EXPONENTIAL ERGODICITY FOR SDES WITH JUMPS AND NON-LIPSCHITZ COEFFICIENTS*

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Abstract. In this paper we show irreducibility and the strong Feller property for transition probabilities of stochastic differential equations with jumps and monotone coefficients. Thus, exponential ergodicity and the spectral gap for the corresponding transition semigroups are obtained.

1. Introduction

Let \((\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})\) be a complete filtered probability space, and \((U, \mathcal{U}, \nu)\) a \(\sigma\)-finite measurable space. Let \(\{W(t)\}_{t \geq 0}\) be a \(d\)-dimensional standard \(\mathcal{F}_t\)-adapted Brownian motion, and \(\{k_t, t \geq 0\}\) a stationary \(\mathcal{F}_t\)-adapted Poisson point process with values in \(U\) and with characteristic measure \(\nu\) (cf. [5]). Let \(N_k((0, t], du)\) be the counting measure of \(k_t\), i.e., for \(A \in \mathcal{U}\)
\[
N_k((0, t], A) := \# \{0 < s \leq t : k_s \in A\},
\]
where \(\#\) denotes the cardinality of a set. The compensator measure of \(N_k\) is given by
\[
\tilde{N}_k((0, t], du) := N_k((0, t], du) - t\nu(du).
\]
In the following, we fix a \(U_0 \in \mathcal{U}\) such that \(\nu(U - U_0) < \infty\), and consider the following stochastic differential equation (SDE) with jumps in \(\mathbb{R}^d\):

\[
X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^{t+} \int_{U_0} f(X_{s^-}, u) \tilde{N}_k(dsdu),
\]

where \(b : \mathbb{R}^d \to \mathbb{R}^d\), \(\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}\) and \(f : \mathbb{R}^d \times U \to \mathbb{R}^d\) are measurable functions. Here, the second integral of the right side in Eq.\((1)\) is taken in the Itô’s sense, and the definition of the third integral is referred to [5]. When \(d = 1\), Youngmif Kwon and Chaniio I. Ff in [7] showed that the transition semigroup of Eq.\((1)\) is strong Feller and irreducible under some smoothness and growth conditions on \(b, \sigma, f\) with nondegenerate diffusion term. If \(b, \sigma, f\) are Lipschitz continuous, Masuda in [8] provided sets of conditions under which the transition semigroup of Eq.\((1)\) fulfils the ergodic theorem for any initial distribution.

In this paper, we study the ergodicity of Eq.\((1)\) under some non-Lipschitz conditions. First of all, recall some notions about the ergodicity. Let \(\{X_t(x), t \geq 0, x \in E\}\) be a family of Markov processes with state space \(E\) being a Hausdorff topology space, and transition
probability $p_t(x, E)$. Then
(i) $p_t$ is called irreducible if for each $t > 0$ and $x \in E$
$$p_t(x, E) > 0 \text{ for any non-empty open set } E \subset E;$$
(ii) $p_t$ is called strong Feller if for each $t > 0$ and $E \in \mathcal{B}(E)$
$$E \ni x \mapsto p_t(x, E) \in [0, 1] \text{ is continuous};$$
(iii) A measure $\mu$ on $(E, \mathcal{B}(E))$ is an invariant measure for $p_t$ if
$$\int_E p_t(x, E) \mu(dx) = \mu(E), \quad \forall t > 0, E \in \mathcal{B}(E).$$

The transition probability $p_t(x, \cdot)$ determines a Markov semigroup $(p_t)_{t \geq 0}$. The theorem below is a classical result combining the above concepts. (cf. [2])

**Theorem 1.1.** Assume a Markov semigroup $(p_t)_{t \geq 0}$ is irreducible and strong Feller. Then there exists at most one invariant measure for it. Moreover, if $\mu$ is the invariant measure, then $\mu$ is ergodic and equivalent to each $p_t(x, \cdot)$ and as $t \to \infty$, $p_t(x, E) \to \mu(E)$ for any Borel set $E$.

Next introduce our non-Lipschitz conditions.

**Hypotheses:**

- **(H1)** There exists $\lambda_0 \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$
$$2\langle x - y, b(x) - b(y) \rangle + \|\sigma(x) - \sigma(y)\|^2 \leq \lambda_0|x - y|^2 \kappa(|x - y|),$$
where $\kappa$ is a positive continuous function, bounded on $[1, \infty)$ and satisfying
$$\lim_{x \to 0} \frac{\kappa(x)}{\log x^{-1}} = \delta < \infty.$$

Here the function $\kappa$ controls the continuity modulus of $b(x)$ and $\sigma(x)$ such that the modulus is non-Lipschitz, for example, $\kappa(x) = C_1 \cdot (\log(1/x) \lor K)^{1/\beta_1}$ for some $\beta_1 > 1$ and $C_1, K > 0$.

- **(H2)** There exists $\lambda_1 > 0$ such that for all $x \in \mathbb{R}^d$
$$|b(x)|^2 + \|\sigma(x)\|^2 \leq \lambda_1(1 + |x|)^2.$$

- **(H3)** $b$ is continuous and there exists $\lambda_2 > 0$ such that
$$\langle \sigma(x)h, h \rangle \geq \sqrt{\lambda_2}|h|^2, \quad x, h \in \mathbb{R}^d.$$  \hspace{1cm} (2)

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$, $|\cdot|$ the length of a vector in $\mathbb{R}^d$ and $\|\cdot\|$ the Hilbert-Schmit norm from $\mathbb{R}^d$ to $\mathbb{R}^d$.

When $\nu(U_0) = 0$ and $b, \sigma$ satisfy the above assumptions and another assumption, Zhang [14] proved that the transition semigroup of Eq. (1) has the exponential ergodicity in the sense that there exists a constant $\beta_2 > 0$ such that for $t > 0$,
$$\|p_t(x_0, \cdot) - \mu\|_{\text{Var}} \leq C_2 \cdot e^{-\beta_2 t},$$
where $\|\cdot\|_{\text{Var}}$ denotes the total variation of a signed measure and $C_2 > 0$ is a constant. Here we require $\nu(U_0) \neq 0$ and follow the same lines as done in [14]. Thus how to treat the term with jumps is our key.

