UNIQUE CONTINUATION FOR THE HEAT OPERATOR WITH POTENTIALS IN WEAK SPACES

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Abstract. We prove strong unique continuation property for the differential inequality
\[ |(\partial_t + \Delta)u(x, t)| \leq V(x, t)|u(x, t)| \]
with \( V \) contained in weak spaces. In particular, we establish the strong unique continuation property for \( V \in L^\infty_t L^{d/2, \infty}_x \), which has been left open since the works of Escauriaza [6] and Escauriaza-Vega [8]. Our results are consequences of the Carleman estimates for the heat operator in the Lorentz spaces.

1. Introduction

We consider the differential inequality
\[ |(\partial_t + \Delta)u(x, t)| \leq |V(x, t)|u(x, t)|, \quad (x, t) \in \mathbb{R}^d \times (0, T). \]

For a differential operator \( P \) on a domain \( \Omega \), the strong unique continuation property (abbreviated to \( \text{sucp} \) in what follows) for \( |Pu| \leq |Vu| \) means that a nontrivial solution \( u \) to \( |Pu| \leq |Vu| \) cannot vanish to infinite order (in a suitable sense) at any point. The \( \text{sucp} \) for second order parabolic operator has been studied by many authors (see [22, 27, 25, 4, 6, 8, 7, 9, 19, 1] and references therein). In particular, the study of \( \text{sucp} \) for the heat operator with time-dependent potentials goes back to Poon [25] and Chen [4], who considered bounded potentials. Escauriaza [6] and Escauriaza and Vega [8] extended the results to unbounded potentials \( V \) under the parabolic vanishing condition: That is to say, for given \( \delta > 0 \) and any \( k \in \mathbb{N} \) there is a constant \( C_k \) such that
\[ |u(x, t)| \leq C_k(|x| + \sqrt{t})e^{(1-\delta)|x|^2/8t}, \quad (x, t) \in \mathbb{R}^d \times (0, T). \]

Here, the growth condition at infinity is necessary since there exists a nonzero solution \( u \) to \( (\partial_t + \Delta)u = 0 \) such that \( u \) vanishes to infinite order in the space-time variables at any point \( (x, 0) \), \( x \in \mathbb{R}^d \) (see, for example, [14, 6]).

The \( \text{sucp} \) for the Laplacian \( -\Delta \) is better understood. Since the pioneering work of Carleman [3], most of subsequent results were obtained by following his idea, the Carleman weighted inequality. In particular, Jerison and Kenig [13] proved the \( \text{sucp} \) for the Laplacian with \( V \in L^{d/2}_{loc}, d \geq 3 \). Their result was extended by Stein [28] to potentials \( V \in L^{d/2, \infty} \) under the assumption that \( \|V\|_{L^{d/2, \infty}} \) is small enough. Later, Wolff [33] showed that the smallness assumption is indispensable if \( V \in L^{d/2, \infty} \). Here, \( \|\cdot\|_{L^{p,r}} \) denotes the norm of the Lorentz space \( L^{p,r}(\mathbb{R}^d) \) (for example, see [29]).

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By the aforementioned results due to Escauriaza [6] and Escauriaza and Vega [8] the sucp for (1.1) is known when $t^{1-d/(2\tau)-1/s}V(x,t) \in L^p_{t,loc}L^\infty_x$ and $\tau, s$ satisfy

\begin{equation}
\frac{d}{2\tau} + \frac{1}{s} \leq 1, \quad 1 \leq \tau, s \leq \infty.
\end{equation}

However, in view of those results concerning the (abovementioned) sucp for the Laplacian, it seems natural to expect that the class of potentials for which the sucp holds can be further expanded to certain weak spaces.

In this paper, we extend the results in [6, 8] to a larger class of potentials, that is to say, $t^{1-d/(2\tau)-1/s}V(x,t) \in L^p_{t,loc}L^\infty_x$, $\frac{d}{2\tau} + \frac{1}{s} \leq 1, d/2 \leq \tau \leq \infty$. As in [28], our result is a consequence of new Carleman estimates for the heat operator in the Lorentz spaces.

**Carleman estimate.** We denote $L^p_tL^b_x = L^p_t(\mathbb{R}^+; L^b_x(\mathbb{R}^d))$. Then, we consider the Carleman inequality for the heat operator of the form

\begin{equation}
\| t^{-\alpha}e^{-\frac{|x|^2}{4t}} q \|_{L^p_tL^b_x} \leq C \| t^{-\alpha+1-\frac{d}{4}(\frac{1}{p}-\frac{1}{q})-(\frac{d}{2}-\frac{2}{q})}(\Delta + \partial_t)q \|_{L^p_tL^\infty_x},
\end{equation}

with $C$ independent of $\alpha$, which holds for $q \in C^\infty_c(\mathbb{R}^{d+1} \setminus \{(0,0)\})$ under a suitable condition on the exponents $p, q, r, s, a$ and $b$. For $\alpha \in \mathbb{R}$, we set

$$\beta = \beta(\alpha, q, s) = 2\alpha - \frac{d}{q} - \frac{2}{s}.$$ 

The estimate (1.4) was formerly considered with $p = a, q = b$. It was Escauriaza [6] who first obtained the estimate (1.4) for some $p = a, q = b, r, s$. More precisely, he showed that the estimate (1.4) holds with the Lebesgue spaces (i.e., $a = p, b = q$) for $p, q$ satisfying $q = p'$ and $1/p - 1/q < 2/d$ if $d \geq 2$, and $1/p - 1/q \leq 1$ if $d = 1$ provided that

$$\text{dist}(\beta, N_0) \geq c$$

for some $c > 0$ where $N_0 := \mathbb{N} \cup \{0\}$. Subsequently, the estimate (1.4) was extended by Escauriaza and Vega [8] to the exponents $p, q$ which lie outside of the line of duality. They obtained the estimate (1.4) for $2d/(d+2) \leq p \leq 2 \leq q \leq 2d/(d-2)$ if $d \geq 3$, and for $1 \leq p \leq 2 \leq q \leq \infty$. 

We extend the previously known results not only to Lorentz spaces but also on a wider range of exponents $p, q, r, s$. To present our result, for $d \geq 3$ we define $A = A(d), B = B(d), D = D(d) \in [1/2, 1] \times [0, 1/2]$ by setting

$$A = \left( \frac{d+2}{2d}, \frac{1}{2} \right), \quad B = \left( \frac{d^2+2d-4}{2(d-1)}, \frac{d-2}{2(d-1)} \right), \quad D = \left( \frac{d+2}{2d}, \frac{d-2}{2d} \right).$$

By $\mathcal{X}$ we denote the closed pentagon with vertices $(\frac{1}{2},\frac{1}{2})$, $A, B, B', A'$ from which the two vertices $B$ and $B'$ are removed. Here, $X' = (1-b, 1-a)$ (the dual point) if $X = (a,b)$. See Figure [1].

**Theorem 1.1.** Let $d \geq 3$ and $(1/p, 1/q) \in \mathcal{X}$. Let $1 \leq r \leq s \leq \infty$ satisfy $(1, 1) \neq (1, \frac{1}{r} \frac{1}{q}), (1, \frac{d}{2} \frac{1}{r} \frac{1}{q}, 0)$, and

\begin{equation}
0 \leq \frac{1}{r} - \frac{1}{s} \leq 1 - \frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right).
\end{equation}
Figure 1. The region of \( p, q \) for which (1.4) holds when \( d \geq 3 \): The dark grayed square stands for the earlier result due to Escauriaza and Vega [8], and the light grayed region for the newly extended range.

