A numerical simulation model for a coupled porous medium and surface fluid system with multiphase flow

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Abstract

We developed a numerical method (ASG method) for simulating a coupled atmospheric gas–surface water system, simulated by Navier–Stokes equations, and groundwater system, simulated by water saturation equations. Here, we derived dimensionless formulas for ASG method to avoid the influence of a dimension of analytical domain, in this study. Further we used the ASG method to simulate the configuration and movement of the infiltration front in an embankment composed of sand and obtained results similar to those obtained by a water tank experiment in another study. In addition, in a practice problem using non-dimensional values, we simulated the movement of gas and water when surface water flowed over an embankment expressed in dimensionless form.

Consequently the ASG method could simulate the movement of water and gas across the interface between a surface system and a porous medium including the practice embankment.

Key Words: Multi-flow, Water saturation equation, Coupling, Porous medium, Surface water

1 Introduction

Several floods occur in Japan each year that cause crevasses to form in levees. When heavy rain fell in the Tokai Region on 11–12 September 2000 (Technical Support Unit of APFM, 2004) and in the Kanto–Tohoku Region on 10 September 2015, some people reported seeing signs of gas spouting from the interior of an embankment before the embankment was overtopped and breached (Maeda and Sakai, 2010). These reports indicate that gas within the embankment interacted with the atmospheric gas surrounding the embankment when water level in a river rises. Moreover when the water of the river flows over the embankment, the gas and water in the embankment interact with the water in the river or overtopping water. The interaction between the gas and/or water in the embankment and that in atmosphere and surface water should be important when a numerical simulation for flowing over the embankment is conducted. Furthermore, the degree of water saturation is related to the internal friction angle and the cohesive force of the material (i.e., soil or sand) composing the embankment, both of which are important factors affecting sliding failure. Therefore, numerical simulations of the flow of water and gas in an embankment, the atmosphere, and the surface water when river water flows over the embankment should consider the distribution of water saturation in the embankment as well as fluid interactions in the atmosphere, surface water,

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and the embankment.

In general, single-phase water flow and two-phase gas–water flow simulations in porous media have been employed to simulate the distribution of water saturation in an embankment when the water level of a river rises in flood. Several studies have used the diffusive wave equation (Delfs et al., 2013; Tosaka, 2000; Weill et al., 2011) or the shallow water equation (Discacciati et al., 2002; Maxwell and Kollet, 2008) and have merged these two types of simulations by considering the flux at the boundary between the groundwater and surface water systems. When these equations are used to simulate surface water, the water is assumed to flow in a direction parallel to the ground and pressure in the surface water is hydrostatic. The Darcy–Brinkman model has also been used to formulate governing equations for a groundwater/surface water system (Maeda and Sakai, 2010; Maeda et al., 2006; Onda et al., 2014). This numerical simulation method uses Navier–Stokes equations for multiple immiscible fluids for the surface system coupled with the Darcy–Brinkman model (Hill and Straughan, 2009) to simulate resistance to flow in a porous medium. However, because this numerical simulation method does not take account of the properties of an unsaturated soil or porous medium, the simulated water saturation changes suddenly at the boundary between saturated and unsaturated zones.

To address vertical and complex flows of water and gas in a surface water system and the precise distribution of water saturation in a porous medium, we earlier developed a numerical method for simulating a coupled atmospheric gas, surface water, and groundwater system that we call the ASG simulation method (Hibi and Tomigashi, 2015a, 2015b; Hibi et al., 2015). This simulation method employs a one-field model for multiple immiscible fluids in surface water, water saturation equations, and global pressure. Notably, global pressure is calculated by a new equation, different from that derived by Chavent and Jaffre (1996) and Dahle et al. (2002) for porous media.

To verify the accuracy of ASG simulation method, we compared simulation results with the results of two column experiments (Hibi et al., 2015). The results showed that the ASG method could precisely simulate the movement of gas and water in the atmosphere, surface water, and the porous medium, and the distribution of water saturation in the porous medium. Furthermore, we confirmed that it was possible to simulate precisely the distribution of water saturation in a trapezoidal embankment by comparing simulation results obtained by the ASG method for two-phase groundwater and gas flow (Hibi et al., 2015) with those obtained by the Modified Picard (MP) simulation method developed by Celia and Bouloutas (1990). The distribution of water saturation simulated by the MP method, which is based on pressure equations and mass balance conservation, was similar to that simulated by the ASG method. Thus, so far, water saturation simulated by the ASG method has been compared only with water saturation obtained by a different simulation method in a two-dimensional domain.

The first aim of this study was to indicate that ASG method did not depend on a dimension of analytical domain for the numerical simulation. Then we derived the dimensionless formulas of ASG method in this study. The next aim was to compare the movement of water and water saturation in an embankment composed of a porous medium as simulated by the AGS method with the results of a water tank experiment conducted by Mizutani et al. (2013). Finally, we applied the ASG method to a practice problem to evaluate its ability to simulate the movements of gas and water when water flows over an embankment. To make the practice problem simulation independent of the dimensions of the analytical domain, we applied a scaling law that allows the Navier-Stokes equations, the equation for the conservation of mass for the surface water–atmosphere system,
and the water saturation equations for groundwater to be expressed in dimensionless form.

2 Theory and procedure of the numerical simulation

2.1 Theory

Hibi and Tomigashi (2015a,b) and Hibi et al. (2015) developed the simulation method used in this study, which employs a one-field model for multiple immiscible fluids in surface water and water saturation equations for porous media.

Consider a stream with typical length \( L \) and typical velocity \( V_{typ} \) (i.e., the real velocity). In the model, non-dimensional divergence \( \nabla \) is defined as the product of \( L \) and the divergence of the prototype \( \nabla' \), that is, divergence in the real rectilinear coordinate system, and the non-dimensional velocity vector \( \mathbf{V} \) (called the velocity vector hereafter) is defined as the ratio of the real velocity of the prototype \( V' \) to \( V_{typ} \), as follows:

\[
\nabla = L \nabla', \quad \mathbf{V} = \mathbf{V}' / V_{typ}
\]

Further, non-dimensional time \( t \) in the model (called time hereafter) is defined in terms of the real time \( t' \) of the prototype, \( V_{typ} \), and \( L \) as follows:

\[
t = (V_{typ} / L) \cdot t'
\]

Non-dimensional fluid pressure \( p \) (called fluid pressure hereafter) can be defined as the ratio of the pressure of the prototype \( p' \), which is the real fluid pressure, to the product of \( \rho_0 \) and \( V_{typ}^2 \):

\[
p = p' / (\rho_0 V_{typ}^2)
\]

where \( \rho_0 \) is the representative fluid density.

Relative density \( \rho \) and relative viscosity \( \mu \) are defined by the following ratios:

\[
\rho = \rho' / \rho_0, \quad \mu = \mu' / \mu_0
\]

where \( \mu_0 \) is representative viscosity, \( \rho' \) and \( \mu' \) are real density and viscosity of the fluid, respectively.

The Reynolds number \( R_{Re} \) and Froude number \( F_r \) are defined by Eqs. (5) and (6), respectively.

\[
R_{Re} = LV_{typ} / (\mu_0 / \rho_0)
\]
\[
F_r = V_{typ} / (|g'|L)^{0.5}
\]

where \(|g'|\) is prototype gravity in real space. Furthermore, non-dimensional gravity \( \mathbf{g} \), which is downward when the vertical coordinate \( z \) is positive upward, equals the following vector derived from the definition of Froude number.

\[
\mathbf{g} = (0, 0, 1/F_r^2)
\]

Although the compressibility of a fluid should generally be considered when the flow of a fluid is simulated numerically, a fluid can be treated as incompressible if the Mach number is very small. If the velocity of the fluid is less than 20 m/s and the velocity of sound is 300 m/s, then the Mach number and its square are less than 0.067 and 0.0045, respectively, and are very small. Thus the law for the conservation of mass can be expressed as in Eq. (8) and applied to the water surface–atmosphere system.

\[
\nabla \cdot \mathbf{V} = 0
\]

Navier–Stokes equations can be used along with the above non-dimensional qualities to describe the surface fluid flow:

\[
\partial \mathbf{V} / \partial t + (\mathbf{V} \cdot \nabla) \mathbf{V} = - (1 / \rho) \nabla p + (\mu / \rho) \left(1 / R_{Re} \right) \nabla^2 \mathbf{V} - \mathbf{g}
\]

