BOUNDINESS OF THE BILINEAR BOCHNER-RIESZ MEANS IN THE NON-BANACH TRIANGLE CASE

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ABSTRACT. In this article, we investigate the boundedness of the bilinear Bochner-Riesz means $S^\alpha$ in the non-Banach triangle case. We improve the corresponding results in [1] in two aspects: Our partition of the non-Banach triangle is simpler and we obtain lower smoothness indices $\alpha(p_1, p_2)$ for various cases apart from $1 \leq p_1 = p_2 < 2$.

1. Introduction

The bilinear Bochner-Riesz means problem originates from the study of the summability of the product of two $n$-dimensional Fourier series. This leads to the study of the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of the bilinear Fourier multiplier operator

$$S^\alpha(f,g)(x) = \int_1^{\infty} \int_{|\xi|^2 + |\eta|^2 \leq 1} (1 - |\xi|^2 - |\eta|^2)^\alpha \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

where $x \in \mathbb{R}^n$, $f$, $g$ are functions on $\mathbb{R}^n$ and $\hat{f}$, $\hat{g}$ are their Fourier transforms. Of course, we hope to obtain a smoothness index $\alpha(p_1, p_2)$ as low as possible such that $S^\alpha$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^p$ for $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$ when $\alpha > \alpha(p_1, p_2)$. Bernicot et al. [1] gave a comprehensive study on this problem. Their results are pretty well when $n = 1$ and in the Banach triangle case when $n \geq 2$. In this article, we investigate the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of $S^\alpha$ in the non-Banach triangle case, i.e. $1 \leq p_1, p_2 \leq \infty$ and $p < 1$, when $n \geq 2$. We improve the corresponding results in [1] in two aspects. Firstly, our partition of the non-Banach triangle is simpler. Secondly, we obtain lower smoothness indices $\alpha(p_1, p_2)$ for various cases apart from $1 \leq p_1 = p_2 < 2$.

Our results are summarized in the following theorem. The pictures also display the comparison of two partitions.

Theorem 1. Assume that $n \geq 2$. Let $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$.

(1)(region I) For $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$ and $p < 1$, if $\alpha > n(\frac{1}{p_1} - \frac{1}{2})$, then $S^\alpha$ is bounded from $L^{p_1} \times L^{p_2}$ to $L^p$; For $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$ and $p < 1$, if $\alpha > n(\frac{1}{p_2} - \frac{1}{2})$, then $S^\alpha$ is bounded from $L^{p_1} \times L^{p_2}$ to $L^p$.

(2)(region II) For $1 \leq p_1 \leq p_2 \leq 2$, if $\alpha > n(\frac{1}{p} - 1) - (\frac{1}{p_1} - \frac{1}{2})$, then $S^\alpha$ is bounded from $L^{p_1} \times L^{p_2}$ to $L^p$; For $1 \leq p_2 \leq p_1 \leq 2$, if $\alpha > n(\frac{1}{p} - 1) - (\frac{1}{p_2} - \frac{1}{2})$, then $S^\alpha$ is bounded from $L^{p_1} \times L^{p_2}$ to $L^p$.

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Remark: Fefferman [2] pointed out that the restriction estimates apply to the study of the boundedness of Bochner-Riesz means. Stein’s earlier result in [4] was improved by using the restriction theorem (see [5]). Our results are obtained without the help of non-trivial restriction estimates. In contrast, the results in [1] depend heavily on the restriction-extension estimates. It seems still a problem that how apply the restriction estimates to obtain a better result about the boundedness of the bilinear Bochner-Riesz means.

2. Preliminaries

We state some facts about the Bessel function and the kernel of the Bochner-Riesz operator. These facts can be found in many references, for example, in [5].

