WHAT KINDS OF KDV-TYPE EQUATIONS ARE ALLOWED BY AN UNEVEN BOTTOM

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ABSTRACT. In this study, we give a survey of derivations of KdV-type equations with an uneven bottom for several cases when small (perturbation) parameters $\alpha, \beta, \delta$ are of different orders. Two kinds of new second order wave equations: the extended KdV and the fifth-order KdV containing terms originated from an uneven bottom are derived for a general case of the bottom function. These results have been obtained for cases $\alpha = O(\beta), \delta = O(\beta^2)$ and $\alpha = O(\beta^2), \delta = O(\beta^2)$. For other four considered cases of ordering of small parameters the wave equations have been derived only for a particular case of linear bottom function and usually only to first order. Numerical simulations show that the effects of uneven bottom appear in a significant way only when wave equations are extended to second order.

1. INTRODUCTION

The Korteveg de Vries equation (KdV in short) [1] belongs to a few most famous equations in mathematical physics. It was originally derived for surface water waves in so-called shallow water wave problem. In the sixties of the last century, rapid development of the theory of nonlinear waves in various physical systems began, which showed that the KdV equation was obtained as the first approximation in the description of many physical phenomena. The range of applications extends, among others, to waves on the surface of liquids, waves in interfaces between various phases of liquids, ion-acoustic waves in plasma, optical impulses in optical fibers and electrical impulses in electrical circuits. There is a vast number of textbooks and monographs referring to studies of these problems, see, e.g. [2–8], to list a few. Wonderful properties of the KdV equation like integrability, a rich variety of analytic solutions and the existence of the infinite number of invariants attracted the attention of physicists, mathematicians, and engineers.

KdV and other KdV-type equations are derived under an important assumption, that the bottom of the fluid container is flat. This assumption is not realistic for most of the situations in the real world, in particular, the bottom of rivers, seas, oceans are non-flat. Despite a big number of efforts in studying nonlinear waves in the case of a non-flat bottom the first KdV-type equations in which terms originating from the bottom profile occur appeared only recently. Among the first papers treating a slowly

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This paper is dedicated to the memory of our friend Professor Eryk Infeld, who recently passed away.
varying bottom are papers by Mei and Le Méhauté [9] and Grimshaw [10]. These authors found that for small amplitudes the wave amplitude varies inversely as the depth but they did not obtain any simple KdV-type equation. Djordjević and Redekopp [11] and later Benilow and Howlin [12] studied the motion of packets of surface gravity waves over an uneven bottom using variable coefficient nonlinear Schrödinger equation (NLS). As a result they found fission of an envelope soliton. Some research groups developed approaches combining linear and nonlinear theories [13–15]. The Gardner equation (sometimes called the forced KdV equation) was also extensively investigated [16–19]. Van Groesen and Pudjaprasetya [20, 21] applied a Hamiltonian approach in which they obtained a forced KdV-type equation. Another widely applied method consists in taking an appropriate average of vertical variables which results in the Green-Naghdi equations [22–24]. An interesting numerical study of the propagation of unsteady surface gravity waves above an irregular bottom is done in [25]. Another study of long wave the propagation over a submerged 2-dimensional bump was recently presented in [26], although according to linear long-wave theory. Several examples of recent studies on the propagation of solitary waves over a variable topography are given in [27–29].

Derivation of wave equations of KdV-type for shallow water problem with an uneven bottom is a difficult task. In 2014, with our co-workers, we considered the nonlinear second order wave equation for shallow water problem with uneven bottom [30,31]. In these papers, besides standard small parameters $\alpha = \frac{a}{H}$ and $\beta = \left(\frac{H}{l}\right)^2$ we introduced the third one defined as $\delta = \frac{a_h}{H}$. In these definitions $a$ denotes the wave amplitude, $H$ the average water depth, $l$ the average wavelength and $a_h$ the amplitude of the variations of the bottom function $h(x)$. The geometry of the considered shallow water problem is presented in Fig. 1.

Then, with standard assumptions for incompressible, inviscid fluid and irrotational motion, one obtains the set of Eulerian equations in dimension variables. Next, introduction of the following

\begin{align*}
\eta(x,t) &\quad \text{undisturbed surface} \\
\alpha = a/H &\quad \text{undisturbed bottom} \\
\beta = (H/l)^2 &\quad h(x) \\
\delta = a_h/H &
\end{align*}

\begin{figure}[h]
\hspace{2cm}
\begin{tikzpicture}
\draw[->, thick] (0,0) -- (5,0) node[anchor=north] {$H$};
\draw[->, thick] (0,0) -- (0,2) node[anchor=east] {$\eta(x,t)$};
\draw[very thick] (0,0) -- (3,0) node[anchor=north] {$h(x)$};
\draw[very thick] (0,0) -- (3,0) node[anchor=north] {$a_h$};
\draw[very thick] (0,0) -- (3,0) node[anchor=north] {$\alpha = a/H$};
\draw[very thick] (0,0) -- (3,0) node[anchor=north] {$\beta = (H/l)^2$};
\draw[very thick] (0,0) -- (3,0) node[anchor=north] {$\delta = a_h/H$};
\end{tikzpicture}
\caption{Schematic view of the geometry of the shallow water wave problem for an uneven bottom.}
\end{figure}
transformation to dimensionless variables

\[\tilde{\phi} = \frac{H}{l a \sqrt{g H}} \phi, \quad \tilde{x} = x/l, \quad \tilde{\eta} = \eta/a, \quad \tilde{z} = z/H, \quad \tilde{t} = t/(l/\sqrt{g H})\]

has made it possible to apply perturbation approach, assuming that appropriate parameters are small.

The set of Euler equations, written in nondimensional variables (tildes are now dropped) has the following form (see, e.g., Eqs. (2)-(5) in [31])

\[
\beta \phi_{xx} + \phi_{zz} = 0, \quad (2)
\]

\[
\eta_t + \alpha \phi_x \eta_x - \frac{1}{\beta} \phi_z = 0, \quad \text{for} \quad z = 1 + \alpha \eta \quad (3)
\]

\[
\phi_t + \frac{1}{2} \alpha \phi_x^2 + \frac{1}{2} \beta \phi_z^2 + \eta - \tau \beta \frac{\eta_{2x}}{(1 + \alpha^2 \beta \eta_x^2)^{3/2}} = 0, \quad \text{for} \quad z = 1 + \alpha \eta \quad (4)
\]

\[
\phi_z - \beta \delta (h_x \phi_x) = 0, \quad \text{for} \quad z = \delta h(x). \quad (5)
\]

Equation (2) is the Laplace equation valid for the whole volume of the fluid. Equations (3) and (4) are so-called kinematic and dynamic boundary conditions at the surface, respectively. The equation (5) represents the boundary condition at the non-flat unpenetrable bottom. In (4), the Bond number \(\tau = \frac{T}{\rho g h^2}\), where \(T\) is the surface tension coefficient. For surface gravity waves this term can be safely neglected, since \(\tau < 10^{-7}\), but it can be important for waves in thin fluid layers. For abbreviation all subscripts in (2)-(5) denote the partial derivatives with respect to particular variables, i.e. \(\phi_t \equiv \frac{\partial \phi}{\partial t}, \eta_{2x} \equiv \frac{\partial^2 \eta}{\partial x^2}\), and so on.

