Super-Minisuperspace and New Variables

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Abstract

We consider the specialization to spatially homogenous solutions of the Jacobson formulation of N=1 canonical supergravity in terms of Ashtekar’s new variables. We find that the classical Poisson algebra of the supersymmetry constraints is preserved by this specialization only for Bianchi type A models. The quantization of supersymmetric Bianchi type A models is carried out in the triad representation. We find the physical states of this quantum theory. Since we are missing a suitable inner product on these physical states, our results are only formal.

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1. Introduction

Witten’s proof of the positivity of energy in general relativity is a strong indication that supergravity has something interesting to say about general relativity [1, 2]. In the same spirit, supergravity can be put to use in quantum cosmology. An interesting application is the investigation of the physical states of the quantum theory of minisuperspace models. In supergravity, the generators of supersymmetry transformations may be seen as ‘square roots’ of the generators of diffeomorphisms that make up the hamiltonian for vacuum general relativity [3]. Thus, modulo factor ordering ambiguities, physical states for supergravity, i.e. states annihilated by the supersymmetry generators, are also physical states for general relativity, although the converse is of course not true. By appealing to supergravity, one would like to identify the class of privileged physical states that are supersymmetry invariant.

In the context of quantum cosmology, using canonical quantization methods, this line of research has been pursued using two different approaches. The first considers the Wheeler-De Witt equation for some vacuum Bianchi model, and then takes its ‘square root’, relying on the analogy between the Wheeler-De Witt equation and the equation of motion for a relativistic particle (see e.g. [4]). This approach can be implemented only within a specific Bianchi model, and it has the disadvantage of introducing fermionic variables without a clear physical interpretation. Recently, it has been exploited in the Ashtekar formulation of canonical gravity in [5].

A second approach, which we will follow, is to consider the specialization of supergravity to the case of spatially homogeneous solutions. We call this specialization super-minisuperspace. It is the generalization to supergravity of Misner’s minisuperspace [6]. This approach has the benefit of providing a straightforward identification of the phase space variables, and of their transformations properties.

Using the triad extended ADM canonical formalism for supergravity [7], various minisuperspace models have been studied, e.g. Bianchi I in [8, 9], Taub in [10], diagonal Bianchi IX in [11, 12]. The results of these investigations indicate that the quantum supersymmetry constraints select only the most symmetrical quan-
tum states. (In [13], D’Eath comes to the same conclusion for physical states with a finite number of fermionic fields for full supergravity. However, both his result and its interpretation are still controversial [14].)

In this paper, we consider the super-minisuperspace of hamiltonian N=1 supergravity in the formulation given by Jacobson [15]. This formulation extends to supergravity the Ashtekar formulation for canonical gravity [16]. The spatial left-handed spin connection, and the densitized spatial triad are used as bosonic phase space variables. The Jacobson formulation shares many of the nice features of Ashtekar’s formulation of gravity. In particular, one is using conjugate phase space variables, and the constraints are put in polynomial form of low order. As a result, the canonical quantization is much easier to carry out than in the triad extended ADM formalism, where one is working with awkward non-conjugate phase space variables. Exploiting these simplifications, we carry out the canonical quantization of super-minisuperspace for all type A Bianchi models. Moreover, we consider the most general factor ordering of the quantum constraints 1.

A key technical assumption in our analysis, and in previous treatments, is that quantum states depend only on a finite number of fermionic fields. This permits one to expand the wavefunction in a finite number of terms corresponding to even powers of the fermionic fields, since they are grassmannian. This assumption is not reasonable if one is interested in the full space of physical states for supergravity, where states with an infinite number of fermionic fields are likely to play a key role [13, 18]. However, the assumption is justified in the super-minisuperspace approximation, where one is dealing only with a finite number of degrees of freedom, and, in any case, if one is interested only in the limit in which the fermionic fields vanish, i.e. general relativity.

Various representations of the quantized theory are possible. In this paper, we focus on the triad representation. Quantum states are represented by wavefunctions that depend on the spatial triad, and on the spatial gravitino field. Our reason for privileging this representation is that it is the closest to the representation used in previous treatments which employed the triad ADM formalism.

