I. INTRODUCTION

Most real world data are only recorded in the rounded figure with a fixed number of significant digits. Strictly this rounding introduces additional systematic uncertainties which must be properly accounted for, in order to infer the property of the intrinsic distribution of the measured quantities. Naively, assuming that there is neither intrinsic uncertainty nor systematic bias, the differences between the true value and the rounded reported value are expected to be distributed evenly over the window of the size of the reporting unit.

In particular, if the variable value \( x \) is rounded to an integer multiple value of the measurement unit as in \( nw \) (where \( w \) is the measurement unit and \( n \) is an integer), then \( (n+\delta−1/2)w < x < (n+\delta+1/2)w \) or \( (n+\delta−1/2)w < x < (n+\delta+1/2)w \). Here the constant \( \delta = 1/2 \) for rounding down to the floor, \( \delta = 1\) for rounding up to the ceiling, and \( \delta = 0 \) for rounding to the nearest integer etc.) whereas the equal signs at the boundary follow the prescribed convention. Then the rounding error (i.e. \( \rho = nw − x \) ) is distributed in the rectangular distribution:

\[
P(\rho) = \begin{cases} 
  w^{-1} & \text{for } -(1/2+\delta)w < \rho < (1/2−\delta)w \\
  0 & \text{elsewhere} 
\end{cases},
\]

(1)

where the distribution at the boundary is determined by the chosen convention – however provided that \( x \) is a real variable in a continuous distribution, the boundaries constitute a null measure set and so the specific choice does not affect the following discussion. For a random variable \( x \), the mean of the rounded values is (with \( \bar{x} \) being the true mean of \( x \))

\[
\bar{m} = \bar{x} + \int_{-(1/2+\delta)w}^{(1/2−\delta)w} \frac{\rho}{w} \, d\rho = \bar{x} − \delta w,
\]

(2)

while the variance is

\[
s^2 = (nw−\bar{m})^2 = (x+\rho)^2 − (\bar{x} + \rho)^2 = \sigma^2 + \rho^2 − \rho^2 + 2(\bar{x}ρ − \bar{x}^2),
\]

(3)

where \( \sigma^2 = \bar{x}^2 − \bar{x}^2 \) is the variance of \( x \), with the variance of the rounding errors given by

\[
\bar{\rho}^2 = \rho^2 = \int_{-(1/2+\delta)w}^{(1/2−\delta)w} \frac{\rho^2}{w} \, d\rho − (\delta w)^2 = \frac{w^2}{12}.
\]

(4)

In other words, provided that the distribution of \( x \) does not affect the rounding (as in \( \sigma = \bar{x} \rho \)), the standard deviation of the rounded values is simply a quadrature sum of the true underlying standard deviation and that of the rounding errors, and the true standard deviation may be estimated from the variance of the rounded values via

\[
\sigma = \left( s^2 − \frac{w^2}{12} \right)^{1/2}.
\]

(5)

However, this result is only valid “on average” sense. That is to say, the underlying distribution of the variable can technically affect the rounding but for an arbitrary unspecified distribution, the expected value of “\( \bar{x} − \bar{x}^2 \)” should be zero and the reported error tends to the quadrature sum of the random error and the rounding error (\( \sigma_{\rho}/w = 1/\sqrt{12} \approx 0.2887 \)).

II. THEORY

Suppose that \( f(x) \) is a probability distribution of a real random variable \( x \) with

\[
\int_{-\infty}^{\infty} dx \, f(x) = 1, \quad \int_{-\infty}^{\infty} dx \, x \, f(x) = \mu, \quad \int_{-\infty}^{\infty} dx \, (x−\mu)^2 \, f(x) = \int_{-\infty}^{\infty} dx \, x^2 \, f(x) − \mu^2 = \sigma^2.
\]

(6)

Next consider the rounding of the measured value of the variable such that, with a fixed constant \( \delta \in [−1/2, 1/2] \) and the measurement unit \( w \), the value \( x \) is read off by an integer multiple of the unit, i.e. \( nw \), where \( n = \lfloor (x/w + 1/2 − \delta) \rfloor \) or \( n = \lceil (x/w − 1/2 − \delta) \rceil \) with \( |x| \) and \( |\delta| \) being the integer floor and ceiling of \( x \). Then the (discrete) distribution of the reported integer \( n \) for the rounded value is found to be

\[
F_n = \int_{(n+\delta+1/2)w}^{(n+\delta+1/2)w} dx \, f(x).
\]

(7)
This distribution is properly normalized: that is,
\[ \sum_{n=-\infty}^{\infty} F_n = \int_{-\infty}^{\infty} dx f(x) = 1, \tag{8} \]
and so we can find the mean and the variance of the rounded variables by calculating
\[ m = \bar{nw} = w \sum_{n=-\infty}^{\infty} nF_n, \]
\[ s^2 = (nw - m)^2 = n^2w^2 - m^2 \]
\[ = w^2 \left[ \sum_{n=-\infty}^{\infty} n^2F_n - \left( \sum_{n=-\infty}^{\infty} nF_n \right)^2 \right]. \tag{9} \]

For some distributions \( f(x) \), the associated discrete distribution \( F_n \) as well as its mean \( m \) and the standard deviation \( s^2 \) of the rounded variable can be computed analytically. However the calculations become quite tedious even for many simple distributions and the computations can only be done numerically for most distributions including the important example such as the normal distribution. Instead here we try to analyze the problem more generally. Henceforth we also assume \( w = 1 \) but the requisite adjustments for any other value of \( w \) are trivial.

