Another proof for the removable singularities of the heat equation

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Abstract
We give two different simple proofs for the removable singularities of the heat equation in \((\Omega \setminus \{x_0\}) \times (0, T)\) where \(x_0 \in \Omega \subset \mathbb{R}^n\) is a bounded domain with \(n \geq 3\). We also give a necessary and sufficient condition for removable singularities of the heat equation in \((\Omega \setminus \{x_0\}) \times (0, T)\) for the case \(n = 2\).

Key words: removable singularities, heat equation
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Singularities of solutions of partial differential equations appear in many problems. For example singularities appears in the study of the solutions of the harmonic map [13] and the harmonic map heat flow [3]. In [14] S. Sato and E. Yanagida studied the solutions for a semilinear parabolic equation with moving singularities. Singularities of solutions also appears in the study of hyperbolic partial differential equations [15] and in the study of the touchdown behavior of the micro-electromechanical systems equation [4], [5], [6].

It is interesting to find the necessary and sufficient condition for the solutions of the equations to have removable singularities. In [8] S.Y. Hsu proved the following theorem.

**Theorem 1.** Let \(n \geq 3\) and let \(0 \in \Omega \subset \mathbb{R}^n\) be a domain. Suppose \(u\) is a solution of the heat equation

\[
  u_t = \Delta u
\]

in \((\Omega \setminus \{0\}) \times (0, T)\). Then \(u\) has removable singularities at \(\{0\} \times (0, T)\) if and only if for any \(0 < t_1 < t_2 < T\) and \(\delta \in (0, 1)\) there exists \(B_{R_0}(0) \subset \Omega\) depending on \(t_1, t_2\) and \(\delta\), such that

\[
  |u(x, t)| \leq \delta|x|^{2-n}
\]
for any $0 < |x| \leq R_0$ and $t_1 \leq t \leq t_2$.

The proof in [8] is based on the Green function estimates of [9] and a careful analysis of the behavior of the solution near the singularities using Dehnelian principle. In this paper we will use the Schauder estimates for heat equation [2, 12], and the technique of [11] and [17] to give two different simple proofs of the above result. We also obtain the following result for the solution of the heat equation in 2-dimension.

**Theorem 2.** Let $0 \in \Omega \subset \mathbb{R}^2$ be a domain. Suppose $u$ is a solution of the heat equation in $(\Omega \setminus \{0\}) \times (0, T)$. Then $u$ has removable singularities at $\{0\} \times (0, T)$ if and only if for any $0 < t_1 < t_2 < T$ and $\delta \in (0, 1)$ there exists $\overline{B_{R_0}}(0) \subset \Omega$ depending on $t_1$, $t_2$ and $\delta$, such that

$$ |u(x, t)| \leq \delta (\log(1/|x|))^{-1} \tag{3} $$

for any $0 < |x| \leq R_0$ and $t_1 \leq t \leq t_2$.

**Remark 3.** Note that the function $\log |x|$ satisfies the heat equation in $(\mathbb{R}^2 \setminus \{0\}) \times (0, \infty)$ but it has non-removable singularities on $\{0\} \times (0, \infty)$ and it does not satisfy (3). Hence (3) is sharp.

We start with some definitions. For any set $A$ we let $\chi_A$ be the characteristic function of the set $A$. Let $0 \in \Omega \subset \mathbb{R}^n$ be a bounded domain. We say that a solution $u$ of the heat equation (1) in $(\Omega \setminus \{0\}) \times (0, T)$ has removable singularities at $\{0\} \times (0, T)$ if there exists a classical solution $v$ of (1) in $\Omega \times (0, T)$ such that $u = v$ in $(\Omega \setminus \{0\}) \times (0, T)$. For any $R > 0$ let $B_R = B_R(0) = \{x : |x| < R\} \subset \mathbb{R}^n$.

**Proof of Theorem 1:** Suppose $u$ has removable singularities at $\{0\} \times (0, T)$. By the same argument as the proof in section 3 of [8] for any $0 < t_1 < t_2 < T$ and $\delta \in (0, 1)$ there exists $\overline{B_{R_0}} \subset \Omega$ depending on $t_1$, $t_2$ and $\delta$, such that (2) holds.

Suppose (2) holds. Then for any $0 < t_1 < t_2 < T$ and $\delta \in (0, 1)$ there exists $\overline{B_{R_0}} \subset \Omega$ depending on $t_1$, $t_2$ and $\delta$, such that (2) holds for any $0 < |x| \leq R_0$ and $t_1 \leq t \leq t_2$.

