ON LINEAR STABILITY OF KAM TORI VIA THE CRAIG-WAYNE-BOURGAIN METHOD

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Abstract. In this paper, we prove the Melnikov’s persistency theorem by combining the traditional Kolmogorov-Arnold-Moser (KAM) technique and the Craig-Wayne-Bourgain (CWB) method. The aim of this paper is twofold. One is to establish the linear stability of the perturbed invariant tori by using the CWB method without the second Melnikov condition. The other one is to illustrate the CWB method in detail and make the CWB method more accessible.

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1. INTRODUCTION

1.1. Background. The celebrated Kolmogorov-Arnold-Moser (KAM) theory concerns with the stable motions in nearly integrable Hamiltonian system. For a smooth Hamiltonian of $n$-degree

$$H = H_0(I) + \varepsilon R(\theta, I)$$

Date: March 4, 2020.

Key words and phrases. Melnikov’s problem, Craig-Wayne-Bourgain method, KAM, linear stability.
with the standard symplectic structure $d\theta \wedge dI$ on $T^n \times \mathbb{R}^n$ and the angle-action variable $(\theta, I)$ belongs to some domain in $T^n \times D \subset T^n \times \mathbb{R}^n$. Assume the unperturbed Hamiltonian $H_0(I)$ is independent of $\theta$ and satisfies the Kolmogorov non-degenerate condition

$$\det(\partial^2 H_0/\partial I^2) \neq 0, \quad I \in D.$$ 

Then the invariant torus $T_0 = \{\omega(I_0) t : t \in \mathbb{R}\} \times \{I_0\}$ with a prescribed Diophantine frequency $\omega(I_0) = \partial H(I_0)/\partial I$ for some $I_0 \in D$ persists under sufficiently small perturbation $\epsilon R(\theta, I)$. This is the well-known KAM theorem ([10, 1, 13]) for the finitely dimensional Hamiltonian system.

It is worthy mentioning that the dimension of the persisting torus equals to the degree of the Hamiltonian system. To explore the existence of those invariant tori whose dimensions are less than the degree of the Hamiltonian, consider the following Hamiltonian

$$E = \langle \omega, y \rangle + \sum_{j=1}^{n} \Omega_j \bar{z}_j \bar{z}_j$$

defined on the phase space $(x, y, z, \bar{z}) \in T^d \times \mathbb{R}^d \times \mathbb{C}^n \times \mathbb{C}^n$ with the symplectic structure $dx \wedge dy + \sqrt{-1} dz \wedge d\bar{z}$. Obviously, one finds that $T_0^d = \{\omega t : t \in \mathbb{R}\} \times \{y = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ is an invariant torus of the Hamiltonian vector field $X_E$. Moreover, the dimension of $T_0^d$ is less than the degree $d+n$ of the Hamiltonian $E$. In 1965, Melnikov [12] announced that, under suitable non-resonant conditions, the lower dimensional invariant tori can persist under sufficiently small Hamiltonian perturbation $\epsilon R(x, y, z, \bar{z})$. In the late 1980’s, Eliasson [9], Pöschel [14], Kuksin [11] provided a complete proof of the problem, well-known nowadays as Melnikov’s persistency theorem.

We briefly explain the main idea of the proof in [9, 14, 11]. Roughly speaking, we expand the perturbation $R$ into Taylor series in $(y, z, \bar{z})$

$$R = R^i(x) + \langle R^i(x), y \rangle + \langle R^i(x), z \rangle + \langle R^i(x), \bar{z} \rangle + \langle R^{iz}(x)z, z \rangle + \langle R^{iz}(x)z, \bar{z} \rangle + \langle R^{iz}(x)\bar{z}, \bar{z} \rangle + O(|y|^2 + |y| \cdot |z| + |z|^3),$$

Then we apply the symplectic transformation to eliminate the items $(I), (II)$ and $(III)$ of lower order. Consequently, $T_0^d$ is an invariant torus of the Hamiltonian $H = E + O(|y|^2 + |y| \cdot |z| + |z|^3)$.

To eliminate the item $(I)$, we need the usual Diophantine condition

$$\langle k, \omega \rangle \neq 0 \quad \text{for all } 0 \neq k \in \mathbb{Z}^d.$$ 

To eliminate the item $(II)$, we need the first Melnikov condition

$$\Omega_j \pm \langle k, \omega \rangle \neq 0, \quad \text{for all } k \in \mathbb{Z}^d, 1 \leq j \leq n.$$ 

To eliminate the expressions $\langle R^{iz}(x)z, z \rangle$ and $\langle R^{iz}(x)z, \bar{z} \rangle$ in $(III)$, we need the second Melnikov condition

$$\Omega_i + \Omega_j \pm \langle k, \omega \rangle \neq 0, \quad \text{for all } k \in \mathbb{Z}^d, 1 \leq i, j \leq n.$$ 

To eliminate $\langle R^{iz}z, \bar{z} \rangle$ in $(III)$, we still need the second Melnikov condition but in the following form

$$\Omega_i - \Omega_j \pm \langle k, \omega \rangle \neq 0, \quad \text{for all } |i - j| + |k| \neq 0, k \in \mathbb{Z}^d, 1 \leq i, j \leq n.$$ 

We see from (1.3) that when $k = 0$ there is $\Omega_i \neq \Omega_j$ for any $i \neq j$, i.e., the multiplicity of the norm frequency should be one, which excludes lots of important applications.
In 1997, Bourgain [4] improved Craig-Wayne’s method [8] to study the Melnikov’s problem, which is completely free from the second Melnikov condition and also applies to infinitely dimensional Hamiltonian system [4, 5]. In his famous book [6] published in 2005, Bourgain developed further the method to prove the existence of invariant torus (or quasi-periodic solution) for NLS and NLW of arbitrary dimension. This method now is known as the Craig-Wayne-Bourgain (CWB) method. The CWB method is less dependent on the Hamiltonian structure. It is essentially based on applying the Newton iteration to solve directly the differential equation for quasi-periodic solutions. However, one has to pay for the price that the homological equation (or the linearized equation) not only has small divisor problem but also contains variable coefficients. Moreover, we are not able to obtain a local norm form around the persisted invariant torus.

Back to the Melnikov’s problem in [4], Bourgain combined the above CWB method with the KAM technique. Taking Taylor expansion of the perturbation and applying the symplectic transformation as before, Bourgain put (II) into unperturbed Hamiltonian $E$ and eliminated sorely (I) and (II), which results in a norm form around the invariant torus $H_{\infty} = E_{\infty} + O(|y|^2 + |y| \cdot |z| + |z|^3)$

with

$$E_{\infty} = \langle \omega, y \rangle + \sum_{j=1}^{n} \Omega_j z_j \bar{z}_j + (III).$$

Obviously, $T^d_0$ is still an invariant torus of $H_{\infty}$. The important thing is that since (III) has been putted into $E_{\infty}$, it avoids completely the usage of the second Melnikov condition. However, to derive such a normal form $H_{\infty}$, we have to solve homological equations with variable coefficients. Moreover, the linear stability of the persisted torus is unknown.

Note that we can actually divide the second Melnikov conditions into two parts (1.2) and (1.3) with (1.2) containing terms of the form $\Omega_i + \Omega_j$. Apparently, part (1.2) has essentially the form of the first Melnikov condition. The true difficulty arises from part (1.3) which involves the terms $\Omega_i - \Omega_j$. Thus we can eliminate terms associated with part (1.2), and put terms corresponding to part (1.3) into the new normal form. As a result, we may obtain a more precise normal form $H'_{\infty} = E'_{\infty} + O(|y|^2 + |y| \cdot |z| + |z|^3)$

with

$$E'_{\infty} = \langle \omega, y \rangle + \sum_{j} \Omega_j z_j \bar{z}_j + \langle R^\bar{z}(x)z, \bar{z} \rangle.$$ 

In particular, $T^d_0 = \{ \omega t : t \in \mathbb{R} \} \times \{ 0 \} \times \{ 0 \} \times \{ 0 \}$ is also an invariant torus of $H'_{\infty}$. Furthermore, the corresponding linearized equation

$$\sqrt{-1} \dot{z} = \Lambda z + R^\bar{z}(x)z, \quad \Lambda = \text{diag}(\Omega_j)$$

admits a $L^2$-conservation law, which implies particularly the linear stability of persisted torus (see Theorem 1.1 and Corollary 1.1 in the following for details).

The aim of this paper is twofold. One is to study the Melnikov’s problem by combining the CWB method and the KAM technique. We show that the linear stability still holds without the second Melnikov condition (1.3). The other one is to explain the CWB method in detail and to make the CWB method more accessible.
Remark 1.1. An alternative method is to put \(|R^{2}(x)|z, \bar{z}\rangle\) into \(E\), where \(|R^{2}(x)| = \int R^{2}(x)dx\). This method has the advantage that the homological equations are of constant coefficients type. The disadvantage is that the second Melnikov conditions are still employed, which seems not applicable to higher spatial dimensional NLS and NLW in infinitely dimensional systems case. This method can be found in an early monograph [2] published in 1969 in Russian. See also [16].

1.2. Main result. Let us recall some basic concepts in the Hamiltonian dynamical systems. Consider a Hamiltonian function \(H = H(x, y, z, \bar{z})\) defined on the phase space \(\mathcal{P} = T^{d} \times \mathbb{R}^{d} \times \mathbb{C}^{n} \times \mathbb{C}^{n}\) with \(T^{d} = \mathbb{R}^{d}/(2\pi\mathbb{Z})^{d}\). We endow the symplectic form

\[
\omega = dx \wedge dy + \sqrt{-1}dz \wedge d\bar{z} = \sum_{j=1}^{d} dx_{j} \wedge dy_{j} + \sqrt{-1} \sum_{k=1}^{n} dz_{k} \wedge d\bar{z}_{k}.
\]

Then the vector field \(X_{H}\) given by \(X_{H} \omega = -dH\) reads

\[
X_{H} = (\partial_{y}H, -\partial_{x}H, \sqrt{-1}\partial_{z}H, -\sqrt{-1}\partial_{\bar{z}}H)^{T}.
\]

The associated Poisson bracket takes the form of

\[
\{F, G\} = \langle F_{x}, G_{y} \rangle - \langle F_{y}, G_{x} \rangle + \sqrt{-1}\langle F_{z}, G_{\bar{z}} \rangle - \sqrt{-1}\langle F_{\bar{z}}, G_{z} \rangle.
\]

Given a function \(F\), the time-1-map of the flow \(X_{E}^{t}\) of the Hamiltonian vector field \(X_{F}\) is symplectic. Moreover,

\[
\frac{d}{dt} G \circ X_{E}^{t} = \{G, F\} \circ X_{F}^{t}.
\]

In this paper, we consider small perturbation of a finite dimensional Hamiltonian in the parameter dependent normal form

\[
E_{0} = \langle \omega_{0}(\xi), y \rangle + \langle \Omega z, \bar{z} \rangle, \quad (x, y, z, \bar{z}) \in T^{d} \times \mathbb{R}^{d} \times \mathbb{C}^{n} \times \mathbb{C}^{n},
\]

where \(\Omega = \text{diag}(\Omega_{j} : 1 \leq j \leq n)\) with \(\Omega_{j} > 0\). The tangent frequency \(\omega_{0}\) depends on \(d\) parameters \(\xi \in \Pi_{0} \subset \mathbb{R}^{d}\), where \(\Pi_{0}\) is a given open set. The associated Hamiltonian vector field \(X_{E_{0}}\) of the normal form \(E_{0}\) is given by

\[
X_{E_{0}} = (\omega_{0}(\xi), 0, \sqrt{-1}\Omega z, -\sqrt{-1}\Omega \bar{z})^{T},
\]

where \((\cdot)^{T}\) represents the transpose of a matrix (or a vector). Obviously, for each \(\xi \in \Pi_{0}\), there is a \(d\)-dimensional invariant torus

\[
\mathcal{T}_{0}^{d} = T^{d} \times \{y = 0\} \times \{z = 0\} \times \{\bar{z} = 0\},
\]

carrying a quasi-periodic flow \(x = \omega_{0}(t) + x_{0}\) with fixed torus frequency \(\omega_{0} = \omega_{0}(\xi)\).

The Melnikov’s problem is to study the persistence of \(\mathcal{T}_{0}^{d}\) under sufficiently small perturbation of the Hamiltonian. We consider perturbation

\[
H = E_{0} + P_{0}
\]

of \(E_{0}\) that are real analytic\(^4\) on some complex neighborhood

\[
(1.4) \quad \mathcal{D}(s, r) : \quad |\text{Im}x|_{\omega} < s, \quad |y| < r^{2}, \quad |z| < r, \quad |\bar{z}| < r
\]

\(^4\)The real analyticity of \(H\) means that \(H\) is analytic on the complex domain \(\mathcal{D}(s, r)\), and takes real value when \(x, y\) are real and \(z, \bar{z}\) are complex conjugated.
of $\mathcal{T}_0^d$ in the complex space $\mathcal{P}_C = (\mathbb{C}^d/2\pi\mathbb{Z}^d) \times \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d$, where $| \cdot |_{\infty}$ denotes the supremum norm and $| \cdot |$ denotes the Euclidean norm. It should be pointed out that $z$ and $\bar{z}$ are independent variables. We also introduce
\[
\mathcal{D}_x(s, r) = \{(x, y, z, \bar{z}) \in \mathcal{D}(s, r) : x, y \in \mathbb{R}^d\},
\]
in which $x, y$ are real but $z$ and $\bar{z}$ stay in the complex space and are complex conjugated.

For $r > 0$, we define the weighted phase norm
\[
|W|_r = |X| + \frac{1}{r^2}|Y| + \frac{1}{r}|Z| + \frac{1}{r}|\bar{Z}|,
\]
for $W = (X, Y, Z, \bar{Z}) \in \mathcal{P}_C$.

For a map $W : \mathcal{D}(s, r) \times \mathcal{O} \to \mathcal{P}_C$, define
\[
|W|_{\mathcal{D}(s, r) \times \mathcal{O}} = \sup_{(u, \xi) \in \mathcal{D}(s, r) \times \mathcal{O}} |W(u, \xi)|
\]
and
\[
|W|_{\mathcal{O} \times \mathcal{D}(s, r)} = \sup_{(u, \xi) \in \mathcal{O} \times \mathcal{D}(s, r)} |\partial_\xi W(u, \xi)|,
\]
where $\partial_\xi$ is the derivative with respect to $\xi$ and $\mathcal{O} \subset \mathbb{C}^d$ is an open set.

Now we state the basic assumptions on the Melnikov’s problem.

**Assumption A** (Analyticity w.r.t. parameters). Assume that $\omega_0$ is real analytic in $\xi$ on $\mathcal{O}_0 \subset \mathbb{C}^d$, where $\mathcal{O}_0 = \mathcal{O}(\Pi_0, \rho_0) = \{z \in \mathbb{C}^d : |z - \xi| < \rho_0 \text{ for some } \xi \in \Pi_0\}$ and $\Pi_0 \subset \mathbb{R}^d$ is an open interval. When saying an open interval in $\mathbb{R}^d$, we always mean any open set of the form $\{(\xi_1, \ldots, \xi_d) : a_j < \xi_j < b_j, 1 \leq j \leq d\}$.

**Assumption B** (Non-degeneracy). There is some absolute constant $C > 0$ such that
\[
\sup_{\xi \in \mathcal{O}_0} |\partial_\xi \omega| < C, \quad \sup_{\xi \in \mathcal{O}_0} |\partial_\xi \omega^{-1}| < C.
\]

**Assumption C** (Regularity). Let $s_0, r_0$ be positive constants. Assume the perturbation $P_0(x, y, z, \bar{z} ; \xi)$ is real analytic in $(x, y, z, \bar{z})$ on the domain $\mathcal{D}(s_0, r_0)$. For each $\xi \in \mathcal{O}_0$, the Hamiltonian vector field
\[
X_{P_0} = (\partial_x P_0, -\partial_y P_0, \sqrt{-1} \partial_z P_0, -\sqrt{-1} \partial_{\bar{z}} P_0)^T
\]
defines near $\mathcal{T}_0^d$ an analytic map
\[
X_{P_0} : \mathcal{D}(s_0, r_0) \subset \mathcal{P}_C \to \mathcal{P}_C.
\]
Also assume that $X_{P_0}$ is real analytic in $\xi \in \mathcal{O}_0$.

**Assumption D** (Reality). For any $(x, y, z, \bar{z}, \xi) \in \mathcal{D}_x(s_0, r_0) \times \Pi_0$, the perturbation $P_0$ satisfies the reality condition, i.e.,
\[
\overline{P_0(x, y, z, \bar{z}, \xi)} = P_0(x, y, z, \bar{z}, \xi),
\]
where the overline denotes the complex conjugate.

\footnote{We say a function is real analytic on some domain in $\mathbb{C}^d$ when it is analytic on that domain and is real for real arguments.}
Theorem 1.1. Suppose \( H = E_0 + P_0 \) satisfies Assumptions A-D and assume the smallness condition
\[
r_0 |X_{P_0}|_{\mathcal{P}(\lambda_0, r_0)\times \mathcal{E}_0} < \varepsilon, \quad r_0 |X_{P_0}|_{\mathcal{P}(\lambda_0, r_0)\times \mathcal{E}_0} < \varepsilon^{1/3}.
\]
Then there is a sufficiently small \( \varepsilon_\ast = \varepsilon_\ast(n, d, r_0, s_0, \rho_0, \Pi_0) > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_\ast \), there is a subset \( \Pi_\infty \subset \Pi_0 \) with
\[
\text{mes} (\Pi_\infty) \geq (1 - O(\varepsilon^{1/2})) \text{mes} (\Pi_0),
\]
and there are a family of embedding \( \Phi : \mathbb{T}^d \times \Pi_\infty \to \mathcal{P} \), a map \( \omega_\ast : \Pi_\infty \to \mathbb{R}^d \) and a matrix function \( B^{\omega_\ast} : \mathbb{T}^d \times \Pi_\infty \to \mathbb{R}^{n\times n} \) such that for each \( \xi \in \Pi_\infty \), the transformation \( \Phi \) and the matrix \( B^{\omega_\ast} \) are real analytic on \( \mathbb{T}_{\varepsilon_\ast/2}^d \) giving rise to
\[
H \circ \Phi|_{\mathbb{T}_{\varepsilon_\ast/2}(\xi)} = \langle \omega_\ast(\xi), y \rangle + \langle \Omega z, \bar{z} \rangle + \langle B^{\omega_\ast}(x; \xi) z, \bar{z} \rangle + O(||y||_1 + ||z||_1^2 + ||\bar{z}||_1^3).
\]

From Theorem 1.1, one readily see that, for each \( \xi \in \Pi_\infty \), the vector \( \Phi|_{\mathbb{T}_{\varepsilon_\ast/2}(\xi)} \) is an analytic embedding of rotational torus with frequency \( \omega_\ast(\xi) \) for the Hamiltonian \( H \) at \( \xi \). Moreover, following the analysis of (1.5), we further obtain the linear stability of the invariant torus.

Corollary 1.1. Under the assumptions of Theorem 1.1, the perturbed invariant tori are linearly stable in the sense that the associated Lyapunov exponent is zero.

Proof. For each \( \xi \in \Pi_\infty \), we consider the Hamiltonian vector field induced by the Hamiltonian \( H \circ \Phi \). We immediately find that
\[
\mathcal{T}_0^d = \mathbb{T}^d \times \{y = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}
\]
is a \( d \)-dimensional invariant torus of the vector field \( X_{H \circ \Phi} \). Then the linearized equation around \( \mathcal{T}_0^d \) is
\[
\begin{cases}
\dot{x} = \omega, \\
\dot{y} = 0, \\
\dot{z} = \sqrt{-1}(\Omega + B^{\omega_\ast}(x)) z, \\
\dot{\bar{z}} = -\sqrt{-1}(\Omega + B^{\omega_\ast}(x)) \bar{z}.
\end{cases}
\]  
(1.5)

Along the trajectory \( z = z(t) \) of (1.5), we have the \( L^2 \)-conservation, i.e.,
\[
\frac{d}{dt} |z|^2 = \frac{d}{dt} \langle z, \bar{z} \rangle = 0,
\]
which implies \( (z, \bar{z}) = 0 \) is a center equilibrium in (1.5). This proves the linear stability of the perturbed invariant torus. \( \square \)

2. The KAM Iterative Lemma

In this section, we establish the KAM Iterative Lemma, upon which our main Theorem 1.1 is an immediate result. To begin with, we summarize the notations and the iterative constants in subsection 2.1 for reader’s quick reference. Next we present and prove the KAM Iterative Lemma in subsection 2.2 and subsection 2.3 respectively. In subsection 2.4 we prove our main Theorem 1.1.
2.1. Notations and the iterative constants. We first introduce some general notations. For two vectors $a, b$ in $\mathbb{R}^d$ or $\mathbb{C}^n$, we denote $(a, b) = \sum_j a_j b_j$. We use the notation $A \setminus B$ for the set theoretical difference. For $k \in \mathbb{Z}^d$ and $U \subset \mathbb{Z}^d$, $k + U$ denotes the set $\{k' = k + p : p \in U\}$. The symbols $\wedge$ and $\vee$ describes the minimal and maximal operators, respectively. The measure of a set $\mathcal{V} \subset \mathbb{R}^d$, denoted by $\text{mes}(\mathcal{V})$, always refers to the Lebesgue measure. By some abuse of notation, we denote by $|\mathcal{V}|$ the diameter of a set $\mathcal{V} \subset \mathbb{R}^d$.

Following the notations in KAM theory, we denote in the sequel various constants by the same letter $C$. Of course, these numbers depend only on the universal constants $d, n, \rho_0, r_0, s_0, \Pi_0$ and could be made explicit by the context where they arise, but need not be. For further simplicity, we write $a \leq b$ in estimates to suppress the multiplicative constant in $Ca < b$. The notation $a \ll b$ indicates $Ca < b$ for sufficiently large $C > 0$ and $a \sim b$ means both $a \ll b$ and $b \ll a$ hold. Furthermore, $e^{-1}$ means $e^{-\delta}$ with some small $\delta > 0$ (the precise meaning of "small" can again be derived from the context), in which the exponent "1−" might be different from line to line.

