FLAT MANIFOLDS ISOSPECTRAL ON p-FORMS

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Abstract. We study isospectrality on p-forms of compact flat manifolds by using the equivariant spectrum of the Hodge-Laplacian on the torus. We give an explicit formula for the multiplicity of eigenvalues and a criterion for isospectrality. We construct a variety of new isospectral pairs, some of which are the first such examples in the context of compact Riemannian manifolds. For instance, we give pairs of flat manifolds of dimension $n = 2p$, $p \geq 2$, not homeomorphic to each other, which are isospectral on p-forms but not on q-forms for $q \neq p$, $0 \leq q \leq n$. Also, we give manifolds isospectral on p-forms if and only if p is odd, one of them orientable and the other not, and a pair of 0-isospectral flat manifolds, one of them Kähler, and the other not admitting any Kähler structure. We also construct pairs, $M, M'$ of dimension $n \geq 6$, which are isospectral on functions and such that $\beta_p(M) < \beta_p(M')$, for $0 < p < n$ and pairs isospectral on p-forms for every $p$ odd, and having different holonomy groups, $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$ respectively.

§1. Introduction

This article is a sequel to [MRI] and [MR2], where we use Sunada’s method ([Su]) to produce many pairs of isospectral, non-homeomorphic Hantzsche-Wendt manifolds. These are natural generalizations to any odd dimension $n$, of the classical 3-dimensional Hantzsche-Wendt manifold (see [Wo]).

The isospectral manifolds obtained by Sunada’s method are always strongly isospectral, hence $p$-isospectral (that is, isospectral on $p$-forms) for all $p$, $0 \leq p \leq n$. The main purpose of the present article is to exhibit many examples of compact flat manifolds which are $p$-isospectral for some (but not all) values of $p$. These examples seem to be new in the context of compact Riemannian manifolds, to our best knowledge. We will study $p$-isospectral manifolds by using the equivariant spectrum on the torus, giving an explicit formula for the multiplicities of the eigenvalues of the Hodge-Laplacian and, as a consequence, a condition for isospectrality on $p$-forms for each $p$ (see Theorem 3.1). This formula will also be used to prove non-isospectrality for some flat manifolds, by computing the multiplicities of specific eigenvalues (see Examples 4.1, 5.1, 5.6). All isospectral manifolds to be constructed, except those in Example 4.2, will be pairwise non-homeomorphic, since they will have non-isomorphic fundamental groups.

We consider first Bieberbach groups with holonomy group $\mathbb{Z}_2^r$ and diagonal holonomy representation. In this case, the formulas for the multiplicities of eigenvalues involve combinatorial coefficients, namely, integral values of Krawtchouk polynomials, which do vanish in some cases. This allows to produce examples of $p$-isospectral manifolds of dimension $2p$ ($p \geq 2$) which are not $q$-isospectral for $q \neq p$. We also construct

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pairs of manifolds isospectral on $p$-forms for every $p$ odd and having different holonomy groups: $\mathbb{Z}_4$ and $\mathbb{Z}_3^2$ respectively (see Examples 4.2 and 5.8). We give two 4-dimensional manifolds, $p$-isospectral for all $p$, with holonomy group $\mathbb{Z}_2^4$, and having different first integral homology group (see Example 4.5).

In Section 5 we look at pairs where the holonomy group is $\mathbb{Z}_4 \times \mathbb{Z}_2$ and the holonomy representation is not diagonal, giving examples with various properties, in particular:

- Two manifolds of dimension $n = 6$ which are 0-isospectral but not isospectral on $p$-forms for any $0 < p < 6$ (Example 5.1; see also Examples 5.4 and 5.5). Moreover one of these manifolds is Kähler and the other does not admit any Kähler structure. By a variation one obtains an isospectral pair where one of the manifolds is hyperkähler and the other is not (Example 5.3).
- Two manifolds which are isospectral on $p$-forms if and only if $p$ is odd, one of them orientable and the other not orientable (Example 5.6; see also Example 4.3).
- Two $n$-manifolds $M, M'$ which are isospectral on functions but not on $p$-forms for $0 < p < n$ and such that $\beta_p(M) < \beta_p(M')$ for each $0 < p < n$ (Example 5.9).

Manifolds which are 0-isospectral and not $p$-isospectral for some $p$ are not very common (see [Pe]). Examples have been given by Ikeda (see [Ik]) for lens spaces, by Gordon (the first example), Gornet (see [Go],[Gt] respectively) in the context of nilmanifolds and by Schueth (see [Sch]) for simply connected manifolds. Flat manifolds yield a very rich family of non-strongly isospectral pairs with a simple construction and having the property that certain of their topological invariants can be easily computed in many cases. Those constructed in this article are quotients of an $n$-torus by free actions of $\mathbb{Z}_2^r$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$ and they often yield isospectral manifolds which are topologically very different from each other, for instance they can be distinguished by their real cohomology (see Remark 5.10). The approach in this article can be applied to study isospectrality with respect to more general differential operators on vector bundles over flat manifolds (see Remark 3.8).

§2. Preliminaries

We begin this section by recalling some known facts on Bieberbach groups (see [Ch] or [Wo]).

A discrete, cocompact, torsion-free subgroup $\Gamma$ of $I(\mathbb{R}^n)$ is called a Bieberbach group. Then $\Gamma$ acts freely on $\mathbb{R}^n$ and $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group $\Gamma$. Any element $\gamma \in I(\mathbb{R}^n)$ decomposes uniquely $\gamma = BL_b$, with $B \in O(n)$ and $b \in \mathbb{R}^n$. The translations in $\Gamma$ form a normal, maximal abelian subgroup of finite index, $L_\Lambda$, with $\Lambda$ a lattice in $\mathbb{R}^n$ which is $B$-stable for each $BL_b \in \Gamma$. The restriction to $\Gamma$ of the canonical projection from $I(\mathbb{R}^n)$ to $O(n)$, mapping $BL_b$ to $B$, has kernel $\Lambda$ and the image is a finite subgroup of $O(n)$, called the point group of $\Gamma$. We shall often make the identification of $F := \Lambda \backslash \Gamma$ with the point group of $\Gamma$. The group $F$ coincides with the linear holonomy group of the Riemannian manifold $M_\Gamma$ and the action of $F$ on $\Lambda$ defines an integral representation of $F$, usually called the holonomy representation.

We now give a proposition which is useful in the construction of Bieberbach groups. It gives necessary and sufficient conditions for a crystallographic group to be torsion-free, in the case of abelian holonomy. It is a natural extension of Proposition 3.1 in [DM] (valid for $F \simeq \mathbb{Z}_2^r$) and will be used in sections 3 and 4 in the construction of isospectral manifolds.
Proposition 2.1. Assume that $\Gamma = \langle \gamma_1, \ldots, \gamma_r, L, \lambda : \lambda \in \Lambda \rangle$ is a subgroup of $\text{Aff}(\mathbb{R}^n)$, with $\gamma_i = B_i L h_i, \lambda \in \mathbb{R}^n$, $B_i \in \text{Gl}(n, \mathbb{R})$ of order $m_i$, $\Lambda$ a lattice in $\mathbb{R}^n$ stable by the $B_i$’s and $\langle B_1, \ldots, B_r \rangle \simeq \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}$. We then have that $\Gamma$ is torsion-free with translation lattice $\Lambda$ if and only if the following two conditions hold:

(i) For each pair $i, j$, $1 \leq i, j \leq r$, $(B_i^{-1} - \text{Id}) b_j - (B_j^{-1} - \text{Id}) b_i \in \Lambda$.

(ii) For each $I = (i_1, \ldots, i_s)$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq r$, let $B_{i_1} L b_{i_1}, \ldots, B_{i_s} L b_{i_s} = B_I L b(I) \in \Gamma$, with $B_I := B_{i_1} \cdots B_{i_s}$ and $b(I) = B_{i_1}^{-1} b_{i_1} + B_{i_2}^{-1} b_{i_2} + \cdots + B_{i_s}^{-1} b_{i_s} - b_s$. If $B_I$ has order $m$, then

$$
\left( \sum_{j=0}^{m-1} B_I^{-j} \right) (b(I) + \lambda) \in \Lambda \setminus \left( \sum_{j=0}^{m-1} B_I^{-j} \right) \Lambda. \tag{2.1}
$$

Finally, if $\Gamma$ satisfies conditions (i) and (ii), then $\Gamma$ is isomorphic to a Bieberbach group with holonomy group $F \simeq \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}$.

Proof. We first show that conditions (i) and (ii) are necessary for $\Gamma$ to be torsion-free with translation lattice $\Lambda$. One computes that $[\gamma_i, \gamma_j] = L_{\mu_{i,j}}$, where $\mu_{i,j} = B_i B_j (B_i^{-1} - \text{Id}) b_i - (B_i^{-1} - \text{Id}) b_j$. Since the translation lattice of $\Gamma$ is assumed to be $\Lambda$, this implies that condition (i) must hold.

Let now $\gamma = B_I L b(I) + \lambda$, with $B_I, b(I)$ as in (ii) and $B_I$ of order $m$. Then, $\gamma^h = B_I^h L v_h(I, \lambda)$, where $v_h(I, \lambda) = \left( B_I^{-(h-1)} + B_I^{-(h-2)} + \cdots + B_I^{-1} + \text{Id} \right) (b(I) + \lambda) \tag{2.2}$

Now, since $B_I$ has order $m$, it follows that $v_m(I, \lambda)$ lies in the translation lattice of $\Gamma$ which is $\Lambda$, and furthermore $v_m(I, \lambda) \neq 0$ because $\Gamma$ is torsion-free, hence condition (ii) must hold.

