Linear pencils and quadratic programming problems with a quadratic constraint

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Abstract

Given bounded selfadjoint operators $A$ and $B$ acting on a Hilbert space $\mathcal{H}$, consider the linear pencil $P(\lambda) = A + \lambda B$, $\lambda \in \mathbb{R}$. The set of parameters $\lambda$ such that $P(\lambda)$ is a positive (semi)definite operator is characterized. These results are applied to solving a quadratic programming problem with an equality quadratic constraint (or a QP1EQC problem).

Keywords: Linear pencil, positive definite operator, quadratic programming

2020 MSC: 15A22, 47A56, 47B50, 47B65

1. Introduction

The ideas considered in this paper are mainly motivated by the following quadratic programming problem with an equality quadratic constraint (QP1EQC):

\textbf{Problem 1.} Given a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, let $A$ and $B$ be bounded selfadjoint operators on $\mathcal{H}$. Given $(w_0, z_0) \in \mathcal{H} \times \mathcal{H}$, analyze the existence of

$$\min \langle A(x - w_0), x - w_0 \rangle \quad \text{subject to} \quad \langle B(x - z_0), x - z_0 \rangle = 0,$$

and if the minimum exists, find the set of arguments at which it is attained.

If $A$ and $B$ are indefinite (i.e. neither positive nor negative semidefinite) operators, this is a problem where the objective function $x \mapsto \langle A(x - w_0), x - w_0 \rangle$ is not convex while the function defining the constraint $x \mapsto \langle B(x - z_0), x - z_0 \rangle$ is sign indefinite. Quadratic programming (QP) with a convex objective function was shown to be polynomial-time solvable. However, QP with an indefinite

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quadratic term is NP-hard in general. There is a lot of literature on quadratically constrained quadratic programming (QCQP) problems, specially in the finite dimensional setting [41, 46, 40, 38, 39].

The standard optimization technique, known as “Lagrange multipliers”, consists in calculating the stationary points of the Fréchet derivatives in order to get candidates for the solutions. In our particular case, a necessary condition for $x_0 \in \mathcal{H}$ to be a solution to Problem 1 is the existence of $\lambda_0 \in \mathbb{R}$ such that

$$(A + \lambda_0 B)x_0 = Aw_0 + \lambda_0 Bz_0 \quad \text{and} \quad A + \lambda_0 B \text{ is positive semidefinite},$$

see e.g. [29, §7.7 Thm. 2] and [1, Prop. 2.4.19]. Hence, the linear pencil $P(\lambda) = A + \lambda B, \lambda \in \mathbb{R}$, is closely related to the above problem.

Therefore, we are interested in describing the conditions under which the sets below are non empty:

$$I_{\geq}(A, B) = \{ \rho \in \mathbb{R} : A + \rho B \text{ is positive semidefinite} \}, \quad (1.1)$$

$$I_{>}(A, B) = \{ \rho \in \mathbb{R} : A + \rho B \text{ is positive definite} \}.$$

The study of necessary and sufficient conditions for the existence of a positive (semi)definite operator in the range of $P(\lambda)$ has attracted a lot of interest in the finite-dimensional case, see [3, 6, 27, 28, 30, 34, 35, 37] among others. In particular, some of the results in Section 3 are inspired by [22].

A large number of spectral problems for operator pencils of the form $P(\lambda) = A + \lambda B$ arises in different areas of applied mathematics, where $A$ and $B$ are symmetric (or selfadjoint) operators acting in a suitable Hilbert space $\mathcal{H}$, see e.g. [32, 11, 24, 43, 36] and the references therein. Moreover, there is a well developed spectral theory for operator pencils with several applications, we only mention here the classical textbooks [4, 31, 18].

Our ultimate objective is to apply the results obtained in Section 3 to the regularization of the following indefinite least squares problem (ILSP) with an equality quadratic constraint:

**Problem 2.** Given a Hilbert space $\mathcal{H}$ and Krein spaces $(\mathcal{K}, [\cdot, \cdot]_\mathcal{K})$ and $(\mathcal{E}, [\cdot, \cdot]_\mathcal{E})$, let $T : \mathcal{H} \to \mathcal{K}$ and $V : \mathcal{H} \to \mathcal{E}$ be bounded operators. Assume that $T$ has closed range and $V$ is surjective. Given $(w_0, z_0) \in \mathcal{K} \times \mathcal{E}$, analyze the existence of

$$\min [Tx - w_0, Tx - w_0]_\mathcal{K}, \text{ subject to } [Vx - z_0, Vx - z_0]_\mathcal{E} = 0,$$

and if the minimum exists, find the set of arguments at which it is attained.

The paper is organized as follows. Section 2 contains some notations used along the paper as well as two classical results that are frequently used.

Given bounded selfadjoint operators $A$ and $B$ on $\mathcal{H}$, in Section 3 we describe the sets $I_{\geq}(A, B)$ and $I_{>}(A, B)$ defined in (1.1). If $Q(B)$ denotes the set of neutral vectors for the quadratic form induced by $B$, it is well-known that $I_{\geq}(A, B) \neq \emptyset$ if and only if

$$\langle Ax, x \rangle \geq 0 \quad \text{for every } x \in Q(B),$$
i.e. the quadratic form induced by \( A \) is nonnegative on \( Q(B) \). Moreover, \( I_\geq(A, B) \) is a closed interval \([\lambda_-, \lambda_+]\) in \( \mathbb{R} \), where the boundary values \( \lambda_- \) and \( \lambda_+ \) are given by

\[
\lambda_- := - \inf \left\{ \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} \mid x \in H, \langle Bx, x \rangle > 0 \right\} \quad \text{and} \quad \lambda_+ := - \sup \left\{ \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} \mid x \in H, \langle Bx, x \rangle < 0 \right\}.
\]

\( I_\geq(A, B) \) is trivially contained in \( I_\geq(A, B) \). If \( \lambda_- \neq \lambda_+ \) and \( I_\geq(A, B) \) is non empty we show that \( I_\geq(A, B) = (\lambda_-, \lambda_+) \), see Theorem 3.8. Also, we prove that a necessary and sufficient condition for \( I_\geq(A, B) \neq \emptyset \) is that the quadratic form induced by \( A \) is uniformly positive on \( Q(B) \), see [39].

In Section 3 we apply the results obtained in Section 3 to the Tikhonov’s regularization of Problem 2. Defining \( L : \mathcal{H} \rightarrow \mathcal{K} \times \mathcal{E} \) by \( Lx = (Tx, Vx) \), the regularized problem can be restated as calculating

\[
\min_{x \in \mathcal{H}} [Lx - (w_0, z_0), Lx - (w_0, z_0)]_\rho,
\]

where \([ \cdot, \cdot ]_\rho\) is an indefinite inner product in \( \mathcal{K} \times \mathcal{E} \) depending on the chosen regularization parameter \( \rho \). Since it is the indefinite least-squares problem (without constraints) associated to the equation \( Lx = (w_0, z_0) \), the existence of solutions to such problem can be characterized in terms of the range of \( L \), see [10, 17].

After presenting some preliminaries on Krein spaces, in Subsection 4.2 we characterize different properties of \( R(L) \) in terms of the operator \( T^\#T + \rho V^\#V \). Then, we consider the linear pencil \( P(\lambda) = T^\#T + \lambda V^\#V \) and we analyze it in its range of positiveness \([\rho_-, \rho_+]\). Finally, we show that if \( R(L) \) is a nondegenerate (resp. pseudo-regular, regular) subspace of \((\mathcal{K} \times \mathcal{E}, [\cdot, \cdot ]_{\rho_0})\) for some \( \rho_0 \in (\rho_-, \rho_+) \), then \( R(L) \) has the same property as a subspace of \((\mathcal{K} \times \mathcal{E}, [\cdot, \cdot ]_{\rho})\) for every \( \rho \in (\rho_-, \rho_+) \).

2. Preliminaries

Along this work \( \mathcal{H} \) denotes a complex (separable) Hilbert space. If \( \mathcal{K} \) is another Hilbert space then \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) is the vector space of bounded linear operators from \( \mathcal{H} \) into \( \mathcal{K} \) and \( \mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H}) \) stands for the algebra of bounded linear operators in \( \mathcal{H} \).

If \( A \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) then \( R(A) \) stands for the range of \( A \) and \( N(A) \) for its nullspace. The next well-known result about the product of closed range operators is frequently used along the paper, see e.g. [23].

**Proposition 2.1.** Given Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \) and \( \mathcal{K} \), let \( A \in \mathcal{L}(\mathcal{K}, \mathcal{H}_2) \) and \( B \in \mathcal{L}(\mathcal{H}_1, \mathcal{K}) \) have closed ranges. Then, \( AB \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) has closed range if and only if \( R(B) + N(A) \) is closed in \( \mathcal{K} \).

