ON THE DIFFERENTIABLE VECTORS FOR
CONTRAGREDIENT REPRESENTATIONS

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Abstract. We establish a few simple results on contragredient repre-
sentations of Lie groups, with a view toward applications to the abstract charac-
terization of some spaces of pseudo-differential operators. In particular, this
method provides an abstract approach to J. Nourrigat’s recent description of
the norm closure of the pseudo-differential operators of order zero.

1. Introduction

In this note we study the abstract characterization of some spaces of pseudo-
differential operators by using a few simple results on the contragredients of Banach
space representations of Lie groups. The applicability of the method based on a
contragredient representation is due to the fact that such a representation may
be discontinuous even if the original representation is continuous; see for instance
the representation (3.1) below, which is discontinuous if \( r = \infty \). In particular,
we provide an abstract approach to J. Nourrigat’s recent description \[No12\] of the
norm closure of the pseudo-differential operators of order zero (see Example 3.3
below) and we also bring additional information on some results from the earlier
literature.

Preliminaries. For any complex Banach space \( Y \) we denote by \( Y^* \) its topological
dual and by \( B(Y)^\times \) the group of invertible elements in the Banach algebra \( B(Y) \) of
all bounded linear operators. If \( G \) is any group, then a Banach space representation
of \( G \) is a group homomorphism \( \pi : G \to B(Y_\pi)^\times \), where \( Y_\pi \) is a complex Banach
space. The contragredient representation of \( \pi \) is the representation
\[
\pi^*: G \to B(Y^*_\pi)^\times, \quad \pi^*(g) := (\pi(g^{-1}))^*,
\]
so that \( Y_{\pi^*} := Y^*_\pi \). If \( \sup_{g \in G} \|\pi(g)\| < \infty \), then we say that \( \pi \) is uniformly bounded,
and in this case also \( \pi^* \) is uniformly bounded.

Now assume that \( G \) is a topological group and for the uniformly bounded rep-
resentation \( \pi : G \to B(Y_\pi)^\times \) define \( Y_{\pi_0} := \{ x \in Y_\pi \mid \pi(\cdot)x \in C(G, Y_\pi) \} \), where \( C \)
indicates the space of continuous mappings. Then \( Y_{\pi_0} \) is a closed linear subspace
of \( Y \) since \( \pi \) is uniformly bounded, and moreover \( Y_{\pi_0} \) is invariant under \( \pi \). Hence
we obtain a strongly continuous representation \( \pi_0 : G \to B(Y_{\pi_0}) \), \( \pi_0(g) := \pi(g)|_{Y_{\pi_0}} \).

By using this construction for the contragredient representation, we define
\[
Y_{\pi^*_0} := \{ \xi \in Y^*_\pi \mid \pi^*(\cdot)\xi \in C(G, Y^*_\pi) \} = \{ \xi \in Y^*_\pi \mid \lim_{g \to 1} \|\pi^*(g)\xi - \xi\| = 0 \}
\]

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and

\[ \pi_0^*: G \to \mathcal{B}(\mathcal{Y}_{\pi_0})^\times, \quad \pi_0^*(g) := \pi^*(g)|_{\mathcal{Y}_{\pi_0}}. \]

If moreover \( G \) is a Lie group, then we also define \( \mathcal{Y}_k := \{ y \in \mathcal{Y} | \pi(\cdot)y \in C^k(G, \mathcal{Y}) \} \) for every integer \( k \geq 0 \), so in particular \( \mathcal{Y}_0 = \mathcal{Y}_{\pi_0} \). Moreover, if the representation \( \pi \) is strongly continuous, that is, \( \mathcal{Y} = \mathcal{Y}_{\pi_0} \), then for every \( k \geq 1 \) and every basis \( \{ X_1, \ldots, X_m \} \) in the Lie algebra \( g \) of \( G \) we have

\[ \mathcal{Y}_k = \bigcap_{1 \leq j_1, \ldots, j_k \leq m} D(\pi(X_{j_1})^* \cdots \pi(X_{j_k})^*) \tag{1.1} \]

(see for instance [Ne10, Th. 9.4]). Here and in what follows we denote by \( D(T) \) the domain of any unbounded operator \( T \).

## 2. The main abstract results

The following theorem can be regarded as a version of (1.1) for some discontinuous representations of Lie groups, namely for the contragredient of any uniformly bounded and strongly continuous representation.

