FACTORIZATION AND WEAK AMENABILITY OF $\mathcal{A}(X)$

Niels Grønbæk

Abstract. We investigate weak amenability of the Banach algebra $\mathcal{A}(X)$ of approximable operators on a Banach space $X$ and its relation to factorization properties of operators in $\mathcal{A}(X)$. We show that if $\mathcal{A}(X)$ is weakly amenable, then either $\mathcal{A}(X)$ is self-induced (a nice factorization property), or $X$ is very special, combining some of the exotic properties of the spaces of Gowers and Maurey [GM] and of Pisier [P1]. In the class of self-induced Banach algebras we show that weak amenability is preserved under an equivalence of Morita type. Using this we extend some results of A. Blanco [B1, B2] about weak amenability of $\mathcal{A}(X)$.

0. Introduction.

Recall that a Banach algebra $\mathfrak{A}$ is called weakly amenable if every bounded derivation $D: \mathfrak{A} \to \mathfrak{A}^*$ is inner, or equivalently if the first bounded Hochschild cohomology group $H^1(\mathfrak{A}, \mathfrak{A}^*)$ vanishes. Recently weak amenability has been investigated for algebras of the type $\mathcal{A}(X)$ for an infinite dimensional Banach space $X$. In [DGG] it is shown that $\mathcal{A}(X)$ is weak amenability when $X = \ell_p(Y)$ with $Y$ reflexive and having the approximation property, or when $X = E \oplus C_p$, where $E$ has the bounded approximation property and $C_p$ denotes any of the universal spaces introduced by W.B. Johnson in [Jo]. In [B2] Blanco introduces a technical property of $X$, so-called trace unbounded triples that allows for taking averages of matrix-like representations of a given finite rank operator. Using this Blanco establishes weak amenability of $\mathcal{A}(X)$ for a wide range of Banach spaces $X$. In [B1] Blanco studies hereditary properties of as well as necessary conditions for weak amenability for algebras $\mathcal{A}(X)$. In this, factorization properties play a crucial role.

In the present paper we shall take a approach almost exclusively related to factorization properties. We show that if $\mathcal{A}(X)$ is weakly amenable, then either $X$ is pathological (probably non-existing), or $\mathcal{A}(X)$ is so-called self-induced. Self-induced Banach algebras constitute the class of Banach algebras for which a Morita theory can naturally be developed. Hence our approach will be to transfer Hochschild cohomology from a few key examples by means of Morita equivalence, i.e. by means of factorization properties. In this way we give a unified approach to some of Blanco’s results with shorter and less technical proofs and in some cases improvements of the statements. We emphasise though, that our aim is to view the question of weak amenability from a more general stand i.e. that of Morita equivalence. Blanco’s results serve here as a source of test cases. In particular, the result [B2] on weak amenability of $\mathcal{A}(T)$ with $T$ the Tsirelson space, remains a challenge.

1991 Mathematics Subject Classification. Primary 47L10, 46M20; Secondary 16D90.

Key words and phrases. Weak amenability, factorization, approximable operators, Morita equivalence.

Typeset by A4S-TeX
1. Preliminaries. For Banach spaces $X$ and $Y$ we consider the following spaces of operators

\[ F(X,Y) = \{ \text{finite rank operators } X \to Y \} \]
\[ A(X,Y) = \{ \text{approximable operators } X \to Y \} \]
\[ N(X,Y) = \{ \text{nuclear operators } X \to Y \} \]
\[ I(X,Y) = \{ \text{integral operators } X \to Y \} \]
\[ B(X,Y) = \{ \text{bounded operators } X \to Y \} \]

As it is customary, we shall write $\text{Operators}(X)$ for $\text{Operators}(X,X)$. We write $| \cdot |_N$, $| \cdot |_I$, and $\| \cdot \|$ for the nuclear, integral, and uniform norm on $N(X,Y)$, $I(X,Y)$, and $B(X,Y)$ respectively. The identity operator on $X$ is denoted by $1_X$, or if the context is clear, simply by $1$.

For Banach spaces $E$ and $F$ their projective tensor product is denoted $E \hat{\otimes} F$.

The tensor algebra of $X$ is $X \hat{\otimes} X^*$ with multiplication given by

\[(x \otimes x^*)(\xi \otimes \xi^*) = x^*(\xi)x \otimes \xi^*, \quad x, \xi \in X; \quad x^*, \xi^* \in X^*.\]

The trace $\text{tr}: X \hat{\otimes} X^* \to \mathbb{C}$ and operator trace $\text{Tr}: X \hat{\otimes} X^* \to N(X)$ are given by

\[ \text{tr}(x \otimes x^*) = x^*(x), \quad \text{Tr}(x \otimes x^*)(\xi) = x^*(\xi)x, \quad x, \xi \in X, x^*, \xi^* \in X^*.\]

Note that $\text{Tr}$ maps onto $N(X)$.

For Banach spaces $X$ and $Y$ we denote the statement '$X$ is isomorphic to $Y$' by $X \cong Y$. For spaces in duality we shall use $\langle \cdot, \cdot \rangle$ to denote the corresponding bilinear form, in particular we shall write $\langle x, x^* \rangle = x^*(x)$ for $x \in X, x^* \in X^*$. We note in particular the trace duality

\[ \langle F, T \rangle = \text{tr}(F^*T), \quad F \in F(X), T \in B(X^*), \]

which isometrically identifies $(X \hat{\otimes} X^*)^*$ and $A(X)^*$ with $(B(X^*), \| \cdot \|)$ and $(I(X^*), | \cdot |_I)$ respectively.

For any normed space $E$ the unit ball is denoted by $E_1$.

Let $X_n$, $n \in \mathbb{N}$ be a sequence of Banach spaces. We denote the $\ell_p$-sums of this sequence by $(\oplus_1^\infty X_n)_p$ for $p = 0, 1 \leq p \leq \infty$. If $X_n = X$ for all $n \in \mathbb{N}$ we simply write $c_0(X)$, or $\ell_p(X)$, $1 \leq p \leq \infty$.

We shall frequently without further reference use the fact (see [D]) that for a Banach space $X$

\[ A(X) \text{ has a bounded left approximate identity } \iff \quad X \text{ has the bounded approximation property.} \]

Hence, if $X$ has the bounded approximation property, we may use Cohen factorization in the Banach algebra $A(X)$.

The definitions of Banach (co)homological concepts are standard and can be found for example in [H] and [J]. We shall only here point to
1.1 Definition. Let $\mathfrak{A}$ be a Banach algebra, let $X$ be a right Banach $\mathfrak{A}$-module, and let $Y$ be a left Banach $\mathfrak{A}$-module. We define

$$X \hat{\otimes} Y = X \hat{\otimes} Y/N,$$

where $\hat{\otimes}$ is the projective tensor product and $N = \text{clspan}\{x.a \otimes y - x \otimes a.y \mid x \in X, y \in Y, a \in \mathfrak{A}\}$. Thus, $X \hat{\otimes} Y$ is the universal object for linearizing bounded, $\mathfrak{A}$-balanced, bilinear maps $X \times Y \to Z$.

