THE FROBENIUS MORPHISM ON FLAG VARIETIES, II

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Abstract. In this paper, which is the sequel to [20], we study the Frobenius pushforward of the structure sheaf on the adjoint varieties in type $A_3$ and $A_4$. We show that this pushforward sheaf decomposes into a direct sum of indecomposable bundles and explicitly determine this set that does not depend of the characteristic. In accordance with the results of [19], this set forms a strong full exceptional collection in the derived category of coherent sheaves. These computations lead to a natural conjectural answer in the general case that we state at the end.

1. Introduction

Let $V$ be a vector space of dimension $n$ over an algebraically closed field $k$ of characteristic $p$, and let $X_n$ denote the partial flag variety $F_{1,n-1,V}$ of type $(1, n-1)$. Recall that given a variety $X$ over $k$, the sheaf of small differential operators $\mathcal{D}^{(1)}_X$ on $X$ is a coherent sheaf of algebras isomorphic to $\text{End}_{\mathcal{O}_X}(\mathcal{F}_*\mathcal{O}_X)$, where $\mathcal{F}$ is the Frobenius morphism. It was shown in [19] that $H^i(X_n, \mathcal{D}^{(1)}_X) = 0$ for $i > 0$ in all characteristics. Coupled with the results of [5], this vanishing theorem gives that for $p > n$ the bundle $\mathcal{F}_*\mathcal{O}_{X_n}$ is tilting on $X_n$.

The proof in [19] was rather implicit, however, as it deduced the higher cohomology vanishing of the sheaf $\mathcal{D}^{(1)}_X$ from the general properties of sheaves of crystalline differential operators that turn out to be particularly nice on the varieties $X_n$. The argument presented in loc.cit. also covers the case of smooth quadrics for odd primes; on the other hand, in the latter case there is an explicit description of the decomposition of the Frobenius pushforward of a line bundle in arbitrary characteristic (see [1] and [16]). The tilting property for appropriate primes follows as well from those decompositions.

The goal of the present paper is to explicitly compute the summands appearing in $\mathcal{F}_*\mathcal{O}_{X_n}$. We use the approach developed in the previous paper [20]. To make the present paper self-contained, we include the preliminary material (without proofs) from loc.cit. that occupies Sections 2–4. The reader familiar with these notions may skip directly to Section 5. In particular, we provide a detailed argument allowing to compute the indecomposable summands of $\mathcal{F}_*\mathcal{O}_{X_n}$ in the case of small ranks $n = 4$ (Theorem 5.1) and $n = 5$ (Theorem 5.2). In consistency with [19], these decompositions show that $\mathcal{F}_*\mathcal{O}_{X_n}$ is a tilting bundle on $X_n$ for $n = 4, 5$ and $p > n$. The approach to construct the decomposition of $\mathcal{F}_*\mathcal{O}_{X_n}$ in these low rank cases is, in fact, uniform; it was to a large extent inspired by the seminal paper [13]. In the final Section 6 we give a conjectural description of the decomposition of $\mathcal{F}_*\mathcal{O}_{X_n}$ in the general case.

Notation. Throughout we fix a perfect field $k$ of characteristic $p > 0$. Given a split semisimple simply connected algebraic group $G$ over $k$, let $T$ denote a maximal torus of $G$, and let $T \subset B$
be a Borel subgroup containing $T$. The flag variety of Borel subgroups in $G$ is denoted $G/B$. Denote $X(T)$ the weight lattice, and let $R$ and $R^\vee$ denote the root and coroot lattices, respectively. Let $S$ be the set of simple roots relative to the choice of a Borel subgroup than contains $T$. The Weyl group $W = N(T)/T$ acts on $X(T)$ via the dot–action: if $w \in W$, and $\lambda \in X(T)$, then $w \cdot \lambda = w(\lambda + \rho) - \rho$. A parabolic subgroup of $G$ is denoted by $P$. For a simple root $\alpha \in S$, denote $P_\alpha \subset G$ the corresponding minimal parabolic subgroup. Given a weight $\lambda \in X(T)$, denote $L_\lambda$ the corresponding line bundle on $G/B$. The half sum of the positive roots (the sum of fundamental weights) is denoted by $\rho$. Given a dominant weight $\lambda \in X(T)$, the induced module $\text{Ind}^G_B \lambda$ is denoted $\nabla_\lambda$, the Weyl module, which is dual to induced module, is denoted $\Delta_\lambda$, and the simple module with the highest weight $\lambda$ is denoted $L_\lambda$. Given a variety $X$ and $n \in \mathbb{N}$, denote $F_n$ the $n$–th iteration of the absolute Frobenius morphism $F_n : X \to X$. For a vector space $V$ over $k$ its $n$-th Frobenius twist $F^n_V$ is denoted $V^{[n]}$. All the functors are supposed to be derived, i.e., given a morphism $f : X \to Y$ between two schemes, we write $f_*, f^*$ for the corresponding derived functors of push–forwards and pull–backs.

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2. Cohomology of line bundles on $G/B$

2.1. Flag varieties of Chevalley groups over $\mathbb{Z}$. Let $G \to \mathbb{Z}$ be a semisimple Chevalley group scheme (a smooth affine group scheme over $\text{Spec}(\mathbb{Z})$ whose geometric fibres are connected semisimple algebraic groups), and $G/B \to \mathbb{Z}$ be the corresponding Chevalley flag scheme (resp., $P \subset G$ the corresponding parabolic subgroup scheme over $\mathbb{Z}$). Then $G/P \to \text{Spec}(\mathbb{Z})$ is flat and the line bundle $L$ on $G/P$ also comes from a line bundle $L$ on $G/P$. Let $k$ be a field of arbitrary characteristic, and $G/B \to \text{Spec}(k)$ be the flag variety obtained by base change along $\text{Spec}(k) \to \text{Spec}(\mathbb{Z})$.

2.2. Bott’s vanishing theorem. We recall first the classical Bott’s theorem (see [9]). Let $G \to \mathbb{Z}$ be a semisimple Chevalley group scheme as above. Assume given a weight $\chi \in X(T)$, and let $L_\chi$ be the corresponding line bundle on $G/B$. The weight $\chi$ is called singular, if it lies on a wall of some Weyl chamber defined by $\langle -, \alpha^\vee \rangle = 0$ for some coroot $\alpha^\vee \in R^\vee$. Weights, which are not singular, are called regular. Let $k$ be a field of characteristic zero, and $G/B \to \text{Spec}(k)$ the corresponding flag variety over $k$. The weight $\chi \in X(T)$ defines a line bundle $L_\chi$ on $G/B$.

Theorem 2.1. [9] Theorem 2]
(a) If $\chi + \rho$ is singular, then $H^i(G/B, L_\chi) = 0$ for all $i$.

(b) If $\chi + \rho$ is regular and dominant, then $H^i(G/B, L_\chi) = 0$ for $i > 0$.

(c) If $\chi + \rho$ is regular, then $H^i(G/B, L_\chi) \neq 0$ for the unique degree $i$, which is equal to $l(w)$. Here $l(w)$ is the length of an element of the Weyl group that takes $\chi$ to the dominant chamber, i.e. $w \cdot \chi \in X_+(T)$. The cohomology group $H^{l(w)}(G/B, L_\chi)$ is the irreducible $G$–module of highest weight $w \cdot \chi$.

2.3. Cohomology of line bundles. Some bits of Theorem 2.1 are still true over $\mathbb{Z}$: if a weight $\chi$ is such that $\langle \chi + \rho, \alpha \vee \rangle = 0$ for some simple root $\alpha$, then the corresponding line bundle is acyclic. Indeed, Lemma from [9, Section 2] holds over fields of arbitrary characteristic.

Theorem 2.2 (Kempf’s vanishing theorem). Let $\chi \in X(T)$, i.e. $\langle \chi, \alpha \vee \rangle \geq 0$ for all simple coroots $\alpha \vee$. Then $H^i(G/B, L_\chi) = 0$ for $i > 0$.

Besides this, however, very little of Theorem 2.1 holds over $\mathbb{Z}$ [11, Part II, Chapter 5]. However, it still holds for weights lying in the interior of the bottom alcove in the dominant chamber [3, Theorem 2.3 and Corollary 2.4]:

Theorem 2.3. If $\chi$ is a weight such that for a simple root $\alpha$ one has $0 \leq \langle \chi + \rho, \alpha \vee \rangle \leq p$ then

\begin{equation}
H^i(G/B, L_\chi) = H^{i+1}(G/B, L_{s_\alpha \cdot \chi}).
\end{equation}

The following theorem is used throughout in all the calculations concerning cohomology of line bundles (see [2, Corollary 3.2]):

Theorem 2.4. Let $\chi$ be a weight. If either $\langle \chi, \alpha \vee \rangle \geq -p$ or $\langle \chi, \alpha \vee \rangle = -ap^n - 1$ for some $a, n \in \mathbb{N}$ and $a < p$ then

\begin{equation}
H^i(G/B, L_\chi) = H^{i-1}(G/B, L_{s_\alpha \cdot \chi}).
\end{equation}

3. Semiorthogonal decompositions, mutations, and exceptional collections

3.1. Semiorthogonal decompositions. Let $k$ be a field. Assume given a $k$–linear triangulated category $\mathcal{D}$, equipped with a shift functor $[1]: \mathcal{D} \to \mathcal{D}$. For two objects $A, B \in \mathcal{D}$ let $Hom^*_\mathcal{D}(A, B)$ be the graded $k$-vector space $\oplus_{i \in \mathbb{Z}} Hom_\mathcal{D}(A, B[i])$. Let $\mathcal{A} \subset \mathcal{D}$ be a full triangulated subcategory, that is a full subcategory of $\mathcal{D}$ which is closed under shifts.