An operator \( A \in \mathcal{L}(\mathcal{H}) \) is positive semidefinite if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \); and it is positive definite if there exists \( \alpha > 0 \) such that \( \langle Ax, x \rangle \geq \alpha \|x\|^2 \) for every \( x \in \mathcal{H} \). The cone of positive semidefinite operators is denoted by \( \mathcal{L}(\mathcal{H})^+ \), and \( GL(\mathcal{H})^+ \) stands for the open cone of positive definite operators.
We consider the order induced by $L(H)^+$ into the real vector space of self-adjoint operators, known as Löwner’s order. If $A, B \in L(H)$ are selfadjoint operators, $A \geq B$ stands for $A - B \in L(H)^+$. In particular, $A \geq 0$ if and only if $A \in L(H)^+$. We say that a selfadjoint operator $A \in L(H)$ is indefinite if it is neither positive nor negative semidefinite, i.e. if there exist $x_+, x_- \in H$ such that $\langle Ax_+, x_+ \rangle > 0$ and $\langle Ax_-, x_- \rangle < 0$.

The following result, due to R. G. Douglas [12], characterizes operator range inclusions.

**Theorem 2.2.** Given Hilbert spaces $H, K_1, K_2$ and operators $A \in L(K_1, H)$ and $B \in L(K_2, H)$, the following conditions are equivalent:

i) the equation $AX = B$ has a solution in $L(K_2, K_1)$;

ii) $R(B) \subseteq R(A)$;

iii) there exists $\lambda > 0$ such that $BB^* \leq \lambda AA^*$.

An immediate consequence of this theorem is that $R(A) \subseteq R(A^{1/2}) \subseteq R(A)$ for $A \in L(H)^+$. Hence, $R(A)$ is closed if and only if $R(A) = R(A^{1/2})$.

**3. The range of positiveness of a linear operator pencil**

In this section we consider linear pencils of the form $P(\lambda) = A + \lambda B$, where $A$ and $B$ are bounded selfadjoint operators acting on a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, and the parameter $\lambda$ runs through $\mathbb{R}$. Inspired by [22], we are interested in describing the conditions under which the sets below are non empty:

\[ I_\geq (A, B) = \{ \rho \in \mathbb{R} : A + \rho B \in L(H)^+ \}, \]
\[ I_> (A, B) = \{ \rho \in \mathbb{R} : A + \rho B \in GL(H)^+ \}. \]

This problem has a long history, which is thoroughly reviewed in [44]. For matrix pencils it can be traced back to [14, 5], and in the case of operator pencils in Hilbert spaces to [8, 25, 24].

Recall the following theorem, taken from [3], which is a version of [24, Thm. 1.1]. Denote by $Q(B)$ the set of neutral vectors for the quadratic form induced by the selfadjoint operator $B$:

\[ Q(B) = \{ x \in H : \langle Bx, x \rangle = 0 \}. \]

**Theorem 3.1.** Let $A, B \in L(H)$ be selfadjoint operators and suppose that $B$ is indefinite. Also, assume that

\[ \langle Ax, x \rangle \geq 0 \quad \text{for every } x \in Q(B). \quad (3.1) \]

Then, for every $y \in H$ such that $\langle By, y \rangle < 0$ and for every $z \in H$ such that $\langle Bz, z \rangle > 0$ we have that

\[ \frac{\langle Ay, y \rangle}{\langle By, y \rangle} \leq \frac{\langle Az, z \rangle}{\langle Bz, z \rangle}. \]
Hence,

\[ \mu_+ = \inf_{\{z \in H : \langle Bz, z \rangle = 1\}} \langle Az, z \rangle > -\infty \quad \text{and} \]
\[ \mu_- = \sup_{\{y \in H : \langle By, y \rangle = -1\}} \langle Ax, x \rangle < +\infty. \]  

(3.2)

Moreover, \( \mu_- \leq \mu_+ \) and for any \( \mu \in [\mu_- , \mu_+] \) the following inequality holds:

\[ \langle Ax, x \rangle \geq \mu \langle Bx, x \rangle \quad \text{for every } x \in H. \]

This implies that (3.1) is a sufficient condition to guarantee that \( I_{\geq}(A, B) \neq \emptyset \), but it is also a necessary one. The next result is a generalization of [22, Thm.2], see also [20, 33].

**Proposition 3.2.** Let \( A, B \in \mathcal{L}(H) \) be selfadjoint operators and suppose that \( B \) is indefinite. Then, \( I_{\geq}(A, B) \neq \emptyset \) if and only if

\[ \langle Ax, x \rangle \geq 0 \quad \text{for every } x \in Q(B). \]

In this case, there exist \( \lambda_- , \lambda_+ \in \mathbb{R} \) such that \( \lambda_- \leq \lambda_+ \) and \( I_{\geq}(A, B) = [\lambda_- , \lambda_+] \).

**Proof.** Assume that \( I_{\geq}(A, B) \neq \emptyset \) and consider \( \rho \in I_{\geq}(A, B) \). Given \( x \in H \) such that \( \langle Bx, x \rangle = 0 \), we have

\[ 0 \leq \langle (A + \rho B)x, x \rangle = \langle Ax, x \rangle + \rho \langle Bx, x \rangle = \langle Ax, x \rangle. \]

Therefore, (3.1) holds. The converse follows from Theorem 3.1 as well as the description of \( I_{\geq}(A, B) \) as an interval. In fact, as a consequence of (3.2), the constants \( \lambda_- \) and \( \lambda_+ \) are given by

\[ \lambda_- := -\mu_+ \quad \text{and} \quad \lambda_+ := -\mu_- . \]

The first assertion in [22, Thm.3] holds true for (bounded) selfadjoint operators acting on a Hilbert space \( (H, \langle \cdot , \cdot \rangle) \), i.e. if \( I_{\geq}(A, B) = \{\lambda_0\} \) then \( B \) is indefinite and \( I_{\geq}(A, B) = \emptyset \). The proof is exactly the same presented there. The last part of [22, Thm.3] can be generalized as follows.

**Proposition 3.3.** Let \( A, B \in \mathcal{L}(H) \) be selfadjoint operators and assume that \( I_{\geq}(A, B) = [\lambda_- , \lambda_+] \) with \( \lambda_- < \lambda_+ \). Then,

i) \( N(A + \lambda B) = Q(A) \cap Q(B) = N(A) \cap N(B) \) for every \( \lambda \in (\lambda_- , \lambda_+) \).

ii) Assume \( I_{\geq}(A, B) \neq \emptyset \). Given \( \lambda \in I_{\geq}(A, B) \), there exist \( S \in GL(H)^+ \), a compact subset \( K \) of \( \mathbb{R} \) and a spectral measure \( \mu \) defined on \( K \) such that

\[ SBS = \int_K t \, d\mu(t) \quad \text{and} \quad SAS = \int_K (1 - \lambda t) \, d\mu(t). \]  

(3.3)
Proof. Assume that \( \lambda_- \neq \lambda_+ \) and consider \( \lambda \in (\lambda_-, \lambda_+) \). Since \( A + \lambda B \in \mathcal{L}(\mathcal{H})^+ \), it is easy to see that
\[
Q(A) \cap Q(B) = N(A + \lambda B) \cap Q(B).
\]

Now, assume that there exists \( x_0 \in N(A + \lambda B) \setminus (Q(A) \cap Q(B)) = N(A + \lambda B) \setminus Q(B) \). Since \( \langle Ax_0, x_0 \rangle + \lambda \langle Bx_0, x_0 \rangle = 0 \) and \( \lambda_- < \lambda < \lambda_+ \),
\[
\sup_{\{x \in \mathcal{H} : \langle Bx, x \rangle < 0\}} \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} = -\lambda_+ < \frac{\langle Ax_0, x_0 \rangle}{\langle Bx_0, x_0 \rangle} < -\lambda_- = \inf_{\{x \in \mathcal{H} : \langle Bx, x \rangle > 0\}} \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle}.
\]

Since \( \langle Bx_0, x_0 \rangle > 0 \) or \( \langle Bx_0, x_0 \rangle < 0 \), (3.3) leads to a contradiction. Therefore,
\[
N(A + \lambda B) = Q(A) \cap Q(B).
\]

In particular, \( N(A + \lambda B) \) does not depend on \( \lambda \). Hence, if \( x \in Q(A) \cap Q(B) \) then \( Ax = -\lambda'Bx \) for any \( \lambda' \in (\lambda_-, \lambda_+) \). This implies that if \( x \in N(A + \lambda B) \) then \( x \in N(A) \cap N(B) \), completing the proof of item i.