A. characteristic function

First let us introduce the characteristic function \( \phi(t) \) of the distribution \( f(x) \): namely,
\[ \phi(t) = \int_{-\infty}^{\infty} dx e^{itx} f(x). \tag{10} \]

The derivatives of the characteristic function then result in
\[ \phi^{(n)}(t) = it^n \int_{-\infty}^{\infty} dx x^n e^{itx} f(x); \]
\[ \phi^{(n)}(0) = it^n \int_{-\infty}^{\infty} dx x^n f(x), \tag{11} \]
and so \( \phi(0) = 1, \phi'(0) = it \) and \( \phi''(0) = -(\sigma^2 + \mu^2) \). We can also define the shifted characteristic function:
\[ \bar{\phi}(t) = e^{-it\mu} \phi(t) = \int_{-\infty}^{\infty} dx e^{it(x-\mu)} f(x) = \int_{-\infty}^{\infty} dx e^{it\tau} \phi(\tau), \]
\[ \bar{\phi}'(t) = e^{-it\mu} \phi'(t) - i\mu \phi(t); \]
\[ \bar{\phi}''(t) = e^{-it\mu} [\phi''(t) - 2i\mu \phi'(t) - \mu^2 \phi(t)]. \tag{12} \]

Then \( \bar{\phi}(0) = \phi(0) = 1, \bar{\phi}'(0) = \phi'(0) - i\mu \phi(0) = 0, \) and \( \bar{\phi}''(0) = \phi''(0) - 2i\mu \phi'(0) - \mu^2 \phi(0) = -\sigma^2 - \mu^2 + 2\mu^2 - \mu^2 = -\sigma^2. \) In other words, the Maclaurin series coefficients of \( \phi(t) \) result in the sequence of the central moments whereas those of \( \bar{\phi}(t) \) result in the moments about the origin. Furthermore, if the distribution is symmetric about its mean \( \mu \) as in \( f(\mu + \varepsilon) = f(\mu - \varepsilon) \) for any \( \varepsilon \in \mathbb{R} \), then
\[ \phi(t) = \int_{-\infty}^{\infty} de^{it\varepsilon} f(\mu - \varepsilon) = \int_{-\infty}^{\infty} de^{i(\varepsilon - t)\mu} f(\mu + \varepsilon) = \Phi(-t); \tag{13} \]
and also \( \bar{\phi}'(n)(t) = (-1)^n \bar{\phi}'(n)(-t) \). That is to say, the shifted characteristic function of a symmetric distribution is an even function. The converse also holds in that, if the characteristic function is in the form of \( \phi(t) = e^{it\mu} \bar{\phi}(t) \) with an even function such that \( \bar{\phi}(t) = \bar{\phi}(-t) \), the distribution must be symmetric about the mean \( \mu \).

B. distribution of rounded values

The characteristic function may also be inverted to recover the distribution via the inverse Fourier transform: that is,
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dr e^{-itx} \phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dr e^{it(\mu-x)} \bar{\phi}(t). \tag{14} \]

Inserting this into equation (7), we find the expression for the discrete distribution \( F_n \) of the rounded variable in terms of the characteristic function \( \phi(t) \): namely,
\[ F_n = \frac{1}{2\pi} \int_{n+\delta-1/2}^{n+\delta+1/2} dx \int_{-\infty}^{\infty} dr e^{-itx} \phi(t) \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dr \phi(t) \int_{n+\delta-1/2}^{n+\delta+1/2} dx e^{-itx} \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dr \phi(t) \int_{n+\delta-1/2}^{n+\delta+1/2} dx e^{-it(\mu-x)} \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dr \phi(t) \int_{n+\delta-1/2}^{n+\delta+1/2} dx e^{it(\mu-\delta-n)}. \tag{15} \]

Here \( \text{sinc}(x) = x^{-1} \sin x \) for \( x \neq 0 \) and \( \text{sinc}(0) = 1 \). In addition we can also define the characteristic function of \( F_n \). Since \( F_n \) is a discrete distribution, its characteristic function is given by
\[ \Phi(t) = \sum_{n=-\infty}^{\infty} e^{int} F_n \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \phi(\tau) \text{sinc} \left( \frac{\tau}{2} \right) e^{-i\tau\delta} \sum_{n=-\infty}^{\infty} e^{it(\tau-n)} \]
\[ = \sum_{k=-\infty}^{\infty} \phi(\tau + 2\pi k) \text{sinc} \left( \frac{\tau}{2} + \pi k \right) e^{-i(\tau+2\pi k)\delta} \]
\[ = \sum_{k=-\infty}^{\infty} \bar{\phi}(\tau + 2\pi k) \text{sinc} \left( \frac{\tau}{2} + \pi k \right) e^{i(\tau+2\pi k)(\mu-\delta)}. \tag{16} \]

Here we have used the Fourier series representation of the so-called Dirac comb distribution: namely,
\[ \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{int} = \sum_{k=-\infty}^{\infty} \delta(t - \tau + 2\pi k). \tag{17} \]
Then the derivative of $\Phi_i$ is found to be

$$
\frac{d\Phi_i}{dt} = \sum_{k=-\infty}^{\infty} \left\{ \phi'(t + 2\pi k) - i\delta \phi(t + 2\pi k) \left[ \frac{\pi}{2} + \pi k \right] \right\} e^{-i(t+2\pi k)\delta}, 
$$

(18)

while the second-order derivative is given by

$$
\frac{d^2\Phi_i}{dt^2} = \sum_{k=-\infty}^{\infty} \left\{ \phi''(t + 2\pi k) - 2i\delta \phi'(t + 2\pi k) - \delta^2 \phi(t + 2\pi k) \right\} \frac{\pi}{2} + \pi k \right\}
$$

$$
+ 2 \left[ \phi'(t + 2\pi k) - i\delta \phi(t + 2\pi k) \right] \frac{d}{dt} \left[ \frac{\pi}{2} + \pi k \right] + \phi(t + 2\pi k) \frac{d^2}{dt^2} \left[ \frac{\pi}{2} + \pi k \right] \right\} e^{-i(t+2\pi k)\delta}. 
$$

(19)

