Spin Structure Functions of $^3$He at Finite $Q^2$

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Abstract

Using recently derived relations between spin-dependent nuclear and nucleon $g_1$ and $g_2$ structure functions at finite $Q^2$, we study nuclear effects in $^3$He in the nucleon resonance and deep inelastic regions. Comparing the finite-$Q^2$ results with the standard convolution formulas obtained in the large-$Q^2$ limit, we find significant broadening of the effective nucleon momentum distribution functions, leading to additional suppression of the nuclear $g_1$ and $g_2$ structure functions around the resonance peaks.
I. INTRODUCTION

The nuclear EMC effect, observed 25 years ago in unpolarized $\mu A$ deep inelastic scattering \cite{1}, demonstrated a dramatic change in the structure function of a nucleon bound in a heavy nucleus relative to that in the deuteron. Since then many ideas have been put forward to describe this modification \cite{2, 3}, although the exact mechanism(s) responsible remains controversial. Recent discussion of the medium modification of nuclear structure functions has focused on polarization, in the expectation that study of deep inelastic scattering from polarized nuclei can provide clues about the nature and origin of the effect. Indeed, recent calculations \cite{4} suggest that the spin-dependent proton $g_1$ structure function undergoes more dramatic change in the nuclear medium than its unpolarized counterpart, $F_2$.

As a purely practical application, polarized nuclear targets are also presently the only source of information about the structure of polarized neutrons, given the absence of free neutron targets. In particular, polarized $^3$He nuclei are commonly used as effective neutron targets \cite{5}. A number of studies of polarized deep inelastic scattering (DIS) from $^3$He nuclei have been made in recent years \cite{6, 7, 8, 9, 10, 11, 12, 13}, which have attempted to quantify the spin dependence of the nuclear effects on bound nucleon structure functions.

The standard formalism used to study nuclear DIS at large Bjorken-$x$ ($x > 0.1$) is the nuclear impulse approximation, in which virtual photon–nucleus scattering proceeds at the sub-nuclear level via virtual photon–nucleon scattering. In the Bjorken limit (where both the energy transfer $\nu$ and four-momentum transfer squared $Q^2$ are large), this formalism allows nuclear spin structure functions to be expressed as convolutions of bound nucleon structure functions and spin-dependent light-cone momentum distributions of nucleon in nuclei, which in this limit are independent of $Q^2$.

In practice, however, many of the experiments with polarized nuclear targets are performed at average $Q^2$ values of between $\sim 1$ and 10 GeV$^2$ \cite{5}. Some analyses of spin-dependent structure functions are made at even smaller $Q^2$: the generalized Gerasimov-Drell-Hearn (GDH) sum rule, for instance, interpolates between the deep inelastic region and the photoproduction limit, where it is given in terms of the nucleon magnetic moment \cite{14}. It is important, therefore, if one is to accurately describe nuclear structure function data at current experimental kinematics, and reliably extract neutron structure information from nuclear targets, that a framework exist within which one can compute nuclear structure...
functions at both high and low $Q^2$.

In a recent paper [15] we evaluated the effects of finite-$Q^2$ kinematics on the $g_1$ and $g_2$ spin structure functions of the deuteron, focusing in particular on the resonance region where the finite-$Q^2$ smearing had significant effects. The resonance region has received considerable interest recently in connection with the phenomenon of Bloom-Gilman duality [16], which relates structure functions in the resonance and DIS regions [17]. In particular, it was shown [15] that at finite $Q^2$ the simple $Q^2$-independent factorization of the convolution approximation breaks down, and the effective nucleon momentum distribution functions acquire an explicit dependence on the scale $Q^2$. In this work we extend this formalism to the case of inclusive scattering from the $^3\text{He}$ nucleus. Inclusive scattering from polarized $^3\text{He}$ in the resonance region was also considered in Ref. [18], although using a different formalism. The differences with our results appear in terms that are higher order in the bound nucleon momentum, which may be important in the low-$Q^2$ region.

In the following section we review the formalism of the nuclear impulse approximation, and outline the derivation of the nuclear hadronic tensor in the approximation of weak nuclear binding. In Sec. III we discuss the general properties of the nuclear spectral function, before turning to the specific case of the $^3\text{He}$ nucleus. The complete formulas for the $g_1$ and $g_2$ structure functions of nuclei in terms of the structure functions of bound nucleons, valid at finite values of $Q^2$, are presented in Sec. IV, which will be particularly useful for studying nuclear effects in the nucleon resonance region. Our equations generalize the Bjorken limit expressions used in earlier analyses, and are consistent with those for $Q^2 \to \infty$. In Sec. IV.B we illustrate the behavior of the spin-dependent nucleon (light-cone) momentum distributions away from the Bjorken limit. Numerical results for the $^3\text{He}$ structure functions are presented in Sec. IV.C, where we compare nuclear effects in both the resonance and deep inelastic regions using our full calculation with those based on various approximations. Finally, in Sec. V we summarize our results and outline future extensions of this work.

II. NUCLEAR SPIN-DEPENDENT STRUCTURE FUNCTIONS

This section outlines the derivation of the basic relations between the nuclear and nucleon hadronic tensors in the nuclear impulse approximation. Starting from a relativistic framework, we systematically apply the nonrelativistic or weak binding approximation for
the nucleon propagator, which enables the nuclear hadronic tensor to be written in terms of an off-shell nucleon truncated tensor and a nonrelativistic nuclear spectral function. While the general formalism is applicable to nuclei with arbitrary spin, we will focus here on the specific case of spin-1/2 targets such as $^3$He. Further details of the derivation can be found in the Appendices.

A. Hadronic tensor

To leading order in the electromagnetic coupling constant, the inclusive differential cross section can be written as a product of leptonic and hadronic tensors. The former describes the lepton–photon interaction, while the latter represents the sum of hadronic matrix elements of the electromagnetic current $J_{\mu}$ over all hadronic final states. From completeness of the final states, the hadronic tensor $W_{\mu\nu}^A$ of the nucleus can be expressed as the Fourier transform of the nuclear matrix element of the commutator of two electromagnetic currents:

$$W_{\mu\nu}^A(P_A, q, S) = \frac{1}{4\pi} \int d^4z \ e^{iq \cdot z} \langle P_A, S| [J_{\mu}(z), J_{\nu}(0)] |P_A, S\rangle , \quad (1)$$

where $q$ is the four-momentum transfer, $P_A$ is the momentum of the target nucleus, and $S$ is the target spin polarization axial-vector, normalized such that $S^2 = -1$ and $P_A \cdot S = 0$. For spin-dependent scattering the relevant component of $W_{\mu\nu}^A$ is antisymmetric in the indices $\mu, \nu$ and can be written in terms of two structure functions, $g_1^A$ and $g_2^A$:

$$W_{\mu\nu}^A(P_A, q, S) = \frac{M_A}{P_A \cdot q} \ i \epsilon_{\mu\nu\alpha\beta} q^\alpha \left[ S^\beta \ (g_1^A + g_2^A) - P_A^\beta \frac{S \cdot q}{P_A \cdot q} g_2^A \right] , \quad (2)$$

where $g_1^A$ and $g_2^A$ are Lorentz-invariant functions of the Bjorken variable $x_A = Q^2/2P \cdot q$ and the photon virtuality $Q^2$, with $M_A$ the nuclear mass. The states are normalized such that $\langle P_A, S|P_A', S'\rangle = 2E_{P_A}(2\pi)^3 \delta(P_A - P_A') \delta_{SS'}$, in which case the structure functions $g_{1,2}^A$ are dimensionless. Hermiticity of the electromagnetic current further ensures that the structure functions are real. The nucleon hadronic tensor is similar to that in Eq. (2).

Calculations of nuclear structure functions are usually framed in the context of the nuclear impulse approximation (IA), in which the virtual photon scatters incoherently from individual nucleons bound in the nucleus. Possible effects which go beyond the impulse approximation include final state interactions (FSI) between the recoiling nucleus and the produced hadronic state, meson exchange currents (MEC), and nuclear shadowing. Both
meson exchange currents and shadowing involve coherent, multiple scattering effects, which are generally restricted to small values of \( x, x < \sim 0.1 \) \[19\]. Also, since it is scalar, direct scattering from a pion in the nucleus does not contribute to spin-dependent structure functions (but can contribute of course to polarization asymmetries).

The effects of FSI and MEC have been considered for quasi-elastic (QE) scattering from \(^3\text{He}\) within a nonrelativistic Faddeev approach in Ref. \[20\]. Comparison with recent data from Jefferson Lab \[21\] found FSI effects to be important at low \( Q^2 (Q^2 \sim 0.05 - 0.2 \text{ GeV}^2)\), and gradually decreasing as \( Q^2 \) increases, bringing the IA calculations closer to the data. In this context we also mention the results of Ref. \[22\], where in a nonrelativistic Green’s function approach analyticity and unitarity requirements were used to argue the FSI effects to cancel in energy-integrated inclusive spin-averaged cross sections. Similar arguments also lead to partial cancellation of FSI effects in inclusive inelastic cross sections. Note that the corresponding FSI effects are significantly stronger for exclusive channels, in which the nucleon is detected in the final state \[22\]. From the existing approaches it is not clear, however, whether FSI effects can be neglected in spin-dependent inclusive cross sections in the resonance region and at higher energies. Computation of the FSI effects here will require extension of the formalism to include relativistic effects and couplings between different open channels — a problem which remains an important challenge.

