Invariant Differential Operators for the Real Exceptional Lie Algebra $F_4''$

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In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact exceptional Lie algebra $F_4''$ which is the split rank one form of the exceptional Lie algebra $F_4$. We classify the reducible Verma modules over $F_4$ which are compatible with this induction. Thus, we obtain the classification of the corresponding invariant differential operators.

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1. Introduction

Invariant differential operators play a very important role in the description of physical symmetries. One of the most important such symmetry is the conformal symmetry. Incidentally, our involvement in this area started in the 70s with the Euclidean conformal group $SO(N,1)$.\textsuperscript{1–3} It is important that this group is of split rank one.\textsuperscript{a}

Our general scheme for constructing invariant differential operators was given some time ago.\textsuperscript{5} After our paper was submitted we learned that similar questions were treated in a different approach (by so-called K-induction).\textsuperscript{6} There were treated the split rank one groups $SU(N,1)$ and $SO(N,1)$.

In recent papers\textsuperscript{7,8} we started the systematic explicit construction of the invariant differential operators. Yet the two remaining split rank one

\textsuperscript{a}Though not in our main exposition flow, we should mention that in parallel were developed the study and applications of the Minkowskian conformal group $SO(N,2)$, cf., e.g.,\textsuperscript{4}
cases, namely, $F_4'$ and $Sp(N,1)$, were not treated until now. Thus, it was important to treat the case $F_4''$ which we do in the present paper.

The first task in our approach is to make the multiplet classification of the reducible Verma modules over the algebra in consideration. Such classification provides the weights of embeddings between the Verma modules via the singular vectors, and thus, the weights of the invariant differential operators.

We have done the multiplet classification for many real non-compact algebras. In the present paper we focus on the complex exceptional Lie algebra $F_4$ and on its split rank one form algebra $F_4''$. Our scheme requires that we use induction from parabolic subalgebras.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of $G$. In the split rank one case $F_4''$ there is only one nontrivial parabolic: $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, where $\mathcal{M} = \mathfrak{so}(7)$, $\dim \mathcal{N} = 15$.

We present the multiplet classification of the reducible Verma modules over $F_4$ which are compatible with the parabolic $\mathcal{P}$ of $F_4''$. We give also the weights of the singular vectors between these modules. By our scheme these singular vectors will produce the invariant differential operators.

2. Preliminaries

2.1. Lie algebra

We start with the complex exceptional Lie algebra $\mathcal{G}^C = F_4$. We use the standard definition of $\mathcal{G}^C$ given in terms of the Chevalley generators $X_i^\pm$, $H_i$, $i = 1, 2, 3, 4(=\text{rank } F_4)$, by the relations:

\begin{align}
[H_j, H_k] &= 0, \quad [H_j, X_k^\pm] = \pm a_{jk} X_k^\pm, \quad [X_j^+, X_k^-] = \delta_{jk} H_j, \\
\sum_{m=0}^{n} (-1)^m \binom{n}{m} (X_j^\pm)^m X_k^\pm (X_j^\pm)^{n-m} &= 0, \quad j \neq k, \quad n = 1 - a_{jk},
\end{align}

where

\begin{equation}
(a_{ij}) = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -2 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix};
\end{equation}

is the Cartan matrix of $\mathcal{G}^C$, $\alpha_j^\vee = \frac{2\alpha_j}{(\alpha_j, \alpha_j)}$ is the co-root of $\alpha_j$, $(\cdot, \cdot)$ is the scalar product of the roots, so that the nonzero products between

\footnote{We are thankful to Joachim Hilgert for pointing out this omission.}
the simple roots are: $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2(\alpha_3, \alpha_3) = 2(\alpha_4, \alpha_4) = 2$, 
$(\alpha_1, \alpha_2) = -1$, $(\alpha_2, \alpha_3) = -1$, $(\alpha_3, \alpha_4) = -1/2$. The elements $H_i$ span the 
Cartan subalgebra $H$ of $G^C$, while the elements $X^\pm_i$ generate the subalgebras $G^\pm$. We shall use the standard triangular decomposition

$$G^C = g_+ \oplus H \oplus g_-, \quad g_\pm = \bigoplus_{\alpha \in \Delta^\pm} g_\alpha,$$

where $\Delta^+, \Delta^-$, are the sets of positive, negative, roots, resp. Explicitly we have that there are roots of two lengths with length ratio 2:1.

The long roots are: $\alpha_1$, $\alpha_2$, $\alpha_1 + \alpha_2$, $\alpha_2 + 2\alpha_3$, $\alpha_1 + \alpha_2 + 2\alpha_3$, $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4$, $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. With the chosen normalization they have length 2.

The short roots are: $\alpha_3$, $\alpha_4$, $\alpha_3 + \alpha_4$, $\alpha_3 + \alpha_4$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $\alpha_2 + 2\alpha_3 + \alpha_4$, $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$, $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$, $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$, $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$, and they have length 1. (Note that the short roots are exactly those which contain $\alpha_3$ and/or $\alpha_4$ with coefficient 1, while the long roots contain $\alpha_3$ and $\alpha_4$ with even coefficients.)

Thus, as well-known, $F_4$ is 52-dimensional ($52 = |\Delta| + \text{rank } F_4$).

