INTEGRATION ORDER REPLACEMENT TECHNIQUE FOR ITERATED ITO STOCHASTIC INTEGRALS AND ITERATED STOCHASTIC INTEGRALS WITH RESPECT TO MARTINGALES

DMITRIY F. KUZNETSOV

ABSTRACT. The article is devoted to the integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. We consider the class of iterated Ito stochastic integrals, for which with probability 1 the formulas of integration order replacement corresponding to the rules of classical integral calculus are correct. The theorems on integration order replacement for the class of iterated Ito stochastic integrals are proved. Many examples of this theorems usage have been considered. These results are generalized for the class of iterated stochastic integrals with respect to martingales.

CONTENTS

1. Introduction 2
2. Formulation of the Theorem on Integration Order Replacement for Iterated Ito Stochastic Integrals of Arbitrary Multiplicity 3
3. Proof of Theorem 1 for the Case of Iterated Ito Stochastic Integrals of Multiplicity 2 6
4. Proof of Theorem 1 for the Case of Iterated Ito Stochastic Integrals of Arbitrary Multiplicity 10
5. Corollaries and Generalizations of Theorem 1 14
6. Examples of Integration Order Replacement Technique for the Concrete Iterated Ito Stochastic Integrals 18
7. Integration Order Replacement Technique for Iterated Stochastic Integrals With Respect to Martingale 22
References 26

Mathematics Subject Classification: 60H05.
Keywords: Iterated Ito stochastic integral, Iterated stochastic integral with respect to martingales, Integration order replacement technique, Ito formula.
In this article, we performed rather laborious work connected with the theorems on integration order replacement for iterated Ito stochastic integrals. However, there may appear a question about a practical usefulness of this theory, since the significant part of its conclusions directly follows from the Ito formula \[1\].

It is not difficult to see that to obtain various relations for iterated Ito stochastic integrals (see, for example, Sect. 6) using the Ito formula, first of all these relations should be guessed. Then it is necessary to introduce corresponding Ito processes and afterwards to use the Ito formula. It is clear that this process requires intellectual expenses and it is not always trivial.

On the other hand, the technique on integration order replacement introduced in this article is formally comply with the similar technique for Riemann integrals, although it is related to Ito integrals, and it provides a possibility to perform transformations naturally (as with Riemann integrals) with iterated Ito stochastic integrals and to obtain various relations for them.

So, in order to implementation of transformations of the specific class of Ito processes, which is represented by iterated Ito stochastic integrals, it is more naturally and easier to use the theorems on integration order replacement, than the Ito formula.

Many examples of these theorems usage are presented in Sect. 6.

Note that in a lot of publications of the author \[2\]-[18] the integration order replacement technique for iterated Ito stochastic integrals has been successfully applied for the proof and development of the method of approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series as well as for the construction of the so-called unified Taylor–Ito and Taylor–Stratonovich expansions.

Let \((\Omega, F, P)\) be a complete probability space and let \(f(t,\omega) : [0, T] \times \Omega \to \mathbb{R}\) be the standard Wiener process defined on the probability space \((\Omega, F, P)\). Further, we will use the following notation: \(f(t,\omega) \overset{\text{def}}{=} f_t\).

Let us consider the family of \(\sigma\)-algebras \(\{F_t, t \in [0, T]\}\) defined on the probability space \((\Omega, F, P)\) and connected with the Wiener process \(f_t\) in such a way that

1. \(F_s \subset F_t \subset F\) for \(s < t\).
2. The Wiener process \(f_t\) is \(F_t\)-measurable for all \(t \in [0, T]\).
3. The process \(f_{t+\Delta} - f_t\) for all \(t \geq 0, \Delta > 0\) is independent with the events of \(\sigma\)-algebra \(F_t\).

Let us introduce the class \(M_2([0,T])\) of functions \(\xi : [0,T] \times \Omega \to \mathbb{R}\), which satisfy the conditions:

1. The function \(\xi(t,\omega)\) is measurable with respect to the pair of variables \((t, \omega)\).
2. The function \(\xi(t,\omega)\) is \(F_t\)-measurable for all \(t \in [0, T]\) and \(\xi(\tau, \omega)\) is independent with increments \(f_{t+\Delta} - f_t\) for \(t \geq \tau, \Delta > 0\).
3. The following relation is fulfilled

\[
\int_0^T M \left\{ (\xi(t,\omega))^2 \right\} dt < \infty.
\]
4. \(M \left\{ (\xi(t,\omega))^2 \right\} < \infty\) for all \(t \in [0, T]\).

For any partition \(\tau_j^{(N)}, j = 0, 1, \ldots, N\) of the interval \([0, T]\) such that

\[
0 = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \quad \max_{0 \leq j \leq N-1} \left| \tau_j^{(N)} - \tau_{j+1}^{(N)} \right| \to 0 \text{ if } N \to \infty
\]
we will define the sequence of step functions

\[ \xi^{(N)}(t, \omega) = \xi_j(\omega) \quad \text{w. p. 1 for} \quad t \in \left[ \tau_j^{(N)}, \tau_{j+1}^{(N)} \right), \]

where \( j = 0, 1, \ldots, N-1 \), \( N = 1, 2, \ldots \). Here and further, w. p. 1 means with probability 1.

Let us define the Ito stochastic integral for \( \xi(t, \omega) \in M_2([0, T]) \) as the following mean-square limit

\[ \lim_{N \to \infty} \sum_{j=0}^{N-1} \xi^{(N)}(\tau_j^{(N)}, \omega) \left( f(\tau_{j+1}^{(N)}, \omega) - f(\tau_j^{(N)}, \omega) \right) = \int_0^T \xi_t \, df_t, \]

which converges to the function \( \xi(t, \omega) \) in the following sense

\[ \lim_{N \to \infty} \int_0^T \mathcal{M} \left( \left| \xi^{(N)}(t, \omega) - \xi(t, \omega) \right|^2 \right) dt = 0. \]

It is well known [1] that the Ito stochastic integral exists as the limit (2) and it does not depend on the selection of sequence \( \xi^{(N)}(t, \omega) \). We suppose that standard properties of the Ito stochastic integral are well known to the reader (see, for example, [1]).

Let us define the stochastic integral for \( \xi_\tau \in M_2([0, T]) \) as the following mean-square limit

\[ \lim_{N \to \infty} \sum_{j=0}^{N-1} \xi^{(N)}(\tau_j^{(N)}, \omega) \left( \tau_j^{(N)} - \tau_{j+1}^{(N)} \right) = \int_0^T \xi_\tau \, d\tau, \]

where \( \xi^{(N)}(t, \omega) \) is any step function from the class \( M_2([0, T]) \), which converges in the sense (3) to the function \( \xi(t, \omega) \).

We will introduce the class \( S_2([0, T]) \) of functions \( \xi : [0, T] \times \Omega \to \mathbb{R} \), which satisfy the conditions:

1. \( \xi(\tau, \omega) \in M_2([0, T]) \).
2. \( \xi(\tau, \omega) \) is the mean-square continuous random process at the interval \([0, T] \).

As we noted above, the Ito stochastic integral exists in the mean-square sense (see (2)), if the random process \( \xi(\tau, \omega) \in M_2([0, T]) \), i.e., perhaps this process does not satisfy the property of the mean-square continuity on the interval \([0, T] \). In this article we will formulate and prove the theorems on integration order replacement for the special class of iterated Ito stochastic integrals. At the same time, the condition of the mean-square continuity of integrand in the innermost stochastic integral will be significant.

Let us introduce the following class of iterated stochastic integrals

\[ J[\phi, \psi^{(k)}]_{T, t} = \int_t^T \psi_1(t_1) \ldots \int_t^{t_k} \psi_k(t_k) \int_t^{t_{k-1}} \phi_\tau \, dw^{(k+1)}_{t_{k+1}} \ldots dw^{(k)}_{t_k} \ldots dw^{(1)}_{t_1}, \]

where \( \phi(\tau) \stackrel{\text{def}}{=} \phi_\tau, \phi_\tau \in S_2([t, T]) \), every \( \psi_l(\tau) \) \((l = 1, \ldots, k)\) is a continuous nonrandom function at the interval \([t, T] \), here and further \( w^{(l)}_\tau = f_\tau \) or \( w^{(l)}_\tau = \tau \) for \( \tau \in [t, T] \) \((l = 1, \ldots, k+1)\), \((\psi_1, \ldots, \psi_k) \stackrel{\text{def}}{=} \psi^{(k)}, \psi^{(1)} \stackrel{\text{def}}{=} \psi_1 \).

We will call the stochastic integral \( J[\phi, \psi^{(k)}]_{T, t} \) as the iterated Ito stochastic integral.
It is well known that for the iterated Riemann integral in the case of specific conditions the formula on integration order replacement is correct. In particular, if the nonrandom functions $f(x)$ and $g(x)$ are continuous at the interval $[a, b]$, then

$$\int_a^b f(x) \int_a^x g(y) dy dx = \int_a^b g(y) \int_y^b f(x) dx dy.$$  

(4)

If we suppose that for the Ito stochastic integral

$$J[\phi, \psi]_{T, t} = \int_t^T \psi_1(s) \int_t^s \phi_\tau dw_\tau^{(1)} dw_\tau^{(2)}$$

the formula on integration order replacement, which is similar to (4), is valid, then we will have

$$\int_t^T \psi_1(s) \int_t^s \phi_\tau dw_\tau^{(2)} dw_\tau^{(1)} = \int_t^T \phi_\tau \int_\tau^T \psi_1(s) dw_s^{(1)} dw_s^{(2)}.$$  

(5)

If, in addition $w_s^{(1)}$, $w_s^{(2)} = f_s$ ($s \in [t, T]$) in (5), then the stochastic process

$$\eta_\tau = \phi_\tau \int_\tau^T \psi_1(s) dw_s^{(1)}$$

does not belong to the class $M_2([t, T])$, and, consequently, for the Ito stochastic integral

$$\int_t^T \eta_\tau dw_\tau^{(2)}$$

on the right-hand side of (5) the conditions of its existence are not fulfilled.

