A weak solution to a perturbed one-Laplace system by $p$-Laplacian is continuously differentiable

Shuntaro Tsubouchi*

Abstract

In this paper we aim to show continuous differentiability of weak solutions to a one-Laplace system perturbed by $p$-Laplacian with $1 < p < \infty$. The main difficulty on this equation is that uniform ellipticity breaks near a facet, the place where a gradient vanishes. We would like to prove that derivatives of weak solutions are continuous even across the facets. This is possible by estimating Hölder continuity of Jacobian matrices multiplied with its modulus truncated near zero. To show this estimate, we consider an approximated system, and use standard methods including De Giorgi’s truncation and freezing coefficient arguments.

Mathematics Subject Classification (2020) 35B65, 35J47, 35J92

Keywords $C^1$-regularity, De Giorgi’s truncation, freezing coefficient method

1 Introduction

In this paper, we consider a very singular elliptic system given by

$$-b \div (|Du|^{-1} \nabla u^i) - \div (|Du|^{p-2} \nabla u^i) = f^i \quad \text{for } i \in \{1, \ldots, N\}$$

in a fixed domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary. Here $b \in (0, \infty)$, $p \in (1, \infty)$ are fixed constants, and the dimensions $n, N$ satisfy $n \geq 2, N \geq 1$. For each $i \in \{1, \ldots, N\}$, the scalar functions $u^i = u^i(x_1, \ldots, x_n)$ and $f^i = f^i(x_1, \ldots, x_n)$ are respectively unknown and given in $\Omega$. The vector $\nabla u^i = (\partial_{x_1} u^i, \ldots, \partial_{x_n} u^i)$ denotes the gradient of the scalar function $u^i$ with $\partial_{x^i} u^i = \partial u^i / \partial x^i$ for $i \in \{1, \ldots, N\}$, $\alpha \in \{1, \ldots, n\}$. The matrix $Du = (\partial_{x^i} u^\alpha)_{i, \alpha}$ denotes the $N \times n$ Jacobian matrix of the mapping $u = (u^1, \ldots, u^n)$. Also, the divergence operator $\div X = \sum_{j=1}^n \partial_{x^j} X^j$ is defined for an $\mathbb{R}^n$-valued vector field $X = (X^1, \ldots, X^n)$ with $X^j = X^j(x_1, \ldots, x_n)$ for $j \in \{1, \ldots, n\}$. This elliptic system is denoted by

$$L_{b,p} u := -b \div (|Du|^{-1}Du) - \div (|Du|^{p-2}Du) = f \quad \text{in } \Omega \quad (1.1)$$

with $f := (f^1, \ldots, f^N)$, or more simply by $-b \Delta_1 u - \Delta_p u = f$. Here the divergence operators $\Delta_1$ and $\Delta_p$, often called one-Laplacian and $p$-Laplacian respectively, are given by

$$\Delta_1 u := \div (|Du|^{-1}Du), \quad \Delta_p u := \div (|Du|^{p-2}Du).$$

In the case $b = 0$, the system (1.1) becomes so called the $p$-Poisson system. For this problem, Hölder regularity of $Du$ is well-established both in scalar and system cases\[10, 19, 27, 33, 36, 37\] (see

*Graduate School of Mathematical Sciences, The University of Tokyo, Japan. The author was partly supported by Grant-in-Aid for JSPS Fellows (No. 22J12394). Email: tsubos@ms.u-tokyo.ac.jp
also [11], [14], [15], [16] for parabolic $p$-Laplace problems). In the limiting case $p = 1$, Hölder continuity of $Du$ generally fails even if $f$ is sufficiently smooth. In fact, even in the simplest case $n = N = 1$, any absolutely continuous non-decreasing function of one variable $u = u(x_1)$ satisfies $-\Delta_1 u = 0$. In particular, even for the one-Laplace equation, it seems impossible in general to show continuous differentiability ($C^1$-regularity) of weak solutions. This problem is substantially because ellipticity of one-Laplace degenerates in the direction of $Du$, which differs from $p$-Laplace. Also, in the multi-dimensional case $n \geq 2$, diffusivity of $\Delta_1 u$ is non-degenerate in directions that are orthogonal to $Du$. It should be mentioned that this ellipticity becomes singular in a facet $\{Du = 0\}$, the place where a gradient vanishes. Since one-Laplace $\Delta_1$ contains anisotropic diffusivity, consisting of degenerate and singular ellipticity, this operator seems analytically difficult to handle in existing elliptic regularity theory.

Therefore, it is a non-trivial problem whether a solution to (1.1), which is a one-Laplace system perturbed by $p$-Laplace, is in $C^1$. In the special case where a solution is both scalar-valued and convex, Giga and the author answered this problem affirmatively in a paper [22]. Most of the arguments therein, based on convex analysis and a strong maximum principle, are rather elementary, although the strategy may not work for the system case. After that work was completed, the authors have found it possible to show $C^1$-regularity of weak solutions without convexity of solutions. Instead, we would like to use standard methods in elliptic regularity theory, including a freezing argument and De Giorgi’s truncation. This approach is valid even for the system problem, and the main purpose of this paper is to establish $C^1$-regularity results in the vectorial case $N \geq 2$.

It is worth mentioning that for the scalar case $N = 1$, $C^1$-regularity results have been established in the author’s recent work [35], where a generalization of the operator $\Delta_1$ is also discussed. Although the basic strategy in this paper is the same with [35], some computations in this paper are rather simpler, since the diffusion operator is assumed to have a symmetric structure, often called the Uhlenbeck structure. It should be recalled that this structure will be often required when one considers everywhere regularity for vector-valued problems. For this reason, we do not try to generalize $\Delta_1$ in this system problem.

More generally, in this paper we would like to discuss an elliptic system

$$\mathcal{L}u := -b\Delta_1 u - \mathcal{L}_p u = f \quad \text{in } \Omega,$$

where $\mathcal{L}_p$ generalizes $\Delta_p$. The detailed conditions of $\mathcal{L}_p$ are described later in Section 1.3.

1.1 Our strategy

We would like to briefly describe our strategy in this paper. It should be mentioned that even for the system problem, the strategy itself is the same as the scalar case [35]. For simplicity, we consider the model case (1.2). Our result is

Theorem 1.1. Let $p \in (1, \infty)$, $q \in (n, \infty]$, and $f \in L^q(\Omega; \mathbb{R}^N)$. Assume that $u$ is a weak solution to the system (1.1). Then, $u$ is continuously differentiable.

The main difficulty on showing $C^1$-regularity is that the system (1.1) becomes non-uniformly elliptic near a facet. To explain this, we compute the Hessian matrix $D^2_\xi E(\xi_0)$ for $\xi_0 \in \mathbb{R}^{Nn} \setminus \{0\}$, where $E(\xi) := b|\xi| + |\xi|^p / p (\xi \in \mathbb{R}^{Nn})$ denotes the energy density. As a result, this $Nn \times Nn$ real symmetric matrix satisfies

$$\text{(ellipticity ratio of } D^2_\xi E(\xi_0)) := \frac{\text{(the largest eigenvalue of } D^2_\xi E(\xi_0))}{\text{(the lowest eigenvalue of } D^2_\xi E(\xi_0))} \leq \frac{1 + b\delta^{1-p}}{p-1} := \mathcal{R}(\delta)$$

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for all $\xi_0 \in \mathbb{R}^n$ with $|\xi_0| > \delta > 0$ with $\delta$ sufficiently close to 0. It should be mentioned that the bound $R(\delta)$ will blow up as $\delta$ tends to 0. In other words, the system (1.1) becomes non-uniformly elliptic as $Du$ vanishes. In particular, it will be difficult to show Hölder continuity of derivatives across a facet of a solution to (1.1). We would like to emphasize that our problem is substantially different from either a non-standard growth problem or a $(p, q)$-growth problem, where ellipticity ratios will become unbounded as a gradient blows up [29], [30]. To see our computation above, however, it will be possible that a mapping $G_0(Du)$ with

$$G_0(\xi) := (|\xi| - \delta) \frac{\xi}{|\xi|}$$

for $\xi \in \mathbb{R}^n, \delta \in (0, 1)$ (1.3)

is $\alpha$-Hölder continuous for some constant $\alpha \in (0, 1)$, which may depend on $\delta$. This observation is expectable because the mapping $G_0(Du)$ is supported in a place $\{|Du| > \delta\}$, where the problem (1.1) becomes uniformly elliptic. Although this $\alpha = \alpha(\delta)$ might degenerate as we let $\delta \to 0$, we are able to conclude that $Du$ is also continuous. In fact, by the definition of $G_0$, it is easy to check that the mapping $G_0(Du)$ uniformly converges to $G_0(Du) = Du$ as $\delta \to 0$. Thus, to prove $C^1$-regularity of solutions, it suffices to prove continuity of the mapping $G_0(Du)$ for each fixed $\delta \in (0, 1)$.

When we show Hölder continuity of $G_0(Du)$, one of the main difficulties is that the system (1.1) becomes very singular near facets of solutions. In particular, it seems impossible to justify regularity on second order Sobolev derivatives of solutions across the facets, based on difference quotient methods. Therefore, we will have to relax the very singular operator $L_{b, p}$ by regularized operators that are non-degenerate and uniformly elliptic, so that higher regularity on Sobolev derivatives are guaranteed. For the model case (1.1), the approximation problem is given by

$$L_{b, p}^\varepsilon u_\varepsilon := -\text{div} \left( \frac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \right) - \text{div} \left( (\varepsilon^2 + |Du_\varepsilon|^2)^{p/2 - 1} Du_\varepsilon \right) = f_\varepsilon \quad \text{for each } \varepsilon \in (0, 1),$$

(1.4)

where $f_\varepsilon$ converges to $f$ in a weak sense. The relaxed operator $L_{b, p}^\varepsilon$ naturally appears when one approximates the density $E$ by

$$E_\varepsilon(z) := b \sqrt{\varepsilon^2 + |z|^2} + \frac{1}{p} \left( \varepsilon^2 + |z|^2 \right)^{p/2} \quad \text{for } \varepsilon \in (0, 1), z \in \mathbb{R}^n.$$

Then, the system (1.4) is uniformly elliptic, in the sense that

$$(\text{ellipticity ratio of } D^2\xi E_\varepsilon(\xi_0)) \leq C \left( 1 + \left( \varepsilon^2 + |\xi_0|^2 \right)^{(1-p)/2} \right) \leq C \left( 1 + \varepsilon^{1-p} \right) \quad \text{for all } \xi_0 \in \mathbb{R}^n,$$

where the positive constant $C$ depends only on $b$, and $p$. This ellipticity ratio appears to be dominated by $\sqrt{\varepsilon^2 + |\xi_0|^2}$, rather than by $|\xi_0|$. In particular, to measure ellipticity ratios for (1.4), it is natural to adapt $V_\varepsilon := \sqrt{\varepsilon^2 + |Du_\varepsilon|^2}$ as another modulus. For this reason, we have to consider another mapping $G_{\delta, \varepsilon}(Du_\varepsilon)$, where

$$G_{\delta, \varepsilon}(\xi) := \left( \sqrt{\varepsilon^2 + |\xi|^2} - \delta \right) \frac{\xi}{|\xi|} \quad \text{for } \xi \in \mathbb{R}^n \quad \text{with } 0 < \varepsilon < \delta.$$

(1.5)

Then, our problem is reduced to a priori Hölder estimates of $G_{2\delta, \varepsilon}(Du_\varepsilon)$, where $u_\varepsilon$ solves (1.4) with $0 < \varepsilon < \delta/4$ and $0 < \delta < 1$. To obtain these a priori estimates, we appeal to a freezing coefficient argument and De Giorgi’s truncation. Roughly speaking, the former method can be applied when $Du_\varepsilon$ does not vanish in a suitable sense, and otherwise the latter is fully used. To judge whether $Du_\varepsilon$ degenerates or not, we will measure superlevel sets of $V_\varepsilon$. It should be mentioned that distinguishing based on sizes of
superlevel sets is found in existing works on $C^{1,\alpha}$-regularity for $p$-Laplace problems both in elliptic and parabolic cases (see e.g., [11 Chapter IX], [28]).

Our method works in the system case, as long as the energy density $E$ is spherically symmetric. Here we recall a well-known fact that when one considers everywhere regularity of a vector-valued solution, a diffusion operator is often required to have a symmetric structure, called the Uhlenbeck structure. For this reason, generalization of the one-Laplace operator $\Delta_1$ is not discussed in this paper (see also [35 §2.4] for detailed explanations). It should be emphasized that this restriction is not necessary in the scalar case. In fact, generalization of $\Delta_1$ is given in the author’s recent work [35], which only focuses on the scalar case, and deals with more general approximation arguments based on the convolution of standard mollifiers.

1.2 Mathematical models and comparisons to related works on very degenerate problems

In Section 1.2, we briefly describe mathematical models.

The equation (1.1) is derived from a minimizing problem of a functional

$$\mathcal{F}(u) := \int_{\Omega} E(Du) \, dx - \int_{\Omega} (f \cdot u) \, dx \quad \text{with} \quad E(z) := b|\xi| + \frac{1}{p}|\xi|^p \quad \text{for} \quad \xi \in \mathbb{R}^{Nn}$$

under a suitable boundary condition. Here $(\cdot | \cdot)$ denotes the standard inner product in $\mathbb{R}^N$. The density $E$ often appears in mathematical modeling of materials, including motion of Bingham fluids and growth of crystal surface.

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In a paper [32], Spohn gave a mathematical model of crystal surface growth under roughening temperatures. From a thermodynamic viewpoint (see [25] and the references therein), the evolution of a scalar-valued function $h = h(x, t)$ denoting the height of crystal in a two-dimensional domain $\Omega$, is modeled as

$$\partial_t h + \Delta \mu = 0 \quad \text{with} \quad \Delta = \Delta_2.$$ 

Here $\mu$ is a scalar-valued function denoting a chemical potential, and considered to satisfy the Euler–Lagrange equation

$$\mu = \frac{-\delta \Phi}{\delta h} \quad \text{with} \quad \Phi(h) = \beta_1 \int_{\Omega} |\nabla h| \, dx + \beta_3 \int_{\Omega} |\nabla h|^3 \, dx, \quad \beta_1, \beta_3 > 0.$$ 

The energy functional $\Phi$ is called a crystal surface energy, whose density is essentially the same as $E$ with $p = 3$. Finally, the resulting evolution equation for $h$ is given by

$$k \partial_t h = \Delta L_{k\beta_1,3} h \quad \text{with} \quad k = \frac{1}{3\beta_3}.$$ 

When $h$ is stationary, then $h$ must satisfy an equation $L_{k\beta_1,3} h = f$, where $f = -k \mu$ is harmonic and therefore smooth. Our Theorem 1.1 implies that a gradient $\nabla h$ is continuous.

When $p = 2$, the density $E$ also appears when modeling motion of a material called Bingham fluid. Mathematical formulations concerning Bingham fluids are found in [17 Chapter VI]. In particular, when the motion of fluids is sufficiently slow, the stationary flow model results in an elliptic system

$$\frac{\delta \Psi}{\delta V} \equiv \mu_2 (L_{\mu_1, \mu_2,2} V) = -\nabla \pi \quad \text{with} \quad \Psi(V) := \mu_1 \int_{\Omega} |DV| \, dx + \frac{\mu}{2} \int_{\Omega} |DV|^2 \, dx.$$ 

Here the unknown $\mathbb{R}^3$-valued vector field $V = V(x)$ denotes velocity of Bingham fluids in a domain $U \subset \mathbb{R}^3$ and satisfies an incompressible condition $\text{div} V = 0$. The other unknown scalar-valued function $\pi = \pi(x)$ denotes the pressure. Bingham fluids contain two different aspects of elasticity and viscosity,
and corresponding to these properties, the positive constants \(\mu_1\) and \(\mu_2\) respectively appear in the energy functional \(\Psi\). From Theorem 1.1 we conclude that if the pressure function \(\pi\) satisfies \(\nabla \pi \in L^q(U; \mathbb{R}^3)\) for some \(q > 3\), then the velocity field \(V\) is continuously differentiable. Also, when one considers a stationary laminar Bingham flow in a cylindrical pipe \(U = \Omega \times \mathbb{R} \subset \mathbb{R}^3\), a scalar problem appears. To be precise, we let \(V\) be of the form \(V = (0, 0, u(x_1, x_2))\), where \(u\) is an unknown scalar function. Clearly this flow \(V\) is incompressible. Under this setting, we have a resulting elliptic equation

\[
L_{\mu_1/\mu_2, 2}u = f \quad \text{in } \Omega \subset \mathbb{R}^2 \quad \text{with} \quad f = -\frac{\pi'}{\mu_2},
\]

where the pressure function \(\pi\) depends only on \(x_3\), and its derivative \(\pi' = -\mu_2 f\) must be constant. Also in this laminar model, from our main result, it follows that the component \(u\) is continuously differentiable.

The density function \(E\) also appears in mathematical modeling of congested traffic dynamics [7]. There this model results in a minimizing problem

\[
\sigma_{\text{opt}} \in \text{arg min} \left\{ \int_{\Omega} E(\sigma) \, dx \bigg| \begin{array}{c} \sigma \in L^p(\Omega; \mathbb{R}^n), \\
-\text{div} \sigma = f \text{ in } \Omega, \sigma \cdot n = 0 \text{ on } \partial \Omega \end{array} \right\} \tag{1.6}
\]

In a paper [5], it is shown that the optimal traffic flow \(\sigma_{\text{opt}}\) of the problem (1.6) is uniquely given by \(\nabla E^*(\nabla v)\), where \(v\) solves an elliptic equation

\[
-\text{div}(\nabla E^*(\nabla v)) = f \in L^q(\Omega) \quad \text{in } \Omega \tag{1.7}
\]

under a Neumann boundary condition. Here

\[
E^*(z) = \frac{1}{p'}(|z| - b)^{p'}
\]

is the Legendre transform of \(E\), and \(p' := p/(p - 1)\) denotes the Hölder conjugate of \(p \in (1, \infty)\).

The problem whether the flow \(\sigma_{\text{opt}} = \nabla E^*(\nabla v)\) is continuous has been answered affirmatively under the assumption \(q \in (n, \infty]\). There it is an interesting question whether vector field \(\mathcal{J}_{b+\delta}(\nabla v)\) with \(\delta > 0\) fixed, defined similarly to (1.3), is continuous. We should note that the equation (1.7) is also non-uniformly elliptic around the set \(|\nabla v| \leq b\), in the sense that the ellipticity ratio of \(\nabla^2 E^*(z_0)\) will blow up as \(|z_0| \to b + 0\). However, continuity of truncated vector fields \(\mathcal{J}_{b+\delta}(\nabla u)\) with \(\delta > 0\) is expectable, since there holds

\[
(\text{ellipticity ratio of } \nabla^2 E^*(z_0)) = \frac{\text{the largest eigenvalue of } \nabla^2 E^*(z_0)}{\text{the lowest eigenvalue of } \nabla^2 E^*(z_0)} \leq (p - 1) \left(1 + (\delta - b)^{-1}\right)
\]

for all \(z_0 \in \mathbb{R}^n\) satisfying \(|z_0| \geq b + \delta\) when \(\delta\) is sufficiently close to 0. This estimate suggests that for each fixed \(\delta > 0\), the truncated vector field \(\mathcal{J}_{b+\delta}(\nabla v)\) should be Hölder continuous. It should be noted that it is possible to show \(\mathcal{J}_{b+\delta}(\nabla v)\) uniformly converges to \(\mathcal{J}_b(\nabla v)\) as \(\delta \to 0\), and thus \(\mathcal{J}_b(\nabla v)\) will be also continuous.

When \(v\) is scalar-valued, continuity of \(\mathcal{J}_{b+\delta}(\nabla v)\) with \(\delta > 0\) was first shown by Santambrogio–Vespri [31] in 2010 for the special case \(n = 2\) with \(b = 1\). The proof therein is based on oscillation estimates on the Dirichlet energy, which works under the condition \(n = 2\) only. Later in 2014, Colombo–Figalli [2] established a more general proof that works for any dimension \(n \geq 2\) and any density function \(E^*\), as long as the zero-levelset of \(E^*\) is sufficiently large enough to define a real-valued Minkowski gauge. This Minkowski gauge becomes a basic modulus for judging uniform ellipticity of equations they treated. Here we would like to note that their strategy will not work for our problem (1.1), since the density function \(E^*\)
seems structurally different from $E$. In fact, in our problem, the zero-levelset of $E$ is only a singleton, and therefore it seems impossible to adapt the Minkowski gauge as a real-valued modulus. The recent work by Bögelein–Duzzaar–Giova–Passarelli di Napoli \(^4\) is motivated by extending these regularity results to the vectorial case. There they considered an approximation problem of the form

\[
-\varepsilon \Delta v - \text{div}(D \varepsilon E^*(Dv)) = f \quad \text{in } \Omega
\]

with $b = 1$. The paper \(^4\) provides a priori Hölder continuity of $G_{1+2\delta}(Dv)$ for each fixed $\delta \in (0, 1)$, whose estimate is independent of an approximation parameter $\varepsilon \in (0, 1)$. It will be worth noting that the modulus $|Dv|$, which measures ellipticity ratio of (1.8), has a symmetric structure. This fact appears to be fit to prove everywhere Hölder continuity of $G_{1+2\delta}(Dv)$ for the system case.

Although our proofs on a priori Hölder estimates are inspired by \(^4\) §4–7, there are mainly three different points between our proofs and theirs. The first is how to approximate systems. To be precise, in vectorial case. There they considered an approximation problem of the form

\[
\|u - v\|_{L^p(\Gamma)} \leq c(L_{\varepsilon}u, \Gamma) \quad \text{for } \varepsilon \in (0, \delta/4)
\]

\( \text{with } \Gamma \text{ a density.} \)

For $\beta_0$-Hölder continuity of $g''_p$, we assume that

\[
|g''_p(\sigma_1) - g''_p(\sigma_2)| \leq \Gamma \mu^{p-4-2\beta_0} |\sigma_1 - \sigma_2|^\beta_0
\]

for all $\sigma_1, \sigma_2 \in [\mu^2/4, 7\mu^2]$ with $\mu \in (0, \infty)$.

A typical example is

\[
g_p(\sigma) := \frac{2\sigma^{p/2}}{p} \quad \text{for } \sigma \in [0, \infty).
\]

In fact, this $g_p$ satisfies (1.9)–(1.11) with $\gamma := \min\{1, p-1\}$, $\Gamma := \max\{1, |p-2|/2\}$. Moreover, we have

\[
|g'''_p(\sigma)| \leq \hat{\Gamma} \sigma^{p/2-3} \quad \text{for } \sigma \in (0, \infty)
\]
with $\Gamma := |(p - 2)(p - 4)|/4$. From this estimate, it is easy to find a constant $\Gamma = \Gamma(p) \in (1, \infty)$ such that (1.12) holds with $\beta_0 = 1$. In this case, the operator $L_\rho$ becomes $\Delta_\rho$. Thus, the system (1.2) generalizes (1.1).

Our main result is the following Theorem 1.2 which clearly yields Theorem 1.1.

**Theorem 1.2.** Let $p \in (1, \infty)$, $q \in (n, \infty]$, and $f \in L^q(\Omega; \mathbb{R}^N)$. Assume that $u$ is a weak solution to (1.2) in a Lipschitz domain $\Omega \subset \mathbb{R}^n$, where $g_p$ satisfies (1.2). Then, for each fixed $\delta \in (0, 1)$ and $x_\delta \in \Omega$ there exists an open ball $B_{r_0}(x_\delta) \subset \Omega$ such that $\mathcal{G}_{2\delta}(Du(x)) \in C^{\alpha}(B_{r_0}/2(x_\delta); \mathbb{R}^{Nn})$. Here the exponent $\alpha \in (0, 1)$ and the radius $r_0 \in (0, 1)$ depend at most on $b, n, N, p, q, \beta_0, \gamma$, $\Gamma$, $\|f\|_{L^q(\Omega)}$, $\|Du\|_{L^p(\Omega)}$, $d_\delta := \text{dist}(x_\delta, \partial\Omega)$, and $\delta$. Moreover, we have

$$
|\mathcal{G}_{2\delta}(Du(x))| \leq \mu_0 \quad \text{for all } x \in B_{r_0}(x_\delta), \quad (1.14)
$$

$$
|\mathcal{G}_{2\delta}(Du(x_1)) - \mathcal{G}_{2\delta}(Du(x_2))| \leq \frac{2^{n/2}\alpha^2\mu_0}{\rho_0^\alpha}|x_1 - x_2|^\alpha \quad \text{for all } x_1, x_2 \in B_{r_0}/2(x_\delta), \quad (1.15)
$$

where the constant $\mu_0 \in (0, \infty)$ depends at most on $b, n, p, q, \gamma, \Gamma, \|f\|_{L^q(\Omega)}$, $\|Du\|_{L^p(\Omega)}$, and $d_\delta$. In particular, the Jacobian matrix $Du$ is continuous in $\Omega$.

