Hilbert transforms along plane curves for vector-valued functions

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Abstract
In this paper we show that Hilbert transforms along a large class of convex curves are bounded on $L^p(\mathbb{R}^2;X)$, where $\frac{5}{3} < p < \frac{5}{2}$, $X$ is some UMD lattice.

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1 Introduction

Let \( X \) be a Banach space and \( \Gamma : \mathbb{R} \to \mathbb{R}^n \) be a curve in \( \mathbb{R}^n \) with \( \Gamma(0) = 0 \), \( n \geq 2 \). For \( f \in C_0^\infty(\mathbb{R}^n; X) \), the Hilbert transform along curve \( \Gamma \) is defined by the following principal-valued integral

\[
\mathcal{H} f(x) = \text{p.v.} \int_{\mathbb{R}} f(x - \Gamma(t)) \frac{dt}{t}.
\]

For \( X = \mathbb{R} \), there has been considerable interest to determine for which curves \( \Gamma \), and which indices \( p \), one has

\[
\| \mathcal{H} f \|_{L^p(\mathbb{R}^n)} \leq A_p \| f \|_{L^p(\mathbb{R}^n)}
\]

for a constant \( A_p \) depending only on \( \Gamma \) and \( p \). This problem has been extensively studied by a large number of authors. More results can be found in [3, 4, 5, 10, 11, 15].

The question of whether these results could be extended to the Lebesgue-Bôchner spaces \( L^p(\mathbb{R}^n; X) \) of vector-valued functions was taken up by several authors recently, where \( X \) is some Banach space. Specially, we want to know for which curves \( \Gamma \), which indices \( p \) and which Banach spaces \( X \), one has \( \| \mathcal{H} f \|_{L^p(\mathbb{R}^n; X)} \leq A_p \| f \|_{L^p(\mathbb{R}^n; X)} \). It was known that the validity of the formal extension of some classical theorems is equivalent to certain probabilistic properties of the Banach space \( X \). Indeed, it was shown by Burkholder and Bourgain in 1980’s that the boundedness of the Hilbert transform on \( L^p(\mathbb{T}; X) \), \( 1 < p < \infty \), is equivalent to the so called UMD-property of \( X \), as well as the extensions of the Marcinkiewicz-Mihlin multiplier theorem. Naturally, we should consider the underlying Banach space \( X \) with UMD-property, i.e. UMD space. Further, present knowledge of the properties and structure of UMD lattices is deeper than for general UMD spaces. So, it is more reasonable to consider the UMD lattice firstly.

For \( 1 < q < \infty \), \( l^q \) is a classical UMD lattice. The boundedness of Hilbert transforms along curves on \( L^p(\mathbb{R}^n; l^q) \) have been well studied. The first is the work done in 1986 by Rubio de Francia, Ruiz and Torra. They obtained the boundedness of \( \mathcal{H} \) on \( L^p(\mathbb{R}^n; l^q) \) for \( 1 < p, q < \infty \), where \( \Gamma \) is a well-curved curve in \( \mathbb{R}^n \) with \( \Gamma(0) = 0 \). Recently, the author proved the analogue result with \( \Gamma \) be a polynomial function or \( \Gamma(0) = 0 \). Unlike above known results, Calderón-Zygmund type argument can not be used anymore, because of the absence of homogeneity. New techniques should be utilized. To present our main result, we need the following notation.

Definition 1.1. Let \( X \) be an UMD lattice, open interval \( (a, b) \subseteq (0, 1) \). \( X \in \mathcal{I}_{(a,b)} \) if there exists \( \theta \in (a, b) \), Hilbert space \( H \) and another UMD lattice \( Y \) such that \( X = [H, Y]_\theta \).

Remark 1.2. 1). Obviously, for \( 1 < q < \infty \), \( l^q \in \mathcal{I}_{(1-\frac{1}{q}, 1)} \) if and only if \( l^{q'} \in \mathcal{I}_{(1-\frac{1}{q'}, 1)} \). In general \( X \in \mathcal{I}_{(a,b)} \) if and only if \( X^* \in \mathcal{I}_{(a,b)} \). Further, if \( (a, b) \subseteq (c, d) \subseteq (0, 1) \), then \( \mathcal{I}_{(a,b)} \subseteq \mathcal{I}_{(c,d)} \).

2). In 1986, Rubio de Francia proved that for any UMD lattice \( X \) there exists \( \theta \in (0, 1) \), Hilbert space \( H \) and another UMD lattice \( Y \) such that \( X = [H, Y]_\theta \). That is every UMD lattice \( X \) belongs to \( \mathcal{I}_{(0,1)} \).
Theorem 1.3. Let $X \in I(0, \frac{1}{5})$, $\gamma(t)$ be a continuous odd function, twice continuously differentiable, increasing and convex for $t \geq 0$. Suppose also that $\gamma''$ is monotone for $t > 0$ and that there exists $C > 0$ so that $\gamma'(t) \leq Ct\gamma''(t)$ for $t > 0$. Then $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^2; X)$ for all $p$ with $\frac{5}{3} < p < \frac{5}{2}$.