In terms of the normalized basis $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ we have:

$$\Delta^+ = \{\varepsilon_i, \ 1 \leq i \leq 4; \ \varepsilon_j \pm \varepsilon_k, \ 1 \leq j < k \leq 4; \ 
\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4), \text{ all signs}\}.$$  \hspace{1cm} (4)

The simple roots are:

$$\pi = \{\alpha_1 = \varepsilon_2 - \varepsilon_3, \ \alpha_2 = \varepsilon_3 - \varepsilon_4, \ \alpha_3 = \varepsilon_4, \ \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}.$$  \hspace{1cm} (5)

For later use we note that the Lie algebra $B_4 = so(9)^C$ may be embedded most easily in $F_4$ as the Lie algebra generated by the 16 roots on the first line of (4). Indeed, the latter form the positive root system of $B_4$ with simple roots $\varepsilon_i - \varepsilon_{i+1}, \ i = 1, 2, 3, \varepsilon_4$.

The Weyl group of $F_4$ is the semidirect product of $S_3$ with a group which itself is the semidirect product of $S_4$ with $(\mathbb{Z}/2\mathbb{Z})^3$, thus, $|W| = 2^7 3^2 = 1152$.

2.2. Verma modules

Let us recall that a Verma module $V^\Lambda$ is defined as the HWM over $G^C$ with highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V^\Lambda$,
induced from the one-dimensional representation $V_0 \cong \mathbb{C}v_0$ of $U(B)$, where $B = H \oplus G_+$ is a Borel subalgebra of $G^\mathbb{C}$, such that:

$$
X v_0 = 0, \quad \forall X \in G_+ \\
H v_0 = \Lambda(H) v_0, \quad \forall H \in H
$$

(6)

Verma modules are generically irreducible. A Verma module $V^\Lambda$ is reducible\(^{14}\) iff there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that

$$
(\Lambda + \rho, \beta^\vee) = m
$$

holds, where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, ($\rho = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4$).

If (7) holds then the reducible Verma module $V^\Lambda$ contains an invariant submodule which is also a Verma module $V^\Lambda'$ with shifted weight $\Lambda' = \Lambda - m\beta$. This statement is equivalent to the fact that $V^\Lambda$ contains a singular vector $v_s \in V^\Lambda$, such that

$$
X v_s = 0, \quad \forall X \in G_+ \\
H v_s = \Lambda'(H) v_s, \quad \forall H \in H
$$

(8)

More explicitly,\(^5\)

$$
v^s_{m,\beta} = P_{m,\beta} v_0.
$$

(9)

The general reducibility conditions (7) for $V^\Lambda$ spelled out for the simple roots in our situation are:

$$
m_1 \equiv m_{\alpha_1} = (\Lambda + \rho, \alpha_1), \quad m_2 \equiv m_{\alpha_2} = (\Lambda + \rho, \alpha_2),
$$

$$
m_3 \equiv m_{\alpha_3} = (\Lambda + \rho, 2\alpha_3), \quad m_4 \equiv m_{\alpha_4} = (\Lambda + \rho, 2\alpha_4)
$$

(10)

If we write

$$
\Lambda = \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3 + \lambda_4 \alpha_4
$$

(11)

then the numbers $\lambda_i$ are expressed through $m_i$ as follows:

$$
\lambda_1 = 2m_1 + 3m_2 + 2m_3 + m_4 - 8, \\
\lambda_2 = 3m_1 + 6m_2 + 4m_3 + 2m_4 - 15, \\
\lambda_3 = 24 - 5m_1 - 10m_2 - 6m_3 - 3m_4, \\
\lambda_4 = \frac{1}{2}(23 - 5m_1) - 5m_2 - 3m_3 - m_4
$$

(12)

The numbers $m_\beta$ from (10) corresponding to the simple roots are called Dynkin labels, while the more general Harish-Chandra parameters are:

$$
m_\beta = (\Lambda + \rho, \beta^\vee), \quad \beta \in \Delta^+
$$

(13)
Note that the expression from (8) \( \Lambda' = \Lambda - m\beta \) may be written using (13) as a Weyl reflection:

\[
s_\beta(\Lambda + \rho) = \Lambda + \rho - m_\beta \beta = \Lambda' + \rho
\]

(14)

Explicitly, the Harish-Chandra parameters (for the non-simple roots) in terms of the Dynkin labels \( m_1, m_2, m_3, m_4 \) are:

\[
\begin{align*}
m_{\alpha_1+\alpha_2} &= m_1 + m_2 \equiv m_{12}, \\
m_{\alpha_2+2\alpha_3} &= m_2 + m_3 \equiv m_{23}, \\
m_{\alpha_1+\alpha_2+2\alpha_3} &= m_1 + m_2 + m_3 \equiv m_{13}, \\
m_{\alpha_1+2\alpha_2+2\alpha_3} &= m_1 + 2m_2 + m_3 \equiv m_{13,2}, \\
m_{\alpha_2+2\alpha_3+2\alpha_4} &= m_2 + m_3 + m_4 \equiv m_{24}, \\
m_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4} &= m_1 + m_2 + m_3 + m_4 \equiv m_{14}, \\
m_{\alpha_1+2\alpha_2+2\alpha_3} &= m_1 + 2m_2 + m_3 + m_4 \equiv m_{14,2}, \\
m_{\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4} &= m_1 + 2m_2 + 2m_3 + m_4 \equiv m_{14,23}, \\
m_{\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4} &= m_1 + 3m_2 + 2m_3 + m_4 \equiv m_{14,23,2}, \\
m_{2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4} &= 2m_1 + 3m_2 + 2m_3 + m_4 \equiv m_{14,13,2} \quad (15)
\end{align*}
\]