This paper is organized as follows.

Section 2 provides approximation problems for the system (1.2), which is based on the relaxation of energy densities. After fixing some notations in Section 2.1, we give a variety of quantitative estimates related to the relaxed densities in Section 2.2. Next in Section 2.3 we will justify that solutions of regularized problem, denoted by $u_\varepsilon$, converges to the original problem (1.2). Our main theorems are proved in Section 2.4 which presents a priori Hölder estimates on Jacobian matrices $Du_\varepsilon$ that are suitably truncated near facets. There we state three basic a priori estimates on approximated solutions, consisting of local Lipschitz bounds (Proposition 2.6), a De Giorgi-type oscillation lemma (Proposition 2.7) and Campanato-type growth estimates (Proposition 2.8). The proofs of Propositions 2.6–2.8 are given later in the remaining Sections 3–5. From these estimates, we will deduce local a priori Hölder estimates of $\mathcal{G}_{2\delta, \varepsilon}(Du_\varepsilon)$, which are independent of an approximation parameter $\varepsilon \in (0, \delta/4)$ (Theorem 2.5). From Proposition 2.4 and Theorem 2.5 we finally give the proof of Theorem 1.2.

Sections 3–5 are devoted to prove Propositions 2.6–2.8. Among them, most of the proofs of Propositions 2.6–2.8 are rather easier. For the reader’s convenience, we would like to provide a brief proof of Proposition 2.6 in the appendix (Section 5). For Proposition 2.7 we briefly describe sketches of the proof in Appendix 3.2 after showing a weak fundamental lemma in Section 3.1. We omit some standard arguments in Section 3.2 concerning De Giorgi’s levelset lemmata, since they are already found in [4, §7] (see also [35, §4.2]). Section 4 is focused on the proof of Proposition 2.8. In Section 4.1 we obtain a variety of energy estimates from the weak formulation deduced in Section 3.1. Section 4.2 establishes a freezing coefficient argument under a good condition telling that a derivative does not vanish. After providing two basic lemmata on our shrinking arguments in Section 4.3, we give the proof of Proposition 2.8 in Section 4.4.

Finally, we would like to mention again that $C^1$-regularity for the scalar case is treated in the author’s recent work [35]. There, approximation of the density function is discussed, which is based on the convolution of Friedrichs’ mollifier and makes it successful to generalize one-Laplace operator. It should be emphasized that this generalization may not work in the system case, when it comes to everywhere continuous differentiability. This is essentially because the general singular operator discussed therein will lack the Uhlenbeck structure, except the one-Laplacian (see [35, §2.4] for further explanations). Also, compared with [35], some of the computations in this paper becomes rather simpler, since we consider a special case where the energy density is spherically symmetric. Some estimates in Section 2.2–2.3 are used without proofs and some proofs in Section 3.2 are omitted. The full computations and proofs of them are given in [35, §2.1 & 4.2].
2 Approximation schemes

The aim of Section 2 is to give approximation schemes for the problem (1.2). The basic idea is that the energy density

\[ E(\xi) = \frac{1}{2} g(|\xi|^2) \quad (\xi \in \mathbb{R}^n) \quad \text{with} \quad g(\sigma) := 2b\sqrt{\sigma} + g_p(\sigma) \quad (0 \leq \sigma < \infty) \quad (2.1) \]

is to be regularized by

\[ E_\varepsilon(\xi) = \frac{1}{2} g_\varepsilon(|\xi|^2) \quad (\xi \in \mathbb{R}^n) \quad \text{with} \quad g_\varepsilon(\sigma) := 2b\sqrt{\varepsilon^2 + \sigma} + g_p(\varepsilon^2 + \sigma) \quad (0 \leq \sigma < \infty) \quad (2.2) \]

for \( \varepsilon \in (0, 1) \) denoting the approximation parameter. Then, the relaxed operator \( \mathcal{L}_\varepsilon \) will be given by

\[ \mathcal{L}_\varepsilon u_\varepsilon := -\text{div} \left( g'_\varepsilon(|Du_\varepsilon|^2)Du_\varepsilon \right) = -\text{div} \left( \frac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \right) - \text{div} \left( g'_{p, \varepsilon}(|Du_\varepsilon|^2)Du_\varepsilon \right), \]

similarly to \( L_{b, p} \) given in (1.4). In Section 2, we will show that this approximation scheme works and deduce some basic estimates on relaxed mapping.

2.1 Notations

We first fix some notations throughout the paper.

We denote \( \mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\} \) by the set of all non-negative integers, and \( \mathbb{N} := \mathbb{Z}_{\geq 0} \setminus \{0\} \) by the set of all natural numbers.

For given \( k \in \mathbb{N} \), we denote \( \langle \cdot | \cdot \rangle_k \) by the standard inner product over the Euclidean space \( \mathbb{R}^k \). That is, for \( k \)-dimensional vectors \( \xi = (\xi_1, \ldots, \xi_k), \eta = (\eta_1, \ldots, \eta_k) \in \mathbb{R}^k \), we define

\[ \langle \xi | \eta \rangle_k := \sum_{j=1}^{k} \xi_j \eta_j \in \mathbb{R}^k. \]

We denote \( |\cdot|_k \) by the Euclidean norm,

\[ |\xi|_k := \sqrt{\langle \xi | \xi \rangle} \in [0, \infty) \quad \text{for} \quad \xi \in \mathbb{R}^k. \]

We also define a tensor product \( \xi \otimes \eta \) as a \( k \times k \) real matrix,

\[ \text{i.e.,} \quad (\xi \otimes \eta)_{j,k} := \langle \xi_j \eta_k \rangle_k = \begin{pmatrix} \xi_1 \eta_1 & \cdots & \xi_1 \eta_k \\ \vdots & \ddots & \vdots \\ \xi_k \eta_1 & \cdots & \xi_k \eta_k \end{pmatrix}. \]

We denote \( \text{id}_k \) by the \( k \times k \) identity matrix. For a \( k \times k \) real matrix \( A \), we define the operator norm

\[ \|A\|_k := \sup \left\{ |Ax|_k \mid x \in \mathbb{R}^k, |x|_k \leq 1 \right\}. \]

We also introduce the positive semi-definite ordering on all real symmetric matrices. In other words, for \( k \times k \) real symmetric matrices \( A, B \), we write \( A \preceq B \) or \( B \succeq A \) when the difference \( B - A \) is positive semi-definite.

For notational simplicity, we often omit the script \( k \) denoting the dimension. In particular, we simply denote the norms by \( |\cdot| \) or \( \|\cdot\| \), and the inner product by \( \langle \cdot | \cdot \rangle \).
We often regard an \( N \times n \) real matrix \( \xi = (\xi_\alpha^i)_{1 \leq \alpha \leq n, 1 \leq i \leq N} \) as an \( Nn \)-dimensional vector by ordering \( \xi = (\xi_1^1, \ldots, \xi_1^n, \ldots, \xi_n^1, \ldots, \xi_n^n) \in \mathbb{R}^{Nn} \). Then, similarly to the above, for given \( N \times n \) matrices \( \xi = (\xi_\alpha^i), \eta = (\eta_\alpha^i) \), we are able to define the inner product \( \langle \xi \mid \eta \rangle_{Nn} \in \mathbb{R} \), and a tensor product \( \xi \otimes \eta \) as an \((Nn) \times (Nn)\) real-valued matrix. We also note that under our setting, the norm \( \| \xi \| \) of a \( N \times n \) real matrix \( \xi \) is identified with the Frobenius norm of \( \xi \).

For a scalar-valued function \( u = u(x_1, \ldots, x_n) \), the gradient of \( u \) is denoted by \( \nabla u := (\partial_{x_\alpha} u)_\alpha \) and is often regarded as an \( n \)-dimensional row vector. For an \( \mathbb{R}^k \)-valued function \( u = (u^1, \ldots, u^k) \) with \( u^i = u^i(x_1, \ldots, x_n) \) for \( i \in \{1, \ldots, k\} \), the Jacobian matrix of \( u \) is denoted by \( D_u := (D_{i} u_1 \cdots D_{i} u_k) \equiv (\partial_{x_\alpha} u^i)_\alpha i \) with \( D_{\alpha} u = (\partial_{x_\alpha} u^i)_i \) for each \( \alpha \in \{1, \ldots, n\} \). These \( D_u \) and \( D_{\alpha} u \) are often considered as an \( k \times n \)-matrix and a \( k \)-dimensional column vector respectively.

For given numbers \( s \in [1, \infty], k \in \mathbb{N}, d \in \mathbb{N} \) and a fixed domain \( U \subset \mathbb{R}^n \), we denote \( L^s(U; \mathbb{R}^d) \) and \( W^{k,s}(U; \mathbb{R}^d) \) respectively by the Lebesgue space and the Sobolev space. To shorten the notations, we often write \( L^s(U) := L^s(U; \mathbb{R}) \) and \( W^{k,s}(U) := W^{k,s}(U; \mathbb{R}) \).

Throughout this paper, we define

\[
g_1(\sigma) := 2b\sqrt{\sigma} \quad \text{for } \sigma \in [0, \infty).
\]

Clearly, \( g_1 \) is in \( C([0, \infty)) \cap C^3((0, \infty)) \) and satisfies

\[
|g_1'(\sigma)| \leq b\sigma^{-1/2} \quad \text{for all } \sigma \in (0, \infty),
\]

\[
|g_1''(\sigma)| \leq \frac{b}{2}\sigma^{-3/2} \quad \text{for all } \sigma \in (0, \infty),
\]

\[
0 \leq g_1' (\sigma + \tau) + 2\sigma \min \{g_1''(\sigma + \tau), 0\} \quad \text{for all } \sigma \in (0, \infty), \tau \in (0, 1).
\]

Also, by the growth estimate

\[
|g_1'''(\sigma)| \leq \frac{3b^2}{4}\sigma^{-5/2} \quad \text{for all } \sigma \in (0, \infty),
\]

it is easy to check that there holds

\[
|g_1''(\sigma_1) - g_1''(\sigma_2)| \leq 24 \cdot b\mu^{-3} |\sigma_1 - \sigma_2|
\]

for all \( \sigma_1, \sigma_2 \in [\mu^2/4, 7\mu^2] \) with \( \mu \in (0, \infty) \). It should be mentioned that unlike the assumption \( (1.11) \), the inequality \( (2.5) \) gives no quantitative monotonicity estimates for the one-Laplace operator \( \Delta_1 \).

We consider the relaxed function \( g_\varepsilon \) given by \( (2.2) \). This function can be decomposed by \( g_\varepsilon = g_{1, \varepsilon} + g_{p, \varepsilon} \) with

\[
g_{1, \varepsilon}(\sigma) := 2b\sqrt{\varepsilon^2 + \sigma}, \quad g_{p, \varepsilon}(\sigma) := g_p(\varepsilon^2 + \sigma)
\]

for \( \sigma \in [0, \infty) \). Corresponding to them, for each \( s \in \{1, p\} \), we define

\[
A_{s, \varepsilon}(\xi) := g_{s, \varepsilon}' (|\xi|^2) \xi \quad \text{for } \xi \in \mathbb{R}^{Nn}
\]

as an \( N \times n \) matrix, and

\[
B_{s, \varepsilon}(\xi) := g_{s, \varepsilon}' (|\xi|^2) \text{id}_{Nn} + 2g_{s, \varepsilon}'' (|\xi|^2) \xi \otimes \xi \quad \text{for } \xi \in \mathbb{R}^{Nn}
\]

as an \( Nn \times Nn \) matrix. Then, the summations

\[
A_\varepsilon(\xi) := A_{1, \varepsilon}(\xi) + A_{p, \varepsilon}(\xi) \quad \text{for } \xi \in \mathbb{R}^{Nn}
\]

(2.7)
and
\[ \mathcal{B}_x(\xi) := \mathcal{B}_{1,x}(\xi) + \mathcal{B}_{p,x}(\xi) \quad \text{for } \xi \in \mathbb{R}^n \]
(2.8)
respectively denote the Jacobian matrices and the Hessian matrices of \( E_x \) defined by (2.2). By direct
calculations, it is easy to check that \( A_{p,x}(\xi) \) converges to \( A_p(\xi) \) for each \( \xi \in \mathbb{R}^n \), where
\[ A_p(\xi) := \begin{cases} g_p'(\|\xi\|^2)\xi & (\xi \neq 0), \\ 0 & (\xi = 0). \end{cases} \]
(2.9)

Finally, we introduce a subdifferential set for the Euclidian norm. We denote \( \partial |\cdot| : (\xi_0) \subset \mathbb{R}^n \) by the
subdifferential of the absolute value function \(|\cdot|_{\mathbb{R}^n} \) at \( \xi_0 \in \mathbb{R}^n \). In other words, \( \partial |\cdot| (\xi_0) \) is the set of all
vectors \( \zeta \in \mathbb{R}^n \) that satisfies a subgradient inequality
\[ |\zeta|_{\mathbb{R}^n} \geq |\xi_0|_{\mathbb{R}^n} + \langle \zeta, \xi - \xi_0 \rangle_{\mathbb{R}^n} \quad \text{for all } \xi \in \mathbb{R}^n. \]
This set is explicitly given by
\[ \partial |\cdot| (\xi_0) = \left\{ \{ \zeta \in \mathbb{R}^n \mid |\zeta| \leq 1 \} \right\} (\xi_0 = 0), \]
(10.2)
\[ \left\{ \{ \xi_0/|\xi_0| \} \right\} (\xi_0 \neq 0), \]
and this formulation is used when one gives the definitions of weak solutions.

2.2 Quantitative estimates on relaxed mappings

In Section 2.2, we would like to introduce quantitative estimates related to the mappings \( \mathcal{G}_{2,\delta, e} \) and \( g_{e}. \)
When deducing these estimates, we often use the assumption that the density function \( E \) is spherically
symmetric. As a related item, we refer the reader to [35] §2.1, which deals with the scalar case without
symmetry of \( E \), and provides full computations of some estimates omitted in Section 2.2.

In this paper, we often assume that
\[ 0 < \delta < 1, \quad \text{and} \quad 0 < e < \frac{\delta}{4}. \]
(2.11)
It should be mentioned that the mapping \( \mathcal{G}_{\delta, e} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) given by (2.5) makes sense as long as
0 < \( e < \delta \) holds. In particular, under the setting (2.11), the mappings \( \mathcal{G}_{\delta, e}, \mathcal{G}_{2,\delta, e} \) are well-defined.
Moreover, Lipschitz continuity of \( \mathcal{G}_{\delta, e} \) follows from (2.11).

**Lemma 2.1.** Let \( \delta, e \) satisfy (2.11). Then the mapping \( \mathcal{G}_{2,\delta, e} \) satisfies
\[ |\mathcal{G}_{2,\delta, e}(z) - \mathcal{G}_{2,\delta, e}(\xi)| \leq c_1|z - \xi| \quad \text{for all } z, z \in \mathbb{R}^n \]
(2.12)
with \( c_1 := 1 + 32/(3 \sqrt{7}) \).

In the limiting case \( \varepsilon = 0 \), Lipschitz continuity like (2.12) is found in [4] Lemma 2.3. Modifying
the arguments therein, we can easily prove Lemma 2.1, the full proof of which is given in [35] Lemma 2.4.

Next, we consider the mappings \( A_{x} = A_{1,x} + A_{p,x} \) and \( \mathcal{B}_{x} = \mathcal{B}_{1,x} + \mathcal{B}_{p,x} \) defined by (2.2)–(2.3),
and describe some basic results including monotonicity and growth estimates. For each \( s \in \{1, p\} \), the
eigenvalues of \( \mathcal{B}_{s,e}(\xi) \) are given by either
\[ \lambda_1(\xi) := g_s'(e^2 + |\xi|^2) \quad \text{or} \quad \lambda_2(\xi) := g_s''(e^2 + |\xi|^2) + 2|\xi|^2 g_s''(e^2 + |\xi|^2). \]
Combining this with (1.19)–(1.11) and (2.3)–(2.5), the mappings \( \mathcal{B}_{p,x} \) and \( \mathcal{B}_{1,x} \) respectively satisfy
\[ \gamma(e^2 + |\xi|^2)^{p/2-1} - 1 id_{\mathbb{R}^n} \leq \mathcal{B}_{p,x}(\xi) \leq 3\gamma(e^2 + |\xi|^2)^{p/2-1} id_{\mathbb{R}^n} \quad \text{for all } \xi \in \mathbb{R}^n. \]
(2.13)
we may let 

\[
O \leq B_{1, \epsilon}(\xi) \leq b(\epsilon^2 + |\xi|^2)^{-1/2} \text{id}_{N_n}
\]

for all \( \xi \in \mathbb{R}^{N_n} \),

(2.14)

where \( O \) denotes the zero matrix. In particular, by elementary computations as in [33] Lemma 3, it is easy to get

\[
\langle A_\epsilon(\xi_1) - A_\epsilon(\xi_0) \mid \xi_1 - \xi_0 \rangle \geq \begin{cases} 
    c(p)\gamma(\epsilon^2 + |\xi_0|^2 + |\xi_1|^2)^{(p-1)/2} |\xi_1 - \xi_0|^2 & (1 < p < 2), \\
    c(p)\gamma |\xi_1 - \xi_0|^p & (2 \leq p < \infty),
\end{cases}
\]

(2.15)

and

\[
|A_{p, \epsilon}(\xi_1) - A_{p, \epsilon}(\xi_0)| \leq \begin{cases} 
    C(p)\Gamma|\xi_1 - \xi_0|^{p-1} & (1 < p < 2), \\
    C(p)\Gamma(\epsilon^p + |\xi_0|^{p-2} + |\xi_1|^{p-2})|\xi_1 - \xi_0| & (2 \leq p < \infty),
\end{cases}
\]

(2.16)

for all \( \xi_0, \xi_1 \in \mathbb{R}^{N_n} \). We often consider a special case where a variable \( \xi \in \mathbb{R}^{N_n} \) may not vanish. On this setting, it is possible to deduce continuity or monotonicity estimates other than (2.15)–(2.16). In fact, following elementary computations given in [35] Lemma 2.2–2.3, from (2.13), we are able to obtain a growth estimate

\[
|A_{p, \epsilon}(\xi_1) - A_{p, \epsilon}(\xi_0)| \leq C(p)\min\{|\xi_0|^{p-2}, |\xi_1|^{p-2}\} |\xi_1 - \xi_0| \geq 2 \min\{|\xi_0|^{-1}, |\xi_1|^{-1}\} |\xi_1 - \xi_0| \geq 2 |\xi_1 - \xi_0| \geq 2 \min\{|\xi_0|^{-1}, |\xi_1|^{-1}\} |\xi_1 - \xi_0| \geq 2 b \min\{|\xi_0|^{-1}, |\xi_1|^{-1}\} |\xi_1 - \xi_0|
\]

(2.17)

for all \( (\xi_0, \xi_1) \in (\mathbb{R}^{N_n} \times \mathbb{R}^{N_n}) \setminus \{(0, 0)\} \) provided \( 1 < p < 2 \), and a monotonicity estimate

\[
\langle A_{p, \epsilon}(\xi_1) - A_{p, \epsilon}(\xi_0) \mid \xi_1 - \xi_0 \rangle \geq C(p)\gamma \max\{|\xi_0|^{p-2}, |\xi_1|^{p-2}\} |\xi_1 - \xi_0|^2
\]

(2.18)

for all \( \xi_0, \xi_1 \in \mathbb{R}^{N_n} \) provided \( 2 \leq p < \infty \). It is worth mentioning that even for \( A_{1, \epsilon} \), there holds a growth estimate

\[
|A_{1, \epsilon}(\xi_1) - A_{1, \epsilon}(\xi_0)| \leq 2 b \min\{|\xi_0|^{-1}, |\xi_1|^{-1}\} |\xi_1 - \xi_0| \geq 2 b \min\{|\xi_0|^{-1}, |\xi_1|^{-1}\} |\xi_1 - \xi_0|
\]

(2.19)

for all \( (\xi_0, \xi_1) \in (\mathbb{R}^{N_n} \times \mathbb{R}^{N_n}) \setminus \{(0, 0)\} \) (see also [35] Lemma 2.1). In fact, without loss of generality, we may let \( |\xi_1| \geq |\xi_0| \). By the triangle inequality, we compute

\[
\frac{|\xi_1|}{\sqrt{\epsilon^2 + |\xi_1|^2}} - \frac{|\xi_0|}{\sqrt{\epsilon^2 + |\xi_0|^2}} = \frac{|\xi_1 - \xi_0|}{\sqrt{\epsilon^2 + |\xi_1|^2}} - \frac{\sqrt{\epsilon^2 + |\xi_1|^2} - \sqrt{\epsilon^2 + |\xi_0|^2}}{\sqrt{\epsilon^2 + |\xi_1|^2}} \leq 2 \frac{|\xi_1 - \xi_0|}{\sqrt{\epsilon^2 + |\xi_0|^2}}
\]

from which (2.19) follows. Here we have used one-Lipschitz continuity of the smooth function \( \sqrt{\epsilon^2 + t^2} (t \in \mathbb{R}) \). For monotonicity of \( A_{1, \epsilon} \), from (2.14) we obtain

\[
\langle A_{1, \epsilon}(\xi_1) - A_{1, \epsilon}(\xi_0) \mid \xi_1 - \xi_0 \rangle \geq 0
\]

(2.20)

for all \( \xi_0, \xi_1 \in \mathbb{R}^{N_n} \). When it comes to a monotonicity estimate that is independent of \( \epsilon \), better estimates than (2.20) seem to be no longer expectable, since ellipticity of \( \Delta_\epsilon u \) degenerates in the direction of \( Du \).

In Lemma 2.2 below, we briefly deduce good estimates that will be used in Section 4.2

**Lemma 2.2.** Let positive constants \( \delta \) and \( \epsilon \) satisfy (2.7). Under the assumptions (1.9)–(1.12), the mappings \( A_\epsilon \) and \( B_\epsilon \) defined by (2.7)–(2.8) satisfy the following:

1. Let \( M \in (\delta, \infty) \) be a fixed constant. Then there exists constants \( C_1, C_2 \in (0, \infty) \), depending at most on \( b, p, \gamma, \Gamma, \delta \) and \( M \), such that we have

\[
\langle A_\epsilon(\xi_1) - A_\epsilon(\xi_0) \mid \xi_1 - \xi_0 \rangle \geq C_1|\xi_1 - \xi_0|^2,
\]

(2.21)

and

\[
|A_\epsilon(\xi_1) - A_\epsilon(\xi_0)| \leq C_2|\xi_1 - \xi_0|,
\]

(2.22)

for all \( \xi_0, \xi_1 \in \mathbb{R}^{N_n} \) satisfying

\[
\delta \leq |\xi_0| \leq M, \quad \text{and} \quad |\xi_1| \leq M.
\]
2. For all $\xi_0, \xi_1 \in \mathbb{R}^n$ enjoying

$$\delta + \frac{\mu}{4} \leq |\xi_0| \leq \delta + \mu \quad \text{and} \quad |\xi_1| \leq \delta + \mu \quad \text{with} \quad \delta < \mu < \infty,$$

we have

$$|B_p(\xi_0)(\xi_1 - \xi_0) - (A_p, e(\xi_1) - A_p, e(\xi_0))| \leq C(b, \beta_0, \Gamma, \delta)\mu^{p-2-\beta_0} |\xi_1 - \xi_0|^{1+\beta_0}.$$  (2.24)

Estimates (2.21)–(2.22) are easy to deduce from (2.15)–(2.20) and $\varepsilon < \delta < M$. We would like to show (2.24), which plays an important role in our proof of regularity estimates. Although an estimate like (2.24) is shown in the author’s recent work [35, Lemmas 2.2 & 2.6], most of our computations herein become rather direct and simple, since the density $E_\varepsilon$ is assumed to be spherically symmetric.

