From the paradoxes of the standard wave-packet analysis to the definition of tunneling times for particles

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November 16, 2018

Abstract

We develop a new variant of the wave-packet analysis and solve the tunneling time problem for one particle. Our approach suggests an individual asymptotic description of the quantum subensembles of transmitted and reflected particles both at the final and initial stage of tunneling. We find the initial states of both subensembles, which are non-orthogonal. The latter reflects ultimately the fact that at the initial stage of tunneling it is impossible to predict whether a particle will be transmitted through or reflected off the barrier. At the same time, in this case, one can say about the to-be-transmitted and to-be-reflected subensembles of particles. We show that before the interaction of the incident packet with the barrier both the number of particles and the expectation value of the particle’s momentum are constant for each subensemble. Besides, these asymptotic quantities coincide with those of the corresponding scattered subensemble. On the basis of this formalism we define individual delay times for both scattering channels. Taking into account the spreading of wave packets, we define the low bound of the scattering time to describe the whole quantum ensemble of particles. All three characteristic times are derived in terms of the expectation values of the position and momentum operators. The condition which must be fulfilled for completed scattering events is derived. We propose also the scheme of a gedanken experiment that allows one to verify our approach.
Introduction

The tunneling time problem is perhaps one of the most long-lived and involved problems of quantum mechanics. Although there exists a variety of proposals (see, for example, reviews [1, 2, 3]) to define characteristic times for a tunneling particle, this question have been continued to be controversial. The first attempts (see, for example, [1, 4, 5, 6, 7]) to solve this problem, on the basis of the one-dimensional Schrödinger equation (OSE), were reduced to the naive analysis of the relative motion of the transmitted (or reflected) and incident wave packets. Such an approach is usually referred to as the standard wave-packet analysis (SWPA). The main result of the SWPA is the development of concepts of the phase and delay times to describe particles with a definite momentum. However, even in this particular case these times proved to be ill-defined. Their rigorous justification leads to the difficulties which remain to be overcome (see [1, 2]). But the most serious difficulties in the SWPA arose in attempt to define tunneling times for wave packets whose width is comparable with the width of a potential barrier. Many authors (see, for instance, [1, 2]) pointed out that some properties of tunneling such wave packets (the dependence of the phase and delay times on the initial distance between the incident wave packet and the potential barrier; the acceleration of the transmitted wave packet by an opaque potential barrier) seem to be paradoxical. Due to this fact the SWPA has been considered (see [2, 3]) as the most unpromising approach to the solution of the tunneling time problem. As a result, all the significant papers published on this problem for the last ten years have been fulfilled out of the framework of the SWPA (see reviews [2, 3] and also [8, 9]). However, as is seen from [2, 3], the alternative approaches themselves meet with the serious difficulties. The basic assumptions (to prescribe the Feynman, Bohmian or Wigner trajectories to a particle; to incorporate in quantum mechanics the conditional probabilities which may be negative by value; to introduce the special ”quantum” clocks for timing the particle’s motion; to admit the averaging procedures (for example, averaging over time intervals) to differ from the averaging over a quantum ensemble; to admit making use, in addition to parameter $t$, of operators with the dimension of time) underlying these approaches are either purely speculative or questionable in many respects. In any case, such assumptions lead the alternative approaches out of the framework of the conventional quantum mechanics and, as a consequence, they themselves need an independent justification. A detailed analysis of this problem can be found in [2, 3, 10] (see also [11]).

We have also to add that none of them explains the mentioned above paradoxical behavior of wave packets. The point is that such an explanation have not merely been a purpose of the alternative approaches. As was said by Landauer and Martin [2], this is “...a responsibility of wave-packet analysis proponents...”. However, in our opinion, such a viewpoint is at least strange because the description of the particle tunneling in terms of wave packets is inherent to quantum theory. So that an exhaustive interpretation of the wave-packet dynamics in tunneling is a task not only for the proponents of the SWPA. All approaches whose aim is to solve the tunneling time problem must explain the behavior of wave packets in tunneling.

Taking into account the above, we see a necessity to return to the SWPA, and clarify the reasons having led to the failure of the earlier approaches in solving the given problem.
We begin our analysis with the remark that there is really the case in the SWPA when no problems arise in timing the particle’s motion. We have in mind the motion of an everywhere free particle. Namely the timing procedure for a free particle was used in defining the phase times (see [1, 4, 5, 6]). The possibility to use such a procedure in the tunneling problem is based on the fact that a tunneling particle moves freely before and after its interaction with the potential barrier.

To avoid some mistakes in the more complex, scattering case, one has to dwell shortly on the free-particle timing procedure. A simple analysis of classical dynamics of particles shows that it is important to distinguish two kinds of characteristic times: 1) the time of arrival of a free particle at some given point, and 2) the duration of passing the particle through some given spatial interval. Defining the first quantity is based in quantum mechanics on the fact that the expectation values of the velocity and position of a free particle coincide with the corresponding characteristics of the “center of mass” (CM) of a wave packet to describe the particle state. As is known, in this case the motion of the CM is described by Newton equations for a free classical particle. Thus, in order to define the (mean) time of arrival of a quantum particle at a given point, one needs simply to follow the CM of the corresponding wave packet. This concept will be used here in defining delay times.

A more distinct difference between the classical and quantum dynamics of free particles arises in determining the characteristic times of the second kind. So, the time a particle spends within some spatial region as well as the time the particle spends to traverse this region are the same classically. However, this is not the case in quantum mechanics. For example, a quantum particle spends no time in the null spatial interval. But there is a non-zero temporal interval when the probability for a particle to traverse this region is large enough. To define the duration of the physical process, one has to follow at least the fronts of a wave packet, rather than its CM. For this purpose, in addition to the expectation value of the position operator at some instant \( t \), one needs to know its mean-square deviation. This concept will be used below in defining the low bound of the scattering time.

The free-motion case reveals the following two aspects of timing a quantum dynamics of particles. Firstly, timing the motion of a quantum particle can be thought only as timing the evolution of different (initial or central) moments of the position (or another Hermitian) operator. This is related ultimately to the fact that quantum mechanics can predict only expectation values of physical quantities. Secondly, there is no necessity to construct special ”quantum clocks” for timing the motion of a free quantum particle. Characteristic times for such a particle (in other words, for the CM or fronts of a free wave packet) can be measured with the help of the same clocks that used in keeping track of a classical free particle.

It is obvious that the above peculiarities of timing a free particle must be true also for a scattering particle when it moves beyond the scattering region (this suggests that the scattering event is a completed scattering). However, in this case a new aspect of timing arises: while a particle interacts with the barrier, to time its motion is impossible. There are two main reasons for this. Firstly, while a wave packet interacts with the barrier, the position and velocity of its CM say nothing about the expectation values of
the position and velocity of a particle: because of the interference, in this case they can be unambiguously related neither to an incident nor transmitted, nor reflected particle. Secondly, any measurement in the barrier region contradicts the conventional setting of the tunneling problem.

As is seen, the free-particle timing procedure itself is simple enough. However, the application of this procedure to a scattering particle, in the case when the wave packet width is comparable with the barrier width, leads to the ill-defined phase times (see, for example, [1, 2, 4, 12, 13]. And the formalism based on this procedure does not provide a proper explanation to the behavior of wave packets in tunneling. As is known, a numerical modelling of the tunneling process for wide (in \(x\)-space) wave packets shows that, though the interaction of a particle with the barrier is elastic, the CMs of the incident, transmitted and reflected wave packets move with the different velocities. As a rule, the transmitted packet moves faster than the incident one (for example, it takes place for an opaque rectangular barrier). In this case the CM of the transmitted packet appears behind of the barrier well before the incident packet arrives at the barrier region, i.e., before forming the reflected packet. The corresponding phase times were found to depend on the initial distance between the barrier and wave packet: the farther from the barrier is at the initial time the CM of the incident packet, the earlier the CM of the transmitted packet appears behind of the barrier.

Thus, according to the SWPA, in the general case the transmitted particles seem to be accelerated (on the average) in tunneling, and the corresponding delay time (positive or negative) depends strongly on the initial distance between the particle and barrier. Note that the observed behavior of wave packets is entirely in accordance with the foundations of quantum mechanics, since it follows from the OSE to describe the tunneling process. At the same time, its interpretation mentioned above appears to be mistaken. It is intuitively evident that a static potential barrier must not accelerate (on the average) a particle, wherever it moves after the scattering event. Besides, the delay time must not depend on the initial distance between the particle and barrier. So that the behavior of wave packets in tunneling requires a reasonable explanation. The lack of such an explanation as well as the lack of well-defined characteristic times have a common root in the SWPA.

At first sight, all the above means that the timing procedure for an everywhere free particle cannot be, in principle, used for a tunneling particle. However, in our opinion, this is not the case. One has only to bear in mind that the incident, transmitted and reflected wave packets describe quantum ensembles with the different number of particles. This fact have not been taken into account in the SWPA. The phase time for transmitted particles was defined in this approach as the difference between the time of arrival of the CM of the transmitted packet at a point far behind of the barrier, and that of the CM of the incident packet at a point far before the barrier. Several authors have been cast doubt on the correctness of such a step. For example, considering the behavior of the incident and transmitted wave packets in the numerical modeling of tunneling, the authors of [14, 15] emphasize that "... arriving peaks do not turn into transmitted peaks ...". In [1, 4, 6] it is pointed to "... the desirability of constructing an "effective" incident packet which would play the role of the counterpart to the transmitted packet...". Moreover, as was pointed out in [13], classical mechanics"... would suggest that the energy distribution
of the transmitted and reflected particles is the same before and after the collision with the barrier...”