We start by recalling some facts about bounded derivations $D: \mathcal{A}(X) \to \mathcal{A}(X)^\ast$. As said above, we identify $\mathcal{A}(X)^\ast$ with $\mathcal{I}(X^\ast)$, and $(X \hat{\otimes} X^\ast)^\ast$ with $\mathcal{B}(X^\ast)$ via trace duality. Consider the diagram

$$
\begin{array}{ccc}
X \hat{\otimes} X^\ast & \xrightarrow{\text{Tr}} & \mathcal{A}(X) \\
\delta \downarrow & & \downarrow D \\
\mathcal{B}(X^\ast) & \xleftarrow{\text{Tr}^\ast} & \mathcal{I}(X^\ast)
\end{array}
$$

Since $X \hat{\otimes} X^\ast$ is biprojective [S] and in particular weakly amenable, the derivation $\delta$ is inner. Hence we have

1.2 Proposition. Let $D: \mathcal{A}(X) \to \mathcal{A}(X)^\ast$ be a bounded derivation. Corresponding to $D$ there is $T \in \mathcal{B}(X^\ast)$ such that

$$(F, D(G)) = \text{tr}((FG - GF)^\ast T), \quad F, G \in \mathcal{F}(X).$$

Consequently $D$ is inner if and only if $T \in \mathcal{I}(X^\ast) + \mathbb{C}1_{X^\ast}$, that is, if and only if there is $\lambda \in \mathbb{C}$ and $K > 0$ such that

$$|\text{tr}(F^\ast T - \lambda F^\ast)| \leq K\|F\|, \quad F \in \mathcal{F}(X).$$

2. Factorization properties and weak amenability.

An important aspect of Morita theory is to provide tools to compare homological properties of Banach algebras using ‘good factorization properties’. In this section we shall extract such factorization properties in order to compare $H^n(\mathfrak{A}, \mathfrak{A}^\ast)$ and $H^n(\mathfrak{B}, \mathfrak{B}^\ast)$ for Banach algebras $\mathfrak{A}$ and $\mathfrak{B}$. Our focus shall be on $n = 1$ and Banach algebras of the type $\mathcal{A}(X)$.

First we make precise what is meant by ‘good factorization’:

2.1 Definition. A Banach algebra $\mathfrak{A}$ is called self-induced if

$$\mathfrak{A} \cong \mathfrak{A} \hat{\otimes} \mathfrak{A}.$$ 

The Banach algebra $\mathcal{A}(X)$ factors approximately through $Y$, if

$$\mathcal{A}(X) \cong \mathcal{A}(Y, X) \hat{\otimes} \mathcal{A}(X, Y),$$

where the isomorphisms are implemented by multiplication.

The usefulness of these factorization properties is that one may define linear maps in terms of balanced bilinear maps. A key example is
\begin{theorem}
Suppose that $\mathfrak{A}$ is self-induced. Let $D: \mathfrak{A} \to \mathfrak{A}^*$ be a bounded derivation. Let $\mathfrak{B}$ be a Banach algebra which contains $\mathfrak{A}$ as a closed 2-sided ideal. Then $D$ may be extended to a bounded derivation $\tilde{D}: \mathfrak{B} \to \mathfrak{A}^*$.

\begin{proof}
Let $T \in \mathfrak{B}$ and consider the bilinear map $\Phi_T: \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ given by
\[ \Phi_T(a, b) = \langle a, D(bT) \rangle - \langle T a, D(b) \rangle, \quad a, b \in \mathfrak{A}. \]
Then $\Phi_T$ is balanced, i.e. $\Phi_T(ac, b) = \Phi_T(a, cb)$, so we may define
\[ \langle ab, \tilde{D}(T) \rangle = \Phi_T(a, b), \quad a, b \in \mathfrak{A}. \]
One checks that this defines a bounded derivation $\mathfrak{B} \to \mathfrak{A}^*$ extending $D$ (in fact the only possible such).

\begin{remark}
In the same way $D$ can be lifted to a derivation $\mathfrak{A} \to \mathfrak{B}^*$.
\end{remark}

\begin{example}
Assume that the multiplication $A(X) \hat{\otimes} A(X) \to A(X)$ is surjective. If $X$ in addition has the approximation property, then $A(X)$ is self-induced. Suppose namely
\[ \sum A_n B_n = 0 \]
with $A_n \to 0$ and $\sum \|B_n\| < \infty$, and let $\varepsilon > 0$. Since $X$ has the approximation property, we may choose $U \in A(X)$ so that
\[ \sup_n \|U A_n - A_n\| \leq \varepsilon. \]
Then
\[ \left\| \sum_{\mathcal{A}(X)} A_n \otimes B_n \right\| \leq \left\| \sum_{\mathcal{A}(X)} U A_n \otimes B_n \right\| + \sum \|A_n - U A_n\| \|B_n\| \leq 0 + \varepsilon \sum \|B_n\|. \]
Thus, an important case occurs, when $X$ has the bounded approximation property, using the bounded approximate identity in $A(X)$ ([D]).

The approximation property is not essential here as will be clear in the course of the paper. However, in the case of nuclear operators self-inducedness and the approximation property is one and the same thing.

\begin{proposition}
The Banach algebra $\mathcal{N}(X)$ is self-induced if and only if $X$ has the approximation property.

\begin{proof}
We define a bounded balanced bilinear form on $\mathcal{N}(X)$ by
\[ \phi(N, M) = \text{tr}(UV) \]
where $U, V \in X \hat{\otimes} X^*$ with $\text{tr}(U) = N$ and $\text{tr}(V) = M$. This is well defined, since if $\text{tr}(U) = 0$, then $U(X \hat{\otimes} X^*) = (X \hat{\otimes} X^*)U = \{0\}$. Suppose now that $\mathcal{N}(X)$ is
Lemma. Then $\phi$ defines a bounded linear functional on $\mathcal{N}(X)$ which agrees with the standard trace on $\mathcal{F}(X)$. But then $X$ must have the approximation property. Conversely, if $X$ has the approximation property, then $\mathcal{N}(X)$ is isometrically isomorphic to the tensor algebra $X \otimes X^*$. Since any rank one tensor $x \otimes x^*$ has a factorization $p(x \otimes x^*)$ with $p$ a norm 1 projection, it follows easily that $X \otimes X^*$ is self-induced.

In order to investigate the relation between derivations and self-inducedness we look at derivations of the following type.

Let $\phi \in \mathfrak{A} \hat{\otimes} \mathfrak{A}^*$. For convenience we shall use the same symbol $\phi$ for the corresponding balanced bilinear functional. Then one checks that

$$\langle a, D(b) \rangle := \phi(a, b) - \phi(b, a)$$

defines a bounded derivation $D: \mathfrak{A} \to \mathfrak{A}^*$.

How does this look in the setting of $\mathfrak{A} = \mathcal{A}(X)$? First we need to describe balanced bilinear functionals.

2.6 Lemma. Let $\phi \in \mathcal{A}(Y, X) \hat{\otimes} \mathcal{A}(X, Y)^*$. Then there is $T \in \mathcal{B}(X^*)$ such that

$$\phi(F, G) = \text{tr}((FG)^*T) \quad F \in \mathcal{F}(Y, X), G \in \mathcal{F}(X, Y)$$

and

$$\sup\{\|\text{tr}((FG)^*T)\| : F \in \mathcal{F}(Y, X), G \in \mathcal{F}(X, Y)\} < \infty.$$
is surjective. Let $\phi \in (\mathcal{A}(Y, X) \hat{\otimes} \mathcal{A}(X, Y))^*$ with corresponding $T \in \mathcal{B}(X^*)$. The assumption ensures that

$$f(\sum U_i V_i) = \text{tr}(\sum U_i V_i)^* T)$$

defines a bounded functional, so that the dual map

$$\mu^*: \mathcal{A}(X)^* \to (\mathcal{A}(Y, X) \hat{\otimes} \mathcal{A}(X, Y))^*$$

is also surjective, i.e. $\mu$ is an isomorphism.

