SU(2) gluon propagators from the lattice – a preview

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High accuracy numerical results for the SU(2) gluonic form factor are previewed for the case of Landau gauge. I focus on the information of quark confinement encoded in the gluon propagator.

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**Introduction.** Two prominent methods to treat non-perturbative Yang-Mills theory will be addressed in this talk: the numerical simulation of lattice gauge theory (LGT) and the approach by means of the Dyson-Schwinger equations (DSE). While LGT covers all non-perturbative effects and, in particular, bears witness of quark confinement (see e.g. [1]), simulations including dynamical quarks are cumbersome despite the recent successes by improved algorithms [2] and the increase of computational power. At the present stage, systems at finite baryon densities are hardly accessible in the realistic case of an SU(3) gauge group [3] (for recent successes see [4]). By contrast, the DSE approach can be easily extended for an investigation of quark physics [5,6] even at finite baryon densities [7]. Unfortunately, the DSE approach requires a truncation of the infinite tower of equations, and this approximation is difficult to improve systematically. In addition, the DSE approach needs gauge fixing which is obscured by Gribov copies. Whether the standard Faddeev-Popov method of gauge fixing is appropriate in non-perturbative studies, is still under debate [8].

In order to merge the advantages of both approaches to low energy Yang-Mills theory, I will firstly address the gluon propagator of pure SU(2) lattice gauge theory in Landau gauge. The result can then be compared with the one provided by the solution of the coupled ghost-gluon Dyson equation [9,10,11,12]. This will allow us to estimate the soundness of the truncations introduced to solve the equations (e.g. vertex ansatz, angular approximation). Secondly, the gluon propagator is an one essential ingredient of the quark DSE. Two options are obvious: a parameterization of the lattice result for the gluon propagator enters the quark DSE. The corresponding solution of this equation provides informations on hadronic observables in quenched approximation i.e. the backreaction of the quarks on the gluonic Greenfunctions is neglected. Once the reliability of the DSE approach to the ghost gluon system has been tested, the second option is to solve a truncated set of coupled ghost-gluon-quark DSEs, thereby, challenging the quenched approximation.

In my talk, I will focus on the gluon propagator as inferred from the lattice calculation, and I will concentrate on the information on quark confinement which might be encoded in the gluon propagator. High accuracy data for the latter are obtained by a numerical method superior to existing techniques. Further informations and details of the numerical method will be presented in a forthcoming publication.

**Lattice definition of the gluon field.** Before identifying the gluonic degrees of freedom in the lattice formulation, I briefly recall the definition of the gluon field in continuum Yang-Mills theory.

The starting point for constructing Yang-Mills theories is the transformation property of the matter fields. In the case of an SU(2) gauge theory, we demand invariance under local SU(2) (say color) transformations of the quark fields

\[ q'(x) = \Omega(x) q(x) , \quad \Omega(x) \in SU(2) . \]  

(1)

In order to construct a gauge invariant kinetic term for the quark fields, one defines the gauge covariant derivative 

\[ D_\mu := \partial_\mu + i A_\mu^a t^a, \]  

where \( t^a \) are the generators of the SU(2) gauge group. Per definitionem, this covariant derivative homogeneously transforms under gauge transformations,

\[ D'_{\mu} q'(x) = \Omega(x) D_{\mu}(x) q(x) \]  

(2)

if the gluon fields transforms as

\[ A'^{a'}_{\mu'}(x) = O^{ab}(x) A^b_{\mu}(x) + f^{abf} O^{cf}(x) \partial_{\mu} O^{fc}, \]  

(3)

\[ O^{ab}(x) := 2 \text{tr}\{\Omega(x) t^a \Omega^\dagger(x) t^b\} , \quad O^{ab}(x) \in SO(3) . \]  

(4)

The crucial observation is that the gluon fields transform according to the *adjoint* representation while the matter fields are defined in the fundamental representation.

Let us compare these definitions of fields with the ones in LGT. In LGT, a discretization of space-time with a lattice spacing \( a \) is instrumental. The 'actors' of the theory are SU(2) matrices \( U_{\mu}(x) \) which are associated with the links of the lattice. These links transform under gauge transformations as

\[ U'_{\mu}(x) = \Omega(x) U_{\mu}(x) \Omega^\dagger(x + \mu) , \quad \Omega(x) \in SU(2) . \]  

(5)

For comparison with the ab initio continuum formulation, I also introduce the adjoint links

\[ \bar{U}^{ab}_{\mu}(x) := 2 \text{tr}\{U_{\mu}(x) t^a \bar{U}^\dagger_{\mu}(x) t^b\} , \]  

(6)

\[ \bar{U}^\mu_{\mu} := O(x) \bar{U}_{\mu}(x) O^T(x) , \quad O(x) \in SO(3) , \]  

(7)
where $O(x)$ was defined in (4).