By applying a semi-Lagrange method with cubic interpolated propagation (CIP) (Yabe, 1991) and the velocity vector \( \mathbf{V}^* \) at the position vector \( \mathbf{R} - \mathbf{V}_t \Delta t \) in the velocity field at time \( t \), where \( \mathbf{R} \) is the position vector at point \( i \), \( \mathbf{V}_t \) is velocity at
time \( t \), and \( \Delta t \) is the time increment, Eq. (9) is transformed by the backward difference method as follows:

\[
\mathbf{V}_{t+\Delta t} - \left( \mu_t / \rho_t \right) (1/R_{so}) \Delta t \nabla \nabla \mathbf{V}_{t+\Delta t} = \nabla \mathbf{V}^* - \left( \Delta t / \rho_t \right) \nabla \left( \nabla \mathbf{p}_{t+\Delta t} + \rho_t \mathbf{g} \right) \tag{10}
\]

where \( \mathbf{V}_{t+\Delta t} \) and \( \mathbf{p}_{t+\Delta t} \) are the velocity and pressure, respectively, of the fluid at time \( t + \Delta t \) and \( \rho_t \) and \( \mu_t \) are the density and viscosity, respectively, of the fluid at time \( t \). \( \left( \mu_t / \rho_t \right) (1/R_{so}) \Delta t \) and \( \left( \Delta t / \rho_t \right) \) in Eq. (10) are constants, because the fluid is incompressible. Consequently, both sides of Eq. (10) can be differentiated in space and transformed as follows:

\[
\nabla \cdot \mathbf{V}_{t+\Delta t} - \left( \mu_t / \rho_t \right) (1/R_{so}) \Delta t \nabla \cdot \left( \nabla \mathbf{V}_{t+\Delta t} \right) = \nabla \cdot \mathbf{V}^* - \left( \Delta t / \rho_t \right) \nabla \cdot \left( \nabla \mathbf{p}_{t+\Delta t} + \rho_t \mathbf{g} \right) \tag{11}
\]

The first term on the left side of Eq. (11), \( \nabla \cdot \mathbf{V}_{t+\Delta t} \) is zero because Eq. (8) should be satisfied at \( t + \Delta t \), and the second term is also zero because \( \nabla \cdot \left( \nabla \mathbf{V}_{t+\Delta t} \right) = \nabla \cdot \left( \nabla \mathbf{V}_{t+\Delta t} \right) = 0 \). As a result, Eq. (11) can be rearranged as follows:

\[
\left( \Delta t / \rho_t \right) \nabla \cdot \left( \nabla \mathbf{p}_{t+\Delta t} + \rho_t \mathbf{g} \right) = \nabla \cdot \mathbf{V}^* \tag{12}
\]

Eqs. (10) and (12) are discretized by the Galerkin finite element method (FEM) as follows:

\[
q_i = (1/\bar{\rho}_i) \sum_{j=1}^{n} N_j \nabla N_i \cdot \nabla N_j dV_j \tag{13a}
\]

\[
\tau_i = (\bar{\mu}_i / R_{so}) \sum_{j=1}^{n} N_j \nabla N_i \cdot \nabla N_j dV_j \mathbf{V}_{j,t+\Delta t} \tag{13b}
\]

Here, \( N_i \) and \( N_j \) are the shape functions of points \( i \) and \( j \), respectively; \( n \) is the number of nodes in the analytical domain; \( \mathbf{V}^* \) is the volume of the analytical domain; \( \mathbf{n} \) is a unit vector normal to the boundary of the analytical domain; \( \mathbf{V}_{i,t+\Delta t} \) is fluid velocity at point \( i \) at time \( t + \Delta t \); \( p_{j,t+\Delta t} \) and \( \mathbf{V}_{j,t+\Delta t} \) are fluid pressure and fluid velocity, respectively, at point \( j \) at time \( t + \Delta t \); \( \bar{\rho}_i \) and \( \bar{\rho}_j \) are the average values of density and viscosity, respectively, in an element of the finite element grid; and \( \mathbf{V}^*_i \) and \( \mathbf{V}^*_j \) are \( \mathbf{V}^* \) at points \( i \) and \( j \), respectively.

The non-dimensional densities of water \( \rho^*_w \) and gas \( \rho^*_g \) relative to the representative density of fluid \( \rho_0 \) are defined as the ratios of the real densities of water \( \rho_w \) and gas \( \rho_g \), respectively, to \( \rho_0 \) as follows:

\[
\rho^*_w = \rho_w / \rho_0, \quad \rho^*_g = \rho_g / \rho_0 \tag{16}
\]

and the non-dimensional viscosities of water \( \mu^*_w \) and gas \( \mu^*_g \) relative to the representative viscosity \( \mu_0 \) are defined as the ratios of the real viscosities of water \( \mu_w \) and the gas \( \mu_g \), respectively, to \( \mu_0 \) as follows:

\[
\mu^*_w = \mu_w / \mu_0, \quad \mu^*_g = \mu_g / \mu_0 \tag{17}
\]

Because a free water surface exists inside some elements of the finite element grid, \( \mu_t \) and \( \rho_t \) must be expressed in terms of the void fraction of water \( S_{aw} \) at point \( i \), which is the ratio of the water volume to the total fluid volume:

\[
\mu_i = S_{aw} \mu_w + (1 - S_{aw}) \mu_g \tag{18}
\]

\[
\rho_i = S_{aw} \rho_w + (1 - S_{aw}) \rho_g \tag{19}
\]

The law of conservation of mass for water can be
expressed as follows:

\[ \frac{\partial S_w}{\partial t} + \nabla \cdot (\rho_w S_w \mathbf{V}) = 0 \]  
(20)

Eq. (20) is transformed to Eq. (21) by considering water to be incompressible and \( \nabla \mathbf{V} = 0 \); thus, change in \( S_w \) represents change in the free water surface as follows:

\[ \frac{\partial S_w}{\partial t} + (\nabla \cdot \nabla) S_w = 0 \]  
(21)

By defining the void fraction of water \( S_w^* \) at the position vector \( \mathbf{r}_i - \mathbf{V}_i \cdot t \Delta t \) for a point \( i \) in the \( S_w \) field at time \( t + \Delta t \), Eq. (21) can be transformed with CIP as follows:

\[ S_{w,i,t+\Delta t} = S_{w,i}^* \]  
(22)

Here, the position vector \( \mathbf{r}_i - \mathbf{V}_i \cdot t \Delta t \) is specified from \( \mathbf{V}_i \cdot t \Delta t \), which is obtained by Eq. (14), and the distribution of \( S_w,i,t,\Delta t \) is obtained by interpolating with CIP for \( S_{w,i}^* \) at \( \mathbf{r}_i - \mathbf{V}_i \cdot t \Delta t \). At the free water surface, \( S_w \) is equal to 0.5.

Water pressure \( p_w \) and gas pressure \( p_g \) in the model are defined in relation to the representative density \( \rho_0 \) and the typical velocity \( V_{typ} \) of the fluid, as in Eq. (3):

\[ p_w = \rho_0 V_{typ}^2 \]  
(23a)

\[ p_g = \rho_0 V_{typ}^2 \]  
(23b)

Furthermore, the non-dimensional intrinsic permeability of the porous medium \( k_i \) can be expressed in terms of the Reynolds number \( R_{re} \), the typical length \( L \), and the real intrinsic permeability of the prototype \( k_i' \) as follows:

\[ k_i = k_i' R_{re} / L^2 \]  
(24)

Darcy’s law describes flows of water and gas in porous media as follows:

\[ \mathbf{V}_w = -(K_{rw} / \mu_w) k_i (\nabla p_w + \rho_w \mathbf{g}) \]  
(25a)

\[ \mathbf{V}_g = -(K_{rg} / \mu_g) k_i (\nabla p_g + \rho_g \mathbf{g}) \]  
(25b)

where \( K_{rw} \) and \( K_{rg} \) are the relative permeabilities of the medium to water and gas, respectively, expressed relative to the intrinsic permeability, and \( \mathbf{V}_w \) and \( \mathbf{V}_g \) are non-dimensional water velocity and gas velocity of a model with the typical velocity \( V_{typ} \).

