ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR COMPETITIVE MODELS WITH FREE BOUNDARIES†

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Abstract. In this paper, we study a competitive model involving two species. When the competition is strong enough, the two species are separated by a free boundary. If the initial data has a positive bound at infinity. We prove that the solution will converge, as \( t \to \infty \), to the traveling wave solution and the free boundary will move to infinity with a constant speed.

1. Introduction

In this paper, we study the asymptotic behavior of solutions for the following competitive model

\[
\begin{aligned}
P_t &= d_1 P_{xx} + f(P), & x < s(t), & t > 0, \\
Q_t &= d_2 Q_{xx} + g(Q), & x > s(t), & t > 0, \\
P(x, t) &= Q(x, t) = 0, & x = s(t), & t > 0, \\
\dot{s}(t) &= -\mu_1 P_x(x, t) - \mu_2 Q_x(x, t), & x = s(t), & t > 0, \\
\end{aligned}
\]

(1.1)

on unbounded domain, where \( x = s(t) \) is the free boundary to be determined together with \( P \) and \( Q \), \( f, g \in C^1([0, \infty)) \) satisfying \( f(0) = g(0) = 0 \).

We mainly consider monostable and bistable type of nonlinearities. More precisely, we call \( f \) a monostable type of nonlinearity (\( f \) is of \( (f_M) \) type, for short), if \( f \in C^1([0, \infty)) \) and

\[
f(0) = 0 < f'(0), \quad f(1) = 0 > f'(1), \quad (1 - s)f(s) > 0 \text{ for } s > 0, s \neq 1;
\]

we say that \( f \) is a bistable type of nonlinearity (\( f \) is of \( (f_B) \) type, for short), if

\[
\begin{aligned}
f &\in C^1([0, \infty)), \quad f(0) = 0 > f'(0), \quad f(1) = 0 > f'(1), \quad \int_0^1 f(s)ds > 0, \\
f(\cdot) < 0 \text{ in } (0, \theta) \cup (1, \infty), \quad f(\cdot) > 0 \text{ in } (\theta, 1) \text{ for some } \theta \in (0, 1), \\
\text{for } F(u) := -2\int_0^u f(s)ds, \quad F(\bar{\theta}) = 0 \text{ for some } \bar{\theta} \in (\theta, 1).
\end{aligned}
\]

In population ecology, the appearance of regional partition of multi-species through strong competition is one interesting phenomena. In [15][16][17], Mimura, Yamada and Yotsutani used...
the following reaction-diffusion equations

\[
\begin{align*}
P_t &= d_1 P_{xx} + f(P), & 0 < x < s(t), & t > 0, \\
Q_t &= d_2 Q_{xx} + g(Q), & s(t) < x < 1, & t > 0, \\
P(x, t) &= Q(x, t) = 0, & x = s(t), & t > 0, \\
s'(t) &= -\mu_1 P_x(x, t) - \mu_2 Q_x(x, t), & x = s(t), & t > 0, \\
P(0, t) &= m_1, & Q(1, t) &= m_2, & t > 0, \\
s(0) &= s_0(0 < s_0 < 1), \\
P(x, 0) &= P_0(x)(0 < x < s_0), & Q(x, 0) &= Q_0(x)(s_0 < x < 1)
\end{align*}
\] (1.2)

to describe regional partition of two species, which are struggling on a boundary to obtain their own habitats. When \( m_1, m_2 > 0 \) and \( f, g \) are monostable nonlinearities. They prove the global existence, uniqueness, regularity and asymptotic behavior of solutions for problem (1.2) in [15]. When \( m_1, m_2 > 0 \) and \( f, g \) are bistable nonlinearities. The author establish the stability for stationary solutions for the free boundary problem and their argument is based on the notion of \( \omega \)-limit set and the comparison principle in [16]. When \( m_1, m_2 = 0 \), i.e. homogeneous Dirichlet boundary conditions are imposed, there is possibility that the free boundary may hit the fixed ends \( x = 0, 1 \) in a finite time. This is a very interesting phenomenon. Therefore, in [17], the author prove that if the free boundary hits the fixed boundary at a finite time \( t = T^* \), the free boundary stays there after \( T^* \).

Problem (1.2) is defined on finite interval \([0, 1]\). A natural question is what will happen if the two species competitive model is defined on \((-\infty, \infty)\)? If the two species competitive model is defined on the entire space, the free boundary \( s(t) \) could move to infinity in different ways and the problem may become more complicated. This is why we are interested in studying problem (1.1). Possibly, the solution will develop into the traveling wave eventually if problem (1.2) is defined on unbounded domain. So, it is necessary to consider the traveling wave solution before we investigate the asymptotic behavior of solutions for problem (1.1). To study the traveling wave solution of problem (1.1) is equivalent to study the solution of the following ordinary differential equations

\[
\begin{align*}
d_1 \phi'' + c \phi' + f(\phi) &= 0, & x \in (-\infty, 0], \\
d_2 \psi'' + c \psi' + g(\psi) &= 0, & x \in [0, \infty), \\
\phi(0) &= \psi(0) = 0, \\
\phi(-\infty) &= \psi(+\infty) = 1, \\
c &= -\mu_1 \phi'(0) - \mu_2 \psi'(0).
\end{align*}
\] (1.3)

Recently, in [3], Chang and Chen prove the existence of a traveling wave solution of (1.3) for logistic type nonlinearities. In [19], we extend the results in [3] to more general nonlinearities. We prove that if \( \alpha > 0 \) is a given constant, then for any \( c \in (c_g^\ast, \hat{c}_f) \), where \( c_g^\ast < 0 \) is the maximal speed when \( g \) is of \((f_M)\) type, or the unique speed when \( g \) is of \((f_B)\) type and \( \hat{c}_f > 0 \) depends only on \( \alpha \) and \( f \), there exists a unique \( \beta(c) > 0 \) such that (1.3) has a unique solution \((\phi, \psi, c)\). Moreover, \( \beta(c) \) is continuous and strictly decreasing in \( c \in (c_g^\ast, \hat{c}_f) \) and

\[
c \rightarrow \hat{c}_f \iff \beta \rightarrow 0, \quad c \rightarrow c_g^\ast \iff \beta \rightarrow \infty, \quad c > 0 \iff \beta < \tilde{\beta};
\]

where \( \tilde{\beta} := \alpha \left( \int_0^1 f(s)ds / \int_0^1 g(s)ds \right)^{1/2} \). If \( \beta > 0 \) is a given constant, we can obtain similar results in the same method.

The main purpose of this paper is to prove that for any initial data \( P_0 \) and \( Q_0 \) in problem (1.1), as long as they have positive lower bound at infinity, the solution of problem (1.1) will converge to a traveling wave as \( t \rightarrow \infty \). Our main result is as follows:
**Theorem 1.1.** Assume \( f, g \) are of \((f_M)\) type and the initial data \( P_0, Q_0 \) satisfy
\[
0 < \liminf_{x \to -\infty} P_0(x) < \limsup_{x \to -\infty} P_0(x) < \infty, \quad 0 < \liminf_{x \to \infty} Q_0(x) < \limsup_{x \to \infty} Q_0(x) < \infty,
\]
then for some constant \( x^* \), the solution of problem \((1.1)\) satisfies
\[
\lim_{t \to \infty} \sup_{(-\infty,s(t))] \vert P(x,t) - \phi(x-ct-x^*) \vert = 0, \quad \lim_{t \to \infty} \sup_{[s(t),\infty)} \vert Q(x,t) - \psi(x-ct-x^*) \vert = 0,
\]
where \((\phi, \psi, c)\) is defined in \((1.3)\). Moreover,
\[
\lim_{t \to \infty} (s(t) - ct - x^*) = 0, \quad \lim_{t \to \infty} s'(t) = c.
\]

**Remark 1.2.** Similar results in Theorem \((1.1)\) also holds when \( f \) and/or \( g \) are of \((f_B)\) type; namely, \((1.5)\) and \((1.6)\) also hold if the initial data \( P, Q \) satisfy one of the following conditions:
\begin{enumerate}[(i)]
\item \( 0 < \liminf_{x \to -\infty} P_0(x) < \limsup_{x \to -\infty} P_0(x) < \infty, \theta < \liminf_{x \to \infty} Q_0(x) < \limsup_{x \to \infty} Q_0(x) < \infty, \)
when \( f \) is of \((f_M)\) type and \( g \) is of \((f_B)\) type;
\item \( \theta < \liminf_{x \to -\infty} P_0(x) < \limsup_{x \to -\infty} P_0(x) < \infty, 0 < \liminf_{x \to \infty} Q_0(x) < \limsup_{x \to \infty} Q_0(x) < \infty, \)
when \( f \) is of \((f_B)\) type and \( g \) is of \((f_M)\) type;
\item \( \theta < \liminf_{x \to -\infty} P_0(x) < \limsup_{x \to -\infty} P_0(x) < \infty, \theta < \liminf_{x \to \infty} Q_0(x) < \limsup_{x \to \infty} Q_0(x) < \infty, \)
when \( f \) is of \((f_B)\) type and \( g \) is of \((f_B)\) type.
\end{enumerate}

**Remark 1.3.** If the diffusion constants \( d_1 = d_2 = 1 \), we can construct Lyapunov functional as in \((11)\) to prove the asymptotic behavior of solutions for problem \((1.1)\). However, through out our paper, we assume that the diffusion constants \( d_1 \neq d_2 \). Thus, the method used in \((11)\) does not work in this case. It is essentially different from the case \( d_1 = d_2 = 1 \). Therefore, in our paper, we use a different way to prove the asymptotic behavior of solutions.

The contents of this paper will be organized as follows: In section \(2\) we give some basic results on global existence of smooth solution for problem \((1.1)\) and comparison principle. In section \(3\) we prove Theorem \((1.1)\).

2. Preliminary results

2.1. The global existence of solutions.

**Theorem 2.1.** If the initial data \( P_0, Q_0 \) satisfy
\[
P_0 \in C^2((-\infty, s_0]), \quad Q_0 \in C^2([s_0, \infty)),
\]
then problem \((1.1)\) has a unique solution
\[
(P, Q, s) \in C^{1+\alpha, (1+\alpha)/2}(\bar{\Omega}_1) \times C^{1+\alpha, (1+\alpha)/2}(\bar{\Omega}_2) \times C^{1+\alpha/2}(0, \infty).
\]

Moreover,
\[
\|P\|_{C^{1+\alpha, (1+\alpha)/2}(\bar{\Omega}_1)} + \|Q\|_{C^{1+\alpha, (1+\alpha)/2}(\bar{\Omega}_2)} + \|s\|_{C^{1+\alpha/2}(0, \infty)} \leq C
\]
where \( \bar{\Omega}_1 = \{(x,t) : x \in (-\infty, s(t)], t > 0\}, \bar{\Omega}_2 = \{(x,t) : x \in [s(t), \infty), t > 0\}, 0 < \alpha < 1, C \)
depends only on \( \|P_0\|_{C^{1+\alpha}((-\infty, s_0])} \) and \( \|Q_0\|_{C^{1+\alpha}([s_0, \infty))} \).

The proof of Theorem \(2.1\) will be given in the Appendix.

