Noncommutative perturbation in superspace

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Abstract

We consider noncommutative $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory. Using the $\mathcal{N} = 1$ superfield formalism and the background field method we compute one-loop four point contributions to the effective action and compare the result with the field theory limit from open string amplitudes in the presence of a constant $B$-field.

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String theory dynamics in the presence of a nonzero $B$-field leads to the appearance of noncommutative geometry, giving rise in the field theory limit to noncommutative gauge theories [1].

Field theories on a noncommutative spacetime can be formulated using standard field theoretical methods, trading the noncommutative property of the geometry with a deformation of the multiplication rule between the fields. In practice one simply replaces the ordinary product between the fields with the $\ast$-product, which absorbs and contains the non local nature of the noncommutative theory. In this framework one can proceed and quantize the theory: the Feynman rules are modified by the appearance of exponential factors at the vertices, but otherwise standard perturbation theory can be applied. Indeed a lot of progress has been made in this direction [2, 3].

Supersymmetric versions of noncommutative field theories have also been considered and in particular their formulation in superspace has been presented [4, 5]. In this letter we address the issue of quantization and perturbation in a superfield-superspace approach. The supersymmetric noncommutative theory is defined on standard superspace, while superfields are multiplied via the $\ast$-product. The $\ast$-operation does not touch the fermionic coordinates and simply introduces derivatives of superfields which are themselves superfields again. Therefore the quantization is performed in a standard manner. The only modifications are in the interaction terms which contain exponential factors from the $\ast$-product. One constructs supergraphs and performs the $D$-algebra in the loops with no new rules as compared to the commutative case. Once this is done, one is left with momentum integrals which are of the same kind as for bosonic noncommutative theories.

As an illustration of the general procedure outlined above, we study the quantization of noncommutative $N = 4$ supersymmetric $U(N)$ Yang-Mills theory in a $\mathcal{N} = 1$ superfield setting. We compute one-loop four point contributions to the effective action using the background field method. The result we obtain is in perfect agreement with the field theory limit from open string amplitudes in the presence of a constant $B$-field [6].

Using the definition of the $\ast$-product for superfields

$$(\phi_1 \ast \phi_2)(x, \theta, \bar{\theta}) \equiv e^{i\Theta_{\mu\nu}\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} \phi_1(x, \theta, \bar{\theta})\phi_2(y, \theta, \bar{\theta})\big|_{y=x}$$

the classical action for the noncommutative $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, in terms of $\mathcal{N} = 1$ superfields, can be written as (we use the notations and conventions adopted in [4])

$$S = \frac{1}{g^2} \text{Tr} \left( \int d^4x \ d^4\theta \ e^{-V} \Phi^i e^V \Phi^i + \frac{1}{2} \int d^4x \ d^2\theta \ W^2 + \frac{1}{2} \int d^4x \ d^2\bar{\theta} \ W^2 \\
+ \frac{1}{3!} \int d^4x \ d^2\theta \ i\epsilon^{ijk} \Phi^i[\Phi^j, \Phi^k] + \frac{1}{3!} \int d^4x \ d^2\bar{\theta} \ i\epsilon^{ijk} \bar{\Phi}^i[\bar{\Phi}^j, \bar{\Phi}^k] \right)_{\ast}$$

where the symbol $|_{\ast}$ implies that the multiplication of superfields is performed as defined in [4]. In (2) the $\Phi^i$ with $i = 1, 2, 3$ denote three chiral superfields, while $W^\alpha = i\tilde{D}^2(e^{-V} D^\alpha e^V)$ is the gauge superfield strength. All the fields are Lie-algebra valued, e.g.
$\Phi^i = \Phi_a T_a$, and $T_a$ are $U(N)$ matrices. The $*$-product in (1) not only maintains explicit supersymmetry; it also preserves gauge invariance. In fact it is easy to show that the action in (2) is invariant under nonlinear gauge transformations, which are just the obvious generalization to the noncommutative case of the standard ones [4, 7].

$$e^V \rightarrow e^{i\bar{\Lambda}} * e^V * e^{-i\Lambda}$$

$$\Phi \rightarrow e^{i\bar{\Lambda}} * \Phi * e^{-i\Lambda} \quad \bar{\Phi} \rightarrow e^{i\bar{\Lambda}} * \bar{\Phi} * e^{-i\bar{\Lambda}}$$

with a gauge parameter $\Lambda$ which is a chiral superfield.

