Linear-Quadratic Stochastic Differential Games on Directed Chain Networks

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Consider the finite system of particles \((X_{t,i}^{(u)}, t \geq 0, \ i = 1, \ldots, N)\) defined by the system of stochastic differential equations

\[
dX_{t,i}^{(u)} = b(t, X_{t,i}^{(u)}, \hat{F}_{t,i}^{(u)}) dt + dW_{t,i}; \quad t \geq 0, \quad i = 1, \ldots, N - 1
\]

where

\[
\hat{F}_{t,i}^{(u)}(\cdot) := u \cdot \delta_{X_{t,i+1}^{(u)}}(\cdot) + (1 - u) \cdot \frac{1}{N} \sum_{j=1}^{N} \delta_{X_{t,j}^{(u)}}(\cdot), \quad i = 1, \ldots, N
\]

with the periodic condition \(X_{t,N+1} = X_{t,1}\).

\(u = 0\): pure Mean Field, and \(u = 1\): pure Directed Chain
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\(u = 0\): pure Mean Field, and \(u = 1\): pure Directed Chain

Ref: *Directed Chain Stochastic Differential Equations* with N. Detering and T. Ichiba, SPTA 2020
Assume $X_{0,i}$ are i.i.d. and independent of the $W$’s. **In the limit** $N \to \infty$, in distribution, one given particle (say the first one) can be described by the following equation for $(X^{(u)}, \tilde{X}^{(u)})$:

$$dX^{(u)}_t = b(t, X^{(u)}_t, F^{(u)}_t) \, dt + dB_t; \quad t \geq 0,$$

driven by a Brownian motion $(B_t, t \geq 0)$, where $F^{(u)}$ is the weighted probability measure

$$F^{(u)}_t(\cdot) := u \cdot \delta_{\tilde{X}^{(u)}_t}(\cdot) + (1 - u) \cdot \mathcal{L}_{X^{(u)}_t}(\cdot),$$

with

$$\text{Law}(X^{(u)}_t, t \geq 0) \equiv \text{Law}(\tilde{X}^{(u)}_t, t \geq 0)$$

$$\sigma(\tilde{X}^{(u)}_t, t \geq 0) \perp \perp \sigma(B_t, t \geq 0).$$
Choosing $b(t, x, \mu) := -\int_{\mathbb{R}} (x - y) \mu(dy)$, the problem becomes:

$$dX_t^{(u)} = u (\tilde{X}_t^{(u)} - X_t^{(u)}) dt + (1 - u)(\mathbb{E}[X_t^{(u)}] - X_t^{(u)}) dt + dB_t$$

For a fixed initial value $X_0^{(u)} = 0$, we have $\mathbb{E}[X_t^{(u)}] = \mathbb{E}[\tilde{X}_t^{(u)}] = 0$
The Linear Case: Infinite Dimensional OU

Choosing \( b(t, x, \mu) := -\int_{\mathbb{R}} (x - y) \mu(dy) \), the problem becomes:

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**Explicitly solvable Gaussian pair** \((X_t^{(u)}, \tilde{X}_t^{(u)})\):

\[
X_t^{(u)} = u \int_0^t e^{-(t-s)} \tilde{X}_s^{(u)} ds + \int_0^t e^{-(t-s)} dB_s
\]

\[
\tilde{X}_t^{(u)} = \sum_{k=1}^{\infty} \int_0^t \mathbb{P}_{1,k}(t-s; u) dW_{s,k}
\]

\[
\mathbb{P}_{1,k}(t-s; u) := \frac{u^{k-1}(t-s)^{k-1}}{(k-1)!} e^{-(t-s)}
\]

where \( (W^k, k \geq 1) \) are independent Brownian motions, independent of the Brownian motion \( B \)
The Linear Case: Summary

The previous formulas lead to explicit computation of variances and covariances.

Different behaviors for different values of $u$ in the linear Gaussian case:

| $u$     | Interaction Type       | Asymptotic Variance | Asymptotic Dependence (Propagation of chaos) |
|---------|------------------------|---------------------|---------------------------------------------|
| $u = 0$ | Pure mean-field        |                     | Independent                                 |
| $u \in (0, 1)$ | Mixed interaction     | Stabilized          | Dependent                                   |
| $u = 1$ | Pure directed chain    | **Explosive**       |                                             |
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Next, we introduce a **game feature** with the following question in mind:

- **Is this dynamics an equilibrium?**
Consider first the pure directed chain network.

\[ dX_i(t) = \alpha_i(t) \, dt + \sigma \, dW_i(t), \]

\[ i = 1, \ldots, N, \]

The system starts at time \( t = 0 \) from \( i_i \).

\( d \) random variables \( X_i(0) = \xi_i \) independent of the Brownian motions and such that \( E(\xi_i) = 0 \).

Player \( i \) chooses its own strategy \( \alpha_i \) in order to minimize its objective function of the form:

\[ J_i(\alpha_1, \ldots, \alpha_N) = E\left\{ \int_0^T \left( \frac{1}{2}(\alpha_i(t))^2 + \epsilon^2(X_{i+1}(t) - X_i(t))^2 \right) dt + c^2 (X_{i+1}(T) - X_i(T))^2 \right\}, \]

for constants \( \epsilon > 0 \) and \( c \geq 0 \), and a BC for \( J_N \):

Periodic BC:

\[ X_{N+1} = X_1 \]

Free BC:

autonomous stochastic control for \( X_N \)

We are looking at an LQ differential game on a directed network since \( X_i \) interacts only with \( X_{i+1} \) through the cost functions.
$N$-Player Directed Chain Game Model

With Yichen Feng and Tomoyuki Ichiba, submitted 2020

Consider first the pure directed chain network.

