Calculating symmetries in Newman-Tamburino metrics

John D. Steele
School of Mathematics, University of New South Wales, Sydney, New South Wales 2052, Australia. email: j.steele@unsw.edu.au

Abstract: In this paper I show that the Newman-Tamburino spherical metrics always admit a Killing vector, correcting a claim by Collinson and French, (1967 J. Math. Phys. 8 701) and also admit a homothety. A similar calculation is given for the limit of the Newman-Tamburino cylindrical metric.

1 Introduction

The Newman-Tamburino metrics are those vacuum solutions of the Einstein equations admitting hypersurface orthogonal geodesic rays with non-vanishing shear and divergence. In the Newman-Penrose formalism this implies that $\Psi_0 = \kappa = 0$, that $\rho$ is real and non-zero and $\sigma \neq 0$. In [1] Newman and Tamburino explicitly gave all such metrics and showed that they fall into two classes: the spherical, with $\rho^2 \neq \sigma \overline{\sigma}$ and the cylindrical with $\rho^2 = \sigma \overline{\sigma}$. In [2] Collinson and French claimed to have shown that the former metrics admit at most one Killing vector, and that happens only in a particular subcase. In fact, the spherical Newman-Tamburino metrics always admit a Killing vector and also always admit a homothety. This preprint is intended to show the full calculations and results when the homothetic equations of [3] are integrated for the Newman-Tamburino spherical metrics. The bulk of sections 3 and 4 come from Maple 9 worksheets, exported to TeX and suitably tidied up for better readability.

Throughout I use the spin coefficient notation of [4]. For example I use $\kappa'$, $\rho'$, $\sigma'$ and $\tau'$ in place of the more traditional $-\nu$, $-\mu$, $-\pi$ and $-\lambda$.

2 Results

The contravariant form of the Newman-Tamburino spherical metric [1] (see also [5], equation (26.21)) is

\[
\begin{align*}
g_{22} &= -\frac{2r^2(\zeta \overline{\zeta})^{1/2}}{R^2} + \frac{2rL}{A} + \frac{2r^3A(\zeta^2 + \overline{\zeta}^2)}{R^4} - \frac{4r^2A^2(\zeta \overline{\zeta})^{3/2}}{R^4} \\
g_{23} &= 4A^2(\zeta \overline{\zeta})^{3/2} x \left[ \frac{L}{2a^3} - \frac{r - 2a}{2a^2 R^2} - \frac{r - a}{R^4} \right] \\
g_{24} &= 4A^2(\zeta \overline{\zeta})^{3/2} y \left[ \frac{L}{2a^3} - \frac{r + 2a}{2a^2 R^2} - \frac{r + a}{R^4} \right] \\
g_{33} &= -\frac{2(\zeta \overline{\zeta})^{3/2}}{(r + a)^2} \\
 g_{44} &= -\frac{2(\zeta \overline{\zeta})^{3/2}}{(r - a)^2} \\
g_{12} &= 1.
\end{align*}
\]
Here our coordinates are $x^1 = u, x^2 = r, x^3 + ix^4 = x + iy = \zeta$ and

$$A(u) = bu + c, \quad L = \frac{1}{2} \log \left( \frac{r + a}{r - a} \right) \quad a = A(\zeta\bar{\zeta})^{1/2} \quad R^2 = r^2 - a^2.$$ 

Here $b$ and $c$ are real constants.

The Collinson and French result (also quoted in [5]) is that there is a Killing vector only in the case where $A$ is constant — in this situation the Killing vector is the obvious $\partial_u$. However, if $b \neq 0$ we can set $c = 0$ by a coordinate change and then the vector

$$K^a = -u\partial_u + r\partial_r + 2x\partial_x + 2y\partial_y.$$ 

is a Killing vector, as will be shown in section 3. This can be checked directly: consider the flow of $K^a$. This scales the coordinates by

$$u \rightarrow \lambda^{-1}u, \quad r \rightarrow \lambda r, \quad \zeta \rightarrow \lambda^2 \zeta$$

for real parameter $\lambda > 0$. Under this scaling it is easy to check that all the contravariant components given above are homogeneous in $\lambda$ (when $A = bu$), and all of the correct degree to make the flow isometric. For example, the $g^{22}$ component is homogeneous of degree 2, and so the metric term $g^{22} \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r}$ is unchanged under the flow.

Also the vector

$$H = r\partial_r + x\partial_x + y\partial_y$$

is a homothety, whatever $A$ is (see section 3). Alternatively, the flow of $H$ is

$$u \rightarrow u, \quad r \rightarrow \lambda r, \quad \zeta \rightarrow \lambda \zeta,$$

and we again find that all the contravariant components given above are homogeneous in $\lambda$, and all of the correct degree to make the flow homothetic. For example, the $g^{22}$ component is homogeneous of degree 1, and so the metric term $g^{22} \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r}$ scales by $\lambda^{-1}$: the same scaling applies to all the metric terms.

Newman and Tamburino [1] also give the following metric, which arises as a limit of the cylindrical case (see also [5] (26.23) for corrections to the $du^2$ coefficient):

$$ds^2 = 2du dr - x^{-2} \left[ b + \log(r^2 x^4) \right] du^2 + 4 \frac{L}{x} du dx - r^2 dx^2 - x^2 dy^2,$$

with the same coordinates as used in the spherical case. The Killing vectors here are obvious ($\partial_u$ and $\partial_y$) and as we shall see there is also a homothetic vector (see section 4)

$$H_2 = 2r\partial_r - x\partial_x + 2y\partial_y.$$ 

One can use the flow of $H_2$ to check it is a homothety as well.
3 The Calculations (spherical case)

The basic information is taken from Collinson and French [2], and Newman and Tamburino [1]. See those papers for those spin coefficients that are not actually calculated here. I have checked in a separate calculation that their results are correct as quoted. I use as coordinates \( u, r, \zeta = x + iy \).

Collinson and French [2] wrote the conformal Killing equations in Newman-Penrose form and used that in their work, although there are a few minor typos in their paper. Here, I will use the formalism of [3], which generalised the ideas of [6] into a form suitable for this task. I will use to the notation of [3] for the components of the homothety

\[
\xi_a = \xi_n \ell^a + \xi_\ell m_a - \zeta_m \overline{m_a},
\]

and its bivector, \( F_{ab} \), with anti-self dual

\[
-F_{ab} = 2\phi_{00} \ell_{[a}m_{b]} + 2\phi_{01} (\ell_{[a}n_{b]} - m_{[a}\overline{m}_{b]} - 2\phi_{11} n_{[a}\overline{m}_{b]}.\]

The tetrad is a standard tetrad (see [1]), based around the Debever-Penrose vector \( \ell^a = \partial_r \), see [1] and [2] for further detail. Since the tetrad is normalised, for the Penrose-Rindler spin coefficients used in [3] we have \( \gamma' = -\epsilon, \beta' = -\alpha \) etc.

In the Maple I use use \( z \) for \( \zeta \) and \( w \) for \( \overline{\zeta} \); \( \text{H1} \) for \( \xi_\ell \) etc. I typically add a \( \text{b} \) for a complex conjugate (\( \text{Hmb} \) is \( \xi_m \)) and a \( \text{1} \) for a dash (\( \text{rho1} \) is \( \rho' \)).