The flows of water and gas should satisfy the law of conservation of mass for gas and water.

\[ \frac{\partial}{\partial t} \{[1 - (S_{rw} + S_{rg})] \Phi S_w \} + \nabla \mathbf{V}_w = 0 \]  
(26a)

\[ \frac{\partial}{\partial t} \{[1 - (S_{rw} + S_{rg})] \Phi S_g \} + \nabla \mathbf{V}_g = 0 \]  
(26b)

where \( S_r \) is the effective saturation of the gas; \( S_{rw} \) and \( S_{rg} \) are the residual water saturation and the residual gas saturation, respectively; and \( \Phi \) is porosity. In Eqs. (26), \( S_w \) is equivalent to the effective water saturation in the porous medium: \( S_w = (S_{wreal} - S_w) / (1 - (S_{rw} + S_{rg})) \), where \( S_{wreal} \) is the real water saturation, which includes the residual water saturation. Similarly, the effective gas saturation is \( S_g = (S_{greal} - S_g) / (1 - (S_{rw} + S_{rg})) \), where \( S_{greal} \) is the real gas saturation, which includes the residual gas saturation. Thus, in Eqs. (26), the water saturation \( S_w \) can range from 0.0 to 1.0 and \( S_w + S_r = 1 \). Hence, \( S_w \) is used for both the void fraction of water in the surface fluid and the water saturation in the porous medium in the numerical simulation of surface water and groundwater.

Therefore, by substituting Eqs. (25) into Eqs. (26), the governing equations for the flows of water and gas in a porous medium can be expressed as Eqs. (27a) and (27b), respectively.

\[ \frac{\partial}{\partial t} \{[1 - (S_{rw} + S_{rg})] \Phi S_w \} + \nabla \mathbf{V}_w = \nabla \cdot [(K_{rw} / \mu_w) k_i (\nabla p_w + \rho_w \mathbf{g})] \]  
(27a)

\[ \frac{\partial}{\partial t} \{[1 - (S_{rw} + S_{rg})] \Phi S_g \} + \nabla \mathbf{V}_g = \nabla \cdot [(K_{rg} / \mu_g) k_i (\nabla p_g + \rho_g \mathbf{g})] \]  
(27b)
Eq. (28), below, is derived by adding both sides of Eqs. (27a) and (27b).

$$\frac{\partial \Phi}{\partial t} = \nabla \cdot k \left[ (K_{\text{up}}/\mu_w) (\nabla p_w + \rho_w \mathbf{g}) + (K_{\text{rg}}/\mu_g) (\nabla p_g + \rho_g \mathbf{g}) \right]$$

(28)

If $\nu_w = K_{\text{up}}/\mu_w$ and $\nu_g = K_{\text{rg}}/\mu_g$, then Eq. (28) can be transformed as follows:

$$\frac{\partial \Phi}{\partial t} = \nabla \cdot k \left[ (\nu_w/\lambda) (\nabla p_w + \rho_w \mathbf{g}) + (\nu_g/\lambda) (\nabla p_g + \rho_g \mathbf{g}) \right]$$

(29)

Then, by defining $\lambda$ as $\lambda = (\nu_w + \nu_g)$, $\nu_w$ and $\nu_g$ inside the square brackets on the right side of Eq. (29) can be divided by $\lambda$, and $k$, outside the square brackets can be multiplied by $\lambda$. As a result, Eq. (29) can be rearranged as follows:

$$\frac{\partial \Phi}{\partial t} = \nabla \cdot k \left[ (\nu_w/\lambda) (\nabla p_w + \rho_w \mathbf{g}) + (\nu_g/\lambda) (\nabla p_g + \rho_g \mathbf{g}) \right]$$

(30)

Here, $f_w$ and $f_g$ are defined as in Eqs. (31) below:

$$f_w = \frac{\nu_w}{\lambda}$$

(31a)

$$f_g = \frac{\nu_g}{\lambda}$$

(31b)

and Eqs. (31) are substituted into Eq. (30), which is rearranged as follows:

$$\frac{\partial \Phi}{\partial t} = \nabla \cdot k \left[ f_w \nabla p_w + f_g \nabla p_g + (f_w \rho_w + f_g \rho_g) \mathbf{g} \right]$$

(32)

Global pressure $P$ is defined as follows:

$$\nabla P = f_w \nabla p_w + f_g \nabla p_g$$

(33)

Then, Eq. (34) is obtained by substituting $P$ into Eq. (32):

$$\frac{\partial \Phi}{\partial t} = \nabla \cdot k \left[ \nabla P + (f_w \rho_w + f_g \rho_g) \mathbf{g} \right]$$

(34)

Eq. (33) is integrated as follows:

$$\int_s \nabla P \, dv = \int_s (f_w \nabla p_w + f_g \nabla p_g) \, dv$$

(35)

The Green theorem is then applied to the integrated Eq. (35) as follows:

$$\int_\Omega P \, d\Omega = \int_\Omega (f_w \rho_w + f_g \rho_g) \, d\Omega - \int_s (p_w \nabla f_w + p_g \nabla f_g) \, dv$$

(36)

Next, $p_{cw} = \rho_g - \rho_w$ and $f_w + f_g = 1$, where $p_{cw}$ is the non-dimensional capillary pressure (called capillary pressure hereafter) between gas and water in the model relative to $\rho_w V_{\text{typ}}^2$, is substituted into Eq. (36) to express the capillary pressure as non-dimensional. Then Eq. (36) can be rearranged as follows:

$$\int_\Omega P \, d\Omega = \int_\Omega (f_w \rho_w + f_g \rho_g) \, d\Omega - \int_s p_{cw} \nabla f_g \, dv$$

(37)

By assuming that the perimeter (boundary) of the analytical domain is $\Omega$ and the direction $n$ is normal to the boundary, Eq. (38) can be derived as follows:

$$\int_\Omega P \, d\Omega = \int_\Omega (f_w \rho_w + f_g \rho_g) \, d\Omega - \int_s p_{cw} \nabla f_g \cdot |J|n \, d\Omega$$

(38)

where $|J|$ is the determinant of the Jacobi matrix for the relation between the vector normal to the boundary of a numerical domain and the rectangular coordinate system. Clearly, $P$ can be derived from Eq. (38) as follows:

$$P = f_w \rho_w + f_g \rho_g - \int_s p_{cw} \nabla f_g \cdot |J| \, n \, d\Omega$$

(39)

By considering the integral to be parallel to the direction normal to the boundary of the numerical domain, Eq. (39) can be rewritten as follows:

$$P = f_w \rho_w + f_g \rho_g - \int_s p_{cw} (\frac{df_g}{dn}) \, d\Omega$$

(40)

By substituting $\frac{df_g}{dn} = (\frac{df_g}{dS_w}) (\frac{dS_w}{dn})$ into Eq. (40) and by integrating the third term on the right side of Eq. (40) with respect to water saturation, we obtain:
\[ P = f_w p_w + f_g p_g - \int_{S_w} p_{cgw} (d\mathbf{f}_g / dS_w) dS_w \] (41)

We integrated the third term in Eq. (41) from \( S_{w,\text{max}} \) (the maximum water saturation in the porous medium, which is equal to 1.0) to \( S_w \) at each calculation point by using the trapezoidal rule.

Though \( \int_{S_w} p_{cgw} (d\mathbf{f}_g / dS_w) dS_w \) on the right side of Eq. (41) is applicable to a porous medium, it is not applicable to an atmosphere or surface water system. Global pressure in the porous medium should be equal to pressure in the surface water when the boundary between the porous medium and the surface water is below the surface water level, though capillary pressure acts in the porous medium, which is equal to global pressure in the porous medium above the surface water level; thus, global pressure in the porous medium should not be equal to the synthetic pressure in the surface water and the atmospheric pressure where the porous medium is unsaturated with water above the surface water level; thus, global pressure in the porous medium at the boundary between the porous medium and the atmosphere where the porous medium is unsaturated with water above the surface water level.

Pressure in the surface water and the atmospheric pressure should be equal to the synthetic pressure \( p = f_w p_w + f_g p_g \), which appear in Eq. (41), in the porous medium at the boundary between the porous medium and the atmosphere or surface water.

