Module Decompositions by Images of Fully Invariant Submodules

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Abstract. Let \( R \) be a ring with identity, \( M \) be a right \( R \)-module and \( F \) be a fully invariant submodule of \( M \). The concept of an \( F \)-inverse split module \( M \) has been investigated recently. In this paper, we approach to this concept with a different perspective, that is, we deal with a notion of an \( F \)-image split module \( M \), and study various properties and obtain some characterizations of this kind of modules. By means of \( F \)-image split modules \( M \), we focus on modules \( M \) in which fully invariant submodules \( F \) are dual Rickart direct summands. In this way, we contribute to the notion of a \( T \)-dual Rickart module \( M \) by considering \( Z_2(\mathit{M}) \) as the fully invariant submodule \( F \) of \( M \). We also deal with a notion of relatively image splitness to investigate direct sums of image split modules. Some applications of image split modules to rings are given.

1. Introduction

Throughout this paper \( R \) denotes an associative ring with identity and modules are unitary right \( R \)-modules unless otherwise stated. For a module \( M \), \( S = \text{End}_R(M) \) is the ring of all right \( R \)-module endomorphisms of \( M \) and \( F \) stands for a fully invariant submodule of \( M \) (i.e., \( f(F) \subseteq F \) for every \( f \in S \)). Maeda [8] and Hattori [5] studied Rickart rings (or principally projective rings), independently. A ring is called right Rickart if every principal right ideal is projective, equivalently, the right annihilator of any single element is generated by an idempotent as a right ideal. A left Rickart ring is defined similarly. Recently, the notion of Rickart rings was generalized to the module theoretic version and investigated in [1] and [6]. A module \( M \) is said to be Rickart if the right annihilator in \( M \) of any single element of \( S \) is generated by an idempotent of \( S \), that is, for any \( f \in S \), \( r_M(f) = \ker f = eM \) for some \( e^2 = e \in S \). In [2], a concept of \( T \)-Rickart modules was defined by considering the second singular (or Goldie torsion) submodule of a module, namely, a module \( M \) is called \( T \)-Rickart if \( \tau_M(f) = \{ m \in M \mid f(m) \in Z_2(M) \} \) is a direct summand of \( M \) for every \( f \in S \). On the other hand, in [15], a module \( M \) is said to be \( F \)-inverse split if \( f^{-1}(F) \) is a direct summand of \( M \) for every \( f \in S \). There are some interesting connections between these classes of modules. For example, in [15], it is proved that \( M \) is \( F \)-inverse split if and only if \( M \) has a decomposition \( M = F \oplus N \) where \( N \) is a Rickart module. Since the second singular submodule \( Z_2(M) \) of \( M \) is fully invariant in \( M \), being a \( T \)-Rickart module and being a \( Z_2(M) \)-inverse split module are the same. Some applications of the notion of an \( F \)-inverse split module \( M \) are presented in [4], [14], [15] and [16] by considering certain fully invariant submodules aside from the second singular submodule.

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As a dual version of Rickart property for modules, in [7], a module $M$ is called dual Rickart if $\text{Im} f$ is a direct summand of $M$ for every endomorphism $f$ of $M$. Motivated by the concepts of dual Rickart modules and $T$-Rickart modules, $T$-dual Rickart modules were introduced in [3], that is, a module $M$ is called $T$-dual Rickart if $f(\overline{Z}(M))$ is a direct summand of $M$ for every $f \in S$ where $\overline{Z}(M) = \bigcap \{\ker: f \in \text{Hom}_R(M,N) \text{ where } N \text{ is small in its injective hull} \}$ which was defined in [13]. With the inspiration of these works, it is of interest to present the notion of $F$-image split modules in a sense of a dual version of $F$-inverse split modules. We say that a module $M$ is $F$-image split if $f(F)$ is a direct summand of $M$ for every $f \in S$.

In the light of aforementioned concepts, it is a reasonable question that what kind of properties does $F$ gain when a module $M$ is splitted by the images of $F$? This question is one of the motivations to deal with the notion of an $F$-image split module $M$. We answer this question in Theorem 2.2, that is, $F$ becomes a dual Rickart module in addition to be a direct summand of $M$. The concept of $T$-dual Rickart modules produces dual Rickart modules by employing the submodule $\overline{Z}(M)$ of $M$. By using the fully invariant submodule $F$ of a module $M$, we produce much more dual Rickart modules for this general setting. Therefore, the concept of $F$-image splitness is more general than that of $T$-dual Rickart modules. These connections make the concept of an $F$-image split module $M$ more attractive to study.

In Section 2, we give some properties and characterizations of $F$-image split modules. We get some results by considering the singular submodule as a fully invariant submodule. We also deal with an $F$-image split module concept for rings and we present some applications about these rings. In Section 3, we focus on when the direct sums of $F$-image split modules $M$ satisfy the same property. In this direction, we study relatively $F$-image splitness. Lastly, in Section 4, we introduce strongly $F$-image split modules and observe a main characterization of these modules.

