HÖLDER CONTINUITY OF THE MINIMIZER OF AN OBSTACLE PROBLEM WITH GENERALIZED ORLICZ GROWTH

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ABSTRACT. We prove local $C^{0,\alpha}$- and $C^{1,\alpha}$-regularity for the local solution to an obstacle problem with non-standard growth. These results cover as special cases standard, variable exponent, double phase and Orlicz growth.

1. INTRODUCTION

The classical obstacle problem is motivated by the description of the equilibrium position of an elastic membrane lying above an obstacle. Its mathematical interpretation is to find minimizers of the elastic energy functional with the addition of a constraint that presents the obstacle. This model leads to the mathematical objects called variational inequalities. In this regard, obstacle problem is deeply related to the study of the calculus of variation and the partial differential equation. It arises in broad applications, such as the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control, and financial mathematics.

The fundamental problem that appears with the study of the obstacle problem is to find the optimal regularity of minimizers. The primary model we have in mind is the non-autonomous minimization problem

\begin{equation}
\min_u \left\{ \int_{\Omega} F(x, \nabla u) \, dx : u \geq \psi \text{ a.e. in } \Omega, u|_{\partial \Omega} = g \right\}
\end{equation}

where $\Omega$ is a bounded open set in $\mathbb{R}^n$ and $\psi$ is the obstacle. If $u$ satisfies (1.1), we say that $u$ is the solution to the obstacle problem. In this paper, we are interested in Hölder regularity properties of solutions to the obstacle problems related to (1.1) with nonstandard growth conditions. In fact, the regularity results for such obstacle problems have been achieved by applying results and techniques developed in the research on the unconstrained case, i.e. when $\psi = -\infty$.

For the unconstrained case, there have been many extensive researches on the regularity theory, including $C^{0,\alpha}$- and $C^{1,\alpha}$-regularity, for the nonstandard growth problem such as the non-autonomous minimization problem (1.1) when $F$ satisfies $(p, q)$-growth conditions, that is, $|z|^p \lesssim F(x, z) \lesssim |z|^q$, $p < q$, starting with Marcellini’s seminal papers [39, 40]. Several model functionals mainly in relation to the Lavrentiev phenomenon were proposed by Zhikov [50] in order to describe the behavior of anisotropic materials in the framework of homogenization and nonlinear elasticity. The main feature of such functionals, including
the variable exponent functionals with
\[ F(x, z) \approx |z|^{p(x)} \text{ for } 1 < \inf_{\Omega} p(x) \leq \sup_{\Omega} p(x) < \infty, \]
and the double phase functionals with
\[ F(x, z) \approx |z|^p + a(x)|z|^q \text{ for } 1 < p \leq q < \infty \text{ and } a(\cdot) \geq 0, \]
is that those integrands \( F(x, z) \) change their ellipticity and growth properties according to the point \( x \). Starting with a higher integrability result of Zhikov \([51]\), regularity problems for minimizers of the variable exponent functionals have been vigorously studied in for instance \([1, 14, 19, 23]\) (see also references in the survey \([43]\)). For the double phase functionals, the regularity theory was developed by Baroni, Colombo and Mingione in a series of remarkable papers \([3, 4, 13]\). We also refer to \([15, 17, 27, 44, 48, 49]\) for regularity results in the various variants and borderline cases. Maximal regularity for Orlicz growth was settled by Lieberman in \([38]\) and for Hölder continuity of the solution assumptions have been relaxed in \([2]\). Other regularity results for Orlicz growth can be found for example in \([11, 12, 18]\). Furthermore, regularity properties have been recently studied for minimizers of the functionals with the generalized Orlicz growth that cover the functionals referred above, see for instance \([31, 32, 33]\).

Our first main result, Theorem 1.5, concerns local \( C^{0,\alpha} \)-continuity of the solution for some \( \alpha \in (0, 1) \). This standard Hölder continuity of solution to an obstacle problem is relevant by the side of maximal regularity results since less assumptions are needed for the functional and the obstacle. We do not require differentiability of the functional and the obstacle is assumed to be merely Hölder continuous rather than having continuous gradient. These lighter results are applied for example in study of removable sets \([8, 37]\).

In the very recent paper \([35]\), \( C^{0,\alpha} \) - and \( C^{1,\alpha} \)- regularity properties were established by Hästö-Ok for minimizers of non-autonomous functionals with sharp and general conditions, covering all the previous results. It is worth pointing out that this result assumed no gap between growth exponents \( p \) and \( q \). Instead their work is based on a carefully crafted continuity assumption (wVA1) for the \( x \) variable in \( \varphi \). This assumption turns out to capture sharp structural assumptions, which ensure regularity, in the important special cases such as double phase and variable exponent cases. Following their results and ideas, we are concerned with the regularity properties for solutions of the obstacle problems under \((p, q)\)-growth condition. These results are collected in Theorem 1.6.

For the obstacle problem, it is noted that the solution inherits the regularity properties from the obstacle. In particular, for the linear obstacle problem, i.e., when \( F(x, z) \approx |z|^2 \) in \((1.1)\), it is well known that the solution has the same regularity as the obstacle \( \psi \), see for instance \([5, 7]\). However this is not usually permitted in the nonlinear cases. Hence extensive research in this direction has been done into the regularity of solutions to the nonlinear obstacle problems. As the first result for the nonlinear functional with standard growth, i.e. when \( F(x, z) \approx |z|^p \) with \( 1 < p < \infty \), Michael-Ziemer \([42]\) proved that the solution is Hölder continuous when the obstacle \( \psi \) is Hölder continuous. Choe \([9]\) established the same result for the gradient of solutions when the gradient of the obstacle \( \psi \) is Hölder continuous. We further refer to \([10, 24, 25, 45, 47]\) for the Hölder regularity results on the nonlinear obstacle problems related to \( p \)-Laplace type functionals and more general functionals. Concerning with the nonstandard growth cases, Hölder type regularity results for obstacle problems with \( p(x) \)-growth were obtained in several papers for instance \([20, 21, 22]\). The \( L^{p(\cdot)} \log L \)-growth case, which can be regarded as a borderline case lying between Orlicz growth and variable exponent growth, was considered in the recent paper \([46]\). Hölder regularity results for the double phase case have been studied for example in \([8]\).
Obstacle problems for functionals having \((p, q)\) growth have been studied in for example \([6, 16]\) with different assumptions. We do not assume any gap between exponents \(p\) and \(q\) or \(C^2\)-regularity of our functional, but instead work with the \((wVA1)\) condition. Results of this paper cover all the previous results in the nonstandard growth cases mentioned above. We would like to point out that our results are new in many borderline cases including the double phase case.

Let us then present our main results. We fix \(\Omega\) to be a bounded open set in \(\mathbb{R}^n\) and consider local minimizers of the functional

\[
(1.2) \quad u \mapsto \int_{\Omega} \varphi(x, |\nabla u|) \, dx
\]

in the class

\[
\mathcal{K}_\psi^\varphi(\Omega) := \{ u \in W^{1,\varphi}(\Omega) \mid u \geq \psi \text{ a.e. in } \Omega \}
\]

for a fixed function \(\psi : \Omega \to [-\infty, \infty)\) which is called the obstacle. We define the related solution of the obstacle problem in the following definition. We note that a solution is also a minimizer and the reverse implication requires differentiability for the \(\Phi\)-function (cf. [30, Theorem 7.6]).

**Definition 1.3.** We say that a function \(u \in \mathcal{K}_\psi^\varphi(\Omega)\) is a solution to the \(\mathcal{K}_\psi^\varphi(\Omega)\)-obstacle problem if it satisfies

\[
\int_{\Omega'} \varphi(x, |\nabla u|) \, dx \leq \int_{\Omega'} \varphi(x, |\nabla w|) \, dx,
\]

where \(u, w \in \mathcal{K}_\psi^\varphi(\Omega) := \{ v \in W^{1,\varphi}(\Omega) \mid v \geq \psi \text{ a.e. in } \Omega \}\) and \(\Omega' \subseteq \Omega\). If \(\varphi \in \Phi_w(\Omega) \cap C^1([0, \infty))\) we say that a function \(u \in \mathcal{K}_\psi^\varphi(\Omega)\) is a solution to the \(\mathcal{K}_\psi^\varphi(\Omega)\)-obstacle problem if it satisfies

\[
\int_{\Omega} \left( \frac{\partial_t \varphi(x, |\nabla u|)}{|\nabla u|} \right) \cdot \nabla (\eta - u) \, dx \geq 0
\]

for all \(\eta \in \mathcal{K}_\psi^\varphi(\Omega)\) with \(\text{supp}(\eta - u) \subseteq \Omega\), which is equivalent to

\[
(1.4) \quad \int_{\Omega} \left( \frac{\partial_t \varphi(x, |\nabla u|)}{|\nabla u|} \right) \cdot \nabla \eta \, dx \geq 0
\]

for all \(\eta \in W^{1,\varphi}(\Omega)\) with a compact support and \(\eta \geq \psi - u\) a.e. in \(\Omega\).

For simplicity, let us define \(\partial_u \varphi = \partial_u \varphi(x, t) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n\) by

\[
\partial_u \varphi(x, t) = \frac{\partial \varphi(x, |t|)}{|t|} t.
\]

Recall that a function \(u \in W^{1,\varphi}_{\text{loc}}(\Omega)\) is called a (local) minimizer of \((1.2)\) if for every open \(\Omega' \subseteq \Omega\) and every \(v \in W^{1,\varphi}(\Omega')\) with \(u - v \in W^{1,\varphi}_{\text{loc}}(\Omega')\) we have

\[
\int_{\Omega'} \varphi(x, |\nabla u|) \, dx \leq \int_{\Omega'} \varphi(x, |\nabla v|) \, dx.
\]

We note that for \(g \in W^{1,\varphi}(\Omega)\) with \(g \geq \psi\) on \(\partial \Omega\), the minimizer of the functional

\[
u \in \{ w \in \mathcal{K}_\psi^\varphi(\Omega) : w = g \text{ on } \partial \Omega \} \mapsto \int_{\Omega} \varphi(x, |\nabla u|) \, dx
\]

is the solution to the obstacle problem of \(\mathcal{K}_\psi^\varphi(\Omega)\) with \(u = g\) on \(\partial \Omega\).

Our first result concerns the local Hölder continuity of the solution. This result does not assume differentiability of our \(\Phi\)-function or \((wVA1)\). Instead, \((A1)\) condition is enough combined with local Hölder continuity of the obstacle function.
Theorem 1.5. Let $\Omega \subset \mathbb{R}^n$ be a domain and $\varphi \in \Phi_w(\Omega)$ satisfy (aInc), (aDec), (A0), and (A1). Let $u$ be a solution to the $K^\psi_\omega(\Omega)$-obstacle problem. Suppose that the obstacle $\psi \in C^{0,\beta}_{\text{loc}}(\Omega)$ for some $\beta \in (0, 1)$. Then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0, 1)$.

The proof is based on constructing classical Harnack’s inequality. First, a supremum estimate of the solution is proved via a use of Caccioppoli type energy estimate and results in [32]. Compared to previous work, we need to take care of an integral average term to match the supremum estimate with the corresponding infimum estimate.

The following second theorem studies maximal regularity of the solution. Here we need to strengthen our assumptions to include $C^{1,\alpha}$-regularity for modulus of continuity.

Theorem 1.6. Let $\varphi \in \Phi_w(\Omega)$ and $\varphi(x, \cdot) \in C^1([0, \infty))$ for any $x \in \Omega$ with $\partial_t \varphi$ satisfying (A0), (Inc)$_{p-1}$, (Dec)$_{q-1}$ for some $1 < p \leq q$. Let $u \in K^\psi_\omega(\Omega)$ be a solution to the $K^\psi_\omega(\Omega)$-obstacle problem and suppose $\psi \in C^{1,\beta}_{\text{loc}}(\Omega)$ for some $\beta \in (0, 1)$.

