D-BRANES AND BIVARIANT K-THEORY

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We review various aspects of the topological classification of D-brane charges in K-theory, focusing on techniques from geometric K-homology and Kasparov’s KK-theory. The latter formulation enables an elaborate description of D-brane charge on large classes of noncommutative spaces, and a refined characterization of open string T-duality in terms of correspondences and KK-equivalence. The examples of D-branes on noncommutative Riemann surfaces and in constant $H$-flux backgrounds are treated in detail. Mathematical constructions include noncommutative generalizations of Poincaré duality and K-orientation, characteristic classes, and the Riemann-Roch theorem.

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1 Introduction

The subject of this paper concerns the intriguing relationship between D-branes and K-theory. As is by now well-known, D-brane charges in string theory are classified by the K-theory of the spacetime $X$ [1]–[7], or equivalently (in the absence of $H$-flux) by the K-theory of the $C^*$-algebra $C_0(X)$ of continuous functions on $X$ vanishing at infinity. D-branes are sources for Ramond-Ramond fields, which are differential forms on spacetime and are correspondingly classified by a smooth refinement of K-theory called the differential K-theory of $X$ [8]–[13]. This topological classification has been used to explain a variety of effects in string theory that ordinary homology or cohomology alone cannot explain, such as the existence of stable non-BPS branes with torsion charges, the self-duality and quantization of Ramond-Ramond fields, and the appearance of certain subtle worldsheet anomalies and Ramond-Ramond field phase factors in the string theory path integral. It has also been used to predict many novel phenomena such as the instability of D-branes wrapping non-contractible cycles, and obstructions to the simultaneous measurement of electric and magnetic Ramond-Ramond fluxes.
The classification of D-branes can be posed as the following problem. Given a closed string background $X$ (a Riemannian spin manifold with possibly other form fields), find all possible states of D-branes in $X$. At the worldsheet level, these states are described as consistent boundary conditions in an underlying boundary superconformal field theory. However, many of these states have no geometrical description. It has therefore proven useful in a variety of contexts to regard D-branes as objects in a suitable category. The classic example of this is in conjunction with topological string theory and Kontsevich’s homological mirror symmetry conjecture, in which B-model D-branes live in a bounded derived category of coherent sheaves, while A-model D-branes are objects in a certain Fukaya category \[14\]. A more recent example has been used to clarify the relationship between boundary conformal field theory and K-theory, and consists in regarding open string boundary conditions in the category of a two-dimensional open/closed topological field theory \[15\]. In the following we will argue that when one combines the worldsheet description with the target space classification in terms of Fredholm modules, one is led to regard D-branes as objects in a certain category of separable $C^\ast$-algebras \[16\]. This is the category underlying Kasparov’s bivariant K-theory (or KK-theory), and it is related to the open string algebras which arise in string field theory \[17, 18\].

The advantages of using the bivariant extension of K-theory are abundant and will be described thoroughly in what follows. It unifies the K-theory and K-homology descriptions of D-branes. It possesses an intersection product which provides the correct framework for formulating notions of duality between generic separable $C^\ast$-algebras, such as Poincaré duality. This can be used to explain the equivalence of the K-theory and K-homology descriptions of D-brane charge. It also leads to a new characterization of open string T-duality as a certain categorical KK-equivalence, which refines and generalizes the more commonly used characterizations in terms of Morita equivalence \[19–21\]. The formalism is also well equipped to deal with examples of “non-geometric” backgrounds which have appeared recently in the context of flux compactifications \[22\]. In certain instances, the noncommutative spacetimes can be viewed \[24\] as globally defined open string versions of Hull’s \textit{T-folds} \[24\], which are backgrounds that fail to be globally defined Riemannian manifolds but admit a local description in which open patches are glued together using closed string T-duality transformations. KK-theory also provides us with a noncommutative version of K-orientation, which generalizes the Freed-Witten anomaly cancellation condition \[5\] and enables us to select the consistent sets of D-branes from our category. Finally, bivariant K-theory yields a noncommutative version of the D-brane charge vector \[1\].

In formulating the notions of D-brane charge and Ramond-Ramond fields on arbitrary $C^\ast$-algebras, one is faced with the problem of developing Poincaré duality and constructing characteristic classes in these general settings. From the mathematical perspective of noncommutative geometry
alone, the formalism thus enables us to develop more tools for dealing with noncommutative spaces in the purely algebraic framework of separable C*-algebras. These include noncommutative versions of Poincaré duality and orientation, topological invariants of noncommutative spaces such as the Todd genus, and a noncommutative version of the Grothendieck-Riemann-Roch theorem which is intimately tied to the formulation of D-brane charge.

2 D-branes and K-homology

We will begin by explaining the topological classification of D-branes using techniques of geometric K-homology [25, 26], following refs. [7, 27]. In this setting, brane charges are expressed in terms of the Chern character in K-homology formulated topologically by the Baum-Douglas construction. Using the Fredholm module description available in analytic K-homology, this will lead to a description of brane charges later on more complicated spaces, in particular on noncommutative spacetime manifolds. Earlier work in this context can be found in refs. [28, 29].

2.1 D-branes and K-cycles

Throughout this paper we will work in the context of Type II superstring theory. Let X be a compact spin*-manifold, with no background H-flux (we will explain in detail later on what we mean precisely by this condition). A D-brane in X may then be defined to be a Baum-Douglas K-cycle \((W, E, f)\) [25], where \(f : W \hookrightarrow X\) is a closed spin*-submanifold called the worldvolume of the brane, and \(E \to W\) is a complex vector bundle with connection called the Chan-Paton gauge bundle. The crucial feature about the Baum-Douglas construction is that \(E\) defines a stable element of the K-theory group \(K_0(W)\).

The set of all K-cycles forms an additive category under disjoint union. The quotient of the set of all K-cycles by Baum-Douglas “gauged equivalence” is isomorphic to the K-homology of \(X\), defined as the collection of stable homotopy classes of Fredholm modules over the commutative C*-algebra \(A = C(X)\) of continuous functions on \(X\). The isomorphism sends a K-cycle \((W, E, f)\) to the unbounded Fredholm module \((H, \rho, \mathcal{D}^{(W)}_E)\), where \(H = L^2(W, S \otimes E)\) is the separable Hilbert space of square integrable \(E\)-valued spinors on \(W\), \(\rho(\phi) = m_{\phi \circ f}\) is the \(*\)-representation of \(\phi \in A\) on \(H\) by pointwise multiplication with the function \(\phi \circ f\), and \(\mathcal{D}^{(W)}_E\) is the \(E\)-twisted Dirac operator associated to the spin* structure on \(W\). The K-homology class \([W, E, f]\) of a D-brane depends only on the K-theory class \([E] \in K_0(W)\) of its Chan-Paton bundle [7]. Actually, to make this map surjective one has to work with more general K-cycles wherein \(W\) is not necessarily a submanifold of spacetime. We will return to this point later on.

It follows that D-branes naturally provide K-homology classes on \(X\), dual to K-theory classes \(f_!(E) \in K^d(X)\), where \(f_!\) is the K-theoretic Gysin map.
and \( d = \dim(X) - \dim(W) \) is the codimension of the brane worldvolume in spacetime. The natural \( \mathbb{Z}_2 \)-grading on K-homology \( K_\bullet(X) \) is by parity of dimension \( \dim(W) = p + 1 \), and the K-cycle \((W, E, f)\) then corresponds to a \( Dp \)-brane. Following ref. [7], we will now describe the Baum-Douglas gauge equivalence relations explicitly, together with their natural physical interpretations.

**Bordism**

Two K-cycles \((W_1, E_1, f_1)\) and \((W_2, E_2, f_2)\) are said to be bordant if there exists a K-cycle with boundary \((M, E, f)\) such that

\[
(\partial M, E|_{\partial M}, f|_{\partial M}) \cong (W_1 \amalg (-W_2), E_1 \amalg E_2, f_1 \amalg f_2),
\]

where \(-W_2\) denotes the manifold \(W_2\) with the opposite spin\(^c\) structure on its tangent bundle \(TW_2\). If \(X\) is locally compact, this relation generates a boundary condition which guarantees that D-branes have finite energy. In particular, it ensures that any K-cycle \((W, E, f)\) is equivalent to the closed string vacuum \((\emptyset, \emptyset, \emptyset)\) (with no D-branes) at “infinity” in \(X\).

**Direct sum**

If \(E_i, \ i = 1, 2\) are complex vector bundles over \(W\), then we identify the K-cycles

\[
(W, E_1 \oplus E_2, f) \sim (W, E_1, f)(W, E_2, f).
\]

This relation reflects gauge symmetry enhancement for coincident branes. The bundle \(E = \bigoplus_i E_i\) is the Chan-Paton bundle associated to a bound state of D-branes with Chan-Paton bundles \(E_i \to W\), bound by open string excitations given by classes of bundle morphisms \([\phi_{ij}] \in \text{Hom}(E_i, E_j)\). Other open string degrees of freedom correspond to classes in \(\text{Ext}^p(E_i, E_j), \ p \geq 1\).

**Vector bundle modification**

Let \((W, E, f)\) be a K-cycle and let \(F \to W\) be a real spin\(^c\) vector bundle of rank \(2n\), with associated bundles of Clifford modules \(S_0(F), S_1(F) \to W\) and their pullbacks \(S_{\pm}(F) \to F\) of rank \(2^{n-1}\). Clifford multiplication induces a bundle map \(\sigma : S_+(F) \to S_-(F)\) which is an isomorphism outside of the zero section. If \(\mathbb{I}\) denotes the trivial real line bundle over \(W\), then upon choosing a Hermitean metric on the fibres of \(F\) we can define the unit sphere bundle

\[
\hat{W} := S(F \oplus \mathbb{I}) \cong B_+(F) \cup_{S(F)} B_-(F)
\]

with bundle projection

\[
\pi : \hat{W} \to W,
\]

where \(B_{\pm}(F)\) are two copies of the unit ball bundle \(B(F)\) of \(F\) whose boundary is the unit sphere bundle \(S(F)\). We can glue \(S_{\pm}(F) = S_{\pm}(F)|_{B_{\pm}}\) together by Clifford multiplication to define the bundle
\[ H(F) = S_+(F) \cup_\sigma S_-(F) \, . \]

The restriction \( H(F)|_{\pi^{-1}(w)} \) is the Bott generator of the \( 2n \)-dimensional sphere \( \pi^{-1}(w) = S^{2n} \) for each \( w \in W \). We impose the equivalence relation

\[(W, E, f) \sim (\hat{W}, H(F) \otimes \pi^*(E), f \circ \pi) \, ,\]

where the right-hand side is called the vector bundle modification of \((W, E, f)\) by \( F \).

This relation can be understood as the K-homology version of the well-known dielectric effect in string theory [30]. To understand this point, consider the simple K-cycle \((W, E, f) = (\text{pt}, \mathbb{C}, i)\), where \( i \) is the inclusion of the point \text{pt} in \( X \). Let \( F = \mathbb{R}^{2n}, n \geq 1 \). Then, with the definitions above, one has \( \hat{W} \cong S^{2n} \) with \( \pi : S^{2n} \to \text{pt} \) the collapsing map \( \varepsilon \). Moreover, \( H(F) = H(F)|_{S^{2n}} \) is the Bott generator of the K-theory group \( K^0(S^{2n}) \). By vector bundle modification, one has an equality of classes of K-cycles given by

\[ [\text{pt}, \mathbb{C}, i] = [S^{2n}, H(F) \otimes \mathbb{C}, i \circ \pi] = [S^{2n}, H(F), \varepsilon] \, . \]

This equality represents the polarization or “blowing up” of a D0-brane (on the left) into a collection of spherical D\((2n)\)-branes (on the right), together with “monopole” gauge fields corresponding to connections on the vector bundles \( H(F) \to S^{2n} \). It is essentially the statement of Bott periodicity.

### 2.2 Tachyon condensation and the Sen-Witten construction

The Sen-Witten construction [2, 31] is the classic model establishing that D-brane charge is classified by K-theory. It relies on the physics of tachyon condensation and the realization of stable D-branes as decay products in unstable systems of spacetime filling branes and antibranes. In K-homology, this construction utilizes the fact that not all K-cycles are associated with submanifolds of spacetime, and correspond generically to non-representable D-branes arising as conformal boundary conditions with no direct geometric realization.

For definiteness, let \( X \) be a locally compact spin manifold of dimension \( \dim(X) = 10 \). Let \( W \subset X \) be a spin\(^c\) submanifold of dimension \( p + 1 \). Then the normal bundle \( \nu_W \to W \) to \( W \) in \( X \) is a real spin\(^c\) vector bundle of rank \( 9 - p \). A D-brane \([M, E, \phi] \in K_*(X)\) is said to wrap \( W \) if \( \dim(M) = p + 1 \) and \( \phi(M) \subset W \). The group of charges of Type IIB Dp-branes (\( p \) odd) wrapping \( W \) may then be computed as the compactly supported K-theory group

\[
K^0(\nu_W) := K^0(B(\nu_W), S(\nu_W)) \\
\cong K_{10}(\nu_W) \\
\cong K_{p+1}(W) \, ,
\]
where the first isomorphism follows from Poincaré duality and the second from the K-homology Thom isomorphism. Upon identifying the total space of $\nu_W$ with a tubular neighbourhood of $W$ in $X$ with respect to a chosen Riemannian metric on $X$, the group $K_{10}(\nu_W)$ classifies 9-branes in $X$. The isomorphism (2) then asserts that this group coincides with the group of $D_p$-branes $[M, E, \phi]$ wrapping $W$. The same calculation carries through for Type IIA $D_p$-branes with $p$ even, starting from the pertinent K-theory group $K^{-1}(\nu_W)$. The original example [2, 31] concerns the charge group of $D_p$-branes in Type IIB string theory on flat space $X = \mathbb{R}^{10}$ given by

$$K_{p+1}(\mathbb{R}^{p+1}) \cong K_0(\mathbb{R}^{10}) := \tilde{K}_0(S^{10})$$

where we have used Bott periodicity. To make this relationship more explicit, we can adapt the Atiyah-Bott-Shapiro (ABS) construction [32] to the setting of geometric K-homology. Given a K-cycle $(W, E, f)$ in $X$, the vector bundle modification relation for $F = \nu_W$ reads

$$[\hat{W}, H(\nu_W) \otimes \pi^*(E), f \circ \pi] = [W, E, f]$$

with $\hat{W}$ diffeomorphic to $X$. Generally, the nowhere vanishing section given by $s : W \to F \oplus \mathbb{R}^1, x \mapsto 0_x \oplus 1$ induces a Gysin homomorphism on K-theory $s^!: K^\bullet(W) \to K^\bullet(\hat{W})$ with

$$s^!(E) = [\pi^*(E) \otimes H(F)]$$

by the K-theory Thom isomorphism. Let $W' \cong \mathbb{B}(\nu_W) \setminus S(\nu_W)$ be a tubular neighbourhood of $f(W)$ with closure $\overline{W}$, retraction $\rho : W' \to W$, and twisted spinor bundles $S^\pm_E := S^\pm(\nu_W) \otimes \rho^*(E) \to \overline{W}$. After a possible K-theoretic stabilization, we can extend the spinor bundles over the complement $X \setminus W'$ to bundles $S^\pm_E \to X$ with K-theory class [2, 4, 32]

$$[S^+_E] - [S^-_E] = s(E),$$

which vanishes over $X \setminus W'$ by Clifford multiplication. Putting everything together finally gives

$$[X, S^+_E, \text{id}_X] - [X, S^-_E, \text{id}_X] = \pm [W, E, f],$$

where the sign depends on whether or not the spin$^c$ structures on $\hat{W}$ and $X$ coincide. This equation is simply the statement of tachyon condensation on the unstable spacetime-filling brane-antibrane system (on the left) to a stable D-brane wrapping $W$ (on the right).

### 2.3 Holonomy on D-branes

In order to cancel certain worldvolume anomalies, it is necessary to introduce Ramond-Ramond flux couplings in the path integral for Type II string
theory [6]. The formalism of geometric K-homology nicely achieves this via a spin<sup>c</sup> cobordism invariant as follows. Introduce “background” D-branes \((W, E, f)\) as K-chains \((\tilde{W}, \tilde{E}, \tilde{f})\) with boundary

\[
\partial(\tilde{W}, \tilde{E}, \tilde{f}) := (\partial \tilde{W}, \tilde{E}|_{\partial \tilde{W}}, \tilde{f}|_{\partial \tilde{W}}) = (W, E, f).
\]

By bordism, such branes have trivial K-homology class and so carry no charge.