Here for the sake of clarity, we have not yet introduced the explicit forms for the derivatives of the sinc function,

$$
\frac{d}{dt} \text{sinc} \left( \frac{t}{2} + \pi k \right) = \frac{1}{2} \frac{\cos(t/2 + \pi k) - \cos(t/2 + \pi k)}{t/2 + \pi k} 
$$

and

$$
\frac{d^2}{dt^2} \text{sinc} \left( \frac{t}{2} + \pi k \right) = \frac{1}{4} \left\{ \frac{1}{2} \frac{\cos(t/2 + \pi k) - \cos(t/2 + \pi k)}{(t/2 + \pi k)^2} \right\}. 
$$

(20)

Next given that

$$
\frac{d^k\Phi_i}{dt^k} = i^k \sum_{n=-\infty}^{\infty} n^k e^{i\pi n} F_n \Rightarrow \sum_{n=-\infty}^{\infty} n^k F_n = \frac{1}{k!} \frac{d^k \Phi_i}{dt^k} \bigg|_{t=0}, 
$$

(21)

we can find that

$$
m = \sum_{n=-\infty}^{\infty} n F_n = \frac{1}{i} \left. \frac{d\Phi_i}{dt} \right|_{t=0} = \mu - \delta + S_0 
$$

$$
s^2 = \sum_{n=-\infty}^{\infty} (n - m)^2 F_n = - \left. \frac{d^2 \Phi_i}{dt^2} \right|_{t=0} = -m^2 
$$

$$
= \sigma^2 + \frac{1}{12} - S_1 - S_0^2, 
$$

(22)

where

$$
S_0 = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \phi(2\pi k)}{2\pi ik} e^{-2\pi ik\delta} 
$$

$$
= \sum_{k=-\infty}^{\infty} \frac{(-1)^k \phi(2\pi k)}{2\pi ik} e^{2\pi ik(\mu - \delta)}, 
$$

(23)

and

$$
S_1 = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\pi k} \left[ \phi'(2\pi k) - i\mu \phi(2\pi k) - \frac{\phi(2\pi k)}{2\pi} \right] e^{-2\pi ik\delta} 
$$

$$
= \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\pi k} \left[ \phi'(2\pi k) - \frac{\phi(2\pi k)}{2\pi} \right] e^{2\pi ik(\mu - \delta)}. 
$$

(24)

Here we have used the fact that sinc$(\pi k) = (\pi k)^{-1} \sin(\pi k) = 0$ for any non-zero integer $k \in \mathbb{Z} - \{0\}$ as well as

$$
\frac{d}{dt} \text{sinc} \left( \frac{t}{2} + \pi k \right) \bigg|_{t=0} = \left\{ \begin{array}{ll}
\frac{(-1)^k}{2\pi k} & k \in \mathbb{Z} - \{0\}, \\
0 & k = 0
\end{array} \right. 
$$

(25)

and

$$
\frac{d^2}{dt^2} \text{sinc} \left( \frac{t}{2} + \pi k \right) \bigg|_{t=0} = \left\{ \begin{array}{ll}
\frac{(-1)^{k+1}}{2(\pi k)^2} & k \in \mathbb{Z} - \{0\}, \\
-\frac{1}{12} & k = 0
\end{array} \right. 
$$

(26)

Equations (22) indeed reproduce the results expected from the elementary arguments given in the introduction with the proviso that the infinite sums, $S_0$ and $S_1$ in equations (23) and (24) are negligible. In other words, if one considers only the $k = 0$ term in the characteristic function of equation (16), we would recover the results that $m = \mu - \delta$ and $s^2 = \sigma^2 + 1/12$. In fact, if one regards $F_n$ to be a continuous distribution over real $n$ and replace the infinite sum $\sum_{n=-\infty}^{\infty} e^{i(t-\tau)n}$ in equation (16) with the integral $\int_{-\infty}^{\infty} e^{i(t-\tau)n} dn$, the Dirac comb $\sum_{n=-\infty}^{\infty} \delta(t - \tau + 2\pi k)$ would be replaced by a single Dirac delta $\delta(t - \tau)$. That is to say, the naive expectation that $m = \mu - \delta$ and $s^2 = \sigma^2 + 1/12$ may be considered as the approximation in the limit of continuous $F_n$. 

C. symmetric distribution

If \( M \in \mathbb{Z} \) is the integer to which the mean \( \mu \) is rounded, \( \mu \in [M + \delta - 1/2, M + \delta + 1/2] \) and \( \chi = \mu - \delta - M \in [-1/2, 1/2] \). Since \( M \) and \( k \) are integers and \( \mu - \delta = M + \chi \), we find

\[
S_0 = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \phi(2\pi k)}{2\pi ik} e^{2\pi ik\chi},
\]

\[
S_1 = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\pi k} \left[ \varphi'(2\pi k) - \frac{\phi(2\pi k)}{2\pi k} \right] e^{2\pi ik\chi},
\]

(27)

which is basically the Fourier series expressions of \( S_0(\chi) \) and \( S_1(\chi) \) for \( \chi \in [-1/2, 1/2] \). Since \( \phi(t) = \bar{\phi}(t) \) and \( \varphi(t) = -\bar{\varphi}(t) \) for a symmetric distribution, these can be further reducible to the real ones:

\[
S_0 = \sum_{k=1}^{\infty} \frac{(-1)^k \phi(2\pi k)}{\pi k} \sin(2\pi k\chi);
\]

\[
S_1 = \sum_{k=1}^{\infty} \frac{2\cdot(-1)^k}{\pi k} \left[ \varphi'(2\pi k) - \frac{\phi(2\pi k)}{2\pi k} \right] \cos(2\pi k\chi)
\]

(28)

\[
= \sum_{k=1}^{\infty} 4\cdot(-1)^k \frac{d}{dt} \left( \frac{\phi(t)}{t} \right) \bigg|_{t=2\pi k} \cos(2\pi k\chi)
\]

if \( f(x) \) is symmetric about its mean.