If not specified, the norm for vectors in real or complex space refers to the Euclidean norm. The norm of a matrix is the induced operator norm on the vectors. For a Fourier series $q(x) = \sum_{k \in \mathbb{Z}^d} \hat{q}(k)e^{\sqrt{-1}(k,x)}$, we define the truncation operator $\Gamma_N$ by

$$ (\Gamma_N q)(x) = \sum_{k \in \mathbb{Z}^d, |k| \leq N} \hat{q}(k)e^{\sqrt{-1}(k,x)}. $$

Next, we define the following iterative constants and domains:

- $s_0 > 0$ and $r_0 > 0$ are fixed and given in Assumption C;
- $A = A(n, d) > 0$ is sufficiently large;
- $l \in \mathbb{N}$ is the number of the KAM iterative steps;
- $e_l = A^{-(\frac{l}{2})^j}$ measures the size of the perturbation at the $l^{th}$ step;
- $l = \frac{2 + (\frac{1}{2} + \frac{1}{2} + \cdots)}{1 + 1 + 1 + \cdots}$ (so $0 < e_l < \frac{1}{4}$ for all $l$);
- $s_l = s_0(1 - e_l)$ (so $s_l > \frac{1}{2}s_0$ for all $l$), which measures the width of the analyticity strip for the angle variable $x$ at the $l^{th}$ step;
- $r_l = r_0(1 - e_l)$ (so $r_l > \frac{1}{2}r_0$ for all $l$), which measure the analyticity radius for the action variable $y$, as well as the normal coordinates $z, \bar{z}$, at the $l^{th}$ step;
- $s_{l+i}^{(j)} = (1 - \frac{1 + j}{100})s_l + \frac{j}{100}s_{l+1}$ ($j = 0, \cdots, 100$) are the intermediate points between $s_l$ and $s_{l+1}$ dividing $[s_l, s_{l+1}]$ into $100$ subintervals with the same length;
- $r_{l+i}^{(j)} = (1 - \frac{1 + j}{100})r_l + \frac{j}{100}r_{l+1}$ ($j = 0, \cdots, 100$) are the intermediate points between $r_l$ and $r_{l+1}$ dividing $[r_l, r_{l+1}]$ into $100$ subintervals with the same length;
- $\mathcal{D}(s_l, r_l) = \{(x, y, z, \bar{z}) \in \mathcal{P}_C : |\text{Im} s| < s_l, |y| < r_l^2, |z| < r_l, |\bar{z}| < r_l\}$ denotes a neighborhood of the torus

$$ \mathcal{T}_0^d = \mathbb{T}^d \times \{y = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}, $$

where $|\cdot|_\infty$ denotes the supremum norm. Obviously,

$$ \mathcal{D}(s_l, r_l) \supset \mathcal{D}(s_{l+1}, r_{l+1}) \supset \cdots \supset \mathcal{D}(s_0, r_0), $$

$$ \mathbb{T}_s^d = \{x \in \mathbb{C}^d/(2\pi\mathbb{Z})^d : |\text{Im} x| < s_l\} $$

denotes a neighborhood of $\mathbb{T}^d$ with strip width $s_l$ and obviously

$$ \mathbb{T}_s^d \supset \mathbb{T}_s^{d+1} \supset \cdots \supset \mathbb{T}_s^{d_0/2}. $$
Given a sequence of open sets $\Pi_l$ in $\mathbb{R}^d$, we denote
\[ \mathcal{O}_l = \mathcal{O}(\Pi_l, A^{-E_l}) = \{ \xi \in \mathbb{C}^d : |\xi - \xi'| < A^{-E_{l'}} \text{ for some } \xi' \in \Pi_l \} \subset \mathbb{C}^d. \]

Finally, we define some matrices depending on the iteration, which are used to solve the homological equations. At the $l$-th KAM iterative step, we define the matrix
\[
T_l = D_l + S_l
\]
defined on $[1, \ldots, n] \times \mathbb{Z}^d$, where $D_l$ is a diagonal matrix
\[
D_l(j, k) = \Omega_j + \langle k, \omega_l \rangle
\]
and $S_l$ is a non-diagonal matrix
\[
S_l((j, k), (j', k')) = (B_l(j', k') + R^2_{l,j', k'})(k - k'),
\]
with $1 \leq j, j' \leq n$ and $k, k' \in \mathbb{Z}^d$. The matrix-valued functions $B_l$ and $R^2_{l,j', k'}$ are defined in the following Iterative Lemma and the hat $(\cdot)(k)$ (or $(\cdot)(k))^\wedge$ denotes the $k$-th Fourier coefficient of the associated function.

For $U \subset \mathbb{Z}^d$, we denote by $T_{l,U}$ the restriction of the matrix $T_l$ on $[1, \ldots, n] \times U$, i.e.,
\[
T_{l,U}((j, k), (j', k')) = T_l((j, k), (j', k')),
\]
when $(j, k), (j', k') \in [1, \ldots, n] \times U$. By some abuse of notation, we sometimes denote by $T_{l,M}$ the restriction of $T_l$ on $[1, \ldots, n] \times ([-M, M]^d \cap \mathbb{Z}^d)$ for any integer $M > 0$. The inverse matrix of $T_{l,M}$ (or $T_{l,A}$), if exists, is always denoted by $G_{l,M}$ (or $G_{l,A}$).

In addition, we define another matrix $T_l$ on $[j = (j_1, j_2) : 1 \leq j_1, j_2 \leq n] \times \mathbb{Z}^d$ by $T_l = D_l + S_l$. The diagonal matrix $D_l$ is defined by $D_l(j, k) = \Omega_j + \langle k, \omega_l \rangle$ and $\Omega_j = \Omega_{j_1} + \Omega_{j_2}$. The non-diagonal matrix $S_l$ is defined in (2.47). As we shall see later, $T_l$ and $T_l$ have essentially the same structure except the difference between the finite indices $j$ and $j$. Similarly, $T_{l,M}$ denotes the restriction of $T_l$ on $[j = (j_1, j_2) : 1 \leq j_1, j_2 \leq n] \times ([-M, M]^d \cap \mathbb{Z}^d)$ and $G_{l,M}$ is the inverse of $T_{l,M}$ whenever the matrix $T_{l,M}$ is invertible.

2.2. The Iterative Lemma. Choose and fix the various constants $C_0, C_1, \cdots, C_7$ such that
\[
C_1 > C_0 \Rightarrow 1, \quad C_2 > 2C_1 + 10, \quad C_4 > C_3 > C_1,
\]
\[
C_3 > C_6 + 2, \quad C_6 > 2C_4, \quad C_7 > (C_4 + 10) \vee C_5.
\]

Unlike the usual KAM theorems, the following Iterative Lemma starts from $l_\epsilon$ with $l_\epsilon = l_\epsilon(\epsilon)$ large enough. To keep the consistency of the notations, we set
\[
H_{l_\epsilon} = H_0, \quad P_{l_\epsilon} = P_0, \quad B_{l_\epsilon} = B_{l_\epsilon-1} = 0, \quad \Pi_{l_\epsilon-1} = \Pi_0, \quad s_{l_\epsilon} = s_0, \quad r_{l_\epsilon} = r_0, \quad \omega_{l_\epsilon} = \omega_{l_\epsilon-1} = \omega_0.
\]

**Lemma 2.1.** Consider a family of Hamiltonian functions $H_l$ ($l \leq l \leq m$),
\[
H_l = E_l + P_l,
\]
defined on $\mathcal{D}(s_l, r_l) \times \mathcal{O}_l$ with $\mathcal{O}_l = \mathcal{O}(\Pi_l, A^{-E_l})$, where
\[
E_l = \langle \omega_l(\xi), y \rangle + \langle \Omega z, \bar{z} \rangle + \langle B_l(x)z, \bar{z} \rangle
\]
is a normal form and the perturbation
\[
P_l = P_{l,\text{low}} + P_{l,\text{high}}, \quad P_{l,\text{high}} = O(|y| \cdot |z| + |y|^2 + |z|^3).
\]
Assume the Hamiltonian $H_l$ and the parameter set $\Pi_l$ satisfy the following properties.
(1.1) The frequency \(\omega_i\) is real analytic on \(\mathcal{O}_i\) and

\[
\sup_{\xi \in \mathcal{O}_i} \left| \frac{\partial \xi \omega_i}{\partial \xi} \right| \leq 1.
\]

Furthermore, we have

\[
\sup_{\xi \in \mathcal{O}_i} |\omega_i - \omega_{i-1}| < \epsilon_i^{1/10}.
\]

(1.2) The matrix \(B_i\) is analytic in \(x \in T^d_x \) and real analytic in \(\xi \in \mathcal{O}_i\). For any fixed \((x, \xi) \in T^d \times \Pi_i\), the matrix \(B_i\) is real symmetry, i.e.,

\[
B_{i;k,j}(x, \xi) = B_{i,j;k}(x, \xi), \quad B_{i;k,j}(x, \xi) = B_{i,j;k}(x, \xi),
\]

in which the indices \(j\) and \(k\) indicate the row or column. Furthermore, we have

\[
\sup_{T^d_x \times \mathcal{O}_i} \left\{ \epsilon^{-1} ||B_i(x, \xi)||, \epsilon^{-1/3} ||\partial \xi B_i(x, \xi)|| \right\} \leq 1,
\]

and

\[
\sup_{T^d_x \times \mathcal{O}_i} |B_i - B_{i-1}| < \epsilon_i^{1/10}.
\]

(1.3) The perturbation \(P_i\) is analytic on \(\mathcal{D}(s_i, r_i) \times \mathcal{O}_i\) and satisfies the reality condition

\[
\overline{P_i(x, y, z, \bar{\bar{z}}, \bar{\xi})} = P_i(x, y, z, \bar{\bar{z}}, \bar{\xi}), \quad \text{for} \ (x, y, z, \bar{\bar{z}}, \bar{\xi}) \in \mathcal{D}(s_i, r_i) \times \Pi_i.
\]

The Hamiltonian vector field

\[
X_{P_i} = (\partial_x P_i, -\partial_y P_i, \sqrt{1-\partial_x P_i} - \sqrt{1-\partial_y P_i})^T,
\]

defines an analytic map

\[
X_{P_i} : \mathcal{D}(s_i, r_i) \subset \mathcal{P}_C \rightarrow \mathcal{P}_C.
\]

and satisfies

\[
r_i|X_{P_i}|_{|\mathcal{D}(s_i, r_i) \times \mathcal{O}_i|} < \epsilon_i, \quad r_i|X_{P_i}|_{|\mathcal{D}(s_i, r_i) \times \mathcal{O}_i|} < \epsilon_i^{1/3},
\]

\[
r_i|X_{P_i}|_{|\mathcal{D}(s_i, r_i) \times \mathcal{O}_i|} \leq \epsilon_i, \quad r_i|X_{P_i}|_{|\mathcal{D}(s_i, r_i) \times \mathcal{O}_i|} \leq \epsilon_i^{1/3}.
\]

(1.4) The parameter set \(\Pi_i\) is the union of a collection \(\Lambda_i\) of disjoint intervals \(\mathcal{J} \subset \mathbb{R}^d\) of size \(A^{-f_1}\), i.e., \(\Pi_i = \bigcup_{\mathcal{J} \notin \Lambda_i} \mathcal{J}\) with \(|\mathcal{J}| = A^{-f_1}\). Moreover, the following properties hold.

(1.4.1) For any interval \(\mathcal{J} \in \Lambda_i\), there is a unique \(\mathcal{J}' \in \Lambda_{i-1}\) such that \(\mathcal{J} < \mathcal{J}'\).

(1.4.2) The parameter set \(\Pi_i\) is contained in

\[
|\xi \in \mathbb{R}^d : |(k, \omega_{i-1}(\xi))| > \sqrt{e} (1 + 2^{-i-1}) |k|^{-r}, 0 \neq |k| \leq A^i\}
\]

\[
\cap \{\xi \in \mathbb{R}^d : |G_{t-1;A'}| < A^{(\log A)^{f_1}}
\]

and

\[
|G_{t-1;A'}(k, k')| < \exp\left(-s_{i-1} - (\log A)^{-8}|k - k'|\right) \text{ for } |k - k'| > (\log A)^{f_2}\}
\]

\[
\cap \{\xi \in \mathbb{R}^d : |G_{t-1;A'}| < A^{(\log A)^{f_1}}
\]

and

\[
|G_{t-1;A'}(k, k')| < \exp\left(-s_{i-1} - (\log A)^{-8}|k - k'|\right) \text{ for } |k - k'| > (\log A)^{f_2},
\]

where \(G_{t-1;A'}\) and \(G_{t-1;A'}\) are defined at the beginning of this section.

(1.4.3) There is the measure estimate

\[
\text{mes} \ (\Pi_{i-1} \setminus \Pi_i) < A^{-c \epsilon_4}.
\]
Then there is an absolute positive constant $\epsilon_\ast > 0$ such that if $0 < \epsilon < \epsilon_\ast$, there is a parameter set $\Pi_{m+1}$ and a change of variables $\Phi_{m+1} : \mathcal{D}(s_{m+1}, r_{m+1}) \times \mathcal{O}_{m+1} \to \mathcal{D}(s_m, r_m)$ being real analytic in space coordinates and also real analytic in $\xi$ on the complex domain $\mathcal{O}_{m+1}$. The transformation is close to the identity in the sense that

$$r_m |\Phi_{m+1} - id|_{\mathcal{D}(s_{m+1}, r_{m+1}) \times \mathcal{O}_{m+1}} < \epsilon_m^{1/3}, \quad r_m |\Phi_{m+1} - id|_{\mathcal{D}(s_{m+1}, r_{m+1}) \times \mathcal{O}_{m+1}} < \epsilon_m^{1/4},$$

where $\mathcal{O}_{m+1} = \mathcal{O}(\Pi_{m+1}, A^{-(m+1)^2})$. Furthermore, the new Hamiltonian $H_{m+1} = H_m \circ \Phi_{m+1}$ has the form of (2.7), and the properties (1.1) – (1.4) hold with $l$ being replaced by $m + 1$.

**Remark 2.1.** Let us briefly explain the property (1.4.2). The parameter set $\Pi_l$ is contained in the intersection of three sets. The first one refers to the Diophantine conditions.

For the second set, we look at the definition of $T_{l-1,A'}$,

$$T_{l-1,A'} = T_{l-1}|_{[-A',A']^d} = (D_{l-1} + S_{l-1})|_{[-A',A']^d},$$

which originates from solving the homological equation of the following form

$$\partial_\omega F^\omega + \sqrt{-1} (\Omega + B(x) + R^\xi(x))F^\omega = \mathcal{E}$$

by the Fourier expansion and the truncation of the Fourier modes. With sufficiently small perturbation, for those initial KAM steps ($l$ is close to $l_*$), the matrices $T_{l-1,A'}$ is diagonally dominated if

$$|D_{l-1}(j,k)|^{-1} = |\Omega_j + (k, \omega_{l-1})|^{-1} \lesssim (1/\epsilon)^{1-}, \quad \text{for all } 1 \leq j \leq n, |k| \leq A'. \tag{2.8}$$

The condition (2.8) corresponds to the first Melnikov condition

$$|\Omega_j + (k, \omega_{l-1})| \neq 0.$$

For the third set in (1.4.2), the construction of $G_{l-1,A'}$ originates from solving

$$\partial_\omega F^\omega + \sqrt{-1} (\Omega + B(x) + R^\xi(x))F^\omega + F^\omega(\Omega + B(x) + R^\xi(x)) = \mathcal{E}.$$

Similarly, for those initial KAM iterations, $G_{l-1,A'}$ is also derived from the dominance of the diagonal matrix $D_{l-1}$, which requires

$$|D_{l-1}(j,k)|^{-1} = |\Omega_{j_1} + \Omega_{j_2} + (k, \omega_{l-1})|^{-1} \lesssim (1/\epsilon)^{1-}, \quad \text{for all } 1 \leq j_1, j_2 \leq n, |k| \leq A'. \tag{2.9}$$

The condition (2.9) corresponds to the second Melnikov condition

$$|\Omega_{j_1} + \Omega_{j_2} + (k, \omega_{l-1})| \neq 0. \tag{2.10}$$

However, only the plus sign in (2.10) occurs in our case, which can be essentially regarded as the first Melnikov condition since $\Omega_{j_1} + \Omega_{j_2}$ never vanishes.

2.3. **Proof of the Iterative Lemma 2.1.** In what follows, we drop the subscript $m$ for simplicity and let

$$\epsilon = \epsilon_m = A^{-\ell_\ast m}, \quad N = A^{m+1}.$$

The intermediate points $s_m^{(j)}$ between $s_m$ and $s_m+1$ are also written by $s^{(j)}$ for $0 \leq j \leq 100$. 

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2.3.1. Derivation of homological equations. Recall that $P^{\text{low}}$ is a polynomial in $y, z, \bar{z}$ of low order and we write $P^{\text{low}} = P^{\text{low}} + (R^{\bar{z}}(x)z, \bar{z})$ with

$$P^{\text{low}} = R^{\bar{z}}(x) + \langle R^{\bar{z}}(x), y \rangle + \langle R^{\bar{z}}(x), z \rangle + \langle R^{\bar{z}}(x)z, \bar{z} \rangle.$$ 

We are looking for a symplectic transformation $\Phi = X_{\Gamma}^{|l=1}$ to eliminate $P^{\text{low}}$ in the Hamiltonian $H$. As a result, we take $F$ in the form of

$$F(x, y, z, \bar{z}) = F^{\bar{z}}(x) + (F^{\bar{z}}(x), y) + (F^{\bar{z}}(x), z) + (F^{\bar{z}}(x)z, \bar{z}) + (F^{\bar{z}}(x)\bar{z}, \bar{z}).$$

Putting the unsolved term $R^{\bar{z}}$ into the normal form $E$, we get a corrected normal form

$$E = E + \langle R^{\bar{z}}, \bar{z} \rangle.$$

Then we have

$$H \circ \Phi = E + \langle [E, F]^{\bar{z}}, \bar{z} \rangle + \langle [P^{\text{high}}, F]^{\bar{z}}, \bar{z} \rangle + \hat{P} + P^{\text{high}} + [P^{\text{high}}, F]^{\text{high}}$$

$$+ P^{\text{low}} + [E, F] + [P^{\text{high}}, F]^{\text{low}},$$

where $[E, F] = [E, F]^{\bar{z}}, [P^{\text{high}}, F]^{\text{low}} = [P^{\text{high}}, F]^{\text{low}} - [P^{\text{high}}, F]^{\bar{z}}$ and

$$\hat{P} = \int_{0}^{1} (1 - t)\langle [E, F] + P^{\text{low}}, F \rangle \circ X_{\Gamma} dt + \int_{0}^{1} (1 - t)\langle [P^{\text{high}}, F] \circ X_{\Gamma} dt, \rangle$$

We aim at solving

$$[F, E] + [F, P^{\text{high}}]^{\text{low}} = P^{\text{low}}.$$  

As usual, we shall employ the truncation technique. Recalling the truncation operator $\Gamma_{N}$ defined in (2.1), we solve (2.12) up to a admissible error and get the following homological equations:

$$\partial_{\omega} F^{x} = \Gamma_{N} R^{x},$$

$$\partial_{\omega} F^{z} + \sqrt{-1} \Gamma_{N}[(\Omega + B(x) + R^{\bar{z}}(x))F^{z}] = \Gamma_{N} \mathcal{E},$$

$$\partial_{\omega} F^{\bar{z}} - \sqrt{-1} \Gamma_{N}[(\Omega + B(x) + R^{\bar{z}}(x))F^{\bar{z}}] = \Gamma_{N} \mathcal{E}',$$

$$\partial_{\omega} F^{y} = \Gamma_{N} \mathcal{R} - \mathcal{R}(0),$$

$$\partial_{\omega} F^{\bar{z}} + \sqrt{-1} \Gamma_{N}[(\Omega + B + R^{\bar{z}})F^{\bar{z}} + F^{\bar{z}}(\Omega + B + R^{\bar{z}})] = \Gamma_{N} \mathcal{J},$$

$$\partial_{\omega} F^{x} + \sqrt{-1} \Gamma_{N}[(\Omega + B + R^{\bar{z}})F^{x} + F^{\bar{z}}(\Omega + B + R^{\bar{z}})] = \Gamma_{N} \mathcal{J}',$$

where

$$\mathcal{E} = R^{z} - P^{z} \partial_{x} F^{x}, \quad \mathcal{E}' = R^{\bar{z}} - P^{\bar{z}} \partial_{x} F^{x},$$

$$\mathcal{R} = R^{\bar{z}} + \sqrt{-1}(P^{x} F^{\bar{z}} - P^{\bar{z}} F^{\bar{z}}) - P^{z} \partial_{x} F^{x},$$

$$\mathcal{J} = R^{\bar{z}} + \sqrt{-1}(P^{x} F^{\bar{z}} - P^{\bar{z}} F^{\bar{z}}) - P^{x} \partial_{x} F^{x} - P^{z} \partial_{x} F^{x},$$

$$\mathcal{J}' = R^{\bar{z}} + \sqrt{-1}(P^{x} F^{\bar{z}} - P^{\bar{z}} F^{\bar{z}}) - P^{\bar{z}} \partial_{x} F^{x} - P^{z} \partial_{x} F^{x}.$$

Without loss of generality, we assume

$$\widetilde{\mathcal{R}}(0) = \int_{\mathcal{T}^{d}} R^{\bar{z}}(x)dx = 0,$$

since the dynamics of the Hamiltonian vector field are unaffected. The homological equations to be solved are divided into four classes. The first one is (2.13), which is well known in the
KAM theory. Upon solving (2.13), we turn to the second homological equations (2.14)-(2.15). Observe that the two equations are complex conjugated if some symmetry is preserved (to be specified later). Moreover, (2.14)-(2.15) contain the variable coefficients \( B(x) \) and \( R^Z(x) \). We will employ the techniques developed in the Anderson localization theory to solve (2.14)-(2.15). Then we come up with the third homological equation (2.16) in which \( \mathcal{H} \) is known from (2.13)-(2.15). The unsolved constant \( \mathcal{H}(0) \) corresponds to the shift of the tangent frequency during the iterations. The last homological equations are (2.17)-(2.18), which are essentially the same to (2.14)-(2.15).

Once (2.13)-(2.18) are solved, we get
\[
H_+ = H \circ \Phi = E_+ + P_+,
\]
where
\[
E_+ = \langle \omega + \mathcal{H}(0), y \rangle + \langle \Omega z, \bar{z} \rangle + \langle (B + R^Z) + \{F, R^Z\}^Z z, \bar{z} \rangle
\]
\[
\equiv \langle \omega_+, y \rangle + \langle \Omega z, \bar{z} \rangle + \langle B_+ z, \bar{z} \rangle
\]
and
\[
P_+ = \rho^{\text{high}} + \{P^{\text{high}}, F\}^{\text{high}} + \hat{P}
\]
(2.23)
\[
= (1 - \Gamma_N)R^x + \langle (1 - \Gamma_N)\mathcal{H}, y \rangle
\]
(2.24)
\[
+ \langle (1 - \Gamma_N)\mathcal{H}, \bar{z} \rangle + \langle (1 - \Gamma_N)\mathcal{H}, \bar{z} \rangle
\]
(2.25)
\[
+ \sqrt{-1}\langle (1 - \Gamma_N)((B + R^Z)F^\bar{Z}), z \rangle - \sqrt{-1}\langle (1 - \Gamma_N)((B + R^Z)F^\bar{Z}), \bar{z} \rangle
\]
(2.26)
\[
+ \langle (1 - \Gamma_N)\mathcal{H}, y \rangle
\]
(2.27)
\[
+ \langle (1 - \Gamma_N)\mathcal{H}, \bar{z} \rangle + \langle (1 - \Gamma_N)\mathcal{H}, \bar{z} \rangle
\]
(2.28)
\[
- \sqrt{-1}\langle (1 - \Gamma_N)((B_+ + R^Z)F^\bar{Z} + F^\bar{Z}(B_+ + R^Z))z, \bar{z} \rangle
\]
(2.29)
\[
- \sqrt{-1}\langle (1 - \Gamma_N)((B_+ + R^Z)F^\bar{Z} + F^\bar{Z}(B_+ + R^Z))\bar{z}, z \rangle
\]
(2.30)
\[
- \sqrt{-1}\langle (1 - \Gamma_N)((B_+ + R^Z)F^\bar{Z} + F^\bar{Z}(B_+ + R^Z))\bar{z}, z \rangle
\]
(2.31)
\[
2.3.2. \text{Reality conditions.} \] In this part, we establish the reality property of the transformation function \( F \) and the new perturbation \( P_+ \).

