On the essential spectrum of $\lambda$-Toeplitz operators over compact Abelian groups

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Abstract. In the recent paper by Mark C. Ho (2014) the notion of a $\lambda$-Toeplitz operator on the Hardy space $H^2(\mathbb{T})$ over the one-dimensional torus $\mathbb{T}$ was introduced and it was shown (under the supplementary condition) that for $\lambda \in \mathbb{T}$ the essential spectrum of such an operator is invariant with respect to the rotation $z \mapsto \lambda z$; if in addition $\lambda$ is not of finite order the essential spectrum is circular. In this paper, we generalize these results to the case when $\mathbb{T}$ is replaced by an arbitrary compact Abelian group whose dual is totally ordered.

Keywords: Compact Abelian group; Totally ordered group; Toeplitz operators; Weighted Composition operators; Essential spectrum

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1 Introduction

In the paper [4] the notion of a $\lambda$-Toeplitz operator on the Hardy space $H^2(\mathbb{T})$ over the one-dimensional torus $\mathbb{T}$ was introduced and it was shown (under the supplementary condition that some modification of the symbol belongs to $L^\infty(\mathbb{T})$) that for $\lambda \in \mathbb{T}$ the essential spectrum of such an operator is invariant with respect to the rotation $z \mapsto \lambda z$; if in addition $\lambda$ is not of finite order the essential spectrum is circular. In this paper, we generalize these results to the case when the group $\mathbb{T}$ is replaced by an arbitrary compact Abelian group whose character group is totally ordered. We use methods and results from [4], [7], [1], and [5].

Throughout the paper, $G$ is a nontrivial compact and connected Abelian group with the normalized Haar measure $m$ and totally ordered character group $X$, and $X_+$ is the positive cone in $X$. It means that in the group $X$ there is a distinguished subsemigroup $X_+$ containing the identity character...
1 and such that $X_+ \cap X_+^{-1} = \{1\}$ and $X_+ \cup X_+^{-1} = X$. The rule $\xi \leq \chi := \xi^{-1}\chi \in X_+$ defines a total order in $X$ which agrees with the structure of the group. We put also $X_- := X \setminus X_+ (= X_+^{-1} \setminus \{1\})$. As an example, let $X$ be an additive subgroup of $\mathbb{R}^n$ endowed with the discrete topology and lexicographical order so that $G$ is its Bohr compactification (or $X = \mathbb{Z}_\infty$, the direct sum of countably many copies of $\mathbb{Z}$ so that $G = \mathbb{T}_\infty$, the infinite-dimensional torus). As is well known, a (discrete) Abelian group $X$ can be totally ordered if and only if it is torsion-free (see, for example, [16]), which in turn is equivalent to the condition that its character group $G$ is connected (see [15]); the total order on $X$ here is not, in general, unique.

Let $H^p(G)$ ($1 \leq p \leq \infty$) denotes the subspace of functions $f \in L^p(G)$ whose Fourier transforms $\hat{f}$ are concentrated on $X_+$. We equip the space $H^2(G)$ with the inner product $\langle \cdot, \cdot \rangle$ induced from $L^2(G)$ (see [16]). Note that the set $X_+(X_+)$ is an orthonormal basis of the space $L^2(G)$ (of the space $H^2(G)$, respectively).

A Toeplitz operator $T\varphi$ on $H^2(G)$ with a symbol $\varphi \in L^\infty(G)$ is defined as follows: $T\varphi f = P_+(\varphi f)$, $f \in H^2(G)$ where $P_+: L^2(G) \to H^2(G)$ is the orthogonal projection. These operators were defined by Murphy in [7] and intensively studied (see, for example, [7] – [13], and [17]). In the paper [5] Toeplitz operators on Banach spaces $H^p(G)$ were considered.

### 2 \quad \lambda$-Toeplitz operators over compact Abelian groups and their connection with Toeplitz operators

**Definition 2.1.** Let $\lambda : X_+ \to \mathbb{C}$. A linear bounded operator $T$ on $H^2(G)$ is said to be $\lambda$-Toeplitz if

$$\langle T(\chi \xi), \chi \eta \rangle = \lambda(\chi)\langle T\xi, \eta \rangle$$

for all $\chi, \xi, \eta \in X_+$.

Recall that a bounded semicharacter of a discrete semigroup $S$ is a nonzero homomorphism from $S$ into the multiplicative semigroup $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$.

**Lemma 2.2.** For every $\lambda$-Toeplitz operator $T \neq 0$ the map $\lambda$ is a bounded semicharacter of the semigroup $X_+$. Moreover, this map extends uniquely to a character of the group $X$ if $|\lambda(\chi)| = 1$ for all $\chi \in X_+$.

