Inversion of the Toeplitz-block Toeplitz matrices and the structure of the corresponding inverse matrices

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Abstract

The results on the inversion of convolution operators and Toeplitz matrices in the 1-D (one dimensional) case are classical and have numerous applications. We consider a 2-D case of Toeplitz-block Toeplitz matrices, describe a minimal information, which is necessary to recover the inverse matrices, and give a complete characterisation of the inverse matrices.

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1 Introduction

The well-known Toeplitz matrices are diagonal-constant matrices whereas Toeplitz-block Toeplitz matrices $T$ are block Toeplitz matrices, the blocks of which also are Toeplitz:

\[ T = \{T_{i-k}\}_{i,k=1}^n, \quad T_r = \{t_{j-l}\}_{j,l=1}^m. \]  

(1.1)

Later we use for Toeplitz-block Toeplitz matrices the acronym TBT. One may consider TBT-matrices as 2-D (two-dimensional) analog of Toeplitz matrices.
Toeplitz matrices (i.e., 1-D Toeplitz matrices) and their continuous analogs (so called convolution operators or operators with difference kernels) are very important in mathematical analysis and various applications (see e.g. [6, 7, 13, 25, 29, 33] and references therein). The inversion of these matrices and operators is connected with the names of N. Wiener, E. Hopf, N. Levinson, M.G. Krein, I.C. Gohberg, V.A. Ambartsumyan, L.A. Sakhnovich and many other mathematicians and applied scientists. The inversion of convolution operators on a semi-axis (of Wiener-Hopf operators) was studied in numerous papers including the brilliant works [14, 20]. The situation with the inversion of finite Toeplitz matrices and of convolution operators on a finite interval is (in many respects) more complicated and essentially different from the case of semi-axis. Important results on this topic were derived, in particular, in [3, 15, 21, 35, 36]. Then, the procedure of recovery of the operator $T^{-1}$, which is inverse to a general-type convolution operator $T$ on interval, from the action of $T^{-1}$ on two functions was given in [31] (see also [32, 33] and references therein), using the method of operator identities. The same method was applied to the general-type finite Toeplitz matrices in [26]. The structure of the matrices and operators $T^{-1}$ was derived in this way as well. The method of operator identities, which was introduced in [30, 31], may be successfully used for the inversion of various other structured matrices and operators. See, for instance, [10] and [29, Appendix D]. Note also that relations of the form $S^*TS = T$ were used for the study of Toeplitz operators in the seminal work [8].

The inversion of Toeplitz and Toeplitz-block Toeplitz (TBT) matrices and their continuous analogs is actively studied in the recent years as well (see e.g. [1, 2, 4, 5, 9, 10, 17, 24, 33, 38] and references therein). However, in spite of some interesting recent and older works [12, 16–19, 22, 23, 37] on the inversion of TBT-matrices and of convolution operators in multidimensional spaces, the structure of the corresponding inverse matrices and operators (and the way to recover these inverses from some minimal information) remained unknown. Our paper deals with this important problem for TBT-matrices (some less complete results on the inversion of convolution operators on a rectangular are presented in [28]).

It is easy to show (see Section 2) that a TBT-matrix $T$ of the form (1.1)
satisfies two matrix identities:

\[ A_p T - T A_p^* = Q_p \quad (p = 1, 2), \]

where \( \text{rank}(Q_1) \leq 2m \), \( \text{rank}(Q_2) \leq 2n \) and the matrices \( A_p \) are discrete analogs of integration operators:

\[
A_p = \left\{ A_{p,i-k} \right\}_{i,k=1}^{n}, \quad A_{1,r} = \begin{cases} 0 & \text{for } r < 0, \\ i/2 I_m & \text{for } r = 0, \\ i I_m & \text{for } r > 0; \end{cases}
\]

\[
A_{2,r} = 0 \quad \text{for } r \neq 0, \quad A_{2,0} = \left\{ a_{j-\ell} \right\}_{j,\ell=1}^{m}; \quad a_r = \begin{cases} 0 & \text{for } r < 0, \\ i/2 & \text{for } r = 0, \\ i & \text{for } r > 0. \end{cases}
\]
In Section 2 we show (see (2.7)) that \( \rho(y, z) \) is easily expressed via
\[
\omega(\lambda, \mu) = 1^*(A_1^* - \mu_1 I)^{-1}(A_2^* - \mu_2 I)^{-1}T^{-1}(A_2 - \lambda_2 I)^{-1}(A_1 - \lambda_1 I)^{-1}1.
\]
(1.7)

Then, we derive an important representation of \( \omega \) in the main Theorem 2.3. This representation is based on the minimal information on \( T^{-1} \) contained in the matrix \( g_{12} \) (or \( g_{21} \), for both matrices see (2.35)) with \( 4mn \) entries. In Section 3 we complete the study of the structure of \( T^{-1} \) (see Theorem 3.3). Some auxiliary results on determinants are derived in Appendix.

The approach works for other important 2-D structured matrices. As usual \( \mathbb{C} \) stands for the complex plane and \( g^r \) stands for the matrix which is the transpose of \( g \). By diag we denote a diagonal (or block diagonal) matrix. For instance, \( \text{diag}\{d_1, d_2\} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \). The notation \( \text{col} \) (column) stands for a column vector or block vector: \( \text{col}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \). The notation \( I_r \) stands for the \( r \times r \) identity matrix and the indices of the matrices \( U_r, J_{2m} \) and \( J_{2n} \) (introduced in (2.37), (2.38), (2.43)) also indicate the orders of the corresponding matrices. In all other cases the indices of the matrices are not related to the order. Many notations were explained before in the text of the Introduction (see, e.g., Notation 1.1).

\section{Representation of the \( \rho \)-polynomial}

1. It is immediate that the matrix \( A_1 \) and the block diagonal matrix \( A_2 \) (given by (1.3) and (1.4)) commute. One easily derives (see e.g. [27, p. 452]) that
\[
(A_1 - \lambda_1 I)^{-1} \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix} = \left( \frac{i}{2} - \lambda_1 \right)^{-1} \begin{bmatrix} I_m \\ \psi(\lambda_1) I_m \\ \cdots \\ \psi(\lambda_1)^{n-1} I_m \end{bmatrix}, \quad \psi(\lambda_1) := \frac{\lambda_1 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}}.
\]
(2.1)
As a special case of (2.1) we have

\[(A_{2,0} - \lambda_2 I_m)^{-1}1_m = \left(\frac{i}{2} - \lambda_2\right)^{-1} \left[\begin{array}{c} 1 \\ \vdots \\ \psi(\lambda_2) \\ \psi(\lambda_2)^{m-1}\end{array}\right]. \tag{2.2}\]

Since \(A_2 = \text{diag}\{A_{2,0}, \ldots, A_{2,0}\}\), relations (1.5), (2.1) and (2.2) imply that

\[\left(\frac{i}{2} - \lambda_2\right)\left(\frac{i}{2} - \lambda_1\right)(A_2 - \lambda_2 I)^{-1}(A_1 - \lambda_1 I)^{-1}1 = h(\psi(\lambda_1), \psi(\lambda_2)). \tag{2.3}\]

Hence, we also have

\[\left(\frac{i}{2} + \mu_2\right)\left(\frac{i}{2} + \mu_1\right)1^*(A_1^* - \mu_1 I)^{-1}(A_2^* - \mu_2 I)^{-1} = h(\psi(\mu_1), \psi(\mu_2))^*. \tag{2.4}\]

Further in the text we assume that \(T\) is invertible. It is immediate from (1.6), (1.7), (2.3) and (2.4) that

\[
\rho(\psi(\lambda_1), \psi(\lambda_2), \psi(\mu_1), \psi(\mu_2)) = \left(\frac{i}{2} + \mu_2\right)\left(\frac{i}{2} + \mu_1\right)
\times \left(\frac{i}{2} - \lambda_2\right)\left(\frac{i}{2} - \lambda_1\right)\omega(\lambda, \mu). \tag{2.5}\]

Recalling that \(\psi\) is given in (2.1), we easily construct the function inverse to \(\psi\) and rewrite the equalities \(y_k = \psi(\lambda_k)\) and \(z_k = \psi(\mu_k)\) in the form

\[\lambda_k = \varphi(y_k), \quad \mu_k = -\varphi(z_k), \quad \varphi(\xi) := \frac{i \xi + 1}{2 \xi - 1}. \tag{2.6}\]

Using (2.6), we rewrite (2.5) in order to express \(\rho(y, z)\) via \(\omega\):

\[\rho(y, z) = \left(\frac{i}{2} - \varphi(z_2)\right)\left(\frac{i}{2} - \varphi(z_1)\right)
\times \left(\frac{i}{2} - \varphi(y_2)\right)\left(\frac{i}{2} - \varphi(y_1)\right)\omega(\varphi(y_1), \varphi(y_2), -\varphi(z_1), -\varphi(z_2)). \tag{2.7}\]

Thus, a representation of \(\rho\) will follow from a representation of \(\omega\).
2. Next, we show that the matrix identities (1.2) hold and consider them in detail. Indeed, according to [27, (1.2)] we have

\[ A_1 T - T A_1^* = i (M_{11} M_{21} + M_{31} M_{41}) \]  

(2.8)

where

\[ M_{11} = \text{col} \left[ \mathcal{M}_{11}^{(1)} \mathcal{M}_{11}^{(2)} \ldots \mathcal{M}_{11}^{(n)} \right], \quad \mathcal{M}_{11}^{(i)} := \frac{1}{2} T_0 + \sum_{s=1}^{i-1} T_s, \]  

(2.9)

\[ M_{21} = [I_m \ 0 \ldots \ 0], \quad M_{31} = M_{21}^*, \]  

(2.10)

\[ M_{41} = \left[ \mathcal{M}_{41}^{(1)} \mathcal{M}_{41}^{(2)} \ldots \mathcal{M}_{41}^{(n)} \right], \quad \mathcal{M}_{41}^{(k)} := \frac{1}{2} T_0 + \sum_{s=1}^{k-1} T_s; \]  

