Casimir energy inside a triangle

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Abstract

For certain class of triangles (with angles proportional to $\frac{\pi}{N}$, $N \geq 3$) we formulate image method by making use of the group $G_N$ generated by reflections with respect to the three lines which form the triangle under consideration. We formulate the renormalization procedure by classification of subgroups of $G_N$ and corresponding fixed points in the triangle. We also calculate Casimir energy for such geometries, for scalar massless fields. More detailed calculation is given for odd $N$.

I. Introduction.

There is a rather restricted class of geometries, for which we have Casimir energies in explicit forms. To calculate the energy momentum tensor one has to solve the boundary problem, that is one has to obtain eigenvalues and eigenfunctions for the field which is confined into the given region. The eigenvalues are usually correspond to the roots of some special functions.

For example for the three dimensional ball \[ \mathbb{B} \] or cylindrical regions \[ \mathbb{C} \] to impose the required radial boundary conditions; one has to deal with Bessel functions and with the roots of them.

For some geometries with plane boundaries the Casimir problem is easier, especially if we can employ the method of images. The original parallel plate geometry, and in general rectangular prisms \[ \mathbb{P} \] are of that type. Using the groups generated by the reflections with respect to the surfaces one can construct the Green functions.

For parallel plate the group is isomorphic to $\mathbb{Z}$, for the three dimensional rectangular prism for example it is $\mathbb{Z}^3$. However if the rectangularity condition is dropped, the groups generated by reflections becomes non-commutative which is the case for present work.

We calculate the Casimir energy for the massless scalar field\[ ^2 \], for a certain class of triangular regions. We restrict our attention on the triangles whose angles proportional to $\frac{\pi}{N}$, where $N$ is a natural number greater than 2. Namely one of the angles is $\frac{\pi}{N}$ and another is $\frac{\pi}{2}$ for even $N$ and $\frac{\pi(N-1)}{2N}$ for odd $N$. We also develop a renormalization technique by reducing the problem of finding divergences to the classification of points of the region and their stability subgroups.

In Section II we investigate the structure of the group $G_N$ generated by the reflections with respect to the three lines which form the triangle. This group will play central role in the construction of the Green function satisfying the

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\[ ^2 \text{It is well known that the energy for electromagnetic field is twice the result one gets for the massless scalar field.} \]
Dirichlet boundary condition, and in the renormalization procedure. We show that this group is isomorphic to the semi-direct product of the dihedral group $D_N$ and a finite dimensional lattices.

Section III is devoted to the construction of the Green function for the massless scalar field which vanishes on the boundary of the triangle.

In Section IV we formulate the renormalization procedure for the Green function in the triangle by classification of points in the triangle and their stability groups. The trivial group consisting of the identity element is the stability group for any points in the triangle. The corresponding term in the Green function is the the free Green function in Minkowski space which makes an infinite contribution to the energy momentum tensor. Stability group of the points on a side of the triangle is generated by the reflection operator with respect to this side. In this way terms with surface divergences is found. Since at the vertices of the triangle the smoothness condition is violated we also have line divergences ( vertex of the triangle in the plane corresponds to the line in the three dimension, that is why we call it line divergence ). We find stability groups of these points and corresponding singular terms in the Green function.

In Section V we give the general expression for the energy momentum tensor in terms of the sum over elementary power functions. We also see that the energy density per unit length in the direction perpendicular to triangle under consideration can be represented as the integral over the boundary of this triangle and the integrand is the elementary power functions. We presented the calculation for odd $N$ in detail.

II. Reflections in a Class of Triangles.
For $N = 3, 4, 5, \ldots$ and $k = 1, 2, \ldots N - 2$ consider the triangles $\Delta^N_k$ in $x^1x^2$-plane formed by the lines

\begin{align*}
L_1 &= \{ \vec{x} \in R^2 : x^2 = 0 \} \\
L_2 &= \{ \vec{x} \in R^2 : x^2 = x^1 \tan v \} \\
L_3 &= \{ \vec{x} \in R^2 : x^2 = (b - x^1) \tan(kv) \}
\end{align*}

where $b$ is the length of the side laying on the line $L_1$ and $v = \frac{\pi}{N}$ is the angle between $L_1$ and $L_2$.