We define the restriction operators

\[ R_\lambda f(x) = \int_{\mathbb{S}^{n-1}} \hat{f}(\lambda \omega) e^{2\pi i x \cdot \lambda \omega} d\sigma(\omega), \quad \lambda > 0, \]

where \( d\sigma \) denote the surface measure on \( \mathbb{S}^{n-1} \). Then

\[ R_1 f(x) = \int_{\mathbb{S}^{n-1}} \hat{f}(\omega) e^{2\pi i x \cdot \omega} d\sigma(\omega) = f * \tilde{d}\sigma, \]

It is known that

\[ \tilde{d}\sigma(x) = \int_{\mathbb{S}^{n-1}} e^{2\pi i x \cdot \omega} d\sigma(\omega) = \frac{2\pi}{|x|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi |x|), \]

where \( J_k \) is the Bessel function given by

\[ J_k(r) = \frac{(\frac{r}{2})^k}{\Gamma(k+\frac{1}{2}) \pi^{\frac{1}{2}}} \int_{-1}^{1} e^{irt}(1-t^2)^{-\frac{k}{2}} dt, \quad k > -\frac{1}{2}. \]

By a dilation argument, we have

\[ R_\lambda f = f * \varphi_\lambda, \quad \lambda > 0, \]

where

\[ \varphi_\lambda(x) = \tilde{d}\sigma(\lambda x) = \frac{2\pi}{(\lambda |x|)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi \lambda |x|). \]
The bilinear Bochner-Riesz means can be written as
\[ S_R^\alpha(f, g)(x) = \int_0^\infty \int_0^\infty \left(1 - \frac{\lambda_1^2 + \lambda_2^2}{R^2}\right)^\alpha R_{\lambda_1}f(x)R_{\lambda_2}g(x) \lambda_1^{n-1}\lambda_2^{n-1} d\lambda_1 d\lambda_2. \]
It is easy to see that
\[ S_R^\alpha(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-x_1)g(x-x_2)S_R^\alpha(x_1, x_2) \, dx_1 \, dx_2, \]
where the kernel is given by
\[ S_R^\alpha(x_1, x_2) = \int_0^\infty \int_0^\infty \left(1 - \frac{\lambda_1^2 + \lambda_2^2}{R^2}\right)^\alpha \varphi_{\lambda_1}(x_1)\varphi_{\lambda_2}(x_2) \lambda_1^{n-1}\lambda_2^{n-1} d\lambda_1 d\lambda_2. \]
Because
\[ S_R^\alpha(x_1, x_2) = R^{2n}S_1^\alpha(Rx_1, Rx_2), \]
by a dilation argument, we know that the \( L^{p_1} \times L^{p_2} \to L^p \) boundedness of the operator \( S_R^\alpha \) is deduced from the \( L^{p_1} \times L^{p_2} \to L^p \) boundedness of the operator \( S_1^\alpha := S_1^\alpha \) as \( 1/p = 1/p_1 + 1/p_2 \).

We note that the kernel of the bilinear Bochner-Riesz operator \( S^\alpha \) on \( \mathbb{R}^n \) coincides with the kernel of the Bochner-Riesz operator on \( \mathbb{R}^{2n} \). This means
\[ S^\alpha(x_1, x_2) = \frac{\Gamma(1+\alpha)}{\pi^\alpha \left| (x_1, x_2) \right|^{n+\alpha}} J_{n+\alpha}(2\pi \left| (x_1, x_2) \right|). \]

The Bessel function satisfies the asymptotic estimate
\[ J_k(r) = O(r^{-\frac{1}{2}}), \quad r \to \infty. \]
As a consequence, we have a basic result: \( S^\alpha \) is bounded from \( L^{p_1} \times L^{p_2} \) into \( L^p \) when \( \alpha > n - \frac{1}{2} \), which is optimal in case of \( (p_1, p_2, p) = (1, 1, \frac{1}{2}) \).

3. PROOF OF RESULTS.

**Lemma 1.** Suppose \( m \in L^\infty(\mathbb{R}) \). Define operator \( T_mf = \int_a^b m(\lambda)R_\lambda f \lambda^{-1} d\lambda \) for \( 0 \leq a < b \). Then, for \( 1 \leq p \leq 2 \), we have
\[ \|T_mf\|_2 \leq C \left((b-a)b^{n-1}\right)^{\frac{1}{2}} \|m\|_\infty \|f\|_p \]