Now we want to study one-loop corrections to the effective action. We perform the quantization directly in superspace and take advantage of the background field method which greatly simplifies the calculations. For ordinary commutative theory it has the additional, main property of keeping explicit the gauge invariance of the result, at every stage of the perturbative computation. For the noncommutative theory defined in (2) we will find that the one-loop effective action is still expressible in terms of field strengths, as the background field method guarantees, but the $*$-product is not maintained. This result confirms what expected from one-loop string theory [6, 8].

The background field quantization has been used efficiently in perturbative calculations for commutative SYM theory [9, 7, 10]. We briefly summarize the method and the results in order to extend them to the noncommutative case.

The quantum-background splitting can be most easily formulated in terms of covariant derivatives. To this end first one rewrites the gauge Lagrangian as

$$\text{Tr} \ W^2 = -\text{Tr} \left( \frac{1}{2} \left[ \nabla_\alpha, \left\{ \nabla_{\bar{\alpha}}, \nabla_\alpha \right\} \right] \right)^2$$

with

$$\nabla_\alpha = e^{-\frac{V}{2}} D_\alpha \ e^\frac{V}{2} \quad \nabla_{\bar{\alpha}} = e^{\frac{V}{2}} \bar{D}_{\bar{\alpha}} \ e^{-\frac{V}{2}}$$

Then one performs the splitting by rewriting them in terms of the quantum prepotential $V$ and background covariant derivatives

$$\nabla_\alpha \rightarrow e^{-\frac{V}{2}} \nabla_\alpha \ e^{\frac{V}{2}} \quad \nabla_{\bar{\alpha}} \rightarrow e^{\frac{V}{2}} \nabla_{\bar{\alpha}} \ e^{-\frac{V}{2}}$$

where now the covariant derivatives are expressed in terms of background connections, i.e.

$$\nabla_\alpha = D_\alpha - i\Gamma_\alpha \quad \nabla_{\bar{\alpha}} = \bar{D}_{\bar{\alpha}} - i\bar{\Gamma}_{\bar{\alpha}} \quad \nabla_\alpha = \partial_\alpha - i\Gamma_\alpha$$

The quantum gauge invariance is fixed through the introduction of background covariantly chiral gauge fixing functions, $\nabla^2 V$ and $\nabla^2 V$. When added to the classical Lagrangian (4) they lead to

$$-\frac{1}{2g^2} \text{Tr} \left[ \left( e^{-V} \nabla^\alpha e^V \right) \nabla^2 \left( e^{-V} \nabla_\alpha e^V \right) + V \left( \nabla^2 \nabla^2 + \nabla^2 \nabla^2 \right) V \right]$$
so that the quantum quadratic gauge Lagrangian can be written as

\[- \frac{1}{2g^2} \text{Tr} \left[ \frac{1}{2} \nabla^a \nabla_a - i W^\alpha \nabla_a - i \bar{W}^{\dot{\alpha}} \bar{\nabla}_{\dot{a}} \right] V \]  \hspace{1cm} (9)

where $\frac{1}{2} \nabla^a \nabla_a$ is the background covariant d’Alembertian and $W^\alpha$ is the background field strength. Since we are interested in one-loop calculations we only need terms in the action which are quadratic in the quantum fields. Thus the expression in (9) suffices: from there one can isolate a free kinetic term plus interactions with the background. Using the definitions of the connections in (7) one finally obtains

\[- \frac{1}{2g^2} \text{Tr} \left[ \frac{1}{2} \partial^a \partial_a - i \Gamma^a \partial_a - \frac{i}{2} \bar{\partial}^a \Gamma_a - \frac{1}{2} \Gamma^a \Gamma_a 
- i W^\alpha (D_a - i \Gamma_a) - i \bar{W}^{\dot{\alpha}} (\bar{D}_{\dot{a}} - i \bar{\Gamma}_{\dot{a}}) \right] V \]  \hspace{1cm} (10)

The quantum vector fields have standard propagators and interactions with the background that one reads from (10).