To take into account these remarks, one needs to decompose the incident wave packet into (to-be-transmitted and to-be-reflected) parts. However, there have been an opinion that this step would contradict the basic principles of quantum theory. For example, Jaworski and Wardlaw [13] pointed out "... that quantum mechanically there is no sense in speaking about transmitted and reflected particles before they are detected as such..." As a result, the above idea remains unrealized. All the known attempts to describe individually transmitted and reflected particles have been made beyond the idea of the individual description of these subensembles at the initial stage of scattering. For example, Steinberg [10] offered to introduce the so-called conditional probabilities (however, being complex, these quantities lead to the serious problem of their interpretation). There is also a radical viewpoint (see, for example, [16]) that an individual description of transmitted and reflected particles is inadmissible in quantum mechanics.

However, as will be shown below, in the frame of the statistical interpretation (SI) of quantum mechanics (see, for example, [17]), the desirable decomposition does not contradict quantum theory. Besides, there is another aspect of the tunneling problem that also requires such a decomposition. We have in mind making use of the term "two-channel scattering" in solving the tunneling time problem. So, according to the definition given in [18, 19], this elastic one-particle one-dimensional scattering must be treated as an one-channel scattering. At the same time, according to Hauge and Stovneng [1], the term "scattering channel" in this problem should be associated with "... distinct final states (transmission and reflection ...)". By this definition, tunneling is a two-channel scattering.

We agree entirely with this viewpoint. The mere fact that investigators attempt to define characteristic times individually for a transmitted and reflected particle means that they distinguish in this task two scattering channels. However, one has to recognize that such a viewpoint needs a proper theoretical basis. Namely, one should to develop a formalism to describe individually the quantum subensembles of transmitted and reflected particles both at the initial and final stages of tunneling. Strictly speaking, it is illegitimately to say, without such a formalism, about two scattering channels in this case. In our opinion, just ignoring this fact leads to the above paradoxes.

Note that there is a variety of scattering problems related to the electron transport (see, for example, [20]) in artificial quantum structures (e.g., in quantum wires) where a similar situation takes place. Right quantum wires which are nonuniform in some limited spatial interval give the most simple generalization of the considered one-dimensional model. More complicated scattering problems arise for branching quantum wires where the number of scattering channels is larger than two. It is obvious that the above difficulties of the one-dimensional case must appear in these models too. To overcome them, one needs to treat the one-dimensional scattering as a two-channel one, and develop a relevant theory.

So, in our opinion, the theory of tunneling must include an individual asymptotic description of the subensembles of transmitted and reflected particles. The purpose of our paper is to elaborate a desirable formalism in the framework of the SI (in our opinion,
solving the tunneling time problem is sensitive to the choice of some interpretation of quantum mechanics). Note that our ideas to solve this problem have been offered in papers [21, 22]. However some aspects of that solution must be corrected and developed further. In particular, we have to study here the role of spreading wave packets, which have been leaved out of account in [21, 22]. This effect is obvious to increase the scattering time. Moreover, when the velocity of the CM is smaller than that of the wave-packet’s fronts, tunneling represents an incompletely scattering.

The paper is organized as follows. In Section 1 we set the tunneling problem. In Section 2 we display explicitly the paradoxes arising in the standard approaches. For this purpose we analyze in detail the behavior of wave packets in the context of the SWPA. In Section 3 we show that the wave function describing the quantum ensemble of tunneling particles contains all information needed for the individual description of both scattering channels at the initial stage of scattering. We find here the corresponding counterparts for the transmitted and reflected packets, which describe individually both scattering channels at early times. In this section we answer also the question, how, in principle, to verify experimentally the formalism presented. In Section 4 we define delay times for transmitted and reflected particles. To estimate the duration of the scattering event, in Section 5 we determine the low bound of the scattering time. Besides, we derive here the condition which must be fulfilled for a completed scattering. In Section 6, the properties of the characteristic times are illustrated in the case of the Gaussian wave packet tunneling through a rectangular barrier. Some useful expressions for the asymptotic expectation values of the position and wave-number operators as well as for their mean-square deviations are presented in Appendix.

1 Setting the problem for a completed scattering

Suppose that a particle moves from the left toward the time-independent potential barrier $V(x)$ which confined to the finite spatial interval $[a, b]$ ($a > 0$); $d = b − a$ is the barrier width. Let the state of a particle be described at $t = 0$ by the normalized wave function $Ψ_0(x)$ belonging to the set $S_∞$ which consists from infinitely differentiable functions vanishing exponentially in the limit $|x| → ∞$. The Fourier-transforms of such functions are known to belong to the set $S_∞$ as well. This property guarantees that the position and momentum operators both are well-defined. Let $< Ψ_0 | ˆx | Ψ_0 > = 0$, $< Ψ_0 | ˆp | Ψ_0 > = ˆh k_0 > 0$, $l_0^2 = < Ψ_0 | ˆx^2 | Ψ_0 >$; here $l_0$ is the wave-packet’s half-width at $t = 0$ ($l_0 << a$); $ˆx$ and $ˆp$ are the operators of the particle’s position and momentum, respectively.

Another important requirement should restrict the rate of spreading the incident wave packet. We must be sure that at early times all particles of the corresponding quantum ensemble move toward the barrier (only a negligible part of particles is assumed to move at $t = 0$ away from the barrier). This means that the packet’s spreading must be sufficiently ineffective (see condition (40) in Section 5).

For our purposes it is useful also to introduce the following notations. Let $ˆH$ be the Hamiltonian of the problem: $ˆH = ˆH_0 + V(x)$, where $ˆH_0$ is the energy operator for a free particle. Besides, let $ˆH_0^{ref}$ be the auxiliary Hamiltonian describing the reflection of a free particle off the absolutely opaque potential wall located at the middle point of the interval
\[ x_{midp} = \frac{(a + b)}{2}. \] Let us introduce also the projection operators \( \hat{\Theta}_+ \) and \( \hat{\Theta}_- \) coinciding with the corresponding \( \theta \)-functions: \( \hat{\Theta}_+ = \theta(x - b), \hat{\Theta}_- = \theta(a - x). \)

In the general form, the solution of the corresponding temporal OSE can be written formally as \( e^{-i\hat{H}t/\hbar}\Psi_0(x) \). To solve explicitly this equation, we will use the transfer matrix method (TMM) [23] that allows one to calculate the tunneling parameters for any system of potential barriers. Then, the (matched) solution describing the incident and reflected waves \( (x < a) \) with the given wave-number \( k \) can be written as

\[
\Psi_{left} = \exp\left[i(kx) + \phi_{ref}(k) \exp(-ikx)\right] \exp[-iE(k)t/\hbar],
\]

where

\[
\phi_{ref}(k) = \sqrt{R(k)} \exp\left[i(2ka + J(k) - F(k) - \frac{\pi}{2})\right];
\]

the solution

\[
\Psi_{right} = \phi_{tr}(k) \exp[i(kx - E(k)t/\hbar)],
\]

represents the transmitted wave \( (x > b) \);

\[
\phi_{tr}(k) = \sqrt{T(k)} \exp[i(J(k) - kd)];
\]

Here \( E(k) = \hbar^2k^2/2m; T(k) \) (the real transmission coefficient) and \( J(k) \) (phase) are even and odd functions of \( k \), respectively; \( F(-k) = \pi - F(k) \).

Thus, for the temporal OSE, the solution to satisfy the above initial condition is given by

\[
\Psi_{left}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f_{inc}(k, t) + f_{ref}(k, t)\right] \exp(i k x) dk
\]

(for the region \( x < a \)) and

\[
\Psi_{right}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_{tr}(k, t) \exp(i k x) dk
\]

(for \( x > b \), where

\[
f_{inc}(k, t) = c \cdot A(k; k_0, l_0) \exp[-iE(k)t/\hbar],
\]

\[
f_{ref}(-k, t) = \phi_{ref}(k)f_{inc}(k, t),
\]

\[
f_{tr}(k, t) = \phi_{tr}(k)f_{inc}(k, t);
\]

c is a normalization constant; the weight function \( A(k; k_0, l_0) \) is the Fourier-transform of \( \Psi_0(x) \). The initial condition both \( \Psi_0(x) \) and \( A(k; k_0, l_0) \) belong to \( S_\infty \). In particular, for \( \Psi_0(x) \) describing the Gaussian wave packet, \( A(k; k_0, l_0) = \exp(-l_0^2(k - k_0)^2) \).

Here functions (3) and (4) present wave packets moving in the out-of-barrier regions. Expression (4) describes the transmitted wave packet. Solution (3) consists of the incident and reflected packets (in fact, the latter appears only on arriving the incident
packet at the barrier region). To study these packets, it is convenient to pass into the $k$-representation. It is obvious that in the case of a completed scattering, at late times, the functions $f_{\text{tr}}(k, t)$ and $f_{\text{ref}}(k, t)$ approximate Fourier-transforms of the functions $\hat{\Theta}_+ e^{-i\hat{H}t/\hbar}\Psi_0(x)$ and $\hat{\Theta}_- e^{-i\hat{H}t/\hbar}\Psi_0(x)$, respectively. The function $f_{\text{inc}}(k, t)$ approximates the Fourier-transform of $e^{-i\hat{H}t/\hbar}\Psi_0(x)$ at early times. A detailed description of these packets in the $k$ representation is presented in Appendix.

As follows from Appendix (see (57), (59) and (60)), for a completed scattering the following normalization conditions must be valid (see also [13]).

For early times
\[ \int_{-\infty}^{a} |\Psi_{\text{left}}(x, t)|^2 dx \approx \int_{-\infty}^{\infty} |f_{\text{inc}}(k, t)|^2 dk = 1; \] (5)

for sufficiently large times
\[ \int_{-\infty}^{a} |\Psi_{\text{left}}(x, t)|^2 dx \approx \int_{-\infty}^{\infty} |f_{\text{ref}}(k, t)|^2 dk = \bar{R}; \] (6)
\[ \int_{b}^{\infty} |\Psi_{\text{right}}(x, t)|^2 dx \approx \int_{-\infty}^{\infty} |f_{\text{tr}}(k, t)|^2 dk = \bar{T}. \] (7)

Here $\bar{T}$ and $\bar{R}$ are the mean values of the transmission and reflection coefficients, respectively: $\bar{T} = <T(k)>_{\text{inc}}$; $\bar{R} = <R(k)>_{\text{inc}}$; angle brackets denote averaging over a relevant wave packet (see (55)). We will assume further that conditions (5)-(7) are satisfied.