In what follows, $\text{Soc}(M)$ and $\overline{Z}(M)$ stand for the socle and the singular submodule of a module $M$, also, $J(R)$ denotes the Jacobson radical of a ring $R$, respectively. For a positive integer $n$, $M_n(R)$ denotes the ring of $n \times n$ matrices over a ring $R$.

2. $F$-image split modules

Throughout this paper, $F$ denotes a fully invariant submodule of a module $M$ under consideration. In this section we study the concept of an $F$-image split module $M$ and get properties about this class of modules. We investigate useful characterizations for this notion. Also, we obtain some results about the ring cases of $F$-image split modules as an application to the ring theory.

Definition 2.1. A module $M$ is called $F$-image split if $f(F)$ is a direct summand of $M$ for every $f \in S$.

It is clear that every semisimple module $M$ is $F$-image split and so every module $M$ over a semisimple ring is $F$-image split. Obviously, every module $M$ is 0-image split. It can be obtained from the definition, a module $M$ is dual Rickart if and only if it is $M$-image split.

We now give an efficient characterization for an $F$-image split module $M$. Thanks to this characterization we can get dual Rickart modules by means of fully invariant submodules.

Theorem 2.2. The following are equivalent for a module $M$.

1. $M$ is an $F$-image split module.
2. $F$ is a dual Rickart direct summand of $M$.

Proof. (2) $\Rightarrow$ (1) Let $f \in S$. As $F$ is a direct summand of $M$, there exists an idempotent $e \in S$ such that $F = eM$. Then, $\text{End}_R(F) = eS$. Since $F$ is dual Rickart, $ef(e)$ is a direct summand of $F$. We claim that $ef(e) = f(F)$. For any $x \in F$, $ef(x) = ef(x) = f(x)$. Therefore, $ef(e) = f(F)$. The rest is clear.

(1) $\Rightarrow$ (2) Let $M$ be $F$-image split. Then, for $1_{M} \in S$, $1_{M}(F) = F$ is a direct summand of $M$. Hence, $F = eM$ for some $e^2 = e \in S$. To see that $F$ is a dual Rickart module, let $f \in \text{End}_R(F) = eS$. Thus, there exists $g \in S$ such that $f = ege$. Since $M$ is $F$-image split, $g(F)$ is a direct summand of $M$. As $F$ is fully invariant, $f(F) = ege(F) = g(F)$. So $f(F)$ is a direct summand of $M$. By modularity condition, $f(F)$ is a direct summand of $F$ and so $F$ is a dual Rickart module. \qed
Corollary 2.3. Let $M$ be an $F$-image split module and $N$ a fully invariant submodule which contains $F$. If every endomorphism of $N$ can be extended to an endomorphism of $M$, then $N$ is $F$-image split.

Corollary 2.4. Every indecomposable $F$-image split module $M$ is either dual Rickart or $F = 0$.

The following corollary is a direct consequence of Corollary 2.4 if we consider the singular submodule as a fully invariant submodule.

Corollary 2.5. Every indecomposable $Z(M)$-image split module $M$ is either nonsingular or singular dual Rickart.

Proof. Let $M$ be an indecomposable $Z(M)$-image split module. By Corollary 2.4, $Z(M) = 0$, i.e., $M$ is nonsingular or $M$ is dual Rickart. In the latter case, $Z(M) = M$. \(\square\)

Example 2.6. Let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ where $F$ is a field. Then, $Z(R_\mathbb{R}) = 0$. Thus, $R_\mathbb{R}$ is $Z(R_\mathbb{R})$-image split.

We approach to Theorem 2.2 in terms of singular submodules.

Theorem 2.7. Let $M$ be a module. Then, the following are equivalent.

1. $M$ is $Z(M)$-image split.
2. $M = Z(M) \oplus N$ where $Z(M)$ is dual Rickart and $N$ is nonsingular.
3. $Z(M)$ is a dual Rickart direct summand of $M$.

Proof. (1) $\Rightarrow$ (2) By hypothesis and Theorem 2.2, $M$ has a decomposition $M = Z(M) \oplus N$ where $Z(M)$ is a dual Rickart module. Since $Z(M)$ is an essential submodule of $Z_2(M)$ and $Z_2(M) = Z(M) \oplus (Z_2(M) \cap N)$, we have $Z(M) = Z_2(M)$. Hence, $N$ is nonsingular since $N \cong M/Z_2(M)$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) $M$ is $Z(M)$-image split by Theorem 2.2. \(\square\)

In the next result we investigate that $F$-image split property for a module $M$ is transferred to direct summands of $M$.

Proposition 2.8. If $M$ is an $F$-image split module and $N$ is a direct summand of $M$, then $N$ is an $(F \cap N)$-image split module.

