Charged anisotropic strange stars in Brans-Dicke gravity with a massive scalar field through embedding approach

S. K. Maurya,1 Ksh. Newton Singh,2 M. Govender,3 and Abdelghani Errehymy4

1Department of Mathematical and Physical Sciences, College of Arts and Science, University of Nizwa, Nizwa, Sultanate of Oman
2Department of Physics, National Defence Academy, Khadakwasla, Pune-411023, India.
3Department of Mathematics, Durban University of Technology, Durban 4000, South Africa
4Laboratory of High Energy Physics and Condensed Matter (LPHEMaC), Department of Physics, Faculty of Sciences Ain Chock, University of Hassan II, B.P. 5366 Maarif, Casablanca 20100, Morocco

In this exposition we seek solutions of the Einstein-Maxwell field equations in the presence of a massive scalar field cast in the Brans-Dicke (BD) formalism which describe charged anisotropic strange stars. The interior spacetime is described by a spherically symmetric static metric of embedding class I. This reduces the problem to a single-generating function of the metric potential which is chosen by appealing to physics based on regularity at each interior point of the stellar interior. The resulting model is subjected to rigorous physical checks based on stability, causality and regularity. We show that our solutions describe compact objects such as PSR J1903+327; Cen X-3; EXO 1785-248 & LMC X-4 to an excellent approximation. Novel results of our investigation reveal that the scalar field leads to higher surface charge densities which in turn affects the compactness and upper and lower values imposed by the modified Buchdahl limit for charged stars. Our results also show that the electric field and scalar field which originate from entirely different sources couple to alter physical characteristics such as mass-radius relation and surface redshift of compact objects. This superposition of the electric and scalar fields is enhanced by an increase in the BD coupling constant, $\omega_{BD}$.

PACS numbers:

I. INTRODUCTION

Einstein’s general relativity (GR) has been fruitful in describing gravitational phenomena on both cosmological and astrophysical scales. The predictions of GR has gone beyond the realms of theory and has been successfully confirmed through a plethora of experiments. With the advancement of technology these predictions have been refined. The perihelion precession of Mercury, one of the first solar system tests of GR has been drastically improved by the collection of data from Mercury MESSENGER which orbited Mercury in 2011[1]. The joint European-Japanese Mercury spacecraft BepiColombo project which launched in 2018 is expected to reveal more precise measurements of the peculiarities of Mercury’s orbit. The first gravitational wave events were detected in September 2015 by the LIGO and Virgo collaborations thus reinforcing the prediction of classical GR. There is no more greater signalling of the Golden Age of astrophysical observations than the 2019 capturing of the image of the black hole at the center of galaxy M87 by Event Horizon Telescope[2].

Despite these confirmations of GR there are still many observations that leave Einstein’s classical gravity theory short. In cosmology researchers are still faced with various problems including the late-time acceleration of the Universe, dark matter and dark energy conundrums, baryon symmetry, horizon problem just to name a few[3]. On the other hand there are outstanding problems in astrophysics some of which include the origin of large surface redshifts in compact objects, the behaviour of matter at extreme densities such as in the core of neutron stars, end-states of continued gravitational collapse amongst others.

To this end researchers in gravitational physics have appealed to modified theories of gravitation in the hope of finding mechanisms that will account for the observations which cannot be resolved by GR. These modified theories must have as their weak field limit Einstein’s general relativity. A simple modification to the standard 4D theory is to accommodate more than just the linear forms of the Riemann tensor, the Ricci tensor and the Ricci scalar in the action principle. It is well-known that incorporation of just linear tensorial quantities produces second order
The accelerated expansion of the universe has led to the development of modified gravity theories. These theories extend the framework of general relativity (GR) by introducing additional scalar fields or modifying the gravitational action. One such theory is the Brans-Dicke theory, which is centered on the notion that the divergence of the energy-momentum tensor is proportional to the divergence of a scalar field contained within a false vacuum. The staticity of the model is broken via quantum tunneling, leading to an emergent universe without invoking exotic matter fields or dissipative processes.

The Brans-Dicke framework has been recently proposed by Labrana and Cossi to explain the inflationary epoch of the universe and the current accelerated phase of the universe. BD gravity has elegantly incorporated Mach’s principle and has been widely utilised in cosmological models. It continues to be one of the more popular theories of modified classical GR and has been extensively utilised in both cosmological and astrophysical contexts.

The Brans-Dicke theory is also called the Jordan-Brans-Dicke gravity theory, which is a theoretical framework that can be represented in Jordan-Brans-Dicke gravity as well as Einstein’s frame. The stability of the model is affected by the anisotropy parameter which is inherently linked to the decoupling constant. The Karmarkar condition has been successfully used in the study of anisotropic compact objects and has been shown to be linked to physical viability tests which are backed by observational data. The Karmarkar condition has been recently proposed by Hansraj and Moodly to demonstrate the nonexistence of a GV solution via the Karmarkar solution which shows that embedding does not allow the interpretation of the generalized Vaidya spacetime as a diffusive medium. In other words, the Karmarkar condition prohibits the GV solution to be interpreted as an atmosphere composed of radiation and diffusive strings of a star undergoing dissipative collapse in the form of a radial heat flux. The Karmarkar condition has been extended to incorporate time-dependent systems which include modelling shear-free, dissipative collapse. The Karmarkar condition has been successfully used in modified gravity theories to investigate contributions from the inclusion of quadratic terms of the tensorial quantities and higher dimensional effects on compactness, stability and surface redshifts of compact objects residing in these theories.
exotic spacetimes.

The role of an equation of state in modeling compact objects has been highlighted in several recent studies. The simple linear equation of state (EoS) which expresses the pressure as a linear function of the fluid density \( p_\alpha = \alpha \rho \) where \( \alpha \geq 0 \) is a constant has been extended to include \( \alpha < 0 \) used in modeling so-called dark stars and phantom fields. The MIT Bag model EoS has gained popularity amongst researchers and has been successfully utilised to model compact objects in classical GR and modified gravity theories \[32, 37\]. The quadratic EoS, polytropic EoS and Chaplygin gas EoS have also led to physically reasonable models of static stars \[35, 40\]. By appealing to results in quantum chromodynamics and quark interactions within the stellar core, the so-called colour-flavoured-Locked (CFL) EoS has been recently used in obtaining models of compact objects which approximate realistic neutron stars, pulsars and strange stars to a very good degree \[11, 12\]. The CFL EoS has also been employed to study the surface tension of neutron stars. This study shows that the surface tension is sensitive to the magnitude of the Bag constant \[13\]. It was observed that larger values of the Bag constant led to stellar models with lower tangential pressures and surface tensions.

The role of the electromagnetic field in the (in)stability of compact objects has occupied the interest of researchers since the discovery of the Schwarzschild solution. The study of charged objects in general relativity has taken a different and refreshing trajectory compared to the early attempts at just finding exact solutions to the Einstein-Maxwell system. It is clear that stellar objects can be endowed with nonzero charge which can give rise to high intensity electric fields \[4, 16\]. The Einstein-Maxwell system can be interpreted as representing an anisotropic fluid with the pressure isotropy condition becoming the definition of the electric field. A recent paper by Maurya & Tello-Ortiz \[47\] highlighted an interesting interplay between the anisotropy parameter and the electric field intensity which provides a mechanism for maintaining stability of the stellar configuration. They found that the force due to the pressure anisotropy initially dominates the Coulombic repulsion closer to the center of the star with the anisotropic force out-growing the Coulombic repulsion towards the surface layers of the star. A similar phenomenon was discovered in a charged compact star of embedding class \[1\]. The effect of charge on stellar characteristics within the framework of Einstein-Gauss-Bonnet (EGB) gravity has been investigated. It is found that mass-radius relation and surface redshifts are modified by the presence of the EGB coupling constant \[13\].

This paper is structured as follows: In §2, we provide the necessary equations within the BD formalism necessary to model a charged compact object. The class 1 embedding condition is derived for the BD framework in §3. The junction conditions required for the smooth matching of the interior spacetime to the Reisner-Nordstrom exterior is presented in §4. In §5 we discuss the regularity of the metric functions and thermodynamical quantities at the center of the stellar configuration and we derive the modified TOV equation in the presence of a massive scalar field, nonzero charge density and pressure anisotropy together with mass-radius relation and moment of inertia thorough \[4, 5\] curves in §5. A detailed discussion of the physical attributes together with the conclusion of our model follows in §6.