The reduced eta-invariant of a K-chain is defined by

\[
\Xi(\tilde{W}, \tilde{E}, \tilde{f}) = \frac{1}{2} \left( \dim \mathcal{H}_E^{(W)} + \eta(D_E^{(W)}) \right) \in \mathbb{R}/\mathbb{Z},
\]

where \(\mathcal{H}_E^{(W)}\) is the space of harmonic \(E\)-valued spinors on \(W\), and \(\eta(D_E^{(W)})\) is the (regulated) spectral asymmetry of the Dirac operator \(D_E^{(W)}\). This invariant is defined up to compact perturbation of the Dirac operator and hence is \(\mathbb{R}/\mathbb{Z}\)-valued. The map \(\Xi\) from K-chains to the group \(\mathbb{R}/\mathbb{Z}\) respects disjoint union, direct sum and vector bundle modification, but not spin<sup>c</sup> cobordism.

To rectify this problem, we introduce the holonomy over the given D-brane background with flat Ramond-Ramond flux \(\xi = [E_0] - [E_1] \in K^{-1}(X, \mathbb{R}/\mathbb{Z})\) by

\[
\Omega(\tilde{W}, \xi, \tilde{f}) = \exp \left[ 2\pi i \left( \Xi(\tilde{W}, \tilde{f}^* E_0, \tilde{f}) - \Xi(\tilde{W}, \tilde{f}^* E_1, \tilde{f}) \right) \right].
\]

This quantity is the desired spin<sup>c</sup> cobordism invariant.

### 2.4 Brane stability

We will now illustrate some of the predictive power of the K-homology classification through two novel sets of examples of brane stability which contradict what ordinary homology theory alone would predict. The first set consists of trivial K-homology classes \([W, W^C, f] = 0\) in \(K_\bullet(X)\), even though the world-volume homology cycle \([W] \neq 0\) in \(H_\bullet(X, \mathbb{Z})\).

The obstructions to extending the homology class \([W]\) to a K-homology class are measured by the Atiyah-Hirzebruch-Whitehead spectral sequence

\[
E^2_{p,q} = H_p(X, K_q(pt)) \implies K_{p+q}(X).
\]

With respect to a cellular decomposition of the spacetime manifold \(X\), for each \(p\) the corresponding filtration groups classify D-branes wrapping \(W\) on the \(p\)-skeleton of \(X\) with no lower brane charges.

The \(r\)-th term in the spectral sequence is determined as the homology of certain differentials \(d^r\). Cycles for which \([W] \notin \ker(d^r)\) for all \(r\) correspond to Freed-Witten anomalous D-branes [33]. On the other hand, if \([W] \in \text{im}(d^r)\) for some \(r\), then the K-homology “lift” of the cycle \([W]\) vanishes and the D-brane is unstable. Cycles contained in the image of \(d^r\) correspond to D-brane
instantons whose charge is not conserved in time along the trajectories of the worldvolume renormalization group flow. The extension problem for the spectral sequence at each term identifies the lower brane charges carried by stable D-branes.

The second set is opposite in character to the first in that now \([W, E, f] \neq 0\) in \(K(X)\) even though \([W] = 0\) in \(H(X, \mathbb{Z})\). This occurs by the process of flux stabilization on spacetimes which are the total spaces of topologically non-trivial fibre bundles \(X \xrightarrow{f} B\). Worldvolume “flux” in this instance corresponds to the characteristic class of the fibration, which provides a conserved charge preventing the D-brane from decaying to the vacuum. Although \(W\) is contractible in \(X\), its class may be non-trivial as an element of \(H(B, \mathbb{Z})\). The obstructions to lifting homology cycles from the base space \(B\) are measured by the Leray-Serre spectral sequence

\[
E^2_{p,q} = H_p(B, K_q(F)) \implies K_{p+q}(X, F).
\]

Let us examine the original example of this phenomenon, that of D-branes in the group manifold of \(SU(2)\), in this language. Spacetime in this case is the total space of the Hopf fibration \(S^3 \xrightarrow{S^1} S^2\), and the spectral sequence computes the K-homology as \(K_i(S^2, \mathbb{Z}) = \mathbb{Z}\). The stable branes are spherical D2-branes, and the stabilizing flux is provided by the first Chern class of the monopole line bundle over \(S^2\). This example readily generalizes to the other Hopf fibrations, and the K-homology framework nicely extends the examples of refs. to spaces with less symmetry.

### 2.5 D-brane charges

We will now describe the cohomological formula for the charge of a D-brane. The mathematical structure of this formula can be motivated by the following simple observation, which we will generalize later on to certain classes of noncommutative spacetimes. The natural bilinear pairing in cohomology is given by

\[
(x, y)_H = \langle x \smile y, [X] \rangle
\] (3)

for cohomology classes \(x, y \in H^*(X, \mathbb{Z})\) in complimentary degrees. Upon choosing de Rham representatives \(\alpha, \beta\) for \(x, y\), this formula corresponds to integration of the product of differential forms \(\int_X \alpha \wedge \beta\). Nondegeneracy of this pairing is the statement of Poincaré duality in cohomology. On the other hand, the natural bilinear pairing in K-theory is provided on complex vector bundles \(E, F \to X\) by the index of the twisted Dirac operator

\[
(E, F)_K = \text{index}(\mathcal{D}_{E \otimes F})
\] (4)

The \(\mathbb{Z}_2\)-graded Chern character ring isomorphism
\[ \text{ch} : K^\bullet(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^\bullet(X, \mathbb{Q}) \]  

is not compatible with these two pairings. However, by the Atiyah-Singer index theorem

\[ \text{index}(D \otimes E \otimes F) = \langle \text{Todd}(X) \sim \text{ch}(E \otimes F), [X] \rangle \]  

we get an isometry with the \( \mathbb{Z}_2 \)-graded modified Chern character group isomorphism

\[ \text{ch} \rightarrow \sqrt{\text{Todd}(X)} \sim \text{ch}, \]

twisted by the square root of the invertible Todd class \( \text{Todd}(X) \in H^{\text{even}}(X, \mathbb{Q}) \) of the tangent bundle \( TX \).

This almost trivial observation motivates the definition of the Ramond-Ramond charge of a D-brane \((W, E, f)\) as \[ Q(W, E, f) = \text{ch}(f_!(E)) \sim \sqrt{\text{Todd}(X)} \in H^\bullet(X, \mathbb{Q}). \]  

In topological string theory, this rational charge vector coincides with the zero mode part of the associated boundary state in the Ramond-Ramond sector. In the D-brane field theory, \( Q(W, E, f) = f_!(D_{\text{WZ}}(W, E, f)) \) is the cohomological Gysin image of the Wess-Zumino class (for vanishing B-field)

\[ D_{\text{WZ}}(W, E, f) = \text{ch}(E) \sim \sqrt{\text{Todd}(W)/\text{Todd}(\nu_W)} \in H^\bullet(W, \mathbb{Q}). \]

This formula interprets the Ramond-Ramond charge as the anomaly inflow on the D-brane worldvolume \( W \). The equivalence of these two formulas follows from the Grothendieck-Riemann-Roch formula

\[ \text{ch}(f_!(E)) \sim \text{Todd}(X) = f_!(\text{ch}(E) \sim \text{Todd}(W)) \]  

together with naturality of the Todd characteristic class. Compatibility with the equivalence relations of geometric K-homology follows easily by direct calculation. In particular, invariance under vector bundle modification is a simple computation showing that the charge of the polarized D-brane \((\hat{W}, s_!(E), f \circ \pi)\) equals \( Q(W, E, f) \).

### 3 D-branes and KK-theory

By merging the worldsheet and target space descriptions of D-branes, we will now motivate a categorical framework for the classification of D-branes using Kasparov’s KK-theory groups. This will set the stage for a noncommutative description of D-branes in a certain category of separable \( C^* \)-algebras. We will then explain various important features of the bivariant version of K-theory, and use them for certain physical and mathematical constructions. The material of this section is based on refs. [16]–[18].
### 3.1 Algebraic characterization of D-branes

The worldsheet description of a D-brane with worldvolume $W \subset X$ is provided by open strings, which may be defined to be relative maps $(\Sigma, \partial \Sigma) \rightarrow (X, W)$ from an oriented Riemann surface $\Sigma$ with boundary $\partial \Sigma$. In the boundary conformal field theory on $\Sigma = \mathbb{R} \times [0,1]$, solutions of the Euler-Lagrange equations require the imposition of suitable boundary conditions, which we will label by $a, b, \ldots$. These boundary conditions are not arbitrary and compatibility with superconformal invariance severely constrains the possible worldvolumes $W$. For example, in the absence of background $H$-flux, $W$ must be spin$c$ in order to ensure the cancellation of global worldsheet anomalies \[5\]. The problem which now arises is that while this is more or less understood at the classical level, there is no generally accepted definition of what is meant by a quantum D-brane. Equivalently, it is not known in general how to define consistent boundary conditions after quantization of the boundary conformal field theory. To formulate our conjectural description of this, we will take a look at the generic structure of open string field theory.

The basic observation is that the concatenation of open string vertex operators defines algebras and bimodules. An $a$-$a$ open string, one with the same boundary condition $a$ at both of its ends, defines a noncommutative algebra $D_a$ of open string fields. The opposite algebra $D_a^\circ$, with the same underlying vector space as $D_a$ but with the product reversed, is obtained by reversing the orientation of the open string. On the other hand, an $a$-$b$ open string, with generically distinct boundary conditions $a, b$ at its two ends, defines a $D_a$-$D_b$ bimodule $E_{ab}$, with the rule that open string ends can join only if their boundary labels are the same. The dual bimodule $E_{ab}^\vee = E_{ba}$ is obtained by reversing orientation, and $E_{aa} = D_a^\circ$ is defined to be the trivial $D_a$-bimodule on which $D_a$ acts by (left and right) multiplication.

We would now like to use these ingredients to define a “category of D-branes” whose objects are the boundary conditions, and whose morphisms $a \rightarrow b$ are precisely the bimodules $E_{ab}$. This requires an associative $\mathbb{C}$-bilinear composition law

$$E_{ab} \times E_{bc} \rightarrow E_{ac}.$$

The problem, however, in the way that we have set things up, is that the operator product expansion of the open string fields is not always well-defined. Elements of the open string bimodule $E_{ab}$ are vertex operators

$$V_{ab} : [0,1] \rightarrow \text{End}(H_{ab})$$

acting on a separable Hilbert space $H_{ab}$. The structure of the vertex operator algebra is encoded in the singular operator product expansion

$$V_{ab}(t) \cdot V_{bc}(t') = \sum_{j=1}^{N} \frac{1}{(t-t')^{h_j}} W_{abcj}(t,t'), \quad t > t',$$
where $W_{abc} : [0,1] \times [0,1] \to \text{End} (\mathcal{H}_{ac})$ and $h_j \geq 0$ are called conformal dimensions. When $h_j > 0$, the leading singularities of the operator product expansion do not give an associative algebra in the usual sense.

### 3.2 Seiberg-Witten limit

Seiberg and Witten \[21\] found a resolution to this problem in the case where spacetime is an $n$-dimensional torus $X = \mathbb{T}^n$ with a constant $B$-field. They introduced a scaling limit wherein both the $B$-field and the string tension $T$ are scaled to infinity in such a way that their ratio $B/T$ remains finite, while the closed string metric $g$ on $\mathbb{T}^n$ is scaled to zero. In this limit the Hilbert space $\mathcal{H}_a$ of the point particle at an open string endpoint is a module for a noncommutative torus algebra $\mathcal{D}_a$, which forms the complete set of observables for boundary conditions of maximal support. The product $\mathcal{D}_a \otimes \mathcal{D}_b$ acts irreducibly on the $\mathcal{D}_a$-$\mathcal{D}_b$ bimodule $\mathcal{E}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b$.

In this case, the composition law

$$V_{ac}(t') = \lim_{t \to t'} V_{ab}(t) \cdot V_{bc}(t')$$

is well-defined since the conformal dimensions scale to zero in the limit as $h_j \sim g/T \to 0$. It extends by associativity of the operator product expansion in the limit to a map

$$\mathcal{E}_{ab} \otimes \mathcal{D}_b \mathcal{E}_{bc} \longrightarrow \mathcal{E}_{ac}.$$ 

Furthermore, there are natural identifications of algebras $\mathcal{D}_a \cong \mathcal{E}_{ab} \otimes \mathcal{D}_b \mathcal{E}_{ba}$ and $\mathcal{D}_b \cong \mathcal{E}_{ba} \otimes \mathcal{D}_a \mathcal{E}_{ab}$. These results all mean that $\mathcal{E}_{ab}$ is a Morita equivalence bimodule, reflecting a $T$-duality between the noncommutative tori $\mathcal{D}_a$ and $\mathcal{D}_b$.

### 3.3 KK-theory

The construction of Section 3.2 above motivates a conjectural framework in which to move both away from the dynamical regime dictated by the Seiberg-Witten limit and into the quantum realm. We will suppose that the appropriate modification consists in replacing $\mathcal{E}_{ab}$ by Kasparov bimodules $(\mathcal{E}_{ab}, F_{ab})$, which generalize Fredholm modules. They coincide with the “trivial” bimodule $(\mathcal{E}_{ab}, 0)$ when $\mathcal{E}_{ab}$ is a Morita equivalence bimodule. We will not enter into a precise definition of these bimodules, which is somewhat technically involved (see ref. \[16\], for example). As we move our way deeper into our treatment we will become better acquainted with the structures inherent in Kasparov’s theory.

Stable homotopy classes of Kasparov bimodules define the $\mathbb{Z}_2$-graded KK-theory group $\text{KK}_*(\mathcal{D}_a, \mathcal{D}_b)$. Classes in this group can be thought of as “generalized” morphisms $\mathcal{D}_a \to \mathcal{D}_b$, in a way that we will make more precise as we go along. In particular, if $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of separable $C^*$-algebras, then it determines a canonical class $[\phi] \in \text{KK}_*(\mathcal{A}, \mathcal{B})$ represented
by the “Morita-type" bimodule \((B, \phi, 0)\). The group \(\text{KK}^\bullet(A, C) = K^\bullet(A)\) is the K-homology of the algebra \(A\), since in this case Kasparov bimodules are the same things as Fredholm modules over \(A\). On the other hand, the group \(\text{KK}^\bullet(C, B) = K_\bullet(B)\) is the K-theory of \(B\).

One of the most powerful aspects of Kasparov’s theory is the existence of a bilinear, associative composition or intersection product

\[
\otimes_B : \text{KK}_i(A, B) \times \text{KK}_j(B, C) \rightarrow \text{KK}_{i+j}(A, C).
\]

We will not attempt a general definition of the intersection product, which is notoriously difficult to define. Later on we will see how it is defined on specific classes of \(C^*\)-algebras. The product is compatible with the composition of morphisms, in that if \(\phi : A \rightarrow B\) and \(\psi : B \rightarrow C\) are homomorphisms of separable \(C^*\)-algebras then

\[
[\phi] \otimes_B [\psi] = [\psi \circ \phi].
\]

The intersection product makes \(\text{KK}_0(A, A)\) into a unital ring with unit \(1_A = [\text{id}_A]\), the class of the identity morphism on \(A\). It can be used to define Kasparov’s bilinear, associative exterior product

\[
\otimes : \text{KK}_i(A_1, B_1) \times \text{KK}_j(A_2, B_2) \rightarrow \text{KK}_{i+j}(A_1 \otimes A_2, B_1 \otimes B_2),
\]

\[
x_1 \otimes x_2 = (x_1 \otimes 1_{A_2}) \otimes_{B_1 \otimes A_2} (1_{B_1} \otimes x_2).
\]

This definition also uses dilation. If \(x = [\phi] \in \text{KK}_j(A, B)\) is the class of the morphism \(\phi : A \rightarrow B\), then \(x \otimes 1_C = [\phi \otimes \text{id}_C] \in \text{KK}_j(A \otimes C, B \otimes C)\) is the class of the morphism \(\phi \otimes \text{id}_C : A \otimes C \rightarrow B \otimes C\).