The gauge-fixing procedure requires the introduction of ghost fields \[\text{[7]}\]. We have two Faddeev-Popov ghosts $c$ and $c'$. Moreover, since we have chosen background-covariantly chiral gauge-fixing functions, we need a Nielsen-Kallosh ghost $b$. They are all background covariantly chiral superfields, i.e. $\nabla_a c = \nabla_{\dot{a}} c' = \nabla_a b = 0$.

For $N = 4$ supersymmetric Yang-Mills we have in addition the three background covariantly chiral matter superfield $\Phi^I$ (see \[\text{[8]}\]). In this case one-loop contributions to the effective action with external vector fields can be easily computed \[\text{[11, 7]}\]. One finds that for a diagram with an arbitrary number of external vector background lines, the loops from the three chiral matter fields are exactly cancelled by the corresponding loops from the three chiral ghosts which have opposite statistics. Only quantum vector loops survive and they give the first nonvanishing result at the level of the four-point function. This is due to the fact that superspace Feynman rules require the presence of two $D$’s and two $\bar{D}$’s for a non zero loop contribution. From the action in (10) we have interactions with the background fields at most linear in the $D$’s. Therefore at least four vertices are needed. The calculation is straightforward and leads to a very simple result: the four-point vector amplitude is given by \[\text{[7]}\]

\[\Gamma = \frac{1}{2} \text{Tr} \int d^2\theta d^2\bar{\theta} \frac{d^4p_1 d^4p_2 d^4p_3 d^4p_4}{(2\pi)^{16}} \delta(\sum p_i) G_0(p_1 \ldots p_4) \left[ W^\alpha(p_1) W^\alpha(p_2) \bar{W}^{\dot{\alpha}}(p_3) \bar{W}^{\dot{\alpha}}(p_4) - \frac{1}{2} W^\alpha(p_1) \bar{W}^{\dot{\alpha}}(p_2) W^\alpha(p_3) \bar{W}^{\dot{\alpha}}(p_4) \right] \]  \hspace{1cm} (11)

where $G_0$ is the four point scalar integral

\[G_0 = \int d^4k \frac{1}{(k + p_1)^2 k^2 (k - p_4)^2 (k + p_1 + p_2)^2} \]  \hspace{1cm} (12)
In order to make contact with corresponding calculations in ordinary Yang-Mills theory, we observe that from (11) we obtain the expected bosonic expression, i.e.

$$\int d^2 \theta \, d^2 \bar{\theta} \left[ W^\alpha_a(p_1) W_{\alpha b}(p_2) W^\dot{\alpha}_c(p_3) W_{\dot{\alpha} d}(p_4) \right. \left. - \frac{1}{2} W^\alpha_a(p_1) W^\dot{\alpha}_b(p_2) W_{\alpha c}(p_3) W_{\dot{\alpha} d}(p_4) \right]$$

$$\rightarrow \frac{1}{4} \left[ F^\mu\nu_a F_{\nu \sigma b} F_{\mu \rho c} F^{\rho \sigma d} + \frac{1}{2} F^\mu\nu_a F_{\nu \sigma b} F_{\rho \sigma c} F_{\mu \rho d} \right. \left. - \frac{1}{4} F^\mu\nu_a F_{\mu \nu b} F_{\rho \sigma c} F_{\rho \sigma d} - \frac{1}{8} F^\mu\nu_a F_{\rho \sigma b} F_{\mu \nu c} F_{\rho \sigma d} \right]$$

(13)

Now we want to repeat the computation for the noncommutative theory.

We go back to the $\mathcal{N} = 4$ Yang-Mills action in (2). Since, as already emphasized, the $\ast$-product does not affect superspace properties, it is clear that the various steps of the background field quantization can be implemented even in this case. We can go all the way to the action in (10) and there too we replace the ordinary multiplication between superfields with the $\ast$-operation. Following what we have done in the commutative example, we consider terms in the effective action with external vector fields. At the one-loop level again we find that ghost contributions cancel matter superfield contributions and one has to deal only with vector quantum loops. As before, for $D$-algebra reasons, the two- and three-point functions are zero and the first nonvanishing result is a loop with four vertices. Thus we focus on this calculation.