In Section 4 of [B1] Blanco discusses necessary conditions for weak amenability. He shows that if $\mathcal{A}(X)$ is weakly amenable, then either $X$ is indecomposable (i.e. $X$ is not the direct sum of two infinite-dimensional Banach spaces) or the trace defines an unbounded bilinear map associated with a decomposition. These considerations are naturally further explored by means of self-inducedness. The Banach spaces of Pisier [P1] for which $\mathcal{A}(X) = \mathcal{N}(X)$ are crucial in this. We note some simple reformulations of this property. But first we need the following estimate of norms, which is essentially an elaboration of the proof of [DU, Theorem VIII.4.12]

2.7 Lemma. Let $T \in \mathcal{I}(X, Y), S \in \mathcal{B}(Y, Z)$. If $S$ is weakly compact, then

$$|ST|_{\mathcal{N}} \leq \|S\| |T|_{\mathcal{I}}$$

Proof. That $ST$ is nuclear is the statement of [DU, Theorem VIII.4.12.(i)]. We note from the proof of this, that $S$ being weakly compact, there is a reflexive space $W$ and operators $A \in \mathcal{B}(Y, W)$ and $B \in \mathcal{B}(W, Z)$ such that $S = BA$ by [DU, Corollary VIII.4.9]. Furthermore, a close inspection of the proof shows that $\|S\| = \inf \{\|B\| \|A\|\}$, where the infimum is taken over such factorizations. Reasoning along with [DU] we get

$$|AT|_{\mathcal{N}} = |AT|_{\mathcal{I}} \leq \|A\| |T|_{\mathcal{I}},$$

so that

$$|ST|_{\mathcal{N}} = |BAT|_{\mathcal{N}} \leq \|B\| |AT|_{\mathcal{N}} \leq \|B\| \|A\| |T|_{\mathcal{I}},$$

Taking the infimum over $\|A\| \|B\|$ gives the wanted estimate.

2.8 Proposition. Let $X$ be a Banach space. Then $(i) \implies (ii) \iff (iii) \implies (iv)$, where

(i) $\mathcal{A}(X) = \mathcal{N}(X)$

(ii) $\mathcal{A}(X) \subseteq \mathcal{I}(X)$

(iii) There is $C > 0$ such that

$$|\text{tr}(AB)| \leq C \|A\| \|B\|, \quad A, B \in \mathcal{F}(X)$$

(iv) the multiplication $\mathcal{A}(X) \hat{\otimes} \mathcal{A}(X) \to \mathcal{A}(X)$ maps onto $\mathcal{N}(X)$. 
In particular, if the multiplication \( A(X) \hat{\otimes} A(X) \to A(X) \) is known to be surjective, then all four are equivalent.

**Proof.** Since in general \( \mathcal{N}(X) \subseteq I(X) \), (i) \( \implies \) (ii) is obvious.

(iii) \( \implies \) (ii): An application of the closed graph theorem shows that the inclusion \( (A(X), \| \cdot \|) \to (I(X), |\cdot|_Z) \) is continuous, thus providing \( C > 0 \) so that \( |A|_Z \leq C\|A\| \) and hence \( |A^*_Z| \leq C\|A\| \) for all \( A \in A(X) \), since \( |A|_Z = |A^*_Z| \) ([DJT, Theorem 5.15]). This gives

\[
|\tr(AB)| = |\tr(B^*A^*)| \leq |A^*_Z|B|B\| \leq C\|A\||B||B\|
\]

for all \( A, B \in \mathcal{F}(X) \).

(iv): By Lemma 2.7 we have for \( A, B \in \mathcal{F}(X) \)

\[
|AB|_\mathcal{N} \leq \|A\|\|B\| \leq C\|A\||B||B||B\|
\]

Hence if \( A = \sum A_n B_n \) with \( A_n, B_n \in A(X) \), \( \sum \|A_n\||B_n\| < \infty \) we have that

\[
\sum |A_n B_n|_\mathcal{N} \leq C \sum \|A_n\||B_n\| < \infty
\]

so that the series is absolutely convergent in the nuclear norm and thus \( A \in \mathcal{N}(X) \). Since the multiplication \( \mathcal{N}(X) \hat{\otimes} \mathcal{N}(X) \to \mathcal{N}(X) \) always is surjective, we arrive at (iv).

We shall now show that weak amenability of \( A(X) \) forces either \( A(X) \) to be self-induced or the underlying space \( X \) to be very peculiar, combining some of the pathological properties of the spaces of Pisier [P1] and Gowers and Maurey [GM].

**2.9 Theorem.** Suppose that \( A(X) \) is not self-induced. Then \( A(X) \) is weakly amenable if and only if both (a) and (b) hold, where

(a) \( A(X) = \mathcal{N}(X) \)

(b) The kernel, \( K \), of the operator trace \( \Tr: X^* \hat{\otimes} X \to \mathcal{N}(X) \) is 1-dimensional.

If (a) and (b) hold then

(c) We have

\[
\mathcal{B}(X) = \mathcal{I}(X) \oplus \mathbb{C}1_X \quad \text{and} \quad \mathcal{B}(X^*) = \mathcal{I}(X^*) \oplus \mathbb{C}1_{X^*}.
\]

(d) If \( X = Y \oplus Z \), then either \( Y \) or \( Z \) is finite dimensional (i.e. \( X \) is indecomposable). Similarly, \( X^* \) is indecomposable.

**Proof.** Suppose that \( A(X) \) is weakly amenable. First we note that multiplication is surjective, since the map \( A(X) \hat{\otimes} A(X) \to A(X): F \otimes G \mapsto FG - GF \) has closed range, and \( A(X) \) has no bounded traces. Hence all four conditions of Proposition 2.8 are equivalent. Now let \( \phi \in (A(X) \hat{\otimes} A(X))^* \) and let \( T \phi \in \mathcal{B}(X^*) \) be the corresponding linear operator according to Lemma 2.6. Since \( A(X) \) is weakly
amenable it follows from the paragraph preceding the lemma, that there is an integral operator $T \in \mathcal{I}(X^*)$ and $\lambda \in \mathbb{C}$ so that

$$T_\phi = T + \lambda 1_{X^*}.$$ 

If $\mathcal{A}(X)$ is not self-induced we may choose $\phi$ so that $\lambda \neq 0$. In this case we have

$$|\text{tr}(AB)| = |\text{tr}(AB)^*| \leq C \|A\| \|B\|, \quad A, B \in \mathcal{F}(X)$$

for appropriate $C > 0$. The statement (a) now follows from Proposition 2.7(i).

In general $\mathcal{N}(X)$ is weakly amenable if and only if $\dim K \leq 1$ ([G1]). Noting that $X$ does not have the approximation property (if it were so, $\mathcal{A}(X)$ would be self-induced, cf. Example 2.3) we arrive at (b). (Recall that $K = \{0\} \iff X$ has the approximation property.)

Setting $K = Cu$ with $u = \sum x_n \otimes x_n^*$ and $\text{tr} u = \sum < x_n, x_n^* > = 1$ the functional $\varphi : \mathcal{B}(X^*) \to \mathbb{C}$ given by

$$\varphi(T) = \sum < x_n, T(x_n^*) >, \quad T \in \mathcal{B}(X^*)$$

is multiplicative with kernel $\mathcal{I}(X^*)$ (see the proof of Corollary 4 of [G1]). Using the trace duality between $X \hat{\otimes} X^*$ and $\mathcal{B}(X^*)$ we find that

$$(*) \quad (\mathcal{N}(X))^* = K^\perp = \ker \varphi$$

Hence, when (a) holds, we get $\ker \varphi = \mathcal{I}(X^*)$, thus proving the last equality in (c). The first equality follows by means of the multiplicative linear functional $T \mapsto \varphi(T^*)$ on $\mathcal{B}(X)$ and the fact that $T$ is integral if and only if $T^*$ is integral [DJT, Theorem 5.15].

