c = 1 String as a Topological Model

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ABSTRACT

The discrete states in the c = 1 string are shown to be the physical states of a certain topological sigma model. We define a set of new fields directly from c = 1 variables, in terms of which the BRST charge and energy-momentum tensor are rewritten as those of the topological sigma model. Remarkably, ground ring generator \(x\) turns out to be a coordinate of the sigma model. All of the discrete states realize a graded ring which contains ground ring as a subset.
1 Introduction

The existence of so called unbroken phase in string theory was conjectured some time ago. Such a phase is naturally expected to be described by an appropriate topological field theory [1, 2].

Even in the conventional formulation one might suspect there is some remnant of the topological degrees of freedom. Since there is no transverse degree of freedom, two dimensional string theory is a good place to look for such things. Actually we already know the physical spectrum of the theory [3, 4]. There are infinite numbers of discrete states which are fairly different from ordinary particle modes. One may naturally bear an idea that the discrete states are the very remnant of topological degrees of freedom we are seeking for.

One of the encouraging hints follows from the gauge theory in four dimension. Let us consider, for instance, the Gupta-Bleuler quantization of free electromagnetic field. In this formulation, physical state is defined by the condition

\[(\partial_{\mu}A^\mu)^{(+)}|\text{phys}\rangle = 0,\]

where \((+)^{(+)}\) stands for the positive frequency part of the operator. This restricts physical polarization to the transverse directions for the generic case of momentum eigenstates. For the vanishing four momentum case, however, there is no restriction on the polarization. Thus we have “discrete states” for the longitudinal and scalar modes. The same kind of states also appear at level 1 in the critical string. Remarkable thing here is that such constant degrees of freedom are responsible for the masslessness of all the components of gauge field. Because in this covariant gauge there still remains a global symmetry \(\delta A^\mu = a^\mu\), where \(a^\mu\) is an arbitrary constant vector, but the vacuum breaks the symmetry with \(\langle A^\mu\rangle = 0\) and corresponding Nambu-Goldstone modes are nothing but four components of the gauge field itself [5, 6]. This mechanism was applied to non-abelian case to try to explain the confinement phase as an unbroken phase of a special gauge symmetry [7].

The above example suggests that the discrete states in the \(c = 1\) string are relevant to the phases in string theory in an analogous way. For example, ghost number\(^1\) one discrete states in relative cohomology are characterized as Fock states which are simultaneously singular and cosingular vectors with respect to the matter part of the Virasoro algebra. This implies that we have similar kind of states also in higher dimensional string including the critical one.

\(^1\)Throughout this paper we use the convention that the \(SL(2, C)\) invariant vacuum has vanishing ghost number.
Of course, in higher dimensional case they are buried in the mass shell, i.e., usual particle modes with particular momenta. But we may consider conversely that such siblings of the discrete states determine the mass shell that possible particle modes belong to. Actually Lorentz invariance and the existence of a particular state on a certain mass shell are enough to show the existence of a particle on that shell. In two dimensional or $c=1$ case, only a massless “tachyon” is a particle mode so that the role of the discrete states in unbroken phase may be rather clear if any. For the $c=1$ string, level of discrete states is not restricted to 1 as in the critical case but extends over all the positive integers. Higher level states have non-vanishing momenta and have not been given a similar interpretation as done for the level 1 or vector field case. We need a more unified view of the discrete states if we consider them as a clue to the phases in string theory. Anyway, it is desired to understand the discrete states from such a viewpoint, and the present paper is, hopefully, the first step toward this end.

In this paper we will show that the discrete states in the $c=1$ string are understood as physical states of a certain topological model. Starting from the $c=1$ string we define new fields in terms of which we can rewrite the BRST charge of the string theory as that of the topological sigma model. There, notably enough, the ground ring generator $x$ plays a role of scalar field of the sigma model (or we should call it a coordinate of a kind of topological string). This relationship is in the operator level not merely in the level of amplitudes as is in $c<1$ string [9]. And all the discrete states are reconstructed through the BRST cohomology of the topological model. They naturally realize a graded ring which contains ground ring as a subset.

It has been argued by several groups that the $c=1$ string can be regarded as a topological model, e.g., topological sigma model [10], twisted $SL(2,R)/U(1)$ model [11] and $SL(2,R)/SL(2,R)$ model [12]. However, the previous attempts were based on comparison of the physical spectrum for topological models with that for the $c=1$ string. It was not clear why the physical spectrum of strings was reproduced from topological models. Our approach is more direct in contrast to these. The $c=1$ string can be viewed as a ‘bosonization’ of the topological sigma model\(^2\), and the coincidence of the physical spectrums for both models is immediate. Although an interpretation of $x$ as a coordinate of a certain three dimensional cone was given in ref.[8], topological field theoretic nature was not clear in that

\(^2\)The observation that string theory can be regarded as bosonization of a topological model was also obtained in ref.[13] for the case of $c=−2$ matter and the $\hat{c}=1$ fermionic case.
time. We emphasize here that the $x$ is directly shown to be a coordinate of a topological sigma model. This point is also a difference from ref. [10] in which the author started from a topological model given a priori and tried to describe only a matter part of the $c = 1$ theory.

In this sense, our observation may be a good starting point toward the direct understanding of the unbroken phase in terms of the field theory on the world sheet. Namely, unbroken phase is described by a topological sigma model and broken phase done by conventional string theory. Alternatively, several backgrounds of strings emerging in the broken phase are governed by a topological model which is possibly related to the unbroken phase.

This paper is organized as follows. In the subsequent section, from the fields of the $c = 1$ string we define new fields such as gauge ghost and anti-ghost with dimension 0 and 1 respectively and a pair of scalar fields which become ingredients of topological sigma model. In use of the definition of these fields, the BRST charge and the energy-momentum tensor of the $c = 1$ string surprisingly turn out to be those of a topological model. In a sense, the $c = 1$ string is a ‘bosonization’ of a topological sigma model. The Lagrangian and gauge fixing of corresponding topological sigma model is discussed in section 3. In section 4 we will show how to understand the discrete state spectrum from the topological theoretic point of view. All the discrete states in the $c = 1$ string are reproduced by taking into account both the picture-changed and dual sectors of the topological sigma model. The final section is devoted to the discussion.

## 2 Topological description of the $c = 1$ string

We begin with rewriting the $c = 1$ string in terms of a topological model. The $c = 1$ string is a model which has a two-dimensional target space with the linear dilaton background. The energy-momentum tensor of the model takes the form

$$T(z) = T^M(z) + T^G(z) = \frac{1}{2} (\partial X)^2 - \frac{1}{2} (\partial \phi)^2 + \sqrt{2} \partial^2 \phi - 2b \partial c - \partial b c,$$

where $X$ and $\phi$ are string coordinates, free bosons with positive signature, while $b$ and $c$ are the diffeomorphism ghosts. In the context of non-critical string, $X$ and $\phi$ represent $c = 1$ matter and the Liouville field respectively. The BRST operator, which governs the physical

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3 We consider the left moving part only.
spectrum of the model, reads
\[ Q = \oint dz \ c \left( T^M + \frac{1}{2} T^G \right) = \oint dz \ (c T^M + bc \partial c). \] (2.2)

Physical states of the \( c = 1 \) string are obtained as cohomology with respect to this BRST operator.