The KK-theory groups have some nice properties, described in refs. [36]–[38], which enable us to define our \(D\)-brane categories. There is an additive category whose objects are separable \(C^*\)-algebras and whose morphisms \(A \rightarrow B\) are precisely the classes in \(\text{KK}_\bullet(A, B)\). This category is a universal category, in the sense that \(\text{KK}^\bullet(-,-)\) can be characterized as the unique bifunctor on the category of separable \(C^*\)-algebras and \(*\)-homomorphisms which is homotopy invariant, compatible with stabilization of \(C^*\)-algebras, and respects split exactness. The composition law in this category is provided by the intersection product. The category is not abelian, but it is triangulated, like other categories of relevance in D-brane physics. It further admits the structure of a “weak” monoidal category, with multiplication given by the spatial tensor product on objects, the external Kasparov product on morphisms, and with identity the one-dimensional \(C^*\)-algebra \(\mathbb{C}\). A diagrammatic calculus in this tensor category is developed in refs. [16] [17].

### 3.4 KK-equivalence

As our first application of the bivariant version of K-theory, we introduce the following notion which will be central to our treatment later on. Any given
fixed element \( \alpha \in KK_d(A,B) \) determines homomorphisms on K-theory and K-homology given by taking intersection products

\[
\otimes_A \alpha : K_j(A) \rightarrow K_{j+d}(B) \quad \text{and} \quad \alpha \otimes_B : K^j(B) \rightarrow K^{j+d}(A).
\]

If \( \alpha \) is invertible, i.e., there exists an element \( \beta \in KK_{-d}(B,A) \) such that \( \alpha \otimes_B \beta = 1_A \) and \( \beta \otimes_A \alpha = 1_B \), then we write \( \beta =: \alpha^{-1} \) and there are isomorphisms

\[
K_j(A) \cong K_{j+d}(B) \quad \text{and} \quad K^j(B) \cong K^{j+d}(A).
\]

In this case the algebras \( A, B \) are said to be \( KK \)-equivalent. From a physical perspective, algebras \( A, B \) are \( KK \)-equivalent if they are isomorphic as objects in the D-brane category described at the end of Section 3.3 above. Such D-branes have the same K-theory and K-homology.

For example, Morita equivalence implies \( KK \)-equivalence, since the discussion of Section 3.2 above shows that the element \( \alpha = \left[ (E_{ab}, 0) \right] \) is invertible. However, the converse is not necessarily true. On the class of \( C^* \)-algebras which are \( KK \)-equivalent to commutative algebras, one has the universal coefficient theorem for \( KK \)-theory given by the exact sequence

\[
0 \rightarrow \text{Ext}_\mathbb{Z}(K_{j+1}(A), K_j(B)) \rightarrow KK_\bullet(A, B) \rightarrow \\
\text{Hom}_\mathbb{Z}(K_\bullet(A), K_\bullet(B)) \rightarrow 0.
\]

(10)

We will make extensive use of this exact sequence in the following.

3.5 Poincaré duality

The noncommutative version of Poincaré duality was introduced by Connes [40] and further developed in refs. [41]–[44]. Our treatment is closest to that of Emerson [43]. Let \( A \) be a separable \( C^* \)-algebra, and let \( A^\circ \) be its opposite algebra. The opposite algebra is introduced in order to regard \( A \)-bimodules as \( (A \otimes A^\circ) \)-modules. We say that \( A \) is a Poincaré duality (PD) algebra if there is a fundamental class \( \Delta \in KK_d(A \otimes A^\circ, \mathbb{C}) = KK_0(A \otimes A^\circ) \) with inverse \( \Delta^\vee \in KK_{-d}(\mathbb{C}, A \otimes A^\circ) = KK_0(A \otimes A^\circ) \) such that

\[
\Delta^\vee \otimes_A A^\circ \Delta = 1_A \in KK_0(A, A),
\]

\[
\Delta^\vee \otimes_A \Delta = (-1)^d 1_{A^\circ} \in KK_0(A^\circ, A^\circ)
\]

for some \( d = 0, 1 \). The subtle sign factor in this definition reflects the orientation of the Bott element \( \Delta^\vee \).

This definition determines inverse isomorphisms

\[
K_i(A) \stackrel{\otimes_A \Delta}{\rightarrow} K^{i+d}(A^\circ) = K^{i+d}(A),
\]

\[
K^i(A) = K^i(A^\circ) \stackrel{\Delta^\vee \otimes_{A^\circ}}{\rightarrow} K_{i-d}(A),
\]
which is the usual requirement of Poincaré duality. More generally, by replacing the opposite algebra $\mathcal{A}^0$ in this definition with an arbitrary separable $C^*$-algebra $\mathcal{B}$, we get the notion of PD pairs $(\mathcal{A}, \mathcal{B})$. Although the class of PD algebras is quite restrictive, PD pairs are rather abundant [17].

As a simple example, consider the commutative algebra $\mathcal{A} = C_0(X) = \mathcal{A}^0$ of continuous functions vanishing at infinity on a complete oriented manifold $X$. Let $\mathcal{B} = C_0(T^*X)$ or $\mathcal{B} = C_0(X, \text{Cliff}(T^*X))$, where $T^*X$ is the cotangent bundle over $X$ and $\text{Cliff}(T^*X)$ is the Clifford algebra bundle of $T^*X$. Then $(\mathcal{A}, \mathcal{B})$ is a PD pair, with $\Delta$ given by the Dirac operator on $\text{Cliff}(T^*X)$.

If in addition $X$ is a spin$^c$ manifold, then $\mathcal{A}$ is a PD algebra. In this case, the fundamental class $\Delta$ is the Dirac operator $D$ on the diagonal of $X \times X$, i.e., the image of the Dirac class $[D] \in K^*(A)$ under the group homomorphism

$$m^* : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{A} \otimes \mathcal{A})$$

induced by the product homomorphism $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, while its inverse $\Delta^\vee$ is the Bott element. Thus in this case the noncommutative version of Poincaré duality agrees with the classical one. We will encounter some purely noncommutative examples later on. See ref. [16] for further examples.

In general, the moduli space of fundamental classes of an algebra $\mathcal{A}$ is isomorphic to the group of invertible elements in the ring $KK_0(\mathcal{A}, \mathcal{A})$ [16]. When $\mathcal{A} = C_0(X)$, this space is in general larger than the space of spin$^c$ structures or K-orientations usually considered in the literature. This follows from the universal coefficient theorem (10), which shows that the moduli space is an extension of the automorphism group $\text{Aut}(K_0(X))$. Similarly, if $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras that are $KK$-equivalent, then the space of all $KK$-equivalences $\alpha$ is a torsor with associated group the invertible elements of $KK_0(\mathcal{A}, \mathcal{A})$.

### 3.6 K-orientation and Gysin homomorphisms

We can treat generic K-oriented maps by generalizing a construction due to Connes and Skandalis in the commutative case [15]. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a $*$-homomorphism of separable $C^*$-algebras in a suitable category. Then a K-orientation for $f$ is a functorial way of associating a class $f! \in KK_d(\mathcal{B}, \mathcal{A})$ for some $d = 0, 1$. This element determines a Gysin “wrong way” homomorphism on K-theory through

$$f! = \otimes_B(f!) : K_\bullet(\mathcal{B}) \longrightarrow K_{\bullet+d}(\mathcal{A}) .$$

If the $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ are both PD algebras with fundamental classes $\Delta_A \in KK_{d_A}(\mathcal{A} \otimes \mathcal{A}^o, \mathbb{C})$ and $\Delta_B \in KK_{d_B}(\mathcal{B} \otimes \mathcal{B}^o, \mathbb{C})$, respectively, then any morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ is K-oriented with K-orientation given by

$$f! = (-1)^d_A \Delta_A^\vee \otimes_{\mathcal{A}^o} [f^o] \otimes_{\mathcal{B}^o} \Delta_B$$
D-branes and bivariant K-theory

and \( d = d_A - d_B \). This construction uses the fact \([16]\) that the involution \( A \to A^o, f \mapsto f^o : A^o \to B^o \) on the stable homotopy category of separable \( C^*\)-algebras and \(*\)-homomorphisms passes to the D-brane category. Functoriality

\[ g! \otimes_B f! = (g \circ f)! \]

for any other \(*\)-homomorphism of separable \( C^*\)-algebras \( g : B \to C \) follows by associativity of the Kasparov intersection product. More general constructions of K-orientations will be encountered later on.

The following construction demonstrates that any D-brane \((W, E, f)\) in \( X \) determines a canonical KK-theory class \( f! \in KK_d(C(W), C(X)) \). Recall that in this instance the normal bundle \( \nu_W = f^*(TX)/TW \) is a spin\(^c\) vector bundle. Let \( i_W! := [[E, F]] \in KK_d(C(W), C(X)) \) be the invertible element associated to the ABS representative of the Thom class of the zero section \( i_W : W \hookrightarrow \nu_W \). Let \( [\Phi] \in KK_0(C_0(\nu_W^*), C_0(\nu_W^*)) \) be the invertible element induced by the isomorphism \( \Phi \) identifying \( W' \) with a neighbourhood of \( i_W(W) \) in \( X \). Let \( j! \in KK_0(C_0(W'), C(X)) \) be the class induced by the extension by zero of the open subset \( j : W' \hookrightarrow X \). Then a K-orientation for \( f \) is given by

\[ f! = i_W! \otimes_{C_0(\nu_W)} [\Phi] \otimes_{C_0(W')} j! \]

In this way our notion of K-orientation coincides with the Freed-Witten anomaly cancellation condition \([5]\). This construction extends to arbitrary smooth proper maps \( \phi : M \to X \), corresponding generally to non-representable D-branes, for which \( TM \oplus \phi^*(TX) \) is a spin\(^c\) vector bundle over \( M \).

4 Cyclic theory

The definition of D-brane charge given in Section 2.5 relied crucially on the connection between the topological K-theory of a spacetime \( X \) and its cohomology through the rational isomorphism provided by the Chern character \([5]\). In the generic noncommutative settings that we are interested in, we need a more general cohomological framework in which to express the D-brane charge. The appropriate receptacle for the Chern character in analytic K-theory is the cyclic cohomology of the given (noncommutative) algebra \( A \) \([40]\). As it is not commonly known material in string theory, in this section we will present a fairly detailed overview of the general aspects of cyclic homology and cohomology. Then we will specialize to the specific bivariant cyclic theory that we will need in subsequent sections. This general formulation will provide a nice intrinsic definition of the D-brane charge, suitable to our noncommutative situations.
4.1 Cyclic homology

Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$. The *universal differential graded algebra* $\Omega^\bullet(\mathcal{A})$ is the universal algebra generated by $\mathcal{A}$ and the symbols $da$, $a \in \mathcal{A}$ with the following properties:

1. $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ is linear;
2. $d$ obeys the Leibniz rule $d(ab) = d(a)b + a d(b)$;
3. $d(1) = 0$; and
4. $d^2 = 0$.

These conditions imply that $d$ is a linear derivation, and elements of $\Omega^\bullet(\mathcal{A})$ are called *noncommutative differential forms* on $\mathcal{A}$, or more precisely on the tensor algebra $T\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}^\otimes n$ of $\mathcal{A}$. We define $\Omega^n(\mathcal{A}) = A$. In degree $n > 0$, the space of $n$-forms is the linear span $\Omega^n(\mathcal{A}) = \text{Span}_\mathbb{C}\{a_0 da_1 \ldots da_n \mid a_0, a_1, \ldots, a_n \in \mathcal{A}\}$, which under the isomorphism $a_0 da_1 \ldots da_n \leftrightarrow a_0 \otimes a_1 \otimes \cdots \otimes a_n$ may be presented explicitly as a vector space by $\Omega^n(\mathcal{A}) \cong \mathcal{A} \otimes (\mathcal{A}/\mathbb{C})^\otimes n$.

The graded vector space $\Omega^\bullet(\mathcal{A})$ then becomes a graded algebra by using the Leibniz rule to define multiplication of forms by

$$
(a_0 da_1 \ldots da_n) \cdot (a_{n+1} da_{n+2} \ldots da_p) = (-1)^n a_0 a_1 da_2 \ldots da_p + \sum_{i=1}^n (-1)^{n-i} a_0 da_1 \ldots d(a_i a_{i+1}) \ldots da_p.
$$

Using this definition the operator $d$ may be extended to a graded derivation on $\Omega^\bullet(\mathcal{A})$.

When the algebra $\mathcal{A}$ is not unital, we apply the above construction to the unitization $\bar{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ of $\mathcal{A}$, with multiplication given by

$$
(a, \lambda) \cdot (b, \mu) = (a b + \lambda b + \mu a, \lambda \mu).
$$

Thus in degree $n > 0$ we have

$$
\Omega^n(\mathcal{A}) := \Omega^n(\bar{\mathcal{A}}) = \mathcal{A}^\otimes (n+1) \oplus \mathcal{A}^\otimes n.
$$

In degree 0 we define $\Omega^0(\mathcal{A}) = \mathcal{A}$.

The cohomology of the differential $d$ on $\Omega^\bullet(\mathcal{A})$ is trivial in positive degree and equal to $\mathbb{C}$ in degree 0. To get interesting homology theory, we need to introduce two other differentials. Let us first define the boundary map

$$
b : \Omega^n(\mathcal{A}) \longrightarrow \Omega^{n-1}(\mathcal{A})
$$
by the formula
\[ b(\omega \, da) = (-1)^{|\omega|} [\omega, a] \]
where \(|\omega| = n - 1\) is the degree of the form \(\omega \in \Omega^{n-1}(A)\). This definition uses the structure of a differential graded algebra on \(\Omega^\bullet(A)\). Using the explicit formula (11) for the product of two forms and assuming that \(\omega = a_0 \, da_1 \cdots da_{n-1}\), this definition may be rewritten in the form
\[
\begin{align*}
\text{(13)}
\end{align*}
\]

The Karoubi operator is the degree 0 operator \(\kappa : \Omega^n(A) \to \Omega^n(A)\) defined by
\[ \kappa(\omega \, da) = (-1)^{|\omega|} \, da \, \omega \]
where \(\omega \in \Omega^{n-1}(A)\). Explicitly, this operator is given by the formula on \(\kappa : A^{\otimes (n+1)} \to A^{\otimes (n+1)}\) through
\[ \kappa(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} + (-1)^n 1 \otimes a_n a_0 \otimes \cdots \otimes a_{n-1}. \]

On the image \(\text{d} \Omega^\bullet(A)\) of the differential \(\text{d}\), this operator is precisely the generator (with sign) of cyclic permutations. With this in mind we introduce the remaining differential
\[ B = \sum_{i=0}^{n} \kappa^i \, \text{d} : \Omega^n(A) \longrightarrow \Omega^{n+1}(A). \]

It is easy to check that the two operators \(b\) and \(B\) anticommute and are nilpotent,
\[ b^2 = b \, B + B \, b = B^2 = 0. \]

The two differentials \(B\) and \(b\) give \(\Omega^\bullet(A)\) the structure of a mixed complex \((\Omega^\bullet(A), b, B)\), which can be organised into a double complex given by the diagram
\[ \text{(14)} \]
which in bidegree \((p, q)\) contains \(\Omega^{p-q}(A)\). The columns in this complex are repeated and we declare all spaces located at \((p, q)\) with \(p - q < 0\) or \(p < 0\) to be trivial. Thus this double complex occupies one octant in the \((p, q)\)-plane. There is a canonical isomorphism \(S\) which by definition is the identity map sending the space \(\Omega^n(A)\) located at \((p+1, q+1)\) to itself located at \((p, q)\). The column at \(p = 0\) is by definition annihilated by \(S\). This operator is Connes’ periodicity operator. It follows from its definition that \(S\) is of degree \(-2\).

The total complex \((\text{Tot} \Omega^\bullet(A), b + B)\) of the bicomplex \((\Omega^\bullet(A), b, B)\) is defined in degree \(n\) by the finite sum

\[
\text{Tot}_n \Omega^\bullet(A) = \bigoplus_{p \geq 0} \Omega^{n-2p}(A).
\]

The Hochschild homology \(\text{HH}_\bullet(A)\) of the algebra \(A\) is defined to be the homology of the complex \((\Omega^\bullet(A), b)\),

\[
\text{HH}_\bullet(A) = H_\bullet(\Omega^\bullet(A), b) .
\]

The cyclic homology \(\text{HC}_\bullet(A)\) of the algebra \(A\) is defined to be the homology of the total complex \((\text{Tot} \Omega^\bullet(A), b + B)\),

\[
\text{HC}_\bullet(A) = H_\bullet(\text{Tot} \Omega^\bullet(A), b + B) .
\]

If we denote by \(I : \Omega^\bullet(A) \rightarrow \text{Tot} \Omega^\bullet(A)\) the inclusion of the first column into the double complex \((14)\), then by using the definition of the Connes periodicity operator it is not difficult to deduce the fundamental relation between Hochschild and cyclic homology given by the long exact sequence

\[
\cdots \rightarrow \text{HH}_{n+2}(A) \xrightarrow{I} \text{HC}_{n+2}(A) \xrightarrow{S} \text{HC}_n(A) \xrightarrow{B} \text{HH}_n(A) \xrightarrow{B} \text{HH}_{n+1}(A) \rightarrow \cdots .
\]

The map \(S\) in this sequence is induced by the periodicity operator which gives rise to a surjection \(S : \text{Tot}_{n+2} \Omega^\bullet(A) \rightarrow \text{Tot}_n \Omega^\bullet(A)\).