Next, consider the family of the distributions sharing the common normal form; namely,

\[
f(x) = \frac{1}{\sigma} F \left( \frac{x - \mu}{\sigma} \right)
\]

(29)

where \( F(u) \) is a fixed non-negative function such that \( \int_{-\infty}^{\infty} du F(u) = 1 \), \( \int_{-\infty}^{\infty} du u F(u) = 0 \), and \( \int_{-\infty}^{\infty} du u^n F(u) = 1 \). Then, for all members of the family, we find \( \phi(t) = e^{\mu t} \Phi(\sigma t) \) and \( \bar{\phi}(t) = \Phi(\sigma t) \) where

\[
\Phi(\tau) = \int_{-\infty}^{\infty} du e^{\mu t} F(u)
\]

(30)

is the characteristic function of the normalized distribution. Here \( f(x) \) is a symmetric distribution if and only if \( F(u) \) and \( \Phi(\tau) \) are even functions: \( F(-u) = F(u) \) and \( \Phi(-\tau) = \Phi(\tau) \). In the limit of \( \sigma = 0 \) essentially \( F(u) = \delta(u) \) we then have \( \phi(t) = \Phi(0) = 1 \) and \( \bar{\phi}(t) = \sigma \Phi'(\sigma t) = 0 \) and so

\[
\lim_{\sigma \to 0} S_0 = \sum_{k=1}^{\infty} \frac{(-1)^k}{\pi k} \sin(2\pi k\chi) = -\chi;
\]

\[
\lim_{\sigma \to 0} S_1 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(\pi k)^2} \cos(2\pi k\chi) = \frac{1}{12} - \chi^2,
\]

(31)

for \( \chi \in [-1/2, 1/2] \). Then it follows that \( m = \mu - \delta - \chi = M \) and \( s^2 = \sigma^2 + 1/12 - (1/12 - \chi^2)^2 - (\chi^2) = \sigma^2 = 0 \), as expected (i.e., every sample point is rounded to the same integer).

Now suppose that \( \Phi(\tau) \) admits an asymptotic expansion;

\[
\Phi(\tau) \sim \frac{1}{|\tau|^s} \sum_{p=0}^{\infty} \Phi_{\omega p} \frac{1}{\tau^{2p}} \quad (\tau \to \infty)
\]

(32)

with a constant \( s > 0 \). Then it follows from equation (28) that

\[
S_0 \approx \sum_{p=0}^{\infty} \Phi_{\omega p} \frac{1}{\tau^{2p}} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2\pi k\chi)}{(\pi k)^{2p+1}}\]

(33)

\[
S_1 \approx \sum_{p=0}^{\infty} (2p + s + 1) \Phi_{\omega p} \frac{1}{\tau^{2p+s}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(2\pi k\chi)}{(\pi k)^{2p+s+2}}.
\]

Here the inner sums on \( k \) converge absolutely for \( s > 0 \) (NB: the sums for an even integer \( s \) are actually reducible to the Bernoulli polynomials) given that

\[
\left| \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2\pi k\chi)}{(\pi k)^{2p+1}} \right| \leq \sum_{k=1}^{\infty} \frac{|(-1)^k \sin(2\pi k\chi)|}{(\pi k)^{2p+s+1}}
\]

\[
\leq \sum_{k=1}^{\infty} \frac{1}{(\pi k)^{2p+s+1}} = \frac{\zeta(2p + s + 1)}{\pi^{2p+s+1}},
\]

(34)

where \( \zeta(\cdot) \) is the Riemann zeta function and similarly

\[
\left| \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(2\pi k\chi)}{(\pi k)^{2p+s+2}} \right| \leq \frac{\zeta(2p + s + 2)}{\pi^{2p+s+2}}.
\]

(35)

If \( \chi = 1/2 \), then \( \cos(2\pi k\chi) = \cos(\pi k) = (-1)^k \) for any integer \( k \) and so the last bound is actually sharp. By contrast, the first bound is not sharp but it suffices for our purposes here. Since \( \lim_{\tau \to \infty} \zeta(\tau) = 1 \) and \( \zeta(\tau) \) for \( \tau > 1 \) is monotonically decreasing, we can conclude that equations (33) is in fact valid asymptotic expansion of \( S_0 \) and \( S_1 \) as \( \sigma \to \infty \). Also it follows that, if \( \lim_{\tau \to \infty} d\ln|\Phi(\tau)|/d\ln|\tau| = -s < 0 \), we have \( S_0 \to \sigma^{-s} \to 0 \) and \( S_1 \to \sigma^{-s} \to 0 \) as \( \sigma \to \infty \) as well as \( m = \mu - \delta + 0(\sigma^{-s}) \) and \( s^2 = \sigma^2 + 1/12 + 0(\sigma^{-s}) \). As a concrete example, consider the bilateral exponential (Laplace) distribution of the mean \( \mu \) and the variance \( \sigma^2 \):

\[
f(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\sqrt{2}|x - \mu|}{\sigma} \right),
\]

(36)

which is easily normalizable so that

\[
F(u) = \frac{e^{-\sqrt{2}|u|}}{\sqrt{2}}, \quad \Phi(\tau) = \left( 1 + \frac{\tau^2}{2} \right)^{-1}.
\]

(37)

Here we find \( \Phi(\tau) = -(\sum_{k=1}^{\infty} (2/\tau^2)^k) \approx 2/\tau^2 \) as \( \tau \to \infty \) and so it should be that \( m = \mu - \delta + 0(\sigma^{-2}) \) and \( s^2 = \sigma^2 + 1/12 + 0(\sigma^{-2}) \). In fact for this case, we know the analytic forms for \( S_0(\chi) \) and \( S_1(\chi) \). That is to say, let us consider the odd function for \( \chi \in [-1/2, 1/2] \) given by

\[
S_0(\chi) = \sum_{k=1}^{\infty} \frac{\sin[(\sqrt{2}/2)\pi k\chi/(2k+1)]}{(2k+1)!\sigma^{2k}} \chi,
\]