Now, we introduce the following set of fields
\[ \begin{align*}
B^+ &= be^{iX^+}, \quad x = (cb + i \partial X^-)e^{iX^+}, \\
C^+ &= ce^{-iX^+}, \quad \bar{p}^+ = e^{-iX^+},
\end{align*} \] (2.3)
where \( X^\pm \) means the light-cone combination of the string coordinates
\[ X^\pm = \frac{1}{\sqrt{2}}(X \pm i \phi). \] (2.4)
The dimension of these operators is 0 for \( C^+ \) and \( x \), and 1 for \( B^+ \) and \( \bar{p}^+ \), respectively.

Operator product expansions (OPE) among them are
\[ \begin{align*}
B^+(z) \ C^+ (w) &\sim \frac{1}{z-w} \sim \ C^+ (z) \ B^+(w), \\
\bar{p}^+(z) \ x (w) &\sim \frac{1}{z-w} \sim -x(z) \ \bar{p}^+(w).
\end{align*} \] (2.5)
(2.6)
The OPE of other combinations are regular. Mutual statistics is therefore fermionic for \( B^+, C^+ \) and bosonic for \( x, \bar{p}^+ \). Note that \( C^+ \) and \( x \) are physical operators, in particular \( x \) is one of the ground ring generators \[8\], whereas \( B^+ \) and \( \bar{p}^+ \) are not physical.

We use these fields to rewrite the \( c = 1 \) string into a topological model; the former can be viewed as a ‘bosonization’ of the latter. The point is that the field contents of \( B^+, C^+, x \) and \( \bar{p}^+ \) are almost the same as those of a topological model which we discuss in the next section. Actually, we can regard this system as a topological sigma model with a pair of complex scalar \( x, \bar{x} \) and gauge ghosts \( B, C \). Necessary identification is
\[ \begin{align*}
B^+ &= B, \ C^+ = C, \ \bar{p}^+ = -\partial \bar{x}.
\end{align*} \] (2.7)
One can consider the \( c = 1 \) string as a realization of the system of fields \( B, C, x \) and \( \bar{p} = -\partial \bar{x} \) through a bosonization \[23\]. This observation is supported by the following facts.

First, the \( SL(2,C) \) vacuum of the \( c = 1 \) string works as the vacuum of the topological sigma model. This can be seen by introducing the mode expansion of the operators:
\[ \begin{align*}
B^+_n &= \oint dz \ z^n B^+(z), \quad x_n = \oint dz \ z^{n-1} x(z), \\
C^+_n &= \oint dz \ z^{n-1} C^+(z), \quad \bar{p}^+_n = \oint dz \ z^n \bar{p}^+(z).
\end{align*} \] (2.8)
These modes act on an operator $\mathcal{O}(z)$ as, for example,

$$B_n^+ \mathcal{O}(w) = \oint dz (z - w)^n B(z) \mathcal{O}(w),$$

(2.9)

where the contour is taken to be encircling the point $w$. Acting these modes on the $SL(2, C)$ vacuum, or equivalently the unit field $\mathbf{1}(z)$, we obtain

$$B_n^+ \mathbf{1} = 0 \text{ for } n \geq 0,$$

$$x_n \mathbf{1} = 0 \text{ for } n \geq 1,$$

$$\bar{p}_n^+ \mathbf{1} = 0 \text{ for } n \geq 0,$$

(2.10)

Gauge ghosts, which have dimension 1 and 0, complex scalar and its derivative act properly on the vacuum as they should.

Second, making use of the definition (2.3), the energy-momentum tensor of the $c = 1$ string turns into that of the topological sigma model. It can be shown that

$$T = -\frac{1}{2} (\partial X)^2 - \frac{1}{2} (\partial \phi)^2 + \sqrt{2} \partial^2 \phi - 2b \partial c - \partial b \partial c$$

$$= i\partial X^+ i\partial X^- + i\partial^2 X^- - i\partial^2 X^+ - 2b \partial c - \partial b \partial c$$

(2.11)

where the composite operators, $\partial x \bar{p}^+$ and $B^+ \partial C^+$, are defined as the finite part in operator product expansion

$$(AB)(w) = \oint dz \frac{1}{z-w} A(z) B(w).$$

(2.12)

By the identification (2.7) mentioned before, one can easily recognize that the last expression is the energy-momentum tensor of the topological sigma model

$$T_{\text{top}} = -\partial x \partial \bar{x} - B \partial C.$$

(2.13)

Third, and most importantly, the BRST operators of both models are also translated into each other. The BRST current of the $c = 1$ string gives rise to that of the topological sigma model

$$c T^M + bc \partial c = C^+ \partial x - \partial \left( c i\partial X^+ + \frac{1}{2} \partial c \right).$$

(2.14)

Total derivative term in the right hand side vanishes upon integration. Thereby we come to an astonishing result, i.e., BRST operator (2.2) of the $c = 1$ string is rewritten as that of the topological sigma model

$$Q_{\text{top}} = \oint dz \ C \partial x.$$

(2.15)

Thus, we have seen the correspondence of the energy-momentum tensor and the BRST operator in both models, which govern physical contents of the model. This fact suggests
a close relationship between the $c = 1$ string and the topological sigma model. In a sense, the $c = 1$ string embodies the topological sigma model as a kind of bosonization of fields. Indeed, as we shall show in the later section, the physical spectrum of the $c = 1$ string, in particular the discrete states, can be characterized as that of the topological model.

Before going into detail, we should remark the $N = 2$ superconformal structure in the $c = 1$ string. As is well-known, the topological sigma model admits a twisted $N = 2$ algebra, the generators of which are

\begin{equation}
T = T_{\text{top}} = -\partial x \partial \bar{x} - B \partial C , \\
G^+ = C \partial x , \\
G^- = -B \partial \bar{x} , \\
J = CB .
\end{equation}

(2.16)

Note that one of the supercurrents $G^+$ is nothing but the BRST current and the $N = 2$ current $J$ is the ghost number current. Since we can consider these fields as that of the $c = 1$ string through a bosonization (2.3), there is also a twisted $N = 2$ algebra in the $c = 1$ string. Using the explicit form (2.3) of the fields, the $N = 2$ generators in the $c = 1$ string read

\begin{equation}
T = -\frac{1}{2} (\partial X)^2 - \frac{1}{2} (\partial \phi)^2 + \sqrt{2} \partial^2 \phi - 2b \partial c - \partial b c , \\
G^+ = c T^M + b c \partial c + \partial \left( c i \partial X^+ + \frac{1}{2} \partial c \right) , \\
G^- = b , \\
J = cb - i \partial X^+ .
\end{equation}