In order to define the gluonic fields from lattice configurations, I exploit the behavior of the (continuum) gluon fields under gauge transformations (see (3)), and identify the lattice gluon fields $\hat{A}_\mu(x)$ as algebra fields of the adjoint representation, i.e.

$$\hat{U}^{ab}_\mu(x) := \left[ \exp\{\epsilon^\ell \hat{A}_\mu^\ell(x) a\} \right]^{ab}, \quad (\epsilon^\ell)^{ac} := \epsilon^{a\ell c},$$  \hspace{1cm} (8)

where the total anti-symmetric tensor $\epsilon^{abc}$ acts as generator for the SU(2) adjoint representation, and where $a$ denotes the lattice spacing.

For later use, it is convenient to have an explicit formula for the (lattice) gluon fields $\hat{A}_\mu(x)$ in terms of the SU(2) link variables $U_\mu(x)$. Usually, these links are given in terms of four vectors of unit length

$$U_\mu(x) = u_\mu^0(x) + i \vec{u}_\mu(x) \vec{\tau}, \quad [u_\mu^0(x)]^2 + [\vec{u}_\mu(x)]^2 = 1,$$

where $\tau^b$ are the Pauli matrices. Using these variables, a straightforward calculation yields

$$\hat{A}_\mu^b(x) a + \mathcal{O}(a^2) = u_\mu^b(x) u_\mu^a(x)^\dagger, \quad \text{without summation over } \mu. \quad (10)$$

I point out that (10) is a novel definition of the lattice gluon field.

Finally, I illustrate the definition (10). Let us assume that we have exploited the gauge degrees of freedom (see (3)) to bring the SU(2) link elements $U_\mu(x)$ as close as possible to the unit matrix,

$$\Omega(x) : \sum_{\{x\}, \mu} \text{tr} U_\mu(x) \rightarrow \text{max}.$$  \hspace{1cm} (11)

In this gauge, I decompose the link variables by

$$U_\mu(x) = Z_\mu(x) \exp\left\{ i \hat{A}_\mu^b(x) t^b a \right\},$$

where $\hat{A}_\mu^b(x)$ is implicitly defined by (8) and $Z_\mu(x) \in \{-1, +1\}$. Indeed, the lattice gluon fields (10) do not change when $U_\mu(x) \rightarrow (-1)U_\mu(x)$. Hence, the fields $\hat{A}_\mu^b(x)$ are relegated to the SO(3) coset space. I here propose to disentangle the information carried by the center and coset parts of the link variables by studying the $Z_\mu(x)$ and $\hat{A}_\mu^b(x)$ correlation functions independently. I stress, however, that in Landau gauge (11) the role of the $Z_\mu(x)$ is de-emphasized ($Z_\mu(x) \rightarrow 1$). In particular, I do not expect a vastly different gluon propagator when other (more standard) definitions of the lattice gluon fields are used (12)(14).

**Gauge fixing.** In the continuum formulation, calculations employing gauge fixed Yang-Mills theory use only gauged gluon fields which satisfies the gauge condition, e.g.

$$\partial_\mu A^{a\mu}_\mu(x) = 0 \quad \text{(Landau gauge)} \quad (13)$$

and rely on the assumption that the Faddeev-Popov determinant corrects the probabilistic weight in an appropriate way. This assumption is true if the gauge condition picks a unique solution $\Omega(x)$ of (13) for a given field $A^{a}_\mu(x)$. Unfortunately, the Landau gauge condition generically admits several solutions depending on the "background field" $\hat{A}_\mu^b(x)$ which is the subject of gauge fixing (Gribov problem). Thus, the above assumption seems not always be justified (8). Further restrictions on the space of possible solutions are required (14).