**Proof.** Let positive constants $\delta$, $\varepsilon$ and vectors $\xi_0, \xi_1 \in \mathbb{R}^n$ satisfy respectively (2.11) and (2.23). We set $\xi_t := \xi_0 + t(\xi_1 - \xi_0) \in \mathbb{R}^k$ for $t \in [0, 1]$. To show (2.24), we claim that

$$|B_p, e(\xi_0)(\xi_1 - \xi_0) - (A_p, e(\xi_1) - A_p, e(\xi_0))| \leq C(p, \beta_0)\Gamma \mu^{p-2-\beta_0} |\xi_1 - \xi_0|^{1+\beta_0},$$

and

$$|B_1, e(\xi_0)(\xi_1 - \xi_0) - (A_1, e(\xi_1) - A_1, e(\xi_0))| \leq Cb \mu^{-2} |\xi_1 - \xi_0|^2.$$  (2.26)

The desired estimate (2.24) immediately follows from (2.25)–(2.26). Here it should be noted that the inequality $\mu^{-2} |\xi_1 - \xi_0|^2 \leq 4^{1-p_0} \delta^{1-p} \mu^{p-2-\beta_0} |\xi_1 - \xi_0|^{1+\beta_0}$ holds by $0 < \delta < \mu < \infty$ and $|\xi_1 - \xi_0| \leq 4\mu$.

We would like to give the proof of (2.25). We first consider the case $|\xi_1 - \xi_0| \leq \mu/2$. Then, by the triangle inequality, it is easy to check that

$$\frac{\mu}{2} \leq |\xi_t| = |\xi_0| + t|\xi_1 - \xi_0| \leq \frac{5\mu}{2}$$

for all $t \in [0, 1]$. In particular, we have

$$\frac{\mu^2}{4} \leq |\xi_t| \leq \max\{\varepsilon^2 + |\xi_1|^2, \varepsilon^2 + |\xi_0|^2\} =: b_t \leq \frac{13\mu^2}{2},$$

where we have used $\varepsilon < \delta < \mu$ to get the last inequality. Also, it is easy to compute

$$||\xi_t \otimes \xi_t| - |\xi_0|\rangle \leq \left(2t|\xi_0|^2 + t^2|\xi_1 - \xi_0|^2\right)|\xi_1 - \xi_0| \leq \frac{9\mu}{2} |\xi_1 - \xi_0|,$$

and

$$b_t - a_t = |\xi_t|^2 - |\xi_0|^2 = |2t\langle \xi_0 | \xi_1 - \xi_0 \rangle + t^2|\xi_1 - \xi_0|^2| \leq \frac{9\mu}{2} |\xi_1 - \xi_0|$$

for all $t \in [0, 1]$, and $||\xi_0 \otimes \xi_0| \leq |\xi_0|^2 \leq (2\mu)^2$. Combining them with (1.10), and (1.12), we are able to check that the operator norm of

$$B_p, e(\xi_t) - B_p, e(\xi_0) = 2g_p''(\varepsilon^2 + |\xi_1|^2)(\xi_t \otimes \xi_t - \xi_0 \otimes \xi_0) + 2g_p''(\varepsilon^2 + |\xi_0|^2)(\xi_0 \otimes \xi_0)$$

$$+ \left[ g_p'(\varepsilon^2 + |\xi_1|^2) - g_p'(\varepsilon^2 + |\xi_0|^2) \right](\xi_0 \otimes \xi_0)$$

$$+ \left[ g_p'(\varepsilon^2 + |\xi_1|^2) - g_p'(\varepsilon^2 + |\xi_0|^2) \right]|d N_n$$

is bounded by $C \Gamma \mu^{p-2-\beta_0} |\xi_1 - \xi_0|^{\beta_0}$ for some constant $C = C(p, \beta_0) \in (0, \infty)$. As a result, we obtain

$$|B_p, e(\xi_0)(\xi_1 - \xi_0) - (A_p, e(\xi_1) - A_p, e(\xi_0))| = \int_0^1 (B_{p, e}(\xi_0) - B_{p, e}(\xi_t)) \cdot (\xi_1 - \xi_0) \, dr$$

12
≤ |ξ₁ - ξ₀| \int_0^1 \| \mathcal{B}_{p,ε} (ξ₁) - \mathcal{B}_{p,ε} (ξ₀) \| \, dt \\
≤ C(p, ℱ_0) Γ^{p-2-F_0} |ξ₁ - ξ₀|^{1+F_0}.

In the remaining case |ξ₁ - ξ₀| > μ/2, it is easy to compute

\begin{align*}
&|\mathcal{B}_{p,ε} (ξ₀) (ξ₁ - ξ₀) - (A_{p,ε} (ξ₁) - A_{p,ε} (ξ₀))| \\
&≤ (\| \mathcal{B}_{p,ε} (ξ₀) \| |ξ₁ - ξ₀| + \| (A_{p,ε} (ξ₁) - A_{p,ε} (ξ₀)) \|) \cdot \left( \frac{2|ξ₁ - ξ₀|}{μ} \right)^{F_0} \\
&≤ C(p, ℱ_0) Γ^{p-2-F_0} |ξ₁ - ξ₀|^{1+F_0}
\end{align*}

by (2.13), (2.16)–(2.17), (2.23) and ε < δ < μ. This completes the proof of (2.25). By similar computations, we are able to conclude (2.26) from (2.4), (2.6), (2.14) and (2.19).

We conclude this section by mentioning that similar estimates hold for another mapping \( G_{p,ε} : \mathbb{R}^N \to \mathbb{R}^N \), defined by

\[ G_{p,ε} (ξ) := h_p' (e^2 + |ξ|^2)ξ \quad \text{for} \ ξ \in \mathbb{R}^N \]

(2.27)

with

\[ h_p (σ) := \frac{2σ^{(p+1)/2}}{p+1} \quad \text{for} \ σ \in [0, ∞). \]

(2.28)

It should be mentioned that this \( h_p \) is the same as \( g_{p+1} \) with \( g_p \) given by (1.13). Hence, similarly to (2.15), we have

\[ \langle G_{p,ε} (ξ₁) - G_{p,ε} (ξ₀) | ξ₁ - ξ₀ \rangle \geq c |ξ₁ - ξ₀|^{p+1} \quad \text{for all} \ ξ₀, ξ₁ \in \mathbb{R}^N \]

with \( c = c(p) \in (0, ∞) \). Moreover, since the mapping \( G_{p,ε} \) is bijective and enjoys \( G_{p,ε} (0) = 0 \) by the definition, the inverse mapping \( G^{-1}_{p,ε} \) satisfies

\[ |G^{-1}_{p,ε} (ξ)| \leq C(p) |ξ|^{1/p} \quad \text{for all} \ ξ \in \mathbb{R}^N \]

(2.29)

with \( C = c^{-1} \in (0, ∞) \). Also, similarly to (2.18), it is possible to get

\[ |G_{p,ε} (ξ₁) - G_{p,ε} (ξ₀)| \geq C(p) \max \{ |ξ₁|^{p-1}, |ξ₂|^{p-1} \} |ξ₁ - ξ₀| \]

(2.30)

for all \( ξ₀, ξ₁ \in \mathbb{R}^N \). The estimates (2.29)–(2.30) will be used in Section 4.1.

### 2.3 Justifications of convergence of solutions

The aim of Section 2.3 is to give an approximating system for (1.2), and justify convergence of weak solutions. We only deal with the special case where the energy density \( E \) is spherically symmetric. In the scalar case, more general approximation problems are discussed in §2.4–2.5, including variational inequality problems and generalization of the total variation energy.

Only in this section, we assume that the exponent \( q \) satisfies

\[
\begin{cases}
\frac{np}{n-p+n} < q ≤ ∞ & (1 < p < n), \\
1 < q ≤ ∞ & (p = n), \\
1 ≤ q ≤ ∞ & (n < p < ∞),
\end{cases}
\]

(2.31)

so that we can use the compact embedding \( W^{1,p} (Ω; \mathbb{R}^N) \hookrightarrow L^q (Ω; \mathbb{R}^N) \) (see e.g., Chapters 4 & 6). Under this setting, we give the definitions of a weak solution to the system (1.2), and of the Dirichlet boundary value problem

\[
\begin{cases}
Lu = f & \text{in} \ Ω, \\
u = u_0 & \text{on} \ ∂Ω.
\end{cases}
\]

(2.32)
Definition 2.3. Let the functions $u_{\ast} \in W^{1,p}(\Omega; \mathbb{R}^N)$, $f \in L^q(\Omega; \mathbb{R}^N)$ be given with $p \in (1, \infty)$ and $q \in [1, \infty]$ satisfying (2.37). A function $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is said to be a weak solution to (1.2) in $\Omega$ when there exists $Z \in L^\infty(\Omega; \mathbb{R}^{Nn})$ such that there hold

$$Z(x) \in \partial \cdot (Du(x)) \quad \text{for a.e. } x \in \Omega,$$

and

$$\int_{\Omega} \langle Z \mid D\phi \rangle \, dx + \int_{\Omega} \langle A_p(Du) \mid D\phi \rangle \, dx = \int_{\Omega} \langle f \mid \phi \rangle \, dx \quad \text{for all } \phi \in W^{1,p}_0(\Omega; \mathbb{R}^N).$$

Here $A_p \in C(\mathbb{R}^N; \mathbb{R}^{Nn})$ and $\partial \cdot (\cdot) \subset \mathbb{R}^{Nn}$ are given by (2.37)-(2.38). When a function $u \in u_{\ast} + W^{1,p}_0(\Omega; \mathbb{R}^N)$ is a weak solution to (1.2) in $\Omega$, $u$ is called a weak solution of the Dirichlet problem (2.22).

It should be recalled that the problem (1.2) is derived from a minimizing problem of the energy functional

$$\mathcal{F}_0(v) := \int_{\Omega} (E(Dv) - \langle f \mid v \rangle) \, dx \quad \text{for } v \in W^{1,p}(\Omega; \mathbb{R}^N)$$

under a suitable boundary condition. We approximate this functional by

$$\mathcal{F}_\varepsilon(v) := \int_{\Omega} (E_{\varepsilon}(Dv) - \langle f_\varepsilon \mid v \rangle) \, dx \quad \text{for } v \in W^{1,p}(\Omega; \mathbb{R}^N), \, \varepsilon \in (0, 1).$$

Here the net $\{f_\varepsilon\}_{0<\varepsilon<1} \subset L^q(\Omega; \mathbb{R}^N)$ satisfies

$$f_\varepsilon \rightharpoonup f \quad \text{in} \quad \sigma\left(L^q(\Omega; \mathbb{R}^N), L^{q'}(\Omega; \mathbb{R}^N)\right).$$

In other words, we only let $f_\varepsilon \in L^q(\Omega; \mathbb{R}^N)$ weakly converge to $f$ when $q$ is finite, and otherwise weak* convergence is assumed. In particular, we may let $\|f_\varepsilon\|_{L^q(\Omega)}$ be uniformly bounded with respect to $\varepsilon$. In Proposition 2.4 we justify convergence of minimizers of relaxed energy functionals.

Proposition 2.4 (A convergence result on approximation problems). Let $p \in (1, \infty), \, q \in [1, \infty]$ satisfy (2.37), and consider functionals $\mathcal{F}_0, \mathcal{F}_\varepsilon (0 < \varepsilon < 1)$ given by (2.35)-(2.36), where $\{f_\varepsilon\}_{0<\varepsilon<1} \subset L^q(\Omega; \mathbb{R}^N)$ satisfies (2.37). For a given function $u_{\ast} \in W^{1,p}(\Omega; \mathbb{R}^N)$, we define

$$u_\varepsilon := \arg \min \left\{ v \in u_{\ast} + W^{1,p}_0(\Omega; \mathbb{R}^N) \mid \mathcal{F}_\varepsilon(v) \right\} \quad \text{for each } \varepsilon \in (0, 1).$$

Then, there exists a unique function $u \in u_{\ast} + W^{1,p}_0(\Omega; \mathbb{R}^N)$ such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^N)$ up to a subsequence. Moreover, the limit function $u$ is a weak solution of the Dirichlet boundary value problem (2.22), and there holds

$$u = \arg \min \left\{ v \in u_{\ast} + W^{1,p}_0(\Omega; \mathbb{R}^N) \mid \mathcal{F}_0(v) \right\}.$$
Proof. We first mention that the right hand sides of (2.38)–(2.39) are well-defined. In fact, to construct minimizers by direct methods ([20], Chapter 8), ([23], Chapter 4), it suffices to prove boundedness and coerciveness of the functional $\mathcal{F}_\varepsilon$ over the space $u_\varepsilon + W_0^{1,p}(\Omega; \mathbb{R}^N)$. To check boundedness, we let $g_p(0) = 0$ without loss of generality. Then the growth estimate $g_p(\sigma) \leq C(p, \Gamma)\sigma^{p/2}$ easily follows from (1.9). In particular, by the continuous embedding $L^q(\Omega; \mathbb{R}^N) \hookrightarrow W^{1,p}(\Omega; \mathbb{R}^N)$, we have

$$\mathcal{F}_\varepsilon(v) \leq K_0 \left(1 + \|Dv\|_{L_p^p(\Omega)}^p + \|f_\varepsilon\|_{L^q(\Omega)} \|v\|_{W^{1,p}(\Omega)}\right)$$

(2.41) for all $\varepsilon \in [0, 1)$ and $v \in W^{1,p}(\Omega; \mathbb{R}^N)$. For a coercive estimate, we have

$$\mathcal{F}_\varepsilon(v) \geq K_1 \|Dv\|_{L_p^p(\Omega)} - K_2 \left(1 + \|f_\varepsilon\|_{L^q(\Omega)} \|u_*\|_{W^{1,p}(\Omega)} + \|f_\varepsilon\|_{L_p^p(\Omega)}\right)$$

(2.42) for all $\varepsilon \in [0, 1)$ and $v \in u_* + W_0^{1,p}(\Omega; \mathbb{R}^N)$. Here the constants $K_1 \in [0, 1)$ and $K_2 \in (1, \infty)$ depend at most on $n, p, q, \gamma$, and $\Omega$, but are independent of $\varepsilon \in [0, 1)$. It is possible to deduce (2.42) by applying the continuous embedding $W^{1,p}(\Omega; \mathbb{R}^N) \hookrightarrow L^q(\Omega; \mathbb{R}^N)$ and the Poincaré inequality

$$\|v - u_*\|_{W^{1,p}(\Omega)} \leq C(n, p, \Omega)\|Dv - Du_*\|_{L_p^p(\Omega)}$$

(2.43) for all $v \in u_* + W_0^{1,p}(\Omega; \mathbb{R}^N)$. The detailed computations to show (2.42) are substantially similar to [35], Proposition 2.11 (see also [34], §3). Uniqueness of minimizers are guaranteed by strict convexity of $E_\varepsilon$ and (2.43). Therefore, we are able to define $u$ and $u_\varepsilon$ satisfying (2.38) and (2.39) respectively.

We also mention that $\{Du_\varepsilon\}_{0 < \varepsilon < 1} \subset L^p(\Omega; \mathbb{R}^{Nn})$ is bounded, and so $\{u_\varepsilon\}_{0 < \varepsilon < 1} \subset W_0^{1,p}(\Omega; \mathbb{R}^N)$ is by (2.43). This is easy to deduce by applying (2.41)–(2.42) to the inequality $\mathcal{F}_\varepsilon(v) \leq \mathcal{F}_\varepsilon(u_*)$ following from (2.38). Hence, by the weak compactness theorem, we may choose a sequence $\{\varepsilon_j\}_{j=1}^\infty \subset (0, 1)$ such that $\varepsilon_j \to 0$ and

$$u_{\varepsilon_j} \rightharpoonup u_0 \quad \text{in} \quad W^{1,p}(\Omega; \mathbb{R}^N)$$

(2.44) for some function $u_0 \in u_* + W_0^{1,p}(\Omega; \mathbb{R}^N)$. Moreover, the condition (2.31) enables us to apply the compact embedding, so that we may let

$$u_{\varepsilon_j} \rightharpoonup u_0 \quad \text{in} \quad L^q(\Omega; \mathbb{R}^N)$$

(2.45) by taking a subsequence if necessary. We claim that

$$I(\varepsilon_j) := \int_\Omega (A_{\varepsilon_j}(Du_{\varepsilon_j}) - A_{\varepsilon_j}(Du_0)) \cdot Du_{\varepsilon_j} - Du_0 \, dx \to 0 \quad \text{as} \quad j \to \infty.$$  

(2.46) Before showing (2.46), we mention that this result yields

$$Du_{\varepsilon_j} \to Du_0 \quad \text{in} \quad L^p(\Omega; \mathbb{R}^{Nn}).$$

(2.47) In fact, when $1 < p < 2$, we apply (2.15), (2.20), and Hölder’s inequality to obtain

$$\|Du_{\varepsilon_j} - Du_0\|_{L_p^p(\Omega)}^p \leq \left(\int_\Omega (\varepsilon_j^2 + |Du_{\varepsilon_j}|^2 + |Du_0|^2)^{p/2 - 1} |Du_{\varepsilon_j} - Du_0|^2 \, dx \right)^{p/2} \cdot \sup_j \left(\int_\Omega (\varepsilon_j^2 + |Du_{\varepsilon_j}|^2 + |Du_0|^2)^{p/2} \, dx \right)^{1-p/2} \to C$$

$$\leq CA^{p/2} \cdot I(\varepsilon_j)^{p/2} \to 0 \quad \text{as} \quad j \to \infty.$$
Here it is noted that $C$ is finite by (2.44). In the remaining case $2 \leq p < \infty$, we simply use (2.15) and (2.20) to get

$$
\|Du_{\varepsilon_j} - Du_0\|_{L^p(\Omega)}^p \leq \frac{I(\varepsilon_j)}{AC(p)} \to 0 \quad \text{as } j \to \infty.
$$

For the proof of (2.46), we decompose $I = I_1 - I_2$ with

$$
I_1(\varepsilon_j) := \int_{\Omega} (A_{\varepsilon_j}(Du_{\varepsilon_j}) - Du_{\varepsilon_j} - Du_0) \, dx, \quad I_2(\varepsilon_j) := \int_{\Omega} (A_{\varepsilon_j}(Du) - Du_{\varepsilon_j} - Du_0) \, dx.
$$

For $I_1$, we test $\phi = u_{\varepsilon_j} - u_0 \in W_0^{1,p}(\Omega; \mathbb{R}^N)$ into (2.40). Then, we obtain

$$
I_1(\varepsilon_j) = \int_{\Omega} (f_{\varepsilon_j} | u_{\varepsilon_j} - u_0) \, dx \to 0 \quad \text{as } j \to \infty
$$

by (2.37) and (2.45). For $I_2$, we note that for every $\xi \in \mathbb{R}^N$, $A_{\varepsilon}(\xi)$ converges to

$$
A_0(\xi) := \left\{
\begin{array}{ll}
g'(|\xi|^2)\xi & (\xi \neq 0), \\
0 & (\xi = 0),
\end{array}
\right.
$$

since $A_{\varepsilon}(\xi) = b(e^2 + |\xi|^2)^{-1/2} \xi + g''_p(e^2 + |\xi|^2)\xi$ satisfies $A_{\varepsilon}(0) = 0$. Also, for every $\varepsilon \in (0, 1)$, it is easy to check that

$$
|A_{\varepsilon}(Du_0)| \leq b + \Gamma\left(e^2 + |Du_0|^2\right)^{\frac{p-1}{2}} \leq C(b, p, \Lambda)(1 + |Du_0|^{p-1}) \quad \text{a.e. in } \Omega.
$$

Hence, by applying Lebesgue’s dominated convergence theorem, we have $A_{\varepsilon}(Du_0) \to A_0(Du_0)$ in $L^{p'}(\Omega; \mathbb{R}^N)$. It should be recalled that the weak convergence $Du_{\varepsilon} \to Du_0$ in $L^p(\Omega; \mathbb{R}^N)$ is already known by (2.44). These convergence results imply $I_2(\varepsilon_j) \to 0$ as $j \to \infty$. Thus, the claim (2.46) is verified.

We would like to prove that the limit $u_0$ is a weak solution of the problem (2.51). First, by (2.47) and [6 Theorem 4.9], we may let

$$
Du_{\varepsilon_j} \to Du_0 \quad \text{a.e. in } \Omega,
$$

and

$$
|Du_{\varepsilon_j}| \leq v \quad \text{a.e. in } \Omega,
$$

by taking a subsequence if necessary. Here a non-negative function $v \in L^p(\Omega)$ is independent of the subscript $j \in \mathbb{N}$. Then,

$$
A_{p,\varepsilon_j}(Du_{\varepsilon_j}) \to A_p(Du_0) \quad \text{in } L^{p'}(\Omega; \mathbb{R}^N)
$$

follows from (2.48) and (2.49). To verify this, we should note that by (2.16), the mapping $A_{p,\varepsilon}$ locally uniformly converges to $A_p$ in $\mathbb{R}^N$. Combining with (2.48), we can check $A_{p,\varepsilon_j}(Du_{\varepsilon_j}) \to A_p(Du_0)$ a.e. in $\Omega$. Also, by (1.9) and (2.49), we get

$$
|A_{p,\varepsilon_j}(Du_{\varepsilon_j})| \leq \Gamma\left(e_{\varepsilon_j}^2 + |Du_{\varepsilon_j}|^2\right)^{p/2-1} |Du_{\varepsilon_j}| \leq \Gamma\left(1 + v^2\right)^{\frac{p-1}{2}} \in L^{p'}(\Omega)
$$

a.e. in $\Omega$. Thus, (2.50) can be deduced by Lebesgue’s dominated convergence theorem. Secondly, since the mapping $Z_j := A_{1,\varepsilon_j}(Du_{\varepsilon_j})$ satisfies $\|Z_j\|_{L^\infty(\Omega)} \leq 1$, up to a subsequence, we may let

$$
Z_j \to Z \quad \text{in } L^\infty(\Omega; \mathbb{R}^N)
$$

(2.51)
for some $Z \in L^\infty(\Omega; \mathbb{R}^N)$ [6, Corollary 3.30]. This limit clearly satisfies $\|Z\|_{L^\infty(\Omega)} \leq 1$. Therefore, to check (2.33), it suffices to prove

$$Z = \frac{D u_0}{|D u_0|} \quad \text{a.e. in } D := \{x \in \Omega \mid D u_0(x) \neq 0\}.$$  

This claim is easy to deduce by (2.48). In fact, (2.48) yields $Z_{e_j} \rightarrow D u_0/|D u_0|$ a.e. in $D$, and hence we are able to conclude $Z_{e_j} \rightharpoonup D u_0/|D u_0|$ in $L^\infty(D; \mathbb{R}^N)$ by Lebesgue’s dominated convergence theorem. Finally, using (2.37) and (2.50)–(2.51), we are able to deduce (2.34) by letting $\varepsilon = e_j$ and $j \rightarrow \infty$ in the weak formulation (2.40).

We mention that $u_0$, a weak solution of (2.32), coincides with $u$ satisfying (2.39). In fact, for arbitrary $\phi \in W^{1,p}_0(\Omega; \mathbb{R}^N)$, we have

$$\|D(u + \phi)\| \geq \|D u\| + \langle Z \mid D \phi \rangle \quad \text{a.e. in } \Omega,$$  

since $Z$ satisfies (2.33). Similarly, we can easily get

$$g_p(|(D(u + \phi))^2|) \geq g_p(|D u|^2) + \langle A_p(D u) \mid D \phi \rangle \quad \text{a.e. in } \Omega.$$  

By testing $\phi \in W^{1,p}_0(\Omega; \mathbb{R}^N)$ into (2.34), we can easily notice $\mathcal{F}_0(u) \leq \mathcal{F}_0(u + \phi)$ for all $\phi \in W^{1,p}_0(\Omega; \mathbb{R}^N)$. In other words, the limit function $u_0 \in u_* + W^{1,p}_0(\Omega; \mathbb{R}^N)$ satisfies (2.39). We recall that a function verifying (2.39) uniquely exists, and thus $u = u_0$. This completes the proof.

### 2.4 Hölder continuity estimates

In Section 2.4, we would like to prove our main theorem (Theorem 1.2) by an approximation argument. In Proposition 2.4, we have already justified that a weak solution to (1.2) can be approximated by a weak solution to

$$-\operatorname{div}(g_p'(|D u_e|^2)D u_e) = f_e,$$  

(2.52)

under a suitable Dirichlet boundary value condition. We should note that the function $u_e$ defined by (2.38) solves (2.52) in the distributional sense, under a boundary condition $u_e = u_*$ on $\partial \Omega$. Since we have already justified convergence for approximated solutions $u_e$ in Proposition 2.4, it suffices to obtain a priori regularity estimates on weak solutions to a regularized system (2.52). The key estimates are given by Theorem 2.5, where the continuity estimates are independent of the approximation parameter $\varepsilon$, so that the Arzelà–Ascoli theorem can be applied.