Within the impulse approximation framework the nuclear hadronic tensor can be written as:

\[
W_A^{\mu\nu}(p_A, q, S) = \sum_{\tau=p,n} \int [dp] \text{Tr} \left[ \mathcal{A}^\tau(p, P_A, S) \tilde{W}^\tau_{\mu\nu}(p, q) \right],
\]

where the integration is performed over the bound nucleon four-momentum \( p \), and we use the shorthand notation \([dp] \equiv d^4p/(2\pi)^4\). Here, and in the following, the index \( \tau \) labels the nucleon isospin state and a sum is taken over protons \((\tau = p)\) and neutrons \((\tau = n)\). The truncated or off-shell nucleon tensor \( \tilde{W}^\tau_{\mu\nu}(p, q) \) describes the inclusive scattering of the virtual photon from an off-mass-shell nucleon, and is a matrix in Dirac space (see Appendix \[B\]). The Dirac matrix \( \mathcal{A}^\tau(p, P_A, S) \) is the imaginary part of the nucleon propagator in the nucleus \( A \) with momentum \( P_A \) and spin \( S \):

\[
\mathcal{A}^\tau_{\alpha\beta}(p, P_A, S) = \int dt \, d^3r \, e^{i(pot-\mathbf{p} \cdot \mathbf{r})} \langle P_A, S | \Psi_\beta(t, \mathbf{r}) \Psi_\alpha^\tau(0) | P_A, S \rangle,
\]

where \( \Psi_\alpha(t, \mathbf{r}) \) is the (relativistic) nucleon field operator, and \( \alpha, \beta \) are Dirac spinor indices. The trace “Tr” in Eq. \[3\] is taken in the nucleon Dirac space.
B. Weak binding approximation (WBA)

The expression for the nuclear tensor in Eq. (3) is covariant and can be evaluated in any frame. It will be convenient, however, to work in the target rest frame, in which the target momentum is \( P_A = (M_A, \mathbf{0}) \) and the spin vector \( S = (0, \mathbf{S}) \), and the momentum transfer to the target defines the \( z \)-axis, \( q = (q_0, \mathbf{0}_\perp, -|q|) \). If the nucleus can be approximated as a nonrelativistic system of weakly bound nucleons with four-momentum \( p \equiv (M + \varepsilon, \mathbf{p}) \), where \( M \) is the nucleon mass, then the nuclear hadronic tensor in Eq. (3) simplifies considerably. This necessarily involves neglecting antinucleon degrees of freedom, and corresponds to bound nucleons in the nucleus having small momentum and energy, \( |p|, |\varepsilon| \ll M \). We refer to this as the “weak binding approximation” (WBA).

To proceed, we perform a nonrelativistic reduction of all Lorentz–Dirac structures in the nucleon hadronic tensor. This can be done by relating the relativistic four-component nucleon field \( \Psi^\tau \) to the nonrelativistic two-component operator \( \psi^\tau \):

\[
\Psi^\tau(p, t) = e^{-iM t} \begin{pmatrix} Z \psi^\tau(p, t) \\ \frac{\mathbf{\sigma} \cdot \mathbf{p}}{2M} \psi^\tau(p, t) \end{pmatrix},
\]

which is valid to order \( p^2/M^2 \). The validity of Eq. (5) relies on the absence of strong fields in the nucleus comparable to the nucleon mass (see e.g. the discussion in the Appendix of Ref. [11]). The nucleon operators in Eq. (5) are taken in a mixed \((p, t)\) representation, \( \psi^\tau(p, t) = \int d^3r \exp(-i\mathbf{p} \cdot \mathbf{r}) \psi^\tau(r, t) \). The renormalization operator \( Z = 1 - p^2/8M^2 \) ensures that the charge (baryon number) is not renormalized when going to the nonrelativistic limit:

\[
\int d^3p \bar{\Psi}^\tau(p, 0) \gamma_0 \Psi^\tau(p, 0) = \int d^3p \bar{\psi}^\dagger^\tau(p, 0) \psi^\tau(p, 0).
\]

One can then define the nuclear spin-dependent spectral function \( \mathcal{P}^\tau \) of a nucleus in terms of the correlator of the nonrelativistic fields \( \psi^\tau \) as:

\[
\mathcal{P}_{\sigma\sigma'}^\tau(\varepsilon, \mathbf{p}, \mathbf{S}) = \int dt e^{-i\varepsilon t} \langle A, S | \psi^{\dagger\tau}_{\sigma'}(p, t) \psi^\tau_{\sigma}(p, 0) | A, S \rangle,
\]

where the expectation value is taken with respect to the nuclear ground state \( |A, S \rangle \), normalized to unity, with polarization \( \mathbf{S} \). The operator \( \psi^{\dagger\tau}_{\sigma}(p, t) \) \( (\psi^\tau_{\sigma}(p, t)) \) creates (annihilates) a nonrelativistic nucleon with isospin \( \tau \), momentum \( \mathbf{p} \) and polarization \( \sigma \), at time \( t \).

Substituting Eq. (4) into Eq. (3), the four-dimensional Dirac spinor matrices reduce to
nonrelativistic two-dimensional spinors (see Appendix C):

$$\frac{1}{M_A} \text{Tr} \left[ \hat{W}^\tau_{\mu\nu}(p, q) A^\tau(p, S) \right] = \frac{1}{M + \varepsilon} \text{tr} \left[ \hat{w}^\tau_{\mu\nu}(p, q) \mathcal{P}^\tau(\varepsilon, p, S) \right],$$

(8)

where “tr” is now a two-dimensional trace in Pauli matrix space. The off-shell truncated nucleon tensor \( \hat{w}^\tau_{\mu\nu}(p, q) \) in the vicinity of the mass-shell can be written in a similar way to the relativistic nucleon tensor in Eq. (2):

$$\hat{w}^\tau_{\mu\nu}(p, q) = \frac{M}{p \cdot q} i \varepsilon_{\mu
u\alpha\beta} q^\alpha \left[ \hat{S}^\beta (g_1^\tau + g_2^\tau) - p^\beta \hat{S} \cdot \frac{q}{p \cdot q} g_2^\tau \right],$$

(9)

where \( g_{1,2} \) are the structure functions of the proton or neutron with four-momentum \( p \). Note that these functions generally depend on 3 variables: \( Q^2, x' = Q^2 / 2 p \cdot q \) and \( p^2 \). The operator \( \hat{S} \) has a structure similar to that of the spin four-vector \((0, \sigma)\) boosted to a frame in which the nucleon has the (nonrelativistic) momentum \( p \):

$$\hat{S} = \left( \sigma \cdot \frac{p}{M}, \sigma + \frac{p (\sigma \cdot p)}{2M^2} \right).$$

(10)

Combining Eqs. (8) and (3) leads then to the relation between the nuclear and nucleon tensors:

$$\frac{1}{M_A} W^A_{\mu\nu}(P_A, q, S) = \sum_{\tau} \int \frac{[dp]}{M + \varepsilon} \text{tr} \left[ \hat{w}^\tau_{\mu\nu}(p, q) \mathcal{P}^\tau(\varepsilon, p, S) \right].$$

(11)

In the derivation of Eq. (8) all terms have been kept to order \( p^2 / M^2 \) and \( \varepsilon / M \), with higher order terms neglected. On the other hand, no assumption has been made about the scale \( Q^2 \), so that Eq. (8) holds for arbitrary values of \( Q^2 \), so long as the nuclear impulse approximation is valid. The relation between the nuclear and nucleon structure functions, which follow from Eq. (11), will be discussed in Sec. IV below.

III. NUCLEAR SPECTRAL FUNCTION

In this section we present a detailed discussion of the nuclear spectral function. After outlining its general properties for an arbitrary nucleus, we then focus on the specific case of \(^3\text{He}\).

A. General properties

The nuclear spectral function \( \mathcal{P}^\tau \) in Eq. (11) is a matrix in the nucleon spin space. The general spin structure of the spectral function can be obtained by expanding \( \mathcal{P}^\tau \) in terms of
the Pauli spin matrices and applying constraints from parity and time-reversal invariance \[ P_{\tau}(\varepsilon, p, S) = \frac{1}{2} \left( f_{0\tau} I + f_{1\tau} \sigma \cdot S + f_{2\tau} T_{ij} S_i \sigma_j \right) , \] (12)

where \( I \) is the 2 \( \times \) 2 identity matrix, \( T_{ij} = \hat{p}_i \hat{p}_j - \frac{1}{3} \delta_{ij} \) is a traceless symmetric tensor, with \( \hat{p}_i = p_i / |p| \) the \( i \)-th spatial component of the momentum, and a sum over repeated indices is implied.

From the hermiticity of the spectral function, the coefficients in Eq. (12) are real functions of the energy \( \varepsilon \) and momentum \( p \), \( f_{m\tau} = f_{m\tau}(\varepsilon, p) \), with \( m = 0, 1, 2 \). The function \( f_{0\tau} \) describes the spin-averaged spectral function, while \( f_{1\tau} \) and \( f_{2\tau} \) characterize the nucleon spin distributions in the nucleus. The function \( f_{0\tau}^p (f_{0\tau}^n) \) is normalized to the number of protons (neutrons) in the nucleus:

\[
\int [dp] \, \text{tr} \left[ P_{\tau}^{p(n)}(\varepsilon, p, S) \right] = \int [dp] \, f_{0\tau}^{p(n)} = Z (A - Z) ,
\] (13)

which follows from Eqs. (6) and (7). The usual nucleon “momentum distribution” in the nucleus is given by the integral of \( f_{0\tau}^\tau \) over \( \varepsilon \):

\[
n_{\tau}(p) = \int \frac{d\varepsilon}{2\pi} \, f_{0\tau}^{\tau}(\varepsilon, p) .
\] (14)

The integrated function \( f_{1\tau}^\tau \) determines the average nucleon polarization in the nucleus:

\[
\langle \sigma_z \rangle^\tau = \int [dp] \, f_{1\tau}^\tau ,
\] (15)

while \( f_{2\tau}^\tau \) is related to the tensor polarization:

\[
\langle T_{zi} \sigma_i \rangle^\tau = \frac{2}{9} \int [dp] \, f_{2\tau}^\tau ,
\] (16)

where we define the nuclear spin vector \( S \) to lie along the z-axis. Using the expansion of \( P_{\tau} \) in Eq. (12), the traces on the right-hand-side of Eq. (11) can be written in terms of the functions \( f_{1,2\tau} \):

\[
\text{tr} \left[ P_{\tau}(\varepsilon, p, S) \hat{S}_0 \right] = \left( f_{1\tau}^\tau + \frac{2}{3} f_{2\tau}^\tau \right) S \cdot v ,
\] (17a)

\[
\text{tr} \left[ P_{\tau}(\varepsilon, p, S) \hat{S}_i \right] = \left[ f_{1\tau}^\tau + \frac{1}{6} v^2 \left( f_{1\tau}^\tau + \frac{2}{3} f_{2\tau}^\tau \right) \right] S_i \\
+ \left[ f_{2\tau}^\tau + \frac{1}{2} v^2 \left( f_{1\tau}^\tau + \frac{2}{3} f_{2\tau}^\tau \right) \right] T_{ij} S_j ,
\] (17b)
where \( v \equiv p/M \) is the nucleon velocity.