**Theorem 2.1.** Let \( G \) be a Lie group with a strongly continuous representation \( \pi : G \to \mathcal{B}(\mathcal{Y}) \) which is also assumed to be uniformly bounded. If \( \{ X_1, \ldots, X_m \} \) is any basis in the Lie algebra \( g \) of \( G \), then

\[ \mathcal{Y}_k \subseteq \bigcap_{1 \leq j_1, \ldots, j_k \leq m} D(\pi(X_{j_1})^* \cdots \pi(X_{j_k})^*) \subseteq \mathcal{Y}_{k-1} \]

for every integer \( k \geq 1 \), and the above inclusions could simultaneously be strict.

For proving the theorem it will be convenient to use the notation

\[ C^k(\pi^*) := \bigcap_{1 \leq j_1, \ldots, j_k \leq m} D(\pi(X_{j_1})^* \cdots \pi(X_{j_k})^*) \]

for an arbitrary integer \( k \geq 1 \). It is clear that \( C^1(\pi^*) \supseteq C^2(\pi^*) \supseteq \cdots \).

The proof will be based on the following auxiliary result, which should be thought of as an embedding lemma on abstract Sobolev spaces.

**Lemma 2.2.** We have \( C^1(\pi^*) \subseteq \mathcal{Y}_{\pi_0^*} \).

**Proof.** For every \( X \in g \) let us denote \( \gamma_X : \mathbb{R} \to G, \gamma_X(t) := \exp_G(tX) \). It follows by [vNe92, Th. 1.3.1] that

\[ D(\pi(X)^*) \subseteq \mathcal{Y}_{\pi^* \circ \gamma_X} = \{ \xi \in \mathcal{Y}^* | \pi^*(\gamma_X(\cdot))\xi \in C(\mathbb{R}, \mathcal{Y}^*) \} \tag{2.1} \]

for arbitrary \( X \in g \). On the other hand, we have

\[ \mathcal{Y}_{\pi^*} = \bigcap_{j=1}^m \mathcal{Y}_{\pi^* \circ \gamma_{X_j}} \tag{2.2} \]
since the inclusion ⊆ is obvious while the inclusion ⊇ holds true for the following reason. For all $t_1, \ldots, t_m \in \mathbb{R}$ and $\xi \in \mathcal{Y}^\pi$ we have

$$\|\pi^* (\gamma_{X_1}(t_1) \cdots \gamma_{X_m}(t_m)) \xi - \xi\|$$

$$\leq \sum_{j=1}^{m} \|\pi^* (\gamma_{X_1}(t_1) \cdots \gamma_{X_{j-1}}(t_{j-1})) (\pi^* (\gamma_{X_j}(t_j)) \xi - \xi)\|$$

$$\leq M \sum_{j=1}^{m} \|\pi^* (\gamma_{X_j}(t_j)) \xi - \xi\|$$

where $M := \sup_{g \in G} \|\pi(g)\|$. Since $\{X_1, \ldots, X_m\}$ is a basis in the Lie algebra $\mathfrak{g}$, it follows that the mapping $\mathbb{R}^m \rightarrow G$, $(t_1, \ldots, t_m) \mapsto \gamma_{X_1}(t_1) \cdots \gamma_{X_m}(t_m)$, is a local diffeomorphism at $0 \in \mathbb{R}^m$, and then the above estimate shows that for every $\xi \in \bigcap_{j=1}^{m} \mathcal{Y}_{\pi^* \gamma_{X_j}}$, we have $\lim_{g \rightarrow 1} \|\pi^*(g) \xi - \xi\| = 0$, hence $\xi \in \mathcal{Y}_{\pi^*}$. This completes the proof of (2.2).

Now, since $D(d\pi(X_1)^*) \cap \cdots \cap D(d\pi(X_m)^*) = C^1(\pi^*)$, the assertion follows by (2.1) and (2.2).

**Proof of Theorem 2.1.** By using Lemma 2.2 and [Po72, Lemma 1.1] we obtain

$$\mathcal{C}^k(\pi^*) \subseteq \bigcap_{1 \leq j_1, \ldots, j_{k-1} \leq m} D(d\pi_0^*(X_{j_1}) \cdots d\pi_0^*(X_{j_{k-1}})) = \mathcal{Y}_{\pi^*}^{k-1}$$

where the latter equality follows by using (1.1) for the strongly continuous representation $\pi_0^*$. The inclusion $\mathcal{Y}_{\pi^*}^{k} \subseteq \mathcal{C}^k(\pi^*)$ can be easily proved by using (1.1) and the fact that for every $X \in \mathfrak{g}$ we have $D(d\pi_0^*(X)) \subset D(d\pi(X)^*)$ and $d\pi(X)^* |_{D(d\pi_0^*(X))} = d\pi_0^*(X)$.