To prove (d) first note that if $P \in \mathcal{B}(X^*)$ is a projection then $\varphi(P)$ is either 1 or 0. A simple applications of the closed graph theorem gives that the integral and uniform norms are equivalent on $\mathcal{I}(X^*)$. Accordingly there is a constant $C > 0$ so that

$$(**) \quad |\text{tr}(A^*T) - \varphi(T) \text{tr} A| \leq C \|A\|, \quad A \in \mathcal{F}(X), \; T \in \mathcal{B}(X^*).$$

If $P \in \mathcal{B}(X)$ is a projection with rank $P = \infty$, we may for each $n \in \mathbb{N}$ choose a projection $Q_n \in \mathcal{F}(X)$ with

$$\text{tr} Q_n = n, \|Q_n\| \leq n^{\frac{1}{2}} , \quad \text{and} \quad P Q_n = Q_n ,$$

cf. [P1, Theorem 1.14]. If this goes along with $(**)$ we must have $\varphi(P^*) = 1$. Thus, if $X$ were decomposable, we would have $2 = 1$. If $P' \in \mathcal{B}(X^*)$ is a projection of infinite rank, choose projections $Q'_{*n} \in \mathcal{F}(X^*)$ as above. We may use local reflexivity to modify the $Q'_{*n}$ to obtain projections $Q_{*n} \in \mathcal{F}(X)$ such that

$$\text{tr} Q_{*n} = n, \|Q_{*n}\| \leq 2n^{\frac{1}{2}} , \quad \text{and} \quad P' Q_{*n} = Q_{*n}^* ,$$

so that also $X^*$ is indecomposable.
3. Weak amenability of self-induced Banach algebras. From now on we shall concentrate on self-induced Banach algebras. In order to compare cohomology of such we shall exploit the double complex of Waldhausen [DI]. First we consider the lower left hand corner of a general double co-complex in the first quadrant:

\[
\begin{array}{c}
0 \rightarrow M^{03} \rightarrow M^{02} \rightarrow M^{12} \rightarrow M^{01} \rightarrow M^{11} \rightarrow M^{21} \rightarrow M^{10} \rightarrow M^{20} \rightarrow M^{30} \\
\end{array}
\]

The upper indices are meant as coordinates in the first quadrant. We shall assume that the diagram is commutative. On the horizontal axis we define the cohomology \( H^1_h \) as kernel modulo image of \( - \rightarrow M^{0n+1} \rightarrow \).

The cohomology on the vertical axis, \( H^1_v \), is defined analogously. We want to compare \( H^1_h \) and \( H^1_v \). In essence this consists of showing that the associated spectral sequence collapses at appropriate \( E^2 \)-terms. However, we give a direct construction of a comparing map using an ad hoc diagram chase.

3.1 Lemma. Consider the diagram \( (D) \). If there is vertical exactness at coordinates \((1,1), (2,0), (3,0), (1,2), \) and \((2,1)\), then we may define a linear map

\[ \mathcal{D}: H^1_v \rightarrow H^1_h \]

such that

- If there is horizontal exactness at \((1,1)\) and \((0,2)\), then \( \mathcal{D} \) is injective.
- If there is horizontal exactness at \((1,2), (2,1), \) and \((0,3)\), then \( \mathcal{D} \) is surjective.

Proof. First we describe a procedure to associate a cocycle at \((2,0)\) to each cocycle at \((0,2)\). We adopt the convention that indices on cochains indicate belonging, i.e. \( m^{ij} \in M^{ij}, \mu^{ij} \in M^{ij} \) etc. Let \( m^{02} \) be a vertical cocycle. The numbers at the
arrows show the progression in the diagram chase.

\[
\begin{array}{c}
0 \\
\downarrow 1.
\end{array}
\quad
\begin{array}{c}
2 \\
\downarrow 4.
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow 5.
\end{array}
\quad
\begin{array}{c}
m^{02} \\
\downarrow 6.
\end{array}
\quad
\begin{array}{c}
m^{12} \\
\downarrow 7.
\end{array}
\quad
\begin{array}{c}
m^{21} \\
\downarrow 8.
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow 9.
\end{array}
\quad
\begin{array}{c}
m^{20} \\
\downarrow 10.
\end{array}
\quad
\begin{array}{c}
m^{30}
\end{array}
\]

The existence of \( m^{11} \) and \( m^{20} \) is due to vertical exactness. Since the vertical map at \((3,0)\) is injective, \( m^{30} = 0 \) so that \( m^{20} \) is a cocycle.

We next show that this actually gives a map into \( H^1_h \). Hence suppose that for the same \( m^{02} \) we have made different choices \( m^{11}_*, m^{20}_* \). Then

\[
\begin{array}{c}
0 \\
\downarrow 1.
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow 2.
\end{array}
\quad
\begin{array}{c}
m^{11} - m^{11}_* \\
\downarrow 3.
\end{array}
\quad
\begin{array}{c}
m^{21} - m^{21}_* \\
\downarrow 4.
\end{array}
\quad
\begin{array}{c}
\mu^{10} \\
\downarrow 5.
\end{array}
\quad
\begin{array}{c}
\mu^{20}
\end{array}
\]

Here \( \mu^{10} \) exists by vertical exactness at \((1,1)\). Since we have vertical injectivity at \((2,0)\), we must have \( \mu^{20} = m^{20} - m^{20}_* \), so \( m^{20} \) and \( m^{20}_* \) are cohomologous.

We next show that this map lifts to the desired map \( \mathcal{D} \). Hence assume that \( m^{02} \) cobounds vertically. Then the procedure gives

\[
\begin{array}{c}
m^{02} \\
\downarrow 1.
\end{array}
\quad
\begin{array}{c}
m^{12} \\
\downarrow 4.
\end{array}
\quad
\begin{array}{c}
m^{11} \\
\downarrow 5.
\end{array}
\quad
\begin{array}{c}
0
\end{array}
\]

i.e. coboundaries go to coboundaries.

Now assume that there is horizontal exactness at places \((1,1)\) and \((0,2)\) and that the procedure \( m^{02} \mapsto m^{20} \) has resulted in a coboundary. This is described in

\[
\begin{array}{c}
m^{11}_* \\
\downarrow 3.
\end{array}
\quad
\begin{array}{c}
m^{21} \\
\downarrow 4.
\end{array}
\quad
\begin{array}{c}
m^{10}_* \\
\downarrow 1.
\end{array}
\quad
\begin{array}{c}
m^{20} \\
\downarrow 2.
\end{array}
\]

\[
\begin{array}{c}
m^{11} \\
\downarrow 3.
\end{array}
\quad
\begin{array}{c}
m^{21} \\
\downarrow 4.
\end{array}
\quad
\begin{array}{c}
m^{10} \\
\downarrow 1.
\end{array}
\quad
\begin{array}{c}
m^{20}
\end{array}
\]

Elements from the procedure are unstarred $m^{ij}$'s. We next get
\[
\begin{array}{c}
\mu^{02} & \xrightarrow{5.} & m^{12} \\
\mu^{01} & \xrightarrow{3.} & m^{11} - m^{11}_* \xrightarrow{2.} 0 \\
\end{array}
\]

The arrow 1. is valid because $m^{11}_*$ is a vertical coboundary, and $\mu^{01}$ exists by horizontal exactness at (1,1). By horizontal injectivity at (0,2) we must have $\mu^{02} = m^{02}_0$, i.e. $m^{02}$ is a coboundary.

Finally assume horizontal exactness at (2,1) and (1,2) and let $m^{20}$ be a horizontal cocycle. Then we get the diagram
\[
\begin{array}{c}
m^{03} & \xrightarrow{12.} & 0 \\
m^{02} & \xrightarrow{9.} & m^{12} \xrightarrow{8.} 0 \\
m^{11} & \xrightarrow{5.} & m^{21} \xrightarrow{4.} 0 \\
m^{20} & \xrightarrow{1.} & 0 \\
\end{array}
\]

Here $m^{11}$ and $m^{02}$ exist due to horizontal exactness at (2,1) and (1,2). Since we have horizontal injectivity at (0,3) we get $m^{03} = 0$, altogether showing that the found $m^{02}$ is a cocycle and by the procedure is taken to the given $m^{20}$, i.e. $\mathcal{D}$ is surjective.

3.2 Remark. Note that in order to define the map $\mathcal{D}$ we did not use the assumptions of vertical exactness at place (1,2) in full. All that is needed, is that the cocycle $m^{12}$ corresponding to the cocycle $m^{02}$ is actually a coboundary.

