PROX-DUAL REGULARIZATION ALGORITHM FOR GENERALIZED FRACTIONAL PROGRAMS

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Abstract. Prox-regularization algorithms for solving generalized fractional programs (GFP) were already considered by several authors. Since the standard dual of a generalized fractional program has not generally the form of GFP, these approaches can not apply directly to the dual problem. In this paper, we propose a primal-dual algorithm for solving convex generalized fractional programs. That is, we use a prox-regularization method to the dual problem that generates a sequence of auxiliary dual problems with unique solutions. So we can avoid the numerical difficulties that can occur if the fractional program does not have a unique solution. Our algorithm is based on Dinkelbach-type algorithms for generalized fractional programming, but uses a regularized parametric auxiliary problem. We establish then the convergence and rate of convergence of this new algorithm.

1. Introduction. In this paper, we will be interested to generalized fractional programs of the form

\[ (P) \quad \lambda_* = \inf_{x \in X} \max_{i \in I} \left\{ \frac{f_i(x)}{g_i(x)} \right\} \]

where \( I = \{1, \ldots, m\} \), \( m \geq 1 \), and \( X \) a non empty subset of \( \mathbb{R}^n \). The functions \( f_i \) and \( g_i \) are defined on an open subset \( K \) containing \( X \), continuous and satisfy 

\[ g_i(x) > 0 \text{ for all } x \in X \text{ and } i \in I. \]

Problems of such type arise in management applications of goal programming, in mathematical economics and numerical analysis, and in telecommunications, information theory and computer science. Applications of single and multi-ratio programming can be found in ([20], [16], [12]).

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There have been several algorithms for solving generalized fractional programs ([7], [8], [9], [6], [4], [2], [3], [13], [18], [19], [1], [15], [22]). The Dinkelbach-type algorithms DT1 and DT2 ([7], [8]) generalize Dinkelbach algorithm [10] to the case $m > 1$. In these algorithms, at each iteration a subproblem must be solved. They are useful to apply if the subproblem is easier to solve than the original problem $(P)$. The algorithms DT1 and DT2 are based on the same principle but DT2 is generally faster than DT1.

The algorithms DTR1 and DTR2 introduced in [13] use the proximal point algorithm (see [14], for ex.), to regularize the parametric auxiliary problem of DT1 and DT2 respectively. These algorithms are useful to surmount the numerical difficulties which can occur if a fractional program does not have a unique solution or the feasible set is unbounded. By using a prox-regularization method that generates a sequence of auxiliary problems with unique solutions, these difficulties can be avoided.

The dual algorithms introduced in [2] and [3] solve a dual problem for convex generalized fractional programs and the main feature of these algorithms is that at each iteration a single-ratio fractional programming problem is solved and the optimal objective value of this fractional programming problem provides a lower bound on the optimal objective value of the original generalized fractional program.

The purpose of this paper is to introduce a prox-regularized dual problem for convex generalized fractional programs and an algorithm to solve this problem. The proposed algorithm combines the dual approach and the prox-regularization method. It is based on two ideas: The first is the proximal point algorithm and the second is the dual approach used in [3] to solve generalized fractional programs. In our contribution, unlike the algorithm used in [3] which calculates exact solutions for the intermediate problems generated by these algorithms, we content our selves by giving approximate ones. That is we prox-regularize the auxiliary problems by applying the method used in Gugat’s paper [13] for explicit fractional generalized program, but in our algorithm we use it for a particular generalized fractional program involving an infinite number of ratios.

Our algorithm generates a sequence of dual values which converges monotonically to the optimal value of $(P)$, and a sequence of dual solutions which converges to a solution of the introduced dual problem of $(P)$. For a class of problems, including linear fractional programs, we establish that this algorithm converges at least linearly.

2. Preliminaries. Before introducing and analyzing the prox-dual regularization method, we will first briefly recall the dual procedure proposed in [3] by Barros et al.

To introduce this algorithm, Barros et al. assume that $(P)$ is a convex generalized fractional programming problem, where the feasible nonempty set $X$ is given by $X := \{x \in S \mid h(x) \leq 0\}$, with $S \subset K$ a compact convex set and $h : \mathbb{R}^n \rightarrow \mathbb{R}^r$ a vector-valued function such that for every $1 \leq j \leq r$, the $j^{th}$ component $h_j$ is convex. Moreover, the continuous functions $f_i, g_i : K \rightarrow \mathbb{R}, i \in I$, verify either of the following convexity/concavity assumptions:

(C1) For every $i \in I$, the function $f_i : K \rightarrow \mathbb{R}$ is convex on $S$ and nonnegative on $X$ and the function $g_i : K \rightarrow \mathbb{R}$ is positive and concave on $S$;

(C2) For every $i \in I$, the function $f_i : K \rightarrow \mathbb{R}$ is convex on $S$ and the function $g_i : K \rightarrow \mathbb{R}$ is positive and affine on $S$. 
The following Slater’s condition is also imposed.

(C3) Let $J$ denote the set of indices $1 \leq j \leq r$ such that the $j^{th}$ component $h_j$ of the vector-valued function $h : \mathbb{R}^n \rightarrow \mathbb{R}^r$ is affine. There exists some $x$ belonging to the relative interior $\text{ri} (S)$ of $S$ satisfying $h_j(x) < 0$, $j \notin J$ and $h_j(x) \leq 0$, $j \in J$.

Remark 1. Notice that if assumption C1 is satisfied and $\lambda \geq 0$ or that assumption C2 is fulfilled and $\lambda \in \mathbb{R}$, then the function $x \mapsto f_i(x) - \lambda g_i(x)$ is convex for all $i \in I$. In addition, if the vector-valued function $h$ is convex then the function $x \mapsto y^\top (f(x) - \lambda g(x)) + z^\top h(x)$ is convex for all $(y, z) \in \mathbb{R}_+^m \times \mathbb{R}_+^r$.

Next we describe this method. For this, let

\[ f(x) = (f_1(x), \ldots, f_m(x))^\top \quad \text{and} \quad g(x) = (g_1(x), \ldots, g_m(x))^\top. \]

The dual problem considered in [3] is

\[ (Q) \quad \sup_{y \in \Sigma, z \geq 0} d(y, z) \]

where the function $d : \Sigma \times \mathbb{R}_+^r \rightarrow \mathbb{R}$ is given by

\[ d(y, z) = \min_{x \in \mathbb{S}} \frac{y^\top f(x) + z^\top h(x)}{y^\top g(x)} \]

and

\[ \Sigma = \left\{ y \in \mathbb{R}^m \mid \sum_{i=1}^m y_i = 1, y_i \geq 0, i = 1, \ldots, m \right\}. \]

For all $(\lambda, y, z) \in \mathbb{R} \times \mathbb{R}_+^m \times \mathbb{R}_+^r$, define the functions

\[ G(\lambda, y, z) = \min_{x \in \mathbb{S}} \left\{ y^\top (f(x) - \lambda g(x)) + z^\top h(x) \right\} \]

and

\[ \hat{G}(\lambda) = \max_{y \in \Sigma, z \geq 0} \min_{x \in \mathbb{S}} \left\{ y^\top (f(x) - \lambda g(x)) + z^\top h(x) \right\}. \]

By means of Lagrange duality, the authors gave in [3] a dual of the parametric problem

\[ (P_\lambda) \quad \inf_{x \in \mathbb{X}} \max_{i \in I} \left\{ f_i(x) - \lambda g_i(x) \right\} \]

appearing in Dinkelbach-type methods. This dual problem is precisely

\[ (Q_\lambda) \quad \sup_{y \in \Sigma, z \geq 0} G(\lambda, y, z). \]

Under the Slater’s condition (C3), the problem $(Q)$ achieves its maximum on $\Sigma \times \mathbb{R}_+^r$ and one has $\vartheta(Q_\lambda) = \vartheta(P_\lambda)$ and $\vartheta(Q) = \vartheta(P)$, ([3], Proposition 1), where for a problem $(R)$, $\vartheta(R)$ designates the optimal value of $(R)$. This result allows us to solve the problem $(P)$ by solving its dual $(Q)$.

Bellow we summarize the dual algorithm.

**Algorithm 2.1.**

1. Take $y_0 \in \Sigma$ and $z_0 \geq 0$. Compute $\lambda_0 = d(y_0, z_0)$ and let $k = 0$.
2. Determine

\[ (y_{k+1}, z_{k+1}) \in \text{argmax}_{y \in \Sigma, z \geq 0} G(\lambda_k, y, z). \]

3. If $\hat{G}(\lambda_k) = 0$, then $(y_{k+1}, z_{k+1})$ is an optimal solution of $(Q)$ and $\lambda_k$ is the optimal value, and stop.
4. Compute $\lambda_{k+1} = d(y_{k+1}, z_{k+1})$, let $k = k + 1$ and go to 2.
Notice that a scaled version of Algorithm 2.1 is obtained in [3] by following the same strategy used to derive DT2.

3. The Regularization of the dual Problem. In this section we propose a prox-regularization of the dual problem presented by Barros et al. We consider the problem \((P)\) with the notations of section 2 and we assume that the assumption (C1) or (C2) is fulfilled and that (C3) holds.