Furthermore, \( p_c \) is defined as \( p_c = \int_{S_u} p_{cgu} (d\mathbf{f}_g / dS_u) dS_u \)

\[ \partial \Phi / \partial t = \nabla \cdot \mathbf{k}_s [\nabla p - \nabla p_c + (f_w p_w + f_g p_g) \mathbf{g}] \] (42)

Eq. (42) is discretized by using the Galerkin FEM and by assuming \( \partial \Phi / \partial t = 0 \) without porous medium matrix compression and considering \( p = f_w p_w + f_g p_g \) as follows:

\[ \sum_{j=1}^{n} \bar{\lambda}_i k_i \int_{V_j} \nabla N_i \cdot \nabla N_j dV p_{j,t,\Delta t} \]

\[ = q_j + \sum_{j=1}^{n} \bar{\lambda}_i k_i \int_{V_j} \nabla N_i \cdot \nabla N_j dV p_{c,j,t} + (f_w p_w + f_g p_g) \int_{V_j} \nabla N_j \cdot \mathbf{g} dV \] (43)

where \( p_{c,j} \) is \( \int_{S_w} p_{cgu} (d\mathbf{f}_g / dS_u) dS_u \) at nodal point \( j \) at time \( t \); \( q_j \) in Eq. (43) is defined as follows:

\[ q_j = \sum_{i=1}^{n} \bar{\lambda}_i k_i \int_{V_j} N_i \cdot [\nabla N_j \cdot (p_{j,t,\Delta t} - p_{c,j})] + (f_w p_w + f_g p_g) \mathbf{g} dV \] (44)

where \( \bar{\lambda}_i \) is the average value of \( \lambda \) in an element at time \( t \), and \( \bar{f}_{w,t} \) and \( \bar{f}_{g,t} \) are the average values of \( f_w \) and \( f_g \), respectively, in an element at time \( t \). In Eq. (15a), \( q_j \) is defined in the surface water and atmosphere, and in Eq. (44), it is defined in the porous medium. Thus, the value of \( p_{j,t,\Delta t} \) in the analytical domain is obtained by solving Eqs. (13) and (43) simultaneously. Then, the total velocity \( \mathbf{V}_T \) in the porous medium is calculated with \( p \) as follows:

\[ \mathbf{V}_T = \mathbf{V}_w + \mathbf{V}_g \]

\[ = - \lambda k_i [\nabla p - \nabla p_c + (f_w p_w + f_g p_g) \mathbf{g}] \] (45)

Eq. (45) is discretized by using the Galerkin FEM, and then \( \mathbf{V}_{T,j,t,\Delta t} \) can be obtained as follows:

\[ \int_{V_j} N_i \mathbf{d}V_{T,j,t,\Delta t} = - \sum_{i=1}^{n} \bar{\lambda}_i k_i \int_{V_j} N_i \cdot \nabla N_j \mathbf{d}V (p_{j,t,\Delta t} - p_{c,j}) + \bar{\lambda}_i k_i (f_w p_w + f_g p_g) \int_{V_j} N_i \mathbf{d}V \] (46)

By substituting \( p_g = p_w + p_{cgw} \), \( \lambda_w = K_{sw} / \mu_w \), and \( \lambda_g = K_{sg} / \mu_g \) into Eqs. (25), we obtain:

\[ \mathbf{V}_w = - \lambda_w k_s [\nabla p_w + \mu_w \mathbf{g}] \] (47a)

\[ \mathbf{V}_g = - \lambda_g k_s (\nabla p_w + \mu_g \mathbf{g}) \] (47b)

We multiply both sides of Eq. (47a) by \( \lambda_g \) and both sides of Eq. (47b) by \( \lambda_w \). Then, by subtracting both sides of Eq. (47a) from both sides of Eq. (47b), we derive Eq. (48):

\[ \lambda_g \mathbf{V}_g - \lambda_w \mathbf{V}_w = - \lambda_g \lambda_w k_s [\nabla p_{cgw} + (\rho_g - \rho_w) \mathbf{g}] \] (48)

\[ \mathbf{V}_w = \mathbf{V}_T - \mathbf{V}_g \] is substituted into Eq. (48); then, because \( \lambda = \lambda_g + \lambda_w \), Eq. (48) can be rearranged as follows:

\[ - \lambda_g \mathbf{V}_T + \lambda \mathbf{V}_g = - \lambda_g \lambda_w k_s [\nabla p_{cgw} + (\rho_g - \rho_w) \mathbf{g}] \] (49)
Both sides of Eq. (49) are divided by \( \lambda \), and \( f_g = \lambda_w / \lambda \), \( f_w = \lambda_f / \lambda \), and \( \kappa = (\lambda_f \lambda_w / \lambda) \) are then substituted into the transformed Eq. (49):

\[
-f_g V_T + V_g = -\kappa k_1 [\nabla p_{gw} + (\rho_g - \rho_w) g]
\]

Consequently, \( V_g \) can be derived from Eq. (50) as follows:

\[
V_g = f/V_T - \kappa k_1 [\nabla p_{gw} + (\rho_g - \rho_w) g]
\]

The governing equation for gas in the porous medium is obtained by substituting \( V_g \) into Eq. (26b) as follows:

\[
[1 - (S_{rw} + S_{rg})] \phi \partial S_w / \partial t = \nabla \cdot \{f/V_T - \kappa k_1 [\nabla p_{gw} + (\rho_g - \rho_w) g]\}
\]

The right side of Eq. (52) can be expanded and rearranged as below, because \( \nabla \cdot V_T = 0 \) when \( \partial \Phi / \partial t = 0 \) without porous medium matrix compression:

\[
(1 - (S_{rw} + S_{rg})) \Phi \partial S_w / \partial t = V_T \cdot \nabla k_1 - \nabla \cdot (\kappa k_1 [\nabla p_{gw} + (\rho_g - \rho_w) g] \Phi \nabla K)
\]

The chain rule is applied with respect to \( S_w \) to the three terms on the left side of Eq. (53):

\[
(1 - (S_{rw} + S_{rg})) \Phi \partial S_w / \partial t = [V_T \cdot (df_g / dS_w) - (\rho_g - \rho_w) k_g (dK / dS_w) - \nabla \cdot (\kappa k_1 [d p_{gw} / dS_w]) \nabla S_w] - \nabla \cdot (\kappa k_1 [d p_{gw} / dS_w] \nabla S_w)
\]

where \( V_{soil} \) and \( D_{soil} \) are defined as

\[
V_{soil} = -[V_T (df_g / dS_w) - k_1 (\rho_g - \rho_w) g (dK / dS_w)] / [1 - (S_{rw} + S_{rg})] \Phi
\]

\[
D_{soil} = [-\kappa k_1 (dp_{ gw} / dS_w)] / [1 - (S_{rw} + S_{rg})] \Phi
\]

Then the saturation equation governing the transport of \( S_w \) is derived by substituting Eqs. (55) into Eq. (54) as follows (Peaceman, 1977):

\[
\frac{\partial S_w}{\partial t} + V_{soil} \cdot \nabla S_w = \nabla \cdot (D_{soil} \nabla S_w)
\]

By applying a semi-Lagrangian method with CIP and the water saturation \( S_{wi} \) at the position vector \( R_i \), \( -V_{Soil, t+\Delta t} \) in the \( S_w \) field at time \( t \), where \( R_i \) is the position vector at a point \( i \) and \( V_{Soil, i} \) is velocity of Eq. (55a) at a nodal point \( i \) at time \( t + \Delta t \), Eq. (56) can be transformed by the backward difference method as follows:

\[
S_{wi, t+\Delta t} = S_{wi, t} + \Delta t \cdot V_{soil, t+\Delta t} \cdot \nabla \cdot (D_{soil} \nabla S_{wi, t+\Delta t})
\]

The Green theorem is applied to Eq. (57), and then Eq. (57) is discretized by using the Galerkin FEM as follows:

\[
\left( \int_{\Omega} \frac{\partial S_{wi, i}}{\partial t} \nabla \cdot \nabla (\nabla N_i \cdot \nabla (\nabla S_{wi, i} + \Delta t \cdot V_{soil, i} \cdot \nabla (D_{soil} \nabla S_{wi, i}))) d\Omega - \int_{\Omega} S_{wi, i} \nabla \cdot (D_{soil} \nabla S_{wi, i} + \Delta t \cdot V_{soil, i} \cdot \nabla (\nabla N_i \cdot \nabla \Phi)) d\Omega \right)
\]

where \( D_{soil, i} \) is the average value of \( D_{soil, i} \) in an element at time \( t \), and \( q_{wi} \) is defined as follows:

\[
q_{wi} = \sum_{i+1}^{n+1} D_{soil, i} [L_{ni} \cdot \nabla (N_i \cdot S_{wi, i, t+\Delta t})] d\Omega
\]

\[
K_{rw}, K_{rg}, \text{ and } p_{gw} \text{ can be calculated from } S_w, \text{ which was obtained with Eq. (58), by using the constitutive relations between water saturation and capillary pressure, Eq. (60) (van Genuchten, 1980), and between water saturation and the relative permeability of a gas or water, Eqs. (61) (Mualem, 1976; Parker and Lenhard, 1987).}

\[
S_w = 1/[1 + (\alpha p_{gw})^\beta]^{\gamma}
\]

where \( \alpha, \beta, \text{ and } \gamma \) are the van Genuchten parameters and \( \gamma = 1 - 1 / \beta \).