As was pointed out above, timing a particle can be made only beyond the scattering region. All needed information about the influence of the potential barrier on a particle is contained, via the tunneling parameters, in expressions (3) and (4). We will consider that these parameters have already been known.

### 2 Paradoxes of the standard wave-packet analysis

In order to display explicitly the basic shortcoming of the SWPA, let us rederive the tunneling times in the context of that approach. The only difference is that we use here the TMM [23].

As is shown in Appendix, the expectation values of the $\hat{x}$-operator for all three packets are given by expressions (see (67) - (69))
\[ < \hat{x} >_{\text{inc}}(t) = m^{-1}\hbar k_0 t \] (8)
for the sufficiently early times, and
\[ < \hat{x} >_{\text{tr}}(t) = m^{-1}\hbar < k >_{\text{tr}} t + d - < J' >_{\text{tr}}, \] (9)
\[ < \hat{x} >_{\text{ref}}(t) = -m^{-1}\hbar < -k >_{\text{ref}} t + 2a + < J' - F' >_{\text{ref}} \] (10)
for the sufficiently large times. Note (see Appendix) that the prime denotes here the derivative with respect to $k$.

Let $Z_1$ be a point to lie at some distance $L_1$ ($L_1 \gg l_0$ and $a - L_1 \gg l_0$) from the left boundary of the barrier, and $Z_2$ be a point to lie at some distance $L_2$ ($L_2 \gg l_0$) from...
its right boundary. Thus, conditions (5) - (7) are fulfilled in this case. Note that the fulfilment of the condition $a - L_1 \gg l_0$ must guarantee that (almost) all particles of the quantum ensemble are localized, at $t = 0$, to the left of the detector placed at the point $Z_1$. In any case, a part of the quantum ensemble to violate this condition must be much smaller than $\tilde{T}$.

Following the SWPA \[1, 4, 5, 6\], let us define the difference between the time of arrival of the CM of the incident packet at the point $Z_1$, and that of the transmitted packet at the point $Z_2$ (this time will be called below as the "transmission time"). Analogously, let the "reflection time" be the difference between the time of arrival of the CM of the incident packet at the point $Z_1$ and that of the reflected wave packet at the same point.

Let $t_1$ and $t_2$ be such instants of time that

$$\langle \hat{x} \rangle_{\text{inc}}(t_1) = a - L_1; \quad \langle \hat{x} \rangle_{\text{tr}}(t_2) = b + L_2. \quad (11)$$

Considering (8) and (9), one can write then the "transmission time" $\Delta t_{\text{tr}}$ ($\Delta t_{\text{tr}} = t_2 - t_1$) for the given interval in the form

$$\Delta t_{\text{tr}} = \frac{m}{\hbar} \left[ \frac{\langle J' \rangle_{\text{tr}} + L_2}{\langle k \rangle_{\text{tr}}} + \frac{L_1}{k_0} + a \left( \frac{1}{\langle k \rangle_{\text{tr}}} - \frac{1}{k_0} \right) \right]. \quad (12)$$

For the reflected packet, let $t'_1$ and $t'_2$ be such instances of time that

$$\langle \hat{x} \rangle_{\text{inc}}(t'_1) = \langle \hat{x} \rangle_{\text{ref}}(t'_2) = a - L_1. \quad (13)$$

From equations (8), (10) and (13) it follows that the "reflection time" $\Delta t_{\text{ref}}$ ($\Delta t_{\text{ref}} = t'_2 - t'_1$) can be written as

$$\Delta t_{\text{ref}} = \frac{m}{\hbar} \left[ \frac{\langle J' - F' \rangle_{\text{ref}} + L_1}{\langle -k \rangle_{\text{ref}}} + \frac{L_1}{k_0} + a \left( \frac{1}{\langle -k \rangle_{\text{ref}}} - \frac{1}{k_0} \right) \right]. \quad (14)$$

Notice that the expectation values of $k$ for all three wave packets coincide only in the limit $l_0 \to \infty$ (i.e., for particles with a well-defined momentum). In the general case these quantities are distinct (it is this asymptotic property of the wave packets that is usually treated \[2\] as the acceleration (or retardation) of a particle in tunneling). From Appendix (see (63)) it follows that the rule

$$\tilde{T} \cdot \langle k \rangle_{\text{tr}} + \tilde{R} \cdot \langle -k \rangle_{\text{ref}} = k_0 \quad (15)$$

must be true.

As is seen, times (12) and (14) cannot serve as characteristic times for a particle. Due to the last term in (12) and (14), the above times depend essentially on the initial distance between the wave packet and barrier, with $L_1$ being fixed (a similar peculiarity arises in \[13\]). These contributions are dominant for the sufficiently large distance $a$. Moreover, one of these terms, either in (12) or in (14), must be negative. For example, it takes place for the transmitted wave packet in the case of the under-barrier tunneling through an opaque rectangular barrier. As was mentioned above, the numerical modelling of the tunneling process (see, for example, \[1, 4, 5, 6\]) shows, in this case, a premature appearance of the CM of the transmitted packet behind of the barrier. This fact have
been treated in [2] as the evidence of the lack of a causal link between the transmitted and incident wave packets. To avoid this effect, the fields of application of the SWPA have been restricted (see [1, 4]) by the limiting case $l_0 \to \infty$. A simple analysis shows however that the last terms in (12) and (14) remain dominant in the limit $l_0 \to \infty$, with the ratio $l_0/a$ being fixed. Note that the limit $l_0 \to \infty$ with a fixed value of $a$ is unacceptable in this analysis, because it contradicts the initial condition $a \gg l_0$ for a completed scattering. Thus, even in the limit $l_0 \to \infty$ the above analysis carried out in the context of the SWPA does not provide the well-defined characteristic times for a particle.

3 Formalism of the individual description of the subensembles of transmitted and reflected particles at the initial stage of tunneling

We think that a principle mistake made in the SWPA, in timing scattering particles, is that the "transmission time" was obtained there from the analysis of the relative motion of the transmitted wave packet with respect to the incident one. This is a physically meaningless step. The point is that the incident packet describes the whole quantum ensemble of particles, while the transmitted packet presents only its part. The incident packet can be used as a reference only for the transmitted and reflected packets taken jointly; in this case the initial and final states both describe the whole ensemble of particles. In the individual description of the transmitted (or reflected) packet, its motion should be compared with the corresponding counterpart which describes the state of transmitted (or reflected) particles at the initial stage of the scattering event. Searching for such counterparts for both subensembles is the following step of our analysis.

Initial states of the subensembles of transmitted and reflected particles

Note that $k$-distributions for both subensembles at early times can be obtained immediately if one takes into account the physical sense of the transmission (and reflection) coefficient as well as the Born interpretation of a wave function. So, the transmission coefficient $T(k)$ is, by definition, the probability for the incident particle with the momentum $\hbar k$ to pass ultimately through the barrier. The expression $|f_{\text{inc}}(k, t)|^2 dk$ is the probability for the particle to be, in $k$ space, at time $t$ in the interval $[k, k + dk]$. Since both the probabilities describe statistically independent events, the expression $T(k)|f_{\text{inc}}(k, t)|^2 dk$ is evident to give the probability that both these opportunities happen jointly for this particle (note that this joint probability does not represent a conditional probability introduced in [10]; the former is always positive unlike the latter). From this it follows that the function $f_{\text{tr}}^{\text{inc}}(k, t)$, where

$$f_{\text{tr}}^{\text{inc}}(k, t) = \sqrt{T(k)} \cdot f_{\text{inc}}(k, t) \exp(i\vartheta(k)_{\text{tr}}),$$

(16)
may be considered as a wave function describing at the initial stage the subensemble of transmitted particles. Similarly, the function \( f^{ref}_{inc}(k, t) \), where

\[
f^{ref}_{inc}(k, t) = \sqrt{R(k)} \cdot f_{inc}(k, t) \exp(i\theta(k)_{ref}),
\]

may be considered as a wave function to describe, at this stage, reflected particles. Here \( \theta_{tr}(k) \) and \( \theta_{ref}(k) \) are arbitrary real functions.

We see that the \( k \)-distributions for the to-be-transmitted and transmitted wave packets are the same. The similar situation takes place for reflected particles. The only difference is that the sign of \( k \) is different after the reflection. This property corresponds to that of a classical scattering pointed out in [13] (see Introduction). To complete the correspondence, one needs to state the initial condition for each scattering channel. Namely, we will consider that the CMs of the to-be-transmitted and to-be-reflected wave packets should start at \( t = 0 \) from the same point, i.e., from the origin.

One can see that the above conditions are not sufficient to find uniquely the phase shifts \( \theta_{tr}(k) \) and \( \theta_{ref}(k) \). It is easily to show that there is a variety of wave packets, with the different functions \( \theta_{tr}(k) \) and \( \theta_{ref}(k) \), which start from the origin and described by the same \( k \)-distribution. However, for the following, of great importance is only the fact that the positions of CMs of all such packets coincide at any moment \( t \).

