DARBOUX TRANSFORMATIONS FOR
MULTIVARIATE ORTHOGONAL POLYNOMIALS

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Abstract. Darboux transformations for polynomial perturbations of a real multivariate measure are
found. The 1D Christoffel formula is extended to the multidimensional realm: multivariate orthogonal
polynomials are expressed in terms of last quasi-determinants and sample matrices. The coefficients of
these matrices are the original orthogonal polynomials evaluated at a set of nodes, which is supposed
to be poised. A discussion for the existence of poised sets and geometrically poised sets is given in
terms of algebraic varieties in the complex affine space.

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1. INTRODUCTION

In a recent paper [14] we studied how the Gauss–Borel or LU factorization of a moment matrix
allows for a better understanding of the links between multivariate orthogonal polynomials (MVOPR)
on a multidimensional real space $\mathbb{R}^D$, $D \geq 1$, and integrable systems of Toda and KP type. In
particular, it was shown how the LU decomposition allows for a simple construction of the three
term relation or the Christoffel–Darboux formula. Remarkably, it is also useful for the construction
of Miwa type expressions in terms of quasi-tau matrices of the MVOPR or the finding of the Darboux
transformation. Indeed, we presented for the first time Darboux transformations for orthogonal
polynomials in several variables, that we called elementary, and its iteration, resulting in a Christoffel
quasi-determinantal type formula. These Darboux transformations allow for the construction of new
MVOPR, associated with a perturbed measure, from the MVOPR of a given non perturbed measure.

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Observe that they also provide a direct method to construct new solutions of the associated Toda–KP type integrable systems.

What we called elementary Darboux transformations in [14] where given as the multiplication of the non perturbed measure by a degree one multivariate polynomial. The \( m \)-th iteration of these so called elementary Darboux transformations leads, therefore, to a perturbation by a multivariate polynomial of degree \( m \). This way of proceeding was motivated by the one dimensional situation, in that case happens that the irreducible polynomials have degree one (the fundamental theorem of algebra). But, in higher dimension the situation is much richer and we do have many irreducible polynomials of higher degree. Therefore, the territory explored for the Darboux transformations in [14] was only a part, significant but incomplete, of a further more vast territory. In this paper, see Theorem 2.1 we give a generalization of the Darboux transformations found in [14] that holds for a perturbation by a polynomial of any degree. This provides us with an elegant quasi-determinantal expression for the new MVOPR which is a broad extension of the 1D determinantal Christoffel formula.

For the construction of the mentioned general Darboux transformation we use multivariate interpolation theory, see [40]. Therefore, we need of poised sets for which the sample matrix is not singular. In this paper we initiate the study of poised sets for general Darboux transformations. We find that the analysis can be split into two parts, one measure-independent part depending exclusively on the relative positions of nodes in the algebraic hypersurface of the generating polynomial, that we refer to as geometrically poised, and another related to the non perturbed measure. The geometrical part, as usual in interpolation theory, requires of the concourse of Vandermonde matrices. In fact, of multivariate Vandermonde matrices, see [40], or multivariate confluent Vandermonde matrices.

With the aid of basic facts in algebraic geometry, see for example [29] or [14] we are able to show, for generating polynomials that can be expressed as the product \( Q = Q_1 \cdots Q_N \) of \( N \) prime factors, –see Theorem 3.1– that there exists, in the complex domain, geometrically poised sets of nodes by forbidding its belonging to any further algebraic hypersurface, different from the algebraic hypersurface of \( Q \), of certain degrees. Moreover, we see that for a perturbation of the measure by a polynomial of the form \( Q = R^d \), geometrically poised sets never exists, and the Darboux transformation as presented in Theorem 2.1 is not applicable. However, with the use of Wronski matrices we can avoid this problem and find an appropriate extension of the Darboux transformation, see Theorem 4.3, for a generating polynomial of the form \( Q = Q_1^{d_1} \cdots Q_N^{d_N} \) where the polynomials \( Q_i \) are irreducible. The discussion on poised sets in this general scenario is given in Theorem 4.4, where again the set of nodes when geometrically poised can not belong to any further algebraic variety of certain type.

The layout of the paper is as follows. Within this introduction we further perform a number observations regarding the historical background and context of the different mathematical issues discussed in this paper. Then, we reproduce, for the reader commodity, some necessary material from [14]. In §2 we give the Darboux transformation generated by a multivariate polynomial, and in §3 we discuss poised sets and introduce geometrically poised sets, giving several conditions for the nodes in order to constitute a geometrically poised set. Finally, in §4 we see that the previous construction fails in some cases, and then we present an extension of the Darboux transformations which overcomes this problem. We also discuss when the set of nodes is geometrically poised.

1.1. Historical background and context.

1.1.1. Darboux transformations. These transformations were introduced in [19] in the context of the Sturm–Liouville theory and since them have been applied in several problems. It was in [38], a paper where explicit solutions of the Toda lattice where found, where this covariant transformation was given the name of Darboux. It has been used in the 1D realm of orthogonal polynomials quite successfully, see for example [17, 16, 17, 37]. In Geometry, the theory of transformations of surfaces preserving some given properties conforms a classical subject, in the list of such transformations
given in the classical treatise by Einsehart [22] we find the Lévy (Lucien) transformation, which later on was named as elementary Darboux transformation and known in the orthogonal polynomials context as Christoffel transformation [47, 46]; in this paper we have denoted it by $T$. The adjoint elementary Darboux or adjoint Lévy transformation $T^{-1}$ is also relevant [38, 20] and is referred some times as a Geronimus transformation [47]. For further information see [13, 28]. For the iteration of elementary Darboux transformations let us mention that the Szegö [46] points out that for $d \mu = dx$ the iteration formula is due to Christoffel [18]. This fact was rediscovered much latter in the Toda context, see for example the formula (5.1.11) in [38] for $W_n^+(N)$.

1.1.2. Multivariate orthogonal polynomials. We refer the reader to monographs [21] and [53]. The recurrence relation for orthogonal polynomials in several variables was studied by Xu in [48], while in [49] he linked multivariate orthogonal polynomials with a commutative family of self-adjoint operators and the spectral theorem was used to show the existence of a three term relation for the orthogonal polynomials. He discusses in [50] how the three term relation leads to the construction of multivariate orthogonal polynomials and cubature formulæ. Xu considers in [51] polynomial subspaces that contain discrete multivariate orthogonal polynomials with respect to the bilinear form and shows that the discrete orthogonal polynomials still satisfy a three-term relation and that Favard’s theorem holds. The analysis of orthogonal polynomials and cubature formulæ on the unit ball, the standard simplex, and the unit sphere [52] lead to conclude the strong connection of orthogonal structures and cubature formulæ for these three regions. The paper [51] presents a systematic study of the common zeros of polynomials in several variables which are related to higher dimensional quadrature. Karlin and McGregor [33] and Milch [39] discussed interesting examples of multivariate Hahn and Krawtchouk polynomials related to growth birth and death processes. A study of two-variable orthogonal polynomials associated with a moment functional satisfying the two-variable analogue of the Pearson differential equation and an extension of some of the usual characterizations of the classical orthogonal polynomials in one variable was found [23].

1.1.3. Quasi-determinants. For its construction we may use Schur complements. Besides its name observe that the Schur complement was not introduced by Issai Schur but by Emilie Haynsworth in 1968 in [30, 31]. In fact, Haynsworth coined that name because the Schur determinant formula given in what today is known as Schur lemma in [15]. In the book [55] one can find an ample overview on the Schur complement and many of its applications. The most easy examples of quasi-determinants are Schur complements. In the late 1920 Archibald Richardson [41, 42], one of the two responsible of Littlewood–Richardson rule, and the famous logician Arend Heyting [32], founder of intuitionist logic, studied possible extensions of the determinant notion to division rings. Heyting defined the designant of a matrix with noncommutative entries, which for $2 \times 2$ matrices was the Schur complement, and generalized to larger dimensions by induction. Let us stress that both Richardson’s and Heyting’s quasi-determinants were generically rational functions of the matrix coefficients. A definitive impulse to the modern theory was given by the Gel’fand’s school [25, 26, 27, 24]. Quasi-determinants where defined over free division rings and it was early noticed that it was not an analog of the commutative determinant but rather of a ratio determinants. A cornerstone for quasi-determinants is the heredity principle, quasi-determinants of quasi-determinants are quasi-determinants; there is no analog of such a principle for determinants. However, many of the properties of determinants extend to this case, see the cited papers. Let us mention that in the early 1990 the Gelfand school [26] already noticed the role quasi-determinants had for some integrable systems. All this paved the route, using the connection with orthogonal polynomials à la Cholesky, to the appearance of quasi-determinants in the multivariate orthogonality context. Later, in 2006 Peter Olver applied quasi-determinants to multivariate interpolation [40], now the blocks have different sizes, and so multiplication of blocks is only allowed if they are compatible. In general, the (non-commutative) multiplication makes sense if the number of columns and rows of the blocks involved fit well. Moreover, we are only permitted to invert diagonal entries that in general makes the minors expansions by columns or rows not
applicable but allows for other result, like the Sylvester’s theorem, to hold in this wider scenario. The last quasi-determinant used in this paper is the one described in [10], see also [14].