**Proof.** From the identity

$$\lambda(\chi_1\chi_2)\langle T\xi, \eta \rangle = \langle T(\chi_1\chi_2\xi), \chi_1\chi_2\eta \rangle = \lambda(\chi_1)\lambda(\chi_2)\langle T\xi, \eta \rangle,$$
and $\lambda(1) = 1$ it follows that $\lambda$ is a nontrivial homomorphism from $X_+$ into the multiplicative semigroup $\mathbb{C}$. Its boundedness follows from the boundedness of $T$. Now, let $|\lambda(\chi)| = 1$ for all $\chi \in X_+$. Every $\theta \in X$ has the form $\xi^{-1}\chi$ where $\xi, \chi \in X_+$, and it is easy to verify that $\lambda$ correctly extends to the character $\lambda$ of the whole group $X$ by the formula $\lambda(\theta) := \lambda(\xi)^{-1}\lambda(\chi)$. □

In the following unless otherwise stipulated we suppose that $|\lambda(\chi)| = 1$ for all $\chi \in X_+$. Taking the Pontrjagin-van Kampen duality theorem and Lemma 2.2 into account we can assume that $\lambda \in G$ and identify $\lambda(\chi)$ with $\chi(\lambda)$ for $\chi \in X$.

To state our first result we need several definitions.

**Definition 2.3.** The function $\varphi \in L^2(G)$ with the Fourier transform

$$\hat{\varphi}(\chi) = \begin{cases} \langle T1, \chi \rangle & \text{if } \chi \in X_+, \\ \langle T\chi^{-1}, 1 \rangle & \text{if } \chi \in X_- \end{cases}$$

is called the symbol of $\lambda$-Toeplitz operator $T$.

By the Bessel’s inequality and Plancherel theorem the above definition is correct because

$$\sum_{\chi \in X} |\hat{\varphi}(\chi)|^2 = \sum_{\chi \in X_+} |\langle T1, \chi \rangle|^2 + \sum_{\xi \in X_+ \setminus \{1\}} |\langle T\xi, 1 \rangle|^2 =$$

$$\sum_{\chi \in X_+} |\langle T1, \chi \rangle|^2 + \sum_{\xi \in X_+ \setminus \{1\}} |\langle T^*1, \xi \rangle|^2 \leq \|T1\|^2 + \|T^*1\|^2.$$

From now on $T_{\lambda,\varphi}$ denotes the $\lambda$-Toeplitz operator with $\lambda \in G$ and the symbol $\varphi$.

**Definition 2.4.** By the modified symbol of $\lambda$-Toeplitz operator $T_{\lambda,\varphi}$ we call the function $\varphi_{\lambda} \in L^2(G)$ with the Fourier transform

$$\hat{\varphi}_{\lambda}(\chi) = \begin{cases} \lambda(\chi)\hat{\varphi}(\chi) & \text{if } \chi \in X_+, \\ \hat{\varphi}(\chi) & \text{if } \chi \in X_- \end{cases}$$

Define also the unitary operator $U_{\lambda}$ on $H^2(G)$ by the formula $U_{\lambda}f(x) := f(\lambda x)$.

**Theorem 2.5.** The modified symbol $\varphi_{\lambda}$ of $\lambda$-Toeplitz operator $T_{\lambda,\varphi}$ belongs to $L^\infty(G)$ and

$$T_{\lambda,\varphi} = U_{\lambda}T_{\varphi_{\lambda}}.$$

In particular, $\|T_{\lambda,\varphi}\| = \|\varphi_{\lambda}\|_\infty$.

Conversely, for every $\lambda \in G$ and $\psi \in L^\infty(G)$ the operator $U_{\lambda}T_{\psi}$ is $\lambda$-Toeplitz with modified symbol $\psi$.

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3In [4] this function was denoted by $\varphi_{\lambda,+}$. 

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Proof. Consider the bounded operator $A = U_\lambda^{-1}T_{\lambda,\varphi}$ on $H^2(G)$. For all $\chi_1, \chi_2 \in X_+$ we have

$$\langle A\chi_1, \chi_2 \rangle = \langle U_\lambda^{-1}T_{\lambda,\varphi}\chi_1, \chi_2 \rangle = \langle T_{\lambda,\varphi}\chi_1, U_\lambda\chi_2 \rangle = \overline{\lambda(\chi_2)}(T_{\lambda,\varphi}\chi_1, \chi_2).$$

Two cases are possible.

1) $\chi_1^{-1}\chi_2 \in X_+$. Then

$$\langle A\chi_1, \chi_2 \rangle = \overline{\lambda(\chi_2)}\lambda(\chi_1^{-1}\chi_2)^{-1}\langle T_{\lambda,\varphi}(\chi_1^{-1}\chi_2)\chi_1, (\chi_1^{-1}\chi_2)\chi_2 \rangle = \lambda(\chi_1^{-1}\chi_2)^{-1}\overline{\lambda(\chi_2)}\langle T_{\lambda,\varphi}\chi_2, (\chi_1^{-1}\chi_2)\chi_2 \rangle = \overline{\lambda(\chi_1^{-1}\chi_2)}(T_{\lambda,\varphi}^{-1}, \chi_1^{-1}\chi_2).$$