For any $0 < |x| \leq R_0$, let

$$ w(y, s) = u(|x|y, |x|^2s) \quad \forall 0 < |y| \leq R_0/|x|, t_1/|x|^2 \leq s \leq t_2/|x|^2. \tag{4} $$

Then $w$ is a solution of (1) in $(\overline{B_1} \setminus \{0\}) \times (|x|^{-2}t_1, |x|^{-2}t_2)$. By (2),

$$ |w(y, s)| \leq \delta(|x||y|)^{2-n} \quad \forall 0 < |y| \leq R_0/|x|, t_1/|x|^2 \leq s \leq t_2/|x|^2. \tag{5} $$

Let $t_1 < t_3 < t_2$. Then

$$ \frac{t_3}{|x|^2} - \frac{t_1}{|x|^2} \geq \frac{t_3 - t_1}{R_0^2} \tag{6} $$
By the parabolic Schauder estimates \([2], [12], (5)\) and \((6)\), there exists a constant \(C_1 > 0\) such that
\[
|\nabla w(y, s)| \leq C_1 \sup_{|x|^{-2} t_1 \leq \tau \leq |x|^{-2} t_2} w(z, \tau) \leq C_2 \delta |x|^{2-n}
\] (7)
holds for any \(2/3 \leq |y| \leq 3/4\), \(t_3/|x|^2 \leq s \leq t_2/|x|^2\) where \(C_2 = 2^{n-2} C_1\). By \((4)\) and \((7)\),
\[
|\nabla u(z, t)| \leq C_2 \delta |z|^{1-n} \quad \forall |z| = \frac{3}{4} |x|, 0 < |x| \leq R_0, t_3 \leq t \leq t_2
\]
\[
\Rightarrow |\nabla u(z, t)| \leq C_2 \delta |z|^{1-n} \quad \forall |z| \leq \frac{3}{4} R_0, t_3 \leq t \leq t_2.
\] (8)

Let \(R_1 = 3/(4R_0)\). We will now use a modification of the proof of Lemma 2.3 of \([1]\) and Lemma 2.1 of \([7]\) to complete the argument. We will first show that \(u\) satisfies \((1)\) in \(\Omega \times (t_1, t_2)\) in the distribution sense. Since \(u\) satisfies \((1)\) in \((\Omega \setminus \{0\}) \times (0, T)\), for any \(0 < \varepsilon < R_1\) and \(\eta \in C_0^\infty(\Omega \times (0, T))\) we have
\[
\int_{\Omega \setminus B_\varepsilon} w \eta dx \bigg|_{t_3}^{t_2} = \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} w \eta \, dx \, dt - \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} \nabla u \cdot \nabla \eta \, dx \, dt
\]
\[
- \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma \, dt
\] (9)
where \(\partial u/\partial n\) is the derivative of \(u\) with respect to the unit outward normal at \(\partial B_\varepsilon\). By \((8)\),
\[
\limsup_{\varepsilon \to 0} \left| \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma \, dt \right| \leq C_2 \delta (t_2 - t_3) |\partial B_1| \|\eta\|_{L^\infty}
\]
Since \(\delta > 0\) is arbitrary, there holds
\[
\lim_{\varepsilon \to 0} \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma \, dt \, dx \, dt = 0.
\] (10)
By \((8)\) and the Lebesgue dominated convergence theorem,
\[
\lim_{\varepsilon \to 0} \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} \nabla u \cdot \nabla \eta \, dx \, dt = \int_{t_3}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \eta \, dx \, dt
\] (11)
Letting \(\varepsilon \to 0\) in \((9)\), by \((10)\) and \((11)\) there holds
\[
\int_{\Omega} w \eta \, dx \bigg|_{t_3}^{t_2} = \int_{t_3}^{t_2} \int_{\Omega} w \eta \, dx \, dt - \int_{t_3}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \eta \, dx \, dt \quad \forall t_3 \in (t_1, t_2).
\] (12)
Hence $u$ is a distribution solution of (1) in $\Omega \times (t_1, t_2)$. By (2) for any $1 \leq p < \frac{n}{n-2}$ there exists a constant $C'_p > 0$ such that
\[
\sup_{t_1 \leq t \leq t_2} \int_{B_{R_0}} u(x, t)^p \, dx \leq C'_p \tag{13}
\]