\[
K_{rw} = S_w^{1/2} [1 - (1 - S_w^{1/\gamma})^{2}]^{1/2}
\]

\[
K_{rg} = (1 - S_w)^{1/2} [1 - (1 - S_w^{1/\gamma})^{2}]^{1/2}
\]

Capillary pressure between gas and water is derived from Eq. (60) as follows:
The calculation of fluid pressure (surface system)  

\[ p_{\text{avg}} = (S_w^{-1/2} - 1)^{1/2}/\alpha \]  

(62)

By substituting \( p = p_w + p_{\text{cgw}} \) into \( p = f_w p_w + f_g p_g \), we obtain \( p = (f_w + f_g)p_w + f_g p_{\text{cgw}} \). Next, we substitute \( f_w + f_g = 1 \) into \( p = (f_w + f_g)p_w + f_g p_{\text{cgw}} \) and then rearrange the result with respect to water pressure in the porous medium \( p_w \) as follows:

\[ p_w = p - f_g p_{\text{cgw}} \]  

(63a)

Consequently, gas pressure in the porous medium \( p_g \) can be obtained from the water pressure calculated using Eq. (63a) and the following equation:

\[ p_g = p_w + p_{\text{cgw}} \]  

(63b)

Furthermore, gas and water velocity in porous media can be obtained with Eqs. (25a) and (25b) and the gas and water pressure, respectively.

The real water saturation in the porous medium must be transformed from the effective gas saturation, obtained by Eq. (58), so that the water saturation of the simulation can be compared with the experimental water front observed by camera. Furthermore the residual saturations of water and gas are needed to calculate \( V_{\text{sol}} \) and \( D_{\text{sol}} \) in Eq. (65). Therefore, it is necessary to specify the residual saturations of water and gas appropriately in the numerical simulations.

2.2 Numerical simulation procedure

We can divide the numerical simulation procedure with the ASG method into three parts: the calculation of fluid pressure (surface system) or synthetic pressure (porous medium) (Box 3 to 9 in Fig. 1), fluid velocity (surface system) or total velocity (porous medium) (Box 10 to 12 in Fig. 1), and the void fraction of water (surface system) or water saturation (porous medium) (Box 13 to 14 in Fig. 1).

First, the position vector \( \mathbf{R}_i - \mathbf{V}_i \Delta t \) must be determined at every nodal point \( i \) from the position vector \( \mathbf{R} \), velocity at time \( t \), and the time increment \( \Delta t \); then, \( \mathbf{V}_i^* \) is obtained by interpolation of the velocity field at time \( t \). Next each above parameters are calculated at every nodal point and averaged per an element. \( p \) values at every point in the porous medium are obtained by integrating \( f_w p_{\text{cgw}}(df_g/dS_w) dS_w \) with the trapezoidal rule after \( p_{\text{cgw}} \) is calculated by Eq. (62) and \( (df_g/dS_w) \) is calculated at each integration point using the water saturation at that point.

We insert the per element average of the values of the parameters calculated above to obtain the fluid pressure in the surface water and atmosphere, and the porous medium into Eqs. (13) and (43); then, we solve the two equations simultaneously to get the fluid pressure and synthetic pressure \( p \) at every point at time \( t + \Delta t \) in the analytical domain. Subsequently, these pressures are inserted into Eqs. (14) and (46) to obtain the fluid velocity in the surface water–atmosphere system, and the total velocity in the porous medium. Then the recent fluid velocity in the surface water–atmosphere system \( \mathbf{V} \) and the total velocity in the porous medium \( \mathbf{V}_f \) at every point and time \( t + \Delta t \) are determined by solving Eqs. (14) and (46) simultaneously.

\( \mathbf{V}_{\text{sol}} \) at every nodal point in the porous medium is calculated with Eq. (55a) from \( (df_g/dS_w) \) and \( (d\gamma/dS_w) \) to obtain the water saturation in the medium. The position vector \( \mathbf{R} - \mathbf{V}_i \Delta t \) with respect to every point in the surface water–atmosphere system and the position vector \( \mathbf{R} - \mathbf{V}_{\text{sol}, t} \Delta t \) with respect to every point in the porous medium are determined, and \( S_w^* \) at the position vector \( \mathbf{R} - \mathbf{V}_i \Delta t \) and at the position vector \( \mathbf{R} - \mathbf{V}_{\text{sol}, t} \Delta t \) is obtained by interpolation in the water saturation field in the porous medium or the void fraction of water field in the surface water–atmosphere system at time \( t \). Finally, the void fraction of water \( S_w^*, t + \Delta t \) at time \( t + \Delta t \) is equal to \( S_w^* \) in Eq. (22) for the surface water–atmosphere system, and the water saturation \( S_w^*, t + \Delta t \) at time \( t + \Delta t \) is obtained by solving Eq. (58) for the porous medium using the calculated values of \( S_w^* \).
If time \( t \) is less than the final time \( t_{\text{max}} \), the time increment \( \Delta t \) is added to time \( t \), and \( V^* \) at the next time step is computed. The entire procedure is repeated until time \( t \) is greater than \( t_{\text{max}} \).

2.3 Conditions at the boundary between the porous medium and the surface water–atmosphere system

Fluid pressure conditions at the boundary between a porous medium and a surface water–atmosphere system as determined by the ASG simulation method results are shown in Fig. 2. Comparison of Eq. (13) and Eq. (43) implicitly indicates that fluid pressure in the surface water–atmosphere system below the surface water level is equal to only the water pressure below the water level in the porous medium, (Eq. f2–11 in Fig. 2); above the water level in the porous medium, at the boundary between the surface water and the porous medium, it is equal to the synthetic pressure (Eqs. f2–7 or f2–8 in Fig. 2). At this boundary, the water pressure in the porous medium saturated with water should be equal to the gas pressure in the atmosphere above the surface water level \((p_{\text{w, porous}} = p_{\text{g, surface}}\text{, not shown in Fig. 2})\). The synthetic pressure in the porous medium unsaturated with water, however, is equal to the gas pressure in the atmosphere where the unsaturated porous medium is in contact with the atmosphere (Eq. f2–7 in Fig. 2), and it is equal to the water pressure in the surface water where the unsaturated porous medium is in contact with the surface water (Eq. f2–8 in Fig. 2).

However, the water saturation changes with the total velocity in the porous medium, which is determined by the gradient of global pressure (Eqs. (41), (45), and (46)). Furthermore, the global pressure in the unsaturated porous medium includes \( p_{\text{s}} \), because capillary pressure acts on fluid there. Thus, the global pressure in the unsaturated porous medium differs from the gas pressure in the atmosphere and from the water pressure in the surface water at the boundary between the unsaturated porous medium and the atmosphere or surface water (Eq. f2–1 to 3 in Fig. 2). On the other hand, global pressure in a porous medium saturated with water should be equal to the synthetic pressure because of \( p_{\text{s}} = 0 \).

Velocity conditions at the boundary between a porous medium and a surface water–atmosphere system are shown in Fig. 3. Where the water in a porous medium saturated with water comes in contact with surface water at the boundary between them, water velocity in the saturated porous medium is equal to that in the surface water (Eq. f3–5 in Fig. 3). At the boundary between the porous medium unsaturated with water and the surface water or atmosphere, however, water in the surface water or gas in the atmosphere encounters both gas and water in the unsaturated porous medium. In this case, the gas velocity or water velocity in the surface water–atmosphere system does not equal either the gas velocity or the water velocity in the unsaturated porous medium because of capillary pressure in the porous medium. Instead, the total velocity, which is the sum of the gas velocity and water velocity in the unsaturated porous medium, should be equal to the water velocity in the surface water (Eq. f3–4 in Fig. 3) or the gas velocity in the atmosphere (Eq. f3–3 in Fig. 3). For this reason, Eq. (14) for the fluid velocity in the surface water–atmosphere system and Eq. (46) for the total velocity in the porous medium are solved simultaneously to take account of the interaction of fluid velocity between the surface water–atmosphere and the porous medium.

