Hermitian Self-Dual Cyclic Codes of Length $p^a$ over GR($p^2, s$)

Somphong Jitman, San Ling, and Ekkasit Sangwisut

Abstract

In this paper, we study cyclic codes over the Galois ring GR($p^2, s$). The main result is the characterization and enumeration of Hermitian self-dual cyclic codes of length $p^a$ over GR($p^2, s$). Combining with some known results and the standard Discrete Fourier Transform decomposition, we arrive at the characterization and enumeration of Euclidean self-dual cyclic codes of any length over GR($p^2, s$). Some corrections to results on Euclidean self-dual cyclic codes of even length over $\mathbb{Z}_4$ in Discrete Appl. Math. 128, (2003), 27 and Des. Codes Cryptogr. 39, (2006), 127 are provided.

1 Introduction

Cyclic and self-dual codes over finite fields have been extensively studied for both theoretical and practical reasons (see [6], [11], and references therein). These concepts have been extended and studied over the ring $\mathbb{Z}_4$ (see [1], [2], and [3]), after it has been proven that some binary non-linear codes, such as the Kerdock, Preparata, and Goethals codes, are the Gray image of linear cyclic codes over $\mathbb{Z}_4$ in [5]. Later on, the study of cyclic and self-dual codes has been generalized to codes over $\mathbb{Z}_{p^r}$ and Galois rings (see [5], [9], [10], [12], and [13]).

In [2], the structure of cyclic and Euclidean self-dual cyclic codes of oddly even length ($2m$, where $m$ is odd) over $\mathbb{Z}_4$ has been studied via the Discrete Fourier Transform decomposition. This idea has been extended to the case of all even lengths in [4]. However, some results concerning Euclidean self-dual cyclic codes over $\mathbb{Z}_4$ in [2] and [4] are not correct. The corrections to these are provided in Section 4. Using a spectral approach and a generalization of the results in [2] and [4], the structure of cyclic codes over $\mathbb{Z}_{p^r}$, for any prime $p$, has been studied in [5]. A nice classification of cyclic and Euclidean self-dual cyclic codes of length $p^a$ over GR($p^2, s$) has been given using a different approach in [9], [10], and [12].

In this paper, we focus on the characterization and enumeration of Hermitian self-dual cyclic codes of length $p^a$ over GR($p^2, s$) and their application to the enumeration of Euclidean self-dual cyclic codes of any length over GR($p^2, s$). Using the standard Discrete Fourier Transform decomposition viewed as an extension of [5], a cyclic code $C$ of any length $n = mp^a$, where $p$ is a prime and $p \nmid m$, over GR($p^2, s$) can be viewed as a product of cyclic codes of length $p^a$ over some Galois extensions of GR($p^2, s$). Euclidean self-dual cyclic codes can be characterized based on this decomposition. Applying some known results concerning cyclic and Euclidean self-dual cyclic codes of length $p^a$ in [9] and [10] and our result on Hermitian self-dual cyclic codes of...
length $p^a$, the number of Euclidean self-dual cyclic codes of arbitrary length over $\text{GR}(p^2, s)$ can be determined. Finally, we point out some mistakes on Euclidean self-dual cyclic codes of even length over $\mathbb{Z}_4$ in [2] and [4]. The corrections to these are given as well.

The paper is organized as follows. Some preliminary concepts and results are recalled in Section 2. In Section 3, we prove the main result concerning the number of Hermitian self-dual cyclic codes of length $p^a$ over $\text{GR}(p^2, s)$. An application to the enumeration of Euclidean self-dual cyclic codes of any length over $\text{GR}(p^2, s)$ and corrections to [2] and [4] are discussed in Section 4. A conclusion is provided in Section 5.

2 Preliminaries

In this section, we recall some definitions and basic properties of cyclic codes over the Galois ring $\text{GR}(p^2, s)$.

2.1 Cyclic Codes over $\text{GR}(p^2, s)$

For a prime $p$ and a positive integer $s$, the Galois ring $\text{GR}(p^2, s)$ is the Galois extension of the integer residue ring $\mathbb{Z}_{p^2}$ of degree $s$. Let $\xi$ be an element in $\text{GR}(p^2, s)$ that generates the Teichmüller set $\mathcal{T}_s$ of $\text{GR}(p^2, s)$. In other words, $\mathcal{T}_s = \{0, 1, \xi, \xi^2, \ldots, \xi^{p^s-2}\}$. Then every element in $\text{GR}(p^2, s)$ has a unique $p$-adic expansion of the form

$$\alpha = a + bp,$$

where $a, b \in \mathcal{T}_s$. If $s$ is even, let $\bar{\cdot}$ denote the automorphism on $\text{GR}(p^2, s)$ defined by

$$\bar{\alpha} = a^{p^{s/2}} + b^{p^{s/2}}.$$

(2.1)

For more details concerning Galois rings, we refer the readers to [14].

A cyclic code of length $n$ over $\text{GR}(p^2, s)$ is a $\text{GR}(p^2, s)$-submodule of the $\text{GR}(p^2, s)$-module $(\text{GR}(p^2, s))^n$ which is invariant under the cyclic shift. It is well known that every cyclic code $C$ of length $n$ over $\text{GR}(p^2, s)$ can be regarded as an ideal in the quotient polynomial ring $\text{GR}(p^2, s)[X]/\langle X^n - 1 \rangle$ and represented by its polynomial representation

$$\left\{ \sum_{i=0}^{n-1} c_i X^i \mid (c_0, c_1, \ldots, c_{n-1}) \in C \right\}.$$

For a given cyclic code $C$ of length $n$ over $\text{GR}(p^2, s)$, denote by $C^E$ the Euclidean dual of $C$ defined with respect to the form

$$\langle u, v \rangle_E := \sum_{i=0}^{n-1} u_i v_i$$

where $u = \sum_{i=0}^{n-1} u_i X^i$ and $v = \sum_{i=0}^{n-1} v_i X^i$. The code $C$ is said to be Euclidean self-dual if $C = C^E$.

In addition, if $s$ is even, we can also consider the Hermitian dual $C^H$ of $C$ defined with respect to the form

$$\langle u, v \rangle_H := \sum_{i=0}^{n-1} u_i \bar{v}_i.$$

The code $C$ is said to be Hermitian self-dual if $C = C^H$.

The goal of this paper is to characterize and enumerate self-dual codes over $\text{GR}(p^2, s)$. For convenience, let $N(\text{GR}(p^2, s), n)$, $N_E(\text{GR}(p^2, s), n)$, and $N_H(\text{GR}(p^2, s), n)$ denote the numbers of cyclic codes, Euclidean self-dual cyclic codes, and Hermitian self-dual cyclic codes of length $n$ over $\text{GR}(p^2, s)$, respectively.
2.2 Some Results on Cyclic Codes of Length $p^a$ over $\text{GR}(p^2, s)$

In this subsection, we recall some results concerning cyclic and Euclidean self-dual cyclic codes of length $p^a$ over $\text{GR}(p^2, s)$ in terms of ideals in $\text{GR}(p^2, s)[X]/(X^{p^a} - 1)$.

In [9], it has been shown that all the ideals in $\text{GR}(p^2, s)[X]/(X^{p^a} - 1)$ have a unique representation.