The original source for most of the definitions and statements in this section is [6]. We follow the expositions of [10, Section 2.1] and [14, Section 2.2].
**Definition 3.1.** The right orthogonal $A^\perp \subset D$ is defined to be the full subcategory

\[(3.1) \quad A^\perp = \{B \in D : \text{Hom}_D(A, B) = 0\}\]

for all $A \in A$. The left orthogonal $^\perp A$ is defined similarly.

**Definition 3.2.** A full triangulated subcategory $A$ of $D$ is called right admissible if the inclusion functor $A \hookrightarrow D$ has a right adjoint. Similarly, $A$ is called left admissible if the inclusion functor has a left adjoint. Finally, $A$ is admissible if it is both right and left admissible.

If a full triangulated category $A \subset D$ is right admissible then every object $X \in D$ fits into a distinguished triangle

\[(3.2) \quad \cdots \rightarrow Y \rightarrow X \rightarrow Z \rightarrow Y[1] \rightarrow \cdots \]

with $Y \in A$ and $Z \in A^\perp$. One then says that there is a semiorthogonal decomposition of $D$ into the subcategories $(A^\perp, A)$. More generally, assume given a sequence of full triangulated subcategories $A_1, \ldots, A_n \subset D$. Denote $\langle A_1, \ldots, A_n \rangle$ the triangulated subcategory of $D$ generated by $A_1, \ldots, A_n$.

**Definition 3.3.** A sequence $(A_1, \ldots, A_n)$ of admissible subcategories of $D$ is called semiorthogonal if $A_i \subset A_j^\perp$ for $1 \leq i < j \leq n$, and $A_i \subset ^\perp A_j$ for $1 \leq j < i \leq n$. The sequence $(A_1, \ldots, A_n)$ is called a semiorthogonal decomposition of $D$ if $\langle A_1, \ldots, A_n \rangle^\perp = 0$, that is $D = \langle A_1, \ldots, A_n \rangle$.

The above definition is equivalent to:

**Definition 3.4.** A semiorthogonal decomposition of a triangulated category $D$ is a sequence of full triangulated subcategories $(A_1, \ldots, A_n)$ in $D$ such that $A_i \subset A_j^\perp$ for $1 \leq i < j \leq n$ and for every object $X \in D$ there exists a chain of morphisms in $D$,

\[
\begin{array}{ccccccc}
0 & = X_n & \rightarrow & X_{n-1} & \rightarrow & X_{n-2} & \rightarrow & \cdots & \rightarrow & X_1 & \rightarrow & X_0 = X \\
& \downarrow [1] & & \downarrow [1] & & \downarrow [1] & & & & \downarrow [1] & & \\
& A_{n-1} & & A_{n-2} & & & & & & A_0 & \\
\end{array}
\]

such that a cone $A_k$ of the morphism $X_k \rightarrow X_{k-1}$ belongs to $A_k$ for $k = 1, \ldots, n$.

3.2. **Mutations.** Let $D$ be a triangulated category and assume $D$ admits a semiorthogonal decomposition $D = \langle A, B \rangle$.

**Definition 3.5.** The left mutation of $B$ through $A$ is defined to be $L_A(B) := A^\perp$. The right mutation of $A$ through $B$ is defined to be $R_B(A) := ^\perp B$.

One obtains semiorthogonal decompositions $D = \langle L_A(B), A \rangle$ and $D = \langle A, R_B(A) \rangle$.

Let $A$ be an admissible subcategory of $D$, and $i : A \rightarrow D$ the embedding functor. It admits a left and a right adjoint functors $D \rightarrow A$, the subcategory $A$ being admissible; denote them $i^*$ and...
Given an object $F \in \mathcal{D}$, define the left mutation $L_A(F)$ and the right mutations $R_A(F)$ of $F$ through $A$ by

\begin{equation}
(3.3) \quad L_A(F) := \text{Cone}(i^!(F) \to F), \quad R_A(F) := \text{Cone}(F \to i^*(F))[-1].
\end{equation}

One proves:

**Lemma 3.1.** \cite{[14]} Lemma 2.7] There are equivalences $L_A : \mathcal{B} \simeq \mathcal{D}/A \simeq L_A(\mathcal{B})$ and $R_A : A \simeq \mathcal{D}/B \simeq R_A(B)$.

**Proposition 3.1.** \cite{[6]} Proposition 2.3] Let $\mathcal{D} = \langle A, B \rangle$ be as above. Right and left mutations are mutually inverse to each other, i.e., $R_A L_A \simeq \text{id}_A$, and $L_B R_B \simeq \text{id}_B$.

**Definition 3.6.** Let $\mathcal{D} = \langle A_1, \ldots, A_n \rangle$ be a semiorthogonal decomposition. The left dual semiorthogonal decomposition $\mathcal{D} = \langle B_n, \ldots, B_1 \rangle$ is defined by

\begin{equation}
(3.4) \quad B_i := L_{A_1} L_{A_2} \cdots L_{A_{i-1}} A_i = L_{\langle A_1, \ldots, A_{i-1} \rangle} A_i, \quad 1 \leq i \leq n.
\end{equation}

The right dual semiorthogonal decomposition $\mathcal{D} = \langle C_n, \ldots, C_1 \rangle$ is defined by

\begin{equation}
(3.5) \quad C_i := R_{A_n} R_{A_{n-1}} \cdots R_{A_{i+1}} A_i = R_{\langle A_{i+1}, \ldots, A_n \rangle} A_i, \quad 1 \leq i \leq n.
\end{equation}

**Lemma 3.2.** \cite{[14]} Lemma 2.10] Let $\mathcal{D} = \langle A_1, \ldots, A_n \rangle$ be a semiorthogonal decomposition such that the components $A_k$ and $A_{k+1}$ are completely orthogonal, i.e., $\text{Hom}_\mathcal{D}(A_k, A_{k+1}) = 0$ and $\text{Hom}_\mathcal{D}(A_{k+1}, A_k) = 0$. Then

\begin{equation}
(3.6) \quad L_{A_k} A_{k+1} = A_{k+1} \quad \text{and} \quad R_{A_{k+1}} A_k = A_k,
\end{equation}

and both the left mutation of $A_{k+1}$ through $A_k$ and the right mutation of $A_k$ through $A_{k+1}$ boil down to a permutation and

\begin{equation}
(3.7) \quad \mathcal{D} = \langle A_1, \ldots, A_{k-1}, A_{k+1}, A_k, A_{k+2}, \ldots A_n \rangle
\end{equation}

is the resulting semiorthogonal decomposition of $\mathcal{D}$.

**3.3. Exceptional collections.** Exceptional collections in $k$–linear triangulated categories are a special case of semiorthogonal decompositions with each component of the decomposition being equivalent to $\mathcal{D}^b(\text{Vect} - k)$. The above properties of mutations thus specialize to this special case. Still, there are new features appearing as shown in Subsection 3.4.

**Definition 3.7.** An object $E \in \mathcal{D}$ of a $k$–linear triangulated category $\mathcal{D}$ is said to be exceptional if there is an isomorphism of graded $k$-algebras...
A collection of exceptional objects \((E_0, \ldots, E_n)\) in \(D\) is called exceptional if for \(1 \leq i < j \leq n\) one has

\[
\text{Hom}_D^\bullet(E_j, E_i) = 0.
\]

Denote \(\langle E_0, \ldots, E_n \rangle \subset D\) the full triangulated subcategory generated by the objects \(E_0, \ldots, E_n\). One proves [6, Theorem 3.2] that such a category is admissible. The collection \((E_0, \ldots, E_n)\) in \(D\) is said to be full if \(\langle E_0, \ldots, E_n \rangle^\perp = 0\), in other words \(D = \langle E_0, \ldots, E_n \rangle\).

If \(A \subset D\) is generated by an exceptional object \(E\), then by (3.3) the left and right mutations of an object \(F \in D\) through \(A\) are given by the following distinguished triangles:

\[
R\text{Hom}_D(E, F) \otimes E \to F \to L\langle E \rangle(F), \quad R\langle E \rangle(F) \to F \to R\text{Hom}_D(F, E)^* \otimes E.
\]

### 3.4. Block collections and block mutations.

The results of this section are needed for the subsequent Theorem 4.2. We follow the exposition of [8, Section 4].

**Definition 3.8.** A \(d\)-block exceptional collection is an exceptional collection \(E = (E_1, \ldots, E_n)\) together with a partition of \(E\) into \(d\) subcollections

\[
E = (E_1, \ldots, E_d),
\]

called blocks, such that the objects in each block \(E_i\) are mutually orthogonal, i.e. \(\text{Hom}^\bullet(E, E') = 0 = \text{Hom}^\bullet(E', E)\) for any \(E, E' \in E_i\).