Firstly, define the terms \( a, a_0 \) (the latter is \( \alpha_0 \) in [2]).

\[
\begin{align*}
\text{a} &:= A(u) z^{1/2} w^{1/2} : \\
\text{a0} &:= 3/4 w^{3/4} z^{-1/4} : \\
\text{a0b} &:= 3/4 z^{3/4} w^{-1/4} .
\end{align*}
\]

Rather than use the explicit definitions for \( L \) and \( R \) in [2] and [1], I will leave them as “unknown” functions and define a routine later that will substitute for their derivatives. I will also use \( Q(u, r, z, w) \) in place of \( 1/R^2 \) to make things more transparent. I define what these functions actually are so we can substitute for them more easily when that become useful. I also define dummy symbols to use in place of the full functional dependence of \( L \) and \( Q \) for ease of readability. I have also suppressed the functional dependence in the Maple output, replacing \( Q(u, r, z, w) \) with \( Q(x^a) \) for example.

\[
\begin{align*}
\text{LL} &:= L(u, r, z, w) : \text{Lis} := 1/2 \log \left( \frac{r + A(u) \sqrt{z} \sqrt{w}}{r - A(u) \sqrt{z} \sqrt{w}} \right) ; \\
\text{QQ} &:= Q(u, r, z, w) : \text{Qis} := 1/((r^2 - A(u)^2) zw) .
\end{align*}
\]

Now we define the routine to simplify derivatives and products and also add a line to collect terms.
The terms $S$ and $S_b$ are $\psi^1_0$ and its conjugate in [2].

\[ S := 2A(u)^2 z^{7/4} w^{3/4} \]

And the curvature component $\Psi_1$ is given in [1].

\[ \Psi_1 := 2A(u)^2 z^{7/4} w^{3/4} (x^a)^2 \]

Now from [2] we have $\kappa = \epsilon = \tau^a = 0$, and $\rho$ and $\sigma$ real — these can also be easily checked by Maple. So by [3](6a) $D\xi_\ell = 0$. Using $\tau = \pi + \beta$ ([2]), [3](6c) becomes

\[ \delta\xi_\ell = \tau\xi_\ell - \rho\xi_m - \sigma\xi_m - \phi_1. \]

We next use equation [3](11), since $\ell^a$ is a Debever-Penrose vector. Unfortunately, [3](11) contains an error — the right hand side is the complex conjugate of what it ought to be. With this correction, we have

\[ \phi_{11} = -\tau\xi_\ell + \rho\xi_m + \sigma\xi_m. \]

Hence $\delta\xi_\ell = 0$ and $\xi_\ell = \xi_\ell(u)$, as found in [2].

Equation (10a) of [3] is

\[ D\phi_{11} = -\xi_\ell \Psi_1, \]

and so integrates to give the $r$ dependence of $\phi_{11}$, here called $p_{11}$. We ignore the factor independent of $r$ when integrating:

\[ \int (-Qis^2, r); \]

\[ -\frac{r}{2A(u)^2 z w (r^2 - A(u)^2 z w)} + \frac{1}{2} \arctanh \left( \frac{r}{A(u) \sqrt{z w}} \right) A(u)^{-3} z^{-1} w^{-1} \frac{1}{\sqrt{z w}} \]

The $\arctanh$ term here is just $L$ and we get

\[ p_{11} := S \cdot Hl(u)/2 + 2A(u)^2 z^{3/4} w^{3/4} \cdot L \cdot (x^a) - \frac{z^{1/4} Hl(u)}{A(u) w^{3/4}} \cdot (x^a) + p_{110}(u, z, w); \]
Here \( p_{110}(u,z,w) \) is the integration constant. Now the spin coefficients — see [2].

\[
\alpha := \text{expand}\left(\text{simplify}\left(\frac{S \cdot LL \cdot r}{2} a^2 + \frac{QQ}{a} (r \cdot a_0 + a \cdot a_0b) - S \cdot QQ / 2, \text{radical})\right)\right);
\]

\[
\beta := -S \cdot LL \cdot QQ / 2 a - QQ \cdot (r \cdot a_0b + a \cdot a0) ;
\]

Now a routine to take conjugates nicely, as we need conjugates to define \( \tau \).

\[
\text{conj} := \text{proc}(XX) \text{subs}(z=w1, w=z1, XX) : \text{subs}(I=-I, %) ;
\]

\[
\text{conj}(\alpha) + \beta ;
\]

\[
\tau := \text{collect}(%, [L(u,r,z,w), Q(u,r,z,w)]) ;
\]

\[
\rho := -r \cdot Q(u,r,z,w) ;
\]

\[
\sigma := a \cdot Q(u,r,z,w) ;
\]

\[
\sigma_1 := \frac{r \cdot LL^2 \cdot S^2}{4 a^4} \cdot QQ + \frac{r \cdot LL \cdot S \cdot Sb}{2 a^4} \cdot QQ - \frac{r \cdot \text{diff}(a,u)}{a} \cdot QQ - \frac{S \cdot Sb}{2 a^3} \cdot QQ ;
\]

\[
\text{unprotect}(\gamma) ;
\]

\[
\gamma := -\frac{r \cdot LL^2 \cdot S \cdot Sb}{4 a^4} \cdot QQ + \frac{S^2 \cdot LL^2}{4 a^3} \cdot QQ + a \cdot r \cdot LL \left( \frac{S \cdot a_0b - Sb \cdot a0}{2 a^3} \cdot QQ \right) + \frac{S \cdot Sb \cdot QQ \cdot LL / 2 a^3 - S \cdot Sb / 4 a^3 \cdot (LL / 2 a^2 - r / 2 a \cdot QQ)} + \frac{(S^2 / 2 a - a \cdot (S \cdot a_0b - Sb \cdot a0)) / 2 a^2 \cdot QQ} ;
\]

We need derivative operators \( \delta \) and \( \delta' \) to find \( \rho' \). Firstly, the components of \( m^a \) come from [2] and [1].

\[
\text{om0} := -A(u) z^{(1/4)} w^{(5/4)} ;
\]

\[
\omega := -w^{3/4} z^{-1/4} L(x^a) + \left( -r A(u) z^{1/4} w^{5/4} + A(u) z^{7/4} w^{3/4} \right) Q(x^a) ;
\]

\[
\text{omegal1} := \text{conj}(\omega) ;
\]

\[
\text{P} := z^{(3/4)} w^{(3/4)} ;
\]

\[
\text{del} := XX \rightarrow \omega \cdot \text{diff}(XX,x) + 2 r \cdot \text{P} \cdot QQ \cdot \text{diff}(XX,w) - 2 a \cdot \text{P} \cdot QQ \cdot \text{diff}(XX,z) ;
\]

\[
\text{del1} := XX \rightarrow \omega_1 \cdot \text{diff}(XX,x) + 2 r \cdot \text{P} \cdot QQ \cdot \text{diff}(XX,z) - 2 a \cdot \text{P} \cdot QQ \cdot \text{diff}(XX,w) ;
\]

To calculate \( \rho' \) we use [4] (4.11.12 e').

\[
\text{expand}(\text{diff}(\sigma_1, r) - \rho \cdot \sigma_1) / \sigma_1 ;
\]

\[
\rho_1 := -A(u) z^2 L(x^a)^2 Q(x^a) - 2 \left( A(u) z w + \frac{z^{3/2} r}{\sqrt{w}} \right) Q(x^a) L(x^a) + A(u) z w \frac{d A}{du} Q(x^a) ;
\]

We check some curvature equations next before we go on.

