First order phase transitions and integrable field theory.
The dilute $q$-state Potts model

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Abstract

We consider the two-dimensional dilute $q$-state Potts model on its first order phase transition surface for $0 < q \leq 4$. After determining the exact scattering theory which describes the scaling limit, we compute the two-kink form factors of the dilution, thermal and spin operators. They provide an approximation for the correlation functions whose accuracy is illustrated by evaluating the central charge and the scaling dimensions along the tricritical line.
1 Introduction

Two-dimensional quantum field theory allows for a large class of non-scale-invariant models which exhibit an infinite number of integrals of motion [1]. These models are integrable, in the sense that the $S$-matrix which determine their on-shell physics in $1 + 1$ dimensions can be computed exactly [2]. The solvability of the models extends off-shell through the possibility of exact determination of the matrix elements of local operators on the asymptotic states [3, 4], which in turn leads to the evaluation of correlation functions in the form of spectral series.

A natural domain of application for quantum field theory is provided by the scaling limit of classical statistical mechanics. It is well known that in two dimensions there exists a number of lattice models which are exactly solvable due to the presence of a family of commuting transfer matrices [5]. The scaling limit of these models is described by an integrable field theory. What is more interesting is that the opposite appears to be not true, a circumstance which attests of an inequivalence of the notions of integrability on the lattice and in the continuous limit. In fact, there are integrable quantum field theories which describe the scaling limit of statistical models which are not solved on the lattice. For these models, integrability seems to emerge as a property of the scaling limit, after that the non-universal, lattice dependent features have been eliminated. Among the examples of this kind which have been studied in more detail, we mention the Ising model in a magnetic field at $T = T_c$ [1, 3, 7], the off-critical $q$-state Potts model with $0 < q \leq 4$ [8, 9], and the off-critical Ashkin-Teller model [10].

In this paper we are interested in the generalisation of the $q$-state Potts model which allows for the presence of annealed vacancies on the lattice [11]. More precisely, we will consider the model on its first order transition surface for $q \leq 4$. The model is not solvable on the lattice but is integrable in the scaling limit because it corresponds to the perturbation of the conformal field theories with $C \leq 1$ by the primary operator $\varphi_{1,3}$ [13]. The $\varphi_{1,3}$-perturbed conformal field theories have been the object of extensive study. In particular, they are known to describe the scaling limit of the RSOS models [14, 15] and the $O(n)$ model for $-2 < n < 2$ [12, 13]. The exact scattering descriptions corresponding to the massive phases of these models were found in Refs. [16, 17] and [18], respectively. Here we will directly determine the $S$-matrix which is suitable for the Potts model interpretation by requiring that it describes a system in which a disordered phase coexists with $q$ ordered phases which are exchanged by permutation symmetry. The cases $q = 2, 3$ were already discussed in Ref. [13].
Working within the particle basis which makes explicit the relevant symmetries of the model is important for dealing with the problem of the determination of the matrix elements of local operators on the asymptotic particle states (form factors). The knowledge of the form factors determines the correlation functions in the form of spectral series. These series, however, are non-trivial mathematical objects for which no resummation technique is presently available. For this reason, it becomes important to be able to judge of the level of accuracy yielded by partial sums. A particularly interesting test is provided by the sum rules which express the ultraviolet data (central charge and scaling dimensions) as integrals of correlation functions computed in the off-critical theory [20, 21]. The general response is that the rate of convergence of the spectral series is remarkably fast. The results of this paper provide, in particular, an illustration of this fact. Of course, the agreement with the predictions of conformal field theory for the ultraviolet limit also represents a non-trivial confirmation of the correctness of the proposed scattering theory.

The layout of the paper is the following. In the next section we recall some basic features of the dilute Potts model before turning to the determination of the S-matrix in section 3. The exact scattering description is then used in section 4 to compute the form factors of the dilution, thermal and spin operators. Section 5 deals with the correlation functions of these operators and the sum rules for the central charge and the scaling dimensions; correlation lengths and interfacial tensions are also discussed. Few final remarks are collected in section 6.

2 The dilute Potts model

The dilute q-state Potts model is defined by the lattice Hamiltonian [11]

$$H = -J \sum_{\langle x, y \rangle} t(x)t(y)\delta_{s(x),s(y)} + \Delta \sum_x t(x). \tag{2.1}$$

Here $s(x)$ is the ordinary Potts spin variable which can assume $q$ different values (colours); the system is clearly invariant under permutations of the colours. $t(x)$ is a lattice gas variable which equals 0 if the site $x$ is empty or 1 if it is occupied. The coupling $J > 0$ (ferromagnetic case) plays the role of the inverse temperature, while $\Delta$ is a chemical potential controlling the vacancy density. The pure Potts model is recovered when $\Delta = -\infty$.

