Recovery of structure of looped jointed objects from multiframes

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Abstract. A method to recover structural parameters of looped jointed objects from multiframes is being developed. Each rigid part of the jointed body needs only to be traced at two (that is at junction) points. This method has been linearized for 4-part loops, with recovery from at least 19 frames.

1 Introduction

In his papers Johansson, [1], drew attention to the capability of a man to reconstruct motion and shape of an object if only two traceable points are available on each rigid part of a jointed object. Most of known algorithm for recovery of structure from motion require more (or even significantly more) traceable features to recover motion of rigid bodies alone (compare [3, 10, 11, 12, 13, 14]).

Several attempts have been therefore made to make use of the jointedness property of jointed objects to utilize the psychological observation of Johansson. Clocksin, [2], tried to recover the structure of joins of an object (the join structure shows which visible points are rigidly connected); He used a heuristic approach based on prediction of motion of rigidly connected points. First, he assumed that all points are rigidly connected and then broke rigid connection for points not fitting the predicted pattern of motion.

Rashid, [5], tried also to recover heuristically the structure of joins. He estimated relative positions and speeds of all pairs of points for a given number of projections (25 or 30). Then he discovered the join structure finding minimal spanning tree of the graph of all points calculating weights of branches from correlation of position and speed of points. A weakness of this approach is missing three dimensional interpretation of motion of points which may lead under some circumstances to erroneous interpretation of joins.

To interpret Johnson’s figures, O’Rourke and Badler, [6], used background knowledge about the join structure of the observed jointed object (human body shape in their case). The model was confronted with the image enabling to reconstruct the position even of invisible parts of the object.
Hoffman and Flinchbaugh, [4], elaborated a method of reconstruction of structure of joins and of 3-dimensional structure of Johansson's figures assuming that the motion of all rigid parts in the image is planar.

Webb and Aggarwal, [9, 7, 8], made also restrictive assumptions - that all traceable points of the object rotate around a fixed rotation axis.

Lee, [12], showed that one can recover structure and motion parameters of two rigidly connected points even assuming only a fixed direction of rotation.

In this paper we investigate a different assumption about the jointed object. We assume that the rigid parts of the object form a loop or loops (see Fig. 1) and that the traceable points are the junction points. We will concentrate on loops composed of 4 rigid parts only and demonstrate that the task can be linearized if 19 images are available. We handle orthogonal projections only. Planarity of motion is not required.

2 The Method

The fundamental approach to the problem of structure and motion from multiframe is to find and utilize one (or more) invariant properties combining visible quantities with global unknowns (not engaging unknowns local to the single image). Surprisingly, for orthogonal projections of looped jointed objects the property is analogous to three-point rigid objects case described in [13].

Let us consider a loop consisting of points \( P_1, P_2, ..., P_n, P_1 \). Then obviously the sum of vectors \( P_1P_2, P_2P_3, ..., P_{n-1}P_n, P_nP_1 \) fulfills the equation:

\[
P_1P_2 + P_2P_3 + ... + P_{n-1}P_n + P_nP_1 = 0. \tag{1}
\]

Let us assume that the image plane is the XY plane (the projection direction
is then along the Z-axis). The above relation is also true for component vectors of these vectors in each direction. Let $P_k P_j$ be the Z-direction component of $P_k P_j$. We have:

$$P_1 P_2 + P_2 P_3 + \ldots + P_{n-1} P_n + P_n P_1 = 0.$$  \hspace{1cm} (2)

But this means that if $||P_k P_j||$ denotes the length of the vector $P_k P_j$ then there exists a combination of +’s and -’s such that we get a result equal to 0 in the equation below:

$$||P_1 P_2|| \pm ||P_2 P_3|| \pm \ldots \pm ||P_{n-1} P_n|| \pm ||P_n P_1|| = 0.$$  \hspace{1cm} (3)

Let us denote by $P'_j$ the projection of point $P_j$. Obviously:

$$||P_k P_j|| = \sqrt{||P_k P_j||^2 - ||P'_k P'_j||^2}.$$  

Hence

$$\sqrt{||P_1 P_2||^2 - ||P'_1 P'_2||^2} \pm \sqrt{||P_2 P_3||^2 - ||P'_2 P'_3||^2} \pm \ldots \pm \sqrt{||P_{n-1} P_n||^2 - ||P'_n P'_1||^2} = 0.$$  \hspace{1cm} (4)

This is an equation in $n$ unknowns (lengths of line segments $P_1 P_2$, $P_2 P_3$, $\ldots$, $P_{n-1} P_n$, $P_n P_1$). From $n$ images we can get in principle $n$ (most probably independent) equations and just find the unknown lengths of rigid edges of the jointed objects. Once these lengths are known, the position of the object in space for each frame is known (up to shift along the Z axis and reflections about the XY plane).