We want to use Lemma 3.1 to establish instances of Morita invariance of Hochschild cohomology, essentially by refining the arguments in [G2]. The definition of Morita equivalence is usually given in terms of functors between categories of modules. We only need a slightly weaker concept (which in the case of Banach algebras with bounded one-sided approximate identities coincides with the full version of Morita equivalence).

3.3 Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be self-induced Banach algebras. Then $\mathfrak{A}$ and $\mathfrak{B}$ are $M$-equivalent, in symbols $\mathfrak{A} \sim_M \mathfrak{B}$, if there are bimodules $\mathfrak{A}P_{\mathfrak{B}}$ and $\mathfrak{B}Q_{\mathfrak{A}}$ and balanced pairings
\[
\begin{align*}
\langle \cdot \rangle : & \mathfrak{A}P_{\mathfrak{B}} \hat{\otimes} Q_{\mathfrak{A}} \to \mathfrak{A} \\
\langle \cdot \rangle : & \mathfrak{B}Q_{\mathfrak{A}} \hat{\otimes} P_{\mathfrak{B}} \to \mathfrak{B} ,
\end{align*}
\]
which are bimodule isomorphisms satisfying
\[
[p \hat{\otimes} q], p' = p, [q \hat{\otimes} p'], [q \hat{\otimes} p], q' = q, [p \hat{\otimes} q], p, q, p', q' \in \mathfrak{A}P_{\mathfrak{B}}, q, q' \in \mathfrak{B}Q_{\mathfrak{A}}.
\]
The double complex to which we shall apply the Theorem 3.1, is the dual complex of the Waldhausen double complex [DI]. We shall for short write \( P \) and \( Q \) instead of \( _\mathfrak{A}P_\mathfrak{B} \) and \( _\mathfrak{A}Q_\mathfrak{A} \). The lower left hand corner of the Waldhausen double complex is

\[
\begin{array}{cccc}
0 & \overset{\mathfrak{B}}{\otimes} & \overset{\mathfrak{B}}{\otimes} & \mathfrak{B} \\
\downarrow & & & \downarrow \\
0 & \overset{\mathfrak{B}}{\otimes} & \mathfrak{B} & \overset{\mathfrak{B}}{\otimes} Q \\
\downarrow & & & \downarrow \\
(W) & 0 & \overset{\mathfrak{B}}{\otimes} Q & \overset{\mathfrak{B}}{\otimes} Q_\mathfrak{A} \\
\downarrow & & & \downarrow \\
0 & \overset{\mathfrak{A}}{\otimes} & \mathfrak{A} & \overset{\mathfrak{A}}{\otimes} \mathfrak{A} \\
\downarrow & & & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

The complexes on the axes are the usual Hochschild complexes. The \( n \)th column is the complex \( _\mathfrak{A}P_\mathfrak{B}C_* (\mathfrak{B}, Q_\mathfrak{A} \otimes_\mathfrak{A} \mathfrak{A}^{(n-1)} ) \) where \( C_* (\mathfrak{B}, Q_\mathfrak{A} \otimes_\mathfrak{A} \mathfrak{A}^{(n-1)} ) \) is the normalized bar resolution of the left \( \mathfrak{B} \)-module \( Q_\mathfrak{A} \otimes_\mathfrak{A} \mathfrak{A}^{(n-1)} \). Similarly, the \( m \)th row is the complex \( _\mathfrak{A}Q_\mathfrak{A}C_* (\mathfrak{A}, P_\mathfrak{B} \otimes_\mathfrak{B} \mathfrak{B}^{(m-1)} ) \). For details, see [G2]. Concerning exactness we have

**3.4 Lemma.** Let \((W^*)\) be the dual double co-complex of \((W)\) and suppose that \( \mathfrak{A} \approx \mathfrak{B} \). Then there is vertical exactness at places \((n, i)\) for \( i = 0, 1 \) and \( n \geq 1 \) and horizontal exactness at places \((i, n)\) for \( i = 0, 1 \) and \( n \geq 1 \) in \((W^*)\). If \( \mathfrak{A} \) has a BAI, then columns of \((W^*)\) are acyclic except possibly on the vertical edge.

**Proof.** With minor modifications the proofs of [G2, Lemma 3.1] and [G3, Theorem 4.6] can be adapted to the present situation, so the reader is referred to these references.

Applying this to algebras of approximable operators we get

**3.5 Theorem.** Suppose that \( \mathcal{A}(X) \) and \( \mathcal{A}(Y) \) are self-induced and that \( \mathcal{A}(X) \overset{\sim}{\sim} \mathcal{A}(Y) \). If \( X \) has the bounded approximation property, then there is an injection

\[
\mathcal{H}^1 (\mathcal{A}(Y), \mathcal{A}(Y)^*) \to \mathcal{H}^1 (\mathcal{A}(X), \mathcal{A}(X)^*).
\]

In particular, if \( X \) has the bounded approximation property and \( \mathcal{A}(X) \) is weakly amenable, then \( \mathcal{A}(Y) \) is weakly amenable.

**Proof.** According to Lemma 3.4, the double co-complex \((W^*)\) corresponding to \( \mathfrak{A} = \mathcal{A}(X), \mathfrak{B} = \mathcal{A}(Y) \) satisfies the hypothesis of Lemma 3.1 to conclude injectivity.

**3.6 Remark.** From [G3, Corollary 4.9] it follows that if \( X \) and \( Y \) both have the bounded approximation property, then \( \mathcal{A}(X) \overset{\sim}{\sim} \mathcal{A}(Y) \) implies

\[
\mathcal{H}^n (\mathcal{A}(Y), \mathcal{A}(Y)^*) \cong \mathcal{H}^n (\mathcal{A}(X), \mathcal{A}(X)^*) \text{ for all } n \in \mathbb{N}.
\]
4. Some illustrative applications.

As mentioned in the introduction our approach will be to establish weak amenability for some key examples and then conclude weak amenability for other Banach algebras by means of the relation $\sim_M$. Towards this end we start by

4.1 Theorem. Let $X$ be an arbitrary Banach space. Then for any $1 \leq p < \infty$

$$H^1(B(\ell_p(X)), A(\ell_p(X)^*) = \{0\}.$$

Proof. Since $\text{cl } A(\ell_p(X))^2 = A(\ell_p(X))$ a derivation $D: B(\ell_p(x)) \to A(\ell_p(X))^*$ is given by its restriction to $A(\ell_p(X))$, that is, there is $T \in B(\ell_p(X)^*)$ such that

$$\langle A, D(S) \rangle = \text{tr}((SA - AS)^*T), \quad A \in F(\ell_p(X)), S \in B(\ell_p(X)).$$

We want to find $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is integral. We start by setting up some notation. We view $B(\ell_p(X))$ as consisting of infinite matrices with each entry an operator from $B(X)$. For $n \in \mathbb{N}$ we let $V_n$ and $H_n$ denote the left and right shifts by $n$ places. We let $\mathfrak{M} = \{W \in F(\ell_p(X)) \mid \exists n \in \mathbb{N}: WH_n = V_nW = 0\}$, i.e. $\mathfrak{M}$ is the dense subalgebra of $A(\ell_p(X))$ consisting of matrices of the form

$$W = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix},$$

where $W$ denotes a finite square matrix with entries from $F(X)$ and the 0's represent infinite 0-matrices of the appropriate size. Let $W \in \mathfrak{M}$ and choose a $(d \times d)$-matrix $W$ to represent $W$. For $N \in \mathbb{N}$ we write $\Delta_N,d(W)$ for the matrix obtained by repeating the matrix $W$ along the diagonal $N$ times, i.e.