Inserting a complete set of intermediate states in Eq. (7) and calculating the transition matrix elements between the ground and intermediate states allows the spectral function to be written as:

\[
P_{\sigma' \sigma}^\tau (\varepsilon, p) = \sum_f \psi_{f, \sigma}^\tau (p) \psi_{f, \sigma'}^\tau (p) \quad 2\pi \delta \left[ \varepsilon + M + E_{A-1}^f (p) - M_A \right],
\]

where the function \( \psi_{f, \sigma}^\tau (p) = \langle (A-1)_f, -p | \psi_{\sigma}^\tau (0) | A \rangle \) gives the probability amplitude to find a nucleon with isospin \( \tau \) and polarization \( \sigma \) in the nuclear ground state and the remaining \( A - 1 \) nucleons in a state with total momentum \( -p \), with the subscript \( f \) labeling all other quantum numbers. The energy of the residual system, including the kinetic energy, is denoted by \( E_{A-1}^f (p) = M_{A-1}^f + p^2 / 2M_{A-1}^f \), where \( M_{A-1}^f \) is the mass of remaining \( (A-1)_f \) nuclear system.

Note that Eq. (18) is written in the target rest frame and defines the nuclear spectral function as a function of nucleon energy \( \varepsilon = p_0 - M \) and momentum \( p \). However, in practice the spectral function is usually considered as a function of the nucleon separation energy \( E \). The spectral function in this case, denoted by \( P(E, p) \), is given by Eq. (18) with the energy conserving \( \delta \)-function replaced by \( \delta (E - E^f) \), where

\[
E^f = M + M_{A-1}^f - M_A
\]

is the energy needed to separate a nucleon from the nucleus \( A \), leaving the residual nuclear system in a state \( (A-1)_f \). For a given (positive) separation energy \( E \), the relation between the two spectral functions \( P(\varepsilon, p) \) and \( P(E, p) \) can be found by defining

\[
\varepsilon (E, p) = -E - \frac{p^2}{2(M_A - M + E)},
\]

as suggested by Eq. (19) and the argument of the energy \( \delta \)-function in Eq. (18). From Eqs. (18) and (20) one then finds the relation

\[
P (E, p) = \left| \frac{\partial \varepsilon (E, p)}{\partial E} \right| P(\varepsilon (E, p), p),
\]

where the derivative factor ensures that the energy integrals of the two spectral functions are equal, i.e., \( \int dE P(E, p) = \int d\varepsilon P(\varepsilon, p) \), and the normalizations in Eqs. (13)–(16) remain valid. In the following we shall take the functions \( f_{m}^{\tau} \) to be functions of \( E \) and \( p \).
For the deuterium nucleus, discussed in Ref. [15], the intermediate states are exhausted by a single proton or neutron. In this case the spectral function is expressed entirely in terms of the deuteron wave function. For the three-nucleon system, however, the calculation of the spectral function is rather more complicated, as we discuss next.

B. $^3$He spectral function

For a $^3$He nucleus, the proton spectral function has two contributions: from the bound ($pn$) intermediate state corresponding to a deuteron, where the separation energy is $E = \varepsilon_d - \varepsilon_{^3}\text{He}$, with $\varepsilon_d = -2.22$ MeV and $\varepsilon_{^3}\text{He} = -7.72$ MeV the deuterium and $^3$He binding energies; and from the ($pn$) continuum scattering states, with energy $E$. The proton functions $f_{m}^{p}$ can be written as

$$f_{m}^{p}(E, p) = f_{m}^{p(d)}(p)\delta(E + \varepsilon_{^3}\text{He} - \varepsilon_d) + f_{m}^{p(\text{cont})}(E, p).$$

(22)

The neutron spectral function, on the other hand, has only the ($pp$) continuum contribution:

$$f_{m}^{n}(E, p) = f_{m}^{n(\text{cont})}(E, p).$$

(23)

From these spectral functions one can compute the effective polarization and tensor polarization of the proton and neutron in the $^3$He nucleus using Eqs. (15) and (16).

More familiar notation expresses the polarizations in terms of probabilities to find the proton and neutron in various orbital states. For $^3$He these correspond to the dominant space-symmetric (spin-isospin antisymmetric) $S$-state ($P_S$); a small (1–2%) admixture of the $L = 0$ mixed-symmetric $S'$ state ($P_{S'}$), which reflects spin-isospin correlations in the nuclear force; and the $L = 2$ $D$-state ($P_D$), generated by the tensor force, in which the nucleon spins are aligned antiparallel to the nuclear spin projection, with probability $\sim 10\%$ [23]. The probability of the $L = 1$ $P$-state is very small ($< 1\%$) and is not considered here. The effective proton and neutron polarizations in $^3$He can then be written as [23]:

$$\langle \sigma_z \rangle^p = -\frac{2}{3} (P_D - P_{S'}),$$

(24a)

$$\langle \sigma_z \rangle^n = P_S - \frac{1}{3} (P_D - P_{S'}),$$

(24b)

where $\langle \sigma_z \rangle^p$ is defined as the total proton polarization (i.e., the sum of the two protons’ polarizations).
In the present analysis we use the spectral functions from Refs. [24, 25] (denoted by “KPSV”) and Ref. [9] (denoted by “SS”), which provide a representative sample of models of the $^3$He nucleus. The KPSV spectral function [24] is obtained using a $^3$He wave function calculated in a variational method with a pair-correlated hyperspherical harmonic basis [26], for various $NN$ potentials, including a three-body force and a Coulomb interaction between the two protons. In our numerical calculations we use the KPSV spectral function with the more recent AV18 $NN$ potential and the $NNN$ Urbana IX interaction [25]. The SS spectral function [9], on the other hand, is obtained by solving the Faddeev equation with the Paris $NN$ potential [27] for the ground state wave function, and constructing its projection onto the deuteron and two-body continuum states.

The neutron polarization from the KPSV (SS) spectral function is $\langle \sigma_z \rangle^n = 0.86$ (0.89). The deuteron pole contribution to the proton polarization is large and negative, $\langle \sigma_z \rangle^{p(d)} = -0.445$ ($-0.453$) for the KPSV (SS) model, but is canceled by the positive continuum contribution, $\langle \sigma_z \rangle^{p(\text{cont})} = 0.386$ (0.409), leaving a very small negative total proton polarization, $\langle \sigma_z \rangle^p = -0.059$ ($-0.044$) for the KPSV (SS) spectral function. Note that the deuteron pole contribution to the spin-averaged spectral function is rather large, providing some 67% (68%) of its normalization in the KPSV (SS) model. The corresponding tensor polarizations of the neutron for the two models are $\langle T_{zi}\sigma_i \rangle^n = 0.045$ [25] and 0.038 [9], respectively. The proton tensor polarizations, $\langle T_{zi}\sigma_i \rangle^p = -0.226$ [25] and $-0.235$ [9], arise mainly from the deuteron pole, with the continuum contributions being $< 2\%$ and $< 10\%$ for the KPSV and SS spectral functions, respectively.

IV. POLARIZED $^3$HE STRUCTURE FUNCTIONS

In this section we apply the formalism developed in Sec. III together with the spectral functions described in Sec. III to compute the $g_1$ and $g_2$ structure functions of $^3$He. We present the complete finite-$Q^2$ formulas for the nuclear structure functions, and also formulate the results in terms of the familiar light-cone convolution formulas. We conclude by performing a detailed analysis of the nuclear effects on the nucleon structure functions in both the resonance and DIS regions.
A. Finite-$Q^2$ convolution

Using the nuclear hadronic tensor in Eq. (11) we can write the nuclear structure functions $g_1^A$ and $g_2^A$ as convolutions of the (off-shell) nucleon spin structure functions and the nuclear spectral function. The nuclear and nucleon structure functions can be related by considering the helicity structure functions in Eqs. (A2a) and (A2c) of Appendix A, and projecting onto appropriate helicity states. The final result for the nuclear $g_1^A$ and $g_2^A$ structure functions at finite $Q^2$ can be summarized as:

$$xg_a^A(x, Q^2) = \int [dp] D_{ab}^\tau(\epsilon, p, \gamma) x'g_b^A(x', Q^2, p^2),$$

where $x' = Q^2/2p \cdot q = x/[1 + (\varepsilon + \gamma p_z)/M]$ is the Bjorken variable for the bound nucleon, $p^2 = (M + \varepsilon)^2 - p^2$ is the off-shell nucleon virtuality, and $a, b = 1, 2$ (in order to simplify notation, summation over repeated indices $b$ and $\tau$ is assumed). The energy-momentum distributions $D_{ab}$ are given by:

$$D_{11} = f_1 + \frac{3 - \gamma^2}{6\gamma^2} (3p_z^2 - 1) f_2 + \frac{v p_z}{\gamma} \left( f_1 + \frac{2}{3} f_2 \right) + v^2 \frac{(3 - \gamma^2)p_z^2 - 1 - \gamma^2}{12\gamma^2} (3f_1 - f_2),$$

$$D_{12} = (\gamma^2 - 1) \left[ -\frac{3p_z^2 - 1}{2\gamma^2} f_2 + \frac{v p_z}{\gamma} \left( f_1 + \left( \frac{5}{2} p_z^2 - \frac{5}{6}\gamma^2 \right) f_2 \right) - v^2 \left( \frac{1 + p_z^2(4\gamma^2 - 3)}{4\gamma^2} f_1 + \frac{5 + 18\gamma^4 - 5p_z^2(3 + 2\gamma^2)}{12\gamma^2} f_2 \right) \right],$$

