Sobolev estimates for fractional parabolic equations with space-time non-local operators

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Abstract
We obtain $L_p$ estimates for fractional parabolic equations with space-time non-local operators
\[
\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in} \ (0, T) \times \mathbb{R}^d,
\]
where $\partial_t^\alpha u$ is the Caputo fractional derivative of order $\alpha \in (0, 1]$, $T \in (0, \infty)$, and
\[
Lu(t, x) = \int_{\mathbb{R}^d} \left( u(t, x + y) - u(t, x) - y \cdot \nabla_x u(t, x) \chi^{(\sigma)}(y) \right) K(t, x, y) \, dy
\]
is an integro-differential operator in the spatial variables. Here we do not impose any regularity assumption on the kernel $K$ with respect to $t$ and $y$. We also derive a weighted mixed-norm estimate for the equations with operators that are local in time, i.e., $\alpha = 1$, which extend the previous results in Mikulevičius and Pragarauskas (J Differ Equ 256(4):1581–1626, 2014) and Zhang (Annales l’IHPAnalyse Nonlinéaire 30:573–614, 2013) by using a quite different method.

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Introduction

This paper is devoted to a study of $L_p$ (and $L_{p,q}$) estimates for non-divergence form equations with space-time non-local operators of the form

$$\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } (0, T) \times \mathbb{R}^d$$

(1.1)

with the zero initial condition $u(0, \cdot) = 0$, where $T \in (0, \infty)$, $\partial_t^\alpha u$ is the Caputo fractional derivative of order $\alpha \in (0, 1]$, and $L$ is an integro-differential operator

$$Lu(t, x) := \int_{\mathbb{R}^d} \left( u(t, x + y) - u(t, x) - y \cdot \nabla_x u(t, x) \chi(\sigma)(y) \right) K(t, x, y) \, dy,$$  

(1.2)

where $\chi(\sigma)(y) = 1_{\sigma \in (1, 2)} + 1_{\sigma = 1 |y| \leq 1}$, and $K$ is nonnegative. A simple example of $L$ is the fractional Laplacian $(-\Delta)^{\sigma/2}$ for $\sigma \in (0, 2)$. We refer the reader to Sect. 2 for the precise definitions of $\partial_t^\alpha u$ and $L$.

Fractional parabolic equations of the form (1.1) have applications in various fields, including physics and probability theory. For example, by viewing the fundamental solution of (1.1) as the probability density of a peculiar self-similar stochastic process evolving in time, we can see that the equation is related to the continuous-time random walk [14].

There are many works regarding $L_p$ (or $L_{p,q}$) estimates for parabolic equations with either fractional derivatives in time or integro-differential operators in space or both. Equations with non-local time derivatives of the form

$$\partial_t^\alpha u - a^{ij}(t, x) D_{ij} u = f(t, x),$$

(1.3)

were studied in [8–10, 15, 16, 26]. In [16], the authors derived mixed-norm estimates for (1.3) with $\alpha \in (0, 2)$ under the assumption that $a^{ij}$ are piecewise continuous in $t$ and uniformly continuous in $x$. In [15], a weighted mixed-norm estimate was obtained when $\alpha \in (0, 2)$ and $a^{ij} = \delta_{ij}$. More general coefficients $a^{ij}$ were studied in [8, 9], where the $L_p$ and the weighted mixed-norm estimates were obtained respectively for $\alpha \in (0, 1)$ under the assumption that $a^{ij}$ have small mean oscillations with respect to the spatial variables. Quite recently, in [10] the weighted mixed-norm estimates are obtained for $\alpha \in (1, 2)$ under the assumption that $a^{ij}$ have small oscillations in $(t, x)$. Furthermore, a weighted mixed-norm regularity theory for (1.3) was obtained in [26] when the coefficients are at least uniformly continuous in both $t$ and $x$.

Equations with non-local operators in $x$ of the form

$$\partial_t u(t, x) - Lu(t, x) = f(t, x)$$

(1.4)

with $L$ defined in (1.2) were studied in [12, 21–23, 28]. In [21], using the Fourier multiplier theorem, the authors derived an $L_p$ estimate for (1.4) under the conditions that $p$ is sufficiently large, and the operator satisfies an ellipticity condition, and $K(t, x, y)|y|^{d+\sigma}$ is homogeneous of order zero and smooth in $y$, and its derivatives in $y$ are continuous in $x$ uniformly in $(t, y)$. In [22], an $L_p$ estimate for (1.4) was obtained under the conditions that $p$ is sufficiently large, and $K(t, x, y)|y|^{d+\sigma}$ is bounded from above and below and Hölder continuous in $x$.
uniformly in \((t, y)\). No regularity assumption is assumed for \(K\) in \(t\) and \(y\). In particular, when \(K = K(t, y)\), i.e., independent of \(x\), and \(f\) is sufficiently smooth, using a probabilistic method, they constructed the solution for (1.4) explicitly as the expected value involving some stochastic processes, and when \(K = K(t, x, y)\), they applied the frozen coefficient argument. For time-independent equations with \(K = K(y)\), such result was obtained earlier in [6] by using a purely analytic method. It has been an open question whether in the case when \(K = K(t, x, y)\), the restriction of \(p\) in [22] can be removed. This problem was resolved in a recent paper [12] by using a weighted estimate and an extrapolation argument. In [28], by a probabilistic representation of the solution, the authors investigated the \(L_p\)-maximal regularity of (1.4) as well as more general equations with singular and non-symmetric Lévy operators.

Sobolev type estimates for equations with space-time non-local operators similar to (1.1) were studied in [17] by deriving fundamental solutions using probability density functions of some stochastic processes. Particularly, the equation

\[ \partial_t^\alpha u - \phi(\Delta) u = f \]  

(1.5)
is considered, where \(\alpha \in (0, 1)\) and \(\phi\) is a Bernstein function satisfying a certain growth condition. The authors obtained a mixed-norm estimate of (1.5) based on estimates of the fundamental solution from [3] together with the Calderón–Zygmund theorem.

In this paper, we obtain the a priori estimate and the unique solvability for (1.1) with space-time non-local operators. Compared to [17], our estimates do not directly rely on the representation of the fundamental solution. However, to apply the method of continuity, we need the results in [12] or [13]. Indeed, for \(K = K(t, y)\), we only require that \(K(t, y)|y|^{d+\sigma}\) is bounded from above and below and measurable, an assumption for which the fundamental solution is unavailable. Furthermore, when \(\alpha = 1\), we obtain the weighted mixed-norm estimate for (1.1), where the weights are taken in both spatial and time variables. The results are further extended to the case when \(K = K(t, x, y)\), under the assumption that \(K\) is Hölder continuous in \(x\) uniformly in \((t, y)\). See Assumptions 2.1 and 2.3 for details. We also consider equations which contain lower-order terms. These, in particular, extend the aforementioned results in [12, 22, 28] to the weighted mixed-norm setting by using a quite different method. Our main theorem reads that for any \(p \in (1, \infty)\) and the kernel \(K\) such that \(K(t, y)|y|^{d+\sigma}\) is bounded from above and below, if \(u\) satisfies (1.1) with the zero initial condition, then we have

\[
\|(1 - \Delta)^{\sigma/2} u\|_{L_p((0,T) \times \mathbb{R}^d)} + \|\partial_t^\alpha u\|_{L_p((0,T) \times \mathbb{R}^d)} \leq N\|f\|_{L_p((0,T) \times \mathbb{R}^d)},
\]

where \(N\) is independent of \(u\) and \(f\). Furthermore, for any \(f \in L_p((0,T) \times \mathbb{R}^d)\), there exists a unique solution \(u\) to (1.1) with the zero initial condition in the appropriate Sobolev spaces defined in Sect. 2. Moreover, when \(\alpha = 1\), the a priori estimate and the unique solvability hold for (1.4) in the appropriate weighted mixed-norm Sobolev spaces with Muckenhoupt weights.

For the proof of the \(L_p\) estimates for (1.1) when \(\alpha \in (0, 1]\), we apply a level set argument together with Lemma A.5 (“crawling of ink spots lemma”) by adapting the argument in [8]. More precisely, we first prove the theorem for \(p = 2\) by using the Fourier transform. Then, we apply a bootstrap argument: assuming the theorem holds for \(p_0\) and estimating the solution by a decomposition, we show that the theorem holds for \(p \in (p_0, p_1)\) for some \(p_1 > p_0\) such that the increment \(p_1 - p_0\) is independent of \(p_0\). Indeed, for \((t_0, x_0) \in (0,T] \times \mathbb{R}^d\), we decompose the solution \(u\) to (1.1) into \(u = w + v\) such that

\[ \partial_t^\alpha w - Lw + \lambda w = \zeta f \quad \text{in} \quad [(t_0 - 1, t_0) \times \mathbb{R}^d) \]  

(1.6)
with the zero initial condition at \( t = t_0 - 1 \), where \( \zeta \) is a suitable cutoff function satisfying \( \zeta = 1 \) in \( Q_1(t_0, x_0) \). Then \( v \) satisfies the homogeneous equation in \( Q_1(t_0, x_0) \). We bound \( w \) by the maximal function of \( f \) using the global \( L_{p_0} \) estimate and prove a higher integrability result for \( v \). See Proposition 3.6. Compared to [8], our proof is more involved since we need to deal with the nonlocality in both space and time. For the proof of the weighted mixed-norm estimates of (1.1) when \( \alpha = 1 \), we derive a mean oscillation estimate of \( (-\Delta)^{\sigma/2}u \), where \( u \) is a solution to (1.1), by using an iteration argument. To this end, for \( t_0 \in (0, T] \), we decompose \( u = w + v \), where

\[
\partial_t w - Lw + \lambda w = f \quad \text{in} \quad (t_0 - 1, t_0) \times \mathbb{R}^d
\]

so that \( v \) satisfies the homogeneous equation in the strip \( (t_0 - 1, t_0) \times \mathbb{R}^d \). Compared to the estimate of \( w \) in the previous case (cf. (1.6)), here we cannot directly apply the global \( L_{p_0} \) estimate since the right-hand side is not compactly supported in \( x \). Our idea is to use a localization and iteration argument. See Proposition 4.3. We also establish a Hölder estimate of \( v \) by using a delicate bootstrap argument: first derive a local estimate by using an estimate of the commutator (Lemma A.3) and then apply the Sobolev embedding theorems (Lemmas A.6, A.7). For this proof, it is crucial that \( v \) satisfies the homogeneous equation in the strip \( (t_0 - 1, t_0) \times \mathbb{R}^d \) instead of just in a cylinder. The weighted mixed-norm estimates then follow by using the Fefferman–Stein theorem for sharp functions and the Hardy–Littlewood theorem for (strong) maximal functions in weighted mixed-norm spaces. For the general cases when \( K = K(t, x, y) \), we use a boundedness result in [24] with a perturbation argument and an extrapolation theorem. Finally, we extend our results to equations in the whole space with non-zero initial conditions, in which we focus on equations in weighted mixed-norm spaces with \( \alpha = 1 \) and power weights in time.

In the future, we plan to extend the results by considering equations in domains, and divergence form equations. Indeed, some interesting work has been done in those directions. For example, solutions for equations in domains were studied in [2] using the probabilistic method. Divergence form equations were also studied in [1, 20] with various assumptions on the kernel \( K \). More precisely, Hölder estimates and \( L_p \) estimates were obtained for the divergence form equations in [1, 20], respectively. Furthermore, there are results about divergence form non-local elliptic equations in [19, 24, 25], where more general operators were considered. For example, in [25], the authors obtained some regularity results for operators with VMO kernels and non-linearity. Another interesting question is whether the weighted mixed-norm estimates still hold for (1.1) when \( \alpha \in (0, 1) \), where the weights are taken over the time or the spatial variables. See Remarks 4.2 and 4.4 for more information.

The remaining part of the paper is organized as follows. In Sect. 2, we introduce notation, definitions, and the main results of the paper. In Sect. 3, we obtain the \( L_p \) estimates using the level set argument. In Sect. 4, we first estimate the mean oscillation of solutions by using the decomposition mentioned above, and then prove the weighted mixed-norm estimates. In Appendix A, we prove miscellaneous lemmas used in the main proofs. In Appendix B, we consider equations with non-zero initial conditions.

## 2 Notation and main results

We first introduce some notation used throughout the paper. For \( \alpha \in (0, 1) \) and \( S \in \mathbb{R} \), we denote
\[ I^\alpha_S u(t, x) = \frac{1}{\Gamma(\alpha)} \int_S (t - s)^{\alpha - 1} u(s, x) \, ds, \quad t \geq S, \quad x \in \mathbb{R}^d, \]

and
\[ \partial_t^\alpha u(t, x) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \partial_t u(s, x) \, ds, \quad t \geq 0, \quad x \in \mathbb{R}^d. \]

It is easily seen that \( \partial_t^\alpha u = \partial_t I_0^{1-\alpha} u \) for a sufficiently smooth \( u \) with \( u(0, x) = 0 \).

We use \( \mathcal{F}(u) \) and \( \hat{u} \) to denote the Fourier transform of \( u \). If \( u = u(t, x) \), then \( \mathcal{F}(u)(t, \xi) \) and \( \hat{u}(t, \xi) \) denote the Fourier transform of \( u \) in \( x \) for a fixed time \( t \). For \( p \in (0, \infty) \) and \( \sigma \in (0, 2) \), recall the definition of the Bessel potential space
\[ H^\sigma_p(\mathbb{R}^d) = \{ u \in L_p(\mathbb{R}^d) : (1 - \Delta)^{\sigma/2} u \in L_p(\mathbb{R}^d) \} \]
and
\[ \| u \|_{H^\sigma_p(\mathbb{R}^d)} = \| (1 - \Delta)^{\sigma/2} u \|_{L_p(\mathbb{R}^d)}, \]
where
\[ (1 - \Delta)^{\sigma/2} u(x) = \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^{\sigma/2} \mathcal{F}(u)(\xi) \right](x). \]

For \( p, q \in (1, \infty) \) and \( -\infty < S < T < \infty \), we define \( L_{p,q}((S, T) \times \mathbb{R}^d) \) to be the set of all measurable functions \( f \) defined on \((S, T) \times \mathbb{R}^d\) satisfying
\[ \| f \|_{L_{p,q}((S, T) \times \mathbb{R}^d)} = \left( \int_S \left( \int_{\mathbb{R}^d} |f(t, x)|^q \, dx \right)^{p/q} \, dt \right)^{1/p} < \infty. \]

When \( p = q \), we write \( L_p((S, T) \times \mathbb{R}^d) = L_{p,p}((S, T) \times \mathbb{R}^d) \). Furthermore, for \( \alpha \in (0, 1] \) and \( \sigma \in (0, 2) \), we denote \( \mathbb{H}^{\alpha,\sigma}_{p,q}(S, T) \) to be the collection of functions such that \( u \in L_p((S, T); H^\sigma_p(\mathbb{R}^d)) \), \( \partial_t^\alpha u \in L_{p,q}((S, T) \times \mathbb{R}^d) \), and
\[ \| u \|_{\mathbb{H}^{\alpha,\sigma}_{p,q}(S,T),T} = \left( \int_S \left( \| u(t, \cdot) \|_{H^\sigma_p(\mathbb{R}^d)}^{p/q} \, dt \right)^{1/p} + \| \partial_t^\alpha u \|_{L_{p,q}((S,T) \times \mathbb{R}^d)} \right)^{1/p}. \]

We write \( u \in \mathbb{H}^{\alpha,\sigma}_{p,q,0}(S,T) := \mathbb{H}^{\alpha,\sigma}_{p,q}(S,T) \) if there exists a sequence of functions \( \{u_n\} \) such that \( u_n \in C^\infty([S, T] \times \mathbb{R}^d) \) with \( u_n(S, x) = 0 \) vanishing for large \( |x| \), and
\[ \| u_n - u \|_{\mathbb{H}^{\alpha,\sigma}_{p,q,0}(S,T)} \to 0 \quad \text{as} \quad n \to \infty. \]

Moreover, for a domain \( \Omega \subset \mathbb{R}^d \) and \( u \) defined on \((S, T) \times \Omega \), we write \( u \in \mathbb{H}^{\alpha,\sigma}_{p,q,0}((S, T) \times \Omega) \) if there exists an extension of \( u \) to \((S, T) \times \mathbb{R}^d \), i.e.,
\[ \| u \|_{\mathbb{H}^{\alpha,\sigma}_{p,q,0}((S, T) \times \Omega)} = \inf \{ \| \overline{u} \|_{\mathbb{H}^{\alpha,\sigma}_{p,q,0}(S,T) : \overline{u} \in \mathbb{H}^{\alpha,\sigma}_{p,q,0}(S,T) \text{ and } \overline{u}|_{(S,T)\times\Omega} = u \} < \infty. \]

We denote \( \mathbb{H}^{\alpha,\sigma}_{p,0}((S, T) \times \Omega) := \mathbb{H}^{\alpha,\sigma}_{p,0}((S, T) \times \Omega) \) if \( p = q \). Furthermore, we take \( \psi(x) = 1/(1 + |x|^{d+\sigma}) \), and denote
\[ \| u \|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))} \triangleq \| \psi u \|_{L_p((0,T);L_1(\mathbb{R}^d))}. \quad (2.1) \]

For \( T \in (0, \infty) \), we denote \((0, T) \times \mathbb{R}^d := \mathbb{R}^d_T \) and we often denote \( \mathbb{H}^{\alpha,\sigma}_{p,q}(T) := \mathbb{H}^{\alpha,\sigma}_{p,q}(0, T) \) and \( \mathbb{H}^{\alpha,\sigma}_{p,q,0}(T) := \mathbb{H}^{\alpha,\sigma}_{p,q,0}(0, T) \). We use the notation \( u \in \mathbb{H}^{\alpha,\sigma}_{p,0,\text{loc}}(\mathbb{R}^d_T) \) to indicate a function satisfying \( u \in \mathbb{H}^{\alpha,\sigma}_{p,0}(0, T) \times B_R \) for all \( R > 0 \).
For $\alpha \in (0, 1), \sigma \in (0, 2), r_1, r_2 > 0$, and $(t, x) \in \mathbb{R}^{d+1}$, we denote the parabolic cylinder by

$$Q_{r_1, r_2}(t, x) = (t-r_1^{\sigma/\alpha}, t) \times B_{r_2}(x)$$

and $Q_r(t, x) = Q_{r, r}(t, x)$, where $B_{r_2}(x) = \{ y \in \mathbb{R}^d : |y-x| < r_2 \}$. We write $B_{r}$ and $Q_{r}$ for $B_r(0)$ and $Q_r(0, 0)$. Furthermore, for $f \in L_{1,\text{loc}}$ defined on $\mathcal{D} \subset \mathbb{R}^{d+1}$ and $(t, x) \in \mathcal{D}$, we define its maximal function and strong maximal function, respectively, by

$$\mathcal{M}f(t, x) = \sup_{Q_r(x, y) \ni (t, x)} \int_{Q_r(x, y)} |f(r, z)| \chi_{\mathcal{D}} dz \, dr$$

and

$$(SMf)(t, x) = \sup_{Q_{r_1, r_2}(x, y) \ni (t, x)} \int_{Q_{r_1, r_2}(x, y)} |f(r, z)| \chi_{\mathcal{D}} dz \, dr.$$ 

Next, for $p \in (1, \infty), k \in \{1, 2, \ldots\}$, let $A_p(\mathbb{R}^k, dx)$ be the set of all non-negative functions $w$ on $\mathbb{R}^k$ such that

$$[w]_{A_p(\mathbb{R}^k)} := \sup_{x_0 \in \mathbb{R}^k, r > 0} \left( \int_{B_r(x_0)} w(x) \, dx \right) \left( \int_{B_r(x_0)} (w(x))^{-\frac{1}{p}} \, dx \right)^{p-1} < \infty,$$

where $B_r(x_0) = \{x \in \mathbb{R}^k : |x-x_0| < r\}$. Furthermore, for a constant $M_1 > 0$, we write $[w]_{p, q} \leq M_1$ if $w = w_1(t)w_2(x)$ for some $w_1$ and $w_2$ satisfying

$$w_1(t) \in A_p(\mathbb{R}, dt), \quad w_2(x) \in A_q(\mathbb{R}^d, dx), \quad \text{and} \quad [w_1]_{A_p(\mathbb{R})}, [w_2]_{A_q(\mathbb{R}^d)} \leq M_1.$$

We denote $L_{p,q,w}(\mathbb{R}^{d+1})$ to be the set of all measurable functions $f$ defined on $\mathbb{R}^{d+1}$ satisfying

$$\|f\|_{L_{p,q,w}(\mathbb{R}^{d+1})} := \left( \int_0^T \left( \int_{\mathbb{R}^d} |f(t, x)|^q w_2(x) \, dx \right)^{p/q} w_1(t) \, dt \right)^{1/p} < \infty.$$

When $p = q$ and $w = 1$, $L_{p,q,w}(\mathbb{R}^{d+1})$ becomes the usual Lebesgue space $L_p(\mathbb{R}^{d+1})$.

We write $N = N(\cdots)$ if the constant $N$ depends only on the parameters in the parentheses.

Next, we present the assumptions for the operators. In this paper, we consider equations which are non-local in both time and space:

$$\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in} \quad \mathbb{R}^{d+1},$$

where $L$ is defined in (1.2). We impose the following assumptions on the kernel $K = K(t, x, y) > 0$.

**Assumption 2.1** (1) There exist some positive constants $\nu, \Lambda > 0$ and a function $m_0 = m_0(t, y) \geq 0$ which is measurable and homogeneous in $y$ with index 0 such that

$$\inf_{\xi \in S^{d-1}} \int_{S^{d-1}} |\xi \cdot y|^{\sigma} m_0(t, y) \mu_{d-1}(dy) \geq \nu \quad \text{for all} \ t,$$

and

$$(2-\sigma) \frac{m_0(t, y)}{|y|^{d+\sigma}} \leq K(t, x, y) \leq (2-\sigma) \frac{\Lambda}{|y|^{d+\sigma}} \quad \text{for all} \ t, x \text{ and } y,$$

where $\mu_{d-1}$ is the standard Lebesgue measure on the unit sphere $S^{d-1}$.
(2) When $\sigma = 1$, for any $r > 0$,
\[
\int_{S^{d-1}} yK(t, x, ry) \mu_{d-1}(dy) = 0, \tag{2.3}
\]

**Remark 2.2** Assumption 2.1 (1) with additional homogeneity, cancellation, and smoothness conditions was introduced in [22] (see also [28]), and it is satisfied, for instance, when
\[
(2 - \sigma)\nu y^{-d-\sigma} \leq K(t, x, y) \leq (2 - \sigma)\Lambda|y|^{-d-\sigma}.
\]
Also, note that (2.3) implies that $\chi^1$ in (1.2) can be replaced by $1_{y \in B_r}$ for any $r > 0$.

**Assumption 2.3** There exist $\beta \in (0, 1)$ and a continuous increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that
\[
|y|^{d+\sigma} |K(t, x_1, y) - K(t, x_2, y)| \leq (2 - \sigma)\omega(|x_1 - x_2|) \tag{2.4}
\]
and
\[
\int_{|y| \leq 1} \omega(|y|)|y|^{-d-\beta} dy < \infty. \tag{2.5}
\]

**Remark 2.4** On one hand, it is easily seen that if $|y|^{d+\sigma} K(t, \cdot, y)$ is Hölder continuous in $x$ with exponent $\beta + \varepsilon$ for any $\varepsilon > 0$ uniformly in $(t, y)$, then (2.5) is satisfied. On the other hand, since $\omega$ is increasing, from (2.5) we see that $\omega(r) \leq Nr^\beta$ for any $r \leq 1/2$, which implies that $|y|^{d+\sigma} K(t, \cdot, y)$ is Hölder continuous in $x$ with exponent $\beta$ uniformly in $(t, y)$.

