NORMALIZERS AND APPROXIMATE UNITS FOR INCLUSIONS OF
C*-ALGEBRAS

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Abstract. For an inclusion of C*-algebras \( D \subseteq A \) with \( D \) abelian, we show that when \( n \in A \) normalizes \( D \), \( n^*n \) and \( nn^* \) commute with \( D \). As a corollary, when \( D \) is a regular MASA in \( A \), every approximate unit for \( D \) is also an approximate unit for \( A \). This permits removal of the non-degeneracy hypothesis from the definition of a Cartan MASA in the non-unital case.

We give examples of singular MASA inclusions: for some, every approximate unit for \( D \) is an approximate unit for \( A \), while for others, no approximate unit for \( D \) is an approximate unit for \( A \). Our results imply that if the unitization of an inclusion \( D \subseteq A \) is a C*-diagonal, then \( D \) is regular in \( A \). In contrast, we give an example of a non-regular inclusion whose unitization is a Cartan inclusion.

If \( D \) is a MASA in \( A \), we ask when \( A \) is a subalgebra of \( B \) with \( D \) a regular MASA in \( B \). When \( D \) is a MASA in \( B(\ell^2(\mathbb{N})) \), no such \( B \) exists.

1. Introduction

Given an abelian C*-algebra \( D \), it is often of interest to construct a larger C*-algebra \( A \) from data involving \( D \). A very well-known example occurs when \( \Gamma \) is a discrete group of automorphisms of \( D \) and \( A \) is taken to be the completion of the convolution algebra \( C_c(\Gamma, D) \) with respect to a C*-norm. Other examples arise from studying an inverse semigroup of partial homeomorphisms of \( \hat{D} \); from this information, one can take \( A \) to be a C*-algebra arising from the groupoid of germs of the inverse semigroup. The resulting algebra \( A \) encodes dynamical properties of the action, which provides a bridge between topological dynamics and C*-algebras.

There are some settings in which the process can be reversed. Remarkable results of Kumjian [9] and Renault [14] show that given a C*-algebra \( A \) and an appropriate maximal abelian *-subalgebra (MASA) \( D \subseteq A \), it is possible to use \( D \) to introduce "coordinates" for \( A \), so that \( A \) may be described as the completion of a convolution algebra of continuous functions on a suitable topological groupoid. (Many other authors have produced results along these lines, but Kumjian and Renault were pioneers.) An examination of the special case where \( A = D \), shows that the Kumjian-Renault process is a generalization of the familiar Gelfand representation of \( D \) as the algebra of continuous functions vanishing at infinity on the locally compact Hausdorff space \( \hat{D} \).

Our interest is with inclusions, which we now define.

Definition 1.1. An inclusion is a pair \((A, D)\) where \( A \) and \( D \) are C*-algebras, \( D \) is abelian, and \( D \subseteq A \). The set of normalizers for an inclusion \((A, D)\) is

\[ N(A, D) := \{ n \in A : nDn^* \cup n^*Dn \subseteq D \} \]

and \((A, D)\) is regular when the linear span of \( N(A, D) \) is norm-dense in \( A \). The inclusion \((A, D)\) is singular if \( N(A, D) = D \). (When \( A \) is not abelian, singular MASA inclusions are as far from being regular as possible.) Finally, when \( D \) is maximal abelian in \( A \), \((A, D)\) is a MASA inclusion.
It is frequently useful to impose a non-degeneracy condition on an inclusion \((A, D)\). Often, as in Renault’s definition of Cartan MASA ([14, Definition 5.1]), this condition is that the inclusion have the approximate unit property, that is, \(D\) contains an approximate unit for \(A\).

When \(A\) has a unit and \(D\) is a MASA in \(A\), the approximate unit property is automatic. However, in the non-unital case, it is not automatic, even when \((A, D)\) is a MASA inclusion, see [14, Section 3.2] or Example [3.5] below. Renault remarks that he included the approximate unit property in the definition of Cartan MASA because the groupoid models he had in mind possess it and, due to the example in [15], it seems to be needed.

The main purpose of this note is to establish Proposition 2.1 and several of its consequences. A surprising corollary of Proposition 2.1 is Theorem 2.6, which shows that when \((A, D)\) is a regular MASA inclusion, a fact which (understandably) appears to have been overlooked in the literature. In [9], Kumjian imposes the approximate unit hypothesis may be removed from the definition of a Cartan MASA, a fact which (understandably) appears to have been overlooked in the literature. In [9], Kumjian imposes a different non-degeneracy condition when defining a \(C^*\)-diagonal in the non-unital setting, but as we note in Proposition 2.10, it is again automatic.

We give examples showing that in some cases, the regularity hypothesis in Theorem 2.6 may be removed, but as noted above, it cannot be removed in general. These examples suggest the problem of determining which MASA inclusions \((A, D)\) are intermediate to a regular MASA inclusion \((B, D)\) in the sense that \(D \subseteq A \subseteq B\). Theorem 3.7 shows that when \(H\) is a separable and infinite dimensional Hilbert space, a MASA inclusion of the form \((B(H), D)\) is never intermediate to a regular MASA inclusion.

The author is grateful to Anna Duwenig for noticing that the proof of [2, Proposition 3.8] implicitly used Proposition 2.1; her observation was the impetus for the present note. We thank Jonathan Brown, Adam Fuller, and Sarah Reznikoff for several helpful conversations and suggestions. Finally, we appreciate the referee’s useful suggestions regarding the structure of the paper.

We conclude this section with a pair of preliminary results, the first of which is folklore.

**Fact 1.2.** Let \((A, D)\) be an inclusion. Suppose \(\rho\) is a state on \(A\) such that \(\rho|_D \in \hat{D}\). Then for any \(a \in A\) and \(d \in D\),

\[\rho(ad) = \rho(da) = \rho(a)\rho(d).\]

**Proof.** We show that \(\rho(ad) = \rho(a)\rho(d)\); the other equality is similar. Let \((u_\lambda)\) be an approximate unit for \(A\). The Cauchy-Schwarz inequality and the hypothesis \(\rho|_D \in \hat{D}\) give

\[|\rho(ad - \rho(d)a)|^2 = \lim_{\lambda} |\rho(ad - \rho(d)au_\lambda)|^2 = \lim_{\lambda} |\rho(a(d - \rho(d)u_\lambda))|^2 \leq \lim_{\lambda} \rho(aa^*)\rho((d - \rho(d)u_\lambda)^*(d - \rho(d)u_\lambda)) = 0.\]

\[\square\]

For regular inclusions, the following observation gives a characterization of the approximate unit property.

**Observation 1.3.** Let \((A, D)\) be an inclusion. The following statements hold.

1. If \((A, D)\) has the approximate unit property, then \(n^*n \in D\) for every \(n \in N(A, D)\).

2. If \((A, D)\) is regular and \(n^*n \in D\) for every \(n \in N(A, D)\), then every approximate unit for \(D\) is also an approximate unit for \(A\).

**Proof.** 1) Let \((u_\lambda)\) be an approximate unit for \(D\) which is also an approximate unit for \(A\). For \(n \in N(A, D)\), \(n^*n = \lim_{\lambda} n^*u_\lambda n \in D\).

2) This is essentially the argument from the proof of [2, Lemma 3.10(2)]. Fix an approximate unit \((u_\lambda)\) for \(D\) and let \(n \in N(A, D)\). As \(n^*n \in D\),

\[(u_\lambda n - n)(u_\lambda n - n)^* = u_\lambda nn^*u_\lambda - nn^*u_\lambda - u_\lambda nn^* + nn^* \to 0,\]
whence $u_\lambda n \to n$. Replacing $n$ with $n^*$ in this argument gives $nu_\lambda \to n$. Hence for any $a \in \text{span} N(A,D)$, $u_\lambda a \to a$ and $au_\lambda \to a$. Since span $N(A,D)$ is dense in $A$, $(u_\lambda)$ is an approximate unit for $A$. 

