Hopf Algebras of Dimension $pq$

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Abstract

Let $H$ be a non-semisimple Hopf algebra with antipode $S$ of dimension $pq$ over an algebraically closed field of characteristic 0 where $p \leq q$ are odd primes. We prove that $\text{Tr}(S^{2p}) = p^2d$ where $d \equiv pq$ (mod 4). As a consequence, if $p, q$ are twin primes, then any Hopf algebra of dimension $pq$ is semisimple.

0 Introduction

Let $p$ be an odd prime and $k$ an algebraically closed field of characteristic 0. If $H$ is a semisimple Hopf algebra of dimension $p^2$ over $k$, then $H$ is isomorphic to $k[Z_{p^2}]$ or $k[Z_p \times Z_p]$ by [Mas96]. A more general result for semisimple Hopf algebras of dimension $pq$, where $p, q$ are odd primes, is obtained by [EG98]. In [Ng02], the author proved that non-semisimple Hopf algebras of dimension $p^2$ over $k$ are Taft algebras and hence completed the classification of Hopf algebras of dimension $p^2$. However, there is no known example of non-semisimple Hopf algebras of dimension $pq$, with $p < q$. In fact, it is shown in [AN01] and [BD02] that there is no non-semisimple Hopf algebra over $k$ of dimension 15, 21, 35, 55, 77, 65, 91 or 143.

By [Ng02], if $p \leq q$ are odd primes and $H$ is a non-semisimple Hopf algebra with antipode $S$ of dimension $pq$, then $S^{4p} = id_H$ and $\text{Tr}(S^{2p}) = p^2d$ for some odd integer $d$. In this paper, we prove that $d \equiv pq$ (mod 4). As a consequence, we prove that if $p, q$ are twin primes, any Hopf algebra of dimension $pq$ over $k$ is semisimple. Recently, Etingof and Gelaki also announce a even more general result [EG03] which covers the cases when $p < q \leq 2p + 1$.

1 Notation and Preliminaries

Throughout this paper, $p \leq q$ are odd primes, $k$ denotes an algebraically closed field of characteristic 0, and $H$ denotes a finite-dimensional Hopf algebra over $k$ with antipode $S$. Its comultiplication and counit are, respectively, denoted by $\Delta$ and $\varepsilon$. We will use Sweedler’s notation [Swe69]:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

A non-zero element $a \in H$ is called group-like if $\Delta(a) = a \otimes a$. The set of all group-like elements $G(H)$ of $H$ is a linearly independent set, and it forms a group under the multiplication of $H$. For the details of elementary aspects for finite-dimensional Hopf algebras,
Let \( \lambda \in H^* \) be a non-zero right integral of \( H^* \) and \( \Lambda \in H \) a non-zero left integral of \( H \). There exists \( \alpha \in \text{Alg}(H, k) = G(H^*) \), independent of the choice of \( \Lambda \), such that \( \Lambda a = \alpha(a) \Lambda \) for \( a \in H \). Likewise, there is a group-like element \( g \in H \), independent of the choice of \( \lambda \), such that \( \beta \lambda = \beta(g) \lambda \) for \( \beta \in H^* \).

Follow [Rad76]:

\[
S^4(a) = g(\alpha \to a \leftarrow \alpha^{-1})g^{-1} \quad \text{for} \quad a \in H ,
\]

where \( \to \) and \( \leftarrow \) denote the natural actions of the Hopf algebra \( H^* \) on \( H \) described by

\[
\beta \to a = \sum a_{(1)} \beta(a_{(2)}) \quad \text{and} \quad a \leftarrow \beta = \sum \beta(a_{(1)}) a_{(2)}
\]

for \( \beta \in H^* \) and \( a \in H \). If \( \lambda \) and \( \Lambda \) are normalized, there are formulae for the trace of any linear endomorphism on \( H \).

**Theorem 1.1** [Rad90, Theorem 1] Let \( H \) be a finite-dimensional Hopf algebra with antipode \( S \) over the field \( k \). Suppose that \( \lambda \) is a right integral of \( H^* \), and that \( \Lambda \) is a left integral of \( H \) such that \( \lambda(\Lambda) = 1 \). Then for any \( f \in \text{End}_k(H) \),

\[
\text{Tr}(f) = \sum \lambda \left( S(\Lambda_{(2)}) f(\Lambda_{(1)}) \right) = \sum \lambda \left( (S \circ f)(\Lambda_{(2)}) \Lambda_{(1)} \right) = \sum \lambda \left( (f \circ S)(\Lambda_{(2)}) \Lambda_{(1)} \right) .
\]

Following [Ng02, Section 2], the index of \( H \) is the least positive integer \( n \) such that \( S^{4n} = id_H \) and \( g^n = 1 \).