Remark 1.4. 1). Theorem 1.3 covers a large class of functions $\gamma(t)$ such as

\[
\gamma(t) = \text{sgn}(t)|t|^{\alpha}, \quad (\alpha \geq 2)
\]

and $\gamma(t) = te^{-1/|t|}$.

The first one is homogeneous, while the another one is without homogeneity. For $5/3 < q < 5/2$, $l^q \in I(0, \frac{1}{5})$. So, we extend the class of curves for which the $L^p(\mathbb{R}^2; l^q)$-result is known for $5/3 < p, q < 5/2$. On the other hand, we also generalized the Theorem 3.1 of Nagel and Wainger in [10] to the Banach-valued setting for $5/3 < p < 5/2$.

2). In fact, we prove a more general version in this paper. Theorem 1.3 is also true if $X$ satisfies the following weaker condition: there exists $\theta \in (0, \frac{1}{5})$, Hilbert space $H$ and UMD space $Y$ with property $(\alpha)$ such that $X = [H, Y]_{\theta}$.

The proof of Theorem 1.3 is given in several steps, which are similar to that in [10]. Some estimates can be found in this reference, we reprove it just for completeness. An analytic family of operators $\mathcal{H}_z$ are introduced so that $\mathcal{H}_0 = \mathcal{H}$. The $L^2(\mathbb{R}^2; H)$ result is equivalent to the boundedness of the multiplier associated to $\mathcal{H}_z$ with $Re(z) < a$, where $H$ is a Hilbert space, $a$ is some positive constant. Van der Corput lemma and integration by part are used to obtain the boundedness. The kernels of $\mathcal{H}_z$ do not have appropriate homogeneity, we have to use the vector-valued Fourier multiplier theorem (See Lemma 2.7) to show that $\mathcal{H}_z$ is bounded on $L^p(\mathbb{R}^2; Y)$ for $p$ with $1 < p < \infty$ and $Re(z) < b$, where $Y$ is a UMD lattice and $b$ is some negative constant. Finally, the theorem follows from above two fact by using generalized analytic interpolation of operators.

2 Proof of the theorem

For $z \in \mathbb{C}$, we define an analytic family of operators $\mathcal{H}_z$ by

\[
\mathcal{H}_z f(\xi, \eta) = m_z(\xi, \eta)f(\xi, \eta),
\]

where $m_z$ are given by

\[
m_z(\xi, \eta) = p.v. \int_{\mathbb{R}} e^{-2\pi i [\xi t + \eta \gamma(t)]} \left[ 1 + \eta^2 \gamma^2(t) \right]^{z} \frac{dt}{t},
\]

Obviously, $\mathcal{H}_0$ is our original operator $\mathcal{H}$.

2.1 The boundedness of $\mathcal{H}_z$ on $L^2(\mathbb{R}^2; H)$.

In this subsection, we prove that

\[
\|\mathcal{H}_z f\|_{L^2(\mathbb{R}^2; H)} \leq C_\delta [1 + |Im(z)|] \|f\|_{L^2(\mathbb{R}^2; H)},
\]

where $Re(z) = \frac{1}{4} - \delta$ for some $\delta > 0$. 

Clearly, the boundedness of $\mathcal{H}_z$ on $L^2(\mathbb{R}^2; H)$ is equivalent to the uniform boundedness of the corresponding multiplier $m_z(\xi, \eta)$. Thus, we just need to show that
\begin{equation}
|m_z(\xi, \eta)| \leq C_\delta \left[ 1 + |Im(z)| \right],
\end{equation}
where the constant $C_\delta$ is independent of $Im(z)$.

In proving (2.1), some of our estimates will require the following lemmas. The first one is detailed in [10]
Lemma 2.1.

**Lemma 2.1.** Let $A < B$, let $h \in L^\infty((A, B)) \cap C^1((A, B))$ and suppose $(h(t)/t)' = 0$ has at most $k$ roots in $(A, B) \setminus (-1, 1)$. Then
\[
\left| \int_A^B \sin(t)h(t) \frac{dt}{t} \right| \leq (2 + 4k)\|h\|_{\infty}.
\]

The second one is Van der Corput lemma which plays an important role in estimating related multipliers.
This lemma appears in several books or papers, cf. e.g. Stein [16], P.332.

**Lemma 2.2.** Suppose $\phi$ is real-valued and smooth in $(a, b)$, and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$. Then
\[
\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq C_k \lambda^{-1/k}
\]
holds when $k \geq 2$, the bound $C_k$ is independent of $\phi$ and $\lambda$.

To estimate $|m_z(\xi, \eta)|$, for fixed $\eta$, we choose $t_0 > 0$ so that $|\eta|\gamma(t_0) = 1$ and decompose $m_z$ as
\[
|m_z(\xi, \eta)| \leq \left| p.v. \int_{|t| < t_0} e^{-2\pi i [\xi \eta \gamma(t)]} \left[ 1 + \eta^2 \gamma^2(t) \right] z \frac{dt}{t} \right| + \left| \int_{|t| \geq t_0} e^{-2\pi i [\xi \eta \gamma(t)]} \left[ 1 + \eta^2 \gamma^2(t) \right] z \frac{dt}{t} \right|.
\]