2. A Commutation Result and Some Consequences

This section contains our main results: a commutation result, Proposition 2.1 and several corollaries. Among these consequences are a simpler definition for Cartan inclusions and clarification of issues surrounding Kumjian’s notion of $C^*$-diagonal.

It is not difficult to produce examples of regular inclusions without the approximate unit property; for instance, take $D$ to be a proper ideal of the non-unital and abelian $C^*$-algebra $A$. By Observation 1.3, there is $n \in N(A,D)$ such that $n^*n \notin D$. Our first result shows that for a general inclusion $(A,D)$ and $n \in N(A,D)$, $n^*n$ is intimately related to $D$ despite the fact that it need not belong to $D$.

**Proposition 2.1.** Let $(A,D)$ be an inclusion. For $n \in N(A,D)$ and $d \in D$,

$$n^*nd = dn^*n \in D \quad \text{and} \quad nn^*d = dnn^* \in D.$$ 

Furthermore, if $\rho_1$ and $\rho_2$ are states on $A$ such that $\rho_1|_D = \rho_2|_D \in \hat{D}$, then $\rho_1(n^*n) = \rho_2(n^*n)$ and $\rho_1(nn^*) = \rho_2(nn^*)$.

In the terminology of [5, Definition 8.2], the final statement of Proposition 2.1 says that every element of $\hat{D}$ is free relative to $n^*n$ and to $nn^*$.

**Proof.** Since $N(A,D)$ is closed under adjoints, it suffices to prove the result for $n^*n$.

Let $h = n^*n$. Then $h \in N(A,D)$ because $N(A,D)$ is closed under products and the adjoint operation. For any $d \in D$, $(d^*hd)^2 = d^*(hddd^*h)d \in D$. Taking the square root gives,

$$d^*hd \in D \quad \text{for all} \quad d \in D.$$ 

Without loss of generality, we may assume that $A \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Let $Q$ be the orthogonal projection onto $D\mathcal{H}$ and fix an approximate unit $(u_\lambda)$ for $D$. Then

$$\text{sot lim } u_\lambda = Q \quad \text{and for } d \in D, \quad dQ = Qd = d.$$ 

Using (2.2) and the fact that on bounded sets, multiplication is jointly continuous in the strong operator topology, we conclude that sot lim $u_\lambda hu_\lambda = QhQ \in D'$ (actually, $QhQ \in D''$).

Furthermore,

$$Q^\perp hQhQ^\perp = \text{sot lim } Q^\perp (hu_\lambda h)Q^\perp = 0,$$

because $h \in N(A,D)$ and $u_\lambda \in D$. Therefore, $0 = Q^\perp hQ = QhQ^\perp$, so $Q$ commutes with $h$. Thus,

$$Qh = hQ = QhQ \in D'.$$

Hence for $d \in D$,

$$dh = d(Qh) = (Qh)d = h(Qd) = hd,$$

so $d$ commutes with $n^*n$.

Next, for any $0 \leq f \in D$,

$$f^2(n^*n)^2 = (n^*n)f^2(n^*n) \in D$$

because $n^*n \in N(A,D)$. Therefore, $f^2(n^*n)^2 \in D$, so $f^2(n^*n) \in D$. Since $D$ is the span of its positive elements, $dn^*n \in D$ for every $d \in D$.

Finally, suppose for $i = 1, 2$ that $\rho_i$ are states on $A$ such that $\rho_1|_D = \rho_2|_D$ is a pure state $\sigma$ on $D$. Choose $k \in D$ such that $\sigma(k) \neq 0$. Applying Fact 1.2 to $\rho_i(kn^*n)$ gives $\rho_1(n^*n) = \rho_2(n^*n)$. 

□
We now give several corollaries of Proposition 2.1. Our first concerns dynamical objects associated to \( n \in N(A, D) \); it extends well-known constructions for inclusions with the approximate unit property to arbitrary inclusions. With Proposition 2.1 in hand, they are routine modifications of those results (e.g. \cite[Proposition 6°]{9} or \cite[Propositions 2.1 and 2.2]{11}).

**Corollary 2.3.** Let \((A, D)\) be any inclusion and fix \( n \in N(A, D) \).

The partial automorphism associated to \( n \): Let \( B \) be an AW*-algebra with \( A \subseteq B \) and let \( n = u_n = |n^*|u \) be the polar decomposition of \( n \) in \( B \). Then \( n_n^*D \) and \( n^*nD \) are ideals in \( D \) and the map \( nn^*d \mapsto n^*dn \) uniquely extends to a *-isomorphism \( \theta_n : \frac{n_n^*D}{n^*nD} \to \frac{n_n^*D}{n^*nD} \) such that for each \( h \in \frac{n_n^*D}{n^*nD} \),

\[
n\theta_n(h) = hn \quad \text{and} \quad u^*hu = \theta_n(h).
\]

The partial homeomorphism associated to \( n \): Let \( \text{dom} \, n := \{ \sigma \in \hat{D} : \sigma(n_n^*D) \neq 0 \} \) and \( \text{ran} \, n := \{ \sigma \in \hat{D} : \sigma(n^*nD) \neq 0 \} \). Then \( \text{dom} \, n \) and \( \text{ran} \, n \) are open subsets of \( D \) and there is a homeomorphism \( \beta_n : \text{dom} \, n \to \text{ran} \, n \) such that for every \( h \in \frac{n_n^*D}{n^*nD} \) and \( \sigma \in \text{dom} \, n \),

\[
\beta_n(\sigma)(h) = \sigma(\theta_n(h)).
\]

For \( \sigma \in \text{dom} \, n \), define \( \sigma(n_n^*n) := \rho(n_n^*n) \), where \( \rho \) is any extension of \( \sigma \) to a state on \( A \). Then \( \sigma(n_n^*n) \neq 0 \) and for \( d \in D \),

\[
\beta_n(\sigma)(d) = \frac{\sigma(n_n^*dn)}{\sigma(n_n^*n)}.
\]

For a \( C^* \)-algebra \( A \), we write \( \hat{A} \) for its unitization and \( M(A) \) for its multiplier algebra. When \((A_1, D_1)\) and \((A_2, D_2)\) are inclusions, a *-homomorphism \( \alpha : A_1 \to A_2 \) is regular if \( \alpha(N(A_1, D_1)) \subseteq N(A_2, D_2) \). We use the notation \( \alpha : (A_1, D_1) \to (A_2, D_2) \) when \( \alpha \) is a regular *-homomorphism. Our next corollary concerns the regularity of the inclusion map of \( A \) into \( \hat{A} \) or another unital \( C^* \)-subalgebra of \( M(A) \).

**Corollary 2.4.** Suppose \((A, D)\) is a MASA inclusion and \( A \) is not unital.

1. The usual embedding \( \iota : A \to \hat{A} \) is a regular *-monomorphism of \((A, D)\) into \((\hat{A}, \hat{D})\).

2. Suppose in addition that \((A, D)\) has the approximate unit property and that \( B \) is a unital \( C^* \)-algebra containing \( A \) as an essential ideal. Let \( D_B := \{ b \in B : bD \cup Db \subseteq D \} \). Then \((B, D_B)\) is a MASA inclusion and \( N(A, D) \subseteq N(B, D_B) \). In fact, for every \( n \in N(A, D) \) and \( b \in D_B \),

\[
nbn^* \in D \quad \text{and} \quad n^*bn \in D.
\]

**Proof.**

1. For \((d, \lambda) \in \hat{D} \) and \( n \in N(A, D) \),

\[
\iota(n)(d, \lambda)\iota(n^*) = (n, 0)(d, \lambda)(n, 0) = (n_d, \iota(n^*\iota(d)) = (n_n^* + \lambda n_n^*, 0).
\]

As \( D \) is a MASA, Proposition 2.1 gives \( n_n^* \in D \), so \( \iota(n)\hat{D} \iota(n^*) \in \hat{D} \). Similarly \( \iota(n)^*\hat{D} \iota(n) \in \hat{D} \), so \( \iota(n) \in N(\hat{A}, \hat{D}) \).