3 Verification of the ASG simulation method based on a two-dimensional problem

We verified the accuracy of the ASG method by comparing the results of an experiment for infiltration of water into an embankment conducted by Mizutani et al. (2013) with the results of a numerical simulation of this experiment with the
Fig. 1 Flowchart of the ASG simulation method procedure.
Fig. 2 Pressure at the boundary between a porous medium and a surface water-atmosphere system in the ASG simulation method.

Fig. 3 Velocity at the boundary between a porous medium and a surface water-atmosphere system in the ASG simulation method.
ASG method. Mizutani et al. (2013) constructed an embankment composed of packed sand across an open channel of 5.00 m length, 0.50 m height, and 0.30 m width. They first installed a layer of basal sand with horizontal dimensions of 1.20 m and 0.30 m and a height of 0.15 m; then on top of the basal sand they constructed a trapezoidal embankment crosswise to the channel of 0.15 m height, 0.1 m width at the top and 1.2 m width at the bottom. In the experiment, water level in the channel on the upstream side of the embankment was raised to 0.30 m, which was equal to the total height of the embankment, so that water infiltrated into the embankment.

In the simulation, a porous medium with the dimensions of the experimental packed sand was specified. A wall made of impermeable material (0.1 m wide) on the downstream side of the porous medium prevented the embankment material from moving into the outlet of the open channel. The two-dimensional analytical domain was of 0.9 m high (three times the height of the embankment) and 2.3 m wide (including a 0.5 m space on each side of the embankment) so that the boundary conditions at the edges of the domain would not influence the distribution of water saturation in the porous medium.

The water flowed into the analytical domain through a space at \( z = 0.0 \) to 0.01 m at the left boundary of the analytical domain. Thus, values of water velocity and the void fraction of water of zero and 1.0, respectively, were imposed there, and the water pressure in this space was specified as the hydrostatic pressure when the water level was \( z = 0.30 \) m. The gas could flow vertically into the analytical domain across the top boundary. The velocity at the bottom boundary was imposed by assuming a slip boundary condition. At the right boundary of the analytical domain, gas could thus flow into or out of the domain. However, at the left boundary of the analytical domain, gas and water could not flow into or out of the numerical analytical domain. At this boundary above \( z = 0.01 \) m, it could flow vertically. (Fig. 4).

The finite element grid for this simulation consisted of 18,201 nodes defining 17,960
rectangular and triangular elements in the analytical domain. The grid elements in the porous medium had a horizontal dimension of 0.01 m and a vertical dimension ranging from 0.005 m to 0.010 m. In the atmosphere, the grid elements had a horizontal dimension ranging from 0.01 m to 0.0125 m and a vertical dimension ranging from 0.005 m to 0.03 m (Fig. 5).

The properties of the fluids and the porous medium used in the simulation are listed in Table 1. Sand with an intrinsic permeability of $2.049 \times 10^{-11} \text{ m}^2$, which for the experiment was measured by conducting a permeability test with water, was used in both the experiment and the numerical simulation. The numerical simulation should employ a water retention curve incorporating the wetting process. However, the van Genuchten parameter values shown in Table 1 for the water retention curves (Eq. 60) were obtained by laboratory experiments of the draining process conducted by Mizutani et al. (2013). Thus, although we employed the experimentally obtained values of the van Genuchten parameters $\alpha$ and $\beta$, in the numerical simulation, we set the residual saturation of gas to 0.115 and the residual saturation of water to zero to simulate the wetting process. In the numerical simulation, we used the porosity of the sand calculated by Mizutani et al. (2013) from the mass and volume of the sand packed into the open channel and the density of the packed sand particles. We employed the densities and viscosities of water and gas when the temperature was 20 °C. The initial gas pressure was set to the static pressure, which was zero at the top of the analytical domain. The initial synthetic pressure was determined as $p = f_w p_w + f_g p_g$ by using the $\alpha$ and $\beta$ values given in Table 1, and the static gas pressure and hydrostatic water pressure, which

| Table 1 Sand (porous medium) and fluid parameter values used in the numerical simulation of the experimental infiltration of water into an embankment. |
| --- | --- | --- | --- |
| Water | Density $\rho_w$ g/cm$^3$ | 0.998 |
| | Viscosity $\mu_w$ kPa·s | 9.324 $\times 10^{-3}$ |
| Gas | Density $\rho_g$ g/cm$^3$ | 0.012 |
| | Viscosity $\mu_g$ kPa·s | 1.800 $\times 10^{-5}$ |
| Packed Toyoura sand | Intrinsic permeability $k_r$ m$^2$ | 2.049 $\times 10^{-11}$ |
| | Porosity $\Phi$ | 0.53 |
| van Genuchten parameters for water retention curve | $\alpha'$ kPa$^{-1}$ | 0.392 |
| | $\beta$ | 3.852 |
| Residual saturation of gas | $S_{rg}$ | 0.115 |
| Residual saturation of water | $S_{rw}$ | 0.000 |

Fig. 5 Schema of the mesh used in the numerical simulation for verifying the ASG method against experimental results.
were computed as the water saturations at \( z = 0.30 \) m (top of the embankment) and at \( z = 0.0 \) (bottom of the porous medium) were 0.0224 and 0.0623, respectively. The experiment was semi-implicitly simulated with a time increment of \( \Delta t = 0.01 \) s.

In the experiment, the infiltration of water into the packed sand was visually observed by means of a video camera and still photographs. Where the water had infiltrated, the sand became darker, and the boundary between the regions with and without infiltration could be recognized. Between the elapsed times of 8 and 10 min, the movement of the simulated water saturation contours from 0.0 to 0.8 was mostly similar to that of the experimental infiltration front. Furthermore, between the elapsed times of 10 and 15 min, the velocity of the simulated movement of the water saturation contours was similar to the movement velocity of the experimental boundary. However, the configuration of the simulated water infiltration front differed slightly from that of the experimental infiltration front. Despite the slight difference in configuration between the experimental and simulated infiltration fronts at the elapsed time of 15 min, we considered both the configuration and movement of the simulated infiltration front to be sufficiently similar to those of the experiment. Therefore, we considered that the numerical simulation with the ASG method could approximately simulate the movement of water in an embankment composed of sand (Fig. 6).

4 Simulation of a practice problem

4.1 Dimensionless parameters

To confirm that it is possible to simulate a practice flow problem with both surface water flow and groundwater flow by the ASG method, we conducted a simulation of flow overtopping an embankment (Fig. 7). The non-dimensional values of the densities and viscosities of water and gas, the intrinsic permeability, porosity, van Genuchten parameters and residual saturations of water and gas are shown in Table 2; the non-dimensional gravity was set to 10. Under the scaling law, the results of this simulation will be consistent with those of other simulations conducted at different scales. For instance, if we assume that the typical length \( L \) is 1 m, the typical velocity \( V_{sg} \) is 1 m/s, the prototype gravity \( |g'| \) is 10 m/s², and the representative density \( \rho_0 \) and viscosity \( \mu_0 \) are 1.0 g/cm³ and 1.0 × 10⁻³ Pa·s, respectively, then \( 1/F_r^2 \) is equal to 10.0, which is equal to the non-dimensional gravity \( |g| \) (Eq. (7)). Then, by Eq. (6), \( F_r = 0.316 \), and by Eq. (5), the Reynolds number \( R_c \) in the surface water is 1.0 × 10⁶. Given these assumptions, therefore, the typical length and the typical velocity are equal to the water depth and the water velocity flowing over the embankment, respectively, and the representative density \( \rho_0 \) and viscosity \( \mu_0 \) are equal to the density and viscosity of water. Furthermore, if it is assumed that the representative density \( \rho_0 \) and viscosity \( \mu_0 \) are 1.0 g/cm³ and 1.0 × 10⁻³ Pa·s, respectively (i.e., equal to those of water), then by using Eqs. (16) and (17) and the non-dimensional density (1.0) and viscosity (1.0) of water and the non-dimensional density (1.2 × 10⁻⁵) and viscosity (1.8 × 10⁻³) of the gas, the density and viscosity of the gas become 1.2 × 10⁻² g/cm³ and 1.8 × 10⁻³ Pa·s, respectively. We can then use Eq. (24) with the non-dimensional intrinsic permeability of the porous medium \( k_s \) (2.0 × 10⁻⁵), the Reynolds number \( R_c \) (1.0 × 10⁶), and the typical length \( L \) (1 m) to calculate the dimensional intrinsic permeability to be 2.0 × 10⁻¹¹ m², which is the approximate intrinsic permeability of a sand. In this simulation, we used the non-dimensional van Genuchten parameters \( \alpha = 0.231 \) and \( \beta = 4.5 \), which were determined by using Eq. (3) and the results of water retention experiments with the water head method and a process draining water from sand (unpublished data), the representative fluid density \( \rho_0 \) (1.0 g/cm³, which is equal to that of water), and the typical velocity \( V \) (1 m/s). Thus, the numerical simulation employed a porosity similar to the porosity of any sand.
4.2 Analytical domain, boundary condition and initial condition

