Effective models for conductance in magnetic fields:
derivation of Harper and Hofstadter models

G. De Nittis* and G. Panati**

* SISSA Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy
denittis@sissa.it

** Dipartimento di Matematica, Università di Roma “La Sapienza”, Roma, Italy
panati@mat.uniroma1.it

27th July 2010

Abstract

Some relevant transport properties of solids do not depend only on the spectrum of the electronic Hamiltonian, but on finer properties preserved only by unitary equivalence, the most striking example being the conductance. When interested in such properties, and aiming to a simpler model, it is mandatory to check that the simpler effective Hamiltonian is approximately unitarily equivalent to the original one, in the appropriate asymptotic regime. In this paper, we consider the Hamiltonian of an electron in a 2-dimensional periodic potential (e.g. generated by the ionic cores of a crystalline solid) under the influence of a uniform transverse magnetic field. We prove that such Hamiltonian is approximately unitarily equivalent to a Hofstadter-like (resp. Harper-like) Hamiltonian, in the limit of weak (resp. strong) magnetic field. The result concerning the case of weak magnetic field holds true in any dimension. Finally, in the limit of strong uniform magnetic field, we show that an additional periodic magnetic potential induces a non-trivial coupling of the Landau bands.

MSC 2010: 81Q15; 81Q20; 81V70; 81S05.

Key words: Hofstadter model, Harper model, magnetic Bloch electron.
Effective models for conductance in magnetic fields

G. De Nittis & G. Panati

Contents

1 Introduction 2

2 Description of the model 7

3 Space-adiabatic theory for the Hofstadter regime 9

3.1 Adiabatic parameter for weak magnetic fields 9

3.2 Separation of scales: the Bloch-Floquet transform 10

3.3 The periodic Hamiltonian and the gap condition 12

3.4 $\tau$-equivariant and special $\tau$-equivariant symbol classes 13

3.5 Semiclassics: quantization of equivariant symbols 14

3.6 Main result: effective dynamics for weak magnetic fields 15

3.7 Hofstadter-like Hamiltonians 23

4 Space-adiabatic theory for the Harper regime 25

4.1 Adiabatic parameter for strong magnetic fields 25

4.2 Separation of scales: the von Neumann unitary 27

4.3 Relevant part of the spectrum: the Landau bands 27

4.4 Symbol class and asymptotic expansion 30

4.5 Semiclassics: the $O(\delta^{4})$-approximated symbol 32

4.6 Main result: effective dynamics for strong magnetic fields 35

4.7 Harper-like Hamiltonians 42

4.8 Coupling of Landau bands in a periodic magnetic potential 43

A Some technical results 45

A.1 Self-adjointness of $H_{BL}$ and $H_{per}$ 45

A.2 Band spectrum of $H_{per}$ 46

A.3 The Landau Hamiltonian $H_{L}$ 47

B Canonical transform for fast and slow variables 49

B.1 The transform $W$ built “by hand” 49

B.2 The integral kernel of $W$ 51

1 Introduction

Schrödinger operators with periodic potentials and magnetic fields have been fascinating physicist and mathematicians for the last decades. Due to the competition between the crystal length scale and the magnetic length scale, these operators reveals striking features as fractal spectrum [GPPO00], anomalous localization [OGM08], quantization of the transverse conductance [TKNN82, ASS83, BSE94], anomalous thermodynamic phase diagrams [Avr04, OA01].

Due to its relevance for the Quantum Hall Effect (QHE) [Mor88, Gra07], we focus on
the two-dimensional Bloch-Landau Hamiltonian

$$H_{BL} := \frac{1}{2m} \left( -i\hbar \nabla_r + t_q \frac{|q|B}{2e} e_\perp \wedge r \right)^2 + \nabla_\Gamma(r), \quad (1)$$

acting in the Hilbert space $\mathcal{H}_{phy} = L^2(\mathbb{R}^2, d^2r)$, $r = (r_1, r_2) \in \mathbb{R}^2$. Here $c$ is the speed of light, $h := 2\pi\hbar$ is the Planck constant, $m$ is the mass and $q$ the charge (positive if $t_q = 1$ or negative if $t_q = -1$) of the charge carrier, $B$ is the strength of the external uniform time-independent magnetic field, $e_\perp = (0, 0, 1)$ is a unit vector orthogonal to the sample, and $\nabla_\Gamma$ is a periodic potential describing the interaction of the carrier with the ionic cores of the crystal. For the sake of a simpler notation, in this introduction we assume that the periodic lattice $\Gamma$ is simply $\mathbb{Z}^2$.

While extremely interesting, a direct analysis of the fine properties of the operator $H_{BL}$ is a formidable task. Thus the need to study simpler effective models which capture the main features of (1) in suitable physical regimes, as for example in the limit of weak (resp. strong) magnetic field. The relevant dimensionless parameter appearing in the problem is $h_B := \Phi_0/2\Phi_B \propto B^{-1}$, where $\Phi_0 = hc/e$ is the magnetic flux quantum, $\Phi_B = \Omega_\Gamma B$ is the flux of the external magnetic field through the unit cell of the periodicity lattice $\Gamma$ (whose area is $\Omega_\Gamma$) and $Z = |q|/e$ is the magnitude of the charge $q$ of the carrier in units of $e$ (the positron charge). It is also useful to introduce the reduced constant $\hbar_B := \hbar_B/2\pi$.

In the limit of weak magnetic field, $\hbar_B \to \infty$, one expects that the relevant features are captured by the well-known Peierls’ substitution [Pei33, Har55, Hof76], thus yielding to consider, for each Bloch band $E_* = E_*(k_1, k_2)$ of interest, the following effective model: in the Hilbert space $L^2(T^2, d^2k)$, $k$ being the Bloch momentum and $T^2 \simeq [0, 2\pi) \times [0, 2\pi)$, one considers the Hamiltonian operator

$$H_{Hof} \psi = E_*(k - \left( \frac{t_q}{\hbar_B} \right) \frac{1}{2} e_\perp \wedge i \nabla_k) \psi, \quad \psi \in L^2(T^2, dk). \quad (2)$$

In physicists’ words, the above Hamiltonian corresponds to replace the variables $k_1$ and $k_2$ in $E_*$ with the symmetric operators (magnetic momenta)

$$\mathcal{X}_1 := k_1 + \frac{i}{2} \left( \frac{t_q}{\hbar_B} \right) \frac{\partial}{\partial k_2}, \quad \mathcal{X}_2 := k_2 - \frac{i}{2} \left( \frac{t_q}{\hbar_B} \right) \frac{\partial}{\partial k_1}. \quad (3)$$

Since $[\mathcal{X}_1, \mathcal{X}_2] \neq 0$ the latter prescription is formal and (2) must be defined by an appropriate variant of the Weyl quantization. The rigorous justification of the Peierls’ substitution and the definition and the derivation of the Hamiltonian (2) are the content of Section 8.

The use of the operator (2) traces back to the pioneering works of R. Peierls [Pei33] and P. G. Harper [Har55], and its spectrum was extensively studied by D. Hofstadter in the celebrated paper [Hol76], where he specialized to the case $E_*(k_1, k_2) = 2 \cos k_1 + 2 \cos k_2$. In view of that, we call Hofstadter-like Hamiltonian any operator in the form (2), while the name Hofstadter Hamiltonian is used only for the special case above. However, this nomenclature is far to be unique. For instance M. A. Shubin in [Shu94] names discrete magnetic Laplacian the same operator (up to a Fourier transform).
As for the case of a strong magnetic field, $\hbar B \to 0$, the periodic potential can be considered a small perturbation of the Landau Hamiltonian, which provides the leading order approximation of $H_{BL}$. To the next order of accuracy in $\hbar B$, to each Landau level there corresponds an effective Hamiltonian, acting in $L^2(\mathbb{R}, dx)$, given (up to a suitable rescaling of the energy scale) by

$$
H_{Har} \psi = V_\Gamma \left( -i(\iota q \hbar B) \frac{\partial}{\partial x}, x \right) \psi, \quad \psi \in L^2(\mathbb{R}, dx),
$$

(4)

where the r.h.s refers to the ordinary ($\iota q \hbar B$)-Weyl quantization of the $\mathbb{Z}^2$-periodic function $V_\Gamma : \mathbb{R}^2 \to \mathbb{R}$. We refer to Section 4 for the definition of the effective Hamiltonian (4). We call Harper-like Hamiltonian any operator of the form (4), using the name Harper Hamiltonian for the special case $V_\Gamma(p, x) = 2 \cos(2\pi p) + 2 \cos(2\pi x)$.

**Remark 1.1** (nomenclature and historical overview). In a remarkable series of papers [HS88, HS90, HS89a] B. Helffer and J. Sjöstrand studied the relation between the spectrum of the operator (4) and the spectra of a one-parameter family of one-dimensional operators on $\ell^2(\mathbb{Z})$ defined by

$$
(h_\beta u)_n := u_{n+1} + u_{n-1} + 2 \cos(2\pi \theta n + \beta) u_n,
$$

(5)

where $\theta \in \mathbb{R}$ is a fixed number (deformation parameter) and $\beta \in [0, 2\pi)$ is the parameter of the family. In the work of the French authors the operator defined by (5) is called Harper operator (and indeed it was introduced by Harper in [Har55]). However, in the last three decades, the operator (5) has been extensively studied by many authors (see [Las05, Las94] for an updated review) with the name of almost-Mathieu operator. To avoid confusion and make the nomenclature clear, we chose to adhere to the most recent convention, using the name almost-Mathieu operator for (5). We thus decided to give credits to Harper’s work by associating his name to the operator (4).

The regime of strong magnetic field was originally investigated by A. Rauh [Rau74, Rau75]. However the correct effective model, the operator (4), was derived firstly by M. Wilkinson [Wil87] and then, rigorously, by J. Bellissard in an algebraic context [Bel88] and by B. Helffer and J. Sjöstrand in [HS89b], inspired by the latter paper. In both these papers the case of weak magnetic field is also considered. In particular, in [HS89b] it is proven that $H_{Har}$ (resp. $H_{Hof}$) has, locally on the energy axis, the same spectrum and the same density of states of $H_{BL}$, as $\hbar B \to 0$ (resp. $\hbar B \to \infty$).

Beyond the spectrum and the density of states, there are other mathematical properties of $H_{BL}$ which reveal interesting physics, as for example the orbital magnetization [GA03, TCVR05, CTVR06] or the transverse (Hall) conductance (1). These properties are not invariant under a loose equivalence relation as isospectrality, then it is important to show that $H_{Har}$ (resp. $H_{Hof}$) is approximately unitarily equivalent to $H_{BL}$ in the appropriate

---

(1) Some authors refer to the “Hall conductivity” rather than to the “Hall conductance”. Indeed, in two space dimensions the conductivity (microscopic quantity) coincides exactly with the conductance (macroscopic quantity), so the two concepts are synonymous. In this sense the quantization of the Hall conductivity is a macroscopic quantum phenomenon.
limit. This is the main goal of this paper. The problem is not purely academic, since there exist examples of two isospectral Schrödinger operators which are however not unitarily equivalent and which exhibit different values of the transverse conductance. In particular, in [DFP10b] we prove that the Hofstadter operator and the Harper operator are isospectral but not unitarily equivalent. One concludes that, in the study of the conductance, it is not enough to prove that the effective model is isospectral to the original Hamiltonian.

We thus introduce the stronger notion of \textbf{unitarily effective model}, referring to the concept of almost-invariant subspace introduced by G. Nenciu \cite{Nen02} and to the related notion of effective Hamiltonian \cite{PST03b, PST03a, Teu03}, which we shortly review. Let us focus on the regime of weak (resp. strong) magnetic field and define

$$
\varepsilon = \varepsilon(B) = 1/\hbar B \quad \text{(resp. } 
\varepsilon(B) = \hbar B) \quad \text{so that } \varepsilon \to 0 \text{ in the relevant limit.}
$$

Let \(\Pi_\varepsilon\) be an orthogonal projector in \(H_{\text{phy}}\) such that, for any \(N \in \mathbb{N}, N \leq N_0\) there exist a constant \(C_N\) such that

$$
\| [\mathbf{H}_{BL}; \Pi_\varepsilon] \| \leq C_N \varepsilon^N \tag{6}
$$

for \(\varepsilon\) sufficiently small. Then \(\text{Ran } \Pi_\varepsilon\) is called an \textbf{almost-invariant subspace} \cite{Nen02, Teu03} at accuracy \(N_0\), since it follows by a Duhammel’s argument that

$$
\| (1 - \Pi_\varepsilon) \, e^{-i s \mathbf{H}_{BL}} \, \Pi_\varepsilon \| \leq C_N |s| \varepsilon^N \tag{8}
$$

and for every \(s \in \mathbb{R}, N \leq N_0\). Granted the existence of such a subspace, we call \textbf{(unitarily) effective Hamiltonian} a self-adjoint operator \(H_{\text{eff}}\) acting on a Hilbert space \(H_{\text{ref}}\) such that there exists a unitary \(U_\varepsilon : \text{Ran } \Pi_\varepsilon \to H_{\text{ref}}\) such that for any \(N \in \mathbb{N}, N \leq N_0\), one has

$$
\| (\Pi_\varepsilon \mathbf{H}_{BL} - U_\varepsilon^{-1} H_{\text{eff}} U_\varepsilon) \Pi_\varepsilon \| \leq C'_N \varepsilon^N. \tag{7}
$$

The estimates (6) and (7) imply that

$$
\| (e^{-i s \mathbf{H}_{BL}} - U_\varepsilon^{-1} e^{-i s H_{\text{eff}} U_\varepsilon}) \Pi_\varepsilon \| \leq C''_N |s| \varepsilon^N. \tag{8}
$$

When the macroscopic time-scale \(t = \varepsilon s\) is physically relevant, the estimate above is simply rescaled. The triple \((H_{\text{ref}}, U_\varepsilon, H_{\text{eff}})\) is, by definition, a unitarily effective model for \(H_{BL}\). To our purposes, it is important to notice that the asymptotic unitary equivalence in (7) assures that the \textbf{topological invariants} related with the spectral projections of \(\Pi, \mathbf{H}_{BL}^{\varepsilon}, \mathbf{H}_{\text{eff}}^{\varepsilon}\) (\(K\)-theory, Chern numbers, . . . ) are equal to those of \(H_{\text{eff}}\), for \(\varepsilon\) sufficiently small \cite{DP09}.

In this paper we prove that in the limit \(\hbar B \to \infty\) the Hofstadter-like Hamiltonian (2) provides a unitarily effective model for \(H_{BL}\) with accuracy \(N_0 = 1\), and we exhibit an iterative algorithm to construct an effective model at any order of accuracy \(N_0 \in \mathbb{N}\) (Theorem 3.12). As for the limit \(\hbar B \to \infty\), up to a rescaling of the energy, the non-trivial leading order (accuracy \(N_0 = 1\)) for the effective Hamiltonian is given by the Harper-like Hamiltonian (4). We also exhibit the effective Hamiltonian with accuracy \(N_0 = 2\), i.e. up to errors of order \(O(\varepsilon^2)\) (Theorem 4.7). Moreover, due to the robustness of the adiabatic techniques, we can generalize the simple model described by (1) to include other potentials, see (9), including in particular a periodic vector potential \(A_F\). This terms produces interesting consequences especially in the Harper regime (see Section 4.8) and it could play a relevant role in the
theory of orbital magnetization. This kind of generalization is new with respect to both [Bel88] and [HS89b].

Our proof is based on the observation that both the Hofstadter and the Harper regime are space-adiabatic limits, and can be treated in the framework of space-adiabatic perturbation theory (SAPT) [PST03b, PST03a, see also [Teu03]. As for the Hofstadter regime, the proof follows ideas similar to the ones in [PST03a]. Our generalization allows however to consider a constant magnetic field (while in [PST03a] the vector potential is assumed in $C_b^\infty(\mathbb{R}^d)$) and to include a periodic vector potential. Moreover the proof extends the one in [PST03a], in view of the use of the special symbol classes defined in Section 3.4. On the contrary, from the discussion of the Harper regime $\hbar B \to 0$ some new mathematical problems emerge. Then, although the “philosophy” of the proof of Theorem 4.7 is of SAPT-type, the technical part is new as it will be explained in Section 4. Notice that the regime of weak magnetic field can also be conveniently approached by using the magnetic Weyl quantization [MP04, MPR05, IMP07, IMP09], a viewpoint which is investigated in [DL10].

For the sake of completeness, we summarize some salient aspects of the SAPT. We refer to specific references (e.g. [Teu03]) for a complete exposition. Let $H$ be the Hamiltonian of a generic physical system which acts on the total (or physical) Hilbert space $\mathcal{H}$. For the SAPT to be applicable, three important ingredients needs: (i) a distinction between fast and slow degrees of freedom which is mathematically expressed by a unitary decomposition of the physical space $\mathcal{H}$ into a product space $\mathcal{H}_s \otimes \mathcal{H}_f$ (or, more generally, a direct integral), the first factor being the space of the slow degrees of freedom and the second the space of the fast degrees of freedom; $\mathcal{H}_s \cong L^2(\mathcal{M})$ for suitable measure space $\mathcal{M}$ is also required; (ii) a dimensionless adiabatic parameter $\varepsilon \ll 1$ that quantifies the separation of scales between the fast and slow degrees of freedom and which measures how far are the slow degrees of freedom to be “classical” in terms of some process of quantization; (iii) a relevant part of the spectrum for the fast dynamics which remains separated from the rest of the spectrum under the perturbation caused by the slow degrees of freedom.

A numerical simulation of the spectrum of the Hofstadter operator, as a function of the parameter $\varepsilon = 1/\hbar B$, leads to a fascinating picture known as Hofstadter butterfly [Hof76]. Since the spectrum has zero measure as a subset of the square, the physically relevant object is its complement, the resolvent set. It has been pointed out by D. Osadchy and J. Avron [OA01] that the open connected regions of the resolvent set (islands) can be associated to different thermodynamic phases (at zero temperature) of a gas of non interacting fermions in a periodic potential, with $\varepsilon \propto B$ and the chemical potential as thermodynamic coordinates. The different phases are labeled by an integer (topological quantum number), interpreted as the value of the transverse conductance of the system in units of $e^2/h$ in the limit of weak magnetic field. The latter integers are conveniently visualized by different colors, thus leading to the colored Hofstadter butterfly [OA01, Avr04]. With this language in mind, the main result of this paper can be reformulated by saying that the Hofstadter-like and Harper-like Hamiltonians are “colour-preserving effective models” for the original Bloch-Landau Hamiltonian. Thus they describe, though in a distorted and approximated way, some aspects of the thermodynamics of the original system.

Acknowledgments. It is a pleasure to thank Y. Avron, J. Bellissard, G. Dell’Antonio,
M. Măntoiu, D. Masoero, H. Spohn and S. Teufel for many useful discussions, and F. Faure for stimulating comments on a previous version of the paper. We are grateful to B. Hellfer for suggesting useful references. Financial support by the INdAM-GNFM project Giovane ricercatore 2009 is gratefully acknowledged.

2 Description of the model

A generalized Bloch-Landau Hamiltonian

The Hamiltonian (1) describes the dynamics of particle with mass $m$ and charge $q$ which interacts with the ionic structure of a two dimensional crystal and with an external orthogonal uniform magnetic field. A more general model is provided by the operator

$$H_{BL} := \frac{1}{2m} \left[ -i\hbar \nabla_r - \frac{q}{c} A_\Gamma (r) - \frac{q}{c} A (r) \right]^2 + V_\Gamma (r) + q \Phi (r)$$

(9)

still called Bloch-Landau Hamiltonian and, with an abuse of notation, still denoted with the same symbol used in (1). The vector-valued function $A := (A_1, A_2)$ is a vector potential corresponding to an (orthogonal) external magnetic field $B = \nabla r \wedge A = (\partial_1 A_2 - \partial_2 A_1) e_\perp$, $\Phi$ is a scalar potential corresponding to a (parallel) external electric field $E = -\nabla r \Phi$ and $A_\Gamma$ and $V_\Gamma$ are internal periodic potentials which describe the electromagnetic interaction with the ionic cores of the crystal lattice. The external vector potential is assumed to have the following structure

$$A (r) = A_0 (r) + A_B (r),$$

(10)

where $A_0$ is a bounded function and $A_B$ describes a uniform orthogonal magnetic field of strenght $B$, i.e. in the symmetric gauge

$$A_B (r) = \frac{B}{2} e_\perp \wedge r = \left( -\frac{B}{2} r_2, \frac{B}{2} r_1 \right), \quad \nabla_r \wedge A_B = B e_\perp, \quad \nabla_r \cdot A_B = 0.$$  

(11)

The evolution of the system is prescribed by the Schrödinger equation

$$i\hbar \frac{d}{ds} \psi (r, s) = H_{BL} \psi (r, s),$$

(12)

where $s$ corresponds to the microscopical time-scale.

Mathematical description of the crystal structure

The periodicity of the crystal is described by a two dimensional lattice $\Gamma \subset \mathbb{R}^2$ (namely a discrete subgroup of maximal dimension of the Abelian group $(\mathbb{R}^2, +)$), thus $\Gamma \simeq \mathbb{Z}^2$. Let $\{a, b\} \subset \mathbb{R}^2$ be two generators of $\Gamma$, i.e.

$$\Gamma = \{ \gamma \in \mathbb{R}^2 : \gamma = n_1 a + n_2 b, \quad n_1, n_2 \in \mathbb{Z} \}.$$  

The fundamental or Voronoi cell of $\Gamma$ is $M_\Gamma := \{ r \in \mathbb{R}^2 \mid r = l_1 a + l_2 b, \quad l_1, l_2 \in [0, 1] \}$ and its area is given by $\Omega_\Gamma = |a \wedge b|$. We fix the orientation of the lattice in such a way that
\( \Omega_\Gamma = (a_1 b_2 - a_2 b_1) > 0. \) We say that a function \( f_\Gamma : \mathbb{R}^2 \to \mathbb{C} \) is \( \Gamma \)-periodic if \( f_\Gamma(r + \gamma) = f_\Gamma(r) \) for all \( \gamma \in \Gamma \) and all \( r \in \mathbb{R}^2 \). The electrostatic and magnetostatic crystal potentials \( V_\Gamma \) and \( A_\Gamma \) are assumed to be \( \Gamma \)-periodic according to the previous definition.

An important notion is that of dual lattice \( \Gamma^* \) which is the set of the vectors \( \gamma^* \in \mathbb{R}^2 \) such that \( \gamma^* \cdot \gamma \in 2\pi \mathbb{Z} \) for all \( \gamma \in \Gamma \). Let \( \{a^*, b^*\} \subset \mathbb{R}^2 \) be defined by the relations \( a^* \cdot a = b^* \cdot b = 1 \) and \( a^* \cdot b = b^* \cdot a = 0 \); these vectors are the generators of the lattice \( \Gamma^* \), i.e.,

\[
\Gamma^* = \{ \gamma^* \in \mathbb{R}^2 : \gamma^* = m_1 2\pi a^* + m_2 2\pi b^*, \quad m_1, m_2 \in \mathbb{Z}\}.
\]

The Brillouin zone \( M_{\Gamma^*} := \{ k \in \mathbb{R}^2 \mid k = k_1 a^* + k_2 b^*, \quad k_1, k_2 \in [0, 2\pi] \} \) is the fundamental cell of the dual lattice \( \Gamma^* \). The explicit expressions for the dual generators \( \{a^*, b^*\} \) in terms of the basis \( \{a, b\} \) is

\[
a^* = e_\perp \wedge b = \frac{1}{\Omega_\Gamma} (b_2, -b_1), \quad b^* = -e_\perp \wedge a = \frac{1}{\Omega_\Gamma} (-a_2, a_1).
\]

It follows from (13) that the surface of the Brillouin zone is \( \Omega_{\Gamma^*} = (2\pi)^2 |a^* \cdot b^*| = (2\pi)^2 / \Omega_\Gamma \).

Given a \( \Gamma \)-periodic function \( f_\Gamma \), we denote its Fourier decomposition as

\[
f_\Gamma(r) = \sum_{\gamma^* \in \Gamma^*} f(\gamma^*) e^{i\gamma^* \cdot r} = \sum_{m_1, m_2 \in \mathbb{Z}} f_{m_1, m_2} e^{i2\pi(m_1 a^* + m_2 b^*) \cdot r}.
\]

A \( \mathbb{Z}^2 \)-periodic function \( f : \mathbb{R}^2 \to \mathbb{C} \) is a function periodic with respect to an orthonormal lattice, namely such that \( f(x_1 + 1, x_2) = f(x_1, x_2 + 1) = f(x_1, x_2) \) for all \( x_1, x_2 \in \mathbb{R} \). If one replaces the two real variables by \( x_1 := a^* \cdot r \) and \( x_2 := b^* \cdot r \) one has that \( f_\Gamma(r) := f(a^* \cdot r, b^* \cdot r) \) is \( \Gamma \)-periodic in \( r \). Every \( \Gamma \)-periodic function can be obtained in this way.

**Assumptions on the regularity of the potentials and self-adjointness**

Let us denote by \( C^n_0(\mathbb{R}^2, \mathbb{R}) \) the space of real-valued \( n \)-times differentiable functions (smooth functions if \( n = \infty \)) with continuous and bounded derivatives up to order \( n \). Concerning the internal potentials \( A_\Gamma \) and \( V_\Gamma \) we need to assume that:

**ASSUMPTION (A_s) [internal potentials, strong form].** The \( \Gamma \)-periodic potential \( V_\Gamma \) and the two components of the \( \Gamma \)-periodic vector potential \( A_\Gamma \) are functions of class \( C_0^\infty(\mathbb{R}^2, \mathbb{R}) \).

Sometime will be enough to consider a weaker version of this assumption, namely:

**ASSUMPTION (A_w) [internal potentials, weak form].** The two components of the \( \Gamma \)-periodic vector potential \( A_\Gamma \) are in \( C_0^1(\mathbb{R}^2, \mathbb{R}) \). The \( \Gamma \)-periodic potential \( V_\Gamma \) verifies the condition \( \int_{M_{\Gamma}} |V_\Gamma(r)|^2 \, d^2 r < +\infty \).

Assumption (A_s) implies that \( V_\Gamma \) is uniformly locally \( L^2 \) and this implies also that \( V_\Gamma \) is infinitesimally bounded with respect to \( -\Delta_r \) (see Theorem XIII.96 in [RS78]). Concerning the external potentials \( A \) and \( \Phi \), we need to assume that:
**Assumption (B) [external potentials].** The scalar potential $\Phi$ is of class $C^\infty_b(\mathbb{R}^2, \mathbb{R})$. The vector potential $\mathbf{A}$ consists of a linear term $\mathbf{A}_B$ of the form (11) plus a bounded term $\mathbf{A}_0$ which is of class $C^\infty_b(\mathbb{R}^2, \mathbb{R})$.

When the external potentials $\mathbf{A}$ and $\Phi$ vanish, the Bloch-Landau Hamiltonian (9) reduces to the periodic Hamiltonian (or Bloch Hamiltonian)

$$H_{\text{per}} := \frac{1}{2m} \left( -i\hbar \nabla_r - \frac{q}{e} A(r) \right)^2 + V(r).$$

The domains of self-adjointness of $H_{\text{BL}}$ and $H_{\text{per}}$ are described in the following proposition. Its proof, together with some basic notion about the Sobolev space $H^2(\mathbb{R}^2)$ and the magnetic-Sobolev space $H^2_M(\mathbb{R}^2)$, is postponed to Section A.1.

**Proposition 2.1.** Let Assumptions (A_w) and (B) hold true. Then both $H_{\text{BL}}$ and $H_{\text{per}}$ are essentially self-adjoint operators on $L^2(\mathbb{R}^2, d^2r)$ with common domain of essential self-adjointness the space of smooth functions with compact support $C^\infty_c(\mathbb{R}^2, \mathbb{C})$. Moreover the domain of self-adjointness of $H_{\text{per}}$ is $H^2(\mathbb{R}^2)$ while the domain of self-adjointness of $H_{\text{BL}}$ is $H^2_M(\mathbb{R}^2)$.

## 3 Space-adiabatic theory for the Hofstadter regime

### 3.1 Adiabatic parameter for weak magnetic fields

The SAPT for a Bloch electron developed in [PST03a] is based on the existence of a separation between the microscopic space scale fixed by the lattice spacing $\ell := \sqrt{\Omega_\Gamma}$, and a macroscopic space scale fixed by the scale of variation of the “slowly varying” external potentials. The existence of such a separation of scales is expressed by introducing a dimensionless parameter $\varepsilon \ll 1$ (adiabatic parameter) to control the scale of variation of the vector potential and the scalar potential $\Phi$ appearing in (9), namely by setting $\mathbf{A} = \mathbf{A}(\varepsilon r)$ and $\Phi = \Phi(\varepsilon r)$. In particular the external magnetic and electric fields are weak compared to the fields generated by the ionic cores.