We have to compute a box supergraph with Feynman rules that can be obtained directly from (10) with the appropriate $\ast$-multiplication inserted. For any noncommutative theory the quadratic part of the action is the same as in the commutative case. Thus we have in momentum space the vector propagators given by

$$\langle V^a(\theta)V^b(\theta') \rangle = \frac{-g^2}{p^2} \delta^{ab} \delta^4(\theta - \theta')$$

(14)

The relevant vertices are ( cf. (10) )

$$\frac{1}{2g^2} \mathrm{Tr} \, V \ast \left[ i W^\alpha D_a + i \bar{W}^\dot{\alpha} \bar{D}_{\dot{a}} \right] \ast V$$

(15)

and one needs two $D$’s and two $\bar{D}$’s for a nonzero completion of the $D$-algebra in the loop. As in (11) we always obtain two $W$’s and two $\bar{W}$’s, a factor with four scalar propagators, and in addition exponential factors from the $\ast$-product at the vertices.
More precisely, using the definition in (1), the three-point interactions can be written in momentum space as

\[
\frac{1}{2g^2} \left( \mathcal{U}(k_1, k_2, k_3) + \tilde{\mathcal{U}}(k_1, k_2, k_3) \right) \equiv \frac{1}{2g^2} V_a(k_1) \left[ i \mathcal{W}^\alpha_b(k_2) D_\alpha + i \tilde{\mathcal{W}}^\alpha_b(k_2) \tilde{D}_\alpha \right] V_c(k_3) 
\]

![Image](image.png)

with momenta flowing into the vertex, \( k_1 + k_2 + k_3 = 0 \) and \( k_i \times k_j \equiv (k_i)_\mu \Theta^{\mu\nu}(k_j)_\nu \). In order to obtain a box diagram we need a forth order term from the Wick expansion, with two \( \mathcal{U} \)'s and two \( \tilde{\mathcal{U}} \)'s, i.e. \( \frac{1}{4(2g^2)^2} \mathcal{U}^2 \tilde{\mathcal{U}}^2 \). Every vertex, when inserted in the loop, gives rise to an untwisted and a twisted term

\[
\mathcal{U}(k_1, k_2, k_3) \rightarrow V_a(k_1) i \mathcal{W}^\alpha_b(k_2) D_\alpha V_c(k_3) \left[ \text{Tr}(T^a T^b T^c) \ e^{-\frac{i}{2}(k_1 \times k_2 + k_2 \times k_3 + k_3 \times k_1)} - \text{Tr}(T^c T^b T^a) \ e^{\frac{i}{2}(k_1 \times k_2 + k_2 \times k_3 + k_3 \times k_1)} \right] 
\]

with the understanding that now the quantum lines have to be Wick contracted in the order in which they appear. As in the commutative case, there are two possible arrangements of the vertices in the loop, i.e. \( \mathcal{U}(1) \mathcal{U}(2) \mathcal{U}(3) \mathcal{U}(4) \) with multiplicity two, and \( \mathcal{U}(1) \mathcal{U}(2) \tilde{\mathcal{U}}(3) \tilde{\mathcal{U}}(4) \) with multiplicity one.

The \( D \)-algebra is trivial and it gives \( D_\alpha D_\beta \tilde{D}_\alpha \tilde{D}_\beta \rightarrow C_{\beta\alpha} C_{\beta\alpha} \) and \( D_\alpha \tilde{D}_\alpha D_\beta \tilde{D}_\beta \rightarrow -C_{\beta\alpha} C_{\beta\alpha} \) respectively for the two arrangements of the vertices. We obtain a result that we can write symbolically as

\[
\frac{1}{32} \left[ \mathcal{U}^a \mathcal{U}_a \mathcal{U}^{\bar{\alpha}} \tilde{U}_{\bar{\alpha}} - \frac{1}{2} \mathcal{U}^a \tilde{U}_a \mathcal{U}^{\bar{\alpha}} \tilde{U}_{\bar{\alpha}} \right] 
\]

where we have defined ( see (17) )

\[
\mathcal{U}^a(k_1, k_2, k_3) \equiv \mathcal{U}_P^a(k_1, k_2, k_3) + \mathcal{U}_T^a(k_1, k_2, k_3) \equiv V_a(k_1) i \mathcal{W}^\alpha_b(k_2) V_c(k_3) 
\]