1.1.4. LU factorization. This technique was the corner stone for Mark Adler and Pierre van Moerbeke when in a series of papers where the theory of the 2D Toda hierarchy and what they called the discrete KP hierarchy was analyzed [11]-[7]. These papers clearly established –from a group-theoretical setup– why standard orthogonality of polynomials and integrability of nonlinear equations of Toda type where so close. In fact, the LU factorization of the moment matrix may be understood as the Gauss–Borel factorization of the initial condition for the integrable hierarchy. In the Madrid group, based on the Gauss–Borel factorization, we have been searching further the deep links between the Theory of Orthogonal Polynomials and the Theory of Integrable Systems. In [8] we studied the generalized orthogonal polynomials [11] and its matrix extensions from the Gauss–Borel view point. In [9] we gave a complete study in terms of factorization for multiple orthogonal polynomials of generalized orthogonal polynomials [1] and its matrix extensions from the Gauss–Borel view point. Based on the Gauss–Borel factorization, we have been searching further the deep links between the Gauss–Borel factorization of the initial condition for the integrable hierarchy. In the Madrid group, we introduced the set

\[ [k] := \{ \alpha \in \mathbb{Z}^D_+ : |\alpha| = k \}, \]

built up with those vectors in the lattice \( \mathbb{Z}^D_+ \) with a given length \( k \). We will use the graded lexicographic order; i.e., for \( \alpha_1, \alpha_2 \in [k] \)

\[ \alpha_1 \succ \alpha_2 \iff \exists p \in \mathbb{Z}_+ \text{ with } p < D \text{ such that } \alpha_{1,1} = \alpha_{2,1}, \ldots, \alpha_{1,p} = \alpha_{2,p} \text{ and } \alpha_{1,p+1} < \alpha_{2,p+1}, \]

and if \( \alpha^{(k)} \in [k] \) and \( \alpha^{(\ell)} \in [\ell] \), with \( k < \ell \) then \( \alpha^{(k)} \prec \alpha^{(\ell)} \). Given the set of integer vectors of length \( k \) we use the lexicographic order and write

\[ [k] = \{ \alpha^{(k)}_1, \alpha^{(k)}_2, \ldots, \alpha^{(k)}_{|k|} \} \text{ with } \alpha^{(k)}_a > \alpha^{(k)}_{a+1}. \]

Here \( |[k]| \) is the cardinality of the set \( [k] \), i.e., the number of elements in the set. This is the dimension of the linear space of homogenous multivariate polynomials of total degree \( k \). Either counting weak compositions or multisets one obtains \( |[k]| = \binom{D+k-1}{k} \). The dimension of the linear space \( \mathbb{R}_k[x_1, \ldots, x_D] \) of multivariate polynomials of degree less or equal to \( k \) is

\[ N_k = 1 + |[2]| + \cdots + |[k]| = \binom{D+k}{D}. \]
Observe that for \( k = 1 \) we have that the vectors \( \alpha^{(1)}_a = e_a \) for \( a \in \{1, \ldots, D\} \) forms the canonical basis of \( \mathbb R^D \), and for any \( \alpha_j \in [k] \) we have \( \alpha_j = \sum_{a=1}^D \alpha_j^a e_a \). For the sake of simplicity unless needed we will drop off the super-index and write \( \alpha_j \) instead of \( \alpha_j^{(k)} \), as it is understood that \( |\alpha_j| = k \).

The dual space of the symmetric tensor powers is isomorphic to the set of symmetric multilinear functionals on \( \mathbb R^D \), \( \text{Sym}^k(\mathbb R^D)^* \cong S((\mathbb R^D)^k, \mathbb R) \). Hence, homogeneous polynomials of a given total degree can be identified with symmetric tensor powers. Each multi-index \( \alpha \in [k] \) can be thought as a weak \( D \)-composition of \( k \) (or weak composition in \( D \) parts), \( k = \alpha_1 + \cdots + \alpha_D \). Notice that these weak compositions may be considered as multisets and that, given a linear basis \( \{ e_a \}_{a=1}^D \) of \( \mathbb R^D \) we have the linear basis \( \{ e_{a_1} \odot \cdots \odot e_{a_k} \}_{a_1, \ldots, a_k \leq D} \) for the symmetric power \( S^k(\mathbb R^D) \), where we are using multisets

\[
1 \leq a_1 \leq \cdots \leq a_k \leq D.
\]

In particular the vectors of this basis \( e_{a_1} \odot \cdots \odot e_{a_k} \), or better its duals \( (e_{a_1}^*) \odot \cdots \odot (e_{a_k}^*) \) are in bijection with monomials of the form \( x_{a_1}^{\alpha_1} \cdots x_{a_k}^{\alpha_k} \).

The lexicographic order can be applied to \( (\mathbb R^D)^{\odot k} \cong \mathbb R^{[k]} \), then we can take a linear basis of \( S^k(\mathbb R^D) \) as the ordered set \( B_c = \{ e^{\alpha_1}, \ldots, e^{\alpha_{[k]}} \} \) with \( e^{\alpha_j} := e_1^{\alpha_{[1]}^j} \otimes \cdots \otimes e_D^{\alpha_{[D]}^j} \) so that \( \chi^{[k]}(x) = \sum_{j=1}^{[k]} x^{\alpha_j} e^{\alpha_j} \).

We consider semi-infinite matrices \( A \) with a block or partitioned structure induced by the graded reversed lexicographic order

\[
A = \begin{pmatrix}
A_{[0],[0]} & A_{[0],[1]} & \cdots \\
A_{[1],[0]} & A_{[1],[1]} & \cdots \\
\vdots & \vdots & \ddots \\
A_{[k],[0]} & A_{[k],[1]} & \cdots \\
\end{pmatrix}, \quad
A_{[k],[\ell]} = \begin{pmatrix}
A_{[1],[\ell]} & A_{[1],[\ell]} & \cdots \\
A_{[2],[\ell]} & A_{[2],[\ell]} & \cdots \\
\vdots & \vdots & \ddots \\
A_{[k],[\ell]} & A_{[k],[\ell]} & \cdots \\
\end{pmatrix} \in \mathbb R^{[k] \times [\ell]}.
\]

We use the notation \( 0_{[k],[\ell]} \in \mathbb R^{[k] \times [\ell]} \) for the rectangular zero matrix, \( 0_{[k]} \in \mathbb R^{[k]} \) for the zero vector, and \( I_{[k]} \in \mathbb R^{[k] \times [k]} \) for the identity matrix. For the sake of simplicity we normally just write 0 or 1 for the zero or identity matrices, and we implicitly assume that the sizes of these matrices are the ones indicated by its position in the partitioned matrix.

**Definition 1.1.** Associated with the measure \( d \mu \) we have the following moment matrix

\[
G := \int_\Omega \chi(x) d \mu(x) \chi(x)^T.
\]

We write the moment matrix in block form

\[
G = \begin{pmatrix}
G_{[0],[0]} & G_{[0],[1]} & \cdots \\
G_{[1],[0]} & G_{[1],[1]} & \cdots \\
\vdots & \vdots & \ddots \\
G_{[n],[0]} & G_{[n],[1]} & \cdots \\
\end{pmatrix}.
\]

Truncated moment matrices are given by

\[
G^{[\ell]} := \begin{pmatrix}
G_{[0],[0]} & \cdots & G_{[0],[\ell-1]} \\
\vdots & \ddots & \vdots \\
G_{[\ell-1],[0]} & \cdots & G_{[\ell-1],[\ell-1]} \\
\end{pmatrix}.
\]
Notice that from the above definition we know that the moment matrix is a symmetric matrix, \(G = G^\top\), which implies that a Gauss–Borel factorization of it, in terms of lower unitriangular and upper triangular matrices, is a Cholesky factorization.

**Proposition 1.1.** If the last quasi-determinants \(\Theta_\ast(G^{[k+1]}), \ k \in \{0, 1, \ldots\}\), of the truncated moment matrices are invertible the Cholesky factorization

\[
(1.1) \quad G = S^{-1}H \left(S^{-1}\right)^\top,
\]

with

\[
S^{-1} = \begin{pmatrix}
\mathbb{I} & 0 & 0 & \cdots \\
(S^{-1})_{[1],[0]} & \mathbb{I} & 0 & \cdots \\
& (S^{-1})_{[2],[0]} & \mathbb{I} & \cdots \\
& & \vdots & \ddots
\end{pmatrix}, \quad H = \begin{pmatrix}
H_{[0]} & 0 & 0 & \cdots \\
0 & H_{[1]} & 0 & \cdots \\
0 & 0 & H_{[2]} & \cdots \\
& \vdots & \vdots & \ddots
\end{pmatrix},
\]

can be performed. Moreover, the rectangular blocks can be expressed in terms of last quasi-determinants of truncations of moment matrix

\[
H_{[k]} = \Theta_\ast(G^{[k+1]}), \quad (S^{-1})_{[k],[\ell]} = \Theta_\ast(G^{[\ell+1]})^{-1} \Theta_\ast(G^{[k+1]}).
\]

We are ready to introduce the MVOPR

**Definition 1.2.** The MVOPR associated to the measure \(d\mu\) are

\[
(1.2) \quad P = S\chi = \begin{pmatrix} P_{[0]} \\ P_{[1]} \\ \vdots \end{pmatrix}, \quad P_{[k]}(x) = \sum_{\ell=0}^{k} S_{[k],[\ell]} \chi_{[\ell]}(x) = \begin{pmatrix} P^{(k)}_{\alpha_{i}} \\ \vdots \\ P^{(k)}_{\alpha_{k}} \end{pmatrix}, \quad P^{(k)}_{\alpha_{i}} = \sum_{\ell=0}^{k} \sum_{j=1}^{[\ell]} S_{\alpha_{i}, \alpha_{j}}^{(k)} x^{\alpha_{j}}.
\]

Observe that \(P_{[k]} = \chi_{[k]}(x) + \beta_{[k]} \chi_{[k-1]}(x) + \cdots\) is a vector constructed with the polynomials \(P_{\alpha_{i}}(x)\) of degree \(k\), each of which has only one monomial of degree \(k\); i. e., we can write \(P_{\alpha_{i}}(x) = x^{\alpha_{i}} + Q_{\alpha_{i}}(x)\), with \(\deg Q_{\alpha_{i}} < k\).