When writing the Hamiltonian (2.1) it is natural to think of the number of colours $q$ as being an integer. However, it is well known that it makes sense and is extremely
useful to consider the analytically continued version of the Potts model in which $q$ is a real parameter. In the pure model, the “monochromatic” limit $q \to 1$ describes isotropic percolation \[22\]. The dilute system simply reduces to an Ising model at $q = 1$.

The phase diagram corresponding to the Hamiltonian \[2.1\] in two dimensions \[23\] is shown in Fig. 1. The system undergoes a ferromagnetic phase transition at a critical value of the temperature $T_c$. It is desordered at $T > T_c$, and exhibits spontaneous magnetisation in the low-temperature phase in which $q$ vacua (one for each colour) are degenerate. In the pure model ($e^\Delta = 0$), the phase transition is second order for $q \leq 4$ and first order for $q > 4$ \[3\]. When the vacancy density becomes non-zero, the transition remains continuous at $q < 4$ up to a critical value $\Delta_c$ of the dilution, and becomes first order for $\Delta > \Delta_c$.

The disordered phase coexists with the $q$ ordered phases along the first order transition lines (dashed in Fig. 1). The line $T = T_c$, $\Delta = \Delta_c$ (upper thick segment in the phase diagram) is a line of tricritical fixed points. Since the transition is always discontinuous for $q > 4$, the tricritical line has to meet the line of fixed points of the pure model (lower thick segment) at $q = 4$.

Let us consider the dilute $q$-state Potts model in the framework of conformal field theory and its perturbations. Both the critical and tricritical line of fixed point are described by conformal field theory \[13\]. Here we are interested in the tricritical one, since it describes the ultraviolet limit of the first order phase transition lines we want to study. It corresponds to the conformal theory with central charge

$$C = 1 - \frac{6}{p(p+1)}, \quad p > 2$$

with $p$ related to $q$ as

$$\sqrt{q} = 2 \cos \frac{\pi}{p}.$$  \[2.3\]

In the continuous field theoretic description, the dilution, thermal and spin variables are described by operators which we denote by $\psi(x)$, $\varepsilon(x)$ and $\sigma_i(x) \ (i = 1, 2, \ldots, q)$, respectively. They correspond to the operators $\varphi_{1,3}$, $\varphi_{1,2}$ and $\varphi_{1/2,2}$ in the conformal field theory classification, and their scaling dimensions along the tricritical line are

$$X_{\psi} = 2 \frac{p - 1}{p + 1},$$  \[2.4\]

$$X_{\varepsilon} = \frac{p - 2}{2(p + 1)},$$  \[2.5\]

$$X_{\sigma} = \frac{p^2 - 4}{8p(p + 1)}.$$  \[2.6\]

The renormalisation group trajectories flowing out of the tricritical line at $T = T_c$ (vertical lines in Fig. 1) are described by the perturbation of the tricritical line by the dilution
operator $\psi(x)$, which is relevant ($X_\psi < 2$) for $q < 4$. The action associated to such trajectories reads

$$A = A_{\text{tricr}} + g \int d^2 x \, \psi(x),$$

(2.7)

where $A_{\text{tricr}}$ denotes the action of the tricritical line and $g \sim \Delta - \Delta_c$ is the coupling measuring the deviation from the critical dilution. Depending on the sign of $g$, the action above describes either the massless flows to the ordinary critical point (at least for $2 \leq q < 4$), or the massive trajectories associated to the first order transition. Both types of trajectories are known to be integrable, as a consequence of the integrability of the $\varphi_{1,3}$ perturbations of conformal field theory [1].

In this paper, we deal with the first order trajectories along which the elementary excitations have a mass $m$ (inverse correlation length) related to the coupling $g$ as

$$m \sim g^{1 \over 2 - X_\psi}, \quad q < 4.$$

(2.8)

As $q$ approaches 4 from below, the perturbation becomes marginally relevant, the correlation length develops an essential singularity as $g \to 0^+$, and the previous relation is replaced by [24]

$$m \sim e^{-A/g}, \quad q = 4$$

(2.9)

where $A$ is a positive constant. Notice that for $q = 1$ one has $C = 1/2$, $X_\psi = 1$ and $X_\varepsilon = 1/8$, so that the action (2.7) describes an Ising model in zero magnetic field.