**Proof.** The operator \( T_m \) can be written as
\[ T_mf = f \ast G_m, \]
where the kernel
\[ G_m(x) = \int_{\mathbb{R}^n} \chi_{[a,b]}(\left| \xi \right|)m(\left| \xi \right|)e^{2\pi i \left< x, \xi \right>} \, d\xi. \]
Applying Plancherel theorem, we have
\[ \|G_m\|_2^2 = \int_{a \leq \left| \xi \right| \leq b} |m(\left| \xi \right|)|^2 \, d\xi \leq C \|m\|_\infty^2 ((b-a)b^{n-1}) \cdot \]
It follows that
\[ \|T_mf\|_2 \leq \|G_m\|_2 \|f\|_1 \leq C \left((b-a)b^{n-1}\right)^{\frac{1}{2}} \|m\|_\infty \|f\|_1. \]
Interpolation with the trivial estimate
\[ \|T_mf\|_2 \leq \|m\|_\infty \|f\|_2, \]
we get
\[ \|T_mf\|_2 \leq C \left((b-a)b^{n-1}\right)^{\left(\frac{1}{p}-\frac{1}{2}\right)} \|m\|_\infty \|f\|_p. \]
Lemma 1 is proved. \qed

**Theorem 2.** Let \( n \geq 2 \). Suppose \( 1 \leq p_1, p_2 \leq 2 \) and \( 1/p = 1/p_1 + 1/p_2 \). If \( \alpha > n(\frac{1}{p} - 1) \), then \( S^\alpha \) is bounded from \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \).

**Proof.** First, we choose a nonnegative function \( \varphi \in C_0^\infty((\frac{1}{2}, 2]) \) satisfying \( \sum_{-\infty}^{\infty} \varphi(2^j s) = 1 \), \( s > 0 \). For each \( j \geq 0 \), we set function

\[
\varphi_j^\alpha(s, t) = (1 - s^2 - t^2)^\alpha \varphi(2^j (1 - s^2 - t^2)),
\]

and define bilinear operator

\[
T_j^\alpha(f, g) = \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1, \lambda_2) R_{\lambda_1} f R_{\lambda_2} g \lambda_1^{\alpha-1} \lambda_2^{\alpha-1} d\lambda_1 d\lambda_2.
\]

It is obvious that

\[
S^\alpha = \sum_{j=0}^{\infty} T_j^\alpha,
\]

and our result would follow if we can show that when \( \alpha > n(\frac{1}{p} - 1) \), there exists some \( \varepsilon > 0 \) such that for each \( j \geq 0 \),

\[
(3.1) \quad \|T_j^\alpha\|_{L^{p_1} \times L^{p_2} \to L^p} \leq 2^{-\varepsilon j}.
\]

Fixing \( j \geq 0 \). In order to prove (3.1), we define \( B_j = \{ x : |x| \leq 2^{j(1+\gamma)} \} \subseteq \mathbb{R}^n \) and split the kernel \( K_j^\alpha \) of \( T_j^\alpha \) into four parts:

\[
K_j^\alpha = K_j^1 + K_j^2 + K_j^3 + K_j^4,
\]

where

\[
K_j^1(x_1, x_2) = K_j^2(x_1, x_2) \chi_{B_j}(x_1) \chi_{B_j}(x_2),
K_j^2(x_1, x_2) = K_j^2(x_1, x_2) \chi_{B_j}(x_1) \chi_{B_j}(x_2),
K_j^3(x_1, x_2) = K_j^3(x_1, x_2) \chi_{B_j}(x_1) \chi_{B_j}(x_2),
K_j^4(x_1, x_2) = K_j^4(x_1, x_2) \chi_{B_j}(x_1) \chi_{B_j}(x_2).
\]

Here \( \chi_A \) stands for the characteristic function of set \( A \) and \( \gamma > 0 \) is to be fixed. Let \( T_j^l \) be the bilinear operator with kernel \( K_j^l \), \( l = 1, 2, 3, 4 \). Then, (3.1) would be the consequence of the estimates

\[
\|T_j^l\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C 2^{-\varepsilon j}, \quad l = 1, 2, 3, 4.
\]

We first consider \( T_j^1 \). Note that the kernel of \( T_j^1 \) is written as

\[
K_j^1(x_1, x_2) = \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1, \lambda_2) \varphi_{\lambda_1}(x_1) \varphi_{\lambda_2}(x_2) \lambda_1^{\alpha-1} \lambda_2^{\alpha-1} d\lambda_1 d\lambda_2
\]

satisfying

\[
|K_j^1(x_1, x_2)| \leq C 2^{-j \alpha} 2^{-j} (1 + 2^{-j} |x_1|)^{-M} (1 + 2^{-j} |x_2|)^{-M}
\]

for each \( M > 0 \). So, by Hölder’s inequality and Young’s inequality, we can get