The velocity potential is seek in the form of power series in the vertical coordinate

\[
\phi(x, z, t) = \sum_{m=0}^{\infty} z^m \phi^{(m)}(x, t), \quad (6)
\]

where \(\phi^{(m)}(x, t)\) are yet unknown functions. The Laplace equation (2) determines \(\phi\) in the form which involves only two unknown functions with the lowest \(m\)-indexes, \(f(x, t) := \phi^{(0)}(x, t)\) and \(F(x, t) := \phi^{(1)}(x, t)\). Hence,

\[
\phi(x, z, t) = \sum_{m=0}^{\infty} \frac{(-1)^m \beta^m m^2 f(x, t)}{(2m)!} z^{2m} + \sum_{m=0}^{\infty} \frac{(-1)^m \beta^{m+1} (m+1) F(x, t)}{(2m+1)!} z^{2m+1}. \quad (7)
\]

The explicit form of this velocity potential reads as

\[
\phi = f - \frac{1}{2} \beta z^2 f_{2x} + \frac{1}{24} \beta^2 z^4 f_{4x} - \frac{1}{720} \beta^3 z^6 f_{6x} + \cdots + \beta z F_x - \frac{1}{6} \beta^2 z^3 F_{3x} + \frac{1}{120} \beta^3 z^5 F_{5x} + \cdots. \quad (8)
\]

Next, one applies the perturbation approach, assuming that parameters \(\alpha, \beta, \delta\) are small. As pointed in [32][33], the proper ordering of small parameters is crucial to obtain appropriate final wave equations. Therefore, for each particular case, the perturbation approach has to be performed separately.
We will be interested in all possible cases of wave equations obtained in the perturbation approach up to second order. So, we can specify the following cases, see Table 1.

**Table 1. Different ordering of small parameters considered in the paper.**

| Case | $\alpha$ | $\beta$ | $\delta$ |
|------|----------|----------|----------|
| 1    | $O(\beta)$ | $O(\beta)$ | $O(\beta)$ |
| 2    | $O(\beta)$ | $O(\beta^2)$ | $O(\beta^2)$ |
| 3    | $O(\beta^2)$ | $O(\beta)$ | $O(\beta^2)$ |
| 4    | $O(\beta^2)$ | $O(\beta^2)$ | $O(\beta^2)$ |
| 5    | $O(\alpha^2)$ | $O(\alpha)$ | $O(\alpha)$ |
| 6    | $O(\alpha^2)$ | $O(\alpha^2)$ | $O(\alpha^2)$ |

Our study is an extension of that done thoroughly by Burde and Sergyeyev in [32]. They considered several cases of a different ordering of two small parameters $\alpha, \beta$, still for the flat bottom case sometimes going up to third or fourth order. The cases studied in [32] were: $\beta = O(\alpha)$, $\beta = O(\alpha^2)$, $\beta = O(\alpha^3), \alpha = O(\beta^2)$ and $\alpha = O(\beta^3)$. The authors showed that different ordering of small parameters implied several kinds of wave equations, previously derived in the literature from different physical assumptions.

The form of velocity potential (8), determined by (6) and the Laplace equation (2), is the same for all considered cases 1-6. The boundary condition at the bottom (5) implies different forms of the $F$ function, depending on the particular ordering of small parameters $\beta, \delta$.

It is worth noticing the important difference between the cases related to a flat bottom and those when the bottom is not even. In the former ones $F = 0$, due to $\delta = 0$ in (5). Therefore, when the Boussinesq equations are used to determine correction terms $Q$ one can always utilize the condition $Q_t = -Q_x$. These facts ensure to derive KdV-type equations up to arbitrary order. This is not possible for a general uneven bottom. In the case of uneven bottom the boundary condition (5) imposes a differential equation on $f$ and $F$ which can be resolved to obtain $F(f(x,t), h(x))$ only in some low orders depending on ordering relations between small parameters. For higher orders that equation cannot be resolved.

In the following sections, we discuss derivations of KdV-type wave equations in cases 1-6. In case 1 and case 2 the wave equations were already derived in [33-35]. We include these cases into the paper for completeness and comparison with the cases 3-6. A few examples of numerical simulations illustrating soliton motion over a linearly sloped bottom are presented in Section 8. The last Section 9 contains conclusions.

In this paper, we focused on derivations of wave equations which include terms from the uneven bottom. We leave the broader numerical studies of derived equations to the next article.
In order to consider perturbation expansion in only one small parameter, in this case, we can set
\[ \alpha = A \beta, \quad \delta = D \beta, \]
where the constants \( A, D \) are of the order of 1.

Substitution of (8) into (5) gives (with \( z = q \beta h(x) \)) the following nontrivial relation between the functions \( F_x \) and \( f \)
\[ F_x - D \beta(hf_x)_x - \frac{1}{2}D^2 \beta^3(h^2f_{2x})_x + \frac{1}{6}D^3 \beta^4(h^3f_{3x})_x + \frac{1}{24}D^4 \beta^6(h^4F_{4x})_x + \cdots = 0. \]

Keeping only terms lower than third order leaves
\[ F_x = D \beta(hf_x)_x, \]
which allows us to express the \( x \)-dependence of the velocity potential through \( f, h \) and their \( x \)-derivatives. With higher order terms it is impossible. The equation (11) determines \( F_x \) up to second order. Since this term enters (8) with the factor \( \beta z \), the velocity potential is determined correctly up to third order in \( \beta \). It is worth to emphasize that due to the presence of the term \( -\frac{1}{\beta} \phi_z \) in (3), the Boussinesq equations resulting from the substitution of (8) into (3) and (4) are correct up to second order. Therefore, the boundary condition at the uneven bottom implies the limit on the order of theory in which the Boussinesq equations can be derived. For the case \( \alpha = O(\beta) \), \( \delta = O(\beta) \) this is second order.

It is easy to see that the form of (10), when considered to higher orders does not allow us for obtaining an explicit expression of \( F_x \) through \( f, h \) and their \( x \)-derivatives.

Substituting (11) into (8) and retaining terms up to third order in \( \beta \) gives the velocity potential as
\[ \phi = f - \frac{1}{2} \beta z^2 f_{2x} + \frac{1}{24} \beta^3 z^4 f_{4x} - \frac{1}{720} \beta^5 z^6 f_{6x} + \beta^2 zD(hf_x)_x - \frac{1}{6} \beta^3 z^3 D(hf_x)_3x. \]
[Due to the term \( \frac{1}{\beta} \phi_z \) in the equation (3), to obtain equations up to second order one has to keep the velocity potential up to third order.]

Inserting (12) into (3) and (4), with \( z = 1 + A \beta \eta_1 \), and retaining terms up to second order yields the set of the Boussinesq equations in the following form (with usual notation \( w = f_x \))
\[ \eta_t + w_x + \beta \left( A(\eta w)_x - \frac{1}{6} w_{3x} - D(hw)_x \right) \]
\[ + \beta^2 \left( -A \frac{1}{2} (\eta w_{2x})_x + \frac{1}{120} w_{5x} + D(hw)_3x \right) = 0, \]
\[ w_t + \eta_x + \beta \left( Aw_x - \frac{1}{2} w_{2xt} - \tau \eta_{3x} \right) \]
\[ + \beta^2 \left( A \left( -(\eta w_{xt})_x + \frac{1}{2} w_x w_{2x} - \frac{1}{2} w_{3x} \right) + \frac{1}{24} w_{4xt} + D(hw)_2x \right) = 0. \]
In the lowest (zero) order the Boussinesq set reduces to
\[ \eta_t + w_x = 0, \quad w_t + \eta_x = 0, \quad \text{implying} \quad w = \eta, \quad \eta_t = -\eta_x, \quad w_t = -w_x. \]

In the first order the Boussinesq set reduces to
\[ \eta_t + w_x + \beta \left( A(\eta w)_x - \frac{1}{6} w_{3x} - D(hw)_x \right) = 0, \tag{16} \]
\[ w_t + \eta_x + \beta \left( Aw_w_x - \frac{1}{2} w_{2xt} - \tau \eta_{3x} \right) = 0. \tag{17} \]

Note that terms originating from an uneven bottom appear in (16) but not in (17). This is the reason why in first order the Boussinesq equations (16)-(17) can be made compatible only for the particular case of the bottom function \( h(x) \).

