On Relatively Prime Subsets and Supersets

Mohamed El Bachraoui

Dept. Math. Sci., United Arab Emirates University, P.O.Box 17551, Al-Ain, UAE
melbachraoui@uaeu.ac.ae

Abstract

A nonempty finite set of positive integers $A$ is relatively prime if $\gcd(A) = 1$ and it is relatively prime to $n$ if $\gcd(A \cup \{n\}) = 1$. The number of nonempty subsets of $A$ which are relatively prime to $n$ is $\Phi(A, n)$ and the number of such subsets of cardinality $k$ is $\Phi_k(A, n)$. Given positive integers $l_1, l_2, m_2, n$ such that $l_1 \leq l_2 \leq m_2$ we give $\Phi([1, m_1] \cup [l_2, m_2], n)$ along with $\Phi_k([1, m_1] \cup [l_2, m_2], n)$. Given positive integers $l, m$, and $n$ such that $l \leq m$ we count for any subset $A$ of $\{l, l+1, \ldots, m\}$ the number of its supersets in $[l, m]$ which are relatively prime and we count the number of such supersets which are relatively prime to $n$. Formulas are also obtained for corresponding supersets having fixed cardinalities. Intermediate consequences include a formula for the number of relatively prime sets with a nonempty intersection with some fixed set of positive integers.

Keywords: Relatively prime sets, Phi function, Möbius inversion.

Subject Class: 11A25, 11B05, 11B75.

1. Introduction

Throughout let $k, l, m, n$ be positive integers such that $l \leq m$, let $[l, m] = \{l, l+1, \ldots, m\}$, let $\mu$ be the Möbius function, and let $\lfloor x \rfloor$ be the floor of $x$. If $A$ is a set of integers and $d \neq 0$, then $\frac{A}{d} = \{a/d : a \in A\}$. A nonempty set of positive integers $A$ is called relatively prime if $\gcd(A) = 1$ and it is called relatively prime to $n$ if $\gcd(A \cup \{n\}) = \gcd(A, n) = 1$. Unless otherwise specified $A$ and $B$ will denote nonempty sets of positive integers. We will need the following basic identity on binomial coefficients stating that for nonnegative integers $L \leq M \leq N$

$$\sum_{j=M}^{N} \binom{j}{L} = \binom{N+1}{L+1} - \binom{M}{L+1}. \quad (1)$$

\footnote{Supported by RA at UAEU, grant: 02-01-2-11/09}
Definition 1. Let
\[
\Phi(A, n) = \#\{X \subseteq A : X \neq \emptyset \text{ and } \gcd(X, n) = 1\},
\]
\[
\Phi_k(A, n) = \#\{X \subseteq A : \#X = k \text{ and } \gcd(X, n) = 1\},
\]
\[
f(A) = \#\{X \subseteq A : X \neq \emptyset \text{ and } \gcd(X) = 1\},
\]
\[
f_k(A) = \#\{X \subseteq A : \#X = k \text{ and } \gcd(X) = 1\}.
\]

Nathanson in [5] introduced \(f(n), f_k(n), \Phi(n), \text{ and } \Phi_k(n)\) (in our terminology \(f([1, n]), f_k([1, n]), \Phi([1, n], n), \text{ and } \Phi_k([1, n], n)\) respectively) and gave their formulas along with asymptotic estimates. Formulas for \(f([m, n]), f_k([m, n]), \Phi([m, n], n), \text{ and } \Phi_k([m, n], n)\) are found in [3, 6] and formulas for \(\Phi([1, m], n)\) and \(\Phi_k([1, m], n)\) for \(m \leq n\) are obtained in [4]. Recently Ayad and Kihel in [2] considered phi functions for sets which are in arithmetic progression and obtained the following more general formulas for \(\Phi([l, m], n)\) and \(\Phi_k([l, m], n)\).