**Lemma 2.2.** Assume \((P, Q, s)\) be the solution of the free boundary problem \((1.1)\) defined for \( t \in (0, T) \) for some \( T \in (0, \infty] \). Then there exists a positive constant \( H \) independent of \( T \) such that
\[
|s'(t)| \leq H \text{ for } 0 < t < T.
\]
Proof. For any fix $a \in (-\infty, \infty)$, with no loss of generality, we may assume $a \in [-l, l](l > 0)$, then we consider the following problem

$$
\begin{aligned}
&d_1 U'' + f(U) = 0, & y < a, t > 0, \\
d_2 V'' + g(V) = 0, & y > a, t > 0, \\
\mu_1 U'(a) + \mu_2 V'(a) = 0, & t > 0, \\
U(a) = V(a) = 0, & \\
U(-l) = 2L_U, V(l) = 2L_V.
\end{aligned}
$$

(2.2)

where $L_U = \sup \{\|P(\cdot, t)\|_{L^\infty} : t > 0\}$ $L_V = \sup \{\|Q(\cdot, t)\|_{L^\infty} : t > 0\}$. We know $(U, V)$ is the solution of (2.2), i.e., $(U, V)$ is a stationary solution of (1.1). Choose $(U, V)$ satisfying $|U'(a)| > w_1, V'(a) > w_2$, where $w_1 = \sqrt{\frac{2}{\mu_1}} \int_0^1 f(s) ds, w_2 = \sqrt{\frac{2}{\mu_2}} \int_0^1 g(s) ds$, then $V'(a) = -\frac{\mu_2}{\mu_1} U'(a) > \frac{\mu_1}{\mu_2} w_1$, i.e., $V'(a) > \max \{w_2, \frac{\mu_1}{\mu_2} w_1\}$. We select $(U, V)$ satisfying $\min V'(a) \geq \max \{\max_{x \in [0, \infty)} Q_0(x), w_2, \frac{\mu_1}{\mu_2} w_1\}, \min |U'(a)| \geq \max \{\max_{x \in (-\infty, 0]} P_0(x), w_1\}$. Denote $|U'(a)| =: a_0 \in [\min |U'(a)|, \max |U'(a)|]$, correspondingly, $V'(a) =: \beta_0 \in [\min V'(a), \max V'(a)]$.

Case 1: $s(t)$ first moves across the point $(a, 0)$ at $x$-axis from the right to the left, $Q(x, t)$ must cross with $V(x)$ at $(a, 0)$. Therefore, $Q_x(s(t), t) \leq V'(a) = \beta_0$, for $s'(t) \leq 0$, we have $-P_x(s(t), t) \leq -U'(a) = a_0$, so $|s'(t)| \leq \mu_1 |P_x(s(t), t)| + \mu_2 |Q_x(s(t), t)| \leq \mu_1 a_0 + \mu_2 \beta_0 =: H$.

Case 1-1: $s(t)$ stop at the point $(a, 0)$, we have $|s'(t)| \leq H$;

Case 1-2: $s(t)$ moves backwards and then moves to the left and cross the point $(a, 0)$ again, this case can be discussed similarly as above;

Case 1-3: $s(t)$ moves across the point $(a, 0)$, then $P(x, t)$ must separate with $U(x)$ temporarily. Even if $s(t)$ moves towards the right and cross the point $(a, 0)$ again, we can discuss this case as above, so $|s'(t)| \leq H$.

Since $a$ is arbitrary, so the conclusion of Lemma follows. \hfill \Box

2.2. Comparison principle.

Lemma 2.3. Suppose that $f, g \in C^1([0, \infty))$ satisfying $f(0) = g(0) = 0$, $T \in (0, \infty)$, $u \in C^1([0, T]) \cap C^2(\overline{D_U})$ with $D_U = \{(x, t) \in \mathbb{R}^2 : x \in (-\infty, s(t)], 0 < t \leq T\}$, $v \in C^1([0, T]) \cap C^2(\overline{D_V})$ with $D_V = \{(x, t) \in \mathbb{R}^2 : x \in [s(t), \infty), 0 < t \leq T\}$ and

$$
\begin{aligned}
&u_x \leq d_1 u_{xx} + f(u), & x \in (-\infty, s(t)], 0 < t \leq T, \\
&v_x \geq d_2 v_{xx} + g(v), & x \in (s(t), \infty), 0 < t \leq T, \\
u(s(t), t) = v(s(t), t) = 0, & 0 < t < T, \\
u'(t) \geq -\mu_1 u_x(s(t), t) - \mu_2 u_v(s(t), t), & 0 < t < T.
\end{aligned}
$$

If

$$
\begin{aligned}
&u(x, 0) \leq P_0(x) \text{ in } (-\infty, s_0], \quad v(x, 0) \geq Q_0(x) \text{ in } [s_0, \infty) \text{ and } s_0 \leq s_0,
\end{aligned}
$$

where $(P, Q, s)$ is a solution of (1.1), then

$$
\begin{aligned}
&u(x, t) \leq P(x, t) \text{ for } x \in (-\infty, s(t)] \text{ and } t \in (0, T], \\
v(x, t) \geq Q(x, t) \text{ for } x \in [s(t), \infty) \text{ and } t \in (0, T], \\
s(t) \leq s(t) \text{ for } t \in (0, T].
\end{aligned}
$$

The proof of Lemma 2.3 is identical to that of Lemma 5.7 in [7], so we omit the details here.

Remark 2.4. The triple $(u, v, s)$ is often called a lower solution of problem (1.1) on $[0, T]$ with initial data $(u(x, 0), v(x, 0), s_0)$. An upper solution can be defined analogously by reversing all the inequalities.
3. The Proof of Theorem 3.1

In this section, $f, g$ are always assumed to be of $(f_M)$ or $(f_B)$ type.

3.1. Converge to 1 uniformly at infinity. We consider the solution of $d_1U'' + f(U) = 0$ with compact supports, if $f$ is of type $(f_M)$ (resp. $(f_B)$), for each $m \in (0, 1)$ (resp. $m \in (\theta, 1)$), consider the trajectory $\Gamma$ given by $d_1U'' = F(U) - F(m)$, which connect $(0, \sqrt{(F(0) - F(m))/d_1})$ and $(0, -\sqrt{(F(0) - F(m))/d_1})$ through $(m, 0)$, and the solution $U_m$ satisfies $d_1U_m'' + f(U_m) = 0 < U_m \leq m$ in $(-L_m, L_m)$, where

$$L_m := \int_0^m \frac{ds}{\sqrt{(F(s) - F(m))/d_1}}, \quad m \in (\theta, 1).$$

**Lemma 3.1.** (1) If $f, g$ are of $(f_M)$ type and the initial data $P_0, Q_0$ satisfy (1.4). $P$ is the solution of the problem

$$\begin{cases}
    P_t - d_1P_{xx} = f(P), & x < s(t), \quad t > 0, \\
    P(s(t), t) = 0, & t > 0, \\
    P(x, 0) = P_0(x), & x \leq s_0,
\end{cases}$$

and $Q$ is the solution of the problem

$$\begin{cases}
    Q_t - d_2Q_{xx} = g(Q), & x > s(t), \quad t > 0, \\
    Q(s(t), t) = 0, & t > 0, \\
    Q(x, 0) = Q_0(x), & x \geq s_0,
\end{cases}$$

then for any $p_0, q_0 \in (0, 1)$, there exist $T, M > 0$ such that

$$P(x, t) \geq 1 - \frac{3}{4}p_0, \quad x \leq -M, \quad t \geq T, \quad Q(x, t) \leq 1 + \frac{1}{2}q_0, \quad x \geq M, \quad t \geq T.$$

(2) If $f, g$ are of type $(f_B)$ and the initial data $P_0, Q_0$ satisfy (iii) in Remark 1.2. $P$ is the solution of (3.2) and $Q$ is the solution of (3.3), then for any $p_0, q_0 \in (0, \frac{4}{3}(1 - \theta))$, there exist $T, M > 0$ such that

$$P(x, t) \geq 1 - \frac{3}{4}p_0, \quad x \leq -M, \quad t \geq T, \quad Q(x, t) \leq 1 + \frac{1}{2}q_0, \quad x \geq M, \quad t \geq T.$$

**Proof.** (1) By Lemma 2.2 $|s'(t)| \leq H$ for $0 < t \leq T_1$, so we can choose $M_1 > 0$ large enough such that $\max_{t \in (0, T_1]} |s'(t)/T_1| \leq M_1$. Since $\liminf_{x \to -\infty} u_0(x) > 0$, so there exists $0 < \sigma < 1$ and $M_2 > M_1 > 0$ such that when $x < -M_2$, we have $u_0(x) \geq \sigma > 0$. Let $\eta(t)$ be the solution of

$$\eta_t = f(\eta) \text{ on } [0, \infty), \quad \eta(0) = \sigma.$$

Since $f(\cdot) > 0$ in $(0, 1)$, for any $p_0 \in (0, 1)$ and $T_2 = \int_{1 - \frac{4}{3}p_0}^{1} \frac{ds}{f(s)}$, we have $\eta(T_2) = 1 - \frac{1}{4}p_0$.

We fix $R = L_1 - \frac{4}{3}p_0$. Let $L \gg R$ be a constant to be determined later (we may assume $L < 9M_2$ with no loss of generality) and $w_0(x)$ be a function satisfying

$$w_0(x) = \sigma \text{ when } x \in (-\infty, -10M_2 + L - 1], \quad w_0(-10M_2 + L) = 0,$$

$$w_0'(x) \leq 0 \text{ when } x \in (-10M_2 + L - 1, -10M_2 + L).$$

Let $w(x, t)$ be the solution of the following problem

$$\begin{cases}
    w_t - d_1w_{xx} = f(w), & \forall \ x \in (-\infty, -10M_2 + L], \quad t > 0, \\
    w(-10M_2 + L, t) = 0, & \forall \ t > 0, \\
    w(x, 0) = w_0(x), & \forall \ x \in (-\infty, -10M_2 + L].
\end{cases}$$
Set \( \rho(x) := \frac{1}{1+(x+10M_2)^2} \) and \( \zeta(x,t) := \rho(x)[w(x,t) - \eta(t)] \) satisfying \( \zeta_t - d_1\zeta_{xx} = 4d_1x\rho\zeta_x + [2d_1\rho + f']\zeta \). Denote \( \Lambda := 2d_1 + \max_{0 \leq t \leq 1} f'(s) \), so we have

\[
\max_{x \in (\infty, -10M_2 + L)} \{ \rho(x)|w(x,t) - \eta(t)| \} \leq e^{\Lambda t} \max_{x \in (\infty, -10M_2 + L)} \{ \rho(x)|w_0(x) - \sigma| \} \leq \frac{e^{\Lambda t}}{1 + (L-1)^2}.
\]

Take \( L = L(\sigma) = 1 + \sqrt{2(1 + R^2)e^{\Lambda T_2}/p_0} - 1 \), when \( x \in (-\infty, -10M_2 + R] \), we have

\[
|w(x,T_2) - \eta(T_2)| \leq \frac{1}{\rho(x)} \frac{e^{\Lambda T_2}}{1 + (L-1)^2} \leq \frac{(1 + R^2)e^{\Lambda T_2}}{1 + (L-1)^2} = \frac{1}{2}p_0.
\]

Consequently, we obtain \( w(x,T_2) \geq \eta(T_2) - \frac{1}{2}p_0 = 1 - \frac{3}{2}p_0 \).