Conversely, assume now that (i) and (ii) hold. We note that by (i), $\gamma_i \gamma_j = \gamma_j \gamma_i L_{\lambda_{i,j}}$ with $\lambda_{i,j} \in \Lambda$ and also, if $\lambda \in \Lambda, \gamma_i L_{\lambda^{-1}} L_{\lambda'} = L_{\lambda'}$, for some $\lambda' \in \Lambda$. Furthermore we have that $\gamma_i^{-1} = \gamma_i^{-1} L_{\lambda_i}$ with $\lambda_i = -B_i \left( \sum_{j=0}^{m_i-1} B_i^{-j} \right) b_i \in \Lambda$, by (ii). Using these facts, one can show that any product of generators in $\Gamma$ can be reordered in such a way that a general element $\gamma \in \Gamma$ can be written $\gamma = \gamma_{i_1} \cdots \gamma_{i_s} L_{\lambda} = B_I L b(1) + \lambda$, for some $I$ as above and $\lambda \in \Lambda$. Now suppose that $\gamma^h = 1$. Then $B_I^h = \text{Id}$ and $v_h(I, \lambda) = 0$. This implies, if $m$ is the order of $B_I$, that $h = mk$. Hence $\gamma^h = (\gamma^m)^k = L_{v_m(I, \lambda)} = L_{k v_m(I, \lambda)}$, thus $k v_m(I, \lambda) = v_{k m}(I, \lambda) = 0$ and therefore $v_m(I, \lambda) = 0$. By (2.2), it now follows that $\left( \sum_{j=0}^{m-1} B_I^{-j} \right) b(I) \in \left( \sum_{j=0}^{m-1} B_I^{-j} \right) \Lambda$, contradicting (ii). This shows that $\Gamma$ is torsion-free.

It only remains to show that the translation lattice of $\Gamma$ equals $\Lambda$. Any element of $\Gamma$ has the form $\gamma = \gamma_1^{l_1} \cdots \gamma_r^{l_r} L \lambda$, with $\lambda \in \Lambda$ and $l_i \geq 0$. If $\gamma = L_m, \mu \in \mathbb{R}^n$, then necessarily $l_j = m_j k_j$ for $1 \leq j \leq r$, by the condition on the $B_j$’s in the statement. Now, for each $j$, $\gamma_j^{l_j} = (\gamma_j^{m_j})^{k_j} = L_{\lambda_j}^{k_j} = L_{k_j} \lambda_j$, with $\lambda_j \in \Lambda$, by (ii). It thus follows that $\mu \in \Lambda$, as was to be shown.

Concerning the last assertion, there exists an inner product in $\mathbb{R}^n$ which is invariant under the holonomy representation of $F$. Thus, conjugation by the positive definite transformation relating this inner product to the canonical one, takes $\Gamma$ into a torsion-free subgroup of $I(\mathbb{R}^n)$. This concludes the proof. □
Remark 2.2. (a) The arguments in the proof actually show that under the assumptions in the statement, $\Gamma$ will have translation lattice $\Lambda$ if and only if condition (i) holds and, for each $1 \leq i \leq r$, $\left(\sum_{j=0}^{m-1} B_{ij}\right) b_i \in \Lambda$. Furthermore, if this is the case, $\Gamma$ will be torsion-free if and only if $\left(\sum_{j=0}^{m-1} B_{ij}\right) b(I) \notin \left(\sum_{j=0}^{m-1} B_{ij}\right) \Lambda$, for any $I$ as above.

(b) In [DM], for $F \cong \mathbb{Z}_2^n$, it was shown that the condition for $\Gamma$ to be torsion-free is $(B_{i_1} \ldots B_{i_n} + \text{Id}) b(i_1, \ldots, i_n) \in \Lambda \setminus (B_{i_1} \ldots B_{i_n} + \text{Id}) \Lambda$ for any $1 \leq i_1 < i_2 < \ldots < i_n \leq r$. This condition is equivalent to the condition that the class defined by $\Gamma$ in $H^2(\mathbb{Z}_2^n, \mathbb{Z})$ be a special class (see [Ch, Thm 2.1, p. 79]).

§3. Equivariant spectrum of flat manifolds

If $M$ is a compact Riemannian manifold, let $\text{spec}^p(M)$ denote the spectrum of the Hodge-Laplace operator acting on smooth $p$-forms on $M$, $0 \leq p \leq n$. For each $p$, $\text{spec}^p(M)$ is a sequence of non-negative real numbers tending to $\infty$. Two Riemannian manifolds $M, M'$ are said to be $p$-isospectral if $\text{spec}^p(M) = \text{spec}^p(M')$. Usually, 0- isospectral manifolds are just called isospectral. Also, $M, M'$ are said to be strongly-isospectral if $\text{spec}_D(M) = \text{spec}_D(M')$, for any natural elliptic differential operator $D$ on $M, M'$ (see [DG, Def. 3.2]).

Our main goal in this section will be to describe the spectrum of the Laplacian on $p$-forms on $M_{\Gamma} = \Gamma \backslash \mathbb{R}^n$, $\Gamma$ a Bieberbach group with translation lattice $\Lambda$. We shall use on $\Lambda \backslash \mathbb{R}^n$ and on $\Gamma \backslash \mathbb{R}^n$ the Riemannian metrics induced by the canonical metric on $\mathbb{R}^n$.

We will first discuss the function case, $p = 0$. We note that if $v \in \Lambda^*$, the dual lattice of $\Lambda$, the function $f_v(x) = e^{2\pi i v \cdot x}$ on $\mathbb{R}^n$ is $\Lambda$-invariant and $-\Delta f_v = 4\pi^2 ||v||^2 f_v$ ($\Delta$ the Laplacian on $\mathbb{R}^n$). Furthermore, if $v, v' \in \Lambda^*$, $v \neq v'$, then $(f_v, f_{v'}) = 0$ and by the Stone-Weierstrass theorem, the set $\{f_v : v \in \Lambda^*\}$ is a complete orthogonal system of $L^2(\Lambda \backslash \mathbb{R}^n)$. Thus, for each $\mu \geq 0$, the eigenspace of $-\Delta$ with eigenvalue $4\pi^2 \mu$ is given by $\mathcal{H}_\mu := \text{span}\{f_v : v \in \Lambda^*_\mu\}$, where $\Lambda^*_\mu = \{v \in \Lambda^* : ||v||^2 = \mu\}$. The spectrum of $-\Delta$ in $\Lambda \backslash \mathbb{R}^n$ is thus determined by the cardinality of the sets $\Lambda^*_\mu$.

If we now look at $M_{\Gamma} = \Gamma \backslash \mathbb{R}^n$, we see that each $\gamma \in \Gamma$ preserves $\mathcal{H}_\mu$ since, for $\gamma = B L_b, B \in O(n), b \in \mathbb{R}^n$, $f_v(\gamma x) = e^{2\pi i v \cdot (Bx + Bb)} = e^{2\pi i B^{-1} v \cdot b} f_{B^{-1}v}(x)$. Hence

$$L^2(\Gamma \backslash \mathbb{R}^n) \simeq L^2(\Lambda \backslash \mathbb{R}^n)^\Gamma = \bigoplus_{\mu} \mathcal{H}_{\mu}^\Gamma.$$ 

Thus $\text{spec}(\Gamma \backslash \mathbb{R}^n) = \{(\mu, d_\mu) : \mu \geq 0$ and $d_\mu > 0\}$, where $d_\mu = \text{dim} \mathcal{H}_{\mu}^\Gamma$, for each $\mu$.

The spectrum on $p$-forms on $\Gamma \backslash \mathbb{R}^n$ is obtained in an entirely similar way. Let $\omega = \sum_J f_J dx_J$ be a $p$-form on $\mathbb{R}^n$, where $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_p}$ for $J = (j_1, \ldots, j_p)$, $1 \leq j_1 < \cdots < j_p \leq n$ and $f_J \in C^\infty(\mathbb{R}^n)$. We have that $\omega$ induces a $p$-form on $\Lambda \backslash \mathbb{R}^n$ if and only if $L^*\omega = \omega$ for any $\lambda \in \Lambda$, that is, if $f_J$ is translation invariant by $\Lambda$, for each $J$.

Since $\Delta_p \omega = \sum_J \Delta f_J dx_J$, it follows that, for each $\mu \geq 0$, an orthogonal basis of the eigenspace $\mathcal{H}_{\mu, p}$ of $-\Delta_p$ with eigenvalue $4\pi^2 \mu$ is

$$\{f_v dx_J : v \in \Lambda^*, ||v||^2 = \mu, J = (j_1, \ldots, j_p), |J| = p\}.$$ 

Furthermore, a simple calculation shows that $||f_v dx_J||^2 = \text{vol}(\Lambda \backslash \mathbb{R}^n)$, for every $v, J$.