By (12) and (13) and an argument similar to the proof of [11] and section 1 of [10] $u \in L^\infty_{loc}(B_{R_0} \times (t_1, t_2))$. We now let $v$ be the solution of

\[
\begin{cases}
  v_t = \Delta v & \text{in } B_{R_1} \times (t_3, t_2) \\
  \frac{\partial v}{\partial n}(x, t) = \frac{\partial u}{\partial n}(x, t) & \text{on } \partial B_{R_1} \times (t_3, t_2) \\
  v(x, t_3) = u(x, t_3) & \text{in } B_{R_1}.
\end{cases} \tag{14}
\]

For any $0 \leq h \in C_0^\infty(B_{R_1})$ and $t_3 < t \leq t_2$ let $\eta$ be the solution of

\[
\begin{cases}
  \eta_t + \Delta \eta = 0 & \text{in } B_{R_1} \times (t_3, t) \\
  \frac{\partial \eta}{\partial n}(x, t) = 0 & \text{on } \partial B_{R_1} \times (t_3, t) \\
  \eta(x, t) = h(x) & \text{in } B_{R_1}.
\end{cases} \tag{15}
\]

By the maximum principle,
\[
0 \leq \eta \leq \|h\|_{L^\infty} \quad \text{in } B_{R_1} \times (t_3, t). \tag{16}
\]

Then by (14) and (15),
\[
\int_{B_{R_1} \setminus B_\varepsilon} (u - v) \eta \, dx \bigg|_{t_3}^{t} = \int_{t_3}^{t} \int_{B_{R_1} \setminus B_\varepsilon} [(u - v)\eta_t + (u - v)\eta] \, dx \, dt
\]
\[
= \int_{t_3}^{t} \int_{B_{R_1} \setminus B_\varepsilon} [(u - v)\eta_t + \Delta(u - v)\eta] \, dx \, dt
\]
\[
= \int_{t_3}^{t} \int_{B_{R_1} \setminus B_\varepsilon} (u - v)(\eta_t + \Delta \eta) \, dx \, dt - \int_{t_3}^{t} \int_{\partial B_\varepsilon} \eta \frac{\partial (u - v)}{\partial n} \, d\sigma \, dt
\]
\[
+ \int_{t_3}^{t} \int_{\partial B_\varepsilon} (u - v) \frac{\partial \eta}{\partial n} \, d\sigma \, dt
\]
\[
= -\int_{t_3}^{t} \int_{\partial B_\varepsilon} \eta \frac{\partial (u - v)}{\partial n} \, d\sigma \, dt + \int_{t_3}^{t} \int_{\partial B_\varepsilon} (u - v) \frac{\partial \eta}{\partial n} \, d\sigma \, dt. \tag{17}
\]

By (2),
\[
\left| \int_{t_3}^{t} \int_{\partial B_\varepsilon} (u - v) \frac{\partial \eta}{\partial n} \, d\sigma \, dt \right| \leq C \varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{18}
\]
By (8) and (16),
\[ \limsup_{\varepsilon \to 0} \left| \int_{t_3}^{t_1} \int_{\partial B_{\varepsilon}} \eta \frac{\partial}{\partial n}(u - v) \, ds \right| dt \leq C\delta. \] (19)

Since \( \delta > 0 \) is arbitrary, by (19) there holds
\[ \lim_{\varepsilon \to 0} \left| \int_{t_3}^{t_1} \int_{\partial B_{\varepsilon}} \eta \frac{\partial}{\partial n}(u - v) \, ds \right| = 0. \] (20)

Letting \( \varepsilon \to 0 \) in (17), by (18) and (20),
\[ \int_{B_{R_1}} (u - v)(x, t) \, h(x) \, dx = \int_{B_{R_1}} (u - v)(x, t_3) \eta(x, t_3) \, dx = 0. \] (21)

We now choose a sequence of functions \( h_i \in C_{0}^{\infty}(B_{R_1}) \) converging to \( \chi_{\{u > v\}} \) a.e. \( x \in B_{R_1} \) as \( i \to \infty \). Putting \( h = h_i \) in (21) and letting \( i \to 0 \),
\[ \int_{B_{R_1}} (u - v)(x, t) \, dx = 0 \quad \forall t_3 < t \leq t_2. \] (22)

By interchanging the role of \( u \) and \( v \) we get
\[ \int_{B_{R_1}} (v - u)(x, t) \, dx = 0 \quad \forall t_3 < t \leq t_2. \] (23)