At the same time

$$\int_t^T \int_t^s df_s \int_t^s ds = \int_t^T (s - t) df_s + \int_t^T (f_s - f_t) ds \quad \text{w. p. 1},$$  

(6)

and we can obtain this equality, for example, using the Ito formula, but (6) can be considered as a result of integration order replacement (see below).

Actually, we can demonstrate that

$$\int_t^T (f_s - f_t) ds = \int_t^T \int_t^s df_\tau ds = \int_t^T \int_\tau^T ds df_\tau \quad \text{w. p. 1}.$$  

Then

$$\int_t^T (s - t) df_s + \int_t^T (f_s - f_t) ds = \int_t^T \int_t^s ds df_\tau + \int_t^T \int_\tau^T ds df_\tau = \int_t^T df_s \int_t^T ds \quad \text{w. p. 1}.$$
The aim of this article is to establish the strict mathematical sense of the formula (5) for the case \( w_s(1), w_s(2) = f_s \ (s \in [t, T]) \) as well as its analogue corresponding to the iterated Ito stochastic integral \( J[\phi, \psi^{(k)}]_{T,t}, \ k \geq 2 \). At that, we will use the definition of the Ito stochastic integral which is more general than (2).

Let us consider the partition \( \tau_j^{(N)}, \ j = 0, 1, \ldots, N \) of the interval \([t, T]\) such that
\[
(7) \quad t = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \quad \max_{0 \leq j \leq N-1} \left| \tau_j^{(N)} - \tau_j^{(N)} \right| \to 0 \text{ if } N \to \infty.
\]

In [19] Stratonovich R.L. introduced the definition of the so-called combined stochastic integral for the specific class of integrated processes. Taking this definition as a foundation, let us consider the following construction of stochastic integral
\[
(8) \quad \lim_{N \to \infty} \sum_{j=0}^{N-1} \phi_{\tau_j} \left( f_{\tau_{j+1}} - f_{\tau_j} \right) \theta_{\tau_{j+1}} \overset{\text{def}}{=} \int_t^T \phi_{\tau} df_{\tau} \theta_{\tau},
\]
where \( \phi_{\tau}, \theta_{\tau} \in S_2([t, T]), \) \( \{\tau_j\}_{j=0}^N \) is the partition of the interval \([t, T]\), which satisfies to the condition (1) (for simplicity we write here and sometimes further \( \tau_j \) instead of \( \tau_j^{(N)} \)).

Further, we will prove existence of the integral \( 8 \) for \( \phi_{\tau} \in S_2([t, T]) \) and \( \theta_{\tau} \) from a little bit narrower class of processes than \( S_2([t, T]) \). In addition, the integral defined by \( 8 \) will be used for the formulation and proof of the theorem on integration order replacement for the iterated Ito stochastic integrals \( J[\phi, \psi^{(k)}]_{T,t}, \ k \geq 1 \).

Note that under the appropriate conditions the following properties of stochastic integrals defined by the formula \( 8 \) can be proved
\[
\int_t^T \phi_{\tau} df_{\tau} g(\tau) = \int_t^T \phi_{\tau} g(\tau) df_{\tau} \quad \text{w. p. 1},
\]
where \( g(\tau) \) is a continuous nonrandom function at the interval \([t, T]\),
\[
\int_t^T (\alpha \phi_{\tau} + \beta \psi_{\tau}) df_{\tau} \theta_{\tau} = \alpha \int_t^T \phi_{\tau} df_{\tau} \theta_{\tau} + \beta \int_t^T \psi_{\tau} df_{\tau} \theta_{\tau} \quad \text{w. p. 1},
\]
\[
\int_t^T \phi_{\tau} df_{\tau} (\alpha \theta_{\tau} + \beta \psi_{\tau}) = \alpha \int_t^T \phi_{\tau} df_{\tau} \theta_{\tau} + \beta \int_t^T \phi_{\tau} df_{\tau} \psi_{\tau} \quad \text{w. p. 1},
\]
where \( \alpha, \beta \in \mathbb{R} \).

At that, we suppose that the stochastic processes \( \phi_{\tau}, \theta_{\tau}, \psi_{\tau} \) are such that the integrals included in the mentioned properties exist.

2. Formulation of the Theorem on Integration Order Replacement for Iterated Ito Stochastic Integrals of Arbitrary Multiplicity

Let us define the stochastic integrals \( \hat{J}[\psi^{(k)}]_{T,s}, \ k \geq 1 \) of the form
\[ \hat{I}[\psi^{(k)}]_{T,s} = \int_s^T \psi_k(t_k)dw^{(k)}_{t_k} \int_{t_k}^T \psi_{k-1}(t_{k-1})dw^{(k-1)}_{t_{k-1}} \cdots \int_{t_2}^T \psi_1(t_1)dw^{(1)}_{t_1} \]

in accordance with the definition (8) by the following recurrence relation

\[ \hat{I}[\psi^{(k)}]_{T,t} \overset{\text{def}}{=} \lim_{N \to \infty} \sum_{l=0}^{N-1} \psi_\tau(l) \Delta w^{(k)}_{\tau_l} \hat{I}[\psi^{(k-1)}]_{T,\tau_{l+1}}, \]

where \( k \geq 1, \hat{I}[\psi^{(0)}]_{T,s} \overset{\text{def}}{=} 1, [s,T] \subseteq [t,T] \), here and further \( \Delta w^{(i)}_{\tau_l} = w^{(i)}_{\tau_{l+1}} - w^{(i)}_{\tau_l}, \ i = 1, \ldots, k+1, \ l = 0, 1, \ldots, N-1. \)

Then, we will define the iterated stochastic integral \( \hat{J}[\phi, \psi^{(k)}]_{T,t}, k \geq 1 \)

\[ \hat{J}[\phi, \psi^{(k)}]_{T,t} = \int_t^T \phi_s dw^{(k+1)}_s \hat{I}[\psi^{(k)}]_{T,s} \]

similarly in accordance with the definition (8)

\[ \hat{J}[\phi, \psi^{(k)}]_{T,t} \overset{\text{def}}{=} \lim_{N \to \infty} \sum_{l=0}^{N-1} \phi_\tau(l) \Delta w^{(k+1)}_{\tau_l} \hat{I}[\psi^{(k)}]_{T,\tau_{l+1}}. \]

Let us formulate the theorem on integration order replacement for iterated Ito stochastic integrals.

**Theorem 1** [20, 21] (also see [2]-[7], [16]-[18]). Suppose that \( \phi_\tau \in \mathcal{S}_2([t,T]) \) and every \( \psi_\tau(l) \) \((l = 1, \ldots, k)\) is a continuous nonrandom function at the interval \([t,T]\). Then, the stochastic integral \( \hat{J}[\phi, \psi^{(k)}]_{T,t} \) \((k \geq 1)\) exists and

\[ J[\phi, \psi^{(k)}]_{T,t} = \hat{J}[\phi, \psi^{(k)}]_{T,t} \ w. \ p. \ 1. \]

3. **Proof of Theorem 1 for the Case of Iterated Ito Stochastic Integrals of Multiplicity 2**

First, let us prove Theorem 1 for the case \( k = 1 \). We have

\[ J[\phi, \psi_1]_{T,t} \overset{\text{def}}{=} \lim_{N \to \infty} \sum_{l=0}^{N-1} \psi_1(\tau_l) \Delta w^{(1)}_{\tau_l} \int_t^{\tau_l} \phi_s dw^{(2)}_s = \]

\[ = \lim_{N \to \infty} \sum_{l=0}^{N-1} \psi_1(\tau_l) \Delta w^{(1)}_{\tau_l} \sum_{j=0}^{l-1} \int_{\tau_j}^{\tau_{j+1}} \phi_s dw^{(2)}_s, \]

\( (10) \)
\[
\hat{J}[\phi, \psi_1]_{T,t} \overset{\text{def}}{=} \lim_{N \to \infty} \sum_{j=0}^{N-1} \phi_{\tau_j} \Delta w_2^{(2)} \int_{\tau_j}^{T} \psi_1(s)dw_1^{(1)} = \\
= \lim_{N \to \infty} \sum_{j=0}^{N-1} \phi_{\tau_j} \Delta w_2^{(2)} \sum_{l=j+1}^{N-1} \int_{\tau_l}^{\tau_{l+1}} \psi_1(s)dw_1^{(1)} = \\
= \lim_{N \to \infty} \sum_{l=0}^{N-1} \int_{\tau_l}^{\tau_{l+1}} \psi_1(s)dw_1^{(1)} \sum_{j=0}^{l-1} \phi_{\tau_j} \Delta w_2^{(2)}.
\]

(11)

It is clear that if the difference \( \varepsilon_N \) of prelimit expressions on the right-hand sides of (10) and (11) tends to zero when \( N \to \infty \) in the mean-square sense, then the stochastic integral \( \hat{J}[\phi, \psi_1]_{T,t} \) exists and

\[
J[\phi, \psi_1]_{T,t} = \hat{J}[\phi, \psi_1]_{T,t} \quad \text{w. p. 1.}
\]

The difference \( \varepsilon_N \) can be presented in the form \( \varepsilon_N = \bar{\varepsilon}_N + \hat{\varepsilon}_N \), where

\[
\bar{\varepsilon}_N = \sum_{l=0}^{N-1} \psi_1(\tau_l) \Delta w_2^{(1)} \sum_{j=0}^{l-1} \int_{\tau_j}^{\tau_{j+1}} (\phi_{\tau_l} - \phi_{\tau_j}) dw_2^{(2)};
\]

\[
\hat{\varepsilon}_N = \sum_{l=0}^{N-1} \int_{\tau_l}^{\tau_{l+1}} (\psi_1(\tau_l) - \psi_1(s)) dw_1^{(1)} \sum_{j=0}^{l-1} \phi_{\tau_j} \Delta w_2^{(2)}.
\]

We will demonstrate that \( \lim_{N \to \infty} \varepsilon_N = 0 \).

