TILING OBJECTS IN SINGULARITY CATEGORIES AND LEVELLED MUTATIONS

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Abstract. We show the existence of tilting objects in the singularity category $D^b_{\text{Sg}}(eAe)$ associated to certain noetherian AS-regular algebras $A$ and idempotents $e$. This gives a triangle equivalence between $D^b_{\text{Sg}}(eAe)$ and the derived category of a finite-dimensional algebra. In particular, we obtain a tilting object if the Beilinson algebra of $A$ is a levelled Koszul algebra. This generalises the existence of a tilting object in $D^b_{\text{Sg}}(SG)$, where $S$ is a Koszul AS-regular algebra and $G$ is a finite group acting on $S$, found by Iyama–Takahashi and Mori–Ueyama. Our method involves the use of Orlov’s embedding of $D^b_{\text{Sg}}(eAe)$ into $D^b(\text{qgr}eAe)$, the bounded derived category of graded tails, and of levelled mutations on a tilting object of $D^b(\text{qgr}eAe)$.

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1. Introduction

The singularity category $D_{\text{Sg}}(X)$ of an algebraic variety $X$ was introduced by Orlov [Orl04] as an invariant which reflects the properties of the singularities of $X$. It draws inspiration from the homological mirror symmetry conjecture [Kon95]. Analogous categories were defined for (graded) noetherian algebras $R$. In this paper we are interested in the graded singularity category, defined as the Verdier localisation

$$D^b_{\text{Sg}}(R) := D^b(\text{gr}R)/D^b(\text{grproj}R).$$

When $R$ is Gorenstein, this category is of particular importance because it is triangle equivalent to $\text{CM}^Z(R)$, the stable category of graded Cohen–Macaulay modules [Buc87, Orl04], which play a central role in representation theory and commutative algebra (see [Yos90] or [LW12] for nice treatments).

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Tilting theory is a generalisation of Morita theory, and is a powerful tool in the study of triangulated categories. In fact, tilting objects induce a triangle equivalence between algebraic triangulated categories and the derived category of finite-dimensional algebras [Kel94]. It is thus natural to ask for which graded Gorenstein ring \( R \) the singularity category \( D_{\text{gr}}(R) \) admits tilting objects. This question is very hard to answer in its full generality and has motivated a lot of interesting research over the last years (e.g. [BIY18, DL16a, DL16b, Han19, HIMO14, Kim18, KST07, KST09, OI13, Yam13]). In this paper, we specialise to the case where \( R = eAe \) is an AS-Gorenstein algebra coming from a noetherian AS-regular algebra \( A \) and an idempotent \( e \).

**Question.** Let \( A \) be a noetherian AS-regular algebra of Gorenstein parameter \( \ell \) and \( e \) an idempotent. When does \( D_{\text{gr}}(eAe) \) admit a tilting object?

In this context, tilting objects were discovered in two different settings. The first one was obtained in \([AIR15]\) in the case where \( eA_0(1-e) = 0 \) and \( A \) is bimodule Calabi–Yau of Gorenstein parameter 1, that is, \( A \) is a preprojective algebra over a higher representation-infinite algebra. The second one was given in \([IT13]\) and later generalised in \([MU16b]\) in the case where \( A = S\#G \) is the skew-group algebra over a Koszul AS-regular algebra \( S \), \( G \leq \text{GrAut} S \) is a finite group whose elements have homological determinant 1, and \( e = \frac{1}{|G|} \sum_{g \in G} g \). In this case, \( eAe \cong S^G \).

We note that these two results are in some sense at the two ends of a spectrum. On the one hand, we obtain a tilting object in the case where the Gorenstein parameter is 1. On the other hand, we get a different object when the Gorenstein parameter is equal to the global dimension. We are thus interested in finding tilting objects for the cases where the Gorenstein parameter is more general. For example, we found in \([Thi20]\) a class of skew-group algebras \( k[x_1, \ldots, x_d]\#G \) which do not admit a grading endowing them with the structure of a bimodule \( d \)-Calabi–Yau algebra of Gorenstein parameter 1, but that naturally have gradings giving them higher Gorenstein parameter which are less than \( d \). It would be interesting to determine whether \( D_{\text{gr}}(S^G) \) admits tilting objects for these gradings.

In \([Ami13]\), the author showed that the result in \([AIR15]\), that is, when \( A \) is a preprojective algebra, can be explained by studying Orlov’s embedding \([Orl09]\) \( \Phi: D_{\text{gr}}(eAe) \rightarrow D^b(qA) \), where the target category is the bounded derived category of graded tails. In fact, by a result in \([Min12]\), \( D^b(qA) \) has a tilting object which induces a tilting object in \( D_{\text{gr}}(eAe) \). A similar technique was also employed in \([Ued08]\) for preprojective algebras of the form \( A = k[x_1, \ldots, x_d]\#G \), where \( G \) is a cyclic group satisfying a certain property.

Inspired by this, we compare two semiorthogonal decompositions:

- \([Orl09]\) \( D^b(qA) \cong \langle qAe, qAe(1), \ldots, qAe(\ell - 1), \Phi(D_{\text{gr}}(eAe)) \rangle \);
- \([MM11]\) \( D^b(qA) \cong \langle qA, qA(1), \ldots, qA(\ell - 1) \rangle \cong D^b(\nabla A) \),

where \( q : GrA \rightarrow qA \) is the natural quotient functor and \( \nabla A \) is the Beilinson algebra. Note that suitable restrictions on \( e \) give us an equivalence \( qA \cong qA \).

Using this strategy, we can extend the equivalence in \([AIR15]\) between \( D_{\text{gr}}(eAe) \) and the derived category of a finite-dimensional algebra to certain algebras of Gorenstein parameter 2.

**Theorem A.** Let \( A \) be a locally finite noetherian \( d \)-AS-regular algebra of Gorenstein parameter 2. Let \( e = e^2 \in A \) be such that

1. \( A/AeA \) is finite-dimensional;
2. \( eAe \) is AS-Gorenstein;
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Then there is a triangle equivalence
\[ D_{S_{\tilde{e}}}^{gr}(eA) \cong D^b((1 - \tilde{e})(\nabla A)(1 - \tilde{e})), \]
where \( \tilde{e} \) is the idempotent induced from \( e \) in \( \nabla A \).

We can also get a second class of tilting objects by mutating an exceptional collection in \( D^b(qgr eA) \). Unfortunately, mutations of tilting objects are often not tilting objects, so in order to obtain a satisfying result, we must restrict to the case where \( \nabla A \) is a levelled Koszul algebra.

The notions of levelled mutations and levelled algebras were introduced in [Hil95] and they offer a natural generalisation of the results obtained in [Bon89]. In particular, levelled mutations of tilting objects in levelled Koszul algebras behave well. With this hypothesis, we obtain the following result.