**Remark 2.5** It is well known that for $\sigma \in (0, 2)$, if $K(t, x, y) = K(y):=c|y|^{-d-\sigma}$, where
\[
c = c(d, \sigma) = \frac{\sigma(2-\sigma)\Gamma(d+\sigma)}{\pi^{d/2}2^{\sigma-\sigma}\Gamma(2-\frac{\sigma}{2})},
\]
then
\[
Lu = \frac{c}{2} \int_{\mathbb{R}^d} \left(u(t, x+y) - u(t, x) - 2u(x)\right)|y|^{-d-\sigma} dy = -(-\Delta)^{\sigma/2}u.
\]

In some cases, the lower bound of the operator is not necessary. Thus, we use $L^1$ to denote operators with the kernel
\[
K_1(t, x, y) = K_1(t, y) \leq \frac{\Lambda}{|y|^{d+\sigma}}, \tag{2.6}
\]
and we assume that (2.3) is satisfied when $\sigma = 1$.

With the assumptions above, we are ready to state the main theorems of this paper.

**Theorem 2.6** Let $\alpha \in (0, 1]$, $\sigma \in (0, 2)$, $T \in (0, \infty)$, $\lambda \geq 0$, and $p \in (1, \infty)$. Suppose that the kernel $K = K(t, y)$ satisfies Assumption 2.1. Then $\partial_t^\alpha - L$ is a continuous operator from $H^{\alpha, \sigma}_{p,0}(T)$ to $L^p(\mathbb{R}^d_T)$. Also, for any $u \in H^{\alpha, \sigma}_{p,0}(T)$ satisfying
\[
\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T, \tag{2.7}
\]
and any $L^1$ satisfying (2.3) when $\sigma = 1$ and (2.6), we have
\[
\|\partial_t^\alpha u\|_{L^p(\mathbb{R}^d_T)} + \|L^1 u\|_{L^p(\mathbb{R}^d_T)} + \lambda \|u\|_{L^p(\mathbb{R}^d_T)} \leq N\|f\|_{L^p(\mathbb{R}^d_T)}. \tag{2.8}
\]
and

\[ \|u\|_{\mathbb{H}_{p,\sigma}^g(T)} \leq N \min\{T^\alpha, \lambda^{-1}\} \|f\|_{L^1_p(\mathbb{R}^d_T)}, \tag{2.9} \]

where \( N = N(d, v, \Lambda, \alpha, \sigma, p) \). Moreover, for any \( f \in L^1_p(\mathbb{R}^d_T) \), there exists a unique solution \( u \in \mathbb{H}_{p,\sigma}^g(T) \) to (2.7).

**Remark 2.7** Indeed, (2.9) follows from (2.8) upon setting \( L^1 = (-\Delta)^{\sigma/2} \), \( L^1 = L \), and using (2.7) and Lemma A.2. Furthermore, by (2.8) with \( \lambda = 0 \), for any \( u \in \mathbb{H}_{p,0}^g(T) \) if

\[ \partial_t^\alpha u + (-\Delta)^{\sigma/2} u = g \quad \text{in} \ \mathbb{R}^d_T, \]

then

\[ \|L^1 u\|_{L^p(\mathbb{R}^d_T)} \leq N \|g\|_{L^p(\mathbb{R}^d_T)}. \tag{2.10} \]

For any \( v \in H^\sigma_p(\mathbb{R}^d) \), by first taking \( T > 0 \) and a nonzero function \( \eta \in C_0^\infty(\mathbb{R}) \) with \( \eta(0) = 0 \), then applying (2.10) to \( u(t, x) = \eta(t/T)v(x) \), and finally sending \( T \to \infty \), we conclude that

\[ \|L^1 v\|_{L^p(\mathbb{R}^d_T)} \leq N \|(-\Delta)^{\sigma/2} v\|_{L^p(\mathbb{R}^d)}. \tag{2.11} \]

Thus, for \( u \in \mathbb{H}_{p,0}^g(T) \) and \( t \in (0, T) \)

\[ \|L^1 u(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq N \|(-\Delta)^{\sigma/2} u(t, \cdot)\|_{L^p(\mathbb{R}^d)}, \]

which implies

\[ \|L^1 u\|_{L^p(\mathbb{R}^d_T)} \leq N \|(-\Delta)^{\sigma/2} u\|_{L^p(\mathbb{R}^d_T)} \tag{2.12} \]

and the continuity of the operator \( \partial_t^\alpha - L \). We refer the reader to [6] for a different proof of (2.11).

Next, we have the following results when \( K = K(t, x, y) \) and for a constant \( M > 0 \),

\[ |b| = \left| (b^1(t, x), \ldots, b^d(t, x)) \right| \leq M, \quad |c| = |c(t, x)| \leq M. \tag{2.13} \]

**Corollary 2.8** Let \( \beta \in (0, 1) \), \( \alpha \in (0, 1] \), \( \sigma \in (0, 2) \), \( T \in (0, \infty) \), and \( p \in (1, \infty) \). Suppose that the kernel \( K = K(t, x, y) \) satisfies Assumptions 2.1 and 2.3, and (2.13) holds. Then \( \partial_t^\alpha - L \) is a continuous operator from \( \mathbb{H}_{p,0}^g(T) \) to \( L^1_p(\mathbb{R}^d_T) \).

1. There exists \( \lambda_0 = \lambda_0(d, v, \Lambda, \alpha, \sigma, p, M, \beta, \omega) \geq 1 \) such that for any \( \lambda \geq \lambda_0 \) and \( u \in \mathbb{H}_{p,0}^g(T) \) satisfying

\[ \partial_t^\alpha u - Lu + \lambda u = f \quad \text{in} \ \mathbb{R}^d_T, \]

we have

\[ \|u\|_{\mathbb{H}_{p,\sigma}^g(T)} \leq N \|f\|_{L^1_p(\mathbb{R}^d_T)}, \]

where \( N = N(d, v, \Lambda, \alpha, \sigma, p, M, \beta, \omega) \) is independent of \( T \).

2. Also, for any \( u \in \mathbb{H}_{p,0}^g(T) \) satisfying

\[ \partial_t^\alpha u - Lu + b^i D_i u 1_{\sigma > 1} + cu = f \quad \text{in} \ \mathbb{R}^d_T, \tag{2.14} \]

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we have

\[ \|u\|_{H^{d \sigma}_p (T)} \leq N \|f\|_{L_p (\mathbb{R}^d_T)}, \]

(2.15)

where \( N = N(d, v, \Lambda, \alpha, \sigma, p, T, M, \beta, \omega) \). Moreover, for any \( f \in L_p (\mathbb{R}^d_T) \), there exists a unique solution \( u \in H^{d \sigma}_p (0, T) \) to (2.14). Furthermore, when \( \sigma = 1 \), if \( \|b\|_{\infty} \) is sufficiently small, then the a priori estimate and the unique solvability hold for

\[ \partial^\alpha u - Lu + b^j D_j u + cu = f \quad \text{in} \ \mathbb{R}^d_T. \]

(2.16)

When \( \alpha = 1 \) and \( \sigma = 1 \), if \( b \) is uniformly continuous, then the a priori estimate and the unique solvability also hold for (2.16).

In the case of \( \alpha = 1 \), i.e., the operator is local in time, we have the following results regarding the weighted mixed-norm.

**Theorem 2.9** Let \( \sigma \in (0, 2) \), \( T \in (0, \infty) \), \( \lambda \geq 0 \), \( p, q \in (1, \infty) \), \( M_1 \in [1, \infty) \), and \( [w]_{p,q} \leq M_1 \). Suppose the kernel \( K = K(t, y) \) satisfies Assumption 2.1. Then \( \partial_t - L \) is a continuous operator from \( \mathbb{H}^{1,\sigma}_{p,q,w,0}(T) \) to \( L_{p,q,w}(\mathbb{R}^d_T) \). Also, for any \( u \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(T) \) satisfying

\[ \partial_t u - Lu + \lambda u = f \quad \text{in} \ \mathbb{R}^d_T, \]

(2.17)

and any \( L^1 \) satisfying (2.3) when \( \sigma = 1 \) and (2.6), we have

\[ \|\partial_t u\|_{L_{p,q,w}(\mathbb{R}^d_T)} + \|L^1 u\|_{L_{p,q,w}(\mathbb{R}^d_T)} + \lambda \|u\|_{L_{p,q,w}(\mathbb{R}^d_T)} \leq N \|f\|_{L_{p,q,w}(\mathbb{R}^d_T)}, \]

(2.18)

and

\[ \|u\|_{\mathbb{H}^{1,\sigma}_{p,q,w}(T)} \leq N \min(T, \lambda^{-1}) \|f\|_{L_{p,q,w}(\mathbb{R}^d_T)}, \]

(2.19)

where \( N = N(d, v, \Lambda, \alpha, \sigma, p, q, M_1) \). Moreover, for any \( f \in L_{p,q,w}(T) \), there exists a unique solution \( u \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(T) \) to (2.17).

With bounded \( b \) and \( c \) satisfying (2.13), we have the following result.

**Corollary 2.10** Let \( \beta \in (0, 1) \), \( \sigma \in (0, 2) \), \( T \in (0, \infty) \), \( p, q \in (1, \infty) \), \( M_1 \in [1, \infty) \), and \( [w]_{p,q} \leq M_1 \). Suppose the kernel \( K = K(t, x, y) \) satisfies Assumptions 2.1 and 2.3. Then \( \partial^\alpha_t - L \) is a continuous operator from \( \mathbb{H}^{1,\sigma}_{p,q,w,0}(T) \) to \( L_{p,q,w}(\mathbb{R}^d_T) \).

1. There exists \( \lambda_0 = \lambda_0(d, v, \Lambda, \sigma, p, q, M_1, M, \beta, \omega) \geq 1 \) such that for any \( \lambda \geq \lambda_0 \) and \( u \in \mathbb{H}^{\alpha,\sigma}_{p,0,0}(T) \) satisfying

\[ \partial_t u - Lu + \lambda u = f \quad \text{in} \ \mathbb{R}^d_T, \]

we have

\[ \|u\|_{\mathbb{H}^{1,\sigma}_{p,q,w}(T)} \leq N \|f\|_{L_{p,q,w}(\mathbb{R}^d_T)}, \]

(2.20)

where \( N = N(d, v, \Lambda, \sigma, p, q, M_1, M, \beta, \omega) \) is independent of \( T \).

2. Also, for any \( u \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(T) \) satisfying

\[ \partial_t u - Lu + b^j D_j u 1_{\sigma > 1} + cu = f \quad \text{in} \ \mathbb{R}^d_T, \]

(2.21)

we have

\[ \|u\|_{\mathbb{H}^{1,\sigma}_{p,q,w}(T)} \leq N \|f\|_{L_{p,q,w}(\mathbb{R}^d_T)}, \]

(2.22)
where \( N = N(d, v, \Lambda, \sigma, p, q, T, M_1, M, \beta, \omega) \). Moreover, for any \( f \in L_{p,q,w}(T) \), there exists a unique solution \( u \in H^{1,\sigma}_{p,q,w,0}(T) \) to (2.21). Furthermore, when \( \sigma = 1 \), if \( \|b\|_\infty \) is sufficiently small or \( b \) is uniformly continuous, then the a priori estimate and the unique solvability hold for

\[
\partial_t u - Lu + b^j D_j u + cu = f \quad \text{in } \mathbb{R}^d_T.
\]

(2.23)

**Remark 2.11** For \( \sigma \in (0, 2) \), due to the presence of \((2 - \sigma)\) in the bounds (2.2) and (2.4) in Assumptions 2.1 and 2.3 respectively, the constant \( N \) in above theorems and corollaries can be chosen to be dependent on \( \tilde{\sigma} \) instead of on \( \sigma \) itself, where

\[
\tilde{\sigma} = \begin{cases} 
(\sigma_0, \sigma_1) & \text{when } 0 < \sigma_0 \leq \sigma \leq \sigma_1 < 1, \\
1 & \text{when } \sigma = 1, \\
\sigma_0 & \text{when } 1 < \sigma_0 \leq \sigma < 2.
\end{cases}
\]

(2.24)

Thus, when \( \sigma > 1 \), \( N \) does not blow up when \( \sigma \uparrow 2 \). To see this, we keep track of the dependence of constants on \( \sigma \) in Lemma A.3 and the proof of Theorem 2.6 when \( p = 2 \). See also Remark 4.7. Moreover, note that Lemmas A.5 and A.7 hold for \( \sigma_0 \), and

\[
\|u\|_{H^{\alpha,\sigma}_p(T)} \leq N \|u\|_{H^{\alpha,\tilde{\sigma}}_p(T)},
\]

where \( N \) can be chosen to be independent of \( \sigma \) by the Mikhlin multiplier theorem (see, for instance, [18]) and [12, Lemma 3.4]. Thus, we can choose the increment of \( p \) in the iteration arguments in the proofs of Theorems 2.6 and 2.9 using \( \sigma_0 \) instead of \( \sigma \). Similar phenomena were observed before, for example in [4, 6].

### 3 Equations in \( L_p \)

In this section, we prove Theorem 2.6 and Corollary 2.8. To prove Theorem 2.6, we use a level set argument and a bootstrap argument with the Sobolev embedding.

#### 3.1 The case of \( p = 2 \) and auxiliary results

In order to apply the bootstrap argument, we start with the case when \( p = 2 \).

**Proof of Theorem 2.6** We first prove the continuity of \( \partial_\alpha^\sigma - L \). By taking the Fourier transform,

\[
\widehat{Lu}(t, \xi) = \hat{u}(t, \xi) \int_{\mathbb{R}^d} (e^{ix \cdot y} - 1 - iy \cdot \xi \chi^{(\sigma)}(y)) K(t, y) dy := \hat{u}(t, \xi)m(\xi),
\]

(3.1)

and in particular, by Remark 2.5,

\[
(-\Delta)^{\sigma/2}u(t, \xi) = \hat{u}(t, \xi) \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) C_y |y|^{-d-\sigma} dy.
\]

(3.2)

By a change of variables \( y \rightarrow y/|\xi| \) and the upper bound of \( K \), it is seen that

\[
|\hat{m}(\xi)| \leq N(d, \Lambda, \tilde{\sigma}) |\xi|^\sigma,
\]

where \( \tilde{\sigma} \) is defined in Remark 2.11. Thus, (3.1) leads to

\[
\|Lu\|_{L_2(\mathbb{R}^d_T)} = \|\widehat{Lu}\|_{L_2(\mathbb{R}^d_T)} \leq N \|\hat{u}(\cdot, \xi)|\xi|^\sigma\|_{L_2(\mathbb{R}^d)} = N \|(-\Delta)^{\sigma/2}u\|_{L_2(\mathbb{R}^d_T)},
\]

(3.3)
where \( N = N(d, \Lambda, \hat{\sigma}) \), which implies the continuity of \( \partial_t^\alpha \hat{u} - L \) from \( \mathbb{H}^{\alpha, 0}_2(T) \) to \( L_2(\mathbb{R}_d^T) \).

Next, we prove the a priori estimate (2.8). With the continuity of the operator and the density of smooth functions in \( \mathbb{H}^{\alpha, 0}_2(T) \), without loss of generality, we assume that \( u \in C_0^\infty([0, T] \times \mathbb{R}_d^T) \) with \( u(0, \cdot) = 0 \). Multiplying \( (-\Delta)^{\sigma/2} u \) to both sides of (2.7) and integrating over \( \mathbb{R}_d^T \), we arrive at

\[
\int_{\mathbb{R}_d^T} \partial_t^\alpha u(-\Delta)^{\sigma/2} u \, dx \, dt = \int_{\mathbb{R}_d^T} L u(-\Delta)^{\sigma/2} u \, dx \, dt + \lambda \int_{\mathbb{R}_d^T} u(-\Delta)^{\sigma/2} u \, dx \, dt = \int_{\mathbb{R}_d^T} f(-\Delta)^{\sigma/2} u. \tag{3.4}
\]

For the first term on the left-hand side of (3.4), due to (3.2) and the fact that it is real,

\[
\int_{\mathbb{R}_d^T} \partial_t^\alpha u(t, x)(-\Delta)^{\sigma/2} u(t, x) \, dx \, dt = \int_{\mathbb{R}_d^T} (-\Delta)^{\sigma/2} u(t, x) \partial_t^\alpha \hat{u}(t, \xi) \, d\xi \, dt = c \int_{\mathbb{R}_d^T} \partial_t^\alpha \Re(\hat{u})(t, \xi)\Re(\hat{u})(t, \xi) \int_{\mathbb{R}_d} (1 - \cos(\xi \cdot y)) |y|^{1-d-\sigma} \, dy \, d\xi \, dt
\]

\[
= c \int_{\mathbb{R}_d^T} \partial_t^\alpha \Re(\hat{u})(t, \xi)\Re(\hat{u})(t, \xi) \int_{\mathbb{R}_d} (1 - \cos(\xi \cdot y)) |y|^{1-d-\sigma} \, dy \, d\xi \, dt
\]

\[
+ c \int_{\mathbb{R}_d^T} \partial_t^\alpha \Im(\hat{u})(t, \xi)\Im(\hat{u})(t, \xi) \int_{\mathbb{R}_d} (1 - \cos(\xi \cdot y)) |y|^{1-d-\sigma} \, dy \, d\xi \, dt. \tag{3.5}
\]

Also, by [8, Proposition 4.1], we have

\[
\partial_t^\alpha \Re(\hat{u})(t, \xi)\Re(\hat{u})(t, \xi) + \partial_t^\alpha \Im(\hat{u})(t, \xi)\Im(\hat{u})(t, \xi) \geq \frac{1}{2} \partial_t^\alpha |\Re(\hat{u})|^2(t, \xi) + \frac{1}{2} \partial_t^\alpha |\Im(\hat{u})|^2(t, \xi) = \frac{1}{2} \partial_t^\alpha |\hat{u}|^2(t, \xi). \tag{3.6}
\]

Thus, by (3.5), (3.6), and the definition of \( \partial_t^\alpha \) together with the fundamental theorem of calculus,

\[
\int_{\mathbb{R}_d^T} \partial_t^\alpha u(t, x)(-\Delta)^{\sigma/2} u(t, x) \, dx \, dt
\]

\[
\geq c \int_{\mathbb{R}_d^T} \frac{1}{2} \partial_t^\alpha |\hat{u}|^2(t, \xi) \int_{\mathbb{R}_d} (1 - \cos(\xi \cdot y)) |y|^{1-d-\sigma} \, dy \, d\xi \, dt
\]

\[
= N(d, \sigma) \int_{\mathbb{R}_d} \int_{\mathbb{R}_d} (1 - \cos(\xi \cdot y)) |y|^{1-d-\sigma} \int_0^T \partial_t^\alpha |\hat{u}|^2(t, \xi) \, dt \, dy \, d\xi
\]

\[
= N(d, \alpha, \sigma) \int_{\mathbb{R}_d} \int_{\mathbb{R}_d} (1 - \cos(\xi \cdot y)) |y|^{1-d-\sigma} \int_0^T (T-s)^{-\alpha} |\hat{u}(s, \xi)|^2 \, ds \, dy \, d\xi \geq 0.
\]

For the second term on the left-hand side of (3.4), by (3.2) and (3.1),

\[
- \int_{\mathbb{R}_d^T} L u(t, x)(-\Delta)^{\sigma/2} u(t, x) \, dx \, dt = - \int_{\mathbb{R}_d^T} \overline{L u(t, \xi)}(-\Delta)^{\sigma/2} u(t, \xi) \, d\xi \, dt
\]

\[
= \int_{\mathbb{R}_d^T} |\hat{u}(t, \xi)|^2 \left( \int_{\mathbb{R}_d} (1 - \cos(\xi \cdot y)) K(t, y) \, dy \right) \int_{\mathbb{R}_d} (1 - \cos(\xi \cdot y)) c |y|^{1-d-\sigma} \, dy \, d\xi \, dt \tag{3.7}
\]

where in the second equality we used the fact that the left-hand side is real. Furthermore, by Assumption 2.1, a change of variables, and an integration in polar coordinates, we have
\[
\int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) K(t, y) \, dy \geq \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y))(2 - \sigma)m_0(t, y) |y|^{-d-\sigma} \, dy
\]
\[
= |\xi|^\sigma \int_{\mathbb{R}^d} (1 - \cos(|\xi|^{-1} \xi \cdot y))(2 - \sigma)m_0(t, y) |y|^{-d-\sigma} \, dy
\]
\[
= |\xi|^\sigma \int_{S^{d-1}} \left( \int_0^\infty (1 - \cos(r|\xi|^{-1} \xi \cdot y))(2 - \sigma)r^{-\sigma-1} \, dr \right)m_0(t, y) \mu_{d-1}(dy)
\]
\[
= |\xi|^\sigma \int_{S^{d-1}} \left( \int_0^\infty (1 - \cos(r))(2 - \sigma)r^{-\sigma-1} \, dr \right)|||\xi|^{-1} \xi \cdot y|^{\sigma}m_0(t, y) \mu_{d-1}(dy)
\]
\[
\geq N(\tilde{\sigma})|\xi|^\sigma.
\]

Therefore, (3.7) and (3.8) lead to
\[
- \int_{\mathbb{R}^d_T} L u(t, x)(-\Delta)^{\sigma/2} u(t, x) \, dx \, dt
\]
\[
\geq N(d, \tilde{\sigma}) v \int_{\mathbb{R}^d_T} |\tilde{u}(t, \xi)|^2 |\xi|^{2\sigma} \, d\xi \, dt = N(d, \tilde{\sigma}, v)(-\Delta)^{\sigma/2} f^2_{L_2(\mathbb{R}^d_T)}.
\]

Similarly, multiplying \( \lambda u \) to both sides of (2.7) and integrating over \( \mathbb{R}^d_T \), we get
\[
\lambda^2 \int_{\mathbb{R}^d_T} u^2 \leq \int_{\mathbb{R}^d_T} f^2.
\]

Thus, (3.4), (3.5), (3.9), and (3.10) together with Young’s inequality yield
\[
\|(-\Delta)^{\sigma/2} u\|_{L_2(\mathbb{R}^d_T)} + \lambda \|u\|_{L_2(\mathbb{R}^d_T)} \leq N(d, \tilde{\sigma}, v)\|f\|_{L_2(\mathbb{R}^d_T)}.
\]

Together with (3.3), we arrive at
\[
\|L^1 u + \lambda |u|\|_{L_2(\mathbb{R}^d_T)} \leq N(d, \tilde{\sigma})\|(-\Delta)^{\sigma/2} u\|_{L_2(\mathbb{R}^d_T)} + \lambda \|u\|_{L_2(\mathbb{R}^d_T)}
\]
\[
\leq N(d, \tilde{\sigma})\|f\|_{L_2(\mathbb{R}^d_T)},
\]

where \( L^1 \) is defined in (2.6). Finally, by [17, Theorem 2.8] together with the method of continuity, we have the existence of solutions. Theorem 2.6 is proved when \( p = 2 \). \( \square \)

**Remark 3.1** Indeed, (3.8) is the only place where we use the lower bound, \( m_0 \) defined in Assumption 2.1, of the kernel \( K \).

Next, assuming that Theorem 2.6 holds, we derive a local estimate. In the following lemma, we denote \( \| \cdot \|_{L^p_r} := \| \cdot \|_{L^p((0, T) \times B_r)} \) for any \( r > 0 \). Also, recall the definition of \( \| \cdot \|_{L^p_p((0, T) \times \hat{L}^1(\mathbb{R}^d, \psi))} \) in (2.1).

