LATTICE DELONE SIMPLICES WITH SUPER-EXPONENTIAL VOLUME

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ABSTRACT. In this short note we give a construction of an infinite series of Delone simplices whose relative volume grows super-exponentially with their dimension. This dramatically improves the previous best lower bound, which was linear.

BACKGROUND

Consider the Euclidean space $\mathbb{R}^d$ with norm $\| \cdot \|$ and a discrete subset $\Lambda \subset \mathbb{R}^d$. A $d$-dimensional polytope $L = \text{conv}\{v_0, \ldots, v_n\}$ with $v_0, \ldots, v_n \in \Lambda$ is called a Delone polytope of $\Lambda$, if there exists an empty sphere $S$ with $S \cap \Lambda = \{v_0, \ldots, v_n\}$. That is, if there is a center $c \in \mathbb{R}^d$ and a radius $r > 0$ such that $\|v_i - c\| = r$ for $i = 0, \ldots, n$, and $\|v - c\| > r$ for the remaining $v \in \Lambda \setminus \{v_0, \ldots, v_n\}$. If the Delone polytope is a simplex, hence $n = d$, we speak of a Delone simplex. Over the past decades Delone polytopes experienced a renaissance in applications like computer graphics and computational geometry, where they are traditionally called Delaunay polytopes due to the French transcription of Делоне.

Historically, Delone polytopes received attention in the study of positive definite quadratic forms (PQFs), in particular in a reduction theory due to Voronoi (cf. [Vor08]). To every $d$-ary PQF $Q$, a point lattice $\Lambda = AZ^d$ with $Q = A^t A$ is associated; $\Lambda$ is a discrete set and uniquely determined by $Q$ up to orthogonal transformations. A $d$-dimensional polytope $L = \text{conv}\{v_0, \ldots, v_n\}$, with $v_0, \ldots, v_n \in \mathbb{Z}^d$, is a Delone polytope of the lattice $\Lambda$ (and also called a Delone polytope of $Q$), if and only if there exists a $c \in \mathbb{R}^d$ and $r > 0$ with $Q[v - c] = \|A(v - c)\|^2 \geq r^2$ for all $v \in \mathbb{Z}^d$ and with equality if and only if $v = v_i$ for $i \in \{0, \ldots, n\}$. The set of all Delone polytopes forms a periodic face-to-face tiling of $\mathbb{R}^d$. It is called the Delone subdivision of $Q$. If all Delone polytopes are simplices we speak of a Delone triangulation. Delone subdivisions form a poset with respect to refinement, in which triangulations are maximally refined elements.

Two Delone polytopes $L$ and $L'$ are unimodularly equivalent, if $L = UL' + t$ for some unimodular transformation $U \in \text{GL}_d(\mathbb{Z})$ and a translation vector $t \in \mathbb{Z}^d$.
Voronoi \cite{Vor08} showed that, up to unimodular equivalence, there exist only finitely many Delone subdivisions in each dimension $d$. Delone simplices may be considered as “building blocks” in this theory and therefore their classification is of particular interest. For more information on the classical theory we refer the interested reader to \cite{SV05} or the original works of Voronoi \cite{Vor08} and Delone \cite{Del37}.

**Bounds for relative volume**

Let $\Lambda \subset \mathbb{R}^d$ be a lattice. A lattice simplex $\text{conv}\{v_0, \ldots, v_n\}$ is called unimodular if $\{v_0, \ldots, v_n\}$ is an affine basis of $\Lambda$. All unimodular simplices are unimodularly equivalent and, in particular, have the same volume. The relative volume (or normalized volume) of a lattice simplex $L$ is the volume of $L$ divided by that of a unimodular simplex of $\Lambda$, so that the relative volume equals $\text{vol}(L) \cdot \frac{d}{d!}$. Equivalently, it equals the index in $\Lambda$ of the sublattice affinely spanned by the vertices of $L$.