Firstly, we make the following assumption on $f$: 
The transition probability is given by

\[ \int_{U_0} \left| f(x,u) - f(y,u) \right|^2 \nu(du) \leq 2|\lambda_0||x-y|^2 \kappa(|x-y|) \]

and for \( q = 2 \) and \( 4 \)

\[ \int_{U_0} |f(x,u)|^q \nu(du) \leq \lambda_1 (1 + |x|)^q. \]

Under \((H_1), (H_2) \) and \((H_f)\), it is well known that there exists a unique strong solution to Eq. (1) (cf. [12, Theorem 170, p.140]). This solution will be denoted by \( X_t(x_0) \). The transition semigroup associated with \( X_t(x_0) \) is defined by

\[ p_t \varphi(x_0) := E \varphi(X_t(x_0)), \quad t > 0, \quad \varphi \in B_b(\mathbb{R}^d), \]

where \( B_b(\mathbb{R}^d) \) stands for the Banach space of all bounded measurable functions on \( \mathbb{R}^d \).

By Girsanov’s theorem on processes with jumps we get the irreducibility under \((H_1)-(H_3) \) and \((H_f)\) (cf. Proposition 2.4).

In order to get strong Feller property, we need the following stronger assumptions on \( b, \sigma, f \):

\((H_1')\) There exists \( \lambda_0 \in \mathbb{R} \) such that for all \( x, y \in \mathbb{R}^d \)

\[ 2\langle x - y, b(x) - b(y) \rangle + ||\sigma_{\lambda_2}(x) - \sigma_{\lambda_2}(y)||^2 \leq \lambda_0|x - y|^2 \kappa(|x - y|), \]

where \( \sigma_{\lambda_2}(x) \) is the unique symmetric nonnegative definite matrix-valued function such that \( \sigma_{\lambda_2}(x) = \sigma(x) - \lambda_2 I \) for the unit matrix \( I \).

\((H_f')\) There exists a positive function \( L(u) \) satisfying

\[ \sup_{x \in U_0} L(u) \leq \gamma < 1 \quad \text{and} \quad \int_{U_0} L(u)^2 \nu(du) < +\infty, \]

such that for any \( x, y \in \mathbb{R}^d \) and \( u \in U_0 \)

\[ |f(x,u) - f(y,u)| \leq L(u)|x-y|, \]

and

\[ |f(0,u)| \leq L(u). \]

**Remark 1.2.** To explain that \((H_1')\) is stronger than \((H_1)\), a matrix result is needed. And we will give a general result in Section 2.

When \( b, \sigma \) and \( f \) satisfy \((H_1'), (H_2), (H_3) \) and \((H_f')\), we obtain strong Feller property by the coupling method (cf. Proposition 2.5).

Finally, to show the ergodicity, the following assumption is needed:

\((H_{b,\sigma,f})\) There exist a \( r \geq 2 \) and two constants \( \lambda_3 > 0, \lambda_4 \geq 0 \) such that for all \( x \in \mathbb{R}^d \)

\[ 2\langle x, b(x) \rangle + ||\sigma(x)||^2 + \int_{U_0} |f(x,u)|^2 \nu(du) \leq -\lambda_3 |x|^r + \lambda_4. \]

We are now in a position to state our main result in the present paper.
**Theorem 1.3.** Assume \((H'_1), \,(H_2), \,(H_3)\) and \((H'_4)\). Then the semigroup \(p_t\) is irreducible and strong Feller. If in addition, \((H_{b,\sigma,f})\) holds, then there exists a unique invariant probability measure \(\mu\) of \(p_t\) having full support in \(\mathbb{R}^d\) such that

(i) if \(r \geq 2\) in \((H_{b,\sigma,f})\), then for all \(t > 0\) and \(x_0 \in \mathbb{R}^d\), \(\mu\) is equivalent to \(p_t(x_0, \cdot)\) and

\[
\lim_{t \to \infty} \|p_t(x_0, \cdot) - \mu\|_{Var} = 0.
\]

(ii) if \(r > 2\) in \((H_{b,\sigma,f})\), then for some \(\alpha, C_3 > 0\) independent of \(x_0\) and \(t\),

\[
\|p_t(x_0, \cdot) - \mu\|_{Var} \leq C_3 \cdot e^{-\alpha t}.
\]

Moreover, for any \(\gamma > 1\) and each \(\varphi \in L^\gamma(\mathbb{R}^d, \mu)\)

\[
\|p_t \varphi - \mu(\varphi)\|_{L^\gamma(\mathbb{R}^d, \mu)} \leq C_4 \cdot e^{-\alpha t / \gamma} \|\varphi\|_{L^\gamma(\mathbb{R}^d, \mu)}, \quad \forall t > 0,
\]

where \(\alpha\) is the same as above, \(\mu(\varphi) := \int_{\mathbb{R}^d} \varphi(x) \mu(dx)\) and \(C_4 > 0\) is a constant depending on \(\gamma\). In particular, let \(L_\gamma\) be the generator of \(p_t\) in \(L^\gamma(\mathbb{R}^d, \mu)\), then \(L_\gamma\) has a spectral gap (greater than \(\alpha / \gamma\)) in \(L^\gamma(\mathbb{R}^d, \mu)\).

The following convention will be used throughout the paper: \(C\) with or without indices will denote different positive constants (depending on the indices) whose values may change from one place to another one.

2. **Proof of Theorem 1.3**

**Lemma 2.1.** Suppose \(A, B\) are two symmetric positive definite matrices satisfying that there exists a \(\lambda > 0\) such that \(\langle Ah, h \rangle \geq \sqrt{\lambda} |h|^2, \langle Bh, h \rangle \geq \sqrt{\lambda} |h|^2\) for \(h \in \mathbb{R}^d\), and \(AB = BA\). Then

\[
\|A - B\| \leq \|A^2 - \lambda I - B^2 - \lambda I\|. \tag{3}
\]

**Proof.** Set

\[
A_\lambda := \sqrt{A^2 - \lambda I}, \quad B_\lambda := \sqrt{B^2 - \lambda I}.
\]

To show (3), we consider the difference of \(\|A - B\|^2\) and \(\|A_\lambda - B_\lambda\|^2\), i.e.

\[
\|A - B\|^2 - \|A_\lambda - B_\lambda\|^2 = \text{tr}((A - B)^2) - \text{tr}((A_\lambda - B_\lambda)^2) = \text{tr}(A^2 - 2AB + B^2) - \text{tr}(A_\lambda^2 - 2A_\lambda B_\lambda + B_\lambda^2) = \text{tr}(A^2 - 2AB + B^2) - \text{tr}(A^2 - \lambda I - 2A_\lambda B_\lambda + B^2 - \lambda I) = \text{tr}(-2AB + 2\lambda I + 2A_\lambda B_\lambda) = 2 [\text{tr}(A_\lambda B_\lambda) - \text{tr}(AB) + \lambda d], \tag{4}
\]

where \(\text{tr}(\cdot)\) stands for the trace of a matrix.