For example, the initial states of the to-be-transmitted and to-be-reflected subensembles, which obey the above requirements, can be defined as follows

\[
\Psi_{inc}^{tr}(x) = \tilde{\Psi}_{inc}^{tr}(x - x_{tr}), \quad \Psi_{inc}^{ref}(x) = \tilde{\Psi}_{inc}^{ref}(x - x_{ref}),
\]

where

\[
\tilde{\Psi}_{inc}^{tr}(x) = \lim_{t \to \infty} e^{i\hat{H}_0 t/\hbar} \hat{\Theta} + e^{-i\hat{H}_0 t/\hbar} \Psi_0(x), \quad \tilde{\Psi}_{inc}^{ref}(x) = \lim_{t \to \infty} e^{i\hat{H}_0^{ref} t/\hbar} \hat{\Theta} - e^{-i\hat{H}_0 t/\hbar} \Psi_0(x);
\]

\[
x_{tr} = \frac{\langle \tilde{\Psi}_{inc}^{tr} | \tilde{\Psi}_{inc}^{tr} \rangle}{\langle \tilde{\Psi}_{inc}^{tr} | \tilde{\Psi}_{inc}^{tr} \rangle}, \quad x_{ref} = \frac{\langle \tilde{\Psi}_{inc}^{ref} | \tilde{\Psi}_{inc}^{ref} \rangle}{\langle \tilde{\Psi}_{inc}^{ref} | \tilde{\Psi}_{inc}^{ref} \rangle}.
\]

It is easy to show that the phases \( \theta(k)_{tr} \) and \( \theta(k)_{ref} \) of the Fourier transforms \( f_{inc}^{tr}(k, 0) \) and \( f_{inc}^{ref}(k, 0) \) of the initial states \( \Psi_{inc}^{tr}(x) \) and \( \Psi_{inc}^{ref}(x) \) read as

\[
\theta(k)_{tr} = J(k) - k < J'(k) >_{tr}, \quad \theta(k)_{ref} = J(k) - F(k) - k < J'(k) - F'(k) >_{ref}.
\]

Functions (18) are evident to be non-orthogonal. In this case the incident wave packet can be presented in the form

\[
f_{inc}(k, t) = f_{inc}^{tr}(k, t) + f_{inc}^{ref}(k, t) + f_{inc}^{int}(k, t)
\]

where

\[
f_{inc}^{int} = \left( 1 - \sqrt{T(k)} \cdot \exp(i\theta(k)_{tr}) - \sqrt{R(k)} \cdot \exp(i\theta(k)_{ref}) \right) \cdot f_{inc}(k, t).
\]
The non-orthogonality of the initial states of the transmitted and reflected subensembles of particles is connected ultimately to the fact that the future of a single particle from the incident quantum ensemble is unpredictable in quantum mechanics. The sorting of particles at the initial stage of scattering is, in fact, purely conditional. In other words, there is an interchange of particles between the subensembles corresponding to the to-be-transmitted and to-be-reflected wave packets. Nevertheless, in spite of this exchange, the number of particles in each subensemble remains constant. One can show that, in this case, the following relations are valid,

\[
<f_{\text{tr inc}}|f_{\text{tr inc}}^\text{tr} > + < f_{\text{ref inc}}|f_{\text{ref inc}}^\text{ref} > = < f_{\text{inc}}|f_{\text{inc}} > .
\]  

(22)

\[
<f_{\text{inc}}|f_{\text{inc}}^\text{tr} > = < f_{\text{tr}}|f_{\text{tr}} > , < f_{\text{inc}}|f_{\text{inc}}^\text{ref} > = < f_{\text{ref}}|f_{\text{ref}} > ;
\]  

(23)

\[
<k >_{\text{inc}}^{\text{tr}} = < k >_{\text{tr}} ; < k >_{\text{inc}}^{\text{ref}} = < -k >_{\text{ref}} ;
\]  

(24)

\[
< \hat{x} >_{\text{inc}}^{\text{tr}} (t) = m^{-1} \hbar < k >_{\text{inc}}^{\text{tr}} \cdot t ;
\]  

(25)

\[
< \hat{x} >_{\text{inc}}^{\text{ref}} (t) = m^{-1} \hbar < k >_{\text{inc}}^{\text{ref}} \cdot t.
\]  

(26)

As is seen, in (22) there are neither terms with \( f_{\text{inc}}^{\text{int}} \) nor terms to describe interference between the three contributions entering decomposition (21). Besides, we have to emphasize that, at this stage, there is no interference between the incident and reflected wave packets (the latter has not yet been formed). Thus, from these expressions it follows that, long before the collision, the whole quantum ensemble of incident particles do indeed consists from two parts (see (22)). Relations (23) suggest that, at this stage, the to-be-transmitted and to-be-reflected parts are described by \( f_{\text{inc}}^{\text{tr}}(k,t) \) and \( f_{\text{inc}}^{\text{ref}}(k,t) \), respectively. For each subensemble, both the number of particles (see (23)) and the absolute value of the average wave-number \( k \) (see (24)) must be the same both before and after the collision. One can easily show also that

\[
\bar{T} \cdot \langle k \rangle_{\text{inc}}^{\text{tr}} + \bar{R} \cdot \langle k \rangle_{\text{inc}}^{\text{ref}} = k_0.
\]  

(27)

Besides, at \( t = 0 \) the CMs of both the packets are located, in accordance with the initial conditions, at the point \( x = 0 \) (see (25) and (26)).

Thus, at the stage long before the scattering event, the incident wave packet can be divided into the to-be-transmitted and to-be-reflected ones whose evolution is described uniquely by relations (22) - (27). This provides solid grounds to say now that tunneling is a two-channel scattering, and (answering the questions in [2]), that the potential barrier does not accelerate (on the average) a particle. By our approach, the absolute value of the average momentum of particles must be conserved asymptotically not only for the whole quantum ensemble, but also for its transmitted and reflected parts. We have to stress that this conservation law was stated in the frame of conventional quantum mechanics, without new guesses.
On the experimental verification of the individual description of scattering channels in tunneling

At first sight, it seems to be strange that, although the need for an individual description of both scattering channels has been recognized (see Introduction), a desirable formalism have not yet been elaborated. In our opinion, the main obstacle which arises here is the problem of the interpretation of quantum mechanics, rather than a purely mathematical one. For example, according to the orthodox interpretation, the above formalism contradicts the causality principle since the knowledge of the initial states for both scattering channels should mean that a particle ”feels” the barrier yet before the interaction (see, for example, [13]). We disagree entirely with such a viewpoint. It itself contradicts Born’s interpretation of a wave function. We think that a proper understanding of the above formalism can be attained only in the framework of the SI which is known to base entirely on the statistical, Born interpretation of a wave function. According to the SI, our formalism does not contradict the causality principle, and can be verified experimentally. Let us consider the last question in detail.

In order to verify our formalism, it is sufficient to check the fact that the ensemble of incident particles can indeed be divided into two parts, with the expectation values of the particle’s position changing in accordance with relations (25) and (26). To perform this check, one needs to obtain the momentum and position (further, in this section, these variables will be denoted by $P$ and $Q$, respectively) distributions for the ensemble of incident particles. For the subensembles of transmitted and reflected particles, one needs to obtain only the $P$-distributions. All the above suggests carrying out the sufficiently large (strictly speaking, infinite) number of identical experiments to provide the needed $P$- and $Q$-data. In order to satisfy conditions (5) - (7) all measurements must be performed far enough from the barrier.

The basic asymptotic property of the subensembles which can be used in this testing is that $P$-distributions for each scattering channel, before and after the scattering event, must be the same. Thus, one needs firstly to obtain experimental $P$-data for the subensembles of transmitted and reflected particles, to extract from them the corresponding $P$-distributions, and then to sort out incident particles into two subsets with the given $P$-distributions. The last step is the most important in this testing. Its peculiarity is that sorting the $P$-data should give two corresponding subsets of the $Q$-data. This means that the above $P$- and $Q$-data must represent the set of $(P,Q)$-pairs. It is naturally to suppose that such pairs can be obtained as a result of a simultaneous measuring of the particle’s position and momentum. According to the SI, ”...there is no conflict with quantum theory in thinking of a particle as having definite [unpredictable] values of both position and momentum...” [17]. Moreover, we have to add that the values of $P$ and $Q$ can be measured simultaneously (jointly) with an arbitrary accuracy (in our opinion, if one assumed that the Heisenberg uncertainty relation does not admit an accurate simultaneous measuring of these quantities, he would then to recognize that this relation cannot be verified in quantum mechanics). We consider that, strictly speaking, there are no restrictions, in the conventional quantum mechanics, to measure jointly these quantities. In particular, this means that one can construct, in principle, such devices that will allow one to measure $P$ and $Q$ simultaneously, without the violation of the connection between the $P$- and $Q$-data.
imposed by the Fourier-transformation. Thus, in line with the terminology introduced in [24], in this question we adhere to the second extreme view. In any case, the possibility to measure jointly $P$ and $Q$ have been accepted, in principle, by many authors (see, for example, [17, 25, 26, 27]).

So, let for the given instant $t$ $(P, Q)$-pairs for incident particles, and $P$-data for scattered particles be available: it is supposed that the total number of the experimental data is sufficiently large and the $(P, Q)$-pairs have been accidentally numerated. So, one may consider the $P$-distributions for the transmitted and reflected subensembles have been known. Let also the $P$-scale be partitioned into sets of intervals with a sufficiently small width $\hbar \Delta k$. Then, let us consider for incident particles the $(P, Q)$-pairs with the $P$ values to belong some interval $[\hbar k, \hbar (k + \Delta k)]$, take from this interval the $T(k)$-th part of these pairs (for example, with the first numbers) and include this part into the to-be-transmitted subensemble. Note that the values of $Q$ must not be taken into account in sorting. Another part should be associated to the to-be-reflected subensemble. All the above should be done for all $\hbar k$-intervals. Then, for incident particles of both scattering channels, we can calculate the expectation values of $P$ and $Q$, for the given $t$, and check lastly expressions (22)-(26). Of course, in doing so, we wait that the $P$- and $Q$-distributions for each subensemble are connected by the Fourier-transformation. Otherwise, they do not correspond some wave functions.

For the following it is important to consider another variant of testing. Namely, let all measurements be carried out at a given point $Z_1$ rather than at a given instant of time. Now we will measure the particle’s momentum $P$ and the value of parameter $t$ at which the incident particle arrives at the given point $Z_1$. Since before the interaction the incident particle is free, the $P$-shape of the corresponding incident wave packet must remain unaltered in time. This means that such measurements should give the same $P$-distribution for the incident packet, as in the above case. Thus, the only difference between both these variants is that now one needs to sort $(P, t)$-pairs instead of $(P, Q)$-pairs. After the mean values of $t$ for each subensemble have been calculated expressions (22)-(26) can be checked. This variant is more suitable for testing characteristic times introduced below.

