THE GROTHENDIECK GROUP OF A CLUSTER CATEGORY

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Abstract. For the cluster category of a hereditary or a canonical algebra, equivalently for the cluster category of the hereditary category of coherent sheaves on a weighted projective line, we study the Grothendieck group with respect to an admissible triangulated structure.

1. Introduction

The cluster category $\mathcal{C} = \mathcal{C}(A)$ of a finite dimensional hereditary algebra $A$ was introduced by Buan, Marsh, Reineke, Reiten and Todorov [1], in order to realize the cluster algebras of Fomin and Zelevinsky [2] via tilting theory.

The construction of the orbit category $\mathcal{C}(A)$, see [7], generalizes to the situation where $A$ is any $k$-algebra of finite global dimension. In this paper, all algebras will be unitary, associative and of finite dimension over an algebraically closed ground field $k$.

We call a triangulated structure $\mathcal{S}$ on $\mathcal{C}$ admissible if the canonical projection functor $\pi : D^b(\text{mod } A) \to \mathcal{C}$ is exact, that is, sends exact triangles to triangles from $\mathcal{S}$. We use the notation $\mathcal{C}_\mathcal{S}$ if we consider $\mathcal{C}$ as a triangulated category with triangulated structure $\mathcal{S}$. We suspect that an admissible triangulated structure for $\mathcal{C}$ may not be unique.

By Keller [7], $\mathcal{C}$ admits an admissible triangulated structure in case $D^b(\text{mod } A)$ is triangle-equivalent to $D^b(\mathcal{H})$ for some hereditary abelian $k$-category $\mathcal{H}$. Assuming $\mathcal{H}$ connected, by Happel’s classification theorem this happens if and only if $A$ is derived equivalent to a hereditary or a canonical algebra, see [3] [6]. In the first case, we can choose $\mathcal{H} = \text{mod } A$ where $A$ is hereditary and in the second $\mathcal{H} = \text{coh } \mathbb{X}$, the category of coherent sheaves over a weighted projective line $\mathbb{X}$, see [3]. In the present paper we focus on the case $\mathcal{H} = \text{coh } \mathbb{X}$, but also deal with the cases $\mathcal{H} = \text{mod } A$ where $A$ is the path algebra of a Dynkin or an extended Dynkin quiver.

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Given an admissible triangulated structure $\mathcal{S}$ on $\mathcal{C}$ we study the Grothendieck group $K_0(\mathcal{C}_S)$ with respect to all triangles in $\mathcal{S}$ and compare it with the Grothendieck group $K_0(\mathcal{C})$ with respect to all induced triangles, that is, the images of exact triangles of $\text{D}^b(\text{mod } A)$ under the projection $\pi$.

Assuming $A$ of finite global dimension, we denote by $\Phi$ the Coxeter transformation on $K_0(\text{D}^b(\text{mod } A))$, that is, the map induced by the Auslander-Reiten translation $\tau$ of $\text{D}^b(\text{mod } A)$. In Section 3 we show the following result.

**Proposition 1.1.** If $A$ is an algebra of finite global dimension and $\mathcal{C} = \mathcal{C}(A)$ then we have $K_0(\mathcal{C}) = \text{Coker}(1 + \Phi)$.

Let $A$ be a heredity algebra of finite representation type or a canonical algebra. In both cases $K_0(\mathcal{C})$ and $K_0(\mathcal{C}_S)$ are shown to be free either over $\mathbb{Z}$ or over $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (independently of the admissible triangulated structure $\mathcal{S}$). We define the dual Grothendieck groups $K_0(\mathcal{C})^*$ and $K_0(\mathcal{C})^*$ as the respective $\mathbb{Z}$- or $\mathbb{Z}_2$-dual. In Section 4 we show our first main result.

**Theorem 1.2.** We have $K_0(\mathcal{C}_S) = \overline{K}_0(\mathcal{C})$ in each of the following three cases:

(i) $A$ is canonical with weight sequence $(p_1, \ldots, p_t)$ having at least one even weight.

(ii) $A$ is tubular.

(iii) $A$ is hereditary of finite representation type.

The remaining canonical cases are covered by the next result.

**Theorem 1.3.** Assume $\mathcal{C} = \mathcal{C}(A)$ is the cluster category of a canonical algebra $A$ with weight sequence $(p_1, \ldots, p_t)$, where all weights $p_i$ are odd. For any admissible triangulated structure $\mathcal{S}$ on $\mathcal{C}$ the Grothendieck group $K_0(\mathcal{C}_S)$ is a non-zero quotient of $\overline{K}_0(\mathcal{C}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Accordingly, if $A$ is canonical (of any weight type), we have $K_0(\mathcal{C}_S) \neq 0$ and Proposition 3.7 yields an explicit basis of $\overline{K}_0(\mathcal{C})$. Since each tame hereditary algebra is derived equivalent to a canonical one, Theorem 1.2 and 1.3 cover also the tame hereditary situation. To prove the two theorems our main device is to provide a categorification of suitable members of the dual Grothendieck group $\overline{K}_0(\mathcal{C})^*$, that is, to realize them by additive functions on $\mathcal{C}_S$ in categorical terms of $\mathcal{C}$ (in a sense defined at the beginning of Section 4). B. Keller informed the authors that his student Y. Palu proved $K_0(\mathcal{C}_S) = \overline{K}_0(\mathcal{C})$ for the admissible structure $\mathcal{S}$ constructed in [7].
In the last section, we consider the cluster category $C(T)$ of an “isolated” tube $T$. We show that there always exists an admissible triangulated structure on $C(T)$ and determine its Grothendieck group explicitly.

2. Notations and definitions

**Definition of cluster categories.** We assume that $A$ is an algebra (we recall that this means a unitary, associative algebra of finite dimension over $k = \overline{k}$) of finite global dimension. We denote by $\text{mod} A$ the category of finitely generated (or equivalently finite-dimensional) right $A$-modules and by $D = \text{Db}(\text{mod} A)$ the bounded derived category of $\text{mod} A$. Since $A$ has finite global dimension, $D$ is a triangulated category, see [4], and we denote by $T$ its suspension functor. Moreover, $D$ has Auslander-Reiten triangles and the Auslander-Reiten translation $\tau$ is an auto-equivalence of $D$.

Denoting $F = \tau^{-1} \circ T$, the cluster category $C = C(A)$ is defined as the *orbit category* $C(A) = D/F \mathbb{Z}$, whose objects are the objects of $D$ and whose morphism spaces are given by

$$\text{Hom}_{C(A)}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_D(X, F^i Y),$$

which are finite dimensional spaces if $A$ is derived equivalent to a hereditary or to a canonical algebra. We denote by $\pi : D \to C(A)$ the canonical projection functor and write occasionally $\pi X$ rather than $X$ for objects in $C$ for emphasis.

**Admissible triangulated structures.** We call a triangulated structure $S$ on $C$ *admissible* if the projection $\pi$ is exact and denote by $C_S$ the category $C$ equipped with $S$. Keller [7] proves the existence of an admissible triangulated structure for $C(A)$ if $\text{Db}(\text{mod} A)$ is triangle equivalent to $\text{Db}(H)$ for some hereditary abelian $k$-category $H$. Then $H$ has a tilting complex, hence by [6, Theorem 1.7] a tilting object. We may assume that $H$ is connected. Passing to a derived equivalent hereditary category we may then assume by Happel’s theorem [5] that $H = \text{mod} H$, where $H$ is a hereditary algebra, or $H = \text{coh} X$, where $X$ is a weighted projective line [3]. In the first case $A$ is derived equivalent to a hereditary, in the second case to a canonical algebra, see paragraph “Canonical algebras” below. Since $C$ — up to equivalence — only depends on $\text{Db}(\text{mod} A)$, we can assume that $A$ itself is hereditary or canonical. Often, we also shall write $C(H)$ instead of $C(A)$ if $\text{Db}(H) \simeq \text{Db}(\text{mod} A)$.
Grothendieck groups. Any $\mathcal{C}$ as above is equipped with the auto-equivalence $\tau: \mathcal{C} \to \mathcal{C}$, induced by the Auslander-Reiten translation of $D^b(\text{mod} \ A)$. A triangle $X \to Y \to Z \to \tau X$ in $\mathcal{C}$ is called induced if it is – up to isomorphism – the image under $\pi$ of an exact triangle in $D^b(\text{mod} \ A)$. Note that $\tau$ takes the role of a suspension functor for $\mathcal{C}$, although the induced triangles usually will not define a triangulated structure on $\mathcal{C}$. We denote by $K_0(\mathcal{C})$ the Grothendieck group of $\mathcal{C}$ with respect to all induced triangles.