2. We may assume \( A \subseteq \mathcal{B}(\mathcal{H}) \) is a non-degenerate \( C^* \)-algebra and \( M(A) = \{ x \in \mathcal{B}(\mathcal{H}) : xA \cup Ax \subseteq A \} \). Since \( A \) is an essential ideal of \( B \),

\[
I_B = I_{\mathcal{H}} \in B \subseteq M(A).
\]

Routine arguments and the approximate unit property yield: \( D \subseteq \mathcal{B}(\mathcal{H}) \) is non-degenerate; \( M(D) \subseteq M(A) \); and \( D_B = B \cap M(D) \) is a MASA in \( B \).

Let \( n \in N(A, D) \) and \( b \in D_B \). Note that \( nb \) belongs to \( N(A, D) \). Since \( D \) is a MASA in \( A \), Proposition 2.1 gives \( nbn^* \in D \). Similarly, \( n^*b^*bn \in D \). Since \( D_B \) is spanned by its positive elements, \( nD_Bn^* \cup n^*D_Bn \subseteq D \).

\( \square \)
Recall that \( n \in N(A, D) \) is free if \( n^2 = 0 \); we write \( N_f(A, D) \) for the collection of free normalizers.

**Corollary 2.5.** Suppose \((A, D)\) is an inclusion and \( n \in N_f(A, D) \). If \( \rho \) is a state on \( A \) such that \( \rho|_D \) is a pure state on \( D \), then \( \rho(n) = 0 \).

**Proof.** Proposition 2.1 gives \( n^*nd = dn^*n \in D \) and \( nn^*d = dnn^* \in D \), so if \( d \in D \) satisfies \( \rho(d) = 1 \), Fact 1.2 gives

\[
\rho(n^*n)\rho(nn^*) = \rho(dn^*n)\rho(nn^*d) = \rho(dn^*n^2d^*) = 0,
\]

whence \( 0 \in \{ \rho(n^*n), \rho(nn^*) \} \).

With \( d \in D \) again satisfying \( \rho(d) = 1 \), \( |\rho(n)|^2 = |\rho(dn)|^2 = |\rho(nd)|^2 \). By the Cauchy-Schwartz inequality,

\[
|\rho(n)|^2 \leq \min\{\rho(n^*n), \rho(nn^*)\} = 0.
\]

\( \Box \)

**Theorem 2.6.** If \((A, D)\) is a regular MASA inclusion, then every approximate unit for \( D \) is an approximate unit for \( A \).

**Proof.** Proposition 2.1 shows that if \( n \in N(A, D) \), then \( n^*n \) commutes with \( D \). As \( D \) is a MASA in \( A \), \( n^*n \in D \). Now apply Observation 1.3.

An immediate consequence of Theorem 2.6 is that the approximate unit property behaves well for intermediate subalgebras of regular MASA inclusions.

**Corollary 2.7.** Suppose \((A, D)\) is a regular MASA inclusion and \( B \) is a norm-closed, but not necessarily selfadjoint, subalgebra satisfying \( D \subseteq B \subseteq A \). Then every approximate unit for \( D \) is an approximate unit for \( B \).

We shall use the following definition of Cartan inclusion in the sequel. By Theorem 2.6 it is equivalent to Renault’s original definition of Cartan inclusion.

**Definition 2.8** (cf. [14, Definition 5.1]). A Cartan inclusion is a regular MASA inclusion \((A, D)\) such that there exists a faithful conditional expectation \( P : A \to D \).

**Remark 2.9.** The class of regular MASA inclusions is considerably broader than the class of Cartan inclusions. Indeed, given a regular MASA inclusion \((A, D)\) with \( A \) unital, there is always a unique pseudo-expectation \( E \) of \( A \) into the injective envelope \( I(D) \) of \( D \) (see [11, Definition 1.3] and [11, Theorem 3.5]), and \((A, D)\) falls into exactly one of the following four classes of unital regular MASA inclusions:

- i) \( E \) is a faithful conditional expectation of \( A \) onto \( D \) (in this case \((A, D)\) is a Cartan inclusion);
- ii) \( E \) is a non-faithful conditional expectation of \( A \) onto \( D \);
- iii) \( E \) is faithful, and there is no conditional expectation of \( A \) onto \( D \);
- iv) \( E \) is not faithful, and there is no conditional expectation of \( A \) onto \( D \).

These classes are non-void, and we now describe or reference constructions for each. The simplest non-trivial example of a Cartan inclusion is obtained by taking \( D \) to be a MASA in \( A = B(C^n) \), but far more interesting examples are abundant in the literature. See [6, Theorem 2.2 and Example 2.3] for a construction of a family of examples in class ii). A crossed product construction found in [11, Section 6] produces examples in class iii). Finally, the direct sum of two inclusions, with one belonging to class ii) and the other in class iii), will yield an example in class iv).

We wish to discuss Kumjian’s notion of \( C^* \)-diagonal in light of our results so far. After Renault’s papers [13, 14] appeared, it became commonplace to define a \( C^* \)-diagonal as an inclusion \((A, D)\) such that \( D \) is a Cartan MASA in \( A \) having the pure state extension property, in the sense of [11, Definition 2.5], that is, every pure state of \( D \) extends uniquely to a pure state of \( A \) and no pure state of \( A \) annihilates \( D \). (The pure state extension property is also called the extension property.)
This definition differs from Kumjian’s original definition of $C^*$-diagonal. Indeed, in his original paper on the topic, Kumjian ([9 Definition 3]) says an inclusion $(A, D)$ with $A$ unital is a $C^*$-diagonal if

(I) there is a faithful conditional expectation $P : A \to D$; and

(II) $\overline{\text{span}} \, N_f(A, D)$ is dense in $\ker P$.

In the non-unital setting, Kumjian defines an inclusion $(A, D)$ to be a $C^*$-diagonal if its unitization $(\tilde{A}, \tilde{D})$ satisfies Conditions (I) and (II).

Let us say that an inclusion $(A, D)$ (where $A$ is not assumed unital) satisfies Kumjian’s conditions if both (I) and (II) hold.

In the unital case, the two notions for $C^*$-diagonal just described are known to be equivalent, but we do not know of a reference where this fact is established. For convenience, we therefore outline a proof in the first part of Proposition 2.10 below.

When $A$ is not unital, it is unclear whether an inclusion $(A, D)$ satisfying Kumjian’s conditions must be a $C^*$-diagonal in the sense that $(\tilde{A}, \tilde{D})$ satisfies Kumjian’s conditions or a $C^*$-diagonal in the sense that $(A, D)$ is a Cartan inclusion with the extension property. The next result shows there is no ambiguity: if $(A, D)$ satisfies Kumjian’s conditions, then $(\tilde{A}, \tilde{D})$ satisfies Kumjian’s conditions and $(A, D)$ is a Cartan inclusion with the extension property.

**Proposition 2.10.** Let $(A, D)$ be an inclusion.

**The Unital Case:** If $A$ is unital, then $(A, D)$ satisfies Kumjian’s conditions if and only if $(A, D)$ is a Cartan inclusion and every pure state on $D$ uniquely extends to a state on $A$.