It is useful to rewrite the ($\varepsilon$-dependent) Hamiltonian (9) in a dimensionless form. The microscopic unit of length being $\ell$, we introduce the dimensionless position vector $x := r/\ell$ and the dimensionless gradient $\nabla_x = \ell \nabla_r$. Moreover, since the vector potential has the dimension of a length times a magnetic field, then $A(\varepsilon x) := \varepsilon / \ell B \mathbf{A}(\varepsilon x)$ is a dimensionless function, with $B$ a dimensional constant which fixes the order of magnitude of the magnetic field due to the external vector potential $\mathbf{A}$. Similarly for $\mathbf{A}_\Gamma$ (with $\varepsilon = 1$). Factoring out the dimensional constants one finds

$$H_{\text{BL}} := \frac{1}{\varepsilon_0} H_{\text{BL}} = \frac{1}{2} \left[ -i \nabla_x - \frac{\varepsilon \Omega_\Gamma B \mathbf{r}}{\varepsilon_0} A(\varepsilon x) - \varepsilon \frac{q \Omega_\Gamma B}{\varepsilon_0} A(\varepsilon x) \right] + V(\varepsilon x) + \phi(\varepsilon x),$$

(16)
where $E_0 := \hbar^2 / m \Omega$ is the natural unit of the energy fixed by the problem, $V_\Gamma(x) := 1 / \varepsilon_0 \partial_x \Phi(\varepsilon x)$ and $\phi(\varepsilon x) := q / \varepsilon_0 \Phi(\varepsilon x)$ are both dimensionless quantities. The constant $h_\Gamma$ will play no particular role in the rest of this paper, so it is reabsorbed into the definition of the dimensionless vector potential $A_\Gamma$, i.e. formally $h_\Gamma = 1$.

Comparing the dimensional Hamiltonian (16) with the original Hamiltonian (9), or observing that the strength of the magnetic field goes to zero (at least linearly) with $\varepsilon$, it is physically reasonable to estimate $\varepsilon h_B \propto 1$. This is rigorously true in the case in which the external electromagnetic field is uniform.

The external force due to $A$ and $\phi$ are of order of $\varepsilon$ and therefore have to act over a time of order $\varepsilon^{-1}$ to produce a finite change, which defines the macroscopic time-scale. The macroscopic dimensionless (slow) time-scale is fixed by $t := \varepsilon E_0 / \hbar s$ where $s$ is the dimensional microscopic (fast) time-scale. With this change of scale the Schrödinger equation (12) reads

$$i \varepsilon \frac{d}{dt} \psi = H_{BL} \psi$$

with $H_{BL}$ given by equation (16).

**Remark 3.1.** Observe that from the definition of the dimensionless periodic potential $A_\Gamma$ and $V_\Gamma$ it follows that they are periodic with respect to the transform $x \mapsto x + \gamma / \ell$. This means that $A_\Gamma$ and $V_\Gamma$ are periodic with respect to a “normalized” lattice whose fundamental cell has surface 1.

### 3.2 Separation of scales: the Bloch-Floquet transform

To make explicit the presence of the linear term of the external vector potential, we can rewrite the (16) as follows

$$H_{BL} = \frac{1}{2} \left[ -i \nabla - A_\Gamma(x) - A_0(\varepsilon x) - t_q \frac{1}{2} e_\perp \wedge \varepsilon x \right]^2 + V_\Gamma(x) + \phi(\varepsilon x),$$

where the adiabatic parameter $\varepsilon$ expresses the separation between the macroscopic length-scale, defined by the external potentials, and the microscopic length-scale, defined by the internal $\Gamma$-periodic potentials. The separation between slow and fast degrees of freedom can be expressed decomposing the physical Hilbert space $\mathcal{H}_{phy} = L^2(\mathbb{R}^2, d^2 x)$ into a product of two Hilbert spaces or, more generally, into a direct integral. To this end, we use the Bloch-Floquet transform [Kuc93]. As in [PST03a] we define the (modified) Bloch-Floquet transform $Z$ of a function $\psi \in \mathcal{S}(\mathbb{R}^2)$ to be

$$(Z \psi)(k, \theta) := \sum_{\gamma \in \Gamma} e^{-i(\theta + \gamma) \cdot k} \psi(\theta + \gamma), \quad (k, \theta) \in \mathbb{R}^2 \times \mathbb{R}^2. \quad (19)$$

Directly from the definition one can check the following periodicity properties:

$$Z \psi)(k, \theta + \gamma) = (Z \psi)(k, \theta) \quad \forall \gamma \in \Gamma \quad (20)$$

$$(Z \psi)(k + \gamma^*, \theta) = e^{-i\theta \cdot \gamma^*} (Z \psi)(k, \theta) \quad \forall \gamma^* \in \Gamma^* \quad (21)$$
Equation (20) shows that for any fixed \( k \in \mathbb{R}^2 \), \( (Z\psi)(k, \cdot) \) is a \( \Gamma \)-periodic function and can be seen as an element of \( \mathcal{H}_\tau := L^2(\mathcal{V}, d^2\theta) \) with \( \mathcal{V} := \mathbb{R}^2/\Gamma \) a two-dimensional slant torus (Voronoi torus). The torus \( \mathcal{V} \) coincides with the translation cell \( M_\Gamma \) endowed with the identification of the opposite edges and \( d^2\theta \) denotes the (normalized) measure induced on \( \mathcal{V} \) by the identification with \( M_\Gamma \). The Hilbert space \( \mathcal{H}_1 \) is the space of fast degrees of freedom, corresponding to the existence of fast and slow degrees of freedom. Equation (21) involves a unitary representation \( \tau : \Gamma^* \to \mathcal{U}(\mathcal{H}_1) \) of the group of the (dual) lattice translations \( \Gamma^* \) on the Hilbert space \( \mathcal{H}_1 \). For every \( \gamma^* \in \Gamma^* \) the unitary operator \( \tau(\gamma^*) \) is the multiplication with \( e^{i\theta^*} \). It will be convenient to introduce the Hilbert space

\[
\mathcal{H}_\tau := \left\{ \psi \in L^2_{loc}(\mathbb{R}^2, d^2k, \mathcal{H}_1) : \psi(k - \gamma^*, \cdot) = \tau(\gamma^*) \psi(k, \cdot) \right\}
\]

(22)
equipped with the inner product \( \langle \psi; \varphi \rangle_{\mathcal{H}_\tau} := \int_{M_\Gamma^*} (\psi(k); \varphi(k))_{\mathcal{H}_1} d^2k \) where \( d^2k := \frac{d^2k}{(2\pi)^2} \) is the normalized measure. There is a natural isomorphism from \( \mathcal{H}_\tau \) to \( L^2(M_{\Gamma^*}, d^2k, \mathcal{H}_1) \) given by restriction from \( \mathbb{R}^2 \) to \( M_{\Gamma^*} \) and with inverse given by \( \tau \)-covariant continuation, as suggested by (21). The Bloch-Floquet transform (19) extends to a unitary map

\[
Z : \mathcal{H}_{\text{phy}} \to \mathcal{H}_{\tau} \simeq L^2(M_{\Gamma^*}, d^2k, \mathcal{H}_1) \simeq L^2(M_{\Gamma^*}, d^2k) \otimes \mathcal{H}_1.
\]

(23)
The Hilbert space \( L^2(M_{\Gamma^*}, d^2k) \) can be seen as the space of slow degrees of freedom and in this sense the transform \( Z \) produces a decomposition of the physical Hilbert space according to the existence of fast and slow degrees of freedom.

We need to discuss how differential and multiplication operators behave under \( Z \). Let \( Q = (Q_1, Q_2) \) be the multiplication by \( x = (x_1, x_2) \) defined on its maximal domain and \( P = (P_1, P_2) = -i\nabla_x \) with domain the Sobolev space \( \mathcal{H}^1(\mathbb{R}^2) \), then from (19) it follows:

\[
ZPZ^{-1} = k \otimes 1_{\mathcal{H}_1} + 1_{L^2(M_{\Gamma^*})} \otimes -i\nabla_\theta, \quad ZQZ^{-1} = i\nabla_k^\tau
\]

(24)
where \( -i\nabla_\theta \) acts on the domain \( \mathcal{H}^1(\mathcal{V}) \) while the domain of the differential operator \( i\nabla_k^\tau \) is the space \( \mathcal{H}_\tau \cap \mathcal{H}^1_{\text{loc}}(\mathbb{R}^2, \mathcal{H}_1) \), namely it consists of vector-valued distributions which are in \( \mathcal{H}^1(M_{\Gamma^*}, \mathcal{H}_1) \) and satisfy the \( \theta \)-dependent boundary condition associated with (21). The central feature of the Bloch-Floquet transform is, however, that multiplication operators corresponding to \( \Gamma \)-periodic functions like \( A_\Gamma \) or \( V_\Gamma \) are mapped into multiplication operators corresponding to the same function, i.e.

\[
ZA_\Gamma(x)Z^{-1} = 1_{L^2(M_{\Gamma^*})} \otimes A_\Gamma(\theta) \quad ZV_\Gamma(x)Z^{-1} = 1_{L^2(M_{\Gamma^*})} \otimes V_\Gamma(\theta).
\]

(25)
Let \( H^Z := ZH_{\text{BL}}Z^{-1} \) be the Bloch-Floquet transform of the Bloch-Landau Hamiltonian (18). According to the relations (24) and (25) one obtains from (18) that

\[
H^Z = \frac{1}{2} \left[ -i\nabla_\theta k - A_\Gamma(\theta) - A_0(i\epsilon \nabla_k) - i_\theta \frac{1}{2} e_\perp \wedge (i\epsilon \nabla_k^\tau) \right]^2 + V_\Gamma(\theta) + \phi(i\epsilon \nabla_k^\tau)
\]

(26)
with domain of self-adjointness \( Z\mathcal{H}^2_\mathcal{M}(\mathbb{R}^2) \subset \mathcal{H}_\tau \), i.e. the image under \( Z \) of the second magnetic-Sobolev space.
3.3 The periodic Hamiltonian and the gap condition

When $\varepsilon = 0$ the Bloch-Landau Hamiltonian (18) reduces to the periodic Hamiltonian

$$H_{\text{per}} = \frac{1}{2} [-i\nabla_x - A_{\Gamma}(x)]^2 + V_\Gamma(x).$$

(27)

According to (26) the Bloch-Floquet transform maps $H_{\text{per}}$ into a fibered operator. In other words, denoting $H_{\text{per}}^Z := H_{\text{per}}Z^{-1}$, one has $H_{\text{per}}^Z = \int_{\mathbb{R}^2} H_{\text{per}}(k) \, dk$ where, for each $k \in M_{\Gamma^*}$,

$$H_{\text{per}}(k) = \frac{1}{2} [-i\nabla_\theta + k - A_{\Gamma}(\theta)]^2 + V_\Gamma(\theta).$$

(28)

The operator $H_{\text{per}}(k)$ acts on $\mathcal{H}_f = L^2(\mathbb{V}, d^2\theta)$ with self-adjointness domain $\mathcal{D} := \mathcal{H}^2(\mathbb{V})$ (the second Sobolev space) independent of $k \in M_{\Gamma^*}$. Moreover it is easy to check that the Bloch-Floquet transform induces the following property of periodicity, called $\tau$-equivariance:

$$H_{\text{per}}([k] - \gamma^*) = \tau(\gamma^*) H_{\text{per}}([k]) \tau(\gamma^*)^{-1} \in \Gamma^* \quad \forall \gamma^* \in \Gamma^*. \quad (29)$$

where the notation $k := [k] - \gamma^*$ denotes the a.e.-unique decomposition of $k \in \mathbb{R}^2$ as a sum of $[k] \in M_{\Gamma^*}$ and $\gamma^* \in \Gamma^*$.

**Remark 3.2** (Analyticity). For any $k \in \mathbb{R}^2$, let $I(k)$ be the unitary operator acting on $\mathcal{H}_f$ as the multiplication by $e^{-i\theta - k}$. Obviously $I(k) = I([k] - \gamma^*) = I([k])\tau(\gamma^*)^{-1}$. A simple computation shows that

$$H_{\text{per}}(k) = I(k) H_{\text{per}}(0) I(k)^{-1}$$

(30)

where the equality holds on the fixed domain of self-adjointness $\mathcal{D} = \mathcal{H}^2(\mathbb{V})$. The $\tau$-equivariance property (29) follows immediately from (30). Moreover from (30) is evident that $H_{\text{per}}(k)$ defines an analytic family (of type A) in the sense of Kato (see [RS78] Chapter XII). Finally a short computation shows

$$(\partial_{kj} H_{\text{per}})(k) = -i I(k) [\theta; H_{\text{per}}(0)] I(k)^{-1} = I(k) (-i \nabla_\theta - A_{\Gamma}(\theta)) j I(k)^{-1}$$

and $(\partial_{kj}^2 H_{\text{per}})(k) = 0$ on the domain $\mathcal{D}$. ♦

The spectrum of $H_{\text{per}}$, which coincides with the spectrum of $H_{\text{per}}^Z$, is given by the union of all the spectra of $H_{\text{per}}(k)$. The following classical results hold true:

**Proposition 3.3.** Let $V_\Gamma$ and $A_{\Gamma}$ satisfy Assumption (A_w), then:

(i) for all $k \in \mathbb{R}^2$ the operator $H_{\text{per}}(k)$ defined by (27) is self-adjoint with domain $\mathcal{D} = \mathcal{H}^2(\mathbb{V})$ and is bounded below;

(ii) $H_{\text{per}}(k)$ has compact resolvent and its spectrum is purely discrete with eigenvalues $\mathcal{E}_n(k) \to +\infty$ as $n \to +\infty$;

(iii) let the eigenvalues be arranged in increasing order and repeated according to their multiplicity for any $k \in M_{\Gamma^*}$, i.e. $\mathcal{E}_1(k) \leq \mathcal{E}_2(k) \leq \mathcal{E}_3(k) \leq \ldots$ then $\mathcal{E}_n(k)$ is a continuous $\Gamma^*$-periodic function of $k$. 

12
The above result differs from the standard theory of periodic Schrödinger operators just for the presence of a periodic vector potential \( A_\Gamma \). Since we were no able to find a suitable reference in the literature, we sketch its proof in Appendix A.1.

We call \( \mathcal{E}_n(\cdot) \) the \( n \)th Bloch band or energy band. The corresponding normalized eigenstates \( \{ \varphi_n(k) \}_{n \in \mathbb{N}} \subset \mathcal{D} \) are called Bloch functions and form, for any \( k \in M_\Gamma^*, \) an orthonormal basis of \( \mathcal{H}_I \). Notice that, with this choice of the labelling, \( \mathcal{E}_n(\cdot) \) and \( \varphi_n(\cdot) \) are continuous in \( k \), but generally they are not smooth functions if eigenvalue crossings are present.

We say that a family of Bloch bands \( \{ \mathcal{E}_n(\cdot) \}_{n \in I} \), with \( I := [I_+, I_-] \cap \mathbb{N} \), is isolated if

\[
\inf_{k \in M_\Gamma^*} \text{dist} \left( \bigcup_{n \in I} \{ \mathcal{E}_n(k) \}, \bigcup_{j \notin I} \{ \mathcal{E}_j(k) \} \right) = C_g > 0. \tag{31}
\]

The existence of an isolated part of the spectrum is a necessary ingredient for an adiabatic theory. We introduce the following:

**Assumption (C) [constant gap condition].** The spectrum of \( H_{\text{per}} \) admits a family of Bloch bands \( \{ \mathcal{E}_n(\cdot) \}_{n \in I} \) which is isolated in the sense of (31).

Let \( P_I(k) \) be the spectral projector of \( H_{\text{per}}(k) \) corresponding to the family of eigenvalues \( \{ \mathcal{E}_n(k) \}_{n \in I} \), then \( P_I^2 := \int_{M_\Gamma^*} \! P_I(k) \, d^2k \) is the projector on the isolated family of Bloch bands labeled by \( I \). In terms of Bloch functions (using the Dirac notation), one has that \( P_I(\cdot) = \sum_{n \in I} |\varphi_n(\cdot)\rangle \langle \varphi_n(\cdot)| \). However, in general, \( \varphi_n(\cdot) \) are not smooth functions of \( k \) at eigenvalue crossing, while \( P_I(\cdot) \) is a smooth function of \( k \) because of the gap condition. Moreover, from the periodicity of \( H_{\text{per}}(\cdot) \), one argues \( P_I([k] - \gamma^*) = \tau(\gamma^*) P_I([k]) \tau(\gamma^*)^{-1} \).

In general the smoothness of \( P_I(\cdot) \) is not enough to assure the existence of family of orthonormal basis for the subspaces \( \text{Ran} P_I(\cdot) \) which varies smoothly (or only continuously) with respect to \( k \in M_\Gamma^* \). Then we need the following assumption.

**Assumption (D) [smooth frame].** Let \( \{ \mathcal{E}_n(\cdot) \}_{n \in I} \) be a family of Bloch bands (\( |I| = m > 1 \) ). We assume that there exists an orthonormal basis \( \{ \psi_j(\cdot) \}_{j=1}^m \) of \( \text{Ran} P_I(\cdot) \) whose elements are smooth and (left) \( \tau \)-covariant with respect to \( k \), i.e. \( \psi_j(\cdot - \gamma^*) = \tau(\gamma^*) \psi_j(\cdot) \) for all \( j = 1, \ldots, m \) and \( \gamma^* \in \Gamma^* \).

Note that it is not required that \( \psi_j(k) \) is an eigenfunction of \( H_{\text{per}}(k) \). However, in the special but important case in which the family of bands consist of a single isolated \( m \)-fold degenerate eigenvalue, i.e. \( \mathcal{E}_n(k) = \mathcal{E}_s(k) \) for every \( n = 1, \ldots, m \), then the Assumption (D) is equivalent to the existence of an orthonormal basis consisting of smooth and \( \tau \)-covariant Bloch functions.

**Remark 3.4** (Time-reversal symmetry breaking). As far as low dimensional models are concerned \( (d \leq 3) \), Theorem 1 in \cite{Pan07} assures that Assumption (D) is true whenever the Hamiltonian \( H_{\text{per}} \) is invariant with respect to the time-reversal symmetry, which is implemented in the Schrödinger representation by the complex conjugation operator. However, the term \( A_\Gamma \neq 0 \) in \( H_{\text{per}} \) generically breaks the time reversal symmetry. Therefore, to consider also the effects due to a periodic vector potential, we need to assume the existence of
a smooth family of frames. Anyway is opinion of the authors that the result in [Pan07] can be extended to the case of a periodic vector potential, at least assuming that $A_\Gamma$ is small in a suitable sense.

Let $k_0$ be a fixed point in $M_\Gamma^r$ and define the projection $\pi_r := P_{\pi}(k_0)$. If the Assumption (C) holds true then $\dim \pi_r = \dim P_{\pi}(k)$ for all $k \in \mathbb{R}^2$. Let $\{\chi_n\}_{n=1}^\infty$ be an orthonormal basis for $\text{Ran} \pi_r$ and define a unitary map

$$u_0(k) := \tilde{u}_0(k) + u_0^+(k), \quad \text{with} \quad \tilde{u}_0(k) := \sum_{1 \leq j \leq \ell} |\chi_j\rangle \langle \psi_j(k)|,$$  

(32)

which maps $\text{Ran} P_{\pi}(k)$ in $\text{Ran} \pi_r$. The definition of this unitary is not unique because the freedom in the choice of the frame and of the orthogonal complement $u_0^+(k)$. From the definition and the $\tau$-covariance of $\psi_j(\cdot)$ one has that $u_0(k) P_{\pi}(k) u_0(k)^{-1} = \pi_r$ and $u_0([k] - \gamma^*) = u_0([k]) \tau(\gamma^*)^{-1}$ (right $\tau$-covariance).

### 3.4 $\tau$-equivariant and special $\tau$-equivariant symbol classes

Proposition 3.3 shows that for all $k \in \mathbb{R}^2$, the operator $H_{\text{per}}(k)$ defines an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}_k$ with dense domain $\mathcal{D} := \mathcal{F}^2(V)$. However the domain $\mathcal{D}$ can be considered itself as a Hilbert space with respect to the Sobolev norm $\| \cdot \|_D := \|f \cdot \Delta_\theta \|_{\mathcal{H}_k}$ and so $H_{\text{per}}(k)$ can be seen as a bounded linear operator from $\mathcal{D}$ to $\mathcal{H}_k$, i.e. as an element of the Banach space $\mathcal{B}(\mathcal{D}, \mathcal{H}_k)$. The map $\mathbb{R}^2 \ni k \mapsto H_{\text{per}}(k) \in \mathcal{B}(\mathcal{D}, \mathcal{H}_k)$ is a special example of a operator-valued symbol. For a summary about the theory of the Weyl quantization of vector-valued symbols, we refer to Appendices A and B in [Teu03]. In what follows we will need the following definition.

**Definition 3.5** (Hörmander symbol classes). A symbol is any map $F$ from the (cotangent) space $\mathbb{R}^2 \times \mathbb{R}^2$ to the Banach space $\mathcal{B}(\mathcal{D}, \mathcal{H}_k)$, i.e. $\mathbb{R}^2 \times \mathbb{R}^2 \ni (k, \eta) \mapsto F(k, \eta) \in \mathcal{B}(\mathcal{D}, \mathcal{H}_k)$. A function $w : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ is said to be an order function if there exists constants $C_0 > 0$ and $N_0 > 0$ such that

$$w(k, \eta) \leq C_0 \left(1 + |k - k'|^2 + |\eta - \eta'|^2\right)^{\frac{N_0}{2}} \frac{N_0}{2} w(k', \eta'),$$  

(33)

for every $(k, \eta), (k', \eta') \in \mathbb{R}^2 \times \mathbb{R}^2$. A symbol $F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2, \mathcal{B}(\mathcal{D}, \mathcal{H}_k))$ is an element of the (Hörmander) symbol class $S^{\alpha, \beta}(\mathcal{B}(\mathcal{D}, \mathcal{H}_k))$ with order function $w$, if for every $\alpha, \beta \in \mathbb{N}^2$ there exists a constant $C_{\alpha, \beta} > 0$ such that $\|\partial_k^\alpha \partial_\eta^\beta F(k, \eta)\|_{\mathcal{B}(\mathcal{D}, \mathcal{H}_k)} \leq C_{\alpha, \beta} w(k, \eta)$ for every $(k, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$.

According to the previous definition, the vector-valued map $H_{\text{per}}(\cdot)$ defines a Hörmander symbol constant in the $\eta$-variables and with order function $v(k, \eta) := 1 + |k|^2$ (see the proof of Proposition 3.10 below). However, as showed by equation (29), the symbol $H_{\text{per}}(\cdot)$ satisfies an extra condition of periodicity.

**Definition 3.6** ($\tau$-equivariant symbols). Let $\Gamma^*$ be a two dimensional lattice (the dual lattice defined in Section 3 for our aims) and $\tau : \Gamma^* \to \mathcal{W}(\mathcal{H}_k)$ the unitary representation
defined in Section 3.2. Denote by \( \tilde{\tau} := \tau \big|_{D} \) the bounded-operator representation of \( \Gamma^* \) in \( D \). A symbol \( F \in S^w(\mathcal{B}(D, \mathcal{H})) \) is said to be \( \tau \)-equivariant if

\[
F(k - \gamma^*, \eta) = \tau(\gamma^*) \ F(k, \eta) \ \tilde{\tau}(\gamma^*)^{-1} \quad \forall \ \gamma^* \in \Gamma^*, k \in \mathbb{R}^2.
\]

The space of \( \tau \)-equivariant symbols is denoted as \( S^w_\tau(\mathcal{B}(D, \mathcal{H})) \).

For the purposes of this work, it is convenient to focus on special classes of symbols. By considering the kinetic momentum function \( \mathbb{R}^2 \times \mathbb{R}^2 \ni (k, \eta) \mapsto \kappa(k, \eta) := k - A(\eta) \in \mathbb{R}^2 \), with \( A \) fulfilling Assumption (B), one defines the minimal coupling map by

\[
(k, \eta) \mapsto j_\kappa(k, \eta) := (\kappa(k, \eta), \eta) \in \mathbb{R}^2 \times \mathbb{R}^2.
\]

**Definition 3.7** (Special \( \tau \)-equivariant symbols). Let \( w \) be an order function, in the sense of (33). We define

\[
S^w_{\kappa, \tau}(\mathcal{B}(D, \mathcal{H})) := \{ \tilde{F} = F \circ j_\kappa : F \in S^w(\mathcal{B}(D, \mathcal{H})) \}.
\]

We refer to \( S^w_{\kappa, \tau}(\mathcal{B}(D, \mathcal{H})) \) as the class of special \( \tau \)-equivariant symbols. The following result shows that special symbols can be considered as genuine \( \tau \)-equivariant symbols with respect to a modified order function. The key ingredient is the linear growth of the kinetic momentum.

**Lemma 3.8.** With the above notations \( S^w_{\kappa, \tau}(\mathcal{B}(D, \mathcal{H})) \subset S^{w'}_\tau(\mathcal{B}(D, \mathcal{H})) \) where \( w' := w \circ j_\kappa \).

**Proof.** If \( F \in S^w_\tau(\mathcal{B}(D, \mathcal{H})) \) then also \( F \circ j_\kappa \) is \( \tau \)-equivariant, indeed \( \kappa(k - \gamma^*, \eta) = \kappa(k, \eta) - \gamma^* \) and \( (F \circ j_\kappa)(k - \gamma^*, \eta) = \tau(\gamma^*) (F \circ j_\kappa)(k, \eta) \ \tilde{\tau}(\gamma^*)^{-1} \). Since \( j_\kappa \) is a smooth function, then also the composition \( F \circ j_\kappa \) is a smooth function. Observing that \( (F \circ j_\kappa)(k, \eta) = F(k - A(\eta), \eta) \) it follows that

\[
(\partial_k j_\kappa(F \circ j_\kappa))(k, \eta) = ((\partial_k F) \circ j_\kappa)(k, \eta)
\]

\[
(\partial_\eta j_\kappa(F \circ j_\kappa))(k, \eta) = ((\partial_\eta F) \circ j_\kappa)(k, \eta) + \sum_{i=1}^{2} (\partial_\eta \kappa_i)(k, \eta) ((\partial_{k_i} F) \circ j_\kappa)(k, \eta)
\]

where \( \partial_\eta \kappa_i \) are bounded functions because Assumption (B). From the first equation it follows that

\[
\|\partial_k j_\kappa(F \circ j_\kappa)(k, \eta)\|_{\mathcal{B}(D, \mathcal{H})} \leq C_{j,0} (w \circ j_\kappa)(k, \eta) \quad j = 1, 2
\]

for suitable positive constants \( C_{j,0} \). Similarly the second equation implies

\[
\|\partial_\eta j_\kappa(F \circ j_\kappa)(k, \eta)\|_{\mathcal{B}(D, \mathcal{H})} \leq [C_{0,j} + K(C_{1,0} + C_{2,0})](w \circ j_\kappa)(k, \eta).
\]

where \( K > 0 \) is a bound for the functions \( \partial_\eta \kappa_i \). By an inductive argument on the number of the derivatives one can proof that the derivatives of \( F \circ j_\kappa \) are bounded by the function \( w' := w \circ j_\kappa \). To complete the proof we need to show that \( w' \) is an order function according to Definition 3.3. This follows by a simple computation using the fact that \( \kappa \) has a linear growth in \( k \) and \( \eta \).

---

(2) Clearly \( \tau(\gamma^*) \) acts as an invertible bounded operator on the space \( D \), but it is no longer unitary with respect to the Sobolev-norm defined on \( D \).
In view of Lemma 3.8 all the results of Appendix B of [Ten03] hold true for symbols in $S^w_{\kappa,\tau}(\mathcal{B}(\mathcal{D}, \mathcal{H}_t))$ and in particular the quantization of a symbol in $S^w_{\kappa,\tau}(\mathcal{B}(\mathcal{D}, \mathcal{H}_t))$ preserves the $\tau$-equivariance and that the pointwise product or the Moyal product of two symbols of order $w_1$ and $w_2$ produce a symbol of order $w_1w_2$ (see [Ten03], Propositions B.3 and B.4).

**Remark 3.9** (Notation). In what follows we will use the short notation $F(\kappa;\eta) := (F \circ j_{\kappa})(k, \eta)$ to denote the special symbol $F \circ j_{\kappa} \in S^w_{\kappa,\tau}(\mathcal{B}(\mathcal{D}, \mathcal{H}_t))$ related to the $\tau$-equivariant symbols $F \in S^w_{\kappa,\tau}(\mathcal{B}(\mathcal{D}, \mathcal{H}_t))$. We emphasize on the use of the semicolon “;” instead the comma “,” and of the symbol of the kinetic momentum $\kappa$ instead the quasi-momentum $k$.