for the long roots, while for the short roots we have:

\[
\begin{align*}
m_{\alpha_2+\alpha_3} &= 2m_2 + m_3 \equiv m_{23,2}, \\
m_{\alpha_1+\alpha_2+\alpha_3} &= 2m_1 + 2m_2 + m_3 \equiv m_{13,12}, \\
m_{\alpha_3+\alpha_4} &= m_3 + m_4 \equiv m_{34}, \\
m_{\alpha_2+\alpha_3+\alpha_4} &= 2m_2 + m_3 + m_4 \equiv m_{24,2}, \\
m_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} &= 2m_1 + 2m_2 + m_3 + m_4 \equiv m_{14,12}, \\
m_{\alpha_2+2\alpha_1+\alpha_4} &= 2m_2 + 2m_3 + m_4 \equiv m_{24,23}, \\
m_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} &= 2m_1 + 4m_2 + 2m_3 + m_4 \equiv m_{14,13,2,2}, \\
m_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} &= 2m_1 + 2m_2 + 2m_3 + m_4 \equiv m_{14,13}, \\
m_{\alpha_1+2\alpha_2+3\alpha_3+\alpha_4} &= 2m_1 + 4m_2 + 3m_3 + m_4 \equiv m_{14,13,23,2}, \\
m_{\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4} &= 2m_1 + 4m_2 + 3m_3 + 2m_4 \equiv m_{14,14,23,2} \quad (16)
\end{align*}
\]

where we have introduce short-hand notation, e.g., \( m_{13,2} \equiv m_{13} + m_2, m_{13,12} \equiv m_{13} + m_{12}, \) etc.

2.3. Structure theory of the real form

The split real form of \( F_4 \) is denoted as \( F_4'' \), sometimes as \( F_4(-20) \). It has rank four. Its maximal compact subalgebra is \( K \cong so(9) \), also...
of rank four. This real form has discrete series representations since \( \text{rank} F'_4 = \text{rank} K \). The number of discrete series is equal to the ratio 
\[ \frac{|W(G^C, H^C)|}{|W(K^C, H^C)|} \]
where \( H \) is a compact Cartan subalgebra of both \( G \) and \( K \). 
Thus, the number of discrete series in our setting is three. They will be identified below.

The Iwasawa decomposition of \( G \equiv F'_4 \), is:

\[ G = K \oplus A \oplus N \quad (17) \]

the Cartan decomposition is:

\[ G = K \oplus Q \quad (18) \]

where we use \( \dim \mathbb{R} Q = 16, \dim \mathbb{R} A = 1, \quad N^+ = N^+, \quad \text{or} \quad N^- = N^- \equiv N^+, \quad \dim \mathbb{R} N^{\pm} = 15. \)

Since \( G \) is of split rank one, thus, the minimal (also maximal) parabolic \( P \) and the corresponding Bruhat decomposition are:

\[ P = M \oplus A \oplus N, \quad M = \text{so}(7) \quad (19) \]

\[ G = M \oplus A \oplus N^+ \oplus N^- \]

Note that the root system of \( M^C = \text{so}(7, \mathbb{C}) = B_3 \) consists of the roots

\[ \Delta^+_3 = \{ \varepsilon_i, \ 2 \leq i \leq 4; \ \varepsilon_j \pm \varepsilon_k, \ 2 \leq j < k \leq 4 \} \quad (20) \]

which are part of (4), while the simple roots are part of (5).

\[ \pi_3 = \{ \alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_3 - \varepsilon_4, \quad \alpha_3 = \varepsilon_4 \} \quad (21) \]

The roots of \( M^C \) are called \( M \)-compact roots of the \( F_4 \) root system (4), the rest are called \( M \)-noncompact roots. The latter give rise to invariant differential operators, as explained below.

More explicitly, the \( M \)-compact roots are:

\[ \begin{align*}
\alpha_1, \ \alpha_2, \ \alpha_1 + \alpha_2 \equiv \alpha_{12}, \ \alpha_2 + 2\alpha_3 \equiv \alpha_{23,3}, \ \alpha_1 + \alpha_2 + 2\alpha_3 \equiv \alpha_{13,3}, \\
\alpha_1 + 2\alpha_2 + 2\alpha_3 \equiv \alpha_{13,23}, \\
\alpha_3, \ \alpha_2 + \alpha_3 \equiv \alpha_{23}, \ \alpha_1 + \alpha_2 + \alpha_3 \equiv \alpha_{13} \quad (22a) \\
\end{align*} \]

