FACTORIZATION OF GROUP DETERMINANT IN SOME GROUP ALGEBRAS

NAOYA YAMAGUCHI

ABSTRACT. We give an analog of Frobenius’ theorem about the factorization of the group determinant on the group algebra of finite abelian groups and we extend it into dihedral groups and generalized quaternion groups. Furthermore, we describe the group determinant of dihedral groups and generalized quaternion groups as a circulant determinant of homogeneous polynomials.

This analog on the group algebra is stronger than Frobenius’s theorem and as a corollary, we obtain a simple expression formula for inverse elements in the group algebra. Furthermore, the commutators of irreducible factors of the factorization of the group determinant on the group algebra corresponding to degree one representations have interesting algebraic properties. From this result, we know that degree one representations form natural pairing.

At the current stage, the extension of Frobenius’ theorem is not represent as a determinant. We expect to find a determinant expression similar to Frobenius’ theorem.

1. INTRODUCTION

In this paper, we give an analog of the factorization of the group determinant on the group algebra, where the group $G$ is a finite abelian group, a dihedral group or a generalized quaternion group. The group determinant $\Theta(G)$ is the determinant of the matrix whose elements are independent variables $x_g$ corresponding to $g \in G$. Frobenius gave the following theorem about the factorization of the group determinant.

**Theorem 1** (Frobenius). Let $G$ be a finite group and $\hat{G}$ be a complete set of irreducible representations of $G$. Then, we have

$$\Theta(G) = \prod_{\varphi \in \hat{G}} \det \left( \sum_{g \in G} \varphi(g)x_g \right)^{\deg \varphi}.$$

Our main results of this paper are non-trivial extensions of this theorem for some group on the corresponding group algebras.

1.1. Results for finite abelian groups. Our main result for finite abelian group is the following theorem.
Theorem 2. Let $G$ be a finite abelian group and $e$ be the unit element of $G$. Then, we have

$$\Theta(G)e = \prod_{\chi \in \hat{G}} \sum_{g \in G} \chi(g)x_gg.$$ 

Let

$$CG = \left\{ \sum_{g \in G} x_gg \mid x_g \in \mathbb{C}, g \in G \right\}$$

be the group algebra of $G$. Note that the equality in Theorem 2 is the equality on the group algebra. Theorem 2 is stronger than Theorem 1. In fact, let $F : CG \to \mathbb{C}$ be a $\mathbb{C}$-linear ring homomorphism such that $F(g) = 1$ for all $g \in G$. Theorem 1 for finite abelian groups follows by applying $F$ to Theorem 2. Moreover, we obtain the following formula for inverse elements in the group algebra.

**Corollary 3.** Let $G$ be a finite abelian group and $\chi_1$ be the trivial representation of $G$. If $\Theta(G) \neq 0$, then we have

$$\left( \sum_{g \in G} x_gg \right)^{-1} = \frac{1}{\Theta(G)} \prod_{\chi \in \hat{G}\setminus\{\chi_1\}} \left( \sum_{g \in G} \chi(g)x_gg \right).$$

1.2. Main results for dihedral groups and generalized quaternion groups.

We give an analog of Theorem 1 on the group algebra for dihedral groups and generalized quaternion groups. Furthermore, we describe the group determinant of these groups as a circulant determinant of homogeneous polynomials. Since the expression for the general case is troublesome, we illustrate the result on a dihedral group $D_3 = \{e, a, a^2, b, ab, a^2b\}$. Let $\langle a \rangle$ be the subgroup generated by $\{a\}$ and $A_h$ be a homogeneous polynomial defined by

$$A_h = \sum_{g \in \langle a \rangle} \chi_2(g)x_gx_{hg}$$

for $h \in G$. We describe $\Theta(D_3)$ as a circulant determinant of $A_h$.

**Theorem 4.** Let $G$ be $D_3$. Then, we have

$$\Theta(D_3) = \prod_{\chi \in \hat{\langle a \rangle}} \sum_{g \in \langle a \rangle} \chi(g)A_g.$$ 

Furthermore, we obtain the analog of Theorem 1 on the group algebra for $D_3$.

**Theorem 5.** Let $G$ be $D_3$, $\omega$ be a primitive third roots of unity and $\chi_2$ be the non-trivial degree one representation. Then, we have

$$\Theta(D_3)e = \alpha_1\alpha_2 \left( A_e + \omega A_a + \omega^2 A_{a^2} \right) \left( A_e + \omega^2 A_a + \omega A_{a^2} \right)$$

$$= \prod_{\chi \in \hat{\langle a \rangle}} \sum_{g \in \langle a \rangle} \chi(g)A_gg,$$

where

$$\alpha_1 = \sum_{g \in D_3} x_gg, \quad \alpha_2 = \sum_{g \in D_3} \chi_2(g)x_{g^{-1}}g.$$ 

Theorem 1 for $D_3$ and Theorem 4 follow by applying $F$ to Theorem 5. Moreover, we obtain the following formula for inverse elements in the group algebra for $D_3$. 
Corollary 6. Let \( G \) be \( D_3 \). If \( \Theta(G) \neq 0 \), then we have

\[
\alpha_1^{-1} = \frac{1}{\Theta(D_3)} \alpha_2 \left( A_e e + \omega A_0 a + \omega^2 A_0^2 a^2 \right) \left( A_e e + \omega^2 A_0 a + \omega A_0^2 a^2 \right).
\]

We can extend the results in Section 1.2 into dihedral groups and generalized quaternion groups.

1.3. Algebraic properties of irreducible factors corresponding to degree one representations. Let \( G \) be a dihedral group \( D_m \) or generalized quaternion group \( Q_m \). The commutators of irreducible factors of the factorization of the group determinant on the group algebra corresponding to degree one representations have the following algebraic properties.

Theorem 7. Let \( G \) be \( D_m \) or \( Q_m \). Then, we have

\[
[\alpha_1, \alpha_2] = 0, \quad [\alpha_3, \alpha_4] = 0,
\]

\[
[\alpha_1, \alpha_3 + \alpha_4] = 0, \quad [\alpha_2, \alpha_3 + \alpha_4] = 0, \quad [\alpha_3, \alpha_1 + \alpha_2] = 0, \quad [\alpha_4, \alpha_1 + \alpha_2] = 0.
\]

where \( \alpha_i (1 \leq i \leq 4) \) is the irreducible factor of the factorization of the group determinant on the group algebra corresponding to the degree one representation. However, the equations for \( \alpha_3 \) and \( \alpha_4 \) do not hold when \( G = D_m \) and \( m \) is odd.

We remark that when \( m \) is odd the number of degree one representation of \( D_m \) is two and when \( m \) is even the number is four. The number of degree one representation of \( Q_m \) is always four. From Theorem \( \text{[7]} \) we know that the degree one representations form natural pairing \( (\chi_1, \chi_2) \) and \( (\chi_3, \chi_4) \).

1.4. Future works. From the above results and Theorem \( \text{[1]} \) we expect to find determinant formulas for matrices with elements on the group algebra.

2. Factorization of group determinant

In this section, we recall the group determinant and the factorization of the group determinant.

2.1. Factorization of group determinant. Let \( G \) be a finite group of order \( n \) and \( \{x_g | g \in G\} \) be independent commuting variables. The group determinant \( \Theta(G) \) is the determinant of the \( n \times n \) matrix \( (x_{g,h}) \) where \( x_{g,h} = x_{gh^{-1}} \) for \( g, h \in G \) and is thus a homogeneous polynomial of degree \( n \) in \( x_g \). Frobenius gave the following theorem about the factorization of the group determinant.