### 3.5 Semiclassics: quantization of equivariant symbols

As explained in Section 3.2, the Bloch-Floquet transform $Z$ provides the separation between the fast degrees of freedom, associated to the Hilbert space $\mathcal{H}_t = L^2(V, d^2\theta)$, and the slow degrees of freedom, associated to the Hilbert $L^2(\mathcal{M}_{\Gamma}, d^2k)$. A fruitful point of view is to consider the slow degrees of freedom “classical” with respect to the “quantum” fast degrees of freedom. Mathematically, this is achieved by recognizing that the Hamiltonian $H_Z$ defined in (26) is the Weyl quantization of an operator-valued “semiclassical” symbol over the classical phase space $\mathbb{R}^2 \times \mathbb{R}^2$. As explained rigorously in the Appendices A and B of [Ten03], the quantization procedure maps an operator-valued symbol $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{B}(\mathcal{D}, \mathcal{H}_t)$ into a linear operator $\text{Op}_\varepsilon(F) : S(\mathcal{D}, \mathcal{D}) \to S(\mathbb{R}^2, \mathcal{H}_t)$, where $S(\mathbb{R}^2, \mathcal{H})$ denotes the space of $\mathcal{H}$-valued Schwartz functions. The quantization procedure concerns only the slow degrees of freedom and at a formal level can be identified with the prescription

$$k \mapsto \text{Op}_\varepsilon(k) := \text{multiplication by } k \otimes \mathbb{I}_{\mathcal{D}}; \quad \eta \mapsto \text{Op}_\varepsilon(\eta) := i\varepsilon \nabla_k \otimes \mathbb{I}_{\mathcal{D}}. \quad (35)$$

Let us consider the operator-valued symbol $H_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{B}(\mathcal{D}, \mathcal{H}_t)$ defined by

$$H_0(k, \eta) := \frac{1}{2} \left[ -i\nabla_\theta + k - A_T(\theta) - A_0(\eta) - i\frac{\varepsilon}{2} e_{\perp} \wedge \eta \right]^2 + V_T(\theta) + \phi(\eta). \quad (36)$$

The symbol $H_0$ does not depend on $\varepsilon$ and in view of Proposition 3.3 it defines an unbounded operator on $\mathcal{H}_t$ with domain of self-adjointness $\mathcal{D} = \mathcal{H}^2(\mathbb{V})$ for all choice of $(k, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$. According to the notation of Section 3.4 and comparing (36) with (28) we can write

$$H_0(k, \eta) = H_{\text{per}}(\kappa(k, \eta)) + \phi(\eta) = (H_\phi \circ j_{\kappa})(k, \eta). \quad (37)$$

where $H_\phi(k, \eta) := H_{\text{per}}(k) + \phi(\eta)$. As suggested by equation (29), $H_\phi$ is a $\tau$-equivariant symbol. Thus the symbol $H_0$ is $\tau$-equivariant with respect to the kinetic momentum $\kappa$. The following result establishes the exact symbol class for $H_0$.

**Proposition 3.10.** If Assumption (A_\omega) and (B) hold true then $H_0 \in S^w_{\kappa,\tau}(\mathcal{B}(\mathcal{D}, \mathcal{H}_t))$ with order function $v(k, \eta) := 1 + |k|^2$.

**Proof.** Using the result of Lemma 3.8 we only need to show that $H_\phi \in S^w(\mathcal{B}(\mathcal{D}, \mathcal{H}_t))$. The later claim is easy to verify, indeed the derivative in $\eta$ are bounded functions, the second
derivative in $k$ is a constant and the derivatives of higher order in $k$ are zero. Then we need only to check the growth of the first derivative in $k$. A simple computation shows that

$$
\| (\partial_{k_j} H_g)(k, \eta) \|_{\mathcal{B}(D, \mathcal{H}_t)} = \| (\partial_{k_j} H_{\text{per}})(k) (1_{\mathcal{H}_t} - \Delta_{\theta})^{-1} \|_{\mathcal{B}(\mathcal{H}_t)}
$$

and since $\partial_{k_j} H_{\text{per}}$ is $\tau$-equivariant (see Remark 3.2), then

$$
\| (\partial_{k_j} H_g)(k, \eta) \|_{\mathcal{B}(D, \mathcal{H}_t)} = \| (\partial_{k_j} H_{\text{per}})([k]) \tau(\gamma^*)^{-1}(1_{\mathcal{H}_t} - \Delta_{\theta})^{-1} \|_{\mathcal{B}(\mathcal{H}_t)}.
$$

Observing that $\tau(\gamma^*)$ is the multiplication by $e^{\theta \gamma^*}$ in $\mathcal{H}_t$ and by a simple computation that $(\partial_{k_j} H_{\text{per}})([k]) \tau(\gamma^*)^{-1} = \tau(\gamma^*)^{-1}[-2\gamma^* + (\partial_{k_j} H_{\text{per}})([k])]$ one has

$$
\| (\partial_{k_j} H_g)(k, \eta) \|_{\mathcal{B}(D, \mathcal{H}_t)} \leq C_1 |\gamma^*| + \| (\partial_{k_j} H_{\text{per}})([k]) \|_{\mathcal{B}(D, \mathcal{H}_t)} \leq C_1 (|k| + C_3) + C_2
$$

where $C_1 = 2\| (1_{\mathcal{H}_t} - \Delta_{\theta})^{-1} \|_{\mathcal{B}(\mathcal{H}_t)}$, $C_2 := \max_{k \in M_{\Gamma^*}} \| (\partial_{k_j} H_{\text{per}})([k]) \|_{\mathcal{B}(D, \mathcal{H}_t)}$ and $|\gamma^*_j| \leq |\gamma^*| = |k - [k]| \leq |k| + C_3$ with $C_3 := \max_{k \in M_{\Gamma^*}} |k|$. The claim follows observing that $1 + |k| \leq 2(1 + |k|^2)$.

---

**Figure 1:** Structure of the spectrum of $H_0(k, \eta)$. The picture shows schematically a “relevant part of the spectrum”, consisting of two energy bands $\{E_*, E_{*+1}\}$, with $E_{*+j}(k, \eta) = E_{*+j}(\kappa(k, \eta)) + \phi(\eta)$. Notice that we assume only a local gap condition, as stated in (38), while in the picture a stronger condition is satisfied: a gap exists when projecting the relevant bands on the vertical axis.

Equation (37) provides information about the dependence on $k$ and $\eta$ of the spectrum of $H_0$. The $n$th eigenvalue $E_n(k, \eta)$ of the operator $H_0(k, \eta)$ is related to the $n$th eigenvalue $E_n(k)$ of the periodic Hamiltonian $H_{\text{per}}(k)$ by the relation $E_n(k, \eta) = E_n(\kappa(k, \eta)) + \phi(\eta)$. The function $E_n : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is still $\Gamma^*$-periodic in $k$ but only oscillating with bounded
variation in $\eta$. Assumption (C) for the family of Bloch bands $\{E_n(\cdot)\}_{n \in \mathcal{I}}$ immediately implies that

$$\inf_{(k,\eta) \in M_{\mathcal{I}} \times \mathbb{R}^2} \text{dist} \left( \bigcup_{n \in \mathcal{I}} \{E_n(k,\eta)\}, \bigcup_{j \not\in \mathcal{I}} \{E_j(k,\eta)\} \right) = C_g > 0. \quad (38)$$

This is the relevant part of the spectrum of $H_0$ which we are interested in.

According to the general theory (see Appendices A and B of [Teu03]), one has that

$$\text{Op}_\varepsilon(H_0) = \frac{1}{2} \left[ -i\nabla_{\theta} + k - A_{\Gamma}(\theta) - A_0(i\varepsilon \nabla_k) - i\eta \frac{1}{2} e_\perp \wedge (i\varepsilon \nabla_k) \right]^2 + V_\Gamma(\theta) + \phi(i\varepsilon \nabla_k) \quad (39)$$

defines a linear operator from $\mathcal{S}(\mathbb{R}^2, D)$ in $\mathcal{S}(\mathbb{R}^2, \mathcal{H}_I)$ and by duality it extends to a continuous mapping $\text{Op}_\varepsilon(H_0) : \mathcal{S}'(\mathbb{R}^2, D) \to \mathcal{S}'(\mathbb{R}^2, \mathcal{H}_I)$ (with an abuse of notation we use the same symbol for the extended operator). The $\tau$-equivariance assures that $\text{Op}_\varepsilon(H_0)\varphi([k] - \gamma^*) = \tau(\gamma^*)\text{Op}_\varepsilon(H_0)\varphi([k])$ (see [Teu03] Proposition B.3). Since $\text{Op}_\varepsilon(H_0)$ preserves $\tau$-equivariance it can then be restricted to an operator on the domain $\mathcal{H}_M^2(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2, D)$ which is the domain of self-adjointness of $H^Z$, according to (26). To conclude that $\text{Op}_\varepsilon(H_0)$, restricted to $\mathcal{H}_M^2(\mathbb{R}^2)$, agrees with $H^Z$ it is enough to recall that $i\nabla_k^\perp$ is defined as $i\nabla_k$ restricted to its natural domain $\mathcal{H}_I^1(\mathbb{R}^2, D) \cap \mathcal{H}_\tau$ and to use the spectral calculus. These arguments justify the following:

**PROPOSITION 3.11.** The Hamiltonian $H^Z$, defined by (26), agrees on its domain of definition with the Weyl quantization of the operator-valued symbol $H_0$ defined by (36).

With a little abuse of notation, we refer to this result by writing $H^Z = \text{Op}_\varepsilon(H_0)$.

### 3.6 Main result: effective dynamics for weak magnetic fields

Let $A_\varepsilon$ and $B_\varepsilon$ be $\varepsilon$-dependent (possibly unbounded) linear operators in $\mathcal{H}$. We write $A_\varepsilon = B_\varepsilon + \mathcal{O}_0(\varepsilon^\infty)$ if: for any $N \in \mathbb{N}$ there exist a positive constant $C_N$ such that

$$\|A_\varepsilon - B_\varepsilon\|_{\mathcal{B}(\mathcal{H})} \leq C_N \varepsilon^N \quad (40)$$

for every $\varepsilon \in [0,\varepsilon_0)$. Notice that, though the operators are unbounded, the difference is required to be a bounded operator.

We refer to Appendices A and B of [Teu03] for the basic terminology concerning pseudodifferential operators, and in particular as for the notions of principal symbol, asymptotic expansion, resummation, Moyal product.

**THEOREM 3.12.** Let Assumptions (A$_w$), (B), (C) and (D) be satisfied and let $\{E_n(\cdot)\}_{n \in \mathcal{I}}$ (with $|\mathcal{I}| = m$) be an isolated family of energy bands for $H_0$ satisfying condition (38). Then:

1. **Almost-invariant subspace:** there exist an orthogonal projection $\Pi_\varepsilon \in \mathcal{B}(\mathcal{H}_\tau)$, with $\Pi_\varepsilon = \text{Op}_\varepsilon(\pi) + \mathcal{O}_0(\varepsilon^\infty)$ and the symbol $\pi(k,\eta) \approx \sum_{j=0}^\infty \varepsilon^j \pi_j(k,\eta)$ having principal part $\pi_0(k,\eta) = P_\varepsilon(k - A(\eta))$, so that

$$[H^Z; \Pi_\varepsilon] = \mathcal{O}_0(\varepsilon^\infty).$$
In particular for any $N \in \mathbb{N}$ there exist a $C_N$ such that
\begin{equation}
\|(1 - \Pi_\varepsilon) e^{-i\varepsilon H_Z} \Pi_\varepsilon\| \leq C_N \varepsilon^N |t| \tag{41}
\end{equation}
for $\varepsilon$ sufficiently small, $t \in \mathbb{R}$.

2. Effective dynamics: let $\mathcal{H}_{\text{ref}} = L^2(M_{\Gamma}, d^2k) \otimes \mathcal{H}_t$, $\pi_\varepsilon$ as defined above (32) and $\Pi_\varepsilon = \Pi L^2(M_{\Gamma, \varepsilon}) \otimes \pi_\varepsilon \in \mathcal{B}(\mathcal{H}_{\text{ref}})$. Then there exist a unitary operator

$$U_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_{\text{ref}}$$

such that

(i) $U_\varepsilon = \text{Op}_\varepsilon(u) + \mathcal{O}_0(\varepsilon^\infty)$, where the symbol $u \simeq \sum_{j=0}^\infty \varepsilon^j u_j$ has principal part $u_0$ given by (32) with $k$ replaced by $\kappa(k, \eta)$;

(ii) $\Pi_\varepsilon = U_\varepsilon \Pi_\varepsilon U_\varepsilon^{-1}$;

(iii) posing $K := \Pi_\varepsilon \mathcal{H}_{\text{ref}}$, one has

$$U_\varepsilon \Pi_\varepsilon H^2 \Pi_\varepsilon U_\varepsilon^{-1} = H^\text{eff}_\varepsilon + \mathcal{O}_0(\varepsilon^\infty) \in \mathcal{B}(K)$$

with $H^\text{eff}_\varepsilon = \text{Op}_\varepsilon(h)$ and $h$ a resummation of the formal symbol $u \upharpoonright \pi \upharpoonright H_0 \upharpoonright \pi \upharpoonright u^{-1}$ (thus algorithmically computable at any finite order). Moreover,

$$\|(e^{-i\varepsilon H_Z} - U_\varepsilon^{-1} e^{-i\varepsilon H^\text{eff}_\varepsilon} U_\varepsilon) \Pi_\varepsilon\| \leq C_N' \varepsilon^N (\varepsilon + |t|). \tag{42}$$

\textbf{Remark 3.13.} The previous theorem and the following proof generalize straightforwardly to any dimension $d \in \mathbb{N}$. We prefer to state it only in the case $d = 2$ in view of the application to the QHE and of the comparison with the results in Section 3, the latter being valid only for $d = 2$.

\textbf{Proof of Theorem 3.12}

Step 1. Almost-invariant subspace

The proof of the existence of the super-adiabatic projection is very close to the proof of Proposition 1 of [PST03a], so we only sketch the strategy and emphasize the main differences with respect to that proof.

First of all, one constructs a formal symbol $\pi \simeq \sum_{j=0}^\infty \varepsilon^j \pi_j$ (the Moyal projection) such that: (i) $\pi \upharpoonright \pi \simeq \pi$; (ii) $\pi^\dagger = \pi$; (iii) $H_0 \upharpoonright \pi \simeq \pi \upharpoonright H_0$ where $\simeq$ denotes the asymptotic equivalence of formal series.

The symbol $\pi$ is constructed recursively at any order $j \in \mathbb{N}$ starting from $\pi_0$ and $H_0$. One firstly show the uniqueness of $\pi$ (see Lemma 2.3. in [PST03b]). The uniqueness allows us to construct $\pi$ locally, i.e. in a neighborhood of some point $z_0 := (k_0, \eta_0) \in \mathbb{R}^2 \times \mathbb{R}^2$. From
the continuity of the map $k \mapsto H_{\text{per}}(k)$ and the condition (31) it follows that there exists a neighborhood $U_{k_0}$ of $k_0$ such that for every $k \in U_{k_0}$ the set $\{E_n(k)\}_{n \in \mathbb{Z}}$ can be enclosed by a positively-oriented circle $\Sigma(k_0) \subset \mathbb{C}$ independent of $k$. Moreover it is possible to choose $\Sigma(k_0)$ in such a way that: it is symmetric with respect to the real axis; dist $\Sigma(k_0)$ positively-oriented circle is bounded by a constant $C_r$ independent of $k$; $\Sigma(k_0 - \gamma^*) = \Sigma(k_0)$ for all $\gamma^* \in \Gamma^*$.

With the notation of Section 3.4 we have $H_0 = H_\phi \circ j_\kappa$ with $H_\phi(k, \eta) := H_{\text{per}}(k) + \phi(\eta)$. Let $\tilde{\Lambda}(k_0, \eta_0) := \Sigma(k_0) + \phi(\eta_0)$ denote the translation of the circle $\Sigma(k_0)$ by $\phi(\eta_0) \in \mathbb{R}$ and pose $\Lambda := \tilde{\Lambda} \circ j_\kappa$. From the smoothness of $\phi$ it follows that there exists a neighborhood $U_{z_0} \subset \mathbb{R}^2 \times \mathbb{R}^2$ of $z_0$ such that dist($\Lambda(z_0)$, $\sigma(H_0(z))$) $\geq \frac{1}{4}C_g$ for all $z \in U_{z_0}$. Moreover $\Lambda(z_0)$ is symmetric with respect to the real axis, has radius bounded by $C_r$ and is $\Gamma^*$-periodic in the variable $\kappa = k - A(\eta)$ (see Figure I).

We proceed by using the Riesz formula, namely by posing

$$\pi_j(z) := \frac{i}{2\pi} \oint_{\Lambda(z_0)} d\lambda R_j(\lambda, z) \quad \text{on} \quad U_{z_0}$$

where $R_j(\lambda, \cdot)$ denotes the $j$-th term in the Moyal resolvent $R(\lambda, \cdot) = \sum_{j=0}^{\infty} \varepsilon^j R_j(\lambda, \cdot)$ (also known as the parametrix), defined by the request that

$$(H_0(\cdot) - \lambda I_D)^j R(\lambda, \cdot) = 1_{\mathcal{H}_t}, \quad R(\lambda, \cdot) (H_0(\cdot) - \lambda I_D) = 1_D \quad \text{on} \quad U_{z_0}.$$  

Each term $R_j$ is computed by a recursive procedure starting from $R_0(\lambda, \cdot) := (H_0(\cdot) - \lambda I_D)^{-1}$, as illustrated in [GMS91]. Following [PST03a] equations (30) and (31) one obtains that

$$R_j(\lambda, z) = -R_0(\lambda, z) L_j(\lambda, z)$$  

(43)

where $L_j$ is the $(j - 1)$-th order obstruction for $R_0$ to be the Moyal resolvent, i.e.

$$(H_0(\cdot) - \lambda I_D)^j \left( \sum_{n=0}^{j-1} \varepsilon^n R_j(\lambda, \cdot) \right) = 1_{\mathcal{H}_t} + \varepsilon^j L_j(\cdot) + O(\varepsilon^{j+1})$$  

(44)

At the first order $L_1 = -\frac{i}{2} \{H_0, R_0\}_{k, \eta}$, with $\{\cdot, \cdot\}_{k, \eta}$ the Poisson brackets.

The technical (and crucial) part of the proof is to show that

$$\pi_j \in S^v_{k, \tau} (\mathcal{B}(\mathcal{H}_t, D)) \cap S^1_{k, \tau} (\mathcal{B}(\mathcal{H}_t))$$

for all $j \in \mathbb{N}$, with $v(k, \eta) := (1 + |k|^2)^2$. By means of the recursive construction each $R_j(\lambda, \cdot)$ inherits the special $\tau$-equivariance from the principal symbol $R_0(\lambda, \cdot) = ((H_\phi \circ j_\kappa)(\cdot) - \lambda I_D)^{-1}$. The special periodicity in $\kappa$ of the domain of integration $\Lambda(\cdot)$ which appears in the Riesz formula assures also the special $\tau$-equivariance of each $\pi_j(\cdot)$.

Since $\|((\partial_\alpha^\tau \pi_j)(z))\|_b \leq 2\pi C_r \sup_{\lambda \in \Lambda(z_0)} \|\partial_\alpha^\tau (R_j(\lambda, z))\|_b$ ($b$ means either $\mathcal{B}(\mathcal{H}_t)$ or $\mathcal{B}(\mathcal{H}_t; D)$, $\alpha \in \mathbb{N}^3$ is a multiindex and $\partial_\alpha^\tau := \partial_{k_1}^\tau \partial_{k_2}^\tau \partial_{k_3}^\tau \partial_{\eta}^\tau$), we need only to prove that $R_j(\lambda, \cdot) \in S^v_{k, \tau} (\mathcal{B}(\mathcal{H}_t, D)) \cap S^1_{k, \tau} (\mathcal{B}(\mathcal{H}_t))$ uniformly in $\lambda$. This is the delicate point of the proof.
First of all, from the definition of $\Lambda(z_0)$ it follows that
\[
\|R_0(\lambda, z)\|_{\mathcal{S}(\mathcal{H}_1)} = [\text{dist}(\lambda, \sigma(H_0(z))))]^{-1} \leq \frac{4}{c_6}
\]
uniformly in $\lambda$. Let $\sigma \in \mathbb{N}^4$, with $|\sigma| = 1$. One observes that $\partial_z^\sigma R_0(\lambda, z) = -R_0(\lambda, z)N_z^\sigma(\lambda, z)$ with $N_z^\sigma(\lambda, z) := \partial_z^\sigma H_0(z) R_0(\lambda, z)$. From the relation
\[
\partial_z^\sigma N_z^\sigma = N_z^{\sigma+\sigma} - N_z^\sigma N_z^\sigma
\]
and an inductive argument, it follows the chain rule
\[
\partial_z^\sigma R_0 = R_0 \sum \omega_{\beta_1} \ldots \omega_{|\alpha|} N_z^{\beta_1} \ldots N_z^{\beta_{|\alpha|}}
\]
where $\beta_1, \ldots, \beta_{|\alpha|} \in \mathbb{N}^4$, $|\alpha| := \alpha_1 + \ldots + \alpha_4$, $\omega_{\beta_1} \ldots \omega_{|\alpha|} = \pm 1$ is a suitable sign function and the sum runs over all the combinations of multiindices such that $\beta_1 + \ldots + \beta_{|\alpha|} = \alpha$ with the convention $N_z^0 = 1$. The chain rule implies that $R_0 \in S_{\kappa, \tau}^1(\mathcal{B}(\mathcal{H}_1))$ provided that
\[
\|N_z^\alpha\|_{\mathcal{S}(\mathcal{H}_1)} = \|\partial_z^\alpha H_0 R_0\|_{\mathcal{S}(\mathcal{H}_1)} \leq C_\alpha \quad \text{uniformly in } \lambda.
\]
The latter condition is true since $\|((\partial_z^\alpha H_0)(k, \eta) R_0(\lambda, k, \eta))\|_{\mathcal{S}(\mathcal{H}_1)} \leq (g \circ j_\kappa)(k, \eta)$, for a suitable $g(k, \eta)$, $\Gamma^*$-periodic in $k$ and bounded in $\eta$; the latter claim can be checked as in Proposition $3.10$.

Similarly to prove that $R_0 \in S_{\kappa, \tau}^v(\mathcal{B}(\mathcal{H}_1, \mathcal{D}))$ we need to show that
\[
\|R_0 N_z^\alpha\|_{\mathcal{S}(\mathcal{H}_1, \mathcal{D})} = \|(1_{\mathcal{H}_1} - \Delta_\theta) R_0 N_z^\alpha\|_{\mathcal{S}(\mathcal{H}_1)} \leq C_\alpha v'(\cdot)
\]
uniformly in $\lambda$. Since $N_z^\alpha$ is bounded on $\mathcal{H}_1$ it is sufficient to show that $\|(1_{\mathcal{H}_1} - \Delta_\theta) R_0(\lambda; z)\|_{\mathcal{S}(\mathcal{H}_1)} \leq C'_\alpha v'(z)$. Observe that $\|(1_{\mathcal{H}_1} - \Delta_\theta) R_0(\lambda, [\kappa] - \gamma^*; \eta)\|_{\mathcal{S}(\mathcal{H}_1)} = \|(1_{\mathcal{H}_1} - \Delta_\theta) \tau(\gamma^*)^{-1} R_0(\lambda, [\kappa]; \eta)\|_{\mathcal{S}(\mathcal{H}_1)}$. The commutation relation
\[
-\Delta_\theta \tau(\gamma^*)^{-1} = \tau(\gamma^*)^{-1} (|\gamma^*|^2 + i2\gamma^* \cdot \nabla_{\theta} - \Delta_\theta)
\]
and the straightforward bound
\[
\| (|\gamma^*|^2 + i2\gamma^* \cdot \nabla_{\theta} - \Delta_\theta) (1_{\mathcal{H}_1} - \Delta_\theta)^{-1} \|_{\mathcal{S}(\mathcal{H}_1)} \leq C(1 + |\gamma^*|^2) \leq C'(1 + |\kappa(k, \eta)|^2)
\]
imply
\[
\|(1_{\mathcal{H}_1} - \Delta_\theta) R_0(\lambda, z)\|_{\mathcal{S}(\mathcal{H}_1)} \leq C'_\alpha v'(z) \|(1_{\mathcal{H}_1} - \Delta_\theta) R_0(\lambda, [\kappa]; \eta)\|_{\mathcal{S}(\mathcal{H}_1)}
\]
with $v' := v \circ j_\kappa$. Finally observe that
\[
\|(1_{\mathcal{H}_1} - \Delta_\theta) R_0(\lambda, [\kappa]; \eta)\|_{\mathcal{S}(\mathcal{H}_1)} \leq f(\kappa; \eta) \leq C''.
\tag{45}
\]
The first inequality above follows by an expansion on the Fourier basis, for fixed $[\kappa]$ and $\eta$: the second follows from the fact that $[\kappa]$ takes values on a compact set and the explicit dependence on $\eta$ is through the bounded function $\phi$. The bound (45) implies that $R_0 \in S_{\kappa, \tau}^v(\mathcal{B}(\mathcal{H}_1, \mathcal{D})) \cap S_{\kappa, \tau}^1(\mathcal{B}(\mathcal{H}_1))$ uniformly in $\lambda$.

To prove that $R_j \in S_{\kappa, \tau}^1(\mathcal{B}(\mathcal{H}_1))$, we observe that for any $\alpha \in \mathbb{N}^d$ one has
\[
\partial_z^\alpha R_j(\lambda, z) = R_0(\lambda, z) M^\alpha_{z,j}(\lambda, z)
\]
where \( M^\alpha_{z,j} \) is a linear combination of terms which are product of \( N^\beta_z \) and \( \partial_z^k L_j \) with \( |\beta|, |\delta| \leq |\alpha| \). Thus it is sufficient to prove that \( L_j \in S^1_{\kappa^*} (\mathcal{B}(\mathcal{H}_t)) \) for every \( j \in \mathbb{N} \). The latter claim is proved by induction on \( j \in \mathbb{N} \). Referring to (43), one has trivially that \( L_1 \in S^1_{\kappa^*} (\mathcal{B}(\mathcal{H}_t)) \). \( L_{j+1} \) is a linear combination of products of \( N^\alpha_z \) (with \( 0 \leq |\alpha| \leq j + 1 \)) and \( M^\beta_{z,i} \) (with \( |\beta| + i = j + 1 \) and \( 0 \leq i \leq j \)). Then the induction hypothesis on \( L_i \) for all \( i = 1, \ldots, j \) implies that \( L_{j+1} \in S^1_{\kappa^*} (\mathcal{B}(\mathcal{H}_t)) \).

Finally observing that \( \| \mathcal{O}_{\kappa}^0 R_j \|_{\mathcal{B}(\mathcal{H}_t, D)} \leq \| M^\alpha_{z,j} \|_{\mathcal{B}(\mathcal{H}_t, D)} \| R_0 \|_{\mathcal{B}(\mathcal{H}_t, D)} \) and using the fact that \( R_0 \in S^0_{\kappa^*} (\mathcal{B}(\mathcal{H}_t, D)) \) it follows that \( R_j \in S^0_{\kappa^*} (\mathcal{B}(\mathcal{H}_t, D)) \cap S^1_{\kappa^*} (\mathcal{B}(\mathcal{H}_t)) \) uniformly in \( \lambda \), for all \( j \in \mathbb{N} \).

As explained in Section 3.4, we can apply the result of Proposition B.4 in [Teu03] to special \( \tau \)-equivariant symbols obtaining \( H_0 \mathbb{Z} \pi \in S^0_{\kappa^*} (\mathcal{B}(\mathcal{H}_t)) \). However the \( \tau \)-equivariance of \( H_0 \mathbb{Z} \pi \) and its derivatives implies that the norms are bounded in \( z \), hence \( H_0 \mathbb{Z} \pi \in S^1_{\kappa^*} (\mathcal{B}(\mathcal{H}_t)) \) which implies by adjointness also \( \pi \mathbb{Z} H_0 \in S^1_{\kappa^*} (\mathcal{B}(\mathcal{H}_t)) \). By construction \( [H^2, \mathcal{O}_\varepsilon(\pi)] = \mathcal{O}_\varepsilon(H_0 \mathbb{Z} \pi - \mathbb{Z} \mathbb{Z} H_0) = \mathcal{O}_0(\varepsilon^\infty) \) where the remainder is bounded in the norm of \( \mathcal{B}(\mathcal{H}_t) \).