II. THE BACKGROUND OF BRANS-DICKE GRAVITY THEORY AND FIELD EQUATIONS

The action of scalartensor theory in BransDicke frame in relativistic units \( G = c = 1 \) is defined as,

\[
S = \frac{1}{16\pi} \int \left[ R\Phi - \frac{\omega_{BD}}{\Phi} \nabla^i \nabla_i \Phi - L(\Phi) \right] \sqrt{-g} \, d^4 x + \int \mathcal{L}_m \sqrt{-g} \, d^4 x + \int \mathcal{L}_e \sqrt{-g} \, d^4 x, \tag{1}
\]

where \( R \), \( g \), \( \mathcal{L}_m \), and \( \mathcal{L}_e \) describe the Ricci scalar, determinant of metric tensor, matter Lagrangian density, and Lagrangian electromagnetic field respectively, while \( \omega_{BD} \) is a dimensionless Dicke coupling constant and \( \Phi \) is a scalar field. Here the function \( L(\Phi) \) depends completely on the scalar field \( \Phi \). In the present case we define this scalar field function \( L(\Phi) \) as,

\[
L(\Phi) = \frac{1}{2} m_\phi^2 \Phi^2 \tag{2}
\]

Now by varying of the action \[11\] with respect to metric tensor \( g_{ij} \) and scalar field \( \Phi \) provides the following field equations and evaluation equation, respectively, which can be written as,

\[
G_{ij} = \frac{1}{\Phi} \left[ 8\pi T^{(m)}_{ij} + 8\pi E_{ij} + T^{(\Phi)}_{ij} \right], \tag{3}
\]

where, \( T^{(m)}_{ij} \) and \( E_{ij} \) denotes the energy-momentum tensor for matter distribution and electromagnetic field tensor, respectively while \( T^{(\Phi)}_{ij} \) represents a scalar tensor appears in the system due to the scalar field \( \Phi \). All the field tensors...
can be written as,

\[ T_{ij}^{(m)} = (\rho + p_r)u_i u_j - p_t g_{ij} + (p_r - p_t) v_i v_j, \]  
\[ E_{ij} = \frac{1}{4\pi} \left( -F^m_i F_n^j + \frac{1}{4} g_{ij} F^\gamma_n F^{\gamma m} \right), \]  
\[ T_{ij}^{(\Phi)} = \Phi_i \delta_j - g_{ij} \Phi + \frac{\omega_{BD}}{\Phi} \left( \Phi_i \Phi_j - g_{ij} \Phi^2 \Phi^\delta \right) - \frac{\mathcal{L}(\Phi)}{\Phi} g_{ij}. \]

Here, \( \Box \) denotes the d’Alembert operator, then \( \Box \Phi \) can be given as,

\[ \Box \Phi = \frac{T^{(m)}}{3 + 2\omega_{BD}} + \frac{1}{3 + 2\omega_{BD}} \left( \Phi \frac{d\mathcal{L}(\Phi)}{d\Phi} - 2\mathcal{L}(\Phi) \right). \]

Here, \( \rho, p_r \) and \( p_t \) denote the energy density, radial pressure and transverse pressure, respectively with \( T^{(m)} \) being the trace of energy tensor \( T_{ij}^{(m)} \). Also, \( u^i = e^{-\xi/2} \delta_0^i \), designating the four-velocity, and \( v^i = e^{-\eta/2} \delta_1^i \), designating the four-vector, are specified as, \( u^i u_i = 1 \) and \( v^i v_i = -1 \), and \( \Box \Phi = \Phi_i = (g)^{-1/2}[(-g)^{-1} \Phi^1]_i \). In addition, the anti-symmetric electromagnetic field tensor \( F_{ij} \) given in Eq. (5) is characterized as

\[ F_{ij} = \nabla_i A_j - \nabla_j A_i \]

for which Maxwell’s equations have been satisfied,

\[ F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \]

with

\[ F^{ik} : k = 4\pi J^i \]

where, \( J^i \) is the electromagnetic 4-current vector. This can be expressed as

\[ J^i = \frac{\sigma}{\sqrt{g_{00}}} \frac{dx^i}{dt} = \sigma v^i, \]

where \( \sigma = e^{\xi/2} J^0(r) \) representing the charge density. It turns out that for a static matter distribution with spherical symmetry, there is only one non-zero component of the electromagnetic 4-current \( J^i \) which is \( J^0 \), a function of the radial distance, \( r \). The \( F^{01} \) and \( F^{10} \) components are the only non-zero components of electromagnetic field tensor expressed in [5] and they are connected by the formula \( F^{01} = -F^{10} \), which characterizes the radial constituent of the electric field. The constituent of the electric field is determined through Eqs. (10) and (11) as follows

\[ F^{01} = -F^{10} = \frac{q}{r^2} e^{-\xi/2}/2 \]

The quantity \( q(r) \) represents the effective charge of a spherical system of radial coordinate, \( r \), subsequently, this electric charge can be characterized by the relativistic Gauss law and corresponding electric field \( E \) explicitly as,

\[ q(r) = 4\pi \int_0^r \sigma r^2 e^{\eta/2} dr = r^2 \sqrt{-F_{10} F^{10}} \]

\[ E^2 = -F_{10} F^{10} = q^2/r^4. \]

It is noted that Doneva et al. [54] and Yazadjiev et al. [51] have already discussed both slowly and rapidly rotating neutron stars by using the above potential function [2]. In order to describe the stellar structure, we assume a static spherically symmetric line element which can cast as,

\[ ds^2 = e^{\xi(r)} dt^2 - e^{\eta(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where \( \xi(r) \) and \( \eta(r) \) are metric potentials and rely just upon the radial distance \( r \) that ensures the staticity of the space-time. By using Eqs. [2] - [13], We obtain the following field equations,
\[ e^{-\eta} \left( \frac{\eta'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{1}{\Phi} \left( 8\pi \rho + \frac{q^2}{r^4} + T_0^{0(\Phi)} \right), \]  
(16)

\[ e^{-\eta} \left( \frac{\xi'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{1}{\Phi} \left( 8\pi \rho_r + \frac{q^2}{r^4} - T_1^{1(\Phi)} \right), \]  
(17)

\[ \frac{e^{-\eta}}{2} \left( \xi'' + \frac{\xi'^2}{2} + \frac{\xi' - \eta'}{2} - \frac{\xi'\eta'}{2} \right) = \frac{1}{\Phi} \left( 8\pi \rho + \frac{q^2}{r^4} - T_2^{2(\Phi)} \right), \]  
(18)

where prime denotes differentiation with respect to \( r \). On the other hand, the scalar tensor components \( T_0^{0(\Phi)}, T_1^{1(\Phi)}, \) and \( T_2^{2(\Phi)} \) in terms of \( \xi \) and \( \eta \) are given as,

\[ T_0^{0(\Phi)} = e^{-\eta} \left[ \Phi'' + \left( \frac{2}{r} - \frac{\eta'}{2} \right) \Phi' + \frac{\omega_{BD}}{2\Phi} \Phi'^2 - e^\eta \frac{\mathcal{L}(\Phi)}{2} \right], \]  
(19)

\[ T_1^{1(\Phi)} = e^{-\eta} \left[ \frac{2}{r} + \frac{\xi'}{2} \right] \Phi' - \frac{\omega_{BD}}{2\Phi} \Phi'^2 - e^\eta \frac{\mathcal{L}(\Phi)}{2} \],  
(20)

\[ T_2^{2(\Phi)} = e^{-\eta} \left[ \Phi'' + \left( \frac{1}{r} - \frac{\eta'}{2} + \frac{\xi'}{2} \right) \Phi' + \frac{\omega_{BD}}{2\Phi} \Phi'^2 - e^\eta \frac{\mathcal{L}(\Phi)}{2} \right]. \]  
(21)

However, from Eqs. (7) and (15) we obtain,

\[ \Box \Phi = -e^{-\eta} \left[ \left( \frac{2}{r} - \frac{\eta'}{2} + \frac{\xi'}{2} \right) \Phi'(r) + \Phi''(r) \right] = \frac{1}{(3 + 2\omega_{BD})} \left[ T^{(m)} + \Phi \frac{d\mathcal{L}(\Phi)}{d\Phi} - 2 \mathcal{L}(\Phi) \right]. \]  
(22)

For more simplicity, extreme temperatures and pressures at the core of massive neutron stars can transform into quark stars with up \((u)\), down \((d)\) and strange star \((s)\) quark flavors. In this regard, we suppose that the MIT bag model rules the matter variables (density and pressure) in the interior of these relativistic massive stars. It is additionally supposed that non-interacting and massless quarks occupy the inside of the stellar structures. Accordingly the MIT Bag model, the quark pressure \( p_r \) can be composed as follows,

\[ p_r = \sum_f p^f - B, \quad f = u, d, s \]  
(23)

where \( p^f \) describes the individual pressures due to each quark flavor which is balanced by Bag constant (or total external Bag pressure) \( B \). The deconfined quarks inside the MIT Bag model have the accompanying total energy density

\[ \rho = \sum_f \rho^f + B, \]  
(24)

where \( \rho^f \) indicates the matter density due to each flavor which is connected to the corresponding pressure by the formula given as \( \rho^f = 3p^f \). Consequently, Eqs. (23) and (24) are consolidated to express the following simplified MIT Bag model,

\[ p_r = \frac{1}{3}(\rho - 4B), \]  
(25)

It should be mentioned here that this specific linear form of the MIT bag model EoS has been applied for portraying the stellar systems made of the strange quark matter distribution in pure GR and modified gravity theories.

Now using the Eqs. (16) and (17) along with EOS (25), we obtain the expression for the electric field as,

\[ \frac{4q^2}{r^4} = \Phi \left[ e^{-\eta} \left( \frac{\eta' - 3\xi'}{r} \right) + \frac{4(1 - e^{-\eta})}{r^2} \right] - (3T_1^{1(\Phi)} + T_0^{0(\Phi)}) - 32\pi B, \]  
(26)
III. BASIC FORMULATION OF CLASS ONE CONDITION AND ITS SOLUTION IN BRANS-DICKE GRAVITY

It is well-known that the embedding of $n$–dimensional space $V^n$ in a pseudo-Euclidean space $E^n$ attracted much consideration as inferred by Eisland [52] and Eisenhart [53]. In the case where a $n$–dimensional space $V^n$ can be isometrically immersed in $(n + m)$–dimensional space, where $m$ is a minimum number of supplementary dimensions, at this stage $V^n$ is said to be $m$–class embedding. Habitually, the metric expressed in (15) provides the four–dimensional spherically symmetric space-time which describes a space-time of class two i.e, when $m = 2$, which shows that it is implemented a six–dimensional pseudo-Euclidean space for embedding. On the other hand, it should be noted that one can reveal a possible parametrization in order to incorporate the space-time expressed in (15) into a five-dimensional pseudo-Euclidean space which leads to class $m = 1$ dubbed as embedding class one [52; 54]. A spherically symmetric space-time in both cases static or non-static to be class-one, the system has to be consistent with the following necessary and suitable conditions:

- A stellar system of symmetric amounts $b_{ij}$ should be determined under the associated Gauss conditions:

\[ R_{ijhk} = \epsilon (b_{ih}b_{jk} - b_{ik}b_{jh}) , \tag{27} \]

where $\epsilon = \pm 1$ everywhere the normal to the variety is time-like (-1) or space-like (+1).