Suppose \( \beta \notin \mathbb{N}_0 \). Then, if \( \frac{1}{p} - \frac{1}{q} < \frac{2}{d} \), \( p \neq 2 \), and \( q \neq 2 \), the estimate (1.4) holds for \( 1 \leq a = b \leq \infty \) with \( C \) depending only on \( p, q, a, b, r, s, \) and \( \text{dist}(\beta, \mathbb{N}_0) \); if

\[
\frac{1}{p} - \frac{1}{q} = \frac{2}{d},
\]

the same estimate (1.4) holds for \( a = b = 2 \).

It is remarkable that Theorem 1.1 gives (1.4) for \( (1/p, 1/q) \) contained in the open line segment \( (B, B') \) (see Figure 1). The exponents \( p, q \) satisfying (1.6) constitute the critical case in that (1.4) is no longer true if \( 1/p - 1/q > 2/d \). (See Remark 2 and the condition (1.5).) Consequently, it is more difficult to obtain the estimate (1.4) for \( p, q \) satisfying (1.6) than that for \( p, q \) satisfying \( 1/p - 1/q < 2/d \). Only the estimate (1.4) with \( (1/p, 1/q) = \mathcal{D} \), \( a = p \), and \( b = q \) was previously shown by Escauriaza and Vega [8].

If \( p, q \) satisfy (1.6) and \( r = s \), then the estimate (1.4) implies the Carleman inequality for the Laplacian (see [8]):

\[
\| |x|^{-\sigma} f \|_{L^r_x} \leq C \| |x|^{-\sigma} \Delta f \|_{L^p_x}, \quad f \in C_c^{\infty}(\mathbb{R}^d \setminus \{0\})
\]

for \( \sigma > 0 \) with \( C > 0 \) depending on \( d, p, q \) and \( \text{dist}(\sigma, \mathbb{N} + \frac{d}{q}) \) if \( \text{dist}(\sigma, \mathbb{N} + \frac{d}{q}) > 0 \). By this implication the estimates in Theorem 1.1 with \( p, q \) satisfying (1.6) gives (1.7) for \( p, q \) satisfying \( (1/p, 1/q) \in (B, B') \). However, it does not extend the previously known range of \( p, q \) for which (1.7) holds. When \( d \geq 5 \), the range of \( p, q \) coincides with that in Kwon and Lee [20], which was obtained by making use of the sharp estimate for the spherical harmonic projection. The optimal range for the estimate (1.7) still remains open.

To obtain \( \text{supp} \) for potentials in \( L^p_{t,loc}L^\infty_x \) we need to obtain (1.4) with \( a = b \). To this end, we are basically relying on real interpolation to upgrade \( L^p_tL^p_x - L^q_tL^q_x \) estimates to these of \( L^p_tL^{p,a}_x - L^q_tL^{q,b}_x \) with \( a = b \). However, such extension of the
Carleman inequality \((1.2)\) to the Lorentz spaces is not so straightforward as in [26] since real interpolation does not behave well in mixed norm spaces (see [3]). In particular, we are only able to obtain \((1.4)\) with \(a = b = 2\) when \(p, q\) satisfy \((1.6)\) (also see Lemma 4.1).

**Strong unique continuation property for the heat operator.** The extension of the Carleman estimate to the Lorentz spaces (Theorem 1.3) allows a larger class of potentials for the strong unique continuation property for the heat operator. In this regard we obtain Theorem 1.2 and Theorem 1.3 below which improve the results in [8]. Once we have the Carleman estimate \((1.4)\), those theorems can be shown by routine adaptation of the argument in [8]. So, we state them without providing proofs.

**Theorem 1.2.** Let \(d \geq 3\), \(0 < T < \infty\), and \(r, s\) satisfy \((1.3)\). Let \((1/p, 1/q) \in \mathbb{S}\) satisfy \(1/p - 1/q = 1/r\). Suppose that \(u \in W^{1,a}((0, T); W^{2,p}(\mathbb{R}^d))\), \(a \leq \min\{2, s\}\), is a solution to the differential inequality \((1.1)\) and suppose that for any \(k \in \mathbb{N}\) there is a constant \(C_k\) such that \((1.2)\) holds for some \(\delta > 0\). Then \(u\) is identically zero on \(\mathbb{R}^d \times (0, T)\) provided that \(\|t^{1 - \frac{s}{r} - \frac{1}{2}r}V\|_{L^s((0, T); L^{s/2}_\infty(\mathbb{R}^d))}\) is small enough.

Most significantly, Theorem 1.2 gives the \(\text{supc}\) with \(V \in L^\infty((0, T); L^{d/2, \infty}_\infty(\mathbb{R}^d))\). This strengthens the result obtained by Escauriaza and Vega [8] under the assumption that \(\|V\|_{L^\infty((0, T); L^{d/2}_\infty)}\) is small enough. Using Wolff’s construction in [33] we can show that the smallness assumption is necessary in general for \(V \in L^\infty((0, T); L^{d/2, \infty}_\infty(\mathbb{R}^d))\) or \(V \in L^{d/2, \infty}(\mathbb{R}^d; L^\infty((0, T)))\). Indeed, Wolff showed that there is a bounded nonzero function \(w\) such that \(|\Delta w| \leq |V_* w|\) with \(V_* \in L^{d/2, \infty}\) and vanishes to infinite order at the origin. Since the function \(w\) in [33] is bounded, considering the time independent function \(u(x, t) := w(x)\) it is easy to see that \(u(x, t)\) satisfies \((1.2)\) and obviously the differential inequality \(|\Delta u + \partial_t u| \leq |V_* u|\).

We also have the following \(\text{supc}\) result for a local solution.

**Theorem 1.3.** Let \(d \geq 3\) and \(r, s\) satisfy \((1.3)\). Suppose that \(u\) is a continuous solution to \(|\Delta u + \partial_t u| \leq |V u|\) on \(B(0, 2) \times (0, 2)\) and suppose that for any \(k \in \mathbb{N}\) there is a constant \(C_k\) such that
\[
\|e^{-|x|^2/8\varepsilon^2}u\|_{L^2((0, x); L^2_k(B(0, 2)))} \leq C_k \varepsilon^k, \quad 0 < \varepsilon < 2.
\]
Then \(u(x, 0)\) vanishes on \(B(0, 2)\) if \(\|t^{1 - \frac{s}{r} - \frac{1}{2}r}V\|_{L^s((0, 2); L^{s/2}_\infty(B(0, 2)))}\) is small enough.