Since \( P_0(x) \geq \sigma > 0 \) for \( x \in (-\infty, -10M_2 + L] \), thus \( P_0(x) \geq w_0(x) \) for \( x \in (-\infty, -10M_2 + L] \).

By comparison principle, we have

\[
P(x,t) \geq w(x,t) \text{ for } x \in (-\infty, -10M_2 + L], \ t \geq 0.
\]

In particular, we get \( P(x,T_2) \geq w(x,T_2) = 1 - \frac{3}{2}p_0 \) for \( x \in [-10M_2 - R, -10M_2 + R] \). By using the same method, we can repeat the above argument to prove \( P(x,T_2) \geq w(x,T_2) = 1 - \frac{3}{2}p_0 \) for \( x \in (-\infty, -10M_2 - R] \) and \([-10M_2 + R, -2M_2] \). Therefore, we have

\[
P(x,t) \geq 1 - \frac{3}{4}p_0, \ x \in (-\infty, -2M_2], \ t \geq T_2.
\]

On the other hand, since \( \limsup_{x \to \infty} Q_0(x) < \infty \), so there exist \( K > 0 \) and \( M_3 > M_1 \) such that \( |v_0(x)| \leq K \) for \( x \in [M_3, \infty) \). Let \( \xi(t) \) be the solution of the following problem

\[
(3.4) \quad \xi_t = g(\xi) \text{ on } [0, \infty), \ \xi(0) = \|v_0\|_{L^\infty} + 1.
\]

Then \( \xi(t) \) is an upper solution of \( (3.3) \). So \( Q(x,t) \leq \xi(t) \) for all \( t \geq 0 \). Since \( g(Q) < 0 \) for \( Q > 1 \), \( \xi(t) \) is a decreasing function converging to \( 1 \) as \( t \to \infty \). Thus, for any \( q_0 \in (0,1) \), there exists \( T_3 > 0 \) such that \( \xi(t) \leq 1 + \frac{1}{2}q_0 \) for \( t \geq T_3 \). In particular, we have \( Q(x,T_3) \leq \xi(T_3) < 1 + \frac{1}{2}q_0 \) for \( x \in [M_3, \infty) \) and so

\[
Q(x,t) \leq 1 + \frac{1}{2}q_0 \text{ for } x \in [M_3, \infty), \ t \geq T_3.
\]

Consequently, if we choose \( T = \max\{T_1, T_2, T_3\} \) and \( M > \max\{2M_2, M_3\} \), we obtain

\[
P(x,t) \geq 1 - \frac{3}{4}p_0, \ x \in (-\infty, -M], \ t \geq T, \quad Q(x,t) \leq 1 + \frac{1}{2}q_0, \ x \in [M, \infty), \ t \geq T.
\]

(2) If \( f \) is of \((fB)\) type, since \( \liminf_{x \to -\infty} u_0(x) > \theta \), so there exists \( m \in (\theta, \bar{\theta}] \), \( M_2 > 0 \) such that \( u_0(x) \geq m > 0 \) for \( x < -M_2 \). Let \( \eta(t) \) be the solution of

\[
\eta_t = f(\eta) \text{ on } [0, \infty), \ \eta(0) = m.
\]

Since \( f(\cdot) > 0 \) in \((\theta, \bar{\theta})\), for any \( p_0 \in (0, \frac{4}{3}(1-\bar{\theta})) \), and \( T_2 = \int_{\frac{1}{4}p_0}^{1 - \frac{1}{4}p_0} \frac{dt}{f(t)} \), we have \( \eta(T_2) = 1 - \frac{1}{4}p_0 \).

Moreover, we know that \( 1 - \frac{1}{4}p_0 > \bar{\theta} + 2\epsilon \) where \( \epsilon = \frac{1-\bar{\theta}}{3} \). Here, we fix \( R = L_{\bar{\theta} + 2\epsilon} \). The following proof is the same as (1), so we omit the details. This completes the proof.

\[\Box\]

3.2. Precise estimates of the solutions in moving coordinates. In the following, we assume that \( (P(x,t), Q(x,t)) \) is the solution of \( (1.1) \). Denote \( u(z,t) = P(z+ct,t) = P(x,t), v(z,t) = Q(z+ct,t) = Q(x,t) \), which satisfies

\[
\begin{align*}
\begin{cases}
u_t - d_1\nu_{z} - cu_z &= f(u), & z < s(t) - ct, \ t > 0, \\
\nu_x - d_2
v_{z} - cv_z &= g(v), & z > s(t) - ct, \ t > 0.
\end{cases}
\end{align*}
\]
Proof. Firstly, we prove the left part of (3.5) and the right part of (3.6).
\[
\phi(z - \rho_1) - p_0 e^{-\vartheta t} \leq u(z, t) \leq \phi(z - \rho_2) + p_0 e^{-\vartheta t},
\]
for all \( z \) and \( t \geq T \). Moreover, \( |s(t) - ct| \) is bounded for all \( t > 0 \).

We define \( u(z, t) = \max\{0, \phi(z - \xi(t) + \alpha(t)) - p(t)\} \), \( v(z, t) = \psi(z + \eta(t) - \beta(t)) + q(t) \) and say \( \xi(t) \) the free boundary of \((u, v, \vartheta)\). By Lemma 2.3, if \((u, v, \vartheta)\) is a lower solution of \((u, v, \vartheta)\), it has to satisfy
\[
\begin{align*}
\tag{3.7}
(\vartheta(t), t) &= v(z(t), t) = 0, \\
\xi'((t), t) &\leq -\mu_1 u(z(t), t) - \mu_2 v(z(t), t) - c, \\
u(z, 0) &\leq u_0(z), v(z, 0) \geq v_0(z), \xi(0) \leq s(0).
\end{align*}
\]
We will check (3.7) one by one. Denote
\[
A[u] := u - d_1 u_{zz} - cu_z - f(u)
= \phi'(\xi' + \alpha') - p' - d_1 \phi'' - c\phi' - f(\phi - p)
= -\xi'\phi' + \alpha'\phi' - p' + f(\phi) - f(\phi - p).
\]
Here we assume that \( \xi' < 0, \alpha' > 0 \), since \( f'(1) < 0 \), we choose \( \varrho > 0, 0 < \delta < 1 \) such that \( f(u) - f(u - p) \leq -\varrho \) for \( |u - 1| < \delta \) and \(|p| < \delta \), when \( \phi \in [1 - \delta, 1] \),
\[
A[u] \leq -p' + f(\phi) - f(\phi - p) = -p' + f'(\varrho) p \leq -p' + \varrho p,
\]
we can choose \( p = p_0 e^{-\vartheta t} \) and \( p_0 < \delta \) such that \( A[u] \leq 0 \). When \( \phi \in [0, 1 - \delta) \), we choose \( \phi' \leq -\gamma \) and \( r > 0 \), then we have
\[
A[u] = -\xi'\phi' + \alpha'\phi' - p' + f(\phi) - f(\phi - p)
\leq \gamma \xi' - p' + f'(\varrho) p
\leq \gamma \xi' - p' + rp.
\]
Therefore, we choose \( \xi' = \frac{\varrho - rp}{\gamma} = -\frac{\varrho + \vartheta}{\gamma} p(< 0) \) such that \( A[u] \leq 0 \). Thus \( \xi(t) = z_1 + z_2 e^{-\vartheta t} \), where \( z_1 = \xi(0) - \frac{\varrho + \vartheta}{\gamma} p_0 \), \( z_2 = \frac{\varrho + \vartheta}{\gamma} p_0 \) and \( \xi(0) \) is a constant to be determined later.

Next, we extend the domain of \( \psi \) from \([0, \infty)\) to \((\infty, \infty)\), define
\[
\psi(x) = \begin{cases} 
\psi(x), & x \in [0, \infty), \\
\lambda_0 - \lambda_0 e^{-\psi(0) x}, & x \in (-\infty, 0],
\end{cases}
\]
where \( \lambda_0 > 0 \) and it satisfies \( \lambda_0 c < \psi'(0) d_2 \).
\[
B[v] := v - d_2 v_{zz} - cv_z - g(v)
= \psi'(\eta' - \beta') + q' - d_2 \psi'' + c\psi' - g(\psi + q)
= \eta' \psi' - \beta' \psi' + q' + g(\psi) - g(\psi + q).
\]
We assume that \( \eta' > 0, \beta' < 0 \). Since \( g'(1) < 0 \), so there exist \( \vartheta, \nu > 0 \) such that \( g(v) - g(v + q) \geq \vartheta q \) for \( |v - 1| < \nu \) and \(|q| < \nu \). Here we choose \( q_0 < \nu \) and \( \vartheta < \vartheta \), when \( \psi \in [1 - \nu, 1] \),
\[
B[v] \geq q' + g(\psi) - g(\psi + q) \geq q' + \vartheta q,
\]
so we choose \( q = q_0 e^{-\vartheta t} \) such that \( B[w] \geq 0 \). When \( \psi \in [0, 1 - \nu) \), we select \( \psi' \geq \lambda > 0 \), \( \tau > 0 \) and \( \lambda < \frac{(\vartheta + \tau)q_0}{\mu_2 \psi'(0)q_0 + \lambda_0 (\mu_1 \phi'(\phi^{-1}(p_0)) - \mu_1 \phi'(0))} \), then we obtain

\[
B[w] = \eta' \psi' - \beta' \psi' + q' + g(\psi) - g(\psi + q) \\
\geq \eta' \psi' + q' + g(\psi) - g(\psi + q) \\
= \eta' \psi' + q' - g'(\nu)q \\
\geq \eta' \psi' + q' - \tau q \\
\geq \lambda \eta' + q' - \tau q.
\]

So we choose \( \eta' = -\frac{\vartheta + \tau}{\lambda} q(> 0) \) such that \( B[w] \geq 0 \). Thus \( \eta(t) = z_3 - z_4 e^{-\vartheta t} \), where \( z_3 = \eta(0) + \frac{\vartheta + \tau}{\lambda} q_0 \) and \( \eta(0) \) is a constant to be determined later.

Denote \( h(z, t) := z + \eta(t) - \beta(t) \), \( w(z, t) := \lambda_0 - \lambda_0 e^{\frac{\psi'(0)}{\lambda_0} h} + q(t) \). \( g(w) = g(0) + g'(\zeta)w \), we denote \( l_0 := \max_{0 \leq \zeta \leq \eta_0} |g'(\zeta)| \), so we have \( g(w) \leq l_0 w \leq \log \).

\[
B[w] = \eta - 2w - \lambda \psi - g(w) \\
= \psi'(0)e^{-\frac{\psi'(0)}{\lambda_0} h} [\eta' - \beta'] + q' + d_2 \frac{\psi'(0)}{\lambda_0} e^{-\frac{\psi'(0)}{\lambda_0} h} - c \psi'(0)e^{-\frac{\psi'(0)}{\lambda_0} h} - g(w) \\
\geq q' + d_2 \frac{\psi'(0)}{\lambda_0} e^{-\frac{\psi'(0)}{\lambda_0} h} - c \psi'(0)e^{-\frac{\psi'(0)}{\lambda_0} h} - \log q \\
= - (\vartheta + l_0) q + \left( \frac{\psi'(0) d_2}{\lambda_0} - c \psi'(0) \right) e^{-\frac{\psi'(0)}{\lambda_0} h}.
\]

Since \( q(t) \) is bounded and \( \lambda_0 c < \psi'(0)d_2 \), we can choose \( q_0 \leq \min \left\{ \frac{\psi'(0) d_2 - c \lambda_0 \psi'(0)}{\lambda_0 (\vartheta + l_0)}, \frac{4}{3} (1 - \vartheta) \right\} \) small such that \( B[w] \geq 0 \).