**Lemma 3.2** Let \( \alpha \in (0, 1) \), \( \sigma \in (0, 2) \), \( T \in (0, \infty) \), \( p \in (1, \infty) \), and \( 0 < r < R < \infty \). Also, let \( \zeta_0 \) be a cutoff function such that
\[
\zeta_0 \in C_0^\infty(B(R + r)/2), \quad \zeta_0 \equiv 1 \quad \text{in } B_r, \quad \text{and } |D\zeta_0| \leq 4/(R - r).
\]

Suppose that Theorem 2.6 holds for this \( p \). If \( u \in H_{p, 0}^{a, \sigma}(T) \), and \( L^1 \) satisfies (2.3) when \( \sigma = 1 \) and (2.6), and
\[
\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T,
\]

\[
\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T.
\]
Moreover, by applying Theorem 2.6 to (3.15), we have
\[ \| \partial_t^\sigma (\xi_0 u) \|_{L^p(\mathbb{R}^d_T)} + \| L^1 (\xi_0 u) \|_{L^p(\mathbb{R}^d_T)} + \lambda \| \xi_0 u \|_{L^p(\mathbb{R}^d_T)} \]
\[ \leq N \| f \|_{p,R} + N \| u \|_{p,R} \frac{R^d/p (1 + R^{d+\sigma})}{(R-r)^{d+\sigma}} \| u \|_{L^p((0,T);L^1(\mathbb{R}^d,\psi))}, \tag{3.12} \]
and
\[ \| \partial_t^\sigma u \|_{p,r} + \| L^1 u \|_{p,r} + \lambda \| u \|_{p,r} \]
\[ \leq N \| f \|_{p,R} + N \| u \|_{p,R} \frac{R^d/p (1 + R^{d+\sigma})}{(R-r)^{d+\sigma}} \| u \|_{L^p((0,T);L^1(\mathbb{R}^d,\psi))}, \tag{3.13} \]
where \( N = N(d,v,\Lambda,\alpha,\sigma,p) \).

**Proof** We first take cutoff functions in the spatial variables as follows. For \( k = 1, 2, \ldots, \)
\[ r_k = r + (R-r) \sum_{j=1}^{k} 2^{-j}, \quad \zeta_k \in C_0^\infty(B_{r_{k+1}}), \]
\[ \zeta_k \in [0,1], \quad \zeta_k = 1 \text{ in } B_{r_k}, \quad |D\zeta_k| \leq \frac{4 \cdot 2^k}{R-r}, \quad \text{and } |D^2 \zeta_k| \leq \frac{16 \cdot 2^{2k}}{(R-r)^2}. \tag{3.14} \]
It follows that \( \zeta_k u \in \mathbb{H}^0_{p,0}(T) \) and
\[ \partial_t^\sigma (\zeta_k u) - L(\zeta_k u) + \lambda (\zeta_k u) = \zeta_k f + \zeta_k Lu - L(\zeta_k u) \text{ in } \mathbb{R}^d_T. \tag{3.15} \]
By (2.12),
\[ \| \partial_t^\sigma u \|_{p,r} + \| L^1 u \|_{p,r} + \lambda \| u \|_{p,r} \]
\[ \leq \| \partial_t^\sigma (\zeta_k u) \|_{p,r} + \| \zeta_k L^1 u \|_{p,r} + \lambda \| \zeta_k u \|_{p,r} \]
\[ \leq \| L^1 (\zeta_k u) - \zeta_k L^1 u \|_{L^p(\mathbb{R}^d_T)} + \| \partial_t^\sigma (\zeta_k u) \|_{L^p(\mathbb{R}^d_T)} \]
\[ + \| L^1 (\zeta_k u) \|_{L^p(\mathbb{R}^d_T)} + \lambda \| \zeta_k u \|_{L^p(\mathbb{R}^d_T)} \]
\[ \leq \| L^1 (\zeta_k u) - \zeta_k L^1 u \|_{L^p(\mathbb{R}^d_T)} + N \| \partial_t^\sigma (\zeta_k u) \|_{L^p(\mathbb{R}^d_T)} \]
\[ + \| (-\Delta)^{\sigma/2} (\zeta_k u) \|_{L^p(\mathbb{R}^d_T)} + \lambda \| \zeta_k u \|_{L^p(\mathbb{R}^d_T)}. \tag{3.16} \]
Moreover, by applying Theorem 2.6 to (3.15), we have
\[ \| \partial_t^\sigma (\zeta_k u) \|_{L^p(\mathbb{R}^d_T)} + \| (-\Delta)^{\sigma/2} (\zeta_k u) \|_{L^p(\mathbb{R}^d_T)} + \lambda \| \zeta_k u \|_{L^p(\mathbb{R}^d_T)} \]
\[ \leq N \| \zeta_k f + \zeta_k Lu - L(\zeta_k u) \|_{L^p(\mathbb{R}^d_T)} \]
\[ \leq N \| f \|_{p,R} + N \| \zeta_k Lu - L(\zeta_k u) \|_{L^p(\mathbb{R}^d_T)}, \tag{3.17} \]
where \( N = N(d,v,\Lambda,\alpha,\sigma,p) \). We refer the reader to Lemma A.3 for the details about the estimates of the commutator term \( \| \zeta_k Lu - L(\zeta_k u) \|_{L^p(\mathbb{R}^d_T)} \). Note that since we only used the upper bound of the kernel of the operator in Lemma A.3, the estimate can be applied to \( \| L^1 (\zeta_k u) - \zeta_k L^1 u \|_{L^p(\mathbb{R}^d_T)} \).

**Case 1**: \( \sigma \in (0,1) \). In this case, (3.13) and (3.12) follow directly from (3.16), (3.17) and (A.3) with \( k = 0 \).

**Case 2**: \( \sigma \in (1,2) \). In this case, by (3.17) and (A.4), we have
Also, note that by Lemma A.1, for any $\varepsilon \in (0, 1)$,

$$N \frac{2^{(\sigma-1)k}}{(R-r)^{\sigma-1}} \| Du \|_{p, r_k+3} \leq N \frac{2^{(\sigma-1)k}}{(R-r)^{\sigma-1}} \| D(\zeta_{k+3}u) \|_{L_p(\mathbb{R}^d_T)}$$

$$\leq N \frac{2^{\sigma k}}{(R-r)^{\sigma}} \varepsilon^{3/(1-\sigma)} \| \zeta_{k+3} u \|_{L_p(\mathbb{R}^d_T)} + \varepsilon^{3} \| (-\Delta)^{\sigma/2} (\zeta_{k+3} u) \|_{L_p(\mathbb{R}^d_T)}$$

$$\leq N \frac{2^{\sigma k}}{(R-r)^{\sigma}} \varepsilon^{3/(1-\sigma)} \| u \|_{p, r} + \varepsilon^{3} \| (-\Delta)^{\sigma/2} (\zeta_{k+3} u) \|_{L_p(\mathbb{R}^d_T)}.$$ (3.19)

Therefore, (3.18) and (3.19) lead to

$$\| \partial_t^\alpha (\zeta_0 u) \|_{L_p(\mathbb{R}^d_T)} + \| (-\Delta)^{\sigma/2} (\zeta_0 u) \|_{L_p(\mathbb{R}^d_T)} + \lambda \| \zeta_0 u \|_{L_p(\mathbb{R}^d_T)}$$

$$\leq N \| f \|_{p, r} + N \frac{2^{\sigma k}}{(R-r)^{\sigma}} \varepsilon^{3/(1-\sigma)} \| u \|_{p, r} + \varepsilon^{3} \| (-\Delta)^{\sigma/2} (\zeta_{k+3} u) \|_{L_p(\mathbb{R}^d_T)}$$

$$+ N \frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}} \varepsilon^{3/(1-\sigma)} \| u \|_{p, r} + \varepsilon^{3} \| (-\Delta)^{\sigma/2} (\zeta_{k+3} u) \|_{L_p(\mathbb{R}^d_T)}.$$(3.20)

Multiplying $\varepsilon^k$ to both sides of (3.20) and taking the sum over $k = 0, 1, \ldots$ lead to

$$\| \partial_t^\alpha (\zeta_0 u) \|_{L_p(\mathbb{R}^d_T)} + \| \lambda (\zeta_0 u) \|_{L_p(\mathbb{R}^d_T)} \sum_{k=0}^{\infty} \varepsilon^k + \sum_{k=0}^{\infty} \varepsilon^k \| (-\Delta)^{\sigma/2} (\zeta_0 u) \|_{L_p(\mathbb{R}^d_T)}$$

$$\leq N \varepsilon^{3/(1-\sigma)} \| u \|_{p, r} \sum_{k=0}^{\infty} (\varepsilon^{2})^k + \sum_{k=0}^{\infty} \varepsilon^{k+3} \| (-\Delta)^{\sigma/2} (\zeta_{k+3} u) \|_{L_p(\mathbb{R}^d_T)}$$

$$+ N \| f \|_{p, r} \sum_{k=0}^{\infty} \varepsilon^k + N \frac{R^{d/p}(1 + R^{d+\sigma})}{(R-r)^{d+\sigma}} \| u \|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))} \sum_{k=0}^{\infty} (\varepsilon^{2})^k.$$ (3.21)

Therefore, by first taking $\varepsilon$ to be sufficiently small so that $\varepsilon^{2^{d+\sigma}} < 1$, and then absorbing the second term on the right-hand side of (3.21) to the left-hand side, we arrive at (3.12) with $(-\Delta)^{\sigma/2}$ in place of $\tilde{L}^1$ on the left-hand side. The estimates for general $L^1$ follow from (2.12). Similarly, it is easily seen that if we replace $\zeta_0 u$ with $\zeta_{k} u$ on the left-hand side of (3.12), the inequality still holds. Thus, by (A.4), (2.12), and (3.19) with $k = 0$,

$$\| L^1 u \|_{p, r} \leq \| L^1 (\zeta_0 u) \|_{L_p(\mathbb{R}^d_T)} + \| L^1 (\zeta_0 u) - \zeta_0 L^1 u \|_{L_p(\mathbb{R}^d_T)}$$

$$\leq \| (-\Delta)^{\sigma/2} (\zeta_0 u) \|_{L_p(\mathbb{R}^d_T)} + N \frac{\| u \|_{p, r}}{(R-r)^{\sigma}} + \| (-\Delta)^{\sigma/2} (\zeta_3 u) \|_{L_p(\mathbb{R}^d_T)}$$

$$+ N \frac{R^{d/p}(1 + R^{d+\sigma})}{(R-r)^{d+\sigma}} \| u \|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))} \leq N \| f \|_{p, r} + N \frac{\| u \|_{p, r}}{(R-r)^{\sigma}} + N \frac{R^{d/p}(1 + R^{d+\sigma})}{(R-r)^{d+\sigma}} \| u \|_{L_p((0,T);L_1(\mathbb{R}^d,\psi)).}$$

Combining this with (3.12), we infer (3.13) when $\sigma \in (1, 2)$.
Remark 3.3

In Lemma 3.2, we assume that we obtain (3.12) when

Multiplying \( \varepsilon^{k-1} \) to both sides of (3.22) and taking the sum over \( k = 1, \ldots \) lead to

Therefore, by first taking \( \varepsilon \) to be sufficiently small so that \( \varepsilon^{2d+1} < 1 \), and then absorbing the second term on the right-hand side of (3.23) to the left-hand side, we obtain

Also, by (2.12) and

we obtain (3.12) when \( \sigma = 1 \). As before, replacing \( \zeta_0 u \) with \( \zeta_3 u \) on the left-hand side of (3.12) together with (3.16) and (A.5), we obtain (3.13). The lemma is proved. \( \square \)

Remark 3.3 In Lemma 3.2, we assume that \( u \in \Pi_{p,0}^{\alpha,\sigma}(T) \). However, in the proof we only used the fact that \( \zeta_k u \in \Pi_{p,0}^{\alpha,\sigma}(T) \). Thus, it suffices to assume that \( u|_{B_R} \in \Pi_{p,0}^{\alpha,\sigma}((0, T) \times B_R) \) for \( u \) defined on \( (0, T) \times \mathbb{R}^d \) satisfying (3.11) in \( (0, T) \times \mathbb{R}^d \).

Lemma 3.2 and the embeddings in Lemmas A.6 and A.7 lead to the following corollary.

Corollary 3.4 Let \( \alpha \in (0, 1], \sigma \in (0, 2), T \in (0, \infty), p \in (1, \infty), 0 < r < R < \infty, \lambda \geq 0, u \in L_p((0, T); L_1(\mathbb{R}^d, \psi)) \) be such that \( u|_{B_R} \in \Pi_{p,0}^{\alpha,\sigma}((0, T) \times B_R) \), and \( q \in (p, \infty) \) satisfy

Let \( f = \partial_t^\alpha u - Lu + \lambda u \) in \( (0, T) \times B_R \).
(1) If \( p \leq d/\sigma + 1/\alpha \), then for any \( l \in \{p, q\} \),

\[
\|u\|_{L_l((0,T) \times B_\rho)} \leq N\|f\|_{L_p((0,T) \times B_R)} + N\left\|u\right\|_{L_p((0,T) \times B_R)} \left(\frac{R}{\rho}\right)^\sigma
\]

\[
+ N\frac{R^{d/p}}{\left(\frac{R}{\rho}\right)^{d+\sigma}}\|u\|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))},
\]

where \( N = N(d, \alpha, \sigma, p, l, T) \).

(2) If \( p > d/\sigma + 1/\alpha \), then there exists \( \tau = \sigma - (d + \sigma/\alpha)/p \in (0, 1) \) such that

\[
\|u\|_{C^\tau; (0,T) \times B_R} \leq N\|f\|_{L_p((0,T) \times B_R)} + N\left\|u\right\|_{L_p((0,T) \times B_R)} \left(\frac{R}{\rho}\right)^\sigma
\]

\[
\left. \right. + N\frac{R^{d/p}}{\left(\frac{R}{\rho}\right)^{d+\sigma}}\|u\|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))},
\]

where \( N = N(d, \alpha, \sigma, p, T) \).

Proof (1) If \( p \leq d/\sigma + 1/\alpha \), recall the definition of \( \zeta_0 \) in Lemma 3.2. By the Sobolev embeddings in Lemma A.6 and Hölder’s inequality, we have

\[
\|u\|_{L_l((0,T) \times B_\rho)} \leq N\|\zeta_0 u\|_{L_1(\mathbb{R}^d)} \leq N\|\zeta_0 u\|_{W^{\alpha, \sigma}(T)}
\]

\[
\leq N\left(\|f\|_{L_p((0,T) \times B_R)} + \frac{\|u\|_{L_p((0,T) \times B_R)}}{\left(\frac{R}{\rho}\right)^\sigma} + \frac{R^{d/p}}{\left(\frac{R}{\rho}\right)^{d+\sigma}}\|u\|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))}\right),
\]

where for the last inequality, we used (3.12) and Remark 3.3.

(2) If \( p > d/\sigma + 1/\alpha \), the proof is similar to that of (1) by using Lemma A.7.

By Lemma 3.2 with a scaling in the spatial coordinates, we derive an estimate that will be used later in Sect. 4.

Corollary 3.5 Let \( \alpha \in (0, 1], \sigma \in (0, 2), T \in (0, \infty), p \in (1, \infty), \) and \( R \in (0, \infty) \). Suppose that Theorem 2.6 holds for this \( p \). If \( u \in H^{\alpha, \sigma}_{\infty, 0}(T) \), and \( L^1 \) satisfies (2.3) when \( \sigma = 1 \) and (2.6), and

\[\partial^\alpha_t u - Lu + \lambda u = f \text{ in } \mathbb{R}^d_T,\]

then

\[
\left(\left|\partial^\alpha_t u\right|^{1/p}(0,T) \times B_{R/2}(x_0)\right) + \left(\left|L^1 u\right|^{1/p}(0,T) \times B_{R/2}(x_0)\right) + \lambda \left(\left|u\right|^{1/p}(0,T) \times B_{R/2}(x_0)\right)
\]

\[
\leq N\left(\|f\|^{1/p}(0,T) \times B_R(x_0) + \right. \sum_{k=0}^{\infty} 2^{-k\alpha} \left(\left|u\right|^{1/p}(0,T) \times B_{2^{-k}R}(x_0)\right) \right),
\]

(3.24)

where \( N = N(d, \nu, \Lambda, \alpha, \sigma, p) \).

Proof By shifting the coordinates, without loss of generality, we assume that \( x_0 = 0 \).

When \( R = 1 \), denoting \( p' = p/(p - 1) \), by the Minkowski inequality and Hölder’s inequality, we have

\[
\|u\|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))}
\]

\[
= \left(\int_0^T \left(\int_{\mathbb{R}^d} \left|u(t, x)\right| \frac{1}{1 + |x|^{d+\sigma}} \, dx\right)^p \, dt\right)^{1/p}.
\]

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\[
\begin{align*}
&\leq \int_{\mathbb{R}^d} \left( \int_0^T |u(t, x)|^p \, dt \right)^{1/p} \frac{1}{1 + |x|^{d+\sigma}} \, dx \\
&\leq \sum_{k=0}^{\infty} \int_{\hat{B}_{2k} \setminus \hat{B}_{2k-1}} \left( \int_0^T |u(t, x)|^p \, dt \right)^{1/p} \frac{1}{1 + |x|^{d+\sigma}} \, dx \\
&\leq \sum_{k=0}^{\infty} \left( \int_{\hat{B}_{2k} \setminus \hat{B}_{2k-1}} \int_0^T |u(t, x)|^p \, dt \, dx \right)^{1/p} \left( \int_{\hat{B}_{2k} \setminus \hat{B}_{2k-1}} \frac{1}{1 + |x|^{d+\sigma}} \, dx \right)^{1/p'} \\
&\leq T^{1/p} \sum_{k=0}^{\infty} \left( 2^{-k\sigma} \int_0^T \int_{B_{2k}} |u(t, x)|^p \, dx \, dt \right)^{1/p} 2^{-k\sigma/p'} \\
&\leq NT^{1/p} \sum_{k=0}^{\infty} 2^{-k\sigma} \left( |u|^p \right)_{(0,T) \times B_{2k}}^{1/p}, \tag{3.25}
\end{align*}
\]

where \( \hat{B}_{2k} = B_{2k} \) for \( k \geq 0 \) and \( \hat{B}_{2^{-1}} = \emptyset \). Thus, when \( R = 1 \), (3.24) follows from (3.13) and (3.25).

For general \( R > 0 \), we take
\[
\bar{u}(t, x) = R^{-\sigma} u(R^{\sigma/\alpha} t, Rx) \quad \text{and} \quad \bar{f}(t, x) = f(R^{\sigma/\alpha} t, Rx). \tag{3.26}
\]

It follows that
\[
\partial_t^\alpha \bar{u} - \bar{L} \bar{u} + \lambda \bar{u} = \bar{f} \quad \text{in} \ (0, \bar{T}) \times \mathbb{R}^d,
\]

where \( \bar{T} = R^{-\sigma/\alpha} T \), and \( \bar{L} \) is the operator with the kernel \( R^{d+\sigma} K(R^{\sigma/\alpha} t, R y) \) satisfying (2.2) with the same \( v \) and \( \Lambda \) as \( K \), the kernel of \( L \). Since
\[
\begin{align*}
(|\partial_t^\alpha \bar{u}|^p)_{(0, \bar{T}) \times B_{1/2}} &+ (|L^1 \bar{u}|^p)_{(0, \bar{T}) \times B_{1/2}} + \lambda (|\bar{u}|^p)_{(0, \bar{T}) \times B_{1/2}} \\
&\leq N (|\bar{f}|^p)_{(0, \bar{T}) \times B_1} + N \sum_{k=0}^{\infty} 2^{-k\sigma} (|\bar{u}|^p)_{(0, \bar{T}) \times B_{2k}},
\end{align*}
\]

we arrive at (3.24) by a change of variables. \( \Box \)

### 3.2 The level set argument

In this subsection, we prove Theorem 2.6 for general \( p \in (1, \infty) \). Recall the equation
\[
\partial_t^\alpha u - Lu + \lambda u = f. \tag{3.27}
\]

We start with a decomposition of the solution.

**Proposition 3.6** Let \( \alpha \in (0, 1], \sigma \in (0, 2), T \in (0, \infty), \lambda \geq 0 \) and \( p \in (1, \infty) \). Suppose that Theorem 2.6 holds for this \( p \), and \( u \in \mathbb{H}^{\alpha, \sigma}_{p, 0}(T) \) satisfies Eq. (3.27). Then there exists \( p_1 = p_1(d, \alpha, \sigma, p) \in (p, \infty) \) satisfying
\[
p_1 - p = \delta(d, \alpha, \sigma) > 0, \tag{3.28}
\]

and the following holds. For any \((t_0, x_0) \in [0, T] \times \mathbb{R}^d\), \( R > 0 \), and \( S = \min\{0, t_0 - R^{\sigma/\alpha}\} \), there exist
\[
w \in \mathbb{H}^{\alpha, \sigma}_{p_1, 0}((t_0 - R^{\sigma/\alpha}, t_0) \times \mathbb{R}^d) \quad \text{and} \quad v \in \mathbb{H}^{\alpha, \sigma}_{p_1, 0}((S, t_0) \times \mathbb{R}^d),
\]

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such that \( u = w + v \) in \( Q_R(t_0, x_0) \),

\[
(\|L^1w\|_{Q_R(t_0, x_0)})^{1/p} + (\|\lambda w\|_{Q_R(t_0, x_0)})^{1/p} \leq N(\|f\|_{Q_{2R}(t_0, x_0)})^{1/p},
\]

(3.29)

\[
(\|L^1v\|_{Q_{R/2}(t_0, x_0)})^{1/p_1} \leq N(\|f\|_{Q_{2R}(t_0, x_0)})^{1/p} + N \sum_{k=0}^{\infty} 2^{-k\sigma} (\|f\|_{(t_0 - R^{\sigma/\alpha}, t_0) \times B_{2kR}(x_0)})^{1/p}
\]

\[
+ N \sum_{k=0}^{\infty} 2^{-k\sigma} (\|L^1u\|_{(t_0 - R^{\sigma/\alpha}, t_0) \times B_{2kR}(x_0)})^{1/p}
\]

\[
+ N \sum_{k=0}^{\infty} 2^{-k\alpha} (\|L^1u\|_{(t_0 - (2^{k+1} + 1)R^{\sigma/\alpha}, t_0) \times B_R(x_0)})^{1/p},
\]

(3.30)

and

\[
(\|\lambda v\|_{Q_{R/2}(t_0, x_0)})^{1/p_1} \leq N(\|f\|_{Q_{2R}(t_0, x_0)})^{1/p} + N \sum_{k=0}^{\infty} 2^{-k\sigma} (\|f\|_{(t_0 - R^{\sigma/\alpha}, t_0) \times B_{2kR}(x_0)})^{1/p}
\]

\[
+ N \sum_{k=0}^{\infty} 2^{-k\sigma} (\|\lambda u\|_{(t_0 - R^{\sigma/\alpha}, t_0) \times B_{2kR}(x_0)})^{1/p}
\]

\[
+ N \sum_{k=0}^{\infty} 2^{-k\alpha} (\|\lambda u\|_{(t_0 - (2^{k+1} + 1)R^{\sigma/\alpha}, t_0) \times B_R(x_0)})^{1/p},
\]

(3.31)

where \( N = N(d, \nu, \Lambda, \alpha, \sigma, \rho) \),

\[
(1 \cdot |p|^1_p)_{Q_{R/2}(t_0, x_0)} := \|L_\infty(Q_{R/2}(t_0, x_0)) \|_{L_\infty(Q_{R/2}(t_0, x_0))} \text{ when } p_1 = \infty,
\]

and \( u, f \) are extended to be zero for \( t < 0 \).

**Proof** The proof of (3.31) would be similar to the proof of (3.30) by replacing \( L^1v \) with \( \lambda v \). Thus, we focus on (3.30).

By a shift of the coordinates and a scaling similar to (3.26), without loss of generality, we assume that \( x_0 = 0 \) and \( R = 1 \).

For any \( t_0 \in [0, T] \), we take a cutoff function \( \zeta \in C_0^\infty((t_0 - 2^{\sigma/\alpha}, t_0 + 2^{\sigma/\alpha}) \times B_2) \) satisfying

\[
\zeta \in [0, 1] \text{ and } \zeta = 1 \text{ in } (t_0 - 1, t_0) \times B_1.
\]

By Theorem 2.6, there exists \( w \in \mathbb{H}_{p,0}^{\alpha, \sigma}((t_0 - 1, t_0) \times \mathbb{R}^d) \) satisfying

\[
\partial_t^\alpha w - Lw + \lambda w = \zeta f \text{ in } (t_0 - 1, t_0) \times \mathbb{R}^d,
\]

and

\[
\|L^1w\|_{L_p((t_0 - 1, t_0) \times \mathbb{R}^d)} \leq N\|\zeta f\|_{L_p((t_0 - 1, t_0) \times \mathbb{R}^d)} \leq N\|f\|_{L_p(Q_2(t_0, 0))},
\]

(3.32)

where \( N = N(d, \nu, \Lambda, \alpha, \sigma, \rho) \). We then obtain the estimate (3.29).

