CASIMIR ELEMENTS AND KERNEL OF WEITZENBÖK DERIVATION.

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Abstract. Let $k[X]:=k[x_0, x_1, \ldots, x_n]$ be a polynomial algebra over a field $k$ of characteristic zero. We offer an algorithm for calculation of kernel of Weitzenbök derivation $d(x_i) = x_{i-1}, \ldots, d(x_0) = 0, i = 1 \ldots n$ that is based on an analogue of the well known Casimir elements of finite dimensional Lie algebras. By using this algorithm, the kernel is calculated in the case $n < 7$.

1. Introduction

Let $k[X]:=k[x_0, x_1, \ldots, x_n]$ be a polynomial algebra over a field $k$ of characteristic zero. For arbitrary derivation $D$ of $k[X]$ denote by $k[X]^D$ a kernel of $D$, that is,

$$k[X]^D := \text{Ker} D := \{ f \in k[X]; D(f) = 0 \}.$$

Let $d$ be Weitzenbök derivation defined by rule $d(x_i) = x_{i-1}, \ldots, d(x_0) = 0, i = 1 \ldots n$. The kernel of the derivation $d$ was actively studied by various authors. The well-known Weitzenbök's theorem implies a finite generation of the derivation $d$. A minimal generating set of algebra $k[X]^d$ for $n \leq 4$ presented in [1] and in [5] for $n \leq 5$. The aim of this paper is to calculate a generating set of the kernel of Weitzenbök derivation in case $n \leq 6$. We offer the general description of a kernel of arbitrary polynomialy derivation $D$ by using a construction which is a commutative analogue to a construction of Casimir elements of finite dimension Lie algebras. Let us recall that a (generalised) Casimir element of a finite dimensional Lie algebra $L$ is called a central element of the universal enveloping algebra $U(L)$ of the follow form

$$\sum_i u_i u_i^*,$$

where $\{u_i\}, \{u_i^*\}$ are a dual bases of a contragradient $L$-modules in $U(L)$ with respect to the adjoint action of the Lie algebra on the algebra $U(L)$. In the case char$(k)=0$ it is well known that every element of the center of an universal enveloping algebra $U(L)$ is a Casimir element. The case char$(k)=p>0$ was studied by present autor in [4].

This paper is organized as follows. In section 2 a conceptions of $D$-module, dual $D$-modules and Casimir elements are introduced. We prove that arbitrary Casimir elements of derivation $D$ belongs to the kernel $k[X]^D$. For any linear derivation $D$ is showed that a theorem inverse to above theorem is true, i.e., any element of kernel $k[X]^D$ is a Casimir element.

In section 3 we study Casimir elements(polynomial) of the Weitzenbök derivation $d$. Since $d$ is linear derivation we see that the problem of finding of the kernel $k[X]^d$ is equivalent to the problem of finding a realisation of a dual $d$-modules in $k[X]$. We get such realisations from any element of the kernel $k[X]^d$ by using two new derivations $e$ and $\hat{d}$ which arising with the natural embedding of $d$-modules into $sl_2$-modules.

In section 4 we introduce a maps $\tau_i : k[X]^d \to k[X]^d$. To any element $z$ of kernel derivation $d$ we assign certain family $\tau_i(z)$ elements of the kernel and studying properties of this correspondence.

In section 5 we present a criterion to verify if a subalgebra of $k[X]^d$ coincides with the whole algebra $k[X]^d$. By using this criterion we offer an algorithm for computing the kernel $k[X]^d$.

By using the algorithm in section 6 we compute and present a list of generating elements of the kernel of of Weitzenbök derivation in the case $n < 7$. The result for $n = 6$ is new.
The aim of this section is to offer a method of constructing elements of $k[X]^D$ where $D$ is an arbitrary derivation of $k[X]$.

**Definition 2.1.** A vector space $V$ is called a $D$-module if $D(V) \subseteq V$.

The derivation $d$ defined by $d(x_n) = x_{n-1}$, $d(x_{n-1}) = x_{n-2}$, $\ldots$, $d(x_0) = 0$ is called the Weitzenböck derivation. Let $X_m$ be a vector space spanned by the elements $x_0, x_1, \ldots, x_m$. Then, obviously, $X_m$ is $d$-module. We denote by $D_V$ a matrix of derivation $D$ in some ordered fixed basis of the space $V$. For example the matrix of derivation $d$ in $X_m$ is Jordan cell $J_{m+1}(0)$.

**Definition 2.2.** $D$-module $V$ is called dual to $D$-module $V^*$ if there are dual bases $\{v_i\}$, $\{v^*_i\}$ of $V$ and $V^*$ such that

$$D_{V^*} = (-D_V)^T.$$ 

The bases $\{v_i\}$, $\{v^*_i\}$ are also called dual bases. From definition 2.2 it follows that the derivation $d$ acts on $X_m^*$ such that $d(x^*_i) = -x_{i+1}, d(x_m) = 0$.

Let $D, \hat{D}$ be derivations of $k[X]$.

**Definition 2.3.** $D$-module $V$ and $\hat{D}$-module $W$ are called isomorphic if there is a linear space isomorphism $\phi : V \rightarrow W$ such that $D \phi = \phi \hat{D}$. In this case we write $V \cong W$.

**Theorem 2.1.** $X_m^* \cong X_m$ as $d$-modules.

**Proof.** Let $\varphi : X_m^* \rightarrow X_m$ be the isomorphism defined by rule $\varphi(x^*_i) = (-1)^{i+1}x_{m-i+1}$. Then

$$d(\varphi(x^*_i)) = d((-1)^{i+1}x_{m-i+1}) = (-1)^{i+1}x_{m-i},$$

and

$$\varphi(d(x^*_i)) = \varphi(-x_{i+1}^*) = (-1)^{i+2}x_{m-i} = (-1)^{i+1}x_{m-i} = d(\varphi(x^*_i)).$$

Hence $\varphi$ is the isomorphism from $d$-module $X_m^*$ to $d$-module $X_m$ and bases $\{x_i\}$, $\{(−1)^{i+1}x_i\}$ are dual ones.

**Definition 2.4.** Suppose $V = \{v_i\}$, $V^* = \{v^*_i\}$ are dual $D$-modules in $k[X]$. The polynomial

$$\Delta(V, V^*) := \sum_i (v_i \cdot v^*_i).$$

is called the Casimir element of a derivation $D$.

