Mechanical Science and Technology 31(12) 1203-1212 DOI: 10.1007/s12206-017-0219-9
[8] Qiang C, Wang F and Guo C 2019 Study on impact stress of abrasive slurry jet in cutting stainless steel The International Journal of Advanced Manufacturing Technology 100(1-4) 297-309 DOI: 10.1007/978-1-4419-7302-3_9
[9] Wang R and Wang M 2010 A two-fluid model of abrasive waterjet Journal of Materials Processing Technology 210(1) 190-196
[10] Wang C Han Z and Liu C 2011 Mechanism analysis and simulation of water-jet propulsion Shenyang Jianzhu Gongcheng Xueyuan Xuebao (Ziran Kexue Ban) 27(1) 196-199 DOI: 10.1080/17686733.2016.1243749
[11] Fu S Z and Xie T Y 2017 Calculating investigation on pulsing flow for underwater radiated noise of the waterjet Chuan Bo Li Xue 21(5) 549-554
[12] Hua W, Zhang W, Li J, Shen T and Li X 2019 Measurement of flow field in waterjet nozzles
with different structures *Instrumentation Mesure Metrologie* 18(4) 381-388 DOI: 10.18280/i2m.180407

[13] Liang Z, Shan S, Liu X and Wen Y 2019 Fuzzy prediction of awj turbulence characteristics by using typical multi-phase flow models *Engineering Applications of Computational Fluid Mechanics* 11(1) 225-257 DOI: 10.1080/19942060.2016.1277556

[14] Basniev K S, Dmitriev N M and Chilingar G V 2012 Mechanics of fluid flow *Beverli* DOI: 10.1002/9781118533628

[15] Zhou X, Niu Z, Li Y, Sun X, Du Q, Jiao K and Xuan J 2019 Investigation of two-phase flow in the compressed gas diffusion layer microstructures *International Journal of Hydrogen Energy* 44(48) 26498-26516 DOI: 10.1016/j.ijhydene.2019.08.108

[16] Hirano T, Van Der Kolk N and Bilandzic A 2010 Hydrodynamics and flow *Lecture Notes in Physics* 785(48) 139-178 DOI: 10.1007/978-3-642-02286-9_4

[17] Wong C Y 2010 Foundation of hydrodynamics for systems with strong interactions *AIP Conference Proceedings "IV Mexican Meeting on Mathematical and Experimental Physics: Relativistic Fluids and Biological Physics"* 39-48 DOI: 10.1063/1.3533204

[18] Chaudhri A, Bell J B, Garcia A L and Donev A 2014 Modeling multiphase flow using fluctuating hydrodynamics *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics* 90(3) DOI: 10.1103/PhysRevE.90.033014

[19] Sun X, Sun B, Wang Z, Chen L and Gao Y 2017 A new model for hydrodynamics and mass transfer of hydrated bubble rising in deep water *Chemical Engineering Science* 173(48) 168-178. DOI: 10.1016/j.ces.2017.07.040

[20] Zahmatkesh I, Emdad H and Alishahi M M 2011 Importance of molecular interaction description on the hydrodynamics of gas mixtures *Scientia Iranica* 18(6) 1287-1296 DOI: 10.1016/j.ces.2011.08.032