Assume that in the first order the function \( w \) has the form
\[ w = \eta + \beta \left( -\frac{1}{4} A \eta^2 + \frac{1}{6} (2 - 3\tau) \eta_{2x} + Q \right). \tag{18} \]
It is easy to see that the first two terms in the correction function assure the KdV equation in the case of the flat bottom. Then inserting (18) into (16)-(17) gives in first order
\[ \eta_t + \eta_x + \beta \left( \frac{3}{2} A \eta_{2x} + \frac{1}{6} (1 - 3\tau) \eta_{3x} - D(h\eta)_x + Q_x \right) = 0 \quad \text{and} \tag{19} \]
\[ \eta_t + \eta_x + \beta \left( \frac{3}{2} A \eta_{2x} + \frac{1}{6} (1 - 3\tau) \eta_{3x} + Q_t \right) = 0, \tag{20} \]
where in (20) we already replaced \( \eta_t \) by \( -\eta_x \). The equations (19) and (20) become compatible when
\[ Q_t = Q_x - D(h\eta)_x. \tag{21} \]

As pointed in [33] this condition can be satisfied in the particular case \( h(x) = kx \), where \( k \) is a constant. In this particular case the correction term in the form
\[ Q = \frac{1}{4} D(2kx\eta + k \int \eta dx) \tag{22} \]
makes the equations (19) and (20) compatible and the resulting first order KdV-type equation has the following form
\[ \eta_t + \eta_x + \beta \left( \frac{3}{2} A \eta_{2x} + \frac{1}{6} (1 - 3\tau) \eta_{3x} - \frac{1}{4} D(k\eta + 2kx\eta_x) \right) = 0. \tag{23} \]

The equation (23) is the new result achieved in [33].

In original notations for small parameters, this equation reads as
\[ \eta_t + \eta_x + \frac{3}{2} A \eta_{2x} + \frac{1}{6} (1 - 3\tau) \beta \eta_{3x} - \frac{1}{4} \delta(k\eta + 2kx\eta_x) = 0. \tag{24} \]
For the case of the flat bottom \((D = \delta = 0)\), this equation reduces to the usual KdV equation

\[ \eta_t + \eta_x + \frac{3}{2} \alpha \eta_x + \frac{1}{6} (1 - 3\tau) \beta \eta_{xx} = 0. \]

When \(h(x)\) is a bounded arbitrary function the equations (19)-(20) cannot be made compatible (see, [34, 35]). Attempts to continue derivation of a wave equation to second order in small parameters show that equations (13)-(14) cannot be made compatible [33]. We confirm this fact.

Let us remind that in the case of the flat bottom (with surface tension neglected) the second order wave equation reduces to well known the so-called extended KdV equation (25)

\[ \eta_t + \eta_x + \frac{3}{2} \alpha \eta_x + \frac{1}{6} \beta \eta_{xx} - \frac{3}{8} \alpha^2 \eta_x^2 + \frac{1}{5} \beta \eta_{xx} + \frac{23}{24} \eta_{xx} + \frac{5}{12} \eta_{xx}^2 + \frac{19}{360} \beta^2 \eta_{xx} = 0 \]

derived for the first time by Marchant and Smyth in [36] and sometimes called KdV2.

### 3. Case 2: \(\alpha = O(\beta), \quad \delta = O(\beta^2)\)

In this case we set

\[ \alpha = A\beta, \quad \delta = D\beta^2. \]

Now, we insert the general form of velocity potential (8) into the bottom boundary condition (5) which in this case is

\[ \phi_z - D\beta^3 (h_x \phi_x) = 0, \quad \text{for} \quad z = D\beta^2 h(x) \]

obtaining relation similar to (10)

\[ F_x - D\beta^2 (h f_x)_x - \frac{1}{2} D^2 \beta^5 (h^2 F_{2x})_x + \frac{1}{6} D^3 \beta^7 (h^3 f_{3x})_x + \frac{1}{24} D^4 \beta^{10} (h^4 F_{4x})_x - \frac{1}{120} D^5 \beta^{12} (h^5 f_{5x})_x + \cdots = 0. \]

Then, neglecting terms of order higher than fourth in \(\beta\) we have

\[ F_x = D\beta^2 (h f_x)_x, \]

which inserted into (8) gives the velocity potential of the form

\[ \phi = f - \frac{1}{2} \beta^2 f_{2x} + \frac{1}{24} \beta^4 f_{4x} - \frac{1}{720} \beta^6 f_{6x} + \cdots + D\beta^3 z (h f_x)_x - \frac{1}{6} D^2 \beta^3 (h f_x)_{3x} + \frac{1}{120} D^3 \beta^5 z^5 (h f_x)_{5x} + \cdots. \]

Since (29) determines \(F_x\) up to third order, the Boussinesq equations can be derived up to third order, as well.
Substituting the velocity potential (30) into (3)-(4) and retaining terms up to second order supplies the Boussinesq equations in the following form

\[
\begin{align*}
\eta_t + w_x + \beta \left(A(\eta w)_x - \frac{1}{6} w_{3x}\right) + \beta^2 \left(-A\frac{1}{2}(\eta w_x)_x + \frac{1}{120} w_{5x} - D(\eta w)_x\right) & = 0, \\
\eta_t + w_x + \beta \left(A(w_x - \frac{1}{2} w_{2x} - \tau \eta_3)\right) & + \beta^2 \left(-A(\eta w_{xt})_x + A\frac{1}{2} w_x w_{2x} - A\frac{1}{2} w_{3x} + \frac{1}{24} w_{4x} \right) & = 0.
\end{align*}
\]

(31) \hspace{1cm} (32)

In the first order, this system reduces to the usual KdV system, with

\[
\begin{align*}
\eta_t + \eta_x + \beta \left(A\frac{3}{2} \eta_3 + \frac{1}{6} (1 - 3 \tau) \eta_{3x}\right) & = 0 \\
\eta_t + \eta_x + \alpha \frac{3}{2} \eta_{2x} + \beta \frac{1}{6} (1 - 3 \tau) \eta_{3x} & = 0
\end{align*}
\]

(33) \hspace{1cm} (34)

in original variables.

Now, we aim to satisfy the Boussinesq system (31)-(32) with the terms of the second order included. Then, we set

\[
\begin{align*}
w & = \eta + \beta \left(-A\frac{1}{4} \eta^2 + \frac{1}{6} (2 - 3 \tau) \eta_{2x}\right) + \beta^2 Q.
\end{align*}
\]

(35)

Next, we insert the trial function (35) into (31) and (32) and retain terms up to second order in \( \beta \). Proceeding analogously as in the case of first order we find the form of the correction function

\[
\begin{align*}
Q & = A^2 \frac{1}{8} \eta^3 + A \frac{3 + 7 \tau}{16} \eta_x^2 + A \frac{2 + \tau}{4} \eta_{2x} + \frac{12 - 20 \tau - 15 \tau^2}{120} \eta_{4x} + \frac{1}{2} D(h\eta).
\end{align*}
\]

(36)

Note, that in order to replace \( t \)-derivatives by \( x \)-derivatives one has to use the properties of the first order equation (34), that is, \( \eta_t = -\eta_x - \beta \left(A\frac{3}{2} \eta_3 + \frac{1 - 3 \tau}{6} \eta_{3x}\right) \) and its derivatives. The form (36) of the correction function makes the Boussinesq system (31)-(32) compatible and allows us to derive explicit form for the wave equation for the case of \( \alpha = O(\beta) \) and \( \delta = O(\beta^2) \). Finally, we obtain

\[
\begin{align*}
w & = \eta + \beta \left(-A\frac{1}{4} \eta^2 + \frac{2 - 3 \tau}{6} \eta_{2x}\right) + \beta^2 \left(A^2 \frac{1}{8} \eta^3 + A \frac{3 + 7 \tau}{16} \eta_x^2 + A \frac{2 + \tau}{4} \eta_{2x} + \frac{12 - 20 \tau - 15 \tau^2}{120} \eta_{4x} + \frac{1}{2} D(h\eta) \right).
\end{align*}
\]

(37)
and

\begin{align}
&\eta_t + \eta_x + \beta \left( A \frac{3}{2} \eta_x + \frac{1}{6} - 3 \tau \eta_{3x} \right) + \beta^2 \left( -A^2 \frac{3}{8} \eta_x^2 + A \frac{23 + 15 \tau}{24} \eta_x \eta_{2x} + A \frac{5 - 3 \tau}{12} \eta_x \eta_{3x} \\
&\quad + \frac{19 - 30 \tau - 45 \tau^2}{360} \eta_{5x} - \frac{1}{2} D(h\eta)_x \right) = 0.
\end{align}

The equation (38) is the nonlinear wave equation for uneven bottom, when \( \alpha = \beta, \delta = O(\beta^2) \), derived consistently within second order perturbation approach.