Proof. Assume that $M = N \oplus K$ for some submodule $K$ of $M$ and $M$ is $F$-image split. By [12], $F = (F \cap N) \oplus (F \cap K)$. Let $\varepsilon : M \to N$ be the canonical homomorphism and $g \in \text{End}_R(N)$. Since $M$ is an $F$-image split module, $g(F)$ is a direct summand of $M$. Also, $g(F) = g(F \cap N) + g(F \cap K) = g(F \cap N)$. As $g(F \cap N) = g(F \cap N) = g(F)$, $g(F \cap N)$ is a direct summand of $M$. Hence, $g(F \cap N)$ is a direct summand of $N$, as asserted. \(\square\)

Corollary 2.9. If $M$ is a $Z(M)$-image split module, then any direct summand $N$ of $M$ is $Z(N)$-image split.

Proof. Let $M$ be a $Z(M)$-image split module and $N$ a direct summand of $M$. Then, $N$ is $(N \cap Z(M))$-image split by Proposition 2.8. Hence, $N$ is $Z(N)$-image split since $N \cap Z(M) = N$. \(\square\)

Proposition 2.10. Let $M$ be a quasi-projective module. Then, $M$ is $F$-image split if and only if for every submodule $K$ of $M$ with $K \subseteq g(F)$ for each $0 \neq g \in \text{End}_R(M)$, $M/K$ is $F/K$-image split.

Proof. Let $M$ be $F$-image split and $f \in \text{End}_R(M/K)$. Since $M$ is a quasi-projective module, there exists $g \in \text{End}_R(M)$ such that the following diagram commutes. The module $M$ being $F$-image split implies that $M = g(F) \oplus L$ for some submodule $L$ of $M$. Then, $M/K = (g(F)/K) + ((L + K)/K)$ and this sum is direct since
\((g(F)/K) \cap ((L + K)/K) = [0 + K]\). Also, it can be shown that \(f(F/K) = g(F)/K\). Hence, \(f(F/K)\) is a direct summand of \(M/K\), and so \(M/K\) is \(F/K\)-image split.

The converse is obvious. □

Recall that \(M\) has the summand sum property (SSP) if the sum of two direct summands is a direct summand of \(M\). Also, \(M\) has the strong summand sum property (SSSP) if the sum of any number of direct summands is again a direct summand of \(M\).

**Proposition 2.11.** For an \(F\)-image split module \(M\), the following statements hold.

1. Let \(K, L\) be direct summands of \(M\) and \(K \subseteq F\). Then, \(K + L\) is a direct summand of \(M\).
2. \(M\) has SSP for direct summands which are contained in \(F\).

**Proof.** It is clear from the proof of [3, Proposition 3.14]. □

**Theorem 2.12.** The following are equivalent for a module \(M\).

1. \(M\) is \(F\)-image split.
2. \(\sum f(F)\) is a direct summand of \(M\) for every finitely generated right ideal \(I\) of \(S\).
3. \(\sum f(F)\) is a direct summand of \(M\) for every finite subset \(I\) of \(S\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(I =< f_1, \ldots, f_n >\) be a finitely generated right ideal of \(S\). Since \(M\) is \(F\)-image split, \(f_i(F)\) is a direct summand of \(M\) for each \(1 \leq i \leq n\). Hence, \(\sum f(F)\) is a direct summand of \(M\) by Proposition 2.11(2).

(2) \(\Rightarrow\) (1) For every \(f \in S\), \(\sum_{g \in I} g(F) = f(F)\) for which \(I = fS\). Hence, the proof is clear.

(1) \(\Leftrightarrow\) (3) It is obvious. □

Now we consider the concept of \(F\)-image splitness for rings. Note that \(I\) is an ideal of a ring \(R\) if and only if it is a fully invariant submodule of \(R_S\).

**Definition 2.13.** Let \(I\) be an ideal of a ring \(R\). Then, \(R\) is called right \(I\)-image split if for every \(f \in \text{End}_R(R_R)\), \(f(I)\) is a direct summand of \(R_R\), i.e., \(R\) is \(I\)-image split as a right \(R\)-module.

The left \(I\)-image splitness for a ring \(R\) can be defined similarly where \(I\) is an ideal of \(R\). The right \(I\)-image split rings need not be left \(I\)-image split as the following example shows, therefore being an \(I\)-image split ring is not left-right symmetric.

**Example 2.14.** Let \(R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}\) where \(F\) is a field. Consider the ideal \(I = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}\) of \(R\). Then, \(R = I \oplus J\) where \(J = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}\) is a right ideal of \(R\) and \(I\) is dual Rickart. Hence, \(R\) is a right \(I\)-image split. Since \(I\) is essential in \(R\) as a left ideal, it is not a direct summand of \(R\) as a left ideal. Therefore, \(R\) is not left \(I\)-image split.
In the next theorem, we characterize a right $I$-image split ring $R$.