**Lemma 2.2.** Assume \( P \) satisfies the following reality condition
\[
\mathcal{H}(x, y, z, \bar{z}, \xi) = P(x, y, z, \bar{z}, \xi), \quad \forall \ (x, y, z, \bar{z}, \xi) \in \mathcal{H}(s, r) \times \Pi.
\]
Let \( R_{ij}^{xz}, R_{ij}^{xz}, \bar{R}_{ij}^{xz} \) be the elements of the matrices \( R^x, R^z, R^{\bar{z}} \) respectively. Then we have that, for real \( x \),
\[
\bar{R}^x(x) = R^x(x), \quad \bar{R}^z(x) = R^z(x), \quad \bar{R}^{\bar{z}}(x) = R^{\bar{z}}(x),
\]
\[
\bar{R}_{ij}^{xz}(x) = R_{ji}^{xz}(x), \quad \bar{R}_{ij}^{xz}(x) = R_{ji}^{xz}(x), \quad \bar{R}_{ij}^{xz}(x) = R_{ji}^{xz}(x),
\]
\[
\bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x), \quad \bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x), \quad \bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x),
\]
\[
\bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x), \quad \bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x), \quad \bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x)
\]
\[
\bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x), \quad \bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x), \quad \bar{P}_{ij}^{xz}(x) = P_{ji}^{xz}(x)
\]

**Proof.** Taking \( n = 1 \) for example, we write
\[
P(x, y, z, \bar{z}, \xi) = \sum_{s,t \geq 0} p_{s,t}^{xz} z^t,
\]
where \( p_{s,t} = p(x, y; \xi) \). It follows from the reality condition of \( P \) that
\[
\bar{p}_{s,t}(x, y; \xi) = p_{t,s}(x, y; \xi), \quad \forall x \in \mathbb{T}^d, z \in \mathbb{R}^d, \xi \in \Pi.
\]
Lemma 2.3. Suppose \( P \) satisfies the reality condition (2.32). If \( F \) is some sub-domain \( \mathcal{D}(s', r') \times \Pi' \) of \( \mathcal{D}(s, r) \times \Pi \) is the unique solution of the homological equations (2.13)-(2.18), then \( F \) satisfies the reality condition:

\[
\overline{F}(x, y, z, \bar{z}; \xi) = F(x, y, z, \bar{z}; \xi), \quad \forall (x, y, z, \bar{z}, \xi) \in \mathcal{D}_R(s', r') \times \Pi'.
\]

Proof. Taking complex conjugation on both sides of (2.13), we see from \( \overline{F^x} = R^x \) that \( \overline{F^x} \) is also a solution to (2.13), which, by the uniqueness of solution, implies that \( \overline{F} = F^x \). For (2.14) and (2.15), it then follows that \( \overline{\xi'} = \xi' \). As a result, if \( (F^x, \xi^x) \) solves (2.14)-(2.15), so does \( (\overline{F^x}, \xi^x) \). Also by using the uniqueness assumption, we have \( \overline{F^x} = F^x \). Similarly, we can show \( \overline{F^y} = F^y, \overline{F^z} = F^z \). Consequently, the reality condition (2.33) of \( F \) holds true. \( \square \)

Proposition 2.1. Under the assumption of Lemma 2.3 we have that the matrix \( B_+ \) is self-adjoint, i.e.,

\[
\overline{B'_+(x; \xi)} = B_+(x; \xi), \quad \forall x \in \mathbb{T}^d, \xi \in \Pi'.
\]

Moreover, the new perturbation \( P_+ \) satisfies the reality condition (2.32) on \( \mathcal{D}_R(s', r') \times \Pi' \).

Proof. This is an immediate result of the reality property of \( P \) and \( F \). \( \square \)

2.3.3. Solutions of the homological equations. In this part, we establish several propositions to solve the homological equations (2.13)-(2.18) in order.

Firstly, we solve the homological equation (2.13), which is very standard in the classical KAM theory.

Proposition 2.2. (Solution of (2.13)) Under the assumptions of Lemma 2.1 there is a parameter set \( V^{(1)} \) with \( \text{mes} (V^{(1)}) < A^{-mr} \) such that for all \( \xi \in \Pi \setminus V^{(1)} \), the Diophantine condition holds

\[
|(k, \omega(\xi))| > \frac{\sqrt{\epsilon}(1 + 2^{-m})}{|k|^r}, \quad 0 \neq |k| \leq N, k \in \mathbb{Z}^d.
\]

Then equation (2.13) has an analytic solution \( F^x \) defined on \( \mathbb{T}^d_{x^{(1)}} \times \mathcal{O}(\Pi \setminus V^{(1)}, 10A^{-(m+1)}\xi') \). Moreover, we have

\[
\sup_{(x,\xi)\in \mathbb{T}^d_{x^{(1)}} \times \mathcal{O}(\Pi \setminus V^{(1)}, 8A^{-(m+1)}\xi')} \{ |F^x|, |\partial_x F^x|, |\partial_\xi F^x|, |\partial_\xi \partial_x F^x| \} < \epsilon^{1-}.
\]
Proof. Recall that $\tilde{R}(0) = 0$. Moreover,
$$|R| \leq 2 \left| \frac{\partial P(x, 0, 0, 0)}{\partial x} \right| \leq |X_{p=x}| \leq \varepsilon$$
and thus $\sup_{x \in \mathcal{T}^2} |r(x)| \leq \varepsilon$. From (m.4.2) in the iterative lemma, there is
$$|k, \omega_{m-1}| > \frac{\sqrt{\varepsilon(1 + 2^{-(m-1)})}}{|k|^r}, \quad 0 \neq |k| \leq A^m.$$
Then by $|\omega - \omega_{m-1}| < \varepsilon_{m-1}/10$ in (m.1), we have
$$|\langle k, \omega_m \rangle| \geq \frac{\sqrt{\varepsilon(1 + 2^{-(m-1)})}}{|k|^r} - A^m \varepsilon_{m-1} > \frac{\sqrt{\varepsilon(1 + 2^{-m})}}{|k|^r}$$
for all $0 \neq |k| \leq A^m$. It remains to exclude the parameter $\xi$ in
$$V^{(1)} = \bigcup_{A^m \leq |k| \leq A^{m+1}} \left\{ \xi \in \Pi_m : |\langle k, \omega(\xi) \rangle| < \frac{\sqrt{\varepsilon(1 + 2^{-m})}}{|k|^r} \right\},$$
where $|\partial_x \omega|$ and $|\partial_x \omega^{-1}|$ are uniformly bounded along the iterations. As a result, the total excluded measure
$$\text{mes } V^{(1)} \leq \sqrt{\varepsilon} \sum_{A^m \leq |k| \leq A^{m+1}} \frac{1}{|k|^r} < A^{-m} \varepsilon \ll A^{-(\log(m+1))^4},$$
which is allowed in the measure estimate in the iterative lemma. Moreover, the Diophantine condition (2.34) remains valid on an $(m+1)^5$-neighborhood of $\Pi_m \setminus V^{(1)}$ due to the fact that $A^{-(m+1)^5} \ll A^{-(10b(m+1)^4)}$. Using the Diophantine condition, the existence of the solution $F^x$ of (2.13), as well as its estimates on derivatives, is well known in KAM theory. We omit the proof here.

Secondly, we solve the homological equations (2.14)-(2.15), by using some techniques from the Craig-Wayne-Bourgain method. From Lemma 2.3, it suffices to solve (2.14) since $F^z = F^z$.

Proposition 2.3. (Solution of (2.14)-(2.15)) Under the assumptions of Lemma 2.7, there exists a parameter set $\Pi^{(1)}_+ \subset \Pi \setminus V^{(1)}$ such that for all $\xi \in \Pi^{(1)}_+$, equations (2.14), (2.15) have a unique solution $(F^z, F^z)$ with $F^z = F^z$, which admits analytic extension to $\mathbb{T}^d_{\varepsilon(5)} \times O(\Pi^{(1)}_+, 8A^{-(m+1)^5})$ and satisfies
$$\sup_{\mathbb{T}^d_{\varepsilon(5)} \times O(\Pi^{(1)}_+, 8A^{-(m+1)^5})} \left| F^z, |\partial_x F^z|, |\partial_x F^z|, |\partial_x \partial_x F^z| \right| < \varepsilon^{-1}.$$  
Moreover, $\Pi^{(1)}_+$ is the union of a family of disjoint intervals $\mathcal{J}'$ with $|\mathcal{J}'| = A^{-(m+1)^5}$. For each $\mathcal{J}'$, there is a unique $\mathcal{J} \in \Lambda_m$ such that $\mathcal{J}' \subset \mathcal{J}$. The total removed set satisfies
$$\text{mes } (\Pi \setminus \Pi^{(1)}_+) < \frac{1}{5} A^{-(\log(m+1))^2}.$$

Proof. Due to Lemma 2.3, we only solve the homological equation (2.14) with $\mathcal{E}$ given by (2.19)
$$\partial_x F^z + \sqrt{-1} \Gamma_N(\Omega + \Gamma_N B(x) + \Gamma_N R^z(x))F^z = \Gamma_NT\mathcal{E},$$
which, by matching the components of the vector-valued functions, turns out to be
$$-\sqrt{-1} \partial_x F^z + \Gamma_N \Omega_j F^z + \Gamma_N \left( \sum_{1 \leq r \leq n} (\Gamma_N B_j(x))F^z(x) + \sum_{1 \leq r \leq n} (\Gamma_N R^z_{j}(x))F^z(x) \right) = -\sqrt{-1} \Gamma_N \mathcal{E}_j.$$
for \(1 \leq j \leq n\). We are looking for solution \(F^j(x)\) with compact support in the Fourier modes

\[
F^j(x) = \sum_{|k| \leq N, k \in \mathbb{Z}^d} \hat{F}^j(k)e^{\sqrt{-1}\langle k, x \rangle}.
\]

Passing to the Fourier transformation, we then get

\[
(2.35) \quad \langle (k, \omega) + \Omega_j \rangle \hat{F}^j(k) + \sum_{1 \leq s \leq n} \sum_{|p| \leq N} (B_{jr} + R_{jr}^{e^2})^\wedge (k - p) \hat{F}^j(p) = -\sqrt{-1} \hat{E}^j(k),
\]

where \(1 \leq j \leq n, |k| \leq N, k \in \mathbb{Z}^d\) and \((\cdot)^\wedge\) stands for the \(k\)-th Fourier coefficient of the indicated function.

Thinking of \(\hat{F}\) as a vector defined on \(\{1, \cdots, n\} \times \mathbb{Z}^d\), we write \((2.35)\) in a matrix form. To this end, we let

\[
T = D + S
\]

where \(D\) is a diagonal matrix

\[
D((j, k), (r, p)) = \langle B_{jr} + R_{jr}^{e^2}, (k - p) \rangle.
\]

and \(S\) is a non-diagonal matrix

\[
S((j, k), (r, p)) = \langle B_{jr} + R_{jr}^{e^2}, (k - p) \rangle.
\]

We denote by \(T_N\) and \(\hat{E}_N\) the restriction of the matrix \(T\) and the restriction of the vector \(\hat{E}\) on \(\{1, \cdots, n\} \times \{k \in \mathbb{Z}^d : |k| \leq N\}\) respectively. With the notations introduced above, equation \((2.35)\) is equivalent to

\[
(2.36) \quad T_N \hat{F}_N = -\sqrt{-1} \hat{E}_N.
\]

Then our goal is to establish the existence and the decay property of the inverse matrix of \(T_N\) (see \((3.29)\)), which is also called the Green’s function estimate. Indeed, we have the following results, whose proof is delayed to the next section.

**Lemma 2.4.** Under the assumptions of Lemma 2.1 there exists a parameter set \(\Pi^{(1)}_+ \subset \Pi \setminus V^{(1)}\) such that for any \(\xi \in \mathcal{O}(\Pi^{(1)}_+, 10A^{-(m+1)C_3})\), the inverse matrix \(G_N = T_N^{-1}\) exists and satisfies

\[
|G_N| < A^{(\log N)^{C_1}},
\]

\[
|G_N(x, y)| < e^{-(x-(\log N)^{-4}) |x-y|} \quad \text{for} \quad |x - y| > (\log N)^{C_2}.
\]

Moreover, \(\Pi^{(1)}_+\) is the union of a family of disjoint intervals \(\mathcal{J}'\) with \(|\mathcal{J}'| = A^{-(m+1)C_3}\). For each \(\mathcal{J}'\), there is a unique \(\mathcal{J} \in \Lambda_m\) such that \(\mathcal{J}' \subset \mathcal{J}\). The total removed set satisfies

\[
\text{mes} (\Pi \setminus \Pi^{(1)}_+) < \frac{1}{3} A^{-(\log(m+1)C_4)}.
\]

Now we apply Lemma 2.4 to solve \((2.37)\) and then to prove Proposition 2.3. Recall the definition of \(\Pi^{(1)}\) in \((2.19)\). Observe that \(R^c = \partial_x P(x, 0, 0, 0)\), and it follows from Cauchy’s estimate that

\[
\sup_{x \in \mathbb{T}^d} |R^c(x)| < \epsilon.
\]

Moreover, \(P^c = \partial_x \partial_x P(x, 0, 0, 0)\) and then

\[
\sup_{x \in \mathbb{T}^d} |P^c| \leq \frac{1}{r} \sup_{x \in \mathbb{T}^d} |\partial_x P(x, 0, z, 0)| < \epsilon.
\]
Moreover, from Cauchy estimate that

\[(2.43)\]

\[
|\hat{\phi}(k)| \lesssim \varepsilon^{1-} e^{-\varepsilon^2|k|}
\]

for any \(k \in \mathbb{Z}^d\).

Back to (2.37), we have

\[
\hat{F}^c = -\sqrt{-1} G_N \hat{\phi}_N,
\]

\[
\hat{F}^c(k) = -\sqrt{-1} \sum_{|p| \leq N, p \in \mathbb{Z}^d} G_N(k, p) \hat{\phi}_N(p).
\]

It then follows that

\[(2.41)\]

\[
|\hat{F}^c(k)| \lesssim \varepsilon^{1-} \sum_{|k-p| \leq (\log N)^{C_2}} A^{(\log N)^{C_1}} e^{-s(\log N)^{C_2}}|k-p| e^{-s(\log N)^{C_2}}|p| + \sum_{|k-p| > (\log N)^{C_2}} e^{-(s-(\log N)^{C_4})|k-p|} e^{-s(\log N)^{C_2}}|p| + C \varepsilon e^{-\varepsilon^2|k|} < \varepsilon^{1-} e^{-\varepsilon^2|k|}.
\]

Passing to the function \(F^c\), we then have, for any \(x \in \mathbb{T}^d_{\varepsilon^3}\),

\[
|F^c(x)| \leq \sum_{k \in \mathbb{Z}^d} |\hat{F}^c(k)| \cdot |e^{\sqrt{-1} (k, x)}| < \varepsilon^{1-}.
\]

The remaining estimates for \(\partial_x F^c\), \(\partial_x F^c\) and \(\partial_x \partial_y F^c\) follow from the Cauchy’s estimate by using the fact that \((m+1)^2 \ll A^{(m+1)^{C_3}} \ll (1/\varepsilon)^{C_4} + \varepsilon^{1-}. This completes the proof of Proposition 2.3 \(\Box\)

**Thirdly**, we solve the homological equation (2.16), which is similar to that of (2.13). To begin with, we need to estimate \(\hat{\Omega}\) defined by (2.20). Observe that \(R^c = \partial_x P(x, 0, 0, 0)\) and it follows from Cauchy estimate that

\[
\sup_{x \in \mathbb{T}^d_{\varepsilon^3}} |R^c(x)| < \varepsilon |X_p| \quad \cdot \quad r < \varepsilon
\]

Moreover, \(P^y = \partial_{yy}^2 P(x, 0, 0, 0)\) and

\[
\sup_{x \in \mathbb{T}^d_{\varepsilon^3}} |P^y(x)| \leq \frac{1}{r^2} \sup_{x \in \mathbb{T}^d_{\varepsilon^3}} |\partial_y P(x, 0, 0, 0)| \leq \varepsilon |X_p| \quad \leq \varepsilon.
\]

Then we see from (2.39), Proposition 2.2 and Proposition 2.3 that

\[(2.42)\]

\[
\sup_{x \in \mathbb{T}^d_{\varepsilon^3}} |\hat{\Omega}| < \varepsilon^{1-}.
\]

Recalling the frequency shift induced by the unsolved term \(\hat{\phi}(0)\), we have

\[(2.43)\]

\[
|\omega_+ - \omega| = |\hat{\phi}(0)| < \varepsilon^{1-}.
\]

**Proposition 2.4. (Solution of (2.16))** Under the assumptions of Lemma 2.7, equation (2.16) has an analytic solution \(F^y\) defined on \(\mathbb{T}^d_{\varepsilon^3} \times O(\Pi^{(1)}\varepsilon, 6A^{-m+1}C_3)\) satisfying

\[
\sup_{(x, \xi) \in \mathbb{T}^d_{\varepsilon^3} \times O(\Pi^{(1)}\varepsilon, 6A^{-m+1}C_3)} \left\{|F^y|, |\partial_x F^y|, |\partial_x \partial_y F^y|, |\partial_x \partial_y F^y|\right\} < \varepsilon^{1-}.
\]
Finally, we solve the homological equations (2.17)-(2.18), which are essentially the same to (2.14)-(2.15). To begin with, we need to estimate \( J \) and \( J' \) defined in (2.21) and (2.22) respectively. From the estimates of \( F^c, F^{cc}, F^{cc} \), it is easy to see

\[
\sup_{x \in \mathbb{R}^d} \| J \| \leq \varepsilon^{1-}.
\]

(2.44)

**Proposition 2.5. (Solution of (2.17)-(2.18))** Under the assumptions of Lemma 2.7, there exists a parameter set \( \Pi_+^{(2)} \subset \mathbb{R}^d \) such that for all \( \varepsilon \in \mathbb{R} \), \( \varepsilon \in \Pi_+^{(2)} \cap \Pi_+^{(1)} \), equations (2.17)-(2.18) have a unique solution \( (F^{cc}, F^{cc}) \) with \( F^{cc} = F^{cc} \), which admits analytic extension to \( \mathbb{T}^d \times \mathcal{O}(\Pi_+, 6A^{-(m+1)^s}) \) and satisfies

\[
\sup_{x \in \mathbb{R}^d} \{ |J^{cc}|, |\partial_x J^{cc}|, |\partial_x \partial_z J^{cc}| \} < \varepsilon^{1-}.
\]

Moreover, \( \Pi_+^{(2)} \) is the union of a family of disjoint intervals \( J' \) with \( |J'| = A^{-(m+1)^s} \). For each \( J' \), there is a unique \( J \in \Lambda_m \) such that \( J' \subset J \). The total removed set satisfies

\[
\text{mes}(\Pi \setminus \Pi_+^{(2)}) < \frac{1}{5} A^{-(\log(m+1)^s)}.
\]

**Proof.** Note that the unknown function \( F^{cc} \) is of matrix value, i.e., \( F^{cc}(x) = (F^{cc}_{ij}(x))_{1 \leq i, j \leq n} \). Writing equation (2.17) into components yields

\[
\partial^2_{x_k} F^{cc}_{ij} + \sqrt{-1} \Gamma_N(\Omega_i + \Omega_j) F^{cc}_{ij} + \sum_{p=1}^{n} \Gamma_N(B^{cc}_{ip} + R^{cc}_{ip}) F^{cc}_{ij} + \gamma^{cc}_{ij} \Gamma_N(B^{cc}_{ip} + R^{cc}_{ip}) = \Gamma_N J^{cc}_{ij}.
\]

Let

\[
j = (i, j), \quad \Omega_j = \Omega_i + \Omega_j, \quad 1 \leq i, j \leq n,
\]

and hence \( j \) is an index taking \( n^2 \) many values. Then (2.45) is equivalent to

\[
\partial^2_{x_k} F^{cc}_{ij} + \sqrt{-1} \Gamma_N(\Omega_i + \Omega_j) F^{cc}_{ij} + \sum_{j'} \Gamma_N(B^{cc}_{ij} + R^{cc}_{ij}) F^{cc}_{ij'} = \Gamma_N J^{cc}_{ij'},
\]

where \( j' = (i', j') \),

\[
B^{cc}_{ij} = \begin{cases} 
B^{cc}_{ii}, & \text{for } j' = j, i' \neq i, \\
B^{cc}_{jj}, & \text{for } j' = j, i' = i, \\
B^{cc}_{ij} + B^{cc}_{jj}, & \text{for } j' \neq j, i' = i, \\
0, & \text{otherwise,}
\end{cases}
\]

and

\[
R^{cc}_{ij} = \begin{cases} 
R^{cc}_{i'i'}, & \text{for } j' = j, i' \neq i, \\
R^{cc}_{i'i'}, & \text{for } j' \neq j, i' = i, \\
R^{cc}_{ii} + R^{cc}_{jj}, & \text{for } j' = j, i' = i, \\
0, & \text{otherwise.}
\end{cases}
\]

We look for solution \( F^{cc}_{ij}(x) \) of (2.46) in the following form

\[
F^{cc}_{ij}(x) = \sum_{|k| \leq N, \lambda \in \mathbb{Z}^d} F^{cc}_{ij}(k)e^{|k| x}.
\]

Expanding (2.46) into Fourier series and matching the coefficients yield

\[
\langle (k, \omega) + \Omega_j \rangle F^{cc}_{ij}(k) + \sum_{j' \neq j} \sum_{k' \in \mathbb{Z}^d, |k'| \leq N} (B^{cc}_{ij} + R^{cc}_{ij})^\lambda(k - k') F^{cc}_{ij'}(k') = -\sqrt{-1} \langle J^{cc}_{ij}(k) \rangle.
\]

(2.47)

where \( k \in \mathbb{Z}^d \) and \( |k| \leq N \).
Writing further (2.47) into a matrix equation like (2.57), we obtain an essentially same matrix $T_N$ except the difference between the finite index $j$ and $j$. Note also the fact that $\Omega_j = \Omega_i + \Omega_j$ with $j = (i, j)$ and $\Omega_j$ never vanishes. Therefore, for $G_N$ defined on $[j = (i, j) : 1 \leq i, j \leq n] \times \{k \in \mathbb{Z}^d : |k| \leq N\}$, we are also able to establish the Green’s function estimate like Proposition 2.3 and obtain the desired parameter set $\Pi^{(2)}_+$. The remaining estimate of $F^\varepsilon$ is the same to that of $F^\xi$ and we omit it here. \hfill \Box

Let $\Pi_{m+1} = \Pi_+ \cap \Pi^{(2)}_+$. One easily finds that $\Pi_{m+1}$ satisfies ((m + 1).4). See Remark 3.1 for more details.