By using theorem 2.1 we obtain following Casimir elements of degree 2 for the Weitzenböck derivation $d$:

$$\Delta(X_k, X_k^*) = \sum_{i=1}^{k} (-1)^{i+1}x_i \cdot x_{k-i+1}.$$ 

It is easy to show that $\Delta(X_k, X_k^*) \in k[X]^D$. In generally the following theorem holds:

**Theorem 2.2.** Suppose $U$ and $U^*$ are two dual $D$-modules in $k[X]$. Then $\Delta(U, U^*) \in k[X]^D$. 

We end this section with the following

**Proof.** Assume that the bases \( \{ u_i \}, \{ u_i^* \} \) are dual ones and \( D_U = \{ \lambda_{ij} \}, \ i, j = 1 \ldots n \). Then
\[
D(u_i) = \sum_{j=1}^{n} \lambda_{ij} u_j, \ D(u_i^*) = \sum_{j=1}^{n} (-\lambda_{ji} u_j).
\]
Therefore
\[
D(\Delta(U, U^*)) = D\left( \sum_{i=1}^{n} u_i u_i^* \right) = \sum_{i=1}^{n} (D(u_i) u_i^* + u_i D(u_i^*)) =
\]
\[
= \sum_{i=1}^{n}\left( \sum_{j=1}^{n} (\lambda_{ij} u_j u_i^* + u_i D(u_j^*)) \right) = \sum_{i=1}^{n}\left( \sum_{j=1}^{n} \lambda_{ji} u_j u_i^* + u_i D(u_j^*) \right) =
\]
\[
= \sum_{i=1}^{n} u_i \left( \sum_{j=1}^{n} \lambda_{ji} u_i^* + D(u_j^*) \right) = 0.
\]
\[\square\]

For any Casimir element one can show that this element does not depend on the choice of dual bases; so that a Casimir element is well defined.

For linear derivation \( D \) of \( k[X] \) a theorem inverse to theorem 2.2 is true. Let us assume that
\[
D(x_i) = \sum_{j=0}^{n} \lambda_{ij} x_j.
\]

**Theorem 2.3.** Let \( z \) be homogeneous polynomial belonging to \( k[X]^D \). Then \( z \) is a Casimir element.

**Proof.** Let us remember that a derivation of the algebra \( k[X] \) of a form \( f_0 \partial_0 + f_1 \partial_1 + \cdots + f_n \partial_n \), \( f_i \in k[X] \) is called a special derivation. It is well-known that the set of all special derivation \( W_n := \text{Der}(k[X]) \) is a Lie algebra with respect to the commutator of a derivations. Taking into account \( D = D(x_0) \partial_0 + D(x_1) \partial_1 + \cdots + D(x_n) \partial_n \) we get
\[
[D, \partial_i] = -\sum_{i=0}^{n} \lambda_{ji} \partial_j.
\]
It is clear that vector space \( X_n \) is \( D \)-module. Without loss of generality, we can assume that \( \partial_n(z) \neq 0 \).

**Lema 2.1.** Let \( z \) be non-vanishing element from \( k[X]^D \). Then the vector space \( Z_D := \langle \partial_0(z), \cdots, \partial_n(z) \rangle \) is \( D \)-module and \( Z_D \cong X_n^* \).

**Proof.** We need to verify duality of the bases elements now. In fact
\[
D(\partial_i(z)) = [D, \partial_i](z) + \partial_i(D(z)) = [D, \partial_i](z) = -\sum_{i=0}^{n} \lambda_{ji} \partial_j(z).
\]
\[\square\]

Now since the modules \( X_n \) and \( Z_D \) are dual we can write their Casimir element. By using Euler’s theorem about homogeneous polynomials we get
\[
\Delta(X_n, Z_D) = x_0 \partial_0(z) + x_1 \partial_1(z) + \cdots + x_n \partial_n(z) = \deg(z) z.
\]
Thus \( z = \frac{1}{\deg(z)} \Delta(X_n, Z_D) \), i.e., \( z \) is a Casimir element. \[\square\]

We end this section with the following

**Conjecture.** For any derivation \( D \) the algebra \( k[X]^D \) is generated by Casimir elements.
3. Casimir elements of Weitzenbök derivation.

From theorem 2.3 it follow that to know the ring $k[X]^D$ we have to know a realisations of $D$ - modules in $k[X]$. Below we offer a way to construct these realisations for Weitzenbök derivation $d$.

**Theorem 3.1.** Any $d$ - module $V = \langle v_0, v_1, \ldots, v_n \rangle$ can be extended to $sl_2$ - module where $sl_2$ is simple three-dimensional Lie algebra over field $k$.

**Proof.** Let us introduce on $V$ two additional derivations $\hat{d}$ and $e$ as follows:

$$\hat{d}(v_i) := (i + 1)(n - i)v_{i+1},$$

$$e(v_i) := (n - 2i)v_i.$$

By straightforward calculation for any $i$ we get:

$$[d, \hat{d}](v_i) = e(v_i),$$

$$[d, e](v_i) = -2d(v_i),$$

$$[\hat{d}, e](v_i) = 2\hat{d}(v_i).$$

Those commutator relations coincide with the commutator relations of bases elements of the simple three-dimensional Lie algebra $sl_2$. Hence the vector space $V_n$ together with operators $d, \hat{d}, e$ is $sl_2$ - module. $\square$

**Definition 3.1.** For any polynomial $z \in k[X]$ a natural number $s$ is called an order of the polynomial $z$ if the number $s$ is the smallest natural number such that

$$\hat{d}^s(z) \neq 0, \hat{d}^{s+1}(z) = 0.$$

We denote an order of $z$ by ord($z$). For example ord ($x_0$) = $n$. By using Leibniz’s formula we get

$$\text{ord}(a \cdot b) = \text{ord}(a) + \text{ord}(b) \text{ for all } a, b \in k[X].$$

For the derivation $e$ every monomial $x_0^{\alpha_1} x_1^{\alpha_2} \cdots x_n^{\alpha_n}$ is an eigenvector with the eigenvalue

$$w(x_0^{\alpha_1} x_1^{\alpha_2} \cdots x_n^{\alpha_n}) = n \left( \sum_i \alpha_i \right) - 2(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n).$$

A homogeneous polynomial is called isobaric if all its monomials have equal eigenvalue.

**Definition 3.2.** An eigenvalue of arbitrary monomial of a homogeneous isobaric polynomial $z$ is called the weight of the polynomial $z$ and denoted by $\omega(z)$.