The system starts at time $t = 0$ from $i$. Random variables $X_i^0 = \xi_i$ independent of the Brownian motions and such that $E(\xi_i) = 0$.

Player $i$ chooses its own strategy $\alpha_i$ in order to minimize its objective function of the form:

$$J_i(\alpha_1, \ldots, \alpha_N) = E\left\{ \int_0^T \left( \frac{1}{2}(\alpha_i^2 + \epsilon^2(X_i^t + 1 - X_i^{t+1}))^2 \right) dt + c^2 (X_i^{T+1} - X_i^T)^2 \right\},$$

for constants $\epsilon > 0$ and $c \geq 0$, and a BC for $J_N$:

Periodic BC: $X_N^{T+1} = X_1^T$
Free BC: autonomous stochastic control for $X_N$

We are looking at an $LQ$ differential game on a directed network since $X_i$ interacts only with $X_{i+1}$ through the cost functions.
Consider first the pure directed chain network.

\[ dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \quad i = 1, \cdots, N, \]

The system starts at time \( t = 0 \) from i.i.d. random variables \( X_0^i = \xi_i \) independent of the Brownian motions and such that \( \mathbb{E}(\xi_i) = 0 \).

Player \( i \) chooses its own strategy \( \alpha_t^i \) in order to minimize its objective function of the form:

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J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E} \left\{ \int_0^T \left( \frac{1}{2}(\alpha_t^i)^2 + \frac{\epsilon}{2}(X_{t+1}^i - X_t^i)^2 \right) dt + \frac{c}{2}(X_{T+1}^i - X_T^i)^2 \right\}
\]

for constants \( \epsilon > 0 \) and \( c \geq 0 \), and a BC for \( J^N \):

**Periodic BC:** \( X^{N+1} = X^1 \)

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$N$-Player Directed Chain Game Model

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Consider first the pure directed chain network.

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for constants $\epsilon > 0$ and $c \geq 0$, and a BC for $J^N$:

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We are looking at an $LQ$ differential game on a directed network since $X^i$ interacts only with $X^{i+1}$ through the cost functions.
Construction of Open-Loop Nash Equilibria

The Hamiltonian for player $i$ is given by:

\[ H^i(x^1, \ldots, x^N, y^{i,1}, \ldots, y^{i,N}, \alpha^1, \ldots, \alpha^N) = \sum_{k=1}^{N} \alpha^k y^{i,k} + \frac{1}{2} (\alpha^i)^2 + \frac{\epsilon}{2} (x^{i+1} - x^i)^2 \]
Construction of Open-Loop Nash Equilibria

The Hamiltonian for player $i$ is given by:

$$H^i(x^1, \ldots, x^N, y^i, 1, \ldots, y^i, N, \alpha^1, \ldots, \alpha^N) = \sum_{k=1}^{N} \alpha^k y^{i,k} + \frac{1}{2} (\alpha^i)^2 + \frac{\epsilon}{2} (x^{i+1} - x^i)^2$$

The adjoint processes $Y^i_t = (Y^i_t; j = 1, \ldots, N)$ and $Z^i_t = (Z^i_t; j, k = 1, \ldots, N)$ for $i = 1, \ldots, N$ solve the BSDE:

$$dY^{i,j}_t = - \partial_{x^j} H^i(X_t, Y^i_t, \alpha_t) dt + \sum_{k=1}^{N} Z^{i,j,k}_t dW^k_t$$

$$= -\epsilon (X^{i+1}_t - X^i_t)(\delta_{i+1,j} - \delta_{i,j}) dt + \sum_{k=1}^{N} Z^{i,j,k}_t dW^k_t,$$

$$Y^{i,j}_T = c(X_{T}^{i+1} - X^i_{T})(\delta_{i+1,j} - \delta_{i,j})$$
The Hamiltonian for player $i$ is given by:

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$$= -\epsilon (X^{i+1}_t - X^i_t)(\delta_{i+1,j} - \delta_{i,j}) dt + \sum_{k=1}^{N} Z^{i,j,k}_t dW^k_t,$$

$$Y^{i,j}_T = c(X^{i+1}_T - X^i_T)(\delta_{i+1,j} - \delta_{i,j})$$

For $j \neq i$ or $i + 1$, $dY^{i,j}_t = \sum_{k=1}^{N} Z^{i,j,k}_t dW^k_t$ and $Y^{i,j}_T = 0$ implies $Z^{i,j,k}_t = 0$. 
By **Pontryagin stochastic maximum principle**, we get an open-loop Nash equilibrium by minimizing the Hamiltonian $H^i$ with respect to $\alpha^i$:

$$\partial_{\alpha^i} H^i = y^i + \alpha^i = 0 \quad \text{leads to the choice: } \hat{\alpha}^i = -y^i.$$ 

Then, one needs to solve the coupled FBSDE system for $X$ and $(Y, Z)$.
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Then, one needs to solve the coupled FBSDE system for $X$ and $(Y, Z)$. Since the problem is **Linear Quadratic** and the system is translation invariant with periodic BC, we make the ansatz:

$$Y^{i,i}_t = \sum_{j=0}^{N-1} \phi_t^{N,j} X^{i+j}_t,$$

for some **deterministic scalar functions** $\phi_t$’s satisfying the terminal conditions: $\phi_T^{N,0} = c$, $\phi_T^{N,1} = -c$, $\phi_T^{N,k} = 0$ for $k \geq 2$. 