\[
\text{diffsbs}(\text{diff}(\tau, r) - \rho \cdot \tau - \sigma_1 \cdot \text{conj}(\tau) - \text{Psi1}) ;
\]

\[
0
\]
\[ \text{diffsbs(diff(alpha,r)-rho*alpha-sigma*beta); } \]  
\[ \text{# [4] 4.11.12h \& i'} \]

\[ \text{0} \]

\[ \text{Psi2:=-diffsbs(diff(rho1,r)-rho1*rho-sigma*sgma1) ;} \]  
\[ \text{# [4] 4.11.12 f'} \]

\[ \Psi_2 = -4A(u)z^5/2L(xa)Q(xa)^2 \sqrt{w} - (2 A(u) z^2 r + 4A(u)z^3 w^3/2) Q(xa)^2 \]

This expression for \( \Psi_2 \) agrees with [1].

\[ \text{diffebs(diff(gamma,r) -beta*conj(tau)-alpha*tau-Psi2); } \]  
\[ \text{# [4] 4.11.12k} \]

\[ \text{0} \]

\[ \text{diffebs(del(rho)-del1(sigma)-rho*(conj(alpha)+beta)+} \]
\[ \text{sigma*(3*alpha-conj(beta) ) + Psi1); } \]  
\[ \text{# [4] 4.11.12 d} \]

\[ \text{0} \]

\[ \text{diffebs(del1(beta)-del(alpha)-rho*rho1+sigma*sigma1+alpha*conj(alpha)+} \]
\[ \text{beta*conj(beta)-2*alpha*beta-Psi2 ); } \]  
\[ \text{# [4] 4.11.12 l} \]

\[ \text{0} \]

Integrating [3] (6g) and using [3](11) (corrected, see above):

\[ \text{Hm1:=-r*p110(u,z,w) + Hl(u)*S/2/a^3*r*LL+Hm0;} \]

\[ Hm1 := -r p110(u,z,w) + \frac{Hl(u)}{A(u)} \sqrt{\frac{z^{1/4} r L(xa)}{w^{3/4}}} + Hm0 \]

\[ \text{Hmb1:=-r*p110b(u,z,w) + Hl(u)*Sb/2/a^3*r*LL+Hmb0;} \]

\[ Hmb1 := -r p110b(u,z,w) + \frac{Hl(u)}{A(u)} \sqrt{\frac{z^{1/4} r L(xa)}{w^{3/4}}} + Hmb0 \]

By [3] (11) the following ought to be zero.

\[ \text{diffebs(p11+tau*Hl(u)-rho*Hm1-sigma*Hmb1):} \]

\[ \text{collect(%/QQ,r);} \]

\[ \left( Hm0 + \frac{z^{3/4} Hl(u)}{w^{1/4}} + A(u) \sqrt{zw} p110b(u,z,w) \right) r \]

\[ - A(u) \sqrt{zw} Hmb0 - (A(u))^2 zw p110(u,z,w) - Hl(u) A(u) w^{5/4} z^{1/4} \]

\[ \expandafter{\text{expand}}(\text{solve(coeff(%,r,1),Hm0))},\text{expand(solve(coeff(%,r,0),Hmb0))} ; \]

\[ - \frac{z^{3/4} Hl(u)}{w^{1/4}} - A(u) \sqrt{zw} p110b(u,z,w), - \sqrt{zw} A(u) p110(u,z,w) - \frac{w^{3/4} Hl(u)}{z^{1/4}} \]

So we get

\[ \text{Hm:=-r*p110(u,z,w)-a*p110b(u,z,w)+Hl(u)*expand(S/2/a^3*(r*LL-a));} \]
> \( H_{mb} = \text{conj}(H_m) \):
> These agree with the components in [2] (their \( V_3 \) and \( V_4 \)). Now we use [3] (10b) to get \( \phi_{01} \).
> 
> \[
> \text{diffeqs}(\text{Psi1*Hm/2/sigma-beta*p11/sigma+del(p11)/2/sigma}):
> \]
> 
> \[
> p_{01} := \text{collect}(%, [Hl(u), Q(u,r,z,w), L(u,r,z,w), r]);
> \]
>
> \[
> \begin{align*}
> p_{01} := & \left( \left( \frac{1}{2} \sqrt{z} \sqrt{wr} + \left( 2 \frac{z^{3/2} r}{\sqrt{w}} + A(u) zw \right) L(x^a) \right) \frac{Q(x^a) - \frac{z(L(x^a))^2}{A(u) w} - \frac{L(x^a)}{2A(u)}}{Hl(u)} \right) \\
> & + \frac{z^{3/4} L(x^a) p_{110}(u,z,w)}{w^{1/4}} + \frac{3w^{3/4} p_{110}(u,z,w)}{4z^{1/4}} \\
> & - \left( A(u) z^{5/4} w^{1/4} r p_{110}(u,z,w) + A(u)^2 z^{7/4} w^{3/4} p_{110b}(u,z,w) \right) Q(x^a) \\
> & + \left( \frac{3p_{110}(u,z,w)}{4A(u) w^{3/4}} + \frac{w^{1/4} \frac{\partial}{\partial w} p_{110}(u,z,w)}{A(u)} \right) z^{1/4} r - z^{3/4} w^{3/4} \frac{\partial}{\partial z} p_{110}(u,z,w) \right) \\
> \end{align*}
> \]
>
> Now [3](10c) and (8a) will give us information on the \( w \) (that is, \( \zeta \)) dependence of \( \phi_{01} \).
> 
> \[
> \text{diffeqs}(\text{Psi1*Hmb-Psi2*Hl(u)+2*rho*p01+2*alpha*p11-del1(p11)}); \# [3]10c
> \]
>
> \[
> \begin{align*}
> \left( \frac{3p_{110}(u,z,w)}{2A(u) w^{3/4}} - 2 \frac{w^{1/4}}{A(u) \partial w} p_{110}(u,z,w) \right) z^{1/4} \\
> \end{align*}
> \]
>
> \[
> \text{dsolve}(% = 0, p_{110}(u,z,w));
> \]
>
> \[
> p_{110}(u,z,w) = \frac{\text{F1}(u,z)}{w^{3/4}}
> \]
>
> \[
> \text{diffeqs}(\text{diff(p01,r)+del1(p11)-2*rho*p01-2*alpha*p11}); \# [3] (8a)
> \]
>
> \[
> \begin{align*}
> \left( \frac{9p_{110}(u,z,w)}{4A(u) w^{3/4}} + 3 \frac{w^{1/4}}{A(u) \partial w} p_{110}(u,z,w) \right) z^{1/4} \\
> \end{align*}
> \]
>
> \[
> \text{dsolve}(% = 0, p_{110}(u,z,w));
> \]
>
> \[
> p_{110}(u,z,w) = \frac{\text{F1}(u,z)}{w^{3/4}}
> \]
>
> Both giving the same result. Now we turn to \( \xi_n \) and [3](6i), which we solve for \( \sigma \xi_\ell \).
> 
> \[
> \text{rhs6i} := \text{diffeqs}((-\text{del}(Hm) - \text{conj}(\text{sigma1})*Hl(u) - Hm*(\text{conj}(\text{alpha}) - \text{beta})));
> \]
>
> The imaginary part ought to be zero as \( \xi_n \) is real, so using results from [3] (10c) and (8a), we find the imaginary part divide out a common nor-zero factor and call what’s left \( X \).
> Imrhs6i:=%~conj(%):
subs(p110(u,z,w)=F(u,z)/w^(-3/4),p110b(u,z,w)=Fb(u,w)/z^(-3/4),%):
X:=expand(%/r/QQ/sqrt(z)/sqrt(w));

\[
X := -4 z^{3/4} A(u) \frac{\partial}{\partial z} F(u, z) + 4 w^{3/4} A(u) \frac{\partial}{\partial w} Fb(u, w) + 3 \frac{A(u) Fb(u, w)}{w^{1/4}} - 3 \frac{A(u) F(u, z)}{z^{1/4}}
\]