It is easy to see that $\omega(a \cdot b) = \omega(a) + \omega(b)$ for a homogeneous isobaric polynomials $a, b$.

**Theorem 3.2.** For arbitrary homogeneous isobaric polynomial $z \in k[X]^d$ a vector space

$$V_m(z) := \langle v_0, v_1, \ldots, v_m \rangle, v_i = \frac{(w(z) - i)!}{i!(\omega(z))!} \hat{d}^i(z), v_0 := z, m = 0 \ldots s,$$

is $d$ - module, moreover $X_m \cong V_m(z)$. Here $\omega(z), s$ are weight and order of $z$.

**Proof.** Let us prove two intermediate relations :

(i) $e(\hat{d}^i(z)) = (\omega(z) - 2i) \hat{d}^i(z),$

(ii) $d(\hat{d}^i(z)) = i(\omega(z) - i + 1) \hat{d}^{i-1}(z).$
The relation (i) is valid for $i = 0$:
\[ e(\hat{d}^0(z)) = e(z) = \omega(z) z. \]
If this relation holds for certain $i$, then
\[
\begin{align*}
    e(\hat{d}^{i+1}(z)) &= [e, \hat{d}](\hat{d}^i(z)) + \hat{d}(e(\hat{d}^i(z))) = -2\hat{d}^{i+1}(z) + \hat{d}((\omega(z) - 2i)\hat{d}^{i+1}(z)) = \\
    &= (\omega(z) - 2(i + 1))\hat{d}^{i+1}(z).
\end{align*}
\]
The relation (ii) is true for $i = 0$. In fact
\[
    d(\hat{d}^0(z)) = d(z) = 0
\]
If this relation holds for certain $i$ then by using (i) we get
\[
    d(\hat{d}^{i+1}(z)) = [d, \hat{d}](\hat{d}^i(z)) + \hat{d}(d(\hat{d}^i(z))) = e(\hat{d}^i(z)) + \hat{d}(i(\omega(z) - i + 1)\hat{d}^{i-1}(z)) = \\
    = (\omega(z) - 2i)\hat{d}^i(z) + i(\omega(z) - i + 1)\hat{d}^i(z) = (i + 1)(\omega(z) - i)\hat{d}^i(z).
\]
Hence the relations are valid for any $i$. Consider now a vector space
\[
    V_m(z) := \langle v_0, v_1, \cdots, v_m \rangle,
\]
where $v_i = \alpha_i(z)\hat{d}^i(z)$ for some undefined $\alpha_i(z) \in k$. For the vector space $V_m(z)$ to be a $d$-module it is enough to have $d(v_i) = v_{i-1}$ for all $i$. Since
\[
    d(v_i) = d(\alpha_i(z)\hat{d}^i(z)) = \alpha_i(z) i(\omega(z) - i)\hat{d}^{i-1}(z),
\]
we get a following recurrence formula for $\alpha_i(z)$:
\[
    i(\omega(z) - i + 1)\alpha_i(z) = \alpha_{i-1}(z), \quad \alpha_1(z) = 1,
\]
by solving it we obtain
\[
    \alpha_i(z) = \frac{(\omega(z) - i)!}{i!\omega(z)!}.
\]

4. Maps $\tau_i$.

For arbitrary homogeneous isobaric polynomial $z \in k[X]^d$ denote by $\tau_i(z)$ the Casimir element:
\[
    \tau_i(z) := \Delta(X_i, V^*_i(z)), \ 0 \leq i \leq \min(\text{ord}(z), n), z \in k[X]^d.
\]
Since every polynomial of $k[X]^d$ is a sum of a homogeneous isobaric polynomials, the map $\tau_i$ can be extended to whole algebra $k[X]^d$ by requiring $\tau_i(u + v) := \tau_i(u) + \tau_i(v)$. Besides we clearly have $\tau_i(\lambda z) = \lambda \tau_i(z)$, $\lambda \in k$. Thus $\tau_i : k[X]^d \rightarrow k[X]^d$ is now a linear map. Note that the maps $\tau_i$ is well defined only on elements $z$ of a kernel such that ord($z$) $\leq i$.

**Theorem 4.1.** Any polynomial of $z \in k[X]^d$ has the form
\[
    z = \frac{1}{\deg(z)}(\tau_n(c(0)) + \tau_{n-1}(c_1) + \ldots + \tau_0(c(n))),
\]
where
\[
    c(0) = \partial_n(z),
\]
\[
    c(i) = \partial_{n-i}(z) + \sum_{k=1}^{i} (-1)^{k+1}c_k(i - k),
\]
\[
    c_k(i) := \frac{(w(c(i)) - k)!}{k! w(c(i))!} d^k(c(i)), c_0(i) := c(i).
\]
Proof. We may assume that \( d(c(i)) = 0 \) for all \( i \). Since \([\partial_n,d] = 0\) we have \( d(c(0)) = d(\partial_n(z)) = \partial_n(d(z)) = 0\). Suppose by induction \( d(c(i)) = 0\). We have to show that \( d(c(i+1)) = 0\). From theorem 3.2 it follows that \( d(c_k(i)) = c_{k-1}(i)\). By definition we have

\[
c(i+1) = \partial_{n-i+1}(z) + \sum_{k=1}^{i+1}(-1)^{k+1}c_k(i+1-k) = \partial_{n-i+1}(z) + c_1(i) + \sum_{k=2}^{i+1}(-1)^{k+1}c_k(i+1-k).
\]

Therefore

\[
d(c(i+1)) = d(\partial_{n-i+1}(z)) + d(\sum_{k=2}^{i+1}(-1)^{k+1}c_k(i+1-k)) + d(c_1(i)) =
\]

\[
= -\partial_{n-i}(z) + \sum_{k=1}^{i+1}(-1)^{k+1}c_k(i+1-k) + c(i) =
\]

\[
= -(\partial_{n-i}(z) + \sum_{k=1}^{i+1}(-1)^{k+1}c_k(i+1-k)) + c(i) = -c(i) + c(i) = 0.
\]

Now since \( \partial_n(z) = c(0) \) and \( \text{ord}(\partial_n(z)) = \text{ord}(z) + n \geq n \) then the Casimir element \( \tau_n(c(0)) \) exists and

\[
\tau_n(c(0)) = x_n c(0) - x_{n-1} c_1(0) + \ldots + (-1)^n x_0 c_n(0).
\]