**Theorem 2.5** (A priori Hölder estimates on truncated Jacobian matrices). Let positive numbers $\delta, \varepsilon$ satisfy (2.11), and $u_e$ be a weak solution to a regularized system (2.52) in $\Omega$ with

$$\|D u_e\|_{L^p(\Omega)} \leq L,$$  

(2.53)

and

$$\|f_e\|_{L^p(\Omega)} \leq F,$$  

(2.54)

for some constants $F, L \in (0, \infty)$. Then, for each fixed $x_0 \in \Omega$, there exist a sufficiently small open ball $B_{r_0}(x_0) \subseteq \Omega$ and an exponent $\alpha \in (0, 1)$, depending at most on $b, n, N, p, q, \gamma, \Gamma, F, L, d_* = \text{dist}(x_0, \partial \Omega)$, and $\delta$, such that $\mathcal{S}_{2\delta, \varepsilon}(D u_e) \in C^\alpha(B_{r_0/2}(x_0); \mathbb{R}^{Nn})$. Moreover, there exists a constant $\mu_0 \in (0, \infty)$, depending at most on $b, n, N, p, q, \gamma, \Gamma, F, L$, and $d_*$, such that

$$\sup_{B_{r_0}(x_0)} \mathcal{S}_{2\delta, \varepsilon}(D u_e) \leq \mu_0,$$  

(2.55)
and
\[ |\zeta_{2\delta,\varepsilon}(Du_e(x_1)) - \zeta_{2\delta,\varepsilon}(Du_e(x_2))| \leq \frac{2n/2\alpha+2\mu_0}{\rho_0^{\alpha}}|x_1 - x_2|^\alpha \] (2.56)
for all \( x_1, x_2 \in B_{\rho_0/2}(x_0) \).

To prove Theorem 2.5 we use three basic propositions, whose proofs are given later in Sections 3.5. Here it is noted that when stating Propositions 2.6–2.8 we have to use another modulus \( V_\varepsilon := \sqrt{e^2 + |Du_e|^2} \), instead of \( |Du_e| \), since we have relaxed the principal divergence operator \( \mathcal{L} \) itself. Along with this, we have already introduced another truncation mapping \( \Omega_{2\delta,\varepsilon} \), instead of \( \Omega_{2\delta} \).

The first proposition states local a priori Lipschitz bounds for regularized solutions.

\textbf{Proposition 2.6} (Local Lipschitz bounds). Let \( \varepsilon \in (0,1) \) and \( u_\varepsilon \) be a weak solution to a regularized system (2.52) in \( \Omega \). Fix an open ball \( B_\rho(x_0) \Subset \Omega \) with \( \rho \in (0,1) \). Then, there exists a constant \( C \in (0, \infty) \) depending at most on \( b, n, p, q, \gamma \), and \( \Gamma \), such that
\[ \text{ess sup}_{B_{\rho/2}(x_0)} |\zeta_{0}(Du_\varepsilon)| \leq C \left[ 1 + \|\zeta_{0}(Du_\varepsilon)\|_{L^p(B_{\rho}(x_0))} \right] \] for all \( \theta \in (0,1) \). Here the exponent \( d \in [n/p, \infty) \) depends at most on \( n, p, \) and \( q \).

These estimates can be deduced by carefully choosing test functions whose supports are separated from facets of approximated solutions. Also, it should be emphasized that local Lipschitz bounds in Proposition 2.6 are expectable, since the density \( E_\varepsilon \) has a \( p \)-Laplace-type structure when it is sufficiently far from the origin. Lipschitz estimates from a viewpoint of asymptotic behaviour of density functions are found in the existing literature [12] (see also [8] as a classical work). For the reader’s convenience, we would like to provide the proof of Proposition 2.6 in the appendix (Section 5).

Hereinafter we assume local uniform boundedness of \( V_\varepsilon \), which is guaranteed by Proposition 2.6. In particular, a scalar function \( |\zeta_{0}(Du_\varepsilon)| \) is uniformly bounded in each fixed subdomain of \( \Omega \). For an open ball \( B_\rho(x_0) \Subset \Omega \) and positive numbers \( \mu \in (0, \infty) \), \( \nu \in (0,1) \), we introduce a superlevel set
\[ S_{\rho,\mu,\nu}(x_0) := \{ x \in B_\rho(x_0) \mid V_\varepsilon - \delta > (1 - \nu)\mu \} \]

The second and third propositions (Propositions 2.7, 2.8) are useful for estimating oscillation of \( \zeta_{2\delta,\varepsilon}(Du_\varepsilon) \). Our analysis herein depends on whether \( V_\varepsilon \) vanishes near a point \( x_0 \) or not, which is judged by measuring the size of the superlevel set \( S_{\rho,\mu,\nu}(x_0) \). Before stating Propositions 2.7, 2.8 we introduce a constant \( \beta \in (0,1) \) by
\[ \beta := \begin{cases} 1 - \frac{n}{q} & (n < q < \infty), \\ \frac{\tilde{\beta}_0}{q} & (q = \infty), \end{cases} \]
where \( \tilde{\beta}_0 \in (0,1) \) is an arbitrary number. The exponent \( \beta \) appears when one considers regularity of solutions to the Poisson system \(-\Delta v = f\) with \( f \in L^q \). In fact, from the classical Schauder theory, it is well-known that this weak solution \( v \) admits local \( C^{1,\beta} \)-regularity.

Propositions 2.7, 2.8 below will be shown in Sections 3.4.

\textbf{Proposition 2.7} (An oscillation lemma). Let \( u_\varepsilon \) be a weak solution to a regularized system (2.52) in \( \Omega \). Assume that positive numbers \( \delta, \varepsilon, \mu, F, M \) satisfy (2.7), (2.54) and
\[ \text{ess sup}_{B_\rho(x_0)} |\zeta_{0}(Du_\varepsilon)| \leq \mu < \mu + \delta \leq M \] (2.57)
for some open ball \( B_\rho(x_0) \Subset \Omega \) with \( \rho \in (0,1) \). Assume that there holds
\[ |S_{\rho/2,\mu,\nu}(x_0)| \leq (1 - \nu)|B_{\rho/2}(x_0)| \] (2.58)
for some constant \( \nu \in (0, 1) \). Then, we have either
\[
\mu^2 < C_* \beta^p, \tag{2.59}
\]
or
\[
\text{ess sup}_{B_{r}(x_0)} |G_{\delta, \epsilon}(Du_{\epsilon})| \leq \kappa \mu. \tag{2.60}
\]

Here the constants \( C_* \in (0, \infty) \) and \( \kappa \in (2^{-\beta}, 1) \) depend at most on \( b, n, N, p, q, \gamma, \Gamma, F, M, \delta \) and \( \nu \).

**Proposition 2.8** (Campanato-type growth estimates). Let \( u_{\epsilon} \) be a weak solution to a regularized system \((2.52)\) in \( \Omega \). Assume that positive numbers \( \delta, \epsilon, \mu, F, M \) satisfy \((2.77), (2.54)\).
\[
\text{ess sup}_{B_{r}(x_0)} V_{\epsilon} \leq \mu + \delta \leq M \tag{2.61}
\]
for some open ball \( B_{r}(x_0) \subseteq \Omega \), and
\[
0 < \delta < \mu. \tag{2.62}
\]

Then, there exist numbers \( \nu \in (0, 1/4), \rho_* \in (0, 1), \) depending at most on \( b, n, N, p, q, \gamma, \Gamma, F, M, \delta \), such that the following statement holds true. If there hold \( 0 < \rho < \rho_* \) and
\[
|S_{\rho, \mu, \nu}(x_0)| > (1 - \nu)|B_{\rho}(x_0)|,
\]

then the limit
\[
\Gamma_{2\delta, \epsilon}(x_0) := \lim_{r \to 0} (G_{2\delta, \epsilon}(Du_{\epsilon}))_{x_0, r} \in \mathbb{R}^N \tag{2.64}
\]
exists. Moreover, there hold
\[
|\Gamma_{2\delta, \epsilon}(x_0)| \leq \mu, \tag{2.65}
\]
and
\[
\int_{B_{r}(x_0)} |G_{2\delta, \epsilon}(Du_{\epsilon}) - \Gamma_{2\delta, \epsilon}(x_0)|^2 \, dx \leq \left( \frac{r}{\rho} \right)^{2\beta} \mu^2 \text{ for all } r \in (0, \rho]. \tag{2.66}
\]

**Remark 2.9.** Let \( \delta \) and \( \epsilon \) satisfy \( 0 < \epsilon < \delta \). Then, for \( \xi \in \mathbb{R}^N \), \( |G_{\delta, \epsilon}(\xi)| \leq \mu \) holds if and only if \( \xi \) satisfies \( \sqrt{\epsilon^2 + |\xi|^2} \leq \mu + \delta \). In particular, the conditions \((2.57)\) and \((2.61)\) are equivalent. Also, it should be noted that the mapping \( G_{2\delta, \epsilon} \) satisfies
\[
|G_{2\delta, \epsilon}(\xi)| \leq (|G_{\delta, \epsilon}(\xi)| - \delta)_+ \leq |G_{\delta, \epsilon}(\xi)| \text{ for all } \xi \in \mathbb{R}^N, \tag{2.67}
\]
which is used in the proof of Theorem 2.5.

By applying Propositions 2.6, 2.8 we would like to prove Theorem 2.5.

**Proof.** For each fixed \( x_* \in \Omega \), we first choose
\[
R := \min \left\{ \frac{1}{2}, \frac{1}{3} \text{ dist } (x_*, \partial \Omega) \right\} > 0,
\]
so that \( B_{2R}(x_*) \subseteq \Omega \) holds. By Proposition 2.6 we may take a finite constant \( \mu_0 \in (0, \infty) \), depending at most on \( b, n, N, p, q, \gamma, \Gamma, F, \) and \( R \), such that
\[
\text{ess sup}_{B_{R}(x_*)} V_{\epsilon} \leq \mu_0. \tag{2.68}
\]
We set \( M := 1 + \mu_0 \), so that \( \mu_0 + \delta \leq M \) clearly holds.
We choose and fix the numbers \( \nu \in (0, 1/4) \), \( \rho_* \in (0, 1) \) as in Proposition 2.8 which depend at most on \( b, n, N, p, q, \gamma, \Gamma, F, M, \) and \( \delta \). Corresponding to this \( \nu \), we choose finite constants \( \kappa \in (2^{-\beta}, 1) \), \( C_* \in [1, \infty) \) as in Proposition 2.7. We define the desired Hölder exponent \( \alpha \in (0, \beta/2) \) by 
\[
\alpha := -\log \kappa/\log 4,
\]
so that the identity \( 4^{-\alpha} = \kappa \) holds. We also put the radius \( \rho_0 \) such that 
\[
0 < \rho_0 \leq \min \left\{ \frac{R}{2}, \rho_* \right\} < 1 \quad \text{and} \quad C_* \rho_0^\beta \leq \kappa^2 \mu^2_0, \quad (2.69)
\]
which depends at most on \( b, n, p, q, \gamma, \Gamma, F, L, M, \) and \( \delta \). We define non-negative decreasing sequences \( \{\rho_k\}_{k=1}^\infty \), \( \{\mu_k\}_{k=1}^\infty \) by setting \( \rho_k := 4^{-k} \rho_0 \), \( \mu_k := \kappa^k \mu_0 \) for \( k \in \mathbb{N} \). By \( 2^{-\beta} < \kappa = 4^{-\alpha} < 1 \) and (2.69), it is easy to check that 
\[
\sqrt{C_* \rho_k^\beta} \leq 2^{-\beta k} \kappa \mu_0 \leq \kappa^{k+1} \mu_0 = \mu_{k+1}, \quad (2.70)
\]
and
\[
\mu_k = 4^{\alpha k} \mu_0 = \left( \frac{\rho_k}{\rho_0} \right)^\alpha \mu_0 \quad (2.71)
\]
for every \( k \in \mathbb{Z}_{\geq 0} \).

We claim that for every \( x_0 \in B_{\rho_0}(x_*) \), the limit
\[
\Gamma_{2\delta,\varepsilon}(x_0) := \lim_{r \to 0} \left\{ \mathcal{S}_{2\delta,\varepsilon}(Du_{\varepsilon}) \right\}_{x_0, r} \in \mathbb{R}^n
\]
exists, and this limit satisfies
\[
\int_{B_r(x_0)} \left| \mathcal{S}_{2\delta,\varepsilon}(Du_{\varepsilon}) - \Gamma_{2\delta,\varepsilon}(x_0) \right|^2 \, dx \leq 4^{2\alpha+1} \left( \frac{r}{\rho_0} \right)^{2\alpha} \mu_0^2 \quad \text{for all } r \in (0, \rho_0]. \quad (2.72)
\]
In the proof of (2.72), we introduce a set
\[
\mathcal{N} := \left\{ k \in \mathbb{Z}_{\geq 0} \mid |S_{\rho_k/2, \mu_k, \varepsilon}(x_0)| > (1-\nu)|B_{\rho_k/2}(x_0)| \right\},
\]
and define \( k_* \in \mathbb{Z}_{\geq 0} \) to be the minimum number of \( \mathcal{N} \) when it is non-empty. We consider the three possible cases: \( \mathcal{N} \neq \emptyset \) and \( \mu_{k_*} > \delta; \mathcal{N} \neq \emptyset \) and \( \mu_{k_*} \leq \delta; \) and \( \mathcal{N} = \emptyset \). Before dealing with each case, we mention that if \( \mathcal{N} \neq \emptyset \), then it is possible to apply Proposition 2.7 with \( (\rho, \mu) = (\rho_k, \mu_k) \) for \( k \in \{0, 1, \ldots, k_* - 1\} \), by the definition of \( k_* \). With (2.70) in mind, we are able to obtain
\[
\text{ess sup}_{B_{\rho_k}(x_0)} \left| \mathcal{S}_{\delta,\varepsilon}(\nabla u_{\varepsilon}) \right| \leq \mu_k \quad \text{for every } k \in \{0, 1, \ldots, k_*\}. \quad (2.73)
\]

In the first case, the conditions \( k_* \in \mathcal{N} \) and \( \mu_{k_*} > \delta \) enable us to apply Proposition 2.8 for the open ball \( B_{\rho_{k_*}/2}(x_0) \) with \( \mu = \mu_{k_*} \). In particular, the limit \( \Gamma_{2\delta,\varepsilon}(x_0) \) exists and this limit satisfies
\[
|\Gamma_{2\delta,\varepsilon}(x_0)| \leq \mu_{k_*}, \quad (2.74)
\]
and
\[
\int_{B_r(x_0)} \left| \mathcal{S}_{2\delta,\varepsilon}(Du_{\varepsilon}) - \Gamma_{2\delta,\varepsilon}(x_0) \right|^2 \, dx \leq \left( \frac{2r}{\rho_{k_*}} \right)^{2\beta} \mu_{k_*}^2 \quad \text{for all } r \in \left(0, \frac{\rho_{k_*}}{2}\right], \quad (2.75)
\]
When \( 0 < r \leq \rho_{k_*}/2 \), we use (2.71), (2.73), and \( \alpha < \beta \) to get
\[
\int_{B_r(x_0)} \left| \mathcal{S}_{2\delta,\varepsilon}(Du_{\varepsilon}) - \Gamma_{2\delta,\varepsilon}(x_0) \right|^2 \, dx \leq \left( \frac{2r}{\rho_{k_*}} \right)^{2\alpha} \left( \frac{\rho_{k_*}}{\rho_0} \right)^{2\alpha} \mu_0^2 = 4^\alpha \left( \frac{r}{\rho_0} \right)^{2\alpha} \mu_0^2.
\]
When \( \rho_{k_0}/2 < r \leq \rho_0 \), there corresponds a unique integer \( k \in \{0, \ldots, k_0\} \) such that \( \rho_{k+1} < r \leq \rho_k \). By (2.73), we compute
\[
\int_{B_r(x_0)} \left| \mathcal{G}_{2\delta, \varepsilon}(Du) - \Gamma_{2\delta, \varepsilon}(x_0) \right|^2 \, dx \leq 2 \left( \text{ess sup} \left| \mathcal{G}_{2\delta, \varepsilon}(Du) \right|^2 + \left| \Gamma_{2\delta, \varepsilon}(x_0) \right|^2 \right)
\leq 4 \mu_k^2 \leq 4 \left( \frac{\rho_k}{\rho'_0} \right)^{2\alpha} \mu_0^2 \leq 4 \left( \frac{4r}{\rho'_0} \right)^{2\alpha} \mu_0^2.
\]

In the second case, we recall (2.67) in Remark 2.9 to notice \( \mathcal{G}_{2\delta, \varepsilon}(Du) = 0 \) a.e. in \( B_{k_0}(x_0) \). Combining with (2.73), we can easily check
\[
\text{ess sup}_{B_{\rho_k}(x_0)} \left| \mathcal{G}_{2\delta, \varepsilon}(Du) \right| \leq \mu_k \quad \text{for every } k \in \mathbb{Z}_{\geq 0},
\]
which clearly yields \( \Gamma_{2\delta, \varepsilon}(x_0) = 0 \). For every \( r \in (0, \rho_0) \), there corresponds a unique integer \( k \in \mathbb{Z}_{\geq 0} \) such that \( \rho_{k+1} < r \leq \rho_k \). By (2.76) and \( \kappa = 4^{-\alpha} \), we have
\[
\int_{B_r(x_0)} \left| \mathcal{G}_{2\delta, \varepsilon}(Du) - \Gamma_{2\delta, \varepsilon}(x_0) \right|^2 \, dx = \int_{B_r(x_0)} \left| \mathcal{G}_{2\delta, \varepsilon}(Du) \right|^2 \, dx
\leq \text{ess sup}_{B_{\rho_k}(x_0)} \left| \mathcal{G}_{2\delta, \varepsilon}(Du) \right| \leq \mu_k^2 = 4^{-2\alpha} \mu_0^2
\leq \left[ 4 \left( \frac{r}{\rho'_0} \right) \right]^{2\alpha} \mu_0^2 = 16^\alpha \left( \frac{r}{\rho'_0} \right)^{2\alpha} \mu_0^2.
\]

In the remaining case \( N = 0 \), it is clear that there holds \( |E_{\rho_k/2, \mu_k, y_k}| \leq (1 - \nu)|B_{\rho_k/2}(x_0)| \) for every \( k \in \mathbb{Z}_{\geq 0} \). Applying (2.70) and Proposition 2.7 with \( (\rho, \mu) = (\rho_k, \mu_k) \) \( (k \in \mathbb{Z}_{\geq 0}) \) repeatedly, we can easily check that
\[
\text{ess sup}_{B_{\rho_k}(x_0)} \left| \mathcal{G}_{\delta, \varepsilon}(Du) \right| \leq \mu_k \quad \text{for every } k \in \mathbb{Z}_{\geq 0}.
\]
In particular, (2.76) clearly follows from this and (2.67). Thus, the proof of (2.72) can be accomplished, similarly to the second case. In any possible cases, we conclude that \( \Gamma_{2\delta, \varepsilon}(x_0) \) exists and satisfies (2.72), as well as
\[
\left| \Gamma_{2\delta, \varepsilon}(x_0) \right| \leq \mu_0 \quad \text{for all } x_0 \in B_{\rho_0}(x_s),
\]
which immediately follows from (2.68).

From (2.72) and (2.77), we would like to show
\[
\left| \Gamma_{2\delta, \varepsilon}(x_1) - \Gamma_{2\delta, \varepsilon}(x_2) \right| \leq \left( \frac{2^{2\alpha + 2\alpha n/2} \mu_0}{\rho'_0} \right)^{\alpha} |x_1 - x_2|^\alpha
\]
for all \( x_1, x_2 \in B_{\rho_0/2}(x_s) \). We prove (2.78) by dividing into the two possible cases. When \( r := |x_1 - x_2| \leq \rho_0/2 \), we set a point \( x_3 := (x_1 + x_2)/2 \in B_{\rho_0}(x_s) \). By \( B_{\rho_0/2}(x_3) \subset B_r(x) \subset B_{\rho_0/2}(x_j) \subset B_{\rho_0}(x) \) for each \( j \in \{1, 2\} \), we use (2.72) to obtain
\[
\left| \Gamma_{2\delta, \varepsilon}(x_1) - \Gamma_{2\delta, \varepsilon}(x_2) \right|^2
\leq 2 \left( \int_{B_{\rho_0/2}(x_3)} \left| \mathcal{G}_{2\delta, \varepsilon}(Du) - \Gamma_{2\delta, \varepsilon}(x_1) \right|^2 \, dx \right.
\]
which yields (2.78). In the remaining case \(|x_1 - x_2| > \rho_0/2\), we simply use (2.77) to get
\[
|\Gamma_{2\delta, \varepsilon}(x_1) - \Gamma_{2\delta, \varepsilon}(x_2)| \leq 2\mu_0 \leq 2 \cdot \frac{2^\alpha |x_1 - x_2|^\alpha}{\rho_0^\alpha} \mu_0,
\]
which completes the proof of (2.78). Finally, since the mapping \(\Gamma_{2\delta, \varepsilon}\) is a Lebesgue representative of \(G_{2\delta, \varepsilon}(Du,v)\) in \(L^p(\Omega; \mathbb{R}^{Nn})\), the claims (2.55)–(2.56) immediately follow from (2.77)–(2.78).

We would like to conclude Section 2 by giving the proof of our main Theorem 1.2.

**Proof.** Let \(u \in W^{1, p}(\Omega; \mathbb{R}^N)\) be a weak solution to (1.2). We choose and fix \(\{f_\varepsilon\}_{0 < \varepsilon < 1} \subset L^q(\Omega; \mathbb{R}^{Nn})\) enjoying (2.37), and for each fixed \(\varepsilon \in (0, 1)\), we define the functional \(\mathcal{F}_\varepsilon\) by (2.50). We set a function
\[
u_\varepsilon := \arg \min \left\{ v \in u + W^{1, p}_0(\Omega; \mathbb{R}^N) \mid \mathcal{F}_\varepsilon(v) \right\}
\]
which is a unique solution of the Dirichlet problem
\[
\begin{cases}
\mathcal{L}^\varepsilon u_\varepsilon &= f_\varepsilon \quad \text{in } \Omega, \\
u_\varepsilon &= u \quad \text{on } \partial \Omega.
\end{cases}
\]
By Proposition 2.14 there exists a decreasing sequence \(\{\varepsilon_j\}_{j=1}^\infty \subset (0, 1)\) such that \(\varepsilon_j \to 0 \quad \text{and} \quad u_{\varepsilon_j} \to u \quad \text{in} \quad W^{1, p}(\Omega; \mathbb{R}^N)\). In particular, we may let (2.48) hold by taking a subsequence if necessary. Also, we may choose finite constants \(L > \|Du\|_{L^p(\Omega)}\) and \(F > \|f_\varepsilon\|_{L^q(\Omega)}\) that satisfy (2.53)–(2.54).

Fix \(x_0 \in \Omega\) and \(\delta \in (0, 1)\) arbitrarily. Then Theorem 2.9 enables us to apply the Arzelà–Ascoli theorem to the net \(\{G_{2\delta, \varepsilon}(Du_{\varepsilon})\}_{0 < \varepsilon < \delta/4} \subset C^\alpha(B_{\rho_0/2}(x_0); \mathbb{R}^{Nn})\). In particular, by taking a subsequence if necessary, we are able to conclude that \(G_{2\delta, \varepsilon}(Du_{\varepsilon_j}) \in C^\alpha(B_{\rho_0/2}(x_0); \mathbb{R}^{Nn})\) uniformly converges to a continuous mapping \(v_{2\delta} \in C^\alpha(B_{\rho_0/2}(x_0); \mathbb{R}^{Nn})\). On the other hand, by (2.48) we have already known that \(G_{2\delta, \varepsilon}(Du_{\varepsilon_j}) \to G_{2\delta}(Du)\) a.e. in \(\Omega\). Hence, it follows that \(v_{2\delta} = G_{2\delta}(Du)\) a.e. in \(B_{\rho_0/2}(x_0)\). Moreover, from (2.55)–(2.56) we conclude that \(G_{2\delta}(Du)\) satisfies (1.14)–(1.15).

We have already proved \(G_{\delta}(Du) \in C^0(\Omega; \mathbb{R}^{Nn})\) for each fixed \(\delta \in (0, 1)\), from which we show continuity of \(Du\). By the definition of \(G_{\delta}\), we can easily check that
\[
\sup_{\Omega} \left| G_{\delta_1}(Du) - G_{\delta_2}(Du) \right| \leq |\delta_1 - \delta_2| \quad \text{for all } \delta_1, \delta_2 \in (0, 1).
\]
In particular, the net \(\{G_{\delta}(Du)\}_{0 < \delta < 1} \subset C^0(\Omega; \mathbb{R}^{Nn})\) uniformly converges to a continuous mapping \(v_0 \in C^0(\Omega; \mathbb{R}^{Nn})\). On the other hand, it is clear that \(G_{\delta}(Du) \to Du\) a.e. in \(\Omega\). Thus, we realize \(v_0 = Du\) a.e. in \(\Omega\), and this completes the proof.