Let us restrict now our treatment to the case of only four traceable points (a loop of four rigid parts). Let us call the traceable points $P, Q, R, S$ and their projections $P'_1, Q'_1, R'_1, S'_1$ respectively (see fig.2). Let us also introduce the following notation for global unknowns:

$$a = PQ^2, b = QR^2, c = RS^2, d = SP^2.$$  \hspace{1cm} (5)

Quantities obtainable (measurable) from $i$’th frame are as follows:

$$A_i = P'_i Q'^2, B_i = Q'_i R'^2, C_i = R'_i S'^2, D_i = S'_i P'^2.$$  \hspace{1cm} (6)

We will occasionally drop the $i$-index if we are talking about only one (current) frame. Under this notation the eq. 4 turns to:

$$\sqrt{a - A_i} \pm \sqrt{b - B_i} \pm \sqrt{c - C_i} \pm \sqrt{d - D_i} = 0.$$  \hspace{1cm} (7)

We will show now how this equation can be turned to "linear" form. Details are given in the Appendix A. The principle is to square out the square roots
to obtain a polynomial in our four unknowns \( a, b, c \) and \( d \). Thereafter auxiliary variables \( x_1 - x_{19} \) are introduced which are themselves polynomials in \( a, b, c, d \) and independent of the current frame. Variables \( x_1, x_2, x_3, x_4 \) are identical with \( a, b, c, d \) resp. and are of primary interest for us (the lengths of links in the loop are then obtained as square roots of \( x_1, x_2, x_3, x_4 \)). If we represent constant expressions obtained as free expressions and as factors for \( x_1 - x_{19} \) as \( f_{i,0} - f_{i,19} \) \((i - \text{the index of the frame})\), we can write the above eq. 7 in the plain form:

\[
\begin{align*}
&f_{i,1}x_1 + f_{i,2}x_2 + f_{i,3}x_3 + f_{i,4}x_4 + f_{i,5}x_5 + f_{i,6}x_6 + f_{i,7}x_7 + f_{i,8}x_8 + f_{i,9}x_9 \\
&+ f_{i,10}x_{10} + f_{i,11}x_{11} + f_{i,12}x_{12} + f_{i,13}x_{13} + f_{i,14}x_{14} + f_{i,15}x_{15} \\
&+ f_{i,16}x_{16} + f_{i,17}x_{17} + f_{i,18}x_{18} + f_{i,19}x_{19} + f_{i,0} = 0.
\end{align*}
\] (8)

We have now one linear equation in 19 variables for each frame. Though the variables and coefficients are dependent, they are not linearly dependent. Hence 19 frames from a free motion of this body may allow us to recover the object parameters \( a, b, c, d \). The subsequent example demonstrates our approach.

3 An Example

We assumed a jointed body with rigid edge lengths \( PQ = 2, QR = 3, RS = 4, SP = 1 \). Then randomly 19 positions of this body have been generated (see fig. 3) using the following degrees of freedom: distance between \( P \) and \( R \), distance between \( Q \) and \( S \), rotations around \( X \), \( Y \) and \( Z \) axes. We measured the distances between projected points in the respective frames which are contained in tab. 1. (The high precision stems from the fact that we simulated data.) Then we calculated the matrix of coefficients \( f_{i,1}, \ldots, f_{i,19}, f_{i,0} \) below for all frames \( i=1..19 \). given in the Appendix B.
Then we solved for $x_1,\ldots,x_{19}$ to obtain the result given below (for most interesting variables):

$$x_1 = 4.00415, \quad x_2 = 8.98225, \quad x_3 = 15.9834, \quad x_4 = 0.999825.$$  

(Original values were $a = 4, b = 9, c = 16, d = 1$ resp.).