**Uniform resolvent estimate for the Hermite operator.** We now consider the resolvent estimate for the Hermite operator \(H = -\Delta + |x|^2\) in \(\mathbb{R}^d\):
\[
(1.8) \quad \|(H - z)^{-1}f\|_q \leq C\|f\|_p, \quad z \in \mathbb{C} \setminus (2\mathbb{N}_0 + d)
\]
with a constant \(C\) independent of \(z\). The estimate has independent interest while it plays an important role in proving Theorem 1.1 (see Lemma 4.1). Since \(H\) has the discrete spectrum \(2\mathbb{N}_0 + d\), \(z \in 2\mathbb{N}_0 + d\) are excluded. In contrast with the operator with a continuous spectrum, it is impossible for \((1.8)\) to hold with \(C\) independent of \(z\), so we need to impose the assumption that
\[
(1.9) \quad \text{dist}(z, 2\mathbb{N}_0 + d) \geq c
\]
for some $1 \gg c > 0$. (See Remark 1.) The estimate (1.8) may be compared with the corresponding estimate for the resolvent of the Laplacian which is due to Kenig, Ruiz, and Sogge [17]. It was shown in [17] that the estimate
\[
\|(-\Delta - z)^{-1}f\|_q \leq C\|f\|_p, \quad z \in \mathbb{C} \setminus (0, \infty)
\]
holds with $C$ independent of $z$ if and only if $1/p - 1/q = 2/d$, $2d/(d + 3) < p < 2d/(d + 1)$ and $d \geq 3$. Also, see [10] for the uniform estimates for more general second order differential operators and [21] for the sharp bounds which depend on $z$. Under the assumption (1.9), the uniform resolvent estimate for $H$ continues to hold with $p, q$ away from the critical line $1/p - 1/q = 2/d$ whereas this can not be true for $-\Delta$ because of scaling structure (see [17][21]).

The uniform estimate (1.8) was obtained by Escauriaza and Vega [8] for $2d/(d + 2) \leq p \leq 2 \leq q \leq 2d/(d - 2)$, $d \geq 3$. However, (1.8) fails to hold if $1/p - 1/q > 2/d$ (see Remark 2) and the proof of (1.8) is more involved if $p, q$ satisfy (1.6). As for such $(p, q)$ of the critical case the estimate has been known only for $(p, q) = (2d/(d + 2), 2d/(d - 2))$. In what follows we establish (1.8) for $(1/p, 1/q) \in (B, B')$. Those estimates in the expanded range are crucial for obtaining (1.4) with $a = b$ when $p, q$ satisfy (1.6).

**Theorem 1.4.** Let $d \geq 3$. Suppose $(1/p, 1/q) \in B$ and (1.10) holds. Then, there is a constant $C > 0$ such that (1.8) holds. Furthermore, if $(1/p, 1/q) = B$ or $B'$, we have restricted weak type (uniform) estimate for $(H - z)^{-1}$.

The proof of the estimate (1.8) with $(p, q) = (2d/(d + 2), 2d/(d - 2))$ in Escauriaza and Vega [8] heavily relies on the uniform bound on the spectral projection operator $\Pi_k$ which is the projection onto the $k$-th eigenspace of the Hermite operator $H$ (see Section 2). In fact, they also used interpolation along an analytic family of operators which are motivated by Mehler’s formula for the Hermite function. However, their argument is not enough to prove (1.8) for $(1/p, 1/q) \in (B, B')$. We develop a different approach which is more direct and significantly simpler. We make use of a representation formula (2.1) for $\Pi_k$ which was observed in [11] and an estimate for the Hermite-Schrödinger propagator $e^{-\alpha tH}$ (see Proposition 2.1) which is a consequence of the representation formula and the endpoint Strichartz estimate [16].

**Organization of the paper.** The rest of this paper is organized as follows. In Section 2 we provide useful properties of the Hermite operator $H$ and the Hermite spectral projection operator $\Pi_k$. We prove boundedness of more general multiplier operator for the Hermite operator in Section 3, which implies Theorem 1.4. Finally, the proof of the Carleman estimate for the heat operator is given in Section 4.

2. Properties of the Hermite operator

For any multi-index $\alpha \in \mathbb{N}_0^d$ the $L^2$-normalized Hermite function $\Phi_\alpha$ which is a tensor product of one dimensional Hermite functions is an eigenfunction of $H$ with eigenvalue $2|\alpha| + d$. Here $|\alpha| := \alpha_1 + \cdots + \alpha_d$. The set $\{\Phi_\alpha : \alpha \in \mathbb{N}_0^d\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$. Thus, for any $f \in L^2(\mathbb{R}^d)$ we have the Hermite expansion $f = \sum_\alpha \langle f, \Phi_\alpha \rangle \Phi_\alpha$. 
We consider the Hermite spectral projection operator $\Pi_k$ which is defined by
\[
\Pi_k f = \sum_{\alpha \in \mathbb{N}_0^d; |\alpha| = k} (f, \Phi_{\alpha}) \Phi_{\alpha}, \quad f \in \mathcal{S}(\mathbb{R}^d).
\]
Then, the Hermite-Schrödinger propagator is given by
\[
e^{-itH} f = \sum_{k \in \mathbb{N}_0} e^{-it(2k+d)}\Pi_k f, \quad f \in \mathcal{S}(\mathbb{R}^d),
\]
which is the solution to the Cauchy problem $(i\partial_t - H)u = 0, u(x, 0) = f(x)$. If $f \in \mathcal{S}(\mathbb{R}^d)$, it is easy to see that $\Pi_k f$ decays rapidly in $k$, thus $\sum_{k=0}^{\infty} e^{-it(2k+d)}\Pi_k f$ converges uniformly. Clearly, $\Pi_k f = \sum_{k' \in \mathbb{N}_0} \frac{1}{2\pi}(\int_{-\pi}^{\pi} e^{it(k-k')} dt)\Pi_{k'} f$. Therefore, we obtain
\[
(2.1) \quad \Pi_k f = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{t\frac{d}{2}(2k+d-H)} f dt
\]
for $f \in \mathcal{S}(\mathbb{R}^d)$. Meanwhile, the operator $e^{-itH}$ has the kernel formula
\[
(2.2) \quad e^{-itH} f(x) = C_d(\sin 2t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{it\frac{(x-y)^2}{2}} \cot 2t - \langle x, y \rangle \csc 2t f(y) dy
\]
for $f \in \mathcal{S}(\mathbb{R}^d)$, which is shown by making use of Mehler’s formula ([26, 30]). Combining this with (2.1) gives an explicit expression of the kernel of $\Pi_k$.

In order to prove the uniform resolvent estimate (Theorem 1.3) we make use of the following mixed norm estimate for $e^{-itH}$, which strengthens the uniform bound (2.4) in a different direction.

**Proposition 2.1.** Let $d \geq 3$ and $(1/p, 1/q) = \mathcal{B}'$. Then, we have
\[
(2.3) \quad \| \int_{-\pi}^{\pi} |e^{-itH} f| dt \|_{p, \infty} \leq C \|f\|_{p, 1}.
\]

Various authors (see [15, 22, 18, 11]) studied the problem of characterizing the sharp asymptotic bound on the operator norm $\|\Pi_k\|_{p \rightarrow q}$ of $\Pi_k$ from $L^p$ to $L^q$ as $k \to \infty$. In particular, Karadzhov [15] showed
\[
(2.4) \quad \|\Pi_k\|_{p \rightarrow q} \leq C
\]
for a constant $C$ when $p = 2$ and $q = 2d/(d-2)$. By duality and $TT^*$-argument, the bound (2.4) with $(p, q) = (2d/(d+2), 2)$ and $(p, q) = (2d/(d+2), 2d/(d-2))$ follows. Interpolating those estimates with the trivial bound $\|\Pi_k\|_{2 \rightarrow 2} \leq 1$, we have (2.4) for $p, q$ satisfying $2d/(d+2) \leq p \leq 2 \leq q \leq 2d/(d-2)$.