**Theorem B.** Let \( A \) be a locally finite noetherian \( d \)-AS-regular algebra of Gorenstein parameter \( \ell \). Assume that \( \nabla A \) is a levelled Koszul algebra. Let \( e = e^2 \in A \) be such that

1. \( A/eA \) is finite-dimensional;
2. \( eAe \) is AS-Gorenstein;
3. \( eA_0e \cong k \).

Then there is a triangle equivalence
\[ D_{S_{\tilde{e}}}^{gr}(eA) \cong D^b((1 - \tilde{e})(\nabla A)^!(1 - \tilde{e})), \]
where \( (\nabla A)^! \) is the Koszul dual of \( \nabla A \).

The Beilinson algebras of the algebras considered in [IT13] and [MU16b] are levelled Koszul and their idempotent satisfies our conditions. When we restrict to their setting, we obtain the same triangle equivalence. Our theorem covers also certain examples where the Gorenstein parameter is not 1 nor equal to the global dimension of \( A \).

**Notation.** Let \( k \) be an algebraically closed field. If \( V \) is a vector space, denote by \( D(V) := \text{Hom}_k(V, k) \) the \( k \)-dual. If \( R \) is a ring, we denote the opposite ring by \( R^{\text{op}} \). Let \( \text{Mod} R \) be the category of right modules and \( \text{Proj} R \) be the category of projective right modules. Let \( \text{Gr} R \) be the category of graded right \( R \)-modules. The categories written with lower case letters denote the respective subcategories of finitely generated modules. Let \( D(-) \) be the derived category and \( D^b(-) \) be the bounded derived category. If \( T \) is a triangulated category, let \( \text{Hom}^*_T(A, B) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}(A, B[i]) \).

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2. **Preliminaries**

2.1. **Koszul algebras.** Let \( A = \bigoplus_{i \geq 0} A_i \) be a positively graded \( k \)-algebra such that \( S := A_0 \) is a finite-dimensional semisimple algebra. Throughout this section, all tensor products are over \( S \). We define Koszul algebras, following mostly [BGS96]. Combined with a levelled property, defined in the next subsection, these algebras behave well under mutation, a tool which we will need.
Definition 2.1. The algebra $A$ is Koszul if $S$, considered as a right graded $A$-module, admits a graded projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to S \to 0$$

such that $P^\ell$ is generated in degree $\ell$.

Koszul algebras are quadratic, that is, there is a graded isomorphism

$$A \cong T_S V/(R),$$

where $V$ is a $S$-bimodule placed in degree 1 and $R$ is a $S$-bimodule such that $R \subset V \otimes V$. Here $(R)$ denotes the 2-sided ideal generated by $R$.

To every Koszul algebra one can associate its Koszul dual.

Definition 2.2. Let $A$ be a Koszul algebra. The Koszul dual, denoted by $A^!$, is the graded algebra

$$A^! := \bigoplus_{i \in \mathbb{N}} \bigoplus_{j \in \mathbb{Z}} \text{Ext}^i_{Gr A}(A_0, A_0(j)).$$

We have the following properties of Koszul duals.

Theorem 2.3. Let $A$ be a Koszul algebra and suppose that all $A_i$ are finitely generated left $A_0$-modules. Then

- $A^!$ is a Koszul algebra;
- There is an isomorphism of graded algebras $A \cong (A^!)^!$.

2.2. Levelled exceptional collections and mutations. We explain the notion of levelled exceptional collections and levelled mutations, which were first studied in [Hil95] as a natural generalisation of concepts introduced in [Bon89]. These will turn out to be essential tools in the proof of our main theorem, the idea being that levelled mutations behave well under a Koszulity assumption.

Let $T$ be a $k$-linear Krull–Schmidt triangulated category, which is of finite type, that is, for any objects $A, B \in T$, the vector space

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_T(A, B[i])$$

is finite-dimensional. Let $S \subset T$ be a full triangulated subcategory. The right orthogonal to $S$, denoted by $S^\perp$, is the full triangulated subcategory

$$S^\perp := \{A \in T \mid \text{Hom}_T(B, A) = 0 \text{ for any } B \in S\}.$$ 

Dually, the left orthogonal to $S$, denoted by $^\perp S$, is the full triangulated subcategory

$$^\perp S := \{A \in T \mid \text{Hom}_T(A, B) = 0 \text{ for any } B \in S\}.$$ 

The subcategory $S$ is said to be right admissible (resp. left admissible) if there exists a functor $T \to S$, which is right (resp. left) adjoint to an embedding $S \to T$. It is admissible if it is both left and right admissible.

Lemma 2.4 ([Orl09], Lemma 1.4). Let $S$ be a full triangulated subcategory of $T$. If $S$ is right (resp. left) admissible, then $T/S \cong S^\perp$ (resp. $T/S \cong ^\perp S$).

For a set $\Omega$ of objects in $T$, denote by $\langle \Omega \rangle$ the smallest full triangulated subcategory containing the elements of $\Omega$ and closed under isomorphism and direct summands.
**Definition 2.5.** A sequence of full triangulated subcategories \( (S_0, \ldots, S_m) \) in \( T \) is called a semiorthogonal decomposition if all \( S_i \) are admissible in \( T \) and there is a sequence of left admissible subcategories \( T_0 = S_0 \subset T_1 \subset \cdots \subset T_m = T \) such that \( S_i \) is left orthogonal to \( T_{i-1} \) in \( T_i \). In this case we write \( T = \langle S_0, \ldots, S_m \rangle \).

The simplest semiorthogonal decompositions come from exceptional collections.

**Definition 2.6.** An object \( E \in T \) is said to be exceptional if \( \text{Hom}_T(E, E[\ell]) = 0 \) when \( \ell \neq 0 \) and \( \text{Hom}_T(E, E) = k \). An exceptional collection \( E \) in \( T \) is a sequence of exceptional objects \( (E_0, \ldots, E_m) \) in \( T \) such that if \( 0 \leq i < j \leq m \), then \( \text{Hom}_T(E_j, E_i[\ell]) = 0 \) for all \( \ell \in \mathbb{Z} \). It is full if \( T = \langle \oplus_{j=1}^m E_j \rangle \).

Moreover, it is strong if, in addition, \( \text{Hom}_T(E_i, E_j[\ell]) = 0 \) for all \( i \) and \( j \) and \( \ell \neq 0 \).

If \( T \) has a full exceptional collection \( E = (E_0, \ldots, E_m) \), then it admits a semiorthogonal decomposition \( (S_0, \ldots, S_m) \), where \( S_\ell = \langle E_\ell \rangle \cong \mathcal{D}^b(k) \).

To every exceptional collection \( E \), we associate a graded finite-dimensional algebra

\[
\text{End}(E) := \bigoplus_{\ell \geq 0} \bigoplus_{j-i=\ell} \text{Hom}_T(E_i, E_j).
\]

This algebra is finite-dimensional and has finite global dimension.