The operator \( \mathcal{O}_\varepsilon(\pi) \) is only approximately a projection, since \( \mathcal{O}_\varepsilon(\varepsilon)^2 = \mathcal{O}_\varepsilon(\pi \mathbb{Z} \pi) = \mathcal{O}_\varepsilon(\pi) + \mathcal{O}_0(\varepsilon^\infty) \). We obtain the super adiabatic projection \( \Pi_\varepsilon \) by using the trick in [NS04]. Indeed, one notices that, for \( \varepsilon \) sufficiently small, the spectrum of \( \mathcal{O}_\varepsilon(\pi) \) does not contain e.g. the points \( \{1/2\} \) and \( \{3/2\} \). Thus, the formula

\[
\Pi_\varepsilon = \frac{i}{2\pi} \int_{|z-1|=1/2} (\mathcal{O}_\varepsilon(\pi) - z)^{-1}. \tag{46}
\]

yields an orthogonal projector such that \( \Pi_\varepsilon = \mathcal{O}_\varepsilon(\pi) + \mathcal{O}_0(\varepsilon^\infty) \).

Finally, equation (41) follows by observing that \( [H, \Pi_\varepsilon] = \mathcal{O}_0(\varepsilon^\infty) \) implies

\[
[e^{-i\frac{t}{\varepsilon} H}; \Pi_\varepsilon] = \mathcal{O}_0(\varepsilon^\infty |t|)
\]
as proved in Corollary 3.3 in [Teu03].

**Step 2. Construction of the intertwining unitary**

The construction of the intertwining unitary follows as in the proof of Proposition 2 of [PST03a]. Firstly one constructs a formal symbol \( u \propto \sum_{j=0}^{\infty} \varepsilon^j u_j \) such that: (i) \( u^1 \mathbb{Z} u = u \mathbb{Z} u^1 = \mathbb{1}_{\mathcal{H}_t} \); (ii) \( u \mathbb{Z} \pi \mathbb{Z} u^1 = \pi_t \).

The existence of such a symbol follows from a recursive procedure starting from \( u_0 \) and using the expansion of \( \pi \propto \sum_{j=0}^{\infty} \varepsilon^j \pi_j \) obtained above. However, the symbol \( u \) which comes out of this procedure is not unique.

Since \( u_0 \) is right \( \tau \)-covariant (see the end of Section 3.3) in \( \kappa \), then one can prove by induction that the symbol is also true for all the symbols \( u_j \) and hence for the full symbol \( u \). Finally, since \( u_0 \in S^1(\mathcal{B}(\mathcal{H}_t)) \) one deduces by induction also \( u_j \in S^1(\mathcal{B}(\mathcal{H}_t)) \) for all \( j \in \mathbb{N} \). The quantization of this symbol is an element of \( \mathcal{B}(\mathcal{H}_e, \mathcal{H}_{ref}) \) satisfying the following properties:
Lemma 3.3 (Step II) in [PST03b] to obtain the true unitary $U_\epsilon$.

Nevertheless $O_\epsilon(u)$ can be modified by an $O_0(\epsilon^\infty)$ term using the same technique of Lemma 3.3 (Step II) in [PST03b] to obtain the true unitary $U_\epsilon$.

Step 3. Effective dynamics

The last step of the proof is identical to the corresponding part (Proposition 3) of [PST03a].

3.7 Hofstadter-like Hamiltonians

We now focus on the special case of a single isolated energy band $E_\epsilon$, i.e. $m = 1$, and we comment on the relation between the effective Hamiltonian, the celebrated Peierls substitution and Hofstadter-like Hamiltonians (see Section 1).

In this special case, $\pi_0(\kappa) = |\psi_\epsilon(\kappa))\langle\psi_\epsilon(\kappa)|$ and $u_0(\kappa) = |\chi\langle\psi_\epsilon(\kappa)| + u_0^\perp$ where $\psi_\epsilon(k)$ is the eigenvector of $H_{\text{per}}(k)$ corresponding to the eigenvalue $E_\epsilon(k)$. Let $h \in S^1(\mathcal{B}(H))$ be a resummation of the formal symbol $u \pi \pi H_0 \pi \pi u^{-1}$. A straightforward computation yields

$$h_0 = u_0 \pi_0 H_0 \pi_0 u_0^\dagger = |\chi\langle\psi_\epsilon|\langle\psi_\epsilon|H_0|\psi_\epsilon\rangle\langle\psi_\epsilon| = E_\epsilon \pi_\tau.$$

Since $\pi_\tau$ is one-dimensional, $h_0$ can be regarded as a scalar-valued symbol with explicit expression

$$h_0(k, \eta) = E_\epsilon(k, \eta) = E_\epsilon(k - A(\eta)) + \phi(\eta).$$

By considering the quantization of the latter, the effective one-band Hamiltonian reads

$$O_\epsilon(h_0) = E_\epsilon(k, i\varepsilon \nabla_k) = E_\epsilon(k - A(i\varepsilon \nabla_k)) + \phi(i\varepsilon \nabla_k).$$

The latter formula corresponds to the momentum-space reformulation of the well-known Peierls substitution [Pei33, AM76].

To illustrate this point, we specialize to the case of a uniform external magnetic field and zero external electric field, setting $\phi = 0$ and $A_0 = 0$ in (36). The leading order contribution to the dynamics in the almost invariant subspace is therefore given by a bounded operator, acting on the reference Hilbert space $L^2(M_\tau, d^2k)$, defined as the quantization (in the sense of Section 3.5) of the function $E_\epsilon \circ j_\epsilon : (k, \eta) \mapsto E_\epsilon(k - A(\eta))$, defined on $\mathbb{T}^d \times \mathbb{R}^d$.

Loosely speaking, the above procedure corresponds to the following “substitution rule”: one may think to quantize the smooth function $E_\epsilon : \mathbb{T}^d \rightarrow \mathbb{R}$ by formally replacing the variables $(k_1, k_2)$ with the operators $(\mathcal{K}_1', \mathcal{K}_2')$ defined by

$$\mathcal{K}_1' := k_1 + \frac{i}{2}(t_1 \varepsilon) \frac{\partial}{\partial k_2}, \quad \mathcal{K}_2' := k_2 - \frac{i}{2}(t_2 \varepsilon) \frac{\partial}{\partial k_1},$$

(48)
regarded as unbounded operators acting in $L^2(M_{f\star}, d^2k)$. To make this procedure rigorous, one can expand $E_*$ in its Fourier series, i.e. $E_*(k) = \sum_{n,m \in \mathbb{Z}} c_{n,m} e^{i2\pi (na + mb)k}$ and define the **Peierls’ quantization** of $E_*$ as the operator obtained by the same series expansion with the phases $e^{i2\pi (na + mb)k}$ replaced by the unitary operators $e^{i2\pi (na + mb)k} \mathcal{H}r$ (the series is norm-convergent, in view of the regularity of $E_*$). This fixes uniquely the prescription for the quantization.

To streamline the notation, one introduces new coordinates $\xi_1 := 2\pi(a \cdot k)$ and $\xi_2 := 2\pi(b \cdot k)$ such that the function $E'_*, E'_*(\xi_1, \xi_2) := E_*(k(\xi))$ becomes $(2\pi \mathbb{Z})^2$-periodic. The change of variables induces a unitary map from $L^2(M_{f\star}, d^2k)$ to $L^2(\mathbb{T}^2, d^2\xi)$ which intertwines the operators (48) with the operators (recall $\varepsilon = 2\pi/h_B$)

$$
\mathcal{H}_1 := \xi_1 + i\pi \left( \frac{\xi_2}{h_B} \right) \frac{\partial}{\partial \xi_2}, \quad \mathcal{H}_2 := \xi_2 - i\pi \left( \frac{\xi_1}{h_B} \right) \frac{\partial}{\partial \xi_1},
$$

so that $2\pi(a \cdot \mathcal{H}r) \mapsto \mathcal{H}_1$ and $2\pi(b \cdot \mathcal{H}r) \mapsto \mathcal{H}_2$.

Let $F : \mathbb{T}^2 \to \mathbb{C}$ be sufficiently regular that its Fourier series $F(\xi_1, \xi_2) = \sum_{n,m \in \mathbb{Z}} f_{n,m} e^{i(n\xi_1 + m\xi_2)}$ is uniformly-convergent. We define the **Peierls quantization** of $F$ as

$$
\hat{F} := \sum_{n,m \in \mathbb{Z}} f_{n,m} e^{i(n\mathcal{H}_1 + m\mathcal{H}_2)}.
$$

Let $U_0 = e^{i\mathcal{H}_1}$ and $V_0 = e^{i\mathcal{H}_2}$ (Hofstadter unitaries), acting on $L^2(\mathbb{T}^2, d^2\xi)$ as

$$
(U_0 \psi)(\xi_1, \xi_2) = e^{i\xi_1} \psi \left( \xi_1, \xi_2 - \pi \frac{\xi_2}{h_B} \right), \quad (V_0 \psi)(\xi_1, \xi_2) = e^{i\xi_2} \psi \left( \xi_1 + \pi \frac{\xi_1}{h_B}, \xi_2 \right).
$$

We regard (50) as the definition of the two unitaries, so there is no need to specify the domain of definition of the generators (49). Thus the Peierls quantization of the function $F$ defines a bounded operator on $L^2(\mathbb{T}^2, d^2\xi)$ given, in terms of the Hofstadter unitaries, by

$$
\hat{F}(U_0, V_0) = \sum_{n,m = -\infty}^{+\infty} f_{n,m} e^{i\pi nm \left( \frac{\xi_2}{h_B} \right)} U_0^n V_0^m,
$$

where the fundamental commutation relation $U_0 V_0 = e^{-i2\pi \left( \frac{\xi_2}{h_B} \right)} V_0 U_0$ has been used. Formula (51) defines a **Hofstadter-like Hamiltonian** with deformation parameter $1/h_B$. Indeed, the special case $H_{Hof} = U_0 + U_0^{-1} + V_0 + V_0^{-1}$ is (up to a unitary equivalence) the celebrated Hofstadter Hamiltonian [Hof76].

Summarizing, we draw the following

**Conclusion 3.14.** *Under the assumption of Theorem 4.7, for every $V_\Gamma \in L^2_{\text{loc}}(\mathbb{R}^2, d^2r)$, in the Hofstadter regime ($h_B \to \infty$), the dynamics generated by the Hamiltonian $H_{BL}([1])$ in the subspace related to a single isolated Bloch band, is approximated up to an error of order $1/h_B$ (and up to a unitary transform) by the dynamics generated on the reference Hilbert space $L^2(\mathbb{T}^2, d^2\xi)$ by a Hofstadter-like Hamiltonian, i.e. by a power series in the Hofstadter unitaries $U_0$ and $V_0$ defined by (50).*
4 Space-adiabatic theory for the Harper regime

4.1 Adiabatic parameter for strong magnetic fields

We now consider the case of a strong external magnetic field. Since we are interested in the limit \( B \to +\infty \) we set \( A_0 = 0 \) and \( \Phi = 0 \) in the Hamiltonian \([9]\). By exploiting the gauge freedom, we choose

\[
\nabla_r \cdot A_r = 0, \quad \int_{\mathcal{M}_r} A_r(r) \, d^2r = 0, \tag{52}
\]

this choice being always possible \([Sob97]\). Let us denote by \( Q_r = (Q_{r_1}, Q_{r_2}) \) the multiplication operators by \( r_1 \) and \( r_2 \) and with \( P_r = (P_{r_1}, P_{r_2}) = -i\hbar \nabla_r \). Taking into account conditions \((52)\) and \( A_0 = 0, \phi = 0 \), the Hamiltonian \((9)\) is rewritten as

\[
H_{BL} = \frac{1}{2m} \left[ \left( P_{r_1} + \frac{qB}{2c} Q_{r_2} \right)^2 + \left( P_{r_2} - \frac{qB}{2c} Q_{r_1} \right)^2 \right] + \tilde{V}_\Gamma(Q_r) + \tilde{W}(Q_r) \tag{53}
\]

where

\[
\tilde{V}_\Gamma(Q_r) = V_\Gamma(Q_r) + \frac{q^2}{2mc^2} |A_r(Q_r)|^2 \tag{54}
\]

\[
\tilde{W}(Q_r, P_r) = -\frac{q}{mc} (A_r)_1 Q_r \left[ P_{r_1} + \frac{qB}{2c} Q_{r_2} \right] - \frac{q}{mc} (A_r)_{2} Q_r \left[ P_{r_2} - \frac{qB}{2c} Q_{r_1} \right] \tag{55}
\]

with \((A_r)_1\) and \((A_r)_2\) the \( \Gamma \)-periodic components of the vector potential \( A_r \). The first of \((52)\) assures that \( W \) is a symmetric operator.

It is useful to define two new pairs of canonical dimensionless operators:

\[
\begin{align*}
K_1 & := -\frac{1}{2\delta} b^* \cdot Q_r - i\frac{\delta}{\hbar} a \cdot P_r \\
K_2 & := \frac{1}{2\delta} a^* \cdot Q_r - i\frac{\delta}{\hbar} b \cdot P_r
\end{align*} \quad \begin{align*}
G_1 & := \frac{1}{2} b^* \cdot Q_r - i\frac{\delta^2}{\hbar} a \cdot P_r \\
G_2 & := \frac{1}{2} a^* \cdot Q_r + i\frac{\delta^2}{\hbar} b \cdot P_r
\end{align*} \tag{56}
\]

where \( \delta := \sqrt{\hbar_B} = \sqrt{\phi_0/2\pi Z e B} \) according to the notation introduced in Section \([1]\) Since \( \delta^2 \propto 1/B \), the limit of strong magnetic field corresponds to \( \delta \to 0 \). We consider \( \delta \) as the \textit{adiabatic parameter} in the Harper regime. A direct computation shows that

\[
[K_1; K_2] = i\hbar \Gamma_{\text{phy}}, \quad [G_1; G_2] = i\hbar \delta^2 \Gamma_{\text{phy}}, \quad [K_j; G_k] = 0, \quad j, k = 1, 2. \tag{57}
\]

These new variables are important for three reasons:

(a) they make evident a separation of scales between the slow degrees of freedom related to the the dynamics induced by periodic potential and the fast degrees of freedom related to the cyclotron motion induced by the external magnetic field. Indeed, for \( V_\Gamma = 0 \), the \textit{fast variables} \((K_1, K_2)\) (the \textit{kinetic momenta}) describe the kinetic energy of the cyclotron motion, while the \textit{slow variables} \((G_1, G_2)\) correspond semiclassically to the center of the cyclotron orbit and are conserved quantities.
(b) The new variables are dimensionless. According to the notation used in Section 3.1 let $H_{BL} := \frac{1}{\varepsilon_0} H_{BL}$ be the dimensionless Bloch-Landau Hamiltonian with $\varepsilon_0 := \frac{\hbar^2}{m\Omega}$. The natural unit of energy is

$$E_0 := \frac{\hbar^2}{m\Omega \Gamma}.$$

(c) The use of the new variables simplifies the expression of the $\Gamma$-periodic functions appearing in $H_{BL}$. Indeed, $a^* \cdot Q_r = G_2 + \delta K_2$ and $b^* \cdot Q_r = G_1 - \delta K_1$, hence if $f_\Gamma$ is any $\Gamma$-periodic function one has

$$f_\Gamma(Q_r) = f(G_2 + \delta K_2, G_1 - \delta K_1)$$

where $f$ is the $\mathbb{Z}^2$-periodic function related to $f_\Gamma$.

In terms of the new variables, the Hamiltonian $H_{BL}$ reads

$$H_{BL} = \frac{1}{\delta^2} \Xi(K_1, K_2) + V(G_2 + \delta K_2, G_1 - \delta K_1) + \frac{1}{\delta} W(K_1, G_1, K_2, G_2)$$

where

$$\Xi(K_1, K_2) := \frac{1}{2\Omega} \left[ |a|^2 K_2^2 + |b|^2 K_1^2 - a \cdot b \{ K_1; K_2 \} \right]$$

is a quadratic function of the operators $K_1$ and $K_2$ ($\{ \cdot; \cdot \}$ denotes the anticommutator), $V$ is the $\mathbb{Z}^2$-periodic function related to the $\Gamma$-periodic function $1/\varepsilon_0 \tilde{V}_\Gamma$ and $W$ denotes the function $1/\varepsilon_0 \tilde{W}$ with respect the new canonical pairs, namely

$$W(K_1, G_1, K_2, G_2) = f_1(G_2 + \delta K_2, G_1 - \delta K_1) K_1 - f_2(G_2 + \delta K_2, G_1 - \delta K_1) K_2$$

where $f_1$ and $f_2$ are the $\mathbb{Z}^2$-periodic dimensionless functions

$$f_1(a^* \cdot r, b^* \cdot r) := 2\pi \frac{Z\Omega}{\Phi_0} (a^* \cdot A_\Gamma)(r) \quad \text{and} \quad f_2(a^* \cdot r, b^* \cdot r) := 2\pi \frac{Z\Omega}{\Phi_0} (b^* \cdot A_\Gamma)(r).$$

An easy computation shows that the first gauge condition of (52) is equivalent to

$$\frac{\partial f_1}{\partial x_1}(x_1, x_2) + \frac{\partial f_2}{\partial x_2}(x_1, x_2) = 0.$$

(62)

Obviously $W$ is a symmetric operator, since $\tilde{W}$ is symmetric.

The problem has a natural time-scale which is fixed by the cyclotron frequency $\omega_c = \frac{|q|B}{mc}$. With respect to the (fast) ultramicroscopic time-scale $\tau := \omega_c s$, equation (12) becomes

$$i \frac{1}{\delta^2} \frac{\partial}{\partial \tau} \psi = H_{BL}(t) \psi, \quad \delta^2 = \frac{\varepsilon_0}{\hbar \omega_c}.$$  

(63)

Thus the physically relevant Hamiltonian is

$$H_{BL}^\delta := \delta^2 H_{BL} = \Xi(K_1, K_2) + \delta W(K_1, G_1, K_2, G_2) + \delta^2 V(G_2 + \delta K_2, G_1 - \delta K_1).$$

(64)
4.2 Separation of scales: the von Neumann unitary

The commutation relations (57) show that \((K_1, K_2)\) and \((G_1, G_2)\) are two pairs of canonical conjugate operators. The Stone-von Neumann uniqueness Theorem (see [BR97, Corollary 5.2.15]) assures the existence of a unitary map \(W\) (called von Neumann unitary)

\[
W : \mathcal{H}_{\text{phy}} \rightarrow \mathcal{H}_w := \mathcal{H}_s \otimes \mathcal{H}_f = L^2(\mathbb{R}, dx_s) \otimes L^2(\mathbb{R}, dx_t)
\]

such that

\[
WG_1 W^{-1} = Q_s = \text{multiplication by } x_s, \quad WG_2 W^{-1} = P_s = -i q \frac{\partial}{\partial x_s} \quad \text{(66)}
\]

\[
WK_1 W^{-1} = Q_f = \text{multiplication by } x_f, \quad WK_2 W^{-1} = P_f = -i q \frac{\partial}{\partial x_f}. \quad \text{(67)}
\]

The explicit construction of the von Neumann unitary \(W\) is described in Appendix B.

Let \(X_j := G_j + (-1)^j \delta K_j\) with \(j = 1, 2\). From (66) and (67) it follows that

\[
X'_1 := WX_1 W^{-1} = Q_s - \delta Q_f, \quad X'_2 := WX_2 W^{-1} = P_s + \delta P_f. \quad \text{(68)}
\]

Since \(X_1\) and \(X_2\) commute, one can use the spectral calculus to define any measurable function of \(X_1\) and \(X_2\). For any \(f \in L^\infty(\mathbb{R}^2, d^2x)\) one defines \(f(X_1, X_2) := \int_{\mathbb{R}^2} f(x_1, x_2) dE_{x_1}^{(1)} dE_{x_2}^{(2)}\) where \(dE^{(j)}\) is the projection-valued measure corresponding to \(X_j\). In view of the unitarity of \(W\), and observing that \(dE^{(j)} := WdE^{(j)} W^{-1}\) is the projection-valued measure of \(X'_j\), one obtains that

\[
Wf(X_1, X_2) W^{-1} = \int_{\mathbb{R}^2} f(x_1, x_2) dE_{x_1}^{(1)} dE_{x_2}^{(2)} = f(X'_1, X'_2).
\]

So the effect of the conjugation through \(W\) on a function \(f\) of the operators \(X_1\) and \(X_2\) formally amounts to replace the operators \(X_j\) with \(X'_j\) inside \(f\).

In view of the above remark, one can easily rewrite \(H^{\delta}_{\text{BL}}\), making explicit the rôle of the fast and slow variables, obtaining

\[
H^{W} := WH^\delta_{\text{BL}} W^{-1} = 1_{\mathcal{H}_s} \otimes \Xi(Q_f, P_f) + \delta W(Q_f, Q_s, P_f, P_s) + \delta^2 V (P_s + \delta P_f, Q_s - \delta Q_f) \quad \text{(69)}
\]

where, according to (61),

\[
W(Q_f, Q_s, P_f, P_s) = f_1 (P_s + \delta P_f, Q_s - \delta Q_f) \quad Q_f - f_2 (P_s + \delta P_f, Q_s - \delta Q_f) \quad P_f. \quad \text{(70)}
\]

4.3 Relevant part of the spectrum: the Landau bands

The existence of a separation between fast and slow degrees of freedom and the decomposition of the physical Hilbert space \(\mathcal{H}_{\text{phy}}\) into the product space \(\mathcal{H}_w = \mathcal{H}_s \otimes \mathcal{H}_f\) are the first two ingredients to develop the SAPT. According to the general scheme, we “replace” the canonical operators corresponding to the slow degrees of freedom with classical variables
which will be re-quantized “a posteriori”. Mathematically, we show that the Hamiltonian $H^W$ acting in $\mathcal{H}_w$ is the Weyl quantization of the operator-valued function (symbol) $H_\delta$,

$$H_\delta(p_s, x_s) := \Xi(Q_t, P_t) + \delta \left( W(Q_t, x_s, P_t, p_s) + \delta^2 V(p_s + \delta P_t, x_s - \delta Q_t) \right). \tag{71}$$

The quantization is defined (formally) by the rules

$$x_s \mapsto \text{Op}_\delta(x_s) := Q_s \otimes 1_{\mathcal{H}_t}, \quad p_s \mapsto \text{Op}_\delta(p_s) := P_s \otimes 1_{\mathcal{H}_t}.$$

For every $(p_s, x_s) \in \mathbb{R}^2$, equation (71) defines an unbounded operator $H_\delta(p_s, x_s)$ which acts in the Hilbert space $\mathcal{H}_t$. To make the quantization procedure rigorous, as explained in Appendix A of [PST03b], we need to consider $H_\delta$ as function from $\mathbb{R}^2$ into some Banach space which is also a domain of self-adjointness for $H_\delta(p_s, x_s)$. We take care of this details in the Section 4.4.

To complete the list of ingredients needed for the SAPT, we need to analyze the spectrum of the principal part of the symbol (71) as $(p_s, x_s)$ varies in $\mathbb{R}^2$. The principal part of the symbol, denoted by $H_0(p_s, x_s)$, is given by (71) when $\delta = 0$, so it reads:

$$H_0(p_s, x_s) := \Xi(Q_t, P_t) = \frac{1}{2\Omega_t} \left[ |a|^2 P_t^2 + |b|^2 Q_t^2 - a \cdot b \{ Q_t; P_t \} \right]. \tag{72}$$

Since the principal symbol is constant on the phase space, i.e. $H_0(p_s, x_s) = \Xi$ for all $(p_s, x_s) \in \mathbb{R}^2$, we are reduced to compute the spectrum of $\Xi$. As well-known (see Remark 4.1 below), the spectrum of $\Xi$ is pure point with $\sigma(\Xi) = \{ \lambda_n := (n + \frac{1}{2}) : n \in \mathbb{N} \}$. We refer to the eigenvalue $\lambda_n$ as the $n$-th Landau level.

The spectrum of the symbol $H_0$ consists of a collection of constant functions $\sigma_n : \mathbb{R}^2 \to \mathbb{R}$, $n \in \mathbb{N}$, $\sigma_n(p_s, x_s) \equiv \lambda_n$, which we call Landau bands. The band $\sigma_n$ is separated by the rest of the spectrum by a constant gap. In the gap condition (analogous to (31)) one can choose $C_g = 1$. Therefore, each finite family of contiguous Landau bands defines a relevant part of the spectrum appropriate to develop the SAPT.

**Remark 4.1** (The domain of self-adjointness). We describe explicitly the domain of self-adjointness of $H_0(p_s, x_s)$. Mimicking the standard theory of Landau levels, one introduces operators

$$a := \frac{i}{\sqrt{2} \Omega_t} \left[ (a_1 + ia_2) P_t - (b_1 + ib_2) Q_t \right] = \frac{i}{\sqrt{2}} [z_a P_t - \overline{z_b} Q_t], \tag{73}$$

$$a^\dagger := \frac{-i}{\sqrt{2} \Omega_t} \left[ (a_1 - ia_2) P_t - (b_1 - ib_2) Q_t \right] = \frac{-i}{\sqrt{2}} [z_a P_t - z_b Q_t], \tag{74}$$

where $z_a := \frac{1}{i}(a_1 - ia_2)$ and $z_b := \frac{1}{i}(b_1 - ib_2)$. It is easy to check that

$$aa^\dagger = \Xi(Q_t, P_t) + \ell q \frac{1}{2} 1_{\mathcal{H}_t}, \quad a^\dagger a = \Xi(Q_t, P_t) - \ell q \frac{1}{2} 1_{\mathcal{H}_t}, \quad [a; a^\dagger] = \ell q 1_{\mathcal{H}_t}. \tag{75}$$

Without loss of generality, we suppose that $\ell q = 1$. Let $\psi_0$ be the ground state defined by $a\psi_0 = 0$. A simple computation shows that $\psi_0(x_t) = Ce^{-\beta(\xi - a)}x_t^2$, where $C > 0$ is a
Figure 2: Structure of the spectrum of $H_0$. The picture shows a “relevant part of the spectrum” consisting of two Landau bands of constant energy $\lambda_*$ and $\lambda_{*+1}$.

normalization constant, and $\alpha \in \mathbb{R}$, $\beta > 0$ are related to the geometry of the lattice $\Gamma$ by $\alpha := a b / 2 |a|^2$ and $\beta := \Omega / 2 |a|^2$. Since $\psi_0$ is a fast decreasing smooth function, the vectors $\psi_n := (n!)^{-\frac{1}{2}} (\mathbf{a}^\dagger)^n \psi_0$, with $n = 0, 1, \ldots$, are well defined. From the algebraic relations (75) it follows straightforwardly that: (i) $\mathbf{a} \psi_n = \sqrt{n} \psi_{n-1}$; (ii) the family of vectors $\{\psi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_f$ called the generalized Hermite basis; (iii) $\Xi \psi_n = \lambda_n \psi_n$; (iv) the spectrum of $\Xi$ is pure point with $\sigma(\Xi) = \{\lambda_n : n \in \mathbb{N}\}$.

Let $\mathcal{L} \subset \mathcal{H}_f$ be the set of the finite linear combinations of the elements of the basis $\{\psi_n\}_{n \in \mathbb{N}}$. The unbounded operators $\mathbf{a}$, $\mathbf{a}^\dagger$ and $\Xi$ are well defined on $\mathcal{L}$ and on this domain $\mathbf{a}^\dagger$ acts as the adjoint of $\mathbf{a}$ and $\Xi$ is symmetric. Both $\mathbf{a}$ and $\mathbf{a}^\dagger$ are closable and we will denote their closure by the same symbols. The operator $\Xi$ is essentially selfadjoint on the domain $\mathcal{L}$ (the deficiency indices are both zero) and so its domain of selfadjointness $\mathcal{F} := \mathcal{D}(\Xi)$ is the closure of $\mathcal{L}$ with respect to the graph norm $\|\psi\|_\mathcal{F}^2 := \|\psi\|_{\mathcal{H}_f}^2 + \|\Xi \psi\|_{\mathcal{H}_f}^2$. The graph norm is equivalent to the more simple regularized norm

$$\|\psi\|_\mathcal{F} := \|\Xi \psi\|_{\mathcal{H}_f}.$$  

The domain $\mathcal{F}$ has the structure of an Hilbert space with hermitian structure provided by the regularized scalar product $(\psi; \varphi)_\mathcal{F} := (\Xi \psi; \Xi \varphi)_{\mathcal{H}_f}$.
4.4 Symbol class and asymptotic expansion

In this section, we firstly identify the Banach space in which the symbol $H_\delta$ defined by (71) takes values and, secondarily, we explain in which sense $H_\delta$ is a “semiclassical symbol” in a suitable Hörmander symbol class. The main results are contained in Proposition 4.2.

