ASSOCIATED PRIMES OF GRADED COMPONENTS OF LOCAL
COHOMOLOGY MODULES

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ABSTRACT. The $i$-th local cohomology module of a finitely generated graded module $M$ over a standard positively graded commutative Noetherian ring $R$, with respect to the irrelevant ideal $R_+$, is itself graded; all its graded components are finitely generated modules over $R_0$, the component of $R$ of degree 0. It is known that the $n$-th component $H^n_{R_+}(M)$ of this local cohomology module $H^n_{R_+}(M)$ is zero for all $n >> 0$. This paper is concerned with the asymptotic behaviour of $\text{Ass}_{R_0}(H^n_{R_+}(M))$ as $n \to -\infty$.

The smallest $i$ for which such study is interesting is the finiteness dimension $f$ of $M$ relative to $R_+$, defined as the least integer $j$ for which $H^j_{R_+}(M)$ is not finitely generated. Brodmann and Hellus have shown that $\text{Ass}_{R_0}(H^f_{R_+}(M))$ is constant for all $n << 0$ (that is, in their terminology, $\text{Ass}_{R_0}(H^f_{R_+}(M))$ is asymptotically stable for $n \to -\infty$). The first main aim of this paper is to identify the ultimate constant value (under the mild assumption that $R$ is a homomorphic image of a regular ring): our answer is precisely the set of contractions to $R_0$ of certain relevant primes of $R$ whose existence is confirmed by Grothendieck’s Finiteness Theorem for local cohomology.

Brodmann and Hellus raised various questions about such asymptotic behaviour when $i > f$. They noted that Singh’s study of a particular example (in which $f = 2$) shows that $\text{Ass}_{R_0}(H^3_{R_+}(R))$ need not be asymptotically stable for $n \to -\infty$. The second main aim of this paper is to determine, for Singh’s example, $\text{Ass}_{R_0}(H^f_{R_+}(R))$ quite precisely for every integer $n$, and thereby, answer one of the questions raised by Brodmann and Hellus.

0. Introduction

Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a positively graded commutative Noetherian ring which is standard in the sense that $R = R_0[R_1]$, and set $R_+ := \bigoplus_{n \in \mathbb{N}} R_n$, the irrelevant ideal of $R$. (Here, $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of non-negative and positive integers respectively; $\mathbb{Z}$ will denote the set of all integers.) Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a non-zero finitely generated graded $R$-module. This paper is concerned with the behaviour of the graded components of the graded local cohomology modules $H^i_{R_+}(M)$ ($i \in \mathbb{N}_0$) of $M$ with respect to $R_+$.

It is known (see [B-S, 15.1.5]) that there exists $r \in \mathbb{Z}$ such that $H^i_{R_+}(M)_n = 0$ for all $i \in \mathbb{N}_0$ and all $n \geq r$, and that $H^i_{R_+}(M)_n$ is a finitely generated $R_0$-module for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. Set

$$f := f_{R_+}(M) = \inf \left\{ i \in \mathbb{N} : H^i_{R_+}(M) \text{ is not finitely generated} \right\},$$

the finiteness dimension of $M$ relative to $R_+$: see [B-S, 9.1.3]. We assume that $f$ is finite. M. Brodmann and M. Hellus have shown in [B-H, Proposition 5.6] that $\text{Ass}_{R_0}(H^f_{R_+}(M)_n)$ is constant for all $n << 0$.

The first part of this paper determines the ultimate constant value under the mild restriction that $R$ is a homomorphic image of a regular (commutative Noetherian) ring; the main result is related to Grothendieck’s Finiteness Theorem for local cohomology, which (under the specified restriction) gives an alternative description of $f$. Let $^\ast \text{Spec}(R)$ denote the set of graded prime ideals of $R$, and $\text{Proj}(R)$ denote the set $\{ p \in ^\ast \text{Spec}(R) : p \not\supseteq R_+ \}$. Write

$$\lambda^{R_+}(M) := \inf \left\{ \text{depth}_{R_p} M_p + \text{ht}(R_+ + p)/p : p \in \text{Proj}(R) \right\}.$$
(We interpret the depth of a zero module as $\infty$.) It is a consequence of Grothendieck’s Finiteness Theorem [8, Exposé VIII, Corollaire 2.3] that, when $R$ is a homomorphic image of a regular ring,

$$f = \lambda^R_{R_+}(M) = \inf \left\{ \text{depth}_{R_+} M_p + \text{ht}(R_+ + p)/p : p \in \text{Proj}(R) \right\}.$$  

(See [B-S, 13.1.17].) The main result of [7] is that, under the assumption that $R$ is a homomorphic image of a regular ring,

$$\left\{ p \cap R_0 : p \in \text{Proj}(R) \text{ and } \text{depth}_{R_+} M_p + \text{ht}(p + R_+)/p = f \right\} = \text{Ass}_{R_0}(H^d_{R_+}(M)_n) \text{ for all } n << 0.$$  

The final [7] is concerned with the asymptotic behaviour of $\text{Ass}_{R_0}(H^d_{R_+}(M)_n)$ as $n \to -\infty$ when $i > f$. Brodmann and Hellus say that $\text{Ass}_{R_0}(H^d_{R_+}(M)_n)$ is asymptotically stable (respectively asymptotically increasing) for $n \to -\infty$ if there exists $n_0 \in \mathbb{Z}$ such that $\text{Ass}_{R_0}(H^d_{R_+}(M)_n) = \text{Ass}_{R_0}(H^d_{R_+}(M)_{n_0})$ (respectively $\text{Ass}_{R_0}(H^d_{R_+}(M)_n) \subseteq \text{Ass}_{R_0}(H^d_{R_+}(M)_{n+1})$) for all $n \leq n_0$. They used an example of A. Singh [8, §4] to show that, when $i > f$, $\text{Ass}_{R_0}(H^d_{R_+}(M)_n)$ need not be asymptotically stable for $n \to -\infty$. In §3 we use Gröbner basis techniques to show that, for Singh’s example

$$R = \mathbb{Z}[X, Y, Z, U, V, W]/(XU + YV + ZW),$$

where the polynomial ring $\mathbb{Z}[X, Y, Z, U, V, W]$ is graded so that its 0-th component is $\mathbb{Z}[X, Y, Z]$ and $U, V, W$ have degree 1, we have

$$\text{Ass}_{R_0}(H^d_{R_+}(R)_n) = \{(X, Y, Z) \cup \{(p, X, Y, Z) : p \in \Pi(d - 2)\} \text{ for all } d \geq 3,$$

where

$$\Pi(d - 2) := \left\{ p : p \text{ is a prime factor of } \binom{d - 2}{i} \text{ for some } i \in \{0, \ldots, d - 2\} \right\}.$$  

It follows that $\text{Ass}_{R_0}(H^d_{R_+}(R)_n)$ is not asymptotically increasing for $n \to -\infty$, and this settles a question raised by Brodmann and Hellus.

1. Asymptotic behaviour at the finiteness dimension

1.1. Notation. The notation introduced in the above [7] will be maintained for the whole paper. We shall only assume that $R$ is a homomorphic image of a regular ring when this is explicitly stated. Here we introduce additional notation.

We use, for $j \in \mathbb{Z}$, the notation $L_j$ to denote the $j$-th component of a $\mathbb{Z}$-graded module $L$, and $(\ast)(j)$ to denote the $j$-th shift functor on the category of graded $R$-modules and homogeneous homomorphisms (by ‘homogeneous’ here, we mean ‘homogeneous of degree zero’). It will be convenient to have available the concepts of the end and beginning (beg($L$)) of the graded $R$-module $L = \bigoplus_{n \in \mathbb{Z}} L_n$, which are defined by

$$\text{end}(L) := \sup \{ n \in \mathbb{Z} : L_n \neq 0 \} \quad \text{and} \quad \text{beg}(L) := \inf \{ n \in \mathbb{Z} : L_n \neq 0 \}.$$  

(Not that end($L$) could be $\infty$, and that the supremum of the empty set of integers is to be taken as $-\infty$; similar comments apply to beg($L$).)

For $p \in \text{Spec}(R)$, we abbreviate $\text{depth}_{R_+} M_p$ by $\text{depth} M_p$ and the projective dimension $\text{proj} \dim_{R_+} M_p$ by $\text{proj} \dim M_p$.

1.2. Lemma. (The notation is as in [7] and [10]) Let $p \in \text{Proj}(R) \cap \text{Ass}_R M$ be such that $\text{ht}(p + R_+)/p = 1$. Set $p_0 = p \cap R_0$. Then $p_0 \in \text{Ass}_{R_0}(H^d_{R_+}(M)_n)$ for all $n < \text{beg}(M)$.

Proof. Set $\overline{M} := M/\Gamma_{R_+}(M)$, and note that, by [B-S, 2.1.12 and 2.1.7(iii)],

$$\text{Ass}_R(\overline{M}) = \text{Proj}(R) \cap \text{Ass}_R M$$

and there is a homogeneous isomorphism $H^d_{R_+}(M) \cong H^d_{R_+}(\overline{M})$. We therefore can, and do, assume that $\Gamma_{R_+}(M) = 0$ in the remainder of this proof.

We now use homogeneous localization at $p + R_+$ to see that is is enough to prove the claim under the additional hypotheses that $R$ is *local with unique *maximal ideal $m$, and that $m_0 := m \cap R_0 = p_0$. The assumptions that $R$ is standard and *local with $m_0 = p_0$, and that $\text{ht}(p + R_+)/p = 1$, ensure that there exists $g_1 \in R_1 \setminus p$, and that, then, $\sqrt{p} + g_1 R = p + R_+$.

Now there exists $t \in \mathbb{Z}$ such that $M$ has a graded $R$-submodule $N$ homogeneously isomorphic to $(R/p)(-t)$. We now consider the ideal transform $D_{R_0}(N)$ of $N$ with respect to $Rg_1$: this is naturally graded, and since $g_1$ is a non-zerodivisor on $R/p$, the description of this ideal transform afforded
Proof. Observe that there are
$R$-modules and homogeneous $R$-homomorphisms.

For part of the proof of our main result of this section, we shall be able to reduce to the case where $R_0$ is a regular local ring and $R = R_0[X_1, \ldots, X_r]$ is a polynomial ring over $R_0$ in which the independent indeterminates $X_1, \ldots, X_r$ all have degree 1. This explains why several subsequent lemmas are concerned with this case.

