1. Introduction

There have been numerous attempts in the literature to generalize results in robust control theory (42; 45) to linear time-varying (LTV) systems (for e.g. (10–13; 30; 33; 37; 39; 40) and references therein). In (12)(13) and (11) the authors studied the optimal weighted sensitivity minimization problem, the two-block problem, and the model-matching problem for LTV systems using inner-outer factorization for positive operators. Abstract solutions involving the computation of induced operators norms of operators are obtained. However, there is no clear indication on how to compute optimal linear LTV controllers. In (40) the authors rely on state space techniques which lead to algorithms based on infinite dimensional operator inequalities which are difficult to solve. These methods lead to suboptimal controllers and are restricted to finite dimensional systems. An extension of these results to uncertain systems is reported in (41) relying on uniform stability concepts. In (9) both the sensitivity minimization problem in the presence of plant uncertainty, and robust stability for LTV systems in the $\ell^\infty$ induced norm is considered. However, their methods could not be extended to the case of systems operating on finite energy signals. In (37) the standard problem of $H^\infty$ control theory for finite-dimensional LTV continuous-time plants is considered. It is shown that a solution to this problem exists if and only if a pair of matrix Riccati differential equations admits positive semidefinite stabilizing solutions. State-space formulae for one solution to the problem are also given.

The gap metric was introduced to study stability robustness of feedback systems. It induces the weakest topology in which feedback stability is robust (6; 7; 31; 32; 38). Extensions of the gap to time-varying systems have been proposed in (33; 34) where a geometric framework was developed. Several results on the gap metric and the gap topology were established, in particular, the concept of a graphable subspace was introduced. In (21) the problem of robust stabilization for LTV systems subject to time-varying normalized coprime factor uncertainty is considered. Operator theoretic results which generalize similar results known to hold for linear time-invariant (infinite-dimensional) systems are developed. In particular, a tight upper bound for the maximal achievable stability margin under TV normalized coprime factor uncertainty in terms of the norm of an operator with a time-varying Hankel structure is computed.

Analysis of time-varying control strategies for optimal disturbance rejection for known time-invariant plants has been studied in (2; 16). A robust version of these problems was
considered in (8; 15) in different induced norm topologies. All these references showed that for time-invariant nominal plants and weighting functions, time-varying control laws offer no advantage over time-invariant ones.

In this paper, we are interested in optimal disturbance rejection for (possibly infinite-dimensional, i.e., systems with an infinite number of states) LTV systems. These systems have been used as models in computational linear algebra and in a variety of computational and communication networks (17). This allows variable number of states which is predominant in networks which can switch on or off certain parts of the system (17), and infinite number of states as in distributed parameter systems.

Using inner-outer factorizations as defined in (3; 11) with respect of the nest algebra of lower triangular (causal) bounded linear operators defined on \(\ell^2\) we show that the problem reduces to a distance minimization between a special operator and the nest algebra. The inner-outer factorization used here holds under weaker assumptions than (12; 13), and in fact, as pointed in ((3) p. 180), is different from the factorization for positive operators used there.

The optimal disturbance attenuation for LTV systems has been addressed using Banach space duality theory in (20; 28). Its robust version which deals with plant uncertainty is addressed in (4; 5; 19) using also duality theory ideas. Furthermore, using the commutant lifting theorem for nest algebras the optimum is shown to be equal to the norm of a compact time-varying Hankel operator defined on the space of causal Hilbert-Schmidt operators. The latter is the “natural” analogue to the Hankel operator used in the LTI case. An operator identity to compute the optimal TV Youla parameter is also provided.

The results are generalized to the mixed sensitivity problem for TV systems as well, where it is shown that the optimum is equal to the operator induced of a TV mixed Hankel-Toeplitz operator generalizing analogous results known to hold in the linear time-invariant (LTI) case (22; 38; 43).

Our approach is purely input-output and does not use any state space realization, therefore the results derived here apply to infinite dimensional LTV systems, i.e., TV systems with an infinite number of state variables (33). Although the theory is developed for causal stable system, it can be extended in a straightforward fashion to the unstable case using coprime factorization techniques for LTV systems discussed in (11; 13).

The rest of the chapter is organized as follows. Section 2 the commutant lifting theorem for nest algebras is introduced. In section 3 the optimal disturbance rejection problem is formulated and solved in terms of a TV Hankel operator. A Generalization to the TV mixed sensitivity problem is carried out in section 4. Section 5 contains some concluding remarks.

**Definitions and notation**

- \(\mathcal{B}(E, F)\) denotes the space of bounded linear operators from a Banach space \(E\) to a Banach space \(F\), endowed with the operator norm

\[
\|A\| := \sup_{x \in E, \|x\| \leq 1} \|Ax\|, \quad A \in \mathcal{B}(E, F)
\]

- \(\ell^2\) denotes the usual Hilbert space of square summable sequences with the standard norm

\[
\|x\|_2^2 := \sum_{j=0}^{\infty} |x_j|^2, \quad x := (x_0, x_1, x_2, \cdots) \in \ell^2
\]
• $P_k$ the usual truncation operator for some integer $k$, which sets all outputs after time $k$ to zero.

• An operator $A \in \mathcal{B}(E, F)$ is said to be causal if it satisfies the operator equation:
  \[ P_k A P_k = P_k A, \forall k \text{ positive integers} \]

• $tr(\cdot)$ denotes the trace of its argument.