Theorem 1. We have
\[
\begin{align*}
(\text{a}) & \quad \Phi([l, m], n) = \sum_{d|n} \mu(d)2^{\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{(l-1)}{d} \right\rfloor}, \\
(\text{b}) & \quad \Phi_k([l, m], n) = \sum_{d|n} \mu(d)\left(\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{(l-1)}{d} \right\rfloor \right)/k.
\end{align*}
\]

2. Relatively prime subsets for \([1, m_1] \cup [l_2, m_2]\)

If \([1, m_1] \cap [l_2, m_2] = \emptyset\), then phi functions for \([1, m_1] \cup [l_2, m_2] = [1, m_2]\) are obtained by Theorem 1. So we may assume that \(1 \leq m_1 < l_2 \leq m_2\).

Lemma 1. Let
\[
\Psi(m_1, l_2, m_2, n) = \#\{X \subseteq [1, m_1] \cup [l_2, m_2] : l_2 \in X \text{ and } \gcd(X, n) = 1\},
\]
\[
\Psi_k(m_1, l_2, m_2, n) = \#\{X \subseteq [1, m_1] \cup [l_2, m_2] : l_2 \in X, \ |X| = k, \text{ and } \gcd(X, n) = 1\}.
\]

Then
\[
\begin{align*}
(\text{a}) & \quad \Psi(m_1, l_2, m_2, n) = \sum_{d|(l_2, n)} \mu(d)2^{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \frac{l_2}{d}}, \\
(\text{b}) & \quad \Psi_k(m_1, l_2, m_2, n) = \sum_{d|(l_2, n)} \mu(d)\left(\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \frac{l_2}{d}\right)/k - 1.
\end{align*}
\]

Proof. (a) Assume first that \(m_2 \leq n\). Let \(\mathcal{P}(m_1, l_2, m_2)\) denote the set of subsets of \([1, m_1] \cup [l_2, m_2]\) containing \(l_2\) and let \(\mathcal{P}(m_1, l_2, m_2, d)\) be the set of subsets \(X\) of \([1, m_1] \cup [l_2, m_2]\) such that \(l_2 \in X\) and \(\gcd(X, n) = d\). It is clear that the set \(\mathcal{P}(m_1, l_2, m_2)\) of cardinality \(2^{m_1+m_2-l_2}\) can be partitioned using the equivalence relation of having the same gcd (dividing \(l_2\) and \(n\)).
Moreover, the mapping $A \mapsto \frac{1}{d} A$ is a one-to-one correspondence between $\mathcal{P}(m_1, l_2, m_2, d)$ and the set of subsets $Y$ of $[1, \lfloor m_1/d \rfloor] \cup [l_2/d, \lfloor m_2/d \rfloor]$ such that $l_2/d \in Y$ and $\gcd(Y, n/d) = 1$. Then
\[
\# \mathcal{P}(m_1, l_2, m_2, d) = \Psi([m_1/d], l_2/d, [m_2/d], n/d).
\]
Thus
\[
2^{m_1 + m_2 - l_2} = \sum_{d \mid (l_2, n)} \# \mathcal{P}(m_1, l_2, m_2, d) = \sum_{d \mid (l_2, n)} \Psi([m_1/d], l_2/d, [m_2/d], n/d),
\]
which by the Möbius inversion formula extended to multivariable functions [3, Theorem 2] is equivalent to
\[
\Psi(m_1, l_2, m_2, n) = \sum_{d \mid (l_2, n)} \mu(d)2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d}
\]
Assume now that $m_2 > n$ and let $a$ be a positive integer such that $m_2 \leq n^a$. As $\gcd(X, n) = 1$ if and only if $\gcd(X, n^a) = 1$ and $\mu(d) = 0$ whenever $d$ has a nontrivial square factor, we have
\[
\Psi(m_1, l_2, m_2, n) = \Psi(m_1, l_2, m_2, n^a)
\]
\[
= \sum_{d \mid (l_2, n^a)} \mu(d)2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d}
\]
\[
= \sum_{d \mid (l_2, n)} \mu(d)2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d}.
\]
(b) For the same reason as before, we may assume that $m_2 \leq n$. Noting that the correspondence $X \mapsto \frac{1}{d} X$ defined above preserves the cardinality and using an argument similar to the one in part (a), we obtain the following identity
\[
\binom{m_1 + m_2 - l_2}{k - 1} = \sum_{d \mid (l_2, n)} \Psi_k([m_1/d], l_2/d, [m_2/d], n/d)
\]
which by the Möbius inversion formula [3, Theorem 2] is equivalent to
\[
\Psi_k(m_1, l_2, m_2, n) = \sum_{d \mid (l_2, n)} \mu(d)\binom{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d}{k - 1},
\]
as desired.