1.3. Lemma. The notation is as in [§1] and [1.1]. In addition, suppose that $(R_0, m_0)$ is a regular local ring of dimension $d$ and that $R = R_0[X_1, \ldots, X_r]$, a polynomial ring graded in the usual way. Suppose that $\mathfrak{p} \in \text{Supp}(M) \cap \text{Proj}(R)$ is such that $\mathfrak{p} \cap R_0 = m_0$. Then

$$\text{depth } M_\mathfrak{p} + \text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} = d + r - \text{proj dim } M_\mathfrak{p}.$$ 

Proof. As $R$ is a catenary domain,

$$\text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} = \text{ht}(\mathfrak{p} + R_+) - \text{ht } \mathfrak{p} = d + r - \text{ht } \mathfrak{p}.$$ 

Moreover, by the Auslander-Buchsbaum-Serre Theorem,

$$\text{depth } M_\mathfrak{p} = \dim R_\mathfrak{p} - \text{proj dim } M_\mathfrak{p} = \text{ht } \mathfrak{p} - \text{proj dim } M_\mathfrak{p}.$$ 

1.4. Lemma. The notation is as in [§1] and [1.1]. In addition, suppose that $(R_0, m_0)$ is a regular local ring of dimension $d$ and that $R = R_0[X_1, \ldots, X_r]$, a polynomial ring graded in the usual way.

Let $(R_0', m_0')$ be a regular flat extension ring of $R_0$ such that $m_0 R_0' = m_0'$. Let $R' = R \otimes_{R_0} R_0'$, which we identify with $R_0'[X_1, \ldots, X_r]$ in the obvious way. Let $M'$ denote the finitely generated graded $R'$-module $M \otimes_R R'$, and let $\mathfrak{p}' \in \text{Proj}(R')$ be such that $\mathfrak{p}' \cap R_0' = m_0'$. Set $\mathfrak{p} := \mathfrak{p}' \cap R$. Then $\mathfrak{p} \in \text{Proj}(R)$ and $\mathfrak{p} \cap R_0 = m_0$; also

$$\text{depth } M_\mathfrak{p} + \text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} \leq \text{depth } M_{\mathfrak{p}'} + \text{ht}(\mathfrak{p}' + R_+)/\mathfrak{p}'.$$ 

Proof. Observe that there are $R_{\mathfrak{p}'}$-isomorphisms

$$M_{\mathfrak{p}'} \cong (M \otimes_R R') \otimes_{R'} R_{\mathfrak{p}'} \cong M \otimes_R R_{\mathfrak{p}'} \cong M_{\mathfrak{p}} \otimes_{R_0} R_{\mathfrak{p}'}.$$ 

As $R_{\mathfrak{p}'}$ is a flat $R_\mathfrak{p}$-algebra, proj dim $M_{\mathfrak{p}'} \leq \text{proj dim } M_\mathfrak{p}$. Hence, by two uses of Lemma 1.3,

$$\text{depth } M_\mathfrak{p} + \text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} = d + r - \text{proj dim } M_\mathfrak{p} \leq d + r - \text{proj dim } M_{\mathfrak{p}'} \leq \text{depth } M_{\mathfrak{p}'} + \text{ht}(\mathfrak{p}' + R_+)/\mathfrak{p}'.$$ 

1.5. Lemma. The notation is as in [§1] and [1.1]. In addition, suppose that $(R_0, m_0)$ is a regular local ring of dimension $d$ such that the field $R_0/m_0$ is algebraically closed and that $R = R_0[X_1, \ldots, X_r]$, a polynomial ring graded in the usual way.

Suppose that $r > 1$, that $f_{R_+}(M) = r$ and that $m_0 \in \text{Ass}_{R_0}(H^r_{R_+}(M)_n)$ for all $n << 0$. Then there exists $y \in R_1 \setminus m_0 R_1$ such that $y$ is a non-zerodivisor on $M/\Gamma_{R_+}(M)$, that $f_{R_+}(M/yM) = r - 1$ and that

$$m_0 \in \text{Ass}_{R_0}(H^{r-1}_{R_+}(M/yM)_n) \quad \text{for all } n << 0.$$
Proof. Set $\overline{M} := M/\Gamma_{R_+}(M)$. For a homogeneous element $y$ of $R$, we have homogeneous isomorphisms
\[ \overline{M}/y\overline{M} \cong M/(yM + \Gamma_{R_+}(M)) \cong (M/yM)/(yM + \Gamma_{R_+}(M))/yM, \]
so that there are homogeneous isomorphisms $H^i_{R_+}(M) \cong H^i_{R_+}(\overline{M})$ and $H^i_{R_+}(M/yM) \cong H^i_{R_+}(\overline{M}/y\overline{M})$ for all $i > 0$. We may therefore replace $M$ by $\overline{M}$. We therefore assume that $\Gamma_{R_+}(M) = 0$ and $\text{Ass}_R M \subseteq \text{Proj}(R)$.

Now let $p \in \text{Ass}_R M$ and set $p_0 := p \cap R_0$. Then, since $R$ is a regular, and therefore catenary, domain, $\text{ht}_{R_0} p_0 + r - \text{ht} p = \text{ht}(p_0 R + R_+) - \text{ht} p$ Therefore, as we have already noted that $m \in R$ is injective for all $R$-modules and homogeneous homomorphisms in which $m$ is an isomorphism when $i > 0$, we must have $\text{ht}_{R_0} p_0 = \text{ht} p$ and $p_0 R_+ \subseteq m_0 R_+$.

It therefore follows that, if we let $U$ denote the subset of $R_1 \setminus m_0 R_1$ defined by
\[ U := \{ a_1 X_1 + a_2 X_2 : (a_1, a_2) \in R_0 \times R_0 \setminus (m_0 \times m_0) \}, \]
then $U \cap p = \emptyset$. Therefore each element of $U$ is a non-zero divisor on $M$.

Set $J := \bigcap_{n \in \mathbb{Z}} \Gamma_{m_0}(H^{-1}_{R_+}(M/xM))^n$. The hypotheses ensure that $J$ is not a finitely generated $R$-module. We shall show that one of the elements of $U$ can be taken for $y$. To achieve this, we suppose that, for each $x \in U$, there exists $n_x \in \mathbb{Z}$ such that, for all $n \leq n_x$, it is the case that $m_0 \notin \text{Ass}_{R_0}(H^{-1}_{R_+}(M/xM))^n$, and we seek a contradiction.

This supposition means that, for each $x \in U$, we have $\Gamma_{m_0}(H^{-1}_{R_+}(M/xM))^n = 0$ for all $n \leq n_x$. Since $\text{ht}_{R_+}(M) = r$, there exists $\bar{n} \in \mathbb{Z}$ such that $H^{-1}_{R_+}(M)^n = 0$ for all $n \leq \bar{n}$. For each $x \in U$, the application of local cohomology with respect to $R_+$ to the exact sequence
\[ 0 \rightarrow M(1) \rightarrow M \rightarrow M/xM \rightarrow 0 \]
shows that $f_{R_+}(M/xM) \geq r - 1$ and leads to an exact sequence of $R_0$-modules
\[ 0 \rightarrow H^{-1}_{R_+}(M/xM)^n \rightarrow H^{-1}_{R_+}(M)^{n-1} \rightarrow H^{-1}_{R_+}(M)^n \]
for each $n \leq \bar{n}$. The left exactness of the functor $\Gamma_{m_0}$ therefore leads to the conclusion that, for each $x \in U$, the map
\[ J_{n-1} = \Gamma_{m_0}(H^{-1}_{R_+}(M)^{n-1}) \rightarrow J_n = \Gamma_{m_0}(H^{-1}_{R_+}(M)^n) \]
is injective for all $n \leq \min(\bar{n}, n_x)$. Hence $(0 : x)$ is an $R$-module of finite length, for all $x \in U$. Since $R_0/m_0$ is algebraically closed, we can now deduce from [2, Corollary (2.2)] that $J$ is an $R$-module of finite length, and this is a contradiction. We have therefore proved that there exists $y \in U$ such that $m_0 \in \text{Ass}_{R_0}(H^{-1}_{R_+}(M/yM))^n$ for infinitely many $n < 0$. This implies that $f_{R_+}(M/yM) \leq r - 1$; therefore, as we have already noted that $f_{R_+}(M/yM) \geq r - 1$, we must have $f_{R_+}(M/yM) = r - 1$. Hence, by [2, Proposition (5.6)], $\text{Ass}_{R_0}(H^{-1}_{R_+}(M/yM))^n$ is asymptotically stable for $n \rightarrow -\infty$; therefore $m_0 \in \text{Ass}_{R_0}(H^{-1}_{R_+}(M/yM))^n$ for all $n < 0$.

\[ \square \]

1.6. Lemma. The notation is as in [3] and [4]. In addition, suppose that $(R_0, m_0)$ is a regular local ring of dimension $d$ and that $R = R_0[\{X_1, \ldots, X_r\}$, a polynomial ring graded in the usual way.

Assume that $f_{R_+}(M) < r$ and that $0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0$ is an exact sequence of finitely generated graded $R$-modules and homogeneous homomorphisms in which $F$ is free. Then
\begin{enumerate}
  \item depth $N$ is $\min \{ \text{ht} p, \text{depth} M_0 + 1 \}$ for all $p \in \text{Supp}(N)$;
  \item for $i \in \mathbb{N}$, the (necessarily homogeneous) connecting homomorphism $H^i_{R_+}(M) \rightarrow H^{i+1}_{R_+}(N)$ induced by the given exact sequence is an isomorphism when $i < r - 1$ and a monomorphism when $i = r - 1$; and
  \item $f_{R_+}(N) = f_{R_+}(M) + 1$.
\end{enumerate}

Proof. Note that $N \neq 0$ because $f_{R_+}(F) = r$.

(i) This is immediate from the exact sequence $0 \rightarrow N_p \rightarrow F_p \rightarrow M_p \rightarrow 0$.

(ii) This is immediate from the fact that $H^i_{R_+}(F) = 0$ for all $i < r$.

(iii) This now follows from part (ii) and the hypothesis that $f_{R_+}(M) < r$. \[ \square \]
1.7. Lemma. Assume that \((R_0, m_0)\) is a regular local ring. Then there exists a regular local flat extension ring \((R'_0, m'_0)\) of \(R_0\) such that \(m_0R'_0 = m'_0\) and \(R'_0/m'_0\) is algebraically closed.

Proof. Denote as usual \(\dim R_0 = d\). Let \((\widehat{R_0}, \widehat{m_0})\) denote the completion of \(R_0\), so that \(\widehat{m_0} = m_0\widehat{R_0}\); of course, this is a regular local flat extension ring of \(R_0\) of dimension \(d\). By [B-M-M, Proposition (2.2)], there exists a (Noetherian) local flat extension ring \((R'_0, m'_0)\) of \(\widehat{R_0}\) such that \(m_0\widehat{R_0} = m'_0\) and \(R'_0/m'_0\) is algebraically closed. Therefore \(m_0R'_0 = m'_0\), so that \(m_0\widehat{R_0}m'_0\) can be generated by \(d\) elements. By flatness, \(\dim R'_0 \geq d\), and so \((R'_0, m'_0)\) is a regular local ring of dimension \(d\). \(\square\)

We are now ready to present our main result of this section.