We find that physical states for quantum Bianchi type A models are in general

\footnote{After this work was completed we became aware of Ref. [17], where the quantization of type A Bianchi models is carried out in the triad ADM formalism.}
of the following form. The bosonic part of the wavefunction annihilated by the quantum constraints is given by

$$\Psi_{(0)} = k_{(0)} h^{\alpha/2} \exp\{2iF\},$$

where $k_{(0)}$ is a constant, $h_{ab}$ is the homogenous metric, $h := \text{det} h_{ab}$. The constant $\alpha$ is determined by the factor ordering chosen. $F$ is the bosonic part of the homogenous specialization of the generating function of the canonical transformation from the ADM variables to Jacobson’s. ($F$ is pure imaginary.) In general, the next terms in the wavefunction, $\Psi_{(2)}$ and $\Psi_{(4)}$, quadratic and quartic in the fermionic fields, respectively, are found to vanish. The exceptions require a fine adjustment of the factor ordering. The last term, of sixth order in the fermionic fields, is given by

$$\Psi_{(6)} = k_{(6)} h^{1/3} [\beta^2][\rho^2]^2,$$

where $k_{(6)}$ is a constant, and $[\beta^2], [\rho^2]$ are Lorentz invariants built with the gravitino field, defined in the text below. This term of the wavefunction has the interesting feature of being independent of the specific Bianchi model, and of the factor ordering.

We emphasize that our results are only formal. Although we identify the space of physical states, we have not identified an inner product on this space. Therefore, we are unable to extract any physical information from these quantum states.

The use of the left-handed connection as phase space variable, implies the use of complex variables. To recover the real phase space, one needs to impose reality conditions. In turn, these reality conditions can be used to select an inner product on the physical states [19, 20]. Since the system is finite-dimensional, it should be possible to carry out this step [21], but we have not tried to do it here, leaving this problem for future work.

We also leave for future work quantization in the connection representation. For this representation, the supersymmetry constraints are second order in the momenta, as for pure gravity. Thus, much of the motivation for looking at supergravity is lost. However, it may still be of interest as a first timid step towards the inclusion of matter fields in the new variables non-perturbative quantization of gravity.
This paper is organized as follows. In sect. 2, we review briefly the Jacobson canonical formulation of N=1 supergravity. In sect. 3, we summarize the kinematics of spatial homogeneity. The specialization to homogenous supergravity is described in sect. 4. In sect. 5, we consider the canonical quantization of Bianchi type A models in the triad representation, and we solve explicitly the quantum supersymmetry constraints.

2. Jacobson’s Hamiltonian N=1 Supergravity

In this section, we briefly recall the Jacobson formulation of hamiltonian N=1 supergravity in terms of Ashtekar’s new variables [15]. We refer the reader to [15] for its derivation from an action principle, and for a thorough discussion.

The canonical coordinates are a complex traceless $SL(2,C)$ spatial connection, $A_{iA}^B$, and a traceless spatial vector density of weight 1, $\tilde{\sigma}^{iAB}$, together with the spatial anti-commuting gravitino field $\psi^A_i$, and its conjugate momentum the anti-commuting $\tilde{\pi}^i_A$. (Small latin letters from the middle of the alphabet denote spatial indices, $i,j,\ldots = 1,2,3$. Capital latin letters denote $SL(2,C)$ indices $A,B,\ldots = 0,1$. These indices are raised and lowered with the anti-symmetric symbol $\epsilon^{AB}$, and its inverse $\epsilon_{AB}$, according to the rules $\lambda^A = \epsilon^{AB} \lambda_B$, $\lambda_A = \lambda^B \epsilon_{BA}$.)

The connection $A_{iA}^B$ is the spatial pull-back of the left-handed spin connection. The vector density $\tilde{\sigma}^{iAB}$ may be interpreted as the (densitized) spatial triad, in the sense that the covariant (doubly densitized) spatial metric is given by $(\det q)^j_i = \tilde{\sigma}^{iAB} \tilde{\sigma}_j^{AB}$. The momentum $\tilde{\pi}^i_A$ is related to the complex conjugate of the spatial gravitino field, $\psi^A_i$.