$$\Delta_N,d(W) = \begin{pmatrix} W & & \\ & \ddots & \\ & & W \end{pmatrix}.$$

Note that

$$\| \begin{pmatrix} \Delta_N,d(W) & 0 \\ 0 & 0 \end{pmatrix} \| = \| W \|.$$

Note also that a given $W \in \mathfrak{M}$ can be represented by different matrices $W$, since we may add 0-rows and 0-columns. In order to prove that $T - \lambda I$ is integral it suffices to prove that there is a constant $C > 0$ such that

$$| \text{tr}(W^*(T - \lambda I)) | \leq C \| W \|, \quad W \in \mathfrak{M}.$$

Let $W \in \mathfrak{M}$. Then

$$| \text{tr}((W - H_nWV_n)^*T) | = | \langle H_nW, D(V_n) \rangle | \leq \| W \| \| D \|.$$

It follows that the sequence $(\text{tr}((H_nWV_n)^*T))$ is bounded. Let LIM be a Banach limit and define a linear functional (possibly unbounded) $f: \mathfrak{M} \to \mathbb{C}$ by

$$f(W) = \text{LIM}(\text{tr}((H_nWV_n)^*T)), \quad W \in \mathfrak{M}.$$
We now prove that $f(UW) = f(WU)$, $U, W \in M$. By including some 0-entries, if necessary, we may suppose that $U$ and $W$ are represented by matrices, $U$ and $V$ respectively, of equal size, say $d \times d$. Let for $n \in \mathbb{N}$ the 0-matrix of size $n \times n$ be denoted $0_n$, and consider the operators in $M$ given by the matrices

$$R_n(W) = \begin{pmatrix} 0_n & 0 & 0 & 0 \\ 0 & 0 & \Delta_{N,d}(W) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_n(U) = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{N,d}(U) \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\|R_n(W)\| = \|W\|$ and $\|S_n(U)\| = \|U\|$. From the identity

$$\langle R_n(W), D(S_n(U)) \rangle = \operatorname{tr} \left( \begin{pmatrix} 0_n & 0 & 0 & 0 \\ 0 & \Delta_{N,d}(WU) & 0 & 0 \\ 0 & 0 & \Delta_{N,d}(UW) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} T \end{pmatrix} \right)$$

and from translation invariance of the Banach limit we get

$$|f(WU - UW)| = \frac{1}{N} \operatorname{LIM}(R_n(W), D(S_n(U))) \leq \frac{1}{N} \|D\| \|W\| \|U\|.$$

Since $N$ is arbitrary, we arrive at $f(WU) = f(UW)$. It follows that there is $\lambda \in \mathbb{C}$ such that $f(W) = \lambda \operatorname{tr}(W)$. This is the $\lambda$ we are looking for:

$$\operatorname{tr} \left( \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}^* (T - \lambda 1) \right) = \operatorname{tr} \left( \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}^* (T - \lambda 1) \right) + \operatorname{tr} \left( \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* (T - \lambda 1) \right) = \operatorname{tr} \left( \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}^* T \right) + \operatorname{tr} \left( \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* T \right) - \lambda \operatorname{tr}(W).$$

Since

$$\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}$$

for an appropriate coordinate projection $P$, we get by taking LIM that

$$|\operatorname{tr}(W^*(T - \lambda 1))| = |\operatorname{tr} \left( \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}^* (T - \lambda 1) \right)| \leq \|D\| \|W\|,$$

which is want we wanted.
4.2 Corollary. Let $X$ be a Banach space. Then

$$
\mathcal{A}(\ell_p(X)) \text{ is weakly amenable} \iff \mathcal{A}(\ell_p(X)) \text{ is self-induced.}
$$

Proof. $\ell_p(X)$ is decomposable, so if $\mathcal{A}(\ell_p(X))$ is weakly amenable, then it is self-induced by Theorem 3.8. If $\mathcal{A}(\ell_p(X))$ is self-induced, then every derivation

$$
D: \mathcal{A}(\ell_p(X)) \to \mathcal{A}(\ell_p(X))^*
$$

can be extended to a derivation $\tilde{D}: \mathcal{B}(\ell_p(X)) \to \mathcal{A}(\ell_p(X))^*$, which by Theorem 4.1 is inner.

4.3 Remark. In the proof of Theorem 4.1 the only properties of $\ell_p(X)$ we used were (i): there is a constant $C > 0$ with $\|\Delta_{N,d}(W)\| \leq C\|W\|$ for all $W$, (ii): $\mathcal{M}$ is dense in $\mathcal{A}(\ell_p(X))$. The latter is equivalent to $\lim_n H_n AV_n = 0$ for all $A \in \mathcal{A}(\ell_p(X))$. Hence there are many other Banach spaces of sequences from $X$ for which the proof works, notably $c_0(X)$. However, in the present paper we shall only make use of the spaces $\ell_p(X)$, $1 \leq p < \infty$.

The next result concerning weak amenability of $\mathcal{A}(L_p(\mu, X))$ strengthens Theorem 4.1 of [B2] by weakening the hypothesis 'X has the bounded approximation property' to 'X has the bounded approximation property'. The data of the space $L_p(\mu, X)$ are a measure space $(\Omega, \Sigma, \mu)$ and a sequence $(\Omega_n)$ of pairwise disjoint sets in $\Sigma$ with $0 < \mu(\Omega_n) < \infty$ (to avoid simply dealing with the case $X \oplus \cdots \oplus X$). Without loss of generality we may further assume that $\mu$ is a probability measure, since every compact set in $L_p(\mu, X)$ has $\sigma$-finite support.

4.4 Theorem. Let $X$ be a Banach space with the bounded approximation property. Then $\mathcal{A}(L_p(\mu, X))$ is weakly amenable.

Proof. Since $X$ has the bounded approximation property, the same is true for the spaces $\ell_p(X)$ and $L_p(\mu, X)$. In particular $\mathcal{A}(\ell_p(X))$ is self-induced and therefore weakly amenable by Corollary 4.2. Thus we may prove the theorem by showing $\mathcal{A}(L_p(\mu, X)) \sim \mathcal{A}(\ell_p(X))$. First we give some notation and well-known facts. A mesh $\mathcal{m} = \{E_n \mid n \in \mathbb{N}\}$ is a partition $\Omega = \bigcup_{n=1}^{\infty} E_n$ into pairwise disjoint measurable sets. A mesh $\mathcal{m}$ defines a norm-1 projection $P_{\mathcal{m}} \in \mathcal{B}(L_p(\mu, \mathcal{m}))$ by the rule

$$
P_{\mathcal{m}}(f) = \sum_{E \in \mathcal{m}, \mu(E) \neq 0} \left( \frac{1}{\mu(E)} \int_E f \, d\mu \right) \xi_E
$$

The set {meshes} is ordered by refinement and $\lim_{m \to \infty} P_{\mathcal{m}} = 1_{\mathcal{A}(L_p(\mu, X))}$ uniformly on compacta. For a mesh-projection $P_{\mathcal{m}}$ the range is isometrically isomorphic to $\ell_\infty^\kappa(X)$, where $\kappa$ is the cardinality of $\{E \in \mathcal{m} \mid \mu(E) > 0\}$. In particular $L_p(\mu, X)$ has a complemented subspace isometric to $\ell_p(X)$, so that

$$
L_p(\mu, X) \cong L_p(\mu, X) \oplus \ell_p(X).
$$

Since $\mathcal{A}(\ell_p(X))$ has a bounded left approximate identity, it follows that $\mathcal{A}(\ell_p(X))$ factors approximately through $\mathcal{A}(L_p(\mu, X))$. To show that $\mathcal{A}(L_p(\mu, X))$ factors approximately through $\mathcal{A}(\ell_p(X))$ first note that $\{P_{\mathcal{m}} G \mid G \in \mathcal{A}(L_p(\mu, X))_{\mathcal{m}}, \mathcal{m} \text{ a mesh}\}$
is dense in $\mathcal{A}(L_p(\mu, X))_1$. It follows that each $A \in \mathcal{A}(L_p(\mu, X))_1$ is the sum of a series

$$A = \sum_{1}^{\infty} 2^{-n} P_n G_n,$$

where the $P_n$’s are mesh-projections and $\{G_n\} \subseteq \mathcal{A}(L_p(\mu, X))_1$. Identifying the ranges of mesh-projections with the appropriate $\ell_p(X)$-spaces we get

$$A = \sum_{1}^{\infty} 2^{-n} \tilde{P}_n \tilde{G}_n,$$

with $\tilde{P}_n \in B(\ell_p(X), L_p(\mu, X))_1, \tilde{G}_n \in \mathcal{A}(L_p(\mu, X), \ell_p(X))_1$. Since $\mathcal{A}(L_p(\mu, X))$ has a bounded left approximate identity, we conclude that $\mathcal{A}(L_p(\mu, X))$ factors approximately through $\mathcal{A}(\ell_p(X))$. Alltogether $\mathcal{A}(L_p(\mu, X)) \sim M \mathcal{A}(\ell_p(X))$.