(i) If $\varphi$ satisfies (wVA1), then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0, 1)$.

(ii) If $\varphi$ satisfies (wVA1) with

$$\omega(r) \leq r^\delta$$

for all $r \in (0, 1]$ and for some $\delta > 0$,

then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0, 1)$.

For proving results in Theorem 1.6, we first obtain the higher integrability of the gradient of solutions to the obstacle problem which implies the reverse Hölder type inequality. Taking into account the regularized Orlicz function $\hat{\varphi}$, which was constructed by [35], we derive comparison estimates for the gradients of solutions to the $K^\hat{\varphi}_\omega(\Omega)$-obstacle problem and to $\hat{\varphi}$-Laplacian equations. The proofs conclude with classical iteration arguments.

2. Preliminaries

2.1. Properties of generalized $\Phi$-functions. For an integrable function $g : U \subset \mathbb{R}^n \to \mathbb{R}$, we define the average of $g$ in $U$ by

$$\bar{g}_U := \frac{1}{|U|} \int_U g \, dx.$$ 

Generalized $\Phi$-functions. In this section we introduce the basic notations, definitions and assumptions for our growth rate. By $\Omega \subset \mathbb{R}^n$ we denote a bounded domain, i.e. an open and connected. If for an open set $\Omega'$ we have $\overline{\Omega'} \subset \Omega$, we denote it simply as $\Omega' \subset \Omega$. By $Q_r$ we mean a cube with side length $2r$ and by $B_r$ a ball with radius $r$.

A function $f$ is said to be almost increasing if there exists $L \geq 1$ such that $f(s) \leq Lf(t)$ for all $s \leq t$. Almost decreasing is defined analogously. By increasing we mean that the inequality holds for $L = 1$ (some call this non-decreasing), similarly for decreasing.

Definition 2.1. We say that $\varphi : \Omega \times [0, \infty) \to [0, \infty]$ is a (generalized) $\Phi$-prefunction if the following hold:

(i) The function $x \mapsto \varphi(x, t)$ is measurable for every $t \in [0, \infty)$.

(ii) The function $t \mapsto \varphi(x, t)$ is non-decreasing for every $x \in \Omega$.

(iii) $\lim_{t \to 0^+} \varphi(x, t) = \varphi(x, 0) = 0$ and $\lim_{t \to \infty} \varphi(x, t) = \infty$ for every $x \in \Omega$. 
A $\Phi$-prefunction $\varphi$ is called a (generalized weak) $\Phi$-function, denoted by $\varphi \in \Phi_w(\Omega)$, if the function $t \mapsto \frac{\varphi(x,t)}{t}$ is almost increasing in $(0, \infty)$ for every $x \in \Omega$, and a (generalized) convex $\Phi$-function, denoted by $\varphi \in \Phi_c(\Omega)$, if the function $t \mapsto \varphi(x,t)$ is left-continuous and convex for every $x \in \Omega$. Additionally, we denote $\varphi \in \Phi_w$ or $\varphi \in \Phi_c$ if $\varphi$ is independent of the space variable $x$.

If $\varphi \in \Phi_c(\Omega)$, we note that there exists its right-derivative $\varphi' = \varphi'(x,t)$, which is non-decreasing and right continuous, satisfying

$$\varphi(x,t) = \int_0^t \varphi'(x,s) \, ds.$$ 

This derivative is also denoted by $\varphi_t$.

Now let us consider $\varphi \in \Phi_w(\Omega)$ and $\gamma > 0$. By $\varphi^{-1}(x, \cdot) : [0, \infty) \to [0, \infty]$ we denote the left-continuous inverse of $\varphi$ defined by

$$\varphi^{-1}(x,t) := \inf\{\tau \geq 0 : \varphi(x,\tau) \geq t\}.$$ 

We define some conditions on $\varphi$ which are related to regularity properties with respect to the $x$-variable and the $t$-variable. We say that $\varphi$ satisfies

(A0) if there exists $L \geq 1$ such that $\frac{1}{L} \leq \varphi^{-1}(x,1) \leq L$ for every $x \in \Omega$.

(A1) if there exists $L \geq 1$ such that for any ball $B_r \in \Omega$ with $|B_r| < 1$,

$$\varphi^+_B(t) \leq L \varphi^-_B(t) \quad \text{for all} \quad t > 0 \quad \text{with} \quad \varphi^-_B(t) \in \left[1, \frac{1}{|B_r|}\right]$$

(wVA1) if for any $\epsilon > 0$, there exists a non-decreasing continuous function $\omega = \omega_\epsilon : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \in \Omega$,

$$\varphi^+_B(t) \leq (1 + \omega(r)) \varphi^-_B(t) + \omega(r) \quad \text{for all} \quad t > 0 \quad \text{with} \quad \varphi^-_B(t) \in \left[\omega(r), \frac{1}{|B_r|^{1-\epsilon}}\right].$$

(aInc), if $t \mapsto \frac{\varphi(x,t)}{t}$ is almost increasing in $(0, \infty)$ with constant $L \geq 1$ uniformly in $x \in \Omega$.

(Inc), if $t \mapsto \frac{\varphi(x,t)}{t}$ is non-decreasing for every $x \in \Omega$.

(aDec), if $t \mapsto \frac{\varphi(x,t)}{t}$ is almost decreasing with constant $L \geq 1$ uniformly in $x \in \Omega$.

(Dec), if $t \mapsto \frac{\varphi(x,t)}{t}$ is non-increasing for every $x \in \Omega$.

We would like to explain these assumptions more informally. Firstly, (A0) condition places us in an "unweighted" space, that is, $\varphi$ is not singular with respect to the spatial variable. Secondly, conditions (A1) and (wVA1) are regularity conditions with respect to the spatial variable. The former one is a jump condition, where as the latter is a refined continuity condition. In many cases $\varepsilon$ could be equal to 0, and this would imply regularity for many special cases such as variable exponent and double phase. However, the weak form catches interesting borderline assumptions for example in double phase case. Last four conditions control the growth of the $\Phi$-function with respect to the $t$-variable. Often we want $\gamma$ in (aInc), to be strictly greater than 1 to exclude $L^1$ case and finite in (aDec), to exclude $L^\infty$ case. The "almost" part is more flexible and is invariant under equivalent $\Phi$-functions. However it allows for local exceptions in growth rate, which (Inc), and (Dec), exclude. We also note that (aDec), implies doubling $\varphi(x,2t) \leq c \varphi(x,t)$.

The notation $f_1 \lesssim f_2$ means that there exists a constant $C > 0$ such that $f_1 \leq Cf_2$. The notation $f_1 \sim f_2$ means that $f_1 \lesssim f_2 \lesssim f_1$ whereas $f_1 \simeq f_2$ means that $f_1(t/C) \leq f_2(t) \leq f_1(Ct)$ for some constant $C \geq 1$. Throughout this paper, we use these notations when the relevant constants $C$ depend on $n$ and constants in our conditions such as (aInc), (Inc), (aDec), (Dec), and (A0).
Generalized Orlicz space. Let \( \varphi \in \Phi_c(\Omega) \) and \( L^0(\Omega) \) be the set of the measurable functions on \( \Omega \). The generalized Orlicz space is defined as the set
\[
L^\varphi(\Omega) := \left\{ f \in L^0(\Omega) : \|f\|_{L^\varphi(\Omega)} \leq \infty \right\}
\]
with the (Luxemburg) norm
\[
\|f\|_{L^\varphi(\Omega)} := \inf \left\{ \lambda > 0 : g_{L^\varphi}\left(\frac{f}{\lambda}\right) \leq 1 \right\},
\]
where \( g_{L^\varphi}(g) \) is the modular of \( g \in L^0(\Omega) \) defined by
\[
g_{L^\varphi}(g) := \int_\Omega \varphi(x, |g(x)|) \, dx.
\]
A function \( f \in L^\varphi(\Omega) \) belongs to the Orlicz–Sobolev space \( W^{1,\varphi}(\Omega) \) if its weak partial derivatives \( \partial_1 f, \ldots, \partial_n f \) exist and belong to \( L^\varphi(\Omega) \) with the norm
\[
\|f\|_{W^{1,\varphi}(\Omega)} := \|f\|_{L^\varphi(\Omega)} + \sum_i \|\partial_i f\|_{L^\varphi(\Omega)}.
\]
Furthermore, we denote by \( W^{1,\varphi}_0(\Omega) \) the closure of \( C_0^\infty(\Omega) \) in \( W^{1,\varphi}(\Omega) \).

Inequalities for generalized Orlicz functions. Lastly we introduce some pointwise inequalities for generalized Orlicz functions. The classical Hölder conjugate exponent for \( \varphi \) is replaced by a conjugate \( \varphi \)-function
\[
\varphi^*(x, t) := \sup_{s \geq 0} \left( ts - \varphi(x, s) \right).
\]
From the definition it is immediate that the generalized Young’s inequality
\[
st \leq \varphi(x, s) + \varphi^*(x, t)
\]
holds for all \( s, t \geq 0 \).

The following lemmas deal with derivatives of \( \Phi \)-functions and their proofs can be found in [35, Proposition 3.8].

**Lemma 2.4.** Let \( \varphi \in \Phi_c \cap C^1([0, \infty)) \) with \( \varphi' \) satisfying (Inc)\(_p\)–1 and (Dec)\(_q\)–1 for some \( 1 < p \leq q \). Then for \( \kappa \in (0, \infty) \) and \( x, y \in \mathbb{R}^n \), the following are satisfied:
1. \( t \varphi'(t) \approx \varphi(t) \) and \( \varphi \) satisfies (Inc)\(_p\) and (Dec)\(_q\);
2. \( \varphi'(|x| + |y|)|x - y|^2 \approx \frac{\varphi'(|x|)}{|x|}x - \frac{\varphi'(|y|)}{|y|}y \cdot (x - y); \)
3. \( \varphi'(|x| + |y|)|x - y|^2 \leq \varphi(|x|) - \varphi(|y|) - \frac{\varphi'(|y|)}{|y|}y \cdot (x - y); \)
4. \( \varphi(|x - y|) \lesssim \kappa [\varphi(|x|) + \varphi(|y|)] + \kappa^{-1} \frac{\varphi'(|x| + |y|)|x + |y|}{|x| + |y|}|x - y|^2. \)

Moreover, if \( \varphi \in C^2((0, \infty)) \), then \( t \varphi''(t) \approx \varphi'(t) \) and \( \frac{\varphi'(|x| + |y|)}{|x| + |y|} \) can be replaced by \( \varphi''(|x| + |y|) \) in (2)–(4).

**Lemma 2.5** (Propositions 3.5 & 3.6 [35]). Let \( \varphi \) be a \( \Phi \)-prefunction.
1. If \( \varphi \) satisfies (aInc)\(_1\), then there exists \( \eta \in \Phi_c(\Omega) \) such that \( \varphi \approx \eta \).
2. If \( \varphi \) satisfies (aDec)\(_1\), then there exists \( \eta \in \Phi_c(\Omega) \) such that \( \varphi \approx \eta^{-1} \). Note that \( \eta^{-1}(x, \cdot) \) is concave.
(3) Let \( p, q \in (1, \infty) \). Then \( \varphi \) satisfies (alnc)\(_p\) or (aDec)\(_q\) if and only if \( \varphi^* \) satisfies (aDec)\(_{\frac{q}{p}}\) or (alnc)\(_{\frac{p}{q}}\), respectively.