Let \(\alpha > 0\). For \(\lambda \in \mathbb{R}, (s, t) \in \Sigma \times \mathbb{R}_+^r\), define the auxiliary problem

\[
(Q(\lambda, s, t)) \quad H(\lambda, s, t) = \sup_{y \in \Sigma, z \geq 0} \left\{ G(\lambda, y, z) - \alpha \| (y, z) - (s, t) \|^2 \right\}.
\]

Proposition 1. 1. The function \((y, z) \mapsto G(\lambda, y, z)\) is concave for all \(\lambda \in \mathbb{R}\) and the function \((\lambda, y, z) \mapsto G(\lambda, y, z)\) is upper semi-continuous over \(\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^r\).

2. For all \((y, z) \in \Sigma \times \mathbb{R}_+^r\), we have \(G(d(y, z), y, z) = 0\) and \(G(\lambda_*, y_*, z_*) = 0\) where \(\lambda_*\) is the optimal value of \((P)\) and \((y_*, z_*)\) is any optimal solution of \((Q)\).

3. The function \((\lambda, s, t) \mapsto H(\lambda, s, t)\) is lower semi-continuous over \(\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}_+^r\).

4. For \(\lambda \in \mathbb{R}\), and \((s, t) \in \Sigma \times \mathbb{R}_+^r\)

\[
H(\lambda, s, t) = \max_{y \in \Sigma, z \geq 0} \left\{ G(\lambda, y, z) - \alpha \| (y, z) - (s, t) \|^2 \right\}.
\]

That is to say that, the function \((y, z) \mapsto G(\lambda, y, z) - \alpha \| (y, z) - (s, t) \|^2\) achieves its maximum over \(\Sigma \times \mathbb{R}_+^r\).

5. Let \((s, t) \in \Sigma \times \mathbb{R}_+^r\). If \(H(\lambda, s, t) = 0\), then

\[
\lambda = \max_{y \in \Sigma, z \geq 0} \min_{x \in S} \left\{ \frac{y^T f(x) + z^T h(x) - \alpha \| (y, z) - (s, t) \|^2}{y^T g(x)} \right\}.
\]

Proof: 1. It is obvious that for all \(\lambda \in \mathbb{R}\) the function \((y, z) \mapsto G(\lambda, y, z)\) is concave, as a lower hull of concave functions ; and that the function \((\lambda, y, z) \mapsto G(\lambda, y, z)\) is upper semi-continuous as a lower hull of continuous functions.

2. Let \((y, z) \in \Sigma \times \mathbb{R}_+^r\). We have

\[
d(y, z) = \min_{x \in S} \left\{ \frac{y^T f(x) + z^T h(x)}{y^T g(x)} \right\} \leq \frac{y^T f(x) + z^T h(x)}{y^T g(x)} \quad \forall x \in S,
\]

implies

\[
y^T f(x) + z^T h(x) - d(y, z)y^T g(x) \geq 0 \quad \forall x \in S.
\]

Thus,

\[
G\left( d(y, z), y, z \right) \geq 0.
\]

On the other hand, since \(S\) is compact, there exists some \(x' \in S\) such that

\[
d(y, z) = \min_{x \in S} \left\{ \frac{y^T f(x) + z^T h(x)}{y^T g(x)} \right\} = \frac{y^T f(x') + z^T h(x')}{y^T g(x')},
\]
which implies that
\[ y^\top f(x') + z^\top h(x') - d(y, z)y^\top g(x') = 0. \]

Thus,
\[ G(d(y, z), y, z) \leq 0. \]

Therefore,
\[ G(d(y, z), y, z) = 0 \quad \forall (y, z) \in \Sigma \times \mathbb{R}^r_+. \]

Now let \((y_*, z_*)\) be an optimal solution of \((Q)\) and \(\lambda_*\) be the optimal value of \((P)\). By definition, \(d(y_*, z_*)\) is the optimal value of \((Q)\). From ([3], Proposition 1), \(\vartheta(P) = \vartheta(Q)\) that is \(\lambda_* = d(y_*, z_*)\). According to the previous result \(G(\lambda_*, y_*, z_*) = 0\).

3. It is clear that the function
\[ (\lambda, s, t) \mapsto \min_{x \in S} \left\{ y^\top (f(x) - \lambda g(x)) + z^\top h(x) \right\} - \alpha \| (y, z) - (s, t) \|^2 \]
is concave and finite. So, it is continuous. Therefore, \((\lambda, s, t) \mapsto H(\lambda, s, t)\) is lower semi-continuous.

4. Since the function
\[ (y, z) \mapsto G(\lambda, y, z) := \min_{x \in S} \left\{ y^\top (f(x) - \lambda g(x)) + z^\top h(x) \right\} \]
is concave and finite for all \(\lambda \in \mathbb{R}\), then the function
\[ (y, z) \mapsto G(\lambda, y, z) - \alpha \| (y, z) - (s, t) \|^2 \]
achieves its maximum on the closed set \(\Sigma \times \mathbb{R}^r_+\) for all \((s, t) \in \Sigma \times \mathbb{R}^r_+\).

5. For all \((\lambda, s, t) \in \mathbb{R} \times \Sigma \times \mathbb{R}^r_+\), \(H(\lambda, s, t) = 0\) implies that
\[ \max_{y \in \Sigma, z \geq 0} \left\{ G(\lambda, y, z) - \alpha \| (y, z) - (s, t) \|^2 \right\} = 0. \]

It follows that for all \((y, z) \in \Sigma \times \mathbb{R}^r_+\) we have
\[ G(\lambda, y, z) - \alpha \| (y, z) - (s, t) \|^2 \leq 0. \]

Then, from the definition of \(G(\lambda, y, z)\), for all \((y, z) \in \Sigma \times \mathbb{R}^r_+\), there exists \(x \in S\) such that
\[ y^\top (f(x) - \lambda g(x)) + z^\top h(x) - \alpha \| (y, z) - (s, t) \|^2 \leq 0. \]

Hence,
\[ \frac{y^\top f(x) + z^\top h(x) - \alpha \| (y, z) - (s, t) \|^2}{y^\top g(x)} \leq \lambda \]

implying that
\[ \min_{x \in S} \left\{ \frac{y^\top f(x) + z^\top h(x) - \alpha \| (y, z) - (s, t) \|^2}{y^\top g(x)} \right\} \leq \lambda. \]

Finally we have
\[ \max_{y \in \Sigma, z \geq 0} \min_{x \in S} \left\{ \frac{y^\top f(x) + z^\top h(x) - \alpha \| (y, z) - (s, t) \|^2}{y^\top g(x)} \right\} \leq \lambda. \]

On the other hand,
\[ \max_{y \in \Sigma, z \geq 0} \left\{ G(\lambda, y, z) - \alpha \| (y, z) - (s, t) \|^2 \right\} = 0 \]
implies that there exists some \((\hat{y}, \hat{z}) \in \Sigma \times \mathbb{R}^+_\mu\) such that
\[
G(\lambda, \hat{y}, \hat{z}) - \alpha \| (\hat{y}, \hat{z}) - (s, t) \|^2 = 0.
\]
Consequently,
\[
\hat{y}^\top (f(x) - \lambda g(x)) + \hat{z}^\top h(x) - \alpha \| (\hat{y}, \hat{z}) - (s, t) \|^2 \geq 0 \quad \forall x \in S,
\]
which implies that
\[
\frac{\hat{y}^\top f(x) + \hat{z}^\top h(x) - \alpha \| (\hat{y}, \hat{z}) - (s, t) \|^2}{\hat{y}^\top g(x)} \geq \lambda, \quad \forall x \in S.
\]
Thus,
\[
\min_{x \in S} \left\{ \frac{\hat{y}^\top f(x) + \hat{z}^\top h(x) - \alpha \| (\hat{y}, \hat{z}) - (s, t) \|^2}{\hat{y}^\top g(x)} \right\} \geq \lambda.
\]
Therefore,
\[
\max_{y \in \Sigma, z \geq 0} \min_{x \in S} \left\{ \frac{y^\top f(x) + z^\top h(x) - \alpha \| (y, z) - (s, t) \|^2}{y^\top g(x)} \right\} \geq \lambda. \tag{2}
\]
From (1) and (2) we deduce that
\[
\lambda = \max_{y \in \Sigma, z \geq 0} \min_{x \in S} \left\{ \frac{y^\top f(x) + z^\top h(x) - \alpha \| (y, z) - (s, t) \|^2}{y^\top g(x)} \right\}
\]
and the assertion holds.

\[\square\]

Let
\[
\delta = \min_{x \in S} \min_{i \in I} g_i(x) \quad \text{and} \quad \Delta = \max_{x \in S} \max_{i \in I} g_i(x).
\]

**Lemma 3.1.** Let \(y \in \Sigma, z \geq 0\) and \(\lambda \leq \mu\). Then
1. \(G(\lambda, y, z) \geq G(\mu, y, z) + (\mu - \lambda)\delta\),
2. \(G(\mu, y, z) \geq G(\lambda, y, z) - (\mu - \lambda)\Delta\).