If $\mathcal{S}$ is an admissible triangulated structure on $\mathcal{C}$ we denote by $K_0(\mathcal{C}_S)$ the Grothendieck group of $\mathcal{C}$ with respect to all triangles from $\mathcal{S}$. Since each induced triangle lies in $\mathcal{S}$ we get a natural epimorphism $K_0(\mathcal{C}) \to K_0(\mathcal{C}_S)$.

Hereditary categories. If $\mathcal{H}$ is hereditary then the derived category admits a simple description: the indecomposable objects of $D^b(\mathcal{H})$ are of the form $T^i X$ for $X \in \mathcal{H}$ indecomposable and some $i \in \mathbb{Z}$. The morphism spaces are given by

$$\text{Hom}_{D^b(\mathcal{H})}(T^i X, T^j Y) = \text{Ext}_H^{j-i}(X, Y), \text{ for } X, Y \in \mathcal{H}. \tag{2.1}$$

In case $\mathcal{H} = \text{coh} \ X$, $\tau$ is an autoequivalence on $\mathcal{H}$ and therefore $\mathcal{H}$ is a fundamental region for the functor $F$, that is, for each indecomposable object $X \in \mathcal{D}$ there exists a unique $Y \in \mathcal{H}$ such that $X = F^i Y$ and therefore, we can identify the objects of $\mathcal{C}$ with the objects of $\mathcal{H}$ up to isomorphism.

We recall that the category $\text{coh} \ X$ has Serre duality, that is, there exists an autoequivalence $\tau$ for which $\text{Ext}_H^1(X, Y) \simeq D \text{Hom}_H(Y, \tau X)$ holds functorially in $X$ and $Y$. Similarly the categories $\mathcal{D} = D^b(\text{coh} \ X)$ and $\mathcal{D} = D^b(\text{mod} \ A)$, for $A$ hereditary, have also Serre duality in the sense that $\text{Hom}_\mathcal{D}(X, TY) \simeq D \text{Hom}_\mathcal{D}(Y, \tau X)$ holds functorially in $X$ and $Y$.

Canonical algebras. Canonical algebras were introduced by C. M. Ringel in [11] as algebras $A = kQ/I$, where the quiver $Q$ is obtained by joining a source 1 with a sink $n$ by $t \geq 2$ arms consisting of $p_1, \ldots, p_t$ arrows respectively, all pointing from 1 to $n$: 

![Diagram of a quiver with arrows connecting nodes 1 to n through intermediate nodes labeled α1, α2, ..., αt.](image-url)
The ideal \( I \) is generated by \( t - 2 \) relations \( \alpha_i^p = \alpha_j^p - \mu_i \alpha_j^1 \) for some pairwise distinct \( \mu_i \in k \) with \( \mu_i \neq 0, 1 \). The sequence \( (p_1, \ldots, p_t) \) is called the weight sequence of \( A \). If \( \sum_{i=1}^t \frac{1}{p_i} = t - 2 \) then \( A \) is called tubular; this happens precisely for the weight sequences \( (2, 2, 2, 3), (3, 3, 3), (2, 4, 4) \) and \( (2, 3, 6) \). We usually omit weights \( p_i = 1 \) from the sequence, hence the weight sequence \( (3) \) means the sequence \( (1, 3) \). We recall that if \( A \) is canonical of weight type \( (p_1, \ldots, p_t) \) then \( D^b(\text{mod} \ A) \simeq D^b(\text{coh} \ X) \) for a weighted projective line \( X \) of weight type \( (p_1, \ldots, p_t) \).

**Tubes.** Let \( X \) be a weighted projective line of weight type \( (p_1, \ldots, p_t) \) and \( \mathcal{H} = \text{coh} \ X \). We denote by \( \mathcal{H}_0 \) the full subcategory of \( \mathcal{H} \) given by the objects of finite length and by \( \mathcal{H}_+ \) the full subcategory of direct sums of indecomposable objects of infinite length. It is known, see [3], that \( \mathcal{H}_0 = \bigoplus_{x \in X} T_x \) is a coproduct of categories, where each \( T_x \) is a tube of rank \( q \), that is a connected, hereditary, uniserial category, which in abstract form can be realized as mod_{\mathbb{Z}^q} k[[X]] \) (that is, as the category of \( \mathbb{Z}^q \)-graded \( k[[X]] \))-modules of finite length). Each tube of \( \mathcal{H}_0 \) has rank one except finitely many (exceptional) tubes having rank \( p_1, \ldots, p_t \), respectively.

Furthermore Hom\((\mathcal{H}_0, \mathcal{H}_+) = 0 \) and for each non-zero object \( M \in \mathcal{H}_+ \) and each \( x \in X \), we have Hom\(_\mathcal{H}(M, T_x) \neq 0 \).

**Formulas for \( K_0(\text{coh} \ X) \).** The Grothendieck group \( K_0(\mathcal{H}) \) of the abelian category \( \mathcal{H} = \text{coh} \ X \) is described in detail in [2, 3]. It is equipped with the Euler form defined by

\[
\langle [X], [Y] \rangle = \dim_k \text{Hom}_\mathcal{H}(X, Y) - \dim_k \text{Ext}^1_\mathcal{H}(X, Y)
\]

on classes of objects \( X, Y \in \mathcal{H} \). It follows from Serre duality that for all \( x, y \in K_0(\mathcal{H}) \) we have \( \langle y, x \rangle = - \langle x, \Phi y \rangle \), where \( \Phi \) is the Coxeter transformation.

We denote by \( L \) the structure sheaf and for each \( i = 1, \ldots, t \) the unique simple sheaf \( S_i \) belonging to the \( i \)-th exceptional tube such that \( \text{Hom}_\mathcal{H}(L, S_i) \neq 0 \). Then \( \text{Hom}_\mathcal{H}(L, S_i) \) is one-dimensional, and \( \text{Hom}_\mathcal{H}(L, \tau^j S_i) = 0 \) for \( j = 1, \ldots, p_i - 1 \). Furthermore, all simple sheaves from homogeneous tubes have the same class in \( K_0(\mathcal{H}) \); we fix one, say \( S_0 \). Now define the following elements of \( K_0(\mathcal{H}) \):

\[
a = [L], \ s_0 = [S_0], \ s_i = [S_i] \text{ for } i = 1, \ldots, t.
\]

Define then the elements \( s_i(j) = \Phi^j s_i \) for \( j \in \mathbb{Z}_{p_i} \). For later use we reproduce some facts from [9].

**Proposition 2.1.** Let \( \mathcal{H} = \text{coh} \ X \) where \( X \) is of weight type \( (p_1, \ldots, p_t) \).
(a) The abelian group $K_0(\mathcal{H})$ is generated by the elements $a$, $s_0$, $s_i(j)$, $i = 1, \ldots, t$ and $j = 0, \ldots, p_i - 1$, subject to the defining relations

$$
\sum_{j=0}^{p_i-1} s_i(j) = s_0, \text{ for } i = 1, \ldots, t.
$$

(b) Define $p = \text{lcm}(p_1, \ldots, p_t)$, $\delta = p \left( t - 2 - \sum_{i=1}^{t} \frac{1}{p_i} \right)$ and $\text{rk}(x) = \langle x, s_0 \rangle$. Then for all $x \in K_0(\mathcal{H})$, we have

$$
\Phi^p x = x + \delta \cdot \text{rk}(x) \cdot s_0
$$

(c) We have

$$
\Phi a = a - \sum_{i=1}^{t} s_i + (t - 2) \cdot s_0
$$

(d) Furthermore, we have $\langle s_i(m), s_j(n) \rangle = 0$ for $i \neq j$, and

$$
\langle s_i(m), s_i(n) \rangle = \begin{cases} 
1 & \text{if } n \equiv m \mod p_i \\
-1 & \text{if } n \equiv m + 1 \mod p_i \\
0 & \text{else}
\end{cases}
$$

for all $i = 1, \ldots, t$.