**Proposition 2.1 ([9] Theorem 3.8).** Every ideal in $\text{GR}(p^2, s)[X]/(X^{p^a} - 1)$ can be uniquely represented in the form of

$$C = \langle (X - 1)^{i_0} + p \sum_{j=0}^{i_1-1} h_j(X - 1)^j, p(X - 1)^{i_1} \rangle,$$

(2.2)

where $i_0, i_1$ are integers such that $0 \leq i_0 < p^a$, $0 \leq i_1 \leq \min\{i_0, p^a-1\}$, $i_0 + i_1 \leq p^a$, and $h_j$ is an element in the Teichmüller set $T_s$ for all $j$.

The integer $i_1$ in Proposition 2.1 is called the *first torsion index* of $C$.

**Corollary 2.2 ([9] Corollary 3.9).** In $\text{GR}(p^2, s)[X]/(X^{p^a} - 1)$, the number of distinct ideals with $i_0 + i_1 = d$, where $0 \leq d \leq p^a$, is

$$\frac{p^{s(\ell+1)} - 1}{p^s - 1},$$

where $\ell = \min\{\left\lfloor \frac{d}{s} \right\rfloor, p^a-1\}$.

The next corollary follows immediately from Corollary 2.2 and [9] Theorem 3.6.

**Corollary 2.3.** The number of all cyclic codes of length $p^a$ over $\text{GR}(p^2, s)$ is

$$N(\text{GR}(p^2, s), p^a) = 2 \left( \sum_{d=0}^{p^a-1} \frac{p^{(\min\{\left\lfloor \frac{d}{s} \right\rfloor, p^a-1\})+1} - 1}{p^s - 1} \right) + \frac{p^{(p^a-1)+1} - 1}{p^s - 1}. \quad (2.3)$$

Combining the results in [9], [10], and [12], the complete enumeration of Euclidean self-dual cyclic codes of length $p^a$ over $\text{GR}(p^2, s)$ can be summarized as follows.

**Proposition 2.4 ([10] Corollary 3.5).** The number of Euclidean self-dual cyclic codes of length $2^a$ over $\text{GR}(2^2, s)$ is

$$N_E(\text{GR}(2^2, s), 2^a) = \begin{cases} 1 & \text{if } a = 1, \\ 1 + 2^a & \text{if } a = 2, \\ 1 + 2^a + 2^{2s+1} \left( \frac{(2^a)(2^{a-2}+1) - 1}{2^a - 1} \right) & \text{if } a \geq 3. \end{cases}$$

If $p$ is an odd prime, then the number of Euclidean self-dual cyclic codes of length $p^a$ over $\text{GR}(p^2, s)$ is

$$N_E(\text{GR}(p^2, s), p^a) = 2 \left( \frac{(p^s)^{\frac{p^a-1}{2}} - 1}{p^s - 1} \right).$$

3 Hermitian Self-Dual Cyclic Codes of Length $p^a$ over $\text{GR}(p^2, s)$

In this section, we assume that $s$ is even and focus on characterizing and enumerating Hermitian self-dual cyclic codes of length $p^a$ over $\text{GR}(p^2, s)$. 
It has been proven in [12, Theorem 2] that the Euclidean dual of the cyclic code $C$ in (2.2) is of the form

$$C^\perp = \left\{(X - 1)^{p^n-i_1} - p(X - 1)^{p^n-i_0-i_1} \sum_{t=0}^{i_1-1} \left( \sum_{j=0}^{t} (-1)^{i_0+j} \left( \frac{i_0 - j}{t-j} \right) h_{ij} \right) (X - 1)^t \right\}$$

$$+ \sum_{t=1}^{K} \left( \sum_{j=1}^{\min(t,p-1)} \left( -1 \right)^{j+1+1} \left( \frac{p - j}{t-j} \right) \left( \frac{p}{j} \right) \right) (X - 1)^{p^n-i_1+1} + p(X - 1)^{p^n-i_0},$$

where $K = \left\lfloor \frac{p^n-i_0+i_1-1}{p^n-i_0} \right\rfloor$ and the empty sum is regarded as zero.

For a subset $A$ of $\text{GR}(p^2,s)[X]/(X^{p^n} - 1)$, let $\overline{A}$ denote the set

$$\left\{ \sum_{i=0}^{p^n-1} a_i X^i \left| \sum_{i=0}^{p^n-1} a_i X^i \in A \right. \right\},$$

where $-^*$ is the automorphism defined in (2.1). Since it is well known that $C^\perp = \overline{C^{\perp}}$, we have

$$C^\perp = \left\{(X - 1)^{p^n-i_1} - p(X - 1)^{p^n-i_0-i_1} \sum_{t=0}^{i_1-1} \left( \sum_{j=0}^{t} (-1)^{i_0+j} \left( \frac{i_0 - j}{t-j} \right) h_{ij} \right) (X - 1)^t \right\}$$

$$+ \sum_{t=1}^{K} \left( \sum_{j=1}^{\min(t,p-1)} \left( -1 \right)^{j+1+1} \left( \frac{p - j}{t-j} \right) \left( \frac{p}{j} \right) \right) (X - 1)^{p^n-i_1+1} + p(X - 1)^{p^n-i_0},$$

If $C = C^\perp$, then $|C| = (p^n)^n$ which implies that $i_0 + i_1 = p^n$. Then we can write

$$C^\perp = \left\{ \begin{array}{ll}
(X - 1)^{i_0} - p \sum_{j=0}^{i_1-1} \left( \sum_{j=0}^{t} (-1)^{i_0+j} \left( \frac{i_0 - j}{t-j} \right) h_{ij} \right) (X - 1)^t, p(X - 1)^{i_1} \\
+ p(X - 1)^{p^n-i_1+1} + p(X - 1)^{i_1} & \text{if } i_1 < \frac{p^n-i_0+1}{2}, \\
(3.1) \end{array} \right.$$
where $\mathbf{x} := (x_1, x_2, \ldots, x_i)^T$, $\mathbf{x}^{p/2} := (x_1^{p/2}, x_2^{p/2}, \ldots, x_i^{p/2})^T$, and $\mathbf{b} := (b_1, b_2, \ldots, b_i)^T$ is a zero vector except for the case $i_1 \geq \frac{p^{s-1} - 1}{2}$ where, for each $1 \leq i \leq i_1$, $b_i$ is defined by

$$b_i = \begin{cases} 1 & \text{if } i = p^{a-1} - i_1 + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.5)$$

Therefore, the cyclic code $C$ in (2.2) is Hermitian self-dual if and only if the matrix equation (3.4) has a solution in $\mathbb{F}_{p^s}$. Moreover, the number of Hermitian self-dual cyclic codes of length $p^s$ over $\text{GR}(p^2, s)$ with first torsion degree $i_1$ equals the number of solutions of (3.4) in $\mathbb{F}_{p^s}$.