To prove item ii, consider a fixed \( \lambda \in I_>(A, B) \) and denote \( S := (A + \lambda B)^{-1/2} \in GL(\mathcal{H})^+ \). Then,
\[
SAS = I - \lambda SBS.
\]
Hence, if \( K = \sigma(SBS) \) and \( \mu \) is the spectral measure of \( SBS \), then (3.3) holds.

It can also be proven following the same lines as in the proof of [22, Thm. 3]. Item ii in the above proposition can be read as a “simultaneous diagonalization via congruence” for selfadjoint operators acting on a Hilbert space.

Our aim is to show that if \( I>(A, B) = [\lambda_-, \lambda_+] \) and \( \lambda_- < \lambda_+ \), then \( I>(A, B) = (\lambda_-, \lambda_+) \). First, we need to prove some technical preliminaries. In the first place, for \( \lambda \in [\lambda_-, \lambda_+] \) consider the seminorm \( \| \cdot \|_\lambda \) defined by
\[
\| x \|_\lambda = ((A + \lambda B)x, x)^{1/2}, \quad x \in \mathcal{H}.
\]

By Proposition 3.3 if \( \lambda \in (\lambda_-, \lambda_+) \) then \( \| \cdot \|_\lambda \) is a norm in the Hilbert space \( \mathcal{H}' := (N(A) \cap N(B))^\perp \).

Lemma 3.4. If \( \lambda_- \neq \lambda_+ \) then \( \| \cdot \|_\lambda \) and \( \| \cdot \|_{\lambda'} \) are equivalent on \( \mathcal{H}' \) for every pair \( \lambda, \lambda' \in (\lambda_-, \lambda_+) \).

Proof. Given \( \lambda, \lambda' \in (\lambda_-, \lambda_+) \) assume that \( \lambda' > \lambda \). For an arbitrary \( x \in \mathcal{H}' \), \( x \neq 0 \),
\[
\| x \|_{\lambda'}^2 = \langle Ax, x \rangle + \lambda' \langle Bx, x \rangle = (\langle Ax, x \rangle + \lambda \langle Bx, x \rangle) + (\lambda' - \lambda) \langle Bx, x \rangle
\]
\[
= \| x \|_\lambda^2 + (\lambda' - \lambda) \langle Bx, x \rangle,
\]
and consequently,

\[
\frac{\|x\|_2^2}{\|x\|_\lambda^2} = 1 + (\lambda' - \lambda) \frac{\langle Bx, x \rangle}{\|x\|_\lambda^2} = 1 + (\lambda' - \lambda) \frac{\langle Bx, x \rangle}{\langle Ax, x \rangle + \lambda \langle Bx, x \rangle}.
\] (3.5)

Now we show that

\[
- \frac{1}{\lambda_+ - \lambda} \leq \frac{\langle Bx, x \rangle}{\langle Ax, x \rangle + \lambda \langle Bx, x \rangle} \leq \frac{1}{\lambda - \lambda_-}.
\] (3.6)

If \( \langle Bx, x \rangle = 0 \), (3.6) is trivially satisfied. If \( \langle Bx, x \rangle > 0 \), then

\[
\frac{\langle Bx, x \rangle}{\langle Ax, x \rangle + \lambda \langle Bx, x \rangle} \leq \sup_{y \in \mathcal{H} : \langle By, y \rangle > 0} \frac{\langle By, y \rangle}{\langle Ay, y \rangle + \lambda \langle By, y \rangle} = \frac{1}{\lambda - \lambda_-}.
\]

Furthermore, since \( \langle Ax, x \rangle + \lambda \langle Bx, x \rangle > 0 \) and \( \langle Bx, x \rangle > 0 \),

\[
- \frac{1}{\lambda_+ - \lambda} < \frac{\langle Bx, x \rangle}{\langle Ax, x \rangle + \lambda \langle Bx, x \rangle}.
\]

A similar argument can be given if \( \langle Bx, x \rangle < 0 \).

As a consequence of (3.6) we have that

\[
\frac{\lambda_+ - \lambda'}{\lambda_+ - \lambda} \leq 1 + (\lambda' - \lambda) \frac{\langle Bx, x \rangle}{\langle Ax, x \rangle + \lambda \langle Bx, x \rangle} \leq \frac{\lambda' - \lambda_-}{\lambda - \lambda_-}.
\] (3.7)

By (3.5) and (3.7),

\[
\left( \frac{\lambda_+ - \lambda'}{\lambda_+ - \lambda} \right)^{1/2} \leq \frac{\|x\|_{\lambda'}_{\lambda}}{\|x\|_{\lambda}} \leq \left( \frac{\lambda' - \lambda_-}{\lambda - \lambda_-} \right)^{1/2},
\]

and consequently, the norms \( \| \cdot \|_{\lambda'} \) and \( \| \cdot \|_{\lambda} \) are equivalent. 

\[\square\]

**Remark 3.5.** With the same procedure used in the proof of Lemma 3.4, it can be shown that for every \( \lambda \in (\lambda_-, \lambda_+) \) and every \( x \in \mathcal{H}' \)

\[
\|x\|_{\lambda_-} \leq \left( \frac{\lambda_+ - \lambda_-}{\lambda_+ - \lambda} \right)^{1/2} \|x\|_{\lambda} \quad \text{and} \quad \|x\|_{\lambda_+} \leq \left( \frac{\lambda_+ - \lambda_-}{\lambda - \lambda_-} \right)^{1/2} \|x\|_{\lambda}.
\] (3.8)

**Corollary 3.6.** Let \( A, B \in \mathcal{L}(\mathcal{H}) \) be selfadjoint operators and suppose that \( B \) is indefinite. Assume \( I_{\geq}(A, B) = [\lambda_-, \lambda_+] \) with \( \lambda_- < \lambda_+ \). Then,

i) \( R((A + \lambda B)^{1/2}) = R((A + \lambda' B)^{1/2}) \) for every \( \lambda, \lambda' \in (\lambda_-, \lambda_+) \);

ii) \( R((A + \lambda \pm B)^{1/2}) \subseteq R((A + \lambda B)^{1/2}) \) for every \( \lambda \in (\lambda_-, \lambda_+) \).
Proof. By Proposition 3.3, we have that \( N(A + \lambda B)^\perp = \mathcal{H}' \) for every \( \lambda \in (\lambda_-, \lambda_+) \). Then, given \( \lambda, \lambda' \in (\lambda_-, \lambda_+) \), \( \lambda' \neq \lambda \), the norm equivalence in Lemma 3.3 can be rephrased as: there exist \( 0 < \alpha < \beta \) such that

\[
\alpha(A + \lambda' B) \leq A + \lambda B \leq \beta(A + \lambda B).
\]

Applying Theorem 2.2

\[
R((A + \lambda B)^{1/2}) = R((A + \lambda B)^{1/2}).
\]

Analogously, the inequalities in (3.8) imply there exist \( \alpha_\pm > 0 \) such that

\[
A + \lambda_\pm B \leq \alpha_\pm (A + \lambda B).
\]

Hence,

\[
R((A + \lambda_\pm B)^{1/2}) \subseteq R((A + \lambda B)^{1/2}).
\]

Corollary 3.7. Let \( A, B \in \mathcal{L}(\mathcal{H}) \) be selfadjoint operators and suppose that \( B \) is indefinite. Assume \( I_\geq(A, B) = [\lambda_-, \lambda_+] \) with \( \lambda_- < \lambda_+ \). If \( A + \lambda_0 B \) has closed range for some \( \lambda_0 \in (\lambda_-, \lambda_+) \), then \( A + \lambda B \) has closed range for every \( \lambda \in (\lambda_-, \lambda_+) \).

The main result of this section shows that, if \( I_\geq(A, B) \) is not empty then \( I_\geq(A, B) \) coincides with the interior of \( I_\geq(A, B) \).

Theorem 3.8. Let \( A, B \in \mathcal{L}(\mathcal{H}) \) be selfadjoint operators and suppose that \( B \) is indefinite. Assume \( I_\geq(A, B) = [\lambda_-, \lambda_+] \) with \( \lambda_- \leq \lambda_+ \). Then, \( I_\geq(A, B) \neq \emptyset \) if and only if there exists \( \alpha > 0 \) such that

\[
\langle Ax, x \rangle \geq \alpha \|x\|^2 \quad \text{for every } x \in \mathcal{H} \text{ such that } \langle Bx, x \rangle = 0. \quad (3.9)
\]

In this case, \( \lambda_- < \lambda_+ \) and \( I_\geq(A, B) = (\lambda_-, \lambda_+) \).

Proof. Assume that \( I_\geq(A, B) \neq \emptyset \) and consider \( \rho \in I_\geq(A, B) \). Then, \( A + \rho B \) is injective and, by Proposition 3.3, \( N(A + \lambda B) = \{0\} \) for every \( \lambda \in (\lambda_-, \lambda_+) \). Also, \( A + \rho B \) has closed range and, by Proposition 3.7, \( R(A + \lambda B) \) is closed for every \( \lambda \in (\lambda_-, \lambda_+) \). Therefore, \( (\lambda_, \lambda_+) \subseteq I_\geq(A, B) \).