In the following, we select \( \alpha(t) = \eta(t) + \phi^{-1}(p(t)) \), \( \beta(t) = \xi(t) + \lambda_0 (\psi'(0))^{-1} \log \left( 1 + \frac{q(t)}{\lambda_0} \right) \), then we have \( \bar{z}(t) = \xi(t) - \eta(t) \).

\[
w(\bar{z}(t), t) = \lambda_0 - \lambda_0 e^{-\frac{\psi'(0)}{\lambda_0} (\xi(t) - \eta(t) + \alpha(t))} + q(t) \\
= \lambda_0 - \lambda_0 e^{-\frac{\psi'(0)}{\lambda_0} (-\lambda_0 (\psi'(0))^{-1} \log (1 + \frac{q(t)}{\lambda_0}))} + q(t) \\
= \lambda_0 - \lambda_0 \left( 1 + \frac{q(t)}{\lambda_0} \right) + q(t) = 0,
\]

and

\[
\phi(\bar{z}(t), t) - p(t) = \phi(\xi(t) - \eta(t) - \xi(t) + \alpha(t)) - p(t) \\
= \phi(\alpha(t) - \eta(t)) - p(t) \\
= \phi(\phi^{-1}(p(t))) - p(t) = 0.
\]

\[
\bar{z}'(t) = \xi'(t) - \eta'(t) = -\frac{\vartheta + \tau}{\gamma} p_0 e^{-\vartheta t} - \frac{\vartheta + \tau}{\lambda} q_0 e^{-\vartheta t} < -\frac{\vartheta + \tau}{\lambda} q_0 e^{-\vartheta t},
\]

\[
-\mu_1 \psi_1(\bar{z}(t), t) - \mu_2 \psi_2(\bar{z}(t), t) - c = -\mu_1 \phi'(p(t)) + \mu_1 (\psi'(0)) - \mu_2 \psi'(0) \lambda_0^{-1} q_0 e^{-\vartheta t}.
\]

Since \( \lambda < \frac{(\vartheta + \tau)q_0}{\mu_2 \psi'(0)q_0 + \lambda_0 (\mu_1 \phi'(\phi^{-1}(p_0)) - \mu_1 \phi'(0))} \), we have

\[
\bar{z}'(t) \leq -\mu_1 \psi_1(\bar{z}(t), t) - \mu_2 \psi_2(\bar{z}(t), t) - c.
\]
Denote
\[
\rho^* := -\eta(0) + \xi(0) - \phi^{-1}(p_0), \quad \varrho^* := -\eta(0) + \xi(0) + \lambda_0(p'(0))^{-1} \log \left(1 + \frac{q_0}{\lambda_0}\right),
\]
\[
\Sigma := \xi(0) - \frac{\rho + r}{\rho\gamma} p_0 - \eta(0) - \frac{\varrho + \tau}{\varrho\gamma} q_0, \quad \rho_1 := -\eta(0) + \xi(0) - \frac{\rho + r}{\rho\gamma} p_0 - \frac{\varrho + \tau}{\varrho\gamma} q_0.
\]
By Lemma 3.1, we can select \(\eta(0)\) positive and large enough while \(\xi(0)\) negative such that
\[
\phi(z - \rho^*) - p_0 \leq 1 - \frac{3}{4}p_0 \leq u(z, t) \quad \text{for} \quad z \in (-\infty, -M] \setminus \{0\}, \quad t \geq T,
\]
\[
\psi(z - \varrho^*) + q_0 \geq 1 + \frac{1}{2}q_0 \geq v(z, t) \quad \text{for} \quad z \in [M, \infty), \quad t \geq T.
\]
Thus, we know that \(s(t) \geq z(t) + \epsilon t\), and \(\liminf_{t \to \infty} (s(t) - ct) \geq \Sigma\). Consequently, \((\tilde{u}, \tilde{v}, \tilde{z})\) is a lower solution of \((u, \tilde{v}, s)\) for \(t \geq T\). It follows that
\[
u(z, t) \leq \psi(z + \eta(t) - \beta(t)) + q(t) \leq \psi(z - \rho_1) + q_0e^{\beta t},
\]
and
\[
u(z, t) \leq \psi(z + \eta(t) - \beta(t)) + q(t) \leq \psi(z - \rho_1) + q_0e^{\beta t}.
\]
Using the same method, we can construct an upper solution in the similar way. The proof is now completed. \(\square\)

**Lemma 3.3.** Under the assumption of Theorem 1.1, there exists a function \(\omega(\epsilon)\), defined for small positive \(\epsilon\), satisfying \(\lim_{\epsilon \to 0} \omega(\epsilon) = 0\). And if there exists \(T > 0\) such that \(|u(z, T) - \phi(z - \rho_0)| < \epsilon\), \(|v(z, T) - \psi(z - \rho_0)| < \epsilon\) for some \(\rho_0\), then
\[
|u(z, t) - \phi(z - \rho_0)| < \omega(\epsilon), \quad |v(z, t) - \psi(z - \rho_0)| < \omega(\epsilon)
\]
for all \(z\) and all \(t \geq T\).

**Proof.** In the proof of Lemma 3.2, we may take \(p_0 = O(\epsilon), \quad q_0 = O(\epsilon), \quad |\rho^* - \rho_0| = O(\epsilon)\) and \(|\varrho^* - \rho_0| = O(\epsilon)\). Hence also \(|\rho_1 - \rho_0| = O(\epsilon)\), \(|\rho_2 - \rho_0| = O(\epsilon)\) and the conclusion follows from Lemma 3.2. \(\square\)

3.3. **Convergence of the solutions.** By Lemma 3.2, we know that there exist \(C, T > 0\) such that
\[-C \leq s(t) - ct \leq C \quad \text{for} \quad t \geq T.\]
Here we assume that \(C > \max\{\rho_1, \rho_2\}\). Denote
\[k(t) = ct - 2C\]
and define
\[\tilde{u}(x, t) = P(x + k(t), t), \tilde{v}(x, t) = Q(x + k(t), t), \tilde{s}(t) = s(t) - k(t), t \geq T.\]
Obviously,
\[C \leq \tilde{s}(t) \leq 3C \quad \text{for} \quad t \geq T.\]
By simple calculation, we know that \((\tilde{u}, \tilde{v}, \tilde{s})\) satisfies
\[
\begin{align*}
\tilde{u}_t - d_1 \tilde{u}_{xx} - c\tilde{u}_x - f(\tilde{u}) &= 0, \quad x \in (-\infty, \tilde{s}(t)), t \geq T, \\
\tilde{v}_t - d_2 \tilde{v}_{xx} - c\tilde{v}_x - g(\tilde{v}) &= 0, \quad x \in (\tilde{s}(t), \infty), t \geq T, \\
\tilde{u}(\tilde{s}(t), t) &= \tilde{v}(\tilde{s}(t), t) = 0, \quad t \geq T, \\
\tilde{s}(t) &= -\mu_1 \tilde{u}_x(\tilde{s}(t), t) - \mu_2 \tilde{v}_x(\tilde{s}(t), t) - c, t \geq T.
\end{align*}
\]
Let \(t_n \to \infty\) be an arbitrary sequence satisfying \(t_n \geq T\) for every \(n \geq 1\).
Define

\[ k_n(t) = k(t + t_n), \quad \tilde{u}_n(x, t) = \tilde{u}(x, t + t_n), \quad \tilde{v}_n(x, t) = \tilde{v}(x, t + t_n), \quad \tilde{s}_n(t) = \tilde{s}(t + t_n). \]

**Lemma 3.4.** Subject to a subsequence,

\[ \tilde{s}_n(t) \to G(t) \in C^{1+\alpha/2}_loc(\mathbb{R}^1), \tilde{u}_n \to \tilde{U} \text{ in } C^{1+\alpha, (1+\alpha)/2}_loc(D_u) \text{ and } \tilde{v}_n \to \tilde{V} \text{ in } C^{1+\alpha, (1+\alpha)/2}_loc(D_v), \]

where \( 0 < \alpha < 1, D_u = \{(x, t): -\infty < x < G(t), t \in \mathbb{R}^1\}, D_v = \{(x, t): G(t) < x < \infty, t \in \mathbb{R}^1\} \)
and \((\tilde{U}(x, t), \tilde{V}(x, t), G(t))\) satisfies

\[
\begin{cases}
\tilde{U}_t - d_1 \tilde{U}_{xx} - c \tilde{U}_x = f(\tilde{U}), & (x, t) \in D_u, \\
\tilde{V}_t - d_2 \tilde{V}_{xx} - c \tilde{V}_x = g(\tilde{V}), & (x, t) \in D_v, \\
\tilde{U}(G(t), t) = \tilde{V}(G(t), t) = 0, & t \in \mathbb{R}^1, \\
G'(t) = -\mu_1 \tilde{U}_x(G(t), t) - \mu_2 \tilde{V}_x(G(t), t) - c, & t \in \mathbb{R}^1.
\end{cases}
\]

(3.8)

**Proof.** By Lemma 2.2 we have \( |s'(t)| \leq H \), for all \( t > 0 \). So, there exists \( \tilde{H} > 0 \) such that \( |\tilde{s}_n(t)| \leq \tilde{H} \) for \( t + t_n \) large and every \( n \geq 1 \).