- The symmetric tensor $b_{ij}$ must fulfill the accompanying Codazzi conditions:

\[ \nabla_h b_{ij} - \nabla_j b_{ih} = 0 . \tag{28} \]

It is worth mentioning here that the overall Codazzi condition as expressed in (23) is not zero. This general form of Codazzi condition has been suggested by Eiesland and Eisenhart [52; 53]. In this respect, we use the line element \( V^n \), the Riemann components can be expressed as follows,

\[
\begin{align*}
R_{0101} &= -c^2 \left( \frac{\epsilon''}{2} - \frac{\eta' \epsilon'}{4} + \frac{\epsilon'^2}{4} \right); \quad R_{1313} = -\frac{r}{2} \eta' \sin^2 \theta; \quad R_{2323} = -\frac{r^2 \sin^2 \theta}{e^\eta} (e^\eta - 1); \\
R_{1202} &= 0, \quad R_{0202} = -\frac{r}{2} \epsilon' e^{\eta - \eta}; \quad R_{1303} = 0; \quad R_{0303} = -\frac{r}{2} \epsilon' e^{\eta - \eta} \sin^2 \theta; \quad R_{1212} = -\frac{r}{2} \eta'.
\end{align*}
\]

Substituting these Riemann components into Gauss’s equation (27) leads to

\[
\begin{align*}
b_{01}b_{33} &= R_{1303} = 0; \quad b_{01}b_{22} = R_{1212} = 0; \quad b_{00}b_{33} = R_{0303}; \quad b_{00}b_{22} = R_{0202}; \\
b_{11}b_{33} &= R_{1313}; \quad b_{22}b_{33} = R_{2323}; \quad b_{11}b_{22} = R_{1212}; \quad b_{00}b_{11} = R_{0101}.
\end{align*}
\]

These relations expressed in (30) leads immediately to the following expressions

\[
\begin{align*}
(b_{00})^2 &= \left( \frac{R_{0202}}{R_{2323}} \right)^2 \sin^2 \theta; \quad (b_{11})^2 = \left( \frac{R_{1212}}{R_{2323}} \right)^2 \sin^2 \theta; \quad (b_{22})^2 = \left( \frac{R_{2323}}{\sin^2 \theta} \right); \quad (b_{33})^2 = \sin^2 \theta \cdot R_{2323}.
\end{align*}
\]

By combining the relations (30) into components of Eq. (31), we found the following relationship in Riemann components

\[ R_{0202}R_{1313} = R_{0101}R_{2323} , \tag{32} \]

subject to $R_{2323} \neq 0 \ [53]$. It ought to be noticed that all the components are given in (31) fulfill the Codazzi equation (28). Moreover, there is a significant point that we might want to refer to: for an overall spherically symmetric space-time, its symmetric tensor $b_{ij}$ can be composed as follows

\[
\begin{align*}
b_{01}b_{22} &= R_{1212} \quad \text{and} \quad b_{00}b_{11} - (b_{01})^2 = R_{0101};
\end{align*}
\]

where $(b_{01})^2 = \sin^2 \theta \cdot (R_{1202})^2 / R_{2323}$. In view of this situation, the embedding Class-one condition known as Kar-markar condition [52; 54] take the following form

\[ R_{0202}R_{1313} = R_{0101}R_{2323} + R_{1202}R_{1303} . \tag{34} \]

Although in our circumstance, the relationship in Riemann components (32) according to the static spherically symmetric line element [13] will be equivalent to (34). This condition plays a fundamental role in describing a
space-time \([15]\) as being of class-one, and also well-known as a necessary and sufficient condition. At this point, by incorporating the Riemann components in expression \([34]\), we accomplish the accompanying equation,

\[
\frac{2\xi''}{\xi'} + \xi' = \frac{\eta^e_0}{e^{\eta}_0 - 1},
\]

with \(e^{\eta}_0 \neq 1\). Consequently, the solution of the differential equation given in \([35]\) has been determined in the case where the space-time \([20]\) to be a class-one. Now, by integrating the equation \([34]\), we find the relationship amongst the gravitational potentials in the following form

\[
\xi(r) = 2 \ln \left[ A + B \int \sqrt{e^{\eta_0} - 1} \, dr \right].
\]

Here \(A\) and \(B\) are integration constants. It ought to be enhanced that the above methodology has been generally utilized in the domain of compact configurations portraying genuine heavenly bodies. In order to find the solution of field Eqs. \([10]-[15]\) in Brans-Dicke gravity under Class I condition \([36]\), we need to find the metric potentials admitting Karmarkar condition and Pandey-Sharma condition. As it is well-known in general, the invariance of the Ricci tensor necessitates that the matter variables viz., energy density \(\rho\), radial pressure \(p_r\), and transverse pressure \(p_t\) ought to be finite at the center. The regularity of the Weyl invariants necessitates that the following two quantities: mass \(m(r)\) and electric charge \(q(r)\) satisfy all the conditions: \(m(0) = q(0) = 0\) and \(m'(r) > 0\) and \(q(0) = 0\), \(q'(r) > 0\) i.e., reach the minimum and maximum values at the center as well as the surface of the celestial body, respectively. In this regard, Maurya and collaborators \([71]\) have already exhibited that the gravitational potential \(\xi(0)\) is equal to a finite constant value, \(q(0) = 0\), \(\xi'(0) = 0\) and \(\xi''(0) > 0\) according to the modelling of charged anisotropic compact celestial bodies. Since both physical quantities viz., energy density \(\rho\), radial pressure \(p_r\), and transverse pressure \(p_t\) are positive finite and continuous and also pursue the condition \(r > 2m(r)\) \([72, 73]\). So, from \(p_t(r) \geq 0\) with the help of the condition \(r > 2m(r)\), we can obtain \(\xi' \neq 0\), which implies that the general function \(\xi(r)\) is regular minimum at the center and a monotone increasing function of radial coordinate \(r\). Consequently, the general function \(\xi(r)\) has should conserve the said physical characteristics. Then again, the gravitational potential function \(e^{\eta}_0\) should fulfill the accompanying form \(e^{\eta}_0 = 1 + O(r^2)\) to ensures the regularity and stability of the compact stellar object. In this way by keeping all the attributes in our mind, we have assumed the \(e^{\eta}_0\) as follow:

\[
e^{\eta_0}(r) = 1 + a r^2 \, e^{br^2}.
\]

Using \([37]\) in \([36]\), we get

\[
\xi(r) = 2 \ln \left[ A + \frac{\sqrt{A} B}{b} \, e^{br^2/2} \right].
\]

To close the solution, one must find the scalar field \(\Phi(r)\) and the electric field intensity \(E(r)\). Following \([60]\), \(\Phi(r) = a e^{\beta \xi(r)}\) and using \([26]\) electric field intensity can be determined as below:

\[
\frac{4q^2}{r^4} = \frac{\alpha^2 m^2}{b^{4\beta}} \left( \sqrt{a} Be^{br^2} + Ab \right)^{4\beta} + \frac{b^2 (1 - \beta)}{(abr^2 e^{br^2} + b)^2} \left( \sqrt{a} Be^{br^2} + Ab \right)^{2(\beta - 1)}
\]

\[
2 a \alpha A^2 b^2 e^{br^2} \left\{ r^2 \left( 2a e^{br^2} + b \right) + 3 \right\} + 2 a \sqrt{a} Ab e^{br^2} f_1(r) + 2 a \alpha B^2 e^{br^2} f_2(r),
\]

where,

\[
f_1(r) = 4a^2 r^2 e^{2br^2} - a e^{br^2} (b(8\beta + 1)r^2 - 6 - b(b^2 + 9) + 3)
\]

\[
f_2(r) = 2a^2 r^2 e^{2br^2} + a e^{br^2} \left[ 2br^2 (\beta \omega - 1 - 4) - 11 + 3 \right]
\]

\[+ b \left[ \beta (b^2 (2\beta \omega - 1) - 3) - 9 \right].\]
Now the physical quantities are found to be

\[ 8\pi \rho = \frac{3\sqrt{8\alpha} b^{2(1-\beta)} \sqrt{\frac{b^2}{a}}}{2 (a r^2 e^{b r^2} + b^2)} \left( \sqrt{\alpha} B \rho^2 + A b \right)^{2(\beta-1)} \left[ -a^{3/2} B^2 \left( 2b r^2 \beta \omega + \beta - 1 \right) - 1 \right] e^{\frac{a r}{b}} + \sqrt{\alpha} B \rho^2 \left[ A^2 b (b r^2 + 1) - B^2 \left( \beta r^2 [2 \beta (\omega + 1) - 1] + \beta - 1 \right) + a A b e^{b r^2} \right] \]

\( \left( 3 b r^2 + 2 \right) - A b^2 B \left( \beta r^2 + \beta - 1 \right) \].