(2.11)

We derive the expressions for \( A_2, 0 T - T A_2^* \) as the special cases of (2.8) (after setting in (2.8) \( m = 1 \) and substituting then \( n = m \)). In view of these expressions, for the matrix \( A_2 = \text{diag} \{ A_{2,0}, \ldots, A_{2,0} \} \) the next identity follows:

\[ A_2 T - T A_2^* = i (M_{12} M_{22} + M_{32} M_{42}) \]  

(2.12)

where \( M_{12}, M_{32} \) and \( M_{22}, M_{42} \) are \( mn \times n \) and \( n \times mn \), respectively, matrices,

\[ M_{12} = \{ \mathcal{M}_{12}^{(i-k)} \}_{i,k=1}^{n, n}, \quad M_{22} = \{ \mathcal{M}_{22}^{(i-k)} \}_{i,k=1}^{n, n}, \]  

(2.13)

\[ M_{32} = M_{22}^*, \quad M_{42} = \{ \mathcal{M}_{42}^{(i-k)} \}_{i,k=1}^{n, n}, \]  

(2.14)

\[ \mathcal{M}_{12}^{(r)} := \text{col} \left[ \frac{1}{2} t_r^{(0)} + \sum_{s=1}^{r-1} t_s^{(0)} + \sum_{s=0}^{m} t_{r}^{(s)} \right], \]  

(2.15)

\[ \mathcal{M}_{22}^{(r)} = 1_n, \quad \text{for } r = 0; \quad \mathcal{M}_{22}^{(r)} = 0, \quad \text{for } r \neq 0, \]  

(2.16)

\[ \mathcal{M}_{42}^{(r)} := \left[ \frac{1}{2} t_r^{(0)} + \sum_{s=0}^{m} t_{r}^{(s)} \right]. \]  

(2.17)

Clearly, the matrices \( M_{1p} \) and \( M_{4p} \) satisfy some operator identities which are similar to the identities (2.8) and (2.12). Indeed, put

\[ \mathcal{A}_1 = \{ a_{j-\ell} \}_{j,\ell=1}^{n}, \quad a_r = \begin{cases} 0 \text{ for } r < 0, \\ i/2 \text{ for } r = 0, \\ i \text{ for } r > 0; \end{cases} \]  

(2.18)
where $\mathcal{A}_{2,0}$ is given in (1.4). We note that $\mathcal{A}_1$ coincides with the special case of $A_1$ where $m = 1$, and $\mathcal{A}_2$ coincides with the special case of $A_2$ where $n = 1$. Similarly to (2.8) and (2.12), one can show that

\begin{equation}
\mathcal{A}_s M_{4p} - M_{4p} \mathcal{A}_s = iQ_s \quad \text{for} \quad s = 1, \quad p = 2 \quad \text{and} \quad s = 2, \quad p = 1; \quad (2.19)
\end{equation}

\begin{equation}
Q_1 = K_{11} M_{21} + 1_n K, \quad Q_2 = K_{12} M_{22} + 1_m K; \quad (2.20)
\end{equation}

\begin{equation}
K = \left[ \frac{1}{2} \mathcal{M}_{42}^{(0)} + \frac{1}{2} \mathcal{M}_{42}^{(0)} + \mathcal{M}_{42}^{(-1)} \quad \ldots \quad \frac{1}{2} \mathcal{M}_{42}^{(0)} + \sum_{k=2}^{n} \mathcal{M}_{42}^{(1-k)} \right], \quad (2.21)
\end{equation}

\begin{equation}
K_{11} = \text{col} \left[ \frac{1}{2} \mathcal{M}_{12}^{(0)} + \frac{1}{2} \mathcal{M}_{12}^{(0)} + \mathcal{M}_{12}^{(-1)} \quad \ldots \quad \frac{1}{2} \mathcal{M}_{12}^{(0)} + \sum_{i=2}^{n} \mathcal{M}_{12}^{(i-1)} \right], \quad (2.22)
\end{equation}

\begin{equation}
K_{12} = \left[ \frac{1}{2} \mathcal{M}_{12}^{(0)} + \frac{1}{2} \mathcal{M}_{12}^{(0)} + \mathcal{M}_{12}^{(-1)} \quad \ldots \quad \frac{1}{2} \mathcal{M}_{12}^{(0)} + \sum_{k=2}^{n} \mathcal{M}_{12}^{(1-k)} \right]. \quad (2.23)
\end{equation}

3. The identities (2.8) and (2.12) may be rewritten in the form

\begin{equation}
A_p T - TA_p = i\Pi_p \hat{\Pi}_p, \quad \Pi_p := \begin{bmatrix} M_{1p} & M_{3p} \end{bmatrix}, \quad \hat{\Pi}_p := \begin{bmatrix} M_{2p} \\ M_{4p} \end{bmatrix}. \quad (2.24)
\end{equation}

Introduce the notations:

\begin{equation}
R := T^{-1}; \quad \Gamma_p := R \Pi_p, \quad \hat{\Gamma}_p := \hat{\Pi}_p R \quad (p = 1, 2); \quad (2.25)
\end{equation}

\begin{equation}
\Gamma := \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}, \quad \hat{\Gamma} := \begin{bmatrix} \hat{\Gamma}_1 \\ \hat{\Gamma}_2 \end{bmatrix}, \quad P_1 := \begin{bmatrix} I_{2m} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 := I_{2(m+n)} - P_1. \quad (2.26)
\end{equation}

We start with the following simple representations of $\omega$.

**Proposition 2.1** Let $\omega(\lambda, \mu)$ be given by (1.7). Then we have

\begin{equation}
\omega(\lambda, \mu) = i(\lambda_p - \mu) u(\mu) P_p \hat{\omega}(\lambda) \quad (p = 1, 2), \quad (2.27)
\end{equation}

where

\begin{equation}
u(\lambda) = 1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} \Gamma
- i \begin{bmatrix} 1_m^* (A_1^* - \mu_2 I_m)^{-1} 0 \\ 1_n^* (A_1^* - \mu_1 I_n)^{-1} 0 \end{bmatrix}, \quad (2.28)
\end{equation}

\begin{equation}
\hat{\omega}(\lambda) = \hat{\Gamma} (A_2 - \lambda_2 I)^{-1} (A_1 - \lambda_1 I)^{-1} 1
+ i \text{col} \left[ 0 \quad (A_2 - \lambda_2 I_m)^{-1} 1_m \quad 0 \quad (A_1 - \lambda_1 I_n)^{-1} 1_n \right]. \quad (2.29)
\end{equation}
Proof. Relations (2.24) and (2.25) yield

\[ RA_p - A_p^* R = i \Gamma_p \hat{\Gamma}_p. \]  

(2.30)

Hence, we have

\[ (\mu_p - \lambda_p) R = R(A_p - \lambda_p I) - (A_p^* - \mu_p I) R - i \Gamma_p \hat{\Gamma}_p, \]

or, equivalently,

\[ (A_p^* - \mu_p I)^{-1} R(A_p - \lambda_p I)^{-1} = (A_p^* - \mu_p I)^{-1} R - R(A_p - \lambda_p I)^{-1} \]

\[ - i(A_p^* - \mu_p I)^{-1} \Gamma_p \hat{\Gamma}_p (A_p - \lambda_p I)^{-1}. \]  

(2.31)

It easily follows from the definitions (2.10), (2.13) and (2.16) of \( M \) that

\[ 1^* (A_1^* - \mu_1 I)^{-1} = 1_m^* M_{21} (A_1^* - \mu_1 I)^{-1} = 1_n^* (A_1^* - \mu_1 I_n)^{-1} M_{22}. \]  

(2.32)

\[ 1^* (A_2^* - \mu_2 I)^{-1} = 1_m^* M_{21} (A_2^* - \mu_2 I)^{-1} = 1_n^* (A_2^* - \mu_2 I_m)^{-1} M_{21}. \]  

(2.33)

Now, set in (2.31) \( p = 1 \), multiply both sides of (2.31) by \( 1^* (A_2^* - \mu_2 I)^{-1} \) from the left and by \( (A_2 - \mu_2 I)^{-1} 1 \) from the right, and take into account that \( A_1 \) and \( A_2 \) commute and that the equalities (1.7), (2.33) and \( M_{31} = M_{21}^* \) hold. Then, we obtain

\[ \omega(\lambda, \mu) = i(\lambda_1 - \mu_1)^{-1} \left( 1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} \Gamma_1 \hat{\Gamma}_1 (A_2 - \lambda_2 I)^{-1} \right. \]

\[ \times (A_1 - \lambda_1 I)^{-1} 1 \]  

(2.34)

\[ + i1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} R M_{31} (A_2 - \lambda_2 I_m)^{-1} 1_m \]

\[ - i1_m^* (A_2^* - \mu_2 I_m)^{-1} M_{21} R (A_2 - \lambda_2 I)^{-1} (A_1 - \lambda_1 I)^{-1} 1 \). \]

Recall the definitions of \( \Pi_p, \hat{\Pi}_p, \Gamma_p \) and \( \hat{\Gamma}_p \) in (2.24) and (2.25). Hence, formula (2.27) follows (for \( p = 1 \)) from (2.28), (2.29) and (2.34).

Formula (2.27) for \( p = 2 \) follows in the same way (as (2.27) for \( p = 1 \)) but this time from (2.31) (with \( p = 2 \)) and from (2.32). \( \blacksquare \)

4. In Proposition 2.1, we recover \( \omega \) (and so \( \rho \) and \( T^{-1} \)) either from \( \Gamma_1 = T^{-1} \Pi_1 \) and \( \hat{\Gamma}_1 = \hat{\Pi}_1 T^{-1} \) or from \( \Gamma_2 = T^{-1} \Pi_2 \) and \( \hat{\Gamma}_2 = \hat{\Pi}_2 T^{-1} \), which means
that only one of the operator identities (either (2.8) or (2.12)) is used. In this paragraph we will recover \( \omega \) either from the \( 2m \times 2n \) matrix \( \hat{\Pi}_1T^{-1}\Pi_2 \) or from the \( 2n \times 2m \) matrix \( \hat{\Pi}_2T^{-1}\Pi_1 \), using both operator identities.