2.3.4. The estimate of new error. Now we are at the stage of estimating the new terms after the symplectic transformation, which are given in (2.23) and (2.25)-(2.30). The majority of them arise from the remaining terms after the truncation.

For $\omega_+$, it follows from (2.43) that

$$|\omega_+ - \omega| = |\Delta \omega| < \varepsilon^{1-}.$$  

For $B_+$, we have

$$B_+ - B = R^\varepsilon - [E_0, F]^\varepsilon - [P^{\text{high}}, F]^\varepsilon$$

in which, for $|\text{Im}|x| \leq s$,

$$|R^\varepsilon(x)| = |\partial_3 P(x, 0, 0, 0)| \leq \frac{1}{r} \sup_{\mathcal{D}(x, r)} |\partial_3 P^{\text{low}}(x, 0, z, 0)| \leq \varepsilon.$$

It then follows from Proposition 2.4 that

$$\sup_{|\text{Im}|x| < \varepsilon^0} |[E_0, F]^\varepsilon| \leq \sup_{|\text{Im}|x| < \varepsilon^0} |\partial_4 B + \partial_4 R^\varepsilon| \cdot |F^\varepsilon| < \varepsilon^{1-} (m + 1)^2 < \varepsilon^{1-}.$$

Similarly, we obtain $\sup_{x \in \mathcal{T}^d_{\epsilon}(\delta)} |[P^{\text{high}}, F]^\varepsilon| < \varepsilon^{1-}$ and then we have

$$\sup_{|\text{Im}|x| \leq \varepsilon^0} |B_+(x) - B(x)| < \varepsilon^{1-},$$

and

$$\sup_{|\text{Im}|x| \leq \varepsilon^0} |B_+(x)| \leq \sup_{|\text{Im}|x| \leq \varepsilon^0} |B_+| + \sum_{l=1}^{m+1} \sup_{|\text{Im}|x| \leq \varepsilon^0} |B_l - B_l - 1| \leq \varepsilon.$$

Next we estimate (2.25)-(2.30). Obviously, using [15, Lemma A.2], we have

$$\sup_{x \in \mathcal{T}^d_{\epsilon}(\delta), \tau \in [0, 1]} \left| \partial_3^\varepsilon [(1 - \Gamma_\delta) R^\varepsilon] \right| \leq \frac{N^d}{{s^2}} e^{-N(s^2 - 2)} \sup_{\tau \in [0, 1]} |R^\varepsilon| < \frac{1}{100} A^{-(m+1)} \leq \frac{1}{100} \varepsilon^{4/3}$$

provided $A \gg 1$. For $\dot{P}$ defined in (2.11), we have

$$\sup_{\mathcal{D}(x, \varepsilon^0), \tau \in [0, 1]} \left| \partial_3^\varepsilon [(1 - \Gamma_\delta) \dot{P}] \right| \leq \frac{N^d e^{-N(s^2 - 2)}}{(s^2 - 2)} C \leq \frac{1}{100} \varepsilon^{4/3},$$

where $\nabla \dot{P} = (\partial_3 \dot{P}, \partial_4 \dot{P}, \partial_5 \dot{P}, \partial_6 \dot{P})$. For (2.26), we see from (2.40) that

$$\sup_{x \in \mathcal{T}^d_{\epsilon}(\delta), \tau \in [0, 1]} \left| \partial_3^\varepsilon [(1 - \Gamma_\delta) \dot{P}] \right| \leq \frac{N^d e^{-N(s^2 - 2)}}{(s^2 - 2)} C \leq \frac{1}{100} \varepsilon^{4/3}.$$
Estimate (2.48) still holds when replacing $\mathcal{E}$ by $\mathcal{E}'$. By (2.42), (2.44) we get
\[
\sup_{x \in T^d_{s, r}, \mathcal{E} \in (0, 1)} \left| \partial_\mathcal{E}^a \left[ \left( 1 - \Gamma_N \right) \mathcal{F} \right], \partial_\mathcal{E}^a \left[ \left( 1 - \Gamma_N \right) \mathcal{J} \right], \partial_\mathcal{E}^a \left[ \left( 1 - \Gamma_N \right) \mathcal{J}' \right] \right| < \frac{1}{100} \mathcal{E}^{4/3},
\]
which controls (2.28) and (2.29).

For (2.27), we see that
\[
(1 - \Gamma_N)((B + R_2^c)F^c) = \sum_{|k| \geq N} ((B + R_2^c)F^c)^\wedge (k)e^{\sqrt{-1}\mathcal{K}(k,x)}.
\]
Since $B_j(x)$ and $R_2^c(x) = \partial_x \partial_z P^\text{low}_{l'}(x, 0, 0)$ are analytic in $x \in T^d_{s, r}$, we have
\[
|\hat{B}(k)| \leq e^{-s|k|}, \quad |\hat{R}(k)| < \frac{1}{l} \sup_{(s', r')} |\partial_r^s \mathcal{E}| \mathcal{E}^{-s|k|} < \mathcal{E}^{-s|k|}.
\]
It follows from (2.49) and (2.41) that
\[
|((B + R_2^c)F^c)^\wedge (k)| = \sum_{|p| \leq N} (B + R_2^c)^\wedge (k - p)\mathcal{F}(p)| \leq \sum_{|p| \leq N} (\mathcal{E} + \mathcal{E})e^{-s|k - p|}\mathcal{E}^{-s|k|} < \mathcal{E}^{-s|k|}.
\]
Then
\[
\sup_{|\mathcal{E}| \leq \mathcal{E}^{6}} \left| (1 - \Gamma_N)((B + R_2^c)F^c) \right| < \mathcal{E}^{-1} \sum_{|k| \geq N} e^{-s|k|}e^{-s|k|} \leq \mathcal{E}^{-1}N^d e^{-s|k|} < \frac{1}{100} \mathcal{E}^{4/3}.
\]
With the margins in our estimate, we further have
\[
\sup_{|\mathcal{E}| \leq \mathcal{E}^{6}, \mathcal{E} \in (0, 1)} \left| \partial_\mathcal{E}^a \left[ \left( 1 - \Gamma_N \right) \left( (B + R_2^c)F^c \right) \right], \partial_\mathcal{E}^a \left[ \left( 1 - \Gamma_N \right) \left( (B + R_2^c)F^c \right) \right] \right| < \frac{1}{100} \mathcal{E}^{4/3}.
\]
The estimate of (2.30)-(2.31) is the same to that of (2.27) and reads
\[
\sup_{|\mathcal{E}| \leq \mathcal{E}^{10}, \mathcal{E} \in (0, 1)} \left\{ \left| \partial_\mathcal{E}^a \left[ \left( 1 - \Gamma_N \right) \left( (B + R_2^c)F^c \right) \right], \partial_\mathcal{E}^a \left[ \left( 1 - \Gamma_N \right) \left( (B + R_2^c)F^c \right) \right] \right| \right\} < \frac{1}{100} \mathcal{E}^{4/3}.
\]
Note that
\[
P^\text{low}_{l'} = P^\text{low} + (2.25) + (2.26) + \cdots + (2.31),
\]
and
\[
P^\text{high}_{l'} = P^\text{high} + \hat{P}^\text{high} + \{P^\text{high}, F\}^\text{high}.
\]
Take
\[
\mathcal{E}_+ = \mathcal{E}^{4/3} = A^{-1/2}m^1, \quad s_+ = s_{m+1}, \quad r_+ = r_{m+1}, \quad \mathcal{O}_+ = \mathcal{O}(\Pi_+, A^{-1/2}),
\]
We obtain from the above analysis that
\[
r_+ |X_{p^\text{low}_{l'}}|_{\mathcal{O}(s_+, r_+) \times \mathcal{O}_+} \leq \mathcal{E}_+
\]
and
\[
r_+ |X_{p^\text{high}_{l'}}|_{\mathcal{O}(s_+, r_+) \times \mathcal{O}_+} \leq \mathcal{E}.
\]
The transformation $\Phi = X_{p^l_{l+1}}$ is also close to the identity in the sense that
\[
r_+ |\Phi - \text{id}|_{\mathcal{O}(s_+, r_+)} \leq \mathcal{E}^{1/3}
\]
and
\[
\sup_{\mathcal{O}(s_+, r_+)} |\nabla F| \leq \mathcal{E}^{1/3}.
\]
2.4. Proof of the main Theorem 1.1. Let the constant $A$ be sufficiently large. In the proof of the iterative lemma, we take extensively advantage of the largeness of the iteration step \( l \). As a result, it suffices to start the iteration from \( l = l_* \), \( l_* = l_*(\epsilon) \gg 1 \) (independent of the iterations) instead of \( l = 0 \).

We then need to verify the induction statements at \( l = l_* \). Recall our imposition (2.6) in the first step

\[
H_{l_*} = H_0, \quad P_{l_{1-1}} = P_{l_*} = P_0, \quad B_{l_*} = B_{l_{1-1}} = 0, \quad \Pi_{l_{1-1}} = \Pi_0, \quad s_{l_*} = s_0, \quad r_{l_*} = r_0, \quad \omega_{l_*} = \omega_{l_{1-1}} = \omega_0.
\]

Obviously, the statements \((l_*,1),(l_*,2)\) and \((l_*,3)\) hold. It suffices to find the set \( \Pi_{l_*} \) such that the statement \((l_*,4)\) holds, which can essentially be described by the Diophantine condition and the first Melnikov condition. The construction of the set \( \Pi_{l_*} \) is given below.

We first pave the set \( \Pi_{l_{1-1}} = \Pi_0 \) into a \( \tilde{\Lambda} \) family of disjoint intervals of diameter \( A^{-l_3^*} \), i.e.,

\[
\Pi_0 = \bigcup \mathcal{J} \in \tilde{\Lambda} \text{ with } |\mathcal{J}| = A^{-l_3^*} \text{ for each } \mathcal{J} \in \tilde{\Lambda}.
\]

If there exists some \( \xi_0 \in \mathcal{J} \) such that

\[
|\langle k, \omega_0(\xi_0) \rangle| > \sqrt{\epsilon}(1+2^{-(l_*-1)})|k|^{-\tau}
\]

violates for some \( |k| \leq A^{l_*} \), then for any \( \xi \in \mathcal{J} \), there is

\[
|\langle k, \omega_0(\xi) \rangle| \leq \sqrt{\epsilon}(1+2^{-(l_*-1)})|k|^{-\tau} + C A^{l_*} A^{-l_3^*} < 2 \sqrt{\epsilon}(1+2^{-(l_*-1)})|k|^{-\tau}
\]

Let

\[
\Lambda^{(1)}_{l_*} = \{ \mathcal{J} \in \tilde{\Lambda} : (\text{2.50}) \text{ holds for all } \xi \in \mathcal{J} \text{ and all } 0 \neq |k| \leq A^{l_*}, \text{ and one easily sees from the twist condition in Assumption B that}
\]

\[
\text{mes \left( \bigcup_{\mathcal{J} \in \Lambda^{(1)}_{l_*}} \mathcal{J} \right) \leq \sqrt{\epsilon}.}
\]

Next we consider the matrix \( T_{l_{1-1}} = D_{l_{1-1}} + S_{l_{1-1}} \) with

\[
D_{l_{1-1}}(j,k) = \Omega_j + \langle k, \omega_0 \rangle
\]

and \( S_{l_{1-1}}((j,k),(j',k')) = \frac{p_{0,jj'}}{\omega_{0,jj'}}(k-k') \) since \( B_{l_{1-1}} = 0 \) (see (2.2)-(2.4)). To describe the first Melnikov’s condition, we take

\[
\Lambda^{(2)}_{l_*} = \{ \mathcal{J} \in \tilde{\Lambda} : |\langle k, \omega_0(\xi) \rangle + \Omega_j| > \sqrt{\epsilon}|k|^{-\tau}
\]

holds for all \( 1 \leq j \leq n, |k| \leq A^{l_*} \text{ and all } \xi \in \mathcal{J} \} \).

For \( \xi \in \bigcup_{\mathcal{J} \in \Lambda^{(2)}_{l_*}} \mathcal{J} \), there is

\[
|D_{l_{1-1}}(j,k)|^{-1} \leq A^{l_*} \epsilon^{-1/2}
\]

and hence the diagonal matrix \( ||D_{l_{1-1}}^{-1}|| \leq A^{l_*} \epsilon^{-1/2} \). Observing that \( ||S_{l_{1-1}}|| \leq \epsilon \), we obtain from the Neumann series that the inverse matrix \( G_{l_{1-1};A^{l_*}} \) of \( T_{l_{1-1};A^{l_*}} \) satisfies

\[
||G_{l_{1-1};A^{l_*}}|| \leq 2A^{l_*} \epsilon^{-1/2} < A^{l_3^*}
\]

if we take \( l_* = l_*(\epsilon) \sim \log A^{l_*} \epsilon \) (more precisely, \( A^{l_*} = \epsilon^{-1/3} \)). Moreover, there is

\[
|G_{l_{1-1};A^{l_*}}(k,k')| < e^{-\omega_0|k-k'|}, \quad \text{for } |k|, |k'| \leq A^{l_*}.
\]
Similarly, letting
\[ \Lambda_{l}^{(3)} = \{ \mathcal{J} \in \tilde{\Lambda} : |(k, \omega_0(\xi)) + \Omega_{j_1} + \Omega_{j_2}| > \sqrt{\varepsilon}|k|^{-r} \] holds for all \( 1 \leq j_1, j_2 \leq n, |k| \leq A_l \) and for all \( \xi \in \mathcal{J} \),
we have (2.53) and (2.54) hold on \( \bigcup_{\mathcal{J} \in \Lambda_{l}^{(3)}} \mathcal{J} \) when replacing \( G_{l-1;A_l} \) by \( G_{l-1;A_l} \). There is also the measure estimate as that in (2.51) for \( \Lambda_{l}^{(2)} \) and \( \Lambda_{l}^{(3)} \), which implies
\[ \text{mes}\left( \bigcup_{\mathcal{J} \in \Lambda_{l}^{(3)} \cap \Lambda_{l}^{(2)} \cap \Lambda_{l}^{(3)}} \mathcal{J} \right) \leq \sqrt{\varepsilon} < A^{-(\log l)^{cs}}. \]
Then, taking
\[ \Pi_{l} = \bigcup_{\mathcal{J} \in \Lambda_{l}^{(3)} \cap \Lambda_{l}^{(2)} \cap \Lambda_{l}^{(3)}} \mathcal{J}, \]
we obtain the desired parameter set in the statement (l,4). This verifies the first step of the iteration in the Iterative Lemma.

Letting \( \Pi_{\infty} = \cap_{l=2}^{\infty} \Pi_{l} \), the convergence of the iteration on the uniform domain \( \mathcal{D}(\frac{m}{12}, \frac{m}{2}) \times \Pi_{\infty} \) is standard and we omit the details. \( \square \)

3. Green’s function estimate

This section is devoted to the proof of Lemma 2.4 in which, for simplicity, we have dropped the iterative subscript \( m \) for some expressions. For reader’s convenience, we recall and explain some notations at the beginning.

Recall that
\[ \varepsilon = A^{-\left(\frac{1}{4}\right)^{m}}, \quad N = A^{m+1}, \quad \Pi = \Pi_{m} = \bigcup_{\mathcal{J} \in \Lambda_{m}} \mathcal{J}, \quad (r, s) = (r_{m}, s_{m}), \quad \omega = \omega_{m}. \]
The matrix \( T = T_{m} \) in Lemma 2.4 (depending on the \( m \)-th iteration) is defined by
\[ T = D + S, \quad D = D_{m}, \quad S = S_{m}, \]
where the diagonal matrix
\[ D_{m}(j, k) = \Omega_{j} + \langle k, \omega_{m} \rangle \]
and non-diagonal matrix
\[ S_{m}((j, k), (j', k')) = (B_{m, jj'} + R_{m, jj'}^{\Omega}(k - k')), \]
with \( \langle \cdot \rangle^{\Omega}(k) \) being the \( k \)-th Fourier coefficient of the associated function. In what follows, we shall also consider those matrices depending on the \( l \)-th iteration. To make a distinction, we recall (2.2)-(2.4) that
\[ T_{l} = D_{l} + S_{l}, \quad l_{s} \leq l \leq m, \]
where \( D_{l} \) and \( S_{l} \) are defined by (3.1) and (3.2) upon replacing \( m \) by \( l \), respectively.

For any set \( U \subset \mathbb{Z}^{d} \), we denote by \( T_{l;U} \) the restriction of \( T_{l} \) on \( \{1, \cdots, n\} \times U \). For any integer \( M > 0 \), we write \( T_{l;M} = T_{l;[-M^{d}, M^{d}]^{d} \cap \mathbb{Z}^{d}} \) by some abuse of notation. As a result, \( T_{N} = T_{m;N} \) in Lemma 2.4 denotes the restriction of \( T_{m} \) on \( \{1, \cdots, n\} \times ([-A^{m+1}, A^{m+1}]^{d} \cap \mathbb{Z}^{d}) \). Our goal in this section is to construct and control the inverse of the matrix \( T_{N} \), i.e., to establish the Green’s function estimate for \( T_{N} \).
By the definition of \( S_l \) in (3.4) (replacing \( m \) by any \( l_* \leq l \leq m \)), one readily sees that \( S_l \) is a Toeplitz matrix with respect to the indices \( k, k' \) in \( \mathbb{Z}^d \), i.e.,

\[
S_l((j, k + p), (j', k' + p)) = S_l((j, k), (j', k'))
\]

for any \( p \in \mathbb{Z}^d \). Moreover, since \( B_l(x) \) and \( R^S_l(x) = \partial_x \partial_{x'} P^0_l(x, 0, 0) \) are analytic in \( x \in \mathbb{T}^d \), we have

\[
|\tilde{B}_l(k)| \leq e^{-x'|k|}, \quad |\tilde{R}^S_l(k)| < \frac{1}{ \sup_{r \in \mathcal{S}(k, r)} |\partial_x P^0_l(e^{-x}|k|) |} e^{-x'|k|}.
\]

Consequently, the matrix \( S_l \) enjoys the off-diagonal exponential decay

\[
|S_l((j, k), (j', k'))| \leq e^{-x|k-k'|}.
\]

Throughout the proof of Lemma 3.1, one easily finds that the spatial indices \( j, j' \) play seldom role in establishing the Green’s function estimate, except those estimates involving absolute constants depending only on \( n \). For that reason, we omit the finite indices and write \( S(k, k') = S((j, k), (j', k')) \) for simplicity.

Now we give an outline of the construction and estimate of the Green’s function \( G_N = T_N^{-1} \).

By the Iterative Lemma, we are able to obtain the Green’s function estimates for \( G_K \) with \( K \sim (\log N)C \). Then we shall apply the large deviation estimate to establish the Green’s function estimate for all \( G_{k_0 + [-M_0, M_0]}^r \) with \( K/2 \leq k_0 \leq N \) and \( M_0 \sim (\log N)C \), in which parameter exclusion should be taken care of by the semialgebraic sets arguments. Finally, we employ a coupling lemma with two scales \((K \text{ and } M_0)\) to prove (3.29) for \( G_N \), in which one should be careful on the loss of the decay rate.

Due to the rapid convergence of the Newton iteration, we can study those suitable matrices \( T_l \) with \( l < m \) and work out the Green’s function estimate for them. Then \( G_K \) and \( G_{k_0 + [-M_0, M_0]}^r \) can be derived directly from \( G_l \) and \( G_{l_0 + [-M_0, M_0]} \) by employing the Neumann series.

We organize this section as follows. In subsection 3.1, we give some auxiliary lemmas, which are frequently used in this section. In subsection 3.2, we apply the large deviation theorem and the multiscale analysis method to establish the estimate of the Green’s function \( G_{M_0}^r \). In subsection 3.3, we employ the semialgebraic set method to give the measure estimate and obtain the desired parameter \( \Pi_{m+1} \) in the Iterative Lemma. Finally, we apply the coupling lemma to prove the estimate of the Green’s function \( G_N \), which completes the proof of Lemma 3.1.

### 3.1. Preliminary

We first give a quantitative lemma here based on the Neumann series, which is frequently used throughout this section. It is worthy mentioning that the matrix \( T \), the integer \( \mathcal{N} \) and \( \epsilon \) in Lemma 3.1 are arbitrary and independent of the KAM iterations.

**Lemma 3.1.** Let \( U \subset \mathbb{Z}^d \) satisfy the diameter \( |U| = \mathcal{N} > 0 \) and let \( T, T' \) be two linear operators on \( \ell^2(\mathbb{Z}^d) \). Denote \( \mathcal{T}_U = R_U T R_U \) with \( R_U \) being the restriction operator on \( U \). Let further \( \alpha > 0, \rho > 0 \) and \( 0 < b < \theta < 1 \).

Assume the following properties hold.

(i) \( \mathcal{G}_U = \mathcal{T}_U^{-1} \) admits the Green’s function estimate

\[
\| \mathcal{G}_U \| \leq e^{N^b}, \quad |\mathcal{G}_U(x, y)| \leq e^{-\alpha|x-y|} \quad \text{for } |x-y| > \mathcal{N}^\theta.
\]

(ii) For all \( x, y \in U \),

\[
|\mathcal{(T}_U' - \mathcal{T}_U)(x, y)| \leq \epsilon e^{-\rho|x-y|}.
\]
Then, if $\varepsilon < e^{-4pN^\theta}$, we have
\[
\|G'_U\| \leq 2\|G_U\|, \\
\|T'_U(x, y)\| \leq 2e^{-(\alpha \wedge \rho)|x-y|}, \quad \text{for } |x - y| > N^\theta,
\]
where $\alpha \wedge \rho = \min(\alpha, \rho)$.