(22a) are long roots, (22b) - short.
The $\mathcal{M}$-noncompact roots are:

\[
\begin{align*}
\alpha_2 + 2\alpha_3 + 2\alpha_4 &\equiv \alpha_{24,23}, \quad \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 \equiv \alpha_{14,34}, \\
\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 &\equiv \alpha_{14,24}, \\
\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 &\equiv \alpha_{14,24,3,3}, \quad \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \equiv \alpha_{14,24,23,3}, \\
2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 &\equiv \alpha_{14,14,23,3} \\
\alpha_4, \quad \alpha_3 + \alpha_4 &\equiv \alpha_{34}, \quad \alpha_2 + \alpha_3 + \alpha_4 \equiv \alpha_{24}, \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv \alpha_{14}, \\
\alpha_2 + 2\alpha_3 + \alpha_4 &\equiv \alpha_{24,3}, \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \equiv \alpha_{24,3,3}, \\
\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 &\equiv \alpha_{14,3}, \quad \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \equiv \alpha_{14,23,3}, \\
\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 &\equiv \alpha_{14,24,3} \\
\end{align*}
\] (23a)

(23a) are long roots, (23b) - short.

Correspondingly, the Dynkin labels $m_1, m_2, m_3$ are called $\mathcal{M}$-compact, while $m_4$ is called $\mathcal{M}$-noncompact.

### 2.4. Elementary representations

Further, let $G, K, P, M, A, N$ are Lie groups with Lie algebras $\mathfrak{g}_0, \mathfrak{k}, \mathfrak{p}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}$.

Let $\nu$ be a (non-unitary) character of $A$, $\nu \in \mathfrak{a}^*$. Let $\mu$ fix a finite-dimensional unitary representation $D^\mu$ of $M$ on the space $V^\mu$.

We call the induced representation $\chi = \text{Ind}^G_P(\mu \otimes \nu \otimes 1)$ an elementary representation of $G$.\(^1\) (These are called also generalized principal series representations (or limits thereof).\(^15\)) Their spaces of functions are:

\[
\begin{align*}
\mathcal{C}_\chi = \{ \mathcal{F} \in C^\infty(G, V^\mu) \mid \mathcal{F}(g \text{man}) = e^{-\nu(H)} \cdot D^\mu(m^{-1}) \mathcal{F}(g) \} \\
\end{align*}
\] (24)

where $\alpha = \exp(H) \in A$, $H \in \mathfrak{a}$, $m \in M$, $n \in N$. The representation action is the left regular action:

\[
(T^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G.
\] (25)

An important ingredient in our considerations are the highest/lowest weight representations of $G$. These can be realized as (factor-modules of) Verma modules $V^\Lambda$ over $G$, where $\Lambda \in (\mathcal{H})^*$, the weight $\Lambda = \Lambda(\chi)$ being determined uniquely from $\chi$.\(^5\)

As we have seen when a Verma module is reducible and (7) holds then there is a singular vector (9). Relatedly, then there exists\(^5\) an invariant differential operator

\[
\mathcal{D}_{m, \beta} : \mathcal{C}_\chi(\Lambda) \longrightarrow \mathcal{C}_\chi(\Lambda - m, \beta)
\] (26)
Given explicitly by:

\[ D_{m,\beta} = P_{m,\beta}(\hat{N}^-) \]  

(27)

where \( \hat{N}^- \) denotes the right action on the functions \( \mathcal{F} \).

Actually, since our ERs are induced from finite-dimensional representations of \( \mathcal{M} \) the corresponding Verma modules are always reducible. Thus, it is more convenient to use generalised Verma modules \( \tilde{V}^A \) such that the role of the highest/lowest weight vector \( v_0 \) is taken by the (finite-dimensional) space \( V_{\mu} v_0 \).

Algebraically, the above is governed by the notion of \( \mathcal{M} \)-compact roots of \( \mathcal{G}^C \). The consequence of this is that (7) is always fulfilled for the \( \mathcal{M} \)-compact roots of \( \mathcal{G}^C \). That is why we consider generalised Verma modules. Relatedly, the invariant differential operators corresponding to \( \mathcal{M} \)-compact roots are trivial.

3. Main multiplets of \( F_4'' \)

Further we classify the generalized Verma modules (GVM) relative to the parabolic subalgebra \( \mathcal{P} \) (19). This also provides the classification of the \( \mathcal{P} \)-induced ERs with the same Casimirs. The classification is done as follows. We group the reducible Verma modules (also the corresponding ERs) related by nontrivial embeddings in sets called multiplets.\(^5,9\) These multiplets may be depicted as a connected graph, the vertices of which correspond to the GVMs and the lines between the vertices correspond to the GVM embeddings (and also the invariant differential operators between the ERs). The explicit parametrization of the multiplets and of their Verma modules (and ERs) is important for understanding of the situation.