Since \( \delta = D\beta^2 \) we can come back to original notations for small parameters, used in [31]. Then equations (37) and (38) take the following forms

\begin{align}
w &= \eta - \frac{1}{4} \alpha \eta^2 + \frac{2 - 3 \tau}{6} \beta \eta_{2x} \\
&\quad + \frac{1}{8} \alpha^2 \eta^3 + \alpha \beta \left( \frac{3 + 7 \tau}{16} \eta_x^2 + \frac{2 + \tau}{4} \eta \eta_{2x} \right) + \frac{12 - 20 \tau - 15 \tau^2}{120} \beta^2 \eta_{4x} + \delta \frac{1}{2} \eta,
\end{align}

\begin{align}
\eta_t + \eta_x + \frac{3}{2} \alpha \eta_{3x} + \frac{3}{6} \beta \eta_{3x} - \frac{3}{8} \alpha^2 \eta^2 \eta_x + \alpha \beta \left( \frac{23 + 15 \tau}{24} \eta_x^2 + \frac{5 - 3 \tau}{12} \eta \eta_{2x} \right) \\
&\quad + \beta^2 \left( \frac{19 - 30 \tau - 45 \tau^2}{360} \eta_{5x} \right) - \delta \frac{1}{2} (h\eta)_x = 0.
\end{align}

Since the terms with \( \delta \), looking as first order ones, are, in fact, of second order the form of the equations (39) and (40) may be misleading.

The equation (40), limited to the case \( \delta = D = 0 \) and \( \tau = 0 \), is known as the extended KdV equation [36] or KdV2 [40]. This equation is nonintegrable. Despite this fact, we with our co-workers found several forms of analytic solutions to KdV2: soliton solutions \( \sim \text{sech}^2[B(x - vt)] \) in [31], cnoidal solutions \( \sim \text{cn}^2[B(x - vt)] \) in [37] and superposition cnoidal solutions \( \sim \text{dn}^2[B(x - vt)] + \sqrt{m} \text{cn}[B(x - vt)] \text{dn}[B(x - vt)] \) in [38,39].

The equation (40) is second order wave equation derived consistently for the case of \( \alpha = O(\beta) \) and \( \delta = O(\beta^2) \) which takes directly into account an uneven bottom.

The wave equation (40) is very similar to the erroneous [31, Eq. (18)]. The latter contains, apart from the leading term from the bottom \( -\frac{1}{2} \delta \eta_x \eta_{2x} \), two other terms which resulted from not fully consistent derivation.

4. Case 3: \( \alpha = O(\beta^2), \ \delta = O(\beta) \)

In this case we set

\begin{align}
\alpha &= A \beta^2, \ \delta = D \beta.
\end{align}

Since \( \delta \) is of the same order as \( \beta \) the formulas (11)-(12) expressing the velocity potential hold. Now we substitute the velocity potential (12) into the kinematic and dynamic boundary conditions at the
unknown surface which in this case are

\begin{equation}
\eta_t + A \beta^2 \phi_x \eta_x - \frac{1}{\beta} \phi_x = 0, \quad \text{for} \quad z = 1 + A \beta^2 \eta,
\end{equation}

\begin{equation}
\phi_t + \frac{1}{2} A \beta^2 \phi^2_x + \frac{1}{2} A \beta \phi_x^2 + \eta - \tau \beta \frac{\eta_{2x}}{(1 + A^2 \beta^2 \eta_x^2)^{3/2}} = 0, \quad \text{for} \quad z = 1 + A \beta^2 \eta.
\end{equation}

Next, we neglect all terms of orders higher than $\beta^2$. The result consists in the following Boussinesq equations (in the meantime the second equation was differentiated by $x$)

\begin{equation}
\eta_t + w_x - \beta \left( D (hw)_x + \frac{1}{6} w_{3x} \right) + \beta^2 \left( A (\eta w)_x + \frac{1}{2} D (hw)_{3x} + \frac{1}{120} w_{5x} \right) = 0,
\end{equation}

\begin{equation}
w_t + \eta_x - \beta \left( \tau \eta_{3x} + \frac{1}{2} w_{2xt} \right) + \beta^2 \left( A w w_x + D (hw)_t 2x + \frac{1}{24} w_{4xt} \right) = 0,
\end{equation}

where the usual notation $w = f_x$ is used.

For the flat bottom case ($D = 0$) the equations (44)-(45) can be made compatible up to any order. Some of them are derived from different physical models. Below we cite this equations keeping terms up to second order in $\beta$ (see, \cite[Eqs. (A.1)-(A.2)]{32})

\begin{equation}
w = \eta + \frac{1}{2} \beta^2 - \frac{3}{6} \eta_{2x} + \beta^2 \left( -\frac{A}{4} \eta^2 + \frac{12 - 20 \tau - 15 \tau^2}{120} \eta_{4x} \right),
\end{equation}

\begin{equation}
\eta_t + \eta_x + \beta \left( \frac{1}{6} - \tau \eta_{3x} + \beta^2 \left( \frac{A}{3} \eta_x + \frac{19 - 30 \tau - 45 \tau^2}{360} \eta_{5x} \right) = 0.
\end{equation}

This result is equivalent to well known the fifth-order KdV equation derived by Hunter and Sheurle in \cite{41} as a model equation for gravity-capillary shallow water waves of small amplitude. Neglecting surface tension (reasonable for shallow water problem) and changing variables by

\begin{equation}
\tilde{x} = \sqrt{\frac{3 \alpha}{2 \beta}} (x - t), \quad \tilde{t} = \frac{1}{4} \sqrt{\frac{3 \alpha^3}{2 \beta}} t,
\end{equation}

one reduces the equation (47) to

\begin{equation}
\eta_{\tilde{t}} + 6 \eta \eta_{\tilde{x}} + 4 \eta_{3\tilde{x}} + P \eta_{5\tilde{x}} = 0, \quad P = \frac{19}{40},
\end{equation}

which is the fifth-order KdV equation obtained in \cite{41} with $P$ defined in a different way. This equation is known to have a rich structure of solitary wave solutions, see, e.g., \cite{42}. As pointed out in \cite{32} the wave equation obtained in third order belongs to the type $K(m, n)$ introduced by Rosenau and Hyman \cite{45} with $m = 4$ and $n = 1$ which in some range of wave velocities admits soliton-like traveling wave solutions.

