THE HOFFMAN-ROSSI THEOREM FOR OPERATOR ALGEBRAS

DAVID P. BLECHER, LUIS C. FLORES, AND BEATE G. ZIMMER

Abstract. We study possible noncommutative (operator algebra) variants of the classical Hoffman-Rossi theorem from the theory of function algebras. In particular we give a condition on the range of a contractive weak* continuous homomorphism defined on an operator algebra $A$, which is necessary and sufficient (in the setting we explain) for a positive weak* continuous extension to any von Neumann algebra containing $A$.

1. Introduction

The Hoffman-Rossi theorem that we are interested in here is a remarkable result from the theory of function algebras [7, Theorem 3.2]. It states that if $A$ is a weak* closed unital subalgebra of $M = L^\infty(\mu)$, for a probability measure $\mu$, and if $\varphi$ is a weak* continuous character (i.e. nontrivial complex-valued homomorphism) on $A$, then $\varphi$ has a weak* continuous positive linear extension to $M$. Since unital linear functionals on $M$ (or on any $C^*$-algebra) are states (that is, contractive and unital linear functionals) if and only if they are positive, this is saying that weak* continuous characters on $A$ have weak* continuous Hahn-Banach extensions to $L^\infty(\mu)$. Or, in the language of von Neumann algebras (see e.g. p. 245, 248–249 in [1]), weak* continuous characters on $A$ have normal state extensions to $L^\infty(\mu)$. In the present paper we study possible noncommutative (operator algebra) variants of this result. An operator algebra is a unital algebra of operators on a Hilbert space, or more abstractly a Banach algebra isometrically isomorphic to such an algebra of Hilbert space operators. (Sometimes one wants to consider an operator space structure on an operator algebra, and replace the word ‘isometrically’ by ‘completely isometrically’ in the last sentence (see e.g. [4] for definitions), but this will not be important in the present paper.) Our main result is a condition on the range of a contractive (or completely contractive) weak* continuous homomorphism $\Phi$ defined on a unital operator algebra $A$, which is necessary and sufficient in the setting explained below, for a positive weak* continuous extension of $\Phi$ to any von Neumann algebra containing $A$ as a weak* closed subalgebra.

2. A von Neumann algebra valued Hoffman-Rossi theorem

There are several natural ways to try to generalize the Hoffman-Rossi theorem to the operator algebra setting. First however we note that the setting of algebras and algebra homomorphisms is crucial. Even in the classical setting, one cannot hope...
that weak* continuous states on operator systems have weak* continuous Hahn-Banach extensions (that is, normal state extensions). A convincing illustration of this is the state \( \varphi_1 \) of evaluation at 1 on the set \( \mathcal{S} \) of polynomials of degree \( \leq 1 \) on \([0,1]\), viewed as a subspace of \( L^\infty([0,1]) \). This is weak* continuous (since \( \mathcal{S} \) is finite dimensional). Any normal state extension to \( L^\infty([0,1]) \) of \( \varphi_1 \) is integration against a positive \( g \in L^1([0,1]) \) with \( \int_{[0,1]} g dt = 1 \). Applying this state to the monomial \( t \) gives \( \int_{[0,1]} t g(t) dt = \varphi_1(t) = 1 \). Hence \( \int_{[0,1]} (1-t) g(t) dt = 0 \), forcing the contradiction \( g = 0 \) a.e.. We will use this example later. With a little more work one can find a weak* continuous state on a unital weak* closed subalgebra of a von Neumann algebra \( M \), which has no normal state extension to \( M \) (for example the algebra in the proof of Proposition 2.3 below).

In the noncommutative (operator algebra) setting we suppose that we have a weak* closed unital subalgebra \( A \) of a von Neumann algebra \( M \), and a weak* continuous unital contractive homomorphism \( \Phi : A \to D \), for a von Neumann algebra \( D \). By the first (resp. second) ‘unital’ here we mean that 1 \( \in A \), where 1 is the identity of \( M \) (resp. \( \Phi(1) = 1 \)). The question we are interested in is when does \( \Phi \) have a weak* continuous contractive (or equivalently, positive) \( D \)-valued linear extension to \( M \)? Sometimes we will add the adjective ‘completely’, for example consider UCP (unital completely positive) extensions–see e.g. [4, Chapter 1] for notation. To obtain such a Hoffman-Rossi theorem one has to have restrictions on the algebras or on \( \Phi \) or its range of \( \Phi \), as we will see below. It is not true in general, for example, when \( D = B(H) \) for a Hilbert space \( H \) (unless \( A \) is also selfadjoint, in which case it may be proved using as one ingredient the well known extendibility of normal representations of von Neumann algebras).

