Limits of Matrix Theory in Curved Space

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Abstract

We study curved space versions of matrix string theory taking as a definition of the theory a gauged matrix sigma model. By analyzing the divergent terms in the loop expansion for the effective action we reduce the problem to a simple matrix generalization of the standard string theory beta function calculation. It is then demonstrated that the model can only be consistent for Ricci flat manifolds with vanishing six-dimensional Euler density.

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1 Introduction

Matrix string theory [1] [2] [3] is hoped to provide a non-perturbative definition of type IIA string theory. The free string states have a clear interpretation in this picture. They are formed from “winding sectors” in which large numbers of eigenvalues form, via twisted boundary conditions, long string configurations composed of an order $N$ number of eigenvalues. The large $N$ limit corresponds to taking a finer and finer discretization of the light-cone string world-sheet into infinitesimal strips, and corresponds to taking the conformal limit of the theory. From this conformal field theory point of view string interactions can be argued to be described by an irrelevant, local CFT operator [3]. Dimensional analysis then requires the operator to be associated with a linear power of the string coupling constant $g_s$. This analysis however leaves unclear what power of $N$, if any is associated with the interaction. Progress has been made in constructing classical solutions corresponding to string interactions [4] [5] [6] but it is not yet known how to calculate the quantum fluctuations to confirm the string interaction weight directly from the SYM theory.

In string theory the classical equations of motion for the effective theory can be found either from tree level string scattering amplitudes or from the consistency conditions (conformal invariance) of the action for a non-interacting string propagating in a curved space with background fields. It is thus worth exploring what can be learnt from the consistency conditions on curved space versions of matrix string theory. In [7] [8] [9] gauged versions of matrix sigma models were proposed as a description for D-brane actions in curved space at finite $N$, and it was suggested that they might also be used as descriptions of matrix theory in curved space. These actions were built from a simple set of axioms, the most important being that they have a $U(N)$ invariance are built from a single trace and for diagonal matrices describe $N$ identical copies of the standard string sigma model. We will describe the matrix string model coupled to a curved background using a slight modification of these axioms.

We study here the conditions imposed on the curved space versions of matrix string theory in which perturbative string theory is hoped to be recovered. The calculations resemble closely the determination of the string beta function, evaluated using the background field method. This was already observed in [1]. We will show below that such models are only consistent for an extremely limited class of manifolds i.e. Ricci flat manifolds with vanishing Euler class. An example of which is provided by the direct product $M = S \times C$ where $S$ is a hyper-Kähler surface. This is for instance the case of the ALE spaces [10].

The calculation is performed in the context of matrix string theory, but this result applies also to the quantum mechanics of D0 branes.

We begin in section 2. by recalling the essential elements of matrix string
theory. In section 3. we review the proposals for D-brane actions/ matrix theory in curved space put forward by Douglas et. al. [7][8][9]. In section 4 we focus on the divergent part of calculation of the effective action and show that it reduces to a simple matrix generalization of the string theory beta function calculation. Having mapped the calculation to that of the string beta function we use known 4-loop results [11] to demonstrate that the effective action can only be consistent for Ricci flat manifolds with vanishing six-dimensional Euler density.

2 \(N = 8\) Two Dimensional Super Yang-Mills and Matrix String Theory

In this section we summarize the essential ingredients of the correspondence between the dimensional reduction of ten dimensional super Yang-Mills theory and type IIA string theory. The two dimensional action reduces to

\[
S = \int d\tau d\sigma \text{Tr} \left[ \frac{1}{2} (D_\alpha X^I)^2 + \frac{i}{2} \Theta^T \gamma^\alpha \Theta - \frac{1}{4} F_{\alpha\beta}^2 + \frac{1}{4g_s^2} [X^I, X^J]^2 + \frac{1}{2} g_s \Theta^T \gamma_I [X^I, \Theta] \right]
\]

(1)

The fields are \(N \times N\) Hermitean matrices. The index \(I\) runs from 1 to 8 and the sixteen fermions split into the 8s and 8c spinorial representations of \(SO(8)\). The string coupling constant of the type IIA string theory is \(g_s\). The coordinate \(\sigma\) lives between 0 and \(2\pi\).