First of all, we consider the first integral. For any $0 < \varepsilon < t_0$,
\[
\left| \int_{\varepsilon < |t| < t_0} e^{-2\pi i [\xi \eta \gamma(t)]} \left[ 1 + \eta^2 \gamma^2(t) \right] z \frac{dt}{t} \right| \leq \left| \int_{\varepsilon < |t| < t_0} e^{-2\pi i \xi t} \left(e^{-2\pi i \eta \gamma(t)} - 1 \right) \left[ 1 + \eta^2 \gamma^2(t) \right] z \frac{dt}{t} \right| + \left| \int_{\varepsilon < |t| < t_0} e^{-2\pi i \xi t} \left[ 1 + \eta^2 \gamma^2(t) \right] z \frac{dt}{t} \right| := I + II.
\]
Notice that $1 \leq 1 + \eta^2 \gamma^2(t) \leq 2$ for $|t| \leq t_0$. Because of the convexity of $\gamma(t)$, we have $\gamma(t) \leq t \gamma'(t)$ for $t > 0$. Thus,
\[
I \leq \int_{\varepsilon < |t| < t_0} \left| e^{-2\pi i \eta \gamma(t)} - 1 \right| \left[ 1 + \eta^2 \gamma^2(t) \right] \frac{Re(z) dt}{|t|} \leq C \int_{\varepsilon < |t| < t_0} |\eta \gamma(t)| \left[ 1 + \eta^2 \gamma^2(t) \right] \frac{dt}{|t|} \leq C_\delta |\eta \gamma(t_0)| \leq C_\delta.
\]

For the integral $II$, an easy calculation shows that
\[
II = 2 \left| \int_{\varepsilon}^{t_0} \sin(2\pi \xi t) \left[ 1 + \eta^2 \gamma^2(t) \right] z \frac{dt}{t} \right|
\leq 2 \left| \int_{\varepsilon}^{t_0} \sin(2\pi \xi t) \left[ 1 + \eta^2 \gamma^2(t) \right] \frac{dt}{t} \right|
+ 2 \left| \int_{\varepsilon}^{t_0} \sin(2\pi \xi t) \left[ 1 + \eta^2 \gamma^2(t) \right] \frac{dt}{t} \right|.
\]
It is trivial that \([1 + \eta^2\gamma^2(t)]^{\frac{1}{4} - \delta} \cos [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))]\) is piecewise monotone in \([\varepsilon, t_0]\). Further, it has at most \(C[1 + |\text{Im}(z)|]\) monotone interval in \([\varepsilon, t_0]\). Indeed, the derivative of \([1 + \eta^2\gamma^2(t)]^{\frac{1}{4} - \delta} \cos [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))]\) is
\[
\frac{2\eta^2\gamma(t)\gamma'(t)}{[1 + \eta^2\gamma^2(t)]^{\frac{1}{4} + \delta}} \left(\frac{1}{4} - \delta\right) \cos [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))] \right) - \text{Im}(z) \sin [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))].
\]
\(\gamma(t)\gamma'(t)\) is nonzero for \(t \in [\varepsilon, t_0]\). So, we consider equation
\[
\left(\frac{1}{4} - \delta\right) \cos [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))] - \text{Im}(z) \sin [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))] = 0.
\]
We suppose \(\text{Im}(z) \neq 0\), otherwise, it is obvious. Then, above equation is
\[(2.2)\]
\[
\tan [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))] = \left(\frac{\frac{1}{4} - \delta}{\text{Im}(z)}\right).
\]
For \(\text{Im}(z) > 0\) and \(t \in [\varepsilon, t_0]\), we have \(\text{Im}(z) \ln (1 + \eta^2\gamma^2(t)) \in (0, \text{Im}(z) \ln 2)\). There are about \(\frac{\text{Im}(z) \ln 2}{\pi} + 1\) periods in interval \((0, \text{Im}(z) \ln 2)\), only one root exists in one period. So, equation \((2.2)\) has at most \(\frac{\text{Im}(z) \ln 2}{\pi} + 1\) roots. That mean that \([1 + \eta^2\gamma^2(t)]^{\frac{1}{4} - \delta} \cos [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))]\) has at most \(C[1 + |\text{Im}(z)|]\) stationary points in \([\varepsilon, t_0]\), that is there are at most \(C[1 + |\text{Im}(z)|]\) monotone intervals. The case \(\text{Im}(z) < 0\) can be treated in the same way. Without loss of generality, we suppose it is monotone, otherwise, we deal with the integral on every monotone interval. By the second mean value theorem and Lemma 2.4 there exists \(t_1 \in (\varepsilon, t_0)\) such that
\[
\left|\int_{\varepsilon}^{t_0} \sin(2\pi t\xi) \left[1 + \eta^2\gamma^2(t)\right]^{\frac{1}{4} - \delta} \cos [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))] \frac{dt}{t}\right| \
\leq \left[1 + \eta^2\gamma^2(\varepsilon)\right]^{\frac{1}{4} - \delta} \left|\int_{\varepsilon}^{t_1} \sin(2\pi \xi t) \frac{dt}{t}\right| \
+ \left[1 + \eta^2\gamma^2(t_0)\right]^{\frac{1}{4} - \delta} \left|\int_{t_1}^{t_0} \sin(2\pi \xi t) \frac{dt}{t}\right| \
\leq C[1 + |\text{Im}(z)|].
\]
In the same way, we can prove that
\[
\left|\int_{\varepsilon}^{t_0} \sin(2\pi t\xi) \left[1 + \eta^2\gamma^2(t)\right]^{\frac{1}{4} - \delta} \sin [\text{Im}(z) \ln (1 + \eta^2\gamma^2(t))] \frac{dt}{t}\right| \leq C[1 + |\text{Im}(z)|].
\]
We next deal with the the part of the integral where \(|t| \geq t_0\). If \(\gamma''(t)\) is monotone increasing, we set
\[
\varphi(t) = -\int_{t}^{\infty} e^{-2\pi i [\xi t + \eta\gamma(s)]} ds,
\]
while if \(\gamma''(t)\) is monotone decreasing, put
\[
\varphi(t) = \int_{0}^{t} e^{-2\pi i [\xi t + \eta\gamma(s)]} ds.
\]
Obviously, \(\varphi'(t) = e^{-2\pi i [\xi t + \eta\gamma(t)]}\). By integration by part, we have
\[
\int_{t_0}^{\infty} e^{-2\pi i [\xi t + \eta\gamma(t)]} \left[1 + \eta^2\gamma^2(t)\right] \frac{dt}{t} = \int_{t_0}^{\infty} \varphi'(t) \left[1 + \eta^2\gamma^2(t)\right] \frac{dt}{t} = \varphi(t) \left[1 + \eta^2\gamma^2(t)\right] \frac{\varphi'(t)}{t} \bigg|_{t_0}^{\infty} \]
\[
- \int_{t_0}^{\infty} \varphi(t) \left[1 + \eta^2\gamma^2(t)\right] \frac{d}{dt} t^{-2} \frac{\varphi'(t)}{t} \frac{dt}{t}.
\]
Notice that \(|\xi + \eta \gamma(s)^n| = |\eta \gamma''(s)| \geq |\eta \gamma''(t)|\) for \(s \in [t, \infty]\) or \(s \in [t_0, t]\). Van der Corput lemma shows that 
\(|\varphi(t)| \leq C[|\eta \gamma''(t)|]^{-1} \frac{2 \eta^2 \gamma^2(t)}{|t|} \) in either case. A combination of \(\gamma(t) \leq t \gamma'(t)\) with the hypothesis \(\gamma'(t) \leq C t \gamma''(t)\)
implies that \(\gamma(t) \leq C t^2 \gamma''(t)\). So, for the boundary terms, \(t \in [t_0, \infty)\), we have