### 3 Scattering theory

In this section we determine the exact scattering theory describing the first order transition lines of the dilute $q$-state Potts model (vertical dashed lines in Fig. 1). In 1+1 dimensions one knows that the elementary excitations of a theory exhibiting degenerate ferromagnetic vacua are kinks interpolating between adjacent vacua. Kink excitations differ from the ordinary particle excitations for the fact that, in general, the composition of multi-kink states is subject to some restrictions. Both the number of elementary kinks and the type of restrictions can be argued from an analysis of the vacuum structure.

Consider the dilute Potts model at $T \leq T_c$. There are $q$ ordered ground states that we denote $\Omega_i$, $i$ being the colour index; they are sent into one another by the permutation group under which the model is invariant. Intuitively, one can think of them as being located at the $q$ vertices of a hypertetrahedron living in the $(q - 1)$-dimensional space of the independent spin components. The disordered vacuum (we call it $\Omega_0$) is located at the
center of the hypertetrahedron. At $T < T_c$ the energy of the disordered vacuum is higher than that of the ordered vacua and the elementary excitations of the scattering theory are kinks $\tilde{K}_{ij}(\theta)$ interpolating among the ordered vacua. They can be associated to the edges of the hypertetrahedron. This scattering theory and its off-shell consequences were studied in Refs. [8, 9] for the case of the pure Potts model. On the first order transition surface ($T = T_c$, $\Delta > \Delta_c$) the disordered vacuum becomes degenerate with the ordered vacua and the $q$ + 1 phases coexist. In this situation the elementary excitations are the kinks $K_{0i}(\theta)$ and $K_{i0}(\theta)$ interpolating among the center of the hypertetrahedron and its vertices, while the kinks we had at $T < T_c$ become composite excitations $\tilde{K}_{ij} \sim K_{i0}K_{0j}$.

Let us fix our attention on the transition surface and consider the space of states constructed on the elementary excitations $K_{0i}, K_{i0}$. The interpretation of kinks as excitations interpolating among adjacent vacua requires that, in a multi-kink state, adjacent vacuum indices belonging to different kinks coincide. In our case, this means that the only allowed kink sequences are of the type

$$\ldots K_{0i}(\theta_1)K_{i0}(\theta_2)K_{0j}(\theta_3)K_{j0}(\theta_4)\ldots .$$ (3.1)

The integrability of the theory ensures that the scattering of these multi-kink states is completely elastic and factorised into the product of elementary two-body amplitudes [2]. Taking into account the vacuum structure described above and the permutation symmetry of the ordered vacua, one immediately realises that the scattering theory has only the four two-kink amplitudes represented in Fig. 2. They can be associated to the commutation relations

$$K_{0i}(\theta_1)K_{i0}(\theta_2) = A_0(\theta_1 - \theta_2)K_{0i}(\theta_2)K_{i0}(\theta_1) + A_1(\theta_1 - \theta_2)\sum_{j \neq i} K_{0j}(\theta_2)K_{j0}(\theta_1),$$ (3.2)

$$K_{i0}(\theta_1)K_{0j}(\theta_2) = \delta_{ij}B_0(\theta_1 - \theta_2)K_{i0}(\theta_2)K_{0i}(\theta_1) + (1 - \delta_{ij})B_1(\theta_1 - \theta_2)K_{i0}(\theta_2)K_{0j}(\theta_1).$$ (3.3)

The amplitudes are related in pairs by crossing symmetry

$$A_0(\theta) = B_0(i\pi - \theta),$$ (3.4)

$$A_1(\theta) = B_1(i\pi - \theta).$$

They also have to satisfy the unitarity conditions which can be formally obtained by commuting once again the r.h.s. of Eqs. (3.2), (3.3)

$$A_0(\theta)A_0(-\theta) + (q - 1)A_1(\theta)A_1(-\theta) = 1,$$ (3.5)

The rapidity $\theta$ parameterises the on-shell energy and momentum of the kinks of mass $m$ as $(p^0, p^1) = (mcosh \theta, m sinh \theta)$.

In principle, $K_{i0}K_{0i}$ could also form bound states corresponding to $\tilde{K}_{ij}$, but we will see that this is not the case.
\begin{align}
A_0(\theta)A_1(-\theta) + (q-2)A_1(\theta)A_1(-\theta) + A_1(\theta)A_0(-\theta) &= 0, \\
B_0(\theta)B_0(-\theta) &= B_1(\theta)B_1(-\theta) = 1.
\end{align}

Finally, the so-called factorisation (or Yang-Baxter) equations are obtained by considering a three-kink initial state and requiring the equality of the final states obtained performing the pair commutations (3.2), (3.3) in the two possible orderings

\begin{align}
A_0B_0A_0 + (q-1)A_1B_1A_1 &= B_0A_0B_0, \\
A_0B_0A_1 + A_1B_1A_0 + (q-2)A_1B_1A_1 &= B_1A_1B_0, \\
A_0B_1A_0 + A_1B_0A_1 + (q-2)A_1B_1A_1 &= B_1A_0B_1, \\
A_0B_1A_1 + A_1B_0A_1 + A_1B_1A_0 + (q-3)A_1B_1A_1 &= B_1A_1B_1;
\end{align}

here the arguments of the three factors in each product are \( \theta, \theta + \theta' \) and \( \theta' \), respectively.