According to [3] the weakly coupled string is to be obtained from the \(g_s \to 0\) limit corresponding to the infra-red limit of the SYM theory. In this regime the matrices commute and describe strings in the light cone frame. The corresponding action evaluated for these configurations is the sum of \(N\) replicas of the light cone Green-Schwarz action. In this limit the matrix coordinates can always be diagonalized using unitary transformations \(U\)

\[X^I = U x^I U.\]

(2)

The matrix \(U\) is defined up to an element \(g\) of the Weyl group of \(U(N)\) permuting the eigenvalues.

\[U(\sigma + 2\pi) = U(\sigma)g, \quad x^I(\sigma + 2\pi) = gx^I(\sigma)g^\dagger\]

(3)

The infra-red regime is then identified with the two-dimensional conformal field theory described by the \(N = 8\) sigma model on the target space

\[S^N R^8 = (R^8)^N / S_N\]

(4)

The freely propagating strings in the light cone frame are identified in the limit \(N \to \infty\) with the cycles of the eigenvalues \(x^I\) under the permutation group.
It is useful at this point to reformulate the action \([1]\) in a way that generalizes easily to a curved background. A supersymmetric and gauge invariant action can be written using the \(d = 4, N = 1\) superfield formalism. The four gauge fields belong to a vector multiplet \(V\) while the bosonic fields belong to three chiral multiplets \(\Phi^i\). The eight bosonic fields of the original action thus split into a group of six belonging to the three chiral multiplets and two obtained by dimensional reduction of the 4d gauge fields belonging to the vector multiplet. This formulation breaks the global \(SO(8)\) symmetry into \(SO(6) \times SO(2)\). The full \(SO(8)\) symmetry is restored by going to the Wess-Zumino gauge. The Lagrangian is

\[
S = \frac{1}{\alpha'} \text{tr} \left( \int d^2x d^4\theta e^{gV} \Phi e^{-gV} \Phi^\dagger + \frac{1}{64g^2} \int d^2x d^4\theta W^2 + \frac{i}{3!\sqrt{\alpha'}} \int d^2x d^2\theta \epsilon_{ijk} \Phi^i \Phi^j \Phi^k + cc \right) 
\]

where \(g^{-2} = \alpha' g^2\) is the YM coupling constant and \(W_\alpha = \bar{D}^2 e^{gV} D_\alpha e^{-gV}\). The two derivative Lagrangian of the 4d \(N = 4\) SYM theory is finite, in particular the beta function vanishes.

In this article we will be focusing uniquely on the two derivative part of the effective action. To set up the formalism for later use let us briefly describe how the calculation proceeds in terms of superfields. This is just a rerun of how the background superfield formalism can be used to perturbatively show the finiteness of \(d = 4, N = 4\) SYM. We send the reader to \([12]\) for fuller details.

The background superfields in our case will be the diagonal field configurations corresponding to long strings. One is interested in separating the superfields into a background configuration and the quantum fluctuating parts. The fluctuating parts comprise terms inside and outside the Cartan subalgebra. It is most convenient to use the background field formalism \([12][13][14]\) a review of which can be found in the appendix. We decompose the vector superfields according to

\[
e^{V_T} = e^{V_B} e^V e^{V_B} 
\]

where \(V_T\) is the total superfield while \(V_B\) is the background configuration and \(V\) the fluctuating part. The background chiral superfields can be split into Cartan \(\Phi_h\) and fluctuating quantum parts \(\phi_h, \phi\)

\[
\Phi = \Phi_h + \phi_h + \phi 
\]

where \(\phi\) does not belong to the Cartan subalgebra. The renormalized Lagrangian is obtained after integrating over the fluctuating parts. This integration is nothing but the renormalization process of the \(N = 4\) SYM theory reduced to two dimensions. It is well known \([13]\) that the two derivative renormalized action is finite. The result of the integration over the fluctuating parts gives the original
action when the fields are given by their background values. This implies that the effective action is

$$S_{\text{eff}} = \frac{1}{\alpha'} (\int d^2 x d^4 \theta \phi_d^d \phi_d + \frac{1}{64} \int d^2 x d^2 \theta W^2)$$

where $$W_\alpha = \bar{D}^2 D_\alpha \bar{V}_B$$. Expanding in component fields and putting to zero the F and D terms yields $$N$$ copies of the flat space string theory action. The crucial point of the present derivation is the finiteness of the $$N = 4$$ SYM theory. In the following we will apply the same method to the matrix string theory in a curved background.

### 3 Curved Space Actions

Candidate formulations for D-brane actions in curved space have been been proposed in [7][8][9]. For small curvatures a single D-brane is described by the Born-Infeld theory. The crucial point is that this contains a $$U(1)$$ gauge field which becomes non-Abelian when $$N$$ D-Branes coincide. In the low energy regime this reduces to a SYM theory on the world-volume of the D-branes. In curved space the D-brane action should combine the non-Abelian nature of the gauge theory and a fraction of the original sixteen supersymmetries preserved by the D-brane configuration.