Assuming \( F \) is differentiable in \( z \) we can split this
> subs(Fb=0,X):%

\[
-4 z^{3/4} A(u) \frac{\partial}{\partial z} F(u, z) - 3 \frac{A(u) F(u, z)}{z^{1/4}}
\]

This is a (real) function of \( u \) and \( w \). We choose the shape of the separation function to simplify the solution to the differential equation slightly.
> dsolve(%=-4*A(u)*G(u),F(u,z));

\[
F(u, z) = \frac{G(u) z + F1(u)}{z^{3/4}}
\]

Check this out:
> subs(F(u,z)=G(u)*z^(1/4)+H(u)/z^(-3/4),Fb(u,w)=G(u)*w^(1/4)+Hb(u)/w^(-3/4),X):
> expand(%);

0
So we define a simplification routine for \( \phi_{11}^0 \).
> P110sbs1:=proc(XX)
> subs(p110(u,z,w)=F(u,z)/w^(-3/4),p110b(u,z,w)=Fb(u,w)/z^(-3/4),XX);
> subs(F(u,z)=G(u)*z^(-1/4)+H(u)/z^(-3/4),Fb(u,w)=G(u)*w^(-1/4)+Hb(u)/w^(-3/4),X);
> expand(%);
> end proc:
And check it works
> P110sbs1(Imrhs6i);

0

Turning to [3](6b) next,
> eqn6b:=H1(u)*(gamma+conj(gamma))-conj(tau)*Hm-tau*conj(Hm)-p01-conj(p01)+psi:
> P110sbs1(diffsbs(%));

\[
\psi - G(u) - 3 \frac{H(u)}{2z} - 3 \frac{Hb(u)}{2w}
\]

This ought to be \( \dot{\xi}_\ell \), a function of \( u \) only, so \( H = 0 \) and we define a new simplification routine and test it out:
> P110sbs2:=proc(XX);
> expand( subs( p110(u,z,w)= (psi-diff(H1(u),u))*z^(-1/4)/w^(-3/4) , p110b(u,z,w)=(psi-diff(H1(u),u))*w^(-1/4)/z^(-3/4),XX));
> collect(%,[L(u,r,z,w),Q(u,r,z,w),r,H1(u),z,w]);end proc:
> P110sbs2(diffsbs(eqn6b));
\[
\frac{d}{du} H_l(u)
\]

This is as it should be. Now we can define \(\xi_n\).

\[P110sbs2(diffsbs(rhs6i+conj(rhs6i))/sigma/2):\]

\[H_n:=collect(\%,[L(u,r,z,w),Q(u,r,z,w),H_l(u),diff(H_l(u),u),r,psi]):\]

We check this against the [2] version, called \(V_2\) there. It is clear from the shape of \(\xi_n (= V_3 \text{ of } [2])\) that \(a_0 \text{ in [2]}\) is my \(\phi_{11}^0\).

\[ay0:=p110(u,z,w): \text{ay0b:=p110b(u,z,w):} \]

\[V2:=r*LL^2*(-S^2-Sb^2)*H_l(u)/4/a^5 - LL^2*S*Sb*H_l(u)/4/a^4 - \]

\[LL*(ay0b*S+ay0*Sb)/2/a + r*(2*a*H_l(u)*diff(a,u)-ay0*Sb-ay0b*S)/2/a^2 + \]

\[(-2*a^3*ay0*S-2*a^3*ay0b*Sb+H_l(u)*S+Sb)/4/a^4+r*LL*(-2*a^3*ay0*S- \]

\[2*a^3*ay0b*Sb- H_l(u)*S+Sb)/4/a^5 + 1/QQ*(-3*ay0*S-3*ay0b*Sb- \]

\[4*a^2*2*P*(diff(p110(u,z,w),w) +diff(p110b(u,z,w),z) ) )/8/a^3: \]

\[expand(H_n-P110sbs2(V2)): \]

\[simplify(subs(psi=0,diff(H_l(u),u)=0,%)); \]

\[0 \]

So our \(\xi_n\) agrees with [2] in the case of their Killing vector (\(\psi = 0\) and \(\xi_\ell\) constant).

However, if \(\xi_\ell\) is not constant, the terms differ:

\[simplify(subs(psi=0,%%)); \]

\[-2 \]

Next, we put our \(\xi_n\) into [3](6d).

\[eqn6d:=diffsbs(diff(H_n,r)-p01-conj(p01)-psi): \]

\[P110sbs2(\%); \]

\[\frac{H_l(u)}{A(u)} \frac{d}{du} A(u) - \frac{d}{du} H_l(u) \]

\[dsolve(\%); \]

\[H_l(u) = C1 A(u) \]

So next a routine to replace \(\xi_\ell(u)\) with a multiple of \(A(u)\), and also to kill off the second derivative of \(A(u)\).

\[Hlsbs:=proc(XX) \]

\[subs(H_l(u)=C*A(u),XX);subs(diff(A(u),u,u)=0,\%);subs(diff(H_l(u),u,u)=0,\%);\%

\[end proc; \]

We now try [3](6j).

\[dell(Hm)+conj(rho1)*H_l(u)+rho*Hn+(conj(beta)-alpha)*Hm-p01+conj(p01)+psi: \]

\[Hlsbs(P110sbs2(diffsbs(\%))); \]

\[0 \]
So that is satisfied. Now for \( \phi_{00} \), which we get from the conjugate of \([3](6f)\).

\[ eqn6f := -\text{del1}(Hn) - (\text{conj}(\beta) + \alpha) Hn - \text{conj}(\rho1) Hnmb - \sigma1 Hm; \]

\[ p00 := \text{diffsbs}(\text{P110sbs2}(\text{diffsbs}(\%))); \]

I've suppressed this component as it's very long, but we check the result with \([3](8d)\).

\[ \text{del1}(p01) + \text{diff}(p00, r) - \rho \text{p00} - \sigma1 \text{p11}; \]

\[ \text{P110sbs2}(\text{diffsbs}(\%)); \]

To go any further we need to get the components of \( n^a = (1, U, X^3, X^4) \) and to define \( D' \). Taking the metric terms from \([1]\) and \([2]\):

\[ gup22 := -2 \times r^2 \times \sqrt{w} \times \sqrt{z} \times QQ + 2 \times r \times LL/A(u) + QQ \times 2 \times (2 \times r \times 3 \times A(u) \times (w^2 + z^2) - 4 \times r^2 \times A(u) \times 2 \times w \times (3/2) \times z \times (3/2)); \]

\[ gup22 + 2 \times \text{omega} \times \text{conj} \times (\text{omega}); \]

\[ U := \text{diffsbs}(\%/2); \]

\[
U := \sqrt{z} \sqrt{w} L \left( x^a \right)^2 + \left( \left( w^2 + z^2 \right) A(u) r - 2 A(u)^2 z^{3/2} w^{3/2} \right) Q \left( x^a \right) - \sqrt{z} \sqrt{w} \\
+ \left( -A(u)^2 z^{5/2} \sqrt{w} + 2 zw A(u) - A(u)^2 \sqrt{z} w^{5/2} \right) Q \left( x^a \right) + \frac{r}{A(u)} \right) L \left( x^a \right)
\]

\[ gup33 := -2 \times z \times (3/2) \times w \times (3/2) / (r + a)^2; \]