Strong full exceptional collections are closely related to tilting objects. We are also interested in the weaker notion of silting objects.

**Definition 2.7.** Let \( U \in T \) be an object. We say that \( U \) is tilting (resp. silting) if \( \text{Hom}_T(U, U[i]) = 0 \) for any \( i \neq 0 \) (resp. \( i > 0 \)) and \( \langle U \rangle = T \).

Note that an exceptional sequence \( E = (E_0, \ldots, E_m) \) is full and strong if and only if \( \oplus_{i=0}^m E_i \) is a tilting object in \( T \). Tilting objects give a nice equivalence for algebraic triangulated categories.

**Theorem 2.8** ([Kel94, Theorem 4.3]). Let \( T \) be an algebraic Krull–Schmidt triangulated category with a tilting object \( U \). If \( \text{gl.dim} \text{End}_T(U) < \infty \), then there is a triangle equivalence

\[
\text{Hom}_T^\bullet(U, -) : T \xrightarrow{\sim} \mathcal{D}^b(\text{End}_T(U)).
\]

We are interested in mutating exceptional collections coming from certain tilting objects in order to get tilting objects in the singularity categories we study.

**Definition 2.9.** Let \( E \in T \) be an exceptional object and \( X \in \perp(E) \). The left mutation of \( X \) through \( E \), denoted by \( L_E(X) \in \perp(E) \), is defined up to isomorphism by the triangle

\[
L_E(X) \to \text{Hom}_T^\bullet(E, X) \otimes E \xrightarrow{ev} X \to L_E(X)[1],
\]

where \( ev \) is the evaluation map. Dually, if \( X \in \perp(E) \), we define the right mutation of \( X \) through \( E \), denoted by \( R_E(X) \in \perp(E) \), by the triangle

\[
X \xrightarrow{\text{coev}} D \text{Hom}_T^\bullet(X, E) \otimes E \to R_E(X) \to X[1],
\]

where \( \text{coev} \) is the coevaluation map. If \( E = (E_0, \ldots, E_t) \) is an exceptional collection and \( X \in \perp E \), then we define

\[
L_E(X) := L_{E_0} \cdots L_{E_t}(X) \in \perp E.
\]

Similarly, if \( X \in E^\perp \), we define

\[
R_E(X) := R_{E_t} \cdots R_{E_0}(X) \in E^\perp.
\]
Definition 2.10. An exceptional collection $E = (E_0, \ldots, E_m)$ is $n$-levelled if there exists a surjective monotonic map $s : \{0, \ldots, m\} \to \{0, \ldots, n\}$ such that

$$\text{Hom}^\bullet_T(E_i, E_j) = 0 \text{ for all } i \neq j \text{ for which } s(i) = s(j).$$

The subcollections $E_i := (E_{i_0}, \ldots, E_{i_p})_{i_0, \ldots, i_p \in s^{-1}(i)}$ are called levels. We define the right levelled mutations on levelled exceptional collections as follows:

$$R_j(E_0, \ldots, E_m) := (E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}}(E_i), E_{i+2}, \ldots, E_m),$$

where, if $E_i = (E_{i_0}, \ldots, E_{i_p})$, then

$$R_{E_{i+1}}(E_i) := (R_{E_{i+1}}(E_{i_0}), \ldots, R_{E_{i+1}}(E_{i_p})).$$

The left levelled mutations $L_i$ are defined similarly:

$$L_i(E_0, \ldots, E_m) := (E_1, \ldots, E_{i-2}, L_{E_{i-1}}(E_i), E_{i-1}, E_{i+1}, \ldots, E_m).$$

Proposition 2.11 ([HI95 Proposition 4.2]). Let $E$ be a levelled exceptional collection. Then $R_jE$ and $L_iE$ are also levelled exceptional. Moreover, if $E$ is full, then these are also full.

Two exceptional collections associated to a levelled exceptional collection $E$ are of particular importance. If $E_i$ is a level and $r \in \mathbb{N}$, define the iterated right mutation

$$R^r(E_i) := R_{E_{i+r}} \cdots R_{E_{i+1}}(E_i) \in R_{i+r-1} \cdots R_i(E).$$

Similarly one can define the iterated left mutation:

$$L^r(E_i) : L_{E_{i-r}} \cdots L_{E_{i-1}}(E_i) \in L_{i-r+1} \cdots L_i(E).$$

Definition 2.12. Let $E = (E_0, \ldots, E_n)$ be a levelled exceptional collection. The levelled right dual collection is defined as

$$E^\vee := (E_n, R_1^1E_{n-1}, \ldots, R_n^0E_0).$$

The levelled left dual collection is defined as

$$\vee E := (L_nE_n, L_{n-1}E_{n-1}, \ldots, E_0).$$

Unfortunately, mutations do not preserve in general the strong property of exceptional collections. In other words, as opposed to silting objects, mutations of tilting objects are often not tilting objects. However, if the endomorphism algebra is levelled and Koszul, then the dual collections remain strong.

Definition 2.13. A quiver $Q$ is ordered if $Q_0 = \{0, \ldots, m\}$ is an ordered set and for all $i \leq j$, $e_iQ_1e_j = \emptyset$. Moreover, we say that $Q$ is $n$-levelled if it is ordered and there exists a surjective monotonic map

$$s : Q_0 \to \{1, \ldots, n\}$$

having the property that if $e_jQ_1e_i \neq \emptyset$, then $s(j) = s(i) + 1$. Finally, an algebra is levelled if it is Morita equivalent to a quiver algebra with a levelled quiver.

If $A$ is an ordered algebra and $\{e_0, \ldots, e_m\}$ is a complete set of primitive orthogonal idempotents in the given ordering, then $\text{D}^b(A)$ admits a full strong exceptional collection

$$E = (P_0, \ldots, P_m),$$

where $P_i = e_iA$. It is levelled if $A$ is a levelled algebra. Note that in this case $A \cong \text{End}(E)$.

We have the following necessary and sufficient conditions for a levelled algebra to be Koszul. Let $S_i = \text{top} e_iA$. 
Lemma 2.14 ([H195 Lemma 3.1]). A levelled algebra $A$ is Koszul if and only if $\text{Ext}^\ell_A(S_j, S_i) \neq 0$ only for $\ell = s(j) - s(i)$.

Proposition 2.15 ([H195 Proposition 4.5]). Let $A$ be a levelled algebra with level function $s : \{0, \ldots, m\} \to \{0, \ldots, n\}$. Let $\mathcal{P} = (P_0, \ldots, P_m)$ be the full strong exceptional collection in $\text{D}^b(A)$ consisting of indecomposable projective $A$-modules. Then,

$$\forall \mathcal{P} = (S_m[-s(m)], S_{m-1}[-s(m-1)], \ldots, S_0),$$

where $S_i := \text{top} P_i$. Moreover, $A$ is Koszul if and only if $\forall \mathcal{P}$ is a full strong exceptional collection. In this case, $\text{End}(\forall \mathcal{P}) \cong A^!$.