**Theorem 2.** We have
\[
(\text{a}) \quad \Phi([1, m_1] \cup [l_2, m_2], n) = \sum_{d \mid n} \mu(d)2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - \lfloor l_2/d \rfloor},
\]
\[
(\text{b}) \quad \Phi_k([1, m_1] \cup [l_2, m_2], n) = \sum_{d \mid n} \mu(d)\binom{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - \lfloor l_2/d \rfloor}{k - 1}.
\]
Proof. (a) Clearly
\[ \Phi([1, m_1] \cup [l_2, m_2], n) = \Phi([1, m_1] \cup [l_2 - 1, m_2], n) - \Psi(m_1, l_2 - 1, m_2, n) \]
\[ = \Phi([1, m_1] \cup [m_1 + 1, m_2], n) - \sum_{i=m_1+1}^{l_2-1} \Psi(m_1, i, m_2, n) \]
\[ = \Phi([1, m_2]) - \sum_{i=m_1+1}^{l_2-1} \Psi(m_1, i, m_2, n) \] (2)
\[ = \sum_{d|n} \mu(d)2^{\left\lfloor \frac{m_1}{d} \right\rfloor} - \sum_{i=m_1+1}^{l_2-1} \sum_{d|n} \mu(d)2^{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \frac{i}{d}}, \]
where the last identity follows by Theorem 1 for \( l = 1 \) and Lemma 1. Rearranging the last summation in (2) gives
\[ \sum_{i=m_1+1}^{l_2-1} \sum_{d|(n,i)} \mu(d)2^{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \frac{i}{d}} = \sum_{d|n} \mu(d)2^{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \frac{i}{d}} \]
\[ = \sum_{d|n} \mu(d)2^{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor} \sum_{j=\left\lfloor \frac{m_1}{d} \right\rfloor + 1}^{\left\lfloor \frac{l_2-1}{d} \right\rfloor} 2^{-j} \]
\[ = \sum_{d|n} \mu(d)2^{\left\lfloor \frac{m_2}{d} \right\rfloor} \left( 1 - 2^{-\left\lfloor \frac{l_2-1}{d} \right\rfloor + \left\lfloor \frac{m_1}{d} \right\rfloor} \right). \] (3)

Now combining identities (2 3) yields the result.

(b) Proceeding as in part (a) we find
\[ \Phi_k([1, m_1] \cup [l_2, m_2], n) = \sum_{d|n} \mu(d)\left( \left\lfloor \frac{m_2}{d} \right\rfloor \right) - \sum_{i=m_1+1}^{l_2-1} \sum_{d|n} \mu(d)\left( \left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \frac{i}{d} \right). \] (4)