Clearly, the relative volume is an invariant with respect to unimodular transformations. Hence, in order to classify possible Delone simplices up to unimodular equivalence, a first question, already raised by Delone in \cite{Del37}, is what is the maximum relative volume $\text{mv}(d)$ of $d$-dimensional Delone simplices. The sequence $\text{mv}(d)$ is (weakly) increasing, since from a Delone simplex for a lattice $\Lambda \subset \mathbb{R}^d$ one can easily construct another of the same relative volume for the lattice $\Lambda \times \mathbb{Z} \subset \mathbb{R}^{d+1}$.

Voronoi knew that up to dimension $d = 4$ all Delone simplices have relative volume 1, while there are Delone simplices of volume 2 in $d = 5$. In \cite{Bar73} Baranovskii proved $\text{mv}(5) = 2$, and later Baranovskii and Ryshkov \cite{BR98} proved $\text{mv}(6) = 3$. Dutour classified all 6-dimensional Delone polytopes in \cite{Dut04}.

Ryshkov \cite{Rys76} was the first who proved that relative volumes of Delone simplices are not bounded when the dimension goes to infinity. More precisely, for every $k \in \mathbb{N}$ he constructed Delone simplices of relative volume $k$ in dimension $2k + 1$, establishing that $\text{mv}(d) \geq \lceil \frac{d-1}{2} \rceil$. This was recently improved to the still linear lower bound $\text{mv}(d) \geq d - 3$ by Erdahl and Rybnikov \cite{ER02}.

In this note we prove the following two lower bounds on $\text{mv}(d)$:

**Theorem 1.** For every pair of dimensions $d_1$ and $d_2$,

\[
\text{mv}(d_1 + d_2) \geq \text{mv}(d_1) \cdot \text{mv}(d_2).
\]

**Theorem 2.** For every dimension of the form $d = 2^n - 1$,

\[
\text{mv}(d) \geq (d + 1)^{(d+3)/2} / 4^d.
\]

Theorem 1 immediately implies exponential lower bounds on $\text{mv}(d)$. For example, $\text{mv}(5) = 2$ gives $\text{mv}(d) \geq 2^{\lfloor d/5 \rfloor} \sim 1.1487^d$. Even better, it is known that $\text{mv}(24) \geq 20480$. Indeed, the Delone subdivision of the Leech lattice, which was determined by Borcherds, Conway, Queen, Parker and Sloane \cite[Chapter 25]{CS88}, contains simplices of relative volume 20480 (the simplex denoted $a_1^{24}a_1$ in their classification). Therefore we obtain:
Corollary 1.

$$mv(d) \geq 20480^{d/24} \sim 1.5123^d.$$ 

Theorem 2 gives a much better lower bound asymptotically:

Corollary 2.

$$\log(mv(d)) \in \Theta(d \log d).$$

Proof. For the lower bound, let $2^n$ be the largest power of two that is smaller or equal to $d + 1$ (so that $2^n \geq (d + 1)/2$). Theorem 2, together with the monotonicity of $mv(d)$, gives:

$$mv(d) \geq mv(2^n - 1) \geq \frac{(2^n)(2^n + 2)/2}{4^{2^n - 1}} \in 2^{\Theta(n 2^n)} = 2^{\Theta(d \log d)}.$$ 

For the upper bound we use the following argument, which is Lemma 14.2.5 in [DL97] (attributed to L. Lovasz): Given a Delone simplex $L$ of some PQF, the volume of the centrally symmetric difference body $L - L = \{v - v' : v, v' \in L\}$ is $\text{vol}(L - L) = \left(\frac{d}{2d}\right) \text{vol}(L)$ (see [RS57]). The polytope $L - L$ does not contain elements of $L \setminus \{0\}$ in its interior. Thus, by Minkowski’s fundamental theorem (see [GL87], §5 Theorem 1) we know that $\text{vol}(L - L) \leq 2^d$. Putting things together we get

$$mv(d) \leq 2^{d d!} \left(\frac{d}{2e}\right)^d.$$ 

More precisely, the arguments in this proof say that

$$\frac{1}{4} \leq \liminf \frac{\log mv(d)}{d \log d} \leq \limsup \frac{\log mv(d)}{d \log d} \leq 1.$$ 

Similarly, Theorem 2 directly implies

$$\frac{1}{2} \leq \limsup \frac{\log mv(d)}{d \log d}.$$ 

We do not know whether $\lim \frac{\log mv(d)}{d \log d}$ exists.