Recently, the authors [11, Theorem 1.2] showed that (2.4) holds on an extended range of $p, q$ for $d \geq 3$ (see [12] for a related result). By means of Proposition 2.1 we can provide a simple alternative proof of this result. Indeed, from (2.1) and Proposition 2.1 it follows that $\|\Pi_k f\|_{q, \infty} \leq C \|f\|_{p, 1}$ if $(1/p, 1/q) = \mathcal{B}'$. By duality, the same estimate also holds for $(1/p, 1/q) = \mathcal{B}$. Interpolating these estimates with the above mentioned estimate (2.4) for $2d/(d+2) \leq p \leq 2 \leq q \leq 2d/(d-2)$ gives the following. (See Figure 1)

**Corollary 2.2.** ([11, Theorem 1.2]) Let $d \geq 3$. For $p, q$ satisfying $(1/p, 1/q) \in \mathcal{F}$ there is a constant $C > 0$, independent of $k$, such that (2.4) holds. Furthermore, the uniform restricted weak type estimate for $\Pi_k$ holds if $(\frac{1}{p'}, \frac{1}{q'}) = \mathcal{B}$ or $\mathcal{B}'$. 
Proof of Proposition 2.1. We make use of the endpoint Strichartz estimate for \( e^{-itH} \): \[
\| e^{-i\frac{t}{4}H} f \|_{L_t^2([−\pi,\pi];L_x^2(\mathbb{R}^d))} \leq C \| f \|_2
\] at \( p_0 = \frac{2d}{d−2} \), which can be shown by the dispersive estimate from [22] and the standard argument in [16] (for example, see [26]). We choose a smooth partition of unity so that \[
\psi^0 + \sum_{j \geq 4} (\psi(2^jt) + \psi(-2^jt) + \psi(2^j(t+\pi)) + \psi(2^j(\pi−t))) = 1
\] for \( t \in (−\pi, \pi) \setminus \{0\} \). Here \( \psi \in C^\infty(\mathbb{R}) \) satisfying \( \sum_j \psi(2^jt) = 1 \) for \( t > 0 \), and \( \psi^0 \) is a smooth function which is supported in the interval \( [−\pi, \pi] \) and vanishes near \( 0, \pi, \) and \( −\pi \).

Set \( \psi_j^\pm = \psi(\pm 2^jt) \) and \( \psi_j^{\pm\pi} = \psi(2^j(\pi ± \cdot)) \). Then, for \( \sigma = ±, ±\pi \), we have \[
\int |\psi_j^\sigma e^{-i\frac{t}{4}H} f| dt \lesssim 2^{\frac{d-2}{2}j^2} \| f \|_1
\] because \( |\psi_j^\sigma e^{-i\frac{t}{4}H} f| \lesssim 2^{\frac{d}{2}j} \| f \|_1 \) by [22]. By using (2.5) and Hölder’s inequality followed by Minkowski’s inequality we also get \[
\| \int |\psi_j^\sigma e^{-i\frac{t}{4}H} f| dt \|_{\frac{d}{2d}} \lesssim 2^{-\frac{d}{2}j} \| f \|_2.
\] Interpolation among those estimates gives \[
\| \int |\psi_j^\sigma e^{-i\frac{t}{4}H} f| dt \|_q \lesssim 2^{(\frac{d}{2}+\frac{d}{2})(1-1)j} \| f \|_p, \quad \sigma = ±, ±\pi
\] if \( (1/p,1/q) \) is contained in the line segment \([1,0), (1/2, (d−2)/2)]\). Bourgain’s summation trick (for example see [11] Lemma 2.4) to the above estimates gives \[
\| \int \sum_j |\psi_j^\sigma e^{-i\frac{t}{4}H} f| dt \|_q,\infty \lesssim \| f \|_{p,1,1}, \quad \sigma = ±, ±\pi
\] for \( (1/p,1/q) = \mathcal{B}' \). By a similar argument, it is easy to show \( \| \int |\psi^0 e^{-i\frac{t}{4}H} f| dt \|_q \lesssim \| f \|_p \) for \( (1/p,1/q) = \mathcal{B}' \). Hence, combining all of those estimates, we get (2.3). \( \square \)

We now consider \( L^p−L^q \) estimate for the operator \( H^{-s} \), \( s > 0 \) which is defined by \( H^{-s}f = \sum_{k=0}^\infty (2k+d)^{-s} \Pi_k f \). The operator can also be written as \( H^{-s}f = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}e^{-tH}f \ dt \)[1] making use of the heat semigroup \( e^{-tH} \) associated to \( H \). By means of the explicit kernel expression of \( e^{-tH} \) which is based on Mehler’s formula (see [31]), Bongioanni and Torrea [2] obtained \( L^p−L^q \) boundedness for \( H^{-s} \). Sharpness of their result was later verified by Nowak and Stempak [24]. Thus, the results completely characterize \( L^p−L^q \) boundedness of \( H^{-s} \).

**Theorem 2.3.** ([2, Theorem 8], [24 Theorem 3.1]) Let \( d \geq 1, 1 < p, q < \infty, \) and \( 0 < s < d/2 \). Then, \( H^{-s} \) is bounded from \( L^p \) to \( L^q \) if and only if \( −2s/d < 1/p − 1/q \leq 2s/d \).

There are weak/restricted weak type estimates in the borderline cases which are not included in the above theorem, and we refer the readers to [24] for more details regarding such endpoint estimates.

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1. For a bounded function \( m \) on \( \mathbb{R}_+ \) the operator \( m(H) \) is formally defined by \( m(H) = \sum_{k \in \mathbb{N}_0} m(2k+d) \Pi_k \).
3. Proof of Theorem 3.1

We consider more general operator \((H - z)^{-m}, m \in \mathbb{N}\) which is given by

\[
(H - z)^{-m} f = \sum_{k=0}^{\infty} \frac{\Pi_k f}{(2k + d - z)^m} = (-2)^{-m} \sum_{k=0}^{\infty} \frac{\Pi_k f}{(\imath \tau + \beta - k)^m}
\]

with \(z = 2\beta + d + 2\imath\) and \(\beta \notin \mathbb{N}\). We prove the following.

**Theorem 3.1.** Let \(d \geq 3\) and let \(m\) be a positive integer. Suppose that \((\ref{eq:gap-condition})\) holds for some \(c > 0\). If \((1/p, 1/q) \in (\mathcal{B}, \mathcal{B}')\), then there is a constant \(C = C(m)\), independent of \(z\), such that

\[
\|(H - z)^{-m} f\|_q \leq C(1 + |\imath \tau z|)^{1-m} \|f\|_p
\]

Furthermore, if \((\frac{1}{p}, \frac{1}{q}) = \mathcal{B}\) or \(\mathcal{B}'\), then we have the restricted weak type estimate

\[
\|(H - z)^{-m} f\|_{q, \infty} \leq C(1 + |\imath \tau z|)^{1-m} \|f\|_{p, 1}.
\]

While the estimates for \(m \geq 2\) are rather straightforward from the \((\ref{eq:gap-condition})\), the proof of \((\ref{eq:theorem-3.1})\) for \(m = 1\) is more involved. This case is handled in Proposition 3.2 below.

**Remark 1.** The gap condition \((\ref{eq:gap-condition})\) is necessary for the uniform estimate \((\ref{eq:theorem-3.1})\) to hold. In fact, \(\|(H - z)^{-m} f\|_{p, q} \geq |2k + d - z|^{-m} \|f\|_p\) if \(f\) is an eigenfunction with eigenvalue \(2k + d\). Therefore, the operator norm can not be bounded as \(z \to 2k + d\) unless \((\ref{eq:gap-condition})\) holds.