The second part of the proposition is a direct consequence of Lemma 2.14. Now suppose $T$ is an algebraic Krull–Schmidt triangulated category. Let $E$ be a full strong exceptional collection in $T$. Using the equivalence of Theorem 2.8, we obtain the following corollary.

Corollary 2.16. If $E$ is a full strong exceptional collection in an algebraic Krull–Schmidt triangulated category and $A := \text{End}(E)$ is an $n$-levelled algebra, then the left dual collection $\forall E$ is also a strong full exceptional collection and $\text{End}(\forall E) \cong A^!$.

We can also obtain a similar statement for the right dual collection as follows. By [BS10 Corollary 2.10], if $E^i \in E_i$, then

$$R^{n-s(i)}(E^i) = S_m^{-1}(L^{s(i)}(E^i)),$$

where $S$ is the Serre functor on $T$, which exists because $T$ has a full exceptional collection [BKS9 Corollary 2.10]. Here, the length of $E$ is $m + 1$ and $S_m := S[-m]$. This implies that

$$\text{Hom}_T^\bullet(R^{n-s(i)}(E^i), R^{n-s(j)}(E^j)) \cong \text{Hom}_T^\bullet(L^{s(i)}(E^j), L^{s(j)}(E^i)),$$

Therefore, if $A$ is levelled Koszul, then $\mathcal{E}^\forall$ is also strong and $\text{End}(\mathcal{E}^\forall) \cong A^!$ as well.

2.3. AS-regular algebras and singularity categories. In this subsection, we give the definition of the objects that we study in this paper.

Definition 2.17. Let $A = \oplus_{i \geq 0} A_i$ be a noetherian locally finite graded algebra. We say that $A$ is $d$-AS-regular (resp. $d$-AS-Gorenstein) of Gorenstein parameter $\ell$ if $\text{gl.dim} A = d$ and $\text{gl.dim} A_0 < \infty$ (resp. $\text{inj.dim}_A A = \text{inj.dim}_{A^{op}} A = d$) and

$$\text{RHom}_A(A_0, A) \cong (DA_0)(\ell)[-d] \text{ in } D(\text{Gr} A_0) \text{ and in } D(\text{Gr} A_0^{op}),$$

where $\text{RHom}_A(A_0, A) := \oplus_{i \in \mathbb{Z}}\text{RHom}_{\text{Gr}} A(A_0, A(i))$.

We now proceed to define the categories that we are interested in.

Definition 2.18. Let $A$ be a positively graded noetherian algebra.

1. We define the quotient abelian category of graded tails $\text{agr} A := \text{gr} A/\text{tors} A$,

where $\text{tors} A$ is the full subcategory consisting of all graded finite-dimensional $A$-modules. Let $q : \text{gr} A \to \text{agr} A$ be the natural quotient functor. The morphisms in $\text{agr} A$ are given by

$$\text{Hom}_{\text{agr} A}(qM, qN) := \lim_{p \to \infty} \text{Hom}_{\text{gr} A}(M_{\geq p}, N).$$
(2) We define the singularity category to be the Verdier localisation
\[ D^g_{Sg}(A) := D^b(gr A)/D^b(grproj A), \]
where \( D^b(grproj A) \) is the triangulated subcategory consisting of objects that are isomorphic to bounded complexes of projectives. We denote by \( \pi : D^b(gr A) \to D^g_{Sg}(A) \) the localisation functor.

(3) Let \( A \) be AS-Gorenstein. We define the graded category of Cohen–Macaulay \( A \)-modules as follows:
\[ CM^Z(A) := \{ M \in gr A \mid Ext^i_A(M, A) = 0 \text{ for any } i > 0 \}. \]
The stable category \( CM^Z(A) \) has the same objects as \( CM^Z(A) \) and the morphisms are given by
\[ \text{Hom}_{CM^Z(A)}(M, N) := \text{Hom}_{CM^Z(A)}(M, N)/[A](M, N), \]
where \([A](M, N)\) consists of the morphisms which factor through \( \text{add } A \).

A famous theorem of Buchweitz [Buc87, Theorem 4.4.1] and Orlov [Orl04, Theorem 3.9] states that if \( A \) is AS-Gorenstein, then there is a triangle equivalence
\[ D^g_{Sg}(A) \cong CM^Z(A). \]

Our leading motivation is to determine when these categories admit tilting objects. By [IT13, Propositions 1.3 & 1.4], they are Krull–Schmidt algebraic triangulated categories, so the existence of tilting objects implies that they are equivalent to the derived category of a finite-dimensional algebra. The first two categories in our definition are related by the following result of Orlov, which was generalised to our setting in [BS15].

**Theorem 2.19** ([Orl09, Theorem 2.5]). Let \( A \) be an AS-Gorenstein algebra of parameter \( \ell \geq 1 \). Then there exists a fully faithful functor
\[ \Phi : D^g_{Sg}(A) \to D^b(qgr A) \]
and a semiorthogonal decomposition
\[ D^b(qgr A) = \langle qA, \ldots, qA(\ell - 1), \Phi(D^g_{Sg}(A)) \rangle. \]

When \( A \) is AS-regular, there is also a semiorthogonal decomposition of \( D^b(qgr A) \), given in [MM11].

**Definition 2.20.** Let \( A \) be an AS-regular algebra of Gorenstein parameter \( \ell \). The Beilinson algebra is defined by
\[ \nabla A := \begin{pmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ A_1 & A_0 & 0 & \cdots & 0 & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{\ell-2} & A_{\ell-3} & A_{\ell-4} & \cdots & A_0 & 0 \\ A_{\ell-1} & A_{\ell-2} & A_{\ell-3} & \cdots & A_1 & A_0 \end{pmatrix}. \]

If \( e \) is an idempotent in \( A \), then we define \( \tilde{e}_i := \text{diag}(0, \ldots, 0, e, 0, \ldots, 0) \) to be the diagonal matrix in \( \nabla A \) with only one non-zero entry in position \((i, i)\) and we set \( \tilde{e} := \sum_{i=0}^{\ell-1} e_i. \)
Example 2.23. Let \( W \) define \( G \) and \( R \) vector space by isolated singularity, then
\[ qU := \oplus_{i=0}^{\ell-1} qA(i) \]
is a tilting object in \( \mathbb{D}^b(qgr A) \) and \( \text{Hom}_{\mathbb{D}^b(qgr A)}(qU) \cong \nabla A \). This implies that there is a triangle equivalence
\[ \mathbb{R} \text{Hom}_{\mathbb{D}^b(qgr A)}(qU, -) : \mathbb{D}^b(qgr A) \to \mathbb{D}^b(\nabla A). \]

In the remainder of the paper, we shall assume the following setting.