(40)

\[ p_r = \frac{1}{3} \left[ \rho(r) - 4B \right]. \]

(41)

\[ 8\pi p_t = \frac{a b^{2(1-\beta)} \sqrt{\frac{b^2}{a}}}{2 (a r^2 e^{b r^2} + b^2)} \left( \sqrt{\alpha} B \rho^2 + A b \right)^{2(\beta-1)} \left[ a^{3/2} B^2 (12 \beta - 1) B r^2 e^{\frac{a r}{b}} + a^2 B^2 e^{b r^2} \left[ 2b r^2 \left( \beta (b r^2 [\beta (\omega + 5) + 1] + 6) + 1 \right) - 5 \right] + \sqrt{\alpha} A B e^{b r^2} \left[ \left( b (5 \beta + 2) \right) r^2 + 17 \beta + 7 \right] \right] \frac{-10 a^{3/2} A B e^{b r^2} - 4 \left( a \right)^4 A B r^2 e^{b r^2} - 2 a^2 A^2 b^2 r^2 e^{b r^2} - a B r^2 A^2 b (3 b r^2 + 5) - B^2 \left( b r^2 [\beta (2 \beta (\omega + 5) + 7) + 2] + 17 \beta + 7 \right) \right] - 2 a^3 B^2 r^2 e^{3 b r^2} - a m^2 \left( a r^2 e^{b r^2} + 1 \right)^2 \]

\( b^{2 \beta} \].

(42)

Now the anisotropy is defined as \( \Delta = p_t - p_r \) and the rest can be found easily.

### IV. JUNCTION CONDITIONS

To describe the complete structure of the self gravitating anisotropic compact object, the interior spacetime must be matched smoothly with the exterior spacetime at the pressure free boundary \( \Sigma \). The exterior spacetime is considered to be the Reisner-Nordstrom spacetime given by,

\[ ds^2_+ = - \left( 1 - \frac{2 M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 - r^2 (d\theta^2 - \sin^2 \theta d\phi^2) + \left( 1 - \frac{2 M}{r} + \frac{Q^2}{r^2} \right) dt^2, \]

where, \( M \) is the total mass. In order to satisfy the smoothness and continuity of geometry for internal spacetime metric \( ds^2_\Sigma \) and external spacetime \( ds^2_+ \) at the boundary surface \( \Sigma \), the following conditions must be fulfilled at the surface

\[ [ds^2_\Sigma] = [ds^2_+]_\Sigma, \quad [K_{ij-}]_\Sigma = [K_{ij+}]_\Sigma, \]

\[ [\Phi(r)-]_\Sigma = [\Phi(r)+]_\Sigma, \quad [\Psi(r)-]_\Sigma = [\Psi(r)+]_\Sigma. \]

(44)

(45)

Here, as usual \(-\) and \(+\) denotes the interior and exterior spacetime, respectively while \( K_{ij} \) represents the curvature. The condition (44) provides \( g_{rr} = g_{rr}^+ \) and \( g_{tr} = g_{tr}^+ \). Then from Eqs. (13) and (14), we get

\[ e^{-\Phi(R)} = 1 - \frac{2 M}{R} + \frac{Q^2}{r^2}, \quad \text{and} \quad e^{\Phi(R)} = 1 - \frac{2 M}{R} + \frac{Q^2}{r^2}. \]

(46)

On the other hand, the spacetime (15) must satisfy the second fundamental form at the surface \( \Sigma \). The continuity of the second fundamental form \( (K_{ij}) \) yields \( [G_{ij} r^j]_\Sigma = 0 \), where \( r^j \) denotes a unit radial vector. Then the field equations together with boundary condition (17) gives,

\[ [T_{ij} r^j]_\Sigma = 0 \implies p_r(R) = 0. \]

(47)
By employing the relationship $e^{-\eta(R)} = 1 - \frac{2M}{R} + \frac{\ell^2}{R^2}$, we get the global mass of the celestial structure as follows

$$M = \frac{aR^3e^{\beta R^2}}{2(1 + aR^2e^{\beta R^2})} + \frac{R^3}{8} \left\{ \frac{\alpha^2m_0^2}{b^4} \left( \sqrt{aBe} \frac{bR^2}{R^2} + Ab \right)^{4\beta} \frac{b^2(1-\beta)}{\left( abR^2e^{\beta R^2} + b \right)^2} \right\}$$

$$\left[ 2\alpha A^2\beta e^{\beta R^2} \left( R^2 \left( 2ae^{\beta R^2} + b \right) + 3 \right) + 2\alpha \sqrt{a} Ab Be^{\frac{1+\beta}{2}} f_1(R) + 2\alpha aB^2e^{\beta R^2} f_2(R) \right].$$

(48)

with

$$f_1(R) = 4a^2R^2e^{2\beta R^2} - ae^{\beta R^2} \left( b(8\beta + 1)R^2 - 6 \right) - b \left( \beta (bR^2 + 9) + 3 \right)$$

$$f_2(R) = 2a^2R^2e^{2\beta R^2} + ae^{\beta R^2} \left[ 2bR^2 \left( \beta \left( bR^2 \left( \beta (\omega - 1) - 1 \right) - 4 \right) - 1 \right) + 3 \right]$$

$$+ b \left[ \beta (bR^2 \left( 2\beta (\omega - 1) - 3 \right) - 9) - 3 \right].$$

Thus, by matching of the gravitational potential functions $e^{\xi(r)}$ and $e^{-\eta(r)}$ at the boundary of the compact celestial body $r = R$, we immediately get the constant $A$ as,

$$A = \frac{1}{\sqrt{1 + aR^2e^{\beta R^2}}} - \frac{\sqrt{6} Be^{\frac{1+\beta}{2}}}{b}.$$  

(49)

whereas by using the second fundamental form expressed in Eq. (17), we will calculate numerically the value of the constant $B$.

V. PHYSICAL VIABILITY OF THE MODEL

A. Central values of the physical parameters

In the interior of the compact star, the central values of all the physical parameters must be finite and non-singular. To strengthen the physical validity of the solution to show its finiteness at $r = 0$. First of all, we find the central values for both metric functions at the center $r = 0$ as: $e^{\eta(0)} = 1$ and $\xi(r) = 2\ln[A + \sqrt{aB/b}]$. Furthermore, the values for density and pressures at the center $r = 0$ is,

$$\rho(0) = B + \frac{3\alpha \sqrt{\alpha} (\sqrt{\alpha} Ab + aB - b(\beta - 1)B)}{16\pi (\sqrt{\alpha} B + Ab)} \left\{ \frac{(\sqrt{\alpha} B + Ab)^2}{b^2} \right\}^\beta$$

(50)

$$p_r(0) = \frac{\sqrt{\alpha} \alpha (\sqrt{\alpha} Ab + aB - b(\beta - 1)B)}{16\pi (\sqrt{\alpha} B + Ab)} \left\{ \frac{(\sqrt{\alpha} B + Ab)^2}{b^2} \right\}^\beta - B$$

(51)

$$p_t(0) = \frac{\alpha}{16\pi (\sqrt{\alpha} B + Ab)} \left\{ \frac{(\sqrt{\alpha} B + Ab)^2}{b^2} \right\}^\beta - \alpha m_0^2 (\sqrt{\alpha} B + Ab) \left\{ \frac{(\sqrt{\alpha} B + Ab)^2}{b^2} \right\}^\beta$$

$$- 5a^{3/2}B - 5aAb + \sqrt{ab(17\beta + 7)B} \right\} + B.$$

(52)

Here we can observe that $p_r(0) \neq p_t(0)$ which leads $\Delta(0) \neq 0$. This situation may create certain problems in stellar modeling as the TOV will be unable to balance at the center. Therefore, in order to nullify the anisotropy at $r = 0$, we must have $p_r(0) = p_t(0)$, which leads to

$$B = \frac{\alpha}{32\pi (\sqrt{\alpha} B + Ab)} \left\{ \frac{(\sqrt{\alpha} B + Ab)^2}{b^2} \right\}^\beta \left\{ 6a^{3/2}B + 6am_0^2 (\sqrt{\alpha} B + Ab) \left\{ \frac{(\sqrt{\alpha} B + Ab)^2}{b^2} \right\}^\beta \right\}$$

$$+ 6aAb - 6\sqrt{ab(3\beta + 1)B} \right\}.$$

(53)
Now the central value of $E^2$ is given by

$$E^2(0) = \frac{1}{4} \left\{ \alpha^2 m_\phi^2 \left( \frac{(\sqrt{a}B + Ab)^2}{b^2} \right)^{2\beta} - 32\pi B + \left( \frac{(\sqrt{a}B + Ab)^2}{b^2} \right)^\beta \frac{6\sqrt{a} \alpha E_1}{\sqrt{a}B + Ab} \right\}. \quad (54)$$

where, $E_1 = [\sqrt{a}Ab + aB - b(3\beta + 1)B]$. As we can see from above Eq. (54) that $E^2(0)$ is not zero explicitly. However, incorporating Eq. (53) in Eq. (54), we will get $E^2(0) = 0$. Now we will discuss the regularity of these physical parameters at the centre $r = 0$.

**B. Regularity**

(i) **Metric functions at the centre, $r = 0$:** we observe from Fig 1 (left panel) that the metric functions at the centre $r = 0$ assume finite values and are smooth and continuous throughout the interior of the stellar configuration. We conclude that the metric functions are free from singularity and positive at the centre.