Next, we take \( S = \min(0, t_0 - 1) \) and \( v = u - w \). Indeed, by taking the zero extension of \( w \) for \( t < t_0 - 1 \), we have \( w \in \mathbb{H}_{p,0}^{\alpha, \sigma}((S, t_0) \times \mathbb{R}^d) \). See [8, Lemma 3.5] for details. Thus, it follows that \( v \in \mathbb{H}_{p,0}^{\alpha, \sigma}((S, t_0) \times \mathbb{R}^d) \) and

\[
\partial_t^\alpha v - Lv + \lambda v = (1 - \zeta)f \text{ in } (S, t_0) \times \mathbb{R}^d.
\]
Furthermore, we take \( \eta \in C^\infty(\mathbb{R}) \) such that

\[
\eta(t) = \begin{cases} 
1 & \text{when } t \in (t_0 - \frac{1}{2})^{\frac{1}{\alpha}}(t_0), \\
0 & \text{when } t \in (t_0 - 1, t_0 + 1)^c,
\end{cases} \quad \text{and } |\eta'| \leq N(\alpha, \sigma). \tag{3.33}
\]

It follows from [8, Lemma 3.6] that \( h := \eta v \in \mathbb{H}_{p,0}^{\alpha,\sigma}((t_0 - 1, t_0) \times \mathbb{R}^d) \) and

\[
\partial_t^\alpha h - Lh + \lambda h = \begin{cases} 
g + \eta(1 - \zeta) f & \text{in } (t_0 - 1, t_0) \times \mathbb{R}^d \quad \text{when } \alpha \in (0, 1), \\
\eta v + \eta(1 - \zeta) f & \text{in } (t_0 - 1, t_0) \times \mathbb{R}^d \quad \text{when } \alpha = 1,
\end{cases} \tag{3.34}
\]

where

\[
g(t, x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_s^t (t - s)^{-\alpha - 1}(\eta(t) - \eta(s))v(s, x) \, ds.
\]

Moreover, we take

\[
\xi \in C^\infty_0(\mathbb{R}^d), \quad \text{supp}(\xi) \subset B_1, \quad \int_{\mathbb{R}^d} \xi = 1, \quad \text{and } \xi(\cdot) := \varepsilon^{-d} \xi(\cdot / \varepsilon).
\]

To estimate \( L^1 v \), we first mollify Eq. (3.34) and then take \( L^1 \) on both sides to get

\[
\partial_t^\alpha (L^1 h^\varepsilon) - L(L^1 h^\varepsilon) + \lambda L^1 h^\varepsilon = \tilde{g}^\varepsilon + L^1 (\eta[(1 - \zeta)f]^\varepsilon) \quad \text{in } (t_0 - 1, t_0) \times \mathbb{R}^d. \tag{3.35}
\]

where \( v^\varepsilon := v \ast_x \xi^\varepsilon \) and

\[
\tilde{g}^\varepsilon(t, x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_s^t (t - s)^{-\alpha - 1}(\eta(t) - \eta(s))L^1 v^\varepsilon(s, x) \, ds,
\]

when \( \alpha \in (0, 1) \) and \( \tilde{g}(t, x) = \eta'Lv^\varepsilon \) when \( \alpha = 1 \).

Next, we take \( p_1 \) such that

\[
1/p_1 = 1/p - \alpha \sigma/(2\alpha d + 2\sigma) \quad \text{if } p \leq d/\sigma + 1/\alpha,
\]

and

\[
p_1 = \infty \quad \text{if } p > d/\sigma + 1/\alpha.
\]

Note that \( p_1 \) satisfies (3.28), i.e., the increment is independent of \( p \). By the fact that \( L^1 h^\varepsilon \in \mathbb{H}_{p,0}^{\alpha,\sigma}((t_0 - 1, t_0) \times \mathbb{R}^d) \) and Corollary 3.4, we have

\[
\|L^1 h^\varepsilon\|_{L^p(Q_{1/2}(t_0, 0))} \leq \|L^1 h^\varepsilon\|_{L^p((t_0 - 1, t_0) \times B_{1/2})} \\
\leq N\|L^1 h^\varepsilon\| + \|\tilde{g}^\varepsilon\| + |L^1 (\eta[(1 - \zeta)f]^\varepsilon)|\|_{L_p((t_0 - 1, t_0) \times B_{3/4})} \\
+ N\|L^1 h^\varepsilon\|_{L^p((t_0 - 1, t_0); L^1(\mathbb{R}^d, \psi))},
\]

which, by taking the limit of \( \varepsilon \rightarrow 0 \), implies that

\[
\|L^1 h\|_{L^p(Q_{1/2}(t_0, 0))} \leq N\|L^1 h\| + \|\tilde{g}\| + |L^1 (\eta[(1 - \zeta)f])|\|_{L_p((t_0 - 1, t_0) \times B_{3/4})} \\
+ N\|L^1 h\|_{L^p((t_0 - 1, t_0); L^1(\mathbb{R}^d, \psi))}, \tag{3.36}
\]

where

\[
\tilde{g}(t, x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_s^t (t - s)^{-\alpha - 1}(\eta(t) - \eta(s))L^1 v(s, x) \, ds
\]
when \( \alpha \in (0, 1) \) and \( \tilde{g}(t, x) = \eta' L v \) when \( \alpha = 1 \). Then, by the definition of \( h \) and (3.36), we obtain

\[
\| L^1 v \|_{L_p(Q_{1/2}(t_0, 0))} = \| L^1 h \|_{L_p(Q_{1/2}(t_0, 0))}
\]
\[
\leq N \| L^1 v \|_{L_p((t_0-1,t_0) \times B_1)} + N \| L^1 ((1 - \zeta) f) \|_{L_p((t_0-1,t_0) \times B_{3/4})}
\]
\[
+ N \| L^1 v \|_{L_p((t_0-1,t_0); L_1(\mathbb{R}^d, \psi))},
\]

(3.37)

where \( N = N(d, v, \Lambda, \alpha, \sigma, p, p_1) \). We estimate the terms on the right-hand side of (3.37) separately as follows.

When \( \alpha \in (0, 1) \), a similar computation as in [8, Proposition 5.1] leads to

\[
\| \tilde{g} \|_{L_p(Q_{1}(t_0, 0))} \leq N \sum_{k=0}^{\infty} 2^{-k\alpha} \left( \| L^1 u \|^p_{(t_0-2^{k+1}+1)^{\sigma/\alpha}, t_0} \times B_1 \right.
\]
\[+ N \sum_{k=0}^{\infty} 2^{-k\alpha} \left( \| L^1 w \|^p_{(t_0-2^{k+1}+1)^{\sigma/\alpha}, t_0} \times B_1 \right.
\]
\[\leq N \sum_{k=0}^{\infty} 2^{-k\alpha} \left( \| L^1 u \|^p_{(t_0-2^{k+1}+1)^{\sigma/\alpha}, t_0} \times B_1 \right.
\]
\[+ N \| L^1 w \|^p_{Q_1(t_0, 0)} \]
\[\leq N \sum_{k=0}^{\infty} 2^{-k\alpha} \left( \| L^1 u \|^p_{(t_0-2^{k+1}+1)^{\sigma/\alpha}, t_0} \times B_1 \right.
\]
\[+ N \| f \|^p_{Q_2(t_0, 0)}. \]

(3.38)

where in the second inequality we used the fact that \( w \) vanishes when \( t < t_0 - 1 \), and in the last inequality, we used (3.32). Recall that when \( \alpha = 1 \), \( \tilde{g} = \eta' L^1 v \). Then it is easily seen that

\[
\| \tilde{g} \|_{L_p(Q_{1}(t_0, 0))} \leq N \| L^1 u \|^p_{(t_0-3^{\sigma}, t_0) \times B_1} + N \| f \|^p_{Q_2(t_0, 0)}. \]

Also, by the fact that \( 1 - \zeta \) vanishes in \( Q_1(t_0, 0) \) together with the Minkowski inequality and Hölder’s inequality, we have

\[
\| L^1 ((1 - \zeta) f) \|_{L_p((t_0-1,t_0) \times B_{3/4})}
\]
\[
\leq N \| \int_{\mathbb{R}^d} |(1 - \zeta(\cdot, \cdot + y)) f(\cdot, \cdot + y)| |y|^{-d-\sigma} \, dy \|_{L_p((t_0-1,t_0) \times B_{3/4})}
\]
\[
\leq N \| \int_{|y|>1/4} |(1 - \zeta(\cdot, \cdot + y)) f(\cdot, \cdot + y)| |y|^{-d-\sigma} \, dy \|_{L_p((t_0-1,t_0) \times B_{3/4})}
\]
\[
\leq N \| \sum_{k=-1}^{\infty} 2^{-kd-k\sigma} \int_{B_{2^k} \setminus B_{2^{k-1}}} |f(\cdot, \cdot + y)| \, dy \|_{L_p((t_0-1,t_0) \times B_{3/4})}
\]
\[
\leq N \sum_{k=0}^{\infty} 2^{-k\sigma} \left( t_0 \int_{t_0-1}^{t_0} \int_{B_{2^k}} |f(t, x)|^p \, dx \, dt \right)^{1/p} \]

(3.39)

Finally, using an estimate similar to (3.25) together with the fact that \( v = u - w \) and (3.32), we obtain

\[
\| L^1 v \|_{L_p((t_0-1,t_0); L_1(\mathbb{R}^d, \psi))}
\]
\[
\leq N \sum_{k=0}^{\infty} 2^{-k\sigma} \left( \| L^1 v \|^p_{(t_0-1,t_0) \times B_{2^k}} \right)
\]
\[ \leq N \sum_{k=0}^{\infty} 2^{-k\sigma} \left( \left( |L^1 u|^p \right)_{(t_0-1, t_0)} \times B_{2^k} + \left( |L^1 w|^p \right)_{(t_0-1, t_0)} \times B_{2^k} \right) \]

\[ \leq N \sum_{k=0}^{\infty} 2^{-k\sigma} \left( \left( |L^1 u|^p \right)_{(t_0-1, t_0)} \times B_{2^k} + N \left( |f|^p \right)_{Q_2(t_0, 0)}^{1/p} \right). \tag{3.40} \]

Thus, combining (3.37), (3.38), (3.39), and (3.40), we arrive at (3.30) when \( R = 1 \). The proof of (3.31) is similar (and actually simpler) by using

\[ \|h\|_{L^p((Q_{1/2}(t_0,0)) \leq N\|h\|_{L^p((t_0-1, t_0)) \times B_{1/4}} + N\|h\|_{L^p((t_0-1, t_0)) \times L^1(\mathbb{R}^d, \psi)} \]

instead of (3.36). We omit the details. The proposition is proved. \( \square \)

Next, with the estimates above, we verify Assumption A.4 of Lemma A.5, which is the key to the level set argument. For \( R > 0 \), \( p \in (1, \infty) \), and \( p_1 = p_1(d, \alpha, \sigma) \) from Proposition 3.6, denote

\[ A(s) = \left\{ (t, x) \in (-\infty, T) \times \mathbb{R}^d : |L^1 u(t, x)| > s \right\}, \]

and

\[ B_{\gamma}(s) = \left\{ (t, x) \in (-\infty, T) \times \mathbb{R}^d : \gamma^{-1/p} (M|f|^p(t, x))^{1/p} + \gamma^{-1/p_1} (SM|L^1 u|^p(t, x))^{1/p} > s \right\}. \tag{3.41} \]

Also, we write

\[ C_R(t, x) = (t - R^{\sigma/\alpha}, t + R^{\sigma/\alpha}) \times B_R(x) \quad \text{and} \quad \tilde{C}_R(t, x) = C_R(t, x) \cap \{ t \leq T \}. \tag{3.42} \]

**Lemma 3.7** Let \( \gamma \in (0, 1), \alpha \in (0, 1), \sigma \in (0, 2), T \in (0, \infty), R > 0, \lambda \geq 0 \) and \( p \in (1, \infty) \). Suppose that Theorem 2.6 holds for this \( p \), and \( u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T) \) satisfies Eq. (3.27). Then, there exists a sufficiently large constant \( \kappa = \kappa(d, \nu, \Lambda, \alpha, \sigma, p) > 1 \) such that for any \( (t_0, x_0) \in (-\infty, T) \times \mathbb{R}^d \) and \( s > 0 \), if

\[ |C_R(t_0, x_0) \cap A(\kappa s)| \geq \gamma |C_R(t_0, x_0)|, \tag{3.43} \]

then we have

\[ \tilde{C}_R(t_0, x_0) \subset B_{\gamma}(s), \]

where \( \tilde{R} = 2^{-1-\alpha/\sigma} R \).

**Proof** By dividing Eq. (3.27) by \( s \), we assume that \( s = 1 \).

First note that since \( u \) vanishes when \( t < 0 \), if \( t_0 + \tilde{R}^{\sigma/\alpha} < 0 \), then

\[ C_R(t_0, x_0) \cap A(\kappa) \subset \left\{ (t, x) \in (-\infty, 0) \times \mathbb{R}^d : |L^1 u(t, x)| > 1 \right\} = \emptyset, \]

and (3.43) is not satisfied. Thus, it suffices to consider the case when \( t_0 + \tilde{R}^{\sigma/\alpha} \geq 0 \).

We argue by contradiction. Suppose that there exist some \( (s, y) \in \tilde{C}_R(t_0, x_0) \) such that

\[ \gamma^{-1/p} (M|f|^p(s, y))^{1/p} + \gamma^{-1/p_1} (SM|L^1 u|^p(s, y))^{1/p} \leq 1. \tag{3.44} \]

Let \( t_1 := \min\{t_0, t_1 - R^{\sigma/\alpha}, T\} \). Then by Proposition 3.6, for \( S = \min\{0, t_1 - R^{\sigma/\alpha}\} \), there exist \( w \in \mathbb{H}_{p,0}^{\alpha,\sigma}((t_1 - R^{\sigma/\alpha}, t_1) \times \mathbb{R}^d) \) and \( v \in \mathbb{H}_{p,0}^{\alpha,\sigma}((S, t_1) \times \mathbb{R}^d) \) such that \( u = w + v \) in \( Q_R(t_1, x_0) \),

\[ \left( |L^1 w|^p \right)_{Q_R(t_1, x_0)}^{1/p} \leq N \left( |f|^p \right)_{Q_2(t_1, x_0)}^{1/p}, \tag{3.45} \]

and
\[
\left( |L^1 v|^{p_1} \right)^{1/p_1}_{Q_{R/2}(t_1,x_0)} \leq N \left( |f|^p \right)^{1/p}_{Q_{2R}(t_1,x_0)} \\
+ N \sum_{k=0}^{\infty} 2^{-k\sigma} \left( \int_{t_1-R^{q/\alpha}}^{t_1} \int_{B_{2^k R}(x_0)} |f(t,x)|^p \, dx \, dt \right)^{1/p} \\
+ N \sum_{k=0}^{\infty} 2^{-k\sigma} \left( \int_{t_1-(2^{k+1}+1)R^{q/\alpha}}^{t_1} \int_{B_R(x_0)} |L^1 u(t,y)|^p \, dy \, dt \right)^{1/p} \\
+ N \sum_{k=0}^{\infty} 2^{-k\sigma} \left( \int_{t_1-R^{q/\alpha}}^{t_1} \int_{B_{2^k R}(x_0)} |L^1 u(t,y)|^p \, dx \, dt \right)^{1/p},
\]

(3.46)

where \( N = N(d, \nu, \Lambda, \alpha, \sigma, p) \). We have

\[
(s, y) \in \mathcal{C}\hat{R}(t_0, x_0) \subset Q_{R/2}(t_1, x_0) \subset Q_{2R}(t_1, x_0),
\]

\[
(s, y) \in \mathcal{C}\hat{R}(t_0, x_0) \subset (t_1 - (2^{k+1}+1)R^{q/\alpha}, t_1) \times B_R(x_0),
\]

and

\[
(s, y) \in \mathcal{C}\hat{R}(t_0, x_0) \subset (t_1 - R^{q/\alpha}, t_1) \times B_{2^k R}(x_0)
\]

for all \( k = 0, 1, \ldots \). By these set inclusions together with (3.44), (3.45), and (3.46), we infer

\[
|L^1 v|^{p_1}_{Q_{R/2}(t_1,x_0)} \leq N \gamma^{1/p_1} \quad \text{and} \quad |L^1 w|^{p_1}_{Q_{R}(t_1,x_0)} \leq N \gamma^{1/p_1},
\]

where \( N = N(d, \nu, \Lambda, \alpha, \sigma, p) \). Then for a constant \( C_1 > 0 \) to be determined, by the Chebyshev inequality,

\[
|\mathcal{C}\hat{R}(t_0, x_0) \cap A(\kappa)|
\]

\[
= |\{ (t, x) \in \mathcal{C}\hat{R}(t_0, x_0) : t \in (-\infty, T) : |L^1 u(t,x)| > \kappa \}|
\]

\[
\leq |\{ (t, x) \in Q_{R/2}(t_1, x_0) : |L^1 u(t,x)| > \kappa \}|
\]

\[
\leq |\{ (t, x) \in Q_{R/2}(t_1, x_0) : |L^1 w(t,x)| > \kappa - C_1 \}|
\]

\[
+ |\{ (t, x) \in Q_{R/2}(t_1, x_0) : |L^1 v(t,x)| > C_1 \}|
\]

\[
\leq \mathcal{O}_{Q_{R/2}(t_1,x_0)}(\kappa - C_1)^{-p} |L^1 w|^p + C_1^{-p_1} |L^1 v|^{p_1} \, dx \, dt 
\]

\[
\leq \frac{N^p \gamma |Q_{R/2}|}{(\kappa - C_1)^p} + \frac{N^{p_1} \gamma |Q_{R/2}|}{C_1^{p_1}} |L^1 v|^{p_1} \, dx \, dt 
\]

\[
\leq N_0(d, \alpha, \sigma) |\mathcal{C}\hat{R}(t_0, x_0)| \gamma \left( \frac{N^p}{(\kappa - C_1)^p} + \frac{N}{C_1^{p_1}} \right) 
\]

By first taking a sufficiently large \( C_1 = C_1(d, \nu, \Lambda, \alpha, \sigma, p) \) such that

\[
N_0(N/C_1)^p < 1/2,
\]

and then taking a large \( \kappa = \kappa(d, \nu, \Lambda, \alpha, \sigma, p) \) so that

\[
N_0N^p/(\kappa - C_1)^p < 1/2,
\]

we obtain

\[
|\mathcal{C}\hat{R}(t_0, x_0) \cap A(\kappa)| < \gamma |\mathcal{C}\hat{R}(t_0, x_0)|.
\]

(3.47)

However, (3.47) contradicts (3.43). The lemma is proved. \( \square \)
Remark 3.8 By replacing $L^1 u$ with $\lambda u$ in the above proof together with (3.31), we conclude that Lemma 3.7 holds for $\mathcal{A}'(s) = \{(t, x) \in (-\infty, T) \times \mathbb{R}^d : |\lambda u(t, x)| > s\}$, and

$$B'_p(s) = \{(t, x) \in (-\infty, T) \times \mathbb{R}^d : \gamma^{-1/p}(\mathcal{M}|f|^p(t, x))^{1/p} + \gamma^{-1/p_1}(\mathcal{S}\mathcal{M}|\lambda u|^p(t, x))^{1/p} > s\}.$$  

We are ready to prove Theorem 2.6 for general $p \in (1, \infty)$.

Proof of Theorem 2.6 Since the existence of solutions for the equation

$$\partial_t^\alpha u + (-\Delta)^{\alpha/2} u = f$$

was derived in [17, Theorem 2.8], by the method of continuity, it suffices to prove the a priori estimate (2.8). Note that by the argument in Remark 2.7, if (2.8) holds for smooth functions, then (2.12) holds for smooth functions. Thus, by the density of smooth functions in $\mathbb{H}^\sigma_{p,0}(T)$, without loss of generality, we assume that $u \in C^\infty_0([0, T] \times \mathbb{R}^d)$ with $u(0, \cdot) = 0$.

We first consider the case when $p \in [2, \infty)$ by using an iterative argument to successively increase the exponent $p$, which is referred to as the bootstrap argument. Recall that we have proved the base case when $p = 2$. Now we assume that the theorem holds for some $p_0 \in [2, \infty)$, and we prove (2.8) for $p \in (p_0, p_1)$, where $p_1 = p_1(d, \alpha, \sigma, p_0)$ is from Proposition 3.6. Note that

$$\|L^1 u\|_{L_p(\mathbb{R}^d)}^p = p \int_0^\infty |A(s)| s^{p-1} ds = pk^p \int_0^\infty |A(\kappa s)| s^{p-1} ds,$$

and Lemma 3.7 together with Lemma A.5 leads to

$$|A(\kappa s)| \leq N(d, \alpha) |B_{\gamma'}(s)|$$

for all $s \in (0, \infty)$, where $\kappa = \kappa(d, v, \Lambda, \alpha, \sigma, p)$ is from Lemma 3.7 and $A, B_{\gamma'}$ are defined in (3.41). Thus, by the Hardy–Littlewood theorem for strong maximal functions,

$$\|L^1 u\|_{L_p(\mathbb{R}^d)}^p \leq Npk^p \gamma \int_0^\infty |B_{\gamma'}(s)| s^{p-1} ds$$

$$\leq N \gamma \int_0^\infty \left| \left\{ (t, x) \in (-\infty, T) \times \mathbb{R}^d : \gamma^{-\frac{1}{p}}(\mathcal{S}\mathcal{M}|L^1 u|^p(t, x))^{\frac{1}{p}} > s/2 \right\} \right| s^{p-1} ds$$

and

$$\leq N \gamma^{-1/p_1} \|L^1 u\|_{L_p(\mathbb{R}^d)}^p + N \gamma^{-1/p_0} \|f\|_{L_p(\mathbb{R}^d)}^p.$$
which implies (2.8) for \( p \in (p_0, p_1) \). We repeat this procedure. Recall that \( p_1 - p \) depends only on \( d, \alpha, \sigma \). Thus in finite steps, we get a \( p_0 \) which is larger than \( d/2 + 1/\alpha \), so that \( p_1 = p_1(d, \alpha, p_0) = \infty \). Therefore, the theorem is proved for any \( p \in [2, \infty) \).