**Theorem 2.15.** Let $R$ be a ring and $I$ be an ideal of $R$. Then, the following are equivalent.

1. For every positive integer $n$, $M_n(R)$ is right $M_n(I)$-image split.
2. $R$ is right $I$-image split.
3. $I$ is a direct summand of $R$ as a right ideal and $\text{End}_R(I)$ is a von Neumann regular ring.
4. For every $e^2 = e \in R$, $eRe$ is right $eIe$-image split.
5. For every finitely generated free $R$-module $M$, $M$ is $I$-image split.

**Proof.** (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (2) are clear.
(2) $\Rightarrow$ (3) Let $R$ be a right $I$-image split ring. By Theorem 2.2, $I$ is a direct summand of $R$ as a right ideal and it is also a dual Rickart module. Then, for every $f \in \text{End}_R(I)$, $\text{Im} f$ is a direct summand of $I$. Since $I/\text{Ker} f \cong \text{Im} f$ and $I$ is projective, $\text{Ker} f$ is a direct summand of $I$. Thus, $\text{End}_R(I)$ is a von Neumann regular ring by [17, Corollary 3.2].
(2) $\Rightarrow$ (1) Let $n$ be a positive integer. Since $I$ is a direct summand of $R$ as a right ideal, there exists a right ideal $J$ of $R$ such that $R = I \oplus J$. Hence, $M_n(I)$ is a right ideal of $M_n(R)$ such that $M_n(R) = M_n(I) \oplus M_n(J)$. Note that $\text{End}_{M_n(R)}(M_n(I)) \cong M_n(\text{End}_R(I))$ is a von Neumann regular ring because $\text{End}_R(I)$ is a von Neumann regular ring as in the proof of (2) $\Rightarrow$ (3). Thus, $M_n(R)$ is right $M_n(I)$-image split by [7, Theorem 3.8].
(2) $\Rightarrow$ (4) Let $R$ be right $I$-image split and $e^2 = e \in R$. Then, $eRe$ is $eIe$-image split. We claim that for every $f \in \text{End}_{R(eRe)}(eRe)$, $f(eIe)$ is a direct summand of $eRe$ as a right ideal. Since $\text{End}_{R(eRe)}(eRe) \cong eRe \cong \text{End}_R(eRe)$, $f(eIe)$ is a direct summand of $eRe$. Hence, there exists a right ideal $J$ of $eRe$ such that $eRe = f(eIe) \oplus J$. Thus, $eRe = f(eIe) + eIe$ and $f(eIe) \subseteq f(eIe) \cap eIe = f(0) = 0$. Therefore, $f(eIe)$ is a direct summand of $eRe$ as a right ideal.
(3) $\Rightarrow$ (5) Let $K$ be a finitely generated free $R$-module. By (3), $I$ is a direct summand of $R$ as a right ideal, and so $I$ is also a direct summand of $K$. Since $\text{End}_K(I)$ is von Neumann regular, $\text{Im} f$ is a direct summand of $I$ for every $f \in \text{End}_R(I)$ by [17, Corollary 3.2]. Hence, $I$ is dual Rickart. Thus, $K$ is $I$-image split by Theorem 2.2. □

**Theorem 2.16.** Let $R$ be a ring and $I$ be an ideal of $R$. Then, the following are equivalent.

1. $\bigoplus_{n=1}^{\infty} R_n$ is $(\bigoplus_{n=1}^{\infty} I_n)$-image split where $R_n = R$ and $I_n = I$ for all $n$.
2. $I$ is a direct summand of $R$ as a right ideal and $\text{End}_R(I)$ is a semisimple ring.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $\bigoplus_{n=1}^{\infty} R_n$ is $(\bigoplus_{n=1}^{\infty} I_n)$-image split where $R_n = R$ and $I_n = I$ for all $n$. By Proposition 2.8, $R$ is right $I$-image split. In particular, $I$ is a direct summand of $R$ as a right ideal. As in the proof of (2) $\Rightarrow$ (3) in Theorem 2.15, one can see that $\text{End}_R(\bigoplus_{n=1}^{\infty} I_n)$ is a von Neumann regular ring. Let $K = \text{End}_R(I)$. Note that $\text{End}_R(\bigoplus_{n=1}^{\infty} I_n) \cong \bigoplus_{n=1}^{\infty} (\bigoplus_{n=1}^{\infty} K_n)$, where $K_n = K$ for all $n$. By [17, Theorem 3.5], $K$ is a semisimple ring.
(2) $\Rightarrow$ (1) Suppose that $I$ is a direct summand of $R$ as a right ideal and $\text{End}_R(I)$ is a semisimple ring. Then, $\bigoplus_{n=1}^{\infty} I_n$ is a direct summand of $\bigoplus_{n=1}^{\infty} R_n$ where $I_n = I$ and $R_n = R$ for all $n$. Let $K = \text{End}_R(I)$. Since $\text{End}_{\bigoplus_{n=1}^{\infty} R_n}(\bigoplus_{n=1}^{\infty} I_n) \cong \bigoplus_{n=1}^{\infty} \text{End}_R(\bigoplus_{n=1}^{\infty} I_n)$ where $K_n = K$ for all $n$, and $K$ is a semisimple ring, we have that $\text{End}_R(\bigoplus_{n=1}^{\infty} I_n)$ is a von Neumann regular ring. Hence, $\text{Im} f$ is a direct summand of $\bigoplus_{n=1}^{\infty} I_n$ for every $f \in \text{End}_R(\bigoplus_{n=1}^{\infty} I_n)$ by [17, Corollary 3.2].
Thus, $\bigoplus_{n=1}^{\infty} I_n$ is a dual Rickart module. Consequently, $\bigoplus_{n=1}^{\infty} R_n$ is $(\bigoplus_{n=1}^{\infty} I_n)$-image split by Theorem 2.2. □