$$D_{21} = -\frac{3p_z^2 - 1}{2\gamma^2} f_2 - \frac{v p_z}{\gamma} \left( f_1 + \frac{2}{3} f_2 \right) - v^2 \frac{3p_z^2 - 1}{12\gamma^2} (3f_1 - f_2),$$

$$D_{22} = f_1 + \frac{2\gamma^2 - 3}{6\gamma^2} (3p_z^2 - 1) f_2 + \frac{v p_z}{\gamma} \left[ (1 - \gamma^2)f_1 + \left( -\frac{5}{6} + \frac{1}{3}\gamma^2 + p_z^2(\frac{3}{2} - \gamma^2) \right) f_2 \right] + v^2 \left[ \frac{p_z^2(3 - 6\gamma^2 + 4\gamma^4) - 1 - 2\gamma^2}{4\gamma^2} f_1 + \frac{5 - 2\gamma^2(1 + 3p_z^2) + 4p_z^2\gamma^4}{12\gamma^2} (3p_z^2 - 1) f_2 \right],$$

where $v = |v| = |p|/M$, $\gamma = |q|/q_0 = \sqrt{1 + (2Mx)^2/Q^2}$ is the “velocity” of the virtual photon, and for clarity the isospin indices have been suppressed. The expressions in Eqs. (26) are derived in the weak binding approximation, making use of the fact that characteristic values of the nucleon velocity $v$ are small, which allows the kinematical factors to be expanded up to order $v^2$.

Note that while in the Bjorken limit the nucleon distributions $D_{ab}$ are independent of $Q^2$, at finite $Q^2$ the scale dependence enters explicitly through the parameter $\gamma$. Furthermore, nuclear effects lead to nonzero off-diagonal distributions $D_{12}$ and $D_{21}$, giving rise to the mixing of different spin structure functions in the convolution integral (25). In the limit of
high $Q^2$ the parameter $\gamma \to 1$ and the distributions \eqref{eq:D2} simplify considerably. In particular, in this limit the function $D_{12} \to 0$, and the convolution formula for $g_1^A$ becomes diagonal. On the other hand, the mixing effect for $g_2^A$ persists even in the $\gamma \to 1$ limit \cite{8,11}. The nuclear $g_1^A$ and $g_2^A$ structure functions can then be computed from Eqs. \eqref{eq:D2} at any value of $Q^2$.

The relation between the nuclear and nucleon spin structure functions for polarized $^3$He has also been discussed in several previous studies \cite{8,9,12,18}. The implementation of the impulse approximation in Refs. \cite{9,12} is different from the present approach, leading to different distribution functions in the nuclear convolution. In particular, in Ref. \cite{9,12} the struck nucleon was assumed to be on-mass-shell, while in our approach the nucleon is by definition off its mass-shell. This leads to different definitions of the Bjorken variable of the struck nucleon $x'$, resulting in significant differences in the numerical analysis. The convolution equation Eq. \eqref{eq:conv} and the corresponding distribution in Eq. \eqref{eq:D2} for $g_1$ are similar to those in Ref. \cite{8,18} to leading order in $\nu$ and in the Bjorken limit $\gamma \to 1$. However, the higher order terms in $\nu$ and $\gamma^2 - 1$ are different.

B. Light-cone distributions

At large $Q^2$ deep inelastic nuclear structure functions are usually written as simple, one-dimensional convolutions of nucleon structure functions and nucleon light-cone momentum distribution functions in the nucleus. We can generalize this convolution to finite $Q^2$ by using Eq. \eqref{eq:conv} and defining effective light-cone momentum distributions $f_{\tau a b}$ as integrals of the functions $D_{\tau a b}$ in Eq. \eqref{eq:D2}:

$$f_{\tau a b}(y, \gamma) = \int [d\varepsilon] D_{\tau a b}(\varepsilon, \mathbf{p}, \gamma) \delta \left( y - 1 - \frac{\varepsilon + \gamma p_z}{M} \right), \quad (27)$$

where the variable $y = (p_0 + \gamma p_z)/M = 1 + (\varepsilon + \gamma p_z)/M = x/x'$ in the Bjorken ($\gamma \to 1$) limit is the light-cone fraction of the nucleus carried by the interacting nucleon. The nuclear structure functions $g_1^A$ and $g_2^A$ can then be written as \cite{15}

$$x g_\alpha^A(x, Q^2) = \int_x^{M \Lambda^2/M} dy \ f_{\tau a b}(y, \gamma) \ x' g_\beta^A(x', Q^2). \quad (28)$$

For inelastic scattering the lower limit of the $y$-integration is $y_{\text{min}} = x/x_{\text{th}}$, where $x_{\text{th}} = [1 + (2Mm_\pi + m_\pi^2)/Q^2]^{-1}$ corresponds to the inelastic threshold for which the invariant mass
of the final state \( W \geq M + m_\pi \). In Eq. (28) we have also suppressed the dependence of the nucleon structure functions on the nucleon virtuality \( p^2 \). We justify this in this analysis since the aim here is to study the role of the finite-\( Q^2 \) smearing on structure functions, the effects of which should be largely independent of details of the nucleon structure function input.

In Figs. 1 and 2 we show the neutron and proton light-cone momentum distribution functions \( f_{nab}(y, \gamma) \) in \(^3\)He, respectively, calculated from the SS spectral function [9], for several values of \( \gamma \) (the results with the KPSV spectral function [24] are similar). For details of the integration over the energy \( \varepsilon \) and momentum \( p \) in Eq. (27) see Ref. [3], Appendix A. The results for \( \gamma = 1 \) correspond to the distributions in the Bjorken limit.

For the neutron, the diagonal distributions \( f_{11}^n \) and \( f_{22}^n \) in Figs. 1(a) and (d) peak around \( y = 1 \), and drop rapidly with increasing \( |y - 1| \). The finite-\( Q^2 \) effects render the distributions broader and the peak smaller with increasing \( \gamma \). The magnitude of the non-diagonal distributions \( f_{12}^n \) and \( f_{21}^n \) in Figs. 1(b) and (c) is considerably smaller than that of the diagonal distributions. In the Bjorken limit the function \( f_{12}^n \) in fact vanishes identically, but becomes finite for \( \gamma > 1 \), leading to nonzero contributions from the nucleon \( g_2^\tau \) structure function to \( g_2^3\)He. The magnitude of \( f_{12}^p \) grows with increasing \( \gamma \), reaching at its peak about 6\% of the diagonal \( f_{11}^n \) for \( \gamma = 2 \). The function \( f_{21}^n \), on the other hand, is finite for all \( \gamma \), so that the nucleon \( g_2^\tau \) structure function contributes to the nuclear \( g_2^3\)He even in the Bjorken limit [18].

For the proton, the \( f_{11}^p \) distribution in Fig. 2(a) is negative in the vicinity of \( y = 1 \) and very small in magnitude compared with the corresponding neutron distribution. The small proton distribution results from a sizable cancellation of the (negative) deuteron pole and (positive) \((pn)\) continuum contributions, as discussed in Sec. III.B. The \( f_{22}^p \) distribution is qualitatively similar, with a peak that is negative at \( y = 1 \) and somewhat more pronounced than for \( f_{11}^p \) for \( \gamma = 1 \), but with greater suppression for larger \( \gamma \). The non-diagonal \( f_{12}^p \) distribution for the proton is similar in magnitude to that for the neutron, vanishing for \( \gamma = 1 \), while the \( f_{21}^p \) distribution remains finite for all \( \gamma \).

The strong \( \gamma \) dependence of the light-cone momentum distribution functions will have important consequences for quasi-elastic scattering, which is given by Eq. (28) with the nucleon structure functions expressed in terms of the elastic nucleon form factors,

\[
 g_1^{\tau(\text{el})}(x, Q^2) = \frac{1}{2} \frac{G_M^\tau(Q^2)(G_E^\tau(Q^2) + \eta G_M^\tau(Q^2))}{1 + \eta} \delta(x - 1), \tag{29a}
\]
FIG. 1: (Color online) Neutron light-cone momentum distribution functions $f_{a|b}^{n}(y, \gamma)$ in $^3\text{He}$, for $\gamma = 1$ (Bjorken limit), 1.5 and 2.
FIG. 2: (Color online) Proton light-cone momentum distribution functions $f_{ab}^p(y, \gamma)$ in $^3$He, for $\gamma = 1$ (Bjorken limit), 1.5 and 2.
\begin{equation}
\tau^{(\text{el})}(x, Q^2) = \frac{1}{2} \frac{\eta \ G^r_M(Q^2) (G^r_E(Q^2) - G^r_M(Q^2))}{1 + \eta} \delta(x - 1),
\end{equation}

where \( \eta = Q^2/4M^2 \), and \( G^r_E \) and \( G^r_M \) are the Sachs electric and magnetic form factors, respectively. The presence of the \( \delta \)-functions in Eqs. \( 29 \) means that the QE contributions to the structure functions are given by products of the \( (Q^2\text{-dependent}) \) elastic form factors and the \( (x\text{- and } \gamma\text{-dependent}) \) light-cone distributions \( f_{ab}^r \). For the dominant neutron \( f_{11}^n \) distribution, for instance, a factor of two difference at the QE peak is evident between the distributions for \( \gamma = 1 \) and \( \gamma = 2 \) (which corresponds to \( Q^2 \sim 1 \text{ GeV}^2 \)). A more detailed discussion of QE scattering in the present formalism will be presented elsewhere \[28\]; in the remainder of this paper we shall focus on inelastic contributions only.