## 3 Weak formulations

### 3.1 A basic weak formulation

We note that for each fixed \(\varepsilon \in (0, 1)\), the regularized system (2.52) is uniformly elliptic. To be precise, by direct computations, we can easily check that the relaxed density \(E_\varepsilon(\xi) = g_\varepsilon(|\xi|^2)\) admits a constant \(C_\varepsilon \in (\gamma, \infty)\), depending on \(\varepsilon\), such that
\[
\gamma \left( \varepsilon^2 + |\xi|^2 \right)^{p/2-1} \text{id}_{\mathbb{R}^n} \leq B_\varepsilon(\xi) = D^2 E_\varepsilon(\xi) \leq C_\varepsilon \left( \varepsilon^2 + |\xi|^2 \right)^{p/2-1} \text{id}_{\mathbb{R}^n} \quad \text{for all } \xi \in \mathbb{R}^{Nn} \quad (3.1)
\]
Hence, it is not restrictive to let \( u_\varepsilon \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{R}^N) \cap W^{2,2}_{\text{loc}}(\Omega; \mathbb{R}^N) \) (see also Remark 5.3 in Section 5). In particular, for each fixed \( B_\rho(x_0) \subset \Omega \), we are able to deduce

\[
\int_{B_\rho(x_0)} \langle A_\varepsilon(Du_\varepsilon) \mid D\phi \rangle \, dx = \int_{B_\rho(x_0)} \langle f_\varepsilon \mid \phi \rangle \, dx
\]

for all \( \phi \in W^{1,1}_{0}(B_\rho(x_0); \mathbb{R}^{Nn}) \). Also, differentiating (2.5.2) by \( x_\alpha \) and integrating by parts, we can deduce an weak formulation

\[
\int_{B_\rho(x_0)} \langle D_\alpha(A_\varepsilon(Du_\varepsilon)) \mid D\phi \rangle \, dx = -\int_{B_\rho(x_0)} \langle f_\varepsilon \mid D_\alpha\phi \rangle \, dx
\]

for all \( \phi \in W^{1,2}_{0}(B_\rho(x_0); \mathbb{R}^N), \alpha \in \{1, \ldots, n\} \). Following [4, Lemma 4.1] (see also [35, Lemma 3.5]), we would like to give a basic weak formulation that is fully used in our computations.

**Lemma 3.1.** Let \( u_\varepsilon \) be a weak solution to the regularized system (2.5.2) with \( 0 < \varepsilon < 1 \). Assume that a non-decreasing and non-negative function \( \psi \in W^{1,\infty}_\text{loc}([0, \infty)) \) is differentiable except at finitely many points. For any non-negative function \( \zeta \in C^1_\text{loc}(B_\rho(x_0)) \), we set

\[
\begin{aligned}
J_1 & : = \int_{B_\rho(x_0)} \langle \mathcal{C}_\varepsilon(Du_\varepsilon) \nabla V_\varepsilon \mid \nabla \zeta \rangle \psi(V_\varepsilon) \nabla V_\varepsilon \, dx, \\
J_2 & : = \int_{B_\rho(x_0)} \langle \mathcal{C}_\varepsilon(Du_\varepsilon) \nabla V_\varepsilon \mid \nabla \nabla \zeta \rangle \psi'(V_\varepsilon) \nabla V_\varepsilon \, dx, \\
J_3 & : = \int_{B_\rho(x_0)} V_\varepsilon^{p-2} |D^2 u_\varepsilon|^2 \zeta \psi(V_\varepsilon) \, dx, \\
J_4 & : = \int_{B_\rho(x_0)} |f_\varepsilon|^2 \psi(V_\varepsilon) V_\varepsilon^{2-p} \zeta \, dx, \\
J_5 & : = \int_{B_\rho(x_0)} |f_\varepsilon|^2 \psi'(V_\varepsilon) V_\varepsilon^{3-p} \zeta \, dx, \\
J_6 & : = \int_{B_\rho(x_0)} |f_\varepsilon| V_\varepsilon^{2-p} \zeta \, dx,
\end{aligned}
\]

with

\[
\mathcal{C}_\varepsilon(Du_\varepsilon) : = g_\varepsilon^1(|Du_\varepsilon|^2) \text{id}_n + 2g_\varepsilon^2(|Du_\varepsilon|^2) \sum_{i=1}^{N} \nabla u_\varepsilon^i \otimes \nabla u_\varepsilon^i.
\]

Then we have

\[
2J_1 + J_2 + \gamma J_3 \leq \frac{1}{\gamma} (nJ_4 + J_5) + 2J_6, \tag{3.4}
\]

Before the proof, we mention that the matrix-valued functions \( \mathcal{C}_\varepsilon(Du_\varepsilon) \) satisfy

\[
\gamma V_\varepsilon^{p-2} \text{id}_n \leq \mathcal{C}_\varepsilon(Du_\varepsilon) \leq \left(bV_\varepsilon^{-1} + 3V_\varepsilon^{p-2}\right) \text{id}_n \quad \text{a.e. in } \Omega, \tag{3.5}
\]

which follows from (1.10)–(1.11) and (2.5)–(2.6).

**Proof.** For notational simplicity we write \( B := B_\rho(x_0) \). For a fixed index \( \alpha \in \{1, \ldots, n\} \), we test \( \phi := \zeta \psi(V_\varepsilon) D_\alpha u_\varepsilon \in W^{1,1}_{0}(B; \mathbb{R}^N) \) into (3.3). Summing over \( \alpha \in \{1, \ldots, n\} \), we have

\[
J_0 + J_1 + J_2
\]

\[
= -\int_{B} \psi(V_\varepsilon) \langle f_\varepsilon \mid Du_\varepsilon \nabla \zeta \rangle \, dx - \int_{B} \zeta \psi'(V_\varepsilon) \langle f_\varepsilon \mid Du_\varepsilon \nabla V_\varepsilon \rangle \, dx - \int_{B} \zeta \psi(V_\varepsilon) \sum_{i=1}^{N} \sum_{\alpha=1}^{n} f_\varepsilon^i \theta_{x_\alpha} u_\varepsilon^i \, dx
\]
and Young’s inequality we can compute Lemma 3.2.

In the resulting weak formulation (3.4), we may discard two non-negative integrals from which (3.4) immediately follows.

As a result, we get \( u_{1} \) and an open ball 3.2 below, we are able to obtain Caccioppoli-type estimates for \{ \}

implies that the scalar function \( \) a.e. in \( \Omega \) a.e. in \( \Omega \). These inequalities imply \( J_{0} \geq \gamma J_{3} \). Clearly, \( |J_{7}| \leq J_{6} \) holds. To compute \( J_{8}, J_{9} \), we recall (3.5). Then by Young’s inequality we can compute

\[
|J_{8}| \leq \frac{\gamma}{2} \int_{B} V_{e}^{p-1} |\nabla V_{e}|^{2} \zeta \psi'(V_{e}) \, dx + \frac{1}{2} \int_{B} |f_{e}|^{2} V_{e}^{3-p} \zeta \psi'(V_{e}) \, dx \leq \frac{1}{2} J_{2} + \frac{1}{2} J_{5},
\]

and

\[
|J_{9}| \leq \sqrt{n} \int_{B} |f_{e}| \zeta \psi(V_{e}) |D^{2} u_{e}| \, dx
\leq \frac{\gamma}{2} \int_{B} V_{e}^{p-1} |D^{2} u_{e}|^{2} \zeta \psi(V_{e}) \, dx + \frac{n}{2} \int_{B} |f_{e}|^{2} \psi(V_{e}) V_{e}^{2-p} \zeta \, dx
= \frac{\gamma}{2} J_{2} + \frac{n}{2} J_{4}.
\]

As a result, we get

\[
J_{1} + J_{2} + \gamma J_{3} \leq \frac{1}{2} J_{2} + \frac{\gamma}{2} J_{3} + \frac{n}{2} J_{4} + \frac{1}{2} J_{5} + J_{6},
\]

from which (3.4) immediately follows. \( \square \)

### 3.2 De Giorgi’s truncation and Caccioppoli-type estimates

In the resulting weak formulation (3.4), we may discard two non-negative integrals \( J_{2}, J_{3} \). Then, (3.4) implies that the scalar function \( V_{e} \) is a subsolution to an elliptic problem. By appealing to convex composition of \( V_{e} \), we would like to prove that another scalar function \( U_{\delta, e} \), defined by

\[
U_{\delta, e} := (V_{e} - \delta)^{2} \in L^{\infty}_{\text{loc}}(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega)
\]

is also a subsolution to a certain uniformly elliptic problem. This is possible since this function is supported in a place \( \{ V_{e} > \delta \} \), where the system (2.52) becomes uniformly elliptic. Thus, as in Lemma 3.2 below, we are able to obtain Caccioppoli-type estimates for \( U_{\delta, e} \).

**Lemma 3.2.** Let \( u_{e} \) be a weak solution to (2.52) with \( 0 < \varepsilon < 1 \). Assume that positive numbers \( \delta, \mu, M, \) and an open ball \( B_{\rho}(x_{0}) \) satisfy (2.57). Then, the scalar function \( U_{\delta, e} \) defined by (3.6) satisfies

\[
\int_{B_{\rho}(x_{0})} (A_{\delta, e}(D u_{e}) \nabla U_{\delta, e} \cdot \nabla \zeta) \, dx \leq C_{0} \left[ \int_{B_{\rho}(x_{0})} |f_{e}|^{2} \zeta \, dx + \int_{B_{\rho}(x_{0})} |f_{e}| |
\]

\( \zeta| \, dx \right]
\]

(3.7) satisfies
for all non-negative function $\zeta \in C^1_c(B_\rho(x_0))$, where $A_{\delta, \epsilon}(Du_{\epsilon})$ is an $n \times n$ matrix-valued function satisfying

$$\gamma_*\text{id}_n \leq A_{\delta, \epsilon}(Du_{\epsilon}) \leq \Gamma_*\text{id}_n \text{ a.e. in } B_\rho(x_0).$$

(3.8)

The constants $C_0 \in (0, \infty)$ and $0 < \gamma_* \leq \Gamma_* < \infty$ depend at most on $b, n, p, \gamma, \Gamma, M,$ and $\delta$. In particular, we have

$$\int_{B_\rho(x_0)} |\nabla [\eta(U_{\delta, \epsilon} - k)_\epsilon]|^2 \, dx \leq C \int_{B_\rho(x_0)} |\nabla [\eta]^2(U_{\delta, \epsilon} - k)^2_\epsilon| \, dx + \mu^2 \int_{A_{\delta, \epsilon}(x_0)} |f_\epsilon|^2 \eta^2 \, dx$$

(3.9)

for all $k \in (0, \infty)$ and for any non-negative function $\eta \in C^1_0(B_\rho(x_0))$. Here $A_{k, \rho}(x_0) := \{x \in B_\rho(x_0) \mid U_{\delta, \epsilon}(x) > k\}$, and the constant $C \in (0, \infty)$ depends on $\gamma_*, \Gamma_*$, and $C_0$.

**Proof.** We choose $\psi(t) := (t - \delta)_+$ for $t \in [0, \infty)$, so that $\psi(V_{\epsilon}) = |\zeta_{\delta, \epsilon}(Du_{\epsilon})|$ holds. From (3.4), we will deduce (3.7). To compute $J_1$, we note that $\psi(V_{\epsilon})$ vanishes when $V_{\epsilon} \leq \delta$, and that the identity $\nabla U_{\delta, \epsilon} = 2\psi(V_{\epsilon})\nabla V_{\epsilon}$ holds. From these we obtain

$$J_1 = \int_{B_\rho(x_0)} \langle V_{\epsilon} G_{\epsilon}(Du_{\epsilon}) \psi(V_{\epsilon}) \nabla V_{\epsilon} \mid \nabla \zeta \rangle \, dx = \frac{1}{2} \int_{B_\rho(x_0)} \langle A_{\delta, \epsilon}(Du_{\epsilon}) \nabla U_{\delta, \epsilon} \mid \nabla \zeta \rangle \, dx$$

with

$$A_{\delta, \epsilon}(Du_{\epsilon}) := \begin{cases} V_{\epsilon} G_{\epsilon}(Du_{\epsilon}) & \text{(if } V_{\epsilon} > \delta), \\ \text{id}_n & \text{(otherwise).} \end{cases}$$

By (3.3), the coefficient matrix $A_{\delta, \epsilon}(Du_{\epsilon})$ satisfies (3.8) with

$$\gamma_* = \min\{1, \gamma \delta^{p-1}\}, \quad \Gamma_* = \max\{1, b + 3\Gamma M^{p-1}\}.$$

By discarding positive integrals $J_2, J_3$, we compute

$$J_1 \leq \frac{1}{2\gamma} \int_{B_\rho(x_0)} |f_\epsilon|^{2} \eta_\epsilon^{\gamma} \langle \psi(V_{\epsilon})V_{\epsilon} + \psi'(V_{\epsilon})V_{\epsilon}^2 \rangle d\zeta \, dx + \int_{B_\rho(x_0)} |f_\epsilon| |\nabla \zeta| \psi(V_{\epsilon})V_{\epsilon} \, dx$$

$$\leq \frac{1}{2\gamma} \int_{B_\rho(x_0)} |f_\epsilon|^{2} \eta_\epsilon^{\gamma} \langle \frac{\mu M + M^2}{\delta^{p-1}} \rangle d\zeta \, dx + \mu M \int_{B_\rho(x_0)} |f_\epsilon| |\nabla \zeta| \, dx,$$

from which (3.7) immediately follows.

The estimate (3.9) is easy to deduce by choosing $\zeta := \eta^2(U_{\delta, \epsilon} - k)_\epsilon \in W^{1,2}_0(B_\rho(x_0))$ into (3.7), and making standard absorbing arguments (see [35] Lemma 3.15 for detailed computations). We should mention that this test function $\zeta$ is admissible by approximation, since it is compactly supported in $B_\rho(x_0)$.

The Caccioppoli-type inequality (3.9) implies that the scalar function $U_{\delta, \epsilon}$ is in a certain De Giorgi class. From this fact, we can conclude two oscillation lemmata for the scalar function $U_{\delta, \epsilon} = |\zeta_{\delta, \epsilon}(\nabla u_{\epsilon})|^2$ (Lemmata 3.3, 3.4).

**Lemma 3.3.** Let $u_{\epsilon}$ be a weak solution to (2.52). Assume that positive numbers $\delta, \mu, F, M,$ and an open ball $B_\rho(x_0)$ satisfy (2.54) and (2.57). Then there exists a number $\hat{\nu} \in (0, 1)$, depending at most on $b, n, N, p, q, \gamma, F, M,$ and $\delta$, such that if there holds

$$\frac{|\{x \in B_{\rho/2}(x_0) \mid U_{\delta, \epsilon}(x) > (1 - \theta)\mu^2\}|}{|B_{\rho/2}(x_0)|} \leq \hat{\nu}$$

(3.10)

for some $\theta \in (0, 1)$, then we have either

$$\mu^2 < \frac{\rho^\beta}{\theta} \text{ or } \text{ess sup}_{B_{\rho/\delta}(x_0)} U_{\delta, \epsilon} \leq \left(1 - \frac{\theta}{2}\right)^2 \mu^2.$$

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Lemma 3.4. Under the assumptions of Proposition [2.7] for every \( i_* \in \mathbb{N} \), we have either
\[
\mu^2 < \frac{2^{i_*}p^\beta}{\nu} \text{ or } \left\{ x \in B_{p/2}(x_0) \left| \frac{U_{\delta, \epsilon}(x)}{|B_{p/2}(x_0)|} > (1 - 2^{-i_*} \nu)\mu^2 \right. \right\} < C_{\tau},
\]
where the constant \( C_{\tau} \in (0, \infty) \) depends at most on \( b, n, N, p, q, \gamma, \Gamma, F, M, \) and \( \delta \).

We omit the proofs of Lemmata 3.3–3.4 because they can be deduced by routine calculations given in [13] Chapter 10, §4–5. For detailed computations, see [4] §7 or [35] §4. By Lemmata 3.3–3.4, we are able to find the desired constants \( C_* \in (1, \infty) \) and \( \kappa \in (2^{-\beta}, 1) \) in Proposition 2.7. To be precise, we choose a sufficiently large number \( i_* \in (C_{\tau}, \nu, \bar{\nu}) \in \mathbb{N} \) such that
\[
\frac{C_{\tau}}{\nu \sqrt{i_*}} \leq \bar{\nu} \quad \text{and} \quad 0 < 2^{-(i_* + 1)} < 1 - 2^{-2\beta}.
\]
Then, by applying Lemmata 3.3–3.4 with \( \theta = 2^{-i_*} \nu \), we can easily check that the constants \( C_* := \theta^{-1} = 2^{i_*} \nu^{-1} \in (1, \infty) \) and \( \kappa := \sqrt{1 - 2^{-(i_* + 1)}} \in (2^{-\beta}, 1) \) satisfy Proposition 2.7 (see also [4] Proposition 3.5), [35] Proposition 3.3).

4 Campanato-type decay estimates

Section 4 provides the proof of Proposition 2.8. The basic idea is that the measure assumption \( (2.63) \) implies that the scalar function \( V_{\epsilon} \) will not degenerate near the point \( x_0 \) if the ratio \( \nu \) is sufficiently close to 0. With this in mind, we appeal to freezing coefficient arguments and shrinking methods to obtain Campanato-type integral growth estimates \( (2.66) \). To prove this, however, we also have to check that an average \( (Du_{\epsilon})_{x_0,r} \) never degenerates even when the radius \( r \) tends to 0. To overcome this problem, we will provide a variety of energy estimates from the assumptions \( (2.62) \) and \( (2.63) \). Most of our computations concerning energy estimates differ from [4] §4–5, since the density function we discuss is structurally different from theirs.

In Section 4, we consider an \( L^2 \)-mean oscillation given by
\[
\Phi(x_0, r) := \int_{B_r(x_0)} |Du_{\epsilon} - (Du_{\epsilon})_{x_0,r}|^2 \, dx \text{ for } r \in (0, \rho]
\]
in an open ball \( B_r(x_0) \subseteq \Omega \). To estimate \( \Phi \), we often use a well-known fact that there holds
\[
\int_{B_r(x_0)} |v - (v)_{x_0,r}|^2 \, dx = \min_{\xi \in \mathbb{R}^N} \int_{B_r(x_0)} |v - \xi|^2 \, dx \quad (4.1)
\]
for every \( v \in L^2(B_r(x_0); \mathbb{R}^N) \). Also, we recall the Poincaré–Sobolev inequality [12] Chapter IX, Theorem 10.1:
\[
\int_{B_r(x_0)} |v|^2 \, dx \leq C(n)r^2 \left( \int_{B_r(x_0)} |Dv|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \quad (4.2)
\]
for all \( v \in W^{1, \frac{2n}{n-2}}(B_r(x_0); \mathbb{R}^N) \) satisfying
\[
\int_{B_r(x_0)} v \, dx = 0,
\]
which is used in Sections 4.1, 4.2.
4.1 Energy estimates

First in Section 4.1 we go back to the weak formulation (3.4), and deduce some energy estimates under suitable assumptions as in Proposition 2.8. Here we would like to show local $L^2$-bounds for the Jacobian matrices of $G_{p,e}(Du_e) = V_{e}^{p-1}Du_e$, where the mapping $G_{p,e}$ is defined by (2.27).

**Lemma 4.1.** Let $u_e$ be a weak solution to (2.52) in $\Omega$. Assume that positive numbers $\delta, \mu, F, M,$ and an open ball $B_{\rho}(x_0) \Subset \Omega$ satisfy (2.57), (2.61)–(2.62) and $0 < \rho < 1$. Then, for all $\nu \in (0, 1)$ and $\tau \in (0, 1)$, the following estimates (4.3)–(4.4) hold:

\[
\int_{B_{\tau \rho}(x_0)} |D[G_{p,e}(Du_e)]|^2 \, dx \leq \frac{C}{\tau^n} \left[ \frac{1}{(1-\tau)^2 \rho^2} + \frac{F^2 \rho^{-\frac{2\mu}{\nu}}}{\nu} \right] \mu^{2p}.
\]  

(4.3)

\[
\frac{1}{|B_{\tau \rho}(x_0)|} \int_{S_{\tau \rho,\nu}(x_0)} |D[G_{p,e}(Du_e)]|^2 \, dS \leq \frac{C}{\tau^n} \left[ \frac{\nu}{(1-\tau)^2 \rho^2} + \frac{F^2 \rho^{-\frac{2\mu}{\nu}}}{\nu} \right] \mu^{2p}.
\]  

(4.4)

Here the constants $C \in (0, \infty)$ given in (4.3)–(4.4) depend at most on $b, n, p, q, \gamma, \Gamma$, and $\delta$.

Following computations given in [35, Lemma 3.9] for the scalar problems, we provide the proof of Lemma 4.1.

**Proof.** We first compute

\[
|D[G_{p,e}(Du_e)]|^2 \leq |h_p'(V_e^2)D^2u_e + 2h_p''(V_e^2)Du_e \otimes D^2u_e Du_e|^2
\]

\[
\leq 2V_e^{2(p-1)}|D^2u_e|^2 + 2(p-1)^2V_e^{2(p-3)}|D^2u_e|^2|Du_e|^4
\]

\[
\leq c_p V_e^{2(p-1)}|D^2u_e|^2 \quad \text{a.e. in } \Omega,
\]

where $c_p := 2(p^2 - 2p + 2) > 0$. With this in mind, we apply Lemma 3.1 with $\psi(t) := t^p \hat{\psi}(t)$ for $t \in [0, \infty)$. Here the function $\hat{\psi}$ will later be chosen as either

\[
\hat{\psi}(t) \equiv 1,
\]  

(4.5)

or

\[
\hat{\psi}(t) := (t-\delta-k)^2,
\]  

(4.6)

for some constant $k > 0$. We choose a cutoff function $\eta \in C^1_c(B_{\rho}(x_0))$ such that

\[
\eta \equiv 1 \quad \text{on } B_{\tau \rho}(x_0) \quad \text{and} \quad |\nabla \eta| \leq \frac{2}{(1-\tau)\rho} \quad \text{in } B_{\rho}(x_0),
\]

and set $\zeta := \eta^2$. Then, the weak formulation (3.4) becomes

\[
\gamma L_1 + L_2
\]

\[
:= \gamma \int_B V_e^{2(p-1)}|D^2u_e|^2 \hat{\psi}(V_e)\eta^2 \, dx
\]

\[
+ \int_B \langle \mathcal{C}_e(D_e) \nabla V_e \mid \nabla \hat{\psi}(V_e)V_e \eta^2 \rangle \, dx
\]

\[
\leq 4 \int_B |\mathcal{C}_e(D_e) \nabla V_e | \nabla \eta| \psi(V_e)V_e \eta \, dx
\]

\[
+ \frac{1}{\gamma} \int_B |f_e|^2 V_e^{2-2p} (n\psi(V_e) + V_e \psi'(V_e)) \eta^2 \, dx
\]

\[
\leq \frac{C}{\tau^n} \left[ \frac{1}{(1-\tau)^2 \rho^2} + \frac{F^2 \rho^{-\frac{2\mu}{\nu}}}{\nu} \right] \mu^{2p}.
\]  

(4.3)

\[
\frac{1}{|B_{\tau \rho}(x_0)|} \int_{S_{\tau \rho,\nu}(x_0)} |D[G_{p,e}(Du_e)]|^2 \, dS \leq \frac{C}{\tau^n} \left[ \frac{\nu}{(1-\tau)^2 \rho^2} + \frac{F^2 \rho^{-\frac{2\mu}{\nu}}}{\nu} \right] \mu^{2p}.
\]  

(4.4)
For $R_1$, we use the Cauchy–Schwarz inequality

\[ |\langle c_e(D_e)x \nabla x | \nabla y \rangle| \leq \sqrt{\langle c_e(D_e)x \nabla x | \nabla y \rangle} \cdot \sqrt{\langle c_e(D_e)y \nabla y | \nabla y \rangle}, \]

which is possible by (3.5). Hence, by Young’s inequality, we have

\[ R_1 \leq L_2 + 4 \int_B \langle c_e(D_e)x \nabla x | \nabla x \rangle \frac{V_x \psi'(V_x)}{\psi'(V_x)} dx. \]

For $R_3$, we apply Young’s inequality to get

\[ R_3 \leq 2 \int_B |f_x|^2 \psi'(V_x) V_x^{3-p} dx + 2 \int_B |\nabla \psi'(V_x)| V_x^2 dx. \]

Therefore, by (3.5), we obtain

\[ L_1 \leq \frac{1}{\gamma} \int_B |\nabla \eta|^2 [2V_x^{p-2} + 4 \left( b V_x^{-1} + 3 V_x^{p-2} \right)] \frac{V_x \psi'(V_x)}{\psi'(V_x)} dx \]

\[ + \frac{1}{\gamma} \left( 2 + \frac{1}{\gamma} \right) \int_B |f_x|^2 \eta^2 \left( \frac{n \psi(V_x) + V_x \psi'(V_x)}{V_x^{p-2}} \right) dx. \]

Here we note that the assumptions (2.61)–(2.62) yield $V_x \leq \delta + \mu \leq 2\mu$ a.e. in $B$, and $\mu^l = \mu^{l-2p} \cdot \mu^{2p} \leq \delta^{l-2p} \mu^{2p}$ for every $l \in (0, 2p)$. Hence it follows that

\[ \left[ 2V_x^{p-2} + 4 \left( b V_x^{-1} + 3 V_x^{p-2} \right) \right] \frac{V_x \psi'(V_x)}{\psi'(V_x)} \leq C(b, \Gamma) \left[ V_x^{p-2} + V_x^{-1} \right] \frac{V_x^{p+2} \tilde{\psi}(V_x)}{p \tilde{\psi}(V_x) + V_x \tilde{\psi}'(V_x)} \]

a.e. in $B$, and

\[ \frac{n \psi(V_x) + V_x \psi'(V_x)}{V_x^{p-2}} = n(p + 1) V_x^2 \tilde{\psi}(V_x) + V_x^3 \tilde{\psi}'(V_x) \]

a.e. in $B$. As a result, we obtain

\[ \int_B \left| D\left[ G_{p, \epsilon}(\nabla \rho_e) \right] \right|^2 \eta^2 \tilde{\psi}(V_x) dx \leq c_p L_1 \]

\[ \leq C \mu^{2p} \left[ \int_B |\nabla \eta|^2 \frac{\tilde{\psi}(V_x)^2}{p \tilde{\psi}(V_x) + V_x \tilde{\psi}'(V_x)} dx + \int_B |f_x|^2 \eta^2 \left[ \tilde{\psi}(V_x) + V_x \tilde{\psi}'(V_x) \right] dx \right] \]

(4.7)

with $C = C(b, n, \mu, \gamma, \Gamma, \delta) \in (0, \infty)$. We will deduce (4.3)–(4.4) from (4.7) by choosing $\tilde{\psi}$ as $4.5$ or $4.6$.