Define

\[ y = \frac{x}{s_n(t)}, \quad \tilde{u}_n(y, t) = \tilde{u}_n(x, t), \quad \tilde{v}_n(y, t) = \tilde{v}_n(x, t), \]

then \((\tilde{u}_n(y, t), \tilde{v}_n(y, t), \tilde{s}_n(t))\) satisfies

\[
\begin{cases}
\tilde{u}_n(t) - \frac{d_1}{s_n(t)} (\tilde{u}_n)'_y - [y \tilde{s}_n(t) + c] \frac{(\tilde{u}_n)_y}{s_n(t)} = f(\tilde{u}_n), & -\infty < y \leq 1, t > T - t_n, \\
(\tilde{v}_n)_y - c \tilde{s}_n(t) \frac{(\tilde{v}_n)_y}{s_n(t)} = g(\tilde{v}_n), & 1 \leq y < \infty, t > T - t_n, \\
\tilde{u}_n(1, t) = \tilde{v}_n(1, t) = 0, & t > T - t_n, \\
\tilde{s}_n(t) = -\mu_1 \frac{(\tilde{u}_n)'_y(1, t)}{s_n(t)} - \mu_2 \frac{(\tilde{v}_n)_y(1, t)}{s_n(t)} - c, & t > T - t_n.
\end{cases}
\]

(3.9)

For any given \( R_0 > 0 \) and \( T_0 \in \mathbb{R}^1 \), by using the interior-boundary \( L^p \) estimates (see Theorem 7.15 in [14]) to (3.9) over \([-R_0 - 1, 1] \times [T_0 - 1, T_0 + 1] \) and \([1, R_0 + 1] \times [T_0 - 1, T_0 + 1] \), we get, for any \( p > 1 \)

\[
\|\tilde{u}_n\|_{W^{1,2}_p([-R_0, 1] \times [T_0, T_0 + 1])} \leq C_{R_0} \text{ for all large } n,
\]

\[
\|\tilde{v}_n\|_{W^{1,2}_p([1, R_0] \times [T_0, T_0 + 1])} \leq C_{R_0} \text{ for all large } n,
\]

where \( C_{R_0} \) is a constant depending on \( R_0 \) and \( p \) but independent of \( n \) and \( T_0 \). Therefore, for any \( \alpha' \in (0, 1) \), we can choose \( p > 1 \) large enough and use the Sobolev embedding theorem (see [13]) to obtain

\[
\begin{align*}
\|\tilde{u}_n\|_{C^{1+\alpha', \frac{1+\alpha'}{2}}([-R_0, 1] \times [T_0, T_0 + 1])} & \leq \tilde{C}_{R_0} \text{ for all large } n, \\
\|\tilde{v}_n\|_{C^{1+\alpha', \frac{1+\alpha'}{2}}([1, R_0] \times [T_0, T_0 + 1])} & \leq \tilde{C}_{R_0} \text{ for all large } n,
\end{align*}
\]

(3.10)

where \( \tilde{C}_{R_0} \) is a constant depending on \( R_0 \) and \( \alpha' \) but independent of \( n \) and \( T_0 \). From (3.9) and (3.10), we deduce that

\[
\|\tilde{s}_n\|_{C^{1+\frac{\alpha}{2}}([T_0, \infty))} \leq C_1 \text{ for all large } n,
\]

where \( C_1 \) is a constant independent of \( n \) and \( T_0 \). Hence by passing to a subsequence, we obtain that, as \( n \to \infty \),

\[
\tilde{u}_n \to \tilde{U} \text{ in } C^{1+\alpha, \frac{1+\alpha}{2}}_{loc}((-\infty, 1] \times \mathbb{R}^1), \quad \tilde{v}_n \to \tilde{V} \text{ in } C^{1+\alpha, \frac{1+\alpha}{2}}_{loc}([1, \infty) \times \mathbb{R}^1),
\]

\[
\tilde{s}_n \to G \text{ in } C^{1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^1),
\]
where $\alpha \in (0, \alpha')$. Moreover, by using (3.9), we know that $\hat{U}, \hat{V}, G)$ satisfies in the $W^{1,2}_p$ sense (and hence classical sense by standard regularity theory),

$$
\begin{align*}
& (\hat{U}), - \frac{d^2}{G(t)} (\hat{U})_{yy} - [yG'(t) + c] \frac{(\hat{U})}{G(t)} = f(\hat{U}), \quad -\infty < y \leq 1, t \in \mathbb{R}^1, \\
& (\hat{V}), - \frac{d^2}{G(t)} (\hat{V})_{yy} - [yG'(t) + c] \frac{(\hat{V})}{G(t)} = g(\hat{V}), \quad 1 \leq y < \infty, t \in \mathbb{R}^1, \\
& \hat{U}(1, t) = \hat{V}(1, t) = 0, \quad t \in \mathbb{R}^1, \\
& G'(t) = -\mu_1 \frac{(\hat{U})}{G(t)} - \mu_2 \frac{(\hat{V})}{G(t)} - c, \quad t \in \mathbb{R}^1.
\end{align*}
$$

Define $\tilde{U}(x, t) = \hat{U} \left( \frac{x}{G(t)}, t \right), \tilde{V}(x, t) = \hat{V} \left( \frac{x}{G(t)}, t \right)$, then $(\tilde{U}, \tilde{V}, G)$ satisfies (3.8) and

$$
\begin{align*}
& \lim_{n \to \infty} \| \tilde{u}_n - \tilde{U} \|_{C^{1+\alpha, (1+\alpha)/2}(D_u)} = 0, \\
& \lim_{n \to \infty} \| \tilde{v}_n - \tilde{V} \|_{C^{1+\alpha, (1+\alpha)/2}(D_v)} = 0.
\end{align*}
$$

Since $C \leq \tilde{s}(t) \leq 3C$ for $t \geq T$, so $C \leq G(t) \leq 3C$ for $t \in \mathbb{R}^1$.

By the proof of Lemma 3.2, we have

$$
\begin{align*}
& \tilde{u}_n(x, t) \leq \phi(x - 2C - \rho_2) + p_0 e^{-\theta(t+t_n)}, \\
& \tilde{v}_n(x, t) \geq \psi(x - 2C - \rho_2) - q_0 e^{-\theta(t+t_n)}.
\end{align*}
$$

Letting $n \to \infty$, we obtain

$$
\begin{align*}
& \tilde{U}(x, t) \leq \phi(x - 2C - \rho_2) \text{ for all } t \in \mathbb{R}^1, x < G(t), \\
& \tilde{V}(x, t) \geq \psi(x - 2C - \rho_2) \text{ for all } t \in \mathbb{R}^1, x > G(t).
\end{align*}
$$

Define

$$
\begin{align*}
R_u = \inf \{ R : \tilde{U}(x, t) \leq \phi(x - R) \text{ for } (x, t) \in D_u \}, \\
R_v = \inf \{ R : \tilde{V}(x, t) \geq \psi(x - R) \text{ for } (x, t) \in D_v \},
\end{align*}
$$

then

$$
\tilde{U}(x, t) \leq \phi(x - R_u) \text{ for } (x, t) \in D_u, \tilde{V}(x, t) \geq \psi(x - R_v) \text{ for } (x, t) \in D_v.
$$

Denote $R_* = \max \{R_u, R_v\}$, hence

$$
\tilde{U}(x, t) \leq \phi(x - R_*) \text{ for } (x, t) \in D_u, \tilde{V}(x, t) \geq \psi(x - R_*) \text{ for } (x, t) \in D_v
$$

and

$$
C \leq \inf_{t \in \mathbb{R}^1} G(t) \leq \sup_{t \in \mathbb{R}^1} G(t) \leq R_* \leq 3C.
$$

**Lemma 3.5.** $R_* = \sup_{t \in \mathbb{R}^1} G(t)$.

**Proof.** We prove it by contradiction, if not, we have $R_* > \sup_{t \in \mathbb{R}^1} G(t)$. Choose $\delta > 0$ such that $G(t) \leq R_* - \delta$ for all $t \in \mathbb{R}^1$.

**Step 1:** $\tilde{U}(x, t) < \phi(x - R_*)$ for all $t \in \mathbb{R}^1$ and $x \leq G(t), \tilde{V}(x, t) > \psi(x - R_*)$ for all $t \in \mathbb{R}^1$ and $x \geq G(t)$.

Otherwise, there exists $(x_0, t_0) \in D_u = \{(x, t) : -\infty < x \leq G(t), t \in \mathbb{R}^1\}$ such that $\tilde{U}(x_0, t_0) = \phi(x_0 - R_*) \geq \phi(-\delta) > 0$, therefore, $x_0 < G(t_0)$. Since $\tilde{U}(x, t) \leq \phi(x - R_*)$ in $D_u$ and $\phi(x - R_*)$ satisfies the equation (3.8), by strong maximum principle, we conclude that $\tilde{U}(x, t) \equiv \phi(x - R_*)$ in $D_{u_0} := \{(x, t) : x < G(t), t \leq t_0\}$, and this contradicts with $G(t) \leq R_* - \delta$. If there exists $(x_0, t_0) \in D_v = \{(x, t) : G(t) \leq x < \infty, t \in \mathbb{R}^1\}$ such that $\tilde{V}(x_0, t_0) = \psi(x_0 - R_*)$. 


Case 1 $x_0 = R_*$. We have $\widetilde{V}(x_0, t_0) = \psi(0) = 0$, so $x_0 = R_* = G(t_0)$, and this contradiction implies the conclusion easily.

Case 2 $x_0 > R_*$. We have $\widetilde{V}(x_0, t_0) = \psi(x_0 - R_*) > \psi(0) = 0$. Since $\widetilde{V}(x, t) \geq \psi(x - R_*)$ in $D_v$ and $\psi(x - R_*)$ satisfies the equation \([3.11]\), by strong maximum principle, we conclude that $\widetilde{V}(x, t) \equiv \psi(x - R_*)$ in $D_{0v} := \{(x, t) : x > G(t), t \leq t_0\}$, so $\widetilde{V}(R_*, t) = \psi(R_* - R_*) = 0$ for $t \leq t_0$ and $R_* = G(t)$ for $t \leq t_0$. This is a contradiction with $G(t) \leq R_* - \delta$.

Step 2: $M_u(x) := \inf_{t \in \mathbb{R}} [\phi(x - R_*) - \widetilde{U}(x, t)] > 0$ for $x \in (-\infty, R_* - \delta]$ and $M_v(x) := \inf_{t \in \mathbb{R}} [\widetilde{V}(x, t) - \psi(x - R_*)] > 0$ for $x \in [R_*, \infty)$. Otherwise, there exist $x_{10} \in (-\infty, R_* - \delta]$ and $x_{20} \in [R_*, \infty)$ such that $M_u(x_{10}) = 0, M_v(x_{20}) = 0$, since the definition of $R_*$ implies $M_u(x) \geq 0$. By step 1, we know $M_u(x_{10})$ and $M_v(x_{20})$ can not be achieved at any finite $t$. Therefore, there exists $s_n \in \mathbb{R}$ with $|s_n| \to \infty$ such that

$$\phi(x_{10} - R_*) = \lim_{n \to \infty} \widetilde{U}(x_{10}, s_n).$$

Define

$$\tilde{U}_n(x, t), \tilde{V}_n(x, t), G_n(t) = (\tilde{U}(x, t + s_n), \tilde{V}(x, t + s_n), G(t + s_n)).$$

Then we can use the same argument as in the proof of Lemma \[3.4\] to show that, by passing to a subsequence, $U_n, V_n, G_n \to (U^*, V^*, G^*)$ with $(U^*, V^*, G^*)$ satisfying

\begin{align}
U^*_t - d_1 U^*_{xx} - cU^*_x &= f(U^*), \quad -\infty < x < G^*(t), \ t \in \mathbb{R}^1, \\
V^*_t - d_2 V^*_{xx} - cV^*_x &= g(V^*), \quad G^*(t) < x < \infty, \ t \in \mathbb{R}^1, \\
U^*(G^*(t), t) &= V^*(G^*(t), t) = 0, \quad t \in \mathbb{R}^1.
\end{align}

Moreover,

\begin{align}
U^*(x, t) \leq \phi(x - R_*), \quad V^*(x, t) \geq \psi(x - R_*), \quad G^*(t) \leq R_* - \delta, \\
U^*(x_{10}, 0) &= \phi(x_{10} - R_*), \quad V^*(x_{20}, 0) = \psi(x_{20} - R_*).
\end{align}

Since $(\phi(x - R_*), \psi(x - R_*))$ satisfies \([3.11]\) with $G^*(t)$ replaced by $R_*$, by strong maximum principle, we have $U^*(x, t) \equiv \phi(x - R_*)$ for $t \leq 0$, $x \leq G^*(t)$, $V^*(x, t) \equiv \psi(x - R_*)$ for $t \leq 0$, $x \geq G^*(t)$, which is clearly impossible. On the other hand, If there exists $\tau_n \in \mathbb{R}$ with $|\tau_n| \to \infty$ such that $\psi(x_{20} - R_*) = \lim_{n \to \infty} \widetilde{V}(x_{20}, \tau_n)$. In the same way, we can derive a contradiction. Therefore, the conclusion follows easily.