Furthermore, taking into account \( \text{deg}(z) z = x_n \partial_n(z) + x_{n-1} \partial_{n-1}(z) + \ldots + x_0 \partial_0(z) \) we get:

\[
\text{deg}(z) z - \tau_n(c(0)) = x_{n-1}(\partial_n(z) + c_1(0)) + x_{n-2}(\partial_{n-1}(z) - c_2(0)) + \ldots + x_0(\partial_0 - (-1)^n c_n(0)).
\]

The coefficient of \( x_{n-1} \) is equal \( c(1) \) hence

\[
\tau_{n-1}(\partial_n(z) + c_1(0)) = x_{n-1} c_0(1) - x_{n-2} c_1(1) + \ldots + (-1)^n x_0 c_{n-1}(1).
\]

Therefore

\[
\text{deg}(z) z - (\tau_n(c(0)) + \tau_{n-1}(c(1))) = x_{n-2}(\partial_{n-2}(z) - c_2(0) + c_1(1)) + \ldots + x_i(\partial_i(z) - (-1)^{i+1} c_i(1)) + \ldots + x_0(\partial_0(z) - (-1)^n c_n(0) - (-1)^{n-1} c_{n-1}(1)) =
\]

\[
= x_{n-2}(c(2)) + \ldots + x_0(\partial_0(z) - (-1)^n c_n(0) - (-1)^{n-1} c_{n-1}(1)).
\]

Finally, we obtain

\[
\text{deg}(z) z - (\tau_n(c(0)) + \tau_{n-1}(c(1))) = x_0(\partial_0(z) + c_1(n) - c_2(n-1) + \ldots + (-1)^{n+1} c_n(0)) = x_0 c(n) = \tau_0(c(n)).
\]

Hence

\[
\text{deg}(z) z = \tau_n(c(0)) + \tau_{n-1}(c(1)) + \ldots + \tau_1(c(n-1)) + \tau_0(c(n)),
\]

and we get

\[
z = \frac{1}{\text{deg}(z)}(\tau_n(c(0)) + \tau_{n-1}(c_1) + \ldots + \tau_0(c(n))).
\]

\[\square\]

**Theorem 4.2.** Let \( z \) be a homogeneous isobaric polynomial of \( k[X]^d \). Then

(i) \( \text{ord}(\tau_i(z)) = n + \text{ord}(z) - 2i \),

(ii) \( \omega(z) = \text{ord}(z) \).
Proof. (i) Let us denote by $s$ the order of $z$. We first show that $\hat{d}^{n+s-2}(\tau_1(z)) \neq 0$ but $
olimits\hat{d}^{n+s-2i+1}(\tau_1(z)) = 0$. Consider now two $\hat{d}$–modules:

$$\hat{X}_m := \langle x_n, \frac{x_{n-1}}{\gamma_{n-1}}, \frac{x_{n-2}}{\gamma_{n-1} \gamma_{n-2}}, \ldots, \frac{x_m}{\gamma_{n-1} \cdots \gamma_{n-m}} \rangle,$$

$$\hat{V}_m(z) := \langle z_s, \frac{z_{s-1}}{\alpha_s(z)}, \frac{z_{s-2}}{\alpha_s(z) \alpha_{s-2}(z)}, \ldots, \frac{z_m}{\alpha_{s-1}(z) \cdots \alpha_{s-m}(z)} \rangle,$$

where $\gamma_i = (i + 1)(n - i)$, $z_i := \alpha_i(z) \hat{d}(z)$, and $\alpha_i = (w(z) - i)! / i! w(z)!$, $m = \min(n, s)$ and $w(z)$ is a weight of the polynomial $z$. Define a linear multiplicative map $\psi : X_m \cdot V_m(z) \longrightarrow \hat{X}_m \cdot \hat{V}_m(z)$ by the rule

$$\psi(x_i) = \frac{x_{n-i}}{\gamma_{n-1} \cdots \gamma_{n-i}}, \psi(z_i) = \frac{z_{n-i}}{\alpha_{n-1}(z) \cdots \alpha_{n-i}(z)}.$$

Since $\hat{x}_i = \gamma_i x_{i+1}$ and

$$\psi(d(x_i)) = \psi(x_{i-1}) = \frac{x_{n-(i-1)}}{\gamma_{n-1} \cdots \gamma_{n-(i-1)}},$$

$$\hat{d}(\psi(x_i)) = \hat{d}\left(\frac{x_{n-i}}{\gamma_{n-1} \cdots \gamma_{n-i}}\right) = \frac{x_{n-(i-1)}}{\gamma_{n-1} \cdots \gamma_{n-(i-1)}},$$

it follows that $\psi(d(x_i)) = \hat{d}(\psi(x_i))$ thus the restriction of $\psi$ to $X_m$ is an isomorphism from $d$–module $X_m$ to $\hat{d}$–module $\hat{X}_m$. Similarly $V_m(z)$ and $\hat{V}_m(z)$ are isomorphic thus $\psi$ is an isomorphism of $d$–module $X_m \cdot V_m(z)$ to $\hat{d}$–module $\hat{X}_m \cdot \hat{V}_m(z)$. From Theorem 3.2 it follows that $\hat{d}$–modules $\hat{X}_m$ and $\hat{V}_m(z)$ are isomorphic. For arbitrary homogeneous isobaric polynomials $z \in k[X]^d$ denote by $\hat{\tau}_i(z)$ the corresponding Casimir element

$$\hat{\tau}_i(z) = \Delta(\hat{X}_i, \hat{V}_i(z)).$$

It is easy to check that $\hat{\tau}_i(z) = \psi(\tau_i(z))$. Let us show that

$$\hat{d}^{n+s-2}(\tau_1(z)) = \frac{(n + s - 2)! (n - 1)! (s - 1)!}{n s} \hat{\tau}_i(z).$$

Since

$$\gamma_0 \cdot \gamma_1 \cdots \gamma_{k-1} = (1 \cdot n)(2 \cdot (n - 1)) \cdots (k \cdot (n - (k - 1))) = [k!]^2 \binom{n}{k},$$

we have

$$\hat{d}^k(x_i) = \gamma_i \gamma_{i+1} \cdots \gamma_{i+k-1} x_{k+i} = \frac{\gamma_0 \gamma_1 \cdots \gamma_{i+k-1} x_{k+i}}{\gamma_0 \gamma_1 \cdots \gamma_{k-1}} x_{k+i} = \frac{[(i + k)!]^2}{[i!]^2} \binom{n}{k+i} x_{k+i}.$$ 