For each integer \(1 < i \leq d\) we can define an operation \(\tau_i\) on \(d\)-block collections in \(D\) by the rule

\[
\tau_i(E_1, \ldots, E_{i-2}, E_{i-1}, E_i, E_{i+1}, \ldots E_d) =
(E_1, \ldots, E_{i-2}, L_{E_{i-1}}(E_i)[-1], E_{i-1}, E_{i+1}, \ldots E_d).
\]

Here, if \(E_i = (E_{a+1}, \ldots, E_b)\) then by definition

\[
L_{E_{i-1}}(E_i) = (L_{E_{i-1}}E_{a+1}, \ldots, L_{E_{i-1}}E_b).
\]

**Remark 3.1.** Note the shift by \([-1]\) in (3.12) at \(L_{E_{i-1}}(E_i)\); this will ensure that in the situations below the block mutations \(\tau_i\) preserve collections of pure objects.
Recall that a Serre functor (see [7]) on a $k$–linear triangulated category $D$ is an autoequivalence $S_D$ of $D$ for which there are natural isomorphisms $\text{Hom}_D(E, F) = \text{Hom}_D(F, S_D(E))^*$ for $E, F \in D$. If a Serre functor exists then it is unique up to isomorphism [7, Proposition 3.4].

Given a smooth algebraic variety $X$ over a field $k$, denote $D^b(X)$ the bounded derived category of coherent sheaves. It is a $k$–linear triangulated category. Let $\omega_X$ be the canonical line bundle on $X$. If $D = D^b(X)$ for a smooth projective variety $X$ of dimension $d$, then $S_D = (\cdot \otimes \omega_X)[d]$.

**Theorem 3.1.** [8, Theorem 4.5] Suppose $E = (E_1, \ldots, E_d)$ is a full $d$–block collection and take $1 < i \leq d$. Suppose $D$ is equipped with a $t$–structure that is preserved by the autoequivalence $S_D[1−d]$. Then

- $E$ pure implies $\tau_i(E)$ pure.
- $E$ pure implies $\tau_i(E)$ strong.

For our needs, Theorem 3.1 means the following. Let $X$ be a smooth variety of dimension $d − 1$, and $D = D^b(X)$ equipped with the standard $t$–structure $(D^b(X)^{\leq 0}, D^b(X)^{\geq 0})$. Assume there exists a $d$–block full exceptional collection in $D^b(X)$ consisting of pure objects, that is, of coherent sheaves in this case. Then the autoequivalence $S_D[1−d]$ is just tensoring with $\omega_X$, thus the condition of Theorem 3.1 is immediately satisfied. In this setting, Theorem 3.1 then means that left and right mutations are, too, exceptional collections consisting of coherent sheaves.

**Lemma 3.3.** [14, Lemma 2.11] Assume given a semiorthogonal decomposition $D = \langle A, B \rangle$. Then

$$L_A(B) = B \otimes \omega_X$$  and  $$R_A(B) = A \otimes \omega_X^{-1}.$$  (3.14)

Let $E$ be a vector bundle of rank $r$ on $X$, and consider the associated projective bundle $\pi : \mathbb{P}(E) \to X$. Denote $O_\pi(−1)$ the invertible line bundle on $\mathbb{P}(E)$ of relative degree $−1$, such that $\pi_*O_\pi(1) = E^*$. One has, [17]:

**Theorem 3.2.** The category $D^b(\mathbb{P}(E))$ has a semiorthogonal decomposition:

$$D^b(\mathbb{P}(E)) = (\pi^*D^b(X) \otimes O_\pi(−r + 1), \ldots, \pi^*D^b(X) \otimes O_\pi(−1), \pi^*D^b(X)).$$  (3.15)

### 3.5. Dual exceptional collections.

**Definition 3.9.** Let $X$ be a smooth variety, and assume given an exceptional collection $(E_0, \ldots, E_n)$ in $D^b(X)$. The right dual exceptional collection $(F_n, \ldots, F_0)$ to $(E_0, \ldots, E_n)$ is defined as

$$F_i := R_{(E_{i+1}, \ldots, E_n)}E_i, \quad \text{for} \quad 1 \leq i \leq n.$$  (3.16)
The left dual exceptional collection \((G_n, \ldots, G_0)\) to \((E_0, \ldots, E_n)\) is defined as

\[
G_i := L_{\langle E_1, \ldots, E_{i-1} \rangle} E_i, \quad \text{for} \quad 1 \leq i \leq n.
\]

**Proposition 3.2.** [10, Proposition 2.15] Let \((E_0, \ldots, E_n)\) be a semiorthogonal decomposition in a triangulated category \(D\). The left dual exceptional collection \(\langle F_n, \ldots, F_0 \rangle\) is uniquely determined by the following property:

\[
\text{Hom}_D^l(E_i, F_j) = \begin{cases} 
  k, & \text{for } l = 0, \ i = j, \\
  0, & \text{otherwise.}
\end{cases}
\]

Similarly, the right dual exceptional collection \(\langle G_n, \ldots, G_0 \rangle\) is uniquely determined by the following property:

\[
\text{Hom}_D^l(G_i, E_j) = \begin{cases} 
  k, & \text{for } l = 0, \ i = j, \\
  0, & \text{otherwise.}
\end{cases}
\]

### 4. Resolutions of the diagonal and the decomposition of \(F_*\mathcal{O}_X\)

Let \(X\) be a smooth variety. Given two (admissible) subcategories \(\mathcal{A}\) and \(\mathcal{B}\) of \(D^b(X)\), define \(\mathcal{A} \boxtimes \mathcal{B} \subset D^b(X)\) to be the minimal triangulated subcategory of \(D^b(X)\) that contains all the objects \((\mathcal{A} \boxtimes \mathcal{B})|A \in \mathcal{A}, B \in \mathcal{B}\). The results of [15] on base change for semiorthogonal decompositions imply:

**Theorem 4.1.** Let \(X\) be as above, and assume given a semiorthogonal decomposition \(\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle\) of \(D^b(X)\). Let \(\langle \mathcal{C}_m, \ldots, \mathcal{C}_1 \rangle\) be the right dual semiorthogonal decomposition of \(D^b(X)\) as in Definition 3.4. Then the structure sheaf of the diagonal \(\Delta_* \mathcal{O}_X\) admits (cf. Definition 3.4) a decomposition

\[
0 = D_m \to D_{m-1} \to \cdots \to D_1 \to D_0 = \Delta_* \mathcal{O}_X \quad \text{in} \quad D^b(X \times X),
\]

such that \(\text{Cone}(D_i \to D_{i-1}) \in \mathcal{A}_i \boxtimes \mathcal{C}_i^\vee \subset D^b(X \times X)\).

**Corollary 4.1.** Let \(X\) be a smooth projective variety, and \((\mathcal{E}_0, \ldots, \mathcal{E}_m)\) be a full exceptional collection in \(D^b(X)\) with \((\mathcal{F}_m, \ldots, \mathcal{F}_1)\) being its right dual. Then the structure sheaf of the diagonal \(\Delta_* \mathcal{O}_X\) admits a decomposition

\[
0 = D_{m+1} \to D_m \to \cdots \to D_1 \to D_0 = \Delta_* \mathcal{O}_X \quad \text{in} \quad D^b(X \times X),
\]

such that a cone of each morphism \(D_i \to D_{i-1}\) is quasiisomorphic to \(\mathcal{E}_i \boxtimes \mathcal{F}_i^\vee\), where \(\mathcal{F}_i^\vee = R\text{Hom}(\mathcal{F}_i, \mathcal{O}_X)\) is the dual object.

In particular, for any object \(\mathcal{G}\) of \(D^b(X)\) there is a spectral sequence

\[
E_1^{p,q} := H^{p+q}(X, \mathcal{G} \otimes F_i^\vee) \otimes \mathcal{E}_p \Rightarrow \mathcal{G}.
\]
Assembling together all the previous definitions and statements, we immediately obtain:

**Theorem 4.2.** Let \( X \) be a smooth variety of dimension \( d - 1 \) over a \( k \) of characteristic \( p \). Fix an \( m \geq 1 \) and consider the \( m \)-th Frobenius morphism \( F_m \). Assume given a \( d \)-block (cf. Definition 3.8) full exceptional collection \( E = (E_{-d+1}, \ldots, E_0) \) in \( D^b(X) \) consisting of coherent sheaves. Furthermore, assume that for any exceptional object \( E \in E_i \) and \( -d + 1 \leq i \leq 0 \), one has \( H^j(X, F_m^* E) = 0 \) for \( j \neq -i \). Denote \( G = (G_0, \ldots, G_{-d+1}) \) the right dual collection. Then

1. The right dual collection \( G = (G_0, \ldots, G_{-d+1}) \) is a \( d \)-block full exceptional collection.
2. For an exceptional vector bundle \( G \in G_i \) the corresponding shift is equal to \(-i\).
3. There is a decomposition of the bundle \( F_n^* O_X \) into the direct sum:

\[
F_n^* O_X = \bigoplus_{i=d}^{i=1} \bigoplus_{E \in E_i, G \in G_{i-d}} H^i(X, F_m^* E) \otimes G^V,
\]

and in the inner sum of (4.2) \( G \) is the right dual object for \( E \) as in Definition 3.9.
4. The terms of \( G \) are, up to a shift, vector bundles on \( X \).
5. Conversely, assume given a decomposition of \( F_n^* O_X \) into the direct sum of vector bundles that form a full exceptional collection \( G = (G_{-d+1}, \ldots, G_0) \) in \( D^b(X) \). Then the multiplicity space at \( G \in G_i \) is given by \( H^i(X, F_n^* E) \), where \( E \) is the right dual object for \( G \).