In the case of the original Hoffman-Rossi theorem we may identify the range of the character with \( D = C^1_A = C^1_M \), in which case the homomorphism we are extending is an idempotent map on \( A \), and is a \( D \)-bimodule map. Thus we will usually restrict our attention to the setting of a weak* continuous unital contractive or completely contractive homomorphism \( \Phi : A \to D \subset A \) which is a \( D \)-bimodule map (or equivalently, is the identity map on \( D \)). We remark that by 4.2.9 in [4], any unital completely contractive projection of a unit operator algebra onto a subalgebra \( D \) is a \( D \)-bimodule map. We then ask for a positive normal extension from \( M \) into \( D \). Note that the latter implies that \( D \) is a von Neumann algebra, and \( \Psi \) is a conditional expectation onto \( D \) (see p. 132–133 in [1] for the main facts about conditional expectations and their relation to bimodule maps and projections maps of norm 1). However \( D \) being a von Neumann algebra is not enough for a weak* continuous positive extension from \( M \) into \( D \), even if \( D \) is also injective and commutative. E.g. Takesaki [9] showed that there need not exist a positive weak* continuous extension from \( M \) onto such \( D \) if \( M \) is not ‘finite’.

The correct Hoffman-Rossi theorem in the setting in the last paragraph that works for all von Neumann algebras \( M \), requires \( D \) to be finite dimensional, or more generally, a (purely) atomic von Neumann algebra (see p. 354 in [1]). This is the next result, and the result after it shows the necessity of the atomic hypothesis.

**Theorem 2.1.** Suppose that \( A \) is a weak* closed unital subalgebra of a von Neumann algebra \( M \). Suppose that \( D \in \mathbb{C} \) or is a weak* closed selfadjoint unital subalgebra of \( A \) which is finite dimensional, or more generally is an atomic von
Neumann algebra. Suppose that Φ : A → D is a weak* continuous unital contractive homomorphism on A that is a D-bimodule map. Then there exists a normal positive map Ψ : M → D extending Φ (hence Ψ is also a conditional expectation).

Proof. First suppose that D = C. Let J be the kernel of Φ in A, then A = J + C1. Suppose that M acts on a Hilbert space K, so that A ⊂ B(K). Consider the amplification π(χ) = x(∞) of the identity representation of M, acting on the countably infinite direct sum K(∞) of copies of K. Then π(J) is reflexive by e.g. [4] Appendix A.1.5, and by definition of reflexive there, since I /∈ π(J), there is a vector ξ ∈ K(∞) with ξ /∈ [π(J)]ξ. Hence [π(A)]ξ ⊗ [π(J)]ξ ≠ (0). Choose η ∈ [π(A)]ξ ⊗ [π(J)]ξ of norm 1. Since π(J)η ∈ π(J)[π(A)]ξ ⊂ [π(J)]ξ it follows that (π(η), η) is a normal state on M annihilating J. Hence its restriction to A is a multiple of Φ, indeed equals Φ since both are states.

Next suppose that D is a type I factor, thus isomorphic to B(l2(I)) for an index set I. We may suppose that M = B(K), and that D is the range of a normal unital *-homomorphism π : B(l2(I)) → B(K). For i ∈ I let pi = π(Eii), where {Eik} are the matrix units in B(l2(I)), and set L = pjK for a fixed j ∈ I. By a matrix unit argument we may suppose that (unitarily) K = L(I) = L ⊗ l2(I), the Hilbert space sum of I copies of L, that M = B(L) ⊗ B(l2(I)), and D = I_L ⊗ B(l2(I)) and pi = I_L ⊗ Eii ∈ D ⊂ A. Also, if Aj = pjApj and B = {T ∈ B(L) : pjTpj ∈ Aj ⊂ A} then it is easy to see that B ∋ B ⊗ Ejj = Aj unitarily, and A = B ⊗ B(l2(I)). For i, j ∈ I, Φ(piapk) = piΦ(a)pk ∈ C Eik. Hence Φ(b ⊗ Eik), viewed as a matrix in \(B(l^2(I))\), is zero except perhaps for its i-k entry, for any b ∈ B. Let φ = πjj ◦ Φ ◦ e_jj, where \(e_j(b) = b ⊗ Ejj\) and \(πjj\) is the state on D that evaluates the j-j entry. Then φ is a character of B. Also, \(Φ = φ = I\), indeed for b ∈ B and i, k ∈ I we have

\[
Φ(b ⊗ Eik) = EijΦ(b ⊗ Ejj)Ejk = Eijφ(b)Ejk = (φ ⊗ I)(b ⊗ Eik).
\]

By the first paragraph, φ extends to a normal state σ on B(L), so that Ψ = σ ⊗ I is a normal UCP map extension of Φ to \(M = B(L) ⊗ B(l^2(I))\).