Let us contrast the continuum gauge fixing with its lattice analog. In a first step, link configurations $U_\mu(x)$ are generated by means of the gauge invariant action without any bias to a gauge condition. In a second step, the gauge-fixed ensemble is obtained by adjusting the gauge matrices $\Omega(x)$ (see (8)) until the gauged link ensemble satisfies the gauge condition. This procedure obviously guarantees the correct probabilistic measure for the gauged configurations, and gauge invariant quantities which are calculated with the gauged configurations evidently coincide with the ones obtained from un-fixed configurations. However, the Gribov problem re-appears as the problem of finding "the name of the gauge". Let me illustrate this last point: The naive Landau gauge condition for the lattice gluon field, i.e.
\[ \hat{\partial}_\mu \hat{A}_\mu^b(x) = 0 \quad (14) \]
is satisfied if we seek an extremum of the variational condition (11). If we restrict the variety of solutions \( \Omega (x) \) which extremize (11) to those solutions which maximize the functional (11), we confine ourselves to the case where the Faddeev-Popov matrix is positive semi-definite. The fraction of the configuration space of gauge fixed fields \( \hat{A}_\mu^b \) is said to lie within the first Gribov horizon. A numerical algorithm which obtains the gauge transformation matrices \( \Omega (x) \) from the condition (11) still samples a particular set of local maxima. Different algorithms might differ in the subset of chosen local maxima, and, hence, implement different gauges. A conceptual solution to the problem is to restrict the configuration space of gauge fixed fields \( \hat{A}_\mu^b \) to the so-called fundamental modular region. In the lattice simulation, this amounts to picking the global maximum of the variational condition (11). In practice, finding the global maximum is a highly non-trivial task. Here, I adopt two extreme cases of gauge fixing: firstly, I will study the gluon propagator of the gauge where an iteration over-relaxation algorithm almost randomly averages over the local maxima of the variational condition (11). This result will then be compared with the gluon propagator of a gauge where a simulated annealing algorithm searches for the global maximum. It will turn out that the gluon propagators of both gauges agree within statistical error bars.

**The numerical simulation:** The link configurations are generated using the Wilson action. I refrain from using a "perfect action" since I am interested in the gluon propagator in the full momentum range; simulations using perfect actions recover a good deal of continuum physics at finite values of the lattice spacing at the cost of a non-local action. For practical simulations, perfect actions are truncated which is poor approximation at high energies where the full non-locality of the action must come into play.

Here, calculations were performed using a \( 16^3 \times 32 \) lattice. The dependence of the lattice spacing on \( \beta \) (renormalization), i.e.

\[ \sigma a^2(\beta) = 0.12 \exp \left\{ \frac{6\pi^2}{11} (\beta - 2.3) \right\}, \quad \sigma := (440 \text{ MeV})^2, \quad (15) \]
is appropriate for \( \beta \in [2.1, 2.6] \) for the achieved numerical accuracy.

Once gauge-fixed ensembles are obtained by implementing a variational gauge condition (see discussion of previous section), the gluon propagator is calculated using

\[ D_{\mu \nu}^a(x-y) = \langle \hat{A}^a_\mu (x) \hat{A}^b_\nu (y) \rangle_{MC}, \quad (16) \]

where the Monte-Carlo average is taken over 200 properly thermalized gauge configurations. Of particular interest is the gluonic form factor \( F(p^2) \) which is defined by

\[ D(p^2) = \int D_{\mu \nu}^{a a}(x) \exp \{ ipx \} \, d^4x, \quad D(p^2) = \frac{F(p^2)}{p^2}. \quad (17) \]

Since in Landau gauge the propagator is diagonal in color and transversal in Lorentz space, the form factor \( F(p^2) \) contains the full information.

| \( \beta \) | \( \text{L [fm]} \) | \( \Lambda [\text{GeV}] \) |
|---|---|---|
| 2.1 | 8.6 | 2.3 |
| 2.2 | 6.6 | 3.0 |
| 2.3 | 5.0 | 4.0 |
| 2.4 | 3.8 | 5.2 |
| 2.5 | 2.9 | 6.8 |

**TABLE I.** Simulation parameters: \( L = 32a \): lattice extension, \( \Lambda \): UV cutoff.
Results I: Landau gauge

The lattice momentum in units of the lattice spacing is given by

\[ p_x a = 2 \sin \left( \frac{\pi}{N} n_x \right), \quad x = 1 \ldots 4 \]  \hspace{1cm} (18)

where \( n_x \) is an integer which numbers the Matsubara frequency and \( N \) is the number of lattice points (in \( x \)-direction).

\[
\begin{array}{cccc}
\text{SU}(2), 16^3 \times 32 & \text{SU}(2), 16^3 \times 32 \\
m_1=0.64 \text{ GeV}, m_s=1.08 \text{ GeV}, m_l=1.2 \text{ GeV} & m_1=0.64 \text{ GeV}, m_s=1.08 \text{ GeV}, m_l=1.2 \text{ GeV} \\
\end{array}
\]

\[
\begin{array}{cccc}
\beta=2.1 & \beta=2.2 & \beta=2.3 & \beta=2.35 \\
\beta=2.4 & \beta=2.5 & \beta=2.6 & \beta=2.45 \\
\beta=2.5 & \beta=2.6 & \beta=2.55 & \beta=2.45 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{DSE} & \text{mass fit} \\
\end{array}
\]

**FIG. 1.** The gluonic form factor \( F(p^2) \) as function of the momentum transfer (left panel: linear scale; right panel: log-log scale). Also shown is the solution of the set of DSEs proposed in [9] which have been solved for the case of \( SU(2) \) [10].