We close this section by giving some applications about $I$-image split rings $R$. 

T. Pekacar Calci et al. / Filomat 35:11 (2021), 3679–3687
Proposition 2.17. If $R$ is a right $Z(R_k)$-image split ring, then it is right nonsingular.

Proof. Let $R$ be a right $Z(R_k)$-image split ring and $x \in Z(R_k)$. Assume that $x \neq 0$ and we reach a contradiction. By definition, $xZ(R_k)$ is a direct summand of $R$. It entails that $xZ(R_k)$ has an idempotent $e$. Hence there exists $t \in Z(R_k)$ such that $e = xt$. Since $x, t \in Z(R_k)$, we have $e \in Z(R_k)$. This is the required contradiction since $r_R(e) = (1 - e)R$ is not essential in $R$. It follows $Z(R_k) = 0$. Therefore, $R$ is right nonsingular. □

Recall that in [10], a right module $M$ is called mininjective if for every simple right ideal $K$ of $R$, each homomorphism $K \to M$ extends to a homomorphism $R \to M$. The next result shows that every module over $Soc(\cdot)$-image split ring is mininjective.

Proposition 2.18. If $R$ is a right $Soc(R_k)$-image split ring, then every right $R$-module is mininjective.

Proof. Let $R$ be a right $Soc(R_k)$-image split ring. Then, $R = Soc(R_k) \oplus K$ for some right ideal $K$ of $R$. Hence, $J(R) = Rad(Soc(R_k)) \oplus Rad(K)$. This yields $J(R) = Rad(K)$ since $Rad(Soc(R_k)) = 0$. Thus, $Soc(R_k) \cap J(R) = 0$ and so every right $R$-module is mininjective by [11, Theorem 2.36]. □

3. Direct sums of $F$-image split modules

A direct sum of $F_i$-image split modules $M_i$ where $i \in I$ for some index set $I$ need not satisfy image splitness as shown in [3, Example 4.1]. In this section, we investigate under which conditions direct sums of $F_i$-image split modules $M_i$ have the same property.

Proposition 3.1. Let $\{M_i\}_{i \in I}$ be a class of $R$-modules for an arbitrary index set $I$. If for every $i \in I$, $M_i$ is a fully invariant submodule of $\bigoplus_{i \in I} M_i$, then $M_i$ is $F_i$-image split for every $i \in I$ if and only if $\bigoplus_{i \in I} M_i$ is $\bigoplus_{i \in I} F_i$-image split.

Proof. Let $\bigoplus_{i \in I} M_i$ be $\bigoplus_{i \in I} F_i$-image split. Then, by Proposition 2.8, $M_i$ is $F_i$-image split for every $i \in I$. For the necessity, let $M = \bigoplus_{i \in I} M_i$, $F = \bigoplus_{i \in I} F_i$, and $f = (f_i) \in S$ where $f_i \in \text{Hom}_R(M, M_i)$. Since for every $i \in I$, $M_i$ is a fully invariant submodule of $\bigoplus_{i \in I} M_i$, $\text{Hom}_R(M, M_i) = 0$ for every $i, j \in I$ with $i \neq j$. By hypothesis, $f_i(F_i)$ is a direct summand of $M_i$ for each $i \in I$. On the other hand, we have $f(F) = \bigoplus_{i \in I} f_i(F_i)$. Hence, $f(F)$ is a direct summand of $M$, as asserted. □

Recall that a module is said to be abelian if every idempotent element of its endomorphism ring is central. By the fact that a module $M$ is abelian if and only if every direct summand of $M$ is fully invariant in $M$, the following result is an immediate consequence of Proposition 3.1.

Corollary 3.2. Let $\{M_i\}_{i \in I}$ be a class of $R$-modules for an arbitrary index set $I$ and $\bigoplus_{i \in I} M_i$ be an abelian module. Then, $M_i$ is $F_i$-image split for all $i \in I$ if and only if $\bigoplus_{i \in I} M_i$ is $\bigoplus_{i \in I} F_i$-image split.