\[
\left| \varphi(t) \frac{1 + \eta^2 \gamma^2(t)}{t} \right| \leq C[|\eta \gamma''(t)|]^{-\frac{1}{2}} \frac{2 \eta^2 \gamma^2(t)}{|t|} \leq C[|\eta \gamma''(t)|]^{-\frac{1}{2}} \leq C[|\eta \gamma(t)|]^{-\frac{1}{2}} \leq C. 
\]

Finally, we consider the integrated terms, it is bounded by

\[
\left| \int_{t_0}^{\infty} \varphi(t) \left[ 1 + \eta^2 \gamma^2(t) \right] \frac{1}{t^2} \left( 2 \eta^2 \gamma(t) \gamma'(t) - 1 - \eta^2 \gamma^2(t) \right) dt \right| 
\leq C \int_{t_0}^{\infty} \left| \eta \gamma''(t) \right|^{-\frac{1}{2}} \left( \left| \eta \gamma(t) \right| \right)^{-\frac{1}{2}} \frac{2 \left| \text{Re}(z) \right| + \left| \text{Im}(z) \right| \eta^2 \gamma(t) \gamma'(t) + 2 \eta^2 \gamma^2(t)}{t^2} \frac{dt}{t^2} \leq C \left[ \left| \eta \gamma''(t) \right|^{-\frac{1}{2}} \left( \left| \eta \gamma(t) \right| \right)^{-\frac{1}{2}} \right] \left( \left| \eta \gamma(t) \right| \right)^{-\frac{1}{2}} \frac{dt}{t^2} \]

\[
+ C \left| \eta \gamma''(t) \right|^{-\frac{1}{2}} \left( \left| \eta \gamma(t) \right| \right)^{-\frac{1}{2}} \frac{\gamma(t)}{t} \frac{dt}{t} \leq C \left[ 1 + \left| \text{Im}(z) \right| \right] \frac{dt}{t} \]

The integral

\[
\int_{-\infty}^{t_0} e^{-2 \pi i \left( t + \eta \gamma(t) \right)} \left[ 1 + \eta^2 \gamma^2(t) \right] \frac{dt}{t}
\]

can be handled similarly. This completes the proof of (2.1).

### 2.2 The boundedness of \(\mathcal{H}_2\) on \(L^q(\mathbb{R}^2; \mathbf{Y})\).

In this subsection, we show that

\[
\| \mathcal{H}_2 f \|_{L^q(\mathbb{R}^2; \mathbf{Y})} \leq C \left[ 1 + \left| \text{Im}(z) \right| \right] \| f \|_{L^4(\mathbb{R}^2; \mathbf{Y})},
\]

where \(\mathbf{Y}\) is a UMD lattice, \(\text{Re}(z) < -1, 1 < q < \infty\), the constant \(C\) depends on \(\text{Re}(z)\) and is independent of \(\text{Im}(z)\).