The subscript “c” denotes the restriction of a subspace of operators to its intersection with causal (see (11; 29) for the definition) operators. “⊕” denotes for the direct sum of two spaces. “⋆” stands for the adjoint of an operator.

2. The commutant lifting theorem

The commutant lifting theorem has been proposed by Sz.Nagy and Foias (35; 36). It has been used successfully to solve several interpolation problems including $H^\infty$ control problems for linear time invariant (LTI) systems (31; 32; 43; 44). In this chapter, we rely on a time-varying version of the commutant lifting theorem which corresponds to nest or triangular algebras. Following (3; 18) a nest $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ is a family of closed subspaces of $\mathcal{H}$ ordered by inclusion. The triangular or nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators $T$ such that $T \mathcal{N} \subseteq \mathcal{N}$ for every element $\mathcal{N}$ in $\mathcal{N}$. A representation of $\mathcal{T}(\mathcal{N})$ is an algebra homomorphism $h$ from $\mathcal{T}(\mathcal{N})$ into the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. A representation is contractive if $\|h(A)\| \leq \|A\|$, for all $A \in \mathcal{T}(\mathcal{N})$. It is weak* continuous if $h(A_i)$ converges to zero in the weak* topology of $\mathcal{B}(\mathcal{H})$ whenever the net $\{A_i\}$ converges to zero in the weak* topology of $\mathcal{B}(\mathcal{H})$. The representation $h$ is said to be unital if $h(I_{\mathcal{H}}) = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$, and $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$. The Sz. Nagy Theorem asserts that any such a representation $h$ has a $\mathcal{B}(\mathcal{H})$-dilation, that is, there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and a positive representation $H$ of $\mathcal{B}(\mathcal{H})$ such that $P_H H(A) |_{\mathcal{H}} = h(A)$, where $P_H$ is the orthogonal projection from $\mathcal{K}$ into $\mathcal{H}$ (3; 18).

We now state the commutant lifting theorem for nest algebras from (3; 18) (see also references therein).

**Theorem 1.** (3; 18) Let

\[
\begin{align*}
  h : \mathcal{T}(\mathcal{N}) &\rightarrow \mathcal{B}(\mathcal{H}) \\
  h^* : \mathcal{T}(\mathcal{N}) &\rightarrow \mathcal{B}(\mathcal{H}')
\end{align*}
\]

be two unital weak* continuous contractive representations with $\mathcal{B}(\mathcal{H})$-dilations

\[
\begin{align*}
  H : \mathcal{B}(\mathcal{H}) &\rightarrow \mathcal{B}(\mathcal{K}) \\
  H^* : \mathcal{B}(\mathcal{H}) &\rightarrow \mathcal{B}(\mathcal{K}')
\end{align*}
\]

respectively. Assume that $X : \mathcal{H} \rightarrow \mathcal{H}'$ is a linear operator with $\|X\| \leq 1$, such that $Xh(A) = h'(A)X$ for all $A \in \mathcal{T}(\mathcal{N})$, that is, $X$ intertwines $h$ and $h'$. Then there exists an operator $Y : \mathcal{K} \rightarrow \mathcal{K}'$ such that

i) $\|Y\| \leq 1$.

ii) $Y$ intertwines $H$ and $H'$, that is, $YH(A) = H'(A)Y$ for all $A \in \mathcal{B}(\mathcal{H})$. 

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iii) $Y$ dilates $X$, that is, $Y : \mathcal{M} \mapsto \mathcal{M}'$, and $P \mathcal{H} / Y |_{\mathcal{M}} = XP \mathcal{H} |_{\mathcal{M}}$, where $\mathcal{H} = \mathcal{M} \ominus \mathcal{N}$ is the orthogonal representation of $\mathcal{H}$ as the orthogonal difference of invariant subspaces for $H|_{T(\mathcal{N})}$, and similarly for $\mathcal{H}'$.

In the next section the optimal disturbance rejection problem is formulated and solved using this Theorem in terms of a TV Hankel operator.

### 3. Time-varying optimal disturbance rejection problem

In this chapter, we first consider the problem of optimizing performance for causal linear time varying systems by considering the standard block diagram for the optimal disturbance attenuation problem represented in Fig. 1, where $u$ represents the control inputs, $y$ the measured outputs, $z$ is the controlled output, $w$ the exogenous perturbations. $P$ denotes a causal stable linear time varying plant, and $K$ denotes a time varying controller. The closed-loop transmission from $w$ to $z$ is denoted by $T_{zw}$. Using the standard TV Youla parametrization of all stabilizing controllers the closed loop operator $T_{zw}$ can be written as 

$$
T_{zw} = T_1 - T_2 QT_3
$$

where $T_1$, $T_2$ and $T_3$ are stable causal LTV operators, that is, $T_1, T_2$ and $T_3 \in B_c(\ell^2, \ell^2)$. Here it is assumed without loss of generality that $P$ is stable, the Youla parameter $Q := K(I + PK)^{-1}$ is then an operator belonging to $B_c(\ell^2, \ell^2)$, and is related in a one-to-one onto fashion to the controller $K$ (29). Note that $Q$ is allowed to be time-varying. If $P$ is unstable it suffices to use the coprime factorization techniques in (11; 39) which lead to similar results. The magnitude of the signals $w$ and $z$ is measured in the $\ell^2$-norm. The performance index which quantifies
optimal disturbance rejection can be written in the following form (20)

\[
\mu := \inf \left\{ \| T_{2w} \| : K \text{ being robustly stabilizing linear time-varying controller} \right\} \\
= \inf_{Q \in \mathcal{B}_c(\ell^2, \ell^2)} \| T_1 - T_2 QT_3 \|
\]