A set of axioms have been proposed in [7][8][9] to describe the possible actions. A particularly natural set of D-brane actions in this context are those obtained from the the dimensional reduction of a 4d ($$N = 1$$) $$U(N)$$ SYM theory to $$d + 1$$ dimensions [7]. The curved background is a 3d complex Kähler manifold whose metric depends on a Kähler potential $$K$$. The vector superfields contain $$(3 - d)$$ real flat coordinates. Notice that the splitting of the background manifold implies that the original $$SO(8)$$ global symmetry is reduced to $$SO(3 - d)$$. The case $$d = 1$$ corresponds to the matrix string theory while $$d = 0$$ is a curved version of the matrix model for M-theory.

In a setting adapted to our purposes the axioms amount to the following four requirements for the D-brane action defined on a 3 dimensional Kähler manifold $$\mathcal{M}$$.

a) The classical moduli space, determined by the vanishing of the D and F terms of the SYM theory, is the symmetric product $$\mathcal{M}^N/S_N$$.

b) The generic unbroken gauge symmetry is $$U(1)^N$$.

c) Given non-coincident branes at points $$p_i \neq p_j$$, all states charged under $$U(1)_i \times U(1)_j$$ have mass $$m_{ij} = d(p_i, p_j)$$ the distance along the shortest geodesic
between the two points.

d) The action is a single trace.

These axioms imply that the action in curved space reads

$$S = \frac{1}{\alpha'} \text{tr}(\int d^{d+1}x d^4 \Phi e^{-g^V} \Phi e^{-g^V} d\theta K(e^{g^V} \Phi e^{-g^V}, \Phi^\dagger) + (\int d^{d+1}x d^4 \theta W(\Phi) + \frac{1}{64g^2} \int d^{d+1}x d^2 \theta W^a W_a + \text{cc}))$$

(9)

The analysis of axiom a) leads to the following form for the superpotential

$$W = \epsilon_{ijk} a^i(\Phi)[\Phi^j, \Phi^k]$$

(10)

where $a^i(\Phi)$ is a holomorphic vector field in the adjoint representation of the gauge group. In the following we will choose such a superpotential but consider a less restrictive set of axioms. Effectively we will relax axiom c) and use the most general Kähler potential allowed by supersymmetry and gauge invariance. We will use the fact that there exists around each point of the moduli space a set of normal Kähler coordinates. These coordinates are such that locally

$$K(z, \bar{z}) = z \bar{z} + \sum \frac{1}{L^\bullet R} K_{I_1 I_p I_{p+1} \ldots I_n} z^{I_1} \ldots z^{I_p} \bar{z}^{I_{p+1}} \ldots \bar{z}^n$$

(11)

The existence of this expansion is guaranteed up to an analytic change of coordinates on the curved manifold. By definition the $K_{I_1 I_p I_{p+1} \ldots I_n}$ are symmetric with respect to arbitrary reorderings of the holomorphic indices and arbitrary reorderings of the antiholomorphic indices. Finally since we are dealing with matrices there is a question of ordering in the Kähler potential. The most natural ansatz is to assume that all terms in the Kähler potential are symmetrized products of matrices, but there could be more general orderings. The fourth order term for example can be written as

$$K_{I K J L} [\delta \Phi^I \Phi^K \Phi^J \Phi^L + \tau \Phi^I \Phi^J \Phi^K \Phi^L],$$

(12)

where $\delta$ and $\tau = 1 - \delta$ are constants. It is also possible for the Kähler potential to contain terms proportional to commutators of matrices since these vanish for the classical moduli space (diagonal matrices). In fact it was found in [9] that imposing the axioms stated above constrains the fourth order term to be the totally symmetrized product ($\delta = 2/3$, $\tau = 1/3$) with no additional terms corresponding to commutators.