1.8. Theorem. Assume that the graded ring \(R\) is a homomorphic image of a regular (commutative Noetherian) ring, and that the non-zero graded \(R\)-module \(M = \bigoplus_{n \in \mathbb{Z}} M_n\) is finitely generated and not \(R_+\)-torsion. Set

\[
 f := f_{R_+}(M) = \inf \left\{ i \in \mathbb{N} : H^i_{R_+}(M) \text{ is not finitely generated} \right\}.
\]

Then

\[
 \text{Ass}_{R_0}(H^i_{R_+}(M)_n) = \{ p \cap R_0 : p \in \text{Proj}(R) \text{ and depth } M_p + \text{ht}(p + R_+)/p = f \} \quad \text{for all } n < 0.
\]

Note. By Grothendieck’s Finiteness Theorem (see [B-S, 13.1.17]), the set on the right-hand side of the final display in the statement of the theorem is non-empty: note that \(f\) is finite. A consequence of this theorem is that set is finite.

Proof. We first show by induction on \(f\) that, for \(p \in \text{Proj}(R)\) with \(\text{ht}(p + R_+)/p = f\), we have \(p \cap R_0 \in \text{Ass}_{R_0}(H^i_{R_+}(M)_n)\) for all \(n < 0\). Now \(\text{ht}(p + R_+)/p \geq 1\); so, if \(f = 1\) and \(\text{depth } M_p + \text{ht}(p + R_+)/p = 1\), then \(p \in \text{Ass}_{R_0} M\). The claim in the case when \(f = 1\) is therefore immediate from Lemma [1.2].

Thus we assume now that \(f > 1\) and make the obvious inductive assumption. One can use homogeneous localization at \(p + R_+\) to see that it is enough to complete the inductive step under the additional hypotheses that \(R\) is *local with unique *maximal ideal \(m\), and that \(m_0 := m \cap R_0 = p_0\).

Set \(\widehat{M} := M/\Gamma_{R_+}(M)\); recall ([B-S, 2.1.7]) that there are homogeneous isomorphisms \(H^i_{R_+}(M) \xrightarrow{\sim} H^i_{R_+}(\widehat{M})\) for each \(i \in \mathbb{N}\). Since \(M_p \cong \widehat{M}_p\), it follows that one may assume, in this inductive step, that \(\Gamma_{R_+}(M) = 0\).

The argument now splits into two cases, according as \(p \in \text{Ass}_R M\) or \(p \not\in \text{Ass}_R M\). In the first case, it follows from [B-S, 15.1.2] that there exists a positive integer \(d\) and a homogeneous element \(q_d \in R_d\) which is a non-zerodivisor on \(M\). Let \(q\) be a minimal prime ideal of \(p + R_d\); necessarily, \(q \in \text{Proj}(R)\) (since \(f > 1\)), and \(q \cap R_0 = m_0\). The catenarity of \(R\) ensures that \(\text{ht}(q + R_+)/q) = f - 1\). It follows from [M, Chapter 6, Lemma 4] that \(q \in \text{Ass}(M/q_dM)\), and so one can use Grothendieck’s Finiteness Theorem (see [B-S, 9.5.2]) to see that \(f_{R_+}(M/q_dM) \leq 0 + \text{ht}(q + R_+)/q) = f - 1\).

In the second case, when \(p \not\in \text{Ass}_R M\), we choose \(q_d\) as follows. First note that, for each \(q' \in \text{Ass}_R M\), we have \(p \cap R_+ \not\subseteq q'\). To see this, suppose that \(p \cap R_+ \subseteq q'\) for some \(q' \in \text{Ass}_R M\). Then \(p \subseteq q'\) (since \(\Gamma_{R_+}(M) = 0\)), so that (since \(q' \cap R_0 \supseteq p \cap R_0 = m_0\), we have

\[
 \text{ht}(q' + R_+)/q')) = \text{ht}(m/q') < \text{ht}(m/p) = \text{ht}(p + R_+)/p).
\]

This implies that \(\text{depth } M_p + \text{ht}(q' + R_+)/q')) < f - 1\), contrary to Grothendieck’s Finiteness Theorem. We have therefore shown that \(p \cap R_+ \not\subseteq q'\). As this is true for all \(q' \in \text{Ass}_R M\), we can now use [B-S, 15.1.2] to see that there exists a positive integer \(d\) and a homogeneous element \(q_d \in p \cap R_d\) which is a non-zerodivisor on \(M\). Note that \(\text{depth}(M/q_dM)_p = \text{depth } M_p - 1\), so that, by Grothendieck’s Finiteness Theorem,

\[
 f_{R_+}(M/q_dM) \leq \text{depth}(M/q_dM)_p + \text{ht}(p + R_+)/p) = f - 1.
\]

Thus, in both cases, we have found a homogeneous element \(q_d\) of \(R\) of positive degree \(d\) which is a non-zerodivisor on \(M\) and is such that \(f_{R_+}(M/q_dM) \leq f - 1\). Application of local cohomology with respect to \(R_+\) to the exact sequence

\[
 0 \longrightarrow M(-d) \xrightarrow{q_d} M \longrightarrow M/q_dM \longrightarrow 0
\]

shows that \(f_{R_+}(M/q_dM) \geq f - 1\), and that, for all \(n << 0\), the \(R_0\)-module \(H^i_{R_+}(M)_n\) has a submodule isomorphic to \(H^i_{R_+}(M/q_dM)_n+\). It therefore follows that \(f_{R_+}(M/q_dM) = f - 1\) (so that \(M/q_dM\) is not \(R_+\)-torsion), and we can apply the inductive hypothesis to \(M/q_dM\).
In our first case, when \( p \in \mathrm{Ass}_R M \), we have already noted that \( q \in \mathrm{Proj}(R) \cap \mathrm{Ass}(M/gdM) \), that \( q \cap R_0 = m_0 \), and that \( \text{depth}(M/gdM)_q + \text{ht}(q + R_+)/q) = f - 1 \). We therefore use \( q \) to draw a conclusion from the inductive hypothesis.

In our second case, when \( p \not\in \mathrm{Ass}_R M \), we noted that \( \text{depth}(M/gdM)_p + \text{ht}(p + R_+)/p) = f - 1 \); in this case, we use \( p \) to draw a conclusion from the inductive hypothesis.

In both cases, the inductive hypothesis yields that \( m_0 \in \mathrm{Ass}_{R_0}(H_{R_0}^{f-1}(M/gdM)_{n+d}) \) for all \( n < 0 \).

Therefore \( m_0 \in \mathrm{Ass}_{R_0}(H_{R_0}^f(M)_n) \) for all \( n < 0 \), and the inductive step is complete.

We have thus proved that

\[
\mathrm{Ass}_{R_0}(H_{R_0}^f(M)_n) \supseteq \{ p \cap R_0 : p \in \mathrm{Proj}(R) \text{ and } \text{depth}_p + \text{ht}(p + R_+)/p) = f \} \quad \text{for all } n < 0.
\]

To complete the proof, we suppose that \( p_0 \in \mathrm{Ass}_{R_0}(H_{R_0}^f(M)_n) \) for all \( n < 0 \); it is enough for us to show that there exists \( p \in \mathrm{Proj}(R) \) with \( p \cap R_0 = p_0 \) and \( \text{depth}_p + \text{ht}(p + R_+)/p) = f \); our first steps in this direction show that all simplifications are possible.

Invert \( R_0/p_0 \); in other words, apply homogeneous localization at \( p_0 + R_+ \). Observe that the hypotheses imply that \( (R_0)_{p_0} \) is a homomorphic image of a regular local ring \( (R'_0, m'_0) \) and that \( R(R_0+p_0) \) is an image of a polynomial ring \( R' := R'_0[X_1, \ldots, X_n] \), graded in the usual way, under a ring homomorphism which is isomorphic in the sense of [B-S, Definition 13.1.2]. Consider \( M \) as a finitely generated graded \( R' \)-module; we can then use the Graded Independence Theorem [B-S, 13.1.6] to see that it is enough for us to establish the existence of a \( R \)-module; we can then use the Graded Independence Theorem [B-S, 13.1.6] to see that it is enough for us to establish the existence of a \( \Gamma \)-homomorphism \( \varphi \) such that \( \varphi = \varphi \) for all \( n \geq 0 \).

By hypothesis, \( (0 : H_{R_0}^f(M) : m_0) \neq 0 \) for all \( n < 0 \). It therefore follows that the graded \( R \)-module

\[
\Gamma_{m_0 R} \left( H_{R_0}^f(M) \right)
\]

is not finitely generated.

Let

\[
0 \to \ast E^0(M) \xrightarrow{d^0} \ast E^1(M) \xrightarrow{d^1} \ast E^2(M) \to \cdots \to \ast E^i(M) \to \cdots
\]

be the minimal \( \ast \)-injective resolution of \( M \), with associated (necessary homogeneous) augmentation homomorphism \( d^1 : M \to \ast E^1(M) \). Since \( \Gamma_{R_0}(E^0(M)) = 0 \), it follows from [3, Theorem 2.4] that \( \Gamma_{R_0}(E^0(M)) = 0 \), so that \( \Gamma_m(E^0(M)) = 0 \). Therefore

\[
H_{R_0}^1(M) \cong \ker(\Gamma_{R_0}(d^1)) \quad \text{and} \quad H_m^1(M) \cong \ker(\Gamma_m(d^1)).
\]

Here, \( \Gamma_{R_0}(d^1) : \Gamma_{R_0}(E^1(M)) \to \Gamma_{R_0}(E^2(M)) \) is the map induced by \( d^1 \), etcetera. Thus

\[
\Gamma_{m_0 R} \left( H_{R_0}^1(M) \right) \cong \Gamma_{m_0 R}(\ker(\Gamma_{R_0}(d^1))) = \ker(\Gamma_m(d^1)) \cong H_m^1(M).
\]

Therefore, \( H_m^1(M) \) is not finitely generated. Hence, by Grothendieck’s Finiteness Theorem (see [B-S, 13.1.17]), there exists \( p \in \ast \text{Spec}(R) \cap \varphi \text{Var}(m) \) such that \( \text{depth}_p + \text{ht}_p/p = 1 \). This means that \( p \in \text{Ass}_R M \) and \( \text{ht}_M/p = 1 \). Note that \( p \not\in R_+ \), because \( \Gamma_{R_0}(M) = 0 \). Therefore \( p_0 := p \cap R_0 = m_0 \), since otherwise \( m_0 \supseteq p_0 + R_+ \supseteq p \) would be a chain of distinct prime ideals of \( R_0 \), contrary to the fact that \( \text{ht}_M/p = 1 \). The claim is therefore proved in the case where \( r = 1 \).