Also, since \( A + \rho B \) is invertible, there exists \( \alpha > 0 \) such that \( \langle (A + \rho B)x, x \rangle \geq \alpha \|x\|^2 \) for every \( x \in \mathcal{H} \). Hence, given \( x \in \mathcal{H} \) such that \( \langle Bx, x \rangle = 0 \) we have

\[
\alpha \|x\|^2 \leq \langle (A + \rho B)x, x \rangle = \langle Ax, x \rangle,
\]

which proves (3.9).

Now, let us show that \( \lambda_\pm \notin I_\geq(A, B) \). If \( \lambda_\pm \in I_\geq(A, B) \), then there exists \( \alpha_\pm > 0 \) such that \( \langle (A + \lambda_\pm B)x, x \rangle \geq \alpha_\pm \|x\|^2 \) for every \( x \in \mathcal{H} \). Since \( |\langle Bx, x \rangle| \leq \|B\| \|x\|^2 \) for all \( x \in \mathcal{H} \), we have that

\[
\left\langle \left( A + (\lambda_\pm \pm \frac{\alpha_\pm}{\|B\|})B \right)x, x \right\rangle = \langle (A + \lambda_\pm B)x, x \rangle \pm \frac{\alpha_\pm}{\|B\|} \langle Bx, x \rangle \geq \langle (A + \lambda_\pm B)x, x \rangle - \alpha_\pm \|x\|^2 \geq 0,
\]

which proves Theorem 3.8.
for every $x \in \mathcal{H}$, i.e. $\lambda_\pm + \frac{\rho}{\|\cdot\|_B} \in I_\geq (A, B)$, which is a contradiction. Thus, $I_\geq (A, B) = (\lambda_-, \lambda_+)$.

Conversely, assume that (3.9) holds for a given $\alpha > 0$. Then, the selfadjoint operators $A' := A - \alpha I$ and $B$ satisfy (3.11). Thus, there exist $\eta_-, \eta_+ \in \mathbb{R}$ such that $\{\lambda \in \mathbb{R} : A + \lambda B \geq \alpha I\} = I_\geq (A', B) = [\eta_-, \eta_+]$. If $\rho \in [\eta_-, \eta_+]$ then $A + \rho B$ is a positive definite operator, i.e. $\rho \in I_\geq (A, B)$.

To conclude this section, we characterize the nullspaces of the operators associated to the extreme values of $I_\geq (A, B)$, i.e. $N(A + \lambda_- B)$ and $N(A + \lambda_+ B)$. To do so, we introduce the following sets:

$$\mathcal{M}_\pm := \left\{ x \in \mathcal{H} : \langle Bx, x \rangle \neq 0 \text{ and } \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} = -\lambda_\pm \right\}. \tag{3.10}$$

**Remark 3.9.** If $\lambda_- < \lambda_+$ then

$$\mathcal{M}_+ = \left\{ x \in \mathcal{H} : \langle Bx, x \rangle > 0 \text{ and } \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} = -\lambda_- \right\},$$

and

$$\mathcal{M}_- = \left\{ x \in \mathcal{H} : \langle Bx, x \rangle < 0 \text{ and } \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} = -\lambda_+ \right\}.$$ 

Indeed, assume that there exists $x_0 \in \mathcal{H}$ such that $\langle Bx_0, x_0 \rangle < 0$ and $x_0 \in M_+$. Then,

$$-\lambda_+ = \sup_{\{x \in \mathcal{H} : \langle Bx, x \rangle < 0\}} \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} \geq \frac{\langle Ax_0, x_0 \rangle}{\langle Bx_0, x_0 \rangle} = -\lambda_-,$$

which is a contradiction to $\lambda_- < \lambda_+$. A similar argument holds to show the alternative description of $\mathcal{M}_-$.

**Lemma 3.10.** Let $A, B \in \mathcal{L}(\mathcal{H})$ be selfadjoint operators and suppose that $B$ is indefinite.

i) if $\lambda_- \neq \lambda_+$ then $N(A + \lambda_\pm B) = \mathcal{M}_\mp \cup (N(A) \cap N(B))$.

ii) If $\lambda_- = \lambda_+$ then $\mathcal{M}_- = \mathcal{M}_+$ and $N(A + \lambda_\pm B) = \mathcal{M}_+ \cup (Q(A) \cap Q(B))$.

**Proof.** First assume that $\lambda_- \neq \lambda_+$, and consider the left boundary $\lambda_-$ (similar considerations hold for the right one, $\lambda_+$. Since $A + \lambda_- B \in \mathcal{L}(\mathcal{H})^+$, we have that $x \in N(A + \lambda_- B)$ if and only if $\langle Ax, x \rangle = -\lambda_- \langle Bx, x \rangle$.

For $x \in \mathcal{M}_+$ or $x \in N(A) \cap N(B)$, $\langle Ax, x \rangle = -\lambda_- \langle Bx, x \rangle$ holds, thus the inclusion $\mathcal{M}_+ \cup (N(A) \cap N(B)) \subseteq N(A + \lambda_- B)$ is immediate.

Now, let $x \in N(A + \lambda_- B)$. On the one hand, if $\langle Bx, x \rangle = 0$ then $\langle Ax, x \rangle = -\lambda_- \langle Bx, x \rangle = 0$ and, by Lemma 3.3 $x \in Q(A) \cap Q(B) = N(A) \cap N(B)$. On the other hand, if $\langle Bx, x \rangle \neq 0$ then

$$\frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} = -\lambda_-.$$
so that \( x \in \mathcal{M}_+ \).

Using the same procedure as above, if we assume that \( \lambda_- = \lambda_+ \) the assertion follows immediately. \( \square \)

If \( \mathcal{H} \) is finite dimensional then \( \mathcal{M}_+ \) and \( \mathcal{M}_- \) are trivially non empty. However, the next example shows that the set \( \mathcal{M}_+ \) may be empty in the infinite dimensional setting. A similar example can be constructed to show that \( \mathcal{M}_- \) may also be empty.

**Example 1.** Let us consider \( \mathcal{H} = \ell^2(\mathbb{N}) \) endowed with the usual inner product:

\[
\langle (x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{N}} \rangle = \sum_{n \in \mathbb{N}} x_n y_n, \quad (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in \mathcal{H}.
\]

If \( A, B \in \mathcal{L}(\mathcal{H}) \) are given by

\[
A(x_n)_{n \in \mathbb{N}} = (y_n)_{n \in \mathbb{N}}, \quad \text{with} \quad y_1 = x_1, \quad \text{and} \quad y_n = -(1 - \frac{1}{n}) x_n \text{ if } n \neq 1,
\]

\[
B(x_n)_{n \in \mathbb{N}} = (z_n)_{n \in \mathbb{N}}, \quad \text{with} \quad z_1 = -\frac{1}{2} x_1, \quad \text{and} \quad z_n = x_n \text{ if } n \neq 1,
\]

then both \( A \) and \( B \) are selfadjoint indefinite operators. Given \( \lambda \in \mathbb{R} \),

\[
\langle (A + \lambda B)(x_n)_{n \in \mathbb{N}}, (x_n)_{n \in \mathbb{N}} \rangle = (1 - \frac{1}{2} \lambda) |x_1|^2 + \sum_{n \neq 1} (\lambda - (1 - \frac{1}{n})) |x_n|^2.
\]

Hence, \( A + \lambda B \in \mathcal{L}(\mathcal{H})^+ \) if and only if \( 1 - \frac{1}{2} \lambda \geq 0 \) and \( \lambda - (1 - \frac{1}{n}) \geq 0 \) for every \( n \in \mathbb{N} \setminus \{1\} \). Therefore,

\[
\lambda_- = 1 \quad \text{and} \quad \lambda_+ = 2.
\]

Observe that \( \mathcal{N}(A + \lambda_- B) = \{0\} \) because

\[
\langle (A + \lambda_- B)(x_n)_{n \in \mathbb{N}}, (x_n)_{n \in \mathbb{N}} \rangle = \frac{1}{2}|x_1|^2 + \sum_{n \neq 1} \frac{1}{n}|x_n|^2,
\]

and it is zero if and only if \( x_n = 0 \) for every \( n \in \mathbb{N} \). Thus, \( \mathcal{M}_+ = \emptyset \).