The fundamental Poisson brackets are given by

$$\{\tilde{\sigma}^{kAB}(x), A_{jCD}(y)\} = \frac{i}{\sqrt{2}} \delta^k_j \delta^{(A}_C \delta^{D)}_B \delta^3(x,y), \quad (1)$$

$$\{\tilde{\pi}^kA(x), \psi^B_j(y)\} = \frac{i}{\sqrt{2}} \delta^k_j \delta^A_B \delta^3(x,y). \quad (2)$$

Note that we differ with Jacobson by a factor of $i/\sqrt{2}$ on the right hand side.

In terms of these variables, the hamiltonian for N=1 (complex) supergravity may
be written in polynomial form, at most quartic in the phase space variables,

$$H = i\sqrt{2} \int d^3x \{ e_{0AB} \mathcal{H}^{AB} + \psi_{0A} S^A + S^\dagger_A \psi_{0A}^\dagger + A_{0AB} \mathcal{J}^{AB} \}. \quad (3)$$

The fields $e_{0AB}, p_{0A}, \psi_{0A}^\dagger, A_{0AB}$ are Lagrange multipliers, that enforce the constraints

$$\mathcal{H}^{AB} : = (\bar{\sigma}^i \bar{\sigma}^j F_{ij})^{BA} + 2(\bar{\pi}^j \bar{\sigma}^k D_{[j} \psi_{k]} \varepsilon^{AB} + 2(\bar{\pi}^j D_{[j} \psi_{k]}) \tilde{\sigma}^{kAB} = 0, \quad (4)$$

$$\mathcal{J}^{AB} : = D_k \sigma^{kAB} - \tilde{\pi}^k (A \psi^B) = 0, \quad (5)$$

$$S^A : = D_k \tilde{\pi}^{kA} = 0, \quad (6)$$

$$S^\dagger_A : = (\bar{\sigma}^i \bar{\sigma}^j D_{[j} \psi_{k]})^A = 0. \quad (7)$$

Here $D_i$ is the covariant derivative of $A_{ij}^{AB}$, with curvature $F_{ij}^{AB} := 2\partial_{[i} A_{j]A}^{AB} + 2A_{[i|A} C A_{j|C}^{B}$. Following Jacobson, we are using the convention that suppressed spinor indices are contracted from upper left to lower right, e.g. $(\bar{\sigma}^i \bar{\sigma}^j)^{AB} = \bar{\sigma}^{iAC} \bar{\sigma}^j_{C}^{B}$. Hermitian conjugation is defined with respect to some Hermitian metric $\eta^{AA'}$.

The constraint $\mathcal{H}^{AB} = 0$ is the generator of diffeomorphisms. Its part symmetric in the indices $AB$ generates diffeomorphisms tangential to the space-like hypersurface, up to an $SL(2, C)$ rotation, and up to a right-handed supersymmetry transformation. $\mathcal{H}^{AB} \varepsilon_{AB}$ generates diffeomorphisms out of the hyper-surface up to a right-handed supersymmetry transformation$^2$. See [13] for its derivation. Basically, it is the appropriate linear combination of the diffeomorphism constraint and a constraint of the form (7) that makes the hamiltonian polynomial in the phase space variables.

The constraint $\mathcal{J}^{AB}$ enforces $SL(2, C)$ covariance, and generalizes the Gauss constraint of vacuum general relativity.

The last two constraints, $S^A, S^\dagger_A$, generate left-handed, and right-handed supersymmetry transformations, respectively. Their Poisson algebra reads,

$$\{ S^A, S^B \} = 0, \quad (8)$$

$^2$The consequences of writing the diffeomorphism constraints in this ‘unified’ form have been studied in Ref. [22].
\[ \{ S^{\dagger A}, S^{\dagger B} \} = C^{AB}_{\ C} S^{\dagger C}, \]  
\[ \{ S^A, S^{\dagger B} \} = \frac{i}{2\sqrt{2}} \mathcal{H}^{AB}, \]

where \( C^{AB}_{\ C} := (i/\sqrt{2})(2\bar{\sigma}^{k(A}_{\ C}\psi^{B)}_{k} - \bar{\sigma}^{kAB}\psi_{kC}) \).