The small overall proton polarization means that the \(^3\text{He}\) nucleus can, to a good approximation, be used as an effective neutron target, although for quantitative computations the proton contribution needs to be accounted for. If one further neglects Fermi motion and binding so that the \( y \) dependence of the distributions \( f_{ab}^r \) is approximated by \( \delta(y - 1) \), and in addition omits the non-diagonal terms \( a \neq b \), then the \(^3\text{He}\) structure functions can be written as simple sums of proton and neutron structure functions weighted by the effective nucleon polarizations in Eqs. \( 24 \):

\begin{equation}
xg_{a}^{^3\text{He}}(x, Q^2) = \langle \sigma_z \rangle^p xg_{a}^p(x, Q^2) + \langle \sigma_z \rangle^n xg_{a}^n(x, Q^2),
\end{equation}

for \( a = 1, 2 \). This simple \textit{ansatz} is often used in experimental analyses to describe spin-dependent \(^3\text{He}\) structure functions. In the next section we test the accuracy of this \textit{ansatz} by comparing the effects of nuclear smearing using the full results in Eqs. \( 25 \) and \( 26 \) with this and other approximations, in both the resonance and deep inelastic regions.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{The calculated \( xg_1 \) and \( xg_2 \) structure functions for \(^3\text{He}\) are shown in Fig. 3 using the MAID parameterization \[29\] for the input free proton and neutron structure functions at \( Q^2 = 1 \text{ GeV}^2 \), and the KPSV \(^3\text{He}\) spectral function \[25\]. The proton contribution to \( xg_{1}^{^3\text{He}} \) in Fig. 3(a) (solid curve) is considerably smaller in magnitude and of opposite sign than the free proton structure function (dotted), reflecting the small negative proton polarization in the \(^3\text{He}\) nucleus, as well as partial cancellation of the deuteron pole (dashed) and \( pn \) continuum contributions (dot-dashed).}
\end{figure}
FIG. 3: (Color online) (a) Proton contribution to $xg_1^3\text{He}$ (solid), including the deuteron pole (dashed) and continuum (dot-dashed) contributions, compared with the free proton structure function (dotted). (b) Neutron contribution to $xg_1^3\text{He}$ (dashed), compared with the free proton structure function (dotted), and the total $xg_1^3\text{He}$ (solid). The MAID parameterization of the free $xg_1^N[29]$ is used at $Q^2 = 1 \text{ GeV}^2$, with the KPSV $^3\text{He}$ spectral function [25]. The corresponding proton and neutron contributions to $xg_2^3\text{He}$ are shown in (c) and (d).

The neutron contribution to $xg_1^3\text{He}$ in Fig. 3(b) (dashed) shows significant nuclear effects which smear the resonance peaks in the free neutron $xg_1$ (dotted) and result in a much less pronounced resonance structure, especially in the $\Delta(1232)$ region at large $x$. The similarity of the total $xg_1^3\text{He}$ (solid) and the neutron contribution (which arises only from $pp$ continuum states) reflects the relatively small size of the proton contribution to the $^3\text{He}$ structure function.

Similar findings are seen for the proton and neutron contributions to the $xg_2^3\text{He}$ structure function in Figs. 3(c) and (d), respectively, which are generally of opposite sign to the $xg_1$
results.

The effects of the smearing of the neutron structure functions bound in $^3$He clearly has dramatic consequences for the comparison of free and bound structure functions. It considerably complicates, for instance, the extraction of free neutron structure functions from $^3$He data [31]. This is in contrast to the effect of smearing in the deep inelastic region, where the structure functions are smooth, and the differences between the neutron and $^3$He $xg_{1,2}$ are much less pronounced. This is illustrated in Fig. 4, where the free neutron and the total $^3$He $xg_1$ and $xg_2$ structure functions are shown in panels (a) and (b), respectively. Here the resonance region is parameterized by the MAID model [29], while the deep inelastic curve is given by the leading twist fit from Ref. [30], both at $Q^2 = 2$ GeV$^2$.

The preceding $^3$He structure functions, in Figs. 3 and 4, have been calculated using the KPSV spectral function [24, 25]. To determine the nuclear spectral function dependence of our finite-$Q^2$ smearing, we compare in Figs. 5(a) and (b) the $xg_{1}^{^3\text{He}}$ and $xg_{2}^{^3\text{He}}$ structure functions, respectively, for the spectral functions of the KPSV [25] (solid) and SS [9] (dashed) models. The free neutron $xg_{1}^{n}$ and $xg_{2}^{n}$ structure functions from the MAID parameterization [29] (dotted) are shown at $Q^2 = 1$ GeV$^2$ for comparison.

Both nuclear models show similarly striking differences between the free neutron and $^3$He structure functions, demonstrating that the qualitative features of the smearing are model-
FIG. 5: (Color online) Dependence of the (a) $x g_1$ and (b) $x g_2$ structure functions of $^3$He on the nuclear spectral function, for the KPSV [25] (dashed) and SS [9] (solid) models, at $Q^2 = 1$ GeV$^2$. The free neutron structure functions from the MAID [29] parameterization are also shown for comparison (dotted).

The free neutron structure functions from the MAID parameterization are also shown. The differences between the two nuclear models are very small, with about 7% stronger smearing at the $\Delta(1232)$ resonance peak for the KPSV spectral function [25], which is due mainly to the slightly larger neutron polarization than in the SS model [9]. The nuclear model dependence in the deep inelastic region, where the structure functions are smooth, is even smaller.

To demonstrate the effects of the finite-$Q^2$ smearing, we compare in Figs. 6(a) and (b) the $x g_1^{^3\text{He}}$ and $x g_2^{^3\text{He}}$ structure functions calculated from the full expressions in Eqs. (25) and (26) (solid), using the MAID parameterization [29] for the free neutron $x g_1^n$, at $Q^2 = 1$ GeV$^2$, with the results of taking the Bjorken limit ($\gamma \to 1$) in the distributions $D_{ab}$ (dashed). The finite-$Q^2$ results display greater smearing compared with the $\gamma = 1$ case, with some 30% additional suppression at the $\Delta(1232)$ resonance peak.

Also shown are the $^3$He structure functions computed using the effective polarizations in Eq. (30). Not surprisingly, since there is no smearing of the nucleon structure functions here, the peaks in $x g_1^{^3\text{He}}$ for the prominent resonances are only slightly reduced from those in the neutron, suggesting that this ansatz provides a poor approximation to the full result for the $x$ dependence at finite $Q^2$. A similar observation was also made in Ref. [18].

This approximation works better at higher $Q^2$ for leading twist structure functions, where
FIG. 6: (Color online) Comparison of the full calculation of the (a) $xg_1^{^3\text{He}}$ and (b) $xg_2^{^3\text{He}}$ structure functions (solid), using the KPSV spectral function [25], with the Bjorken limit results (dashed), and with the effective polarizations approximation (dot-dashed), for the MAID [29] parameterization at $Q^2 = 1 \text{ GeV}^2$. Panels (c) & (d) are as in (a) & (b), but for the BB [30] parameterization at $Q^2 = 2 \text{ GeV}^2$. The free neutron structure functions are shown for comparison (dotted).

The effects of smearing are not as severe. This is illustrated in Figs. 6(c) and (d) for $xg_1^{^3\text{He}}$ and $xg_2^{^3\text{He}}$ computed with the BB leading twist parameterization [30] at $Q^2 = 2 \text{ GeV}^2$. In this case the effective polarization ansatz overestimates the nuclear effects in $^3\text{He}$ at intermediate $x$ ($0.3 \lesssim x \lesssim 0.6$) by as much as 30% compared with the free neutron $g_{1,2}^n$. On the other hand, applying the $Q^2$-dependent smearing or that in the Bjorken limit results in a smaller difference, around 10 – 20% at $x \sim 0.5$. With the ever increasing precision of nuclear structure function data in modern experiments [5], it is thus highly questionable that either effective polarizations or Bjorken limit convolution are sufficiently reliably methods to account for nuclear corrections at $Q^2 \sim \text{few-GeV}^2$ scales.
V. CONCLUSION

The $^3$He nucleus has for some time been used as an effective neutron target in spin-dependent deep inelastic scattering experiments [5]. This is justified by the small proton polarization in polarized $^3$He, as our analysis using several realistic spectral functions [9, 24, 25] has confirmed. While most previous theoretical analyses have focused on nuclear effects in the DIS region at large values of the final state invariant mass $W$ and $Q^2$, the new high-precision data in the relatively unexplored nucleon resonance region at intermediate $Q^2 \sim 1 \text{ GeV}^2$ has motivated a fresh look at nuclear effects which explicitly take into account kinematical $Q^2$ corrections. The main purpose of the present work has been to examine in detail the role of the nuclear corrections in polarized $^3$He at finite values of $Q^2$, with emphasis on the resonance region.

Using the weak binding approximation, in which the lepton–nucleon scattering amplitude is expanded to order $p^2$ and $\varepsilon$ in the nucleon momentum and energy, we derive expressions for the nuclear $g_{1,2}^\text{He}$ structure functions in terms of the nucleon $g_{1,2}^\tau$ structure functions, valid at all $Q^2$. Unlike earlier analyses using convolution formulas with $Q^2$-independent nucleon light-cone momentum distributions, we find that inclusion of kinematical $Q^2$ corrections breaks this simple factorization, giving rise to effective momentum distribution functions which explicitly depend on the virtual photon velocity $\gamma = |\mathbf{q}|/q_0$.

Our results show the smearing effects of the nucleon light-cone momentum distributions are significantly more dramatic in the nucleon resonance region than in the deep inelastic region, consistent with the earlier findings of Ref. [18], resulting in much less pronounced resonance structure in $g_{1,2}^\text{He}$, especially in the $\Delta(1232)$ region. This poses a greater challenge for the extraction of the free neutron structure functions from nuclear data in the resonance region than in DIS kinematics, where the differences between the neutron and $^3$He structure functions are relatively small.

The main effect that we find in this analysis is the broadening of the peak in the nucleon light-cone momentum distribution functions with increasing $\gamma$, leading to additional suppression of the nuclear structure functions around the resonance peaks at finite $Q^2$ relative to the Bjorken limit results. For $Q^2 = 1 \text{ GeV}^2$, for example, the finite-$Q^2$ results are some 30% smaller in magnitude at the $\Delta(1232)$ peak compared with the $\gamma = 1$ case.