(2.17)

This time, the supercurrent $G^+$ is not the BRST current itself but modified with total derivative terms. Clearly, these terms vanish upon integration and $Q = \oint G^+$. We have shown that the existence of the twisted $N = 2$ superconformal algebra (SCA) in the $c = 1$ string follows from the fact that the $c = 1$ model can be viewed as a bosonization of the topological sigma model in which the $N = 2$ SCA is manifest. In addition to this, we can show there also exist a series of twisted $N = 2$ SCA, with the BRST current as $G^+$, in the $c = 1$ string. The essential point is that we can modify $G^+$ (2.16) in the topological sigma model by a total derivative term $\partial (C x)$ to yield a series of twisted $N = 2$ SCA’s, the
generators of which are
\[ T = \partial x \bar{p} - B \partial C , \]
\[ G^+ = (1 + \lambda) C \partial x + \lambda C \partial x , \]
\[ G^- = B \bar{p} , \]
\[ J = (1 + \lambda) CB - \lambda x \bar{p} , \]

where the parameter \( \lambda \) is related to the central charge \( c \) of the untwisted algebra
\[ c = 3(1 + 2\lambda) . \]

Of course, we have the counterpart of these \( N = 2 \) SCA in the \( c = 1 \) string and one obtains the following result
\[ T = -\frac{1}{2} (\partial X)^2 - \frac{1}{2} (\partial \phi)^2 + \sqrt{2} \partial^2 \phi - 2b \partial c - \partial b c , \]
\[ G^+ = c T^M + bc \partial c + \partial \left( c i \partial X^+ + \lambda c i \partial X^- + \frac{1}{2} (1 + 2\lambda) \partial c \right) , \]
\[ G^- = b , \]
\[ J = cb - (1 + 2\lambda) i \partial X^+ + \lambda i \partial X^- . \]

The previous case corresponds to \( \lambda = 0 \) and has the untwisted central charge \( c = 3 \). Since we modify \( G^+ \) by a total derivative term, the BRST operator is again obtained as \( \oint G^+ \). This modification, which is necessary in order to form the \( N = 2 \) algebra, of the BRST current is essentially the same one as recently reported, \( \lambda = -1 \) corresponds to that in ref.\[14\] while \( \lambda = 1 \) does ref.\[11\].

So far, we consider the topological model consisting of \( B^+, C^+, x \) and \( \bar{p}^+ \). In the \( c = 1 \) string, however, there exists another realization of the topologic al sigma model, the fundamental fields of which take the following form;
\[ B^- = b e^{-iX^-} , \quad y = (cb - i \partial X^+) e^{-iX^-} , \]
\[ C^- = c e^{iX^-} , \quad \bar{p}^- = e^{iX^-} . \]

Again, \( y \) is one of the ground ring generators. It is evident that the argument for the ‘+’-fields is equally applied to these ‘−’-fields.

### 3 Topological sigma model

In this section we construct a topological sigma model which reproduces BRST charge and energy-momentum tensor appeared in the previous section.
The basic fields of our model are complex scalar \( x(z, \bar{z}) \) and \( \bar{x}(z, \bar{z}) \), and Lagrangian is taken to be \( \mathcal{L}_0 = 0 \). Apparently this theory has a gauge symmetry \( \delta x = \epsilon, \delta \bar{x} = \bar{\epsilon} \). Physical contents of topological field theory are encoded in the gauge fixing of the symmetry. According to the standard procedure \[15\] we introduce a pair of ghost \( C, \bar{C} \), anti-ghost \( B, \bar{B} \) and Nakanishi-Lautrup field \( \lambda, \bar{\lambda} \), and define nilpotent BRST transformation:

\[
\begin{align*}
\delta_B x &= -C \\
\delta_B C &= 0 \\
\delta_B B &= \lambda \\
\delta_B \bar{\lambda} &= 0 \\
\delta_{\bar{B}} x &= -\bar{C} \\
\delta_{\bar{B}} \bar{C} &= 0 \\
\delta_{\bar{B}} B &= \bar{\lambda} \\
\delta_{\bar{B}} \lambda &= 0
\end{align*}
\] (3.1)

where we omit a Grassmann parameter of the transformation. Then the gauge fixing term of the Lagrangian is given by the BRST transform which depends on the gauge choice. We adopt the following gauge fixed Lagrangian:

\[
\mathcal{L} = \mathcal{L}_0 + \delta_B \left( -\frac{1}{2} B \bar{\lambda} - \frac{1}{2} \bar{B} \lambda + B \partial_z \bar{x} + \bar{B} \partial_{\bar{z}} x \right) = -\lambda \bar{\lambda} + \lambda \partial_z \bar{x} + \bar{\lambda} \partial_{\bar{z}} x + B \partial_z C + \bar{B} \partial_{\bar{z}} \bar{C}. \] (3.2)

After integrating out \( \lambda \) and \( \bar{\lambda} \), we get

\[
\mathcal{L} = \partial_z x \partial_{\bar{z}} \bar{x} + B \partial_z C + \bar{B} \partial_{\bar{z}} \bar{C}. \] (3.3)

By utilizing the equations of motion, we can separate left mover and right mover of the fields

\[
\begin{align*}
x &= x(z) + x(\bar{z}), \\
C &= C(z), \\
B &= B(z), \\
\bar{x} &= \bar{x}(z) + \bar{x}(\bar{z}), \\
\bar{C} &= \bar{C}(\bar{z}), \\
\bar{B} &= \bar{B}(\bar{z}).
\end{align*}
\] (3.4)

Then we obtain the energy-momentum tensor and BRST charge for each sector

\[
\begin{align*}
T(z) &= -\partial_z x(z) \partial_{\bar{z}} \bar{x}(z) - B(z) \partial_z C(z), \\
\bar{T}(\bar{z}) &= -\partial_{\bar{z}} x(\bar{z}) \partial_z \bar{x}(\bar{z}) - \bar{B}(\bar{z}) \partial_{\bar{z}} \bar{C}(\bar{z}), \\
Q &= \oint dz \, C(z) \partial_z x(z), \\
\bar{Q} &= \oint d\bar{z} \, \bar{C}(\bar{z}) \partial_{\bar{z}} \bar{x}(\bar{z}).
\end{align*}
\] (3.5-3.8)

These operators are what appeared in the previous section in rewriting the \( c = 1 \) string. So, we can say that this topological sigma model accounts for the physical contents of the \( c = 1 \) string. More precisely, it is the \( p = 0 \) sector, where \( p \) is the momentum conjugate to \( \bar{x} \), of this model that corresponds to the \( c = 1 \) string, because there is no \( \log z \) term in the mode expansion (2.8) of \( x \).
The Lagrangian (3.3) was considered in a foresighted paper [1] as a candidate of topological string. At that time only a vacuum was recognized as a physical state. But more careful analysis shows that there is a set of zero modes, being physical, which gives rich structure to the theory. Also this is an essential point to relate the theory with the conventional string formulation. Reconstructing physical states of the $c = 1$ string from the viewpoint of the topological theory will be given in the next section.