5. **The Incidence Varieties in Types A\(_3\) and A\(_4\)**

Recall some notation from [20, Section 6.1].

**Definition 5.1.** Given a semisimple algebraic group \( G \) and a dominant weight \( \omega \) of \( G \), for \( 1 \leq i \leq l \), where \( l = \dim(\nabla_{\omega}) \) the bundle \( \Psi^\omega_1 \) is set to be the pull-back of \( \Omega^i_{\mathbb{P}(\nabla_{\omega})}(i) \) along the morphism \( G/\mathbb{B} \to \mathbb{P}(\nabla_{\omega}) \) defined by a (semi)-ample line bundle \( L_{\omega} \).

By definition, the bundles \( \Psi^\omega_1 \) fit into short exact sequences:

\[
0 \to \Psi^\omega_1 \to \nabla_{\omega} \otimes O_{G/\mathbb{B}} \to L_{\omega} \to 0,
\]

Given a vector space \( V \) of dimension \( n \) over \( k \), the incidence variety \( X_{n,1}^1 \) is defined to be the variety of partial flags of type \((1, n-1)\) in \( V \). It is a partial flag variety of the group \( SL_n \) with the

\(^1\) Also called the adjoint variety as being isomorphic to the orbit of the highest weight vector in the adjoint representation of \( SL_n \).
Picard group isomorphic to $\mathbb{Z}^2$. Its canonical sheaf $\omega_{X_n}$ is isomorphic to $\mathcal{L}_{-(n-1)(\omega_1+\omega_{n-1})}$, where $\omega_1, \omega_2, \ldots, \omega_{n-1}$ are the fundamental weights of $\text{SL}_n$.

In this section we work out in detail the groups $\text{SL}_4$ and $\text{SL}_5$. Thus, starting with $\text{SL}_4$, denote $\omega_1, \omega_2, \omega_3$ the fundamental weights, and let $\alpha_1, \alpha_2, \alpha_3$ be the simple roots. For each simple root $\alpha_i$ let $P_{\hat{\alpha}_i} \supset B$ denote the corresponding minimal parabolic subgroup. The homogeneous spaces $\text{SL}_4/P_{\hat{\alpha}_i}$ can then be identified with varieties of partial flags $0 \subset V_1 \subset V_{i-1} \subset V_{i+1} \subset V$. There are the tautological bundles $U_i$ for $i = 1, 2, 3$ on $\text{SL}_4/B$.

The homogeneous space $\text{SL}_4/P_{\hat{\alpha}_2}$ can be identified with the partial flag variety $F_{1,3,4} =: X_4$. The line bundles $\mathcal{L}_{\omega_1}, \mathcal{L}_{\omega_3}$ and $\mathcal{L}_{\omega_1+\omega_3}$ on $X_4$ give rise to morphisms to $\mathbb{P}(\nabla_{\omega_1}), \mathbb{P}(\nabla_{\omega_3}),$ and $\mathbb{P}(\nabla_{\omega_1+\omega_3})$, respectively. As in (5.1), related to these morphisms are the short exact sequences:

\begin{align*}
(5.2) \quad & 0 \to \Psi_1^{\omega_1} \to \nabla_{\omega_1} \otimes \mathcal{O} \to \mathcal{L}_{\omega_1} \to 0, \\
(5.3) \quad & 0 \to \Psi_1^{\omega_3} \to \nabla_{\omega_3} \otimes \mathcal{O} \to \mathcal{L}_{\omega_3} \to 0, \\
\text{and} \quad & 0 \to \Psi_1^{\omega_1+\omega_3} \to \nabla_{\omega_1+\omega_3} \otimes \mathcal{O} \to \mathcal{L}_{\omega_1+\omega_3} \to 0.
\end{align*}

Further, denote $E$ the quotient bundle $U_3/U_1$. Identifying the bundle $U_3$ with $\Psi_1^{\omega_3}$ and $U_1$ with $\mathcal{L}_{-\omega_1}$, we obtain a short exact sequence:

\begin{align*}
(5.5) \quad & 0 \to \mathcal{L}_{-\omega_1} \to \Psi_1^{\omega_3} \to E \to 0.
\end{align*}

Denoting $\pi : \text{SL}_4/B \to X_4$ the projection, one also obtains the bundle $E$ as an extension:

\begin{align*}
(5.6) \quad & 0 \to \mathcal{L}_{\omega_1-\omega_2} \to \pi^*E \to \mathcal{L}_{\omega_2-\omega_3} \to 0.
\end{align*}

We first going to produce a semiorthogonal decomposition of $D^b(X_4)$ that will satisfy the conditions of Theorem 4.2. To this end, consider the semiorthogonal decomposition of $D^b(X_4) = \langle \tilde{A}_-, \tilde{A}_0, \tilde{A}, A_0, A_1, A_2 \rangle$ given by the following block structure (the fact that it is indeed a semiorthogonal decomposition will be proven in Lemma 5.1 below):
We then consequently mutate the blocks $A_1$ (resp., $A_2$) to the left through the block $A_0$ (resp., through the subcategory $\langle A_0, A_1 \rangle$), while mutating the block $\tilde{A}_{-1}$ to the right through $\tilde{A}_0$ and leaving the block $\tilde{A}$ intact to obtain the following decomposition:

\[
\begin{array}{ccccccc}
\tilde{A}_{-1} & \tilde{A}_0 & \tilde{A} & A_0 & A_1 & A_2 \\
\| & \| & \| & \| & \| & \\
\mathcal{L}_{-2\omega_1-\omega_3} & \mathcal{L}_{-\omega_1-\omega_3} & \mathcal{L}_{-\omega_1} & \mathcal{E} \otimes \mathcal{L}_{-\omega_1} & \mathcal{O}_{X_4} & \mathcal{L}_{\omega_1} & \mathcal{L}_{\omega_1+\omega_3} \\
\mathcal{L}_{-\omega_1-2\omega_3} & \mathcal{L}_{-\omega_1} & \mathcal{E} \otimes \mathcal{L}_{-\omega_1} & \mathcal{L}_{-\omega_3} & \mathcal{L}_{\omega_1} & \mathcal{L}_{\omega_1+\omega_3} & \mathcal{L}_{2\omega_1} \\
\end{array}
\]

We then consequently mutate the blocks $A_1$ (resp., $A_2$) to the left through the block $A_0$ (resp., through the subcategory $\langle A_0, A_1 \rangle$), while mutating the block $\tilde{A}_{-1}$ to the right through $\tilde{A}_0$ and leaving the block $\tilde{A}$ intact to obtain the following decomposition:

\[
\begin{array}{ccccccc}
\mathcal{C}_{-5} & \mathcal{C}_{-4} & \mathcal{C}_{-3} & \mathcal{C}_{-2} & \mathcal{C}_{-1} & \mathcal{C}_0 \\
\| & \| & \| & \| & \| & \\
\mathcal{L}_{-\omega_1-\omega_3} & \mathcal{L}_{-\omega_1} & \mathcal{E} \otimes \mathcal{L}_{-\omega_1} & \mathcal{O}_{X_4} & \mathcal{L}_{\omega_1} & \mathcal{L}_{\omega_1+\omega_3} & \mathcal{L}_{2\omega_1} \\
\end{array}
\]

Here $\Psi_{2,\omega_1,\omega_3}^\omega$ is the result of the left mutation of $\mathcal{L}_{\omega_1+\omega_3}$ through the subcategory $\langle A_0, A_1 \rangle$.

\[
\begin{array}{ccccccc}
(\Psi_{1}^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1-\omega_3} & (\Psi_{1}^{\omega_3})^* \otimes \mathcal{L}_{-\omega_1-\omega_3} & \mathcal{L}_{-\omega_1} & \mathcal{E} \otimes \mathcal{L}_{-\omega_1} & \mathcal{L}_{-\omega_3} & \mathcal{L}_{\omega_1} & \mathcal{L}_{\omega_1+\omega_3} \\
(\Psi_{1}^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1-\omega_3} & (\Psi_{1}^{\omega_3})^* \otimes \mathcal{L}_{-\omega_1-\omega_3} & \mathcal{L}_{-\omega_1} & \mathcal{E} \otimes \mathcal{L}_{-\omega_1} & \mathcal{L}_{-\omega_3} & \mathcal{L}_{\omega_1} & \mathcal{L}_{\omega_1+\omega_3} \\
\end{array}
\]

\[
\Psi_{2,\omega_1,\omega_3}^\omega := \mathcal{L}_{\langle A_0, A_1 \rangle}(\mathcal{L}_{\omega_1+\omega_3}).
\]

**Remark 5.1.** The automorphism of $\text{SL}_4$ interchanging the two simple roots $\alpha_1$ and $\alpha_3$ induces an automorphism of $X_4$. Hence the sought–for collection should be invariant under this automorphism as well. Since $\mathcal{E} \otimes \mathcal{L}_{-\omega_1} = \mathcal{E}^* \otimes \mathcal{L}_{-\omega_3}$, the above collection is indeed invariant.