In particular $\hat{d}^{n-1}(x_0) = [(n - 1)!]^2 n \cdot x_{n-1}$, $\hat{d}^{n-1}(x_1) = \frac{[(n)!]^2}{n} x_n$. Likewise,

$$\hat{d}^k(z_i) = \frac{[(i + k)!]^2}{[i!]^2} \binom{s}{k+i} x_{z+i}.$$ 

Taking into account $\hat{d}^{n+1}(x_0) = \hat{d}^{s+1}(z_0) = 0$ and $\tau_1(z) = x_0 \cdot z_1 - x_1 \cdot z_0$, we obtain
\[ \hat{d}^{n+s-2}(x_0 z_1 - x_1 z_0) = \binom{n+s-2}{n} \hat{d}^n(x_0) \cdot \hat{d}^{s-2}(z_1) + \binom{n+s-2}{n-1} \hat{d}^{n-1}(x_0) \cdot \hat{d}^{s-1}(z_1) - \binom{n+s-2}{n-1} \hat{d}^{n-1}(x_1) \cdot \hat{d}^{s-1}(z_0) - \binom{n+s-2}{n-2} \hat{d}^{n-2}(x_1) \cdot \hat{d}^{s}(z_0) = \]

\[ = x_{n-1} z_s \left( \binom{n+s-2}{n} \cdot [(n-1)!]^2 \frac{[(s-1)!]^2}{s} - \binom{n+2-2}{n-2} \cdot [(n-1)!]^2 \frac{[(s-1)!]^2}{n} \right) + \]

\[ + x_n \tau_{s-1} \left( \binom{n+s-2}{n} [n!]^2 [(s-1)!]^2 - \binom{n+s-2}{n-1} \cdot [(s-1)!]^2 \frac{[(n-1)!]^2}{n} \right) = \]

\[ = (n+s-2)! \cdot (n-1)! x_{n-1} z_s - (n+s-2)! \cdot n \cdot (s-1)! x_n z_{s-1} = \]

\[ = (n+s-2)! (n-1)! (s-1)! x_{n-1} z_s - n x_n z_{s-1} = \]

\[ = \frac{(n+s-2)! (n-1)! (s-1)!}{ns} \left( x_{n-1} z_s - x_n \frac{z_{s-1}}{s} \right) = \frac{(n+s-2)! (n-1)! (s-1)!}{ns} \tau_1(z). \]

In the general case, it can be shown (the proof is routine) that

\[ \hat{d}^{n+s-2i}(\tau_i(z)) = \frac{(n+s-2i)! (n-i)! (s-i)!}{n(n-1) \cdots (s-i)! (s-1) \cdots (s-i)} \tau_i(z) \neq 0, \]

whenever \( \tau_i(z) \neq 0 \), but \( \hat{d}^{n+s-2i+1}(\tau_i(z)) = \hat{d}(\hat{d}^{n+s-2i}(\tau_i(z))) = 0 \) since \( \tau_i(z) \) belongs to the kernel of the derivation \( \hat{d} \), therefore the order of non-vanished polynomial \( \tau_i(z) \) is equal \( n+s-2i \).

\( (ii) \) It is a direct corollary of the relation \( d(\hat{d}^i(z)) = i (\omega(z) - i + 1) \hat{d}^{i-1}(z) \).

\[ \square \]

5. An algorithm of computing of the kernel \( k[X]^d \)

For any subalgebra \( T \subseteq k[X]^d \) we write \( \tau(T) \) for the subalgebra generated by the elements \( \tau_i(z), z \in T, i \leq \text{ord}(z) \). The map \( \tau \) allow us to arrange the following iteration process for calculation of the algebra \( k[X]^d \): for arbitrary subalgebra \( B \) of algebra \( k[X]^d \) denote by \( \overline{B} \) a subalgebra generated by elements \( B \cup \tau(B) \). For every integer \( m \geq 0 \) define the following sequence \( B_m \) of subalgebras of \( k[X]^d \)

\[
\begin{cases}
B_0 = k[x_0] \\
B_m = \overline{B}_{m-1}.
\end{cases}
\]

We get a increasing chain of the subalgebras \( B_i \)

\[ B_0 \subseteq B_1 \subseteq B_2 \ldots \]

Now it is possible to state the following proposition

**Theorem 5.1.** There exists some \( k \) such that \( B_k = B_{k+1} = k[X]^d \).

This theorem is a direct corollary of finite generation of the Weitzenbök derivation and of the follow theorem:
Theorem 5.2. If $T$ is a subalgebra of $k[X]^d$ with the property that $x_0 \in T$ and $\tau(T) \subseteq T$, then $T = k[X]^d$.

Proof. It is enough to show that $k[X]^d \subseteq T$. The proof is by induction on the degree of a polynomial $z \in T$. Under the condition of the theorem we have $x_0 \in T$. Let us assume that the theorem is true for all polynomials of degree less or equal to $s$. Suppose deg($z$) = $s + 1$; then from theorem 4.1 it follows that $z$ can be expressed as a sum of polynomials $\tau_i(z')$ where $z'_i$ are elements of the degree $s$. Then by the induction hypothesis, we have $z \in T$ therefore $k[X]^d \subseteq T$. 

For the realization of the algorithm it is necessary to be able to calculate algebra $\tau(B_i)$ knowing generating set for algebra $B_i$. Let us give a definition of a irreducible polynomials of algebra $k[X]^d$. We will say that $x_0$ is the only irreducible polynomial of depth unity, and that $x_0, K_1, \cdots, K_s$ form a complete set of irreducible polynomials of depth $< m$ if every polynomials of $B_i$, $i < m$ is a polynomial in $x_0, K_1, \cdots, K_s$, over field $k$. Polynomials which are not irreducible we will call as reducible. A several cases of reducibility considered in the following theorem.

Theorem 5.3. Suppose that $u, v \in k[X]^d$ are irreducible polynomials of depth $i$; then

(i) If ord($u$) = 0 then $\tau_k(uv)$ is reducible of the depth $i$;
(ii) $\tau_i(uv)$ is reducible for all $i \leq \min(n, \text{ord}(v))$;
(iii) Polynomial $\tau_i(u_1 u_2 \cdots u_{i+1})$, $u_i \in B_i$ is always reducible.