By the proof of [4, Theorem 7.4.10, p.433], one can obtain \(A = U M U^*\) and \(B = U N U^*\), where \(U \in \mathbb{R}^{d \times d}\) is real orthogonal, \(M = \text{diag}(\eta_1, \ldots, \eta_d)\), \(N = \text{diag}(\mu_1, \ldots, \mu_d)\), and \(\eta_i, \mu_i\) are eigenvalues of \(A\) and \(B\), respectively, and larger than \(\sqrt{\lambda}\). Moreover, \(A_\lambda = U M_\lambda U^*\) and \(B_\lambda = U N_\lambda U^*\), where \(M_\lambda = \text{diag}(\sqrt{\eta_1^2 - \lambda}, \ldots, \sqrt{\eta_d^2 - \lambda})\), \(N_\lambda = \text{diag}(\sqrt{\mu_1^2 - \lambda}, \ldots, \sqrt{\mu_d^2 - \lambda})\). Thus,

\[
\text{tr}(A_\lambda B_\lambda) = \text{tr}(U M_\lambda U^* U N_\lambda U^*) = \text{tr}(U M_\lambda N_\lambda U^*) = \sum_{i=1}^d \sqrt{\eta_i^2 - \lambda} \sqrt{\mu_i^2 - \lambda}.
\]
By the same deduction as the above one, we have $\text{tr}(AB) = \sum_{i=1}^{d} \eta_{i} \mu_{i}$. So, the right hand side of (4) can be written as $2 \sum_{i=1}^{d} (\sqrt{\eta_{i}^{2} - \lambda \mu_{i}^{2} - \lambda} - (\eta_{i} \mu_{i} - \lambda))$.

Noting that
\[
\left( \sqrt{\eta_{i}^{2} - \lambda \mu_{i}^{2} - \lambda} - (\eta_{i} \mu_{i} - \lambda) \right)^{2} = (\eta_{i}^{2} - \lambda)(\mu_{i}^{2} - \lambda) - (\eta_{i} \mu_{i} - \lambda)^{2}
\]
\[
= \eta_{i}^{2} \mu_{i}^{2} - \eta_{i}^{2} \lambda - \lambda \mu_{i}^{2} + \lambda^{2} - \eta_{i}^{2} \mu_{i}^{2} + 2 \lambda \eta_{i} \mu_{i} - \lambda^{2}
\]
\[
= -\lambda(\eta_{i} - \mu_{i})^{2} \leq 0,
\]
we get $\sqrt{\eta_{i}^{2} - \lambda \mu_{i}^{2} - \lambda} \leq \eta_{i} \mu_{i} - \lambda$. Thus, it holds that $\|A - B\|^{2} - \|A_{\lambda} - B_{\lambda}\|^{2} \leq 0$ and (3).

To show the irreducibility, we firstly estimate $X_{t}(x_{0})$.

**Lemma 2.2.** Suppose that $b, \sigma$ and $f$ satisfy (H$_{1}$)-(H$_{2}$) and (H$_{f}$). Then for any $T > 0$
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |X_{t}|^{2} \right] \leq C,
\]
where $C$ depends on $x_{0}, \lambda_{1}$ and $T$.

**Proof.** Applying Itô’s formula to Eq. (1), we have
\[
|X_{t}|^{2} = |x_{0}|^{2} + \int_{0}^{t} 2\langle X_{s}, b(X_{s}) \rangle ds + \int_{0}^{t} 2\langle X_{s}, \sigma(X_{s}) \rangle dW_{s}
\]
\[
+ \int_{0}^{t} \int_{U_{0}} \left( |X_{s-} + f(X_{s-}, u)|^{2} - |X_{s-}|^{2} \right) \tilde{N}_{k}(ds, du) + \int_{0}^{t} \|\sigma(X_{s})\|^{2} ds
\]
\[
+ \int_{0}^{t} \int_{U_{0}} \left( |X_{s} + f(X_{s}, u)|^{2} - |X_{s}|^{2} - 2\langle X_{s}, f(X_{s}, u) \rangle \right) \nu(du) ds.
\]

By BDG inequality, Young’s inequality and mean theorem, one can obtain that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |X_{t}|^{2} \right] \leq |x_{0}|^{2} + \mathbb{E} \int_{0}^{T} 2|x_{s}||b(X_{s})|ds + CE \left( \int_{0}^{T} |X_{s}|^{2} \|\sigma(X_{s})\|^{2} ds \right)^{\frac{1}{2}}
\]
\[
+ CE \left[ \int_{0}^{T} \int_{U_{0}} [\|f(X_{s-}, u)\|^{2} + \|f(X_{s-}, u)\| |X_{s-}|^{2}] \tilde{N}_{k}(ds, du) \right]^{\frac{1}{2}}
\]
\[
+ \mathbb{E} \int_{0}^{T} \|\sigma(X_{s})\|^{2} ds + \mathbb{E} \int_{0}^{T} \int_{U_{0}} |f(X_{s}, u)|^{2} \nu(du) ds
\]
\[
\leq |x_{0}|^{2} + \mathbb{E} \left( 2 \sup_{t \in [0, T]} |X_{t}| \int_{0}^{T} |b(X_{s})| ds \right) + \mathbb{E} \int_{0}^{T} \|\sigma(X_{s})\|^{2} ds
\]
\[
+ CE \left( \sup_{t \in [0, T]} |X_{t}|^{2} \int_{0}^{T} \|\sigma(X_{s})\|^{2} ds \right)^{\frac{1}{2}} + \mathbb{E} \int_{0}^{T} \int_{U_{0}} |f(X_{s}, u)|^{2} \nu(du) ds
\]
where Gronwall’s inequality yields

\[ H \]