(4) If \( \varphi \) satisfies (alnc)\(_p\) and (aDec)\(_q\), then for any \( s, t \geq 0 \) and \( \kappa \in (0, 1) \),
\[
\int_{Q_r} \varphi \left( x, \frac{\beta |u|}{r} \right)^s \, dx \leq \int_{Q_r} \varphi(x, |\nabla u|) \, dx + \frac{\{\nabla u \neq 0\} \cap Q_r}{|Q_r|}
\]
for any \( u \in W^{1,1}_0(Q_r) \) with \( \|\nabla u\|_{L^r} < 1 \). If additionally \( s \leq p \), then
\[
\int_{Q_r} \varphi \left( x, \beta \frac{|u - u_Q|}{r} \right) \, dx \leq \left( \int_{Q_r} \varphi(x, |\nabla u|)^{\frac{s}{r}} \, dx \right)^{\frac{1}{s}} + 1
\]
for any \( u \in W^{1,1}(Q_r) \); in the case (A1), we need that \( \|\nabla u\|_{L^s} \leq M \), and the implicit constant depends on \( M \). The average \( u_Q \) can be replaced by \( u_{\Omega} \) for some cube or ball \( Q \subset Q_r \) with \( |Q| > \mu |Q_r| \), in which case the constant depends also on \( \mu \).

2.2. Essential supremum and infimum estimates. Our first main theorem proven in Section 3 is local Hölder continuity of a solution to an obstacle problem with Hölder continuous obstacle \( \psi \). This follows the classical route of estimating supremum and infimum of minimizer \( u \) with its integral averages. We prove the essential supremum result in the context of equations to highlight that the result follows also in that case. The only difference this makes is in the proof of Caccioppoli type inequality in Proposition 2.13. For minimizers, the proof can be modified from similar result in [32].

For the following we write
\[
A(k, r) := Q_r \cap \{ u > k \}.
\]

We first recall the supremum bounds for the local minimizer \( u \) of the \( \varphi \)-energy.

Proposition 2.9 (Proposition 5.5 and Corollary 5.8 [32]). Let \( \varphi \in \Phi_w(\Omega) \) satisfy (alnc)\(_p\), (aDec)\(_q\), (A0), and (A1). Suppose that \( u \in W^{1,1}_0(\Omega) \) satisfies \( \varphi(\nabla u) \leq 1 \) for \( Q_{2r} \subset \Omega \). Suppose that \( u \) satisfies the Caccioppoli inequality
\[
\int_{A(\ell, r)} \varphi(x, |\nabla (u - \ell)|) \, dx \leq \int_{A(\ell, 2r)} \varphi(x, \frac{(u - \ell)}{r}) \, dx
\]
for any $\ell \geq 0$. Then $u_+$ is bounded and

$$\text{ess sup}_{Q_{r/2}} u_+ \lesssim \left( \int_{Q_r} u_+^q \, dx \right)^{1/q} + |u_{Q_{r/2}}| + r \tag{2.11}$$

for any $Q_r \subset \Omega$. The term $|u_{Q_{r/2}}|$ can be omitted if $u$ is non-negative.

Furthermore, if $u \in L^\infty(Q_r)$ satisfies (2.11) without the term $|u_{Q_{r/2}}|$, then

$$\text{ess sup}_{Q_{r/2}} u_+ \lesssim \left( \int_{Q_r} u_+^h \, dx \right)^{1/h} + r \tag{2.12}$$

for any $h \in (0, \infty)$.

We need similar supremum and infimum bounds for the solutions of the obstacle problem. We start with the supremum estimate and base our proof on [32, Section 5]. Therefore we only need to prove that solution to the obstacle problem satisfies the Caccioppoli type energy estimate (2.10). Note that in the case of obstacle problems, we need to restrict possible values of $u$ using the obstacle $\psi$.

**Proposition 2.13.** Let $\varphi \in \Phi^\varphi_\theta(\Omega) \cap C^1([0, \infty))$ satisfy (aDec)$_\theta$, (aLoc)$_\theta$, (A0), and (A1). Let $u \in K^\varphi_\psi(\Omega)$ be a solution to the $K^\varphi_\psi(\Omega)$-obstacle problem. Then if $\psi \in W^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$ and $\theta \in \left[ \frac{1}{2}, 1 \right)$ we have

$$\text{ess sup}_{Q_{2r}} (u - \ell)_+ \lesssim (1 - \theta)^{-4nq^2} \left[ \left( \int_{Q_r} (u - \ell)_+^q \, dx \right)^{1/q} + |(u - \ell)_{Q_{r/2}}| \right] + r \tag{2.14}$$

for any $Q_{2r} \subset \Omega$ and $\ell \geq \sup_{Q_{2r}} \psi$, provided that $q_{L^\varphi(\Omega)}(|\nabla u|) \leq 1$. The term $|(u - \ell)_{Q_{r/2}}|$ can be omitted if $u - \ell$ is non-negative.

Furthermore, if $u \in L^\infty(Q_r)$ satisfies (2.14) without the term $|(u - \ell)_{Q_{r/2}}|$, then

$$\text{ess sup}_{Q_{r/2}} (u - \ell)_+ \lesssim \left( \int_{Q_r} (u - \ell)_+^h \, dx \right)^{1/h} + r \tag{2.15}$$

for any $h \in (0, \infty)$.

**Proof.** If $v := u - \ell$ satisfies Caccioppoli inequality (2.10), then Proposition 2.9 implies the desired bound. Hence it suffices to show that $v$ satisfies (2.10) for $\ell \geq \sup_{Q_{2r}} \psi$.

Consider any cubes $Q_\sigma \subset Q_\varphi \subset Q_{2r}$. Let $k \geq 0$ and let $\tau \in C^0(\Omega)$ be a cut off function such that $0 \leq \tau \leq 1$ in $Q_\varphi$, $\tau = 1$ in $Q_\sigma$ and $|\nabla \tau| \leq \frac{c(n)}{\sigma - \varphi}$.

Since $(v - k)_+ = (u - \ell - k)_+ \leq u - \psi$, we take a test function $\eta = -(v - k)_+ \tau^q$ in (1.4) to discover that

$$\int_{A(k + \ell, \varphi)} \partial \varphi(x, |\nabla u|) \cdot \nabla (- (v - k)_+ \tau^q) \, dx$$

$$= - \int_{A(k + \ell, \varphi)} \left[ \partial \varphi(x, |\nabla u|) \cdot \nabla u \right] \tau^q \, dx - q \int_{A(k + \ell, \varphi)} \left[ \partial \varphi(x, |\nabla u|) \cdot \nabla \tau \right] (v - k)_+ \tau^{q - 1} \, dx \geq 0,$$

where $A(k + \ell, \varphi) := Q_\varphi \cap \{ u > \ell + k \}$. Then from Lemma 2.5 (5) and the fact that $\varphi^*(x, \varphi(x, t)) \approx \varphi^*(x, \varphi(x, t)/t) \approx \varphi(x, t)$ and $\varphi^*$ satisfies (aLoc)$_{\varphi^*}$, we deduce with
\( (\text{aDec}) \) that
\[
\int_{A(k+\ell,\varrho)} \varphi(x, |\nabla u|) \tau^q \, dx \lesssim \int_{A(k+\ell,\varrho)} |\partial \varphi(x, |\nabla u|)| \nabla \tau \,(v-k)_+ \tau^{q-1} \, dx
\]
\[
\leq q \int_{A(k+\ell,\varrho)} |\partial \varphi(x, |\nabla u|)| \nabla \tau \,(v-k)_+ \tau^{q-1} \, dx
\]
\[
\lesssim \kappa \int_{A(k+\ell,\varrho)} \varphi^*(x, |\partial \varphi(x, |\nabla u|)| \tau^{q-1}) \, dx + \frac{1}{\kappa^{q-1}} \int_{Q_\varrho} \varphi(x, |\nabla \tau|) \,(v-k)_+ \, dx
\]
\[
\lesssim \kappa \int_{A(k+\ell,\varrho)} \varphi(x, |\nabla u|) \tau^q \, dx + c_\kappa \int_{Q_\varrho} \varphi \left( x, \frac{(v-k)_+}{\varrho - \sigma} \right) \, dx
\]
for any \( \kappa \in (0, 1) \). Therefore, by choosing \( \kappa \) sufficiently small, we conclude
\[
\int_{Q_\varrho} \varphi(x, |\nabla (v-k)_+|) \tau^q \, dx \lesssim \int_{Q_\varrho} \varphi \left( x, \frac{(v-k)_+}{\varrho - \sigma} \right) \, dx. \quad \square
\]

Since solution to an obstacle problem is also a superminimizer, the standard arguments provide the following weak Harnack inequality, see [32, Corollary 6.4].

**Proposition 2.16.** Let \( \varphi \in \Phi_w(\Omega) \) satisfy \((\text{alnc})_p\), \((\text{aDec})_q\), \((A0)\), and \((A1)\). Suppose that \( u \in W^{1,\varphi}_{\text{loc}}(\Omega) \) is a non-negative solution of the \( K^\varphi_w(\Omega)\)-obstacle problem. Then there exists \( h_0 > 0 \) such that
\[
\left( \int_{Q_{r/2}} u^{h_0} \, dx \right)^{1/h_0} \lesssim \text{ess inf}_{Q_{r/2}} u + r
\]
when \( Q_{2r} \subset \Omega \) and \( \varrho_{L^w(Q_{2r})}(|\nabla u|) \leq 1 \).

### 3. Hölder continuity

Now we are ready to prove local Hölder continuity of the solution to an obstacle problem with Hölder continuous obstacle. Compared to higher regularity results in later sections, we assume the weaker condition \((A1)\) instead of \((\text{wVA1})\).

**Proof of Theorem 1.5.** Since \( \psi \in C^{0,\beta}_{\text{loc}}(\Omega) \) for some \( \beta \in (0, 1) \), we note that for any \( Q_{2r} \subset \Omega \) with \( r < 1 \), there exists a constant \( [\psi]_\beta > 0 \) such that
\[
|\psi(x) - \psi(y)| \leq [\psi]_\beta |x - y|^{\beta} \quad \text{for all } x, y \in Q_{2r}.
\]
From Propositions 2.13-2.16, we note that \( u \) is locally bounded in \( \Omega \). Then for \( Q_{2r} \subset \Omega \), we define
\[
\widetilde{u}(r) := \text{ess sup}_{x \in Q_{r}} u(x), \quad \underline{u}(r) := \text{ess inf}_{x \in Q_{r}} u(x), \quad \overline{\psi}(r) := \text{ess sup}_{x \in Q_{r}} \psi(x), \quad \underline{\psi}(r) := \text{ess inf}_{x \in Q_{r}} \psi(x).
\]