Proof. (i) Let $i \leq \text{ord}(v)$ and $V_i(v) = \langle v_0 v_1, \ldots, v_i \rangle$. If $\hat{d}(u) = 0$ then we have $V_i(uv) = \langle u v_0, u v_1, \ldots, u v_i \rangle$. Thus

$$\tau_i(uv) = \Delta(X_i, V_i(uv)) = \sum_{k=1}^{i} (-1)^{k+1} x_k u v_{i-k+1} = u \tau_i(v).$$

(ii) Let us show that for $i \leq \min(n, \text{ord}(v))$ and for certain $c_k' \in k[X]^d$ the relation

$$\tau_i(uv) = u \tau_i(v) + \sum_{k=1}^{i-1} \tau_{i-k}(c_k'),$$

is true. In fact

$$\tau_i(uv) - u \tau_i(v) = \sum_{k=0}^{i} (-1)^{i-k} \alpha_{i-k}(uv)x_{i-k}\hat{d}^k(uv) - u \left( \sum_{k=0}^{i} (-1)^{i-k} \alpha_{i-k}(v)x_{i-k}\hat{d}^k(v) \right) =$$

$$= x_i uv + \sum_{k=1}^{i} (-1)^{i-k} \alpha_{i-k}(uv)x_{i-k}\hat{d}^k(uv) - u \left( x_i v + \sum_{k=1}^{i} (-1)^{i-k} \alpha_{i-k}(v)x_{i-k}\hat{d}^k(v) \right) =$$

$$= \sum_{k=1}^{i} (-1)^{i-k} x_{i-k}(\alpha_{i-k}(uv)\hat{d}^k(uv) - \alpha_{i-k}(v)u \hat{d}^k(v)).$$

The polynomial of the right hand side belongs to the kernel as difference of two elements of the kernel. To conclude the proof, it remains to apply theorem 4.1

(iii) If among the polynomials $u_1, u_2, \ldots, u_{i+1}$ there is a polynomial of the order zero then $\tau_i(u_1 u_2 \cdots u_{i+1})$ is reducible by (i). If all of them have non-vanishing orders then the order of $u_1 u_2 \cdots u_i$ is $\geq i$ and $\tau_i(u_1 u_2 \cdots u_{i+1})$ is reducible by (ii). 

Definition 5.1. A polynomial $z \in \tau(B_m)$ is called acceptable for algebra $B_m$ if $z$ is irreducible and $z \notin B_m.$
Let \( B_m = k[f_1, \ldots, f_r, g_1, \ldots, g_s] \) where \( \{g_i\} \) are acceptable for \( B_{m-1} \). From theorem 4.3 it is follows

**Theorem 5.4.** Polynomial \( \tau_i(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_r^{\alpha_r} g_1^{\beta_1} \cdots g_s^{\beta_s}) \) can not be an acceptable polynomial for \( B_m \) if any of follow conditions holds:

1. \( \sum \alpha_k + \sum \beta_k > i \).
2. \( \sum \beta_k = 0 \).
3. Some of \( f_i, g_k \) have the order equal to zero but \( \alpha_i \neq 0, \beta_k \neq 0 \).
4. \( f_1^{\alpha_1} f_2^{\alpha_2} f_r^{\alpha_r} g_1^{\beta_1} g_s^{\beta_s} \) can be expressed as product of two polynomials one of them has order greater than \( i \).

**Definition 5.2.** A triple of integer numbers

\[
[\deg(z), \ord(z), \frac{n \deg(z) - \omega(z)}{2}]
\]

is called the signature of a polynomial \( z \) and denoted by \([z]\).

From theorem 4.2 it is follow that \([u, v] = [u] + [v]\)

Now let us offer a verification algorithm if a polynomial \( z \) is already calculated where \( \{g_i\} \) are acceptable polynomials for \( B_m \) of algebra \( k[X]^d \).

1. Set up the following system of equation

\[
[z] = \alpha_1 [f_1] + \alpha_2 [f_2] + \cdots + \alpha_m [f_m]
\]

2. If this system has no an integer positive solutions then \( z \) doesn’t belong to subalgebra \( B \).
3. Suppose \( \{\alpha^{(i)} = (\alpha_1^{(i)}, \ldots, \alpha_m^{(i)}), i = 1..k\} \) is the set of all positive integer solutions of the system. Set up the new system of equations:

\[
z = \beta_1 f_1^{\alpha^{(1)}} + \beta_2 f_2^{\alpha^{(2)}} + \cdots + \beta_k f_m^{\alpha^{(k)}},
\]

where

\[
f_1^{\alpha^{(1)}} f_2^{\alpha^{(2)}} \cdots f_m^{\alpha^{(m)}} = f_1^{\alpha_1^{(i)}} f_2^{\alpha_2^{(i)}} \cdots f_m^{\alpha_m^{(i)}}.
\]

4. If the system has non-vanishing solutions then \( z \in B \) otherwise \( z \not\in B \).

From above we obtain the following algorithm for computing of kernel \( k[X]^d \). Let \( \{B\} \) be a generating set of some algebra \( B \).

1. \( \{B_1\} = \{x_0\} \).
2. Suppose that algebra \( \{B_i\} = \{f_1, \ldots, f_r\} \cup \{g_1, \ldots, g_s\} \) is already calculated where \( \{g_1, \ldots, g_s\} \) are acceptable polynomials for \( B_i \).
3. Consider a finite sets of elements of \( \tau(B_i) \) which could be an acceptable for \( B_i \):

\[
B_i^{(m)} := \{\tau_k(f_1^{\alpha_1} \cdots f_r^{\alpha_r} g_1^{\beta_1} \cdots g_s^{\beta_s}), \sum \alpha_q + \sum \beta_q = m, m \leq n, k \leq n\}
\]

4. By using previous algorithm we compute a set \( H \) of acceptable polynomials of \( B_i^{(m)} \), \( m \leq n \).
5. If \( H = \emptyset \) then \( k[X]^d = B_i \) else \( B_{i+1} = \{f_1, \ldots, f_r\} \cup \{g_1, \ldots, g_s\} \cup H \).
6. A calculation $k[X]^d$ for $n < 7$.

Denote by $t$ the variable $x_0$.

**Theorem 6.1.** $B_1 = k[t, \tau_2(t), \tau_4(t), \ldots, \tau_{2^n}(t)], n > 2$

**Proof.** It is clear that an acceptable polynomials for $B_1$ can only be any one of the following polynomials $\tau_i(t), i \leq n$. It is easy to check that for odd $i$ we have $\tau_i(t) = 0$. To prove that the set of polynomials $t, \tau_2(t), \tau_4(t), \ldots, \tau_{2^n}(t)$ is the minimal generating set for subalgebra $B_1$ it is enough to prove that there are not any linear relations for the polynomials $t^2, \tau_2(t), \tau_4(t), \ldots, \tau_{2^n}(t)$. The proof follows obviously from the fact that no two polynomials have the same orders. \hfill \Box

The following computations were all done with Maple.