Suppose that \( H \) is a finite-dimensional Hopf algebra of odd index \( n > 1 \), and that \( \omega \in k \) is a primitive \( n \)th of unity. Since \( g^n = 1 \) and \( \alpha \) is an algebra map, \( \alpha(g) \) is a \( n \)th root of unity. There exists a unique element \( x(\omega, H) \in \mathbb{Z}_n \) such that

\[
\alpha(g) = \omega^{x(\omega, H)}. \]

Following the notation in [Ng02], we let

\[
H^\omega_{a,i,j} = \{ u \in H \mid S^2(u) = (-1)^a \omega^i u \text{ and } ug = \omega^j u \}
\]

for any \( (a, i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n \). Since the \( r(g) \in \text{End}_k(H) \), defined by \( r(g)(a) = ag \) for \( a \in H \), commutes with \( S^2 \), we have

\[
H = \bigoplus_{a \in \mathbb{Z}_n} H^\omega_a \quad (1.2)
\]
where $\mathcal{K}_n$ denotes the group $\mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n$.

Using the eigenspace decomposition of $H$ in (1.2), the diagonalization of a left integral $\Lambda$ of $H$ admits the following form (cf. [Ng02]),

$$
\Delta(\Lambda) = \sum_{a \in \mathcal{K}_n} \left( \sum_{u_a \otimes v_{-a+x}} \right)
$$

(1.3)

where $\sum_{u_a \otimes v_{-a+x}} \in H_{\omega_a} \otimes H_{\omega_{-a+x}}$ and $x = (0, -x(\omega, H), x(\omega, H))$ in $\mathcal{K}_n$.

In the sequel, we will call the expression in equation (1.3) the **normal form** of $\Delta(\Lambda)$ associated with $\omega$. We will simply write $u_a \otimes v_{-a+x}$ for the sum $\sum_{u_a \otimes v_{-a+x}}$ in the normal form of $\Delta(\Lambda)$.

Let $E_{\omega}^a$, $a \in \mathcal{K}_n$, be the set of orthogonal projections associated with the decomposition (1.2). Then

$$
\dim(H_{\omega}^a) = \text{Tr}(E_{\omega}^a)
$$

and we have the following lemma.

**Lemma 1.2** Let $H$ be a finite-dimensional Hopf algebra with the antipode $S$ of odd index $n > 1$ over $k$, and $\omega \in k$ a primitive $n$th root of unity. Let $x = x(\omega, H) \in \mathbb{Z}_n$ and $x = (0, -x, x)$. Then we have

$$
\dim(H_{\omega}^a) = \dim(H_{\omega}^{x-a})
$$

for all $a \in \mathcal{K}_n$.

**Proof.** Let $\Lambda$ be a left integral for $H$ and let $\lambda$ be a right integral for $H^*$ such that $\lambda(\Lambda) = 1$. Using the normal form of $\Delta(\Lambda)$ associated with $\omega$ in (1.3) and Theorem 1.1 we have

$$
\text{Tr}(E_{\omega}^a) = \sum_{b \in \mathcal{K}_n} \lambda(S(v_{-b+x})E_{\omega}^a(u_b)) = \lambda(S(v_{-a+x})u_a) .
$$

By Theorem 1.1 again, we also have

$$
\text{Tr}(E_{\omega}^{x-a}) = \sum_{b \in \mathcal{K}_n} \lambda(S(E_{-a+x}^{\omega}(v_{-b+x}))u_b) = \lambda(S(v_{-a+x})u_a) .
$$

Therefore, $\text{Tr}(E_{\omega}^a) = \text{Tr}(E_{\omega}^{x-a})$. Since $\dim(H_{\omega}^a) = \text{Tr}(E_{\omega}^a)$ for any $a \in \mathcal{K}_n$, the result follows.

**Theorem 1.3** [Ng02] Let $H$ be a Hopf algebra of dimension $pq$ over $k$ with antipode $S$, where $p \leq q$ are odd primes. Then the index of $H$ and the order of $S^4$ are equal to $p$, and $\text{Tr}(S^{2p}) = p^2d$ for some odd integer $d$.