From (3.3), we observe that $M(p^s, i_1)$ has the following properties. For $1 \leq j, i \leq i_1$, let $m_{ij}$ denote the entry in the $i$th row and $j$th column of $M(p^s, i_1)$. Then, for integers $1 \leq k < j < i \leq i_1$, we have

$$m_{ij} = (-1)^{i_0 + j - 1}(i_0 - j + 1),$$

and hence,

$$m_{ij}m_{jk} = (-1)^{i_0 + j - 1}(i - k)(j - k)m_{ik}, \quad (3.6)$$

**Lemma 3.1.** Let $i$ be an integer such that $1 \leq i < i_1$. Then

$$\sum_{j=1}^{i} m_{i+1,j}b_j = 0 \text{ in } \mathbb{F}_{p^s}. \quad \Box$$

**Proof.** If $i_1 < \frac{p^{a-1} - 1}{2}$, then $b_i = 0$ for all $1 \leq i \leq i_1$, and hence, the result follows.

Assume that $i_1 \geq \frac{p^{a-1} - 1}{2}$. If $i < \frac{p^{a-1} + 1}{2}$, then $b_j = 0$ for all $1 \leq j \leq i$, and hence, $\sum_{j=1}^{i} m_{i+1,j}b_j = 0$ in $\mathbb{F}_{p^s}$. Assume that $i \geq \frac{p^{a-1} + 1}{2}$. Then, by (3.5), we have

$$\sum_{j=1}^{i} m_{i+1,j}b_j = m_{i+1,1} = \frac{p^a - p^{a-1}}{2}(i_0 + 1 - i)$$

$$= 0 \text{ in } \mathbb{F}_{p^s},$$

by Lucas’s Theorem (see [3, Theorem 26]) and the fact that $1 \leq i_0 - i < p^{a-1}$. $\Box$

In order to determine the number of solutions of (3.4) in $\mathbb{F}_{p^s}$, we recall two maps which are important tools.

1. The trace map $\text{Tr} : \mathbb{F}_{p^s} \rightarrow \mathbb{F}_{p^{s/2}}$ defined by $\alpha \mapsto \alpha + \alpha^{p^{s/2}}$ for all $\alpha \in \mathbb{F}_{p^s}$.

2. The map $\Psi : \mathbb{F}_{p^s} \rightarrow \mathbb{F}_{p^s}$ defined by $\Psi(\alpha) = \alpha^{p^{s/2}} - \alpha$ for all $\alpha \in \mathbb{F}_{p^s}$.

It is well known that $\text{Tr}$ is $\mathbb{F}_{p^{s/2}}$-linear and it is not difficult to see that $\Psi$ is also $\mathbb{F}_{p^{s/2}}$-linear. If $p = 2$, then $\text{Tr} = \Psi$. Moreover, we have the following properties.

**Lemma 3.2.** For each $\alpha \in \mathbb{F}_{p^s}$, $\Psi(\alpha) = 0$ if and only if $\alpha \in \mathbb{F}_{p^{s/2}}$. Moreover,

$$\Psi \circ \text{Tr} \equiv 0 \equiv \text{Tr} \circ \Psi. \quad (3.7)$$

**Proof.** It follows immediately from the definitions. $\Box$
Lemma 3.3. The following statements hold.

i) For each \( a \in \Psi(F_{p^s}) \), \(|\Psi^{-1}(a)| = p^{s/2}\).

ii) For each \( a \in \text{Tr}(F_{p^s}) \), \(|\text{Tr}^{-1}(a)| = p^{s/2}\).

Proof. We note that, for each \( a \in \Psi(F_{p^s}) \) and \( b \in F_{p^s} \), \( b \in \Psi^{-1}(a) \) if and only if \( b + \ker(\Psi) = \Psi^{-1}(a) \). Since \( \ker(\Psi) = F_{p^s/2} \), the statement i) follows.

Similarly, for each \( a \in \text{Tr}(F_{p^s}) \) and \( b \in F_{p^s} \), \( b \in \text{Tr}^{-1}(a) \) if and only if \( b + \ker(\text{Tr}) = \text{Tr}^{-1}(a) \). Since \( \text{Tr} \) is a surjective \( F_{p^s/2} \)-linear map from \( F_{p^s} \) to \( F_{p^s/2} \), we have \(|\ker(\text{Tr})| = |F_{p^s}|/|F_{p^s/2}| = p^{s/2}\), and hence, ii) follows.

\[ \square \]

Proposition 3.4. Let \( s \) be an even positive integer and let \( i_1 \) be a positive integer such that \( i_1 \leq p^{a-1} \). Then the number of solutions of (3.4) in \( F_{p^{i_1}} \) is \( p^{i_1/2} \).

Proof. From \( [10] \), \( M(p^a, i_1) \) has 4 presentations depending on the parity of \( p \) and \( i_1 \). We therefore separate the proof into 4 cases.

Case 1. \( p \) is odd and \( i_1 = 2i_1 + 1 \) is odd.

From \( [10] \), the matrix \( M(p^a, i_1) \) can be written as

\[
M(p^a, i_1) = \begin{pmatrix}
2 & 0 & 0 & 0 & \cdots & 0 \\
* & 0 & 0 & 0 & \cdots & 0 \\
* & * & 2 & 0 & \cdots & 0 \\
* & * & * & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & * & \cdots & 2
\end{pmatrix},
\]

where \(*\)'s denote entries of \( M(p^a, i_1) \) defined in (3.3).

From (3.4) and (3.8), we have

\[
\text{Tr}(x_1) = b_1,
\]

(3.9)

and

\[
\Psi(x_2i) = b_{2i} - \sum_{j=1}^{2i-1} m_{2i,j}x_j,
\]

(3.10)

for all integers \( 1 \leq i \leq \mu_1 \).

This implies that (3.4) has a solution if and only if the right hand sides of (3.9) and (3.11) are in \( F_{p^s/2} \) and the right hand side of (3.10) is in \( \Psi(F_{p^s}) \). In this case, we have

\[
x_1 \in \text{Tr}^{-1}(b_1),
\]

and

\[
x_{2i} \in \Psi^{-1}\left(b_{2i} - \sum_{j=1}^{2i-1} m_{2i,j}x_j\right),
\]

(3.11)
and

\[ x_{2i+1} \in \text{Tr}^{-1} \left( b_{2i+1} - \sum_{j=1}^{2i} m_{2i+1,j} x_j \right) \]

for all \( 1 \leq i \leq \mu_1 \). Hence, by Lemma 3.3 the number of solutions of (3.11) is \( p^{\mu_1/2} \).

By Lemma 3.2, it suffices to show that the images under \( \Psi \) of the right hand sides of (3.9) and (3.11) are 0 and the image under the trace map of the right hand side of (3.10) is 0.

From (3.9), we have \( \Psi(b_i) = 0 \) since \( b_i \in \{0, 1\} \subseteq F_{p^{\nu}} \). Let \( 1 \leq i \leq \mu_1 \) be an integer. From (3.10), we have

\[
\text{Tr} \left( b_{2i} - \sum_{j=1}^{2i-1} m_{2i,j} x_j \right) = \text{Tr}(b_{2i}) - \sum_{j=1}^{2i-1} m_{2i,j} \text{Tr}(x_j)
\]

\[
= 0 - \left( \sum_{j=1}^{i} m_{2i,2j-1} \text{Tr}(x_{2j-1}) + \sum_{j=1}^{i-1} m_{2i,2j} \text{Tr}(x_{2j}) \right),
\]

since \( p^{a-1} - i_1 + 1 \) is odd and then \( b_{2i} = 0 \) for all \( i = 1, 2, \ldots, \mu_1 \),

\[
= - \left( \sum_{j=1}^{i} m_{2i,2j-1} \text{Tr}(x_{2j-1}) + \sum_{j=1}^{i-1} m_{2i,2j} \Phi(x_{2j}) + 2x_{2j} \right),
\]

since \( \text{Tr}(\alpha) = \Phi(\alpha) + 2\alpha \) for all \( \alpha \in F_{p^\nu} \),

\[
= - \sum_{j=1}^{2i-1} m_{2i,j} b_j + \sum_{j=1}^{2i-1} \sum_{k=1}^{j-1} m_{2i,j} m_{j,k} x_k - \sum_{j=1}^{i-1} 2m_{2i,2j} x_{2j}
\]

