\textbf{*-SUPER POTENT DOMAINS}

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Abstract. For a finite-type star operation \( \ast \) on a domain \( R \), we say that \( R \) is \( \ast \)-super potent if each maximal \( \ast \)-ideal of \( R \) contains a finitely generated ideal \( I \) such that (1) \( I \) is contained in no other maximal \( \ast \)-ideal of \( R \) and (2) \( J \) is \( \ast \)-invertible for every finitely generated ideal \( J \supseteq I \). Examples of \( t \)-super potent domains include domains each of whose maximal \( t \)-ideals is \( t \)-invertible (e.g., Krull domains). We show that if the domain \( R \) is \( \ast \)-super potent for some finite-type star operation \( \ast \), then \( R \) is \( t \)-super potent, we study \( t \)-super potency in polynomial rings and pullbacks, and we prove that a domain \( R \) is a generalized Krull domain if and only if it is \( t \)-super potent and has \( t \)-dimension one.

Introduction

Dedekind domains are characterized as those domains having all nonzero ideals invertible. On the other hand, if \( D \) is a Dedekind domain with quotient field \( K \), then the domain \( R := D + (x^2, x^3)K[[x^2, x^3]] \) has invertibility strictly above \( M := (x^2, x^3)R \), but \( M \) itself is not invertible in \( R \). Similarly, it is well known that Krull domains are characterized as those domains having all nonzero ideals \( t \)-invertible (definitions reviewed below), while in the example above, one has \( t \)-invertibility only above a certain level. The goal of this paper is to explore one form of this kind of \( (t) \)-invertibility.

Now the \( t \)-operation is a particular example of a star operation, and it is useful to generalize to arbitrary finite-type star operations. Let \( R \) be a domain with quotient field \( K \). Denoting by \( \mathcal{F}(R) \) the set of nonzero fractional ideals of \( R \), a map \( \ast : \mathcal{F}(R) \to \mathcal{F}(R) \) is a star operation on \( R \) if the following conditions hold for all \( A, B \in \mathcal{F}(R) \) and all \( c \in K \setminus (0) \):

1. \((cA)\ast = cA\ast \) and \( R^\ast = R\);
2. \( A \subseteq A^\ast \), and, if \( A \subseteq B \), then \( A^\ast \subseteq B^\ast \); and
3. \( A^{\ast\ast} = A^\ast \).

An ideal \( I \) satisfying \( I^\ast = I \) is called a \( \ast \)-ideal. Other than the \( d \)-operation (\( I^d = I \) for all nonzero fractional ideals \( I \)), the best known star operation is the \( v \)-operation: for \( I \in \mathcal{F}(R) \), put \( I^{-1} = \{ x \in K \mid xI \subseteq R \} \) and \( I^v = (I^{-1})^{-1} \). For any star operation \( \ast \), we may define an associated star operation \( \ast_f \) merely by setting, for \( I \in \mathcal{F}(R) \), \( I^{\ast_f} = \bigcup J^\ast \), where the union is taken over all finitely generated subideals \( J \) of \( I \), and we say that \( \ast \) has finite type if \( \ast = \ast_f \). The \( t \)-operation is then given by \( t = v_f \). It is well known that for a finite-type star operation \( \ast \) on
a domain $R$, each $*$-ideal is contained in a maximal $*$-ideal, that is, a star ideal maximal in the set of $*$-ideals; maximal $*$-ideals are prime; and $R = \bigcap R_P$, where the intersection is taken over the set of maximal $*$-ideals $P$. A nonzero ideal $I$ is $*$-invertible if $(II^{-1})^* = R$. For star operations $\star_1, \star_2$, we say that $\star_1 \leq \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$ for each $I \in \mathcal{F}(R)$.

Generalizing notions from [3][4], we call a nonzero finitely generated ideal $I$ $*$-rigid if it is contained in a unique maximal $*$-ideal and $*$-super rigid if, in addition, $J$ is $*$-invertible for each finitely generated ideal $J \supseteq I$. We say that a maximal $*$-ideal $M$ is $*$-potent ($*$-super potent) if $M$ contains a $*$-rigid ($*$-super rigid) ideal and that the domain $R$ is $*$-potent ($*$-super potent) if each maximal $*$-ideal of $R$ is $*$-potent ($*$-super potent). It is clear that any domain each of whose maximal ideals is invertible is $d$-super potent, as is any valuation domain. On the other hand, a Krull domain may not (even) be $d$-potent (e.g., a polynomial ring in two indeterminates over a field) but is $t$-super potent.

For the remainder of the introduction, we assume that all star operations mentioned have finite type. In Section 1 we lay out many of the basic properties of $*$-super potency. In Corollary 1.6 we show that if $\star_1 \leq \star_2$ and $R$ is $\star_1$-super potent, then it is also $\star_2$-super potent; in particular, since $d \leq \star \leq t$ for all (finite-type) $\star$, we have that $t$-super potency is the weakest type of super potency. In Theorem 1.10 we obtain a local characterization: $R$ is $*$-super potent if and only if $R$ is $*$-potent and $R_M$ is $d$-super potent for each maximal $*$-ideal $M$ of $R$. In Theorem 1.11 we establish, among other things, that if $I$ is a $*$-super rigid ideal, then $\bigcap_{n=1}^{\infty} (I^n)^*$ is prime. In Section 2 we study local super potency and show that a (non-field) local domain $(R, M)$ is $d$-super potent if and only there is a prime ideal $P \subseteq M$ for which $P = PR_P$ and $R/P$ is a valuation domain. In a brief Section 3 we show that $t$-potency and $t$-super potency extend from $R$ to the polynomial ring $R[X]$. Section 4 is devoted to determining how $t$-potency and $t$-super potency behave in a commonly studied type of pullback diagram, and these results are used to provide several examples. In Section 5 the main result is a characterization of Ribenboim’s generalized Krull domains [25], those domains that may be expressed as a locally finite intersection of essential rank-one valuation domains: the domain $R$ is a generalized Krull domain if and only if it is $t$-super potent and every maximal $t$-ideal of $R$ has height one.

1. Basic results on $*$-super potency

From now on, we use $R$ to denote a domain and $K$ to denote its quotient field. We begin by repeating the definition of $*$-(super) potency.

**Definition 1.1.** Let $*$ be a finite-type star operation on the domain $R$. Call a finitely generated ideal $I$ of $R$ $*$-rigid if it is contained in exactly one maximal $*$-ideal of $R$ and $*$-super rigid if, in addition, each finitely generated ideal $J \supseteq I$ is $*$-invertible. We then say that a maximal $*$-ideal of $R$ is $*$-potent ($*$-super potent) if it contains a $*$-rigid ($*$-super rigid) ideal and that $R$ itself is $*$-potent ($*$-super potent) if each maximal $*$-ideal of $R$ is $*$-potent ($*$-super potent).

**Remark 1.2.** Recall that for a star operation $*$ on $R$, a $*$-ideal $A$ is said to have finite type if $A = B^*$ for some finitely generated ideal $B$ of $R$. In [5] a finite type $t$-ideal $J$ was dubbed rigid if it is contained in exactly one maximal $t$-ideal. For such a $J$, we have $J = I^t$ for some finitely generated subideal $I$ of $J$, and, since it
is more convenient to work with the subideal \( I \), we apply the “rigid” terminology to \( I \) instead of \( J \). Moreover, we want to consider finite-type star operations other than the \( t \)-operation (e.g., the \( d \)-operation!), and therefore prefer “\( t \)-rigid” in place of “rigid.” Similarly, we replace “potent” with “\( t \)-potent.”

In [28, 29] Wang and McCasland studied the \( w \)-operation in the context of strong Mori domains. Motivated by this, Anderson and Cook associated to any star operation \( * \) on a domain \( R \) a finite-type star operation \( *_w \), given by \( A^{*w} = \{ x \in K \mid xB \subseteq A \text{ for some finitely generated ideal } B \text{ of } R \text{ with } B^* = R \} \) [3]. We always have \( A^{*w} = \bigcap \{ AR_P \mid P \in \star_f \text{-Max}(R) \} \) [3, Corollary 2.10], from which it follows that \( A^{*w} R_P = AR_P \) for each \( P \in \star_f \text{-Max}(R) \). (Recall that \( \star_f \) is the finite-type star operation associated to \( * \) given by \( A^{\star_f} = \bigcup B^* \), where the union is taken over all finitely generated subideals \( B \) of \( A \).) We also have \( \star_f \text{-Max}(R) = \star_w \text{-Max}(R) \) [3, Theorem 2.16]. For the “original” \( w \)-operation, we have \( w = v_w = t_w \) (hence the notation \( *_w \)).

The following is an easy consequence of the definitions.

**Proposition 1.3.** Let \( \star_1, \star_2 \) be finite-type star operations on a domain \( R \) for which \( \star_1 \text{-Max}(R) = \star_2 \text{-Max}(R) \). Then \( \star_1 \)-rigidity (\( \star_1 \)-potency, \( \star_1 \)-super potency) coincides with \( \star_2 \)-rigidity (\( \star_2 \)-potency, \( \star_2 \)-super potency). In particular, \( R \) is \( \star_1 \)-potent (\( \star_1 \)-super potent) if and only if \( R \) is \( \star_2 \)-potent (\( \star_2 \)-super potent).