![Image](image.png)

\[
\left[ \text{Tr}(T^a T^b T^c) \ e^{-\frac{i}{2}(k_1 \times k_2 + k_2 \times k_3 + k_3 \times k_1)} - \text{Tr}(T^c T^b T^a) \ e^{\frac{i}{2}(k_1 \times k_2 + k_2 \times k_3 + k_3 \times k_1)} \right] 
\]

The \( V \) quantum lines must be contracted in the consecutive order as they appear in (18). Substituting (13) in (18), we obtain the sum of sixteen terms: two of them, i.e. the ones which contain all untwisted \( P \) and all twisted \( T \) vertices correspond to planar diagrams. All the others, i.e. the ones with two \( P \) and two \( T \) vertices (a total of six), the ones with one \( P \) and three \( T \)’s (a total of four) and the ones with one \( T \) and three \( P \)’s (a total of four), correspond to nonplanar graphs. Now we analyze these contributions in some detail.

As anticipated above, the planar diagrams correspond to terms from (18), which contain either four untwisted vertices \( \mathcal{U}_P \) or four twisted vertices \( \mathcal{U}_T \). For the \( U(N) \) gauge matrices
we use the relation $T_{ij}^a T_{kl}^a = \delta_{il} \delta_{jk}$, and obtain the following contribution to the effective action

$$\Gamma_{\text{planar}} = \frac{N}{32} \int d^2 \theta \, d^2 \bar{\theta} \, \frac{d^4 p_1 \, d^4 p_2 \, d^4 p_3 \, d^4 p_4}{(2\pi)^{16}} \, \delta(\sum_i p_i) \, G_0(p_1 \ldots p_4)$$

$$\left( W^\alpha(a) \left( W_\alpha(p_1) W_\alpha(b) \bar{W}_\bar{\alpha}(p_2) \bar{W}^\bar{\alpha}(p_3) - \frac{1}{2} W^\alpha(a) \bar{W}^\bar{\alpha}(p_2) W_\alpha(p_3) \bar{W}^\bar{\alpha}(p_4) \right) \right)$$

$$\left( \bar{W}^\bar{\alpha}(p_2) W_\alpha(p_3) W^\alpha(p_4) \right)$$

$$\left( \frac{1}{2} W^\alpha(a) \bar{W}^\bar{\alpha}(p_2) W_\alpha(p_3) \bar{W}^\bar{\alpha}(p_4) - \frac{1}{2} W^\alpha(a) \bar{W}^\bar{\alpha}(p_2) W_\alpha(p_3) \bar{W}^\bar{\alpha}(p_4) \right)$$

with $G_0$ defined in (12). Comparing (20) with the result in the commutative theory we find that the only difference is given by the exponential factors which depend on $\Theta$ and on the external momenta. In fact the exponentials are such to reconstruct the $\ast$-product between the field strengths, yielding

$$\Gamma_{\text{planar}} = \frac{N}{32} \int d^2 \theta \, d^2 \bar{\theta} \, \frac{d^4 p_1 \, d^4 p_2 \, d^4 p_3 \, d^4 p_4}{(2\pi)^{16}} \, \delta(\sum_i p_i) \, G_0(p_1 \ldots p_4)$$

$$\left( W^\alpha(a) \ast W_\alpha(p_2) \ast \bar{W}^\bar{\alpha}(p_3) \ast \bar{W}^\bar{\alpha}(p_4) + \bar{W}^\bar{\alpha}(p_2) \ast W^\alpha(p_3) \ast W^\alpha(p_4) \ast \bar{W}^\bar{\alpha}(p_1) \right)$$

$$- \frac{1}{2} W^\alpha(a) \ast \bar{W}^\bar{\alpha}(p_2) \ast W_\alpha(p_3) \ast \bar{W}^\bar{\alpha}(p_4) - \frac{1}{2} W^\alpha(a) \ast \bar{W}^\bar{\alpha}(p_2) \ast W_\alpha(p_3) \ast \bar{W}^\bar{\alpha}(p_4)$$

(20)

Making use of the relation in (13) with appropriate $\ast$-products implemented, we can obtain the bosonic expression corresponding to (21). Now we turn to the study of the nonplanar supergraphs.