The result of our classification is as follows. The multiplets of GVMs (and ERs) induced from \( \mathcal{P} \) are parametrized by four positive integers - the Dynkin labels. Each multiplet contains 24 GVMs (ERs). Their signatures
are given as follows:

\[
\begin{align*}
\lambda_0^- &= \{m_1, m_2, m_3, m_4\} \\
\lambda_a^- &= \{m_1, m_2, m_{34}, -m_4\} \\
\lambda_b^- &= \{m_1, m_{23}, m_4, -m_{34}\} \\
\lambda_c^- &= \{m_{12}, m_{23}, m_4, -m_{24,2}\} \\
\lambda_d^- &= \{m_2, m_{13}, m_4, -m_{14,12}\} \\
\lambda_e^- &= \{m_{13}, m_2, m_{34}, -m_{24,23}\} \\
\lambda_f^- &= \{m_{23}, m_{12}, m_{34}, -m_{14,13}\} \\
\lambda_g^- &= \{m_{14}, m_2, m_3, -m_{24,23}\} \\
\lambda_h^- &= \{m_{23}, m_1, m_{24,2}, -m_{14,13,2,2}\} \\
\lambda_i^- &= \{m_{24}, m_{12}, m_3, -m_{14,13}\} \\
\lambda_j^- &= \{m_2, m_1, m_{24,23}, -m_{14,13,23,2}\} \\
\lambda_k^- &= \{m_{24}, m_1, m_{23,2}, -m_{14,13,2,2}\} \\
\lambda^+_k &= \{m_{24}, m_1, m_{23,2}, -m_{14,13,23,2}\} \\
\lambda^+_j &= \{m_2, m_1, m_{24,23}, -m_{14,14,23,2}\} \\
\lambda^+_i &= \{m_{24}, m_{12}, m_3, -m_{14,13,23,2}\} \\
\lambda^+_h &= \{m_{23}, m_1, m_{24,2}, -m_{14,14,23,2}\} \\
\lambda^+_g &= \{m_{14}, m_2, m_3, -m_{14,13,23,2}\} \\
\lambda^+_f &= \{m_{23}, m_{12}, m_{34}, -m_{14,14,23,2}\} \\
\lambda^+_e &= \{m_{13}, m_2, m_{34}, -m_{14,14,23,2}\} \\
\lambda^+_d &= \{m_2, m_{13}, m_4, -m_{14,14,23,2}\} \\
\lambda^+_c &= \{m_{12}, m_{23}, m_4, -m_{14,14,23,2}\} \\
\lambda^+_b &= \{m_1, m_{23}, m_4, -m_{14,14,23,2}\} \\
\lambda^+_a &= \{m_1, m_2, m_{34}, -m_{14,14,23,2}\} \\
\lambda^+_0 &= \{m_1, m_2, m_3, -m_{14,14,24,2}\}
\end{align*}
\]

These multiplets are presented in Fig. 1. On the figure each arrow represents an embedding between two Verma modules, \(V^\Lambda\) and \(V^{\Lambda'}\), the arrow pointing to the embedded module \(V^{\Lambda'}\). Each arrow carries a number \(n\), \(n = 1, 2, 3, 4\), which indicates the level of the embedding, \(\Lambda' = \Lambda - n \beta\).\(^{10}\) By our construction it also represents the invariant differential operator \(D_{n,\beta}\), cf. (26).

Further, we note that there is an additional symmetry w.r.t. to the dashed line in Fig. 1. It is relevant for the ERs and indicates the inte-
gral intertwining Knapp-Stein (KS) operators\(^16\) acting between the spaces \(C_{\chi^\pm}\) in opposite directions:

\[
G_{KS}^+ : C_{\chi^-} \to C_{\chi^+}, \quad G_{KS}^- : C_{\chi^+} \to C_{\chi^-} \quad (29)
\]

Note that the KS opposites are induced from the same irreps of \(M\).

This symmetry may be more explicit if we change the parametrization:

\[
\{m_1, m_2, m_3, m_4\} \rightarrow [m_1, m_2, m_3; c] \quad (30)
\]

so that the action of the KS operators on this signature is:

\[
G_{KS}^\pm : [m_1, m_2, m_3; c] \rightarrow [m_1, m_2, m_3; -c] \quad (31)
\]

This enables us to write the multiplet in a more compact way:

\[
\chi_{\pm} = \left[ m_1, m_2, m_3; \pm (m_{14,24} + m_{13,23}/2) \right] \quad (32)
\]

\[
\chi_{a} = \left[ m_1, m_2, m_{34}; \pm \frac{1}{2} m_{14,13,23,2} \right]
\]

\[
\chi_{b} = \left[ m_1, m_{23}, m_4; \pm \frac{1}{2} m_{14,13,2,2} \right]
\]

\[
\chi_{c} = \left[ m_{12}, m_{23}, m_4; \pm \frac{1}{2} m_{14,13} \right]
\]

\[
\chi_{d} = \left[ m_2, m_{13}, m_4; \pm \frac{1}{2} m_{24,23} \right]
\]

\[
\chi_{e} = \left[ m_{13}, m_2, m_{34}; \pm \frac{1}{2} m_{14,12} \right]
\]

\[
\chi_{f} = \left[ m_{23}, m_{12}, m_{34}; \pm \frac{1}{2} m_{24,2} \right]
\]

\[
\chi_{g} = \left[ m_{14}, m_2, m_3; \pm \frac{1}{2} m_{13,12} \right]
\]

\[
\chi_{h} = \left[ m_{23}, m_1, m_{24,2}; \pm \frac{1}{2} m_{34} \right]
\]

\[
\chi_{i} = \left[ m_{24}, m_1, m_3; \pm \frac{1}{2} m_{23,2} \right]
\]

\[
\chi_{j} = \left[ m_2, m_1, m_{24,23}; \pm \frac{1}{2} m_4 \right]
\]

\[
\chi_{k} = \left[ m_{24}, m_1, m_{23,2}; \pm \frac{1}{2} m_3 \right]
\]

Note that if in (32) we denote generically

\[
\chi_{\pm} = \{m_1, m_2, m_3, m_4^\pm\} = [m_1, m_2, m_3; c^\pm] \quad (33)
\]

then there is the relation

\[
|c^+| + |c^-| = |m_4^+| + |m_4^-| . \quad (34)
\]

Standardly we take \(\Lambda_0^-\) as the top Verma module (ER), since it is not embedded in any other Verma module. Relatedly, the VM \(\Lambda_0^+\) has no embedded Verma modules - it is the KS opposite of \(\Lambda_0^-\). We have indicated also the other KS opposites by denoting their signatures with “\(\pm\)”. 