Readers who are not interested in technical details can jump directly to the next section. For the definitions of the Hörmander classes $S^1(\mathcal{B}(\mathcal{H}))$ and $S^1(\mathcal{B}(\mathcal{F};\mathcal{H}))$ we refer to Section 3.4.

**Proposition 4.2.** Assume that Assumption (A$_n$) holds true. Then for all $(p_\delta, x_\delta) \in \mathbb{R}^2$ the operator $H_\delta(p_\delta, x_\delta)$ is essentially self-adjoint on the dense domain $\mathcal{L} \subset \mathcal{H}_\delta$ consisting of finite linear combinations of generalized Hermite functions, and its domain of self-adjointness is the domain $\mathcal{F}$ on which the operator $H_0 = \Xi$ is self-adjoint. Finally, $H_\delta$ is in the Hörmander class $S^1(\mathcal{B}(\mathcal{F};\mathcal{H}))$.

In particular, $H_\delta(p_\delta, x_\delta)$ is a bounded operator from the Hilbert space $\mathcal{F}$ to the Hilbert space $\mathcal{H}_\delta$ for all $(p_\delta, x_\delta) \in \mathbb{R}^2$. The proof of the Proposition 4.2 follows from the Kato-Rellich Theorem showing that for any $(p_\delta, x_\delta) \in \mathbb{R}^2$ the operator $H_\delta(p_\delta, x_\delta)$ differs from $H_0$ by a relatively bounded perturbation. The latter claim will be proved in Lemmas 4.3 and 4.4 below.

In view of Assumption (A$_n$), $\tilde{V}_\Gamma \in C^\infty_b(\mathbb{R}^2, \mathbb{R})$ so its Fourier series

$$\tilde{V}_\Gamma(r) = \sum_{n,m \in \mathbb{Z}} w_{n,m} e^{i2\pi n a^* \cdot r} e^{i2\pi m b^* \cdot r}$$

converges uniformly and moreover $\sum_{n,m=-\infty}^{+\infty} |m|^{\alpha_1} |n|^{\alpha_2} |w_{n,m}| \leq C_\alpha$ for all $\alpha \in \mathbb{N}^2$.

Let $V$ be the $\mathbb{Z}^2$-periodic function related to $1/\delta\tilde{V}_\Gamma$, as in Section 4.1. In view of (58) one has

$$\mathcal{W}_{\tilde{E}_0}(X_1, X_2)W^{-1} = V(P_s + \delta P_l, Q_s - \delta P_l) = \sum_{n,m = -\infty}^{+\infty} v_{n,m} e^{i2\pi (nP_s + mQ_s)} e^{i2\pi \delta (nP_l - mQ_l)}$$

(76)

with $v_{n,m} := 1/\delta w_{n,m}$ and where we used the fact that fast and slow variables commute and $[Q_s; P_s] = \delta^2 [Q_l; P_l]$. The operator (76) can be seen as the Weyl quantization of the operator-valued symbol

$$V_\delta(p_\delta, x_\delta) := \sum_{n,m = -\infty}^{+\infty} v_{n,m} e^{i2\pi (nP_s + mQ_s)} e^{i2\pi \delta (nP_l - mQ_l)}$$

(77)

with quantization rule

$$\text{Op}_\delta \left(e^{i2\pi (nP_s + mQ_s)}\right) = e^{i2\pi (nP_l + mQ_s)} \otimes 1_{\mathcal{H}_\delta}. \quad (78)$$

**Lemma 4.3.** Let Assumption (A$_n$) hold true. Then $V_\delta \in S^1(\mathcal{B}(\mathcal{H})) \cap S^1(\mathcal{B}(\mathcal{F};\mathcal{H}))$. In particular $V_\delta(p_\delta, x_\delta)$ is a bounded self-adjoint operator on $\mathcal{H}_\delta$ for all $(p_\delta, x_\delta) \in \mathbb{R}^2$.  

Proof. It is sufficient to show that \( V_\delta \in S^1(\mathcal{B}(\mathcal{H}_i)) \) since
\[
\| \partial^\alpha V_\delta \|_{\mathcal{B}(\mathcal{H}_i)} = \| (\partial^\alpha V_\delta) \Xi^{-1} \|_{\mathcal{B}(\mathcal{H}_i)} \leq 2 \| \partial^\alpha V_\delta \|_{\mathcal{B}(\mathcal{H}_i)}
\]
in view of \( \| \Xi^{-1} \|_{\mathcal{B}(\mathcal{H}_i)} = 2 \). Let \( \alpha := (\alpha_1, \alpha_2) \in \mathbb{N}^2 \), then
\[
\| \partial_p^\alpha \partial_x^\beta V_\delta(p_s, x_s) \|_{\mathcal{B}(\mathcal{H}_i)} \leq (2\pi)^{|\alpha|} \sum_{n,m=\pm \infty}^{+\infty} |n|^{|\alpha_1|} |m|^{|\alpha_2|} |v_{n,m}| \leq \frac{(2\pi)^{|\alpha|}}{\mathcal{E}_0} C_\alpha
\]
for all \((p_s, x_s) \in \mathbb{R}^2\), as a consequence of the unitarity of \( e^{i2\pi(nP_1-mQ_0)} \). The self-adjointness follows by observing that \( \{v_{n,m}\} \) are the Fourier coefficients of a real function.

Assumption \((A_n)\) implies that the \( \Gamma \)-periodic functions \( a^* \cdot A_T \) and \( b^* \cdot A_T \) are elements of \( C_0^\infty(\mathbb{R}^2, \mathbb{R}) \). By the same arguments above, one proves that the operators \( f_j(P_s + \delta P_t, Q_s - \delta P_t), j = 1, 2 \), appearing in \((70)\), are the Weyl quantization of the operator-valued functions
\[
f^{(j)}_\delta(p_s, x_s) := \sum_{n,m=\pm \infty}^{+\infty} f^{(j)}_{n,m} e^{i2\pi(nP_1+mQ_0)} e^{i2\pi\delta(nP_1-mQ_0)} \quad j = 1, 2
\]
according to \((78)\). The coefficients \( \frac{f^{(j)}_{n,m}}{2\pi \Xi(nP_1+mQ_0)} \) are the Fourier coefficients of \( a^* \cdot A_T \) if \( j = 1 \) and of \( b^* \cdot A_T \) if \( j = 2 \). Thus, equation \((70)\) shows that the operator \( W(Q_t, Q_s, P_t, P_s) \) coincides with the Weyl quantization of the operator-valued symbol
\[
W_\delta(p_s, x_s) := f^{(1)}_\delta(p_s, x_s) Q_t + f^{(2)}_\delta(p_s, x_s) P_t,
\]
defined, initially, on the dense domain \( \mathcal{L} \).

Lemma 4.4. Let Assumption \((A_n)\) hold true. Then \( f^{(j)}_\delta \in S^1(\mathcal{B}(\mathcal{H}_i)) \cap S^1(\mathcal{B}(\mathcal{F}; \mathcal{H}_i)) \), for \( j = 1, 2 \). For all \((p_s, x_s) \in \mathbb{R}^2\), the bounded operators \( f^{(j)}_\delta(p_s, x_s) \) are self-adjoint while \( W_\delta(p_s, x_s) \) is symmetric on the dense domain \( \mathcal{L} \) and infinitesimally bounded with respect to \( \Xi \). Finally \( W_\delta \in S^1(\mathcal{B}(\mathcal{F}; \mathcal{H}_i)) \).

Proof. As in the first part of the proof of Lemma 4.3, one proves that \( f^{(j)}_\delta \in S^1(\mathcal{B}(\mathcal{H}_i)) \cap S^1(\mathcal{B}(\mathcal{F}; \mathcal{H}_i)) \) and its self-adjointness. The operator \( W_\delta(p_s, x_s) \) is a linear combination of \( Q_t \) and \( P_t \), which are densely defined on \( \mathcal{L} \), multiplied by bounded operators. Using \((62)\) one checks by a direct computation that \( W_\delta(p_s, x_s) \) acts as a symmetric operator on \( \mathcal{L} \). Since \( Q_t \) and \( P_t \) are infinitesimally bounded with respect to \( \Xi \), then the same holds true for \( W_\delta(p_s, x_s) \), \((p_s, x_s) \in \mathbb{R}^2\). The last claim follows by observing that
\[
\| \partial^\alpha (f^{(j)}_\delta X_t) \|_{\mathcal{B}(\mathcal{F}; \mathcal{H}_i)} = \| (\partial^\alpha f^{(j)}_\delta) X_t \Xi^{-1} \|_{\mathcal{B}(\mathcal{H}_i)} \leq \| X_t \Xi^{-1} \|_{\mathcal{B}(\mathcal{H}_i)} \| \partial^\alpha f^{(j)}_\delta \|_{\mathcal{B}(\mathcal{H}_i)},
\]
with \( j = 1, 2 \) and \( X_t = Q_t \) or \( P_t \). Since \( \| X_t \Xi^{-1} \|_{\mathcal{B}(\mathcal{H}_i)} \leq C \) and \( f^{(j)}_\delta \in S^1(\mathcal{B}(\mathcal{H}_i)) \), the claim is proved.

Lemmas 4.3 and 4.4 together with the fact that \( H_0 \equiv \Xi \) is clearly in \( S^1(\mathcal{B}(\mathcal{F}; \mathcal{H}_i)) \) imply the last part of Proposition 4.2.
4.5 Semiclassics: the $O(\delta^4)$-approximated symbol

In this section we consider the asymptotic expansion for the symbol $H_\delta$ in the parameter $\delta$. The Fourier expansion $^{(77)}$ for $V_\delta$ and the similar expression for $W_\delta$, namely

$$W_\delta(p_s, x_s) = \sum_{n,m=-\infty}^{+\infty} e^{i2\pi(nP_t +mQ_t)} e^{i2\pi\delta(nP_t -mQ_t)} \left[ f_{n,m}^{(1)} Q_t + f_{n,m}^{(2)} P_t \right],$$

suggest a way to expand the symbol $H_\delta$ in powers of $\delta$. By inserting the expansion $e^{i2\pi\delta I_{n,m}} = \sum_{j=0}^{+\infty} (i2\pi\delta)^j I_{n,m}^j$, with $I_{n,m} := nP_t - mQ_t$, in $(77)$ and $(81)$ and by exchanging the order of the series one obtains the formal expansions

$$V_\delta(p_s, x_s) \simeq \sum_{j=0}^{+\infty} \delta^j V_{j+2}(p_s, x_s) \quad \text{and} \quad W_\delta(p_s, x_s) \simeq \sum_{j=0}^{+\infty} \delta^j W_{j+1}(p_s, x_s) \quad (82)$$

where

$$V_{j+2}(p_s, x_s) := \left(\frac{i2\pi}{j!}\right)^j \sum_{n,m=-\infty}^{+\infty} e^{i2\pi(nP_s +mQ_s)} I_{n,m}^j \nu_{n,m} \quad (83)$$

$$W_{j+1}(p_s, x_s) := \left(\frac{i2\pi}{j!}\right)^j \sum_{n,m=-\infty}^{+\infty} e^{i2\pi(nP_s +mQ_s)} I_{n,m}^j \left[ f_{n,m}^{(1)} Q_t + f_{n,m}^{(2)} P_t \right]. \quad (84)$$

In view of (62), one easily shows that the operators $W_j$ are (formally) symmetric.

The justification of the formal expansions above requires some cautions: (i) we need to specify the domains of definitions of the unbounded operators $I_{n,m}^j$ and consequently the domains of definitions of $V_j$ and $W_j$; (ii) we need to justify the exchange of the order of the series in the equations (77) and (81).

As for (i), one notices that

$$I_{n,m} = \alpha_{n,m} a + \overline{\alpha_{n,m}} a^\dagger, \quad \alpha_{n,m} := \frac{nz_b - mz_a}{\sqrt{2}}. \quad (85)$$

For all $(n, m) \in \mathbb{Z}^2$ the operators $I_{n,m}$ are essentially self-adjoint on the invariant dense domain $\mathcal{L}$ (their deficiency indices are both zero). The powers $I_{n,m}^j$ are also well defined and essentially self-adjoint on $\mathcal{L}$, as consequence of the Nelson Theorem (Theorem X.39 in RS75) since the set $\{\psi_n\}_{n\in\mathbb{N}}$ of the generalized Hermite functions is a total set of analytic vectors for every $I_{n,m}$ (see Example 2 Section X.6 of RS75). The domain of self-adjointness for $I_{n,m}^j$ is the closure of $\mathcal{L}$ with respect the corresponding graph norm.

The operator $V_j(p_s, x_s)$ defined by equation (83) is an homogeneous polynomial of degree $j - 2$ in $a$ and $a^\dagger$. It is symmetric (hence closable) and essentially self-adjoint on the invariant dense domain $\mathcal{L}$. Analogously, the operators

$$M_{n,m}^j := I_{n,m}^j \left[ f_{n,m}^{(1)} Q_t + f_{n,m}^{(2)} P_t \right] = I_{n,m}^j \left[ g_{n,m} a + \overline{g_{n,m}} a^\dagger \right], \quad g_{n,m} := \frac{z_af_{n,m}^{(1)} + z_b f_{n,m}^{(2)}}{\sqrt{2}} \quad (86)$$

for all $(n, m) \in \mathbb{Z}^2$ and $(a, b) \in \mathbb{C}^2$, are essentially self-adjoint on $\mathcal{L}$ with respect the graph norm.
which appear in the right-hand side of equation (84), are essentially self-adjoint on \( \mathcal{L} \) since the set of the generalized Hermite functions provides a total set of analytic vectors. Thus we answered to point (i).

Since the generalized Hermite functions are a total set of analytic vectors for all \( I_{n,m} \) then the series \( \sum_{j=0}^{+\infty} \frac{(2\pi\delta)^j}{j!} I_{n,m}^j \psi \) converges in norm for every \( \psi \in \mathcal{L} \). From this observation, the fact that the series of coefficients \( v_{n,m}, f_{n,m}^{(1)} \) and \( f_{n,m}^{(2)} \) are absolutely convergent and that \( Q_t \) and \( P_t \) leave invariant the domain \( \mathcal{L} \), one argues that for all \( \psi \in \mathcal{L} \) the double series which defines \( \tilde{V}_\delta(p_s, x_s) \psi \) and \( W_\delta(p_s, x_s) \psi \) are absolutely convergent, hence the order of the sums can be exchanged. Thus the series appearing on the r.h.s. of (82) agrees with \( \tilde{V}_\delta \) (respectively, with \( W_\delta \)) on the dense domain \( \mathcal{L} \). By a density argument, the equality in (82) holds true on the full domain of definition of \( \tilde{V}_\delta \) (which is \( \mathcal{H}_t \)) and \( W_\delta \) respectively.

In view of the above, we write the “semiclassical expansion” of the symbol \( H_\delta \) as:

\[
H_\delta(p_s, x_s) = \Xi + \sum_{j=1}^{+\infty} \delta^j H_j(p_s, x_s), \quad H_j(p_s, x_s) := W_j(p_s, x_s) + V_j(p_s, x_s)
\]  (87)

with \( V_1 = 0 \).

Proposition 4.2 shows that the natural domain for the full symbol \( H_\delta(p_s, x_s) \) is the domain \( \mathcal{F} \) of self-adjointness of \( \Xi \). However, if we want to truncate the series (87) at the \( j \)-th order, we must be careful in the determination of the domain of definition of the single terms and to control the remainder. Every term in the expansion (87) is essentially self-adjoint on \( \mathcal{L} \). However, the \( j \)-th order term \( H_j \) is the sum of two homogeneous polynomials in \( Q_t \) and \( P_t \) (or equivalently in \( a \) and \( a^\dagger \) ), \( W_j \) of degree \( j \) and \( V_j \) of degree \( j - 2 \). Since \( W_j = 0 \) if \( A_\Gamma = 0 \), one obtains

\[
\deg H_j = \begin{cases} j, & \text{if } A_\Gamma \neq 0 \\ j - 2, & \text{if } A_\Gamma = 0 \end{cases}
\]

where \( \deg H_j \) means the degree of \( H_j \) as a polynomial in \( Q_t \) and \( P_t \). If \( \deg H_j > 2 \) then the operator \( H_j \) is not bounded by the principal symbol \( \Xi \), and in this sense it cannot be considered as a “small perturbation” in the sense of Kato. Moreover, some other problems appear (see Remark 4.8). In order to avoid these problems, we truncate the expansion (87) up to the polynomial term of degree 2, i.e. up to order \( \delta^2 \) if \( A_\Gamma = 0 \) and up to order \( \delta^4 \) if \( A_\Gamma \neq 0 \).

Hereafter let \( \zeta \) be the indicator function of the periodic vector potential, defined as

\[
\zeta = \begin{cases} 0, & \text{if } A_\Gamma \neq 0 \\ 1, & \text{if } A_\Gamma = 0. \end{cases}
\]

Let \( \tilde{H}_\delta(p_s, x_s) := \Xi + \sum_{j=1}^{2(1+\zeta)} \delta^j H_j(p_s, x_s) \), namely

\[
\tilde{H}_\delta^1(p_s, x_s) = \Xi + \delta^2 \sum_{n,m=-\infty}^{+\infty} v_{n,m} e^{i2\pi(np_s + mx_s)} \left( 1_{\mathcal{H}_t} + i2\pi \delta I_{n,m} + \frac{1}{2} (i2\pi)^2 \delta^2 I_{n,m}^2 \right)
\]  (88)

\[
\tilde{H}_\delta^0(p_s, x_s) = \Xi + \delta \sum_{n,m=-\infty}^{+\infty} e^{i2\pi(np_s + mx_s)} \left[ M_{n,m}^0 + \delta (i2\pi M_{n,m}^{1} + v_{n,m} 1_{\mathcal{H}_t}) \right].
\]  (89)

33
We call $\tilde{H}_\delta^2$ the approximated symbol up to order $\delta^{2(1+\varepsilon)}$. As a consequence of the Kato-Rellich theorem we have the following result:

**Proposition 4.5.** Under Assumption (Aₜ) there exists a constant $\delta_0$ such that for every $\delta < \delta_0$ and for every $(p_s, x_s) \in \mathbb{R}^2$ the operator $\tilde{H}_\delta^2(p_s, x_s)$ (both for $\varepsilon = 0$ or 1) is self-adjoint on the domain $\mathcal{F}$ and bounded from below. Moreover $\tilde{H}_\delta^2 \in S^1(\mathcal{B}(\mathcal{F}; \mathcal{H}_t))$.

**Proof.** As proved in Lemma A.2, a and $a^\dagger$ are infinitesimally bounded with respect to $\Xi$. This fact and Assumption (Aₜ), which assures the fast decay of the coefficients $v_{n,m}$ and $g_{n,m}$ (see (86)), imply that the operators

$$\sum_{n,m=-\infty}^{+\infty} v_{n,m} e^{2\pi i (np_s + mx_s)} (1_{\mathcal{H}_t} + i 2\pi \delta I_{n,m}), \quad \sum_{n,m=-\infty}^{+\infty} e^{i 2\pi (np_s + mx_s)} [M^0_{n,m} + \delta v_{n,m} 1_{\mathcal{H}_t}]$$

are infinitesimally bounded with respect to $\Xi$ and are elements of $S^1(\mathcal{B}(\mathcal{F}; \mathcal{H}_t))$.

The operators $I_{n,m}^2$ and $M^1_{n,m}$ are only bounded (and not infinitesimally bounded) with respect to $\Xi$. First of all it is easy to check that $\|a^2 \psi\|_{\mathcal{H}_t} \leq 3\|\Xi \psi\|_{\mathcal{H}_t}$ for every $\psi \in \mathcal{F}$, where $a^2$ means $a$ or $a^\dagger$. Then, for every $\psi \in \mathcal{F}$,

$$\|I_{n,m}^2 \psi\|_{\mathcal{H}_t}^2 \leq |\alpha_{n,m}|^2 \left( \|a^2 \psi\|_{\mathcal{H}_t} + \|a^\dagger a^\dagger \psi\|_{\mathcal{H}_t} + \|a^2 \psi\|_{\mathcal{H}_t} \right)^2 \leq 27 d^4 4 \|\Xi \psi\|_{\mathcal{H}_t}^2 \tag{90}$$

where we used the inequality $(\alpha + \beta + \gamma)^2 \leq 3(\alpha^2 + \beta^2 + \gamma^2)$, the identity $\{a; a^\dagger\} = 2\Xi$ and the bound $|\alpha_{n,m}|^2 \leq d^2/d(n^2 - m^2)$ with $d^2 := \max\{|a|, |b|\}$. Assumption (Aₜ) assures that the operator $\delta^4 2\pi^2 \sum_{n,m=-\infty}^{+\infty} v_{n,m} e^{2\pi i (np_s + mx_s)} I_{n,m}^2$, which appears in (88), is bounded by $\Xi$ by a constant $\delta^4 C$, with $C \propto \sum_{n,m=-\infty}^{+\infty} v_{n,m}(n^2 + m^2)$, and is in $S^1(\mathcal{B}(\mathcal{F}; \mathcal{H}_t))$. The claim for $\tilde{H}_\delta^1$ follows from the Kato-Rellich theorem fixing $\delta_0 := C^{-\frac{1}{4}}$.

The claim for $\tilde{H}_\delta^0$ follows in the same way proving an inequality of the type (90) for $M^1_{n,m} = c_{n,m} a^2 + c_{n,m} a^\dagger a^\dagger + 2\Re(d_{n,m}) \Xi + \iota_3 \Im(d_{n,m})$ where $c_{n,m} := \alpha_{n,m} g_{n,m}$ and $d_{n,m} := \alpha_{n,m} \bar{g}_{n,m}$. Observe that the series of coefficients $g_{n,m}$ decays rapidly, then also the serie $c_{n,m}$ and $d_{n,m}$ have a fast decay and in particular are bounded. This implies that in the inequality of type (90) we can find a global constant which does not depend on $n$ and $m$. ■

It is useful to have explicit expressions of the first terms $H_j$, in terms of $a$ and $a^\dagger$. From equations (87), (88) and (89), using the Fourier expansion of the derivatives of $V$ and $g := 1/\sqrt{\varepsilon}(z_1 f_1 + z_2 f_2)$, it is easy to check the following:

- **Case 1: $A_{\Gamma} = 0$** - In this situation

$$\tilde{H}_\delta^1 = \Xi + \delta^2 H_2 + \delta^3 H_3 + \delta^4 H_4$$

with

$$H_2(p_s, x_s) = V(p_s, x_s) 1_{\mathcal{H}_t} \tag{91}$$

$$H_3(p_s, x_s) = -\frac{1}{\sqrt{2}} [D_z(V) a + D_{\Xi}(V) a^\dagger] \tag{92}$$

$$H_4(p_s, x_s) = \frac{1}{4} \left[ |D_z|^2(V) 2\Xi + D_z^2(V) a^2 + D_z^2(V) a^\dagger^2 \right] \tag{93}$$
where $D_z$ is the differential operator defined by $D_z := \left(z_a \frac{\partial}{\partial x_s} - z_b \frac{\partial}{\partial p_s}\right)$ and $D_{\bar{\tau}}$ is obtained by replacing $z_a$ and $z_b$ with $\bar{z}_a$ and $\bar{z}_b$. Since $V$ is real, $D_{z}(V) = D_{\bar{\tau}}(V)$, which shows that $H_{3}$ is symmetric. The explicit expression of the second order differential operator $|D_z|^2 := D_z \circ D_{\bar{\tau}}$ is

$$|D_z|^2 = \frac{1}{\Omega_{\Gamma}} \left( |a|^2 \frac{\partial^2}{\partial x_s^2} - 2a \cdot b \frac{\partial^2}{\partial x_s \partial p_s} + |b|^2 \frac{\partial^2}{\partial p_s^2} \right).$$

(94)

- **Case 2:** $A_{\Gamma} \neq 0$ - In this situation

$$\tilde{H}_0^\delta = \Xi + \delta H_1 + \delta^2 H_2$$

with

$$H_1(p_s, x_s) = g(p_s, x_s) a + \bar{g}(p_s, x_s) a^\dagger$$

$$H_2(p_s, x_s) = V \ 1_{\mathcal{H}_t} - \sqrt{2} D_{z}(\bar{g}) \Xi - \frac{1}{\sqrt{2}} \left[ D_{z}(g) a^2 + D_{\bar{\tau}}(\bar{g}) a^\dagger a^\dagger \right].$$

(96)

In the computation of (96) we used the first of the gauge conditions (52) which assures that $D_{z}(g) = 1\sqrt{2}\Omega_{\Gamma}$$

$$D_{z}(\bar{g}) = \frac{1}{\sqrt{2}\Omega_{\Gamma}} \left[ |a|^2 \frac{\partial f_1}{\partial x_s} + a \cdot b \left( \frac{\partial f_2}{\partial x_s} - \frac{\partial f_1}{\partial p_s} \right) - |b|^2 \frac{\partial f_2}{\partial p_s} \right]$$

is a real function. From the definition of $g$, $f_1$ and $f_2$ it follows that

$$g(a^* \cdot r, b^* \cdot r) = \pi \sqrt{2} \frac{Z^2}{\Phi_0} \left[ (A_{\Gamma})_1 - i(A_{\Gamma})_2 \right](r),$$

(97)

namely $g$ is the dimensionless $\mathbb{Z}^2$-periodic function related to the $\Gamma$-periodic function $(A_{\Gamma})_1 - i(A_{\Gamma})_2$, up to a multiplicative constant.

### 4.6 Main result: effective dynamics for strong magnetic fields

#### Preliminary estimates on the remainder

The difference $\mathcal{R}_0^\delta := H_\delta - \tilde{H}_0^\delta$ is a self-adjoint element of $S^1(\mathcal{B}(\mathcal{F}; \mathcal{H}_t))$, which we call the **remainder symbol**. To develop the SAPT for the Harper regime we need to estimate the order of the remainder symbol. The next result shows essentially that

$$H_\delta(p_s, x_s) = \tilde{H}_0^\delta(p_s, x_s) + \mathcal{O}(\delta^{2(\ell+1)}), \quad \pi_r H_\delta(p_s, x_s) \pi_r = \pi_r \tilde{H}_0^\delta(p_s, x_s) \pi_r + \mathcal{O}(\delta^{2(\ell+1)+1}),$$

(98)

where

$$\pi_r := \sum_{i=1}^{m} |\psi_k_i \rangle \langle \psi_k_i|$$

(99)

is the projection on the subspace spanned by the finite family of generalized Hermite functions $\{\psi_k_i\}_{i=1}^m$. In other words, the error done by replacing the true symbol $H_\delta$ with the
approximated symbol $\tilde{H}_\delta^s$ (which has order $2(\delta + 1)$ in $\delta$) is of the same order of the approximated symbol, so in this sense $\tilde{H}_\delta^s$ is not a good approximation for $H_\delta$. On the other side, what we need to develop the SAPT is to control the operator $\pi_r H_\delta \pi_r$, which is well approximated by $\pi_r \tilde{H}_\delta^s \pi_r$ up to an error of order $2(\delta + 1) + 1$ in $\delta$.

**Proposition 4.6.** Let Assumption $(A_\delta)$ hold true. Then $\mathcal{R}_\delta^s$ has order $O(\delta^{2(\gamma + 1)})$, i.e. there exist a constant $C$ such that $\|\mathcal{R}_\delta^s(p_s, x_s)\|_{\mathcal{S}(\mathcal{D}, \mathcal{H}_t)} \leq C\delta^{2(\gamma + 1)}$ for all $(p_s, x_s) \in \mathbb{R}^2$. Moreover $\|\mathcal{R}_\delta^s \pi_r\|_{\mathcal{S}(\mathcal{H}_t)} = \|\pi_r R_\delta^s\|_{\mathcal{S}(\mathcal{H}_t)} \leq C\delta^{2(\gamma + 1) + 1}$, for all $(p_s, x_s) \in \mathbb{R}^2$, i.e. $\mathcal{R}_\delta^s \pi_r, \pi_r \mathcal{R}_\delta^s$ and $[\mathcal{R}_\delta^s; \pi_r]$ are $\mathcal{B}(\mathcal{H}_t)$-valued symbols of order $O(\delta^{2(\gamma + 1) + 1})$.