Rearranging the last summation on the right of (4) gives
\[ \sum_{i=m_1+1}^{l_2-1} \sum_{d|(n,i)} \left( \left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \frac{i}{d} \right) = \sum_{d|n} \mu(d) \sum_{j=\left\lfloor \frac{m_1}{d} \right\rfloor + 1}^{\left\lfloor \frac{l_2-1}{d} \right\rfloor} \left( \left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - j \right) \]
\[ = \sum_{d|n} \mu(d) \sum_{i=\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \left\lfloor \frac{l_2-1}{d} \right\rfloor}^{\left\lfloor \frac{m_1}{d} \right\rfloor} \frac{i}{k-1} \]
\[ = \sum_{d|n} \mu(d) \left( \left( \left\lfloor \frac{m_1}{d} \right\rfloor \right) - \left( \left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \left\lfloor \frac{l_2-1}{d} \right\rfloor \right) \right), \] (5)
where the last identity follows by formula (1). Then identities (4 5) yield the desired result. □
Definition 2. Let
\[ \varepsilon(A, B, n) = \#\{ X \subseteq B : X \neq \emptyset, X \cap A = \emptyset, \text{ and } \gcd(X, n) = 1 \}, \]
\[ \varepsilon_k(A, B, n) = \#\{ X \subseteq B : \#X = k, X \cap A = \emptyset, \text{ and } \gcd(X, n) = 1 \}. \]

If \( B = [1, n] \) we will simply write \( \varepsilon(A, n) \) and \( \varepsilon_k(A, n) \) rather than \( \varepsilon(A, [1, n], n) \) and \( \varepsilon_k(A, [1, n], n) \) respectively.

Theorem 3. If \( l \leq m < n \), then
\[ (a) \quad \varepsilon([l, m], n) = \sum_{d|n} \mu(d)2^{|(l-1)/d| + n/d - \lfloor m/d \rfloor}, \]
\[ (b) \quad \varepsilon_k([l, m], n) = \sum_{d|n} \mu(d)\left(\frac{|(l-1)/d| + n/d - \lfloor m/d \rfloor}{k}\right). \]

Proof. Immediate from Theorem 2 since
\[ \varepsilon([l, m], n) = \Phi([1, l - 1] \cup [m + 1, n], n) \text{ and } \varepsilon_k([l, m], n) = \Phi_k([1, l - 1] \cup [m + 1, n], n). \]

\[ \square \]

3. Relatively prime supersets

In this section the sets \( A \) and \( B \) are not necessary nonempty.

Definition 3. If \( A \subseteq B \) let
\[ \overline{\Phi}(A, B, n) = \#\{ X \subseteq B : X \neq \emptyset, A \subseteq X, \text{ and } \gcd(X, n) = 1 \}, \]
\[ \overline{\Phi_k}(A, B, n) = \#\{ X \subseteq B : A \subseteq X, \#X = k, \text{ and } \gcd(X, n) = 1 \}, \]
\[ \overline{\mathcal{F}}(A, B) = \#\{ X \subseteq B : X \neq \emptyset, A \subseteq X, \text{ and } \gcd(X) = 1 \}, \]
\[ \overline{\mathcal{F}_k}(A, B) = \#\{ X \subseteq B : \#X = k, A \subseteq X, \text{ and } \gcd(X) = 1 \}. \]

The purpose of this section is to give formulas for \( \overline{\mathcal{F}}(A, [l, m]), \overline{\mathcal{F}}(A, [l, m]), \overline{\Phi}(A, [l, m], n), \) and \( \overline{\Phi_k}(A, [l, m], n) \) for any subset \( A \) of \([l, m]\). We need a lemma.