**Proof.** It is easy to see $T'_U = T_U(Id + G_U(T' - T))$ and we write
\[
\Delta = G_U(T' - T).
\]
Then by assumptions, $\|\Delta\| \leq 1/2$, which together with Neumann series argument implies $\|G'_U\| \leq 2\|G_U\|$. For any integer $s \geq 1$, we compute
\[
\Delta^s(x, y) = \sum_{j_0, \ldots, j_{s-1} \in U} \Delta(k_0, k_1)\Delta(k_1, k_2) \cdots \Delta(k_{s-1}, k_s)
\]
where $k_0 = x$ and $k_s = y$. If $|k_j - l_j| > N^\theta$, there is
\[
|G_U(k_j, l_j)| \cdot |(T' - T)(l_j, k_{j+1})| < \varepsilon e^{-(\alpha \wedge \rho)|k_j - k_{j+1}|},
\]
and if $|k_j - l_j| \leq N^\theta$, there is
\[
|G_U(k_j, l_j)| \cdot |(T' - T)(l_j, k_{j+1})| < \varepsilon e^{N^\theta + \rho N^\theta - \rho |k_j - k_{j+1}|}.
\]
It follows from $\varepsilon < e^{-4pN^\theta}$ that
\[
|\Delta^s(x, y)| < (CN)^{2ds} \varepsilon e^{N^\theta + \rho N^\theta} e^{-\rho |x-y|} < e^{-2pN^\theta \varepsilon} e^{-\rho |x-y|}
\]
and thus
\[
\left| \sum_{s=1}^{\infty} \Delta^s(x, y) \right| < 2e^{-2pN^\theta} e^{-\rho |x-y|}.
\]
Finally, for any $x, y \in U$ we have
\[
|G'_U(x, y)| < |G_U(x, y)| + \sum_{l \in U} \sum_{s=1}^{\infty} |\Delta^s(x, l)| \cdot |G_U(l, y)|
\]
\[
< |G_U(x, y)| + \sum_{l \in U, |l-y| \leq N^\theta} \sum_{s=1}^{\infty} |\Delta^s(x, l)| \cdot |G_U(l, y)|
\]
\[
+ \sum_{l \in U, |l-y| > N^\theta} \sum_{s=1}^{\infty} |\Delta^s(x, l)| \cdot |G_U(l, y)|
\]
\[
< |G_U(x, y)| + (CN)^d e^{-2pN^\theta} e^{-\rho |x-y|} + (CN)^d e^{-2pN^\theta} e^{N^\theta + \rho N^\theta} e^{-\rho |x-y|}
\]
\[
< |G_U(x, y)| + \frac{1}{10} e^{-\rho |x-y|}.
\]
As a result, whenever $|x - y| > N^\theta$,
\[
|G'_U(x, y)| < 2e^{-(\alpha \wedge \rho)|x-y|}.
\]
This completes the proof. $\square$
Next we describe quantitatively the variation of $T_i$, which enables us to apply Lemma 3.1 in what follows.

**Lemma 3.2.** Let $l < l' < m$ and consider the linear operator $T_i, T_{i'}$ defined in (3.3). Let further $T = T_{i,A^p}$ and $T' = T_{i',A^{p'}}$ be the restriction of $T_i, T_{i'}$ on $[-A^p, A^{p'}][d]$. Then, we have

$$|(T' - T)(k, k')| \leq A^p \cdot \varepsilon_i^{1/10} \exp(-s_{p'}|k - k'|).$$

**Proof.** By definition we have

$$(T_i - T_{i'})(k, k') = (\omega_i - \omega_{i'}, k) \cdot \delta_{kl} + (B_i - B_{i'})^\omega (k - k') + (R_i^{z_{kl}} - R_{i'}^{z_{kl}})^\omega (k - k'),$$

where $\delta_{kl}$ equals to one if $k = k'$ and vanishes otherwise. By the Iterative Lemma, there is

$$|\omega_i - \omega_{i'}| \leq \varepsilon_i^{1/10}.$$

Moreover,

$$B_i - B_{i'} = \sum_{r = 1}^{r - 1} R_i^{z_{kl}} + \{N_i, F_i\}^{z_{kl}} + \{F_{i'}^{r_{high}}, F_{i'}\}^{z_{kl}}$$

and

$$R_i^{z_{kl}} - R_{i'}^{z_{kl}} = \partial_{z^l} P_{i, r}^{\text{low}}(x, 0, 0, 0) - \partial_{z^l} P_{i', r}^{\text{low}}(x, 0, 0, 0).$$

The property $\sup_{x \in T_{i,r}^{z_{kl}}} |R_i^{z_{kl}}(x) - R_{i'}^{z_{kl}}(x)| \leq \varepsilon_i$ ensures

$$\sup_{x \in T_{i,r}^{z_{kl}}} |R_i^{z_{kl}}(x)| \leq \varepsilon_i.$$ 

Since $R_i^{z_{kl}} = \partial_{z^l} P_{i, r}^{\text{low}}(x, 0, 0, 0)$, there is also

$$\sup_{x \in T_{i,r}^{z_{kl}}} |R_i^{z_{kl}}(x)| \leq \varepsilon_i.$$ 

and

$$\sup_{x \in T_{i', r}^{z_{kl}}} |[E_i, F_i]^{z_{kl}}(x) + \{F_{i'}^{r_{high}}, F_{i'}\}^{z_{kl}}(x)| \leq \varepsilon_i^{1/3}(l + 1)^C < \varepsilon_i^{1/4}.$$ 

Hence

$$\sup_{x \in T_{i', r}^{z_{kl}}} |B_i(x) - B_{i'}(x)| < \varepsilon_i^{1/10}$$

and the conclusion follows. \qed

We finally cite here a decomposition lemma in [6, Lemma 9.9].

**Lemma 3.3.** Let $S \subset [0, 1]^{2n}$ be a semi-algebraic set of degree $B$ and $\mes_{2n}(S) < \eta, \log B \ll \log \frac{1}{\eta}$. We denote $(\omega, x) \in [0, 1]^n \times [0, 1]^{n}$ the product variable. Fix $\varepsilon > \eta^{1/2}$. Then there is a decomposition

$$S = S_1 \cup S_2,$$

$S_1$ satisfying

$$|\text{Proj}_{\omega} S_1| < B^C \varepsilon$$

and $S_2$ satisfying the transversality property

$$\mes_n(S_2 \cap L) < B^C \varepsilon^{-1} \eta^{1/2}$$

for any $n$-dimensional hyperplane $L$ such that $\max_{0 \leq j \leq n-1} |\text{Proj}_L(e_j)| < \frac{1}{100} \varepsilon$ (we denote $(e_0, \cdots, e_{n-1})$ the $\omega$-coordinate vectors.)
3.2. Estimate of $G_{M_0}^\sigma$. In this part, our goal is to establish the following type of Green’s function estimate

\begin{align}
\|G_{U(k_0)}\| &< e^{M_0^d}, \\
|G_{U(k_0)}(k, k')| &< e^{-\alpha''|k-k'|} \quad \text{for }|k-k'| > M_0^d,
\end{align}

(3.6)

for all $T_{U(k_0)}$ with $K/2 \leq |k_0| \leq N$, where $0 < b < \theta < 1$, $\alpha'' > 0$ is to be specified, $U(k_0) = k_0 + [-M_0, M_0]^d$ and $M_0 = (\log N)^{\tilde{c}_1}, \ \log K = (\log M_0)^{\tilde{c}_2}$.

As mentioned before, we shall work on some $T_{l_0; U(k_0)}$ with $l_0 < m$ rather than on $T_{m; U(k_0)}$ directly, due to the rapid convergence of the Newton iteration. This can be resolved by a simple application of Neumann series (see Lemma 3.1). Indeed, choosing

\begin{align}
l_0 = C_8 \log M_0, \quad \text{with } C_8 > \frac{1 + \log 10}{\log \frac{4}{3}},
\end{align}

(3.7)

we see from Lemma 3.2 that

$$|T_{U(k_0)}(k, k') - T_{l_0; U(k_0)}(k, k')| \leq \epsilon_{l_0}^{1/10} \cdot N \exp(-s|k-k'|), \quad s = s_m.$$ 

Suppose (3.6) is valid for $G_{l_0; U(k_0)}$, then, by verifying

$$\epsilon_{l_0}^{1/10} \cdot N < \epsilon_{l_0}^{1/20} < \frac{1}{100} e^{-M_0^d},$$

it follows from Lemma 3.1 that Green’s function estimate (3.16) also holds for $G_{U(k_0)}$ up to a constant multiplication. To this end, we shall establish (3.6) for $G_{l_0; U(k_0)}$ in what follows.

Recalling the Toeplitz property (3.4) for $T_l$, $l_0 \leq l \leq m$, we denote

$$T_l^\sigma = D_l^\sigma + S_l,$

(3.8)

where $S_l$ is defined in (3.2) (replacing $m$ by $l$) and $D_l^\sigma$ takes the form of

$$D^\sigma_l(j, k) = \sigma + \langle k, \omega_l \rangle + \Omega_j, \quad 1 \leq j \leq n, \ k \in \mathbb{Z}^d.$$ 

Observe by the Toeplitz property that

$$T_{l_0; U(k_0)} = T_{l_0; U(k_0)}^{\sigma=0} = T_{l_0; M_0}^{\sigma=\langle \omega_l, 0 \rangle}.$$ 

Then it suffices to establish the Green’s function estimate of $T_{l_0, M_0}^{\sigma}$ for $

\sigma \in \{(k, \omega_{l_0}) : K/2 \leq k \leq N\}.$

The lemma below is the core of our analysis in this part, which is independent of the Iterative Lemma and whose proof is delayed to the appendix. To formulate it, we need to introduce the elementary regions. An elementary region is defined to be a set $U$ of the form

$$U = R \setminus (R + z)$$

where $z \in \mathbb{Z}^d$ is arbitrary and $R$ is a block in $\mathbb{Z}^d$, i.e.,

$$R = \{y = (y_1, \ldots, y_d) \in \mathbb{Z}^d : y_i \in [x_i - M_i, x_i + M_i], i = 1 \cdots, d\}.$$

The diameter of an elementary region $U$ is denoted by $|U|$. The set of all elementary regions of diameter $M$ is denoted by $\mathcal{ER}(M)$. The class of elementary regions consists of $d$-dimensional rectangles, L-shaped regions and $(d-1)$-dimensional rectangles with normal vector parallel to the axis.
Lemma 3.4. Consider the matrix $T^{\sigma} = D^\sigma + S$, where $\sigma \in \mathbb{R}$ and $D^\sigma$ is a diagonal matrix with

$$D^\sigma(j,k) = \langle k, \omega \rangle + \sigma + \Omega_j, \quad 1 \leq j \leq n, \ k \in \mathbb{Z}^d,$$

and we omit the finite index $j$ for simplicity. Let $N_0, N_0^C$ be sufficiently large and let the various constants below satisfy

$$0 < \beta \ll 1, \quad 1 - \frac{\beta}{10} < b < \theta < 1, \quad \alpha_0 > 0, \quad \rho > 0.$$

Assume the following properties hold.

(i) The matrix $S$ satisfies the Toeplitz property with respect to the $k$-index and

$$|S(x,y)| < \epsilon e^{-\rho |x-y|}, \quad x \neq y.$$

(ii) The frequency $\omega$ satisfies Diophantine condition

$$|\langle k, \omega \rangle| > \nu |k|^{-\tau}, \quad 0 < \nu < 1, \tau > d + 1.$$

(iii) For any $N_0 < N_0^c < N_0$ and any elementary region $U_0 \in ER(N_0)$, the Green’s function estimate

$$\|G_{U_0}\| < e^{N_0^b}, \quad |G_{U_0}(x,y)| < e^{-\mu |x-y|}, \quad \text{for } |x-y| > N_0^c,$$

holds for all $\sigma$ except in a set $E_0(U_0)$ of measure at most $e^{-N_0^3}$. Then for any large $N > N_0^c$ and any elementary region $U \in ER(N)$, the Green’s function estimate

$$\|G_U\| < e^{N^b}, \quad |G_U(x,y)| < e^{-\mu |x-y|}, \quad \text{for } |x-y| > N^d,$$

holds for all $\sigma \in \mathbb{R}$ outside of a set $E = E(U)$ with

$$\text{mes}(E) < e^{-N^3},$$

where $\alpha > (\alpha_0 \wedge \rho) - (\log N_0)^{-8}$.

Proof. We prove the proposition by the method of inductions on $l$. The initial steps ($l_* \leq l \leq l_C$) are essentially a direct application of the Neumann series provided the perturbation is small enough and we omit it. See also the similar arguments in subsection 2.4.

Proposition 3.6. Under the assumptions of Lemma 2.7 we consider a family of matrices $T^\sigma_l$ defined by (3.8). Let $q(l) = \frac{\log 2}{2 \log d} l$. Then for all $l_* \leq l \leq m$ and any elementary region $U \in ER(A^{q(l)})$, there is

$$\|G_{U}^\sigma\| < e^{A^{q(l)a}}, \quad |G_{U}^\sigma(k,k')| < e^{-a'(l)|k-k'|}, \quad \text{for } |k-k'| > A^{q(l)a},$$

for all $\sigma$ except in a set $E_l = E_l(U)$ with $\text{mes}(E_l) < e^{-A^q \beta^3}$, where $a'(l) > s_{l+1}$.

Proof. We prove the proposition by the method of inductions on $l$. The initial steps ($l_* \leq l \leq l_C$) are essentially a direct application of the Neumann series provided the perturbation is small enough and we omit it. See also the similar arguments in subsection 2.4.
Assume by induction that the property (3.11) holds with \( l < m \). We need to establish (3.11) for \( l + 1 \) and any \( U \in \mathcal{E}\mathcal{R}(A^{q(l+1)}) \). Observe first by similar computations in Lemma 3.2 that, for any \( V \in \mathcal{E}\mathcal{R}(A^{q(l)}) \), there is

\[
|\langle T_{l+1}^\sigma - T_l^\sigma \rangle(k, k')| < e^{\frac{1}{20}} \exp(-s_{l+1}|k - k'|).
\]

Since \( e^{\frac{1}{20}} = A^{-\frac{1}{4}\beta} < e^{-A^{q(l)}} \), it follows from Lemma 3.1 that

\[
||G_{l+1}^\sigma|| < e^{-A^{q(l)}},
\]

\[
|G_{l+1}^\sigma(k, k')| < e^{-A^{q(l)}} |k - k'| \quad \text{for } |k - k'| > A^{q(l)},
\]

essentially holds for all \( \sigma \) except in a set \( \mathcal{E}_l \) with \( (\mathcal{E}_l) < e^{-A^{q(l)}\beta^2} \). Now we apply Lemma 3.4 by taking

\[
T^\sigma = T_l^\sigma, \quad N = A^{q(l+1)}, \quad N_0 = A^{q(l)}, \quad \rho = s_{l+1}, \quad \omega = \omega_{l+1}.
\]

Then we obtain (3.11) for \( l + 1 \) with \( \alpha'(l + 1) > s_{l+1} - (\log A^{q(l)})^{-8} > s_{l+2} \). This completes the proof of the induction statements. \( \square \)

Recall that \( l_0 \) and \( M_0 \) are fixed in (3.7) (depending on \( N \)). Back to \( T_{l_0; M_0}^\sigma \), we have

**Proposition 3.7.** Under the assumption of the Iterative Lemma 2.1, we have

\[
||G_{l_0; M_0}^\sigma|| < e^{M_0}.
\]

(3.12)

\[
|G_{l_0; M_0}^\sigma(k, k')| < \exp(-\alpha''|k - k'|) \quad \text{for } |k - k'| > M_0.
\]

except for all \( \sigma \in \mathbb{R} \) outside of a set \( \mathcal{E}_{l_0} \) with \( (\mathcal{E}_{l_0}) < e^{-M_0^\beta} \), where \( \alpha'' > s_0 \) and \( 0 < \beta \ll 1 \).

**Proof.** For fixed \( l_0 = l_0(N) \sim \log(m + 1) \), we define \( N_0, l_0' \) and \( l_0 \) in order as follows

\[
N_0 = 2 \exp(l_0^{1/4}), \quad N_0 = A^{l_0'}, \quad l_0 = \frac{2 \log A}{\log 2} l_0' \sim \log N_0.
\]

(3.13)

By Proposition 3.6 for \( l_0 \sim (\log(m + 1))^{1/4} \) and \( N_0 \) satisfying (3.13), there is, for any \( U_0 \in \mathcal{E}\mathcal{R}(N_0) \), the estimate

\[
|G_{l_0; U_0}^\sigma| < e^{N_0},
\]

(3.14)

\[
|G_{l_0; U_0}^\sigma(k, k')| < e^{-\alpha'(l_0)|k - k'|}, \quad \text{for } |k - k'| > N_0.
\]

holds for all \( \sigma \) except in a set \( \mathcal{E}_{N_0} \) with \( (\mathcal{E}_{N_0}) < e^{-N_0^\beta} \), where \( \alpha'(l_0) > s_{l_0 + 1} \). It then follows from Lemma 3.1 that the Green’s function estimate (3.14) essentially holds when replacing \( G_{l_0; U_0}^\sigma \) by \( G_{l_0; U_0}^\sigma \), since \( e^{l_0} e^{N_0} = A^{-\frac{1}{2}l_0} A^{\frac{1}{2}N_0} < 1 \). By (3.4.2) in the Iterative Lemma, we have

\[
|\langle k, \omega_{l_0}(\xi) \rangle| > \frac{\nu}{|k|}, \quad 0 \neq |k| \leq M_0, \quad k \in \mathbb{Z}^d,
\]

(3.15)

where \( \nu \sim \sqrt{\epsilon} \) and \( \tau > d + 1 \). Taking

\[
T^\sigma = T_l^\sigma, \quad N = M_0, \quad \rho = s_{l_0}, \quad \omega = \omega_{l_0}(\xi),
\]

and applying Lemma 3.4, we have

\[
||G_{l_0; M_0}^\sigma|| < e^{M_0},
\]

(3.16)

\[
|G_{l_0; M_0}^\sigma(k, k')| < \exp(-\alpha''|k - k'|) \quad \text{for } |k - k'| > M_0,
\]
except for all $\sigma \in \mathbb{R}$ outside of a set $\mathcal{E}_{M_0}$ with $\mes(\mathcal{E}_{M_0}) < e^{-M_0}$, where
\[
\alpha'' = (\alpha' \wedge s_{i_0}) - (\log N_0)^{-s} > s_{i_0} - (\log N_0)^{-s} > s_{m+1/10} > s = s_m.
\]

This completes the proof. \hfill \Box

3.3. Elimination of $\sigma$ and measure estimate. In this part, we shall eliminate the additional parameter $\sigma$ and establish the Green’s function estimates for all $G_{k_0; U(k_0)}$ with $K/2 \leq |k_0| \leq N$ and $U(k_0) = (k_0 + [-M_0, M_0]^d) \cap \mathbb{Z}^d$. This requires a further parameter exclusion, whose measure is estimated by the decomposition theorem for semialgebraic sets. For that reason, we need to give a semialgebraic description for the breakdown of the Green’s function estimate. The main result in this part is presented below.

**Lemma 3.5.** Under the assumption of Lemma 2.1 there exists a measurable set $\Pi_{+}^{(1)} \subset \mathbb{R}^d$ such that for all $x \in \mathcal{O}(\Pi_{+}^{(1)}, A^{-(m+1)\beta/3})$ there is
\[
\| G_{k_0; U(k_0)} \| < e^{M_0}, \quad 0 < b < 1,
\]
\[
| G_{k_0; U(k_0)}(k, k') | < e^{-|k-k'|}, \quad \text{for } |k-k'| > M_0^0,
\]
for all $K/2 \leq |k_0| \leq N$. Moreover, $\Pi_{+}^{(1)}$ is the union of a family of disjoint intervals $\mathcal{J}'$ with $| \mathcal{J}' | = A^{-(m+1)\beta/3}$. For each $\mathcal{J}'$, there is a unique $\mathcal{J} \in \Lambda_m$ such that $\mathcal{J}' \subset \mathcal{J}$. The total removed set satisfies
\[
\mes(\Pi \setminus \Pi_{+}^{(1)}) < \frac{1}{3} A^{-(\log(m+1)\beta/3)}.
\]

**Remark 3.1.** In the proof of Lemma 3.5 we shall pave $\Pi$ into a collection of intervals of diameter $A^{-(m+1)\beta/3}$, with the shrunken parameter set $\Pi_{+}^{(1)} \subset \Pi$ being a sub-collection. When solving (2.17)-(2.18) by the same method, we would obtain another set $\Pi_{+}^{(2)}$ which is also a sub-collection of the $A^{-(m+1)\beta/3}$-intervals paving $\Pi$. Then we obtain the desired set $\Pi_+ = \Pi_{+}^{(1)} \cap \Pi_{+}^{(2)}$, which satisfies $\mes(\Pi \setminus \Pi_+) < A^{-(\log(m+1)\beta/3)}$. Moreover, $\Pi_+$ is a $\Lambda_+$ collection of disjoint $A^{-(m+1)\beta/3}$-intervals. For each $\mathcal{J}' \in \Lambda_+$, there is a unique $\mathcal{J} \in \Lambda$ such that $\mathcal{J}' \subset \mathcal{J}$.

**Proof.** We divide the proof into three steps. Step one is devoted to the truncation of parameters in the Green’s function estimate, which enables us to make a semialgebraic description. Step two is devoted to the elimination of the additional parameter $\sigma$. Step three is devoted to the construction of the desired parameter set and establishing the associated measure estimate.