Combining (7), (8), (9), (H) for the fifth term in the right hand side, we use Young’s inequality and (H_f) to get

\[
CE \left[ \int_0^T \int_{U_0} |f(X_{s-}, u)|^4 N_k(ds, du) \right]^{\frac{1}{2}} 
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} |X_t|^2 \right] + C E \int_0^T |b(X_s)|^2 ds + C E \int_0^T ||\sigma(X_s)||^2 ds 
+ CE \left[ \int_0^T \int_{U_0} |f(X_{s-}, u)|^4 N_k(ds, du) \right]^{\frac{1}{2}} + E \int_0^T \int_{U_0} |f(X_s, u)|^2 \nu(du)ds 
+ CE \left[ \int_0^T \int_{U_0} |f(X_{s-}, u)|^2 |X_s|^2 N_k(ds, du) \right]^{\frac{1}{2}}. \tag{7}
\]

For the fifth term in the right hand side, we use Young’s inequality and (H_f) to get

\[
CE \left[ \int_0^T \int_{U_0} |f(X_{s-}, u)|^4 N_k(ds, du) \right]^{\frac{1}{2}} 
\leq CE \left[ \sup_{t \in [0,T]} (1 + |X_{t-}|)^2 \right] \int_0^T \int_{U_0} \frac{|f(X_{s-}, u)|^4}{(1 + |X_{s-}|)^2} N_k(ds, du) \right]^{\frac{1}{2}} 
\leq \frac{1}{8} E \left[ \sup_{t \in [0,T]} (1 + |X_{t-}|)^2 \right] + CE \int_0^T \int_{U_0} \frac{|f(X_{s-}, u)|^4}{(1 + |X_{s-}|)^2} \nu(du)ds 
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} |X_{t-}|^2 \right] + \frac{1}{4} + CE \int_0^T (1 + |X_s|)^2 ds. \tag{8}
\]

To the seventh term in the right hand side, the similar method yields that

\[
CE \left[ \int_0^T \int_{U_0} |f(X_{s-}, u)|^2 |X_{s-}|^2 N_k(ds, du) \right]^{\frac{1}{2}} 
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} |X_{t-}|^2 \right] + CE \int_0^T \int_{U_0} |f(X_s, u)|^2 \nu(du)ds. \tag{9}
\]

Combining (7), (8), (9), (H_2) and (H_f), we obtain

\[
E \left[ \sup_{t \in [0,T]} |X_t|^2 \right] \leq 4|x_0|^2 + 1 + CE \int_0^T (1 + |X_s|)^2 ds 
\leq 4|x_0|^2 + 1 + CE \int_0^T 2(1 + |X_s|^2) ds 
\leq 4|x_0|^2 + (1 + 2CT) + 2C \int_0^T E \left[ \sup_{s \in [0,t]} |X_s|^2 \right] dt.
\]

Gronwall’s inequality yields

\[
E \left[ \sup_{t \in [0,T]} |X_t|^2 \right] \leq C,
\]

where C depends on x_0, \lambda_1 and T. \hfill \Box
Secondly, construct some auxiliary processes. For any $T > 0$, let $t_0 \in (0, T)$, whose value will be determined below. Set for any $n \in \mathbb{N}$

$$X^n_{t_0} = X_{t_0} I_{[|X_{t_0}| \leq n]}.$$ 

And then by Lemma 2.2

$$\lim_{n \to \infty} \mathbb{E}|X^n_{t_0} - X_{t_0}|^2 = 0.$$ 

For $t \in [t_0, T]$ and $y \in \mathbb{R}^d$, define

$$J^n_t = \frac{T - t}{T - t_0} X^n_{t_0} + \frac{t - t_0}{T - t_0} y,$$

$$h^n_t = \frac{y - X^n_{t_0}}{T - t_0} - b(J^n_t).$$

Thus,

$$J^n_{t_0} = X^n_{t_0}, \quad J^n_T = y,$$

and $J^n_t$ satisfies the following equation:

$$J^n_t = X^n_{t_0} + \int_{t_0}^t b(J^n_s) ds + \int_{t_0}^t h^n_s ds, \quad t \in [t_0, T].$$

Next, we introduce the following equation:

$$Y_t = X_{t_0} + \int_{t_0}^t b(Y_s) ds + \int_{t_0}^t h^n_s ds + \int_{t_0}^t \sigma(Y_s) dW_s$$

$$+ \int_{t_0}^{t+} \int_{t_0}^t f(Y_{s-u}, u) \tilde{N}_k(duds), \quad t \in [t_0, T].$$

**Lemma 2.3.** Suppose that $b, \sigma$ and $f$ satisfy (H1)-(H2) and (Hf). Then

$$\mathbb{E}|Y_T - y|^2 \leq \left[ \mathbb{E}|X_{t_0} - X^n_{t_0}|^2 + C(T - t_0) \right] e^{-\lambda_0(T - t_0)}.$$  \hfill (10)

**Proof.** Set $Z_t := Y_t - J^n_t$, and then $Z_t$ satisfies the following equation

$$Z_t = X_{t_0} - X^n_{t_0} + \int_{t_0}^t (b(Y_s) - b(J^n_s)) ds + \int_{t_0}^t \sigma(Y_s) dW_s$$

$$+ \int_{t_0}^{t+} \int_{t_0}^t f(Y_{s-u}, u) \tilde{N}_k(duds), \quad t \in [t_0, T].$$

By Itô’s formula we obtain

$$|Z_t|^2 = |X_{t_0} - X^n_{t_0}|^2 + \int_{t_0}^t 2\langle Z_s, b(Y_s) - b(J^n_s) \rangle ds + \int_{t_0}^t 2\langle Z_s, \sigma(Y_s) dW_s \rangle$$

$$+ \int_{t_0}^t \int_{t_0}^t [|Z_s + f(Y_{s-u}, u)|^2 - |Z_s|^2] \tilde{N}_k(ds, du) + \int_{t_0}^t ||\sigma(Y_s)||^2 ds$$

$$+ \int_{t_0}^t \int_{t_0}^t [|Z_s + f(Y_s, u)|^2 - |Z_s|^2 - 2\langle Z_s, f(Y_s, u) \rangle] \nu(du) ds.$$