Observe that Proposition 1.3 may be applied to \( * \) and \( *_w \) for any finite-type star operation \( * \) on a domain \( R \). In particular, the proposition may be applied to \( t \)- and \( w \)-operations.

**Proposition 1.4.** Let \( \star_1 \leq \star_2 \) be finite-type star operations on a domain \( R \). If \( M \in \star_1 \text{-Max}(R) \cap \star_2 \text{-Max}(R) \) and \( M \) is \( \star_1 \)-potent, then \( M \) is \( \star_2 \)-potent.

**Proof.** Let \( M \in \star_1 \text{-Max}(R) \cap \star_2 \text{-Max}(R) \) with \( M \) \( \star_1 \)-potent, and let \( I \) be a \( \star_1 \)-rigid ideal contained in \( M \). Suppose that \( I \subseteq N \) for some maximal \( \star_2 \)-ideal \( N \) of \( R \). Since \( N^{\star_1} \subseteq N^{\star_2} \neq R \), there is a maximal \( \star_1 \)-ideal \( N' \) of \( R \) for which \( N \subseteq N' \). Since \( I \subseteq N' \), this forces \( N' = M \) and hence \( N = M \). Therefore, \( I \) is also \( \star_2 \)-rigid, and hence \( M \) is \( \star_2 \)-potent.

With respect to Proposition 1.3 it is not true that for finite-type star operations \( \star_1 \leq \star_2 \) on a domain \( R \) and \( M \in \star_1 \text{-Max}(R) \cap \star_2 \text{-Max}(R) \) with \( M \) \( \star_2 \)-potent, we must also have \( M \) \( \star_1 \)-potent—see Example 1.3 below. It is also not the case that for \( \star_1 \leq \star_2 \) and \( M \) a \( \star_1 \)-potent maximal \( \star_1 \)-ideal, we must have that \( M \) is a maximal \( \star_2 \)-ideal. (Let \( R = k[x,y] \), a polynomial ring in two variables over a field. Then \( M = (x,y) \) is a \( d \)-potent maximal \( (d) \)-ideal but is not a \( t \)-ideal.) More interestingly, it is not the case that, for \( \star_1 \leq \star_2 \), \( R \) \( \star_1 \)-potent implies \( R \) \( \star_2 \)-potent, as we show in Example 1.4 below.

The situation is better for super potency:

**Theorem 1.5.** Let \( \star_1 \leq \star_2 \) be finite-type star operations on a domain \( R \). If \( M \) is a \( \star_1 \)-super potent maximal \( \star_1 \)-ideal of \( R \), then \( M \) is also a \( \star_2 \)-super potent maximal \( \star_2 \)-ideal of \( R \).

**Proof.** Let \( M \) be a \( \star_1 \)-super potent maximal \( \star_1 \)-ideal of \( R \), and let \( A \subseteq M \) be a \( \star_1 \)-super rigid ideal. We first show that \( M^{\star_2} \neq R \). If, on the contrary, \( M^{\star_2} = R \), then there is a finitely generated ideal \( B \subseteq M \) with \( B^{\star_2} = R \). Let \( C := A + B \). Then \( C \) is \( \star_1 \)-invertible, whence \( (C^*C^{-1})^{\star_2} = (CC^{-1})^{\star_2} \geq (CC^{-1})^{\star_1} = R \). Since
$C^{*2} = R$, this yields $C^{-1} = (C^{-1})^{*2} = R$. However, the equation $(CC^{-1})^{*1} = R$ then forces $C^{*1} = R$, the desired contradiction. Thus $M^{*2} \neq R$ and, since $M^{*2}$ is a $\star_1$-ideal, we must have $M^{*2} = M$. Then, again since $\star_2$ ideals are also $\star_1$-ideals, it must be the case that $M$ is a maximal $\star_2$-ideal. That $M$ must be $\star_2$-super potent now follows easily, since for any finitely generated ideal $I \supseteq A$, $\star_1$-invertibility of $I$ implies $\star_2$-invertibility.

As a consequence of the preceding result, we have that the weakest type of super potency is $t$-super potency:

**Corollary 1.6.** Let $\star$ be a finite-type star operation on a domain $R$.

(1) If $M$ is a $\star$-super potent maximal $\star$-ideal of $R$, then $M$ is a $t$-super potent maximal $t$-ideal of $R$.

(2) If $R$ is $\star$-super potent, then $R$ is $t$-super potent. □

The converse of Corollary 1.6(2) is false: if $k$ is a field, then the polynomial ring $k[X, Y]$, being a Krull domain, is $t$-super potent but is not $d$-super potent. However, we do not know whether one can have a maximal ideal $M$ of a domain such that $M$ is a $t$-super potent maximal $t$-ideal but is not $d$-super potent.

Now let $R$ be a domain and $T$ a flat overring of $R$. According to [27] Proposition 3.3, if $\star$ is a finite-type star operation on $R$, then the map $\star_T : IT \mapsto I^*T$ is a well-defined finite-type star operation on $T$. In the following result, we study how (super) potency extends to flat overrings. We assume standard facts about flat overrings (including the fact, used above, that each fractional ideal of $T$ is extended from a fractional ideal of $R$); these follow readily from [26].

**Lemma 1.7.** Let $R$ be a domain, $T$ a flat overring of $R$, $\star$ a finite-type star operation on $R$, and $P$ the set of $\star$-primes $P$ of $R$ maximal with respect to the property $PT \neq T$. Then:

(1) $\star_T$-Max($T$) = \{PT | P ∈ P\}.

(2) If $M$ is a $\star$-(super) potent maximal $\star$-ideal of $R$ for which $MT \neq T$, then $MT$ is a $\star_T$-(super) potent maximal $\star_T$-ideal of $T$. (In fact, if $M$ is as hypothesized and $I \subseteq M$ is $\star$-(super) rigid, then $IT$ is $\star_T$-(super) rigid in $T$.)

**Proof.** Let $P ∈ P$. Then $(PT)^{\star T} = P^{\star T} = PT$, that is, $PT$ is a $\star_T$-ideal of $T$. Moreover, if $Q$ is a prime of $R$ for which $QT$ is a maximal $\star_T$-ideal of $T$ containing $PT$, then $Q^* \subseteq Q^*T \cap R = (QT)^{\star T} \cap R = QT \cap R = Q$; that is, $Q$ is a $\star$-ideal of $R$ containing $P$. Since $P ∈ P$, we have $(Q = P$ and hence) $QT = PT$. Therefore $PT$ is a maximal $\star_T$-ideal of $T$. Conversely, let $P$ be a prime of $R$ for which $PT$ is a maximal $\star_T$-ideal of $T$. Then $P^* \subseteq P^*T \cap R = (PT)^{\star T} \cap R = PT \cap R = P$, and so $P$ is a $\star$-ideal of $R$. Suppose that $P \subseteq Q$, where $Q$ is a $\star$-prime of $R$ and $QT \neq T$. Then $QT$ is a $\star$-ideal of $T$ (since, $(QT)^{\star T} = Q^*T = QT$) containing $PT$, whence $(QT = PT$ and hence) $Q = P$. This proves (1).

Let $M$ be a $\star$-potent maximal $\star$-ideal of $R$ such that $MT \neq T$. Then $MT$ is a maximal $\star_T$-ideal of $T$ by (1). Now let $I$ be a $\star$-rigid ideal contained in $M$, and suppose that $IT \subseteq NT$, where $N$ is a prime ideal of $R$ for which $NT$ is a maximal $\star_T$-ideal of $T$. Then $N^* \neq R$, whence $N \subseteq N'$ for some maximal $\star$-ideal $N'$ of $R$. Since $I$ is contained in no maximal $\star$-ideal of $R$ other than $M$, we must have $N' = M$. However, this yields $N \subseteq M$ and hence $NT = MT$. It follows that $IT$ is $\star_T$-rigid in $T$. 
Now assume that $M$ is $\star$-super potent and that $I \subseteq M$ is $\star$-super rigid. Let $J$ be a finitely generated ideal of $R$ for which $JT \supseteq IT$. Replacing $J$ with $I + J$ if necessary, we may assume that $J \supseteq I$. Then $J$ is $\star$-invertible, whence, in particular, $JJ^{-1} \not\subseteq M$. This, in turn, yields $(JT)(T : JT) \not\subseteq MT$. Since $MT$ is the only maximal $\star_T$-ideal of $T$ containing $JT$, $JT$ is $\star_T$-invertible. Therefore, $IT$ is $\star_T$-super potent. This completes the proof of (2).

Remark 1.8. Suppose that $(R, M)$ is local and that $\star$ is a star operation on $R$ for which $M$ is a $\star$-ideal. Then if $I$ is a $\star$-invertible ideal of $R$, we cannot have $II^{-1} \subseteq M$, and hence $I$ is actually (invertible and hence) principal. In particular, if $\star$ is of finite-type and $I \subseteq M$ is $\star$-super rigid, then $I$ is principal. We shall use this fact often in the sequel.

Lemma 1.9. Let $(R, M)$ be a local domain. The following statements are equivalent.

1. $M$ is a $\star$-super potent maximal $\star$-ideal for some finite-type star operation $\star$ on $R$.
2. $M$ is a $t$-super potent maximal $t$-ideal.
3. $M$ is $d$-super potent.
4. $M$ is a $\star$-super potent maximal $\star$-ideal for every finite-type star operation on $R$.