**The Non-Unital Case:** Suppose $A$ is not unital. The following are equivalent.

1. $(A, D)$ satisfies Kumjian's conditions.
2. $(A, D)$ is a Cartan inclusion such that every pure state of $D$ has a unique extension to a state on $A$.
3. $(A, D)$ is a Cartan inclusion such that every pure state of $D$ has a unique extension to a state on $A$ and no pure state of $A$ annihilates $D$.
4. $(\tilde{A}, \tilde{D})$ is a Cartan inclusion such that every pure state of $\tilde{D}$ extends uniquely to a state on $\tilde{A}$.
5. $(\tilde{A}, \tilde{D})$ satisfies Kumjian's conditions.

**Proof.** The Unital Case. Suppose $A$ is unital.

It follows from [9 Proposition 4] that if $(A, D)$ satisfies Kumjian’s conditions, then it is a Cartan inclusion and every pure state on $D$ uniquely extends to a state on $A$. (Throughout [9], Kumjian makes the blanket assumption that all $C^*$-algebras are separable, but his proof of [9 Proposition 4] is also valid when $A$ is not separable.)

Now suppose $(A, D)$ is a Cartan inclusion such that every pure state on $D$ extends uniquely to a state on $A$. Let $P$ be the conditional expectation of $A$ onto $D$ and let $n \in N(A, D)$. An application of [4 Proposition 3.10] (with $x = I$) shows that $\{nP(n^*n), n^*P(n)\} \subseteq D$. A computation now yields $n - P(n) \in N(A, D)$. Since $(A, D)$ is a regular inclusion and $P$ is contractive, to show that $\overline{\text{span}} \, N_f(A, D)$ is dense in $\ker P$, it is enough to show that $N(A, D) \cap \ker P \subseteq \overline{\text{span}} \, N_f(A, D)$.

Fix $n \in N(A, D) \cap \ker P$. For $\sigma \in \text{dom} \, n$, [4 Proposition 3.12] or [9 Lemma 9] gives $\beta_n(\sigma) \neq \sigma$.

To show $n \in \overline{\text{span}} \, N_f(A, D)$ use a partition of unity argument. Here are some of the details.

Let $X$ be the (compact) set of all pure states on $D$ and for $\varepsilon > 0$, let $X_\varepsilon := \{\sigma \in X : \sigma(n^*n) \geq \varepsilon^2\}$. For $\sigma \in X_\varepsilon$, the fact that $\beta_n(\sigma) \neq \sigma$ implies that we may find $a_\sigma \in m^*D$ such that: $\sigma(a_\sigma) \neq 0$, $0 \leq a_\sigma \leq 1$, and $a_\sigma \beta_n(a_\sigma) = 0$. Using compactness of $X_\varepsilon$, select a finite subset $\{b_j\}_{j=1}^k \subseteq \{a_\sigma : \sigma \in X_\varepsilon\}$ such that the Gelfand transform of $b := \sum_{j=1}^k b_j$ does not vanish on $X_\varepsilon$. If $m = \min\{\sigma(b) : \sigma \in X_\varepsilon\}$
and \( Z := \{ \sigma \in X : \sigma(b) \leq m/2 \} \), then \( X_\epsilon \cap Z = \emptyset \). If \( Z \neq \emptyset \), use Urysohn’s lemma to find \( h \in D \) with \( 0 \leq h \leq 1 \) such that \( h|_Z = 0 \) and \( h|_{X_\epsilon} = 1 \); if \( Z = \emptyset \), let \( h = 1 \). For \( 1 \leq j \leq k \) and \( \sigma \in X \), let

\[
f_j(\sigma) = \begin{cases} \sigma(h) \frac{\sigma(b_j)}{\sigma(b)} & \sigma \notin Z \\ 0 & \sigma \in Z. \end{cases}
\]

Then \( f_j \in C(X) \), so there exists \( d_j \in D \) with \( \hat{d}_j = f_j \). We have produced a collection \( \{d_j\}_{j=1}^k \subseteq nn^\ast D \) such that for each \( j \), \( 0 \leq d_j \leq 1 \) and \( d_j \theta_n(d_j) = 0 \). Hence \( d_j n \in N_f(A, D) \) for \( 1 \leq j \leq k \).

Furthermore, \( d := \sum_{j=1}^k d_j \) satisfies \( 0 \leq d \leq 1 \) and \( \sigma(d) = 1 \) for each \( \sigma \in X_\epsilon \). Then

\[
||dn - n||^2 = \sup_{\sigma \in X} \sigma((d - I)^2 nn^\ast) = \sup_{\sigma \in (X \setminus X_\epsilon)} \sigma((d - I)^2 nn^\ast) < \epsilon^2.
\]

As \( dn = \sum_{j=1}^k d_j n \in \text{span} N_f(A, D) \), we conclude \( n \in \text{span} N_f(A, D) \). Thus the unital case holds.

**The Non-Unital Case.** For the remainder of the proof, assume \( A \) is not unital.

(1) \( \Rightarrow \) (2). Since \( A = D + \ker P \), \( (A, D) \) is regular. Corollary 2.5 and condition (II) imply that every pure state on \( D \) extends uniquely to a (necessarily pure) state on \( A \). We cannot immediately conclude that \( D \) is a MASA in \( A \) because of the possibility that there is a pure state on \( A \) which annihilates \( D \). Condition (I) is key to showing \( D \) is a MASA.

**Claim:** \( D \) is a MASA in \( A \).

**Proof.** Let \( D_1 \subseteq A \) be a MASA containing \( D \) and let \( S := \{ \rho \in \hat{D}_1 : \rho|_D \neq 0 \} \). Suppose \( a \in D_1 \) and \( \rho(a) = 0 \) for every \( \rho \in \hat{D}_1 \setminus S \). The fact that every pure state of \( D \) extends uniquely to a pure state on \( D_1 \) implies that for every \( \rho \in S \), \( \rho(a) = \rho(P(a)) \), and clearly if \( \rho \in \hat{D}_1 \setminus S \), \( \rho(a) = 0 = \rho(P(a)) \). Therefore, \( a = P(a) \), that is, \( a \in D_1 \). It follows that \( D = \{ a \in D_1 : \rho(a) = 0 \text{ for all } \rho \in \hat{D}_1 \setminus S \} \), so \( D \) is an ideal in \( D_1 \). If \( 0 \leq h \in D_1 \) and \( hd = 0 \) for all \( d \in D \), then \( 0 \leq P(h)^2 = P(hP(h)) = 0 \). Thus \( P(h) = 0 \), so \( h = 0 \) by faithfulness of \( P \). It follows that \( D \) is an essential ideal in \( D_1 \).

Fix \( a \in D_1 \). Then for any \( d \in D \), \( (a - P(a))d = ad - P(ad) = 0 \). As \( D \) is an essential ideal, \( a = P(a) \). Therefore \( D = D_1 \), so \( D \) is a MASA in \( A \).

Thus \( (A, D) \) is a Cartan inclusion such that each pure state of \( D \) extends uniquely to a pure state of \( A \).

(2) \( \Rightarrow \) (3). Let \( (u_\lambda) \) be an approximate unit for \( D \). Theorem 2.6 ensures that \( (u_\lambda) \) is an approximate unit for \( A \). Thus, for any pure state \( \rho \) on \( A \), \( \lim_\lambda \rho(u_\lambda) = 1 \), so (3) holds.

(3) \( \Rightarrow \) (4). It is readily seen that the map \( \tilde{P} : \hat{A} \to \hat{D} \) given by \( \tilde{P}(a, \lambda) = (P(a), \lambda) \) is a faithful conditional expectation. That \( (\hat{A}, \hat{D}) \) is regular follows from Corollary 2.4 and [11, Remarks 2.6(iii)] gives the extension property for \( (\hat{A}, \hat{D}) \), so (4) holds.

(4) \( \Rightarrow \) (5). This follows from the unital case.