2) $\chi_1^{-1}\chi_2 \in X_-$. Then

$$\langle A\chi_1, \chi_2 \rangle = \overline{\lambda(\chi_2)}\langle T_{\lambda,\varphi}\chi_2(\chi_1\chi_2^{-1}), \chi_2 \rangle = \langle T_{\lambda,\varphi}(\chi_1\chi_2^{-1}), 1 \rangle.$$  

So in both cases $\langle A\chi_1, \chi_2 \rangle = a(\chi_1^{-1}\chi_2)$ for all $\chi_1, \chi_2 \in X_+$ where

$$a(\chi) = \begin{cases} \overline{\lambda(\chi)}\langle T_{\lambda,\varphi}^{-1}, 1 \rangle & \text{if } \chi \in X_+, \\ \langle T_{\lambda,\varphi}\chi^{-1}, 1 \rangle & \text{if } \chi \in X_- \end{cases}.$$  

By [5, Theorem 1], we have $A = T_\psi$ with $\psi \in L^\infty$ and $\hat{\psi} = a$, $||T_\psi|| = ||\psi||_\infty$. Since $a = \varphi_\lambda$, the first statement is proved.

Finally, if $T = U_\lambda T_\psi$ for some $\lambda \in G$, $\psi \in L^\infty$, then for all $\chi, \xi, \eta \in X_+$ we have, since $U_\lambda^{-1}\xi = \overline{\lambda(\xi)}\zeta$ ($\zeta \in X_+$),

$$\langle T(\chi\xi), \chi\eta \rangle = \langle T_\psi(\chi\xi), U_\lambda^{-1}(\chi\eta) \rangle = \lambda(\chi)\overline{\lambda(\eta)}\langle T_\psi(\chi\xi), \chi\eta \rangle = \lambda(\chi)\lambda(\eta)\langle T_\psi\xi, \eta \rangle = \lambda(\chi)\langle T_\psi\xi, \eta \rangle,$$

Thus $T = T_{\lambda,\varphi} = U_\lambda T_{\varphi,\lambda}$ for some $\varphi, \lambda \in L^\infty(G)$. So $U_\lambda T_\psi = U_\lambda T_{\varphi,\lambda}$, which implies $\psi = \varphi_\lambda$ (see [7, Theorem 3.2]). The proof is complete. □

In the following, $S_\chi$ ($\chi \in X_+$) denotes the isometry $f \mapsto \chi f$ of the space $H^2(G)$.

**Corollary 2.6.** A linear bounded operator $T$ on $H^2(G)$ is $\lambda$-Toeplitz if and only if

$$S_\chi^*TS_\chi = \lambda(\chi)T$$

for all $\chi \in X_+$.

**Proof.** The necessity can be verified directly. To prove the sufficiency, consider the operator $A := U_\lambda^{-1}T$. If we replace $T$ by $U_\lambda A$ in (1), we get for all $\chi, \xi, \eta \in X_+$

$$\langle S_\chi^*U_\lambda AS_\chi\xi, \eta \rangle = \lambda(\chi)\langle U_\lambda A\xi, \eta \rangle.$$
In other words,

\[ \langle A(\chi \xi), U^{-1}_\lambda(\chi \eta) \rangle = \lambda(\chi) \langle A \xi, U^{-1}_\lambda \eta \rangle. \]

Using \( U^{-1}_\lambda \zeta = \overline{\lambda(\zeta)} \zeta \) (\( \zeta \in X_+ \)), we obtain

\[ \langle A(\chi \xi), \chi \eta \rangle = \langle A \xi, \eta \rangle. \]

It follows [7, Theorem 3.10] that \( A = T_\psi \) for some \( \psi \in L^\infty(G) \). □

Remark. As it was mentioned in [4] for the classical case \( G = \mathbb{T} \) the problem of studying operators with the property (1) was posed in [3, p. 629 – 630].

Corollary 2.7. Every \( \lambda \)-Toeplitz operator is uniquely determined by the pair \( (\lambda, \varphi) \) where \( \lambda \in G \) and \( \varphi \in L^2(G) \) is such that \( \varphi_\lambda \in L^\infty(G) \).

Proof. The necessity for the condition \( \varphi_\lambda \in L^\infty(G) \) was proved above. Conversely, if the pair \( (\lambda, \varphi) \) meets all the conditions of this corollary, the operator \( T := U_\lambda T_\varphi \) is \( \lambda \)-Toeplitz with a symbol \( \varphi \) and it is obvious in view of the preceding theorem that two \( \lambda \)-Toeplitz operators with the same symbol \( \varphi \) coincide. □

Let \( C(G) \) denotes the algebra of continuous functions on \( G \) and \( C(G)^{-1} \) the group of invertible elements of \( C(G) \). To state and prove the next corollary (and several other results below), we need the notion of the rotation index for functions in some subgroup of \( C(G)^{-1} \) given in [5]. We begin with the definition of the rotation index for a character of the group \( G \) (the symbol \( \sharp F \) will denote the number of elements of a finite set \( F \) in what follows).