These next two terms are the components of \( m^a \).

\[ \text{gup44} := -2 \times z \times (3/2) \times w \times (3/2) / (r - a)^2; \]

\[ \text{xi3} := P \times (r - a) \times QQ; \text{xi4} := I \times P \times (r + a) \times QQ; \]

\[ \text{xi3} := z^{3/4} w^{3/4} (r - A(u) \sqrt{z} \sqrt{w}) Q \left( x^a \right) \]

\[ \text{xi4} := i z^{3/4} w^{3/4} (r + A(u) \sqrt{z} \sqrt{w}) Q \left( x^a \right) \]

\[ \text{xi3} \times \text{conj} \times (\text{xi4}) + \text{xi4} \times \text{conj} \times (\text{xi3}); \]

# checking

\[ 0 \]

\[ \text{simplify}((\text{subs}(\% \times (u, r, z, w) = \text{Qis}, \text{xi3} \times \text{conj} \times (\text{xi3}) \times 2 + \text{gup33}))); \]

\[ 0 \]

\[ \text{simplify}((\text{subs}(\% \times (u, r, z, w) = \text{Qis}, \text{xi4} \times \text{conj} \times (\text{xi4}) \times 2 + \text{gup44})); \]

\[ 0 \]

\[ \text{gup23} := 4 \times A(u) \times (3/2) \times w \times (3/2) \times (z + w) / 2 \times (LL / 2 / a^3 - (r - 2 \times a) \times QQ / 2 / a^2 \times (r - a) \times QQ); \]

\[ \text{gup24} := 4 \times A(u) \times (3/2) \times w \times (3/2) \times (z - w) / 2 \times I \times (LL / 2 / a^3 - (r + 2 \times a) \times QQ / 2 / a^2 \times (r + a) \times QQ); \]

\[ \text{gup23} + \text{omega} \times \text{conj} \times (\text{xi3}) + \text{conj} \times (\text{omega}) \times \text{xi3}; \]

\[ \text{X3} := \text{diffsbs}(\%); \]
\[ X_3 := \left( \left( -z^{3/2} \sqrt{w} - \sqrt{z} w^{3/2} \right) r + A(u) z w^2 + A(u) z^2 w \right) Q(x^a) + \frac{w}{A(u)} + \frac{z}{A(u)} \right) L(x^a) \\
+ \left( \left( -z^{3/2} \sqrt{w} - \sqrt{z} w^{3/2} \right) r + A(u) z w^2 + A(u) z^2 w \right) Q(x^a) \]

\[
> \text{gup24+omega*conj(xi4)+omega1*xi4:} \\
> \text{X4:=diffsbs(factor(\%))};
\]

\[ X_4 := i \left( \left( -z^{3/2} \sqrt{w} + \sqrt{z} w^{3/2} \right) r + z w^2 A(u) - z^2 w A(u) \right) Q(x^a) + \frac{w}{A(u)} - \frac{z}{A(u)} \right) L(x^a) \\
+ \left( \left( i \sqrt{z} w^{3/2} + iz^{3/2} \sqrt{w} \right) r + iz^2 w A(u) - iz w^2 A(u) \right) Q(x^a) \]

As a double check we firstly define the (contravariant) tetrad and then check against the metric terms.

\[
> \text{ell:=<0,1,0,0>:en:=<1,U,X3,X4>:} \\
> \text{em:=<0,omega,xi3,xi4>:emb:=map(conj,em):} \\
> \text{ell.Transpose(en)-em.Transpose(emb):} \\
> \%+Transpose(\%): \\
> \text{g:=map(diffsbs,\%):} \\
> \text{diffsbs(g[2,2]-gup22); 0} \\
> \text{diffsbs(g[2,3]-gup23); 0} \\
> \text{diffsbs(g[2,4]-gup24); 0} \\
> \text{diffsbs(simplify(g[3,3]-gup33)); 0} \\
> \text{diffsbs(simplify(g[4,4]-gup44)); 0} \\
\]

For a second check we apply two of the commutators \([4] (4.11.11)\) to \(r\) and check what we get.

\[
> \text{diff(U,r)+gamma+conj(gamma)-tau*conj(omega)-conj(tau)*omega:} \\
> \text{diffsbs(\%); 0} \\
> \text{diff(X3,r)-tau*conj(xi3)-conj(tau)*xi3:} \\
> \text{diffsbs(\%); 0} \\
\]
Since all this checks out we go ahead and define $D'$.

\[
\begin{align*}
D1 & := \text{proc}(XX) \\
& \text{diff}(XX, u) + \text{diff}(XX, r) \times U + (X3 + I \times X4) \times \text{diff}(XX, z) + (X3 - I \times X4) \times \text{diff}(XX, w); \\
& \text{P110sbs2}(\text{diffsbs}(%)); \\
& \text{end proc}:
\end{align*}
\]

We make use of $D'$ firstly to find the last spin coefficient, $\kappa'$, using [4](4.11.12g).

\[
\begin{align*}
\kappa1 & := -\left( z^{5/4} w^{1/4} r - \frac{z^{11/4} A(u)}{w^{1/4}} \right) Q(x^a) L(x^a)^3 \\
& - \left( -2 \frac{z^{9/4}}{w^{3/4}} + 2 z^{1/4} w^{5/4} \right) r - 3 z^{7/4} w^{3/4} A(u) \right) Q(x^a) L(x^a)^2 + \\
& \left( \frac{dA}{du} z^{1/4} w^{5/4} - 2 z^{5/4} w^{1/4} \right) r - A(u) \left( z^{7/4} w^{3/4} \frac{dA}{du} + 4 z^{3/4} w^{7/4} \right) \right) Q(x^a) L(x^a) \\
& - \left( -2 z^{7/4} w^{3/4} A(u) + w^{7/4} z^{3/4} A(u) \frac{dA}{du} - 2 z^{5/4} w^{1/4} r \frac{dA}{du} \right) Q(x^a)
\end{align*}
\]

Now to look at the equations that involve $D'$. Firstly [3] (6h):

\[
\begin{align*}
eqn6h & := D1(Hm) + \text{conj}(kappa1) \times Hl(u) + \tau \times Hn + (\text{conj}(\gamma) - \gamma) \times Hm - \text{conj}(p00): \\
& \text{P110sbs2}(\text{diffsbs}(%)); \text{factor}(Hlsbs(%)); \\
& 0
\end{align*}
\]

Then we look at [3] (6c):

\[
\begin{align*}
eqn6c & := D1(Hn) + (\gamma + \text{conj}(\gamma)) \times Hn + \kappa1 \times Hm + \text{conj}(\kappa1) \times Hmb: \\
& \text{P110sbs2}(\text{diffsbs}(%)); \\
& \text{factor}(Hlsbs(%)); \\
& 0
\end{align*}
\]

and [3] (8c)

\[
\begin{align*}
eq8c & := \text{diffsbs}(\text{del}(p01) + D1(p11) - \sigma p00 - 2 \times \tau \times p01 - (\rho1 + 2 \times \gamma) \times p11): \\
& \text{factor}(Hlsbs(P110sbs2(%))); \\
& 0
\end{align*}
\]

and [3] (8b)

\[
\begin{align*}
D1(p01) & + \text{del}(p00) - (\tau - 2 \times \beta) \times p00 - 2 \times \rho1 \times p01 - \kappa1 \times p11: \\
& \text{P110sbs2}(\text{diffsbs}(%)); \\
& \text{factor}(Hlsbs(%)); \\
& 0
\end{align*}
\]

and finally [3] (10d).
Next we consider what happens if we have a Killing vector ($\psi = 0$) with $\xi_\ell$ zero ($C = 0$)

\[ \text{subs}(C=0,\psi=0,Hlsbs(P110sbs2(Hm))); \]

\[ 0 \]

\[ \text{subs}(C=0,\psi=0,Hlsbs(P110sbs2(Hn))); \]

\[ 0 \]