(2)

The performance index (2) will be transformed into a shortest distance minimization between a certain bounded linear operator and a subspace to be specified shortly. In order to do so, following (11) define a nest \( \mathcal{N} \) as a family of closed subspaces of the Hilbert space \( \ell^2 \) containing \( \{0\} \) and \( \ell^2 \) which is closed under intersection and closed span. Let \( Q_n := I - P_n \) for \( n = -1, 0, 1, \ldots \), where \( P_{-1} := 0 \) and \( P_{\infty} := I \). Then \( Q_n \) is a projection, and we can associate to it the following nest \( \mathcal{N} := \{ Q_n \ell^2, \ n = -1, 0, 1, \ldots \} \). In this case the triangular or nest algebra \( \mathcal{T}(\mathcal{N}) \) is the set of all operators \( T \) such that \( TN \subseteq N \) for every element \( N \) in \( \mathcal{N} \). That is

\[
\mathcal{T}(\mathcal{N}) = \{ A \in \mathcal{B}(\ell^2, \ell^2) : P_n A(I - P_n) = 0, \forall n \} \\
= \{ A \in \mathcal{B}(\ell^2, \ell^2) : (I - Q_n)AQ_n = 0, \forall n \}
\]

(3)

Note that the Banach space \( \mathcal{B}_c(\ell^2, \ell^2) \) is identical to the nest algebra \( \mathcal{T}(\mathcal{N}) \). For \( N \) belonging to the nest \( \mathcal{N} \), \( N \) has the form \( Q_n \ell^2 \) for some \( n \). Define

\[
N^- = \bigcap \{ N' \in \mathcal{N} : N' < N \} \\
N^+ = \bigcup \{ N' \in \mathcal{N} : N' > N \}
\]

(4)

(5)

where \( N' < N \) means \( N' \subset N \), and \( N' > N \) means \( N' \supset N \). The subspaces \( N \cap N^- \) are called the atoms of \( \mathcal{N} \). Since in our case the atoms of \( \mathcal{N} \) span \( \ell^2 \), then \( \mathcal{N} \) is said to be atomic (3).

The early days of \( H^\infty \) control theory saw solutions based on the so-called inner-outer factorizations of functions belonging to the Hardy spaces \( H^2 \) and \( H^\infty \), and their corresponding matrix valued counterparts for multi-input multi-output (MIMO) systems (22; 23). Generalizations in the context of nest algebras have been proposed in (1; 3) as follows: An operator \( A \) in \( \mathcal{T}(\mathcal{N}) \) is called outer if the range projection \( P(R_A) \), \( R_A \) being the range of \( A \) and \( P \) the orthogonal projection onto \( R_A \), commutes with \( N \) and \( AN \) is dense in \( N \cap R_A \) for every \( N \in \mathcal{N} \). A partial isometry \( U \) is called inner in \( \mathcal{T}(\mathcal{N}) \) if \( U^*U \) commutes with \( \mathcal{N} \) (1; 3; 11). In our case, \( A \in \mathcal{T}(\mathcal{N}) = \mathcal{B}_c(\ell^2, \ell^2) \) is outer if \( P \) commutes with each \( Q_n \) and \( A(AQ_n)\ell^2 \) is dense in \( Q_n \ell^2 \cap A\ell^2 \). \( U \in \mathcal{B}_c(\ell^2, \ell^2) \) is inner if \( U \) is a partial isometry and \( U^*U \) commutes with every \( Q_n \). Applying these notions to the time-varying operator \( T_2 = T_{2i}T_{2o} \), where \( T_{2i} \) and \( T_{2o} \) are inner outer operators in \( \mathcal{B}_c(\ell^2, \ell^2) \), respectively. Similarly, the operator \( T_3 \) can be factored as \( T_3 = T_{3o}T_{3i} \) where \( T_{3i} \in \mathcal{B}_c(\ell^2, \ell^2) \) is inner, \( T_{3o} \in \mathcal{B}_c(\ell^2, \ell^2) \) is outer. The performance index \( \mu \) in (2) can then be written as

\[
\mu = \inf_{Q \in \mathcal{B}_c(\ell^2, \ell^2)} \| T_1 - T_{2i}T_{2o}QT_{3o}T_{3i} \|
\]

(6)

Following the classical \( H^\infty \) control theory (22; 23; 45), we assume (A1) that \( T_{2o} \) and \( T_{3o} \) are invertible both in \( \mathcal{B}_c(\ell^2, \ell^2) \).