**Proof.** 1. We have
\[
G(\lambda, y, z) = \min_{x \in S} \{ y^\top (f(x) - \lambda g(x)) + z^\top h(x) \}
\]
\[
= \min_{x \in S} \{ y^\top (f(x) - \lambda g(x) - \mu g(x) + \mu g(x)) + z^\top h(x) \}
\]
\[
= \min_{x \in S} \{ y^\top (f(x) - \mu g(x) + (\mu - \lambda)g(x)) + z^\top h(x) \}
\]
\[
\geq \min_{x \in S} \{ y^\top (f(x) - \mu g(x)) + z^\top h(x) \} + \min_{x \in S} \{ (\mu - \lambda)y^\top g(x) \}
\]
\[
= G(\mu, y, z) + \min_{x \in S} \{ (\mu - \lambda)y^\top g(x) \}
\]
\[
= G(\mu, y, z) + (\mu - \lambda) \min_{x \in S} y^\top g(x)
\]
\[
\geq G(\mu, y, z) + (\mu - \lambda)\delta.
\]
2. We have
\[
G(\mu, y, z) = \min_{x \in S} \{ y^\top (f(x) - \mu g(x)) + z^\top h(x) \} \\
= \min_{x \in S} \{ y^\top (f(x) - \mu g(x) - \lambda g(x) + \lambda g(x)) + z^\top h(x) \} \\
= \min_{x \in S} \{ y^\top (f(x) - \lambda g(x) - (\mu - \lambda) g(x)) + z^\top h(x) \} \\
\geq \min_{x \in S} \{ y^\top (f(x) - \lambda g(x)) + z^\top h(x) \} - (\mu - \lambda) \max_{x \in S} \{ y^\top g(x) \} \\
\geq G(\lambda, y, z) - (\mu - \lambda) \Delta. 
\]

\[\square\]

Now we introduce our algorithm.

**Algorithm 3.1.** Choose a sequence of nonnegative numbers \(\{\eta_k\}\) such that
\[
\sum_{k=0}^{\infty} \sqrt{\eta_k} < \infty.
\]

1. If assumption C1 holds, take \(d_0 = 0\) and a point \((y_0, z_0) \in \Sigma \times \mathbb{R}^r_+\). Else choose a point \((y_0, z_0) \in \Sigma \times \mathbb{R}^r_+\) and calculate
\[
d_0 := d(y_0, z_0) = \min_{x \in S} \left\{ \frac{y_0^\top f(x) + z_0^\top h(x)}{y_0^\top g(x)} \right\}.
\]

Let \(k = 0\).

2. Given \((y_k, z_k)\) and \(d_k\), calculate \((y_{k+1}, z_{k+1}) \in \Sigma \times \mathbb{R}^r_+\) such that
\[
G(d_k, y_{k+1}, z_{k+1}) - \alpha \| (y_{k+1}, z_{k+1}) - (y_k, z_k) \|^2 \geq \max\{0, H(d_k, y_k, z_k) - \eta_k\}.
\]

3. Calculate
\[
d_{k+1} := d(y_{k+1}, z_{k+1}) = \min_{x \in S} \left\{ \frac{y_{k+1}^\top f(x) + z_{k+1}^\top h(x)}{y_{k+1}^\top g(x)} \right\},
\]

set \(k = k + 1\) and go to 2.

The important step in our algorithm is to solve the minimax problem
\[
(Q(d_k, y_k, z_k)) \quad H(d_k, y_k, z_k) = \max_{y \in \Sigma, z \geq 0} \left\{ G(d_k, y, z) - \alpha \| (y, z) - (y_k, z_k) \|^2 \right\}
\]
where \(d_k = d(y_k, z_k)\).

Remark that the point \((y_{k+1}, z_{k+1})\) in Step 2 is well defined since \(H(d_k, y_k, z_k) \geq 0\). Indeed, from the definition of \(H(d_k, y_k, z_k)\) we have
\[
H(d_k, y_k, z_k) \geq G(d_k, y, z) - \alpha \| (y, z) - (y_k, z_k) \|^2 \quad \text{for all} \quad (y, z) \in \Sigma \times \mathbb{R}^r_+,
\]
and with \((y, z) = (y_k, z_k)\) we get \(H(d_k, y_k, z_k) \geq 0\) since \(G(d_k, y_k, z_k) = 0\) by Proposition 1, 2.

Before analyzing the convergence and the rate of convergence of the algorithm, we will give an equivalent simpler problem than \((Q(d_k, y_k, z_k))\).

We have
\[
G(d_k, y, z) = \min_{x \in S} \{ y^\top (f(x) - d_k g(x)) + z^\top h(x) \}.
\]

Remark that
\[
y^\top (f(x) - d_k g(x)) + z^\top h(x) - \alpha \| (y, z) - (y_k, z_k) \|^2 \\
= y^\top (f(x) - d_k g(x) + 2\alpha y_k) + z^\top (h(x) + 2\alpha z_k) - \alpha \| (y, z) \|^2 - \alpha \| (y_k, z_k) \|^2,
\]
and let
\[ L_k(x, y, z) = y^\top (f(x) - d_kg(x) + 2\alpha y_k) + z^\top (h(x) + 2\alpha z_k) - \alpha \|(y, z)\|^2. \]

Then, we can remark that
\[ H(d_k, y_k, z_k) = \max_{y \in \Sigma, z \geq 0} \min_{x \in S} \left\{ L_k(x, y, z) - \alpha \|(y_k, z_k)\|^2 \right\}. \quad (3) \]

Let
\[ c = \left( 1, \ldots, 1, 0, \ldots, 0 \right)^\top, \]
\[ \gamma = \left( \frac{y}{z} \right), \text{ with } (y, z) \in \Sigma \times \mathbb{R}_r^+ \]
and
\[ \phi_k(x) = \left( \begin{array}{c} f(x) - d_kg(x) + 2\alpha y_k \\ h(x) + 2\alpha z_k \end{array} \right). \]

With these notations, we have
\[ L_k(x, \gamma) = -\alpha \|\gamma\|^2 + \gamma^\top \phi_k(x), \quad (4) \]
and
\[ H(d_k, y_k, z_k) = \max_{\gamma \geq 0} \min_{x \in S} \left\{ L_k(x, \gamma) - \alpha \|(y_k, z_k)\|^2 \right\}. \quad (5) \]

**Remark 2.** In order to ensure the convexity of the functions \( f_i - d_kg_i \) in the case where the hypothesis C1 is fulfilled, we need to have \( d_k \geq 0 \). We will show later that the sequence \( \{d_k\} \) is monotonically increasing. Therefore, to have \( d_k \geq 0 \) for all \( k \in \mathbb{N} \) when the hypothesis C1 is fulfilled, it suffices to take \( d_0 = 0 \).

**Proposition 2.** Suppose that either assumption C1 is satisfied and \( d_k \geq 0 \), or C2 is fulfilled. Then we have
\[ H(d_k, y_k, z_k) = \inf_{x \in S, \mu \in \mathbb{R}_{++}, \nu \geq 0} \left\{ \alpha \|t\|^2 + \mu - \alpha \|(y_k, z_k)\|^2 \mid \phi_k(x) - \mu c - 2\alpha t \leq 0 \right\}. \]

**Proof.** Let \( x \in S \). Consider the minimization problem
\[
\begin{cases}
\inf \alpha \|\gamma\|^2 - \gamma^\top \phi_k(x) \\
c^\top \gamma = 1, \\
\gamma \geq 0.
\end{cases}
\]
Its Lagrangian is defined on \( \mathbb{R}^{m+r} \times \mathbb{R} \times \mathbb{R}^{m+r} \) by
\[ l_k(\gamma, \mu, \nu) = \alpha \|\gamma\|^2 - \gamma^\top \phi_k(x) + \mu(c^\top \gamma - 1) - \nu^\top \gamma, \]
and its dual problem is
\[
\begin{cases}
sup_{\mu \in \mathbb{R}} \inf_{\nu \geq 0} l_k(\gamma, \mu, \nu) \quad (6)
\end{cases}
\]
Let \( \gamma \) be a critical point of \( l_k(\cdot, \mu, \nu) \). Then
\[ \nabla_\gamma l_k(\gamma, \mu, \nu) = 0 \]
which implies that
\[ 2\alpha \gamma - \phi_k(x) + \mu c - \nu = 0 \]
and
\[ \gamma = \frac{1}{2\alpha} (\phi_k(x) - \mu c + \nu). \]
Let 
\[ t = \frac{1}{2\alpha} (\phi_k(x) - \mu c + \nu). \]

Then,
\[ \inf_{\gamma \in \mathbb{R}^{m+r}} l_k(\gamma, \mu, \nu) = \alpha \|t\|^2 - t^\top \phi_k(x) + \mu (c^\top t - 1) - \nu^\top t. \]

Thus,
\[ \sup_{\mu \in \mathbb{R}} \inf_{\gamma \in \mathbb{R}^{m+r}, \nu \geq 0} l_k(\gamma, \mu, \nu) \]
\[ = \sup_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}, \nu \geq 0} \left\{ \alpha \|t\|^2 - t^\top \phi_k(x) + \mu (c^\top t - 1) - \nu^\top t \mid t = \frac{1}{2\alpha} (\phi_k(x) - \mu c + \nu) \right\}. \quad (6) \]

Replacing \( \phi_k(x) \) by \( 2\alpha t + \mu c - \nu \) we get
\[ \alpha \|t\|^2 - t^\top \phi_k(x) + \mu (c^\top t - 1) - \nu^\top t = -\alpha \|t\|^2 - \mu. \]