Beside the rank function $\text{rk}(x) = \langle x, s_0 \rangle$ we also define the degree function by

$$
\text{deg}(x) = \sum_{j=0}^{p-1} \langle \Phi^j a, x - \text{rk}(x) a \rangle
$$

where $p = \text{lcm}(p_1, \ldots, p_t)$. It is characterized by the properties $\text{deg}(L) = 0$, $\text{deg}(S_0) = p$ and $\text{deg}(\tau^j S_i) = \frac{p}{p_i}$ for $i = 1, \ldots, t$ and $j \in \mathbb{Z}$.

**Discriminant and slope.** Let $\mathcal{H} = \text{coh}(\mathbb{X})$ be of weight type $(p_1, \ldots, p_t)$ and put $p = \text{lcm}(p_1, \ldots, p_t)$. The discriminant

$$
\delta_{\mathcal{H}} = p \left( (t - 2) - \sum_{i=1}^{t} \frac{1}{p_i} \right)
$$

is an invariant of $\mathcal{H}$ deciding on the complexity of the classification problem for $\mathcal{H}$, hence for $\mathcal{C}(\mathcal{H})$, see [3]. For $\delta_{\mathcal{H}} < 0$ the category $\mathcal{H}$ is derived equivalent to the category mod $A$ for the path algebra $kQ$ of an extended Dynkin quiver, and each such algebra $kQ$ has this property. For $\delta_{\mathcal{H}} = 0$ we are dealing with the tubular weights, and for $\delta_{\mathcal{H}} > 0$ the classification problem for $\mathcal{H}$ is wild. For this and the following statements we refer to [3].
Each bundle $E$ has a line bundle filtration $0 = E_1 \subset E_1 \subset \cdots \subset E_r = E$ where each $E_i/E_{i-1}$ is a line bundle. For each non-zero bundle $E$ its slope $\mu(E) = \deg(E)/\text{rk}(E)$ is a rational number such that 

$$\mu(\tau E) = \mu(E) + \delta_H$$

holds. By means of line bundle filtrations for $E$ and $F$ it follows that $\text{Hom}_H(E, F) = 0$ if $\mu(E) - \mu(F)$ is sufficiently large. In particular, for $\delta_H > 0$ (resp. $\delta_H < 0$) we have $\text{Hom}_H(\tau^n E, F) = 0$ (resp. $\text{Hom}_H(E, \tau^n F) = 0$ for $n \gg 0$).

3. Grothendieck group with respect to induced triangles

In this section, we describe the Grothendieck group $K_0(C)$ with respect to the induced triangles. Let $\mathcal{D} = \text{D}^b(\text{mod } A)$. Then the Coxeter transformation $\Phi : K_0(\mathcal{D}) \to K_0(\mathcal{D})$ is given by $\Phi([X]) = [\tau X]$ for any object $X$ of $\mathcal{D}$.

**Proof of Proposition 3.1.** The projection $\pi : \mathcal{D} \to \mathcal{C}$ sends exact triangles to induced triangles, hence yields an epimorphism

$$K_0(\mathcal{D}) \to \overline{K}_0(\mathcal{C}), [X] \mapsto [\pi X].$$

We have $[F^{-1}X] = -[\tau X]$ in $K_0(\mathcal{D})$, hence $[\pi X] = [\pi F^{-1}X] = -[\pi \tau X]$ in $\overline{K}_0(\mathcal{C})$ showing that $\pi(1 + \Phi) = 0$. In order to prove the exactness of

$$K_0(\mathcal{D}) \xrightarrow{1+\Phi} K_0(\mathcal{D}) \xrightarrow{\pi} \overline{K}_0(\mathcal{C}) \to 0$$

it therefore suffices to show that each morphism $\lambda : K_0(\mathcal{D}) \to G$, for $G$ an abelian group, with $\lambda(1+\Phi) = 0$ induces a morphism $\overline{\lambda} : \overline{K}_0(\mathcal{C}) \to G$ with $\overline{\lambda} = \overline{\lambda \pi}$.

By the assumption $\lambda(1+\Phi) = 0$ the corresponding function $\lambda : \mathcal{D} \to G$ is constant on $F$-orbits and additive on exact triangles of $\mathcal{D}$, hence induces a function $\overline{\lambda} : \mathcal{C} \to G$ which is additive on induced triangles.

**Explicit description of $\overline{K}_0(\mathcal{C})$.** Write $\mathbb{Z}_m$ for $\mathbb{Z}/m\mathbb{Z}$. We have the following general description of $\overline{K}_0(\mathcal{C})$.

**Proposition 3.1.** Let $A$ be any algebra of finite global dimension and let $\mathcal{C} = \mathcal{C}(A)$. Then $\overline{K}_0(\mathcal{C})$ has a unique expression as $\mathbb{Z}^r \oplus \bigoplus_{i=1}^s (\mathbb{Z}_{m_i} \oplus \mathbb{Z}_{m_i})$, for natural numbers $r, s$ and positive $m_1, \ldots, m_s$ such that $m_i$ divides $m_{i+1}$ for all $i$. Moreover, any such group occurs as $\overline{K}_0(\mathcal{C}(H))$, where $H$ is the hereditary path algebra given by the following quiver.
Before we enter the proof we need some preparatory lemmas. If \( Q \) is a quiver, we denote by \( B_Q \) the adjacency matrix of \( Q \), that is, \( (B_Q)_{ij} \) denotes the number of arrows in \( Q \) from \( i \) to \( j \). Let \( C \) be the Cartan matrix of \( A \), that is, a matrix representing the Euler form. Since \( A \) has finite global dimension, \( C \) has determinant \( \pm 1 \) and \( \Phi = -C^{-1}C^{tr} \).

**Lemma 3.2.** Let \( A = kQ \) be a hereditary algebra and \( C = C(A) \). Then we have \( K_0(C) = \text{Coker}(B_Q - B_Q^{tr}) \).

**Proof.** Since \( A \) is finite-dimensional, \( Q \) can not contain an oriented cycle. Hence the vertices of \( Q \) can be ordered such that \( C \) is upper triangular. Thus we see that \( B_Q \) is nilpotent, hence \( C = 1 + B_Q + B_Q^2 + B_Q^3 + \ldots \) is a finite sum and \( C^{-1} = 1 - B_Q \). Therefore \( 1 + \Phi = (C^{-tr} - C^{-1})C^{tr} = ((1 - B_Q)^{tr} - (1 - B_Q))C^{tr} = (B_Q - B_Q^{tr})C^{tr} \), which shows that \( \text{Coker}(1 + \Phi) = \text{Coker}(B_Q - B_Q^{tr}) \), thus the result follows by Proposition 1.1. \( \square \)

We call an arrow \( \alpha : v \to w \) of a quiver \( Q \) a source-arrow if \( v \) is a source of \( Q \) and \( \alpha \) is the unique arrow of \( Q \) starting in \( v \). Similarly an arrow \( \alpha : w \to v \) is a sink-arrow if \( v \) is a sink and \( \alpha \) the unique arrow ending in \( v \). In both cases we denote by \( Q_{-\alpha} \) the quiver obtained from \( Q \) by removing the vertices \( v \) and \( w \) and all arrows starting or ending in \( v \) or \( w \). The situation of a source-arrow is depicted as follows.

\[
Q:\begin{array}{c}
v \\
\alpha \\
m_1 \text{ arrows} \\
\vdots \\
m_s \text{ arrows} \\
\end{array}
\]

The next result is quite useful for calculating \( K_0(C) \) in practice.

**Lemma 3.3.** Let \( Q \) be a quiver with an arrow \( \alpha \), which is a source- or a sink-arrow. Denote \( H = kQ \) and \( H' = kQ_{-\alpha} \). Then we have \( K_0(C(H)) \simeq K_0(C(H')) \).
**Proof.** Assume that \( \alpha \) is a source arrow (the case where \( \alpha \) is a sink-arrow is similar). By renumbering the vertices, we can assume that \( \alpha \) is the arrow \( 1 \to 2 \). Then we have

\[
B_Q - B^\text{tr}_Q = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & \rho^\text{tr} \\
0 & -\rho & B_{Q-\alpha} - B^\text{tr}_{Q-\alpha}
\end{bmatrix}.
\]

Adding multiples of the first row to the rows 3, \ldots, \( n \) and simultaneously adding (the same) multiples of the second column to the columns 3, \ldots, \( n \) we obtain a transformation matrix \( T \) and a block diagonal matrix

\[
T(B_Q - B^\text{tr}_Q)T^\text{tr} = \text{diag}(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B_{Q-\alpha} - B^\text{tr}_{Q-\alpha})
\]

and the result follows by Lemma 3.2. \( \square \)

**Proof of Proposition 3.1.** If \( C \) denotes the Cartan matrix of \( A \) then \( 1 + \Phi = (C^\text{tr} - C^{-1})C^\text{tr} \). Now \( S = (C^\text{tr} - C^{-1}) \) is skew-symmetric. Clearly, we have \( \text{Coker}(1 + \Phi) = \text{Coker} S \).