In order to do it we will analyze four cases:

1. \( w^{(2)}_{\tau} = f_{\tau}, \Delta w_1^{(1)} = \Delta f_{\tau_l} \).
2. \( w^{(2)}_{\tau} = \tau, \Delta w_1^{(1)} = \Delta \tau_l \).
3. \( w^{(2)}_{\tau} = f_{\tau}, \Delta w_1^{(1)} = \Delta \tau_l \).
4. \( w^{(2)}_{\tau} = \tau, \Delta w_1^{(1)} = \Delta \tau_l \).

Consider the well known standard moment properties of stochastic integrals [1]

\[
\mathbb{M} \left\{ \left| \int_{t_0}^{t} \xi_\tau df_{\tau} \right|^2 \right\} = \int_{t_0}^{t} \mathbb{M} \left\{ |\xi_\tau|^2 \right\} d\tau,
\]
where $\xi_\tau \in M_2([t_0, t])$.

For Case 1 using standard moment properties for the Ito stochastic integral as well as mean-square continuity (which means uniform mean-square continuity) of the process $\phi_\tau$ on the interval $[t, T]$, we obtain

$$M \left\{ \int_t^{t_0} \xi_\tau \, d\tau \right\}^2 \leq (t - t_0) \int_t^{t_0} M \{ |\xi_\tau|^2 \} \, d\tau,$$

(12)

where $\xi_\tau \in M_2([t_0, t])$.

i.e. $M \left\{ |\xi_N|^2 \right\} 	o 0$ when $N \to \infty$. Here $\Delta \tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N - 1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on $\tau$), $|\psi_1(\tau)| < C$.

Let us consider Case 2. Using the Minkowski inequality, uniform mean-square continuity of the process $\phi_\tau$ as well as the estimate (12) for the stochastic integral, we have

$$M \left\{ |\xi_N|^2 \right\} = \sum_{k=0}^{N-1} \psi_1^2(\tau_k) \Delta \tau_k \sum_{j=0}^{k-1} \sum_{j=0}^{\tau_j} \left( M \left\{ \phi_\tau - \phi_\tau \right\}^2 \right) <$$

$$< C^2 \varepsilon \sum_{k=0}^{N-1} \Delta \tau_k \sum_{j=0}^{k-1} \Delta \tau_j < C^2 \varepsilon \frac{(T - t)^3}{3},$$

i.e. $M \left\{ |\xi_N|^2 \right\} \to 0$ when $N \to \infty$. Here $\Delta \tau_j < \delta(\varepsilon)$, $j = 0, 1, \ldots, N - 1$ ($\delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ and it does not depend on $\tau$), $|\psi_1(\tau)| < C$.

For Case 3 using the Minkowski inequality, standard moment properties for the Ito stochastic integral as well as uniform mean-square continuity of the process $\phi_\tau$, we find

$$M \left\{ |\xi_N|^2 \right\} \leq \sum_{k=0}^{N-1} |\psi_1(\tau_k)| \Delta \tau_k \left( M \left\{ \sum_{j=0}^{k-1} \left( \int_{\tau_j}^{\tau_{j+1}} (\phi_\tau - \phi_\tau) \, d\tau \right) \right\}^{1/2} \right)^2 \leq$$

$$< C^2 \varepsilon \sum_{k=0}^{N-1} \Delta \tau_k \left( \sum_{j=0}^{k-1} \Delta \tau_j \right)^2 < C^2 \varepsilon \frac{(T - t)^3}{3},$$
\[
\begin{align*}
&= \left( \sum_{k=0}^{N-1} |\psi_1(\tau_k)| \Delta \tau_k \left( \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} M \{ |\phi_\tau - \phi_{\tau_j}|^2 \} \, d\tau \right) \right)^{1/2} \\
&< C^2 \epsilon \left( \sum_{k=0}^{N-1} \Delta \tau_k \left( \sum_{j=0}^{k-1} \Delta \tau_j \right) \right)^{1/2} < C^2 \epsilon \frac{(T-t)^3}{9},
\end{align*}
\]

i.e. \( M \{ |\hat{E}_N|^2 \} \to 0 \) when \( N \to \infty \). Here \( \Delta \tau_j < \delta(\epsilon), j = 0, 1, \ldots, N-1 \) (\( \delta(\epsilon) > 0 \) exists for any \( \epsilon > 0 \)
and it does not depend on \( \tau \), \( |\psi_1(\tau)| < C \).

Finally, for Case 4 using the Minkowski inequality, uniform mean-square continuity of the process \( \phi_\tau \) as well as the estimate \((12)\) for the stochastic integral, we obtain

\[
M \{ |\hat{E}_N|^2 \} \leq \left( \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} |\psi_1(\tau_k)| \Delta \tau_k \left( \sum_{j=0}^{k-1} \left( \int_{\tau_j}^{\tau_{j+1}} |\phi_\tau - \phi_{\tau_j}| \, d\tau \right)^2 \right)^{1/2} \right)^2 <
\]

\[
< C^2 \epsilon \left( \sum_{k=0}^{N-1} \Delta \tau_k \sum_{j=0}^{k-1} \Delta \tau_j \right)^2 < C^2 \epsilon \frac{(T-t)^4}{4},
\]

\[
i.e. \ M \{ |\hat{E}_N|^2 \} \to 0 \text{ when } N \to \infty. \]Here \( \Delta \tau_j < \delta(\epsilon), j = 0, 1, \ldots, N-1 \) (\( \delta(\epsilon) > 0 \) exists for any \( \epsilon > 0 \)
and it does not depend on \( \tau \), \( |\psi_1(\tau)| < C \).

Thus, we have proved that

\[
\text{l.i.m. } \hat{E}_N = 0.
\]

Analogously, taking into account the uniform continuity of the function \( \psi_1(\tau) \) on the interval \([t, T]\),
we can demonstrate that

\[
\text{l.i.m. } \hat{E}_N = 0.
\]

Consequently,

\[
\text{l.i.m. } \epsilon_N = 0.
\]

Theorem 1 is proved for the case \( k = 1 \).

**Remark 1.** Proving Theorem 1, we used the fact that if the stochastic process \( \phi_t \) is mean-square continuous at the interval \([t, T]\), then it is uniformly mean-square continuous at this interval, i.e. \( \forall \ \epsilon > 0 \ \exists \delta(\epsilon) > 0 \) such that for all \( t_1, t_2 \in [t, T] \) satisfying the condition \( |t_1 - t_2| < \delta(\epsilon) \) the inequality

\[
M \{ |\phi_{t_1} - \phi_{t_2}|^2 \} < \epsilon
\]

is fulfilled (here \( \delta(\epsilon) \) does not depend on \( t_1 \) and \( t_2 \)).

**Proof.** Suppose that the stochastic process \( \phi_t \) is mean-square continuous at the interval \([t, T]\), but
not uniformly mean-square continuous at this interval. Then for some \( \epsilon > 0 \) and \( \forall \delta(\epsilon) > 0 \ \exists \ t_1, t_2 \in [t, T] \) such that \( |t_1 - t_2| < \delta(\epsilon) \), but
\[ M \left\{ |\phi_{t_1} - \phi_{t_2}|^2 \right\} \geq \varepsilon. \]

Consequently, for \( \delta = \delta_n = 1/n \ (n \in \mathbb{N}) \) \( \exists \ t_1^{(n)}, t_2^{(n)} \in [t, T] \) such that
\[ |t_1^{(n)} - t_2^{(n)}| < \frac{1}{n}, \]
but
\[ M \left\{ |\phi_{t_1^{(n)}} - \phi_{t_2^{(n)}}|^2 \right\} \geq \varepsilon. \]

The sequence \( t_1^{(n)} \ (n \in \mathbb{N}) \) is bounded, consequently, according to the Bolzano–Weierstrass Theorem, we can choose from it the subsequence \( t_1^{(k_n)} \ (n \in \mathbb{N}) \) that converges to a certain number \( \tilde{t} \) (it is simple to demonstrate that \( \tilde{t} \in [t, T] \)). Similarly to it and in virtue of the inequality
\[ |t_1^{(n)} - t_2^{(n)}| < \frac{1}{n}, \]
we have \( t_2^{(k_n)} \to \tilde{t} \) when \( n \to \infty \).

According to the mean-square continuity of the process \( \phi_t \) at the moment \( \tilde{t} \) and the elementary inequality \( (a+b)^2 \leq 2(a^2 + b^2) \), we obtain
\[
0 \leq M \left\{ |\phi_{t_1^{(k_n)}} - \phi_{t_2^{(k_n)}}|^2 \right\} \leq 2 \left( M \left\{ |\phi_{t_1^{(k_n)}} - \phi_{\tilde{t}}|^2 \right\} + M \left\{ |\phi_{t_2^{(k_n)}} - \phi_{\tilde{t}}|^2 \right\} \right) \to 0
\]
when \( n \to \infty \). Then
\[
\lim_{n \to \infty} M \left\{ |\phi_{t_1^{(k_n)}} - \phi_{t_2^{(k_n)}}|^2 \right\} = 0.
\]
It is impossible by virtue of the fact that
\[ M \left\{ |\phi_{t_1^{(n)}} - \phi_{t_2^{(n)}}|^2 \right\} \geq \varepsilon > 0. \]

The obtained contradiction proves the required statement.