(38)

\[
= \frac{\sinh[(\sqrt{2}/2)\pi \chi/2]}{2\sinh[1/(\sqrt{2}\sigma)]} - \chi,
\]

where \( B_{2n}(z) \) is the Bernoulli Polynomial. Then we find that

\[
\int_{-1/2}^{1/2} d\chi S_0(\chi) \sin(2\pi k\chi) = \frac{(-1)^k}{2\pi k[1 + 2(\pi k\sigma)^2]}.
\]

(39)
for a positive integer $k$. It follows that the first infinite sum in equation (28) with $\phi(t) = \Phi(\sigma t) = [1 + (\sigma t)^2]^{-1}$ is the Fourier (sine) series expansion for $S_0(\chi)$ in equation (38). In other words, if we sample the random variable $x$ distributed according to equation (36) and round it to an integer $n$ such that $n + \delta - 1/2 \leq x < n + \delta + 1/2$ or $n + \delta - 1/2 \leq x \leq n + \delta + 1/2$ with a fixed $\delta \in [-1/2, 1/2]$, the mean of $n$ is

$$m = \mu - \delta + \bar{S}_0 = M + \frac{\sinh(\sqrt{2}\chi/\sigma)}{2\sinh[1/(\sqrt{2}\sigma)]} \approx \mu - \delta - \frac{\chi(1 - 4\chi^2)}{12\sigma^2} + \sigma(\sigma^{-4}),$$

where $M$ is the integer to which $\mu$ is rounded and $\chi = \frac{\mu - \delta - \bar{S}_0}{\sigma}$. Similarly we can also establish that the second infinite sum in equation (28) with the same $\phi(t)$ is a Fourier series representation of

$$S_1(\chi) = \sum_{k=1}^{2^k B_{2k+1} (1/2 + \chi)} \frac{(k+1)(2k)! \sigma^{2k}}{2k+1} = \sigma^2 + \frac{\chi^2}{12} - \frac{\chi^4}{4} + \frac{\chi}{\sigma^2} + \sigma(\sigma^{-4}),$$

and so the variance of the rounded integers sampled over the random variables with the distribution in equation (36) is

$$\sigma^2 = \frac{2\cosh[1/(\sqrt{2}\sigma)] \cosh(\sqrt{2}\chi/\sigma) - \sinh(\sqrt{2}\chi/\sigma)}{4\sinh^2[1/(\sqrt{2}\sigma)]} \approx \sigma^2 + \frac{1}{12} - \frac{7}{480} - \frac{\chi^2}{4} + \frac{\chi}{2} \frac{1}{\sigma^2} + \sigma(\sigma^{-4}).$$

(42)

### III. EXPECTATION VALUES FOR UNSPECIFIED MEAN

The results so far have concerned the distributions with a known fixed mean. Here instead we consider the cases of unspecified means. That is to say, let us calculate the expectation values for the (difference to the true) mean and the variance of the rounded variables averaged over distributions with all possible means. In practice, this is achieved by averaging over $x \in [-1/2, 1/2]$ and so $\langle m - \mu \rangle = -\delta + \bar{S}_0$ and $\langle s^2 \rangle = \sigma^2 + 1/12 - \langle S_1 \rangle - \langle S_0 \rangle$, where $\langle S_0 \rangle = \int_{-1/2}^{1/2} \chi S_0(\chi)$ and so on. However, we have $\langle 2\pi k \chi \rangle = \langle \sin(2\pi k \chi) \rangle = \langle \cos(2\pi k \chi) \rangle = 0$ for any non-zero integer $k$ and thus $\langle S_k \rangle = \langle S_1 \rangle = 0$ given equations (27) and (28). As for $\langle S_0 \rangle$, let us first note $\langle S_0 \rangle = \langle S_0 \rangle^2$. Rather from equation (27),

$$\langle S_0 \rangle^2 = \sum_{k,p=\pm \infty \atop k,p \neq 0} (-1)^{k+p} \frac{\phi(2\pi k) \phi(2\pi p)}{(2\pi)^2 k p} (e^{2\pi i (k+p) x}) \langle e^{2\pi i (k+p) x} \rangle$$

$$= \sum_{k=-\infty \atop k \neq 0}^{\infty} \frac{\phi(2\pi k) \phi(-2\pi k)}{(2\pi k)^2} = \sum_{k=1}^{\infty} \frac{\phi(2\pi k))^2}{2(2\pi k)^2},$$

where we have used the fact that $\bar{\phi}(-t)$ is the complex conjugate of $\phi(t)$ for any real $x$ (see eq. (12)). The same result for the symmetric distributions may also be derived from equation (28) given $\langle \sin^2(2\pi \chi) \rangle = 1/2$ and $\langle \sin(2\pi \chi) \sin(2\pi \chi) \rangle = 0$ for positive integers $k \neq p$. Consequently

$$\langle m \rangle = \mu - \delta, \quad \langle s^2 \rangle = \sigma^2 + \frac{1}{2} - \sum_{k=1}^{\infty} \frac{\phi(2\pi k)^2}{2(2\pi k)^2},$$

with $\lim_{\sigma \to 0} \langle S_0 \rangle = \xi(2)/(2\pi^2) = 1/12$ (given $\phi(t) = 1$ for $\sigma = 0$) and $\langle s^2 \rangle = \sigma^2 = 0$ in the limit of $\sigma = 0$.