As in the case $p = 0$, one has that a form $\omega$ pushes down to $\Gamma \backslash \mathbb{R}^n$ if and only if $\gamma^* \omega = \omega$ for each $\gamma \in \Gamma$, hence the eigenspace of $-\Delta_p$ on $p$-forms on $\Gamma \backslash \mathbb{R}^n$ with eigenvalue $4\pi^2 \mu$
We note that \( \sum O \) and \( \Gamma \) are isospectral on \( p \)-forms. Thus if \( \mu, B, \) is a Bieberbach group with holonomy group \( F \), for each \( \mu \), \( B \), we have:

\[
\gamma^*(f_v \ dx_J) = \gamma^* f_v \ B^* dx_J = e^{2\pi i B^{-1} v. b} f_{B^{-1} v} \sum_{|J'| = p} c_{J,J'}(B) dx_{J'}
\]

Thus \( \langle \gamma^*(f_v \ dx_J), f_v \ dx_J \rangle = e^{2\pi i v. b} c_{J,J}(B) \delta_{Bv,v} \ vol(\Lambda \setminus \mathbb{R}^n) \). Hence

\[
\text{tr } \gamma^*|_{\mathcal{H}_{p,\mu}} = \text{vol}(\Lambda \setminus \mathbb{R}^n)^{-1} \sum_{v:|v|^2 = \mu} \sum_{J:|J| = p} \langle \gamma^*(f_v dx_J), f_v dx_J \rangle = \sum_{\gamma \in \Lambda_0^*} e^{2\pi i v. b} \sum_{|J| = p} c_{J,J}(B).
\]

We note that \( \sum_{|J| = p} c_{J,J}(B) = \text{tr } \tau_p(B) \), where \( \tau_p \) is the canonical representation of \( O(n) \) on \( \Lambda^p(\mathbb{R}^n) \). We shall write \( \text{tr}_p(B) := \text{tr } \tau_p(B) \) (\( \text{tr}_0(B) = 1 \)). Now for each \( B \in F \) and \( \mu \geq 0 \), we set \( e_{\mu,B}(\Gamma) := \sum_{v:|v|^2 = \mu} e^{2\pi i v. b} \). Since, for \( BL_b \in \Gamma \), \( b \) is uniquely determined by \( B \mod \Lambda \), \( e_{\mu,B}(\Gamma) \) is well defined. We have thus proved:

**Theorem 3.1.** If \( \Gamma \) is a Bieberbach group with holonomy group \( F \), for each \( \mu \geq 0 \) and \( 0 \leq p \leq n \), the multiplicity of the eigenvalue \( 4\pi^2 \mu \) of \( -\Delta_p \) is given by

\[
d_{p,\mu}(\Gamma) = |F|^{-1} \sum_{B \in F} \text{tr}_p(B) e_{\mu,B}(\Gamma). \tag{3.1}
\]

Let \( \Gamma \) and \( \Gamma' \) be Bieberbach groups with holonomy groups \( F \) and \( F' \) and translation lattice \( \Lambda \). Let \( 0 \leq p \leq n \). If there is a bijection \( \Phi : B \leftrightarrow B' \) from \( F \) onto \( F' \) such that, for each \( \mu, B \),

\[
\text{tr}_p(B) e_{\mu,B}(\Gamma) = \text{tr}_p(B') e_{\mu,B'}(\Gamma')
\]

then \( M_\Gamma \) and \( M_{\Gamma'} \) are isospectral on \( p \)-forms.

In the examples in this paper we shall always use \( \Lambda \) equal to the canonical lattice, hence we will have \( \Lambda^* = \Lambda \), \( \text{vol}(\Lambda \setminus \mathbb{R}^n) = 1 \), all \( f_v dx_J \)'s will have norm one and the eigenvalues will be of the form \( 4\pi^2 \mu \) with \( \mu \in \mathbb{N}_0 \).

**Remark 3.2.** In the notation of Theorem 3.1 we see that if the bijection \( \Phi \) preserves \( \text{tr}_p \), then 0-isospectral implies \( p \)-isospectral. In particular this is always the case if \( F = F' \) and if one can take \( \Phi = \text{Id} \), then 0-isospectral implies \( p \)-isospectral for all \( p \). On the other hand, we shall see that \( p \)-isospectral need not imply 0-isospectral.

**Remark 3.3.** If \( B \in O(n) \) then \( \text{tr}_p(B) = \det(B) \text{tr}_{n-p}(B) \), hence (3.1) implies the well known fact that if \( M_\Gamma \) is orientable then \( d_{p,\mu} = d_{n-p,\mu} \) for all \( \mu \in \mathbb{N}_0 \) (see [BGM, p. 238]). In particular, if \( M_\Gamma \) and \( M_{\Gamma'} \) are orientable, then they are \( p \)-isospectral if and only if they are \((n-p)\)-isospectral.

**Remark 3.4.** We note that if \( \mu = 0 \) we have that

\[
\mathcal{H}_{p,0}^\Gamma = \langle dx_J : |J| = p \rangle^\Gamma = \langle dx_J : |J| = p \rangle^F \simeq \Lambda^p(\mathbb{R}^n)^F,
\]

hence \( d_{p,0} = \dim \Lambda^p(\mathbb{R}^n)^F \), which equals \( \beta_p(M_\Gamma) \) by [Hi, Lemma 2.2]. The equality \( d_{p,0} = \beta_p(M_\Gamma) \) follows also from the Hodge-de Rham theorem (see [BGM, p. 238]).
Definition 3.5. Given a set $I$ of $h$ elements, and a subset $I_o$ of $I$, with $|I_o| = j$, define for $l \leq h$, $w_{l,j}(h) = \sum_{|L|\leq l} (-1)^{|L\cap I_o|}$.

It is easy to see that $w_{l,j}(h)$ depends only on $h, j, l$ and not on $I$ and $I_o$. Furthermore, for each fixed $t$, $0 \leq t \leq \min(j, l)$, the number of subsets $L \subset I$ such that $|L \cap I_o| = t$ equals $({l \choose t})({h-j \choose i-t})$. Therefore, we have the formula

$$w_{l,j}(h) = \sum_{t=0}^{\min(j,l)} (-1)^t {j \choose t} {h-j \choose l-t} = K^h_l(j).$$

(3.2)

where $K^h_l(x) := \sum_{t=0}^{l}(-1)^t({x \choose t})({h-x \choose l-t})$ is the (binary) Kravtchouk polynomial ($P^h_l(x)$ in the notation of [KL]).

Remark 3.6. In the case when the holonomy representation diagonalizes in an orthonormal basis of the lattice (and we may assume this basis is the canonical basis), we can rewrite expression (3.1) in a more explicit form. In this case, $Bx = \pm e_i$, hence $B^*dx = (-1)^{|I_o\cap I|}dx$ where $I_B := \{ i : Bc_i = e_i \}$. Thus $\text{tr}_p(B) = \sum_{|J|\leq p}(-1)^{|J\cap I|} = K^n_p(n - n_B)$, where $I_B = \{ i : i \notin I_B \}, n_B = |I_B|$. Therefore we have

$$d_{p,\mu} = |F|^{-1} \sum_{v: \|v\|^2 = \mu} \sum_{\gamma \in \Lambda \setminus \Gamma} K^n_p(n - n_B) e^{2\pi i v.\gamma}.$$ (3.3)

The following lemma will be useful. These facts can be found in [KL] or [ChS], but we include a short proof for completeness.

Lemma 3.7. If $1 \leq i, j \leq n$, then we have that $K^n_i(j) = (-1)^j K^n_{i-1}(j)$ and $K^n_i(j) = (-1)^j K^n_{i-1}(n-j)$. Hence $K^n_i(n/2) = K^n_{n/2}(j) = 0$, for $n$ even and $i, j$ odd.

Proof. Let $I_o \subset I$ be such that $|I_o| = j, |I| = n$. If $L'$ denotes the complementary subset of $L \subset I$, $|L| = l$, then clearly $(-1)^{|L\cap I_o|} = (-1)^{|I_o|}(-1)^{|L'\cap I_o|}$. Since $L \rightarrow L'$ gives a bijection between the subsets of $I$ of cardinality $l$ and those of cardinality $n-l$, it follows that

$$K^n_l(j) = \sum_{L: L \subset I \atop |L| = l} (-1)^{|I_o|}(-1)^{|L'\cap I_o|} = (-1)^j \sum_{L': L' \subset I \atop |L'| = n-l} (-1)^{|I_o|} = (-1)^j K^n_{n-l}(j).$$

Similarly, since $(-1)^{|L\cap I_o|} = (-1)^{|L|}(-1)^{|L'\cap I_o|}$, we have that

$$K^n_l(j) = \sum_{L: L \subset I \atop |L| = l} (-1)^{|I|}(-1)^{|L'\cap I_o|}(-1)^l \sum_{L': L' \subset I \atop |L'| = l} (-1)^{|I_o|} = (-1)^l K^n_l(n-j). \quad \Box$$

Remark 3.8. We conclude this section by stating a representation theoretic generalization of Theorem 3.1.

Let $\Gamma$ be a Bieberbach group, $K = SO(n)$ and $G = I(\mathbb{R}^n)$. Let $V$ be a finite dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ and let $\rho \times \tau$ a unitary representation of $\Gamma \times K$ on
Wendt manifolds, recalling several results on isospectrality proved in [MR1] and [MR2].

In particular, if \( p > 1 \) and \( \tau = \tau_p \) (the exterior representation of \( SO(n) \) on \( \Lambda^p(\mathbb{R}^n) \)), we get the exterior vector bundle \( \Lambda^p(M_\Gamma) \) over \( M_\Gamma \) and \( \Delta_{p,\tau} \) coincides with the Hodge-Laplacian, since the Ricci tensor is zero.

If \( d_{p\times \tau, \mu}(\Gamma) \) denotes the multiplicity of the eigenvalue \( 4\pi^2 \mu \) of \( \Delta_{p,\tau} \), an argument similar to that in the proof of Theorem 3.1 shows that

\[
d_{p\times \tau, \mu}(\Gamma) = |F|^{-1} \sum_{\gamma = BL_\mu \in \Lambda \setminus \Gamma} \text{tr}(\rho(\gamma)\tau(B)) e_{\mu,B}(\Gamma).
\]

In particular, if \( \rho = 1 \) and \( \tau = \tau_p \), we get the formula (3.1) for the multiplicity of the eigenvalues of the Hodge-Laplacian acting on \( p \)-forms.

§4. \( p \)-isospectral flat manifolds with diagonal holonomy

This section will be devoted to give examples of isospectral flat manifolds with diagonal holonomy, showing in particular that \( p \)-isospectrality for some \( p > 0 \) need not imply \( q \)-isospectrality for \( q \neq p \) (Example 4.1). In Example 4.2 we give pairs of \( p \)-isospectral flat \( n \)-manifolds \((n \geq 4)\) with holonomy group \( \mathbb{Z}_2 \), which can be seen as analogues of the well known pairs of isospectral, non-isometric tori. In 4.4 we discuss the case of Hantzsche-Wendt manifolds, recalling several results on isospectrality proved in [MR1] and [MR2]. In Example 4.5 we give a pair of 4-manifolds, \( p \)-isospectral for all \( p \), whose construction is elementary and having different first integral homology groups.