Finally, we define the periodic cyclic homology. For this, we need to consider a complex that is a completion, in a certain sense, of the total complex used in the construction of cyclic homology. Thus we put

\[
\widehat{\Omega}^\bullet(A) = \prod_{n \geq 0} \Omega^n(A) .
\]

Elements of this space are inhomogenous forms \((\omega_0, \omega_1, \ldots, \omega_n, \ldots)\), where \(\omega_n \in \Omega^n(A)\), with possibly infinitely many non-zero components. We shall regard this space as being \(\mathbb{Z}_2\)-graded with the decomposition into even and odd degree forms given by

\[
\widehat{\Omega}^{\text{even}}(A) = \prod_{n \geq 0} \Omega^{2n}(A) \quad \text{and} \quad \widehat{\Omega}^{\text{odd}}(A) = \prod_{n \geq 0} \Omega^{2n+1}(A) .
\]
A typical element of $\hat{\Omega}^{\text{even}}(A)$ is a sequence $(\omega_0, \omega_2, \ldots, \omega_{2n}, \ldots)$, and similarly for $\hat{\Omega}^{\text{odd}}(A)$. Then the periodic cyclic homology $HP_\bullet(A)$ of the algebra $A$ is defined to be the homology of the $\mathbb{Z}_2$-graded complex

$$\cdots \xrightarrow{b+B} \hat{\Omega}^{\text{even}}(A) \xrightarrow{b+B} \hat{\Omega}^{\text{odd}}(A) \xrightarrow{b+B} \hat{\Omega}^{\text{even}}(A) \xrightarrow{b+B} \cdots .$$

The Connes operator $S$ also provides a relation between the cyclic and periodic cyclic homology in the following way. For every $n$, there is a surjection

$$T_{2n} : \hat{\Omega}^{\text{even}}(A) \longrightarrow \text{Tot}_{2n} \Omega^\bullet(A)$$

which sends a form $(\omega_0, \omega_2, \ldots)$ to its truncation $(\omega_0, \omega_2, \ldots, \omega_{2n})$. For various values of $n$ these surjections are compatible with the periodicity operator $S$ in the sense that there is a commutative diagram

$$\begin{array}{cccc}
\text{Tot}_{2n+2} \Omega^\bullet(A) & \xrightarrow{T_{2n+2}} & \hat{\Omega}^{\text{even}}(A) & \xrightarrow{S} \\
\text{Tot}_{2n} \Omega^\bullet(A) & \xrightarrow{T_{2n}} & & \\
\end{array}$$

(17)

An even periodic cycle is a sequence of the type described above which is annihilated by the operator $b+B$, i.e., applying this operator creates the zero chain in $\hat{\Omega}^{\text{odd}}(A)$ as in the diagram

$$\begin{array}{cccc}
\vdots & \downarrow b & 0 & \xleftarrow{B} \omega_4 \\
& & 0 & \xleftarrow{B} \omega_2 \\
& & & 0 \xleftarrow{B} \omega_0 \\
& & & 0 \\
\end{array}$$

The vertical map in degree 0 is the zero map. The truncation of this cycle in, say, degree 2 creates an element of $\text{Tot}_2 \Omega^\bullet(A)$ which is a cycle in the cyclic complex.
The zero map in the upper left corner appears due to the definition of the differential in the cyclic complex. It kills the leftmost column (where \( p = 0 \)), which in this case is the column where \( \omega_2 \) is located.

Thus, for any \( n \) the truncation map \( T_{2n} \) sends an even periodic cycle to a cyclic \( 2n \)-cycle and so induces a map \( T_{2n} : \text{HP}_{\text{even}}(\mathcal{A}) \to \text{HC}_{2n}(\mathcal{A}) \). From the diagram (17) it follows that these maps are compatible with the periodicity operator \( S \) and we obtain a surjection

\[
\text{HP}_{\text{even}}(\mathcal{A}) \longrightarrow \lim_{\mathcal{S}} \text{HC}_{2n}(\mathcal{A}).
\]

There is a complementary map in odd degree whose construction is identical to the one just described. This map is not an isomorphism in general. Its kernel is equal to \( \lim_{\mathcal{S}} \text{HC}_{\ast + 2n + 1}(\mathcal{A}) \), where \( \lim_{\mathcal{S}} \) is the first derived functor of the inverse limit functor.

We will now consider a key example which illustrates the importance of these constructions. Let \( \mathcal{A} = C^\infty(\mathcal{X}) \) be the algebra of smooth functions on a smooth paracompact spacetime manifold \( \mathcal{X} \). Then the action of the boundary map (13) is trivial and the mixed complex \((\Omega^\bullet(\mathcal{A}), b, B)\) reduces to the complexified de Rham complex \((\Omega^\bullet(\mathcal{X}), d)\), where \( d \) is the usual de Rham exterior derivative on \( \mathcal{X} \). Equivalently, there is a natural surjection \( \mu : (\Omega^\bullet(\mathcal{A}), b, B) \to (\Omega^\bullet(\mathcal{X}), 0, d) \) of mixed complexes. The Connes-Pflaum version of the Hochschild-Kostant-Rosenberg theorem asserts that the map \( \mu \) is a quasi-isomorphism, i.e., it induces equality of the Hochschild homology \((15)\) with the de Rham complex. Explicitly, the map \( \mu : \Omega^n(\mathcal{A}) \to \Omega^n(\mathcal{X}) \) is implemented by sending a noncommutative \( n \)-form to a differential \( n \)-form as

\[
\mu(f^0 \, df^1 \cdots df^n) = \frac{1}{n!} f^0 \, df^1 \wedge \cdots \wedge df^n
\]

for \( f^i \in C^\infty(\mathcal{X}) \). It follows that the Hochschild homology of the algebra \( C^\infty(\mathcal{X}) \) gives the de Rham complex,

\[
\text{HH}_n(C^\infty(\mathcal{X})) \cong \Omega^n(\mathcal{X}),
\]

which implies that the periodic cyclic homology computes the periodic de Rham cohomology as

\[
\text{HP}_{\text{even}}(C^\infty(\mathcal{X})) \cong \text{H}_{\text{dR}}^{\text{even}}(\mathcal{X}) \quad \text{and} \quad \text{HP}_{\text{odd}}(C^\infty(\mathcal{X})) \cong \text{H}_{\text{dR}}^{\text{odd}}(\mathcal{X}).
\]

(18)
It is in this sense that we may regard cyclic homology as a generalization of de Rham cohomology to other (possibly noncommutative) settings.

4.2 Cyclic cohomology

As one would expect, by considering the duals of the chain spaces introduced in Section 4.1 above, one obtains the cohomology theories corresponding to the three cyclic-type homology theories defined there. A Hochschild $n$-cochain on the algebra $A$ is a linear form on $\Omega^n(A)$, or equivalently an $(n+1)$-multilinear functional $\varphi$ on $A$ which is simplicially normalized in the sense that $\varphi(a_0, a_1, \ldots, a_n) = 0$ if $a_i = 1$ for any $i$ such that $1 \leq i \leq n$. With the collection of all $n$-cochains denoted $C^n(A) = \text{Hom}_\mathbb{C}(\Omega^n(A), \mathbb{C})$, we form the Hochschild cochain complex $(C^\bullet(A), b)$ with coboundary map

$$b : C^n(A) \longrightarrow C^{n+1}(A)$$

given by the transpose of the differential $b$ as

$$b\varphi(a_0, a_1, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \ldots, a_n).$$

The cohomology of this complex is the Hochschild cohomology $\text{HH}^\bullet(A) = H^\bullet(C^\bullet(A), b)$, the dual theory to Hochschild homology defined in eq. (15).

Similarly, the operator $B$ transposes to the cochain complex $C^\bullet(A)$ and the cyclic cohomology $HC^\bullet(A)$ is defined as the cohomology of the complex $((\text{Tot} \Omega^\bullet(A))^\vee, b + B)$. The dual of the periodic complex is the complex which in even degree is spanned by finite sequences $(\varphi_0, \varphi_2, \ldots, \varphi_{2n})$ with $\varphi_1 \in C^1(A)$, and similarly in odd degree. The periodic cyclic cohomology $HP^\bullet(A)$ is the cohomology of this complex. The long exact sequence relating Hochschild and cyclic homology has an obvious dual sequence that links Hochschild and cyclic cohomology. The transpose of the periodicity operator provides an injection $S : HC^n(A) \rightarrow HC^{n+2}(A)$ of cyclic cohomology groups and therefore gives rise to two inductive systems of abelian groups, one running through even degrees and the other through odd degrees. One has

$$HP^\bullet(A) = \lim_{\longrightarrow} HC^{n+2n}(A).$$

This formal approach to cyclic cohomology, while very useful, hides two important features of the theory. Firstly, it seems to imply that cyclic cohomology is secondary to cyclic homology. In fact, it turns out that many
geometric and analytic situations provide natural examples of cyclic cocycles [10]. Secondly, this approach does not explain why cyclic cohomology is indeed cyclic. For this, we note that a Hochschild 0-cocycle \( \tau \in \text{Hom}(A, C) \) on the algebra \( A \) is a trace, i.e., \( \tau(a_0 a_1) = \tau(a_1 a_0) \). This tracial property is extended to higher orders via the following notion. Let \( \lambda: C^n(A) \to C^n(A) \) be the operator defined by

\[
\lambda(\phi(a_0, a_1, \ldots, a_n)) = (-1)^n \phi(a_n, a_0, \ldots, a_{n-1}).
\]

Then an \( n \)-cochain \( \varphi \in C^n(A) \) is said to be cyclic if it is invariant under the action of the cyclic group, \( \lambda \varphi = \varphi \). The set of cyclic \( n \)-cochains is denoted \( C^n_{\text{cyc}}(A) \). One can prove that the cohomology of this complex is isomorphic to the cohomology of the complex we have used above to define cyclic cohomology, and so we can alternatively define the cyclic cohomology \( HC^\bullet(A) \) of the algebra \( A \) as the cohomology of the cyclic cochain complex \( (C^\bullet_{\text{cyc}}(A), b) \).

\[
HC^\bullet(A) = H^\bullet(C^\bullet_{\text{cyc}}(A), b).
\]

An important class of cyclic cocycles is obtained as follows. Consider the algebra \( A = C^\infty(X) \) of smooth functions on a compact oriented manifold \( X \) of dimension \( d \). Put

\[
\varphi_X(f^0, f^1, \ldots, f^d) = \frac{1}{d!} \int_X f^0 \, df^1 \wedge \cdots \wedge df^d
\]

for \( f^i \in A \). Then \( \varphi_X \) is a cyclic \( d \)-cocycle. More generally, one can associate in this way a cyclic \((d - k)\)-cocycle with any closed \( k \)-current \( C \) on \( X \). In particular, the Chern-Simons coupling \( \langle C \sim Q(W, E, f), [X] \rangle \) on a D-brane \((W, E, f)\) is an inhomogeneous cyclic cocycle of definite parity for any closed cochain \( C \) associated to a Ramond-Ramond field on \( X \).

### 4.3 Local cyclic cohomology

Thus far we have not considered the possibility that the algebra \( A \) might be equipped with a topology. A major weakness of cyclic cohomology compared to K-theory is that it depends very sensitively on the domain of algebras. For instance, \( A = C^\infty(X) \) is the commutative, nuclear Fréchet algebra of smooth functions on the spacetime manifold \( X \) equipped with its standard semi-norm topology. More generally, we can allow \( A \) to be a complete multiplicatively convex algebra, i.e., \( A \) is a topological algebra whose topology is given by a family of submultiplicative semi-norms. In such cases the definition of the algebra \( \mathcal{A}^\bullet(A) \) of noncommutative differential forms will involve a choice of a suitably completed topological tensor product \( \otimes \). The correct choice is forced by the topology on \( A \) and the corresponding continuity properties of the multiplication map \( m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \). For nuclear Fréchet algebras \( \mathcal{A} \), there is a unique topology which is compatible with the tensor product.
structure on $A \otimes A$. In our later considerations we will often consider the situation in which $A = B^\infty$ is a suitable smooth subalgebra of a separable C*-algebra $B$. Local cyclic cohomology is best suited to deal with these and other classes of algebras, and it moreover has a useful extension to a bivariant functor. The bivariant cyclic cohomology theories were introduced to provide a target for the Chern character from KK-theory, which we describe in the next section.

The space of cochains in this theory is a certain deformation of the space of maps $\text{Hom}_C(\check{\Omega}^\bullet(A), \check{\Omega}^\bullet(B))$ with the $\mathbb{Z}_2$-grading induced from the spaces of inhomogeneous forms over the algebras $A$ and $B$. Alternatively, we can define

$$\text{Hom}_C(\check{\Omega}^\bullet(A), \check{\Omega}^\bullet(B)) = \lim_{\leftarrow} \lim_{\rightarrow} \text{Hom}_C(\bigoplus_{i \leq n} \Omega^i(A), \bigoplus_{j \leq m} \Omega^j(B)).$$

This is a $\mathbb{Z}_2$-graded vector space equipped with a differential $\partial$ that acts on cochains $\varphi$ by the graded commutator

$$\partial \varphi = [\varphi, b + B].$$

The local version of this theory is defined by using a deformation of the tensor algebra called the $X$-complex, which is the $\mathbb{Z}_2$-graded completion of $\Omega^\bullet(A)$ given by

$$X^\bullet(TA) : \Omega^0(TA) = TA \xrightarrow{\text{odd}} \Omega^1(TA) \xrightarrow{b} \Omega^2(TA) := \frac{\Omega^1(TA)}{[\Omega^1(TA), \Omega^1(TA)]}.$$

Puschnigg’s completion of $X^\bullet(TA)$, let

$$\check{X}^\bullet(TA) : \check{\Omega}^{even}(A) \xrightarrow{\text{odd}} \check{\Omega}^{odd}(A),$$

then defines the $\mathbb{Z}_2$-graded bivariant local cyclic cohomology

$$\text{HL}_\bullet(A, B) = \text{H}_\bullet(\text{Hom}_C(\check{X}^\bullet(TA), \check{X}^\bullet(TB)), \partial).$$

The main virtues of Puschnigg’s cyclic theory for our purposes is that it is the one “closest” to Kasparov’s KK-theory, in the sense that it possesses the following properties. It is defined on large classes of topological and bornological algebras, i.e., algebras together with a chosen family of bounded subsets closed under forming finite unions and taking subsets, and for separable C*-algebras. It defines a bifunctor $\text{HL}_\bullet(-, -)$ which is homotopy invariant, split exact and satisfies excision in each argument. It possesses a bilinear, associative composition product

$$\otimes : \text{HL}_i(A, B) \times \text{HL}_j(B, C) \rightarrow \text{HL}_{i+j}(A, C).$$

It also carries a bilinear, associative exterior product
defined using the projective tensor product which maps onto the minimal $C^*$-algebraic tensor product on the category of separable $C^*$-algebras. In general, without any extra assumptions, this tensor product differs from the usual spatial tensor product, but at least in the examples we consider later on this problem can always be fixed. Thus in what follows, we will not distinguish between the algebraic tensor product $\otimes$ and its topological completion.