Theorem 8 (Frobenius). Let \( \hat{G} \) be a complete set of irreducible representations of \( G \). Then, we have

\[
\Theta(G) = \prod_{\varphi \in \hat{G}} \det \left( \sum_{g \in G} \varphi(g)x_g \right)^{\deg \varphi}.
\]

3. Results for finite abelian groups

In this section, we give an analog of Frobenius' theorem for finite abelian groups about factorization of the group determinant on the group algebra.
3.1. Preparation for main results of finite abelian groups. We prepare to explain main results of finite abelian groups. Let \( \{ \varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(s)} \} \) be a complete set of irreducible representations of \( G \), \( M_k(\mathbb{C}) \) be a set of \( k \times k \) matrices with entries in \( \mathbb{C} \) and \( L(G) \) be the linear space of all complex functions on \( G \). The set \( L(G) \) is a ring with addition taken pointwise and convolution as multiplication [1, Theorem 5.2.3].

**Definition 9** (Fourier transform). Define

\[
T : L(G) \rightarrow M_{d_1}(\mathbb{C}) \times M_{d_2}(\mathbb{C}) \times \cdots \times M_{d_s}(\mathbb{C})
\]

by

\[
T f = (\hat{f}(\varphi^{(1)}), \hat{f}(\varphi^{(2)}), \ldots, \hat{f}(\varphi^{(s)})).
\]

where

\[
\hat{f}(\varphi^{(k)}) = \sum_{g \in G} \varphi^{(k)}(g)f(g),
\]

the bar denotes complex conjugation. We call \( T f \) the Fourier transform of \( f \).

**Theorem 10** ([1], Theorem 5.5.6). The Fourier transform

\[
T : L(G) \rightarrow M_{d_1}(\mathbb{C}) \times M_{d_2}(\mathbb{C}) \times \cdots \times M_{d_s}(\mathbb{C})
\]

is an isomorphism of rings.

Let

\[
\mathbb{C}G = \left\{ \sum_{g \in G} x_g g \mid x_g \in \mathbb{C}, g \in G \right\}
\]

be the group algebra of \( G \).

**Lemma 11.** Let \( G \) be a finite group. Then, we have

\[
L(G) \cong \mathbb{C}G.
\]

as an isomorphism of rigns.

3.2. Results for finite abelian groups. We give the following theorem that is an analog of Theorem 8 for finite abelian groups on the group algebra.

**Theorem 12.** Let \( G \) be a finite abelian group and \( e \) be the unit element of \( G \). Then, we have

\[
\Theta(G)e = \prod_{\chi \in \hat{G}} \sum_{g \in G} \chi(g)x_g g.
\]

Proof. Let \( G = \{ g_1, g_2, \ldots, g_n \} \) and \( \hat{G} = \{ \chi_1, \chi_2, \ldots, \chi_n \} \). By definition, we have

\[
\alpha = \sum_{i=1}^{n} x_{g_i} g_i \in \mathbb{C}G.
\]

Then, for \( \chi_k \in \hat{G} \), we define

\[
\alpha_{\chi_k} = \sum_{i=1}^{n} \chi_k(g_i)x_{g_i} g_i \in \mathbb{C}G, \quad \alpha^*_{\chi_k} = \sum_{i=1}^{n} \chi_k(g_i)x_{g_i} \in \mathbb{C}.
\]

We can regard \( \alpha_{\chi_k} \) as an element in \( L(G) \) by Lemma 11; then we have

\[
\alpha_{\chi_k}(g_i) = \chi_k(g_i)x_{g_i}.
\]
The \( l \)-th component of the Fourier transform \( T \) of \( \alpha \chi \) is

\[
\hat{\alpha \chi}(\chi l) = \sum_{i=1}^{n} \chi_l(g_i) \alpha \chi(g_i)
\]

\[
= \sum_{i=1}^{n} \chi_l(g_i) \chi_k(g_i) x_{g_i}
\]

\[
= \sum_{i=1}^{n} (\chi_k(\chi l)^{-1})(g_i) x_{g_i}
\]

\[
= \alpha_{\chi_k^{-1}}.
\]

Therefore, by Theorem 8, we have

\[
T(\alpha \chi_1) T(\alpha \chi_2) \cdots T(\alpha \chi_n) = \alpha_{\chi_1}^* \alpha_{\chi_2}^* \cdots \alpha_{\chi_n}^* (1,1,\ldots,1)
\]

By Theorem 10 we have

\[
\alpha_{\chi_1} \alpha_{\chi_2} \cdots \alpha_{\chi_n} = \Theta(G)e.
\]

\[\square\]

Theorem 12 is stronger than Theorem 8 for finite abelian groups. In fact, let \( F : \mathbb{C}G \to \mathbb{C} \) be a \( \mathbb{C} \)-linear ring homomorphism such that \( F(g) = 1 \) for all \( g \in G \), Theorem 8 for finite abelian groups follows by applying \( F \) to Theorem 12. Moreover, we obtain the following formula for inverse elements in the group algebra.

**Corollary 13.** Let \( G \) be a finite abelian group and \( \chi_1 \) be the trivial representation of \( G \). If \( \Theta(G) \neq 0 \), then we have

\[
\left( \sum_{g \in G} x_g g \right)^{-1} = \frac{1}{\Theta(G)} \prod_{\chi \in \hat{G} \setminus \{\chi_1\}} \left( \sum_{g \in G} \chi(g) x_g g \right).
\]

### 4. Recall dihedral groups and generalized quaternion groups

In this section, we recall dihedral groups and generalized quaternion groups.

#### 4.1. Recall dihedral groups.

We recall dihedral group \( D_m \) given by the presentation

\[
D_m = \langle a, b \mid a^m = e, b^2 = e, b^{-1}ab = a^{-1} \rangle.
\]

**Lemma 14.** Every element \( g \in D_m \) can be written uniquely as \( g = a^k b^l \) where \( 0 \leq k < m \), and \( l = 0,1 \). Namely, the order of \( D_m \) is \( 2m \).

We have the following list of irreducible representations of \( D_m \) ([2, Theorem 3]). Let \( \omega \) be a primitive \( m \)-th roots of unity, for \( 1 \leq k \leq m - 1 \).
(1) When \( m \) is odd, and \( 1 \leq l \leq \frac{m-1}{2} \).

\[
\begin{array}{|c|c|c|c|}
\hline
& e & a^k & a^kb \\
\hline
\chi_1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & -1 \\
\varphi_l & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \omega^{lk} & 0 \\ 0 & \omega^{-lk} \end{bmatrix} & \begin{bmatrix} 0 & \omega^{lk} \\ \omega^{-lk} & 0 \end{bmatrix} \\
\hline
\end{array}
\]

(2) When \( m \) is even, and \( 1 \leq l \leq \frac{m}{2} - 1 \).

\[
\begin{array}{|c|c|c|c|}
\hline
& e & a^k & a^kb \\
\hline
\chi_1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & -1 \\
\chi_3 & 1 & (-1)^k & i(-1)^k \\
\chi_4 & 1 & (-1)^k & i(-1)^{k+1} \\
\varphi_l & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \omega^{lk} & 0 \\ 0 & \omega^{-lk} \end{bmatrix} & \begin{bmatrix} 0 & \omega^{lk} \\ -\omega^{-lk} & 0 \end{bmatrix} \\
\hline
\end{array}
\]

4.2. Recall generalized quaternion groups. We recall generalized quaternion group \( Q_m \) given by the presentation

\[
Q_m = \langle a, b \mid a^{2m} = e, b^2 = a^m, b^{-1}ab = a^{-1} \rangle.
\]

**Lemma 15.** Every element \( g \in Q_m \) can be written uniquely as \( g = a^k b^l \) where \( 0 \leq k < 2m \), and \( l = 0, 1 \). Namely, the order of \( Q_m \) is \( 4m \).