Physical units for the momentum can be obtained by using (15). Calculations with different \( \beta \)-values correspond to simulations with a different UV-cutoff \( \Lambda := \pi/a(\beta) \). In order to obtain the renormalized gluon propagator, the gluonic wave function renormalization is chosen to yield a finite (given) value for the form factor at fixed momentum transfer.

Figure 1 shows my result for the form factor \( F(p^2) \) where the condition (13) was implemented with an iteration over-relaxation method. The data obtained with different \( \beta \)-values are identical within numerical accuracy, thus signaling a proper extrapolation to the continuum limit.

At high momentum the lattice data are consistent with the behavior known from perturbation theory,

\[ F(p^2) \propto 1/ \left[ \log \frac{p^2}{\mu^2} \right]^{13/22}, \quad p^2 \gg \mu^2 \approx (1 \text{ GeV})^2. \]  \hspace{1cm} (19)

The lattice data are compared with the solution of the gluon-ghost DSE [11]. From the DSE studies one expects a scaling law at small momentum

\[ F(p^2) \propto \left[ p^2 \right]^{2\kappa}, \quad p^2 \ll \mu^2 \approx (1 \text{ GeV})^2. \]  \hspace{1cm} (20)

Depending on the truncation of the Dyson tower of equations and on the angular approximation of the momentum loop integral, one finds \( \kappa = 0.92 \) [12] or \( \kappa = 0.77 \) [13] or \( \kappa \to 1 \) [14]. The lattice data are consistent with \( \kappa = 0.5 \) corresponding to an infrared screening by a gluonic mass. Also shown is the coarse grained "mass fit" (\( \mu = 1 \text{ GeV} \))

\[ F(p^2) = \frac{p^2}{p^2 + m_1} \left[ \frac{\mu^4}{p^2 + m_2} + \frac{s}{\log \left( \frac{m_1^2}{p^2} + \frac{p^2}{m_2} \right)^{13/22}} \right] \]  \hspace{1cm} (21)

\*

* I thank Chr. Fischer for communicating his DSE solution for the SU(2) case prior to publication.
which nicely reproduces the lattice data within the statistical error bars.

**Results II: gluon propagator and confinement** In order to get a handle on the information of quark confinement encoded in the gluon propagator in Landau gauge, I now change by hand the SU(2) Yang-Mills theory to a theory which does not confine quarks. It is instructive to compare the gluon propagator of the modified theory with the SU(2) result (see figure [1]).

In the Maximal Center gauge [16], the mechanism of quark confinement can be understood by a percolation of vortices which acquire physical relevance in the continuum limit [17]. An intuitive picture in terms of vortex physics is also available for the deconfinement phase transition at finite temperatures [18]. Reducing the full Yang-Mills configurations to their vortex content still yields the full string tension [17]. Vice versa, removing these vortices from the Yang-Mills ensemble results in a vanishing string tension. This observation was used in [19] to verify the impact of the vortices on chiral symmetry breaking.

![Graph](image)

**FIG. 2.** The static quark anti-quark potential (left panel) and the corresponding gluonic form factors (right panel); DSE solution from [10].

The static quark anti-quark potential in figure 2 demonstrates that a removal of the center vortices produces a non-confining theory. Figure 2 also shows the gluonic form factor obtained from the modified ensemble. The striking feature is that the strength of the form factor in the infra-red momentum range is drastically reduced.

**Results III: the Gribov noise** Finally, let us check how strongly the gluonic form factor $F(p^2)$ depends on the choice of gauge, i.e. on the sample of maxima of the variational condition (11) selected by the algorithm. For this purpose, I adopt an extreme point of view by comparing the gauge implemented by the iteration over-relaxation (IA) algorithm with the gauge obtained by simulated annealing (SA). The results of the form factor in both cases are shown in figure 3. I find, in agreement with [13], that, in the case of the gluonic form factor, the Gribov noise is comparable with statistical noise for data generated with $\beta \in [2.1, 2.5]$ (scaling window).
FIG. 3. The gluonic form factor $F(p^2)$ for a $10^3 \times 20$ lattice in the gauge IA and SA, respectively (left panel) and compared with previous results (IA, $16^3 \times 32$) (right panel).

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