Assumption (A1) can be relaxed by assuming instead that the outer operators \( T_{2o} \) and \( T_{3o} \) are bounded below (see Lemma (1) p. 220). Assumption (A1) guarantees that the map \( Q \mapsto T_{2o}\mathcal{B}_c(\ell^2, \ell^2)T_{3o} \) is bijective. Under this assumption \( T_{2i} \) becomes an isometry and \( T_{3i} \)
a co-isometry in which case $T_2^*T_2 = I$ and $T_3^*T_3^* = I$. The operators $T_{2o}$ and $T_{3o}$ can be absorbed into the Youla operator $Q$, and expression (6) is then equivalent to

$$\mu = \inf_{Q \in \mathcal{B}_c(\ell^2, \ell^2)} \| T_2^*T_1T_3^* - Q \| \tag{7}$$

Expression (7) is the distance from the operator $T_2^*T_1T_3^* \in \mathcal{B}(\ell^2, \ell^2)$ to the nest algebra $\mathcal{B}_c(\ell^2, \ell^2)$. It is the shortest distance from the bounded linear operator $T_2^*T_1T_3^*$ to the space of causal bounded linear operators $\mathcal{B}_c(\ell^2, \ell^2)$, which is a subspace of $\mathcal{B}(\ell^2, \ell^2)$. In the sequel, the commutant lifting theorem is used to solve the minimization (7) in terms of a time varying version of Hankel operators. First, let $C_2$ denote the special class of compact operators on $\ell^2$ called the Hilbert-Schmidt or Schatten 2-class [3; 14] under the norm,

$$\|A\|_2 := \left( \text{tr}(A^*A) \right)^{\frac{1}{2}} \tag{8}$$

Note that $C_2$ is a Hilbert space under the inner product (3)

$$(A, B) = \text{tr}(B^*A), \quad \forall \ A, B \in C_2 \tag{9}$$

Define the space

$$\mathcal{A}_2 := C_2 \cap \mathcal{B}_c(\ell^2, \ell^2) \tag{10}$$

Then $\mathcal{A}_2$ is the space of causal Hilbert-Schmidt operators. This space can be viewed as the TV counterpart of the standard Hardy space $H^2$ in the standard $H^\infty$ theory. Define the orthogonal projection $P$ of $C_2$ onto $\mathcal{A}_2$. $P$ is the lower triangular truncation, and is analogous to the standard positive Riesz projection (for functions on the unit circle) for the LTI case. Following (27) an operator $X$ in $\mathcal{B}(\ell^2, \ell^2)$ determines a Hankel operator $H_X$ on $\mathcal{A}_2$ if

$$H_X A = (I - P)XA, \quad \text{for } A \in \mathcal{A}_2 \tag{11}$$

We shall show that the shortest distance $\mu$ is equal to the norm of a particular LTV Hankel operator using the time varying version of the commutant lifting theorem in Theorem 1, thus generalizing a similar result in the LTI setting. Let $H_B$ be the Hankel operator $(I - P)B^*P$ associated with the symbol $B := T_2^*T_1T_3^*$. The Hankel operator $H_B$ belongs to the Banach space of bounded linear operators on $C_2'$, namely, $\mathcal{B}(C_2, C_2')$. We have then the following Theorem which relates the optimal distance minimization $\mu$ to the induced norm of the Hankel operator $H_{T_2^*T_1T_3^*}$.

**Theorem 2.** Under assumptions (A1) the following holds:

$$\mu = \| H_{T_2^*T_1T_3^*} \| = \| (I - P)T_2^*T_1T_3^*P \| \tag{12}$$

**Proof.** Following (3; 18) let $H_1 = \mathcal{A}_2$ and $H_2 = C_2 \ominus \mathcal{A}_2$ the orthogonal complement of $\mathcal{A}_2$ in $C_2$. Define the representations $h$ and $h'$ of $\mathcal{A}_2$ by

$$h(A) = R_A|_{H_1}, \quad A \in \mathcal{B}_c(\ell^2, \ell^2) \tag{13}$$

$$h'(A) = (I - P)R_A|_{H_2}, \quad A \in \mathcal{B}_c(\ell^2, \ell^2) \tag{14}$$
where now $R_A$ denotes the right multiplication associated to the operator $A$ defined on the specified Hilbert space, i.e., $R_A B = BA, B \in A_2$. The representation $h(\cdot)$ and $h'(\cdot)$ have dilations $H = H'$ given by

$$H(A) = H'(A) = R_A \text{ on } C_2, \ A \in B_c(\ell^2, \ell^2)$$ (15)

Let $M := B_c(\ell^2, \ell^2), N = \{0\}, M' := C_2, N' := A_2$, and $H_1 = M \otimes N, H_2 = M' \otimes N'$ are orthogonal representations of $H_1$ and $H_2$ of invariant subspaces under $H|_{B_c(\ell^2, \ell^2)}$, that is, $R_A B_c(\ell^2, \ell^2) \subset B_c(\ell^2, \ell^2)$. Now we have to show that the operator $H_{T_2} T_1 T_{3c1}'$ intertwines $h$ and $h'$, that is, if $B := T_2 T_1 T_{3c1}'$, then $h'(A) H_B = H_B h(A)$ holds for for all $A \in B_c(\ell^2, \ell^2), h'$.

Applying the Commutant Lifting Theorem for representations of nest algebras implies that $H_B$ has a dilation $\tilde{H}_B$ that intertwines $H$ and $H'$, i.e., $\tilde{H}_B H(A) = H'(A) \tilde{H}_B, \forall A \in B(\ell^2, \ell^2)$. By Lemma 4.4. in (18) $\tilde{H}_B$ is a left multiplication operator acting from $A_2$ into $C_2 \oplus A_2$. That is, $\tilde{H}_B = L_K$ for some $K \in B(\ell^2, \ell^2)$, with $\|L_K\| = \|K\| = \|\tilde{H}_X\| = \|H_B\|$ by Lemma 4.5. (18).