So,
\[ \sup_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}, \nu \geq 0} \left\{ \alpha \|t\|^2 - t^\top \phi_k(x) + \mu (c^\top t - 1) - \nu^\top t \mid t = \frac{1}{2\alpha} (\phi_k(x) - \mu c + \nu) \right\}.
\]
\[ = \sup_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}, \nu \geq 0} \left\{ -\alpha \|t\|^2 - \mu \mid 2\alpha t + \mu c - \nu - \phi_k(x) = 0 \right\}. \quad (7) \]

Combining (6) and (7), we obtain
\[ \sup_{\mu \in \mathbb{R}} \inf_{\gamma \in \mathbb{R}^{m+r}, \nu \geq 0} l_k(\gamma, \mu, \nu) = \sup_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}, \nu \geq 0} \left\{ -\alpha \|t\|^2 - \mu \mid 2\alpha t + \mu c - \nu - \phi_k(x) = 0 \right\} 
\]
\[ = \sup_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}} \left\{ -\alpha \|t\|^2 - \mu \mid \phi_k(x) - \mu c - 2\alpha t \leq 0 \right\} 
\]
\[ = -\inf_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}} \left\{ \alpha \|t\|^2 + \mu \mid \phi_k(x) - \mu c - 2\alpha t \leq 0 \right\}. \quad (8) \]

Using ([17], Corollary 28.2.2 and Theorem 28.4), we get
\[ \sup_{\mu \in \mathbb{R}} \inf_{\gamma \in \mathbb{R}^{m+r}, \nu \geq 0} l_k(\gamma, \mu, \nu) = \min_{c, \gamma_1 = 1, \gamma \geq 0} \left\{ \alpha \|\gamma\|^2 - \gamma^\top \phi_k(x) \right\}. \quad (9) \]

Then using (8) and (9), we obtain
\[ \min_{c, \gamma_1 = 1, \gamma \geq 0} \left\{ \alpha \|\gamma\|^2 - \gamma^\top \phi_k(x) \right\} = -\inf_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}} \left\{ \alpha \|t\|^2 + \mu \mid \phi_k(x) - \mu c - 2\alpha t \leq 0 \right\}. \]

So, for all \( x \in S \)
\[ \max_{c, \gamma_1 = 1, \gamma \geq 0} \left\{ -\alpha \|\gamma\|^2 + \gamma^\top \phi_k(x) \right\} = \inf_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}} \left\{ \alpha \|t\|^2 + \mu \mid \phi_k(x) - \mu c - 2\alpha t \leq 0 \right\}. \]
Thus,
\[
\min_{x \in S} \max_{c, \gamma \geq 1} \left\{ -\alpha \|y\|^2 + \gamma^T \phi_k(x) \right\} = \min_{x \in S} \inf_{t \in \mathbb{R}^{m+r}, \mu \in \mathbb{R}} \left\{ \alpha \|t\|^2 + \mu |\phi_k(x) - \mu c - 2\alpha t| \right\} = \inf_{x \in S, \mu \in \mathbb{R}, t \in \mathbb{R}^{m+r}} \left\{ \alpha \|t\|^2 + \mu |\phi_k(x) - \mu c - 2\alpha t| \right\}.
\]

(10)

Remark that if hypothesis C1 is satisfied then \(d_k \geq 0\), by our assumption, and if hypothesis C2 is fulfilled then the functions \(g_i\) are affine for all \(i \in I\). In all cases, the functions \(x \mapsto f_i(x) - d_k g_i(x)\) are convex for all \(i \in I\).

Now, the function \(x \mapsto -\alpha \|y\|^2 + \gamma^T \phi_k(x)\) is convex for all \(\gamma \in \mathbb{R}^{m+r}\), the function \(\gamma \mapsto -\alpha \|\gamma\|^2 + \gamma^T \phi_k(x)\) is concave for all \(x \in S\), and the sets \(S\) and \(\{\gamma \in \mathbb{R}^{m+r} | c^T \gamma = 1, \gamma \geq 0\}\) are convex and \(S\) is compact. Then, Sion’s theorem ([21], Theorem 3.4 and Corollary 3.3) implies that
\[
\min_{x \in S} \max_{c, \gamma \geq 1} \left\{ -\alpha \|\gamma\|^2 + \gamma^T \phi_k(x) \right\} = \max \min_{c, \gamma \geq 1} \left\{ -\alpha \|\gamma\|^2 + \gamma^T \phi_k(x) \right\}.
\]

Referring to (4), (5) and the last equality we obtain
\[
H(d_k, y_k, z_k) = \min_{x \in S} \max_{c, \gamma \geq 1} \left\{ -\alpha \|\gamma\|^2 + \gamma^T \phi_k(x) - \alpha \|(y_k, z_k)\|^2 \right\}.
\]

Therefore, this equality with (10) give
\[
H(d_k, y_k, z_k) = \inf_{x \in S, \mu \in \mathbb{R}, t \in \mathbb{R}^{m+r}} \left\{ \alpha \|t\|^2 + \mu - \alpha \|(y_k, z_k)\|^2 | \phi_k(x) - \mu c - 2\alpha t \leq 0 \right\}.
\]

Proposition 3. With the hypotheses of Proposition 2, let \((\tilde{x}_k, \tilde{y}_k, \tilde{t}_k) \in S \times \mathbb{R} \times \mathbb{R}^{m+r}\) be an optimal solution of the problem
\[
\inf_{x \in S, \mu \in \mathbb{R}, t \in \mathbb{R}^{m+r}} \left\{ \alpha \|t\|^2 + \mu - \alpha \|(y_k, z_k)\|^2 | \phi_k(x) - \mu c - 2\alpha t \leq 0 \right\}.
\]

(11)

Let \((\tilde{y}_k, \tilde{z}_k) \in \mathbb{R}^m \times \mathbb{R}^r\) be such that \(\tilde{t}_k = (\tilde{y}_k, \tilde{z}_k)\). Then \((\tilde{y}_k, \tilde{z}_k)\) is the solution of \((Q (d_k, y_k, z_k))\).

Proof. Let \(L_k(x, \mu, t, \gamma)\) denote the Lagrangian associated to (11). Then
\[
L_k(x, \mu, t, \gamma) = \alpha \|t\|^2 + \mu + \gamma^T (\phi_k(x) - \mu c - 2\alpha t).
\]

Using ([17], Theorem 28.2 and Theorem 28.4) we get
\[
\sup_{\gamma \geq 0} \inf_{x \in S, \mu \in \mathbb{R}, t \in \mathbb{R}^{m+r}} L_k(x, \mu, t, \gamma) = \inf_{x \in S, \mu \in \mathbb{R}, t \in \mathbb{R}^{m+r}} \left\{ \alpha \|t\|^2 + \mu |\phi_k(x) - \mu c - 2\alpha t| \right\} = H(d_k, y_k, z_k) + \alpha \|(y_k, z_k)\|^2,
\]

where the last equality follows from Proposition 2.

From (3), we have
\[
H(d_k, y_k, z_k) = \sup_{y \in \Sigma, z \geq 0} \inf_{x \in S} \left\{ L_k(x, y, z) - \alpha \|(y_k, z_k)\|^2 \right\}
\]
and then

\[
\sup_{\gamma \geq 0} \inf_{x \in S, \mu \in \mathbb{R}} L_k(x, \mu, t, \gamma) = \sup_{y \in \Sigma, z \geq 0} \inf_{x \in S} L_k(x, y, z) = \max_{c} \inf_{\gamma \geq 0} L_k(x, \gamma) = \max_{c} \inf_{\gamma \geq 0} \left\{ -\alpha \|y\|^2 + \gamma^T \phi_k(x) \right\}.
\]

It follows that if \( \tilde{\gamma}_k \) is an optimal solution of the problem

\[
\sup_{\gamma \geq 0} \left\{ \inf_{x \in S, \mu \in \mathbb{R}} L_k(x, \mu, t, \gamma) \right\}
\]

and if \((\tilde{y}_k, \tilde{z}_k) \in \mathbb{R}^m \times \mathbb{R}^r\) is such that \( \tilde{\gamma}_k = \left( \begin{array}{c} \tilde{y}_k \\ \tilde{z}_k \end{array} \right) \), then \((\tilde{y}_k, \tilde{z}_k)\) is the optimal solution of \((Q(d_k, y_k, z_k))\).

It suffices now to show that if \( \tilde{\gamma}_k \) is an optimal solution of (12) then \( \tilde{\gamma}_k = \tilde{t}_k \).