Next we consider two cases: \( \underline{u}(r) \geq \overline{\psi}(r) \) and \( \underline{u}(r) \leq \overline{\psi}(r) \). For the first case we can use Proposition 2.13 with \( \ell = \underline{u}(r) \). As \( u - \underline{u}(r) \) is always nonnegative in \( Q_{r} \), we can omit the average term and this yields an equivalent form of (2.15)
\[
(3.1) \quad \overline{u}(r/4) - \underline{u}(r) = \text{ess sup}_{Q_{r/4}} \left( u - \underline{u}(r) \right) \lesssim \left[ \int_{Q_{r/2}} \left( u - \underline{u}(r) \right)^h \, dx \right]^{1/h} + r
\]
for any \( h \in (0, \infty) \).
For the second case, applying Proposition 2.13 with $\ell = \overline{\psi}(r)$ and local Hölder continuity of $\psi$, we get
\[
\esssup_{Q_{\theta r/2}} (u - \overline{\psi}(r)) \leq \esssup_{Q_{\theta r/2}} (u - \psi(r)) \leq \esssup_{Q_{\theta r/2}} (u - \overline{\psi}(r)) + [\psi]_{\beta} r^\beta
\]
\[
\lesssim (1 - \theta)^{-4nq^2} \left[ \left( \int_{Q_{\theta r/2}} (u - \overline{\psi}(r))^q \, dx \right)^{1/q} + |(u - \overline{\psi}(r))_{Q_{\theta r/2}}| + r + r^\beta \right]
\]
Combining the two previous estimates we have
\[
\esssup_{Q_{\theta r/2}} (u - \overline{\psi}(r)) \lesssim (1 - \theta)^{-4nq^2} \left( \int_{Q_{\theta r/2}} (u - \overline{\psi}(r))^q \, dx \right)^{1/q} + r^\beta.
\]
Let us briefly focus on the average term. Again, using the Hölder continuity of the obstacle and the inequality $\overline{u}(r) \lesssim \overline{\psi}(r)$ combined with Hölder inequality to increase the exponent we get
\[
|(u - \overline{\psi}(r))_{Q_{\theta r/2}}| = \left| \int_{Q_{\theta r/2}} u - \overline{\psi}(r) \, dx \right| = \left| \int_{Q_{\theta r/2}} u - \psi(r) + \psi(r) - \overline{\psi}(r) \, dx \right|
\]
\[
\lesssim \left| \int_{Q_{\theta r/2}} u - \overline{\psi}(r) \, dx \right| + |\psi(r) - \overline{\psi}(r)| \leq \int_{Q_{\theta r/2}} u - \overline{\psi}(r) \, dx + [\psi]_{\beta} r^\beta
\]
\[
\lesssim \int_{Q_{\theta r/2}} (u - \overline{\psi}(r))_+ \, dx + 2[\psi]_{\beta} r^\beta \leq \int_{Q_{\theta r/2}} u - \overline{u}(r) \, dx + 2[\psi]_{\beta} r^\beta
\]
\[
\lesssim \left[ \int_{Q_{\theta r/2}} (u - \overline{u}(r))^q \, dx \right]^{1/q} + 2[\psi]_{\beta} r^\beta.
\]
Combining the two previous estimates we have
\[
\esssup_{Q_{\theta r/2}} (u - \overline{u}(r)) \lesssim (1 - \theta)^{-4nq^2} \left[ \int_{Q_{\theta r/2}} (u - \overline{u}(r))^q \, dx \right]^{1/q} + r^\beta.
\]
Therefore, performing the iteration argument in the same way as the proof of [32, Corollary 5.8], we obtain
\[
\esssup_{Q_{r/4}} (u - \overline{u}(r)) \lesssim \left[ \int_{Q_{r/2}} (u - \overline{u}(r))^h \, dx \right]^{1/h} + r^\beta
\]
for any $h \in (0, \infty)$.

Combining the inequalities (3.1)-(3.2) and choosing $h = h_0$ where $h_0$ is given in Proposition 2.16 we get
\[
\overline{u}(r/4) - \overline{u}(r) \lesssim \left[ \int_{Q_{r/2}} (u - \overline{u}(r))^{h_0} \, dx \right]^{1/h_0} + r^\beta.
\]
Since for a fixed $r$, the function $u - \overline{u}(r)$ is a nonnegative solution to the $K_{\psi - \overline{u}(r)}$-obstacle problem, Proposition 2.16 yields that
\[
\left[ \int_{Q_{r/2}} (u - \overline{u}(r))^{h_0} \, dx \right]^{1/h_0} \lesssim u(r/4) - u(r) + r.
\]
Combining this with (3.3) we arrive at
\[
\overline{u}(r/4) - \overline{u}(r) \lesssim C(u(r/4) - u(r)) + cr^\beta
\]
for some constants $C, c > 0$. Again there are two cases. First if $(C + 1)(\overline{u}(r/4) - \underline{u}(r)) \leq \overline{u}(r) - \underline{u}(r)$, then
\[
\overline{u}(r/4) - \underline{u}(r/4) \leq \overline{u}(r/4) - \underline{u}(r) \leq \frac{C}{C + 1}(\overline{u}(r) - \underline{u}(r)) + cr^\beta.
\]
On the other hand, if $(C + 1)(\overline{u}(r/4) - \underline{u}(r)) > \overline{u}(r) - \underline{u}(r)$, then
\[
\overline{u}(r/4) - \underline{u}(r/4) = \overline{u}(r/4) - \underline{u}(r) - (\overline{u}(r/4) - \underline{u}(r))
< \overline{u}(r/4) - \underline{u}(r) - \frac{1}{C + 1}(\overline{u}(r) - \underline{u}(r))
\leq \frac{C}{C + 1}(\overline{u}(r) - \underline{u}(r)).
\]
From both cases we arrive to a conclusion that
\[
\text{osc}(u, r/4) := \overline{u}(r/4) - \underline{u}(r/4) \leq \frac{C}{C + 1}(\overline{u}(r) - \underline{u}(r)) + cr^\beta
= \frac{C}{C + 1} \text{osc}(u, r) + cr^\beta.
\]

4. Hölder continuity for the gradient

From the rest of the paper we assume that $\psi \in C^{1,\beta}_{\text{loc}}(\Omega)$ for some $\beta \in (0, 1)$. Then we note that for any $\Omega' \Subset \Omega$, there exists a constant $[\nabla \psi]_\beta > 0$ such that
\[
|\nabla \psi(x) - \nabla \psi(y)| \leq |\nabla \psi|_\beta |x - y|^\beta \quad \text{for all} \quad x, y \in \Omega'.
\]

In the previous section, we proved the first result in Theorem 1.5 with cubes because of the simple proof based on the previous results, especially, supremum and infimum estimates with cubes. However, from now on we use balls instead of cubes in order to prove our second result in Theorem 1.6. It can be also proved with cubes by the same approach we use here, but seems to be more complicated because in the process of its proof it has to be mixed cubes and balls.

4.1. Higher integrability of the gradient. We start this section by introducing some higher integrability results of $\nabla u$.

**Lemma 4.1** (Higher Integrability, Theorem 1.1 [31]). Let $\varphi \in \Phi_u(\Omega)$ satisfy (A0), (A1), (alnc)$_p$ and (aDec)$_q$ with constant $L \geq 1$ and $1 < p \leq q < \infty$. If $u \in W^{1,\varphi}_{\text{loc}}(\Omega)$ is a minimizer of
\[
\int_{\Omega} \varphi(x, |\nabla u|) \, dx,
\]
then there exist $\sigma_0 = \sigma_0(n, p, q, L) > 0$ and $c_1 = c_1(n, p, q, L) \geq 1$ such that
\[
\left( \int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx \right)^{1/(1+\sigma_0)} \leq c_1 \left( \int_{B_{2r}} \varphi(x, |\nabla u|) \, dx + 1 \right)
\text{for any} \quad B_{2r} \Subset \Omega \quad \text{with} \quad \|\nabla u\|_{L^\varphi(B_{2r})} \leq 1.
\]

**Lemma 4.3** (Reverse Hölder type inequality, Lemma 4.7 [35]). Assume that $u \in W^{1,\varphi}_{\text{loc}}(\Omega)$ satisfies (4.2) for some $B_{2r} \Subset \Omega$. For every $t \in [0, 1]$ there exists $c = c(c_1, t, q) > 0$ such that
\[
\left( \int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx \right)^{1/(1+\sigma_0)} \leq c \left[ \left( \int_{B_{2r}} \varphi(x, |\nabla u|) \, dx \right)^t + 1 \right].
\]
Moreover, if \( \varphi \) satisfies (A0), (A1) and (aDec), with \( L \geq 1 \) and \( q > 1 \), and \( \| \nabla u \|_{L^p(B_{2r})} \leq 1 \), then
\[
\int_{B_r} \varphi(x, |\nabla u|) \, dx \leq \left( \int_{B_{2r}} \varphi(x, |\nabla u|)^{(1+\sigma_0)} \, dx \right)^{\frac{1}{1+\sigma_0}} c \left( \varphi_{B_{2r}} \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1 \right)
\]
for some \( c = c(\sigma_1, q, L) \).

**Lemma 4.4** (Higher Integrability for the obstacle problem). Let \( \varphi \in \Phi_{\psi}(\Omega) \cap C^1([0, \infty)) \) satisfy (A0), (A1), (aDec), and (aDec), with constant \( L \geq 1 \) and \( 1 < p \leq q < \infty \). Assume that \( u \) is a solution to the \( K_{\varphi}(\Omega) \)-obstacle problem and \( B_{2r} \subset \Omega \) with \( r > 0 \) satisfying \( q_{L^p(B_{2r})}(|\nabla u|), q_{L^p(B_{2r})}(|\nabla \psi|) \leq 1 \). Then there exists \( \sigma_0 \in (0, 1) \) such that \( \varphi(\cdot, |\nabla u|) \in L^{1+\sigma_0}(B_r) \), and for any \( \sigma \in (0, \sigma_0] \),
\[
\int_{B_{2r}} \varphi(x, |\nabla u|)^{1+\sigma} \, dx \leq c_2 \left[ \left( \int_{B_{2r}} \varphi(x, |\nabla u|) \, dx \right)^{1+\sigma} + \int_{B_{2r}} \varphi(x, |\nabla \psi|)^{1+\sigma} \, dx + 1 \right]
\]
for some \( c_2 = c_2(n, p, q, L) > 0 \).

**Proof.** Let \( \tau \in C_0^\infty(\mathbb{R}^n) \) be a cut off function such that \( 0 \leq \tau \leq 1, \tau \equiv 1 \) in \( B_r \) and \( |\nabla \tau| \leq \frac{c(n)}{r} \). Since \( \psi - \psi_{B_{2r}} - u + \bar{u}_{B_{2r}} \geq \psi - u \), we note that \( \eta := \tau^q(\psi - \psi_{B_{2r}} - u + \bar{u}_{B_{2r}}) \geq \psi - u \) in \( B_{2r} \) where \( q \) is given in (aDec). Then we take \( \eta \) as a test function in (1.4) to have that
\[
\int_{B_{2r}} \partial \varphi(x, \nabla u) \cdot \nabla \eta \, dx \geq 0,
\]
which implies that
\[
\int_{B_{2r}} [\partial \varphi(x, \nabla u) \cdot \nabla u] \tau^q \, dx \leq \int_{B_{2r}} [\partial \varphi(x, \nabla u) \cdot \nabla \psi] \tau^q \, dx
\]
\[
+ q \int_{B_{2r}} [\partial \varphi(x, \nabla u) \cdot \nabla \tau] \tau^{q-1}(\psi - \psi_{B_{2r}} - u + \bar{u}_{B_{2r}}) \, dx.
\]

Then applying Lemma 2.4 (2) and Lemma 2.5 (4)–(5),
\[
\int_{B_{2r}} \varphi(x, |\nabla u|) \, dx \leq \int_{B_{2r}} [\partial \varphi(x, \nabla u) \cdot \nabla u] \tau^q \, dx
\]
\[
\leq \int_{B_{2r}} [\partial \varphi(x, \nabla u) \cdot \nabla \psi] \tau^q \, dx
\]
\[
+ q \int_{B_{2r}} [\partial \varphi(x, \nabla u) \cdot \nabla \tau] \tau^{q-1}(\psi - \psi_{B_{2r}} - u + \bar{u}_{B_{2r}}) \, dx
\]
\[
\leq \kappa \int_{B_{2r}} \varphi^*(x, |\partial \varphi(x, \nabla u)|) \, dx + c(\kappa) \int_{B_{2r}} \varphi(x, |\nabla \psi|) \, dx
\]
\[
+ q\kappa \int_{B_{2r}} \varphi^*(x, |\partial \varphi(x, \nabla u)|) \, dx + q c(\kappa) \int_{B_{2r}} \varphi(x, \tau^{q-1}|\nabla \tau| \psi - \psi_{B_{2r}} - u + \psi_{B_{2r}}) \, dx
\]
\[
\leq \kappa \int_{B_{2r}} \varphi(x, |\nabla u|) \, dx + \int_{B_{2r}} \varphi(x, |\nabla \psi|) \, dx
\]
\[
+ \int_{B_{2r}} \varphi \left( x, \frac{|u - \bar{u}_{B_{2r}}|}{r} \right) \, dx + \int_{B_{2r}} \varphi \left( x, \frac{|\psi - \psi_{B_{2r}}|}{r} \right) \, dx.
\]
for any $\kappa \in (0, 1)$. Therefore, by choosing $\kappa$ sufficiently small, we have
\[
\int_{B_r} \varphi(x, |\nabla u|) \, dx \lesssim \int_{B_r} \varphi\left(x, \frac{|u - \overline{u}_{2r}|}{r}\right) \, dx
\]
\[
+ \int_{B_r} \varphi(x, |\nabla \psi|) \, dx + \int_{B_r} \varphi\left(x, \frac{|\psi - \overline{\psi}_{2r}|}{r}\right) \, dx.
\]

