Bouquets, vertex covers and the projective dimension of graphs

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Abstract

We characterize the maximum cardinality of a minimal vertex cover of a graph in terms of bouquet subgraphs.

1 Introduction

Let $G$ be a finite simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. Let $k$ be a fixed field and $S = k[x_1, \ldots, x_n]$. The edge ideal of $G$ is a squarefree monomial ideal $I(G) \subseteq S$ given by

$$I(G) = (x_ix_j : \{x_i, x_j\} \in E(G)).$$

The projective dimension of $G$ is defined as the length of the minimal free resolution of $S/I(G)$ and, it is denoted by $\text{pd}(G)$. The Betti numbers of $G$ are the ranks of modules in a minimal free resolution of $S/I(G)$. A current research topic in commutative algebra is to express or bound the invariants of minimal free resolution of a graph in terms of its combinatorial properties (see, for example, [1] – [9]). Many authors introduced new graph parameters and notions in this context.

A subset $C$ of vertices of $G$ is called a vertex cover of $G$ if every edge in $G$ contains an element of $C$. A vertex cover $C$ is minimal if no proper subset of $C$ is a vertex cover of $G$. We write $\alpha'_0(G)$ for the maximum cardinality of a minimal vertex cover of $G$. There is a one-to-one correspondence between the minimal vertex covers of $G$ and the minimal prime ideals of $I(G)$ given by

$$C \text{ is a minimal vertex cover of } G \iff (x_i : i \in C) \text{ is a minimal prime ideal of } I(G).$$

Therefore the parameter $\alpha'_0(G)$ coincides with the big height of $I(G)$, which is the maximum height of the minimal prime ideals of $I(G)$. It is well known that the maximum cardinality of a minimal vertex cover of a graph is a lower bound for its projective dimension. In fact, it is a sharp bound for the following cases:

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Theorem 1.1. For a graph $G$, the equality $\text{pd}(G) = \alpha'_{0}(G)$ holds if

(a) $S/I(G)$ is sequentially Cohen-Macaulay.
(b) $G$ is a chordal graph.
(c) $G$ is a vertex decomposable graph.

In this work, we will give a new characterization of $\alpha'_{0}(G)$, or equivalently, the big height of $I(G)$.

2 Semi-strongly disjoint bouquets and minimal vertex covers

A bouquet is a graph $B$ with $V(B) = \{r, f_1, \ldots, f_d\}$ and $E(B) = \{\{r, f_i\} : i = 1, \ldots, d\}$ where $d \geq 1$. Then the vertex $r$ is called the root of $B$ and the vertices $f_i$ flowers of $B$. Suppose that $\mathcal{B} = \{B_1, \ldots, B_j\}$ is a set of bouquets of $G$. Then

$$F(\mathcal{B}) = \{f : f \text{ is a flower of some bouquet of } \mathcal{B}\},$$
$$R(\mathcal{B}) = \{r : r \text{ is a root of some bouquet of } \mathcal{B}\}.$$

In [7] Kimura introduced the following definition in order to study Betti numbers of chordal graphs.

**Definition 2.1** (Definition 5.1, [7]). A set $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ of bouquets of $G$ is said to be semi-strongly disjoint in $G$ if the following conditions hold.

1. $V(B_k) \cap V(B_\ell) = \emptyset$ for all $k \neq \ell$.
2. Any two vertices belonging to $R(\mathcal{B})$ are not adjacent in $G$.

Also the number

$$d'_G = \max\{\#F(\mathcal{B}) : \mathcal{B} \text{ is a semi-strongly disjoint set of bouquets of } G\}.$$

is the maximum number of flowers that a semi-strongly disjoint set of bouquets of $G$ can have.

**Lemma 2.2.** If $H$ is an induced subgraph of $G$ then $\alpha'_0(H) \leq \alpha'_0(G)$.

**Proof.** By induction it suffices to prove the given statement when the order of $G$ is one more than the order of $H$. Suppose that $u$ is the only vertex of $G$ which do not belong to $H$. Let $C = \{v_1, \ldots, v_s\}$ be a minimal vertex cover of $H$ of maximum cardinality. If every edge which is incident to $u$ in $G$ has an endpoint in $C$, then clearly $C$ is also a minimal vertex cover of $G$. Hence $s \leq \alpha'_0(G)$ follows. Therefore we assume that there exists an edge $\{u, w\}$ of $G$ such that $w \notin C$. Then clearly $\{v_1, \ldots, v_s, u\}$ is a vertex cover of $G$. To see that it is a minimal one, first note that $u$ is not redundant as $w \notin C$. By minimality of $C$ in $H$ none of $v_i$ is redundant in $\{v_1, \ldots, v_s, u\}$ for $i = 1, \ldots, s$. Hence $s + 1 \leq \alpha'_0(G)$ and the result follows.