For this, let \((\tilde{x}_k, \tilde{\mu}_k, \tilde{t}_k)\) be an optimal solution of (11) and let \( \tilde{\gamma}_k \) be an optimal solution of (12). Then

\[
\inf_{x \in S, \mu \in \mathbb{R}} L_k(x, \mu, \tilde{t}_k) = L_k(\tilde{x}_k, \tilde{\mu}_k, \tilde{t}_k, \tilde{\gamma}_k).
\]

Since the function

\[
t \mapsto L_k(x, \mu, t, \tilde{\gamma}_k) := \alpha \|t\|^2 - 2\alpha \tilde{\gamma}_k^T t + \mu(1 - \tilde{\gamma}_k^T c) + \tilde{\gamma}_k^T \phi_k(x)
\]

achieves its minimum at the unique point \( t = \tilde{\gamma}_k \) for all \((x, \mu) \in S \times \mathbb{R}\), then it follows that

\[
\inf_{x \in S, \mu \in \mathbb{R}} L_k(x, \mu, t, \tilde{\gamma}_k) = \inf_{x \in S, \mu \in \mathbb{R}} \left\{ \inf_{t \in \mathbb{R}^{m+r}} L_k(x, \mu, t, \tilde{\gamma}_k) \right\} = \inf_{x \in S, \mu \in \mathbb{R}} L_k(x, \mu, \tilde{t}_k, \tilde{\gamma}_k) = L_k(\tilde{x}_k, \tilde{\mu}_k, \tilde{t}_k, \tilde{\gamma}_k)
\]

and \( \tilde{t}_k = \tilde{\gamma}_k \), which gives the desired result. \( \square \)

4. Convergence and rate of convergence of algorithm 3.1. To prove convergence and give rate of convergence of our algorithm, we begin showing some intermediate results.

**Lemma 4.1.** The sequence \( \{d(y_k, z_k)\} \) is increasing, bounded from above by \( \lambda_* \).

**Proof.** From the definition of \((y_{k+1}, z_{k+1})\) in Algorithm 3.1, we have

\[
G(d(y_k, z_k), y_{k+1}, z_{k+1}) - \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2 \geq 0.
\]

So,

\[
G(d(y_k, z_k), y_{k+1}, z_{k+1}) \geq \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2.
\]
On the other hand
\[ G(d(y_k, z_k, y_{k+1}, z_{k+1}) = \min_{x \in S} \left\{ y_{k+1}^T (f(x) - d(y_k, z_k)g(x)) + z_{k+1}^T h(x) \right\} \]
\[ \geq \alpha \| (y_{k+1}, z_{k+1}) - (y_k, z_k) \|^2 \]
implies that
\[ \min_{x \in S} \left\{ y_{k+1}^T (f(x) - d(y_k, z_k)g(x)) + z_{k+1}^T h(x) \right\} \geq 0. \]
Thus,
\[ y_{k+1}^T (f(x) - d(y_k, z_k)g(x)) + z_{k+1}^T h(x) \geq 0 \quad \forall x \in S. \]
It follows that
\[ y_{k+1}^T f(x) + z_{k+1}^T h(x) \geq d(y_k, z_k) y_{k+1}^T g(x) \quad \forall x \in S, \]
and then
\[ \frac{y_{k+1}^T f(x) + z_{k+1}^T h(x)}{y_{k+1}^T g(x)} \geq d(y_k, z_k) \quad \forall x \in S. \]
This implies that
\[ \min_{x \in S} \left\{ y_{k+1}^T f(x) + z_{k+1}^T h(x) \right\} \geq d(y_k, z_k). \]
Finally,
\[ d(y_{k+1}, z_{k+1}) \geq d(y_k, z_k), \]
and the sequence \( \{d(y_k, z_k)\} \) is increasing.

Now, let \( x \in X \). Then, there exists \( r \in I \) such that
\[ \max_{i \in I} \left\{ \frac{f_i(x)}{g_i(x)} \right\} = \frac{f_r(x)}{g_r(x)}, \]
which implies that
\[ \frac{f_i(x)}{g_i(x)} \leq \frac{f_r(x)}{g_r(x)} \quad \forall i \in I. \]
Since \( g_i > 0 \) on \( X \), we have
\[ f_i(x) g_r(x) \leq f_r(x) g_i(x) \quad \forall i \in I. \]
Thus, for all \( y \in \Sigma \),
\[ \sum_{i=1}^{m} y_i f_i(x) g_r(x) \leq \sum_{i=1}^{m} y_i f_r(x) g_i(x), \]
implying that
\[ g_r(x) y^T f(x) \leq f_r(x) y^T g(x). \]
It follows that
\[ \frac{y^T f(x)}{y^T g(x)} \leq \frac{f_r(x)}{g_r(x)}. \]
This implies that
\[ \min_{x \in X} \left\{ \frac{y^T f(x)}{y^T g(x)} \right\} \leq \frac{f_r(x)}{g_r(x)}. \]
On the other hand, we have \( X \subset S \), then for all \( z \in \mathbb{R}_+^r \),
\[ \min_{x \in S} \left\{ \frac{y^T f(x) + z^T h(x)}{y^T g(x)} \right\} \leq \min_{x \in X} \left\{ \frac{y^T f(x) + z^T h(x)}{y^T g(x)} \right\} \leq \min_{x \in X} \left\{ \frac{y^T f(x)}{y^T g(x)} \right\}. \]
The last inequality follows from the fact that \( h(x) \leq 0 \) on \( X \) and \( z \geq 0 \). Thus,

\[
\min_{x \in S} \left\{ \frac{y^\top f(x) + z^\top h(x)}{y^\top g(x)} \right\} \leq \frac{f_r(x)}{g_r(x)}.
\]

Therefore,

\[
\min_{x \in S} \left\{ \frac{y^\top f(x) + z^\top h(x)}{y^\top g(x)} \right\} \leq \min_{x \in X} \max_{i \in I} \left\{ \frac{f_i(x)}{g_i(x)} \right\} = \lambda_*.
\]

Hence,

\[
d(y, z) \leq \lambda_*,
\]

and in particular

\[
d(y_k, z_k) \leq \lambda_* \quad \forall k \in \mathbb{N}.
\]

Which means that \( d(y_k, z_k) \) is bounded from above by \( \lambda_* \).

\[\text{Proposition 4.} \]

Let

\[
\hat{d} = \min_{x \in S} \left\{ \frac{a^\top f(x) + b^\top h(x)}{a^\top g(x)} \right\}
\]

with \((a, b) \in \Sigma \times \mathbb{R}_+^r\). If \( H(\hat{d}, a, b) = 0 \) then \( \hat{d} = \lambda_* \) and \((a, b) \) solves \((Q)\).

\[\text{Proof.} \]

We have

\[
\hat{d} = d(a, b) \leq \lambda_*.
\]

Let

\[
d + \nu = \lambda_*
\]

with \( \nu \geq 0 \), and let \((y_*, z_*)\) be an optimal solution of \((Q)\). For \( \lambda \in ]0, 1[ \) consider

\[
(y(\lambda), z(\lambda)) = \lambda(y_*, z_*) + (1 - \lambda)(a, b).
\]

Then \((y(\lambda), z(\lambda)) \in \Sigma \times \mathbb{R}_+^r\). Since

\[
\hat{d} = \min_{x \in S} \left\{ \frac{a^\top f(x) + b^\top h(x)}{a^\top g(x)} \right\}
\]

then by Proposition 1, 2.,

\[
G(\hat{d}, a, b) = 0.
\]

We also have

\[
H(\hat{d}, a, b) = 0.
\]

Which implies that

\[
0 \geq G(\hat{d}, y(\lambda), z(\lambda)) - \alpha \| (y(\lambda), z(\lambda)) - (a, b) \|^2.
\]

The concavity of the function \( G(\hat{d}, \ldots) \) implies

\[
0 \geq \lambda G(\hat{d}, y_*, z_*) + (1 - \lambda)G(\hat{d}, a, b) - \alpha \| (y(\lambda), z(\lambda)) - (a, b) \|^2.
\]

The last inequality and \((13)\) imply

\[
0 \geq \lambda G(\hat{d}, y_*, z_*) - \alpha \| (y(\lambda), z(\lambda)) - (a, b) \|^2,
\]

which reduces to

\[
0 \geq \lambda G(\hat{d}, y_*, z_*) - \alpha \lambda^2 \| (y_*, z_*) - (a, b) \|^2.
\]

Applying Lemma 3.1 with \( \lambda = \hat{d} \) and \( \mu = \lambda_* \) to the last inequality we get

\[
0 \geq \lambda \left[ G(\lambda_*, y_*, z_*) + (\lambda_* - \hat{d}) \delta \right] - \alpha \lambda^2 \| (y_*, z_*) - (a, b) \|^2.
\]
So, we have
\[
0 \geq \delta \lambda (\bar{\lambda} - \bar{d}) - \alpha \lambda^2 \|(y_*, z_*) - (a, b)\|^2
= \delta \lambda v - \alpha \lambda^2 \|(y_*, z_*) - (a, b)\|^2.
\]
Thus,
\[
\alpha \lambda \|(y_*, z_*) - (a, b)\|^2 \geq \delta v, \quad \forall \lambda \in [0, 1].
\]
By passing to limit as \(\lambda\) tends to 0, we get \(v = 0\) and \(\bar{d} = \bar{\lambda}\). And since
\[
\hat{d} = \min_{x \in S} \left\{ \frac{a^\top f(x) + b^\top h(x)}{a^\top g(x)} \right\}
\]
then \((a, b)\) is an optimal solution of \((Q)\).