Now suppose that \( r \geq 2 \), and that the desired result has been proved for smaller values of \( r \). Note that, by Grothendieck’s Finiteness Theorem, it is enough for us to show that there exists \( p \in \text{Proj}(R) \cap \varphi \text{Var}(m_0) \) with \( \text{depth}_p + \text{ht}_p/p = f \).

By Lemma [1, 13.1.8] there exists a regular local flat extension ring \( R'_0, m'_0 \) of \( R_0 \) such that \( m_0 R'_0 = m'_0 \) and \( R'_0/m'_0 \) is algebraically closed. Let \( R' = R \otimes_{R_0} R_0' \), which we identify with \( R'_0[X_1, \ldots, X_n] \) in the obvious way. Let \( M' \) denote the finitely generated graded \( R' \)-module \( M \otimes_{R_0} R' \). It follows from [B-S, 13.1.8 and 15.2.2] that \( f_{R_0'}(M') = f \) and \( m'_0 \in \text{Ass}_{R_0'}(H_{R_0'}^{f-1}(M')_{n+d}) \) for all \( n < 0 \).

Suppose that we have found \( p' \in \text{Proj}(R') \cap \varphi \text{Var}(m'_0 R') \) such that \( \text{depth}_{R'}(p' + R'_+)/p' \leq f \). Set \( p := p' \cap R \). Then it follows from Lemma [1, 14] that \( p \in \text{Proj}(R) \cap \varphi \text{Var}(m_0 R) \) and

\[
\text{depth}_p + \text{ht}(p + R_+)/p \leq \text{depth}_{R'}(p' + R'_+)/p' \leq f.
\]

Therefore we can, and do, assume for the remainder of this proof that \( R_0/m_0 = \text{algebraically closed} \).
We now proceed by descending induction on \( f \). Note that \( f \leq r \). By Lemma 1.6(i), there exists \( y_r \in R_1 \setminus \mathfrak{m}_0R_1 \) such that \( y_r \) is a non-zerodivisor on \( M/\Gamma_{R_+}(M) \), that \( f_{R_+}(M/y_rM) = r - 1 \) and that
\[
\mathfrak{m}_0 \in \text{Ass}_{R_0}(H^{r-1}_{\pi R_+}(M/y_rM)_n) \quad \text{for all } n << 0.
\]
Since the image of \( y_r \) in \( R_1/\mathfrak{m}_0R_1 \) is non-zero, there exist \( y_1, \ldots, y_{r-1} \in R_1 \) such that \( R_1 \) is generated over \( R_0 \) by \( y_1, \ldots, y_{r-1}, y_r \). Note that \( R = R_0[y_1, \ldots, y_{r-1}, y_r] \) and that \( y_1, \ldots, y_{r-1}, y_r \) are algebraically independent over \( R_0 \). Therefore, we can, and do, assume that \( y_r = X_r \).

We can consider \( M/X_rM \) as a finitely generated graded module over \( R/X_rR \). The Graded Independence Theorem [B-S, 13.1.6] shows that \( f_{(R/X_rR)+}(M/X_rM) = r - 1 \) and that
\[
\mathfrak{m}_0 \in \text{Ass}_{R_0}(H^{r-1}_{(R/X_rR)+}(M/X_rM)_n) \quad \text{for all } n << 0.
\]
Since \( R/X_rR \) is (homogeneously) isomorphic to \( R_0[X_1, \ldots, X_{r-1}] \), we may apply the inductive hypothesis to deduce that there exists \( \mathfrak{p} \in \text{Proj}(R/X_rR) \cap \text{Var}(\mathfrak{m}_0R_0) \) with
\[
\text{depth}(M/X_rM)_\mathfrak{p} + \text{ht}(\mathfrak{p} + (R/X_rR)_+) / \mathfrak{p} \leq r - 1.
\]
Let \( \mathfrak{p} \) be the inverse image of \( \mathfrak{p} \) under the natural ring homomorphism \( R \to R/X_rR \). Then \( X_r \in \mathfrak{p} \) and \( \mathfrak{p} \in \text{Proj}(R) \cap \text{Var}(\mathfrak{m}_0R) \); also depth \( M_\mathfrak{p} = \text{depth}(M/X_rM)_\mathfrak{p} + 1 \) (because \( M_\mathfrak{p} \cong (M/\Gamma_{R_+}(M))_\mathfrak{p} \)) and
\[
\text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} = \text{ht}(\mathfrak{p} + (R/X_rR)_+)/\mathfrak{p}.
\]
Hence depth \( M_\mathfrak{p} + \text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} \leq r \). Thus we have found a \( \mathfrak{p} \) with the required properties in the case where \( f > r \).

Now suppose that \( f < r \) and that the desired result has been proved for larger values of \( f \) (for this value of \( r \)). There is an exact sequence \( 0 \to N \to F \to M \to 0 \) of finitely generated graded \( R \)-modules and homogeneous homomorphisms in which \( F \) is free. By Lemma 1.6(iii), we have \( f_{R_+}(N) = f + 1 \); by part (ii) of the same lemma, \( \mathfrak{m}_0 \in \text{Ass}_{R_0}(H^{r-1}_{\pi R_+}(N)_n) \) for all \( n << 0 \). Therefore, by the inductive hypothesis, there exists \( \mathfrak{p} \in \text{Proj}(R) \cap \text{Var}(\mathfrak{m}_0R) \) with depth \( N_\mathfrak{p} \leq \text{depth}(M_\mathfrak{p} + \text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} \leq f + 1 \).

Note that depth \( M_\mathfrak{p} \leq \text{ht}(\mathfrak{p} + R_+) \); we consider the cases where depth \( M_\mathfrak{p} < \text{ht}(\mathfrak{p} + R_+) \) and depth \( M_\mathfrak{p} = \text{ht}(\mathfrak{p} + R_+) \) separately. When depth \( M_\mathfrak{p} < \text{ht}(\mathfrak{p} + R_+) \), it follows from Lemma 1.6(i) that depth \( N_\mathfrak{p} = \text{depth}(M_\mathfrak{p} + 1) \); therefore depth \( M_\mathfrak{p} + \text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} \leq f + 1 \). In the other case, depth \( M_\mathfrak{p} = \text{ht}(\mathfrak{p} + R_+) \), so that, again by Lemma 1.6(i), depth \( N_\mathfrak{p} = \text{ht}(\mathfrak{p} + R_+) \). Therefore, in this case,
\[
\text{ht}(\mathfrak{m}_0 + R_+) = \text{ht}(\mathfrak{p} + R_+)
\]
\[
= \text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} + \text{ht}(\mathfrak{p})
\]
\[
= \text{ht}(\mathfrak{p} + R_+)/\mathfrak{p} + \text{depth}(N_\mathfrak{p} \leq f + 1 \leq r).
\]
Therefore \( \mathfrak{m}_0 = 0 \) and \( R_0 \) is a field. In this case, the desired conclusion is clear from the graded version of Grothendieck’s Finiteness Theorem (see [B-S, 13.1.17]). The proof is now complete.

2. Further examination of Singh’s example

In [S, §4], A. K. Singh showed that the ring \( R' := \mathbb{Z}[X, Y, Z, U, V, W]/(XU + YV + ZW) \), where \( X, Y, Z, U, V, W \) are independent indeterminates over \( \mathbb{Z} \), has the property that \( \text{Ass}_{R'}(H^2_{\pi R'}(R')) \) is infinite, where \( \pi \) is the ideal generated by the images of \( U, V, W \), Brodmann and Hellus [B-H, (5.7)(A)] observed that Singh’s argument leads to an interesting conclusion about graded components of graded local cohomology modules: we can consider \( \mathbb{Z}[X, Y, Z, U, V, W] \) as a positively graded ring with 0th component \( \mathbb{Z}[X, Y, Z] \) and \( U, V, W \) each assigned degree 1; \( R' \) inherits a structure as a standard positively graded ring with \( R'_+ = \mathfrak{a} \); the argument Singh used to prove his result mentioned above actually shows that
\[
\{ \mathfrak{p} \cap \mathbb{Z} : \mathfrak{p} \in \text{Ass}_{R'}(H^2_{\pi R'}(R')) \}
\]
is an infinite set, and Brodmann and Hellus noted that this implies that \( \text{Ass}_{R_0}(H^2_{\pi R_+}(R'_+)_n) \) is not asymptotically stable for \( n \to \infty \).

Our aim in the rest of this paper is to use Gröbner basis techniques on Singh’s example to identify precisely the set \( \text{Ass}_{R_0}(H^2_{\pi R_+}(R'_+)_n) \) for each \( n \leq -3 \), and to then deduce that \( \text{Ass}_{R_0}(H^2_{\pi R_+}(R'_+)_n) \) is not asymptotically increasing for \( n \to \infty \).

2.1. Notation. Throughout the rest of the paper, the symbol \( L \) will denote either a field or a principal ideal domain (PID), and \( R \) will denote the polynomial ring \( L[X, Y, Z, U, V, W] \), graded so that \( U, V, W \) have degree 1 and \( X, Y, Z \) have degree 0; thus \( R_0 = L[X, Y, Z] \). We shall set \( F := XV + YV + ZW \), and \( R' := R/FR \), again a standard positively graded ring. The natural map \( R \to R' \) maps \( R_0 \) isomorphically onto \( R'_0 \), and so we shall identify elements of \( R_0 \) with their natural images in \( R'_0 \). In the case where
$L = \mathbb{Z}$, the rings $R$ and $R'$ are those occurring in Singh’s example mentioned above. However, it will be helpful in another context to have some calculations available in the case where $L$ is the rational field, for example.