Let $\tilde{\psi}$ satisfy (4.8). Then by (4.7) and Hölder’s inequality, we have

\[ \int_{B_{\rho_\epsilon}(x_0)} \left| D\left[ G_{p, \epsilon}(\nabla \rho_e) \right] \right|^2 dx \]
Then, from (2.68), it follows that
\[ \text{and Lemma 4.2. Let} \]
\[ \square \text{for some constant} \]
\[ \text{Next, we let} \]
\[ \text{Proof.} \]
\[ \text{For completeness, we provide the proof of Lemma 4.2, which can be accomplished similarly to [35, II.10]} \]
\[ \text{To prove (4.8), we consider integrals I, II given by} \]
\[ \Phi(x_0, \tau \rho) \leq C_\tau \mu^2 \frac{\nu \sqrt{n}}{\tau^n} \left[ \frac{\nu^{2/n}}{(1-\tau)^2} + \frac{F^2 \rho^{2\beta}}{\nu} \right], \]
\[ \text{where the constant} \]
\[ \text{For completeness, we provide the proof of Lemma 4.2 which can be accomplished similarly to [35 Lemma 3.10] for the scalar case.} \]
\[ \frac{1}{|B_{\tau \rho}(x_0)|} \int_{B_{\tau \rho}(x_0) \cap S_{\rho, \mu, v}(x_0)} |Du_E - \xi|^2 \, dx, \]
\[ \frac{1}{|B_{\tau \rho}(x_0)|} \int_{B_{\tau \rho}(x_0) \cap S_{\rho, \mu, v}(x_0)} |Du_E - \xi|^2 \, dx, \]
\[ \leq C(b, n, p, q, \gamma, \Gamma, \delta) \mu^{2p} \left[ \frac{1}{p} \int_B |\nabla \eta|^2 \, dx + \int_B |f_E|^2 \eta^2 \, dx \right] \]
\[ \leq C(b, n, p, q, \gamma, \Gamma, \delta) \mu^{2p} \left[ \frac{1}{(1-\tau)^2 \rho^2} + \frac{F^2 \rho^{2\beta}}{\nu} \right] |B_{\rho}(x_0)|. \]
\[ \leq C(b, n, p, q, \gamma, \Gamma, \delta) \mu^{2p} \left[ \frac{1}{(1-\tau)^2 \rho^2} + \frac{F^2 \rho^{2\beta}}{\nu} \right] |B_{\rho}(x_0)|. \]
Then, recalling (2.29), we obtain
\[ I \leq \frac{|B_p(x_0) \setminus S_{\rho,\mu,v}(x_0)|}{\tau^n |B_p(x_0)|} \text{ess sup}_{B_p(x_0)} (V_\epsilon + |\xi|)^2 \leq \frac{C(p)v\mu^2}{\tau^n}. \]

The last inequality is easy to check by (2.61)-(2.62). In fact, we have
\[ |G_{p,\epsilon}(Du_\epsilon)| = V_{\epsilon}^{p-1}|Du_\epsilon| \leq V_\epsilon^p \leq (\delta + \mu)^p \leq (2\mu)^p \quad \text{a.e. in } B_p(x_0). \]

Then, recalling (2.29), we obtain
\[ |\xi| \leq C(p)|G_{p,\epsilon}(\xi)|^{1/p} = C(p)|(G_{p,\epsilon}(Du_\epsilon))_{\infty,\tau\rho}|^{1/p} \leq C(p) \text{ess sup}_{B_p(x_0)}|G_{p,\epsilon}(Du_\epsilon)|^{1/p} \leq C(p)\mu, \]
and hence it is possible to estimate I as above. Before computing II, we recall (2.11) and the definition of \( S_{\rho,\mu,v}(x_0) \). Then, we are able to compute
\[ \frac{\delta}{4} + |Du_\epsilon| \geq \epsilon + |Du_\epsilon| \geq V_\epsilon \geq \delta + (1-\nu)\mu \quad \text{a.e. in } S_{\rho,\mu,v}(x_0). \] (4.9)

In particular, there holds
\[ |Du_\epsilon| \geq \frac{3}{4} \mu \quad \text{a.e. in } S_{\rho,\mu,v}(x_0) \]
by \( 0 < \nu < 1/4 \). With this in mind, we apply (2.30) to obtain
\[ II \leq \frac{C(p)}{\mu^{2(p-1)} |B_\tau(x_0)|} \int_{B_\tau(x_0) \cap S_{\rho,\mu,v}(x_0)} |G_{p,\epsilon}(Du_\epsilon) - G_{p,\epsilon}(\xi)|^2 \, dx \]
\[ \leq \frac{C(p)}{\mu^{2(p-1)} (\tau\rho)^2} \int_{B_\tau(x_0)} |D[G_{p,\epsilon}(Du_\epsilon)]|^2 \frac{1}{n^2} \, dx \]
\[ = : \frac{C(p)}{\mu^{2(p-1)}} (\tau\rho)^2 \, III_{1+2/n}. \]

Here we have applied (4.2) to the function \( G_{p,\epsilon}(Du_\epsilon) - G_{p,\epsilon}(\xi) \). The integral III can be decomposed by \( III = III_1 + III_2 \) with
\[ III_1 := \frac{1}{|B_\tau(x_0)|} \int_{B_\tau(x_0) \cap S_{\rho,\mu,v}(x_0)} |D[G_{p,\epsilon}(Du_\epsilon)]|^2 \frac{1}{n^2} \, dx, \]
and
\[ III_2 := \frac{1}{|B_\tau(x_0)|} \int_{S_{\rho,\mu,v}(x_0)} |D[G_{p,\epsilon}(Du_\epsilon)]|^2 \frac{1}{n^2} \, dx. \]

To control these integrals, we apply Lemma 4.4. For \( III_1 \), we use Hölder’s inequality and (4.3) to obtain
\[ III_1^{1+2/n} \leq \left[ \frac{|B_p(x_0) \setminus S_{\rho,\mu,v}(x_0)|}{|B_p(x_0)|} \right]^{2/n} \int_{B_\tau(x_0)} |D[G_{p,\epsilon}(Du_\epsilon)]|^2 \, dx \]
\[ \leq C(b, n, p, q, \gamma, \Gamma, \delta) \frac{\nu^{2/n} \mu^{2p}}{\tau^n} \frac{1}{(1-\tau)^2 \rho^2 + F^2 \rho^{-\frac{2n}{\gamma}}} \]
where we should note \( |B_p(x_0) \setminus S_{\rho,\mu,v}(x_0)| \leq v |B_p(x_0)| \) by (2.63). Similarly for \( III_2 \), we can compute
\[ III_2^{1+2/n} \leq \left[ \frac{|S_{\rho,\mu,v}(x_0)|}{|B_\tau(x_0)|} \right]^{2/n} \frac{1}{|B_\tau(x_0)|} \int_{S_{\rho,\mu,v}(x_0)} |D[G_{p,\epsilon}(Du_\epsilon)]|^2 \, dx \]

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\[ \leq C(b, n, p, q, \gamma, \Gamma, \delta) \frac{\mu^{2p}}{\tau^n} \left[ \frac{\nu}{(1-\tau)^2 \rho^2} + \frac{F^2 \rho^{-2\eta}}{\nu} \right]. \]

by Hölder’s inequality and \((4.4)\). Finally, we use \((4.1)\) to obtain

\[ \Phi(x_0, \tau \rho) \leq \int_{B_{\tau \rho}(x_0)} |Du_x - \xi|^2 \, dx = I + II \]

\[ \leq \frac{C(p) \nu \mu^2}{\tau^n} + \frac{C(n, p) \mu^2 \rho^{-2} (\tau \rho)^2 (III_1^{1+2/n} + III_2^{1+2/n})}{\nu} \]

\[ \leq \frac{C \mu^2}{\tau^n} \left[ \nu + \frac{\nu \mu^2}{\tau^2} + \frac{1+\nu \mu^2}{\nu} + \frac{1+\nu \mu^2}{\nu} \right] \]

with \( C \in (0, \infty) \) depending at most on \( b, n, p, q, \gamma, \Gamma, \) and \( \delta \). By \( 0 < \tau < 1, 0 < \nu < 1/4 \) and \( n \geq 2 \), we are able to find a constant \( C_1 = C_1(b, n, p, q, \gamma, \Gamma, \delta) \in (0, \infty) \) such that \((4.8)\) holds.

\( \square \)

### 4.2 Higher integrability estimates and freezing coefficient arguments

In Section \(4.2\), we appeal to freezing coefficient methods when an average \((Du_x)_{x_0, \rho}\) does not vanish. We introduce a harmonic mapping \( v_x \) near \( x_0 \), and obtain an error estimate for a comparison function \( u_x - v_x \).

When computing errors from a comparison function, we have to deduce higher integrability estimates on \(|Du_x - (Du_x)_{x_0, \rho}|\), which can be justified by applying so called Gehring’s lemma.

**Lemma 4.3** (Gehring’s Lemma). Let \( B = B_R(x_0) \subset \mathbb{R}^n \) be an open ball, and non-negative function \( g, h \) satisfy \( g \in L^s(B), h \in L^\tilde{s}(B) \) with \( 1 < s < \tilde{s} \leq \infty \). Suppose that there holds

\[ \int_{B_r(z_0)} g^\tau \, dx \leq \tilde{C} \left[ \left( \int_{B_{2r}(z_0)} g \, dx \right)^s + \int_{B_{2r}(z_0)} h^s \, dx \right] \]

for all \( B_r(z_0) \subset B \). Here \( \tilde{C} \in (0, \infty) \) is a constant independent of \( z_0 \in B \) and \( r > 0 \). Then there exists a sufficiently small positive number \( \zeta = \zeta(b, s, \tilde{s}, n) \) such that \( g \in L^\zeta_{\text{loc}}(B) \) with \( \sigma_0 := s(1+\zeta) \in (s, \tilde{s}) \). Moreover, for each \( \sigma \in (s, \sigma_0) \), we have

\[ \left( \int_{B_{2r}(z_0)} g^{\sigma} \, dx \right)^{1/\sigma} \leq \tilde{C} \left[ \left( \int_{B_{2r}(z_0)} g^s \, dx \right)^{1/\sigma} + \left( \int_{B_{2r}(z_0)} h^\sigma \, dx \right)^{1/\sigma} \right], \]

where the constant \( \tilde{C} \in (0, \infty) \) depends at most on \( \sigma, n, s, \tilde{s} \) and \( \tilde{C} \).

The proof of Gehring’s lemma is found in [\text{38} \text{ Theorem 3.3}], which is based on ball decompositions [\text{38} \text{ Lemma 3.1}] and generally works for a metric space with a doubling measure (see also [\text{23} \text{ §6.4}]).

By applying Lemma 4.3, we prove Lemma 4.4.

**Lemma 4.4** (Higher integrability lemma). Let \( u_x \) be a weak solution to (2.52). Assume that positive numbers \( \delta, \epsilon, \mu, M, F \) and an open ball \( B_{\rho}(x_0) \Subset \Omega \) satisfy (2.71), (2.54) and (2.61)–(2.62). Fix a matrix \( \xi \in \mathbb{R}^{Nn} \) with

\[ \delta + \frac{\mu}{4} \leq |\xi| \leq \delta + \mu. \]

Then, there exists a constant \( \vartheta = \vartheta(b, n, N, p, q, \gamma, \Gamma, M, \delta) \) such that

\[ 0 < \vartheta \leq \min \left\{ \frac{1}{2}, \beta, \beta_0 \right\} < 1, \]

\( \square \)

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and
\[
\int_{B_{r/2}(x_0)} |Du_x - \xi|^2(1+\theta) \, dx \leq C \left( \int_{B_{r}(x_0)} |Du_x - \xi|^2 \, dx \right)^{1+\theta} + F^{2(1+\theta)} \rho^{2\beta(1+\theta)}. \tag{4.12}
\]

Here the constant \( C \in (0, \infty) \) depends at most on \( b, n, N, \rho, q, \gamma, \Gamma, M, \delta. \)

**Proof.** It suffices to prove
\[
\int_{B_{r/2}(x_0)} |Du_x - \xi|^2 \, dx \leq \hat{C} \left( \int_{B_{r}(x_0)} |Du_x - \xi|^2 \, dx \right)^{\frac{n+2}{n}} + \int_{B_{r/2}(x_0)} |\rho f_x|^2 \, dx \tag{4.13}
\]
for any \( B_r(z_0) \subset B := B_\rho(x_0), \) where \( \hat{C} = \hat{C}(b, n, p, \gamma, \Gamma, M, \delta) \in (0, \infty) \) is a constant. In fact, this result enables us to apply Lemma \( \text{[4.3]} \) with \( (s, \tilde{s}) := (1 + 2/n, q(n+2)/2n), \) \( g := |Du_x - \xi|^\frac{2n}{n+2} \in L^s(B), \) \( h := |\rho f_x|^\frac{2n}{n+2} \in L^\tilde{s}(B), \) we are able to find a small exponent \( \theta = \theta(n, q, C_\ast) > 0 \) and a constant \( C = C(n, q, \vartheta, C_\ast) \in (0, \infty) \) such that there hold \( \text{(4.11)} \) and
\[
\int_{B_{r/2}(x_0)} |Du_x - \xi|^2(1+\theta) \, dx \leq C \left[ \left( \int_{B_{r}(x_0)} |Du_x - \xi|^2 \, dx \right)^{1+\theta} + \int_{B_{r/2}(x_0)} |\rho f_x|^{2(1+\theta)} \, dx \right].
\]

We note that \( \theta \leq \beta \) from \( \text{(4.11)} \) yields \( 2(1+\theta) \leq q, \) and therefore \( |\rho f_x|^{2(1+\theta)} \) is integrable in \( B_\rho(x_0). \) Moreover, by Hölder’s inequality, we have
\[
\int_{B_{r/2}(x_0)} |\rho f_x|^{2(1+\theta)} \, dx \leq C(n, \theta) F^{2(1+\theta)} \rho^{2\beta(1+\theta)},
\]
from which \( \text{(4.12)} \) easily follows.

To prove \( \text{(4.13)} \), for each fixed ball \( B_r(z_0) \subset B, \) we set a function \( w_x \in W^{1,\infty}(B_r(z_0); \mathbb{R}^N) \) by
\[
w_x(x) := u_x(x) - (u_x)_{z_0}, \quad \text{for } x \in B_r(z_0).
\]
Clearly \( Dw_x = Du_x - \xi \) holds. We choose a cutoff function \( \eta \in C_c(B_r(z_0)) \) satisfying
\[
\eta \equiv 1 \quad \text{on } B_{r/2}(z_0), \quad \text{and } 0 \leq \eta \leq 1, \quad |\nabla \eta| \leq \frac{4}{r} \quad \text{in } B_r(z_0),
\]
and test \( \phi := \eta^2 w_x \in W^{1,1}_0(B_\rho(x_0); \mathbb{R}^N) \) into \( \text{(3.2)} \). Then we have
\[
0 = \int_{B_r(z_0)} \langle A_x(Du_x) - A_x(\xi), |D\phi| \rangle \, dx - \int_{B_{r/2}(z_0)} \langle f_x, \phi \rangle \, dx
\]
\[
= \int_{B_r(z_0)} \eta^2 \langle A_x(Du_x) - A_x(\xi), Du_x - \xi \rangle \, dx
\]
\[
+ 2 \int_{B_r(z_0)} \eta \langle A_x(Du_x) - A_x(\xi), w_x \otimes \nabla \eta \rangle \, dx - \int_{B_{r/2}(z_0)} \eta^2 \langle f_x, w_x \rangle \, dx
\]
\[
= : J_1 + J_2 + J_3.
\]

Here it should be mentioned that since there clearly hold \( \delta \leq |\xi| \leq M, \) and \( |Du_x| \leq M \) a.e. in \( B, \) we are able to apply \( \text{(2.21)} \) and \( \text{(2.22)} \) in Lemma \( \text{[2.2]} \) to \( J_1 \) and \( J_2 \) respectively. Then, by applying Young’s
inequality to $J_2$ and $J_3$, and making a standard absorbing argument (see [35] Lemma 3.7) for detailed computations, we are able to obtain

$$
\int_{B_{r/2}(z_0)} |Dw_x|^2 \, dx \leq 2^n \int_{B_r(z_0)} |D(\eta w_x)|^2 \, dx
$$

$$
\leq C \int_{B_r(z_0)} \left( |\nabla \eta|^2 + \frac{\eta^2}{r^2} \right) |w_x|^2 \, dx + r^2 \int_{B_{r/2}(z_0)} \eta^2 |f_x|^2 \, dx
$$

$$
\leq C \left[ r^{-2} \int_{B_r(z_0)} |w_x|^2 \, dx + \int_{B_{r/2}(z_0)} |\rho f_x|^2 \, dx \right]
$$

$$
\leq C^2 \left( \left[ \int_{B_{r/2}(z_0)} |Dw_x| \right]^{2m/2m} \ dx + \left[ \int_{B_r(z_0)} |\rho f_x|^2 \, dx \right]^{1/2} \right)^2
$$

for some constant $\hat{C} \in (0, \infty)$ depending on $n$, $C_1$, and $C_2$. Here we have applied (4.2) to the function $w_x$ to obtain the last inequality. Recalling $Dw_x = Du_x - \zeta$, we finally conclude that (4.13) holds for any open ball $B_r(z_0) \subset B$.

From Lemma 4.4, we would like to deduce a comparison estimate in Lemma 4.5.

**Lemma 4.5.** Let $u_x$ be a weak solution to (2.52). Assume that positive numbers $\delta, \varepsilon, \mu, M, F$, and an open ball $B_\rho(x_0) \subset \Omega$ satisfy (2.77), (2.52), (2.61)–(2.62), and

$$
\delta + \frac{\mu}{4} \leq |(Du_x)_{x_0, \rho}| \leq \delta + \mu. \quad (4.14)
$$

Consider the Dirichlet boundary value problem

$$
\begin{cases}
-\text{div}(B_x((Du_x)_{x_0, \rho}) Dv_x) = 0 & \text{in} \ B_{\rho/2}(x_0), \\
v_x = u_x & \text{on} \ \partial B_{\rho/2}(x_0).
\end{cases} \quad (4.15)
$$

Then, there exists a unique function $v_x \in \text{W}^{1,2}_0(\Omega; \mathbb{R}^N)$ that solves (4.15). Moreover, we have

$$
\int_{B_{\rho/2}(x_0)} |Du_x - Dv_x|^2 \, dx \leq C \left\{ \frac{\Phi(\rho, \delta, \mu)}{\mu^2} \Phi(\rho, \delta, \mu) + (F^2 + F^2(1 + \theta)) \right\}^{1/2}, \quad (4.16)
$$

and

$$
\int_{B_{\rho/2}(x_0)} |Dv_x - (Du_x)_{x_0, \rho}|^2 \, dx \leq C \int_{B_{\rho/2}(x_0)} |Dv_x - (Du_x)_{x_0, \rho}|^2 \, dx \quad (4.17)
$$

for all $\tau \in (0, 1/2]$. Here the exponent $\theta$ is given by Lemma 4.4, and the constants $C \in (0, \infty)$ in (4.16)–(4.17) depends at most on $b$, $n$, $P$, $q_i$, $\gamma$, $\Gamma$, $M$, and $\delta$.

Before showing Lemma 4.5, we mention that our analysis on perturbation arguments is based on the assumption (4.14). It is easy to estimate $|((Du_x)_{x_0, \rho})$ by above. In fact, by (2.61) we have

$$
|((Du_x)_{x_0, \rho})| \leq \int_{B_{\rho}(x_0)} |Du_x| \, dx \leq \int_{B_{\rho}(x_0)} V_x \, dx \leq \delta + \mu.
$$

To estimate the value $|((Du_x)_{x_0, \rho})$ by below, however, we have to make careful computations, which are based on the measure assumption (2.63) and energy estimates as in Section 4.1. The condition $|((Du_x)_{x_0, \rho})| \geq \delta + \mu/4$ is to be justified later in Sections 4.3, 4.4.
Proof. We set \( \xi := (Du_\varepsilon)_{x_0, \rho} \in \mathbb{R}^N \). By (2.11), (2.62) and (4.14), it is easy to check \( \delta/4 \leq \mu/4 \leq \sqrt{\varepsilon^2 + |\xi|^2} \leq \delta + (\delta + \mu) \leq \delta + M \). Hence, the matrix \( \mathcal{B}_\varepsilon(\xi) \) admits a constant \( m \in (0, 1) \), depending at most on \( b, p, \gamma, \Gamma, M \), and \( \delta \), such that there holds \( m \delta N \leq \mathcal{B}_\varepsilon(Du_\varepsilon) \leq m^{-1} \delta N \). In particular, the matrix \( \mathcal{B}_\varepsilon(\xi) \) satisfies the Legendre condition, and hence unique existence of the Dirichlet problem (4.15) follows from \([21]\) Theorem 3.39. Moreover, since the coefficient matrix \( \mathcal{B}_\varepsilon(\xi) \) is constant and satisfies the Legendre–Hadamard condition, it is easy to find a constant \( C = C(n, N, m) \in (0, \infty) \) such that (4.17) holds (see e.g., \([2]\) Lemma 2.17, \([21]\) Proposition 5.8).