In all cases we shall use Proposition 2.1 in the construction of Bieberbach groups. Condition (i) will hold automatically since the holonomy representation is diagonal and \( b_j \in \frac{1}{2} \Lambda \) (actually the use of Proposition 2.1 in [DM] would suffice in this section).

Example 4.1. We will give, for each \( p \geq 2 \), a family of Bieberbach groups of dimension \( n = 2p \), with holonomy group \( \mathbb{Z}_2 \), which, generically, are pairwise \( q \)-isospectral if and only if \( q = p \).

If \( k \) is odd, \( 1 \leq k \leq n - 1 \), we set \( C_k = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \), and define \( \Gamma_k = \langle C_k L_{2k}, L_\Lambda \rangle \). Clearly, \( (C_k + Id)^{\frac{k}{2}} = e_1 \in \Lambda \setminus (C_k + Id)\Lambda \), hence Proposition 2.1 applies. We thus get \( p \) Bieberbach groups with holonomy group \( \mathbb{Z}_2 \) which are pairwise non-isomorphic to each other, since the holonomy representations are pairwise non-semi-equivalent. We note that the contribution of the identity element to the sum in (3.3) is the same for all \( \Gamma_k \)'s. Now the vanishing of the \( K^p_{\mu}(j) \)'s for \( j \) odd, by Lemma 3.7, implies that the contribution of the second element of \( F \simeq \mathbb{Z}_2 \) to the sum in (3.3) is zero, hence all \( \Gamma_k \)'s are pairwise \( p \)-isospectral.

Furthermore, it is easy to see that these manifolds are not 0-isospectral, for instance by computing \( d_1(\Gamma_k) = d_{0,1}(\Gamma_k) \), the multiplicity of the eigenvalue \( 4\pi^2 \). Indeed, by (3.3) we have that

\[
d_1(\Gamma_k) = 2^{-1} \left( 2n + \sum_{v: \|v\| = 1 \atop C_k v = v} e^{\pi iv.e_1} \right) = 2^{-1} (2n - 2 + 2(k - 1)) = n + k - 2.
\]
Similarly we see that the $\Gamma_k$’s are pairwise not $n$-isospectral, by computing $d_{n,1}(\Gamma_k)$. In this case by (3.3):

$$d_{n,1}(\Gamma_k) = 2^{-1} \left( 2n + \sum_{\|v\|=1} K_n^{|I_{C_k}|} e^{\pi i v \cdot e_1} \right) = 2^{-1} \left( 2n + (-1)(-2+2(k-1)) \right) = n-k+2.$$ 

We note that generically, the $\Gamma_k$’s are pairwise not $q$-isospectral for $q \neq p$. We shall verify this in the case of the pair $\Gamma_1$ and $\Gamma_3$ by showing that $\beta_q(\Gamma_1) \neq \beta_q(\Gamma_3)$, for $1 \leq q \leq n-1$, $q \neq p$.

If $q$ is even, then we compute that

$$\beta_q(\Gamma_1) = \binom{n-1}{q}, \quad \beta_q(\Gamma_3) = 3 \binom{n-3}{q-2} + \binom{n-3}{q}.$$ 

Since $(\frac{n-1}{q}) = \binom{n-3}{q-1} + \binom{n-3}{q-2}$, it follows that $\beta_q(\Gamma_1) = \beta_q(\Gamma_3)$ if and only if $(\frac{n-3}{q-1}) = (\frac{n-3}{q-2})$, which occurs if and only if $q = \frac{n}{2}$.

In the case when $q$ is odd we have $\beta_q(\Gamma_1) = \binom{n-1}{q-1}, \quad \beta_q(\Gamma_3) = 3 \binom{n-3}{q-1} + \binom{n-3}{q-3}$. An argument entirely similar to the previous one in the case $q$ even, shows that equality occurs only if $q = \frac{n}{2}$. Remark 3.4 now implies that $M_{\Gamma_1}$ and $M_{\Gamma_3}$ are not $q$-isospectral for $1 \leq q \leq n-1$, $q \neq p$.

**Example 4.2.** The family in Example 4.1 can be enlarged by considering groups $\Gamma_{k,j} := \langle C_k L e_1, \ldots, L e_j, L_A \rangle$, for $k$ odd, $1 \leq j \leq k \leq n-1$, $n = 2p$. The $\Gamma_{k,j}$ with $1 \leq j \leq k$, are Bieberbach groups, which for fixed $k$ are pairwise non isometric to each other, since they are pairwise not isospectral. Indeed, by (3.3),

$$d_1(\Gamma_{k,j}) = \frac{1}{2} (2n + 2(k-2j)) = n + k - 2j.$$ 

However, by arguing exactly as in Example 4.1, we conclude that, for $p = \frac{n}{2}$, they are all $p$-isospectral to each other.

We shall next describe, for any $n$ even, a family of $n$-dimensional Bieberbach groups $\Gamma'_h$, all isomorphic to each other, and such that the corresponding manifolds are isospectral on $q$-forms for each $q$ odd, and they are pairwise non isometric.

For any $h$ with $1 \leq h \leq \frac{n}{2}$, we set $C := \text{diag}(1, \ldots, 1, -1, \ldots, -1)$, and define $\Gamma_h := \langle C L e_1 + \ldots + e_h \rangle$. We thus get $\frac{n}{2}$ Bieberbach groups with holonomy group $\mathbb{Z}_2$ which are all isomorphic to each other. Indeed, if $T_h$ is the linear transformation of $\mathbb{R}^n$ fixing $e_j$, for $j \geq 2$ and such that $T_h e_1 = \sum_{j=1}^h e_j$, then $T_h$ conjugates $\Gamma_1$ onto $\Gamma_h$.

Since, by Lemma 3.7, $K_q^{|I_{\mathbb{Z}_2}|}(\frac{n}{2}) = 0$ for any $q$ odd, it follows that the associated flat manifolds are isospectral on $q$-forms for $q$ odd, by (3.3). One can again see that they are not isospectral on functions (hence they are not isometric) by computing the multiplicity of the first non zero eigenvalue. These manifolds are quotients of a flat torus of dimension $n$ by free actions of $\mathbb{Z}_2$. In particular, in dimension $n = 4$, we get two very simple manifolds which are diffeomorphic, non isometric, and $q$-isospectral for $q = 1, 3$. We note that these examples can be seen as analogues of the well known isospectral, non isometric tori, which exist for $n \geq 4$ (see [CS] and references therein).
Example 4.3. We now construct a pair of flat 9-manifolds which are $p$-isospectral only for one odd value of $p$ and one even value of $p$. Moreover, one of them will be orientable and the other not orientable. We use $n = 9$ since it is the first odd dimension such that some of the coefficients $K_1^n(j)$ vanish. Actually, one computes easily using (3.2) that $K_1^3(j) = 0$ for $(l, j) = (2, 3); (2, 6); (3, 2); (3, 7); (6, 2); (6, 7); (7, 3)$ and (7, 6).

We let $B := \text{diag}(1, 1, 1, -1, -1, -1, -1, -1, -1)$, $B' := \text{diag}(1, 1, 1, 1, 1, -1, -1, -1, -1)$.

We define $\Gamma := \langle BL_{2}^{n}, L_{\Lambda} \rangle$ and $\Gamma' := \langle B'L_{2}^{n}, L_{\Lambda} \rangle$, $\Lambda$ the canonical lattice. As in Example 4.1 we see that Proposition 2.1 applies. We note that $M_{\Gamma} = \Gamma \backslash \mathbb{R}^9$ is orientable but $M_{\Gamma'} = \Gamma' \backslash \mathbb{R}^9$ is not orientable.

The vanishing of $K_2^3(6)$ and $K_2^3(6)$ imply that $tr_2(B) = tr_7(B) = 0$. Similarly, $tr_2(B') = tr_7(B') = 0$ since $K_2^3(3) = K_2^3(3) = 0$. It follows from Theorem 3.1 that $M_{\Gamma}$ and $M_{\Gamma'}$ are isospectral on $p$-forms for $p = 2$ and 7.

The non isospectrality for the other values of $p$ can be seen as follows. By Remark 3.4, the computation of the Betti numbers shows the non $p$-isospectrality for $p = 1, 3, 5$ and 9. Indeed, one computes that the Betti numbers are: 1, 3, 18, 46, 60, 60, 46, 18, 3, 1 for $M$ and 1, 6, 18, 38, 60, 66, 46, 18, 3, 0 for $M'$, respectively.

For the remaining values of $p$ (that is, 0, 4, 6 and 8) one can argue as in Example 4.1, showing that $d_1(\Gamma) < d_1(\Gamma')$, which proves non isospectrality on functions. Since $tr_p(B) = tr_p(B') \neq 0$ for $p = 4, 6$ and 8, this implies that $d_{p, 1}(\Gamma) \neq d_{p, 1}(\Gamma')$ for these values of $p$.

