Noncommutative Unification of General Relativity and Quantum Mechanics.

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Abstract

In Gen. Rel. Grav. (36, 111-126 (2004); in press, gr-qc/0410010) we have proposed a model unifying general relativity and quantum mechanics based on a noncommutative geometry. This geometry was developed in terms of a noncommutative algebra $A$ defined on a transformation groupoid $\Gamma$ given by the action of a group $G$ on a space $E$. Owing to the fact that $G$ was assumed to be finite it was possible to compute the model in full details. In the present paper we develop the model in the case when $G$ is a noncompact group. It turns out

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that also in this case the model works well. The case is important since to obtain physical effects predicted by the model we should assume that $G$ is a Lorentz group or some of its representations. We show that the generalized Einstein equation of the model has the form of the eigenvalue equation for the generalized Ricci operator, and all relevant operators in the quantum sector of the model are random operators; we study their dynamics. We also show that the model correctly reproduces general relativity and the usual quantum mechanics. It is interesting that the latter is recovered by performing the measurement of any observable. In the act of such a measurement the model “collapses” to the usual quantum mechanics.

KEY WORDS: General relativity; quantum mechanics; unification theory; noncommutative geometry; groupoid.

1 Introduction

Noncommutative geometry plays an increasingly important role in the present search for quantum gravity (from a host of papers let as quote at least a few [1, 2, 3, 4, 6, 14, 15, 19]). It has also recently been recognized that it is a useful tool in superstring theory (the classical paper is [18], and a book [13]). In a series of papers ([8, 9, 10]), we have proposed our own approach to the unification of general relativity and quantum mechanics based on noncommutative geometry. Our starting point is the standard method of changing a differential manifold (space-time) $M$ into a noncommutative space [5, p. 86]. It is done by implementing the following steps: (1) we represent $M$ as a quotient space $N/R$ where $N$ is a suitable space and $R$ a suitable equivalence relation; (2) then we change from $N/R$ to a suitably organized subset $\mathcal{R}$ of $N \times N$; we call this the “pairing process”; (3) we define a suitable algebra on $\mathcal{R}$; and finally, (4) we extract information about $N/R$ from this algebra.

We implement this strategy as follows. Let $M$ be a space-time manifold. The natural way to present $M$ as a quotient space is with the help of the frame bundle over $M$. Let $\pi_M : E \to M$ be the frame bundle with the structure group $G$, then $M = E/G$. To perform the "pairing process" let us notice that the group $G$ acts (to the right) on $E$ (along the fibres), $E \times G \to E$. We can equip $E \times G$ with the groupoid structure. This groupoid is called a
transformation groupoid, and will be denoted by $\Gamma$ (its description is given in Sect. 2). Now, we define a (noncommutative) compactly supported, smooth, complex valued algebra $\mathcal{A}$ on $\Gamma$ with convolution as multiplication. Then we construct, in terms of this algebra, the (noncommutative) geometry of the groupoid $\Gamma$ which is a generalization of the usual space-time geometry (on $M$). The regular representation of the algebra $\mathcal{A}$ on a bundle of Hilbert spaces gives us the “quantum sector” of the model.

To smooth out some inaccuracies and avoid conceptual traps in which our previous work was involved we have tested the method on a simpler model in which the group $G$ was finite ([11, 12, 17]). It has turned out that this simplified model works well. Let us notice, however, that if a finite group $G$ acts freely on a space $E$ then $G$ must be a cyclic group isomorphic with $\mathbb{Z}^n$ where $n = |G|$. Indeed, for $G \ni g \neq e$ the set $\{gp, g^2p, \ldots, g^n p\}$ is bijective with $G$ and, as it can be easily seen, $g^n = e$. However, we should notice that the fact that the group $G$ is Abelian does not entail the commutativity of the groupoid algebra $\mathcal{A}$. Therefore, our model with a finite group $G$ could serve well as an “exercise model”, but to have a more physically realistic approach we must change to an infinite group $G$. This is exactly what we do in the present paper. Throughout this paper it is assumed that $G$ is a noncompact group. This is an important case since to obtain physical effects predicted by our model we should assume that $G$ is a Lorentz group or some of its representations.

In Sect. 2, we briefly present the groupoid $\Gamma$ and its algebra $\mathcal{A}$, and we establish notation. The geometry of the groupoid $\Gamma$ is based on the module of derivations of the algebra $\mathcal{A}$. In Sect. 3, we study the structure of this module and, in Sects. 4 and 5, we develop the differential geometry of the groupoid $\Gamma$. This enables us to formulate, in Sect. 6, generalized Einstein’s equation. It turns out that it has the form of the eigenvalue equation for the generalized Ricci operator. We also show that the standard space-time geometry is obtained by suitably “averaging” elements of $\mathcal{A}$. In Sect. 7, we study the quantum sector of the model, and show that all relevant operators are random operators. We also investigate their generalized dynamics. The transition from our model to the usual quantum mechanics is presented in Sect. 8. Interestingly, it is the act of measurement of any observable that reduces our model to the usual quantum mechanics. We thus can say that from the perspective of our model quantum mechanics is but a theory of making measurements.
The present paper focuses on mathematical aspects of the proposed model; its physical aspects will fuller be discussed in a forthcoming paper.

2 Preliminaries

Let $\Gamma = E \times G$, where $E$ is the frame bundle over space-time $M$ with the structural group $G$, such that $G$ is a noncompact semisimple Lie group acting on $E$, be a transformation groupoid, and $\mathcal{A} = C^\infty_c(\Gamma, \mathbb{C})$ the noncommutative algebra of smooth, compactly supported, complex valued functions on $\Gamma$ with convolution as multiplication. Let further $\gamma_1 = (p_1, g_1), \gamma_2 = (p_2, g_2) \in \Gamma$, and $p_2 = p_1 g$. We assume the convention $\gamma_1 \circ \gamma_2 = (p_1, g_1 g_2)$, and consequently

$$ (f_1 * f_2)(\gamma) = \int_{\Gamma_{d(\gamma)}} f_1(\gamma_1) f_2(\gamma_1^{-1} \gamma) d\gamma_1 $$

for $f_1, f_2 \in \mathcal{A}$, where $d(\gamma) = d(p, g) = p$.