**Lemma 4.2.** We have
\[
\lim_{k \to \infty} \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\| = 0.
\]

**Proof.** We use the notation \(d_k = d(y_k, z_k)\). We have
\[
G(d_k, y_{k+1}, z_{k+1}) - \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2 \geq H(d_k, y_k, z_k) - \eta_k.
\]
So,
\[
G(d_k, y_{k+1}, z_{k+1}) - \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2 \geq G(d_k, y_k, z_k) - \eta_k.
\]
Using Proposition 1, 2., we deduce
\[
G(d_k, y_{k+1}, z_{k+1}) - \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2 \geq -\eta_k.
\]
Thus,
\[
G(d_k, y_{k+1}, z_{k+1}) + \eta_k \geq \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2.
\]
Applying Lemma 3.1, with \(\lambda = d_k\) and \(\mu = d_{k+1}\), we obtain
\[
G(d_{k+1}, y_{k+1}, z_{k+1}) + (d_{k+1} - d_k) \Delta + \eta_k \geq \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2.
\]
Since \(G(d_{k+1}, y_{k+1}, z_{k+1}) = 0\) by Proposition 1, 2., then
\[
(d_{k+1} - d_k) \Delta + \eta_k \geq \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2.
\]
From Lemma 4.1 the sequence \(\{d_k\}\) is convergent. Therefore,
\[
\lim_{k \to \infty} \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\| = 0.
\]

**Lemma 4.3.** Let \((\bar{y}_{k+1}, \bar{z}_{k+1})\) be the optimal solution of \((Q(d_k, y_k, z_k))\). Then, for all \((y, z) \in \Sigma \times \mathbb{R}_+^r\) we have
\[
G(d_k, \bar{y}_{k+1}, \bar{z}_{k+1}) - G(d_k, y, z)
\geq
-2\alpha \|(\bar{y}_{k+1}, \bar{z}_{k+1}) - (y_k, z_k), (y, z) - (\bar{y}_{k+1}, \bar{z}_{k+1})\|
= -\alpha \|(y, z) - (y_k, z_k)\|^2 + \alpha \|(y, z) - (\bar{y}_{k+1}, \bar{z}_{k+1})\|^2
+ \alpha \|(y_k, z_k) - (\bar{y}_{k+1}, \bar{z}_{k+1})\|^2.
\]

**Proof.** See for example ([11], Proposition 2.2, p. 37).
Lemma 4.4. Let \( \{\mu_k\}, \{\beta_k\} \) be sequences of nonnegative reals such that
\[
\sum_{j=1}^{\infty} \mu_j < \infty, \quad \sum_{j=1}^{\infty} \beta_j < \infty,
\]
and let \( \{u_k\} \) be a sequence of reals such that
\[
u_{k+1} \leq (1 + \mu_k)u_k + \beta_k.
\]
Then, the sequence \( \{u_k\} \) converges to some \( u \in \mathbb{R} \cup \{-\infty\} \).

\begin{proof}
Recall that we use the notations \( d_k = d(y_k, z_k) \) for all \( k \in \mathbb{N} \), and
\[
\delta = \min_{x \in S} \min_{i \in I} g_i(x) \text{ and } \Delta = \max_{x \in S} \max_{i \in I} g_i(x).
\]
Using Lemma 3.1, with \((y_*, z_*)\) any optimal solution of \((Q)\), we obtain
\[
G(d_k+1, y_*, z_*) \geq G(\lambda_*, y_*, z_*) + (\lambda_* - d_k+1)\delta.
\]
Since \( G(\lambda_*, y_*, z_*) = 0 \) by Proposition 1, 2., then
\[
G(d_k+1, y_*, z_*) \geq (\lambda_* - d_k+1)\delta.
\]
Using again Lemma 3.1 and Proposition 1, 2., we get for all \( k \geq 1 \),
\[
0 = G(d_k, y_k, z_k) \geq G(d_{k-1}, y_k, z_k) - (d_k - d_{k-1})\Delta,
\]
which implies that
\[
0 \geq G(d_{k-1}, y_k, z_k) - (d_k - d_{k-1})\Delta.
\]
Let \((\bar{y}_{k+1}, \bar{z}_{k+1})\) be the optimal solution of \((Q(d_k, y_k, z_k))\). Then the definition of \((y_{k+1}, z_{k+1})\) implies that
\[
G(d_k, y_{k+1}, z_{k+1}) - \alpha \| (y_{k+1}, z_{k+1}) - (y_k, z_k) \|^2
\geq \ G(d_k, \bar{y}_{k+1}, \bar{z}_{k+1}) - \alpha \| (\bar{y}_{k+1}, \bar{z}_{k+1}) - (y_k, z_k) \|^2 - \eta_k.
\]
With \((y, z) = (y_*, z_*)\), Lemma 4.3 and the last inequality imply that
\[
G(d_k, y_{k+1}, z_{k+1}) - \alpha \| (y_{k+1}, z_{k+1}) - (y_k, z_k) \|^2
\geq \ G(d_k, \bar{y}_{k+1}, \bar{z}_{k+1}) - \alpha \| (\bar{y}_{k+1}, \bar{z}_{k+1}) - (y_k, z_k) \|^2 - \eta_k.
\]
Rearranging terms, we obtain
\[
\| (y_*, z_*) - (y_k, z_k) \|^2 - \| (y_*, z_*) - (\bar{y}_{k+1}, \bar{z}_{k+1}) \|^2
\geq \ \frac{1}{\alpha} \left[ G(d_k, y_*, z_*) - G(d_k, y_{k+1}, z_{k+1}) - \eta_k \right].
\]
Since \( G(d_k, y_*, z_*) \geq 0 \), then by (14) we obtain
\[
\| (y_*, z_*) - (y_k, z_k) \|^2 - \| (y_*, z_*) - (\bar{y}_{k+1}, \bar{z}_{k+1}) \|^2
\geq \ \frac{1}{\alpha} \left[ - G(d_k, y_{k+1}, z_{k+1}) + G(d_{k-1}, y_k, z_k) + (d_{k-1} - d_k)\Delta - \eta_k \right].
\]
Let
\[
\sigma_k = \frac{1}{\alpha} \left[ - G(d_k, y_{k+1}, z_{k+1}) + G(d_{k-1}, y_k, z_k) + (d_{k-1} - d_k)\Delta - \eta_k \right].
\]
Notice that $\sigma_k < 0$ and that
\[
\|(y_*, z_*) - (\bar{y}_{k+1}, \bar{z}_{k+1})\|^2 - \|(y_*, z_*) - (y_k, z_k)\|^2 \leq -\sigma_k. \tag{16}
\]
By summing in (15) over $k = 1$ to $k = N$, we get
\[
\alpha \sum_{k=1}^{N} \sigma_k = \sum_{k=1}^{N} \left[ G(d_{k-1}, y_k, z_k) - G(d_k, y_{k+1}, z_{k+1}) + (d_{k-1} - d_k)\Delta - \eta_k \right]
\]
\[
= G(d_0, y_1, z_1) - G(d_N, y_{N+1}, z_{N+1}) + (d_0 - d_N)\Delta - \sum_{k=1}^{N} \eta_k,
\]
Lemma 3.1 with $\mu = d_{N+1}$, $\lambda = d_N$, $y = y_{N+1}$ and $z = z_{N+1}$ implies that
\[
\alpha \sum_{k=1}^{N} \sigma_k \geq G(d_0, y_1, z_1) - G(d_{N+1}, y_{N+1}, z_{N+1}) - (d_{N+1} - d_N)\Delta
\]
\[
+ (d_0 - d_N)\Delta - \sum_{k=1}^{N} \eta_k
\]
\[
= G(d_0, y_1, z_1) + (d_0 - d_{N+1})\Delta - \sum_{k=1}^{N} \eta_k,
\]
where the equality holds since $G(d_{N+1}, y_{N+1}, z_{N+1}) = 0$ by Proposition 1, 2. Since $d_{N+1} \leq \lambda$, then
\[
\alpha \sum_{k=1}^{N} \sigma_k \geq G(d_0, y_1, z_1) + (d_0 - \lambda)\Delta - \sum_{k=1}^{N} \eta_k
\]
implying that
\[
\sum_{k=1}^{N} \eta_k - G(d_0, y_1, z_1) - (d_0 - \lambda)\Delta \geq -\alpha \sum_{k=1}^{N} \sigma_k > 0
\]
and that
\[
0 < -\sum_{k=1}^{\infty} \sigma_k < \infty.
\]
With $(y, z) = (y_{k+1}, z_{k+1})$, Lemma 4.3 implies
\[
G(d_k, \bar{y}_{k+1}, \bar{z}_{k+1}) - G(d_{k+1}, y_{k+1}, z_{k+1})
\geq -\alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2 + \alpha \|(y_{k+1}, z_{k+1}) - (\bar{y}_{k+1}, \bar{z}_{k+1})\|^2
\]
\[
+ \alpha \|(\bar{y}_{k+1}, \bar{z}_{k+1}) - (y_k, z_k)\|^2. \tag{17}
\]
The definition of $(y_{k+1}, z_{k+1})$ gives
\[
\eta_k \geq G(d_k, \bar{y}_{k+1}, \bar{z}_{k+1}) - G(d_{k+1}, y_{k+1}, z_{k+1}) + \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2
\]
\[
- \alpha \|(\bar{y}_{k+1}, \bar{z}_{k+1}) - (y_k, z_k)\|^2.
\]
The last inequality and (17) imply
\[
\|(y_{k+1}, z_{k+1}) - (\bar{y}_{k+1}, \bar{z}_{k+1})\|^2 \leq \frac{\eta_k}{\alpha}. \tag{18}
\]
The Schwartz inequality implies
\[
\|(y_*, z_*) - (y_k, z_k)\|^2 \leq \|(y_k, \bar{z}_k) - (y_*, z_*)\|^2 + \|(\bar{y}_k, \bar{z}_k) - (y_k, z_k)\|^2
\]
\[
+ 2 \|(y_k, \bar{z}_k) - (y_*, z_*)\| \|(\bar{y}_k, \bar{z}_k) - (y_k, z_k)\|.  
\]
So, combining the last inequality and (18) we get
\[ \| (y_*, z_*) - (y_k, z_k) \|^2 \leq \| (\bar{y}_k, \bar{z}_k) - (y_*, z_*) \|^2 + \| (\bar{y}_k, \bar{z}_k) - (y_*, z_*) \|^2 \sqrt{\frac{\eta_k - 1}{\alpha} + \frac{\eta_k - 1}{\alpha}}. \]
The inequality \( \| (\bar{y}_k, \bar{z}_k) - (y_*, z_*) \| \leq 1 + \| (\bar{y}_k, \bar{z}_k) - (y_*, z_*) \|^2 \) and the last inequality imply that
\[ \| (y_*, z_*) - (y_k, z_k) \|^2 \leq \| (\bar{y}_k, \bar{z}_k) - (y_*, z_*) \|^2 \left( 1 + 2 \sqrt{\frac{\eta_k - 1}{\alpha}} \right) + 2 \sqrt{\frac{\eta_k - 1}{\alpha}} + \eta_k - 1. \]
Inequality (16) and the last inequality imply
\[ \| (y_*, z_*) - (\bar{y}_{k+1}, \bar{z}_{k+1}) \|^2 \leq \| (\bar{y}_k, \bar{z}_k) - (y_*, z_*) \|^2 \left( 1 + 2 \sqrt{\frac{\eta_k - 1}{\alpha}} \right) + 2 \sqrt{\frac{\eta_k - 1}{\alpha}} + \eta_k - 1 - \sigma_k. \] (19)