The interesting bracket is the last one. The mixed Poisson bracket gives the ‘unified’ diffeomorphism constraint (4). In this sense, one can think of the supersymmetry constraints as a ‘square root’ of the diffeomorphism constraints [3].

As in the case of pure general relativity, the use of the self-dual spin connection as field variable implies the use of complex coordinates on the real phase space of supergravity. Therefore, to recover the physical phase space, one needs to introduce appropriate reality conditions. One can require \( \bar{\sigma}^{iAB} \) to be Hermitian with respect to some Hermitian metric \( n^{AA'} \). This gives reality conditions for the fermionic fields, that amount to the relation of \( \bar{\pi}^{iA} \) with the complex conjugate of \( \psi^{iA} \). To impose reality condition on the connection two options are available. The first is non-polynomial, and may be written as,

\[ A_i^{AB} = \Gamma_i^{AB} + i\Pi_i^{AB}, \]

where \( \Gamma_i^{AB} \) is the Hermitian (torsionful) spin connection compatible with \( \bar{\sigma}^{iAB} \), and \( \Pi_i^{AB} \) is Hermitian, and determined by the extrinsic curvature. The second option is to impose polynomial conditions equivalent to (11) by requiring that the reality of the (doubly densitized) metric \( q_{ij} \) be preserved in time [23].

3. Bianchi formalism

In this section, we summarize the kinematics of spatial homogeneity. We follow closely the analogous specialization of the new variables formulation of hamiltonian vacuum general relativity given by Ashtekar and Pullin in [24].

As in the ADM treatment of Bianchi models (see e.g. [25, 26]), we consider a kinematical triad of vectors, \( X_i^a \), which commute with the three Killing vectors on the spatial hypersurface \( \Sigma \). (Latin letters from the beginning of the alphabet, \( a, b, c, \ldots \) label the triad vectors.) The triad satisfies,

\[ [X_a, X_b]^i = C_{ab}^c X_c^i, \]
where $C_{ab}^c$ denote the structure constants of the Bianchi type under consideration. The basis dual to $X^i_a$, defined with $X^i_a \chi^b_i = \delta^b_a$, satisfies,

$$2\partial_{[i}^a \chi^a_{j]} = -C^a_{bc} \chi^b_i \chi^c_j.$$  

(13)

Without loss of generality, one may set

$$C_{ab}^c = \epsilon_{abd} M^{dc} + 2\delta^c_{[a} V_{b]},$$

(14)

with $M^{ab}$ symmetric. From the Jacobi identities, it follows that $M^{ab}V_b = 0$. The Bianchi classification is given by the vanishing or not of $V_a$, and the rank and the signature of $M^{ab}$, subject to this condition. Models with $V_a = 0$, $V_a \neq 0$, are called type A, and type B, respectively. In the type A models, the most popular are Bianchi I, selected by $M^{ab} = 0$, and Bianchi IX, selected by $M^{ab} = \delta^{ab}$.

4. Super-Minisuperspace

The Jacobson phase space variables for $N=1$ supergravity may be expanded with respect to the kinematical triad $X^i_a$ (or $\chi^b_i$), as,

$$A_{iA}{}^B = A_{aA}{}^B \chi^a_i,$$

$$\tilde{\sigma}^{iAB} = (\text{det} \chi) \sigma^{aAB} X^i_a,$$

$$\psi^i_A = \psi^a_i \chi^a_i,$$

$$\tilde{\pi}^{iA} = (\text{det} \chi) \pi^{aA} X^i_a,$$

where $\text{det} \chi$ denotes the determinant of $\chi^a_i$, which is introduced in order to de-densitize $\tilde{\sigma}^{iAB}$ and $\tilde{\pi}^{iA}$. Note that $\psi^a_i$ and $\pi^{bB}$ are anti-commuting. All of the spatial dependence of the fields is contained in the kinematical quantities $X^i_a$ and $\chi^b_i$. The phase space has been reduced from 30 degrees of freedom per space point to only 30 global degrees of freedom.