**Proposition 1.2.** The MVOPR satisfy

\[
(1.3) \quad \int_{\Omega} P_{[k]}(x) d\mu(x) (P_{[\ell]}(x))^\top = \int_{\Omega} P_{[k]}(x) d\mu(x) (\chi_{[\ell]}(x))^\top = 0, \quad \ell = 0, 1, \ldots, k - 1,
\]

\[
(1.4) \quad \int_{\Omega} P_{[k]}(x) d\mu(x) (P_{[k]}(x))^\top = \int_{\Omega} P_{[k]}(x) d\mu(x) (\chi_{[k]}(x))^\top = H_{[k]}.
\]

Therefore, we have the following orthogonality conditions

\[
\int_{\Omega} P^{(k)}_{\alpha_{i}}(x) P^{(j)}_{\alpha_{j}}(x) d\mu(x) = \int_{\Omega} P^{(k)}_{\alpha_{i}}(x) x^{\alpha_{j}} d\mu(x) = 0,
\]

for \(\ell = 0, 1, \ldots, k - 1, i = 1, \ldots, [[k]]\) and \(j = 1, \ldots, [[\ell]]\), with the normalization conditions

\[
\int_{\Omega} P_{\alpha_{i}}(x) P_{\alpha_{j}}(x) d\mu(x) = \int_{\Omega} P_{\alpha_{i}}(x) x^{\alpha_{j}} d\mu(x) = H_{\alpha_{i}, \alpha_{j}}, \quad i, j = 1, \ldots, [[k]].
\]

**Definition 1.3.** The shift matrices are given by

\[
\Lambda_{\alpha} = \begin{pmatrix}
0 & (\Lambda_{\alpha})_{[0],[1]} & 0 & 0 & \cdots \\
0 & 0 & (\Lambda_{\alpha})_{[1],[2]} & 0 & \cdots \\
0 & 0 & 0 & (\Lambda_{\alpha})_{[2],[3]} & \cdots \\
& \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

\(^1\)Lower triangular with the block diagonal populated by identity matrices.
where the entries in the non zero blocks are given by
\[(\Lambda_a)^{(k)}_{\alpha_i^{(k)} \alpha_j^{(k+1)}} = \delta_{\alpha_i^{(k)} + e_a \alpha_j^{(k+1)}}, \quad a = 1, \ldots, D, \quad i = 1, \ldots, [k], \quad j = 1, \ldots, [k + 1].\]

We also use
\[\Lambda := (\Lambda_1, \ldots, \Lambda_D)^\top,\]

**Proposition 1.3.**

1. The shift matrices commute among them
\[\Lambda_a \Lambda_b = \Lambda_b \Lambda_a.\]
2. We also have the “eigenvalue” type properties
\[(1.5) \quad \Lambda_a \chi(x) = x_a \chi(x).\]
3. The moment matrix \(G\) satisfies
\[(1.6) \quad \Lambda_a G = G(\Lambda_a)^\top.\]

Using these properties one derives three term relations or Christoffel–Darboux formulæ, but as this is not the subject of this paper we refer the interested reader to our paper [14].

## 2. Extending the Christoffel formula to the multivariate realm

In this section a Darboux transformation for MVOPR is found. Here we use polynomial perturbation of the measure, but to ensure that the procedure works we need perturbations that factor out as \(N\) different prime polynomials. Latter we will discuss how we can modify this to include the most general polynomial perturbation.

**Definition 2.1.** Given a multivariate polynomial, which we call generating polynomial,
\[Q = \sum_{j=0}^{m} Q^{(j)}(x) \quad \text{deg} Q^{(j)} = j \quad Q^{(m)} \neq 0,\]
the corresponding Darboux transformation of the measure is the following perturbed measure
\[T \, d\mu(x) = Q(x) \, d\mu(x).\]

Observe that, if we want a positive definite perturbed measure, we must request to \(Q\) to be positive definite in the support of the original measure.

**Definition 2.2.** We introduce the resolvent matrix
\[\omega := (TS) Q(\Lambda) S^{-1}\]
given in terms of the lower unitriangular matrices \(S\) and \(TS\) of the Cholesky factorizations of the moment matrices \(G = S^{-1} H (S^{-1})^\top\) and \(TG = (TS)^{-1} (TH)(TS^{-1})^\top\).

In terms of block superdiagonals the resolvent \(\omega\) can be expressed as follows
\[\omega = \begin{align*}
Q^{(m)}(\Lambda) \\
&\quad \text{m-th superdiagonal} \\
+ (T\beta) Q^{(m-1)}(\Lambda) - Q^{(m-1)}(\Lambda) \beta \\
&\quad \text{(m - 1)-th superdiagonal} \\
\vdots \\
+ (TH) H^{-1} \\
&\quad \text{diagonal}
\end{align*}\]
Proposition 2.1. The MVOPR satisfy \( Q(x)TP(x) = \omega P(x) \). Consequently, for any element \( p \) in the algebraic hypersurface \( Z(Q) := \{ x \in \mathbb{R}^D : Q(x) = 0 \} \) we have the important relation
\[
\omega[k],[k+m]P[k+m](p) + \omega[k],[k+m-1]P[k+m-1](p) + \cdots + \omega[k],[k]P[k](p) = 0. 
\]
Proof. We have
\[
\omega P(x) = (TS)Q(\Lambda)S^{-1}S\chi(x) \\
= (TS)Q(\Lambda)\chi(x) \\
= Q(x)(TS)\chi(x) \\
= Q(x)(TP)(x). 
\]
Finally, when this formula is evaluated at a point in the algebraic hypersurface of \( Q \) we obtain that the MVOPR at such points are vectors in the kernel of the resolvent. \( \square \)

To deal with this equation we consider

Definition 2.3. A set of nodes
\[
\mathcal{N}_{k,m} := \{ p_j \}_{j=1}^{r_k,m} \subset \mathbb{R}^D 
\]
is a set with \( r_{k,m} = N_{k,m-1} - N_{k-1} = |[k]| + \cdots + |[k + m - 1]| \) vectors in \( \mathbb{R}^D \). Given these nodes we consider the corresponding sample matrices
\[
\Sigma^m_k := \begin{pmatrix} 
P[k](p_1) & \cdots & P[k](p_{r_k,m}) \\
\vdots & \ddots & \vdots \\
P[k+m-1](p_1) & \cdots & P[k+m-1](p_{r_k,m}) 
\end{pmatrix} \in \mathbb{R}^{r_k,m \times r_k,m},
\]
\[
\Sigma_{[k,m]} := (P[k+m](p_1), \ldots, P[k+m](p_{r_k,m})) \in \mathbb{R}^{[k+m] \times r_k,m}.
\]

Lemma 2.1. When the set of nodes \( \mathcal{N}_{k,m} \subset Z(Q) \) belong to the algebraic hypersurface of the polynomial \( Q \) the resolvent coefficients satisfy
\[
\omega[k],[k+m]\Sigma_{[k,m]} + (\omega[k],[k], \cdots, \omega[k],[k+m-1]\Sigma^m_k = 0.
\]
Proof. Is a direct consequence of (2.1). \( \square \)

Definition 2.4. We say that \( \mathcal{N}_{k,m} \) is a poised set if the sample matrix is non singular
\[
\det \Sigma^m_k \neq 0.
\]

Theorem 2.1. For a poised set of nodes \( \mathcal{N}_{k,m} \subset Z(Q) \) in the algebraic hypersurface of the generating polynomial \( Q \) the Darboux transformation of the orthogonal polynomials can be expressed in terms of the original ones as the following last quasi-determinantal expression
\[
TP[k](x) = \frac{(Q(\Lambda))_{[k],[k+m]}}{Q(x)} \Theta_*(\Sigma^m_k \begin{pmatrix} 
P[k](x) \\
\vdots \\
P[k+m-1](x) \\
P[k+m](x) 
\end{pmatrix}).
\]

Proof. Observe that Lemma 2.1 together with \( \omega[k],[k+m] = (Q(\Lambda))_{[k],[k+m]} \) implies
\[
(\omega[k],[k], \cdots, \omega[k],[k+m-1]) = -(Q(\Lambda))_{[k],[k+m]}\Sigma_{[k,m]}(\Sigma^m_k)^{-1}.
\]
and from \( Q(x)TP(x) = \omega P(x) \) the result follows. \( \square \)
3. Poised sets and geometrically poised sets

To construct Darboux transformations in the multivariate setting we need of poised sets in order to find invertible sample matrices with the original polynomials as interpolating functions. When is this possible? Let us start a discussion on this question. First, we introduce two important matrices in the study of poised sets

**Definition 3.1.** We consider the Vandermonde type matrix

\[ V^{\text{m}}_k := \left( \chi^{[k+m]}(p_1), \ldots, \chi^{[k+m]}(p_{r_{k,m}}) \right) \in \mathbb{R}^{N_{k,m} \times r_{k,m}}, \]

made up of truncated of multivariate monomials \( \chi^{[k+m]}(x) \) evaluated at the nodes. We also consider the following truncation \( S^{\text{m}}_m \in \mathbb{R}^{r_{k,m} \times N_{k,m} - 1} \) of the lower unitriangular factor \( S \) of the Gauss–Borel factorization of the moment matrix

\[
S^{\text{m}}_m := \begin{pmatrix}
S[k,0] & S[k,1] & \cdots & I[k] & 0[k,k+1] & \cdots & 0[k,k+m-1] \\
S[k+1,0] & S[k+1,1] & \cdots & S[k+1,k] & I[k+1] & \cdots & 0[k+1,k+m-1] \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
S[k+m-1,0] & S[k+m-1,1] & \cdots & S[k+m-1,k+m-2] & I[k+m-1] 
\end{pmatrix}.
\]

They are relevant because

**Lemma 3.1.** We have the following factorization

\[ \Sigma^{\text{m}}_k = S^{\text{m}}_k V^{\text{m}}_k. \]

From where it immediately follows that

**Proposition 3.1.** The following relations between linear subspaces

\[ \text{Ker } V^{\text{m}}_k \subset \text{Ker } \Sigma^{\text{m}}_k, \quad \text{Im } \Sigma^{\text{m}}_k \subset \text{Im } S^{\text{m}}_k = \mathbb{R}^{r_{k,m}}, \]

hold true.