Let us first use a very naive argument to justify the link between the matrix string theory on a curved background and the type IIA string theory in curved space. Substituting the diagonal matrices describing the moduli space $\mathcal{M}$ in
the action leads to a sum of $N$ copies of the $U(1)$ gauged sigma model in 2d. The gauge part of the action describing the flat component of the background manifold decouples and one is left with $N$ copies of the sigma model defined by the background curved manifold

$$\frac{1}{\alpha'} \int d^2 x d^4 \theta K(\Phi, \bar{\Phi})$$

(13)

where $\Phi$ represents one of the $N$ components. The analysis of this action reveals that there are two dimensional UV divergences. These logarithmic divergences can be cancelled up to three loop order by imposing that the Ricci tensor vanishes

$$R_{I\bar{J}} = 0$$

(14)

This is the usual Einstein equation as deduced from the conformal invariance of string theory. At four loop order this is not true anymore, the beta function is non-zero for Ricci-flat manifolds. The divergence is proportional to

$$R_{hkmn} R_{rs}^h n (R_{srm}^k + R_{kmrs}^k)$$

(15)

when expressed in terms of the underlying real coordinates. This is equivalent to the result obtained from the calculation of the four-graviton scattering for type IIA theory. This leads to a correction of the effective 10d supergravity action and the familiar $R^4$ term.

It seems therefore that a naive application of matrix theory in curved space leads to the correct identification of the string equations. This is misleading as a detailed analysis expounded in the following will show.

4 The Effective Action in a Curved Background

In the previous section we have defined the curved background version of the matrix string theory. This involves an explicit splitting between the six curved coordinates represented by a non-linear sigma model coupled to $SU(N)$ YM fields and the two coordinates obtained by dimensional reduction of the four dimensional YM gauge fields. We are interested in the equivalence between this theory and string theory in a curved background. In particular we have seen that a naive calculation of the effective action for diagonal configurations leads to the string equations. In this section we reexamine this issue by properly integrating over the background fluctuations to arrive at an effective action for the diagonal configuration. We will focus solely on the divergent contributions to the Kähler potential. We will show that the resulting effective action can only be consistent for a very limited class of manifolds.
4.1 Superfield reduction

Separating the chiral superfields and the vector superfields into diagonal and off-diagonal parts the effective action for the diagonal fields is obtained by integrating over the off-diagonal elements $\phi$ and $v$ and the fluctuations of the diagonal parts $\phi_d$ and $v_d$. The resulting effective action possesses a modified Kähler potential $K_R$ in such a way that

$$S_{\text{eff}} = \frac{1}{\alpha'} \int d^2 x d^4 \theta K_R(\Phi_d, \bar{\Phi}_d)$$

(16)

The superpotential is not renormalized and vanishes for diagonal configurations. The renormalized Kähler potential is obtained after removing the UV divergences leading to poles in $\frac{1}{\epsilon}$ when using dimensional regularization. These poles correspond to the logarithmic divergences of the sigma models in two dimensions.

Let us see in more details how this is implemented in the $N = 2$ two dimensional SYM context. It is well known that the flat space action with simple quadratic Kähler potential (5) is finite [13] so it is only diagrams containing higher order terms in the Kähler potential that can lead to divergences. The simplest such term is

$$K_{IJKL} \Phi^I e^{-gv} \Phi^J e^{gv} \Phi^K e^{-gv} \Phi^L e^{gv},$$

(17)

along with its symmetrized partners. Each field $\Phi$ can be split into a background part and a fluctuation part (7). The fields $v$ (see equation(6)) are the fluctuating part of the vector superfields and the exponentials can then be expanded to arbitrary order. All fluctuation fields are contracted, with the loop diagrams being easily deduced from the usual rules for the propagators and the vertices. Below we give the relevant propagators and vertices, with their appropriate factors of $D^2$ and $\bar{D}^2$ (33). Since we are only interested in the divergent part of the Feynman diagram expansion we can simply replace the background covariant derivatives $D^2$ (33) by ordinary fermionic derivatives $D^2$. The diagonal background fields lead to the off-diagonal quantum fields acquiring a mass. Again since we are only interested here in the ultraviolet divergences we ignore these masses. In this context we can also ignore the propagators between two chiral fields and those between two antichiral fields. Likewise the contribution from the superpotential vertices can be ignored as we will discuss below. We are thus

Figure 1: Propagators and vertices
left with the Feynman vertices and propagators shown in figure 1. These connect together the quantum fields in the expansion of the Kähler potential such as (17). A typical Kähler potential vertex is illustrated below. Only the quantum fields are represented. All chiral legs carry a factor $D^2$ and all anti-chiral legs a factor of $D^2$.

![Figure 2: Typical Kähler potential vertex](image)

The simplest divergent contributions are those coming from connecting together chiral and antichiral fields, and with no vector superfields participating. These contributions are matrix generalizations of the divergent diagrams calculated in string theory in the determination of the beta function by the background field method. We also, however, have to consider the contribution of loops containing one or more vector superfields and/or superpotential insertions.