\[
\|T_j^1(f, g)\|_p \leq C \|f\|_{p_1} \|g\|_{p_2} \int_{|x_1| \geq 2^{j(1+\gamma)}} (1 + 2^{-j} |x_1|)^{-M} dx_1
\]

\[
\times \int_{|x_2| \geq 2^{j(1+\gamma)}} (1 + 2^{-j} |x_2|)^{-M} dx_2
\]

\[
\leq C \left( 2^{jM} 2^{-j(1+\gamma)(M-n)} \right)^{\frac{1}{2}} \|f\|_{p_1} \|g\|_{p_2}.
\]
Choosing \( M \) large enough such that
\[
M \gamma > (1 + \gamma)n,
\]
it follows that
\[
\|T^3_j\|_{L^p \times L^p \to L^p} \leq C 2^{-\varepsilon j}
\]
for some \( \varepsilon > 0 \). Similarly, for \( T^3_j \), we have
\[
\|T^3_j(f, g)\|_p \leq C \|f\|_{p_1} \|g\|_{p_2} \int_{|x_1| \leq 2^{j(1+\gamma)}} (1 + 2^{-j} |x_1|)^{-M} \, dx_1
\]
\[
\times \int_{|x_2| \geq 2^{j(1+\gamma)}} (1 + 2^{-j} |x_2|)^{-M} \, dx_2
\]
\[
\leq C 2^{j(1+\gamma)n} 2^{jM} 2^{-j(1+\gamma)(M-n)} \|f\|_{p_1} \|g\|_{p_2}.
\]
Choosing large \( M \) satisfying
\[
M \gamma > 2(1 + \gamma)n,
\]
it follows that
\[
\|T^3_j\|_{L^p \times L^p \to L^p} \leq C 2^{-\varepsilon j}.
\]
for some \( \varepsilon > 0 \). Obviously, (3.3) also holds for \( T^j_j \).

Now, it remains to estimate \( T^j_j \). For any fixed \( y \in \mathbb{R}^n \), we set \( B_j(y, R) = \{ x : |x - y| \leq R 2^{j(1+\gamma)} \} \) with \( R > 0 \), and split the functions \( f \) and \( g \) into three parts respectively: \( f = f_1 + f_2 + f_3 \), \( g = g_1 + g_2 + g_3 \), where
\[
f_1 = f \chi_{B_j(y, \frac{3}{4})}, \quad g_1 = g \chi_{B_j(y, \frac{3}{4})},
\]
\[
f_2 = f \chi_{B_j(y, \frac{3}{4}) \setminus B_j(y, \frac{5}{4})}, \quad g_2 = g \chi_{B_j(y, \frac{3}{4}) \setminus B_j(y, \frac{5}{4})},
\]
\[
f_3 = f \chi_{\mathbb{R}^n \setminus B_j(y, \frac{5}{4})}, \quad g_3 = g \chi_{\mathbb{R}^n \setminus B_j(y, \frac{5}{4})}.
\]
Assume that \( |x - y| \leq \frac{1}{4} 2^{j(1+\gamma)} \). Since that \( f_3 \) is supported on \( \mathbb{R}^n \setminus B_j(y, \frac{5}{4}) \), then \( f_3 \neq 0 \) leads to
\[
|x - x_1 - y| \geq \frac{5}{4} 2^{j(1+\gamma)}.
\]
It follows that
\[
|x_1| \geq \frac{1}{2} 2^{j(1+\gamma)} \quad \text{and} \quad |x_2| \geq \frac{1}{2} 2^{j(1+\gamma)}.
\]
Note that the kernel \( K^j_j \) is supported on \( B_j \times B_j \). Hence, \( T^j_j(f_3, g) = 0 \). In the same way, we have \( T^j_j(f_2, g_3) = 0 \). Since that \( f_2 \) and \( g_2 \) are supported on \( B_j(y, \frac{5}{4}) \setminus B_j(y, \frac{3}{4}) \), then \( f_2, g_2 \neq 0 \) yields that
\[
|x_1| \geq \frac{1}{2} 2^{j(1+\gamma)} \quad \text{and} \quad |x_2| \geq \frac{1}{2} 2^{j(1+\gamma)}.
\]
Repeating the proof of (3.2), we get that
\[
\|T^j_j(f_2, g_2)\|_{L^p(B_j(y, \frac{3}{4}))} \leq C 2^{-\varepsilon j} \|f_2\|_{p_1} \|g_2\|_{p_2}
\]
\[
\leq C 2^{-\varepsilon j} \|f\|_{L^{p_1}(B_j(y, \frac{5}{4}))} \|g\|_{L^{p_2}(B_j(y, \frac{5}{4}))}.
\]
Both side of this inequality are the fuction of $y$. So, taking the $L^p$ norm with respect to $y$ and using Hölder’s inequality, we have that