Hence we cannot have both $\psi$ and $C$ zero. This is the Collinson and French result: only one Killing vector at most. We have a look at the homothety.

\[ \text{Hl}(u)\ast \text{en} + \text{Hn} \ast \text{ell} - \text{Hm} \ast \text{emb} - \text{conj}(\text{Hm}) \ast \text{em}; \]

\[ \text{map}(\text{diffsbs},\%); \]

\[ \text{map}(\text{P110sbs2},\%); \]

\[ \text{map}(\text{Hlsbs},\%); \]

\[ \text{subs}(z=x+I*y,w=x-I*y,K):K:=\text{map}(\text{expand},\%); \]

\[ K := \begin{pmatrix} \frac{C A(u)}{} \\ -r C \frac{dA}{du} + 2 r \psi \\ 2 \psi x - 2 C \frac{dA}{du} x \\ 2 \psi y - 2 C \frac{dA}{du} y \end{pmatrix} \]

So the obvious Killing vector if $A$ is constant:

\[ \text{KK}:=\text{subs}(\psi=0,C=1/B,\text{diff}(A(u),u)=0,A(u)=B,K):\%; \]

\[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]

The new Killing vector in the other case:

\[ \text{KK2}:=\text{subs}(\psi=0,A(u)=u*B,C=1/B,K):\text{map}(\text{simplify},\%); \]

\[ \begin{pmatrix} u \\ -r \\ -2x \\ -2y \end{pmatrix} \]

And the proper homothety for both cases:

\[ \text{HH}:=\text{subs}(C=0,\psi=1,K); \]
\[
\begin{pmatrix}
0 \\
2r \\
2x \\
2y
\end{pmatrix}
\]

We now turn to the remaining curvature equations and Bianchi identities. To make life easy, we define the weighted derivative operators, [4] section 4.14.

\[
\text{thorn} := X \to \text{diffsbs} (\text{diff}(X,r)):
\]

\[
\text{thorn1} := \text{proc}(X,p,q):
\]

\[
\text{local } i;
\]

\[
\text{D1}(X) - p \cdot \gamma X - q \cdot \text{conj}(\gamma) X;
\]

\[
\text{diffsbs}(\text{expand}(\%));
\]

\[
\text{end proc:}
\]

\[
\text{edth} := \text{proc}(X,p,q):
\]

\[
\text{local } i;
\]

\[
\text{del}(X) - p \cdot \beta X - q \cdot \text{conj}(\alpha) X;
\]

\[
\text{diffsbs}(\text{expand}(\%));
\]

\[
\text{end proc:}
\]

\[
\text{edth1} := \text{proc}(X,p,q):
\]

\[
\text{local } i;
\]

\[
\text{del1}(X) - p \cdot \alpha X - q \cdot \text{conj}(\beta) X;
\]

\[
\text{diffsbs}(\text{expand}(\%));
\]

\[
\text{end proc:}
\]

As a check on the calculations, we can run through the curvature equations, [4] (4.12.32), some of which we've used already, some of which will give us Ψ\(_3\) and Ψ\(_4\). The only ones that do not give zero are (b') and (c'), the first of which gives us Ψ\(_4\):

\[
\text{thorn1(sigma1,-3,1)-edth1(kappa1,-3,-1)-sigma1*(rho1+conj(rho1))}
\]

\[
\text{Psi4:=Hlsbs(diffsbs(\%));}
\]

\[
\Psi_4 := -8 \, L \left( x^a \right)^3 \, z^4 \, Q \left( x^a \right)^2 \, A(u)^2 \\
+ \left( -12 \frac{r z^{7/2} A(u)}{\sqrt{w}} - 32 A(u)^2 z^3 w \right) Q \left( x^a \right)^2 - 8 z^2 Q(x^a) \right) L \left( x^a \right)^2 \\
+ \left( -32 A(u)^2 z^2 w^2 - 8 \left( A(u) \right)^2 z^3 w \frac{dA}{du} - 24 r z^{5/2} \sqrt{w} A(u) \right) Q \left( x^a \right)^2 \\
+ \left( -8 z w - 8 \frac{dA}{du} z^2 \right) Q \left( x^a \right) L \left( x^a \right) - 8 \frac{dA}{du} z w Q \left( x^a \right) \\
+ \left( -12 r A(u) z^{3/2} w^{3/2} - 8 z w^3 A(u)^2 - 8 A(u)^2 z^2 w^2 \frac{dA}{du} A(u) \right) Q(x^a)^2
\]

And (c') gives Ψ\(_3\) (as do several others):
\[
\Psi_3 := 6Q(x^a)^2 L(x^a)^2 A(u)^2 z^{13/4} w^{1/4} \\
+ \left( \left( 14 A(u)^2 z^{9/4} w^{5/4} + 6 \frac{A(u) z^{11/4} r}{w^{1/4}} \right) Q(x^a)^2 + 2 Q(x^a) z^{5/4} w^{1/4} \right) L(x^a) \\
+ \left( 2 A(u)^2 z^{9/4} w^{5/4} \frac{d A}{d u} + 6 z^{5/4} w^{9/4} A(u)^2 + 6 A(u) z^{7/4} w^{3/4} r \right) Q(x^a)^2 \\
+ 2 \frac{d A}{d u} z^{5/4} w^{1/4} Q(x^a)
\]

Now we check the leading terms (in inverse powers of \( r \)) of our \( \Psi_3 \) and \( \Psi_4 \) and compare to [1].

\[
\Psi_3 := \Psi_3 \text{ subs}(L(u,r,z,w)=Lis,Q(u,r,z,w)=Qis,Hlsbs(Psi4)):
\]

\[
T4:=\text{subs}(r=1/R,%):
\]

\[
\text{series}(T4,R=0,3) \text{ assuming R::positive;}
\]

\[
-8 \frac{d A}{d u} z w R^2 + O(R^3)
\]

Here the leading term agrees with [1]. Next \( \Psi_3 \)

\[
\Psi_3 := \Psi_3 \text{ subs}(L(u,r,z,w)=Lis,Q(u,r,z,w)=Qis,Hlsbs(Psi3)):
\]

\[
T3:=\text{subs}(r=1/R,%):
\]

\[
\text{series}(T3,R=0,4) \text{ assuming R::positive;}
\]

\[
2 \frac{d A}{d u} z^{5/4} w^{1/4} R^2 + 8 z^{7/4} w^{3/4} A(u) R^3 + O(R^4)
\]

We find that the leading term agrees with [1], but in the second term the powers of \( z = \zeta \) and \( w = \zeta \) are wrong in [1]. We can also check that the Bianchi identities, [4] (4.12.36-39) are satisfied (and they are).