Since $H^3_{R'_+}(R')$ is homogeneous isomorphic to $H^3_{R'_+}(R/FR)$, we can use the exact sequence

$$H^3_{R'_+}(R)(-1) \overset{F}{\rightarrow} H^3_{R'_+}(R) \rightarrow H^3_{R'_+}(R/FR) \rightarrow 0$$

of graded $R$-modules and homogeneous homomorphisms (induced from the exact sequence

$$0 \rightarrow R(-1) \overset{F}{\rightarrow} R \rightarrow R/FR \rightarrow 0$$

to study $H^3_{R'_+}(R')$. Furthermore, we can realize $H^3_{R'_+}(R)$ as the module $R_0[U^-,V^-,W^-]$ of inverse polynomials described in [E.S. 12.4.1]: this graded $R$-module has end $-3$, and, for each $d \geq 3$, its $(-d)$-th component is a free $R_0$-module of rank $(n+1)^3$ with base $\langle U^\alpha V^\beta W^\gamma \rangle - \alpha, -\beta, -\gamma \in \mathbb{N}, \alpha + \beta + \gamma = -d$. We plan to study the graded components of $H^3_{R'_+}(R/FR)$ by considering the cokernels of the $R_0$-homomorphisms

$$F_{-d} : R_0[U^-,V^-,W^-]_{-d-1} \rightarrow R_0[U^-,V^-,W^-]_{-d} \quad (d \geq 3)$$
given by multiplication by $F$. In order to represent these $R_0$-homomorphisms between free $R_0$-modules by matrices, we specify an ordering for each of the above-mentioned bases by declaring that

$$U^\alpha V^\beta W^\gamma < U^\alpha' V^\beta' W^\gamma'$$

(where $\alpha_i, -\beta_i, -\gamma_i \in \mathbb{Z}$ and $\alpha_1 + \beta_1 + \gamma_1 = n < 3$ for $i = 1, 2$) precisely when $\alpha_1 > \alpha_2$ or $\alpha_1 = \alpha_2$ and $\beta_1 > \beta_2$. For example, this ordering on our base for $R_0[U^-,V^-,W^-]_{-5}$ is such that

$$U^{-1}V^{-1}W^{-3} < U^{-1}V^{-2}W^{-2} < U^{-1}V^{-3}W^{-1} < U^{-2}V^{-2}W^{-1} < U^{-2}V^{-2}W^{-1} < U^{-3}V^{-1}W^{-1}.$$ 

We shall frequently need to consider an $R_0$-homomorphism from the free $R_0$-module $R_0^n$ (regarded as consisting of column vectors) to $R_0^m$ (where $m$ and $n$ are positive integers) given by left multiplication by an $m \times n$ matrix $C$ with entries in $R_0$. In these circumstances, we shall also use $C$ to denote the homomorphism; its image $\text{Im} C$ is just the submodule of $R_0^n$ generated by the columns of $C$, for if $(e_i)_{i=1,\ldots,n}$ denotes the standard base for $R_0^n$, then $C e_j$ is just the $j$-th column of $C$ (for $1 \leq j \leq n$).

The theory of Gröbner bases is well developed for ideals in polynomial rings over a field (see, for example, [A-L, Chapter 3]). It is straightforward to combine the methods from these two parts of the theory to produce a theory of Gröbner bases for ideals in polynomial rings in finitely many indeterminates with coefficients in a principal ideal domain, and for submodules of finite free modules over polynomial rings in finitely many indeterminates over a field (see, for example, [A-L, Chapter 3]).

In this paper, we use the lexicographical term order with $X > Y > Z$ in $R_0$, and for each $n \in \mathbb{N}$ we set $> \omega$ to be the ‘term-over-position’ extension of this order to $R_0^n$ defined as follows: a monomial in $R_0^n$ is a column vector of the form $m e_j$, where $m$ is a monomial in $R_0$ and $e_j$ is the $j$-th standard base vector of $R_0^n$; and $m_1 e_{j_1} > m_2 e_{j_2}$ (for monomials $m_1, m_2$ of $R_0$ and $j_1, j_2 \in \{1, \ldots, n\}$) if and only if

$$m_1 > m_2 \quad \text{or} \quad m_1 = m_2 \quad \text{and} \quad j_1 < j_2.$$

If $A$ is an $m \times n$ matrix with entries in $R_0$ (we shall say ‘over $R_0$’) and $f, h \in R_0^n$, then we shall say that $f$ reduces to $h$ modulo $A$, denoted by $f \rightarrow^A h$, when $f$ reduces to $h$ modulo the set of columns of $A$ (see [A-L, Definition 3.5.8], but modify that definition to imitate [A-L, Definitions 4.1.1 and 4.1.6] in the case where $L$ is a PID).

We shall denote the leading monomial, leading coefficient and leading term of $f \in R_0^n$ by $\text{lm}(f)$, $\text{lc}(f)$ and $\text{lt}(f)$ respectively.

We shall use $I_n$ to denote the $n \times n$ identity matrix. For each $n \in \mathbb{N}$, we let $A_n$ denote the $n \times (n+1)$ matrix given by

$$A_n = \begin{bmatrix} Z & Y & 0 & \ldots & 0 \\ 0 & Z & Y & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \ldots & 0 & Z & Y \end{bmatrix}.$$ 

2.2. Lemma. Let $d \in \mathbb{N}$ with $d \geq 3$.

(i) With the notation of [2], the $R_0$-homomorphism

$$F_{-d} : R_0[U^-,V^-,W^-]_{-d-1} \rightarrow R_0[U^-,V^-,W^-]_{-d}$$

is a PID.
given by multiplication by \( F \) is represented, relative to the bases specified in \( \Box \) listed in increasing order, by the \( (d-1) \times (d-1) \) matrix

\[
T_d := \begin{bmatrix}
A_{d-2} & Xl_{d-2} & 0 & \ldots & 0 \\
0 & A_{d-3} & Xl_{d-3} & 0 & \ldots & 0 \\
& \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & A_1 & Xl_1
\end{bmatrix},
\]

where \( A_{d-2}, \ldots, A_1 \) are as defined in \( \Box \).

(ii) Each associated prime in \( \text{Ass}_{R_0}(H_{R_0}^d(R_d) - d) \) contains \( X, Y \) and \( Z \).

(iii) We have \( (X, Y, Z) \in \text{Ass}_{R_0}(H_{R_0}^d(R_d) - d) \).

Proof. (i) This follows from the fact that, for negative integers \( \alpha, \beta, \gamma \),

\[
F(U^{\alpha}V^\beta W^\gamma) = X(1 - \delta_{\alpha, -1})U^{\alpha+1}V^\beta W^\gamma + Y(1 - \delta_{\beta, -1})U^{\alpha}V^{\beta+1}W^\gamma + Z(1 - \delta_{\gamma, -1})U^{\alpha}V^\beta W^{\gamma+1},
\]

where \( \delta_{i, j} \) is Kronecker’s delta.

(ii) Consider the last column of \( T_d \) to see that \( Xe^{(d-1)} \in \text{Im} T_d \); therefore \( XYe^{(d-1)}, XZe^{(d-1)} \in \text{Im} T_d \), so that \( X^2e^{(d-1)}, X^2e^{(d-1)} \in \text{Im} T_d \) in view of the next-to-last and second-to-last columns of \( T_d \); we can now continue in this way to see that each element of \( \text{Coker} T_d = \text{Coker} F_{-d} \) is annihilated by \( X^{d-2} \). By symmetry, \( Y^{d-2} \) and \( Z^{d-2} \) also annihilate \( \text{Coker} F_{-d} = \text{Coker} T_d \).

(iii) It is clear from part (i) that \( (\text{Im} F_{-3}) : R_0 U^{-1}V^{-1}W^{-1} = (X, Y, Z) \). Hence \( (0 : R_0 \text{Coker} F_{-3}) = (X, Y, Z) \). Now multiplication by \( U^{d-3} \) induces an \( R_0 \)-epimorphism \( \text{Coker} F_{-d} \rightarrow \text{Coker} F_{-3} \), so that, in view of the above proof of part (ii), we have

\[
(X^{d-2}, Y^{d-2}, Z^{d-2}) \subseteq (0 : R_0 \text{Coker} F_{-d}) \subseteq (0 : R_0 \text{Coker} F_{-3}) = (X, Y, Z).
\]

Therefore \( (X, Y, Z) \) is a minimal member of the support of \( \text{Coker} F_{-d} \). \( \square \)

2.3. Lemma. Let \( k, m, n, q \in \mathbb{N}_0 \) with \( m, n > 0 \). Let \( A = [a_{ij}] \) be an \( m \times n \) matrix with entries in \( L[Y, Z] \), let \( f \in L[Y, Z]^n \), and let \( M \) and \( M' \) denote the \( (k + n + m + q) \)-rowed block matrices over \( R_0 \) given by

\[
M := \begin{bmatrix}
0 & 0 \\
Xl_n & f \\
A & 0 \\
0 & 0
\end{bmatrix} \quad \text{and} \quad M' := \begin{bmatrix}
0 & Xl_n \\
A & 0
\end{bmatrix},
\]

in which the first \( k \) and last \( q \) rows are all zero.

Then each \( S \)-polynomial of two columns of \( M \) is either 0 or reduces modulo \( M' \) to

\[
\begin{bmatrix}
0 \\
0 \\
\pm Af \\
0
\end{bmatrix}
\]

(in which the lowest \( 0 \)’ stands for the \( q \times 1 \) zero matrix), and these column matrices do arise from \( S \)-polynomials in this way.

Proof. Suppose \( f \neq 0 \), and let \( f = \sum_{j=1}^{t} c_{i_j} T_{i_j} e_{i_j} \) be an expression for \( f \) as a sum of terms, where the \( T_{i_j} \) are monomials in \( Y \) and \( Z \) and the \( c_{i_j} \) are elements of \( L \); suppose that \( \text{lt}(f) = c_{i_h} T_{i_h} e_{i_h} \). Let \( m_j \) denote the \( j \)th column of \( M \), for each \( j = 1, \ldots, n + 1 \).

Since \( f \in L[Y, Z] \), we have \( \text{lcm}(T_{i_j}, X) = T_{i_h} X \). All \( S \)-polynomials of two columns of \( M \) are zero except possibly for those of \( m_{i_h} \) and \( m_{n+1} \). Note that

\[
m_{n+1} = \sum_{j=1}^{t} c_{i_j} T_{i_j} e_{i_j+k} \quad \text{and} \quad m_i = Xe_{i+k} + \sum_{p=1}^{m} a_{p, i} e_{p+k+n} \quad (1 \leq i \leq n).
\]
We have
\[ S(m_i, m_{n+1}) = \frac{e_i T_{n+k}}{X} m_{i} - \frac{e_i T_{n+k}}{X} m_{n+1} = c_i T_{n+k} m_i - X m_{n+1} \]
\[ = c_i T_{n+k} e_{i+k} + \sum_{p=1}^{m} a_{p+i} c_i T_{n+k} e_{p+k+n} - \sum_{j=1}^{t} c_j X T_{j} e_{i+k} \]
\[ = \sum_{p=1}^{m} a_{p+i} c_i T_{n+k} e_{p+k+n} - \sum_{j=1}^{t} c_j X T_{j} e_{i+k} \]
\[ = \sum_{j=1}^{t} \sum_{p=1}^{m} a_{p+i} c_i T_{j} e_{p+k+n} = \begin{bmatrix} 0 \\ 0 \\ A_1 \end{bmatrix}, \]
as claimed.

\[ \square \]