**5.1 n = 1.**

Since $\tau_1(t) = 0$ we get $\tau(B_1) = 0$. Therefore, we have $B_2 = B_1$ thus $k[X]^d = k[t]$.

**5.2 n = 2.**

By using theorem 6.1 we have $B_1 = k[t, dv]$ where $dv := \tau_2(t) = tx_2 - 2x_2^2$. Since $\text{ord}(dv) = 0$; then $B_1$ has no an acceptable elements. Therefore $B_2 = B_1$ and $k[X]^d = k[t, dv]$.

**5.3 n = 3.**

We have $B_1 = k[t, dv]$ where $dv = \tau_2(t)$. Since $\text{ord}(dv) = 3 + 3 - 2 \cdot 2 = 2$, we obtain

$$B_1^{(1)} = \{\tau_1(dv), \tau_2(dv)\},$$

$$B_1^{(2)} = \{\tau_3(dv^2)\}.$$ However by a straightforward calculation we obtain $\tau_3(dv^2)=0$ and $\tau_2(dv)=0$. Denote the remaining element by $tr := \tau_1(dv)$. By using an algorithm of the section 5 we get $tr \notin B_1$ and thus $B_2 = k[t, dv, tr]$. Since $\text{ord}(tr) = 3$ we see that an acceptable elements for $B_2$ can only be the polynomial $c = \tau_3(tr)$. Note $\text{ord}(c) = 0$ and $B_2$ has no an elements of order zero. Hence $c \notin B_2$ and $B_3 = k[t, dv, tr, c]$. Since $\text{ord}(c) = 0$, then $B_3$ has no any acceptable elements. Hense $B_4 = B_3$ thus $k[X]^d = k[t, dv, tr, c]$ where

$$dv = -2tx_2 + x_1^2,$$

$$tr = -3tx_1x_2 + 3t^2x_3 + x_1^3,$$

$$c = -18tx_1x_2x_3 + 8t^2x_2^3 + 9x_3^2t^2 + 6x_1^3x_3 - 3x_1^2x_2^2.$$

**5.4 n = 4.**

$B_1 = k[t, d_1, d_2]$, where

$$d_1 = \tau_2(t) \quad [d_1] = [2, 4, 4],$$

$$d_2 = \tau_4(t) \quad [d_2] = [2, 0, 12].$$

It is easy to see that for subalgebra $B_1$ only polynomial $\tau_i(d_1), i = 1..4$ can be an acceptable polynomial. By direct calculation we obtain $\tau_2(d_1) = td_2, \tau_3(d_1) = 0$. Put

$$tr_1 = \tau_1(d_1) \quad [tr_1] = [3, 6, 8],$$

$$tr_2 = \tau_4(d_1) \quad [tr_2] = [3, 0, 18].$$
The signatures of $t^3, td_1, td_2$ are equal $[3, 8, 0], [3, 8, 4], [3, 4, 12]$ therefore $tr_1, tr_2$ are not in $B_1$, hence $B_2 = k[t, d_1, d_2, tr_1, tr_2]$. Since $\text{ord}(d_2) = \text{ord}(tr_2) = 0$ we see that an acceptable elements can only be the follow elements $\tau_i(tr_1), i = 1..4$. Take into account $\tau_2(tr_1) = 0, \tau_2(tr_4) = 0$ and

$$
\begin{align*}
\tau_1(tr_1) &= d_2 t^2 + d_1, \\
\tau_3(tr_2) &= d_1 d_2 - t \cdot tr_2,
\end{align*}
$$

we have $\tau(B_2) \subseteq B_2$, $B_3 = B_2$ and $k[X]^{d} = k[t, d_1, d_2, tr_1, tr_2]$ where

$$
\begin{align*}
d_1 &= -2tx_2 + x_1^2, \\
d_2 &= -2tx_4 + 2x_1x_3 - x_2^2, \\
tr_1 &= -3tx_1x_2 + 3x_3t^2 + x_1^3, \\
tr_2 &= 12x_2tx_4 - 9x_3^2t - 6x_1^2x_4 + 6x_1x_2x_3 - 2x_2^3.
\end{align*}
$$

5.5 $n = 6$.

We have $B_2 = k[t, d_1, d_2]$ where

$$
\begin{align*}
d_1 &= \tau_2(t), [d_1] = [2, 6, 2], \\
d_2 &= \tau_4(t), [d_2] = [2, 2, 4].
\end{align*}
$$

The following 10 polynomials can be acceptable polynomials for $B_2$ :

$$
\begin{align*}
B_2^{(1)} &= \{\tau_i(d_1), i = 1..5; \tau_i(d_2), i = 1, 2\}, \\
B_2^{(2)} &= \{\tau_i(d_2), i = 3, 4\}, \\
B_2^{(3)} &= \{\tau_5(d_3)\}.
\end{align*}
$$

By direct calculation we obtain

$$
\begin{align*}
\tau_2(d_1) &= -\frac{6}{9}td_2, & \tau_5(d_1) &= 0 \\
\tau(d_2) &= 5\tau_3(d_1), & \tau_2(d_2) &= -\frac{5}{9}\tau_4(d_1).
\end{align*}
$$

Make the denotations,

$$
\begin{align*}
tr_1 &= \tau_4(d_1), & [tr_1] &= [3, 3, 6], \\
tr_2 &= \tau_3(d_1), & [tr_2] &= [3, 5, 5], \\
tr_3 &= \tau_1(d_1), & [tr_3] &= [3, 9, 3], \\
p_1 &= \tau_4(d_2), & [p_1] &= [5, 1, 12], \\
p_2 &= \tau_3(d_2), & [p_2] &= [5, 3, 11], \\
si_1 &= \tau_5(d_2), & [si_1] &= [7, 1, 17].
\end{align*}
$$