Let us first consider the vector superfield contribution. A closed vector superfield loop is automatically zero since it is proportional to $\delta(0)$ in superspace. A single loop formed from a vector propagator and a chiral propagator can be seen to be ultraviolet finite. It has one momentum integral, contributing $p^2$, four propagators, contributing $p^{-4}$ and four $D$s. Dimensionally the four $D$s are the equivalent of two momenta however they are used up in the identity $\delta_{12}D_1^2D_2^2\delta_{12} = 16\delta_{12}$ in integrating over the $\theta$ variables (see for example [13][14]). The total diagram thus has divergence $-2$.

One can proceed systematically in this way to find the divergence for a general loop diagram. If one has $L$ loops with $P$ propagators of any type, $C$ of which are chiral-antichiral propagators, we have the degree of divergence

$$\text{div} = 2L - 2P + 2C - 2L = -2(P - C).$$

The first $2L$ comes from the integral over the $2L$ loop momenta and the $-2P$ from the propagators. We are only considering the vertices shown in figures 1 and 2 so all chiral-antichiral propagators come with four factors of $D$, equivalent to two momenta and hence the contribution $2C$. The final $-2L$ in (18) comes from the fact that for each loop we need four $D$s to obtain a non-zero result when performing the final $\theta$ integral connected with a loop. We thus see that for divergent diagrams $P = C$.

Finally superpotential insertions always reduce the degree of divergence since one of their internal legs has no factors of $D$. We are thus left with examining
the divergences due to chiral diagrams with no superpotential insertions and no
gauge fields.

4.2 Chiral diagrams

We have thus reduced the calculation of the divergent part of the loop expansion
for the effective action to a matrix generalization of the string theory beta function
calculation.

As stated above it is known that in string theory the one, two or three loops
divergent contributions disappear for Ricci flat manifolds whilst at four loops
there is a correction that only disappears for manifolds with a vanishing six
dimensional Euler density. The divergences lead to the famous $R^4$ term being
added to the low energy effective action for the massless modes of the string.
In other words Ricci flatness is a low order approximation corrected by terms of
higher order in $\alpha'$. For the curved space versions of matrix theory however there are two types of
chiral diagrams. Firstly there are those coming from the expansion of the Kähler
potential in terms of the diagonal fluctuations only. This is nothing but $N$ copies
of the two dimensional sigma model with values in a six dimensional complex
Kählerian manifold. Secondly there are diagrams involving the off-diagonal part
of $\phi$. These will lead to divergent terms involving one or more diagonal elements
i.e. to terms consisting of products of traces. Since these are not included in the
original action they have to be set to zero. In other words we find that each loop
order has to be individually set to zero. This is a more stringent restriction than
in string theory.

Retaining only at each loop order the contribution due to the diagonal matrices
is a simple generalization of the string theory beta function result (we study
this question below) we see that, in particular, the four loop term has to be set
to zero. This implies that the curved manifold must be Ricci flat with a vanishing
Euler class. This is for instance the case of products $M \times C$ where $M$ is hyper-
Kähler. In particular the ALE spaces are good candidates for a description of
matrix string theory in curved space.

This result thus restricts quite severely the range of applicability of matrix
sigma models as descriptions of matrix theory in curved space. Up to now we have
only considered the terms due to the diagonal matrices. This is not sufficient to
guarantee the finiteness of the model. We now turn to off diagonal contributions.
We will only examine them at the one loop order.

4.3 1 loop contribution

It is not immediately obvious that Ricci flat metrics lead to vanishing one, two
and three loop contributions for the matrix sigma model. Indeed the contribution
of the off-diagonal matrices needs to be carefully examined. It is important to measure their relevance at least for the first non-trivial term. Failure of the cancellation process at this level would almost certainly lead to the conclusion that matrix theory can only be consistent for flat space. The first non-trivial test involves the sixth order term in the expansion of the Kähler potential. We show that, by a particular choice of ordering and the addition of a particular commutator term (that vanishes on the classical moduli space), this contribution will disappear for Ricci flat metrics. The condition to be satisfied for this to be the case is identical to one of the mass conditions deduced in [9].