To prove (4.16), we first check \( l_0 \mu^{p-2} \text{id}_{Nn} \leq \mathcal{B}_\varepsilon(\xi) \) for some constant \( l_0 = l_0(\rho) \in (0, 1) \). This can be easily deduced by \( \mu/4 \leq \sqrt{\varepsilon^2 + |\xi|^2} \leq \delta + (\delta + \mu) \leq 5\mu \). Since \( v_\varepsilon \) satisfies a weak formulation

\[
\int_B \langle \mathcal{B}_\varepsilon(\xi) Du_\varepsilon - Dv_\varepsilon \rangle \cdot \, D\phi \, dx = 0 \quad \text{for all } \phi \in W^{1,2}_0(B; \mathbb{R}^N),
\]

where we write \( B := B_\rho/2(x_0) \) for notational simplicity, combining with (3.2), we have

\[
\int_B \langle \mathcal{B}_\varepsilon(\xi) (Du_\varepsilon - Dv_\varepsilon) \rangle \cdot \, D\phi \, dx

= \int_B \langle \mathcal{B}_\varepsilon(\xi) (Du_\varepsilon - \xi) - (A_\varepsilon(Du_\varepsilon) - A_\varepsilon(\xi)) \rangle \cdot \, D\phi \rangle \, dx + \int_B \langle f_\varepsilon \mid \phi \rangle \, dx
\]

for all \( \phi \in W^{1,2}_0(B; \mathbb{R}^N) \). The assumptions (2.61)–(2.62) and (4.14) enable us to use (2.24) in Lemma 2.2. As a result, we are able to find a constant \( C \in (0, \infty) \), depending at most on \( b, p, \beta_0, \gamma, \Gamma, M \), and \( \delta \), such that

\[
\int_B \langle \mathcal{B}_\varepsilon(\xi) (Du_\varepsilon - Dv_\varepsilon) \rangle \cdot \, D\phi \, dx \leq C \mu^{p-2-\beta_0} \int_B |Du_\varepsilon - \xi|^{1+\beta_0} |D\phi| \, dx + \int_B |f_\varepsilon||\phi| \, dx
\]

for all \( \phi \in W^{1,2}_0(B; \mathbb{R}^N) \). Testing \( \phi \equiv u_\varepsilon - v_\varepsilon \in W^{1,2}_0(B; \mathbb{R}^N) \) into this weak formulation and using the Cauchy–Schwarz inequality, we compute

\[
l_0 \gamma \mu^{p-2} \int_B |Du_\varepsilon - Dv_\varepsilon|^2 \, dx

\leq C \mu^{p-2-\beta_0} \left( \int_B |Du_\varepsilon - \xi|^{2(1+\beta_0)} \, dx \right)^{1/2} \left( \int_B |Du_\varepsilon - Dv_\varepsilon|^2 \, dx \right)^{1/2}

+ C(n, q) F_\rho^{\beta+2} \left( \int_B |Du_\varepsilon - Dv_\varepsilon|^2 \, dx \right)^{1/2}.
\]

Here we have also applied the Poincaré inequality to the function \( u_\varepsilon - v_\varepsilon \in W^{1,2}_0(B; \mathbb{R}^N) \). Thus, we obtain

\[
\int_B |Du_\varepsilon - Dv_\varepsilon|^2 \, dx \leq C \left[ \frac{1}{\mu^{2\beta_0}} \int_B |Du_\varepsilon - \xi|^{2(1+\beta_0)} \, dx + \mu^{2(2-p)} F^2 \rho^{2\beta} \right]
\]

for some constant \( C \in (0, \infty) \) depending at most on \( b, n, p, q, \beta_0, \gamma, \Gamma, M \), and \( \delta \). Since \( \xi \) clearly satisfies (4.10), we are able to apply Lemma 4.3. Also, by (2.61)–(2.62) and (4.11), it is easy to check that \( 2(1+\theta) \leq 2(1+\beta_0) \) and

\[
|Du_\varepsilon - \xi| \leq 2(\delta + \mu) \leq 4\mu \quad \text{a.e. in } B_\rho(x_0).
\]

With these results in mind, we use Hölder’s inequality and (4.12) to obtain

\[
\int_B |Du_\varepsilon - Dv_\varepsilon|^2 \, dx \leq C \left[ \frac{c(n, \theta)}{\mu^{2\theta}} \int_B |Du_\varepsilon - \xi|^{2(1+\theta)} \, dx + \mu^{2(2-p)} F^2 \rho^{2\theta} \right]
\]

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for some $C \in (0, \infty)$. Here it is mentioned that $0 < \rho \leq 1$ and $\delta < \mu < M$ hold by (2.61)–(2.62), and therefore (4.16) is verified.

### 4.3 Key lemmata in shrinking methods

In Section 4.3, we provide two lemmata on shrinking methods. To prove these, we use results from Sections 4.1, 4.2.

The first lemma (Lemma 4.6) states that an average $(Du_\varepsilon)_{x_0, \rho} \in \mathbb{R}^{Nn}$ does not vanish under suitable settings. This result makes sure that our freezing coefficient argument given in Section 4.2 will work.

**Lemma 4.6.** Let $u_\varepsilon$ be a weak solution to (2.52) in $\Omega$. Assume that positive numbers $\delta$, $\varepsilon$, $\mu$, $F$, $M$, and an open ball $B_\rho(x_0) \subset \Omega$ satisfy (2.71), (2.54), and (2.61)–(2.62). Then, for each fixed $\theta \in (0, 1/16)$, there exist numbers $\nu \in (0, 1/4)$, $\hat{\rho} \in (0, 1)$, depending at most on $b$, $n$, $N$, $p$, $q$, $\gamma$, $F$, $M$, $\delta$, and $\theta$, such that the following statement is valid. If both $0 < \rho < \hat{\rho}$ and (2.63) hold, then we have

$$|(Du_\varepsilon)_{x_0, \rho}| \geq \delta + \frac{\mu}{2},$$

and

$$\Phi(x_0, \rho) \leq \theta \mu^2.$$  

Although the proof of Lemma 4.6 is inspired by [4] Lemma 5.5, it should be emphasized that on our regularized problem (2.52), we have to deal with two different moduli $|Du_\varepsilon|$ and $V_\varepsilon = \sqrt{\varepsilon^2 + |Du_\varepsilon|^2}$, which is substantially different from [4]. Therefore, as mentioned in Section 1.2 in the proof of Lemma 4.6, we have to carefully utilize (2.11), so that $\varepsilon$ can be suitably dominated by $\delta$. Also, it should be mentioned that our proof is substantially the same with [35] Lemma 3.12, although there are some differences on ranges of $\varepsilon$ or $\theta$.

**Proof.** We will later choose constants $\tau \in (0, 1)$, $\nu \in (0, 1/4)$, $\hat{\rho} \in (0, 1)$. By (4.1), we have

$$\Phi(x_0, \rho) \leq \int_{B_\rho(x_0)} |Du_\varepsilon - (Du_\varepsilon)_{x_0, \tau \rho}|^2 \, dx = J_1 + J_2$$

with

$$\begin{align*}
J_1 & := \frac{1}{|B_\rho(x_0)|} \int_{B_{\tau \rho}(x_0)} |Du_\varepsilon - (Du_\varepsilon)_{x_0, \tau \rho}|^2 \, dx, \\
J_2 & := \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0) \setminus B_{\tau \rho}(x_0)} |Du_\varepsilon - (Du_\varepsilon)_{x_0, \tau \rho}|^2 \, dx.
\end{align*}$$

For $J_1$, we apply Lemma 4.2 to obtain

$$J_1 = \tau^n \Phi(x_0, \tau \rho) \leq C_\tau \mu^2 \left( \frac{\nu^{2/n}}{(1 - \tau)^2} + \frac{F^2}{\nu} \right)$$

with $C_\tau = C_\tau(b, n, N, p, \gamma, \Gamma, M, \delta) \in (0, \infty)$. For $J_2$, we use (2.61)–(2.62) to get $|Du_\varepsilon| \leq V_\varepsilon \leq \delta + \mu \leq 2\mu$ a.e. in $B_\rho(x_0)$, and hence $|(Du_\varepsilon)_{x_0, \tau \rho}| \leq 2\mu$. These inequalities yield

$$J_2 \leq 8 \mu^2 \cdot \frac{|B_\rho(x_0) \setminus B_{\tau \rho}(x_0)|}{|B_\rho(x_0)|} = 8 \mu^2 (1 - \tau^n) \leq 8n \mu^2 (1 - \tau),$$

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where we have used \(1 - \tau^n = (1 + \tau + \cdots + \tau^{n-1})(1 - \tau) \leq n(1 - \tau)\). Hence, we obtain
\[
\Phi(x_0, \rho) \leq C_1 \mu^2 \left[ \frac{\nu^{2/n}}{(1 - \tau)^2} + \frac{F^2}{\nu} \hat{\rho}^{2\beta} \right] + 8n(1 - \tau)\mu^2.
\]
We first fix
\[
\tau := 1 - \frac{\theta}{24n} \in (0, 1), \quad \text{so that there holds} \quad 8n(1 - \tau) = \frac{\theta}{3}.
\]
Next we choose \(\nu \in (0, 1/4)\) sufficiently small that it satisfies
\[
\nu \leq \min \left\{ \left( \frac{\theta(1 - \tau)^2}{3C_1} \right)^{n/2}, \frac{1 - 4\sqrt{\theta} + \nu}{11} \right\},
\]
so that we have
\[
\frac{C_1 \nu^{2/n}}{(1 - \tau)^2} \leq \frac{\theta}{3}, \quad \text{and} \quad \sqrt{\theta} \leq \frac{1 - 11\nu}{4}.
\]
Corresponding to this \(\nu\), we choose and fix sufficiently small \(\hat{\rho} \in (0, 1)\) satisfying
\[
\hat{\rho}^{2\beta} \leq \frac{\nu \theta}{3C_1(1 + F^2)},
\]
which yields \(C_1 F^2 \hat{\rho}^{2\beta}/\nu \leq \theta/3\). Our settings of \(\tau, \nu, \hat{\rho}\) clearly yield (4.19).

To prove (4.18), we recall (4.9) and use (2.63). Then we obtain
\[
\int_{B_\nu(x_0)} |Du_x| \, dx \geq \frac{|S_{\rho, \nu, \nu(x_0)}|}{|B_\rho(x_0)|} \cdot \text{ess inf} \ |Du_x| \
\geq (1 - \nu) \cdot \left( (1 - \nu) \mu + \frac{3}{4} \delta \right) > 0.
\]

On the other hand, by the triangle inequality, the Cauchy–Schwarz inequality and (4.19), it is easy to get
\[
\left| \int_{B_\rho(x_0)} |Du_x| \, dx - |(Du_x)_{x_0, \rho}| \right| \leq \int_{B_\rho(x_0)} |Du_x| \, dx - |(Du_x)_{x_0, \rho}| \leq \sqrt{\Phi(x_0, \rho)} \leq \sqrt{\theta} \mu.
\]

Again by the triangle inequality, we obtain
\[
|(Du_x)_{x_0, \rho}| \geq \int_{B_\rho(x_0)} |Du_x| \, dx - \int_{B_\rho(x_0)} |Du_x| \, dx - |(Du_x)_{x_0, \rho}| \geq \left( (1 - \nu)^2 - \sqrt{\theta} \right) \mu + \frac{3}{4}(1 - \nu)\delta.
\]

By (2.02) and our choice of \(\nu\), we can check that
\[
\left( (1 - \nu)^2 - \sqrt{\theta} \right) \mu + \frac{3}{4}(1 - \nu)\delta - \left( \delta + \frac{\mu}{2} \right) = \left( \frac{1}{2} - 2\nu + \nu^2 - \sqrt{\theta} \right) \mu - \left( \frac{1}{4} + \frac{3}{4} \right) \delta \geq \left( \frac{1 - 11\nu}{4} - \sqrt{\theta} \right) \mu \geq 0,
\]
which completes the proof of (4.18). \(\Box\)
The second lemma (Lemma 4.7) is a result from perturbation arguments given in Section 4.2.

**Lemma 4.7.** Let \( u_\varepsilon \) be a weak solution to (2.52) in \( \Omega \). Assume that positive numbers \( \delta, \varepsilon, \mu, F, M, \) and an open ball \( B_\rho(x_0) \) satisfy (2.11), (2.53), (2.61), (2.62), and \( 0 < \rho < 1 \). Let \( v \) be the constant in Lemma 4.4 and

\[
\Phi(x_0, \rho) \leq \frac{\sigma_\rho^2}{\tau^2} \mu^2 \quad (4.20)
\]

hold for some \( \tau \in (0, 1/2) \), then we have

\[
\Phi(x_0, \tau \rho) \leq C \left[ \tau^2 \Phi(x_0, \rho) + \frac{\rho^{2\beta}}{\tau^n} \mu^2 \right] \quad (4.21)
\]

Here the constant \( C \), depends at most on \( b, n, N, p, q, \beta_0, \gamma, \Gamma, F, M, \) and \( \delta \).

**Proof.** Let \( v \in u_\varepsilon + W^{1,2}_0(B_{\rho/2}(x_0) \cap \Omega) \) be the unique solution of (4.15). We use (4.11) to get

\[
\Phi(x_0, \tau \rho) \leq \int_{B_{\tau \rho}(x_0)} |Du_\varepsilon - (Dv_\varepsilon)_{x_0, \tau \rho}|^2 \, dx
\]

\[
\leq \frac{2}{(2\tau)^n} \int_{B_{\rho/2}(x_0)} |Du_\varepsilon - Dv_\varepsilon|^2 \, dx + 2 \int_{B_{\tau \rho}(x_0)} |Dv_\varepsilon - (Dv_\varepsilon)_{x_0, \tau \rho}|^2 \, dx,
\]

where we have used \( |B_{\rho/2}(x_0)| = (2\tau)^{-n} |B_{\tau \rho}(x_0)| \). For the second average integral, (4.1) and (4.17) yield

\[
\int_{B_{\tau \rho}(x_0)} |Dv_\varepsilon - (Dv_\varepsilon)_{x_0, \tau \rho}|^2 \, dx \leq C \left[ \tau^2 \int_{B_{\rho/2}(x_0)} |Du_\varepsilon - Dv_\varepsilon|^2 \, dx + \tau^2 \Phi(x_0, \rho) \right]
\]

with \( C = C(n, N, p, q, \gamma, \Gamma, M, \delta) \in (0, \infty) \). By (2.62), (4.16) and (4.20), we are able to compute

\[
\Phi(x_0, \tau \rho) \leq C \left[ \int_{B_{\rho/2}(x_0)} |Du_\varepsilon - Dv_\varepsilon|^2 \, dx + \tau^2 \Phi(x_0, \rho) \right]
\]

\[
\leq C \left[ \left( \frac{\Phi(x_0, \rho)}{\mu^2} \right)^\theta \cdot \frac{\Phi(x_0, \rho)}{\tau^n} + \frac{F^2 + F^{2(1+\theta)}}{\tau^n} \rho^{2\beta} + \tau^2 \Phi(x_0, \rho) \right]
\]

\[
\leq C \left[ \tau^2 \Phi(x_0, \rho) + \left( F^2 + F^{2(1+\theta)} \right) \frac{\rho^{2\beta}}{\tau^n} \left( \frac{\mu}{\delta} \right)^2 \right]
\]

\[
\leq C \left( b, n, N, p, q, \beta_0, \gamma, \Gamma, F, M, \delta \right) \left[ \tau^2 \Phi(x_0, \rho) + \frac{\rho^{2\beta}}{\tau^n} \mu^2 \right],
\]

which completes the proof. \( \square \)

### 4.4 Proof of Proposition 2.8

We would like to prove Proposition 2.8 by shrinking arguments. A key point of the proof, which is inspired by [4] Proposition 3.4, is to justify that an average \( (Du_\varepsilon)_{x_0, r} \in \mathbb{R}^n \) never vanishes even when \( r \) tends to 0, so that Lemma 4.7 can be applied. To verify this, we make careful computations, found in the proof of Lemma 4.6.

**Proof.** We set a constant \( \theta \) as in Lemma 4.4. We will determine a sufficiently small constant \( \tau \in (0, 1/2) \), and corresponding to this \( \tau \), we will put the desired constants \( \rho \in (0, 1) \) and \( \nu \in (0, 1/4) \).
We first assume that
\[
0 < \tau < \max\{ \tau^\beta, \tau^{1-\beta}\} < \frac{1}{16}, \quad \text{and therefore} \quad \theta := \frac{n\beta}{n^2} \in \left(0, \frac{1}{16}\right) \quad (4.22)
\]
Throughout the proof, we let \( \nu \in (0, 1/6) \) and \( \beta \in (0, 1) \) be sufficiently small constants satisfying Lemma 4.6 with \( \theta \) defined by (4.22). We also assume that \( \rho_* \) is so small that there holds

\[
0 < \rho_* \leq \beta < 1. \quad (4.23)
\]
Assume that the open ball \( B_\rho(x_0) \) satisfies \( 0 < \rho < \rho_* \), and \( (4.23) \) holds for the constant \( \nu \in (0, 1/6) \). We set a non-negative decreasing sequence \( \{\rho_k\}_{k=0}^\infty \) by \( \rho_k := \tau^k \rho \). We will choose suitable \( \tau \) and \( \rho_* \) such that there hold
\[
\left|(Du_\delta)_{x_0, \rho_k}\right| \geq \delta + \left[\frac{1}{2} - \frac{1}{8} \sum_{j=0}^{k-1} 2^{-j}\right] \mu \geq \delta + \frac{\mu}{4} \quad (4.24)
\]
and
\[
\Phi(x_0, \rho_k) \leq \tau^{2\beta k} \tau^{-\frac{n\beta}{n^2}} \mu^2, \quad (4.25)
\]
for all \( k \in \mathbb{Z}_{\geq 0} \), which will be proved by mathematical induction. For \( k = 0, 1 \), we apply Lemma 4.6 to deduce (4.18)–(4.19) with \( \theta = \tau^{\frac{n\beta}{n^2}} \). In particular, we have
\[
\Phi(x_0, \rho) \leq \tau^{\frac{n\beta}{n^2}} \mu^2, \quad (4.26)
\]
and hence (4.25) is obvious when \( k = 0 \). From (4.18), we have already known that (4.24) holds for \( k = 0 \). Also, (4.18) and (4.26) enable us to apply Lemma 4.7 to obtain
\[
\Phi(x_0, \rho_1) \leq C_\ast \left[ 2^2 \Phi(x_0, \rho) + \frac{\rho_1^{2\beta}}{\tau^n} \mu^2 \right] \leq C_\ast \tau^{2(1-\beta)} \cdot 2^{2\beta} \tau^{-\frac{\rho_1 \beta}{n^2}} \mu^2 + \frac{C_\ast \rho_1^{2\beta}}{\tau^n} \mu^2,
\]
where \( C_\ast \in (0, \infty) \) is a constant as in Lemma 4.7 depending at most on \( b, n, N, p, q, \beta_0, \gamma, \Gamma, F, M, \) and \( \delta \). Now we assume that \( \tau \) and \( \rho_* \) satisfy
\[
C_\ast \tau^{2(1-\beta)} \leq \frac{1}{2}, \quad (4.27)
\]
and
\[
C_\ast \mu^2 \leq \frac{1}{2} \tau^{n+2\beta+\frac{n\beta}{n^2}}, \quad (4.28)
\]
so that (4.25) holds for \( k = 1 \). In particular, by (4.11), (4.22), (4.26) and the Cauchy–Schwarz inequality, we obtain
\[
\left|(Du_\delta)_{x_0, \rho_1} - (Du_\delta)_{x_0, \rho_0}\right| \leq \int_{B_{\rho_1}(x_0)} |Du_\delta - (Du_\delta)_{x_0, \rho_0}| \, dx \leq \left( \int_{B_{\rho_1}(x_0)} |Du_\delta - (Du_\delta)_{x_0, \rho_0}|^2 \, dx \right)^{1/2} = \tau^{-\nu} \Phi(x_0, \rho)^{1/2} \leq \tau^{\frac{n\beta}{n^2}} \mu \leq \tau \mu \leq \frac{1}{8} \mu.
\]

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Combining this result with (4.18), we use the triangle inequality to get
\[ |(Du_x)_{x_0, \rho_k}| \geq |(Du_x)_{x_0, \rho_0}| - |(Du_x)_{x_0, \rho_{k+1}}| \geq \left( \delta + \frac{\mu}{2} \right) - \frac{\mu}{8}, \]
which means that (4.24) holds true for \( k = 1 \). Next, we assume that the claims (4.24)–(4.25) are valid for an integer \( k \geq 1 \). Then \( \Phi(x_0, \rho_k) \leq \tau^\beta \mu^2 \) clearly holds. Combining this result with the induction hypothesis (4.24), we have clarified that the solution \( u_x \) satisfies assumptions of Lemma 4.7 in a smaller ball \( B_{\rho_k}(x_0) \subset B_{\rho}(x_0) \). By Lemma 4.7 (4.27), and the induction hypothesis (4.25), we compute
\[
\Phi(x_0, \rho_{k+1}) \leq C \left[ \tau^2 \Phi(x_0, \rho_k) + \frac{\rho_k^{2\beta}}{\tau^n} \mu^2 \right]
\leq C \tau^{2(1-\beta)} \cdot \tau^{2\beta(k+1)} \cdot \frac{\rho_k^{2\beta}}{\tau^n} \mu^2 + \frac{C_\epsilon \rho_k^{2\beta}}{\tau^n} \cdot \tau^{2\beta k} \mu^2
\leq \tau^{2\beta(k+1)} \cdot \frac{\rho_k^{2\beta}}{\tau^n} \mu^2,
\]
which means that (4.25) holds true for \( k + 1 \). Also, by the Cauchy–Schwarz inequality and the induction hypothesis (4.25), we have
\[
|(Du_x)_{x_0, \rho_{k+1}} - (Du_x)_{x_0, \rho_k}| \leq \int_{B_{\rho_{k+1}}(x_0)} |Du_x - (Du_x)_{x_0, \rho_k}| \, dx
\leq \left( \int_{B_{\rho_{k+1}}(x_0)} |Du_x - (Du_x)_{x_0, \rho_k}| \, dx \right)^{1/2}
\leq \tau^{\beta k} \cdot \frac{n \rho_k^{2\beta}}{\tau^n} \mu \leq 2^{-k} \cdot \frac{1}{8} \mu.
\]
Here we have also used (2.22). Therefore, by the induction hypothesis (4.24) and the triangle inequality, we get
\[
|(Du_x)_{x_0, \rho_{k+1}}| \geq |(Du_x)_{x_0, \rho_k}| - |(Du_x)_{x_0, \rho_{k+1}} - (Du_x)_{x_0, \rho_k}|
\geq \delta + \frac{1}{2} - \frac{1}{8} \sum_{j=0}^{k-1} 2^{-j} \mu - \frac{1}{8} \cdot 2^{-k} \mu,
\]
which implies that (4.24) is valid for \( k + 1 \). This completes the proof of (4.24)–(4.25).