Here, Sobolev–Poincaré inequality (2.8) yields that for $1 < s \leq p$,
\[
\int_{B_r} \varphi\left(x, \frac{|u - \overline{u}_{2r}|}{r}\right) \, dx \lesssim \left(\int_{B_r} \varphi(x, |\nabla u|)^s \, dx\right)^{\frac{1}{s}} + 1,
\]
because
\[
\int_{B_r} \varphi\left(x, |\nabla u|\right)^s \, dx \lesssim \int_{B_r} \varphi\left(x, |\nabla u|\right) \, dx + 1 \lesssim 2.
\]
Similarly, we have that
\[
\int_{B_r} \varphi\left(x, \frac{|\psi - \overline{\psi}_{2r}|}{r}\right) \, dx \lesssim \left(\int_{B_r} \varphi(x, |\nabla \psi|)^s \, dx\right)^{\frac{1}{s}} + 1 \lesssim \int_{B_r} \varphi(x, |\nabla \psi|) \, dx + 1,
\]
because
\[
\int_{B_r} \varphi\left(x, |\nabla \psi|\right)^s \, dx \lesssim \int_{B_r} \varphi\left(x, |\nabla \psi|\right) \, dx + 1 \lesssim 2.
\]
Hence, we conclude that
\[
\int_{B_r} \varphi(x, |\nabla u|) \, dx \lesssim \left(\int_{B_r} \varphi(x, |\nabla u|)^s \, dx\right)^{\frac{1}{s}} + \int_{B_r} \varphi(x, |\nabla \psi|) \, dx + 1.
\]
Since $\nabla \psi \in L^\infty_{\text{loc}}(\Omega)$, by Gehring’s lemma (see [28, Theorem 6.6 and Corollary 6.1, pp. 203–204]), we obtain the desired estimates, when we denote the implicit constant by $c_2$. □

We can upgrade the higher integrability for the obstacle problem like in the case for minimizers. We obtain the reverse Hölder type inequality (4.7) from (4.5) by the same argument as in [28, Lemma 6.12, pp. 205]. The second inequality (4.8) can be proved by the same way as in the proof of Lemma 4.7 of [35].

**Lemma 4.6 (Reverse Hölder type inequality).** Under the same hypotheses as in Lemma 4.4, for every $t \in (0, 1]$, there exists $c_t = c_t(c_2, t, q) > 0$ such that

\[
\left(\int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx\right)^{1/(1+\sigma_0)} \leq c_t \left[\left(\int_{B_r} \varphi(x, |\nabla u|)^t \, dx\right)^{1/t} + \left(\int_{B_r} \varphi(x, |\nabla \psi|)^{1+\sigma_0} \, dx\right)^{1/(1+\sigma_0)} + 1\right].
\]

Additionally, if $\varphi$ satisfies (A0), (A1), (aDec)$_q$ with constant $L \geq 1$ and $q > 1$, and $\|\nabla u\|_{L^q(B_{2r})} \leq 1$, then

\[
\int_{B_r} \varphi(x, |\nabla u|) \, dx \lesssim \left(\int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx\right)^{1/(1+\sigma_0)} \leq c \left[\int_{B_{2r}} \varphi(x, |\nabla u|) \, dx + \left(\int_{B_{2r}} \varphi(x, |\nabla \psi|)^{1+\sigma_0} \, dx\right)^{1/(1+\sigma_0)} + 1\right],
\]
where $c \geq 1$ depends on $c_2, q$ and $L$.

4.2. **Comparison estimates.** Now we are ready to prove some essential estimates for obtaining higher regularity. The first step is to construct a suitable reference problem. To that end, we introduce a regularized Orlicz function $\tilde{\varphi}$ and use that for Orlicz type equations, which are known to have solutions with $C^{1,0}_{\text{loc}}$-regularity.

Assume that $\varphi \in \Phi_c(\Omega) \cap C^1([0, \infty))$ satisfies (wVA1) with $\varphi'$ satisfying (Inc)$_{p-1}$ and (Dec)$_{q-1}$ for some $1 < p \leq q$.

From (wVA1), we note that $\varphi$ satisfies a stronger version of (A1) i.e., there exist $L \geq 1$ and a non-decreasing, bounded, continuous function $\omega : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \subset \Omega$

$$\varphi^+(t) \leq L \varphi^-(t) \quad \text{for all} \quad t > 0 \quad \text{with} \quad \varphi^+(t) \in \left[ \omega(r), \frac{1}{B_r} \right].$$

We further assume that $\varphi'$ satisfies (A0) with the same constant $L \geq 1$, (Inc)$_{p-1}$ and (Dec)$_{q-1}$ for some $1 < p \leq q$.

In order to use higher integrability results we need to assume that $|\nabla u|$ is small in sense of norms. This is achieved by considering as the integration domain a small enough ball. To quantify this smallness we henceforth assume that

$$r \leq \frac{1}{2}, \quad \omega(2r) \leq \frac{1}{L}$$

and

$$|B_{2r}| \leq \min \left\{ \frac{1}{2L}, 2^{-\frac{2}{1+\alpha_0}} \left( \int_{\Omega'} \varphi(x, |\nabla u|)^{1+\alpha_0} dx \right)^{-\frac{2+\alpha_0}{\alpha_0}}, |\Omega'| \varphi(x, |\nabla \psi|)^{1+\alpha_0} dx \right\},$$

where $\Omega' \subset \Omega$ is fixed, $B_{2r} = B(x_0, 2r) \subset \Omega'$, and $\sigma_0 \in (0, 1)$ is given by (4.2). These assumptions guarantee that

$$\int_{B_{2r}} \varphi(x, |\nabla u|) dx \leq \int_{B_{2r}} \varphi(x, |\nabla u|)^{1+\sigma_0} + 1 dx$$

$$\leq |B_{2r}| \left( \int_{B_{2r}} \varphi(x, |\nabla u|)^{1+\sigma_0} dx \right)^{\frac{2+\sigma_0}{2(1+\sigma_0)}} + |B_{2r}| \leq \frac{1}{2} + \frac{1}{2} = 1.$$

We give a quick sketch and collect properties of a regularized Orlicz function $\tilde{\varphi}$ constructed from a generalized Orlicz function $\varphi$. The detailed construction and the proofs can be found in [35, Section 5].

We start by defining

$$\varphi^\pm(t) := \varphi^\pm_{B_{2r}}(t), \quad t_1 := (\varphi^-)^{-1}(\omega(2r)) \quad \text{and} \quad t_2 := (\varphi^-)^{-1}(|B_{2r}|^{-1}).$$

Note that $t_1 \leq 1 \leq t_2$ from the assumptions (4.9) and (A0). Now we can define

$$\tau_{B_{2r}}(t) := \begin{cases} a_1 \left( \frac{t}{t_1} \right)^{p-1}, & \text{if } 0 \leq t < t_1, \\ \varphi'(x_0, t) & \text{if } t_1 \leq t \leq t_2, \\ a_2 \left( \frac{t}{t_2} \right)^{p-1}, & \text{if } t_2 < t < \infty, \end{cases}$$
where $a_1 := \varphi'(x_0,t_1)$ and $a_2 := \varphi'(x_0,t_2)$ are chosen so that $\tau_{B_{2r}}$ is continuous. We continue defining

$$\varphi_{B_{2r}}(t) := \int_0^t \tau_{B_{2r}}(s) \, ds.$$  

Finally, for $\eta \in C_0^\infty(\mathbb{R})$ with $\eta \geq 0$, supp $\eta \subset (0, 1)$ and $\|\eta\|_1 = 1$, we define the regularized Orlicz function $\hat{\varphi}(0) := 0$ and

\begin{equation}
\hat{\varphi}(t) := \int_0^\infty \varphi_{B_{2r}}(t\sigma) \eta(\sigma - 1) \, d\sigma = \int_0^\infty \varphi_{B_{2r}}(s) \eta_{t}(s - t) \, ds, \text{ where } \eta_{t}(t) := \frac{1}{t} \eta\left(\frac{t}{t}\right).
\end{equation}

This function $\hat{\varphi}$ has the following properties.

**Lemma 4.14 (Lemma 5.10 [35]).** Let $\hat{\varphi}$ be the regularized Orlicz function. Then

1. $\varphi_{B_{2r}}(t) \leq \hat{\varphi}(t) \leq (1 + cr)\varphi_{B_{2r}}(t)$ for all $t > 0$ with $c > 0$ depending only on $q$, and $0 \leq \hat{\varphi}(t) - \varphi(x_0,t) \leq cr \varphi^{-}(t)$ for all $t \in [t_1, t_2]$;

2. $\hat{\varphi} \in C^1([0, \infty))$ and it satisfies (Inc)$_p$, (Dec)$_q$ and (A0), and $\varphi'$ satisfies (Inc)$_{p-1}$ and (Dec)$_{q-1}$ and (A0). In particular $\hat{\varphi}'(t) \approx t \hat{\varphi}''(t)$ for all $t > 0$;

3. $\hat{\varphi}(t) \leq c \varphi(x,t)$ for all $(x,t) \in B_{2r} \times [1, \infty)$, and so $\hat{\varphi}(t) \leq \varphi(x,t) + 1$ for all $(x,t) \in B_{2r} \times [0, \infty)$.

Here the constant $c$ and the implicit constants depend only on $n, p, q$ and $L$.

4.3. **Comparison functions.** Let us first consider the following comparison principle for $\hat{\varphi}$.

**Lemma 4.15.** Assume that $w \in W^{1,\varphi}(\Omega)$ satisfies

\[
\left\{ \begin{array}{ll}
-\text{div} \left( \frac{\varphi'(\|\nabla \psi\|)}{\|\nabla \psi\|} \nabla \psi \right) & \leq -\text{div} \left( \frac{\varphi'(\|\nabla w\|)}{\|\nabla w\|} \nabla w \right) \\
\psi & \leq w \\
\end{array} \right. \quad \text{in } B_r,
\]

in the weak sense, that is, $(\psi - w)_+ \in W^{1,\hat{\varphi}}(B_r)$ and

\[
\int_{B_r} \left( \frac{\varphi'(\|\nabla \psi\|)}{\|\nabla \psi\|} \nabla \psi - \frac{\varphi'(\|\nabla w\|)}{\|\nabla w\|} \nabla w \right) \cdot \nabla \eta \, dx \leq 0 \quad \text{for all } \eta \in W^{1,\hat{\varphi}}(B_r) \text{ with } \eta \geq 0.
\]

Then we have $\psi \leq w$ a.e. in $B_r$.