Furthermore, the method of effective polarizations, which involves no smearing at all,
results in only $\sim 10 - 15\%$ suppression of the $^{3}\text{He}$ structure functions compared with the neutron. This suggests that the effective polarization ansatz provides a rather poor approximation to the $g_{1/2}^{^{3}\text{He}}$ structure functions at finite $Q^{2}$. With the high precision of polarized $^{3}\text{He}$ structure function data in recent and upcoming experiments \cite{21, 32}, it is important therefore to account for the correct $Q^{2}$ dependence when analyzing data at moderate $Q^{2}$ and $W$ kinematics.

The formalism presented here provides the necessary framework in which the finite-$Q^{2}$ corrections can be quantified within the impulse approximation. The analysis can be extended by considering additional effects beyond the impulse approximation, such as multiple scattering, or final state interactions, which may be more relevant in quantitative analyses of data at low $Q^{2}$ \cite{21, 32}. Even within the impulse approximation, the possible modification of the intrinsic structure of the nucleon in the nuclear medium, which in our framework is represented by the nucleon off-mass-shell dependence of the bound nucleon structure functions, should be taken into account. While currently not very well constrained, such effects can be investigated using the methods described for example in Refs. \cite{3, 4, 10, 11, 33, 34, 35, 36}.

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**APPENDIX A: HELICITY AMPLITUDES AND STRUCTURE FUNCTIONS**

This appendix summarizes the relations between structure functions and virtual photon helicity amplitudes, which are useful for extracting specific structure functions from the hadronic tensor. For completeness, we consider both the spin-dependent $g_{1}$ and $g_{2}$ and spin-averaged $F_{1}$ and $F_{2}$ structure functions.

Projecting the hadronic tensor $W^{\mu\nu}$ of a target with spin $S$ onto states with definite
photon polarizations, the virtual photon helicity amplitudes can be written:

\[ W^{(h,h')} (S) = e_{\mu}^{(h)}* W^{\mu \nu} (S) e_{\nu}^{(h')} , \]  

(A1)

where the polarization vector \( e_{\mu}^{(h)} \) describes a virtual photon with helicity \( h \). In a reference frame in which the momentum transfer is \( q = (q_0, 0_\perp, -|q|) \), the polarization vectors are given by \( e^{(\pm 1)} = (0, 1, \pm i, 0)/\sqrt{2} \) for right- \( (h = +1) \) and left- \( (h = -1) \) polarized photons, and \( e^{(0)} = (q_z, 0_\perp, q_0)/Q \) for longitudinally polarized photons, where \( Q = (Q^2)^{1/2} \).

Evaluating the amplitudes \( W^{(h,h')} \) for specific helicities \( h,h' \), one finds:

\[ W^{(\pm 1,\pm 1)} = F_1 - S_z \left[ g_1 + (1 - \gamma^2) g_2 \right] , \]  

(A2a)

\[ W^{(0,0)} = F_L = \frac{\gamma^2 F_2}{2x} - F_1 , \]  

(A2b)

\[ W^{(0,\pm 1)} = -\frac{Q}{\sqrt{2}q_0} (S_x \pm i S_y)(g_1 + g_2) , \]  

(A2c)

where \( \gamma = |q|/q_0 \). We also note that for transversely polarized photons the off-diagonal terms vanish, \( W^{(1,-1)} = W^{(-1,1)} = 0 \), and the LT-interference terms for the right- and left-polarized photons are related as \( W^{(\pm 1,0)} = -W^{(0,\pm 1)} \).

One can further define the structure functions \( w_{3/2} \equiv W^{(1,1)}(S_z = +1) \) and \( w_{1/2} \equiv W^{(1,1)}(S_z = -1) \), which correspond to the projections of the total photon–nucleon spin in the photon momentum direction equal to 3/2 and 1/2, respectively. Using the inequality \( |w_{1/2} - w_{3/2}| \leq w_{1/2} + w_{3/2} \), together with the Schwarz inequality for the off-diagonal helicity amplitude, \( |W^{(0,1)}|^2 \leq W^{(0,0)} W^{(1,1)} \), leads then to the following constraints on the \( g_1 \) and \( g_2 \) structure functions:

\[ g_1 + (1 - \gamma^2) g_2 \leq F_1 , \]  

(A3)

\[ (\gamma^2 - 1)(g_1 + g_2)^2 \leq 2RF_1^2 \]  

(A4)

where \( R = F_L/F_1 \) is the ratio of longitudinal to transverse structure functions for unpolarized scattering.

**APPENDIX B: OFF-SHELL NUCLEON ELECTROMAGNETIC TENSOR**

Here we present a detailed derivation of the most general structure of the truncated electromagnetic tensor \( \hat{W}_{\mu \nu}(p,q) \) for an off-mass-shell nucleon, representing the forward
Compton scattering of a virtual photon (momentum $q$) from a nucleon (momentum $q'$), with the nucleon legs amputated. For earlier discussions of the truncated hadronic tensor see Refs. [33, 35] and [10, 11] for unpolarized and polarized structure functions, respectively. The hadronic tensor for an on-shell nucleon ($p^2 = M^2$) can be written in terms of $\hat{W}_{\mu\nu}$ as:

$$W_{\mu\nu}(p, q, S) = \frac{1}{2} \text{Tr} \left[ (\not{p} + M)(1 + \gamma_5 \not{S}) \hat{W}_{\mu\nu}(p, q) \right], \quad (B1)$$

where $S$ is the nucleon spin four-vector, orthogonal to the nucleon four-momentum $p$, $S \cdot p = 0$.

The number of independent structure functions (Lorentz–Dirac structures) which describe $\hat{W}_{\mu\nu}$ is determined by the requirements of the time-reversal ($T$) and parity ($P$) invariance of the electromagnetic interaction, and hermiticity ($H$) of the electromagnetic current operator, $J_\mu(x) = J_\mu^\dagger(x)$. These can be summarized as [10]:

$$\hat{W}_{\mu\nu}(p, q) \overset{T}{=} \left( T \hat{W}_{\mu\nu}(p', q') T^\dagger \right)^* , \quad (B2a)$$
$$\hat{W}_{\mu\nu}(p, q) \overset{P}{=} \mathcal{P} \hat{W}_{\mu\nu}(p', q') \mathcal{P}^\dagger , \quad (B2b)$$
$$\hat{W}_{\mu\nu}(p, q) \overset{H}{=} \gamma_0 \hat{W}_{\nu\mu}(p, q) \gamma_0 , \quad (B2c)$$

where $p'^\mu = p_\mu = (p_0, -\vec{p})$, $q'^\mu = (q_0, -\vec{q})$, and $T$ and $\mathcal{P}$ are time-reversal and parity operators, respectively. In the Dirac representation they are given in terms of the Dirac matrices as $T = -i\gamma_5 \mathcal{C}$ and $\mathcal{P} = \gamma_0$, where $\mathcal{C} = i\gamma_0 \gamma^2$ is the charge conjugation operator. The asterisk $^*$ in Eq. (B2a) denotes complex conjugation.

For an on-shell nucleon the requirements (B2), together with current conservation and the Dirac equation, result in 2 independent Lorentz structures for the symmetric electromagnetic tensor, $W_{\{\mu\nu\}} = (W_{\mu\nu} + W_{\nu\mu})/2$, and 2 independent structures for the antisymmetric tensor $W_{[\mu\nu]} = (W_{\mu\nu} - W_{\nu\mu})/(2i)$. These are parameterized in terms of the usual structure functions as:

$$W_{\{\mu\nu\}}(p, q) = -F_1 \tilde{g}_{\mu\nu} + F_2 \frac{\vec{p} \vec{p} \nu}{p \cdot q}, \quad (B3)$$
$$W_{[\mu\nu]}(p, q, S) = \frac{M}{p \cdot q} \varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[ S^\beta (g_1 + g_2) - p^\beta S \cdot q g_2 \right], \quad (B4)$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is totally antisymmetric tensor with $\varepsilon_{0123} = 1$ and

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}. \quad (B5a)$$
\[ \tilde{p}_\mu = p_\mu - q_\mu \frac{p \cdot q}{q^2} . \]  

(B5b)

For an off-shell nucleon, the most general Lorentz–Dirac form of \( \hat{W}_{\mu\nu}(p, q) \) can be written as:

\[ \hat{W}_{\mu\nu}(p, q) = C_{\mu\alpha}^S I + C_{\mu\alpha}^V \gamma^\alpha + C_{\mu\alpha}^P \gamma_5 \gamma^\alpha + C_{\mu\alpha\beta\gamma}^T \sigma^{\alpha\beta} , \]  

(B6)

where the coefficients are constructed from the nucleon momentum \( p \), the momentum transfer \( q \), the metric tensor \( g_{\mu\nu} \), and the antisymmetric tensor \( \epsilon_{\mu\nu\alpha\beta} \). The requirement of hermiticity ensures that the symmetric, \( C_{\{\mu\nu\}} \), and antisymmetric, \( C_{[\mu\nu]} \), combinations are real for all the coefficients in Eq. (B6). The discrete symmetries in Eqs. (B2) and current conservation impose further requirements on the individual Dirac structures.

- In the Dirac scalar sector, Eqs. (B2) become:

\[ C_{\mu\nu}^{S*}(p, q) \overset{T}{=} C_{\mu\nu}(p', q') , \]  

(B7a)

\[ C_{\mu\nu}^{S}(p, q) \overset{P}{=} C_{\mu\nu}(p', q') . \]  

(B7b)

There are 4 independent \( T \)-even and \( P \)-even symmetric Lorentz structures in this case: \( g_{\mu\nu} \), \( p_\mu p_\nu \), \( q_\mu q_\nu \), and \( p_\mu q_\nu \), where we use the notation \( a_{\{\mu} b_{\nu\}} \equiv (a_\mu b_\nu + a_\nu b_\mu)/2 \) and \( a_{[\mu} b_{\nu]} \equiv (a_\mu b_\nu - a_\nu b_\mu)/2 \). The current conservation condition, \( q_\mu C^{S\mu\nu} = 0 \), reduces the number of independent tensors to 2, for which we choose \( \tilde{g}_{\mu\nu} \) and \( \tilde{p}_\mu \tilde{p}_\nu \).