We have the following list of irreducible representations of \( Q_m \). Let \( \omega \) be a primitive \( 2m \)-th roots of unity, for \( 1 \leq k \leq 2m - 1 \).

(1) When \( m \) is odd, and \( 1 \leq l \leq m - 1 \).

\[
\begin{array}{|c|c|c|c|}
\hline
& e & a^k & a^kb \\
\hline
\chi_1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & -1 \\
\chi_3 & 1 & (-1)^k & i(-1)^k \\
\chi_4 & 1 & (-1)^k & i(-1)^{k+1} \\
\varphi_l & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \omega^{lk} & 0 \\ 0 & \omega^{-lk} \end{bmatrix} & \begin{bmatrix} 0 & \omega^{lk} \\ -\omega^{-lk} & 0 \end{bmatrix} \\
\hline
\end{array}
\]

(2) When \( m \) is even, and \( 1 \leq l \leq m - 1 \).

\[
\begin{array}{|c|c|c|c|}
\hline
& e & a^k & a^kb \\
\hline
\chi_1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & -1 \\
\chi_3 & 1 & (-1)^k & (-1)^k \\
\chi_4 & 1 & (-1)^k & (-1)^{k+1} \\
\varphi_l & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \omega^{lk} & 0 \\ 0 & \omega^{-lk} \end{bmatrix} & \begin{bmatrix} 0 & \omega^{lk} \\ -\omega^{-lk} & 0 \end{bmatrix} \\
\hline
\end{array}
\]
5. Main results for dihedral groups and generalized quaternion groups

In this section, we describe the group determinant of dihedral groups and generalized quaternion groups as a circulant determinant of homogeneous polynomials and we give an analog of Frobenius’ theorem for dihedral groups and generalized quaternion groups about factorization of the group determinant on the group algebra.

5.1. Describe some group determinant as a circulant determinant. Let $G$ be $D_m$ or $Q_m$, $\langle a \rangle$ be the subgroup generated by $\{a\}$ and define

$$\chi'_l(a^k) = \omega^{lk}$$

for $0 \leq l \leq |\langle a \rangle| - 1$ where $|H|$ is the order of the group $H$. Notice that $\chi'_l$ is a degree one representation of $\langle a \rangle$. Let $A_h$ be a homogeneous polynomial defined by

$$A_h = \sum_{g \in \langle a \rangle} (x_g x_{hg} - x_{gb} x_{hgb^{-1}})$$

for $h \in G$. This homogeneous polynomial $A_h$ has the following properties.

**Lemma 16.** Let $G$ be $D_m$ or $Q_m$. Then, we have

1. If $h \in G \setminus \langle a \rangle$, $A_h = 0$.
2. For all $h \in G$, $A_h = A_{h^{-1}}$.

**Proof.** Element $h \in G \setminus \langle a \rangle$ can be written as $a^k b$ for $0 \leq k \leq |\langle a \rangle| - 1$. If $g \in \langle a \rangle$, then $bg = g^{-1} b$ and we can write $g$ as $a^k g^{-1}$. From these, we have

$$A_{a^k b} = \sum_{g \in \langle a \rangle} x_g x_{a^k bg} - \sum_{g \in \langle a \rangle} x_{gb} x_{a^k bg^{-1}}$$

$$= \sum_{g \in \langle a \rangle} x_{a^k g^{-1} a^k b a^k g^{-1}} - \sum_{g \in \langle a \rangle} x_{gb} x_{a^k g^{-1}}$$

$$= \sum_{g \in \langle a \rangle} x_{a^k g^{-1} g} x_{gb} - \sum_{g \in \langle a \rangle} x_{a^k g^{-1} g} x_{gb}$$

$$= 0.$$

This proves the first claim. If $h \in G \setminus \langle a \rangle$, then, we have $h^{-1} \in G \setminus \langle a \rangle$. (2) follows from first claim. We assume that $h \in \langle a \rangle$. We can write $g$ as $h^{-1} g$ in the first sum and $g$ as $g = h^{-1} gb^{-2}$ in the second. From this and $b^{-3} = b$, we have

$$A_h = \sum_{g \in \langle a \rangle} x_g x_{hg} - \sum_{g \in \langle a \rangle} x_{gb} x_{hgb^{-1}}$$

$$= \sum_{g \in \langle a \rangle} x_{(h^{-1} g)} x_{h(h^{-1} g)} - \sum_{g \in \langle a \rangle} x_{(h^{-1} gb^{-2})} x_{h(h^{-1} gb^{-2})} b^{-1}$$

$$= \sum_{g \in \langle a \rangle} x_{h^{-1} g} x_g - \sum_{g \in \langle a \rangle} x_{h^{-1} gb^{-1} g} x_{gb^{-3}}$$

$$= \sum_{g \in \langle a \rangle} x_g x_{h^{-1} g} - \sum_{g \in \langle a \rangle} x_{h^{-1} gb^{-1} g} x_{gb}$$

$$= A_{h^{-1}}.$$

This completes the proof. □
Lemma 17. Let $G$ be $D_m$ or $Q_m$ and $\chi_l (1 \leq l \leq 4)$ be a degree one representation of Section 4. Then, we have

$$\sum_{g \in G} \chi_1(g)x_g \sum_{g' \in G} \chi_2(g')x_{g'} = \sum_{h \in (a)} A_h,$$

$$\sum_{g \in G} \chi_3(g)x_g \sum_{g' \in G} \chi_4(g')x_{g'} = \sum_{h \in (a)} \chi'_1(h)A_h.$$

This lemma will be proved later.

Lemma 18. Let $G$ be $D_m$ or $Q_m$ and $\varphi_l$ be degree two representations of Section 4. Then, we have

$$\det \left( \sum_{g \in G} \varphi_l(g)x_g \right) = \sum_{h \in (a)} \chi'_1(h)A_h.$$

Proof. Define the function $\delta$ by

$$\delta(G) = \begin{cases} 1 & G = D_m, \\ -1 & G = Q_m. \end{cases}$$

We obtain

$$\det \left( \sum_{g \in G} \varphi_l(g)x_g \right) = \det \left( \sum_{g \in (a)} \varphi_l(g)x_g + \sum_{g \in G \setminus (a)} \varphi_l(g)x_g \right)$$

$$= \det \left( \sum_{g \in (a)} \varphi_l(g)x_g + \sum_{g \in (a)} \varphi_l(gb)x_{gb} \right)$$

$$= \det \left[ \sum_{g \in (a)} \chi'_1(g)x_g \sum_{g \in (a)} \chi'_1(g)x_{gb} \right]$$

$$= \sum_{h \in (a)} \sum_{g \in (a)} \chi'_1(h)\chi'_1(h)x_{gb}\chi_hx_{gb} - \delta(G) \sum_{g \in (a)} \sum_{h \in (a)} \chi'_1(g^{-1}h)x_{gb}x_{hb}.$$ (1)