The poisedness of \( N_{k,m} \) can be reworded as

\[ \text{Ker } \Sigma^{\text{m}}_k = \{0\}, \]

or equivalently

\[ \dim \text{Im } \Sigma^{\text{m}}_k = r_{k,m}. \]

**Proposition 3.2.** For poised set \( N_{k,m} \) the multivariate Vandermonde \( V^{\text{m}}_k \) is a full column rank matrix; i.e., \( \dim \text{Im } V^{\text{m}}_k = r_{k,m} \).

**Proof.** For a set to be poised we need that \( \text{Ker } \Sigma^{\text{m}}_k = \{0\} \), but \( \text{Ker } V^{\text{m}}_k \subset \text{Ker } \Sigma^{\text{m}}_k \) and consequently, \( \dim \text{Ker } V^{\text{m}}_k = 0 \), and, as \( \dim \text{Ker } V^{\text{m}}_k + \dim \text{Im } V^{\text{m}}_k = r_{k,m} \), full column rank of the Vandermonde matrix is needed for a set to be poised. \( \square \)

3.1. Geometrically poised sets. Facing the relevance of the Vandermonde matrix for the existence of poised set we introduce

**Definition 3.2.** We say that a set \( N_{k,m} \) is geometrically poised if the Vandermonde matrix \( V^{\text{m}}_k \) has full column rank.

The name is suggested by the observation that this condition imposes constraints of geometrical type, i.e. on the relative position of the nodes in the space. Moreover,

**Proposition 3.3.** A non geometrically poised set is non poised for every Borel measure \( \mu \).
It could happen that a set is geometrically poised and still is non poised. Indeed, as $S^m_k$ is a full row rank matrix we have $\dim \ker S^m_k = N_{k+m-1} - r_{k,m} = N_{k-1}$; it could happen that even thought $\dim \ker V^m_k = 0$ we had $\ker V^m_k \cap \operatorname{Im} V^m_k \neq \{0\}$. This possibility does depend on the form of the MVOPR and its associated measure, and is not of an intrinsic geometric nature as the nontriviality of the kernel of the Vandermonde matrix. For example, suppose that one of the MVOPR, say $P_{\alpha_i}$ with $k \leq |\alpha_i| < k + m$, belongs to the principal ideal $(Q)$, then we will have a row of zeroes in $\Sigma^m_k$ and the nodes are non poised even if they were geometrically poised.

The study of the orthogonal complement of the rank; i.e, the linear subspace $(\operatorname{Im} V^m_k)^\perp \subset \mathbb{R}^{N_{k+m-1}}$ of vectors orthogonal to the image $\operatorname{Im} V^m_k$ where $v \in (\operatorname{Im} V^m_k)^\perp$ if $v^T V^m_k = 0$, gives a better insight on the structure of the rank of the Vandermonde matrix. As $\operatorname{Im} V^m_k \oplus (\operatorname{Im} V^m_k)^\perp = \mathbb{R}^{N_{k+m-1}}$ we have the dimensional formula

$$\dim (\operatorname{Im} V^m_k)^\perp + \dim (\operatorname{Im} V^m_k) = N_{k+m-1}.$$

**Proposition 3.4.** The set $\mathcal{N}_{k,m}$ is geometrically poised if and only if

$$\dim (\operatorname{Im} V^m_k)^\perp = N_{k+m-1} - r_{k,m} = N_{k-1}.$$ 

### 3.2. Algebro-geometrical aspects.

Algebraic geometry will guide us in the search of geometrically poised sets in the algebraic hypersurface of the $m$-th degree polynomial $Q$, $\mathcal{N}_{k,m} \subset Z(Q)$. We need to abandon the real field $\mathbb{R}$ and work in its the algebraical closure $\mathbb{C}$; i.e., we understand $Q$ as a complex polynomial with real coefficients and consider its zero set as an algebraic hypersurface in the $D$-dimensional complex affine space $\mathbb{C}^D$; we also change the notation from $x \in \mathbb{R}^D$ to $z \in \mathbb{C}^D$.

**Definition 3.3.** For a multivariate polynomial $V$ of total degree $\deg V < k + m - 1$ its the principal ideal is $(V) := \{z^\alpha V(z) : \alpha \in \mathbb{Z}^D \} \subset \mathbb{C}[z_1, \ldots, z_D]$ and for its intersection with the polynomials of degree less or equal than $k + m - 1$ we employ the notation

$$(V)_{k+m-1} = (V) \cap \mathbb{C}[z_1, \ldots, z_D] = \mathbb{C}\{z^\alpha V(z)\}_{0 \leq |\alpha| < k + m - \deg V}.$$ 

It happens that the elements in the orthogonal complement of the rank of the Vandermonde matrix are polynomials with zeroes at the nodes.

**Proposition 3.5.** As linear spaces the orthogonal complement of the rank of the Vandermonde matrix $(\operatorname{Im} V^m_k)^\perp$ and the space of polynomials of degree less than $k + m$ and zeroes at $\mathcal{N}_{k,m}$ are isomorphic. 

**Proof.** The linear bijection is

$$v = (v_i)_{i=0}^{N_{k+m-1}-1} \in (\operatorname{Im} V^m_k)^\perp \leftrightarrow V(z) = \sum_{i=0}^{N_{k+m-1}-1} v_i z^{\alpha_i}$$

where $V(z)$ do have zeroes at $\mathcal{N}^m_k \subset Z(V)$. Now, we observe that a vector $v = (v_i)_{i=0}^{N_{k+m-1}-1} \in (\operatorname{Im} V^m_k)^\perp$ can be identified with the polynomial $V(z) = \sum_i v_i z^{\alpha_i}$ which cancels, as a consequence of $v^T V^m_k = 0$, at the nodes. \hfill $\square$

Thus, given this linear isomorphism, for any polynomial $V$ with $\deg V < k + m$ with zeroes at $\mathcal{N}^m_k$ we write $V \in (\operatorname{Im} V^m_k)^\perp$.

**Proposition 3.6.** Given a polynomial $V \in (\operatorname{Im} V^m_k)^\perp$ then

$$(V)_{k+m-1} \subset (\operatorname{Im} V^m_k)^\perp,$$

or equivalently

$$(V)_{k+m-1} \supset \operatorname{Im} V^m_k.$$
Proposition 3.7. For a prime polynomial \( Q \) the node set \( \mathcal{N}_{k,m} \subset Z(Q) \subset \mathbb{C}^D \) is geometrically poised if the nodes do not belong to any further complex algebraic hypersurface of degree \( < k + m \) different from \( Z(Q) \).

Proof. When the polynomial \( Q \) is prime (equivalently, as we are dealing with and UFD an irreducible polynomial) its principal ideal \( (Q) \) is a prime ideal, i.e., if the product \( P_1 P_2 \) of two polynomials \( P_1 \) and \( P_2 \) belong to \( (Q) \) then either \( P_1 \) or \( P_2 \) belong to \( (Q) \). Therefore, if some polynomial \( P \) vanishes at the irreducible algebraic hypersurface \( Z(Q) = \{ z \in \mathbb{C}^D : Q(z) = 0 \} \), according to the Hilbert’s Nullstellensatz it exists some \( m \in \mathbb{N} \) such that \( P^m \in (Q) \), but this is a prime ideal and therefore is radical: \( P \) itself must belong to it. Thus,

\[
\mathcal{N}_{k,m} \subset Z(Q) \iff (\text{Im } V^m_k)^\perp \supseteq (Q)_{k+m-1},
\]

and consequently \( \dim (\text{Im } V^m_k)^\perp \geq \dim (Q)_{k+m-1} = N_{k-1}. \) The equality is achieved if we ensure that there is no other polynomial \( P \) with \( \deg P \leq k + m - 1 \) such that \( \mathcal{N}_{k,m} \subset Z(P) \); i.e., \( \text{Im}(V^m_k) = (Q)_{k+m-1} \).

An extension of this result to a more general situation given by the product of different prime factors is

Theorem 3.1. Let \( Q = Q_1 \cdots Q_N \) be the product of \( N \) different irreducible polynomials with \( \deg Q_a = m_a, \ a \in \{1, \ldots, N\} \), and \( \deg Q = m = \sum_{a=1}^N m_a \). Then, the set \( \mathcal{N}_{k,m} \subset Z(Q) = \bigcup_{a=1}^N Z(Q_a) \) is geometrically poised if the nodes do not belong to any further complex algebraic hypersurface of degree smaller than \( k + m \) and different from \( Z(Q) \).

Proof. It is very similar to the previous proof but now we deal with a reducible algebraic hypersurface. Given a subset \( Y \subset \mathbb{C}^D \) we define the corresponding ideal \( I(Y) = \{ P \in \mathbb{C}[z_1, \ldots, z_D] : P(z) = 0 \forall z \in Y \} \); then, \( I \left( \bigcup_{a=1}^N Y_a \right) = \bigcap_{a=1}^N I(Y_a) \) and therefore \( I(Z(Q)) = \bigcap_{a=1}^N I(Z(Q_a)) \). But, according to the Hilbert’s Nullstellensatz and the prime character of each factor \( Q_a \) (every prime ideal is radical) we can write

\[
I(Z(Q)) = \sqrt{(Q)} = \bigcap_{a=1}^N (Q_a) = (Q)
\]

where \( \sqrt{(Q)} \) is the radical of the principal ideal of \( Q \). Thus, we conclude

\[
(\text{Im } V^m_k)^\perp \supseteq (Q)_{k+m-1}
\]

and deduce

\[
\dim (\text{Im } V^m_k)^\perp \geq N_{k-1}.
\]

The equality is achieved whenever we can ensure that there is no further algebraic hypersurface of degree less than \( k + m \), different from \( Z(Q) \), to which the nodes also belong; i.e., \( \text{Im}(V^m_k) = (Q_1 \cdots Q_N)_{k+m-1} \).