For \(\mathcal{B}, t_0 > 0\), let \(C(\mathcal{B}, t_0)\) denote the class of functions on \(\mathbb{R}\) which satisfy

\[
|G(n)| \leq \mathcal{B}, \quad n \in \mathbb{Z};
\]

\[
\sum_{k=1}^{\infty} |G(k) + G(-k)| \leq \mathcal{B};
\]

\[
\sum_{k=1}^{\infty} |kG(k) - (k + 1)G(k + 1)| \leq \mathcal{B};
\]

\[
\left| \frac{d}{dt} \right|^l G(t) \right| \leq \mathcal{B}(1 + |t|)^{-l-1}, \quad t_0 < |t|
\]

for \(0 \leq l \leq (d + 2)/2\). Particular examples satisfying the conditions \((\ref{eq:3.2})-\(\ref{eq:3.5}\)) are \(G_{\mu,\tau}(t) = 1/(\imath \tau + t + \mu)\) where \((\mu, \tau) \in (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R}\) and \(|(\mu, \tau)| \geq c\) for some small \(c > 0\).

**Proposition 3.2.** Let \(d \geq 3\) and \((1/p, 1/q) \in (\mathcal{B}, \mathcal{B}')\). Suppose that \(G\) is in \(C(\mathcal{B}, t_0)\). Then, for any \(n \in \mathbb{N}_0\), there is a constant \(C\), depending only on \(\mathcal{B}\) and \(t_0\), such that

\[
\left\| G\left(\frac{2n + d - H}{2}\right) f \right\|_q \leq C \|f\|_p.
\]

Furthermore, if \((1/p, 1/q) = \mathcal{B}\) or \(\mathcal{B}'\), the restricted weak type \((p, q)\) estimate holds for \(G(\frac{2n + d - H}{2})\) with a uniform bound.

**Proof.** Let \(p_*\) and \(q_*\) be given by \((1/p_*, 1/q_*) = \mathcal{B}'\). In order to show Proposition 3.2 it is sufficient to show the restricted weak type \((p_*, q_*)\) estimate for \(G(\frac{2n + d - H}{2})\).

Note that the adjoint operator \(G(\frac{2n + d - H}{2})^*\) is given by \(G(\frac{2n + d - H}{2})^* f = \sum_{k=0}^{\infty} G(n-\)
Then, clearly $G \in C(B, t_0)$. Hence, the same argument shows that restricted weak type $(p_*, q_*)$ estimate holds for $G(\frac{2n+d-H}{2})^\ast$. This in turn gives the restricted weak type estimate $(q'_*, p'_*)$ for $G(\frac{2n+d-H}{2})$ by duality. Real interpolation between these two (restricted weak type) estimates for $G(\frac{2n+d-H}{2})$ yields the desired estimates for $(1/p, 1/q) \in (B, B')$.

No differentiability assumption is made on $G$ for $|t| \leq t_0$. So, we handle the cases $n \geq n_0$ and $n < n_0$ separately, where $n_0$ is an integer satisfying $n_0 \geq 2t_0$. We first consider the case $n \geq n_0$. Recalling $G(\frac{2n+d-H}{2}) = \sum_{k=0}^{\infty} G(n-k)\Pi_k$, we decompose

$$G(\frac{2n+d-H}{2}) =: J_n + K_n,$$

where

$$J_n := \sum_{k=0}^{\infty} G(n-k)\phi\left(\frac{n-k}{n}\right)\Pi_k,$$

$$K_n := \sum_{k=0}^{\infty} G(n-k)\left(1 - \phi\left(\frac{n-k}{n}\right)\right)\Pi_k.$$

Here, we choose a non-negative smooth even function $\phi$ on $\mathbb{R}$ such that $\phi(t) = 1$ on $[-1/2, 1/2]$, $\phi = 0$ if $1 \leq |t|$, and $\phi$ is non-increasing on the half-line $t > 0$. This monotonicity assumption plays an important role in estimating a sum of trigonometric functions.

The $J_n$ is the major contribution to the estimate (3.6) and is to be handled by the integral formula for $\Pi_k$ and Lemma 2.1. The second $K_n$ behaves like the operator $H^{-1}$, which is actually bounded from $L^p - L^q$ on a larger range of $p, q$. We consider $J_n$ first.

We set

$$I_1 = \sum_{k=1}^{n} G(k)\phi(k/n)(\Pi_{n-k} - \Pi_{n+k}),$$

$$I_2 = \sum_{k=1}^{n} (G(-k) + G(k))\phi(k/n)\Pi_{n+k}.$$

Since $\phi$ is even function and supported in $[-1, 1]$, after reindexing by $(n-k) \rightarrow k$ we see $J_n = \sum_{k=1}^{n} G(k)\phi(k/n)\Pi_{n-k} + G(0)\Pi_n + \sum_{k=1}^{n} G(-k)\phi(k/n)\Pi_{n+k}$. Thus,

$$J_n = I_1 + I_2 + G(0)\Pi_n.$$

By (3.2), (3.3), and the uniform restricted weak type $(p_*, q_*)$ estimate for $\Pi_k$ in Corollary 2.2, it follows that $\|G(0)\Pi_n f\|_{q, \infty} \lesssim B\|f\|_{p, 1}$ and $\|I_2\|_{q, \infty} \lesssim B\|f\|_{p, 1}$. So, it suffices to deal with the first term $I_1$. Using the formula (2.1), we note

$$\Pi_{n-k} f - \Pi_{n+k} f = -\frac{i}{\pi} \int_{-\pi}^{\pi} \sin(tk)e^{i\frac{t}{2}(2n+d-H)} f dt.$$

Thus, we have

$$I_1 f = \int_{-\pi}^{\pi} \zeta_n(t)e^{-i\frac{t}{2}H} f dt,$$

where

$$\zeta_n(t) = -\frac{i}{\pi} e^{i\frac{t}{2}(2n+d)} \sum_{k=1}^{n} G(k)\sin(tk)\phi(k/n), \quad -\pi \leq t \leq \pi.$$
Using Proposition 2.1, it is sufficient to show

\[ |\zeta_n(t)| \leq C \]

with \( C \) independent of \( n \) and \( G \). By the property of \( \phi \) which we have chosen, it is clear that \( |\zeta_n(t)| \lesssim |\sum_{k=\lfloor t/\pi \rfloor + 1}^{n} \sin(tk)G(k)| + |\sum_{k=\lfloor t/\pi \rfloor + 1}^{n} \sin(tk)G(k)\phi(k/n)| \). Bound-
edness of the second term is easy to show. Indeed, since the condition (3.5) holds

we have

\[ \|K_1\|_{p,q} \lesssim 1 \quad \text{if} \quad 1 < p, q < \infty \]

and \( 1/p - 1/q = 2/d \). Hence, we have

\[ \|K_n\|_{p,q} \lesssim \|H^{-1}\|_{p,q} \|m_n(H)\|_{p,q} \lesssim 1 \]

with the implicit constant independent of \( n \).