\[
= 0 + \sum_{k=1}^{2i-2} \left( \sum_{j=k+1}^{2i-1} m_{2i,j} m_{j,k} \right) x_k - \sum_{j=1}^{i-1} 2m_{2i,2j} x_{2j}, \text{ by Lemma 3.1}
\]

\[
= \sum_{k=1}^{2i-2} \left( \sum_{j=k+1}^{2i-1} (-1)^{j+k-1} \binom{2i-1}{j-k} m_{2i,k} \right) x_k - \sum_{j=1}^{i-1} 2m_{2i,2j} x_{2j}, \text{ by (3.6)}\]

\[
= \sum_{k=1}^{2i-2} \left( m_{2i,k} \sum_{j=1}^{2i-k-1} (-1)^{j+k-1} \binom{2i-1}{j} \right) x_k - \sum_{j=1}^{i-1} 2m_{2i,2j} x_{2j}
\]

\[
= \sum_{k=1}^{i} \left( m_{2i,2k-1} \sum_{j=1}^{2i-2k} (-1)^{j} \binom{2i-2k+1}{j} \right) x_{2k-1}
\]

\[
\quad + \sum_{k=1}^{i-1} \left( m_{2i,2k} \sum_{j=1}^{2i-2k} (-1)^{j} \binom{2i-2k}{j} \right) x_{2k} - \sum_{j=1}^{i-1} 2m_{2i,2j} x_{2j}
\]

\[
= \sum_{k=1}^{i} \left( m_{2i,2k-1} \left( -1 - (-1)^{2i-2k+1} \right) \right) x_{2k-1}
\]

\[
\quad + \sum_{k=1}^{i-1} \left( (-1) \left(-1^{2i-2k} \right) m_{2i,2k} \right) x_{2k} - \sum_{j=1}^{i-1} 2m_{2i,2j} x_{2j}
\]

\[
= 0 \text{ in } F_{p^\nu}.
\]

(3.12)
From (3.11), we have

\[ \Psi \left( b_{2i+1} - \sum_{j=1}^{2i} m_{2i+1,j} x_j \right) = \Psi(b_{2i+1}) - \sum_{j=1}^{2i} m_{2i+1,j} \Psi(x_j) \]

\[ = 0 - \left( \sum_{j=1}^{i} m_{2i+1,2j-1} \Psi(x_{2j-1}) + \sum_{j=1}^{i} m_{2i+1,2j} \Psi(x_{2j}) \right), \]

since \( b_{2i+1} \in \mathbb{F}_{p^r} \) for all \( i = 1, 2, \ldots, \mu_1 \),

\[ = - \left( \sum_{j=1}^{i} m_{2i+1,2j-1} (\text{Tr}(x_{2j-1}) - 2x_{2j-1}) + \sum_{j=1}^{i} m_{2i+1,2j} \Psi(x_{2j}) \right), \]

since \( \Psi(\alpha) = \text{Tr}(\alpha) - 2\alpha \) for all \( \alpha \in \mathbb{F}_{p^r} \),

\[ = - \sum_{j=1}^{2i} m_{2i+1,j} b_j + \sum_{j=1}^{2i-1} \sum_{k=1}^{j} m_{2i+1,j} m_{j,k} x_k + \sum_{j=1}^{i} 2m_{2i+1,2j-1} x_{2j-1} \]

\[ = 0 + \sum_{k=1}^{2i-1} \left( \sum_{j=k+1}^{2i} (-1)^j (2i - k + 1) m_{2i+1,k} \right) x_k \]

\[ + \sum_{j=1}^{i} 2m_{2i+1,2j-1} x_{2j-1}, \text{ by Lemma 3.1} \]

\[ = \sum_{k=1}^{i} \left( \sum_{j=1}^{2i-2k-1} (-1)^j (2i - 2k + 2) m_{2i+1,2k-1} \right) x_{2k-1} \]

\[ + \sum_{k=1}^{i} \left( \sum_{j=1}^{2i-2k} (-1)^j (2i - 2k + 1) m_{2i,2k} \right) x_{2k} + \sum_{j=1}^{i} 2m_{2i+1,2j-1} x_{2j-1} \]

\[ = \sum_{k=1}^{i} \left( (-1) - (-1)^{2i-2k+2} m_{2i+1,2k-1} \right) x_{2k-1} \]

\[ + \sum_{k=1}^{i} (\sum_{j=1}^{2i-2k+1} (-1)^j m_{2i-2k}) x_{2k} + \sum_{j=1}^{i} 2m_{2i+1,2j-1} x_{2j-1} \]

\[ = 0 \text{ in } \mathbb{F}_{p^{r/2}}. \] (3.13)

This case is completed.

**Case 2.** \( p \) is odd and \( i_1 = 2\mu_1 \) is even.
From [10], we have
\[
M(p^a, i_1) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
* & 2 & 0 & 0 & \cdots & 0 \\
* & * & 0 & 0 & \cdots & 0 \\
* & * & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & 2 \\
\end{pmatrix},
\] (3.14)
where *’s denote entries of \(M(p^a, i_1)\) defined in (3.3).

From (3.4) and (3.14), we have
\[
\Psi(x_{2i_1-1}) = b_{2i_1-1} - \sum_{j=1}^{2i_1-2} m_{2i_1-1,j} x_j 
\] (3.15)
and
\[
\text{Tr}(x_{2i_1}) = b_{2i_1} - \sum_{j=1}^{2i_1-1} m_{2i_1,j} x_j 
\] (3.16)
for all integers \(1 \leq i \leq \mu_1\).

This implies that (3.4) has a solution if and only if the right hand side of (3.15) is in \(\Psi(F_{p^e})\) and the right hand side of (3.16) is in \(F_{p^{e+1}}\). In this case, by Lemma 3.3, the number of solutions of (3.4) is \(p^{e\mu_1}/2\).

By Lemma 3.2, it is sufficient to show that the image under the trace map of the right hand side of (3.15) and the image under \(\Psi\) of the right hand side of (3.16) are 0. Using computations similar to those in (3.12) and (3.13), the desired properties can be concluded.

Case 3. \(p = 2\) and \(i_1 = 2\mu_1 + 1\) is odd.

From [10], we have
\[
M(p^a, i_1) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
* & 0 & 0 & \cdots & 0 \\
* & * & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & 0 \\
\end{pmatrix},
\] (3.17)
where *’s denote entries of \(M(p^a, i_1)\) defined in (3.3).

From (3.4) and (3.17), we conclude that
\[
\text{Tr}(x_1) = b_1, 
\] (3.18)
\[
\text{Tr}(x_{2i}) = b_{2i} + \sum_{j=1}^{2i-1} m_{2i,j} x_j, 
\] (3.19)
and
\[
\text{Tr}(x_{2i+1}) = b_{2i+1} + \sum_{j=1}^{2i-1} m_{2i+1,j} x_j 
\] (3.20)
for all integers \(1 \leq i \leq \mu_1\).
Similar to Cases 1 and 2, we need to show that the right hand sides of (3.18), (3.19), and (3.20) are in \(F_{2^2}\), or equivalently, the images under the trace map of the right hand sides of (3.18), (3.19), and (3.20) are 0. Clearly, the right hand side of (3.18) is \(b_1 \in \{0, 1\} \subseteq F_{2^2}\) and \(\text{Tr}(b_1) = 0\). Let \(1 \leq i \leq \mu_1\) be an integer.