Using the skew-normal form of \( S \), see [10, Theorem IV.1], we obtain \( S' = U^\text{tr}SU \) for some \( U \in \mathbb{GL}_n(\mathbb{Z}) \), where \( S' = \text{diag}(B_0, B_1, \ldots, B_s) \) is a block-diagonal matrix with the following blocks: \( B_0 \) is the zero matrix of size \( r \times r \) and for \( i = 1, \ldots, s \),

\[
B_i = \begin{bmatrix} 0 & m_i \\ -m_i & 0 \end{bmatrix}
\]

where \( m_i \) divides \( m_{i+1} \) for all \( i = 1, \ldots, s-1 \). Therefore \( \text{Im} S \simeq \text{Im} S' \simeq \bigoplus_{i=1}^r (m_i \mathbb{Z})^2 \) and we obtain \( K_0(C) \simeq \text{Coker} S' \simeq \mathbb{Z}^r \oplus \bigoplus_{i=1}^s (\mathbb{Z}_{m_i} \oplus \mathbb{Z}_{m_i}) \) as desired.

Let \( H \) be the hereditary algebra defined by the quiver in Proposition 3.1 and denote \( H' = kQ_{-\alpha} \). By Lemma 3.3 we have \( K_0(C(H)) \simeq K_0(C(H')) \). Now, the claim is obvious for \( H' \) since \( B_{Q-\alpha} - B^\text{tr}_{Q-\alpha} = \text{diag}(B_0, B_1, \ldots, B_s) \) is the block-diagonal matrix as above. \( \square \)

**The hereditary case.**

**Proposition 3.4.** If \( A \) is a hereditary algebra whose quiver is a tree then \( K_0(C(A)) \) is a free abelian group.

**Proof.** Any tree can be reduced to a disjoint union of \( r \) vertices, for some \( r \), by cutting off source- and sink-arrows. Hence, we get \( K_0(C(A)) \simeq \mathbb{Z}^r \) by Lemma 3.3. \( \square \)
Proposition 3.5. Let $A$ be a connected hereditary representation-finite algebra, that is, the underlying graph of its quiver is a Dynkin diagram $\Delta$. Then, we have the following description.

$$\overline{K}_0(C) = \begin{cases} 0, & \text{if } \Delta = A_n, E_n \text{ with } n \text{ even} \\ \mathbb{Z}, & \text{if } \Delta = A_n, D_n, E_7 \text{ with } n \text{ odd} \\ \mathbb{Z}^2, & \text{if } \Delta = D_n \text{ with } n \text{ even} \end{cases}$$

Proof. This follows immediately using Lemma 3.3. \qed

The canonical case. We now assume that $A$ is canonical of weight type $(p_1, \ldots, p_t)$ and $H$ is the associated category of coherent sheaves. We put $C = C(A) = C(H)$ and start by describing $\overline{K}_0(C)$ by generators and defining relations.

Proposition 3.6. The abelian group $\overline{K}_0(C)$ is generated by the elements $\overline{a}, \overline{s}_0, \overline{s}_1, \ldots, \overline{s}_t$ subject to the following defining relations.

$$(3.1) \quad 2\overline{s}_0 = 0,$$

$$(3.2) \quad 2\overline{a} = \sum_{i=1}^{t} (\overline{s}_i - \overline{w}),$$

$$(3.3) \quad \overline{s}_0 = \frac{1 - (-1)^{p_i}}{2} \overline{s}_i, \text{ for } i = 1, \ldots, t.$$

Proof. We recall from Proposition 2.1(a), that $K_0(H)$ is the abelian group generated by $\{a, s_0, s_i(j) | i = 1, \ldots, t \text{ and } j = 0, \ldots, p_i - 1\}$ subject to the defining relations (2.2). Therefore $\overline{K}_0(C) = K_0(H)/\text{Im}(1 + \Phi)$ is the abelian group generated by the same generators with the relations (2.2) and the additional relations

$$(3.4) \quad \overline{a} + \Phi \overline{a} = 0,$$

$$(3.5) \quad \overline{s}_0 + \Phi \overline{s}_0 = 0 \text{ and}$$

$$(3.6) \quad \overline{s}_i(j) + \Phi \overline{s}_i(j) = 0 \text{ for } i = 1, \ldots, t \text{ and } j = 1, \ldots, p_i,$$

which altogether form a system of defining relations. Using Proposition 2.1(c), we can rewrite (3.4) as (3.2). Using $\Phi s_0 = s_0$ we rewrite (3.5) as (3.1). Using $\Phi s_i(j) = s_i(j + 1)$ and (2.2) we obtain

$$\overline{s}_0 = \sum_{j=0}^{p_i-1} (-1)^j \overline{s}_i$$

which can be rewritten in the form (3.3). Thus, since $\Phi s_i(j) = s_i(j+1)$, the group $\overline{K}_0(C)$ is generated by $\overline{a}, \overline{s}_0, \overline{s}_i = \overline{s}_i(0)$ for $i = 1, \ldots, t$ subject to the defining relations (3.1), (3.2) and (3.3). \qed
Proposition 3.7. Let $\mathcal{H} = \text{coh} \mathcal{X}$ with weight sequence $(p_1, \ldots, p_t)$ where $p_1, \ldots, p_r$ are even and $p_{r+1}, \ldots, p_t$ are odd. Further let $\mathcal{C} = \mathcal{C}(\mathcal{H})$.

(i) If $r \geq 1$ then $\overline{K}_0(\mathcal{C})$ is the free abelian group on $\overline{\mathfrak{a}}, \overline{s}_2, \ldots, \overline{s}_r$.

(ii) If $r = 0$ (that is, all weights $p_i$ are odd) then $\overline{K}_0(\mathcal{C}) \simeq \mathbb{Z}\overline{\mathfrak{a}} \oplus \mathbb{Z}\overline{s}_0 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. Let first $r \geq 1$. Then, by (3.3), we have $\overline{s}_0 = \frac{1-(-1)^{p_1}}{2}\overline{s}_1 = 0$ and for $i > r$, we obtain $\overline{s}_i = 0$, again by (3.3). Therefore $\overline{s}_1 = 2\mathfrak{a} - \sum_{i=2}^r \overline{s}_i$ because of (3.2). It follows that $\overline{\mathfrak{a}}, \overline{s}_2, \ldots, \overline{s}_r$ generate $\overline{K}_0(\mathcal{C})$ without relations.

Now let $r = 0$. Then we obtain from (3.3) that $\overline{s}_i = \overline{s}_0$ for all $i = 1, \ldots, t$. Therefore we get that $\overline{K}_0(\mathcal{C})$ is generated by $\overline{\mathfrak{a}}$ and $\overline{s}_0$ with the remaining defining relations $2\overline{\mathfrak{a}} = 0$ and $2\overline{s}_0 = 0$.

The dual Grothendieck groups. In the sequel the Grothendieck groups $\overline{K}_0(\mathcal{C})$ and $K_0(\mathcal{C}_S)$ are free over $\mathbb{Z}$ or $\mathbb{Z}_2$, respectively. We define dual Grothendieck groups $\overline{K}_0(\mathcal{C})^*$ and $K_0(\mathcal{C}_S)^*$ forming the respective $\mathbb{Z}$- or $\mathbb{Z}_2$-duals.

We first deal with the $\mathbb{Z}$-free case. Since the Cartan matrix has determinant $\pm 1$, the Euler form induces an isomorphism $K_0(\mathcal{H}) \xrightarrow{\sim} K_0(\mathcal{H})^*$, $\mathbf{y} \mapsto \langle \mathbf{y}, - \rangle$. A linear form $\lambda : K_0(\mathcal{H}) \to \mathbb{Z}$ induces a linear form $\overline{\lambda} : \overline{K}_0(\mathcal{C}) \to \mathbb{Z}$ if and only if $\lambda \circ (1 + \Phi) = 0$.