Setting 2.22. Let \( A = \oplus_{i \geq 0} A_i \) be a locally finite noetherian \( d \)-AS-regular algebra of Gorenstein parameter \( \ell \), with \( \ell \geq 1 \). Let \( e = e^2 \in A \) be such that
a) \( A/AeA \) is finite-dimensional;

b) \( eAe \) is \( d \)-AS-Gorenstein of parameter \( \ell \).

Condition [a] implies that the functor
\[ \Psi : \text{gr } A \to \text{gr } eAe \]
\[ (1.1) \]
induces an equivalence \( \Psi : qgr A \cong qgr eAe \) [Ami13, Proof of Corollary 3.3]. Condition [b] allows us to invoke Orlov’s semiorthogonal decomposition to study \( \mathbb{D}^b(eAe) \). Note that it follows automatically from [b] when \( A \) is bimodule Calabi–Yau of Gorenstein parameter \( \ell \) [Ami13, Proof of Theorem 4.3]. For more information on bimodule Calabi–Yau algebras, we refer to [AIR15].

An important fact is that they are a special class of AS-regular algebras [MU16a, Theorem 3.5].

Before stating our main running example, we define skew-group algebras. If \( R \) is an algebra and \( G < \text{Aut } R \) is a subgroup, then the skew-group algebra, denoted by \( R \# G \), is defined as a vector space by \( R \# G = R \otimes_k kG \) with multiplication
\[ (a \otimes g)(b \otimes h) = ag(b) \otimes gh. \]

We define \( G \# R \) in a similar way.

Example 2.23. Let \( S = k[x_1, \ldots, x_d] \), \( G < SL(n, k) \) be finite and \( e = \frac{1}{G} \sum_{g \in G} g \). Let \( A = S \# G \) be the skew-group algebra. Then there exists many gradings endowing it with a structure of AS-regular algebra, for example by putting the variables in degree 1. The reader can find other examples of gradings in the next sections. In this case, \( eAe \cong S^G \), the invariant ring, and it is well-known that, since \( G < SL(d, k) \), the ring \( S^G \) is AS-Gorenstein. If, in addition, \( S^G \) is an isolated singularity, then \( A \) and \( e \) satisfy condition [a] in Setting 2.22.

In this paper, our examples will be skew-group algebras as described above. We therefore explain here how to construct the relevant quivers.

Definition 2.24. Let \( S = k[x_1, \ldots, x_d] \) be the polynomial ring and \( G = \frac{1}{r}(a_1, \ldots, a_d) < SL(d, k) \) be the cyclic group generated by the diagonal matrix \( \text{diag}(\xi^{a_1}, \ldots, \xi^{a_d}) \), where \( \xi \) is an \( r \)th root of unity and \( 0 \leq a_j < r \). Let \( A := S \# G \) be the skew-group algebra. Then \( A \cong kQ/(R) \), where \( Q \) is the McKay quiver, whose vertices are given by \( \mathbb{Z}/r\mathbb{Z} \). Furthermore, there are arrows
\[ x_j : i \to i + a_j \]
for each \( i \in \mathbb{Z}/r\mathbb{Z} \) and \( 1 \leq j \leq d \). The relations are generated by \( x_ix_j - x_jx_i = 0 \).

If \( A \) is endowed with a grading giving it the structure of an AS-regular algebra of Gorenstein parameter \( \ell \), then we can deduce from [MI13, Proposition 7.13] how to compute the quiver of
the finite-dimensional algebra $\nabla A$. Indeed, it is described by the $\ell$-folded McKay quiver, whose vertices are given by $\mathbb{Z}/r\mathbb{Z} \times \{0, \ldots, \ell - 1\}$. If $x_j$ is an arrow in degree $\delta$ in the quiver of $A$, then we have arrows

$$x_j : (i, p) \rightarrow (i + a_j, p + \delta)$$

for each $0 \leq p \leq \ell - 1$ such that $p + \delta \leq \ell - 1$, $i \in \mathbb{Z}/r\mathbb{Z}$ and $1 \leq j \leq d$. We denote $(i, p)$ by $i^p$.

The idempotent $\tilde{e} \in \nabla A$ induced from $e$ is then given by $e^0 + \ldots + e^{\ell-1}$.

As mentioned in the introduction, two partial answers were given to our motivating question.

**Theorem 2.25 ([AIR15, Theorem 4.1]).** Let $A$ be a noetherian bimodule Calabi–Yau algebra of Gorenstein parameter $1$. Let $e$ be an idempotent such that

1. $A/AeA$ is finite-dimensional;
2. $eA_0(1 - e) = 0$.

Then $\pi A e$ is a tilting object in $D_{gr} S_{g}(eAe)$ and there is a triangle equivalence

$$D_{gr} S_{g}(eAe) \cong D^{b}((1 - e)A_0(1 - e)).$$

Note that since the Gorenstein parameter is $1$, we have that $A_0 = \nabla A$ and $e = \tilde{e}$.

**Theorem 2.26 ([IT13, Theorem 1.7]; [MU16b, Theorem 4.17]).** Let $S$ be a noetherian AS-regular Koszul algebra over $k$ of dimension $d \geq 2$. Let $G \leq \mathrm{GrAut} S$ be a finite subgroup such that char $k$ does not divide $|G|$ and let $e = \frac{1}{|G|} \sum_{g \in G} g$. Assume in addition that

a) $S\# G/(e)$ is finite-dimensional over $k$;
b) $S^{G} \cong eS\# Ge$ is AS-Gorenstein.

Then there is a triangle equivalence

$$D_{sg}(S^{G}) \cong D^{b}((1 - \tilde{e})(G\# \nabla(S^{G}))(1 - \tilde{e})).$$

The fact that $S\# G/(e)$ is finite-dimensional is equivalent to the notion of being a noncommutative graded isolated singularity. Moreover, $S^{G}$ is AS-Gorenstein if every element of $G$ has homological determinant $1$ (see [IZ00]). Note that, in this setting, the Gorenstein parameter of $S$ is $d$.

In both cases the algebras satisfy Setting 2.22. Theorem 2.25 describes a situation in which the Gorenstein parameter is $1$, whereas in Theorem 2.26, the Gorenstein parameter always equals the global dimension. The goal of this paper is to extend these two results by comparing the two semiorthogonal decompositions of $D^{b}(qgr eAe)$. In particular, our generalisation will cover certain examples where the Gorenstein parameter is not $1$, nor equal to the global dimension.