(5) \( \Rightarrow \) (1). Let \( \tilde{P} : \hat{A} \to \hat{D} \) be the faithful conditional expectation. As in the proof of (1) implies (2) above, Corollary 2.5 and Condition (II) show that \( (\hat{A}, \hat{D}) \) has the extension property ([10, Proposition 4] also gives this). Therefore \( (\hat{A}, \hat{D}) \) is a MASA inclusion. Hence \( (A, D) \) is also a MASA inclusion.

If \( v := (a, \lambda) \in N_f(\hat{A}, \hat{D}) \), then \( \lambda = 0 \) and \( a \in N_f(A, D) \). This shows \( N_f(\hat{A}, \hat{D}) \subseteq \iota(N_f(A, D)) \). Part (1) of Corollary 2.4 gives \( \iota(N_f(A, D)) \subseteq N_f(\hat{A}, \hat{D}) \). Thus, \( N_f(\hat{A}, \hat{D}) = \iota(N_f(A, D)) \), whence

\[
\iota(\text{span} N_f(A, D)) = \text{span} N_f(\hat{A}, \hat{D}) = \ker \tilde{P}.
\]

Since

\[
\hat{A} = \ker \tilde{P} + \iota(D) + C(0, 1)
\]
and \( \ker \tilde{P} \subseteq \iota(A) \), \( \tilde{P} \) leaves \( \iota(A) \) invariant. Therefore, \( P := \iota^{-1} \circ \tilde{P} \circ \iota \) is a faithful conditional expectation of \( A \) onto \( D \). Since \( \iota(\ker \tilde{P}) = \ker \tilde{P} = \iota(\overline{\text{span}} N_f(A, D)) \), we obtain \( \ker P = \overline{\text{span}} N_f(A, D) \). Thus \((A, D)\) satisfies Kumjian’s conditions and the proof is complete.

Proposition 2.10 shows the following definition of \( C^*\)-diagonal is equivalent to Kumjian’s original definition, regardless of whether \( A \) is unital.

**Definition 2.11.** An inclusion \((A, D)\) is a \( C^*\)-diagonal if it satisfies Kumjian’s conditions, or equivalently, if \((A, D)\) is a Cartan inclusion for which every pure state of \( D \) uniquely extends to a state on \( A \).

### 3. Examples

The purpose of this section is to give a variety of examples. We begin with an example concerning the unitization of an inclusion.

Let \((A, D)\) be an inclusion with \( A \) non-unital. Proposition 2.10 shows \((A, D)\) is a \( C^*\)-diagonal if and only if its unitization \((\tilde{A}, \tilde{D})\) is a \( C^*\)-diagonal. This is easily seen to be true for the class of MASA inclusions, but as we shall see momentarily, it is not true for the class of Cartan inclusions. While a routine argument (sketched in the proof of Proposition 3.2) shows that \((\tilde{A}, \tilde{D})\) is a Cartan inclusion whenever \((A, D)\) is Cartan, our first example shows the converse fails: it is possible for the unitization of a non-regular inclusion \((A, D)\) to be a Cartan inclusion.

**Example 3.1.** Let \( S \) be the unilateral shift, let \( B = C^*(S) \) be the Toeplitz algebra, let \( D_1 = C^*(\{I\} \cup \{S^n S^m : n, m \in \mathbb{N}\}) \), and let \( \mathcal{K} \) be the compact operators. Then \((B, D_1)\) is a Cartan inclusion.

Identify \( B/\mathcal{K} \) with \( C(\mathbb{T}) \), and let \( q : B \to C(\mathbb{T}) \) be the quotient map. Fix \( \omega \in \mathbb{T} \) and for \( b \in B \), let \( \tau_{\omega}(b) = q(b)(\omega) \). Then \( \tau_{\omega} \) is a multiplicative linear functional on \( B \). Let

\[
A = \ker \tau_{\omega} \quad \text{and} \quad D = D_1 \cap A.
\]

The map \( \tilde{A} \ni (x, \lambda) \mapsto x + \lambda I \in B \) is an isomorphism which carries \( \tilde{D} \) onto \( D_1 \). Thus \((\tilde{A}, \tilde{D})\) is a Cartan inclusion (and in particular a MASA inclusion). Since \( D \subseteq \mathcal{K} \) and \( A \) contains the non-compact operator \( S - \omega I \), \((A, D)\) does not have the approximate unit property. Theorem 2.6 implies \((A, D)\) cannot be a regular inclusion, so \((A, D)\) is not a Cartan inclusion.

We now note that the behavior displayed in Example 3.1 cannot occur when \((A, D)\) is regular.

**Proposition 3.2.** Suppose \((A, D)\) is a regular inclusion, with \( A \) not unital. Then \((A, D)\) is a Cartan inclusion if and only if \((\tilde{A}, \tilde{D})\) is a Cartan inclusion.

**Proof.** We again use \( \iota : A \to \tilde{A} \) for the map \( x \mapsto (x, 0) \). Suppose \((\tilde{A}, \tilde{D})\) is a Cartan inclusion. Then \((A, D)\) is a regular MASA inclusion, so Theorem 2.6 ensures \( D \) contains an approximate unit \( (u_\lambda) \) for \( A \). Let \( \tilde{P} : \tilde{A} \to \tilde{D} \) be the conditional expectation. For \( x \in A \),

\[
\tilde{P}(\iota(x)) = \lim \tilde{P}(\iota(u_\lambda x)) = \lim \iota(u_\lambda) \tilde{P}(\iota(x)).
\]

Thus \( \iota(A) \) is invariant under \( \tilde{P} \), and, as \( \iota \) is one-to-one, \( P := \iota^{-1} \circ \tilde{P} \circ \iota \) is a conditional expectation of \( A \) onto \( D \). As \( \tilde{P} \) is faithful, so is \( P \), whence \((A, D)\) is a Cartan inclusion.

We sketch the converse. Suppose \((A, D)\) is a Cartan inclusion with conditional expectation \( P : A \to D \). Then \((\tilde{A}, \tilde{D})\) is a regular MASA inclusion. Define \( \tilde{P} : \tilde{A} \to \tilde{D} \) by \( \tilde{P}(x, \lambda) = (P(x), \lambda) \). For \((x, \lambda) \in \tilde{A} \), the fact that \( P(x^* x) \geq P(x)^* P(x) \) gives

\[
P((x, \lambda)^*(x, \lambda)) \geq (P(x)^* P(x) + \overline{x} x + \lambda x^* \lambda)^2 = \tilde{P}(x, \lambda)^* \tilde{P}(x, \lambda) \geq 0.
\]

Then \( \tilde{P} \) is a faithful conditional expectation, so \((\tilde{A}, \tilde{D})\) is a Cartan inclusion. \( \square \)
Next we give examples of singular MASA inclusions, some of which have the approximate unit property, while others do not. Corollary 3.4 is our key tool for constructing non-regular inclusions with the approximate unit property: we make appropriate choices of subalgebras intermediate to a regular MASA inclusion. Theorem 3.7 gives an example of a MASA inclusion which is not intermediate to a regular MASA inclusion. These results lead us to pose a number of questions.

If $(A, D)$ is a MASA inclusion and $B$ is an intermediate $C^*$-subalgebra, $D \subseteq B \subseteq A$, regularity properties of $(B, D)$ cannot be deduced from regularity of $(A, D)$. Indeed, $(B, D)$ may be

(a) regular: take $D$ to be the $n \times n$ diagonal matrices and $D \subseteq B \subseteq M_n(\mathbb{C})$;

(b) singular: take $D$ to be a non-atomic MASA in $\mathcal{B}(\mathcal{H})$, let $A = \overline{\text{span}}(\mathcal{B}(\mathcal{H}), D)$ and $B = D + \mathcal{K}$, see [6] Corollary 3.9 and Proposition 2.9; or

(c) something peculiar: [3, Example 5.1] gives an example of a Cartan pair $(A, D)$ and an intermediate non-regular $C^*$-subalgebra $B$ having full support in the Renault twist $\Sigma \rightarrow G$ associated with $(A, D)$.

