A LIOUVILLE THEOREM FOR THE COMPLEX MONGE-AMPERE EQUATION

YU WANG

Abstract. In this note, we derive a Liouville theorem for the complex Monge-Ampère equation from the small perturbation result of O. Savin [4]. Let $dx$ stands for the standard Lebesgue measure, our result states that if a plurisubharmonic function $u$ solves

$$(i\partial \bar{\partial} u)^n = dx, \quad \text{on } \mathbb{C}^n$$

and $u$ satisfies the growth condition

$$u = \frac{1}{2} |x|^2 + o(|x|^2), \quad \text{as } x \to \infty,$$

then $u$ differs from $|x|^2/2$ by a linear function.

1. Introduction

In this note, we consider the global solutions of the complex Monge-Ampère equation. Denote the Lebesgue measure by $dx$, our result states:

**Theorem 1.1.** If the plurisubharmonic function $u$ is a viscosity solution of

$$(i\partial \bar{\partial} u)^n = dx, \quad \text{on } \mathbb{C}^n$$

and $u$ satisfies the growth condition

$$u = \frac{1}{2} |x|^2 + o(|x|^2), \quad \text{as } x \to \infty,$$

then

$$u = \frac{1}{2} |x|^2 + l(x)$$

where $l(x)$ is a linear function.

By a linear change of coordinates, one can replace $|x|^2/2$ by every quadratic polynomial $P$ such that

$$(i\partial \bar{\partial} P)^n = dx.$$

We have stated the theorem in terms of viscosity solutions for our convenience. It is also valid for pluripotential solutions, as viscosity and pluripotential solutions are equivalent (see [2, 5]).
Unlike the real Monge-Ampère equation, global solutions of (1.1) cannot be classified without any restriction on solution’s growth at infinity. Consider the following example due to Blocki [1]: the function

$$u = |z| (1 + |w|^2)$$

satisfies (1.1) on $\mathbb{C}^2$ in viscosity sense. However $u$ is clearly not the pull back of a quadratic polynomial by a holomorphic mapping. In fact, $u$ is not even $C^2$ at the points $\{(z, w) : z = 0\}$. We also notice that, along the diagonal direction $z = w$

$$u(x) \sim |x|^3,$$

as $x = (z, w) \to \infty$.

A disadvantage of Theorem 1.1 (and our proof) is that we have not been able to handle the case $u - |x|^2 / 2$ has exactly quadratic growth, i.e.,

$$\frac{|x|^2}{2C} \leq u - |x|^2 / 2 \leq \frac{C}{2} |x|^2, \quad C > 1.$$

The author believe that $u$ is a quadratic polynomial in this case.

The study of the complex Monge-Ampère equation is largely motivated by the study of Kähler geometry. From the geometric point of view, our theorem implies the following rigidity statement.

**Corollary 1.2.** Suppose that $g$ is a Ricci-flat Kähler metric on $\mathbb{C}^n$. Let $\varphi$ be its Kähler potential and $\mu_g$ be the induced measure. Denote $|B_1|$ the Lebesgue measure of the unit ball.

If $\mu_g$ is comparable with Lebesgue measure, i.e.,

$$C^{-1} dx \leq \mu_g \leq C dx$$

and

$$\varphi = \left( \frac{\mu_g(B_1)}{|B_1|} \right)^{1/n} \frac{|x|^2}{2} + o(|x|^2) \quad \text{as} \quad x \to \infty,$$

then

$$\left( \frac{|B_1|}{\mu_g(B_1)} \right)^{1/n} g$$

is the Euclidean metric.

The above statement would be more satisfactory if one can replace the analytic condition (1.4) by a pure geometric condition.

To end the introduction, we would like to mention that if one replace the condition (1.2) in Theorem 1.1 by

$$u = \frac{1}{2} |x|^2 + O(1)$$

then the conclusion can be derived from an unpublished result of Kolodziej [3]. This reference is pointed out to the author by S. Dinew. The method in this paper is independent from the work of Kolodziej.
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2. Preliminaries

In this section, we recall some basic facts regarding the complex Monge-Ampère operator and the statement of Savin’s small perturbation theorem.

2.1. Complex Monge-Ampère equation in real Hessian. Let $\text{Sym}(2n)$ be the space of $2n \times 2n$ symmetric matrices equipped with the standard spectral normal

$$\|M\| = \max\{|\lambda_i|\}, \lambda_i, \text{ eigenvalue of } M \in \text{Sym}(2n).$$

Let $\text{Herm}(n)$ be the space of $n \times n$ Hermitian matrices. Denote the $n \times n$ identity matrix by $I_n$.

The space $\mathbb{C}^n$ can be identify to $\mathbb{R}^{2n}$ equipped with the complex structure

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$ 

This identification induces an embedding $\iota$ of $\text{Herm}(n)$ to $\text{Sym}(2n)$

$$\iota : H = A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$ 

Moreover, we have

$$\iota(\text{Herm}(n)) = \{M \in \text{Sym}(2n) | [M, J] = 0\}.$$ 

From now on, we shall identify a Hermitian matrix and its image under the embedding $\iota$.