Definition 2.8. In each of the following cases, we define the rotation index of a character \( \chi \in X \) as follows:

1) \( \text{ind} \chi = \sharp(X_+ \setminus \chi X_+) \) if \( \chi \in X_+ \) and the set \( X_+ \setminus \chi X_+ \) is finite;
2) \( \text{ind} \chi = \text{ind} \chi_1 - \text{ind} \chi_2 \) if \( \chi = \chi_1 \chi_2^{-1} \), where \( \chi_j \in X_+ \), where both sets \( X_+ \setminus \chi_j X_+ \) are finite, \( j = 1, 2 \).

In the other cases we assume that the character has no index.

We denote the set of characters that have an index by \( X^i \). It follows from [5, Theorem 2] that \( X^i \) is a cyclic subgroup of \( X \) and it is nontrivial if and only if \( X \) contains the smallest strictly positive element.

Definition 2.9. Consider a function \( \varphi \in C(G)^{-1} \) of the form \( \chi \exp(g) \), where \( g \in C(G) \) and \( \chi \in X \) (the Bohr-van Kampen decomposition). If \( \chi \in X^i \), then we set \( \text{ind} \varphi = \text{ind} \chi \). Otherwise we assume that the function \( \varphi \) has no index.

We denote the set of functions in \( C(G)^{-1} \) which have an index by \( \Phi(G) \). Thus, \( \Phi(G) = X^i \exp(C(G)) \).

We recall that, for a bounded operator \( T \) on a Banach space \( Y \), the symbol \( T \in \Phi_+(Y) \) means that the image \( \text{Im} T \) is closed and the kernel \( \text{Ker} T \) is finite-dimensional, whereas the symbol \( T \in \Phi_-(Y) \) means that the quotient space
Y/ImT is finite-dimensional; the operators in $\Phi_+(Y) \cup \Phi_-(Y)$ are referred to as semi-Fredholm operators, whereas those in $\Phi_+(Y) \cap \Phi_-(Y)$ are called Fredholm operators on $Y$.

Corollary 2.10. Let $\varphi_\lambda \in C(G)$. The $\lambda$-Toeplitz operator $T_{\lambda, \varphi}$ is Fredholm if and only if $\varphi_\lambda \in \Phi(G)$. In this case,

$$\text{Ind}T_{\lambda, \varphi} = -\text{ind}\varphi_\lambda.$$  

**Proof.** It follows from the above theorem and [5, Theorem 4]. □

Corollary 2.11. If an operator $T_{\lambda, \varphi}$ is semi-Fredholm, then its modified symbol $\varphi_\lambda$ is invertible in the algebra $L^\infty(G)$.

**Proof.** It follows from the above theorem and [5, Theorem 3]. □

Corollary 2.12. A $\lambda$-Toeplitz operator is compact if and only if it is zero.

**Proof.** It follows from the above theorem and [7, Theorem 3.5]. □

Corollary 2.13. A linear bounded operator $T$ on $H^2(G)$ is $\lambda$-Toeplitz if and only if $\chi_2(\lambda)(T\chi_1, \chi_2)$ depends on $\chi_1^{-1}\chi_2$ only ($\chi_1, \chi_2 \in X_+)$.

**Proof.** It follows from the above theorem and [5, Theorem 1]. □

## 3 Rotational invariance for the essential spectrum

In the following, $\sigma_e(T)$ stands for the essential (Fredholm) spectrum of an operator $T$, $\rho_e(T) = \mathbb{C} \setminus \sigma_e(T)$.

**Lemma 3.1.** Let $\lambda$ be a nonvanishing complex-valued function defined on the set $X_+ \cap X^i$, $T$ a linear bounded operator on $H^2(G)$ such that $S^\ast_\chi(TS_\chi) = \lambda(\chi)T$ for all $\chi \in X_+ \cap X^i$. Suppose that $T - \mu$ is Fredholm. Then for every $\chi \in X_+ \cap X^i$ the operator $T - \lambda(\chi)^{-1}\mu$ is also Fredholm and

$$\text{Ind}(T - \mu) = \text{Ind}(T - \lambda(\chi)^{-1}\mu).$$

Consequently, for every $\chi \in X_+ \cap X^i$ we have $\lambda(\chi)^{-1}\rho_e(T) \subseteq \rho_e(T)$.

**Proof.** First note that $S_\chi (= T_\chi)$ is Fredholm for every $\chi \in X_+ \cap X^i$ by [5, Theorem 4]. Now the operator $T - \lambda(\chi)^{-1}\mu$ is Fredholm, since

$$T - \lambda(\chi)^{-1}\mu = \lambda(\chi)^{-1}S^\ast_\chi(T - \mu)S_\chi.$$  

We conclude also that $\rho_e(T) \subseteq \lambda(\chi)\rho_e(T)$, and

$$\text{Ind}(T - \lambda(\chi)^{-1}\mu) = \text{Ind}S^\ast_\chi + \text{Ind}(T - \mu) + \text{Ind}S_\chi = \text{Ind}(T - \mu).$$ □

By definition, put $\tau(x) = \lambda^{-1}x$, $x \in G$. 