Let us first focus on the 1-loop contribution. For $N = 1$ (this amounts to the 1 loop string beta function calculation) the correction to the Kähler potential is given by

$$
\delta K^{1L} = \frac{1}{\epsilon} \ln(\det g), \quad \text{with} \quad g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi}),
$$

where $(\epsilon = d - 2)$ we are using dimensional regularization. If we write this in powers of the background field $\Phi$ we arrive at the expansion shown diagrammatically in figure 3. Each vertex corresponds to a term in the expansion of the Kähler potential. The external lines correspond to the number of background fields. The first line thus corresponds to the Ricci tensor $R_{I\bar{J}} = \delta^{KK} K_{IK\bar{J}K}$ evaluated at the special point about which we have chosen the normal coordinates.\footnote{The series starts at order $\Phi^2$ since this is the first relevant contribution inside the full superspace integral.} The second line corresponds to a correction of order $\Phi^3$ etc. Saying that the metric is Ricci flat at the point $\Phi = 0$ amounts to having the first term equal to zero. Saying that it is Ricci flat everywhere implies that every line, (the coefficient for each power of $\Phi$) is zero.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{1 loop expansion}
\end{figure}

\[3\]
Now we consider general $N$. The 1 loop contribution now reads

$$\delta K^{1L} = \frac{1}{\epsilon} \sum_{ij} \ln(\det(g_{ij})), \quad \text{with} \quad (g_{ij})_{ij} = \frac{\partial^2}{\partial \Phi_I^I \partial \Phi_J^J} K(\Phi, \bar{\Phi}), \quad (20)$$

where the determinant is taken over the indices $I$ and $\bar{J}$. As discussed in section (4.2) this contribution has to be set to zero. Equivalently this leads to the condition that for each $i, j$

$$\det(g_{ij}) = 1. \quad (21)$$

This is precisely one of the mass conditions deduced in [9].

This condition first becomes nontrivial for the third line of figure (3) which represents the sum of a term coming from all possible connected contractions of the two fourth order terms in the Kähler potential and the contraction of single sixth order term with itself.

There were two mass conditions found in [9]. Imposing them both constrained the fourth order term to be the totally symmetrized product

$$K^{(4)} = K_{IKJL}[\frac{2}{3} \Phi^I \Phi^K \Phi^J \Phi^L + \frac{1}{3} \Phi^I \Phi^J \Phi^K \Phi^L]. \quad (22)$$

For the sixth order term it was found in [9] that the symmetrized trace was no longer sufficient to satisfy the mass conditions, and an explicit sixth order term which did satisfy them was constructed. This term was somewhat complicated. Since in fact there exists a very simple solution to the condition (21), we think it worth presenting below.

The contraction of two symmetrized fourth order terms of the form (22) gives rise to a contribution proportional to

$$\delta K_{K_4K_4} = \delta^{MM} \delta^{PP}(K_{IMKP}K_{PLM} + \frac{1}{2}K_{IMMP}K_{PLMM}) \frac{1}{9} \mathcal{P}(I, K, \bar{J}, \bar{L}) + \delta^{MM} \delta^{PP} K_{IMJP}K_{KLMP} \frac{8}{9} \mathcal{Q}(I, K, \bar{J}, \bar{L}), \quad (23)$$

where $\mathcal{P}$ and $\mathcal{Q}$ are polynomials in the diagonal chiral fields. They are given by

$$\mathcal{P}(I, K, \bar{J}, \bar{L}) = (2N + 7) \sum_i I_i K_i \bar{J}_i \bar{L}_i + \sum_{i \neq j} [2I_i K_i \bar{J}_j \bar{L}_i + \frac{1}{2} I_i K_j \bar{J}_i \bar{L}_j + \frac{1}{2} I_i K_j \bar{J}_j \bar{L}_i + I_i K_i \bar{J}_j \bar{L}_i + I_i K_j \bar{J}_i \bar{L}_j + I_i K_j \bar{J}_j \bar{L}_i], \quad (24)$$

$$\mathcal{Q}(I, K, \bar{J}, \bar{L}) = \sum_{i \neq j} [I_i K_i \bar{J}_j \bar{L}_j - I_i K_j \bar{J}_i \bar{L}_i] \quad (25)$$

where for compactness we denote the chiral fields solely by their complex indices, i.e. $I_i = \Phi_i^I, \bar{J}_i = \Phi_i^J$ etc. We have split the result up into two polynomials for reasons that will become clear shortly. The two possible contractions give...
rise to polynomials that are almost exactly equivalent. The first contraction in \((23)\) differs from the second in that the coefficients of two of the terms in the second line of \(P\) are interchanged. The addition of the polynomial \(Q\) serves to interchange these two coefficients.