For the case $D \neq 0$ limitation of equations (44)-(45) to first order yields

\begin{equation}
\eta_t + w_x - \beta \left( D (hw)_x - \frac{1}{6} w_{3x} \right) = 0 \quad \text{and} \quad \eta_x + w_t - \beta \left( \tau \eta_{3x} + \frac{1}{2} w_{2xt} \right) = 0.
\end{equation}
Since in zeroth order \( \eta = w \), \( \eta_t = -\eta_x \), \( w_t = -w_x \), one assumes that in the first order

\[
(49) \quad w = \eta + \beta \left( \frac{2 - 3\tau}{6} \eta_{2x} + Q \right),
\]

where the first part of the correction term is already known from (46) and \( Q \) is responsible for first order correction related to the bottom term in (48). Then, substitute (49) into equations (48) and retain terms only to the first order. This yields

\[
(50) \quad \eta_t + \eta_x + \beta \left( Q_x - D(h\eta)_x + \frac{1 - 3\tau}{6} \eta_{3x} \right) = 0 \quad \text{and} \quad \eta_t + \eta_x + \beta \left( Q_t + \frac{1 - 3\tau}{6} \eta_{3x} \right) = 0.
\]

Compatibility of these equations requires

\[
(51) \quad Q_x - Q_t = D(h\eta).
\]

This is the same condition as (21) which cannot be satisfied for general form of \( h(x) \) but can be satisfied for the particular case \( h(x) = kx \). In this particular case \( Q \) given by (22) makes the equations (50) compatible. So, with

\[
 w = \eta + \beta \left( \frac{2 - 3\tau}{6} \eta_{2x} + \frac{1}{4} D \left( 2kx\eta + k \int \eta \, dx \right) \right)
\]

we obtain the resulting first order KdV-type equation in the following form

\[
(52) \quad \eta_t + \eta_x + \beta \left( \frac{1 - 3\tau}{6} \eta_{3x} - \frac{1}{4} D(k\eta + 2kx\eta_x) \right) = 0.
\]

Note, that the equation (52), in the case of the flat bottom \( (D = 0) \), is reduced to the linear dispersive one. Therefore the equation (52) has no soliton solutions.

Can we extend the derivation to second order?

In second order we utilize the knowledge of (46) and assume

\[
(53) \quad w = \eta + \beta \left( \frac{2 - 3\tau}{6} \eta_{2x} + \frac{1}{4} D \left( 2kx\eta + k \int \eta \, dx \right) \right) + \beta^2 \left( -A \frac{1}{4} \eta^2 + \frac{12 - 20\tau - 15\tau^2}{120} \eta_{4x} + Q \right) + \frac{1}{2} Dk \left( \frac{7 + 4\tau}{8} \eta_{2x} + \frac{1 + 6\tau}{12} \eta_{3x} \right) + D^2 k^2 \left( \frac{5}{4} \eta + \frac{1}{2} \eta_x - \int \eta \, dx \right) = 0,
\]

where \( Q \) represents the correction term originating from the bottom variations. With \( w \) given by (53) the first Boussinesq equation (44) yields

\[
(54) \quad \eta_t + \eta_x + \beta \left[ \frac{1 - 3\tau}{6} \eta_{3x} - \frac{1}{4} Dk \left( \eta + 2x\eta_x \right) \right] + \beta^2 \left[ Q_x + \frac{3}{2} A \eta \eta_x - \frac{17 - 30\tau}{360} \eta_{5x} \right] + Dk \left( \frac{7 + 4\tau}{8} \eta_{2x} + \frac{1 + 6\tau}{12} x \eta_{3x} \right) + D^2 k^2 \left( \frac{5}{4} \eta + \frac{1}{2} x^2 \eta_x - \int \eta \, dx \right) = 0.
\]
whereas the second one (55), after a proper replacement of $t$-derivatives by $x$-derivatives according to (52), gives

$$(55) \quad \eta_t + \eta_x + \beta \left[ \frac{1}{6} \eta_{3x} - \frac{1}{4} Dk (\eta + 2x\eta_x) \right] + \beta^2 \left[ Q_t + \frac{3}{2} A\eta_{1x} + \frac{11 - 18\tau - 18\tau^2}{72} \eta_{5x} + \frac{3}{4} \int \eta \, dx + \frac{11 + 12\tau}{18} \int x\eta_x \, dx \right] = 0.$$  

It appears that the equations (54)-(55) cannot be made compatible for any $Q(x, \eta)$. Therefore in this case of ordering of small parameters second order wave equation for uneven bottom cannot be derived.

5. CASE 4: $\alpha = O(\beta^2), \quad \delta = O(\beta^2)$

In this case we set

$$(56) \quad \alpha = A\beta^2, \quad \delta = D\beta^2.$$  

Since $\delta = O(\beta^2)$, the forms of the function $F$ and the velocity potential are given by (29)-(30). The Boussinesq set receives in this case the following form

$$(57) \quad \eta_t + w_x - \beta \frac{1}{6} w_{3x} + \beta^2 \left( A(w\eta)_x + \frac{1}{120} w_{5x} - D(hw)_x \right) = 0,$$

$$w_t + \eta_x - \beta \left( \frac{1}{2} w_{2xt} + \tau \eta_{3x} \right) + \beta^2 \left( Aw_{x} + \frac{1}{24} w_{4xt} \right) = 0.$$  

In first order $w$ in the form

$$(59) \quad w = \eta + \beta \frac{2 - 3\tau}{6} \eta_{2x}$$

makes the equations (57)-(58) compatible, with the result

$$(60) \quad \eta_t + \eta_x + \beta \frac{1 - 3\tau}{6} \eta_{3x} = 0.$$  

In second order we look for $w$ in the form

$$(61) \quad w = \eta + \beta \frac{2 - 3\tau}{6} \eta_{2x} + \beta^2 \left( - \frac{1}{4} A \eta^2 + \frac{12 - 20\tau - 15\tau^2}{120} \eta_{4x} + DQ \right).$$  

In (61) we already used the part of second order correction term known to make compatible Boussinesq’s set for the flat bottom (see, e.g., [32 Eqs. (A.8)-(A.9)]. Substitution of (61) into (57)-(58) gives

$$(62) \quad \eta_t + \eta_x + \beta \frac{1 - 3\tau}{6} \eta_{3x} + \beta^2 \left( Q_x - D(h\eta)_x + \frac{3}{2} A\eta_{1x} + \frac{19 - 30\tau - 45\tau^2}{360} \eta_{5x} \right) = 0 \quad \text{and}$$

$$(63) \quad \eta_t + \eta_x + \beta \frac{1 - 3\tau}{6} \eta_{3x} + \beta^2 \left( Q_t + \frac{3}{2} A\eta_{1x} + \frac{19 - 30\tau - 45\tau^2}{360} \eta_{5x} \right) = 0,$$
respective. (In (63) $t$-derivatives are already properly replaced by $x$-derivatives.) Assuming $Q_t = -Q_x$ the correction $Q$ is obtained as

\[(64)\]

\[Q = \frac{1}{2} D h \eta.\]

So, finally in second order with

\[(65)\]

\[w = \eta + \beta \frac{2 - 3\tau}{6} \eta_{2x} + \beta^2 \left( -\frac{1}{4} A \eta^2 + \frac{12 - 20\tau - 15\tau^2}{120} \eta_{4x} + \frac{1}{2} D(h \eta) \right)\]

one obtains the wave equation containing a direct term from the bottom

\[(66)\]

\[\eta_t + \eta_x + \beta \frac{1 - 3\tau}{6} \eta_{3x} + \beta^2 \left( \frac{3}{2} \eta_{xx} + \frac{19 - 30\tau - 45\tau^2}{360} \eta_{5x} - \frac{1}{2} D(h \eta)_x \right) = 0.\]

In the original parameters the equations (65)-(66) receive the form

\[(67)\]

\[w = \eta + \beta \frac{2 - 3\tau}{6} \eta_{2x} - \frac{1}{4} \alpha \eta^2 + \beta^2 \frac{12 - 20\tau - 15\tau^2}{120} \eta_{4x} + \frac{1}{2} \delta(h \eta),\]

\[(68)\]

\[\eta_t + \eta_x + \beta \frac{1 - 3\tau}{6} \eta_{3x} + \frac{3}{2} \eta_{xx} + \beta^2 \frac{19 - 30\tau - 45\tau^2}{360} \eta_{5x} - \frac{1}{2} \delta(h \eta)_x = 0.\]

This is, besides the equation (40), another second order wave equation obtained for a general bottom function $h(x)$.

For $D = \delta = 0$, that is, the flat bottom, the equations (65)-(66) reduce to the equations (46)-(47) from section 4.