**Remark 2.27.** One other possible approach to finding tilting objects in $D_{sg}(eAe)$ is to consider the preprojective algebra $\Pi$ over $\nabla A$. From [MM11], this algebra is an AS-regular algebra of parameter $1$. Moreover, there is an equivalence of categories

$$\mathrm{Gr} A \cong \mathrm{Gr} \Pi.$$

It could thus be tempting to apply Theorem 2.25 to find a tilting object. However, the corresponding idempotent $\tilde{e} \in \Pi$ does not satisfy in general condition (2) of the statement. Therefore this approach does not work directly.
3. A silting object

Let $A = \bigoplus_{i \geq 0} A_i$ and $e = e^2 \in A$ be as in Setting 2.22. Let $e' := (1 - e)$. In this section, we give a silting object in $D_{Sg}(eAe)$. This object is also tilting in some specific cases, which we describe. Let

$$U := \bigoplus_{i=0}^{\ell-1} A(i) \in \text{gr} A$$

and define

$$D^b(qgr eAe) := D^b(qgr eAe)/\langle q\text{e}Ae, \ldots, q\text{e}Ae(\ell - 1)\rangle.$$  

Using Orlov’s semiorthogonal decomposition

$$D^b(qgr eAe) \cong \langle q\text{e}Ae, \ldots, q\text{e}Ae(\ell - 1), \Phi(D_{Sg}^g(eAe)) \rangle,$$

we have that

$$\Phi(D_{Sg}^g(eAe)) \cong \langle q\text{e}Ae, \ldots, q\text{e}Ae(\ell - 1) \rangle.$$  

Then, by Lemma 2.21 we obtain a triangle equivalence

$$D_{Sg}^g(eAe) \cong D^b(qgr eAe).$$

**Theorem 3.1.** The object $q\text{e}Ue$ is a silting object in $D^b(qgr eAe)$. It is tilting if either

a) [AIR15] Theorem 4.1, [Ami13] Theorem 4.3] $\ell = 1$ and $eA_0e' = 0$ or $e'A_0e = 0$, or

b) $\ell = 2$ and $eA_0e' = e'A_0e = 0$.

**Remark 3.2.** In [IT13] Theorem 1.6], the authors proved b) in the case $A = k[x, y]/#G$, where $k[x, y]$ is the polynomial ring, $G < SL(2, k)$ is finite, $e = \frac{1}{|G|} \sum_{g \in G} g$ and the grading is induced by putting the variables in degree 1. In their setting, we have that $A_0 = kG$ is semisimple, so $eA_0e' = e'A_0e = 0$.

**Proof.** By Theorem 2.21 $qU$ is a tilting object in $D^b(qgr A)$. Moreover, recall that the functor

$$\Psi: \text{gr} A \to \text{gr} eAe$$

$$M \mapsto Me$$

described in (2.4), induces an equivalence of categories

$$qgr A \overset{\sim}{\longrightarrow} qgr eAe.$$  

Thus, $qUe$ is a tilting object in $D^b(qgr eAe)$. Now, each $q\text{e}Ae(i)$ is a projective summand of $qUe$. Therefore, by [IY18] Theorem 3.6], $q\text{e}Ue$ is a silting object in

$$D^b(qgr eAe) \cong D_{Sg}^g(eAe).$$

Now assume that $\ell = 1$ and $eA_0e' = 0$, the case where $e'A_0e = 0$ being similar. Then,

$$(3.1) \quad \text{Hom}_{D^b(qgr eAe)}(q\text{e}Ae, q\text{e}Ae) \cong eA_0e' = 0,$$

so the tilting object $qUe = q\text{e}Ae \in D^b(qgr eAe)$ gives rise to a semiorthogonal decomposition

$$D^b(qgr eAe) \cong \langle q\text{e}Ae, q\text{e}'Ae \rangle.$$  

Therefore, $q\text{e}'Ue = q\text{e}'Ae$ is a tilting object in

$$\langle q\text{e}'Ae \rangle \cong \langle q\text{e}Ae \rangle \cong D^b(qgr eAe)/\langle q\text{e}Ae \rangle = D^b(qgr eAe),$$

since it is a direct summand of $qUe$.
Similarly, if $\ell = 2$ and $eA_0e' = e'A_0e = 0$, then, in addition to (3.1), we have
\[
\text{Hom}_{\mathcal{D}^b(qgr \ A e)}(q_0 A e, q_0 A e') \cong e'A_0 e = 0.
\]
Thus, the tilting object $qUe = qAe \oplus qAe(1)$ gives rise to a semiorthogonal decomposition
\[
\mathcal{D}^b(qgr \ A e) \cong \langle qAe, qe' A e, qe' A e(1), qeAe(1) \rangle,
\]
so $qe' A e \oplus qe' A e(1) \oplus qeAe(1)$ is a tilting object in
\[
\langle qe' A e, qe' A e(1), qeAe(1) \rangle \cong q_{\perp} \cong \mathcal{D}^b(qgr \ A e)/\langle qeAe \rangle
\]
and $qe' Ue = qe' A e \oplus qe' A e(1)$ is a tilting object in
\[
\langle qe' A e, qe' A e(1) \rangle \cong (qeAe(1))^{\perp} \cong \mathcal{D}^b(qgr \ A e)/\langle qeAe, qeAe(1) \rangle = \mathcal{D}^b(qgr \ A e),
\]
where the right orthogonal is taken in $\mathcal{D}^b(qgr \ A e)/\langle qeAe \rangle$. \hfill \Box

We now compute the endomorphism ring of the silting object we found.

**Lemma 3.3.** There is an isomorphism of graded $k$-algebras
\[
\text{End}_{\mathcal{D}^b(qgr \ A e)}(qe' Ue) \cong (1 - \bar{e})(\nabla A)(1 - \bar{e}).
\]

**Proof.** We have that
\[
\text{End}_{\mathcal{D}^b(qgr \ A e)}(qe' Ue) \cong (1 - \bar{e}) \text{End}_{\mathcal{D}^b(qgr \ A e)}(qeUe)(1 - \bar{e}) \cong (1 - \bar{e})(\nabla A)(1 - \bar{e}),
\]
where the last isomorphism is given in Theorem 3.1. \hfill \Box

**Corollary 3.4.** If $A$ satisfies hypothesis a) or b) in Theorem 3.1, then there is a triangle equivalence
\[
\mathcal{D}_{\text{sg}}^{gr}(eAe) \cong \mathcal{D}^b((1 - \bar{e})(\nabla A)(1 - \bar{e})�).
\]

**Proof.** This is a direct application of Theorem 2.8. The category $\mathcal{D}_{\text{sg}}^{gr}(eAe)$ is an algebraic Krull–Schmidt triangulated category. Moreover, $(1 - \bar{e})(\nabla A)(1 - \bar{e})$ is an ordered finite-dimensional algebra, so it has finite global dimension. \hfill \Box