First introduce matrices

\[
g_{12} = i\hat{\Pi}_1T^{-1}\Pi_2 - i \begin{bmatrix} 1_m^1 & 0 \\ K_{12} & 0 \end{bmatrix}, \quad g_{21} = i\hat{\Pi}_2T^{-1}\Pi_1 - i \begin{bmatrix} 1_n^1 & 0 \\ K_{11} & 0 \end{bmatrix},
\]

(2.35)

where \( K_{11} \) and \( K_{12} \) are given in (2.22) and (2.23), respectively. The connection between \( g_{12} \) and \( g_{21} \) is described by the simple formula

\[
g_{21} = -U_{2n}J_{2n}g_{12}^\tau J_{2m}U_{2m},
\]

(2.36)

where \( g_{12}^\tau \) is the transpose of \( g_{12} \) and

\[
U_{2n} = \{\delta_{2n-i-k+1}\}_{i,k=1}^{2n}, \quad J_{2n} = \text{diag}\{I_n,-I_n\}, \quad U_{2m} = \{\delta_{2m-i-k+1}\}_{i,k=1}^{2m}, \quad J_{2m} = \text{diag}\{I_m,-I_m\}.
\]

(2.37)

(2.38)

The validity of (2.36) is shown in the proof of the main Theorem 2.3 and means that given \( g_{12} \) we easily construct \( g_{21} \) and vice versa.

Next, we introduce the matrix function

\[
G(\lambda) := \begin{bmatrix}
A_2 - \lambda_2 I_m & 0 & g_{12} \\
0 & A_2 - \lambda_2 I_m & 0 \\
g_{21} & A_1 - \lambda_1 I_n & 0 \\
A_1 - \lambda_1 I_n & 0 & A_1 - \lambda_1 I_n
\end{bmatrix},
\]

(2.39)

**Remark 2.2** We introduce a polynomial \( \theta(\lambda) \) via the determinant of \( G(\lambda) \):

\[
\theta(\lambda) = \sqrt{\det(G(\lambda))},
\]

(2.40)

where the branch of the root in (2.40) is chosen so that \( \sqrt{\det(G(\lambda))} \) is a polynomial such that the coefficient corresponding to the term \( \lambda_1^1\lambda_2^n \) equals \(-1\). The existence of the polynomial \( \theta(\lambda) \) and the way to construct it explicitly (when \( G(\lambda) \) of the form (2.39) is given) are shown in Lemma A.2.

In the following main theorem, we express \( \hat{u} \) and \( u \) given by (2.29) and (2.28), respectively, via \( G(\lambda) \) (that is, via \( g_{12} \) or \( g_{21} \)). Then, one can apply Proposition 2.1 in order to construct \( \omega \) and \( \rho \).
Theorem 2.3 Let a TBT-matrix $T$ be invertible and let the corresponding matrix $g_{12}$ or $g_{21}$ (of the form (2.35)) be given.

Then, $\omega(\lambda, \mu)$ of the form (1.7) admits representations (2.27), where $u$ and $\hat{u}$ (which are introduced in (2.28) and (2.29)) may be recovered from the relations

$$\hat{u}(\lambda) = -i\left(\lambda_1 - \frac{i}{2}\right)^{-n}(\lambda_2 - \frac{i}{2})^{-m}\theta(\lambda)G(\lambda)^{-1}\col[0 \ 1_m \ 0 \ 1_n], \quad (2.41)$$

$$u(\mu) = \left(\frac{\mu_1 - \frac{i}{2}}{\mu_1 + \frac{i}{2}}\right)^n\left(\frac{\mu_2 - \frac{i}{2}}{\mu_2 + \frac{i}{2}}\right)^m\hat{u}(\mu)^{\tau}\text{diag}\{J_{2m}U_{2m}, J_{2n}U_{2n}\}, \quad (2.42)$$

$G$ is expressed in (2.39) via $g_{12}$ or, equivalently, via $g_{21}$ (using (2.36)), and $\theta$ is given in (2.40).

Proof. Step 1. In this step we prove the auxiliary equalities (2.42) and (2.36). For this purpose, recall the definitions (2.37) and (2.38) and also set

$$U = U_{mn} = \{\delta_{mn-i-k+1}\}_{i,k=1}^{mn}, \quad U_m = \{\delta_{m-i-k+1}\}_{i,k=1}^{m}. \quad (2.43)$$

It is easy to see that $UTU = T^{\tau}$, and so

$$URU = R^{\tau} \quad (R = T^{-1}). \quad (2.44)$$

Similarly to $UTU = T^{\tau}$ we have

$$U_mT_r^{\tau} = T_rU_m, \quad T_r^{\tau}U_m = U_mT_r,$$

which (in view of the definitions (2.9)–(2.11)) yields

$$UM_{41}^{\tau} + M_{11}U_m = TM_{31}U_m, \quad M_{11}^{\tau}U + U_mM_{41} = U_mM_{21}T. \quad (2.45)$$

Using the definitions (2.13)–(2.17), we derive (in the same way as (2.45)) the equalities

$$UM_{42}^{\tau} + M_{12}U_n = TM_{32}U_n, \quad M_{12}^{\tau}U + U_nM_{42} = U_nM_{22}T. \quad (2.46)$$

Now, let us show that

$$\hat{\Gamma}_1^{\tau} = UT_1U_{2m}J_{2m} + [0 \ M_{31}], \quad \hat{\Gamma}_2^{\tau} = UT_2U_{2n}J_{2n} + [0 \ M_{32}]. \quad (2.47)$$
Indeed, (2.24), (2.25), (2.44) and (2.45) imply that
\[
\hat{\Gamma}_1^r = URU \begin{bmatrix} M_{21}^r & M_{11}^r \end{bmatrix} = UR \begin{bmatrix} M_{31} U_m & -M_{11} U_m + T M_{31} U_m \end{bmatrix} \\
= UR \begin{bmatrix} M_{11} & M_{31} \end{bmatrix} U_m J_{2m} + \begin{bmatrix} 0 & UM_{31} U_m \end{bmatrix},
\]
and the first equality in (2.47) follows. Here we used the immediate equalities
\[
M_{21}^r = M_{31}, \quad UM_{31} U_m = M_{31}.
\] (2.48)

Later we will also need an analog of (2.48) (for the matrices \(M_s\)), namely,
the equalities
\[
M_{32}^r = M_{22}, \quad U_n M_{22} U = M_{22}.
\] (2.49)

Taking into account (2.46), we (similarly to the proof of the first equality in (2.47)) derive the second equality in (2.47). For
\[
F(\mu_1, \mu_2) := \begin{bmatrix} A_1^* - \mu_1 I \end{bmatrix}^{-1} \begin{bmatrix} A_2^* - \mu_2 I \end{bmatrix}^{-1},
\] (2.50)
we will need the following relation:
\[
F(\mu_1, \mu_2) = \left( \frac{\mu_1 - \frac{i}{2}}{\mu_1 + \frac{i}{2}} \right)^n \left( \frac{\mu_2 - \frac{i}{2}}{\mu_2 + \frac{i}{2}} \right)^m \begin{bmatrix} A_1^* + \mu_1 I \end{bmatrix}^{-1} \begin{bmatrix} A_2^* + \mu_2 I \end{bmatrix}^{-1} U. \quad (2.51)
\]

In order to obtain (2.51), we take into account (2.4) and rewrite (2.50) in the form
\[
F(\mu_1, \mu_2) = \{F_r(\mu_1, \mu_2)\}_{r=1}^{mn}
\] (2.52)
\[
= \left( \mu_1 + \frac{i}{2} \right)^{1-i} \left( \mu_2 + \frac{i}{2} \right)^{1-j} \left\{ \left( \frac{\mu_1 - \frac{i}{2}}{\mu_1 + \frac{i}{2}} \right)^{i-1} \left( \frac{\mu_2 - \frac{i}{2}}{\mu_2 + \frac{i}{2}} \right)^{j-1} \right\},
\]
where \(i\) and \(j\) for the \(r\)th entry (of the row vector function) in the braces above are chosen similarly to the way it is done in (1.5). Using the definition (2.50) of \(F\), substituting \(-\mu_p\) instead of \(\mu_p\) (\(p = 1, 2\)) into (2.52) and applying \(U\) from the right, we derive
\[
\begin{bmatrix} A_1^* + \mu_1 I \end{bmatrix}^{-1} \begin{bmatrix} A_2^* + \mu_2 I \end{bmatrix}^{-1} U
\] (2.53)
\[
= \left( \mu_1 - \frac{i}{2} \right)^{1-i} \left( \mu_2 - \frac{i}{2} \right)^{1-j} \left\{ \left( \frac{\mu_1 + \frac{i}{2}}{\mu_1 - \frac{i}{2}} \right)^{n-i} \left( \frac{\mu_2 + \frac{i}{2}}{\mu_2 - \frac{i}{2}} \right)^{m-j} \right\}.
\]
According to (2.53), the right hand sides of (2.51) and (2.52) are equal, and so (2.51) follows from (2.52).