6. CASE 5: $\beta = O(\alpha^2), \quad \delta = O(\alpha)$

In this case we set

\[(69)\]

\[\beta = B \alpha^2, \quad \delta = D \alpha.\]

Now, we have to express all perturbation equations with respect to parameter $\alpha$. Then the velocity potential (8) can be rewritten as

\[(70)\]

\[\phi = f - \frac{1}{2} B \alpha^2 z^2 f_{2x} + \frac{1}{24} B^2 \alpha^4 z^4 f_{4x} - \frac{1}{720} B^3 \alpha^6 z^6 f_{6x} + \cdots\]

\[+ B \alpha^2 z F_x - \frac{1}{6} B^2 \alpha^4 z^3 F_{3x} + \frac{1}{120} B^3 \alpha^6 z^5 F_{5x} + \cdots.\]

The boundary condition at the bottom (5) takes now the following form

\[(71)\]

\[\phi_z - B D \alpha^3 (h_x \phi_x) = 0 \quad \text{for} \quad z = D \alpha h(x).\]

Applying this equation to $\phi$ given by (70) implies

\[(72)\]

\[F_x = \alpha D(h f_x)_x + \alpha^4 BD^2(h^2 F_{2x})_x - \alpha^5 \frac{1}{6} BD^3(h^3 f_{3x})_x + O(\alpha^8).\]
This equation allows us to express $F_x$ through $h$, $f$ and their derivatives only when terms of the fourth and higher orders are neglected

$$F_x = \alpha D(h f_x)_x + O(\alpha^4).$$

This formula allows us to express $\phi$ through only one unknown function $f$ and its derivatives. Note that next terms in $F_x$ would enter in $\phi$ in sixth order. Therefore we can express the velocity potential containing terms from the bottom function only up to fifth order

$$\phi = f - \frac{1}{2} B \alpha^2 z^2 f_{2x} + \frac{1}{24} B^2 \alpha^4 z^4 f_{4x} + BD \alpha^3 z(h f_x)_x - \frac{1}{6} B^2 D \alpha^5 z^3(h f_x)_{3x} + O(\alpha^6).$$

This form of the velocity potential implies the Boussinesq set as

$$\eta_t + w_x + \alpha [Q_x - D(h \eta)_x] - \frac{1}{6} \alpha^2 B w_{3x} + \alpha^3 B \frac{1}{2} [D(h w)_{3x} - (\eta w_{2x})_x] = 0,$$

$$w_t + \eta_x + \alpha w w_x - \alpha^2 B \left[ \tau \eta_{3x} + \frac{1}{2} w_{2xt} \right] + \alpha^3 B \left[ -(\eta w_{xt})_x + \frac{1}{2} w_x w_{2x} - \frac{1}{2} w w_{3x} + D(h w_t)_{2x} \right] = 0,$$

where terms up to $\alpha^3$ are retained. Note that for the uneven bottom ($\delta, D \neq 0$) consistent Boussinesq’s set cannot be derived in orders higher than $\alpha^3$. Due to the term $\phi_z/\beta = \phi_z/(B \alpha^2)$ in the kinematic boundary condition at the surface (3) and relations (72)-(73) we cannot extend equations (75)-(76) with the potential (74) to higher orders. When the bottom is flat, there is no such limitation.

Begin with first order Boussinesq’s set. Assuming $w = \eta + \alpha Q$ one obtains from (75)-(76)

$$\eta_t + \eta_x + \alpha [Q_x + 2 \eta \eta_x - D(h \eta)_x] = 0,$$

$$\eta_t + \eta_x + \alpha [Q_t + \eta \eta_x] = 0.$$

Equations (75)-(76) become compatible in first order when

$$Q_x - Q_t = -D(h \eta)_x - \eta \eta_x.$$

This condition cannot be satisfied for arbitrary bounded bottom function $h(x)$. Therefore, for a general $h(x)$ the set (75)-(76) cannot be made compatible already in first order.

The condition (79) is similar to the condition (21) which appeared for the case $\alpha = O(\beta), \delta = O(\beta)$, discussed in Section 2. Therefore, analogously as in that case equations (77)-(78) can be made compatible for the same particular case of linear bottom function $h(x) = kx$ but only when $D = 1$ ($\delta = \alpha$). It is easy to check that for $Q = \frac{1}{4} \left( 2 k x \eta + k \int \eta dx - \eta^2 \right)$ one gets

$$w = \eta + \alpha \frac{1}{4} \left( 2 k x \eta + k \int \eta dx - \eta^2 \right),$$

which, when $D = 1$, implies both from (77) and (78), the same wave equation

$$\eta_t + \eta_x + \alpha \left( 3 \eta_{2x} - \frac{1}{4} k \eta - \frac{1}{2} k x \eta_x \right) = 0.$$
In second order we assume
\begin{equation}
(82) \quad w = \eta + \alpha \frac{1}{4} \left( 2kx \eta + k \int \eta dx - \eta^2 \right) + \alpha^2 Q.
\end{equation}

Substitution (82) into (75)-(76) and use the condition \( Q_t = -Q_x \) allows us to obtain
\begin{equation}
(83) \quad Q = \frac{1}{8} \eta^3 + B \frac{2 - 3\tau}{6} \eta_{2x} + \int \left( \frac{3}{8} k^2 x^2 \eta_x + \frac{11}{16} k^2 x \eta - \frac{3}{4} k x \eta_{2x} - \frac{9}{32} k \eta^2 + \frac{5}{32} k^2 \int \eta dx + \frac{1}{16} k^2 \int x \eta_x dx \right) dx.
\end{equation}

The correction term in (82) gives for both equations (75)-(76)
\begin{equation}
(84) \quad \eta_t + \eta_x + \alpha \left( \frac{3}{2} \eta \eta_x - \frac{1}{4} k \eta - \frac{1}{2} k x \eta_x \right) + \alpha^2 \left( - \frac{3}{8} \eta^2 \eta_x + B \frac{1 - 3\tau}{6} \eta_{2x} - \frac{1}{8} k^2 x^2 \eta_x - \frac{9}{16} k^2 x \eta \right) + \frac{23}{32} k \eta^2 + \frac{3}{4} k x \eta_x + \frac{1}{16} k^2 \int x \eta_x dx - \frac{3}{32} k^2 \int \eta dx + \frac{1}{4} k \eta_x \int \eta dx \right) = 0.
\end{equation}

This second order wave equation for the particular case of the linear bottom function \( h = kx \) and additionally \( \delta = \alpha \) is a partial integro-differential equation. It seems to be very difficult to analyze.

Assuming flat bottom \((k = 0)\), the equations (82)-(84) reduce to
\begin{equation}
(85) \quad w = \eta - \alpha \frac{1}{4} \eta^2 + \alpha^2 \left( \frac{1}{8} \eta^3 + B \frac{2 - 3\tau}{6} \eta_{2x} \right),
\end{equation}
\begin{equation}
(86) \quad \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \alpha^2 \left( - \frac{3}{8} \eta^2 \eta_x + B \frac{1 - 3\tau}{6} \eta_{2x} \right) = 0.
\end{equation}

It is worth noticing that the equation (85) is well known as the Gardner equation which is a linear combination of the KdV and the modified KdV equation. It was derived for internal waves in a two-layer liquid with a density jump at the interface [46, 47]. The Gardner equation is integrable and possesses solitary wave solutions. However, the Gardner solitons may have different properties from the KdV solitons, see, e.g., [18].