We now discuss on the distribution of nodes along the different irreducible components of the algebraic hypersurface of \( Q \).
Proposition 3.8. In a geometrically poised set, \( D > 1 \), the number \( n_a \) of nodes in the irreducible algebraic hypersurface \( Z(Q_a) \) fulfill
\[
k + m_a \leq n_a \leq r_{k+m-m_a,m_a}, \quad n_1 + \cdots + n_N = r_{k,m}.
\]

Proof. Assume that a number \( M_a \) smaller than \( r_a := k + m_a \), \( M_a < r_a \), of nodes lay in the irreducible algebraic hypersurface \( Z(Q_a) \), i.e., the number of nodes in its complementary algebraic hypersurface \( Z(Q_1 \cdots Q_{a-1} Q_{a+1} \cdots Q_N) \) is bigger than \( r_{k,m} - r_a \). Then, the set of nodes belong to the algebraic hypersurface \( \text{different of } Z(Q) \) of degree \( m-m_a+M_a < k+m \) of the polynomial \( Q_1 \cdots Q_{a-1} Q_{a+1} \cdots Q_N \pi_1 \cdots \pi_{M_a} \), where \( \pi_j \) is a degree one polynomial with a zero at the \( j \)-th node that belongs to \( Z(Q_a) \), where we have taken care that \( \pi_1 \cdots \pi_{M_a} \notin (Q_a) \), which for \( D > 1 \) can be always be achieved. Therefore, we need \( M_a \geq k + m_a \) to avoid this situation and to have a geometrically poised set.

The maximum rank of the Vandermonde submatrix built up with the columns corresponding to the evaluation of \( \chi \) at the nodes in \( Z(Q_a) \), recalling that \( \dim(Q_a)_{k+m-1} - N_{k+m-m_a} \), is \( N_{k+m-1} - N_{k+m-1-m_a} = r_{k+m-m_a,m_a} \).

Notice, that we need to put \( k+m \) nodes at each irreducible component \( Z(Q_a) \), for \( a \in \{1, \ldots, N\} \), hence we impose conditions on \( Nk + m \) nodes. But, do we have enough nodes? The positive answer for \( D > 1 \) can be deduced as follows.

Proposition 3.9. The bound \( r_{k,m} > Nk + m \) holds.

Proof. We have \( r_{k,m} = \lceil [k + m - 1] \rceil + \cdots + \lceil k \rceil \), thus a rude lower bound of nodes (for \( D > 1 \)) is
\[
r_{k,m} > m \lceil k \rceil = m \left( \frac{k + D - 1}{D - 1} \right) = m \left( 1 + \frac{k}{D - 1} \right) \cdots \left( 1 + \frac{k}{2} \right) (1 + k)
\]
\[
> m(k + 1)
\]
\[
> Nk + m.
\]
\[
\]

But, what happens with this condition for \( D = 1 \)? Now, we have \( r_{k,m} = m \) nodes, \( m_a = 1 \), each \( Z(Q_i) \) is a single point in \( \mathbb{C} \) and \( N = m \). In this case, the reasoning that lead to the construction of the polynomial \( Q_1 \cdots Q_{a-1} Q_{a+1} \cdots Q_N \pi_1 \cdots \pi_{M_a} \) in the previous proof is not applicable; first \( M_a = 1 \), given that all prime factors are degree one polynomials and second, the polynomial \( \pi_1 \) must be \( Q_a \) and therefore the product leads to the polynomial \( Q \) and no further constraint must be considered.

The maximum \( n_a \) is greater than the minimum number of nodes of that type \( r_{k+m-m_a,m_a} > m_a(k + m - m_a + 1) > k + m_a \). Moreover, the sum of the maximum ranks exceeds the number of nodes, and the full column rank condition is reachable:

Proposition 3.10. We have \( \sum_{i=1}^{N} r_{k+m-m_a,m_a} \geq r_{k,m} \).

Proof. For \( N = 2 \) we need to show that \( r_{k+m_2,m_1} + r_{k+m_1,m_2} > r_{k,m_1+m_2} \) or
\[
| [k + m_2 + m_1 - 1] | + \cdots + | [k + m_2] | + | [k + m_2 + m_1 - 1] | + \cdots + | [k + m_1] | > | [k + m_2 + m_1 - 1] | + \cdots + | [k + m_2] | + | [k + m_2 - 1] | + \cdots + | [k] |
\]
which is obvious. Then, for \( N = 3 \) we need to prove that \( r_{k+m_3,m_1} + r_{k+m_1,m_3} + r_{k+m_1,m_3} > r_{k,m_1+m_2+m_3} \), but using the already proven \( N = 2 \) case we have \( r_{k+m_3,m_1} + r_{k+m_1,m_3} + r_{k+m_1,m_3} > r_{k+m_3,m_1+m_2} \) and using the \( N = 2 \) equation again we do have \( r_{k,m_3,m_1+m_2} + r_{k+m_1,m_3} > r_{k,m_1+m_2+m_3} \), as desired. An induction procedure gives the result for arbitrary \( N \). \( \square \)
In the next picture we illustrate the case \( N = 2 \) of two prime polynomials of degrees \( m_1 \) and \( m_2 \). The blue rectangle gives the possible values for the number of nodes \((n_1, n_2)\) corresponding \( n_i \) to the prime polynomial \( Q_i \) according to the bounds

\[
k + m_1 \leq n_1 \leq r_{k+m_2,m_1}, \quad k + m_2 \leq n_2 \leq r_{k+m_1,m_2}.
\]

The blue diagonals \( n_1 + n_2 = K \) are ordered according (we assume for the degrees that \( m_1 \leq m_2 \) and therefore \( r_{k+m_2,m_1} \leq r_{k+m_1,m_2} \)) to the chain of inequalities

\[
2k + m \leq \max(r_{k+m_2,m_1} + k + m_2, r_{k+m_1,m_2} + k + m_1) \leq r_{k,m} \leq r_{k+m_2,m_1} + r_{k+m_1,m_2}.
\]

Notice that \( \max(r_{k+m_2,m_1} + k + m_2, r_{k+m_1,m_2} + k + m_1) \leq r_{k,m} \) follows \( r_{k,m} = r_{k+m_i,m_j} + r_{k,m_i} \geq \max_{i \neq j} r_{k+m_i,m_j} + m_i(k+1) \geq \max r_{k+m_i,m_j} + k + m_i \), where \((i,j) = (1,2), (2,1)\). Therefore, the striped triangle is the area where the couples \((n_1, n_2)\) of number of nodes belong. We have drawn the passing of the line \( n_1 + n_2 = r_{k,m} \) trough it, and show the integer couples in that segment, those will be the possible distributions of nodes among the zeroes of both prime polynomials.

4. Darboux Transformations for a General Perturbation

We begin with a negative result

**Proposition 4.1.** Poised sets do not exist for \( Q = R^d, \ d \in \{2,3,\ldots\} \), for any given polynomial \( R \).

**Proof.** Now \( Q = R^d, \ d \in \{2,3,4,\ldots\} \) deg \( Q = d \) deg \( R \), for some polynomial \( R \). In this case \( Z(Q) = Z(R) \), but \( \dim(R)_{k+m-1} = N_{k-1+(d-1)\deg R} > N_k \) and consequently the set is not geometrically poised. \( \square \)
We now discuss a method to overcome this situation. We will generalize the construction of nodes, sample matrices and poised sets. In this manner we are able to give explicit Christoffel type formulae for the Darboux transformation of more general generating polynomials. We consider multi-Wronski type matrices and multivariate confluent Vandermonde matrices.

4.1. Discussion for the arbitrary power of a prime polynomial. Now we take \( Q = \mathcal{R}^d \), \( \deg \mathcal{R} = n \) and \( \deg Q = dn \), so that \( Z(Q) = Z(\mathcal{R}) \) with \( \mathcal{R} \) to be a prime polynomial. From Proposition 2.1 we know that \( \omega P(x) = \mathcal{R}^d(x)TP(x) \). To analyze this situation we consider a set of linearly independent vectors \( \{ n_i^{(j)} \}_{i=1}^{\rho_j} \subset \mathbb{R}^{[j]} \cong (\mathbb{R}^D)^{\rho_j} \), \( \rho_j \leq ||j|| \); here \( n_i^{(j)} = (n_i^{(j)}拉_{\alpha \in [j]}) \), and to each of these vectors we associate the following homogeneous linear differential operator

\[
\frac{\partial^i}{\partial n_i^{(j)}} = \sum_{|\alpha|=j} n_i^{(j)} \frac{\partial^i}{\partial x^\alpha}.
\]

From the Leibniz rule we infer

**Proposition 4.2.** For any element \( p \) in the algebraic hypersurface \( Z(\mathcal{R}) := \{ x \in \mathbb{R}^D : \mathcal{R}(x) = 0 \} \) we have

\[
\omega[k,[k+nd]] \frac{\partial^i}{\partial n_i^{(j)}}(p) + \omega[k,[k+nd-1]] \frac{\partial^i}{\partial n_i^{(j)}}(p) + \cdots + \omega[k,[k]] \frac{\partial^i}{\partial n_i^{(j)}}(p) = 0,
\]

for \( j \in \{0,1,\ldots,d-1\} \) and \( i \in \{1,\ldots,\rho_j\} \).