If $\Phi(t)$ is given by the same function admitting the asymptotic expansion of equation (32), we find

$$\langle S_0 \rangle = \sum_{p=0}^{\infty} \frac{\xi(2p+2)}{2^{2p+2} \pi^{2p} + 2^{p+2}}$$

and so $\langle S_0 \rangle \sim \sigma^{-2s} \to 0$ and $\langle s^2 \rangle = \sigma^2 + 1/12 + \sigma(\sigma^{-2s})$ (also $\leq \langle s^2 \rangle \leq \sigma^2 + 1/12$ as $\sigma \to \infty$ for $\Phi(\tau) \sim \tau^{-s}$ as $\tau \to \infty$). That is to say, $\langle s^2 \rangle$ typically tends to the limiting value $\lim_{\sigma \to \infty} \langle \sigma^2 - \langle \chi^2 \rangle \rangle = 1/12$ about twice much faster than the individual $s^2$ does. For example, with the bilateral exponential distribution given in equation (39), we specifically have

$$\langle S_0^2 \rangle = \sum_{p=0}^{\infty} \frac{2\langle (2\pi k)^2 \rangle}{(2\pi k)^2} = \sum_{p=0}^{\infty} \frac{2\langle (2\pi k)^2 \rangle}{(2\pi k)^2} = \sum_{p=0}^{\infty} \frac{2\langle (2\pi k)^2 \rangle}{(2\pi k)^2}$$

$$= \sum_{p=0}^{\infty} \frac{2\langle (2\pi k)^2 \rangle}{(2\pi k)^2} = \sum_{p=0}^{\infty} \frac{2\langle (2\pi k)^2 \rangle}{(2\pi k)^2} = \sum_{p=0}^{\infty} \frac{2\langle (2\pi k)^2 \rangle}{(2\pi k)^2}$$

where $B_k$ is the Bernoulli number, and so

$$\langle s^2 \rangle = \frac{2 + 3\sqrt{2} \sinh(\sqrt{2}/\sigma)}{16\sinh^2[1/(\sqrt{2}\sigma)]}$$

$$= \sigma^2 + \frac{1}{12} - \frac{1}{\sigma^2} \sum_{p=0}^{\infty} \frac{2\langle (2\pi k)^2 \rangle}{(2\pi k)^2}$$

(47)

that is, $\langle s^2 \rangle \approx \sigma^2 + 1/12 - 1/(7560\sigma^4) + \sigma(\sigma^{-6})$, which contrasts to equation (42).

### A. distributions with a compact support

Suppose that $f(x)$ is

$$f(x) = \begin{cases} \frac{1}{2\sqrt{3}\sigma} & \mu - \sqrt{3}\sigma \leq x \leq \mu + \sqrt{3}\sigma \\ 0 & \text{elsewhere} \end{cases}$$

(48)
i.e. the uniform distribution over a compact interval, the normal form of which is

$$F(u) = \begin{cases} 
\frac{1}{2\sqrt{3}} & (-\sqrt{3} \leq u \leq \sqrt{3}) \\
0 & \text{elsewhere}
\end{cases}$$

where $\Delta_\pm = \mu - \delta / 2 \pm \sqrt{3}\sigma - m_\pm$ is the fractional part of $\mu \pm \sqrt{3}\sigma - m_\pm$ and $m_\pm = (1/2 - \mu - \delta \pm \sqrt{3}\sigma)$ is the integer to which the upper/lower limit of the compact support (i.e. $\mu \pm \sqrt{3}\sigma$) is rounded. In addition, $\lambda = (\Delta_+ + \Delta_-) / 2$ and $\xi = (\Delta_+ - \Delta_-) / 2$. Also used are $-1)^k \sin(2\pi k\chi) = \sin(2\pi k(1/2 + \mu - \delta))$ for any integer $k$ given $\chi = -\delta - M$ with an integer $M$, and $B_2(x) = x^2 - x + 1 / 6$. Similarly, 

$$S_1 = \sum_{k=1}^\infty (-1)^k \left[ \frac{\cos(2\sqrt{3}\sigma\pi k)}{(\pi k)^2} - \frac{\sin(2\sqrt{3}\sigma\pi k)}{\sqrt{3}\sigma(\pi k)^3} \right] \cos(2\pi k\chi)$$

$\Phi(\tau) = \int_{-\sqrt{3}}^{\sqrt{3}} e^{i\tau u} du = \sin(\sqrt{3}\tau) / \sqrt{3}\tau$.

We then find for the compact uniform distribution that

$$\langle S_0 \rangle = \frac{\sin^2(2\sqrt{3}\pi\sigma \xi)}{4\pi^2(\pi k)^2} = \sum_{k=1}^\infty \frac{1 - \cos(4\sqrt{3}\pi\sigma k)}{48\sigma^2(\pi k)^4}$$

$$= \frac{\pi^2(1 - \xi^2)^2}{144\sigma^2}$$

where $\xi = 2\sqrt{3}\sigma - [2\sqrt{3}\sigma] \in (0, 1)$ is the fractional part of $2\sqrt{3}\sigma$ (which is the width of the support)”. Here we have used the Fourier series expansion of the Bernoulli polynomial (for $0 \leq \xi \leq 1$) of the even order

$$B_2n(\xi) = (-1)^{n+1}(2n)! \sum_{k=1}^\infty \cos(2\pi k\xi)/(\pi k)^{2n}$$

with $\zeta(4) = \pi^4 / 90$ and $B_2(\xi) = \xi(\xi - 1)^2 / 3$. That is to say, while the remainder $(S_0) = \sigma^2 + 1 / 12 - (\zeta^2)$ falls off “on average” like $\sigma^{-2}$ as $\sigma \to \infty$, the actual behavior includes the periodic modulation superimposed on the asymptotic scale-free decay. This is due to the compact support on the underlying distribution: note that a compact distribution $F(u)$ typically results in an oscillatory $\Phi(\tau)$, and $(S_0)$ is basically the sum on the regular sampling of the latter. The resulting modulation of $(S_0)$ may be understood as a sort of interference patterns between the width of the compact support and the unit intervals for the rounded integer values. However, unless the variance of the underlying continuous distribution $f(x)$ is known a priori, the averaged asymptotic behavior of $(S_0)$ as $\sigma \to \infty$ can be used to evaluate $\sigma^2$ from $\sigma^2$ in practice within a reasonable accuracy (provided $\sigma^2 >> 1$). If one is in fact only interested in the averaged asymptotic behavior, we can further average $(S_0)$ in equation (50) over $\xi \in [0, 1]$ and get $(S_0) = \zeta(4)/(48\sigma^2 \pi^4) = (4320\sigma^2)^{-1}$. For a general distribution with a compact support, one may obtain the averaged asymptotic behavior for $(S_0)$ in equation (49) by assuming that any sum of the form $\sum \sin(\alpha \xi) / k^n$ or $\sum \cos(\alpha \xi) / k^n$ (where $\alpha$ is a fixed real constant) also vanishes on average.