■
Lemma 1.4 Suppose that $H$ is a non-semisimple Hopf algebra of dimension $pq$ over $k$ where $p \leq q$ are odd primes, and that $\omega \in k$ is a primitive $p$th root of unity. Let $g$ and $\alpha$ be the distinguished group-like elements of $H$ and $H^*$ respectively. If $g$ is non-trivial, then the integer $d$ in Theorem 1.3 is given by

$$ \dim(H_{0,i,j}^\omega) - \dim(H_{1,i,j}^\omega) = d $$

for all $i, j \in \mathbb{Z}_p$. Moreover, if both $g$ and $\alpha$ are not trivial, then

$$ \dim(H_{a,i,j'}^\omega) = \dim(H_{a,i,j}^\omega) $$

for any $a \in \mathbb{Z}_2$ and $i, j, j' \in \mathbb{Z}_p$.

Proof. If $\alpha$ is trivial and $g \neq 1$, then by [Ng02, Lemma 4.3],

$$ \dim(H_{0,i,j}^\omega) - \dim(H_{1,i,j}^\omega) = d $$

If both $g$ and $\alpha$ are non-trivial, then by the proof of [Ng02, Proposition 5.3], $H$ is isomorphic to the biproduct

$$ R \times B \quad (1.4) $$

as Hopf algebras (cf. [Rad85]) where $B = k[g]$ and $R$ is a right $B$-comodule subalgebra of $H$. It is shown in [AS98, section 4] that $R$ is invariant under $S^2$. Moreover, in the identification $H \cong R \otimes B$ given by multiplication, one has

$$ S^2 = T \otimes id_B \quad (1.5) $$

where $T$ is the restriction of $S^2$ on $R$. Let

$$ R_{a,i} = \{ x \in R \mid S^2(x) = (-1)^a \omega^i x \} $$

for any $(a, i) \in \mathbb{Z}_2 \times \mathbb{Z}_p$. It follows from the proof of [Ng02, Proposition 5.3] that

$$ \dim(R_{0,i}) - \dim(R_{1,i}) = d $$

By (1.4),

$$ H_{a,i,j}^\omega = R_{a,i} \otimes e_j $$

for all $(a, i, j) \in \mathcal{K}_p$ where $e_j$ is the central idempotent of $B$ such that $e_j g = \omega^i e_j$. Thus,

$$ \dim(H_{a,i,j}^\omega) = R_{a,i} $$

for all $(a, i, j) \in \mathcal{K}_p$ and hence

$$ \dim(H_{0,i,j}^\omega) - \dim(H_{1,i,j}^\omega) = d $$

\[\blacksquare\]
2 Proofs of Main Results

Lemma 2.1 Let $H$ be a finite-dimensional Hopf algebra with antipode $S$ of odd index $n > 1$ over $k$, and $\omega \in k$ a primitive $n$th root of unity. Let $\ell \in \mathbb{Z}_n$ such that $2\ell = x(\omega, H)$. Then

$$\dim(H^\omega_{1,-\ell,\ell})$$

is even.

Proof. Let $V$ be space of all $f \in H^*$ such that $f(u) = 0$ for $u \in H^\omega_{a,i,j}$ whenever $(a, i, j) \neq (1, -\ell, \ell)$. Obviously, $V$ is isomorphic to $(H^\omega_{1,-\ell,\ell})^*$ and so $\dim(V) = \dim(H^\omega_{1,-\ell,\ell})$. Let $\Lambda$ be a non-zero left integral of $H$ and

$$\Delta(\Lambda) = \sum_{a \in \mathbb{K}_n} u_a \otimes v_{-a+x}$$

the normal form of $\Delta(\Lambda)$ associated with $\omega$ where $x = (0, -2\ell, 2\ell)$. Then

$$(f, h) = (f \otimes h)\Delta(\Lambda)$$

defines a non-degenerate bilinear form on $H^*$. Let $f \in V$ such that $(f, h) = 0$ for all $h \in V$. For any $h' \in H^*$, there exists $h \in V$ such that $h'(u) = h(u)$ for all $u \in H^\omega_{1,-\ell,\ell}$. Thus

$$(f, h') = \sum_{a \in \mathbb{K}_n} f(u_a)h'(v_{-a+x}) = f(u_{1,-\ell,\ell})h'(v_{1,-\ell,\ell}) = (f, h) = 0.$$