Let us notice that the center of the algebra $\mathcal{A}$ vanishes, $Z(\mathcal{A}) = \{0\}$, but $\mathcal{A}$ is a module over $Z = \pi_M^* C^\infty(M)$ (here $\pi_M : E \to M$ is the bundle projection). Functions of $Z$, in general, are not compactly supported. However, they do act on $\mathcal{A}$ in the following way: $\alpha : Z \times \mathcal{A} \to \mathcal{A}$ by

$$ \alpha(f, a)(p, q) = f(p) a(p, g), $$

$f \in Z, a \in \mathcal{A}$. Now, let us define the distribution

$$ \tilde{f}(p, g) = f(p) \delta_e(g) $$

where $\delta$ is the Dirac distribution, $g \in G$, and $e$ is the unit of $G$. $\tilde{f}$ convolutes well with functions of $\mathcal{A}$. Indeed, let $a \in \mathcal{A}$; we have

$$ (\tilde{f} * a)(p, g) = \int_G \tilde{f}(p, g_1) a(p g_1, g_1^{-1} g) d g_1 = f(p) a(p, g) \in \mathcal{A}. $$

(Here we have used the integral notation for the distribution action on test functions.)

Let $\mathcal{G} = E \times E$ be the space of the pair groupoid, where $E$ is, as before, the total space of the frame bundle over space-time $M$, i.e. $\mathcal{G} = \{(x, p_1, p_2)$ \footnote{Let us notice that the Lorentz group is noncompact and semisimple.}.
$p_1, p_2 \in E$ and $\pi_M(p_1) = (\pi_M(p_2)) = x$, and the algebra $\tilde{A} = C^\infty(G, \mathbb{C})$ with convolution as multiplication. The composition law reads $(x, p_1, p_2) \circ (x, p_2, p_3) = (x, p_1, p_3)$, $p_1, p_2, p_3 \in E_x, x \in M$, and the convolution is defined accordingly.

**Proposition 1** The mapping $J : \tilde{A} \to A$, given by

$$J(f)(\gamma) = f(\pi_M(p), p, pg),$$

for $f \in \tilde{A}, \gamma = (p, g)$, is an isomorphism of algebras.

**Proof** Let $\tilde{f}_1, \tilde{f}_2 \in \tilde{A}$; we have

$$(\tilde{f}_1 \ast \tilde{f}_2)(x, p_1, p_2) = \int_{E_x} \tilde{f}_1(x, p_1, p_3)\tilde{f}_2(x, p_3, p_2)dp_3.$$

We notice that the fiber $E_x$, for every $x \in M$, is diffeomorphic with the group $G$, and consequently there is a measure on $E_x$ induced from the Haar measure on $G$. After making the substitution $p_3 = p_1g_1, p_2 = p_1g$, we obtain

$$(\tilde{f}_1 \ast \tilde{f}_2)(x, p_1, g) = \int_G \tilde{f}_1(x, p_1, p_1g_1)\tilde{f}_2(x, p_1g_1, p_1g)dg_1$$

which can be rewritten as

$$(f_1 \ast f_2)(\gamma) = \int_{\Gamma_{d(\gamma)}} f_1(\gamma_1)f_2(\gamma_1^{-1} \circ \gamma)d\gamma_1. \quad \Box$$

### 3 Module of derivations

Among derivations of the algebra $A$ on the groupoid $\Gamma = E \times G$ we can distinguish three types: horizontal derivations, vertical derivations, and inner derivations of $A$; we denote them by $\text{Der}_{\text{Hor}}A$, $\text{Der}_{\text{Ver}}A$, and $\text{Inn}A$, respectively. We shall study them in turn.

**Lemma 1** Let $X \in \mathcal{X}(E)$ be a right invariant vector field (on a principal bundle), i.e., $(R_g)_pX(p) = X(pg)$ for every $g \in G$. Its lifting to $\Gamma$, $\tilde{X}(p, g) = (t_g)_*pX(p)$, where the inclusion $t_g : E \times G$ is defined by $t_g(p) = (p, g)$, is a derivation of the algebra $(A, \ast)$.
Proof

\[
\bar{X}(p,g)(f \ast h)(p,g) = [\bar{X}(p)(f \ast h)](\iota_g(p)) = \\
\int_G [\bar{X}(p)f(p,g_1)]h(pg_1,g_1^{-1}g)dg_1 + \\
\int_G f(p,g_1)[(X)(p)h(pg_1,g_1^{-1}g)]dg_1 = \\
\int_G [(\bar{X}))(p,g_1)f(p,g_1)h(pg_1,g_1^{-1}g)dg_1 + \\
\int_G f(p,g_1)[\bar{X}(pg_1,g_1^{-1}g)]h(pg_1,g_1^{-1}g)dg_1 = \\
(\bar{X}f \ast h + f \ast \bar{X}h)(p,g).
\]

We have employed here the right invariance property. \(\square\)

3.1 Horizontal Derivations

The group \(G\) acts freely and transitively on the fibres of \(E\). Consequently, the \(G\)-right-invariant vector fields on \(E\) are determined by their values at a single point of every fiber. Therefore, they can be identified with the cross sections \(\Sigma = TE/G\) of the bundle. Let us consider the mapping

\[ (\pi_M)_*: TE \to TM. \]

Since \((\pi_M)_*\) is \(G\)-invariant, it induces the mapping

\[ \pi_M : \Sigma \to TM. \]

Let us denote \(\rho = (\pi_M)_*\), and consider the exact sequence of vector bundles

\[
\rho \\
0 \to \ker \rho \xrightarrow{j} \Sigma \xrightarrow{\sigma} TM \to 0.
\]

The mappings \(j\) and \(\rho\) are homomorphisms of vector bundles, and \(j\) is an inclusion. The homomorphism of vector bundles \(\sigma : TM \to \Sigma\) is a connection in the principal bundle \(\pi_M : E \to M\) if it splits this sequence, i.e., if \(\rho \circ \sigma = \)
id_{TM}. In our case, such \( \sigma \) always exists although it is not unique. With
the help of \( \sigma \) we lift a vector field \( X \in \mathcal{X}(M) \) from \( M \) to \( \Sigma \), i.e., \( \tilde{X}(p) = \sigma(X(\pi_M(p))), \pi_M(p) = x \in M \), and we consider \( \tilde{X} \) as a \( G \)-right-invariant vector field on \( E \). Finally, we lift this field, with the help of the inclusion \( \iota_g \), to the groupoid \( \Gamma \). We thus obtain
\[
\tilde{X}(p, g) = (\iota_g)_*p \tilde{X}(p) \in \mathcal{X}(E \times G)
\]
for every \( (p, g) \in \Gamma \). Vector fields \( \tilde{X} \in \mathcal{X}(\Gamma) \), obtained in this way, inherit from \( \sigma \) the right invariance property. Lemma 1 evidently applies to such vector fields. Moreover, we have the following proposition.