Partition $u$ and $\hat{u}$ into the row and column (respectively) vector blocks:

$$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} \quad (u_1, \hat{u}_1 \in \mathbb{C}^{2m}, \quad u_2, \hat{u}_2 \in \mathbb{C}^{2n}).$$

(2.54)

Taking into account (2.26), (2.28), (2.29), (2.47) and (2.51), we see that

$$u_1(\mu) = \left( \frac{\mu_1 - \frac{i}{2}}{\mu_1 + \frac{i}{2}} \right)^n \left( \frac{\mu_2 - \frac{i}{2}}{\mu_2 + \frac{i}{2}} \right)^m \hat{u}_1(\mu)^\tau J_{2m} U_{2m}.$$  \hspace{1cm} (2.55)

Here, we also used the relation

$$1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} M_{31}$$

$$- i \left( \frac{\mu_1 - \frac{i}{2}}{\mu_1 + \frac{i}{2}} \right)^n \left( \frac{\mu_2 - \frac{i}{2}}{\mu_2 + \frac{i}{2}} \right)^m 1_m (A_2^* + \mu_2 I_m)^{-1} U_m = i 1^*_m (A_2^* - \mu_2 I_m)^{-1},$$

which easily follows from (2.1) and (2.2). The next equality, that is,

$$u_2(\mu) = \left( \frac{\mu_1 - \frac{i}{2}}{\mu_1 + \frac{i}{2}} \right)^n \left( \frac{\mu_2 - \frac{i}{2}}{\mu_2 + \frac{i}{2}} \right)^m \hat{u}_2(\mu)^\tau J_{2n} U_{2n}$$

(2.57)

is proved in the same way as (2.55). Finally, relations (2.54), (2.55) and (2.57) yield (2.42).

Relations (2.45) and (2.46) also help to prove (2.36). Indeed, from the definitions (2.35) and (2.24) we have

$$U_{2n} J_{2n} g_{12} J_{2m} U_{2m} = - i \begin{bmatrix} U_n M_{32}^T \\ U_n M_{42}^T \end{bmatrix} U T^{-1} U \begin{bmatrix} M_{11} U_m & - M_{31} U_m \\ 0 & 0 \end{bmatrix}$$

$$- i \begin{bmatrix} 0 & \hat{1}_m^* \\ -U_n K_{12}^T U_m & 1_m \end{bmatrix}.$$  \hspace{1cm} (2.58)

Using the equalities (2.45) and (2.46) (as well as (2.48) and (2.49)), we rewrite (2.58) in the form

$$U_{2n} J_{2n} g_{12} J_{2m} U_{2m} = - i \begin{bmatrix} M_{22} \\ M_{42} \end{bmatrix} T^{-1} \begin{bmatrix} M_{11} & M_{31} \end{bmatrix} + Z,$$  \hspace{1cm} (2.59)
where
\[
Z = i\begin{bmatrix} M_{22} \\ M_{42} \end{bmatrix} T^{-1} \begin{bmatrix} TM_{31} & 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ M_{22}T \end{bmatrix} T^{-1} \begin{bmatrix} M_{11} & M_{31} \end{bmatrix} - i \begin{bmatrix} 0 \\ M_{22}T \end{bmatrix} T^{-1} \begin{bmatrix} TM_{31} & 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ -U_nK_{12}^T U_m \end{bmatrix} \begin{bmatrix} 0 \\ 1_n 1^*_m \end{bmatrix}
\] (2.60)

We partition \( Z \) into for blocks \( Z = \{ Z_{ps} \}_{p,s=1}^2 \) and easily derive
\[
Z_{11} = iM_{22}M_{31} = i1_n 1^*_m, \quad Z_{22} = i(M_{22}M_{31} - i1_n 1^*_m) = 0, \tag{2.61}
\]
\[
Z_{12} = 0, \quad Z_{21} = i(M_{42}M_{31} + M_{22}(M_{11} - TM_{31}) + U_nK_{12}^T U_m). \tag{2.62}
\]

In view of (2.48), taking the transpose of the second equality in (2.46) we obtain \( M_{11} - TM_{31} = -UM_{41}^TU_m \). Thus, \( Z_{21} \) admits representation
\[
Z_{21} = i(M_{42}M_{31} - M_{22}UM_{41}^TU_m + U_nK_{12}^T U_m). \tag{2.63}
\]

Direct calculations show that
\[
M_{42}M_{31} - M_{22}UM_{41}^TU_m + U_nK_{12}^T U_m = K_{11}, \tag{2.64}
\]
where \( K_{11} \) is introduced in (2.22). According to (2.63) and (2.64), one may rewrite (2.62) as
\[
Z_{12} = 0, \quad Z_{21} = iK_{11}. \tag{2.65}
\]

Clearly, the definition of \( g_{21} \) in (2.35) and relations (2.59), (2.61) and (2.65) imply the equality (2.36).

Step 2. In the Step 2 we prove the basic for our proof equality
\[
G(\lambda)\hat{u}(\lambda) = -i\left(\lambda_1 - \frac{i}{2}\right)^{-n} \left(\lambda_2 - \frac{i}{2}\right)^{-m} \theta(\lambda) \text{col} \begin{bmatrix} 0 & 1_m & 0 & 1_n \end{bmatrix}, \tag{2.66}
\]
which is equivalent to (2.41). For this purpose, we rewrite (2.30) in the form
\[
\Gamma_p \hat{\Gamma}_p = i((A_p^* - \lambda_p I)R - R(A_p - \lambda_p I)) \quad (p = 1, 2). \tag{2.67}
\]

It is easy to see that
\[
(A_1^* - \lambda_1 I_n)^{-1}M_{22} = M_{22}(A_1^* - \lambda_1 I)^{-1}, \tag{2.68}
\]
\[
(A_2^* - \lambda_2 I_m)^{-1}M_{21} = M_{21}(A_2^* - \lambda_2 I)^{-1}, \tag{2.69}
\]
\[
1_m^* M_{21} = 1_n^* M_{22} = 1^*. \tag{2.70}
\]
Since $M_{3p} = M^*_2p$, equalities (2.68)–(2.70) imply that

\[ M_{31}(A_2 - \lambda_2 I_m)^{-1}1_m = (A_2 - \lambda_2 I)^{-1}1, \]  
\[ M_{32}(A_1 - \lambda_1 I_n)^{-1}1_n = (A_1 - \lambda_1 I)^{-1}1. \]  

In view of the definitions (2.24)–(2.26) of the matrices $\hat{G}$, $\Gamma_p$ and $\Pi_p$, and in view of the definition of $\hat{u}_p(\lambda)$ (see (2.29) and (2.54)), formulas (2.67), (2.71) and (2.72) yield

\[ (A^*_p - \lambda_p I)^{-1} \Gamma_p \hat{u}_p(\lambda) = i R (A_2 - \lambda_2 I)^{-1}(A_1 - \lambda_1 I)^{-1}1. \]  

Indeed, in the case $p = 1$ we have

\[ (A_1^* - \lambda_1 I)^{-1} \Gamma_1 \hat{u}_1(\lambda) = i R (A_2 - \lambda_2 I)^{-1}(A_1 - \lambda_1 I)^{-1}1 \]
\[ - i (A_1^* - \lambda_1 I)^{-1} R (A_2 - \lambda_2 I)^{-1}1 + i (A_1^* - \lambda_1 I)^{-1} R M_{31} (A_2 - \lambda_2 I_m)^{-1}1_m, \]

and (2.73) follows. In the same way (2.73) is derived for $p = 2$.

Taking into account (2.29), (2.54) and equality $\hat{\Pi}_p R = \hat{\Gamma}_p$, we see that (2.73) implies the formula

\[
\begin{bmatrix}
\mathfrak{A}_2(\lambda_2) & i \mathfrak{A}_2(\lambda_2) \hat{\Pi}_1 (A^*_2 - \lambda_2 I)^{-1} \Gamma_2 \\
i \mathfrak{A}_1(\lambda_1) \hat{\Pi}_2 (A^*_1 - \lambda_1 I)^{-1} \Gamma_1 & \mathfrak{A}_1(\lambda_1)
\end{bmatrix} \hat{u}(\lambda) = i \begin{bmatrix}
0 \\
1_m \\
0 \\
1_n
\end{bmatrix},
\]

(2.74)

where

\[
\mathfrak{A}_1(\lambda_1) = \begin{bmatrix}
A^*_1 - \lambda_1 I_n & 0 \\
0 & A_1 - \lambda_1 I_n
\end{bmatrix}, \quad \mathfrak{A}_2(\lambda_2) = \begin{bmatrix}
A^*_2 - \lambda_2 I_m & 0 \\
0 & A_2 - \lambda_2 I_m
\end{bmatrix}.
\]

Using the last equality in (2.24) and formulas (2.19), (2.68) and (2.69), we obtain the following equalities (for $p = 1$, $s = 2$ and for $p = 2$, $s = 1$):

\[
\hat{\Pi}_p (A^*_s - \lambda_s I)^{-1} = \mathfrak{A}_s(\lambda_s)^{-1} \left( \hat{\Pi}_p + i \begin{bmatrix}
0 \\
Q_s
\end{bmatrix} (A^*_s - \lambda_s I)^{-1} \right).
\]

(2.75)
Moreover, using again (2.73) we derive

$$\left[ \begin{array}{c} 0 \\ Q_s \end{array} \right] (A^*_s - \lambda_s I)^{-1} \Gamma_s \hat{u}_s(\lambda) = i \left[ \begin{array}{c} 0 \\ Q_s \end{array} \right] R(A_2 - \lambda_2 I)^{-1}(A_1 - \lambda_1 I)^{-1} \mathbf{1}. \quad (2.76)$$

Since $Q_s$ is given by (2.20) (where $K$ is a row) and since $M_{2s}R$ equals $[I_m \ 0] \hat{\Gamma}_1$ or $[I_n \ 0] \hat{\Gamma}_2$ (depending on $s$), one may rewrite the right hand side of (2.76):

$$\left[ \begin{array}{c} 0 \\ Q_1 \end{array} \right] (A^*_1 - \lambda_1 I)^{-1} \Gamma_1 \hat{u}_1(\lambda) = i \left[ \begin{array}{c} 0 \\ K_{11} \\ 0 \end{array} \right] \hat{u}_1(\lambda) + i\tilde{\theta}(\lambda) \left[ \begin{array}{c} 0 \\ 1_m \end{array} \right], \quad (2.77)$$

$$\left[ \begin{array}{c} 0 \\ Q_s \end{array} \right] (A^*_s - \lambda_s I)^{-1} \Gamma_s \hat{u}_s(\lambda) = i \left[ \begin{array}{c} 0 \\ K_{12} \\ 0 \end{array} \right] \hat{u}_s(\lambda) + i\tilde{\theta}(\lambda) \left[ \begin{array}{c} 0 \\ 1_m \end{array} \right], \quad (2.78)$$

$$\tilde{\theta}(\lambda) := KR(A_2 - \lambda_2 I)^{-1}(A_1 - \lambda_1 I)^{-1} \mathbf{1}. \quad (2.79)$$

In particular, we again used above the definition (2.29) of $\hat{u}$.