In the following, we introduce relatively $F$-image splitness in order to a more comprehensively study on direct sums of $F_i$-image split modules $M_i$ where $i \in I$ for some index set $I$.

Definition 3.3. A module $M$ is called $F$-image split module relative to $N$ (or shortly, $N$-$F$-image split) if for each $f \in \text{Hom}_R(M, N)$, $f(F)$ is a direct summand of $N$.

Theorem 3.4. Let $M$ and $N$ be $R$-modules. Then, $M$ is an $F$-image split module relative to $N$ if and only if for every direct summand $L$ of $M$ and every submodule $K$ of $N$, $L$ is $(L \cap F)$-image split relative to $K$. 

T. Pekacar Calci et al. / Filomat 35:11 (2021), 3679–3687
Proof. Let \( L \) be a direct summand of \( M, K \) a submodule of \( N \) and \( f \in \text{Hom}_R(L, K) \). Then, \( L = eM \) for some \( e \in I = 5 \) and \( f \in \text{Hom}_R(M, N) \). Since \( M \) is \( N \)-image split, \( fe(F) \) is a direct summand of \( N \). As \( fe(F) \subseteq K \), \( fe(F) \) is a direct summand of \( K \). We claim that \( fe(F) = f(L \cap F) \). To see that \( x \neq f(L \cap F) \), there exists \( y \in L \cap F \) such that \( f(y) = x \). Since \( y \in L = eM \), \( x = f(y) \in fe(F) \). Hence, \( f(L \cap F) \subseteq fe(F) \). It is clear that \( fe(F) \subseteq f(L \cap F) \). Thus, \( f(L \cap F) \) is a direct summand of \( K \), as asserted. The converse is clear. \( \square \)

The next result is obtained as an immediate consequence of Theorem 3.4 if we take into account of the fully invariant submodule \( Z(M) \) of a module \( M \).

**Corollary 3.5.** [3, Corollary 3.13] Let \( M \) be an \( R \)-module. Then, the following are equivalent.

1. \( M \) is \( T \)-dual Rickart.
2. For any submodule \( N \) of \( M \), each direct summand \( L \) of \( M \) is \( T \)-dual Rickart relative to \( N \).
3. If \( L \) and \( N \) are direct summands of \( M \), then for any \( \varphi \in \text{Hom}_R(L, N) \), \( \varphi(Z(M)) \) is a direct summand of \( N \).

**Proposition 3.6.** Let \( \{ M_i \}_{i \in I} \) be a class of \( R \)-modules for an index set \( I \) and \( N \) an \( R \)-module with a fully invariant submodule \( F \) of \( \bigoplus_{i \in I} M_i \). Then, the following hold.

1. Let \( N \) have SSP and \( I \) be finite. Then, \( \bigoplus_{i \in I} M_i \) is \( N \)-image split if and only if \( M_i \) is \( N-(F \cap M_i) \)-image split for all \( i \in I \).
2. Let \( N \) have SSSP and \( I \) be arbitrary. Then,
   
   (a) \( \bigoplus_{i \in I} M_i \) is \( N \)-image split if and only if \( M_i \) is \( N-(F \cap M_i) \)-image split for all \( i \in I \).
   
   (b) \( \prod_{i \in I} M_i \) is \( N \)-image split if and only if \( M_i \) is \( N-(F \cap M_i) \)-image split for all \( i \in I \).

**Proof.** (1) Let \( \bigoplus_{i \in I} M_i \) be \( N \)-image split, then \( M_i \) is \( N-(F \cap M_i) \)-image split for all \( i \in I \) by Theorem 3.4. To see the converse statement let \( f : \bigoplus_{i \in I} M_i \to N \) and \( \iota_i : M_i \to \bigoplus_{i \in I} M_i \) be a inclusion. Then, \( f = f_i \in \text{Hom}(M_i, N) \).

It can be seen that \( f(F) = \sum_{i \in I} f_i(F \cap M_i) \). By hypothesis, \( f_i(F \cap M_i) \) is a direct summand of \( N \) for all \( i \in I \). Since \( N \) has SSP, \( \sum_{i \in I} f_i(F \cap M_i) \) is a direct summand of \( N \). Hence, \( f(F) \) is a direct summand of \( N \) as asserted.

The proof of (2) is similar to that of (1). \( \square \)

**Corollary 3.7.** Let \( \{ M_i \}_{i \in I} \) be \( R \)-modules where \( I = \{ 1, 2, \ldots, n \} \) and \( F \) a fully invariant submodule of \( \bigoplus_{i \in I} M_i \). Then, for each \( j \in I \), \( \bigoplus_{i \in I} M_i \) is \( M_j \)-image split if and only if \( M_i \) is \( M_j-(F \cap M_i) \)-image split for all \( i \in I \).