The actions of the reflections $Q_j$ with respect to the lines $L_j$, $j = 1, 2, 3$ on the vector

$$\vec{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

are given by

$$Q_1 \vec{x} = p \vec{x}, \quad Q_2 \vec{x} = rp \vec{x}, \quad Q_3 \vec{x} = pr^k \vec{x} + \vec{x}_0,$$

where

$$r = \begin{pmatrix} \cos 2v & -\sin 2v \\ \sin 2v & \cos 2v \end{pmatrix}, \quad p = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{x}_0 = (1 - pr^k) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
obtained from the formulas (5), and from the properties of the rotation \( r \) and reflection \( p \) operators

\[
r^N = 1, \quad p^2 = 1, \quad pr = r^{N-1}p, \quad r^k x_0 = -p x_0.
\]  

(7)

Some of the obvious relations are

\[
Q_j^2 = 1, \quad (Q_1 Q_2)^N = 1,
\]  

(8)

from which we conclude that the reflections \( Q_1 \) and \( Q_2 \) generate the finite subgroup

\[
D_N = \{r^s, pr^s, s = 0, 1, \ldots N - 1\}
\]  

(9)

which is the dihedral group of dimension \( 2N \). Consider the linear space \( V_N \), which consists of the vectors

\[
\vec{\xi} = \sum_{s=0}^{N-1} n_s \vec{x}_s,
\]  

(10)

where \( n_s \) are integers and

\[
\vec{x}_s = r^s \vec{x}_0.
\]  

(11)

The equalities

\[
r \vec{x}_s = \vec{x}_{s+1}, \quad p \vec{x}_s = \vec{x}_{N-s+k}
\]  

(12)

imply that \( D_N \) is the automorphism group of the linear space \( V_N \). The action of \( D_N \) is given in the natural way

\[
\pi(q) \vec{\xi} = q \vec{\xi}; \quad q \in D_N.
\]  

(13)

Since \( V_N \) is a vector space over the integer numbers, unlike the spaces over the real numbers, the dimension \( |V_N| \) is not necessarily equal to the dimension of the vectors \( \vec{x}_s \). It may be larger, that is in our case may be greater than two. For example the dimensions of \( V_5, V_8 \) are four; while the dimensions of \( V_3, V_6 \) and \( V_4 \) are two\(^3\). A detailed discussion of this problem is given in the Appendix.

The group \( G_N \) is the subgroup of the semidirect product group \( D_N \rtimes V_N \). In fact for any element \( g \in G_N \) one can find the pair of elements \( q \in D_N \) and \( \vec{\xi} \in V_N \) as

\[
g \vec{x} = q \vec{x} + \vec{\xi} = (q, \vec{\xi}) \vec{x}; \quad \vec{x} \in R^2.
\]  

(14)

In particular

\[
Q_1 = (p, 0), \quad Q_2 = (rp, 0), \quad Q_3 = (pr^k, \vec{x}_0).
\]  

(15)

\( G_N \) contains two subgroups: \( D_N \) and the one generated by \( Q_3 \). Since \( V_N \) does not contain invariant subspaces with respect to (13) we conclude that there is no subgroup in the semidirect product group which contains \( D_N \) and the group generated by \( Q_3 \) simultaneously. This fact implies that \( G_N \) is isomorphic to \( D_N \rtimes V_N \). In the special case of \( |V_N| = 2 \) this group is called the wallpaper group \[4\].

\(^3\)the vector spaces \( V_3 \equiv V_6 \) and \( V_4 \) are known as the hexagonal and square lattices \[4\]
III. Construction of the Green Function in the Triangles without obtuse Angles

Consider the representation of the group $G_N$ in the space of functions on the four dimensional Minkowski space

$$T(g)f(x) = f(gx).$$

Here the action of the group $G_N$ is given by substitution $\vec{x} \rightarrow x$, $\vec{\xi} \rightarrow \xi$, $p \rightarrow P$, $r \rightarrow R$ where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which are $4 \times 4$ matrices and

$$\xi = \begin{pmatrix} 0 \\ \vec{\xi} \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} x^0 \\ \vec{x} \\ x^3 \end{pmatrix}$$

which are 4 dimensional column vectors.

Using (15) one can verify that the operator

$$O = \sum_{n \in \mathbb{Z}} \sum_{s=0}^{N-1} (T((R^s, \xi)) - T((PR^s, \xi)))$$

satisfies the following property

$$T(Q_j)O = -O.$$  \hspace{1cm} (20)

In (19) $n = (n_0, n_1, \ldots, n_{|V_N|-1})$ is multyindex and

$$\xi = \sum_{t=0}^{|V_N|-1} n_t x_t$$

where

$$x_s = \begin{pmatrix} x^0 \\ \vec{x}_s \\ x^3 \end{pmatrix}$$

and $\vec{x}_s$ are the base vectors described in the previous section.

