ON CONSTRUCTING SPECIAL LAGRANGIAN SUBMANIFOLDS BY GLUING

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Abstract. The purpose of this paper is to give an application of the gluing theorem for special Lagrangian submanifolds of a Calabi-Yau 3-fold. In [2], a gluing theorem is proved to smooth a codimension-two singularity of a particular special Lagrangian submanifold. In this paper we will show that this theorem can be applied to more general cases where different special Lagrangians are intersecting and gives a way of constructing new special Lagrangian submanifolds. As an example we will show that a smooth special Lagrangian submanifold can be obtained from five copies of $\mathbb{RP}^3$ intersecting pairwise in a quintic.

1. Introduction

Due to SYZ conjecture [4], one important problem in Mirror Symmetry is to find a suitable compactification of the moduli space of special Lagrangian submanifolds. In particular, one should understand the singularities of this moduli space. For this purpose, in [2], we showed that a particular special Lagrangian submanifold with an irreducible singularity can be a limit point in this space by proving the following theorem:

**Theorem 1.1.** [2] Given a connected immersed special Lagrangian submanifold $L^3$ of a Calabi-Yau manifold $X^6$ with a particular irreducible, orthogonal self intersection $K$ of codimension-two (singularity of type $z_1z_2 = 0$) it can be approximated by a sequence of smooth special Lagrangian submanifolds and therefore $L$ is a limit point in the moduli space.

In this paper we will show that Theorem 1.1 can be applied to more general cases where different special Lagrangians are intersecting and hence we can construct new special Lagrangian submanifolds nearby by gluing. In particular we will modify Theorem 1.1 as follows:

**Theorem 1.2.** Given a singular special Lagrangian submanifold $L$ which is invariant under a geometric $\mathbb{Z}_m$ action and consists of pairwise, orthogonal and cyclic intersections of special Lagrangian submanifolds $L_1, ..., L_m$ of a three dimensional Calabi-Yau manifold $X$, it can be approximated by a sequence of smooth special Lagrangian submanifolds.

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approximated by a sequence of smooth special Lagrangian submanifolds obtained by smoothing each codimension-two intersections $K_i$ (singularity of type $z_1z_2 = 0$)

2. The Eigenvalue Estimate

In this section we will explain how Theorem 1.1 can be generalized to pairwise intersections of several special Lagrangian submanifolds.

In what follows, $X$ will denote a 3-dimensional Calabi-Yau manifold with $m$ different special Lagrangians $L_1, ..., L_m$ intersecting pairwise (locally $z_i \cdot \overline{z}_{i+1} = 0$ for $z_i \in L_i$, $1 \leq i \leq m - 1$) and perpendicularly with respect to the induced metric along curves $K_i$ for each $i$. Moreover we will assume that $L_m$ intersects $L_1$ to complete the cycle and $Z_m$ is acting isometrically on $X$ and the singular special Lagrangian $L = \cup L_i$ is invariant under this action.

In [2], for a given singular special Lagrangian submanifold $L$ and a gluing parameter $\delta$, we first construct an approximate special Lagrangian submanifold $H_\delta$ in an open ball $V$ around the singular set $K$ and use the Implicit Function Theorem to prove that there exists a true special Lagrangian submanifold nearby. In order to prove the existence of a true special Lagrangian we need to get a uniform estimate for the right inverse for the linearized operator $D_\delta$. Here we need to do the same for each intersection.

Remark 2.1: Note that since we assume $K$ is irreducible, Theorem 1.1 is not strong enough to be applied directly to the case where we glue two arbitrary special Lagrangian submanifolds. Even though we need this irreducibility condition only in proving the eigenvalue estimate for the linearized operator and that other parts of the proof do not require this condition it is a crucial restriction on the gluing model. Without this assumption Theorem 1.1 is not true. However, one can automatically obtain irreducibility requirement if singular special Lagrangian is replaced by a group of intersecting totally symmetric special Lagrangian submanifolds which complete a cycle. Totally symmetric means that $Z_m$ acting isometrically on $X$ and $L = \cup L_i \subset X$ is invariant under this action. Completing a cycle means that if $L$ consists of $m$ special Lagrangian submanifolds $L_1, ..., L_m$ then $L_i$ intersects $L_{i+1}$ along curves $K_i$ for $1 \leq i \leq m - 1$ and $L_m$ intersects $L_1$ along $K_m$.