To prove (2.3), we need a vector-valued Fourier multiplier theorem. We have to recall some definitions before we present the theorem.

**Definition 2.3.** A Banach space \(\mathbf{X}\) is an UMD space if the \(\mathbf{X}\)-valued martingale difference sequences on any probability space \((\Omega, \mathcal{A}, \mathbb{P})\) are unconditional on \(L^p(\Omega; \mathbf{X})\) for some (equivalently, all) \(p \in (1, \infty)\). That is, \(\mathbf{X}\) is UMD if there is a constant \(C\) such that

\[
\left( E \left| \sum_{k=1}^{N} \epsilon_k d_k \right|^p \right)^{1/p} \leq C \left( E \left| \sum_{k=1}^{N} d_k^p \right| \right)^{1/p},
\]

for all \(N \in \mathbb{N}\), all fixed signs \(\epsilon_k \in \{-1, 1\}\), every increasing sequence \((\mathcal{A}_k)_{k=1}^{N}\) of sub-\(\sigma\)-algebras of \(\mathcal{A}\), and for every adapted sequence \(d_k \in L^p(\mathcal{A}_k; \mathbf{X})(1 \leq k \leq N)\) with the martingale difference property \(E[d_k|\mathcal{A}_{k-1}] = 0(1 \leq k \leq N)\).
Remark 2.4. For $1 < p, q < \infty$, the reflexive Lebesgue spaces $L^p$, Lorentz spaces $L^{p,q}$ and Schatten-von Neumann classes $S_q$ are well known examples of UMD spaces. Further, if $X$ is an UMD space, so are its dual $X^*$ and the Böchner spaces $L^p(\mu, X)$, where $\mu$ is a measure and $1 < p < \infty$. See [13] for a survey of UMD spaces.

Definition 2.5. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\varepsilon_k$, $k \in \mathbb{Z}$, independent random variables with distribution $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$, and $\varepsilon'_l$, $l \in \mathbb{Z}$ is an independent identical sequence. A Banach space satisfies property $(\alpha)$ if there is a constant $C < \infty$ such that

$$
E \left| \sum_{k,l=1}^{N} \varepsilon_k \varepsilon'_l \alpha_{k,l} x_{kl} \right|_X \leq C \left| \sum_{k,l=1}^{N} \varepsilon_k \varepsilon'_l x_{kl} \right|_X
$$

for all $N \in \mathbb{N}$, all vectors $x_{kl} \in X$ and scalars $|\alpha_{k,l}| \leq 1$ ($1 \leq k, l \leq N$).

Remark 2.6. The commutative $L^p$ spaces satisfy property $(\alpha)$ for all $1 \leq p < \infty$. Also, this property is inherited from $X$ by $L^p(\mu, X)$ for $p \in [1, \infty)$. Every Banach space with a local unconditional structure and finite cotype, in particular every Banach lattice, has property $(\alpha)$.

Let $m : \mathbb{R}^n \to \mathbb{C}$ be a bounded function, we associate operators $T_m$ defined on the test functions $f \in \mathcal{S}(\mathbb{R}^n; X)$ by

$$
T_m f(x) = (mf)'(x).
$$

As a vector-valued Fourier multiplier theorem, we state the following vector-valued Mikhlin theorem. The sufficiency part was proved by Štrkalj and Weis [14], the optimality of those conditions was obtained by Weis and Hytönen [7].

Lemma 2.7. The Marcinkiewicz-Lizorkin condition $|\xi^\beta| |D^\lambda m(\xi)| \leq C$ for all $\beta \in \{0, 1\}^n$ is sufficient for $T_m \in \mathcal{L}(L^p(\mathbb{R}^n; X))$, $n > 1$, if and only if $X$ is an UMD space with property $(\alpha)$.

In view of Lemma 2.7 and Remark 2.6 for $n = 2$, it suffices to show that the following functions

$$
m_z(\xi, \eta), \xi \frac{\partial m_z}{\partial \xi}(\xi, \eta), \eta \frac{\partial m_z}{\partial \eta}(\xi, \eta), \xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta)
$$

are uniformly bounded on $\mathbb{R}^2$ for $Re(z) < -1$.

The uniform boundedness of $m_z(\xi, \eta)$ is trivial, it can be established by minor modification of the proof of (2.1). Without repetition, we omit the proof.

2.2.1 The boundedness of $\xi \frac{\partial m_z}{\partial \xi}(\xi, \eta)$.

Integration by part implies that

$$
\xi \frac{\partial m_z}{\partial \xi}(\xi, \eta) = -2\pi i \int_{\mathbb{R}} e^{-2\pi i \xi (t+\eta \gamma(t))} \xi \left[1 + \eta^2 \gamma^2(t)\right]^{-1} dt
$$

$$
= \int_{\mathbb{R}} \frac{d}{dt} \left(e^{-2\pi i \xi t} e^{-2\pi i \eta \gamma(t)} \left[1 + \eta^2 \gamma^2(t)\right]^{-1} \right) dt
$$