**Proof.** If for each \( j \in I \), \( \bigoplus_{i \in I} M_i \) is \( M_j \)-image split, then \( M_i \) is \( M_j-(F \cap M_i) \)-image split for all \( i \in I \) by Theorem 3.4. To see the converse statement, let \( M_i \) be \( M_j-(F \cap M_i) \)-image split for all \( i \in I \). Then, \( M_i \) is \( F \cap M_i \)-image split. Hence, \( M_i \) has SSP for direct summands which are contained in \( F \cap M_i \) by Proposition 2.11. Thus, the rest can be proved similar to the proof of Proposition 3.6(1). \( \square \)

**Theorem 3.8.** Let \( \{ M_i \}_{i \in I} \) be a class of \( R \)-modules, \( N \) an \( R \)-module where \( I = \{ 1, 2, \ldots, n \} \) and assume that \( M_i \) is \( M_j \)-projective for all \( i \geq j \in I \). Then, an \( R \)-module \( N \) is \( \bigoplus_{i \in I} M_i \)-image split if and only if \( N \) is \( M_j \)-image split for all \( j \in I \).

**Proof.** The sufficiency is clear by Theorem 3.4. Let \( N \) be an \( M_j \)-image split for all \( j \in I \). We use induction on \( n \). Let \( n = 2 \), \( f \) be a homomorphism from \( N \) to \( M_1 \oplus M_2 \) and \( \pi_i : M_1 \oplus M_2 \to M_1 \) a natural projection where \( i = 1, 2 \). Since \( N \) is \( M_2 \)-image split, \( \pi_2 f(F) \) is a direct summand of \( M_2 \). Hence, \( M_1 \oplus \pi_2 f(F) \) is a direct summand of \( M_1 \oplus M_2 \). To see \( M_1 + f(F) = M_1 \oplus \pi_2 f(F) \), let \( z + y \in M_1 + f(F) \). Then, \( y = \pi_1 y + \pi_2 y \). Hence,
$z + y = z + \pi_1 y + \pi_2 y \in M_1 + \pi_2 f(F)$. For the reverse inclusion, let $x + y \in M_1 \oplus \pi_2 f(F)$. Then, there exists $z \in F$ such that $y = \pi_2 f(z)$. Hence, $x + y = x + \pi_2 f(z) + \pi_1 f(z) - \pi_1 f(z) = x - \pi_1 f(z) + f(z) \in M_1 + f(F)$, as asserted.

By hypothesis, $M_2$ is $M_1$-projective and since $\pi_2 f(F)$ is $M_1$-projective. Then, there exists a submodule $K$ of $f(F)$ such that $M_1 + f(F) = M_1 \oplus K$ by [9, Lemma 4.47]. Hence, $f(F) = K \oplus (M_1 \cap f(F))$. Thus, $\pi_1 f(F) = M_1 \cap f(F)$ since $K \cap M_1 = 0$. Hence, $f(F) = K \oplus \pi_1 f(F)$ which is a direct summand of $K \oplus M_1$. Since $K \oplus M_1 = M_1 \oplus \pi_2 f(F)$, $f(F)$ is a direct summand of $M_1 \oplus M_2$. Thus, $N$ is $(M_1 \oplus M_2)$-$F$-image split. Consequently, we complete the rest of the proof by induction on $n$. □

**Corollary 3.9.** Let $\{M_i\}_{i \in I}$ be a class of $R$-modules where $I = \{1, 2, \ldots, n\}$ and assume that $M_i$ is $M_j$-projective for all $i \geq j \in I$. Then, $\bigoplus_{i \in I} M_i$ is $F$-image split if and only if $M_i$ is $M_j$-$F$-$M_i$-image split for all $i, j \in I$.

**Proof.** The sufficiency is clear by Theorem 3.4. For the necessity, let $M_i$ be $M_j$-$F$-$M_i$-image split for all $i, j \in I$. Then, $\bigoplus_{i \in I} M_i$ is $M_j$-$F$-image split by Corollary 3.7. Hence, $\bigoplus_{i \in I} M_i$ is $F$-image split by Theorem 3.8. □

4. **Strongly $F$-image split modules**

In this section, we deal with a module $M$ for which $f(F)$ is not only a direct summand but also a fully invariant submodule for every $f \in S$.

**Definition 4.1.** An $R$-module $M$ is called strongly $F$-image split if for every $f \in S$, $f(F)$ is a fully invariant direct summand of $M$.

It is obvious that $M$ being a strongly $F$-image split module implies that it is $F$-image split. We now investigate when the converse holds.

**Theorem 4.2.** The following are equivalent for a module $M$.

1. $M$ is strongly $F$-image split.
2. $M$ is $F$-image split and each direct summand of $M$ which is contained in $F$ is fully invariant.
3. $F$ is a dual Rickart and abelian direct summand of $M$.