The free water surface, except at the right-side boundary of the analytical domain, was initially flat at a height of 3.0 above the basal porous medium, and the discharge of the surface water on the right side of the embankment at the right-side boundary of the domain was simulated by dropping the free water surface at this boundary to the upper surface of the basal porous medium up to an elapsed time of 36,000 after the free water surface began to drop. As a result, the free water surface everywhere on the right side of the embankment dropped to the upper surface of the basal porous medium. The water retention curve equation should reflect a draining process until the elapsed time of 36,000. In the numerical simulation, the free water surface on the left side of the

![Fig. 6 Comparison between the experimentally observed water infiltration front (solid lines) and the distribution of water saturation contours obtained by the numerical simulation (colors) at elapsed times of (a) 5, (b) 8, (c) 10, and (d) 15 min.](image)
embankment was raised to the height of 6.0 above the upper surface of the basal porous medium; consequently, the surface water on the left side of the embankment flowed over the embankment. Accordingly, because surface water infiltrated into the embankment, the water retention curve should reflect a wetting process when the water overtopped the embankment. Therefore, in the numerical simulation, the water retention curve equation should employ draining before the embankment is overtopped and wetting after it is overtopped. However, this numerical simulation cannot employ a water retention curve equation that describes both wetting and draining. Thus, in this numerical simulation, a water retention curve reflecting only draining was used, although it should be used only before the embankment is overtopped. Therefore, the numerical simulation included a residual water saturation of 0.187 and a residual gas saturation of 0.0 in the water retention curve equation. These residual saturations were determined by the results of our previous moisture water retention experiments.

The non-dimensional analytical domain of the numerical simulation for flow overtopping the embankment was 100 wide and 30 high. At the bottom of the analytical domain, the thickness of the layer of porous medium was 10. A trapezoidal embankment with a non-dimensional height of 5 and a width of 5 at the top and of 25 at the bottom was placed above the center of the basal porous medium; the gradient of the two slopes of the embankment was one in two. First, the simulation was carried out until the elapsed time of 36,000 in order to reach a steady state with the depth of the surface water 3 on the left side and 0 on the right side of the analytical domain. Subsequently, a rapid rise of the free water surface at the left-side boundary of the domain was simulated until the depth of the surface water on the left side of the embankment was 6.0, and the surface water flowed from the left side of the embankment over the embankment to the right side of the analytical domain. After the overtopping of the embankment, the numerical simulation was conducted until the porous medium of the embankment was mostly saturated with water.

The lower boundary of the analytical domain (below the basal porous medium) was set as impermeable. Further, at the upper boundary of the analytical domain, gas did not flow into or out of the analytical domain at the upper boundary of the domain in this numerical simulation. The depth of the surface water at the left boundary of the analytical domain was set to 3 until the elapsed time of 3600, and to 6 over an elapsed time 36,000 after the free water surface began to drop at the right-side boundary of the domain. In contrast, at the right boundary of the domain, the surface water depth was set throughout the simulation to 0. Where, we specified that the fluid pressure was zero at the upper boundary of the analytical domain. The gas pressure and water pressure were static and hydrostatic, respectively, with the free water surface as noted above. Fluid was then allowed to flow into the analytical domain, and its flow was accelerated by imposing a fluid pressure at the left boundary of the domain (Fig. 7).

Initial fluid velocity was set to zero throughout

| Table 2 | Porous medium and fluid parameter values used in the numerical simulation of water overtopping an embankment. |
|---------|------------------------------------------------------------------------------------------------------------------|
| Water   | Density $\rho_w$ 1.000                                                                                       |
|         | Viscosity $\mu_w$ 1.000                                                                                       |
| Gas     | Density $\rho_g$ $1.2 \times 10^{-2}$                                                                       |
|         | Viscosity $\mu_g$ $1.8 \times 10^{-2}$                                                                       |
| Porous medium | Intrinsic permeability $k_s$ $2.0 \times 10^{-3}$                  |
|         | Porosity $\Phi$ 0.400                                                                                         |
|         | van Genuchten parameters for water retention curve $\alpha$ 0.231                                           |
|         | $\beta$ 4.500                                                                                                 |
|         | Residual saturation of gas $S_{rg}$ 0.000                                                                   |
|         | Residual saturation of water $S_{rw}$ 0.187                                                                  |
the domain. The initial free water surface was level to the left of the embankment and its height above the bottom of the domain was 13, the same as the water level at the left boundary. The initial free water surface to the right of the embankment was 10, the same as the water level at the right boundary of the domain. Thus, the initial void fraction of water/water saturation was 0.0 in the atmosphere and 1.0 in the surface water and porous medium (except in the embankment). Furthermore, the initial gas pressure and water pressure were static and hydrostatic, respectively. The initial water pressure in the embankment was set to hydrostatic pressure, consistent with the surface water level to the left of the embankment of 13 above the bottom of the domain. Further, the initial gas pressure in the embankment was set to static pressure when the gas pressure was zero at the upper boundary of the domain. The initial water saturation in the porous medium was calculated from the capillary pressure by using Eq. (60).

The finite element grid for this simulation consisted of 35,805 nodes defining 35,440 rectangular and triangular elements in the analytical domain. The analytical domain was horizontally and vertically divided at intervals of 0.10 to 0.5 and 0.125 to 1.0, respectively (Fig. 8).

4.3 Results of the simulation

The void fraction of water and water saturation, as well as the synthetic pressure, water pressure, and gas pressure, mostly did not change with the further passage of time after the elapsed time of 30,000 after the beginning of drawdown of the surface water on the right side of the embankment. Further, the flow of surface water, groundwater, and gas had reached a steady state by the elapsed time of 36,000 (Figs. 9 and 10). The distributions of water saturation, synthetic pressure, gas pressure, and water pressure in the porous medium at the elapsed time of 36,000 thus represent the initial steady state before surface water overflowed the embankment. The steady-state water saturation contours from 0.4 to 0.9 were distributed in a narrow band about 0.4 wide; the water saturation contours extending from the free water surface at the left side of the embankment to its right side were linear straight. There was a seepage surface with a height of 0.50 above the basal porous

Fig. 7 Schema of the analytical domain and the imposed boundary conditions for numerical simulation of water overtopping an embankment.
medium on the right slope of the embankment, and groundwater in the porous medium flowed out to the atmosphere through this seepage surface. The width between the 0.4 and 0.9 water saturation contours in the center of the embankment was 0.40, similar to the water head, which was obtained by dividing the difference between the capillary pressure at a water saturation of 0.4 and that at a water saturation of 0.9 on the water retention curve, by $\rho_w|g|$. The height of the seepage surface from the upper surface of the basal porous medium, 0.50, was also similar to the height of the water head of 0.43, which was obtained by dividing the entry pressure of water (equal to the capillary pressure when water saturation first decreased below 1.0), by $\rho_w|g|$. Therefore, the numerical simulation might be expected to reproduce the proper distribution of the water saturation (Figs. 9a).

At the elapsed time of 1500 after the surface water began to overtop the embankment, the water saturation of the top surface and the surface of the upper half of the embankment slope exceeded 0.9; these surfaces became mostly saturated with water by infiltration of the surface water that was overtopping the embankment. At this elapsed time, however, there was also a region unsaturated with water (water saturation from 0.2 to 0.4) on the right side of the embankment. Because the water saturation was smallest at the top of embankment before the embankment was overtopped, the water saturation value was smallest, 0.196, in the vicinity of the top of the region unsaturated with water at the elapsed time of 1500 (Fig. 9b).