Let $\varphi$ be a $C^2$-function on $\mathbb{C}^n$. Recall that

$$(i\partial\bar{\partial}\varphi)^n = \det(2\varphi_{z_i\bar{z}_k}) \, dx.$$ 

Denote the real Hessian of $\varphi$ by $D^2\varphi$. It is easy to see that

$$\iota(2\varphi_{z_i\bar{z}_k}) = \frac{1}{2} (D^2\varphi + J^T D^2\varphi J)$$

and

$$\det(2\varphi_{z_i\bar{z}_k}) = \det^{1/2} \left[ \frac{1}{2} (D^2\varphi + J^T D^2\varphi J) \right].$$ 

The above discussion immediately implies the following lemma
Lemma 2.1. Let the function \( F : \text{Sym}(2n) \to \mathbb{R} \) be given by

\[
F(M) := \begin{cases} 
\det^{1/2} \left[ \frac{1}{2} (M + J^T MJ) + I \right] - 1 & M + J^T MJ \geq -I \\
-1 & \text{otherwise.}
\end{cases}
\]

If \( u \) is a viscosity solution of (1.1), then

\[
w := u - \frac{|x|^2}{2}
\]

is a viscosity solution of \( F(D^2w) = 0 \), on \( \mathbb{R}^{2n} \).

Proof. Let \( P \) be a quadratic polynomial that touches \( w \) from above, then \( P + \frac{|x|^2}{2} \) touches \( u \) from above. Since \( u \) is a plurisubharmonic function,

\[
\frac{1}{2} \left( D^2P + J^T D^2P J \right) \geq -I
\]

Since \( u \) is a viscosity subsolution of (1.1),

\[
\det^{1/n} \left[ \frac{1}{2} \left( D^2 \varphi + J^T D^2 \varphi J \right) + I \right] = \det \left[ 2 \left( P + \frac{|x|^2}{2} \right) \right] \geq 1.
\]

Therefore, we conclude that \( w \) is a viscosity subsolution of \( F(D^2w) = 0 \). Let \( P \) be a quadratic polynomial that touches \( w \) from below. It suffices to consider the case

\[
\frac{1}{2} \left( D^2P + J^T D^2P J \right) > -I.
\]

Since otherwise, \( F(D^2P) = -1 < 0 \). Again \( P + \frac{|x|^2}{2} \) touches \( u \) from below. By the fact that \( u \) is also a viscosity supersolution of (1.1), we have

\[
\det^{1/n} \left[ \frac{1}{2} \left( D^2 \varphi + J^T D^2 \varphi J \right) + I \right] = \det \left[ 2 \left( P + \frac{|x|^2}{2} \right) \right] \leq 1.
\]

In turn \( F(D^2P) \leq 0 \) and \( w \) is a viscosity supersolution of \( F(D^2w) = 0 \). \( \square \)

2.2. Small perturbation theorem. We recall the small perturbation theorem due to Savin [4]. The following definition will be convenient.

Definition 2.2. Given constants \( \delta, \theta, K > 0 \), the family \( \mathcal{F}_{\delta, \theta, K} \) consists of functions \( F : \text{Sym}(2n) \to \mathbb{R} \) that satisfy the following conditions:

- **H1:** For every \( M \in \text{Sym}(2n) \)
  
  \[ F(M + P) \geq F(M), \quad \forall P \geq 0. \]

- **H2:** \( F(0) = 0 \).

- **H3:** For every \( M \) with \( \|M\| \leq \delta \)
  
  \[ \theta^{-1} \|P\| \geq F(M + P) - F(M) \geq \theta \|P\|, \quad \forall P \geq 0, \|P\| \leq \delta. \]
**H4:** $F$ is twice differentiable in the set $\{M \mid \|M\| \leq \delta\}$ and
\[ |D^2 F(M)| \leq K. \]

We state the following version of the small perturbation theorem.

**Theorem 2.3** (Savin, 2007). Given constant $\delta > 0$, if
\[ F \in \mathcal{F}_{\delta, \theta, K}, \text{ for some } \theta, K > 0, \]
then there exist constant $\mu$ only depending on $n, \delta, \theta, K$ such that, if $u \in C(B_1)$ is a viscosity solution of $F(D^2 u) = 1$ and
\[ \|u\|_{L^\infty(B_1)} \leq \mu, \]
then $u$ is $C^2(B_{1/2})$ and
\[ \|D^2 u\|_{L^\infty(B_{1/2})} \leq \delta. \]

We end this section with the following lemma.

**Lemma 2.4.** Let $F$ be the function on $\text{Sym}(2n)$ given by (2.1). There exists constants $\theta, K$ only depends on $n$ such that
\[ F \in \mathcal{F}_{\delta, \theta, K}, \forall \delta < 1/3. \]

**Proof.** The fact that $F$ satisfies $\text{H1}$ and $\text{H2}$ of Definition 2.2 follows immediately from its expression.