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Lemma 3.2. For any \(\lambda\)-Toeplitz operator \(T_{\lambda,\varphi}\) we have
\[
T_{\lambda,\varphi}^k = U_{\lambda}^k T_{\varphi \lambda \circ \tau^k} \cdots T_{\varphi \lambda}
\]
for \(k = 1, 2, \ldots\).

Proof. First we prove that
\[
T_f U_{\lambda}^k = U_{\lambda}^k T_f \circ \tau^k
\]
for \(f \in L^\infty(G), k = 1, 2, \ldots\).
Indeed, using [5, Theorem 1] we have for \(\chi, \eta \in X_+\)
\[
\langle T_f U_{\lambda}^k \chi, \eta \rangle = \lambda(\chi) \lambda(\eta) \langle T_f \chi, \eta \rangle = \lambda(\chi) \lambda(\eta) \langle T_f \chi, \eta \rangle = \lambda(\chi) \lambda(\eta) \langle T_f \chi, \eta \rangle.
\]
and, on the other hand,
\[
\langle U_{\lambda}^k T_{f \circ \tau^k} \chi, \eta \rangle = \langle T_{f \circ \tau^k} \chi, U_{\lambda}^{-1} \eta \rangle = \lambda(\eta) \langle T_{f \circ \tau^k} \chi, \eta \rangle = \lambda(\eta) \langle T_{f \circ \tau^k} \chi, \eta \rangle.
\]
Now in view of Theorem 2.5 the statement of the Lemma follows by induction. \(\square\)

In what follows we put \(\Phi_{\lambda} = \prod_{j=0}^{q-1} \varphi_{\lambda \circ \tau^j}.\)

Our main result is the following.

Theorem 3.3. Let \(\lambda \in G\).
1) \(\sigma_e(T_{\lambda,\varphi}) = \lambda(\chi) \sigma_e(T_{\lambda,\varphi})\) for all \(\chi \in X^i\).
2) If the number \(\lambda(\chi)\) is not of finite order in \(T\) for some \(\chi \in X_i\), then \(\sigma_e(T_{\lambda,\varphi})\) is circular.
3) Suppose \(\lambda\) is of order \(q\) in \(G\), and the number \(\lambda(\chi_0)\) is a primitive root of 1 of order \(q\) for some \(\chi_0 \in X^i\). Then
\[
\text{Ind}(T_{\lambda,\varphi} - \mu) = q \text{Ind}(T_{\varphi \lambda \circ \tau^q \cdots \tau^q - \mu^q})\]
for all \(\mu \in \rho_e(T_{\lambda,\varphi})\).
4) Let \(\varphi \in H^\infty(G)\). Then \(T_{\lambda,\varphi}\) is a weighted shift operator on \(H^2(G)\) and
\[
\sigma(T_{\lambda,\varphi}) = \lambda(\chi) \sigma(T_{\lambda,\varphi})\]
for all \(\chi \in X\). Consequently, \(\sigma(T_{\lambda,\varphi})\) is circular if \(\lambda\) is not of finite order in \(G\).
5) Let \(\varphi \in H^\infty(G)\). Suppose \(\lambda\) is of order \(q\) in \(G\) and the number \(\lambda(\chi_0)\) is a primitive root of 1 of order \(q\) for some \(\chi_0 \in X^i\). Then
\[
(i) \quad \sigma_e(T_{\lambda,\varphi}) = \mu \in \mathbb{C} : \mu^q \in \sigma_e(T_{\varphi \lambda})\]
(ii) \(\sigma(T_{\lambda,\varphi}) = \mu \in \mathbb{C} : \mu^q \in \sigma(T_{\varphi \lambda})\).

Proof. To prove 1), we can assume that \(X^i \neq \{1\}\). First suppose that
the number \(\lambda(\chi)\) is of finite order in \(T\) for some \(\chi \in X^i \setminus \{1\}\). It follows, since the group \(X^i\) is cyclic [5, Theorem 2], that the number \(\lambda(\chi)\) is of finite order.

\(\text{In [4] this function was denoted by } \Phi_{\lambda+}.\)
for every \( \chi \in X^i \). Let \( \chi \in X_+ \cap X^i \) and \( \lambda(\chi) \) is of order \( r \). Then Corollary 2.6 and Lemma 3.1 imply

\[
\rho_e(T_{\lambda, \varphi}) \geq \lambda(\chi)^{-1} \rho_e(T_{\lambda, \varphi}) \geq \cdots \geq \lambda(\chi)^{-r} \rho_e(T_{\lambda, \varphi}) = \rho_e(T_{\lambda, \varphi}).
\]

Since, by [5, Theorem 2, 2)], \( X^i = (X_+ \cap X^i) \cup (X_+ \cap X^i)^{-1} \), this proves 1) in the case of finite order.