Given the gluing parameters $\delta_1, ..., \delta_m$ we can smooth the singularities of the form $z_1z_{i+1} = 0$ for each intersection inside open neighbourhoods $V_i$ around $K_i$ and construct approximate special Lagrangians $H_\delta^i$.
which agree with $L_i$ and $L_{i+1}$ outside a tubular neighbourhood of their intersection $K_i$. As before one can show that for each $H^i$ the linearized operator for the special Lagrangian equation is also $\Psi_i \cdot \Delta^i_{\delta_i}$ for each intersection $K_i$ where $\Psi_i$ is a small function for small values of $\delta_i$. Therefore it is sufficient to check the invertibility of $\Delta^i_{\delta_i}$. Also note that $Z_m$ is acting on $L$ isometrically and therefore it is sufficient to check the invertibility for only one intersection.

Next, we will modify the eigenvalue estimates for each Laplacian operator $\Delta^i_{\delta_i}$ which are needed in the gluing theorem [2], and in section 3 we will apply this to an example.

**Lemma 2.1.** There are constants $C_i > 0$ ($1 \leq i \leq m$) independent of the gluing parameters $\delta_i$, such that for $\delta_i$ sufficiently small, the first (nonzero) eigenvalues $\lambda_1(\Delta^i_{\delta_i})$ of $\Delta^i_{\delta_i}$ are bounded below by $C_i$.

**Proof:** As in [2], we prove it by contradiction. Note that we have $m$ Laplacian operators $\Delta^1_{\delta_1}, ..., \Delta^m_{\delta_m}$ for $H^1, ..., H^m$ and since $Z_m$ is acting isometrically we can assume that they are all equivalent. Therefore the analysis reduces to the case where two special Lagrangian submanifolds are intersecting as before, [2].

Suppose that the lemma is not true for $\Delta^1_{\delta_1}$ in $L_1 \cup L_2$. Then we may assume that the first eigenvalue $\lambda_1(\Delta^1_{\delta_1})$ converges to zero as $\delta_1$ tends to zero. Since we have the equivalence coming from the isometric action we can drop the index 1 in $\delta_1$. Let $\phi_\delta$ be the eigenfunction of $\lambda_1(\Delta^1_{\delta_1})$ satisfying

$$\int_{H^1_{\delta_1}} |\phi_\delta|^2 = 1 \text{ and } \int_{\overline{H^1_{\delta_1}}} \phi_\delta = 0 \text{ and } \Delta^1_{\delta_1} \phi_\delta = \lambda_{1,\delta} \phi_\delta .$$

where $\lambda_{1,\delta}$ determines the dependence of the first eigenvalue on the gluing parameter $\delta$ and $\overline{H^1_{\delta_1}}$ is the connected union of smoothed approximate special Lagrangians $H^1_{\delta_1}$.

For small compact sets away from singularity, the $L^2_{\delta}$ norm is uniformly equivalent to the usual $L^2$ norm. On these compact sets there exists a subsequence of $\phi_n$ that converges smoothly to a limit $\Delta \phi_0 = 0$. Following the same argument for the sequence of compact sets, and passing to a diagonal subsequence, we obtain a nonzero eigenfunction $\phi_0$ as the limit defined in the complement of the singularity satisfying

$$\int |\phi_0|^2 = 1 \text{ and } \int \phi_0 = 0 .$$
We now explain why $\phi_0$ cannot be zero. If $\phi_0 = 0$ then for very small $\delta$, $\phi_\delta$ will be very small everywhere (almost zero) which contradicts the fact that

$$||\phi_\delta||_{L^2} \leq ||\phi_\delta||_{L^\infty} \text{ and our initial assumption } ||\phi_\delta||_{L^2} = 1.$$ 