The sixth order term in the Kähler potential has the general form

\[
K_{IKM\bar{J}\bar{L}\bar{N}}(\alpha I KM\bar{J}\bar{L}\bar{N} + \beta_1 IKJ\bar{M}\bar{L}\bar{N} + \beta_2 IKJ\bar{L}M\bar{N} + \gamma IJK\bar{L}M\bar{N}),
\]  

where for now we are neglecting possible commutator terms. The coefficient \(K_{IKM\bar{J}\bar{L}\bar{N}}\) is symmetric under arbitrary interchange of holomorphic indices and under interchange of antiholomorphic indices. Again we denote, for compactness, \(\Phi_{ij} = I_i\) etc. The totally symmetrized product (within a trace) corresponds to \(\alpha = \beta_1 = \beta_2 = 3/10\) and \(\gamma = 1/10\).

Ignoring the possible commutator terms we sum over all possible contractions of a holomorphic and anti-holomorphic index in equation \((26)\) and look for coefficients \(\alpha, \beta_1, \beta_2\) and \(\gamma\) for which the resulting polynomial resembles as much as possible equation \((23)\). We find that, for \(\alpha = 1\) and all other constants zero,

\[
\delta K_6 = \delta^{M\bar{M}} K_{IKM\bar{J}\bar{L}\bar{N}} P(I, K, J, L),
\]  

where the polynomial \(P\) is defined in \((24)\). To complete the sixth order term we now add a term which upon contraction will cancel with \(Q\) polynomial contribution in equation \((23)\). The term required is

\[
-\frac{4}{3} \delta^{PP} K_{IMJ\bar{P}K\bar{P}L\bar{N}}(MIK\bar{J}N\bar{L} - MI\bar{J}\bar{N}KL + NIKMJ\bar{L} - NI\bar{J}MK\bar{L}).
\]  

It is relatively easy to see that contracting in all possible ways a holomorphic with an anti-holomorphic index does indeed give the term proportional to \(Q\). Firstly contracting \(K\) with \(J\) is zero by construction and furthermore contracting \(K\) with \(L\) or \(N\) automatically gives zero since the result is proportional to the Ricci tensor. One thus only needs to consider the contractions between \(IM\) and \(L\bar{N}\), which by symmetry reduces to contractions between \(M\) and \(\bar{N}\).

The total sixth order term is thus

\[
K_6 = K_{IKM\bar{J}L\bar{N}} IKM\bar{J}L\bar{N} - \frac{4}{3} \delta^{PP} K_{IMJ\bar{P}K\bar{P}L\bar{N}}(MIK\bar{J}N\bar{L} - MI\bar{J}\bar{N}KL
+ NIKMJ\bar{L} - NI\bar{J}MK\bar{L}).
\]  

The only other possible terms that could be added to this are terms which disappear under the contractions being considered above, i.e. terms which are zero when there are less than three non diagonal matrices. Such terms can only be constructed from the product of three commutators. Presumably the difference between the result \((25)\) and the complicated form presented in \([9]\) amounts to the addition of such terms.
5 D0 Branes in a Curved Background

The analysis for string matrix theory in a curved background can be applied to the quantum mechanics of D0 branes. This is obtained by further reducing the $N = 4$ 4d SYM theory to 0+1 dimensions. This describes the evolution of D0 branes in a six dimension Kählerian background. Using the background field method one can study the effective action obtained from integrating out the off-diagonal fields and the fluctuations of the diagonal part. The analysis is similar to the calculations presented in the previous section. The only difference being the dimension of the loop integrals; one integrates over a single momentum variable. All the loop integrals are therefore UV finite.

Integrating out the background fields thus leads to non-local, (in spacetime) finite terms coupling two or more diagonal elements/D0 branes. As discussed in the previous section, by a judicious choice of matrix ordering, it might be possible to ensure that all such terms are zero for Ricci flat metrics at the one, two and three loop level. At four loops however Ricci flatness is insufficient to cancel the non-local terms and the manifold has to be further restricted to have vanishing six dimensional Euler class. This is the case for a hyper-Kähler surface.

6 Conclusions

We have shown that the 1 loop calculation for the effective action for matrix string theory in a curved space has divergences corresponding to non-local terms connecting together two or more diagonal elements. These terms arise from simple matrix generalizations of the string theory beta function calculation. They correspond to powers of traces and, since the original action is postulated to contain a single trace, cannot be renormalized into a redefinition of the Kähler potential. The condition that these terms vanish is identical to one of the two mass conditions imposed on the Kähler potential in the analysis of Douglas et. al. \[9\]. At lowest nontrivial order it is possible to find particular matrix orderings and commutator terms that satisfy the condition.

However the fact that the four loop term cannot be renormalized into the Kähler potential means that these models have a limited range of applicability, only being consistent for Ricci flat manifolds with vanishing six dimensional Euler density.