7. CASE 6: \( \beta = O(\alpha^2), \ \delta = O(\alpha^2) \)

In this case we set
\begin{equation}
(87) \quad \beta = B \alpha^2, \ \delta = D \alpha^2.
\end{equation}

The velocity potential is expressed, as in the section 5, by (70), but the boundary condition at the bottom (5) takes now the form
\begin{equation}
(88) \quad \phi_z - B D \alpha^4 (h_x \phi_x) = 0, \ \text{for} \ z = D \alpha^2 h(x).
\end{equation}

So, from (70) and (88) one gets
\[ F_x = \alpha^2 D (h f_x)_x + \frac{1}{2} \alpha^6 B^2 D^2 (h^2 F_{2x})_x - \frac{1}{6} \alpha^8 B^3 D^3 (h^3 f_{3x})_x + O(\alpha^{12}) = 0. \]
Neglecting higher order terms we can use

\[ F_x = \alpha^2 D(hf_x)_x + O(\alpha^6), \]

which ensures the expression of \( \phi \) through only one unknown function \( f \) and its derivatives. Note that next terms in \( F_x \) would enter in \( \phi \) in \( \alpha^8 \) order. Then we can express the velocity potential as

\[ \phi = f - \frac{1}{2} B \alpha^2 z^2 f_{2x} + \frac{1}{24} B^2 \alpha^4 z^4 f_{4x} - \frac{1}{720} B^3 \alpha^6 z^6 f_{6x} + BD \alpha^4 z(hf_x)_x - \frac{1}{6} B^2 \alpha^6 z^3(hf_x)_{3x} + O(\alpha^8). \]

With potential given by (90) we obtain from (3) and (4) the Boussinesq set

\[
\begin{align*}
\eta_t + w_x + \alpha(\eta w)_x - \alpha^2 \left( \frac{1}{2} B w_{3x} + D(hw)_x \right) - \alpha^3 \frac{1}{2} B (\eta w_{2x})_x + \alpha^4 \left( \frac{1}{2} BD(hw)_{3x} - \frac{1}{2} B (\eta^2 w_{2x})_x + \frac{1}{120} B^2 w_{5x} \right) &= 0, \\
w_t + \eta_x + \alpha w w_x - \alpha^2 B \left( \tau \eta_{3x} + \frac{1}{2} w_{2xt} \right) + \alpha^3 B \left( -(\eta w_{xt})_x + \frac{1}{2} w_x w_{2x} - \frac{1}{2} w w_{3x} \right) + \alpha^4 B \left( D(h\eta w)_{2x} + w_x (\eta w)_x - w(\eta w_{2x})_x - \frac{1}{2} (\eta^2 w_{xt})_x + \frac{1}{24} B w_{4xt} \right) &= 0,
\end{align*}
\]

respectively, where terms up to \( \alpha^4 \) are retained.

In first order \( w = \eta + \alpha(\frac{-1}{4} \eta^2) \) makes (91)-(92) compatible giving

\[ \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x = 0. \]

In second order one assumes \( w \) in the form

\[ w = \eta - \frac{1}{4} \alpha \eta^2 + \alpha^2 \left( \frac{1}{8} \eta^3 + \frac{2 - 3 \tau}{6} \eta_{2x} + Q \right), \]

which substituted to (91)-(92) gives (after neglect of terms of higher orders)

\[ \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \alpha^2 \left( -\frac{3}{8} \eta^2 \eta_x - \frac{1 - 3 \tau}{6} B \eta_{3x} - \frac{1}{2} D(h\eta)_x + Q \right) = 0 \]

from (91) and

\[ \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \alpha^2 \left( -\frac{3}{8} \eta^2 \eta_x - \frac{1 - 3 \tau}{6} B \eta_{3x} + Q \right) = 0 \]

from (92). Compatibility of equations (94) and (95) requires

\[ Q_x - Q_t = \frac{1}{2} D(h\eta)_x, \]

the condition analogous to those in (21) and (79). This equation cannot be fulfilled for a general bounded bottom function \( h(x) \). However, analogously as in the case considered in section 2 (see, eqs.
and section 6 (see, eqs. (79)-(80)), the condition (96) can be satisfied in the particular case
\( h(x) = k x \). In this case
\[
Q = \frac{1}{2} D (k x \eta + \frac{1}{2} k \int \eta \, dx),
\]
and the wave equation is
\[
\eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \alpha^2 \left( \frac{3}{8} \eta^2 \eta_x - \frac{1}{6} \frac{3 \tau}{\beta} \eta_{3x} - \frac{1}{2} D (k x \eta_x + \frac{1}{2} k \eta) \right) = 0.
\]
In original parameters the wave equation reads as
\[
\eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \frac{3}{8} \alpha^2 \eta^2 \eta_x - \frac{1}{6} \frac{3 \tau}{\beta} \eta_{3x} - \frac{1}{2} \delta (k x \eta_x + \frac{1}{2} k \eta) = 0.
\]
The equation (99) is another second order wave equation derived consistently for the particular ordering of small parameters, however, for the particular case of the linear bottom function.

For the flat bottom case, \( \delta = D = 0 \), the equations (97)-(99) reduce to the equations (85)-(86), where the last one is the Gardner equation (see the last paragraph in section 6).

8. Examples of numerical simulations

In this section, we show some examples of motion of solitons, initially moving over an even bottom when they enter the region where the bottom is no longer even. These results are obtained by numerical time evolution according to wave equations derived in previous sections. In order to be able to compare the influence of the bottom on the soliton movement, in all cases presented in this section, we assumed the same bottom shape in the form of a linear function. We discuss here only three cases: Case 1, where KdV solitons exist, Case 2, where KdV2 solitons were discovered by us in [31] and Case 4, where KdV5 solitons are known, as well [43, 44].

In the calculations we use our finite difference code described in detail in [31]. The cases 1 and 2 in which \( \alpha = O(\beta) \) are appropriate for shallow water waves where surface tension can be safely neglected. This is because when average water depth is given in meters then \( \tau < 10^{-7} \). Therefore in the simulations presented in subsections 8.1 and 8.2 we set \( \tau = 0 \) in the corresponding wave equations.

8.1. Case 1. In this case \( \alpha = O(\beta), \delta = O(\beta) \), we performed calculations according to the first order equation (23). The bottom function is taken in the form
\[
h(x) = \begin{cases} 
0 & \text{for } x \leq 0, \\
\frac{1}{20}x & \text{for } 0 < x \leq 20 \\
1 & \text{for } x > 20.
\end{cases}
\]
The shape of the bottom is displayed in presented figures below the snapshots of soliton’s motion, not in the same scale.
In Fig. 2, we display the numerical simulation obtained for the following parameters: $\alpha = 0.2424, \beta = 0.3, \delta = 0.2$. As initial condition we took the KdV soliton

$$\eta(x, t) = a \ \text{sech}^2 \left[ B(x - x_0 - vt) \right],$$

where $B = \sqrt{\frac{3\alpha}{4\beta}}$ and $v = 1 + \frac{\alpha}{2} a$,

with $x_0 = 0, t = 0$ and the amplitude $a = 1$ (to compare with the Case 2).

If we come back to dimension variables and assume $H = 2 \text{ m}$ then the surface elevation in meters is obtained by multiplying by $\alpha$. The horizontal coordinates have to be multiplied by $l = \frac{H}{\sqrt{\beta}} \approx 3.65 \text{ m}$. So, the range $x \in [0, 20]$ in the displayed figures corresponds to the interval (approximately) $[0, 73]$ in meters. The time increment in presented Figs. 2 and 3 is $dt = 2.5$ (in dimensionless units) what corresponds to approximately 0.825 second. Note that $v = 1.1212\sqrt{gH} \approx 4.96 \text{ m/s}$.

8.2. Case 2. In this case, $\alpha = O(\beta), \delta = O(\beta^2)$, the appropriate wave equation is the equation (40). It is worth to remind that when $\delta = 0$, the equation (40) reduces to the extended KdV (KdV2). Since KdV2 possesses exact soliton solution [31], we use this solution as the initial condition in the example presented in Fig. 3.

Contrary to the KdV equation which leaves one parameter freedom for the coefficients of the exact solutions (therefore KdV permits for solitons of different amplitudes), the parameters $\alpha, \beta$ of the
Figure 3. Time evolution of the KdV2 soliton entering the region with linear bottom function $h(x) = kx$ obtained in numerical integration of the equation (40) with parameters $\alpha = 0.2424$, $\beta = 0.3$, $\delta = 0.2$.