To investigate the impact of observational errors, the same experiment has been repeated assuming that the precision of measurement of position of projected points is up to three leading digits. The results were:

$$x_1 = 3.88039, \quad x_2 = 8.55283, \quad x_3 = 15.1446, \quad x_4 = 0.976372.$$  

4 Discussion

The example demonstrates the principal possibility of linear recovery of shape parameters on a 4 part looped rigid object. Advantages and disadvantages of the approach are visible. The advantage is the need to solve a linear equation system only instead of multivariable high order non-linear one. No restrictions are posed on the motion pattern of the object or on relative motion of its jointed parts. The disadvantages are: the great number of frames needed (nearly as much as used by Rashid [5]), and the danger of rounding errors.

A more close study should be devoted to the impact of imprecision of position of projections of traceable points. Experiments with reduction of precision of up to three digits did not prove totally destructive.

One may wonder whether it is possible to apply the same technique for longer loops of rigid parts. The eq. (4) can be always "squared out" to obtain an algebraic equation which in turn may be converted to a linearized form. However, the number of variables will explode making practical application not feasible. We can see this already when comparing the case of 3-part loops (see [13]) and the 4-part loops (as described above). Linearization in the former case increases the number of necessary equations from 3 to 4, and in the latter case from 4 to 19.

Still another question is the validity of the looped object model. Elsewhere we demonstrate that for rigid bodies a test can be applied from frame to frame to check whether or not the points belong to the same rigid body. Regrettably, we cannot construct such a two-frame test for general looped objects. Only after we have determined all the edge lengths, we can check whether or not they fit geometrical requirements for each frame considered (whether they can close a loop or not). If we want to guess which $n$ points constitute a looped jointed object, we need to use some clues from the image, e.g. the fact that points are connected in the image by a straight line or by a curved line.

We clearly can always run at risk of not being able to recover the structure at all (the matrix of the equation system may be singular). This can happen if the sequence of images is partially non-informative, e.g. if the object does
Figure 3: 19 orthogonal projections of a freely moving four part jointed object $PQRS$. 
Table 1: Distances between projections of points $P, Q, R, S$ measured for frames 1-19 in a simulation run.

| Frame | $P'Q'$  | $Q'R'$  | $R'S'$  | $S'P'$  |
|-------|---------|---------|---------|---------|
| 1     | 1.95661 | 1.44393 | 3.13125 | 0.961803|
| 2     | 1.93014 | 2.91888 | 3.97348 | 0.956746|
| 3     | 1.77619 | 1.90047 | 2.0017  | 0.974922|
| 4     | 1.91128 | 1.42811 | 2.28367 | 0.998392|
| 5     | 1.9462  | 2.92254 | 3.98388 | 0.989842|
| 6     | 1.99945 | 2.97997 | 3.90664 | 0.884827|
| 7     | 1.81095 | 1.96477 | 3.02693 | 0.865462|
| 8     | 1.98808 | 2.58903 | 3.41279 | 0.614067|
| 9     | 1.99991 | 2.11134 | 2.9322  | 0.852155|
| 10    | 1.98413 | 2.76734 | 3.58316 | 0.929779|
| 11    | 1.80399 | 2.84376 | 3.97649 | 0.940047|
| 12    | 1.71086 | 2.55302 | 3.93766 | 0.986488|
| 13    | 1.75558 | 2.47493 | 3.85948 | 0.949584|
| 14    | 1.8613  | 1.78228 | 2.53998 | 0.99849 |
| 15    | 1.99357 | 1.47441 | 2.84904 | 0.993398|
| 16    | 1.75743 | 2.2873  | 2.61136 | 0.990972|
| 17    | 1.66138 | 1.41837 | 2.12216 | 0.930475|
| 18    | 1.99832 | 2.68278 | 3.63783 | 0.915393|
| 19    | 1.84269 | 2.09825 | 1.97617 | 0.831109|

not move from frame to frame (so we get all images identical), or if it is only shifted and not rotated, or if it is rotated only around the axis perpendicular to the projection plane, etc. Also the looped object may in fact behave like a rigid body, but in this case there exists a simple test to detect this.

Last but not least we shall ask whether looped jointed objects may be observed in the reality. Though such objects are hardly observed in nature, many man-made mechanisms possess a structure which operate like jointed looped objects (think e.g. of the coupling system for Apollo space ships).

5 Conclusion

- A method has been outlined to recover structural parameters of looped jointed objects from multiframes.