**Remark 2.3.** Lemma 2.2 is not necessarily true if $H$ is a subgraph of $G$. See for example Figures 1 and 2.
The next result was proved for the special case of vertex decomposable graphs in [5] (see Theorem 3.8 in [5]).

**Theorem 2.4.** For any graph $G$, the equality $\alpha'_0(G) = d'_G$ holds.

**Proof.** Let us assume that $G$ has no isolated vertices since deleting isolated vertices from a graph $G$ does not change $\alpha'_0(G)$ or $d'_G$. First we show that $\alpha'_0(G) \geq d'_G$. Let $\mathcal{B}$ be a semi-strongly disjoint set of bouquets of $G$ which has the maximum number of flowers. Let $G_{\mathcal{B}}$ be the induced subgraph of $G$ on $F(\mathcal{B}) \cup R(\mathcal{B})$. Since $F(\mathcal{B}) \cap R(\mathcal{B}) = \emptyset$, the set $F(\mathcal{B})$ is a vertex cover of $G_{\mathcal{B}}$. As no two vertices belonging to $R(\mathcal{B})$ are adjacent in $G$, the set $F(\mathcal{B})$ is a minimal vertex cover of $G_{\mathcal{B}}$. Therefore by Lemma 2.2 we get $\alpha'_0(G) \geq \alpha'_0(G_{\mathcal{B}}) \geq |F(\mathcal{B})| = d'_G$ as desired.

Next we show that $\alpha'_0(G) \leq d'_G$. Let $C$ be a minimal vertex cover of $G$ of maximum cardinality. We will construct a set $\mathcal{B}$ of semi-strongly disjoint bouquets of $G$ such that $|F(\mathcal{B})| = |C|$. First note that by minimality of $C$, for every $v \in C$ there exists a vertex $u \notin C$ which is adjacent to $v$ in $G$. Pick a vertex $r_1 \notin C$. Let $\{f_1^1, ..., f_{d_1}^1\}$ be the set of vertices of $C$ which are adjacent to $r_1$. Let $B_1$ be the bouquet with root $r_1$ and flowers $f_1^1, ..., f_{d_1}^1$. If $C = F(B_1)$ then $\mathcal{B} = \{B_1\}$ and we are done. Otherwise we keep constructing new bouquets inductively as follows. Suppose that we have bouquets $B_1, ..., B_i$ such that $F(B_1) \cup ... \cup F(B_i)$ is a proper subset of $C$. Then there exists $f_{i+1}^1 \in C \setminus (F(B_1) \cup ... \cup F(B_i))$. By minimality of $C$ there exists $r_{i+1} \notin C$ which is adjacent to $f_{i+1}^1$. Let $B_{i+1}$ be the bouquet with root $r_{i+1}$ and let the flower set $F(B_{i+1}) = \{f_{i+1}^1, ..., f_{d_{i+1}}^{i+1}\}$ be the subset of $C \setminus (F(B_1) \cup ... \cup F(B_i))$ which consists of neighbours of $r_{i+1}$ in $G$. Note that by construction of $B_{i+1}$ we have $r_{i+1} \notin \{r_1, ..., r_i\}$. Now if $C = F(B_1) \cup ... \cup F(B_{i+1})$ then we claim that $\mathcal{B} = \{B_1, ..., B_{i+1}\}$ is a semi-strongly disjoint set of bouquets of $G$. To see this, observe that $V(B_k) \cap V(B_{\ell}) = \emptyset$ for all $k \neq \ell$ by construction. Also for all $k \neq \ell$, the pair $\{r_k, r_{\ell}\}$ is not an edge of $G$ since $r_k, r_{\ell} \notin C$ and $C$ is a vertex cover of $G$. Hence the proof is completed for such a case. If $F(B_1) \cup ... \cup F(B_{i+1})$ is a proper subset of $C$ then we continue the process and it will stop at some step since $G$ is a finite graph.

**Corollary 2.5.** If $G$ is a graph then $pd S/I(G) \geq d'_G = \alpha'_0(G)$.

**Remark 2.6.** The authors of [5] generalized Definition 2.1 and the parameter $d'_G$ to hypergraphs (see Definition 3.2 in [5]). However with their definition, for a hypergraph $\bar{\mathcal{H}}$, the parameters $d'_G$ and $\alpha'_0(\bar{\mathcal{H}})$ are not always equal. What would be a natural generalization
of semi-strongly disjoint bouquets to hypergraphs in the sense that the parameter $d'_H$ still characterizes $\alpha'_0(H)$?

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