By the elapsed time of 3000 after the embankment was overtopped, the area of the unsaturated region on the left side decreased and became mostly saturated, because the surface water on the left side of the embankment infiltrated the embankment. The left-side surface water continued to infiltrate into the embankment until the elapsed time of 13,500, when water saturation everywhere in the embankment was more than 0.9. Thus, the area of the region on the left side of the embankment unsaturated with water decreased as the elapsed time increased and the region finally disappeared altogether.

At the elapsed time of 4500 after the embankment was overtopped, the water saturation was more than 0.9 at the surface of the right-side slope of the embankment, but the thickness of the region with a water saturation of more than 0.9 on the right-side slope was very thin, ranging from 0.125 to 0.25. With the passage of elapsed time, the region with a water saturation of more than 0.9 expanded to the lower half of the right-side embankment slope, and the unsaturated region, where the water saturation did not exceed 0.4,
Fig. 9  The free water surface and the distribution of water saturation in a porous medium when surface water flowed over an embankment in a practice simulation problem.
was gradually eliminated. The minimum water saturation in the region unsaturated with water became higher with the passage of elapsed time and was more than 0.9 at the elapsed time of 13,500 after water began to overtop the embankment. At the elapsed time of 10,500 after water began to overtop the embankment, a region unsaturated with water existed only in the vicinity of the right top of the embankment; subsequently, this region finally disappeared as the embankment became mostly saturated with water (Figs. 9b–g).

In the steady-state conditions before surface water overtopped the embankment, gas pressure was roughly zero in the unsaturated region of the embankment and was similar to atmospheric pressure. At this time, water pressure was at the minimum of \(-41\) at the top and on both upper slopes of the embankment, and it ranged from about \(-41\) in the upper part of the embankment to about zero above the free water surface. Further, at this time synthetic pressure \(p = f_w p_w + f_g p_g\) in the embankment should be equal to the water pressure in regions saturated with water because \(f_w\) and \(f_g\) were 1.0 and 0.0, respectively, when the water saturation was 1.0. In contrast, the synthetic pressure was similar to the gas pressure in the region where the water saturation was not more than 0.4 because there \(f_w\) and \(f_g\) were close to 0.0 and 1.0, respectively. The synthetic pressure over 1.0 water saturation contour in the embankment ranged from the water pressure to the gas pressure as \(f_w\) decreased and \(f_g\) increased, and these changes depended on the water saturation. In this region, because the water pressure was below zero and the gas pressure was close to zero, the synthetic pressure was below zero and its minimum was \(-1.8\) (Fig. 10a).

The water pressure in the surface water increased when the free water surface on the left side of the embankment was raised to a height of 6.0 above the upper surface of the basal porous medium. Consequently, at the elapsed time of 1500 after the surface water began to flow over the embankment, the water pressure in the embankment had increased from \(-41.0\) to \(-3.9\). Further, the gas pressure had also increased from about zero to a value exceeding 10, and it was higher on the left side of the region unsaturated with water. The synthetic pressure would also have increased on the left side of this unsaturated region as the gas pressure increased. However, the gas pressure and synthetic pressure were close to zero on the right slope of the embankment (Fig. 10b).

The gas pressure increased with the passage of elapsed time until the water saturation was 0.9 and the embankment was mostly saturated with water; at the elapsed time of 13,500 after the water began to overtop the embankment, the gas pressure and the water pressure had both reached a static state. The water pressure was always less than 0.0 in the unsaturated region where the water saturation was below 0.9. When the unsaturated region where the water saturation was 0.9 finally disappeared from the embankment, the water pressure was hydrostatic in the embankment. Because the capillary pressure between water and gas approached zero as the water saturation increased, the gas pressure was similar to the water pressure at the elapsed time of 13,500 when the water saturation was close to 1.0. As noted above, the gas pressure in the region unsaturated with water increased as the surface water infiltrated into the embankment and was very high except below the downstream slope of the overtopped embankment, where the gas pressure was smaller than it was on the left slope of the embankment. Thus, the direction of the gradient of gas pressure in the embankment was toward the right slope of the embankment. Because this gas pressure gradient caused the gas in the embankment to be pushed out into the right-side surface water, the surface water was able to infiltrate the embankment even though the downstream slope of the overtopped embankment was covered with surface water (Fig. 10c–g).

Figs. 11 and 12 show vectors of total fluid...
velocity, water velocity, and gas velocity at the elapsed times of 1500 and 4500, respectively, after water began to overtop the embankment.

The scale of the vectors in Figs. 11 and 12 differs between the porous medium and the atmosphere-surface water system. At the elapsed time of 1500,

Fig. 10 Distributions of synthetic pressure, water pressure, and gas pressure in the practice simulation with the ASG method in which the surface water flowed over an embankment.
the maximum total velocity, water velocity, and gas velocity in the porous medium were $1.0 \times 10^{-2}$, $4.8 \times 10^{-4}$, and $1.0 \times 10^{-2}$, respectively. In contrast, the maximum fluid velocity in the surface water was 6.8. Similarly, at the elapsed time of 4500, the maximum fluid velocity in the surface water and the total velocity, water velocity, and gas velocity in the porous medium were $6.9$, $3.5 \times 10^{-4}$, $1.9 \times 10^{-4}$, and $3.6 \times 10^{-4}$, respectively.

As shown in Figs. 11 and 12, in the porous medium, the total velocity was the sum of the gas velocity and the water velocity. The surface water on the left side of the embankment flowed into the embankment through the left-side embankment slope at elapsed times of both 1500 (Fig. 11b) and 4500 (Fig. 12b). Further, the directions of the water and gas velocity vectors at the elapsed time of 1500 (Fig. 11b and c) confirms that water infiltrating the embankment pushed gas out of the porous medium into the surface water through the right-side slope of the embankment. At the elapsed time of 4500, gas in the region unsaturated with water, which was smaller in area than it was at the elapsed time of 1500, was also discharged into

Fig. 11 Vectors of (a) total velocity, (b) water velocity, and (c) gas velocity in the analytical domain at the elapsed time of 1500 after water began to overflow the embankment. Velocity vectors in the surface water and atmosphere are red, and those in the porous medium are green.
the right-side surface water from this unsaturated region. Further, the surface water on the top and right side of the embankment flowed into the region unsaturated with water (Figs. 12b and c).

As shown above, the ASG method could simulate the movement of water and gas across the boundary between the surface system, which consisted of surface water and atmosphere, and the porous medium, because the void fraction of water in the surface system and the water saturation in the porous medium migrated in the numerical simulation. The ASG method could also simulate the interaction between water pressure and gas pressure in the surface system and the porous medium; the gas pressure in the porous medium rose as the surface water flowed into the porous medium, and the result also showed that the direction of the water velocity vectors was opposite to that of the gas velocity vectors in the region unsaturated with water. Accordingly, the numerical simulation with the ASG method could take account of the interaction between the surface water and the water in the porous medium with respect to the movement of fluid, fluid pressure,
and fluid velocity.

5 Conclusions

We previously developed a numerical method for simulating a coupled atmospheric gas, surface water, and groundwater system (ASG method). In this study, the equations employed in the ASG method are non-dimensional.

Comparison between results obtained by the AGS method and results of an experiment with a water tank conducted by Mizutani et al. (2013) showed that the configuration and movement of the infiltration front obtained by the simulation with the ASG method were similar to the experimental results. Therefore, numerical simulation with the ASG method could approximately simulate the movement of water in an embankment.

Moreover, we simulated the movement of gas and water when surface water flowed over an embankment in a practice problem with arbitrary dimensions to evaluate the ability of the ASG method. The result clearly showed that the ASG method could simulate the movement of water and gas across the boundary between the surface system and the porous medium and could also handle the interaction between water pressure and gas pressure in the surface system and the porous medium. Further, the ASG method also simulated the direction of water velocity vectors to be opposite that of the gas velocity vectors in the region unsaturated with water. Accordingly, the practice problem results clearly showed that the ASG method could take account of the interaction between surface water and water in the porous medium with respect to the movement of fluid, fluid pressure, and fluid velocity.

The accuracy of the ASG method should be further evaluated in the future by conducting additional comparisons between simulation results and experimental results. Furthermore, the ability of the ASG method in applications to additional types of problems needs to be investigated.

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