**Step one.** From the Iterative Lemma, we know that $T_{k_0}^\sigma$ is analytic in $\xi \in \mathcal{O}_{l_0} = \mathcal{O}(\Pi_{l_0}, A^{-\delta_{\mathcal{J}_0}})$ with $\Pi_{l_0} = \bigcup_{\mathcal{J} \in \Lambda_{l_0}} \mathcal{J}$ and $| \mathcal{J} | = A^{-\delta_{\mathcal{J}_0}}$. Fix any $\mathcal{J} \in \Lambda_{l_0}$ and denote the center of $\mathcal{J}$ by $\xi_0$. Recall that $T_{k_0; M_0}^\sigma = D_{l_0}^\sigma + S_{l_0}$ with
\[
D_{l_0}^\sigma(j, k; \xi) = \sigma + \langle k, \omega_{l_0}(\xi) \rangle + \Omega_j,
\]
and
\[
S_{l_0}(k, k') = (B_{l_0}(\xi) + R_{l_0}^2(\xi)) \chi(k - k').
\]
Let \( p = A_0^{\ell_3} \) with \( C_3 \) given by (2.5). By Taylor’s formula, we denote
\[
\omega_l^p(\xi) = \sum_{i \leq p} \frac{\omega_l^0(\xi_0)}{i!}(\xi - \xi_0)^i,
\]
\[
B_l^p(\xi) = \sum_{i \leq p} \frac{B_l^0(\xi_0)}{i!}(\xi - \xi_0)^i, \quad R_l^{\xi;\xi}(\xi) = \sum_{i \leq p} \frac{(R_l^{\xi;\xi})^0(\xi_0)}{i!}(\xi - \xi_0)^i,
\]
\[
D_{l_0}^{\sigma;\xi}(j, k; \xi) = \sigma + \langle k, \omega_l^\xi(\xi) \rangle + \Omega_j, \quad S_l^\xi(k, k') = (B_l^\xi(\xi) + R_l^{\xi;\xi}(\xi))(k - k'),
\]
\[
T_{l_0}^{\sigma;\xi}(k, k') = D_{l_0}^{\sigma;\xi}(x) + S_l^\xi(k, k').
\]
As a result, \( T_{l_0}^{\sigma;\xi} \) is a polynomial function in \( \xi \), whose degree
\[
\deg_\xi T_{l_0}^{\sigma;\xi}(k, k') \leq p.
\]
Obviously, the truncation error satisfies
\[
|(T_{l_0;M_0}^{\sigma} - T_{l_0;M_0}^{\sigma;\xi})(k, k')| \leq |M_0|\omega_l^\xi - \omega_l^\xi| + (B_l^\xi - B_l^\xi) + (R_l^{\xi;\xi} - R_l^{\xi;\xi})| \exp(-s_l|k - k'|).
\]
For \( |\xi - \xi_0| \leq k|\mathcal{J}| = \kappa A^{-\ell_3} \) with \( \kappa \approx 1 \) to be specified, we see from Cauchy’s estimate that
\[
(3.18) \quad |\omega_l^\xi - \omega_l^\xi| \leq \sup_{\xi \in \mathcal{J}} |\omega_l^\xi| \cdot \frac{|\xi - \xi_0|^{p+1}}{|\mathcal{J}|^p \cdot (|\mathcal{J}| - |\xi - \xi_0|)} \leq \frac{k^{p+1}}{1 - \kappa}.
\]
Since \( B_l^\xi \) and \( R_l^{\xi;\xi} \) stay uniformly bounded in their analytical domain, there is also
\[
\sup_{|\xi - \xi_0| \leq k|\mathcal{J}|} \left( |B_l^\xi - B_l^\xi|, |R_l^{\xi;\xi} - R_l^{\xi;\xi}| \right) \leq \frac{k^{p+1}}{1 - \kappa}.
\]
On the one hand, letting
\[
\mathcal{Y}_1 = \bigcup_{\mathcal{J} \in A_{l_0}} \{ \xi \in \mathcal{J} : \kappa A^{-\ell_3} < |\xi - \xi_0| \leq A^{-\ell_3} \} \subset \mathbb{R}^d,
\]
we have
\[
\text{mes } \mathcal{Y}_1 \leq_d \frac{1}{|\mathcal{J}|^d} \cdot (1 - \kappa)|\mathcal{J}| \leq (1 - \kappa)A^{(d-1)\ell_3}.
\]
Taking
\[
\kappa = 1 - A^{-(\log(m+1)\ell_3)}
\]
with \( C_6 \) given by (2.5), then
\[
\text{mes } (\mathcal{Y}_1) \leq A^{-(\log(m+1)\ell_3)}A^{(d-1)(\log(m+1)\ell_3)} < \frac{1}{100}A^{-(\log(m+1)\ell_4)}
\]
provided \( m \) is large.

On the other hand, for \( |\xi - \xi_0| < \kappa|\mathcal{J}| \), there is
\[
|(T_{l_0;M_0}^{\sigma} - T_{l_0;M_0}^{\sigma;\xi})(k, k')| \leq M_0 \frac{k^{p+1}}{1 - \kappa} \exp(-s_l|k - k'|).
\]
By noticing \( p = A_0^{\ell_3} \) and \( C_5 = C_6 + 2 \), we get
\[
M_0 \frac{k^{p+1}}{1 - \kappa} = M_0(1 - A^{-(\log(m+1)\ell_3)})^{p+1}A^{(\log(m+1)\ell_3)} < e^{-M_0} = e^{-C(m+1)\ell_5}.
\]
In conclusion, we have
\[(3.19) \quad |(T^{\sigma}_{l_0;M_0^0} - T^{\sigma}_{l_0;M_0})(k, k')| < e^{-M_0^0}e^{-\alpha \cdot |k-k'|} \quad \text{for } |k| \leq M_0, |k'| \leq M_0.\]

**Step two.** By Lemma 3.1 and Proposition 3.7, we also essentially have
\[(3.20) \quad ||G^{\sigma}_{l_0;M_0}(k, k')|| < e^{M_0^0},\]
\[(3.20) \quad |G^{\sigma}_{l_0;M_0}(k, k')| < \exp(-\alpha' |k-k'|) \quad \text{for } |k-k'| > M_0^0,\]
except for all \(\sigma \in \mathbb{R}\) outside of a set \(\mathcal{E}_{M_0}\) with \(\text{mes } \mathcal{E}_{M_0} < e^{-M_0^0}\). Using the formula
\[G^{\sigma}_{l_0;M_0}(k, k') = (T^{\sigma}_{l_0;M_0})(k, k')/ \det T^{\sigma}_{l_0;M_0}\]
with \((\cdot)^*\) being the adjoint matrix, we consider the set \(\mathcal{E}\) of the triplets \((\xi, \omega^\xi_{l_0^0}, \sigma)\) such that
\[|(k, \omega^\xi_{l_0^0})| > \frac{\nu}{|k|}, \quad 0 \neq |k| \leq M_0, \quad k \in \mathbb{Z}^d\]
and (3.20) fails. Obviously, \(\mathcal{E} \subset (\mathcal{J} \cap \mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}\) is a semi-algebraic set of degree at most \(M_0^0 p = M_0^0 A^{(\log(m+1))^{C_7}}\). Since \(T\) is restricted to \([-M_0, M_0]^d\), we may restrict \(\sigma\) to be in \([-CM_0, CM_0]\). Otherwise, \(T\) is diagonal dominated and it suffices to apply Neumann series to \(T^\sigma\) to get the desired estimate. We decompose \([-CM_0, CM_0]\) into intervals of length 1 and identify each of them with \([0, 1]\). Then \(\mathcal{E}\) is divided into \(CM_0\) sub-intervals \(\mathcal{E}'\).

Let \(\epsilon\) (in Lemma 3.3) be \(\epsilon = 2/K\) and
\[
\log K = (\log M_0)^{C_7}.
\]
We apply the decomposition Lemma 3.3 to the semialgebraic set \(\mathcal{E}'\) by identifying the algebraic curve \((\xi, \omega^\xi_{l_0^0})\) with an interval. Then we obtain
\[
\mathcal{E}' = \mathcal{E}'_1 \cup \mathcal{E}'_2
\]
with
\[
\text{Proj}_{(\xi, \omega^\xi_{l_0^0} \in \mathcal{E}_0)} \mathcal{E}'_2 < M_0^0 A^{\log(m+1)} \epsilon < A^{\log(m+1)} e^{-\log(m+1)} < A^{-\log(m+1)} \epsilon^{C_7},
\]
since \(C_7 > (C_4 + 10) \vee C_4\) in our choice (2.5) of constants.

Moreover, for any \(|k| > K/2\) and the hyperplane \(L_k = \{(\xi, \omega^\xi_{l_0^0}, (k, \omega^\xi_{l_0^0})\})\), there is
\[
\text{mes } (\mathcal{E}' \cap L_k) < M_0^0 A^{\log(m+1)} \epsilon^{-1} e^{-M_0^0} e^{-2(d)} < A^{\log(m+1)} \epsilon^{-1} e^{-M_0^0} e^{-2(d)} < e^{-(m+1)^2C_0}.
\]
Therefore, taking into consideration of each \(\mathcal{E}'\), there is a set \(\mathcal{Y}_2 \subset \mathcal{J}\) satisfying
\[
\text{mes } (\mathcal{Y}_2) < C M_0 \left(A^{-\log(m+1)} + A^{\log(m+1)} e^{-2(d)}\right) < A^{-\log(m+1)} \epsilon^{C_7},
\]
such that (3.20) holds for all \(\sigma = (k, \omega^\xi_{l_0^0})\) with \(K/2 \leq |k| \leq N\).

**Step three.** We divide \(\mathcal{J}\) into a sequence of disjoint sub-intervals with each interval of diameter \(A^{-(m+1)^2}\), i.e., \(\mathcal{J} = \mathcal{J} \cup \mathcal{J}'\) with \(|\mathcal{J}'| = A^{-(m+1)^2}\). Then for any \(\xi \in \mathcal{J}\) but lying outside the boundaries of the subintervals, there is a unique \(\mathcal{J}'\) such that \(\xi \in \mathcal{J}'\). Suppose

---

We omit the constant multiplier induced by Lemma 3.1 which finally can be absorbed by the margins in our estimates. See Lemma 3.3 for example.
Let the integers $\xi \in S_2$, i.e., (3.20) fails for $\xi$ and for some $\sigma = \langle k, \omega^{\xi}_0 \rangle$, $K/2 \leq |k| \leq N$. We have that, for all $\xi' \in S'$, (3.20) with the above $\sigma$ also fails but with a smaller constant due to Neumann series (actually, $M_0^{\sigma} A^{-(m+1)f_3} \ll e^{-M_0^\sigma}$), i.e., for some $\sigma = \langle k, \omega^{\xi}_0 \rangle$, there is
\[
\|G_{l_0,M_0^\sigma}(\xi')\| > \frac{1}{2} e^{M_0^\sigma},
\]
(3.21)
or
\[
\|G_{l_0,M_0^\sigma}(k', \xi')\| > \frac{1}{2} \exp(-\alpha' |k - k'|) \quad \text{for some } |k - k'| > M_0^\sigma.
\]
As before, (3.21) also has a semialgebraic description. Denoting by $S'_2$ the set of all $\xi \in S$ such that (3.21) holds for some $\sigma = \langle k, \omega^{\xi}_0 \rangle$, we have $\text{mes}(S'_2) < A^{-(\log (m+1)k_3)^4 + \varepsilon}$. Moreover, we have
\[
\Delta S = \left\{ \xi' \in S' : \exists \xi \in S' \subset S \text{ s.t. } (3.20) \text{ fails for some } \sigma = \langle k, \omega^{\xi}_0 \rangle \right\} \subset S'_2.
\]
As a result, $S \setminus \Delta S$ is the union of a sequence of intervals with each interval of diameter $A^{-(m+1)f_3}$ and $\text{mes}(\Delta S) < A^{-(\log (m+1)k_3)^4}$.

Letting $S$ range over $\Lambda_{l_0}$, the total measure of the set $\Delta S$ removed from $\Pi_{l_0}$ fulfills
\[
\text{mes}(\cup_{S \in \Lambda_{l_0}} \Delta S) < A_{l_0}^{\varepsilon} A^{-(\log (m+1)k_3)^4 + \varepsilon} < \frac{1}{100} A^{-(\log (m+1)k_3)^4}
\]
in view of $C_4 > C_3$ in (2.5).

Let $\Pi_+ = \cup_{S \in \Lambda_{l_0} \Pi \cap (S \setminus \Delta S)}$. Then $\Pi_+$ is a collection $A^{(1)}_{\Pi_+}$ of disjoint intervals with diameter $A^{-(m+1)f_3}$. Since $\Pi \subset \Pi_{l_0}$, for each interval $S' \in A^{(1)}_{\Pi_+}$, there is a unique $S \in A$ such that $S' \subset S$. On $\Pi_+$, (3.20) essentially holds (up to a constant multiplication by applying the Neumann series). From (3.19), we see that (3.17) essentially holds on $\Pi_+$. Since $C_3 > C_1 > C_0$ in (2.5), it then follows from Lemma 3.1 that (3.17) remains valid on $\partial (\Pi_+, A^{-(m+1)f_3})$ by verifying $A^{-(m+1)f_3} < A^{-M_0^\sigma}$.

This completes the proof of Lemma 3.5.

\[\square\]

3.4. Estimate of $G_N$. In this part, we shall establish the Green’s function estimate for $G_N$. As we mentioned before, we shall apply a coupling lemma involving two scales, which is independent of the KAM iteration.

Lemma 3.6. Let the matrix $T = D + \epsilon S$ defined on $[-N, N]^d \cap \mathbb{Z}^d$ satisfy
\[
|S(x,y)| < e^{\rho|x-y|}, \quad x \neq y.
\]
Let the integers $0 < 2M_0 < K < N$ be sufficiently large and the various constants below satisfy
\[
C_1 > C_0 > 10, \quad C_2 > 2C_1 + 10,
\]
\[
M_0 \sim (\log N)^{C_0}, \quad \log \log K \sim \log \log M_0,
\]
\[
0 < \frac{\rho_0}{2} < \rho < \bar{\alpha}, \quad \frac{\rho_0}{2} < \alpha < \bar{\alpha},
\]
\[
0 < b < \theta < 1 - \frac{8}{C_0}.
\]
Assume
- there is Green’s function estimate on $G_K$
\[
\|G_K\| \leq A^{(\log K)^{C_1}},
\]
\[
|G_K(x,y)| \leq e^{-\alpha|x-y|}, \quad \text{for } |x - y| > (\log K)^{C_2}.
\]
for each $|k_0| > K/2$, there is

$$\|G_{k_0+[-M_0,M_0]^d}\| \leq e^{M_0^d},$$

$$|G_{k_0+[-M_0,M_0]^d}(x,y)| \leq e^{-\gamma|x-y|}, \quad \text{for } |x - y| > M_0^d.$$  

The we have the Green’s function estimate

$$\|G_N\| < A^{(\log N)^{c_1}},$$

$$|G_N(x,y)| < e^{-\gamma|x-y|}, \quad \text{for } |x - y| > (\log N)^{C_2},$$

where $\gamma > (\alpha \wedge \rho) - (\log N)^{-8}$.

**Remark 3.2.** The above lemma also appeared in [6, Chapter 18] and [3, Lemma 5.1]. One should be very careful to establish Lemma 3.6 when taking into account the loss of regularity in the KAM iteration. In [6] and [3], the off-diagonal exponential decay for $G_{M_0}(k,k')$ is valid when $|k - k'| > \frac{1}{10} M_0$, rather than $|k - k'| > M_0^d$ in our imposition. We remark that in [6], this might lead to a great loss of regularity at each KAM step, which possibly impedes us to get a uniform analyticity domain for the angle variable. In [3], there is no such trouble since the matrix therein is of short range. This is the main reason why we establish the Green’s function estimate for those $|k - k'| > M_0^d$.

**Proof.** The proof is based on the application of the resolvent identity. We divide the proof into two parts which are on the norm control and the exponential decay estimate, respectively.

1. **Estimate of the norm.** For any fixed $x \in [-N, N]^d$, we define

$$U(x) = \begin{cases} 
[-K, K]^d, & \text{if } |x| \leq \frac{K}{2}, \\
(x + [-M_0, M_0]^d) \cap [-N, N]^d, & \text{if } |x| > \frac{K}{2}.
\end{cases}$$

For $|x| \leq K/2$, we have

$$\text{dist}(x, [-N, N]^d \setminus U(x)) \geq \frac{K}{2},$$

and for $|x| > K/2$, we have

$$\text{dist}(x, [-N, N]^d \setminus U(x)) > M_0.$$

Compute by the resolvent identity

$$|G_N(x,y)| \leq |G_{U(x)}(x,y)| \chi_{U(x)}(y) + \sum_{w \in U(x), v \in U(x)} |G_{U(x)}(x,w)| e^{-\rho|w-v|} |G_N(v,y)|.$$  

When $|x| \leq K/2$, we have

$$G_N(x,y) \leq |G_{U(x)}(x,y)| \chi_{U(x)}(y) + \varphi_K e^{-(\rho \wedge \alpha)|x-y|} |G_N(v,y)|$$

for some $|v| > K$, where

$$\varphi_K = 2K^d N^d A (\log K)^{c_1} e^{(\log K)^{c_2}} < N^{2d}.$$  

Since $|x - v| > K/2$, there is

$$\varphi_K e^{-(\rho \wedge \alpha)|x-y|} < N^{2d} e^{-\rho_0 K/4} < \frac{1}{10}.$$  

When $|x| > K/2$, we have

$$|G_N(x,y)| \leq |G_{U(x)}(x,y)| \chi_{U(x)}(y) + \varphi_L e^{-\rho_0 |v-y|} |G_N(v,y)|$$

for some $|v| > K$, where

$$\varphi_L = 2K^d N^d A (\log K)^{c_1} e^{(\log K)^{c_2}} < N^{2d} e^{-\rho_0 K/4} < \frac{1}{10}.$$  

When $|x| > K/2$, we have

$$|G_N(x,y)| \leq |G_{U(x)}(x,y)| \chi_{U(x)}(y) + \varphi_L e^{-\rho_0 |v-y|} |G_N(v,y)|$$

for some $|v| > K$, where

$$\varphi_L = 2K^d N^d A (\log K)^{c_1} e^{(\log K)^{c_2}} < N^{2d} e^{-\rho_0 K/4} < \frac{1}{10}.$$
for some \( v \) satisfying \(|v - x| > M_0\), where
\[
\varphi_{M_0} = 2M_0^dN^d e^{M_0^d} e^{pM_0^d} < e^{2pM_0^d}.
\]
Moreover,
\[
\varphi_{M_0} e^{-p|v-x|} < e^{2pM_0^d} e^{-pM_0} < \frac{1}{10}.
\]
In conclusion, we have
\[
|\mathcal{G}_N(x, y)| < (A^{(\log K)^{c_1}} + e^{M_0^d}) + \frac{1}{4} \max_{v \in [-N, N]^d} |\mathcal{G}_N(v, y)|.
\]
which further implies
\[
\max_{v \in [-N, N]^d} |\mathcal{G}_N(x, y)| < 2(A^{(\log K)^{c_1}} + e^{M_0^d})
\]
for any \( y \in [-N, N]^d \). By Schur’s criterion, we finally get
\[
(3.25) \quad ||\mathcal{G}_N|| < 2N^d (A^{(\log K)^{c_1}} + e^{M_0^d}) < A^{(\log N)^{c_1}}
\]
by our assumptions on the constants.

2. Exponential decay estimate.
For any \(|x|, |y| \leq N\), we apply (3.23) and (3.24) to take iterations. At each step, there are four cases. See table 1. When \(|x - y| > (\log N)^{c_2}\), the iteration would start from A2 or A4. Note also 10(\log N)^{c_2} < K and log K \sim \log \log N.

A sequence of iterations should obey the following rule
\[
\cdots \to (A2 \to A4) \to A4 \to \cdots, \\
or \quad \cdots \to (A2 \to A4) \to (A2 \to A4) \to \cdots, \\
or \quad \cdots \to A4 \to (A2 \to A4) \to \cdots, \\
or \quad \cdots \to A4 \to A4 \to \cdots.
\]
The iteration would stop in the following way
\[
\cdots A4 \to A1 / A3, \\
or \quad \cdots A2 \to A3.
\]
Assume we are able to iterate (A2 \to A4) for \( p \) times and iterate A4 alone for \( q \) times. Then we have
\[
(3.26) \quad |\mathcal{G}_N(x, y)| < (p + q - 1)\varphi_{M_0^d} p^{q-1} e^{-(\alpha p)|x-y|} + \varphi_{M_0^d} p^{q} e^{-(\alpha p)|x-y|} |\mathcal{G}_N(v_2p+q, y)|
\]
and
\[ |x - v_{2p+q}| > \frac{p}{2} + qM_0. \]

Let
\[ p\frac{K}{2} + q'M_0 = 10|x - y| \]
and thus \( p' \leq 20|x - y|/K \) and \( p' + q' \leq 10|x - y|/M_0 \). Moreover, we have
\[
\log \varphi_{K}^{p'} \leq \log N^{2dp'} \leq \frac{40d \log N}{K} |x - y| < \frac{1}{10(\log N)^{8}} |x - y|
\]
and also
\[
\log \varphi_{M_0}^{p' + q'} \leq \frac{20p}{M_0 - p} |x - y| < \frac{1}{10(\log N)^{8}} |x - y|.
\]

If \((p, q) = (p', q')\), it follows from (3.26) that
\[
|G_N(x, y)| < \frac{1}{2} \exp\left( -\left(\alpha \wedge \rho - \frac{1}{5(\log N)^{8}}\right)|x - y| \right)
+ \exp\left( \frac{1}{5(\log N)^{8}} |x - y| \right) A^{(\log N)^{C_1} e^{-10(\alpha \wedge \rho)|x - y|}}
< \exp\left( -\left(\alpha \wedge \rho - \frac{1}{(\log N)^{8}}\right)|x - y| \right)
\]
since \(|x - y| > (\log N)^{C_2} \) and \( C_2 > 2C_1 + 10 \).

If we stop the iteration before \((p, q)\) arriving at \((p', q')\), then we have
\[
\varphi_{K}^{p} \varphi_{M_0}^{p + q} e^{-\left(\alpha \wedge \rho\right)|x - v_{2p+q}|} |G_N(v_{2p+q}, y)| < \exp\left( \frac{1}{5(\log N)^{8}} |x - y| \right) A^{(\log N)^{C_1} e^{-10(\alpha \wedge \rho)|x - y|}}\varphi_{M_0}^{p}
< \frac{1}{2} \exp\left( -\left(\alpha \wedge \rho - \frac{1}{(\log N)^{8}}\right)|x - y| \right),
\]
which together with (3.26) implies
\[
|G_N(x, y)| < \exp\left( -\left(\alpha \wedge \rho - \frac{1}{(\log N)^{8}}\right)|x - y| \right).
\]
This completes the proof. \(\square\)

Now we turn to establish the Green’s function estimates on \(G_N\). Recall the two scales \(0 < M_0 < K\) satisfying
\[ M_0 = (\log N)^{C_0}, \quad K = (\log M_0)^{C_7}. \]
Moreover, we take \(l_0\) and \(l_1\) such that
\[ l_0 = C_8 \log M_0, \quad K = A^{l_1}, \]
with \(C_8 > (1 + \log 10)/(\log \frac{5}{2})\). By the Iterative Lemma [2.1] we have
\[ \|G_{l_0; x}\| < A^{l_1}, \quad |G_{l_0; x}(k, k')| < e^{-\alpha_1 |k - k'|} \quad \text{for} \quad |k - k'| > l_1^2 \sim (\log K)^{C_2}, \]
for any $\xi \in \Pi_i$. Using (2.29), Lemma 3.2 and Lemma 3.4 we have
\[(3.28)\]
$$\|G_k\| < A^{(\log K)^{C_1}},$$
$$|G_k(k, k')| < e^{-s|k-k'|}, \quad \text{for } |k - k'| > (\log K)^{C_2},$$
since
$$\epsilon_i^{1/10} \cdot N + \epsilon_i^{1/10} < A^{-\epsilon_i^{C_1}}.$$ (Indeed, $\log \log \epsilon_i^{-1} \sim I_1 \sim (\log (m + 1))^{C_1} \gg \log (m + 1) \sim \log \log N$.) Moreover, by verifying $A^{-(m+1)^{C_3}} < A^{-(\log K)^{C_1}}$, it follows again from Lemma 3.1 that (3.28) remains valid on $\mathcal{O}(\Pi, A^{-(m+1)^{C_3}})$ and hence on $\mathcal{O} \Pi, A^{-(m+1)^{C_3}}$.