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**Note:** One needs $\phi_{N,i,j}^t$ in the (non-stationary) free BC case.
By **Pontryagin stochastic maximum principle**, we get an open-loop Nash equilibrium by minimizing the Hamiltonian $H^i$ with respect to $\alpha^i$:

$$\partial_{\alpha^i} H^i = y_{i,i}^i + \alpha^i = 0 \quad \text{leads to the choice:} \quad \hat{\alpha}^i = -y_{i,i}^i.$$ 

Then, one needs to solve the coupled FBSDE system for $X$ and $(Y, Z)$. Since the problem is **Linear Quadratic** and the system is translation invariant with periodic BC, we make the ansatz:

$$Y_{t,i}^i = \sum_{j=0}^{N-1} \phi_t^{N,j} X_{t+j}^i,$$

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**Note that one needs $\phi_t^{N;i,j}$ in the (non-stationary) free BC case.**
The **forward equations** become

\[ dX_t^i = - \sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j} dt + \sigma dW_t^i \]

Differentiating the ansatz \( Y_t^{i,i} = \sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j} \), we get:

\[
\begin{align*}
    dY_t^{i,i} &= \sum_{j=0}^{N-1} \left[ X_t^{i+j} \phi_t^{N,j} dt + \phi_t^{N,j} dX_t^{i+j} \right] \\
    &= \sum_{j=0}^{N-1} X_t^{i+j} \phi_t^{N,j} dt - \sum_{j=0}^{N-1} \phi_t^{N,j} \sum_{k=0}^{N-1} \phi_t^{N,k} X_t^{i+j+k} dt + \sum_{j=0}^{N-1} \sigma \phi_t^{N,j} dW_t^{i+j}
\end{align*}
\]
Comparing with the **backward equations**, the **martingale terms** give:

\[ Z_{t}^{i,i,0} = 0; \quad Z_{t}^{i,i,k} = \sigma_{t}^{N,N+k-i} \text{ for } 1 \leq k < i; \quad Z_{t}^{i,i,k} = \sigma_{t}^{N,k-i} \text{ for } i \leq k \leq N. \]
Comparing with the **backward equations**, the martingale terms give:

\[ Z^i,i,0_t = 0; \ Z^i,i,k = \sigma_t \phi^{N,N+k-i} \] for \( 1 \leq k < i \); \[ Z^i,i,k = \sigma_t \phi^{N,k-i} \] for \( i \leq k \leq N \).

From the **drift terms**:

\[ \dot{\phi}^N = \phi^N \cdot \dot{\phi}^N + \sum_{i=1}^{N-1} \phi^N \cdot \phi^{N,N-i} = \epsilon, \quad \phi^N_T = c, \]

\[ \dot{\phi}^N_1 = \phi^N \cdot \dot{\phi}^N_1 + \phi^N \cdot \dot{\phi}^N_0 + \sum_{i=2}^{N-1} \phi^N \cdot \phi^{N,N+1-i} + \epsilon, \quad \phi^N_T = -c, \]

\[ \dot{\phi}^N_k = \sum_{j=0}^{k} \phi^N \cdot \phi^{N,N-k-j} + \sum_{i=k+1}^{N-1} \phi^N \cdot \phi^{N,N+k-i}, \quad \phi^N_T = 0, \]

\[ \dot{\phi}^N_{N-1} = \sum_{j=0}^{N-1} \phi^N \cdot \phi^{N,N-1-j}, \quad \phi^N_T = 0, \]
Comparing with the backward equations, the martingale terms give:

\[ Z_{t}^{i,i,0} = 0; \quad Z_{t}^{i,i,k} = \sigma\phi_{t}^{N,N+k-i} \quad \text{for } 1 \leq k < i; \quad Z_{t}^{i,i,k} = \sigma\phi_{t}^{N,k-i} \quad \text{for } i \leq k \leq N. \]

From the drift terms:

\[
\begin{align*}
\dot{\phi}_{t}^{N,0} &= \phi_{t}^{N,0} \cdot \phi_{t}^{N,0} + \sum_{i=1}^{N-1} \phi_{t}^{N,i} \cdot \phi_{t}^{N,N-i} - \epsilon, \quad \phi_{T}^{N,0} = c, \\
\dot{\phi}_{t}^{N,1} &= \phi_{t}^{N,0} \cdot \phi_{t}^{N,1} + \phi_{t}^{N,1} \cdot \phi_{t}^{N,0} + \sum_{i=2}^{N-1} \phi_{t}^{N,i} \cdot \phi_{t}^{N,N+1-i} + \epsilon, \quad \phi_{T}^{N,1} = -c, \\
\dot{\phi}_{t}^{N,k} &= \sum_{j=0}^{k} \phi_{t}^{N,j} \cdot \phi_{t}^{N,k-j} + \sum_{i=k+1}^{N-1} \phi_{t}^{N,i} \cdot \phi_{t}^{N,N+k-i}, \quad \phi_{T}^{N,k} = 0, \\
\dot{\phi}_{t}^{N,N-1} &= \sum_{j=0}^{N-1} \phi_{t}^{N,j} \cdot \phi_{t}^{N,N-1-j}, \quad \phi_{T}^{N,N-1} = 0, \quad \text{(or free BC)}
\end{align*}
\]
Comparing with the *backward equations*, the martingale terms give:

\[ Z_t^{i,i,0} = 0; \quad Z_t^{i,i,k} = \sigma \phi_t^{N,N+k-i} \quad \text{for } 1 \leq k < i; \quad Z_t^{i,i,k} = \sigma \phi_t^{N,k-i} \quad \text{for } i \leq k \leq N. \]