In order to facilitate the use of factorization properties, the generalization given by Blanco in [B2] of Johnson’s $C_p$-spaces is very useful. We quote it here:

**4.5 Definition.** A Banach space $J$ is called a Johnson space if it has the form $(\oplus_{1}^{\infty} G_n)_p, p = 0, 1 \leq p < \infty$, where $(G_n)_n$ is a sequence of finite-dimensional Banach spaces such that for each $i \in \mathbb{N}$ the set $\{n \in \mathbb{N} | G_n \cong X_i \text{ isometrically}\}$ is infinite.

Let $J = (\oplus G_n)_p$ be a Johnson space. A Banach space $X$ is called a $J$-space if there is $\lambda \geq 1$ such that for every finite-dimensional subspace $E$ of $X$, there is a subspace $G$ of $X$ containing $E$ such that the Banach-Mazur distance $d(G, G_i) \leq \lambda$ for some $i \in \mathbb{N}$.

The usefulness of these notions lies in

**4.6 Proposition.** Let $J = (\oplus_{1}^{\infty} G_n)_p$ be a Johnson space, and let $X$ be a $J$-space. Then $\mathcal{A}(J)$ is weakly amenable, and $\mathcal{A}(X)$ factors approximately through $\mathcal{A}(J)$.

**Proof.** It is an immediate consequence of Corollary 4.2 that $\mathcal{A}(J)$ is weakly amenable. Let $A \in \mathcal{F}(X)$ and choose $\text{range}(A) \subseteq G$ and corresponding $G_i$ in accordance with the definition of $X$ being a $J$-space. This gives a factorization

$$G_i \overset{i}{\longrightarrow} J$$

$$V \overset{}{\downarrow} \quad \quad \downarrow U$$

$$X \overset{A}{\longrightarrow} X$$

with $\|U\|\|V\| \leq \lambda\|A\|$. The claim now follows from Lemma 2.6.

The next result is Proposition 3.3 of [B2].

**4.7 Proposition [Blanco].** Let $J = (\oplus_{1}^{\infty} G_n)_p$ be a Johnson space, and let $X$ be a $J$-space. Then $\mathcal{A}(X \oplus J)$ is weakly amenable.

**Proof.** Since $J$ has the bounded approximation property we obviously have that $\mathcal{A}(J)$ factors through $\mathcal{A}(X \oplus J)$ and from Proposition 4.2 it follows that $X \oplus J$ factors approximately through $J$, i.e. $\mathcal{A}(X \oplus J) \sim M \mathcal{A}(J)$. Since $\mathcal{A}(J)$ is weakly amenable by Corollary 4.2, the proof is concluded by an appeal to Theorem 3.5.
As a final illustration, we shall look at the James spaces $\mathcal{J}_p$. Blanco [B2] shows that $\mathcal{A}(\mathcal{J}_p)$ is weakly amenable by showing that there is a Johnson space $J_p$ such that $\mathcal{J}_p$ is a $J_p$-space and $\mathcal{J}_p \cong \mathcal{J}_p \oplus J_p$, whence the result follows from Proposition 4.7. Using the relation $\sim$ makes it possible to extend this result to vector-valued James spaces. We start by briefly recalling basic properties of the spaces $\mathcal{J}_p$ following the notation of [B2]. Let $1 < p < \infty$ and let $(\alpha_n) \in \mathbb{C}^\mathbb{N}$. Define $\| \cdot \|_\mathcal{J}_p$ by

$$
\| (\alpha_n) \|_\mathcal{J}_p = \sup \{ \left( \sum_{n=1}^{m-1} |\alpha_{i_n} - \alpha_{i_{n+1}}|^p \right)^{\frac{1}{p}} \mid i_1 < \cdots < i_m, \ m \geq 2 \}.
$$

Then $\mathcal{J}_p = \{ (\alpha_n) \in \mathbb{C}^\mathbb{N} \mid \| (\alpha_n) \|_\mathcal{J}_p < \infty, \lim_n \alpha_n = 0 \}.$

With this norm $\mathcal{J}_p$ is a Banach space. The sequences $e_n = (\delta_{kn})_k$ form a normalized, 1-unconditional basis, $e$, for $\mathcal{J}_p$, the canonical basis. We now define vector valued James spaces. But we start with a general setting which is a special case of the spaces described in [Lau].

**4.8 Definition.** Let $E$ be a Banach space with a normalized 1-unconditional basis $b = \{ b_1, \ldots \}$, and let $X$ be any Banach space. Then we define $E \overline{\otimes}_b X$ as

$$
E \overline{\otimes}_b X = \{ (x_n) \in X^\mathbb{N} \mid \sum_{n=1}^{\infty} \| x_n \| b_n \in E \}
$$

with the norm $\| (x_n) \|_b = \| \sum_{n=1}^{\infty} \| x_n \| b_n \|$. The $X$-valued James $p$-space is $\mathcal{J}_p(X) = E \overline{\otimes}_b X$, where $e$ is the canonical basis. In this case we use the notation $\| \cdot \|_{\mathcal{J}_p(X)} = \| \cdot \|_e$.

It is straightforward to verify that $(E \overline{\otimes}_b X, \| \cdot \|_b)$ is a Banach space, which may be viewed as a completion of the algebraic tensor product $E \otimes X$, if we identify $(\sum_{n} \alpha_n b_n) \otimes x \in E \otimes X$ with $(\alpha, x) \in E \overline{\otimes}_b X$. Note that in this picture $\| \cdot \|_b$ is a cross-norm: $\| (\sum_{n} \alpha_n b_n) \otimes x \|_b = \| \sum_{n} \alpha_n b_n \| \| x \|$. In accordance with this picture consider $S \in \mathcal{B}(E)$, $T \in \mathcal{B}(X)$. If the linear map $S \otimes T : E \otimes X \rightarrow E \otimes X$ extends to a bounded operator $E \overline{\otimes}_b X \rightarrow E \overline{\otimes}_b X$, the latter will be denoted $S \overline{\otimes}_b T$.

The result above by Blanco is the case $X = \mathbb{C}$ of the following theorem.

**4.9 Theorem.** Let $X$ be a Banach space. If $X$ has the bounded approximation property, then the Banach algebra $A(\mathcal{J}_p(X))$ is weakly amenable for every $1 < p < \infty$.