**Proposition 2** Vector fields \( \tilde{X} \in \mathcal{X}(\Gamma) \) form a \( \mathbb{Z} \)-submodule of the \( \mathbb{Z} \)-module \( \text{Der}A \) of derivations of the algebra \( A \). They will be called horizontal derivations of \( A \) and denoted by \( \text{Der}_{\text{Hor}}A \).

**Proof** Let \( a, b \in A \). One readily checks, taking into account the right invariance of \( \tilde{X} \), that \( f \tilde{X}(a \ast b), f \in \mathbb{Z}, a, b \in A \), satisfies the Leibniz rule. We shall show that \( f \tilde{X} \in \text{Der}_{\text{Hor}}A \). Indeed, let \( f_0 \in C^\infty(M) \) be such that \( f = \pi_M^*f_0 \), and \( X' := f_0X, X \in \mathcal{X}(M) \). We have
\[
\tilde{X}' = \pi_M^*f_0\tilde{X} = f\tilde{X},
\]
and by acting on both sides with \( \iota_g \) we obtain \( \tilde{X}' = f\tilde{X} \).

### 3.2 Vertical Derivations

Let us consider all right invariant vertical vector fields on \( E \), i.e., all right invariant vector fields \( \tilde{X} \in \mathcal{X}(E) \) such that \( (\pi_M)_*(\tilde{X}) = 0 \). Such vector fields lifted to \( \Gamma \) are, on the strength of Lemma 1, derivations of the algebra \( (A, \ast) \); we shall call them vertical derivations of this algebra, i.e.
\[
\tilde{X}(p, g) = (\iota_g)_*p \tilde{X}(p) \in \text{Der}_{\text{Ver}}A
\]
for every \( g \in G \).

Let us notice that \( \tilde{X} \in \mathcal{X}(E) \) can be regarded as cross sections of the vector bundle \( \text{ker}\rho \) and, as it can be easily seen, \( \text{Der}_{\text{Ver}}A \) is a \( \mathbb{Z} \)-submodule of the \( \mathbb{Z} \)-module \( \text{Der}A \).
3.3 Inner Derivations

The set of inner derivations of the algebra $\mathcal{A}$ is defined as follows

$$\text{Inn}\mathcal{A} = \{\text{ada} : a \in \mathcal{A}\}$$

where $(\text{ada})(b) := a \ast b - b \ast a$.

**Lemma 2** The mapping $\Phi : \mathcal{A} \to \text{Inn}\mathcal{A}$, given by $\Phi(a) = \text{ada}$, is an isomorphism of Lie algebras (and also of $\mathbb{Z}$-moduli).

**Proof** It can be easily seen that

$$[\text{ada}, \text{adb}] = \text{ad}[a, b] \in \mathcal{A},$$

i.e., $\text{Inn}\mathcal{A}$ is a Lie algebra and $\Phi$ is a Lie algebra homomorphism. Then we have: $\Phi(a) = \Phi(b) \Rightarrow [a, c] = [b, c]$, for every $c \in \mathcal{A}$. Hence $[a - b, c] = 0$ since $a - b \in \mathcal{Z}(\mathcal{A}) = \{0\}$. Therefore, $a = b$. We also see that

$$\Phi(fa) = \text{ad}(fa) = f\text{ada} = f\Phi(a)$$

for every $f \in \mathbb{Z}$. □

As we have seen in the proof of this Lemma, the fact that $\mathcal{Z}(\mathcal{A}) = \{0\}$ plays an important role in the entire structure.

3.4 Some Properties of Derivations

By *differential algebra* we understand a pair $(\mathcal{A}, \text{Der}\mathcal{A})$ where $\mathcal{A}$ is a not necessarily commutative algebra and $\text{Der}\mathcal{A}$ a (sub)module of its derivations. In the following, we will base the construction of a noncommutative geometry of the transformation groupoid $\Gamma$ on the differential algebra $(\mathcal{A}, \text{Der}\mathcal{A})$ where $\mathcal{A}$ is, as above, $C^\infty_c(\Gamma, \mathbb{C})$, and

$$\text{Der}\mathcal{A} = \text{Der}_{\text{Hor}}\mathcal{A} \oplus \text{Der}_{\text{Ver}}\mathcal{A} \oplus \text{Inn}\mathcal{A}.$$

Let $\tilde{X}_1, \tilde{Y}_1 \in \text{Der}_{\text{Hor}}\mathcal{A}$, $\tilde{X}_2, \tilde{Y}_2 \in \text{Der}_{\text{Ver}}\mathcal{A}$, $\text{ada}, \text{adb} \in \text{Inn}\mathcal{A}$. We have the following properties:

1. $[\tilde{X}_1, \tilde{Y}_1] = [\tilde{X}_1, \tilde{Y}_1]$. This follows from the fact that $\tilde{X} = \sigma(X)$ which implies that $[\tilde{X}_1, \tilde{Y}_1] = [\tilde{X}_1, \tilde{Y}_1]$. 

8
2. $[\bar{X}_2, \bar{Y}_2] = [\bar{X}_2, Y_2].$

3. $[ada, adb] = ad[a, b] \in \text{Inn}A,$ see proof of Lemma 2.

4. $[\bar{X}_1, \bar{X}_2] = 0,$ since cross sections of the vector bundle $\Sigma$ form a Lie algebra which splits into the sum of two Lie subalgebras, and the fields $\bar{X}_1$ and $\bar{X}_2$ belong to different subalgebras.

5. $[\bar{X}_1, ada] = ad\bar{X}_1(a),$ by simple computations.