From the drift terms:

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\dot{\phi}_t^{N,0} &= \phi_t^{N,0} \cdot \phi_t^{N,0} + \sum_{i=1}^{N-1} \phi_t^{N,i} \phi_t^{N,N-i} - \epsilon, \quad \phi_T^{N,0} = c, \\
\dot{\phi}_t^{N,1} &= \phi_t^{N,0} \cdot \phi_t^{N,1} + \phi_t^{N,1} \cdot \phi_t^{N,0} + \sum_{i=2}^{N-1} \phi_t^{N,i} \phi_t^{N,N+1-i} + \epsilon, \quad \phi_T^{N,1} = -c, \\
\dot{\phi}_t^{N,k} &= \sum_{j=0}^{k} \phi_t^{N,j} \phi_t^{N,k-j} + \sum_{i=k+1}^{N-1} \phi_t^{N,i} \phi_t^{N,N+k-i}, \quad \phi_T^{N,k} = 0, \\
\dot{\phi}_t^{N,N-1} &= \sum_{j=0}^{N-1} \phi_t^{N,j} \phi_t^{N,N-1-j}, \quad \phi_T^{N,N-1} = 0, \quad \text{(or free BC)}
\end{align*}
\]

Matrix Riccati equation:

\[ \dot{\Phi}^N(t) = \Phi^N(t)\Phi^N(t) - E, \quad \Phi^N(T) := C \]
Convergence as $N \to \infty$: numerical results

Blue line: $\phi_t^{N,0} \to 1$. Orange line $\phi_t^{N,1} \to -\frac{1}{2}$, Other lines: $\phi_t^{N,k} \to 0$ for $k \geq 2$. Left: $N = 4$. Right: $N = 100$. 
Convergence as $N \to \infty$: numerical results

$$N - 1 \sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$$

for different values of $N$
By either **taking the limit** $N \to \infty$ in the previous system,

\[
\dot{\phi}_0(t) = \phi_0(t) \cdot \phi_0(t) - \epsilon, \quad \phi_0(T) = c, \\
\dot{\phi}_1(t) = 2 \phi_0(t) \cdot \phi_1(t) + \epsilon, \quad \phi_1(T) = -c, \\
k \geq 2 : \dot{\phi}_k(t) = \phi_0(t) \cdot \phi_k(t) + \phi_1(t) \cdot \phi_k(t) - \phi_{k-1}(t) + \cdots + \phi_{k-1}(t) \cdot \phi_1(t) + \phi_k(t) \cdot \phi_0(t), \quad \phi_k(T) = 0.
\]

Define $S_t(z) = \sum_{k=0}^{\infty} z^k \cdot \phi_k(t)$ where $0 \leq z < 1$ and $\phi_k(t)$ to avoid confusion.

Then $\dot{S}_t(z) = (S_t(z))^2 - \epsilon (1 - z)$,

$S_T(z) = c (1 - z)$.

Taking $z = 0$ gives $\phi(0)$ explicitly, and $z = 1$ implies $\sum_{k=1}^{\infty} \phi_k(t) = -\phi(0)$.
The $\infty$-Player Directed Chain Game

By either taking the limit $N \to \infty$ in the previous system, or by considering directly the $\infty$-player game,
The $\infty$-Player Directed Chain Game

By either taking the limit $N \to \infty$ in the previous system, or by considering directly the $\infty$-player game, one obtains:

\[
\begin{align*}
\dot{\phi}_t^0 &= \phi_t^0 \cdot \phi_t^0 - \epsilon, & \phi_T^0 &= c, \\
\dot{\phi}_t^1 &= 2\phi_t^0 \cdot \phi_t^1 + \epsilon, & \phi_T^1 &= -c, \\
\dot{\phi}_t^k &= \phi_t^0 \cdot \phi_t^k + \phi_t^1 \cdot \phi_t^{k-1} + \cdots + \phi_t^{k-1} \cdot \phi_t^1 + \phi_t^k \cdot \phi_t^0, & \phi_T^k &= 0.
\end{align*}
\]
By either **taking the limit** \( N \to \infty \) **in the previous system**, or by **considering directly the \( \infty \)-player game**, one obtains:

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\dot{\phi}_t^0 &= \phi_t^0 \cdot \phi_t^0 - \epsilon, \quad \phi_T^0 = c, \\
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\dot{\phi}_t^k &= \phi_t^0 \cdot \phi_t^k + \phi_t^1 \cdot \phi_t^{k-1} + \cdots + \phi_t^{k-1} \cdot \phi_t^1 + \phi_t^k \cdot \phi_t^0, \quad \phi_T^k = 0.
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Define \( S_t(z) = \sum_{k=0}^{\infty} z^k \cdot \phi_t^{(k)} \) where \( 0 \leq z < 1 \) and \( \phi_t^{(k)} = \phi_t^k \) to avoid confusion.