We now prove by example that the inclusion in the statement can be strict for $k = 1$. To this end let $G = \mathbb{R}$, $\mathcal{Y}$ be the space of trace-class operators on $L^2(\mathbb{R})$, and consider the regular representation $\rho: \mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{R}))$, $\rho(t)f = f(.-t)$. Then define $\pi: \mathbb{R} \rightarrow \mathcal{B}(\mathcal{Y})$, $\pi(t)A = A \rho(t)A \rho(t)^{-1}$ and for every $\phi \in L^\infty(\mathbb{R})$ denote by $\phi(Q)$ the multiplication-by-$\phi$ operator on $L^2(\mathbb{R})$, so that $\phi(Q) \in \mathcal{B}(L^2(\mathbb{R})) \simeq \mathcal{Y}^\pi$. It was noted in [ABC96] Ex. 6.2.7 that $\phi(Q) \in \mathcal{Y}_{\pi^*}^k$, if and only the first $k$ derivatives of $\phi$ exist, are bounded, and the $k$-th derivative is also uniformly continuous on $\mathbb{R}$.

On the other hand, if we denote by $P = -i\frac{d}{dt}$ the infinitesimal generator of $\rho$, then it is easily checked that $\phi(Q) \in C^1(\pi^*)$ if and only if the commutator $[\phi(Q), P]$ belongs to $\mathcal{B}(L^2(\mathbb{R}))$, hence by using also [ABC96] Prop. 5.1.2(b) and again [ABC96] Ex. 6.2.7 we see that the latter commutator condition is equivalent to the fact that $\phi$ is bounded and satisfies the Lipschitz condition globally on $\mathbb{R}$. Therefore there exist $\phi, \psi \in L^\infty(\mathbb{R})$ such that $\phi(Q) \in C^1(\pi^*) \setminus \mathcal{Y}_{\pi^*}^2$, and $\psi(Q) \in \mathcal{Y}_{\pi^*}^2 \setminus C^1(\pi^*)$. This completes the proof.

**Corollary 2.3.** In the setting of Theorem 2.1 the linear subspace

$$\bigcap_{k \geq 1} \bigcap_{1 \leq j_1, \ldots, j_k \leq m} D(d\pi(X_{j_1})^* \cdots d\pi(X_{j_k})^*)$$

is dense in $\mathcal{Y}_{\pi^*}^\infty$. 


Proof. It follows by Theorem 2.1 that this linear subspace is equal to the space of smooth vectors for the strongly continuous representation \( \pi_0 \), hence it is dense in the representation space \( Y_0 \) (see [Ga47]).

3. Applications

In this section we will develop a more general version of the example used in the proof of Theorem 2.1. Let \( G \) be a Lie group with a continuous unitary representation \( \rho: G \to B(\mathcal{H}) \). If \( 1 \leq p < \infty \), denote by \( \mathcal{S}_p(\mathcal{H}) \) the \( p \)-th Schatten ideal, and let \( \mathcal{S}_\infty(\mathcal{H}) := B(\mathcal{H}) \) and \( \mathcal{S}_0(\mathcal{H}) \) be the ideal of all compact operators on \( \mathcal{H} \). It is well known that if \( p, q \in \{0\} \cup [1, \infty] \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p \neq \infty \), then there exists an isometric linear isomorphism \( \mathcal{S}_p(\mathcal{H})^* \cong \mathcal{S}_q(\mathcal{H}) \) defined by the duality pairing

\[
\langle \cdot, \cdot \rangle: \mathcal{S}_q(\mathcal{H}) \times \mathcal{S}_p(\mathcal{H}) \to \mathbb{C}, \quad \langle Y, V \rangle := \text{Tr}(YV).
\]

The representation \( \rho(q) \) can thus be regarded as the contragredient representation of the strongly continuous representation \( \rho(r) \), where

\[
(\forall r \in \{0\} \cup [1, \infty]) \quad \rho(r): G \to B(\mathcal{S}_r(\mathcal{H})), \quad \rho(r)(g)Y = \rho(g)Y \rho(g)^{-1} \quad (3.1)
\]

(see also [BB10]).

Here is a consequence of the results from the previous section. In the special case of the Heisenberg group, this establishes a direct relationship between the classical characterizations of pseudo-differential operators from [Be77] and [Co79].