PROOF OF THEOREM 1

Theorem 1 follows from the fact that an orthogonal product of simplices decomposes into simplices with relative volume being the product of the individual relative volumes. Let us be more precise: Let $L_1 = \text{conv}\{v_0, \ldots, v_{d_1}\} \subseteq \mathbb{R}^{d_1}$ be a Delone simplex of the lattice $\Lambda_1$ with relative volume $mv(d_1)$, and let $L_2 = \text{conv}\{w_0, \ldots, w_{d_2}\} \subseteq \mathbb{R}^{d_2}$ be a Delone simplex of the lattice $\Lambda_2$ with relative volume $mv(d_2)$. Then, the direct product $L_1 \times L_2 = \text{conv}\{(v_i, w_j) : i = 0, \ldots, d_1, j = 0, \ldots, d_2\} \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is a $(d_1 + d_2)$-dimensional Delone polytope of the lattice $\Lambda = \Lambda_1 \times \Lambda_2$. Let $\Lambda_1', \Lambda_2'$ and $\Lambda' = \Lambda_1' \times \Lambda_2'$ denote the lattices affinely generated by the vertices of $L_1$, $L_2$ and $L_1 \times L_2$, respectively. By the classical theory of Voronoi (see [SV05], Proposition 5.1 and Proposition 5.4) we know that by a suitable infinitesimal change of the PQF $Q$ that induces $\Lambda$ the Delone
polytope $L_1 \times L_2$ is triangulated into Delone simplices, which hence have vertices in the sublattice $\Lambda'$ (more precisely, in a perturbation of it). Since the index of $\Lambda'$ in $\Lambda$ is precisely $\text{mv}(d_1) \text{mv}(d_2)$, these Delone simplices have relative volume at least that number.

**Remark.** The product of two simplices $L_1$ and $L_2$ is a **totally unimodular** polytope, meaning that all the simplices spanned by a subset of its vertices have the same volume. In particular, the Delone simplices that we obtain in the last step of the proof have relative volume exactly $\text{mv}(d_1) \text{mv}(d_2)$ (cf., for example, [Hai91]). As a consequence, every triangulation of $L_1 \times L_2$ consists of exactly $\binom{d_1 + d_2}{d_1}$ simplices.

**Proof of Theorem 2**

Remember that a Hadamard matrix of order $d$ is a $d \times d$ matrix with elements $+1$ and $-1$ in which distinct columns are orthogonal. Hadamard matrices exist at least whenever $d$ is a power of two and conjecturally whenever $d$ is a multiple of four (cf. [CS88], Chapter 3, 2.13; for a recent survey on Hadamard matrices see [EK05]).

By multiplying columns with $\pm 1$ we normalize columns of a Hadamard matrix $H$ so that the entries of the first row are all $+1$. Then, from $H$ we get the $(d - 1) \times d$ matrix $\tilde{H}$ with elements $0$ and $1$ by deleting the first row and replacing $+1$ by $0$, and $-1$ by $1$. The columns of $\tilde{H}$ form the vertex set of a regular $(d - 1)$ simplex with edge length $\sqrt{d}/2$ and volume $\det(H)/2^{d-1} = d^{d/2}/2^{d-1}$. This is the maximum volume of a simplex contained in the $(d - 1)$ unit cube $[0,1]^{d-1}$. Such a simplex is called a $(d - 1)$ Hadamard simplex.

A standard construction for Hadamard matrices of order $2^n$ is as follows

$$H_1 = (1), \quad H_{2^n} = \begin{pmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H_{2^{n-1}} & H_{2^{n-1}} \end{pmatrix}.$$ 

Let $d = 2^n - 1$ and consider the matrix $\tilde{H}_{2^n}$ associated to the particular Hadamard matrix $H_{2^n}$ constructed this way. It is well known, and easy to see, that the columns of $\tilde{H}_{2^n}$ form an $n$-dimensional subspace of $\mathbb{F}_2^d$. This subspace is also known as the $[d,n]$ binary simplex code or the dual of the $[d,d-n]$ binary Hamming code.