The Poisson brackets take the form

$$\{\sigma^{aAB}, A^{bMN}_i \} = \frac{i}{\sqrt{2}} \delta^a_b \delta^{(M}_N \delta^{B)}_i,$$

(15)

$$\{\pi^{aA}, \psi^{bM} \} = \frac{i}{\sqrt{2}} \delta^a_b \delta^{M}_i.$$  

(16)
Inserting the expansion in the supersymmetry constraints, they reduce to

\[
S^A := (C_{ba}^\, \epsilon^{AB} + A_a^{AB}) \pi^a_B = 0, \tag{17}
\]

\[
S^{\dagger A} := -\frac{1}{2} C_{ab}^\, \epsilon^{D} + A_a^{CD} \delta^C_b \psi_c D \sigma^{|a|\{AB\}} \sigma^{b|} C = 0. \tag{18}
\]

The diffeomorphism and \(SL(2, C)\) constraints become

\[
H^{AB} := -\epsilon^{a b} C_{ab}^\, C_{c}^{A} \pi^a_B + A_a^{AB} C_{c}^{A} \pi^a_B + 2 \epsilon^{a b} C_{ab}^\, C_{c}^{A} \sigma^{|a|\{AB\}} \sigma^{b|} C = 0. \tag{19}
\]

\[
J^{AB} := C_{ba}^\, \pi^a_B + 2 A_a^{C} \sigma^{|a|\{AB\}} C - \pi^a_A \psi_B = 0. \tag{20}
\]

Note that we have rescaled the constraints by the appropriate power of \(det \chi\) to de-densitize them.

In general, it is not guaranteed that the truncation to the homogenous solutions will preserve the Poisson algebra of the constraints. For our purposes, it is sufficient to consider the algebra of the supersymmetry generators. It is automatic that \(\{S^A, S^B\} = 0\). A short calculation shows that also the right-handed generators continue to close into themselves, \(i.e\). that \(\{S^{\dagger A}, S^{\dagger B}\} = C^{A B} C (\sigma \psi) S^{i C}\). On the other hand, the mixed bracket yields

\[
\{S^A, S^{\dagger B}\} = \frac{i}{2 \sqrt{2}} \left( H^{AB} + \frac{i}{2 \sqrt{2}} C_{ba}^\, C_{cd}^\, a (\sigma^c \sigma^d)^{BA} + \frac{i}{\sqrt{2}} C_{bc}^\, b (\sigma^c \sigma^d) A_d^{BA} \right). \tag{21}
\]

The extra terms on the right hand side vanish when \(C_{ba}^\) = 0, \(i.e\). for type A Bianchi models.

The supersymmetry constraints fail to close for Bianchi type B. For this reason, in addition to the (possibly related) well known difficulties one encounters in giving a hamiltonian formulation of these models, in the following we will restrict ourselves to Bianchi models of type A, \(i.e\). we will assume that \(C_{ba}^\) = 0.

For this case, the supersymmetry constraints reduce further to

\[
S^{\dagger A} = -\frac{1}{2} \epsilon_{abd} M^{dc} (\sigma^a \sigma^b \psi^c)^A + (\sigma^{|a|\} A_a \psi_B)^A, \tag{22}
\]

\[
S^A = A_a^{AB} \pi^a_B. \tag{23}
\]
At this point, it is tempting to specialize further, e.g. by considering $\sigma^{aAB}$, $A_{aAB}$ diagonal, when considered as three by three matrices $[27, 12]$. We will resist this temptation, since it complicates, rather than simplifies, the form of the supersymmetry constraints.

5. Quantization. Triad Representation

In this section, we turn to the quantization of type A Bianchi models. In the triad representation, quantum states may be represented by wavefunctions that depend on the triad, and on the gravitino field, $\Psi = \Psi(\sigma, \psi)$. The Poisson bracket (15) turns into commutators of operators, while (16) into anti-commutators. The variables $\sigma^{aAB}$ and $\psi^A_a$ are considered as ‘position’ operators. We use the standard notation of denoting operators with a hat. Their momenta may be represented with

$$\hat{A}^A_{aB} \Psi = \frac{1}{\sqrt{2}} \frac{\delta \Psi}{\delta \sigma^a_{AB}},$$