- This method has been linearized for 4-part loops. The number of necessary frames is 19.

- Extension of linearization approach to longer loops is possible in principle, but will lead to unrealistic requirements concerning the number of necessary frames.
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Appendix A

Below we sketch transformations required to 'linearize' the equation (7):
\[
\sqrt{a - A_i} \pm \sqrt{b - B_i} \pm \sqrt{c - C_i} \pm \sqrt{d - D_i} = 0. \tag{9}
\]

First we square the equation to obtain:
\[
(a - A_i) + (b - B_i) - (c - C_i) - (d - D_i) \pm 2\sqrt{(a - A_i)(b - B_i)} \pm 2\sqrt{(c - C_i)(d - D_i)} = 0. \tag{10}
\]

We square again to obtain:
\[
(a - A_i)^2 + (b - B_i)^2 + (c - C_i)^2 + (d - D_i)^2
- 2(a - A_i)(b - B_i) - 2(a - A_i)(c - C_i) - 2(a - A_i)(d - D_i) -
2(b - B_i)(c - C_i) - 2(b - B_i)(d - D_i) - 2(c - C_i)(d - D_i)
+ 8\sqrt{(a - A_i)(b - B_i)(c - C_i)(d - D_i)} = 0. \tag{11}
\]

And after the third squaring we finally eliminate square root and get:
\[
+(a - A_i)^4 + (b - B_i)^4 + (c - C_i)^4 + (d - D_i)^4
+ 6(a - A_i)^2(b - B_i)^2 + 6(a - A_i)^2(c - C_i)^2 + 6(a - A_i)^2(d - D_i)^2
+ 6(b - B_i)^2(c - C_i)^2 + 6(b - B_i)^2(d - D_i)^2 + 6(c - C_i)^2(d - D_i)^2
- 40(a - A_i)(b - B_i)(c - C_i)(d - D_i)
- 4(a - A_i)^3(b - B_i) - 4(a - A_i)^3(c - C_i) - 4(a - A_i)^3(d - D_i)
- 4(a - A_i)(b - B_i)^3 - 4(b - B_i)^3(c - C_i) - 4(b - B_i)^3(d - D_i)
- 4(a - A_i)(c - C_i)^3 - 4(b - B_i)(c - C_i)^3 - 4(c - C_i)^3(d - D_i)
- 4(a - A_i)(d - D_i)^3 - 4(b - B_i)(d - D_i)^3 - 4(c - C_i)(d - D_i)^3
+ 4(a - A_i)^2(b - B_i)(c - C_i) + 4(a - A_i)^2(b - B_i)(d - D_i)
+ 4(a - A_i)^2(c - C_i)(d - D_i)
+ 4(a - A_i)(b - B_i)^2(c - C_i) + 4(a - A_i)(b - B_i)^2(d - D_i)
+ 4(b - B_i)^2(c - C_i)(d - D_i) + 4(a - A_i)(b - B_i)(c - C_i)^2
+ 4(a - A_i)(c - C_i)^2(d - D_i) + 4(b - B_i)(c - C_i)^2(d - D_i)
+ 4(a - A_i)(b - B_i)(d - D_i)^2 + 4(a - A_i)(c - C_i)(d - D_i)^2
+ 4(b - B_i)(c - C_i)(d - D_i)^2 = 0. \tag{12}
\]

Finally, a rearrangement leads to equation (indices i dropped):
\[
(A^3 + B^3 + C^3 + D^3)
+ 6A^2B^2 + 6A^2C^2 + 6A^2D^2 + 6B^2C^2 + 6B^2D^2 + 6C^2D^2 - 40ABCDD
- 4A^3B - 4A^3C - 4A^3D - 4AB^3 - 4B^3C - 4B^3D
- 4AC^3 - 4BC^3 - 4CD^3 - 4ABD^3 - 4BD^3 - 4C^3
+ 4A^2BC + 4A^2BD + 4A^2CD + 4ABC^2 + 4AB^2D + 4B^2CD
+ 4ABC^2 + 4AC^2D + 4BC^2D + 4ABD^2 + 4ACD^2 + 4BCD^2)
+ a(-4A^3 - 12AB^2 - 12AC^2 - 12AD^2 + 40BCD + 4B^3 + 4C^3 + 4D^3
\]
Let us introduce variables:

\[ x_1 = a, \quad x_2 = b, \quad x_3 = c, \quad x_4 = d, x_5 = a^2, \quad x_6 = b^2, \quad x_7 = c^2, \quad x_8 = d^2, \]
\[ x_9 = (ab), \quad x_{10} = (ac), \quad x_{11} = (ad), x_{12} = (bc), \quad x_{13} = (bd), \quad x_{14} = (cd), \]
\[ x_{15} = (-4a^3 - 12ab^2 - 12ac^2 - 12ad^2 + 40bcd + 4b^3 + 4c^3 + 4d^3 + 12a^2b + 12a^2c + 12a^2d - 4b^2 - 4b^2d - 4c^2 - 4c^2d - 4cd^2 - 4abcd - 8abc - 8abd - 8acd) \]
\[ + B(-4a^3 - 12a^2b - 12b^2c - 12bd^2 + 40acd + 4a^3 + 4b^3 + 4c^3 + 4d^3 + 12ab^2 + 12b^2c + 12b^2d - 4a^2c - 4a^2d - 4ac^2 - 4cd^2 - 8abc - 8abd - 8acd) \]
\[ + C(-4c^3 - 12b^2c - 12cd^2 + 40abd + 4a^3 + 4b^3 + 4d^3 + 12ac^2 + 12bc^2 + 12c^2d - 4a^2b - 4a^2d - 4ab^2 - 4b^2d - 4acd^2 - 8abc - 8abd - 8acd) \]
\[ + D(-4d^3 - 12b^2d - 12c^2d + 40abc + 4a^3 + 4b^3 + 4c^3 + 12ad^2 + 12bd^2 + 12cd^2 - 4a^2b - 4a^2c - 4ab^2 - 4b^2c - 4ac^2 - 4bc^2 - 4acd - 8abc - 8abd - 8acd) \]
\[ + E(a^4 + b^4 + c^4 + d^4 + 6a^2b^2 + 6a^2c^2 + 6b^2d^2 + 6b^2c^2 + 6c^2d^2 - 40abcd + 4a^3b - 4a^3c - 4b^3d - 4ac^3 - 4bc^3 - 4ac^3d - 4ad^3 - 4cd^3 + 4a^2bc + 4a^2bd + 4a^2cd + 4ab^2c + 4ab^2d + 4b^2cd + 4abc^2 + 4ac^2d + 4acd^2 + 4bcd^2) \].

(13)
\[-4b^2 c - 4b^2 d - 4bc^2 - 4c^2 d - 4bd^2 - 4cd^2 - 8abc - 8abd - 8acd\],
\[x_{16} = (-4b^3 - 12a^2 b - 12bc^2 - 12bd^2 + 40acd + 4a^3 + 4c^3 + 4d^3 + 12ab^2 + 12b^2 c + 12b^2 d - 4a^2 c - 4a^2 d - 4ac^2 - 4c^2 d - 4ad^2 - 4bcd - 8abc - 8abd - 8acd),
\[x_{17} = (-4a^3 - 12a^2 c - 12b^2 c - 12cd^2 + 40abd + 4a^3 + 4b^3 + 4c^3 + 12ac^2 + 12bc^2 + 12c^2 d - 4a^2 b - 4a^2 d - 4ab^2 - 4b^2 d - 4ad^2 - 4b^2 d - 4bd^2 - 8abc - 8acd - 8bcd),
\[x_{18} = (-4d^3 - 12a^2 d - 12b^2 d - 12c^2 d + 40abc + 4a^3 + 4b^3 + 4c^3 + 12ad^2 + 12bd^2 + 12cd^2 + 4a^2 b - 4a^2 c - 4ab^2 - 4b^2 c - 4c^2 d - 8abc - 8abd - 8acd - 8bcd),
\[x_{19} = (a^4 + b^4 + c^4 + d^4 + 6a^2 b^2 + 6a^2 c^2 + 6a^2 d^2 + 6b^2 c^2 + 6b^2 d^2 + 6c^2 d^2 - 40abcd - 4a^3 b - 4a^3 d - 4ab^3 - 4b^3 c - 4b^3 d - 4ac^3 - 4bc^3 - 4c^3 d - 4ad^3 - 4bd^3 - 4cd^3 + 4a^2 bc + 4a^2 bd + 4a^2 cd + 4ab^2 c + 4ab^2 d + 4b^2 cd + 4abc^2 + 4abc d + 4ab^2 d + 4acd^2 + 4bcd^2)\]