$$
\left( \int_{\mathbb{R}^n} \int_{B_j(y, \frac{1}{4})} \left| T_j^1(f_2, g_2)(x) \right|^p \, dx \, dy \right)^{\frac{1}{p}} \leq C 2^{-\varepsilon j} \left( \int_{\mathbb{R}^n} \int_{B_j(y, \frac{1}{4})} \left| f(x) \right|^{p_1} \, dx \, dy \right)^{\frac{1}{p_1}} \left( \int_{\mathbb{R}^n} \int_{B_j(y, \frac{1}{4})} \left| g(x) \right|^{p_2} \, dx \, dy \right)^{\frac{1}{p_2}}.
$$

Changing variable and exchanging the order of integration, the left side equals to

$$
\left( \int_{\mathbb{R}^n} \int_{B_j(y, \frac{1}{4})} \left| T_j^1(f_2, g_2)(x) \right|^p \, dx \, dy \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^n} \int_{|x| \leq \frac{1}{4} 2^{j(1+\gamma)}} \left| T_j^1(f_2, g_2)(x+y) \right|^p \, dx \, dy \right)^{\frac{1}{p}} = \left( \int_{|x| \leq \frac{1}{4} 2^{j(1+\gamma)}} \int_{\mathbb{R}^n} \left| T_j^1(f_2, g_2)(x+y) \right|^p \, dy \, dx \right)^{\frac{1}{p}} = \left( \frac{1}{4} 2^{j(1+\gamma)} \right)^{\frac{n}{p}} \left\| T_j^1(f_2, g_2) \right\|_p.
$$

Similarly, the right side equals to

$$
C 2^{-\varepsilon j} \left( \frac{5}{4} 2^{j(1+\gamma)} \right)^{\frac{n}{p_1}} \| f \|_{p_1} \left( \frac{5}{4} 2^{j(1+\gamma)} \right)^{\frac{n}{p_2}} \| g \|_{p_2} = C 2^{-\varepsilon j} \left( \frac{5}{4} 2^{j(1+\gamma)} \right)^{\frac{n}{p}} \| f \|_{p_1} \| g \|_{p_2}.
$$

This yields that

(3.4) \quad \left\| T_j^1(f_2, g_2) \right\|_p \leq C 2^{-\varepsilon j} \| f \|_{p_1} \| g \|_{p_2}.

Since that $f_1$ is supported on $B_j(y, \frac{1}{4})$, then $f_1, g_2 \neq 0$ implies that

$$
|x_1| \leq 2^{j(1+\gamma)} \quad \text{and} \quad |x_2| \geq \frac{1}{2} 2^{j(1+\gamma)}.
$$

Repeating the proof of (3.3), we have

$$
\left\| T_j^1(f_1, g_2) \right\|_{L^p(B_j(y, \frac{1}{4}))} \leq C 2^{-\varepsilon j} \| f_1 \|_{p_1} \| g_2 \|_{p_2} \leq C 2^{-\varepsilon j} \| f \|_{L^{p_1}(B_j(y, \frac{1}{4}))} \| g \|_{L^{p_2}(B_j(y, \frac{1}{4}))}.
$$

Taking the $L^p$ norm with respect to $y$ as above, we get that

(3.5) \quad \left\| T_j^1(f_1, g_2) \right\|_p \leq C 2^{-\varepsilon j} \| f \|_{p_1} \| g \|_{p_2}.

Obviously, (3.5) also holds for $T_j^1(f_2, g_1)$.