The local cyclic cohomology reduces to other cyclic theories under suitable conditions, such as the periodic cyclic cohomology for non-topological algebras and even Fréchet algebras, Meyer’s analytic theories for bornological algebras [48], and Connes’ entire cyclic cohomology for Banach algebras. It thus possesses the same algebraic properties as the usual bivariant cyclic cohomology theories, and in this sense it unifies cyclic homology and cohomology. A particularly useful property which we will make extensive use of is the following. Let $\mathcal{A}$ be a Banach algebra with the metric approximation property, and let $\mathcal{A}^\infty$ be a smooth subalgebra of $\mathcal{A}$. Then the inclusion $\mathcal{A}^\infty \hookrightarrow \mathcal{A}$ induces an invertible element of $\text{HL}_0(\mathcal{A}^\infty, \mathcal{A})$. Thus in this case the algebras $\mathcal{A}^\infty$ and $\mathcal{A}$ are $\text{HL}$-equivalent.

Let us consider again the illustrative example of the algebra of functions $\mathcal{A} = C(X)$ on a compact oriented manifold $X$ with $\dim(X) = d$. In this case the inclusion of the smooth subalgebra $C^\infty(X) \hookrightarrow C(X)$ gives an isomorphism [48] $\text{HL}(C(X)) \cong \text{HL}(C^\infty(X)) \cong \text{HP}(C^\infty(X))$ with the periodic cyclic cohomology. The Puschnigg complex coincides with the periodic complexified de Rham complex $(\Omega^\bullet(X), d)$. Using the isomorphism [18] we then arrive at the isomorphism of $\mathbb{Z}_2$-graded groups

$$\text{HL}_\bullet(C(X)) \cong H^\bullet_{\text{dR}}(X).$$

The cyclic $d$-cocycle [19] under this isomorphism induces the orientation fundamental class $\Xi = m^*[\varphi] \in \text{HL}^d(C(X) \otimes C(X))$ corresponding to the orientation cycle $[X]$ of the manifold $X$.

5 D-brane charge on noncommutative spaces

In this section we will generalize the Minasian-Moore formula [7] for the Ramond-Ramond charge of a D-brane to large classes of separable $C^*$-algebras representing generic noncommutative spacetimes. This will require a few mathematical constructions of independent interest on their own. In particular, we will develop noncommutative versions of the characteristic classes appearing in eq. [7] and show how they are related through a generalization of the Grothendieck-Riemann-Roch theorem [9].
5.1 Chern characters

We will begin by exhibiting the fundamental Chern character maps which link K-theory and periodic cyclic homology, K-homology and periodic cyclic cohomology, and more generally KK-theory and bivariant cyclic cohomology. They provide explicit cyclic cocycles for Fredholm modules, and establish crucial links between duality in KK-theory and in bivariant cyclic cohomology which will be the crux of some of our later constructions. We begin with a description of the Chern character in K-theory. Let \( A \) be a unital Fréchet algebra over \( \mathbb{C} \). Acting on the K-theory of the algebra \( A \), we construct the homomorphism of abelian groups

\[
\text{ch} : K_0(A) \to \text{HP}_{\text{even}}(A)
\]

as follows. Let \([p] \in K_0(A)\) be the Murray-von Neumann equivalence class of an idempotent matrix \( p \in M_r(A) = A \otimes M_r(\mathbb{C}) \), i.e., a projection \( p = p^2 \). Then the Chern character assigns to \([p]\) an even class in the periodic cyclic homology of \( A \) represented by the even periodic cycle

\[
\text{ch}^\#(p) = \text{Tr}_r(p) + \sum_{n \geq 1} \frac{(2n)!}{n!} \text{Tr}_r\left((p - \frac{1}{2}) dp^{2n}\right)
\]

valued in \( \Omega_{\text{even}}(A) \), where \( \text{Tr}_r : M_r(A) \to A \) is the ordinary \( r \times r \) matrix trace. One readily checks that it gives a cycle in the reduced \((b,B)\) bi-complex of cyclic homology that we described in Section 4.1, i.e., \((b + B)\text{ch}^\#(p) = 0\).

When \( A = C^\infty(X) \) with \( X \) a smooth compact manifold, it coincides with the usual Chern-Weil character \( \text{Tr} \exp(F/2\pi i) \) defined in terms of the curvature \( F \) of the canonical Grassmann connection of the corresponding complex vector bundle \( E \to X \). The Chern map (20) becomes an isomorphism on tensoring with \( \mathbb{C} \).

For applications to the description of D-brane charges in cyclic theory, it is more natural to use cyclic cohomology classes corresponding to elements in K-homology. Let \((\mathcal{H}, \rho, F)\) be an \((n+1)\)-summable even Fredholm module over the algebra \( A \) with \( n \) even. This means that \([F, \rho(a)] \in \mathcal{L}^{n+1} \) for all \( a \in A \), where \( \mathcal{L}^p = \mathcal{L}^p(\mathcal{H}) := \{ T \in \mathcal{K}(\mathcal{H}) \mid \text{Tr}_\mathcal{H}(T^p) < \infty \} \) is the \( p \)-class Shatten ideal of compact operators. Then the character of the Fredholm module is the cyclic \( n \)-cocycle \( \tau^n \) given by

\[
\tau^n(a_0, a_1, \ldots, a_n) = \text{Tr}_\mathcal{H}(\gamma \rho(a_0) [F, \rho(a_1)] \cdots [F, \rho(a_n)])
\]

where \( \gamma \) is the grading involution on \( \mathcal{H} \) defining its \( \mathbb{Z}_2 \)-grading into \( \pm 1 \) eigenspaces of \( \gamma \). One checks closure \( b \tau^n = 0 \) and cyclicity \( \lambda \tau^n = (-1)^n \tau^n \).

Since \( \mathcal{L}^{p_1} \subset \mathcal{L}^{p_2} \) for \( p_1 \leq p_2 \), we can replace \( n \) by \( n + 2k \) with \( k \) any integer in this definition, and so only the (even) parity of \( n \) is fixed. Thus for any \( k \geq 0 \), one gets a sequence of cyclic cocycles \( \tau^{n+2k} \) with the same parity. The cyclic cohomology classes of these cocycles are related by Connes’ periodicity operator \( S \) in \( \text{HC}^{n+2k+2}(A) \), and therefore the sequence \( (\tau^{n+2k})_{k \geq 0} \) determines
a well-defined class \( ch^\sharp(\mathcal{H}, \rho, F) \) called the Chern character of the even Fredholm module \((\mathcal{H}, \rho, F)\) in the even periodic cyclic cohomology \( HP^{\text{even}}(A) \). Thus we get a map

\[
ch^\sharp : K^0(A) \longrightarrow HP^{\text{even}}(A)
\]

which becomes an isomorphism after tensoring over \( \mathbb{C} \). See ref. [16] for an extension of this definition to unbounded and infinite-dimensional Fredholm modules.

Our main object of interest is the Chern character in KK-theory. A cohomological functor which complements the bivariant KK-theory is provided by the local bivariant cyclic cohomology \( HL_\bullet(A, B) \) that we introduced in Section 4.3. Since both \( KK_\bullet(A, B) \) and \( HL_\bullet(A, B) \) are homotopy invariant, stable and satisfy excision, the universal property of KK-theory implies that there is a natural bivariant \( \mathbb{Z}_2 \)-graded Chern character homomorphism

\[
ch : KK_\bullet(A, B) \longrightarrow HL_\bullet(A, B)
\]

which enjoys the following properties:

1. \( ch \) is multiplicative, i.e., if \( \alpha \in KK_i(A, B) \) and \( \beta \in KK_j(B, C) \) then

\[
ch(\alpha \otimes_B \beta) = ch(\alpha) \otimes_B ch(\beta);
\]

2. \( ch \) is compatible with the exterior product; and

3. \( ch([\phi]_{KK}) = [\phi]_{HL} \) for any for any algebra homomorphism \( \phi : A \to B \).

The last property implies that the Chern character sends invertible elements of KK-theory to invertible elements of bivariant cyclic cohomology. In particular, every PD pair for KK-theory is also a PD pair for HL-theory, but not conversely (due to e.g. torsion). However, in the following it will be important to consider distinct fundamental classes \( \Xi \neq ch(\Delta) \) in local cyclic cohomology. If \( A, B \) obey the universal coefficient theorem (10) for KK-theory, then there is an isomorphism

\[
HL_\bullet(A, B) \cong \text{Hom}_\mathbb{C}(K_\bullet(A) \otimes_\mathbb{Z} \mathbb{C}, K_\bullet(B) \otimes_\mathbb{Z} \mathbb{C}) .
\]

If the K-theory \( K_\bullet(A) \) is finitely generated, then this is also equal to

\[
HL_\bullet(A, B) \cong KK_\bullet(A, B) \otimes_\mathbb{Z} \mathbb{C} .
\]

5.2 Todd classes

Let \( A \) be a PD algebra with fundamental K-homology class \( \Delta \in K^d(A \otimes A^o) \), and fundamental cyclic cohomology class \( \Xi \in HL^d(A \otimes A^o) \). Then we define the Todd class of \( A \) to be the element

\[
\text{Todd}(A) := \Xi^\vee \otimes_{A^o} ch(\Delta) \in HL_0(A, A) .
\]
The Todd class is invertible with inverse given by
\[ \text{Todd}(\mathcal{A})^{-1} = (-1)^d \text{ch}(\Delta^{\vee}) \otimes_{\mathcal{A}} \Xi. \]

More generally, one defines the Todd class \( \text{Todd}(\mathcal{A}) \) for PD pairs of algebras \((\mathcal{A}, \mathcal{B})\) by replacing \( \mathcal{A}^o \) with \( \mathcal{B} \) above [16].

The Todd class depends "covariantly" on the choices of fundamental classes in the respective moduli spaces [16]. For any other fundamental class \( \Delta_1 \) for K-theory of \( \mathcal{A} \), one has \( \Xi^{\vee} \otimes_{\mathcal{A}} \text{ch}(\Delta_1) = \text{ch}(\ell) \otimes_{\mathcal{A}} \text{Todd}(\mathcal{A}) \) where \( \ell = \Delta^{\vee} \otimes_{\mathcal{A}} \Delta_1 \) is an invertible element in \( \text{KK}_0(\mathcal{A}, \mathcal{A}) \). Conversely, if \( \ell \) is an invertible element in \( \text{KK}_0(\mathcal{A}, \mathcal{A}) \), then \( \ell \otimes_{\mathcal{A}} \Delta \) is a fundamental class for K-theory of \( \mathcal{A} \) for any fundamental class \( \Delta \). In particular, if \( \mathcal{A}, \mathcal{B} \) are KK-equivalent \( C^* \)-algebras, with the KK-equivalence implemented by an invertible element \( \alpha \) in \( \text{KK}_*(\mathcal{A}, \mathcal{B}) \), then their Todd classes are related through
\[ \text{Todd}(\mathcal{B}) = \text{ch}(\alpha)^{-1} \otimes_{\mathcal{A}} \text{Todd}(\mathcal{A}) \otimes_{\mathcal{A}} \text{ch}(\alpha). \] (21)

The following example provides the motivation behind this definition. Let \( \mathcal{A} = C(X) \) where \( X \) is a compact complex manifold. Then \( \mathcal{A} \) is a PD algebra, with KK-theory fundamental class \( \Delta \) provided by the Dolbeault operator \( \partial \) on \( X \times X \), and HL-theory fundamental class \( \Xi \) provided by the orientation cycle \([X]\). By the universal coefficient theorem [16], one has an isomorphism \( \text{HL}_0(\mathcal{A}, \mathcal{A}) \cong \text{End}(H^*(X, \mathbb{Q})) \). Then \( \text{Todd}(\mathcal{A}) = \cup \text{Todd}(X) \) is cup product with the usual Todd characteristic class \( \text{Todd}(X) \in H^*(X, \mathbb{Q}) \) of the tangent bundle of \( X \).

5.3 Grothendieck-Riemann-Roch theorem

Let \( f : \mathcal{A} \to \mathcal{B} \) be a K-oriented morphism of separable \( C^* \)-algebras. The Grothendieck-Riemann-Roch formula compares the class \( \text{ch}(f!) \) with the HL-theory orientation class \( f^* \) in \( \text{HL}_d(\mathcal{B}, \mathcal{A}) \). If \( \mathcal{A}, \mathcal{B} \) are PD algebras, then one has \( d = d_{\mathcal{A}} - d_{\mathcal{B}} \) and
\[ \text{ch}(f!) = (-1)^{d_{\mathcal{B}}} \text{Todd}(\mathcal{B}) \otimes_{\mathcal{B}} (f^*) \otimes_{\mathcal{A}} \text{Todd}(\mathcal{A})^{-1}. \] (22)

This formula is proven by expanding out both sides using the various definitions, along with associativity of the Kasparov intersection product [16]. It leads to the commutative diagram
\[
\begin{array}{ccc}
K_*(\mathcal{B}) & \xrightarrow{f^*} & K_*(\mathcal{A}) \\
\text{ch} \otimes_{\mathcal{B}} \text{Todd}(\mathcal{B}) & & \text{ch} \otimes_{\mathcal{A}} \text{Todd}(\mathcal{A}) \\
\text{HL}_*(\mathcal{B}) & \xrightarrow{f_*} & \text{HL}_*(\mathcal{A})
\end{array}
\]
generalizing eq. [9].
As an example of the applicability of this formula, suppose that $\mathcal{A}$ is unital with even degree fundamental class. Then there is a canonical $K$-oriented morphism $\lambda: \mathbb{C} \rightarrow \mathcal{A}$, $z \mapsto z \cdot 1$ which induces a homomorphism on $K$-theory $\lambda!: K_0(\mathcal{A}) \rightarrow \mathbb{Z}$ with

$$\lambda!(\xi) = \lambda_*(\text{ch}(\xi) \otimes_\mathcal{A} \text{Todd}(\mathcal{A}))$$

for $\xi \in K_0(\mathcal{A})$. When $\mathcal{A} = C(X)$, with $X$ a compact spin$^c$ manifold, then $\lambda!(\xi) = \text{index}(\mathcal{D}_\xi)$ for $\xi \in K^0(X)$ and eq. (23) is just the Atiyah-Singer index theorem (6). Generally, when $\xi = \mathcal{A}$ is the trivial rank one module over $\mathcal{A}$, then $\lambda!(\xi)$ defines a characteristic numerical invariant of $\mathcal{A}$, which we may call the Todd genus of the algebra $\mathcal{A}$.

### 5.4 Isometric pairing formulas

Suppose that $\mathcal{A}$ is a PD algebra with symmetric fundamental classes $\Delta$ and $\Xi$, i.e., $\sigma(\Delta)^o = \Delta$ in $K^d(\mathcal{A} \otimes \mathcal{A}^o)$, where $\sigma: \mathcal{A} \otimes \mathcal{A}^o \rightarrow \mathcal{A}^o \otimes \mathcal{A}$ is the flip involution $x \otimes y^o \mapsto y^o \otimes x$, and similarly for $\Xi$ in $HL^d(\mathcal{A} \otimes \mathcal{A}^o)$. In this case we can define a symmetric bilinear pairing on the $K$-theory of $\mathcal{A}$ by

$$(\alpha, \beta)_K = (\alpha \otimes \beta)^o \otimes_{\mathcal{A} \otimes \mathcal{A}^o} \Delta \in KK_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$$

for $\alpha, \beta \in K_*(\mathcal{A})$. It coincides with the index pairing (1) when $\mathcal{A} = C(X)$, for $X$ a spin$^c$ manifold with fundamental class given by the Dirac operator $\Delta = \mathcal{D} \otimes \mathcal{D}$, as then

$$(\alpha, \beta)_K = \mathcal{D}_\alpha \otimes_{C(X)} \beta = \text{index}(\mathcal{D}_\alpha \otimes \beta)$$

by definition of the intersection product on KK-theory. Similarly, one has a symmetric bilinear pairing on local cyclic homology given by

$$(x,y)_{HL} = (x \otimes y^o) \otimes_{\mathcal{A} \otimes \mathcal{A}^o} \Xi \in HL_0(\mathbb{C}, \mathbb{C}) = \mathbb{C},$$

generalizing the pairing (3).