**Lemma 5.1.** Let $p > 2$. Then the collection of subcategories $\mathcal{C} = \langle \mathcal{C}_{-5}, \mathcal{C}_{-4}, \mathcal{C}_{-3}, \mathcal{C}_{-2}, \mathcal{C}_{-1}, \mathcal{C}_0 \rangle$ is a semiorthogonal decomposition of $\text{D}^b(X_4)$ satisfying the conditions of Theorem 4.2.

**Proof.** The proof is broken up into a few separate statements which are found below. \qed
5.0.1. Orthogonality. We first observe that since the collection \( \mathcal{C} \) is obtained by mutating the collection \( \mathcal{A} = \langle \tilde{A}_1, \tilde{A}_0, \tilde{A}, A_0, A_1, A_2 \rangle \), the semiorthogonality of \( \mathcal{C} \) is equivalent to that of the collection \( \mathcal{A} \). The necessary orthogonalties between the line bundles in \( \mathcal{A} \) follow immediately from Theorems 2.2 and 2.3. To ensure the necessary orthogonalties for the bundle \( \mathcal{E} \otimes L_{-\omega_1} \), use sequences (5.5) and (5.6), and once again Theorems 2.2 and 2.3.

\[ 0 \to L_{-2\omega_1} \to \Psi_1^{\omega_3} \otimes L_{-\omega_1} \to \mathcal{E} \otimes L_{-\omega_1} \to 0, \]

and

\[ 0 \to \Psi_1^{\omega_3} \otimes L_{-\omega_1} \to \nabla \omega_1 \otimes L_{-\omega_1} \to L_{-\omega_1 + \omega_3} \to 0, \]

Considering the resolution

\[ 0 \to \psi_2^{\omega_3} \to \nabla \omega_2 \otimes O \to \nabla \omega_1 \otimes L_{\omega_1} \to L_{2\omega_3} \to 0, \]

and tensoring it with \( L_{-\omega_1} \), we obtain that \( L_{2\omega_3 - \omega_1} \) is in \( \mathcal{A} \), in virtue of an isomorphism \( \psi_2^{\omega_3} \otimes L_{-\omega_1} = (\psi_1^{\omega_3})^* \otimes L_{-\omega_1 - \omega_3} \). Hence, \( D^b(\mathbb{P}(\nabla \omega_1)) \otimes L_{-\omega_1} \) is also contained in \( \mathcal{A} \).

Clearly, \( L_{-2\omega_1 + \omega_3} \) and \( L_{-2\omega_1} \) are in \( \mathcal{A} \). Considering the short exact sequence

\[ 0 \to L_{-2\omega_1 + \omega_3} \to \psi_1^{\omega_3} \otimes L_{-\omega_1 + \omega_3} \to \mathcal{E}^* \to 0, \]

that is obtained from (5.9) by tensoring with \( L_{\omega_3} \) and using an isomorphism \( \mathcal{E}^* = \mathcal{E} \otimes L_{-\omega_1 + \omega_3} \).

Now \( \mathcal{E}^* \) is in \( \mathcal{A} \); this is seen by taking the dual of (5.5). Taking into account that \( \psi_1^{\omega_3} \otimes L_{-\omega_1 + \omega_3} \in (L_{-\omega_1 + \omega_3}, L_{-\omega_1 + 2\omega_3}) \), we conclude that \( L_{-2\omega_1 + \omega_3} \) is in \( \mathcal{A} \). Finally, assuming that there is a non–trivial object in the right orthogonal to \( L_{-\omega_1 - 2\omega_3} \), we see that it must be quasiisomorphic to \( L_{-2\omega_1 - 2\omega_3} \otimes V^* \) for some graded vector space \( V^* \). However, \( \text{Hom}^\bullet(\psi_1^{\omega_1 + \omega_3}, L_{-2\omega_1 - 2\omega_3} \otimes V^*) = V^*[5] \), since \( \omega_X = L_{-3\omega_1 - 3\omega_3} \). Hence, \( V^* = 0 \) and the statement follows.

5.0.2. Fullness. The variety \( X_3 \) is a projective bundle over \( \mathbb{P}(\nabla \omega_1) \) that is canonically isomorphic to \( \mathbb{P}(\psi_1^{\omega_3}) \) with relative Picard group being generated by \( L_{-\omega_1} \) (cf. sequence (5.5)). By Theorem 3.2, it is sufficient to prove that the full triangulated subcategory generated by \( \mathcal{A} \) contains the subcategories \( D^b(\mathbb{P}(\nabla \omega_1)) \otimes L_{-2\omega_1}, D^b(\mathbb{P}(\nabla \omega_1)) \otimes L_{-\omega_1}, \) and \( D^b(\mathbb{P}(\nabla \omega_1)) \).

5.0.3. Cohomology of Frobenius pull–backs. Let us now check the property \( \text{H}^j(X_4, F^*(?)) = 0 \) for \( j \neq i \) for a bundle \( ? \in \mathcal{C}_i, i = 0, \ldots, 5 \). The non–trivial verifications here concern the two bundles in the block \( \mathcal{C}_4 \), the bundle \( \mathcal{E} \otimes L_{-\omega_1} \) in \( \mathcal{C}_3 \), and \( \psi_2^{\omega_1 + \omega_3} \) in \( \mathcal{C}_1 \).

Claim 5.1. One has \( \text{H}^i(X_4, F^* \psi_2^{\omega_1 + \omega_3}) = 0 \) for \( i \neq 2 \).
Proof. Recall that $\Psi_i^{\omega_1, \omega_3}$ is defined to be the left mutation of $\Psi_i^{\omega_1} \cup \omega_3$ through the subcategory $(\Psi_i^{\omega_1}, \Psi_i^{\omega_3})$. Thus, there is a resolution:

\begin{equation}
0 \to \Psi_2^{\omega_1, \omega_3} \to \Psi_1^{\omega_1} \otimes \nabla_{\omega_3} \oplus \Psi_{\omega_3}^{\omega_1} \otimes \nabla_{\omega_1} \to \nabla_{\omega_1 + \omega_3} \otimes \mathcal{O} \to \mathcal{L}_{\omega_1 + \omega_3} \to 0.
\end{equation}

Clearly, $H^i(X_4, F^* \Psi_1^{\omega_1}) = 0$ for $i \neq 1$ and $k = 1, 3$ (see [20, Proposition 6.1]). Thus, one obtains $H^i(X_4, F^* \Psi_2^{\omega_1, \omega_3}) = 0$ for $i \neq 1, 2$.

It follows from the subsequent Proposition 5.1 and Claims 5.2 and 5.3 that $H^i(X_4, F^* \Psi_2^{\omega_1, \omega_3}) = 0$.

Indeed, by Lemma 5.1 and Theorem 4.1 one obtains, upon identifying the right dual collection with

\begin{equation}
H \to \cdots \to 1 \to L \to 0.
\end{equation}

\begin{equation}
H \to \cdots \to 0.
\end{equation}

Claim 5.2. One has $H^i(X_4, F^* (\mathcal{E} \otimes \mathcal{L}_{-\omega_1})) = 0$ for $i \neq 3$.

Proof. Recall the short exact sequences:

\begin{equation}
0 \to \mathcal{L}_{-\omega_2} \to \pi^* \mathcal{E} \otimes \mathcal{L}_{-\omega_1} \to \mathcal{L}_{-\omega_1 + 2\omega_3} \to 0,
\end{equation}

and,

\begin{equation}
0 \to \mathcal{L}_{-2\omega_1} \to \mathcal{L}_{-\omega_1} \to \mathcal{E} \otimes \mathcal{L}_{-\omega_1} \to 0.
\end{equation}

Finally, tensoring (5.3) with $\mathcal{L}_{-\omega_1 + \omega_3}$, obtain

\begin{equation}
0 \to \Psi_{\omega_3} \otimes \mathcal{L}_{-\omega_1} \to \nabla_{\omega_3} \otimes \mathcal{L}_{-\omega_1} \to \mathcal{L}_{-\omega_1 + \omega_3} \to 0.
\end{equation}