4 Discrete states as physical states of a topological model

In this section, we study the physical states of the $c = 1$ string in the light of the topological sigma model. As is well-known [3, 4], the $c = 1$ physical spectrum consists of a massless propagating degree of freedom, so-called tachyon, and the discrete states which appear at the special value of momenta. We restrict our attention to the discrete states and the discrete tachyon; tachyon states with momentum multiple of $1/\sqrt{2}$ in the $X$ direction. This choice corresponds to the self-dual radius of $X$.

In the previous sections, we have shown that the $c = 1$ string can be regarded as a bosonization of the topological sigma model. The energy-momentum tensor and the BRST operator of the topological sigma model turn into those of the $c = 1$ string. Therefore, one can naturally expect that the physical spectrum of the $c = 1$ string can be understood as that of the topological model. As we shall see, this is the case.

We first examine the physical spectrum of the topological sigma model which consists of the following fields

\begin{align*}
B(z) &= \sum_n B_n z^{-n-1}, & x(z) &= \sum_n x_n z^{-n}, \\
C(z) &= \sum_n C_n z^{-n}, & \bar{p}(z) &= \sum_n \bar{p}_n z^{-n-1},
\end{align*}

(4.1)

where $n$ takes integer value. The commutation relations among them are

\begin{align*}
\{B_m, C_n\} &= \delta_{m+n} , & [x_m, \bar{p}_n] &= -\delta_{m+n} .
\end{align*}

(4.2)

These modes act on the vacuum $|0\rangle$ as

\begin{align*}
B_n |0\rangle &= 0 \text{ for } n \geq 0 , & x_n |0\rangle &= 0 \text{ for } n \geq 1 , \\
C_n |0\rangle &= 0 \text{ for } n \geq 1 , & \bar{p}_n |0\rangle &= 0 \text{ for } n \geq 0 .
\end{align*}

(4.3)
The Fock space $\mathcal{F}_0$ of the topological model is constructed on this vacuum by applying the creation modes.

Physical states in this Fock space is defined as cohomology of the BRST operator $Q$

$$Q = \int dz C(z) \partial x(z) = \sum_n nC_n x_{-n}. \quad (4.4)$$

From eq.(4.3), one can see that the vacuum $|0\rangle$ is annihilated by this BRST operator, $Q|0\rangle = 0$. Hence, the vacuum is physical, and the physical spectrum is built on it by applying physical modes. So, we need to know which mode becomes physical, or how the modes (4.1) are transformed into each other under the action of the BRST operator. Using the commutation relations (4.2), one obtains the following result

$$B_n \xrightarrow{Q} -nx_n \quad \bar{p}_n \xrightarrow{Q} -nC_n. \quad (4.5)$$

Note that almost all the modes form doublets under the BRST transformation, i.e., they are unphysical. For $n = 0$, however, this doublet decouples into two singlets and we have four physical modes, $B_0, x_0, \bar{p}_0$ and $C_0$, but two of them annihilate the vacuum as one can see from eq.(4.3). Therefore we have two physical modes, $x_0$ and $C_0$, which create the physical states upon the vacuum $|0\rangle$. The physical spectrum in the Fock space $\mathcal{F}_0$ is thus spanned by two kinds of elements

$$x_0^n|0\rangle \text{ and } C_0 x_0^n|0\rangle \quad n = 0, 1, 2, \cdots. \quad (4.6)$$

Next, we examine how these physical states are expressed in the Fock space of the $c = 1$ string. The $c = 1$ string realizes a bosonization of the topological sigma model which is carried out by identifying the fields $B, C, x$ and $\bar{p}$ with $B^+, C^+, x$ and $\bar{p}^+$ (See eq.(2.3)) in the $c = 1$ string. Through this bosonization, the physical states obtained above are expressed as fields in the $c = 1$ string, where, as was explained in Section 2, the unit field $1(z)$ works as the vacuum. By simple calculation, we can show that

$$x_0 \mathbf{1} = (cb + i \partial X^-)e^{iX^+},$$
$$C_0^+ \mathbf{1} = ce^{-iX^+}, \quad C_0^+ x_0 \mathbf{1} = c i \partial X^- + \partial c. \quad (4.7)$$

One can readily see that these indeed belong to the physical spectrum of the $c = 1$ string; $x_0 \mathbf{1}$ is the ground ring generator, while $C_0^+ \mathbf{1}$ and $C_0^+ x_0 \mathbf{1}$ are a discrete tachyon and one of the level 1 discrete operators, respectively.
This correspondence of physical spectrum for the topological sigma model and the $c = 1$ string holds in general. More precisely, we will show that the $c = 1$ expressions, $x_0^m \mathbf{1}$ and $C_0^+ x_0^m \mathbf{1}$, reproduce a part of the physical spectrum of the $c = 1$ string. Since the translation from the topological sigma model to the $c = 1$ string is simply a bosonization, it is clear that these $c = 1$ expressions are annihilated by the BRST operator of the $c = 1$ string. Not obvious thing is whether these operators are BRST exact or not. If it is impossible to write them as BRST exact ones in the $c = 1$ Fock space, they turn out to be the physical operators of the $c = 1$ string. In fact, non-triviality of them can be shown as follows.

Let us suppose that $x_0^m \mathbf{1}$ is written as a BRST-exact operator $x_0^m \mathbf{1} = Q \alpha$. Acting $\bar{p}_0^+$ to this operator and using the commutation relation (4.2), one can strip off $x_0$ one by one to reach the vacuum $\mathbf{1}$. Since the BRST operator $Q$ commutes with $\bar{p}_0^+$, this means that the vacuum itself is BRST-exact; a wrong statement. Hence, we can conclude that $x_0^m \mathbf{1}$ is a non-trivial physical operator. Non-triviality of the operators $C_0^+ x_0^m \mathbf{1}$ can be shown in the same way. This time, we use $B_0^+$ as an ‘annihilation’ operator. If $C_0^+ x_0^m \mathbf{1}$ is BRST-exact, $x_0^m \mathbf{1}$ would be also written as a BRST-exact one by applying $B_0^+$ to it, which is inconsistent with the previous result.

Thus, we have shown that the physical states $x_0^m \mathbf{1}$ and $C_0 x_0^m |0\rangle$ of the topological sigma model give rise to the physical operators of the $c = 1$ string through the bosonization (2.3). The structure of the physical spectrum realized in the $c = 1$ Fock space is depicted in Fig.1. However, these are not enough to cover all the physical spectrum of the $c = 1$ string. Both the operators $x_0^m \mathbf{1}$ and $C_0^+ x_0^m \mathbf{1}$ have momentum such as $\exp(inX^+)$, while there also exist physical states with momentum $\exp(imX^+ + inX^-)$. How can we explain these states from the point of view of the topological model?