A solution to Eqs. (3.4)–(3.11) can be found along the following lines. It is easily seen that they fix \( A_1(0) = 0, A_0(0) = B_0(0) = B_1(0) = \pm 1 \). Then, remembering also crossing and unitarity, we know that \( R \equiv B_1/B_0 \) satisfies \( R(\theta)R(-\theta) = 1, R(0) = 1 \) and \( R(i\pi) = 0 \). The analysis of the cases \( q = 2, 3 \) in Ref. [19] suggests the ansatz \( R(\theta) = \sinh \lambda(i\pi - \theta)/\sinh \lambda(i\pi + \theta) \). Then Eq. (3.4) immediately fixes \( 2 \cos \pi \lambda = \sqrt{q} \) or, in view of (2.3), \( \lambda = 1/p \). Notice that the restriction \( q \leq 4 \) automatically emerges from the consistency requirements of the scattering theory. This is expected since the scaling limit of the lattice model can no longer be defined for \( q > 4 \).

Having determined the ratio \( A_1(\theta)/A_0(\theta) = R(i\pi - \theta) \), we can reduce Eq. (3.5) to the form

\[ A_1(\theta)A_1(-\theta) = -\frac{1}{4 \cos^2 \frac{\pi}{p}} \frac{\sinh^2 \frac{\theta}{p} \sinh \frac{1}{p}(i\pi + \theta) \sinh \frac{1}{p}(i\pi - \theta)}{\sinh \frac{1}{p}(2i\pi - \theta)} . \]

Solving this equation together with \( A_1(i\pi - \theta)A_1(i\pi + \theta) = 1 \) leads to the final result

\begin{align}
A_0(\theta) &= \frac{e^{-i\gamma \theta}}{\sqrt{q}} \frac{\sinh \frac{1}{p}(2i\pi - \theta)}{\sinh \frac{1}{p}(i\pi - \theta)} S_0(\theta), \\
A_1(\theta) &= \frac{e^{-i\gamma \theta}}{\sqrt{q}} \frac{\sinh \frac{1}{p} \theta}{\sinh \frac{1}{p}(i\pi - \theta)} S_0(\theta), \\
B_0(\theta) &= \frac{e^{i\gamma \theta}}{\sqrt{q}} \frac{\sinh \frac{1}{p}(i\pi + \theta)}{\sinh \frac{1}{p}(i\pi - \theta)} S_0(\theta), \\
B_1(\theta) &= e^{i\gamma \theta} S_0(\theta),
\end{align}

with

\[ \gamma = \frac{1}{2\pi} \ln q, \]
\[ S_0(\theta) = -\prod_{n=0}^{\infty} \frac{\Gamma \left( 1 + \frac{2}{p}(n + \frac{1}{2}) + \frac{\theta}{\pi p} \right) \Gamma \left( 1 + \frac{2}{p}n - \frac{\theta}{\pi p} \right)}{\Gamma \left( 1 + \frac{2}{p}(n + \frac{1}{2}) - \frac{\theta}{\pi p} \right) \Gamma \left( 1 + \frac{2}{p}n + \frac{\theta}{\pi p} \right)} \times \frac{\Gamma \left( \frac{2}{p}(n + 1) - \frac{\theta}{\pi p} \right) \Gamma \left( \frac{2}{p}(n + \frac{1}{2}) + \frac{\theta}{\pi p} \right)}{\Gamma \left( \frac{2}{p}(n + 1) + \frac{\theta}{\pi p} \right) \Gamma \left( \frac{2}{p}(n + \frac{1}{2}) - \frac{\theta}{\pi p} \right)} \]
\[ = -\exp \left\{ i \int_0^{\infty} dx \frac{\sinh(p - 1)\frac{x}{2}}{x} \frac{\sin \frac{px}{2}}{\cosh \frac{x}{2} \sin \frac{\theta}{\pi}} \right\} . \]  

(3.18)

The amplitudes above satisfy the factorisation equations (3.8)–(3.11). They are free of poles in the physical strip \( \text{Im} \theta \in (0, \pi) \) and then the theory does not possess bound states. It can be checked that at \( q = 1 \) the amplitudes \( A_0 \) and \( B_0 \) reduce to \(-1\), as expected for the thermal Ising model\(^3\). At \( q = 4 \) the amplitudes (3.13)–(3.16) become linear combinations of those of Sine-Gordon solitons\(^4\) at the point \( \beta = 8\pi \) where the term \( \cos \beta \varphi \) which perturbs the gaussian fixed point becomes marginally relevant. This scattering theory is known to imply the relation (2.9) among the mass and the coupling constant \( \beta \). These checks should be sufficient to get rid of the doubts concerning the usual CDD ambiguities\(^5\).