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) follow from Theorem 1.6 and (4) $\Rightarrow$ (1) is trivial. Assume (2), and let $A$ be a $t$-super rigid ideal contained in $M$ and $B$ a finitely generated ideal containing $A$. Then $B$ is $t$-invertible and hence principal (Remark 1.5). Therefore, $A$ is $d$-super rigid, as desired.

Since the extension (as defined above) of the $d$-operation on $R$ to a flat overring $T$ is the $d$-operation on $T$, we shall write “$d$” instead of “$d_T$” in this case.

It is now an easy matter to characterize $\star$-super potency locally:

Theorem 1.10. Let $\star$ be a finite-type star operation on $R$ and $M$ a maximal $\star$-ideal of $R$. Then:

1. $M$ is $\star$-super potent if and only if $M$ is $\star$-potent and $MR_M$ is $d$-super potent.
2. $R$ is $\star$-super potent if and only if $R$ is $\star$-potent and $R_P$ is $d$-super potent for each maximal $\star$-ideal $P$ of $R$.

Proof. It suffices to prove (1). Assume that $M$ is $\star$-potent and $MR_M$ is $d$-super potent. Then there is a finitely generated ideal $A$ of $R$ such that $AR_M$ is $d$-super rigid and a $\star$-rigid ideal $B$ of $R$ contained in $M$. Suppose that $C$ is a finitely generated ideal of $R$ with $C \supseteq A + B$. Then $CR_M$ is principal, and $CR_N = R_N$ for each maximal $\star$-ideal $N \neq M$. It follows that $C$ is $\star$-invertible. Thus $(A + B$ is $\star$-super rigid and hence) $M$ is $\star$-super potent. For the converse, if $M$ is $\star$-super potent, then $M$ is certainly $\star$-potent. Moreover, $MR_M$ is $\star_{R_M}$-super potent by Lemma 1.7 and hence $d$-super potent by Lemma 1.9.

In spite of Theorem 1.10 (and Lemma 1.7), $\star$-super potency does not in general localize at non-maximal $\star$-primes—see Example 1.6 below. Also, observe that if $R$ is a non-Dedekind almost Dedekind domain, then $R_M$ is $d$-super potent for each maximal $(t)$-ideal $M$, but $R$ is not $t$-potent. (Hence the $\star$-potency assumption is necessary in Theorem 1.10(1,2).)
Theorem 1.11. Let \( \ast \) be a finite-type star operation on a domain \( R \), \( M \) a \( \ast \)-super potent maximal \( \ast \)-ideal of \( R \), and \( I \) a \( \ast \)-super rigid ideal of \( R \) contained in \( M \).

1. If \( A \) is a finitely generated ideal for which \( A^\ast \supseteq I \), then \( A \) is \( \ast \)-super rigid.
2. If \( J \) is a \( \ast \)-super rigid ideal contained in \( M \), then \( I \subseteq J^\ast \) or \( J \subseteq I^\ast \).
3. If \( J \) is a \( \ast \)-super rigid ideal contained in \( M \), then \( IJ \) is also a \( \ast \)-super rigid ideal.
4. \( I^n \) is \( \ast \)-super rigid for each positive integer \( n \).
5. If \( R \) is local with maximal ideal \( M \), then \( I \) is comparable to each ideal of \( R \), and \( \bigcap_{n=1}^{\infty} I^n \) is prime.
6. \( I^* = IR_M \cap R \).
7. \( \bigcap_{n=1}^{\infty} (I^n)^* \) is prime.
8. If \( P \) is a prime ideal of \( R \) with \( P \subseteq M \) and \( I \not
subseteq P \), then \( P \subseteq \bigcap_{n=1}^{\infty} (I^n)^* \).

Proof. (1) Let \( A \) be a finitely generated ideal with \( A^\ast \supseteq I \). Then \( A \) is \( \ast \)-rigid. Let \( B \) be a finitely generated ideal with \( B \supseteq A \). Set \( C := I + B \). Then \( C \) is \( \ast \)-invertible, and, since \( C^* = (I^* + B^*)^* \subseteq (A^* + B^*)^* = B^* \), we have \( C^* = B^* \), and hence \( B \) is \( \ast \)-invertible. Therefore, \( A \) is \( \ast \)-super rigid.

(2) Let \( J \) be a \( \ast \)-super rigid ideal contained in \( M \), and set \( C := I + J \). Then \( C \) is \( \ast \)-invertible, and we have \( (IC^{-1} + JC^{-1})^* = R \). Note that \( IC^{-1} \supseteq I \) and \( JC^{-1} \supseteq J \), and hence \( IC^{-1} \not
subseteq M \) or \( JC^{-1} \not
subseteq M \). Since \( IC^{-1}, JC^{-1} \) can be contained in no maximal \( \ast \)-ideal of \( R \) other than \( M \), we must have \( (IC^{-1})^* = R \) or \( (JC^{-1})^* = R \), that is, \( C^* = I^* \) or \( C^* = J^* \). The conclusion follows easily.

(3) Again, let \( J \) be a \( \ast \)-super rigid ideal contained in \( M \), and let \( C \) be a finitely generated ideal containing \( IJ \). Since \( I \) is \( \ast \)-invertible, \( I^{-1} = A^* \) for some finitely generated ideal \( A \). This yields \( (CA)^* \supseteq (IJ)^* = J^* \supseteq J \), and hence \( CA \) is \( \ast \)-invertible. It follows that \( C \) is \( \ast \)-invertible.

(4) This follows from (3).

(5) Assume that \( R \) is local with maximal ideal \( M \). By Lemma 1.13 (and its proof) \( I \) is \( d \)-super rigid and therefore principal (Remark 1.18), say \( I = (c) \). Choose \( r \in M \setminus \{c\} \). Then \( (c,r) \) is principal, and, since \( R \) is local, \( (c,r) = (r) \), i.e. \( c \in (r) \). It follows that \( I \) is comparable to each ideal of \( R \). Now suppose, by way of contradiction, that \( a,b \in R \) with \( ab \in (c^n) \) and \( a,b \notin (c^n) \). Choose \( n,m \) with \( a \in (c^n) \setminus (c^{n+1}) \) and \( b \in (c^m) \setminus (c^{m+1}) \). Then \( a/c^n, b/c^m \notin (c) \), whence, by the claim, \( c \in (a/c^n) \cap (b/c^m) \). Hence \( c^{n+m+2} \in (ab) \subseteq (c^{n+m+3}) \), yielding the contradiction that \( 1 \in (c) \). Hence \( \bigcap_{n=1}^{\infty} I^n \) is prime.

(6) We have \( I^* \subseteq I^* R_M \cap R = (IR_M)^{\ast_{NM}} \cap R = IR_M \cap R \) (since \( IR_M \) is principal). On the other hand, \( IR_N = R_N \) for \( N \in \mathcal{N} := \ast_{-\operatorname{Max}}(R) \setminus \{M\} \), and hence \( I^* \supseteq I^{\ast_{NM}} = IR_M \cap (\bigcap_{N \in \mathcal{N}} IR_N) = IR_M \cap R \).

(7) By (4), Lemma 1.14, and (the proof of) Lemma 1.19, \( I^n R_M \) is \( d \)-super rigid for each \( n \). Using (6), we have \( \bigcap_{n=1}^{\infty} (I^n)^* = \bigcap_{n=1}^{\infty} (I^n R_M \cap R) = (\bigcap_{n=1}^{\infty} I^n R_M) \cap R \), which is prime by (5).

(8) Let \( P \) be as described. Since \( IR_M \not
subseteq PR_M \), we have by (5) and (6) that \( P \subseteq \bigcap_{n=1}^{\infty} I^n R_M \cap R = \bigcap_{n=1}^{\infty} (I^n)^* \). \( \square \)

We record the following useful consequence of Theorem 1.11.

Corollary 1.12. If \( M \) is a \( t \)-super potent ideal of height one in a domain \( R \), then \( R_M \) is a valuation domain. In particular, a one-dimensional local \( d \)-super potent domain is a valuation domain.
Proof. We begin with the “in particular” statement. Let $R$ be a one-dimensional local $d$-super potent domain, $I$ a $d$-super rigid ideal of $R$, and $J$ a finitely generated ideal of $R$. Then $J \supseteq I^n$ for some positive integer $n$. Since $I^n$ is $d$-super rigid by Theorem 1.11, $J$ must be (invertible and hence) principal. It follows that $R$ is a valuation domain. Now assume that $M$ is $t$-super potent of height one in a domain $R$. By Theorem 1.10, $R_M$ is $d$-super potent and is therefore a valuation domain by what has just been proved. \( \square \)

It is easy to see that the requirement on the height of $M$ in Corollary 1.12 is necessary–take $R$ to be any local non-valuation domain having principal maximal ideal and dimension at least two.

2. THE LOCAL CASE

Let $(R, M)$ be a local domain and $\star$ a finite-type star operation on $R$. Recall from Lemma 1.9 that $R$ is $\star$-super potent if and only if $R$ is $d$-super potent. We shall characterize and study local $d$-super potency.

As in [7] we say that a prime ideal $P$ of a domain $R$ is divided if $P = PR_P$. Domains in which each prime ideal is divided were introduced and briefly studied in [1], apparently motivated by considerations from [15]. Recall that if $P$ is a prime ideal of a domain $R$, then $R + PR_P$ is called the CPI-extension of $R$ with respect to $P$ [6]. (“CPI” is short for “complete pre-image.”) The next lemma follows easily from arguments in [1, 7, 6].