Now suppose that the number \( \lambda(\chi) \) is not of finite order in \( T \) for some (and therefore for all) \( \chi \in X_+ \setminus \{1\} \) and choose \( \chi_1 \in (X_+ \cap X^i) \setminus \{1\} \). Let \( \mu \in \rho_e(T_{\lambda, \varphi}) \). There is an arc \( J \) in the circle \( \{z : |z| = |\mu|\} \) such that \( \mu \in J \subset \rho_e(T_{\lambda, \varphi}) \). Since \( \{\lambda(\chi_1)^k : k = 1, 2, \ldots\} \) is dense in the circle \( T = \{z : |z| = 1\} \), Lemma 3.1 yields

\[
\rho_e(T_{\lambda, \varphi}) \supseteq \bigcup_{k=1}^{\infty} \lambda(\chi_1)^k J = \{z : |z| = |\mu|\}.
\]

This proves 2). Moreover, formula (2) implies \( \lambda(\chi) \mu \in \rho_e(T_{\lambda, \varphi}) \) and therefore \( \rho_e(T_{\lambda, \varphi}) \subseteq \lambda(\chi) \rho_e(T_{\lambda, \varphi}) \) for all \( \chi \in X^i \). Combining this result and Lemma 3.1, we obtain the assertion 1) for \( \chi \in X_+ \cap X^i \) and therefore for all \( \chi \in X^i \). Thus, 1) is proved in the case of infinite order as well.

3) Consider the factorization

\[
T_{\lambda, \varphi}^q - \mu^q = \prod_{k=0}^{q-1} (T_{\lambda, \varphi} - \lambda(\chi_0)^k \mu).
\]

By 1), \( \lambda(\chi_0)^k \mu \in \rho_e(T_{\lambda, \varphi}) \). Hence the operator \( T_{\lambda, \varphi}^q - \mu^q \) is Fredholm. Since, by Lemma 3.2, \( T_{\lambda, \varphi}^q = T_{\varphi(\circ \tau^{-1})} \ldots T_{\varphi_\lambda} \), we conclude in view of Lemma 3.1 and Corollary 2.6 that

\[
\text{Ind}(T_{\varphi(\circ \tau^{-1})} \ldots T_{\varphi_\lambda} - \mu^q) = \sum_{k=0}^{q-1} \text{Ind}(T_{\lambda, \varphi} - \lambda(\chi_0)^k \mu) = q \text{Ind}(T_{\lambda, \varphi} - \mu).
\]

4) First note that, by Theorem 2.5, for \( \varphi \in H^\infty(G) \) the \( \lambda \)-Toeplitz operator has the form

\[
T_{\lambda, \varphi} f(x) = \varphi(x) f(\lambda x)
\]

and so is a weighted shift operator on \( H^2(G) \) (we refer the reader to [11] for the general theory of weighted shift operators). It follows that for all \( \chi \in X \)

\[
S_\chi^{-1} T_{\lambda, \varphi} S_\chi = \chi(\lambda) T_{\lambda, \varphi}
\]
(actually, both sides of this equality are bounded operators and coincide on the basis $X_+$ of $H^2(G)$). In turn, the last equality implies

$$\chi(\lambda)\mu - T_{\lambda, \varphi} = \chi(\lambda)S_{\chi}^{-1}(\mu - T_{\lambda, \varphi})S_{\chi}.$$ 

This proves the first statement. Now the second statement follows from the fact that the set $\{\lambda(\chi) : \chi \in X\}$ is dense in $\mathbb{T}$ if $\lambda$ is an element of infinite order (see, e. g., [1, p. 119]).

5) (i) Since $\varphi \in H^\infty(G)$, we have $\varphi_\lambda \circ \tau_j \in H^\infty(G)$ and therefore $T_{\lambda, \varphi}^q = T_{\Phi_\lambda}$ by Lemma 3.2. Hence, by the spectral mapping theorem,

$$\sigma_e(T_{\lambda, \varphi})^q = \sigma_e(T_{\Phi_\lambda}), \text{ and } \sigma(T_{\lambda, \varphi})^q = \sigma(T_{\Phi_\lambda}). \quad (4)$$

First of all, (4) implies that

$$\sigma_e(T_{\lambda, \varphi}) \subseteq \{ \mu \in \mathbb{C} : \mu^q \in \sigma_e(T_{\Phi_\lambda}) \}.$$ 

Now let $\mu \in \mathbb{C}, \mu^q \in \sigma_e(T_{\Phi_\lambda})$. Then, by (4), one can find $\nu \in \sigma_e(T_{\Phi_\lambda})$ such that $\nu^q = \mu^q$. Using 1), we get for some $k \in \{1, \ldots, q\}$ that $\mu = \nu \lambda(\chi_0)^k \in \sigma_e(T_{\lambda, \varphi})$. This proves (i).