Finally, we turn to the remaining integrability conditions, [3] (10e) to (10h).

\[
P110sbs2(\text{diffsbs}(\Psi_2*Hmb-\text{diff}(p00,r)-\Psi_3*Hl(u))); \quad # \text{ [3] 10e}
\]

\[
0
\]

\[
\Psi_3*Hm-\Psi_2*Hn-2*\rho_1*p01+2*\beta*p00+\delta(p00): \quad # \text{ [3] 10f}
\]

\[
P110sbs2(\text{diffsbs}(%));
\]

\[
0
\]

\[
\Psi_3*Hmb-\Psi_4*Hl(u)+2*\sigma_1*p01-2*\alpha*p00-\delta1(p00): \quad # \text{ [3] 10g}
\]

\[
P110sbs2(\text{diffsbs}(%));
\]

\[
0
\]

\[
\Psi_4*Hm-\Psi_3*Hn-2*\kappa_1*p01+2*\gamma*p00+D1(p00): \quad ## \text{ [3] 10h}
\]
So we see that all the homothetic and Killing equations are satisfied and we have shown that there is always a Killing vector in these metrics and also always a homothety.

4 The Calculations (limit cylindrical case)

Since neither [1] not [2] give the spin coefficients for the limit cylindrical metric, we will need to calculate them using Maple’s tensor package. Note that we use the corrected version of this metric, see [5] equation (26.23)

```maple
> with(tensor):
> coord:=[u,r,x,y]:g_c:=array(1..4,1..4,symmetric,sparse):
> g_c[1,1]:=-expand(simplify((b+log(r^2*x^4))/x^2/2)
> assuming r::positive,x::positive);
> g_c[1,2]:=1:g_c[3,3]:=-2*r^2:g_c[4,4]:=-2*x^2:g_c[1,3]:=2*r/x:
> g:=create([-1,-1],eval(g_c)):
```

Next we calculate all the relevent tensors.

```maple
> tensorsGR(coord,g,gup,'detg', 'C1','C2','Rm','Rc', 'R','G','C');
> read 'PRcoeff':
> md:=create([-1],vector([0,0,r,-I*x])):mup:=raise(gup,md,1):
> mbd:=create([-1],vector([0,0,r,I*x])):mbup:=raise(gup,mbd,1):
> ld:=create([-1],vector([1,0,0,0])):lup:=raise(gup,ld,1):
> nd:=create([-1],vector([g_c[1,1]/2,1,2*r/x,0])):nup:=raise(gup,nd,1):
> his:=linalg[stackmatrix](ld[compts],nd[compts],md[compts],mbd[compts]):
> h:=create([1,-1],op(his)):
> spins:=PRspin(g,h,C2,coord):
> crv:=PRcurve(g,h,C,Rc,coord);
```
From these two calculations we find that the non-zero spin coefficients are

\[
\tau = \beta = \tau' = -\frac{1}{2rx}, \quad \rho = \sigma = -\frac{1}{2r}, \quad \gamma = -\frac{1}{4rx^2}, \quad \rho' = \sigma' = \frac{b + \log(r^2x^4)}{8rx^2};
\]

and the non-zero curvature components are

\[
\Psi_1 = \frac{1}{2r^2x}; \quad \Psi_2 = \frac{1}{2r^2x^2}; \quad \Psi_3 = \frac{b + \log(r^2x^4)}{8rx^2}.
\]

Now we define the derivative operators \(D, D', \delta\) and \(\delta'\):

\[
\begin{align*}
D &:= XX \to \text{add}(\text{compts}[i] \cdot \text{diff}(XX, \text{coord}[i]), i = 1..4): \\
D1 &:= XX \to \text{add}(\text{compts}[i] \cdot \text{diff}(XX, \text{coord}[i]), i = 1..4): \\
\delta &:= XX \to \text{add}(\text{compts}[i] \cdot \text{diff}(XX, \text{coord}[i]), i = 1..4): \\
\delta1 &:= XX \to \text{add}(\text{compts}[i] \cdot \text{diff}(XX, \text{coord}[i]), i = 1..4):
\end{align*}
\]

Now to find the Killing vectors. Using [3] (6a), (6c) and (11) gives \(\xi_\ell = \xi_\ell(u)\). Then from [3] (10a) we get \(\phi_{11}\), and find that \(\phi_{01}(u, x, y)\), the integration constant, is real by [3] (6g), which also gives \(\xi_m\). So

\[
\begin{align*}
\text{Hm} &:= -r \cdot p110(u, x, y) + i \cdot \text{Hm0}(u, x, y): \\
\text{Hmb} &:= -r \cdot p110(u, x, y) - i \cdot \text{Hm0}(u, x, y): \\
p11 &:= \frac{Hl(u)}{2x} + p110(u, x, y); \ # \text{note that p110 is real}
\end{align*}
\]

We also solve [3] (10b) for \(\phi_{01}\):

\[
\begin{align*}
\text{crv}[\Psi1] \cdot \text{Hm} - 2 \cdot \text{spins}[\sigma] \cdot p01 - (\text{spins}[\beta] - \text{spins}[\alpha]) \cdot p11 + \text{del}(p11):
\end{align*}
\]

Now looking at [3] (8a), using the fact that \(\kappa = 0\):

\[
\begin{align*}
\text{diff}(p01, r) + \text{del1}(p11) - 2 \cdot \text{spins}[\rho] \cdot p01 -
\end{align*}
\]

So \(\phi_{11}\) is independent of \(y\). Now looking at [3] (6b):

\[
\begin{align*}
D1(Hl(u)) + 2 \cdot \text{spins}[\epsilon1] \cdot Hl(u) + \text{spins}[\tau1] \cdot \text{Hm} + \text{spins}[\tau1] \cdot \text{Hmb} + p01
\end{align*}
\]

\[
\begin{align*}
\text{collect}(%, r);
\end{align*}
\]
\[
\frac{d}{du} \mathcal{H}(u) + \frac{\partial}{\partial x} p_{110}(u, x, y) - \psi
\]

So we solve this for \(\phi_{11}^0\), recalling that \(\phi_{11}^0\) is independent of \(y\), and use it to redefine \(\phi_{11}, \phi_{01}\) and \(\xi_m\).

\[
p_{11} := \frac{\mathcal{H}(u)}{2/x/r} + (\psi - \text{diff}(\mathcal{H}(u), u)) \times + p_0(u): \quad \text{# note that } p_0 \text{ is real}
\]

\[
p_{01} := \text{expand}\left(\text{subs}\left(p_{110}(u, x, y) = (\psi - \text{diff}(\mathcal{H}(u), u)) \times + p_0(u), p_{01}\right)\right):
\]

\[
\mathcal{H}_m := \text{expand}\left(\text{subs}\left(p_{110}(u, x, y) = (\psi - \text{diff}(\mathcal{H}(u), u)) \times + p_0(u), \mathcal{H}_m\right)\right);
\]

Turning to \([3] \ (10c)\)

\[
\text{crv}[\Psi_2] \ast \mathcal{H}(u) - \text{crv}[\Psi_1] \ast \mathcal{H}_m - 2 \ast \text{spins}[\rho] \ast p_{01} - (\text{spins}[\alpha] - \text{spins}[\beta_1]) \ast p_{11} + \text{del1}(p_{11})
\]

\[
\text{expand}(\%);
\]

\[
\text{0}
\]

Now the right hand side of \([3] \ (6d)\) is

\[
-2 \ast \text{spins}[\epsilon] \ast \mathcal{H}_n - \text{spins}[\tau_1] \ast \mathcal{H}_m - \text{spins}[\tau_1] \ast \mathcal{H}_mb + p_{01} + \text{subs}(I=-I, p_{01}) + \psi:
\]

\[
\text{expand}(\%);
\]

\[
\frac{d}{du} \mathcal{H}(u) - 2 \frac{p_0(u)}{x} - \frac{\mathcal{H}(u)}{2x^2 r}
\]