In order to answer this question, let us remember that a bosonized model provides us with much larger Fock space than the original one has. We can realize several representations with distinct vacua within this enlarged Fock space. Since the $c = 1$ string is a bosonization for the topological sigma model, the $c = 1$ Fock space also admits representations other than we already considered, e.g., picture-changed one. As we shall show in the following, it is this picture-changed representation that generates the rest of the physical states in the $c = 1$ string. So, we turn to the determination of the physical spectrum of the topological sigma model in the picture-changed sector.

The picture-changed Fock space $\mathcal{F}_\theta$ is constructed on the picture-changed vacuum $|\theta\rangle$
with bosonic and fermionic sea level shifted by $\theta$

\[
B_n |\theta\rangle = 0 \quad \text{for} \quad n \geq \theta, \quad x_n |\theta\rangle = 0 \quad \text{for} \quad n \geq 1 + \theta, \quad (4.8)
\]

\[
C_n |\theta\rangle = 0 \quad \text{for} \quad n \geq 1 - \theta, \quad \bar{p}_n |\theta\rangle = 0 \quad \text{for} \quad n \geq -\theta.
\]

The BRST operator $Q$ (4.4) acts on this vacuum as

\[
Q |\theta\rangle = -\theta C_{-\theta} x_0 |\theta\rangle. \quad (4.9)
\]

The vacuum $|\theta\rangle$ is not physical unless $\theta = 0$, i.e., the canonical sector which we treated before. However, if we saturate this vacuum with $C_{-\theta}$, we obtain a physical state

\[
QC_{-\theta}|\theta\rangle = -C_{-\theta}Q|\theta\rangle = \theta C_{-\theta}^2 x_0 |\theta\rangle = 0. \quad (4.10)
\]

This state is a picture-changed analogue of the ‘up’ vacuum in the fermionic $BC$-system and we denote it as $|\theta, \uparrow\rangle = C_{-\theta}|\theta\rangle$. One can construct the physical spectrum in the picture-changed sector by applying the physical modes on this ‘up’ vacuum $|\theta, \uparrow\rangle$.

The structure of the spectrum changes drastically whether $\theta$ is an integer or not. First, we consider the case of fractional $\theta$. BRST transformation property of the modes is the same as in the canonical sector $\theta = 0$ (4.3). However, the suffix $n$ of the modes does not take integer value if $\theta$ is fractional. All the modes therefore form BRST doublets to leave no physical modes, since decoupling of the doublets occurs only when $n = 0$. Hence, we have only one physical state $|\theta, \uparrow\rangle$ in these sectors. Next, we treat the case of integer $\theta$. This time, there exist physical modes $B_0, C_0, x_0$ and $\bar{p}_0$, and we can construct physical spectrum by acting these modes on the ‘up’ vacuum $|\theta, \uparrow\rangle$. However, two of them annihilate the ‘up’ vacuum according to the sign of $\theta$. From eq.(4.8) it follows that

\[
C_0 |\theta, \uparrow\rangle = \bar{p}_0 |\theta, \uparrow\rangle = 0 \quad \text{for} \quad \theta \geq 0, \\
B_0 |\theta, \uparrow\rangle = x_0 |\theta, \uparrow\rangle = 0 \quad \text{for} \quad \theta < 0. \quad (4.11)
\]

Therefore, the physical spectrum in the picture-changed sector for integer $\theta$ consists of the following states

\[
x^n_0 |\theta, \uparrow\rangle \quad \text{and} \quad B_0 x^n_0 |\theta, \uparrow\rangle \quad \text{for} \quad \theta = 0, 1, 2, \cdots \\
\bar{p}^n_0 |\theta, \uparrow\rangle \quad \text{and} \quad C_0 \bar{p}^n_0 |\theta, \uparrow\rangle \quad \text{for} \quad \theta = -1, -2, \cdots. \quad (4.12)
\]

Let us see how these states are expressed in the $c = 1$ Fock space through the bosonization (2.3). The picture-changed vacuum $|\theta\rangle$ is realized as a vertex operator which we denote $1^\theta$.

\[
1^\theta = e^{-i\theta X^-}. \quad (4.13)
\]
One can easily verify that this operator has the required property (4.8) as the picture-changed vacuum. Picture-changed Fock space $\mathcal{F}_\theta$ is constructed on this vacuum by applying the mode operators $B^+_n, C^+_n, x_n$ and $\bar{p}^+_n$. Since these modes have momentum such as $\exp(\pm iX^+)$, momentum of the resulting operators takes the form $\exp(-i\theta X^- + inX^+)$, $n = 0, \pm 1, \pm 2, \ldots$. As was noticed in the last paragraph, $1^\theta$ is not physical except $\theta = 0$. Physical ground state, the ‘up’ vacuum $|\theta, \uparrow\rangle = C^-\theta |\theta\rangle$, gives rise to

$$C^+\theta 1^\theta = \oint dz \, z^{-\theta-1} C^+(z) 1^\theta = c e^{-iX^+ - i\theta X^-}, \quad (4.14)$$

which is nothing but the on-shell tachyon vertex operator and clearly physical in the $c = 1$ Fock space. We have shown that the physical spectrum for the topological sigma model consists of only the ‘up’ vacuum if $\theta$ is fractional. On the other hand, it is well-known that the physical states with momentum $\exp(-i\theta X^- + inX^+)$ in the $c = 1$ string exist only on the tachyon shell in the case of fractional $\theta$ (no discrete states). As was seen above, in the context of the topological sigma model, this fact is naturally explained as missing of the physical modes, $B^+_0, C^+_0, x_0$ and $\bar{p}^+_0$, in the fractional picture sector. In the case of integer $\theta$, we have physical modes and the spectrum is built on the ‘up’ vacuum $c e^{-iX^+ - i\theta X^-}$ by applying $B^+_0, C^+_0, x_0$ and $\bar{p}^+_0$. The physical states (4.12) give rise to

$$x_0^n C^+\theta 1^\theta \quad \text{and} \quad B^+_0 x_0^n C^+\theta 1^\theta \quad n = 0, 1, 2, \ldots \quad \text{for} \quad \theta = 0, 1, 2, \ldots$$
$$ (\bar{p}^+_0)^n C^+\theta 1^\theta \quad \text{and} \quad C^+_0 (\bar{p}^+_0)^n C^+\theta 1^\theta \quad n = 0, 1, 2, \ldots \quad \text{for} \quad \theta = -1, -2, \ldots. \quad (4.15)$$

We explain this structure of the $c = 1$ spectrum with some concrete examples.