Assume that \( I_>(A, B) \neq \emptyset \) and let \( \rho \in I_>(A, B) \). Considering the selfadjoint operator \( G \in \mathcal{L}(\mathcal{H}) \) given by

\[
G := (A + \rho B)^{-1/2} B (A + \rho B)^{-1/2},
\]

the pencil \( P(\lambda) = A + \lambda B \) is then congruent to \( I + (\lambda - \rho)G \). Indeed,

\[
A + \lambda B = (A + \rho B)^{1/2} (I + (\lambda - \rho)G) (A + \rho B)^{-1/2}.
\]

This reduction is frequently used in the operator pencils context, see e.g. [18, Chapter IV] and [21]. We end this section by characterizing the range of positiveness of operator pencils of such form, when \( G \) is an indefinite operator.
Given an indefinite selfadjoint operator \( G \in \mathcal{L}(\mathcal{H}) \), consider its canonical decomposition as the difference of two positive operators, i.e. consider the decomposition

\[
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus N(G),
\]

(3.11)

and let \( G_\pm \in \mathcal{L}(\mathcal{H}_\pm) \) be such that \( G = \begin{pmatrix} G_+ & 0 \\ 0 & -G_- \end{pmatrix} \) with respect to (3.11).

The following statement result describes the range of positiveness of the operator pencil \( I + \eta G \) in terms of the the norms of the operators \( G_+ \) and \( G_- \).

**Proposition 3.11.** Given a selfadjoint operator \( G \in \mathcal{L}(\mathcal{H}) \), if \( G \) is indefinite then

\[
I \geq (I, G) = [\lambda_-, \lambda_+],
\]

(3.11)

and

\[
I > (I, G) = (\lambda_-, \lambda_+).
\]

Proof. By Proposition 3.2 and Theorem 3.8, \( I \geq (I, G) = \lambda_- \), \( \lambda_+ \) with \( \lambda_- < \lambda_+ \) and \( I_>(I, G) = \lambda_+ \). Given \( x \in \mathcal{H} \), assume that it is decomposed as \( x = x^+ + x^- + x^0 \), where \( x^ \pm \in \mathcal{H}_\pm \) and \( x^0 \in N(G) \). Also, assume that \( x^+ \neq 0 \).

Then,

\[
\frac{\langle Gx, x \rangle}{\|x\|^2} = \frac{\langle G_+x^+, x^+ \rangle - \langle G_-x^-, x^- \rangle}{\|x^+\|^2 + \|x^-\|^2 + \|x^0\|^2} \leq \frac{\langle G_+x^+, x^+ \rangle}{\|x^+\|^2} = \frac{\langle Gx^+, x^+ \rangle}{\|x^+\|^2},
\]

(3.12)

Since \( \mathcal{H}_+ \setminus \{0\} \subseteq \{ x \in \mathcal{H} : \langle Gx, x \rangle > 0 \} \), (3.12) implies that

\[
\sup_{x \in \mathcal{H}_+: \langle Gx, x \rangle > 0} \frac{\langle Gx, x \rangle}{\|x\|^2} = \sup_{x \in \mathcal{H}_+ \setminus \{0\}} \frac{\langle Gx, x \rangle}{\|x\|^2} = \|G_+\| < +\infty,
\]

and consequently,

\[
\lambda_- = \inf_{x \in \mathcal{H}_+: \langle Gx, x \rangle > 0} \frac{\langle x, x \rangle}{\|Gx, x\|} = \left( \sup_{x \in \mathcal{H}_+: \langle Gx, x \rangle > 0} \frac{\langle Gx, x \rangle}{\|x\|^2} \right)^{-1} = -\|G_+\|^{-1}.
\]

A similar procedure with vectors in \( \mathcal{H}_- \) proves that \( \lambda_+ = \|G_-\|^{-1} \).

4. On the regularization of a QP1EQC

Along this section we apply the results obtained before to the regularization of the indefinite least-squares problem with a quadratic constraint presented as Problem 2. Hereafter, given a Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \), and Krein spaces \( (\mathcal{K}, [\cdot, \cdot]_K) \) and \( (\mathcal{E}, [\cdot, \cdot]_E) \), let \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) and \( V \in \mathcal{L}(\mathcal{H}, \mathcal{E}) \) be closed range operators.

Given a real constant \( \rho \neq 0 \), the regularization of Problem 2 consists in minimizing the functional

\[
x \mapsto [Tx - w_0, Tx - w_0]_K + \rho [Vx - z_0, Vx - z_0]_E.
\]
Defining the following indefinite inner product on $\mathcal{K} \times \mathcal{E}$:

$$[\ (y, z), (y', z')\ ]_\rho = [y, y']_\mathcal{K} + \rho [z, z']_\mathcal{E}, \quad y, y' \in \mathcal{K} \text{ and } z, z' \in \mathcal{E}, \quad (4.1)$$

it is easy to see that $(\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_\rho)$ is a Krein space. Also, if $L \in \mathcal{L}(\mathcal{H}, \mathcal{K} \times \mathcal{E})$ is defined by

$$Lx = (Tx, Vx), \quad x \in \mathcal{H}, \quad (4.2)$$

the regularized problem can be restated as calculating

$$\min_{x \in \mathcal{H}} [Lx - (w_0, z_0), Lx - (w_0, z_0)]_\rho, \quad (4.3)$$

which is the indefinite least-squares problem (without constraints) associated to the equation $Lx = (w_0, z_0)$. The existence of solutions to such problem was characterized in [16, 17] in terms of different properties of the range of $L$.

We now present some preliminaries on Krein spaces.

4.1. Preliminaries on Krein spaces

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject and the proofs of the results below see [2, 3, 7, 13, 42].

An indefinite inner product space $(\mathcal{F}, [\cdot, \cdot])$ is a (complex) vector space $\mathcal{F}$ endowed with a Hermitian sesquilinear form $[\cdot, \cdot] : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$.

A vector $x \in \mathcal{F}$ is positive, negative, or neutral if $[x, x] > 0$, $[x, x] < 0$, or $[x, x] = 0$, respectively. The set of positive vectors in $\mathcal{F}$ is denoted by $\mathcal{P}^{++}(\mathcal{F})$, and the set of nonnegative vectors in $\mathcal{F}$ by $\mathcal{P}^+(\mathcal{F})$. The sets of negative, nonpositive and neutral vectors in $\mathcal{F}$ are defined analogously, and they are denoted by $\mathcal{P}^{-+}(\mathcal{F})$, $\mathcal{P}^-(\mathcal{F})$, and $\mathcal{P}^0(\mathcal{F})$, respectively.

Likewise, a subspace $\mathcal{M}$ of $\mathcal{F}$ is positive if every $x \in \mathcal{M}$, $x \neq 0$ is a positive vector in $\mathcal{F}$; and it is nonnegative if $[x, x] \geq 0$ for every $x \in \mathcal{M}$. Negative, nonpositive and neutral subspaces are defined mutatis mutandis.

If $\mathcal{S}$ is a subset of an indefinite inner product space $\mathcal{F}$, the orthogonal companion to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{ x \in \mathcal{F} : [x, s] = 0 \text{ for every } s \in \mathcal{S} \}.$$ 

It is easy to see that $\mathcal{S}^{[\perp]}$ is always a subspace of $\mathcal{F}$.

**Definition.** An indefinite inner product space $(\mathcal{H}, [\cdot, \cdot])$ is a *Krein space* if it can be decomposed as a direct (orthogonal) sum of a Hilbert space and an anti Hilbert space, i.e. there exist subspaces $\mathcal{H}_{\pm}$ of $\mathcal{H}$ such that $(\mathcal{H}_+, [\cdot, \cdot])$ and $(\mathcal{H}_-, [\cdot, \cdot])$ are Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-, \quad (4.4)$$

and $\mathcal{H}_+$ is orthogonal to $\mathcal{H}_-$ with respect to the indefinite inner product. Sometimes we use the notation $[\cdot, \cdot]_\mathcal{H}$ instead of $[\cdot, \cdot]$ to emphasize the Krein space considered.
A pair of subspaces $H_\pm$ as in (4.4) is called a fundamental decomposition of $H$. Given a Krein space $H$ and a fundamental decomposition $H = H_+ \oplus H_-$, the direct (orthogonal) sum of the Hilbert spaces $(H_+[,\cdot,\cdot])$ and $(H_-,\langle\cdot,\cdot\rangle)$ is denoted by $(H,\langle\cdot,\cdot\rangle)$.

If $H = H_+ \oplus H_-$ and $H = H'_+ \oplus H'_-$ are two different fundamental decompositions of $H$, the corresponding associated inner products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ turn out to be equivalent on $H$. Therefore, the norm topology on $H$ does not depend on the chosen fundamental decomposition.