From (3.19), we have
\[
\text{Tr} \left( b_{2i} + \sum_{j=1}^{2i-1} m_{2i,j} x_j \right) = 2i-1 \sum_{j=1}^{2i-1} m_{2i,j} \text{Tr}(x_j)
\]
\[
= \sum_{j=1}^{2i-1} m_{2i,j} \left( b_j + \sum_{k=1}^{j-1} m_{j,k} x_k \right)
\]
\[
= \sum_{j=1}^{2i-1} m_{2i,j} b_j + \sum_{j=1}^{2i-1} \sum_{k=1}^{j-1} m_{2i,j} m_{j,k} x_k
\]
\[
= 0 + \sum_{k=1}^{2i-2} \left( \sum_{j=k+1}^{2i-1} m_{2i,j} m_{j,k} \right) x_k, \text{ by Lemma 3.1}
\]
\[
= \sum_{k=1}^{2i-2} \left( \sum_{j=k+1}^{2i-1} m_{2i,j} m_{j,k} \right) x_k, \text{ by (3.16)}
\]
\[
= \sum_{k=1}^{2i-2} \left( m_{2i,k} \sum_{j=1}^{2i-k-1} \left( \begin{array}{c} 2i-k \\ j \end{array} \right) \right) x_k
\]
\[
= \sum_{k=1}^{2i-2} \left( m_{2i,k} (2^{2i-k} - 2) \right) x_k
\]
\[
= 0 \text{ in } F_{2^2/2}.
\]

Applying a similar computation to (3.20) yields
\[
\text{Tr} \left( b_{2i+1} + \sum_{j=1}^{2i+1} m_{2i+1,j} x_j \right) = 0 \text{ in } F_{2^2/2}.
\]

**Case 4.** \(p = 2\) and \(i_1 = 2\mu_1 + 2\) is even.

From (10), we have
\[
M(p^a, i_1) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
* & 1 & 0 & 0 & 0 & \cdots & 0 \\
* & * & 0 & 0 & 0 & \cdots & 0 \\
* & * & * & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & * & * & \cdots & 0
\end{pmatrix},
\]

where *’s denote entries of \(M(p^a, i_1)\) defined in (3.3).

From (3.4) and (3.22), it follows that
\[
\text{Tr}(x_1) = b_1,
\]
\[
\text{Tr}(x_2) = b_2,
\]

From (3.23) and (3.24), it follows that
\[ \text{Tr}(x_{2i+1}) = b_{2i+1} + \sum_{j=1}^{2i} m_{2i+1,j}x_j, \]  
\[ \text{Tr}(x_{2i+2}) = b_{2i+2} + \sum_{j=1}^{2i} m_{2i+2,j}x_j \]  
for all integers \( 1 \leq i \leq \mu_1. \)

Similar to the previous cases, we need to show that the right hand sides of (3.23), (3.24), (3.25), and (3.26) are in \( \mathbb{F}_{2^{r_2}} \). Equivalently, the images under the trace map of the right hand sides of (3.23), (3.24), (3.25), and (3.26) are 0. Clearly, the traces of the right hand sides of (3.23) and (3.24) are 0 since \( b_1, b_2 \in \{0, 1\} \subseteq \mathbb{F}_{2^{r_2}}. \)

For each integer \( 1 \leq i \leq \mu_1 \), using computations similar to those in (3.21), the trace of the right hand sides of (3.25) and (3.26) become 0 in \( \mathbb{F}_{2^{r_2}}. \)

The proof is now completed. \( \square \)

The next theorem follows immediately from Propositions 2.1 and 3.3.

**Theorem 3.5.** Let \( p \) be a prime and let \( s, a \) be positive integers such that \( s \) is even. Then

\[ N_{H}(\text{GR}(p^2, s), p^a) = \sum_{i=0}^{p^{a-1}} p^{i(2^{a+1})/2 - 1} \]  

4 Euclidean Self-Dual Cyclic Codes of Arbitrary Length over \( \text{GR}(p^2, s) \)

In this section, we focus on cyclic codes of any length \( n \) over \( \text{GR}(p^2, s) \). We generalize the decomposition in [4] (see also [2] and [5]) to this case. Combining with the results in [9], [10], and in the previous section, we characterize and enumerate Euclidean self-dual cyclic codes of any length \( n \) over \( \text{GR}(p^2, s) \). Some corrections to [2] and [4] are also provided.

Write \( n = mp^a \), where \( p \mid m \) and \( a \geq 0 \). Let \( \text{GR}(p^2, s) \) be an \( \mathbb{F}_p \)-linear space. Let \( \sim \) be an involution on \( \text{GR}(p^2, s) \) that fixes \( \text{GR}(p^2, s) \) and that maps \( u \) to \( u^{-1}(mod p^a) \) for all \( 0 \leq i \leq p^a \), and extend \( \sim \) naturally to \( \text{GR}(p^2, s) \) for all positive integers \( v \). It is not difficult to verify that the map \( \Phi : (\text{GR}(p^2, s))^m \to \text{GR}(p^2, s)[X]/(X^{mp^a} - 1) \) defined by

\[ \left( \sum_{j=0}^{p^a-1} c_{0,j}u^j, \sum_{j=0}^{p^a-1} c_{1,j}u^j, \ldots, \sum_{j=0}^{p^a-1} c_{m-1,j}u^j \right) \mapsto \sum_{i=0}^{m-1} \sum_{j=0}^{p^a-1} c_{i,j}X^{i+jm} \]

is a \( \text{GR}(p^2, s) \)-module isomorphism.

The following lemma is an obvious generalization of [4, Lemma 5.1].

**Lemma 4.1.** Let \( d = (d_0, d_1, \ldots, d_{m-1}) \) and \( d' = (d'_0, d'_1, \ldots, d'_{m-1}) \) be elements in \( (\text{GR}(p^2, s))^m \). Then \( [d, d'] := \sum_{i=0}^{m-1} d_i d'_i = 0 \) if and only if \( \langle X^{mi}\Phi(d), \Phi(d') \rangle = 0 \) for all \( 0 \leq i \leq p^a - 1 \).

The lemma implies that, for cyclic codes \( C_1 \) and \( C_2 \) of length \( n \) over \( \text{GR}(p^2, s) \), \( C_2 = C_1^\perp \) if and only if \( \Phi^{-1}(C_2) \) is the dual of \( \Phi^{-1}(C_1) \) under the form \([\cdot, \cdot]\). In particular, \( C_1 \) is Euclidean self-dual if and only if \( \Phi^{-1}(C_1) \) is self-dual under the form \([\cdot, \cdot]\).
4.1 Decomposition

We generalize the Discrete Fourier Transform decomposition for cyclic codes over $\mathbb{Z}_4$ in [4] to cyclic codes over GR($p^2$, $s$) as follows.

Let $M$ be the multiplicative order of $p^s$ modulo $m$ and let $\zeta$ denote a primitive $m$th root of unity in GR($p^2$, $s$M). The Discrete Fourier Transform of

$$c(X) = \sum_{i=0}^{m-1} \sum_{j=0}^{p^s-1} c_{i,j} X^{i+jm} \in \text{GR}(p^2, s)[X]/(X^{mp^s} - 1)$$

is the vector

$$(\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{m-1}) \in R_{p^2}(u, sM)^m$$

with

$$\hat{c}_h = \sum_{i=0}^{m-1} \sum_{j=0}^{p^s-1} c_{i,j} u^{m'i+j} \zeta^{hi}$$

for all $0 \leq h \leq m-1$, where $mm' \equiv 1 \pmod{p^s}$.