By Lemma 4.3. (18) $K = B - Q, \exists Q \in B_c(\ell^2, \ell^2)$ with $\|K\| = \|H_B\|$ as required.

By Theorem 2.1. (26) the Hankel operator $H_B$ is a compact operator if and only if $B$ belongs to the space $B_c(\ell^2, \ell^2) + \mathcal{K}$, where $\mathcal{K}$ is the space of compact operators on the Hilbert space $\ell^2$. A basic property of compact operators on Hilbert spaces is that they have maximizing vectors, that is, there exists at least one operator $A_0 \in A_2, \|A_0\|_2 = 1$ such that $H_B$ achieves its induced norm at $A_0$. That is,

$$\|H_B A_0\|_2 = \|H_B\| \|A_0\|_2 = \|H_B\|$$ (19)

We can then deduce from (7) and (12) an operator identity for the minimizer, that is, the optimal TV Youla parameter $Q_0$ follows

$$Q_0 A_0 = T_{21} T_1 T_{3c1} A_0 - H T_{21} T_1 T_{3c1} A_0$$

where the unknown is $Q_0$.

In the next section the mixed sensitivity problems for LTV systems is formulated and solved using the commutant lifting theorem.

4. The time-varying mixed sensitivity problem

The mixed sensitivity problem for stable plants (42; 45) involves the sensitivity operator $T_1 := \left( \begin{array}{c} W \\ 0 \end{array} \right)$, the complementary sensitivity operator $T_2 := \left( \begin{array}{c} W \\ V \end{array} \right) P$ and $T_3 := I$ which are all assumed to belong to $B_c(\ell^2, \ell^2 \times \ell^2)$, and is given by the minimization (13; 20; 46)

$$\mu_o = \inf_{Q \in B_c(\ell^2, \ell^2)} \left\| \left( \begin{array}{c} W \\ 0 \end{array} \right) - \left( \begin{array}{c} W \\ V \end{array} \right) PQ \right\|$$ (20)

where $\| \cdot \|$ stands for the operator norm in $B(\ell^2, \ell^2 \times \ell^2)$ given by

$$\|B\| = \sup_{\|x\|_2 \leq 1, x \in \ell^2} \left( \|B_1 x\|_2^2 + \|B_2 x\|_2^2 \right)^{1/2}, \quad B = \left( \begin{array}{c} B_1 \\ B_2 \end{array} \right)$$ (21)
The optimization problem (20) can be expressed as a shortest distance problem from the operator $T_1$ to the subspace $S = T_2P \mathcal{B}_c(\ell^2, \ell^2)$ of $\mathcal{B}(\ell^2, \ell^2 \times \ell^2)$.

To ensure closedness of $S$, we assume that $W^*W + V^*V > 0$, i.e., $W^*W + V^*V$ as an operator acting on $\ell^2$ is a positive operator. In this case, there exists an outer spectral factorization $\Lambda_1 \in \mathcal{B}_c(\ell^2, \ell^2)$, invertible in $\mathcal{B}_c(\ell^2, \ell^2)$ such that $\Lambda_1^*\Lambda_1 = W^*W + V^*V$ (11). Consequently, $\Lambda_1 P$ as a bounded linear operator in $\mathcal{B}_c(\ell^2, \ell^2)$ has an inner-outer factorization $U_1G$, where $U_1$ is inner and $G$ an outer operator defined on $\ell^2$.

Next we assume (A2) $G$ is invertible, so $U_1$ is unitary, and the operator $G$ and its inverse $G^{-1} \in \mathcal{B}_c(\ell^2, \ell^2)$. The assumption (A2) is satisfied when, for e.g., the outer factor of the plant is invertible. Let $R := T_2\Lambda_1^{-1}U_1$, assumption (A2) implies that the operator $R^*R \in \mathcal{B}(\ell^2, \ell^2)$ has a bounded inverse, this ensures closedness of $S$. It follows from Corollary 2 (1), that the self-adjoint operator $R^*R$ has a spectral factorization of the form: $R^*R = \Lambda^*\Lambda$, where $\Lambda$, $\Lambda^{-1} \in \mathcal{B}_c(\ell^2, \ell^2)$.

Define the operator $R_2 := R\Lambda^{-1}$, then $R_2^*R_2 = I$, and $S$ has the equivalent representation, $S = R_2\mathcal{B}_c(\ell^2, \ell^2)$. After "absorbing" $\Lambda$ into the free parameter $Q$, the optimization problem (20) is then equivalent to:

$$\mu_o = \inf_{Q \in \mathcal{B}_c(\ell^2, \ell^2)} \|T_1 - R_2Q\|$$  \hspace{1cm} (22)

The minimization problem (22) gives the optimal mixed sensitivity with respect to controller design (as represented by $Q$). It is solved in terms of a projection of a multiplication operator. If the minimization (22) is achieved by a particular $Q_o$, we call it optimal.