From the last sequence we conclude that $H^i(X_4, F^* (\Psi_{\omega_3}^{\omega_1} \otimes \mathcal{L}_{-\omega_1})) = 0$ for $i \neq 3$. Indeed, $H^i(X_4, \nabla_{\omega_3} \otimes \mathcal{L}_{-p\omega_1 + \omega_3}) = 0$ for $i \neq 3$ (for $p = 2, 3$ it is acyclic). On the other hand, one has $H^i(X_4, \mathcal{L}_{-p\omega_1 + \omega_3}) = 0$ for $i \neq 2$. Indeed, $s_{\alpha_2} \cdot s_{\alpha_1} \cdot (-p\omega_1 + p\omega_3) = (p - 3)\omega_2 + 2\omega_3$ (for $p = 2$ it is acyclic). Therefore, from the middle sequence we obtain that $H^i(X_4, F^* (\mathcal{E} \otimes \mathcal{L}_{-\omega_1})) = 0$ for $i \neq 2, 3$. On the other hand, from the first sequence we see that $H^i(X_4, F^* (\mathcal{E} \otimes \mathcal{L}_{-\omega_1})) = 0$ for
$i \neq 3, 4$ (we use throughout the fact that computing the cohomology of a coherent sheaf on $X_4$ is equivalent to doing that on the full flag variety $\text{SL}_4/\mathcal{B}$, the functor $\pi^*: \text{D}^b(X_4) \rightarrow \text{D}^b(\text{SL}_4/\mathcal{B})$ being fully faithful). Indeed, $H^i(\text{SL}_4/\mathcal{B}, \mathcal{L}_{-p\omega_1}) = 0$ for $i \neq 4$ (for $p = 2, 3$ it is acyclic), while $H^i(\text{SL}_4/\mathcal{B}, \mathcal{L}_{-p\omega_1 + p\omega_2 - \omega_3}) = 0$ for $i \neq 3$ (for $p = 2$ it is trivial for $i \neq 2$, and isomorphic to $k$ in this degree, and for $p = 3$ it is acyclic): indeed, $s_{\alpha_2} \cdot s_{\alpha_3} \cdot s_{\alpha_1} \cdot (-p\omega_1 + p\omega_2 - \omega_3) = \omega_1 + (p - 4)\omega_2 + \omega_3$. □

**Claim 5.3.** One has $H^i(X_4, F^*((\Psi_1^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1 - \omega_3})) = H^i(X_4, F^*((\Psi_1^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1 - \omega_3})) = 0$ for $i \neq 4$.

**Proof.** It is sufficient to prove the statement for the first group. On the one hand, one has a short exact sequence (note that $(\Psi_1^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1 - \omega_3} = T_F(\nabla_{\omega_1}) \otimes \mathcal{L}_{-2\omega_1 - \omega_3}$):

\begin{equation}
0 \rightarrow \mathcal{L}_{-2\omega_1 - \omega_3} \rightarrow \nabla_{\omega_1} \otimes \mathcal{L}_{-\omega_1 - \omega_3} \rightarrow (\Psi_1^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1 - \omega_3} \rightarrow 0,
\end{equation}

from which we conclude that $H^i(X_4, F^*((\Psi_1^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1 - \omega_3})) = 0$ for $i \neq 4, 5$. On the other hand, we have an isomorphism $(\Psi_1^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1} = \Psi_2^{\omega_1}$ and the resolution

\begin{equation}
0 \rightarrow \Psi_2^{\omega_1} \rightarrow \nabla_{\omega_2} \otimes \mathcal{O} \rightarrow \nabla_{\omega_3} \otimes \mathcal{L}_{\omega_1} \rightarrow \mathcal{L}_{2\omega_1} \rightarrow 0.
\end{equation}

Tensoring it with $\mathcal{L}_{-\omega_3}$, one obtains:

\begin{equation}
0 \rightarrow (\Psi_1^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1 - \omega_3} \rightarrow \nabla_{\omega_2} \otimes \mathcal{L}_{\omega_3} \rightarrow \nabla_{\omega_3} \otimes \mathcal{L}_{\omega_1 - \omega_3} \rightarrow \mathcal{L}_{2\omega_1 - \omega_3} \rightarrow 0.
\end{equation}

We see that $H^i(X_4, \nabla_{\omega_2} \otimes \mathcal{L}_{-p\omega_3}) = 0$ for $i \neq 3$ (for $p = 2, 3$ it is acyclic), and $H^i(X_4, \nabla_{\omega_3} \otimes \mathcal{L}_{-p\omega_1 - p\omega_3}) = 0$ for $i \neq 2$. Finally, $H^i(X_4, \mathcal{L}_{2p\omega_1 - p\omega_3}) = 0$ for $i \neq 2$. Indeed, $s_{\alpha_2} \cdot s_{\alpha_3} \cdot (2p\omega_1 - p\omega_3) = (p + 2)\omega_1 + (p - 3)\omega_2$. Therefore, $H^5(X_4, F^*((\Psi_1^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1 - \omega_3})) = 0$, and the statement follows. □

**Proposition 5.1.** The right dual decomposition $\tilde{C}$ to (5.7) consists of the following subcategories:

\begin{equation}
\begin{array}{cccccc}
\tilde{C}_0 & \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 & \tilde{C}_5 \\
\otimes_X & \otimes & \otimes & \otimes & \otimes & \otimes \\
\mathcal{O}_{X_4} & \mathcal{L}_{\omega_1} & \mathcal{L}_{\omega_1 + \omega_3} & \mathcal{L}_{2\omega_1 + 2\omega_3} & \mathcal{L}_{\omega_1 + 2\omega_3} & \mathcal{L}_{2(\omega_1 + \omega_3)} \\
\mathcal{L}_{\omega_3} & \mathcal{L}_{\omega_1 + \omega_3} & \mathcal{L}_{2\omega_3} & \mathcal{L}_{\omega_1 + 2\omega_3} & \mathcal{L}_{2\omega_1 + \omega_3} & \mathcal{L}_{2(\omega_1 + \omega_3)} \\
\mathcal{L}_{2\omega_3} & \mathcal{L}_{\omega_1 + \omega_3} & \mathcal{L}_{2\omega_1 + \omega_3} & \mathcal{L}_{\omega_1 + 2\omega_3} & \mathcal{L}_{2\omega_1 + \omega_3} & \mathcal{L}_{2(\omega_1 + \omega_3)} \\
\end{array}
\end{equation}
where $\tilde{G}$ is a vector bundle of rank 4 fitting into a unique non-split short exact sequence:

\begin{equation}
0 \to L_{2\omega_3} \to \tilde{G} \to \Psi_1^{\omega_3} \otimes L_{2\omega_1 + \omega_3} \to 0.
\end{equation}

**Proof.** The right dual objects to all the generators of $\mathcal{A}$ but $E \otimes L_{-\omega_1}$ are calculated rather straightforwardly. To compute the right dual to $E \otimes L_{-\omega_1}$, it is sufficient to compute its left mutation through the subcategory generated by $\langle \tilde{A}_0, \tilde{A}_1 \rangle$ and then use Lemma 3.3. One computes $\text{Hom}^*(L_{-\omega_1 - \omega_3}, E \otimes L_{-\omega_1}) = \nabla_{\omega_2}$. One then obtains the commutative diagram depicted below:

\begin{equation}
\begin{array}{ccccccccc}
0 & \rightarrow & \Psi_2^{\omega_1} \otimes L_{-\omega_1 - \omega_3} & \rightarrow & \mathcal{F} & \rightarrow & L_{-2\omega_3} & \rightarrow & 0 \\
0 & \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \nabla_{\omega_2} \otimes L_{-\omega_1 - \omega_3} & \rightarrow & \nabla_{\omega_2} \otimes L_{-\omega_1 - \omega_3} & \rightarrow & 0 \\
0 & \downarrow & \downarrow & \rightarrow & \Psi_1^{\omega_1} \otimes L_{-\omega_3} & \rightarrow & \mathcal{E} \otimes L_{-\omega_1} & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\end{equation}

Thus, $L_{\mathcal{A}_0}(E \otimes L_{-\omega_1}) = \mathcal{F}[1]$, where $\mathcal{F}$ is the extension from the top row of the above diagram that corresponds to a unique non-split extension $\text{Ext}^1(L_{-2\omega_3}, \Psi_2^{\omega_1} \otimes L_{-\omega_1 - \omega_3}) = k$. Further, as $L_{-2\omega_1 - \omega_3}$ and $L_{-\omega_1 - 2\omega_3}$ are mutually orthogonal, the left mutation of $\mathcal{F}$ through $\langle L_{-2\omega_1 - \omega_3}, L_{-\omega_1 - 2\omega_3} \rangle$ is found from the triangle

\begin{equation}
\cdots \rightarrow \nabla_{\omega_1} \otimes (L_{-2\omega_1 - \omega_3} \oplus L_{-\omega_1 - 2\omega_3}) \rightarrow \mathcal{F} \rightarrow \tilde{G}[1] \rightarrow \cdots
\end{equation}

as one calculates $\text{Hom}^*(L_{-2\omega_1 - \omega_3}, \mathcal{F}) = \text{Hom}^*(L_{-\omega_1 - 2\omega_3}, \mathcal{F}) = \nabla_{\omega_1}$ from the same top row of the above diagram. Consider the diagram
The bottom row of the above diagram is the defining short exact sequence for $F$, the middle one being the split extension. One thus finds the left mutation $L_{\tilde{A}_{-1}}(F)$ to be isomorphic to $G[1]$, where $G$ is defined by the top row and is obtained as a unique non-split extension corresponding to $\text{Ext}^1(\Psi_1^{\omega_1} \otimes L_{-\omega_1-3\omega_3}, L_{-3\omega_1-\omega_3}) = k$. Finally, the right dual to $E \otimes L_{-\omega_1}$ is seen to be isomorphic to

$$R_\langle A_0 A_1 A_2 \rangle (E \otimes L_{-\omega_1}) = L_{\langle \tilde{A}_{-1} A_0 \rangle} (E \otimes L_{-\omega_1}) \otimes L_{3(\omega_1+\omega_3)} = \tilde{G}[2-5] = \tilde{G}[-3].$$

For consistency of the above calculations, one computes $\text{Hom}^\bullet(\tilde{G}, E \otimes L_{-\omega_1}) = k[-3]$, while $\text{Hom}^\bullet(\tilde{G}, -) = 0$ for all other exceptional generators of the decomposition $C$, thus verifying by Proposition 3.2 that the object $\tilde{G}[-3]$ is the right dual to $E \otimes L_{-\omega_1}$.