Lemma 2.1. Let $P$ be a prime ideal of a domain $R$. Then the following statements are equivalent.

1. $P$ is divided.
2. $P$ is comparable to each principal ideal of $R$.
3. $P$ is comparable to each ideal of $R$.
4. $R$ is the CPI-extension of $R$ with respect to $P$.

Theorem 2.2. Let $(R, M)$ be a local domain, not a field. Then $R$ is $d$-super potent if and only if there is a divided prime $P \subset M$ such that $R/P$ is a valuation domain.

Proof. Suppose that $R$ is $d$-super potent, and let $I \subseteq M$ be $d$-super rigid. Then $I = (c)$ for some $c \in M$. Moreover, by Theorem 1.11(4,5), $P := \bigcap (c^n)$ is prime, and, for each positive integer $m$, $(c^m)$ is $d$-super rigid and hence comparable to each ideal of $R$. Let $a \in M \setminus P$. Then $a \notin (c^k)$ for some $k$, whence $P \subseteq (c^k) \subseteq (a)$. Hence $(a)$ is $d$-super rigid. This shows both that $P$ is divided (Lemma 2.1) and that any two principal ideals generated by elements of $M \setminus P$ must be comparable (since each is a $d$-super rigid ideal). It follows that $R/P$ is a valuation domain.

Now assume that $P$ is a divided prime properly contained in $M$ and that $R/P$ is a valuation domain. Let $a \in M \setminus P$. Since $P$ is divided, we have $P \subset (a)$ (Lemma 2.1). Suppose that $I = (a_1, \ldots, a_n)$ is a finitely generated ideal containing $(a)$. Then $I/P \supseteq (a)/P$ in the valuation domain $R/P$, and it follows that $(I/P$ and hence) $I$ is principal. Therefore, $(a)$ is super rigid. \( \square \)

Recall from Corollary 1.12 that a one-dimensional $d$-super potent domain is a valuation domain. Of course, this is also an immediate corollary of Theorem 2.2 as is the following result in the two-dimensional case.

Corollary 2.3. If $R$ is a two-dimensional local $d$-super potent domain, then $R$ has exactly two nonzero prime ideals. \( \square \)
It is trivial that a Noetherian domain $R$ is $\star$-potent for any star operation $\star$ on $R$. As another consequence of Theorem 2.2, we have a characterization of Noetherian $t$-super potent domains:

**Corollary 2.4.** Let $R$ be a Noetherian domain.

1. If $M$ is a $t$-super potent maximal $t$-ideal of $R$, then $ht(M) = 1$.
2. If $R$ is $t$-super potent, then $R$ is a Krull domain.

**Proof.** (1) Let $M$ be a $t$-super potent maximal $t$-ideal of $R$. Then $R_M$ is a $d$-super potent Noetherian domain, and hence we may as well assume that $R$ is local with $d$-super potent maximal ideal $M$. By Theorem 2.2 there is a divided prime $P \subseteq M$ such that $R/P$ is a Noetherian valuation domain. Moreover, if we choose $a \in M \setminus P$ and shrink $M$ to a prime $Q$ minimal over $a$, then $Q \supseteq (a) \supseteq P$. By the principal ideal theorem, we must have $ht(Q) = 1$, and hence $P = (0)$. But then $R$ is a Noetherian valuation domain, and we must have $ht(M) = 1$. For (2), suppose that $R$ is $t$-super potent. By (1) $R_M$ is a Noetherian valuation domain for each $M \in t$-Max($R$), and hence the representation $R = \bigcap \{R_M \mid M \in t$-Max($R$)$\}$ shows that $R$ is (completely) integrally closed and therefore a Krull domain.

**Remark 2.5.** (1) Recall from [16] that a nonzero element $a$ of a domain $R$ is said to be comparable if $(a)$ compares to each ideal of $R$ under inclusion. By [16, Theorem 2.3] and Theorem 2.2, non-field local $d$-super potent domains coincide with domains that admit nonzero, nonunit comparable elements. Moreover, again by [16, Theorem 2.3], for such a domain $R$, the ideal $P_0 := \bigcap \{(c) \mid c$ is a nonzero comparable element of $R\}$ is a divided prime and is such that $R/P_0$ is a valuation domain, and $P_0$ is the (unique) smallest prime $L$ of $R$ such that $L$ is divided and $R/L$ is a valuation domain.

(2) With the notation above, the following statements are equivalent: (a) $R$ is a valuation domain, (b) $R_{P_0}$ is a valuation domain, and (c) $P_0 = (0)$: the implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are clear, and (b) $\Rightarrow$ (c) by the remark following Theorem 2.3 of [16].

(3) As explained in the just-mentioned remark in [16], every local domain $(R, M)$ that admits a nonzero, nonunit comparable element arises as a pullback

$$
\begin{array}{ccc}
R & \longrightarrow & V \\
\downarrow & & \downarrow \\
T & \xrightarrow{\varphi} & T/M = k,
\end{array}
$$

where $(T, M)$ is a local domain and $V$ is a valuation domain with quotient field $k$ (in which case we have $T = R_M$). In particular, if $T$ is a two-dimensional Noetherian domain, it must have infinitely many height-one primes and hence so must $R$. Thus Corollary 2.3 does not extend to higher dimensions; indeed, the primes of a local $d$-super potent domain need not be linearly ordered (e.g. $R = \mathbb{Z}_p + (x, y)\mathbb{Q}[x, y]$, where $p$ is prime and $x, y$ are indeterminates).

We end this section with an attempt to globalize local $\star$-super potency.

**Lemma 2.6.** Let $M, N, P$ be primes in a domain $R$ with $P \subseteq M \cap N$, and assume that $PR_M$ is divided in $R_M$ and that $R_M/PR_M$ and $R_N$ are valuation domains. Then $R_M$ is a valuation domain.
Proof. Since \( R_N \) is a valuation domain, so is \( R_P = (R_N)_{PR_N} \). However, we also have \( R_P = (R_M)_{PR_M} \), and since \( R_M = \varphi^{-1}(R_M/PR_M) \), where \( \varphi : R_P \to R_P/PR_P \) is the canonical projection, \( R_M \) is a valuation domain by [14, Proposition 18.2(3)]. \( \square \)

**Definition 2.7.** Let \( R \) be a domain, and let \( P \subseteq M \) be prime ideals of \( R \). We say that \( P \) belongs to \( M \) if \( PR_M = PR_P \) and \( R_M/PR_M \) is a valuation domain.

Note that, by Theorem 2.2, a prime ideal \( M \) of a domain \( R \) contains a belonging prime if and only if \( R_M \) is \( d \)-super potent. Moreover, if \( M \) contains a belonging prime, then it contains a smallest one by Remark 2.9.

**Lemma 2.8.** Let \( R \) be a domain, let \( M, N \) be prime ideals of \( R \), and suppose that there is a prime belonging to both \( M \) and \( N \). Then the smallest prime of \( R \) that belongs to \( N \) also belongs to \( M \) (and vice versa).

**Proof.** Let \( P \) belong to both \( M \) and \( N \), and let \( Q \) be the smallest prime belonging to \( N \). We have \( Q \subseteq P \). Applying Lemma 2.6 to \( R/Q \) yields that \( R_M/QR_M = (R/Q)_{M/Q} \) is a valuation domain. Also, since \( QR_N \) is divided in \( R_N \),

\[
QR_Q = QR_N \subseteq QR_P \subseteq PR_P = PR_M \subseteq R_M.
\]

Hence \( QR_Q = QR_Q \cap R_M = QR_M \). Therefore, \( Q \) belongs to \( M \). \( \square \)

**Remark 2.9.** Let \( * \) be a finite-type star operation on a domain \( R \), and assume that \( R \) is \( * \)-super potent. Then each maximal \( * \)-ideal of \( R \) contains a belonging prime by Theorems 1.10 and 2.2. Define \( \sim \) on \( *\-\text{Max}(R) \) by \( M \sim N \) if \( M \) and \( N \) contain a common belonging prime. It is perhaps interesting that \( \sim \) is an equivalence relation: it is clearly reflexive and symmetric, and transitivity follows easily from Lemma 2.8.

Observe that the relation described above forces a certain amount of “independence” in \( *\-\text{Max}(R) \): if \( M, N \) are two maximal \( * \)-ideals in the \( * \)-super potent domain \( R \) with \( M \not\sim N \), then \( M \not\subset P \) and \( Q \not\subset P \). We give a simple example illustrating this.

**Example 2.10.** Let \( F \) be a field, and \( x, y \) indeterminates. Set \( V = F(x)[y]_yF(x)[y]_y \), \( T = F(y)[x^2, x^3, x^2y, x^3y] \), \( R_1 = V + P \), where \( P \) is the maximal ideal of \( T \), \( R_2 = F(x)[y]_{y+1}F(x)[y]_y \), and \( R = R_1 \cap R_2 \). Then \( R_1 \) and \( R_2 \) are \( d \)-super potent (both have principal maximal ideals). Denote the maximal ideal of \( R_1 \) by \( M_1 \). Then \( M := M_1 \cap R \) and \( N := (y + 1)R_2 \cap R \) are the maximal ideals of \( R \), and by [23, Theorem 3], we have \( R_M = R_1 \) and \( R_N = R_2 \). The domain \( R \) is therefore \( d \)-super potent by Theorem 1.10, and it is clear that \( (0) \) belongs to \( N \) and that \( P \) (but not \( (0) \)) belongs to \( M \).