For \( p \in (1, 2) \), we use a duality argument. Again, we assume that \( u \in C_0^\infty([0, T] \times \mathbb{R}^d) \) with \( u(0, x) = 0 \) and prove (2.8). Let \( L^* \) be the operator with the kernel \( K(-t, -y) \), where \( K \) is the kernel of \( L \). For \( p' = p/(p-1) \) and \( \phi \in L_{p'}(\mathbb{R}_T^d) \), there exist \( w \in H_{p',0}^\alpha((-T, 0) \times \mathbb{R}^d) \) satisfying

\[
\partial_t^\alpha w - L^* w + \lambda w = \phi(-t, x) \quad \text{in } (-T, 0) \times \mathbb{R}^d
\]

and

\[
\|L^1 w\|_{L_{p'}((-T,0) \times \mathbb{R}^d)} \leq N \|\phi(-t, x)\|_{L_{p'}((-T,0) \times \mathbb{R}^d)} = N \|\phi\|_{L_{p'}(\mathbb{R}_T^d)},
\]

where \( \partial_t^\alpha w = \partial_t I_T^{- \alpha} w \) and \( N = N(d, \nu, \Lambda, \alpha, \sigma, p) \). It follows that

\[
\int_0^T \int_{\mathbb{R}^d} \phi L^1 u \, dx \, dt = \int_{-T}^0 \int_{\mathbb{R}^d} \phi(-t, x)L^1 u(-t, x) \, dx \, dt
\]

\[
= \int_{-T}^T \int_{\mathbb{R}^d} (\partial_t^\alpha w(t, x) - L^* w(t, x))L^1 u(-t, x) \, dx \, dt
\]

\[
= \int_0^T \int_{\mathbb{R}^d} (\partial_t^\alpha u(t, x) - Lu(t, x))(L^1)^* w(-t, x) \, dx \, dt
\]

\[
= \int_0^T \int_{\mathbb{R}^d} f(t, x)(L^1)^* w(-t, x) \, dx \, dt \leq N \|f\|_{L_{p'}(\mathbb{R}_T^d)} \|\phi\|_{L_{p'}(\mathbb{R}_T^d)}. \tag{3.48}
\]

Note that for the third equality, if \( w \) is smooth, then we apply the Plancherel theorem to the integral of \( L^* w(t, x)L^1 u(-t, x) \) and apply an integration by parts to the integral of \( \partial_t^\alpha w(t, x)L^1 u(-t, x) \) using the zero initial conditions of \( w \) and \( u \). If \( w \) is not necessarily smooth, we take \( w_k \in C_0^\infty([-T, 0] \times \mathbb{R}^d) \) with \( w_k(-T, 0) = 0 \) such that

\[w_k \to w \quad \text{in } H_{p',0}^\alpha((-T, 0) \times \mathbb{R}^d).\]

Similarly, we have

\[
\int_0^T \int_{\mathbb{R}^d} \phi \lambda u \, dx \, dt = \int_0^T \int_{\mathbb{R}^d} f(t, x)\lambda w(-t, x) \, dx \, dt \leq N \|f\|_{L_{p'}(\mathbb{R}_T^d)} \|\phi\|_{L_{p'}(\mathbb{R}_T^d)}. \tag{3.49}
\]

Thus, by (3.48) and (3.49), we arrive at (2.8). As before, by the method of continuity, the solvability and thus theorem are proved for the remaining case when \( p \in (1, 2) \).

**Proof of Corollary 2.8** With Theorem 2.6, the proof of Corollary 2.8 proceeds similar to that of Corollary 2.10, using the frozen coefficient argument and a partition of unity. We refer the reader to the proof of Corollary 2.8 at the end of Sect. 4.

### 4 Equations in \( L_{p,q,w} \) when \( \alpha = 1 \)

In this section, we prove Theorem 2.9 by deriving a mean oscillation estimate of \( u \in H_{p_0}^1((-\infty, T) \times \mathbb{R}^d) \) satisfying

\[
\partial_t u - Lu + \lambda u = f \quad \text{in } (-\infty, T) \times \mathbb{R}^d.
\]
In particular, for \( t_0 \in (-\infty, T) \), we are going to decompose \( u = w + v \) in \((t_0 - 1, t_0) \times \mathbb{R}^d\) and estimate them separately, where \( w \in \mathbb{H}_{p_0,0}^{1,\sigma}((t_0 - 1, t_0) \times \mathbb{R}^d)\) satisfies
\[
\partial_t w - L w + \lambda w = f \quad \text{in} \quad (t_0 - 1, t_0) \times \mathbb{R}^d
\] (4.1) and
\[
\partial_t v - L v + \lambda v = 0 \quad \text{in} \quad (t_0 - 1, t_0) \times \mathbb{R}^d. \] (4.2)

Note that the decomposition above differs from the one in the previous section.

Throughout this section, We denote \( L^1 \) to be an operator satisfying (2.3) when \( \sigma = 1 \) and (2.6). Furthermore, for \( r > 0 \) and \( z = (z_1, \ldots, z^d) \in \mathbb{R}^d \), the cube centered at \( z \) with radius \( r \) is defined as
\[
C_r(z) := (z^1 - r/2, z^1 + r/2) \times (z^2 - r/2, z^2 + r/2) \times \ldots \times (z^d - r/2, z^d + r/2).
\]

We start with the estimate of \( v \). In the following proposition, using the Sobolev embedding and an iteration argument, we bound the Hölder semi-norm of \( L^1 v \), in particular \((-\Delta)^{\sigma/2} v\), by a sum of its \( L_{p_0} \) norms taken over cubes of fixed size centered at integer points. For the convenience of computation, we use cubes instead of balls in the spatial coordinates. Note that since \( B_{r/2} \subset C_r \subset B_{\sqrt{d}r/2} \), we can replace the norms taken over balls with norms taken over cubes in Corollary 3.4.

**Proposition 4.1** Let \( \sigma \in (0, 2) \), \( p_0 \in (1, \infty) \), \( t_0 \in \mathbb{R} \), and \( v \in \mathbb{H}_{p_0}^{1,\sigma}((-\infty, t_0) \times \mathbb{R}^d) \) satisfy (4.2). Then for any \( r > 0 \),

1. for any \( p \in (p_0, \infty) \) and \( x_0 \in \mathbb{R}^d \), we have
\[
(|L^1 v|^p)_{Q_r(z_0, x_0)}^{1/p} \leq N \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} (|L^1 v|^p)_{(t_0 - r, t_0)}^{1/p_0} \times C_r(z + x_0)
\] (4.3)
and
\[
(|\lambda v|^p)_{Q_r(z_0, x_0)}^{1/p} \leq N \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} (|\lambda v|^p)_{(t_0 - r, t_0)}^{1/p_0} \times C_r(z + x_0), \] (4.4)

where \( N = N(d, \nu, \Lambda, \sigma, p_0, p) \).

2. For any \( x_0 \in \mathbb{R}^d \), there exists \( \tau = \tau(d, \sigma) \in (0, 1) \) such that
\[
[L^1 v]_{C^{\tau,s}_{Q_r(2(t_0, x_0))}} \leq N r^{-\tau} \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} (|L^1 v|^p)_{(t_0 - r, t_0)}^{1/p_0} \times C_r(z + x_0)
\] (4.5)
and
\[
[\lambda v]_{C^{\tau,s}_{Q_r(2(t_0, x_0))}} \leq N r^{-\tau} \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} (|\lambda v|^p)_{(t_0 - r, t_0)}^{1/p_0} \times C_r(z + x_0),
\] (4.6)

where \( N = N(d, \nu, \Lambda, \sigma, p_0) \).

**Proof** The proofs of (4.4) and (4.6) are similar to the proofs of (4.3) and (4.5) by replacing \( L^1 v \) with \( \lambda v \). Thus, we focus on (4.3) and (4.5).

By scaling and shifting the coordinates, we assume that \( r = 1 \) and \( x_0 = 0 \). Moreover, for \( m = 1, 2, \ldots \), we take \( p_m = p_m(d, \sigma, p_0) \) satisfying
\[
1/p_m = 1/p_0 - m\sigma/(d + \sigma)
\] (4.7)
whenever the right-hand side is positive.
We first prove (4.3). We take cutoff function \( \eta \in C^\infty(\mathbb{R}) \) satisfying (3.33). It follows that \( \eta v \in H^1_{p_0,0}( (t_0 - 1, t_0) \times \mathbb{R}^d) \) and
\[
\partial_t (\eta v) - L(\eta v) + \lambda (\eta v) = \partial_t (\eta v) - \eta \partial_t v = v \eta' \quad \text{in} \quad (t_0 - 1, t_0) \times \mathbb{R}^d. \tag{4.8}
\]
Taking \( L^1 \) on both sides of (4.8) leads to
\[
\partial_t (\eta L^1 v) - L(\eta L^1 v) + \lambda (\eta L^1 v) = L^1 v \eta' \quad \text{in} \quad (t_0 - 1, t_0) \times \mathbb{R}^d. \tag{4.9}
\]
Note that if \( v \) is not regular enough, we can first mollify the equation and then take the limit of the estimate as in (3.35). Moreover, note that by the Minkowski inequality
\[
\|L^1 v\|_{L^p_{p_0}((t_0 - 1, t_0); L^1(\mathbb{R}^d, \psi))} \leq \int_{\mathbb{R}^d} \left( \int_{t_0 - 1}^{t_0} |L^1 v(t, x)|^p dt \right)^{1/p} dx \leq \sum_{z \in \mathbb{Z}^d} \left( \int_{C_1(z)} \left( \int_{t_0 - 1}^{t_0} |L^1 v(t, x)|^p dt \right)^{1/p} \right)^{1/p} \leq N(d, \sigma, p_0) \sum_{z \in \mathbb{Z}^d} \left( 1 + |z|^{d+\sigma} \right)^{-1} \left( \int_{C_1(z)} |L^1 v(t, x)|^p dt \right)^{1/p} \leq N \sum_{z \in \mathbb{Z}^d} \left( 1 + |z|^{d+\sigma} \right)^{-1} \left( |L^1 v|^p_{p_0, \rho_0, (t_0 - 1, t_0) \times C_1(z)} \right)^{1/p} \leq N(d, \sigma, p_0) \sum_{z \in \mathbb{Z}^d} \left( 1 + |z|^{d+\sigma} \right)^{-1} \left( |L^1 v|^p_{p_0, (t_0 - 1, t_0) \times C_1(z)} \right)^{1/p}.
\]
Hence, by applying Corollary 3.4 to (4.9), we have
\[
\|L^1 v\|_{L^p_{p_1}(Q_{1/2}(t_0, 0))} \leq \|\eta L^1 v\|_{L^p_{1}(((t_0 - 1, t_0) \times B_{1/2})} \leq N\|L^1 v\|_{L^p_{p_0}((t_0 - 1, t_0) \times B_{1})} + \|L^1 v\|_{L^p_{p_0}((t_0 - 1, t_0); L^1(\mathbb{R}^d, \psi))} \leq N \sum_{z \in \mathbb{Z}^d} \left( 1 + |z|^{d+\sigma} \right)^{-1} \left( |L^1 v|^p_{p_0, \rho_0, (t_0 - 1, t_0) \times C_1(z)} \right)^{1/p} \leq N \sum_{z \in \mathbb{Z}^d} \left( 1 + |z|^{d+\sigma} \right)^{-1} \left( |L^1 v|^p_{p_0, (t_0 - 1, t_0) \times C_1(z)} \right)^{1/p},
\]
where \( N = N(d, v, \Lambda, \sigma, p_0) \). If \( p \leq p_1 \), then we arrive at (4.3) by Hölder’s inequality.

Otherwise, note that with a minor modification of the above estimates, we conclude that \( v, (-\Delta)^{\sigma/2} v, \partial_t v \in L_{p_1}((t_0 - 1/2^\sigma, t_0) \times B_{3/4}) \).

Furthermore, we take a cutoff function
\[
\zeta \in C^\infty(\mathbb{R}^d), \quad \zeta = 1 \quad \text{in} \quad B_{1/2}, \quad \text{and} \quad \zeta = 0 \quad \text{in} \quad B^c_{3/4}.
\]
By taking the mollification (in both the time and the spatial variables) of \( \zeta v \) as the defining sequence, we conclude that
\[
\zeta v \in H^1_{p_1, \sigma}((t_0 - 1/2^\sigma, t_0) \times \mathbb{R}^d). \tag{4.11}
\]
Moreover, similar to (4.10) with a shift of the coordinates and some minor modifications, we have
\[ \|L^1 v\|_{L^p_1((t_0-(1/2)^\sigma,t_0)) \times B_{r/2}(x_0)} \leq N \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} \left( \|L^1 v\|_{(t_0-(1/2)^\sigma,t_0)}^{1/p_0} \right)_{(t_0-1,t_0)} \times C_1(z+x_0). \] (4.12)

Therefore, by taking another cutoff function in time together with (4.11), Corollary 3.4, and (4.12), we obtain

\[ \|L^1 v\|_{L^p_2(Q_1/4(t_0,0))} \leq N \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} \left( \|L^1 v\|_{(t_0-(1/2)^\sigma,t_0)}^{1/p_1} \right)_{(t_0-1/2,t_0)} \times C_1(z), \]

where for the last inequality, we used

\[ \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} (1 + |y - z|^{d+\sigma})^{-1} \leq N(d, \sigma)(1 + |y|^{d+\sigma})^{-1} \sum_{z \in \mathbb{Z}^d} \left( (1 + |z|^{d+\sigma})^{-1} + (1 + |y - z|^{d+\sigma})^{-1} \right) \]

\[ \leq N(d, \sigma)(1 + |y|^{d+\sigma})^{-1}. \]

We repeat the above process with \( p_m, m = 3, 4, 5, \ldots \), finite many times until \( p_i \geq p \) or \( p_i > d/\sigma + 1 \). Note that the number of iterations depends only on \( d, \sigma \), and \( p \) by (4.7). Thus, by a covering argument, we obtain (4.3).

For (4.5), if \( p_0 \geq 2(d/\sigma + 1) \), then by Corollary 3.4, we can replace the left-hand side of the inequality (4.10) with \( \|L^1 v\|_{C^{1/\alpha,1}(Q_1/2(t_0,0))} \) to get (4.5). Otherwise, we apply the iteration argument as above until \( p_k \geq 2(d/\sigma + 1) \), and by a covering argument, we arrive at (4.5). The proposition is proved. \( \square \)

**Remark 4.2** It is worth noting that the operator we considered in Proposition 4.1 was local in time, i.e., \( \alpha = 1 \). For general \( \alpha \not= 1 \), it is not clear to us whether one can derive a similar estimate. In particular, we cannot get an expression as simple as (4.9) (see the extra \( \tilde{g} \) term on (3.35)), and the second inequality of (4.10) no longer holds.

Next, we estimate the non-homogeneous part, which satisfies the zero initial condition at \( t = t_0 - 1 \). In the following proposition, we denote \( \| \cdot \|_{p_0,\Omega} := \| \cdot \|_{L^p_0((t_0-1,t_0) \times \Omega)} \) for any \( \Omega \subset \mathbb{R}^d \) and \( \| \cdot \|_{p_0} := \| \cdot \|_{L^p_0((t_0-1,t_0) \times \mathbb{R}^d)}. \)

**Proposition 4.3** Let \( t_0 \in \mathbb{R} \) and \( w \in H^{1,\sigma}_{p_0,0}((t_0 - 1, t_0) \times \mathbb{R}^d) \) be such that

\[ \partial_t w - Lw + \lambda w = f \quad \text{in} \quad (t_0 - 1, t_0) \times \mathbb{R}^d. \]

Then, for any \( x_0 \in \mathbb{R}^d \) and \( R \geq 1 \), we have

\[ \left( |L^1 w|_{p_0} \right)_{(t_0-1,t_0)} \times B_{R/2}(x_0) + \lambda \left( |w|_{p_0} \right)_{(t_0-1,t_0)} \times B_{R/2} \leq N \sum_{k=0}^{\infty} 2^{-k\sigma} \left( |f|_{p_0} \right)_{(t_0-1,t_0)} \times B_{2kR}(x_0), \]

(4.13)
where \( N = N(d, \nu, \Lambda, \sigma, p_0) \).

**Proof** By shifting the coordinates, we assume that \( x_0 = 0 \). It follows from Corollary 3.5 that

\[
(|L^1 w|^{p_0})_{(t_0-1,t_0)}^{1/p_0} + \lambda (|w|^{p_0})_{(t_0-1,t_0)}^{1/p_0} \leq N (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} + N \sum_{k=0}^{\infty} 2^{-k\sigma} (|w|^{p_0})_{(t_0-1,t_0)}^{1/p_0} \times B_{2^k R}.
\]

Thus,

\[
:= N (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} + N \sum_{k=0}^{\infty} 2^{-k\sigma} A_k. \tag{4.14}
\]

It remains to estimate \( A_k \). By Lemma A.2 and Corollary 3.5, for each \( k = 0, 1, \ldots \),

\[
A_k \leq N (|\partial_t w|^{p_0})_{(t_0-1,t_0)}^{1/p_0} B_{2^k R} \leq N (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} B_{2^k+1 R} + N \sum_{j=0}^{\infty} 2^{-(j+k+1)\sigma} (|w|^{p_0})_{(t_0-1,t_0)}^{1/p_0} \times B_{2^{j+k+1} R} \leq N (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} B_{2^k+1 R} + N \sum_{j=0}^{\infty} 2^{-j\sigma} A_j, \tag{4.15}
\]

where \( N \) is independent of \( k \). By first multiplying both sides of (4.15) by \( 2^{-\sigma k} \) and then summing over \( k = k_0, k_0 + 1, \ldots \), for some integer \( k_0 \) to be determined, we obtain

\[
\sum_{k=k_0}^{\infty} 2^{-\sigma k} A_k \leq N \sum_{k=k_0}^{\infty} 2^{-\sigma(k+1)} (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} B_{2^{k+1} R} + N \sum_{k=k_0}^{\infty} 2^{-\sigma k} \sum_{j=k+1}^{\infty} 2^{-j\sigma} A_j \leq N \sum_{k=k_0}^{\infty} 2^{-\sigma(k+1)} (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} B_{2^{k+1} R} + N \sum_{j=k_0+1}^{\infty} 2^{-\sigma j} A_j.
\]

Picking \( k_0 \) sufficiently large so that \( N 2^{-\sigma k_0} \leq 1/2 \), we have

\[
\sum_{k=k_0}^{\infty} 2^{-\sigma k} A_k \leq N \sum_{k=k_0}^{\infty} 2^{-\sigma k} (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} B_{2^k R}.
\]

Therefore, by induction,

\[
\sum_{k=0}^{\infty} 2^{-\sigma k} A_k \leq N \sum_{k=0}^{\infty} 2^{-\sigma k} (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} B_{2^k R}. \tag{4.16}
\]

Indeed, if there exists \( N = N(d, \nu, \Lambda, \sigma, p_0) \) such that

\[
\sum_{k=j}^{\infty} 2^{-\sigma k} A_k \leq N \sum_{k=j}^{\infty} 2^{-\sigma k} (|f|^{p_0})_{(t_0-1,t_0)}^{1/p_0} B_{2^k R}
\]

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for $j = 1, 2, \ldots, k_0$, then, by (4.15),

$$
\sum_{k=j-1}^{\infty} 2^{-\sigma k} A_k \leq 2^{-\sigma (j-1)} A_{j-1} + \sum_{k=j}^{\infty} 2^{-\sigma k} A_k
$$

$$
\leq N 2^{-\sigma j} \left( |f|_{p_0} \right)^{1/p_0} \|_{(0_0-1, T_0)} \times B_{2^{-j}R} + N \sum_{k=j}^{\infty} 2^{-k \sigma} A_k
$$

$$
\leq N \sum_{k=j-1}^{\infty} 2^{-\sigma k} \left( |f|_{p_0} \right)^{1/p_0} \|_{(0_0-1, T_0)} \times B_{2^{-k}R}.
$$

Therefore, by (4.14) and (4.16), we arrive at (4.13), and the lemma is proved. \hfill \Box

**Remark 4.4** It is worth noting that although the operator we considered in Proposition 4.3 was local in time, i.e., $\alpha = 1$, the proof still works for any $\alpha \in (0, 1]$ since the embedding used in (4.15) still holds in this case.

With the estimates of $v$ and $w$, we are ready to prove a mean oscillation estimate for $u$.

**Proposition 4.5** Let $\sigma \in (0, 2)$, $T \in (0, \infty)$, $p_0 \in (1, \infty)$, and $u \in \mathbb{H}^{1, \sigma}_{p_0, T} ((-\infty, T) \times \mathbb{R}^d)$ satisfy

$$
\partial_t u - Lu + \lambda u = f \text{ in } (-\infty, T) \times \mathbb{R}^d.
$$

Then, for any $(t_0, x_0) \in (-\infty, T] \times \mathbb{R}^d$, $r \in (0, \infty)$, and $\kappa \in (0, 1/4)$, we have

$$
(\| L^1 u - (L^1 u)_{Q_\kappa(t_0, x_0)} \|_{Q_\kappa(t_0, x_0)} + (\| u - (\lambda u)_{Q_\kappa(t_0, x_0)} \|_{Q_\kappa(t_0, x_0)})
$$

$$
\leq N \kappa^{-\tau} \sum_{k=0}^{\infty} 2^{-\sigma k} \left( \left( \| L^1 u \|^{p_0}_{Q_{\kappa(t_0, x_0)}} \right)^{1/p_0} + \left( \| u \|^{p_0}_{Q_{\kappa(t_0, x_0)}} \right)^{1/p_0} \right)
$$

$$
+ N \kappa^{-1-d+\sigma/p_0} \sum_{k=0}^{\infty} 2^{-\sigma k} \left( |f|_{p_0} \right)^{1/p_0} \|_{(0_0-1, T_0)} \times B_{2^{-k}R}(x_0),
$$

(4.17)

where $\tau = \tau(d, \sigma, p_0)$ and $N = N(d, \nu, \Lambda, \sigma, p_0)$.

**Proof** By shifting the coordinates and scaling, we assume $x_0 = 0$ and $r = 1$. Moreover, by Theorem 2.6, there exists $w \in \mathbb{H}^{1, \sigma}_{p_0, 0}((t_0 - 1, t_0) \times \mathbb{R}^d)$ satisfying (4.1). Also, $v := u - w \in \mathbb{H}^{1, \sigma}_{p_0, 0}((-\infty, t_0) \times \mathbb{R}^d)$ satisfies (4.2).