$$\hat{\pi}^A_a \Psi = \frac{1}{\sqrt{2}} \frac{\delta \Psi}{\delta \psi^A_a}.$$ 

In the translation of the supersymmetry constraints to their quantum version, the issue of factor ordering arises in the second term of (22). Schematically, three possibilities are available: (i) $\sigma \sigma A$; (ii) $\sigma A \sigma$; (iii) $A \sigma \sigma$. To accommodate this ambiguity, we write the quantum version of (22) in the form

$$\hat{S}^A \Psi(\sigma, \psi) = \left[ -\frac{1}{2} \varepsilon_{abcd} M^{dc}(\sigma^a \sigma^b \psi_c)^A + \frac{1}{\sqrt{2}} (\sigma^{[a} \sigma^{b]} \frac{\delta}{\delta \sigma^a} \psi_b)^A + \frac{1}{\sqrt{2}} \alpha \sigma^{aAC} \psi_a C \right] \Psi = 0$$

where the constant $\alpha$ characterizes the factor ordering, e.g. $\alpha = 0, 1, 2$ for the orderings (i), (ii), (iii), respectively.

The left-handed generator (23) takes the form

$$\hat{S}^A \Psi(\sigma, \psi) = \frac{1}{2} \frac{\delta^2 \Psi}{\delta \sigma^a_{AB} \delta \psi^B_a} = 0,$$

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In addition, a physical state must also satisfy
\[ \hat{J}^{AB}\Psi = 0, \] (26)
i.e. be invariant under \( SL(2, C) \) rotations. It follows that the wavefunction must be an \( SL(2, C) \) scalar. (In \( \hat{J}^{AB} \), we assume an appropriate factor ordering so that it generates \( SL(2, C) \) rotations of the wavefunction.)

As mentioned in the introduction, since the hamiltonian system is finite dimensional, the wavefunction may be expanded in even powers of the gravitino fields, symbolically as follows,
\[ \Psi(\sigma, \psi) = \Psi(0)(\sigma) + \Psi(2)(\sigma, \psi) + \Psi(4)(\sigma, \psi) + \Psi(6)(\sigma, \psi), \] (27)
where the subscript indicates the number of gravitino fields. Only even powers appear because of \( SL(2, C) \) invariance. Since the super-symmetry generators do not mix fermionic number, this decomposition permits us to solve the quantum constraints (24, 25, 26) order by order.

5.1 Bosonic states

The first term in the wavefunction (27) is \( \Psi(0)(\sigma) \). From \( SL(2, C) \) invariance, it can depend on \( \sigma^a_{AB} \) only in the combination \( h^{ab} := tr(\sigma^a\sigma^b) \), i.e. \( \Psi(0) = \Psi(0)(h) \).

In this representation, \( \Psi(0) \) satisfies (25) automatically. We are left with (24). The gravitino field is arbitrary, so we can peel it off. Multiplying through by \( \sigma_{e}^{AD} \) gives, after some index reshuffling,
\[ \left[ -\frac{1}{2}\varepsilon_{abc}M^{cd}Tr(\sigma^{a}\sigma^{b}\sigma^{c}) + \frac{1}{\sqrt{2}}Tr(\sigma^{c}\sigma^{[a}\delta_{\sigma^{d}]}\frac{\delta}{\delta\sigma^{a}}) + \frac{1}{\sqrt{2}}\alpha h^{de}\right]\Psi(0) = 0. \] (28)

Using the identities
\[ \sqrt{2}Tr(\sigma^{a}\sigma^{b}\sigma^{c}) = -h^{-1/2}\varepsilon^{abc}, \]
\[ 2\sigma^{[a|AB]\sigma^{d]}\sigma^{a]}_{B(C\sigma^{e]D)}A} = h^{ea}\sigma^{d}_{CD} - h^{ed}\sigma^{a}_{CD}, \]
where we denote with \( h \) the determinant of \( h_{ab} \), and the chain rule, \( Tr(\sigma^{a}\delta/\delta\sigma^{b}) = 2h^{ac}\delta/\delta h^{bc} \), equation (28) can be put in the form,
\[ [(h^{ab}h^{cd} - h^{ac}h^{bd})\frac{\delta}{\delta h^{cd}} + \alpha h^{ab} + h^{-1/2}M^{ab}]\Psi(0) = 0. \] (29)
This equation can be easily integrated to give

\[ \Psi(0)(h) = k(0) h^{\alpha/2} \exp\{2iF\}, \quad (30) \]

where \( k(0) \) is a constant, and

\[ F := -\frac{i}{2} h^{-1/2} h_{ab} M^{ab}. \quad (31) \]