**Proof.** (Case $\gamma = 1$) The explicit expression of the remainder symbol is

\[ \mathcal{R}_\delta^s(p_s, x_s) = \delta^2 \sum_{n, m = -\infty}^{+\infty} v_{n,m} e^{i2\pi(np_s + mx_s)} \left[ e^{i2\pi \delta I_{n,m}} - \left( 1 + i2\pi \delta I_{n,m} + \frac{1}{2}(i2\pi)^2 \delta^2 I_{n,m}^2 \right) \right]. \]

(100)

and from (100) it follows that $\|\mathcal{R}_\delta^s(p_s, x_s)\|_{\mathcal{S}(\mathcal{D}, \mathcal{H}_t)} \leq \delta^2 \sum_{n, m = -\infty}^{+\infty} |v_{n,m}| \Lambda_{n,m}$ with

\[ \Lambda_{n,m} := \sup_{\psi \in \mathcal{H}_t \setminus \{0\}} \frac{\left\| \left[ e^{i2\pi \delta I_{n,m}} - \left( 1 + i2\pi \delta I_{n,m} + \frac{1}{2}(i2\pi)^2 \delta^2 I_{n,m}^2 \right) \right] \Xi^{-1}\psi \right\|_{\mathcal{H}_t}}{\|\psi\|_{\mathcal{H}_t}}, \]

(101)

since $\|\psi\|_{\mathcal{F}} := \|\Xi \psi\|_{\mathcal{H}_t}$ and $\mathcal{F} = \Xi^{-1}\mathcal{H}_t$. The operators $I_{n,m}$ are essentially self-adjoint on $\mathcal{L}$ and we denote their closure with the same symbol. Since the operators $I_{n,m}^2$ are positive, we can consider the resolvent operators $R_{n,m} := (I_{n,m}^2 + 1)^{-1}$. Let suppose that

\[ \zeta_{n,m}(\delta) := \left\| \left[ e^{i2\pi \delta I_{n,m}} - \left( 1 + i2\pi \delta I_{n,m} + \frac{1}{2}(i2\pi)^2 \delta^2 I_{n,m}^2 \right) \right] R_{n,m} \right\|_{\mathcal{S}(\mathcal{H}_t)}, \]

(102)

for all $n, m \in \mathbb{Z}$, with $\sup_{\delta} \zeta(\delta) < +\infty$. Then equation (101) would imply

\[ \Lambda_{n,m} \leq (\delta^2 \sum_{n, m = -\infty}^{+\infty} |v_{n,m}|(|n| + |m|)^2) \delta^2 \zeta(\delta) \leq C_2 \delta^2 \zeta(\delta). \]

for suitable positive constants $C_1$ and $C_2$.

It remains to prove the inequality (102) and the estimate on $\zeta(\delta)$. By spectral calculus one has that $\zeta_{n,m}(\delta) = \sup_{t \in \sigma(I_{n,m})} |Z_\delta(t)| \leq \sup_{t \in \mathbb{R}} |Z_\delta(t)| =: \zeta(\delta)$ where

\[ Z_\delta(t) := 4\pi^2 \delta^2 \frac{e^{i2\pi \delta t} - (1 + i2\pi \delta t - \frac{1}{2}(2\pi \delta t)^2)}{(2\pi \delta t)^2 + 4\pi^2 \delta^2}. \]
After some manipulations and the change of variable $\tau := 2\pi \delta t$ one has that
\[
G_\delta(\tau) := \frac{1}{4\pi^4 \delta^4} \left| Z_\delta \left( \frac{\tau}{2\pi \delta^2} \right) \right|^2 \leq \frac{\tau^4 + 4\tau^2 \cos(\tau) - 8\tau \sin(\tau) - 8 \cos(\tau) + 8}{\tau^4} < C_3.
\]
Thus $\zeta(\delta)^2 = 4\pi^4 \delta^4 \sup_{\tau \in R} G_\delta(\tau) \leq 4\pi^4 C_3 \delta^4$, hence $\|\mathcal{R}_3^1(p_s, x_s)\|_{\mathcal{A}(D, H)} \leq C \delta^3$. This concludes the first part of the proof.

Since $\|\mathcal{R}_3^1 \|_{\mathcal{A}(H)} \leq \sum_{i=1}^{m} \|\mathcal{R}_3^1\|_{(\psi_k)^i} \|\|_{\mathcal{A}(H)}$, then it is enough to show that for any Hermite vector $\psi_k$ the inequality $\|\mathcal{R}_3^1\|_{(\psi_k)^i} \|\|_{\mathcal{A}(H)} \leq C_k \delta^3$ holds true. Observing that $\|\mathcal{R}_3^1\|_{(\psi_k)^i} \|\|_{\mathcal{A}(H)} = \|\mathcal{R}_3^1\|_{\psi_k} \|\|_{\mathcal{H}}$, one deduces

\[
\lim_{\delta \to 0} \delta^{-5} \|\mathcal{R}_3^1\|_{(\psi_k)^i} \|\|_{\mathcal{A}(H)} = \lim_{\delta \to 0} \left\| \sum_{n,m=-\infty}^{+\infty} v_{n,m} e^{2\pi i (np \cdot mx_0)} \left( \sum_{j=3}^{+\infty} \left( (2\pi i)^j I_{n,m}^j \right) \right) \psi_k \right\|_{\mathcal{H}} 
\leq \frac{4}{3} \pi^3 \sum_{n,m=-\infty}^{+\infty} |v_{n,m}| |I_{n,m}|^3 \psi_k \|_{\mathcal{H}} \leq 32 \pi^3 C' \|A^3 \psi_k\|_{\mathcal{H}} = \frac{32}{3} \sqrt{(k+3)!} \pi^3 C' =: C_k
\]

where $C' := \sum_{n=m=-\infty}^{+\infty} |v_{n,m}| |\alpha_n\|_3^3$ is finite in view of Assumption (A).

This shows that for all $\delta \in [0, \delta_0)$ (for a suitable $\delta_0 > 0$) the norm $\|\mathcal{R}_3^1\|_{\mathcal{A}(H)}$ is bounded by $C_k \delta^3$ and so it follows that $\|\mathcal{R}_3^1 \|_{\mathcal{A}(H)} \leq m C \delta^5$ with $C := \max_{i,\ldots,m}(C_k)$. Finally $\|\pi_1 \mathcal{R}_3^1\|_{\mathcal{A}(H)} = (\|\mathcal{R}_3^1\|_{\mathcal{A}(H)})^{1/3} \|\mathcal{A}(H)\| = \|\mathcal{R}_3^1\|_{\mathcal{A}(H)}$.

(Case $\delta = 0$) The proof proceeds as in the previous case. Divide the remainder symbol in two terms $\mathcal{R}_3^0 = \mathcal{R}_3^0 + \mathcal{R}_3^1$.

\[
\mathcal{R}_3^0(p_s, x_s) := \delta \sum_{n,m=-\infty}^{+\infty} e^{2\pi i (np \cdot mx_0)} \left( e^{2\pi i I_{n,m}} - 1 \right) \mathcal{H}_t
\]
\[
\mathcal{R}_3^1(p_s, x_s) := \delta^2 \sum_{n,m=-\infty}^{+\infty} v_{n,m} e^{2\pi i (np \cdot mx_0)} \left( e^{2\pi i I_{n,m}} - 1 \right).
\]

The control of $\mathcal{R}_3^0$ is easy, indeed $\|\mathcal{R}_3^0\|_{\mathcal{A}(D, H)} \leq 2 \|\mathcal{R}_3^1\|_{\mathcal{A}(H)} \leq 4 C \delta^2$ where $C := \sum_{n,m=-\infty}^{+\infty} |v_{n,m}|$. Moreover (with the same technique used for the case $\delta = 1$), one can check that for any Hermite vector $\psi_k$ the function $t_1(\delta) := \frac{1}{\delta} \|\mathcal{R}_3^1(p_s, x_s)\|_{\mathcal{H}}$ is bounded by a constant $C_k > 0$ in a suitable interval $[0, \delta_0)$. This assures that $\|\mathcal{R}_3^0\|_{\pi_1 \|\mathcal{A}(H)}$ is of order $O(\delta^3)$.

To control $\mathcal{R}_3^0$ we need to estimate $\Sigma_{n,m} := \| e^{2\pi i I_{n,m}} - 1 \| H_t \leq 2 \| e^{2\pi i I_{n,m}} \|_{\mathcal{H}_t} \leq 4 C \delta^2$ where $C := \sum_{n,m=-\infty}^{+\infty} |v_{n,m}|$. Let $R_{n,m}$ be the resolvent $(I_{n,m} + i \mathcal{H}_t)^{-1}$. It is easy to check that $\| R_{n,m} \|_{\mathcal{A}(H)}$ is bounded by a linear expression in $|n|$ and $|m|$. Indeed, as proved in Proposition 4.5 both $M_{n,m}^0$ and $M_{n,m}^1$ are bounded by $\Xi$. By spectral calculus $\| (e^{2\pi i I_{n,m}} - 1 \mathcal{H}_t - i \mathcal{H}_t) R_{n,m}^2 \|_{\mathcal{A}(H)}$ is bounded by the maximum in $\tau$ of the function $F_\delta(\tau) := 4 \pi^2 \delta^2 \frac{r^2 - 2 \tau \sin(\tau) - \cos(\tau)}{\tau^2}$.

The last part follows observing that $M_{n,m}^0$ is a linear combinations of $a$ and $a^\dagger$ and so they act splitting a Hermite vector $\psi_k$ as $C_{k,m}^0 \psi_{k-1} + d_{k,m}^0 \psi_{k+1}$ where for a fixed $k$ the coefficients depend on $f_{n,m}^{(k)}$. To conclude the proof it is sufficient to notice that $t_0(\delta) := \frac{1}{\delta} \| e^{2\pi i I_{n,m}} - 1 \mathcal{H}_t - i \mathcal{H}_t \|_{\mathcal{H}_t}$ is bounded by a constant $C_k > 0$ in a suitable interval $[0, \delta_0)$. ■
Derivation of the adiabatically decoupled effective dynamics

We recall that the Weyl quantization of the symbol $H_\delta$ is the Hamiltonian $H^{\text{W}}$, namely $\text{Op}_\delta(H_\delta) = H^{\text{W}}$. As for the approximated symbol $\tilde{H}_\delta^2$, we pose $\tilde{H}_\delta^2 := \text{Op}_\delta(H_\delta^2)$. Both $H^{\text{W}}$ and $\tilde{H}_\delta^2$ are bounded operators from $L^2(\mathbb{R}, dx_s) \otimes \mathcal{F}$ to $\mathcal{H}_w := L^2(\mathbb{R}, dx_s) \otimes \mathcal{H}_t$.

**Theorem 4.7.** Let Assumption (A$_w$) be satisfied. Let $\{\sigma_n(\cdot)\}_{n \in \mathbb{Z}}$, with $\mathbb{Z} = \{n, \ldots, n + m - 1\}$, be a family of Landau bands for $\Xi$ and let $\pi_r := \sum_{n \in \mathbb{Z}} |\psi_n\rangle \langle \psi_n|$ be the spectral projector of $H_0 = \Xi$ corresponding to the set $\{\sigma_n(p_s, x_s)\}_{n \in \mathbb{Z}}$. Then:

1. **Almost-invariant subspace:** there exists an orthogonal projection $\Pi_0^2 \in \mathcal{B}(\mathcal{H}_w)$, with $\Pi_0^2 = \text{Op}_\delta(\pi) + O_0(\delta^{\infty}), \pi(p_s, x_s) \propto \sum_{j=0}^{\infty} \delta^j \pi_j(p_s, x_s)$, and $\pi_0(p_s, x_s) \equiv \pi_r$, such that
   \[ [\tilde{H}_\delta^2; \Pi_0^2] = O_0(\delta^{\infty}), \quad [H^{\text{W}}; \Pi_0^2] = O_0(\delta^{2(\varepsilon + 1)}). \tag{103} \]

2. **Effective dynamics:** let $\Pi_r := 1_{\mathcal{H}_0} \otimes \pi_r \in \mathcal{B}(\mathcal{H}_w)$ and $\mathcal{K} := \text{Ran} \Pi_r \simeq L^2(\mathbb{R}, dx_s) \otimes \mathbb{C}^w$. Then there exists a unitary operator $U_\delta^2 \in \mathcal{B}(\mathcal{H}_w)$ such that
   
   (i) $U_\delta^2 = \text{Op}_\delta(u) + O_0(\delta^{\infty})$, where the symbol $u \propto \sum_{j=0}^{\infty} \delta^j u_j$ has principal part $u_0 \equiv 1_{\mathcal{H}_t}$;  
   (ii) $\Pi_r = U_\delta^2 \Pi_0^2 U_\delta^{\varepsilon - 1}$;  
   (iii) Let $h^\varepsilon$ in $S^1(\mathcal{B}(\mathcal{H}_0))$ be a resummation of the formal symbol $\pi \pi^\dagger \tilde{H}_\delta^2 \pi \pi^\dagger u^\dagger$ and define the effective Hamiltonian by $H_{\varepsilon}^\delta := \text{Op}_\delta(h^\varepsilon)$. Since $[H_{\varepsilon}^\delta; \Pi_r] = 0$, $H_{\varepsilon}^\delta$ is a bounded operator on $\mathcal{K}$. Then
   \[ U_\delta^2 \Pi_r^2 H^{\text{W}} \Pi_0^2 U_\delta^{\varepsilon - 1} = H_{\varepsilon}^\delta + O_0(\delta^{2(\varepsilon + 1)} + 1) \in \mathcal{B}(\mathcal{K}). \tag{104} \]

2'. **Effective dynamics for a single Landau band when $A_\Gamma = 0$:** Consider a single Landau band $\sigma_s(\cdot) = \lambda_s$, so that $\pi_r = |\psi_s\rangle \langle \psi_s|$. Then, up to the order $\delta^1$, one has that
   \[ H_{\varepsilon}^\delta = \lambda_s 1_{\mathcal{H}_s} + \delta \lambda_s Y(P_s, Q_s) + \frac{\lambda_s}{2} \frac{1}{2} Y(P_s, Q_s) + O_0(\delta^2) \tag{105} \]
   where $V(P_s, Q_s) := \text{Op}_\delta(V)$ is the Weyl quantization of the $\mathbb{Z}^2$-periodic function $V(p_s, x_s)$ related to the $\Gamma$-periodic potential $V_\Gamma$, while $Y(P_s, Q_s) := \text{Op}_\delta(|D_z|^2(V))$ is the Weyl quantization of the function $|D_z|^2(V)(p_s, x_s)$ defined through the differential operator $H_{\varepsilon}^\delta$.

The derivation of the effective dynamics when $A_\Gamma \neq 0$ will be considered in Section 4.8.

**Proof of Theorem 4.7**

**Step 1. Almost-invariant subspace**

As explained in the first part of proof of the Theorem 3.12 one constructs a formal symbol $\pi$ (the Moyal projection) such that: (i) $\pi \pi^\dagger \pi \propto \pi$; (ii) $\pi \pi^\dagger = \pi$; (iii) $\tilde{H}_\delta^2 \pi \propto \pi \tilde{H}_\delta^2$. Such a
symbol $\pi \asymp \sum_{j=0}^{\infty} \delta^j \pi_j$ is constructed recursively order by order starting from $\pi_0 = \pi_r$ and $\tilde{H}_\delta^z$ and it is unique (see Lemma 2.3. in \cite{PST03b}). The recursive relations are

$$\pi_n := \pi_n^D + \pi_n^{OD}$$

where the \textit{diagonal part} is $\pi_n^D := \pi_r G_n \pi_r + (\mathbb{1}_H - \pi_r) G_n (\mathbb{1}_H - \pi_r)$ with

$$G_n := \left[ \left( \sum_{j=0}^{n-1} \delta^j \pi_j \right) - \left( \sum_{j=0}^{n-1} \delta^j \pi_j \right) \right]_n.$$  \hfill (107)

The \textit{off-diagonal part} is defined by the implicit relation $[H_0; \pi_n^{OD}] = -F_n$ where

$$F_n := \left[ \tilde{H}_\delta^z \left( \sum_{j=0}^{n-1} \delta^j \pi_j + \delta^n \pi_n^D \right) - \left( \sum_{j=0}^{n-1} \delta^j \pi_j + \delta^n \pi_n^D \right) \right]_n.$$ \hfill (108)

The uniqueness allows us to construct $\pi$ only locally and this local construction is explained in the second part of Lemma 2.3 in \cite{PST03b}. In our case we can choose a $\alpha$-independent positively oriented complex circle $\Lambda \subset \mathbb{C}$, symmetric with respect to the real axis, which encloses the family of (constant) spectral bands $\{\sigma_n(\cdot) = \lambda_n\}_{n \in \mathbb{Z}}$ and such that $\text{dist}(\Lambda, \sigma(H_0)) \geq \frac{1}{2}$ (see Figure \[\text{2}\]). For all $\lambda \in \Lambda$ we construct recursively the \textit{Moyal resolvent} (or \textit{parametrix}) $R^n(\lambda; \cdot) := \sum_{j=0}^{\infty} \delta^j R_j^\circ(\lambda; \cdot)$ of $\tilde{H}_\delta^z$, following the same technique explained during the proof of Theorem \[3.12\]. The approximants of the symbol $\pi$ are related to the approximants of the Moyal resolvent by the usual Riesz formula $\pi_j(z) := \frac{i}{2\pi} \oint_\Lambda d\lambda R_j^\circ(\lambda; z)$ where $z := (p_x, x_s) \in \mathbb{R}^2$. Some care is required to show (iii) since, by construction, $\tilde{H}_\delta^z \parallel \pi$ takes values in $\mathcal{B}(\mathcal{H}_l)$ while $\pi \parallel \tilde{H}_\delta^z$ takes values in $\mathcal{B}(\mathcal{F})$. To solve this problem one can use the same argument proposed in Lemma 7 of \cite{PST03a}.

The technical and new part of the proof consist in showing that $\pi \in S^1(\mathcal{B}(\mathcal{H}_l)) \cap S^1(\mathcal{B}(\mathcal{H}_l, \mathcal{F}))$. The Riesz formula implies $\| (\delta^2 \pi_j(z)) \| \leq 2 \pi \sup_{\lambda \in \Lambda} \| \delta^2 R_j^\circ(\lambda; z) \|$, for all $\alpha \in \mathbb{N}^2$ \(b\) means either $\mathcal{B}(\mathcal{H}_l)$ or $\mathcal{B}(\mathcal{H}_l, \mathcal{F})$ and $\partial^2 := \partial^\alpha_\pi \partial^\alpha_\pi$ since $\Lambda$ does not depend on $\alpha$. Then we need only to show that $R^\circ_j(\lambda; \cdot) \in S^1(\mathcal{B}(\mathcal{H}_l)) \cap S^1(\mathcal{B}(\mathcal{H}_l, \mathcal{F}))$. The choice of $\lambda$ assures $\| R^\circ_0(\lambda; z) \|_{\mathcal{B}(\mathcal{H}_l)} = \| (\Xi - \lambda I_{\mathcal{H}_l})^{-1} \|_{\mathcal{B}(\mathcal{H}_l)} \leq 2$. Moreover $\partial^2_\pi R^\circ_0(\lambda; z) = 0$ for all $\alpha \neq 0$ and this implies that $R^\circ_0(\lambda; z) \in S^1(\mathcal{B}(\mathcal{H}_l))$ uniformly in $\lambda$. Since $\| R^\circ_0(\lambda; z) \|_{\mathcal{B}(\mathcal{H}_l, \mathcal{F})} = \| (\Xi - \lambda I_{\mathcal{H}_l})^{-1} \|_{\mathcal{B}(\mathcal{H}_l)} \leq \infty$ one concludes that $R^\circ_0(\lambda; z) \in S^1(\mathcal{B}(\mathcal{H}_l, \mathcal{F}))$ uniformly in $\lambda$.

By means of equation \[43\], one has $R^\circ_j = -R^\circ_j L^j$ where $L^j$ is the \textit{j}-th order obstruction for $R^\circ_0$ to be the Moyal resolvent. In view of this recursive relation, the proof of $R^\circ_j \in S^1(\mathcal{B}(\mathcal{H}_l))$ for all $j \in \mathbb{N}$ is reduced to show that $L^j \in S^1(\mathcal{B}(\mathcal{H}_l))$ for all $j \in \mathbb{N}$.

The first order obstruction, computed by means of \[44\], is

$$L^1_j(\lambda; z) = \delta^{-1} [ (\tilde{H}_\delta^z(z) - \lambda I_{\mathcal{H}_l}) \parallel R^\circ_0(\lambda; z) - I_{\mathcal{H}_l} ]_1 = H_1(z) R^\circ_0(\lambda; z) - \frac{i}{2} (\Xi; R^\circ_j(\lambda; z) \parallel p_{x,x_s}.$$ 

Since $\Xi$ and $R^\circ_0$ do not depend on $z \in \mathbb{R}^2$ it follows that $L^1_j = H_1 R^\circ_0$. The operator $H_1$ is linear in $a$ and $a^\dagger$ (with all its derivative) if $z = 0$ or $H_1 = 0$ if $z = 1$. In both cases $H_1$ (with
its derivatives) is infinitesimally bounded with respect to $\Xi$ (see Lemma A.2). This shows that $L^2_\lambda \in S^1(\mathcal{B}(\mathcal{H}_i))$ (but not in $S^1(\mathcal{B}(\mathcal{H}_i, F))$ if $\xi = 0$).

We proceed by induction assuming that $L^2_j \in S^1(\mathcal{B}(\mathcal{H}_i))$ for all $j \leq m \in \mathbb{N}$. The $(m+1)$-th order obstruction $L^2_{m+1}$ can be computed by means of equation (14) and the Moyal formula for the expansion of $\xi$ (see equation (A.9) in [Teu03]). After some manipulations, one gets

$$L^2_{m+1}(\lambda; z) = \frac{1}{(2\pi)^{m+1}} \sum_{\alpha_1, \alpha_2 = 0}^{\alpha_1 + \alpha_2 = m+1} \frac{(-1)^{|\alpha_1| + 1}}{\alpha_1 ! \alpha_2 !} \left( \partial^{\alpha_1}_{x_i} \partial^{\alpha_2}_{p_i} H^2_{\alpha_i} R^2_0 \right)(\lambda; z) \left( \partial^{\alpha_1}_{x_i} \partial^{\alpha_2}_{p_i} L^2_1 \right)(\lambda; z).$$

Since $H_i R^2_0 \in S^1(\mathcal{B}(\mathcal{H}_i))$ uniformly in $\lambda$ (see Proposition 4.6) then $L^2_{m+1} \in S^1(\mathcal{B}(\mathcal{H}_i))$, and this concludes the inductive argument.

Finally to prove $R^2_j \in S^1(\mathcal{B}(\mathcal{H}_i, F))$, observe that $\|\partial^\alpha_z R^2_j\|_{\mathcal{B}(\mathcal{H}_i, F)} = \|\Xi R^2_0 \left( \partial^\alpha_z L^2_1 \right)\|_{\mathcal{B}(\mathcal{H}_i)} \leq C_\alpha \|\Xi R^2_0\|_{\mathcal{B}(\mathcal{H}_i)} < +\infty$ for all $j, \alpha \in \mathbb{N}$.

**Remark 4.8.** It clearly emerges from the proof that the order $\delta^{2(z+1)}$ is the best approximation which can be obtained with this technique. The obstruction is the condition $H_i R^2_0 \in S^1(\mathcal{B}(\mathcal{H}_i))$, which can be satisfied by the resolvent $R^2_0 := (\Xi - \zeta I_i)^{-1}$ only for $0 \leq r \leq 2(z + 1)$.

Proposition A.9 of [Teu03] assures that $\tilde{H}^2_\delta \pi \in S^1(\mathcal{B}(\mathcal{H}_i))$ and, by adjointness, also $\pi \tilde{H}^2_\delta \pi \in S^1(\mathcal{B}(\mathcal{H}_i))$. By construction $[\tilde{H}^2; \mathrm{Op}_\delta(\pi)] = \mathrm{Op}_\delta([\tilde{H}^2; \pi]\pi) = \mathcal{O}_0(\delta^\infty)$ where $[\tilde{H}^2; \pi]\pi := \tilde{H}^2 \pi - \pi \tilde{H}^2 = \mathcal{O}(\delta^\infty)$ denotes the Moyal commutator. Observing that $[H; \pi]\pi = [\tilde{H}^2, \pi]\pi = [\tilde{H}^2, \pi] + \mathcal{O}(\delta^\infty)$ and since Proposition 4.6 implies $[\mathcal{R}^2_0; \pi] = [\mathcal{R}^2_0, \pi]\pi + \mathcal{O}(\delta^\infty)$, it follows $[H; \pi]\pi = \mathcal{O}(\delta^\infty)$ which implies after the quantization $[H; \pi][H; \pi] = \mathcal{O}(\delta^\infty)$.

The last step is to obtain the true projection $\Pi^2_\delta$ (the super adiabatic projection) from $\mathrm{Op}_\delta(\pi)$ by means of the formula (46). Since $\Pi^2_\delta - \mathrm{Op}_\delta(\pi) = \mathcal{O}(\delta^\infty)$, one recovers the estimates (103).

**Step 2. Construction of the intertwining unitary**

The construction of the intertwining unitary follows as in the proof of Theorem 3.1 of [PST03b]. Firstly one constructs a formal symbol $u \simeq \sum_{j=0}^{\infty} \delta^j u_j$ such that: (i) $u^I u = u u^I = 1_{\mathcal{H}_i}$; (ii) $u u = \pi u u^I = \pi$.

The existence of such a symbol follows from a recursive procedure starting from $u_0$ (which can be fixed to be $1_{\mathcal{H}_i}$ in our specific case) and using the expansion of $\pi \simeq \sum_{j=0}^{\infty} \delta^j \pi_j$ obtained above. However, the symbol $u$ which comes out of this procedure is not unique. The recursive relations are

$$u_n := a_n + b_n \quad \text{with} \quad a_n := -\frac{1}{2} A_n, \quad b_n := [\pi; B_n] \quad (109)$$
where
\[
A_n := \left[ \left( \sum_{j=0}^{n-1} \delta^j u_j \right) \# \left( \sum_{j=0}^{n-1} \delta^j u_j \right)^\dagger - \mathbf{1}_{\mathcal{H}_\delta} \right]_n
\](110)
and
\[
B_n := \left[ \left( \sum_{j=0}^{n-1} \delta^j u_j + \delta^n a_n \right) \# \pi \# \left( \sum_{j=0}^{n-1} \delta^j u_j + \delta^n a_n \right)^\dagger - \pi_t \right]_n
\](111)
Since \(u_0 = \mathbf{1}_{\mathcal{H}_\delta} \in S^1(\mathcal{B}(\mathcal{H}_\delta))\), then it follows by induction that \(u_j \in S^1(\mathcal{B}(\mathcal{H}_\delta))\) for all \(j \in \mathbb{N}\).

The quantization of \(u\) is an element of \(\mathcal{B}(\mathcal{H}_w)\) but it is not a true unitary. Nevertheless \(\text{Op}_\delta(u)\) can be modified by an \(O(\delta^\infty)\) term using the same technique of Lemma 3.3 (Step II) in [PST03b] to obtain the true unitary \(\text{U}_\delta\).

**Step 3. Effective dynamics**

By construction \([H^\dagger_{\text{eff}}; \Pi_t] = \text{Op}_\delta([\hbar^z; \pi_t]) = [U^\dagger_{\delta} \Pi^z_{\delta} U^z_{\delta}; \Pi_t] = 0\) since \(\Pi_t = U^\dagger_{\delta} \Pi^z_{\delta} U^z_{\delta}\).

Moreover equation (104) follows observing that \(U^\dagger_{\delta} \Pi^z_{\delta} H^{\text{VW}} \Pi^z_{\delta} U^z_{\delta} - H^\dagger_{\text{eff}}\) coincides with the quantization of \(u^\dagger_{\delta} \pi \# \mathcal{R}^2_{\delta} \# \pi \# u\) which is a symbol of order \(O(\delta^{2(\omega+1)+1})\).

**Step 4. The case of a single Landau band when \(A_{\Gamma} = 0\)**

We need to expand the Moyal product \(h^{z=1} = u^z \pi \# \tilde{H}_\delta^1 \# \pi \# u^\dagger = \pi_t \# u^z \tilde{H}_\delta^1 \# u^\dagger \# \pi_t + O(\delta^\infty)\) up to the order \(\delta^4\). To compute the various terms of the expansion \(h^{z=1} \approx \sum_{j=0}^{\infty} \delta^j h_j\) it is useful to define \(\chi_j := [u^z \tilde{H}_\delta^1 \# u^\dagger]_j\), so that \(h_j = \pi_t \chi_j \pi_t\). Observing that
\[
u^z \tilde{H}_\delta^1 - \left( \sum_{j=0}^{m-1} \delta^j \chi_j \right) \# u = \left( u^z \tilde{H}_\delta^1 \# u^\dagger - \sum_{j=0}^{m-1} \delta^j \chi_j \right) \# u + O(\delta^\infty) = \delta^m \chi_m + O(\delta^{m+1})
\]
one obtains the useful formula
\[
\chi_m = \left[ u^z \tilde{H}_\delta^1 - \left( \sum_{j=0}^{m-1} \delta^j \chi_j \right) \# u \right]_m
\](112)
At the zeroth order one finds \(h_0 = \pi_0 u_0 H_0 u_0^\dagger \pi_0 = \pi_t \# \pi_t = \lambda_s \pi_t\) since \(u_0 = \mathbf{1}_{\mathcal{H}_\delta}\) and \(\pi_0 = \pi_t\). Its quantization is the operator \(\text{Op}_\delta(h_0) = \lambda_s \mathbf{1}_{\mathcal{H}_\delta}\) acting on \(\mathcal{K} = L^2(\mathbb{R}, dx_s)\).