In principle we can also calculate $S_0$ and $S_1$ first, and subsequently sum over the proper interval. For the uniform distribution in equation (49), equation (50) results in

$$\langle S_0 \rangle = \frac{-1)^k \sin(2\sqrt{3}\pi\sigma k)}{2\sqrt{3}\pi\sigma(\pi k)^2} \sin(2\pi k\chi)$$

$$= \sum_{k=1}^\infty \frac{\cos(2\pi \Delta_\pm k) - \cos(2\pi \Delta_\pm k)}{4\sigma^2(\pi k)^2}$$

$$= \frac{B_2(\Delta_-) - B_2(\Delta_+)}{4\sqrt{3}\sigma}$$

where $\Delta_\pm = \mu - \delta / 2 \pm \sqrt{3}\sigma - m_\pm$ is the fractional part of $\mu \pm \sqrt{3}\sigma - \delta / 2$ and $m_\pm = (1/2 - \mu - \delta \pm \sqrt{3}\sigma)$ is the integer to which the upper/lower limit of the compact support (i.e. $\mu \pm \sqrt{3}\sigma$) is rounded. In addition, $\lambda = (\Delta_+ + \Delta_-) / 2$ and $\xi = (\Delta_+ - \Delta_-) / 2$. Also used are $-1)^k \sin(2\pi k\chi) = \sin(2\pi k(1/2 + \mu - \delta))$ for any integer $k$ given $\chi = -\delta - M$ with an integer $M$, and $B_2(x) = x^2 - x + 1 / 6$. Similarly, 

$$S_1 = \sum_{k=1}^\infty (-1)^k \left[ \frac{\cos(2\sqrt{3}\pi\sigma k)}{(\pi k)^2} - \frac{\sin(2\sqrt{3}\pi\sigma k)}{\sqrt{3}\sigma(\pi k)^3} \right] \cos(2\pi k\chi)$$

further utilizing $B_3(x) = x^3 - 3x^2 / 2 + x / 2$ and the Fourier series for the odd-order Bernoulli polynomial:

$$\langle S_0 \rangle = \frac{\sin^2(2\sqrt{3}\pi\sigma \xi)}{4\pi^2(\pi k)^2} = \sum_{k=1}^\infty \frac{1 - \cos(4\sqrt{3}\pi\sigma k)}{48\sigma^2(\pi k)^4}$$

$$= \frac{\pi^2(1 - \xi^2)^2}{144\sigma^2}$$

where $\Delta_\pm = \mu - \delta / 2 \pm \sqrt{3}\sigma - m_\pm$ (continued)
which recovers equation (50). Since \( S_0 \) is an odd function of \( \lambda \), it is immediately obvious that \( S_0 = 0 \). As for \( S_1 \), two respective integrals for \( \xi = \xi_1 / 2 \geq 0 \) and \( \xi = (\xi - 1) / 2 < 0 \) exactly cancel each other and so \( S_1 = 0 \).

Since \( \chi \) (i.e. the offset of the mean from an integer value) determines both \( \Delta_\pm \) (i.e. the offsets of the boundary points of the support from integer values) once \( \sigma \) (which specifies the width of the support) is fixed, \( \Delta_+ \) and \( \Delta_- \) are not in fact independent from each other. Nevertheless it is still formally possible to consider \( S_0 \) in equation (52) as a function of the pair of independent variables \( (\Delta_+, \Delta_-) \in [0, 1)^2 \) and average \( S_0^2 \) over this whole rectangular domain. Following the coordinate transform \( (\Delta_+, \Delta_-) \to (\lambda, \xi) \), the resulting average is shown to be identical to further averaging the \( \chi \)-average of equation (55) over \( \xi \in (-1/2, 1/2) \) or equivalently \( \xi \in [0, 1] \). That is to say, if one assumes that both upper and lower boundaries of the compact support are randomly placed relative to the integer values (and independent from each other), the resulting expectation value (averaged over all possible such placements) recovers only the “slow” asymptotic decay behavior of \( \langle s^2 \rangle \) for the rounded random variables on a compact support while averaging off the “fast” modulation due to the interference between the width of the support and the integer signposts.

**IV. NORMAL DISTRIBUTIONS**

Finally we would like to consider the case of \( f(x) \) being the Gaussian normal distribution:

\[
f(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right),
\]

or in the standard form

\[
F(u) = \frac{e^{-u^2/2}}{\sqrt{2\pi}}, \quad \Phi(\tau) = e^{-\tau^2/2}.
\]

Then with \( \check{\phi}(\tau) = \Phi(\sigma\tau) \), we have

\[
S_0 = \sum_{k=1}^{\infty} (-1)^k \sin(2\pi k \chi) \frac{e^{-2(\pi\sigma)^2}}{\pi k};
S_1 = \sum_{k=1}^{\infty} (-1)^k \cos(2\pi k \chi) \frac{d}{d\kappa} \left( \frac{e^{-2(\pi\sigma)^2}}{\kappa} \right)_{\kappa=k}
= \sum_{k=1}^{\infty} (-1)^{k+1} \cos(2\pi k \chi) \frac{1 + (2\pi \sigma)^2}{(2\pi k \sigma)^2} e^{-2(\pi\sigma)^2}.
\]