Example 4.4. An interesting case of flat manifolds with diagonal holonomy representation is that of Hantzsche-Wendt manifolds. These are compact flat $n$-manifolds with holonomy group $\mathbb{Z}_2^{n-1}$, which are rational homology spheres. We will recall some basic facts from [MR1]. Let $n$ be odd. A Hantzsche-Wendt group (or HW group) $\Gamma$ is an $n$-dimensional Bieberbach group with holonomy group $F \simeq \mathbb{Z}_2^{n-1}$ and such that the action of every $B \in F$ diagonalizes on a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of $\Lambda$ with $\det(\chi) = 1$. The holonomy group $F$ can thus be identified to the diagonal subgroup $\{B : Bv_i = \pm v_i, 1 \leq i \leq n, \det B = 1\}$. Since $n \geq 3$, for each pair $j \neq k$, there exists $B \in F$ such that $Bv_j = v_j, Bv_k = -v_k$, therefore it follows that all the vectors $v_i$ must be orthogonal to each other. Hence, after conjugation of $\Gamma$ by an element in $O(n, \mathbb{R})$ we may change the translation lattice of $\Gamma$ into a lattice generated by vectors $e_i, 1 \leq i \leq n$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$. For the purposes of constructing isospectral flat manifolds, we will assume as usual, that $\Lambda = \sum_{i=1}^{n} \mathbb{Z}e_i$, the canonical lattice, which is the one which allows more symmetries. The corresponding manifold $M_\Gamma := \Gamma \backslash \mathbb{R}^n$ is called a Hantzsche-Wendt (or HW) manifold.

We denote by $B_i$ the diagonal matrix fixing $e_i$ and such that $B_i e_j = -e_j$ (if $j \neq i$), for each $1 \leq i \leq n$. Clearly, $F$ is generated by $B_1, B_2, \ldots, B_{n-1}$. Furthermore $B_n = B_1 B_2 \ldots B_{n-1}$.

If $\Gamma$ is an HW group, then $\Gamma = \langle B_1 L_{b_1} \ldots B_{n-1} L_{b_{n-1}}, L_{\lambda} : \lambda \in \Lambda \rangle$, for some $b_i \in \mathbb{R}^n$, $1 \leq i \leq n - 1$, where it may be assumed that the components of $b_i, b_{ij} \in \{0, \frac{1}{n} \}$ for $1 \leq i, j \leq n$. The matrix $A := [b_{ij}]$ plays a main role in the study of HW groups. If $I = \{i_1, \ldots, i_s\}$, with $1 \leq i_1 < \cdots < i_s \leq n$, we write $B_I = B_{i_1} \ldots B_{i_s}$. Since $b_{ij} \in \{0, \frac{1}{n} \}$ for all $i, j$, it follows that any $\gamma \in \Gamma$ can be written uniquely $\gamma = B_1 L_{b_s(I)} L_{\lambda}$ with $s$ odd, $\lambda \in \Lambda$ and $b_s(I) \in \{0, \frac{1}{n} \}$, for $1 \leq j \leq n$. Condition (i) in Proposition 2.1 is automatic, while (ii) says that $(B_I + Id) b_s(I) \in \Lambda \setminus (B_I + Id) \Lambda$, for any $I$ as above.

We now rewrite formula (3.3) for the multiplicities of eigenvalues $d_{p, \mu}$ in the present case. For each $v \in \Lambda$ we set $I_v := \{j : v.e_j \neq 0\}$, $I_v^{\text{odd}} := \{j : v.e_j \text{ is odd}\}$. One has that $B_I e_j = -e_j$ if and only if $j \notin I$, since $|I|$ is odd. Hence $B_I v = v$ if and only if
Formula (4.1) can be used to construct many isospectral pairs of Hantzsche-Wendt manifolds. In [MR1] and [MR2] by using Sunada’s theorem, we have shown that the number of pairs of isospectral HW manifolds non homeomorphic to each other grows exponentially with $n$. In [MR2] we showed that there are exactly 62 diffeomorphism classes of HW manifolds in dimension 7, giving a full set of representatives and using (4.1) to list all isospectral classes for $p = 0$. From this classification it follows in particular that isospectrality implies Sunada-isospectrality, hence $p$-isospectrality for all $p$, for HW manifolds in dimension $n = 7$. Also, there is evidence that there exist large isospectral sets, for instance there are several families consisting of 10 pairwise isospectral HW manifolds in dimension 9.

In the more general setting of flat manifolds with diagonal holonomy representation, we have proved by combinatorial methods that 0-isospectrality implies Sunada-isospectrality, hence $p$-isospectrality for all $p$. As we have seen, there are some exceptions to this rule, due to the vanishing of some of the combinatorial numbers $K^n_p(j)$, which allow to have $p$-isospectrality for certain values of $p$ only. Even in the case of HW manifolds, there exist manifolds of dimension 9 which are isospectral on 3-forms but not 0-isospectral. We discuss these topics in another paper (see [MR3]).

Example 4.5. Among the known examples of isospectral, non homeomorphic manifolds in low dimensions one can mention those in [Vi] (hyperbolic 3-manifolds), [Ik] (lens spaces, for $n$ odd, $n \geq 5$), [Go] (nilmanifolds, $n \geq 5$) and [DM] (flat manifolds with $n \geq 5$). We shall now give a pair of isospectral flat manifolds $M_{\Gamma}$, $M_{\Gamma'}$ of dimension $n = 4$, with holonomy group $\mathbb{Z}_2^4$, whose construction is elementary. We will give their Betti numbers, and will show that $H_1(M_{\Gamma}, \mathbb{Z})$ and $H_1(M_{\Gamma'}, \mathbb{Z})$ are not isomorphic.

Now let $B_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $b_1 = \frac{\epsilon_a + \epsilon_4}{2}$, $b_2 = \frac{\epsilon_3}{2}$, $b'_1 = \frac{\epsilon_4}{2}$, $b'_2 = \frac{\epsilon_1}{2}$, and let $\Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}, \mathbb{Z}_4^4 \rangle$ and $\Gamma' = \langle B_1 L_{b'_1}, B_2 L_{b'_2}, \mathbb{Z}_4^4 \rangle$. One has that $B_1 L_{b_1} B_2 L_{b_2} = B_1 B_2 L_{\epsilon_a + \epsilon_4 + \epsilon_3}$, $\Lambda$, with $\lambda = -e_2 - e_4 \in \Lambda$. One easily sees that condition (ii) in Proposition 2.1 holds, hence $\Gamma$ and $\Gamma'$ are Bieberbach groups (they correspond respectively to those denoted by 5/1/2/7 and 5/1/2/9 in [BBNWZ]).

A direct computation shows that $H_1(M_{\Gamma}, \mathbb{Z}) \simeq \mathbb{Z}/[\Gamma, \Gamma] = \mathbb{Z} \oplus \mathbb{Z}_2^3$ and $H_1(M'_{\Gamma}, \mathbb{Z}) \simeq \mathbb{Z}/[\Gamma', \Gamma'] = \mathbb{Z} \oplus \mathbb{Z}_2^3$ (see alternately the table in [RT] §7, the manifolds with parameters $r = m_1 = m_2 = m_3 = 1$ and special classes $(0, 1, 1, 1)$ and $(h_1, 1, 0, 0)$ respectively). The Betti numbers of $M_{\Gamma}$ and $M'_{\Gamma}$ are immediately computed by using that $\beta_j = \dim \Lambda^j(\mathbb{R}^4)^F$ for $0 \leq j \leq 4$ (Remark 3.4). One has that $\Lambda(\mathbb{R}^4)^F = \langle 1, e_1, e_2 \wedge e_3 \wedge e_4, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \rangle$, hence $\beta_j = 1$ for $j \neq 2$ and $\beta_2 = 0$, for both, $M_{\Gamma}$ and $M'_{\Gamma}$.

We shall verify next, using Theorem 3.1 with $\Phi = \text{Id}$, that $M_{\Gamma}$ and $M_{\Gamma'}$ are isospectral. We first see that for each $\mu \in \mathbb{N}_0$ and $B \in F$, $\epsilon_{\mu,B}(\Gamma) = \epsilon_{\mu,B}(\Gamma')$. By Remark 3.2 it will then follow that $M_{\Gamma}$ and $M_{\Gamma'}$ are $p$-isospectral for all $p$. 

In the first place, if $B = \text{Id}$, one always has that $e_{\mu,\text{Id}} = \text{card}\{v : ||v||^2 = \mu\}$. If $B \neq \text{Id}$ one has:

$$
eq \sum_{||v||^2 = \mu, v \in \langle e_1, e_2 \rangle} e^{2\pi i \frac{e_1}{\mu} v} = \sum_{||v||^2 = \mu, v \in \langle e_1, e_2 \rangle} e^{2\pi i \frac{\mu}{e_1} v} = e_{\mu, B_1}(\Gamma');$$

$$e_{\mu, B_2}(\Gamma) = \sum_{||v||^2 = \mu, v \in \langle e_1, e_3 \rangle} e^{2\pi i \frac{e_1}{\mu} v} = \sum_{||v||^2 = \mu, v \in \langle e_1, e_3 \rangle} e^{2\pi i \frac{\mu}{e_1} v} = e_{\mu, B_2}(\Gamma'),$$

where the central equality of the second line holds since the sum is symmetric with respect to $e_1, e_3$. In the case of $B_1$ we have used the fact that one can always disregard any vector perpendicular to the space fixed by $B$ (in this case $\frac{e_3}{\mu}$). The equality in the case of $B_1B_2$ results from combining these two observations. Indeed

$$e_{\mu, B_1B_2}(\Gamma) = \sum_{||v||^2 = \mu, v \in \langle e_1, e_4 \rangle} e^{2\pi i \frac{e_1}{\mu} v} = \sum_{||v||^2 = \mu, v \in \langle e_1, e_4 \rangle} e^{2\pi i \frac{\mu}{e_1} v} = e_{\mu, B_1B_2}(\Gamma').$$

We observe that one can show that the manifolds in the present example are actually Sunada-isospectral, by using the same methods as in [MR1, §3], for instance.