Corollary 3.1. In the above setting, pick any basis \( \{X_1, \ldots, X_m\} \) in the Lie algebra \( \mathfrak{g} \) of \( G \). Assume \( 1 \leq q \leq \infty \) and denote

\[
\Psi_q(\rho) := \{Y \in \mathcal{S}_q(\mathcal{H}) \mid \rho(q)(\cdot)Y \in C^\infty(G, \mathcal{S}_p(\mathcal{H}))\}.
\]

Then the following assertions hold:

1. The linear subspace \( \Psi_q(\rho) \) is precisely the set of all \( Y \in \mathcal{S}_q(\mathcal{H}) \) for which

\[
[d\rho(X_{j_1}), \ldots, [d\rho(X_{j_k}), Y] \ldots] \in \mathcal{S}_q(\mathcal{H})
\]

for arbitrary \( k \geq 1 \) and \( j_1, \ldots, j_k \in \{1, \ldots, m\} \).

2. If \( 1 \leq q < \infty \), then \( \Psi_q(\rho) \) is dense in \( \mathcal{S}_q(\mathcal{H}) \). If \( q = \infty \), then \( \Psi_q(\rho) \) contains the ideal of compact operators on \( \mathcal{H} \) and is dense in the norm-closed subspace \( \{Y \in B(\mathcal{H}) \mid \rho(q)(\cdot)Y \in C(G, B(\mathcal{H}))\} \) of \( B(\mathcal{H}) \).

Proof. We have that

\[
C^1(\rho(q)) = \{Y \in \mathcal{S}_q(\mathcal{H}) \mid [d\rho(X_j), Y] \in \mathcal{S}_q(\mathcal{H}) \text{ for } j = 1, \ldots, m\}.
\]

Then both assertions are special cases of Theorem 2.1 and Corollary 2.3.

We can now prove a corollary which shows that the first two conditions in [Mc00, Th. 1] are equivalent irrespective of the unitary representation involved therein. This also shows that the \( C^\infty \) part of the relation between differentiability and existence of commutators suggested after [Co95, Eq. (8.4)] holds true although the \( C^1 \) part of that suggestion fails to be true, since the following corollary would be false with the class \( C^\infty \) replaced by \( C^k \) for any \( k < \infty \). In fact, recall from the proof of Theorem 2.1 that the corresponding inclusions are strict in a special instance of the present setting, which is precisely the special instance referred to in [Co95].

Corollary 3.2. If \( Y \in B(\mathcal{H}) \) then the above mapping \( \rho(\infty)(\cdot)Y: G \to B(\mathcal{H}) \) is of class \( C^\infty \) with respect to the norm operator topology on \( B(\mathcal{H}) \) if and only if it is \( C^\infty \) with respect to the strong operator topology.
Proof. The mapping $\rho^{(\infty)}(\cdot)Y: G \to B(\mathcal{H})$ is smooth with respect to any topology on $B(\mathcal{H})$ if and only if it is smooth on any neighborhood of $1 \in G$. On the other hand, just as in the proof of [ABG96 Prop. 5.1.2(b)], one can see that this mapping is smooth with respect to the strong operator topology on $B(\mathcal{H})$ if and only if the iterated commutator condition in Corollary 3.1 is satisfied, hence the conclusion follows by Corollary 3.1, where the smoothness of $\rho^{(\infty)}(\cdot)Y$ is understood with respect to the norm operator topology on $\mathcal{S}_\infty(\mathcal{H}) = B(\mathcal{H})$. □

Example 3.3. Let $G = \mathbb{H}_{2n+1}$ be the $(2n + 1)$-dimensional Heisenberg group with the Schrödinger representation $\rho: G \to B(\mathcal{H})$. As recalled in [No12] for $1 \leq p \leq \infty$, the set $\Psi_p(\rho)$ of the above Corollary 3.1 is precisely the set of pseudo-differential operators on $L^2(\mathbb{R}^n)$ corresponding to the space of symbols $\{a \in C^\infty(\mathbb{R}^{2n}) \mid (\forall \alpha \in \mathbb{N}^{2n}) \partial^\alpha a \in L^p(\mathbb{R}^{2n})\}$ (see also [BB12] for similar results on more general nilpotent Lie groups). Thus our Corollary 3.1 leads to the main results of [No12].

Example 3.4. The above Corollary 3.1 also provides additional information in the setting of pseudo-differential operators on a compact manifold acted on by a Lie group, as studied for instance in [Ta97] and [Me00]. Thus, it follows that the notions of $U$-smoothness and $A$-smoothness from [Ta97 Sect. 2] actually coincide.

Remark 3.5. It would be interesting to extend the above result of Ex. 3.3 to the setting of the magnetic Weyl calculus of [IMP10]. Such an extension is likely to require infinite-dimensional Lie groups.

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