Remark: Note that the pairs $\chi_j^\pm$ and $\chi_k^\pm$ are related by KS operators, but in each case the operator $G_{KS}^\pm$ is degenerated into a differential operator, namely, we have

\begin{align}
\Lambda_j^- \xrightarrow{m_4 \alpha_{14,24,3}} \Lambda_j^+ \\
\Lambda_k^- \xrightarrow{m_3 \alpha_{14,24,3}} \Lambda_k^+
\end{align}

4. Reduced multiplets

4.1. Main reduced multiplets

The above multiplets do not exhaust the relevant invariant differential operators. There are reduced multiplets which are obtained as we go to the walls of the relevant Weyl chambers. First, there are four main reduced multiplets $M_k$, $k = 1, 2, 3, 4$, which may be obtained by setting the parameter $m_k = 0$.

The main reduced multiplet $M_1$ contains 18 GVMs (ERs). Their signatures are given as follows:
\[ \chi_0^- = \{0, m_2, m_3, m_4\} \]  
\[ \chi_a^- = \{0, m_2, m_34, -m_4\} \]  
\[ \chi_b^- = \{0, m_{23}, m_4, -m_{34}\} \]  
\[ \chi_c^- = \{m_2, m_{23}, m_4, -m_{24,2}\} = \chi_d^- \]  
\[ \chi_e^- = \{m_{23}, m_2, m_{34}, -m_{24,23}\} = \chi_f^- \]  
\[ \chi_g^- = \{m_{24}, m_2, m_3, -m_{24,23}\} = \chi_i^- \]  
\[ \chi_h^- = \{m_{23}, 0, m_{24,2}, -m_{24,23,2,2}\} \]  
\[ \chi_k^- = \{m_2, 0, m_{24,23}, -m_{24,23,2,2}\} \]  
\[ \chi_l^- = \{m_{24}, 0, m_{23,2}, -m_{24,23,2,2}\} \]  
\[ \chi_j^+ = \{m_2, 0, m_{24,23}, -m_{24,24,2}\} \]  
\[ \chi_k^+ = \{m_{23}, 0, m_{24,2}, -m_{24,24,2}\} \]  
\[ \chi_g^+ = \{m_{24}, m_2, m_3, -m_{24,24,23,2,2}\} = \chi_i^+ \]  
\[ \chi_e^+ = \{m_{23}, m_2, m_{34}, -m_{24,24,23,2,2}\} = \chi_f^+ \]  
\[ \chi_c^+ = \{m_2, m_{23}, m_4, -m_{24,24,23,2,2}\} = \chi_d^+ \]  
\[ \chi_h^+ = \{0, m_{23}, m_4, -m_{24,24,23,2}\} \]  
\[ \chi_k^+ = \{0, m_{24}, m_3, -m_{24,24,23,2}\} \]  
\[ \chi_a^+ = \{0, m_{23}, m_4, -m_{24,24,23,2}\} \]  
\[ \chi_d^+ = \{0, m_2, m_{34}, -m_{24,24,23,2}\} \]

Note that only six of these GVMs (ERs), namely, \( \chi_c^-, \chi_e^-, \chi_g^-, \chi_e^+, \chi_g^+, \chi_f^+ \), are induced by finite-dimensional representations of \( \mathcal{M} \). We give the latter also in the more compact notation:

\[ \chi_{i+} = [m_2, m_{23}, m_4; \pm \frac{1}{2} m_{24,23}] \]  
\[ \chi_{i-} = [m_{23}, m_2, m_{34}; \pm \frac{1}{2} m_{24,2}] \]  
\[ \chi_{g-} = [m_{24}, m_2, m_3; \pm \frac{1}{2} m_{23,2}] \]