Lemma 2. If \( A \subseteq [1, m] \), then
\[ (a) \quad \overline{\Phi}(A, [1, m], n) = \sum_{d|(A, n)} \mu(d)2^{|m/d| - \#A}, \]
\[ (b) \quad \overline{\Phi_k}(A, [1, m], n) = \sum_{d|(A, n)} \mu(d)\left(\frac{|m/d| - \#A}{k - \#A}\right) \text{ whenever } \#A \leq k \leq m. \]
Proof. If \( A = \emptyset \), then clearly
\[
\Phi(A, [1, m], n) = \Phi([1, m], n) \quad \text{and} \quad \Phi_k(A, [1, m], n) = \Phi_k([1, m], n)
\]
and the identities in (a) and (b) follow by Theorem 1 for \( l = 1 \). Assume now that \( A \neq \emptyset \). If \( m \leq n \), then
\[
2^{m - \#A} = \sum_{d|\Phi(A, [1, \lfloor m/d \rfloor], n/d)} \Phi\left(\frac{A}{d}, [1, \lfloor m/d \rfloor], n/d\right)
\]
and
\[
\binom{m - \#A}{k - \#A} = \sum_{d|\Phi_k(A, [1, \lfloor m/d \rfloor], n/d)} \mu(d) \Phi_k\left(\frac{A}{d}, [1, \lfloor m/d \rfloor], n/d\right)
\]
which by Möbius inversion \([3, \text{Theorem 2}]\) are equivalent to the identities in (a) and in (b) respectively. If \( m > n \), let \( a \) be a positive integer such that \( m \leq na \). As \( \gcd(X, n^a) = 1 \) if and only if \( \gcd(X, n) = 1 \) and \( \mu(d) = 0 \) whenever \( d \) has a nontrivial square factor we have
\[
\Phi(A, [1, m], n) = \Phi(A, [1, m^n], n) = \sum_{d|\Phi(A, [1, m], n^a)} \mu(d) 2^{[m/d] - \#A}
\]
and
\[
\Phi_k(A, [1, m], n) = \sum_{d|\Phi_k(A, [1, m], n^a)} \mu(d) 2^{[m/d] - \#A - \#A^k}
\]
The same argument gives the formula for \( \Phi_k(A, [1, m], n) \).

\[\square\]

**Theorem 4.** If \( A \subseteq [l, m] \), then

(a) \( \Phi(A, [l, m], n) = \sum_{d|\Phi(A, [l, m], n^a)} \mu(d) 2^{[m/d] - [(l - 1)/d] - \#A} \)

(b) \( \Phi_k(A, [l, m], n) = \sum_{d|\Phi_k(A, [l, m], n^a)} \mu(d) \binom{[m/d] - [(l - 1)/d] - \#A}{k - \#A} \)

whenever \( \#A \leq k \leq m - l + 1 \).

**Proof.** If \( A = \emptyset \), then clearly
\[
\Phi(A, [l, m], n) = \Phi([l, m], n)
\]
and
\[
\Phi_k(A, [l, m], n) = \Phi_k([l, m], n)
\]
and the identities in (a) and (b) follow by Theorem 1.
Assume now that \( A \neq \emptyset \). Let
\[
\Psi(A, l, m, n) = \#\{X \subseteq [l, m] : A \cup \{l\} \subseteq X, \text{ and } \gcd(X, n) = 1\}.
\]
Then
\[ 2^{m-l-\#A} = \sum_{d|\gcd(A,l)} \Psi\left(\frac{A}{d}, l/d, \lfloor m/d \rfloor, n/d\right), \]
which by Möbius inversion [3, Theorem 2] means that
\[ \Psi(A, l, m, n) = \sum_{d|\gcd(A,l,n)} \mu(d) 2^{\lfloor m/d \rfloor - l/d - \#A}. \]  
(6)

Then combining identity (6) with Lemma 2 gives
\[ \Phi(A, [l, m], n) = \Phi([A, [1, m], n) - \sum_{i=1}^{l-1} \Psi(i, m, A, n) = \sum_{d|\gcd(A,n)} \mu(d) 2^{\lfloor m/d \rfloor - l/d - \#A} \]
(7)

This completes the proof of (a). Part (b) follows similarly.

As to \( \overline{f}(A, [l, m]) \) and \( \overline{f}_k(A, [l, m]) \) we similarly have:

**Theorem 5.** If \( A \subseteq [l, m] \), then

(a) \( \overline{f}(A, [l, m]) = \sum_{d|\gcd(A)} \mu(d) 2^{\lfloor m/d \rfloor - \lfloor \frac{l-1}{d} \rfloor - \#A} \),

(b) \( \overline{f}_k(A, [l, m]) = \sum_{d|\gcd(A)} \mu(d) \left( \lfloor \frac{m}{d} \rfloor - \lfloor \frac{l-1}{d} \rfloor - \#A \right), \) whenever \( \#A \leq k \leq m - l + 1 \).