It follows from (H1)-(H2) and (Hf) that

$$\mathbb{E}|Z_t|^2 = \mathbb{E}|X_{t_0} - X^n_{t_0}|^2 + \mathbb{E} \int_{t_0}^t 2\langle Z_s, b(Y_s) - b(J^n_s) \rangle ds + \mathbb{E} \int_{t_0}^t ||\sigma(Y_s)||^2 ds$$
where \(\rho\) Gronwall’s inequality admits us to obtain that
\[
\sup_{t \in [0,T]} \mathbb{E}|X_t| \leq C \int_0^t (1 + |Y_s|)^2 \mathbb{E} ds.
\]

There exists a \(\delta > 0\) such that
\[
x^2 \kappa(x) \leq \rho_\delta(x^2), \quad x > 0,
\]
where \(\rho_\delta : \mathbb{R} \to \mathbb{R}_+\) is a concave function given by
\[
\rho_\delta(x) := \begin{cases} x \log x^{-1}, & x \leq \delta, \\ (\log x - 1)x + \eta, & x > \delta. \end{cases}
\]

Next, estimate \(\mathbb{E}|Y_t|^2\). For \(t \in [t_0, T]\), by Hölder’s inequality, one can get
\[
|Y_t|^2 \leq 5|X_t|^2 + 5T^{\frac{t}{2}} \int_{t_0}^t |b(Y_s)|^2 ds + 5T^{\frac{t}{2}} \int_{t_0}^t |h_s^n|^2 ds + 5 \int_{t_0}^t \sigma(Y_s) dW_s
\]
\[
+ 5 \left(\int_{t_0}^{t+} \int_{U_0} f(Y_{s-}, u) \tilde{N}_1(dsd)\right)^2.
\]

Moreover, by Burkholder’s inequality, \((H_2)\) and \((H_f)\), it holds that
\[
\mathbb{E}|Y_t|^2 \leq 5\mathbb{E}|X_t|^2 + 5T^{\frac{t}{2}} \mathbb{E} \int_{t_0}^t |b(Y_s)|^2 ds + 5T^{\frac{t}{2}} \mathbb{E} \int_{t_0}^t |h_s^n|^2 ds
\]
\[
+ 5 \int_{t_0}^t \|\sigma(Y_s)\|^2 ds + 5 \int_{t_0}^{t+} \int_{U_0} |f(Y_{s-}, u)|^2 \nu(du) ds
\]
\[
\leq 5\mathbb{E}|X_t|^2 + C + C \int_{t_0}^t \mathbb{E}|Y_s|^2 ds.
\]

Gronwall’s inequality admits us to obtain that
\[
\sup_{t \in [t_0, T]} \mathbb{E}|Y_t|^2 \leq C,
\]
where \(C\) is independent of \(t_0\).

Combining \((11)-(13)\), by Jensen’s inequality and the Bihari inequality (cf. [14, Lemma 2.1]), we have
\[
\mathbb{E}|Y_T - y|^2 \leq \left[\mathbb{E}|X_{t_0} - X_{t_0}^n|^2 + C(T - t_0)\right] e^{-|\lambda_0(T - t_0)|}.
\]

The proof is completed. \(\square\)

**Proposition 2.4.** Suppose that \(b, \sigma\) and \(f\) satisfy \((H_1)-(H_3)\) and \((H_f)\). Then the transition probability \(p_t\) is irreducible.

**Proof.** To prove the irreducibility, it suffices to prove that for each \(T > 0\) and \(x_0 \in \mathbb{R}^d\),
\[
p_T(x_0, B(y, a)) = P(X_T(x_0) \in B(y, a)) = P(|X_T(x_0) - y| < a) > 0,
\]
or equivalently
\[
P(|X_T(x_0) - y| \geq a) < 1,
\]
for any \(y \in \mathbb{R}^d\) and \(a > 0\).
First of all, study the process $Y_t$. Define

$$Y_t := X_t, \quad t \in [0, t_0]$$

and then

$$Y_t = x_0 + \int_0^t b(Y_s) ds + \int_0^t I_{(s \geq t_0)} h^n_s ds + \int_0^t \sigma(Y_s) dW_s$$

$$+ \int_{t_0}^{t_+} \int_{U_0} f(Y_{s-}, u) \tilde{N}_k(dsdu), \quad t \in [0, T].$$

Set

$$H_t := I_{(t > t_0)} \sigma(Y_t)^{-1} h^n_t,$$

$$\xi_t := \exp \left\{ \int_0^t \langle dW_s, H_s \rangle - \frac{1}{2} \int_0^t |H_s|^2 ds \right\}.$$

By (H3), we obtain that $|H_t|^2$ is bounded, which yields that $E\xi_T = 1$ by Novikov’s criteria.

And then define

$$\bar{W}_t := W_t + \int_0^t H_s ds,$$

$$Q := \xi_T P.$$

Thus by [12, Theorem 132] we know that $Q$ is a probability measure, $\bar{W}_t$ is a $Q$-Brownian motion and $\tilde{N}_k((0, t], du)$ is a Poisson martingale measure under $Q$ with the same compensator $\nu(du)t$. Moreover, $Y_t$ is the solution of the following equation

$$Y_t = x_0 + \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) d\bar{W}_s + \int_{t_0}^{t_+} \int_{U_0} f(Y_{s-}, u) \tilde{N}_k(dsdu).$$

By the uniqueness in law of Eq.(1) we attain that the law of $\{X_t, t \in [0, T]\}$ under $P$ is the same to that of $\{Y_t, t \in [0, T]\}$ under $Q$. Therefore, to obtain (14), it is sufficient to prove $Q(\{|Y_T - y| \geq a\}) < 1$, and furthermore, $P(\{|Y_T - y| \geq a\}) < 1$ by equivalency of $Q, P$.