This suggests to extend the set of nodes and the sample matrices

**Definition 4.1.** We consider the splitting into positive integers \( r_{k,nd} = N_{k+nd-1} - N_{k-1} = ||k|| + \cdots + ||k+dn-1|| = \sum_{j=0}^{d-1} \sum_{i=1}^{\rho_j} \nu_i^{(j)} \), and for each \( j \in \{0,1,\ldots,d-1\} \) we consider the following set of distinct nodes

\[
N_{i}^{(j)} := \{ p_{i,l}^{(j)} \}_{l=1}^{\nu_i^{(j)}} \subset \mathbb{R}^D,
\]

where we allow for non empty intersections between these sets of nodes and we denote its union by \( N_{k,nd} = \bigcup_{j=0}^{d-1} \bigcup_{i=1}^{\rho_j} N_{i}^{(j)} \). We also need of the above mentioned set of linearly independent vectors \( \{ n_i^{(j)}\}_{i=1}^{\rho_j} \subset \mathbb{R}^{[j]} \cong (\mathbb{R}^D)^{\rho_j} \), \( \rho_j \leq ||j|| \), \( j \in \{0,\ldots,d-1\} \). The partial blocks of the homogeneous sample matrices are

\[
(\Sigma_k^{nd})_{i}^{(j)} := \begin{pmatrix}
\frac{\partial^i}{\partial n_i^{(j)}}(p_{i,1}^{(j)}) & \cdots & \frac{\partial^i}{\partial n_i^{(j)}}(p_{i,\nu_i^{(j)}}^{(j)})
\end{pmatrix} \in \mathbb{R}^{r_{k,nd} \times \nu_i^{(j)}},
\]

\[
(\Sigma_{k,nd})_{i}^{(j)} := \begin{pmatrix}
\frac{\partial^i}{\partial n_i^{(j)}}(p_{i,1}^{(j)}) & \cdots & \frac{\partial^i}{\partial n_i^{(j)}}(p_{i,\nu_i^{(j)}}^{(j)})
\end{pmatrix} \in \mathbb{R}^{[k+nd]\times \nu_i^{(j)}},
\]

in terms of which we write the homogenous sample matrices

\[
(\Sigma_k^{nd})^{(j)} := (\Sigma_k^{nd})_{1}^{(j)}, \ldots, (\Sigma_k^{nd})_{\rho_j}^{(j)} \in \mathbb{R}^{r_{k,nd}\times \sum_{i=1}^{\rho_j} \nu_i^{(j)}},
\]

\[
(\Sigma_{k,nd})^{(j)} := (\Sigma_{k,nd})_{1}^{(j)}, \ldots, (\Sigma_{k,nd})_{\rho_j}^{(j)} \in \mathbb{R}^{[k+nd]\times \sum_{i=1}^{\rho_j} \nu_i^{(j)}},
\]
which allow us to define the multivariate Wronski type sample matrices
\[ \Sigma_k^{nd} := \left( (\Sigma_k^{nd})_1^{(0)}, \ldots, (\Sigma_k^{nd})_1^{(d-1)} \right) \in \mathbb{R}^{r_k,nd \times r_k,nd}, \]
\[ \Sigma_{[k,nd]} := \left( (\Sigma_{[k,nd]})_1^{(0)}, \ldots, (\Sigma_{[k,nd]})_1^{(d-1)} \right) \in \mathbb{R}^{[k+nd] \times [k+nd]}. \]

**Definition 4.2.** We say that \( N_{k,nd} \) is a poised set if the sample matrix is non singular
\[ \det \Sigma_k^{nd} \neq 0. \]

**Theorem 4.1.** For a poised set of nodes \( N_{k,nd} \subset Z(Q) \) in the algebraic hypersurface of the prime polynomial \( R \) the transformed orthogonal polynomials can be expressed in terms of the original ones as according to the quasi-determinantal expression
\[ TP_{[k]}(x) = \frac{(R(A)^d)[k,nd]}{R(x)^d} \Theta \begin{pmatrix} \Sigma_k^{nd} & P_{[k]}(x) \\ \vdots & \vdots \\ \Sigma_{[k,nd]} & P_{[k+nd-1]}(x) \end{pmatrix}. \]

**Proof.** Proposition 4.2 gives
\[ \omega[k,nd] \Sigma_{[k,nd]} + (\omega[k,nd], \ldots, \omega[k,nd]) \Sigma_k^{nd} = 0 \]
so that
\[ (\omega[k,nd], \ldots, \omega[k,nd]) = -(R(A)^d)[k,nd] \Sigma_{[k,nd]} (\Sigma_k^{nd})^{-1}. \]
and \( R(x)^d TP(x) = \omega P(x) \) gives the result. \( \Box \)

To discuss the existence of geometrically poised sets we allow the nodes to be complex.

**Definition 4.3.** We introduce the partial derived Vandermonde matrices
\[ (\mathcal{V}_k^{nd})^{(j)} := \left( \frac{\partial^j x[k+nd]}{\partial n_i^{(j)}}(p_{i,1}^{(j)}), \ldots, \frac{\partial^j x[k+nd]}{\partial n_i^{(j)}}(p_{i,\nu_i}^{(j)}) \right) \in \mathbb{C}^{N_{k+nd-1} \times \nu_i^{(j)}}, \]
for \( j \in \{0, \ldots, d-1\} \) and \( i \in \{1, \ldots, \rho_i\} \), the derived Vandermonde matrix is
\[ (\mathcal{V}_k^{nd})^{(j)} := \left( (\mathcal{V}_k^{nd})_1^{(j)}, \ldots, (\mathcal{V}_k^{nd})_{\rho_i}^{(j)} \right) \in \mathbb{C}^{N_{k+nd-1} \times \sum_{i=1}^{\rho_i} \nu_i^{(j)}}, \]
and the multivariate confluent Vandermonde matrix
\[ \mathcal{V}_k^{nd} := \left( (\mathcal{V}_k^{nd})^{(0)}, (\mathcal{V}_k^{nd})^{(1)}, \ldots, (\mathcal{V}_k^{nd})^{(d-1)} \right) \in \mathbb{C}^{N_{k+nd-1} \times [k+nd]}. \]

As in the previous analysis we have \( \Sigma_k^m = S_k^m \mathcal{V}_k^m \), where \( S_k^m \) given in (3.1), and \( \text{Ker} \mathcal{V}_k^{nd} \subset \text{Ker} \Sigma_k^{nd}. \) For \( N_{k,nd} \) to be poised we must request to \( \mathcal{V}_k^{nd} \) to be a full column rank matrix; i.e., \( \dim \text{Im} \mathcal{V}_k^{nd} = r_{k,nd}. \)

**Definition 4.4.** We say that a set \( N_{k,nd} \subset \mathbb{C}^D \) is geometrically poised if the confluent Vandermonde matrix \( \mathcal{V}_k^{nd} \) has full column rank.

A non geometrically poised set is non poised for every Borel measure \( \mu \) and a poised set requires \( \dim(\text{Im} \mathcal{V}_k^{nd}) = N_{k-1}. \)

**Theorem 4.2.** The node set \( N_{k,nd} \subset \mathbb{C}^D \) is geometrically poised if it does not belong to any algebraic variety, different from \( Z(R) \), of the polynomials
\[ \left\{ \frac{\partial^j V}{\partial n_i^{(j)}} \right\}_{j=0, \ldots, d-1, i=1, \ldots, \rho_i} \] for some polynomial \( V \) with \( \deg V \leq k + nd - 1. \)
Proof. A vector \( v = (v_i)^{N_{k+n d-1}}_{i=1} \in (\text{Im } V^d)^\perp \) if for the corresponding polynomial \( V = \sum_{i=1}^{N_{k+n d-1}} v_i x^{\alpha_i} \) the polynomials \( \frac{\partial^j V}{\partial n^{(j)}_i} \) cancel at \( q_i^{(j)} \). Remarkably, \( \frac{\partial^j (x^{\alpha} R^d)}{\partial n^{(j)}_i} (p) = 0, \ j = 1, \ldots, d-1 \) for \( i \in \{1, \ldots, \rho_j\} \) and \( p \in Z(\mathcal{R}) \). Hence, we conclude that
\[
(\mathcal{R}^d)_{k+n d-1} \subseteq (\text{Im } V^m)^\perp,
\]
and, as \( \text{dim}(\mathcal{R}^d)_{k+n d-1} = N_{k-1} \), the set is geometrical poised if \( \text{Im } V^m = (\mathcal{R}^d)_{k+n d-1} \).

\[\square\]

Corollary 4.1. For a geometrically poised set
- we can not take the vectors \( n^{(j)}_i \in \mathbb{R}^{[j]} \) such that for a given \( p \in \{1, \ldots, d\} \) the polynomial \( \frac{\partial^j (R^p)}{\partial n^{(j)}_i} \) cancels at \( Z(\mathcal{R}) \).
- we can not pick up the nodes from an algebraic hypersurface of degree less than or equal to \( \lfloor \frac{k-1}{d} \rfloor + n \).
- the following upper bounds must hold
\[
\nu_0^{(0)} \leq r_{k+n (d-1), n}, \quad \nu_1^{(j)} \leq r_{k+n d-1 d_i^{(j)}, d_i^{(j)}},
\]
where \( d_i^{(j)} := \deg (\partial^j (R^d)) \).

Proof. When \( \frac{\partial^j (R^p)}{\partial n^{(j)}_i} \) cancels at \( Z(\mathcal{R}) \) then \( \text{dim}(\text{Im } V^m)^\perp \geq N_{k+(d-p)n} \) and the set is not geometrically poised. Given a polynomial \( W \), \( \deg W \leq \lfloor \frac{k-1}{d} \rfloor + n \), of the described type we see that \( V = W^d \), \( \deg V \leq k - 1 + m \), is a polynomial such that \( \frac{\partial^j V}{\partial n^{(j)}_i} \) cancels at \( Z(W) \) and again the set is not geometrically poised. All the columns in the Vandermonde block \( (V^m)^{(0)} \) (we have remove the subindex because for \( j = 0 \) there is only one and no need to distinguish among several of them) imply no directional partial derivatives, so that \( (\mathcal{R}_i)^{k+n d-1} \supseteq \text{Im}(V^m)^{(0)} \) and the maximum achievable rank for this block is \( N_{k+n d-1} - N_{k+n (d-1), n} = r_{k+n (d-1), n} \). For \( j = 1, \ldots, d-1 \) the columns used in the construction of the block \( (V_i^m)^{(j)} \) imply directional partial derivatives \( \frac{\partial^j}{\partial n^{(j)}_i} \) and consequently \( \left( \frac{\partial^j (R^d)}{\partial n^{(j)}_i} \right)^{k+n d-1} \supseteq \left( \text{Im } (V^m)^{(j)} \right) \); hence, the maximum rank is \( N_{k+n d-1} - N_{k+n d-1 - d_i^{(j)}} \).

\[\square\]

4.2. The general case. We now consider the general situation of a polynomial in several variables, i.e., \( \mathcal{Q} = \mathcal{R}_1^{d_1} \cdots \mathcal{R}_N^{d_N} \) where \( \mathcal{R}_i \), \( \deg \mathcal{R}_i = m_i \), are different prime polynomials; we have for the degree of the polynomial \( \deg \mathcal{Q} m = n_1 d_1 + \cdots + n_D d_N \). As with the study of the product of \( N \) different prime polynomials developed in \( \S \), \( 3 \) we have \( Z(\mathcal{Q}) = \bigcup_{i=1}^{N} Z(\mathcal{R}_i) \).