4. PROOF OF THEOREM 1 FOR THE CASE OF ITERATED ITO STOCHASTIC INTEGRALS OF ARBITRARY MULTIPLICITY

Let us prove Theorem 1 for the case \( k > 1 \). In order to do it we will introduce the following notations
\[ I[\psi_q^{(r+1)}]_{\theta, s} \overset{\text{def}}{=} \int_s^\theta \psi_q(t_1) \cdots \int_s^{t_r} \psi_q^{(r+1)}(t_{r+1}) du^{(q+r)}_{t_{r+1}} \cdots du_{t_1}^{(q)}. \]
INTEGRATION ORDER REPLACEMENT TECHNIQUE

\( J[\phi, \psi^{(r+1)}]_{[s, t]} \) \( \overset{\text{def}}{=} \int_s^t \psi_q(t_1) \ldots \psi_q(t_{r+1}) \int_s^{t_{r+1}} \phi_s dw_t^{(q+r+1)} dt 

G[\psi^{(r+1)}]_{n, m} = \sum_{j = m}^{n-1} \sum_{j+1 = m}^{j+r} \cdots \sum_{j+r = m}^{j+q} \prod_{l = q}^{r+q} I[\psi^l]_{\tau_{j+1}, \tau_j},

(\psi_q, \ldots, \psi_q^{(r+1)}) \overset{\text{def}}{=} \psi_q^{(r+1)}, \quad (\psi_1, \ldots, \psi_1^{(r+1)}) \overset{\text{def}}{=} \psi_1^{(r+1)},

Note that according to notations introduced above

\( I[\psi]_{s, t} = \int_s^t \psi(\tau) dW^{(l)} \).

To prove Theorem 1 for \( k > 1 \) it is enough to show that

\( J[\phi, \psi^{(k)}]_{[T, t]} = \lim_{N \to \infty} S[\phi, \psi^{(k)}]_N = \hat{J}[\phi, \psi^{(k)}]_{[T, t]} \) w. p. 1,

where

\( S[\phi, \psi^{(k)}]_N = G[\psi^{(k)}]_{N, 0} \sum_{j = 0}^{j_k - 1} \phi_{\tau_l} \Delta u_{\tau_l}^{(k+1)}, \)

where \( \Delta u_{\tau_l}^{(k+1)} = u_{\tau_{l+1}}^{(k+1)} - u_{\tau_l}^{(k+1)} \).

First, let us prove the right equality in (13). We have

\( \hat{J}[\phi, \psi^{(k)}]_{[T, t]} = \lim_{N \to \infty} \sum_{l = 0}^{N-1} \phi_{\tau_l} \Delta u_{\tau_l}^{(k+1)} \hat{I}[\psi^{(k)}]_{[T, \tau_{l+1}}. \)

On the basis of the inductive hypothesis we obtain that

\( \hat{I}[\psi^{(k)}]_{[T, \tau_{l+1}} = \hat{I}[\psi^{(k)}]_{[T, \tau_{l+1}} \) w. p. 1,

where \( \hat{I}[\psi^{(k)}]_{[T, s} \) is defined in accordance with (9) and

\( I[\psi^{(k)}]_{[T, s} = \int_s^T \psi_1(t_1) \ldots \psi_k(t_k) dw_k^{(k)} dw_{k-1}^{(k-1)} \ldots dw_1^{(1)}. \)

Let us note that when \( k \geq 4 \) (for \( k = 2, 3 \) the arguments are similar) due to additivity of the Ito stochastic integral the following equalities are correct
\[ I[\psi^{(k)}]_{T, \tau_{l+1}} = \sum_{j_1=l+1}^{N-1} \int_{\tau_{j_1}}^{\tau_{j_1+1}} \psi_1(t_1) \int_{\tau_{j_1+1}}^{t_1} \psi_2(t_2) I[\psi^{(k-2)}]_{t_2, \tau_{l+1}} \, dw^{(2)}_{t_2} \, dw^{(1)}_{t_1} = \]

\[ \sum_{j_1=l+1}^{N-1} \int_{\tau_{j_1}}^{\tau_{j_1+1}} \psi_1(t_1) \left( \sum_{j_2=l+1}^{N-1} \int_{\tau_{j_2+1}}^{t_1} \psi_2(t_2) I[\psi^{(k-2)}]_{t_2, \tau_{l+1}} \, dw^{(2)}_{t_2} \, dw^{(1)}_{t_1} \right) \]

(16)

\[ = \ldots = G[\psi^{(k)}]_{N, l+1} + H[\psi^{(k)}]_{N, l+1} \quad \text{w. p. 1}, \]

where

\[ H[\psi^{(k)}]_{N, l+1} = \sum_{j_1=l+1}^{N-1} \int_{\tau_{j_1}}^{\tau_{j_1+1}} \psi_1(s) \int_{\tau_{j_1}}^{s} \psi_2(\tau) I[\psi^{(k-2)}]_{\tau, \tau_{l+1}} \, dw^{(2)}_{\tau} \, dw^{(1)}_{s} + \]

\[ + \sum_{r=2}^{k-2} G[\psi^{(r-1)}]_{N, l+1} \sum_{j_r=l+1}^{N-1} \int_{\tau_{j_r}}^{\tau_{j_r+1}} \psi_r(s) \int_{\tau_{j_r}}^{s} \psi_{r+1}(\tau) I[\psi^{(k-r-1)}]_{\tau, \tau_{l+1}} \, dw^{(r+1)}_{\tau} \, dw^{(r)}_{s} + \]

(17)

\[ + G[\psi^{(k-2)}]_{N, l+1} \sum_{j_{k-2}=l+1}^{N-1} I[\psi^{(2)}]_{j_{k-2}, \tau_{j_{k-2}+1}, \tau_{j_{k-1}}} \cdot \]

Let us substitute (16) into (15) and (15) into (14). Then w. p. 1

(18)

\[ \tilde{J}[\phi, \psi^{(k)}]_{T, \ell} = \lim_{N \to \infty} \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta w^{(k+1)}_{\tau_l} \left( G[\psi^{(k)}]_{N, l+1} + H[\psi^{(k)}]_{N, l+1} \right). \]

Since

(19)

\[ \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{j_1-1} \cdots \sum_{j_k=0}^{j_{k-1}-1} a_{j_1 \ldots j_k} = \sum_{j_k=0}^{N-1} \sum_{j_{k-1}=j_k+1}^{N-1} \cdots \sum_{j_1=j_2+1}^{N-1} a_{j_1 \ldots j_k}, \]

where \( a_{j_1 \ldots j_k} \) are scalars, then

(20)

\[ G[\psi^{(k)}]_{N, l+1} = \sum_{j_k=l+1}^{N-1} \cdots \sum_{j_1=j_{k+1}+1}^{N-1} \prod_{l=1}^{k} I[\psi_l]_{\tau_l+1, \tau_l}. \]

Let us substitute (20) into

\[ \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta w^{(k+1)}_{\tau_l} G[\psi^{(k)}]_{N, l+1} \]

and again use the formula (19). Then
Let us suppose that the limit
\[ \lim_{N \to \infty} S[\phi, \psi^{(k)}]_N \]
exists (its existence will be proved further).

Then from (21) and (18) it follows that for proof of the right equality in (13) we have to demonstrate
that w. p. 1
\[ \lim_{N \to \infty} \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta w_{\tau_l}^{(k+1)} H[\psi^{(k)}]_{N, l+1} = 0. \]

Analyzing the second moment of the prelimit expression on the left-hand side of (23) and taking into
account (17), the independence of \( \phi_{\tau_l}, \Delta w_{\tau_l}^{(k+1)}, \) and \( H[\psi^{(k)}]_{N, l+1} \) as well as the standard estimates
for second moments of stochastic integrals and the Minkowski inequality, we find that (23) is correct.

Thus, by the assumption of existence of the limit (22) we obtain that the right equality in (13) is
fulfilled.

Let us demonstrate that the left equality in (13) is also fulfilled.

We have
\[ J[\phi, \psi^{(k)}]_{T, t} = \lim_{N \to \infty} \sum_{l=0}^{N-1} \psi_1(\tau_l) \Delta w_{\tau_l}^{(1)} J[\phi, \psi^{(k-1)}]_{\tau_l, t}. \]

Let us use for the integral \( J[\phi, \psi^{(k-1)}]_{\tau_l, t} \) in (24) the same arguments, which resulted to the relation
(16) for the integral \( I[\psi^{(k)}]_{T, \tau_l+1} \). After that let us substitute the expression obtained for the integral
\( J[\phi, \psi^{(k-1)}]_{\tau_l, t} \) into (24).