The exponent of the prefactor \( h^{\alpha/2} \) can be adjusted by changing the factor ordering. Until an inner product is available, we cannot appeal to physical arguments to select a preferred factor ordering.

The quantity \( F \) is the bosonic part of the homogenous specialization of the generating functional of the canonical transformation from the triad extended ADM phase space to Jacobson’s [28].

It is interesting to observe that we could have chosen as well the ‘momentum’ representation for the fermionic fields. In this representation, for a bosonic wavefunction, \( \hat{S}^{*A} \Psi(0)(\sigma) = 0 \) is satisfied automatically. \( \hat{S}^{*A} \Psi(0) = 0 \), implies \( \delta \Psi(0) / \delta \sigma^{aAB} = 0 \), i.e. that \( \Psi(0) \) is constant. This should serve as a reminder that physical states can take different forms, depending crucially on the representation chosen. In the following, for concreteness, and to remain close to previous treatments, we will stick to the ‘position’ representation.

5.2 Fermionic states

We turn now to the solution of (24, 25, 26) for wavefunctions that depend on the gravitino field.

Our first step is to identify the irreducible spin 1/2 and spin 3/2 parts of the gravitino field. Let,

\[ \psi^{ABC} := \psi_a^{A} \sigma^{aABC} = \psi^{A(BC)}. \quad (32) \]

Then,

\[ \psi^{ABC} = \rho^{ABC} + \epsilon^{A(B} \beta^{C)} \quad (33) \]
where \( \rho^{ABC} = \rho^{(ABC)} \), represents the spin 3/2 part, and \( \beta^A = (2/3)\psi_B^{AB} \), the spin 1/2 part. In what follows it will be a key fact that \( \rho^{ABC} \) and \( \beta^A \) can be specified independently.

The wavefunction can depend on the gravitino field only in Lorentz invariant combinations. The only non-vanishing possibilities are

- second order : \( [\rho^2] := \rho^{ABC} \rho_{ABC} \), \( [\beta^2] := \beta^A \beta_A \),
- fourth order : \( [\rho^2]^2 \), \( [\rho^2][\beta^2] \),
- sixth order : \( [\rho^2]^2[\beta^2] \).

Note that in terms of the phase space variables one has,

\[
[\rho^2] = \frac{2}{3} \left\{ \psi^A_{\psi A B} \psi^{B A}_{\psi B} h_{ab} - (2h)^{-1/2} \psi^A_{\psi A B} \psi^{B E}_{\psi B} \epsilon^{abc}_{\sigma_{c A B}} \right\},
\]

\[
[\beta^2] = \frac{2}{9} \left\{ \psi^A_{\psi A B} \psi^{B A}_{\psi B} h_{ab} + \sqrt{2} h^{-1/2} \psi^A_{\psi A B} \psi^{B E}_{\psi B} \epsilon^{abc}_{\sigma_{c A B}} \right\}.
\]

Then, the most general expression for wavefunctions that depend on the gravitino field is of the form

\[
\Psi^{(2)}(\sigma, \psi) = F_1(h) [\rho^2] + F_2(h) [\beta^2],
\]

\[
\Psi^{(4)}(\sigma, \psi) = G_1(h) [\rho^2]^2 + G_2(h) [\rho^2][\beta^2],
\]

\[
\Psi^{(6)}(\sigma, \psi) = H(h) [\rho^2]^2 [\beta^2],
\]

where the functions \( F, G, H \) depend on \( \sigma^{a A B} \) only in the combination \( h_{ab} = Tr(\sigma^a \sigma^b) \).