Lemma 3.8. A linear form $\lambda = \langle \mathbf{y}, - \rangle$ satisfies $\lambda \circ (1 + \Phi) = 0$ if and only if $\Phi \mathbf{y} = -\mathbf{y}$. In particular, in this case $\text{rk} \mathbf{y} = 0$ and $\text{deg} \mathbf{y} = 0$.

Proof. We have $\langle \mathbf{y}, - \rangle \circ (1 + \Phi) = 0$ if and only if $\langle \mathbf{y}, \Phi^{-1}\mathbf{x} \rangle + \langle \mathbf{y}, \Phi\Phi^{-1}\mathbf{x} \rangle = 0$ for all $\mathbf{x} \in K_0(\mathcal{H})$, and since $\langle \mathbf{y}, \Phi\mathbf{x} \rangle = \langle \Phi^{-1}\mathbf{y}, \mathbf{x} \rangle$ this is equivalent to $\langle \mathbf{y} + \Phi\mathbf{y}, - \rangle = 0$. Since the Cartan matrix has determinant $\pm 1$ the assertion follows.

For any abelian group $G$ define $G_2 = G \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Furthermore let $\text{rk}_2, \text{deg}_2 : K_0(\mathcal{H})_2 \to \mathbb{Z}_2$ be the functions induced by $\text{rk}$ and $\text{deg}$. Similarly define $\langle -, - \rangle_2 : K_0(\mathcal{H})_2 \times K_0(\mathcal{H})_2 \to \mathbb{Z}_2$ to be induced by the Euler form.

Proposition 3.9. Let $\mathcal{H} = \text{coh} \mathcal{X}$ with weight sequence $(p_1, \ldots, p_t)$ and set $\mathcal{C} = \mathcal{C}(\mathcal{H})$. The group $(\Phi)$ acts on $K_0(\mathcal{H})$ by $\Phi \mathbf{y} = -\mathbf{y}$.

(i) If there is at least one even weight $p_i$ then there is an isomorphism $K_0(\mathcal{H})^{(\Phi)} \xrightarrow{\sim} \overline{K}_0(\mathcal{C})^*$, $\mathbf{y} \mapsto \langle \mathbf{y}, - \rangle$.
which gives rise to an exact sequence

\[ 0 \to K_0(C)^* \to K_0(\mathcal{H}) \to K_0(\mathcal{H}) \to K_0(C) \to 0. \]

(ii) If all weights are odd, then there is an isomorphism

\[ K_0(\mathcal{H})_2^{(\Phi)} \cong K_0(\mathcal{C})^*, \ y \mapsto \langle y, - \rangle_2 \]

which gives rise to an exact sequence

\[ 0 \to K_0(\mathcal{C})^* \to K_0(\mathcal{H})_2 \to K_0(\mathcal{H}) \to K_0(\mathcal{C}) \to 0. \]

Proof. Part (i) follows from Lemma 3.8 and the proof of (ii) is similar using reduction modulo 2.

If \( x \in K_0(\mathcal{H}) \) is a \( \Phi \)-periodic object with period \( q_x \), we define

\[ v(x) = \sum_{j=0}^{q_x-1} (-1)^j \Phi^j x \]

and if \( q_x \) is even, we define

\[ h(x) = \sum_{j=0}^{q_x-1} \Phi^{2j} x. \]

Proposition 3.10. Let \( \mathcal{H} = \text{coh} \mathcal{X} \) with weight sequence \( (p_1, \ldots, p_t) \) where \( p_1, \ldots, p_r \) are even and \( p_{r+1}, \ldots, p_t \) are odd. Further let \( \mathcal{C} = \mathcal{C}(\mathcal{H}) \).

(i) If \( r \geq 1 \) then

\[ \langle v(s_1), - \rangle, \langle h(s_2) - h(s_1), - \rangle, \ldots, \langle h(s_r) - h(s_1), - \rangle \]

is a \( \mathbb{Z} \)-basis of \( K_0(\mathcal{C})^* \).

(ii) If \( r = 0 \) (that is, all weights \( p_i \) are odd) then \( \text{rk}_2 \) and \( \text{deg}_2 \) is a \( \mathbb{Z}_2 \)-basis of \( K_0(\mathcal{C})^* \).

Proof. (i) Clearly \( (1 + \Phi)v(s_1) = 0 \) since \( p_1 \) is even. Furthermore, \( (1 + \Phi)h(s_i) = s_0 \) for \( i = 1, \ldots, r \) and therefore, by the Proposition 3.9 we get that \( (3.7) \) are indeed elements of \( K_0(\mathcal{C})^* \). From the formulas

\[ \langle v(s_1), a \rangle = 1, \quad \langle v(s_1), s_h \rangle = 0 \]

\[ \langle h(s_j) - h(s_1), a \rangle = 0, \quad \langle h(s_j) - h(s_1), s_h \rangle = \delta_{jh} \]

it follows that \( (3.7) \) forms a \( \mathbb{Z} \)-basis of \( K_0(\mathcal{C})^* \).

(ii) We know that \( \alpha, s_0 \) is a \( \mathbb{Z}_2 \)-basis of \( K_0(\mathcal{C}) \) by Proposition 3.7. We have \( \Phi s_0 = s_0 \) and therefore \( \text{rk}_2 = \langle -, s_0 \rangle_2 \) defines a linear form on \( K_0(\mathcal{H}) \).
Since $s_i = s_0$ for $i = 1, \ldots, t$, we get from Proposition 2.1(d) that $\Phi a = a \mod 2$. Hence we get
\[
\deg_2(x) = \sum_{j=0}^{p} \langle \Phi^j a, x - \text{rk}(x)a \rangle_2 = \langle a, x - \text{rk}(x)a \rangle_2 = \langle a, x \rangle_2 + \text{rk}_2(x).
\]
Thus, also $\deg$ induces a linear map $\deg_2 : K_0(C) \to \mathbb{Z}_2$. Since \(\text{rk}_2(s_0) = 0\) and $\deg_2(a) = 0$, it follows that $\text{rk}_2$, $\deg_2$ form a $\mathbb{Z}_2$-basis of $K_0(C)^*$.

□

4. Additive functions on $\mathcal{C}_S$

Cutting technique. For a finite dimensional $k$-vector space $V$ let $|V|$ (resp. $|V|_2$) denote its $k$-dimension (resp. its $k$-dimension modulo two). We put $\mu_E(X) = |\text{Hom}_C(E, X)|$ and write $\overline{\mu}_E(X)$ for $\mu_E(X)$ modulo two.

In the sequel we identify members $\lambda$ from the dual Grothendieck group $K_0(C)^*$ with mappings $\lambda$ defined on $\mathcal{C} = \mathcal{C}(\mathcal{H})$ with values in $\mathbb{Z}$, respectively in $\mathbb{Z}_2$, that are additive on induced triangles. We call $\lambda$ realizable if, depending on the case considered, it has the form $\mu_E - \mu_F$ (resp. $\overline{\mu}_E$) with $E$ and $F$ from $\mathcal{C}$. The realizable functions form a subgroup of $K_0(C)^*$. Note that usually neither $\mu_E$ nor $\overline{\mu}_E$ (respectively $\mu_E - \mu_F$) are realizable. Our next proposition shows how to construct realizable functions which additionally belong to $K_0(C_S)^*$ for an admissible triangulated structure $\mathcal{S}$ on $\mathcal{C}$.

For any object $U \in \mathcal{D} = \text{D}^b(\mathcal{H})$ and any positive integer $q$ define the function $\lambda^{(q)}_U$ on the objects $Y$ of $\mathcal{C}$ by
\[
\lambda^{(q)}_U : \mathcal{C} \to \mathbb{Z}, \quad \lambda^{(q)}_U(Y) = \sum_{i=0}^{q-1} (-1)^i |\text{Hom}_C(\pi U, T^i Y)|
\]
and set $\overline{\lambda}^{(q)}_U : \mathcal{C} \to \mathbb{Z}_2$, $Y \mapsto \lambda^{(q)}_U(Y) \mod 2$

Proposition 4.1. Suppose that $U$ is an object in $\text{D}^b(\mathcal{H})$ such that for some positive integer $q$ we have $\tau^q X \simeq F^m X$ for some $m \in \mathbb{Z}$.