3. **Polynomial rings over \( t \)-super potent domains**

We begin with some well-known facts about \( t \)-ideals in polynomial rings. Recall that if \( R \) is a domain and \( Q \) is a nonzero prime of \( R[X] \) for which \( Q \cap R = (0) \), then \( Q \) is called an upper to zero.

**Lemma 3.1.** Let \( R \) be a domain.

(1) An ideal \( A \) of \( R \) is a \( t \)-ideal if and only if \( A[X] \) is a \( t \)-ideal of \( R[X] \).
(2) If \( Q \) is maximal \( t \)-ideal of \( R[X] \), then \( Q = P[X] \) for some maximal \( t \)-ideal of \( R \) or \( Q \) is an upper to zero in \( R[X] \).

(3) An ideal \( M \) of \( R \) is a maximal \( t \)-ideal if and only if \( M[X] \) is a maximal \( t \)-ideal of \( R[X] \).

(4) If \( Q \) is an upper to zero in \( R[X] \) and is also a maximal \( t \)-ideal, then \( Q \) is \( t \)-super potent.

**Proof.** For (1) see [19] Proposition 4.3. Let \( Q \) be a maximal \( t \)-ideal of \( R[X] \). By [19] Proposition 1.1, \( Q = (Q \cap R)[X] \) or \( Q \) is an upper to zero. It then follows from (1), that if \( Q = (Q \cap R)[X] \), then \( P := Q \cap R \) must be a maximal \( t \)-ideal of \( R \). This gives (2), and (3) follows from (1) and (2). Now suppose that \( Q \) is an upper to zero and also a maximal \( t \)-ideal in \( R[X] \). Then \( Q = fK[X] \cap R[X] \) for some polynomial \( f \in Q \) such that \( f \) is irreducible in \( K[X] \). By [19] Theorem 1.4 there is an element \( g \in Q \) such that \( c(g)^v = R \) (where \( c(g) \), the content of \( g \), is the ideal of \( R \) generated by the coefficients of \( g \)), and it is easy to see via (1) and (2) that the ideal \((f, g)\) of \( R[X] \) is contained in no maximal \( t \)-ideal of \( R[X] \) other than \( Q \). Hence \( Q \) is \( t \)-potent and therefore by Theorem [1,10] also \( t \)-super potent since \( R[X]_Q \) is a valuation domain. Hence (4) holds.

**Theorem 3.2.** Let \( R \) be a domain. Then \( R \) is \( t \)-(super) potent if and only if \( R[X] \) is \( t \)-(super) potent.

**Proof.** Suppose that \( R \) is \( t \)-potent, and let \( Q \) be a maximal \( t \)-ideal of \( R[X] \). By Lemma [5,12], \( Q \) is either an upper to zero or \( Q = P[X] \) with \( P \) a maximal \( t \)-ideal of \( R \). If \( Q \) is an upper to zero, it is \( t \)-super potent by Lemma [5,14]. If \( Q = P[X] \) with \( P \in t\text{-Max}(R) \), then there is a \( t \)-rigid ideal \( I \) of \( R \) contained in \( P \), and it is easy to see that \( I[X] \) is \( t \)-rigid in \( R[X] \). Hence \( R[X] \) is \( t \)-potent.

Now assume that \( R \) is \( t \)-super potent. Then \( R[X] \) is \( t \)-potent by what has already been proved. Hence, by Theorem [1,10] it suffices to show that \( R[X]_Q \) is \( d \)-super potent for each maximal \( t \)-ideal \( Q \) of \( R[X] \). To this end, let \( Q \) be a maximal \( t \)-ideal of \( R[X] \). Again by Lemma [5,14], we may as well assume that \( Q = P[X] \) with \( P \) a maximal \( t \)-ideal of \( R \). We shall show that \( R[X]_Q \) satisfies the requirements of Theorem [2,2]. Since \( R[X]_Q = R_P[X]_P R_P[X] \) and \( R_P \) is \( d \)-super potent, we change notation and assume that \( R \) is local with \( d \)-super potent maximal ideal \( P \), and we wish to show that \( R[X]_{P[X]} \) is \( d \)-super potent. By Theorem [2,2] there is a prime \( L \) of \( R \) such that \( L \subseteq P \), \( R/L \) is a valuation domain, and \( L = LR_P \). Then \( R[X]_{P[X]} / LR[X]_{P[X]} = (R/L)[X]_{(P/L)[X]} \), which is a valuation domain. Finally, we must show that \( LR[X]_{L[X]} = LR[X]_{P[X]} \). Let \( f, g, g \in R[X] \) with \( c(g) \subseteq L \) and \( f \in R[X] \setminus L[X] \). If \( f \notin P[X] \), then \( g/f \in LR[X]_{P[X]} \), as desired. Suppose that \( f \in P[X] \). Since \( L = LR_P \) and \( f \notin L[X] \), \( c(f) \supseteq c(g) \), and, since \( R/L \) is a valuation domain, \( c(f) = (b) \) for some \( b \in P \setminus L \). Note that \( b^{-1}f \in R[X] \setminus P[X] \). Also, since \( b^{-1}g \cdot b \in L[X] \) and \( b \notin L \), \( b^{-1}g \in L[X] \). Thus \( g/f = b^{-1}g/(b^{-1}f) \in LR[X]_{P[X]} \), as desired.

For the converse, first assume that \( R[X] \) is \( t \)-potent, and let \( P \) be a maximal \( t \)-ideal of \( R \). Then \( P[X] \) is a maximal \( t \)-ideal of \( R[X] \), and we may find a \( t \)-rigid ideal \( A \subseteq P[X] \). Let \( I \) denote the ideal of \( R \) generated by the coefficients of the polynomials in a finite generating set of \( A \). Then \( I \) is a finitely generated ideal of \( R \) contained in \( P \), and since \( A \subseteq I[X] \subseteq P[X] \) yields that \( I[X] \) is \( t \)-rigid in \( R[X] \), it is clear that \( I \) is \( t \)-rigid in \( R \). Hence \( R \) is \( t \)-potent. Finally, suppose that \( R[X] \) is \( t \)-super potent. Using the notation above, we may assume that \( A \) is \( t \)-super rigid,
whence $I[X]$ is also $t$-super rigid. If $J$ is a finitely generated ideal of $R$ containing $I$, then $J[X]$ is a finitely generated ideal of $R[X]$ containing $I[X]$; this yields that $J[X]$ is $t$-invertible in $R[X]$, from which it follows easily that $J$ is $t$-invertible in $R$. Hence $t$-super potency of $R[X]$ implies $t$-super potency of $R$. □

Remark 3.3. It is interesting to note that in the proof above, it was easy to show that $R[X]_{P[R[X]]}/LR[X]_{P[R]}$ is a valuation domain using only the fact that $R/L$ is a valuation domain, but the proof that $LR[X]_{L[X]} = LR[X]_{P[X]}$ used not only the assumption that $L = LR$ but also the assumption that $R/L$ is a valuation domain. Here is an example that shows the necessity of the latter assumption. Let $F$ be a field, $k = F(u)$, $u$ an indeterminate, $V$ a 2-dimensional valuation domain of the form $k + P$ with height-one prime $L$, and $R = F + P$. According to [14, Theorem 19.15 and its proof], denoting the common quotient field of $R$ and $V$ by $K$, $Q := (X - u)K[X] \cap R[X]$ is an upper to zero in $R[X]$ satisfying $Q \subseteq P[X]$. We have $L = LV_L = LR_L$. However, $R/L$ is not a valuation domain, and we claim that we do not have $LR[X]_{P[X]} = LR[X]_{L[X]}$. To see this choose $a \in L$, $a \neq 0$, and $c \in P \setminus L$. Then $a/(cX - cu) \in LR[X]_{L[X]}$. Suppose that we can write $a/(cX - cu) = g/f$ with $g \in L[X]$ and $f \in R[X] \setminus P[X]$. We have $af = g(cX - cu)$, so that $f = a^{-1}g(cX - cu) \in (X - u)K[X] \cap R[X] = Q \subseteq P[X]$, a contradiction. This verifies the claim.

4. Pullbacks

Let $T$ be a domain, $M$ a maximal ideal of $T$, $\varphi : T \to k := T/M$ the natural projection, and $D$ a proper subring of $k$. Then let $R = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms.

$$
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
T & \varphi \longrightarrow & T/M = k.
\end{array}
$$

We list some properties that we shall need.

Lemma 4.1. Consider the pullback diagram above.

1. $T$ is a flat $R$-module if and only if $k$ is the quotient field of $D$.
2. If $I$ is a nonzero finitely generated ideal of $D$, then $\varphi^{-1}(I)$ is a finitely generated ideal of $R$.
3. $t$-$\text{Max}(R) = \{N \cap R \mid N \in t$-$\text{Max}(T)$, $N \nsubseteq M\} \cup \{\varphi^{-1}(P') \mid P' \in t$-$\text{Max}(D)\}$. (By convention, if $D$ is a field, then (0) is a maximal $t$-ideal of $D$, in which case $M$ is a maximal $t$-ideal of $R$).
4. If $N$ is a prime ideal of $T$ that is incomparable to $M$, then $R_{N \cap R} = T_N$.
5. Assume that $D$ is not a field. If $I$ is a $t$-invertible ideal of $R$ with $I \supseteq M$, then $\varphi(I)$ is a $t$-invertible ideal of $D$. Conversely, if $I'$ is a $t$-invertible ideal of $D$, then $\varphi^{-1}(I')$ is a $t$-invertible ideal of $R$.