$$
= e^{-2\pi i \xi (t+\eta \gamma(t))} \left[1 + \eta^2 \gamma^2(t)\right]^{-1} + 2\pi i \eta \int_{\mathbb{R}} e^{-2\pi i \xi |t+\eta \gamma(t)|} \gamma(t) \left[1 + \eta^2 \gamma^2(t)\right]^{-1} dt
$$

$$
- 2\pi \xi^2 \int_{\mathbb{R}} e^{-2\pi i \xi |t+\eta \gamma(t)|} \left[1 + \eta^2 \gamma^2(t)\right]^{-1} \gamma(t) \gamma'(t) dt.
$$
Note that $Re(z) < -1$, for $t \in \mathbb{R}$, we have $\left| \left[ 1 + \eta^2\gamma^2(t) \right] \right|^2 = \left[ 1 + \eta^2\gamma^2(t) \right] Re(z) \leq 1$. The boundary terms is bounded by 1.

For $Re(z) < -1$, making the change of variables $u = |\eta|\gamma(t)$, we obtain

$$
\left| \eta \int_{t} e^{-2\pi i[\xi + \eta \gamma(t)]} \gamma'(t) \left[ 1 + \eta^2\gamma^2(t) \right] z \, dt \right| \leq \int_{\mathbb{R}} \gamma'(t) |\eta| \left[ 1 + \eta^2\gamma^2(t) \right] Re(z) dt \\
\leq \int_{\mathbb{R}} \left( 1 + u^2 \right) Re(z) \, du \leq \pi.
$$

In the similar way, the second integrated term can be dominated by

$$
\left| \eta^2 \int_{\mathbb{R}} e^{-2\pi i[\xi + \eta \gamma(t)]} \left[ 1 + \eta^2\gamma^2(t) \right] \gamma'(t) dt \right| \leq 2 |z| \int_{0}^{\infty} \left[ 1 + \eta^2\gamma^2(t) \right] Re(z)^{-1} \eta^2\gamma(t) \gamma'(t) dt \\
\leq |z| \int_{0}^{\infty} \left( 1 + u \right) Re(z)^{-1} du \\
\leq \frac{|z|}{Re(z)} \leq 1 + |Im(z)|.
$$

Therefore, for $Re(z) < -1$,

$$
\left| \xi \frac{\partial m_z}{\partial \eta}(\xi, \eta) \right| \leq C \left[ 1 + |Im(z)| \right] .
$$

### 2.2.2 The boundedness of $\eta \frac{\partial m_z}{\partial \eta}(\xi, \eta)$.

Integrating by parts, we obtain

$$
\eta \frac{\partial m_z}{\partial \eta}(\xi, \eta) = -2\pi i \text{p.v.} \int_{\mathbb{R}} e^{-2\pi i[\xi + \eta \gamma(t)]} \eta \gamma(t) \left[ 1 + \eta^2\gamma^2(t) \right] z \frac{dt}{t} \\
+ 2 \pi i \text{p.v.} \int_{\mathbb{R}} e^{-2\pi i[\xi + \eta \gamma(t)]} \eta^2 \gamma^2(t) \left[ 1 + \eta^2\gamma^2(t) \right] \gamma'(t) \frac{dt}{t}.
$$

To estimate above two integrals, we follow the argument similar to that in the proof of (2.1). For the first integral, for any $\varepsilon > 0$, it suffices to bound the following two parts

$$
\int_{|t| < \varepsilon} |\eta| |\gamma(t)| \left[ 1 + \eta^2\gamma^2(t) \right] Re(z) \frac{dt}{|t|} \quad \text{and} \quad \int_{|t| \geq \varepsilon} |\eta| |\gamma(t)| \left[ 1 + \eta^2\gamma^2(t) \right] Re(z) \frac{dt}{|t|}.
$$

Recall that $t_0 > 0$ was chose so that $|\eta| |\gamma(t_0)| = 1$, and $\gamma(t) \leq \gamma(t_0)$ because of the convexity. Thus,

$$
\int_{\varepsilon < |t| < t_0} |\eta| |\gamma(t)| \left[ 1 + \eta^2\gamma^2(t) \right] Re(z) \frac{dt}{|t|} \leq 2 |\eta| \int_{0}^{t_0} \frac{\gamma(t)}{t} dt \leq 2 |\eta| \int_{0}^{t_0} \gamma'(t) dt \leq 2.
$$

For $Re(z) < -1$, an elementary calculation shows that

$$
\int_{|t| \geq t_0} |\eta| |\gamma(t)| \left[ 1 + \eta^2\gamma^2(t) \right] Re(z) \frac{dt}{|t|} \leq 2 |\eta|^2 Re(z) + 1 \int_{t_0}^{\infty} \gamma^2 Re(z) \left( \gamma(t) \right) \frac{dt}{t} \leq \frac{2}{|2 Re(z) + 1|} \leq 2.
$$