**Proof.** (1) $\Rightarrow$ (2) Let $N$ be a direct summand of $M$ with $N \subseteq F$. Then, there exists $e^2 = e \in S$ such that $N = eM$. It can be shown that $e(F) = N$. By hypothesis, $e(F)$ is fully invariant in $M$. Thus, $N$ is fully invariant in $M$.

(2) $\Rightarrow$ (3) By Theorem 2.2 and (2), $F$ is a dual Rickart direct summand of $M$. To see that $F$ is an abelian module, let $L$ be a direct summand of $F$. Then, $L$ is a direct summand of $M$. Hence, $L$ is a fully invariant submodule of $M$ by hypothesis. We show that $L$ is fully invariant in $F$. We have $F = eM$ for some $e^2 = e \in S$, and so $\text{End}_R(F) = eS$. Let $f \in \text{End}_R(F)$. Then, there exists $g \in \text{End}_R(M)$ such that $f = ege$. Hence, $f(L) = ege(L) \subseteq L$ since $L$ is fully invariant in $M$. Thus, we have every direct summand of $F$ is fully invariant in $F$. So, $F$ is abelian.

(3) $\Rightarrow$ (1) Let $f \in S$. By Theorem 2.2, $M$ is $F$-image split. Hence, $f(F)$ is a direct summand of $M$. We need to show that $f(F)$ is fully invariant in $M$. We have $F = eM$ for some $e^2 = e \in S$, and so $\text{End}_R(F) = eS$. Thus, $eF(F)$ is a fully invariant direct summand of $F$ by hypothesis. Since $F$ is fully invariant in $M$, $eF(F) = f(F)$. Therefore, $f(F)$ is a fully invariant submodule of $M$ because $F$ and $f(F)$ is fully invariant in $M$ and $F$, respectively. So, $M$ is strongly $F$-image split, as claimed. □

The following example shows that an $F$-image split module $M$ need not be strongly $F$-image split in general.

**Example 4.3.** Let $n$ be a positive integer with $n \geq 2$ and $M$ a vector space over a field $K$ of dimension $n$. Then, $M$ is semisimple and so it is dual Rickart. Hence, $M$ is $M$-image split. But it is not abelian. Thus, $M$ is not strongly $M$-image split.
We end the paper by observing some basic results concerning direct summands and direct sums of strongly $F$-image split modules.

**Proposition 4.4.** Let $M$ be a strongly $F$-image split module. Then, every direct summand $N$ of $M$ is strongly $(N \cap F)$-image split.

**Proof.** Let $K \subseteq N \cap F$ be a direct summand of $N$. Since $N$ is a direct summand of $M$, $M = N \oplus T$ for some submodule $T$ of $M$. By Proposition 2.8, $T$ is $(T \cap F)$-image split. Then, there exists a submodule $T'$ of $T$ such that $T = (T \cap F) \oplus T'$. Hence, $M = K \oplus K' \oplus (T \cap F) \oplus T'$ for some submodule $K'$ of $N$. Also, $K \oplus (T \cap F)$ is contained in $F$ since $K \subseteq N \cap F$. Thus, $K \oplus (T \cap F)$ is fully invariant in $M$ by hypothesis and Theorem 4.2. To see $K$ is fully invariant in $N$, let $f \in \text{End}_R(N)$. Then, $(f \oplus 1_T)(K \oplus (T \cap F)) \subseteq K \oplus (T \cap F)$. Hence, $f(K) \subseteq K$, as asserted. \qed

**Theorem 4.5.** Let $(M_i)_{i \in I}$ be a class of $R$-modules for an arbitrary index set $I$ and $M = \bigoplus_{i \in I} M_i$. Then, $M$ is strongly $F$-image split if and only if for each $i \in I$, $M_i$ is strongly $(F \cap M_i)$-image split and $\text{Hom}_R(F \cap M_i, F \cap M_j) = 0$ for every $i, j \in I$ with $i \neq j$.

**Proof.** Let $M$ be a strongly $F$-image split module. Then, for every $i \in I$, $M_i$ is strongly $(F \cap M_i)$-image split by Proposition 4.4. Since $F$ is fully invariant in $M$, $F = \bigoplus_{i \in I} (F \cap M_i)$. Also, $F$ is a dual Rickart and abelian module by hypothesis and Theorem 4.2. Hence, for every $i \in I$, $F \cap M_i$ is fully invariant in $F$. Thus, $\text{Hom}_R(F \cap M_i, F \cap M_j) = 0$ for every $i, j \in I$ with $i \neq j$. To see the converse statement, let $f = \{f_{ij}\} \in \text{End}_R(\bigoplus_{i \in I} M_i)$ where $f_{ij} \in \text{End}_R(M_j, M_i)$. Then, $f(F) = \bigoplus_{i \in I} f_i(F \cap M_i)$ by hypothesis, and so $f(F)$ is a direct summand of $M$. Since for each $i \in I$, $f_i(F \cap M_i)$ is fully invariant in $M_i$, $f(F)$ is fully invariant in $M$. \qed

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