We can write $h$ as $hg$ in the first sum and $h$ as $hgb^{-2}$ in the second. Then (1) equals

$$\sum_{h \in (a)} \chi'_1(h) \sum_{g \in (a)} x_gx_{hg} - \delta(G) \sum_{g \in (a)} \sum_{h \in (a)} \chi'_1(g^{-1}(gb^{-2}))x_{gb}x_{(gb^{-2})b}$$

(2) $$= \sum_{h \in (a)} \chi'_1(h) \sum_{g \in (a)} x_gx_{hg} - \delta(G) \chi'_1(b^{-2}) \sum_{h \in (a)} \chi'_1(h) \sum_{g \in (a)} x_{gb}x_{gb^{-1}}.$$ 

It is easy to see that $\delta(G)\chi'_1(b^{-2}) = 1$. Then, (2) equals

$$\sum_{h \in (a)} \chi'_1(h) \left( \sum_{g \in (a)} (x_gx_{hg} - x_{gb}x_{gb^{-1}}) \right) = \sum_{h \in (a)} \chi'_1(h)A_h$$

as required. □
Lemma 19. Let $\chi'$ be a degree one representation of $\langle a \rangle$. Then, we have

$$\sum_{g \in \langle a \rangle} \chi'(g)A_g = \sum_{g \in \langle a \rangle} \chi(g)A_g.$$ 

Proof. This follows from Lemma 16. \qed

Let $\hat{H}$ be a complete set of irreducible representations of the group $H$. We describe $\Theta(G)$ as a circulant determinant of $A_h$.

Theorem 20. Let $G$ be $D_m$ or $Q_m$. Then, we have

$$\Theta(G) = \prod_{\chi' \in \hat{\langle a \rangle}} \sum_{g \in \langle a \rangle} \chi'(g)A_g.$$ 

Proof. First, let $G = D_m$ and $m$ be odd. By Theorem 8 and Lemma 17 we have

$$\Theta(G) = \left( \sum_{g \in \langle a \rangle} A_g \right)^{m-1} \left( \sum_{g \in \langle a \rangle} \chi'(g)A_g \right)^2.$$

From Lemma 19 we have

$$\Theta(G) = \prod_{l=0}^{m-1} \sum_{g \in \langle a \rangle} \chi'_l(g)A_g.$$ 

In other case, by Theorem 8 and Lemma 17 we have

$$\Theta(G) = \left( \sum_{g \in \langle a \rangle} A_g \right) \left( \sum_{g \in \langle a \rangle} \chi'_{\lceil |a|/2 \rceil}(g)A_g \right) \prod_{l=1}^{\lfloor |a|/2 \rfloor - 1} \left( \sum_{g \in \langle a \rangle} \chi'_l(g)A_g \right)^2.$$ 

From Lemma 19 we have

$$\Theta(G) = \prod_{l=0}^{\lfloor |a|/2 \rfloor - 1} \sum_{g \in \langle a \rangle} \chi'_l(g)A_g.$$ 

\qed
5.2. Main results for dihedral groups and generalized quaternion groups.

Let \( G \) be \( D_m \) or \( Q_m \) and \( \chi_l(1 \leq l \leq 4) \) be a degree one representation of Section 4. By definition, we have

\[
\begin{align*}
\alpha_1 &= \sum_{g \in G} x_g g, \\
\alpha_2 &= \sum_{g \in (a)} \chi_2(g)x_{g^{-1}}g + \sum_{g \in G \setminus (a)} \chi_2(g)x_g g, \\
\alpha_3 &= \sum_{g \in G} \chi_3(g)x_g g, \\
\alpha_4 &= \sum_{g \in (a)} \chi_4(g)x_{g^{-1}}g + \sum_{g \in G \setminus (a)} \chi_4(g)x_g g.
\end{align*}
\]

**Lemma 21.** The following formula holds.

\[
\alpha_1 \alpha_2 = \sum_{h \in (a)} A_h h.
\]

**Proof.** We have

\[
\begin{align*}
\alpha_1 \alpha_2 &= \sum_{g' \in G} x_{g'} g' \sum_{g \in (a)} \chi_2(g)x_{g^{-1}}x_{g^{-1}}g + \sum_{g' \in G} x_{g'} g' \sum_{g \in G \setminus (a)} \chi_2(g)x_g g \\
&= \sum_{g' \in G} \sum_{g \in (a)} \chi_2(g)x_{g'}x_{g^{-1}}g'g + \sum_{g' \in G} \sum_{g \in G \setminus (a)} \chi_2(g)x_{g'}x_{g}g'g \\
&= \sum_{g' \in G} \sum_{g \in (a)} x_{g'}x_{g^{-1}}g'g + \sum_{g' \in G} \sum_{g \in G \setminus (a)} \chi_2(gb)x_{g'}x_{gb}g'gb \\
&= \sum_{g' \in G} \sum_{g \in (a)} x_{g'}x_{g^{-1}}g'g - \sum_{g' \in G} \sum_{g \in G \setminus (a)} x_{g'}x_{gb}g'gb \\
&= \sum_{g' \in G} \sum_{g \in (a)} x_{g'}x_{g^{-1}}g'g - \sum_{g' \in G} \sum_{g \in G \setminus (a)} x_{g'}x_{gb}g'g^{-1} \\
&= \sum_{g' \in G} \sum_{g \in (a)} x_{g'}x_{g^{-1}}g'g - \sum_{g' \in G} \sum_{g \in G \setminus (a)} x_{g'}x_{gb}g'g^{-1} \\
&= \sum_{g' \in G} \sum_{g \in (a)} x_{g'}x_{g^{-1}}g'g - \sum_{g' \in G} \sum_{g \in G \setminus (a)} x_{g'}x_{gb}g'g^{-1} \\
&= \sum_{g' \in G} \sum_{g \in (a)} x_{g'}x_{g^{-1}}g'g.
\end{align*}
\]

We can make the change of variables \( h = g'g \) to compute, then (3) equals

\[
\begin{align*}
\sum_{h \in G} \sum_{g \in (a)} x_{hg^{-1}}x_{g^{-1}}h - \sum_{h \in G} \sum_{g \in G \setminus (a)} x_{hg^{-1}b^{-1}}x_{g^{-1}h} \\
&= \sum_{h \in G} \sum_{g \in (a)} x_{hg}h - \sum_{h \in G} \sum_{g \in G \setminus (a)} x_{hgb^{-1}}x_{gb}h \\
&= \sum_{h \in (a)} (x_g x_{hg} - x_{gb} x_{hgb^{-1}})h \\
&= \sum_{h \in (a)} A_h h
\end{align*}
\]

as required. \( \square \)
Lemma 22. The following formula holds.

\[ \alpha_3 \alpha_4 = \sum_{h \in \langle a \rangle} \chi_{\langle a \rangle}^\prime (h) A_h. \]

Proof. We have

\[ \alpha_3 \alpha_4 = \sum_{g' \in G} \sum_{g \in \langle a \rangle} \chi_3(g') x_{g'g} \sum_{g' \in G} \chi_4(g) x_{g^{-1}g} + \sum_{g' \in G} \chi_3(g') x_{g'g} \sum_{g \in G \setminus \langle a \rangle} \chi_4(g) x_g g \]

\[ = \sum_{g' \in G} \sum_{g \in \langle a \rangle} \chi_3(g') \chi_4(g) x_{g' x_{g^{-1}g}} g' g + \sum_{g' \in G} \sum_{g \in G \setminus \langle a \rangle} \chi_3(g') \chi_4(g) x_{g' x_g g}. \]