Further, using the Minkowski inequality and standard estimates for second moments of stochastic
integrals it is easy to obtain that
\[ J[\phi, \psi^{(k)}]_{T, t} = \lim_{N \to \infty} R[\phi, \psi^{(k)}]_N \quad \text{w. p. 1}, \]
where
\[ R[\phi, \psi^{(k)}]_N = \sum_{j_1=0}^{N-1} \psi_1(\tau_{j_1}) \Delta w_{\tau_{j_1}}^{(1)} G[\psi^{(k-1)}]_{j_1, 0} \sum_{l=0}^{j_1-1} \int_{\tau_l}^{\tau_{j_1+1}} \phi_{\tau} dw_{\tau}^{(k+1)}. \]

We will demonstrate that
\[ \lim_{N \to \infty} R[\phi, \psi^{(k)}]_N = \lim_{N \to \infty} S[\phi, \psi^{(k)}]_N \quad \text{w. p. 1}. \]

It is easy to see that
\[ R[\phi, \psi^{(k)}]_N = U[\phi, \psi^{(k)}]_N + V[\phi, \psi^{(k)}]_N + S[\phi, \psi^{(k)}]_N \quad \text{w. p. 1}, \]
where
\[
U[\phi, \psi^{(k)}]_N = \sum_{j_1=0}^{N-1} \psi_1(\tau_{j_1}) \Delta w_{\tau_{j_1}}^{(1)} G[\psi_2^{(k-1)}]_{j_1,0} \sum_{l=0}^{j_k-1} I[\Delta \phi]_{\tau_{j_1+1}, \tau_l}
\]

\[
V[\phi, \psi^{(k)}]_N = \sum_{j_1=0}^{N-1} I[\Delta \psi_1]_{\tau_{j_1+1}, \tau_{j_1}} G[\psi_2^{(k-1)}]_{j_1,0} \sum_{l=0}^{j_k-1} \phi_{\tau_l} \Delta w_{\tau_{l+1}}^{(k+1)}
\]

\[
I[\Delta \psi_1]_{\tau_{j_1+1}, \tau_{j_1}} = \int_{\tau_{j_1}}^{\tau_{j_1+1}} (\psi_1(\tau_{j_1}) - \psi_1(\tau)) d\tau
\]

\[
I[\Delta \phi]_{\tau_{j_1+1}, \tau_l} = \int_{\tau_l}^{\tau_{j_1+1}} (\phi_{\tau_l} - \phi_{\tau_{j_1+1}}) d\tau
\]

Using the Minkowski inequality, standard estimates for second moments of stochastic integrals, the condition that the process \(\phi_{\tau}\) belongs to the class \(S_2([t, T])\) as well as continuity (which means uniform continuity) of the function \(\psi_1(\tau)\), we obtain that

\[
l.i.m. \lim_{N \to \infty} V[\phi, \psi^{(k)}]_N = l.i.m. \lim_{N \to \infty} U[\phi, \psi^{(k)}]_N = 0 \text{ w. p. 1.}
\]

Then, considering (27), we obtain (26). From (26) and (25) it follows that the left equality in (13) is fulfilled.

Note that the limit (22) exists because it is equal to the stochastic integral \(J[\phi, \psi^{(k)}]_{T,t}\), which exists under the conditions of Theorem 1. So, the chain of equalities (13) is proved. Theorem 1 is proved.

### 5. Corollaries and Generalizations of Theorem 1

Denote \(D_k = \{(t_1, \ldots, t_k) : t \leq t_1 < \ldots < t_k \leq T\}\). We will use the same symbol \(D_k\) to denote the open and closed domains corresponding to the domain \(D_k\). However, we always specify what domain we consider (open or closed).

Suppose that the following conditions are fulfilled:

AI. \(\xi_{\tau} \in S_2([t, T])\).

AII. \(\Phi(t_1, \ldots, t_{k-1})\) is a continuous nonrandom function in the closed domain \(D_{k-1}\).

Let us define the following stochastic integrals

\[
\hat{J}[\xi, \Phi^{(k)}]_{T,t} = \int_t^T \xi_{t_k} d\omega_{t_k}^{(1)} \ldots \int_{t_3}^T d\omega_{t_2}^{(12)} \int_{t_2}^T \Phi(t_1, t_2, \ldots, t_{k-1}) d\omega_{t_1}^{(1)} \overset{\text{def}}{=} \sum_{N=1}^{N-1} \xi_{\tau_{j_1}} \Delta w_{\tau_{j_1}}^{(1)} \int_{\tau_{j_1+1}}^{T} d\omega_{t_k}^{(1)} \ldots \int_{t_3}^T d\omega_{t_2}^{(1)} \int_{t_2}^T \Phi(t_1, t_2, \ldots, t_{k-1}) d\omega_{t_1}^{(1)}
\]

\[
\overset{\text{def}}{=} \lim_{N \to \infty} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta w_{\tau_l}^{(1)} \int_{\tau_l+1}^{T} d\omega_{t_k}^{(1)} \ldots \int_{t_3}^T d\omega_{t_2}^{(1)} \int_{t_2}^T \Phi(t_1, t_2, \ldots, t_{k-1}) d\omega_{t_1}^{(1)}
\]
for \( k \geq 3 \) and

\[
\hat{J}[\xi, \Phi]^{(2)}_{T,t} = \int_0^T \xi_{t_2} dw_{t_2}^{(i_2)} \int_0^{T_{t_2}} \Phi(t_1) dw_{t_1}^{(i_1)} \text{ def}
\]

\[
\text{def} = \lim_{N \to \infty} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta w_{\tau_l}^{(i_2)} \int_{\tau_l+1}^T \Phi(t_1) dw_{t_1}^{(i_1)}
\]

for \( k = 2 \). Here \( w_r^{(i)} = f_r^{(i)} \) if \( i = 1, \ldots, m \) and \( w_r^{(0)} = \tau, f_r^{(i)} (i = 1, \ldots, m) \) are \( F_\tau \)-measurable for all \( \tau \in [0, T] (0 \leq t < T) \) independent standard Wiener processes, \( i_1, \ldots, i_k = 0, 1, \ldots, m \).

Let us denote

\[
J[\xi, \Phi]^{(k)}_{T,t} = \int_t^{t_k-1} \Phi(t_1, \ldots, t_{k-1}) \xi_{t_k} dw_{t_k}^{(i_k)} \ldots dw_{t_1}^{(i_1)}, \quad k \geq 2,
\]

where the right-hand side of (28) is the iterated Ito stochastic integral.

Let us introduce the following iterated stochastic integrals

\[
\hat{J}[\Phi]^{(k-1)}_{T,t} = \int_t^{T_{t_k-1}} \ldots \int_t^{T_{t_3}} \Phi(t_1, \ldots, t_{k-1}) \xi_{t_k} dw_{t_k}^{(i_k)} \ldots dw_{t_3}^{(i_3)} \text{ def}
\]

\[
\text{def} = \lim_{N \to \infty} \sum_{l=0}^{N-1} \xi_{\tau_l} \Delta w_{\tau_l}^{(i_k-1)} \int_{\tau_l+1}^{T_{t_k-2}} \ldots \int_{\tau_l+1}^{T_{t_3}} \Phi(t_1, \ldots, t_{k-1}) \xi_{t_k} dw_{t_k}^{(i_k-1)} \ldots dw_{t_3}^{(i_3)}
\]

\[
J'[\Phi]^{(k-1)}_{T,t} = \int_t^{T_{t_k-2}} \ldots \int_t^{T_{t_3}} \Phi(t_1, \ldots, t_{k-1}) \xi_{t_k} dw_{t_k}^{(i_k-1)} \ldots dw_{t_3}^{(i_3)}, \quad k \geq 2.
\]

Similarly to the proof of Theorem 1 it is easy to demonstrate that under the condition AII the stochastic integral \( \hat{J}[\Phi]^{(k-1)}_{T,t} \) exists and

\[
J'[\Phi]^{(k-1)}_{T,t} = \hat{J}[\Phi]^{(k-1)}_{T,t} \quad \text{w. p. 1.}
\]

Moreover, using (29) the following generalization of Theorem 1 can be proved similarly to the proof of Theorem 1.

**Theorem 2** [20], [21] (also see [2]-[7], [16]-[18]). Suppose that the conditions AII of this section are fulfilled. Then, the stochastic integral \( \hat{J}[\xi, \Phi]^{(k)}_{T,t} \) exists and for \( k \geq 2 \)

\[
J[\xi, \Phi]^{(k)}_{T,t} = \hat{J}[\xi, \Phi]^{(k)}_{T,t} \quad \text{w. p. 1.}
\]

Let us consider the following stochastic integrals
\[ I = \int_t^T dt_1^{(i_2)} \int_t^{t_2} \Phi_1(t_1, t_2) dt_1^{(i_1)}, \quad J = \int_t^{t_2} \int_t^T dt_2^{(i_2)} \Phi_2(t_1, t_2) dt_2^{(i_1)}. \]

If we consider
\[ \int_t^{t_2} \Phi_1(t_1, t_2) dt_1^{(i_1)} \]
as the integrand of \( I \) and
\[ \int_t^{t_2} \Phi_2(t_1, t_2) dt_2^{(i_1)} \]
as the integrand of \( J \), then, due to independence of these integrands we may mistakenly think that \( M\{IJ\} = 0 \).

But it is not the fact. Actually, using the integration order replacement technique in the stochastic integral \( I \), we have w. p. 1
\[ I = \int_t^{t_2} \int_t^{t_1} \Phi_1(t_1, t_2) dt_1^{(i_2)} dt_1^{(i_1)} = \int_t^{t_2} \int_t^T \Phi_2(t_1, t_2) dt_2^{(i_2)} dt_2^{(i_1)}. \]

So, using the standard properties of the Ito stochastic integral [1], we get
\[ M\{IJ\} = \mathbf{1}_{\{i_1=i_2\}} \int_t^{t_2} \int_t^{t_1} \Phi_1(t_2, t_1) \Phi_2(t_1, t_2) dt_1 dt_2, \]
where \( \mathbf{1}_{\{A\}} \) is the indicator of the set \( A \).

Let us consider the following statement.

**Theorem 3** [20, 21] (also see [2]-[7], [16]-[18]). Let the conditions of Theorem 1 are fulfilled and \( h(\tau) \) is a continuous nonrandom function at the interval \([t, T]\). Then

\[ \int_t^T \phi_\tau dw_\tau^{(k+1)} h(\tau) \hat{I}[\psi^{(k)}]|_{T, \tau} = \int_t^T \phi_\tau h(\tau) dw_\tau^{(k+1)} \hat{I}[\psi^{(k)}]|_{T, \tau} \text{ w. p. 1,} \]

where stochastic integrals on the left-hand side of (30) as well as on the right-hand side of (30) exist.