4 Form factors

The \( S \)-matrix is not an object of primary interest for statistical mechanics. The link with the off-shell physics is provided by the matrix elements of the local operators on the asymptotic particle states. They are known as form factors and are exactly computable in integrable quantum field theories \( \text{\cite{3, 4}} \). In this section we consider the matrix elements computed on two-kink states; they are relatively easy to determine and prove sufficient to provide accurate quantitative information on the correlation functions.

We are interested in the operators \( \Theta(x) \sim \psi(x), \varepsilon(x) \) and \( \sigma_j(x) \), \( \Theta(x) \) denoting the trace of the stress-energy tensor. They all couple to states with zero topological charge, i.e., excitations which begin and end on the same vacuum. Depending on whether the

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\(^3\)More precisely, we used this condition at \( q = 1 \) to fix the overall sign of the \( S \)-matrix which is left undetermined by the general equations.

\(^4\)For generic values of \( p \), the function \( S_0(\theta) \) appears in the Sine-Gordon \( S \)-matrix. This analogy reflects the well known circumstance that \( \varphi_{1,3} \)-perturbed conformal field theories are related to suitable restrictions of the Sine-Gordon model \cite{25, 17}.

\(^5\)For \( q = 2, 3 \), the amplitudes (3.13)–(3.16) do not completely coincide with those proposed in Ref. \cite{19}. It can be checked that the infinite products of gamma functions in \cite{19}, although formal solutions of all the equations, in fact converge to zero.
latter is the disordered vacuum $\Omega_0$ or one of the ordered vacua $\Omega_i$, we have the two-kink form factors (Fig. 3)

$$\langle \Omega_0 | \Phi(0) | K_{0i}(\theta_1) K_{i0}(\theta_2) \rangle \equiv F_{0i}^\Phi(\theta_1 - \theta_2),$$  \hspace{1cm} (4.1)$$

$$\langle \Omega_i | \Phi(0) | K_{i0}(\theta_1) K_{0i}(\theta_2) \rangle \equiv F_{i0}^\Phi(\theta_1 - \theta_2),$$  \hspace{1cm} (4.2)$$

where $\Phi$ generically denotes one of the above mentioned operators. More specifically, the operators $\Theta$ and $\varepsilon$ are invariant under permutations of the colours, and we can write

$$F_{0i}^\Phi(\theta) = F_{-}^\Phi(\theta), \quad \Phi = \Theta, \varepsilon$$  \hspace{1cm} (4.3)$$

$$F_{i0}^\Phi(\theta) = F_{+}^\Phi(\theta), \quad \Phi = \Theta, \varepsilon$$  \hspace{1cm} (4.4)$$

The spin operators $\sigma_j(x)$ are related to the lattice variables $s(x)$ as

$$\sigma_j(x) = \delta_{s(x),j} - \frac{1}{q}, \quad j = 1, 2, \ldots, q$$  \hspace{1cm} (4.5)$$

the constant being subtracted to ensure the vanishing of the order parameter $\langle \sigma_j \rangle$ in the disordered phase. When considering the matrix elements (4.1), (4.2) with $\Phi = \sigma_j$, we only need to distinguish whether the colour indices $i$ and $j$ are equal or different. Since $\sum_{j=1}^{q} \sigma_j = 0$, we can write

$$F_{0i}^{\sigma_j}(\theta) = \frac{q\delta_{ij} - 1}{q - 1} F_{-}^{\sigma}(\theta),$$  \hspace{1cm} (4.6)$$

$$F_{i0}^{\sigma_j}(\theta) = \frac{q\delta_{ij} - 1}{q - 1} F_{+}^{\sigma}(\theta).$$  \hspace{1cm} (4.7)$$

The basic properties of form factors on kink states were discussed in Ref. [9]. The relations (3.2), (3.3) immediately lead to the equations

$$F_{0i}^{\Phi}(\theta) = A_0(\theta) F_{0i}^\Phi(-\theta) + A_1(\theta) \sum_{j \neq i} F_{0j}^\Phi(-\theta),$$  \hspace{1cm} (4.8)$$

$$F_{i0}^{\Phi}(\theta) = B_0(\theta) F_{i0}^\Phi(-\theta).$$  \hspace{1cm} (4.9)$$