Definition 4.5. Consider the splitting \( r_{k,n_1 d_1 + \cdots + n_D d_N} = N_{k+n_1 d_1 + \cdots + n_D d_N-1} - N_{k-1} = |\{k\}| + \cdots + |\{k + n_1 d_1 + \cdots + n_D d_N - 1\}| = \sum_{a=1}^{N} \sum_{j=0}^{d_a-1} \sum_{i=1}^{\rho_a} \nu_i^{(a,j)} \), and for each \( j \in \{0, 1, \ldots, d_a - 1\} \) the following set of different nodes
\[
\mathcal{N}_i^{(a,j)} := \{ p_i^{(a,j)} \}_{i=1}^{\nu_i^{(a,j)}} \in \mathbb{R}^d, \quad \mathcal{N}_a := \bigcup_{j=0}^{d_a-1} \bigcup_{i=1}^{\rho_j} \mathcal{N}_i^{(a,j)}
\]

\(^2\text{Here } [x] \text{ is the floor function and gives the greatest integer less than or equal to } x.\)
where for a fixed \( a \in \{1, \ldots, N\} \) we allow for non empty intersections among sets with different values of \( j \); denote its union by \( \mathcal{N}_{k, n_1 d_1, \ldots, n_N d_N} = \bigcup_{a=1}^{N} \mathcal{N}_a \) and pick a set of linearly independent vectors \( \{ \mathbf{n}_{i}^{(a,j)} \}_{i=1}^{\rho_a} \subset \mathbb{R}^{||j||} \), \( \rho_j \leq ||j||, j \in \{0, \ldots, d_a - 1\} \). The associated \( i \)-th homogeneous blocks of the sample matrices are

\[
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{i}^{(a,j)} := \left( \begin{array}{ccc}
\frac{\partial^i P_{[k]}(\mathbf{n}_{i}^{(a,j)})}{\partial \mathbf{n}_{i}^{(a,j)}}(\mathbf{p}_{1}^{(a,j)}) & \cdots & \frac{\partial^i P_{[k]}(\mathbf{n}_{i}^{(a,j)})}{\partial \mathbf{n}_{i}^{(a,j)}}(\mathbf{p}_{\rho_a(i,j)}^{(a,j)}) \\
\vdots & \ddots & \vdots \\
\frac{\partial^i P_{[k+n_1 d_1, \ldots, n_N d_N-1]}(\mathbf{n}_{i}^{(a,j)})}{\partial \mathbf{n}_{i}^{(a,j)}}(\mathbf{p}_{1}^{(a,j)}) & \cdots & \frac{\partial^i P_{[k+n_1 d_1, \ldots, n_N d_N-1]}(\mathbf{n}_{i}^{(a,j)})}{\partial \mathbf{n}_{i}^{(a,j)}}(\mathbf{p}_{\rho_a(i,j)}^{(a,j)})
\end{array} \right),
\]

with \( \left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{i}^{(a,j)} \in \mathbb{R}^{r_{k,n_1 d_1, \ldots, n_N d_N} \times \ell_{i}^{(a,j)}} \) and \( \left( \Sigma_{[k,n_1 d_1, \ldots, n_N d_N]}^{(a,j)} \right)_{i} \in \mathbb{R}^{[k+n_1 d_1, \ldots, n_N d_N] \times \ell_{i}^{(a,j)}} \), the homogenous sample matrices are

\[
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)^{(a,j)} := \left( \begin{array}{ccc}
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{1}^{(a,j)} & \cdots & \left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{\rho_a}^{(a,j)} \\
\vdots & \ddots & \vdots \\
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{1}^{(a,j)} & \cdots & \left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{\rho_a}^{(a,j)}
\end{array} \right) \in \mathbb{R}^{r_{k,n_1 d_1, \ldots, n_N d_N} \times \sum_{i=1}^{\rho_a} \ell_{i}^{(a,j)}},
\]

and the partial multivariate Wronski type sample matrices are

\[
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{a} := \left( \begin{array}{ccc}
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{1}^{(a,0)} & \cdots & \left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{(a,d_a-1)} \\
\vdots & \ddots & \vdots \\
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{1}^{(a,0)} & \cdots & \left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{(a,d_a-1)}
\end{array} \right) \in \mathbb{R}^{r_{k,n_1 d_1, \ldots, n_N d_N} \times \sum_{a=1}^{N} \sum_{i=1}^{\rho_a} \ell_{i}^{(a,j)}},
\]

that are matrices in \( \mathbb{R}^{r_{k,n_1 d_1, \ldots, n_N d_N} \times \sum_{a=0}^{d_a-1} \sum_{i=1}^{\rho_a} \ell_{i}^{(a,j)}} \) and in \( \mathbb{R}^{[k+n_1 d_1, \ldots, n_N d_N] \times \sum_{j=0}^{a-1} \sum_{i=1}^{\rho_a} \ell_{i}^{(a,j)}}, \) respectively; finally, consider the complete sample matrices collecting all nodes for different \( a \in \{1, \ldots, N\} \)

\[
\Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} := \left( \begin{array}{c}
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{1} \\
\vdots \\
\left( \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \right)_{N}
\end{array} \right),
\]

\[
\Sigma_{[k,n_1 d_1, \ldots, n_N d_N]} := \left( \begin{array}{c}
\left( \Sigma_{[k,n_1 d_1, \ldots, n_N d_N]} \right)_{1} \\
\vdots \\
\left( \Sigma_{[k,n_1 d_1, \ldots, n_N d_N]} \right)_{N}
\end{array} \right).
\]

We use the word partial in the sense that they are linked to one of the involved prime polynomials. We now proceed as we have done in previous situations just changing nodes and sample matrices as we have indicated. Then,

**Definition 4.6.** We say that \( \mathcal{N}_{k,n_1 d_1, \ldots, n_N d_N} \) is a poised set if the sample matrix is non singular

\[
\det \Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \neq 0.
\]

**Theorem 4.3.** For a poised set of nodes \( \mathcal{N}_{k,n_1 d_1, \ldots, n_N d_N} \) in the algebraic hypersurface \( \bigcup_{a=1}^{N} Z(\mathcal{Q}_a) \) the transformed orthogonal polynomials can be expressed in terms of the original ones as the following last quasi-determinantal expression

\[
TP_{[k]}(x) = \frac{\left( \prod_{a=1}^{N} (\mathcal{R}_a(\Lambda))^{d_a} \right) \left[ k, [k+n_1 d_1, \ldots, n_N d_N] \right]}{\mathcal{R}_1(x)^{d_1} \cdots \mathcal{R}_N(x)^{d_N}} \Theta^{*} \left( \begin{array}{c}
\Sigma_{k}^{n_1 d_1, \ldots, n_N d_N} \left[ k, [k+n_1 d_1, \ldots, n_N d_N] \right] \\
\vdots \\
\Sigma_{[k,n_1 d_1, \ldots, n_N d_N]} \left[ k, [k+n_1 d_1, \ldots, n_N d_N-1] \right]
\end{array} \right) \left( \begin{array}{c}
P_{[k]}(x) \\
\vdots \\
P_{[k+n_1 d_1, \ldots, n_N d_N-1]}(x)
\end{array} \right).
\]

**Proof.** Proposition 4.2 gives

\[
\omega_{[k],[k+n_1 d_1, \ldots, n_N d_N]} \Sigma_{[k,n_1 d_1, \ldots, n_N d_N]} + \cdots + \omega_{[k],[k+n_1 d_1, \ldots, n_N d_N-1]} \Sigma_{[k,n_1 d_1, \ldots, n_N d_N]} = 0,
\]

\[
\omega_{[k],[k+n_1 d_1, \ldots, n_N d_N]} \Sigma_{[k,n_1 d_1, \ldots, n_N d_N]} + \cdots + \omega_{[k],[k+n_1 d_1, \ldots, n_N d_N-1]} \Sigma_{[k,n_1 d_1, \ldots, n_N d_N]} = 0.
\]
so that
\[ (\omega[k], k, \ldots, \omega[k], k+n_1d_1+\cdots+n_Nd_N-1) \]
\[ = -\left( \prod_{a=1}^{N} (\mathcal{R}_a(A))^{d_a} \right) [k, k+n_1d_1+\cdots+n_Nd_N] \sum_{k, n_1d_1+\cdots+n_Nd_N} \left( \sum_k \right)^{-1} \]
and \( \mathcal{R}_1(x)^{d_1} \cdots \mathcal{R}_N(x)^{d_N} TP(x) = \omega P(x) \) gives the result. \( \square \)

As in previous discussion we shift from the field \( \mathbb{R} \) to its algebraic closure \( \mathbb{C} \), laying the algebraic varieties and nodes in the \( D \)-dimensional complex affine space.