Define the Mattson-Solomon polynomial of $c(X)$ to be

$$\hat{c}(Z) = \sum_{h=0}^{m-1} \hat{c}_{m-h \pmod{m}} Z^h.$$  

Then the following lemma can be obtained as an extension of [4] Lemma 3.1.

**Lemma 4.2 (Inversion formula).** Let $c(X) \in \text{GR}(p^2, s)[X]/(X^{mp^s} - 1)$ with $\hat{c}(Z)$ its Mattson-Solomon polynomial as defined above. Then

$$c(X) = \Phi \left( (1, u^{-m'}, u^{-2m'}, \ldots, u^{-(m-1)m'}) \ast \frac{1}{m}(\hat{c}(1), \hat{c}(\zeta), \ldots, \hat{c}(\zeta^{m-1})) \right),$$

where $\ast$ indicates the componentwise multiplication.

For $0 \leq h \leq m-1$, denote by $S_{p^s}(h)$ the $p^s$-cyclotomic coset of $h$ modulo $m$, i.e.,

$$S_{p^s}(h) = \{hp^s \pmod{m} \mid i \in \mathbb{Z} \},$$

and denote by $m_h$ the size of $S_{p^s}(h)$. Then $m_h$ is the multiplicative order of $p^s$ modulo ord($h$), where ord($h$) denotes the additive order of $h$ modulo $m$. Since ord($h$) = ord($-h$), we have $m_h = m-h$ for all $0 \leq h \leq m-1$.

The $p^s$-cyclotomic coset $S_{p^s}(h)$ is said to be self-inverse if $S_{p^s}(-h) = S_{p^s}(h)$, or equivalently, $-h \in S_{p^s}(h)$. In this case, the size $m_h$ of $S_{p^s}(h)$ is 1 or even (see [4] Remark 2.6]). Moreover, we have $-h = h$ if $m_h = 1$, and $-h = p^{m_h}/2h$ otherwise. We note that $S_{p^s}(0)$ and $S_{p^s}(\frac{m}{2})$ (if $m$ is even) are self-inverse.

**Remark 4.3.** We have the following observations for the coefficients of the Discrete Fourier Transform.

i) If $S_{p^s}(h)$ is self-inverse of size 1, then

$$\hat{c}_{m-h} = \hat{c}_h.$$  

ii) If $S_{p^s}(h)$ is self-inverse of size 2, then $-h = p^{m_h}h$ and

$$\hat{c}_{m-h} = \sum_{i=0}^{m-1} \sum_{j=0}^{p^s-1} c_{i,j} u^{m'i+j} \zeta^{-hi}$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{p^s-1} c_{i,j} u^{m'i+j} \zeta^{p^s-i}$$

$$= \hat{c}_h.$$  

(4.2)
By (4.1) and (4.2), we conclude the lemma.

Using computations similar to those in [4, Equation (21)], we have

Proof. Let $C$ be a cyclic code of length $n = mp^d$ over $\text{GR}(p^2, s)$. The characterization and enumeration of Euclidean self-dual cyclic codes of length $n$ are established, where $p$ is a prime such that $p \mid m$ and $a \geq 0$.

**Lemma 4.4.** The ring $\text{GR}(p^2, s)[X]/\langle X^{mp^a} - 1 \rangle$ is isomorphic to

$$\prod_{h \in J_0 \cup J_1 \cup J_2} R_{p^2}(u, sm_h)$$

via the ring isomorphism

$$c(X) \mapsto (\hat{c}_h)_{h \in J_0 \cup J_1 \cup J_2}.$$ If $C$ is a cyclic code of length $n = mp^a$ over $\text{GR}(p^2, s)$, then $C$ is isomorphic to

$$\prod_{h \in J_0 \cup J_1 \cup J_2} C_h,$$

where $C_h$ is an ideal in $R_{p^2}(u, sm_h)$ for all $h \in J_0 \cup J_1 \cup J_2$.

### 4.2 Euclidean Self-Dual Cyclic Codes

In this subsection, we consider the Euclidean dual of cyclic codes of length $n = mp^a$ over $\text{GR}(p^2, s)$. The characterization and enumeration of Euclidean self-dual cyclic codes of length $n$ are established, where $p$ is a prime such that $p \mid m$ and $a \geq 0$.

**Lemma 4.5.** Let $c(X)$ and $c'(X)$ be polynomials in $\text{GR}(p^2, s)[X]/\langle X^{mp^a} - 1 \rangle$ with Mattson-Solomon polynomials $\hat{c}(Z) = \sum_{h=0}^{m-1} \hat{c}_h Z^h$ and $\hat{c}'(Z) = \sum_{h=0}^{m-1} \hat{c}'_h Z^h$, respectively. Let $(d_0, d_1, \ldots, d_{m-1}) = \Phi^{-1}(c(X))$ and $(d'_0, d'_1, \ldots, d'_{m-1}) = \Phi^{-1}(c'(X))$. Then

$$\sum_{i=0}^{m-1} d_id'_i = \frac{1}{m} \left( \sum_{j \in I_0} \hat{c}_j \hat{c}'_j + \sum_{h \in I_1} \hat{c}_h \hat{c}'_h + \sum_{k \in I_2} \hat{c}_k \hat{c}'_{m-k \pmod{m}} \right).$$

where $\hat{\ }$ is defined as in [4.1] in the appropriate Galois extension.

**Proof.** Using computations similar to those in [4.1] Equation (21), we have

$$\sum_{i=0}^{m-1} d_id'_i = \frac{1}{m} \left( \sum_{i=0}^{m-1} \hat{c}_i \hat{c}'_{m-i \pmod{m}} \right)$$

$$= \frac{1}{m} \left( \sum_{j \in I_0} \hat{c}_j \hat{c}'_{m-j \pmod{m}} + \sum_{h \in I_1} \hat{c}_h \hat{c}'_{m-h \pmod{m}} + \sum_{k \in I_2} \hat{c}_k \hat{c}'_{m-k \pmod{m}} \right).$$

By [4.1] and [4.2], we conclude the lemma.

Based on the isomorphism defined in Lemma 4.4, we determine the Euclidean duals of cyclic codes over $\text{GR}(p^2, s)$ as follows.
Proposition 4.6. Let $C$ be a cyclic code of length $n = mp^a$ over $\text{GR}(p^2, s)$ decomposed as in (4.5), i.e.,

$$C \cong \prod_{j \in J_0} C_j \times \prod_{h \in J_1} C_h \times \prod_{k \in J_2} C_k.$$  

(4.5)

Then

$$C^\perp = \prod_{j \in J_0} C_j^\perp \times \prod_{h \in J_1} C_h^\perp \times \prod_{k \in J_2} C_{m-k \pmod m}^\perp.$$  

(4.6)

Moreover, $C$ is Euclidean self-dual if and only if $C_j$ is Euclidean self-dual for all $j \in J_0$, $C_h$ is Hermitian self-dual for all $h \in J_1$, and $C_k = C_{m-k \pmod m}^\perp$ for all $k \in J_2$.

Proof. Let $D$ be a cyclic code of length $n = mp^a$ over $\text{GR}(p^2, s)$ such that

$$D \cong \prod_{j \in J_0} C_j^\perp \times \prod_{h \in J_1} C_h^\perp \times \prod_{k \in J_2} C_{m-k \pmod m}^\perp.$$  

Consider $m = 1$ in Lemma 4.3, we have the following facts. For each positive integer $\nu$ and $a, b \in R_{i\mu}(u, s
\nu)$, we have $(a, b)_{\mathbb{R}} = 0$ if $\overline{a}b = 0$, and $(a, b)_H = 0$ if $\overline{a}b = 0$. Therefore, by Lemmas 4.3 and 4.4, we have $D \subseteq C^\perp$. The equality follows from their cardinalities.