**Proof.** According to Theorem 1, the iterated stochastic integral on the right-hand side of (30) exists. In addition

\[ \int_t^T \phi_\tau h(\tau) dw_\tau^{(k+1)} \hat{I}[\psi^{(k)}]|_{T, \tau} = \int_t^T \phi_\tau dw_\tau^{(k+1)} h(\tau) \hat{I}[\psi^{(k)}]|_{T, \tau} - \text{ lim}_{N \to \infty} \sum_{j=0}^{N-1} \phi_{\tau_j} \Delta h(\tau_j) \Delta w_{\tau_j}^{(k+1)} \hat{I}[\psi^{(k)}]|_{T, \tau_j+1} \text{ w. p. 1,} \]
where $\Delta h(\tau_j) = h(\tau_{j+1}) - h(\tau_j)$.

Using the arguments which resulted to the right equality in (13), we obtain

$$\lim_{N \to \infty} \sum_{l=0}^{N-1} \phi_{\tau_l} \Delta h(\tau_l) \Delta w_{\tau_l}^{(k+1)} I[\psi^{(k)}]_{T, \tau_{l+1}} =$$

$$= \lim_{N \to \infty} G[\psi^{(k)}]_{N, 0} \sum_{l=0}^{j_k-1} \phi_{\tau_l} \Delta h(\tau_l) \Delta w_{\tau_l}^{(k+1)} \quad \text{w. p. 1.} \quad (31)$$

Using the Minkowski inequality, standard estimates for second moments of stochastic integrals as well as continuity of the function $h(\tau)$, we obtain that the second moment of the prelimit expression on the right-hand side of (31) tends to zero when $N \to \infty$. Theorem is proved.

Let us consider one corollary of Theorem 1.

**Theorem 4** [20, 21] (also see [2]-[7], [16]-[18]). Under the conditions of Theorem 3 the following equality is fulfilled

$$\int_t^T h(t_1) \int_t^T \phi_{\tau} dw_{\tau}^{(k+2)} \int_t^T \phi_{\tau} dw_{\tau}^{(k+1)} I[\psi^{(k)}]_{T, t_1} =$$

$$= \int_t^T \phi_{\tau} dw_{\tau}^{(k+2)} \int_t^T h(t_1) dw_{t_1}^{(k+1)} I[\psi^{(k)}]_{T, t_1} \quad \text{w. p. 1.} \quad (32)$$

Moreover, the stochastic integrals in (32) exist.

**Proof.** Using Theorem 1 two times, we obtain

$$\int_t^T \phi_{\tau} dw_{\tau}^{(k+2)} \int_t^T h(t_1) dw_{t_1}^{(k+1)} I[\psi^{(k)}]_{T, t_1} =$$

$$= \int_t^T \psi_1(t_1) \ldots \int_t^T \psi_k(t_k) \int_t^T \rho_{\tau} dw_{\tau}^{(k+1)} dw_{1_k}^{(k)} \ldots dw_{1_s}^{(1)} =$$

$$= \int_t^T \rho_{\tau} dw_{\tau}^{(k+1)} \int_t^T \psi_k(t_k) dw_{t_k}^{(k)} \ldots \int_t^T \psi_1(t_1) dw_{t_1}^{(1)} \quad \text{w. p. 1,}$$

where

$$\rho_{\tau} \overset{\text{def}}{=} h(\tau) \int_t^T \phi_{s} dw_{s}^{(k+2)}.$$
As we mentioned above, the formulas from this section could be obtained using the Ito formula. However, the method based on Theorem 1 is more simple and familiar, since it deals with usual rules of the integration order replacement for Riemann integrals.

Using the integration order replacement technique for iterated Ito stochastic integrals (Theorem 1), we obtain the following equalities which are fulfilled w. p. 1

$$\int_t^{T} \int_t^{t_2} df_{t_1} dt_2 = \int_t^{T} (T-t_1) df_{t_1},$$

$$\int_t^{T} \cos(t_2 - T) \int_t^{t_2} df_{t_1} dt_2 = \int_t^{T} \sin(T-t_1) df_{t_1},$$

$$\int_t^{T} \sin(t_2 - T) \int_t^{t_2} df_{t_1} dt_2 = \int_t^{T} \cos(T-t_1) - 1) df_{t_1},$$

$$\int_t^{T} e^{\alpha(t_2 - T)} \int_t^{t_2} df_{t_1} dt_2 = \frac{1}{\alpha} \int_t^{T} \left(1 - e^{\alpha(t_1 - T)}\right) df_{t_1}, \quad \alpha \neq 0,$$

$$\int_t^{T} (t_2 - T)^{\alpha} \int_t^{t_2} df_{t_1} dt_2 = -\frac{1}{\alpha + 1} \int_t^{T} (t_1 - T)^{\alpha+1} df_{t_1}, \quad \alpha \neq -1,$$

$$J_{(100)}^{T,t} = \frac{1}{2} \int_t^{T} (T-t_1)^2 df_{t_1},$$

$$J_{(010)}^{T,t} = \int_t^{T} (t_1 - t)(T-t_1) df_{t_1},$$

$$J_{(110)}^{T,t} = \int_t^{T} (T-t_2) \int_t^{t_2} df_{t_1} df_{t_2},$$

$$J_{(101)}^{T,t} = \int_t^{T} \int_t^{t_2} (t_2 - t_1) df_{t_1} df_{t_2},$$

$$J_{(1011)}^{T,t} = \int_t^{T} \int_t^{t_2} \int_t^{t_3} (t_2 - t_1) df_{t_1} df_{t_2} df_{t_3},$$

(33)
\begin{align*}
J_{(1101)}(T,t) &= \int_{t}^{T} \int_{t}^{t_2} \int_{t}^{t_3} dt_1 dt_2 dt_3, \\
J_{(1110)}(T,t) &= \int_{t}^{T} (T-t) \int_{t}^{t_2} \int_{t}^{t_3} dt_1 dt_2 dt_3, \\
J_{(1100)}(T,t) &= \frac{1}{2} \int_{t}^{T} (T-t_2)^2 \int_{t}^{t_2} dt_1 dt_2, \\
J_{(1001)}(T,t) &= \frac{1}{2} \int_{t}^{T} (t_2-t_1)^2 dt_1 dt_2, \\
J_{(1000)}(T,t) &= \frac{1}{3!} \int_{t}^{T} (T-t_1)^3 dt_1, \\
J_{(0110)}(T,t) &= \int_{t}^{T} (T-t_2) \int_{t}^{t_2} (t_2-t_1) dt_1 dt_2, \\
J_{(0110)}(T,t) &= \int_{t}^{T} (T-t_2) \int_{t}^{t_2} (t_1-t) dt_1 dt_2, \\
J_{(0101)}(T,t) &= \frac{1}{2} \int_{t}^{T} (T-t_2) (t_2-t_1) dt_1 dt_2, \\
J_{(0100)}(T,t) &= \frac{1}{2} \int_{t}^{T} (T-t_2)^2 (t_1-t) dt_1 dt_2, \\
J_{(0010)}(T,t) &= \frac{1}{2} \int_{t}^{T} (T-t_2)^2 (t_1-t) dt_1 dt_2, \\
J_{(0001)}(T,t) &= \frac{1}{2} \int_{t}^{T} (T-t_1)^2 (t_1-t) dt_1 dt_2, \\
J_{(0000)}(T,t) &= \frac{1}{3!} \int_{t}^{T} (T-t_1)^3 dt_1, \\
J_{(10\underbrace{\ldots}_{k-1}0)\underbrace{\ldots}_{k-1}0}(T,t) &= \frac{1}{(k-1)!} \int_{t}^{T} (T-t_1)^{k-1} dt_1, \\
J_{(11\underbrace{\ldots}_{k-2}0\underbrace{\ldots}_{k-2}0)(T,t) &= \frac{1}{(k-2)!} \int_{t}^{T} (T-t_2)^{k-2} \int_{t}^{t_2} dt_1 dt_2. 
\end{align*}
\[
\begin{align*}
J_{(1\ldots1\ 0)}^{T,t} & = \int_{t}^{T} (T - t_{1})J_{(1\ldots1)}^{(t_{1},t_{2})} dt_{1}, \\
J_{(1\ 0\ldots0)}^{T,t} & = \frac{1}{(k-2)!} \int_{t}^{T} \int_{t}^{t_{2}} (t_{2} - t_{1})^{k-2} dt_{1} dt_{2}, \\
J_{(10\ldots1)}^{T,t} & = \int_{t}^{T} \int_{t}^{t_{2}} \ldots \int_{t}^{t_{k-2}} (t_{k-1} - t_{k-2}) dt_{1} \ldots dt_{k-1}, \\
J_{(1\ldots1\ 01)}^{T,t} & = \int_{t}^{T} \int_{t}^{t_{k-1}} \int_{t}^{t_{k-2}} \ldots \int_{t}^{t_{2}} (t_{k-1} - t_{k-2}) dt_{1} \ldots dt_{k-1}, \\
J_{(10)}^{T,t} + J_{(01)}^{T,t} & = (T - t)J_{(1)}^{T,t}, \\
J_{(11)}^{T,t} + J_{(101)}^{T,t} + J_{(011)}^{T,t} & = (T - t)J_{(11)}^{T,t}, \\
J_{(001)}^{T,t} + J_{(010)}^{T,t} + J_{(100)}^{T,t} & = \frac{(T - t)^2}{2} J_{(1)}^{T,t}, \\
J_{(1100)}^{T,t} + J_{(1010)}^{T,t} + J_{(1001)}^{T,t} + J_{(0110)}^{T,t} & + J_{(0101)}^{T,t} + J_{(0011)}^{T,t} = \frac{(T - t)^2}{2} J_{(11)}^{T,t}, \\
J_{(1000)}^{T,t} + J_{(0100)}^{T,t} + J_{(0010)}^{T,t} + J_{(0001)}^{T,t} & = \frac{(T - t)^3}{3!} J_{(1)}^{T,t}, \\
J_{(1110)}^{T,t} + J_{(1101)}^{T,t} + J_{(1011)}^{T,t} + J_{(0111)}^{T,t} & = (T - t)J_{(111)}^{T,t}, \\
\sum_{l=1}^{k} J_{(1\ldots1\ 0\ldots0)}^{T,t} & = \frac{1}{(k-1)!} (T - t)^{k-1} J_{(1)}^{T,t}, \\
\sum_{l=1}^{k} J_{(1\ldots1\ 0\ldots1)}^{T,t} & = (T - t)J_{(1\ldots1)}^{T,t},
\end{align*}
\]
\[
\sum_{\underset{i \in \{0,1\}, i=1,...,k}{l_1+...+l_k=m}} J_{(l_1...l_k)T,t} = (T-t)^{k-m} \frac{\prod_{i=1}^{m} J_{(i...1)T,t}}{(k-m)!}
\]

where

\[
J_{(l_1...l_k)T,t} = \int_t^T \int_t^{l_2} \int_t^{l_3} dw_{l_1} \cdots dw_{l_k},
\]