One thus obtains:

**Theorem 5.1.** The bundle $F_* \mathcal{O}_{X_4}$ decomposes into the direct sum of vector bundles with indecomposable summands being isomorphic to:

$$\mathcal{O}_{X_4}, \mathcal{L}_{-\omega_1}, \mathcal{L}_{-\omega_3}, \mathcal{L}_{-2\omega_1}, \mathcal{L}_{-\omega_1-\omega_3}, \mathcal{L}_{-2\omega_3},$$

$$(\Psi_2^{\omega_1})^* \otimes \mathcal{L}_{-\omega_1-2\omega_3}, \tilde{G}^*, (\Psi_2^{\omega_3})^* \otimes \mathcal{L}_{-2\omega_1-\omega_3}, \mathcal{L}_{-\omega_1-2\omega_3}, \mathcal{L}_{-2\omega_1-\omega_3}, \mathcal{L}_{-2(\omega_1+\omega_3)}$$

The multiplicity spaces at each indecomposable summand are isomorphic, respectively, to the cohomology of the Frobenius pull-back $F^*$ applied to the right dual of the given summand that is found from (5.7).

5.1. **Type $A_4$.** Following the previous section, consider
Likewise, we consequently mutate the blocks $A_i$ for $i = 1, 2, 3$ to the left through the subcategory generated by $\langle A_0, \ldots, A_{i-1} \rangle$, while mutating the blocks $\tilde{A}_{-i}$ for $i = 1, 2$ to the right through the subcategory generated by $\langle \tilde{A}_{-i+1}, \ldots, \tilde{A}_0 \rangle$ obtaining the mutated block structure:

\begin{align*}
\begin{array}{cccccccc}
\tilde{A}_{-2} & \tilde{A}_{-1} & \tilde{A}_0 & \tilde{A} & A_0 & A_1 & A_2 & A_3 \\
\| & \| & \| & \| & \| & \| & \| & \|
\end{array}
\end{align*}

\[ L_{-3\omega_1 - \omega_4} \quad L_{-2\omega_1 - \omega_4} \quad L_{-\omega_1} \quad E \otimes L_{-\omega_1} \quad \Lambda^2 E \otimes L_{-\omega_1} \quad L_{-\omega_4} \quad O_{X_4} \quad L_{\omega_1} \quad L_{\omega_1 + \omega_4} \quad L_{2\omega_1 + \omega_4} \quad L_{3\omega_1}
\]

\[
(5.26) \quad C_{-7} \quad C_{-6} \quad C_{-5} \quad C_{-4} \quad C_{-3} \quad C_{-2} \quad C_{-1} \quad C_0
\]

\begin{align*}
\begin{array}{cccccccc}
L_{-\omega_1 - \omega_4} & \Phi_{\omega_1}^1 & \Phi_{\omega_1}^2 & \Phi_{\omega_1}^4 \\
\| & \| & \| & \|
\end{array}
\end{align*}

\[
\begin{array}{cccccccc}
L_{\omega_1} & E \otimes L_{-\omega_1} & \Lambda^2 E \otimes L_{-\omega_1} & L_{-\omega_4} \\
\| & \| & \| & \|
\end{array}
\end{align*}

\[
\begin{array}{cccccccc}
\Psi_{\omega_1}^1 & \Psi_{\omega_1}^2 & \Psi_{\omega_1}^3 & \Psi_{\omega_1}^4 \\
\| & \| & \| & \|
\end{array}
\end{align*}

where $\Psi_{k}^{i\omega_1,(k-i)\omega_4}$ for $k = 1, 2, 3$ and $0 \leq i \leq k$ denotes the left mutation of $L_{i\omega_1 + (k-i)\omega_4}$ through $\langle A_0, \ldots, A_{k-1} \rangle$, while $\Phi_{k}^{i\omega_1,(k-i)\omega_4}$ for $k = 1, 2$ and $0 \leq i \leq k$ (shortened to $\Phi_{k}^{\omega_1}$ or to $\Phi_{k}^{\omega_4}$ for $i = 0$ or $i = k$) denotes, correspondingly, the right mutation of $L_{(-k+i+1)\omega_1 -(i+1)\omega_4}$ through $\langle \tilde{A}_{-k+1}, \ldots, \tilde{A}_0 \rangle$ (in fact, there are isomorphisms $\Phi_{k}^{i\omega_1,(k-i)\omega_4} = (\Psi_{i}^{\omega_1,(k-i)\omega_4})^* \otimes L_{-\omega_1 - \omega_4}$ for $k = 1, 4$ and $0 \leq i \leq k$).

\[2\text{Note that there are isomorphisms } \Psi_{k}^{i\omega_1,0} = \Psi_{k}^{\omega_1} \text{ and } \Psi_{k}^{0,\omega_4} = \Psi_{k}^{\omega_4} \text{ in the notation of Definition } 5.1 \text{ which justifies the notation in (5.26).} \]
Similarly, one has the following

**Lemma 5.2.** Let $p > 2$. Then the collection of subcategories $\mathcal{C} = \langle C_7, C_6, \ldots, C_0 \rangle$ as above is a semiorthogonal decomposition of $\text{D}^b(X_n)$ satisfying the conditions of Theorem 4.2.

**Proof.** The proof is essentially analogous to that of Lemma 5.1. The new feature is the appearance of the second exterior power of the bundle $E$ whose Frobenius pull–back is supposed to have the single non–trivial cohomology in the prescribed degree (specifically, in degree 4 in the considered case). More generally, one has:

**Claim 5.4.** One has $H^k(X_n, F^*(\Lambda^i E \otimes \mathcal{L}_{-\omega_1})) = 0$ for $k \neq n - 1$ and $i = 1, n - 3$.

**Proof.** Recall the defining short exact sequence for $\mathcal{E} \otimes \mathcal{L}_{-\omega_1}$:

$$0 \to \mathcal{L}_{-2\omega_1} \to \Psi_1^{\omega_{n-1}} \otimes \mathcal{L}_{-\omega_1} \to \mathcal{E} \otimes \mathcal{L}_{-\omega_1} \to 0. \tag{5.27}$$

Considering the short exact sequence

$$0 \to \Psi_1^{\omega_n} \otimes \mathcal{L}_{-\omega_1} \to \nabla_{\omega_1} \otimes \mathcal{L}_{-\omega_1} \to \mathcal{L}_{-\omega_1+n_1} \to 0, \tag{5.28}$$

one sees that $H^k(X_n, F^*(\Psi_1^{\omega_n} \otimes \mathcal{L}_{-\omega_1})) = 0$ for $k \neq n - 1$. Indeed, one has $H^k(X_n, \mathcal{L}_{-p\omega_1}) = H^k(\mathbb{P}(\nabla_{\omega_1}), \mathcal{L}_{-p\omega_1}) = 0$ for $k \neq n - 1$, while

$$s_{\alpha_{n-2}} : s_{\alpha_{n-3}} : \cdots : s_{\alpha_1} \cdot (-p\omega_1 + p\omega_{n-1}) = (p - n)\omega_{n-2} + (n - 1)\omega_{n-1}. \tag{5.29}$$

Hence, $H^k(X_n, \mathcal{L}_{-p\omega_1+p\omega_{n-1}}) = 0$ for $k \neq n - 2$, and the statement follows. From sequence (5.27) one then obtains $H^k(X_n, F^*(\mathcal{E} \otimes \mathcal{L}_{-\omega_1})) = 0$ for $k \neq n - 2, n - 1$. On the other hand, the bundle $\pi^*\mathcal{E} \otimes \mathcal{L}_{-\omega_1}$ is filtered with the set of graded factors being isomorphic to

$$\mathcal{L}_{-\omega_2}, \mathcal{L}_{-\omega_1+\omega_2}, \cdots, \mathcal{L}_{-\omega_1+n_1-\omega_2}. \tag{5.30}$$

Using Theorems 2.3 and 2.2, one checks that $H^{n-2}(X_n, \mathcal{L}_\lambda) = 0$, where $\lambda$ is any weight from the set (5.30), and thus Claim 5.4 for $i = 1$ follows.