If $(H,\langle\cdot,\cdot\rangle_H)$ and $(K,\langle\cdot,\cdot\rangle_K)$ are Krein spaces, $L(H,K)$ stands for the vector space of linear transformations which are bounded with respect to any of the associated Hilbert spaces $(H,\langle\cdot,\cdot\rangle_H)$ and $(K,\langle\cdot,\cdot\rangle_K)$. Given $T \in L(H,K)$, the adjoint operator of $T$ (in the Krein spaces sense) is the unique operator $T^\# \in L(K,H)$ such that

$$\langle Tx, y \rangle_K = \langle x, T^\# y \rangle_H, \quad x \in H, y \in K. \quad (4.5)$$

We frequently use that if $T \in L(H,K)$ and $M$ is a closed subspace of $K$ then

$$T^\# (M^{|\dag}\rangle) = T^{-1}(M|\rangle_K). \quad (4.6)$$

Given a subspace $M$ of a Krein space $H$, the isotropic part of $M$ is defined by $M^\circ := M \cap M^{[\dag]}$. Then, $M$ is nondegenerate if $M^\circ = \{0\}$.

A subspace $M$ of a Krein space $H$ is pseudo-regular if $M + M^{[\dag]}$ is a closed subspace of $H$, and it is regular if $M + M^{[\dag]} = H$. Regular subspaces are examples of nondegenerate subspaces, but pseudo-regular subspaces can be degenerate ones. However, if $M$ is a pseudo-regular subspace then there exists a regular subspace $R$ such that

$$M = M^\circ[M] \oplus R, \quad (4.6)$$

where $[\dag]$ stands for the direct orthogonal sum with respect to the indefinite inner product $[\cdot,\cdot]$. In fact, any closed subspace $R$ such that $M = M^\circ + R$ satisfies (4.6) and turns out to be a regular subspace of $H$, see e.g. [15]. Note that a subspace $M$ is regular if and only if it is pseudo-regular and nondegenerate.

The following propositions can be found in [17, Lemma 3.4] and [3, Chapter 1, §7], respectively.

**Proposition 4.1.** Given Krein spaces $H$ and $K$, let $T \in L(H,K)$ with closed range. Then, $R(T)$ is pseudo-regular if and only if $R(T^\# T)$ is closed.

A subspace $M$ of a Krein space $(H,\langle\cdot,\cdot\rangle)$ is uniformly positive if there exists $\alpha > 0$ such that

$$[x,x] \geq \alpha \|x\|^2 \quad \text{for every } x \in M,$$

where $\|\cdot\|$ is the norm of any associated Hilbert space. Uniformly negative subspaces are defined mutatis mutandis.

**Proposition 4.2.** Let $M$ be a subspace of a Krein space $H$. Then, $M$ is closed and uniformly positive (resp. negative) if and only if $M$ is regular and nonnegative (resp. nonpositive).
4.2. The operator pencil associated to the QP1EQC

As we mentioned before, the regularization of Problem 2 is the indefinite least-squares problem (without constraints) determined by the indefinite inner product $\langle \cdot, \cdot \rangle_\rho$ in $K \times E$ and the operator $L$, see (4.1), (4.2), and (4.3). In particular, if $R(L)$ is a (closed) nonnegative pseudo-regular subspace of $K \times E$ and $(w_0, z_0) \in R(L) + R(L)^{\perp}$ the vectors attaining (4.3) are exactly the solutions of the normal equation:

$$L^\# (Lx - (w_0, z_0)) = 0,$$

where $L^\#$ is the adjoint of $L$ with respect to the indefinite inner product $\langle \cdot, \cdot \rangle_\rho$, see [17, Theorem 3.5].

The adjoint operator of $L$ is given by

$$L^\#(y, z) = T^\# y + \rho V^\# z, \quad (y, z) \in K \times E.$$ 

Indeed, if $x \in H$ and $(y, z) \in K \times E$, we have that

$$[Lx, (y, z)]_\rho = [Tx, y]_K + \rho [Vx, z]_E = \langle x, T^\# y \rangle + \rho \langle x, V^\# z \rangle$$

Moreover, it is easy to see that

$$L^\# L = T^\# T + \rho V^\# V,$$

and the normal equation turns into

$$(T^\# T + \rho V^\# V) x = T^\# w_0 + \rho V^\# z_0.$$ 

Hence, the variation of the regularization parameter $\rho$ brings into consideration the study of the operator pencil $P(\lambda) = T^\# T + \lambda V^\# V$ determined by the selfadjoint operators $T^\# T$ and $V^\# V$.

From the inclusions $N(L) \subseteq N(L^\# L)$ and $R(L^\# L) \subseteq R(L^\#)$ it is immediate that

$$N(T) \cap N(V) \subseteq N(L^\# L) \quad \text{and} \quad R(L^\# L) \subseteq N(T)^\perp + N(V)^\perp.$$ 

The next result describes in detail different properties of $R(L)$ as a subspace of the Krein space $(K \times E, [\cdot, \cdot]_\rho)$, in terms of the operators $T$ and $V$.

**Proposition 4.3.** Given closed range operators $T \in \mathcal{L}(H, K)$ and $V \in \mathcal{L}(H, E)$, consider the regularization operator $L \in \mathcal{L}(H, K \times E)$. Then,

i) $R(L)$ is closed if and only if $N(T) + N(V)$ is a closed subspace;

ii) $R(L)$ is nondegenerate if and only if $N(L^\# L) = N(T) \cap N(V)$;

iii) $R(L)$ is pseudo-regular if and only if $N(T) + N(V)$ and $R(L^\# L)$ are closed subspaces.
Proof. i. On the one hand, \( R(L) \) is closed if and only if \( R(L^\#) \) is closed. On the other hand,
\[
R(L^\#) = R(T^\#) + R(V^\#) = N(T)^\perp + N(V)^\perp.
\]
Then, \( R(L) \) is closed if and only if \( N(T)^\perp + N(V)^\perp \) is closed. By \cite{10}, this is also equivalent to \( N(T) + N(V) \) being a closed subspace.

ii. First, let us show that \( R(L)^\circ = L(N(L^\#L)) \). To this end, note that
\[
N(L^\#L) = L^{-1}(N(L^\#)) = L^{-1}(N(L)[\perp]) = L^{-1}(R(L)^\circ).
\]
Applying \( L \) to both sides of the above equality, we get that \( L(N(L^\#L)) = R(L)^\circ \). Therefore, \( R(L)^\circ = \{0\} \) if and only if
\[
N(L^\#L) \subseteq N(L) = N(T) \cap N(V).
\]
Since the inclusion \( N(L) \subseteq N(L^\#L) \) always holds, the desired equivalence follows.

iii. By item i, \( L \) is a closed range operator if and only the sum of the nullspaces of \( T \) and \( V \) is closed in \( \mathcal{H} \). But, for a closed range operator \( L \), the pseudo-regularity of \( R(L) \) is equivalent to the closedness of \( R(L^\#L) \), see Proposition \cite{13}.

As a corollary, in the particular case that \( V \) is surjective, the regularity of the range of \( L \) can be also characterized by a range inclusion.

**Corollary 4.4.** Assume that \( V \) is surjective and consider \( \rho \neq 0 \). Then, \( R(L) \) is a regular subspace of \( (\mathcal{K} \times \mathcal{E},[\cdot, \cdot]_\rho) \) if and only if
\[
R(T^\#) \subseteq R(L^\#L).
\]

**Proof.** If \( R(L) \) is a regular subspace, then \( R(L) \) is nondegenerate and pseudo-regular. Applying Proposition \cite{13}, it follows that \( N(L^\#L) = N(T) \cap N(V) \) and \( R(L^\#L) \) is closed. Hence,
\[
R(T^\#) = N(T)^\perp \subseteq (N(T) \cap N(V))^\perp = N(L^\#L)^\perp = R(L^\#L).
\]

Conversely, assume that \( R(T^\#) \subseteq R(L^\#L) \). Then \( R(T^\#T) \subseteq R(T^\#) \subseteq R(L^\#L) \), or equivalently, \( R(V^\#) = R(V^\#V) \subseteq R(L^\#L) \). Hence, \( R(L^\#L) = R(T^\#) + R(V^\#) = R(L^\#) \) and given \((y, z) \in \mathcal{K} \times \mathcal{E}\) there exists \( x_0 \in \mathcal{H} \) such that \( Lx_0 \in \mathcal{E} \), or equivalently, \((y, z) \in N(L^\#) = R(L)[\perp] \). Therefore, \((y, z) = Lx_0 + ((y, z) - Lx_0) \in R(L) + R(L)[\perp] \).

Considering that \( R(T^\#) = N(T)^\perp \) and \( R(V^\#) = N(V)^\perp \), the proof of the above corollary implies the following.

**Corollary 4.5.** Assume that \( V \) is surjective. Given \( \rho \neq 0 \), \( R(L) \) is a regular subspace of \( (\mathcal{K} \times \mathcal{E},[\cdot, \cdot]_\rho) \) if and only if
\[
R(L^\#L) = N(T)^\perp + N(V)^\perp.
\]
The above results give rise to a natural question: if the range of \( L \) has some of the listed properties for a particular choice of \( \rho \), does the property remain under perturbations of the parameter? The next section is devoted to analyzing this question.