Then, using (3.28) on $\mathcal{O}(\Pi, A^{-(m+1)^{C_3}})$ and Lemma 3.5 we obtain from Lemma 3.6 that
\[(3.29)\]
$$\|G_N\| < A^{(\log N)^{C_1}},$$
$$|G_N(k, k')| < e^{-(s-(\log N)^{-s})|k-k'|}, \quad \text{for } |k - k'| > (\log N)^{C_2}$$
holds on $\mathcal{O}(\Pi, A^{-(m+1)^{C_3}})$. Note that $s - (\log N)^{-s} > s^1 > s_+ = s_{m+1}$. This completes the proof of Lemma 2.4.

4. Appendix A: Large deviation theorem

The appendix is devoted to the proof of the large deviation theorem (Lemma 3.4), which can be read independently. The proof follows exactly the same line in [7] and we prove it here for completeness. It is worthy noticing that the notations below are also independent of the main body of the paper. For that reason, we write simply $T$ by $T$ and so on.

4.1. Notations and phrases. We consider matrix defined on $\mathbb{Z}^d$. For $m = (m_1, \cdots, m_d), n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$, we define the distance by
$$|m - n| = \max_{1 \leq j \leq d} |m_j - n_j|.$$ For $\Lambda \subset \mathbb{Z}^d$, we denote the diameter of $\Lambda$ by $|\Lambda|$. For a matrix $A$ defined on $\mathbb{Z}^d$, we denote by $\|A\|$ the operator norm induced by the $\ell^2$ norm of a vector in $\mathbb{Z}^d$. The inverse of a matrix is always denoted by $G$.

When applying in resolvent identity, we shall control the Green’s function $G_\Lambda$ with $\Lambda$ being the difference of two boxes in $\mathbb{Z}^d$. For that reason, as in [7], we introduce the elementary regions. An elementary region is defined to be a set $\Lambda$ of the form
$$\Lambda = R \setminus (R + z)$$
where $z \in \mathbb{Z}^d$ is arbitrary and $R$ is a block in $\mathbb{Z}^d$, i.e.,
$$R = \{y = (y_1, \cdots, y_d) \in \mathbb{Z}^d : y_i \in [x_i - a_i, x_i + a_i], i = 1 \cdots, d\}. $$
The size of an elementary region $\Lambda$ is simply its diameter. For any integer $M > 0$, the set of all elementary regions of size $M > 0$ is denoted by $\mathcal{ER}(M)$ and are also referred as $M$-regions. The class of elementary regions consists of $d$-dimensional rectangles, L-shaped regions and $(d - 1)$-dimensional rectangles with normal vector parallel to the axis.

Note that these regions play only a role in the application of resolvent identity in the presence of interior corners, but basically have no effect on the other parts of the argument.
Given a elementary region $\Lambda$, we consider exhaustion $\{S_j(m)\}_{j=0}^l$ of $\Lambda$ of width $2M$ centered at $m \in \Lambda$ defined inductively by
\begin{equation}
S_0(m) = Q_M(m) \cap \Lambda, \quad Q_M(m) = \{n \in \mathbb{Z}^d : |n - m| \leq M\},
\end{equation}
\begin{equation}
S_j(m) = \bigcup_{n \in S_{j-1}(m)} (Q_{2M}(n) \cap \Lambda), \quad \text{for} \quad 1 \leq j \leq l,
\end{equation}
where $l$ is maximal such that $S_l \neq \Lambda$. Define the annulus between the exhaustion by
\begin{equation}
A_j(m) = S_j(m) \setminus S_{j-1}(m), \quad 0 \leq j \leq l
\end{equation}
with $S_{-1}(m) = \emptyset$. We have the following two simple observations:
- Except the possible exception of a single annulus, $Q_M(n) \cap A_j(m)$ is an elementary region for all $n \in A_j(m)$. The exceptional annulus is the one that contains the unique interior corner of $\Lambda$ (i.e., the corner lying in the interior of the hull of $\Lambda$).
- Any two cubes $Q_M(n_1)$ and $Q_M(n_2)$ with centers $n_1$ and $n_2$ lying in nonadjacent annuli are disjoint.

4.2. Coupling Lemma for long range operators. We present and prove two kinds of coupling lemmas.

**Lemma 4.1.** Let $T$ be a matrix defined on a finite set $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| = N$. Let the various constants below satisfy $0 < \theta < 1$, $0 < b < 1$, $0 < \tau < 1$, $b\tau < \theta$, $\alpha > 0$, $\rho > 0$. Further let
\begin{equation}
(\log N)^{10/b} < M < N^\tau.
\end{equation}
Assume the following properties hold.

(i) The matrix $T$ exhibits the off diagonal exponential decay
\begin{equation}
|T(x, y)| < e^{-\rho |x - y|}, \quad x \neq y, \quad x, y \in \Lambda.
\end{equation}

(ii) For every $m \in \Lambda$, there is a subinterval $U(m) \subset \Lambda$ containing $m$ with
\begin{equation}
|U(m)| = M \quad \text{and} \quad \text{dist} (m, \Lambda \setminus U(m)) > \frac{M}{2}
\end{equation}
such that
\begin{equation}
\|G_{U(m)}\| < e^{M^\rho}
\end{equation}
and
\begin{equation}
|G_{U(m)}(x, y)| < e^{-\alpha |x - y|} \quad \text{for} \quad |x - y| > N^\theta, \quad x, y \in U(m).
\end{equation}
Then, there is
\begin{equation}
\|G_\Lambda\| < 2N^d e^{M^\rho} < e^{N^\theta},
\end{equation}
\begin{equation}
|G_\Lambda(x, y)| < e^{-\alpha' |x - y|} \quad \text{for} \quad |x - y| > N^\theta.
\end{equation}
provided $N$ is large enough, i.e., $N \geq N(\alpha, b, d, \rho, \theta, \tau)$. Moreover, the decay rate $\alpha' \geq (\alpha \wedge \rho) - (\log N)^{-50}$.

The proof of Lemma 4.1 is similar to that of Lemma 3.6 and is much simpler. We omit it here.
Lemma 4.2. Let $T$ be a matrix defined on $\Lambda_0 \in E\mathcal{R}(N) \subset \mathbb{Z}^d$. Let the various constants below satisfy
\begin{equation}
0 < \tau < b \leq \theta < 1, \quad \theta \geq \frac{1 - 2\tau}{1 - \tau}, \tag{4.7}
\end{equation}
and let $$N^r < M_0 < 2N^r.$$ Assume the following properties hold.

(i) The matrix $T$ exhibits the off-diagonal exponential decay
\begin{equation}
|T(m, n)| < e^{-\rho|m-n|}, \quad m, n \in \Lambda_0, m \neq n. \tag{4.8}
\end{equation}
(ii) For any $\Lambda \in E\mathcal{R}(L)$, $\Lambda \subset \Lambda_0$ with any $N^r < L < N$, there is a bounded inverse
\begin{equation}
\|G_\Lambda\| < e^{L^b}. \tag{4.9}
\end{equation}

We say an elementary region $\Lambda \in E\mathcal{R}(L), \Lambda \subset \Lambda_0$ is good if in addition to (4.9) the Green’s function exhibits the off diagonal decay
\begin{equation}
|G_\Lambda(m, n)| < e^{-\alpha(L)|m-n|}, \quad m, n \in \Lambda, |m-n| > L^\theta. \tag{4.10}
\end{equation}

Otherwise $\Lambda$ is called bad.

(iii) For any family $\mathcal{F}$ of pairwise disjoint bad $M'$-regions in $\Lambda_0$ with $M_0 + 1 \leq M' \leq 2M_0 + 1$,
\begin{equation}
\#\mathcal{F} < \frac{N^b}{M_0}. \tag{4.11}
\end{equation}

Then, there is
\begin{equation}
|G_{\Lambda_0}(m, n)| < e^{-\alpha'|m-n|}, \quad \text{for all} \quad m, n \in \Lambda_0, |m-n| > N^\theta.
\end{equation}
provided $N$ is sufficiently large, i.e., $N \geq N(b, d, \tau, \theta)$. Moreover, $\alpha' \geq (\alpha \wedge \rho) - N^{-\delta}$ for some $\delta = \delta(b, d, \tau, \theta) > 0$ and $\alpha = \alpha(M_0)$.

Proof. The proof is based on an iteration procedure. In the first step, we give a detailed analysis on the off diagonal decay of the Green’s function at small scale $M_1$. Then we list an induction statement, whose proof is basically same to that in the first step and hence is omitted. By the finitely many iterations, we obtain the Green’s function estimate at large scale $N$.

The first step. Let $M_1 = [M_0^d]$ with $\lambda > 1$ and consider $\Lambda_1 \in E\mathcal{R}(M_1), \Lambda_1 \subset \Lambda_0$. Fix any $m \in \Lambda_1$ and let $\{S_j(m)\}_{j=0}^l$ be the exhaustion of $\Lambda_1$ of width $2M_0$ and centered at $m$ (see (4.1) and the associated annuli in (4.2)).

We say an annulus $A_j(m)$ is good if for any $n \in A_j(m)$ both $Q_{M_0}(n) \cap A_j(m)$ and $Q_{M_0}(n) \cap \Lambda_1$ are good regions in the sense of (4.9) and (4.10). Otherwise $A_j(m)$ is bad. Note that there is at most one annulus $A_{j_0}$ (consisting the interior corner of $\Lambda_1$) such that $Q_{M_0}(n) \cap A_{j_0}(n)$ possibly fails to be an elementary region. In this case, $A_{j_0}$ is counted among the bad annuli. Note also that for good annuli, the diameter of those $Q_{M_0}(n) \cap A_j(m)$ ranges from $M_0 + 1$ to $2M_0 + 1$.

With the above definition of good and bad annuli, we say that an elementary region $\Lambda_1 \in E\mathcal{R}(M_1), \Lambda_1 \subset \Lambda_0$ is "GOOD" if, for any $m \in \Lambda_1$, there are at most $B_1 = \kappa \frac{M_0'}{M_0}$ many bad annuli for the associated exhaustion centered at $m$, where $\kappa$ will be determined below. Otherwise, the $M_1$-region $\Lambda_1$ is called "BAD".

\footnote{We use "GOOD" here to make a difference from the goodness of an elementary region as in (4.9) and (4.10).}
Let $\mathcal{F}_1$ be an arbitrary family of pairwise disjoint “BAD” $M_1$-regions contained in $\Lambda_0$. If $\Lambda_1 \in \mathcal{F}_1$, we can find an exhaustion of $\Lambda_1$ centered at some $m \in \Lambda_1$ such that there are at least $\frac{1}{2}B_1$ many nonadjacent annuli. Each bad annuli $A_j(m)$ contains a bad $M'$-region ($Q_{M_0} \cap A_j(m)$ or $Q_{M_0} \cap \Lambda_1$) with $M_0 \leq M' \leq 2M_0 + 1$, which does not intersect with that in the nonadjacent bad annulus. As a result, we have

$$\# \mathcal{F}_1 < \frac{\# \mathcal{F}}{B_1/2} < \frac{2N^b}{kM_1}.$$  

Consider a "GOOD" region $\Lambda_1 \in \mathcal{ER}(M_1)$ and fix any pair $m, n \in \Lambda_1, |m - n| > M_1^b$. Let \{\$S_j(m)\}_{j=0}^\Lambda$ and \{\$A_j(m)\}_{j=0}^\Lambda$ be the associated exhaustion and annuli of $\Lambda_1$ of width $2M_0$ centered at $m \in \Lambda_1$. Let $A_j, A_{j+1}, \cdots, A_{j+s}$ be adjacent good annuli and denote

$$U = \bigcup_{i=j}^{j+s} A_i.$$ 

Obviously, $\#U \geq 2M_0(s + 1)$. We claim the following Green’s function estimate on $G_U$

$$|G_U(x, y)| < e^{\beta(M_0 - |x - y|)} \quad \text{for all} \quad x, y \in U,$$

where

$$\beta = \alpha \wedge \rho = \alpha(M_0) \wedge \rho.$$

Usually $U$ is no longer an elementary region and thus (4.9) is not applicable to get a norm estimate on $G_U$. Nevertheless, we can invoke Lemma 4.1 to estimate $\|G_U\|$. For any $n \in A_i \subset U$, by definition both $Q_{M_0}(n) \cap \Lambda_1$ and $Q_{M_0}(n) \cap A_i$ are good. Following the notations in Lemma 4.1, we take $U(n) = Q_{M_0}(n) \cap \Lambda_1$ when $Q_{M_0}(n) \subset U$ and take $U(n) = Q_{M_0}(n) \cap A_i$ when $Q_{M_0}(n) \setminus U \neq \emptyset$. Then we have

$$\|G_U\| < 2M_1^b e^{(2M_0 + 1)^\beta}.$$ 

Next we repeat the same analysis as Lemma 4.1 and obtain

$$|G_U(x, y)| < e^{\beta M_0} e^{-\beta |x - y|}, \quad \text{for} \quad |x - y| > M_0$$

as long as

$$1 < \lambda < 2 - (b \lor \beta).$$

Then the claim (4.12) is an immediate result of (4.13) and (4.14).

Now we are back to establish the off diagonal estimate for a good $M_1$-region $\Lambda_1$, i.e., to establish

$$|G_{\Lambda_1}(m, n)| < e^{-\beta |m - n|} \quad \text{for} \quad |m - n| > M_1^b, m, n \in \Lambda_1.$$ 

Recall the exhaustion of $\Lambda_1$ of width $2M_0$ centered at $m$. Suppose $S_0(m)$ is good and write an exhaustion

$$S_0(m) \subset J_0 \subset J_1 \subset \cdots \subset J_g = \Lambda_1$$

satisfying

- $J_{s+1} \setminus J_s$ is the union of adjacent bad annuli (resp. union of adjacent good annuli) if $s$ is even (resp. if $s$ is odd);
The exhaustion is maximal in the sense that if
\[ J_{s+1} \setminus J_s = \bigcup_{j=j_s}^{j_{s+1}} A_j, \quad s \in 2\mathbb{N}, \]
then \( A_{j_s-1}, A_{j_{s+1}} \) are good and \( A_j \) is bad for all \( j_s \leq j \leq j_{s+1} \). The case of \( s \) being odd is similar;

- \( J_s \) is the elementary region in \( \Lambda_1 \) for all \( 1 \leq s \leq g \).

By the "GOOD" property of \( \Lambda_1 \), there is
\begin{equation}
(4.15) \sum_{s \text{ even}} (j_{s+1} - j_s) < \kappa \frac{M_1^0}{M_0},
\end{equation}
and hence
\begin{equation}
(4.16) \quad g < 2\kappa \frac{M_1^0}{M_0}.
\end{equation}

To begin with, we see from (4.12) that
\[ |G_{J_0}(m, y)| < e^{\beta(2M_0 - |m - \gamma|)} \quad \text{for} \quad y \in J_0. \]
Take
\begin{equation}
(4.17) \quad \varphi_0 = e^{2\beta M_0}
\end{equation}
and we assume by induction that
\begin{equation}
(4.18) \quad |G_J(m, y)| < \varphi_s e^{-\beta |m - y|} \quad \text{for} \quad y \in J_s.
\end{equation}

If \( s + 1 \) is odd, \( J_{s+1} \setminus J_s \) is made up of bad annuli. For any \( y \in J_{s+1} \), we apply the resolvent identity
\begin{equation}
(4.19) \quad |G_{J_{s+1}}(m, y)| < |G_J(m, y)| e^{-|m - \gamma|} + \sum_{z \in J_s, z' \in J_{s+1} \setminus J_s} |G_J(m, z)| e^{-\beta |z - \gamma|} |G_{J_{s+1}}(z', y)|
\end{equation}
\[ < \varphi_s e^{-\beta |m - \gamma|} + \varphi_s \sum_{z \in J_s, z' \in J_{s+1} \setminus J_s} e^{-\beta |z - \gamma|} |G_{J_{s+1}}(z', y)|
\end{equation}
\[ < \varphi_s e^{-\beta |m - \gamma|} + \varphi_s M_1^{2d} e^{M_0} \max_{z' \in J_{s+1} \setminus J_s} e^{-\beta |z' - \gamma|}.
\]

If \( y \in J_s \), then \( |m - z'| \geq |m - y| \) and if \( y \in J_{s+1} \setminus J_s \), there is
\[ |m - z'| \geq \text{dist}(m, \partial S_{j_s-1}) \geq |m - \gamma| - 2M_0(j_{s+1} - j_s).
\]

Consequently, we have
\begin{equation}
(4.20) \quad |G_{J_{s+1}}(m, y)| < (1 + M_1^{2d} e^{M_0} e^{2\beta M_0(j_{s+1} - j_s)}) \varphi_s e^{-\beta |m - \gamma|}
\end{equation}

If \( s + 1 \) is even, \( J_{s+1} \setminus J_s \) is made up of good annuli. For any \( y \in J_s \), we repeat the resolvent identity analysis in (4.19) and obtain from \( |m - z'| \geq |m - y| \) that
\begin{equation}
(4.21) \quad |G_{J_{s+1}}(m, y)| < \varphi_s e^{-\beta |m - \gamma|}(1 + M_1^{2d} e^{M_0}).
\end{equation}

If \( y \in J_{s+1} \setminus J_s \), we have
\[ |G_{J_{s+1}}(m, y)| < \sum_{z' \in J_s, z' \in J_{s+1} \setminus J_s} |G_{J_{s+1}}(m, z')| e^{-\beta |z' - \gamma|} |G_{J_{s+1}}(z', y)|.
\]
Applying (4.12) with $U = J_{s+1} \setminus J_s$ to $G_{J_{s+1}\setminus J_s}(z, y)$ and applying (4.21) to $G_{J_{s+1}}(m, z')$ we have
\begin{equation}
|G_{J_{s+1}}(m, y)| < M_1^{2d} e^{M_1^t/2} \varphi, (1 + M_1^{2d} e^{M_1^t/2}) e^{\beta M_0 e^{-\beta |m-y|}}.
\end{equation}

In conclusion, we can take
\begin{equation}
\varphi_{s+1} = \begin{cases} e^{\beta M_0 (j_{s+1} - j_s)} \varphi_s, & s \text{ is even}; \\ e^{\beta M_0 \varphi_s}, & s \text{ is odd}. \end{cases}
\end{equation}

By (4.15), (4.16) and (4.17), we get
\begin{equation}
\varphi < e^{\beta M_0} < e^{15 \beta M_0^t}.
\end{equation}

Suppose $S_0(m)$ is bad, then $\varphi_0 < e^{M_1^t} e^{\beta M_0^t}$ and (4.24) is also valid by the same analysis. Therefore, we prove the induction statement (4.18) and get
\begin{equation}
|G_{\Lambda_1}(m, n)| < e^{15 \beta M_0^t} e^{-\beta |m-n|}.
\end{equation}

Since $|m - n| > M_1^t$, it follows that
\begin{equation}
|G_{\Lambda_1}(m, n)| < e^{-\alpha_1 |m-n|}, \quad \alpha_1 = \beta(1 - 15\kappa),
\end{equation}
which establishes the off diagonal decay of $G_{\Lambda_1}$ for a "GOOD" elementary region $\Lambda_1 \in \mathcal{ER}(M_1)$.

**Induction statement**: Let $\kappa < 10^{-2}$ be specified later and let $\lambda$ satisfy
\begin{equation}
1 < \lambda < 2 - (b \lor \theta) = 2 - \theta, \quad b\lambda < 1.
\end{equation}
Indeed, due to our choice of $b \leq \theta$, we have $2 - \theta < 1/b$.

Define inductively $M_t = \{M_{t-1}^t, t \leq t_s$ and $t_s$ is specified later. Consider $\Lambda_t \in \mathcal{ER}(M_t), \Lambda_t \subset \Lambda_0$. Fix any $m \in \Lambda_t$ and let $\{S_j(m)\}^{j=0}_{j=0}$ be the exhaustion of $\Lambda_t$ of width $2M_{t-1}$ and centered at $m$. Let $\{A_j\}^{j=0}_{j=0}$ be the associated annuli.

We say an annulus $A_j(m)$ is good if for any $n \in A_j(m)$ both $Q_{M_{t-1}}(n) \cap A_j(m)$ and $Q_{M_{t-1}}(n) \cap \Lambda_t$ are good regions in the sense of (4.9) and (4.10) but with the decay rate $\alpha_{t-1} = \beta_{t-1}(1 - 15\kappa) = \beta(1 - 15\kappa)^{t-1}$ and $\beta_{t-1} = \beta_{t-1} \lor \rho$. Otherwise $A_j(m)$ is bad. We say that an elementary region $\Lambda_t \in \mathcal{ER}(M_t), \Lambda_t \subset \Lambda_0$ is "GOOD" if, for any $m \in \Lambda_t$, there are at most $B_t = \kappa \frac{M_t^t}{M_{t-1}}$ many bad annuli for the associated exhaustion centered at $m$. Otherwise, the $M_t$-region $\Lambda_t$ is called "BAD". Let $\mathcal{F}_{t-1}$ be the family of pairwise disjoint "BAD" $M_{t-1}$-regions contained in $\Lambda_0$.

Assume
\begin{equation}
\# \mathcal{F}_{t-1} < \left( \frac{2}{k} \right)^{t-1} \frac{N^b}{M_{t-1}^b} \left( M_{t-2} M_{t-3} \cdots M_1 \right)^{1-\theta}
\end{equation}
and (4.9) holds for all $N^r < L \leq N$. Then for any "GOOD" $M_t$-region $\Lambda_t \in \mathcal{ER}(M_t)$, the Green’s function $G_{\Lambda_t}$ exhibits off diagonal decay
\begin{equation}
|G_{\Lambda_t}(m, n)| < e^{-\alpha_t |m-n|}, \quad |m - n| > M^\theta, m, n \in \Lambda_t,
\end{equation}
with $\alpha_t = \beta(1 - 15\kappa)^t$. Moreover, denoting by $\mathcal{F}_t$ the family of pairwise disjoint "BAD" $M_t$-regions contained in $\Lambda_0$, there is
\begin{equation}
\# \mathcal{F}_t < \left( \frac{2}{k} \right)^{t} \frac{N^b}{M_{t}^b} \left( M_{t-1} M_{t-2} \cdots M_1 \right)^{1-\theta}.
\end{equation}

The proof of the above statement is the same to that in the first step and is omitted.
Off diagonal estimate of $G_N$. In order to reach size $N = M_t$, the number $t_*$ of steps should satisfy
\[ \lambda^{t_*} \log M = \log N \]
hence $\lambda^{t_*} \sim \frac{1}{\log M}$. It then suffices to show that $[-N, N]^d$ is a "GOOD" $M_t$-region, which is of course valid if
\[ \left( \frac{2}{\lambda} \right)^{t_* - 1} \frac{N^b}{M_{t_* - 1}^\theta} (M_{t_* - 2} \cdots M_1)^{1 - \theta} < \frac{N^\theta}{M_{t_* - 1}}. \]
Obviously, (4.26) is equivalent to
\[ \kappa > 2 \left( \frac{1}{2N^\gamma} \right)^{1/t_*}, \quad \gamma = \theta - b - \frac{\lambda r}{\lambda - 1} (\lambda^{t_* - 1} - 1)(1 - \theta). \]
To keep $\gamma > 0$, it suffices to take
\[ \lambda > \frac{1 - b}{\theta(1 - \tau) + \tau - b} \]
which is compatible with $\lambda < 2 - \theta$ according to our choice of $\theta \geq (1 - 2\tau)/(1 - \tau)$.
Take
\[ \kappa = \kappa_N = 4N^{-\frac{\theta \log \lambda}{\log \tau}} \]
and then
\[ |G_N(m, n)| < e^{-\alpha_N |m-n|}, \quad |m-n| > N^\theta. \]
The conclusion is valid with some choice of $\delta = \delta(b, d, \tau, \theta, \lambda(b, \tau, \theta))$ such that
\[ \alpha' = \alpha_{t_*} = \beta(1 - 15\kappa)^{1/4} > (\alpha(M_0) \wedge \rho) - N^{-\delta}. \]
This completes the proof. \[\square\]

4.3. Matrix-valued Cartan’s theorem. The following matrix-valued Cartan’s theorem as well as its proof is given in \[6\].