If \( g \in \langle a \rangle \), then we have \( \chi_3(g) \chi_4(g^{-1}) = 1 \). We have

\[ \sum_{g' \in G} \sum_{g \in \langle a \rangle} \chi_3(g') \chi_4(g) x_{g' x_{g^{-1}g}} g' g = \sum_{h \in G} \sum_{g \in \langle a \rangle} \chi_3(hg^{-1}) \chi_4(g) x_{hg^{-1} x_{g^{-1}h}} = \sum_{h \in G} \sum_{g \in \langle a \rangle} \chi_3(h) \chi_4(g) x_{hg x_g h} = \sum_{h \in G} \sum_{g \in \langle a \rangle} \chi_3(h) x_{g} x_{hg} h. \]

We have

\[ \sum_{g' \in G} \sum_{g \in G \setminus \langle a \rangle} \chi_3(g') \chi_4(g) x_{g' x_{g} g} g = \sum_{g' \in G} \sum_{g \in \langle a \rangle} \chi_3(g') \chi_4(g) x_{g} x_{gb} g' g = \chi_4(b) \sum_{g' \in G} \sum_{g \in \langle a \rangle} \chi_3(g') \chi_4(g) x_{g'} x_{gb} g' g^{-1} = \chi_4(b) \sum_{g' \in G} \sum_{g \in \langle a \rangle} \chi_3(g') \chi_4(g) x_{g' b^{-1}} x_{gb} g' g^{-1}. \]

(4)

It is easy that to see \( \chi_3(b^{-1}) \chi_4(b) = -1 \), then (4) equals

\[ - \sum_{g' \in G} \sum_{g \in \langle a \rangle} \chi_3(g') \chi_4(g) x_{g' b^{-1}} x_{gb} g' g^{-1} = - \sum_{h \in G} \sum_{g \in \langle a \rangle} \chi_3(hg^{-1}) \chi_4(g^{-1}) x_{hg^{-1} b^{-1} x_{g^{-1}h}} = - \sum_{h \in G} \sum_{g \in \langle a \rangle} \chi_3(h) \chi_4(g) x_{hg b^{-1} x_{gb} h}. \]

(5)

If \( g \in \langle a \rangle \), then we have \( \chi_3(g) \chi_4(g) = 1 \) and (5) equals

\[ - \sum_{h \in G} \sum_{g \in \langle a \rangle} \chi_3(h) x_{hb^{-1}} x_{gb} h. \]
We compute
\[
\alpha_3 \alpha_4 = \sum_{h \in G} \chi_3(h) \sum_{g \in \langle a \rangle} (x_g x_{hg} - x_{gb} x_{hgb^{-1}}) h
\]
\[
= \sum_{h \in G} \chi_3(h) A_h h
\]
\[
= \sum_{h \in \langle a \rangle} \chi_3(h) A_h h
\]
\[
= \sum_{h \in \langle a \rangle} \chi'_{\langle a \rangle} (h) A_h h
\]
as required.

\[\square\]

Lemmas 21 and 22 are stronger than Lemma 17. In fact, Lemma 17 follows by applying F to Lemmas 21 and 22.

**Theorem 23.** Let \( G \) be \( D_m \) or \( Q_m \) and \( e \) be the unit element of \( G \). Then, we have
\[
\Theta(G)e = \prod_{\chi' \in \hat{\langle a \rangle}} \sum_{g \in \langle a \rangle} \chi'(g) x_g g.
\]

**Proof.** The group \( \langle a \rangle \) is a finite abelian group and by Theorem 12 there exists a \( C \) such that
\[
Ce = \prod_{\chi' \in \hat{\langle a \rangle}} \sum_{g \in \langle a \rangle} \chi'(g) x_g g.
\]

By Theorem 20 and the mapping \( F \), we have
\[
C = \Theta(G).
\]

This completes the proof. \[\square\]

Moreover, we obtain the following formula for inverse elements in the group algebra for \( D_m \) and \( Q_m \).

**Corollary 24.** Let \( G \) be \( D_m \) or \( Q_m \). If \( \Theta(G) \neq 0 \),

1. When \( G = D_m \) and \( m \) is odd, we have
\[
\alpha_1^{-1} = \frac{1}{\Theta(G)^2} \prod_{\chi' \in \hat{\langle a \rangle} \setminus \{ \chi_0 \}} \sum_{g \in \langle a \rangle} \chi'(g) x_g g.
\]
2. In other cases, we have
\[
\alpha_1^{-1} = \frac{1}{\Theta(G)} \alpha_2 \alpha_3 \alpha_4 \prod_{\chi' \in \hat{\langle a \rangle} \setminus \{ \chi_0 \}} \sum_{g \in \langle a \rangle} \chi'(g) x_g g.
\]

Note that the group algebra is non-commutative, so the order of the factors is important.
6. Algebraic properties of irreducible factors corresponding to degree one representations

In this section, we follow the commutators of irreducible factors of the factorization of the group determinant on the group algebra for \(D_m\) and \(Q_m\) corresponding to degree one representations have interesting algebraic properties. From this result, we know that degree one representations from natural pairing.

6.1. Transformations of irreducible factors corresponding to degree one representations. In this subsection, we obtain transformations of irreducible factors corresponding to degree one representations. We need it to prove for algebraic properties of irreducible factors corresponding to degree one representations.

Lemma 25. Let \(G\) be \(D_m\) or \(Q_m\). Under the change of variable

\[
x_g \mapsto \begin{cases} 
\chi_2(g)x_g^{-1} & \text{if } g \in \langle a \rangle, \\
\chi_2(g)x_g & \text{otherwise.}
\end{cases}
\]

Then \(\alpha_1\) becomes \(\alpha_2\) by and vice versa, the same holds for \(\alpha_3\) and \(\alpha_4\).

Proof. From the definition of \(\alpha_1\), the factor \(\alpha_1\) becomes \(\alpha_3\) under the change of variable. The factor \(\alpha_2\) becomes

\[
\sum_{g \in \langle a \rangle} \chi_2(g)\chi_2(g^{-1})x_g + \sum_{g \in G \setminus \langle a \rangle} \chi_2(g)\chi_2(g)x_g = \sum_{g \in \langle a \rangle} x_g + \chi_2(b)^2 \sum_{g \in G \setminus \langle a \rangle} x_g \\
= \sum_{g \in \langle a \rangle} x_g + \sum_{g \in G \setminus \langle a \rangle} x_g \\
= \sum_{g \in G} x_g = \alpha_1.
\]

The factor \(\alpha_3\) becomes

\[
\sum_{g \in \langle a \rangle} \chi_3(g)\chi_2(g^{-1})x_g + \sum_{g \in G \setminus \langle a \rangle} \chi_3(g)\chi_2(g)x_g \\
= \sum_{g \in \langle a \rangle} x_g - \sum_{g \in G \setminus \langle a \rangle} x_g \\
= \sum_{g \in \langle a \rangle} \chi_4(g)x_g^{-1} + \sum_{g \in G \setminus \langle a \rangle} \chi_4(g)x_g \\
= \alpha_4.
\]