Thus

$$B_3 = k[t, d_1, d_2, tr_{1-3}, p_{1-2}, si_1].$$

By using an algorithm of section 5 one can show that this polynomial system is the generating set for $B_3$ Now write down polynomials which can be acceptable for $B_2$ :

$$
\begin{align*}
B_2^{(1)} &= \{\tau_3(tr_1), \tau_5(tr_2), \tau_4(tr_3), \tau_5(tr_3)\} \\
B_2^{(2)} &= \{\tau_4-5(d_2 tr_1), \tau_5(d_2 p_2), \tau_5(tr_4^2)\} \\
B_2^{(3)} &= \{\tau_5(d_2 p_1), \tau_5(d_2^2 tr_1), \tau_5(d_2^2 si_1), \tau_5(tr_1 p_1 si_1), \tau_5(p_1 p_2 si_1)\} \\
B_2^{(4)} &= \{\emptyset\}
\end{align*}
$$
By direct calculation we obtain that the following polynomials are equal to zero: $\tau_5(tr_1), \tau_5(d_2 tr_1), \tau_5(tr_1^2), \tau_5(d_2 p_1), \tau_5(tr_1 p_1 s_1)$. Put

\[
\begin{align*}
c_1 & := \tau_5(tr_2) & [c_1] & = [4, 0, 10], \\
c_2 & := \tau_4(tr_3) & [c_2] & = [4, 4, 8], \\
c_3 & := \tau_5(tr_3) & [c_3] & = [4, 6, 7], \\
s_1 & := \tau_4(d_2 tr_1) & [s_1] & = [6, 2, 14], \\
v_1 & := \tau_5(d_2 p_2) & [v_1] & = [8, 0, 20], \\
v_2 & := \tau_5(d_3^2 tr_1) & [s_1] & = [8, 2, 19], \\
dv & := \tau_5(dv_2 s_1) & [dv] & = [12, 0, 30], \\
vis & := \tau_5(p_1 p_2 s_1) & [vis] & = [18, 0, 45].
\end{align*}
\]

Thus

\[ B_3 = k[t, d_1, d_2, tr_1, tr_2, tr_3, c_1, c_2, c_3, p_1, p_2, s_1, s_1, v_1, v_2, dv, vis] \]

The following polynomials are acceptable for $B_3$:

\[
\begin{align*}
p_3 & := \tau_2(c_3) & [p_3] & = [5, 7, 9], \\
s_2 & := \tau_1(p_1) & [s_2] & = [6, 4, 13], \\
s_1 & := \tau_1(s_1) & [s_1] & = [7, 5, 15], \\
dev & := \tau_2(v_2) & [dev] & = [9, 3, 21], \\
od & := \tau_5(s_1 d_2) & [od] & = [11, 1, 27], \\
trn & := \tau_5(v_2 d_3^2) & [trn] & = [13, 1, 32].
\end{align*}
\]

As above we can show

\[ B_4 = k[t, d_1, tr_1, tr_2, tr_3, c_1, c_2, c_3, p_1, p_2, d_2, s_1, s_1, d_1, s_1, v_1, v_2, dev, od, dv, trn, vis], \]

Similarly we obtain $\tau(B_4) \subset B_4$ hence the indicated polynomial set of 23 polynomials is a minimal generating set for $k[\tau]^d$.

**5.6 n = 6.** We have $B_2 = k[t, d_1, d_2, d_3]$ where

\[
\begin{align*}
d_1 & := \tau_0(t) & [p_3] & = [2, 0, 6], \\
d_2 & := \tau_4(t) & [s_2] & = [2, 4, 4], \\
d_3 & := \tau_2(t) & [s_1] & = [2, 8, 2],
\end{align*}
\]

Further $B_3 = k[t, d_1, tr_1, tr_2, p_1]$ where

\[
\begin{align*}
tr_1 & := \tau_6(d_3) & [p_3] & = [3, 2, 8], \\
tr_2 & := \tau_4(d_3) & [s_2] & = [3, 6, 6], \\
tr_3 & := \tau_3(d_3) & [s_1] & = [3, 8, 6], \\
tr_4 & := \tau_1(d_3) & [s_1] & = [3, 12, 3], \\
p_1 & := \tau_6(d_2^2) & [p_1] & = [5, 2, 14], \\
p_2 & := \tau_5(d_2^2) & [p_1] & = [5, 4, 13].
\end{align*}
\]
In the same way we obtain $B_4 = k[t, d_{1-3}, tr_{1-4}, c_{1-4}, p_{1-2}, s_{1-3}, si_{1-2}, vi, dev, de_{1-2}, dvan]$ where

\[
\begin{align*}
  c_1 &:= \tau_6(tr_2) & [c_1] &= [4, 0, 12], \\
  c_2 &:= \tau_6(tr_3) & [c_2] &= [4, 4, 10], \\
  c_3 &:= \tau_6(tr_4) & [c_3] &= [4, 6, 9], \\
  c_4 &:= \tau_4(tr_4) & [c_4] &= [4, 10, 7], \\
  s_1 &:= \tau_6(d_2 tr_1) & [p_1] &= [6, 0, 18], \\
  s_2 &:= \tau_3(d_2 tr_1) & [p_1] &= [6, 6, 15], \\
  s_3 &:= \tau_1(p_1) & [p_1] &= [6, 6, 15], \\
  s_4 &:= \tau_4(tr_2^2) & [si_1] &= [7, 2, 20], \\
  s_5 &:= \tau_3(tr_1^2) & [si_2] &= [7, 4, 19], \\
  vi &:= \tau_6(d_2 p_2) & [vi] &= [8, 2, 23], \\
  dev &:= \tau_4(tr_1 p_2) & [dev] &= [9, 4, 25], \\
  de_1 &:= \tau_6(tr_1^3) & [de_1] &= [10, 0, 30], \\
  de_2 &:= \tau_5(tr_1^3) & [de_2] &= [10, 2, 29], \\
  dvan &:= \tau_5(tr_2^2 p_1) & [dvan] &= [12, 2, 35].
\end{align*}
\]

Acceptable polynomials for $B_3$ are only these two polynomials:

\[
\begin{align*}
  p_3 &:= \tau_4(c_4) & [p_3] &= [5, 8, 11], \\
  pt &:= \tau_6(si_1 si_2) & [pt] &= [15, 0, 45].
\end{align*}
\]

The polynomial $pt$ has degree 15 order 0 and consist of 1370 terms. A weight of $pt$ is an odd number whereas the weights of all other generating polynomials of weight zero are even numbers. Therefore $pt$ is irreducible. We can show that $B_4 = k[t, d_{1-3}, tr_{1-4}, c_{1-4}, p_{1-3}, s_{1-3}, si_{1-2}, vi, dev, de_{1-2}, dvan, pt]$.

As above we may obtain $\tau(B_3) \subset B_4$ hence the indicated polynomial set of 26 polynomials is a minimal generating set for $k[X]^d$.

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