Hence by (22) and (23),
\[ \int_{B_{R_1}} |v - u|(x, t) \, dx = 0 \quad \forall t_3 < t \leq t_2 \]
\[ \Rightarrow \quad u(x, t) = v(x, t) \quad \forall 0 < |x| \leq R_1, t_3 < t \leq t_2. \] (24)

Hence \( u \) has removable singularities on \( \{0\} \times (t_3, t_2) \). Since \( 0 < t_1 < t_3 < t_2 < T \) is arbitrary, \( u \) has removable singularities on \( \{0\} \times (0, T) \) and the theorem follows.

**Proof of Theorem 2**: Theorem 2 follows by an argument very similar to the proof of Theorem 1 but with (3) replacing (2) in the argument.

**An alternate proof of Theorems 1 and 2**: We will show that when (2) (respectively (3)) holds, then \( u \) has removable singularities at \( \{0\} \times (0, T) \). Suppose (2) holds if \( n \geq 3 \) and (3) holds if \( n = 2 \). We first observe that by the previous argument for any \( 0 < t_1 < t_2 < T \) \( u \) satisfies (12) and \( u \in L_{loc}^{\infty}(\Omega \times (0, T)) \). Let \( \overline{B}_{R_1} \subset \Omega \) and let \( w \) be the solution of
\[
\begin{cases}
  & w_t = \Delta w \quad \text{in} \ B_{R_1} \times (t_1, t_2) \\
  & w = u \quad \text{on} \ \overline{B}_{R_1} \times \{t_1\} \cup \partial B_{R_1} \times (t_1, t_2).
\end{cases}
\]

By the maximum principle,
\[ \|w\|_{L^{\infty}} \leq \|u\|_{L^{\infty}(B_{R_1} \times (t_1, t_2))} < \infty. \] (25)
For any $\varepsilon > 0$, let
\[
    w_\varepsilon = \begin{cases} 
        w - u + \varepsilon |x|^{2-n} & \text{if } n \geq 3 \\
        w - u + \varepsilon \log(R_1/|x|) & \text{if } n = 2.
    \end{cases}
\]

Then $w_\varepsilon$ satisfies
\[
    \begin{cases}
        w_{\varepsilon,t} = \Delta w_\varepsilon & \text{in } (B_{R_1} \setminus \{0\}) \times (t_1, t_2) \\
        w_\varepsilon \geq u & \text{on } \partial B_{R_1} \times (t_1, t_2) \cup \overline{B_{R_1}} \times \{t_1\}.
    \end{cases}
\]

By (2), (3), and (25) there exists a constant $0 < r_0 < R_1$ such that
\[
    w_\varepsilon \geq 0 \quad \text{on } \partial B_{r_1} \times [t_1, t_2]
\]
for all $0 < r_1 \leq r_0$. By the maximum principle in $(B_{R_1} \setminus B_{r_1}) \times (t_1, t_2)$,
\[
    w_\varepsilon \geq 0 \quad \text{in } (B_{R_1} \setminus B_{r_1}) \times (t_1, t_2)
\]
\[
    \Rightarrow \begin{cases}
        w - u + \varepsilon |x|^{2-n} \geq 0 & \forall r_1 \leq |x| \leq R_1, t_1 \leq t \leq t_2 \quad \text{if } n \geq 3 \\
        w - u + \varepsilon \log(R_0/|x|) \geq 0 & \forall r_1 \leq |x| \leq R_1, t_1 \leq t \leq t_2 \quad \text{if } n = 2
    \end{cases}
\]
\[
    \Rightarrow w \geq u \quad \forall 0 < |x| \leq R_1, t_1 \leq t \leq t_2 \quad \text{as } r_1 \to 0, \varepsilon \to 0. \tag{26}
\]

Similarly by considering the function
\[
    v_\varepsilon = \begin{cases} 
        w - u - \varepsilon |x|^{2-n} & \text{if } n \geq 3 \\
        w - u - \varepsilon \log(R_1/|x|) & \text{if } n = 2
    \end{cases}
\]
and applying the maximum principle and letting $\varepsilon \to 0$ we get
\[
    w \leq u \quad \forall 0 < |x| \leq R_1, t_1 \leq t \leq t_2. \tag{27}
\]

By (26) and (27) we get (24) and Theorem 2 and Theorem 3 follows.

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