We now consider the case \( n < n_0 \), which is much simpler to show than the case \( n \geq n_0 \). To prove (3.6), we break \( G(\frac{2n+d-H}{2}) \) as follows:

\[ G(\frac{2n+d-H}{2}) = \overline{J}_n + \overline{K}_n, \]

where

\[ \overline{J}_n = \sum_{k=0}^{\infty} G(n-k)\phi(k/2n_0)\Pi_k, \]

\[ \overline{K}_n = \sum_{k=0}^{\infty} G(n-k)(1-\phi(k/2n_0))\Pi_k. \]

Clearly, the multiplier \( G((2n+d-H)/2) (1-\phi((2n+d-H)/2n_0)) \) of the operator \( \overline{K}_n \) satisfies the condition (3.5). So, in the same manner as in the above we obtain the bound \( \|\overline{K}_n\|_{p,q} \lesssim 1 \) if \( 1 < p, q < \infty \) and \( 1/p - 1/q = 2/d \). By the condition

---

2 This can be seen by approximating Dirichlet’s kernel, or again by summation by parts.
Let \(n < n_0\) and Corollary \(3.2\) it follows that \(\|\tilde{f}_{n}f\|_{q, \infty} \leq B \sum_{k=0}^{2n_0} \|\Pi_k f\|_{q, \infty} \lesssim \|f\|_{p, 1}\) uniformly in \(n \leq n_0\). This completes the proof of Proposition \(3.2\).

We are ready to prove Theorem \(3.1\).

**Proof of Theorem \(3.1\):** Let \(p_s\) and \(q_s\) be given by \((1/p_s, 1/q_s) = B'\). As in the proof of Proposition \(3.1\), it is enough to show the restricted weak type \((p_s, q_s)\) estimate for \((H - z)^{-m}\) with bound \(C(1 + |\text{Im } z|)^{1 - m}\) since the adjoint operator of \((H - z)^{-m}\) is given by \((H - \overline{z})^{-m}\). We can handle \((H - \overline{z})^{-m}\) in the exactly same way to obtain the restricted weak type \((p_s, q_s)\) estimate with bound \(C(1 + |\text{Im } z|)^{1 - m}\). By duality and interpolation, we get all the desired estimates.

By Corollary \(2.2\), we have the estimate \(\|\Pi_k f\|_{q, \infty} \leq C\|f\|_{p, 1}\) with \(C\) independent of \(k\). Using this estimate, for \(m \geq 2\) we get

\[
\|(H - z)^{-m} f\|_{q, \infty} \lesssim \sum_{k=0}^{\infty} |2k + d - z|^{-m}\|f\|_{p, 1} \lesssim (1 + |\text{Im } z|)^{1-m}\|f\|_{p, 1}
\]

because \(\sum_{k=0}^{\infty} |2k + d - z|^{-m} \leq C_m (1 + |\text{Im } z|)^{1-m}\) with \(C_m\) independent of \(z\) for \(m \geq 2\) if \(1.9\) holds. Thus we need only to show

\[
\|(H - z)^{-1} f\|_{q, \infty} \leq C\|f\|_{p, 1}.
\]

If \(\text{Re } z > d - 1\), \(z = 2(n + \mu) + d + 2i\tau\) for some \(n \in \mathbb{N}_0\), \(\mu \in (-\frac{1}{2}, \frac{1}{2})\), and \(\tau \in \mathbb{R}\) satisfying \(|(\mu, \tau)| \geq c/2\) because of \(1.9\). We note that

\[
(H - z)^{-1} = G_{\mu, \tau}(\frac{2n + d - H}{2})
\]

where \(G_{\mu, \tau}(t) = 1/(i\tau + t + \mu)\). It is easy to see that \(G_{\mu, \tau} \in \mathcal{C}(\mathcal{B}, 1)\) for some \(\mathcal{B} > 0\) provided that \(\mu \in (-\frac{1}{2}, \frac{1}{2})\), and \(\tau \in \mathbb{R}\) satisfy \(|(\mu, \tau)| \geq c/2\). Thus, by Proposition \(3.2\), the estimate \(3.8\) holds uniformly in \(z\). For the remaining case, i.e., \(\text{Re } z < d - 1\), \(z\) clearly stays away from the eigenvalues of \(H\), so \((H - z)^{-1}\) behaves like \(H^{-1}\). More precisely, we obtain the uniform estimate \(3.8\) repeating the same argument as in the case \(n < n_0\) of the proof of Proposition \(3.2\). This completes the proof.

The uniform resolvent estimate in Theorem \(1.4\) is a special case of the following.

**Corollary 3.3.** Let \(d \geq 3\) and \(m\) be a positive integer, and let \(p, q\) be given as in Theorem \(1.7\). Then, there is a constant \(C = C(m)\) such that

\[
\|(H - z)^{-m} f\|_q \leq C(1 + |\text{Im } z|)^{\frac{2}{d} - \frac{2}{q}}\|f\|_p
\]

provided \(1.9\) holds. Furthermore, if \((\frac{2}{p}, \frac{2}{q}) = \mathcal{B} \) or \(B'\), we have the restricted weak type estimate for \((H - z)^{-m}\) with bound \(C(1 + |\text{Im } z|)^{\frac{2}{d} - \frac{2}{q}}\).

**Proof.** By Theorem \(3.1\) we have the estimate \(3.9\) for \((1/p, 1/q) \in (\mathcal{B}, B')\). In view of interpolation, it is enough to show \(3.9\) with \((p, q) = (2, 2)\), \((\frac{2}{p}, \frac{2}{q})\), or \((\frac{2}{p}, \frac{2}{q})\). These estimates are easy to show using orthogonality between the projection operators \(\Pi_k\). In fact, we have

\[
\|(H - z)^{-m} f\|_2 \leq \left(\sum_{k=0}^{\infty} |2k + d - z|^{-2m}\|\Pi_k f\|_2^2\right)^{1/2}.
\]
So, taking the supremum over \( k \) of \( |2k + d - z|^{-2m} \), we obtain (3.9) when \( p = q = 2 \). We note that \( \sum_{k=0}^{\infty} |2k + d - z|^{-2m} \leq C(1 + |\text{Im} z|)^{-2m+1} \) with \( C \) independent of \( z \) as long as (1.9) holds. Applying the uniform \( L^{\frac{2d}{d+2}} \)-\( L^2 \) estimate in Corollary 2.2, we get (3.9) with \( p = \frac{2d}{d+2} \) and \( q = 2 \). Since the adjoint of \((H - z)^{-m}\) is \((H - \bar{z})^{-m}\), the estimate (3.9) with \((p,q) = \left( \frac{2d}{d+2}, 2 \right) \) implies that with \((p,q) = \left( 2, \frac{2d}{d+2} \right) \) by duality.

\[
\begin{align*}
\text{4. Proof of Theorem 1.1} \\
\end{align*}
\]

We now prove the estimate (1.4) by adapting the argument in Escauriaza and Vega [8] (also see [6]) which deduces Carleman estimate for the heat operator from the uniform resolvent estimate for the Hermite operator. We are basically relying on real interpolation as in [28]. However, there are some nontrivial issues which are related to the shortcoming of the real interpolation between mixed norm spaces.