**Example 3.5.** Let $A = k[x_1, x_2, x_3]\# G$, where $G = \frac{1}{3}(1, 2, 2) < SL(3, k)$. We refer to Definition 2.24 for explanations on how to construct the following quivers. We have that $A \cong kQ/(R)$, where $Q$ is the McKay quiver

\[
\begin{array}{ccccc}
1 & \rightarrow & 1 & \rightarrow & 1 \\
\downarrow & & \downarrow & & \\
2 & \rightarrow & x_1 & \rightarrow & x_1 \\
\end{array}
\]
and the relations are given by $x_i x_j - x_j x_i = 0$. We endow the algebra $A$ with a grading by putting the thick arrows in degree 1 and the other arrows in degree 0. Let $e$ be the idempotent corresponding to vertex 0. Then $A$ is 3-AS-regular of Gorenstein parameter 2. Moreover, $eAe \cong$
$S^G$ is an isolated singularity, so $A/AeA$ is finite-dimensional. We also have that $S^G$ is $AS$-Gorenstein, because $G < SL(3,k)$. Finally, $eA_0e' = e'A_0e = 0$, so, by Theorem 3.1, $qe'Ae \oplus qe'Ae(1)$ is a tilting object in $\mathbb{D}^b(qgr eAe)$. We thus have a triangle equivalence

$$\mathbb{D}^g_{Sg}(S^G) \cong \mathbb{D}^b((1 - \tilde{e})(\nabla A)(1 - \tilde{e})).$$

The algebra $\nabla A$ has the following quiver

![Quiver diagram]

and the relations are induced from the relations in $A$. The idempotent $\tilde{e}$ induced from $e$ corresponds to the vertices in the boxes. Thus, $(1 - \tilde{e})(\nabla A)(1 - \tilde{e})$ is given by the following quiver

![Quiver diagram]

4. A tilting object from levelled mutations

In the previous section, we found a tilting object in some specific cases, and only when the Gorenstein parameter was 1 or 2. In this section, we use mutations to find a different tilting object in the case where $\nabla A$ is a Koszul levelled algebra. Our strategy is still to compare the semiorthogonal decompositions of Orlov and Minamoto–Mori.

**Theorem 4.1.** Let $A = \oplus_{i \geq 0} A_i$ and $e$ be as in Setting 2.22. In addition, suppose that $eA_0e \cong k$ and that $\nabla A$ is a Koszul levelled algebra. There is a triangle equivalence

$$\mathbb{D}^g_{Sg}(eAe) \cong \mathbb{D}^b((1 - \tilde{e})(\nabla A)'(1 - \tilde{e})).$$
Remark 4.2. As opposed to the situation in Theorem 2.26, the algebra $A$ itself is not necessarily Koszul with respect to the grading that endows it with a structure of AS-regular algebra.

Proof. The idea is to compare two semiorthogonal decompositions of $\mathcal{D}^b(\text{qgr} eAe)$, mentioned in the preliminaries:

- [MM11] Theorem 4.12 | $\mathcal{D}^b(\text{qgr} eAe) = \langle qAe, qAe(1), \ldots, qAe(\ell - 1) \rangle$;
- [Orl09] Theorem 2.5 | $\mathcal{D}^b(\text{qgr} eAe) = \langle qeAe, qeAe(1), \ldots, qeAe(\ell - 1), \Phi(\mathcal{D}^b_{\text{sg}}(eAe)) \rangle$.

Recall from Theorem 2.21 that

$$\text{Hom}_{\mathcal{D}^b(\text{qgr} eAe)}(\oplus_{i=0}^{\ell-1} qAe(i), \oplus_{i=0}^{\ell-1} qAe(i)) \cong \nabla A.$$  

Moreover, the object $T := \oplus_{i=0}^{\ell-1} qAe(i)$ is tilting, so there is a triangle equivalence

$$F := \text{RHom}_{\mathcal{D}^b(\text{qgr} eAe)}(T, -) : \mathcal{D}^b(\text{qgr} eAe) \xrightarrow{\sim} \mathcal{D}^b(\nabla A).$$

Since $\nabla A$ is levelled, the indecomposable projective $\nabla A$-modules give rise to a levelled full strong exceptional collection

$$\mathcal{D}^b(\nabla A) = \langle P_0, \ldots, P_m \rangle =: \mathbb{P}.$$  

This induces a levelled structure on a full strong exceptional collection

$$\mathcal{D}^b(\text{qgr} eAe) = \langle E_0, \ldots, E_m \rangle = \langle E_0, \ldots, E_n \rangle =: \mathbb{E},$$

where each $E_j = F^{-1}(P_j)$ is an exceptional direct summand of $qAe(r)$ for some $0 \leq r \leq \ell - 1$, and $E_j$ is the collection at level $j$. In particular, since

$$\text{Hom}_{\mathcal{D}^b(\text{qgr} eAe)}(qeAe, qeAe) \cong eA_0e \cong k,$$

we have that $qeAe(i)$ is exceptional for all $0 \leq i \leq \ell - 1$, so $qeAe(i) = E_{j_i}$ for some $0 \leq j_i \leq m$.

Denote by $s : \{0, \ldots, m\} \to \{0, \ldots, n\}$ the level function on this collection. We now perform right mutations on $\mathbb{E}$ to obtain a full exceptional collection

$$R_0^s \mathbb{E} := (qeAe, E_{s(j_0)+1}, \ldots, E_n, R^{n-s(j_0)}(E'_{s(j_0)}), R^{n-s(j_0)+1}(E_{s(j_0)-1}), \ldots, R^n(E_0))$$

which generates $\mathcal{D}^b(\text{qgr} eAe)$. Here, $E'_{s(j_0)}$ is the collection obtained from $E_{s(j_0)}$ by removing $qeAe$. Similarly, denote by $R_0^r \mathbb{E}$ the subcollection consisting of the objects of $R_0^s \mathbb{E}$, except for $qeAe$. Then, by Lemma 2.2.1

$$\langle R_0^s \mathbb{E} \rangle \cong \frac{1}{\langle qeAe \rangle} \cong \mathcal{D}^b(\text{qgr} eAe)/\langle qeAe \rangle.$$  

Comparing with Orlov’s semiorthogonal decomposition, we thus conclude that

$$\langle R_0^s \mathbb{E} \rangle \cong \langle qeAe(1), \ldots, qeAe(\ell - 1), \Phi(\mathcal{D}^b_{\text{sg}}(eAe)) \rangle.$$  

We now do right mutations on $R_0^s \mathbb{E}$ to obtain a full exceptional collection

$$R_1 R_0^s \mathbb{E} := (qeAe(1), E_{s(j_1)+1}, \ldots, E_n, R^{n-s(j_1)}(E'_{s(j_1)}), R^{n-s(j_1)+1}(E_{s(j_1)-1}), \ldots, R^{n-s(j_1)-1}(E_{s(j_1)+1}), R^{n-s(j_1)}(E'_{s(j_1)}), R^{n-s(j_1)+1}(E_{s(j_1)-1}), \ldots, R^n(E_0)).$$

Then, by the same reasoning, we have an equivalence

$$\langle R_1 R_0^s \mathbb{E} \rangle \cong \langle qeAe(2), \ldots, qeAe(\ell - 1), \Phi(\mathcal{D}^b_{\text{sg}}(eAe)) \rangle.$$  