It is obvious that if we define a function $O f(x)$, it must vanish on the lines $L_j$ of reflections $Q_j$; the fact that we make use in the construction of the Green function inside the triangle $\Delta_k^N$, satisfying the Dirichlet boundary conditions. Since the operator $O$ commutes with the Klein- Gordon operator ( which is invariant under translations, rotations and reflections ) the function

$$K(x, x') \equiv OG(x, x') = \sum_{n \in \mathbb{Z}} \sum_{s=0}^{N-1} (G(R^s x + \xi, x') - G(PR^s x + \xi, x')),$$  \hspace{1cm} (23)

satisfies the equation

$$\eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x'^\nu} K(x, x') = O \delta(x - x').$$  \hspace{1cm} (24)
for any $x, x' \in \mathbb{R}^2 \times \Delta_k^N$, and the boundary condition

$$K(x, x') |_{x \in \mathbb{R}^2 \times \partial \Delta_k^N} = 0,$$

(25)

where $\partial \Delta_k^N$ is the boundary of the triangle $\Delta_k^N$. $G$ is the Green function in the Minkowski space with the metric $\eta = (1, -1, -1, -1)$

$$G(x, x') = -\frac{1}{4\pi^2} \frac{1}{|x - x'|^2}.$$

(26)

The function $K(x, x')$ is the Green function if the right hand side of (24) is delta function

$$O \delta(x - x') = \delta(x - x')$$

(27)

for any $x, x' \in \mathbb{R}^2 \times \Delta_k^N$. The above condition implies that for any $(q, \vec{\zeta}) \neq (1, 0)$ and for any $\vec{x}, \vec{x}' \in \Delta_k^N$

$$\delta(q \vec{x} + \vec{\zeta} - \vec{x}') = 0$$

(28)

must be satisfied. In other words any points inside the triangle should be the representative of different orbits of the coset space $\mathbb{R}^2/G_N$. The orbits of the coset space $\mathbb{R}^2/D_N$ are

$$[\vec{x}] = \{r^s \vec{x}, pr^s \vec{x} : s = 0, \ldots, N - 1\}.$$

(29)

It is clear that we can identify this coset space with region $X$ between two lines $L_1$ and $L_2$ including the boundaries. For any orbit in $\mathbb{R}^2/D_N$ there exist a unique representative in $X$. Since the group $G_N$ is generated by the elements of $D_N$ and $Q_3$ the problem of constructing the coset space $\mathbb{R}^2/G_N$ reduces to finding the subspaces $Y$ of $X$ such that the reflection $Q_3$ maps $Y$ into $X$. Consider the area between three lines $L_j$, which is the triangle under consideration. The previous condition implies that the two angles $k\upsilon$ and $s\upsilon$ of the triangle between the lines $L_1, L_3$ and $L_2, L_3$ must be less than or equal to $\pi/2$. The restrictions

$$k\upsilon \leq \frac{\pi}{2}, \quad s\upsilon \equiv \pi - (k + 1)\upsilon \leq \frac{\pi}{2}$$

(30)

with solutions

$$k = \left\{ \begin{array}{ll} \frac{N}{2}, & \text{for even } N \\ \frac{N - 1}{2}, & \text{for odd } N \end{array} \right.$$

(31)

imply that for triangles without obtuse angle the function $K(x, x')$ in (24) is indeed the Green function. Note that the equations (30) have also been solved by $k = \frac{N - 2}{2}$ for even $N$. In this case $s = \frac{N}{2}$. For $k = \frac{N}{2}$ we have $s = \frac{N - 2}{2}$. Therefore this solution is congruent to the previous one; that is $\Delta_k^{N - 2}$ goes to $\Delta_k^{N - 2}$ when the length $b$ goes to $b \cos \upsilon$.

Finally we like to remark that, for massive fields, instead of $G(x, x')$ one has to put the Green function for the massive scalar fields in the Minkowski space in (23).
IV. Renormalization of the Green Function

In polygonal regions there are three types of singular terms that to be sub- 
tracted to obtain renormalized Green function: Free space term, surface and 
vertex terms.

Inspecting (23) we observe that the term

\[ T(g)G(x, x') = G(gx, x') \]  

leads singularity whenever \( gx = x \); that is, the singularities arise at the 
elements of the group \( G_N \) which leave the points fixed. The renormalization 
problem is then reduced to the classification of the points of the region and 
their stability subgroups.

The identity element (which is the trivial subgroup) leaves all points 
fixed. The term \( T((1, 0))G(x, x') \) in the (23) therefore gives the volume sin-
gularity and is the free Green function.

The points on the line \( L_j \) are left fixed by the reflection \( Q_j \). The group 
generated by \( Q_j \) is then the stability subgroup for the line \( L_j \). Since the 
identity element of the two-dimensional reflection group is already employed 
in the volume renormalization, the surface singularity terms in (23) are

\[ K_S(x, x') = \sum_{j=1}^{3} T(Q_j)G(x, x'). \]  

To discuss the vertex singularities, let us first consider the vertex at the 
intersection point of the lines \( L_1 \) and \( L_2 \). The \( N \) dimensional subgroup 
generated by the element \( Q_1Q_2 \) is the stability subgroup of this vertex. The 
divergence term at the vertex we consider is

\[ K_{L_1L_2}(x, x') = \sum_{j=1}^{N-1} T((Q_1Q_2)^j)G(x, x'). \]  

The element \( Q_1Q_3 \) generates the stability subgroup of the vertex at the in-
tersection point of the lines \( L_1 \) and \( L_3 \). Due to restriction (31) and \( Q_1Q_3 = (r^k, -r^k x_0) \) we conclude that the dimension of this group is 2 for even \( N \) 
and \( N \) for odd \( N \). Therefore we have

\[ K_{L_1L_3}(x, x') = \sum_{j=1}^{L-1} T((Q_1Q_3)^j)G(x, x'), \]  

where \( L \) is the dimension of the stability group, that is \( L = 2 \) if \( N \) is even 
and \( L = N \) if \( N \) is odd.