(14)

and constants (factors):

\[f_{i,0} = \(A^4 + B^4 + C^4 + D^4 + 6A^2 B^2 + 6A^2 C^2 + 6A^2 D^2 + 6B^2 C^2 + 6B^2 D^2\),\]
\[+ 6C^2 D^2 - 4A^2 BCD - 4AB^2 C^2 - 4AC^2 D^2 - 4AD^2 C^2 - 4BCD^2 - 4ACD^2 - 4B^2 CD^2 - 4B^2 D^2 - 4CD^2 - 4BCD - 4BD^2 - 4CD^2),\]
\[f_{i,1} = (-4A^3 - 12AB^2 - 12AC^2 - 12AD^2 + 40BCD + 4B^3 + 4C^3 + 4D^3 + 12A^2 D + 12A^2 B - 8ABC - 8ABD - 8ACD - 4B^2 C - 4C^2 D - 4BD^2 - 4CD^2),\]
\[f_{i,2} = (-4B^3 - 12BC^2 - 12BD^2 - 12A^2 B + 40ACD + 4A^3 + 4C^3 + 4D^3 + 12B^2 D + 12B^2 C + 12AB^2 - 8ABC - 8ABD - 8BCD - 4A^2 C - 4A^2 D - 4AC^2 - 4C^2 D - 4AD^2 - 4CD^2),\]
\[f_{i,3} = (-4C^3 - 12CD^2 - 12A^2 C - 12B^2 C + 40ABD + 4A^3 + 4B^3 + 4D^3 + 12C^2 D + 12BC^2 + 12AC^2 - 8ABC - 8ACD - 8BCD - 4A^2 B - 4A^2 D - 4AB^2 - 4B^2 D - 4AD^2 - 4BD^2),\]
\[f_{i,4} = (-4D^3 - 12A^2 D - 12B^2 D - 12C^2 D + 40ABC + 4A^3 + 4B^3 + 4C^3 + 12D^2 D + 12BD^2 + 12AD^2 - 8ABD - 8ACD - 8BCD - 4A^2 B - 4A^2 C - 4AB^2 - 4B^2 C - 4ACD - 4BC^2),\]
\[f_{i,5} = (+6A^2 + 6B^2 + 6C^2 + 6D^2 - 12AB - 12AC - 12AD + 4BC + 4BD + 4CD),\]
\[f_{i,6} = (+6B^2 + 6A^2 + 6C^2 + 6D^2 - 12AB - 12BC - 12BD + 4AC + 4AD + 4CD),\]
\[f_{i,7} = (+6C^2 + 6A^2 + 6B^2 + 6D^2 - 12AC - 12BC - 12CD + 4AB + 4AD + 4BD),\]
\[f_{i,8} = (+6D^2 + 6A^2 + 6B^2 + 6C^2 - 12AD - 12BD - 12CD + 4AB + 4AC + 4BC),\]
\[f_{i,9} = (+24AB - 40CD - 12A^2 - 12B^2 + 8AC + 8AD + 8BC + 8BD + 4C^2 + 4D^2),\]
\[f_{i,10} = (+24AC - 40BD - 12A^2 - 12C^2 + 8AB + 8AD + 4B^2 + 8BC + 8CD + 4D^2),\]
\[f_{i,11} = (+24AD - 40BC - 12A^2 - 12D^2 + 8AB + 8AC + 4B^2 + 4C^2 + 8BD + 8CD),\]
\[f_{i,12} = (+24BC - 40AD - 12B^2 - 12C^2 + 4A^2 + 8AB + 8BD + 8AC + 8CD + 4D^2),\]
\[f_{i,13} = (+24BD - 40AC - 12B^2 - 12D^2 + 4A^2 + 8AB + 8BC + 4C^2 + 8AD + 8CD),\]
\[ f_{i,14} = (+24CD - 40AB - 12C^2 - 12D^2 + 4A^2 + 4B^2 + 8AC + 8BC + 8AD + 8BD), \]
\[ f_{i,15} = A, \quad f_{i,16} = B, \quad f_{i,17} = C, \quad f_{i,18} = D, f_{i,19} = 1. \] (15)