Similarly, the second integral can be controlled by

$$
\left| z \int_{\mathbb{R}} e^{-2\pi i[\xi + \eta \gamma(t)]} \eta^2 \gamma^2(t) \left[ 1 + \eta^2\gamma^2(t) \right] \gamma'(t) \frac{dt}{t} \right| \\
\leq 2 |z| \int_{0}^{t_0} \eta^2 \gamma^2(t) \frac{dt}{t} + 2 |z| \int_{t_0}^{\infty} \eta^2 \gamma^2(t) \left[ \eta^2 \gamma^2(t) \right] Re(z)^{-1} \frac{dt}{t} \\
\leq 2 |z| |\eta|^2 \int_{0}^{t_0} \gamma(t) \gamma'(t) dt + 2 |z| |\eta^2 Re(z)| \int_{t_0}^{\infty} \gamma^2 Re(z)^{-1}(t) \gamma'(t) dt \\
\leq |z| + \frac{|z|}{Re(z)} \leq 2 |Re(z)| [1 + |Im(z)|] .
$$
Therefore, for $\text{Re}(z) < -1$,

$$\left| \frac{\partial m_z}{\partial \xi}(\xi, \eta) \right| \leq C \left[ 1 + |\text{Im}(z)| \right].$$

### 2.2.3 The boundedness of $\xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta)$.

To take care of $\xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta)$, we note that it can be written as

$$\xi \eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta) = -4\pi^2 \xi \eta \int_{\mathbb{R}} e^{-2\pi i \xi \eta \gamma(t)} \left[ 1 + \eta^2 \gamma^2(t) \right] \gamma(t) \, dt$$

For the first term, integrating by parts, we obtain

$$4\pi^2 \xi \eta \int_{\mathbb{R}} e^{-2\pi i \xi \eta \gamma(t)} \left[ 1 + \eta^2 \gamma^2(t) \right] \gamma(t) \, dt$$

which can be written as

$$2\pi i \int \frac{d}{dt} \left( e^{-2\pi i \xi \eta \gamma(t)} \left[ 1 + \eta^2 \gamma^2(t) \right] \gamma(t) \right) dt$$

Obviously, for $\text{Re}(z) < -1$, $|2\pi i e^{-2\pi i \xi \eta \gamma(t)} \eta \gamma(t) [1 + \eta^2 \gamma^2(t)]^2| \leq 2\pi |\eta| |\gamma(t)| \left[ 1 + \eta^2 \gamma^2(t) \right]^{\text{Re}(z)} \leq 2\pi$. So, the boundary terms is bounded by $2\pi$.

For the first integrated terms, making the change of variables $u = \eta^2 \gamma^2(t)$, we have

$$\left| \int_{\mathbb{R}} e^{-2\pi i \xi \eta \gamma(t)} \eta \gamma'(t) [1 + \eta^2 \gamma^2(t)]^{\text{Re}(z)} dt \right| \leq 2 \int_{0}^{\infty} \left[ 1 + \eta^2 \gamma^2(t) \right]^{\text{Re}(z)} \eta \gamma(t) \gamma'(t) dt$$

The second integrated terms can be treated in the same way, let $u = \eta \gamma(t)$,

$$\left| \int_{\mathbb{R}} e^{-2\pi i \xi \eta \gamma(t)} \eta \gamma'(t) [1 + \eta^2 \gamma^2(t)]^{\text{Re}(z)} dt \right| \leq \int_{0}^{\infty} (1 + u^2) \text{Re}(z) \, du \leq \pi.$$

Similarly, a trivial calculation shows that

$$\left| \int_{\mathbb{R}} e^{-2\pi i \xi \eta \gamma(t)} \eta \gamma'(t) [1 + \eta^2 \gamma^2(t)]^{\text{Re}(z)} dt \right| \leq 2|z| \int_{0}^{\infty} u^2 (1 + u^2)^{\text{Re}(z) - 1} du \leq \pi|z|.$$
Obviously, for $\text{Re}(z) < -1$, $t \in \mathbb{R}$, $\mid z e^{-2\pi i [\lambda t + \eta z(t)]} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{-1} \mid \leq |z|$. The boundary term is dominated by $4\pi |z|$. 

For the first integrated terms, by making the change of variables $u = \eta \gamma(t)$, we have the estimate

$$\left| \int_{\mathbb{R}} e^{-2\pi i [\lambda t + \eta z(t)]} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{-1} dt \right| \leq |z| \int_{\mathbb{R}} u^2 (1 + u^2)^{\text{Re}(z)-1} du \leq \pi |z|.$$ 

To estimate the second integrated terms, we make the transformation $u = \eta^2 \gamma(t)$ and get

$$\left| \int_{\mathbb{R}} e^{-2\pi i [\lambda t + \eta z(t)]} \eta^2 \gamma^2(t) [1 + \eta^2 \gamma^2(t)]^{-1} dt \right| \leq |z| \int_{0}^{\infty} (1 + u)^{\text{Re}(z)-1} du \leq \frac{|z|}{|\text{Re}(z)|}.$$

Similarly, the third integrated terms can be treated as

$$\left| \int_{\mathbb{R}} e^{-2\pi i [\lambda t + \eta z(t)]} [1 + \eta^2 \gamma^2(t)]^{-2} \eta^4 \gamma^3(t) \gamma'(t) dt \right| \leq |z(z - 1)| \int_{0}^{\infty} (1 + u)^{\text{Re}(z)-1} du \leq \frac{|z(z - 1)|}{|\text{Re}(z)|}.$$ 

Note that for $\text{Re}(z) < -1$, we have the following elementary estimates

$$|z| \leq |\text{Re}(z)||1 + |\text{Im}(z)|| \quad \text{and} \quad |z - 1| \leq |\text{Re}(z) - 1||1 + |\text{Im}(z)||.$$ 

Finally, combining above eight estimates, we obtain

$$\left| \xi \eta \frac{\partial^2 m_x}{\partial \xi \partial \eta}(\xi, \eta) \right| \leq C [1 + \text{Im}(z)]^2.$$

### 2.3 Analytic interpolation

To complete the proof of our main result, we need a Banach valued analytic interpolation theorem, which can be found in [0].