If $\mathcal{A}$ satisfies the universal coefficient theorem (10), then one has an isomorphism $HL_*(\mathcal{A}, \mathcal{A}) \cong \text{End}(HL_*(\mathcal{A}))$. If $HL_*(\mathcal{A})$ is a finite-dimensional vector space, $n := \dim_\mathbb{C}(HL_*(\mathcal{A})) < \infty$, then we may use the universal coefficient theorem to identify the Todd class $\text{Todd}(\mathcal{A})$ with an invertible matrix in $GL(n, \mathbb{C})$. In this case the square root $\sqrt{\text{Todd}(\mathcal{A})}$ may be defined using the usual Jordan normal form of linear algebra, and then reinterpreted as a class in $HL_*(\mathcal{A}, \mathcal{A})$ again by using the universal coefficient theorem. This square root is not unique, but we assume that it is possible to fix a canonical choice. Under these circumstances, we can define the modified Chern character

$$\text{ch} \otimes_\mathcal{A} \sqrt{\text{Todd}(\mathcal{A})} : K_*(\mathcal{A}) \rightarrow HL_*(\mathcal{A})$$

which is an isometry of the inner products (24) and (25) [16, 17].
Suppose now that $A$, $D$ represent noncommutative D-branes with $A$ as above, with a given K-oriented morphism $f : A \to D$ and Chan-Paton bundle $\xi \in K_*(D)$. In this case there is a noncommutative version of Minasian-Moore formula (7) given by

$$Q(D, \xi, f) = \text{ch}(f_!(\xi)) \otimes_A \sqrt{\text{Tod}(A)} \in \text{HL}_*(A).$$

More generally, consider a D-brane in the noncommutative spacetime $A$ described by a Fredholm module over $A$ representing a K-homology class $\mu \in K^*(A)$. It has a “dual” charge given by

$$Q(\mu) = \sqrt{\text{Tod}(A)}^{-1} \otimes_A \text{ch}(\mu) \in \text{HL}^*(A).$$

This vector satisfies the isometry rule [16]

$$\Xi^\vee \otimes_{A \otimes A^*} (Q(\mu) \otimes Q(\nu)^\vee) = \Delta^\vee \otimes_{A \otimes A^*} (\mu \otimes \nu^\vee),$$

and reproduces the noncommutative Minasian-Moore formula (27) in the case when $\mu = f_!(\xi) \otimes_A \Delta$ is dual to the Chan-Paton bundle $\xi$.

6 Noncommutative D2-branes

In this section we will apply our general formalism to the example of D-branes on noncommutative Riemann surfaces, as defined in refs. [49, 50]. Consider a collection of D2-branes wrapping a compact, oriented Riemann surface $\Sigma_g$ of genus $g \geq 1$ with a constant $B$-field. This example generalizes the classic example of D-branes on the noncommutative torus $T^2_\theta$, obtained for $g = 1$.

The fundamental group of $\Sigma_g$ admits the presentation

$$\Gamma_g = \left\{ U_j, V_j, j = 1, \ldots, g \mid \prod_{j=1}^g [U_j, V_j] = 1 \right\}.$$

Its group cohomology is $H^2(\Gamma_g, U(1)) \cong \mathbb{R}/\mathbb{Z}$, and so for each $\theta \in [0, 1)$ there is a unique $U(1)$-valued two-cocycle $\sigma_\theta$ on $\Gamma_g$, representing the holonomy of the $B$-field on $\Sigma_g$. The reduced twisted group C*-algebra $A_\theta := C^*_r(\Gamma_g, \sigma_\theta)$ is isomorphic to the algebra generated by unitary elements $U_j, V_j, j = 1, \ldots, g$ obeying the single relation

$$\prod_{j=1}^g [U_j, V_j] = \exp(2\pi i \theta).$$

When $\theta$ is irrational, the degree 0 K-theory is $K_0(A_\theta) \cong K^0(\Sigma_g) = \mathbb{Z}^2$, with generators $e_0 = [1]$ and $e_1$ satisfying $\text{Tr}(e_1) = \theta$, where $\text{Tr} : C^*_r(\Gamma_g, \sigma_\theta) \to \mathbb{C}$ is the evaluation at the identity element $1_{\Gamma_g}$ of $\Gamma_g$. The degree 1 K-theory
is given by $K_1(A_\theta) \cong K^1(\Sigma_g) = \mathbb{Z}^{2g}$, with basis $U_j, V_j$. There is a smooth subalgebra $A_\theta^\infty \hookrightarrow A_\theta$ such that $HL_\bullet(A_\theta) \cong HL_\bullet(A_\theta^\infty) \cong HP_\bullet(A_\theta^\infty)$ \cite{16, 17}.

The algebra $A_\theta$ is a PD algebra, with Bott class given by

$$\Delta^\vee = e_0 \otimes e_0^\alpha - e_1 \otimes e_0^\alpha + \sum_{j=1}^g (U_j \otimes V_j^\alpha - V_j \otimes U_j^\alpha).$$

Let $\mu_\theta : K^\bullet(\Sigma_g) \overset{\cong}{\rightarrow} K_\bullet(C^*_r(\Gamma_g, \sigma_\theta))$ be the twisted Kasparov isomorphism, and let $\nu_\theta$ be its analog in periodic cyclic homology. The commutative diagram of isomorphisms

$$
\begin{array}{ccc}
K^\bullet(\Sigma_g) & \overset{\mu_\theta}{\longrightarrow} & K_\bullet(A_\theta) \\
\downarrow \text{ch} & & \downarrow \text{ch}_{\sigma_\theta} \\
H^\bullet(\Sigma_g, \mathbb{Z}) & \overset{\nu_\theta}{\longrightarrow} & HL_\bullet(A_\theta)
\end{array}
$$

then serves to show that the Todd class is given by $\text{Todd}(A_\theta) = \nu_\theta(\text{Todd}(\Sigma_g))$. This construction thus leads to the charge vector for a wrapped noncommutative D2-brane $(D, \xi, f)$, with K-oriented morphism $f : A_\theta \rightarrow D$ and Chan-Paton bundle $\xi \in K_\bullet(D)$, defined by

$$Q_\theta(D, \xi, f) = \nu_\theta\left(\text{ch}(\mu_\theta^{-1} \circ f_!(\xi)) \sim \sqrt{\text{Todd}(\Sigma_g)}\right) \in HL_\bullet(A_\theta).$$

This formula incorporates the contribution from the constant B-field in the usual way \cite{21, 51}.

7 D-branes and $H$-flux

In this section we will consider in some detail the example of D-branes in a compact, even-dimensional oriented manifold $X$ with constant background Neveu-Schwarz $H$-flux. In this case, it is well-known \cite{2, 52} that one should replace spacetime $X$ by a noncommutative $C^*$-algebra $CT(X, H)$, the stable continuous trace $C^*$-algebra with spectrum $X$ and Dixmier-Douady invariant $H$ \cite{53}. This algebra has the property that it is locally Morita equivalent to spacetime, but not in general globally equivalent to it.

7.1 Projective bundles and twisted K-theory

We will start by describing twisted K-theory, the appropriate receptacle for the classification of D-brane charge in $H$-flux backgrounds, in the spirit of Atiyah and Segal \cite{54} (glossing over many topological details, as before). Let $\mathcal{H}$ be a fixed, separable Hilbert space of dimension $\geq 1$. We will denote the
associated projective space of $\mathcal{H}$ by $\mathbb{P} = \mathbb{P}(\mathcal{H})$. It is compact if and only if $\mathcal{H}$ is finite-dimensional. Let $PU = PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ be the projective unitary group of $\mathcal{H}$ equipped with the compact-open topology. A projective bundle over $X$ is a locally trivial bundle of projective spaces, i.e., a fibre bundle $P \to X$ with fibre $\mathbb{P}(\mathcal{H})$ and structure group $PU(\mathcal{H})$. An application of the Banach-Steinhaus theorem shows that we may identify projective bundles with principal $PU(\mathcal{H})$-bundles (and the pointwise convergence topology on $PU(\mathcal{H})$).

If $G$ is a topological group, let $G_X$ denote the sheaf of germs of continuous functions $G \to X$, i.e., the sheaf associated to the constant presheaf given by $U \mapsto F(U) = G$. Given a projective bundle $P \to X$ and a sufficiently fine good open cover $\{U_i\}_{i \in I}$ of $X$, the transition functions between trivializations $P|_{U_i}$ can be lifted to bundle isomorphisms $g_{ij}$ on double intersections $U_{ij} = U_i \cap U_j$ which are projectively coherent, i.e., over each of the triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$ the composition $g_{ki} g_{jk} g_{ij}$ is given as multiplication by a $U(1)$-valued function $f_{ijk} : U_{ijk} \to U(1)$. The collection $\{(U_{ij}, f_{ijk})\}$ defines a $U(1)$-valued two-cocycle called a B-field on $X$, which represents a class $B_P$ in the sheaf cohomology group $H^2(X, U(1)_X)$. On the other hand, the sheaf cohomology $H^1(X, PU(\mathcal{H})_X)$ consists of isomorphism classes of principal $PU(\mathcal{H})$-bundles, and we can consider the isomorphism class $[P] \in H^1(X, PU(\mathcal{H})_X)$. There is an isomorphism $H^1(X, PU(\mathcal{H})_X) \cong H^2(X, U(1)_X)$ provided by the boundary map $[P] \mapsto B_P$. There is also an isomorphism

$$H^2(X, U(1)_X) \cong H^3(X, \mathbb{Z}_X) \cong H^3(X, \mathbb{Z}).$$

The image $\delta(P) \in H^3(X, \mathbb{Z})$ of $B_P$ is called the Dixmier-Douady invariant of $P$. When $\delta(P) = [H]$ is represented in $H^3(X, \mathbb{R})$ by a closed three-form $H$ on $X$, called the $H$-flux of the given B-field $B_P$, we will write $P = P_H$. One has $\delta(P) = 0$ if and only if the projective bundle $P$ comes from a vector bundle $E \to X$, i.e., $P = \mathbb{P}(E)$. By Serre’s theorem every torsion element of $H^3(X, \mathbb{Z})$ arises from a finite-dimensional bundle $P$. Explicitly, consider the commutative diagram of exact sequences of groups given by

$$0 \longrightarrow \mathbb{Z}_n \longrightarrow SU(n) \longrightarrow PU(n) \longrightarrow 0 \quad (28)$$

where we identify the cyclic group $\mathbb{Z}_n$ with the group of $n$-th roots of unity. Let $P$ be a projective bundle with structure group $PU(n)$, i.e., with fibres $\mathbb{P}(\mathbb{C}^n)$. Then the commutative diagram of long exact sequences of sheaf cohomology groups associated to the commutative diagram (28) of groups implies that the element $B_P \in H^2(X, U(1)_X)$ comes from $H^2(X, (\mathbb{Z}_n)_X)$, and therefore its order divides $n$. 

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One also has \( \delta(P_1 \otimes P_2) = \delta(P_1) + \delta(P_2) \) and \( \delta(P^\vee) = -\delta(P) \). This follows from the commutative diagram

\[
\begin{array}{c}
0 & \rightarrow & U(1) \times U(1) & \rightarrow & U(\mathcal{H}_1, \mathcal{H}_2) & \rightarrow & PU(\mathcal{H}_1, \mathcal{H}_2) & \rightarrow & 0 \\
0 & \downarrow & 0 & \downarrow & 0 & \downarrow & 0 & \downarrow & 0 \\
0 & \rightarrow & U(1) & \rightarrow & U(\mathcal{H}_1 \otimes \mathcal{H}_2) & \rightarrow & PU(\mathcal{H}_1 \otimes \mathcal{H}_2) & \rightarrow & 0
\end{array}
\]

and the fact that \( P^\vee \otimes P = \mathbb{P}(E) \) where \( E \) is the vector bundle of Hilbert-Schmidt endomorphisms of \( P \). Putting everything together, it follows that the cohomology group \( H^3(X, \mathbb{Z}) \) is isomorphic to the group of stable equivalence classes of principal \( PU(\mathcal{H}) \)-bundles \( P \rightarrow X \) with the operation of tensor product.

We are now ready to define the twisted K-theory of the manifold \( X \) equipped with a projective bundle \( P \rightarrow X \), such that \( P_x = \mathbb{P}(\mathcal{H}) \) for all \( x \in X \). We will first give a definition in terms of Fredholm operators, and then provide some equivalent, but more geometric definitions. Let \( \mathcal{H} \) be a \( \mathbb{Z}_2 \)-graded Hilbert space. We define \( \text{Fred}^0(\mathcal{H}) \) to be the space of self-adjoint degree 1 Fredholm operators \( T \) on \( \mathcal{H} \) such that \( T^2 - 1 \in \mathcal{K}(\mathcal{H}) \), together with the subspace topology induced by the embedding \( \text{Fred}^0(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H}) \times \mathcal{K}(\mathcal{H}) \) given by \( T \mapsto (T, T^2 - 1) \) where the algebra of bounded linear operators \( \mathcal{B}(\mathcal{H}) \) is given the compact-open topology and the Banach algebra of compact operators \( \mathcal{K} = \mathcal{K}(\mathcal{H}) \) is given the norm topology.

Let \( P = P_H \rightarrow X \) be a projective Hilbert bundle. Then we can construct an associated bundle \( \text{Fred}^0(P) \) whose fibres are \( \text{Fred}^0(\mathcal{H}) \). We define the \textit{twisted K-theory group of the pair} \( (X, P) \) to be the group of homotopy classes of maps

\[
K^0(X, \mathcal{H}) = \left[ X, \text{Fred}^0(P_H) \right].
\] (29)

The group \( K^0(X, \mathcal{H}) \) depends functorially on the pair \( (X, P_H) \), and an isomorphism of projective bundles \( \rho : P \rightarrow P' \) induces a group isomorphism \( \rho_* : K^0(X, \mathcal{H}) \rightarrow K^0(X, \mathcal{H}') \). Addition in \( K^0(X, \mathcal{H}) \) is defined by fibre-wise direct sum, so that the sum of two elements lies in \( K^0(X, \mathcal{H}_2) \) with \( [H_2] = \delta(P \otimes \mathbb{P}(\mathbb{C}^2)) = \delta(P) = [H] \). Under the isomorphism \( \mathcal{H} \otimes \mathbb{C}^2 \cong \mathcal{H} \), there is a projective bundle isomorphism \( P \rightarrow P \otimes \mathbb{P}(\mathbb{C}^2) \) for any projective bundle \( P \) and so \( K^0(X, \mathcal{H}_2) \) is canonically isomorphic to \( K^0(X, \mathcal{H}) \). When \( [H] \) is a non-torsion element of \( H^3(X, \mathbb{Z}) \), so that \( P = P_H \) is an infinite-dimensional bundle of projective spaces, then the index map \( K^0(X, \mathcal{H}) \rightarrow \mathbb{Z} \) is zero, i.e., any section of \( \text{Fred}^0(P) \) takes values in the index zero component of \( \text{Fred}^0(\mathcal{H}) \).

Let us now describe some other models for twisted K-theory which will be useful in our physical applications later on. A definition in algebraic K-theory may given as follows. A bundle of projective spaces \( P \) yields a bundle \( \text{End}(P) \) of algebras. However, if \( \mathcal{H} \) is an infinite-dimensional Hilbert space, then one has natural isomorphisms \( \mathcal{H} \cong \mathcal{H} \oplus \mathcal{H} \) and
as left $\text{End}(\mathcal{H})$-modules, and so the algebraic $K$-theory of the algebra $\text{End}(\mathcal{H})$ is trivial. Instead, we will work with the Banach algebra $K(\mathcal{H})$ of compact operators on $\mathcal{H}$ with the norm topology. Given that the unitary group $U(\mathcal{H})$ with the compact-open topology acts continuously on $K(\mathcal{H})$ by conjugation, to a given projective bundle $P_H$ we can associate a bundle of compact operators $\mathcal{E}_H \to X$ given by

$$\mathcal{E}_H = P_H \times_{PU} \mathcal{K}$$

with $\delta(\mathcal{E}_H) = [H]$. The Banach algebra $A_H := C_0(X, \mathcal{E}_H)$ of continuous sections of $\mathcal{E}_H$ vanishing at infinity is the continuous trace $C^*$-algebra $\text{CT}(X, H)$ [53]. Then the twisted $K$-theory group $K^*(X, H)$ of $X$ is canonically isomorphic to the algebraic $K$-theory group $K_*(A_H)$.

We will also need a smooth version of this definition. Let $A_H^\infty$ be the smooth subalgebra of $A_H$ given by the algebra $\text{CT}^\infty(X, H) = C^\infty(X, \mathcal{E}_H^1)$, where $\mathcal{E}_H^1 = P_H \times_{PU} \mathcal{L}^1$. Then the inclusion $\text{CT}^\infty(X, H) \hookrightarrow \text{CT}(X, H)$ induces an isomorphism $K_*(\text{CT}^\infty(X, H)) \cong K_*(\text{CT}(X, H))$ of algebraic $K$-theory groups. Upon choosing a bundle gerbe connection [55, 56], one has an isomorphism $K_*(\text{CT}^\infty(X, H)) \cong K^*(X, H)$ with the twisted $K$-theory [29] defined in terms of projective Hilbert bundles $P = P_H$ over $X$.