§5. ISOSPECTRAL, NON-STRONGLY ISOSPECTRAL FLAT MANIFOLDS

In the present section we shall construct several examples of pairs of non-homeomorphic manifolds with holonomy group $\mathbb{Z}_4 \times \mathbb{Z}_2$, which are isospectral on $p$-forms for some $p$, $0 \leq p \leq n$, but not for all values of $p$. The use of non-diagonal holonomy representation will allow to construct various isospectral pairs with new properties, as described in the Introduction. The procedure in most cases will be to define a bijection $\Phi : F \to F'$ as in Theorem 3.1, which does not preserve the value of $\text{tr}_p(B)$, for some values of $p$.

Example 5.1. Let $\tilde{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and let $\text{Id}_2$ be the identity transformation on $\mathbb{R}^2$. One has that $\tilde{J}^4 = \text{Id}_2$.

Let $\Gamma = \langle B_1L_{b_1}, B_2L_{b_2}, \Lambda \rangle$, and $\Gamma' = \langle B'_1L_{b'_1}, B'_2L_{b'_2}, \Lambda \rangle$, where $\Lambda = \mathbb{Z}^6$ and

$$B_1 = \begin{bmatrix} \tilde{J} & \tilde{J} \\ \tilde{J} & \text{Id}_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -\text{Id}_2 & \text{Id}_2 \\ \text{Id}_2 & \text{Id}_2 \end{bmatrix},$$

$$B'_1 = \begin{bmatrix} \tilde{J} & 1 \\ -1 & -1 \end{bmatrix}, \quad B'_2 = \begin{bmatrix} -\text{Id}_2 & -1 \\ -1 & -1 \end{bmatrix},$$

$$b_1 = \frac{e_5}{4}, \quad b_2 = \frac{e_6}{2}, \quad b'_1 = \frac{e_6}{4}, \quad b'_2 = \frac{e_4 + e_5}{2}.$$
Condition (ii) holds for $\Gamma$ since $(\sum_{j=0}^{m-1} B^{-j}) b = mb \in \Lambda \setminus \left(\sum_{j=0}^{m-1} B^{-j}\right) \Lambda$. For instance, if $B = B_1$, we get $(\sum_{j=0}^{3} B^{-j}) b_1 = e_5 \in \mathbb{Z}^6 \setminus 4\mathbb{Z}e_5 \oplus 4\mathbb{Z}e_6$. In the case of $\Gamma'$ one can argue in a similar way, using the fact that $B'e_6 = e_6$ for any $B'L_{m'} \in \Gamma'$.

It follows that $\Gamma$ and $\Gamma'$ are Bieberbach groups with translation lattice $\mathbb{Z}^6$ and holonomy group $\mathbb{Z}_4 \times \mathbb{Z}_2$. Since the holonomy representations are not semiequivalent to each other, $\Gamma$ and $\Gamma'$ are not isomorphic (see [Ch. p. 81]).

One has that $F = \{B^i_1B^j_2: 0 \leq i \leq 3, 0 \leq j \leq 1\}$, $F' = \{B'^i_1B'^j_2: 0 \leq i \leq 3, 0 \leq j \leq 1\}$. In the following table we give, in a visual way, a list of the non trivial representantives of $\Lambda \setminus \Gamma$ and $\Lambda \setminus \Gamma'$. One represents each element of $F$ and $F'$ by a column, indicating the (non zero) translational components modulo $\Lambda$ by means of subindices. For example $B_1L_{b_1}$ will be represented by the column $(\tilde{J}, \tilde{J}, 1, 1, 1)^t$, where $\frac{1}{4}$ in the fifth component indicates that $b_1 = \frac{e_5}{4}$. As each translational component is only determined modulo $\Lambda$, one may choose it, in each case, in such a way that its coordinates lie in $[0, 1)$.

\[
\begin{array}{ccc}
B_1 & B^2_1 & B^3_1 & B_2 & B_1B_2 & B^2_1B_2 & B^3_1B_2 \\
\tilde{J} & -Id_2 & -\tilde{J} & -Id_2 & -\tilde{J} & Id_2 & \tilde{J} \\
\tilde{J} & -Id_2 & -\tilde{J} & Id_2 & \tilde{J} & -Id_2 & -\tilde{J} \\
1_4 & 1_2 & 1_4 & 1 & 1_4 & 1_2 & 1_4 \\
1 & 1 & 1 & 1_2 & 1_2 & 1_2 & 1_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
B^i_1 & B^{i^2}_1 & B^{i^3}_1 & B'_2 & B'_1B'_2 & B'^{i^2}_1B'_2 & B'^{i^3}_1B'_2 \\
\tilde{J} & -Id_2 & -\tilde{J} & -Id_2 & -\tilde{J} & Id_2 & \tilde{J} \\
1 & 1 & 1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & 1_2 & -1_2 & 1_2 & -1_2 \\
-1 & 1 & -1 & -1_2 & 1_2 & -1_2 & 1_2 \\
1_4 & 1_2 & 1_4 & 1 & 1_4 & 1_2 & 1_4 \\
\end{array}
\]

We now analyze isospectrality by using Theorem 3.1. We define a bijection $\Phi: F \to F'$ by $\Phi(B^i_1B^j_2) = B'^{i^j}_1B'^{j^i}_2$, if $i = 1, 3$ or $i = 2, j = 1$ and by $\Phi(B^2_1) = B'_2$, $\Phi(B_2) = B'^2_1$. One observes that $\Phi$ preserves, except for a permutation of the coordinates, the spaces fixed by the matrices $B_i$ and the projections of the translational components onto these fixed spaces. For example, since $\ker(B_1 - Id) = \langle e_5, e_6 \rangle$, $b_1 = \frac{e_5}{4}$ and for $B'_1 = \Phi(B_1)$ one has $\ker(B'_1 - Id) = \langle e_3, e_6 \rangle$, $b'_1 = \frac{e_6}{4}$.

One observes that the spaces fixed by $B_1$ and $B'_1$ are the same up to a permutation map which sends $e_3$ to $e_5$, $e_5$ to $e_6$, $e_6$ to $e_3$ and leaves the other $e_i$’s fixed. In the case of $B'^2_1$ and $B'_2 = \Phi(B^2_1)$ one has $\ker(B'_1 - Id) = \langle e_5, e_6 \rangle$, and the translational component of $B'^2_1$ is $\frac{3}{2}$ while $\ker(B'2 - Id) = \langle e_4, e_6 \rangle$ and $b'_2 = \frac{2e_4 + 3e_6}{2}$ which, when projected onto the fixed space of $B'_2$ yields $\frac{3}{2}$. Taking into account these considerations, with an argument analogous to that of Example 4.5, one verifies that $e_{\mu, B}(\Gamma) = e_{\mu, \Phi(B)}(\Gamma')$, $\forall B \in F$, hence $M_{\Gamma}$ and $M_{\Gamma'}$ are isospectral on functions. Since they are orientable, it follows that they are also 6-isospectral.

We now see that $M_{\Gamma}$ and $M_{\Gamma'}$ are not $p$-isospectral, for $1 \leq p \leq 5$. In the first place we
observe that $B_1^2$ and $B_2'$ are conjugate by a permutation matrix, and the same happens with $B_2$ and $B_1^2$ and with $B_1^2 B_2$ and $B_1^2 B_2'$. As a consequence, $\text{tr}_p(B_1^2) = \text{tr}_p(B_2')$, $0 \leq p \leq 6$, hence, since $e_{\mu,B_1^2}(\Gamma) = e_{\mu,B_2'}(\Gamma')$, it follows that $B_1^2$ and $B_2'$ give the same contribution to the sum in (3.1), for each $p$. The same is true for the two remaining pairs.

Since $\tilde{J}$ and $-\tilde{J}$ are conjugate it turns out that all 4 elements of order 4 in $F$ (resp. $F'$): $B_1, B_1^3, B_1 B_2$ and $B_1^3 B_2$ (resp. $B_1', B_1^3, B_1' B_2'$ and $B_1^3 B_2'$) are conjugate to each other, hence their $p$-traces are the same, for each $p$. The $p$-traces of $B_1$ and $B_1'$ are respectively given by

| $p$ | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|
| $\text{tr}_p(B_1)$ | 2  | 3  | 4  | 3  | 2  |
| $\text{tr}_p(B_1')$ | 0  | -1 | 0  | -1 | 0  |

For instance, if $p = 2$, $B_1$ induces a linear transformation in $\Lambda^2(\mathbb{R}^6)$ such that the associated matrix in the basis $\{e_i \wedge e_j : i < j\}$ has only three non-zero diagonal entries corresponding to $e_1 \wedge e_2$, $e_3 \wedge e_4$ and $e_5 \wedge e_6$, all of them equal to 1. Hence $\text{tr}_2(B_1) = 3$. However, in the case of $B_1'$ the non-zero diagonal entries correspond to $e_1 \wedge e_2$, $e_3 \wedge e_6$, $e_4 \wedge e_5$ (equal to 1) and to $e_3 \wedge e_4$, $e_3 \wedge e_5$, $e_4 \wedge e_6$ and $e_5 \wedge e_6$ (equal to $-1$). Hence, $\text{tr}_2(B_1') = -1$. The remaining $p$-traces for $B_1$ and $B_1'$ are computed similarly. Furthermore, by Remark 3.3, $\text{tr}_p(B) = \text{det}(B) \text{tr}_{n-p}(B)$ for any $B \in O(n)$, hence in general it suffices to compare the multiplicities $d_{p,\mu}$ for $p \leq \left\lfloor \frac{n}{2} \right\rfloor$.