(ii) **The density as a function as the radial coordinate is displayed in Fig. 1 (right panel).** We observe that the density assumes a finite value at the centre and decreases monotonically towards the stellar surface.

(iii) **Pressure at the centre $r = 0$:** Fig. 2 (left panel) shows us that both the radial and tangential pressure are regular at the centre of the star and decrease smoothly towards the boundary.

(iv) **The anisotropy parameter is presented in Fig. 2 (right panel) and it is clear that $\Delta > 0$ at each interior point of the stellar configuration.** The anisotropy parameter vanishes at the center of the star and increases to a maximum for some finite radius, $r < \Sigma$ where $r_\Sigma$ denotes the boundary of the star. A positive value for $\Delta$ ($p_r > p_t$) signifies a repulsive force due to anisotropy. It is clear that the increase in the anisotropy parameter especially towards the surface layers lead to greater stability in these regions.

![Figure 1: The metric potentials and matter density are plotted against $r$ by taking $a = 0.006083/km^2$, $A = -0.15398$, $m_\phi = 0.002$, $B = 0.06$, $\omega_{BD} = 5$, $b = 0.00785/km^2$ and $\beta = -0.2$ for PSR J1903+327.](image)

![Figure 2:](image)

**C. Equation of state**

The role of the equation of state (EoS) has been demonstrated in many models of compact objects within the framework of classical general relativity and modified theories of gravitation. A barotropic EoS of the form $p = p(\rho)$ points strongly to the type of matter making up the star. Recently, the colour-flavoured locked-in EoS was utilised to model compact objects. This particular EoS is a generalisation of the MIT bag model and attempts to connect the microphysics to macrophysics of the fluid configuration. Fig. 3 (left panel) shows the variation of the ratio $p/\rho$ with
FIG. 2: The pressure and anisotropy are plotted against $r$ by taking $a = 0.006083/km^2$, $A = -0.15398$, $m_\phi = 0.002$, $B = 0.06$, $\omega_{BD} = 5$, $b = 0.00785/km^2$ and $\beta = -0.2$ for PSR J1903+327.

$r/R$. We note that the pressure is less than the density at each interior point of the configuration. This ratio is also positive everywhere inside the star.

D. Electric field

The trend in the electric field is shown in Fig. 3 (right panel). We observe that the electric field vanishes at the centre of the star and increases monotonically towards the surface. It is well-known that intense electric fields can lead to instabilities within the stellar core. The presence of charge as high as $10^{20}$ coulombs can generate quasi-static equilibrium states. These high charge densities are linked to very intense electric fields which in turn induce pair production within the star thus leading to an unstable core.

FIG. 3: The $p/\rho$ and electric field intensity are plotted against $r$ by taking $a = 0.006083/km^2$, $A = -0.15398$, $m_\phi = 0.002$, $B = 0.06$, $\omega_{BD} = 5$, $b = 0.00785/km^2$ and $\beta = -0.2$ for PSR J1903+327.
E. Energy conditions

In order for physical admissibility of our models the solution should satisfy the following energy conditions, viz., (i) null energy condition (NEC), (ii) weak energy condition (WEC) and (iii) strong energy condition (SEC). In order to satisfy the above energy conditions, the following inequalities must be hold simultaneously at each interior point of the charged fluid sphere:

\[ \text{NEC} : \rho + \frac{E^2}{8\pi} \geq 0, \quad \text{WEC} : \rho + p_i + \frac{E^2}{8\pi} \geq 0, \quad \text{SEC} : \rho + \sum p_i + \frac{E^2}{4\pi} \geq 0. \] (55)

It is clear from Fig. 4 (left panel) that all three energy conditions are satisfied at each interior point of the configuration.

F. TOV equation

It is well known that in the absence of any dissipative effects such as heat flow the equilibrium of a gravitationally bounded charged fluid configuration is characterised by the resultant of the gravitational force, \( F_g \), the hydrostatic force, \( F_h \) and the force due to anisotropy, \( F_a \) and the electrostatic interaction, \( F_e \) vanishing at each interior point of the star. The modified TOV equation in Brans-Dicke gravity is given by

\[ -p_r' - \frac{\xi'}{2} (p_r + \rho) + \frac{2}{r} (p_t - p_r) + \left( T^{1\Phi}_1 \right)' - \frac{\xi'}{2} \left( T^{1\Phi}_1 - T^{0\Phi}_0 \right) + \frac{2}{r} \left( T^{1\Phi}_1 - T^{2\Phi}_2 \right) + \frac{qq'}{4\pi r^4} = 0, \] (56)

where,

\[ F^{BD}_h = -p_r' + \left( T^{1\Phi}_1 \right)'; \quad F^{BD}_g = -\frac{\xi'}{2} (p_r + \rho + T^{1\Phi}_1 - T^{0\Phi}_0); \]
\[ F^{BD}_a = \frac{2}{r} (p_t - p_r + T^{1\Phi}_1 - T^{2\Phi}_2); \quad F^{BD}_e = \frac{qq'}{4\pi r^4}. \] (57)

We note the contribution of the scalar field to the hydrostatic, gravitational and anisotropic forces respectively. In a recent study, Herrera [66] pointed out an interesting observation regarding the nonappearance of the tangential pressure in the gravitational force term. In the case of \( p_t > p_r \), anisotropic spheres are more compact than their isotropic counterparts. In Fig. 4 (right panel), we can see that the combined forces of electric, hydrostatic and anisotropic counter-balanced the gravity so that the configuration is under equilibrium. It can also be seen that when increasing the scalar field contribution (by increasing the scalar parameter \( \alpha \)) all the forces also increases. This ability can make the system to support more masses i.e. the equation of state will be stiffened.

G. Causality

In order to prevent superluminal speeds within the stellar fluid we require that the speed of sound be less than the speed of light everywhere inside the star. The speed of sound for the charged fluid sphere should be monotonically decreasing from centre to the boundary of the star \( (v = \sqrt{dp/d\rho} < 1) \). It is clear from Fig. 5 (left panel) that speed of sound is less than 1 throughout the interior of the matter distribution. This implies that our fluid model fulfills causality requirements.

H. Stability factor

We observe from Fig. 5 (right panel) that our model satisfies the Herrera cracking condition \(-1 < v_t^2 - v_r^2 < 0\). Abreu et al. [67] showed that stable and unstable patches can arise within the stellar fluid and their existence depends on the relative sound speeds in the radial and tangential directions. In particular, potential unstable regions occur when the tangential component of the speed of sound exceeds the radial component.
FIG. 4: The energy conditions and TOV-equation are plotted against $r/R$ by taking $a = 0.006083/km^2$, $A = -0.15398$, $m_\phi = 0.002$, $B = 0.06$, $\omega_{BD} = 5$, $b = 0.00785/km^2$ and $\beta = -0.2$ for PSR J1903+327.

FIG. 5: The speed of sound and stability factor are plotted against $r$ by taking $a = 0.006083/km^2$, $A = -0.15398$, $m_\phi = 0.002$, $B = 0.06$, $\omega_{BD} = 5$, $b = 0.00785/km^2$ and $\beta = -0.2$ for PSR J1903+327.

I. Stability through adiabatic index

Within the Newtonian formalism of gravitation, it is also well known that there has no upper mass limit if the EoS has an adiabatic index $\Gamma > 4/3$ where

$$\Gamma = \frac{p + \rho}{\rho} \frac{dp}{d\rho}$$

the definition of which arises from an assumption within the Harrison-Wheeler formalism [68]. A perturbative study of dissipative collapse by Chan et al. [69] in which gravitational collapse proceeds from an initially static configuration Eq. (58) follows from the EoS of the unperturbed, static matter distribution. Eq. (58) is modified in the presence of anisotropic fluids (radial and transverse stresses are unequal) and we can write

$$\Gamma > \frac{4}{3} - \left[ \frac{4}{3} \frac{p_r - p_t}{\rho p_r} \right]_{\text{max}}$$

It is well-known that a bounded charged configuration can be treated as an anisotropic system. In the special case of isotropic pressure ($p_r = p_t$) the classical Newtonian result holds from Eq. (59). Observations of Eq. (59) indicate
that instability is increased when \( p_r < p_t \) and decreases when \( p_r > p_t \). Fig. 6 (left panel) confirms that our model is stable against radial perturbations at each interior point within the stellar fluid.

### J. Stability through Mass-central density \((M - \rho_c)\) curve

Now, we focus on the \( M - \rho_c \) function dubbed as static stability criterion which is a noteworthy thermodynamically quantity in order to give more insight into the stability of the compact celestial structure. This static stability criterion has been developed and made more accessible by Harrison and co-workers \[75\] and Zeldovich & Novikov \[76\] after suggestions by Chandrashekhar \[77\] in order to portray the stability of gaseous celestial configuration according to radial pulsations. In this respect, the formula associated between the gravitational mass \( M \) and the central density \( \rho_c \) is given as follows,

\[
\frac{\partial M(\rho_c)}{\partial \rho_c} > 0,
\]

which must be satisfied in order to describe the solutions of static and stable celestial configurations. Or otherwise unstable if

\[
\frac{\partial M(\rho_c)}{\partial \rho_c} < 0,
\]

under radial perturbation. We present the variation of the gravitational mass \( M \) with respect to central density \( \rho_c \) in Fig. 6 (right panel). It shows that in the present study the gravitational mass \( M \) is an increasing function with regard to central density for the several different parametric values of \( \alpha \), viz., \( \alpha = 0.5, 0.55, 0.6, 0.65 \), by tuning \( \omega_{BD} \) to 5. This confirms the static stability criterion of the stellar system against radial perturbations. We can see that the solution takes its stability with an increase of all different parametric values of \( \alpha \) and we found also that the celestial bodies become more massive according to increasing central density.