**Proof.** By taking $\eta = (\psi - w)_+$ as a test function to the above weak formulation, we see

\[
\int_{B_r \cap \{\psi > w\}} \left( \frac{\varphi'(\|\nabla \psi\|)}{\|\nabla \psi\|} \nabla \psi - \frac{\varphi'(\|\nabla w\|)}{\|\nabla w\|} \nabla w \right) \cdot (\nabla \psi - \nabla w) \, dx \leq 0.
\]
Then using first (4) and then (2) of Lemma 2.4, we obtain
\[
\int_{B_r \cap \{ \psi > w \}} \tilde{\varphi}(|\nabla \psi - \nabla w|) \, dx \\
\lesssim \kappa \int_{B_r \cap \{ \psi > w \}} \tilde{\varphi}(|\nabla \psi|) + \tilde{\varphi}(|\nabla w|) \, dx \\
+ \kappa^{-1} \int_{B_r \cap \{ \psi > w \}} \frac{\tilde{\varphi}'(|\nabla \psi| + |\nabla w|)}{|\nabla \psi| + |\nabla w|} |\nabla \psi - \nabla w|^2 \, dx \\
\lesssim \kappa \int_{B_r \cap \{ \psi > w \}} \tilde{\varphi}(|\nabla \psi|) + \tilde{\varphi}(|\nabla w|) \, dx \\
+ \kappa^{-1} \int_{B_r \cap \{ \psi > w \}} \left( \frac{\tilde{\varphi}'(|\nabla \psi|)}{|\nabla \psi|} \nabla \psi - \frac{\tilde{\varphi}'(|\nabla w|)}{|\nabla w|} \nabla w \right) \cdot (\nabla \psi - \nabla w) \, dx \\
\leq \kappa \int_{B_r \cap \{ \psi > w \}} \tilde{\varphi}(|\nabla \psi|) + \tilde{\varphi}(|\nabla w|) \, dx
\]
for any \( \kappa \in (0, 1) \). Since \( \kappa \) is arbitrary, we have that \( \psi \leq w \) a.e. in \( B_r \). \( \square \)

Next we define two equations and corresponding solutions \( w \) and \( v \) to which we compare our solution \( u \) to the obstacle problem (1.4). We will prove some energy estimates of \( w \) and \( v \) with respect to \( u \) and regularized Orlicz function \( \tilde{\varphi} \) in Lemma 4.18.

Let \( u \in K^\varphi(\Omega) \) be a solution to the \( K^\varphi(\Omega) \)-obstacle problem and \( B_{2r} \subset \Omega \) with \( r > 0 \) satisfying (4.9). We consider the minimizer \( w \in W^{1,\tilde{\varphi}}(B_r) \) of
\[
\int_{B_r} \tilde{\varphi}(|\nabla w|) \, dx \quad \text{with } w = u \text{ on } \partial B_r,
\]
and the minimizer \( v \in W^{1,\tilde{\varphi}}(B_r) \) of
\[
\int_{B_r} \tilde{\varphi}(|\nabla v|) \, dx \quad \text{with } v = w \text{ on } \partial B_r.
\]
Note that \( w \in W^{1,\tilde{\varphi}}(B_r) \) is the unique weak solution of
\[
(4.16) \quad \begin{cases}
-\text{div} \left( \frac{\tilde{\varphi}'(|\nabla w|)}{|\nabla w|} \nabla w \right) = -\text{div} \left( \frac{\tilde{\varphi}'(|\nabla \psi|)}{|\nabla \psi|} \nabla \psi \right) & \text{in } B_r, \\
w = u & \text{on } \partial B_r,
\end{cases}
\]
and \( v \in W^{1,\tilde{\varphi}}(B_r) \) is the unique weak solution of
\[
(4.17) \quad \begin{cases}
-\text{div} \left( \frac{\tilde{\varphi}'(|\nabla v|)}{|\nabla v|} \nabla v \right) = 0 & \text{in } B_r, \\
v = w & \text{on } \partial B_r.
\end{cases}
\]
From now on, we set \( \overline{|\nabla \psi|} := \sup_{\Omega} |\nabla \psi| \) for simplicity.

Lemma 4.18. For \( w \) and \( v \) defined in (4.16) and (4.17) we have the energy estimates:
\[
(4.19) \quad \int_{B_r} \tilde{\varphi}(|\nabla w|) \, dx \leq c \left( \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + \int_{B_r} \tilde{\varphi}(|\nabla \psi|) \, dx \right) \leq c \left( \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right)
\]
and
\[
(4.20) \quad \int_{B_r} \tilde{\varphi}(|\nabla v|) \, dx \leq c \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx \leq c \left( \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right),
\]
where \( c = c(n, p, q, L, \overline{|\nabla \psi|}) > 0 \).
Proof. Testing with \( w - u \in W^1_0(B_r) \) to the weak formulation, the equation \( (4.16) \) yields

\[
\int_{B_r} \frac{\varphi'(\|\nabla w\|)}{\|\nabla w\|} \nabla w \cdot \nabla w \, dx = \int_{B_r} \frac{\varphi'(\|\nabla w\|)}{\|\nabla w\|} \nabla w \cdot \nabla u + \frac{\varphi'(\|\nabla \psi\|)}{\|\nabla \psi\|} \nabla \psi \cdot \nabla (w - u) \, dx.
\]

Applying \( \varphi'(t) \approx \tilde{\varphi}(t) \) in Lemma 2.4 (1), Lemma 2.5 (4)-(5), and \( (a\text{Dec})_q \), we obtain

\[
\int_{B_r} \frac{\varphi'(\|\nabla w\|)}{\|\nabla w\|} \nabla w \cdot \nabla w \, dx \leq c \int_{B_r} \frac{\varphi'(\|\nabla w\|)}{\|\nabla w\|} |\nabla u| + \frac{\varphi'(\|\nabla \psi\|)}{\|\nabla \psi\|} (|\nabla w| + |\nabla u|) \, dx 
\leq \frac{1}{2} \int_{B_r} \tilde{\varphi}(|\nabla w|) \, dx + c \int_{B_r} \tilde{\varphi}(|\nabla u|) + \varphi(|\nabla \psi|) \, dx.
\]

Recalling Lemma 2.4 (1) again, we can move the first term on the right-hand side to the left-hand side. Finally, estimating \( |\nabla \psi| \) by \( \overline{\nabla w} \) we have proven \( (4.19) \).

The second energy estimate \( (4.20) \) follows immediately since \( v \) as a solution minimizes the corresponding energy integral and \( v \) and \( w \) have the same boundary values in the Sobolev sense. \( \square \)

As \( v \) is the solution of the \( \varphi \)-Laplacian equation, it is known to have \( C^{1,\alpha_0} \)-regularity for some \( \alpha_0 > 0 \) from [38] (see also [35, Lemma 4.12]). Additionally, for any \( B(x_0, \rho) \subset B_r \), we have

\[
(4.21) \quad \sup_{B(x_0, \rho/2)} |\nabla v| \leq c \int_{B(x_0, \rho)} |\nabla v| \, dx
\]

and for any \( \tau \in (0, 1) \)

\[
(4.22) \quad \int_{B(x_0, \tau \rho)} |\nabla v - (\nabla v)_{B(x_0, \tau \rho)}| \, dx \leq c \tau^{\alpha_0} \int_{B(x_0, \rho)} |\nabla v| \, dx,
\]

where \( \alpha_0 \in (0, 1) \) and \( c = c(n, p, q) > 0 \).

### 4.4. Calderón–Zygmund type estimates

We also need the following reverse type estimate and Calderón-Zygmund type estimate for the equation \( (4.16) \). Here we rely on similar results in [35, Lemma 4.15]. As the proofs of these results are almost identical to the original one, we give just sketches of the proofs. The second lemma is an application suitable for the obstacle problem of the more abstract Calderón–Zygmund type estimates outlined in the first lemma.

The following is Calderón-Zygmund type estimates with a ball \( B_r \) with radius \( r \).

**Lemma 4.23.** Let \( \varphi \in \Phi_c \cap C^1([0, \infty)) \cap C^2((0, \infty)) \) with \( \varphi' \) satisfying \( (\text{Inc})_{p-1} \) and \( (\text{Dec})_{q-1} \) for some \( 1 < p \leq q \), and \( |B_r| \leq 1 \). If \( w \) is a solution to \( (4.16) \), then there exists a constant \( c = c(n, p, q, p_1, B_1, L) > 0 \) such that

\[
(4.24) \quad \|\varphi(\|\nabla w\|)\|_{L^p(B_r)} \leq c \left( \|\varphi(\|\nabla \psi\|)\|_{L^p(B_r)} + \|\varphi(\|\nabla u\|)\|_{L^p(B_r)} \right)
\]

for any \( \theta \in \Phi_c(B_r) \) satisfying \( (A0), (A1), (a\text{Inc})_{p_1} \) and \( (a\text{Dec})_{B_1} \) with constant \( L \geq 1 \) and \( 1 < p_1 \leq B_1 \).

Moreover, fix \( \kappa > 0 \) and assume that \( \int_{B_r} \theta(x, \varphi(\|\nabla u\|)) + \theta(x, \varphi(\|\nabla \psi\|)) \, dx \leq \kappa \). Then

\[
(4.25) \quad \int_{B_r} \theta(x, \varphi(\|\nabla w\|)) \, dx \leq c \left( \frac{\kappa}{\kappa + \frac{1}{\pi}} + 1 \right) \left( \int_{B_r} \theta(x, \varphi(\|\nabla u\|)) + \theta(x, \varphi(\|\nabla \psi\|)) \, dx + 1 \right).
\]

Here \( c \geq 0 \) depends on \( n, p, q, p_1, B_1 \) and \( L \).
Proof. In the same way to [35, Theorem B.1 in Appendix B] (also see [41]), we obtain that for any $s \in (0, \infty)$ and Muckenhoupt weight $\mu \in A_s$,

$$\int_{\Omega} \varphi(|\nabla w|)^s \mu(x) \, dx \leq c \left( \int_{\Omega} \varphi(|\nabla \psi|)^s \mu(x) \, dx + \int_{\Omega} \varphi(|\nabla u|)^s \mu(x) \, dx \right)$$

for some constant $c = c(n, p, q, s, [\mu]_{A_s}) > 0$ but we need to replace (B.2) by

$$\text{div} \left( \frac{\varphi'(|\nabla w|)}{\sqrt[1+\sigma_0]{\varphi(1+\sigma_0)}} \nabla w \right) = \text{div} \left( \frac{\varphi'(|\nabla \psi|)}{\sqrt[1+\sigma_0]{\varphi(1+\sigma_0)}} \nabla \psi \right) \quad \text{in} \quad \Omega \quad \text{with} \quad w = u \quad \text{on} \quad \partial \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a Reifenberg flat domain. In addition, to compare this equation with an equation having zero boundary values on $\partial \Omega$ in a local region near boundary, we add the one more assumption

$$\int_{\Omega} \varphi(|\nabla \psi|) \, dx \leq \delta$$

to (B.7) and continue similarly with the proofs in Lemmas B.5 and B.11 to derive (4.26). Hence (4.26) with $\Omega = B_r$ implies the desired norm inequality (4.24) via the extrapolation result for the generalized Orlicz function in [29, Corollary 5.3.4].

The modular inequality (4.25) follows by modifying the proof of [35, Lemma 4.15] slightly, that is replacing the definition of constant $M$ with

$$M := (\theta^{-1})^{-1} \left( \int_{B_r} \theta(x, \varphi(|\nabla u|)) + \theta(x, \varphi(|\nabla \psi|)) \, dx \right).$$

Then in similar fashion we obtain

$$\int_{B_r} \varphi(|\nabla u|) \, dx \leq 1 \Rightarrow \|\varphi(|\nabla u|)\|_{L^{1}(B_r)} + \|\varphi(|\nabla \psi|)\|_{L^{1}(B_r)} \leq 2$$

and the rest follows line by line. \qed

**Lemma 4.27.** We have that

$$\left( \int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx \right)^{\frac{1}{1+\sigma_0}} \lesssim \varphi \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1$$

and

$$\int_{B_r} \varphi(x, |\nabla w|) \, dx \leq \left( \int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx \right)^{\frac{2}{2+\sigma_0}} \lesssim \left( \int_{B_r} \varphi(x, |\nabla w|)^{1+\sigma_0} \, dx + 1 \right)^{\frac{2}{2+\sigma_0}},$$

where the implicit constant depends on $n, p, q, L$ and $\nabla \psi$.