**Lemma 4.1.** Let \( 1 < p \leq 2 \leq q < \infty \), \( 1 \leq r, s \leq \infty \), \( 1 \leq a \leq b \leq \infty \), and let \( 0 \leq \gamma \leq 1 \) and \( \beta \notin \mathbb{N}_0 \) be a real number. Suppose that the estimate

\[
\left\| \sum_{k=0}^{\infty} \frac{\Pi_k f}{(\beta + k)^m} \right\|_{p,a} \leq C_m(1 + |\tau|)^{\gamma-m} \| f \|_{p,a}
\]

holds for \( m = 1, 2, 3 \) with \( C_m \) independent of \( \tau \in \mathbb{R} \) and \( \beta \) provided \( \text{dist}(\beta, \mathbb{N}_0) \geq c \) for some \( c > 0 \). Then, if \( \text{dist}(\beta, \mathbb{N}_0) \geq c \) for some \( c > 0 \), the estimate (1.4) holds uniformly in \( \beta \) whenever the following hold:

- \( \gamma < 1 \), \( 0 \leq \frac{1}{p} - \frac{1}{s} \leq 1 - \gamma \), and \( \frac{1}{p} \neq \left( \frac{1}{r}, \frac{1}{s} \right) \neq (1, \gamma), (1 - \gamma, 0) \).
- \( \gamma = 1 \), \( a = b = 2 \), and \( 1 < r = s < \infty \).

Lemma 4.1 was implicit in [8] with the Lebesgue spaces instead of the Lorentz spaces. The extra condition \( a = b = 2 \) when \( \gamma = 1 \) is due to limitation of the real interpolation in mixed norm spaces. Once we have Lemma 4.1 the proof of Theorem 1.1 is rather simple.

**Proof of Theorem 1.1.** Let \( (1/p, 1/q) \) be in \( \mathfrak{T} \). By real interpolation between the estimates in Corollary 3.3 and inclusion relations between Lorentz spaces, we get (4.1) with \( \gamma = \frac{\alpha(1/p - 1/q)}{2} \) for any \( 1 \leq a \leq b \leq \infty \) if \( p \neq 2 \) and \( q \neq 2 \). Thus Lemma 4.1 gives the estimate (1.4) in the Lorentz spaces if the exponents satisfy the condition in Theorem 1.1.

The estimate (1.4) is equivalent to the Sobolev type inequality

\[
\| h \|_{L^r(\mathbb{R}, L^q_x)} \leq C \| (\Delta - |x|^2 + \partial_t + 2\beta + d)h \|_{L^r(\mathbb{R}, L^{q,a}_x)}, \quad h \in C_c^\infty(\mathbb{R}^{d+1}).
\]

One can easily see this by following the argument in [6]. Especially, if \( r = s \), the inequality (4.2) implies \( \| f \|_q \leq C \| (\Delta - |x|^2 + 2\beta + d) f \|_p \) for \( f \in C_c^\infty(\mathbb{R}^d) \) which is, in fact, a special case of (1.8) where \( z = 2\beta + d \notin 2\mathbb{N}_0 + d \). Indeed, let \( f_1 \) be a compactly supported smooth function on \( \mathbb{R} \) with \( f_1(0) = 1 \). Then, the above estimate follows by applying (4.2) to the function \( h(x,t) = f(x)f_1(t/R)R^{-1/r}, R > 1 \) and letting \( R \to \infty \).
Remark 2. When \( r = s \), the implication from (1.2) to (1.8) with \( z = 2\beta + d \notin 2\mathbb{N}_0 + d \) can be used to show that the Carleman estimate (1.4) holds only if

\[
\frac{1}{p} - \frac{1}{q} \leq \frac{2}{d}.
\]

By the Marcinkiewicz multiplier theorem for the Hermite operator (4.3) \((H - z)^{-1}H\) with \( z = 2\beta + d \notin 2\mathbb{N}_0 + d \) is bounded on \( L^p \), \( 1 < p < \infty \). Thus, we see that the Carleman estimate (1.4) implies the estimate \( \|H^{-1}u\|_q \lesssim \|u\|_p \) for \( u \in C_c^\infty (\mathbb{R}^d) \). By Theorem 2.3 the inequality holds only if \( 1/p - 1/q \leq 2/d \).

Proof of Lemma 4.1. To prove Lemma 4.1 we basically rely on the argument in [6][8], so we shall be brief. By scaling, it is easy to see that (1.4) is equivalent to (1.2). See [6] for the details. Thus, we need to show (1.2) by replacing \( h \) with \((\Delta - |x|^2 + \partial_t + 2\beta + d)^{-1}g\). Applying the projection operator \( \Pi_k \) in \( x \)-variables and taking Fourier transform in \( t \), we see the operator \( S_\beta := (\Delta - |x|^2 + \partial_t + 2\beta + d)^{-1} \) is given by

\[
S_\beta g(x,t) = \int_{\mathbb{R}} K_\beta(t-s)(g(\cdot,s))(x)ds,
\]

where the operator valued kernel \( K_\beta \) is given by

\[
K_\beta(t)(f) = \frac{1}{2} \int_{\mathbb{R}} e^{2\pi i t\tau} \sum_{k=0}^{\infty} \frac{\Pi_k(f)}{\pi i \tau + \beta - k} d\tau, \quad f \in C_c^\infty (\mathbb{R}^d).
\]

To prove (1.4), it is enough to show

\[
\|S_\beta g\|_{L^r(\mathbb{R};L^q_{\mathbb{R}})} \lesssim \|g\|_{L^r(\mathbb{R};L^q_{\mathbb{R}})}, \quad g \in C_c^\infty (\mathbb{R}^{d+1})
\]

with implicit constant independent of \( \beta \) as long as \( \text{dist}(\beta,N_0) \geq c \) for some \( c > 0 \).

We regard \( S_\beta \) as a vector valued convolution operator. Let us first consider the case \( \gamma < 1 \) which is easier. Let \( \phi \in C_c^\infty([-1,1]) \) such that \( \phi(t) = 1 \) on \([-1/2,1/2] \). Breaking the integral with functions \( \phi(\tau t) \), \( 1 - \phi(\tau t) \) and using integration by parts and (2.1), it is easy to see that \( \|K_\beta(t)\|_{L_r^p \to L_q^r} \lesssim \min\{|t|^{-\gamma},|t|^{-2}\} \). Since \( \gamma < 1 \), for \( r, s \) satisfying \( 0 \leq \frac{1}{r} - \frac{1}{s} < 1 - \gamma \) and \((\frac{1}{r},\frac{1}{s}) \neq (1,\gamma), (1-\gamma,0) \) we obtain (1.3) by Young’s convolution inequality and the Hardy-Littlewood-Sobolev inequality.

We now turn to the case \( \gamma = 1 \). We claim that the kernel \( K_\beta \) satisfies the Hörmander condition

\[
(4.4) \quad \sup_{s \neq 0} \int_{|t| > 2|s|} \|K_\beta(t-s) - K_\beta(t)\|_{L^p \to L^q} dt \leq \epsilon < \infty,
\]

where \( \epsilon \) is depending only on the constant \( c > 0 \) such that \( \text{dist}(\beta,N_0) \geq c \). To show (4.4), it is sufficient to show \( \|K_\beta'(t)\|_{L^p \to L^q} \lesssim |t|^{-\frac{3}{2}} \). By integration by parts we have

\[
(-2\pi it)^2 K_\beta'(t) = 2^2(\pi i)^3 \int_{-\infty}^{\infty} \tau e^{2\pi i t\tau} \sum_{k=0}^{\infty} \frac{1}{(\pi i \tau)^3 \Pi_k} d\tau.
\]

The assumption (1.4) (with \( \gamma = 1 \) and \( m = 3 \)) gives \( \|(-2\pi it)^2 K_\beta'(t)\|_{L^p \to L^q} \lesssim 1 \) uniformly in \( t \) and \( \beta \) satisfying \( \text{dist}(\beta,N_0) \geq c \), which proves the claim (4.1). Thanks

---

3. If \( \|K_\beta'(t)\|_{L^q_{\mathbb{R}} \to L^p_{\mathbb{R}}} \lesssim |t|^{-2}, \|K_\beta(t-s) - K_\beta(t)\|_{L^q_{\mathbb{R}} \to L^p_{\mathbb{R}}} = \|\int_t^{t-s} K_\beta'(\sigma) d\sigma\|_{L^q_{\mathbb{R}} \to L^p_{\mathbb{R}}} \lesssim |s||t|^{-2} \). This clearly yields (4.3).
to \((4.1)\) and the usual vector valued singular integral theory, in order to prove \((4.3)\) for \(1 < r = s < \infty\), it suffices to obtain the estimate \((4.3)\) with \(r = s = 2\) and \(a = b = 2\).