By the non-degeneracy of $(\cdot, \cdot)$, $f = 0$. Therefore, $(\cdot, \cdot)$ induces a non-degenerate bilinear form on $V$. Using [Rad94, Theorem 3(d)], we have

$$\Delta^{op}(\Lambda) = \sum_{(a,i,j) \in \mathbb{K}_n} (-1)^a \omega^{-i-j} \left( \sum_{a,j} u_{a,i,j} \otimes v_{a,-2\ell-i,2\ell-j} \right).$$

Therefore, for any $f, h \in V$,

$$(h, f) = (f \otimes h)\Delta^{op}(\Lambda) = -f(u_{1,-\ell,\ell})h(v_{1,-\ell,\ell}) = -(f, h).$$

Hence, $V$ admits a non-degenerate alternating form and so $\dim(V)$ is even. ■

If $H$ is a finite-dimensional Hopf algebra of index $n > 1$, we define

$$H_- := \{ u \in H \mid S^{2n}(u) = -u \},$$
$$H_+ := \{ u \in H \mid S^{2n}(u) = u \}.$$

Corollary 2.2 Suppose $H$ is a finite-dimensional Hopf algebra with antipode $S$ of odd index $n > 1$. Then, the subspace $H_-$ is of even dimension.
Proof. Let $\omega \in k$ be an $n$th of unity and $\ell \in \mathbb{Z}_n$ such that $2\ell = x(\omega, H)$. We then have

$$H_-= \bigoplus_{i,j \in \mathbb{Z}_n} H_{1,i,j}^\omega = H_{1,-\ell,\ell}^\omega \oplus \left( \bigoplus_{\text{some } i,j \in \mathbb{Z}_n \atop (i,j) \neq (-\ell,\ell)} H_{1,i,j}^\omega \oplus H_{1,-2\ell-i,2\ell-j}^\omega \right).$$

It follows from Corollary 1.2 and Lemma 2.1, dim$(H_-)$ is even. ■

**Theorem 2.3** Let $H$ be a non-semisimple Hopf algebra with antipode $S$ of dimension $pq$ where $p \leq q$ are odd primes. Then

$$\text{Tr}(S^{2p}) = p^2d \quad \text{and} \quad d \equiv pq \pmod{4}.$$  

**Proof.** By Theorem 1.3, $H$ is of index $p$ and Tr$(S^{2p}) = p^2d$ for some odd integer $d$. Since

$$\dim(H_+) + \dim(H_-) = pq$$

and

$$\text{Tr}(S^{2p}) = \dim(H_+) - \dim(H_-) = p^2d,$$

we have

$$\dim(H_-) = p(q - pd)/2.$$  

By Corollary 2.2, $p(q - pd) \equiv 0 \pmod{4}$ or $d \equiv pq \pmod{4}$. ■

**Theorem 2.4** For any pair of twin primes $p < q$, if $H$ is a Hopf algebra of dimension $pq$, then $H$ is semisimple.

**Proof.** Suppose there is a non-semisimple Hopf algebra $H$ of dimension $pq$. By [LR88], $H^*$ is also non-semisimple. Since dim$(H)$ is odd, by [LR95, Theorem 2.1], $H$ and $H^*$ cannot be both unimodular. By duality, we may simply assume that $H^*$ is not unimodular. It follows from Theorem 1.3 that $|G(H)| = p$ and so

$$\dim(C) \geq p$$

where $C$ is the coradical of $H$. If dim$(C) = p$, then $H$ is pointed and hence, by [Ste97, Corollary 4], $H$ is semisimple. Therefore, dim$(C) > p$ and so we have

$$\text{Tr}(S^{2p}|_{H/C}) \geq -(pq - \dim(C)) > -pq + p = -p^2 - p.$$  

It follows from [LR88, Lemma 3.2] that

$$\text{Tr}(S^{2p}|_C) \geq p.$$  

Thus, we have

$$\text{Tr}(S^{2p}) = \text{Tr}(S^{2p}|_C) + \text{Tr}(S^{2p}|_{H/C}) > -p^2.$$  

(2.1)

Since $pq \equiv -1 \pmod{4}$, by Theorem 2.3

$$\text{Tr}(S^{2p}) = -p^2$$

but this contradicts (2.1). ■

**Acknowledgement**

The author would like to thank P. Etingof for his useful suggestion for Theorem 2.4 and informing me his recent work [EG03] with S. Gelaki.
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