**The case of $\theta = 1$**

The physical ground state is realized by one of the discrete tachyon

$$C^+_{-1} 1^{\theta=1} = c e^{-iX^+ - iX^-} = c e^{-\sqrt{2}iX}. \quad (4.16)$$

Low level states $B_0|\theta = 1, \uparrow\rangle$ and $x_0|\theta = 1, \uparrow\rangle$ turn into

$$B^+_0 C^+_{-1} 1^{\theta=1} = - (cb - i\partial X^+) e^{-iX^-},$$
$$x_0 C^+_{-1} 1^{\theta=1} = \left( c i\partial X^+ + \partial c i\partial X^+ + c i\partial^2 X^- + bc\partial c + \frac{1}{2} \partial^2 c \right) e^{-iX^-}. \quad (4.17)$$

One can readily see these are the ground ring generator, often denoted as $y$, and a level 2 discrete operator, respectively. Besides these two, there are infinite numbers of physical operators $B^+_0 x_0^n C^+_{-1} 1^{\theta=1}$ and $x_0^n C^+_{-1} 1^{\theta=1}$. Since $B^+$ lowers the $bc$-ghost number by one while
\(x\) leaves it unchanged, operators \(B_0^+ x_0^n C_{-1}^+ 1^{\theta=1}\) and \(x_0^n C_{-1}^+ 1^{\theta=1}\) have bc-ghost number 0 and 1, respectively. Non-triviality of these operators in the \(c = 1\) Fock space can be shown in the same way as in the canonical sector \(\theta = 0\). Hence, we can conclude that the physical states \(B_0 x_0^n |\theta = 1, \uparrow\rangle\) in the topological sigma model give rise to the physical operators with ghost number 0, \textit{i.e.}, ground ring elements, whereas \(x_0^n |\theta = 1, \uparrow\rangle\) correspond to the discrete operators with ghost number 1.

It is easy to extend this analysis to other sectors with positive \(\theta\). The physical ground state is realized by the on-shell tachyon vertex operator. Physical spectrum on it turns into the discrete operators in the \(c = 1\) string; ground ring elements and operators with ghost number 1.

\textbf{the case of \(\theta = -1\)}

Momentum of operators in this sector takes the form \(\exp(iX^- + inX^+\). Since these momenta correspond to the tachyon shell, it is expected that the resulting physical operators are on-shell tachyons. This can be shown explicitly as follows.

The physical ground state \(|\theta = -1, \uparrow\rangle\) turns into

\[
C_1^+ 1^{\theta=-1} = e^{iX^-+iX^+} = e^{\sqrt{2}i}
\]

which is the cosmological constant operator. Physical spectrum in this sector is again constructed by applying the physical modes on it. This time, we can use \(C_0^+\) and \(\bar{p}_0^+\) as creation modes (See eq.(4.11)). By simple calculation, one obtains

\[
(p_0^+) n C_1^+ 1^{\theta=-1} = e^{-i(n+1)X^-+iX^+},
\]

\[
C_0^+ (p_0^+) n C_1^+ 1^{\theta=-1} = \partial c e^{-i(n+2)X^-+iX^+}.
\]

The former belongs to the discrete tachyon while the latter corresponds to its counterpart in the absolute cohomology.

\textbf{the case of \(\theta = -2\)}

The \(\theta = -1\) case is, in a sense, exceptional, since its Fock space is embedded entirely within the tachyon shell. Typical case for negative \(\theta\) starts from \(\theta = -2\).

Physical ground state is a discrete tachyon

\[
C_2^+ 1^{\theta=-2} = e^{-iX^++2iX^-},
\]
and physical spectrum is constructed in the same way as \( \theta = -1; \) \((\bar{p}_0^+)^nC_2^+ \, 1^{\theta=-2}\) and \(C_0(\bar{p}_0^+)^nC_2^+ \, 1^{\theta=-2}\). BRST non-triviality of these operators can be shown by using \(B_0^+\) and \(x_0\) as annihilation operators. Therefore, these operators are physical also in the \(c = 1\) Fock space and reproduce the \(c = 1\) discrete operators with \(bc\)-ghost number 1 and 2. Explicit form of low-lying operators reads

\[
\begin{align*}
\bar{p}_0^+C_2^+ \, 1^{\theta=-2} &= -c i \partial X^+ e^{-2iX^+ + 2iX^-}, \\
C_0^+C_2^+ \, 1^{\theta=-2} &= \left( \frac{1}{2} \partial^2 c c - \partial c c i \partial X^+ \right) e^{-2iX^+ + 2iX^-},
\end{align*}
\]

which are indeed the discrete operators with momentum \(e^{-2iX^+ + 2iX^-} = e^{2\sqrt{2} \phi}\).

We can apply the same argument to other sectors with negative \(\theta\); physical states \((\bar{p}_0)^n|\theta, \uparrow\rangle\) and \(C_0(\bar{p}_0)^n|\theta, \uparrow\rangle\) in the topological sigma model turn into the \(c = 1\) discrete operators with \(bc\)-ghost number 1 and 2, respectively.

We have seen that the physical states of the topological sigma model, together with its picture-changed sector, turn into the \(c = 1\) discrete states. It is summarized in Fig.2(a) how these sectors are embedded in the \(c = 1\) Fock space. However, a half of the discrete states is still missing in this description of the \(c = 1\) string. At vanishing momentum, for example, we obtained two physical operators, the unit field \(1\) and a level 1 discrete operator \(C_0^+x_01\), while there exist four physical states in the \(c = 1\) string; one state with ghost number 0, two with 1 and one with 2. This holds also for other momenta. At each momentum, there exist, at most, two physical states (4.12) of the topological sigma model, while we have four discrete states at each momentum (except the tachyon shell, on which we have two physical states). In order to describe the physical spectrum of the \(c = 1\) string completely (i.e., including the elements of the absolute cohomology) in terms of the topological model, we have to characterize another half of the spectrum also as that of the topological model.

This problem is resolved by taking into account dual representations of the topological sigma model. Let us suppose a Fock space of the bosonic ghost system. It is well-known that one can not find the dual vacuum, i.e., a state having non-vanishing norm with the vacuum, in this Fock space itself. We need another Fock space to evaluate, for example, expectation value of operators.

It is the same situation that we encounter in our analysis of the topological sigma model. Within the Fock spaces \(\mathcal{F}_\theta\) which are constructed on the vacua \(|\theta\rangle\), it is impossible to find the dual vacua \(|\theta^*\rangle\) such that \(\langle \theta^* | \theta \rangle \neq 0\). However, we are treating this model in the bosonized
form; the \( c = 1 \) string. The \( c = 1 \) Fock space is large enough to admit the dual representation as well as the picture-changed one. The dual vacuum \( \mathbf{1}_\theta^\dagger \) conjugate to \( \mathbf{1}_\theta^\dagger = e^{-i\theta X^-} \) is realized as

\[
\mathbf{1}_\theta^\dagger = \frac{1}{2} c \partial c \, e^{-2iX^+ + i(\theta + 2)X^-} \quad (4.22)
\]
in the \( c = 1 \) Fock space. One can construct the dual spectrum on this vacuum, which is characterized as conjugate to the physical spectrum in the original sector. So, a sector dual to the \( \theta \)-picture, which is located at \( \exp(inX^+ - i\theta X^-) \), has momentum \( \exp(inX^+ + i(\theta + 2)X^-) \).