**Theorem 3.** Introduce the orthogonal projection $\Pi$ as follows

$$\Pi : A_2 \oplus A_2 \mapsto (A_2 \oplus A_2) \ominus R_2A_2$$

Under assumptions (A2) the following holds:

$$\mu_o = \|\Pi T_1\|$$ \hspace{1cm} (23)

**Proof.** Denote by $S := (A_2 \oplus A_2) \ominus R_2A_2$. That is, $S$ is the orthogonal complement of the subspace $R_2A_2$ in $A_2 \oplus A_2$, and define the operator

$$\Xi : A_2 \rightarrow S$$

$$\Xi := \Pi T_1$$  \hspace{1cm} (24)

We shall show with the help of the commutant lifting theorem that

$$\mu_o = \|\Xi\|$$ \hspace{1cm} (25)

To see this we need, as before, a representation of $\mathcal{B}_c(\ell^2, \ell^2)$, that is, an algebra homomorphism, say, $h(\cdot)$ (respectively $h'(\cdot)$), from $\mathcal{B}_c(\ell^2, \ell^2)$, into the algebra $B(A_2, A_2)$ (respectively $\mathcal{B}_c(S, S)$), of bounded linear operators from $A_2$ into $A_2$ (respectively from $S$ into $S$). Define the representations $h$ and $h'$ by

$$h : \mathcal{B}_c(\ell^2, \ell^2) \rightarrow B(A_2, A_2), \quad h' : \mathcal{B}_c(\ell^2, \ell^2) \rightarrow \mathcal{B}_c(S, S)$$

$$h(A) := R_A, \quad A \in \mathcal{B}_c(\ell^2, \ell^2), \quad h'(A) := \Pi R_A, \quad A \in \mathcal{B}_c(\ell^2, \ell^2)$$  \hspace{1cm} (26)
where now $R_A$ denotes the right multiplication associated to the operator $A$ defined on the specified Hilbert space. By the Sz. Nagy dilation Theorem there exist dilations $H$ (respectively $H'$) for $h$ (respectively $h'$) given by

$$H(A) = R_A \text{ on } A_2 \text{ for } A \in \mathcal{B}_c(\ell^2, \ell^2)$$

$$H'(A) = R_A \text{ on } A_2 \oplus A_2 \text{ for } A \in \mathcal{B}_c(\ell^2, \ell^2)$$

The spaces $A_2$ and $S$ can be written as orthogonal differences of subspaces invariant under $H$ and $H'$, respectively, as

$$A_2 = A_2 \oplus \{0\}, \quad S = A_2 \oplus A_2 \oplus R_2 A_2$$

Now we have to show that the operator $\Xi$ intertwines $h$ and $h'$, that is, $h'(A)\Xi = \Xi h(A)$ for all $A \in \mathcal{B}_c(\ell^2, \ell^2)$,

$$h'(A)\Xi = \Pi R_A \Pi T_1 |_{A_2} = \Pi R_A \Pi T_1 |_{A_2} = \Pi R_A T_1 |_{A_2} = \Pi T_1 R_A |_{A_2} = \Xi h(A)$$

Applying the commutant lifting theorem for representations of nest algebras implies that $\Xi$ has a dilation $\Xi'$ that intertwines $H$ and $H'$, i.e., $\Xi' H(A) = H'(A)\Xi', \forall A \in \mathcal{B}(\ell^2, \ell^2)$. By Lemma 4.4. in (18) $\Xi'$ is a left multiplication operator acting from $A_2$ into $A_2 \oplus A_2$, and causal. That is, $\Xi' = L_K$ for some $K \in \mathcal{B}_c(A_2, A_2 \oplus A_2)$, with $\|K\| = \|\Xi'\| = \|\Xi\|$. Then $\Xi = \Pi T_1 = \Pi K$, which implies that $\Pi(T_1 - K) = 0$. Hence, $(T_1 - K)f \in R_2 A_2$, for all $f \in A_2$. That is, $(T_1 - K)f = R_2 g$, $g \in A_2$, which can be written as $R_2(T_1 - K)f = g \in A_2$. In particular, $R_2(T_1 - K)f \in \mathcal{B}_c(\ell^2, \ell^2)$, for all $f \in \mathcal{B}_c(\ell^2, \ell^2)$ of finite rank. By Theorem 3.10 (3) there is a sequence $F_n$ of finite rank contractions in $\mathcal{B}_c(\ell^2, \ell^2)$ which converges to the identity operator in the strong *-topology. By an approximation argument it follows that $R_2(T_1 - K) \in \mathcal{B}_c(\ell^2, \ell^2)$. Letting $Q := R_2^*(T_1 - K)$ we have $g = Qf$. We conclude that $T_1 - K = R_2 Q$, that is, $T_1 - R_2 Q = K$, with $\|K\| = \|\Xi\|$, and the Theorem is proved.

The orthogonal projection $\Pi$ can be computed as

$$\Pi = I - R_2 \mathcal{P} R_2^*$$

where $I$ is the identity operator on $A_2 \oplus A_2$, $R_2^*$ is the adjoint operator of $R_2$. To see that (30) holds note that for any $Y \in A_2 \oplus A_2$, we have

$$(I - R_2 \mathcal{P} R_2^*)^2 Y = (I - R_2 \mathcal{P} R_2^*)(I - R_2 \mathcal{P} R_2^*)Y$$

$$= (I - R_2 \mathcal{P} R_2^* - R_2 \mathcal{P} R_2^* + R_2 \mathcal{P} R_2^* \mathcal{P} R_2^*)Y$$