4 Delay times for the subensembles of transmitted and reflected particles

Now, for both scattering channels, we can offer new definitions of characteristic times to describe the influence of the barrier on a particle. For this purpose we have again to consider the particle’s motion in a wide spatial interval containing the barrier region. In particular, let us calculate the (average) transmission time, $\tau_{tr}$, spent by a particle in the interval $[Z_1, Z_2]$. We have to take into account the fact that before the interaction the motion of the to-be-transmitted wave packet is described by the function $e^{-i\hat{H}t/\hbar}\Psi_{tr}^{inc}(x)$ with the Fourier transform $f_{inc}^{tr}(k, t)$ (see expression (16), (18) - (20)). Then the time-of-arrivals $t_1$ and $t_2$ to correspond the extreme points $Z_1$ and $Z_2$, respectively, must obey the equations
\[
\langle \hat{x} \rangle_{\text{inc}}^{\text{tr}}(t_{1}^{\text{tr}}) = a - L_1; \quad \langle \hat{x} \rangle_{\text{tr}}(t_{2}^{\text{tr}}) = b + L_2.
\] (28)

From this it follows that
\[
\tau_{\text{tr}}(L_1, L_2) \equiv t_{2}^{\text{tr}} - t_{1}^{\text{tr}} = \frac{m}{\hbar} \langle \hat{k} \rangle_{\text{tr}} < J' + L_1 + L_2 >.
\] (29)

Similarly, let the reflection time, \(\tau_{\text{ref}}^{(-)}\), be the difference \(t_{2}^{\text{ref}} - t_{1}^{\text{ref}}\) where
\[
\langle \hat{x} \rangle_{\text{inc}}^{\text{ref}}(t_{1}^{\text{ref}}) = \langle \hat{x} \rangle_{\text{ref}}(t_{2}^{\text{ref}}) = a - L_1,
\] (30)

(remind that the to-be-reflected packet is described at the initial stage of scattering by \(e^{-i\hat{H}_0 t/\hbar}\Psi_{\text{inc}}(x)\) with the Fourier transform \(f_{\text{inc}}^\text{ref}(k, t)\) (see expression (17), (18) - (20))

One can easily show that
\[
\tau_{\text{ref}}^{(-)}(L_1) = \frac{m}{\hbar} \langle \hat{k} \rangle_{\text{ref}} < J' - F' >_{\text{ref}} + 2L_1 >.
\] (31)

As was shown in [23, 28], the sign of the phase \(F'\) is opposite for waves impinging the barrier from the right. This case is similar to that when the wave moves from the left to the potential barrier \(V_{\text{inv}}(x)\) such that \(V_{\text{inv}}(x) = V(a + b - x)\). The corresponding reflection time \(\tau_{\text{ref}}^{(+)}\) can be written as
\[
\tau_{\text{ref}}^{(+)}(L_1) = \frac{m}{\hbar} \langle -\hat{k} \rangle_{\text{ref}} < J' + F' >_{\text{ref}} + 2L_1 >.
\] (32)

One has to stress once more that these quantities cannot be treated, at \(L_1 = L_2 = 0\), as the transmission and reflection times for the barrier region. There is no experiment to measure their values in this case, as the disposition of devices at the boundaries of the barrier does not provide a reliable identification of transmitted and reflected particles: conditions (5) - (7) are not fulfilled in this case.

Note that times (30) - (32) are not suitable to describe the influence of the barrier on a particle, because they include the contributions of the out-of-barrier regions. However, they enables one to define time delays to characterize the relative motion of a scattered and corresponding free particles. To determine these quantities, we have to take into account the fact that beyond the scattering region the mean velocity of a free particle (taken as a reference for each scattering channel) should coincide with that of a scattered particle. Then for transmitted particles the delay time \(\tau_{\text{del}}^{\text{tr}}\) can be written, in terms of wave functions (19) and the tunneling parameters, as
\[
\tau_{\text{del}}^{\text{tr}} = \frac{m}{\hbar} \langle \hat{k} \rangle_{\text{tr}} < J' >_{\text{tr}} - d = \frac{\hat{\Psi}_{\text{inc}}^{\text{tr}} | \hat{\Psi}_{\text{inc}}^{\text{tr}} >}{\Psi_{\text{inc}}^{\text{tr}} | \Psi_{\text{inc}}^{\text{tr}} >} / \frac{\langle \hat{\Psi}_{\text{inc}}^{\text{tr}} | \hat{\xi} | \hat{\Psi}_{\text{inc}}^{\text{tr}} >}{\hat{\Psi}_{\text{inc}}^{\text{tr}} | \Psi_{\text{inc}}^{\text{tr}} >}.
\] (33)

Similarly, for reflected particles the delay time \(\tau_{\text{del}}^{(-)}\) can be written as
\[
\tau_{\text{del}}^{(-)} = \frac{m}{\hbar} \langle -\hat{k} \rangle_{\text{ref}} < J' - F' >_{\text{ref}} - d = \frac{\hat{\Psi}_{\text{inc}}^{\text{ref}} | \hat{\Psi}_{\text{inc}}^{\text{ref}} >}{\Psi_{\text{inc}}^{\text{ref}} | \Psi_{\text{inc}}^{\text{ref}} >} / \frac{\langle \hat{\Psi}_{\text{inc}}^{\text{ref}} | \hat{\xi} | \hat{\Psi}_{\text{inc}}^{\text{ref}} >}{\hat{\Psi}_{\text{inc}}^{\text{ref}} | \Psi_{\text{inc}}^{\text{ref}} >}.
\] (34)
The delay time for the corresponding inverted barrier $V_{\text{inv}}(x)$ is given by

$$\tau_{\text{del}}^{(+)} = \frac{m}{\hbar} \langle -k \rangle_{\text{ref}} (\langle J' + F' \rangle_{\text{ref}} - d)$$

Expressions (33) and (34) (or (35) for the inverted barrier) should be considered as the definitions of delay times in our approach. As is seen, these times do not depend on the initial distance between the incident packet and barrier. Besides, they do not equal to zero for the $\delta$-potential, since $\langle J' \rangle_{\text{tr}}$ and $\langle J' + F' \rangle_{\text{ref}}$ do not vanish in this case.

It is evident that $x_{\text{tr}}$ (see expressions (20)) can be treated as the spatial delay for the subensemble of transmitted particles, with $x_{\text{tr}} = \langle J' \rangle_{\text{tr}} - d$. Similarly, $x_{\text{ref}}$ with $x_{\text{ref}} = \langle J' - F' \rangle_{\text{ref}} - d$ can be treated as the spatial delay for reflected particles. In the case of the inverted barrier, the spatial delay is equal to $\langle J' + F' \rangle_{\text{ref}} - d$.

It is useful to compare respectively expressions (29) and (31) with (4.8) and (4.9) presented in [1]. Unlike our definitions, the integral in expression (4.8) is divergent at the point $k = 0$ if $A(0; k_0, l_0) \neq 0$ and $T(0) \neq 0$. For expression (4.9) a similar situation arises if $A(0; k_0, l_0) \neq 0$ and $R(0) \neq 0$. See also [16, 29].

5 The low bound of the scattering time

It should be noted that the above delays for both subensembles are accumulated during the interaction of a particle with the barrier. In this case the delay times themselves say nothing about the duration of this process. Thus, one has to define the third characteristic time of tunneling, which will be further referred to as the low bound of the scattering time. It is obvious that this quantity cannot be defined individually for each scattering channel, because it should describe just the very stage of the one-dimensional scattering when a particle cannot be identified as incident, transmitted or reflected one. Besides, in this case one should keep in mind that the scattering event lasts until the probability to find a particle in the barrier region is noticeable. This means that in defining this quantity, one should take into account spreading the corresponding wave packet. It is obvious that the scattering time does not coincide with the time of staying the particle in the barrier region (see Introduction). Formally speaking, to define the scattering time, one needs to find such temporal interval for which conditions (5) - (7) are violated.

So, let us define such instant of time, $t_{\text{start}}$, at which the distance between the CM of the incident packet and the left boundary of the barrier is equal to the half-width of this packet, i.e.,

$$(a - \langle \hat{x} \rangle_{\text{inc}} (t_{\text{start}}))^2 = \langle (\delta \hat{x})^2 \rangle_{\text{inc}} (t_{\text{start}}).$$

We will treat the instant $t_{\text{start}}$ as the time of starting the scattering event. We will assume that for $t \leq t_{\text{start}}$ conditions (5) - (7) are fulfilled yet with a sufficient accuracy.

As regards the end of the scattering event, one has to take into account that the transmitted and reflected particles move, on this stage, in the disjoint spatial regions, i.e., their wave packets do not interfere with each other. To define the corresponding instant of time $t_{\text{end}}$, one has to use the quantities $S_{\text{tr+ref}}$ and $\langle (\delta \hat{x})^2 \rangle_{\text{tr+ref}}$ characterizing jointly both these packets (see (72) and (80) in Appendix). Here $S_{\text{tr+ref}}$ is a total mean distance
between the CMs of the transmitted and reflected packets, and the corresponding nearest boundaries of the barrier; \( < (\delta \hat{x})^2 >_{tr+ref} \) is a total mean-square deviation of \( \hat{x} \) averaged over the transmitted and reflected packets.

Thus, let \( t_{end} \) be the instant of time at which

\[
S^2_{tr+ref}(t_{end}) = < (\delta \hat{x})^2 >_{tr+ref}(t_{end}).
\]

This definition suggests that for \( t \geq t_{end} \) conditions (5) - (7) are fulfilled with a sufficient accuracy.