Finally, suppose that D is atomic, so \(D \cong \bigoplus_{i ∈ I} B(H_i)\) for Hilbert spaces \(H_i\). Let \(p_i\) be the central projection in \(D\) corresponding to the identity in \(B(H_i)\). We have that \(Φ(p_iap_j) = p_iΦ(a)p_j = 0\) for \(i ≠ j\) (since \(p_i\) is central in \(D\)) Thus \(Φ = Φ ◦ Δ\) where \(Δ\) is the UCP map on \(M\) defined by \(Δ(x) = \sum p_i xp_i\). Let \(Φ_i = Φ|_{p_iAp_i}\). By the last paragraph \(Φ_i\) extends to a normal UCP map \(σ_i : p_iMp_i → p_iDp_i\). We obtain a normal UCP map extension \(Ψ = (Φ_iσ_i) ◦ Δ\) of \(Φ\) to \(M\). By Tomiyama’s well known theorem (see p. 132–133 in [1]) on projections of norm 1, \(Ψ\) is necessarily a \(D\)-bimodule map and conditional expectation.

Remarks. 1) An alternative proof of the case \(D = C\) of the last result, which is slightly longer, comes from following a method of Cassier to prove a similar result for characters on singly generated commutative dual operator algebras [5]. We sketch the details: We may suppose that \(M = B(H)\) for a Hilbert space \(H\). There is a trace class operator \(r\) on \(H\) such that \(tr(xy) = Φ(x)\) for all \(x ∈ A\). It is well known that \(r = ab^*\) for Hilbert-Schmidt operators \(a, b\) on \(H\). Let \(E\) (resp. \(F\)) be the closure in the Hilbert-Schmidt norm of \(Aa\) (resp. \(Ja\)). It is easy to see that \(E = F + Ca\). Let \(c = P_E(b)\), the projection onto \(E\) of \(b\), so that \(c = Φa + f\) for some \(f ∈ F, a ∈ C\). For \(x ∈ A\) we have

\[
tr(xcc^*) = atr(xac^*) + tr(xfc^*) = atr(xab^*) + tr(xfb^*),
\]
the latter by definition of $c$, since $xa, xf \in E$. Since $0 = \Phi(y) = \text{tr}(yr) = \text{tr}(yab^*)$ for $y \in J$, it follows that $b \perp F$. Hence the last number in the last displayed equation is zero, and so $\text{tr}(xcc^*) = \alpha \text{tr}(xr) = \alpha \Phi(x)$. Setting $x = 1$ yields $\alpha = \text{tr}(cc^*)$. If $\alpha = 0$ then $c = 0$ and $b \perp E$, giving the contradiction $1 = \Phi(1) = \text{tr}(1ab^*) = 0$. Hence $\Psi(x) = \frac{1}{\bar{\alpha}} \text{tr}(xcc^*)$ is a normal state extension to $B(H)$ of $\Phi$.

2) A very short proof of the original Hoffman-Rossi theorem (which has hitherto had quite an elaborate proof) may be given based on the method in the last remark. Simply replace $H$ by $L^2(\mu)$ for the probability measure $\mu$ in the Hoffman-Rossi theorem, and replace trace by integral, and take the element $r$ above in $L^1$ with $\int xr \, d\mu = \Phi(x)$ for all $x \in A$. Write $r = ab$ for some $a, b \in L^2(\mu)$, and follow the last remark to obtain normal state extension $\Psi(x) = \frac{1}{\bar{\alpha}} \int x|c|^2 \, d\mu$ for $x \in L^\infty(\mu)$.

**Proposition 2.2.** Suppose that $A$ is a unital weak* closed subalgebra of $M = B(H)$, and suppose that $D$ is a weak* closed unital selfadjoint subalgebra of $A$. Suppose that $\Phi : A \to D$ is a weak* continuous unital homomorphism on $A$ that is a $D$-bimodule map. Suppose that there exists a normal positive map $\Psi : M \to D$ extending $\Phi$. Then $D$ is an atomic von Neumann algebra.

**Proof.** Note that such $\Psi$ would necessarily be a normal norm 1 projection onto $D$. However a von Neumann algebra which is the range of such a projection on $B(H)$ is atomic ([II, Theorem IV.2.2.2]).

The following example shows the importance of the selfadjointness of $D$ in finding any positive $B(H)$-valued extension of a homomorphism.