#### FIG. 6: The adiabatic index and mass-central density curves are plotted against \( r/R \) by taking \( \alpha = 0.006083/km^2 \), \( A = -0.15398 \), \( m_\phi = 0.002 \), \( B = 0.06 \), \( \omega_{BD} = 5 \), \( b = 0.0785/km^2 \) and \( \beta = -0.2 \) for PSR J1903+327.

### K. Effective mass and compactness parameter for the charged compact star

As a starting point we recall that the maximal absolute limit of mass-to-radius \((M/R)\) ratio for a static spherically symmetric isotropic fluid model is given by \( 2M/R \leq 8/9 \) \[61\]. In the case of charged fluid spheres \[62\] showed that there exists a lower bound for the mass-radius ratio

\[
\frac{Q^2 (18R^2 + Q^2)}{2R^2 (12R^2 + Q^2)} \leq \frac{M}{R},
\]
for the constraint $Q < M$.

However this upper bound of the mass-radius ratio for charged compact star was generalized by [63] who proved that

$$\frac{M}{R} \leq \left[ \frac{2R^2 + 3Q^2}{9R^2} + \frac{2}{9R} \sqrt{R^2 + 3Q^2} \right]. \quad (63)$$

The Eqs. (62) and (63) imply that

$$\frac{Q^2 (18R^2 + Q^2)}{2R^2 (12R^2 + Q^2)} \leq \frac{M}{R} \leq \left[ \frac{2R^2 + 3Q^2}{9R^2} + \frac{2}{9R} \sqrt{R^2 + 3Q^2} \right]. \quad (64)$$

The effective mass of the charged fluid sphere can be determined as:

$$m_{\text{eff}} = 4\pi \int_0^R \left( \rho + \frac{E^2}{8\pi} \right) r^2 \, dr = \frac{R}{2} \left[ 1 - e^{-\eta(R)} \right] \quad (65)$$

where $e^{-\eta}$ is given by the equation (37) and compactness $u(r)$ is defined as:

$$u(R) = \frac{m_{\text{eff}}(R)}{R} = \frac{1}{2} \left[ 1 - e^{-\eta(R)} \right] \quad (66)$$

L. Redshift

The maximum possible surface redshift for a bounded configuration with isotropic pressure is $Z_s = 4.77$. In the work of Bowers and Liang they showed that this upper bound can be exceeded when the radial and transverse pressures are different [64]. In particular, when $\Delta > 0$ ($p_r > p_t$) the surface redshift is greater than its isotropic counterpart. Studies show that for strange quark stars the surface redshift is higher in low mass stars with the difference being as high as 30% for a 0.5 solar mass star and 15% for a 1.4 solar mass star. It appears that higher redshift predictions in low mass stars appear to be an anomaly. In a recent study Chandras et al. [65] have used gravitational redshift measurements to determine the mass-radius ratio of white dwarfs. Using data of over three thousand catalogue white dwarfs they were able to determine the mass-radius relation over a wide range of stellar masses. Their improved technique entailed the cancelling of random Doppler shifts by averaging out the apparent radial velocities of white dwarfs with similar radii enabling them to measure the associated gravitational redshift.

The gravitational surface red-shift ($Z_s$) is given as:

$$Z_s = (1 - 2u)^{-1/2} - 1, \quad (67)$$

From Eq. (67), we note that the surface redshift depends upon the compactness $u$, which implies that the surface redshift for any star cannot be arbitrarily large because compactness $u$ satisfies the Buchdhal maximal allowable mass-radius ratio. However, the value for the surface redshift for the different compact objects have been calculated as follows: (i) 0.243 for PSR J1903+32, (ii) 0.217 for Cen X-3, (iii) 0.190 for EXO 1785-248, (iv) 0.155 for LMC X-4. The graphical behavior for gravitational redshift for PSR J1903+32 is shown by Fig. (7).

| TABLE I: Comparative study of lower bound, Mass-radius ratio, upper bound, compactness ($u = M_{\text{eff}}/R$) and surface red-shift of the star for fix values of $\alpha = 0.5$ and $\omega_{BD} = 5$. |
|---|---|---|---|---|
| Objects | $Q(R) \times 10^{19} \text{C}$ | Lower bound | Mass-radius ratio ($M/R$) | Upper bound $Z_s$ |
| PSR J1903+32 | 4.655 | 0.134 | 0.351 | 0.558 | 0.243 |
| Cen X-3 | 4.155 | 0.113 | 0.314 | 0.541 | 0.217 |
| EXO 1785-248 | 3.617 | 0.093 | 0.279 | 0.524 | 0.190 |
| LMC X-4 | 2.925 | 0.069 | 0.231 | 0.504 | 0.155 |
FIG. 7: The gravitational redshift profile is plotted against \( r/R \) for \( A = -0.15398, \ m_\phi = 0.002, \ B = 0.06, \ \omega_{BD} = 5, \ b = 0.00785/\text{km}^2 \) and \( \beta = -0.2 \) for different values of \( \alpha \) for PSR J1903+327.

TABLE II: Comparative study of lower bound, Mass-radius ratio, upper bound, compactness \((u = M_{\text{eff}}/R)\) and surface red-shift of the star for fix values of \( \alpha = 0.65 \) and \( \omega_{BD} = 5 \).

| Objects       | \( Q(R) \) \times 10^{19} \text{C} | Lower bound \( \frac{Q^2}{2R^2(18R^2+Q^2)} \) | Mass-radius \( \frac{M/R}{2R^2+3Q^2+2R\sqrt{R^2+3Q^2}} \) | Upper bound \( \frac{M/R}{9R^2} \) | \( z_s \) |
|---------------|-----------------------------------|---------------------------------|---------------------------------|---------------------------------|--------|
| PSR J1903+327 | 5.31                              | 0.175                           | 0.378                           | 0.590                           | 0.243  |
| Cen X-3       | 4.737                             | 0.147                           | 0.338                           | 0.568                           | 0.217  |
| EXO 1785-248  | 4.124                             | 0.120                           | 0.297                           | 0.546                           | 0.190  |
| LMC X-4       | 3.335                             | 0.089                           | 0.245                           | 0.521                           | 0.155  |

M. The effect of scalar field parameter \( \alpha \), and BD-parameter \( \omega_{BD} \) on the \( M - R \) and \( M - I \) curves

In this section, we examine the \( M - R \) and \( M - I \) diagrams resulted from our stellar model in the background of BD gravity with a massive field via the embedding approach. In this respect, we provide an instructive explanation of the influences included by the choices made on different parameters viz., the parameter \( \alpha \), BD-parameter \( \omega_{BD} \) and the total external bag pressure \( B \) (or bag constant), in order to give a more achievable scenario and efficient astrophysical stellar system. On the other hand, for determining the stiffness of an EoS, we can analyze the moment of inertia \( I \) associating with a static celestial solution which could give a precise instrument via adopting the Bejger & Haensel concept \[74\], given by,

\[
I = \frac{2}{5} \left( 1 + \frac{(M/R) \cdot \text{km}}{M_\odot} \right) MR^2.
\]  

(68)

Our survey on \( M - R \) and \( M - I \) curves is highly significant for the stellar systems which clearly show the state (more or less) of compact celestial bodies via the maximum bound of the total mass as well as the efficacy and the sensitivity

TABLE III: Comparative study of lower bound, Mass-radius ratio, upper bound, compactness \((u = M_{\text{eff}}/R)\) and surface red-shift of the star for fix values of \( \alpha = 0.5 \) and \( \omega_{BD} = 20 \).