Proof. Statement (1) is well-known (see [11, Proposition 1.11]), (2) is part of [9, Corollary 1.7]), and (3) follows from [11, Theorems 2.6, 2.18] (but the ideas are from [9]). For (4), see, e.g. [11, Theorem 1.9], and for (5), see [11, Theorem 2.18 and Proposition 2.20]. □
Theorem 4.2. Consider the pullback diagram above. Then $R$ is $t$-potent if and only if each of the following conditions holds:

1. $D$ is $t$-potent (or a field).
2. $N$ is $t$-potent for each $N \in \text{Max}(T)$ with $N \nsubseteq M$.
3. If $D$ is a field and $M$ is a $t$-ideal of $T$, then $M$ is $t$-potent in $T$.

Proof. Suppose that $R$ is $t$-potent. If $D$ is not a field and $P' \in \text{Max}(D)$, then by Lemma 4.1(3), $P := \varphi^{-1}(P') \in \text{Max}(R)$, whence there is a $t$-rigid ideal $I$ contained in $P$. Then, again using Lemma 4.1(3), it is easy to see that $\varphi(I)$ is a $t$-rigid ideal of $D$ contained in $P'$. Hence $D$ is $t$-potent. Now let $N \in \text{Max}(T), N \nsubseteq M$. Then (Lemma 4.1(3)) $N \cap R \in \text{Max}(R)$ and hence there is a $t$-rigid ideal $J$ contained in $N \cap R$. (In particular, $J \nsubseteq M$.) Then $JT$ is a $t$-rigid ideal of $T$ contained in $N$ (Lemma 4.1(3)), and $N$ is $t$-potent. This holds whether $D$ is a field or not. If $D$ is a field, then $M$ is a maximal $t$-ideal and hence $t$-potent in $R$, and there is a $t$-rigid ideal $I$ of $R$ contained in $M$. If $M$ is a (maximal) $t$-ideal of $T$, then $IT$ is a $t$-rigid ideal of $T$ contained in $M$, and hence $T$ is $t$-potent in this case.

For the converse, let $P \in \text{Max}(R)$. If $P \supseteq M$, then $P = \varphi^{-1}(P')$ for some $P' \in \text{Max}(D)$. By assumption, there is a $t$-rigid ideal $C$ of $D$ contained in $P'$, and Lemma 4.1(2,3) then implies that $\varphi^{-1}(C)$ is a $t$-rigid ideal of $R$ contained in $P$.

Next, suppose that $P = M$. Then $D$ is a field. By assumption $M$ is $t$-potent in $T$ or $M$ is not a $t$-ideal of $T$. In the first case, let $A_1$ be a $t$-rigid ideal of $T$ contained in $M$. In the second case, we have $M^{t^x} = T$ (where $t^x$ is the $t$-operation on $T$), and there is a finitely generated subideal $A_2$ of $M$ with $(A_2)^{t^x} = T$. In either case, there is a finitely generated subideal $A$ of $M$ with $A \nsubseteq N$ for each $N \in \text{Max}(T) \setminus \{M\}$. For such an $A$ we have $A = IT$ for some finitely generated ideal $I$ of $R$, and it is clear from the conditions satisfied by $A$ that $I$ is $t$-rigid in $R$.

Finally, suppose that $P$ is incomparable to $M$. Then $P = N \cap R$ for some $N \in \text{Max}(T)$ with $N \nsubseteq M$. By assumption there is a $t$-rigid ideal $B$ of $T$ contained in $N$, and we may assume that $B$ contains an element $t \in T \setminus M$. Now $\varphi(t) \neq 0$, whence there is an element $t' \in T$ with $\varphi(tt') = 1$. This implies that $tt' \in R$, and, since $1 - tt' \in M$, it is clear that $tt' \notin Q$ for each ideal $Q$ of $R$ such that $Q \supseteq M$. We consider three cases:

Case 1. Suppose that $k$ is the quotient field of $D$. Then $T$ is flat over $R$ (Lemma 4.1), and hence $B = JT$ for some finitely generated ideal $J$ of $R$. By construction, $J$ is a $t$-rigid ideal of $R$ contained in $P = N \cap R$ in this case.

Case 2. Suppose that $D$ is a field. Then, arguing as in the “$P = M$” situation above, there is a finitely generated subideal $A$ of $M$ with $A \nsubseteq L$ for each $L \in \text{Max}(T) \setminus \{M\}$, and $A = IT$ for some finitely generated ideal $I$ of $R$. Write $I = \sum_{i=1}^n Ra_i$ and $B = \sum_{j=1}^m Tb_j$, and let $J = \sum Ra_ib_j$. Then $JT = IB \nsubseteq L$ for $L \in \text{Max}(T) \setminus \{M, N\}$. It then follows easily that $J + Rtt'$ is a $t$-rigid ideal of $R$ contained in $P = N \cap R$.

Case 3. Suppose that $k$ is not the quotient field of $D$ and that $D$ is not a field, and put $S := \varphi^{-1}(F)$, where $F$ is the quotient field of $D$. By what has already been proved, $S$ is $t$-potent. It then follows that $P = N \cap R$ is $t$-potent by Case 1 above. This completes the proof. \qed

We next give an example, promised immediately after Proposition 4.4, of finitetype star operations $*_1 \leq *_2$ on a domain $R$ and a $*_2$-potent maximal $*_2$-ideal that is $*_1$-maximal but not $*_1$-potent.
Example 4.3. Let $F \subseteq k$ be fields, $T = k[x_1, x_2, \ldots]$ a polynomial ring in countably many variables, and $R = F + M$, where $M$ is the maximal ideal of $T$ generated by the $x_i$. In $R$, $M$ is both a maximal $(d)$-ideal and a maximal $t$-ideal. Since $T$ is a Krull domain, it is $t$-potent, whence so is $R$ by Theorem 1.10. In particular, $M$ is $t$-potent in $R$. However, it is clear that each finitely generated subideal of $M$ is contained in infinitely many maximal ideals of $(T$ and hence of) $R$, and so $M$ is not $d$-potent.

Theorem 4.4. Consider the pullback diagram at the beginning of this section. Then $R$ is $t$-super potent if and only if $D$ is $t$-super potent and not a field, and each maximal $t$-ideal of $T$ not contained in $M$ is $t$-super potent.

Proof. Assume that $R$ is $t$-super potent, and suppose, by way of contradiction, that $D$ is a field. Then we have the following associated pullback diagram

$$
\begin{array}{ccc}
R_M & \longrightarrow & D \\
\downarrow & & \downarrow \\
T_M & \longrightarrow & k
\end{array}
$$

Choose a $a \in M$, $a \neq 0$, and let $t \in T_M \setminus R_M$. Then $at \notin aR_M$ since $t \notin R_M$, and $a \notin atR_M$ since $t^{-1} \notin R_M$. However, this implies that $aR_M + atR_M$ is not principal, and hence that $aR_M$ is not $d$-super rigid. It follows that $MR_M$ is not $d$-super potent, and then, by Theorem 1.10 that $M$ is not $t$-super potent, the desired contradiction. Thus $D$ is not a field.

In the rest of the proof, we freely use Lemma 4.11. Let $P'$ be a maximal $t$-ideal of $D$. Then $P := \varphi^{-1}(P')$ is a maximal $t$-ideal of $R$ properly containing $M$ and therefore contains a $t$-super rigid ideal $I$. It is clear that $I' := \varphi(I)$ is contained in $P'$ and in no other maximal $t$-ideal of $D$. Let $J' \supseteq I'$ be a finitely generated ideal of $D$. Then, since $P$ is $t$-super potent, $J := \varphi^{-1}(J')$ is a $t$-invertible ideal of $R$, and hence $\varphi(J) = J'$ is $t$-invertible in $D$. Therefore, $D$ is $t$-super potent.

Now let $N \not\subseteq M$ be a maximal $t$-ideal of $T$. Then $N \cap R$ is a maximal $t$-ideal of $R$, and hence $T_N = R_{N \cap R}$ is $d$-super potent by Theorem 1.10. Therefore, since $N$ is $t$-potent by Theorem 1.12, $N$ is $t$-super potent by Theorem 1.10.

For the converse, let $P \in t$-$\text{Max}(R)$. If $P \supseteq M$, then $\varphi(P)$ is $t$-super potent in $D$, and we can argue more or less as above to see that $P$ is $t$-super potent in $R$. Since $D$ is not a field, the only other possibility is $P = N \cap R$, where $N \in t$-$\text{Max}(T)$, $N \not\subseteq M$. In this case, $t$-super potency of $N$ in $T$ yields $d$-super potency of $NT_N = (N \cap R)R_N$ (Theorem 1.10). Since $N \cap R$ is $t$-potent by Theorem 1.12, we may again apply Theorem 1.10 to conclude that $P = N \cap R$ is $t$-super potent. \(\square\)

From Theorems 4.12 and 4.14 we can determine $t$-(super) potency in a large class of domains that appear frequently in the literature:

Corollary 4.5. Let $D$ be a subdomain of the field $k$ and $x$ an indeterminate. Let $R = D + xk[x]$ or $D + xk[[x]]$. Then

1. $R$ is $t$-potent if and only if $D$ is $t$-potent (or a field).
2. $R$ is $t$-super potent if and only if $D$ is $t$-super potent and not a field.