In each of the latter two cases, $(B, D)$ is a non-regular MASA inclusion with the approximate unit property.

Before continuing, we give a method for constructing singular MASA inclusions. Interestingly, the proof in the non-unital case uses Corollary 2.4. We use the notation $J \trianglelefteq B$ to indicate $J$ is a norm-closed, two-sided ideal in $B$.

**Lemma 3.3.** Let $(A, D)$ be an inclusion with $D$ a MASA in $A$. If $J \trianglelefteq A$ satisfies $J \cap D = \{0\}$, then $(D + J, D)$ is a singular inclusion.

**Proof.** For notational purposes, let $B = D + J$. The case when $B$ is unital is [6, Proposition 2.9].

Now suppose $B$ is not unital. Then $\tilde{D}$ is a MASA in $\tilde{B}$, $J \trianglelefteq \tilde{B}$, and $J \cap \tilde{D} = J \cap D = \{0\}$. As $D$ is a MASA in $\tilde{B}$, Corollary 2.4 and the unital case give

$$N(B, D) \subseteq N(\tilde{B}, \tilde{D}) = \tilde{D}.$$ 

Thus, $N(B, D) \subseteq B \cap \tilde{D} = D$, so $(B, D)$ is a singular inclusion. 

**Example 3.4.** Here is a class of examples of singular MASA inclusions having the approximate unit property. Let $\mathcal{K}$ be a separable infinite dimensional Hilbert space, and suppose $D \subseteq \mathcal{B}(\mathcal{H})$ is a $C^*$-algebra whose double commutant is a non-atomic MASA in $\mathcal{B}(\mathcal{H})$. Letting $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ be the compact operators, note that $D \cap \mathcal{K} \subseteq D' \cap \mathcal{K} = \{0\}$. Set $A = D + \mathcal{K}$.

Then $D$ is a MASA in $A$ and Lemma 3.3 shows it is singular in $A$. If $(u_\lambda)$ is an approximate unit for $D$, then $\text{sot lim } u_j = I$, so for every $K \in \mathcal{K}$, the nets $(u_\lambda K)$ and $(K u_\lambda)$ norm-converge to $K$. It follows that $(u_\lambda)$ is an approximate unit for $A$.

We next present our example of a singular MASA inclusion $(A, D)$, with $A$ separable, such that no approximate unit for $D$ is an approximate unit for $A$. While the approximate unit property also fails for the MASA inclusion $(A_0, C_0)$ found in [15] Section 3.2, that example differs significantly from ours: $C_0$ is generated by minimal projections, the expectation of $A_0$ onto $C_0$ is faithful, and $C_0$ is not singular in $A_0$.

**Example 3.5.** Given any $C^*$-algebra $\mathfrak{A}$, $\ell^\infty(\mathfrak{A})$ will denote the $C^*$-algebra of all bounded sequences in $\mathfrak{A}$ (with the usual pointwise operations and supremum norm).

Let $\mathcal{H} = L^2([0, 1])$ (Lebesgue measure) and fix a set of vectors

$$\{\xi_j : j \in \mathbb{N}\} \subseteq \mathcal{H} \setminus \{0\} \quad \text{such that} \quad \{\xi_j\}_{j \in \mathbb{N}} = \mathcal{H}.$$ 

For each $j \in \mathbb{N}$, let $p_j \in \mathcal{B}(\mathcal{H})$ be the rank-one projection onto $\mathbb{C}\xi_j$. Notice that for any $x \in \mathcal{B}(\mathcal{H})$, 

$$\|x\| = \sup_{j} \|xp_j\| = \sup_{j} \|p_jx\|.$$
Let $M$ be the collection of all multiplication operators,
\[ \mathcal{H} \ni \xi \mapsto f\xi, \quad \text{where} \quad f \in C_0(0, 1), \]
and let $p$ be the projection in $\ell^\infty(\mathcal{K})$ whose $j$th term is $p_j$, that is,
\[ p : j \mapsto p_j. \]

We now describe the inclusion. Take $D$ to be the set of all constant sequences in $\ell^\infty(M)$, and define
\[ A = C^*(\{p\} \cup D) \subseteq \ell^\infty(M + \mathcal{K}). \]

$A$ is separable because $D \simeq M$.

**Claim:** $D$ is a singular MASA in $A$.

**Proof.** Since $M \cap \mathcal{K} = (0)$, the map $\Phi : M + \mathcal{K} \to M$ given by $\Phi(m + k) = m$ (where $m \in M$ and $k \in \mathcal{K}$), is a well-defined *-homomorphism. Next, let $\Delta : \ell^\infty(M + \mathcal{K}) \to \ell^\infty(M)$ be the *-epimorphism given by
\[ (\Delta(x))(j) = \Phi(x(j)) \quad j \in \mathbb{N}. \]

Let us show that $\Delta(A) = D$. Since $\Delta|_D = id|_D$, it suffices to show $\Delta(A) \subseteq D$. To do this, let $X$ be the collection of all finite products where each factor is taken from $\{p\} \cup D$ and at least one of the factors is $p$. Let
\[ Y = \text{span} X \subseteq A, \]
and note that for $y \in Y$ and $j \in \mathbb{N}$, $y(j) \in \mathcal{K}$. The definitions of $\Delta$ and $A$ show that $Y \subseteq \ker \Delta$ and $D + Y$ is dense in $A$. Therefore, given $a \in A$ and $\varepsilon > 0$, we may find $d \in D$ and $y \in Y$ so that $\|a - (d + y)\| < \varepsilon$. As $\Delta(d + y) = d$, we obtain $\|\Delta(a) - d\| < \varepsilon$, showing that $\Delta(a)$ can be approximated as closely as desired by an element of $D$. Therefore, $\Delta(a) \in D$.

Now suppose $a \in A$ commutes with $D$. Then for each $j \in \mathbb{N}$, $a(j)$ commutes with $M$. As $M$ is a MASA in $M + \mathcal{K}$, $a(j) \in M$. This gives $a = \Delta(a) \in D$. Therefore, $D$ is a MASA in $A$.

Since $\Delta|_A$ is a homomorphism, $J := \ker(\Delta|_A)$ is an ideal of $A$ satisfying $D \cap J = (0)$. Also $D + J = A$ (because for $a \in A$, $a = \Delta(a) + (a - \Delta(a))$. An application of Lemma 3.3 shows $(A, D)$ is a singular MASA inclusion.

Now suppose $(u_\lambda)$ is an approximate unit for $D$. Let $v_\lambda = u_\lambda(1) \in M$ (recall elements of $D$ are constant sequences in $M$). Note that $\|v_\lambda - I_{\mathcal{K}}\| = 1$ because $v_\lambda$ is the multiplication operator determined by an element $f_\lambda \in C_0(0, 1)$ with $0 \leq f_\lambda$ and $\|f_\lambda\| \leq 1$. Fixing $\lambda$, we find
\[ \|pu_\lambda - p\| = \sup_j \|p_ju_\lambda - p_j\| = \sup_j \|p_j(v_\lambda - I_{\mathcal{K}})\| = \|v_\lambda - I_{\mathcal{K}}\| = 1. \]

Therefore, $(u_\lambda)$ is not an approximate unit for $A$.

Examples 3.4 and 3.5 and the unpredictable behaviour of regularity for intermediate inclusions (see the discussion between Example 3.4 and Lemma 3.3) motivate the following.

**Question 3.6.** Suppose $D \subseteq A$ is a MASA inclusion. Under what circumstances is there a $C^*$-algebra $B$ with $D \subseteq A \subseteq B$ such that $D$ is a regular MASA in $B$?