By (3.1) the expressions of $d_{p,\mu}(\Gamma)$ and $d_{p,\mu}(\Gamma')$ are as follows:

$$d_{p,\mu}(\Gamma) = [F]^{-1} \left( \text{tr}_p(\text{Id}) e_{\mu,\text{Id}}(\Gamma) + \sum_{B:B \neq \text{Id}} \text{tr}_p(B) e_{\mu,B}(\Gamma) \right) = [F]^{-1} \left( \begin{pmatrix} n \\ p \end{pmatrix} \left| \left\{ v : \|v\|^2 = \mu \right\} \right| + \text{tr}_p(B_1) \sum_{B: \text{ord}(B) = 4} e_{\mu,B}(\Gamma) + \sum_{B: \text{ord}(B) = 2} \text{tr}_p(B) e_{\mu,B}(\Gamma) \right).$$

$$d_{p,\mu}(\Gamma') = [F']^{-1} \left( \begin{pmatrix} n \\ p \end{pmatrix} \left| \left\{ v : \|v\|^2 = \mu \right\} \right| + \text{tr}_p(B_1') \sum_{B': \text{ord}(B') = 4} e_{\mu,B'}(\Gamma') + \sum_{B': \text{ord}(B') = 2} \text{tr}_p(B') e_{\mu,B'}(\Gamma') \right).$$

By the previous observations it turns out that the only possible difference between both expressions is in the second terms. Since $\sum_{\text{ord}(B) = 4} e_{\mu,B}(\Gamma) = \sum_{\text{ord}(B) = 4} e_{\mu,B'}(\Gamma')$ and, as seen above, $\text{tr}_p(B_1) \neq \text{tr}_p(B_1')$, for $1 \leq p \leq 5$, the proof will be complete if we verify that $\sum_{\text{ord}(B) = 4} e_{\mu,B}(\Gamma) \neq 0$, for some value of $\mu$. The fixed space of each $B \in F$ with $\text{ord}(B) = 4$ is $\langle e_5, e_6 \rangle$. We take $\mu = 8$. The vectors $v \in \langle e_5, e_6 \rangle \cap \Lambda$ with $\|v\|^2 = 8$ are $v = \pm 2e_5 \pm 2e_6$ and one has that:

$$\sum_{\text{ord}(B) = 4} e_{2\pi i b \cdot (\pm 2e_5 \pm 2e_6)} = e^{2\pi i \frac{2\pi}{4}(\pm 2e_5 \pm 2e_6)} + e^{2\pi i \frac{2\pi}{4}(\pm 2e_5 \pm 2e_6)} + e^{2\pi i \frac{\pi}{4}(\pm 2e_5 \pm 2e_6)} + e^{2\pi i \frac{\pi}{4}(\pm 2e_5 \pm 2e_6)} = -4.$$
Remark 5.2. We note also that the non isospectrality in the previous example can be obtained by comparing \( d_{p,0}(\Gamma) \) and \( d_{p,0}(\Gamma') \), which can be determined directly by Remark 3.4.

One has that \( d_{p,0}(\Gamma) = \beta_p(\Gamma) = \dim \Lambda^p(\mathbb{R}^n)^F \) and analogously for \( d_{p,0}(\Gamma') \). The calculation of the \( F \) and \( F' \)-invariants gives respectively:

| \( p \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|---|
| \( \beta_p(\Gamma) \) | 1 | 2 | 3 | 4 | 3 | 2 | 1 |
| \( \beta_p(\Gamma') \) | 1 | 1 | 1 | 2 | 1 | 1 | 1 |

Example 5.3. One notes that the group \( \Gamma \) defined in Example 5.1 has the property that the holonomy action commutes with the invariant Kähler structure defined by \( J_1 = \begin{bmatrix} \bar{j} & j \\ j & \bar{j} \end{bmatrix} \) in \( \mathbb{R}^6 \). It follows that \( M_\Gamma \) inherits a Kähler structure. On the other hand \( M_{\Gamma'} \) (as in 5.1) is isospectral to \( M_\Gamma \) but the holonomy action does not commute with \( J_1 \) and actually, \( M_{\Gamma'} \) does not admit any Kähler structure since \( \beta_1(M_{\Gamma'}) = 1 \) is odd (see [We]).

One may use the previous pair to obtain a hyperkähler manifold isospectral to a non-hyperkähler one, as follows. By duplication of the tables of \( \Gamma \) and \( \Gamma' \) in Example 5.1 (graphically, this means placing an identical second copy of each table below the first one) we obtain Bieberbach groups \( \Gamma_2 \) and \( \Gamma_2' \) of dimension 12, with holonomy groups \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) and holonomy representations \( \rho \oplus \rho \) and \( \rho' \oplus \rho' \), where \( \rho, \rho' \) denote the holonomy representation of \( \Gamma \) and \( \Gamma' \), respectively. By using the same bijection as before we see that \( M_{\Gamma_2} \) and \( M_{\Gamma_2'} \) are isospectral.

We extend the complex structure \( J_1 \) to \( \mathbb{R}^{12} \) by \( J_{2,1} = \begin{bmatrix} J_1 & 0 \\ 0 & -J_1 \end{bmatrix} \), and furthermore we have a second complex structure \( J_{2,2} = \begin{bmatrix} 0 & \text{Id}_6 \\ -\text{Id}_6 & 0 \end{bmatrix} \) on \( \mathbb{R}^{12} \), which anticommutes with \( J_{2,1} \), hence \( J_{2,1} \) and \( J_{2,2} \) define a hyperkähler structure on \( \mathbb{R}^{12} \). Since the holonomy action of \( \Gamma_2 \) clearly commutes with \( J_{2,1} \) and \( J_{2,2} \), \( M_{\Gamma_2} \) now inherits a hyperkähler structure. On the other hand, \( \beta_1(M_{\Gamma_2'}) = 2 \), the dimension of the fixed space of the holonomy action. Hence \( M_{\Gamma_2} \) can not carry a hyperkähler structure, since \( \beta_1 \) is not divisible by 4.

Remark 5.4. We may extend the holonomy representations \( \rho \) and \( \rho' \) of \( F \) and \( F' \) respectively, in Example 5.1 to \( \rho \oplus \tau \) and \( \rho' \oplus \tau \) respectively, where \( \tau \) is a sum of characters \( \chi_h, 1 \leq h \leq k \), with values in \( \{1,-1\} \). If we keep the same \( b_i, b_i', 1 \leq i \leq 2 \), clearly the resulting groups \( \Gamma \) and \( \Gamma' \) are torsion-free and of dimension \( k+6 \). Visually what has been done is adding \( k \) rows of 1’s and \(-1\)’s to the table in Example 5.1 (corresponding to the characters \( \chi_1, \ldots, \chi_k \)). If, furthermore, one chooses \( \chi_h \) in such a way that the action of the second generator is trivial (that is, in such a way that all the new entries in the fourth column are 1’s), then \( M_{\Gamma} \) and \( M_{\Gamma'} \) are isospectral by Theorem 3.1, with the bijection \( \Phi \) chosen as in Example 5.1.

Example 5.5. If, as explained in Remark 5.4 one adds to the table in Example 5.1 only one character \( \chi \) represented by the row \((-1,1,-1,1,-1,1,-1)\) one obtains isospectral manifolds \( M_{\Gamma} \) and \( M_{\Gamma'} \) of dimension 7, both non orientable, with holonomy group \( \mathbb{Z}_4 \times \mathbb{Z}_2 \), and which are not \( p \)-isospectral for \( p = 1, 2, 5, 6 \). This follows from the values of the Betti numbers, which are given by:
Example 5.6. In the present case we add the character \((-1,1,-1,1,-1,1,-1)\) twice, to \(\Gamma\) and the characters \((-1,1,-1,1,-1,1,1,1)\) and \((1,1,1,1,1,1,1)\) to \(\Gamma'\) in Example 5.1, respectively. One thus gets the following tables:

| \(p\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|---|---|---|---|---|---|---|
| \(\beta_p(M_\Gamma)\) | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 |
| \(\beta_p(M_{\Gamma'})\) | 1 | 1 | 2 | 4 | 3 | 3 | 2 | 0 |

It is easy to verify that \(M_\Gamma\) and \(M_{\Gamma'}\) are \(p\)-isospectral for \(p = 3\), by the first statement in Remark 3.2, hence also for \(p = 4\) and 7, by the first statement in Remark 3.3. We note that both manifolds satisfy \(\beta_p(M) \neq \beta_{n-p}(M)\), hence \(\text{spec}^p(M) \neq \text{spec}^{n-p}(M)\), for every \(p\), in contrast to the orientable case (see Remark 3.4).

We see that \(M_\Gamma\) is orientable while \(M_{\Gamma'}\) is not. We shall now show that \(M_\Gamma\) and \(M_{\Gamma'}\) are isospectral on \(p\)-forms if and only if \(p\) is odd.