These GVMs are related in the following way:

\[ \Lambda_e^- \xrightarrow{m_3 \alpha_{24}} \Lambda_e^- \xrightarrow{m_4 \alpha_{24,3}} \Lambda_g^- \]

\[ \downarrow \quad \downarrow \quad \downarrow \]  

\[ \Lambda_e^+ \xleftarrow{m_3 \alpha_{24,3}} \Lambda_e^+ \xleftarrow{m_4 \alpha_{14}} \Lambda_g^+ \]

where the up-down arrows designate the \( G^\pm \) KS operators.
The reduced multiplets of type $M_2$ also contain 18 GVMs (ERs). We give only the signatures of those that are induced by finite-dimensional representations of $\mathcal{M}$:

$$
\chi^{-}_c = \{m_1, m_3, m_4, -m_{34}\} = \chi^{-}_b
$$
$$
\chi^{-}_f = \{m_3, m_1, m_{34}, -m_{1,34,1.3}\} = \chi^{-}_h
$$
$$
\chi^{-}_k = \{m_{34}, m_1, m_3, -m_{1,34,1.3}\} = \chi^{-}_i
$$
$$
\chi^{+}_c = \{m_{34}, m_1, m_3, -m_{1,34,1,3,3}\} = \chi^{+}_i
$$
$$
\chi^{+}_f = \{m_3, m_1, m_{34}, -m_{1,34,1,3,3}\} = \chi^{+}_h
$$
$$
\chi^{+}_c = \{m_1, m_3, m_4, -m_{1,34,1,3,3,2}\} = \chi^{+}_b
$$

or in the more compact notation:

$$
\chi^{\pm}_c = [m_1, m_3, m_4; \pm (m_{13} + m_4/2)]
$$
$$
\chi^{\pm}_f = [m_3, m_1, m_{34}; \pm \frac{1}{2}m_{34}]
$$
$$
\chi^{\pm}_k = [m_{34}, m_1, m_3; \pm \frac{1}{2}m_3]
$$

These GVMs are related in the following way:

$$
\Lambda^{-}_c \quad \Lambda^{-}_f \quad \Lambda^{+}_k
$$

$$
\Leftrightarrow
\Leftrightarrow
\Leftrightarrow
$$

$$
\Lambda^{-}_c \quad \Lambda^{-}_f \quad \Lambda^{+}_k
$$

Note that the Remark before (35) is again valid for the pair $\chi^{\pm}_k$.

The reduced multiplets of type $M_3$ contain 15 GVMs (ERs). Those induced by finite-dimensional representations of $\mathcal{M}$ are:

$$
\chi^{-}_b = \{m_1, m_2, m_4, -m_{4}\} = \chi^{-}_a
$$
$$
\chi^{-}_c = \{m_{12}, m_2, m_4, -2m_2 - m_4\} = \chi^{-}_e
$$
$$
\chi^{-}_d = \{m_2, m_{12}, m_4, -m_{12} - m_4\} = \chi^{-}_f
$$
$$
\chi^{-}_j = \{m_2, m_1, m_{2,2,4}, -m_{12, 12, 2,2,4}\} = \chi^{-}_h
$$
$$
\chi^{-}_k = \{m_{2,4}, m_1, m_{2,2}, -m_{12,12,2,2,4}\} = \chi^{-}_i
$$
$$
\chi^{+}_j = \{m_2, m_1, m_{2,2,4}, -m_{12,12,2,2,4}\} = \chi^{+}_h
$$
$$
\chi^{+}_d = \{m_2, m_{12}, m_4, -m_{12,12,2,2,4}\} = \chi^{+}_f
$$
$$
\chi^{+}_c = \{m_{12}, m_2, m_4, -m_{12,12,2,2,4}\} = \chi^{+}_e
$$
$$
\chi^{+}_b = \{m_1, m_2, m_4, -m_{12,12,2,2,4}\} = \chi^{+}_a
$$
or

\[
\begin{align*}
\chi_b^\pm &= [m_1, m_2, m_4; \pm (m_1 + 2m_2 + m_4/2)] \\
\chi_c^\pm &= [m_{12}, m_2, m_4; \pm (m_{12} + m_4/2)] \\
\chi_d^\pm &= [m_2, m_{12}, m_4; \pm (m_2 + m_4/2)] \\
\chi_j^\pm &= [m_2, m_1, m_{2,4}; \pm \frac{1}{2}m_4] \\
\chi_k^\pm &= [m_{2,4}, m_1, m_{2,2}; 0]
\end{align*}
\]

These GVMs are related in the following way:

\[
\begin{align*}
\Lambda^-_b &\xrightarrow{m_{20}^{24,34}} \Lambda^-_c &\xrightarrow{m_{10}^{14,34}} \Lambda^-_d &\xrightarrow{m_{20}^{24,24}} \Lambda^-_j &\xrightarrow{m_{40}^{24,3}} \Lambda^-_k \\
\downarrow &\downarrow &\downarrow &\downarrow &\parallel
\end{align*}
\] (43)

\[
\begin{align*}
\Lambda^+_b &\xleftarrow{m_{20}^{14,14,23,3}} \Lambda^+_c &\xleftarrow{m_{10}^{14,24,23,3}} \Lambda^+_d &\xleftarrow{m_{20}^{24,24,3,3}} \Lambda^+_j &\xleftarrow{m_{40}^{44}} \Lambda^+_k
\end{align*}
\]

Note that the Remark before (35) is again valid for the pair \(\chi_j^\pm\).