6. $[\bar{X}_2, ada] = ad\bar{X}_2(a),$ by simple computations.

4 Geometry of $\text{Der}_{V er}A$ and $\text{Inn}A$

Because of the decomposition of the $Z$-module $\text{Der}A$ into three parts, the metric on $\text{Der}A$

$$G : \text{Der}A \times \text{Der}A \to Z$$

also decomposes into three parts. If $u = u_1 + u_2 + u_3,$ and $u_1 \in \text{Der}_{Hor}A,$ $u_2 \in \text{Der}_{V er}A,$ $u_3 \in \text{Inn}A,$ and analogously for $v = v_1 + v_2 + v_3,$ then

$$G(u,v) = \bar{g}(u_1, v_1) + \bar{k}(u_2, v_2) + h(u_3, v_3)$$

where $\bar{g} : \text{Der}_{Hor}A \times \text{Der}_{Hor}A \to Z$ is evidently the lifting of the metric $g : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$ on space-time $M,$ i.e.,

$$\bar{g}(u_1, v_1) = \pi^*_M(g(x, y))$$

where $x, y \in \mathcal{X}(M).$ We assume that the metrics $\bar{k} : \text{Der}_{V er}A \times \text{Der}_{V er}A \to Z$ and $h : \text{Inn}A \times \text{Inn}A \to Z$ are of the Killing type. Their form will be determined below.

The preconnection is given by the Koszul formula

$$\begin{align*}
(\nabla^*_w v)w &= \frac{1}{2} [u(G(v, w)) + v(G(u, w)) - w(G(u, v)) \\
&+ G(w, [u, v]) + G(v, [w, u]) - G(u, [v, w])].
\end{align*}
$$

(1)

Let us now consider a more general situation which will later be specified to that in our model. Let $(A, *)$ be an algebra over $C,$ $Z = \mathcal{Z}(A)$ its center,
and $\mathcal{V}$ a $Z$-module of derivations of the algebra $\mathcal{A}$. What follows is also valid if $\mathcal{Z}(\mathcal{A}) = \{0\}$ and $Z = \pi^*_M(C^\infty(M))$ as in our model. We further assume that elements of $Z$ play the role of constants for derivations of $\mathcal{V}$, i.e., $\mathcal{V}(Z) = \{v(f) = 0 : v \in \mathcal{V}, f \in Z\}$. Let us consider a metric $g : \mathcal{V} \times \mathcal{V} \to Z$; we assume the $Z$-2-linearity and symmetry of $g$, but not necessarily its nondegeneracy. Let us denote $\mathcal{V}^* = \text{Hom}(\mathcal{V}, Z)$, and $u^* = g(u, \cdot) = \Phi_g(u)$ is a one-form corresponding to the derivation $v \in \mathcal{V}$.

The symmetric two-form $g$ determines the preconnection $\nabla^* : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ by the Koszul formula (2) (with $\mathcal{G}$ replaced by $g$). Since, by assumption, $\mathcal{V}(Z) = \{0\}$, one has $\nabla^*_u(fv) = f\nabla^*_u v$, and $\nabla^*$ is a $Z$-2-linear mapping, i.e. a tensor of $(2,1)$ type. Moreover, from the Koszul formula it follows (even if $g$ is degenerate) that $w(g(u, v)) = (\nabla^*_w u)(v) + (\nabla^*_w v)(u)$

In the Koszul formula the first three terms vanish, and if we assume that $g(v, [w, u]) = g(u, [v, w])$, \(\text{(2)}\) which — as we shall see below — is valid in our case, we obtain an interesting result

\[ (\nabla^*_u v)(w) = g(\frac{1}{2}[u, v], w) \]

showing that there is a strict dependence between the (pre)connection and the metric. We should only look for a mapping $\nabla : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ that would be $g$-consistent with $\nabla^* : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$, i.e. satisfying the condition

\[ (\nabla^*_u v)(w) = g(\nabla u v, w) \]

for every $u, v, w \in \mathcal{V}$. By comparison with (2), it immediately follows that

\[ \nabla u v = \frac{1}{2}[u, v] \]

for every $u, v \in \mathcal{V}$. Moreover, for a nondegenerate tensor $g$ the mapping $\nabla$, $g$-consistent with $\nabla^*$, is unique. Indeed, since from

\[ (\nabla^*_u v)(w) = g(\frac{1}{2} \nabla u v, w) = g(\frac{2}{2} \nabla u v, w) \],
for every $u, v \in \mathcal{V}$, it follows that

$$\frac{1}{2} \nabla_u v = \frac{2}{2} \nabla_u v.$$  

It turns out that if $g$ is nondegenerate, it has all properties required for connection. Let us check, for instance,

$$(\nabla_{u_1+u_2}^* v)(w) = (\nabla_{u_1}^* v)(w) + (\nabla_{u_2}^* v)(w)$$

$$= g(\nabla_{u_1} v + \nabla_{u_2} v, w)$$

$$= g(\nabla_{u_1+u_2} v - \nabla_{u_1} v - \nabla_{u_2} v, w) = 0,$$

and, from the nondegeneracy of $g$, one has

$$\nabla_{u_1+u_2} v = \nabla_{u_1} v + \nabla_{u_1} v.$$

Therefore, we have proved the following proposition

**Proposition 3** Let $\mathcal{V}$ be a $\mathbb{Z}$-module of derivations of an algebra $(\mathcal{A}, \cdot)$ such that $\mathcal{V}(\mathbb{Z}) = \{0\}$. For every symmetric nondegenerate tensor $g : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}$, there exists one and only one connection $g$-consistent with the preconnection $\nabla^*$. It is given by

$$\nabla_u v = \frac{1}{2} [u, v]. \square$$

In the following, we shall assume the metric of the Killing form

$$g(u, v) = \text{Tr}(u \circ v).$$

It satisfies the $g$-consistency condition. Indeed, from the trace definition we have

$$\text{Tr}[w \circ u, v \circ u] = \text{Tr}([w, u] \circ v) + \text{Tr}([w, v] \circ u) = 0.$$

We now return to our model, and assume the above kind of metric for both $\text{Der}_{\text{Ver}} \mathcal{A}$ and $\text{Inn} \mathcal{A}$, but in both these cases the trace should be defined differently.