It holds by Chebyshev’s inequality and Lemma 2.3 that

$$P(\{|Y_T - y| \geq a\}) \leq \left[ E|X_{t_0} - X_{t_0}^n|^2 + C(T - t_0) \right] e^{-\langle \lambda_0, (T-t_0) \rangle} / a^2.$$

Choosing $n$ large enough and $t_0$ close enough to $T$, we have

$$P(\{|Y_T - y| \geq a\}) < 1.$$

The proof is completed. \[\square\]

Next we use the coupling method to prove strong Feller property. Set $a(x) := \sigma(x)\sigma(x)$. And then the infinitesimal generator of Eq.(1) is given by

$$L\psi(x) = b^i(x)\partial_i \psi(x) + \frac{1}{2} \sigma^j(x)\partial_{ij} \psi(x)$$

$$+ \int_{U_0} \left( \psi(x + f(x, u)) - \psi(x) - f^i(x, u)\partial_i \psi(x) \right) \nu(du),$$
for $\psi \in C^2_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Recall that an operator $\bar{L}$ on $\mathbb{R}^{2d}$ is called a coupling operator of $L$ if $\bar{L}$ satisfies the marginal condition:

$$
(\bar{L}\psi)(x, y) = L\psi(x), \quad \psi \in C^2_b(\mathbb{R}^d), \quad x, y \in \mathbb{R}^d,
$$

where $\psi$ is regarded as a function in $C^2_b(\mathbb{R}^{2d})$. For any $\delta \in (0, 1)$ and $|x_0 - y_0| < \delta$, we define

$$
u(x, y) := \lambda_2(I - 2u_\delta(x, y)u_\delta(x, y)* + \sigma_\lambda_2(x)\sigma_\lambda_2(y)*),
$$

for $x, y \in \mathbb{R}^d, x \neq y$, where $\alpha \in (0, 1)$ (its value will be determined below). Thus the operator given by

$$
\bar{L}\psi(x, y) = b^i(x)\partial_{x_i}\psi(x, y) + b^i(x)\partial_{y_i}\psi(x, y)
$$

is a coupling operator of $L$.

By the analysis similar to [13, Section 3.1], there exist a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; (\tilde{\mathcal{F}}_t)_{t \geq 0})$ and a $\mathbb{R}^{2d}$-valued process $\tilde{Z}^t_\mathbf{1}$ ($\mathbb{R}^{2d}$ is one point compactification of $\mathbb{R}^{2d}$) such that $\tilde{Z}^t_\mathbf{1} = (x_0, y_0)$ and for each $\psi \in C^2_b(\mathbb{R}^{2d})$

$$
\left\{ \psi(\tilde{Z}^t_{t \lor e}) - \psi(\tilde{Z}^t_0) - \int_0^{t \lor e} \tilde{L}\psi(\tilde{Z}^t_s)ds, \quad t \geq 0 \right\}
$$

is an $(\tilde{\mathcal{F}}_t)$-local martingale, where $e$ is the explosion time of the process $\tilde{Z}^t_1 = (\tilde{X}^t, \tilde{Y}^t)$, i.e.

$$
e = \lim_{n \to \infty} e_n, \quad e_n = \inf \left\{ t > 0 : \left( |\tilde{X}^t| + |\tilde{Y}^t| \right) > n \right\}.
$$

Since $\tilde{X}^t$ and $\tilde{Y}^t$ are associated with the operator $L$ starting from $x_0$ and $y_0$, respectively, we have $e = \infty$ by $(H_2)$ and $(H_f)$. Consider the coupling time $\tau$ of $\tilde{Z}^t_1$, i.e.

$$
\tau := \inf \{ t > 0 : |\tilde{X}^t| = 0 \},
$$

and define

$$
\tilde{X}_t = \begin{cases} 
\tilde{X}^t, & t < \tau, \\
\tilde{Y}^t, & t \geq \tau.
\end{cases}
$$

And then for any $\psi \in C^\infty_c(\mathbb{R}^d)$,

$$
\psi(\tilde{X}_t) - \psi(x_0) - \int_0^t L\psi(\tilde{X}_s)ds = \psi(\tilde{X}_{t \lor \tau}) - \psi(x_0) - \int_0^{t \lor \tau} L\psi(\tilde{X}_s)ds + \psi(\tilde{X}_t) - \psi(\tilde{X}_{t \lor \tau}) - \int_{t \lor \tau}^t L\psi(\tilde{X}_s)ds
$$
Proposition 2.5. Under \((H_1), (H_2), (H_3)\) and \((H_4')\), the semigroup \(p_t\) of \(X_t\) is strong Feller.

Proof. For any \(\varphi \in B_b(\mathbb{R}^d)\), by the definition of \(\bar{X}_t\)
\[
|p_t\varphi(x_0) - p_t\varphi(y_0)| = |\bar{E}(\varphi(\bar{X}_t(x_0))) - \bar{E}(\varphi(\bar{Y}_t(y_0)))| \\
\leq |\bar{E}[\varphi(\bar{X}_t(x_0))1_{t<\tau}] - \bar{E}[\varphi(\bar{Y}_t(y_0))1_{t<\tau}]| \\
+ |\bar{E}[\varphi(\bar{X}_t(x_0))1_{t\geq\tau}] - \bar{E}[\varphi(\bar{Y}_t(y_0))1_{t\geq\tau}]| \\
\leq 2||\varphi||_0 \bar{P}(t < \tau),
\]
where \(\bar{E}\) is the expectation with respect to \(\bar{P}\).

Next we estimate \(\bar{P}(t < \tau)\).

Define
\[
S_\delta := \inf\{t \geq 0 : |\bar{X}_t - \bar{Y}_t| > \delta\}.
\]
Setting \(g(r) := \frac{r}{1+r}, r \geq 0\) and \(\psi(x, y) := g(|x-y|)\), one can obtain by (16)
\[
\bar{E}g(|\bar{X}_{t\wedge\tau \wedge n \wedge S_\delta} - \bar{Y}_{t\wedge\tau \wedge n \wedge S_\delta}|) \\
= g(|x_0 - y_0|) + \bar{E} \int_0^{t\wedge\tau \wedge n \wedge S_\delta} \frac{G(\bar{X}_s, \bar{Y}_s)}{2} g''(|\bar{X}_s - \bar{Y}_s|) ds \\
+ \bar{E} \int_0^{t\wedge\tau \wedge n \wedge S_\delta} \text{tr} \left[ \frac{G(\bar{X}_s, \bar{Y}_s) - \bar{G}(\bar{X}_s, \bar{Y}_s) + 2F(\bar{X}_s, \bar{Y}_s)}{2|\bar{X}_s - \bar{Y}_s|} g'(|\bar{X}_s - \bar{Y}_s|) ds \\
+ \bar{E} \int_0^{t\wedge\tau \wedge n \wedge S_\delta} \int_{U_0} \left[ g \left( \bar{X}_s + f(\bar{X}_s, u) - \bar{Y}_s - f(\bar{Y}_s, u) \right) - g \left( |\bar{X}_s - \bar{Y}_s| \right) \\
- g'(|\bar{X}_s - \bar{Y}_s|) \left( u(\bar{X}_s, \bar{Y}_s), f(\bar{X}_s, u) - f(\bar{Y}_s, u) \right) \right] \nu(du) ds \right) \\
=: g(|x_0 - y_0|) + I_1 + I_2 + I_3,
\]
where
\[
G(x, y) = a(x) + a(y) - c(x, y) - c(x, y)^*, \\
\bar{G}(x, y) = \langle u(x, y), G(x, y)u(x, y) \rangle, \\
F(x, y) = \langle x - y, b(x) - b(y) \rangle,
\]
and
\[
g'(r) = \frac{1}{(1+r)^2}, \quad g''(r) = -\frac{2}{(1+r)^3}.
\]
Noting that by \((H_2)\)
\[
\bar{G}(x, y) = \langle u(x, y), (a(x) + a(y) - 2\lambda_2 - \sigma_{\lambda_2}(x)\sigma_{\lambda_2}(y) - \sigma_{\lambda_2}(y)\sigma_{\lambda_2}(x))u(x, y) \rangle
\]
Thus