The conjugate operator to the physical ground state \( C_\theta^+ \mathbf{1}_\theta^\dagger = c \, e^{-iX^+ - i\theta X^-} \) takes the following form

\[
B_\theta^+ \mathbf{1}_\theta^\dagger = c \partial c \, e^{-iX^+ + i(\theta + 2)X^-}, \quad (4.23)
\]
which is the absolute cohomology counterpart of the on-shell tachyon. For fractional \( \theta \), we have only this operator in the spectrum. For integer \( \theta \), on the other hand, there are infinite series of the physical operators. Therefore, we have corresponding conjugate operators in the case of integer \( \theta \). From eq. (4.13), one can write them down as follows

\[
(\tilde{p}_0^\dagger)^n B^\pm_{-\theta} \mathbf{1}_\theta^\dagger \quad \text{and} \quad C^+_0 (\tilde{p}_0^\dagger)^n B^\pm_{-\theta} \mathbf{1}_\theta^\dagger \quad n = 0, 1, 2, \ldots \quad \text{for} \quad \theta = 0, 1, 2, \ldots
\]

\[
x_0^n B^\pm_{-\theta} \mathbf{1}_\theta^\dagger \quad \text{and} \quad B^+_0 x_0^n B^\pm_{-\theta} \mathbf{1}_\theta^\dagger \quad n = 0, 1, 2, \ldots \quad \text{for} \quad \theta = -1, -2, \ldots. \quad (4.24)
\]

These are just what we are seeking for. The physical spectrum (1.13) together with its dual (4.24) has the same multiplicity at each momentum as the discrete states (See Fig.2(b)). Since they are physical and mutually independent, we can conclude that they reproduce all the discrete states of the \( c = 1 \) string in the absolute cohomology. We confirm this fact by examining the explicit form of the dual operators (4.24) for several \( \theta \).

**the case of dual to \( \theta = -2 \)**

First, we take the case of \( \theta = -2 \). This sector has the same momentum \( \exp(inX^+) \) as the canonical sector constructed on the unit field \( \mathbf{1}_{\theta=0} \). The dual ground state \( B^+_2 \mathbf{1}_{\theta=-2} = c \partial c \, e^{-iX^+} \) is the partner of the discrete tachyon \( C^+_0 \mathbf{1}_{\theta=0} = c \, e^{-iX^+} \) in the absolute cohomology. Low-lying operators read

\[
B^+_0 B^+_2 \mathbf{1}_{\theta=-2} = -c i \partial X^+ + \partial c, \quad x_0 B^+_2 \mathbf{1}_{\theta=-2} = c \partial c(i \partial X^+ + i \partial X^-) + \frac{1}{2} c \partial^2 c. \quad (4.25)
\]

These are the level 1 discrete operators with the ghost number 1 and 2, respectively. Note that these operators are independent of those we obtained in the canonical sector (4.7).
Hence, together with the canonical sector, all the level 1 operators, the unit field $1$, two discrete operators with ghost number 1 and one discrete operator with ghost number 2, at the vanishing momentum are reproduced as the physical states of the topological sigma model.

**the case of dual to $\theta = -1$**

We next examine the case of $\theta = -1$. By simple calculation, the explicit form of the dual operators (4.24) for $\theta = -1$ can be written as

\[
\begin{align*}
x_0^n B_1^+ 1_{\theta=-1} &= n! c \partial_c e^{i(n-1)X^++iX^-}, \\
B_0^+ x_0^n B_1^+ 1_{\theta=-1} &= -n! c e^{inX^++iX^-},
\end{align*}
\]

(4.26)

which are the discrete tachyons missing in the $\theta = -1$ sector (4.19). We obtain all the discrete tachyons on the tachyon shell $\exp(i nX^++iX^-)$ considering both sectors; $\theta = -1$ and its dual.

**the case of dual to $\theta = 0$**

As the final example, we take the case of $\theta = 0$, which has the same momentum as the $\theta = -2$ sector. One obtains

\[
\begin{align*}
\bar{p}_0^+ B_0^+ 1_{\theta=0} &= \partial_c c i \partial X^+ e^{-2iX^++2iX^-}, \\
C_0^+ B_0^+ 1_{\theta=0} &= -\frac{1}{2} \partial^2 c \partial_c c e^{-2iX^++2iX^-}.
\end{align*}
\]

(4.27)

These are again discrete operators with ghost number 2 and 3, respectively, and reproduce all the level 1 operators at momentum $e^{-2iX^++2iX^-} = e^{2\sqrt{2}\phi}$ together with the physical operators (4.21) in the $\theta = -2$ sector.

We have shown that, through a bosonization, the physical spectrum of the topological sigma model turns into that of the $c = 1$ string by taking into account the picture-changed sectors and their dual. All the discrete states and the discrete tachyons in the $c = 1$ spectrum has been reproduced as the physical states of the topological model. Our result is summarized in Figs.2 and 3. Fig.2 shows the location of each sector in the $c = 1$ Fock space, while Fig.3 illustrates how picture-changed sectors and their dual are embedded in the $c = 1$ Fock space to reproduce the spectrum of the discrete states.

Before closing this section, we point out that our description of the $c = 1$ discrete states (4.15), (4.24) can be fit into more concise form by utilizing the ‘$-$’-fields (2.21). Similarly to
the ‘+’-fields, zero modes $B_0^-, C_0^-, y_0$ and $\bar{p}_0^-$ are physical in the $c = 1$ Fock space. However, they do not exist within the Fock space of the topological sigma model, since we adopt the ‘+’-fields (2.3) in realizing the fields of the topological model. It may be possible that these physical modes play the role of the ‘picture-changing’ operators analogous to those in the fermionic string theories [10].

In fact, we can show that $y_0$ maps the physical operators in the $\theta$ sector to those in the $\theta + 1$ sector. By straightforward calculation, one obtains

$$y_0 C_{\theta}^{+} 1^\theta = (\theta + 1) C_{-(\theta+1)}^{+} 1^{\theta+1},$$

which means that the physical ground states $C_{\theta}^{+} 1^\theta = c e^{-iX^+-i\theta X^-}$ are mapped to that in the adjacent sector by $y_0$. From this fact, it follows that the physical operators $x_0^m C_{\theta}^{+} 1^\theta$ and $B_0^+ x_0^m C_{\theta}^{+} 1^\theta$ in the $\theta$ picture turn into those in the $\theta + 1$ picture, since $y_0$ commutes with $B_0^+$ and $x_0$ up to BRST-exact term

$$\{y_0, B_0^+\} = 0,$$

$$\{y_0, x_0\} = \{Q, \oint dz \frac{1}{z^2} (-b e^{i(X^+-X^-)})\}.$$  

Similarly, $C_0^-$ relates the ordinary sector with the dual sector. The physical ground state $C_{\theta}^{+} 1^\theta$ is mapped to the dual ground state $B_{\theta+1}^{+} 1_{\theta-1}^{\theta-1}$

$$C_{\theta}^- C_{\theta}^{+} 1^\theta = \partial c e^{-iX^+-i(\theta-1)X^-} = -B_{\theta+1}^{+} 1_{\theta-1}^{\theta-1},$$

and the physical operators $x_0^m C_{\theta}^{+} 1^\theta$ and $B_0^+ x_0^m C_{\theta}^{+} 1^\theta$ in the $\theta$ picture turn into the dual operators $x_0^m B_{\theta+1}^{+} 1_{\theta-1}^{\theta-1}$ and $B_0^+ x_0^m B_{\theta+1}^{+} 1_{\theta-1}^{\theta-1}$.