Next, by Hölder’s inequality and Proposition 4.3,

$$
(\| L^1 u - (L^1 u)_{Q_\kappa(t_0, 0)} \|_{Q_\kappa(t_0, 0)} + (\| u - (\lambda u)_{Q_\kappa(t_0, 0)} \|_{Q_\kappa(t_0, 0)})
$$

$$
\leq (\| L^1 v - (L^1 v)_{Q_\kappa(t_0, 0)} \|_{Q_\kappa(t_0, 0)} + (\| v - (\lambda v)_{Q_\kappa(t_0, 0)} \|_{Q_\kappa(t_0, 0)})
$$

$$
\leq (\| L^1 v - (L^1 v)_{Q_\kappa(t_0, 0)} \|_{Q_\kappa(t_0, 0)} + (\| v - (\lambda v)_{Q_\kappa(t_0, 0)} \|_{Q_\kappa(t_0, 0)})
$$

$$
+ N \kappa^{-1-d+\sigma/p_0} \sum_{k=0}^{\infty} 2^{-\sigma k} \left( |f|_{p_0} \right)^{1/p_0} \|_{(0_0-1, T_0)} \times B_{2^{-k}R}.
$$

(4.18)

For the first two terms on the right-hand side of (4.18), by Proposition 4.1, there exists $\tau \in (0, 1)$ such that
\[
\left( |L^1 u - (L^1 v)_{Q_1(x_0)}|, \lambda v - (\lambda v)_{Q_1(x_0)} \right)_{Q_1(x_0)} + \left( |\lambda v - (\lambda v)_{Q_1(x_0)}|, L^1 v \right)_{Q_1(x_0)}
\]
\[
\leq \kappa \left[ (L^1 v)_{C^{\gamma+\eta}(Q_{1/4}(x_0))} + \kappa \left[ (\lambda v)_{C^{\gamma+\eta}(Q_{1/4}(x_0))} \right)_{\mathbb{R}^d} \right.
\]
\[
\leq N \kappa \sum_{z \in \mathbb{Z}^d} (1 + |z|^{\sigma + \eta})^{-1} \left( |(L^1 u)^{p_0}_{(0-1,0)}| + \left( |\lambda v|^{p_0}_{(0-1,0)} \right)_{\mathbb{R}^d} \right)
\]
\[
\leq N \kappa \sum_{z \in \mathbb{Z}^d} (1 + |z|^{\sigma + \eta})^{-1} \left( |L^1 u|^{p_0}_{(0-1,0)} \right)_{\mathbb{R}^d} + \sum_{z \in \mathbb{Z}^d} (1 + |z|^{\sigma + \eta})^{-1} \left( |\lambda v|^{p_0}_{(0-1,0)} \right)_{\mathbb{R}^d} + \sum_{z \in \mathbb{Z}^d} (1 + |z|^{\sigma + \eta})^{-1} \left( |(L^1 u)|^{p_0}_{(0-1,0)} \right)_{\mathbb{R}^d}
\]
\[
= N \kappa \|I_1\| + N \kappa \|I_2\| + N \kappa \|I_3\|.
\]

To estimate \( I_1, I_2, \) and \( I_3, \) for \( R > r \geq 1, \) we denote \( N_R \) to be the number of \( z \in \mathbb{Z}^d \) lying in \( B_R, \) and \( N_{r,R} \) to be the number of \( z \in \mathbb{Z}^d \) lying in \( B_R \setminus B_r^o. \) Note that
\[
N_{r,R} \leq N_R \quad \text{and} \quad |B_{R-\sqrt{d}/2}| \leq N_R \leq |B_{R+\sqrt{d}/2}| \leq N(d) R^d,
\]
and for \( k = 0, 1, 2, 3, \ldots, \) we have
\[
N_{2^k, 2k+1} \geq N(d) 2^{kd}.
\]

Then, by Hölder’s inequality for \( I_{p_0}, (4.20), \) and \( (4.21), \) we have
\[
I_1 - (|L^1 u|^{p_0}_{(0-1,0)} \times C_1(0)) \leq N \sum_{k=0}^{\infty} \sum_{|z|=2^k} 2^{-kd-k\sigma} \left( |L^1 u|^{p_0}_{(0-1,0)} \right)_{\mathbb{R}^d} + \sum_{z \in \mathbb{Z}^d} (1 + |z|^{\sigma + \eta})^{-1} \left( |\lambda v|^{p_0}_{(0-1,0)} \right)_{\mathbb{R}^d}
\]
\[
\leq N \sum_{k=0}^{\infty} 2^{-kd-k\sigma} |N_{2^k, 2k+1}| \sum_{|z|=2^k} |N_{(0-1,0)}| \times |C_1(z)|
\]
\[
\leq N \sum_{k=0}^{\infty} 2^{-kd-k\sigma} \left( |L^1 u|^{p_0}_{(0-1,0)} \right)_{B_{2^k+1, \sqrt{d}/2}}.
\]

Thus,
\[
I_1 \leq N \sum_{k=0}^{\infty} 2^{-kd-k\sigma} \left( |L^1 u|^{p_0}_{(0-1,0)} \right)_{B_{2^k}}.
\]

Similarly, by replacing \( L^1 u \) with \( \lambda u, \) we conclude that
\[
I_2 \leq N \sum_{k=0}^{\infty} 2^{-kd-k\sigma} \left( |\lambda u|^{p_0}_{(0-1,0)} \right)_{B_{2^k}}.
\]
For $I_3$, by Proposition 4.3, we have

$$I_3 \leq N \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} \sum_{j=0}^{\infty} 2^{-j\sigma} (|f|_{p_0})^{1/p_0} (t_0 - 1, t_0) \times B_{2j} \sqrt{d}(z)$$

$$\leq N \sum_{z \in \mathbb{Z}^d} (1 + |z|^{d+\sigma})^{-1} \sum_{j=0}^{\infty} 2^{-j\sigma} (|f|_{p_0})^{1/p_0} (t_0 - 1, t_0) \times C_{2j}(z).$$

(4.24)

For $j \geq 0$, $z = (z^1, \ldots, z^d)$, $i = (i^1, \ldots, i^d) \in \mathbb{Z}^d$ and each component of $i$ takes value in $\{0, 1, \ldots, 2^j - 1\}$, we denote

$$z_{j,i} = (z^1 - (2^j - 1)/2 + i^1, \ldots, z^d - (2^j - 1)/2 + i^d).$$

Since $C_1(z_{j,i})$ is disjoint with respect to $i$ for fixed $z$ and $j$, and it is also disjoint with respect to $z$ for fixed $i$ and $j$, we have

$$\sum_{|z| = 2^k}^{2^{k+1}} \int_{(t_0 - 1, t_0)} \int_{C_{2j}(z)} |f|_{p_0} = \sum_{|z| = 2^k}^{2^{k+1}} \sum_i \int_{(t_0 - 1, t_0)} \int_{C_1(z_{j,i})} |f|_{p_0}$$

$$\leq 2^{\min(k,j)d} \int_{(t_0 - 1, t_0)} \int_{B_{2j - 1} \sqrt{d} + 2^{k+1}} |f|_{p_0}.$$  

(4.25)

By (4.24), (4.21), Hölder’s inequality, and (4.25),

$$I_3 - N \sum_{j=0}^{\infty} 2^{-j\sigma} (|f|_{p_0})^{1/p_0} (t_0 - 1, t_0) \times C_{2j}(0)$$

$$\leq N \sum_{k=0}^{\infty} \sum_{|z| = 2^k}^{2^{k+1}} 2^{-kd - k\sigma} \sum_{j=0}^{\infty} 2^{-j\sigma} (|f|_{p_0})^{1/p_0} (t_0 - 1, t_0) \times C_{2j}(z)$$

$$\leq N \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-k\sigma - j\sigma} \sum_{|z| = 2^k}^{2^{k+1}} N_{2k+2}^{-1} (|f|_{p_0})^{1/p_0} (t_0 - 1, t_0) \times C_{2j}(z)$$

$$\leq N \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-k\sigma - j\sigma} \left( 2^{-kd - jd} \sum_{|z| = 2^k}^{2^{k+1}} \int_{(t_0 - 1, t_0)} \int_{C_{2j}(z)} |f|_{p_0} \right)^{1/p_0}$$

$$\leq N \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-k\sigma - j\sigma} \left( 2^{-kd - jd} 2^{\min(k,j)d} \int_{(t_0 - 1, t_0)} \int_{B_{2j - 1} \sqrt{d} + 2^{k+1}} |f|_{p_0} \right)^{1/p_0}$$

$$\leq N \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-k\sigma - j\sigma} \left( 1_k \geq j (|f|_{p_0})^{1/p_0} (t_0 - 1, t_0) \times B_{2k} + 1_k < j (|f|_{p_0})^{1/p_0} (t_0 - 1, t_0) \times B_{2j} \right)$$. 

$$\leq N \sum_{k=0}^{\infty} 2^{-k\sigma} (|f|_{p_0})^{1/p_0} (t_0 - 1, t_0) \times B_{2k}.$$
which implies that

\[ I_3 \leq N \sum_{k=0}^{\infty} 2^{-k\alpha} (|f|^p)_{(t_0-1,t_0) \times B_{2k}}. \quad (4.26) \]

Thus, combining (4.18), (4.19), (4.22), (4.23), and (4.26), we arrive at (4.17), and the proposition is proved. \( \square \)

Next, we introduce the dyadic cubes as follows: for each \( n \in \mathbb{Z} \), pick \( k(n) \in \mathbb{Z} \) such that

\[ k(n) \leq \sigma n < k(n) + 1, \]

and let

\[ Q^n_i = \left[ \frac{i_0}{2^k(n)} + T, \frac{i_0 + 1}{2^k(n)} + T \right] \times \left[ \frac{i_1}{2^n}, \frac{i_1 + 1}{2^n} \right] \times \cdots \times \left[ \frac{i_d}{2^n}, \frac{i_d + 1}{2^n} \right] \]

and

\[ C_n := \left\{ Q^n_i = Q^n_{(i_0, \ldots, i_d)} : i = (i_0, \ldots, i_d) \in \mathbb{Z}^{d+1}, i_0 \leq -1 \right\}. \]

Furthermore, denote the dyadic sharp function of \( g \) by

\[ g^\text{dy}_d(t, x) = \sup_{n < \infty} \int_{Q^n_{i \ni (t, x)}} \left| g(s, y) - g|_n(t, x) \right| dy \, ds, \]

where

\[ g|_n(t, x) = \int_{Q^n_i} g(s, y) \, dy \, ds \quad \text{for} \quad (t, x) \in Q^n_i. \]

**Proof of Theorem 2.9 Step 1. The a priori estimate.**

We first prove (2.18) under the assumption that \( u \) is compactly supported in the spacial variables. For the given \( w_1 \in A_p(\mathbb{R}, dt) \) and \( w_2 \in A_q(\mathbb{R}^d, dx) \), using reverse Hölder’s inequality for \( A_p \) weights, we pick \( \gamma_1 = \gamma_1(d, p, M_1) \) and \( \gamma_2 = \gamma_2(d, q, M_1) \) such that \( p - \gamma_1 > 1, q - \gamma_2 > 1 \), and

\[ w_1 \in A_{p-\gamma_1}(\mathbb{R}, dt), \quad w_2 \in A_{q-\gamma_2}(\mathbb{R}^d, dx). \]

Take \( p_0 = p_0(d, p, q, M_1) \in (1, \infty) \) so that

\[ p_0 = \min \left\{ \frac{p}{p - \gamma_1}, \frac{q}{q - \gamma_2} \right\} > 1. \]

Note that

\[ w_1 \in A_{p-\gamma_1}(\mathbb{R}, dt) \subset A_{p_0}(\mathbb{R}, dt) \]

and

\[ w_2 \in A_{q-\gamma_2}(\mathbb{R}^d, dx) \subset A_{q_0}(\mathbb{R}^d, dx). \]

From these inclusions and the fact that \( u \in H_{p,q,w,0}^{1,\sigma}(\mathbb{R}^d) \), it follows that \( u \in H^{1,\sigma}_{p_0,0,\text{loc}}(\mathbb{R}^d) \). See the proof of [7, Lemma 5.10] for details. Furthermore, since \( u \) is compactly supported, we have \( u \in H^{1,\sigma}_{p_0,0}(\mathbb{R}^d) \), and \( u \in H^{1,\sigma}_{p_0}((-\infty, 0) \times \mathbb{R}^d) \) by taking the zero extension for \( t < 0. \)
Therefore, by Proposition 4.5, for any $r \in (0, \infty)$, $\kappa \in (0, 1/4)$, and $(t_1, x_0) \in (0, T) \times \mathbb{R}^d$, we have

$$\|L^{1}u - (L^{1}u)_{Q_{r}(t_1, x_0)}\|_{Q_{r}(t_1, x_0)} + \|\lambda u - (\lambda u)_{Q_{r}(t_1, x_0)}\|_{Q_{r}(t_1, x_0)}$$

$$\leq N \kappa^{\tau} \sum_{k=0}^\infty 2^{-k\sigma} \left[ \|L^{1}u\|_{(t_1-r^\sigma, t_1) \times B_{2\kappa r}(x_0)}^{1/p_0} + \|\lambda u\|_{(t_1-r^\sigma, t_1) \times B_{2\kappa r}(x_0)}^{1/p_0} \right]$$

$$+ N \kappa^{-(d+\sigma)/p_0} \sum_{k=0}^\infty 2^{-k\sigma} \left( \|f\|_{(t_1-r^\sigma, t_1) \times B_{2\kappa r}(x_0)}^{1/p_0} \right)$$

(4.27)

where $\tau = \tau(d, \sigma, p, M_1)$ and $N = N(d, v, \Lambda, \sigma, p, q, M_1)$.

Note that for any $(t_0, x_0) \in \mathbb{R}_T^d$, $n \in \mathbb{Z}$, and $Q^n_i$ containing $(t_0, x_0)$, there exist $r = r(d, \sigma, n) > 0$ and $t_1 = \min(T, t_0 + r^\sigma/2) \in (-\infty, T]$ such that

$$Q^n_i \subset Q_r(t_1, x_0)$$

and $|Q_r(t_1, x_0)| \leq N(d, \sigma)|Q^n_i|$. Therefore, by (4.27), we have

$$\int_{Q^n_i \cap (t_0, x_0)} |L^{1}u(s, y) - (L^{1}u)_{[t_0, x_0]}| + |\lambda u(s, y) - (\lambda u)_{[t_0, x_0]}| \, dy \, ds$$

$$\leq N \kappa^{\tau} \sum_{k=0}^\infty 2^{-k\sigma} \left[ \|L^{1}u\|_{(t_1-r^\sigma, t_1) \times B_{2\kappa r}(x_0)}^{1/p_0} + \|\lambda u\|_{(t_1-r^\sigma, t_1) \times B_{2\kappa r}(x_0)}^{1/p_0} \right]$$

$$+ N \kappa^{-(d+\sigma)/p_0} \sum_{k=0}^\infty 2^{-k\sigma} \left( \|f\|_{(t_1-r^\sigma, t_1) \times B_{2\kappa r}(x_0)}^{1/p_0} \right)$$

$$\leq N \left( \kappa^{\tau} |\mathcal{M}L^{1}u|_{p_0} \right)^{1/p_0} + \kappa^{\tau} |\mathcal{M}\lambda u|_{p_0}^{1/p_0} + \kappa^{-(d+\sigma)/p_0} |\mathcal{M}f|_{p_0}^{1/p_0} \right)(t_0, x_0),$$

where $N$ is independent of $n$ and $\kappa$, and for the last inequality, we used that for any function $g$ and $A \subset B$,

$$\left| \int_A g - \int_B g \right| \leq \frac{|B|}{|A|} \int_B |g - (g)_B|.$$ 

Therefore,

$$(L^{1}u)_{d_3}(t_0, x_0) + (\lambda u)_{d_3}(t_0, x_0)$$

$$\leq N \left( \kappa^{\tau} |\mathcal{M}L^{1}u|_{p_0} \right)^{1/p_0} + \kappa^{\tau} |\mathcal{M}\lambda u|_{p_0}^{1/p_0} + \kappa^{-(d+\sigma)/p_0} |\mathcal{M}f|_{p_0}^{1/p_0} \right)(t_0, x_0).$$

Then, by the weighted sharp function theorem [7, Corollary 2.7] and the weighted maximal function theorem for strong maximal functions [9, Theorem 5.2],

$$\|L^{1}u\|_{p,q,w} + \|\lambda u\|_{p,q,w} \leq N \kappa^{\tau} \|L^{1}u\|_{p,q,w} + N \kappa^{\tau} \|\lambda u\|_{p,q,w} + N \kappa^{-(d+\sigma)/p_0} \|f\|_{p,q,w},$$

where $N = N(d, v, \Lambda, \sigma, p, q, M_1)$. Therefore, by taking a sufficiently small $\kappa < 1/4$ such that

$$N \kappa^{\tau} < 1/2,$$

we get

$$\|L^{1}u\|_{L^{p,q,w}} + \lambda \|u\|_{L^{p,q,w}} \leq N \|f\|_{L^{p,q,w}}.$$
Since the space of functions that are compactly supported in the spacial variables is dense in \( H^{1,\sigma}_{p,q,w,0}(T) \), we obtain (2.18) and the continuity of \( \partial_t - L \). Then (2.19) follows from Eq. (2.17) together with the triangle inequality.

Step 2. Existence of solutions.

By the method of continuity, it suffices to prove the existence of solutions for the simple equation with \( L = -(-\Delta)^{\sigma/2} \) and \( \lambda = 0 \). By the a priori estimate proved in Step 1 and the density of smooth functions in Lebesgue spaces, without loss of generality, we assume that \( f \in C^0_0((0, \infty) \times \mathbb{R}^d) \). Let

\[
u(t, x) = \int_0^t \int_{\mathbb{R}^d} \zeta(t-s, x-y) f(s, y) \, dy \, ds,
\]

where \( \hat{\zeta}(t, \xi) = e^{-t|\xi|^2} \). Then by [17, Lemma 4.1],

\[\partial_t \nu + (-\Delta)^{\sigma/2} \nu = f.\]

When \( p = q \), it follows from [13, Theorem 5.14], a Fourier multiplier theorem, that

\[\|\partial_t \nu\|_{L_{p,w}(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2} \nu\|_{L_{p,w}(\mathbb{R}^d_T)} \leq N \|f\|_{L_{p,w}(\mathbb{R}^d_T)}\]

For general \( p \) and \( q \), using the same definition of \( u \) as in (4.28) together with the extrapolation theorem in [7, Theorem 2.5] or [5, Theorem 1.4], we have

\[\|\partial_t u\|_{L_{p,q,w}(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2} u\|_{L_{p,q,w}(\mathbb{R}^d_T)} \leq N \|f\|_{L_{p,q,w}(\mathbb{R}^d_T)},\]

which implies that \( u \in H^{1,\sigma}_{p,q,w,0}(\mathbb{R}^d_T) \). The theorem is proved.

Next, we prove Corollary 2.10 using the frozen coefficient argument and a partition of unity.

**Corollary 4.6** Let \( \beta \in (0, 1) \), \( \alpha \in (0, 1) \), \( \sigma \in (0, 2) \), \( T \in (0, \infty) \), \( p \in (1, \infty) \), \( \lambda \geq 0 \), \( M_1 \in [1, \infty) \), and \( [w]_{p,p} \leq M_1 \). There exists \( r_0 = r_0(d, \nu, \Lambda, \alpha, \sigma, p, \beta, \omega, M_1) > 0 \) such that under Assumptions 2.1 and 2.3, for any \( u \in H^{1,\sigma}_{p,q,w,0}(\mathbb{R}^d_T) \) supported on \((0, T) \times B_{r_0}(x_1)\) for some \( x_1 \in \mathbb{R}^d \) and satisfies

\[\partial_t u - Lu + \lambda u = f\] in \( \mathbb{R}^d_T \),

we have

\[\|\partial_t u\|_{L_{p,w}(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2} u\|_{L_{p,w}(\mathbb{R}^d_T)} + \lambda \|u\|_{L_{p,w}(\mathbb{R}^d_T)} \leq N \|f\|_{L_{p,w}(\mathbb{R}^d_T)} + N \|u\|_{L_{p,w}(\mathbb{R}^d_T)},\]

where \( N = N(d, \nu, \Lambda, \sigma, p, M_1, \beta, \omega) \).

**Proof** This follows from [12, Lemma 5.1] and Theorem 2.9 with the frozen coefficient argument. See the proof of [12, Lemma 5.2] for details.

**Proof of Corollary 2.10** First, the continuity of \( \partial_t - L \) follows by [12, Corollary 2.5]. By Theorem 2.6 together with the method of continuity, it remains to prove the a priori estimate (2.15).

We first consider the case when \( b = c = 0 \) and \( p = q \) with general weight \( w(t, x) = w_1(t)w_2(x) \). We take \( r_0 \) from Corollary 4.6, and use a partition of unity argument with respect to the spatial variables. Let \( \xi \in C^0_0(B_{r_0}) \) satisfy

\[\|\xi\|_{L_p} = 1, \quad \|D_\nu \xi\|_{L_p} \leq N, \quad \|D^2 \xi\|_{L_p} \leq N, \quad \text{and} \quad \xi(x) = \xi(x - z),\]

\( \xi \) Springer
where \( z \in \mathbb{R}^d \) and \( N = N(d, \nu, \Lambda, \sigma, p, \beta, \omega) \). It follows that
\[
\partial_t (\zeta_z u) - L(\zeta_z u) + \lambda \zeta_z u = \zeta_z f + \zeta_z L u - L(\zeta_z u).
\]

By Corollary 4.6, for any \( \varepsilon \in (0, 1) \),
\[
\| \partial_t u \|_{L^p(w(\mathbb{R}^d_T \mathbb{R}^d_x)))}^p + \| (-\Delta)^{\sigma/2} u \|_{L^p(w(\mathbb{R}^d_T \mathbb{R}^d_x)))}^p + \| \lambda u \|_{L^p(w(\mathbb{R}^d_T \mathbb{R}^d_x)))}^p
\]
\[
= \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( |\zeta_z(x)\partial_t u(t, x)|^p + |\zeta_z(x)\lambda u(t, x)|^p \right) w(t, x) w(t, x) \, dz \, dx \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)(-\Delta)^{\sigma/2} u(t, x)|^p w(t, x) \, dx \, dz \, dt
\]
\[
\leq N \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( |\partial_t (\zeta_z u(t, x))|^p + |\lambda \zeta_z u(t, x)|^p \right) w(t, x) \, dx \, dz \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)(-\Delta)^{\sigma/2} u(t, x) - (-\Delta)^{\sigma/2} (\zeta_z u)(t, x)|^p w(t, x) \, dx \, dz \, dt
\]
\[
\leq N \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( |\zeta_z(x) f(t, x)|^p + |\zeta_z(x) u(t, x)|^p \right) w(t, x) \, dx \, dz \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)L u(t, x) - L(\zeta_z u)(t, x)|^p w(t, x) \, dx \, dz \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)(-\Delta)^{\sigma/2} u(t, x) - (-\Delta)^{\sigma/2} (\zeta_z u)(t, x)|^p w(t, x) \, dx \, dz \, dt
\]
\[
\leq N \| f \|_{L^p(w(\mathbb{R}^d_T \mathbb{R}^d_x)))}^p + N \| u \|_{L^p(w(\mathbb{R}^d_T \mathbb{R}^d_x)))}^p + 1_{\sigma > 1} N \| Du \|_{L^p(w(\mathbb{R}^d_T \mathbb{R}^d_x)))}^p + 1_{\sigma = 1} \| Du \|_{L^p(w(\mathbb{R}^d_T \mathbb{R}^d_x)))}^p.
\]

Indeed, for the last inequality, to estimate
\[
I := \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)L u(t, x) - L(\zeta_z u)(t, x)|^p w_2(x) w_1(t) \, dx \, dz \, dt,
\]
we consider the following three cases depending on the value of \( \sigma \). When \( \sigma \in (0, 1) \),
\[
|\zeta_z(x) L u(t, x) - L(\zeta_z u)(t, x)|
\]
\[
\leq N \left( \int_{B_1} + \int_{B_1^c} \right) |\zeta_z(x + y) - \zeta_z(x)| u(t, x + y) \, dy \, dy =: I_1^1 + I_2^1.
\]

Since
\[
I_1^1 \leq \int_{B_1} u(t, x + y) \, dy \, dy \, dy \, dy,
\]
by the Minkowski inequality, [12, Lemma 3.2], and the weighted Hardy-Littlewood maximal function theorem, we conclude that
\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (I_1^1(t, x, z))^p w_2(x) w_1(t) \, dx \, dz \, dt
\]
\[
\leq \| D\zeta \|_{L^p(\mathbb{R}^d)} \int_0^T \int_{\mathbb{R}^d} \left( \int_{B_1} u(t, x + y) \, dy \, dy \, dy \, dy \right)^p w_2(x) w_1(t) \, dx \, dt,
\]
\[
\leq N \int_0^T \int_{\mathbb{R}^d} M_x u(t, x)^p w_2(x) w_1(t) \, dx \, dt \leq N \|u\|_{L_p, w}(\mathbb{R}^d),
\]
and
\[
\int_0^T \int_{\mathbb{R}^d} \left( I_1(t, x, z) \right)^p w_2(x) w_1(t) \, dx \, dz \, dt,
\]
\[
\leq N \|\zeta\|_{L_p(\mathbb{R}^d)}^p \int_0^T \int_{\mathbb{R}^d} \left( \int_{B_1^0} |u(t, x + y)| \|y\|^{-d-\sigma} \, dy \right)^p w_2(x) w_1(t) \, dy \, dt
\]
\[
\leq N \|\zeta\|_{L_p(\mathbb{R}^d)}^p \int_0^T \int_{\mathbb{R}^d} M_x u(t, x)^p w_2(x) w_1(t) \, dy \, dt
\]
\[
\leq N \|u\|_{L_p, w}(\mathbb{R}^d).
\]
Here \( M_x \) stands for the maximal function of \( u \) with respect to the \( x \) variable. Furthermore, applying a similar decomposition as in the proof of Lemma A.3 together with the maximal function as above, when \( \sigma \in (1, 2) \),
\[
I \leq N \|u\|_{L_p, w}(\mathbb{R}^d)^p + 1_{1 > 1} N \| Du \|_{L_p, w}(\mathbb{R}^d)^p,
\]
and when \( \sigma = 1 \),
\[
I \leq N(\varepsilon) \|u\|_{L_p, w}(\mathbb{R}^d)^p + \varepsilon \| Du \|_{L_p, w}(\mathbb{R}^d)^p.
\]

Therefore, by (4.29) together with Lemma A.1 or picking a sufficiently small \( \varepsilon \) when \( \sigma = 1 \), we obtain
\[
\| \partial_t u \|_{L_p, w}(\mathbb{R}^d) + \| (-\Delta)^{\sigma/2} u \|_{L_p, w}(\mathbb{R}^d) + \lambda \|u\|_{L_p, w}(\mathbb{R}^d)
\]
\[
\leq N \|f\|_{L_p, w}(\mathbb{R}^d) + N \|u\|_{L_p, w}(\mathbb{R}^d). \tag{4.30}
\]

By taking \( \lambda_0 = \max(2N, 1) \), for all \( \lambda \geq \lambda_0 \), we have (2.20). On the other hand, when \( \lambda = 0 \), by dividing \((0, T)\) into sufficiently small sub-intervals, taking cutoff functions in time, and applying an induction argument as in [9, Theorem 2.2], we get
\[
\|u\|_{L_p, w}(\mathbb{R}^d) \leq N(T) \|f\|_{L_p, w}(\mathbb{R}^d),
\]
which together with (4.30) implies (2.22).