The factor \(\alpha_4\) becomes

\[
\sum_{g \in \langle a \rangle} \chi_4(g)\chi_2(g^{-1})x_g + \sum_{g \in G \setminus \langle a \rangle} \chi_4(g)\chi_2(g)x_g \\
= \sum_{g \in \langle a \rangle} x_g - \sum_{g \in G \setminus \langle a \rangle} x_g \\
= \sum_{g \in \langle a \rangle} \chi_3(g)x_g + \sum_{g \in G \setminus \langle a \rangle} \chi_3(g)x_g \\
= \alpha_3.
\]

This completes the proof. \(\square\)
Lemma 26. Let $G$ be $D_m$ or $Q_m$. Under the change of variable $x_g \mapsto \chi_3(g)x_g$, the factor $\alpha_1$ becomes $\alpha_3$, the factor $\alpha_2$ becomes $\alpha_4$, and $\alpha_3$ becomes

$$\sum_{g \in \langle a \rangle} x_g g - \sum_{g \in G \setminus \langle a \rangle} x_g g$$

in the case that $m$ is odd and $\alpha_1$ otherwise. The factor $\alpha_4$ becomes

$$\sum_{g \in \langle a \rangle} x_{g^{-1}} g + \sum_{g \in G \setminus \langle a \rangle} x_g g$$

in the case that $m$ is odd and $\alpha_2$ otherwise. Namely, the factor $\alpha_3 + \alpha_4$ becomes $\alpha_1 + \alpha_2$.

Proof. In the definition of $\alpha_1$, we obtain $\alpha_3$. The factor $\alpha_2$ becomes

$$\sum_{g \in \langle a \rangle} \chi_2(g)\chi_3(g^{-1})x_{g^{-1}} g + \sum_{g \in G \setminus \langle a \rangle} \chi_2(g)\chi_3(g)x_g g$$

$$= \sum_{g \in \langle a \rangle} \chi_3(g)x_{g^{-1}} g - \sum_{g \in G \setminus \langle a \rangle} \chi_3(g)x_g g$$

$$= \sum_{g \in \langle a \rangle} \chi_4(g)x_{g^{-1}} g + \sum_{g \in G \setminus \langle a \rangle} \chi_4(g)x_g g$$

$$= \alpha_4.$$  

The factor $\alpha_3$ becomes

$$\sum_{g \in G} \chi_3(g)^2 x_g = \sum_{g \in \langle a \rangle} x_g g + \chi_3(b)^2 \sum_{g \in G \setminus \langle a \rangle} x_g g.$$  

When $m$ is even, right hand side is equal to $\alpha_1$, otherwise is equal to

$$\sum_{g \in \langle a \rangle} x_g g - \sum_{g \in G \setminus \langle a \rangle} x_g g.$$  

The factor $\alpha_4$ becomes

$$\sum_{g \in \langle a \rangle} \chi_4(g)\chi_3(g^{-1})x_{g^{-1}} g + \sum_{g \in G \setminus \langle a \rangle} \chi_4(g)\chi_3(g)x_g g$$

$$= \sum_{g \in \langle a \rangle} x_{g^{-1}} g - \chi_3(b)^2 \sum_{g \in G \setminus \langle a \rangle} x_g g,$$  

When $m$ is odd, right hand side is equal to $\alpha_2$, otherwise is equal to

$$\sum_{g \in \langle a \rangle} x_{g^{-1}} g + \sum_{g \in G \setminus \langle a \rangle} x_g g.$$  

This completes the proof. □

6.2. Algebraic properties of irreducible factors corresponding to degree one representations. In this subsection, we follow the commutators of irreducible factors of the factorization of the group determinant on the group algebra for $D_m$ and $Q_m$ corresponding to degree one representations have interesting algebraic properties. From this result, we know that degree one representations form natural pairing.
Lemma 27. If \( h \in \langle a \rangle \), then we have
\[
\sum_{g \in \langle a \rangle} x_{gb}^{-1}g^{-1}h = \sum_{g \in \langle a \rangle} x_{gb}x_{hgb}^{-1}.
\]

Proof. We compute
\[
\sum_{g \in \langle a \rangle} x_{gb}^{-1}g^{-1}h = \sum_{g \in \langle a \rangle} x_{b^{-1}g^{-1}h}x_{gb} = \sum_{g \in \langle a \rangle} x_{h^{-1}gb^{-1}}x_{gb} = \sum_{g \in \langle a \rangle} x_{h^{-1}gb^{-1}x_{gb}} = \sum_{g \in \langle a \rangle} x_{gb}x_{hgb}^{-1}
\]
as required. \( \Box \)

Lemma 28. If \( h \in G \setminus \langle a \rangle \), then we have
\[
\sum_{g \in G \setminus \langle a \rangle} x_gx_{g^{-1}h} = \sum_{g \in \langle a \rangle} x_gx_{gh}.
\]

Proof. We compute
\[
\sum_{g \in G \setminus \langle a \rangle} x_gx_{g^{-1}h} = \sum_{g \in \langle a \rangle} x_{gb}x_{b^{-1}g^{-1}h}x_{gb} = \sum_{g \in \langle a \rangle} x_{gb}x_{b^{-1}gb^{-1}}x_{gb} = \sum_{g \in \langle a \rangle} x_{gb}x_{b^{-1}gb^{-1}x_{gb}} = \sum_{g \in \langle a \rangle} x_{gb}x_{hgb}^{-1}
\]
as required. \( \Box \)

Define \([a, b] = ab - ba\) where \( a, b \in CG\). This is equal to zero if and only if \( a \) and \( b \) commute.

Theorem 29. The following formula holds.
\[
[a_1, a_2] = 0.
\]
Proof. We have
\[
\alpha_2 \alpha_1 = \left( \sum_{g \in (a)} x_{g^{-1} g} - \sum_{g' \in G} x_g g' \right) \left( \sum_{g' \in G} x_{g' g'} \right)
\]
\[
= \sum_{g' \in G} \left( \sum_{g \in (a)} x_{g^{-1} g} x_{g' g'} - \sum_{g \in G \setminus (a)} x_{g' g'} \right)
\]
\[
= \sum_{h \in G} \left( \sum_{g \in (a)} x_{g^{-1} g} x_{g^{-1} h} - \sum_{g \in G \setminus (a)} x_{g} x_{g^{-1} h} \right) h.
\]
By Lemmas 25 and 28 we have
\[
\alpha_2 \alpha_1 = \sum_{h \in G} \left( \sum_{g \in (a)} x_{g x_{h g^{-1}}} - \sum_{g \in (a)} x_{g} x_{h g^{-1}} \right) = \alpha_1 \alpha_2.
\]
This completes the proof. \qed

Theorem 30. The following formula holds.
\[
[a_3, a_4] = 0.
\]
Proof. This theorem follows from Lemma 29 and Theorem 29. \qed