The various nonplanar diagrams can be collected in two distinct groups. There are graphs in which two vertices are twisted. In this first class the $U(N)$ matrices produce a factor like Tr$(T^p T^q)$Tr$(T^r T^s)$. Then there are the graphs in which one (or equivalently three) of the four vertices are twisted. For them the trace on the gauge matrices factorizes as Tr$(T^p)$Tr$(T^q T^r T^s)$. In both cases the phases from the $\ast$-product at the vertices will contain a dependence on the loop momentum.

We illustrate the procedure considering nonplanar graphs in the first group, from a structure of the vertices like $PPTT$. Looking at (18) we find that these contributions come from

$$U^\alpha_p U^\alpha_{p_2} \bar{U}_\bar{T}^\bar{\alpha} \bar{U}_\bar{T}_{\bar{\alpha}} = \frac{1}{2} U^\alpha_p U^\alpha_{p_2} U^\alpha_{p_3} \bar{U}_\bar{T}_{\bar{\alpha}} \bar{U}_\bar{T}_{\bar{\alpha}}$$

(22)

With the external momenta in the order $p_1, p_2, p_3, p_4$, the exponentials from the $\ast$-operation at the vertices give

$$e^{-\frac{i}{4}(-p_1 \times k)} \, e^{-\frac{i}{4}(-p_2 \times (k + p_1))} \, e^{\frac{i}{4}[(p_1 + p_2 + p_4) \times (k - p_4)]} \, e^{\frac{i}{4}(k \times p_4)}$$

(23)
We find a contribution to the effective action of the form

\[
\Gamma_{PPTT} = \frac{1}{32} \int d^2 \theta \, d^2 \bar{\theta} \frac{d^4 p_1 \, d^4 p_2 \, d^4 p_3 \, d^4 p_4}{(2\pi)^{16}} \delta(\sum p_i)
\]

\[
\int d^4 k \frac{e^{i(p_1+p_2) \times k}}{(k + p_1)^2 k^2 (k - p_4)^2 (k + p_1 + p_2)^2}
\]

\[
e^{-\frac{2}{3} p_1 \times p_2} e^{\frac{2}{3} p_1 \times p_4} \left[ \text{Tr}(\mathbf{W}_\alpha(p_1) \mathbf{W}_\alpha(p_2)) \text{Tr}(\mathbf{W}_\hat{\alpha}(p_3) \mathbf{W}_\hat{\alpha}(p_4)) \right. \\
\left. - \frac{1}{2} \text{Tr}(\mathbf{W}_\alpha(p_1) \mathbf{W}_\hat{\alpha}(p_2)) \text{Tr}(\mathbf{W}_\alpha(p_3) \mathbf{W}_\hat{\alpha}(p_4)) \right]
\]

(24)

We introduce a mass IR regulator, in order to avoid divergences which would arise in the zero limit of the external \( p_i \) momenta, and we perform the loop integration using Schwinger parameters for the denominator factors from the propagators. In this way we obtain

\[
I_0 = \int d^4 k \frac{e^{i(p_1+p_2) \times k}}{[(k + p_1)^2 + m^2][(k - p_4)^2 + m^2][(k + p_1 + p_2)^2 + m^2]}
\]

\[
= \int_0^\infty \prod_{i=1}^4 d\alpha_i \, e^{-\alpha m^2} \int d^4 k \, e^{-\alpha k^2} \, e^{-i k \times (p_1+p_2)} \frac{1}{[\alpha p_1 \times p_2 - \alpha p_4 \times (p_1+p_2)]}
\]

\[
e^{-\frac{1}{\alpha} [(\alpha_2 + \alpha_3) p_1^2 - (\alpha_1 + \alpha_2 + \alpha_3) \alpha_4 p_2^2 - (\alpha_1 + \alpha_2 + \alpha_4) \alpha_3 p_4^2]}
\]

\[
e^{-\frac{2}{\alpha} [(\alpha_2 + \alpha_3) \alpha_4 p_1 \cdot p_2 + (\alpha_1 + \alpha_4) \alpha_3 p_1 \cdot p_4 + (\alpha_2 + \alpha_4) \alpha_3 p_2 \cdot p_4]}
\]