\(l_i = 1\) when \(w_{l_i}^{(i)} = f_{t_i}\) and \(l_i = 0\) when \(w_{l_i}^{(i)} = t_i\) \((i = 1, \ldots, k)\), \(f_t\) is a standard Wiener process.

Let us consider two examples and show explicitly the technique on integration order replacement for iterated Ito stochastic integrals.

**Example 1.** Let us prove the equality \(33\). Using Theorems 1 and 3, we obtain

\[
J_{(110)T,t} \overset{\text{def}}{=} \int_t^T \int_t^{l_2} \int_t^{l_3} df_{l_1} df_{l_2} dt_3 =
\]

\[
= \int_t^T df_{l_1} \int_t^{l_2} df_{l_2} \int_t^{l_3} dt_3 =
\]

\[
= \int_t^T df_{l_1} \int_t^{l_2} df_{l_2} (T-t_2) =
\]

\[
= \int_t^T df_{l_1} \int_t^{l_2} (T-t_2) df_{l_2} =
\]

\[
= \int_t^{(T-t_2)} df_{l_1} df_{l_2} \text{ w. p. } 1.
\]

**Example 2.** Let us prove the equality \(34\). Using Theorems 1 and 3, we obtain

\[
J_{(1010)T,t} \overset{\text{def}}{=} \int_t^T \int_t^{l_3} \int_t^{l_2} \int_t^{l_4} df_{l_1} dt_2 df_{l_3} dt_4 =
\]

\[
= \int_t^T df_{l_1} \int_t^{l_2} \int_t^{l_3} \int_t^{l_4} df_{l_2} df_{l_3} dt_4 =
\]

\[
= \int_t^T df_{l_1} \int_t^{l_2} \int_t^{l_3} \int_t^{l_4} df_{l_2} dt_4 =
\]

\[
= \int_t^T df_{l_1} \int_t^{l_2} \int_t^{l_3} (T-t_3) =
\]
7. Integration Order Replacement Technique for Iterated Stochastic Integrals With Respect to Martingale

In this section, we will generalize the theorems on integration order replacement for iterated Ito stochastic integrals to the class of iterated stochastic integrals with respect to martingale.

Let \((\Omega, F, \mathbb{P})\) be a complete probability space and let \(\{F_t, t \in [0, T]\}\) be a nondecreasing family of \(\sigma\)-algebras defined on the probability space \((\Omega, F, \mathbb{P})\). Suppose that \(M_t, t \in [0, T]\) is an \(F_t\)-measurable martingale for all \(t \in [0, T]\), which satisfies the condition \(M \{\|M_t\| < \infty\}. Moreover, for all \(t \in [0, T]\) there exists an \(F_t\)-measurable and nonnegative w. p. 1 stochastic process \(\rho_t, t \in [0, T]\) such that

\[
\mathbb{M} \left\{ \left( M_s - M_t \right)^2 \mid F_t \right\} = \mathbb{M} \left\{ \int_t^s \rho_r \, dr \mid F_t \right\} \quad \text{w. p. 1,}
\]

where \(0 \leq t < s \leq T\).

Let us consider the class \(H_2(\rho, [0, T])\) of stochastic processes \(\phi_t, t \in [0, T]\), which are \(F_t\)-measurable for all \(t \in [0, T]\) and satisfy the condition

\[
\mathbb{M} \left\{ \int_0^T \phi_t^2 \rho_t \, dt \right\} < \infty.
\]
For any partition $\tau_j^{(N)}$, $j = 0, 1, \ldots, N$ of the interval $[0, T]$ such that

\begin{equation}
0 = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \quad \max_{0 \leq j \leq N-1} |\tau_j^{(N)} - \tau_{j+1}^{(N)}| \to 0 \text{ if } N \to \infty
\end{equation}

we will define the sequence of step functions

$$
\varphi^{(N)}(t, \omega) = \varphi_j(\omega) \text{ w. p. 1 for } t \in \left[\tau_j^{(N)}, \tau_{j+1}^{(N)}\right),
$$

where $j = 0, 1, \ldots, N - 1$, $N = 1, 2, \ldots$

Let us define the stochastic integral with respect to martingale for $\varphi(t, \omega) \in H_2(\rho, [0, T])$ as the following mean-square limit [1]

$$
l.i.m. \sum_{j=0}^{N-1} \varphi^{(N)}(\tau_j^{(N)}, \omega) \left(M \left(\tau_{j+1}^{(N)}, \omega\right) - M \left(\tau_j^{(N)}, \omega\right)\right) \overset{\text{def}}{=} \int_0^T \varphi \, dM_t,
$$

where $\varphi^{(N)}(t, \omega)$ is any step function from the class $H_2(\rho, [0, T])$, which converges to the function $\varphi(t, \omega)$ in the following sense

$$
\lim_{N \to \infty} \int_0^T \mathbb{E}\left\{\left|\varphi^{(N)}(t, \omega) - \varphi(t, \omega)\right|^2\right\} \rho \, dt = 0.
$$

It is well known [1] that the stochastic integral

$$
\int_0^T \varphi \, dM_t
$$

exists and it does not depend on the selection of sequence $\varphi^{(N)}(t, \omega)$.

Let $H_2(\rho, [0, T])$ be the class of stochastic processes $\varphi$, $\tau \in [0, T]$, which are mean-square continuous for all $\tau \in [0, T]$ and belong to the class $H_2(\rho, [0, T])$.

Let us consider the following iterated stochastic integrals

\begin{equation}
S[\phi, \psi^{(k)}]_{T,t} = \int_t^T \psi_1(t_1) \ldots \int_t^{t_{k-1}} \psi_k(t_k) \int_t^{t_k} \phi \, dM_{\tau}^{(k+1)} \, dM_t^{(k)} \ldots dM_{t_1}^{(1)},
\end{equation}

\begin{equation}
S[\psi^{(k)}]_{T,t} = \int_t^T \psi_1(t_1) \ldots \int_t^{t_{k-1}} \psi_k(t_k) \, dM_t^{(k)} \ldots dM_{t_1}^{(1)}.
\end{equation}

Here $\phi \in \hat{H}_2(\rho, [t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau)$ are continuous nonrandom functions at the interval $[t, T]$, $M_{\tau}^{(l)} = M_{\tau}$ or $M_{\tau}^{(l)} = \tau$ if $\tau \in [t, T]$, $l = 1, \ldots, k + 1$, $M_\tau$ is the martingale defined above.

Let us define the iterated stochastic integral $\hat{S}[\psi^{(k)}]_{T,s}$, $0 \leq t \leq s \leq T$, $k \geq 1$ with respect to martingale

$$
\hat{S}[\psi^{(k)}]_{T,s} = \int_s^T \psi_k(t_k) \, dM_{t_k}^{(k)} \ldots \int_s^{t_2} \psi_1(t_1) \, dM_{t_1}^{(1)}
$$

by the following recurrence relation
For example, if we not require the boundedness of the process \( \rho \), then the stochastic integral

\[
(38) \quad \tilde{S}[\psi^{(k)}]_{T,t} \equiv 1 \text{.i.m.} \sum_{i=0}^{N-1} \psi_k(\tau_i) \Delta M_{\tau_i}^{(k)} \tilde{S}[\psi^{(k-1)}]_{T,\tau_{i+1}},
\]

where \( k \geq 1 \), \( \tilde{S}[\psi^{(0)}]_{T,t} \equiv 1 \), \([s, T] \subseteq [t, T]\), and further \( \Delta M_{\tau_i}^{(i)} = M_{\tau_{i+1}}^{(i)} - M_{\tau_i}^{(i)} \), \( i = 1, \ldots, k + 1 \), \( l = 0, 1, \ldots, N - 1 \), \( \{\tau_i\}_{i=0}^N \) is the partition of the interval \([t, T]\), which satisfies the condition similar to (35), another notations are the same as in (36), (37).

Further, let us define the iterated stochastic integral \( \tilde{S}[\phi, \psi^{(k)}]_{T,t}, k \geq 1 \) of the form

\[
\tilde{S}[\phi, \psi^{(k)}]_{T,t} = \int_T^t \phi_s dM_{s}^{(k+1)} \tilde{S}[\psi^{(k)}]_{T,s},
\]

by the equality

\[
\tilde{S}[\phi, \psi^{(k)}]_{T,t} \equiv 1 \text{.i.m.} \sum_{i=0}^{N-1} \phi_{\tau_i} \Delta M_{\tau_i}^{(k+1)} \tilde{S}[\psi^{(k)}]_{T,\tau_{i+1}},
\]

where the sense of notations included in (36)–(38) is saved.