Thanks to the super-exponential decay \( \propto e^{-2(\pi\sigma)^2} \) in \( k \) (NB: the \( k = 2 \) term is suppressed relative to the \( k = 1 \) term by \( \sim e^{-6(\pi\sigma)^2} \); if \( \sigma = 1 \), note \( e^{-6\pi^2} \approx 2 \times 10^{-20} \)!!), these sums (which are actually in the form of the Jacobi theta function and its antiderivatives) converge extremely quickly and are completely dominated by their respective first terms unless \( \sigma \ll 1 \). Alternatively we may construct a more formal argument by bracketing the infinite sums following the integral convergence test. In particular, we first observe that

\[
|S_0| \leq \sum_{k=1}^{\infty} e^{-2(\pi\sigma)^2} = I_0, \quad |S_1| \leq \sum_{k=1}^{\infty} \frac{1 + (2\pi\sigma)^2}{(2\pi k \sigma)^2} e^{-2(\pi\sigma)^2} = I_1
\]

but the summands now are strictly decreasing positive functions of \( k \geq 1 \). Hence the integral convergence test indicates

\[
\frac{E_1(2\pi^2\sigma^2)}{2\pi} \leq I_0 \leq \frac{e^{-2(\pi\sigma)^2}}{\pi} + E_1(2\pi^2\sigma^2),
\]

\[
\frac{e^{-2(\pi\sigma)^2}}{\pi^2} \leq I_1 \leq \frac{(2\pi\sigma)^2 + 2}{2\pi} e^{-2(\pi\sigma)^2},
\]

where \( E_1(x) = \int_1^\infty \frac{e^{-t}}{t} dt \) is the analytic exponential integral. Note that \( I_0 \) for \( \sigma = 0 \), which results in the harmonic series, actually diverges and so the first bounds are only valid for \( \sigma > 0 \), but it has been already shown that \( S_0 = -\chi \) if \( \sigma = 0 \). Given the asymptotic expansion \( e^{\sigma}E_1(\chi) \sim \sum_{k=0}^{\infty} (-1)^k k! / x^{k+1} \) as \( x \to \infty \), the first bounds may also be replaced by the purely elementary functions. In particular, for \( x > 0 \), we find

\[
e^{\sigma}E_1(x) = \int_1^\infty \frac{e^{(1-t)/x}}{t} dt < \int_1^\infty \frac{dt}{e^{-t(x-1)}} = \frac{1}{x},
\]

i.e. \( E_1(x) < e^{-x}/x \) for \( x > 0 \), and so follows that

\[
|S_0| \leq I_0 \leq \sum_{k=1}^{\infty} \frac{e^{-2(\pi\sigma)^2}}{\kappa}.
\]

That is to say, for a sufficiently large \( \sigma \), both sums \( I_0 \) and \( I_1 \) are completely dominated by their respective first terms, and \( S_0 \sim O(e^{-2(\pi\sigma)^2}) \) and \( S_1 \sim O(\sigma^2 e^{-2(\pi\sigma)^2}) \) as \( \sigma \to \infty \). In conclusion, the mean \( m \) and the variance \( s^2 \) of the rounded variables drawn from the normal distribution (of the mean \( \mu \) and the variance \( \sigma^2 \)) behave like

\[
m \approx \mu - \delta + O(e^{-2(\pi\sigma)^2});
\]

\[
s^2 \approx \sigma^2 + 1 + O(e^{-2(\pi\sigma)^2}),
\]

as \( \sigma \to \infty \), but the remainder in most practical purposes can be safely ignored provided \( \sigma \geq 1 \) (NB: \( e^{-\pi^2}/\pi \approx 8.5 \times 10^{-10} \)).

As for the expectation value averaged over \( \mu \) at fixed \( \sigma^2 \), if we consider the sum

\[
\langle S_0^2 \rangle = \sum_{k=1}^{\infty} \frac{e^{-2(\pi\sigma)^2}}{(2\pi k)^2},
\]

the integral test is still applicable:

\[
\int_1^\infty \frac{e^{-2(\pi\sigma)^2}}{2(\pi x)^2} dx \leq \langle S_0^2 \rangle \leq \int_1^\infty \frac{e^{-2(\pi\sigma)^2}}{2(\pi x)^2} dx + \int_1^\infty \frac{e^{-2(\pi\sigma)^2}}{2(\pi x)^2} dx.
\]

While here the integral is technically reducible to the incomplete gamma function (or an expression involving the error function), it is sufficient for our purpose to note

\[
e^a \int_0^a \frac{e^{-ax^2}}{x^2} dx = \int_0^a \frac{e^{-at} dt}{2(t+1)^{3/2}} < \int_0^a \frac{e^{-at} dt}{2} = \frac{1}{2a}.
\]
for $a > 0$. Hence it follows that, for $\sigma > 0$,
\[
\langle s_0^2 \rangle = \sigma^2 + \frac{1}{2} - \langle s^2 \rangle \leq \left[ \frac{1}{2} + \frac{1}{(4\pi\sigma)^2} \right] \frac{e^{-(2\pi\sigma)^2}}{\pi^2}; \quad (67)
\]
\[
\therefore \langle s^2 \rangle \simeq \sigma^2 + \frac{1}{12} + \mathcal{O}(e^{-(2\pi\sigma)^2}) \quad (68)
\]
with an even faster-decaying (cf. $e^{-(2\pi)^2}/\pi^2 \approx 7.2 \times 10^{-19}$) remainder term. In summary, the rounding errors for the normally distributed random variables can for most practical applications be considered as independent from the intrinsic dispersion unless the intrinsic dispersion itself is quite smaller than the rounding unit.

1. A. R. Tricker, “Effects of Rounding on the Moments of a Probability Distribution,” The Statistician, 33, 381–390 (1984)
2. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, “NIST Handbook of Mathematical Functions,” (Cambridge Univ. Press, Cambridge. 2010) [http://dlmf.nist.gov]