| Objects       | \( Q(R) \) \times 10^{19} \text{C} | Lower bound \( \frac{Q^2}{2R^2(18R^2+Q^2)} \) | Mass-radius \( \frac{M/R}{2R^2+3Q^2+2R\sqrt{R^2+3Q^2}} \) | Upper bound \( \frac{M/R}{9R^2} \) | \( z_s \) |
|---------------|-----------------------------------|---------------------------------|---------------------------------|---------------------------------|--------|
| PSR J1903+327 | 6.024                             | 0.224                           | 0.412                           | 0.629                           | 0.243  |
| Cen X-3       | 5.303                             | 0.184                           | 0.363                           | 0.598                           | 0.217  |
| EXO 1785-248  | 4.557                             | 0.147                           | 0.315                           | 0.568                           | 0.190  |
| LMC X-4       | 3.608                             | 0.105                           | 0.255                           | 0.534                           | 0.155  |
to the stiffness of an EoS. In this regard, from Figs. [S] and [D] we show the variation of the total mass \( M \) in \([M_\odot]\) versus the total radial coordinate \( R \) in [km] and the maximum moment of inertia \( I \) in \([10^{45} g - \text{cm}^2]\), for all chosen values of the parameters \( \alpha \), \( \omega_{BD} \) and \( B \). In the present BD gravity stellar model via the embedding approach, we obtain from \( M - R \) curves featured in Fig. [S] (left panel) that due to \( \alpha \) and \( B \) by setting \( \omega_{BD} \) to 5, as both parameters \( \alpha \) and \( B \) increased, the most extreme value of mass \( M \) increases with the increasing total radial coordinate \( R \), which generates us with more massive compact celestial bodies. Moreover, from the \( M - R \) curves illustrated in Fig. [S] (left panel) that due to \( \omega_{BD} \) by fixing \( \alpha \) to 0.5 (Solid) with \( B = 57.82 MeV/fm^3 \) and 0.65 (Dashed) with \( B = 75.17 MeV/fm^3 \) respectively, as if \( \omega_{BD} \) is increasing, then the corresponding radius \( R \) decreases and the most extreme value of mass \( M \) also decreases, which gives us also a celestial system less compact and less massive when \( \omega_{BD} \) increases corresponding to fix \( \alpha = 0.50 \) with \( B = 57.82 MeV/fm^3 \) and \( \alpha = 0.65 \) with \( B = 75.17 MeV/fm^3 \), respectively. This shows that the parameters \( \alpha \), \( M \) and \( \omega_{BD} \) will affect the maximum mass limit as well as compactness of the objects. On the other hand, the variation of the maximum moment of inertia \( I \) with respect to the total mass \( M \) due to the impact of \( \alpha \), \( B \) and \( \omega_{BD} \) has been featured in Figs. [S] (right panel) and [D] (right panel). From these plots, we can see that the maximum moment of inertia \( I \) is always increasing with increasing the mass until up to the most extreme value of mass and decreasing rapidly with decreasing the mass. Consequently, we can infer that the stiffness of EoS is better in the case where \( \alpha = 0.65 \) and \( \omega_{BD} = 5 \) with respect to all other cases, i.e., when \( \omega_{BD} = 10, \ 15 \), \( \alpha = 0.5, \ 0.55, \ 0.6, \ 0.65 \). It is worth mentioning here that the Bag constant \( B \) is changing with only \( \alpha \) therefore \( B \) will also feature in the \( M - I \) curves. Finally, we would like to mention here that we have discovered a good agreement with observational data for four compact celestial objects namely, PSR J1903+327, Cen X-3, EXO 1785-248, LMC X-4 in our resulting \( M - R \) and \( M - I \) curves. It is clear how all the parameters introduced by the BD gravity stellar model with the massive field via the embedding approach have a large effect on the various physical parameters of the celestial configuration.

\[ \text{FIG. 8: The } M - R \text{ and } M - I \text{ curves are plotted against } r \text{ by taking } a = 0.006083/km^2, \ A = -0.15398, \ m_\odot = 0.002, \ B = 0.06, \ \omega_{BD} = 5, \ b = 0.00785/km^2 \text{ and } \beta = -0.2 \text{ for different values of } \alpha. \]

| Objects          | Predicted \( R \) km | \( I \times 10^{45} \ g \ - \text{cm}^2 \) |
|------------------|-----------------------|------------------------------------------|
|                  | \( \alpha \)          | \( \alpha \)                               |
| PSR J1903+327    | 1.667                 | 0.5 0.55 0.6 0.65                         |
|                  |                        | 0.5 0.55 0.6 0.65                        |
| Cen X-3          | 1.49                  | 8.62 9.22 9.73 10.19                     |
|                  |                        | 1.12 1.27 1.41 1.54                      |
| EXO 1785-248     | 1.3                   | 8.07 8.91 8.99 9.34                     |
|                  |                        | 0.75 0.83 0.90 0.99                      |
| LMC X-4          | 1.04                  | 7.50 7.91 8.30 8.63                     |
|                  |                        | 0.51 0.57 0.62 0.67                      |

TABLE IV: Predicted radii and MI for some compact stars for different values of \( \alpha \) with \( \omega = 5 \) correspond to Fig. [S]
leads to higher densities. This observation supports the fact that the radial and tangential stresses also increase as \( \alpha \) increases. The anisotropy parameter is also strengthened in the presence of larger scalar fields. Since \( \Delta > 0 \) throughout the stellar configuration, the repulsive force due to anisotropy helps stabilise the more compact configurations. An interesting observation is the increase in electric field intensity with an increase in scalar field intensity. Although these fields emanate from totally different sources there appears to be a ‘coupling’ which manifests in the formation of more compact stellar configurations. It has been shown that higher order gravity theories predict more compact objects compared to their 4D counterparts. The compactification is attributed to higher dimensional effects rather

\[
M - R \quad \text{and} \quad M - I \quad \text{curves are plotted against } r \quad \text{by taking} \quad a = 0.006083/\text{km}^2, \quad A = -0.15398, \quad m_\phi = 0.002, \quad B = 0.06, \quad \alpha = 0.5 (\text{solid}) \quad \text{and} \quad 0.65 (\text{dashed}), \quad b = 0.00785/\text{km}^2 \quad \text{and} \quad \beta = -0.2 \quad \text{for different values of } \omega_{BD}.
\]

**TABLE V: Effects of \( \alpha \) and \( \omega_{BD} \) on mass, radius and moment of inertia corresponds to Fig. 8**

| \( \alpha \) | \( M_{\text{max}}/M_\odot \) | \( R \) km | \( I \times 10^{45} \text{ g cm}^{-2} \) | \( B \) MeV / fm\(^3\) |
|---|---|---|---|---|
| 0.50 | 1.90 | 8.45 | 1.27 | 57.82 |
| 0.55 | 2.14 | 9.29 | 1.75 | 63.61 |
| 0.60 | 2.37 | 10.14 | 2.33 | 69.39 |
| 0.65 | 2.62 | 11.02 | 3.02 | 75.17 |

**VI. DISCUSSION AND CONCLUSION**

It is clear from the graphical analyses of our solution that the model of charged anisotropic strange star within the framework of Brans-Dicke gravity with a massive scalar field describes realistic stellar objects. The stellar model presented here obeys all the conditions required for hydrostatic equilibrium, stability and causality. Of particular interest is the contribution of the scalar field to the thermodynamical and gravitational properties of the stellar model. In Fig. 1 (right panel) we observe that an increase in \( \alpha \) which corresponds to a larger scalar field intensity leads to higher densities. This observation supports the fact that the radial and tangential stresses also increase as \( \alpha \) increases. The anisotropy parameter is also strengthened in the presence of larger scalar fields. Since \( \Delta > 0 \) throughout the stellar configuration, the repulsive force due to anisotropy helps stabilise the more compact configurations. An interesting observation is the increase in electric field intensity with an increase in scalar field intensity. Although these fields emanate from totally different sources there appears to be a ‘coupling’ which manifests in the formation of more compact stellar configurations. It has been shown that higher order gravity theories predict more compact objects compared to their 4D counterparts. The compactification is attributed to higher dimensional effects rather

**TABLE VI: Effects of \( \alpha \) and \( \omega_{BD} \) on mass, radius and moment of inertia corresponds to Fig. 9**

| \( \alpha \) | \( \omega_{BD} \) | \( M_{\text{max}}/M_\odot \) | \( R \) km | \( I \times 10^{45} \text{ g cm}^{-2} \) | \( B \) MeV / fm\(^3\) |
|---|---|---|---|---|---|
| 0.50 | 5 | 1.90 | 8.45 | 1.27 | 57.82 |
| 10 | 1.56 | 6.98 | 0.73 | |
| 15 | 1.24 | 5.84 | 0.39 | |
| 20 | 0.945 | 4.90 | 0.21 | |
| 0.65 | 5 | 2.63 | 11 | 2.96 | 75.17 |
| 10 | 2.20 | 9.11 | 1.77 | |
| 15 | 1.77 | 7.53 | 0.97 | |
| 20 | 1.36 | 6.44 | 0.54 | |
than exotic matter states such as dark matter. Our model provides an alternative mechanism for the existence of more compact objects than their classical relativistic counterparts. Recent studies of observational data obtained via gravitational redshifts have determined the the mass-radius relation of white dwarfs to a higher degree of accuracy. These results help constrain the equation of state of these compact objects thus giving us an handle on the matter composition and microphysics at play within the stellar fluid. Tables 1-3 display the upper bound and lower bound limits imposed by the modified Buchdahl limit for charged compact objects in Brans-Dicke theory.

In Tables 1 -3, we have generated values for the surface charge for well-known compact objects PSR J1903+327; Cen X-3; EXO 1785-248 & LMC X-4 when \( \alpha = 0.5 \) and \( \alpha = 0.65 \) respectively with \( \omega_{BD} = 5 \). It is clear from the data that an increase in \( \alpha \) is accompanied by an increase in the surface charge as well as surface charge density. The increase in surface charge density is higher in more compact objects. This is expected as the charge contributes to the overall mass of the stellar body. We observe that the upper and lower bounds arising from the Buchdahl limit are modified by a change in \( \alpha \). Tables 1-3, show that a decrease in compactness of approximately 30\% (from PSR J1903+327 to LMC X-4) is accompanied by a change in the lower bound as high as 50\% for both \( \alpha = 0.5 \) and \( \alpha = 0.65 \) while the upper bound changes by approximately 10\% and 12\% respectively for the two values of \( \alpha \) displayed here. In Table 3, we present surface charge densities and the Buchdahl limit for \( \alpha = 0.5 \) and \( \omega_{BD} = 20 \). A comparison of Tables 2 and 3 show the contributions attributed to the Brans-Dicke modification to the classical Einstein gravity theory, i.e., a change in \( \omega_{BD} \). It is clear that surface charge density, lower and upper bounds imposed by the Buchdahl limits are all affected by an increase in the Brans-Dicke coupling constant. We observe that a decrease in compactness of the stellar object of approximately 30\% is accompanied by a 54\% decrease in the lower bound and a corresponding decrease of approximately 16\% in the upper bound associated with the Buchdahl limit when \( \omega_{BD} \) is decreased by 75\%. Let us now turn our attention to the surface redshift for the different parameter sets displayed in Tables 1-3. It is clear that the surface redshift decreases as the compactness decreases. We observe that surface redshift values obtained in Tables 1-3 for the stellar objects displayed here are consistent with the acceptable upper bound for for relativistic stars \( (Z < 5.211) \) [70]. Another well-known characteristic named as the static stability criterion or \( M - \rho_c \) function plays a crucial role in ensuring the stability of spherically symmetric static celestial systems under radial pulsation has been well-satisfied. We can also notice from the data drawn in \( M - \rho_c \) curve that the celestial configurations become more massive according to increasing central density.