Then

\[
\dot{S}_t(z) = (S_t(z))^2 - \epsilon(1 - z), \quad S_T(z) = c(1 - z).
\]
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By either taking the limit \(N \to \infty\) in the previous system, or by considering directly the \(\infty\)-player game, one obtains:

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\begin{align*}
\dot{\phi}_t^0 &= \phi_t^0 \cdot \phi_t^0 - \epsilon, \quad \phi_T^0 = c, \\
\dot{\phi}_t^1 &= 2\phi_t^0 \cdot \phi_t^1 + \epsilon, \quad \phi_T^1 = -c, \\
k \geq 2: \quad \dot{\phi}_t^k &= \phi_t^0 \cdot \phi_t^k + \phi_t^1 \cdot \phi_t^{k-1} + \cdots + \phi_t^{k-1} \cdot \phi_t^1 + \phi_t^k \cdot \phi_t^0, \quad \phi_T^k = 0.
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Define \(S_t(z) = \sum_{k=0}^{\infty} z^k \cdot \phi_t^{(k)}\) where \(0 \leq z < 1\) and \(\phi_t^{(k)} = \phi_t^k\) to avoid confusion. Then

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Taking \(z = 0\) gives \(\phi_t^{(0)}\) explicitly, and \(z = 1\) implies \(\sum_{k=1}^{\infty} \phi_t^{(k)} = -\phi_t^{(0)}\).
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By either taking the limit $N \to \infty$ in the previous system, or by considering directly the $\infty$-player game, one obtains:

\[
\begin{align*}
\dot{\phi}_0^t &= \phi_0^t \cdot \phi_0^t - \epsilon, \quad \phi_0^T = c, \\
\dot{\phi}_1^t &= 2\phi_0^t \cdot \phi_1^t + \epsilon, \quad \phi_1^T = -c, \\
\dot{\phi}_k^t &= \phi_0^t \cdot \phi_k^t + \phi_1^t \cdot \phi_{k-1}^t + \cdots + \phi_{k-1}^t \cdot \phi_1^t + \phi_k^t \cdot \phi_0^t, \quad \phi_k^T = 0.
\end{align*}
\]

Define $S_t(z) = \sum_{k=0}^{\infty} z^k \cdot \phi_k^t$ where $0 \leq z < 1$ and $\phi_k^t = \phi_k^t$ to avoid confusion. Then

\[
\dot{S}_t(z) = (S_t(z))^2 - \epsilon(1-z), \quad S_T(z) = c(1-z).
\]

Taking $z = 0$ gives $\phi_t^{(0)}$ explicitly, and $z = 1$ implies $\sum_{k=1}^{\infty} \phi_t^{(k)} = -\phi_t^{(0)}$.

The model under equilibrium can be written as an $\infty$-dim OU:

\[
dX_t^i = -\sum_{j=0}^{\infty} \phi_j^t X_t^{i+j} dt + \sigma dW_t^i
\]

\[
= \phi_0^t \left[ \sum_{j=1}^{\infty} \left( -\frac{\phi_j^t}{\phi_0^t} \right) X_t^{i+j} - X_t^i \right] dt + \sigma dW_t^i,
\]

which is not exactly the model presented at the beginning.
Catalan Markov Chain

To simplify the presentation we assume $\epsilon = 1$, and we take the limit $T \to \infty$ so that the $\phi_t^k$'s become constant numbers.

They satisfy the recurrence relation:

$\phi_0 = 1$, $\phi_1 = -\frac{1}{2}$, and $\sum_{k=0}^{n} \phi_k \phi_{n-k} = 0$ for $n \geq 2$, which is related to the recurrence relation of Catalan numbers.

Using the moment generating function method, we get the constant solution (or the Catalan functions):

$\phi_0 = 1$, $\phi_1 = -\frac{1}{2}$, $\phi_k = -\frac{(2^k - 3)!}{(k-2)! k! 2^{k-2}}$ for $k \geq 2$.

Let $p_1 = -\phi_1 = \frac{1}{2}$, $p_k = -\phi_k = \frac{(2^k - 3)!}{(k-2)! k! 2^{k-2}}$ for $k \geq 2$. We consider a continuous-time Markov chain $M(\cdot)$ in the state space $\mathbb{N}$ with generator matrix $Q$.
Catalan Markov Chain

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Catalan Markov Chain

To simplify the presentation we assume $\epsilon = 1$, and we take the limit $T \to \infty$ so that the $\phi^k_t$'s become constant numbers. They satisfy the recurrence relation: $\phi^0 = 1$, $\phi^1 = -\frac{1}{2}$, and $\sum_{k=0}^{n} \phi^k \phi^{n-k} = 0$ for $n \geq 2$, which is related to the recurrence relation of Catalan numbers. Using the moment generating function method, we get the constant solution (or the Catalan functions): $\phi^0 = 1$, $\phi^1 = -\frac{1}{2}$,

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Let $p_1 = -\phi^1 = \frac{1}{2}$, $p_k = -\phi^k = \frac{(2k-3)!}{(k-2)!k!2^{2k-2}}$ for $k \geq 2$. We consider a continuous-time Markov chain $M(\cdot)$ in the state space $\mathbb{N}$ with generator matrix $Q =$

$$
\begin{pmatrix}
-1 & p_1 & p_2 & p_3 & \cdots \\
0 & -1 & p_1 & p_2 & \ddots \\
0 & 0 & -1 & p_1 & \ddots \\
& & & & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots
\end{pmatrix}.
$$
The infinite particle system \((X^1_t, X^2_t, \cdots)\) is represented as the solution of the stochastic evolution equation:

\[
dX_t = Q X_t dt + dW_t,
\]

with its solution \(X_t = e^{tQ}x_0 + \int_0^t e^{(t-s)Q} dW_s\), assuming \(X_0 = 0\) w.l.o.g.
Equilibrium Dynamics