For antisymmetric Lorentz structures, the structure \( p_{[\mu q_\nu]} \) is \( T \)-odd. Furthermore, Eqs. (B7) suggest that any antisymmetric \( T \)-even structure is necessarily \( P \)-odd. The only such structure, i.e. \( \epsilon_{\mu\nu}(pq) \), is forbidden in the electromagnetic interaction (although it appears in the weak-current interaction for the \( F_3 \) structure function). Therefore there are no \( T \)-even and \( P \)-even antisymmetric Lorentz structures in the Dirac scalar sector.

- In the Dirac vector sector, from Eqs. (B2) one has:

\[ C_{\mu\nu}^{V*}(p, q) \overset{T}{=} C_{\mu\nu}(p', q') , \]  

(B8a)

\[ C_{\mu\nu}^{V}(p, q) \overset{P}{=} C_{\mu\nu}(p', q') . \]  

(B8b)

Here there are 10 independent \( T \)-even and \( P \)-even symmetric Lorentz structures: \( g_{\mu\nu} p_\alpha \), \( g_{\mu\nu} q_\alpha \), \( p_{(\mu} g_{\nu)} p_\alpha \), \( q_{(\mu} g_{\nu)} p_\alpha \), \( p_{(\mu} p_{\nu)} p_\alpha \), \( p_{(\mu} p_{\nu)} q_\alpha \), \( q_{(\mu} q_{\nu)} p_\alpha \), \( q_{(\mu} q_{\nu)} q_\alpha \), \( p_{(\mu} q_{\nu)} p_\alpha \), and \( p_{(\mu} q_{\nu)} q_\alpha \). Of these, 5

1 We define \( \epsilon_{\mu\nu\alpha}(b) \equiv \epsilon_{\mu\nu\alpha\beta} b^\beta \) and \( \epsilon_{\mu\nu}(ab) \equiv \epsilon_{\mu\nu\alpha\beta} a^\alpha b^\beta \) for any four-vectors \( a \) and \( b \).
are ruled out by current conservation, \( q_{\mu} C^{V(\mu\nu)\alpha} = 0 \), leaving 5 independent \( T \)-even and \( P \)-even symmetric Lorentz structures. As a convenient basis we take \( \tilde{g}_{\mu\nu}p_\alpha, \tilde{g}_{\mu\nu}q_\alpha, \tilde{p}(\mu g_\nu)^\alpha, \tilde{p}_\mu\tilde{p}_\nu p_\alpha \), and \( \tilde{p}_\mu\tilde{p}_\nu q_\alpha \). Similar to the Dirac scalar case, the antisymmetric \( T \)-even tensors are necessarily \( P \)-odd and therefore do not contribute to the electromagnetic interaction.

- In the Dirac pseudoscalar sector, Eqs. (B2) imply:
  \[
  C_{\mu\nu}^P(p, q) \overset{T}{=} -C_{\mu\nu}^P(p', q'), \\
  C_{\mu\nu}^P(p, q) \overset{P}{=} -C_{\mu\nu}^P(p', q').
  \] (B9a, B9b)

There is no \( T \)-even symmetric solution to Eqs. (B9). Furthermore, one can verify that antisymmetric \( T \)-even solutions can only be \( P \)-odd. The only such structure is \( p[\mu q_\nu] \), however, this cannot be matched with current conservation. No \( T \)-even and \( P \)-even solutions can therefore be found to Eqs. (B9), so that the Dirac pseudoscalar sector does not contribute to the expansion in Eq. (B6).

- In the Dirac axial vector sector, one has:
  \[
  C_{\mu\nu}^A(p, q) \overset{T}{=} C_{\mu\nu}^A(p', q'), \\
  C_{\mu\nu}^A(p, q) \overset{P}{=} -C_{\mu\nu}^A(p', q').
  \] (B10a, B10b)

One immediately observes from Eqs. (B10) that \( C_{\{\mu\nu\}}^A = 0 \). There are 3 independent current conserving structures in the antisymmetric tensor \( C_{\{\mu\nu\}}^A : \epsilon_{\mu\nu\alpha}(q), \epsilon_{\mu\nu}(pq)p_\alpha \), and \( \epsilon_{\mu\nu}(pq)q_\alpha \).

- Finally, in the Dirac tensor sector, the transformations are:
  \[
  C_{\mu\nu\alpha\beta}^T(p, q) \overset{T}{=} -C_{\mu\nu\alpha\beta}^T(p', q'), \\
  C_{\mu\nu\alpha\beta}^T(p, q) \overset{P}{=} C_{\mu\nu\alpha\beta}^T(p', q').
  \] (B11a, B11b)

from which one concludes that symmetric tensors must vanish, \( C_{\{\mu\nu\}}^T = 0 \). There are 6 antisymmetric structures \( C_{[\mu\nu]}^T = 0 \) which can be constructed from the product of the four-vectors \( p \) and \( q \) and the metric tensor \( g_{\lambda\sigma} \): \( p[p,q][\alpha q_\beta], p[p,q][\alpha q]\sigma, g_p[p][\alpha q_\beta], g_p[p][\alpha q]\sigma, q[p,q][\alpha q_\beta], q[p,q][\alpha q]\sigma, g_{\rho\alpha}[q_\beta], g_{\rho\alpha}[q]\sigma \). Furthermore, a number of other possible structures involving bilinear combinations of the fully antisymmetric tensor can be constructed, such as \( \epsilon_{\mu\nu\rho}(q)\epsilon_{\alpha\beta}\sigma(p), \epsilon_{\mu\nu}(pq)\epsilon_{\alpha\beta}(pq) \), etc.

\(^2\) There are nontrivial symmetric \( T \)-even and \( P \)-odd solutions to Eqs. (B11), however, these do not contribute to the electromagnetic tensor.
These structures are not all independent, however. In particular, one can show that all the Lorentz tensors bilinear in the antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$ can be rewritten as linear combinations of tensors constructed from antisymmetrized products of the vectors $p$ and $q$ and the metric tensor $g_{\alpha\beta}$. A direct analysis reveals that there are in fact only 6 independent combinations obeying Eqs. (B11), which are reduced by 3 additional constraints from the current conservation condition, $q^\mu C^T_{[\mu\nu]} 0 = 0$. The 3 remaining basis structures in the Dirac tensor channel are then chosen to be: $\epsilon_{\mu\nu\sigma}(q)\epsilon_{\alpha\beta}^\sigma(q)$, $\epsilon_{\mu\nu\sigma}(q)\epsilon_{\alpha\beta}^\sigma(p)$, and $\epsilon_{\mu\nu}(pq)\epsilon_{\alpha\beta}(pq)$. In constructing the explicit form of the Dirac expansion in Eqs. (B11) we also use the identity $i\epsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta} = 2\gamma_5\sigma^{\mu\nu}$.

Collecting the above results, we conclude that the symmetric part of the truncated nucleon electromagnetic tensor is determined by the scalar and vector terms in the expansion (B6), while the antisymmetric part receives contributions from the axial vector and the tensor terms. The symmetric tensor $\hat{W}_{[\mu\nu]}$ generally involves 7 independent Lorentz–Dirac structures (2 Dirac scalar and 5 Dirac vector structures), which we write as:

$$\hat{W}_{[\mu\nu]}(p, q) = \frac{1}{2M} \left( C^1_S I + C^1_{V'} p + C^1_V q \right) g_{\mu\nu} + \frac{1}{2M} \left( C^2_S + C^2_{V'} p + C^2_V q \right) \frac{p\cdot q}{p\cdot q} + \frac{C^2_V}{2M} \tilde{p}(u\tilde{g}_\nu)_{\alpha\gamma} \gamma^\alpha,$$

(B12)

where the coefficients are real scalar functions of the invariants $q^2$, $p\cdot q$, and $p^2$, and the normalization factor $1/(2M)$ is introduced to simplify subsequent expressions for the structure functions.

The antisymmetric tensor $\hat{W}_{[\mu\nu]}$ is constructed similarly from 6 Lorentz–Dirac structures (3 Dirac axial-vector and 3 Dirac tensor structures), which can be written as:

$$\hat{W}_{[\mu\nu]}(p, q) = \frac{1}{2p\cdot q} \epsilon_{\mu\nu\alpha\beta} q^\alpha \left[ C^\gamma_A \gamma_5 \gamma^\beta + p^\beta \gamma_5 \left( C^p_A \gamma^\beta + C^q_A \gamma^\beta \right) + i p^\beta C^p_A T \gamma_5 \sigma^{\rho\lambda} p_\rho q_\lambda + i \gamma_5 \sigma^{\beta\lambda} \left( C^p_A T p_\lambda + C^q_A T q_\lambda \right) \right],$$

(B13)

where again the coefficients are the scalar functions of the invariants $q^2$, $p\cdot q$, and $p^2$.

Substituting Eqs. (B12) and (B13) into Eq. (B1) we reproduce Eqs. (B3) and (B4) for an on-shell nucleon, with the structure functions given in terms of the coefficient functions as:

$$F_1 = C^1_S + C^1_{V'} M + C^1_V \frac{p\cdot q}{M},$$

(B14a)

$$F_2 = C^2_S + C^2_{V'} M + \left( C^2_V + C^2_V \right) \frac{p\cdot q}{M},$$

(B14b)
\[ g_1 = -C_A^\gamma + C_T^p - C_A^{p \cdot q} - C_T^{p \cdot q} M p \cdot q , \quad (B14c) \]
\[ g_2 = (C_A^q M + C_T^{p q} M^2) \frac{p \cdot q}{M} . \quad (B14d) \]

Note that the term proportional to \( \gamma_5 p \) in Eq. \( B13 \) gives a vanishing contribution to the spin structure functions \( g_{1,2} \) because of the condition \( p \cdot s = 0 \), so that only five of the possible six structures in Eq. \( B13 \) contribute to the physical nucleon structure functions.