For general \( b \) and \( c \) and \( p = q \), we have
\[
\partial_t u - Lu = f - b^i D_i u 1_{\sigma \geq 1} - cu.
\]

Thus, when \( \sigma > 1 \) (4.30) follows from the case when \( b = c = 0 \) and Lemma A.1. If \( \sigma = 1 \), then
\[
\| \partial_t u \|_{L_p, w}(\mathbb{R}^d) + \| Du \|_{L_p, w}(\mathbb{R}^d)
\]
\[
\leq N \|f\|_{L_p, w}(\mathbb{R}^d) + N \|u\|_{L_p, w}(\mathbb{R}^d) + N \|b Du\|_{L_p, w}(\mathbb{R}^d). \tag{4.31}
\]

If \( b \) is sufficiently small such that \( N \|b\|_{L_\infty(\mathbb{R}^d)} \leq 1/2 \), then by absorbing the last term on the right-hand side of (4.31) to the left, we obtain (4.30). On the other hand, if \( \alpha = 1 \) and \( b \) is uniformly continuous, we take
\[
B(t) = \int_0^t b(s, 0) \, ds, \quad \bar{b}(t, x) = b(t, x - B(t)), \quad \bar{c}(t, x) = c(t, x - B(t)),
\]
\[
\square \]
and
\[ \tilde{u}(t, x) = u(t, x - B(t)), \quad \tilde{f}(t, x) = f(t, x - B(t)). \]

It follows that
\[ \partial_t \tilde{u}(t, x) - \tilde{L} \tilde{u}(t, x) + (\tilde{b}^i(t, x) - b^i(t, 0)) D_i \tilde{u}(t, x) + \tilde{c} \tilde{u}(t, x) = \tilde{f}(t, x), \]
where \( \tilde{L} \) is the operator with the kernel \( \tilde{K}(t, x, y) = K(t, x - B(t), y) \). Then, by moving the last two terms on the left-hand side of (4.32) to the right-hand side and applying the estimate to \( \tilde{u} \) together with a change of variables, we have
\[ \| Du \|_{L_p, w(\mathbb{R}^d_f)} \leq N\| f \|_{L_p, w(\mathbb{R}^d_f)} + N\| u \|_{L_p, w(\mathbb{R}^d_f)} + N\| (b - b(\cdot, 0)) Du \|_{L_p, w(\mathbb{R}^d_f)}, \]
which implies that there exists \( r_0 > 0 \) depending on the modulus of continuity of \( b \) such that if \( u \) is supported on \( (0, T) \times B_{r_0} \), then
\[ \| Du \|_{L_p, w(\mathbb{R}^d_f)} \leq N\| f \|_{L_p, w(\mathbb{R}^d_f)} + N\| u \|_{L_p, w(\mathbb{R}^d_f)}. \]

Therefore, by a partition of unity and (4.31), we arrive at (4.30), and (2.22) follows.

Finally, for general \( p, q \), we use the extrapolation theorems in [7, Theorem 2.5] or [5, Theorem 1.4]. The Corollary is proved. \( \square \)

**Remark 4.7** Indeed, \( r_0 \) and \( N \) in Corollary 4.6 can be taken to be dependent on \( \tilde{\sigma} \) instead of \( \sigma \) due to the presence of \( (2 - \sigma) \) in the bound (2.4) in Assumption 2.3. Thus, based on the proof above, in Corollary 2.10 the dependence of \( N \) on \( \sigma \) can be reduced to the dependence on \( \tilde{\sigma} \).

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**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Appendix A: Miscellaneous lemmas**

**Lemma A.1** (An interpolation inequality in the spatial variables) Let \( \sigma \in (1, 2) \), \( p \in (1, \infty) \), and \( u \in H^\sigma_p(\mathbb{R}^d) \). For any \( \varepsilon > 0 \),
\[ \| Du \|_{L_p(\mathbb{R}^d)} \leq N(d, \sigma, p) \varepsilon^{1/(1 - \sigma)} \| u \|_{L_p(\mathbb{R}^d)} + \varepsilon \| (-\Delta)^{\sigma/2} u \|_{L_p(\mathbb{R}^d)}. \]  

**(A.1)**

**Proof** By taking \( f := u + (-\Delta)^{\sigma/2} u \), we have
\[ \mathcal{F}(Du)(\xi) = \frac{i\xi}{1 + |\xi|^\sigma} \mathcal{F}(f)(\xi) = m(\xi) \mathcal{F}(f)(\xi). \]

Since \( \sigma > 1 \), the multiplier \( m(\xi) \) satisfies
\[ \sup_{\xi \in \mathbb{R}^d} |\xi|^{\sigma} |\partial^\gamma m| \leq N(d, \sigma, p), \]
for any multi-index \( \gamma \) of length \(|\gamma| \leq d/2 + 1 \). Thus, by the Mikhlin multiplier theorem, we have
\[ \| Du \|_{L_p(\mathbb{R}^d)} \leq N\| f \|_{L_p(\mathbb{R}^d)} \leq N\| u \|_{L_p(\mathbb{R}^d)} + N\| (-\Delta)^{\sigma/2} u \|_{L_p(\mathbb{R}^d)}, \]
\[ \Box \]
where $N = N(d, \sigma, p)$. Then taking $v(\cdot) := u(\varepsilon \cdot)$ and applying (A.2) to $v$ yield
\[ \varepsilon \| Du \|_{L_p(\mathbb{R}^d)} \leq N \| u \|_{L_p(\mathbb{R}^d)} + N \varepsilon^\sigma \| (-\Delta)^{\sigma/2} u \|_{L_p(\mathbb{R}^d)}, \]
which implies (A.1). The lemma is proved. □

**Lemma A.2** Let $\alpha \in (0, 1)$, $T \in (0, \infty)$, $p, q \in (1, \infty)$. For any $u \in \mathbb{H}^{\alpha, \sigma}_{p, q, 0}(T)$, we have
\[ \| u \|_{L_p, q(\mathbb{R}^d_T)} \leq NT^\alpha \| \partial_t^\alpha u \|_{L_p, q(\mathbb{R}^d_T)}, \]
where $N = N(\alpha, p, q)$.

**Proof** Note that $u = I^\alpha \partial_t^\alpha u$. Therefore, the result follows from [9, Lemma 5.5]. □

**Lemma A.3** Let $L$ satisfy (2.3) when $\sigma = 1$ and (2.6). Recall the definition of $\bar{\sigma}$ in (2.24). For $k = 0, 1, \ldots$, let $\xi_k$ be the cutoff functions defined in (3.14). Under the same assumptions and the same notation as in Lemma 3.2, we have the following estimates for $I_k := \| \xi_k Lu - L(\xi_k u) \|_{L_p(\mathbb{R}^d_T)}$:

1. When $\sigma \in (0, 1)$,
\[ I_k \leq N \frac{2^{\sigma k}}{(R - r)^{\sigma}} \| u \|_{p, R} + N \frac{2^{(d+\sigma)k}}{(R - r)^{d+\sigma}} R^{d/p} (1 + R^{d+\sigma}) \| u \|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))}; \]

2. when $\sigma \in (1, 2)$,
\[ I_k \leq N \frac{2^{(\sigma-1)k}}{(R - r)^{\sigma-1}} \| Du \|_{p, r_k+3} + N \frac{2^{\sigma k}}{(R - r)^{\sigma}} \| u \|_{p, R} \]
\[ + N \frac{2^{(d+\sigma)k}}{(R - r)^{d+\sigma}} R^{d/p} (1 + R^{d+\sigma}) \| u \|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))}; \]

3. when $\sigma = 1$, for any $\varepsilon \in (0, 1)$,
\[ I_k \leq \varepsilon^3 \| Du \|_{p, r_k+3} + N \frac{2^k}{R - r} \varepsilon^3 \| u \|_{p, R} \]
\[ + N R^{d/p} \left( 1 + \frac{2^{(d+1)k} \varepsilon^{-3(d+1)}}{(R - r)^{d+1}} (1 + R^{d+1}) \right) \| u \|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))}, \]
where $N = N(d, \Lambda, \bar{\sigma})$ and $\| \cdot \|_{p, r} := \| \cdot \|_{L_p((0,T) \times B_r)}$ for any $r > 0$.

**Proof** In this proof, we use $N$ to denote a constant which may depend on $d$, $\Lambda$, and $\bar{\sigma}$. Note that $N$ does not depend on $v$, the lower bound of the operator.

Next, let $\bar{r}_k = r_{k+3} - r_{k+2}$, and note that
\[ L(\xi_k u)(t, x) - \xi_k Lu(t, x) \]
\[ = \int_{\mathbb{R}^d} \left( (\xi_k(x + y) - \xi_k(x)) u(t, x + y) - y \cdot \nabla \xi_k(x) u(t, x) \chi^{(\sigma)}(y) \right) K(t, y) dy. \]

(A.6)

We are going to split the proof into 3 cases depending on the value of $\sigma$.

**Case 1:** $\sigma \in (0, 1)$. In this case, by (A.6),
\[ |L(\xi_k u)(t, x) - \xi_k Lu(t, x)| \]
leq N \int_{\mathbb{R}^d} |\zeta_k(x+y) - \zeta_k(x)| |u(t, x+y)| |y|^{-\sigma} \, dy \\
leq N \left( \int_{B_{r_k}^c} + \int_{B_{r_k}^c} \right) |\zeta_k(x+y) - \zeta_k(x)| |u(t, x+y)| |y|^{-\sigma} \, dy =: I_{k,1}^1 + I_{k,2}^1.

For \( I_{k,1}^1 \), since \( y \in B_{r_k} \), we have

\[ |\zeta_k(x+y) - \zeta_k(x)| \leq \|D\zeta_k\|_{L_\infty} |y|^{1+d} \leq N \frac{2^{k}}{R - r} |y|^{1+d} < r_{k+2}, \tag{A.7} \]

which together with the Minkowski inequality implies that

\[ \|I_{k,1}^1\|_{L_p(\mathbb{R}^d)} \leq N \frac{2^{k}}{R - r} \int_{B_{r_k}^c} |u(\cdot, y)|_{p, r_{k+2}} |y|^{1-d} \, dy \\
\leq N \frac{2^{k}}{R - r} \|u\|_{p, R} \int_{B_{r_k}^c} |y|^{1-d} \, dy \\
\leq N (d, \sigma_1) \frac{2^{k}}{R - r} \|u\|_{p, R} \leq N \frac{2^{\sigma k}}{(R - r)^{\sigma}} \|u\|_{p, R}. \tag{A.8} \]

On the other hand,

\[ I_{k,2}^1 \leq \int_{B_{r_k}^c} (1_{|x+y| < r_{k+1}} + 1_{|x| < r_{k+1}}) |u(t, x+y)| |y|^{-\sigma} \, dy =: I_{k,2,1}^1 + I_{k,2,2}^1, \]

\[ \|I_{k,2,1}^1\|_{L_p(\mathbb{R}^d)} \leq \int_{B_{r_k}^c} |y|^{-d} \, dy \|u\|_{p, R} \leq N (d, \sigma_0) \tilde{r}^{-\sigma} \|u\|_{p, R} \leq N \frac{2^{\sigma k}}{(R - r)^{\sigma}} \|u\|_{p, R}. \tag{A.9} \]

By Hölder’s inequality and the Minkowski inequality,

\[ \|I_{k,2,2}^1\|_{L_p(\mathbb{R}^d)} \leq N (d, \sigma_0) \tilde{r}^{-d+\sigma} \left( \int_0^T \|I_{k,2,2}^1(t, \cdot)\|^p_{L_\infty(B_{r_k}^c)} \, dt \right)^{1/p} \]

\[ \leq N (d, \sigma_0) \frac{1 + \tilde{r}^{d+\sigma}}{\tilde{r}^{d+\sigma}} \left( \int_0^T \|u(t, \cdot)\|^p_{L_1(\mathbb{R}^d, \psi)} \, dt \right)^{1/p} \]

\[ = N (d, \sigma_0) \frac{1 + \tilde{r}^{d+\sigma}}{\tilde{r}^{d+\sigma}} \|u\|^p_{L_p((0,T);L_1(\mathbb{R}^d, \psi))} \]

\[ \leq N \frac{2^{d+\sigma} k}{(R - r)^d} R^{d/p} (1 + R^{d+\sigma}) \|u\|_{L_p((0,T);L_1(\mathbb{R}^d, \psi))}. \tag{A.10} \]

where for the second inequality, we used the fact that if \(|y| \geq \tilde{r}_k\) and \(|x| < r_{k+1}\) then

\[ \frac{1 + |x+y|^{d+\sigma}}{|y|^{d+\sigma}} \leq \frac{1 + (r_{k+1} + \tilde{r}_k)^{d+\sigma}}{\tilde{r}_k^{d+\sigma}} \leq \frac{1 + r_{k+1}^{d+\sigma}}{\tilde{r}_k^{d+\sigma}}. \]

Combining (A.8), (A.9), and (A.10), we conclude

\[ I_k \leq \|I_{k,1}^1\|_{L_p(\mathbb{R}^d)} + \|I_{k,2}^1\|_{L_p(\mathbb{R}^d)} \]

\[ \leq N \frac{2^{\sigma k}}{(R - r)^{\sigma}} \|u\|_{p, R} + N \frac{2^{d+\sigma} k}{(R - r)^d} R^{d/p} (1 + R^{d+\sigma}) \|u\|_{L_p((0,T);L_1(\mathbb{R}^d, \psi))}. \]
Case 2: $\sigma \in (1, 2)$. In this case, by (A.6),
\[ |L(\zeta_k u)(t, x) - \zeta_k Lu(t, x)| \]
\[ \leq N(\Lambda)(2 - \sigma) \int_{\mathbb{R}^d} |\zeta_k(x + y) - \zeta_k(x)||u(t, x + y) - u(t, x)||y|^{-d - \sigma} \, dy 
+ N(\Lambda)(2 - \sigma) \int_{\mathbb{R}^d} |\zeta_k(x + y) - \zeta_k(x)| - y \cdot \nabla \zeta_k(x)||u(t, x)||y|^{-d - \sigma} \, dy 
=: I_{k,1}^2 + I_{k,2}^2. \]

By (A.7) and the fundamental theorem of calculus,
\[ I_{k,1}^2 \leq N(2 - \sigma) \left( \int_{B_{r_k}} + \int_{B_{r_k}^c} \right) |\zeta_k(x + y) - \zeta_k(x)||u(t, x + y) - u(t, x)||y|^{-d - \sigma} \, dy 
\leq N(2 - \sigma) \left( \int_{B_{r_k}} + \int_{B_{r_k}^c} \right) 2^k \frac{2^{k - 1}}{(R - r)^{d - 1}} |\nabla u(t, x + sy)||y|^{2 - d - \sigma} \, ds \, dy 
+ N \int_{B_{r_k}^c} \left( 1_{|x+y|<r_{k+1}} + 1_{|x|<r_{k+1}} \right) |u(t, x + y) - u(t, x)||y|^{-d - \sigma} \, dy. \]

Similar to (A.8), (A.9), and (A.10), we have
\[ \|I_{k,1}^2\|_{L_p(\mathbb{R}^d_T)} \leq N \left( \frac{2(\sigma - 1)k}{(R - r)^{2\sigma - 1}} \right) \|Du\|_{p, r_{k+1}} + N \frac{2^{\sigma k}}{(R - r)^{\sigma}} \|u\|_{p, R} \]
\[ + N \frac{2^{2k}}{(R - r)^{2\sigma}} R^{d/p}(1 + R^{d+\sigma}) \|u\|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))}. \]  
(A.11)

For $I_{k,2}^2$, when $y \in B_{r_k}$, the mean value theorem leads to
\[ |\zeta_k(x + y) - \zeta_k(x)| - y \cdot \nabla \zeta_k(x)| \]
\[ \leq \|D^2 \zeta_k\|_{L^\infty} |y|^2 \int_{|x|<r_{k+2}} \leq N \frac{2^{2k}}{(R - r)^2} |y|^2 \int_{|x|<r_{k+2}}. \]

Thus,
\[ I_{k,2}^2 \leq N(2 - \sigma) \left( \int_{B_{r_k}} + \int_{B_{r_k}^c} \right) |\zeta_k(x + y) - \zeta_k(x)| - y \cdot \nabla \zeta_k(x)||u(t, x)||y|^{-d - \sigma} \, dy 
\leq N(2 - \sigma) \frac{2^{2k}}{(R - r)^2} |u(t, x)||y|^{2 - d - \sigma} \, dy 
+ N \int_{B_{r_k}^c} \left( 1_{|x+y|<r_{k+1}} + 1_{|x|<r_{k+1}} \right) (1 + \frac{2^k}{(R - r)|y|}) |u(t, x)||y|^{-d - \sigma} \, dy, \]
which, similar to (A.8), (A.9), and (A.10), implies that
\[ \|I_{k,2}^2\|_{L_p(\mathbb{R}^d_T)} \leq N \frac{2^{\sigma k}}{(R - r)^{\sigma}} \|u\|_{p, R} \]
\[ + N \frac{2^{2k}}{(R - r)^{2\sigma}} R^{d/p}(1 + R^{d+\sigma}) \|u\|_{L_p((0,T); L_1(\mathbb{R}^d, \psi))}. \]  
(A.12)

By (A.11) and (A.12),
The estimates of $I_k$ follow that Assumption A.4. For given $N$, $R$, and $r$, it holds
\[
I_k \leq \|I_{k,1}^2\|_{L_p(\mathbb{R}^d)} + \|I_{k,2}^2\|_{L_p(\mathbb{R}^d)} + \|I_{k,3}^2\|_{L_p(\mathbb{R}^d)} + \|I_{k,4}^2\|_{L_p(\mathbb{R}^d)} \leq N \frac{2k}{(R-r)^{\sigma-1}} \|Du\|_{p,r,k+3} + N \frac{2^{\sigma k}}{(R-r)^{\sigma}} \|u\|_{p,R}
\]
\[
+ N \frac{2^{(d+1)\sigma k}}{(R-r)^{d+\sigma}} R^{d/p} (1 + R^{d+\sigma}) \|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\]

Case 3: $\sigma = 1$. In this case, by (A.6) and (2.3), for any $\delta_k > 0$ to be determined,
\[
|L(\xi_k u)(t,x) - \xi_k Lu(t,x)| \leq N \int_{B_{\delta_k}^c} |\xi_k(x+y) - \xi_k(x)||u(t,x+y) - u(t,x)||y|^{-d-1} \, dy
\]
\[
+ N \int_{B_{\delta_k}} |\xi_k(x+y) - \xi_k(x)| - y \cdot \nabla \xi_k(x)||u(t,x)||y|^{-d-1} \, dy
\]
\[
+ N \int_{B_{\delta_k}^c} |\xi_k(x+y) - \xi_k(x)||u(t,x+y)||y|^{-d-1} \, dy =: I_{k,1}^3 + I_{k,2}^3 + I_{k,3}^3.
\]

The estimates of $I_{k,1}^3$, $I_{k,2}^3$, and $I_{k,3}^3$ are similar to that of $I_{k,1}^2$, $I_{k,2}^2$, and $I_{k,1}^2$ respectively. It follows that
\[
\|I_{k,1}^3\|_{L_p(\mathbb{R}^d)} \leq N \frac{2k}{(R-r)} \delta_k \|Du\|_{p,r,k+3}, \quad \|I_{k,2}^3\|_{L_p(\mathbb{R}^d)} \leq N \frac{2^{2k}}{(R-r)^2} \delta_k \|u\|_{p,R},
\]
and
\[
\|I_{k,3}^3\|_{L_p(\mathbb{R}^d)} \leq N \delta_k^{-1} \|u\|_{p,R} + NR^{d/p} \frac{1 + (R + \delta_k)^{d+1}}{\delta_k^{d+1}} \|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))},
\]
where $N$ is independent of the choice of $\delta_k$. For any $\varepsilon \in (0,1)$, let $\delta_k = \varepsilon^3 \gamma_k/N$. Then,
\[
I_k \leq \|I_{k,1}^3\|_{L_p(\mathbb{R}^d)} + \|I_{k,2}^3\|_{L_p(\mathbb{R}^d)} + \|I_{k,3}^3\|_{L_p(\mathbb{R}^d)} \leq \varepsilon^3 \|Du\|_{p,r,k+3} + N \frac{2k}{R-r} \varepsilon^{-3} \|u\|_{p,R}
\]
\[
+ NR^{d/p} \left(1 + \frac{2^{(d+1)k} \varepsilon^{-3(d+1)}}{(R-r)^{d+1}} (1 + R^{d+1}) \right) \|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\]

The lemma is proved. \hfill \Box

**Assumption A.4** For given $E \subset F \subset (-\infty, T)$, if
\[
|C_R(t,x) \cap E| \geq \gamma |C_R(t,x)|
\]
for $(t,x) \in (-\infty, T] \times \mathbb{R}^d$ and $R > 0$, then
\[
\tilde{C}_R(t,x) \subset F,
\]
where $C_R(t,x)$ and $\tilde{C}_R(t,x)$ are defined in (3.42).

**Lemma A.5** (Crawling of ink spots) Let $\gamma \in (0,1)$ and $|E| < \infty$. Suppose that $E \subset F \subset (-\infty, T) \times \mathbb{R}^d$ satisfy Assumption A.4. Then
\[
|E| \leq N(d,\alpha) \gamma |F|.
\]
Proof See [8, Lemma A.20]

Lemma A.6 (Sobolev embeddings) Let $\alpha \in (0, 1)$, $\sigma \in (0, 2)$, $T \in (0, \infty)$, $p \in (1, \infty)$, and $u \in H^{\alpha, \sigma}_{p,0}(T)$.

1) If $p < d/\sigma + 1/\alpha$, then there exists $q \in (p, \infty)$ satisfying

$$1/q = 1/p - \alpha \sigma / (\alpha d + \sigma)$$

such that for all $l \in [p, q]$,

$$\|u\|_{L^l(\mathbb{R}^d_T)} \leq N\|u\|_{H^{\alpha, \sigma}_{p,0}(T)},$$

where $N = N(d, \alpha, \sigma, p, l, T)$.

2) If $p = d/\sigma + 1/\alpha$, then the same estimate holds for $l \in [p, \infty)$.

3) If $p > d/\sigma + 1/\alpha$, then

$$\|u\|_{L^\infty(\mathbb{R}^d_T)} \leq N\|u\|_{H^{\alpha, \sigma}_{p,0}(T)},$$

where $N = N(d, \alpha, \sigma, p, T)$.

Proof (1) By the mixed derivative theorem [27], we have

$$H^{\alpha, \sigma}_{p,0}(T) \hookrightarrow H^{\alpha(1-\theta)}(0, T); H^\sigma_p(\mathbb{R}^d)$$

for all $\theta \in [0, 1]$.

Let $\theta = \alpha d / (\alpha d + \sigma)$ and

$$1/q = 1/p - \alpha(1-\theta) = 1/p - \sigma \theta / d = 1/p - \alpha \sigma / (\alpha d + \sigma).$$

The Sobolev embeddings

$$H^{\alpha(1-\theta)}(0, T) \hookrightarrow L^q(0, T)$$

and $H^\sigma_p(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$,

imply that

$$\|u\|_{L^q(\mathbb{R}^d_T)} \leq N\left(\int_0^T \|u(t, \cdot)\|_{H^\sigma_p(\mathbb{R}^d)}^q dt\right)^{1/q}$$

$$\leq N(T)\|u\|_{H^{\alpha(1-\theta)}(0, T); H^\sigma_p(\mathbb{R}^d)} \leq N\|u\|_{H^{\alpha, \sigma}_{p,0}(T)}.$$ 

Since (A.13) holds for $l = p$, it holds for all $l \in [p, q]$ by Hölder’s inequality.