So we have a nonzero function $\phi_0$ in the limit and since $\lambda_\delta \to 0$ we get $\Delta_0 \phi_0 = 0$. On a compact manifold the only harmonic functions are constant functions. Therefore $\phi_0$ should be some nonzero constant. On one component $\phi_\delta$ will converge to a constant and on the other component it will converge to another constant. Since $Z_m$ is acting isometrically on $L$ these two constants should be same and since $\int \phi_0 = 0$ this is only possible if $\phi_\delta$ converges to zero. This contradicts the fact that $\phi_0$ is nonzero.

One other possibility is the case when the eigenfunctions get trapped in the neck region and as the gluing parameter $\delta$ goes to 0 they converge to maps which are identically zero everywhere but blow up at one point. Here there is no need to study the concentration problem in the neck area because the analysis follows exactly the same way for each intersection as before [2].

Hence we can modify Theorem 1.1 as follows:

**Theorem 2.2.** Given a singular special Lagrangian submanifold $L$ which is invariant under a geometric $Z_m$ action and consists of pairwise, orthogonal and cyclic intersections of special Lagrangian submanifolds $L_1, \ldots, L_m$ of a three dimensional Calabi-Yau manifold $X$ (as in figure 1), it can be approximated by a sequence of smooth special Lagrangian submanifolds obtained by smoothing each codimension-two intersections $K_i$ (singularity of type $z_1 \overline{z}_2 = 0$)

![Figure 1. $L = \bigcup L_i$](image)

**Remark 2.2.** Here we assumed that $L_1, \ldots, L_m$ are intersecting orthogonally with respect to the induced metric but in the next example
we have to verify that this is true. We will do this by averaging the metric with some finite group and making it invariant under this group action.

3. The Example

In this section we will apply Theorem 2.2 to five intersecting copies of \( \mathbb{R}P^3 \), \[3\], to obtain new special Lagrangian submanifolds in a quintic.

Let \( X \) be a 3-dimensional Calabi-Yau manifold defined as a degree five hypersurface in \( \mathbb{C}P^4 \) given as follows:

\[
X = \{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0\} \subset \mathbb{C}P^4
\]

We will first write five different anti-holomorphic involutions \( f_1, \ldots, f_5 \) on \( X \). Then we will find the fixed point sets of these involutions and call them \( F_1, \ldots, F_5 \). By a theorem of R. Bryant \[1\], \( F_1, \ldots, F_5 \) will be five different special Lagrangian submanifolds of the quintic \( X \). Each of them can be visualized as the real part of \( X \) and is diffeomorphic to \( \mathbb{R}P^3 \). In our example they also intersect pairwise as in figure 2. Moreover we will write a \( \mathbb{Z}_5 \) action on \( \mathbb{C}P^4 \) which acts isometrically on \( X \).

Next, we will justify this figure. Let \( \xi = a + ib, (a, b \in \mathbb{R}) \) be the fifth root of unity. For \( z_i = (x_i, y_i) \), let \( L_1, L_2, L_3, L_4, L_5 \) be defined as follows:

\[
L_1 = \left\{ \text{fixed point set of the involution} \begin{array}{l}
    z_0 \rightarrow \overline{z}_0, \ z_1 \rightarrow \overline{\xi z}_1, \ z_2 \rightarrow \overline{\xi^2 z}_2, \ z_3 \rightarrow \overline{\xi^3 z}_3, \ z_4 \rightarrow \overline{\xi^4 z}_4 \\
    \end{array}\right\}
= \{x_0, x_1, x_2, x_3, x_4 \mid x_0^5 + (x_1 + i\frac{1-a}{6}x_1)^5 + (x_2 + i\frac{1-a}{6}x_2)^5 + x_3^5 + x_4^5 = 0\}
\]