Lemma 31. The following formula holds.
\[
[a_1, a_3] = \sum_{g \in G} \sum_{g' \in G} \chi_3(g^{-1} h) x_g x_{g^{-1} h} h - \sum_{h \in G} \sum_{g \in G} \chi_3(g) x_g x_{g^{-1} h} h.
\]
Proof. We compute
\[
[a_1, a_3] = \sum_{g \in G} x_g \sum_{g' \in G} \chi_3(g') x_{g' g'} - \sum_{g \in G} \chi_3(g) x_g \sum_{g' \in G} x_{g' g'}
\]
\[
= \sum_{g \in G} \sum_{g' \in G} \chi_3(g') x_g x_{g' g'} - \sum_{g \in G} \sum_{g' \in G} \chi_3(g) x_g x_{g' g'}
\]
\[
= \sum_{g \in G} \sum_{h \in G} \chi_3(g^{-1} h) x_g x_{g^{-1} h} h - \sum_{g \in G} \sum_{h \in G} \chi_3(g) x_g x_{g^{-1} h} h
\]
as required. \qed

Lemma 32. Let \( h \) be \( a^k \) with \( k \) odd. Then, we have
\[
\sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1} h} = 0.
\]
Proof. We compute
\[
\sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1} h} = \sum_{g \in (a)} \chi_3(g^{-1} h) x_{g^{-1} h} x_g
\]
\[
= - \sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1} h}
\]
as required. \qed
Lemma 33. The coefficient of \( h = a^k \) in \([\alpha_1, \alpha_3]\) is

\[
(-1)^k \sum_{g \in G \setminus \langle a \rangle} \chi_3(g^{-1})x_gx_g^{-1}h - \sum_{g \in G} \chi_3(g)x_gx_g^{-1}h.
\]

Proof. We have

\[
\sum_{g \in G} \chi_3(g^{-1})x_gx_g^{-1}h - \sum_{g \in G} \chi_3(g)x_gx_g^{-1}h = (-1)^k \sum_{g \in G} \chi_3(g^{-1})x_gx_g^{-1}h - \sum_{g \in G} \chi_3(g)x_gx_g^{-1}h = (-1)^k \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_g^{-1}h + (-1)^k \sum_{g \in G \setminus \langle a \rangle} \chi_3(g)x_gx_g^{-1}h - \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_g^{-1}h.
\]

From Lemma 32, this completes the proof. \(\square\)

By Lemma 33, we know that \([\alpha_1, \alpha_3] \neq 0\).

From Lemma 25, we have \([\alpha_2, \alpha_4] \neq 0\).

Namely, we have to be careful with the factors in Corollary 24.

Lemma 34. The coefficient of \( h = a^kb \) in \([\alpha_1, \alpha_3]\) is

1. For the case that \( m \) is odd.

\[
\begin{cases}
(1 + i) \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_{gh} - (1 + i) \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_g^{-1}h, & \text{if } k \text{ is odd.} \\
(1 - i) \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_{gh} - (1 - i) \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_g^{-1}h, & \text{if } k \text{ is even.}
\end{cases}
\]

2. For the case that \( m \) is even.

\[
\begin{cases}
-2 \sum_{g \in G} \chi_3(g)x_gx_g^{-1}h, & \text{if } k \text{ is odd.} \\
0, & \text{if } k \text{ is even.}
\end{cases}
\]
Proof. (1) For the case that \( m \) is odd, we compute
\[
\sum_{g \in G} \chi_3(g^{-1}h)x_gx_{g^{-1}h} - \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h}
\]
\[
= (-1)^k \sum_{g \in G} \chi_3(g^{-1})x_gx_{g^{-1}h} - \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h}
\]
\[
= (-1)^k \sum_{g \in (a)} \chi_3(g^{-1})x_gx_{g^{-1}h} + (-1)^k \sum_{g \in G \setminus (a)} \chi_3(g^{-1})x_gx_{g^{-1}h}
\]
\[
- \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h} - \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h}
\]
\[
= (-1)^k \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h} + (-1)^k \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h}
\]
\[
- \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h} - i \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h}
\]
\[
= (-1)^k \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h} + (-1)^k \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h}
\]
\[
- \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h} - i \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h}
\]
\[
= (-1)^k \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h} + \sum_{g \in (a)} \chi_3(g)x_gx_g
\]
\[
- \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h} + (-1)^{k+1} \sum_{g \in (a)} \chi_3(g)x_gx_g
\]
as required.

(2) For the case that \( m \) is even, we compute
\[
\sum_{g \in G} \chi_3(g^{-1}h)x_gx_{g^{-1}h} - \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h}
\]
\[
= (-1)^k \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h} - \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h}
\]
as required.

\[\square\]

Lemma 35. The following formula holds.
\[
[a_1, a_4] = \sum_{h \in G} \sum_{g \in (a)} \chi_3(g)x_gx_{hg}\ h - \sum_{h \in G} \sum_{g \in (a)} \chi_3(g)x_gx_{h^{-1}g}h
\]
\[
- \sum_{h \in G} \sum_{g \in (a)} \chi_3(g)x_gx_{gh}h + \sum_{h \in G} \sum_{g \in (a)} \chi_3(g)x_gx_{g^{-1}h}h.
\]
Proof. We compute

\[
[\alpha_1, \alpha_4] = \sum_{g' \in G} x_{g'} g' \left( \sum_{g \in G(a)} \chi_4(g)x_{g^{-1}g} + \sum_{g \in G \setminus G(a)} \chi_4(g)x_g g \right) - \left( \sum_{g \in G(a)} \chi_4(g)x_{g^{-1}g} + \sum_{g \in G \setminus G(a)} \chi_4(g)x_g g \right) \sum_{g' \in G} x_{g'} g' \\
= \sum_{g' \in G} \sum_{g \in G(a)} \chi_4(g)x_{g' x^{-1}g} g' + \sum_{g \in G \setminus G(a)} \sum_{g' \in G} \chi_4(g)x_{g' x_g} g' g \\
- \sum_{g \in G(a)} \sum_{g' \in G} \chi_4(g)x_{g^{-1}x_g} g' g - \sum_{g \in G \setminus G(a)} \sum_{g' \in G} \chi_4(g)x_g x_{g' g} g' \\
= \sum_{h \in G} \sum_{g \in G(a)} \chi_3(g)x_{h g^{-1} x_g^{-1}h} + \sum_{h \in G \setminus G(a)} \chi_3(g)x_{h g^{-1} x_g^{-1}h} \\
- \sum_{g \in G(a)} \sum_{h \in G} \chi_3(g)x_{g^{-1}x_g} x_{g^{-1}h} + \sum_{g \in G \setminus G(a)} \chi_3(g)x_{g x_{g^{-1}h}} \\
- \sum_{h \in G} \sum_{g \in G(a)} \chi_3(g)x_g x_{h g^{-1}h} + \sum_{h \in G} \sum_{g \in G \setminus G(a)} \chi_3(g)x_{g x_{g^{-1}h}}
\]

as required.

\[\square\]

Lemma 36. The coefficient of \(h = a^k\) in \([\alpha_1, \alpha_4]\) is

\[\sum_{g \in G \setminus G(a)} \chi_3(g^{-1})x_g x_{g^{-1}h} + \sum_{g \in G \setminus G(a)} \chi_3(g)x_g x_{g^{-1}h}.\]

Proof. We compute

\[
\sum_{g \in G(a)} \chi_3(g)x_g x_{h g} - \sum_{g \in G \setminus G(a)} \chi_3(g)x_g x_{h g^{-1}} \\
- \sum_{g \in G(a)} \chi_3(g)x_g x_{g h} + \sum_{g \in G \setminus G(a)} \chi_3(g)x_g x_{g^{-1}h} \\
= - \sum_{g \in G \setminus G(a)} \chi_3(g)x_g x_{h g^{-1}} + \sum_{g \in G \setminus G(a)} \chi_3(g)x_g x_{g^{-1}h} \\
= - \sum_{g \in G \setminus G(a)} \chi_3(g^{-1}h)x_{g^{-1}h x h^{-1}g} + \sum_{g \in G \setminus G(a)} \chi_3(g)x_{g x_{g^{-1}h}} \\
= (-1)^{k+1} \sum_{g \in G \setminus G(a)} \chi_3(g^{-1})x_g x_{g^{-1}h} + \sum_{g \in G \setminus G(a)} \chi_3(g)x_g x_{g^{-1}h}
\]

as required.

\[\square\]
From Lemma \[36\] we know that
\[\alpha_1, \alpha_4 \neq 0.\]
From Theorems \[29\] and \[30\] we think that degree one representations form natural pairing \((\chi_1, \chi_2)\) and \((\chi_3, \chi_4)\). Furthermore, we will have another interesting algebraic properties soon.