Using Theorem 4.2, it is easy to give examples of $t$-super potent domains with non-$t$-super potent localizations:
Example 4.6. In the notation of Theorem 4.4, assume that $R$ is $t$-super potent.

(1) If the quotient field of $D$ is $F \neq k$, then $R_M$ is not $t$-super potent. We may take $R$ integrally closed or not.

(2) If $T$ is a one-dimensional local non-valuation domain, then $R_M$ is not $t$-super potent.

Proof. (1) In this case, let $S = \varphi^{-1}(F)$. Then we have the pullback diagram

$$
\begin{array}{ccc}
R_M & \rightarrow & F \\
\downarrow & \downarrow & \\
T_M & \rightarrow & k,
\end{array}
$$

whence $R_M$ is not $t$-super potent by Theorem 4.4. Let $x$ be an indeterminate, and $z$ an element of a field $k \supseteq Q$. Let $D = \mathbb{Z}$ and $T = \mathbb{Q}(z)[[x]]$ (so that $R = \mathbb{Z} + x\mathbb{Q}(z)[[x]]$). In this case, if $z$ is an indeterminate, then $R$ is integrally closed. On the other hand, if $z = \sqrt{2}$, then $R$ is not integrally closed.

(2) If $k$ is not the quotient field of $D$, this follows from (1). If $k$ is the quotient field of $D$, then $R_M = T$ is not $t$-super potent by Corollary 1.12. □

5. $t$-dimension one

The primary goal of this section is to characterize generalized Krull domains using $t$-super potency. We recall some definitions. First, a set $\mathcal{P}$ of prime ideals in a domain $R$ is a defining family if $R = \bigcap_{P \in \mathcal{P}} R_P$. A defining family has finite character (or is locally finite) if each nonzero element $a \in R$ lies in at most finitely many elements of $\mathcal{P}$. (Thus, in this terminology, if $\star$ is a finite-type star operation on $R$, then $R$ has finite $\star$-character if the defining family of maximal $\star$-ideals of $R$ has finite character.) A prime $P$ of $R$ is essential if $R_P$ is a valuation domain, and $R$ itself is an essential domain if it possess a defining family of essential primes. Finally, $R$ is a generalized Krull domain if $R$ possesses a finite character defining family of height-one essential primes. For convenience we begin with a lemma, much of which comes from [2] (and no doubt all of which is well known).

Lemma 5.1. Let $R$ be a domain and $\mathcal{P}$ a defining family for $R$. Define $\star$ by $A^\star = \bigcap_{P \in \mathcal{P}} AR_P$ for each nonzero fractional ideal $A$ of $R$. Then:

(1) $\star$ is a star operation on $R$.
(2) If $I$ is an integral ideal of $R$ for which $I^\star \neq R$, then $I \subseteq P$ for some $P \in \mathcal{P}$.
(3) $P^\star = P$ for each $P \in \mathcal{P}$.
(4) If $\mathcal{P}$ has finite character, then $\star$ has finite type.
(5) If $\star$ has finite type, then:
   (a) For each $P \in \mathcal{P}$, there is a maximal element $Q$ of $\mathcal{P}$ such that $P \subseteq Q$. Hence if $\mathcal{P}'$ denotes the set of maximal elements in $\mathcal{P}$, then $A^\star = \bigcap_{P \in \mathcal{P}'} AR_P$ for each nonzero fractional ideal $A$ of $R$.
   (b) Each proper $t$-ideal of $R$ is contained in some $P \in \mathcal{P}$.
   (c) $\star$-Max($R$) = $\mathcal{P}'$.
   (d) If ht($P$) = 1 for each $P \in \mathcal{P}$ and $Q$ denotes the set of height-one primes of $R$, then $\mathcal{P} = t$-Max($R$) = $\star$-Max($R$) = $Q$.

Proof. Statements (1, 2, 3, 4) are in [2]. For (5a), Zorn’s lemma applies since the union $P$ of a chain of elements of $\mathcal{P}$ satisfies $P^\star = P$ and by (2) $P \subseteq Q$ for
some $Q \in \mathcal{P}$. The "hence" statement follows easily. Statement (5b) follows from (2) in view of the fact that a proper $t$-ideal is also a proper $*$-ideal. For (5c), if $Q \in \text{Max}^*(R)$, then $Q \subseteq P$ for some $P \in \mathcal{P}'$ by (2). But then $Q = P'$ by (3). Hence $\text{Max}^*(R) \subseteq \mathcal{P}'$. The reverse inclusion is trivial. Finally, (5d) follows easily from (5a,b,c) and the fact that height-one primes are $t$-primes.

\begin{remark}
With the notation of Lemma 5.1, let $R$ be an almost Dedekind domain with exactly one non-invertible maximal ideal $M$, and let $\mathcal{P}$ denote the set of maximal ideals other than $M$. Then, as is well known, $\mathcal{P}$ is a defining family for $R$, but the associated star operation does not have finite type. Indeed, conclusions (5b,d) fail to hold in this case: $M$ is a $t$-ideal but $M \not\subseteq P$ for all $P \in \mathcal{P}$.

For our next result, recall that if $*$ is a finite-type star operation on a domain $R$, then $R$ is said to have $*$-dimension one if each maximal $*$-ideal of $R$ has height one.

\begin{theorem}
Let $R$ be a domain and $*$ a finite-type star operation on $R$. Assume that $R$ has $*$-dimension one and that $R$ is $*$-potent. Then $R$ has finite $*$-character.
\end{theorem}

\begin{proof}
Denote the set of maximal $*$-ideals of $R$ by $\{M_\gamma\}_{\gamma \in \Gamma}$. For each $\gamma$, choose a $*$-rigid ideal $I_\gamma$ contained in $M_\gamma$. Now suppose, by way of contradiction, that $a$ is a nonzero element of $R$ and $\Lambda$ is an infinite subset of $\Gamma$ with $a \in M_\lambda$ for $\lambda \in \Lambda$ and $a \notin M_\gamma$ for $\gamma \in \Gamma \setminus \Lambda$. For $\lambda \in \Lambda$, $R_{M_\lambda}$ is one-dimensional, and hence there is an element $s_\lambda \in R \setminus M_\lambda$ and a positive integer $n_\lambda$ for which $s_\lambda I_{M_\lambda}^{n_\lambda} \subseteq (a)$. By construction, $(a, \{s_\lambda\})^* = R$, whence $(a, s_1, \ldots, s_k)^* = R$ for some finite subset $\{1, \ldots, k\}$ of $\Lambda$. On the other hand, $(a, s_1, \ldots, s_k)I_{M_1}^{n_1} \cdots I_{M_k}^{n_k} \subseteq (a)$, and since $\Lambda$ is infinite, there is a maximal $*$-ideal $M \in \{M_\lambda\}_{\lambda \in \Lambda} \setminus \{M_1, \ldots, M_k\}$ with $a \in M$. However, $(a, s_1, \ldots, s_k) \not\subseteq M$ (since $(a, s_1, \ldots, s_k)^* = R$ and $I_j \not\subseteq M$ for $j = 1, \ldots, k$, the desired contradiction.
\end{proof}

The assumption on the $*$-dimension in Theorem 5.3 is necessary; for example, the Prüfer domain $\mathbb{Z} + X \mathbb{Q}[X]$ is $d$-(super) potent but does not have finite $d$-character (note that $d = t$ here).

Theorem 5.3 is, at first glance, a generalization of part of [4, Corollary 1.7], which states that a $t$-potent domain of $t$-dimension one has finite $t$-character. In fact, Theorem 5.3 actually follows from [4, Corollary 1.7]. Indeed, if $R$ is as in Theorem 5.3, then Lemma 5.1 shows that $t\text{-Max}(R) = \text{Max}^*(R)$. (However, it is not generally the case that $* = t$.) We have included the proof given above, since it seems much more conceptual than the one given in [4].