2.4. **Theorem.** Consider the matrix
\[ T_d := \begin{bmatrix} A_{d-1} & X & \ldots & 0 \\ 0 & A_{d-3} & X & \ldots & 0 \\ & 0 & A_{d-5} & X & \ldots \\ & & 0 & A_{d-7} & X \\ & & & 0 & A_1 \end{bmatrix} \]
of \(2.2\). Define matrices \(G_{d-2}, G_{d-3}, \ldots, G_1\) by descending induction as follows: let \(G_{d-2}\) be a \((d-2)\)-rowed matrix with entries in \(L[Y, Z]\) whose columns include those of \(A_{d-2}\) and provide a Gröbner basis for \(\text{Im} \, A_{d-2}\); for \(i \in \mathbb{N}\) with \(d-2 > i \geq 1\), on the assumption that \(G_{i+1}\) has been defined as an \((i+1)\)-rowed matrix with entries in \(L[Y, Z]\), let \(G_i\) be an \(i\)-rowed matrix with entries in \(L[Y, Z]\) whose columns include those of \(A_i G_{i+1}\) and provide a Gröbner basis for \(\text{Im} \, A_i G_{i+1}\). Then

(i) the columns of
\[ T'_d := \begin{bmatrix} A_{d-2} & X & \ldots & 0 & G_{d-2} & 0 & \ldots & 0 \\ 0 & A_{d-3} & X & \ldots & 0 & G_{d-3} & 0 & \ldots \\ & 0 & A_{d-5} & X & 0 & G_{d-5} & 0 & \ldots \\ & & 0 & A_{d-7} & 0 & G_{d-7} & 0 & \ldots \\ & & & 0 & A_1 & X & 0 & \ldots \\ & & & & 0 & A_1 & X & 0 & \ldots & G_1 \end{bmatrix} \]
form a Gröbner basis for \(\text{Im} \, T'_d = \text{Im} \, T_d\); and

(ii) the columns of
\[ H_d := \begin{bmatrix} A_{d-2} & 0 & 0 & \ldots & 0 \\ 0 & A_{d-3} A_{d-2} & 0 & \ldots & 0 \\ 0 & 0 & A_{d-4} A_{d-3} A_{d-2} & \ldots & 0 \\ & \vdots & & \ddots & \vdots \\ & & 0 & \ldots & A_1 A_2 \ldots A_{d-2} \end{bmatrix} \]
generate \(\text{Im} \, T_d \cap L[Y, Z]^{(d-1)}\).

**Proof.** (i) Let \(s = S(f, g)\) be a non-zero \(S\)-polynomial of two columns \(f\) and \(g\) of \(T'_d\). There are various cases to consider.
First of all, if \( f \) and \( g \) have leading terms in one of the first \( d - 2 \) rows of \( T'_d \), then either \( s \xrightarrow{T'_d} 0 \) because the columns of \( [A_{d-2}G_{d-2}] \) form a Gröbner basis, or else \( s \) reduces modulo \( T'_d \) to a column of
\[
\begin{pmatrix}
0 \\
\pm A_{d-3}G_{d-2} \\
0 \\
\vdots \\
0
\end{pmatrix}
\]
by Lemma 2.3; since the columns of \( G_{d-3} \) include the columns of \( A_{d-3}G_{d-2} \), it follows that \( s \xrightarrow{T'_d} 0 \) in this case also.

Now suppose that \( i \in \mathbb{N} \) with \( d - 2 > i > 1 \) and that \( f \) and \( g \) have leading terms in row
\[
k + \sum_{j=i+1}^{d-2} j \quad \text{for some } k \in \{1, \ldots, i\}.
\]

Then either \( s \xrightarrow{T'_d} 0 \) because the columns of \( G_i \) form a Gröbner basis, or else \( s \) reduces modulo \( T'_d \) to a column of
\[
\begin{pmatrix}
0 \\
\vdots \\
0 \\
\pm A_{i-1}G_i \\
0 \\
\vdots \\
0
\end{pmatrix}
\]
(where the block \( A_{i-1}G_i \) is in the rows corresponding to those where the blocks \( A_{i-1} \) and \( G_{i-1} \) are positioned in \( T'_d \)), by Lemma 2.3 again; since the columns of \( G_{i-1} \) include the columns of \( A_{i-1}G_i \), it follows that \( s \xrightarrow{T'_d} 0 \) in this case also.

Finally, suppose that \( f \) and \( g \) have leading terms in the last row of \( T'_d \). In this case, either \( s \xrightarrow{T'_d} 0 \) because the columns of \( G_1 \) form a Gröbner basis, or \( s = S \left( Xe_{(d-1)}, he_{(d-1)} \right) \) for some \( h \in L[Y, Z] \); in the latter case, \( s \) reduces to 0 in one step modulo \( \left\{ Xe_{(d-1)} \right\} \).

Thus, in all cases, \( s \xrightarrow{T'_d} 0 \). Hence, by (the analogue of) [A-L, Theorem 3.5.19], the columns of \( T'_d \) form a Gröbner basis.

To complete the proof of part (i), it only remains for us to show that \( \text{Im} T'_d = \text{Im} T_d \). It is easy to see by descending induction on \( i \) that, for each \( i = d - 2, d - 3, \ldots, 1 \), the columns of \( G_i \) include the columns of \( A_iA_{i+1} \ldots A_{d-2} \) and form a Gröbner basis (over \( Z[Y, Z] \)) for \( \text{Im} A_iA_{i+1} \ldots A_{d-2} ; \) hence (over both \( Z[Y, Z] \) and \( Z[X, Y, Z] \))
\[
\text{Im} A_iA_{i+1} \ldots A_{d-2} = \text{Im} A_iG_{i+1} = \text{Im} G_i \quad \text{for all } i = d - 3, d - 4, \ldots, 1.
\]

By Lemma 2.3, for such an \( i \), each column of
\[
\begin{pmatrix}
0 \\
\vdots \\
0 \\
A_iG_{i+1} \\
\vdots \\
0
\end{pmatrix}
\]
(in which the block $A_iG_{i+1}$ occupies the rows corresponding to those occupied by $A_i$ in $T_d$) can by
gained as a result of reducing modulo $T_d$ the $S$-polynomial of a column of $T_d$ and a column of
\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
G_{i+1} \\
0 \\
\vdots \\
0
\end{bmatrix}
\]
(in which the block $G_{i+1}$ occupies the rows corresponding to those occupied by $A_{i+1}$ in $T_d$). The claim
in part (i) now follows from another use of descending induction.

(ii) Since the lexicographical order we are using on $L[X, Y, Z]$ is an elimination order with $X$ greater
than $Y$ and $Z$, it follows from (the analogue of) [A-L, Theorem 3.6.6] that the inter-
section of the set of columns of $T_d'$ with $L[Y, Z]^{(d-1)\choose 2}$ provides a Gröbner basis for $\text{Im} T_d \cap L[Y, Z]^{(d-1)\choose 2}$.

Therefore the columns of
\[
\begin{bmatrix}
A_{d-2} & G_{d-2} & 0 & \ldots & 0 \\
0 & 0 & G_{d-3} & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & G_1
\end{bmatrix}
\]
form a Gröbner basis for $\text{Im} T_d \cap L[Y, Z]^{(d-1)\choose 2}$, and the claim in part (ii) follows from this.

The following lemma provides motivation for part (ii) of the above theorem.

2.5. Lemma. Consider the matrices $T_d$ of 2.2 and $H_d$ of 2.4. Let $r \in L \setminus \{0\}$. Then $r$ annihilates a non-
zero element of $\text{Coker} T_d$ if and only if $r$ annihilates a non-zero element of the quotient $L[Y, Z]$-module
$\text{Coker} H_d$ of $L[Y, Z]^{(d-1)\choose 2}$.

Proof. Suppose that $r$ annihilates a non-zero element of $\text{Coker} T_d$. Thus there exists a $v \in R^{(d-1)\choose 2} \setminus \text{Im} T_d$ such that $rv \in \text{Im} T_d$. We can and do assume that $v$ has been chosen so that its leading term is minimal
among the leading terms of all possible such columns. But $Xe_1, Xe_2, \ldots, Xe_{(d-1)\choose 2}$ are all leading terms
of $T_d$, and so $v$ does not involve $X$. In view of this and the fact, established in 2.4, that $\text{Im} H_d \subseteq \text{Im} T_d$, we have $v \in L[Y, Z]^{(d-1)\choose 2} \setminus \text{Im} H_d$. Furthermore, $rv \in \text{Im} T_d \cap L[Y, Z]^{(d-1)\choose 2}$, and, by 2.4,
this is the $L[Y, Z]$-submodule of $L[Y, Z]^{(d-1)\choose 2}$ generated by the columns of $H_d$.

The converse is even easier.

2.6. Proposition. For each integer $i = 0, \ldots, n-1$, $A_{i+1}A_{i+2} \ldots A_n$ is the $(i+1) \times (n+1)$ matrix
\[
\begin{bmatrix}
Z^{n-i} & \ldots & (n-1\choose i)Z^{n-i-j}Y^j & \ldots & Y^{n-i} & 0 & 0 & \ldots \\
0 & Z^{n-i} & \ldots & (n-1\choose i)Z^{n-i-j}Y^j & \ldots & Y^{n-i} & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & (n-1\choose i)Z^{n-i-j}Y^j & \ldots & Y^{n-i}
\end{bmatrix}
\]

Proof. The result follows from an easy reverse induction on $i$.

The particular case of 2.6 in which $i = 0$ yields the following.

2.7. Corollary. $A_1A_{i+2} \ldots A_n$ is the $1 \times (n+1)$ matrix whose $(1, i+1)$-th entry is $(n\choose i)Y^iZ^{n-i}$ for all
$i = 0, \ldots, n$.

2.8. Proposition. Let $r, k \in \mathbb{N}$, and let $Q_{r,r+k}$ be the $r \times (r+k)$ matrix with entries in $L[Y, Z]$ given
by
\[
Q_{r,r+k} := 
\begin{bmatrix}
Z^k & \ldots & (k\choose j)Z^{k-j}Y^j & \ldots & Y^k & 0 & 0 & \ldots \\
0 & Z^k & \ldots & (k\choose j)Z^{k-j}Y^j & \ldots & Y^k & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & (k\choose j)Z^{k-j}Y^j & \ldots & Y^k
\end{bmatrix}
\]
let $c_j$ denote the $j$-th column of $Q_{r,r+k}$ (for $j = 1, \ldots, r + k$), and let $\tilde{Q}_{r,r+k}$ be the result of evaluating $Q_{r,r+k}$ at $Y = Z = 1$. Thus

$$
\tilde{Q}_{r,r+k} := \begin{bmatrix}
1 & \cdots & \binom{k}{j} & \cdots & 1 & 0 & 0 & \cdots \\
0 & 1 & \cdots & \binom{k}{j} & \cdots & 1 & 0 & \cdots \\
\vdots & & \ddots & & \vdots & & \vdots & \\
0 & 0 & 0 & \cdots & \binom{k}{j} & \cdots & 1 & \\
\end{bmatrix}.
$$

Consider $L[Y,Z]$ as an $\mathbb{N}_0^2$-graded ring in which $L[Y,Z]_{(0,0)} = L$ and $\deg Y^i Z^j = (i + j, i)$. Turn the free $L[Y,Z]$-module

$$L[Y,Z]_r = L[Y,Z]e_1 \oplus \cdots \oplus L[Y,Z]e_r$$

into an $\mathbb{N}_0^2$-graded module over the $\mathbb{N}_0^2$-graded ring $L[Y,Z]$ in such a way that $\deg e_i = (0, i)$ for $i = 1, \ldots, r$. All references to gradings in the rest of this proposition and its proof refer to this $\mathbb{N}_0^2$-grading.