### 4.3. Varying the regularization parameter

Now we analyze the properties of the range of the regularization operator \( L \) in the family of Krein spaces \((\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_\rho)\) obtained under the variation of the regularization parameter \( \rho \).

**Proposition 4.6.** If \( R(L) \) is a closed subspace of \((\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_{\rho_0})\) for some \( \rho_0 \neq 0 \), then \( R(L) \) is a closed subspace of \((\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_\rho)\) for each \( \rho \neq 0 \).

**Proof.** This is an immediate consequence of item \( i \) in Proposition 4.3, because the closedness of \( N(T) + N(V) \) does not depend on \( \rho \). 

In the following we introduce an interval of admissible values for the regularization parameter, which is determined by Problem 2. Let \( \mathcal{C}_V \) denote the set of neutral vectors of the quadratic form associated to \( V^\# V \):

\[
\mathcal{C}_V = Q(V^\# V) = \{ y \in \mathcal{H} : [V y, V y] = 0 \}.
\]

If \( V^\# V \) is a positive (or negative) semidefinite operator in \( \mathcal{H} \), then \( \mathcal{C}_V \) coincides with \( N(V) \). But, if \( V^\# V \) is indefinite, the set \( \mathcal{C}_V \) is strictly larger than \( N(V) \). From now on \( V^\# V \) is assumed to be indefinite; i.e. neither positive nor negative semidefinite. A necessary condition for the existence of solutions for Problem 2 is that \( T \) maps the set \( \mathcal{C}_V \) into the set of nonnegative vectors of the Krein space \( \mathcal{K} \), see [19, Corollary 3.2].

In the following we rewrite several results from the previous section in the Krein space setting. The next result can be interpreted as another manifestation of Farkas’ lemma, see [40, 45].

**Proposition 4.7.** \( T(\mathcal{C}_V) \) is a nonnegative set of \( \mathcal{K} \) if and only if there exists \( \rho \in \mathbb{R} \) such that \( T^\# T + \rho V^\# V \in \mathcal{L}(\mathcal{H})^\dagger \).

If it is also assumed that \( T^\# T \) is indefinite, it is easy to see that \( T(\mathcal{C}_V) \) is a nonnegative set of \( \mathcal{K} \) if and only if \( R(L) \) is nonnegative in \((\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_\rho)\) for some \( \rho \neq 0 \).

Let us also consider the subsets of \( \mathcal{H} \) where the quadratic form associated to \( V^\# V \) takes positive and negative values:

\[
\mathcal{P}^+(V) := \{ x \in \mathcal{H} : [V x, V x] > 0 \} \quad \text{and} \quad (4.7)
\]

\[
\mathcal{P}^-(V) := \{ x \in \mathcal{H} : [V x, V x] < 0 \}.
\]

The next corollary determines the interval for the parameter \( \rho \) in which \( R(L) \) is a nonnegative subspace of \((\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_\rho)\).
Corollary 4.8. Assume that $T(C_V)$ is a nonnegative set of $\mathcal{K}$, and define

$$
\rho_- := - \inf_{x \in \mathcal{P}^+(V)} \frac{\langle Tx, Tx \rangle}{\langle Vx, Vx \rangle} \quad \text{and} \quad \rho_+ := - \sup_{x \in \mathcal{P}^-(V)} \frac{\langle Tx, Tx \rangle}{\langle Vx, Vx \rangle}.
$$

(4.8)

Then $\rho_- < +\infty$, $\rho_+ > -\infty$ and $\rho_- \leq \rho_+$. Moreover, $T^\# T + \rho V^\# V \in \mathcal{L}(\mathcal{H})^+$ if and only if $\rho \in [\rho_-, \rho_+]$.

Hereafter we make the following assumptions.

Hypotheses 4.9. $T^\# T$ and $V^\# V$ are indefinite operators acting in $\mathcal{H}$ and $T(C_V)$ is a nonnegative subset of $\mathcal{K}$.

The indefiniteness of $T^\# T$ implies that $0 \notin [\rho_-, \rho_+]$, i.e. the interval $[\rho_-, \rho_+]$ is either contained in the positive or in the negative real axis. In fact, it can be contained in either one of them, as the following example illustrates.

Example 2. Consider surjective operators $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $V \in \mathcal{L}(\mathcal{H}, \mathcal{E})$ such that $T^\# T$ and $V^\# V$ can be represented, according to some decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, as

$$
T^\# T = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \quad \text{and} \quad V^\# V = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},
$$

with $\alpha, \beta \in \mathbb{R}$. If $\rho \in \mathbb{R}$, and $x \in \mathcal{H}$ is written as $x = x_+ + x_-$ with $x_\pm \in \mathcal{H}_\pm$, then

$$
\langle (T^\# T + \rho V^\# V)x, x \rangle = (\alpha + \rho)\|x_+\|^2 + (\beta - \rho)\|x_-\|^2.
$$

Thus, $T^\# T + \rho V^\# V \in \mathcal{L}(\mathcal{H})^+$ if and only if $\alpha + \rho > 0$ and $\beta - \rho > 0$. Suppose that $\alpha = -1$ and $\beta = 2$, then it is immediate that $\rho_- = 1$ and $\rho_+ = 2$. If we now consider the case when $\alpha = 2$ and $\beta = -1$, then $\rho_- = -2$ and $\rho_+ = -1$.

Remark 4.10. If we denote $C_T = Q(T^\# T) = \{x \in \mathcal{H} : \langle Tx, Tx \rangle_x = 0\}$, Hypotheses 4.9 also imply that $V(C_T)$ is either a nonnegative or a nonpositive set of $\mathcal{E}$, depending on whether $[\rho_-, \rho_+]$ is contained in the positive or in the negative real axis. Indeed, if $\rho \in [\rho_-, \rho_+]$ then

$$
\rho [Vx, Vx] = \langle (T^\# T + \rho V^\# V)x, x \rangle \geq 0, \quad \text{for all} \quad x \in C_T.
$$

Now, we apply the results obtained in Section 3 to the operator pencil determined by the selfadjoint operators $A := T^\# T$ and $B := V^\# V$. We start by analyzing the behaviour of the nullspace of $L^\# L = T^\# T + \rho V^\# V$ for each $\rho \in [\rho_-, \rho_+]$. As a consequence of Proposition 3.3 if $\rho \neq \rho_+$ then the nullspace of $L^\# L$ does not depend on $\rho$ when $\rho \in (\rho_-, \rho_+)$.  

Lemma 4.11. If $\rho_+ \neq \rho_-$ then

$$
N(T^\# T + \rho V^\# V) = C_T \cap C_V = N(T^\# T) \cap N(V) \quad \text{for all} \quad \rho \in (\rho_-, \rho_+).
$$
4.4. Pseudo-regularity and regularity

Assume that \( \rho_- \neq \rho_+ \). To characterize the nullspace of the operators associated to the extreme values of the interval, note that the sets defined in (4.10) are given by:

\[
\mathcal{M}_+ = \left\{ x \in \mathcal{P}^+(V) : \frac{[Tx, Tx]}{V_x, V_x} = -\rho_- \right\}
\]

and

\[
\mathcal{M}_- = \left\{ x \in \mathcal{P}^-(V) : \frac{[Tx, Tx]}{V_x, V_x} = -\rho_+ \right\}.
\]

**Lemma 4.12.** The following conditions are satisfied:

i) if \( \rho_- \neq \rho_+ \), then \( N(T^\#T + \rho_\pm V^\#V) = \mathcal{M}_\pm \cup (N(T^\#T) \cap N(V)) \);

ii) if \( \rho_- = \rho_+ \), then \( \mathcal{M}_+ = \mathcal{M}_- \) and \( N(T^\#T + \rho_\pm V^\#V) = \mathcal{M}_\pm \cup (C_T \cap C_V) \).

Combining Lemma 4.11 with item ii in Proposition 4.13 we have:

**Proposition 4.13.** If \( \rho_- \neq \rho_+ \) then the following conditions are equivalent:

i) \( R(L) \) is a nondegenerate subspace of \((K \times \mathcal{E}, [\cdot, \cdot]_\rho)\) for some \( \rho \in (\rho_-, \rho_+) \);

ii) \( R(L) \) is a nondegenerate subspace of \((K \times \mathcal{E}, [\cdot, \cdot]_\rho)\) for all \( \rho \in (\rho_-, \rho_+) \);

iii) \( N(T^\#T) \cap N(V) = N(T) \cap N(V) \).

4.4. Pseudo-regularity and regularity

In order to prove the main result of this section, we first show that pseudo-regularity is an invariant property in \((\rho_-, \rho_+)\).