Corollary 2.7 shows the approximate unit property is a necessary condition for $(A, D)$ to be an intermediate inclusion arising from a regular MASA inclusion $(B, D)$. In particular, the inclusion of Example 3.3 fails to be such an intermediate inclusion. The inclusion described in item (b) at the start of the present section is intermediate to a regular inclusion, so some choices for $D$ in Example 3.4 yield singular inclusions intermediate to a regular MASA inclusion. We do not know whether every inclusion described in Example 3.4 is intermediate to a regular MASA inclusion, however for such an inclusion $(D + \mathcal{K}, D)$, Proposition 2.1 suggests that $\text{span}\{nd : n \in N(\mathcal{B}(\mathcal{K}), D), d \in D\}$ might be a plausible candidate for the regular algebra $B$. 

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There are MASA inclusions \((A, D)\) with \(A\) unital for which \(A\) cannot be an intermediate sub-algebra of a regular MASA inclusion \((B, D)\). In fact, it can happen that \(A\) contains no MASA \(D\) with this property. The following result gives an example.

**Theorem 3.7.** Let \(\mathcal{H}\) be a separable, infinite dimensional Hilbert space and suppose \(D\) is a MASA in \(\mathcal{B}(\mathcal{H})\). There is no regular MASA inclusion \((B, D)\) with \(D \subseteq \mathcal{B}(\mathcal{H}) \subseteq B\).

**Proof.** Let \(P \in D\) be the strong operator topology sum of the minimal projections of \(D\).

Suppose first that \(P < I\). Since \(PD\) is an atomic MASA in \(\mathcal{B}(P\mathcal{H})\), there is a unique (and faithful) conditional expectation \(E_P : \mathcal{B}(P\mathcal{H}) \to PD\). Also, \(P^\perp D\) is a non-atomic MASA acting on \(\mathcal{B}(P^\perp \mathcal{H})\). By [7, Theorem 2] there are multiple conditional expectations of \(\mathcal{B}(P^\perp \mathcal{H})\) onto \(P^\perp D\). If \(E_{P^\perp}\) is any conditional expectation of \(\mathcal{B}(P^\perp \mathcal{H})\) onto \(P^\perp D\), then \(\mathcal{B}(\mathcal{H}) \ni x \mapsto E_P(PxP) + E_{P^\perp}(P^\perp xP^\perp)\) is a conditional expectation of \(\mathcal{B}(\mathcal{H})\) onto \(D\). Therefore, there are multiple conditional expectations of \(\mathcal{B}(\mathcal{H})\) onto \(D\). Since \(D\) is an abelian von Neumann algebra, it is injective, and hence each conditional expectation of \(\mathcal{B}(\mathcal{H})\) onto \(D\) is a pseudo-expectation, see [11, Definition 1.3]. Since every regular MASA inclusion has a unique pseudo-expectation [11, Theorem 3.5] and the unique pseudo-expectation property is hereditary from above [12, Proposition 2.6] it follows that \(\mathcal{B}(\mathcal{H})\) is not an intermediate algebra for a regular MASA inclusion \((B, D)\).

The case \(P = I\) must be handled differently because in this case, there is a unique and faithful conditional expectation \(E : \mathcal{B}(\mathcal{H}) \to D\). The proof will be accomplished in several steps. Before embarking, note that we may assume \(\mathcal{H} = \ell^2(\mathbb{N})\) and that \(D\) is the collection of all operators diagonal with respect to an orthonormal basis \(\{\zeta_j\}_{j \in \mathbb{N}}\) for \(\ell^2(\mathbb{N})\). Also, for non-zero vectors \(\eta_1, \eta_2 \in \mathcal{H}\), we denote the rank-one operator \(\varnothing \ni \xi \mapsto \langle \xi, \eta_2 \rangle \eta_1\) by \(\eta_1 \eta_2^*\).

**Claim 1:** \((\mathcal{B}(\mathcal{H}), D)\) is not regular\(^1\)

**Proof.** By [12, Example 3.10], \((\mathcal{B}(\mathcal{H})/\mathcal{K}, D/(D \cap \mathcal{K}))\) is a MASA inclusion with a unique pseudo-expectation, which is actually a conditional expectation \(\Delta : \mathcal{B}(\mathcal{H})/\mathcal{K} \to D/(D \cap \mathcal{K})\). As noted in [12, Example 3.10], \(\Delta\) is not faithful.

Suppose now that \((\mathcal{B}(\mathcal{H}), D)\) is regular. Then \((\mathcal{B}(\mathcal{H})/\mathcal{K}, D/(D \cap \mathcal{K}))\) is also regular, so it is a regular MASA inclusion. By [11, Theorem 3.15], the left kernel of \(\Delta\) is an ideal \(\mathcal{L} \subseteq \mathcal{B}(\mathcal{H})/\mathcal{K}\). But \(\mathcal{B}(\mathcal{H})/\mathcal{K}\) is simple, so \(\mathcal{L} = 0\). Hence \(\Delta\) is faithful, yet as we have already observed, it is not. Thus \((\mathcal{B}(\mathcal{H}), D)\) is not regular.

If \((B, D)\) is a regular MASA inclusion with \(D \subseteq \mathcal{B}(\mathcal{H}) \subseteq B\), then there is a unique conditional expectation of \(B\) onto \(D = I(D)\), [11, Theorem 3.5]. The point of the following is that we can reduce to the case where the expectation of \(B\) onto \(D\) is faithful, that is, when \((B, D)\) is a \(C^*\)-diagonal.  

**Claim 2:** Suppose \((B, D)\) is a regular MASA inclusion with \(D \subseteq \mathcal{B}(\mathcal{H}) \subseteq B\). Then there exists a \(C^*\)-diagonal \((C, D)\) with \(D \subseteq \mathcal{B}(\mathcal{H}) \subseteq C\).

**Proof.** As \((B, D)\) is a regular MASA inclusion and \(D\) is injective, the unique pseudo-expectation \(\tilde{E}\) for \((B, D)\) (see [11, Theorem 3.5]) is a conditional expectation whose restriction to \(\mathcal{B}(\mathcal{H})\) is \(E\). We simplify notation and write \(E : B \to D\) instead of using \(\tilde{E}\).

Since \(D\) is an injective \(C^*\)-algebra, [11, Theorem 2.21] shows that every pure state on \(D\) extends uniquely to a pure state of \(B\). Let \(\mathcal{L}(B, D) := \{b \in B : E(b^*b) = 0\}\). By [11, Theorem 3.15], \(\mathcal{L}(B, D)\) is an ideal of \(B\) having trivial intersection with \(D\). Then \(\mathcal{L}(B, D) \cap \mathcal{B}(\mathcal{H})\) is an ideal in \(\mathcal{B}(\mathcal{H})\) also having trivial intersection with \(D\). But \(\mathcal{L}(B, D) \cap \mathcal{B}(\mathcal{H}) \in \{\mathcal{K}, (0)\}\), so as \(\mathcal{K} \cap D \neq (0)\), we conclude \(\mathcal{L}(B, D) \cap \mathcal{B}(\mathcal{H}) = (0)\).

\(^1\)Claim 1 was also established using very different methods by Katavolos and Paulsen in [8, Proposition 19]; we learned of their argument after finding the proof presented here.
Let \( C = B/\mathcal{L}(B, D) \) and let \( q : B \to C \) be the quotient map. By [4, Theorem 4.8], \( (C, q(D)) \) is a \( C^* \)-diagonal. Since \( \mathcal{L}(B, D) \cap \mathcal{B}(\mathcal{H}) = (0) \), we may identify \( (\mathcal{B}(\mathcal{H}), D) \) with \( (q(\mathcal{B}(\mathcal{H})), q(D)) \), and doing so, we obtain \( D \subseteq \mathcal{B}(\mathcal{H}) \subseteq C \).

\[ \text{Claim 3: Suppose } (B, D) \text{ is a } C^* \text{-diagonal with } D \subseteq \mathcal{B}(\mathcal{H}) \subseteq B. \text{ Then } \mathcal{K} \text{ is an essential ideal in } B. \]

**Proof.** Let \( n \in N(B, D) \) be non-zero and fix \( j \in \mathbb{N} \) satisfying \( n\zeta_j\zeta_j^* \neq 0 \). We aim to show that \( v := n\zeta_j\zeta_j^* \) belongs to \( \mathcal{B}(\mathcal{H}) \). As \( v \in N(B, D) \), Corollary 2.3 shows there is a unique \( * \)-isomorphism \( \theta_v : vv^*D \to v^*vD \) extending the map \( vv^*D \ni vv^*h \mapsto v^*hv \); further, for every \( h \in vv^*D \), \( v\theta_v(h) = hv \).