Either of the above equations has two roots. A simple analysis shows that one should take the smallest root, in the case of equation (36), and the biggest root, in the case of (37). So, the searched-for solutions to (36) and (37) can be written in the form

\[
t_{start} = \frac{m}{\hbar} \cdot \frac{ak_0 - \sqrt{l_0^2k_0^2 + (a^2 - l_0^2) < (\delta k)^2 >_{inc}}}{k_0^2 - < (\delta k)^2 >_{inc}},
\]

(remind that \( a \gg l_0 \));

\[
t_{end} = \frac{m}{\hbar} \cdot \frac{\bar{b}k_0 - \chi + \sqrt{l_0^2k_0^2 + \chi^2 - 2k_0b\chi + (b^2 - l^2) < (\delta k)^2 >_{tr+ref}}}{k_0^2 - < (\delta k)^2 >_{tr+ref}}
\]

(see (81)-(84)). The low bound of the scattering time \( \tau_{scatt} \) can be defined now as the difference \( t_{end} - t_{start} \).

A simple analysis shows that this quantity is strictly positive when the inequality

\[
(k_0^2 - < (\delta k)^2 >_{inc}) > 0
\]

is fulfilled. It should be considered as a condition for a completed scattering. It guarantees that (almost) all incident particles start at \( t = 0 \) toward the barrier, and that the transmitted and reflected packets occupy, in the limit \( t \to \infty \), the disjoint spatial regions. For a completed scattering the hierarchy \( t_{end} > t_{start} > 0 \) is obvious to take place. The second inequality is guaranteed by (40). The first one must be valid, since the number of particles in the whole quantum ensemble is constant. Or, in other words, the quantum ensemble of particles cannot exit a region before entering it. In this case it is important to note that, in the limit \( k_0^2 \to < (\delta k)^2 >_{inc} \),

\[
t_{start} = \frac{m(a^2 - l_0^2)}{2\hbar k_0a} \approx \frac{ma}{2\hbar k_0}.
\]

That is, expression (38) has no singularity in this limit.

One can easily show that there is the optimal value of \( l_0 \) at which \( \tau_{scatt} \) is minimal. The point is that, in the limit \( l_0 \to \infty \), the scattering time grows together with \( l_0 \), and, at small values of this parameter, this time is large because of the fast spreading of the wave packet. If requirement (40) is violated, the transmitted and reflected packets must be overlapped at \( t \to \infty \) due to their spreading. As a result, the scattering event becomes incomplete. In this case the process of scattering a particle mimics that of escaping the particle from the barrier region. Our approach allows one to define the scattering time
for the incomplete scattering too. However this question is beyond the framework of our paper.

In the limit \( l_0 \to \infty \), expressions (38) and (39) are essentially simplified. In this case we have \( l \approx l_0, k_0^2 \gg < (\delta k)^2 >_{\text{inc}} \). Thus, the spreading effect may be neglected for narrow (in \( k \)-space) wave packets. Taking account only of the dominant terms in (38) and (39), we obtain

\[
\tau_{\text{scatt}} = \frac{m}{\hbar k_0} (2l_0 + <J'>_{\text{inc}} - <RF'>_{\text{inc}}).
\]

(41)

For the corresponding inverted barrier the last term in (41) has an opposite sign. This term is nonzero only for asymmetrical potential barriers.

One has to stress once again that the above definition yields the low bound of the scattering time. For it does not take into account the "tails" of wave packets. This means that, if \( T \ll 1 \) (or \( R \ll 1 \)), the to-be-transmitted (or to-be-reflected) wave packet may occur in the scattering region when \( t \not\in [t_{\text{start}}, t_{\text{end}}] \). To correct \( \tau_{\text{scatt}} \) for these cases, one needs to "reduce" the size of the packet’s tails. Namely, for these cases a part of particles in the "tails" must be smaller than \( \min(T, R) \). It is obvious that such a correction should increase the scattering time. However, we think this is superfluous. The point is that the quantity \( \tau_{\text{scatt}} \) defined above is sufficient to estimate the duration of the scattering process for the body of the quantum ensemble of particles. As regards its small part (transmitted or reflected), the instants of time \( t_1^{\text{tr}} \) and \( t_2^{\text{tr}} \) (or \( t_1^{\text{ref}} \) and \( t_2^{\text{ref}} \)) given by equations (28) (or (30)) provide a sufficient information about its motion.

6 Tunneling the Gaussian wave packet through rectangular barriers

To display some properties of tunneling, we have considered rectangular potential barriers, and investigated in detail the tunneling parameters of a particle whose initial state is described by the Gaussian wave packet (GWP). The weight function \( A(k; k_0, l_0) \) in (3) is defined in this case by the expression

\[
A(k; k_0, l_0) = \exp(-l_0^2(k - k_0)^2).
\]

As was pointed out above, in the general case the asymptotic value of the average wave-number of transmitted particles differs from that of reflected particles. One can show that for the GWP

\[
<k>_{\text{tr}} = k_0 + \frac{<T'>_{\text{inc}}}{4l_0^2 <T>_{\text{inc}}};
\]

(42)

\[
<-k>_{\text{ref}} = k_0 + \frac{<R'>_{\text{inc}}}{4l_0^2 <R>_{\text{inc}}}. \]

(43)

Supposing that

\[
<k>_{\text{tr}} = k_0 + (\Delta k)_{\text{tr}}, \quad <-k>_{\text{ref}} = k_0 + (\Delta k)_{\text{ref}},
\]

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we can rewrite relations (42) and (43) in the form

\[ \bar{T} \cdot (\Delta k)_{tr} = -\bar{R} \cdot (\Delta k)_{ref} = \frac{< T' >_{inc}}{4l_0^2}. \]  

(44)

Note that \( R' = -T' \). The first equation in (44) coincides with rule (15) to be valid for any wave packet.

Let us also derive several useful correlations for the mean-square deviations of the \( \hat{k} \) operator. Using the relations (62) in Appendix, one can show that for the GWP

\[ < (\delta k)^2 >_{inc} = \frac{1}{4l_0^2}; \]

\[ < (\delta k)^2 >_{tr} = \frac{< T(k)(k - k_0)^2 >_{inc}}{< T >_{inc}} - \frac{(\Delta k)_{tr} < T' >_{inc}}{2l_0^2 - < T >_{inc}} + (\Delta k)_{tr}^2; \]

\[ < (\delta k)^2 >_{ref} = \frac{< R(k)(k - k_0)^2 >_{inc}}{< R >_{inc}} - \frac{(\Delta k)_{ref} < R' >_{inc}}{2l_0^2 - < R >_{inc}} + (\Delta k)_{ref}^2. \]

Further calculations (see (83)) yield

\[ < (\delta k)^2 >_{tr+ref} = \frac{1}{4l_0^2} \left( 1 - \frac{< T' >_{inc}^2}{4l_0^2 TR} \right). \]  

(45)

Now we dwell shortly on the numerical analysis of some peculiarities of tunneling (a more detailed numerical analysis is supposed to be done in the following). All calculations have been carried out on the basis of the TMM [23] (see also [22]). To clear up the role of the spatial localization of a tunneling particle, we have investigated the main features of the \( l_0 \)-dependence of the tunneling parameters for the particular cases when \( V(x) = V_0 = 0.3eV, E_0 = 0.02eV; m = 0.067m_e \) where \( m_e \) is the electron mass. Fig.1 shows the ratios of the mean wave numbers of the transmitted and reflected packets to that of the incident packet versus \( \log(d/l_0) \). As is seen, for particles with the well-defined momentum or position, the expectation values of the momentum for the transmitted and incident packets are the same. The reason is that in the case of the well-defined momentum the incident packet consists, in fact, of a single wave. In the second case the particle tunnels through the barrier without reflection. The point is that the contribution of high-energy harmonics into the GWP, for which the barrier is more transparent, grows together with \( d/l_0 \). As a result, the transmission coefficient increases too (see Fig. 1). In the domain \( 0 \leq \log(d/l_0) \leq 2 \) a situation arises when the contributions of the transmitted and reflected waves are approximately equal. In this case it takes place the most distortion of the packet’s shape as well as the maximal difference between the expectation values of the momentum for the transmitted and incident wave packet. In the case of a many-barrier structure, noticeable variations of \( T(d/l_0) \) can be observed if there are single-wave resonances near the point \( E_0 (E_0 = E(k_0)) \). However, in any case, \( T \to 1, \) in the limit \( l_0 \to 0, \) irrespective of the barrier’s shape and value of \( k_0. \)
Now we address to the spatial delays $\langle J' - d \rangle_{tr}$ and $\langle J' - d \rangle_{ref}$. As was shown in [28], for the rectangular barrier of height $V_0$ and width $d$, $F' \equiv 0$ and the derivative $J'$ is determined by the expressions

$$J' = \frac{2(\kappa^2 - k^2)k^2 \kappa d + (k^2 + \kappa^2)^2 \sinh(2\kappa d)}{\kappa[4k^2\kappa^2 + (k^2 + \kappa^2)^2 \sinh^2(\kappa d)]},$$

(46)

where $\kappa = \sqrt{2m(V_0 - E)}/\hbar^2$, $E < V_0$;

$$J' = \frac{2(k^2 + \kappa^2)k^2 \kappa d - (k^2 - \kappa^2)^2 \sin(2\kappa d)}{\kappa[4k^2\kappa^2 + (k^2 - \kappa^2)^2 \sin^2(\kappa d)]},$$

(47)

where $\kappa = \sqrt{2m(E - V_0)/\hbar^2}$, $E \geq V_0$.

This enables one to explain the numerical data obtained for $\langle J' - d \rangle_{tr}$ and $\langle J' - d \rangle_{ref}$. As is seen from Fig. 2, both quantities are equal only for particles with the well-defined momentum. It is important that in the limit $l_0 \to 0$ the (average) spatial delay for a transmitted particle vanishes. This property, taking place for any barrier, is due to the fact that the average energy of a particle grows infinitely in this limit (such a particle passes freely through the barrier). As regards reflected particles, for rectangular barriers the spatial delay tends to $[J'(k) - d]|_{k=0}$.