Finally, we propose a general definition based on $K$-theory with coefficients in a sheaf of rings. It parallels the bundle gerbe approach to twisted $K$-theory [56]. Let $\mathcal{B}$ be a Banach algebra over $\mathbb{C}$. Let $\mathcal{E}(\mathcal{B}, X)$ be the category of continuous $\mathcal{B}$-bundles over $X$, and let $C(X, \mathcal{B})$ be the sheaf of continuous maps $X \to \mathcal{B}$. The ring structure in $\mathcal{B}$ equips $C(X, \mathcal{B})$ with the structure of a sheaf of rings over $X$. We can therefore consider left (or right) $C(X, \mathcal{B})$-modules, and in particular the category of locally free $C(X, \mathcal{B})$-modules. Using the section functor in the usual way, for $X$ compact there is an equivalence of additive categories

$$\mathcal{E}(\mathcal{B}, X) \cong \text{LF}(C(X, \mathcal{B})) \ .$$

(30)

Since these are both additive categories, we can apply the Grothendieck functor to each of them and obtain the abelian groups $K(\text{LF}(C(X, \mathcal{B})))$ and $K(\mathcal{E}(\mathcal{B}, X))$. The equivalence of categories [30] ensures that there is a natural isomorphism of groups

$$K(\text{LF}(C(X, \mathcal{B}))) \cong K(\mathcal{E}(\mathcal{B}, X)) \ .$$

(31)

This motivates the following general definition. If $\mathcal{A}$ is a sheaf of rings over $X$, then we define the $K$-theory of $X$ with coefficients in $\mathcal{A}$ to be the abelian group

$$K(X, \mathcal{A}) := K(\text{LF}(\mathcal{A})) \ .$$

For example, consider the case $\mathcal{B} = \mathbb{C}$. Then $C(X, \mathbb{C})$ is just the sheaf of continuous functions $X \to \mathbb{C}$, while $\mathcal{E}(\mathbb{C}, X)$ is the category of complex vector bundles over $X$. Using the isomorphism of $K$-theory groups [31] we then have
K(X, C(X, C)) := K(LF(\text{C}(X, C))) \cong K(E(\text{C}, X)) = K^0(X).

The definition of twisted K-theory uses another special instance of this general construction. For this, we define an Azumaya algebra over X of rank m to be a locally trivial algebra bundle over X with fibre isomorphic to the algebra of \( m \times m \) complex matrices over \( \mathbb{C}, \mathbb{M}_m(\mathbb{C}) \). An example is the algebra \( \text{End}(E) \) of endomorphisms of a complex vector bundle \( E \to X \). We can define an equivalence relation on the set \( A(X) \) of Azumaya algebras over X in the following way. Two Azumaya algebras \( A, A' \) are called equivalent if there are vector bundles \( E, E' \) over X such that the algebras \( A \otimes \text{End}(E), A' \otimes \text{End}(E') \) are isomorphic. Then every Azumaya algebra of the form \( \text{End}(E) \) is equivalent to the algebra of functions \( C(X) \) on X. The set of all equivalence classes is a group under the tensor product of algebras, called the Brauer group \( \text{Br}(X) \), see ref. [57].

If \( A \) is an Azumaya algebra bundle, then the space of continuous sections \( C(X, A) \) of \( X \) is a ring and we can consider the algebraic K-theory group \( K(A) := K_0(C(X, A)) \) of equivalence classes of projective \( C(X, A) \)-modules, which depends only on the equivalence class of \( A \) in the Brauer group \([58]\).

Under the equivalence \([50]\), we can represent the Brauer group \( \text{Br}(X) \) as the set of isomorphism classes of sheaves of Azumaya algebras. Let \( A \) be a sheaf of Azumaya algebras, and \( \text{LF}(A) \) the category of locally free \( A \)-modules. Then as above there is an isomorphism

\[
K(X, C(X, A)) \cong K(\text{Proj}(C(X, A)))
\]

where \( \text{Proj}(C(X, A)) \) is the category of finitely-generated projective \( C(X, A) \)-modules. The group on the right-hand side is the group \( K(A) \). For given \( [H] \in \text{tor}(H^3(X, \mathbb{Z})) \) and \( A \in \text{Br}(X) \) such that \( \delta(A) = [H] \), this group can be identified as the twisted K-theory group \( K^0(X, H) \) of X with twisting \( A \). This definition is equivalent to the description in terms of bundle gerbe modules, and from this construction it follows that \( K^0(X, H) \) is a subgroup of the ordinary K-theory of \( X \). If \( \delta(A) = 0 \), then \( A \) is equivalent to \( C(X) \) and we have \( K(A) := K_0(C(X)) = K^0(X) \). The projective \( C(X, A) \)-modules over a rank \( m \) Azumaya algebra \( A \) are vector bundles \( E \to X \) with fibre \( \mathbb{C}^m \cong (\mathbb{C}^m)^{\oplus n} \), which is naturally an \( \mathbb{M}_m(\mathbb{C}) \)-module. This is a projective module and all projective \( C(X, A) \)-modules arise in this way \([57]\).

We will now describe the connection to twisted cohomology, following refs. [56] [59]. Upon choosing a bundle gerbe connection, one has an isomorphism of \( \mathbb{Z}_2 \)-graded cohomology groups

\[
\text{HP}_\bullet(CT^\infty(X, H)) \cong H^\bullet(X, H) = H^\bullet(\Omega^\bullet(X), d - H \wedge)
\]
where the right-hand side is the $H$-twisted cohomology of $X$. The Chern-Weil representative, in terms of differential forms on $X$, of the canonical Connes-Chern character

$$\text{ch} : K\bullet\left(CT^\infty(X, H)\right) \rightarrow HP\bullet\left(CT^\infty(X, H)\right)$$

then leads to the twisted Chern character

$$\text{ch}_H : K\bullet(X, H) \rightarrow H\bullet(X, H).$$

### 7.2 Isometric pairing formulas

The Clifford algebra bundle $\text{Cliff}(T^*X)$ is an Azumaya algebra over $X$ with Dixmier-Douady invariant $\delta(\text{Cliff}(T^*X)) = w_3(X)$, the third Stiefel-Whitney class of the tangent bundle of $X$ [60]. Consider the algebra

$$B_H := CT(X, w_3(X) - H) \cong C_0(X, \mathcal{E}_H \otimes \text{Cliff}(T^*X)).$$

Then $(A_H, B_H)$ is a PD pair with fundamental class $\Delta = \mathcal{D} \otimes \mathcal{D}$ [17] [44]. The restriction of the algebra $A_H \otimes B_H$ to the diagonal of $X \times X$ is isomorphic to the algebra $CT(X, w_3(X)) \otimes K$, which is Morita equivalent to the algebra of continuous sections $C_0(X, \text{Cliff}(T^*X))$.

Under the isomorphism $K^0(X, w_3(X)) \cong K_0(C_0(X, \text{Cliff}(T^*X)))$, the tensor product of projective bundles defines a bilinear pairing on twisted K-theory groups given by

$$K^\bullet(X, H) \otimes K^\bullet(X, w_3(X) - H) \rightarrow K^d(X, w_3(X)) \xrightarrow{\text{index}} \mathbb{Z}.$$

On the other hand, since the torsion class $w_3(X)$ is trivial in de Rham cohomology, there is an isomorphism $H^\bullet(X, w_3(X)) \cong H^\bullet(X, \mathbb{R})$ and hence the cup product defines a bilinear pairing on twisted cohomology groups via the mapping

$$H^\bullet(X, H) \otimes H^\bullet(X, w_3(X) - H) \rightarrow H^{\text{even}}(X, \mathbb{R}).$$

The fundamental cyclic cohomology class $\Xi$ of the PD pair $(A_H, B_H)$ may thus be identified with the orientation cycle $[X]$.

In this case the Todd class $\text{Todd}(A_H)$ may be identified with the Atiyah-Hirzebruch genus $\widehat{A}(X)$ of the tangent bundle $TX$, and the modified Chern character $\text{ch}_H$ is $\text{ch}_H \wedge \sqrt{\widehat{A}(X)}$. Note that when $X$ is a spin$^c$ manifold, then $w_3(X) = 0$ and the algebra $C_0(X, \text{Cliff}(T^*X))$ is Morita equivalent to $C(X)$ [60]. In this instance $B_H = CT(X, -H) = A_H^\perp$ is the opposite algebra of $A_H$, and the restriction of $A_H \otimes B_H$ to the diagonal of $X \times X$ is stably isomorphic to the algebra of functions $C(X)$. 
7.3 Twisted K-cycles and Ramond-Ramond charges

If spacetime $X$ is a spin manifold, then any D-brane $(W, E, f)$ in $X$ determines canonical element \[ f! \in \text{KK}_{d}(CT(W, f^*[H] + w_3(\nu_W)), CT(X, H)) \, . \]

Since $w_3(\nu_W) = w_3(W)$ in this case \([2, 5]\), we may identify the D-brane algebra $D = CT(W, f^*[H] + w_3(W))$ and the corresponding Chan-Paton bundle is an element $E \in K^0(W, f^*[H]+w_3(W))$. There are two particularly interesting special classes of such twisted D-branes.

The first class is determined by the usual requirement that the worldvolume $W$ be a spin$^c$ manifold, as in the ordinary Baum-Douglas construction. This instance was first considered in ref. \([62]\). Then $w_3(W) = 0$, the algebra $D$ is the restriction of $A_H$ to $W$, and $E \in K^0(W, f^*[H])$. The geometric $K$-homology equivalence relations are then completely analogous to those of the untwisted case in Section 2.1 \([62]\). When the $H$-flux defines a non-torsion element in $H^3(X, \mathbb{Z})$, the Chan-Paton bundle $E$ is a projective bundle of infinite rank, corresponding to an infinite number of wrapped branes on $W$. When $H$ defines an $n$-torsion element, then the $B$-field $B_H$ incorporates the contribution from the $\mathbb{Z}_n$-valued ’t Hooft flux necessary for anomaly cancellation on the finite system of $n$ spacetime-filling branes and antibranes in the $H$-flux background \([57]\).

The second class is in some sense opposite to the first one, and it is more physical in that it is tied to the Freed-Witten anomaly cancellation condition \([5]\)
\[ f^*[H] + w_3(W) = 0 \, . \] (32)

In this case $E \in K^0(W)$ and the D-brane algebra $D$ is (stably) commutative. The mathematical meaning of this limit is that it makes the worldvolume $W$ into a “twisted spin$^c$” manifold, which may be defined precisely as follows. By Kuiper’s theorem, the unitary group $U(\mathcal{H})$ of an infinite-dimensional Hilbert space $\mathcal{H}$ is contractible (both in the norm and compact-open topologies). Thus the projective unitary group $PU(\mathcal{H})$ has the homotopy type of an Eilenberg-Maclane space $K(\mathbb{Z}, 2)$, and its classifying space $BPU(\mathcal{H})$ is an Eilenberg-Maclane space $K(\mathbb{Z}, 3)$.

It follows that any element of $H^3(X, \mathbb{Z})$ corresponds to a map $F : X \to BPU(\mathcal{H})$, and hence to the projective bundle which is the pullback by $F$ of the universal bundle over $BPU(\mathcal{H})$.

It follows that $K(\mathbb{Z}, 3)$ is a classifying space for the third cohomology, \[ H^3(X, \mathbb{Z}) \cong [X, K(\mathbb{Z}, 3)] \, , \]

and so we can represent an $H$-flux by a continuous map $H : X \to K(\mathbb{Z}, 3)$. Taking a universal $K(\mathbb{Z}, 2)$-bundle over $K(\mathbb{Z}, 3)$ and pulling it back through $H$ to $X$, we get a $K(\mathbb{Z}, 2)$-bundle $P_H$ over $X$. Consider the $K(\mathbb{Z}, 2)$-bundle
\[ \text{BU}(1) \rightarrow B\text{Spin}^c \rightarrow BSO \]

with classifying map \( \beta \circ w_2 : BSO \rightarrow B\text{BU}(1) = K(\mathbb{Z}, 3) \), the Bockstein homomorphism of the second Stiefel-Whitney class. The action of \( K(\mathbb{Z}, 2) \) on \( B\text{Spin}^c \) induces a principal \( B\text{Spin}^c \)-bundle \( Q = P_H \times_{K(\mathbb{Z}, 2)} B\text{Spin}^c \), i.e., a sequence of bundles \( Q_n = P_H \times_{K(\mathbb{Z}, 2)} B\text{Spin}^c(n) \), with corresponding universal bundles \( UQ_n = (P_H \times_{K(\mathbb{Z}, 2)} E\text{Spin}^c(n)) \times_{\text{spin}^c(n)} \mathbb{R}^n \). The homotopy groups of the associated Thom spectrum

\[ \text{Thom}(UQ) = P_+ \bigwedge_{K(\mathbb{Z}, 2)_+} MS\text{Spin}^c \]

are the \( H \)-twisted spin\(^c \) bordism groups of \( X \). Using this one can deduce that a compact manifold \( W \) is \( H \)-twisted \( K \)-oriented if it is an oriented manifold with a continuous map \( f : W \rightarrow X \) such that the Freed-Witten condition \([42]\) holds. We say that a pair \((W, f)\), with \( W \) a compact oriented manifold and \( f : W \rightarrow X \) a continuous map, is \( H \)-twisted spin\(^c \) if it satisfies this cancellation.

A choice of \( H \)-twisted spin\(^c \) structure is a choice of a two-cochain \( c \) such that, at the cochain level, \( \delta(c) = \beta \circ w_2(W) - f^*[H] \). This follows from the following geometric fact. Let \( \alpha : P \rightarrow P \) be an automorphism of a projective bundle \( P \rightarrow X \) with infinite-dimensional separable fibres. It induces a line bundle \( L_\alpha \rightarrow X \). For \( x \in X \), the non-zero elements of \( (L_\alpha)_x \) are the linear isomorphisms \( E_x \rightarrow E_x \) which induce \( \alpha|_{P_x} \), where \( P_x = \mathbb{P}(E_x) \). Then the assignment \( \text{Aut}(P) \rightarrow H^2(X, \mathbb{Z}), \alpha \mapsto [L_\alpha] \) identifies the group of connected components \( \pi_0(\text{Aut}(P)) \) with the group \( H^2(X, \mathbb{Z}) \) of isomorphism classes of line bundles over \( X \). This follows from the identification of the automorphism group \( \text{Aut}(P) \) of the bundle \( P \) with the space of sections of the endomorphism bundle \( \text{End}(P) \), i.e., the space of maps \( X \rightarrow PU(H) \), which is an Eilenberg-Maclane space \( K(\mathbb{Z}, 2) \). For a more extensive treatment of these issues, see refs. \([63, 64]\).

This leads us to the following notion. Let \((W, f)\) be a manifold (not necessarily \( H \)-twisted spin\(^c \)). A vector bundle \( V \rightarrow W \) is said to be an \( H \)-twisted spin\(^c \) vector bundle if \( f^*[H] = w_3(V) \). The choice of a specific \( H \)-twisted spin\(^c \) structure on \( V \) is made as above by choosing an appropriate two-cochain. The notion of an \( H \)-twisted spin\(^c \) manifold is just the special case \( V = TW \) of this latter one. The analogs of the Baum-Douglas gauge equivalence relations for geometric twisted K-homology may be straightforwardly written down in the obvious way using projective Hilbert bundles instead of vector bundles. In the construction of the unit sphere bundle \([41]\), we assume that \( w_3(\hat{W}) = \pi^*(f^*[H]) \). Then \((\hat{W}, f \circ \pi)\) is an \( H \)-twisted spin\(^c \) manifold. The rest of the construction proceeds by using the untwisted Thom class \( H(F) \in K^i(\hat{W}) \). See ref. \([64]\) for the relation to a description involving bundle gerbe modules.
There are more general twistings one may consider which are still physically meaningful. Suppose that $[H] \in \mathbb{Z}_n \subset H^3(X, \mathbb{Z})$ and fix an element $y \in H^3(X, \mathbb{Z}_n)$. Then we may consider bordism of manifolds $(W, f)$, where the worldvolume $W$ is a compact oriented manifold and $f : W \to X$ is a continuous map satisfying
\[ f^*[H] = w_3(W) + f^*(\beta(y)) , \quad (33) \]
with $\beta$ the Bockstein homomorphism. The condition (33) is the most general form of the Freed-Witten anomaly cancellation condition for a system of $n$ spacetime-filling brane-antibrane pairs [57]. With this more general kind of twisting, one can also consider bordism of manifolds $(W, f)$, where $W$ is a compact spin$^c$ manifold as before and $f : W \to X$ is a continuous map satisfying $f^*[H] = f^*(\beta(y))$. The equivalences between the various forms of the geometric twisted K-homology group $K_*^w(X, H)$ follows from the equivalences among the corresponding twisted K-theories. In any of these cases, one arrives at the twisted D-brane charge vector
\[ Q_H(W, E, f) = \text{ch}_H(f_!(E)) \wedge \sqrt{\hat{A}(X)} \in H^*(X, H) . \]
Only when $[H]$ is a torsion class does the Ramond-Ramond charge correspond to an element of the ordinary (untwisted) cohomology of the spacetime manifold $X$.