**Lemma 37.** The coefficient of \(h = a^k b\) in \([\alpha_1, \alpha_4]\) is

1. For the case that \(m\) is odd.
   \[
   \begin{cases}
   -(1 + i) \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + (1 + i) \sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1}h}, & \text{if } k \text{ is odd.} \\
   -(1 - i) \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + (1 - i) \sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1}h}, & \text{if } k \text{ is even.}
   \end{cases}
   \]
2. For the case that \(m\) is even.
   \[
   \begin{cases}
   2 \sum_{g \in G} \chi_3(g) x_g x_{g^{-1}h}, & \text{if } k \text{ is odd.} \\
   0, & \text{if } k \text{ is even.}
   \end{cases}
   \]

**Proof.**
1. For the case that \(m\) is odd, we compute
   \[
   \begin{align*}
   &\sum_{g \in (a)} \chi_3(g) x_g x_{gh} - \sum_{g \in G \setminus (a)} \chi_3(g) x_g x_{gh^{-1}} \\
   &\quad - \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + \sum_{g \in G \setminus (a)} \chi_3(g) x_g x_{g^{-1}h} \\
   &= \sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1}h} - \sum_{g \in (a)} \chi_3(g b) x_{gb} x_{gh^{-1}g^{-1}} \\
   &\quad - \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + \sum_{g \in (a)} \chi_3(g b) x_{gb} x_{g^{-1}h} \\
   &= \sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1}h} - i \sum_{g \in (a)} \chi_3(g) x_{gb} x_{a^k g^{-1}} \\
   &\quad - \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + i \sum_{g \in (a)} \chi_3(g) x_{gb} x_{a^{-k} g} \\
   &= \sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1}h} - i \sum_{g \in (a)} \chi_3(1) x_{ga^k h} x_{gh^{-1}} \\
   &\quad - \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + (-1)^k i \sum_{g \in (a)} \chi_3(g) x_{gh} x_g \\
   &= \sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1}h} - (-1)^k i \sum_{g \in (a)} \chi_3(g) x_{gh} x_{g^{-1}} \\
   &\quad - \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + (-1)^k i \sum_{g \in (a)} \chi_3(g) x_g x_{gh}
   \end{align*}
   \]
as required.
2. For the case that \(m\) is even, we compute
   \[
   \begin{align*}
   &\sum_{g \in (a)} \chi_3(g) x_g x_{gh} - \sum_{g \in G \setminus (a)} \chi_3(g) x_g x_{gh^{-1}} \\
   &\quad - \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + \sum_{g \in G \setminus (a)} \chi_3(g) x_g x_{g^{-1}h} \\
   &= \sum_{g \in (a)} \chi_3(g) x_g x_{g^{-1}h} - \sum_{g \in (a)} \chi_3(g) x_{gh} x_{g^{-1}} \\
   &\quad - \sum_{g \in (a)} \chi_3(g) x_g x_{gh} + \sum_{g \in G \setminus (a)} \chi_3(g) x_g x_{gh^-1}
   \end{align*}
   \]
\[- \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_{gh} + \sum_{g \in G \backslash \langle a \rangle} \chi_3(g)x_gx_{g^{-1}h} \]
\[= \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_{g^{-1}h} - \sum_{g \in \langle a \rangle} \chi_3(gb)x_{gb}x_{hb^{-1}g^{-1}} \]
\[\quad - \sum_{g \in \langle a \rangle} \chi_3(gb)x_{b^{-1}gb}x_{gh} + \sum_{g \in G \backslash \langle a \rangle} \chi_3(g)x_gx_{g^{-1}h} \]
\[= \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h} - \sum_{g \in \langle a \rangle} \chi_3(g)x_{gb}x_{a^kg^{-1}} \]
\[\quad - \sum_{g \in \langle a \rangle} \chi_3(gb)x_{b^{-1}g^{-1}a^kbg} \]
\[= \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h} - \sum_{g \in \langle a \rangle} \chi_3(ga^{-k})x_{ga^kb}x_{g^{-1}h} \]
\[\quad - \sum_{g \in \langle a \rangle} \chi_3(ga^{-k}b)x_{b^{-1}g^{-1}a^kbg}x_{gb} \]
\[= \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h} + (-1)^{k+1} \sum_{g \in \langle a \rangle} \chi_3(g)x_gx_{g^{-1}h} \]
\[\quad + (-1)^{k+1} \sum_{g \in G \backslash \langle a \rangle} \chi_3(g)x_{g^{-1}h}x_g \]
\[= \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h} + (-1)^{k+1} \sum_{g \in G} \chi_3(g)x_gx_{g^{-1}h} \]

as required. \[\square\]

We have the following algebraic properties.

**Theorem 38.** The formulas

\[
[\alpha_1, \alpha_3 + \alpha_4] = 0,
\]
\[
[\alpha_2, \alpha_3 + \alpha_4] = 0,
\]
\[
[\alpha_3, \alpha_1 + \alpha_2] = 0,
\]
\[
[\alpha_4, \alpha_1 + \alpha_2] = 0
\]

hold.

**Proof.** By Lemmas \[33\] \[34\] \[36\] and \[37\] we have

\[
[\alpha_1, \alpha_3 + \alpha_4] = 0.
\]

From Lemmas \[25\] and \[26\] this completes the proof. \[\square\]
Acknowledgments

I am deeply grateful to Prof. Hiroyuki Ochiai and Prof. Minoru Itoh who provided the helpful comments and suggestions. Also, I would like to thank my college in the Graduate School of Mathematics of Kyushu University, in particular Cid Reyes, Tomoyuki Tamura, Yuka Suzuki for comments and suggestions. I would also like to express my gratitude to my family for their moral support and warm encouragements.

REFERENCES

[1] Benjamin Steinberg, Representation Theory of Finite Groups. Springer, 2012.
[2] C. Shan, C. Hong, T. Guoping, Augmentation quotients for complex representation rings of dihedral groups, Frontiers of Mathematics in China. Vol. 7, pp. 1-18, 2012.
[3] K. W. Jonson, On the group determinant, Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 109, pp. 299-311, 1991.
[4] Z. Qingxia, Y. Hong, On the Structure of Augmentation Quotient Groups for the Generalized Quaternion Group, Algebra Colloquium. Vol. 19, pp. 137-148, 2012.

Naoya Yamaguchi
Graduate School of Mathematics
Kyushu University
Nishi-ku, Fukuoka 819-0395
Japan
n-yamaguchi@math.kyushu-u.ac.jp