The reduced multiplets of type \(M_4\) also contain 15 GVMs (ERs). Those induced by finite-dimensional representations of \(\mathcal{M}\) are:

\[
\begin{align*}
\chi_a^- &= \{m_1, m_2, m_3, 0\} = \chi_0^- \\
\chi_c^- &= \{m_{13}, m_2, m_3, -2m_{23}\} = \chi_9^- \\
\chi_f^- &= \{m_{23}, m_{12}, m_3, -2m_{13}\} = \chi_i^- \\
\chi_k^- &= \{m_{23}, m_1, m_{23,2}, -2m_{13} - 2m_2\} = \chi_h^- \\
\chi_j^- &= \{m_2, m_1, m_{23,23}, -m_{13,13,23,2}\} = \chi_j^+ \\
\chi_k^+ &= \{m_{23}, m_1, m_{23,2}, -m_{13,13,23,2}\} = \chi_h^+ \\
\chi_f^+ &= \{m_{23}, m_{12}, m_3, -m_{13,13,23,2}\} = \chi_i^+ \\
\chi_c^+ &= \{m_{13}, m_2, m_3, -m_{13,13,23,2}\} = \chi_9^+ \\
\chi_a^+ &= \{m_1, m_2, m_3, -m_{13,13,23,2}\} = \chi_0^+
\end{align*}
\]

or

\[
\begin{align*}
\chi_a^\pm &= [m_1, m_2, m_3; \pm (m_{13,2} + m_3/2)] \\
\chi_c^\pm &= [m_{13}, m_2, m_3; \pm (m_{12} + m_3/2)] \\
\chi_f^\pm &= [m_{23}, m_{12}, m_3; \pm (m_2 + m_3/2)] \\
\chi_k^\pm &= [m_{23}, m_1, m_{23,2}; \pm \frac{1}{2}m_3] \\
\chi_j^\pm &= [m_2, m_1, m_{23,23}; 0]
\end{align*}
\] (45)
These GVMs are related in the following way:

$$\begin{align*}
\Lambda^-_a & \quad \Lambda^-_c \quad \Lambda^+_f \quad \Lambda^-_k \quad \Lambda^+_j \\
\downarrow & \quad \downarrow & \quad \uparrow & \quad \uparrow & \quad \parallel \\
\Lambda^+_a & \quad \Lambda^+_c \quad \Lambda^+_f \quad \Lambda^+_k \quad \Lambda^+_j
\end{align*}$$

Again the Remark before (35) is valid for the pair $\chi^\pm_k$.

### 4.2. Further reduced multiplets

We list only those multiplets that contain GVMs (ERs) that are induced by finite-dimensional representations of $\mathcal{M}$:

**Type M13:**

$$\begin{align*}
\chi^-_c &= \{m_2, m_2, m_4, -2m_2 - m_4\} \\
\chi^+_c &= \{m_2, m_2, m_4, -4m_2 - 2m_4\} \\
\chi^\pm_c &= [m_2, m_2, m_4; \pm (m_2 + m_4/2)]
\end{align*}$$

The two ERs of the above doublet are related only by the KS operators.

**Type M24:**

$$\begin{align*}
\chi^-_k &= \{m_3, m_1, m_3, -2m_1 - 2m_3\} \\
\chi^+_k &= \{m_3, m_1, m_3, -2m_1 - 3m_3\} \\
\chi^\pm_k &= [m_3, m_1, m_3; \pm \frac{1}{2}m_3]
\end{align*}$$

Again the Remark before (35) is valid for the pair $\chi^\pm_k$.

**Type M34:** Finally, we have a singlet:

$$\chi^\pm_k = \{m_2, m_1, 2m_2, -2m_1 - 4m_2\} = [m_2, m_1, 2m_2; 0]$$

### 5. Concluding remarks

Matters are arranged so that in every main multiplet only the ER with signature $\chi^-_0$ contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $F''_4$ with signature $\{m_1, m_2, m_3, m_4\}$. Thus, the main multiplets are in 1-to-1 correspondence with the finite-dimensional representations of $F''_4$. 

The subspace $E$ is annihilated by the operator $G^+$, and is the image of the operator $G^-$. The subspace $E$ is annihilated also by the invariant differential operator $D_{m_1 \alpha_4}$ acting from $\chi^{\pm}_0$ to $\chi_a^-$. When all $m_i = 1$ then $\dim E = 1$, and in that case $E$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore in that case the conformal weight is zero: $d = \frac{4}{2} + c|_{m_i=1} = 0$.

In the conjugate ER $\chi_0^+$ there is a unitary discrete series subrepresentation in an infinite-dimensional subspace $D_0$. It is annihilated by the operator $G^-$, and is in the image of the operator $G^+$ acting from $\chi^{\pm}_0$ and in the image of the invariant differential operator $D_{m_4}^{m_4}$ acting from $\chi_a^+$. Two more occurrences of discrete series are in the infinite-dimensional subspaces $D_a, D_b$ of the ERs $\chi^+_a, \chi^+_b$, resp. As above they are annihilated by the operator $G^-$, and are in the images of the operator $G^+$ acting from $\chi^+_a, \chi^+_b$, resp. Furthermore the subspace $D_a$ is in the image of the operator $D_{m_3}^{m_4}$ acting from $\chi^+_a$ and is annihilated by the invariant differential operator $D_{m_4}^{m_4}$. Furthermore the subspace $D_b$ is in the image of the operator $D_{m_2}^{m_4}$ acting from $\chi^+_b$ and is annihilated by the invariant differential operator $D_{m_3}^{m_3}$.

After the present paper the only split rank one case that is not treated yet is $Sp(N,1)$.

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Fig. 1. Main multiplets for $F_4^{\prime\prime}$