The second part is clear. \(\square\)

Since $J_2$ has been chosen such that $h \in J_2$ if and only if $-h \in J_2$, we can write $J_2$ as a disjoint union $J_2 = J_2' \cup J_2''$, where $J_2' \subseteq J_2$ and $J_2'' = \{-h \mid h \in J_2'\}$.

Corollary 4.7. The number of Euclidean self-dual codes of length $n = mp^a$ over $\text{GR}(p^2, s)$ is

$$(N_E(\text{GR}(p^2, s), p^a))^{\delta(m)} \times \prod_{h \in J_1} N_H(\text{GR}(p^2, sm_h), p^a) \times \prod_{k \in J_2'} N(\text{GR}(p^2, sm_k), p^a),$$

where

\[
\delta(m) = \begin{cases} 
1 & \text{if } m \text{ is odd}, \\
2 & \text{if } m \text{ is even}, 
\end{cases}
\]

and the empty product is regarded as 1.

Proof. From Proposition 4.6, a code $C$ decomposed as in (4.5) is Euclidean self-dual if and only if $C_j$ is Euclidean self-dual for all $j \in J_0$, $C_h$ is Hermitian self-dual for all $h \in J_1$, and $C_k = C_{m-k \pmod m}^\perp$ for all $k \in J_2$.

The number of Euclidean (resp., Hermitian) self-dual cyclic codes of length $p^a$ corresponding to $J_0$ (resp., $J_1$) is $\prod_{j \in J_0} N_E(\text{GR}(p^2, s), p^a)$ (resp., $\prod_{h \in J_1} N_H(\text{GR}(p^2, sm_h), p^a)$). The number of choices of cyclic codes of length $p^a$ corresponding to $J_2'$ is $\prod_{k \in J_2'} N(\text{GR}(p^2, sm_k), p^a)$. Since $C_k = C_{m-k \pmod m}^\perp$ for all $k \in J_2$, there is only one possibility for codes corresponding to $J_2''$.

Therefore, the number of Euclidean self-dual codes of length $n = mp^a$ over $\text{GR}(p^2, s)$ is

$$\prod_{j \in J_0} N_E(\text{GR}(p^2, s), p^a) \times \prod_{h \in J_1} N_H(\text{GR}(p^2, sm_h), p^a) \times \prod_{k \in J_2'} N(\text{GR}(p^2, sm_k), p^a).$$

Since $|J_0| = 1$ if $m$ is odd and $|J_0| = 2$ if $m$ is even, the result follows. \(\square\)

We note that, for each positive integer $t$, $\{0\}$, $p\text{GR}(p^2, st)$, and $\text{GR}(p^2, st)$ are all cyclic codes of length 1 over $\text{GR}(p^2, s)$ and $p\text{GR}(p^2, st)$ is the only Euclidean self-dual cyclic code. Hence, $N_E(\text{GR}(p^2, st), 1) = 1$, and $N(\text{GR}(p^2, st), 1) = 3$ for all positive integers $t$. If $st$ is even, then $p\text{GR}(p^2, st)$ is the only Hermitian self-dual cyclic code, and hence, $N_H(\text{GR}(p^2, st), 1) = 1$.

For $a \geq 1$, the numbers $N_E$, $N_H$, and $N$ have been determined in Corollary 2.3 and Theorem 3.5 respectively.

Therefore, the number in Corollary 4.7 is completely determined.
Corollary 4.8. Let \( m \) be a positive integer be such that \( p \nmid m \). Then \( N_E(\text{GR}(p^2, s), mp) = 1 \) if and only if \( m = 1 \) and \( p = 2 \).

Proof. Assume that \( m > 1 \) or \( p \) is odd.

Case 1. \( m > 1 \).

If \( J_2 \) is not empty, then, by Corollaries 2.3 and 4.7, \( N_E(\text{GR}(p^2, s), mp) \geq N_E(\text{GR}(p^2, sm_h), p) > 1 \) for all \( h \in J_2 \). Assume that \( J_2 \) is empty. Since \( m > 1 \), \( J_1 \) is not empty. Then, by Theorem 3.5 and Corollary 4.7, \( N_E(\text{GR}(p^2, s), mp) \geq N_E(\text{GR}(p^2, sm_h), p) > 1 \) for all \( h \in J_1 \).

Case 2. \( p \) is odd.

By Proposition 2.4 and Corollary 4.7, \( N_E(\text{GR}(p^2, s), mp) \geq N_E(\text{GR}(p^2, s), p) = 2 > 1 \).

Conversely, we assume that \( m = 1 \) and \( p = 2 \). Then, by Proposition 2.4, \( N_E(\text{GR}(p^2, s), mp) = N_E(\text{GR}(4, s), 2) = 1 \).

Some examples of the numbers of Euclidean self-dual cyclic codes of small lengths over \( \mathbb{Z}_4 \), over \( \mathbb{Z}_9 \), and over \( \text{GR}(4, 2) \), are given in Tables 1, 2, and 3, respectively.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|
| \( N_E(\mathbb{Z}_4, n) \) | 1 | 1 | 1 | 3 | 1 | 3 | 1 | 11 | 1 | 5 |

| \( n \) | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| \( N_E(\mathbb{Z}_4, n) \) | 1 | 21 | 1 | 13 | 1 | 15 | 1 | 16 | 1 | 19 |

| \( n \) | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| \( N_E(\mathbb{Z}_4, n) \) | 1 | 33 | 1 | 34 | 1 | 35 | 1 | 36 | 1 | 37 |

| \( n \) | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| \( N_E(\mathbb{Z}_4, n) \) | 1 | 1019 | 1 | 289 | 1 | 1533 | 1 | 513 | 1 | 3751 |

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| \( N_E(\mathbb{Z}_9, n) \) | 1 | 1 | 2 | 1 | 1 | 4 | 1 | 1 | 8 | 1 |

| \( n \) | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| \( N_E(\mathbb{Z}_9, n) \) | 1 | 16 | 1 | 1 | 20 | 1 | 1 | 64 | 1 | 1 |

| \( n \) | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| \( N_E(\mathbb{Z}_9, n) \) | 56 | 1 | 1 | 544 | 1 | 1 | 242 | 1 | 1 | 400 |

| \( n \) | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| \( N_E(\mathbb{Z}_9, n) \) | 1 | 1 | 1472 | 1 | 1 | 2560 | 1 | 1 | 15488 | 1 |

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| \( N_E(\text{GR}(4, 2), n) \) | 1 | 1 | 1 | 5 | 1 | 9 | 1 | 37 | 1 | 25 |

| \( n \) | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| \( N_E(\text{GR}(4, 2), n) \) | 1 | 225 | 1 | 69 | 1 | 677 | 1 | 621 | 1 | 2205 |

| \( n \) | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| \( N_E(\text{GR}(4, 2), n) \) | 1 | 1029 | 1 | 29193 | 1 | 4225 | 1 | 22125 | 1 | 99225 |

| \( n \) | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| \( N_E(\text{GR}(4, 2), n) \) | 1 | 174757 | 1 | 83521 | 1 | 995625 | 1 | 262149 | 1 | 4302397 |
In general, the numbers of Euclidean self-dual cyclic codes of any length over $\text{GR}(p^2, s)$, where $p$ is a prime and $s$ is a positive integer, can be computed using the formula in Corollary 2.3, Proposition 2.4, Theorem 3.5, and Corollary 4.7.