It turns out that, in general, the only physical states of second and fourth order are the trivial ones, \( \Psi^{(2)} = 0 = \Psi^{(4)} \).

It is convenient at this point to express \( \Psi \) in terms of the independent quantities \( \beta^A, \rho^{ABC} \),

\[
\sqrt{2} \hat{S}^{ia} \Psi = - h^{-1/2} M^{ab} \sigma^A_{a b} \sigma^C_{b d} \rho_{BCD} \Psi + \sigma^{a A B} \rho^{C D} \frac{\delta \Psi}{\delta \sigma^{a C D}}
+ \frac{1}{2} \beta^A [-h^{-1/2} M^{ab} h_{ab} + 3\alpha + \sigma^{a C D} \frac{\delta}{\delta \sigma^{a C D}}] \Psi = 0.
\]

It turns out that, in general, the only physical states of second and fourth order are the trivial ones, for example,

\[
\Psi^{(2)} = 0 = \Psi^{(4)}.
\]
The reason is that these states must satisfy both quantum supersymmetry equations. When one isolates all the independent terms in these equations, they turn out to be too many relations to be satisfied at the same time. The exceptions are given by special cases like Bianchi I, and a specific factor ordering.

For concreteness, we show explicitly how it goes for $\Psi^{(2)}$. The computation for $\Psi^{(4)}$ follows the same pattern, and comes to the same conclusion.

Consider first,

$$\hat{S}^A \Psi^{(2)} = 2\sigma^{aAB}\sigma^{bCD}\rho_{BCD}\delta F_1 + \frac{2}{3}\beta^A[h^{ab}\delta F_2 + 4F_2 - 3F_1] = 0 \quad (39)$$

Since $\rho^{ABC}$ and $\beta^A$ can be specified independently, their coefficients must vanish separately. The vanishing of the $\rho^{ABC}$ coefficient implies

$$h^{ab}\delta F_2 \propto h_{ab}, \quad (40)$$

while the vanishing of the $\beta$ coefficient gives

$$h^{ab}\delta F_2 \delta h_{ab} + 4F_2 - 3F_1 = 0. \quad (41)$$

It is not necessary to integrate these equations. Using them, (24) takes the form,

$$\sqrt{2}\hat{S}^A \Psi^{(2)} = -[\rho^2]h^{-1/2}M^{ab}\sigma^A_a\sigma^CD_b \rho_{BCD}F_1$$

$$- \frac{1}{2}[\rho^2]\beta^A \{[h^{-1/2}M^{ab}h_{ab} + 3\alpha + 2]F_1 + \frac{1}{6}F_2 + h^{ab}\delta F_1 \delta h_{ab}\}$$

$$- \beta^2 \{[h^{-1/2}M^{ab}\sigma^AB_a\sigma^CD_b \rho_{BCD}F_2 + \sigma^{aAB}\sigma^{bCD}\rho_{BCD}\delta F_2 \delta h_{ab}\} = 0. \quad (42)$$

Each line of this equation must vanish separately. If $M^{ab} \neq 0$, it follows immediately that $F_1 = F_2 = 0$, i.e. that $\Psi^{(2)} = 0$. For $M^{ab} = 0$, i.e. Bianchi I, there is a factor ordering such that $F_1 = F_2 = k^{(2)}h^{1/3}$, with $k^{(2)}$ a constant.

We come now to the last term in the wavefunction (27), i.e. $\Psi^{(6)}$. Note that $\hat{S}^A \Psi^{(6)} = 0$ is identically satisfied, since it is of seventh order in six anti-commuting quantities. We are left with

$$\hat{S}^A \Psi^{(6)} = + 4[\rho^2][\beta^2]\sigma^{aAB}\sigma^{bCD}\rho_{BCD}\delta H_{\delta h_{ab}}$$

$$- 2\beta^A[\rho^2]^2 \{\frac{1}{3}h^{ab}\delta H_{\delta h_{ab}} + 3H\} = 0. \quad (42)$$
Each line must vanish separately. Integrating gives

\[ H = k_{(6)} h^{1/3}, \]  

where \( k_{(6)} \) is a constant. Note that this term in the wavefunction is independent both of the specific Bianchi type A, and of the factor ordering.

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