**APPENDIX C: HADRONIC TENSOR IN THE WBA**

In this appendix we discuss the nucleon hadronic tensor in the weak binding approximation and derive the reduction of the four-dimensional spinor trace to the two-dimensional trace in Eq. (8). To this end we consider the traces \( \text{Tr}[O, A] \) with the basis Dirac operators from Eqs. (B12) and (B13). Using the notation \( p = (M + \varepsilon, p) \) for the nucleon four-momentum \( p \) and the relation between the four-dimensional and two-dimensional spinors in Eq. (5), the traces can be written as:
\[
\frac{1}{2M_A} \text{Tr} [O, A(p, S)] = \frac{1}{p_0} \text{tr} [O^{\text{WBA}} P(\varepsilon, p, S)] , \quad (C1)
\]
where the relativistic (\( A \)) and nonrelativistic (\( P \)) spectral functions are given by Eqs. (4) and (7), respectively. The results for the operators \( O^{\text{WBA}} \) in the weak binding approximation are listed in Table II. These results are derived by systematically expanding in \( 1/M \) and keeping terms to order \( p^2/M^2 \sim \varepsilon/M \). To this order the nucleon 3-momentum and off-shell mass are related by \( 1 - p^2/(2M^2) = (p^2)^{1/2}/p_0 \).

| \( O \)       | \( O^{\text{WBA}} \)             |
|-------------|----------------------------------|
| \( I \)     | \( (p^2)^{1/2} \)                |
| \( \gamma^\alpha \) | \( p^\alpha \)                   |
| \( \gamma_5 \gamma^\alpha \) | \( -\hat{S}^\alpha \) \( (p^2)^{1/2} \) |
| \( i\gamma_5 \sigma^{\alpha\beta} p_\beta \) | \( \hat{S}^\alpha p^2 \)           |
| \( i\gamma_5 \sigma^{\alpha\beta} q_\beta \) | \( (p \cdot q)\hat{S}^\alpha - (\hat{S} \cdot q)p^\alpha \) |

**TABLE I: Nonrelativistic transformation of the basis Dirac structures.** The spin operator \( \hat{S} \) is defined by Eq. (10).
Using Eq. (C1) together with Table II then leads to Eq. (8) for the generic hadronic tensor \( \hat{W}_{\mu\nu} \) discussed in Appendix B. The spin-dependent nucleon hadronic tensor \( \hat{w}_{\mu\nu} \) is given by Eq. (9) (a similar analysis of the unpolarized case is given in Ref. [3]). The structure functions for an off-shell nucleon with momentum \( p \) are then given by:

\[
F_1 = C_S \left( \frac{p^2}{M} \right)^{1/2} + C_V p^2 M + C_{Vq} \frac{q \cdot p}{M}, \tag{C2a}
\]

\[
F_2 = C_S \left( \frac{p^2}{M} \right)^{1/2} + C_V q^2 M + \left( C_{Vq} + C_V \right) \frac{q \cdot p}{M}, \tag{C2b}
\]

\[
g_1 = -C_A \left( \frac{p^2}{M} \right)^{1/2} + C_T q^2 M - \left( C_A \left( \frac{p^2}{M} \right)^{1/2} + C_{Tq} q^2 \right) \frac{q \cdot p}{M}, \tag{C2c}
\]

\[
g_2 = \left( C_A \left( \frac{p^2}{M} \right)^{1/2} + C_{Tq} q^2 + C_T \right) \frac{p \cdot q}{M}, \tag{C2d}
\]

where the coefficients are scalar functions of \( p \cdot q, Q^2, \) and \( p^2 \). One can easily verify that in the limit \( p^2 \to M^2 \), Eqs. (C2) reduce to their correct on-shell limits in Eqs. (B14). Note that the expressions in Eqs. (C2) are valid in the vicinity of the nucleon mass shell where nucleon virtuality \( p^2 - M^2 \) is small. Finally, we observe that although in general the off-shell nucleon tensor is described by 6 (7) independent structure functions for spin-dependent (spin-averaged) scattering, it can nevertheless be characterized by the same number of independent structure functions as on-shell.

**APPENDIX D: NUCLEAR SPECTRAL FUNCTION IN TERMS OF WAVE FUNCTIONS**

Here we discuss the relations between the operator definition of the nuclear spectral function (7) and a more traditional definition in terms of the matrix elements of the wave functions, as used for instance in Refs. [9, 24]. Note that the Fock state of a nucleus \( A \) containing nonrelativistic bound nucleons moving with momentum \( P \) can be written as a product of the nucleon creation operators acting on the vacuum state convoluted with the nuclear wave function:

\[
| A, P \rangle = \frac{1}{\sqrt{A!}} \int [dr]_A \Psi_{P,A}(\{r\}_A) \psi_1^\dagger(1) \cdots \psi_A^\dagger(0) |0\rangle, \tag{D1}
\]

where \( \psi(i) = \psi_{\sigma_i}(r_i) \) is a short-hand notation for the nucleon field operator at coordinate \( r_i \) with polarization \( \sigma_i \) and isospin \( \tau_i \). The nuclear wave function \( \Psi_{P,A} \) depends on the set of coordinates, spin, and isospin of the \( A \) nucleons, which are symbolically denoted by
\( \{r\}_A \), and \([dr]_A\) is a symbolic notation for the integration over coordinates and the sum over the spin and isospin variables. The nuclear wave function also depends on the nuclear momentum as well as on other quantum numbers, including nuclear spin and isospin, which are symbolically denoted by \( A \). For a nonrelativistic system the dependence of the wave function on the center-of-mass momentum and intrinsic variables can be factorized according to:

\[
\Psi_{\mathbf{P}, A}(\{r\}_A) = \exp(i\mathbf{P} \cdot \mathbf{R}_A)\Phi_A(\{\rho\}_A),
\]

where \( \mathbf{R}_A = \sum_{i=1}^{A} \mathbf{r}_i / A \) is the position of the center-of-mass of \( A \) particles (we neglect the mass difference for protons and neutrons), and the coordinate \( \mathbf{\rho}_i = \mathbf{r}_i - \mathbf{R}_A \) describes the position of the \( i \)-th particle relative to the nuclear center-of-mass. In the set of \( \mathbf{\rho}_i \) only \( A - 1 \) coordinates are independent because of the condition \( \sum_{i=1}^{A} \mathbf{\rho}_i = 0 \). The intrinsic wave function \( \Phi_A \) is independent of the nuclear momentum \( \mathbf{P} \) and depends on the relative distances between bound nucleons. Note also that the integration in Eq. (D1) can be written in terms of the integration over the center-of-mass position and intrinsic coordinates as \( [dr]_A = d\mathbf{R}_Ad\mathbf{\rho}_1 \cdots d\mathbf{\rho}_A \delta(\sum_{i=1}^{A} \mathbf{\rho}_i) \).

We now consider the matrix elements \( \psi_f(\mathbf{p}, \sigma, \tau) \) in Eq. (7). Using translational invariance we can write

\[
\psi_f(\mathbf{p}, \sigma, \tau) = \frac{1}{V} \int d\mathbf{r} e^{-i\mathbf{p} \cdot \mathbf{r}} \langle (A - 1)_f, -\mathbf{p} | \psi^*_\sigma(\mathbf{r}) | A \rangle ,
\]

where \( V = \int d\mathbf{r} \) is normalization volume. Applying Eq. (D1), we can then write Eq. (D3) as the overlap integral of the wave functions of the nuclear states. Using antisymmetry of the wave functions under permutation of bound nucleons, we then have

\[
\psi_f(\mathbf{p}, \sigma_1, \tau_1) = \frac{\sqrt{A}}{V} \int d\mathbf{r}_1 [dr]_{A-1} e^{-i\mathbf{p} \cdot \mathbf{r}_1} \Psi^*_{-\mathbf{p}, (A-1)_f}(\{r\}_{A-1} \} A) \Psi_{0, A}(\mathbf{r}_1, \sigma_1, \tau_1; \{r\}_{A-1} \} A),
\]

which corresponds exactly to the matrix elements in the definition of the nuclear spectral function in Refs. \[9, 24\].

In the case of three-nucleon system, the Jacobi coordinates \( \mathbf{x} = \mathbf{r}_3 - \mathbf{r}_2 \) and \( \mathbf{y} = (\mathbf{r}_2 + \mathbf{r}_3)/2 - \mathbf{r}_1 \) are often chosen as independent variables for the wave function. Using Eq. (D2) we separate the center-of-mass motion of the residual two-nucleon state with the center-of-mass coordinate \( \mathbf{R}_2 = (\mathbf{r}_2 + \mathbf{r}_3)/2 \), which leads to:

\[
\psi_f(\mathbf{p}, \sigma_1, \tau_1) = \sqrt{3} \int d\mathbf{x} d\mathbf{y} e^{i\mathbf{p} \cdot \mathbf{y}} \Phi^*_2(\mathbf{x}; \sigma_2, \sigma_3; \tau_2, \tau_3) \Phi_3(\mathbf{x}, \mathbf{y}; \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3),
\]
where $\Phi_{2f}$ and $\Phi_3$ are the center-of-mass wave functions of the residual two-nucleon system and the three-body bound state, respectively. The sum over the spin and isospin variables $\sigma_2, \sigma_3$ and $\tau_2, \tau_3$ is implicitly understood. The spectrum of the residual two-body states is characterized by a set of quantum numbers $f$, which includes the energy $E_f$, orbital angular momentum $(L_f, L_{zf})$ and spin $(S_f, S_{zf})$, as well as the isospin $(T_f, T_{zf})$.

In the case of the deuteron residual state, $\Phi_{2f}$ corresponds to the deuteron wave function with total angular momentum 1, spin 1 and isospin 0. For the other channels, the function $\Phi_{2f}$ describes a continuum two-nucleon state. The details of calculation of $^3$He and $^3$H bound state wave function and spectral function can be found in Refs. [9, 24] and references therein.

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