Let
\[ \mu_k = 2 \sqrt{\frac{\eta_k - 1}{\alpha}} \text{ and } \beta_k = 2 \sqrt{\frac{\eta_k - 1}{\alpha}} + \eta_k - 1 - \sigma_k. \]
Since \( \sum_{k=1}^{\infty} \eta_k < \infty \) and \( \sum_{k=1}^{\infty} -\sigma_k < \infty \) then
\[ \sum_{k=1}^{\infty} \mu_k < \infty \text{ and } \sum_{k=1}^{\infty} \beta_k < \infty. \]
With the notations above, (19) becomes
\[ \| (\bar{y}_{k+1}, \bar{z}_{k+1}) - (y_*, z_*) \|^2 \leq (1 + \mu_k) \| (\bar{y}_k, \bar{z}_k) - (y_*, z_*) \|^2 + \beta_k. \] (20)
Let
\[ u_k = \| (\bar{y}_k, \bar{z}_k) - (y_*, z_*) \|^2. \]
Inequality (20) and Lemma 4.4 imply that \( \{u_k\} \) is convergent, and inequality (18) implies that the sequence \( \{\| (y_k, z_k) - (y_*, z_*) \|^2 \} \) is also convergent, which yields the assertion.

**Theorem 4.6.** The sequence \( \{d(y_k, z_k)\} \) converges to \( \lambda_* \) and the sequence \( \{(y_k, z_k)\} \) converges to a solution of \( (Q) \).

**Proof.** Since the set \( S \) is compact and the function
\[ (x, y, z) \mapsto \frac{y^T f(x) + z^T h(x)}{y^T g(x)} \]
is continuous on \( S \times \Sigma \times \mathbb{R}_+^r \) then the function \( d(\ldots) \) defined by
\[ d(y, z) = \min_{x \in S} \left\{ \frac{y^T f(x) + z^T h(x)}{y^T g(x)} \right\} \]
is continuous on \( \Sigma \times \mathbb{R}_+^r \).

On the other hand, by Lemma 4.5, the sequence \( \{(y_k, z_k)\} \) is bounded, and so we can choose a subsequence \( \{(y_k, z_k)\}_{k \in K} \) converging to some \( (\tilde{y}, \tilde{z}) \in \Sigma \times \mathbb{R}_+^r \). Since the sequence \( \{d(y_k, z_k)\} \) converges by Lemma 4.1, it follows that
\[ \lim_{k \to \infty} d(y_k, z_k) = d(\tilde{y}, \tilde{z}). \]
Let
\[ \tilde{d} = d(\tilde{y}, \tilde{z}) := \min_{x \in S} \left\{ \frac{\tilde{y}^T f(x) + \tilde{z}^T h(x)}{\tilde{y}^T g(x)} \right\}. \]
According to Proposition 1, 3., the function $H$ is lower semi-continuous on the set $]-\infty, \lambda_\ast| \times \Sigma \times \mathbb{R}^r_+$. Thus, with the notation $d_k = d(y_k, z_k)$ we have
\begin{equation}
H(\bar{d}, \bar{y}, \bar{z}) \leq \liminf_{k \to \infty} H(d_k, y_k, z_k).
\end{equation}
(21)

Lemma 4.2 implies that we also have
\[ (\bar{y}, \bar{z}) = \lim_{k \to \infty} (y_{k+1}, z_{k+1}). \]
By Proposition 1, 1., the function $G$ is upper semi-continuous on $]-\infty, \lambda_\ast| \times \Sigma \times \mathbb{R}^r_+$. Thus,
\begin{equation}
G(\bar{d}, \bar{y}, \bar{z}) \geq \limsup_{k \to \infty} G(d_k, y_{k+1}, z_{k+1}),
\end{equation}
which implies that
\begin{align*}
G(\bar{d}, \bar{y}, \bar{z}) &\geq \limsup_{k \to \infty} \left( G(d_k, y_{k+1}, z_{k+1}) - \alpha \|(y_k, z_k) - (y_{k+1}, z_{k+1})\|^2 + \eta_k \right) \\
&\geq \limsup_{k \to \infty} H(d_k, y_k, z_k) \\
&\geq \liminf_{k \to \infty} H(d_k, y_k, z_k).
\end{align*}
Then using (21), we get
\begin{equation*}
G(\bar{d}, \bar{y}, \bar{z}) \geq H(\bar{d}, \bar{y}, \bar{z}).
\end{equation*}
Hence
\begin{equation*}
G(\bar{d}, \bar{y}, \bar{z}) = H(\bar{d}, \bar{y}, \bar{z}).
\end{equation*}
Since
\[ \bar{d} = \min_{x \in S} \left\{ \bar{y}^\top f(x) + \bar{z}^\top h(x) \right\} \]
we obtain by Proposition 1, 2.,
\begin{equation*}
G(\bar{d}, \bar{y}, \bar{z}) = 0.
\end{equation*}
So,
\[ H(\bar{d}, \bar{y}, \bar{z}) = 0. \]
It follows from Proposition 4 that $\bar{d} = \lambda_\ast$ and that $(\bar{y}, \bar{z})$ solves $(Q)$.
Now, let
\[ u_k = \|(\bar{y}_k, \bar{z}_k) - (\bar{y}, \bar{z})\|^2. \]
Since
\[ (\bar{y}, \bar{z}) = \lim_{k \to \infty} (y_k, z_k), \]
then, inequality (18) implies
\[ \lim_{k \to \infty} u_k = 0 \]
Since $(\bar{y}, \bar{z})$ is a solution of $(Q)$, then as in the proof of Lemma 4.5, we can see that the inequality (20) is satisfied with $(\bar{y}, \bar{z})$ in the place of $(y_\ast, z_\ast)$. So, Lemma 4.4 with $\{u_k\}$ as defined above, implies that the sequence $\{(\bar{y}_k, \bar{z}_k)\}$ converges to $(\bar{y}, \bar{z})$. Finally, the inequality (18) implies that $\{(y_k, z_k)\}$ converges to $(\bar{y}, \bar{z})$, a solution of $(Q)$. □
In the following we will analyze the rate of convergence of Algorithm 3.1. For this, let
\[ \Gamma = \Sigma \times \mathbb{R}^r_+ \quad \text{and} \quad \Gamma^* = \arg\max_{(y,z) \in \Sigma \times \mathbb{R}^r_+} G(\lambda_*, y, z). \]

Next, we will denote by \((H)\) the following assumption:

\((H)\) \quad \exists \rho > 0, \exists K > 0, \quad \text{such that} \\
\quad -G(\lambda_*, y, z) \geq K \dist((y, z), \Gamma^*)^2 \quad \text{for all} \quad (y, z) \in B(\Gamma^*, \rho) \cap \Gamma

where
\[ B(\Gamma^*, \rho) = \bigcup_{(\bar{y}, \bar{z}) \in \Gamma^*} B((\bar{y}, \bar{z}), \rho), \]

\[ B((\bar{y}, \bar{z}), \rho) = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^r \mid \|(y, z) - (\bar{y}, \bar{z})\| \leq \rho\} \]

and
\[ \dist((y, z), \Gamma^*) = \inf_{(\bar{y}, \bar{z}) \in \Gamma^*} \|(\bar{y}, \bar{z}) - (y, z)\|. \]

**Proposition 5.** Assume that \( f - \lambda_* g \) and \( h \) are linear maps, and \( S \) is a polyhedral set. Then assumption \((H)\) is satisfied.