This is \(D\xi_n\), so we integrate

\[
\int(\%);\]

\[
r \frac{d}{du} \mathcal{H}(u) - 2 \frac{r}{x} p_0(u) - \frac{\mathcal{H}(u) \ln(r)}{2x^2}
\]

\[
\mathcal{H}_n := \% + \mathcal{H}_n u(x, y):
\]

Turning to \([3] \ (6i)\),

\[
\text{del}(\mathcal{H}_m) + \text{spins}[\sigma_1] \ast \mathcal{H}(u) + \text{spins}[\sigma] \ast \mathcal{H}_n + (\text{spins}[\alpha_1] + \text{spins}[\alpha]) \ast \mathcal{H}_m:
\]

The coefficients of \(r\) are independent, so we collect the terms.

\[
\text{collect}(\%);\]

\[
-\psi + \frac{1}{2} \frac{d}{du} \mathcal{H}(u) - \frac{p_0(u)}{2x} - \frac{1}{2x} \frac{\partial}{\partial y} \mathcal{H}_m u(u, x, y)
\]

\[
- \left(\frac{1}{2} \frac{\partial}{\partial x} \mathcal{H}_m u(u, x, y) + \frac{\mathcal{H}(u) b}{8x^2} + \frac{\mathcal{H}(u) \ln(x)}{2x^2} - \frac{\mathcal{i} \mathcal{H}_m u(u, x, y)}{2x} + \frac{1}{2} \mathcal{H}_n u(u, x, y)\right)r^{-1}
\]

The imaginary part of the \(r^{-1}\) term implies \(\xi_0^0 = xf(u, y)\), for some function \(f(u, y)\), so:
X := collect(subs(Hm0(u,x,y)=x*f(u,y), %), x):

Y := solve(op(5, X), Hn0(u,x,y));

\[
Y := -Hl(u) \frac{b + 4 \ln(x)}{4x^2}
\]

expand(subs(Hn0(u,x,y)=Y,X));

\[
-\psi + \frac{1}{2} dHl(u) - \frac{p0(u)}{2x} - \frac{1}{2} \frac{\partial f(u,y)}{\partial y}
\]

XX := rhs(dsolve(%, f(u,y)))*x;

\[
XX := \left(-2y\psi + y \frac{dHl(u)}{du} - \frac{yp0(u)}{x} + F1(u)\right)x
\]

Where \(F1\) is an arbitrary function. This \(XX\) is \(\xi^0_m\). So

\[
Hm := \left(-\psi + \frac{d}{du}Hl(u)\right)x - p0(u) + \left[i\left(-2\psi + \frac{d}{du}Hl(u)\right)y + iF1(u)\right]x - iy p0(u)
\]

\[
Hn := \left(\frac{d}{du}Hl(u)\right)x - 2 \frac{p0(u)}{x} - \frac{Hl(u)}{4x^2} \left[2 \ln(r) + b + 4 \ln(x)\right]
\]

p01 := expand(subs(Hm0(u,x,y)=XX,p01));

\[
p01 := \frac{p0(u)}{2x} + \frac{i\psi}{r} - \frac{iy}{2r} \frac{d}{du}Hl(u) + \frac{iy}{2r} p0(u) - \frac{i}{2r} F1(u) - \frac{Hl(u)}{4x^2r}
\]

Returning to the integrability conditions, we look at [3] (10d)

\[
eq 10d := crv[\Psi2]*Hm - crv[\Psi1]*Hn - 2*spins[\tau]*p01 - 2*spins[\gamma]*p11 + D1(p11):
\]

\[
\text{expand(\%);}\]

\[
\frac{p0(u)}{2x} + \frac{d}{du}p0(u) - x \frac{d^2}{du^2}Hl(u)
\]

So by comparing coefficients we have

\[
p0(u) := 0; Hl := x \mapsto k0*x + k1
\]
And a quick check shows that eqn10d (3 (10d)) is satisfied. Next, the conjugate of [3] (6h) will give us $\phi_{00}$.

\[ D1(Hmb)+\text{spins}[\tau]*Hn+(\text{spins}[\gamma]+\text{spins}[\epsilon1])*Hmb: \]

\[ p00:=\text{collect}(\text{expand}(\%),[\psi,k1,k0]); \]

\[ p00 := \left( -\frac{b}{4x} - \frac{\ln(r)}{2x} - \frac{\ln(x)}{x} \right) \psi + \left( \frac{\ln(r)}{4rx^3} + \frac{\ln(x)}{2rx^3} + \frac{b}{8rx^3} \right) k1 + \left( \frac{ub}{8rx^3} + \frac{b}{4x} - \frac{1}{2x} + \frac{\ln(r)}{2x} + \frac{\ln(x)}{x} + \frac{u \ln(x)}{2rx^3} + \frac{\ln(r)u}{4rx^3} \right) k0 - i \left( \frac{d}{du} F1(u) \right) x \]

We next check some further integrability conditions, [3] (10e) first.

\[ \text{crv}[\Psi3]*Hl(u)-\text{crv}[\Psi2]*Hmb-2*\text{spins}[\tau1]*p01+2*\text{spins}[\epsilon]*p00 + \text{diff}(p00,r): \]

\[ \text{expand}(\%); \]

\[ 0 \]

And then [3] (8d).

\[ e8d:=\text{diff}(p00,r)+\text{del1}(p01)-\text{spins}[\rho]*p00-2*\text{spins}[\tau1]*p01-\text{spins}[\sigma1]*p11: \]

\[ \text{expand}(\%); \]

\[ -i \frac{x}{2r} \left( \frac{d}{du} F1(u) \right) \]

So the integrability function $\_F1$ is constant:

\[ \_F1(u) := k3; \text{expand}(e8d); \]

\[ \_F1(u) := k3 \]

0

Also [3] (6e) is

\[ \text{eqn6e} := \text{expand}(D1(Hn)+2*\text{spins}[\gamma]*Hn); \]

\[ \text{eqn6e} := -\frac{k0}{2x^2} \]

Thus $k0 = 0$, and the coefficients simplify as follows:

\[ \text{Hl}(u);\text{Hn};\text{Hm}; \]

\[ \frac{k1}{2x^2} - \frac{\ln(r)k1}{4x^2} - \frac{k1 (b + 4 \ln(x))}{4x^2} - r x \psi + (-2 i y \psi + i k3) x \]
\( -\frac{b}{4x} - \frac{\ln(r)}{2x} - \frac{\ln(x)}{x} \) \( \psi \) + \( \frac{\ln(r)}{rx^3} + \frac{\ln(x)}{2rx^3} + \frac{b}{8rx^3} \) \( k1 \)

\[
\begin{align*}
\frac{iy\psi}{r} - \frac{ik3}{2r} - \frac{k1}{4x^2r} \\
\frac{k1}{2rx} + x\psi
\end{align*}
\]

All the remaining homothetic equations and integrability equations are satisfied, and we are left with the general homothetic vector:

\[
\text{lin_com}(H1(u), nup, Hn, lup, -Hm, mbup, -Hmb, mup);
\]

\[
\text{TABLE ([index\_char = [1], compts = vector ([k1, 2 r\psi, -x\psi, 2 y\psi - k3])])}
\]

That is,

\[
k1\partial_u + k3\partial_y + \psi (2r\partial_r - x\partial_x + 2y\partial_y).
\]

5 Acknowledgments

Maple is a registered trademark of Waterloo Maple Inc.

6 References

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