Step 3: Reaching a contradiction. Choose $\epsilon_0 > 0$ small, $R_{10} < 0$ large negative and $R_{20} > 0$ large positive such that

$$\phi(x - R_*) \geq 1 - \epsilon_0 \text{ for } x \leq R_{10}, \quad \psi(x - R_*) \geq 1 - \epsilon_0 \text{ for } x \geq R_{20}$$

and

$$f'(u) < 0 \text{ for } u \in [1 - 2\epsilon_0, 1 + 2\epsilon_0], \quad g'(v) < 0 \text{ for } v \in [1 - \frac{1}{2}\epsilon_0, 1 + 2\epsilon_0].$$

Then choose $\epsilon \in (0, \epsilon_0)$ small such that

$$\tilde{V}(x, t) \geq \psi(x - R_* + \epsilon),$$

$$\phi(R_{10} - R_* + \epsilon) \geq \phi(R_{10} - R_*) - M_u(R_{10}), \quad \psi(R_{20} - R_* + \epsilon) \leq \psi(R_{20} - R_*) + M_v(R_{20}),$$

$$\phi(x - R_* + \epsilon) \geq 1 - 2\epsilon_0 \text{ for } x \leq R_{10}, \quad \psi(x - R_* + \epsilon) \geq 1 - \frac{1}{2}\epsilon_0 \text{ for } x \geq R_{20}.$$
and

$$\begin{align*}
\begin{cases}
\ddot{V}_t - d_2\dddot{V}_{xx} - c\dot{V}_x &= g(\dot{V}), &x > R_{20}, \ t > 0, \\
\dot{V}(R_{20}, t) &= \psi(R_{20} - R_* + \epsilon), &t > 0, \\
\dot{V}(x, 0) &= \psi(x - R_* + \epsilon), &x > R_{20}.
\end{cases}
\end{align*}$$

(3.14)

Since 1 is the upper solution of (3.13) and (3.14), the unique solution of (3.13) and (3.14) are decreasing in $t$. Obviously, $\bar{U}(x, t) = \phi(x - R_* + \epsilon), \bar{V}(x, t) = \psi(x - R_* + \frac{\epsilon}{2})$ are the lower solution of (3.13) and (3.14). By comparison principle, we have

$$\phi(x - R_* + \epsilon) \leq \bar{U}(x, t) \leq 1 \text{ for all } t > 0, \ x < R_{10},$$

$$\psi(x - R_* + \frac{\epsilon}{2}) \leq \bar{V}(x, t) \leq 1 \text{ for all } t > 0, \ x > R_{20}.$$

Therefore,

$$\mathcal{U}(x) := \lim_{t \to \infty} \bar{U}(x, t) \geq \phi(x - R_* + \epsilon) \text{ for all } x < R_{10},$$

$$\mathcal{V}(x) := \lim_{t \to \infty} \bar{V}(x, t) \geq \psi(x - R_* + \frac{\epsilon}{2}) \text{ for all } x > R_{20}.$$

Moreover, $(\mathcal{U}(x), \mathcal{V}(x))$ satisfies

$$\begin{align*}
-\ddot{d}_1\dddot{u}_{xx} - c\ddot{u}_x &= f(\mathcal{U}) \text{ in } (-\infty, R_{10}), \mathcal{U}(-\infty) = 1, \mathcal{U}(R_{10}) = \phi(R_{10} - R_* + \epsilon), \\
-\ddot{d}_2\dddot{v}_{xx} - c\ddot{v}_x &= g(\mathcal{V}) \text{ in } (R_{20}, \infty), \mathcal{V}(\infty) = 1, \mathcal{V}(R_{20}) = \phi(R_{20} - R_* + \frac{\epsilon}{2}).
\end{align*}$$

(3.15)

Write $\Phi(x) = \phi(x - R_* + \epsilon), \Psi(x) = \psi(x - R_* + \frac{\epsilon}{2})$. We notice that $(\Phi(x), \Psi(x))$ also satisfies (3.15). Moreover,

$$1 - 2\epsilon_0 \leq \Phi(x) \leq \mathcal{U}(x) \leq 1 \text{ for } x \leq R_{10},$$

$$\frac{1}{2} - \epsilon_0 \leq \Psi(x) \leq \mathcal{V}(x) \leq 1 \text{ for } x \geq R_{20}.$$

Hence $W_1(x) = \mathcal{U}(x) - \Phi(x) \geq 0$, $W_2(x) = \mathcal{V}(x) - \Psi(x) \geq 0$ and there exist $c_1(x), c_2(x) < 0$ such that

$$-d_1W_1'' = c_1(x)W_1 \text{ in } (-\infty, R_{10}], \ W_1(R_{10}) = 0,$$

$$-d_2W_2'' = c_2(x)W_2 \text{ in } [R_{20}, \infty), \ W_2(R_{20}) = 0,$$

and by the maximum principle we deduce, for any $R_1 < R_{10}, R_2 > R_{20}$,

$$W_1(x) \leq W_1(R_{10}) \text{ in } [R_1, R_{10}], \ W_2(x) \leq W_2(R_{20}) \text{ in } [R_{20}, R_2].$$

Letting $R_1 \to -\infty$ and $R_2 \to \infty$ we deduce that $W_1(x) \leq 0$ in $(-\infty, R_{10}]$ and $W_2(x) \leq 0$ in $[R_{20}, \infty)$. It follows that $W_1 \equiv 0, W_2 \equiv 0$. Hence

$$\mathcal{U}(x) \equiv \Phi(x) = \phi(x - R_* + \epsilon), \mathcal{V}(x) \equiv \Psi(x) = \psi(x - R_* + \frac{\epsilon}{2}).$$

We now look at $(\bar{U}(x, t), \bar{V}(x, t))$, since $\bar{U}(x, t)$ satisfies the first equation in (3.13), and for any $t \in \mathbb{R}^1$,

$$\bar{U}(x, t) \leq 1, \ \bar{U}(R_{10}, t) \leq \phi(R_{10} - R_* - M_{R_{10}}) \leq \phi(R_{10} - R_* + \epsilon).$$

In the same way, we know

$$\bar{V}(x, t) \geq \psi(x - R_* + \epsilon), \ \bar{V}(R_{20}, t) \geq \psi(R_{20} - R_* + M_{R_{20}}) \geq \psi(R_{20} - R + \epsilon).$$

Consequently, we use the comparison principle to deduce that

$$\bar{U}(x, t + s) \leq \bar{U}(x, t) \text{ for all } t > 0, x < R_{10}, s \in \mathbb{R}^1,$$
\[ \tilde{V}(x, t + s) \geq \tilde{V}(x, t) \text{ for all } t > 0, x > R_{20}, s \in \mathbb{R}^1. \]

Or equivalently,
\[ \tilde{U}(x, t) \leq \tilde{U}(x, t - s) \text{ for all } t > s, x < R_{10}, s \in \mathbb{R}^1, \]
\[ \tilde{V}(x, t) \geq \tilde{V}(x, t - s) \text{ for all } t > s, x > R_{20}, s \in \mathbb{R}^1. \]

Letting \( s \to -\infty \) we obtain
\begin{equation}
\tilde{U}(x, t) \leq U(x) = \phi(x - R_\ast + \epsilon) \text{ for all } x < R_{10}, t \in \mathbb{R}^1,
\end{equation}
\[ \tilde{V}(x, t) \geq V(x) = \psi(x - R_\ast + \frac{\epsilon}{2}) \text{ for all } x > R_{20}, t \in \mathbb{R}^1. \]

By Step 2 and the continuity of \( M_u(x), M_v(x) \) in \( x \), we have
\[ M_u(x) \geq \sigma > 0 \text{ for } x \in [R_{10}, R_\ast - \delta], \]
\[ M_v(x) \geq \sigma > 0 \text{ for } x \in [R_\ast, R_{20}]. \]

If we choose \( \epsilon_1 \in (0, \epsilon) \) is small enough we have
\[ \phi(x - R_\ast + \epsilon_1) \geq \phi(x - R_\ast) - \sigma \text{ for } x \in [R_{10}, R_\ast - \delta], \]
\[ \psi(x - R_\ast + \frac{\epsilon_1}{2}) \leq \psi(x - R_\ast) + \sigma \text{ for } x \in [R_\ast, R_{20}], \]

and so
\[ \tilde{U}(x, t) - \phi(x - R_\ast + \epsilon_1) \leq \sigma - M_u(x) \leq 0 \text{ for } x \in [R_{10}, R_\ast - \delta], t \in \mathbb{R}^1, \]
\[ \tilde{V}(x, t) - \psi(x - R_\ast + \frac{\epsilon_1}{2}) \geq M_v(x) - \sigma \geq 0 \text{ for } x \in [R_\ast, R_{20}], t \in \mathbb{R}^1. \]

Therefore we can combine with (3.16) to obtain
\[ \tilde{U}(x, t) - \phi(x - R_\ast + \epsilon_1) \leq 0 \text{ for } x \in (-\infty, R_\ast - \delta], t \in \mathbb{R}^1, \]
\[ \tilde{V}(x, t) - \psi(x - R_\ast + \frac{\epsilon_1}{2}) \geq 0 \text{ for } x \in [R_\ast, \infty), t \in \mathbb{R}^1, \]

for all small \( \epsilon_1 \in (0, \epsilon) \), which contradicts the definition of \( R_u, R_v \). The proof is complete. \( \square \)

**Lemma 3.6.** There exists a sequence \( \{s_n\} \subset \mathbb{R}^1 \) such that

\( G(t + s_n) \to R_\ast \) as \( n \to \infty \) uniformly for \( t \) in compact subset of \( \mathbb{R}^1 \),

\( \tilde{U}(x, t + s_n) \to \phi(x - R_\ast) \) as \( n \to \infty \) uniformly for \( (x, t) \) in compact subset of \( (-\infty, R_\ast] \times \mathbb{R}^1 \),

\( \tilde{V}(x, t + s_n) \to \psi(x - R_\ast) \) as \( n \to \infty \) uniformly for \( (x, t) \) in compact subset of \( [R_\ast, \infty) \times \mathbb{R}^1 \).

**Proof.** There are two possibilities:

(i) \( R_\ast = \sup_{t \in \mathbb{R}^1} G(t) \) is achieved at some finite time \( t = s_0 \),

(ii) \( R_\ast > G(t) \) for all \( t \in \mathbb{R}^1 \) and \( G(s_n) \to R_\ast \) along some unbounded sequence \( s_n \).