**Proof.** Let
\[ C = (-f + \lambda_* g)(S) \times (-h(S)). \]
The support function of \( C \) is defined, for all \((y, z) \in \mathbb{R}^m \times \mathbb{R}^r\), by
\[ \delta^*((y, z) \mid C) := \sup_{x \in S} \{y^T (-f(x) + \lambda_* g(x)) - z^T h(x)\} \]
\[ = -G(\lambda_*, y, z). \]

Theorem 19.3 in [17] implies that \( C \) is a polyhedral set. Corollary 19.2.1 in [17] then implies that \( \delta^*((y, z) \mid C) \) is polyhedral. Following [5] Theorem 3.5 and Corollary 3.6), we deduce that,
\[ \exists K > 0 \quad \text{such that} \quad -K \dist((y, z), \Gamma^*) \geq G(\lambda_*, y, z), \quad \text{for all} \quad (y, z) \in \Gamma, \]
since for all \((y_*, z_*) \in \Gamma^*, G(\lambda_*, y_*, z_*) = 0\). Then, for \(0 < \rho < 1\) and \((y, z) \in B(\Gamma^*, \rho) \cap \Gamma\), we have \(\dist((y, z), \Gamma^*) \leq 1\) and thus \(\dist((y, z), \Gamma^*) \geq \dist((y, z), \Gamma^*)^2\).

It follows that
\[ -K \dist((y, z), \Gamma^*)^2 \geq G(\lambda_*, y, z) \quad \text{for all} \quad (y, z) \in B(\Gamma^*, \rho) \cap \Gamma. \]

**Theorem 4.7.** Assume that \( G(\lambda_*, \ldots) \) satisfies the assumption \((H)\). If \( \alpha \) is sufficiently small and \( \lim_{k \to \infty} \frac{d(\eta_k, z_k)}{d(k^{-1} - \lambda_*)} = 0 \), then the sequence \{\(d(\eta_k, z_k)\)\} converges linearly, in the sense that
\[ \limsup_{k \to \infty} \frac{d_{k+1} - \lambda_*}{d_k - \lambda_*} \leq 1 - \frac{\delta}{\Delta + \frac{\alpha}{K}}. \]

**Proof.** The definition of \((y_{k+1}, z_{k+1})\) gives, for all \((y, z) \in \Sigma \times \mathbb{R}^r_+\)
\[ G(d_k, y_{k+1}, z_{k+1}) \geq G(d_k, y, z) - \alpha \|(y, z) - (y_k, z_k)\|^2 - \eta_k. \]

Since \{\((y_k, z_k)\)\} \(\subset \Gamma\) converges to some \((y_*, z_*) \in \Gamma^*\), then for \(k\) large, \((y_k, z_k) \in B(\Gamma^*, \rho) \cap \Gamma\).

Now, let \((\bar{y}_k, \bar{z}_k) \in \Gamma^* \) be such that
\[ \|(y_k, z_k) - (\bar{y}_k, \bar{z}_k)\| = \dist((y_k, z_k), \Gamma^*). \]
The last equality and the assumption \((H)\) imply that
\[-K\| (\tilde{y}_k, \tilde{z}_k) - (y_k, z_k) \|^2 \geq G(\lambda_* y_k, z_k).
\]
The definition of \((\tilde{y}_k, \tilde{z}_k)\) implies that \(G(\lambda_* \tilde{y}_k, \tilde{z}_k) = 0\). Then, using Lemma 3.1 and the last inequality we get
\[G(d_k, \tilde{y}_k, \tilde{z}_k) \geq (\lambda_* - d_k)\delta,
\]
and
\[G(d_k, y_{k+1}, z_{k+1}) \leq (d_{k+1} - d_k)\Delta,
\]
which implies
\[d_{k+1} - \lambda_* \geq (d_k - \lambda_*) \left(1 - \frac{\delta}{\Delta} + \frac{\alpha}{K}\right) - \eta_k.
\]
Thus,
\[\limsup_{k \to \infty} \frac{d_{k+1} - \lambda_*}{d_k - \lambda_*} \leq 1 - \frac{\delta}{\Delta} + \frac{\alpha}{K},
\]
and the assertion holds for \(\alpha < \frac{\delta K}{\Delta}\). \(\square\)

5. **Numerical tests.** In the following numerical examples, we implemented the algorithms on a personal computer equipped with Matlab R2010A and we use Matlab subroutines linprog and quadprog.

We consider generalized linear fractional programs of the form
\[
\inf_{x \in X} \left\{ \max_{1 \leq i \leq m} \frac{A_i x + a_i}{B_i x + b_i} \right\}
\]
where
\[X = S = \{ x \in \mathbb{R}^n \mid Cx \leq \xi, \ x \geq 0 \},
\]
\[A_i^T, B_i^T \in \mathbb{R}^n \text{ and } a_i, b_i \in \mathbb{R} : C a \ p \times n \text{ matrix and } \xi \in \mathbb{R}^p.\]

We notice by \(A\) and \(B\) (resp. \(a\) and \(b\)) the matrices (resp. vectors) whose rows are the \(A_i\)'s and \(B_i\)'s respectively (resp. whose components are \(a_i\) and \(b_i\) respectively).

The data \(A_i, B_i, a_i, b_i, C\) and \(\xi\) are generated as follows:
- each element of the vector \(A_i\) is uniformly drawn from \([-15, 45]\). Similarly \(a_i\) is uniformly drawn from \([-30, 0]\).
- each element of the vector \(B_i\) is uniformly drawn from \([0, 10]\). Similarly \(b_i\) is drawn uniformly from \([1, 5]\).
- the elements of the matrix \(C\) are uniformly distributed within \([0, 10]\). Similarly the elements of the vector \(\xi\) are uniformly distributed within \([0, 1]\).

Later, we will write the sets \(S\) and \(X\) as follows
\[X = S = \{ x \in \mathbb{R}^n \mid \tilde{C}x \leq \tilde{\xi} \}.
\]
We will compare Algorithm 3.1 to Algorithm [3]. The stopping criterion for Algorithm 3.1 is to reach the accuracy
\[y_{k+1}^T (A - d_k B) x_{k+1} + a - d_k b] + z_{k+1}^T \left[ \tilde{C} x_{k+1} - \tilde{\xi} \right] - \alpha \|(y_{k+1}, z_{k+1}) - (y_k, z_k)\|^2 \leq 10^{-8},
\]
where \((x_{k+1}, y_{k+1}, z_{k+1})\) is obtained from a solution of (11).
Observe that if we set $f_i(x) = A_i x + a_i$, $g_i(x) = B_i x + b_i$ for $i = 1, \ldots, m$, and $h(x) = C x - \xi$, then
\[
y^\top ((A - dB)x + a - db) + z^\top [C x - \xi] = y^\top (f(x) - dg(x)) + z^\top h(x).
\]

It follows that if
\[
y_{k+1}^\top [(A - d_k B)x_{k+1} + a - d_k b] + z_{k+1}^\top [C x_{k+1} - \xi] - \alpha \| (y_{k+1}, z_{k+1}) - (y_k, z_k) \|^2 \leq 10^{-8},
\]

then
\[
y_{k+1}^\top (f(x_{k+1}) - d_k g(x_{k+1})) + z_{k+1}^\top h(x_{k+1}) - \alpha \| (y_{k+1}, z_{k+1}) - (y_k, z_k) \|^2 \leq 10^{-8}.
\]

This implies that
\[
H(d_k, y_k, z_k) \leq 10^{-8}.
\]

But we know from Proposition 4 and the proof of Theorem 4.6 that $H(\bar{d}, \bar{y}, \bar{z}) = 0$ implies that $\bar{d} = \lambda_s$ and $(\bar{y}, \bar{z})$ is a solution of the dual $(Q)$. This justifies the use of the previous stopping criterion.

For Algorithm [3], we use the stopping criterion $G(d_k, y_{k+1}, z_{k+1}) \leq 10^{-8}$. With the same arguments as previously, we see that if $G(\bar{d}, \bar{y}, \bar{z}) = 0$ then $\bar{d} = \lambda_s$ and $(\bar{y}, \bar{z})$ is a solution of the dual $(Q)$.

During these numerical tests, the two algorithms will be tested for different sizes ($n = 5$, $m = 5$, $p = 5$), ($n = 10$, $m = 10$, $p = 5$), ($n = 20$, $m = 10$, $p = 5$) and ($n = 50$, $m = 10$, $p = 5$), where $n$ is the number of variables, $m$ is the number of ratios and $p$ is the number of constraints.

In these tests, we analyze the behavior of Algorithm 3.1 with respect to the regularizing parameter $\alpha$ on sets of ten problems, and in the same time, we test the efficiency of the two algorithms. The results are reported in the tables 1-4.
As we can observe from these results, generally the number of iterations decreases when the regularization parameter $\alpha$ is small.

On the other hand, the first algorithm requires more time than the algorithm [3], in favor of more regular auxiliary problems. This is expected because our algorithm treats simultaneously primal and dual variables. But generally, both algorithms solve the problems with the same number of iterations when the regularization parameter is small.

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