KdV2 equation fix the coefficients of the unique soliton solution. So, for the evolution shown in Fig. 3 the initial condition has the the same form (101) but with coefficients: $a \approx \frac{0.2424}{\alpha}$, $B \approx \sqrt{0.6\alpha}$ and $v \approx 1.11455$. The parameter $\alpha = 0.2424$ assures the amplitude equal one.

For comparison of the KdV2 soliton motion according to the equation (40) to the KdV soliton motion according to the equation (23) we used the same values of the parameters $\alpha, \beta, \delta$. Using $\delta = 0.2 = D \beta^2$ with $D \approx 2.22$, not much different from unity, does not contradict to the assumption $\delta = O(\beta^2)$.

8.3. Case 4. In this case, $\alpha = O(\beta^2)$, $\delta = O(\beta^2)$, the appropriate wave equation is the equation (68). It is worth to remind that when $\delta = 0$, the equation (68) reduces to the fifth-order KdV or KdV5 ([41],[44]). The fifth-order KdV has exact soliton solution ([43],[44]), so we use this solution as the initial condition in the example presented in Fig. 4. The explicit form of this solution is

\begin{equation}
\eta(x,t) = A \text{Sech}^4[B(x - vt)],
\end{equation}
where

\[ A = \frac{1050(1 - 3\tau)^2}{1521(19 - 30\tau - 45\tau^2)\alpha}, \]

\[ B^2 = -\frac{15(1 - 3\tau)}{13(19 - 30\tau - 45\tau^2)\beta}, \]

and \[ v = 1 + \frac{540(1 - 3\tau)^2}{169(19 - 30\tau - 45\tau^2)}. \]

Note that \( B^2 > 0 \) occurs for \( \tau > \frac{1}{3} \). So, for illustration we chose \( \tau = 0.36 \). The profile of this solutions is plotted in Fig. 4 with red line. The numerical evolution of this wave according to the equation (68) for \( \delta = 0 \) (flat bottom) confirms that it moves with the constant shape and constant velocity.

**Figure 4.** Time evolution of the fifth order KdV soliton entering the region with linear bottom function \( h(x) = kx \) obtained in numerical integration of the equation (68) with parameters \( \alpha = 0.2424, \beta = 0.3, \delta = 0.2 \).

Snapshots of numerical evolution of the fifth order KdV soliton (102) according to the equation (68) for \( \delta = 0.2, \alpha = 0.2424, \beta = 0.3 \) are displayed in Fig. 4. In dimensionless coordinates this setup is the same as in previously discussed examples in subsections 8.1 and 8.2. Coming back to dimension variables we realize that it is the system of completely different scale. First, taking \( T = 72 \text{ mN/m} \) as water surface tension one obtains water depth \( h = \sqrt{\frac{T}{\rho g \tau}} \approx 0.0045 \text{ m} \). Next, with \( \beta = 0.3 \) the dimensionless interval \( x \in [0, 20] \) corresponds to \( [0, 0.164] \text{ m} \). Indeed, the equation (68) describes
soliton motion in capillary-gravity case with the uneven bottom. In general, the properties of this motion in dimensionless variables are similar to those observed in subsections 8.1 and 8.2.

![Figure 5](image)

**Figure 5.** Solitons’ maxima versus their positions from Fig. 2 - red symbols, from Fig. 3 - green symbols and from Fig. 4 - blue symbols. In the case of KdV5 displayed are the rescaled absolute values.

8.4. Brief comparison. In all three cases displayed in Figs. 2, 3 and 4 the solitons move initially over the flat bottom with undisturbed shapes and constant initial velocities. Next, moving over the slope all solitons experience an amplitude increase and a corresponding decrease of velocity. The deformation of the profile, that is a lowering of the water level behind the soliton is almost imperceptible.

Surprisingly the amplitude of KdV2 soliton increases much more that that of the KdV soliton. Even more suprisingly, the amplitude of KdV5 soliton grows (in dimensionless variables) much more rapidly.

This is clearly visible in Fig. 5 in which the solitons’ maxima max[η] versus their positions (read from the calculated data) are plotted. Since in KdV5 case the amplitudes are negative we plot their absolute values multiplied by $1/|A(t = 0)|$ to compare the changes with respect to initial values.

For Cases 2 and 4 we have calculated numerical evolution of solitons’ motion with several different cases of bottom profiles, no longer the linear ones. More detail studies of these cases are left for future papers.

9. Conclusions

The results obtained in the paper can be summarized in the table 2.
TABLE 2. Wave equations for uneven bottom derived in this study. For abbreviation "bt" denotes the term originating from the bottom. PIDE is the abbreviation for Partial Integro-Differential Equation.

| Case | bottom function | 1st order equation | 2nd order equation |
|------|-----------------|-------------------|-------------------|
| 1    | linear $h = kx$ | KdV + bt (24)     | does not exist    |
| 2    | general $h = h(x)$ | KdV (52)         | KdV2 + bt (40)    |
| 3    | linear $h = kx$ | (52)              | does not exist    |
| 4    | general $h = h(x)$ | KdV without $\eta\eta_x$-term (60) | 5th ord. KdV + bt (68) |
| 5    | linear $h = kx$ | (82)              | PIDE (84)         |
| 6    | linear $h = kx$ | KdV without $\eta_3x$-term (93) | (99)              |

The presented study led us to the following conclusions.

(1) Physically relevant nonlinear wave equation for the case of uneven bottom can be consistently derived only for two cases of relations between small parameters, namely when $\alpha = O(\beta)$, $\delta = O(\beta^3)$ (Case 2) and $\alpha = O(\beta^2)$, $\delta = O(\beta^2)$ (Case 4). In the first one we obtained second order wave equation (40) which besides all terms of the extended KdV equation [36] contains the term $-\frac{1}{2} \delta (h\eta)_x$ originating from the bottom variations. The extended KdV equation possesses analytic solutions of several kinds [31, 37–39] which can be used in studies of numerical evolution of such waves when they move over the finite region of the uneven bottom of the arbitrary shape. In the second one the resulting second order wave equation (68) is the fifth-order KdV supplemented by the same term $-\frac{1}{2} \delta (h\eta)_x$ originating from the bottom variations. Since analytic solutions to fifth-order KdV are known one can use them to study numerical evolution of such waves when they enter the region of bottom changes.

(2) In other considered cases the Boussinesq equations can be made compatible, giving the final wave equations, only for particular case of the linear bottom function $h(x) = kx$. With this particular bottom shape the first order equation (23) or (24) for the Case 1: $\alpha = O(\beta)$, $\delta = O(\beta)$ was derived in [33]. This equation is the usual KdV supplemented by the term $-\frac{1}{2} \delta k (\eta + 2x\eta_x)$ originated from the uneven bottom.

(3) For the Case 3: $\alpha = O(\beta^2)$, $\delta = O(\beta)$ the resulting first order wave equation (52), which contains a bottom term, is linear. The extension to second order is not possible even for a linear bottom function.

(4) For the Case 5: $\beta = O(\alpha^2)$, $\delta = O(\alpha)$ the first order equation (81) can be obtained only for even more particular case $\delta = \alpha$. This equation contains the nonlinear term, the bottom term but the dispersion term is absent. The extension to second order is not possible.
(5) For the Case 6: $\beta = O(\alpha^2), \delta = O(\alpha^2)$ the first order equation (93) has the form of KdV but without dispersion term. Second order equation (99) can be obtained only for the particular case $h(x) = kx$.

(6) Numerical examples show that the effects of uneven bottom can be taken into account correctly only when the wave equations are extended to second order. It was possible for Cases 2 and 4. The soliton’s form appears to be very robust against substantial changes of the bottom despite changes of amplitudes, widths and velocities.

Studies of numerical evolution according to derived equations are planned in a near future.

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