The two matrix elements $F_{0j}^\Phi(\theta)$ and $F_{j0}^\Phi(\theta)$ are related by crossing

$$F_{0j}^\Phi(\theta + 2i\pi) = F_{j0}^\Phi(-\theta),$$  \hspace{1cm} (4.10)$$

and in general have a simple pole at $\theta = i\pi$ with residue

$$-i \text{Res}_{\theta=i\pi} F_{0j}^\Phi(\theta) = i \text{Res}_{\theta=i\pi} F_{j0}^\Phi(\theta) = \langle \Omega_0 | \Phi | \Omega_0 \rangle - \langle \Omega_j | \Phi | \Omega_j \rangle.$$

(4.11)
The vacuum expectation values of $\Theta(x)$ give the vacuum energy densities and then are identical for the $q + 1$ degenerate vacua. For the other two operators, colour symmetry leads to

$$
\langle \Omega_0 | \varepsilon | \Omega_0 \rangle = U_0, \quad \langle \Omega_i | \varepsilon | \Omega_i \rangle = U_1, \quad (4.12)
$$

$$
\langle \Omega_0 | \sigma_j | \Omega_0 \rangle = 0, \quad \langle \Omega_i | \sigma_j | \Omega_i \rangle = \frac{q \delta_{ij} - 1}{q - 1} M. \quad (4.13)
$$

Here, $U_0 - U_1$ gives the discontinuity in the internal energy across the first order transition surface, and $M$ is the spontaneous magnetisation in the ordered phases. As a last necessary condition, the two-kink form factors are subject to the asymptotic bound

$$
\lim_{\theta \to +\infty} F_{0i}^{\Phi}(\theta) \leq \text{constant} \ e^{X_{\Phi}/2}, \quad (4.14)
$$

with an analogous relation for $F_{i0}^{\Phi}(\theta)$.

The above requirements uniquely determine the solutions

$$
F_{\pm}^{\varepsilon}(\theta) = -\frac{4i\pi}{p} \frac{m^2 e^{\pm i\pi(\pi + i\theta)}}{\sinh \frac{1}{p}(\theta - i\pi)} F_0(\theta), \quad (4.15)
$$

$$
F_{\pm}^{\sigma}(\theta) = \pm i(U_1 - U_0) \frac{e^{\pm \frac{\pi}{2}(\pi + i\theta)}}{p \sinh \frac{1}{p}(\theta - i\pi)} F_0(\theta), \quad (4.16)
$$

$$
F_{\pm}^{\Phi}(\theta) = \pm \frac{M}{2\Upsilon_{\pm}(i\pi)} \frac{e^{\pm \frac{\pi}{2}(\pi + i\theta)}}{\cosh \frac{\theta}{2}} \Upsilon_{\pm}(\theta) F_0(\theta), \quad (4.17)
$$

with

$$
F_0(\theta) = -i \sinh \frac{\theta}{2} \exp \left\{ \int_0^\infty \frac{dx}{x} \sinh (1 - p) \frac{x}{2} \sin^2 (i\pi - \theta) \right\}, \quad (4.18)
$$

$$
\Upsilon_{+}(\theta) = \exp \left\{ 2 \int_0^\infty \frac{dx}{x} \sinh \left( \frac{x}{2} - 1 \right) \sin^2 \left( 2i\pi - \theta \right) \right\}, \quad (4.19)
$$

$$
\Upsilon_{-}(\theta) = \Upsilon_{+}(\theta + 2i\pi). \quad (4.20)
$$

The functions $F_0(\theta)$ and $\Upsilon_{+}(\theta)$ are solutions of the equations

$$
F_0(\theta) = S_0(\theta) F_0(-\theta), \quad F_0(\theta + 2i\pi) = F_0(-\theta), \quad (4.21)
$$

$$
\Upsilon_{+}(\theta) = \frac{\sinh \frac{1}{p}(i\pi + \theta)}{\sinh \frac{1}{p}(i\pi - \theta)} \Upsilon_{+}(-\theta), \quad \Upsilon_{+}(\theta + 4i\pi) = \Upsilon_{+}(-\theta), \quad (4.22)
$$

and behave as

$$
F_0(\theta) \sim \exp \left[ \left( 1 + \frac{1}{p} \right) \theta \frac{1}{4} \right], \quad (4.23)
$$

$$
\Upsilon_{+}(\theta) \sim \exp \left[ \left( 1 - \frac{2}{p} \right) \theta \frac{1}{4} \right], \quad (4.24)
$$

when $\theta \to +\infty$.