We first define the metric for $\text{Der}_{\text{Ver}} \mathcal{A}$. We assume that $G$ is a semisimple group. In this case, the Killing form reads

$$\mathcal{B}(V, W) = \text{Tr}(\text{ad}V \circ \text{ad}W),$$
for $V, W \in g$, where $g$ is the Lie algebra of the group $G$, and $\mathcal{B}$ is nondegenerate. The tangent space to any fiber $E_x$, $x \in M$, is isomorphic to $g$. Therefore, the metric $\bar{k} : \text{Der}_{\text{Ver}}A \times \text{Der}_{\text{Ver}}A \to Z$ is given by

$$\bar{k}(\bar{X}, \bar{Y}) = \mathcal{B}(X(\pi_M(p)), Y(\pi_M(p))).$$

To define the metric for $\text{Inn}A$, let us first define the trace for the algebra $\tilde{A}$ (which, by Proposition 1, is isomorphic to the algebra $A$), $\text{Tr} : A \to Z$, by

$$(\text{Tr}a)(x) = \int_G a(x, g, g) dg.$$ 

It has the following properties: (i) $\text{Tr}(a + b) = \text{Tr}a + \text{Tr}b$, (ii) $\text{Tr}(fa) = f \text{Tr}a$, (iii) $\text{Tr}(a * b) = \text{Tr}(b * a)$, for $a, b \in A$, $f \in Z$. From the last property it follows that

$$\text{Tr}([a, b]) = 0,$$

and, of course, $\text{Tr} \circ \text{ad}a = 0$.

Let us now turn to the submodule $\text{Inn}A$. We should notice that, on the strength of Lemma 2, we also have the connection $\tilde{\nabla} : A \times A \to A$ on $A$ given by

$$\tilde{\nabla}_a b = \frac{1}{2}[a, b].$$

We define the metric $h : \text{Inn}A \times \text{Inn}A \to Z$ by

$$h(\text{ad}a, \text{ad}b) = \text{Tr}(a * b),$$

and the corresponding connection is

$$\nabla_{\text{ad}a} \text{ad}b = \frac{1}{2}[\text{ad}a, \text{ad}b].$$

We shall show that the metric $h$ is nondegenerate. Indeed, let us assume that

$$\text{Tr}(a * b) = \int_G \int_G a(x, g_1, g_2)b(x, g_2, g_1)dg_2dg_1 = 0.$$ 

If $a \neq 0$, then the support of this function is not of the measure zero, and by choosing the function $b(x, g_2, g_1) = a(x, g_1, g_2)$, we obtain

$$\int_G \int_G a^2(x, g_1, g_2)dg_2dg_1 \neq 0.$$ 

We conclude that if the metric is of the trace type (either $\mathcal{B}$ or $\text{Tr}$), the formula (2) is valid (for $g = \bar{k}$, or $g = h$).
5 Curvature

Let us introduce the following abbreviations:

\[ V_1 = \text{Der}_{\text{Hor}} A, \quad V_2 = \text{Der}_{\text{Ver}} A, \quad V_3 = \text{Inn} A. \]

We continue to develop the geometry for \( V_i, \ i = 1, 2, 3 \). The curvature is

\[ i R: V_i \times V_i \times V_i \to V_i, \]

\[ i R (u, v) w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w. \]

If \( j = 2, 3 \), we have

\[ j R (u, v) w = \frac{1}{2} [u, \frac{1}{2} [v, w]] \]

\[ - \frac{1}{2} [v, \frac{1}{2} [u, w]] - \frac{1}{2} [[u, v], w] \]

\[ = - \frac{1}{4} [[u, v], w]. \]

Here we have made use of the Jacobi identity.

For every endomorphism \( T : V_i \to V_i, \ i = 1, 2 \), there exists \( \text{Tr} T \in Z \) satisfying the usual trace conditions. We thus can define

\[ i R^m_{uwm}: V_i \to V_i, \]

\[ i R_{uw} (v) = i R (u, v) w, \]

and

\[ i \text{ric}: V_i \times V_i \to Z \]

\[ i \text{ric} (u, v) = \text{Tr} i R_{uw}. \]

We also define the \textit{adjoint Ricci operator}

\[ i \text{ric} (u, w) = \bar{G} (\bar{R} (u), w) \tag{3} \]

where we have introduced the notation: \( \bar{g} = \frac{1}{2} G, \ \bar{k} = \frac{2}{3} \). If the metric \( \bar{G} \) is nondegenerate, there exists the unique \( \bar{R} \) satisfying eq. (3) for every \( v \in V_i \).
The curvature scalar is \( \mathfrak{r} = \text{Tr } \mathcal{R} \in Z \).

In the module \( V_2 \) there exists the usual trace operator which, in the local basis, can be written as the trace of the operator matrix. Therefore,

\[
2 \mathcal{R}_{uv} (w) = \frac{1}{4} [w, [u, v]] = \frac{1}{4} (\text{ad} w \circ \text{ad} u)(v),
\]

and we have

\[
2 \mathcal{R}_{uv} = \frac{1}{4} (\text{ad} w \circ \text{ad} u)
\]

for every \( u, w \in V_2 \), and

\[
\text{Tr } 2 \mathcal{R}_{uv} = \frac{1}{4} \text{Tr}(\text{ad} w \circ \text{ad} u).
\]

Hence,

\[
2 \text{ric } (u, w) = \frac{1}{4} k(u, w)
\]

for every \( u, w \in V_2 \), which can be regarded as a generalized Killing form. By analogy, we postulate

\[
3 \text{ric } (u, w) = \alpha h(u, v)
\]

for every \( u, w \in V_3 \).