but $R_2^* R_2 = I$ and $\mathcal{P}^2 = \mathcal{P}$, therefore

$$(I - R_2 \mathcal{P} R_2^*)^2 Y = (I - R_2 \mathcal{P} R_2^*)Y$$

This shows that $(I - R_2 \mathcal{P} R_2^*)$ is a projection. The adjoint of $(I - R_2 \mathcal{P} R_2^*)$, $(I - R_2 \mathcal{P} R_2^*)^*$, is clearly equal to $(I - R_2 \mathcal{P} R_2^*)$ showing that it is an orthogonal projection. Now we need to show that the null space of $(I - R_2 \mathcal{P} R_2^*)$ is $R_2 A_2$. Let $Z \in A_2 \oplus A_2$ such that $(I - R_2 \mathcal{P} R_2^*)Z = 0$, so $Z = R_2 \mathcal{P} R_2^* Z$. But $R_2^* Z \in \ell_2$, then $\mathcal{P} R_2^* Z \in A_2$, implying that $Z \in R_2 A_2$. We have showed
that the null space of the projection \((I - RPR_2^*)\) is a subset of \(R_2A_2\). Conversely, let \(Z \in A_2\), then
\[
(I - RPR_2^*)R_2Z = R_2Z - R_2PZ = R_2Z - R_2Z = 0
\]
(34)
hence \(R_2Z\) belongs to the null space of \((I - RPR_2^*)\), and (30) holds.
The operator \(\Xi\) has the following explicit form
\[
\Xi = (I - R_2PR_2^*)T_1
\]
(35)
which leads to the explicit solution
\[
\mu_0 = \| (I - R_2PR_2^*)T_1 \|
\]
(36)
The expression generalizes the solution of the mixed sensitivity problem in the LTI case (25; 43; 46) to the LTV case. This result also applies to solve the robustness problem of feedback systems in the gap metric (38) in the TV case as outlined in (11; 21; 33), since the latter was shown in (11) to be equivalent to a special version of the mixed sensitivity problem (20).

5. Conclusion

The optimal disturbance rejection and the mixed sensitivity problems for LTV systems involve solving shortest distance minimization problems posed in different spaces of bounded linear operators. LTV causal and stable systems form a nest algebras, this allows the commutant lifting theorem for nest algebras to be applied and solve both problems in term of abstract TV Hankel and a TV version generalization of Hankel-Topelitz operators under fairly weak assumptions. Future work includes investigation of numerical solutions based on finite dimensional approximations, and computation of the corresponding controllers.

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6. References

[1] Arveson W. Interpolation problems in nest algebras, Journal of Functional Analysis, 4 (1975) pp. 67-71.
[2] Chapellat H., Dahleh M. Analysis of time-varying control strategies for optimal disturbance rejection and robustness, IEEE Transactions on Automatic Control, 37 (1992) 1734-1746.
[3] Davidson K.R. Nest Algebras, Longman Scientific & Technical, UK, 1988.
[4] Djouadi S.M. Optimal Robust Disturbance Attenuation for Continuous Time-Varying Systems, Proceedings of CDC, vol. 4, (1998) 3819-3824. 1181-1193.
[5] Djouadi S.M. Optimal Robust Disturbance Attenuation for Continuous Time-Varying Systems, International journal of robust and non-linear control, vol. 13, (2003) 1181-1193.
[6] G. Zames and A.K. El-Sakkary, Unstable Systems and Feedback: The Gap Metric, in Proc. of the Allerton Conference, pp. 380-385, 1980.
[7] S.Q. Zhu, M.L.J. Hautus and C. Praagman, Sufficient Conditions for Robust BIBO Stabilisation: Given by the Gap Metric, Systems and Control Letters, vol. 11, pp. 53-59, 1988.
[8] Khammash M., Dahleh M. Time-varying control and the robust performance of systems with structured norm-bounded perturbations, *Proceedings of the IEEE Conference on Decision and Control*, Brighton, UK, 1991.

[9] Khammash M., J.B. Pearson J.B. Performance robustness of discrete-time systems with structured uncertainty, *IEEE Transactions on Automatic Control*, 36 (1991) 398-412.

[10] A. Ichikawa, Quadratic Games and $H^\infty$-Type Problems for Time-Varying Systems, Int. J. Control, vol. 54, pp. 1249-1271, 1991.

[11] Feintuch A. *Robust Control Theory in Hilbert Space*, Springer-Verlag, vol. 130, 1998.

[12] Feintuch A., Francis B.A. Uniformly Optimal Control of Linear Time-Varying Systems, Systems & Control Letters, vol. 5, (1984) 67-71.

[13] Feintuch A., Francis B.A. Uniformly Optimal Control of Linear Feedback Systems, Systems & Control Letters, vol. 21, (1985) 563-574.

[14] Schatten R. *Norm Ideals of Completely Continuous Operators*, Springer-Verlag, Berlin, Gottingen, Heidelberg, 1960.

[15] Shamma J.S. Robust stability with time-varying structured uncertainty, *Proceedings of the IEEE Conference on Decision and Control*, 3163-3168, 1992.

[16] Shamma J.S., Dahleh M.A. Time-varying versus time-invariant compensation for rejection of persistent bounded disturbances and robust stabilization, *IEEE Transactions on Automatic Control*, 36 (1991) 838-847.

[17] P. De Wilde A-J. Van der Veen, Time-Varying Systems and Computations, Springer-Verlag, 1998.