Therefore, together with the result for $y_0$, all the physical operators with positive Liouville momentum are obtained starting from those in the canonical sector, $x_0^n \mathbf{1}$ and $C_0^+ x_0^n \mathbf{1}$, by successive application of the picture-changing operators $y_0$ and $C_0^-$. The expressions (4.15) and (4.24) for the discrete operators can be rewritten (modulo BRST exact terms) into the following form

$$\text{ghost number 0} \quad x_0^m y_0^n \mathbf{1},$$
$$\text{ghost number 1} \quad C_0^+ x_0^m y_0^n \mathbf{1}, \quad C_0^- x_0^m y_0^n \mathbf{1},$$
$$\text{ghost number 2} \quad C_0^+ C_0^- x_0^m y_0^n \mathbf{1},$$

for the operators with positive Liouville momentum. Another half of the spectrum is obtained by taking the dual of this. So, we have expressions

$$\text{ghost number 3} \quad (\bar{p}_0^+)^m (\bar{p}_0^-)^n \mathbf{1}_s,$$
ghost number 2 \[ B_0^+ (\bar{p}_0^+)^m (\bar{p}_0^-)^n \mathbf{1}_s , \ B_0^- (\bar{p}_0^+)^m (\bar{p}_0^-)^n \mathbf{1}_s , \quad (4.33) \]

ghost number 1 \[ B_0^+ B_0^- (\bar{p}_0^+)^m (\bar{p}_0^-)^n \mathbf{1}_s . \]

for those with negative Liouville momentum.

5 Discussions

We have shown that the BRST operators and the energy-momentum tensor of the \( c = 1 \) string are at the same time those of a topological sigma model in which the ground ring generator \( x \) is one of the basic fields. In a sense, the \( c = 1 \) string can be viewed as a bosonization of the topological sigma model. As a result, all the discrete states in the \( c = 1 \) string have been reproduced as physical states of the topological model by taking into account the picture-changed sectors of the latter together with their dual. Corresponding physical operators form a graded ring which contains the ground ring of ghost number zero sector as a subset.

An important problem is to clarify a nature of the topological sigma model which accounts for the discrete states in the \( c = 1 \) string.

As is well-known, the \( c = 1 \) string has a rich structure which is realized by the discrete states with several ghost numbers. The discrete operators with ghost number 0 form the ground ring [8] generated by two operators \( x \) and \( y \), while those with ghost number 1 are related to the \( w_\infty \)-currents [17]. The latter acts on the ground ring as an area-preserving diffeomorphism of the \( xy \)-plane [8]. It seems that the operators \( x \) and \( y \) are on the equal footing in the \( c = 1 \) string. On the other hand, our topological sigma model utilizes only \( x \) as a target space coordinate. The operator \( y \) does not appear in the topological model itself but is in one of the picture-changed sectors and works as a picture-changing operator as is pointed out in Section 4. \( x \) and \( y \) thus play distinct roles from the point of view of the topological sigma model. The status of \( y \) should be figured out in order to understand the meaning of the topological model.

As is mentioned above, the \( w_\infty \)-currents generate an area-preserving diffeomorphism of the \( xy \)-plane, which is a symmetry of the \( c = 1 \) string. If we consider all the elements of the absolute cohomology we get the full diffeomorphism instead of the area-preserving one. Then a natural question arises; what in the topological theory originates such a diffeomorphism? In order to answer to this question, remember the gauge fixing of the topological model adopted
in Section 3. Before fixing the gauge, the model possesses large symmetry $\delta x = \epsilon$ where $\epsilon$ is an arbitrary function. Our gauge fixing kills all the degrees of freedom but zero mode. So we have arbitrary transformation of $x_0$ as a residual symmetry. Apparently such transformation could depend on $x_0$ itself. Thus the model possesses diffeomorphism symmetry of $x_0$:

$$\delta x_0 = \epsilon(x_0).$$  \hspace{1cm} (5.1)

Since the operator $x_0$ maps a physical state to a physical one, the diffeomorphism of $x$ causes mixing of the physical states. The operator $y_0$ similarly maps a physical state to that in the adjacent picture. Therefore we can formally say that the diffeomorphism of $y$ induces mixing of the physical states among different pictures. However, geometrical meaning of this symmetry in the topological model is rather obscure in contrast with the case of $x$.

Lastly, we comment on the relation of the topological sigma model to the vacuum of string theories. At the beginning of this paper, we argued that the ‘discrete state’ in the critical string can be viewed as carrying information about the background geometry of strings. Since our topological model organizes all the discrete states in the $c = 1$ string, it is expected that the topological sigma model is also related to the background of strings. In fact, appearance of discrete states is not restricted to the $c = 1$ case but common feature in string theories with two-dimensional target space, e.g., black hole [18] and $N = 1$ fermionic case [19]. It may be possible that discrete states in these cases are also governed by a topological model. If we can identify these topological models for several backgrounds with that for the $c = 1$ string, our topological model is, in a sense, universal, and could be regarded as parametrizing the background of two-dimensional string theories. The (second-quantized) wave function of the topological sigma model may play an important role in such a description of strings.
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Figure captions

**Fig.1** Physical spectrum of the topological sigma model realized in the $c = 1$ Fock space. The horizontal axis represents momentum conjugate to $X^+$ and the vertical axis does $bc$-ghost number.

**Fig.2** Various sectors of the topological sigma model realized in the $c = 1$ Fock space: (a) picture-changed sectors, (b) dual sectors. The solid lines represent the tachyon shell. Each dot corresponds to the location of the physical states of the topological model and reproduces all the discrete states in the $c = 1$ string.

**Fig.3** The structure of the physical spectrum of the $c = 1$ string in terms of the topological sigma model. The vertical axis shows the $bc$-ghost number. The horizontal direction stands for the momentum conjugate to $X^+$ and corresponds to the solid lines in Fig.2. Each pattern similar to Fig.1 represents a spectrum built on one of the vacua. Exactly the same multiplicity of the discrete states in the $c = 1$ string is reproduced.
Fig. 1
Fig. 2

(a)

(b)
(a) the case of $\theta \geq 0$

(b) the case of $\theta = -1$

(c) the case of $\theta < 0$

Fig. 3