Since \( v^*v = n^*n\zeta_j\zeta_j \), we see that \( \overline{v^*v}D = v^*vD = C\zeta_j\zeta_j^* \). Hence \( \overline{v^*v}D \) is also one-dimensional, so there exists some \( k \in \mathbb{N} \) so that

\[
\overline{v^*v}D = v^*vD = C\zeta_k\zeta_k^*.
\]

This implies that \( \theta_v(\zeta_k\zeta_k^*) = \zeta_j\zeta_j^* \), hence \( \theta_v(\zeta_k\zeta_k^*) = \zeta_k\zeta_k^*v \), that is,

\[
v\zeta_j\zeta_j^* = \zeta_k\zeta_k^*v.
\]

Now set \( u = \zeta_k\zeta_k^* \). Since \( u\zeta_j\zeta_j^* = \zeta_k\zeta_k^*u \), it follows that \( u^*v \) commutes with \( D \). As \( D \) is a MASA in \( B \), \( u^*v \in D \). A computation shows \( u^*vv^*u \neq 0 \), so \( u^*v \neq 0 \). Hence there is \( 0 \neq c \in \mathbb{C} \) such that \( u^*v = c\zeta_j\zeta_j^* \). Then

\[
u^*vv^* = c\zeta_j\zeta_j^*n^*.
\]

Taking adjoints and dividing by \( c \) gives \( n\zeta_j\zeta_j^* = c^{-1}v^*\zeta_k\zeta_k^* \) showing that \( n\zeta_j\zeta_j^* \) is a multiple of a rank-one partial isometry in \( N(\mathcal{B}(\mathcal{H}), D) \).

This holds for every \( j \in \mathbb{N} \), so it follows that for any choice of \( i, j \in \mathbb{N} \),

\[
n\zeta_i\zeta_j^* = n(\zeta_i\zeta_j^*)(\zeta_i\zeta_j^*) \in \mathcal{K}.
\]

As \( \text{span}\{\zeta_i\zeta_j^* : i, j \in \mathbb{N}\} \) is dense in \( \mathcal{K} \), we see that \( n\mathcal{K} \subseteq \mathcal{K} \). Replacing \( n \) with \( n^* \) in the arguments above implies \( \mathcal{K}n \subseteq \mathcal{K} \), so

\[
\mathcal{K}n \cup n\mathcal{K} \subseteq \mathcal{K}.
\]

Since \( D \) is regular in \( B \), we conclude that \( \mathcal{K} \subseteq B \).

Suppose \( J \subseteq B \) and \( J \cap \mathcal{K} = (0) \). Arguing as in the second paragraph of the proof of Claim 2, \( J \cap \mathcal{B}(\mathcal{H}) = (0) \). Let \( E : B \to D \) be the conditional expectation. Taking \( v = I \) in [4, Proposition 3.10] we obtain

\[
E(J) \subseteq J \cap D \subseteq J \cap \mathcal{B}(\mathcal{H}) = (0).
\]

The faithfulness of \( E \) now gives \( J = (0) \). Thus \( \mathcal{K} \subseteq B \) is an essential ideal.

We now complete the proof of the part of Theorem 3.7 assuming the sum of the atoms of \( D \) is the identity. Arguing by contradiction, suppose \( (B, D) \) is a regular MASA inclusion such that \( D \subseteq \mathcal{B}(\mathcal{H}) \subseteq B \). Apply Claim 2 to obtain a \( C^* \)-diagonal \( (C, D) \) with \( D \subseteq \mathcal{B}(\mathcal{H}) \subseteq C \). By Claim 3, \( \mathcal{K} \) is an essential ideal of \( C \), so every element of \( C \) is an element of the multiplier algebra of \( \mathcal{K} \), that is, \( C \subseteq \mathcal{B}(\mathcal{H}) \), so \( C = \mathcal{B}(\mathcal{H}) \). Therefore, the inclusion \( (\mathcal{B}(\mathcal{H}), D) = (C, D) \) is regular, contradicting Claim 1.

\[ \Box \]

It follows from the proof of Theorem 3.7 that whenever \( D \) is a MASA in \( \mathcal{B}(\mathcal{H}) \), then \( (\mathcal{B}(\mathcal{H}), D) \) is not regular. It would be desirable to exhibit an operator \( T \) which does not belong to the norm closure of the span of \( N(\mathcal{B}(\mathcal{H}), D) \).

When \( (A, D) \) is a MASA inclusion with \( A \) non-unital, some desirable properties of \( (A, D) \) (e.g. regularity or the unique state extension property) pass to \( (\hat{A}, \hat{D}) \). Using the notation and context
of part (2) of Corollary 2.4, we now observe that this need not be the case when \((\tilde{A}, \tilde{D})\) is replaced with \((B, D_B)\), even when the original inclusion \((A, D)\) is well-behaved. Here is an example.

**Example 3.8.** Let \(\mathcal{H} = \ell^2(\mathbb{N})\) and let \(D\) be the set of all multiplication operators \(\ell^2(\mathbb{N}) \ni (\zeta_n)_{n \in \mathbb{N}} \mapsto (d_n \zeta_n)_{n \in \mathbb{N}}, \) where \((d_n)_{n \in \mathbb{N}} \in \mathbb{C}_0\). Then \((\mathcal{H}, D)\) is a \(C^*\)-diagonal.

We consider three choices of a unital algebra \(B\) containing \(\mathcal{K}\) as an essential ideal.

1. Take \(B = \tilde{\mathcal{K}}\). With this choice, \((B, D_B) = (\tilde{\mathcal{K}}, \tilde{D})\) is a \(C^*\)-diagonal.
2. Let \(S\) be the unilateral shift and let \(B = C^*(S)\) be the Toeplitz algebra. Then \(D_B = C^*(\{S^n S^m : n, m \in \mathbb{N}\} \cup \{I\}) \simeq \tilde{D}\). Here \((B, D_B)\) is a Cartan pair, but not a \(C^*\)-diagonal because the unique state extension property fails.
3. Finally, take \(B = M(\mathcal{K}) = \mathcal{B}(\mathcal{H})\). Then \(D_B \simeq \ell^\infty\) is an atomic MASA in \(B\). By Theorem 3.7, the inclusion \((B, D_B)\) is not regular. As is now well-known, the remarkable paper [10] gives the solution to the Kadison-Singer problem, so \((B, D_B)\) has the extension property.

**Problem 3.9.** Suppose \((A, D)\) is a \(C^*\)-diagonal and \(B\) is a unital \(C^*\)-algebra containing \(A\) as an essential ideal. Find conditions on \((A, D)\) and \(B\) which ensure that \((B, D_B)\) is regular, or that \((B, D_B)\) has the extension property.

Let \(X\) be a locally compact, non-compact abelian group, fix \(x_0 \in X\), let \(\Gamma\) be the subgroup generated by \(x_0\), and let \(\Gamma\) act on \(C_0(X)\) by translation. A class of \(C^*\)-diagonals which may provide insight into Problem 3.9 are those of the form \((C_0(X) \rtimes \Gamma, C_0(X))\).

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