8 Correspondences and T-duality

In this final section we shall apply our formalism to a new description of topological open string T-duality [16, 17]. The description is based on the formulation of KK-theory in terms of correspondences [45, 65, 66]. Amongst other things, this leads to an explicit construction of the various structures inherent in Kasparov’s bivariant K-theory, and moreover admits a natural noncommutative generalization [17].

8.1 Correspondences

Let $X, Y$ be smooth manifolds, and set $KK_d(X, Y) := KK_d(C_0(X), C_0(Y))$. Elements of the group $KK_d(X, Y)$ can be represented by correspondences

\[ (Z, E) \]

\[ \begin{array}{c}
X \\
\downarrow \scriptstyle f
\end{array} \quad \begin{array}{c}
\downarrow \scriptstyle g \\
Y
\end{array} \]

where $Z$ is a smooth manifold, $E$ is a complex vector bundle over $Z$, the map $f : Z \to X$ is smooth and proper, $g : Z \to Y$ is a smooth K-oriented map, and $d = \dim(Z) - \dim(Y)$. This diagram defines a morphism
\[ g(f^*(-) \otimes E) \in \text{Hom}(K^*(X), K^{*+d}(Y)) \]

implemented by the KK-theory class \([f] \otimes_{C_0(Z)} [[E]] \otimes_{C_0(Z)} (g!), \) where \([[E]]\) is the KK-theory class in \(KK_0(Z, Z) \cong \text{End}(K^*(Z))\) of the vector bundle \(E\) defined by tensor product with the K-theory class \([E]\) of \(E\) (this ignores the extension term in the universal coefficient theorem (10)). The collection of all correspondences forms an additive category under disjoint union. The group \(KK_d(X, Y)\) is then obtained as the quotient space of the set of correspondences by the equivalence relation generated by suitable notions of cobordism, direct sum and vector bundle modification, analogous to those of Section 2.1 [17].

The correspondence picture of KK-theory gives a somewhat more precise realization of the notion, introduced categorically in Section 3.3, of Kasparov bimodules as “generalized” morphisms of \(C^*\)-algebras. It provides a geometric presentation of the analytic index for families of elliptic operators on \(X\) parametrized by \(Y\). The limiting case \(KK_d(X, pt) = K_d(X)\) is the geometric K-homology of \(X\) as described in Section 2, since in this case a correspondence is simply a Baum-Douglas K-cycle \((Z, E, f)\) over \(X\). On the other hand, the group \(KK_d(pt, Y) = K^d(Y)\) is the K-theory of \(Y\), obtained via an ABS-type construction of the charge of the D-brane \((Z, E, g)\) in \(Y\) using the spin\(^c\) structure on the bundle \(TZ \oplus g^*(TY)\).

One of the great virtues of this formalism is that it gives an explicit description of the intersection product in KK-theory, which as mentioned in Section 3.3 is notoriously difficult to define. In the notation above it is a map

\[ \otimes_M : KK(X, M) \times KK(M, Y) \longrightarrow KK(X, Y) \]

which sends two correspondences

\[ f \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
8.2 T-duality and KK-equivalence

The correspondence picture is reminiscent of the Fourier-Mukai transform, which is related to T-duality on spacetimes compactified on tori $X = M \times \mathbb{T}^n$ in the absence of a background $H$-flux. In this case the T-dual is topologically the same space $M \times \mathbb{T}^n$, and the mechanism implementing the T-duality is given by the smooth analog of the Fourier-Mukai transform [67]. Let $\mathbb{T}^n$ be an $n$-torus, and let $\mathbb{T}^n \cong \text{Pic}^0(\mathbb{T}^n)$ be the corresponding dual $n$-torus. Recall that the Poincaré line bundle $\mathcal{P}_0 \to \mathbb{T}^n \times \hat{\mathbb{T}}^n$ is the unique line bundle such that $\mathcal{P}_0|_{\mathbb{T}^n \times \{\hat{t}\}} \in \text{Pic}^0(\mathbb{T}^n)$ is the flat line bundle corresponding to $\hat{t} \in \hat{\mathbb{T}}^n$ and whose restriction $\mathcal{P}_0|_{\{0\} \times \hat{\mathbb{T}}^n}$ is trivial.

This data defines a diagram

$$
\begin{array}{ccc}
M \times \mathbb{T}^n & \xrightarrow{p_1} & M \times \hat{\mathbb{T}}^n \\
\downarrow & & \downarrow \\
M \times \hat{\mathbb{T}}^n & \xrightarrow{p_2} & M \times \mathbb{T}^n
\end{array}
$$

where $p_1, p_2$ are canonical projections and $\mathcal{P}$ is the pullback of the Poincaré line bundle to $M \times \mathbb{T}^n \times \hat{\mathbb{T}}^n$. The smooth analog of the Fourier-Mukai transform is the isomorphism of $K$-theory groups

$$T_1 : K^\bullet(M \times \mathbb{T}^n) \xrightarrow{\cong} K^\bullet(M \times \hat{\mathbb{T}}^n)$$

given by

$$T_1(-) = (p_2)_!(p_1^*(-) \otimes \mathcal{P}) .$$

We conclude that topological open string T-duality is a correspondence. In this case, the correspondence represents an invertible element of KK-theory, i.e., a KK-equivalence.

The Fourier-Mukai transform can be rephrased in a satisfactory manner, entirely in terms of noncommutative geometry, as a crossed product algebra $C_0(M \times \mathbb{T}^n) \rtimes \mathbb{R}^n$, where the action of the group $\mathbb{R}^n$ on $C_0(M \times \mathbb{T}^n)$ is just the given action of $\mathbb{R}^n$ on $\mathbb{T}^n$ by translations and the trivial action on $M$. By Rieffel’s version of the Mackey imprimitivity theorem [68], one sees that the crossed product $C^*$-algebra $C_0(M \times \mathbb{T}^n) \rtimes \mathbb{R}^n$ is Morita equivalent to

$$C_0(M) \otimes C^*(\mathbb{R}^n) \cong C_0(M \times \hat{\mathbb{T}}^n) .$$

Thus the T-dual of the $C^*$-algebra $C_0(M \times \mathbb{T}^n)$ is obtained by taking the crossed product of the algebra with $\mathbb{R}^n$. The Connes-Thom isomorphism then defines a KK-equivalence

$$\alpha \in KK_n(M \times \mathbb{T}^n, M \times \hat{\mathbb{T}}^n)$$

which is just the families Dirac operator. Moreover, Takai duality gives a Morita equivalence
\[ (C_0(M \times \mathbb{T}^n) \times \mathbb{R}^n) \times \mathbb{R}^n \sim C_0(M \times \mathbb{T}^n), \]

showing that the T-duality transformation is topologically of order 2.

The reason for making this reformulation in terms of noncommutative geometry is that it extends to the case when spacetime \( X \) is a principal torus bundle \( \pi : E \rightarrow M \) of rank \( n \) in the presence of a background \( H \)-flux. In this instance the T-dual is a crossed product algebra \( CT(E, H) \rtimes \mathbb{R}^n \), which is generally a bundle of rank \( n \) noncommutative tori fibred over \( M \) \[69\]. This requires that \( H \) restrict to zero in the cohomology of the torus fibers and that the action of \( \mathbb{R}^n \) on the continuous trace \( C^* \)-algebra \( CT(X, H) \) is a lift of the given action of \( \mathbb{R}^n \) on \( X \). That such a lift exists is a non-trivial result proven in ref. \[69\]. This crossed product algebra is a noncommutative \( C^* \)-algebra, but it need not be a continuous trace algebra. In ref. \[23\] it was shown, by checking the open string metric, that in some cases these algebras are globally defined, open string versions of T-folds. The correspondence picture in this context appears to nicely describe the doubled torus formalism for T-folds, as we will see below. When \( \pi_*[H] = 0 \), the T-dual algebra is isomorphic to a continuous trace \( C^* \)-algebra \( CT(\hat{E}, \hat{H}) \) and represents a geometrically dual spacetime in the usual sense.

### 8.3 Noncommutative correspondences

The discussion at the end of Section 8.2 above motivates the following noncommutative generalization of the correspondence picture of Section 8.1 above \[17\]. Let \( A, B \) be separable \( C^* \)-algebras. We will represent elements of \( KK(A, B) \) by noncommutative correspondences

\[ \begin{array}{ccc}
A & \xrightarrow{f} & (C, \xi) \\
& \searrow & \downarrow g \\
& B & \nearrow \\
\end{array} \]

where \( C \) is a separable \( C^* \)-algebra and \( \xi \in KK(C, C) \), whereas \( f : A \rightarrow C \) and \( g : B \rightarrow C \) are homomorphisms with \( g \) K-oriented. The intersection product gives an element \([f] \otimes_C \xi \otimes_C (g!) \in KK(A, B)\), with associated K-theory morphism \( g^!(f_*(-) \otimes_C \xi) \in \text{Hom}(K_*(A), K_*(B))\). Every class in \( KK_d(A, B) \) comes from a noncommutative correspondence, in fact from one with trivial \( \xi = 1_C \). The representation of the intersection product in this instance uses amalgamated products of \( C^* \)-algebras \[17\].

Let us consider the class of examples mentioned earlier, focusing for simplicity on the simplest case where spacetime \( X \) is a principal circle bundle \( \pi : E \rightarrow M \) in a background \( H \)-flux. The T-dual is another principal circle bundle \( \hat{\pi} : \hat{E} \rightarrow \hat{M} \) with characteristic class \( c_1(\hat{E}) = \pi_*[H] \). The Gysin sequence for \( E \) defines the T-dual \( \hat{H} \) as \( \mathbb{H}^3(\hat{E}, \mathbb{Z}) \) with
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c_1(E) = \tilde{\pi}_*[\tilde{H}] and [H] = [\tilde{H}] in H^3(E \times_M \tilde{E}, \mathbb{Z}). This data defines a noncommutative correspondence

\[
(CT(E \times_M \tilde{E}, H), \xi) \rightarrow f \rightarrow g \rightarrow \left(CT(\tilde{E}, \tilde{H}) \right)
\]

where \(\xi\) is an analogue of the Poincaré line bundle. It determines a KK-equivalence \(\alpha \in KK_1(CT(E, H), CT(\tilde{E}, \tilde{H}))\). See ref. [17] for further examples of noncommutative correspondences.

8.4 Axiomatic T-duality and D-brane charge

Inspired by the above results, we now give an axiomatic definition of T-duality in K-theory that any definition of the T-dual \(T(A)\) of a C*-algebra \(A\) should satisfy. These axioms include the requirements that the Ramond-Ramond charges of \(A\) should be in bijective correspondence with the Ramond-Ramond charges of \(T(A)\), and that T-duality applied twice yields a C*-algebra which is physically equivalent to the C*-algebra that we started out with. For this, we postulate the existence of a suitable category of separable C*-algebras, possibly with extra structure (for example the \(\mathbb{R}^n\)-actions used above). Its objects \(A\) are called T-dualizable algebras and satisfy the following requirements:

1. The map \(A \mapsto T(A)\) from \(A\) to the T-dual of \(A\) is a covariant functor;
2. There is a functorial map \(A \mapsto \alpha_A\), where the invertible element \(\alpha_A\) defines a KK-equivalence in KK\((A, T(A))\); and
3. The algebras \(A, T(T(A))\) are Morita equivalent, with associated KK-equivalence given by the invertible element \(\alpha_A \circ_{T(A)} \alpha_{T(A)}\).

Let us consider a class of examples generalizing those already presented in this section. Let \(A\) be a \(G\)-C*-algebra, where \(G\) is a locally compact, abelian vector Lie group (basically \(\mathbb{R}^n\)). Then the algebra \(T(A) = A \rtimes G\) satisfies the axioms above [16], thanks to the Connes-Thom isomorphism and Takai duality (here we tacitly identify \(G\) with its Pontrjagin dual \(\hat{G}\)).

The assumption made above that the T-dual \(T(A)\) is a C*-algebra is very strong and it is not always satisfied, as seen in ref. [70]. Yet even in that case, the axioms above are satisfied, provided one also allows more general algebras belonging to a category studied there. There is also an analogous axiomatic definition of T-duality in local cyclic cohomology [16], relevant to the duality transformations of Ramond-Ramond fields.

A crucial point about the formulation in terms of bivariant K-theory is that it provides a refinement of the usual notion of T-duality. For instance, for a suitable class of algebras the universal coefficient theorem [10]...
expresses the KK-theory group $\text{KK}^*(\mathcal{A},\mathcal{B})$ as an extension of the group $\text{Hom}_\mathbb{Z}(\text{K}^*(\mathcal{A}),\text{K}^*(\mathcal{B}))$ by $\text{Ext}_\mathbb{Z}(\text{K}^*_+(\mathcal{A}),\text{K}^*(\mathcal{B}))$. The extension group can lead to important torsion effects not present in the usual formulations of T-duality.

We close by studying the invariance of the noncommutative D-brane charge vector (27) under T-duality. As is well known [30], the T-duality invariance of Ramond-Ramond couplings on D-branes is a subtle issue which requires further conditions to be imposed on the structures involved. The present formalism yields a systematic and general way to establish these criteria.

If the D-brane algebra $\mathcal{D}$ is a PD algebra, then by the Grothendieck-Riemann-Roch formula (22) one has

$$Q(\mathcal{D},\xi,f) = \text{ch}(\xi) \otimes_{\mathcal{D}} \text{Todd}(\mathcal{D}) \otimes_{\mathcal{D}} (f^*) \otimes_{\mathcal{A}} \sqrt{\text{Todd}(\mathcal{A})^{-1}}.$$  

Suppose that there is a local cyclic cohomology class $\Lambda \in \text{HL}(\mathcal{D},\mathcal{D})$ such that

$$(f^*) \otimes_{\mathcal{A}} \sqrt{\text{Todd}(\mathcal{A})^{-1}} = \Lambda \otimes_{\mathcal{D}} (f^*).$$

Then there is a noncommutative version of the Wess-Zumino class (8) in $\text{HL}^*(\mathcal{D})$ given by

$$D_{\text{WZ}}(\mathcal{D},\xi,f) = \text{ch}(\xi) \otimes_{\mathcal{D}} \text{Todd}(\mathcal{D}) \otimes_{\mathcal{D}} \Lambda.$$  

Consider a pair of D-branes $(\mathcal{D},\xi,f)$ and $(\mathcal{D}',\xi',f')$ which are KK-equivalent, with the equivalence determined by an invertible element $\alpha$ in $\text{KK}(\mathcal{D},\mathcal{D}')$ and $\xi' = \xi \otimes_{\mathcal{D}} \alpha$. If

$$\Lambda' = \text{ch}(\alpha)^{-1} \otimes_{\mathcal{D}} \Lambda \otimes_{\mathcal{D}} \text{ch}(\alpha)$$

then by eq. (21) one has $D_{\text{WZ}}(\mathcal{D}',\xi',f') = D_{\text{WZ}}(\mathcal{D},\xi,f) \otimes_{\mathcal{D}} \text{ch}(\alpha)$. It follows that

$$D_{\text{WZ}}(\mathcal{D}'',\xi'',f'') = D_{\text{WZ}}(\mathcal{D},\xi,f) \otimes_{\mathcal{D}} \text{ch}(\alpha \otimes_{\mathcal{D'}} \alpha')$$

in $\text{HL}^*(\mathcal{D}'') \cong \text{HL}^*(\mathcal{D})$. This formula expresses the desired T-duality covariance under the conditions spelled out above.

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