As mentioned in the paragraph following Proposition 1.4, it is possible to have finite-type star operations $*_1 \leq *_2$ on a domain $R$ with $R$ $*_1$-potent but not $*_2$-potent. Indeed, this phenomenon can occur in a 2-dimensional Prüfer domain. We are grateful to the referee for suggesting the following construction.

\begin{example}
Let $T$ be the absolute integral closure of $\mathbb{Z}[X]$. Since $\mathbb{Z}[X]$ is a Krull domain, $T$ is a Prüfer domain, as was shown by H. Prüfer [24] (see also the more recent paper by F. Lucius [22]). Moreover, it follows (Krull [21, Satz 9]) that, since $\mathbb{Z}[X]$ has $t$-dimension one, so does $T$. Now let $R$ be the localization of $T$ at a maximal ideal lying over $(2,X)$ in $\mathbb{Z}[X]$. Then $R$ is a (local and hence) $t$-potent domain of $t$-dimension one. However, $R$ does not have finite $t$-character (since the ring of algebraic integers does not have finite character [13, Proposition 42.8]), and hence $R$ is not $t$-potent by Theorem 5.3 (or [4, Corollary 1.7]).
In [13], Gilmer introduced the notion of sharpness. The definition amounts to the following. Call a maximal ideal $M$ of a domain $R$ \textit{sharp} if $\bigcap \{R_N \mid N \in \Max(R), N \neq M\} \nsubseteq R_M$, and call $R$ \textit{sharp} if each maximal ideal of $R$ is sharp. In [13] Gilmer focused on one-dimensional domains and proved that a sharp almost Dedekind is a Dedekind domain. (In a later paper, Gilmer and Heinzer [10] extended the ideas to higher dimensions, primarily in the setting of Prüfer domains.) The notion of sharpness was extended to star operations $\ast$ of finite type in [12] Remark 1.4: a maximal $\ast$-ideal $M$ of a domain $R$ is $\ast$-sharp if $\bigcap R_N \nsubseteq R_M$, where the intersection is taken over all maximal $\ast$-ideals $N \neq M$. Hence, for our purposes it is convenient to relabel “sharp” as ”$d$-sharp.” It is relatively easy to prove that a $t$-potent maximal $t$-ideal must be $t$-sharp (see below), but this cannot be extended to arbitrary finite-type star operations. In particular, it is not true for the $d$-operation, as can be seen by observing that maximal ideals of $k[x, y]$ are $d$-potent (as are the maximal ideals of any Noetherian domain) but are not $d$-sharp: if $M$ is maximal in $R := k[x, y]$ and $u \in \bigcap \{R_N \mid N \in \Max(R), N \neq M\}$, then we must have $u \in R$, lest $(R :_R Ru)$ be contained in a height one prime and hence in infinitely many maximal ideals.

**Proposition 5.5.** Let $R$ be a domain.

1. If $M$ is a $t$-potent maximal $t$-ideal of $R$, then $M$ is $t$-sharp.

2. If $\ast$ is a finite-type star operation on $R$ and $M$ is a $\ast$-super potent maximal $\ast$-ideal of $R$, then $M$ is $\ast$-sharp.

**Proof.** (1) This follows easily from [12] proof of Theorem 1.2]. Here is a direct proof: Choose a $t$-rigid ideal of $R$ contained in $M$. Since $I^\prime = I^\prime \subseteq M$, $I^{-1} \neq R$. Choose $u \in I^{-1} \setminus R$. Then $I \subseteq (R :_R R u)$, and hence $(R :_R R u) \nsubseteq N$ for each $N \in t\Max(R)$ with $N \neq M$. On the other hand, since $u \notin R$, we must have $(R :_R R u) \subseteq M$. Hence $u \in \bigcap \{R_N \mid N \in t\Max(R), N \neq M\} \setminus R_M$, as desired.

(2) Let $\ast$ be a finite-type star operation on $R$, $M$ be a $\ast$-super potent maximal $\ast$-ideal, and $I$ a $\ast$-super rigid ideal contained in $M$. Since $M$ is also a (maximal) $t$-ideal (Theorem 1.5], we have $I^{-1} \neq R$. Then, as in the proof of (1), if we choose $u \in I^{-1} \setminus R$, then $u \in \bigcap \{R_N \mid N \in t\Max(R), N \neq M\} \setminus R_M$. \hfill \Box

We observe that a $t$-sharp maximal $t$-ideal need not be $t$-potent ([12] Example 1.5]). To force $t$-sharpness to imply $t$-potency, we add a finiteness condition. Recall that a fractional $t$-ideal $I$ of a domain $R$ has \textit{finite type} if $I = J^\nu$ for some finitely generated fractional ideal $J$. We then say that $R$ is \textit{$v$-coherent} if $I^{-1}$ has finite type for each finitely generated fractional ideal $I$ of $R$. (The notion of $v$-coherence, with a different name, was introduced by El Abidine [8].) We then have from [12] Theorem 1.6] that a $v$-coherent $t$-sharp domain is $t$-potent. The next result is immediate.

**Corollary 5.6.** A $v$-coherent $t$-sharp domain of $t$-dimension one has finite $t$-character. \hfill \Box

The above-mentioned theorem of Gilmer follows easily:

**Corollary 5.7.** [13] Theorem 3] Let $R$ be a $d$-sharp almost Dedekind domain. Then $R$ is a Dedekind domain.

**Proof.** Any Prüfer domain is $v$-coherent. Moreover, the $d$- and $t$-operations coincide in a Prüfer domain. Hence $R$ has finite character by Corollary 5.6 and it is well known that this implies that $R$ is a Dedekind domain. \hfill \Box
We now turn to the characterization of generalized Krull domains. Since these domains are completely integrally closed, the next result will prove useful. (Recall that a domain $R$ with quotient field $K$ is completely integrally closed if, whenever $a \in R$ and $u \in K$ are such that $au^n \in R$ for each positive integer $n$, then $u \in R$.)

**Lemma 5.8.** Let $R$ be a completely integrally closed domain and $M$ a $t$-super potent maximal $t$-ideal of $R$. Then $ht(M) = 1$.

**Proof.** We proceed contrapositively. Suppose that $M$ is a $t$-super potent maximal $t$-ideal of $R$ and that $P$ is a nonzero prime properly contained in $M$. Choose a $t$-super rigid ideal $I \subseteq M$ with $I \nsubseteq P$. By Theorem 1.11, $P \subseteq \bigcap (I^n)^*$, and hence $\bigcap (I^n)^* \neq (0)$. Therefore, $R$ is not completely integrally closed by [5, Corollary 3.4]. □

Recall that a domain $R$ is a Prüfer $v$-multiplication domain (PvMD) if each nonzero finitely generated ideal of $R$ is $t$-invertible; it is well known that $R$ is a PvMD if and only if each maximal $t$-ideal of $R$ is essential (note that the set of maximal $t$-ideals is always a defining family).

**Theorem 5.9.** The following statements are equivalent for a domain $R$.

1. $R$ is a generalized Krull domain.
2. $R$ is a $t$-potent essential domain of $t$-dimension one.
3. $R$ is a $t$-potent PvMD of $t$-dimension one.
4. $R$ is a completely integrally closed $t$-super potent domain.
5. $R$ is a $t$-super potent domain of $t$-dimension one.

**Proof.** (1) $\Rightarrow$ (2): Assume (1), and let $P$ be a finite character defining family of height-one essential primes. By Lemma 5.1, $P$ is in fact the set of maximal $t$-ideals of $R$. (This also follows from [13, Corollary 43.9]). Hence $R$ has $t$-dimension one. Also, $R$ is $t$-potent since $P$ has finite character.

(2) $\Rightarrow$ (3): Assume (2), and let $P$ be a defining family of essential primes. For $P \in P$, $PR_P$ is a $t$-prime in the valuation domain $R_P$, and it is well known (see, e.g., [20, Lemma 3.17]) that this implies that $P$ is a $t$-prime of $R$. Then, since $R$ has finite $t$-character by Theorem 5.3, $P$ also has finite character and is therefore the entire set of $t$-primes (Lemma 5.1). Therefore, $R_P$ is a valuation domain for each $t$-prime $P$, that is, $R$ is a PvMD.

(3) $\Rightarrow$ (4): Let $R$ be a $t$-potent PvMD of $t$-dimension one. Then $R_P$ is a rank-one valuation domain for each $t$-prime $P$, and hence $R = \bigcap R_P$ is completely integrally closed. Also, since $R$ is $t$-potent and $t$-locally $d$-super potent, $R$ is $t$-super potent by Theorem 1.10.

(4) $\Rightarrow$ (5): This follows from Lemma 5.8.

(5) $\Rightarrow$ (1): Assume (5). Then $R$ has finite $t$-character by Theorem 5.3, and $R_M$ is a valuation domain for each maximal $t$-ideal $M$ by Corollary 1.12. Hence $R$ is a generalized Krull domain. □

One upshot of Theorem 5.9 is that a $t$-super potent domain of $t$-dimension one must be completely integrally closed. Note that without the restriction on the $t$-dimension, a $t$-super potent domain need not even be integrally closed (Example 4.6).

We close with a brief discussion regarding the connection between PvMDs and $t$-super potent domains. Observe that a $t$-potent PvMD is automatically $t$-super...
potent, but a P\text{vMD} need not be $t$-potent. (For example, a non-Dedekind almost Dedekind domain is a P\text{vMD} but is not $t$-potent (note that $d=t$ in this situation).) In a P\text{vMD}, all nonzero finitely generated ideals are $t$-invertible, while in a $t$-super potent domain one has $t$-invertibility only “above” $t$-super rigid ideals. Since, as is well known, if $I$ is a $t$-invertible ideal in a domain $R$, then both $I$ and $I^{-1}$ have finite type, a natural question arises: if $R$ is both $t$-super potent and $v$-coherent, must $R$ be a P\text{vMD}? Even if we add the condition that $R$ be integrally closed, the answer is “no,” as is shown by the ring $R := \mathbb{Z} + x\mathbb{Q}(z)[[x]]$ of Example 4.6 (with $z$ an indeterminate): $R$ is $t$-super potent by Theorem 4.4, is $v$-coherent by [10, Theorem 3.5], is integrally closed by standard pullback results, but is not a P\text{vMD} [9, Theorem 4.1].

We thank the referee for numerous suggestions that have greatly improved this paper.

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