**Proposition 4.14.** Assume that \( \rho_- \neq \rho_+ \). If \( R(L) \) is a pseudo-regular subspace of \((K \times \mathcal{E}, [\cdot, \cdot]_{\rho_0})\) for some \( \rho_0 \in (\rho_-, \rho_+) \), then it is a pseudo-regular subspace of \((K \times \mathcal{E}, [\cdot, \cdot]_\rho)\) for all \( \rho \in (\rho_-, \rho_+) \).

**Proof.** Assume that \( R(L) \) is a pseudo-regular subspace of \((K \times \mathcal{E}, [\cdot, \cdot]_{\rho_0})\) for some \( \rho_0 \in (\rho_-, \rho_+) \). By item iii in Proposition 4.13 we have that \( N(T) + N(V) \) and \( R(T^\#T + \rho_0 V^\#V) \) are closed subspaces of \( \mathcal{H} \). If \( \rho \in (\rho_-, \rho_+) \), applying Corollary 3.7 \( R(T^\#T + \rho V^\#V) \) is also a closed subspace of \( \mathcal{H} \). By item iii in Proposition 4.13 it follows that \( R(L) \) is also a pseudo-regular subspace of \((K \times \mathcal{E}, [\cdot, \cdot]_\rho)\). \( \Box \)

**Corollary 4.15.** Suppose that \( \rho_- \neq \rho_+ \). If \( R(L) \) is a regular subspace of \((K \times \mathcal{E}, [\cdot, \cdot]_{\rho_0})\) for some \( \rho_0 \in (\rho_-, \rho_+) \), then it is a regular subspace of \((K \times \mathcal{E}, [\cdot, \cdot]_\rho)\) for all \( \rho \in (\rho_-, \rho_+) \).

**Proof.** Since a subspace is regular if and only if it is nondegenerate and pseudo-regular, the statement is an immediate consequence of Proposition 4.13 and Proposition 4.14. \( \Box \)
Now, we deepen into the analysis of the case in which $R(L)$ is a regular subspace of $(K \times E, [\cdot, \cdot])$ for some $\rho \in [\rho_-, \rho_+]$. First, we show that this regularity is equivalent to $\mathcal{C}_V$ being a uniformly positive subset of $\mathcal{H}$ with respect to the form induced by $T^\# T$. For the sake of simplicity, from now on we assume that

$$N(T) \cap N(V) = \{0\},$$

which is not necessarily true in the general case but does not imply a loss of generality. With minor adjustments the following results can be expressed for the general case.

**Proposition 4.16.** Assume that $N(T) \cap N(V) = \{0\}$. Then, the following conditions are equivalent:

i) there exists $\alpha > 0$ such that $[Ty, Ty] \geq \alpha \|y\|^2$ for every $y \in \mathcal{C}_V$;

ii) $T(\mathcal{C}_V)$ is a uniformly positive set of $\mathcal{K}$ and $N(T) + N(V)$ is a closed subspace;

iii) there exists $\rho \in \mathbb{R}$ such that $T^\# T + \rho V^\# V$ is a positive definite operator;

iv) there exists $\rho \in \mathbb{R}$ such that $R(L)$ is a (closed) uniformly positive subspace of $(K \times E, [\cdot, \cdot], \rho)$.

**Proof.** The equivalence $i \leftrightarrow iii$ follows from Theorem 3.8 considering $A = T^\# T$ and $B = V^\# V$.

$i \rightarrow ii$. Suppose that there exists $\alpha > 0$ such that $[Ty, Ty] \geq \alpha \|y\|^2$ for every $y \in \mathcal{C}_V$. In particular, this implies that $T(N(V))$ is closed. Then, by Proposition 2.1, $N(T) + N(V)$ is also closed.

Also, $T(\mathcal{C}_V)$ is a uniformly positive set of $\mathcal{K}$ because

$$[Ty, Ty] \geq \alpha \|y\|^2 \geq \frac{\alpha}{\|Ty\|^2} \|Ty\|^2, \quad \text{for every } y \in \mathcal{C}_V.$$

$ii \rightarrow iv$. Assume that there exists $\alpha > 0$ such that $[Ty, Ty] \geq \alpha \|Ty\|^2$ for every $y \in \mathcal{C}_V$. Then, if $A := T^\# T - \alpha T^* T$ and $B := V^\# V$ the inequality is equivalent to

$$\langle Ay, y \rangle \geq 0 \quad \text{for every } y \in \mathcal{H} \text{ such that } \langle By, y \rangle = 0.$$

Hence, by Proposition 3.3, there exists $\rho \in \mathbb{R}$ such that $T^\# T - \alpha T^* T + \rho V^\# V = A + \rho B \geq 0$. So, $\alpha T^* T \leq L^\# L$. Then, by Theorem 2.2, we have that

$$N(T)^\perp = R(T^*) \subseteq R((L^\# L)^{1/2}). \quad (4.9)$$

Now, we want to prove that $L^\# L$ is invertible, which implies that $R(L)$ is regular. To do so, we make use of the following result borrowed from [12]: if $C \in \mathcal{L}(\mathcal{H})^+$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{S} \cap N(C) = \{0\}$ and $C(\mathcal{S})$ is closed, then $\mathcal{H} = \mathcal{S}^\perp + C(\mathcal{S})$, see Proposition 3.3 and Lemma 3.8 in [12].

For our purpose, consider the positive semidefinite operator $L^\# L$ and the closed subspace $N(T)$. On the one hand, the closedness of $N(T) + N(V)$ implies
that $R(L) = N(T)^\perp + N(V)^\perp$ is also a closed subspace of $\mathcal{H}$. Moreover, by Proposition 2.1, $L^\#L(N(T)) = V^\#V(N(T))$ is a closed subspace of $\mathcal{K}$. Hence, combining (4.9) with the above mentioned result,

$$\mathcal{H} = N(T)^\perp + L^\#L(N(T)) \subseteq R((L^\#L)^{1/2}).$$

Also, it is immediate that $N(L^\#L) = N(T) \cap N(V) = \{0\}$. Hence, $(L^\#L)^{1/2}$ is invertible. In particular, $R(L^\#L) = \mathcal{H}$ and

$$\mathcal{K} \times \mathcal{E} = (L^\#)^{-1}(R(L^\#L)) = R(L) + N(L^\#) = R(L) + R(L)^{1/2},$$
i.e. $R(L)$ is a regular subspace of $\mathcal{K} \times \mathcal{E}$.

**iv $\rightarrow$ iii.** Suppose that there exists $\rho \in \mathbb{R}$ such that $R(L)$ is a (closed) uniformly positive subspace of $(\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_\rho)$. Then, $R(L^\#L) = \mathcal{H}$ and Proposition 4.3 ensures that $N(L^\#L) = N(T) \cap N(V) = \{0\}$. Hence, $L^\#L$ is a positive definite operator.

As a consequence of the proof, $\rho \in \mathbb{R}$ satisfies item iii if and only if it satisfies item iv. Then, combining the above proposition with Theorem 3.8, we have the following result.

**Theorem 4.17.** If $\rho \in [\rho_-, \rho_+]$ is such that $R(L)$ is a regular subspace of $(\mathcal{K} \times \mathcal{E}, [\cdot, \cdot]_\rho)$, then $\rho_- \neq \rho_+$ and $\rho \in (\rho_-, \rho_+)$. Therefore, the regular case cannot occur in the boundary of the interval. The following corollary relates the lower bound that determines the uniformly positiveness of $T(CV)$ (as a subset of $\mathcal{K}$) with a closed interval contained in $(\rho_-, \rho_+)$ where the pencil is uniformly bounded from below.

**Corollary 4.18.** Given $\alpha > 0$, the following conditions are equivalent:

i) $\langle Ty, Ty \rangle \geq \alpha \|y\|^2$ for every $y \in CV$;

ii) there exists an interval $[\eta_-, \eta_+]$ contained in $(\rho_-, \rho_+)$ such that $T^\#T + \rho V^\#V \geq \alpha I$ for every $\rho \in [\eta_-, \eta_+]$.

**Proof.** Firstly, $i \rightarrow ii$ follows from the proof of Theorem 3.8 considering $A := T^\#T$ and $B := V^\#V$. Secondly, if $\rho \in [\eta_-, \eta_+]$ it is immediate that, for every $y \in CV$,

$$\langle Ty, Ty \rangle = \langle T^\#Ty, y \rangle = \langle (T^\#T + \rho V^\#V)y, y \rangle \geq \alpha \langle y, y \rangle = \alpha \|y\|^2.$$

**Acknowledgements**

The authors gratefully acknowledge the support of CONICET through the grant PIP 11220200102127CO. F. Martínez Pería also acknowledges the support from UNLP 11X829 and PICT 2015-1505.
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