The “Ricci scalar” can be determined from the generalized Einstein equation

\[
3 \text{ric } (u, v) - \frac{1}{2} rh(u, v) = 0
\]

or

\[
\alpha h(u, v) - \frac{1}{2} rh(u, v) = 0.
\]

Hence we obtain

\[
(\alpha - \frac{1}{2} r) h(u, v) = 0,
\]

and

\[
\alpha = \frac{1}{2} r.
\]

We can symbolically regard \( r \in Z \) as a trace of the Ricci operator \( \mathcal{R} \). The Ricci 2-form is thus proportional to the metric tensor \( h \), and the proportionality coefficient (up to factor 2) is a counterpart of the Ricci curvature scalar.
A counterpart of eq. (3) for $V_3$ is

$$\text{ric} (u, v) = h(\mathcal{R} (u), v).$$

(6)

Hence

$$\alpha h(u, v) = h(\mathcal{R} (u), v),$$

or

$$h(\alpha u, v) = h(\mathcal{R} (u), v),$$

and finally,

$$\mathcal{R} (u) = \alpha u.$$

Let us notice that for a commutative algebra we have $\alpha = 0$, and the sector corresponding to $V_3$ vanishes. Therefore, the coefficient $\alpha$ could be regarded as a “measure” of noncommutativity.

This concludes the construction of the noncommutative groupoid geometry. The transition from this geometry to the usual space-time geometry can be done by the following “averaging” procedure. If $a \in \mathcal{A}$ then we have the isomorphism $a(p, g) = \tilde{a}(x, g_1, g_2)$, and we define

$$\langle \tilde{a} \rangle (x) = \int_G \tilde{a}(x, g, g)dg.$$

It is clear that $\langle \tilde{a} \rangle \in C^\infty_c(M)$, and from the algebra $C^\infty_c(M)$ one can reconstruct the usual space-time geometry together with the usual Einstein equations [7].

6 Generalized Einstein’s equation

We have all geometric quantities necessary to write the counterpart of Einstein’s equation on the groupoid $\Gamma$. We stipulate that in the noncommutative regime at the fundamental level, there is only a “pure noncommutative geometry”, and all necessary “matter terms” will somehow emerge out of it. We thus assume that there is no counterpart of the energy-momentum tensor and, consequently, the generalized Einstein’s equation is of the form

$$\mathcal{R} - \frac{1}{2} \text{r.id}_\mathcal{V} = 0$$

(7)
where $\mathcal{R}$ is the Ricci operator defined by eq. (3) (superscript $i = 1, 2$ is omitted but presupposed), and $r = \text{Tr}\mathcal{R}$.

It is clear that eq. (7), for $V = V_1$, is a “lifting” of the usual Einstein equation on space-time $M$ to the groupoid $\Gamma$, and every $\bar{g}$ that solves this equation on $M$ solves also eq. (7).

Let us now consider the case $V = V_2$. By comparing eq. (3) with eq. (4) and noticing that $\frac{1}{4}\bar{k}(u, v) = k(\frac{1}{4}u, v)$, one obtains

$$\frac{3}{4}\mathcal{R} = \frac{1}{4}\text{id}_{V_2}.$$ 

Similarly, for $V = V_3$, by taking into account eq. (5) and comparing it with eq. (4), we obtain

$$\frac{3}{4}\mathcal{R} = \alpha \text{id}_{V_3}.$$ 

Let us consider the $G$-orthogonal sum $V = V_1 \oplus V_2 \oplus V_3$. For the Ricci operator $\mathcal{R} : V \rightarrow V$ we have $\mathcal{R}(V_i) \subseteq V_i, i = 1, 2, 3$, and $\mathcal{R}|_{V_i} = \mathcal{R}_i$. This leads to the eigenvalue equation

$$\mathcal{R}(u) = \lambda u$$

for $u \in V$. This eigenvalue problem has the following solutions:

1. $\lambda_1 = \frac{1}{2}r$ where $r$ is the Ricci scalar curvature for the metric tensor $\bar{g}$.
   
   We thus have the equation
   $$\mathcal{R}(u) - \frac{1}{2}ru = 0$$
   
   for $u \in V_1$, and each such $u$ satisfies this equation. It can be easily checked that this equation reduces to the equation $\mathcal{R} = 0$ on space-time $M$.

2. $\lambda_2 = \frac{1}{4}$ which leads to the equation
   $$\mathcal{R}(u) - \frac{1}{4}u = 0$$
   
   for $u \in V_2$.

3. $\lambda_3 = \alpha$ leading to the equation
   $$\mathcal{R}(u) - \alpha u = 0$$
   
   for $u \in V_3$. In the commutative case $\alpha = 0$ and we obtain $\mathcal{R}(u) = 0$ (on the groupoid $\Gamma$).
7 Quantum sector

The quantum sector of our model is obtained by the regular representation of the groupoid algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}^p = L^2(\Gamma^p), \ p \in E,$

$$\pi_p : \mathcal{A} \to \mathcal{B}(\mathcal{H}^p),$$

where $\mathcal{B}(\mathcal{H}^p)$ denotes the algebra of bounded operators on $\mathcal{H}^p$, given by

$$(\pi_p(a)\psi)(\gamma) = \int_{\Gamma_{d(\gamma)}} a(\gamma_1)\psi(\gamma_1^{-1} \circ \gamma)d\gamma_1$$

where $a \in \mathcal{A}, \psi \in \mathcal{H}^p, \gamma, \gamma_1 \in \Gamma$. Let us notice that the Haar measure on the group $G$, transferred to the fibres of $\Gamma$, forms a Haar system on $\Gamma$.

We shall show that every element $a \in \mathcal{A}$ generates a random operator $r_a$ on $(\mathcal{H}^p)_{p \in E}$. By a random operator $r$ we mean a family of operators $(r_p)_{p \in E}$, i.e., a function

$$r : E \to \bigsqcup_{p \in E} \mathcal{B}(\mathcal{H}^p)$$

such that

1. the function $r$ is measurable in the following sense: if $\xi_p, \eta_p \in \mathcal{H}^p$ then the function

$$E \ni p \mapsto (r(p)\xi_p, \eta_p) \in \mathbb{C}$$

is measurable with respect to the manifold measure on $E$;

2. the function $r$ is bounded with respect to the norm:

$$||r|| = \text{ess sup} ||r(p)||$$

where ess sup means the “supremum modulo zero measure sets”.

Random operator $r$ acts, in fact, on cross sections of the Hilbert bundle $\mathcal{H} = \bigsqcup_{p \in E} \mathcal{H}^p$.