For \(\eta \in C_c^\infty(\mathbb{R})\) we define \(\eta(D_t)\) by \(\mathcal{F}_t(\eta(D_t)g)(x, \tau) = \eta(\tau)\mathcal{F}_t g(x, \tau)\) where \(\mathcal{F}_t\) denotes the Fourier transform in \(t\). We use the following Littlewood-Paley type inequality in the Lorentz spaces.

**Lemma 4.2.** Let \(1 < p, r < \infty\). Suppose \(\eta\) is a smooth function supported in \([2^{-2}, 1]\) which satisfies \(\sum_{j=\infty}^{\infty} |\eta(2^{-j}t)|^2 \sim 1\) for all \(t > 0\). Then we have

\[
\|g\|_{L_t^\alpha(\mathbb{R}; L_r^{p'})} \lesssim \left( \sum_{j \in \mathbb{Z}} \|\eta(2^{-j}|D_t|)g\|^{2r/2}_{L_t^\alpha(\mathbb{R}; L_r^{p'})} \right)^{1/2} \lesssim \|g\|_{L_t^\alpha(\mathbb{R}; L_r^{p'})}.
\]

**Proof.** It is sufficient to show the second inequality in \((4.5)\) because the first inequality follows from the second via the standard polarization argument and duality. For any \(1 < p, r < \infty\) we have \(\|\sum_{j \in \mathbb{Z}} \eta(2^{-j}|D_t|)g\|^{2r/2}_{L_t^\alpha(\mathbb{R}; L_r^{p'})} \lesssim \|g\|_{L_t^\alpha(\mathbb{R}; L_r^{p'})}\) by means of the usual Littlewood-Paley inequality and the vector valued singular integral theorem (see \([3\) Lemma 2.1]). We interpolate these estimates using the real interpolation in the mixed-norm spaces, especially,

\[
(\ell_p(\mathbb{R}; L^{q_0}), \ell_p(\mathbb{R}; L^{q_1}))_\theta,p = \ell_p(\mathbb{R}; L^{\theta q})
\]

whenever \(p_0, q_0, p_1, q_1 \in [1, \infty)\) and \((1/p, 1/q) = (1-\theta)(1/p_0, 1/q_0) = \theta(1/p_1, 1/q_1)\) with \(\theta \in (0, 1)\) (see \([5, 23]\)). Therefore, we obtain the second inequality in \((4.3)\). \(\square\)

We now note that \(\psi(2^{-j}|D_t|)S_{\beta}g(x, t) = \int_{\mathbb{R}} K_{\beta,j}(t-s)g(s)(x)dt\) where

\[
K_{\beta,j}(t)f(x) := \frac{1}{2} \int_{\mathbb{R}} e^{2\pi i t \tau} \psi\left(\frac{\tau}{2}\right) \sum_{|k|=0}^{\infty} \frac{1}{\pi i \tau + \beta - k} \Pi_k f(x)dt.
\]

Using \((4.1)\) with \(a = b = 2\) and integration by parts, we note that \(\|K_{\beta,j}(t)\|_{L^2(\mathbb{R}; L^{2^j})} \leq C 2^j(1+2^j|t|)^{-2}\) with \(C\) independent of \(j\) and \(\beta\) if \(\text{dist}(\beta, \mathbb{N}_0) \geq c > 0\). Thus, Young’s convolution inequality gives

\[
\|\psi(2^{-j}|D_t|)S_{\beta}g\|_{L^2(\mathbb{R}; L^{2^j})} \lesssim \|g\|_{L^2(\mathbb{R}; L^{2^j})}
\]

with the implicit constant independent of \(j\) and \(\beta\). To get the desired \((4.3)\) with \(r = s = 2\), we combine this inequality and Lemma 4.2. Since \(2 \leq q < \infty\), the space \(L^{(q/2)}/2(2)\) is normable. So,

\[
\|\psi(2^{-j}|D_t|)S_{\beta}g\|_{L^2(\mathbb{R}; L^{2^j})} \lesssim \|g\|_{L^2(\mathbb{R}; L^{2^j})}
\]

\[
\sum_{j} \left(\sum_{j \in \mathbb{Z}} |h_j|^{2r/2}\right)^{1/2} \lesssim \sum_{j} \left(\sum_{j \in \mathbb{Z}} \|\eta(2^{-j}|D_t|)g\|^{2r/2}_{L_t^\alpha(\mathbb{R}; L_r^{p'})} \right)^{1/2}
\]

Since \(S_{\beta}g = \sum_{j \in \mathbb{Z}} \psi(2^{-j}|D_t|)S_{\beta}g\), applying Lemma 4.2 and then \((4.3)\), we have

\[
\|S_{\beta}g\|_{L^2(\mathbb{R}; L^{2^j})} \lesssim \left(\sum_{j \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \|\eta(2^{-j}|D_t|)g\|^{2r/2}_{L_t^\alpha(\mathbb{R}; L_r^{p'})} \right)^{1/2}\right)^{1/2}.
\]

Let \(\tilde{\psi} \in C_c([2^{-2}, 1])\) such that \(\psi \tilde{\psi} = \psi\), so \(\psi(2^{-j}|D_t|)S_{\beta}g = \psi(2^{-j}|D_t|)S_{\beta} \tilde{\psi}(2^{-j}|D_t|)g\). Using \((4.0)\) followed by \((4.5)\), we get

\[
\|S_{\beta}g\|_{L^2(\mathbb{R}; L^{2^j})} \lesssim \left(\sum_{j \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \|\tilde{\psi}(2^{-j}|D_t|)g\|^{2r/2}_{L_t^\alpha(\mathbb{R}; L_r^{p'})} \right)^{1/2}\right)^{1/2}.
\]
By duality the inequality (4.7) is equivalent to \((\sum_j \|h_j\|_{L^p_{(R;L^q_{x})}}^2)^{1/2} \lesssim \|\sum_j |h_j|^2\|_{L^p_{(R;L^q_{x})}}^{1/2}\) for \(1 < p \leq 2\). Thus, using Lemma 4.2 we get
\[
\|S\beta g\|_{L^2(R;L^p_{x})} \lesssim \|\left(\sum_{j \in Z} |2^{-j} |D_t||g|^2\right)^{1/2}\|_{L^2(R;L^p_{x})} \lesssim \|g\|_{L^2(R;L^p_{x})}.
\]
This completes the proof. \(\square\)

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