We define \( \Psi_{2\delta, \epsilon}(x_0, r) := \int_{B_r(x_0)} |f_{2\delta, \epsilon}(Du_x) - f_{2\delta, \epsilon}(Du_x)|_{x_0, r} \, dx \) for \( r \in (0, \rho] \), and we set a sequence of vectors \( \{\Gamma_k\}_{k=0}^\infty \subset \mathbb{R}^{N_n} \) by
\[
\Gamma_k := f_{2\delta, \epsilon}(Du_x)_{x_0, \rho_k} \text{ for } k \in \mathbb{Z}_{\geq 0}.
\]
Let \( c_\star > 0 \) be the constant satisfying (2.12). For each \( k \in \mathbb{Z}_{\geq 0} \), we apply (4.1), (4.11) and (4.25)–(4.26) to get
\[
\Psi_{2\delta, \epsilon}(x_0, \rho_k) \leq \int_{B_{\rho_k}(x_0)} |f_{2\delta, \epsilon}(Du_x) - f_{2\delta, \epsilon}(Du_x)|_{x_0, \rho_k} \, dx
\leq c_\star^2 \cdot \Phi(x_0, \rho_k).
\]
\[ \leq c_1^2 \tau^{2\beta k} \frac{2^n}{\tau^{k+1}} \mu^2 \leq c_1^2 \tau^{2\beta k+2(n+2)} \mu^2. \]

Here we let \( \tau \) satisfy
\[ \tau \leq \frac{1}{\sqrt{2c_1}} \] (4.29)

to get
\[ \Psi_{2\delta,\varepsilon}(x_0, \rho_k) \leq \tau^{2n+2} \tau^{\beta k} \mu^2 \] for all \( k \in \mathbb{Z}_{\geq 0}. \) (4.30)

By (4.30) and the Cauchy–Schwarz inequality, we have
\[
|\Gamma_{k+1} - \Gamma_k| \leq \int_{B_k} |G_{2\delta,\varepsilon}(Du_x) - \Gamma_k| \, dx \\
\leq \left( \int_{B_k} |G_{2\delta,\varepsilon}(Du_x) - \Gamma_k|^2 \, dx \right)^{1/2} \\
\leq \tau^{n/2+1} \tau^{\beta k} \mu
\]

for all \( k \in \mathbb{Z}_{\geq 0}. \) In particular, for all \( k, l \in \mathbb{Z}_{\geq 0} \) with \( k < l, \) we have
\[
|\Gamma_l - \Gamma_k| \leq \sum_{i=k}^{l-1} |\Gamma_{i+1} - \Gamma_i| \leq \sum_{i=k}^{l-1} \tau^{n/2+1} \tau^{\beta i} \mu \\
\leq \tau^{n/2+1} \mu \sum_{i=k}^{\infty} \tau^{\beta i} = \tau^{n/2+1} \frac{\tau^{\beta k}}{1 - \tau^{\beta}} \mu.
\]

The setting (4.22) clearly yields \( \tau^\beta \leq 1/2. \) Hence, we have
\[ |\Gamma_k - \Gamma_l| \leq 2\tau^{n/2+1} \tau^{\beta k} \mu \] for all \( k, l \in \mathbb{Z}_{\geq 0} \) with \( k < l, \) (4.31)

which implies that \( \{\Gamma_k\}_{k=0}^\infty \) is a Cauchy sequence in \( \mathbb{R}^{N\mu}. \) Therefore the limit
\[ \Gamma_\infty := \lim_{k \to \infty} \Gamma_k \in \mathbb{R}^{N\mu} \]

exists. Moreover, by letting \( l \to \infty \) in (4.31), we have
\[ |\Gamma_k - \Gamma_\infty| \leq 2\tau^{n/2+1} \tau^{\beta k} \mu \] for every \( k \in \mathbb{Z}_{\geq 0}. \)

Combining this result with (4.30), we obtain
\[
\int_{B_{\rho_k}(x_0)} |G_{2\delta,\varepsilon}(Du_x) - \Gamma_\infty|^2 \, dx \leq 2 \int_{B_{\rho_k}(x_0)} \left[ |G_{2\delta,\varepsilon}(Du_x) - \Gamma_k|^2 + |\Gamma_k - \Gamma_\infty|^2 \right] \, dx \\
\leq 2 \left( \tau^{2n+2} \tau^{2\beta k} \mu^2 + 4\tau^{n+2} \tau^{2\beta k} \mu^2 \right) \\
\leq 10\tau^{2(1-\beta)} \cdot \tau^{n+2\beta(k+1)} \mu^2 \\
\leq \tau^{n+2\beta(k+1)} \mu^2.
\] (4.32)

Here we have used \( \tau^{2(1-\beta)} \leq 1/10, \) which immediately follows from (4.22). For each \( r \in (0, \rho), \) there corresponds a unique \( k \in \mathbb{Z}_{\geq 0} \) such that \( \rho_{k+1} < r \leq \rho_k. \) Then by (4.32), we have
\[
\int_{B_r(x_0)} |G_{2\delta,\varepsilon}(Du_x) - \Gamma_\infty|^2 \, dx \leq \tau^{-n} \int_{B_{\rho_k}(x_0)} |G_{2\delta,\varepsilon}(Du_x) - \Gamma_\infty|^2 \, dx
\]
\[ \leq \tau^{2\beta(k+1)} \mu^2 \leq \left( \frac{r}{\rho} \right)^{2\beta} \mu^2 \]  

(4.33)

for all \( r \in (0, \rho] \). By the Cauchy–Schwarz inequality, we also have

\[
\left| (\mathcal{H}_{2,\delta, e}(Du))_{x_0,r} - \Gamma_{\infty} \right| \leq \iint_{B_r(x_0)} \left| (\mathcal{H}_{2,\delta, e}(Du)) - \Gamma_{\infty} \right|^2 \, dx \\
\leq \left( \iint_{B_r(x_0)} \left| (\mathcal{H}_{2,\delta, e}(Du)) - \Gamma_{\infty} \right|^2 \, dx \right)^{1/2} \leq \left( \frac{r}{\rho} \right)^{\beta} \mu.
\]

for all \( r \in (0, \rho] \). This result yields

\[
\Gamma_{2,\delta, e}(x_0) := \lim_{r \to 0} \left( \mathcal{H}_{2,\delta, e}(Du) \right)_{x_0,r} = \Gamma_{\infty},
\]

and hence the desired estimate (2.66) clearly follows from (4.33). It is noted that (2.61) and (2.67) imply \( \mathcal{H}_{2,\delta, e}(Du) \leq \mu \) a.e. in \( B_{\rho}(x_0) \), and therefore (2.65) is obvious.

Finally, we mention that we may choose a sufficiently small constant \( \tau = \tau(C, \beta) \in (0, 1/2) \) verifying (4.22), (4.27), and (4.29). Corresponding to this \( \tau \), we take sufficient small numbers \( \nu \in (0, 1/6) \), \( \rho \in (0, 1) \) as in Lemma 4.7 depending at most on \( b, n, N, p, q, \gamma, \Gamma, F, M, \delta, \) and \( \theta = \tau \frac{\mu}{2} \). Then, we are able to determine a sufficiently small radius \( \rho_\ast = \rho_\ast(C, \beta, \rho) \in (0, 1) \) verifying (4.23) and (4.28), and this completes the proof. \( \Box \)

5 Appendix: Local Lipschitz bounds

In Section 5 we would like to provide the proof of Proposition 2.6 for the reader’s convenience.

Before showing Proposition 2.6 we recall two basic lemmata (see [24, Lemma 4.3] and [26, Chapter 2, Lemma 4.7] for the proofs).

Lemma 5.1. Assume that a non-negative bounded function \( H: [0, 1] \to (0, \infty) \) admits constants \( \theta \in [0, 1) \) and \( \alpha, A, B \in (0, \infty) \) such that

\[
H(t) \leq \theta H(s) + \frac{A}{(s-t)^\alpha} + B
\]

holds whenever \( 0 \leq t < s \leq 1 \). Then we have

\[
H(t) \leq C(\alpha, \theta) \left[ \frac{A}{(s-t)^\alpha} + B \right]
\]

for any \( 0 \leq t < s \leq 1 \).

Lemma 5.2. Assume that a sequence \( \{a_m\}_{m=0}^\infty \subset (0, \infty) \) satisfies

\[
a_{m+1} \leq CB^ma_m^{1+\varsigma}
\]

for all \( m \in \mathbb{Z}_{\geq 0} \). Here \( C, \varsigma \in (0, \infty) \) and \( B \in (1, \infty) \) are constants. If \( a_0 \) satisfies

\[
a_0 \leq C^{-1/\varsigma}B^{-1/\varsigma^2},
\]

then \( a_m \to 0 \) as \( m \to \infty \).
The proof of Proposition 2.6 is based on De Giorgi’s truncation. More sophisticated computations than ours concerning local Lipschitz bounds by De Giorgi’s truncation are given in [3, Theorem 1.13], where external force terms are assumed to be less regular than $L^q (n < q \leq \infty)$. Compared with [3], our proof of Proposition 2.6 is rather elementary, since we only deal with the case where the external force term $f_x$ is in a Lebesgue space $L^q$ with $q \in (n, \infty]$. To control $f_x$ by the $L^q$-norm, we appeal to standard absorbing arguments as in [24, Theorem 4.1, Method 1] (see also [24, §4.3] for scalar problems).

**Proof.** For notational simplicity, we write $B_r := B_r (x_0) \text{ for } r \in (0, \rho]$, and $B := B_{\rho} (x_0)$. We fix a constant $k := 1 + \| f_x \|^{1/(p-1)}_{L^p (B)} \geq 1$, and define a superlevel set

$$A(l, r) := \{ x \in B_r \mid (V_e (x) - k) \rho > l \}$$

for $l \in (0, \infty)$ and $r \in (0, \rho]$. Then, by (3.5) and $k \geq 1$, there holds

$$\gamma V_e^{p-2} \hat{id}_n \leq \mathcal{E}_e (D u_x) \leq \tilde{\Gamma} V_e^{p-2} \hat{id}_n \quad \text{a.e. in } A(0, \rho) \tag{5.1}$$

with $\tilde{\Gamma} := b + 3\Gamma$.

We first claim that there holds

$$\int_B |\nabla (\eta W_l) |^2 \, dx \leq C (b, n, p, \gamma, \Gamma) \left[ \int_B |\nabla \eta |^2 W_l^2 \, dx + \int_{A(l, \rho)} f_k \left( W_l^2 + l^p W_l + l^{2p} \right) \, dx \right] \tag{5.2}$$

for all $l \in (k^p, \infty)$ and for any non-negative function $\eta \in C_0^1 (B)$. Here the non-negative functions $W_l$ and $f_k$ are respectively given by $W_l := ( (V_e - k) \rho - l )_+$, and

$$f_k := \frac{| f_x |^2}{V_e^{2(p-1)} \chi_{A(0, \rho)} \in L^{q/2} (B)} \quad \text{so that} \quad f_k V_e^{2(p-1)} = | f_x |^2 \chi_{A(0, \rho)}$$

holds a.e. in $B$. To prove (5.2), we apply Lemma 3.1 with $\psi (\sigma) := ((\sigma - k) \rho - l)_+$ and $\zeta := \eta^2$, so that $W_l = \psi (V_e)$ holds. Under this setting, we discard a non-negative term $J_3$, and carefully compute the other integrals. To compute $J_1$, we may use (5.1), since $W_l$ vanishes outside the set $A(l, \rho)$. When estimating $J_4, J_5, J_6$, we use

$$\frac{V_e}{V_e - k} = 1 + \frac{k}{V_e - k} \leq 1 + \frac{k}{l^{1/p}} \leq 2 \quad \text{a.e. in } A(l, \rho),$$

and

$$V_e^p \leq 2^{p-1} ((V_e - k)^p + k^p) \leq 2^p (W_l + l) \quad \text{a.e. in } A(l, \rho),$$

which are easy to deduce by $l > k^p$. Here we also note that the identity $W_l + l = (V_e - k)^p$ holds a.e. in $A(l, \rho)$. Combining these, we can compute

$$p \gamma \int_B \eta^2 | \nabla V_e |^2 V_e^{p-1} (V_e - k)^{p-1} \chi_{A(l, \rho)} \, dx$$

$$\leq J_2 \leq 2 J_1 + \frac{1}{\gamma} (n |J_4| + |J_5|) + 2 |J_6|$$

$$\leq 4 \tilde{\Gamma} \int_B \eta W_l V_e^{p-1} | \nabla u_x | \eta \, dx$$

$$+ \frac{1}{\gamma} \left[ n \int_{A(l, \rho)} f_k W_l V_e^{p} \eta^2 \, dx + p \int_{A(l, \rho)} f_k (V_e - k)^p V_e^{p} \frac{V_e}{V_e - k} \, dx \right]$$

$$+ 4 \int_{A(l, \rho)} |f_x| | \nabla \eta | W_l V_e \eta \, dx$$

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\[
\frac{pY}{2} \int_B \eta^2 |\nabla V_e|^2 V_e^{-1} (V_e - k)^{p-1} \chi_{A(l, \rho)} \, dx \\
+ C(b, n, p, \gamma, \Gamma) \left[ \int_B W_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, \rho)} f_k \left( W_l^2 + lW_l + l^2 \right) \, dx \right]
\]

by Young’s inequality. By an absorbing argument, we obtain

\[
\int_B \eta^2 |\nabla W_l|^2 \, dx = p^2 \int_B \eta^2 |\nabla V_e|^2 (V_e - k)^{2(p-1)} \chi_{A_l} \, dx \\
\leq p^2 \int_B \eta^2 |\nabla V_e|^2 V_e^{-1} (V_e - k)^{p-1} \chi_{A_l} \, dx \\
\leq C(b, n, p, \gamma, \Gamma) \left[ \int_B W_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, \rho)} f_k \left( W_l^2 + lW_l + l^2 \right) \, dx \right]
\]

from which (5.2) immediately follows.

Next, we would like to find a constant \( C_\bullet = C_\bullet (b, n, N, p, q, \gamma, \Gamma) \in (0, \infty) \) and an exponent \( \zeta = \zeta(n, q) \in (0, 2/n) \) such that

\[
\int_{A(l, \hat{l})} W_l^2 \, dx \leq C \left[ \frac{1}{(r - \hat{r})^2 + \hat{r}^2} \frac{1}{(\hat{l} - l)^2} \left( \int_{A(l, r)} W_l^2 \, dx \right)^{1 + \zeta} \right] (5.3)
\]

holds for arbitrary numbers \( l, \hat{l}, r, \hat{r}, \rho \) enjoying \( 0 < \hat{r} < r \leq \hat{\rho} \leq \rho \) and \( 0 < l_0 := k^p + C_\bullet \| W_{k^p} \|_{L^2(B_{\hat{l}})} \leq l < \hat{l} < \infty \). As a preliminary for proving (5.3), we would like to verify that

\[
\int_{A(l, r)} \eta W_l \, dx \leq C \left[ |A(l, r)|^{\zeta} \int_{A(l, r)} W_l^2 |\nabla \eta|^2 \, dx + l^2 |A(l, r)|^{1 + \zeta} \right] (5.4)
\]

holds for all \( r \in (0, \hat{\rho}], l \in [l_0, \infty) \), and for any non-negative function \( \eta \in C^1_c(B_{\hat{l}}) \) with \( 0 \leq \eta \leq 1 \). Here the constants \( C \in (0, \infty) \) in (5.3)–(5.4) depend at most on \( b, n, p, q, \gamma \) and \( \Gamma \). To deduce (5.4), we should mention that \( \| f_k \|_{L^1(B_{\hat{l}})} \leq 1 \) is clear by the definitions of \( k \) and \( f_k \). For simplicity, we consider the case \( n \geq 3 \), where we can use the Sobolev embedding \( W^{1,2}_{0} \left( B_{\hat{l}} \right) \hookrightarrow L^{2^*} \left( B_{\hat{l}} \right) \) with \( 2^* = 2n/(n - 2) \). Combining with Hölder’s inequality and Young’s inequality, by (5.2) we get

\[
\int_{A(l, r)} \eta^2 f_k \left( W_l^2 + lW_l + l^2 \right) \, dx \leq \left( \sigma + C(n) |A(l, r)|^{2/n - 2/q} \right) \int_{A(l, r)} |\nabla (\eta W_l)|^2 \, dx \\
+ l^2 \left( \frac{C(n)}{\sigma} |A(l, r)|^{1 + 2/n - 4/q} + |A(l, r)|^{1 - 2/q} \right)
\]

for any \( \sigma > 0 \). By the definition of \( A(l, r) \) and the Cauchy–Schwarz inequality, we can easily check

\[
|A(l, r)| \leq \frac{1}{l - k^p} \int_{A(l, r)} W_{k^p} \, dx \leq \frac{|A(l, r)|^{1/2}}{l - k^p} \| W_{k^p} \|_{L^2(B_{\hat{l}})}.
\]

Hence, it follows that

\[
|A(l, r)| \leq \left( \frac{\| W_{k^p} \|_{L^2(B_{\hat{l}})}^2}{l - k^p} \right)^2.
\]

Thus, we can choose and fix sufficiently small \( \sigma \in (0, 1) \) and suitably large \( C_\bullet \in (0, \infty) \), both of which depend at most on \( b, n, p, q, \gamma \) and \( \Gamma \), so that an absorbing argument can be made. Moreover, by our choice of \( C_\bullet \), we may let \( |A(l, r)| \leq 1 \). As a result, we are able to conclude (5.4) with \( \zeta := 2/n - 2/q \in (0, 2/n) \).
when \( n \geq 3 \). In the remaining case \( n = 2 \), we fix a sufficiently large constant \( \kappa \in (2, \infty) \), and use the Sobolev embedding \( W^{1,2}_0(B_r) \hookrightarrow L^\kappa(B_r) \). By similar computations, we can find a constant \( C_* \), depending at most on \( b, n, p, q, \gamma, \Gamma \) and \( \kappa \), such that \( (5.4) \) holds for some exponent \( \varsigma = \varsigma(n, q, \kappa) \in (0, 1 - 2/q) \). Now we would like to prove \( (5.3) \). Let \( l, \hat{l} \) and \( r, \hat{r}, \hat{\rho} \) satisfy respectively \( \hat{l} < l \leq \hat{\rho} < \infty \) and \( 0 < \hat{r} < r \leq \hat{\rho} \). Corresponding to the radii \( r, \hat{r}, \hat{\rho} \), we fix a non-negative function \( \eta \in C^1_c(B_r) \) such that

\[
\eta \equiv 1 \quad \text{on } B_{\hat{r}} \quad \text{and} \quad |\nabla \eta| \leq \frac{2}{r - \hat{r}} \quad \text{in } B_r.
\]

Since \( W_{\hat{l}} \geq l - l \) holds a.e. in \( A(\hat{l}, \rho) \), it is easy to check that

\[
|A(\hat{l}, \rho)| \leq \frac{1}{(l - l)^2} \int_{A(l, \rho)} W_{\hat{l}}^2 \, dx.
\]

Also, the inclusion \( A(l, \hat{r}) \subset A(l, r) \) and the inequality \( W_{\hat{l}} \leq W_l \) yield

\[
\int_{A(l, \hat{r})} W_{\hat{l}}^2 \, dx \leq \int_{A(l, r)} W_{\hat{l}}^2 \, dx.
\]

From these inequalities and \( (5.4) \), we can easily conclude \( (5.3) \).

From \( (5.3) \), we would like to complete the proof of Proposition 2.6. We fix arbitrary \( \theta \in (0, 1), \hat{\rho} \in (0, \rho) \). For each \( m \in \mathbb{Z}_{\geq 0} \), we set

\[
l_m := l_0 + L_0 (1 - 2^{-m}), \quad \rho_m := [\theta + 2^{-m} (1 - \theta)] \hat{\rho}, \quad a_m := \|W_{l_m}\|_{L^2(B_{\rho_m})},
\]

where the constant \( L_0 \in (0, \infty) \) is to be chosen later. Then, by \( (5.3) \), we get

\[
a_{m+1} \leq C \left[ \frac{2^{m+1}}{(1 - \theta)r} \right] 2^{\gamma(m+1)} L_0^\varsigma a_m \leq \frac{\hat{C}}{(1 - \theta)r} L_0^{-\varsigma} 2^{(1+\varsigma)m} a_m^{1+\varsigma}
\]

for every \( m \in \mathbb{Z}_{\geq 0} \), where \( \hat{C} \in (0, \infty) \) depends at most on \( b, n, p, q, \gamma \) and \( \Gamma \). We set \( L_0 \) by

\[
L_0 := C_0 \|W_{k^p}\|_{L^2(B_{\hat{\rho}})} \quad \text{with} \quad C_0 := \left[ \frac{\hat{C}}{(1 - \theta)\hat{\rho}} \right]^{1/\varsigma} 2^{1+\varsigma},
\]

so that we obtain

\[
a_0 = \|W_{l_0}\|_{L^2(B_{\hat{\rho}})} \leq \|W_{k^p}\|_{L^2(B_{\hat{\rho}})} \leq \left[ \frac{\hat{C} L_0^{-\varsigma}}{(1 - \theta)\hat{\rho}} \right]^{-1/\varsigma} \left[ 2^{1+\varsigma} \right]^{-1/\varsigma}.
\]

By Lemma 5.2 we have \( a_m \to 0 \) as \( m \to \infty \). In particular, it follows that

\[
\int_{A(l_0 + L_0, \theta \hat{\rho})} W_{l_0 + L_0}^2 \, dx = 0,
\]

which implies \( \|W_{k^p}\|_{L^\infty(B_{\theta \hat{\rho}})} \leq (C_* + C_0) \|W_{k^p}\|_{L^2(B_{\hat{\rho}})} \). As a consequence, we obtain

\[
\|W_{k^p}\|_{L^\infty(B_{\theta \hat{\rho}})} \leq C(b, n, p, q, \gamma, \Gamma) \left( \left[ \frac{\|W_{k^p}\|_{L^2(B_{\hat{\rho}})}}{[(1 - \theta)\hat{\rho}]^{d/2}} \right]^{p d/2} \right)
\]

for all \( \theta \in (0, 1), \hat{\rho} \in (0, \rho) \)

with \( d := 2/(p \varsigma) \in [n/p, \infty) \). By \( \|W_{k^p}\|_{L^2(B_{\hat{\rho}})} \leq \|W_{k^p}\|^{1/2}_{L^1(B)} \|W_{k^p}\|^{1/2}_{L^\infty(B_{\hat{\rho}})} \) and Young’s inequality, we get

\[
\|W_{k^p}\|_{L^\infty(B_{\theta \hat{\rho}})} \leq \frac{1}{2} \|W_{k^p}\|_{L^\infty(B_{\hat{\rho}})} + C(b, n, p, q, \gamma, \Gamma) \left( \left[ \frac{\|W_{k^p}\|_{L^1(B)}}{[(1 - \theta)\hat{\rho}]^{d/2}} \right]^{p d} \right)
\]

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for all $\theta \in (0, 1)$, $\hat{\rho} \in (0, \rho]$. By applying Lemma 5.1 with $H(s) \coloneq \|W_{k^p}\|_{L^\infty(B_{\hat{\rho}^d})}$. As a result, we are able to deduce

$$\|W_{k^p}\|_{L^\infty(B_{\hat{\rho}^d})} \leq C(b, n, p, q, \gamma) \frac{\|W_{k^p}\|_{L^1(B)}}{[(1-\theta)\rho]^d} \quad \text{for all } \theta \in (0, 1).$$

By the definition of $W_{k^p}$, we get

$$\text{ess sup}_{B_{\hat{\rho}^d}} V_{k^p} \leq k + \text{ess sup}_{B_{\hat{\rho}^d}} (W_{k^p} + k^n)^{1/p} \leq 2k + C(b, n, p, q, \gamma) \frac{\|W_{k^p}\|_{L^1(B)}}{[(1-\theta)\rho]^d}.$$ 

By $k = 1 + \|f_e\|_{L^1(B)}^{1/(p-1)}$ and $\|W_{k^p}\|_{L^1(B)} \leq \|V_{k^p}\|_{L^p(B)}$, we complete the proof. \hfill \Box

**Remark 5.3** (Higher regularity of regularized solutions). In this paper, we have often used $u_e \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{R}^N) \cap W^{2,2}_{\text{loc}}(\Omega; \mathbb{R}^N)$. Following [10], [18] and [23, Chapter 8], we briefly describe how to improve this regularity (see also [26, Chapters 4-5]). There it is noted that it is not restrictive to let $f_e \in L^\infty(\Omega; \mathbb{R}^N)$, or even $f_e \in C^\infty(\Omega; \mathbb{R}^N)$, since our approximation arguments work as long as (2.37) holds.

First, appealing to a standard difference quotient method as in [18, Theorem 2] or [23, §8.2], we are able to get

$$\int_\Omega V_{k^p}^{p-2} |D^2 u_e|^2 \, dx \leq C(e, \omega, \|f_e\|_{L^\infty(\Omega)}) < \infty \quad \text{for every } \omega \subset \Omega,$$

which is possible by (3.5). Secondly, we appeal to the Uhlenbeck structure to prove $Du_e \in L^{\infty}_{\text{loc}}(\Omega; \mathbb{R}^{Nn})$. We test

$$\phi \coloneq \eta^2 V_{k^p}^{l} D_\alpha u_e \quad \text{with} \quad \eta \in C^1(B_\rho(x_0)), \quad 0 \leq l < \infty$$

into the weak formulation (3.3) for each $\alpha \in \{1, \ldots, n\}$. We should note that when $l = 0$ this test function is admissible by (3.5), and all of the computations in Lemma 5.1 make sense. By (3.4) and standard computations given in [10, §3], [23, §8.3], we can improve local integrability of $u_e$, and in particular we may test the same $\phi$ with larger $l > 0$. By Moser’s iteration arguments, we are able to conclude $V_e \in L^{\infty}_{\text{loc}}(\Omega)$, and hence $Du_e \in L^{\infty}_{\text{loc}}(\Omega; \mathbb{R}^{Nn})$ follows. Finally, $u_e \in W^{2,2}_{\text{loc}}(\Omega; \mathbb{R}^N)$ is clear by (5.3) and $Du_e \in L^{\infty}_{\text{loc}}(\Omega; \mathbb{R}^{Nn})$.

These computations above are substantially dependent on an approximation parameter $e \in (0, 1)$. When it comes to local uniform $L^\infty$-bounds of $Du_e$, the strategy based on Moser’s iteration may not work, since the test function $\phi$ may intersect with the facet of $u_e$. In contrast, for the scalar problem, local uniform $L^\infty$-bounds of gradients $\nabla u_e = (\partial_{x_1} u_e, \ldots, \partial_{x_n} u_e)$ for weak solutions to (2.52) is successfully deduced by truncating the term $\partial_{x_n} u_e$ in a place $|\partial_{x_n} u_e| \leq k$ for some $k \geq 1$. This modification forces us to adapt another modulus that is different from $V_e = \sqrt{e^2 + |Du_e|^2}$ and that may lack spherical symmetry.

For this reason, it appears that the proof of a priori Lipschitz bounds based on Moser’s iteration works only in the scalar problem. In the proof of Proposition 2.6 we have appealed to De Giorgi’s truncation, since we do not have to choose another non-symmetric modulus.
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