2) Using the embedding

$$H^{\alpha(1-\theta)}(0, T) \hookrightarrow L^1(0, T)$$

and $H^\sigma_p(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$,

the proof is similar to that of (1).

3) Due to the facts that $\alpha(1-\theta)p > 1$ and $\sigma \theta p > d$, and the Sobolev embeddings

$$H^{\alpha(1-\theta)}(0, T) \hookrightarrow L^\infty(0, T)$$

and $H^\sigma_p(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$,

we have

$$\|u\|_{L^\infty(\mathbb{R}^d_T)} \leq N(T)\|u\|_{H^{\alpha(1-\theta)}(0, T); H^\sigma_p(\mathbb{R}^d)} \leq N\|u\|_{H^{\alpha, \sigma}_{p,0}(T)}.$$

Next, we prove an embedding into Hölder spaces. Note that we only use the result when $\alpha = 1$ for this paper.
Lemma A.7 (A Hölder estimate) Let \( \alpha \in (0, 1] \), \( \sigma \in (0, 2) \), \( T \in (0, \infty) \), \( p \in (1, \infty) \), \( p \in (d/\sigma + 1/\alpha, 2d/\sigma + 2/\alpha) \), and \( u \in \mathbb{H}^{\sigma, \sigma}_{p, 0}(T) \). There exists \( \tau = \sigma - (d + \sigma/\alpha)/p \in (0, 1) \) such that

\[
\|u\|_{C^{\sigma/\alpha, \tau}(\mathbb{R}^d_T)} \leq N \|u\|_{\mathbb{H}^{\sigma, \sigma}_{p, 0}(T)},
\]

where \( N = N(d, \alpha, \sigma, p, T) \).

**Proof** Without loss of generality, we assume that \( u \in C^\infty_0([0, T] \times \mathbb{R}^d) \) with \( u(0, \cdot) = 0 \). Denote

\[
K = \sup \left\{ \frac{|u(t_1, x) - u(t_2, y)|}{|t_1 - t_2|^{\sigma/\alpha} + |x - y|^\tau} : (t_1, x), (t_2, y) \in \mathbb{R}^d_T, 0 < |t_1 - t_2|^{\sigma/\alpha} + |x - y| \leq 1 \right\}
\]

and

\[
\rho = \varepsilon(|t_1 - t_2|^{\sigma/\alpha} + |x - y|),
\]

where \( \varepsilon \in (0, 1) \) is a constant to be determined.

Note that by Lemma A.6 and the fact that \( p > d/\sigma + 1/\alpha \), it suffices to prove

\[
K \leq N\|u\|_{\mathbb{H}^{\sigma, \sigma}_{p, 0}(T)}. \tag{A.14}
\]

To this end, we write

\[
|u(t_1, x) - u(t_2, y)| \leq |u(t_1, x) - u(t_2, x)| + |u(t_2, x) - u(t_2, y)| := J_1 + J_2. \tag{A.15}
\]

For \( J_1 \), by taking \( z \in B_\rho(x) \) and the triangle inequality, we have

\[
J_1 \leq |u(t_1, x) - u(t_1, z)| + |u(t_2, x) - u(t_2, z)| + |u(t_1, z) - u(t_2, z)|
\leq 2K \rho^\tau + |u(t_1, z) - u(t_2, z)|
\leq 2K \rho^\tau + N(\alpha, \sigma) |t_1 - t_2|^{\alpha - 1/p} \|\partial^\alpha u(\cdot, z)\|_{L_p(0, T)}, \tag{A.16}
\]

where for the last inequality, we used [8, Lemma A.14] together with the fact that \( \alpha > 1/p \). Then by taking the average over \( B_\rho(x) \) on both sides of (A.16) together with Hölder’s inequality, we get

\[
J_1 \leq 2K \rho^\tau + N(d, \alpha, p) |t_1 - t_2|^{\alpha - 1/p} \rho^{-d/p} \|\partial^\alpha u\|_{L_p(\mathbb{R}^d_T)}
\leq 2K \rho^\tau + N(d, \alpha, p) \varepsilon^{\sigma/(\alpha p) - \sigma} \rho^{\tau} \|u\|_{\mathbb{H}^{\sigma, \sigma}_{p, 0}(T)}. \tag{A.17}
\]

On the other hand, for \( s \in I := (t_2 - \rho^{\sigma/\alpha}, t_2 + \rho^{\sigma/\alpha}) \cap (0, T) \),

\[
J_2 \leq |u(t_2, x) - u(s, x)| + |u(t_2, y) - u(s, y)| + |u(s, x) - u(s, y)|
\leq 2K \rho^\tau + |u(s, x) - u(s, y)|
\leq 2K \rho^\tau + N(d, \sigma, p) |x - y|^{\sigma - d/p} \|u(s, \cdot)\|_{H^p_{\sigma}(\mathbb{R}^d)}, \tag{A.18}
\]

where for the last inequality, we used the embedding \( H^p_{\sigma}(\mathbb{R}^d) \hookrightarrow C^{\sigma - d/p}(\mathbb{R}^d) \). Then by taking the average over \( I \) on both sides of (A.18) together with Hölder’s inequality, we obtain

\[
J_2 \leq 2K \rho^\tau + N(d, \sigma, p) |x - y|^{\sigma - d/p} \rho^{-\sigma/(\alpha p)} \|u\|_{L_p((0, T); H^p_{\sigma}(\mathbb{R}^d))}
\leq 2K \rho^\tau + N(d, \sigma, p) \varepsilon^{d/p - \sigma} \rho^{\tau} \|u\|_{\mathbb{H}^{\sigma, \sigma}_{p, 0}(T)}. \tag{A.19}
\]

Note that \( N \) is independent of \( T \) because we can assume that \( |I| \geq \rho^{\sigma/\alpha} \) by extending \( u(t, \cdot) \) to be zero for \( t < 0 \).
Theorem B.1
Let
Thus, by picking sufficiently small $N$ where $N = N(d, \alpha, \sigma, p)$, which implies that
\[ K \leq 4\epsilon^\tau K + N(\epsilon^{-d/p} + \epsilon^{-\sigma/(\alpha p)})\rho^\tau \|u\|_{L_p^q}\sigma(T). \]

Thus, by picking sufficiently small $\epsilon$ so that $4\epsilon^\tau < 1/2$, we arrive at (A.14), and the lemma is proved.

\[ \Box \]

Appendix B: Equations with non-zero initial conditions

In this section, we consider equations with non-zero initial conditions. In particular, we focus on the case when $\alpha = 1$, i.e., the order of the derivative in time is 1, with $A_p$ weights (c.f. Corollary 2.10),
\[
\begin{cases}
\partial_t u - Lu + b^i D_i u 1_{\sigma > 1} + cu = f & \text{in } \mathbb{R}^d_T \\
u(0, \cdot) = u_0(\cdot) & \text{in } \mathbb{R}^d,
\end{cases}
\]
where $u_0$ is taken from an appropriate trace space defined below and $L$, $b^i$, $c$ satisfy the assumptions in Corollary 2.10. When $\sigma = 1$ and either $\|b\|_\infty$ is sufficiently small or $b$ is uniformly continuous, a similar result holds for (2.23) by the same proof. For general $\alpha \in (0, 1)$ when there are no $A_p$ weights, see [17, Lemma 5.4].

Before we introduce the results, we define the trace space as follows. For $p, q \in (1, \infty)$, a non-integer $\theta > 0$ with $k$ being the integer part of $\theta$, and $w_2 \in A_q(\mathbb{R}^d)$, we define
\[ B_{p, q, w_2}(\mathbb{R}^d) := \{u \in L_q, w_2(\mathbb{R}^d) : \|u\|_{B_{p, q, w_2}(\mathbb{R}^d)} < \infty\}, \]
where
\[
\|u\|_{B_{p, q, w_2}(\mathbb{R}^d)} = [u]_{B_{p, q, w_2}(\mathbb{R}^d)} + \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L_q, w_2(\mathbb{R}^d)}
\]
and
\[
[u]_{B_{p, q, w_2}(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \|D^k u(\cdot + y) - D^k u(\cdot)\|_p^p |y|^{-d-p(\theta-k)} \, dy \right)^{1/p}.
\]
Note that when $w_2 \equiv 1$, $B_{p, q, w_2}(\mathbb{R}^d) = B_{p, q}(\mathbb{R}^d)$, the usual Besov space. When $\theta > 0$ is a non-integer, we denote $X_0 = B_{p, q, w_2}(\mathbb{R}^d)$, and when $\theta_0$ is a nonnegative integer, we denote $X_0 = B_{\theta_0, q, w_2}(\mathbb{R}^d)$ for arbitrary $\theta > 0$.

Our result for equations with non-zero initial conditions is

Theorem B.1 Let $\sigma \in (0, 2)$, $T \in (0, \infty)$, $p, q \in (1, \infty)$, $M_1 \in [1, \infty)$, $\mu \in (-1, p - 1)$, and $w = w_1(t)w_2(x)$, where $w_1(t) = t^\mu$ and $w_2(x) = A_q(\mathbb{R}^d, dx)$ with $[w_2]_{A_q} \leq M_1$. Suppose that the kernel $K = K(t, x, y)$ satisfies Assumptions 2.1 and 2.3, and (2.13) holds. Let $\theta = \sigma - \sigma(1 + \mu)/p(\in (0, \sigma))$, and recall the definitions of the spaces $X_0$ above. For any $f \in L_{p, q, w}(\mathbb{R}^d_T)$ and $u_0 \in X_0$, there exists a unique function $u$ in $\mathbb{R}^d_T$ satisfying (B.1) with the estimate
\[
\|\partial_t u\|_{L_{p, q, w}} + \|u\|_{L_{p, q, w}} + \|(-\Delta)^{\sigma/2} u\|_{L_{p, q, w}} \leq N \|f\|_{L_{p, q, w}} + N\|u_0\|_{X_0},
\]
where $\|\cdot\|_{L_{p, q, w}(\mathbb{R}^d_T)}$ and $N = N(d, v, \Lambda, \sigma, p, q, T, M_1, M, \beta, \omega, \mu)$.
To prove Theorem B.1, we apply Corollary 2.10 with the right-hand side
\[ \tilde{f} := f + LU - b^i D_i U 1_{\sigma > 1} - cU - \partial U, \]
where \( U \) is given in Lemma B.2. Indeed, by Corollary 2.10, there exists a unique \( w \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(\mathbb{R}^d) \) satisfying
\[ \partial_t w - Lw + b^i D_i w 1_{\sigma > 1} + cw = \tilde{f} \quad \text{in} \quad \mathbb{R}^d. \]
Then, it follows that \( u := w + U \) is a solution to the equation (B.1), and the estimates (B.2) follows from (2.22) and (B.3).

**Lemma B.2** Under the assumptions of Theorem B.1, for any \( u_0 \in \mathcal{X}_0 \), there exists a function \( U \in \mathbb{H}^1_T \) such that \( U(0, \cdot) = u_0(\cdot) \) in the trace sense and
\[ \|\partial_t U\|_{p,q,w} + \|U\|_{p,q,w} + \|(-\Delta)^{\sigma/2} U\|_{p,q,w} \leq N\|u_0\|_{\mathcal{X}_0}, \]
where \( w \) is defined in Theorem B.1 and \( N = N(d, \sigma, p, q, M_1, T, \mu) \).

**Proof** Without loss of generality, we assume that \( u_0 \) is smooth with compact support. Also, note that for \( \sigma > 0 \), the operator \( \partial_t + (-\Delta)^{\sigma/2} \) has the fundamental solution
\[ p_\sigma(t, x) = C(d, \sigma) \int_{\mathbb{R}^d} e^{-t|\xi|^\sigma} e^{i\xi \cdot x} \, d\xi, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \]
which satisfies the scaling property
\[ p_\sigma(t, x) = t^{-d/\sigma} p(1, t^{-1/\sigma} x). \]
It follows that \( U(t, \cdot) := p(t, \cdot) * u_0(\cdot) \) satisfies the equation
\[
\begin{cases}
\partial_t U + (-\Delta)^{\sigma/2} U = 0 & \text{in} \quad (0, \infty) \times \mathbb{R}^d \\
U(0, \cdot) = u_0(\cdot) & \text{in} \quad \mathbb{R}^d.
\end{cases}
\]
By the well-known estimate
\[ |p_\sigma(1, x)| \leq N(d, \sigma)(1_{|x| > 1}|x|^{-\sigma-d} + 1_{|x| \leq 1}) \]
(see, for instance, [17, Lemma 3.8] or [11, Theorem 1.2]), (B.5), and a dyadic decomposition, we have for any \( t > 0 \) and \( x \in \mathbb{R}^d \),
\[ |U(t, x)| \leq N \mathcal{M}_x u_0(x), \]
where \( N \) is independent of \( t, x \). Therefore, by the Hardy-Littlewood maximal function theorem with \( A_p \) weights, we get
\[ \|U\|_{p,q,w} \leq N(T)\|u_0\|_{\mathcal{X}_0}. \]
Furthermore, by (B.5) and [11, Lemma 3.1], for \( \eta > 0 \), we have the pointwise estimate
\[
|(-\Delta)^{\eta/2} p_\sigma(t, x)| = |t^{-d/\sigma - \eta/\sigma} (-\Delta)^{\eta/2} p_\sigma(1, t^{-1/\sigma} x)|
\leq N(d, \eta, \sigma)|t^{-d/\sigma - \eta/\sigma} (1_{|y| > 1}|y|^{-\eta-d} + 1_{|y| \leq 1})|, \]
where \( y = t^{-1/\sigma} x \). It is easily seen that \((-\Delta)^{\eta/2} p_\sigma(t, \cdot) \in L^1(\mathbb{R}^d)\), and by taking the Fourier transform together with the definition (B.4), we have
\[
\int_{\mathbb{R}^d} (-\Delta)^{\eta/2} p_\sigma(t, x) \, dx = 0 \quad \forall t > 0. \]
Case 1. $\theta \in (0, 1)$. In this case, by taking $\eta = \sigma$ in (B.9), we have
\[
(-\Delta)^{\sigma/2} U(t, x) = \int_{\mathbb{R}^d} (-\Delta)^{\sigma/2} p_{\sigma}(t, y)(u_0(x - y) - u_0(x)) \, dy,
\]
which together with the Minkowski inequality implies that
\[
\|(-\Delta)^{\sigma/2} U(t, \cdot)\|_{L^q L^w_{\text{w}}} \leq \int_{\mathbb{R}^d} \|(-\Delta)^{\sigma/2} p_{\sigma}(t, y)\|_{L^q L^w_{\text{w}}} \, dy.
\]
Therefore, for any $\beta \in (0, 1)$, by Hölder’s inequality,
\[
\|(-\Delta)^{\sigma/2} U\|_{L^p L^q L^w_{\text{w}}} \leq \int_0^T \left( \int_{\mathbb{R}^d} \|(-\Delta)^{\sigma/2} p_{\sigma}(t, y)\|_{L^q L^w_{\text{w}}} \, dy \right)^p \, dt
\]
\[
\leq N \int_0^T \left( \int_{\mathbb{R}^d} \|(-\Delta)^{\sigma/2} p_{\sigma}(t, y)\|_{L^q L^w_{\text{w}}} \, dy \right)^p \, J_1 \, dt,
\]
where
\[
J_1 := \left( \int_{\mathbb{R}^d} \|(-\Delta)^{\sigma/2} p_{\sigma}(t, y)\|_{L^q L^w_{\text{w}}} \, dy \right)^p - 1.
\]
A computation using the estimate (B.8) leads to
\[
J_1 \leq N t^{d(\sigma - 1)\beta+d(p-1)/\sigma} \left( \int_{\mathbb{R}^d} \|(-\Delta)^{\sigma/2} p_{\sigma}(1, y)\|_{L^q L^w_{\text{w}}} \, dy \right)^p - 1,
\]
where the second inequality holds provided that $\beta < 1$ is sufficiently close to 1 so that
\[
(-\sigma - d)\beta (p-1) + d < 0.
\]
Using the estimate (B.10) and (B.11) together with the Fubini theorem, we arrive at
\[
\|(-\Delta)^{\sigma/2} U\|_{L^p L^q L^w_{\text{w}}} \leq N \int_{\mathbb{R}^d} J_2 \|u_0(\cdot - y) - u_0(\cdot)\|^p_{L^q L^w_{\text{w}}} \, dy,
\]
where
\[
J_2 := \int_0^T \|(-\Delta)^{\sigma/2} p_{\sigma}(t, y)\|_{L^q L^w_{\text{w}}} \, dy \leq \int_0^\infty \cdots \, dt.
\]
To estimate $J_2$, we first apply the scaling property (B.5) and then split the integral into two integrals on $(0, |y|^\sigma)$ and $(|y|^\sigma, \infty)$, which together with the estimate (B.8) implies that
\[
J_2 \leq N \int_0^{|y|^\sigma} t^{(-d/\sigma-1)\beta+p(1-\beta)} t^{(-d/\sigma-1)\beta+d(p-1)/\sigma+\mu} \, dt
\]
\[
+ N \int_0^\infty t^{(-d/\sigma-1)\beta+p(1-\beta)} t^{(-d/\sigma-1)\beta+d(p-1)/\sigma+\mu} \, dt
\]
\[
:= J_{2,1} + J_{2,2}.
\]
A computation leads to
\[
J_{2,1} \leq N |y|^{-d-p\theta}
\]
provided that
\[
(-d/\sigma - 1)\beta p + d(p-1)/\sigma + \mu + 1 > 0,
\]
and
\[ J_{2,2} \leq N |y|^{d-p+1} \]  
(B.17)

provided that
\[ -p - d/\sigma + \mu + 1 < 0. \]  
(B.18)

Note that (B.18) is satisfied because \( \theta > 0 \), and (B.12) and (B.16) are satisfied by taking
\[ \beta = \frac{d(p-1)}{(\sigma+d)p} + \epsilon \]
for a sufficiently small \( \epsilon > 0 \).

Thus, by combining (B.13), (B.14), (B.15), and (B.17), we conclude that
\[ \|(-\Delta)^{\sigma/2} U \|_{L^p_{L^q,w}} \leq N \int_{\mathbb{R}^d} |u_0(\cdot - y) - u_0(\cdot)| \| \nabla u_0 \|_{L^q_{L^r,w}} |y|^{-d-p+1} \, dy = N \| u_0 \|_{X^0}. \]  
(B.19)

Case 2. \( \theta \in (1, 2) \). In this case, by taking \( \eta = \sigma - 1 \) in (B.9), we have
\[ (-\Delta)^{\sigma/2} U(t, x) = \int_{\mathbb{R}^d} (-\Delta)^{(\sigma-1)/2} p_\sigma(t, y)((-\Delta)^{1/2} u_0(x - y) - (-\Delta)^{1/2} u_0(x)) \, dy, \]
which together with Hölder’s inequality implies that for any \( \beta \in (0, 1) \),
\[ \|(-\Delta)^{\sigma/2} U \|_{L^p_{L^q,w}} \leq N \int_0^T \left( \int_{\mathbb{R}^d} (-\Delta)^{(\sigma-1)/2} p_\sigma(t, y)|\nabla u_0(\cdot - y) - \nabla u_0(\cdot)|^p \, dy \right)^{1/p} dt, \]
where
\[ J_1 := \left( \int_{\mathbb{R}^d} (-\Delta)^{(\sigma-1)/2} p_\sigma(t, y)|\nabla u_0(\cdot - y) - \nabla u_0(\cdot)|^p \, dy \right)^{1/p}. \]

A computation using the estimate (B.8) leads to
\[ J_1 \leq N t^{-(d/\sigma - (\sigma-1)/\sigma)\beta p + d(p-1)/\sigma} \left( \int_{\mathbb{R}^d} (-\Delta)^{(\sigma-1)/2} p_\sigma(1, y)\beta p/(p-1) \, dy \right)^{p-1} \leq N t^{-(d/\sigma - (\sigma-1)/\sigma)\beta p + d(p-1)/\sigma}, \]
where the second inequality holds provided that \( \beta < 1 \) is sufficiently close to 1 so that
\[ (-(\sigma - 1) - d)\beta p/(p-1) + d < 0. \]  
(B.20)

By the Fubini theorem, we then get
\[ \|(-\Delta)^{\sigma/2} U \|_{L^p_{L^q,w}} \leq N \int_{\mathbb{R}^d} \int_0^T \| \nabla u_0(\cdot - y) - \nabla u_0(\cdot) \|_{L^q_{L^r,w}} \, dy \, dt, \]  
(B.21)

where
\[ J_2 := \int_0^T \|(-\Delta)^{(\sigma-1)/2} p_\sigma(t, y)|\nabla u_0(\cdot - y) - \nabla u_0(\cdot)|^p \, dt \leq \int_0^\infty \cdots \, dt. \]

Similar to Case 1, we first apply the scaling property (B.5) and then split the integral above into two integrals on \((0, |y|^\sigma)\) and \((|y|^\sigma, \infty)\), which together with the estimate (B.8) implies that
\[ J_2 \leq N \int_0^{|y|^\sigma} t^{-(d/\sigma - (\sigma-1)/\sigma)\beta p + d(p-1)/\sigma + \mu} t^{-1/\sigma} y^{-(\sigma+1-d)(1-\beta) p} \, dt \]
\[ + N \int_0^\infty t\left(\frac{-d}{\sigma} - \frac{(\sigma - 1)}{\sigma} p + d(p - 1)/\sigma + \mu \right) dt \]

\[ := J_{2,1} + J_{2,2}. \quad \text{(B.22)} \]

A computation leads to

\[ J_{2,1} \leq N|y|^{-d-p(\theta-1)} \quad \text{(B.23)} \]

provided that

\[ \beta = \frac{d(p-1)}{(\sigma-1)+d} + \epsilon \quad \text{for a sufficiently small } \epsilon > 0. \quad \text{(B.24)} \]

d and

\[ J_{2,2} \leq N|y|^{-d-p(\theta-1)} \quad \text{(B.25)} \]

provided that

\[ - p(\sigma - 1)/\sigma - d/\sigma + \mu + 1 < 0. \quad \text{(B.26)} \]

Note that (B.26) is satisfied because \( \theta > 1 \), and (B.20) and (B.24) are satisfied by taking

\[ \beta = \frac{d(p-1)}{(\sigma-1)+d} + \epsilon \]

for a sufficiently small \( \epsilon > 0 \). Thus, by combining (B.21), (B.22), (B.23), and (B.25), we conclude that

\[ \|(-\Delta)^{\sigma/2} U\|_{L_{p,q,w}}^p \leq N \int_{\mathbb{R}^d} |Du_0(\cdot - y) - Du_0(\cdot)|^p_{L_{q,w_2}} |y|^{-d-p(\theta-1)} dy = N\|u_0\|_{X_0}^p. \quad \text{(B.27)} \]

**Case 3.** \( \theta = 1 \). In this case, we take \( \mu' \in (-1, \mu) \) and \( \bar{w} = \bar{w}_1(t)w_2(x) \), where \( \bar{w}_1(t) = t^{\mu'} \). By Case 2, we have

\[ \|(-\Delta)^{\sigma/2} U\|_{L_{p,q,w}(\mathbb{R}^d)} \leq N\|u_0\|_{X_0}. \]

Also, since \( t^{\mu} \leq N(T)t^{\mu'} \) for \( t \in (0, T) \), we arrive at

\[ \|(-\Delta)^{\sigma/2} U\|_{L_{p,q,w}} \leq N\|u_0\|_{X_0}. \quad \text{(B.28)} \]

Finally, the estimates (B.19), (B.27), (B.28), (B.7), and Eq. (B.6) lead to (B.3). \( \square \)

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