(i) For all $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$, the component $(L[Y,Z])_{(i,j)}$ is a free $L$-module with base

$$(Y^{j-r}Z^{i+j-r}e_p)_{p = \max\{j-i,1\}, \ldots, \min\{j,r\}}.$$

(Of course, we interpret a free module with an empty base as 0.)

(ii) $\operatorname{Im} Q_{r,r+k}$ is a graded submodule of $L[Y,Z]_r$, and, for all $i \in \mathbb{N}_0$, $j \in \mathbb{N}$,

$$(\operatorname{Im} Q_{r,r+k})_{(i,j)} = \begin{cases}
0 & \text{if } i < k, \\
\sum_{\sigma = \max\{j+k-i,1\}}^{\min\{j,r+k\}} L Y^{j-\sigma} Z^{i+j-\sigma-k} e_\sigma & \text{if } i \geq k,
\end{cases}$$

for $0 \leq i < k$ (and $j \in \mathbb{N}_0$), the component $(\operatorname{Coker} Q_{r,r+k})_{(i,j)}$ is a free $L$-module, and for $k \leq i \leq 2k + r - 1$ and $1 \leq j \leq k + r - 1$, the component $(\operatorname{Coker} Q_{r,r+k})_{(i,j)}$, as an $L$-module, is isomorphic to the cokernel of a submatrix of $\tilde{Q}_{r,r+k}$ made up of the (consecutive) columns of that matrix numbered

$$\max\{j + k - i, 1\}, \max\{j + k - i, 1\} + 1, \ldots, \min\{j, r + k\}.$$

Proof. (i) This is immediate from the fact that, for $\alpha, \beta \in \mathbb{N}_0$ and $\rho \in \{1, \ldots, r\}$, we have $\deg Y^\alpha Z^\beta e_\rho = (\alpha + \beta, \alpha + \rho)$.

(ii) Note that $c_j$ is a homogeneous element of $L[Y,Z]_r$ of degree $(k,j)$ (for all $j = 1, \ldots, k+r$). Hence $\operatorname{Im} Q_{r,r+k}$ is a graded submodule of $L[Y,Z]_r$, and a homogeneous element of $\operatorname{Im} Q_{r,r+k}$ is expressible as a $L[Y,Z]$-linear combination of the columns of $Q_{r,r+k}$ in which all the coefficients are homogeneous. Note that $\deg Y^\alpha Z^\beta c_\gamma = (\alpha + \beta, k + \alpha + \gamma)$ (for $\alpha, \beta \in \mathbb{N}_0$ and $\gamma \in \{1, \ldots, r\}$). Hence $(\operatorname{Im} Q_{r,r+k})_{(i,j)} = 0$ if $i < k$, while $(\operatorname{Im} Q_{r,r+k})_{(i,j)} = \sum_{\sigma = \max\{j+k-i,1\}}^{\min\{j,r+k\}} L Y^{j-\sigma} Z^{i+j-\sigma-k} e_\sigma$ if $i \geq k$.

Notice that the vectors $Z^k e_1, Z^{k+1} e_2, \ldots, Z^{k+r-1} e_r$ and $Y^{k+r-1} e_1, Y^{k+r-2} e_2, \ldots, Y^k e_r$ are all in $\operatorname{Im} Q_{r,r+k}$: for any $1 \leq s \leq r$ multiply the $s$-th column of $Q_{r,r+k}$ by $Z^{s-1}$ and reduce with respect to $Z^k e_1, Z^{k+1} e_2, \ldots, Z^{k+s-2} e_{s-1}$ to obtain $Z^{k+s-1} e_s \in \operatorname{Im} Q_{r,r+k}$; a similar argument shows that $Y^s e_1, Y^{s+k-1} e_{r-1}, \ldots, Y^{k+r-1} e_1 \in \operatorname{Im} Q_{r,r+k}$. Therefore, if $i \geq 2k + r$ or $j \geq k + r$, then

$$Y^{j-r} Z^{i-j+r} e_p \in \operatorname{Im} Q_{r,r+k}$$

for all $\rho = \max\{j-i,1\}, \ldots, \min\{j,r\}$ since $Y^{j-r} e_p \in \operatorname{Im} Q_{r,r+k}$ if $j \geq k + r$, while if $j < k + r$ and $i \geq 2k + r$, then $i - j \geq k + 1$ and $Z^{i-j} e_p \in \operatorname{Im} Q_{r,r+k}$. Thus, for $i \geq 2k + r$ or $j \geq k + r$, all the members of the base found in part (i) for the free $L$-module $L[Y,Z]_r$ lie in $\operatorname{Im} Q_{r,r+k}$.

(iii) All the claims of this except the final one now follow from the previous parts. To deal with the final one, suppose that $k \leq i \leq 2k + r - 1$ and $1 \leq j \leq k + r - 1$. We shall use ‘overlines’ to denote natural images in cokernels of elements of free modules. It will be convenient to abbreviate $\min\{j, r + k\}$ by $\gamma$ and $\max\{j + k - i, 1\}$ by $\beta$; the conditions imposed on $i$ and $j$ ensure that $\beta \leq \gamma$. 


By part (i), $(\text{Coker} \, Q_{r, r+k})_{(i, j)}$ is generated, as $\mathbb{Z}$-module, by
\[
\{Y^{j-p}Z^{i-j+p}e_p: \rho = \max\{j - i, 1\}, \ldots, \min\{j, r\}\}.
\]
(Note that, once again, the conditions imposed on $i$ and $j$ ensure that $\max\{j - i, 1\} \leq \min\{j, r\}$.) The fact that, for each $\sigma = \beta, \ldots, \gamma$, we have $Y^{j-\sigma}Z^{i-j+\sigma}k e_{\sigma} = 0$ shows that the $\sigma$-th column of $Q_{r, r+k}$ leads to a column of relations on the generators displayed above. Furthermore, part (ii) shows that every column of relations on those generators is an $L$-linear combination of the columns of relations arising (in this way) from the $\beta$-th, $(\beta + 1)$-th, ..., $\gamma$-th columns of $Q_{r, r+k}$. 

2.9. Remark. Note that, with the notation of 2.8 (and provided $r > 1$), we have
\[ Q_{r-1, r+k} = Q_{r-1, r}Q_{r, r+k}. \]

2.10. Remark. Let $B$ be a matrix with integer entries and positive rank $d$; let $p$ be a prime number. Then it follows from the theory of the Smith normal form that $p\mathbb{Z} \in \text{Ass}_{\mathbb{Z}}(\text{Coker} \, B)$ if and only if the ideal generated by the $d \times d$ minors of $B$ is contained in $p\mathbb{Z}$.

In view of Remark 2.10 and Proposition 2.8, we are going, in the case when $L = \mathbb{Z}$, to be interested in the value of the determinant of a square matrix (with integer entries) of the form
\[
\Omega := \begin{bmatrix}
(k)_{i} & (k)_{i+1} & \cdots & (k)_{i+s-1} \\
(k+1)_{i} & (k+1)_{i+1} & \cdots & (k+1)_{i+s-2} \\
\vdots & \vdots & \ddots & \vdots \\
(k+s-1)_{i} & (k+s-1)_{i+1} & \cdots & (k+s-1)_{i+s-1}
\end{bmatrix},
\]
where $k, s \in \mathbb{N}$, $i \in \mathbb{N}_0$ and we use the convention that a binomial coefficient $\binom{5}{3}$ is $0$ if either $\eta < 0$ or $\eta > \xi$. The value of this determinant was known to V. van Zeipel in 1865 [Z]; the calculation is described in [Mr, Chapter XX]. For the convenience of the reader, we indicate a route to the answer.

2.11. Proposition. (See van Zeipel [Z].) Let $\Omega$ be as displayed above. Then
\[ \det \, \Omega = \prod_{j=0}^{s-1} \binom{k+s-1-j}{i+j}. \]

Proof. Add the penultimate row of $\Omega$ to the last row; in the result, add the $(r - 2)$-th row to the $(r - 1)$-th, and continue in this way until the first row has been added to the second. In this way one sees that
\[
\det \, \Omega = \begin{vmatrix}
(k)_{i} & (k)_{i+1} & \cdots & (k)_{i+s-1} \\
(k+1)_{i} & (k+1)_{i+1} & \cdots & (k+1)_{i+s-2} \\
\vdots & \vdots & \ddots & \vdots \\
(k+s-1)_{i} & (k+s-1)_{i+1} & \cdots & (k+s-1)_{i+s-1}
\end{vmatrix}.
\]
Now repeat the same sequence of elementary row operations, except that, this time, stop after the second row has been added to the third; then do a further such sequence, this time stopping after the third row has been added to the fourth. Continue in this way to see that
\[
\det \, \Omega = \begin{vmatrix}
(k)_{i} & (k)_{i+1} & \cdots & (k)_{i+s-1} \\
(k+1)_{i} & (k+1)_{i+1} & \cdots & (k+1)_{i+s-2} \\
\vdots & \vdots & \ddots & \vdots \\
(k+s-1)_{i} & (k+s-1)_{i+1} & \cdots & (k+s-1)_{i+s-1}
\end{vmatrix}.
\]
We proceed by induction on $i$. When $i = 0$, it is clear from the initial form of $\Omega$ that $\det \, \Omega = 1$, and the claim is true. We therefore assume that $i > 0$, and make the obvious inductive assumption.
With reference to the last display, take out a factor $k + j - 1$ from the $j$-th row $(1 \leq j \leq s)$ and a factor $1/(i + l - 1)$ from the $l$-th column $(1 \leq l \leq s)$ to see that
\[
\det \Omega = \frac{k(k + 1) \ldots (k + s - 1)}{i(i + 1) \ldots (i + s - 1)} \begin{vmatrix}
(k_{i-1}^{k-1}) & (k_{i}^{k-1}) & \ldots & (k_{i+k-2}^{k-1}) \\
(k_{i}^{k}) & (k_{i+1}^{k}) & \ldots & (k_{i+k-2}^{k}) \\
(k_{i+1}^{k}) & (k_{i+1}^{k}) & \ldots & (k_{i+k-2}^{k}) \\
\vdots & \vdots & \ddots & \vdots \\
(k_{i+s-2}^{k}) & (k_{i+s-2}^{k}) & \ldots & (k_{i+s-2}^{k})
\end{vmatrix}.
\]