Finally let us consider the third vertex which is the intersection point of 
the lines \( L_2 \) and \( L_3 \). The stability group of this point is generated by the 
element \( Q_2Q_3 \). One can verify that the dimension \( D \) of this group is

\[ D = \begin{cases} 
N & \text{for even } N \\
\frac{N}{2} & \text{for odd } \frac{N}{2} \\
\frac{N}{2} & \text{for odd } N 
\end{cases} \]  

and the corresponding singular line terms are

\[ K_{L_2L_3}(x, x') = \sum_{j=1}^{D-1} T((Q_2Q_3)^j)G(x, x'). \]
Collecting all the above terms we arrive at

\[ K_L(x, x') = \sum_{j=1}^{N-1} (T((Q_1 Q_2)^j) + T((Q_1 Q_3)^j) + T((Q_2 Q_3)^j)) G(x, x'). \] (38)

for odd \( N \); and,

\[ K_L(x, x') = (T(Q_1 Q_3) + \sum_{j=1}^{N-1} (T((Q_1 Q_2)^j) + T((Q_2 Q_3)^j))) G(x, x'). \] (39)

for even \( \frac{N}{2} \); and

\[ K_L(x, x') = (T(Q_1 Q_3) + \sum_{j=1}^{N-1} T((Q_1 Q_2)^j) + \sum_{j=1}^{\frac{N}{2}-1} T((Q_2 Q_3)^j))) G(x, x'). \] (40)

for odd \( \frac{N}{2} \). Subtracting all divergences from (23) we obtain the renormalized Green function

\[ K_r(x, x') = K(x, x') - G(x, x') - K_S(x, x') - K_L(x, x'). \] (41)

Before closing this section we like to emphasize that if the method of images is applicable to a geometry, the stability group classification is quite reliable approach to the renormalization.

V. Energy Momentum Tensor

The energy momentum tensor for conformally coupled massless scalar field is given by [5]

\[ T_{\mu\nu} = \frac{2}{3} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{6} \eta_{\mu\nu} \partial_\rho \phi \partial^\rho \phi - \frac{1}{3} \phi \partial_\mu \partial_{\nu} \phi + \frac{1}{12} \eta_{\mu\nu} \partial_\rho \phi \partial^\rho \phi. \] (42)

Since the vacuum expectation value of the product of two scalar fields is the Green function, we can express the energy momentum tensor in the region we study as

\[ T_{\mu\nu} = \lim_{x \to x'} \left[ \frac{1}{3} (\partial_{\mu} \partial_{\nu}^\prime + \partial_{\mu}^\prime \partial_{\nu}) - \frac{\eta_{\mu\nu}}{6} \eta^{\sigma\rho} \partial_\sigma \partial_\rho - \frac{1}{6} (\partial_\mu \partial_\nu + \partial_\mu^\prime \partial_\nu^\prime) \right] K_r(x, x'). \] (43)

where \( \partial_{\mu} \equiv \frac{\partial}{\partial x_{\mu}} \), and \( K_r(x, x') \) is given by (41).

The vacuum energy density in particular is given by the following expression

\[ T_{00} = \frac{1}{6\pi^2} \sum_{(n, s)} \left[ \frac{|(pr^s + 1) \xi^2|^2}{|r^s - 1|^2 + |\xi|^4} + \frac{2 + \cos(2sv)}{\xi} \right]. \] (44)

The summation runs over the indices \( n = (n_0, n_1, \ldots, n_{|V^N|-1}) \in Z^{|V^N|} \), and \( s = 0, 1, \ldots, N - 1 \). The terms corresponding to the singularities described in the previous section should be dropped in (44). The term in the second summation in (44) with \( (n = 0, s = 0) \) is the energy of the free vacuum. The terms with \( (n = 0, s = 0) \), \( (n = 0, s = 1) \) and \( (n = (1,0,\ldots,0), s = k) \) in the first summation of (44) are surface divergence terms. Note that due to the \( |(pr^s + 1) \xi^2|^2 \) factor they are automatically zero. This is not surprising.
since it is known that the energy momentum tensor for conformally cou pled scalar field is finite on flat surfaces [6]. Terms \((n = 0, s = 1, 2, \ldots, N - 1)\) in the second summation of (44) are line divergent ones in the vertex which is the intersection point of the lines \(L_1\) and \(L_2\). Using the results of the previous section one may also find the vertex divergence terms for other two vertices.