Finally, we consider $T_j^1(f_1, g_1)$. Because $f_1, g_1 \neq 0$ implies that

$$
|x_1| \leq 2^{j(1+\gamma)} \quad \text{and} \quad |x_2| \leq 2^{j(1+\gamma)},
$$

we have

(3.6) \quad T_j^1(f_1, g_1)(x) = T_j^1(f_1, g_1)(x), \quad x \in B_j(y, \frac{1}{4}).
Using Hölder’s inequality and (3.6), it follows that

\[ T_j^\alpha(f, g) = \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1, \lambda_2) R_{\lambda_1} f R_{\lambda_2} g \lambda_1^{n-1} \lambda_2^{n-1} d\lambda_1 d\lambda_2 \]

\[ = \frac{1}{2} \int_{[-1,1]^2} \varphi_j^\alpha(\lambda_1, \lambda_2) R_{\lambda_1} f R_{\lambda_2} g \lambda_1^{n-1} \lambda_2^{n-1} d\lambda_1 d\lambda_2. \]

Because for any fixed \( s \in [-1, 1] \), the function

\[ t \mapsto \varphi_j^\alpha(|s|, |t|) \]

is supported in \([-1, 1]\) and vanishes at endpoints \( \pm 1 \), so we can expand this function in Fourier series by considering a periodic extension on \( \mathbb{R} \) of period 2. Then, we have

\[ \varphi_j^\alpha(|s|, |t|) = \sum_{k \in \mathbb{Z}} \gamma_{j,k}^\alpha(s) e^{i\pi k t} \]

with Fourier coefficients

\[ \gamma_{j,k}^\alpha(s) = \frac{1}{2} \int_{-1}^1 \varphi_j^\alpha(|s|, |t|) e^{-i\pi k t} dt. \]

It is easy to see that for any \( 0 < \delta < \alpha \),

\[ \sup_{s \in [-1, 1]} |\gamma_{j,k}^\alpha(s)| (1 + |k|)^{1+\delta} \leq C 2^{-j(\alpha - \delta)}. \]

\( T_j^\alpha \) can be expressed by

\[ T_j^\alpha(f, g) = C \int_{[-1,1]^2} \varphi_j^\alpha(\lambda_1, \lambda_2) R_{\lambda_1} f R_{\lambda_2} g \lambda_1^{n-1} \lambda_2^{n-1} d\lambda_1 d\lambda_2 \]

\[ = C \sum_{k \in \mathbb{Z}} \int_{[-1,1]^2} \gamma_{j,k}^\alpha(\lambda_1) e^{i\pi k \lambda_2} R_{\lambda_1} f R_{\lambda_2} g \lambda_1^{n-1} \lambda_2^{n-1} d\lambda_1 d\lambda_2 \]

\[ = C \sum_{k \in \mathbb{Z}} \int_{[-1,1]}^1 \gamma_{j,k}^\alpha(\lambda_1) R_{\lambda_1} f \lambda_1^{n-1} d\lambda_1 \int_{-1}^1 e^{i\pi k \lambda_2} R_{\lambda_2} g \lambda_2^{n-1} d\lambda_2. \]

Applying Cauchy-Schwartz’s inequality and Lemma 1, we get that

(3.7) \[ \|T_j^\alpha(f, g)\|_1 \]

\[ \leq C \sum_{k \in \mathbb{Z}} \left( \int_{-1}^1 \gamma_{j,k}^\alpha(\lambda_1) R_{\lambda_1} f \lambda_1^{n-1} d\lambda_1 \right)^2 \left( \int_{-1}^1 e^{i\pi k \lambda_2} R_{\lambda_2} g \lambda_2^{n-1} d\lambda_2 \right)^2 \]

\[ \leq C \sum_{k \in \mathbb{Z}} (1 + |k|)^{-1-\delta} \left( \sup_{s \in [-1, 1]} |\gamma_{j,k}^\alpha(s)| (1 + |k|)^{1+\delta} \right) \|f\|_{p_1} \|g\|_{p_2}. \]