As for the first order \((m = 1)\), \(\chi_1 = u_1 H_0 + u_0 H_1 - \chi_0 u_1 + [u_0 \# H_0]_1 - [\chi_0 \# u_0]_1 = [u_1; \Xi]\) since \(\chi_0 = u_0 H_0 u_0^{-1} = \Xi\) and \(H_1 = 0\). Then \(h_1 = \pi_t [u_1; \Xi] \pi_t = \lambda_s (\pi_t u_1 \pi_t - \pi_t u_1 \pi_t) = 0\), hence \(\text{Op}_\delta(h_1) = 0\).

At the second order \((m = 2)\), one obtains after some manipulations \(\chi_2 = H_2 + u_2 \Xi - \Xi u_2 - \chi_1 u_1\) which implies \(h_2 = \pi_t H_2 \pi_t - \pi_t \chi_1 u_1 \pi_t\). We need to compute \(u_1\). Using equations (109), (110) and (111) one obtains that \(-2a_1 := [u_0 \# u_0^\dagger - \mathbf{1}_{\mathcal{H}_\delta}]_1 = 0\) and \(b_1 := [\pi_t; B_1]_1 \) with \(B_1 = [u_0 \# \pi \# u_0^* - \pi_t]_1 = \pi_t\) since \(a_1 = 0\). To compute \(\pi_t\) we use equations (106), (107) and
(108). Since $G_1 = [\pi, \pi, \pi] = 0$ it follows that $\pi^D = 0$. In the case of a single energy band in the relevant part of the spectrum, the implicit relation which defines $\pi^{\OD}_n$ can be solved, obtaining the useful equation

$$\pi^{\OD}_n = \pi_i F_n (\Xi - \lambda_s 1_{H_i})^{-1} (1_{H_i}^{\dagger} - \pi_i) - (1_{H_i} - \pi_i) (\Xi - \lambda_s 1_{H_i})^{-1} F_n \pi_i.$$  

(113)

Since $F_1 = [\hat{H}_0^{1/2}, \pi_i, \pi_i] = H_1 \pi_i - \pi_i H_1 = 0$, being $H_1 = 0$, it follows $B_1 = \pi_1 = \pi_i^{\OD} = 0$ and consequently $u_1 = b_1 = 0$. Then $h_2 = \pi_i H_2 \pi_i = V \pi_i$, according to (91), and its quantization defines on $\mathcal{K}$ the operator $\mathcal{O}_b(h_2) = V(P_s, X_s)$.

Considering (112) at the third order $(m=3)$ and using $u_1 = 0$, one obtains after some computations $\chi_3 = H_3 + u_3 \Xi - \Xi u_3 - \chi_1 u_2$ which implies $h_3 = \pi_i H_3 \pi_i - \pi_i \chi_1 u_2 \pi_i$. Thus we need to compute $u_2$. Since $u_1 = 0$, it follows $-2 a_2 = [u_0 \pi \pi, u_0^\dagger 1_{H_i}] = 0$, $B_2 = [u_0 \pi \pi, u_0^\dagger - \pi_i - \pi_i] = 2 \pi_2$ and $b_2 = [\pi_i; \pi_2]$. Since $\pi_1 = 0$, one has that $G_2 = [\pi_i \pi, \pi_i - \pi_i] = 0$ which implies $\pi^D_2 = 0$. To compute $\pi^{\OD}_2$ we need $F_2 = [\hat{H}_0^{1/2}, \pi_i - \pi_i \hat{H}_0^{1/2}] = [H_2; \pi_i] = [1_{H_i}; \pi_i] = 0$, where $H_2 = V 1_{H_i}$ has been used. Then $B_2 = \pi_2 = \pi_2^{\OD} = 0$ and consequently $u_2 = b_2 = 0$. Therefore $h_3 = \pi_i H_3 \pi_i$, and equation (92) implies that $\pi_i H_3 \pi_i = 0$ in view of $\pi_i^{\dagger} = \pi_i$ and consequently $\pi^{\dagger} = \pi^{\dagger}$. Then $\mathcal{O}_b(h_3) = 0$.

To compute the fourth order, we do not need to compute $u_3$ and $\pi_3$. Indeed, by computing (112) at the fourth order $(m=4)$ one finds $\chi_4 = H_4 + u_4 \Xi - \Xi u_4 + u_3 H_1 - \chi_3 u_1 = H_4 + u_4 \Xi - \Xi u_4$ since $H_1 = u_1 = u_2 = 0$. Then $h_4 = \pi_i H_4 \pi_i = \frac{\lambda_s}{2} |D_2(V)|^2 \pi_i$, according to equation (93), and its quantization yields $\mathcal{O}_b(h_4) = \frac{\lambda_s}{2} Y(P_s, Q_s)$.

### 4.7 Harper-like Hamiltonians

The first term in (105) is a multiple of the identity, and therefore does not contribute to the dynamics as far as the expectation values of the observables are concerned. The leading term, providing a non-trivial contribution to the dynamics at the original microscopic time scale $s \Delta = \delta^2 s$, is the bounded operator $V(P_s, Q_s)$ acting on the reference Hilbert space $L^2(\mathbb{R}, dx_s)$. This operator is the Weyl quantization of the $\mathbb{Z}^2$-periodic smooth function $V$ defined on the classical phase space $\mathbb{R}^2$. Hereafter we write $x_s \equiv x$ to simplify the notation.

The quantization procedure can be reformulated by introducing the unitary operators $\mathcal{U}_\infty := e^{-i2\pi Q}$ and $\mathcal{V}_\infty := e^{-i2\pi P}$ (Harper unitaries), acting on $L^2(\mathbb{R}, dx)$ as (recall $\delta^2 = h/2\pi$)

$$(\mathcal{U}_\infty \psi)(x) = e^{-i2\pi x} \psi(x), \quad (\mathcal{V}_\infty \psi)(x) = \psi(x - i_4 h_B).$$  

(114)

For any $\mathbb{Z}^2$-periodic function $F(p, x) = \sum_{n,m=\infty} f_{n,m} e^{-i2\pi(np + mx)}$ whose Fourier series is uniformly convergent, the $h_B$-Weyl quantization of $F$ is given by the formula

$$\hat{F}(\mathcal{U}_\infty, \mathcal{V}_\infty) = \sum_{n,m=\infty} f_{n,m} e^{-i2\pi(nq h_B)} \mathcal{V}_\infty \mathcal{U}_\infty.$$  

(115)

where the fundamental commutation relation $\mathcal{U}_\infty \mathcal{V}_\infty = e^{-i2\pi h_B} \mathcal{V}_\infty \mathcal{U}_\infty$ has been used. Formula (115) defines a Harper-like Hamiltonian with deformation parameter $h_B$. Indeed,
the special case $H_{\text{Har}} = U\infty + U\infty^{-1} + V\infty + V\infty^{-1}$ is the celebrated Harper Hamiltonian, namely the operator acting on $L^2(\mathbb{R}, dx)$ as

$$
(H_{\text{Har}}\psi)(x) = \psi(x + h_B) + \psi(x - h_B) + 2\cos(2\pi x)\psi(x). 
$$

(116)

In analogy with Section 3, we summarize the discussion in the following conclusion

**Conclusion 4.9.** Under the assumptions of Theorem 4.7, for every $V_\Gamma \in C^\infty_0(\mathbb{R}^2, \mathbb{R})$, in the Harper regime ($h_B \to 0$), the dynamics generated by the Hamiltonian $H_{\text{BL}}$ restricted to the spectral subspace corresponding to a Landau level $\lambda_s$ is approximated up to an error of order $h_B$ (and up to a unitary transform and an energy rescaling) by the dynamics generated on the reference Hilbert space $L^2(\mathbb{R}, dx)$ by a Harper-like Hamiltonian, i.e. by a power series in the Harper unitaries $U\infty$ and $V\infty$, defined by (114).

### 4.8 Coupling of Landau bands in a periodic magnetic potential

According to Theorem 4.7, the first non-trivial term which describes the effective dynamics in the almost invariant subspace related to a single Landau band $\lambda_s$ is of order $\delta^2 \propto h_B$. An important ingredient in the proof is that $A_\Gamma = 0$ implies $H_1 = 0$. Moreover, the second non-trivial correction appears at order $\delta^4 \propto h_B^2$ although $H_3 \neq 0$. Indeed, the correction at order $\delta^3$ vanishes since $H_3$, defined by (92), is linear in $a$ and $a^\dagger$, hence $\langle \psi_s | H_3 | \psi_{s+1} \rangle = 0$. This observation suggests that for a family of Landau bands which contains two contiguous bands $\{\lambda_s, \lambda_{s+1}\}$ one has, in general, a second non-trivial correction of order $\delta^3 \propto h_B^3/2$ for the effective dynamics. Indeed, in this case one has $\pi_s H_3 \pi_r \neq 0$ since $\langle \psi_s | H_3 | \psi_{s+1} \rangle$ is generally non zero. Nevertheless, also in this case, the first non-trivial correction is of order $\delta^2$.

Is there any mechanism to produce a non-trivial correction in the effective dynamics with leading order $\delta \propto h_B^{1/2}$? An affirmative answer requires $H_1 \neq 0$, and the latter condition is satisfied if we include in the Hamiltonian $H_{\text{BL}}$ the effect of a $\Gamma$-periodic vector potential $A_\Gamma$ (i.e. $\xi = 0$). Since in this situation $H_1$ is linear in $a$ and $a^\dagger$, to obtain a non-trivial effect we need to consider a spectral subspace which contains at least two contiguous Landau bands.

Our goal is to derive the (non-trivial) leading order for the effective Hamiltonian in this framework. According to the notation of Theorem 4.7, we need to expand the Moyal product

$$
K^{=0} = u^2 \pi^2 \bar{H}^{0}_{\delta} \pi^2 u^1 = \pi_r^2 u^2 \bar{H}^{0}_{\delta} \pi^2 u^1 \pi_r + O(\delta^\infty) 
$$

up to the first order $\delta$. The symbols $\pi \equiv \pi_+ + O(\delta)$ and $u \equiv 1_{\mathcal{H}_1} + O(\delta)$ are derived as in the general construction showed in the proof of Theorem 4.7.

Now $\mathcal{K} := \text{Ran} \, \Pi_\Gamma \simeq L^2(\mathbb{R}, dx_\delta) \otimes \mathbb{C}^2$.

Expanding at zero order one finds $h_0 = \pi_0 u_0 H_0 u_0^{-1} \pi_0 = \pi_r \Xi \pi_r = \pi_r \Xi = \Xi \pi_r$ and its quantization is the operator on $\mathcal{K}$ defined by

$$
\text{Op}_\delta(h_0) = \begin{pmatrix} (n_r + \frac{3}{2}) 1_{\mathcal{H}_r} & 0 \\ 0 & (n_r + \frac{1}{2}) 1_{\mathcal{H}_r} \end{pmatrix} = (n_r + 1) 1_{\mathcal{K}} + \begin{pmatrix} \frac{1}{2} 1_{\mathcal{H}_r} & 0 \\ 0 & -\frac{1}{2} 1_{\mathcal{H}_r} \end{pmatrix}
$$

(117)

As for the next order, from equation (112) it follows $\chi_1 = H_1 + u_1 H_0 - H_0 u_1$ (we use $\chi_0 = H_0$) which implies $h_1 = \pi_r \chi_1 \pi_r = \pi_r H_1 \pi_r + \pi_r [u_1; \Xi] \pi_r$. To conclude the computation
we need $u_1$ and $\pi_1$. Using the recursive formulas (106), (107), (108), (109), (110) and (111), one obtains $-2a_1 := [u_0^\# u_0^\dagger - 1_{H\ell}]_1 = 0$, $b_1 := [\pi_1; B_1]$ and $B_1 = [u_0^\# \pi^\# u_0^\dagger - \pi^\dagger]_1 = \pi_1$ since $a_1 = 0$. Observing that $G_1 = [\pi^\# \pi - \pi^\dagger]_1 = 0$, it follows that $\pi^\dagger = 0$ and so $u_1 = [\pi_4; \pi_1] = [\pi_4; \pi_1^{\text{OD}}]$ which implies $\pi u_1 \pi = 0$. Finally $\pi_4 [u_1; \Xi]\pi = \pi_4 [u_1; \Xi_4\pi_4] = 0$ and so $h_1 = \pi_4 H_1 \pi_4$. According to (95) the quantization of $h_1$ is an operator which acts on $K$ as

$$\text{Op}_\delta(h_1) = \sqrt{n_s + 1} \begin{pmatrix} 0 & G(P_s, Q_s) \\ G(P_s, Q_s)^\dagger & 0 \end{pmatrix} \tag{118}$$

where the operator $G(P_s, Q_s)$ is defined on $L^2(\mathbb{R}, dx_s)$ as the Weyl quantization of the $\mathbb{Z}^2$-periodic function $g$ defined by equation (97). Summarizing, we obtained the following result:

**Theorem 4.10** (Effective Hamiltonian with a periodic magnetic potential). Under the assumptions of Theorem $\ref{thm:main}$, in the case $A_\Gamma \neq 0$ the dynamics in the spectral subspace related to a family of two contiguous Landau bands $\{\sigma_{s+j}(\cdot) = \lambda_{s+j} \mid j = 0, 1\}$ is approximated by the effective Hamiltonian $H^{\delta}_{\text{eff}} := \text{Op}_\delta(h^{(\delta=0)})$ on the reference space $K = L^2(\mathbb{R}, dx_s) \otimes \mathbb{C}^2$ which is given, up to errors of order $\delta^2$, by

$$H^{\delta}_{\text{eff}} = (n_s + 1)1_{\mathcal{K}} + \sqrt{n_s + 1} \begin{pmatrix} \frac{1}{2\sqrt{n_s + 1}} 1_{\mathcal{H}_s} & \delta G(P_s, Q_s) \\ \delta G(P_s, Q_s)^\dagger & -\frac{1}{2\sqrt{n_s + 1}} 1_{\mathcal{H}_s} \end{pmatrix} + \mathcal{O}_0(\delta^2), \tag{119}$$

according to the notation introduced in (117) and (118).

Equation (97) shows that $g(p_s, x_s) = g_1(p_s, x_s) - ig_2(p_s, x_s)$ where the function $g_1$ and $g_2$ are related to the component $(A_\Gamma)_1$ and $(A_\Gamma)_2$ of the $\Gamma$-periodic vector potential by the relation $g_j(a^* \cdot r, b^* \cdot r) = \pi \sqrt{2L_0}\sqrt{g_0}(A_\Gamma)_j(r)$, $j = 1, 2$. Let $G_j(P_s, Q_s)$ be the Weyl quantization of $g_j$. By introducing the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_\perp = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{120}$$

one can rewrite the effective Hamiltonian (119) in the form

$$H^{\delta}_{\text{eff}} = \left((n_s + 1)1_{\mathbb{C}^2} + \frac{1}{2}\sigma_\perp\right) \otimes 1_{\mathcal{H}_s} + \delta \sqrt{n_s + 1} \sum_{j=1}^{2} \sigma_j \otimes G_j(P_s, Q_s) + \mathcal{O}_0(\delta^2). \tag{121}$$

Clearly, the operator $G_j(P_s, Q_s)$ are Harper-like Hamiltonians and can be represented as a power series of the Harper unitaries $\mathcal{U}_\infty$ and $\mathcal{V}_\infty$ of type (115). In this case the coefficients in the expansion are (up to a multiplicative constant) the Fourier coefficients of the components $(A_\Gamma)_j$ of the $\Gamma$-periodic vector potential.

The determination of the spectrum of $H^{\delta}_{\text{eff}}$ can be reduced to the (generally simpler) problem of the computation of the spectrum of $G\mathcal{G}$.

**Proposition 4.11.** Let $H^{\delta=1}_{\text{eff}}$ be the first order approximation of the effective Hamiltonian (119) (or (121)). Then

$$\sigma(H^{\delta=1}_{\text{eff}}) = (n_s + 1) + S_+ \cup S_-, \quad S_\pm := \{\pm \sqrt{1/4 + \delta^2(n_s + 1)} \lambda : \lambda \in \sigma(G\mathcal{G})\}$$

where $\sigma(G\mathcal{G}) = \sigma(G\mathcal{G}) \cup \{0\}$ if $\{0\} \in \sigma(G\mathcal{G}) \setminus \sigma(G\mathcal{G}) \cap \sigma(G\mathcal{G}) = \sigma(G\mathcal{G})$ otherwise.
Since we include in our analysis a periodic vector potential $A$, some technical results

A.1 Self-adjointness of $H_{\text{BL}}$ and $H_{\text{per}}$

The second Sobolev space $H^2(\mathbb{R}^2)$ is defined to be the set of all $\psi \in L^2(\mathbb{R}^2)$ such that $\partial_{x_1} \partial_{x_2} \psi \in L^2(\mathbb{R}^2)$ in the sense of distributions for all $n := (n_1, n_2) \in \mathbb{N}^2$ with $|n| := n_1 + n_2 \leq 2$. One can prove that $H^2(\mathbb{R}^2)$ is the closure of $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ with respect to the Sobolev norm $\| \cdot \|_{H^2} := \| (1 - \Delta_2) \cdot \|_{L^2}$ and has a Hilbert space structure. Similarly, the second magnetic-Sobolev space $H^2_M(\mathbb{R}^2)$ is defined to be the set of all $\psi \in L^2(\mathbb{R}^2)$ such that $D_1^{n_1} D_2^{n_2} \psi \in L^2(\mathbb{R}^2)$ in the sense of distributions for all $n \in \mathbb{N}^2$ with $|\alpha| \leq 2$, where $D_1 := (\partial_{x_1} + \frac{i}{\hbar} x_2)$ and $D_2 := (\partial_{x_2} - \frac{i}{\hbar} x_1)$. One can prove that $H^2_M(\mathbb{R}^2)$ is the closure of $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ with respect to the magnetic-Sobolev norm $\| \cdot \|_{H^2_M} := \| (1 - \Delta_M) \cdot \|_{L^2}$, where $\Delta_M := D_1^2 + D_2^2$ is the magnetic-Laplacian. Moreover, $H^2_M(\mathbb{R}^2)$ has a natural Hilbert space structure. For further details see Section IX.6 and IX.7 of [RS75] and Chapter 7 of [LL01].

Proof of Proposition 2.1

We prove the claim for the dimensionless operators, namely we fix all the physical constants equal to 1 in (9) and (15).

- Step 1. First of all we prove that $H_{\text{per}}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ and self-adjoint on $H^2(\mathbb{R}^2)$. Notice that

$$H_{\text{per}} = \frac{1}{2} [-i \nabla_x - A_\Gamma(x)]^2 + V_\Gamma(x) = -\frac{1}{2} \Delta_x + T_1 + \frac{1}{2} T_2$$ (122)
with $T_1 := i A_T \cdot \nabla_x$ and $T_2 := i (\nabla_x \cdot A_T) + |A_T|^2 + 2 V_T$. The free Hamiltonian $-\frac{1}{2} \Delta_x$ is a self-adjoint operator with domain $\mathcal{H}^2(\mathbb{R}^2)$, essentially self-adjoint on $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ and from Assumption (A_w) it follows that $T_2$ is an infinitesimally bounded with respect to $-\frac{1}{2} \Delta_x$ (notice that $T_2 - 2 V_T$ is bounded). The symmetric operator $T_1$ is unbounded with domain $\mathcal{D}(T_1) \supset \mathcal{H}^2(\mathbb{R}^2)$. Let $\psi \in \mathcal{H}^2(\mathbb{R}^2)$, then

$$\left\| (A_T)_j \partial_x \psi \right\|_{L^2}^2 \leq \left\| (A_T)_j \right\|_\infty^2 \int_{\mathbb{R}^2} \lambda_j^2 \left| \hat{\psi}(\lambda) \right|^2 \, d^2 \lambda$$

with $\hat{\psi}(\lambda)$ the Fourier transform of $\psi(x)$. For every $a > 0$, if $b = \frac{1}{2a}$ then $\lambda_j^2 \leq (a |\lambda|^2 + b)^2$. It follows that $\left\| (A_T)_j \partial_x \psi \right\|_{L^2}^2 \leq C \left\| (a |\lambda|^2 + b) \hat{\psi} \right\|_{L^2}^2$ which implies that for all $\alpha' > 0$ (arbitrary small) there exists a $b'$ (depending on $\alpha'$) such that

$$\left\| (A_T)_j \partial_x \psi \right\|_{L^2}^2 \leq \alpha' \left\| \lambda_j^2 \hat{\psi} \right\|_{L^2}^2 + b' \left\| \hat{\psi} \right\|_{L^2}^2 = \alpha' \left\| \Delta_x \psi \right\|_{L^2}^2 + b' \left\| \psi \right\|_{L^2}^2.$$

This inequality implies that $T_1$ is infinitesimally bounded with respect to $-\frac{1}{2} \Delta_x$ and the thesis follows form the Kato-Rellich Theorem (see [RS75] Theorem X.12).

**Step 2.** The Bloch-Landau Hamiltonian is

$$H_{BL} := \frac{1}{2} \left\{-i \nabla_x - A_T(x) - A(x) \right\}^2 + V_T(x) + \phi(x). \tag{123}$$

Assumptions (A_w) and (B) imply that $(A_T + A)_j \in C^1(\mathbb{R}^2, \mathbb{R})$, $j = 1, 2$, and $V_T + \phi \in L^2(\mathbb{R}^2)$ and this assures that $H_{BL}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ (see [RS75] Theorem X.34). Let $A = A_0 + A_B$ be the decomposition of the external vector potential with $A_0$ smooth and bounded and $A_B = \frac{1}{2} (- x_2, x_1)$. By posing $D := \nabla_x - i A_B$, $\Delta_M := |D|^2$ and $A_b := A_T + A_0$, the Hamiltonian $H_{BL}$ reads

$$H_{BL} = -\frac{1}{2} \Delta_M - A_b \cdot D + \frac{1}{2} T.$$

where $T := i (\nabla_x \cdot A_B) + |A_B|^2 + 2 (V_T + \phi)$. The operator $T - 2 V_T$ is bounded and the observation that $A_b \cdot D$ is infinitesimally bounded with respect to $-\frac{1}{2} \Delta_M$ is an immediate consequence of Lemma [A.2] the assumption $\int_{\mathbb{R}^2} |V_T(x)|^2 \, d^2 x < +\infty$ implies that $V_T$ is uniformly locally $L^2$ and hence, infinitesimally bounded with respect to $-\Delta_x$ (see Theorem XIII.96 in [RS78]). As proved in [AHS75] (Theorem 2.4) this is enough to claim that $V_T$ is aslo infinitesimally bounded with respect to $-\Delta_M$. Therefore, by the Kato-Rellich Theorem it follows that the domain of self-adjointness of $H_{BL}$ coincides with the domain of self-adjointness of the magnetic-Laplacian, which is $\mathcal{H}^2_M(\mathbb{R}^2)$.

### A.2 Band spectrum of $H_{\text{per}}$

We describe the spectral properties of the periodic Hamiltonian. The Bloch-Floquet transform maps unitarily $H_{\text{per}}$ in $H^2_{\text{per}} \equiv \int_{M_T} H_{\text{per}}(k) \, d^2 k$. Then to have information about the spectrum of $H_{\text{per}}$ we need to study the spectra of the family of Hamiltonians

$$H_{\text{per}}(k) = \frac{1}{2} \left\{-i \nabla_\theta + k - A_T(\theta) \right\}^2 + V_T(\theta) = -\frac{1}{2} \Delta_\theta + T_1(k) + \frac{1}{2} T_2(k)$$
where \( T_1(k) := i(A_L - \frac{k}{2}) \cdot \nabla_\theta \) and \( T_2(k) := i(\nabla_\theta \cdot A_L) + |k|^2 + |A_L|^2 + 2V_L \) are operators acting on the Hilbert space \( H_L := L^2(\mathbb{V}, d\theta) \) with \( \mathbb{V} := \mathbb{R}^2/\Gamma \) (Voronoï torus).

**Proof of Proposition 3.3.**

- (i) The operator \(-1/2\Delta_\theta\) on the Hilbert space \( H_\ell \), is essentially self-adjoint on \( C^\infty(\mathbb{V}) \), has domain of self-adjointness \( D := \mathcal{H}^2(\mathbb{V}) \) and its spectrum is pure point with \( \{e^{i\theta^*}\}_{\gamma^* \in \Gamma^*} \) a complete orthogonal system of eigenvectors. If Assumption (A\_\omega) holds true then \( T_2(k) \) infinitesimally bounded with respect to \(-1/2\Delta_\theta\), hence the Kato-Rellich Theorem implies that \( H_{\text{per}}(k) \) is essentially self-adjoint on \( C^\infty(\mathbb{T}^2) \) and self-adjoint on the domain \( D \). Moreover since \(-1/2\Delta_\theta\) is bounded below then also \( H_{\text{per}}(k) \) is bounded below.

- (ii) For all \( \zeta \) in the resolvent set of \(-1/2\Delta_\theta\) the resolvent operator \( r_0(\zeta) := (-1/2\Delta_\theta - \zeta I_{H_\ell})^{-1} \) is a compact operator. Since \( T_1(k) + 1/2T_2(k) \) is a bounded perturbation of \(-1/2\Delta_\theta\) it follows that \( H_{\text{per}}(k) \) has compact resolvent (see [RS78] Theorem XIII.68) and moreover it has a purely discrete spectrum with eigenvalues \( \mathcal{E}_n(k) \to +\infty \) as \( n \to +\infty \) (see [RS78] Theorem XIII.64).

- (iii) The continuity of the function \( \mathcal{E}_n(\cdot) \) follows from the perturbation theory of discrete spectrum (see [RS78] Theorem XII.13). Indeed, as discussed in Remark 3.2, \( H_{\text{per}}(\cdot) \) is an analytic family (of type A) in the sense of Kato. Finally, since \( H_{\text{per}}(k - \gamma^*) = \tau(\gamma^*)H_{\text{per}}(k)\tau(\gamma^*)^{-1} \), with \( \tau(\gamma^*) \) a unitary operator, then \( \mathcal{E}_n(\cdot) \) are \( \Gamma^* \)-periodic.

\[ \Box \]

### A.3 The Landau Hamiltonian \( H_L \)

The **Landau Hamiltonian** is the operator

\[
H_L := -\frac{1}{2} \Delta_M = \frac{1}{2} \left( K_1^2 + K_2^2 \right) = \frac{1}{2} \left[ \left( -i \frac{\partial}{\partial x_1} + \frac{1}{2} x_2 \right)^2 + \left( -i \frac{\partial}{\partial x_2} - \frac{1}{2} x_1 \right)^2 \right]
\]  

(124)

where \( K_j := -iD_j \), with \( j = 1, 2 \), are the **kinetic momenta**. The Landau Hamiltonian \( H_L \) is essentially self-adjoint on \( C^\infty(\mathbb{R}^2; \mathbb{C}) \subset L^2(\mathbb{R}^2) \) (see [RS75] Theorem X.34) and its domain of self-adjointness is exactly the second magnetic-Sobolev space \( \mathcal{H}^2_M(\mathbb{R}^2) \) defined in Section 3.1. To describe the spectrum of \( H_L \) is helpful to introduce another pair of operators: \( G_1 := -i\partial_{x_1} - \frac{1}{2} x_2 \) and \( G_2 := i\partial_{x_2} - \frac{1}{2} x_1 \). The operators \( K_1, K_2, G_1, G_2 \) are all essentially self-adjoint on \( C^\infty(\mathbb{R}^2; \mathbb{C}) \) (they have **deficiency indices** equal to zero) and on this domain the following commutation relations hold true

\[
[K_1; K_2] = [G_1; G_2] = i1, \quad [G_j; K_i] = 0.
\]  

(125)

The last of the (124) implies \([G_j; H_L] = 0\) hence the operators \( G_1 \) and \( G_2 \) are cause for the degeneration of the spectral eigenspaces of \( H_L \). It is a common lore to introduce the **annihilation operator** \( a := i/\sqrt{2}(K_2 - iK_1) \) (its adjoint \( a^\dagger \) is called **creation operator**) and the **degeneration operator** \( g := i/\sqrt{2}(G_2 - iG_1) \). They fulfill the following (bosonic) commutation relation

\[
[a; a^\dagger] = [g; g^\dagger] = 1, \quad [g; H_L] = [g^\dagger; H_L] = 0, \quad [a; H_L] = a, \quad [a^\dagger; H_L] = -a^\dagger.
\]  

(126)
The last two relations follow from the equality \( H_L = a a^\dagger - \frac{1}{2} I = a^\dagger a + \frac{1}{2} I \). Define the ground state \( \psi_0 \in L^2(\mathbb{R}^2) \) as the normalized solution of \( g \psi_0 = 0 = a \psi_0 \), i.e. \( \psi_0(x) = C e^{-\frac{1}{2} |x|^2} \). The generalized Hermite function of order \((n, m)\) is defined to be \( \psi_{n,m} := \frac{1}{\sqrt{n!m!}} (g^t)^m (a^\dagger)^n \psi_0 \). We will denote by \( \mathcal{F} \subset L^2(\mathbb{R}^2) \) the set of the finite linear combinations of the vectors \( \psi_{n,m} \) and we will call it the Hermite domain. Clearly \( \mathcal{F} \subset S(\mathbb{R}^2) \) (the Schwartz space).