(i) If $q$ is even then $\lambda^{(q)}_X$ is additive on each triangle of an admissible triangulated structure on $\mathcal{C}$.

(ii) If $q$ is odd, then $\overline{\lambda}^{(q)}_X$ is additive on each triangle of an admissible triangulated structure on $\mathcal{C}$.

Proof. Identify $U$ with its image in $\mathcal{C}$. Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \text{T}X$ be a triangle in $\mathcal{C}$ with respect to an admissible triangulated structure.
Application of the functor $\text{Hom}_C(U, -)$ gives a long exact sequence

$$0 \to K \to \text{Hom}_C(U, X) \to \text{Hom}_C(U, Y) \to \text{Hom}_C(U, Z) \to \cdots$$

$$\cdots \to \text{Hom}_C(U, \tau^{q-1} \tau^{-1} X) \to \text{Hom}_C(U, \tau^{q-1} \tau^{-1} Y) \to \text{Hom}_C(U, \tau^{q-1} \tau^{-1} Z) \to K' \to 0,$$

where $K = \text{Ker}(C(U, \alpha))$ and

$$(4.2) \quad K' = \text{Ker}(C(U, \tau^q \alpha)) \cong \text{Ker}(C(\tau^{-q} U, \alpha)) \cong \text{Ker}(C(U, \alpha)) = K.$$

The alternating sum of the dimensions of the spaces in the sequence equals zero. If $q$ is even we hence get

$$(4.3) \quad \lambda^{(q)}_U(X) - \lambda^{(q)}_U(Y) + \lambda^{(q)}_U(Z) = 0.$$ 

Therefore $\lambda^{(q)}_U$ is a linear form on $K_0(C)$. If $q$ is odd then this holds modulo 2. $\square$

**The even canonical case.** We first study the case where $\mathcal{H} = \text{coh} X$ with weight sequence $(p_1, \ldots, p_t)$. In this case we have linear forms which are defined by “periodic” elements which lie in tubes: If $U \in \mathcal{H}$ is indecomposable lying in a tube of rank $q$ then $\tau^q U \cong U$.

Assume that $p_1, \ldots, p_r$ are even and $p_{r+1}, \ldots, p_t$ are odd. Let $\mathcal{T}_1, \ldots, \mathcal{T}_r$ be the exceptional tubes in $\mathcal{H}_0$ of rank $p_1, \ldots, p_r$, respectively, and recall that $S_i$ is a simple object from $\mathcal{T}_i$. By Proposition 4.1 the functions $\lambda_i = \lambda^{(p_i)}_{S_i}$ are additive on the triangles of any admissible triangulated structure $\mathcal{S}$ on $C$.

If $x$ is an element in $K_0(\mathcal{H})$, denote by $\hat{x}$ its image in $K_0(\mathcal{C}_S)$.

**Proposition 4.2.** Assume that the number $r$ of even weights $p_i$ is non-zero, then the linear forms $\lambda_i$ $(i = 1, \ldots, r)$ are realizable, linearly independent over $\mathbb{Z}$ and $K_0(\mathcal{C}_S) = \overline{K_0}(C) \cong \mathbb{Z}^r$.

**Proof.** Linear independence of $\lambda_1, \ldots, \lambda_r$ follows from $\lambda_i(S_j) = 2\delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker symbol. We conclude that $\hat{s}_1, \ldots, \hat{s}_r$ are linearly independent and hence also $\hat{a}, \hat{s}_2, \ldots, \hat{s}_r$, since $2\hat{a} = \sum_{i=1}^m \hat{s}_i$. Therefore $K_0(\mathcal{C}_S) = \overline{K_0}(C) \cong \mathbb{Z}^r$ follows from Proposition 3.7 (i). $\square$

**The odd canonical case.** We adopt the notations of the previous section. Recall that $S_0$ denotes a simple object from a homogeneous tube in $\mathcal{H}_0$.  

**Proposition 4.3.** Let $C = C(\mathcal{H})$, where $\mathcal{H}$ is the category of coherent sheaves on a weighted projective line of weight type $(p_1, \ldots, p_t)$, where all weights $p_i$ are odd. Then the following holds:
(i) Always $\text{rk}_2$ is a non-zero realizable member of $K_0(C_S)^* \subseteq \overline{K}_0(C)^*$.  
(ii) For $\delta_H \neq 0$ the subgroup of realizable members of $\overline{K}_0(C)^*$ agrees with the subgroup $\langle \text{rk}_2 \rangle$ generated by the rank modulo two.  
(iii) For $\delta_H = 0$, that is for weight type $(3, 3, 3)$, we have equality

$$K_0(C_S)^* = \overline{K}_0(C)^* = Z_2 \text{rk}_2 \oplus Z_2 \deg_2,$$

and each member of $\overline{K}_0(C)^*$ is realizable.

Proof. (i) and (iii): We invoke Proposition 3.10 and use that $\text{rk}_2$ can be realized as $\lambda_0 = \lambda_{S_0}^{(3)}$ where $\lambda_0(L) = 1 \mod 2$ and $\lambda_0(S_0) = 0$.

In the tubular case, only the weight type $(3, 3)$ matters, thus the structure sheaf $L$ lies in a tube of $\tau$-period three. Hence $\deg_2$ is realized by $\lambda_{L}^{(3)}$ where $\lambda_{L}^{(3)}(S_0) = 1 \mod 2$.

(ii): Assume the function $\overline{\mu}_E(X) = |\text{Hom}_C(E, X)|_2$ with $E$ from $\mathcal{H}$ is additive on induced triangles. We are going to show that $\overline{\mu}_E$ is a multiple of $\text{rk}_2$. Since $X \to X \to 0 \to \tau X$ is an induced triangle, we get

$$\overline{\mu}_E(X) = \overline{\mu}_E(\tau X) = \overline{\mu}_{\tau^{-1}E}(X)$$

for each object $X$ of $\mathcal{H}$. Since $p = \text{lcm}(p_1, \ldots, p_t)$ is odd, setting $E = \bigoplus_{j=0}^{p-1} \tau^j E$ we thus obtain $\overline{\mu}_E = \overline{\mu}_E$.

Next, we use a decomposition $E = E_+ \oplus E_0$ of $E$ into a bundle $E_+$ and an object $E_0$ of finite length. Invoking that $\tau^p$ acts as the identity on finite length objects of $\mathcal{H}$, we see that $E_0 = \bigoplus_{j=0}^{p-1} \tau^j E$ is fixed under $\tau$. The expression $\overline{\mu}_E(X) = |\text{Hom}_H(E_0, X)|_2 + |\text{Ext}_H^1(E_0, \tau^{-1}X)|_2$ hence agrees with $\langle E_0, X \rangle_2$, and $\overline{\mu}_{E_0}$ is a multiple of $\text{rk}_2$. By part (i) the function $\text{rk}_2$ is additive on induced triangles, we conclude that the same holds for $\overline{\mu}_{E_+}$. From now on, we may hence assume that $E$ is a bundle. Note that

$$\overline{\mu}_E(X) = \overline{\mu}_E(X) = \langle \langle E, X \rangle \rangle_2 + \Delta_E(X),$$

where $\langle \langle E, X \rangle \rangle = \sum_{j=0}^{p-1} \langle \tau^j E, X \rangle$, $\langle \langle E, X \rangle \rangle_2 = \langle \langle E, X \rangle \rangle \mod 2$ and

$$\Delta_E(X) = \sum_{j=0}^{p-1} \left( |\text{Ext}_H^1(\tau^j E, X)|_2 + |\text{Ext}_H^1(\tau^{j+1} E, X)|_2 \right)$$

$$= |\text{Ext}_H^1(E, X)|_2 + |\text{Ext}_H^1(\tau^p E, X)|_2.$$

By the Riemann-Roch formula,

$$(4.4) \quad \langle \langle E, X \rangle \rangle = -\frac{p}{2} \delta_H \text{rk}(E) \text{rk}(X) + \left| \begin{array}{cc} \text{rk}(E) & \text{rk}(X) \\ \text{deg}(E) & \text{deg}(X) \end{array} \right|,$$
see [9], the function \( \langle\langle E, -\rangle\rangle_2 \) is a linear combination of \( \text{rk}_2 \) and \( \text{deg}_2 \). Hence \( \langle\langle E, -\rangle\rangle_2 \) is a member of \( K_0(C)^* \), implying that \( \Delta_E \) also belongs to \( K_0(C)^* \). By construction, \( \Delta_E \) vanishes on \( S_0 \). By means of a line bundle filtration of \( E \), Serre duality implies that \( \Delta_E(L') = 0 \) for any line bundle \( L' \) of sufficiently large degree, and we deduce from Proposition 3.7 that \( \Delta_E = 0 \). If \( E \) is of even rank, then the function \( \langle\langle E, -\rangle\rangle_2 \), hence also the function \( \mu_E \), is a multiple of \( \text{rk}_2 \), proving the claim in this case.