Using Hölder’s inequality and (3.6), it follows that

\[ \|T_j^1(f_1, g_1)\|_{L^p(B_j(y, \frac{1}{4}))} \leq 2^{in(1+\gamma)(\frac{1}{p}-1)} \|T_j^1(f_1, g_1)\|_{L^1(B_j(y, \frac{1}{4}))} \]

\[ = 2^{in(1+\gamma)(\frac{1}{p}-1)} \|T_j^\alpha(f_1, g_1)\|_{L^1(B_j(y, \frac{1}{4}))} \]

\[ \leq C 2^{-j(\alpha - \delta)} 2^{in(1+\gamma)(\frac{1}{p}-1)} \|f_1\|_{p_1} \|g_1\|_{p_2} \]

\[ \leq C 2^{-j(\alpha - \delta)} 2^{in(1+\gamma)(\frac{1}{p}-1)} \|f\|_{L^p(B_j(y, \frac{1}{4}))} \|g\|_{L^p(B_j(y, \frac{1}{4}))}. \]
Taking the $L^p$ norm with respect to $y$ yields that
\begin{equation}
\|T^1_j(f_1, g_1)\|_p \leq C 2^{-j(\alpha - \delta)} 2^{j(n(1+\gamma) - \frac{1}{p})} \|f\|_{p_1} \|g\|_{p_2}.
\end{equation}
Combining (3.4), (3.5) and (3.8), we conclude that
\begin{equation*}
\|T^1_j(f, g)\|_p \leq C 2^{-j(\alpha - \delta)} 2^{j(n(1+\gamma) - \frac{1}{p})} \|f\|_{p_1} \|g\|_{p_2}.
\end{equation*}
Therefore, whenever $\alpha > n(\frac{1}{p} - 1)$, we can choose $\gamma, \delta > 0$ such that
\begin{equation*}
\alpha > n(1 + \gamma)\left(\frac{1}{p} - 1\right) + \delta,
\end{equation*}
which implies that there exists an $\varepsilon > 0$ such that
\begin{equation*}
\|T^1_j\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C 2^{-\varepsilon j}.
\end{equation*}
The proof of Theorem 2 is completed.

As a consequence of Theorem 2, we can give another proof of the estimate in case of $(p_1, p_2, p) = (1, \infty, 1)$, which was already obtained in [1].

**Corollary 1.** If $\alpha > \frac{n}{2}$, then $S^\alpha$ is bounded from $L^1 \times L^\infty$ to $L^1$.

**Proof.** We keep the notations in the proof of Theorem 2. The proof of Theorem 2 is valid apart from the estimate of $T^1_j(f_1, g_1)$. According to (3.7), for any $0 < \delta < \alpha$,
\begin{equation*}
\|T^\alpha_j(f, g)\|_1 \leq C 2^{-j(\alpha - \delta)} \|f\|_1 \|g\|_2,
\end{equation*}
we have
\begin{equation*}
\|T^1_j(f_1, g_1)\|_{L^1(B_j(y, \frac{1}{4}))} = \|T^\alpha_j(f_1, g_1)\|_{L^1(B_j(y, \frac{1}{4}))} \leq C 2^{-j(\alpha - \delta)} \|f\|_{L^1(B_j(y, \frac{1}{4}))} \|g\|_{L^2(B_j(y, \frac{1}{4}))} \leq C 2^{-j(\alpha - \delta)} 2^{j(1+\gamma) \frac{n}{2}} \|f\|_{L^1(B_j(y, \frac{1}{4}))} \|g\|_{L^\infty(B_j(y, \frac{1}{4}))}.
\end{equation*}
It follows that
\begin{equation*}
\|T^1_j(f_1, g_1)\|_1 \leq C 2^{-j(\alpha - \delta)} 2^{j(1+\gamma) \frac{n}{2}} \|f\|_1 \|g\|_\infty.
\end{equation*}
Thus, when $\alpha > \frac{n}{2}$, we can choose $\gamma, \delta > 0$ such that $\alpha > \frac{(1+\gamma)n}{2} + \delta$, which yields that there exists $\varepsilon > 0$ such that
\begin{equation*}
\|T^1_j(f_1, g_1)\|_1 \leq C 2^{-\varepsilon j} \|f\|_1 \|g\|_\infty.
\end{equation*}
The proof is completed.

Now we have obtained the estimate for $S^\alpha$ at some specific triples of points $(p_1, p_2, p)$ like
\begin{equation*}
(1, 1, \frac{1}{2}), (1, 2, \frac{2}{3}), (2, 1, \frac{2}{3}), (2, 2, 1), (1, \infty, 1), (\infty, 1, 1).
\end{equation*}
In fact, our new result is essentially the estimate in case of $(p_1, p_2) = (1, 2, \frac{2}{3})$. The results in Theorem 1 can be obtained by using of the bilinear interpolation via complex method adapted to the setting of analytic families or real method in [3], which was described in [1].

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