\[
L_2 = \left\{ \text{fixed point set of the involution} \begin{array}{l}
    z_0 \rightarrow \overline{z}_0, \ z_1 \rightarrow \overline{z}_1, \ z_2 \rightarrow \overline{\xi z}_2, \ z_3 \rightarrow \overline{\xi^2 z}_3, \ z_4 \rightarrow \overline{\xi^3 z}_4 \\
    \end{array}\right\}
\]

\[
L_3 = \left\{ \text{fixed point set of the involution} \begin{array}{l}
    z_0 \rightarrow \overline{z}_0, \ z_1 \rightarrow \overline{\xi^2 z}_1, \ z_2 \rightarrow \overline{\xi^3 z}_2, \ z_3 \rightarrow \overline{\xi^4 z}_3, \ z_4 \rightarrow \overline{\xi^5 z}_4 \\
    \end{array}\right\}
\]

\[
L_4 = \left\{ \text{fixed point set of the involution} \begin{array}{l}
    z_0 \rightarrow \overline{z}_0, \ z_1 \rightarrow \overline{\xi^3 z}_1, \ z_2 \rightarrow \overline{\xi^4 z}_2, \ z_3 \rightarrow \overline{\xi^5 z}_3, \ z_4 \rightarrow \overline{\xi^1 z}_4 \\
    \end{array}\right\}
\]

\[
L_5 = \left\{ \text{fixed point set of the involution} \begin{array}{l}
    z_0 \rightarrow \overline{z}_0, \ z_1 \rightarrow \overline{\xi^4 z}_1, \ z_2 \rightarrow \overline{\xi^5 z}_2, \ z_3 \rightarrow \overline{\xi^1 z}_3, \ z_4 \rightarrow \overline{\xi^2 z}_4 \\
    \end{array}\right\}
\]
\[ \{x_0, x_1, x_2, x_3, x_4 \mid x_0^5 + x_1^5 + (x_2 + i \frac{1-a}{b} x_2)^5 + (x_3 + i \frac{1-a}{b} x_3)^5 + x_4^5 = 0 \} \]

Let \( L_3 = \left\{ \begin{array}{c}
\text{fixed point set of the involution} \\
z_0 \to \overline{z_0}, z_1 \to \overline{z_1}, z_2 \to \overline{z_2}, z_3 \to \overline{z_3}, z_4 \to \overline{z_4}
\end{array} \right\} \]

\[ \{x_0, x_1, x_2, x_3, x_4 \mid x_0^5 + x_1^5 + x_2^5 + (x_3 + i \frac{1-a}{b} x_3)^5 + (x_4 + i \frac{1-a}{b} x_4)^5 = 0 \} \]

\[ \{x_0, x_1, x_2, x_3, x_4 \mid (x_0 + i \frac{1-a}{b} x_0)^5 + x_1^5 + x_2^5 + x_3^5 + (x_4 + i \frac{1-a}{b} x_4)^5 = 0 \} \]

\[ \{x_0, x_1, x_2, x_3, x_4 \mid (x_0 + i \frac{1-a}{b} x_0)^5 + (x_1 + i \frac{1-a}{b} x_1)^5 + x_2^5 + x_3^5 + x_4^5 = 0 \} \]

The sets of intersection are as follows:

Since

\[ L_1 \cap L_2 = \left\{ \begin{array}{c}
z_0 + z_1 + z_2 + z_3 + z_4 = 0 \\
z_0 = \overline{z_0}, z_1 = \overline{z_1}, z_2 = \overline{z_2}, z_3 = \overline{z_3}, z_4 = \overline{z_4}
\end{array} \right\} \]

this implies

\[ K_1 = L_1 \cap L_2 = \{ z_1 = 0, z_3 = 0, z_0^5 + z_2^5 + z_4^5 = 0 \} \cong S^1, \]

and similarly we get

\[ K_2 = L_2 \cap L_3 = \{ z_2 = 0, z_4 = 0, z_0^5 + z_1^5 + z_3^5 = 0 \} \cong S^1, \]

\[ K_3 = L_3 \cap L_4 = \{ z_0 = 0, z_3 = 0, z_1^5 + z_2^5 + z_4^5 = 0 \} \cong S^1, \]

\[ K_4 = L_4 \cap L_5 = \{ z_1 = 0, z_4 = 0, z_0^5 + z_2^5 + z_3^5 = 0 \} \cong S^1, \]

\[ K_5 = L_5 \cap L_1 = \{ z_0 = 0, z_2 = 0, z_1^5 + z_3^5 + z_4^5 = 0 \} \cong S^1, \]

and for the other pairs we get

\[ L_1 \cap L_3 = \{ z_1 = 0, z_2 = 0, z_3 = 0, z_4 = 0, z_5 = 0 \} = \emptyset, \]

and similarly

\[ L_1 \cap L_4 = \emptyset, L_2 \cap L_4 = \emptyset, L_2 \cap L_5 = \emptyset, \text{ and } L_3 \cap L_5 = \emptyset. \]
Next, we will describe the $\mathbb{Z}_5 = \{g_0, g_1, g_2, g_3, g_4\}$ action which keeps $L = \bigcup L_i$ invariant.

For all $i = 0, 1, ..., 4$, $g_i$ induces a map $\tilde{g}_i : \mathbb{CP}^4 \to \mathbb{CP}^4$ defined as:

$\tilde{g}_0 = \text{id}$

$\tilde{g}_1 : (z_0, z_1, z_2, z_3, z_4) \to (z_4, z_0, z_1, z_2, z_3)$,

$\tilde{g}_2 : (z_4, z_0, z_1, z_2, z_3) \to (z_3, z_4, z_0, z_1, z_2)$

$\tilde{g}_3 : (z_3, z_4, z_0, z_1, z_2) \to (z_2, z_3, z_4, z_0, z_1)$

$\tilde{g}_4 : (z_2, z_3, z_4, z_0, z_1) \to (z_1, z_2, z_3, z_4, z_0)$ are cyclic permutations.

Since the involutions $f_i : \mathbb{CP}^4 \to \mathbb{CP}^4$ satisfy $f_i \circ \tilde{g}_i = \tilde{g}_i \circ f_{i+1}$ for all $i$ and the quintic $X$ is invariant under the maps $\tilde{g}_i$, this implies that $\tilde{g}_i : L_i = F_i \cap X \to L_{i+1} = F_{i+1} \cap X$ where $F_i$ are the fixed point sets of the involutions $f_i$. One can also easily show that $\tilde{g}_i$ will take the intersections $K_i = L_i \cap L_{i+1}$ to $K_{i+1} = L_{i+1} \cap L_{i+2}$. These will imply that $L = \bigcup L_i$ is invariant under the $\mathbb{Z}_5$ action.

As we mentioned in Remark 2.2 we need to verify that these special Lagrangian submanifolds intersect orthogonally with respect to the induced metric. Let $\tilde{G}$ be the group generated by the group of antiholomorphic involutions and $\mathbb{Z}_5$. By construction it is a finite group and we can average the ambient metric so that it is invariant under the group action generated by $\tilde{G}$. This invariance and representation theory will then imply that $L_i \cap L_{i+1}$ orthogonally for each $i$.

Then applying Theorem 2.2 we can smooth the singularities and obtain a smooth special Lagrangian submanifold which agrees with $L_1, ..., L_5$ outside the balls $V_1, ..., V_5$.

**Remark 3.1.** By taking involutions appropriately in other Calabi Yau manifolds one can construct different special Lagrangian submanifolds using the same gluing process. These examples will be discussed somewhere else.

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