Let us formulate the theorem on integration order replacement for the iterated stochastic integrals with respect to martingale, which is the generalization of Theorem 1.

**Theorem 5** (20, 21) (also see 22–7, 14–18). Let \( \phi_\tau \in \tilde{H}_2(\rho, [t, T]) \), every \( \psi_t(\tau) (l = 1, \ldots, k) \) is a continuous nonrandom function at the interval \([t, T]\), and \( |\rho_\tau| \leq K < \infty \) w. p. 1 for all \( \tau \in [t, T] \). Then, the stochastic integral \( \tilde{S}[\phi, \psi^{(k)}]_{T,t} \) exists and

\[
S[\phi, \psi^{(k)}]_{T,t} = \tilde{S}[\phi, \psi^{(k)}]_{T,t} \quad \text{w. p. 1}.
\]

The proof of Theorem 5 is similar to the proof of Theorem 1.

**Remark 2.** Let us note that we can propose another variant of the conditions in Theorem 5. For example, if we not require the boundedness of the process \( \rho_\tau \), then it is necessary to require the fulfillment of the following additional conditions:

1. \( M\{|\rho_\tau|\} < \infty \) for all \( \tau \in [t, T] \).
2. The process \( \rho_\tau \) is independent with the processes \( \phi_\tau \) and \( M_\tau \).

**Remark 3.** Note that it is well known the construction of stochastic integral with respect to the Wiener process with integrable process, which is not an \( F_\tau \)-measurable stochastic process — the so-called Stratonovich stochastic integral (19).

The stochastic integral \( S[\phi, \psi^{(k)}]_{T,t} \) is also the stochastic integral with integrable process, which is not an \( F_\tau \)-measurable stochastic process. However, under the conditions of Theorem 5

\[
S[\phi, \psi^{(k)}]_{T,t} = \tilde{S}[\phi, \psi^{(k)}]_{T,t} \quad \text{w. p. 1},
\]

where \( S[\phi, \psi^{(k)}]_{T,t} \) is a usual iterated stochastic integral with respect to martingale. If, for example, \( M_\tau, \tau \in [t, T] \) is the Wiener process, then the question on connection between stochastic integral \( \tilde{S}[\phi, \psi^{(k)}]_{T,t} \) and Stratonovich stochastic integral is solving as a standard question on connection between Stratonovich and Ito stochastic integrals (19).
Let us consider several statements, which are the generalizations of theorems formulated in the previous sections.

Assume that $D_k = \{(t_1, \ldots, t_k): t \leq t_1 < \ldots < t_k \leq T\}$ and the following conditions are met:

BI. $\xi_t \in H_2(\rho, [t, T])$.

BIII. $\Phi(t_1, \ldots, t_{k-1})$ is a continuous nonrandom function in the closed domain $D_{k-1}$ (recall that we use the same symbol $D_{k-1}$ to denote the open and closed domains corresponding to the domain $D_{k-1}$; however, we always specify what domain we consider (open or closed)).

Let us define the following stochastic integrals with respect to martingale

\[
\hat{S}[\xi, \Phi]_{T,t}^{(k)} = \int_{t}^{T} \xi_t \, dM_{t_k}^{(k)} \cdots \int_{t_3}^{T} dM_{t_2}^{(2)} \int_{t_2}^{T} \Phi(t_1, t_2, \ldots, t_{k-1}) \, dM_{t_1}^{(1)} \overset{\text{def}}{=} 
\]

\[
\overset{\text{def}}{=} \lim_{N \to \infty} \sum_{l=0}^{N-1} \xi_{t_l} \Delta M_{t_l}^{(k)} \int_{t_{l+1}}^{T} dM_{t_{k-1}}^{(k-1)} \cdots \int_{t_3}^{T} dM_{t_2}^{(2)} \int_{t_2}^{T} \Phi(t_1, t_2, \ldots, t_{k-1}) \, dM_{t_1}^{(1)}
\]

for $k \geq 3$ and

\[
\hat{S}[\xi, \Phi]_{T,t}^{(2)} = \int_{t}^{T} \xi_t \, dM_{t_2}^{(2)} \int_{t_2}^{T} \Phi(t_1) \, dM_{t_1}^{(1)} \overset{\text{def}}{=} 
\]

\[
\overset{\text{def}}{=} \lim_{N \to \infty} \sum_{l=0}^{N-1} \xi_{t_l} \Delta M_{t_l}^{(2)} \int_{t_{l+1}}^{T} \Phi(t_1) \, dM_{t_1}^{(1)}
\]

for $k = 2$, where the sense of notations included in $[36] - [38]$ is saved. Moreover, the stochastic process $\xi_\tau$, $\tau \in [t, T]$ belongs to the class $H_2(\rho, [t, T])$.

In addition, let

\[
S[\xi, \Phi]_{T,t}^{(k)} = \int_{t}^{T} \cdots \int_{t}^{T} \Phi(t_1, \ldots, t_{k-1}) \xi_{t_k} \, dM_{t_k}^{(k)} \cdots dM_{t_1}^{(1)}, \quad k \geq 2,
\]

where the right-hand side of $[39]$ is the iterated stochastic integral with respect to martingale.

Let us introduce the following iterated stochastic integrals with respect to martingale

\[
\hat{S}[\Phi]_{T,t}^{(k-1)} = \int_{t}^{T} dM_{t_{k-1}}^{(k-1)} \cdots \int_{t_3}^{T} dM_{t_2}^{(2)} \int_{t_2}^{T} \Phi(t_1, t_2, \ldots, t_{k-1}) \, dM_{t_1}^{(1)} \overset{\text{def}}{=} 
\]

\[
\overset{\text{def}}{=} \lim_{N \to \infty} \sum_{l=0}^{N-1} \Delta M_{t_l}^{(k-1)} \int_{t_{l+1}}^{T} dM_{t_{k-2}}^{(k-2)} \cdots \int_{t_3}^{T} dM_{t_2}^{(2)} \int_{t_2}^{T} \Phi(t_1, t_2, \ldots, t_{k-1}) \, dM_{t_1}^{(1)},
\]

\[
S'[\Phi]_{T,t}^{(k-1)} = \int_{t}^{T} \cdots \int_{t}^{T} \Phi(t_1, \ldots, t_{k-2}) \, dM_{t_{k-1}}^{(k-1)} \cdots dM_{t_1}^{(1)}, \quad k \geq 2.
\]
It is easy to demonstrate similarly to the proof of Theorem 5 that under the condition BII the stochastic integral $\tilde{S}[\Phi]_{T,t}^{(k-1)}$ exists and

$$S'[\Phi]_{T,t}^{(k-1)} = \tilde{S}[\Phi]_{T,t}^{(k-1)} \text{ w. p. 1.}$$

In its turn, using this fact we can prove the following theorem similarly to the proof of Theorem 5.

**Theorem 6** [20], [21] (also see [2]-[7], [16]-[18]). Let the conditions BI, BII of this section are fulfilled and $|\rho_\tau| \leq K < \infty$ w. p. 1 for all $\tau \in [t,T]$. Then, the stochastic integral $\hat{S}[\xi, \Phi]_{T,t}^{(k)}$ exists and for $k \geq 2$

$$S[\xi, \Phi]_{T,t}^{(k)} = \hat{S}[\xi, \Phi]_{T,t}^{(k)} \text{ w. p. 1.}$$

Theorem 6 is the generalization of Theorem 2 for the case of iterated stochastic integrals with respect to martingale.

Let us consider two statements.

**Theorem 7** [20], [21] (also see [2]-[7], [16]-[18]). Let the conditions of Theorem 5 are fulfilled and $h(\tau)$ is a continuous nonrandom function at the interval $[t,T]$. Then

$$\int_{t}^{T} \phi_\tau dM^{(k+1)}_{\tau} h(\tau) \hat{S}[\psi^{(k)}]_{T,\tau} = \int_{t}^{T} \phi_\tau h(\tau) dM^{(k+1)}_{\tau} \hat{S}[\psi^{(k)}]_{T,\tau} \text{ w. p. 1}$$

and stochastic integrals on the left-hand side of (40) as well as on the right-hand side of (40) exist.

**Theorem 8** [20], [21] (also see [2]-[7], [16]-[18]). Under the conditions of Theorem 7

$$\int_{t}^{T} h(t_1) \int_{t}^{T} \phi_\tau dM^{(k+2)}_{\tau} dM^{(k+1)}_{t_1} \hat{S}[\psi^{(k)}]_{T,t_1} = \int_{t}^{T} \phi_\tau dM^{(k+2)}_{\tau} \int_{t}^{T} h(t_1) dM^{(k+1)}_{t_1} \hat{S}[\psi^{(k)}]_{T,t_1} \text{ w. p. 1.}$$

Moreover, the stochastic integrals in (41) exist.

The proofs of Theorems 7 and 8 are similar to the proofs of Theorems 3 and 4 correspondingly.

**Remark 4.** The integration order replacement technique for iterated Ito stochastic integrals (Theorems 1–4) [2]-[15], [16]-[18] has been successfully applied for construction of the so-called unified Taylor–Ito and Taylor–Stratonovich expansions [16]-[18] (see references therein) as well as for proof and development of the mean-square approximation method for iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series [16]-[18] (see references therein).

**References**

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations. Kiev, Naukova Dumka Publ., 1968. 354 pp.
[21] Kuznetsov D.F. Integration order replacement in iterated stochastic integrals with respect to martingale. Preprint. St.-Petersburg: SPbGTU Publ., 1999 , 11 pp. Available at: [http://www.sde-kuznetsov.spb.ru/99c.pdf](http://www.sde-kuznetsov.spb.ru/99c.pdf)

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde.kuznetsov@inbox.ru