It is interesting also to consider the case of the under-barrier tunneling, providing that the barrier’s width grows but the wave-packet width is fixed. It is the very case which is usually analyzed in the literature (e.g., see [1, 13, 30, 31]) to demonstrate the ”superluminal” propagation of particles in tunneling. From (46) it follows that for $E < V_0$ and $\kappa d \gg 1$,

$$J' \approx 2\kappa^{-1},$$

(48)

i.e., in this case, for a particle with a well-defined momentum, this quantity does not depend on $d$. For $E > V_0$ and $\kappa d \gg 1$,

$$J' \approx \frac{2(k^2 + \kappa^2)k^2 d}{4k^2\kappa^2 + k_{top}^2 \sin^2(\kappa d)},$$

(49)

Besides, for $E = V_0$

$$J' = \frac{2}{3} \cdot \frac{9 + 2k_{top} d^2}{4 + k_{top} d^2} \cdot d,$$

(50)

where $k_{top} = \sqrt{2mV_0/\hbar^2}$. From (48) - (50) it follows that for large enough $d$ the function $J'(k)$ is smooth for $E < V_0$ and rapidly oscillates for $E > V_0$. Besides, it grows sharply in going from the ”under-barrier” domain to the ”above-barrier” one. A simple analysis shows that the wider the barrier, the larger the contribution of waves with $E > V_0$ to the transmitted packet. In this case the transmission coefficient decreases until waves with $E < V_0$ dominate in the transmitted wave packet. However, when these waves have been filtered off this packet, $T$ and $< k >_{tr}$ does not depend on $d$ (see Fig. 3 and Fig. 4). These quantities, as functions of $d$ which was varied from 5 nm to 200 nm, have been calculated for $l_0 = 15$ nm; other parameters are the same as for fig.1. As is seen, for the sufficiently
large $d$, $< k >_{tr} \approx k_{top}$. That is, the average energy of particles transmitted through a wide opaque rectangular barrier exceeds slightly the barrier height. Everywhere in the $d$-interval investigated the average energy of reflected particles coincides approximately with that of incident particles; the point is that here $\overline{R} \approx 1$.

Besides, we found (see Fig. 5) that in the "under-barrier" domain $< J' >_{tr} \approx< J' >_{ref} \approx 2.86$ nm. In the "above-barrier" domain, the behavior of $< J' >_{tr} (d)$ is described qualitatively by expression (49). As regards reflected particles, $< J' >_{ref}$ amounts about 2.86 nm in both these domains. For the case considered, $\tau_{scatt} \approx 0.143$ ps.

**Conclusion**

In this paper we have developed a new variant of the wave-packet analysis and proposed a solution to the tunneling time problem. Its key point is a formalism of the individual asymptotic description of transmitted and reflected particles at the stage preceding the scattering event. We have shown that this formalism does not contradict the principles of quantum mechanics, and it can be verified experimentally. According to our approach, the tunneling process, being an elastic one-dimensional scattering of one particle on the time-independent potential barrier, must be treated as a two-channel scattering. For each scattering channel, the momentum distribution must conserve asymptotically.

It is shown that the above formalism gives a good grounds to solve the tunneling time problem. For each scattering channel we have defined delay times which describe the relative motion of the scattered and corresponding free particle moving, outside the scattering region, with the same average velocity. In addition, to characterize the duration of the interaction of a particle with the potential barrier, we define the low bound of the scattering time. It describes jointly both scattering channels. As is shown, there is an optimum initial width of the incident wave packet, leading to the minimal value of the scattering time. This time is strictly positive and describes the main part of particles of the quantum ensemble.

All three characteristic times are defined for wave packets of an arbitrary width. They are obtained in terms of the expectation values of the position and momentum operators (in this sense, characteristic times in the given approach are quantities of a secondary importance). It is important that all integrals arising in our formalism have no singularity at $k = 0$, however the transmission coefficient and function $A(k; k_0, l_0)$ depend on $k$ in the vicinity of $k = 0$. This property may be useful, in particular, in studying the "ultra Hartmann effect" (see [29]).

As is shown, the scattering process is completed, if only the CM of the incident wave packet moves quickly enough (see condition (40). Otherwise, the problem of scattering a particle off the potential barrier turns, in fact, into that of escaping the particle from the barrier region. Note that it is quite possible to adapt our formalism to this case too.
Appendix: the asymptotic properties of a wave function in the \( k \)-representation

The peculiarity of a completed scattering is that at the initial instant of time the incident packet is located entirely to the left of the barrier. After the scattering event there is two packets moving away from the barrier. In the limit \( t \to \infty \), they are located in the disjoint spatial regions.

It is suitable to present the wave functions describing the incident, transmitted and reflected packets in the form

\[
\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk f(k,t) e^{ikx},
\]

where \( f(k,t) \in S_\infty \);

\[
f(k,t) = M(k;k_0,l_0) \exp(i\xi(k,t));
\]

\( M(k;k_0,l_0) \) and \( \xi(k,t) \) are the real functions. In particular, for the incident packet

\[
M_{\text{inc}}(k;k_0,l_0) = cA(k;k_0,l_0); \quad \xi_{\text{inc}}(k,t) = -\frac{\hbar k^2 t}{2m}.
\]

For the transmitted and reflected packets we have

\[
M_{\text{tr}}(k;k_0,l_0) = \sqrt{T(k)}M_{\text{inc}}(k;k_0,l_0); \quad \xi_{\text{tr}}(k,t) = \xi_{\text{inc}}(k,t) + J(k) - kd;
\]

\[
M_{\text{ref}}(-k;k_0) = \sqrt{R(k)}M_{\text{inc}}(k;k_0,l_0); \quad \xi_{\text{ref}}(-k,t) = \xi_{\text{inc}}(k,t) + 2ka + J(k) - F(k) - \frac{\pi}{2}.
\]

In the case of a completed scattering the above packets provide the asymptotic behavior of a wave function.

For any Hermitian operator \( \hat{Q} \), Fourier transformation (51)-(54) enables one to determine the evolution of the expectation value \( <\hat{Q}> \),

\[
<\hat{Q}> = \frac{<\Psi|\hat{Q}|\Psi>}{<\Psi|\Psi>},
\]

at the stages preceding and following the scattering event, where \( \Psi \) is one of the above wave packets. That is, both for the whole quantum ensemble and for its subensembles, we will calculate the expectation values of \( \hat{Q} \) by means of the same rule (55).

Strictly speaking, for the incident and reflected packets, the integrals in (55) should be calculated over the interval \((-\infty,a]\). For the transmitted packet, one should integrate over the interval \([b,\infty)\). The point is that expressions (51)-(54) for these packets are valid only for the corresponding spatial region and corresponding stage of scattering. However, taking into account that the body of each packet is located, in the limit \( t \to \infty \) or \( t \to -\infty \), in its "own" spatial region, we may extend the integration in (55) onto the whole \( OX \)-axis. Due to this step the description of these packets becomes very simple. At the same time, the mistake introduced in the formalism is expected to be negligible: the farther is the packet from the barrier at the initial time, the smaller is this mistake.
It vanishes in the limits \( t \to \infty \). Thus, the asymptotic wave function for a completed scattering (for \( \Psi_0 \in S_\infty \)) may be studied in the \( k \)-representation. Now, on the basis of this representation, we can find the main characteristics of all three packets, which are desirable for the following.

**Normalization**

Note that

\[
< \Psi | \Psi > = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk dk' f^*(k', t) f(k, t) \exp[i(k-k')x] = \\
= \int_{-\infty}^{\infty} dk |f(k, t)|^2 = \int_{-\infty}^{\infty} dk M^2(k; k_0, l_0). \tag{56}
\]

For each packet we have then the following norms. Since the particle is located at the initial time to the left of the barrier, we have

\[
< \Psi_0 | \Psi_0 > = \int_{-\infty}^{\infty} dk M^2_{\text{inc}}(k; k_0, l_0) = 1. \tag{57}
\]

Then, allowing for (52), we have

\[
c^{-2} = \int_{-\infty}^{\infty} dk A^2(k; k_0, l_0). \tag{58}
\]

For the transmitted packet,

\[
< f | f >_{\text{tr}} = \int_{-\infty}^{\infty} dk M^2_{\text{tr}}(k; k_0, l_0) = \int_{-\infty}^{\infty} dk T(k) M^2_{\text{inc}}(k; k_0, l_0) \equiv < T(k) >_{\text{inc}} \equiv \bar{T}. \tag{59}
\]

For the reflected packet,

\[
< f | f >_{\text{ref}} = \int_{-\infty}^{\infty} dk M^2_{\text{ref}}(k; k_0, l_0) = \int_{-\infty}^{\infty} dk R(k) M^2_{\text{inc}}(-k; k_0). \tag{60}
\]

Having made an obvious change of variables, we obtain

\[
< f | f >_{\text{ref}} = < R(k) >_{\text{inc}} \equiv \bar{R}.
\]

From (57) - (60) it follows that

\[
\bar{T} + \bar{R} = 1.
\]

**The expectation values of the operators \( \hat{k}^n \) (\( n \) is the positive number)**

Considering (51), one can find for all the packets that

\[
< \Psi | \hat{k} | \Psi > = -i < \Psi | \frac{\partial \Psi}{\partial x} > = \int_{-\infty}^{\infty} dk M^2(k; k_0, l_0)k
\]
(i.e., $\hat{k}$ is a multiplication operator in this case). Then for any value of $n$ we have

$$\langle \Psi | \hat{k}^n | \Psi \rangle = \langle f | \hat{k}^n | f \rangle = \int_{-\infty}^{\infty} dk M^2(k; k_0, l_0) k^n. \quad (61)$$

Now we can treat the individual packets. From (61) and (53) it follows that

$$\langle f_{tr} | \hat{k}^n | f_{tr} \rangle = \langle f_{inc} | T(k) \hat{k}^n | f_{inc} \rangle.$$

In a similar way we find also that

$$\langle f_{ref} | \hat{k}^n | f_{ref} \rangle = (-1)^n \langle f_{inc} | R(k) \hat{k}^n | f_{inc} \rangle,$$

and, hence,

$$\langle T(k) \hat{k}^n \rangle_{inc} = \bar{T} \langle k^n \rangle_{tr}, \quad \langle R(k) \hat{k}^n \rangle_{inc} = (-1)^n \bar{R} \langle k^n \rangle_{ref}. \quad (62)$$