It can be easily seen that the family of operators $r_a = (\pi_p(a))_{p \in E}$ is a random operator. Indeed, if $\xi_p, \eta_p \in L^2(\Gamma^p)$ then we have the scalar product

$$\langle \int_{\Gamma^p} \pi_p(a)\xi_p, \eta_p \rangle = \int_{\Gamma^p} \left( \int_{\Gamma_{d(\gamma)}} a(\gamma_1)\xi_p(\gamma_1^{-1} \circ \gamma)d\gamma_1 \right) \overline{\eta_p(\gamma)}d\gamma,$$
and the Haar measure is transferred from $G$ to $\Gamma_p$ for each $p \in E$. Therefore, condition (1) is satisfied.

To check condition (2) let us introduce the isomorphisms of Hilbert spaces $I_p : L^2(G) \to H^p$ given by the formula

$$(I_p \psi)(pg^{-1}, g) = \psi(g)$$

for $\psi \in L^2(G)$. Let us consider the operators $\tilde{\pi}_p(a) = I_p^{-1} \circ \pi_p(a) \circ I_p$. It is clear that $||\pi_p(a)|| = ||\tilde{\pi}_p(a)||$. Let us also notice that

$$\tilde{\pi}_{pg}(a) = R_g^{-1} \circ \tilde{\pi}_p(a) \circ R_g$$

(where $R_g$ denotes the right translation operator in the space $L^2(G)$), which entails the (unitary) invariance of the norm $||\pi_{pg}(a)|| = ||\pi_p(a)||$.

Hence, the norm $||\pi_p(a)||$ depends only on $x = \pi_M(p) \in M$; therefore, the function $x \mapsto ||\pi_p(a)||$ is well defined, compactly supported and continuous (in its dependence on $x$) on $M$.

Let $M$ denote the set of all equivalence classes (modulo equality almost everywhere) of random operators $r_a, a \in A$. It forms a von Neumann algebra; we shall call it the algebra of random operators of the groupoid $\Gamma$, or simply the von Neumann algebra of the groupoid $\Gamma$. We shall show that $M$ is a semifinite algebra and, consequently, that it admits a “modular evolution”, just like in the model with a finite group $G$ [17]. To this end, let us first recall some important concepts.

A von Neumann algebra $M$ is semifinite if there exists a faithful, normal, semifinite weight $\varphi$ on $M$ which is a trace.

- A linear functional $\varphi : M \to \mathbb{C}$ is a state on $M$ if $\varphi(r) \geq 0$ for every $r \in M_+$, where $M_+ = \{x \cdot x^* : x \in M\}$ is the subset of positive elements of $M$ and $\varphi(1) = 1$.

- A functional $\varphi : M_+ \to [0, \infty]$ is a weight if $\varphi$ is additive, i.e., $\varphi(x + y) = \varphi(x) + \varphi(y)$, and positively homogeneous, i.e., $\varphi(\lambda x) = \lambda \varphi(x)$, for $\mathbb{R} \ni \lambda \geq 0$, $x, y \in M$. We additionally assume: $\lambda + \infty = \infty$, $\lambda \cdot \infty = \infty$ if $\lambda \neq 0$, and $\lambda \cdot \infty = 0$ if $\lambda = 0$. Let us notice that every state defines a weight.
• A weight $\varphi$ is **faithful** if for $r \in \mathcal{M}_+$ one has: $\varphi(r) = 0 \Rightarrow r = 0$.

• The sufficient and necessary condition for a weight $\varphi$ to be **normal** is: $\varphi(x) = \sum_i \omega_i$ for a family $\{\omega_i\}$ of normal states, i.e., $\omega(r) = \text{Tr}(\rho r)$, $\text{Tr}(\rho) = 1$ [20].

• Let us define: $D\varphi := \{x \in \mathcal{M}_+ : \varphi(x) < \infty\}$ and $\mathcal{M}_\varphi := \text{Span}_\mathbb{C}(D\varphi)$, i.e. $\mathcal{M}_\varphi$ is the space of $\mathbb{C}$-linear combinations of elements of $D\varphi$. A weight $\varphi$ is **semifinite** if $\mathcal{M}_\varphi$ is $\sigma$-weakly dense in $\mathcal{M}$ [20, p. 56].

• A weight $\varphi$ is a **trace** if $\varphi(r^* \cdot r) = \varphi(r \cdot r^*)$, for every $r \in \mathcal{M}$.

**Proposition 4** The von Neumann algebra $\mathcal{M}$ of the groupoid $\Gamma$ is semifinite.

**Proof** We can consider the von Neumann algebra $\mathcal{M}$ as an algebra of bounded operators on the Hilbert space $H = L^2_G(E, \mathcal{H})$ of $G$-covariant square-integrable sections of the bundle $\mathcal{H}$. $H$ is isomorphic to $L^2_G(E, L^2(G)) \simeq L^2(M \times G)$. The latter space is separable ($M \times G$ is a locally compact manifold). We choose the Hilbert basis $\{\psi_k\}_{k=1}^\infty$ in $H$, and define the weight $\varphi : \mathcal{M}_+ \to [0, \infty]$ by

$$\varphi(r) = \sum_{k=1}^\infty (r\psi_k, \psi_k).$$

This weight is clearly faithful and trace. It is also normal since $\varphi_i = \sum_{i=1}^\infty \omega_i$ where $\omega_i$ is given by $\omega_i(r) = \text{Tr}(r \rho_i)$ with $\rho_i$ being the projection onto the basis vector $\psi_i \in H$.

To show that $\varphi$ is semifinite, let us notice that we have the net of finite-dimensional projections $P_\alpha$ such that $\varphi(P_\alpha) < \infty$ and $\lim P_\alpha = 1$, in strong topology, i.e., for every $h \in H$ one has $P_\alpha h = h$. And this is the necessary and sufficient condition for $\varphi$ to be semifinite [20, p. 57].