Formulas (2.75), (2.77) and (2.78) yield

$$i \mathfrak{A}_2(\lambda_2) \hat{\Pi}_1(A^*_2 - \lambda_2 I)^{-1} \Gamma_2 \hat{u}_2(\lambda) = i \left( \hat{\Pi}_1 \Gamma_2 - \left[ \begin{array}{c} 0 \\ K_{12} \\ 0 \end{array} \right] \right) \hat{u}_2(\lambda) \quad (2.80)$$

$$- i\tilde{\theta}(\lambda) \text{col} \left[ \begin{array}{c} 0 \\ 1_m \end{array} \right], \quad (2.80)$$

$$i \mathfrak{A}_1(\lambda_1) \hat{\Pi}_2(A^*_1 - \lambda_1 I)^{-1} \Gamma_1 \hat{u}_1(\lambda) = i \left( \hat{\Pi}_2 \Gamma_1 - \left[ \begin{array}{c} 0 \\ K_{11} \\ 0 \end{array} \right] \right) \hat{u}_1(\lambda) \quad (2.81)$$

$$- i\tilde{\theta}(\lambda) \text{col} \left[ \begin{array}{c} 0 \\ 1_n \end{array} \right]. \quad (2.81)$$

It follows from (2.74), (2.80) and (2.81) that

$$\tilde{G}(\lambda) \hat{u}(\lambda) = i (1 + \tilde{\theta}(\lambda)) \text{col} \left[ \begin{array}{c} 0 \\ 1_m \\ 0 \\ 1_n \end{array} \right], \quad \tilde{G}(\lambda) = \{ \tilde{g}_ps(\lambda) \}_{p,s=1}^2 \quad (2.82)$$

$$\tilde{g}_{11}(\lambda) := \mathfrak{A}_2(\lambda_2), \quad \tilde{g}_{12}(\lambda) \equiv \tilde{g}_{12} := i \hat{\Pi}_1 \Gamma_2 - i \left[ \begin{array}{c} 0 \\ K_{12} \\ 0 \end{array} \right], \quad (2.83)$$

$$\tilde{g}_{21}(\lambda) \equiv \tilde{g}_{21} := i \hat{\Pi}_2 \Gamma_1 - i \left[ \begin{array}{c} 0 \\ K_{11} \\ 0 \end{array} \right], \quad \tilde{g}_{22}(\lambda) := \mathfrak{A}_1(\lambda_1). \quad (2.84)$$

According to formulas (2.35), (2.39), (2.83) and (2.84), we have

$$G(\lambda) = \tilde{G}(\lambda) + i \text{col} \left[ \begin{array}{c} 1_m \\ 0 \\ -1_n \end{array} \right] \left[ \begin{array}{c} 1^*_m \\ 0 \\ -1^*_n \end{array} \right]. \quad (2.85)$$
Taking into account the definitions in (2.24)–(2.26), the definition of $\hat{u}$ in (2.29) and equalities (2.70), we obtain
\[
\begin{bmatrix}
1^*_m & 0 & -1^*_n & 0
\end{bmatrix} \bar{\Gamma} = 0, \quad \begin{bmatrix}
1^*_m & 0 & -1^*_n & 0
\end{bmatrix} \hat{u}(\lambda) \equiv 0. \tag{2.86}
\]
Relations (2.82), (2.85) and (2.86) imply that
\[
G(\lambda)\hat{u}(\lambda) = i(1 + \tilde{\theta}(\lambda)) \text{col} [0 \ 1_m \ 0 \ 1_n]. \tag{2.87}
\]
Moreover, it is shown in the Appendix (see Lemma A.3) that
\[
\left(\lambda_1 - \frac{i}{2}\right)^{-n} \left(\lambda_2 - \frac{i}{2}\right)^{-m} \theta(\lambda) = -(1 + \tilde{\theta}(\lambda)). \tag{2.88}
\]
Equalities (2.87) and (2.88) yield (2.66), which proves the theorem. ■

3 The structure of the operator $T^{-1}$

In the previous section we have shown (see Theorem 2.3) that given an invertible TBT-matrix we can recover $\rho$ and $T^{-1}$ from the matrix $g_{12}$ or from $g_{21}$ of the form (2.35). Here, we complete the description of the structure of $R = T^{-1}$. Namely, we prove that given an arbitrary $2m \times 2n$ matrix $g_{12}$ or an arbitrary $2n \times 2m$ matrix $g_{21}$ and using the same formulas as in Section 2 we recover some matrix $R$. Moreover, if this $R$ is invertible, then $T = R^{-1}$ is a TBT-matrix.

First, using (2.36) and (2.39) we construct $G(\lambda)$. In view of (2.36) and (2.39), we easily derive
\[
G(\lambda) = \text{diag}\{-U_{2m}J_{2m}, U_{2n}J_{2n}\}G(\lambda)^* \text{diag}\{-J_{2m}U_{2m}, J_{2n}U_{2n}\}. \tag{3.1}
\]
Next, we obtain $\hat{u}$ and $u$ via relations (2.41) and (2.42), where $\theta(\lambda) = \sqrt{\det (G(\lambda))}$. The function $\omega$ is expressed in (2.27) via $\hat{u}$ and $u$.

Lemma 3.1 Let $g_{12}$ or $g_{21}$ be given and let the vector functions $\hat{u}$ and $u$ be constructed via (2.41) and (2.42) (using also relations (2.36) and (2.39)).

Then $\omega(\lambda, \mu)$, which we obtain from (2.27), is the same for both cases $p = 1$ and $p = 2$. 16
Proof. The statement of the lemma is equivalent to the relation
\[ u(\mu)((\lambda_2 - \mu_2)P_1 - (\lambda_1 - \mu_1)P_2)\tilde{u}(\lambda) = 0 \]  
\[ (3.2) \]
In view of (2.41) and (2.42), the equality (3.2) is equivalent to the equality
\[ \begin{bmatrix} 0 & 1^\tau_m & 0 & 1^\tau_n \end{bmatrix} (G(\mu)^{-1}) \text{diag} \{ J_{2m}U_{2m}, J_{2n}U_{2n} \} \]
\[ \times ((\lambda_2 - \mu_2)P_1 - (\lambda_1 - \mu_1)P_2)G(\lambda)^{-1}\text{col}[0 \quad 1_m \quad 0 \quad 1_n] = 0. \]  
\[ (3.3) \]
Taking into account (3.1), we rewrite (3.3) as
\[ \begin{bmatrix} 0 & 1^\tau_m & 0 & 1^\tau_n \end{bmatrix} \text{diag} \{-J_{2m}U_{2m}, J_{2n}U_{2n}\}G(\mu)^{-1} \]
\[ \times ((\lambda_2 - \mu_2)P_1 + (\lambda_1 - \mu_1)P_2)G(\lambda)^{-1}\text{col}[0 \quad 1_m \quad 0 \quad 1_n] = 0. \]  
\[ (3.4) \]
Since
\[ G(\mu) - G(\lambda) = (\lambda_2 - \mu_2)P_1 + (\lambda_1 - \mu_1)P_2, \]
one may rewrite (3.4) in the form
\[ \begin{bmatrix} 0 & 1^\tau_m & 0 & 1^\tau_n \end{bmatrix} \text{diag} \{-J_{2m}U_{2m}, J_{2n}U_{2n}\} \]
\[ \times (G(\lambda)^{-1} - G(\mu)^{-1})\text{col}[0 \quad 1_m \quad 0 \quad 1_n] = 0. \]  
\[ (3.5) \]
Therefore, it remains to prove that
\[ \begin{bmatrix} 0 & 1^\tau_m & 0 & 1^\tau_n \end{bmatrix} \text{diag} \{-J_{2m}U_{2m}, J_{2n}U_{2n}\}G(\nu)^{-1}\text{col}[0 \quad 1_m \quad 0 \quad 1_n] = 0, \]  
\[ (3.6) \]
and the equalities (3.5) and (3.2) will follow. Finally, taking the transpose of the left hand side of (3.6) and using the relations (3.1) and \( U_rJ_r = -J_rU_r \) (for the cases \( r = 2m \) and \( r = 2n \)) we derive
\[ \begin{bmatrix} 0 & 1^\tau_m & 0 & 1^\tau_n \end{bmatrix} \text{diag} \{-J_{2m}U_{2m}, J_{2n}U_{2n}\}G(\nu)^{-1}\text{col}[0 \quad 1_m \quad 0 \quad 1_n] \]
\[ = \begin{bmatrix} 0 & 1^\tau_m & 0 & 1^\tau_n \end{bmatrix} (G(\nu)^{-1}) \text{diag} \{-U_{2m}J_{2m}, U_{2n}J_{2n}\} \text{col}[0 \quad 1_m \quad 0 \quad 1_n] \]
\[ = -\begin{bmatrix} 0 & 1^\tau_m & 0 & 1^\tau_n \end{bmatrix} \text{diag} \{-J_{2m}U_{2m}, J_{2n}U_{2n}\}G(\nu)^{-1}\text{col}[0 \quad 1_m \quad 0 \quad 1_n], \]
which yields (3.6). ■