(ii) In view of 4) the proof of this equality is similar to that of (i). □

Remark. For $f \in C(G)$ spectra $\sigma_e(T_j)$ and $\sigma(T_j)$ were calculated in [5]. For $f \in H^\infty(G)$ the spectrum $\sigma(T_j)$ equals to $\sigma_{H^\infty(G)}(f)$, the spectrum of the element $f$ in the algebra $H^\infty(G)$ [7, Theorem 3.12].

Remark. Without the assumption that $\lambda(\chi_0)$ is a primitive root of 1 of order $q$ the conclusions of the part 5) of the above theorem are false as the following simple example shows. Let $G = \mathbb{T}$, $\lambda = 1, q = 2$, $\varphi(z) = z + 2$. Then $T_{\lambda, \varphi} = T_\varphi$ is an analytic Toeplitz operator and therefore $\sigma_e(T_{\lambda, \varphi}) = \varphi(\mathbb{T}) = \mathbb{T} + 2, \sigma(T_{\lambda, \varphi}) = \text{cl}(\varphi(\mathbb{D})) = \overline{\mathbb{D}} + 2$ (see, e. g., [11], p. 98, p. 93). On the other hand, here $\Phi_\lambda(z) = (z + 2)^2$. It follows that $\sigma_e(T_{\lambda, \varphi}) \neq \{ \mu \in \mathbb{C} : \mu^2 \in \Phi_\lambda(\mathbb{T}) \}$ and $\sigma(T_{\lambda, \varphi}) \neq \{ \mu \in \mathbb{C} : \mu^2 \in \text{cl}(\Phi_\lambda(\mathbb{D})) \}$.

4 λ-Toeplitz operators with Arens - Singer symbols

Definition 4.1. The uniform algebra $A(G) := H^\infty(G) \cap C(G)$ is called the Arens - Singer algebra (of the group $G$).

We have the following corollary of Theorem 3.3.

Corollary 4.2. Let $\varphi_\lambda \in A(G)$. Suppose $\lambda$ is of order $q$ in $G$, and the number $\lambda(\chi_0)$ is a primitive root of 1 of order $q$ for some $\chi_0 \in X_+^i$. Then

$$\text{Ind}(T_{\lambda, \varphi} - \mu) = -q^{-1} \text{Ind}(\Phi_\lambda - \mu^q)$$
for all $\mu \in \rho_e(T_{\lambda,\varphi})$;

**Proof.** It follows from the statement 3) of Theorem 3.3 and [5, Theorem 4], since $T_{\varphi,\lambda} \sigma_{q-1} \ldots T_{\varphi,\lambda} - \mu^q = T_{\varphi,\lambda} - \mu^q$ for $\varphi \in A(G)$. □

Recall that a topological group $G$ is called *monothetic with a generator* $\lambda$ if the set $\{\lambda^n : n \in \mathbb{Z}\}$ is dense in $G$. For example, the tori $T^m, m \leq \aleph_0$ are (compact and connected) monothetic groups with a countable base of the topology.

The following fact is a corollary of [1, Theorem 4.4, and Theorem 5.22].

**Lemma 4.3.** Let $G$ be monothetic with a countable base of the topology and $\lambda$ a generator of $G$. If $\varphi \in A(G)$, then $r(T_{\lambda,\varphi})$, the spectral radius of $T_{\lambda,\varphi}$, is

$$\exp \int_G \log |\varphi| dm.$$ 

**Proof.** Since $T_{\lambda,\varphi}$ is a weighted shift operator, we have by [1, Theorem 4.4] $r(T_{\lambda,\varphi}) = \max_{\nu \in \Lambda} \exp \int_G \log |\varphi| d\nu,$

where $\Lambda$ denotes the set of all measures on the Shilov boundary $\partial A(G)$ of the algebra $A = A(G)$, which are ergodic with respect to some homeomorphism $\alpha$ of $\partial A(G)$ associated with $T_{\lambda,\varphi}$. But it is known that $\partial A(G) = G$ [2] and it is easy to verify that $\alpha(x) = \lambda x$ for all $x \in G$. By [1, Theorem 5.22], for $\nu \in \Lambda$ there is such $x_0 \in G$ that

$$\int_G f d\nu = \int_G f(gx_0) dm(g) = \int_G f dm$$

for all $f \in C(G)$. Thus, $\Lambda = \{m\}$ which completes the proof. □

The next theorem is a partial generalization of [4, Proposition 3.5].

**Theorem 4.4.** Let $G$ be monothetic with a countable base of the topology and $\lambda$ a generator of $G$. Suppose that $\varphi \in A(G) \cap C(G)^{-1}$.

1) If $\varphi$ is invertible in $A(G)$, then $\sigma(T_{\lambda,\varphi}) = \sigma_e(T_{\lambda,\varphi}) = \{\mu : |\mu| = r(T_{\lambda,\varphi})\}$.