Linearity implies that the following is a $d$-dimensional sublattice of $\mathbb{Z}^d$:

$$\Lambda(\tilde{H}_{2^n}) = \{(v_1, \ldots, v_d) \in \mathbb{Z}^d : (v_1 \mod 2, \ldots, v_d \mod 2) \in \tilde{H}_{2^n}\}.$$ 

(This procedure is “Construction A” in [CS88], Chapters 5 and 7). The lattice $\Lambda(\tilde{H}_{2^n})$ has determinant $2^{d-n}$ in $\mathbb{Z}^d$, since it contains $2^n$ of the $2^d$ vertices in every lattice unit cube. Moreover, the Hadamard simplex defined by $\tilde{H}_{2^n}$ is a Delone simplex in $\Lambda(\tilde{H}_{2^n})$ because the sphere around the unit cube does not contain more lattice points than the columns of $\tilde{H}_{2^n}$. The relative volume of the Hadamard simplex is

$$\frac{(d + 1)(d+1)/2}{2^{d-n}} = \frac{(d + 1)(d+3)/2}{4^d}.$$
REMARKS ON RELATED PROBLEMS

(1) The determination of \( mV(d) \) is clearly related (but not equivalent) to that of what is the minimal number of translational orbits of Delone simplices in Delone triangulations of \( d \)-dimensional lattices. Let us denote this number \( dt(d) \). Since the sum of volumes of representatives from each orbit must equal \( d! \), the upper bound for \( mV(d) \) implies

\[
dt(d) \geq \binom{2d}{d} 2^{-d} = \Omega(2^d / \sqrt{d}).
\]

For an upper bound, we can use essentially the same trick as in the proof of Theorem 1 to obtain:

**Proposition.** There is a constant \( c < 1 \) such that \( dt(d) \leq c^d d! \), for every \( d \geq 5 \).

**Proof.** Start with Delone triangulations with the minimal number of translational orbits \( dt(d_1) \) and \( dt(d_2) \) in dimensions \( d_1 \) and \( d_2 \), and consider the product. As before, a perturbation of the product lattice produces a Delone triangulation with \( \binom{d_1 + d_2}{d_1} dt(d_1) \ dt(d_2) \) translational orbits, from which we deduce

\[
\frac{dt(d_1 + d_2)}{(d_1 + d_2)!} \leq \frac{dt(d_1)}{d_1!} \cdot \frac{dt(d_2)}{d_2!}.
\]

Hence,

\[
\left( \frac{dt(d_1 + d_2)}{(d_1 + d_2)!} \right)^{\frac{1}{d_1 + d_2}} \leq \max \left\{ \left( \frac{dt(d_1)}{d_1!} \right)^{\frac{1}{d_1}}, \left( \frac{dt(d_2)}{d_2!} \right)^{\frac{1}{d_2}} \right\}.
\]

Since in every dimension \( d \geq 5 \) there are Delone simplices of volume greater than one, we have constants \( c_d < 1 \) such that \( dt(d) = c_d^d d! \). It suffices now to take as the global constant \( c \) for the statement the minimum of \( c_5, c_6, c_7, c_8 \) and \( c_9 \). \( \square \)

Of course, in order to get good (asymptotic) values of the constant in the statement, one needs to compute the number of translational orbits of Delone simplices in “good lattices”.

(2) A similar problem that has attracted attention both for theoretical and applied reasons is the determination of the minimum number of simplices in a triangulation of the \( d \)-dimensional cube \([0, 1]^d\). In fact, our proof of Theorem 1, as well as the upper bound in the previous remark, is essentially an adaptation of the technique used by Haiman [Hai91] to construct “simple and relatively efficient triangulations of the \( n \)-cube”.