It remains to deal with the case that the rank of \( E \) is odd, where we deduce a contradiction from the assumption that \( \mu_E \) belongs to \( K_0(C)^* \). Invoking \( \mu_E = \mu_E \), we have shown that \( \Delta_E = 0 \). Note that \( \Delta_E = 0 \) asserts that

\[
|\text{Ext}^1_H(\overline{E}, X)|_2 = |\text{Ext}^1_H(\tau^{np}\overline{E}, X)|_2
\]

for each \( n \in \mathbb{Z} \) and each object \( X \) from \( \mathcal{H} \).

Case \( \delta_H > 0 \): Clearly, the functions \( \langle\langle \overline{E}, -\rangle\rangle = \langle\langle E, -\rangle\rangle \) and \( \langle\langle \tau^{np}\overline{E}, -\rangle\rangle = \langle\langle \tau^{np}E, -\rangle\rangle \) are additive on induced triangles. They agree modulo two on \( S_0 \) and by formula (4.5) also on each line bundle \( L' \) of large negative degree. It then follows from Proposition 3.7 that \( \langle\langle \overline{E}, -\rangle\rangle_2 = \langle\langle \tau^{np}\overline{E}, -\rangle\rangle_2 \).

By means of a line bundle filtration for \( \overline{E} \) we obtain for each integer \( n \gg 0 \) two line bundles \( L_1 \) and \( L_2 \) of consecutive degrees \( d \) and \( d + 1 \) such that

\[
\text{Hom}_H(\tau^{np}\overline{E}, L_i) = 0 \quad \text{and} \quad \text{Ext}^1_H(\overline{E}, L_i) = 0 \quad \text{for } i = 1, 2.
\]

By (4.4) we get \( \langle\langle \overline{E}, L_i\rangle\rangle_2 = \alpha + \text{deg}_2(L_i) \) for some \( \alpha \in \mathbb{Z}_2 \), only depending on \( E \). We then choose one of the \( L_i \) such that (4.6) and further \( \langle\langle \overline{E}, L_i\rangle\rangle_2 = 1 \) holds. Invoking (4.5) we obtain the contradiction

\[
1 = \langle\langle \overline{E}, L_i\rangle\rangle_2 = \langle\langle \tau^{np}\overline{E}, L_i\rangle\rangle_2 = \text{Hom}_H(\tau^{np}\overline{E}, L_i)|_2 + \text{Ext}^1_H(\tau^{np}\overline{E}, L_i)|_2 = 0.
\]

Case \( \delta_H < 0 \): The proof is similar, choosing \( n \ll 0 \).

The Dynkin case. Now, let \( A \) be a connected hereditary representation-finite algebra whose quiver has as underlying graph the Dynkin diagram \( \Delta \). Then \( \Delta \) is a star with length of the arms \( p_1, \ldots, p_t \) (where \( t \leq 3 \)) and the Auslander-Reiten quiver of \( \text{D}^b(\text{mod} \ A) \) is \( \mathbb{Z}\Delta \) whose \( \tau \)-orbits correspond to the vertices of \( \Delta \). In this case there are no tubes. Nevertheless we find “periodic” objects. Let \( m \) be the Coxeter number.
of \( \Delta \), that is, the order of the Coxeter transformation \( \Phi \). We have

\[
  m = \begin{cases} 
    n + 1 & \text{if } \Delta = A_n, \\
    2(n - 1) & \text{if } \Delta = D_n, \\
    12, 18, 30 & \text{if } \Delta = E_6, E_7, E_8, \text{ respectively}.
  \end{cases}
\]

Since \( K_0(\mathcal{C}) = 0 \) in the cases \( \Delta = A_n \) (n even), \( E_6, E_8 \) by Proposition 3.5, we restrict our attention to the remaining cases. Note that then \( m \) is always an even number.

**Proposition 4.4.** In the cases \( \Delta = A_n \) with \( n \) odd or \( \Delta = E_7 \), let \( M \) be an indecomposable object of \( \text{D}^b(\text{mod } A) \) lying in a \( \tau \)-orbit as indicated in the following picture.

\[
  \Delta = A_n : \quad \Delta = E_7 : \quad M
\]

Then \( \lambda_M^{(m+2)} \) is a non-zero realizable function, which is additive on all triangles.

In the case \( \Delta = D_n \) (\( n \geq 4 \)) one can choose indecomposable objects \( M_1 \) and \( M_2 \) of \( \text{D}^b(\text{mod } A) \) lying in the two \( \tau \)-orbits as indicated in the following picture

\[
  \Delta = D_n : \quad M_1 \quad M_2
\]

such that the functions \( \lambda_i = \lambda_{M_i}^{(m+2)} \) for \( i = 1, 2 \) are non-zero realizable and, for \( n \) even, linearly independent.

**Proof.** For any indecomposable \( A \)-module \( U \) we have \( \tau^m U \simeq T^{-2}U \) in \( \mathcal{D} = \text{D}^b(\text{mod } A) \) which implies \( \tau^{m+2}U \simeq F^{-2}U \). By Proposition 4.1 the function \( \lambda_U^{(m+2)} \) is additive on triangles in \( \mathcal{C} \) with respect to any admissible triangulated structure on \( \mathcal{C} \).

Let \( \mathcal{H} = \text{mod } A \). For indecomposable objects \( M \) and \( N \) in \( \mathcal{D} \) we have (identifying them with their images in \( \mathcal{C} \))

\[
  \lambda_M^{(m+2)}(N) = \sum_{i=0}^{m+1} (-1)^i |\text{Hom}_\mathcal{C}(\tau^{-i}M, N)|
\]

\[
  = \sum_{j \in \mathbb{Z}} \sum_{i=0}^{m+1} (-1)^j |\text{Hom}_\mathcal{D}(M, \tau^{-j-i}T^jN)|.
\]
Setting $\mu_j(M, N) = \sum_{i=0}^{m-1} (-1)^i |\text{Hom}_D(M, \tau^j T^j N)|$ we have $\mu_j(M, N) = 0$ for $j < 0$ and $M, N \in \mathcal{H} \cup \tau^{-\mathcal{H}}$.

In the case $\Delta = A_n$ ($n$ odd) we get $\lambda_{(m+2)}^M(M) = 2$, where $M$ is as indicated above. Indeed, $\mu_0(M, M) = 1 = \mu_1(M, M)$ and $\mu_j(M, M) = 0$ for $j \geq 2$.

In the case $\Delta = E_7$ we have $\lambda_{(m+2)}^M(M) = 6$. Indeed, $\mu_0(M, M) = 1$, $\mu_1(M, M) = 3$, $\mu_2(M, M) = 2$ and $\mu_j(M, M) = 0$ for $j \geq 3$.