Lemma 3.2 Let the conditions of Lemma 3.1 hold. Then \( \omega(\lambda, \mu) \) determines a unique matrix \( R \) such that
\[ \omega(\lambda, \mu) = 1^*(A_1^* - \mu_1I)^{-1}(A_2^* - \mu_2I)^{-1}R(A_2 - \lambda_2I)^{-1}(A_1 - \lambda_1I)^{-1}1. \]  
\[ (3.8) \]
Proof. According to Lemma A.2, the entries of the matrix function \( \theta(\lambda) G(\lambda)^{-1} \), where \( \theta(\lambda) = \sqrt{\text{det}(G(\lambda))} \), are polynomials. Moreover, formula (2.39) yields

\[
\lim_{\lambda_1 \to \infty} G(\lambda)^{-1} = \begin{bmatrix}
\text{diag}\{ (A_2 - \lambda_2 I_m)^{-1}, (A_2 - \lambda_2 I_m)^{-1} \} & 0 \\
0 & 0
\end{bmatrix}, \quad (3.9)
\]

\[
\lim_{\lambda_2 \to \infty} G(\lambda)^{-1} = \begin{bmatrix}
0 & 0 \\
0 & \text{diag}\{ (A_1 - \lambda_1 I_n)^{-1}, (A_1 - \lambda_1 I_n)^{-1} \}
\end{bmatrix}, \quad (3.10)
\]
as well as the asymptotic relation

\[
\theta(\lambda) = -q(\lambda)(1 + o(1)), \quad q(\lambda) := \left( \lambda_1 - \frac{i}{2} \right)^n \left( \lambda_2 - \frac{i}{2} \right)^m, \quad (3.11)
\]
when either \( \lambda_1 \to \infty \) or \( \lambda_2 \to \infty \). Thus, for the vector polynomial

\[
v(\lambda) = \theta(\lambda) G(\lambda)^{-1} \begin{bmatrix} 0 \\ 1_m \\ 0 \\ 1_n \end{bmatrix} + q(\lambda) \begin{bmatrix} 0 \\ (A_2 - \lambda_2 I_m)^{-1} 1_m \\ 0 \\ (A_1 - \lambda_1 I_n)^{-1} 1_n \end{bmatrix}, \quad (3.12)
\]
we have

\[
\lim_{\lambda_1 \to \infty} \left( \lambda_1 - \frac{i}{2} \right)^{-n} v(\lambda) = 0, \quad \lim_{\lambda_2 \to \infty} \left( \lambda_2 - \frac{i}{2} \right)^{-m} v(\lambda) = 0. \quad (3.13)
\]
In other words, the degrees of \( \lambda_1 \) in the entries of \( v \) are less than \( n \) and the degrees of \( \lambda_2 \) in the entries of \( v \) are less than \( m \).

Similar to the partitioning (2.54) for \( \hat{u} \), partition \( v \) into the blocks \( v = \text{col} \left[ v_1 \ v_2 \right] \). Using (2.27), (2.40), (2.41) and (3.12), we obtain

\[
\Omega(\lambda, \mu) := q(\lambda) \left( \mu_1 + \frac{i}{2} \right)^n \left( \mu_2 + \frac{i}{2} \right)^m \omega(\lambda, \mu)
\]

\[
= -\frac{i}{\lambda_1 - \mu_1} V_1(\mu)^\tau J_{2m} U_{2m} V_1(\lambda) = -\frac{i}{\lambda_2 - \mu_2} V_2(\mu)^\tau J_{2n} U_{2n} V_2(\lambda),
\]

where \( V_1(\lambda) = v_1(\lambda) - q(\lambda) \begin{bmatrix} 0 \\ (A_2 - \lambda_2 I_m)^{-1} 1_m \end{bmatrix} \) and

\[
V_2(\lambda) = v_2(\lambda) - q(\lambda) \begin{bmatrix} 0 \\ (A_1 - \lambda_1 I_n)^{-1} 1_n \end{bmatrix}. \quad \text{By virtue of (3.14), we have}
\]

\[
(\lambda_2 - \mu_2) V_1(\mu)^\tau J_{2m} U_{2m} V_1(\lambda) = (\lambda_1 - \mu_1) V_2(\mu)^\tau J_{2n} U_{2n} V_2(\lambda),
\]

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where $\lambda_p - \mu_p$ are irreducible polynomials. Hence, $V_1(\mu)^T J_{2m} U_{2m} V_1(\lambda)$ may be factored into the product of polynomials with $\lambda_1 - \mu_1$ as one of the factors (and a similar statement is valid for $V_2(\mu)^T J_{2n} U_{2n} V_2(\lambda)$ and $\lambda_2 - \mu_2$). Thus, we see that $\Omega(\lambda, \mu)$ introduced in (3.14) is a polynomial.

Then, the degrees of $\lambda_1$ and $\mu_1$ in the terms of $\Omega$ are less than $n$ and the degrees of $\lambda_2$ and $\mu_2$ are less than $m$ (since the degrees of $\lambda_1$ in the entries of $v(\lambda)$ are less than $n$ and the degrees of $\lambda_2$ in the entries of $v(\lambda)$ are less than $m$). In order to derive that, we also take the factors $(\lambda_p - \mu_p)^{-1}$ in (3.14) into account.

It is easy to see that the polynomials

$$p_s(\nu) = \left(\nu - \frac{i}{2}\right)^{s-1} \left(\nu + \frac{i}{2}\right)^{r-s} \quad (1 \leq s \leq r)$$

are linearly independent polynomials (in one variable $\nu$). Therefore, the mentioned above bounds on the degrees of the variables in $\Omega(\lambda, \mu)$ show that $\Omega(\lambda, \mu)$ may be presented as a linear combination of the products

$$p_{s_1}(\lambda_1) p_{s_2}(\mu_1) p_{s_3}(\lambda_2) p_{s_4}(\mu_2) \quad (1 \leq s_1, s_2 \leq n, \ 1 \leq s_3, s_4 \leq m). \quad (3.15)$$

On the other hand, relations (1.5), (2.3) and (2.4) (where $\psi$ is introduced in (2.1)) show that the multiplication of the right hand side of (3.8) by $q(\lambda) \left(\mu_1 + \frac{i}{2}\right)^n \left(\mu_2 + \frac{i}{2}\right)^m$ brings us a polynomial, which is a linear combination of the same products (3.15) with the entries of $R$ as the coefficients. It is immediate that there is a unique matrix $R$ such that (3.8) holds. □

**Theorem 3.3** Let a $2m \times 2n$ matrix $g_{12}$ or a $2n \times 2m$ matrix $g_{21}$ be given and let the vector functions $\tilde{u}$ and $u$ be constructed via (2.41) and (2.42) (using also relations (2.36) and (2.39)). The matrix $\omega(\lambda, \mu)$ given (in terms of $u$ and $\tilde{u}$) by (2.27) determines a unique matrix $R$ such that (3.8) holds. Assume that $\det R \neq 0$.

Then, the matrix $T = R^{-1}$ is a TBT-matrix.

**Proof.** Step 1. The unique recovery of $R$ from $g_{12}$ or, equivalently, from $g_{21}$ is described in Lemma 3.2. Now, consider again $v(\lambda)$ given by (3.12). Since the degrees of $\lambda_1$ in the entries of $v$ are less than $n$ and the degrees of $\lambda_2$ in
the entries of \( v \) are less than \( m \), one can introduce \( \hat{\Gamma} \) by the equality

\[
q(\lambda)\hat{\Gamma}(A_2 - \lambda_2 I)^{-1}(A_1 - \lambda_1 I)^{-1}1 = -iv(\lambda).
\]  

(3.16)

We partition \( \hat{\Gamma} \) into the blocks \( \hat{\Gamma}_1 \) and \( \hat{\Gamma}_2 \) (as in (2.26)), and express matrices \( \Gamma_p \) \((p = 1, 2)\) via \( \hat{\Gamma}_p \) using equalities

\[
\Gamma_1 = U \left( \hat{\Gamma}_1 - [0 \ M_{31}] \right) J_{2n} U_{2n}, \quad \Gamma_2 = U \left( \hat{\Gamma}_2 - [0 \ M_{32}] \right) J_{2n} U_{2n},
\]

(3.17)

which are equivalent to (2.47).

Step 2. The basic step in the theorem’s proof is the proof of the matrix identities

\[
RA_p - A_p^* R = i\Gamma_p \hat{\Gamma}_p \quad (p = 1, 2).
\]

(3.18)

Clearly, identities (3.18) are equivalent to the identities

\[
(\mu_p - \lambda_p) R = R(A_p - \lambda_p I) - (A_p^* - \mu_p I) R - i\Gamma_p \hat{\Gamma}_p.
\]

Hence, in view of (1.5), (1.6) and (2.4), the identities (3.18) are equivalent to the equalities

\[
1^*(A_1^* - \mu_1 I)^{-1}(A_2^* - \mu_2 I)^{-1} R(A_2 - \lambda_2 I)^{-1}(A_1 - \lambda_1 I)^{-1} 1 = \lim_{\lambda_1 \to \infty} \left( -\lambda_1 \omega(\lambda_1, \mu) \right) \hat{u}_1(\lambda). \]

(3.20)
According to (2.41) and (3.12) we have

$$\hat{u}_1(\lambda) = -i \left( q(\lambda)^{-1} v_1(\lambda) - \left[ (A_2 - \lambda_2 I_m)^{-1} 0 \right] \right).$$

(3.21)

Hence, taking into account the first equality in (3.12) and the second equality in (3.11), we derive

$$\lim_{\lambda_1 \to \infty} \hat{u}_1(\lambda) = i \left[ (A_2 - \lambda_2 I_m)^{-1} 0 \right].$$

(3.22)

In view of (3.22), we rewrite (3.20) in the form

$$1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} R (A_2 - \lambda I) = u_1(\mu) \text{col} \left[ 0 \ (A_2 - \lambda_2 I_m)^{-1} 1_m \right].$$

(3.23)