Let $S$ denote the open strip $\{ z : 0 < \text{Re}(z) < 1 \} \subset \mathbb{C}$ and $A(\tilde{S})$ the algebra of bounded continuous functions on $\tilde{S}$ that are analytic on the open strip $S$. Let $(A_0, A_1)$ and $(B_0, B_1)$ be two compatible couples of Banach spaces, $\{ T_z \}_{z \in S}$ is a family of linear operators on $A_0 \cap A_1$ into $B_0 + B_1$. $\mathcal{F}^w(B_0, B_1)$ is the Banach space of all functions $f : \tilde{S} \rightarrow B_0 + B_1$ such that

(i) $\langle b^*, f(z) \rangle \in A(\tilde{S})$ for any $b^* \in (B_0 + B_1)^*$,

(ii) $f(z) \in B_0$ for $\text{Re}(z) = 0$ and $f(z) \in B_1$ for $\text{Re}(z) = 1$,

(iii) $\| f \|_{\mathcal{F}^w} = \sup \{ \| f(j + it) \|_{B_j} : j = 0, 1, -\infty < t < \infty \} < \infty$.

**Lemma 2.8.** If there exist constants $M_0$ and $M_1$ such that

$$\| T_{z + u} a \|_{B_j} \leq M_j \| a \|_{A_j}, \quad j = 0, 1, -\infty < t < \infty, \quad a \in A_0 \cap A_1,$$

and

$$T_{z} a \in \mathcal{F}^w(B_0, B_1) \quad \text{for any} \quad a \in A_0 \cap A_1.$$

Then $T_{\theta}$ has a unique extension to a bounded operator mapping $[A_0, A_1]_{\theta}$ into $[B_0, B_1]_{\theta}$ and $\| T_{\theta} \| \leq M_0^{-1 - \theta} M_1^{\theta}$.

Let us fix $\theta \in (0, \frac{1}{4})$, Hilbert space $H$ and UMD lattice $Y$. Let $T_{z} f(x) = e^{\theta M_{\gamma}(z)} f(x)$. Note that $|e^{z^2}| = e^{\text{Re}(z)^2 - \text{Im}(z)^2}$, then there exists a constant $M_0$ which is independent of $\text{Im}(z)$ such that

$$\| T_{z} f \|_{L^2(\mathbb{R}^2; H)} \leq C_0 e^{-\text{Im}(z)^2} [1 + \text{Im}(z)] \| f \|_{L^2(\mathbb{R}^2; H)} \leq M_0 \| f \|_{L^2(\mathbb{R}^2; H)} \quad \text{when} \quad \text{Re}(z) = \frac{1}{4} - \delta.$$
Also, for UMD lattice $Y$ and $q \in (1, \infty)$, there exists a constant $M_1$ which is independent of $\Im(z)$ such that 

$$\|T_z f\|_{L^q(\mathbb{R}^2; Y)} \leq M_1 \|f\|_{L^q(\mathbb{R}^2; Y)} \quad \text{when} \quad \Re(z) < -1.$$ 

This inequality also holds in particular with $Y = H$.

For $\frac{5}{3} < p \leq 2$, there exists $1 < q < \infty$ and $\theta_0 \in (0, \frac{1}{3})$ so that 

$$\frac{1}{p} = \frac{1 - \theta_0}{2} + \frac{\theta_0}{q} \quad \text{and} \quad \left(\frac{1}{4} - \delta\right)(1 - \theta_0) + (-1 - \varepsilon_0)\theta_0 =: \sigma_1 \in (0, \frac{1}{4})$$ 

for some $\varepsilon_0 > 0$ and $0 < \delta < \frac{1}{4}$. By interpolation of analytic operators, we have 

$$\|T_z f\|_{L^p(\mathbb{R}^2; H)} \leq C(\varepsilon) \|f\|_{L^p(\mathbb{R}^2; H)} \quad \text{for} \quad \Re(z) = \sigma_1 \in (0, 1/4).$$

For fixed $\theta$ and appropriate $\sigma_1$, we choose $\varepsilon_1 > 0$ such that $(1 - \theta)\sigma_1 + \theta(1 - \varepsilon_1) = 0$. Further, $L^p(\mathbb{R}^2; X) = [L^p(\mathbb{R}^2; H), L^p(\mathbb{R}^2; Y)]_{\theta}$. Using interpolation of analytic operators once more, we obtain 

$$\|\mathcal{H} f\|_{L^p(\mathbb{R}^2; X)} \leq C \|f\|_{L^p(\mathbb{R}^2; X)},$$

for $\frac{5}{3} < p \leq 2$. The duality implies the result for $2 \leq p < \frac{5}{2}$. This completes the proof of Theorem 1.3.

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