In the case $\Delta = D_n$ let $M_1$ and $M_2$ be in the AR quiver lying in the following slice.

$$
\begin{array}{c}
M_1 \\
\rightarrow \\
M_2
\end{array}
$$

Let $\lambda_1 = \lambda_{M_1}^{(m+2)}$ and $\lambda_2 = \lambda_{M_2}^{(m+2)}$ where $M_1$ and $M_2$ are as indicated above. Let $M, M' \in \{M_1, M_2\}$ with $M \neq M'$. It is easy to see that $\text{Hom}_D(M, \tau^i M) \neq 0$ if and only if $i$ is even and $-(n-2) \leq i \leq 0$. Similarly, $\text{Hom}_D(M, \tau^i M') \neq 0$ if and only if $i$ is odd and $1 \leq i \leq n-1$. Moreover,

$$
\tau^{-(n-1)} M \simeq \begin{cases} 
TM & n \text{ even}, \\
TM' & n \text{ odd}.
\end{cases}
$$

Using this we get $\mu_0(M, M) = 1$, $\mu_0(M, M') = 0$, and

| $n$ even | $n$ odd |
|----------|----------|
| $\frac{n}{2}$ | $-\frac{n-2}{2}$ |
| $\frac{n}{2}$ | $-\frac{n-1}{2}$ |
| $\frac{n-2}{2}$ | $-\frac{n-3}{2}$ |
| $\frac{n-2}{2}$ | $-\frac{n-1}{2}$ |

and $\mu_j(M, M) = 0 = \mu_j(M, M')$ for $j \geq 3$. Consequently, for even $n$ one has $\lambda_1(M_1) = n$, $\lambda_1(M_2) = -(n-2)$, $\lambda_2(M_1) = -(n-2)$ and $\lambda_2(M_2) = n$, and linear independence of $\lambda_1$ and $\lambda_2$ follows. If $n$ is odd, then $\lambda_1(M_1) = n-1 = \lambda_2(M_2)$ and $\lambda_1(M_2) = -(n-1) = \lambda_2(M_1)$.

**Proof of Theorem 1.2.** For case (i) the assertion follows from Proposition 4.2, for case (ii) it follows from Proposition 4.3 and for (iii) it follows from the fact that by Proposition 1.1, $K_0(C_S)$ is a quotient of $\overline{K_0}(C)$ and by Proposition 4.4 and 3.5 both are free of the same rank.

**Proof of Theorem 1.3.** This follows immediately from Propositions 4.2 and 4.3.
5. Cluster tubes

Existence of admissible structures. Let $\mathcal{T}$ be a tube of rank $q$. We consider the cluster category $\mathcal{C} = \mathcal{C}(\mathcal{T})$ as the orbit category $D^b(\mathcal{T})/F^Z$, where again $F = \tau^{-1}T$, where $\tau$ is the Auslander-Reiten translation and $T$ the suspension functor. We call $\mathcal{C}(\mathcal{T})$ the cluster tube of rank $q$. Since $\mathcal{T}$ has no tilting object, we can not invoke Keller’s result [7] directly to conclude that $\mathcal{T}$ has an admissible triangulated structure. We now show that $\mathcal{T}$ admits an admissible structure anyway.

**Proposition 5.1.** The cluster tube $\mathcal{C}(\mathcal{T})$ of rank $q$ admits an admissible triangulated structure.

**Proof.** Let $X$ be a weighted projective line of weight type $(q) = (1, q)$ and let $\mathcal{H} = \text{coh} X$. Recall the definitions of $\mathcal{H}_0$ and $\mathcal{H}_+$ from Section 2. We may view $\mathcal{T}$ as a full subcategory of $\mathcal{H}_0$, which is even exact because of (2.1). Therefore $\mathcal{C}(\mathcal{T})$ is a full subcategory of $\mathcal{C}(\mathcal{H})$. By [7], there exists an admissible triangulated structure $S$ on $\mathcal{C}(\mathcal{H})$. We denote by $S'$ the subclass of $S$ given by all triangles $X \to Y \to Z \to TX$ such that $X, Y, Z \in \mathcal{C}(\mathcal{T})$. It is clear that once we show that $S'$ is a triangulated structure on $\mathcal{C}(\mathcal{T})$ then it is admissible. Since $\mathcal{T}$, and then also $\mathcal{C}(\mathcal{T})$, is closed under direct sums and summands in $\mathcal{H}$, we only have to verify that $X, Y \in \mathcal{C}(\mathcal{T})$ implies $Z \in \mathcal{C}(\mathcal{T})$ for any triangle $X \to Y \to Z \to TX$ in $S$.

By the preceding remark, we can assume that $X, Y \in \mathcal{T}$ and $Z \in \mathcal{H}$. Write $Z = Z_+ \oplus Z_0$ where $Z_0 \in \mathcal{H}_0$ and $Z_+ \in \mathcal{H}_+$. Let $W \in \mathcal{H}$ be a simple object in some homogeneous tube $\mathcal{T}' \neq \mathcal{T}$. Applying the functor $\text{Hom}_\mathcal{C}(-, W)$ to the triangle $X \to Y \to Z \to TX$, we get an exact sequence

$$\text{Hom}_\mathcal{C}(TX, W) \to \text{Hom}_\mathcal{C}(Z, W) \to \text{Hom}_\mathcal{C}(Y, W)$$

whose end terms are zero, because $\mathcal{T}$ and $\mathcal{T}'$ are orthogonal in $\mathcal{H}$ and $\mathcal{C}(\mathcal{H})$. Therefore $\text{Hom}_\mathcal{C}(Z, W) = 0$, in particular $\text{Hom}_\mathcal{H}(Z, W) = 0$. Hence $Z_+ = 0$ and $Z_0 \not\in \mathcal{T}'$. Since we can vary $\mathcal{T}' \subset \mathcal{H}_0$ we also see that $Z = Z_0 \in \mathcal{T}$. \qed

The Grothendieck group of a cluster tube. Let $\mathcal{T}$ be a tube and $\mathcal{C} = \mathcal{C}(\mathcal{T})$ its cluster category. As in [1.3] one shows $K_0(\mathcal{C}) = \text{Coker}(1 + \Phi)$. We call an admissible triangulated structure on $\mathcal{T}$ an induced triangulated structure if it is obtained from an embedding of $\mathcal{T}$ in $\text{coh} X$ as explained in the previous paragraph.

**Proposition 5.2.** Let $\mathcal{T}$ be a tube of rank $q$. 


(i) If $q$ is even then for any admissible triangulated structure $S$ on $\mathcal{C} = \mathcal{C}(T)$ we have $K_0(\mathcal{C}_S) = K_0(\mathcal{C}) \cong \mathbb{Z}$.

(ii) If $q$ is odd then for any induced triangulated structure $S$ on $\mathcal{C} = \mathcal{C}(T)$ we have $K_0(\mathcal{C}_S) = K_0(\mathcal{C}) \cong \mathbb{Z}_2$.

Proof. (i) If $S$ is a simple object in $T$ then $K_0(T)$ is the free group generated by the elements $s(j) = [\tau^j S]$, for $j \in \mathbb{Z}_q$. Therefore $\overline{s}(j) = -\overline{s}(j+1)$ in $K_0(\mathcal{C})$ and $K_0(\mathcal{C})$ is generated by $\overline{s} = \overline{s}(0)$ without relation. This shows $K_0(\mathcal{C}) = \mathbb{Z}[\overline{s}] \cong \mathbb{Z}$.

Finally, we can define $\lambda_S^{(q)} : \mathcal{C} \rightarrow \mathbb{Z}$ as in \eqref{4.1} which defines a linear form $\lambda : K_0(\mathcal{C}_S) \rightarrow \mathbb{Z}$ with $\lambda(S) = 2$. Thus $K_0(\mathcal{C}_S)$ has at least rank one and (i) follows.

(ii) Let $S \in T$ be a simple object. Then $K_0(\mathcal{C})$ is generated by $\overline{s}$, where $s = [S]$, and we have $2\overline{s} = 0$. We show that $\overline{s}$ induces a non-trivial element in $K_0(\mathcal{C}_S)$. For any object $X$ in $\mathcal{C}$ define

$$\lambda(X) = \sum_{j=0}^{q-1} |\text{Hom}_\mathcal{C}(L, \tau^j X)|_2$$

For an object $X$ in $T$ we have $\lambda(\pi X) = \deg_2(X)$. Indeed, since $\tau^q X \cong X$

$$\lambda(\pi X) = \sum_{j=0}^{q-1} |\text{Hom}_T(L, \tau^j X)|_2 \pm \sum_{j=0}^{q-1} |\text{Ext}^1_T(L, \tau^{j-1} X)|_2$$

$$= \sum_{j=0}^{q-1} \langle L, \tau^j X \rangle_2 = \deg_2(X).$$

In particular, $\lambda(\pi S) = 1 \neq 0$. Now, $\lambda$ is additive on triangles in $\mathcal{C}$, which is shown with a version of the cutting technique similar to the proof of Proposition \[4.1\] in order to show that $K \cong K'$ like in \[4.2\] we use that $\tau^q$ is the identity functor on $T$. □

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