Integrating the above energy density over the triangle \(\Delta^N_k\) we arrive at the energy density \(E\) per unit length in \(x^3\) direction. First we observe the existence of two exact forms

\[
\omega^s_1 = \frac{2 + \cos(2sv)}{2 | (r^s - 1) \vec{x} + \vec{\xi} |^4} dx^1 \wedge dx^2, \quad (45)
\]

\[
\omega^s_2 = \frac{| (pr^s + 1) \vec{\xi} |^2}{| (pr^s) |^6} dx^1 \wedge dx^2. \quad (46)
\]

They can be rewritten as \(\omega^s_j = d\Omega^s_j\), where

\[
\Omega^s_1 = D_s \frac{((r^s - 1) \vec{x} + \vec{\xi})^2 dx^1 - ((r^s - 1) \vec{x} + \vec{\xi}) dx^2}{| (r^s - 1) \vec{x} + \vec{\xi} |^4}, \quad s \neq 0 \quad (47)
\]

\[
\Omega^s_2 = \frac{| (pr^s + 1) \vec{\xi} |^2}{| (pr^s - 1) \vec{x} + \vec{\xi} |^6} (r^{\frac{s}{2}} \vec{x})^1 d(r^{\frac{s}{2}} \vec{x})^2, \quad (48)
\]

where \(D_s = \frac{2 + \cos(2sv)}{6(1 - \cos(2sv))}\). By making use of the Stokes theorem one can convert the integration over the triangle \(\Delta^N_k\) into the integral over its boundary \(\partial \Delta^N_k\). The energy density per unit length in \(x^3\) direction is then

\[
E = -\frac{1}{6\pi^2} \langle J_0 + J_1 - J_2 \rangle \quad (49)
\]

where \(S^N_k\) is the area of the triangle \(\Delta^N_k\) and

\[
J_0 = \sum_{n \in \mathbb{Z}^N} \frac{3S^N_k}{| \vec{\xi} |^4} \quad (50)
\]

\[
J_1 = \sum_{n \in \mathbb{Z}^N} \sum_{s=1}^{N-1} \int_{\partial \Delta^N_k} \Omega^s_1 \quad (51)
\]

\[
J_2 = \sum_{n \in \mathbb{Z}^N} \sum_{s=0}^{N-1} \int_{\partial \Delta^N_k} \Omega^s_2 \quad (52)
\]

In \(J_j\) we take summation over all values of \(n\) and \(s\). The brackets \(\langle \rangle\) in (49) means that we have to drop the singularity terms. \(J_j\) may be divergent if \(| V_N |\) is greater than three. However their difference should be finite. We have to treat each term as formal series. Then collecting them together we obtain the final result.

Let us restrict our attention to the case of odd \(N\), for which the vector space \(V_N\) appears to be invariant under the half angle rotations \(r^{\frac{s}{2}}\). This can be shown from the identity

\[
r^{\frac{s}{2}} \vec{\zeta} = -\vec{\xi}. \quad (53)
\]

Using the relations

\[
(pr^s \pm 1) = r^{-\frac{s}{2}} (p \pm 1) r^{\frac{s}{2}}, \quad (r^s - 1) = -a_s r^{\frac{s}{2}} r^{\frac{s}{2}}, \quad (54)
\]
with $a_s = 2\sin(sv)$ and making the reparametrization in the multyindex $n$ which is equivalent to the change of variable $r^2 \xi \rightarrow \xi$ we arrive at

$$J_1 = - \sum_{n \in \mathbb{Z}^{N}} \sum_{s=1}^{N-1} D_s \int_{\partial \Delta} \frac{(a_s \xi^2 - \xi^2)^2 dz^1 - (a_s \xi^2 - \xi^2)^1 dz^2}{|a_s \xi^2 - \xi^2|^4} \left| a_s \xi^2 - \xi^2 \right|^4$$

$$J_2 = \sum_{n \in \mathbb{Z}^{N}} \sum_{s=0}^{N-1} \int_{\partial \Delta_s} \frac{|(p+1) \xi^2|^2}{|(p-1) \vec{y} + \xi|^6} y^1 dy^2.$$ (55)