[18] J.A. Ball, Commutant Lifting and Interpolation: The Time Varying Case, Integral Equat. and Operator Theory, vol. 25, pp. 377-405, 1996.

[19] S.M. Djouadi and C.D. Charalambous, Time-Varying Optimal Disturbance Minimization in Presence of Plant Uncertainty, SIAM Journal on Optimization and Control, Vol. 48, No. 5, pp. 3354-3367, 2010

[20] S.M. Djouadi, Disturbance rejection and robustness for LTV Systems, Proceedings of the American Control Conference, pp. 3648-2653-6080, June 14-26, 2006.

[21] S. M. Djouadi, Commutant Lifting for Linear Time-Varying Systems, American Control Conference Proceedings, St. Louis, MO, June 10-12, 2009, pp. 4067-4072.

[22] Francis B.A., Doyle J.C. Linear Control Theory with an $H^\infty$ Optimality Criterion, SIAM J. Control and Optimization, vol. 25, (1987) 815-844.

[23] Francis B.A. *A Course in $H^\infty$ Control Theory*, Springer-Verlag, 1987.

[24] Zhou K., Doyle J.C., Glover K. *Robust and Optimal Control*, Prentice Hall, 1996.

[25] M. Verma and E. Jonckheere, $L^\infty$-Compensation with Mixed Sensitivity as a Broadband Matching Problem, Systems & Control Letters, vol. 4, pp. 125-129, 1984.

[26] Power S. Commutators with the Triangular Projection and Hankel Forms on Nest Algebras, J. London Math. Soc., vol. 2, (32), (1985) 272-282.

[27] Power S.C. Factorization in Analytic Operator Algebras, J. Func. Anal., vol. 67, (1986) 413-432.

[28] S.M. Djouadi and C.D. Charalambous, On Optimal Performance for Linear-Time Varying Systems, Proc. of the IEEE 43th Conference on Decision and Control, Paradise Island, Bahamas, pp. 875-880, December 14-17, 2004.

[29] Feintuch A., Saeks R. *System Theory: A Hilbert Space Approach*, Academic Press, N.Y., 1982.

[30] Peters M.A. and Iglesias P.A., Minimum Entropy Control for Time-varying Systems, Boston, Birkhäuser, 1997.
[31] T.T. Georgiou and M.C. Smith, Optimal Robustness in the Gap Metric, IEEE Trans. on Automatic Control, vol. 35, No. 6, pp. 673-686, 1990.
[32] Georgiou T., On the Computation of the Gap Metric, Systems and Control Letters, vol. 11, pp. 253-257, 1988.
[33] Foias C., Georgiou T. and Smith M.C., Robust Stability of Feedback Systems: A Geometric Approach Using The Gap Metric, SIAM J. Control and Optimization, vol. 31, No.6, pp. 1518-1537, 1993.
[34] Foias C., Georgiou T. and Smith M.C., Geometric Techniques for Robust Stabilization of Linear Time-Varying Systems, Proc. of the 29th CDC, pp. 2868-2873, December 1990.
[35] B. Sz. Nagy and C. Foias, Dilatation des commutants des commutateurs d'operateurs, C.R. Academie Sci. paris, Serie A, 266, pp. 493-495, 1968.
[36] B. Sz. Nagy, C. Foias, H. Bercovici, and L. Kerchy, Harmonic Analysis of Operators on Hilbert Space, Springer, 2010.
[37] Ravi R., Nagpal K.M. and Khargonekar P.P., $H^\infty$ Control of Linear Time-Varying Systems: A State Space Approach, SIAM J. Control and Optimization, vol. 29, No.6, pp. 1394-1413, 1991.
[38] T.T. Georgiou and M.C. Smith, Robust Stabilization in the Gap Metric: Controller Design for Distributed Plants, IEEE Trans. on Automatic Control, vol. 37, No. 8, pp. 1133-1143, 1992.
[39] Dale W.N. and Smith M.C., Stabilizability and Existence of System Representation for Discrete-Time-Varying Systems, SIAM J. Control and Optimization, vol. 31, No.6, pp. 1538-1557, 1993.
[40] Dullerud G.E. and Lall S., A new approach for analysis and synthesis of time-varying systems IEEE Trans. on Automatic Control, Vol.: 44 , Issue: 8 , pp. 1486 - 1497, 1999.
[41] C. Pirie and G.E. Dullerud, Robust Controller Synthesis for Uncertain Time-Varying Systems. SIAM J. Control Optimization., Vol. 40, No. 4, pp. 1312Ü1331, 2002.
[42] J.C. Doyle, B.A. Francis and A.R. Tannenbaum, Feedback Control Theory, Macmillan, NY, 1990.
[43] C. Foias, H. Ozbay and A.R. Tannenbaum, Robust Control of Infinite Dimensional Systems, Springer-Verlag, Berlin, Heidelberg, New York, 1996.
[44] C. Foias, A.E. Frazho, I. Gohberg, and M.A. Kaashoek, Metric constrained interpolation, commutant lifting and systems. Operator Theory: Advances and Applications, 100. Birkhäuser Verlag, Basel, 1998
[45] Zhou K. and Doyle J.C., Essentials of Robust Control, Prentice Hall, 1998.
[46] Djouadi S.M. and Birdwell J.D., On the Optimal Two-Block $H^\infty$ Problem, Proceedings of the American Control Conference, pp. 4289 - 4294, June 2005.
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