\[ H = \left( u(x, y), \frac{|x_0 - y_0|^\alpha}{\delta^\alpha} u(x, y) u(x, y) u(x, y) \right) \]

\[ = \left( u(x, y), (\sigma_{\lambda_2}(x) - \sigma_{\lambda_2}(y))^2 u(x, y) \right) + 4\lambda_2 \frac{|x_0 - y_0|^\alpha}{\delta^\alpha} \]

\[ \geq 4\lambda_2 \frac{|x_0 - y_0|^\alpha}{\delta^\alpha}, \]

and

\[ \text{tr}(G(x, y)) - \bar{G}(x, y) = \text{tr}(a(x) + a(y) - 2\lambda_2 - \sigma_{\lambda_2}(x)\sigma_{\lambda_2}(y) - \sigma_{\lambda_2}(y)\sigma_{\lambda_2}(x)) \]

\[ + \text{tr}(4\lambda_2 \frac{|x_0 - y_0|^\alpha}{\delta^\alpha} u(x, y) u(x, y)^* - \bar{G}(x, y)) \]

\[ \leq \|\sigma_{\lambda_2}(x) - \sigma_{\lambda_2}(y)\|^2. \]

Thus

\[ I_1 \leq - \frac{4\lambda_2}{\delta^\alpha(1 + \delta)^3} \cdot |x_0 - y_0|^\alpha \cdot \tilde{E}(t \wedge \tau \wedge e_n \wedge S_\delta). \]  \hspace{1cm} (18)

By \((H_4')\) we have

\[ I_2 \leq \tilde{E} \int_0^{t \wedge \tau \wedge e_n \wedge S_\delta} \frac{|\lambda_0|}{2} |\bar{X}_s - \bar{Y}_s| \kappa(|\bar{X}_s - \bar{Y}_s|) ds. \]  \hspace{1cm} (19)

Next, deal with \(I_3\). Mean theorem and \((H_f')\) admit us to get

\[ g(|x - y + f(x, u) - f(y, u)|) - g(|x - y|) = g'(|x - y|)(u(x, y), f(x, u) - f(y, u)) \]

\[ = g''(|x - y + \theta(f(x, u) - f(y, u))|)|x - y + \theta(f(x, u) - f(y, u))|^{-2} \]

\[ \cdot |x - y + \theta(f(x, u) - f(y, u))| |x - y + \theta(f(x, u) - f(y, u))|^{-2} \]

\[ + g'(|x - y + \theta(f(x, u) - f(y, u))|) \left[ |x - y + \theta(f(x, u) - f(y, u))|^{-1} \right. \]

\[ \cdot |f(x, u) - f(y, u)|^2 - |x - y + \theta(f(x, u) - f(y, u))| f(x, u) - f(y, u) |^2 \]

\[ \left. - |x - y + \theta(f(x, u) - f(y, u))|^{-3} \right] \]

\[ \leq \frac{L(u)^2}{1 - L(u)} |x - y| \]

\[ \leq C L(u)^2 |x - y|, \]

where \(0 < \theta < 1\). So

\[ I_3 \leq \tilde{E} \int_0^{t \wedge \tau \wedge e_n \wedge S_\delta} C |\bar{X}_s - \bar{Y}_s| ds. \]  \hspace{1cm} (20)

Combining (18), (19) and (20), one can obtain that

\[ \frac{1}{1 + \delta} \tilde{E}|\bar{X}_{t \wedge \tau \wedge e_n \wedge S_\delta} - \bar{Y}_{t \wedge \tau \wedge e_n \wedge S_\delta}| \]

\[ \leq |x_0 - y_0| - \frac{4\lambda_2}{\delta^\alpha(1 + \delta)^3} \cdot |x_0 - y_0|^\alpha \cdot \tilde{E}(t \wedge \tau \wedge e_n \wedge S_\delta) \]

\[ + \frac{|\lambda_0|}{2} \int_0^{t \wedge \tau \wedge e_n \wedge S_\delta} \tilde{E}_{\rho_\delta}(|\bar{X}_s - \bar{Y}_s|) ds + C \int_0^{t \wedge \tau \wedge e_n \wedge S_\delta} \tilde{E}|\bar{X}_s - \bar{Y}_s| ds \]
\[ \begin{aligned} &\leq |x_0 - y_0| + \left(\frac{|\lambda_0|}{2} + C\right) \int_0^{t \land \tau \land e_n \land S_\delta} \rho_\delta(\tilde{E}|\tilde{X}_s - \tilde{Y}_s|) \, ds \\
&\quad - \frac{4\lambda_2}{\delta^{\alpha}(1 + \delta)^3} \cdot |x_0 - y_0|^\alpha \cdot \tilde{E}(t \land \tau \land e_n \land S_\delta), \quad (21) \end{aligned} \]

where the second step bases on Jensen’s inequality. By the Bihari inequality (cf. [14, Lemma 2.1]), we get that for any \( t > 0 \) and \( |x_0 - y_0| \leq \delta \),

\[ \tilde{E}|\tilde{X}_{t \land \tau \land e_n \land S_\delta} - \tilde{Y}_{t \land \tau \land e_n \land S_\delta}| \leq (1 + \delta)|x_0 - y_0|^{\exp\left\{-(1+\delta)(\frac{|\lambda_0|}{2} + C)t\right\}}. \quad (22) \]