**Definition 4.7.** We introduce the partial derived Vandermonde matrices \( i \)
\[ (V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N}) := \left( \frac{\partial j \chi^{[k+n_1d_1+\cdots+n_Nd_N]}(P_{(a,j)}^{(a,j)})}{\partial i^{(a,j)}} \right), \ldots, \left( \frac{\partial j \chi^{[k+n_1d_1+\cdots+n_Nd_N]}(P_{(a,j)}^{(a,j)})}{\partial i^{(a,j)}} \right), \]
that belong to \( n \mathbb{C}^{N_k+n_1d_1+\cdots+n_Nd_N-1 \times r_{(a,j)}} \). For \( j \in \{0, \ldots, d_a-1\} \) and \( i \in \{1, \ldots, \rho_{a,j}\} \), the derived Vandermonde matrix is
\[ (V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N}) := \left( V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \right) \in \mathbb{C}^{N_k+n_1d_1+\cdots+n_Nd_N-1 \times \sum_{k=1}^{\rho_{a,j}} \nu_{(a,j)}}, \]
and the partial multivariate confluent Vandermonde matrix in \( \mathbb{C}^{N_k+n_1d_1+\cdots+n_Nd_N-1 \times r_{k,n_1d_1+\cdots+n_Nd_N}} \),
\[ (V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N}) := \left( V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \right) \in \mathbb{C}^{N_k+n_1d_1+\cdots+n_Nd_N-1 \times \sum_{k=1}^{\rho_{a,j}} \nu_{(a,j)}}, \]
Finally, the complete confluent Vandermonde matrix in \( \mathbb{C}^{N_k+n_1d_1+\cdots+n_Nd_N} \),
\[ (V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N}) := \left( V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \right) \in \mathbb{C}^{N_k+n_1d_1+\cdots+n_Nd_N-1 \times r_{k,n_1d_1+\cdots+n_Nd_N}}. \]

Again we find the factorization \( \sum_{k=1}^{\rho_{a,j}} \nu_{(a,j)} = S_k^{n_1d_1+\cdots+n_Nd_N} V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \), with \( S_k \) as in (3.1), so that \( \operatorname{Ker} V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \subseteq \operatorname{Ker} S_k^{n_1d_1+\cdots+n_Nd_N} \). For \( N_k^{n_1d_1+\cdots+n_Nd_N} \) to be poised we must request to \( V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \) to have full column rank matrix: \( \dim \operatorname{Im} V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} = N_k-1 \).

**Definition 4.8.** The set \( N_{k,n_1d_1+\cdots+n_Nd_N} = N_1 \cup \cdots \cup N_N \subseteq \mathbb{C}^D \) is geometrically poised if the complete confluent Vandermonde matrix \( V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \) is a full column rank matrix.

As before a non geometrically poised set is non poised for every Borel measure \( \mu \) and for a poised set we need to have \( \dim \operatorname{Im} V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} = N_k-1 \).

**Theorem 4.4.** A set \( N_{k,n_1d_1+\cdots+n_Nd_N} \subseteq \mathbb{C}^D \) is geometrically poised if it does not belong to any algebraic hypersurface, different from \( \bigcap_{a=1}^{N} Z(\mathcal{R}_a) \), of a polynomial \( V \), \( \deg V \leq k+n_1d_1+\cdots+n_Nd_N-1 \), and each \( N_a \) does not belong to the algebraic variety of \( \left\{ \frac{\partial j V}{\partial n_i^{(a,j)}} \right\}_{j=0,1,\ldots,d_a-1}^{i=1,\ldots,\rho_{a,j}} \).

**Proof.** A vector \( v = (v_i)_{i=1}^{N_k+n_1d_1+\cdots+n_Nd_N-1} \) \( \in \left( \operatorname{Im} V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \right)^\perp \) if for the corresponding polynomial \( V = \sum_{i=1}^{N_k+n_1d_1+\cdots+n_Nd_N-1} v_i x^\alpha \) the polynomials \( \frac{\partial j V}{\partial n_i^{(a,j)}} \) cancel at \( N_{(a,j)} \). Notice that the polynomial \( V = x^\alpha \prod_{a=1}^{N} \mathcal{R}_a^{d_a} \) is such \( \frac{\partial j V}{\partial n_i^{(a,j)}} \) do cancel at \( \bigcup_{b=1}^{N} Z(\mathcal{R}_b) \) for \( j = 0 \) and also at \( Z(\mathcal{R}_b) \) for \( j \in \{1, \ldots, d_a-1\} \) and \( i \in \{1, \ldots, \rho_{a,j}\} \). Hence,
\[ \left( \prod_{a=1}^{N} \mathcal{R}_a^{d_a} \right)_{k+n_1d_1+\cdots+n_Nd_N-1} \subseteq \left( \operatorname{Im} V_{(a,j)}^{n_1d_1+\cdots+n_Nd_N} \right)^\perp. \]
which considered at the light of the condition \( \dim(\prod_{a=1}^{N} \mathcal{R}_a^{d_a})_{k+n_1d_1+\cdots+n_Nd_N-1} = N_{k-1} \) implies that no further constraint can be allowed or the full column rank would not be achievable. Thus, the set geometrical poised if \( \text{Im} \mathcal{V}_{k}^{n_1d_1+\cdots+n_Nd_N} = (\mathcal{R}_{k+n_1d_1+\cdots+n_Nd_N-1}^{d_k})^\perp \).

**Proposition 4.3.** In order to have a geometrical poised set the polynomial \( \frac{\partial^j(\mathcal{R}_a^{p_1} \cdots \mathcal{R}_N^{p_N})}{\partial \mathbf{n}_i^{(a,j)}} \) can not cancel at \( Z(\mathcal{R}_1 \cdots \mathcal{R}_N) \) for \( 0 \leq p_1 < d_1, \cdots, 0 \leq p_N < d_N \). Moreover, when the set of nodes is geometrical poised we can ensure the following bounds for the node subset cardinals

\[
|\mathcal{N}_a| \geq \left\lceil \frac{k}{d_a} \right\rceil + n_a,
\]

\[
|\mathcal{N}^{(a,0)}| \leq r_{k+n_1d_1+\cdots+n_Nd_N-n_a,n_a},
\]

\[
|\mathcal{N}_i^{(a,j)}| \leq r_{k+n_1d_1+\cdots+n_Nd_N-d_i^{(a,j)},n_a}.
\]

Here

\[
d_i^{(a,j)} := \deg \frac{\partial^j(\mathcal{R}_a^{d_a})}{\partial \mathbf{n}_i^{(a,j)}}
\]

and the function \( \lceil x \rceil \) gives the smallest integer \( \geq x \).

**Proof.** If \( \frac{\partial^j(\mathcal{R}_a^{p_1} \cdots \mathcal{R}_N^{p_N})}{\partial \mathbf{n}_i^{(j)}} \) cancels at \( Z(\mathcal{R}) \) then

\[
\dim(\text{Im} \mathcal{V}_{k}^{n_1d_1+\cdots+n_Nd_N})^\perp \geq N_{k+n_1d_1+\cdots+n_Nd_N-(p_1n_1+\cdots+p_Nn_N)}
\]

and the set is not geometrically poised.

To prove (4.1) let us consider first order polynomials \( \pi_{i,l}^{(a,j)}(z) \) which cancels at \( \mathbf{p}_{i,l}^{(a,j)} \), \( \pi_{i,l}^{(a,j)}(\mathbf{p}_{i,l}^{(a,j)}) = 0 \), and construct the polynomial \( \Pi_a = \left( \prod_{l=1}^{a-1} \pi_{i,l}^{(a,j)} \right)_{i=1}^{d_a} \), \( \deg \Pi_a = |\mathcal{N}_a| \). Then, the polynomial

\[
V = \mathcal{R}_1^{d_1} \cdots \mathcal{R}_{a-1}^{d_{a-1}} \mathcal{R}_a^{d_a+1} \cdots \mathcal{R}_N^{d_N} \Pi_a^\perp, \deg V = n_1d_1 + \cdots + n_Nd_N + (|\mathcal{N}_a| - n_a)d_a, \text{has the nodes among its zeroes } \mathcal{N}_a^{n_1d_1+\cdots+n_Nd_N} \subset Z(V) \text{ and } \frac{\partial^jV}{\partial \mathbf{n}_i^{(j)}} \text{ do cancel at } \mathcal{N}_i^{(b,j)} \text{ for all } b \in \{1, \ldots, N\} \text{. Thus, we should request } n_1d_1 + \cdots + n_Nd_N + (|\mathcal{N}_a| - n_a)d_a \geq k + n_1d_1 + \cdots + n_Nd_N \text{; i.e., } (|\mathcal{N}_a| - n_a)d_a \geq k \text{ and the result follows.}
\]

For (4.2) observe that all the columns in the Vandermonde block \( \mathcal{V}_{k}^{n_1d_1+\cdots+n_Nd_N}(a,0) \) imply no directional partial derivatives and are evaluated at nodes which belong to \( Z(\mathcal{R}_a) \). Therefore,

\[
(\mathcal{R}_a^k)^\perp_{k+n_1d_1+\cdots+n_Nd_N-1} \supseteq \text{Im}(\mathcal{V}_{k}^{n_1d_1+\cdots+n_Nd_N}(a,0))
\]

and the maximum rank achievable for this block is \( N_{k+n_1d_1+\cdots+n_Nd_N-1} - N_{k+n_1d_1+\cdots+n_Nd_N-n_a-1} = r_{k+n_1d_1+\cdots+n_Nd_N-n_a,n_a} \). The columns of the block \( \mathcal{V}_{k}^{n_1d_1+\cdots+n_Nd_N}(a,j) \) are vulcanized to \( Z(\frac{\partial^j(\mathcal{R}_a^{d_a})}{\partial \mathbf{n}_i^{(a,j)}}) \) and consequently \( \frac{\partial^j(\mathcal{R}_a^{d_a})^\perp}{\partial \mathbf{n}_i^{(j)}}_{k+n_1d_1+\cdots+n_Nd_N-1} \supseteq \text{Im}(\mathcal{V}_{k}^{n_1d_1+\cdots+n_Nd_N}(a,j)) \); hence, the maximum possible rank for this block is \( N_{k+n_1d_1+\cdots+n_Nd_N-1} - N_{k+n_1d_1+\cdots+n_Nd_N-1-d_i^{(a,j)}} \). \( \square \)

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