(25)

where we have defined

\[
p \circ p \equiv p_{\mu} \Theta^2_{\nu \mu} p_\nu
\]

(26)

and

\[
\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4
\]

(27)

Introducing new integration variables

\[
\lambda \equiv \alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad \xi_i = \frac{\alpha_i}{\alpha} \quad i = 1, 2, 3, 4
\]

(28)
we obtain

$$I_0 = \int_0^\infty d\lambda \int_0^1 d\xi_1 \int_0^1 d\xi_2 \int_0^1 d\xi_3 \int_0^1 d\xi_4 \frac{e^{-\frac{1}{2}L(p_1+p_2)\xi_4}}{\sqrt{2\pi}e^{-\frac{1}{2}L(p_1+p_2)\xi_4}} e^{-\frac{1}{2}L(p_1+p_2)\xi_4} e^{-\frac{1}{2}L(p_1+p_2)\xi_4} \cdot e^{-\frac{1}{2}L(p_1+p_2)\xi_4}$$

This result reproduces the field theory limit obtained from string amplitudes in the presence of a nonzero $B$ field [4, 12]. The contributions from all the other nonplanar supergraphs can be written in straightforward manner. A more detailed presentation and a complete analysis will be given elsewhere [13].

In the low energy approximation $p_i \cdot p_j$ small, with $p_i \times p_j$ finite, the integration on the $\lambda$ and $\xi_i$ variables can be performed exactly [6]. The final result from diagrams containing two $P$ and two $T$ vertices is given by [13].

$$\Gamma_{2P2T} \rightarrow \frac{1}{32} \int d^2\theta d^2\bar{\theta} \frac{d^4p_1 d^4p_2 d^4p_3 d^4p_4}{(2\pi)^{16}} \delta(\sum p_i) e^{-\frac{1}{2}p_1 \times p_2} e^{-\frac{1}{2}p_3 \times p_4}$$

$$\int_0^\infty d\lambda \int_0^1 d\xi_1 \int_0^1 d\xi_2 \int_0^1 d\xi_3 \frac{\sin\left(\frac{p_1 \times p_2}{2}\right)}{\sqrt{2\pi} \sqrt{p_1 \times p_2}}$$

$$K_2(2m\sqrt{(p_1 + p_2) \cdot (p_1 + p_2)})$$

$$= \frac{1}{2} \left[ \text{Tr}(\mathbf{W}^\alpha(p_1)\mathbf{W}_\alpha(p_2)) \text{Tr}(\mathbf{W}^\beta(p_3)\mathbf{W}_\beta(p_4)) \right]$$

$$+ \text{Tr}(\mathbf{W}^\alpha(p_1)\mathbf{W}^\beta(p_2)) \text{Tr}(\mathbf{W}_\alpha(p_3)\mathbf{W}_\beta(p_4))$$

$$- \text{Tr}(\mathbf{W}^\alpha(p_1)\mathbf{W}^\beta(p_2)) \text{Tr}(\mathbf{W}_\alpha(p_3)\mathbf{W}_\beta(p_4))$$

where $K_2$ is the modified Bessel function.

The low-energy effective action contribution in (30) cannot be rewritten, as it was the case for the planar diagrams, in terms of *-products of field strengths. Moreover gauge invariance under the transformations in (3) is not maintained. On the other hand we are reassured that nothing went wrong in the quantization procedure, since our perturbative field theory result is in complete agreement with the field theory limit from one-loop
four-point scattering on $D3$-branes as computed in [6, 8]. It would be interesting to see if one can implement gauge invariant operators perturbatively, using techniques suggested in [14, 15]. In any event it appears that nonlocality, which is an intrinsic property of the noncommutative theory, might require a modified and deeper understanding of the concept of gauge invariance.

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