The fact that the von Neumann algebra $\mathcal{M}$ is semifinite ensures that it admits a modular group of automorphisms [20, Chapt. 2]. In our case, this group can be defined for a state (the assumption that $\varphi$ is a weight was necessary only to prove that $\mathcal{M}$ is semifinite). Let us consider a functional of the form

$$\varphi(r) = \int_E \text{tr}(\hat{\rho}(p)r(p)) d\mu_E(p)$$

where $\hat{\rho}(p)$ is a positive operator of trace class in $\mathcal{B}(\mathcal{H}^p)$, for every $p \in E$. Let $\{e_i\}$ be a basis in $\mathcal{H}^p$ such that $\hat{\rho}(p)e_i = \lambda_i(p)e_i$, $\lambda_i > 0$. We also postulate
\[ \sum_{i=0}^{\infty} \lambda_i(p) = \lambda(p) < \infty \]

for almost every \( p \in E \), and \( \lambda(\cdot) \in L^1(E) \) with

\[ \int_E \lambda(p) d\mu_E(p) = 1. \]

With these conditions the functional \( \varphi \) is a state, and it satisfies all conditions of the Tomita-Takesaki theorem. We thus can write the state dependent evolution of random operators \( r \in \mathcal{M} \) as

\[ \sigma_\varphi^s(r(p)) = e^{isH^\varphi(p)}r(p)e^{-isH^\varphi(p)} \]

where \( H(p) = \log\hat{\rho}(p) \) and \( \log\hat{\rho}(p)e_i = (\log\lambda_i(p))e_i \). After differentiating the above equation it can be rewritten as

\[ \frac{d}{ds}\bigg|_{s=0} r_a(p, s) = i[H^\varphi(p), r_a(p)]. \quad (8) \]

This is a generalization of the Heisenberg equation of the standard quantum mechanics with the only difference that now the dynamics depends on the state \( \varphi \). The fact we have just proved that the von Neumann algebra \( \mathcal{M} \) is semifinite, has serious consequences in this respect. The Dixmier-Takesaki theorem [5, p. 470] states that if \( \mathcal{M} \) is semifinite then every state-dependent evolution is inner equivalent to the trivial one, i.e.,

\[ U_s \sigma_\varphi^s(r(p)) U_s^* = r(p) \]

for every \( s \in \mathbb{R} \), where \( U_s \) is an unitary element of \( \mathcal{M} \). This means that the state-independent evolution, obtained by the Connes-Nicodym-Radon construction [20, p. 74], is trivial. To overcome this difficulty we should assume that the group \( G \) is a locally compact non-unimodular group. We will return to this problem in a forthcoming paper.

However, a dependence of dynamics on a state need not be a drawback when we are dealing with the Planck level. The theory of von Neumann algebras can be regarded as a noncommutative counterpart of the measure theory. In the commutative case there is only one interesting measure (the
Lebesgue measure), whereas in the noncommutative case there is a great variety of measures (see, for instance [21]). Each pair \((\mathcal{M}, \varphi)\), where \(\mathcal{M}\) is a von Neumann algebra and \(\varphi\) a state on \(\mathcal{M}\) (usually assumed to be faithful and normal), is both a dynamic object and a probabilistic object. In this context, the fact that there are as many \(\varphi\)-dependent dynamics as are generalized probabilistic measures \(\varphi\) seems quite natural.

8 Transition to quantum mechanics

Let \(a^*(\gamma) = a(\gamma^{-1}), a \in \mathcal{A}, \gamma \in \Gamma\), and let us denote by \(\mathcal{A}_H\) the subset of all Hermitian elements of \(\mathcal{A}\), i.e. such that \(a^* = a\). If \(a \in \mathcal{A}_H\) then \(\pi_p(a) \in (\mathcal{B}(\mathcal{H}^p))_H\) since \(\pi_p\) is a \(*\)-representation of the algebra \(\mathcal{A}\). In the following we shall consider the random operators of the form \(r_a(p) = \pi_p(a)\). Such operator is Hermitian if \((r_a(p)\psi, \varphi) = (\psi, r_a(p)\varphi)\). Moreover, it is a compact operator for \(a \in \mathcal{A}\), since \(a\) has compact support. On the strength of the spectral theorem for Hermitian compact operators in a separable Hilbert space, there exists in \(\mathcal{H}^p\) an orthonormal countable Hilbert basis of eigenvectors \(\{\psi_i\}_{i \in I}\) of the Hermitian operator \(r_a(p)\). We can write its eigenvalue equation as

\[
  r_a(p)\psi_i(p) = \lambda_i(p)\psi_i(p).
\]

Let us notice that this equation is valid “for every \(p \in E\)” which reflects the fact that the random operator \(r_a\) is a family of functions indexed by \(p \in E\). Therefore, with respect to a random operator it is meaningful to speak only about its eigenfunction \(\lambda_i : E \to \mathbb{R}\) (not about its eigenvalue). However, every concrete measurement is always performed in a given local frame \(p \in E\), and when such a measurement has been done the eigenfunction \(\lambda_i\) collapses to the eigenvalue \(\lambda_i(p)\). Let us observe that from the perspective of the local measurement it looks as if the measurement result were a random effect, but in fact it is but a value of a well determined function \(\lambda_i(p)\) at a given \(p\). Its “randomness” comes from a subtler source, namely from the fact that \(r_a\) is a random operator. This is our model’s version of the so often discussed “collapse of the wave function”.

Let us also notice that every act of measurement, performed at \(p\), singles out the isomorphism \(I_p^{-1} : \mathcal{H}^p \to L^2(G)\) which reproduces the usual quantum mechanics (on \(G\)). For instance, to obtain the quantum evolution for \(a \in \mathcal{A}\),
we apply $I_p^{-1}$ to the left hand side of equation (8), and $I_p$ to its right hand side. In this way, we obtain

$$
\frac{d}{dt}\big|_{t=0} \tilde{\pi}(a(t)) = i[\tilde{H}^\varphi, \tilde{\pi}(a(t))]
$$

where $\tilde{\pi}(a) = I_p^{-1} \circ \pi_p \circ I_p$ and $\tilde{H}^\varphi = I_p^{-1} \circ H^\varphi_p \circ I_p$, we have also put $s = t$. This is the Heisenberg equation of the standard quantum mechanics with the only difference that it depends on the state $\varphi$. In more realistic models, to which the Connes-Nikodym-Radon construction applies, even this difference disappears (see remarks at the end of the preceding section).

The above results seem to be important as far as the interpretation of quantum mechanics is concerned. Its peculiarities are largely due to the fact that it is but a part of a larger structure, out of which it is cut off by every act of measurement. When such an act is performed the larger structure “collapses” to its substructure known as quantum mechanics. Quantum mechanics turns out to be but a theory of making measurements within our model.

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