The case $i = n - 3$ is similar to the above. One has $\Lambda^{n-3}\mathcal{E} \otimes \mathcal{L}_{-\omega_1} = \mathcal{E} \otimes \det(\mathcal{E}) \otimes \mathcal{L}_{-\omega_1} = \mathcal{E} \otimes \mathcal{L}_{-\omega_{n-1}}$, since $\det(\mathcal{E}) = \mathcal{L}_{\omega_1-\omega_{n-1}}$. Consider the short exact sequence

$$0 \to \mathcal{E} \otimes \mathcal{L}_{-\omega_{n-1}} \to (\Psi_1^{\omega_{n-1}})^* \otimes \mathcal{L}_{-\omega_{n-1}} \to \mathcal{L}_{\omega_1-\omega_{n-1}} \to 0, \tag{5.31}$$

One obtains $H^k(X_n, F^*((\Psi_1^{\omega_{n-1}})^* \otimes \mathcal{L}_{-\omega_{n-1}})) = 0$ for $k \neq n - 2, n - 1$, while (cf. 5.29) $H^k(X_n, \mathcal{L}_{p(\omega_1-\omega_{n-1})}) = 0$ for $k \neq n - 2$. Thus, $H^k(X_n, F^*(\mathcal{E} \otimes \mathcal{L}_{-\omega_{n-1}})) = 0$ for $k \neq n - 2, n - 1$. Arguing as in (5.30), one ensures that $H^{n-2}(X_n, F^*(\mathcal{E} \otimes \mathcal{L}_{-\omega_{n-1}})) = 0$, hence the statement.

\[\square\]
5.1.1. **The right dual collection.** The terms of right dual collection to (5.26) are calculated as in Proposition 5.1, most of these can be obtained immediately from the construction. As for the bundles \( E \otimes L^{-\omega_1} \) and \( \Lambda^2(E \otimes L^{-\omega_1}) \), repeating the calculation from Proposition 5.1 one obtains:

\[
\text{(5.32)} \quad L_{(\tilde{A}_1, \tilde{A}_0)}(E \otimes L^{-\omega_1}) = G,
\]

where \( G \) is obtained as a unique non-trivial extension

\[
\text{(5.33)} \quad 0 \to L_{-3\omega_1-\omega_4} \to G \to \Psi_{1}^{\omega_1} \otimes L_{-\omega_1-2\omega_4} \to 0.
\]

This sequence immediately gives that the left mutation \( L_{(\tilde{A}_1, \tilde{A}_0)}(E \otimes L^{-\omega_1}) \) is isomorphic to \( \Psi_{1}^{\omega_1} \otimes L_{-\omega_1-2\omega_4} \). Further, \( \text{Hom}^*(L_{-2\omega_1-2\omega_4}, \Psi_{1}^{\omega_1} \otimes L_{-\omega_1-2\omega_4}) = \nabla_{\omega_2} \), and one obtains

\[
\text{(5.34)} \quad 0 \to \Psi_{1}^{\omega_1} \otimes L_{-2\omega_1-2\omega_4} \to \nabla_{\omega_2} \otimes L_{-2\omega_1-2\omega_4} \to \Psi_{1}^{\omega_1} \otimes L_{-\omega_1-2\omega_4} \to 0;
\]

that is, \( L_{(\tilde{A}_1, \tilde{A}_0)}(\nabla_{\omega_2} \otimes L_{-2\omega_1-2\omega_4}) = \Psi_{1}^{\omega_1} \otimes L_{-2\omega_1-2\omega_4} [1] \). Finally, one calculates the group \( \text{Hom}^*(L_{-\omega_1-3\omega_4}, \Psi_{2}^{\omega_1} \otimes L_{-2\omega_1-2\omega_4} [1]) = k \), and thus the left mutation \( L_{(\tilde{A}_1, \tilde{A}_0)}(\Psi_{2}^{\omega_1} \otimes L_{-2\omega_1-2\omega_4} [1]) = L_{(\tilde{A}_1, \tilde{A}_0)}(E \otimes L^{-\omega_1}) \) is given by a unique non-trivial extension

\[
\text{(5.35)} \quad 0 \to \Psi_{2}^{\omega_1} \otimes L_{-2\omega_1-2\omega_4} \to \nabla_{\omega_2} \otimes L_{-2\omega_1-2\omega_4} \to \Psi_{2}^{\omega_1} \otimes L_{-\omega_1-2\omega_4} \to 0.
\]

Denote \( \tilde{H} = \nabla_{\omega_2} \otimes \omega_{X_5}^{-1} = \nabla_{\omega_2} \otimes L_{4(\omega_1+\omega_4)} \). Finally, by Lemma 3.3 one has

\[
\text{(5.36)} \quad R_{(A_0, A_1, A_2, A_3)}(E \otimes L^{-\omega_1}) = L_{(\tilde{A}_1, \tilde{A}_0)}(E \otimes L^{-\omega_1}) \otimes \omega_{X_5}^{-1} = \tilde{H}[3 - 7] = \tilde{H}[-4].
\]

Observe that \( \Lambda^2E \otimes L_{-\omega_1} = E^* \otimes L_{-\omega_4} \), and that the bundles \( E^* \otimes L_{-\omega_4} \) and \( E^* \otimes L_{-\omega_4} \) are interchanged under the automorphism of the Dynkin diagram \( A_4 \). Thus, the right dual bundle of \( \Lambda^2E \otimes L_{-\omega_1} \) is isomorphic to a unique non-trivial extension

\[
\text{(5.37)} \quad 0 \to \Psi_{2}^{\omega_4} \otimes L_{2\omega_1+2\omega_4} \to \tilde{K} \to L_{\omega_1+3\omega_4} \to 0.
\]

Similarly to Theorem 5.1, one obtains:
Theorem 5.2. The bundle $F_* \mathcal{O}_{X_n}$ decomposes into the direct sum of vector bundles with indecomposable summands being isomorphic to:

$$
\mathcal{O}_{X_n}, \mathcal{L}^{-\omega_1}, \mathcal{L}^{-\omega_4}, \mathcal{L}^{-2\omega_1}, \mathcal{L}^{-\omega_1-\omega_1}, \mathcal{L}^{-2\omega_4}, \mathcal{L}^{-3\omega_1}, \mathcal{L}^{-2\omega_1-\omega_4}, \mathcal{L}^{-\omega_4}, (\Psi_3^{\omega_1})^* \otimes \mathcal{L}^{-\omega_1-3\omega_1}, \mathcal{H}^k, \mathcal{K}^*, (\Psi_3^{\omega_1})^* \otimes \mathcal{L}^{-3\omega_1-\omega_4},
$$

$$
\mathcal{L}^{-\omega_1-3\omega_1}, \mathcal{L}^{-2\omega_1-2\omega_4}, \mathcal{L}^{-3\omega_1-\omega_4}, \mathcal{L}^{-2\omega_1-3\omega_4}, \mathcal{L}^{-3\omega_1-2\omega_4}, \mathcal{L}^{-3(\omega_1+\omega_4)}
$$

The multiplicity spaces at each indecomposable summand are isomorphic, respectively, to the cohomology of $F^*$ of the corresponding terms of (5.26).

6. The General Case

Given the results of Section 5, one naturally arrives at the following conjecture:

Conjecture 6.1. Let $n \in \mathbb{N}$. Consider the following collection of subcategories $\tilde{\mathcal{A}}, \mathcal{B}, \mathcal{A}$ of $D^b(X_n)$, where

(6.1) \[ \mathcal{A} = \langle \mathcal{A}_k \rangle, \quad 0 < k \leq n - 1, \]

where $\mathcal{A}_0 = \langle \mathcal{O}_{X_n} \rangle$, and $\mathcal{A}_k$ for $k < 0$ is defined inductively as the left mutation of the subcategory generated by $\mathcal{L}_{-\omega_1+(k-i)\omega_{n-1}}$ for $0 \leq i \leq k$ through the subcategory generated by $\mathcal{A}_l$ for $0 \leq l < k$. The subcategory $\mathcal{B}$ is generated by an exceptional collection

(6.2) \[ \mathcal{B} = \langle \Lambda^i \mathcal{E} \otimes \mathcal{L}^{-\omega_1} \rangle, \quad 0 \leq i \leq n - 2. \]

Finally, the subcategory $\tilde{\mathcal{A}}$ is defined to be

(6.3) \[ \tilde{\mathcal{A}} = \langle \tilde{\mathcal{A}}_k \rangle, \quad -n + 2 < k \leq 0, \]

where $\tilde{\mathcal{A}}_0 = \langle \mathcal{L}^{-\omega_1-\omega_n} \rangle$, and $\tilde{\mathcal{A}}_k$ for $k < 0$ is defined inductively as the right mutation of the subcategory generated by $\mathcal{L}_{-(k+i)\omega_{n-1}} \otimes \mathcal{L}^{-\omega_1-\omega_n}$ for $0 \leq i \leq -k$ through the subcategory generated by $\tilde{\mathcal{A}}_l$ for $k < l \leq 0$.

Then the collection of subcategories $(\tilde{\mathcal{A}}, \mathcal{B}, \mathcal{A})$ is a semiorthogonal decomposition of $D^b(X_n)$ that satisfies the conditions of Theorem 4.2. The set of indecomposable summands of $F_* \mathcal{O}_{X_n}$ consists of the terms of the right dual decomposition to $(\tilde{\mathcal{A}}, \mathcal{B}, \mathcal{A})$.

Given the above considerations, proving the conjecture essentially reduces to the following statement: $H^k(X_n, F^*(\Lambda^i \mathcal{E} \otimes \mathcal{L}^{-\omega_1})) = 0$ for $k \neq n - 1$ and $0 \leq i \leq n - 2$ (that is, the cohomology of Frobenius pull–backs of the bundles from the block $\mathcal{B}$ in (6.2) can be non–trivial only in the prescribed degree).
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