Here we have used the short notation $\Delta_0 \equiv \Delta_{k}^N$ and $\Delta_s = r^2 \Delta_0$, that is the triangle $\Delta_s$ is $\Delta_0$ rotated by the angle $sv$. We also used the symmetry $s \rightarrow N - s$ in (55) to reduce the summation over $s$. Denote by $a_0$, $a_1$ and $c_0$ the sides of the triangle $\Delta_0$ laying on the lines $L_1$, $L_2$ and $L_3$. Then $a_s$, $a_{s+1}$ and $c_s$ will be the sides of the triangle $\Delta_s$, that is $a_{s+1} = r^2 a_s$ and $c_{s+1} = r^2 c_s$. Since the orientation on the side $a_{s+1}$ of the triangle $\Delta_s$ is opposite to the one on the side $a_{s+1}$ of $\Delta_{s+1}$ we have

$$J_2 = \sum_{n \in \mathbb{Z}^{N}} \int_U \frac{|(p+1) \xi^2|^2}{|(p-1) \vec{y} + \xi|^6} y^1 dy^2,$$ (56)

where the integration contour $U = a_0 \cup b_0 \cup b_1 \cup \cdots b_{N-1} \cup a_N$ oriented anti clock wise. On the sides $a_0$, $a_N$ and $c_N$, we have $y^2 = const$, that is these sides do not make contribution in $J_2$. We also observe that reflection operator $-p$ with respect to the $y^2$ axis send $c_j$ to $c_{N-j-1}$ with opposite orientation. Since the one form in the integral change sign under reflection $\vec{y} \rightarrow -p \vec{y}$ one can rewrite the above expression as

$$J_2 = 2 \sum_{n \in \mathbb{Z}^{N}} \sum_{j=0}^{N-3} \int_{C_j} \frac{|(p+1) \xi^2|^2}{|(p-1) \vec{y} + \xi|^6} y^1 dy^2.$$ (57)

For given $c_j$ we construct closed contour which is parallel to the $y^1$-axis. They intersect $y^2$ axis at the points $b \sin jv$ and $b \sin(j+1)v$. The interval between these two points, $c_j$ and two intervals between them, which are parallel to $y^1$-axes form the desired contour which we denote by $C_j$. Since at $y^1 = 0$ and $y^2 = const$ the one form in the above integral is zero. We then have

$$J_2 = 2 \sum_{n \in \mathbb{Z}^{N}} \sum_{j=0}^{N-3} \int_{C_j} \frac{|(p+1) \xi^2|^2}{|(p-1) \vec{y} + \xi|^6} (y^1 - A_j y^2 - B_j) dy^2$$ (58)

where we have added the exact forms, which are the second and third terms in the bracket, for integral of the exact form over the closed form is zero. We choose the coefficients $A_j$ and $B_j$ to satisfy

$$y^1 - A_j y^2 - B_j = 0,$$ (60)

for $\vec{y} \in c_j$, that is to make the value of the one form zero on the side $c_j$. We have

$$A_j = \frac{\cos v(k-j)}{\sin v(k-j)}, \quad B_j = b \frac{\sin v k}{\sin v(k-j)}$$ (61)
where $k = \frac{N-1}{2}$. Non zero contribution to the integral comes only from the integration over the interval laying on the $y^2$ axis:

$$J_2 = 8 \sum_{n \in Z^{V_N}} \sum_{j=0}^{\frac{N-3}{2}} \int_{b \sin j v}^{b \sin (j+1) v} \frac{(\xi^1)^2(A_j t + B_j) dt}{((\xi^1)^2 + (2t - \xi^2)^2)^{\frac{3}{2}}} \quad (62)$$

or

$$J_2 = \frac{1}{2b^2 \sin^2 k \nu} \sum_{n \in Z^{V_N}} \sum_{j=0}^{\frac{N-3}{2}} \frac{n_j^2}{\sin(k-j)\nu} \int_{f_j}^{f_{j+1}} dx \frac{1 - x \cos((k-j)\nu)}{(\eta^2 + (x - \eta_2)^2)^{\frac{3}{2}}}, \quad (63)$$

where $f_j = \frac{\sin j \nu}{\sin k \nu}$ and $\vec{\zeta} = 2b \sin k \nu \vec{\eta}'$ or

$$\vec{\eta}' = \sum_{t=0}^{|V_N| - 1} n_t \vec{x}'_t, \quad \vec{x}'_t = r^{t+\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (64)$$

(We also used lower indices for the vector $\vec{\eta}'$).

Now consider the expression $\left[ 62 \right]$. Rotation $r_{\nu}^2$ maps interval $a_{\nu}^N$ on $a_{\nu}^{N-1}$ which has the opposite orientations. Since the expression under the integral is invariant under transformation $\vec{z} \to r_{\nu}^2 \vec{z}$, the contributions from these intervals cancel each other. $J_1$ is nonzero on the interval $c_{\nu}$. Let us make change of variable $\vec{y}' = r_{\nu}^{-\frac{N+1}{2}} \vec{z}'$. Then $c_{\nu}$ goes to $c_{-\frac{1}{2}}$ on which $y^1 = b \cos \frac{\nu}{2}$ and $y^2 \in [-b \sin \frac{\nu}{2}, b \sin \frac{\nu}{2}]$. Therefore