**Proof.** Since $(wVA1)$ implies $(A1)$, we have from (4.8) in Lemma 4.6 that

$$\left( \int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx \right)^{\frac{1}{1+\sigma_0}} \lesssim \varphi^{-1} \left( \int_{B_{2r}} |\nabla u| \, dx \right) + \left( \int_{B_{2r}} \varphi(x, |\nabla \psi|)^{1+\sigma_0} \, dx \right)^{\frac{1}{1+\sigma_0}} + 1.$$
Using $\nabla \psi$ we can absorb the second term on the right-hand side to the implicit constant. When $\int_{B_{2r}} |\nabla u| \, dx \leq 1$, we obtain (4.28) by (A0). In the other case that $\int_{B_{2r}} |\nabla u| \, dx > 1$, we have

$$1 < \int_{B_{2r}} |\nabla u| \, dx \leq (\varphi^{-1})^{-1} (\int_{B_{2r}} \varphi^{-1}(|\nabla u|) \, dx) \leq (\varphi^{-1})^{-1} (|B_{2r}|^{-1}) = t_2,$$

where $t_2$ is defined in (4.11). Thus, Lemma 4.14 (1) yields that

$$\left(\int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx\right)^{1/(1+\sigma_0)} \lesssim \varphi^{-1} \left(\int_{B_{2r}} |\nabla u| \, dx\right) + 1 \lesssim \varphi \left(x_0, \int_{B_{2r}} |\nabla u| \, dx\right) + 1 \lesssim \tilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx\right) + 1,$$

which is (4.28).

In order to prove (4.29), let us consider the function $\theta \in \Phi_w(B_r)$ which is given in

$$\theta(x, t) := [\varphi(x, \varphi^{-1}(t))]^{1+\frac{\sigma_0}{2}}$$

for any fixed $\sigma_0 \in (0, 1)$. Applying the Calderón-Zygmund estimates (4.25) and local boundedness of $|\nabla \psi|$ we have that

$$\int_{B_r} \varphi(x, |\nabla w|)^{1+\frac{\sigma_0}{2}} \, dx = \int_{B_r} \theta(x, \varphi(|\nabla w|)) \, dx \lesssim \int_{B_r} \theta(x, \varphi(|\nabla u|)) \, dx + \int_{B_r} \theta(x, \varphi(|\nabla \psi|)) \, dx + 1 = \int_{B_r} \varphi(x, |\nabla u|)^{1+\frac{\sigma_0}{2}} \, dx + \int_{B_r} \varphi(x, |\nabla \psi|)^{1+\frac{\sigma_0}{2}} \, dx + 1 \lesssim \int_{B_r} \varphi(x, |\nabla u|)^{1+\frac{\sigma_0}{2}} \, dx + 1.$$

The desired inequality (4.29) follows by taking the $\frac{2}{2+\sigma_0}$-th root from both sides and applying Hölder’s inequality on the left hand side. \hfill $\Box$

4.5. Comparison estimates. Now we are ready to prove comparison estimates between different solutions. These are the main ingredients for the proof of Theorem 1.6. Much of the proof is similar to the case without any obstacle and these omitted details can be found in [35].

**Lemma 4.30.** Suppose that $w$ is a solution to the equation (4.16), $u$ is a solution to the equation (1.4) and $B_{2r} \subseteq \Omega$. Then

$$\int_{B_r} |\nabla u - \nabla w| \, dx \leq c(\omega(2r)^{p/q} + r^\gamma + r^\beta)^{\frac{1}{2r}} \left(\int_{B_{2r}} |\nabla u| \, dx + 1\right),$$

for some $\gamma = (n, p, q, L) \in (0, 1)$ and some constant $c = c(n, p, q, L, |\nabla \psi|, \varphi, \nabla \psi) > 0$, where $\omega$ is from the assumption (wVA1).

**Proof.** From (4.16), we see that

$$\int_{B_r} \frac{\varphi'(|\nabla w|)}{|\nabla w|} \nabla w \cdot (\nabla w - \nabla u) \, dx = \int_{B_r} \frac{\varphi'(|\nabla \psi|)}{|\nabla \psi|} \nabla \psi \cdot (\nabla w - \nabla u) \, dx,$$

by taking $w - u \in W^{1, \varphi}_0(B_r)$ as a test function to the weak formulation.
Recalling Lemma 4.15 and setting \( w = u \) in \( \Omega 
 B_r \), we have that \( w \in W^{1,\varphi}(\Omega) \) and \( w \geq \psi \) a.e. in \( \Omega \), and so \( w \in \mathcal{K}_{\psi}^*(\Omega) \). Therefore by taking \( \eta := w - u \) as a test function in (1.4),

\[
(4.32) \quad \int_{B_r} \frac{\partial_t \varphi(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot (\nabla w - \nabla u) \, dx \geq 0.
\]

From Lemma 2.4 (3) and (4.31), we then derive that

\[
\int_{B_r} \varphi''(|\nabla u| + |\nabla w|)|\nabla u - \nabla w|^2 \, dx
\]

\[
\leq \int_{B_r} \varphi(|\nabla u|) - \varphi(|\nabla w|) - \frac{\varphi'(|\nabla w|)}{|\nabla w|} \nabla w \cdot (\nabla u - \nabla w) \, dx
\]

\[
= \int_{B_r} \varphi(|\nabla u|) - \varphi(|\nabla w|) \, dx + \int_{B_r} \varphi'(|\nabla w|) \nabla w \cdot (\nabla w - \nabla u) \, dx
\]

\[
= \int_{B_r} \varphi(|\nabla u|) - \varphi(x, |\nabla u|) \, dx + \int_{B_r} \varphi(x, |\nabla u|) - \varphi(x, |\nabla w|) \, dx
\]

\[
+ \int_{B_r} \varphi(x, |\nabla w|) - \varphi(|\nabla w|) \, dx + \int_{B_r} \varphi'(|\nabla w|) \nabla w \cdot (\nabla w - \nabla u) \, dx.
\]

Here, applying Lemma 2.4 (3) pointwise again with (4.32), we see that

\[
\int_{B_r} \varphi(x, |\nabla u|) - \varphi(x, |\nabla w|) \, dx
\]

\[
\leq \int_{B_r} \frac{\partial_t \varphi(x, |\nabla u|)}{|\nabla w|} \nabla u \cdot (\nabla u - \nabla w) \, dx - \int_{B_r} \frac{\partial_t \varphi(x, |\nabla u| + |\nabla w|)}{|\nabla u| + |\nabla w|} |\nabla u - \nabla w|^2 \, dx
\]

\[
\leq - \int_{B_r} \frac{\partial_t \varphi(x, |\nabla u| + |\nabla w|)}{|\nabla u| + |\nabla w|} |\nabla u - \nabla w|^2 \, dx \leq 0
\]

since \( \varphi \) is increasing by assumption. Combining the above two inequalities, we obtain that

\[
\int_{B_r} \varphi(|\nabla u|) - \varphi(|\nabla w|) \, dx + \varphi(|\nabla w|) - \varphi(|\nabla w|) \, dx
\]

\[
+ \int_{B_r} \varphi'(|\nabla w|) \nabla w \cdot (\nabla w - \nabla u) \, dx.
\]

On the other hand, using Lemma 2.4 (4) and (4.19), we infer that

\[
\int_{B_r} \varphi(|\nabla u| - \nabla w|) \, dx
\]

\[
\leq K \int_{B_r} \varphi(|\nabla u| + |\nabla w|) \, dx + \kappa^{-1} \int_{B_r} \varphi''(|\nabla u| + |\nabla w|)|\nabla u - \nabla w|^2 \, dx
\]

\[
\leq K \left( \int_{B_r} \varphi(|\nabla u|) \, dx + 1 \right) + \kappa^{-1}(I_1 + I_2 + I_3)
\]
for any $\kappa \in (0, \infty)$. Estimating $I_2$ the same way as in the proof of [35, Lemma 6.2] we obtain
\[
|I_2| \lesssim \int_{B_r} |\varphi(x, |\nabla w|) - \tilde{\varphi}(|\nabla w|)| \, dx \\
\lesssim (\omega(2r)^{p/q} + r + r^{\frac{n\sigma_0}{4(2+\sigma_0)}}) \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1 \\
\lesssim (\omega(2r)^{p/q} + r) \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1
\]
where $\gamma = \min\{1, \frac{n\sigma_0}{4(2+\sigma_0)}\}$. Note that this is the only place, where the assumption (wVA1) is needed. The estimate for $I_1$ is analogous to $I_2$ with $\nabla u$ instead of $\nabla w$.

We continue with estimating $I_3$. We start by calculating
\[
\partial \tilde{\varphi}(|x|) - \partial \tilde{\varphi}(y) = \int_0^1 \frac{d}{dt} \partial \tilde{\varphi}(tx + (1-t)y) \, dt \lesssim |x - y| \int_0^1 \tilde{\varphi}''(|x| + |y|) \, dt \\
\approx |x - y| \tilde{\varphi}''(|x| + |y|) \frac{|x| + |y|}{|x| + |y|}.
\]
Identifying $x = \nabla \psi$ and $y = \nabla \psi(x_0)$, where $x_0$ is the center of $B_r$, we have
\[
|I_3| = \left| \int_{B_r} \frac{\tilde{\varphi}'(|\nabla \psi|)}{|\nabla \psi|} \nabla \psi \cdot (\nabla w - \nabla u) \, dx \right| \\
= \left| \int_{B_r} \left[ \frac{\tilde{\varphi}'(|\nabla \psi|)}{|\nabla \psi|} \nabla \psi - \frac{\tilde{\varphi}'(|\nabla \psi(x_0)|)}{|\nabla \psi(x_0)|} \nabla \psi(x_0) \right] \cdot (\nabla w - \nabla u) \, dx \right| \\
\lesssim \int_{B_r} \frac{\tilde{\varphi}'(|\nabla \psi| + |\nabla \psi(x_0)|)}{|\nabla \psi| + |\nabla \psi(x_0)|} |\nabla \psi - \nabla \psi(x_0)| |\nabla w - \nabla u| \, dx.
\]
Since $\tilde{\varphi}'$ satisfies (Inc)$_{p-1}$, we note that if $p \geq 2$, then $\tilde{\varphi}'(t)/t$ is non-decreasing for $t \in (0, \infty)$. Then
\[
|I_3| \lesssim \int_{B_r} \frac{\tilde{\varphi}'(|\nabla \psi| + |\nabla \psi(x_0)|)}{|\nabla \psi| + |\nabla \psi(x_0)|} |\nabla \psi - \nabla \psi(x_0)| |\nabla w - \nabla u| \, dx \\
\lesssim r^\beta \int_{B_r} |\nabla u| + |\nabla w| \, dx \lesssim r^\beta \left( \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right).
\]
If $p < 2$, we deduce that
\[
|I_3| \lesssim \int_{B_r} \frac{\tilde{\varphi}'(|\nabla \psi| + |\nabla \psi(x_0)|)}{|\nabla \psi| + |\nabla \psi(x_0)|} |\nabla \psi - \nabla \psi(x_0)| |\nabla w - \nabla u| \, dx \\
\lesssim \int_{B_r} \frac{\tilde{\varphi}'(|\nabla \psi| + |\nabla \psi(x_0)|)}{|\nabla \psi| + |\nabla \psi(x_0)|} (|\nabla \psi| + |\nabla \psi(x_0)|)^{2-p} |\nabla \psi - \nabla \psi(x_0)|^{p-1} |\nabla w - \nabla u| \, dx \\
= \int_{B_r} \frac{\tilde{\varphi}'(|\nabla \psi| + |\nabla \psi(x_0)|)}{|\nabla \psi| + |\nabla \psi(x_0)|}^{p-1} |\nabla \psi - \nabla \psi(x_0)|^{p-1} |\nabla w - \nabla u| \, dx \\
\lesssim r^{\beta(p-1)} \int_{B_r} |\nabla u| + |\nabla w| \, dx \lesssim r^{\beta(p-1)} \left( \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right),
\]
since $\tilde{\varphi}'(t)/t^{p-1}$ is non-decreasing for $t \in (0, \infty)$. In turn, we conclude that
\[
|I_3| \lesssim r^\beta \left( \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right),
\]
where the implicit constant depends on $n, p, q, L, |\nabla \psi|_\beta, \nabla \psi$. 
Note from (4.28) that
\[
\int_{B_r} \varphi(|\nabla u|) \, dx \lesssim \int_{B_r} \varphi(x, |\nabla u|) \, dx + 1
\]
\[
\lesssim \left( \int_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0} \, dx \right)^{1\over 1+\sigma_0} + 1 \lesssim \varphi \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1.
\]
Hence, for any \( \kappa \in (0, 1) \), we conclude that
\[
\int_{B_r} \varphi(|\nabla u - \nabla w|) \, dx 
\lesssim \kappa \left( \int_{B_r} \varphi(|\nabla u|) \, dx + 1 \right) + \kappa^{-1} (\omega(2r) + r^\gamma + r^\beta) \left[ \varphi \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1 \right]
\]
\[
\lesssim (\kappa + \kappa^{-1}[\omega(2r)^{p/q} + r^\gamma + r^\beta]) \left[ \varphi \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1 \right].
\]
By taking \( \kappa := (\omega(2r)^{p/q} + r^\gamma + r^\beta)^{1\over 2} \), we then conclude that
\[
\int_{B_r} \varphi(|\nabla u - \nabla w|) \, dx \lesssim (\omega(2r)^{p/q} + r^\gamma + r^\beta)^{1\over 2} \left[ \varphi \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1 \right].
\]
In turn, by Jensen’s inequality and (aDec)\( _\eta \) of \( \tilde{\varphi} \), we obtain that
\[
\varphi \left( \int_{B_{2r}} |\nabla u| \, dx \right) \lesssim \tilde{\varphi} \left( \int_{B_{2r}} |\nabla u - \nabla w| \, dx \right)
\]
\[
\lesssim \tilde{\varphi} \left( (\omega(2r)^{p/q} + r^\gamma + r^\beta)^{1\over 2} \left[ \varphi \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1 \right] \right).
\]
which implies the claim since \( \tilde{\varphi} \) is strictly increasing. \( \square \)