We shall use an argument similar to that of Example 5.1 with the same bijection \(\Phi : F \to F'\). We have again that, for each \(p\), the contribution of \(B_1^2\) to \(d_{p,\mu}(\Gamma)\) in (3.1) is the same as the contribution of \(B_2'\) to \(d_{p,\mu}(\Gamma')\) and in the same way, the contributions of \(B_2\) and \(B_1'^2\) are the same, and also those of \(B_1^2B_2\) and \(B_1'^2B_2'\). Therefore, the analysis of isospectrality is reduced to the comparison of the contributions of the elements of order 4 in \(F\) and \(F'\). We also observe that \(\text{tr}_p(B)\) (resp. \(\text{tr}_p(B')\)) is the same for all 4 elements of order 4 in \(F\) (resp. \(F'\)), for each \(p\). Hence, \(M_\Gamma\) and \(M_{\Gamma'}\) will be \(p\)-isospectral if and only if, for each \(\mu\),
\[ \text{tr}_p(B_1) \sum_{B: \text{ord}(B)=4} e_{\mu,B}(\Gamma) = \text{tr}_p(B'_1) \sum_{B': \text{ord}(B')=4} e_{\mu,B'}(\Gamma'). \] (5.1)

We get the following values of the \( p \)-traces, whose verification will be omitted since it is similar to the calculation in Example 5.1:

| \( p \) | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \text{tr}_p(B_1) \) | 1   | 0   | 0   | 0   | −2  | 0   | 0   | 0   | 1   |
| \( \text{tr}_p(B'_1) \) | 1   | 0   | −2  | 0   | 0   | 2   | 0   | −1  |     |

Since \( \text{tr}_p(B_1) = \text{tr}_p(B'_1) = 0 \), \( \forall p \) odd, it turns out that \( M_\Gamma \) and \( M_{\Gamma'} \) are isospectral on \( p \)-forms for all \( p \) odd. We now show that they are not \( p \)-isospectral for \( p \) even.

\[
\sum_{B: \text{ord}(B)=4} e^{2\pi i b}(\pm \varepsilon_j) = \sum_{j=5,6} \left( e^{2\pi i \frac{5}{6}}(\pm \varepsilon_j) + e^{2\pi i \frac{3}{6}}(\pm \varepsilon_j) + e^{2\pi i \left( \frac{5}{6} + \frac{3}{6} \right)}(\pm \varepsilon_j) \right) = 0.
\]

\[
\sum_{B': \text{ord}(B')=4} e^{2\pi i b'}(\pm \varepsilon_j) = \sum_{1 \leq j \leq 6} e^{2\pi i b'}(\pm \varepsilon_j) + \sum_{B': \text{ord}(B')=4} e^{2\pi i b'}(\pm \varepsilon_8) = 0 + 4 = 4.
\]

Therefore one concludes that \( \sum_{B: \text{ord}(B)=4} e_{1,B}(\Gamma) = 0 \) and \( \sum_{B': \text{ord}(B')=4} e_{1,B'}(\Gamma') = 8 \) and according to the tables giving \( \text{tr}_p(B_1) \) and \( \text{tr}_p(B'_1) \) one obtains that \( M_\Gamma \) and \( M_{\Gamma'} \) are not \( p \)-isospectral for \( p = 0, 2, 6 \) and 8. For \( p = 4 \) one can take \( \mu = 8 \) and using the calculations in Example 5.1, one obtains

\[
\sum_{\text{ord}(B)=4} e_{8,B}(\Gamma) = -16.
\]

Since \( \text{tr}_4(B_1) = -2 \) and \( \text{tr}_4(B'_1) = 0 \) it turns out that \( d_{4,8}(\Gamma) \neq d_{4,8}(\Gamma') \) hence \( M_\Gamma \) and \( M_{\Gamma'} \) are not isospectral on 4-forms.

Again, one could also have concluded the non \( p \)-isospectrality of \( M_\Gamma \) and \( M_{\Gamma'} \) for \( p \) even, \( p > 0 \) by computing their Betti numbers:

| \( p \) | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \beta_p(M_\Gamma) \) | 1   | 2   | 4   | 6   | 6   | 6   | 4   | 2   | 1   |
| \( \beta_p(M_{\Gamma'}) \) | 1   | 2   | 3   | 6   | 7   | 6   | 5   | 2   | 0   |

**Example 5.7.** If one takes \( \Gamma \) as in Example 5.6 and \( \Gamma' \) by adding twice the trivial character to \( \Gamma' \) in Example 5.1, it turns out that \( M_\Gamma \) and \( M_{\Gamma'} \) are \( p \)-isospectral for \( p = 2 \) and 6 but not for the remaining values of \( p \). Indeed, by arguing as in the previous examples, we see that \( p \)-isospectrality occurs if and only if (5.1) holds in this case. Since
one verifies that $\text{tr}_2(B_1) = \text{tr}_2(B'_1) = 0$, then $M_\Gamma$ and $M_{\Gamma'}$ are isospectral on 2-forms. Since $M_\Gamma$ and $M_{\Gamma'}$ are both orientable, by Remark 3.3, they are isospectral on 6-forms. The non isospectrality for $p \neq 2, 6$ can be verified as in the previous examples.

**Example 5.8.** In the present example, we shall give two flat manifolds of dimension 4 which are isospectral on $p$-forms for $p$ odd and having different holonomy groups: $\mathbb{Z}_2^2$ and $\mathbb{Z}_4$, respectively.

Let $\Gamma = \langle B_1Lb_1, B_2Lb_2, \Lambda \rangle$, and $\Gamma' = \langle B'Lb', \Lambda \rangle$, where $\Lambda = \mathbb{Z}_4$ and

\[
B_1 = \begin{bmatrix}
1 & 1 \\
1 & -1 \\
-1 & 1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 & -1 \\
1 & -1 \\
-1 & 1
\end{bmatrix}, \quad B' = \begin{bmatrix}
\tilde{J} & -1 \\
-1 & 1
\end{bmatrix},
\]

\[
b_1 = \frac{e_1}{2}, \quad b_2 = \frac{e_4}{2}, \quad b' = \frac{e_4}{4}.
\]

By a verification of the conditions (i) and (ii) in Proposition 2.1, as in Example 5.1 (but simpler), we get that $\Gamma$ and $\Gamma'$ are Bieberbach groups with holonomy groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$, respectively.

We shall give two tables listing the non trivial elements in $F$ and $F'$, together with subindices indicating the non zero translational components: for instance, since $b_1 = \frac{e_1}{2}$, we write $\frac{1}{2}$ as a subindex of the first diagonal element of $B_1$.

\[
\begin{array}{ccc}
B_1 & B_2 & B_1B_2 \\
\frac{1}{2} & 1 & \frac{1}{2} \\
1 & -1 & -1 \\
-1 & -1 & 1 \\
-1 & \frac{1}{2} & -\frac{1}{2}
\end{array}
\quad
\begin{array}{ccc}
B' & B'2 & B'3 \\
\tilde{J} & -\text{Id}_2 & -\tilde{J} \\
-1 & 1 & -1 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4}
\end{array}
\]

It is not hard to verify that $\text{tr}_p(B) = 0$ for any $B \in F, F', B \neq \text{Id}$ and $p = 1, 3$, hence Theorem 3.1 implies that $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral for $p = 1, 3$.

By using the above pair as a starting point, we have constructed, for each $m \geq 2$, a pair of flat $2m$-manifolds with holonomy groups $\mathbb{Z}_2^m$ and $\mathbb{Z}_4 \times \mathbb{Z}_2^{m-2}$ respectively, which are $p$-isospectral for any $p$ odd. For brevity we shall omit their description.

**Example 5.9.** Adding, as in Remark 5.4, $k$ trivial characters to the groups in Example 5.1, one obtains 0-isospectral manifolds $M_\Gamma \times T^k$ and $M_{\Gamma'} \times T^k$, of dimension $n = k + 6$.

Calculating the Betti numbers by the Künneth formula one gets:

\[
\beta_h(M_\Gamma \times T^k) = \max(h, 6) \sum_{i=0}^{\max(h, 6)} \beta_i(M_\Gamma) \binom{k}{h-i},
\]

and similarly for $M_{\Gamma'} \times T^k$, under the convention that $\binom{n}{m} = 0$, if $m > n$. From the table of Betti numbers in Example 5.1 one concludes that

\[
\beta_h(M_\Gamma \times T^k) > \beta_h(M_{\Gamma'} \times T^k) \quad (5.2)
\]

for $0 < h < n$. As a consequence, for any $n \geq 6$, one obtains manifolds which are isospectral on functions but not on $p$-forms for $0 < p < n$. 
Remark 5.10. As it was already mentioned, examples of isospectral, non homeomorphic manifolds, which are not isospectral on \( p \)-forms have been constructed by A. Ikeda (see [Ik]) for lens spaces \((n \geq 5)\), by C. Gordon, R. Gornet (see [Go], [Gt]) in the context of nilmanifolds \((n \geq 5)\) and D. Schueth ([Sch]).

It is an open question whether, for every given subset \( I \) of \( \{0, 1, \ldots, n\} \), there exist compact manifolds which are \( p \)-isospectral if and only if \( p \in I \). For each \( k \geq 0 \), Ikeda has constructed lens spaces which are isospectral on \( p \)-forms for \( 0 \leq p \leq k \), but not on \((k+1)\)-forms. Also, we have seen that flat manifolds allow to give many examples of pairs which are isospectral for certain values of \( p \) only. However, even though more examples can be given, the construction of flat manifolds which are \( p \)-isospectral for only a fixed set of values of \( p \), is still a complicated matter, since one has to keep control of all values of the \( p \)-traces, \( t_p \), at the same time.

We have also seen that the isospectral manifolds given in many of the examples above are topologically quite different, in particular for those in Example 5.9 one has \( \beta_p(M) > \beta_p(M') \) for \( 0 < p < n \) (hence \( \text{spec}^p M \neq \text{spec}^p M' \), for \( 0 < p < n \)). We note that all spherical space forms (hence all Ikeda’s lens spaces) have the same de Rham cohomology as the sphere \( S^n \) and their topological distinction is more delicate (see [Gi]). Also, almost all of the isospectral nilmanifolds in the references mentioned above are of the form \( \Gamma \backslash N, \Gamma' \backslash N \), where \( \Gamma \) and \( \Gamma' \) are lattices in the same simply connected nilpotent Lie group \( N \), hence, by a theorem of Nomizu, their real cohomology coincides with the Lie algebra cohomology of \( n \), the Lie algebra of \( N \) (hence they both have the same Betti numbers). To our best knowledge, examples of 0-isospectral nilmanifolds with Betti numbers showing property (5.2) (hence not \( p \)-isospectral for \( 0 < p < n \)) have not yet been given. Their construction would involve finding an isospectral pair \( \Gamma \backslash N, \Gamma' \backslash N' \) with \( N \neq N' \), where one can compute, or at least compare, the dimensions of \( H^p(n) \) and \( H^p(n') \) for each value of \( p \) and this would not seem such a simple matter.

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