**Lemma A.1.** With the notation above:

(i) the set \( \{ \psi_{n,m} : n, m = 0, 1, 2, \ldots \} \) is a complete orthonormal basis for \( L^2(\mathbb{R}^2) \) and so \( \mathcal{F} \) is a dense domain;

(ii) the spectrum of \( H_L \) is pure point and is given by \( \{ \lambda_n := (n + \frac{1}{2}) : n = 0, 1, 2, \ldots \} \), moreover \( H_L \psi_{n,m} = \lambda_n \psi_{n,m} \) for every \( m = 0, 1, 2, \ldots \) (degeneration index);

(iii) \( H_L \) is essentially self-adjoint on \( \mathcal{F} \) and the closure of \( \mathcal{F} \) with respect to the magnetic-Sobolev norm coincides with the magnetic-Sobolev space \( \mathcal{H}^s_M(\mathbb{R}^2) \).

**Proof.** - (i) Let \( W : L^2(\mathbb{R}^2, d^2x) \rightarrow L^2(\mathbb{R}, du) \otimes L^2(\mathbb{R}, dv) \) be the unitary map such that the conjugate pairs \( (K_1, K_2) \) and \( (G_1, G_2) \) are transformed by \( W \) ...\( W^{-1} \) into the canonical pairs \((u, -i \partial_u)\) and \((v, -i \partial_v)\). The existence of such a unitary \( W \) will be discussed in Appendix \[ \Box \]. Obviously \( a \mapsto \tilde{a} = \sqrt{2} (u + \partial_u), g \mapsto \tilde{g} = \sqrt{2} (v + \partial_v) \) and \( \tilde{\psi}_0 := W \psi_0 \) is the solution of \( \tilde{g} \tilde{\psi}_0 = \tilde{a} \tilde{\psi}_0 = 0 \), namely \( \tilde{\psi}_0(u, v) = h_0(u) \otimes h_0(v) \) where \( h_0(t) := e^{-\frac{1}{2} |t|^2} \) is the 0-th Hermite function. Then \( \tilde{\psi}_{n,m}(u, v) := (W \psi_{n,m})(u, v) = h_n(u) \otimes h_m(v) \) which shows that the functions \( \tilde{\psi}_{n,m} \) define an orthonormal basis for \( L^2(\mathbb{R}, du) \otimes L^2(\mathbb{R}, dv) \) since the Hermite functions \( h_n \) are an orthonormal system for \( L^2(\mathbb{R}) \). The claim follows since \( W \) is a unitary map.

- (ii) Clearly \( H_L \psi_0 = (a^\dagger a + \frac{1}{2} I) \psi_0 = \frac{1}{2} \psi_0 \) and from relations \( (126) \) it follows that \( H_L \psi_{n,m} = \frac{1}{\sqrt{n!m!}} \frac{1}{\sqrt{n!m!}} (g^t)^m (a^\dagger)^n \psi_0 = \frac{1}{2} \psi_{n,m} + \frac{1}{\sqrt{n!m!}} (g^t)^m (a^\dagger a)^n \psi_0 = \lambda_n \psi_{n,m} \). Then the generalized Hermite functions \( \psi_{n,m} \) are a complete set of orthonormal eigenvectors for \( H_L \). This proves that the spectrum of \( H_L \) is pure point.

- (iii) The operator \( H_L \) is essentially self-adjoint in \( \mathcal{F} \) since the deficiency indices are both zero. This implies that last part of the claim.

**Lemma A.2.** The operators \( K_1, K_2, a, a^\dagger \) are infinitesimally bounded with respect to \( H_L \).

**Proof.** Since \( K_1 = \sqrt{2} (a + a^\dagger) \) and \( K_2 = \sqrt{2} (a - a^\dagger) \) it is enough to prove the claim for \( a \) and \( a^\dagger \). Let \( \psi := \sum_{n,m=0}^{+\infty} c_{n,m} \psi_{n,m} \in \mathcal{H}^2_M(\mathbb{R}^2) \). An easy computation shows that

\[
\| a \psi \|_{L^2}^2 = \sum_{n,m=0}^{+\infty} |c_{n,m}|^2 n, \quad \| a^\dagger \psi \|_{L^2}^2 = \sum_{n,m=0}^{+\infty} |c_{n,m}|^2 (n + 1) .
\]

Since \( n \leq n + 1 \leq 2 (n + \frac{1}{2}) \leq a (n + \frac{1}{2})^2 + \frac{1}{a} \) holds true for any \( a > 0 \) (arbitrarily small), then

\[
\| a^\dagger \psi \|_{L^2}^2 \leq a \sum_{n,m=0}^{+\infty} |c_{n,m}|^2 \left( n + \frac{1}{2} \right)^2 + b \sum_{n,m=0}^{+\infty} |c_{n,m}|^2 = a \| H_L \psi \|_{L^2}^2 + b \| \psi \|_{L^2}^2 .
\]
with \( b := \frac{1}{a} + \frac{1}{2} \) where \( a \) denotes either \( a \) or \( a^\dagger \).

## B Canonical transform for fast and slow variables

This appendix is devoted to the concrete realization of the von Neumann unitary \( \mathcal{W} \) introduced (in abstract way) in Section 4.2. The unitary \( \mathcal{W} \) maps the fast and slow variables, which satisfy canonical commutation relation, into a set of canonical Schrödinger operators. In Section B.1 we derive a general version of the transform \( \mathcal{W} \) "by hand", as a composition of three sequential transforms. In Section B.2 we compute the integral kernel of \( \mathcal{W} \).

### B.1 The transform \( \mathcal{W} \) built "by hand"

Let \( \mathcal{H} := L^2(\mathbb{R}^2, d^2 r) \) be the initial Hilbert space, with \( r := (r_1, r_2) \). Let \( Q_r := (Q_{r_1}, Q_{r_2}) \) where \( Q_{r_j} \) is the multiplication operator by \( r_j \) and \( P_r := (P_{r_1}, P_{r_2}) \) where \( P_{r_j} := -i\hbar \partial_{r_j} \), with \( j = 1, 2 \). Consider the fast and slow operators

\[
\begin{align*}
(K_1) & := -\frac{\alpha}{2\beta} v \cdot Q_r - \frac{\alpha \beta}{\hbar} w^* \cdot P_r \\
(K_2) & := \frac{\alpha}{2\beta} w \cdot Q_r - \frac{\alpha \beta}{\hbar} v^* \cdot P_r
\end{align*}
\]

and

\[
\begin{align*}
(G_1) & := \frac{1}{2} v \cdot Q_r - \frac{\beta^2}{\hbar} w^* \cdot P_r \\
(G_2) & := \frac{1}{2} w \cdot Q_r + \frac{\beta^2}{\hbar} v^* \cdot P_r
\end{align*}
\]

with \( \alpha, \beta \in \mathbb{C} \) and \( v, w, v^*, w^* \in \mathbb{R}^2 \) such that \( v \cdot v^* = w \cdot w^* = 1, v^* \cdot w = v \cdot w^* = 0 \) and \( |v \wedge w| = \ell^2 > 0 \).

**Remark B.1.** The choice \( v = b^*, w = a^*, \alpha = \sqrt{\bar{\delta}} q \) and \( \beta = \sqrt{\bar{\delta}} \) defines the operators (56), while the choice \( v = v^* = (0, -1), w = w^* = (-1, 0), \alpha = \beta = 1 \) defines the kinetic momenta and the related conjugate operators introduced in Section A.3.

Observing that \([a \cdot Q_r + b \cdot P_r, c \cdot Q_r + d \cdot P_r] = i \hbar (a \cdot d - b \cdot c) \mathbb{1}_\mathcal{H}\) one deduce that the operators (127) verify the following canonical commutation relations (CCR)

\[
[K_1, K_2] = i \alpha^2 \mathbb{1}_\mathcal{H}, \quad [G_1, G_2] = i \beta^2 \mathbb{1}_\mathcal{H}, \quad [K_i, G_j] = 0, \quad i, j = 1, 2.
\]

The Stone-von Neumann uniqueness theorem (see [BR97] Corollary 5.2.15) assures the existence of a unitary map \( \mathcal{W} \) (von Neumann unitary)

\[
\mathcal{W} : \mathcal{H} \longrightarrow \mathcal{H}_w := \mathcal{H}_a \otimes \mathcal{H}_f := L^2(\mathbb{R}, dx_a) \otimes L^2(\mathbb{R}, dx_f)
\]

such that

\[
\begin{align*}
\mathcal{W}G_1\mathcal{W}^{-1} & := Q_s = \text{multiplication by } x_s, \quad \mathcal{W}G_2\mathcal{W}^{-1} := P_s = -i \beta^2 \frac{\partial}{\partial x_s} \\
\mathcal{W}K_1\mathcal{W}^{-1} & := Q_t = \text{multiplication by } x_t, \quad \mathcal{W}K_2\mathcal{W}^{-1} := P_t = -i \alpha^2 \frac{\partial}{\partial x_t}
\end{align*}
\]
In other words, \((Q_s, P_s)\) is a pair of operators which defines a Schrödinger representation on the Hilbert space \(\mathcal{H}_s := L^2(\mathbb{R}, dx_s)\) while the pair \((Q_t, P_t)\) defines a Schrödinger representation on the Hilbert space \(\mathcal{H}_t := L^2(\mathbb{R}, dx_t)\). Our purpose is to give an explicit construction for \(W\). First of all consider the change of coordinates \((r_1, r_2) \mapsto (k_1 := \frac{w_r}{T}, k_2 := \frac{w_r}{T})\). The inverse transforms are defined by \(r_1(k) = \frac{1}{\ell}(w_2 k_1 - w_2 k_2)\) and \(r_2(k) = \frac{1}{\ell}(w_1 k_1 - w_2 k_1)\). The map \(J : L^2(\mathbb{R}^2, d^2 r) \to L^2(\mathbb{R}^2, d^2 k)\) defined by \((J \psi)(k) := \psi(r(k))\) if \(\psi \in L^2(\mathbb{R}^2, dr)\) is unitary since the change of coordinates is invertible and isometric. Moreover \(JQ_r J^{-1}\) acts on \(L^2(\mathbb{R}^2, dk)\) as the multiplication by \(r_j(k)\), while \(J P_r J^{-1} = \frac{w}{\ell} P_1 + \frac{w}{\ell} P_2\) where \(P_k := -i\hbar \partial_k\), with \(j = 1, 2\). Then

\[
\begin{align*}
JG_1 J^{-1} & := \frac{\ell}{2} Q_{k_1} - \frac{\beta^2}{h} \frac{1}{\ell} P_{k_2}, & JG_2 J^{-1} & := \frac{\ell}{2} Q_{k_2} + \frac{\beta^2}{h} \frac{1}{\ell} P_{k_1} \\
JK_1 J^{-1} & := -\frac{\alpha}{\beta} \frac{\ell}{2} Q_{k_1} - \frac{\alpha}{\beta} \frac{1}{\ell} P_{k_2}, & JK_2 J^{-1} & := \frac{\alpha}{\beta} \frac{\ell}{2} Q_{k_2} - \frac{\alpha}{\beta} \frac{1}{\ell} P_{k_1}.
\end{align*}
\]

Let \(F_{2, \mu} : L^2(\mathbb{R}, dk_2) \to L^2(\mathbb{R}, d\zeta_2)\) be the \(k_2\)-Fourier transform of weight \(\mu\), defined by

\[
(F_{2, \mu} \psi)(\zeta_2) := \sqrt{\frac{\mu}{2\pi}} \int_{\mathbb{R}} e^{i\mu \xi k_2} \psi(k_2) \, dk_2;
\]

and let \(\Pi_1 : L^2(\mathbb{R}, dk_1) \to L^2(\mathbb{R}, d\zeta_1)\) be the \(k_1\)-parity operator defined by \((\Pi_2 \psi)(\zeta_1) := \psi(-\zeta_1)\) (namely by the change of coordinates \(k_1 \mapsto \zeta_1\)). Let \(\Pi\) the unitary map which identifies \(L^2(\mathbb{R}^2, d^2 k)\) with \(L^2(\mathbb{R}, dk_1) \otimes L^2(\mathbb{R}, dk_2)\). Let \(\zeta := (\zeta_1, \zeta_2)\) whith \(\zeta_j\) the multiplication operator by \(\zeta_j\) and \(P_\zeta := (P_{\zeta_1}, P_{\zeta_2})\) where \(P_{\zeta_1} := -i\hbar \partial_{\zeta_1}\), with \(j = 1, 2\). One can check that

\[
F_{2, \mu} Q_{k_2} F_{2, \mu}^{-1} = -\frac{1}{\mu \hbar} P_{\zeta_1}, \quad F_{2, \mu} P_{k_2} F_{2, \mu}^{-1} = \mu \hbar Q_{\zeta_2}, \quad \Pi_1 Q_{k_3} \Pi_1^{-1} = -Q_{\zeta_1}, \quad \Pi_1 P_{k_1} \Pi_1^{-1} = -P_{\zeta_1}.
\]

Fix \(\mu := -\frac{\ell^2}{2\pi}\), then the unitary map \(\mathcal{L} := (\Pi_1 \otimes F_{2, \mu}) \circ \Pi \circ J : \mathcal{H} \to L^2(\mathbb{R}, d\zeta_1) \otimes L^2(\mathbb{R}, d\zeta_2)\) acts on the operators \((127)\) in the following way

\[
\begin{align*}
\mathcal{L}G_1 \mathcal{L}^{-1} & := -\frac{\ell}{2} (Q_{\zeta_1} - Q_{\zeta_2}), & \mathcal{L}G_2 \mathcal{L}^{-1} & := -\frac{\beta^2}{\ell \hbar} (P_{\zeta_1} - P_{\zeta_2}) \\
\mathcal{L}K_1 \mathcal{L}^{-1} & := \frac{\alpha}{\beta} \frac{\ell}{2} (Q_{\zeta_1} + Q_{\zeta_2}), & \mathcal{L}K_2 \mathcal{L}^{-1} & := \frac{\alpha}{\beta} \frac{\ell}{h} (P_{\zeta_1} + P_{\zeta_2}).
\end{align*}
\]

Now we can consider the change of coordinates \((\zeta_1, \zeta_2) \mapsto (x_s, x_t)\) defined by

\[
\begin{align*}
x_s & = -\frac{\ell}{2} (\zeta_1 - \zeta_2) \\
x_t & = \frac{\alpha}{\beta} \frac{\ell}{2} (\zeta_1 + \zeta_2)
\end{align*}
\]

\[
\begin{align*}
\zeta_1 & = -\frac{\ell}{\alpha} (x_s - \frac{\beta}{\alpha} x_t) \\
\zeta_2 & = \frac{\ell}{\alpha} (x_s + \frac{\beta}{\alpha} x_t)
\end{align*}
\]

The jacobian of this transformation is \(\left| \frac{\partial (\zeta_1, \zeta_2)}{\partial (x_s, x_t)} \right| = \frac{2\ell}{\beta} \left| \frac{\beta}{\alpha} \right| := C\), then the map \((\mathcal{R} \psi)(x_s, x_t) := \sqrt{C} \psi(\zeta(x_s, x_t))\) defines a unitary map \(\mathcal{R} : L^2(\mathbb{R}^2, d^2 \zeta) \to L^2(\mathbb{R}^2, dx_s \, dx_t)\). With a direct computation one can check that \(\mathcal{R} Q_{\zeta_1} \mathcal{R}^{-1}\) acts on \(L^2(\mathbb{R}^2, dx_s \, dx_t)\) as the multiplication by \(\zeta_j(x_s, x_t)\), while \(\mathcal{R} P_{\zeta_1} \mathcal{R}^{-1} = \frac{\hbar \ell}{2 \pi \beta} \left( \frac{1}{\beta} \alpha P_s + P_t \right)\), with \(j = 1, 2\). This shows that the unitary map \(W := \mathbb{I} \circ \mathcal{R} \circ \Pi^{-1} \circ \mathcal{L} := \mathbb{I} \circ \mathcal{R} \circ \Pi^{-1} \circ (\Pi_1 \otimes F_{2, \mu}) \circ \Pi \circ J\) is the von Neumann unitary that verifies the relations \((130)\) and \((131)\).
B.2 The integral kernel of $W$

The unitary operators $J : L^2(\mathbb{R}^2, d^2k) \to L^2(\mathbb{R}^2, d^2r)$ and $R : L^2(\mathbb{R}^2, d^2\zeta) \to L^2(\mathbb{R}^2, dx_s dx_t)$ related to the change of coordinates $(r_1, r_2) \mapsto (k_1, k_2)$ and $(\zeta_1, \zeta_2) \mapsto (x_s, x_t)$ can be written as integral operators

$$(J \psi)(k) = \int_{\mathbb{R}^2} J(r; k) \psi(r) \, d^2r, \quad (R \varphi)(x_s, x_t) = \int_{\mathbb{R}^2} R(\zeta, x_s, x_t) \varphi(\zeta) \, d^2\zeta$$

with distributional integral kernels

$$J(r_1, r_2; k_1, k_2) := \delta \left( r_1 - \frac{1}{\ell} (w_2 k_1 - v_2 k_2) \right) \delta \left( r_2 + \frac{1}{\ell} (w_1 k_1 - v_1 k_2) \right)$$

$$R(\zeta_1, \zeta_2; x_s, x_t) := \sqrt{C} \delta \left( \zeta_1 + \frac{1}{\ell} (x_s - \frac{\beta}{\alpha} x_t) \right) \delta \left( \zeta_2 - \frac{1}{\ell} (x_s + \frac{\beta}{\alpha} x_t) \right)$$

with $C = \frac{2}{\ell} |\beta/\alpha|$. The $k_1$-parity operator $\Pi_1$ can be written as an integral operator with the distributional kernel $\delta(r_1 + k_1)$ while the integral kernel of the $k_2$-Fourier transform $F_{2,\mu}$, with $\mu := -\frac{\ell^2}{2\pi}$, is $\frac{\ell}{2|\beta|\sqrt{\pi}} e^{i\frac{\ell^2}{2\pi} \zeta k_2}$. Then the unitary map $\Pi^{-1} \circ (\Pi_1 \otimes F_{2,\mu}) \circ \Pi : L^2(\mathbb{R}^2, d^2k) \to L^2(\mathbb{R}^2, d^2\zeta)$ as the integral distributional kernel

$$L(k_1, k_2; \zeta_1, \zeta_2) := \delta(\zeta_1 + k_1) \frac{\ell}{2|\beta|\sqrt{\pi}} e^{i\frac{\ell^2}{2\pi} \zeta k_2}.$$

Summarizing the total transform $W : L^2(\mathbb{R}^2, d^2r) \to L^2(\mathbb{R}^2, dx_s dx_t)$ (up to the obvious identification $\Pi$) can be expressed as an integral operator

$$(W \psi)(x_s, x_t) = \int_{\mathbb{R}^2} W(r; x_s, x_t) \psi(r) \, d^2r$$

with a (total) integral distributional kernel

$$W(r_1, r_2; x_s, x_t) := \frac{\ell}{\sqrt{2\pi |\alpha\beta|}} \delta \left( v \cdot r - \left( x_s - \frac{\beta}{\alpha} x_t \right) \right) e^{i\frac{\pi}{2\beta}(x_s + \frac{\beta}{\alpha} x_t)}.$$
References

[AHS78] J. E. Avron, I. Herbst, and B. Simon. Schrödinger Operators with Magnetic Fields I. General Interactions. *Duke Math. J.*, **45** (4): 847–883, 1978.

[AM76] N. W. Ashcroft and N. D. Mermin. *Solid State Physics*. Saunders, 1976.

[ASS83] J. E. Avron, R. Seiler, and B. Simon. Homotopy and Quantization in Condensed Matter Physics. *Phys. Rev. Lett.*, **51**: 51–53, 1983.

[Avr04] J. E. Avron. *Colored Hofstadter butterflies*. Ph. Blanchard, G. Dell’Antonio (eds.), Multiscale Methods in Quantum Mechanics: Theory and Experiments. Birkhäuser, 2004.

[Bel88] J. Bellissard. $C^*$-algebras in solid state physics. 2D electrons in a uniform magnetic field, volume II of *Operator algebras and applications*. University Press, 1988.

[BR97] O. Bratteli and D. W. Robinson. *Equilibrium States, Models in Quantum Statistical Mechanics*, volume II of *Operator Algebras and Quantum Statistical Mechanics*. Springer-Verlag, 1997.

[BSE94] J. Bellissard, H. Schulz-Baldes, and A. van Elst. The Non Commutative Geometry of the Quantum Hall Effect. *J. Math. Phys.*, **35**: 5373–5471, 1994.

[CTVR06] D. Ceresoli, T. Thonhauser, D. Vanderbilt, and R. Resta. Orbital magnetization in crystalline solids: multi-band insulators, Chern insulators, and metals. *Phys. Rev. B*, **74**: 024408, 2006.

[DL10] G. De Nittis and M. Lein. Applications of magnetic $\Psi$DO techniques to Space-adiabatic Perturbation Theory. Preprint arXive.org/1006.3103, 2010.

[GA03] O. Gat and J. E. Avron. Semiclassical analysis and the magnetization of the Hofstadter model. *Phys. Rev. Lett.*, **91**: 186801, 2003.

[GMS91] C. Gérard, A. Martinez, and J. Sjöstrand. A mathematical approach to the effective Hamiltonian in perturbed periodic problems. *Comm. Math. Phys.*, **142**: 217–244, 1991.

[GPPO00] V. A. Geyler, I. Yu. Popov, A. V. Popov, and A. A. Ovechkina. Fractal spectrum of periodic quantum systems in a magnetic field. *Chaos, solitons and fractals*, **11**: 281–288, 2000.

[Gra07] G. M. Graf. Aspects of the Integer Quantum Hall Effect. *Proceedings of Symposia in Pure Mathematics*, **76.1**: 429–442, 2007.

[Har55] P. G. Harper. Single band motion of conduction electrons in a uniform magnetic field. *Proc. Phys. Soc. A*, **68**: 874–892, 1955.
[Hof76] D. R. Hofstadter. Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields. *Phys. Rev. B*, 14: 2239–2249, 1976.

[HS88] B. Helffer and J. Sjöstrand. Analyse semi-classique pour l’équation de Harper (avec application à l’étude de Schrödinger avec champ magnétique). *Mém. Soc. Math. France*, 34: 1761–1771, 1988.

[HS89a] B. Helffer and J. Sjöstrand. Analyse semi-classique pour l’équation de Harper III. *Mém. Soc. Math. France*, 39, 1989.

[HS89b] B. Helffer and J. Sjöstrand. Équation de Schrödinger avec champ magnétique et équation de Harper, volume 345 of *Lecture Notes in Physics*. Springer-Verlag, 1989.

[HS90] B. Helffer and J. Sjöstrand. Analyse semi-classique pour l’équation de Harper II (comportement semi-classique près d’un rationnel). *Mém. Soc. Math. France*, 40, 1990.

[IMP07] V. Iftimie, M. Măntoiu, and R. Purice. Magnetic pseudodifferential operators. *Publ. RIMS.*, 43 no. 3: 585–623, 2007.

[IMP09] V. Iftimie, M. Măntoiu, and R. Purice. The magnetic formalism; new results. *Contemporary Mathematics*, 500: 123–138, 2009.

[Kuc93] P. Kuchment. *Floquet Theory for Partial Differential Equations*. Operator Theory: Advances and Applications. Birkhäuser, 1993.

[Las94] Y. Last. Zero measure spectrum for the almost Mathieu operator. *Comm. Math. Phys.*, 164: 421–432, 1994.

[Las05] Y. Last. *Spectral theory of Sturm-Liouville operators on infinite intervals: A review of recent developments*. Sturm-Liouville Theory (W. O. Amrein, A. M. Hinz, and D. B. Pearson, eds.). Birkhäuser, 2005.

[LL01] E. H. Lieb and M. Loss. *Analysis*. American Mathematical Society, 2001.

[Mor88] G. Morandi. *Quantum Hall Effect*. Bibliopolis, 1988.

[MP04] M. Măntoiu and R. Purice. The Magnetic Weyl calculus. *J. Math. Phys.*, 45: 1394, 2004.

[MPR05] M. Măntoiu, R. Purice, and S. Richard. Twisted Crossed Products and Magnetic Pseudodifferential Operators. in *Advances in Operator Algebras and Mathematical Physics*, pages 137–172, 2005.

[Nen02] G. Nenciu. On asymptotic perturbation theory for Quantum Mechanics: Almost invariant subspaces and gauge invariant magnetic perturbation theory. *J. Math. Phys.*, 43: 1273–1298, 2002.
[NS04] G. Nenciu and V. Sordoni. Semiclassic limit for multistate Klein-Gordon systems: almost invariant subspaces, and scattering theory. *J. Math. Phys.*, 45: 3676, 2004.

[OA01] D. Osadchy and J. E. Avron. Hofstadter butterfly as quantum phase diagram. *J. Math. Phys.*, 42: 5665–5671, 2001.

[OGM08] P. M. Ostrovsky, I. V. Gornyi, and A. D. Mirlin. Theory of anomalous quantum Hall effects in graphene. *Phys. Rev. B*, 77: 195430, 2008.

[Pan07] G. Panati. Triviality of Bloch and Bloch-Dirac bundles. *Ann. Henri Poincaré*, 8: 995–1011, 2007.

[Pei33] R. Peierls. Zur Theorie des Diamagnetismus von Leitungselektronen. Z. Phys., 80: 763–791, 1933.

[PST03a] G. Panati, H. Spohn, and S. Teufel. Effective dynamics for Bloch electrons: Peierls substitution and beyond. *Comm. Math. Phys.*, 242: 547–578, 2003.

[PST03b] G. Panati, H. Spohn, and S. Teufel. Space-Adiabatic Perturbation Theory. *Adv. Theor. Math. Phys.*, 7: 145–204, 2003.

[Rau74] A. Rauh. Degeneracy of Landau levels in crystals. *Phys. Status Solidi B*, 65: K131–K135, 1974.

[Rau75] A. Rauh. On the broadening of Landau levels in crystals. *Phys. Status Solidi B*, 69: K9–K13, 1975.

[RS75] M. Reed and B. Simon. *Fourier Analysis, Self-Adjointness*, volume II of *Methods of Modern Mathematical Physics*. Academic Press, 1975.

[RS78] M. Reed and B. Simon. *Analysis of Operators*, volume IV of *Methods of Modern Mathematical Physics*. Academic Press, 1978.

[Shu94] M. A. Shubin. Discrete Magnetic Laplacian. *Commun. Math. Phys.*, 164: 259–275, 1994.

[Sob97] A. V. Sobolev. Absolute continuity of the periodic magnetic Schrödinger operator. *Invent. Math.*, 137: 85–112, 1997.

[TCVR05] T. Thonhauser, D. Ceresoli, D. Vanderbilt, and R. Resta. Orbital magnetization in periodic insulators. *Phys. Rev. Lett.*, 95:137205, 2005.

[Teu03] S. Teufel. *Adiabatic Perturbation Theory in Quantum Dynamics*, volume 1821 of *Lecture Notes in Mathematics*. Springer-Verlag, 2003.

[TKNN82] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. Nijs. Quantized Hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.*, 49: 405–408, 1982.

[Wil87] M. Wilkinson. An exact effective Hamiltonian for a perturbed Landau level. *J. Phys. A*, 20: 1761–1771, 1987.