The infinite particle system \((X_t^1, X_t^2, \cdots)\) is represented as the solution of the stochastic evolution equation:

\[
dX_t = Q X_t dt + dW_t,
\]

with its solution \(X_t = e^{tQ}x_0 + \int_0^t e^{(t-s)Q}dW_s\), assuming \(X_0 = 0\) w.l.o.g. Taking advantage of the Jordan block structure of \(Q^2\):

\[
Q^2 = \begin{pmatrix}
1 & -1 & 0 & \cdots \\
0 & 1 & -1 & \ddots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix},
\]

one can show that

\[
X_t^j = \sum_{k=j}^{\infty} \int_0^t \frac{(t-s)^{2(k-j)}}{(k-j)!} \rho_{k-j}(- (t-s)^2) e^{-(t-s)} dW_s^k,
\]

where the functions \(\rho_k\)'s are given by
Variance Stabilization

\[ \rho_k(-\nu^2) = \frac{e^{\nu}}{2^k \nu^k} \sqrt{\frac{2\nu}{\pi}} K_{k-\frac{1}{2}}(\nu); \quad k \geq 1, \quad \rho_0 \equiv 1, \]

where \( K_n(x) \) is the modified Bessel function of the second kind, i.e.,

\[ K_n(x) = \int_0^\infty e^{-x \cosh t} \cosh(nt) dt; \quad n > -1, \ x > 0. \]
Variance Stabilization

\[ \rho_k(-\nu^2) = \frac{e^\nu}{2^{k\nu}k^\nu} \sqrt{\frac{2\nu}{\pi}} K_{k-\frac{1}{2}}(\nu); \quad k \geq 1, \quad \rho_0 \equiv 1, \]

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\[ K_n(x) = \int_0^\infty e^{-x \cosh t} \cosh(nt) dt; \quad n > -1, \ x > 0. \]

We have

\[ \text{Var}(X^1_t) = \sum_{k=1}^\infty \int_0^t \frac{2}{\pi (k!)^2 4^k} \left( K_{k-\frac{1}{2}}(\nu) \right)^2 d\nu + \frac{1 - e^{-2t}}{2} \]

and we derive that the variance stabilizes

\[ \lim_{t \to \infty} \text{Var}(X^1_t) = \frac{1}{\sqrt{2}} < \infty \]
\[
\rho_k(-\nu^2) = \frac{e^\nu}{2^{k\nu_k}} \sqrt{\frac{2\nu}{\pi}} K_{\frac{1}{2}k}(\nu) ; \quad k \geq 1, \quad \rho_0 \equiv 1 ,
\]
where \( K_n(x) \) is the modified Bessel function of the second kind, i.e.,

\[
K_n(x) = \int_0^\infty e^{-x \cosh t} \cosh(nt) dt ; \quad n > -1, \; x > 0.
\]

We have

\[
\text{Var}(X_t^1) = \sum_{k=1}^{\infty} \int_0^t \frac{2}{\pi} \frac{\nu^{2k+1}}{(k!)^2 4^k} (K_{\frac{1}{2}k}(\nu))^2 d\nu + \frac{1 - e^{-2t}}{2}
\]

and we derive that the \textbf{variance stabilizes}

\[
\lim_{t \to \infty} \text{Var}(X_t^1) = \frac{1}{\sqrt{2}} < \infty
\]

but we also show that \textbf{asymptotic dependence persists}.
Mixed Games

As before:

\[ dX^i_t = \alpha^i_t dt + \sigma dW^i_t; \quad i = 1, 2, \ldots. \]

By choosing \( \alpha^i_t \), player \( i \) tries to minimize:

\[
J^i(\alpha^1, \ldots) = \mathbb{E}\left\{ \int_0^T \left[ \frac{1}{2}(\alpha^i_t)^2 + u \frac{\epsilon}{2} (X^{i+1}_t - X^i_t)^2 + (1 - u) \frac{\epsilon}{2} (m_t - X^i_t)^2 \right] dt + u \frac{c}{2} (X^{i+1}_T - X^i_T)^2 + (1 - u) \frac{c}{2} (m_T - X^i_T)^2 \right\},
\]

for some constants \( \epsilon > 0, c \geq 0 \) and \( u \in [0, 1] \).

If \( u < 1 \), \( m_t \) is thought of as a candidate for the limit of \( \frac{1}{N} \sum_{i=1}^N X^i_t \) as \( N \to \infty \).
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+ u \frac{c}{2}(X^{i+1}_T - X^i_T)^2 + (1 - u) \frac{c}{2}(m_T - X^i_T)^2 \right\},
\]

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One needs to be careful and start with the \( N \)-player game or simply set \( m_t = \mathbb{E}\{X^i_t\} \).
Open-Loop Nash Equilibria

The generalized Hamiltonian for individual $i$ is given by:

$$H^i(x^1, x^2, \ldots, y^{i,1}, y^{i,2}, \ldots, \alpha^1, \alpha^2, \ldots) = \sum_{k=1}^{\infty} \alpha^k y^{i,k} + \frac{1}{2} (\alpha^i)^2 + u \frac{\epsilon}{2} (x^{i+1} - x^i)^2 + (1 - u) \frac{\epsilon}{2} (m_t - x^i)^2$$

Then, one writes the generalized BSDEs for the adjoint processes and the ansatz:

$$Y_t^{i,i} = u \sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j - (1 - u) \psi_t (m_t - X_t^i)$$
Open-Loop Nash Equilibria

The generalized Hamiltonian for individual \( i \) is given by:

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\]

Then, one writes the generalized BSDEs for the adjoint processes and the ansatz:

\[
Y^{i,i}_t = u \sum_{j=i}^{\infty} \phi^{j-i}_t X^j_t - (1 - u) \psi_t(m_t - X^i_t)
\]