$$J_1 = \sum_{n \in Z^{V_N}} \sum_{s=1}^{\frac{N-1}{2}} D_s \int_{-b \sin s \frac{\nu}{2}}^{b \sin s \frac{\nu}{2}} \frac{(ba_s \cos \frac{\nu}{2} - \xi^1) dy}{((ba_s \cos \frac{\nu}{2} - \xi^1)^2 + (a_s y - \xi^2)^2)^{\frac{3}{2}}} \quad (65)$$

or

$$J_1 = \frac{\sin \frac{\nu}{2}}{8b^2 \sin^3 k \nu} \left( \sum_{n \in Z^{V_N}} \sum_{s=1}^{\frac{N-1}{2}} D_s \int_{-f_s}^{f_s} \frac{(f_s \cos \frac{\nu}{2} - \eta_1) dx}{((f_s \cos \frac{\nu}{2} - \eta_1)^2 + (x \sin \frac{\nu}{2} - \eta_2)^2)^{\frac{3}{2}}} \right). \quad (66)$$

Note that in the above expression we have dropped the term $s = \frac{N-1}{2}$ by the reparametrization $\vec{\eta}' \to \vec{\eta}' + \vec{x}'_0$ we can rewrite it as

$$- \frac{1}{8b^2 \sin^3 k \nu} D_{\frac{N+1}{2}} \sum_{n \in Z^{V_N}} \int_{-1}^{1} \frac{\eta_1 dx}{\eta_1^2 + ((x - 1) \sin \frac{\nu}{2} - \eta_2)^2} \quad (67)$$

which is odd function in $\eta_1$ variable. Therefore it is zero.

Finally let us consider the special case when $N = 3$ and $k = 1$. We have $v = \frac{\pi}{3}$ and $\vec{\zeta} = \sqrt{3}b \vec{\eta}'$ with

$$\vec{\eta}' = \frac{n_0}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} + \frac{n_1}{2} \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \quad (68)$$

that is $|V_3| = 2$. We have $S_1^3 = \frac{\sqrt{3}}{4}b^2$, $J_1 = 0$ and

$$J_0 = \frac{\sqrt{3}}{12b^2} \sum_{n \in Z^2} \frac{1}{|\vec{\eta}'|^4}, \quad (69)$$
\[ J_2 = \frac{2}{3\sqrt{3}b^2} \sum_{n \in \mathbb{Z}^2} \eta_1^2 \int_0^1 dx \frac{2 - x}{(\eta_1^2 + (x - \eta_2)^2)^3}. \]  

(70)

The expressions

\[ \sum_{n \in \mathbb{Z}^2} \eta_1^2 \int_0^1 dx \frac{x - \eta_2}{(\eta_1^2 + (x - \eta_2)^2)^3} = -\frac{1}{4} \sum_{n \in \mathbb{Z}^2} \frac{1}{(\eta_1^2 + (x - \eta_2)^2)^2} \bigg|_0^1 \]  

(71)

\[ \sum_{n \in \mathbb{Z}^2} \eta_1^2 \int_0^1 dx \frac{1}{(\eta_1^2 + (x - \eta_2)^2)^3} = \sum_{n \in \mathbb{Z}^2} F(\eta_1, x - \eta_2) \bigg|_0 \]  

(72)

due to the symmetry \( \eta_1 \to \eta_1 \) and \( \eta_2 \to \eta_2 + 1 \) is zero. Here

\[ F(x, y) = \int_{-\infty}^{y} \frac{dz}{(z^2 + x^2)^3}. \]  

(73)

Therefore we are left with

\[ J_2 = -\frac{2}{3\sqrt{3}b^2} \sum_{n \in \mathbb{Z}^2} \frac{1}{\eta_1^2} f(\eta_2), \]  

(74)

where

\[ f(x) = \int_{-\infty}^{x} \frac{dz}{(z^2 + 1)^3}. \]  

(75)

The vacuum energy density per unit length in \( x^3 \) direction is then

\[ E = -\frac{2}{3\sqrt{3}\pi b^2} \sum_{(l,m) \neq (0,0)} \frac{1}{(3l^2 + m^2)^2} + \frac{4}{9\sqrt{3}b^2} f\left(\frac{m}{\sqrt{3}l}\right). \]  

(76)

VI Discussion

We have calculated the Casimir energy for a class of triangles without obtuse angle. We applied the method of images. Unlike the case of parallel plate or rectangular prisms, the group generated by reflections is not abelian; thus, the employment of the image method for triangles is not trivial.

Renormalization procedure is observed to be equivalent to the classification of the points in the triangle and their stability subgroups. To renormalize the Green function we subtract the terms corresponding these stability subgroups. Identity element is the stability subgroup for all points, reflections and bi-product of reflections generate the stability subgroups of points on the planes involving the sides, and of the lines passing through the vertices respectively.