**Proof.** For each $n \in \mathbb{N}$, define a closed subspace of $\mathcal{J}_p(X)$ by $\mathcal{J}_{p,n}(X) = \{ x_1 e_1 + \cdots + x_n e_n \mid x_1, \ldots, x_n \in X \}$. Let $(G_k)$ be a sequence of Banach spaces obtained by repeating each $\mathcal{J}_{p,n}(X)$ infinitely many times. Define a Banach space by $J_p(X) = (\oplus G_k)_p$. We prove that $A(\mathcal{J}_p(X)) \sim M A(J_p(X))$. Invoking Theorem 3.5 and Corollary 4.2, the claim follows, since both spaces $\mathcal{J}_p(X)$, $J_p(X)$ have the bounded approximation property and $\ell_p(J_p(X)) \cong J_p(X)$. Let $P_n : \mathcal{J}_p(X) \rightarrow J_p(X)$
be the canonical projection onto \( \mathcal{J}_{p,n}(X) \). Then \( P_n A \to A \) for all \( A \in \mathcal{A}(\mathcal{J}_p(X)) \). Each \( P_n \) having an obvious factorization

\[
P_n = \iota_n Q_n, \quad Q_n \in \mathcal{A}(\mathcal{J}_p(X), J_p(X)), \quad \iota_n \in \mathcal{A}(J_p(X), \mathcal{J}_p(X)), \quad \|Q_n\| = \|\iota_n\| = 1,
\]
it follows that each \( A \in \mathcal{A}(\mathcal{J}_p(X)) \) has a decomposition

\[
A = \sum_{1}^{\infty} T_n S_n, \quad S_n \in \mathcal{B}(\mathcal{J}_p(X), J_p(X)), T_n \in \mathcal{B}(J_p(X), \mathcal{J}_p(X)),
\]

with \( \sum \|S_n\||T_n\| \leq 2\|A\| \). Since \( \mathcal{J}_p(X) \) has the bounded approximation property, we may write \( A = A_1 A_2 A_3, \quad A_1, A_2, A_3 \in \mathcal{A}(\mathcal{J}_p(X)) \). It follows that \( \mathcal{A}(\mathcal{J}_p(X)) \) factors approximately through \( \mathcal{A}(J_p(X)) \).

To prove that \( \mathcal{A}(J_p(X)) \) factors approximately through \( \mathcal{A}(\mathcal{J}_p(X)) \) we just note that Blanco’s decomposition of \( \mathcal{J}_p \) works equally well for \( \mathcal{J}_p(X) \) with the same proof, so that we have \( \mathcal{J}_p(X) \cong J_p(X) \oplus J_p(X) \). Since \( J_p(X) \) has the bounded approximation property, we may factor as desired.

5. Conclusion. As demonstrated, many questions of weak amenability of Banach algebras (notably of the type \( \mathcal{A}(X) \)) can be approached using factorization of Morita equivalence type. This has been illustrated by giving a framework behind much of the reasoning in Blanco’s papers \([B1]\) and \([B2]\). We would like to raise some questions related to this.

5.1 Question. In Proposition 4.2 of \([B1]\) Blanco shows that if \( P \) is a Banach space such that \( P \) and \( P^* \) both have cotype 2, then \( \mathcal{A}(\ell_2(P)) \) is weakly amenable. Part of his argument consists in using a factorization theorem by Pisier (Theorem 4.1 of \([P1]\)), which combined with Lemma 2.6 shows that \( \mathcal{A}(\ell_2(P)) \) is self-induced. However, \( P \) can be chosen such that \( \mathcal{A}(P) \) is not weakly amenable.

In the same paper Blanco constructs a reflexive space \( E \) with an unconditional basis such that \( \mathcal{A}(E) \) is not weakly amenable. Since \( E \) has the bounded approximation property, \( \mathcal{A}(\ell_p(E)) \) is self-induced and hence weakly amenable for \( 1 \leq p < \infty \).

The spaces \( \ell_p(X) \) have the form \( \ell_p \mathcal{\hat{\otimes}} X \) and are tight tensor products in the sense of \([GJW]\). Thus we may view \( \mathcal{A}(\ell_p(X)) \) as a tensor product \( \mathcal{A}(\ell_p) \mathcal{\hat{\otimes}} \mathcal{A}(X) \). The preceding paragraphs can be phrased as a stabilizing effect of the functor \( \mathcal{A}(\ell_p) \mathcal{\hat{\otimes}} - \), in liking with stabilizing in the theory of \( C^* \)-algebras. This leads to

Is \( \mathcal{A}(\ell_2(X)) \) self-induced for all \( X \)? Is \( \mathcal{A}(\ell_p(X)), \quad 1 \leq p < \infty \)? A test case would be the space constructed in \([P2]\) for which multiplication is not surjective.

5.2 Question. Our reasoning has relied on matrix-structures with a certain uniformity loosely speaking enabling us to shift matrices around. Hence the Tsirelson space \( T \), for which Blanco established weak amenability of \( \mathcal{A}(T) \), presents a possible shortcoming. But in proving that \( \mathcal{A}(T) \) is weak amenability, it suffices to prove that \( \mathcal{A}(X) \) is weakly amenable for some ‘nice’ \( X \) such that \( \mathcal{A}(T) \mathcal{\sim}_M \mathcal{A}(X) \). This leads to

What are the spaces \( X \) such that \( \mathcal{A}(X) \) and \( \mathcal{A}(T) \) are Morita equivalent? Which among these have an unconditional basis?
REFERENCES

[B1] A. Blanco, On the weak amenability of \( A(X) \) and its relation with the approximation property (to appear).

[B2] ______, Weak amenability of \( A(E) \) and the geometry of \( E \), J. London Math. Soc. (2) 66 (2002), 721–740.

[DGG] H. G. Dales, F. Ghahramani, and N. Grønbæk, Derivations into iterated duals of Banach algebras, Studia Math. 128 (1998), 19–54.

[DI] K. Dennis and K. Igusa, Hochschild homology and the second obstruction for pseudo-isotopy, Lect. Notes in Math. 906, Springer Verlag, 1982, pp. 7–58.

[DJT] J. Diestel, H. Jarchow, and A. Tonge, Absolutely summing operators, Cambridge University Press, Cambridge, 1995.

[DU] J. Diestel and J. J. Uhl, Jr., Vector Measures, (Math. Surveys, No. 15), Amer. Math. Soc., Providence, R.I., 1977.

[D] P. G. Dixon, Left approximate identities in algebras of compact operators on Banach spaces, Proc. Royal Soc. Edinburgh 104A (1989), 169–175.

[GM] W.T. Gowers and B. Maurey, The unconditional basic sequence problem, J. Amer. Math. Soc. 6 (1993), 851–874.

[G1] N. Grønbæk, Amenability and weak amenability of tensor algebras and algebras of nuclear operators, J. Austral. Math. Soc., Ser. A 51 (1991), 483–488.

[G2] ______, Morita equivalence for Banach algebras, J. Pure Appl. Algebra 99 (1995), 183–219.

[G3] ______, Morita equivalence for self-induced Banach algebras, Houston J. Math. 22 (1996), 109–140.

[G,J&W] N. Grønbæk, B. E. Johnson and G. A. Willis, Amenability of Banach algebras of compact operators, Isr. J. Math. 87 (1994), 289–324.

[H] A. Ya. Helemskii, The homology of Banach and topological algebras, Kluwer, Dordrecht, 1986.

[J] B. E. Johnson, Cohomology in Banach algebras, Mem. Amer. Math. Soc. 127 (1972).

[Jo] W. B. Johnson, Factoring compact operators, Israel J. Math. 9 (1972), 337–345.

[Lau] N. J. Laustsen, Matrix multiplication and composition of operators on the direct sum of an infinite sequence of Banach spaces, Math. Proc. Camb. Phil. Soc. 131 (2001), 165–183.

[P1] G. Pisier, Factorization of Linear Operators and Geometry of Banach Spaces, (Regional Conference Series in Mathematics, No. 60), Amer. Math. Soc., Providence, Rhode Island, 1986.

[P2] ______, On a question of Niels Grønbæk, Math. Proc. R. Ir. Acad. 100A, No.1 (2000), 55-58.

[S] Yu. V. Selivanov, Biprojective Banach algebras, Math. USSR Izvestija 15 (1980), 387–399.

Department of Mathematics, Institute for Mathematical Sciences, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

E-mail address: gronbaek@math.ku.dk