**Proposition 2.3.** There exists a finite von Neumann algebra $M$, and a commutative finite dimensional weak* closed unital subalgebra $A$ of $M$, which in turn has a unital subalgebra which may be identified (completely isometrically) with a unital subalgebra $D$ of the $2 \times 2$ matrices $M_2$, and a weak* continuous completely contractive unital homomorphism $\Phi : A \to D \subset M_2$, such that $\Phi$ has no contractive or positive weak* continuous linear extension from $M$ to $M_2$ or to $D$. Moreover if $\Phi$ is viewed as a map $A \to D \subset A$ then $\Phi$ is an idempotent $D$-bimodule map.

**Proof.** Let $\varphi_1$ be the weak* continuous state on $S \subset L^\infty([0, 1])$ in the example in the first paragraph of this section. Let $M = M_2(L^\infty([0, 1]))$, the $W^*$-algebra of $2 \times 2$ matrices with entries in $L^\infty([0, 1])$. Let $A$ be the subalgebra of $M_2(L^\infty([0, 1]))$ consisting of upper triangular matrices with scalars (constant functions) as the main diagonal entries and elements from $S$ in the 1-2 entry. This is four dimensional. It is also weak* closed in $M$ since any finite dimensional subspace is closed in any linear topology. This is related to [II, Lemma 2.7.7]. Define $\Phi : A \to M_2$ to be the map that applies $\varphi_1$ in the 1-2 entry, and leaves other entries ‘unchanged’. That is $\Phi$ is ‘evaluation’ at 1. This is easily seen to be a weak* continuous unital homomorphism. Also, $\Phi$ is (completely) contractive by [II, Proposition 2.2.11]. Let $D$ be the range of $\Phi$, the upper triangular $2 \times 2$ matrices. We may also view $D \subset A$ by identifying an upper triangular matrix with the same matrix in $A$, but with 1-2 entry multiplied by the monomial $t$. Then $\Phi$ viewed as a map $A \to D \subset A$ is an idempotent $D$-bimodule map. Suppose that $R : M \to M_2$ was a weak* continuous contractive extension of $\Phi$. Then $R$ is positive (since it is well known that contractive unital maps on $C^*$-algebras are positive). The restriction of $R$ to matrices that are only nonzero in their 1-2 entry, followed by the projection onto the 1-2 entry, defines a weak* continuous contractive functional $\psi$ on $L^\infty([0, 1])$. Finally, it is clear that
ψ extends φ₁, contradicting the first paragraph of this section. This contradiction shows that our extension \( R \) cannot be positive or contractive.

One may adjust \( A \) in the proof above to be three dimensional by taking the main diagonal entries of matrices in \( A \) to be equal. We also remark that if one insists on bimodule map extensions then one may get counterexamples with \( M \) finite dimensional (see e.g. [8, Example 3.5]).

We also remark that very strong forms of the noncommutative Hoffman-Rossi theorem hold for Arveson’s maximal subdiagonal subalgebras of \( \sigma \)-finite von Neumann algebras [3] (or more generally for algebras having some of the Gleason-Whitney properties GW1, GW2, GW from the start of Section 4 in [2], or their variants for states, and studied further in e.g. [3] Section 5) in the \( \sigma \)-finite case). In these setting it is not necessary to require the von Neumann algebra \( D \) to be a subalgebra of \( A \) or that \( \Phi \) is a homomorphism or \( D \)-bimodule map. Since our paper is about homomorphisms we will just mention some results of the first author and Labuschagne along these lines from a forthcoming work. Suppose that \( \Phi : A \to D \subset B(H) \) is a unital weak* continuous contractive linear map. If \( A \) has the Gleason-Whitney type property that every weak* continuous state on \( A \) has a unique normal state extension to \( M \), then \( \Phi \) has a normal positive extension \( M \to D \). If \( A \) has the Gleason-Whitney type property that every state extension to \( M \) of a weak* continuous state on \( A \) is normal, and \( \Phi \) is a unital weak* continuous complete contraction then \( \Phi \) has a normal UCP extension \( M \to B(H) \). The latter follows by applying the latter Gleason-Whitney type property to the composition of vector states on \( B(H) \) with a fixed UCP extension \( \Psi : M \to B(H) \).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204-3008, USA
E-mail address, David Blecher: dblecher@math.uh.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 765212, USA
E-mail address, Luis Flores: lcfvd6@mail.missouri.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, TEXAS A&M UNIVERSITY–CORPUS CHRISTI,
CORPUS CHRISTI, TX 78412-5825, USA
E-mail address, Beate Zimmer: beate.zimmer@tamucc.edu