Continuing in this fashion for every $qeAe(i)$, we obtain an equivalence

$$\langle R_{\ell-1} R_0^s \mathbb{E} \rangle \cong \mathcal{D}^b(\text{qgr} eAe)/\langle qeAe, \ldots, qeAe(\ell - 1) \rangle \cong \mathcal{D}^b_{\text{sg}}(eAe).$$
Note that $R'_{t-1} \cdots R'_{0}E$ is a subcollection of the right dual collection $E^\vee$, from which the objects $R'^{-s(j_i)}(qeAe(i))$ are removed. By Corollary [2.16], since $\nabla A$ is levelled Koszul, we have that $E^\vee$ is a full strong exceptional collection and

$$\text{End}(E^\vee) \cong (\nabla A)^!.$$  

Therefore the collection $R'_{t-1} \cdots R'_{0}E$ is also strong. Now, by uniqueness of the Serre functor, there is a commutative diagram

$$\begin{array}{ccc}
D^b(qgr eAe) & \xrightarrow{F} & D^b(\nabla A) \\
\downarrow S^1_m & & \downarrow S^1_m \\
D^b(qgr eAe) & \xrightarrow{F} & D^b(\nabla A)
\end{array}$$

where, by abuse of notation, $S_m = S[-m]$ denotes the shifted Serre functor in $D^b(qgr eAe)$ and $D^b(\nabla A)$. Moreover,

$$F(qeAe(i)) = \text{Hom}_{D^b(qgr eAe)}(\bigoplus_{i=0}^{\ell-1} qAeAe(i), qeAe(i))$$

$$\cong \tilde{e}_i \text{Hom}_{D^b(qgr eAe)}(\bigoplus_{i=0}^{\ell-1} qAeAe(i), \bigoplus_{i=0}^{\ell-1} qAeAe(i))$$

$$\cong \tilde{e}_i eAe,$$

where $\tilde{e}_i$ is defined in [2.20]. Thus,

$$R'^{-s(j_i)}(qeAe(i)) \cong S^{-1}_m(L^{s(j_i)}(qeAe(i))) \xrightarrow{F} S^{-1}_m(L^{s(j_i)}(\tilde{e}_i \nabla A)) \cong S^{-1}_m(\text{top}(\tilde{e}_i \nabla A) [-s(j_i)]).$$

We conclude that

$$\text{End}(R'_{t-1} \cdots R'_{0}E) \cong \text{End}_{D^b(\nabla A)} \left( \bigoplus_{j: j \neq j_i} \text{top}(P_j) [-s(j)] \right)$$

$$\cong (1 - \tilde{e})(\nabla A)^!(1 - \tilde{e}),$$

so, applying theorem [2.28] we obtain a triangle equivalence

$$D^g_{S^e}(eAe) \cong D^b((1 - \tilde{e})(\nabla A)^!(1 - \tilde{e})).$$

\[\square\]

Perhaps the most interesting consequence of this theorem is that the equivalence of Mori–Ueyama, cited in Theorem [2.26] can be obtained as a special case.

**Corollary 4.3** ([MU16b Theorem 4.17]). Let $S$ be a noetherian $AS$-regular Koszul algebra over $k$ of dimension $d \geq 2$. Let $G \leq \text{GrAut } S$ be a finite subgroup such that char $k$ does not divide $|G|$ and let $e = \frac{1}{|G|} \sum_{g \in G} g$. Finally, assume that $S^G \cong eS#Ge$ is AS-Gorenstein and $S#G/(e)$ is finite-dimensional over $k$. Then there is a triangle equivalence

$$D^g_{S^e}(S^G) \cong D^b((1 - \tilde{e})(G#\nabla(S^e))(1 - \tilde{e})).$$

**Proof.** By [MU16b Lemma 2.21], the algebra $A := S#G$ is AS-regular, so $A$ and the idempotent $e$ satisfy the conditions of Setting [2.22]. Moreover, we have that $eA_0e = ekGe \cong k$. Also, $S$ is Koszul, so $\nabla A \cong (\nabla S)#G$ is Koszul.

Since $A$ itself is also Koszul, there is a graded isomorphism $A \cong T_{A \circ A}A/(R)$. In particular, the Gorenstein parameter of $A$ is $d$. Moreover, $A$ is Morita equivalent to a quiver algebra $\tilde{A} = T_{kQ_0}kQ_1/\langle \bar{R} \rangle$ and $\nabla A$ is Morita equivalent to $\nabla \tilde{A} = T_{kQ_0}kQ_1/\langle \bar{R} \rangle$. Suppose that $Q_0 =
So the conclusions of Theorems 2.26 and 4.1 are the same in this context.

Therefore, all the hypotheses of our main theorem are fulfilled.

Finally, by [MU16b, Proposition 2.14],

\[ (\nabla(S\#G))' \cong ((\nabla S)\#G)' \cong G\#(\nabla S)' \cong G\#(\nabla S), \]

so the conclusions of Theorems 2.26 and 4.1 are the same in this context. \(\square\)

We conclude with an example where the Gorenstein parameter is not equal to 1 nor to the global dimension.

**Example 4.4.** Let \( A = k[x_1, x_2, x_3, x_4] \# G \), where \( G = \frac{1}{4}(1, 1, 3, 3) < SL(4, k) \). The following quivers are described in Definition 2.24. The quiver of \( A \) is given by:

![Quiver diagram](image)

with relations \( x_i x_j - x_j x_i = 0 \). We give \( A \) a grading by putting the thick arrows in degree 1 and the other arrows in degree 0. In this case, \( A \) is a 4-AS-regular algebra of Gorenstein parameter 2. It is important to mention that \( A \) is not Koszul with respect to the grading that endows it with the structure of an AS-regular algebra, so it does not fit in the setting of [MU16b].

Let \( e \) be the idempotent corresponding to vertex 0. Then \( eAe \cong S^G \) is an isolated singularity, so \( A/AeA \) is finite-dimensional. Moreover, since \( G < SL(4, k) \), \( S^G \) is AS-Gorenstein. The quiver of \( \nabla A \) is given by:

![Quiver diagram](image)

and the relations are induced from the relations in \( A \). The induced idempotent \( \tilde{e} \) is the one corresponding to the vertices in the boxes. This is a Koszul levelled algebra, so by Theorem 4.1.
there is a triangle equivalence
\[ \text{D}^\text{gr}_{Sg}(S^G) \cong \text{D}^b((1 - \bar{e})(\nabla A)^1(1 - \bar{e})). \]

The quiver of \((1 - \bar{e})(\nabla A)^1(1 - \bar{e})\) is given by

\[
\begin{array}{cccc}
2 & 0 & 1 & 0 \\
\downarrow & & & \downarrow \\
\downarrow & x_1 & x_2 & x_3 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

and the relations are given by \(x_i x_j + x_j x_i = 0\).

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