The analysis of this article did not depend on the size of the matrices and it is hard to see any hidden subtleties in the taking of the large $N$ limit that might change the analysis for infinite $N$. \[4\]

\[4\]Some subtleties in the large $N$ limit have been pointed out in \[15\] but they are infra-red effects not ultra-violet.
It is also not at all obvious how to modify the gauged matrix sigma models to have a more general applicability. The addition by hand of powers of traces to cancel the divergences would be ad hoc and it is not clear how the inclusion of higher derivative terms could improve the problem. It seems likely that there is something more fundamental missing from the description. Certainly one is all too aware of the lack of a basic principle to guide us and the lack of a solid set of fundamental building blocks from which to construct actions. Perhaps this is another sign [16, 17] that matrix variables are insufficient to describe curved space, even for infinite $N$.

7 Appendix

The superfield reduction of section 4 was based on the background superfield method first devised in [12] to discuss properties of $N = 4$ SYM theories. For the matrix string theory on a curved background this method is crucial to integrate out the background fields and obtain the effective action of the background diagonal fields. The idea is a simple superfield generalization of the background field method used to quantize gauge theories. We refer the reader to [12] [13] [14] for fuller details.

Recall that the gauge transformation of $N = 1$ vector superfields are given by

$$e^V ightarrow e^\Lambda e^V e^{-\Lambda}$$

(30)

where $\Lambda$ is a chiral superfield satisfying $\bar{D}_\alpha \Lambda = 0$ i.e. the chiral superfields are annihilated by the fermionic covariant derivatives. Let us now assume that the vector superfield is the sum of a background configuration and a fluctuating part. The most natural splitting would be to define the vector field as the sum of these two contributions. However the quantization of the theory with a background is easier if one splits

$$e^V = e^{\frac{v}{\lambda^2}} e^{-\frac{v}{\lambda^2}}$$

(31)

where $v$ is the fluctuating part and $V_B$ the background equation. To first order in the Campbell-Hausdorff expansion this leads to

$$V \sim V_B + v$$

(32)

as expected. The chiral part of the Lagrangian is constructed using the background covariant derivatives

$$D_\alpha = e^{-\frac{v}{\lambda^2}} D_\alpha e^{\frac{v}{\lambda^2}}$$

(33)
and its conjugate. Indeed one defines background covariantly chiral superfield as the solutions of
\[ \mathcal{D}_\alpha \Phi = 0 \] (34)
This constraint is solved defining
\[ \Phi = e^{\frac{V}{2}} \tilde{\Phi} e^{-\frac{V}{2}} \] (35)
and \( \tilde{\Phi} \) is a chiral superfield.

These ingredients can be used to write the Lagrangian in a way that naturally separates the background fields from the fluctuating parts. The chiral superfield containing the field strengths of the gauge fields is
\[ e^{\frac{V}{2}} W_\alpha e^{-\frac{V}{2}} = i \left[ \mathcal{D}_\beta, \{ e^{-v} \mathcal{D}_\alpha e^v, \mathcal{D}_\beta \} \right] \] (36)

The chiral part of the Lagrangian can be written in terms of the background chiral fields and the fluctuating part of the gauge fields. For instance the canonical term becomes
\[ \text{tr}(e^V \tilde{\Phi} e^{-V} \tilde{\Phi}^\dagger) = \text{tr}(e^v \Phi e^{-v} \Phi^\dagger) \] (37)
Similarly the whole Kähler potential is a function of \( v, \Phi \) and \( \tilde{\Phi} \) only. Likewise the superpotential becomes a function \( W(\Phi) \) of \( \Phi \) only. Finally the background chiral fields can be split into a background part and a fluctuating part
\[ \Phi = \Phi_B + \phi \] (38)
where both fields are background chiral.

The advantage of redefining the Lagrangian in such a way is that the effective action obtained after integrating over the fluctuations around the background field is gauge invariant. The effective action is obtained after fixing the gauge. The gauge fixing term is chosen to be
\[ \mathcal{D}^2 V = \bar{f}, \quad \mathcal{D}^2 V = f \] (39)
where \( f \) is a background chiral field. This leads to the introduction of three ghosts, the two Fadeev-Popov ghosts and the Kallosh-Nielsen ghost. Using all these ingredients one can apply the usual procedure to calculate super-Feynman diagrams.

The fact that the fields are covariantly chiral leads to more complicated propagators than is the case for ordinary chiral fields but for the divergent diagrams we consider in this article the distinction is unimportant, allowing us to use standard superfield Feynman rules.
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