In case (i), we have \( G'(s_0) = 0 \), since \( \tilde{U}(x, t) \leq \phi(x - R_\ast) \) for \( x \leq G(t) \), \( \tilde{V}(x, t) \geq \psi(x - R_\ast) \) for \( x \geq G(t) \) and \( t \in \mathbb{R}^1 \), with \( \tilde{U}(G(s_0), t) = \phi(G(s_0) - R_\ast) = \phi(0) = 0 \), \( \tilde{V}(G(s_0), t) = \psi(G(s_0) - R_\ast) = \psi(0) = 0 \), by strong maximum principle and Hopf boundary lemma, we have
\[ \tilde{U}_x(G(s_0), t) > \phi'(0) \text{ unless } \tilde{U}(x, t) \equiv \phi(x - R_\ast) \text{ in } D_{0u} = \{(x, t) : x \leq G(t), t \leq s_0\}, \]
\[ \tilde{V}_x(G(s_0), t) > \psi'(0) \text{ unless } \tilde{V}(x, t) \equiv \psi(x - R_\ast) \text{ in } D_{0v} = \{(x, t) : x \geq G(t), t \leq s_0\}. \]

On the other hand, we know
\[ -\mu_1 \tilde{U}_x(G(s_0), s_0) - \mu_2 \tilde{V}_x(G(s_0), s_0) - c = G'(s_0) = 0, \]
but in view of
\[ -\mu_1 \tilde{U}_x(G(s_0), s_0) < -\mu_1 \phi'(0), \quad -\mu_2 \tilde{V}_x(G(s_0), s_0) < -\mu_2 \psi'(0), \]
By Lemma 3.6, there exists
\[\mathcal{U}(x,t) = \phi(x - R_s)\] for all \(x \leq G(t), t \in \mathbb{R}^1\),
\[\mathcal{V}(x,t) = \psi(x - R_s)\] for all \(x \geq G(t), t \in \mathbb{R}^1\),
so we choose \(s_n \equiv s_0\) and the conclusion of the lemma holds. In case (ii), we consider the following sequence
\[U_n(x,t) = \tilde{U}(x,t + l_n), \quad V_n(x,t) = \tilde{V}(x,t + l_n), \quad G_n(t) = G(t + l_n).\]
Using the same method as in the proof of Lemma 3.4, we can choose a subsequence such that
\[U_n \to U \text{ in } C_{loc}^{1+\alpha,1+\alpha/2}(D_u), \quad V_n \to V \text{ in } C_{loc}^{1+\alpha,1+\alpha/2}(D_v), \quad G_n \to G \text{ in } C_{loc}^1(\mathbb{R}^1),\]
and \((U, V, G)\) satisfies (3.8). Moreover,
\[G(t) \leq R_s, \quad G(0) = R_s.\]
Hence we are back to case (i) and thus \(U(x,t) \equiv \phi(x - R_s)\) in \(D_u\), \(V(x,t) \equiv \psi(x - R_s)\) in \(D_v\), and \(G \equiv R_s\). The conclusion of the lemma follows easily. \(\square\)

**Lemma 3.7.** There exists \(T_k \to \infty\) such that
\[\bar{s}(t + T_k) \to R_s\] as \(k \to \infty\) uniformly for \(t\) in compact subset of \(\mathbb{R}^1\),
\[\bar{u}(x,t + T_k) \to \phi(x - R_s)\] as \(k \to \infty\) uniformly for \((x,t)\) in compact subset of \((-\infty, R_s] \times \mathbb{R}^1\),
\[\bar{v}(x,t + T_k) \to \psi(x - R_s)\] as \(k \to \infty\) uniformly for \((x,t)\) in compact subset of \([R_s, \infty) \times \mathbb{R}^1\).

**Proof.** For \(k \geq 1\), define
\[D^g = [-k,k], \quad D^u = [-k,R_s], \quad D^v = [R_s,k].\]
By Lemma 3.6, there exists \(s_{nk}\) such that
\[|G(t + s_{nk}) - R_s| \leq \frac{1}{k}\] for \(t \in D^g\),
\[|\bar{U}(x,t + s_{nk}) - \phi(x - R_s)| \leq \frac{1}{k}\] for \((x,t) \in D^u \times D^g\),
\[|\bar{V}(x,t + s_{nk}) - \psi(x - R_s)| \leq \frac{1}{k}\] for \((x,t) \in D^v \times D^g\).
By Lemma 3.4, there exists \(t_{mk}\) such that \(t_{mk} + s_{nk} > k\) and
\[|\bar{s}(t + t_{mk} + s_{nk}) - G(t + s_{nk})| < \frac{1}{k}\] for \(t \in D^g\),
\[|\bar{u}(x,t + t_{mk} + s_{nk}) - \bar{U}(x,t + s_{nk})| \leq \frac{1}{k}\] for \((x,t) \in D^u \times D^g\),
\[|\bar{v}(x,t + t_{mk} + s_{nk}) - \bar{V}(x,t + s_{nk})| \leq \frac{1}{k}\] for \((x,t) \in D^v \times D^g\).
Therefore we can take \(T_k = t_{mk} + s_{nk}\), then as \(k \to \infty, T_k \to \infty\) and
\[|\bar{s}(t + T_k) - R_s| \leq \frac{2}{k} \to 0\] uniformly for \(t\) in compact subset of \(\mathbb{R}^1\),
\[|\bar{u}(x,t + T_k) - \phi(x - R_s)| \leq \frac{2}{k} \to 0\] uniformly for \((x,t)\) in compact subset of \((-\infty, R_s] \times \mathbb{R}^1\),
\[|\bar{v}(x,t + T_k) - \psi(x - R_s)| \leq \frac{2}{k} \to 0\] uniformly for \((x,t)\) in compact subset of \([R_s, \infty) \times \mathbb{R}^1\).
3.4. The Proof of Theorem 1.1. We now apply Lemma 3.3, which indicate that once \( \tilde{u} \) is close to \( \phi(x-R_s) \) and \( \tilde{v} \) is close to \( \psi(x-R_s) \) for some \( T_k \), it remains close for all later time. Thus
\[
\tilde{u}(x,t) \to \phi(x-R_s), \quad \tilde{v}(x,t) \to \psi(x-R_s) \text{ as } t \to \infty.
\]
Denote \( x^* = R_s - 2C \), we have
\[
P(x,t) \to \phi(x-ct-x^*), \quad Q(x,t) \to \psi(x-ct-x^*) \text{ as } t \to \infty.
\]
Moreover, \( |s(t) - ct - x^*| \to 0 \) as \( t \to \infty \).

Next, we prove \( s'(t) \to c \).

If \( s'(t) \) does not converge to \( c \), there exist an \( \epsilon > 0 \) and a sequence \( \{t_k\}_{k=1}^\infty \) such that \( |s'(t_k) - c| > \epsilon \) for \( k \) and \( t_k \to \infty \) as \( k \to \infty \). With no loss of generality, we assume \( s'(t_k) > c + \epsilon \). Since \( |s(t) - ct - x^*| \to 0 \) as \( t \to \infty \), we can make estimate about \( \tilde{u} \) and \( \tilde{v} \) at \( x = x^* \)
\[
\tilde{u}(x^*, t_k) = \tilde{u}(s(t_k) - ct_k, t_k) + O(s(t_k) - ct_k - x^*)
\]
\[
= \tilde{u}(s(t_k) - ct_k + 2C, t_k) - \tilde{u}_x(s(t_k) - ct_k + 2C, t_k)(2C) + o((2C)^2) + O(s(t_k) - ct_k - x^*)
\]
\[
= \tilde{u}x(s(t_k) - ct_k + 2C, t_k)(2C) + o((2C)^2) + O(s(t_k) - ct_k - x^*)
\]
and
\[
\tilde{v}(x^*, t_k) = -\tilde{v}_x(s(t_k) - ct_k + 2C, t_k)(2C) + o((2C)^2) + O(s(t_k) - ct_k - x^*).
\]

Using the same method, we have
\[
\phi(-2C) = \phi(0) - \phi'(0)(2C) + o((2C)^2)
\]
and
\[
\psi(-2C) = \psi(0) - \psi'(0)(2C) + o((2C)^2).
\]

Obviously, \( O(s(t_k) - ct_k - x^*) \to 0 \) as \( k \to \infty \).

\[
\liminf_{k \to \infty} (\mu_1(\tilde{u}(x^*, t_k) - \phi(-2C)) + \mu_2(\tilde{v}(x^*, t_k) - \psi(-2C)))
\]
\[
> (c + \epsilon)(2C) + o((2C)^2) - 2cC - o((2C)^2)
\]
\[
= 2cC + o((2C)^2)
\]
\[
> \epsilon C > 0
\]

However, this is a contradiction with the asymptotic behavior of \( \tilde{u} \) and \( \tilde{v} \). Thus \( s'(t) \to c \). This completes the proof of Theorem 1.1. \( \square \)

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[3] C. H. Chang and C. C. Chen, Traveling wave solution of a free boundary problem for a two-species competitive model, Commun. Pure Appl. Anal., 12 (2013), 1065–1074.
[4] X. F. Chen, B. D. Lou, M. L. Zhou and T. Giletti, Long Time Behavior of Solutions of a reaction-diffusion equation on unbounded intervals with Robin boundary conditions, (to appear).
Assume Problem (1.1) can be transformed as follows:

Let

\[ P(x,t) = P(y + s(t),t) = u(y,t), \quad Q(x,t) = Q(y + s(t),t) = v(y,t), \]

Then

\[ P_t = u_t - s'(t)u_y, \quad Q_t = v_t - s'(t)v_y, \quad P_x = u_y, \quad Q_x = v_y, \quad P_{xx} = u_{yy}, \quad Q_{xx} = v_{yy}. \]

Problem (1.1) can be transformed as follows:

\[
\begin{cases}
  u_t - d_1 u_{yy} - s'(t)u_y = f(u), & y < 0, \quad t > 0, \\
  v_t - d_2 v_{yy} - s'(t)v_y = g(v), & y > 0, \quad t > 0, \\
  u(0,t) = v(0,t) = 0, & t > 0, \\
  s'(t) = -\mu_1 u_y(0,t) - \mu_2 v_y(0,t), & t > 0, \\
  s(0) = s_0, \quad s_0 \in (-\infty, \infty), \\
  u(y,0) = P_0(y + s_0)(y < 0), \quad v(y,0) = Q_0(y + s_0)(y > 0).
\end{cases}
\]  

Assume

\[ P_0 \in C^2((-\infty, s_0]), \quad Q_0 \in C^2([s_0, \infty)), \quad P_0(s_0) = Q_0(s_0) = 0. \]
Define
\[ D_{1T} = (-\infty, 0) \times (0, T), \quad D_{2T} = (0, \infty) \times (0, T), \quad D_{1T} = (-\infty, 0] \times [0, T], \quad D_{2T} = [0, \infty) \times [0, T], \]
\[ D_1 = \{ u(y, t) \in C(D_{1T}), u(y, 0) = P_0(y + s_0) \}, \quad \tilde{D}_{1T} = \{ u \in D_1, \sup_{y \leq 0, t \in [0, T]} |u(y, t) - u(y, 0)| \leq 1 \}, \]
\[ D_2 = \{ v(y, t) \in C(D_{2T}), v(y, 0) = Q_0(y + s_0) \}, \quad \tilde{D}_{2T} = \{ v \in D_2, \sup_{y \geq 0, t \in [0, T]} |v(y, t) - v(y, 0)| \leq 1 \}, \]
\[ D_3 = \{ s \in C^1[0, T], s(0) = s_0, s'(0) = -\mu_1 P_0'(s_0) - \mu_2 Q_0'(s_0) \}, \quad \tilde{D}_{3T} = \{ s \in D_3, \sup_{t \in [0, T]} |s'(t) - s'(0)| \leq 1 \}, \]
then \( D = \tilde{D}_{1T} \times \tilde{D}_{2T} \times \tilde{D}_{3T} \) is a completed metric space with the distance defined as follows:
\[ d((u_1, v_1, s_1), (u_2, v_2, s_2)) = \|u_1 - u_2\|_{C(\tilde{D}_{1T})} + \|v_1 - v_2\|_{C(\tilde{D}_{2T})} + \|s_1' - s_2'\|_{C([0, T])}, \]
for \( s_1, s_2 \in \tilde{D}_{3T}, s_1(0) = s_2(0) = s_0 \), we have \( \|s_1 - s_2\|_{C([0, T])} \leq T \|s_1' - s_2'\|_{C([0, T])} \).