Let $D_{ij} F$ be the differentiation of $F$ with respect to the $ij$-entry of a matrix variable. By direct calculation, we have
\[ |D_{ij} F(M)| = \left| \left( D_{ij} \det^{1/2} \right) \left[ \frac{1}{2} (M + J^T MJ) + I \right] \right| \]
\[ |D_{ij,kl}^2 F| = \left| \left( D_{ij,kl}^2 \det^{1/2} \right) \left[ \frac{1}{2} (M + J^T MJ) + I \right] \right|. \]

The fact that $F$ satisfies $\text{H3}$ and $\text{H4}$ then follows from
\[ \|M + J^T MJ\| \leq 2 \|M\| \]
and
\[ \left| D_{ij} \det^{1/2}(N) \right|, \left| D_{ij,kl}^{1/2} \det^{1/2}(N) \right| \leq C(n), \forall N, \frac{1}{3} I \leq N \leq 3I. \]
3. Proof of the Main Statements

The Theorem 1.1 follows from Theorem 2.3 via a simple scaling argument.

Proof of Theorem 1.1. By a translation of coordinate, it suffices to show that $u$ is $C^2$ and

$$D^2 u(0) = I.$$ 

Consider the function

$$(3.1)\quad w_R(x) := \frac{1}{R^2} u(Rx) - \frac{1}{2} |x|^2$$

Claim:

$$(3.2)\quad \|w_R\|_{L^\infty(B_1)} \leq \frac{o(R^2)}{R^2}$$

Consider the domain $B_1$, both $u(Rx)/R^2$ and $|x|^2/2$ satisfies

$$(i\partial\bar{\partial}\varphi)^n = dx$$

in $B_1$. By the comparison principle of the complex Monge-Ampère operator, we conclude that

$$\left\| \frac{1}{R^2} u(Rx) - \frac{|x|^2}{2} \right\|_{L^\infty(B_1)} \leq \left\| \frac{1}{R^2} u(Rx) - \frac{|x|^2}{2} \right\|_{L^\infty(\partial B_1)} = \frac{1}{R^2} \left\| u - |x|^2/2 \right\|_{L^\infty(\partial B_R)}.$$ 

The claim (3.2) then follows from the assumption (1.2).

Now, let $F$ be the operator given by (2.1) and $\theta, K$ be the constants given by Lemma 2.4. For every $\delta \in (0, 1/3)$, let $\mu_\delta$ be the constant produced by Theorem 2.3 with respect to $F_{\delta, \theta, K}$.

By Lemma 2.4 $F \in F_{\delta, \theta, K}$. It is easy to see that

$$w_R = \frac{1}{R^2} \left( u(Rx) - \frac{1}{2} |Rx|^2 \right)$$

satisfies $F(D^2w) = 0$ for any $R > 0$. Moreover, by the claim (3.2), we can take $R$ large so that

$$\|w_R\|_{L^\infty(B_1)} \leq \mu_\delta.$$ 

Therefore, we can apply Theorem 2.3 to conclude that $w_R$ is $C^2$ in $B_{1/2}$ and

$$\|D^2 w_R(0)\| \leq \delta.$$ 

It follows then $u$ is $C^2$ in $B_{R/2}$ and

$$\|D^2 u(0) - I\| \leq \delta.$$ 

The desired conclusion follows by letting $\delta$ tend to 0. \[\square\]

Next, we prove Corollary 1.2.
Proof of Corollary 1.2. Since $g$ is Ricci flat, we have
\[ \Delta \log \det(\varphi_{zi\bar{z}_k}) = 0, \text{ on } \mathbb{C}^n \]
Since $\mu_g$ is comparable with Lebesgue measure, we have
\[ C^{-1} \leq \det(\varphi_{zi\bar{z}_k}) \leq C. \]
Therefore, $\log \det(\varphi_{zi\bar{z}_k})$ is a bounded harmonic function on entire $\mathbb{C}^n$. Henceforth,
\[ \log \det(\varphi_{zi\bar{z}_k}) = \text{constant}. \]
It follows then $\varphi$ satisfies
\[ (i\partial \bar{\partial} \varphi) = \frac{\mu_g(B_1)}{|B_1|} dx, \text{ on } \mathbb{C}^n \]
Along with the assumption (1.4), we see
\[ \tilde{\varphi} := \left( \frac{|B_1|}{\mu_g(B_1)} \right)^{1/n} \varphi \]
satisfies the hypotheses of Theorem 1.1. Therefore, we conclude that
\[ \tilde{\varphi} = \frac{1}{2} |x|^2 + l(x). \]
The desired conclusion follows. \hfill \Box

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Department of Mathematics, Columbia University, Room 509, MC 4406, 2990 Broadway, New York, NY 10027, USA
E-mail address: yuwang@math.columbia.edu