Moreover, relations (2.42) and (3.22) imply that

$$\lim_{\mu_1 \to \infty} u_1(\mu) = -i \left( \left( \mu_2 - \frac{i}{2} \right)^m \left( \mu_2 + \frac{i}{2} \right)^{-m} \right) \left[ 0 \ 1^*_m (A_2^* - \mu_2 I_m)^{-1} \right] U_{2m}$$

$$= i \left( \left( \mu_2 - \frac{i}{2} \right)^m \left( \mu_2 + \frac{i}{2} \right)^{-m} \right) \left[ 0 \ 1^*_m (A_2^* + \mu_2 I_m)^{-1} U_m \ 0 \right].$$

(3.24)

Using (2.2) in order to calculate the right hand side of (3.24), after easy transformations we have

$$\lim_{\mu_1 \to \infty} u_1(\mu) = -i \left[ 1^*_m (A_2^* - \mu_2 I_m)^{-1} \ 0 \right].$$

(3.25)

By virtue of (2.27), (3.8) and (3.25), we obtain a result

$$1^* (A_2^* - \mu_2 I)^{-1} R (A_2 - \lambda_2 I)^{-1} (A_1 - \lambda I) = \lim_{\mu_1 \to \infty} \left( - \mu_1 \omega(\lambda, \mu) \right) \left[ 1^*_m (A_2^* - \mu_2 I_m)^{-1} \ 0 \right] \hat{u}_1(\lambda).$$

(3.26)

It remains to simplify the expression

$$1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} \Gamma_1 \tilde{\Gamma}_1 (A_2 - \lambda_2 I)^{-1} (A_1 - \lambda_1 I)^{-1} 1.$$

(3.27)
According to (3.16) and (3.21) we derive
\[
\hat{\Gamma}_1 (A_2 - \lambda_2 I)^{-1} (A_1 - \lambda_1 I)^{-1} \mathbf{1} = -i q(\lambda)^{-1} v(\lambda) = \hat{u}_1(\lambda) - i \text{col } [0 \ (A_2 - \lambda_2 I_m)^{-1} \mathbf{1}_m].
\] (3.28)

From (2.50), (2.51) and (3.17) we also see that
\[
\begin{aligned}
\Gamma_1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} \mathbf{1} = & \ 
\hat{u}_1(\lambda) - i \text{col } [0 \ (A_2 - \lambda_2 I_m)^{-1} \mathbf{1}_m].
\end{aligned}
\]

Using (2.42), we rewrite (3.31): 
\[
\begin{aligned}
\Gamma_1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} \mathbf{1} = & \ 
\hat{u}_1(\lambda) - i \text{col } [0 \ (A_2 - \lambda_2 I_m)^{-1} \mathbf{1}_m].
\end{aligned}
\]

Finally, denoting the right hand side of (3.19) (where \(p = 1\) and \(s = 2\)) by \(Y(\lambda, \mu)\) and taking into account (3.23), (3.26), (3.28) and (3.32) we obtain
\[
\begin{aligned}
Y(\lambda, \mu) = & \left( u_1(\mu) \text{col } [0 \ (A_2 - \lambda_2 I_m)^{-1} \mathbf{1}_m] - \Gamma_1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} \mathbf{1} \right) \hat{u}_1(\lambda) \\
& - i \left( u_1(\mu) + i \left[ \Gamma_1^* (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1} \mathbf{1} \right] \right) \hat{u}_1(\lambda) \\
& \times \left( \hat{u}_1(\lambda) - i \text{col } [0 \ (A_2 - \lambda_2 I_m)^{-1} \mathbf{1}_m] \right) \bigg/ (\mu_1 - \lambda_1).
\end{aligned}
\] (3.33)
Collecting similar terms in (3.33) and using (2.27), we derive

\[ Y(\lambda, \mu) = i(\lambda_1 - \mu_1)^{-1}u_1(\mu)\hat{u}_1(\lambda) = \omega(\lambda, \mu). \]  
(3.34)

That is, (3.19) (and so (3.18)) with \( p = 1 \) is equivalent to (3.8), which proves (3.18) for \( p = 1 \). In the same way, (3.18) is proved for \( p = 2 \).

Step 3. Next, we prove that

\[ \Gamma_1 \left[ \begin{array}{c} 0 \\ I_m \end{array} \right] = RM_{31}, \quad \left[ I_m \ 0 \right] \hat{\Gamma}_1 = M_{21}R, \]  
(3.35)

where \( M_{21} \) and \( M_{31} \) have the special forms (2.10) and do not depend on \( R \).

Indeed, the identity (3.18) for \( p = 1 \) implies that

\[ (A_1^* - \mu_1 I)^{-1}R - R(A - \mu_1 I)^{-1} \equiv i(A_1^* - \mu_1 I)^{-1}\Gamma_1\hat{\Gamma}_1(A - \mu_1 I)^{-1}, \]

and, in particular, we have

\[ M_{21}(A_1^* - \mu_1 I)^{-1}RM_{31} - M_{21}R(A - \mu_1 I)^{-1}M_{31} \]

\[ \equiv iM_{21}(A_1^* - \mu_1 I)^{-1}\Gamma_1\hat{\Gamma}_1(A - \mu_1 I)^{-1}M_{31}. \]  
(3.36)

Furthermore, equality (3.2) yields

\[ u_1(\mu)\hat{u}_1(\lambda) \equiv 0 \quad \text{for} \quad \lambda_1 = \mu_1. \]  
(3.37)

On the other hand, using (2.33) in order to rewrite the left hand sides of (3.28) and (3.32), we obtain the following expressions for \( \hat{u}_1 \) and \( u_1 \), respectively:

\[ \hat{u}_1(\lambda) = \left( \hat{\Gamma}_1(A_1 - \lambda_1 I)^{-1}M_{31} + i \left[ \begin{array}{c} 0 \\ I_m \end{array} \right] \right) (A_2 - \lambda_2 I_m)^{-1}1_m, \]  
(3.38)

\[ u_1(\mu) = 1_m^*(A_2^* - \mu_2 I_m)^{-1}(M_{21}(A_1^* - \mu_1 I)^{-1}\Gamma_1 - i \left[ I_m \ 0 \right]) . \]  
(3.39)

Substituting (3.38) and (3.39) into (3.37), we see that

\[ (M_{21}(A_1^* - \mu_1 I)^{-1}\Gamma_1 - i \left[ I_m \ 0 \right]) \left( \hat{\Gamma}_1(A_1 - \mu_1 I)^{-1}M_{31} + i \left[ 0 \\ I_m \right] \right) \equiv 0. \]  
(3.40)
It is immediate from (3.36) and (3.40) that

\[ M_{21}(A_1^* - \mu_1 I)^{-1} \left( R M_{31} - \Gamma_1 \begin{bmatrix} 0 \\ I_m \end{bmatrix} \right) \equiv \left( M_{21} R - [I_m \ 0] \hat{\Gamma}_1 \right) (A - \mu_1 I)^{-1} M_{31}. \]  

(3.41)

Taking into account the expressions for \( M_{21}(A_1^* - \mu_1 I)^{-1} \) and for \((A - \mu_1 I)^{-1} M_{31}\) (see (2.1) and (2.10)), we conclude from (3.41) that (3.35) holds.

In a similar way, one proves that

\[ \Gamma_2 \begin{bmatrix} 0 \\ I_n \end{bmatrix} = R M_{32}, \quad [I_n \ 0] \hat{\Gamma}_2 = M_{22} R, \]  

(3.42)

where \( M_{22} \) and \( M_{32} \) are introduced in (2.13), (2.14), (2.16).

Since we assumed that \( R \) is invertible, relations (3.18), (3.35) and (3.42) yield the matrix identities

\[ A_p T - T^* A_p = i TT_p \hat{\Gamma}_p T = i(M_{1p} M_{2p} + M_{3p} M_{4p}) \quad (T := R^{-1}), \]  

(3.43)

where \( M_{2p} \) and \( M_{3p} \) are given by (2.10), (2.13), (2.14), (2.16), and

\[ M_{11} = TT_1 \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \quad M_{12} = TT_2 \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \]

\[ M_{41} = [0 \ I_m] \hat{\Gamma}_1 T, \quad M_{42} = [0 \ I_n] \hat{\Gamma}_2 T. \]

In the case \( p = 1 \) (and with \( M_{21} \) and \( M_{31} \) given by (2.10)), the identity (3.43) means that \( T \) is a block Toeplitz matrix (i.e., \( T = \{T_{j-k}\}_{j,k=1}^{n} \)). Indeed, when the right hand side of (3.43) is fixed, the matrix \( T \) satisfying (3.43) is unique, and there is always a block Toeplitz matrix \( T \) determined by (3.43) with \( p = 1 \). One can construct this \( T \) by rewriting (3.43) \((p = 1)\) in the form

\[ A_1 T - T^* A_1 = i(\tilde{M}_{11} M_{21} + \tilde{M}_{31} M_{41}); \quad \tilde{M}_{11} = M_{11} + \beta M_{31}, \quad \tilde{M}_{41} = M_{41} - \beta M_{21}, \]

where \( \beta \) is chosen so that the first blocks of \( \tilde{M}_{11} \) and \( \tilde{M}_{41} \) coincide. After that one writes down the blocks of \( \tilde{M}_{11} \) and \( \tilde{M}_{41} \) in the form of the last equalities.
in (2.9) and (2.11), respectively. In this way, one determines the blocks $T_{i-k}$ of the block Toeplitz matrix $T$ satisfying (3.43) with $p = 1$.

Similarly, the identity (3.43), where $p = 2$, means that the blocks $T_{ik}$ of $T$ are Toeplitz. Thus, $T$ is TBT-matrix.