The computational machinery presented in the case \( u = 0 \) can be extended to the case \( u > 0 \) by using a killed Catalan Markov chain. That leads to computation of variances and covariances.
Asymptotic Behavior

\[ \text{Var}(X_t) = \sum_{k=1}^{\infty} \int_0^t \frac{2u^{2k}}{\pi(k!)^2 4^k} \nu^{2k+1} (K_{k-\frac{1}{2}}(\nu))^2 d\nu + \frac{1 - e^{-2t}}{2} \]

\[ \longrightarrow \frac{1}{2} (1 - \frac{u^2}{2})^{-\frac{1}{2}} \quad \text{as} \quad t \to \infty \]
Asymptotic Behavior

\[
\text{Var}(X_t) = \sum_{k=1}^{\infty} \int_0^t \frac{2u^{2k}}{\pi(k!)^2} \nu^{2k+1}(K_k-\frac{1}{2}(\nu))^2 d\nu + \frac{1-e^{-2t}}{2}
\]

\[
\rightarrow \frac{1}{2} \left(1 - \frac{u^2}{2}\right)^{-\frac{1}{2}} \quad \text{as} \quad t \rightarrow \infty
\]

To summarize:

| $u$       | Interaction Type      | Asymptotic Variance   | Asymptotic Dependence |
|-----------|-----------------------|-----------------------|-----------------------|
| $u = 0$   | Pure mean-field       | Stabilized            | Independent           |
| $u \in (0, 1)$ | Mixed interaction    | Stabilized            | Dependent             |
| $u = 1$   | Pure directed chain   | **Stabilized**        | Dependent             |
Figure: Directed Tree Network with $d$ direct descendants
Extension: Directed Tree Networks

At the $n^{th}$ generation:

$$dX_{t}^{n,k} = \alpha_{t}^{n,k} dt + \sigma dW_{t}^{n,k}, \quad 0 \leq t \leq T$$

Objective:

$$J^{n,k} = \mathbb{E} \left\{ \int_{0}^{T} \left[ \frac{1}{2} (\alpha_{t}^{n,k})^2 + \frac{\epsilon}{2} (\overline{X}_{t}^{n+1,k} - X_{t}^{n,k})^2 \right] dt + \frac{c}{2} (\overline{X}_{T}^{n+1,k} - X_{T}^{n,k})^2 \right\},$$

where $\overline{X}_{.}^{n,k} := \frac{1}{d} \sum_{i=(k-1)d+1}^{kd} X_{.}^{n,i}$ for some constants $\epsilon > 0$ and $c \geq 0$ and for $n, k \geq 1$. 
Extension: Directed Tree Networks

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$$dX_t^{n,k} = \alpha_t^{n,k} dt + \sigma dW_t^{n,k}, \quad 0 \leq t \leq T$$

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where $X_t^{n,k} := \frac{1}{d} \sum_{i=(k-1)d+1}^{kd} X_t^{n,i}$ for some constants $\epsilon > 0$ and $c \geq 0$ and for $n, k \geq 1$.

**Hamiltonian:**

$$H^{n,k} = \sum_{m=1}^{M_n} \sum_{l=1}^{d^{m-1}} \alpha^{m,l} Y_t^{n,k;m,l} + \frac{1}{2} (\alpha_t^{n,k})^2 + \frac{\epsilon}{2} \left( X_t^{n+1,k} - X_t^{n,k} \right)^2,$$

where only finitely many $Y_t^{n,k;m,l}$'s will be non-zero for every given $(n, k)$. Here, $M_n \in \mathbb{N}$ represents a depth of this finite dependence depending on $n$ with $M_n > n$ for $n \geq 1$. 
The **adjoint processes** $Y_{t}^{n,k} = (Y_{t}^{n,k;m,l}; m \in \mathbb{N}, 1 \leq l \leq d^{m-1})$ and $Z_{t}^{n,k} = (Z_{t}^{n,k;m,l;p,q}; m, p \in \mathbb{N}, 1 \leq l \leq d^{m-1}, 1 \leq q \leq d^{p-1})$ for $n \in \mathbb{N}, 1 \leq k \leq d^{n-1}$ are defined as the solutions of BSDEs

\[
dY_{t}^{n,k;m,l} = -\epsilon (\bar{X}_{t}^{n+1,k} - X_{t}^{n,k})(\delta^{n+1,k}_{m,l} - \delta^{n,k}_{m,l})dt + \sum_{p=1}^{\infty} \sum_{q=1}^{d^{p-1}} Z_{t}^{n,k;m,l;p,q} dW_{t}^{p,q},
\]

\[
Y_{T}^{n,k;m,l} = \partial_{X_{m,l}} g_{n,k}(X_{T}) = c(\bar{X}_{T}^{n+1,k} - X_{T}^{n,k})(\delta^{n+1,k}_{m,l} - \delta^{n,k}_{m,l}),
\]

where $\delta^{n,k}_{m,l} := 1$, if $(n, k) = (m, \ell)$; 0, otherwise, and $\delta^{n,k}_{m,l} := \frac{1}{d} \sum_{i=(k-1)d+1}^{kd} \delta^{n,i}_{m,l}$. 
The adjoint processes \( Y_{t}^{n,k} = (Y_{t}^{n,k;m,l}; m \in \mathbb{N}, 1 \leq l \leq d^{m-1}) \) and \( Z_{t}^{n,k} = (Z_{t}^{n,k;m,l;p,q}; m, p \in \mathbb{N}, 1 \leq l \leq d^{m-1}, 1 \leq q \leq d^{p-1}) \) for \( n \in \mathbb{N}, 1 \leq k \leq d^{n-1} \) are defined as the solutions of BSDEs