Now use the inductive hypothesis. \qed

2.12. Corollary. In the situation of Proposition 2.11, we have
\[
\det \Omega = \frac{\prod_{j=0}^{s-1} (k+s-j)}{\prod_{j=0}^{s-1} (k-j)}.
\]

Proof. First note that, for $j \in \{1, \ldots, s-1\}$, we have
\[
\binom{k + s - 1 - j}{i} = \binom{k + s - 1}{i + j} \frac{(i+1) \ldots (i+j)}{(k+s-j) \ldots (k+s-1)}.
\]

It therefore follows from Proposition 2.11 that
\[
\det \Omega = \prod_{j=0}^{s-1} \binom{k+s-1-j}{i} \binom{i+j}{i} \frac{1}{\prod_{j=0}^{s-1} (k+s-j) \ldots (k+s-1)}
\]
\[
= \left( \prod_{j=0}^{s-1} \binom{k+s-1-j}{i+j} \right) \left( \prod_{j=0}^{s-1} \frac{j!(k+s-j-1)!}{(k+s-1)!} \right)
\]
\[
= \prod_{j=0}^{s-1} \binom{k+s-1-j}{i+j} \prod_{j=0}^{s-1} \frac{j!(k+s-j-1)!}{(k+s-1)!}
\]
\[
\prod_{j=0}^{s-1} \binom{k+s-1-j}{i+j} \prod_{j=0}^{s-1} \frac{j!(k+s-j-1)!}{(k+s-1)!}.
\]
\qed

2.13. Notation. For each $n \in \mathbb{N}$, we set
\[
\Pi(n) := \left\{ p : p \text{ is a prime factor of } \binom{n}{i} \text{ for some } i \in \{0, \ldots, n\} \right\}.
\]

2.14. Corollary. Let $r, k \in \mathbb{N}$ and consider the matrix $\tilde{Q}_{r,r+k}$ of 2.8. Let $\Delta$ be a submatrix of $\tilde{Q}_{r,r+k}$ formed by $c$ ($> 0$) consecutive columns of that matrix; set $s := \min\{c,r\}$. If $p \in \mathbb{Z}$ is a prime number such that every $s \times s$ minor of $\Delta$ is contained in $p\mathbb{Z}$, then $p \in \Pi(r+k-1)$.

Proof. We argue by induction on $r$. Note that $Q_{1,1+k}$ is the $1 \times (1+k)$ matrix
\[
\begin{bmatrix}
1 & \binom{k}{i} & \ldots & \binom{k}{j} & \ldots & 1
\end{bmatrix},
\]
and so the result is clear in this case.

Now suppose that $r > 1$ and that the result has been established, for all values of $k$, for smaller values of $r$.

If $s = r$, then there is an $r \times r$ submatrix of $\tilde{Q}_{r,r+k}$ of the form
\[
\Omega := \begin{bmatrix}
\binom{k}{i-1} & \binom{k}{i} & \ldots & \binom{k}{i+r-1} \\
\binom{k}{i} & \binom{k}{i} & \ldots & \binom{k}{i+r-2} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{k}{i-r+1} & \binom{k}{i-r+2} & \ldots & \binom{k}{i}
\end{bmatrix},
\]
where $i \in \{0, \ldots, k\}$, such that $\det \Omega \in p\mathbb{Z}$. It now follows from Corollary 2.12 that $p$ is a factor of $\binom{k+r-1}{i-1}$ for some $l \in \{0, \ldots, k+r-1\}$. 

Now suppose that \( s = c < r \). Set \( D' := \overline{Q}_{r-1,r}D \). As \( \overline{Q}_{r-1,r+k} = \overline{Q}_{r-1,r}\overline{Q}_{r,r+k} \) by \( \text{2.4} \), it follows that \( \Delta' := \overline{Q}_{r-1,r}\Delta \) is the \((r-1) \times c\) submatrix of \( \overline{Q}_{r-1,r+k} \) involving the same columns as \( \Delta \). But

\[
\overline{Q}_{r-1,r} = \begin{pmatrix}
1 & 1 & 0 & \ldots & 0 \\
0 & 1 & 1 & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1 & 1
\end{pmatrix},
\]

and so the rows of \( \Delta' \) are the sums of consecutive rows of \( \Delta \). Therefore any \( s \times s \) minor of \( \Delta' \) is the sum of \( 2^s \) determinants, each one being either obviously zero or an \( s \times s \) minor of \( \Delta \). Hence every \( s \times s \) minor of \( \Delta' \) is contained in \( p\mathbb{Z} \), and so, by the inductive hypothesis, \( p \in \Pi(r-1+k+1-1) \), that is, \( p \in \Pi(r+k-1) \).

\[
\begin{proof}
\begin{aligned}
& \text{2.15. Lemma. The set of integers } \{ \#\Pi(n) : n \in \mathbb{N} \} \text{ is unbounded.} \\
& \text{Proof. Let } (p_n)_{n} \in \mathbb{N} \text{ be an enumeration of the prime numbers. Then, for each } n \in \mathbb{N}, \text{ we have } \\
p_1p_2\ldots p_n = \begin{pmatrix} p_1p_2\ldots p_n \\ 1 \end{pmatrix} \in \Pi(p_1p_2\ldots p_n).
\end{aligned}
\end{proof}
\]

\[
\begin{proof}
\begin{aligned}
& \text{2.16. Lemma. Let } p \in \mathbb{Z} \text{ be a prime number. Then the sets } \\
& \{ j \in \mathbb{N} : j \geq 3 \text{ and } p \in \Pi(j-2) \} \text{ and } \\
& \{ j \in \mathbb{N} : j \geq 3 \text{ and } p \not\in \Pi(j-2) \}
\end{aligned}
\]

are both infinite.

\[
\begin{proof}
\begin{aligned}
& \text{Proof. If } p \text{ divides } j-2 \in \mathbb{N}, \text{ then } p \in \Pi(j-2) \text{ because } p \text{ divides } \binom{j-2}{1} = j-2; \text{ hence the first set is infinite.}
\end{aligned}
\]

To prove that the second set is infinite it is enough to show that \( p \not\in \Pi(p^k-1) \) for all \( k \geq 1 \). Let \( T \) be an indeterminate; working modulo \( p \) we have \((1 + T)p^k-1(1 + T) = (1 + T)p^k-1 + Tp^k \) and if we compare the coefficients of \( T^i \) on both sides of this congruence we see that, for \( 0 < i \leq p^k - 1 \),

\[
\binom{p^k-1}{i} + \binom{p^k-1}{i-1} = 0
\]

and since \( p \) does not divide \( \binom{p^k-1}{i} \), \( i \) shows that \( p \) does not divide \( \binom{p^k-1}{i} \) for all \( i \) with \( 0 \leq i \leq p^k - 1 \).

\[
\begin{proof}
\begin{aligned}
& \text{2.17. Theorem. Let } R' \text{ denote the ring } \mathbb{Z}[X,Y,Z,U,V,W]/(XU + YV + ZW) \text{ (considered by Singh) } \\
& \text{graded in the manner described in } \text{2.4}; \text{ let } -d \in \mathbb{Z} \text{ with } d \geq 3; \text{ and let } p \in \mathbb{Z} \text{ be a prime number. Then}
\end{aligned}
\]

\[
\begin{aligned}
& \text{(i) } p\mathbb{Z} \in \ass_{\mathbb{Z}}(H^3_{R'_+}(R')_{-d}) \text{ if and only if } p \in \Pi(d-2); \\
& \text{(ii) } \ass_{R'_0}(H^3_{R'_+}(R')_{-d}) = \{(X,Y,Z) \} \cup \{(q,X,Y,Z) : q \in \Pi(d-2)\}; \\
& \text{(iii) the set of integers } \{ \#(\ass_{R'_0}(H^3_{R'_+}(R')_{-j})) : j \geq 3 \} \text{ is unbounded}; \\
& \text{(iv) the sets } \\
& \{ j \in \mathbb{Z} : j \geq 3 \text{ and } (p,X,Y,Z) \in \ass_{R'_0}(H^3_{R'_+}(R')_{-j}) \}
\end{aligned}
\]

are both infinite; and

\[
\begin{aligned}
& \text{(v) } \ass_{R'_0}(H^3_{R'_+}(R')_n) \text{ is not asymptotically increasing for } n \to -\infty.
\end{aligned}
\]

\[
\begin{proof}
\begin{aligned}
& \text{Proof. (i) It follows from Lemma 2.2 that } p\mathbb{Z} \in \ass_{\mathbb{Z}}(H^3_{R'_+}(R')_{-d}) \text{ if and only if } p\mathbb{Z} \in \ass_{\mathbb{Z}}(\coker T_d); \\
& \text{furthermore, by Lemma 2.3, this is the case if and only if } p\mathbb{Z} \in \ass_{\mathbb{Z}}(\coker H_d), \text{ where the matrix } \\
& H_d \text{ is as defined in Theorem 2.4. It therefore follows from 2.4 and the notation introduced in 2.8 that } \\
& p\mathbb{Z} \in \ass_{\mathbb{Z}}(H^3_{R'_+}(R')_{-d}) \text{ if and only if if } \\
& p\mathbb{Z} \in \bigcup_{i=1}^{d-2} \ass_{\mathbb{Z}}(\coker Q_{i,d-1}).
\end{aligned}
\end{proof}
\]
Suppose that $p \in \Pi(d-2)$, so that there exists $j \in \{1, \ldots, d-3\}$ such that $p$ is a factor of $d-2$.

Then it follows from Theorem 2.8(iii) that (for example) $p \mathbb{Z} \in \text{Ass}_{\mathbb{Z}}(\text{Coker } Q_{1,d-1})_{d-2,j+1}$.

Conversely, suppose that $p \mathbb{Z} \in \text{Ass}_{\mathbb{Z}}(\text{Coker } Q_{i,d-1})$, where $i \in \{1, \ldots, d-2\}$. We use Theorem 2.8(iii) to see that $p \mathbb{Z} \in \text{Ass}_{\mathbb{Z}}(\text{Coker } \Delta)$, where $\Delta$ is a submatrix of $Q_{i,d-1}$ formed by $c (> 0)$ consecutive columns of that matrix; set $s := \min\{c, i\}$. It follows from Proposition 2.11 that $\Delta$ has rank $s$, and therefore from Remark 2.10 that the ideal generated by the $s \times s$ minors of $\Delta$ is contained in $p \mathbb{Z}$. Therefore $p \in \Pi(d-2)$ by Corollary 2.14.

(ii) This is now immediate from part (i) and Lemma 2.2.
(iii) This is a consequence of part (ii) and Lemma 2.13.
(iv) This is now immediate from part (ii) and Lemmas 2.3 and 2.16.
(v) This is a consequence of parts (ii) and (iv).  \[\square\]

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