As a consequence, the next correlation is obvious to be valid

$$\langle k^n \rangle_{inc} = \bar{T} \langle k^n \rangle_{tr} + \bar{R} \langle (-k)^n \rangle_{ref}. \quad (63)$$

The expectation values of the operator $\hat{x}$

We begin again with the expressions to be common for all three packets. We have

$$\langle \Psi | \hat{x} | \Psi \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk dk' f^*(k', t) f(k, t) x \exp[i(k - k')x]. \quad (64)$$

Substituting $-i \frac{\partial}{\partial k} \exp(i(k - k')x)$ for the expression $x \exp(i(k - k')x)$, and integrating in parts, we find that

$$\langle \Psi | \hat{x} | \Psi \rangle = i \int_{-\infty}^{\infty} dk f^*(k, t) \frac{\partial f(k, t)}{\partial k} =$$

$$= i \int_{-\infty}^{\infty} dk M(k; k_0, l_0) \frac{dM(k; k_0, l_0)}{dk} - \int_{-\infty}^{\infty} dk M^2(k; k_0, l_0) \frac{\partial \xi(k, t)}{\partial k}. \quad (65)$$

Since the first term here is equal to

$$\frac{i}{2} M^2(k; k_0, l_0) |_{-\infty}^{\infty} = 0,$$

we have

$$\langle \Psi | \hat{x} | \Psi \rangle = - \int_{-\infty}^{\infty} dk M^2(k; k_0, l_0) \frac{\partial \xi(k, t)}{\partial k} \equiv - \langle f | \frac{\partial \xi(k, t)}{\partial k} | f \rangle. \quad (66)$$

For the incident and transmitted packets, taking into account expressions (52) and (53) for $\xi(k, t)$, we obtain

$$\langle \hat{x} \rangle_{inc} = \frac{\hbar t}{m} \langle k \rangle_{inc}. \quad (67)$$
\[ <\hat{x}>_{tr} = \frac{ht}{m} < k >_{tr} - < J'(k) >_{tr} + d. \] (68)

Since the functions \( J'(k) \) and \( F'(k) \) are even, from (54) it follows that

\[ <\hat{x}>_{ref} = 2a + < J'(k) - F'(k) >_{ref} - \frac{ht}{m} < -k >_{ref}. \] (69)

Let, at the instant \( t \), \( S_{tr} \) be the distance between the CM of the transmitted packet and the nearest boundary of the barrier, i.e., \( S_{tr} = < \hat{x}>_{tr} - b \). Similarly, let \( S_{ref} \) be the distance between the CM and the corresponding barrier’s boundary for the reflected packet at the same instant: \( S_{ref} = a - < \hat{x}>_{ref} \). From (68) and (69) it follows that

\[ S_{tr}(t) = \frac{ht}{m} < k >_{tr} - < J'(k) >_{tr} - a, \] (70)

\[ S_{ref}(t) = \frac{ht}{m} < -k >_{ref} - < J'(k) - F'(k) >_{ref} - a. \] (71)

Let us define now the average distance \( S_{tr+ref}(t) \) describing the both packets jointly:

\[ S_{tr+ref}(t) = \bar{T}S_{tr}(t) + \bar{R}S_{ref}(t). \] (72)

Considering (70), (71) and (63), we get

\[ S_{tr+ref}(t) = \frac{ht}{m} < k >_{inc} - \bar{b}, \] (73)

where \( \bar{b} = a + < J'(k) >_{inc} - < R(k)F'(k) >_{inc} \) (note that \( < J'(k) >_{inc} = d \), and \( < F'(k) >_{inc} = 0 \) when \( V(x) = 0 \)).

**The mean-square deviations in x-space**

Let us derive firstly the expression to be common for all packets. We have

\[ <\Psi|\hat{x}^2|\Psi> = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dkdk' f^*(k', t)f(k, t)x^2 \exp[i(k - k')x]. \]

Substituting the expression \(-\frac{\partial^2}{\partial k^2} \exp(i(k-k')x)\) for that \( x^2 \exp(i(k-k')x) \), and integrating in parts, we find

\[ <\Psi|\hat{x}^2|\Psi> = -\int_{-\infty}^{\infty} dkf^*(k, t)\frac{\partial^2 f(k, t)}{\partial k^2}. \]

Since

\[ \frac{\partial^2 f(k, t)}{\partial k^2} = \left[M'' - M(\xi')^2 + i(2M'\xi' + M\xi'')\right]e^{ik}, \]

we have

\[ <\Psi|\hat{x}^2|\Psi> = \int_{-\infty}^{\infty} dk M[M(\xi')^2 - M''] - i \int_{-\infty}^{\infty} dk[(M^2)'\xi' + M^2\xi''] \] (74)

(hereinafter the prime denotes the derivative with respect to \( k \) if the functions of two variables are written without the independent variables). One can easily show that the last integral in (74) is equal to zero. Therefore
\[ < \Psi | \hat{x}^2 | \Psi > = \int_{-\infty}^{\infty} dk M^2(k; k_0, l_0)[\xi(k, t)]^2 + \int_{-\infty}^{\infty} dk [M'(k; k_0, l_0)]^2. \] (75)

Let, for any operator \( \hat{Q} \), \( < (\delta \hat{Q})^2 > \) be the mean-square deviation \( < \hat{Q}^2 > - < \hat{Q} >^2 \); \( \delta \hat{Q} = \hat{Q} - < \hat{Q} > \). Then for the operator \( \hat{x} \) we have

\[ < (\delta \hat{x})^2 > = < (\ln' M)^2 > + < (\delta \xi')^2 > . \] (76)

Now we are ready to determine these quantities for each packet. Using (76) and expressions (52)-(54), one can show that for incident packet

\[ < (\delta \hat{x})^2 >_{inc} = < (\ln' A)^2 >_{inc} + \frac{h^2 l^2}{m^2} < (\delta k)^2 >_{inc} \] (77)

(the first term here, in accordance with the initial condition, is equal to \( l_0^2 \)); for the transmitted packet

\[ < (\delta \hat{x})^2 >_{tr} = < (\ln' M_{tr})^2 >_{tr} + < (\delta J')^2 >_{tr} - \]

\[ -2 \frac{\hbar t}{m} < (\delta J') (\delta k) >_{tr} + \frac{h^2 l^2}{m^2} < (\delta k)^2 >_{tr}; \] (78)

for the reflected packet

\[ < (\delta \hat{x})^2 >_{ref} = < (\ln' M_{ref})^2 >_{ref} + < (\delta J' - \delta F')^2 >_{ref} + \]

\[ +2 \frac{\hbar t}{m} < (\delta J' - \delta F') (\delta k) >_{ref} + \frac{h^2 l^2}{m^2} < (\delta k)^2 >_{ref} . \] (79)

Let us determine now the mean-square value of \( (\delta \hat{x})^2 \) averaged jointly over the transmitted and reflected packets:

\[ < (\delta \hat{x})^2 >_{tr+ref} = \bar{T} < (\delta \hat{x})^2 >_{tr} + \bar{R} < (\delta \hat{x})^2 >_{ref} \] (80)

(note that \( < (\delta \hat{x})^2 >_{tr+ref} \neq < (\delta \hat{x})^2 >_{inc} \) because \( < \hat{x} >_{tr+ref} \neq < \hat{x} >_{ref} \) in the general case.) Considering (77)-(79), we can reduce this expression to the form

\[ < (\delta \hat{x})^2 >_{tr+ref} = l^2 - 2 \frac{\hbar t}{m} \chi + \frac{h^2 l^2}{m^2} < (\delta k)^2 >_{tr+ref}, \] (81)

where

\[ l^2 = \bar{T} < (\ln' M_{tr})^2 >_{tr} + \bar{R} < (\ln' M_{ref})^2 >_{ref} \]

\[ + \bar{T} < (\delta J')^2 >_{tr} + \bar{R} < (\delta J' - \delta F')^2 >_{ref}, \]

\[ \chi = \bar{T} < (\delta J') (\delta k) >_{tr} + \bar{R} < (\delta J' - \delta F') (\delta k) >_{ref}, \] (82)

\[ < (\delta k)^2 >_{tr+ref} = \bar{T} < (\delta k)^2 >_{tr} + \bar{R} < (\delta k)^2 >_{ref}. \] (83)
The first two terms in the expression for $l^2$ may be rewritten, using (53), (54) and the correlation $T' + R' = 0$, as

$$l^2 = \langle (\ln' A)^2 \rangle_{inc} - \frac{1}{4} \langle (\ln' T)(\ln' R) \rangle_{inc} +$$

$$+\bar{T} \langle (\delta J')^2 \rangle_{tr} + \bar{R} \langle (\delta J' - \delta F')^2 \rangle_{ref};$$

(84)

here the second term is positive and not singular both at the resonances and at the point $k = 0$. Since $\langle (\ln' A)^2 \rangle_{inc} = l_0^2$, we have $l^2 > l_0^2$. 


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Figure captions

Figure 1. The transmission coefficient ($\circ$) as well as the ratios $<k>_{tr}/k_0$ (solid line) and $<-k>_{ref}/k_0$ (dashed line) versus $\lg(d/l_0)$, for the rectangular barriers with $V_0 = 0.3$ eV, $d = 5$ nm; $E_0 = 0.02$ eV, $m = 0.067m_e$; where $m_e$ is the electron’s mass.

Figure 2. The spatial delays $<J'-d>_{tr}$ and $<J'-d>_{ref}$ for the transmitted (solid line) and reflected (dashed line) particles, respectively. The barrier’s and particle’s parameters are the same as for fig.1.

Figure 3. $\ln(T)$ versus $d$, for $l_0 = 15$ nm; the other parameters are the same as for fig.1.

Figure 4. The ratios $<k>_{tr}/k_0$ versus $d$, for the same parameters (see fig.3).

Figure 5. The dependence of $<J'>_{tr}$ on $d$, for the same parameters (see fig.3).
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