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A Auxiliary results

Here, we study the expression $\sqrt{\det G(\lambda)}$. The fact that the ring $\mathbb{C}[x_1,...x_n]$ of polynomials (in several variables) with complex coefficients is factorial, that is, that $\mathbb{C}[x_1,...x_n]$ is a unique factorisation domain, is well known (see, e.g., [34, p. 72]), and it is essential in our further considerations. We will also need some interrelations between the minors of the given matrices and their cofactor matrices (see, e.g., [11])

First note that in view of (2.36) and (2.39) we obtain (3.1) (both in Sections 2 and 3). Thus, we have a skew-symmetric matrix function

$$\tilde{G}(\lambda) := \text{diag}\{J_{2m}U_{2m}, -J_{2n}U_{2n}\} G(\lambda) = -\tilde{G}(\lambda)^\tau.$$ (A.1)

Clearly, (A.1) yields

$$\det \tilde{G}(\lambda) = \det G(\lambda).$$ (A.2)

Notation A.1 The notation $D_{k_1k_2...k_s}^{i_1i_2...i_s}(\lambda)$ stands for the determinant of the matrix, cut down from $\tilde{G}(\lambda)$ by removing its rows with the numbers $i_1, i_2, \ldots, i_s$ and its columns with the numbers $k_1, k_2, \ldots, k_s$ ($i_j \neq i_\ell$ and $k_j \neq k_\ell$ for $j \neq \ell$).

In our notations, $D(\lambda) = \det \tilde{G}(\lambda)$ (if $s = 0$), and we set $D_{122m}^{122m}(\lambda) = 1$.

According to the well-known property of the minors (see, e.g., formula (4) in §2 of Ch. 1 [11] and set there $p = 2$), which we apply to the matrix cut down from $\tilde{G}(\lambda)$ by removing the rows and columns with the same numbers $i_1, i_2, \ldots, i_{r-1}$ ($1 \leq r \leq 2m + 2n - 1$), we have

$$D_{i_1...i_r-1i_r}^{i_1...i_r-1i_r}(\lambda)D_{i_1...i_r-1i_r+1}^{i_1...i_r-1i_r+1}(\lambda) - D_{i_1...i_r-1i_r}^{i_1...i_r-1i_r}(\lambda)D_{i_1...i_r-1i_r+1}^{i_1...i_r-1i_r+1}(\lambda)$$

$$= D_{i_1...i_r-1}^{i_1...i_r-1}(\lambda)D_{i_1...i_r-1i_r}^{i_1...i_r-1i_r}(\lambda).$$ (A.3)
We note that either the numbers of the rows and columns, which are cut down, or the numbers of the rows and columns, which are preserved, are given for minors in the literature (and in this respect our notations differ from the notations in [11]).

The skew-symmetric structure of $\tilde{G}$ (see (A.1)) implies for the odd values of $r$ the equalities

$$D_{i_1\ldots i_r-1 i_r}^{i_1\ldots i_r} (\lambda) = D_{i_1\ldots i_r-1 i_r+1}^{i_1\ldots i_r} (\lambda) = 0, \quad D_{i_1\ldots i_r-1 i_r}^{i_1\ldots i_r} (\lambda) = -D_{i_1\ldots i_r-1 i_r+1}^{i_1\ldots i_r} (\lambda).$$

(A.4)

From (A.3) and (A.4) we derive (for the odd values of $r$) an important relation:

$$D_{i_1\ldots i_r-1}^{i_1\ldots i_r} (\lambda) = \left( D_{i_1\ldots i_r-1}^{i_1\ldots i_r} (\lambda) \right)^2 \left( D_{i_1\ldots i_r-1 i_r+1}^{i_1\ldots i_r} (\lambda) \right)^{-1}. \quad (A.5)$$

Lemma A.2 Let $G(\lambda)$ of the form (2.39) be given and let (2.36) hold. Then, $\sqrt{\det G(\lambda)}$ is a polynomial, and it is presented by the formula

$$\sqrt{\det G(\lambda)} = \alpha \frac{D_{i_2}^{i_1 \ldots i_s \ldots i_{s+1}} (\lambda) \ldots D_{i_1 \ldots i_r - 1 i_r}^{i_1 \ldots i_r} (\lambda)}{D_{i_1 i_2 i_4}^{i_3 i_5} (\lambda) \ldots D_{i_1 \ldots i_s - 1 i_s}^{i_1 \ldots i_s} (\lambda)}, \quad (A.6)$$

where $\alpha = \pm 1$ and we fix $\alpha$ (in accordance with Remark 2.2) so that the coefficient corresponding to $\lambda^n \lambda^{m}$ in $\sqrt{\det G(\lambda)}$ equals $-1$. (Here, $r + 3 = s + 1 = 2m + 2n$ if $2m + 2n$ is divisible by 4, and $r + 1 = s + 3 = 2m + 2n$ if $2m + 2n$ is not.)

Moreover, the entries of $\sqrt{\det G(\lambda)} G(\lambda)^{-1}$ are polynomials as well.

Proof. Setting $r = 1$ in (A.5) and taking into account Notation A.1 and equality (A.2), we obtain

$$\det G(\lambda) = \det \tilde{G}(\lambda) = \left( D_{i_2}^{i_1} (\lambda) \right)^2 / D_{i_1 i_2}^{i_3 i_4} (\lambda). \quad (A.7)$$

In the same way, we set $r = 3$ in (A.5) and express $D_{i_1 i_2}^{i_3 i_4}$ via determinants of smaller matrices. Next, we set $r = 5, \ldots$, and after a final number of steps we have

$$\det G(\lambda) = \left( \frac{D_{i_2}^{i_1 \ldots i_s i_{s+1}} (\lambda) \ldots D_{i_1 \ldots i_r - 1 i_r}^{i_1 \ldots i_r} (\lambda)}{D_{i_1 i_2 i_4}^{i_3 i_5} (\lambda) \ldots D_{i_1 \ldots i_s - 1 i_s}^{i_1 \ldots i_s} (\lambda)} \right)^2. \quad (A.8)$$
Factorising the nominator and denominator of the fraction on the right hand side of (A.8) into the products of irreducible polynomials, rewriting (A.8) in the form

\[
\left( D_{i_1i_2i_3} \cdots D_{i_{s-1}i_{s+1}} (\lambda) \right)^2 \det G(\lambda) = \left( D_{i_1}(\lambda) D_{i_1 \cdots i_4}(\lambda) \cdots D_{i_{r-1}i_r}(\lambda) \right)^2, \tag{A.9}
\]

and taking into account the uniqueness of the factorisation, we see that \( \det G(\lambda) \) is a square of some polynomial, that is, there is a polynomial \( \sqrt{\det G(\lambda)} \). Moreover, (A.8) yields (A.6).

Similarly to \( \det G(\lambda) \), one can deal with \( D_{i_1i_2}(\lambda) \) and show that

\[
D_{i_1i_2}(\lambda) = p(\lambda)^2 \tag{A.10}
\]

for some polynomial \( p(\lambda) \). According to relations (A.7) and (A.10) and to the first formula in (A.4), we have

\[
D_{i_1}(\lambda) / \sqrt{\det G(\lambda)} = 0; \quad D_{i_2}(\lambda) / \sqrt{\det G(\lambda)} = \bar{\alpha}p(\lambda), \tag{A.11}
\]

where \( \bar{\alpha} \) equals either 1 or \(-1\). Clearly, the expressions on the left hand side of the equalities in (A.11) coincide (up to the sign) with the entries of \( \sqrt{\det G(\lambda)} G(\lambda)^{-1} \) (and all the entries may be obtained by the proper choice of \( i_1 \) and \( i_2 \)). Therefore, the entries of \( \sqrt{\det G(\lambda)} G(\lambda)^{-1} \) are, indeed, polynomials. ■

**Lemma A.3** Let the conditions of Theorem 2.3 hold and let \( \tilde{\theta} \) be given by (2.79). Then we have

\[
\theta(\lambda) = -q(\lambda)(1 + \tilde{\theta}(\lambda)), \tag{A.12}
\]

where \( q \) is given in (3.11).

**Proof.** In view of (2.1), (2.2) and (2.29), \( q(\lambda)\tilde{u}(\lambda) \) is a vector polynomial. Furthermore, if \( q(\lambda)\tilde{u}(\lambda) \) turns at some point \( \lambda = c \) to zero, then formulas (1.7)–(2.2) and (2.27) yield

\[
1^*(A_1^* - \mu_1 I)^{-1}(A_2^* - \mu_2 I)^{-1}T^{-1}h_c \equiv 0 \quad \text{for} \quad h_c \neq 0 \quad (h_c \in \mathbb{C}^{mn}). \tag{A.13}
\]
However, (A.13) is impossible for any nonzero $h_c$ because the span of $1^\ast (A_1^* - \mu_1 I)^{-1} (A_2^* - \mu_2 I)^{-1}$ coincides with $\mathbb{C}^{mn}$ (and $\det T^{-1} \neq 0$). Hence, $q \hat{u}$ does not have zeros.

It follows from (2.87) that
\[
(q \hat{u})(\lambda) = P(\lambda) G(\lambda)^{-1} \col [0 \quad 1_m \quad 0 \quad 1_n], \quad P(\lambda) := iq(\lambda)(1 + \tilde{\theta}(\lambda)),
\]
where (according to (2.79)) $P(\lambda)$ is a polynomial of degree $m + n$ with the coefficient $i$ before $\lambda_1^n \lambda_2^m$. Since $P(\lambda) G(\lambda)^{-1} \col [0 \quad 1_m \quad 0 \quad 1_n]$ is a vector polynomial and does not have zeros, all other polynomials $\tilde{P}(\lambda)$, such that $\tilde{P}(\lambda) G(\lambda)^{-1} \col [0 \quad 1_m \quad 0 \quad 1_n]$ is a vector polynomial, may be factored into the product of $P$ and some other polynomial. On the other hand, it is stated in Lemma A.2 that $\sqrt{\det G(\lambda)} G(\lambda)^{-1}$ is a vector polynomial, $\theta(\lambda) = \sqrt{\det G(\lambda)}$ is a polynomial such that the coefficient before $\lambda_1^n \lambda_2^m$ equals $-1$ (and the degree of $\sqrt{\det G(\lambda)}$ equals $m + n$). It is immediate that $\sqrt{\det G(\lambda)} = iP(\lambda)$. Now, (A.12) follows from the definition of $P$ in (A.14).

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