On one hand, substituting (22) into (21) yields

\[ \tilde{E}(t \land \tau \land e_n \land S_\delta) \leq \frac{\delta^{\alpha}(1 + \delta)^3}{4\lambda_2} \left[ |x_0 - y_0|^{1-\alpha} + \left(\frac{|\lambda_0|}{2} + C\right)t \right] \rho_\delta \left((1 + \delta)|x_0 - y_0|^{\exp\left\{-(1+\delta)(\frac{|\lambda_0|}{2} + C)t\right\}}\right) \cdot |x_0 - y_0|^{-\alpha}. \]

Letting \( n \to \infty \), one can obtain by Levy’s theorem

\[ \tilde{E}((2t) \land \tau \land S_\delta) \leq C_{t,\lambda_0,\delta} \cdot |x_0 - y_0|^{\exp\left\{-(1+\delta)(|\lambda_0|+2C)t\right\}}/6. \quad (23) \]

On the other hand, it follows from (22)

\[ \delta \tilde{P}(t \land \tau \land e_n > S_\delta) \leq \tilde{E}\left[|\tilde{X}_{t \land \tau \land e_n \land S_\delta} - \tilde{Y}_{t \land \tau \land e_n \land S_\delta}| \cdot I_{t \land \tau \land e_n > S_\delta}\right] \leq \tilde{E}|\tilde{X}_{t \land \tau \land e_n \land S_\delta} - \tilde{Y}_{t \land \tau \land e_n \land S_\delta}| \leq (1 + \delta)|x_0 - y_0|^{\exp\left\{-(1+\delta)(\frac{|\lambda_0|}{2} + C)t\right\}}. \]

Letting \( n \to \infty \), we have

\[ \tilde{P}((2t) \land \tau > S_\delta) \leq \frac{1 + \delta}{\delta}|x_0 - y_0|^{\exp\left\{-(1+\delta)(|\lambda_0|+2C)t\right\}}. \quad (24) \]

Finally, by (23) and (24) it holds that

\[ \tilde{P}(t < \tau) = \tilde{P}(t < \tau, S_\delta > t) + \tilde{P}(t < \tau, S_\delta \leq t) \leq \tilde{P}((2t) \land \tau \land S_\delta > t) + \tilde{P}((2t) \land \tau > S_\delta) \leq \frac{1 + \delta}{\delta}|x_0 - y_0|^{\exp\left\{-(1+\delta)(|\lambda_0|+2C)t\right\}} \leq C_{t,\lambda_0,\delta} \cdot |x_0 - y_0|^{\exp\left\{-(1+\delta)(|\lambda_0|+2C)t\right\}}/6. \]

Thus, the proof is completed. \(\square\)

We now give

**Proof of Theorem 1.3**
By Itô’s formula and mean theorem we get
\[ E|X_t|^2 = |x_0|^2 + \int_0^t E2\langle X_s, b(X_s) \rangle ds + \int_0^t E\|\sigma(X_s)\|^2 ds \]
\[ + \int_0^t \int_{U_0} E \left[ |X_s + f(X_s, u)|^2 - |X_s|^2 - 2\langle X_s, f(X_s, u) \rangle \right] \nu(du) ds \]
\[ = |x_0|^2 + \int_0^t E2\langle X_s, b(X_s) \rangle ds + \int_0^t E\|\sigma(X_s)\|^2 ds \]
\[ + \int_0^t \int_{U_0} E|f(X_s, u)|^2 \nu(du) ds. \] (25)

By \((H_{b,\sigma,f})\), one obtains
\[ \frac{1}{t} \int_0^t E|X_s|^r ds \leq \frac{x_0}{\lambda_3 t} + \frac{\lambda_4}{\lambda_3}. \]

Set
\[ \mu_T(A) := \frac{1}{T} \int_0^T p_t(x_0, A) dt, \]
for any \(T > 0\) and \(A \in \mathcal{B}(\mathbb{R}^d)\). And we have by Chebyshev’s inequality
\[ \mu_T(B^c(0, R)) = \frac{1}{T} \int_0^T p_t(x_0, B^c(0, R)) dt \]
\[ \leq \frac{1}{TR^r} \int_0^T E|X_t|^r dt \]
\[ \leq \frac{1}{R^r} \left( \frac{x_0}{\lambda_3 T} + \frac{\lambda_4}{\lambda_3} \right). \]

Thus, for any \(\varepsilon > 0\), \(\mu_T(B(0, R)) > 1 - \varepsilon\) for \(R\) being large enough. Hence, \(\{\mu_T, T > 0\}\) is tight and its limit is an invariant probability measure \(\mu\). As we have just proved, \(p_t\) is irreducible and strong Feller, then by Theorem \[\text{[11]}\] \(\mu\) is equivalent to each \(p_t(x, \cdot)\) and as \(t \to \infty\), \(p_t(x, E) \to \mu(E)\) for any Borel set \(E\).

If \(r > 2\), to [25], taking derivatives with respect to \(t\) and using \((H_{b,\sigma,f})\) and Hölder’s inequality give
\[ \frac{dE|X_t|^2}{dt} \leq -\lambda_3 E|X_t|^r + \lambda_4 \]
\[ \leq -\lambda_3 (E|X_t|^2)^\frac{r}{2} + \lambda_4. \]

Let \(f(t)\) solve the following ODE:
\[ \begin{cases} f'(t) = -\lambda_3 f(t)^\frac{r}{2} + \lambda_4, \\ f(0) = |x_0|^2. \end{cases} \]

By the comparison theorem on ODE and [11] Lemma 1.2.6, p.32, we have for some \(C > 0\)
\[ E|X_t|^2 \leq f(t) \leq C \left[ 1 + t^{1-r/2} \right], \]
where the right side is independent of \(x_0\).
Since \( p_t \) is irreducible and strong Feller, we have for any \( b, a > 0 \) and \( t > 0 \)
\[
\inf_{x_0 \in B(0,b)} p_t(x_0, B(0,a)) > 0.
\]
The second conclusion then follows from [3, Theorem 2.5 (b) and Theorem 2.7].

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