The situation for this problem is that the best lower bound known for the minimum size \( t(d) \) of a triangulation of \([0, 1]^d\) is

\[
\sqrt{\frac{6}{d + 1}} \sim \sqrt{\frac{6}{\sqrt{2} (d + 1)^{2 + 1}}} \leq \frac{\sqrt{d}}{d!},
\]

where \( \sqrt{\frac{6}{d + 1}} \) is the best lower bound known for the minimum size \( t(d) \) of a triangulation of \([0, 1]^d\) is
obtained by Smith [Smi00]. The best (asymptotic) upper bound is

\[ \lim_{d \to \infty} \sqrt[\sqrt{d}] {\frac{t(d)}{d!}} \leq 0.816, \]

due to Orden and Santos [OS03]. For more information on this topic see [Zon05].

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REFERENCES

[Bar73] E.P. Baranovskii, Volumes of L-simplexes of five-dimensional lattices, Math. Notes 13 (1973), 460–466, translation from Mat. Zametki 13, 771–782 (1973).
[BR98] E.P. Baranovskii and S.S. Ryshkov, Repartitioning complexes in n-dimensional lattices (with full description for \( n \leq 6 \)), pages 115–124 in Voronoi’s Impact on Modern Science, Book II (P. Engel, H. Syta, eds.; Institute of Math., Kyiv 1998 = Vol.21 of Proc. Inst. Math. Nat. Acad. Sci. Ukraine).
[CS88] J.H. Conway and N.J.A. Sloane, Sphere packings, lattices and groups, Springer-Verlag, New York, 1988.
[Del37] B.N. Delone, The geometry of positive quadratic forms, Uspekhi Mat. Nauk 3 (1937), 16–62, in Russian.
[DL97] M.M. Deza and M. Laurent, Geometry of cuts and metrics, Springer-Verlag, Berlin, 1997.
[Dut04] M. Dutour, The six-dimensional Delaunay polytopes, European J. Combin. 25 (2004), 535–548.
[EK05] S. Eliahou, and M. Kervaire, A survey on modular Hadamard matrices, Discrete Mathematics 302:1–3 (2005), 85–106.
[ER02] R. Erdahl and K. Rybnikov, An infinite series of perfect quadratic forms and big Delaunay simplices in \( \mathbb{Z}^n \), Tr. Mat. Inst. Steklova 239 (2002), 170–178.
[GL87] P.M. Gruber and C.G. Lekkerkerker, Geometry of numbers, North–Holland, Amsterdam, 1987.
[Hai91] H. Haiman, A simple and relatively efficient triangulation of the \( n \)-cube, Discrete Comput. Geom. 6 (1991), 287–289.
[OS03] D. Orden and F. Santos, Asymptotically efficient triangulations of the \( d \)-cube, Discrete Comput. Geom., 30:4 (2003), 509–528.
[RS57] C.A. Rogers and G.C. Shephard, The difference body of a convex body, Arch. Math. 8 (1957), 220–233.
[Rys76] S.S. Ryshkov, The perfect form \( \Lambda_n^* \): the existence of lattices with a nonfundamental division simplex, and the existence of perfect forms which are not Minkowski-reducible to forms having identical diagonal coefficients, J. Sov. Math. 6 (1976), 672–676, translation from Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova 33, 65–71 (1973).
[SV05] A. Schürmann and F. Vallentin, Computational approaches to lattice packing and covering problems, Discrete Comp. Geom. (2005), to appear. cf. arXiv:math.MG/0403272.
[Smi00] W. D. Smith, A lower bound for the simplexity of the \( N \)-cube via hyperbolic volumes, in Combinatorics of convex polytopes (K. Fukuda and G. M. Ziegler, eds.), European J. Combin., 21:1 (2000), 131–137.
[Vor08] G.F. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Deuxième Mémoire, recherches sur les paralléloèdres primitifs., J. Reine Angew. Math. 134 (1908), 198–287, and 136 (1909), 67–181.
[Zon05] C. Zong, What is known about unit cubes, Bull. of the Amer. Math. Soc. 42:2 (2005), 181–211.
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