2) If $\varphi$ is not invertible in $A(G)$, then $\sigma(T_{\lambda,\varphi}) = \{\mu : |\mu| \leq r(T_{\lambda,\varphi})\}$.

**Proof.** 1) Let $\varphi^{-1} \in A(G)$. Since $\varphi$ is invertible in $H^\infty(G)$, the analytic Toeplitz operator $T_{\varphi}$ is invertible and $T_{\varphi}^{-1} = T_{\varphi^{-1}}$. Formula (3) implies that $T_{\lambda,\varphi} = T_{\varphi} U_{\lambda}$. Whence, this operator is invertible and

$$T_{\lambda,\varphi}^{-1} = U_{\lambda^{-1}} T_{\varphi^{-1}} = T_{\lambda^{-1},\varphi^{-1} \sigma}.$$
Moreover, by Lemma 4.3,

\[ r(T_{\lambda,\varphi}^{-1}) = \exp \int_G \log |\varphi^{-1}(\lambda^{-1}x)| dm(x) = \frac{1}{r(T_{\lambda,\varphi})}. \]

For \( 0 < |\mu| < r(T_{\lambda,\varphi}) \) we have

\[ T_{\lambda,\varphi} - \mu = \mu T_{\lambda,\varphi}(\mu^{-1} - T_{\lambda,\varphi}^{-1}). \]

It follows that the operator \( T_{\lambda,\varphi} - \mu \) is invertible, since \( |\mu^{-1}| > 1/r(T_{\lambda,\varphi}) = r(T_{\lambda,\varphi}^{-1}) \). But the spectrum of \( T_{\lambda,\varphi} \) is circular by the assertion 4) of Theorem 3.3, since \( \lambda \) is not of finite order in \( G \). This proves that \( \sigma(T_{\lambda,\varphi}) \) is a circle centered at the origin.

Now let us suppose that \( \sigma(T_{\lambda,\varphi}) \neq \sigma_\\epsilon(T_{\lambda,\varphi}) \). Then the complement in \( \mathbb{C} \) of \( \sigma_\\epsilon(T_{\lambda,\varphi}) \) is connected, and by [15, Corollary XI.8.5 and Theorem II.1.1] the points of \( \sigma(T_{\lambda,\varphi}) \setminus \sigma_\\epsilon(T_{\lambda,\varphi}) \) are isolated points of \( \sigma(T_{\lambda,\varphi}) \). This contradiction concludes the proof of the first statement.

2) First let us prove that \( 0 \in \sigma(T_{\lambda,\varphi}) \). Indeed, the function \( \varphi \) is not invertible in \( H^\infty(G) \), since \( \varphi \in C(G)^{-1} \) and \( A(G) = H^\infty(G) \cap C(G) \). It follows that \( T_\varphi \) is not invertible. Assume the converse. Then \( H^2(G) = T_\varphi H^2(G) = \varphi H^2(G) \), and therefore \( \psi \varphi = 1 \) for some \( \psi \in H^2(G) \). This implies that \( H^2(G) = \psi \varphi H^2(G) = \psi H^2(G) \) and hence \( \psi \in H^\infty(G) \) by [5, Lemma 2]. Thus we arrive at a contradiction. Because of the equality \( T_{\lambda,\varphi} = T_\varphi U_\lambda \), the operator \( T_{\lambda,\varphi} \) is not invertible along with \( T_\varphi \).

Since, by Theorem 3.3, the set \( \sigma(T_{\lambda,\varphi}) \) is circular, it remains to prove that the set \( |\sigma(T_{\lambda,\varphi})| := \{ |\mu| : \mu \in \sigma(T_{\lambda,\varphi}) \} \) is connected. But it follows from [1, Corollary 7.4] that the number of connected components of \( |\sigma(T_{\lambda,\varphi})| \) do not exceed the number of connected components of \( M \), the maximal ideal space of \( A(G) \). In turn, the space \( M \) can be identified with \( \hat{X}_+ \), the space of bounded semicharacters of the semigroup \( X_+ \) [2]. Since \( X_+ \cap X_+^{-1} = \{ 1 \} \), the connectedness of \( \hat{X}_+ \) follows from [6, Lemma 3]. □

**Remark.** In the case of \( G = \mathbb{T} \) it is known that under the conditions of the part 2) of Theorem 4.4 \( \sigma_\\epsilon(T_{\lambda,\varphi}) \) is a circle centered at the origin, too [4, Proposition 3.4]. This is not the case for groups distinct from \( \mathbb{T} \) because \( 0 \in \sigma_\\epsilon(T_{\lambda,\chi}) \) if \( \chi \in X_+ \setminus X^i \) (by [5, Corollary 1], \( X_+ \setminus X^i \neq \emptyset \) if \( G \neq \mathbb{T} \)). In fact, in this case, the operator \( T_\chi \) is not Fredholm by [5, Theorem 4]. Hence, the operator \( T_{\lambda,\chi} = T_\chi U_\lambda \) is not Fredholm as well.
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