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6 The form factor of $\Theta(x)$ is normalised by the condition $F_{0i}^{\Phi}(i\pi) = 2\pi m^2$, $m$ being the mass of the kinks.
5 Correlation functions

In the S-matrix approach, correlation functions are expressed as spectral series over complete sets of intermediate asymptotic states. The two-kink form factors we determined above are what is needed to compute the first non-trivial term of these series for the operators $\Theta$, $\varepsilon$ and $\sigma_j$. Depending on the phase in which correlations are computed, for the two-point case we have

$$\langle \Omega_0|\Phi_1(x)\Phi_2(0)|\Omega_0 \rangle_c = \sum_{i=1}^{q} \int_{\theta_1>\theta_2} \frac{d\theta_1 d\theta_2}{2\pi^2} F_{0i}^\Phi \left( \theta_1 - \theta_2 \right) F_{0i}^\Phi \left( \theta_2 - \theta_1 \right) e^{-|x|E_2} + O(e^{-4m|x|}) \quad (5.1)$$

$$\langle \Omega_i|\Phi_1(x)\Phi_2(0)|\Omega_i \rangle_c = \int_{\theta_1>\theta_2} \frac{d\theta_1 d\theta_2}{2\pi^2} F_{\alpha i}^\Phi \left( \theta_1 - \theta_2 \right) F_{\alpha i}^\Phi \left( \theta_2 - \theta_1 \right) e^{-|x|E_2} + O(e^{-4m|x|}) \quad (5.2)$$

where $\langle \cdots \rangle_c$ denotes connected correlators and $E_2 = m(\cosh \theta_1 + \cosh \theta_2)$ is the energy of the two-kink asymptotic state. The spectral series over form factors are known to converge quite rapidly (see \cite{20, 21} and references therein). A very effective way to test this property is to use the exact sum rules \cite{20, 21}

$$C = \frac{3}{4\pi} \int d^2x \, |x|^2 \langle \Omega_\alpha|\Theta(x)\Theta(0)|\Omega_\alpha \rangle_c ,$$

$$X_\Phi = -\frac{1}{2\pi \langle \Omega_\alpha|\Phi|\Omega_\alpha \rangle} \int d^2x \, \langle \Omega_\alpha|\Theta(x)\Phi(0)|\Omega_\alpha \rangle_c ,$$

(\(\alpha = 0, 1, \ldots, q\)) to recover the conformal data (central charge and scaling dimensions) from the off-critical theory. The results obtained for $C$, $X_\varepsilon$ and $X_\sigma$ by plugging into these sum rules the two-kink approximations (5.1), (5.2) for the correlators are shown in Figs. 4, 5, 6 and compared with the exact formulas; the numerical values corresponding to integer $q$ are given in the Table. The growth of the scaling dimensions of the considered operators with increasing $q$ leads to more severe ultraviolet singularities of the exact correlators which account for the decreasing accuracy of the two-kink approximation (which of course cannot reproduce the singularities).

At $q = 1$, the two-kink computation gives exact results for $C$ and $X_\varepsilon$ but not for $X_\sigma$. This can be understood as follows. With an obvious simplification of notation, the $2(n+1)$-kink contribution to the correlator $\langle \Omega_i|\Theta(x)\Phi(0)|\Omega_i \rangle$ can be written as

$$\sum_{i_1, \ldots, i_n=1}^{q} \langle \Omega_i|\Theta|K_{0i_1}K_{0i_1'}K_{i_10} \ldots K_{i_n0}K_{i_n0'}|\Phi|\Omega_i \rangle ,$$

or, for our present purpose, as

$$\langle \Omega_i|\Theta|K_{0i_1}K_{0i_1'} \ldots K_{i_00}K_{i_00'}|\Phi|\Omega_i \rangle + \sum_{i_1, \ldots, i_n} \langle \Omega_i|\Theta|K_{0i_1} \ldots K_{i_n0}K_{i_00'}|\Phi|\Omega_i \rangle ,$$

(5.6)
with the prime denoting the omission in the sum of the term with \( i_1 = i_2 = \ldots = i_n = i \).