\[
dY_{t}^{n,k;m,l} = -\epsilon(X_{t}^{n+1,k} - X_{t}^{n,k})(\delta_{m,l}^{n,k} - \delta_{m,l})dt + \sum_{p=1}^{d^{p-1}} \sum_{q=1}^{d^{q-1}} Z_{t}^{n,k;m,l;p,q}dW_{t}^{p,q},
\]

\[
Y_{T}^{n,k;m,l} = \partial_{x_{m,l}}g_{n,k}(X_{T}) = c(X_{T}^{n+1,k} - X_{T}^{n,k})(\delta_{m,l}^{n,k} - \delta_{m,l}),
\]

where \( \delta_{m,l}^{n,k} := 1 \), if \((n, k) = (m, l)\); 0, otherwise, and \( \overline{\delta}_{m,l}^{n,k} = \frac{d}{d_{i}=(k-1)d+1} \delta_{m,l}^{n,i} \).

**Open-loop Nash equilibrium plus ansatz:**

\[
\hat{\alpha}_{t}^{n,k} = -Y_{t}^{n,k;n,k} = - \sum_{m=n}^{\infty} \phi_{t}^{m-n} \sum_{j=0}^{d^{m-n}-1} X_{t}^{m,d^{m-n}k-j}
\]
Riccati equations:

\[
\dot{\phi}_k^t = \sum_{j=0}^k \phi_j^t \phi_{k-j}^t - \epsilon \left( \delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right), \quad \phi_k^T = c \left( \delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right).
\]

WLOG, we assume \( \epsilon = 1 \) and \( \sigma = 1 \). By taking \( T \to \infty \), we look at the stationary long-time behavior of the Riccati system. Then the system gives the recurrence relation:

\[
\phi_0 = 1, \quad \phi_1 = -\frac{1}{2d}, \quad \sum_{k=0}^{\infty} \phi_{k-j} \phi_j^n = 0 \quad \text{for} \quad k \geq 0.
\]

By using a moment generating function method, we obtain the stationary solution:

\[
\phi_0 = 1, \quad \phi_1 = -\frac{1}{2d}, \quad \phi_k = -\frac{(2k-3)!}{(k-2)! k!} \cdot \frac{1}{d} \quad \text{for} \quad k \geq 2.
\]

As in the case \( d = 1 \), we can use a Catalan Markov chain to derive explicit formulas, in particular for the asymptotic variance:

\[
\lim_{t \to \infty} \text{Var}(X_1, 1_t) = \sqrt{\frac{2}{2} \cdot \left( 1 + \left( \frac{d-1}{d} \right)^{1/2} \right) - \frac{1}{2}} \in \left( \frac{1}{2}, \sqrt{2} \right] .
\]
Riccati equations:

\[ \dot{\phi}_t^k = \sum_{j=0}^{k} \phi_t^j \phi_t^{k-j} - \epsilon \left( \delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right), \quad \phi_T^k = c \left( \delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right). \]

WLOG, we assume \( \epsilon = 1 \) and \( \sigma = 1 \). By taking \( T \to \infty \), we look at the stationary long-time behavior of the Riccati system. Then the system gives the recurrence relation: \( \phi^0 = 1, \phi^1 = -1/(2d) \) and \( \sum_{j=0}^{k} \phi^j \phi^{k-j} = 0 \) for \( k \geq 0 \). By using a moment generating function method, we obtain the stationary solution:

\[ \phi^0 = 1, \quad \phi^1 = -\frac{1}{2d}, \quad \text{and} \quad \phi^k = -\frac{(2k-3)!}{(k-2)!k!2^{2k-2}} \cdot \frac{1}{d^k} \quad \text{for} \quad k \geq 2 \]
Extensions: Directed Tree Networks

Riccati equations:

\[ \dot{\phi}_k(t) = \sum_{j=0}^{k} \phi_j(t) \phi_{k-j}(t) - \epsilon \left( \delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right), \quad \phi_T^k = c \left( \delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right). \]

WLOG, we assume \( \epsilon = 1 \) and \( \sigma = 1 \). By taking \( T \to \infty \), we look at the stationary long-time behavior of the Riccati system. Then the system gives the recurrence relation: \( \phi^0 = 1, \phi^1 = -1/(2d) \) and \( \sum_{j=0}^{k} \phi^j \phi^{k-j} = 0 \) for \( k \geq 0 \). By using a moment generating function method, we obtain the stationary solution:

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As in the case \( d = 1 \), we can use a Catalan Markov chain to derive explicit formulas, in particular for the asymptotic variance:

\[ \lim_{t \to \infty} \text{Var}(X_{t^1}^1) = \frac{\sqrt{2}}{2} \cdot \left( 1 + \left( \frac{d-1}{d} \right)^{1/2} \right)^{-1/2} \in \left( \frac{1}{2}, \frac{\sqrt{2}}{2} \right]. \]
Stochastic Games on Stochastic Directed Networks
Work in Progress

- Stochastic Games on **Stochastic Directed Networks**
- **Bi-directional Chain** Interaction
Work in Progress

- Stochastic Games on **Stochastic Directed Networks**
- **Bi-directional Chain** Interaction
- And more ...
THANKS FOR YOUR ATTENTION

and

STAY HEALTHY!