**Lemma 4.33.** Suppose that \( w \) is a solution to the equation (4.16), \( v \) is a solution to the equation (4.17) and \( B_{2r} \subseteq \Omega \). Then
\[
\int_{B_r} |\nabla w - \nabla v| \, dx \lesssim cr^{\beta/2} \left( \int_{B_{2r}} |\nabla u| \, dx + 1 \right)
\]
for some constant \( c = c(n, p, q, L, [\nabla \psi]_\beta, \nabla \psi) > 0 \).

**Proof.** Since \( w - v \in W^{1, \varphi}_0(B_r) \), we infer from (4.16) and (4.17) that
\[
\int_{B_r} \left[ \frac{\varphi'(|\nabla w|)}{|\nabla w|} \nabla w - \varphi'(|\nabla v|) \nabla v \right] \cdot (\nabla w - \nabla v) \, dx
\]
\[
= \int_{B_r} \left[ \frac{\varphi'(|\nabla \psi|)}{|\nabla \psi|} \nabla \psi - \varphi'(|\nabla \psi(x_0)|) \nabla \psi(x_0) \right] \cdot (\nabla w - \nabla v) \, dx,
\]
where \( x_0 \) is the center of \( B_r \). In a similar way to estimate \( I_3 \) in the previous lemma, we see that
\[
\int_{B_r} \left[ \frac{\varphi'(|\nabla \psi|)}{|\nabla \psi|} \nabla \psi - \varphi'(|\nabla \psi(x_0)|) \nabla \psi(x_0) \right] \cdot (\nabla w - \nabla v) \, dx
\]
\[
\lesssim r^\beta \left( \int_{B_r} \varphi(|\nabla u|) \, dx + 1 \right).
\]
Therefore, by Lemma 2.4 (2) & (4) with (4.19) and (4.20), we obtain that
\[
\int_{B_r} \tilde{\varphi}(|\nabla w - \nabla v|) \, dx \\
\lesssim \kappa \left( \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right) + \kappa^{-1} r^\beta \left( \int_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right),
\]
which implies that
\[
\int_{B_r} \tilde{\varphi}(|\nabla w - \nabla v|) \, dx \lesssim r^\beta \left[ \tilde{\varphi} \left( \int_{B_{2r}} |\nabla u| \, dx \right) + 1 \right]
\]
by taking \( \kappa := r^\beta \). In the same way as in the proof of the previous lemma, the claim follows. \( \square \)

The following is the well-known technical iteration lemma (cf. [35, Lemma 7.1] and [26, Lemma 2.1, Chapter 3]).

**Lemma 4.34.** Let \( g : [0, r] \to [0, \infty) \) be a non-decreasing function. Suppose that
\[
g(r) \leq C \left( \left( \frac{\rho}{r} \right)^n + \epsilon \right) g(r) + Cr^n
\]
for all \( 0 < \rho < r \leq r_0 \) with non-negative constant \( C \). Then for any \( \mu \in (0, n) \) there exist \( \epsilon_1 = \epsilon_1(n, C, \mu) > 0 \) such that if \( \epsilon < \epsilon_1 \), we have
\[
g(r) \leq c \left( \frac{\rho}{r} \right)^{n-\mu} \left( g(r) + r^{n-\mu} \right),
\]
where \( c \) is a constant depending on \( n, C \) and \( \mu \).

Now we prove our main results. In the results, \( \omega \) is the function from (wVA1) for \( \epsilon = \epsilon_0 \) and \( L \geq 1 \) is the constant from (A0). We also remark that (wVA1) can be replaced by the conditions (A1) and (wVA1) with fixed \( \epsilon > 0 \) which is sufficiently small depending on \( n, p, q, L \).

**Proof of Theorem 1.6 (i).** Let \( r_0 \in (0, 1) \) be a sufficiently small number which will be determined later. Consider any \( B_{r_0} \subset \Omega' \subset \Omega \) assuming that \( r_0 > 0 \) satisfies (4.9) with \( r = r_0 \), and let \( 0 < 2r \leq r_0 \).

In the same way as in the proof of Lemma 4.30, we note from (4.20) that
\[
\int_{B_r} |\nabla v| \, dx \lesssim \int_{B_{2r}} |\nabla u| + 1 \, dx.
\]

For simplicity, we write \( \omega_0(r) := (\omega(2r)^{p/q} + r^\gamma + r^\beta)^{1/2} \). Then if \( \rho < \frac{r}{2} \), by Lemmas 4.30, 4.33 and (4.21), we have that
\[
\int_{B_\rho} |\nabla u| \, dx \lesssim \int_{B_{2\rho}} |\nabla u - \nabla w| \, dx + \int_{B_r} |\nabla w - \nabla v| \, dx + \int_{B_\rho} |\nabla v| \, dx \\
\lesssim \omega_0(r) \int_{B_{2\rho}} |\nabla u| + 1 \, dx + g^n \sup_{B_{\rho/2}} |\nabla v| \\
\lesssim \omega_0(r_0) \int_{B_{2\rho}} |\nabla u| + 1 \, dx + \rho g^n \int_{B_r} |\nabla v| \, dx \\
\lesssim \left( \omega_0(r_0) + \left( \frac{\rho}{r} \right)^n \right) \int_{B_{2\rho}} |\nabla u| \, dx + r^n.
\]

Otherwise, i.e. if \( \frac{r}{2} \leq \rho < 2r \), the above estimate is clear because \( \frac{1}{2} \leq \frac{\rho}{r} \).
Therefore, by choosing $r_0$ so small that $\omega_0(r_0) = (\omega(r_0) + r_0^\gamma + r_0^\beta)^{\frac{1}{\gamma}} < \epsilon_1$, where $\epsilon_1$ is given in Lemma 4.34, we have from Lemma 4.34 that for any $\mu \in (0, n),$

\[(4.35) \quad \int_{B_{r_0}} |\nabla u| \, dx \lesssim \left( \frac{r}{r_0} \right)^{n-\mu} \left( \int_{B_{r_0}} |\nabla u| \, dx + r_0^{n-\mu} \right)\]

for all $B_{r_0} \subset \Omega'$ with $r \in (0, r_0)$. In addition, $B_{r_0} \subset \Omega'$ is arbitrary and the implicit constant is universal. Hence, we take $1 - \mu = \alpha$ to obtain that $u \in C^{0,\alpha}_{loc}(\Omega')$ by Morrey type embedding.

\[\square\]

**Proof of Theorem 1.6 (ii).** Fix $\Omega' \Subset \Omega$. Recall from (4.22) that for any $0 < \varrho < \frac{r}{2}$

\[
\int_{B_{\varrho}} |\nabla v - (\nabla u)_{B_{\varrho}}| \, dx \leq c \left( \frac{\varrho}{r} \right)^{\alpha^0} \int_{B_{\varrho}} |\nabla v| \, dx
\]

for some $\alpha^0 > 0$. From (4.35), we note that for $\mu \in (0, 1),$

\[
\int_{B_{r_2}} |\nabla u| \, dx \leq c_\mu r^{-\mu}
\]

for all $B_{r_2} \subset \Omega'$ with $r \in (0, \frac{r_0}{2})$, where $r_0$ is from the proof of Theorem 1.6 (i) and the constant $c_\mu \geq 1$ depends on $n, p, q, L, r_0$, and $\mu$. Let us consider a small $r < \frac{r_0}{2}$ which will be determined later. From Lemmas 4.30,4.33, we then derive that for $0 < \varrho < \frac{r}{2},$

\[
\int_{B_{\varrho}} |\nabla u - (\nabla u)_{B_{\varrho}}| \, dx \leq 2 \int_{B_{\varrho}} |\nabla u - (\nabla v)_{B_{\varrho}}| \, dx
\]

\[
\leq 2 \int_{B_{\varrho}} |\nabla u - \nabla w| \, dx + 2 \int_{B_{\varrho}} |\nabla w - \nabla v| \, dx + 2 \int_{B_{\varrho}} |\nabla v - (\nabla v)_{B_{\varrho}}| \, dx
\]

\[
\lesssim \left( \frac{r}{\varrho} \right)^{n} \int_{B_{r_2}} |\nabla u - \nabla w| \, dx + \left( \frac{r}{\varrho} \right)^{\alpha^0} \int_{B_{\varrho}} |\nabla v - \nabla v_{B_{\varrho}}| \, dx
\]

\[
\lesssim \left( r_0 \right)^{\alpha^0} \left( \omega(2r)^{p/q} + r^\gamma + r^\beta \right)^{\frac{1}{\gamma}} \left( \int_{B_{2r_2}} |\nabla u| \, dx + 1 \right) + \left( \frac{\varrho}{r} \right)^{\alpha^0} \int_{B_{\varrho}} |\nabla v| \, dx
\]

\[
\lesssim \left( r^\delta_0 \left( \frac{r}{\varrho} \right)^{n} + \left( \frac{\varrho}{r} \right)^{\alpha^0} \right) \left( \int_{B_{2r_2}} |\nabla u| \, dx + 1 \right) \leq c_\mu r^{-\mu} \left( r^\delta_0 \left( \frac{r}{\varrho} \right)^{n} + \left( \frac{\varrho}{r} \right)^{\alpha^0} \right),
\]

where $\delta_0 := \frac{1}{2} \min \{ \frac{p}{q}, \gamma, \beta \}$. In the same way as in the proof of Theorem 7.4 in [35], we obtain $\nabla u \in C^{0,\alpha}_{loc}(\Omega')$ for some $\alpha > 0$.

\[\square\]

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