Lemma 4.3. Let $A(\sigma)$ be a matrix valued function defined on $\sigma \in [-\delta, \delta]$ with $A(\sigma)(m, n) \in \mathbb{C}$ for $m, n \in \Lambda_0 \subset \mathbb{Z}^d, |\Lambda_0| = N$. Assume
\[ (i)\ A(\sigma) \text{ is real analytic in } \sigma, \text{ and there is a holomorphic extension to a strip} \]
\[ (4.27)\ \ |Re z| < \delta, \quad |Im z| < \gamma \]
satisfying
\[ (4.28)\ \ ||A(z)|| < B_1. \]
\[ (ii)\ For each $\sigma \in [-\delta, \delta]$, there is a subset $\Lambda \subset \Lambda_0$ such that \]
\[ (4.29)\ \ ||\Lambda^c| < M \]
\[ (4.30)\ \ ||(R_{\Lambda}A(\sigma)R_{\Lambda})^{-1}|| < B_2. \]
\[ (iii)\ \ mes \{ \sigma \in [-\delta, \delta] : ||A(\sigma)^{-1}|| > B_3 \} < 10^{-3} \gamma(1 + B_1)^{-1}(1 + B_2)^{-1}. \]
Then, letting
\[ \kappa < (1 + B_1 + B_2)^{-10M}, \]
we have
\[ \text{mes}\left\{ \sigma \in \left[ -\frac{\delta}{2}, \frac{\delta}{2} \right] : \|A(\sigma)^{-1}\| > \frac{1}{\kappa} \right\} < \exp\left\{ -\frac{c \log \kappa^{-1}}{M \log(M + B_1 + B_2 + B_3)} \right\}. \]

**Corollary 4.2.** Let \( T(\sigma) \) be a matrix valued function defined on \( \sigma \in [-\delta, \delta] \) with \( T(\sigma)(m, n) \in \mathbb{C} \) for \( m, n \in \Lambda_0 \subset [-N, N]^d \subset \mathbb{Z}^d, |\Lambda_0| = N \). Let the various constants below satisfy
\[ 0 < \alpha, b, \beta, \rho, \theta < 1, \quad 0 < \tau < \frac{9}{10(1 + d)^\beta}, \quad C > 1 \]
and further let
\[ (\log N)^2 < M < N^\tau. \]
Assume
(i) \( T(\sigma) \) is real analytic in \( \sigma \), and exhibits the off diagonal decay
\[ |T(\sigma)(m, n)| < e^{-\rho|m-n|}, \quad m \neq n. \]
Moreover, there is a holomorphic extension to a strip
\[ |\Re z| < \delta, \quad |\Im z| < \gamma \]
satisfying
\[ \|T(z)\| < N^{-c}. \]
(ii) For each \( \sigma \in [-\delta, \delta] \), the set \( \Omega(\sigma) \) of bad sites satisfies
\[ \#\Omega(\sigma) < N^{1-\beta}. \]
Here we say \( m \in \Lambda_0 \) is a good site if \( Q_M(m) \cap \Lambda_0 \) and the restriction of \( T(\sigma) \) on \( Q = Q_M(m) \) is invertible. Also
\[ \|(R_Q T(\sigma) R_Q)^{-1}\| < e^{M^\theta}, \]
and
\[ |(R_Q T(\sigma) R_Q)^{-1}(x, y)| < e^{-\alpha|x-y|}, \quad x, y \in Q_M(M), |x - y| > M^\theta. \]
Otherwise \( m \) is called a bad site.

(iii) \[ \text{mes}\left\{ \sigma \in [-\delta, \delta] : \|T(\sigma)^{-1}\| > e^{N^\beta} \right\} < e^{-10M}. \]
Then, we have
\[ \text{mes}\left\{ \sigma \in \left[ -\frac{\delta}{2}, \frac{\delta}{2} \right] : \|T(\sigma)^{-1}\| > e^{N^{1-\frac{\beta}{2}}} \right\} < e^{-N^{\frac{\beta}{2}}}. \]

\(^{\text{4}}\)Note that for those sites at the corner of \( \Lambda_0 \), the size of \( Q_M \cap \Lambda_0 \) might have very small diameter. For that reason, we think of all corners of \( \Lambda_0 \) as bad sites.
**Proof.** Fix $\sigma \in [\delta, \delta]$. Consider a paving of $\Lambda_0$ by $Q_M(x)$ with $x \in 2M\mathbb{Z}^d$. Let

$$\Lambda = \bigcup_{Q_M(x) \cap \Omega(\sigma) = 0} Q_M(x) \cap \Lambda_0.$$ 

From (4.37), we have

$$\# \Lambda^c < 2^d M^d N^{1-\beta}.$$ 

By Lemma 4.1, we get a norm control on $G_{\Lambda} = (R_\Lambda T(\sigma) R_\Lambda)^{-1}$

$$\| G_{\Lambda} \| < 2N^d e^M.$$ 

Next we employ 4.3 to prove the corollary. Obviously, $B_1 = N^C$. Then Lemma 4.3 (i) holds with $B_2 = 2N^d e^M$. Letting $B_3 = e^{N^b}$, we have

$$e^{-10M} < 10^{-3} \gamma B_1^{-1} B_2^{-1} \sim \gamma N^C e^{-M}$$

and thus Lemma 4.3 (iii) holds. Noticing that

$$\kappa = e^{-N^{1-\beta} / 10} < (B_1 + B_2)^{-10^{d} M^d N^{1-\beta}} \sim e^{-N^{1-\beta}}$$

then the conclusion (4.40) is an immediate result of (4.33) as long as $N$ is large enough. □

4.4. Multi-scale analysis.

**Lemma 4.4.** Let $T(\sigma)$ be a matrix valued function defined on $\sigma \in [\delta, \delta]$ with $T(\sigma)(m, n) \in \mathbb{C}$ for $m, n \in \Lambda_0$, $\Lambda_0 \in \mathcal{E} \mathcal{R}(N)$. Let the various constants below satisfy

$$0 < \beta \ll 1, \quad \alpha > 0, \quad \rho > 0, \quad 0 < 1 - \frac{\beta}{10} < b \leq \theta < 1,$$

and further let

$$M = [N^{\beta^6}], \quad L_0 = [N^{\frac{\beta}{10}}].$$

Assume the following properties hold.

(i) $T(\sigma)$ is real analytic in $\sigma$ and satisfies (4.34), (4.35) and (4.36).

(ii) For any $I \in \mathcal{E} \mathcal{R}(L_0)$, except for $\sigma$ in a set $\delta(I)$ of measure at most $e^{-L_0^\beta}$,

$$\| (R_I T(\sigma) R_I)^{-1} \| < e^{L_0^b}$$

and

$$\| (R_I T(\sigma) R_I)^{-1} (m, n) \| < e^{-\rho(L_0) |m-n|} \quad \text{for $m, n \in I, |m-n| > L_0^\beta$.}$$

(iii) Define again $\Omega(\sigma)$ the set of bad sites in $\Lambda_0$ by condition (4.38) and (4.39). Assume further that for any $J \in \mathcal{E} \mathcal{R}(L)$ such that

$$L > N^{\pi},$$

we have

$$\# J \cap \Omega(\sigma) < L^{1-\beta}.$$
Then we have
\[ \|G_{\Lambda_0}\| < e^{\lambda_0} \]
and
\[ |G_{\Lambda_0}(m,n)| < e^{-\alpha'|m-n|}, \quad \text{for all} \quad m,n \in \Lambda_0, |m-n| > N^0 \]
extcept for \( \sigma \in [-\frac{\delta}{2}, \frac{\delta}{2}] \) in a set of measure at most \( e^{-N^{\beta^2}} \), where \( 0 < c = c(d) < 1 \) is an absolute constant. Moreover, the decay rate \( \alpha' > (\alpha \wedge \rho) - (\log N)^{-8} \).

**Proof.** Let
\begin{equation}
\mathcal{E}_0 = \bigcup_{I \in \mathcal{E}(L_0)} \mathcal{E}(I).
\end{equation}
It follows that
\[ \text{mes} \ \mathcal{E}_0 < N^d e^{-L_0^\beta}. \]
For any \( \sigma \in [-\delta, \delta] \ \setminus \ \mathcal{E} \), all \( L_0 \)-regions in \( \Lambda_0 \) are good in the sense of (4.42) and (4.43). Using Lemma 4.1 and taking
\[ M \equiv L_0, \quad N \equiv L, \quad \tau \equiv \frac{\beta}{20}, \]
we obtain that for any \( J \in \mathcal{E}(L) \), \( J \subset \Lambda_0 \) with \( L > N^\beta \), there is
\[ \|G_J(\sigma)\| < 2L^d e^{l_0^\beta} < e^{l_0^\beta}. \]
In other words, for any such \( J \), we have
\[ \text{mes} \ \{ \sigma \in [-\delta, \delta] : \|G_J(\sigma)\| > e^{l_0^\beta} \} < N^d e^{-L_0^\beta} < e^{-10M}. \]
Combining (4.44) and taking
\[ M \equiv M, \quad N \equiv L, \quad \tau \equiv \beta^4, \]
it follows from the Cartan’s estimate Corollary 4.2 that
\[ \text{mes} \ \{ \sigma \in [-\delta/2, \delta/2] : \|G_J(\sigma)\| > e^{l_1^\beta} \} < e^{l_1^\beta}. \]
Denoting
\begin{equation}
\mathcal{E} = \bigcup_{J \in \mathcal{E}(L), L > N^{\beta^5}} \{ \sigma \in [-\delta/2, \delta/2] : \|G_J(\sigma)\| > e^{l_1^\beta} \},
\end{equation}
then \( \mathcal{E} \) is the desired exceptional set satisfying
\[ \text{mes} \ \mathcal{E} < N^{d+1} e^{-l_1^\beta} < e^{-N^{\beta^2}} \]
with \( 0 < c < \frac{1}{100} \) depending on \( d \).

Fix any \( \sigma \in \mathcal{E}^c = [-\delta, \delta] \ \setminus \ \mathcal{E} \) in what follows. We shall apply Lemma 4.2 to prove the results.

Observe first that, for \( \Lambda \in \mathcal{E}(L), \ L > N^\beta \) with \( \Lambda \subset \Lambda_0 \), it follows from (4.46) that
\[ \|G_\Lambda(\sigma)\| < e^{l_1^\beta} < e^{l^b}. \]
Next, for \( I \in \mathcal{E}(N^\beta), \ I \subset \Lambda_0 \) with \( I \cap \Omega(\sigma) = \emptyset \), applying Lemma 4.1 by taking
\[ M \equiv M, \quad N \equiv N^\beta, \quad \tau = \beta^4, \]
we get
\[ |G_1(\sigma)(x,y)| < e^{-\tilde{\alpha}|x-y|}, \quad \text{for } |x-y| > N^\theta \beta, x, y \in I, \]
where
\[ \tilde{\alpha} > (\alpha \land \rho) - (\log N^{\beta/5})^{-50} > (\alpha \land \rho) - (\log N)^{-10}. \]
As a result, we call the above region \( I \) is good and call a \( N^\beta \)-region bad if it contains a bad site in \( \Omega(\sigma) \).

Finally recalling (4.44), there are at most \( N^{1-\alpha} \) bad M-sites in \( \Lambda_0 \). The set \( \mathcal{F} \) of disjoint bad \( N^\beta \)-region satisfies
\[ \# \mathcal{F} < N^{1-\beta} < \frac{N^b}{N^\beta} \]
since \( b > 1 - \frac{\beta}{10} \). Then the conclusion follows from Lemma 4.2 by taking
\[ M_0 \equiv N^\beta, \quad N \equiv N, \quad \tau \equiv \frac{\beta}{5}. \]
The arithmetical condition (4.7) is valid since \( \theta \geq b > 1 - \frac{\beta}{10} \) and \( \beta \ll 1 \).

The decay rate \( \alpha' \) satisfies
\[ \alpha' > \tilde{\alpha} - N^{-\delta} > \alpha \land \rho - (\log N)^{-8}. \]

\[ \square \]

4.5. Large deviation theorem. Consider the matrix
\[ (4.47) \quad T^\sigma = D^\sigma + \varepsilon S \]
where \( D^\sigma \) is a diagonal matrix with
\[ (4.48) \quad D^\sigma_{\pm, kk} = \pm (\langle k, \lambda' \rangle + \sigma) - \mu_j. \]
\( S \) satisfies the Toeplitz property and \( \| S \| < 1 \). The one dimensional parameter \( \sigma \) is defined on some open set \( \mathcal{J} \subset \mathbb{R} \). Let \( 0 < \beta \ll 1 \) and \( 0 < 1 - \frac{\beta}{10} < b < \theta < 1 \).

Assume that
\[ (4.49) \quad |S(x,y)| < e^{-\rho|x-y|}, \quad \text{for some } \rho > 0. \]

Obviously, for any large \( N \), \( T^\sigma_N \) has a holomorphic extension of \( \sigma \) on \( \mathcal{J} \) to the complex domain
\[ \{ \sigma \in \mathbb{C} : \text{dist}(\sigma, \mathcal{J}) < 1 \} \]
such that
\[ \| T^\sigma_N \| < N^C. \]
Note that \( \sup_{\sigma \in \mathcal{J}} |\sigma| \sim N \). Otherwise, for \( |\sigma| > 100N \), the matrix \( T^\sigma_N \) is diagonal dominated and a simple application of Neumann series yields a desired Green’s function estimate of \( G^\sigma_N \).

Assume \( N_0 \) is sufficiently large and the property
\[ (4.50) \quad "N_0 \text{-good}" : \begin{cases} \| G^\sigma_{N_0} \| < e^{N_0^{\beta}}, \\ |G^\sigma_{N_0}(x,y)| < e^{-\alpha_0|x-y|}, \quad \text{for } |x-y| > N_0^{\beta}, |x| \leq N_0, |y| \leq N_0 \end{cases} \]
holds for all \( \sigma \) except in a set \( \mathcal{E}_0 \) of measure at most \( e^{-N_0^{\beta}} \).
Indeed, for low scale $N_0$, the "$N_0$-good" property can be derived from a simple application of Neumann series. We show some details here. Consider

$$|D^\sigma_{\pm,j,k}| = |\langle k, \lambda \rangle + \sigma \pm \mu_j| < \varepsilon_1,$$

which is valid for $\sigma$ lying in an interval of size $2\varepsilon_1$. Then, denoting

$$E_0 = \left\{ \sigma \in \mathbb{R} : \min_{1 \leq j \leq d, |k| \leq N_0} |D^\sigma_{\pm,j,k}| < \varepsilon_1 \right\},$$

there is

$$\text{mes } E_0 < 8dN_0^d\varepsilon_1.$$ 

For $\sigma \in J \setminus E_0$, 

$$\|D^\sigma_{N_0}\|^{-1} < \frac{1}{\varepsilon_1}.$$ 

Assume

(4.51) \quad 0 < \varepsilon < e^{-4\sigma N_0^d}, \quad \varepsilon_1 \sim e^{-N_0^d}.

By Lemma 3.1, we obtain that

$$\|G^\sigma_{N_0}\| = \|\left(T^\sigma_{N_0}\right)^{-1}\| < \frac{2}{\varepsilon_1},$$

and

$$|G^\sigma_{N_0}(x, y)| < e^{-\varepsilon|x-y|}.$$ 

To ensure that

$$\text{mes } E_0 < e^{-N_0^d}$$

for some $0 < \gamma < 1$, we take

$$1 - \frac{\beta}{10} < \gamma < b.$$ 

Consequently, the "$N_0$-good" property holds (with $\alpha_0 = \rho$) for all $\sigma$ except in a set $E_0$ of measure at most $e^{-N_0^d} < e^{-N_0^d}$. Observe also that the matrix element of $T^\sigma$ is at most linear in $\sigma$. Hence $E_0(N_0)$ is a semi-algebraic set in $\sigma$ of degree at most $N_0^{C_{\rho}}$.

Let $N_0 \gg 1$ and let

$$N_0 = N_0^{100\rho^2}, \quad N_0 < \overline{N}_0 < N_0^{C_\rho},$$

where $C_\rho$ is to be determined later.

We apply Lemma 4.4 to get the Green’s function estimate at larger scales. Following the notations in Lemma 4.4, we take

$$L_0 \in [\overline{N}_0, \overline{N}_0]$$

and define

$$N \equiv \left[ L_0^{100\rho^2} \right], \quad M \equiv L_0^{100\rho^2} (= N_0^{\rho^2}).$$

where $0 < \beta \ll 1$ is a fixed constant.

For any $\Lambda_0 \in \mathcal{ER}(N)$, we establish the Green’s function estimate on $G^\sigma_{\Lambda_0}$. For any $I \in \mathcal{ER}(L_0)$ and $I \subset \Lambda_0$, it follows from the previous arguments that, if $\varepsilon < e^{-4\rho \overline{N}_0}$, then $G^\sigma_I = G^\sigma_{I+(k, \lambda)}$ satisfies the "$L_0$-good" property except $\sigma + \langle k, \lambda \rangle \in E_0$, where $k, l \in [-L_0, L_0]^d$ and $I' \in \mathcal{ER}(L_0)$. The exceptional set $E(I)$ is characterized by

$$E(I) = E_0(L_0) - \langle k, \lambda \rangle.$$
and thus
\[ \text{mes } \mathcal{E}(I) < e^{-\beta_0^T} < e^{-\beta_0^d}. \]
This verifies conditions (4.42) and (4.43).

For any \( J \in \mathcal{E}(L) \) and any \( N > L > N^{\beta/d} \), we compute \( \#(J \cap \Omega(\sigma)) \), where \( \Omega(\sigma) \) is the set of the \( M \)-bad sites in \( \Lambda_0 \). Roughly speaking,
\[
n \in \Omega(\sigma) \Leftrightarrow (n + [-M, M]^d) \cap \Lambda_0 \text{ is a } M \text{-bad region}
\]
and this verifies (4.44).

Recall that \( \mathcal{E}_0(M) \) is a semi-algebraic set of degree at most \( M^{C(d)} \). The number of connected components of \( \mathcal{E}_0(M) \) does not exceed \( M^C \). The constant \( C = C(d) \) might differ from line to line. Moreover, the size of each component of \( \mathcal{E}_0(M) \) is less than \( \eta = e^{-M^r} \). Fix a component \([a - \frac{\tau}{2}, a + \frac{\tau}{2}]\) and consider the set
\[ H = \{ n \in J : |\sigma + \langle n, \lambda' \rangle - a| < \eta/2 \}. \]
For two different \( n, n' \in H \), we have
\[ |\langle n - n', \lambda' \rangle| < \eta. \]

Assume \( \lambda' \) is diophantine
\[ |\langle k, \lambda' \rangle| > \frac{\nu}{|k|^d}, \quad 0 \neq k \in \mathbb{Z}^d, 0 < \nu < 1, \tau > d + 1. \]
Then
\[ |n - n'| > \nu \left( \frac{1}{\eta} \right)^{1/\tau} = \nu e^{M^r/\tau} \gg N = M^{1/\beta}. \]
whenever \( N_0 \) is large. As a result, we have
\[ \#(J \cap \Omega(\sigma)) < M^C = N^{C_\beta^d} < (N^{\beta})^{\gamma C^\beta} < L^{1-\beta}. \]
and this verifies (4.44).

Let
\[ N_1 = N_0^{100/\beta^2}, \quad \overline{N}_1 = N_0^{100/\beta^2}. \]
By Lemma 4.4, we have that for any \( N_1 < N_1 < \overline{N}_1 \) and any \( \Lambda_0 \in \mathcal{E}(N_1) \), the property
\[ (4.52) \quad "N_1 \text{ good}: \begin{cases} \|G_{\Lambda_0}^0\| < e^{N_0^4}, \\ |G_{\Lambda_0}^0(x, y)| < e^{-\alpha_1|x-y|}, \quad \text{for } |x - y| > N_1^0, x, y \in \Lambda_0 \end{cases} \]
except for \( \sigma \) is a set \( \mathcal{E}_1 = \mathcal{E}_1(N_1) \) of measure
\[ \mathcal{E}_1(N_1) < e^{-N_1^0} N_1^d < e^{-N_1^d}, \]
where \( \alpha_1 = (\alpha_0 \wedge \rho) - (\log N_1) - 8. \)
To iterate on, we impose the condition that
\[ N_1 < \overline{N}_0 \]
which results in
\[ \frac{100}{\beta^2} < C. \]
We only write out the iteration statement, whose proof is essential the same to that from the scale \( N_0 \) to \( N_1 \). The following statement holds for all \( k \geq 0. \)
For any $N_{k+1}/\beta = N_k < N_k = \overline{N}_{k-1}/\beta^2$ and any $\Lambda_0 \in \mathcal{ER}(N_k)$, the property

\begin{equation}
N_k - \text{good} : \begin{cases} 
\|G_{\Lambda_0}^{\sigma}\| < e^{N_0^2}, \\
|G_{\Lambda_0}^{\sigma}(x, y)| < e^{-\alpha_k|x-y|}, \text{ for } |x-y| > N_k^\theta, x, y \in \Lambda_0
\end{cases}
\end{equation}

holds except for $\sigma$ is a set $\mathcal{E}_k = \mathcal{E}_k(N_k)$ of measure

$$\mathcal{E}_k < e^{-N_0^3}$$

where $\alpha_k = (\alpha_0 \land \rho) - (\log N_0)^{-8} - \cdots - (\log N_k)^{-8}$. One easily finds that

$$\lim_{k \to \infty} \alpha_k > \rho - (\log N_0)^{-8}.$$ 

Since $N_{k+1} < \overline{N}_k$, we are able to iterate constantly and proves Lemma 3.4.

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LINEAR STABILITY OF KAM TORI

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