Next we will prove the existence and uniqueness result by using the contraction mapping theorem. Applying \( L^p \) theory and Sobolev embedding theorem, for any \((u, v, s) \in D\), the following problem
\[
\begin{align*}
\{ \quad & \bar{u}_t - d_1 \bar{u}_{yy} - s'(t) \bar{u}_y = f(u), & y < 0, & t > 0, \\
& \bar{u}(0, t) = 0, & t > 0, \\
& \bar{u}(y, 0) = P_0(y + s_0), & y \leq 0,
\end{align*}
\]
has a unique solution \( \bar{u} \in C^{1+\alpha, 1+\alpha/2}(\tilde{D}_{1T}) \).

In the same way, the following problem
\[
\begin{align*}
\{ \quad & \bar{v}_t - d_2 \bar{v}_{yy} - s'(t) \bar{v}_y = g(v), & y > 0, & t > 0, \\
& \bar{v}(0, t) = 0, & t > 0, \\
& \bar{v}(y, 0) = Q_0(y + s_0), & y \geq 0,
\end{align*}
\]
has a unique solution \( \bar{v} \in C^{1+\alpha, 1+\alpha/2}(\tilde{D}_{2T}) \). Moreover, we have
\[ \|\bar{u}\|_{C^{1+\alpha, 1+\alpha/2}(\tilde{D}_{1T})} + \|\bar{v}\|_{C^{1+\alpha, 1+\alpha/2}(\tilde{D}_{2T})} \leq C_1, \]
where \( C_1 \) depends on \( \alpha, |s'(0)|, \|P_0\|_{C^{1+\alpha}((-\infty, 0])}, \|Q_0\|_{C^{1+\alpha}(0, \infty)} \). Define new free boundary
\[ \bar{s}(t) = -\mu_1 \int_0^t \bar{u}_y(0, \tau)d\tau - \mu_2 \int_0^t \bar{v}_y(0, \tau)d\tau, \]
so, \( \bar{s}'(t) = -\mu_1 \bar{u}_y(0, t) - \mu_2 \bar{v}_y(0, t), \bar{s}(0) = 0, \bar{s}' \in C^2([0, T]) \) and \( \|\bar{s}'\|_{C^2([0, T])} \leq C_2 \), where \( C_2 \) depends on \( \alpha, |s'(0)|, \|P_0\|_{C^{1+\alpha}((-\infty, 0])}, \|Q_0\|_{C^{1+\alpha}(0, \infty)} \).

Defining a mapping \( F : D \rightarrow C(\tilde{D}_{2T}) \times C(\tilde{D}_{2T}) \times C^1([0, T]) \), by
\[ F(u, v, s) = (\bar{u}, \bar{v}, \bar{s}), \]
Obviously, \((u, v, s)\) is the fixed point of \( F \) if and only if \((u, v, s)\) is the solution of (3.17). Noting that
\[ |\bar{u}(y, t) - u(y, 0)| \leq \sup_{y \leq 0, t \in [0, T]} \frac{|\bar{u}(y, t) - u(y, 0)|}{t^{1+\alpha}} \leq T^{1+\alpha} \|\bar{u}\|_{C^{1+\alpha, 1+\alpha/2}(\tilde{D}_{1T})}. \]
Therefore, for any $y \leq 0, t \in [0, T]$, we have
\[
\sup_{y \leq 0, t \in [0, T]} |\overline{v}(y, t) - u(y, 0)| \leq C_1 T^{1+\alpha}.
\]
In the same way, we know that
\[
\sup_{y \geq 0, t \in [0, T]} |\overline{v}(y, t) - v(y, 0)| \leq C_1 T^{1+\alpha}.
\]
Moreover,
\[
\overline{v}'(t) - s'(0) = -\mu_1 \overline{y}_y(0, t) - \mu_2 \overline{y}_y(0, t) + \mu_1 u_y(0, 0) + \mu_2 v_y(0, 0),
\]
Consequently,
\[
\sup_{t \in [0, T]} |\overline{v}'(t) - s'(0)| \leq \mu_1 \sup_{t \in [0, T]} |\overline{y}_y(0, t) - u_y(0, 0)| + \mu_2 \sup_{t \in [0, T]} |\overline{y}_y(0, t) - v_y(0, 0)|
\]
\[
\leq \mu_1 T^{\frac{\alpha}{2}} \|\overline{v}\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{D}_1 T)} + \mu_2 T^{\frac{\alpha}{2}} \|\overline{v}\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{D}_2 T)}
\]
\[
\leq (\mu_1 + \mu_2) C_1 T^{\frac{\alpha}{2}}.
\]
Choose $T \leq \min\{((\mu_1 + \mu_2) C_1)^{-\frac{2}{\alpha}}, C_1^{-\frac{2}{1+\alpha}}\}$, then the mapping $F$ maps $D$ into itself.

Next, we prove $F$ is a contraction mapping if $T$ is sufficiently small. For $(u_i, v_i, s_i) \in D$, denote $(\overline{u}_i, \overline{v}_i, \overline{s}_i) = F(u_i, v_i, s_i)$, then
\[
\|\overline{u}_i\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{D}_1 T)} \leq C_1, \|\overline{v}_i\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{D}_2 T)} \leq C_1, \|\overline{s}_i\|_{C^\frac{\alpha}{2}(0, T)} \leq C_2.
\]
Denote $U = \overline{u}_1 - \overline{u}_2$, then
\[
U_t - d_i U_{yy} - s'_i(t) U_y = f(u_1) - f(u_2) + (s'_i(t) - s'_2(t)) \overline{u}_{2y}, \quad y > 0, \ t > 0,
\]
\[
U(0, t) = 0, \quad U(y, 0) = 0,
\]
Applying $L^p$ estimate and Sobolev embedding theorem
\[
\|\overline{u}_1 - \overline{u}_2\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{D}_1 T)} \leq C_3 \left(\|u_1 - u_2\|_{C(\overline{D}_1 T)} + \|s_1 - s_2\|_{C([0, T])}\right),
\]
where $C_3$ depends on $\alpha, |s'(0)|, \|P_0\|_{C^{1+\alpha}((\infty, 0])}$ and $\|Q_0\|_{C^{1+\alpha}([0, \infty))}$.

In the same way, let $V = \overline{v}_1 - \overline{v}_2$, then
\[
\|\overline{v}_1 - \overline{v}_2\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{D}_2 T)} \leq C_4 \left(\|v_1 - v_2\|_{C(\overline{D}_2 T)} + \|s_1 - s_2\|_{C([0, T])}\right),
\]
where $C_4$ depends on $\alpha, |s'(0)|, \|P_0\|_{C^{1+\alpha}((\infty, 0])}$ and $\|Q_0\|_{C^{1+\alpha}([0, \infty))}$. Assume $T \leq 1$, we have
\[
\|\overline{v}_1 - \overline{v}_2\|_{C^\frac{\alpha}{2}([0, T])} \leq C_5 \left(\|u_1 - u_2\|_{C(\overline{D}_1 T)} + \|v_1 - v_2\|_{C(\overline{D}_2 T)} + \|s_1 - s_2\|_{C([0, T])}\right),
\]
where $C_5$ depends on $\alpha, |s'(0)|, \|P_0\|_{C^{1+\alpha}((\infty, 0])}$, $\|Q_0\|_{C^{1+\alpha}([0, \infty))}$, $\mu_1$ and $\mu_2$. 

So, if we take
\[ T := \min \left\{ 1, \left( (\mu_1 + \mu_2)C_1 \right) - \frac{\Delta}{2}, C_1^{\frac{1}{1+\alpha}}, \left( \frac{1}{2C_5} \right)^{\frac{1}{2}} \right\}, \]
then
\[
\| \bar{u}_1 - \bar{u}_2 \|_{C(D_{1T})} + \| \bar{v}_1 - \bar{v}_2 \|_{C(D_{2T})} + \| \bar{s}'_1 - \bar{s}'_2 \|_{C([0,T])} \leq T^{\frac{1+\alpha}{2}} \left( \| \bar{u}_1 - \bar{u}_2 \|_{C_1^{1+\alpha} (D_{1T})} + \| \bar{v}_1 - \bar{v}_2 \|_{C_1^{1+\alpha} (D_{2T})} \right) + T^{\frac{\Delta}{2}} \| \bar{s}'_1 - \bar{s}'_2 \|_{C^\Delta ([0,T])} \]
\[
\leq C_5 T^{\frac{\Delta}{2}} \left( \| u_1 - u_2 \|_{C(D_{1T})} + \| v_1 - v_2 \|_{C(D_{2T})} + \| s'_1 - s'_2 \|_{C([0,T])} \right) \]
\[
\leq \frac{1}{2} \left( \| u_1 - u_2 \|_{C(D_{1T})} + \| v_1 - v_2 \|_{C(D_{2T})} + \| s'_1 - s'_2 \|_{C([0,T])} \right). \]

For such \( T \), \( \mathcal{F} \) is a contraction mapping on \( D \). By contraction mapping theorem, \( \mathcal{F} \) has a unique fixed point \((u, v, s) \in D\). \( \square \)

**Theorem 3.9.** Under the assumption of Theorem 2.1, the unique solution of the problem (1.1) exists, and it can be extended to \([0, T_{\text{max}}] \), where \( T_{\text{max}} \leq \infty \).

**Proof.** In order to prove the present theorem, we argue it indirectly. Assume that \( T_{\text{max}} < \infty \). Since \( |s'(t)| \leq H \) in \([0, T_{\text{max}}]\), using bootstrap argument and Schauder's estimate yields a priori bound of \( |P(x, t)|_{C^{1+\alpha}([-\infty, s(t)])} + |Q(x, t)|_{C^{1+\alpha}([s(t), \infty])} \) for all \( t \in [0, T_{\text{max}}] \). Let the bound be \( C_6 \).

It follows from the proof of Theorem 3.8 that there exists a \( \tau \) depending only on \( T_{\text{max}}, C_1 \) such that the solution of the problem (1.1) with the initial time \( T_{\text{max}} - \frac{\tau}{2} \) can be extended uniquely to the time \( T_{\text{max}} - \frac{\tau}{2} + \tau \) that contradicts the assumption. This completes the proof. \( \square \)