Further, we tested the state of the compact celestial bodies as well as the efficacy and the sensitivity to the stiffness of an EoS by studying the \( M - R \) and \( M - I \) diagrams generated from our celestial model. From \( M - R \) curves that due to \( \alpha \), \( \omega_{BD} \) and \( B \) we envisaged two cases. The first case corresponds to \( \alpha \) and \( B \) by setting \( \omega_{BD} \) to 5, one can see that when both parameters \( \alpha \) and \( B \) increases, the maximum value of mass \( M \) increases with the increasing radius \( R \), which produces more massive compact celestial bodies. The second case corresponds to \( \omega_{BD} \) by setting \( (\alpha, B) = (0.5, 57.82) \) & \( (0.65, 75.17) \), we can observe that when \( \omega_{BD} \) increases, the maximum value of mass \( M \) and corresponding \( R \) decreases, which gives us a celestial system less compact and less massive for increasing with fix \( \alpha \) and \( B \). Moreover, from \( M - I \) curves, we can see that the maximum moment of inertia \( I \) is always increasing with an increase in the mass until up to the maximum value of mass and decreasing rapidly with decreasing the mass under the effect of \( \alpha \), \( \omega_{BD} \) and \( B \). In this respect, we can conclude that the stiffness of EoS is better in the case of \( \alpha = 0.65 \) and \( \omega_{BD} = 5 \) while compared to all other cases, i.e., when \( \omega_{BD} = 10, 15, 20 \) and \( \alpha = 0.5, 0.55, 0.6, 0.65 \). We also found a good agreement with observational data on \( M - R \) and \( M - I \) diagrams for four compact celestial bodies viz., PSR J1903+327, Cen X-3, EXO 1785-248, LMC X-4 and many others can be adapted. The tables (4)-(6) display the values of physical parameters such as maximum mass, radius and moment of inertia corresponding the Fig. [8] and [9] for different values of \( \alpha \), \( \omega_{BD} \), and \( B \). With the above rigorous analyses of the gravitational and thermodynamical behaviour of our solution we ascertain that our model meets the necessary requirements for a physically realizable self-gravitating compact object.

In this study we have generated a model of a compact charged stellar object within the Brans-Dicke gravity framework in the presence of a massive scalar field. In addition, the matter composition of the stellar interior obeys the MIT Bag model equation of state. Our model satisfies all the criteria for a genre of compact objects which include strange stars. The highlight of our work is the interplay between the electric and scalar fields which originate from completely different sources combine to affect physical characteristics of the model. The Brans-Dicke coupling constant also affects stellar characteristics such as compactness, redshift and the bounds required by the modified Buchdahl limit for charged stars. In addition, we observed that \( M - R \) and \( M - I \) curves are sensitive to changes in \( \alpha \), \( \omega_{BD} \) and the Bag constant which in turn points to a change in stiffness of the stellar fluid. We believe that this is a novel feature in our model which inherently connects the macrophysics (scalar field, electric field and BD coupling constant to the microphysics (Bag constant). It would be interesting to compare and contrast our findings.
to phenomenological features derived in higher dimensional gravity theories such as Einstein-Gauss-Bonnet gravity.

[1] R. S. Park, W. M. Folkner, A. S. Konopliv, J. G. Williams, D. E. Smith, and M. T. Zuber, Astron. J., 153, 121 (2017)
[2] The Event Horizon Telescope Collaboration, Astron. J., 875, L1 (2019)
[3] P. Fleury, C. Clarkson, and R. Maartens, JCAP, 2017, 1 (2017)
[4] N. Dadhich, R. Durka, N. Merino, and O. Miskovic, Phys. Rev. D, 93, 064009 (2016)
[5] N. Dadhich and S. Chakraborty, Rev. D, 95, 064059 (2017)
[6] T. Harko, F. S. N. Lobo, S. Nojiri, Shin'Ichi and S. D. Odintsov, Phys. Rev. D, 84, 024020 (2011)
[7] S. Hansraj, A. and P. Channmee, Annals of Physics, 400, 320 (2019)
[8] A. A. Coley and G. F. R. Ellis, Class. Quantum Grav., 875, L1 (2019)
[9] P. Fleury, C. Clarkson, and R. Maartens, JCAP, 2017, 1 (2017)
[10] N. Dadhich, R. Durka, N. Merino, and O. Miskovic, Phys. Rev. D, 93, 064009 (2016)
[11] N. Dadhich and S. Chakraborty, Rev. D, 95, 064059 (2017)
[12] T. Harko, F. S. N. Lobo, S. Nojiri, Shin'Ichi and S. D. Odintsov, Phys. Rev. D, 84, 024020 (2011)
[13] S. Hansraj, A. and P. Channmee, Annals of Physics, 400, 320 (2019)
[14] A. A. Coley and G. F. R. Ellis, Class. Quantum Grav., 875, L1 (2019)
[15] P. Fleury, C. Clarkson, and R. Maartens, JCAP, 2017, 1 (2017)
[16] N. Dadhich, R. Durka, N. Merino, and O. Miskovic, Phys. Rev. D, 93, 064009 (2016)
[17] N. Dadhich and S. Chakraborty, Rev. D, 95, 064059 (2017)
[18] T. Harko, F. S. N. Lobo, S. Nojiri, Shin'Ichi and S. D. Odintsov, Phys. Rev. D, 84, 024020 (2011)
[19] S. Hansraj, A. and P. Channmee, Annals of Physics, 400, 320 (2019)
[20] A. A. Coley and G. F. R. Ellis, Class. Quantum Grav., 875, L1 (2019)
[21] P. Fleury, C. Clarkson, and R. Maartens, JCAP, 2017, 1 (2017)
[22] N. Dadhich, R. Durka, N. Merino, and O. Miskovic, Phys. Rev. D, 93, 064009 (2016)
[23] N. Dadhich and S. Chakraborty, Rev. D, 95, 064059 (2017)
[24] T. Harko, F. S. N. Lobo, S. Nojiri, Shin'Ichi and S. D. Odintsov, Phys. Rev. D, 84, 024020 (2011)
[25] S. Hansraj, A. and P. Channmee, Annals of Physics, 400, 320 (2019)
[26] A. A. Coley and G. F. R. Ellis, Class. Quantum Grav., 875, L1 (2019)
[27] P. Fleury, C. Clarkson, and R. Maartens, JCAP, 2017, 1 (2017)
[28] N. Dadhich, R. Durka, N. Merino, and O. Miskovic, Phys. Rev. D, 93, 064009 (2016)
[29] N. Dadhich and S. Chakraborty, Rev. D, 95, 064059 (2017)
[30] T. Harko, F. S. N. Lobo, S. Nojiri, Shin'Ichi and S. D. Odintsov, Phys. Rev. D, 84, 024020 (2011)
[31] S. Hansraj, A. and P. Channmee, Annals of Physics, 400, 320 (2019)
[32] A. A. Coley and G. F. R. Ellis, Class. Quantum Grav., 875, L1 (2019)
[59] K. Lake, Gen. Relativ. Gravit. 49, 134 (2017).
[60] W. F. Bruckman and E. Kazes, Phys. Rev. D 16, 261 (1977).
[61] H.A. Buchdahl, Phys. Rev. D, 116, 1027 (1959).
[62] C.G. Böhmer and T. Harko, Class. Quantum Gravit. 23, 6479 (2006).
[63] H. Andréasson, Commun. Math. Phys. 288, 715 (2009).
[64] R. L. Bowers and E. P. T. Liang Astrophys. J. 188, 657 (1974).
[65] V. Chandra, H-Chih Hwang, N. L. Zakamska and S. Cheng, ApJ, in press (2020).
[66] L. Herrera, Phys. Rev. D, 101, 104024 (2020).
[67] H. Abreu et al., Class. Quantum Gravity, 24, 4631 (2007).
[68] B. K. Harrison, K. S. Throne, M. Wakano and J. A. Wheeler, Gravitation Theory and Gravitational Collapse (University of Chicago press, 1965).
[69] R. Chan, L. Herrera and N. O. Santos, MNRAS 265, 533 (1993).
[70] B.V. Ivanov, Phys. Rev. D, 65, 104011 (2002).
[71] S.K. Maurya, Y.K. Gupta, S. Ray, Eur. Phys. J. C, 77, 360 (2017).
[72] T.W. Baumgarte, A.D. Rendall, Class. Quantum Gravit., 10, 327 (1993).
[73] M. Mars, M. Merc Martn-Prats, Phys. Lett. A, 218, 147 (1996).
[74] M. Bejger, P. Haensel, A. & A., 396, 917 (2002).
[75] B. K. Harrison et al, Gravitational Theory and Gravitational Collapse, University of Chicago Press, Chicago, (1966).
[76] Y. B. Zeldovich and I. D. Novikov, Relativistic Astrophysics1: Stars and Relativity, University of Chicago Press, Chicago, (1971).
[77] S. Chandrasekhar, Phys. Rev. Lett. 12, 114 (1964); S. Chandrasekhar, Ap. J. 140, 417 (1964).