We close this section by formulas for relatively prime sets which have a nonempty intersection with \( A \).

**Definition 4.** Let
\[ \overline{e}(A, B, n) = \# \{ X \subseteq B : X \cap A \neq \emptyset \text{ and } \gcd(X, n) = 1 \}, \]
\[ \overline{e}_k(A, B, n) = \# \{ X \subseteq B : \#X = k, X \cap A \neq \emptyset \text{ and } \gcd(X, n) = 1 \}, \]
\[ \overline{e}(A, B) = \# \{ X \subseteq B : X \cap A \neq \emptyset \text{ and } \gcd(X) = 1 \}, \]
\[ \overline{e}_k(A, B) = \# \{ X \subseteq B : \#X = k, X \cap A \neq \emptyset \text{ and } \gcd(X) = 1 \}. \]
Theorem 6. We have

(a) \( \varepsilon(A, [l, m], n) = \sum_{\emptyset \neq X \subseteq A} \sum_{d \mid (X, n)} \mu(d) 2^{\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor \cdot \#X}, \)

(b) \( \varepsilon_k(A, [l, m], n) = \sum_{\emptyset \neq X \subseteq A} \sum_{d \mid (X, n)} \mu(d) \left( \frac{\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor}{k} \cdot \#X \right), \)

(c) \( \varepsilon(A, B) = \sum_{\emptyset \neq X \subseteq A} \sum_{d \mid \gcd(X)} \mu(d) 2^{\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor \cdot \#X}, \)

(d) \( \varepsilon_k(A, B) = \sum_{\emptyset \neq X \subseteq A} \sum_{d \mid \gcd(X)} \mu(d) \left( \frac{\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor}{k} \cdot \#X \right). \)

Proof. These formulas Follow by Theorems 4, 5 and the facts that

\[ \varepsilon(A, [l, m], n) = \sum_{\emptyset \neq X \subseteq A} \Phi(X, [l, m], n), \]

\[ \varepsilon_k(A, [l, m], n) = \sum_{\emptyset \neq X \subseteq A \atop \#X \leq k} \Phi_k(X, [l, m], n), \]

\[ \varepsilon(A, [l, m]) = \sum_{\emptyset \neq X \subseteq A} \sum_{d \mid (X, n)} \mu(d) \left( \frac{\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor}{k} \cdot \right), \]

\[ \varepsilon_k(A, [l, m]) = \sum_{\emptyset \neq X \subseteq A \atop \#X \leq k} \sum_{d \mid \gcd(X)} \mu(d) \left( \frac{\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor}{k} \cdot \right). \]

\[ \square \]

References

[1] Mohamed Ayad and Omar Kihel, On the Number of Subsets Relatively Prime to an Integer, Journal of Integer Sequences, Vol. 11, (2008), Article 08.5.5.

[2] Mohamed Ayad and Omar Kihel, On Relatively Prime Sets, Integers 9, (2009), 343-352.

[3] Mohamed El Bachraoui, The number of relatively prime subsets and phi functions for sets \{m, m + 1, \ldots, n\}, Integers 7 (2007), A43, 8pp.

[4] Mohamed El Bachraoui, On the Number of Subsets of [1, m] Relatively Prime to n and Asymptotic Estimates, Integers 8 (2008), A 41, 5 pp.
[5] Melvyn B. Nathanson, *Affine invariants, relatively prime sets, and a phi function for subsets of \( \{1, 2, \ldots, n\} \),* Integers 7 (2007), A01, 7pp.

[6] Melvyn B. Nathanson and Brooke Orosz, *Asymptotic estimates for phi functions for subsets of \( \{m + 1, m + 2, \ldots, n\} \),* Integers 7 (2007), A54, 5pp.