4.3 A Note on Euclidean Self-dual Cyclic Codes of Even Length over $\mathbb{Z}_4$

In this subsection, we reconsider cyclic codes of even length over $\mathbb{Z}_4$ which have been studied in [2] and [4]. This case can be viewed as a special case of the previous two subsections where $p = 2$ and $s = 1$. We discovered that [2] Lemma 7, Lemma 9, and Corollary 2] and [4] Lemma 5.2, Corollary 5.4, Proposition 5.8, Corollary 5.9, and Section 6] contain errors. Corrections to these errors are discussed as follows.

Let $n$ be an even positive integer written as $n = m2^a$, where $m$ is odd and $a \geq 1$. Then, in the decomposition (4.3), we have $J_0 = \{0\}$ and every cyclic code $C$ of length $n$ over $\mathbb{Z}_4$ can be viewed as

$$C \cong C_0 \times \prod_{h \in J_1} C_h \times \prod_{k \in J_2} C_k,$$

(4.7)

where $C_0$, $C_h$, and $C_k$ are cyclic codes of length $2^a$ over appropriate Galois extensions of $\mathbb{Z}_4$.

We note that all the errors in [2] and [4] are caused by misinterpretation of the orthogonality in [2] Lemma 6] and [4] Equation (21)] (see also (4.4)). This effects the components in the decomposition of the Euclidean dual of $C$ in (4.7) that relates to the 2-cyclotomic cosets of elements in $J_1$. The errors are pointed out in terms of our notations. The readers may refer to the original statements via the number cited. The corrections to these are discussed as well.

• The Euclidean dual of $C$ in (4.7).

In [2] Lemma 7 and Lemma 9] and [4] Lemma 5.2 and Corollary 5.4], an incorrect interpretation has been made as follows.

"$C_{\perp E} \cong C_{\perp E}^0 \times \prod_{h \in J_1} C_{\perp E}^h \times \prod_{k \in J_2} C_{\perp E}^{m-k (mod m)}$".

The Euclidean dual of $C$ in (4.7) can be viewed as a special case of Proposition 4.6 where $p = 2$ and $s = 1$, i.e.,

$$C_{\perp E} \cong C_{\perp E}^0 \times \prod_{h \in J_1} C_{\perp E}^h \times \prod_{k \in J_2} C_{\perp E}^{m-k (mod m)}.$$

• The number of Euclidean self-dual cyclic codes over $\mathbb{Z}_4$.

Since the determination of the Euclidean dual of a cyclic code in [2] and [4] is not correct, in [2] Lemma 9] and [4] Proposition 5.8], an incorrect statement about the number of Euclidean self-dual cyclic codes of length $n = m2^a$ over $\mathbb{Z}_4$ has been proposed as follows.

"The number of Euclidean self-dual cyclic codes of length $n = m2^a$ over $\mathbb{Z}_4$ is

$$\prod_{h \in \{0\} \cup J_1} N_E(\text{GR}(4, m_h), 2^a) \times \prod_{k \in J_2} N(\text{GR}(4, m_k), 2^a).$"

The correct statement can be viewed as a special case of Corollary 4.7 where $p = 2$ and $s = 1$. 

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Corollary 4.9. The number of Euclidean self-dual codes of length $n = m2^a$ over $\mathbb{Z}_4$ is

$$N_E(\mathbb{Z}_4, 2^a) \times \prod_{h \in J_i} N_H(\text{GR}(4, m_h), 2^a) \times \prod_{k \in J_2} N(\text{GR}(4, m_k), 2^a).$$

where the empty product is regarded as 1.

- A unique Euclidean self-dual code of length $n = 2m$ over $\mathbb{Z}_4$.

In [2, Corollary 2] and [4, Corollary 5.9], it has been claimed that

"If there exists an integer $e$ such that $-1 \equiv 2^e \mod n$, then there is only one Euclidean self-dual cyclic code of length $2n$ where $n$ is odd, namely $2(\mathbb{Z}_4)^2n$.

This claim is not correct and the correct statement can be viewed as a special case of Corollary 4.8 with $m = 2$ and $s = 1$. It can be restated as follows.

Corollary 4.10. Let $m$ be an odd positive integer. Then $N_E(\mathbb{Z}_4, 2m) = 1$ if and only if $m = 1$.

This error in [4] led to an incorrect claim that there is only one Euclidean self-dual cyclic codes of length $n$ in the cases where $n = 6$ and $n = 10$ (see [4, Section 6]). Actually, in Table 1 the numbers of such codes are 3 and 5, respectively. Here, the codes are also provided.

Example 4.11. The Euclidean self-dual cyclic codes of length 6 and 10 over $\mathbb{Z}_4$ are determined as follows.

- $n = 6$. Then $m = 3$ and the 2-cyclotomic cosets modulo 3 are $\{0\}$ and $\{1, 2\}$. Both the 2-cyclotomic cosets are self-inverse, which implies that $I_0 = \{0\}$ and $I_1 = \{1, 2\}$, and hence, $J_0 = \{0\}$ and $J_1 = \{1\}$. Therefore,

$$\mathbb{Z}_4[X]/\langle X^6 - 1 \rangle \cong R_4(u, 1) \times R_4(u, 2).$$

By Proposition 2.3 and Theorem 3.5, $N_E(\mathbb{Z}_4, 2) = 1$ and $N_H(\text{GR}(4, 2), 2) = 3$. Therefore, by Corollary 4.7, we have $N_E(\mathbb{Z}_4, 6) = 3$, given by the codes $\langle 2 \rangle \times \langle 2 \rangle$, $\langle 2 \rangle \times \langle 1 + u + 2\xi \rangle$, and $\langle 2 \rangle \times \langle 1 + u + 2\xi^2 \rangle$, where $\xi$ is the generator of a Teichmüller set $T_2$.

- $n = 10$. Then $m = 5$ and the 2-cyclotomic cosets modulo 5 are $\{0\}$ and $\{1, 2, 3, 4\}$. Both the 2-cyclotomic cosets are self-inverse, which implies that $I_0 = \{0\}$ and $I_1 = \{1, 2, 3, 4\}$, and hence, $J_0 = \{0\}$ and $J_1 = \{1\}$. Therefore,

$$\mathbb{Z}_4[X]/\langle X^{10} - 1 \rangle \cong R_4(u, 1) \times R_4(u, 4).$$

By Proposition 2.3 and Theorem 3.5, $N_E(\mathbb{Z}_4, 2) = 1$ and $N_H(\text{GR}(4, 4), 2) = 5$. Therefore, by Corollary 4.7, we have $N_E(\mathbb{Z}_4, 10) = 5$.

5 Conclusion

The complete characterization and enumeration of Hermitian self-dual cyclic codes of length $p^a$ over $\text{GR}(p^2, s)$ has been established. Using the Discrete Fourier Transform decomposition, we have characterized the structure of Euclidean self-dual cyclic codes of any length over $\text{GR}(p^2, s)$.

The enumeration of such codes has been given through this decomposition, our results on Hermitian self-dual codes, and some known results on cyclic codes of length $p^a$ over $\text{GR}(p^2, s)$. Based on the established characterization and enumeration, we have corrected mistakes in some earlier results on Euclidean self-dual cyclic codes of even length over $\mathbb{Z}_4$ in [2] and [4].
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