We hope that the technique we used which essentially is based on the employment of the groups generated by reflections from the surfaces, can be employed for other polygonal regions. We also hope that it may even be possible to study some other geometries with smooth boundaries, as the limiting case of the suitable polygonal regions. For example for an elliptical region such a process may not be as hopeless as dealing with the roots of Mathieu functions.

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Appendix

To find the dimension $|V_N|$ of $V_N$ one has to investigate the non zero independent solutions of the equation

$$\sum_{s=0}^{N-1} n_s x^s = 0 \quad (A.1)$$

Assume that we have found all independent solutions $n^t = (n^t_0, n^t_1, \ldots, n^t_{N-1})$, $t = 1, \ldots m$ then $|V_N| = N - m$. From (11) we observe that the equation (A.1) can be rewritten in the form

$$\sum_{s=0}^{N-1} n_s r^s = 0. \quad (A.2)$$

Let $N$ be a prime number. The periodicity condition $r^N = 1$ and the identity

$$\frac{1 - x^N}{1 - x} = 1 + x + \cdots + x^{N-1} \quad (A.3)$$

which is valid for $x \neq 1$ imply the solution $n^0 = (1, 1, \ldots, 1)$ of (A.1). Let $N = M^l$, where $M$ is a prime and $l$ is a natural numbers. Then the operators $R_p = r^{N/M^l}$, $p = 1, 2, \ldots, l$ satisfy the periodicity condition $R_p^{M^l} = 1$. This implies $\frac{N}{M} + \frac{N}{M^2} + \cdots + \frac{N}{M^{l-1}} + 1$ relations of the form

$$r^{s_p}(1 + R_p + \cdots + R_p^{M^l-1}) = 0 \quad (A.4)$$

or

$$r^{s_p} + r^{N/M^p} + r^{N/M^{p+1}} + \cdots + r^{N/M^{l-1}} = 0. \quad (A.5)$$

We denote the corresponding solutions by $n^{s_p}$, where $p = 1, 2, \ldots, l$ and $s_p = 0, 1, \ldots, N/M^p - 1$. The rank of the matrix $(n^{s_p})$ appears to be $\frac{N}{M}$ and we choose solutions $n^{s_1}$ as independent set. We demonstrate it for $N = 2^3$. We have four relations

$$r^{s_1} + r^{4+s_1} = 0, \quad s_1 = 0, 1, 2, 3 \quad (A.6)$$

two relations

$$r^{s_2} + r^{2+s_2} + r^{4+s_2} + r^{6+s_2} = 0, \quad s_2 = 0, 1 \quad (A.7)$$

and one relation

$$1 + r + \cdots + r^7 = 0. \quad (A.8)$$

We see that the relations (A.7) reduce to

$$r^{s_2} + r^{4+s_2} = 0, \quad r^{(2+s_2)} + r^{4+(2+s_2)} = 0. \quad (A.9)$$

Summation of four relations (A.9) leads the relation (A.8). Therefore we have four independent solutions. The general case can be proved in a similar fashion. Let now $N = M_1^iM_2^j \cdots M_f^j$, where $M_j$ are prime numbers such that $M_1 < M_2 < \cdots < M_f$. In this case we have $\frac{N}{M_1} + \frac{N}{M_2} + \cdots + \frac{N}{M_f}$ solutions $n^{s_j}$:

$$r^{s_j} + r^{\frac{N}{M_j}+s_j} + \cdots + r^{\frac{N}{M_j}(M_j-1)+s_j} = 0 \quad (A.10)$$
where $j = 1, 2, \ldots f$ and $s^j = 0, \ldots \frac{N}{M_j} - 1$. The rank of the matrix $A_N = (n^s)$ gives the number of independent relations. The matrix $A_N$ which consist of $\frac{N}{M_1} + \frac{N}{M_2} + \cdots + \frac{N}{M_f}$ rows and $N$ columns can be shown to have the following form

$$A_N = \begin{pmatrix}
I_1 & I_1 & \cdots & I_1 \\
I_2 & I_2 & \cdots & I_2 \\
\vdots & \vdots & \ddots & \vdots \\
I_f & I_f & \cdots & I_f
\end{pmatrix},$$

(A.11)

where $I_j$ is $\frac{N}{M_j} \times \frac{N}{M_j}$ unit matrix. Note that the number of $I_j$ matrices in $j^{th}$ row is $M_j$. Assume that the relations described above exhaust all relations of the form (A.1). Then we have $|V_N| = N - \text{rank}(A_N)$. For example the vector spaces $V_3, V_4$ and $V_6$ has dimension two. We conjecture the following result

$$|V_N| = N - 1, \text{ for a prime } N \quad (A.12)$$

$$|V_N| = N - \frac{N}{M}, \text{ for } N = M^i \text{ and a prime } M. \quad (A.13)$$

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