Then, if we represent constant expressions as \( f_{i,0} - f_{i,19} \) (i - the index of the frame), we can write the above eq. (13) as:

\[ f_{i,1}x_1 + f_{i,2}x_2 + f_{i,3}x_3 + f_{i,4}x_4 + f_{i,5}x_5 + f_{i,6}x_6 + f_{i,7}x_7 + f_{i,8}x_8 + f_{i,9}x_9 + f_{i,10}x_{10} + f_{i,11}x_{11} + f_{i,12}x_{12} + f_{i,13}x_{13} + f_{i,14}x_{14} + f_{i,15}x_{15} + f_{i,16}x_{16} + f_{i,17}x_{17} + f_{i,18}x_{18} + f_{i,19}x_{19} + f_{i,0} = 0 \] (16)

**Appendix B**

In Table 2 we give coefficients of equations obtained for our example.

The solution was:

\[ x_1 = 4.00415, x_2 = 8.98225, x_3 = 15.9834, x_4 = 0.999825, \]
\[ x_5 = 9.36547, x_6 = 73.8661, x_7 = 248.763, x_8 = -5.91918, x_9 = 29.1833, \]
\[ x_{10} = 57.1582, x_{11} = -2.78924, x_{12} = 136.854, x_{13} = 2.10081, x_{14} = 9.15377, \]
\[ x_{15} = -7655.73, x_{16} = -5115.85, x_{17} = 3834.1, x_{18} = 15333.2, x_{19} = -13.2689 \]

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| $f_1$ | 407.58 | -6299.13 | -365.78 | -247.129 | -6290.81 | -8257.51 | -1259.51 |
|------|--------|-----------|---------|-----------|----------|-----------|----------|
| $f_2$ | -23.3146 | -6299.13 | -365.78 | -247.129 | -6290.81 | -8257.51 | -1259.51 |
| $f_3$ | 1006.6 | -6299.13 | -365.78 | -247.129 | -6290.81 | -8257.51 | -1259.51 |
| $f_4$ | 2345.37 | -6299.13 | -365.78 | -247.129 | -6290.81 | -8257.51 | -1259.51 |
| $f_5$ | 233.006 | -6299.13 | -365.78 | -247.129 | -6290.81 | -8257.51 | -1259.51 |
| $f_6$ | 532.301 | 237.559 | -34.0638 | 144.302 | 243.085 | 138.853 | 207.413 |
| $f_7$ | -54.8179 | 237.559 | -34.0638 | 144.302 | 243.085 | 138.853 | 207.413 |
| $f_8$ | 785.302 | 237.559 | -34.0638 | 144.302 | 243.085 | 138.853 | 207.413 |
| $f_9$ | 496.262 | 237.559 | -34.0638 | 144.302 | 243.085 | 138.853 | 207.413 |
| $f_{10}$ | -156.663 | 237.559 | -34.0638 | 144.302 | 243.085 | 138.853 | 207.413 |
| $f_{11}$ | -342.622 | -3285.47 | -270.569 | -114.827 | -3286.55 | -3374.7 | -676.524 |
| $f_{12}$ | -746.125 | -3285.47 | -270.569 | -114.827 | -3286.55 | -3374.7 | -676.524 |
| $f_{13}$ | -681.854 | -3285.47 | -270.569 | -114.827 | -3286.55 | -3374.7 | -676.524 |
| $f_{14}$ | 9.80493 | 8.51988 | 3.6179 | 2.03949 | 8.54125 | 8.88019 | 3.86034 |
| $f_{15}$ | 6.70309 | 4.45775 | 7.65817 | 8.08695 | 6.51791 | 6.12529 | 3.17654 |
| $f_{16}$ | 6.70309 | 4.45775 | 7.65817 | 8.08695 | 6.51791 | 6.12529 | 3.17654 |
| $f_{17}$ | 6.70309 | 4.45775 | 7.65817 | 8.08695 | 6.51791 | 6.12529 | 3.17654 |
| $f_{18}$ | 6.70309 | 4.45775 | 7.65817 | 8.08695 | 6.51791 | 6.12529 | 3.17654 |
| $f_{19}$ | 6.70309 | 4.45775 | 7.65817 | 8.08695 | 6.51791 | 6.12529 | 3.17654 |

Table 2: Factors $f_0$ – $f_{19}$ for the frames from the example from section 3