The first term in (5.6) involves only two vacua and at \( q = 1 \) is computed on the physical excitations of the thermal Ising model, which behave as free fermions (scattering amplitude equal \(-1\)). As a consequence, the \( 2(n+1) \)-kink form factors \( \langle \Omega_i | \Theta | K_{i_0} K_{0i} \ldots K_{i_0} K_{0i} \rangle \) vanish at \( q = 1 \) for any \( n > 0 \). Consider now the primed sum in (5.6). It involves \( q^n - 1 \) terms which, if \( \Phi \) is invariant under colour permutations, are all identical and finite. Therefore, we conclude that, at \( q = 1 \), (5.5) is identically zero for any \( n > 0 \) whenever \( \Phi \) is a colour singlet operator (e.g. \( \Phi = \Theta, \epsilon \)). The situation is more subtle when \( \Phi = \sigma_i \). In fact, although the number of terms in the primed sum vanishes as \( q \to 1 \), now the sum contains contributions which diverge in the same limit\(^7\). In this way, the correlator \( \langle \Omega_i | \Theta(x) \sigma_i(0) | \Omega_i \rangle_{q=1} \) receives a finite contribution even from excitations which are unphysical at \( q = 1 \). This is not that surprising if one considers that the operator \( \sigma_i \) itself is unphysical at \( q = 1 \) (there are no Potts spin degrees of freedom) and only makes sense as an analytic continuation.

We conclude with two remarks concerning correlation lengths and interfacial tensions. The “true” correlation lengths \( \xi_0, \xi_i \) in the disordered and ordered phases are defined through the large distance decay of the spin-spin correlation functions

\[
\langle \Omega_\alpha | \sigma_i(x) \sigma_i(0) | \Omega_\alpha \rangle_c \sim e^{-|x|/\xi_\alpha}, \quad \alpha = 0, i.
\]

Since this asymptotic behaviour is given by Eqs. (5.1), (5.2), one immediately concludes that

\[
\xi_0 = \xi_i = \frac{1}{2m}.
\]

In two dimensions the interfacial tension between two coexisting phases coincides with the mass of the excitation which interpolates between them. Then, denoting \( \sigma_{i0} \) the ordered-disordered tension and \( \sigma_{ij} \) (\( i \neq j \)) the ordered-ordered tension, we have

\[
\sigma_{i0} = \frac{\sigma_{ij}}{2} = m.
\]

6 Conclusion

In this paper we applied the \( S \)-matrix program to the two-dimensional dilute \( q \)-state Potts model on its first order phase transition surface for \( q \leq 4 \). We have shown how integrability allows an exact scattering description of the scaling limit and how this leads
to the computation of correlation functions for the interesting operators through the form factor approach. The sum rules for the central charge and the scaling dimensions have been used to show the remarkable accuracy provided by the two-kink approximation for the correlators. Of course, in the same approximation, the correlation functions computed here can be used to evaluate other universal quantities which are characteristic of the off-critical model and are not known from conformal field theory, for example the universal combinations of critical amplitudes. For the pure model, this was done in Ref. [9].

We have shown how the results (5.8) and (5.9) for the correlation lengths in the different coexisting phases and the interfacial tensions are simple consequences of the basic structure of the scattering theory associated to the statistical model. Although these results were obtained for $q \leq 4$, there are reasons to believe that they hold true for $q > 4$. As a matter of fact, in the pure system at $q > 4$, Eq. (5.3) is known to hold [27], and Eq. (5.8) received strong support form Monte Carlo simulations [28].

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| $q$ | 1   | 2     | 3     | 4     |
|-----|-----|-------|-------|-------|
| $C$ | 1/2 | 7/10  | 6/7   | 1     |
|     | 1/2 | 0.699 | 0.853 | 0.987 |
| $X_\epsilon$ | 1/8 | 1/5   | 2/7   | 1/2   |
|      | 1/8 | 0.202 | 0.295 | 0.560 |
| $X_\sigma$ | 5/96 | 3/40  | 2/21  | 1/8   |
|      | 0.0514 | 0.0734 | 0.0919 | 0.101 |

**Table.** Central charge and scaling dimensions of the thermal and spin operators in the tricritical $q$-state Potts model. The results of the two-kink approximation are shown below the exact values.
Figure 1. Phase diagram of the dilute $q$-state Potts model in the space of temperature, vacancy density and number of states.
Figure 2. The two-kink scattering amplitudes $A_0, A_1, B_0, B_1 (i \neq j)$.

Figure 3. The two-kink form factors $F_{00}^\Phi$ and $F_{i0}^\Phi$ of a colour singlet operator $\Phi$. 
Figure 4. Central charge of the tricritical $q$-state Potts model. Exact formula $^{[2,3]}$ (continuous line) and two-kink approximation (dots).
Figure 5. Scaling dimension of the thermal operator in the tricritical $q$-state Potts model. Exact formula (2.3) (continuous line) and two-kink approximation (dots).
Figure 6. Scaling dimension of the spin operators in the tricritical $q$-state Potts model. Exact formula (2.6) (continuous line) and two-kink approximation (dots).