Onset of Vortices in Thin Superconducting Strips and Wires

I. Aranson, M. Gitterman and B. Ya. Shapiro

Department of Physics and Jack and Pearl Resnick Institute of Advanced Technology,
Bar Ilan University, Ramat Gan 52900, Israel

Abstract

Spontaneous nucleation and the consequent penetration of vortices into thin superconducting films and wires, subjected to a magnetic field, can be considered as a nonlinear stage of primary instability of the current-carrying superconducting state. The development of the instability leads to the formation of a chain of vortices in strips and helicoidal vortex lines in wires. The boundary of instability was obtained analytically. The nonlinear stage was investigated by simulations of the time-dependent generalized Ginzburg-Landau equation.

74.60.Ge, 74.76.-w
Finite resistance in massive type-II superconductors subjected to a supercritical magnetic field $H_{c1}$ is caused by the motion of vortices \[1,2,3,4\]. The order parameter is suppressed at a distance whose magnitude comparable to the correlation length $\xi$ within the core of the vortex, and the induced currents and fields decay at the penetration length $\lambda$. The dynamics of vortices both in massive three-dimensional systems and in two-dimensional films has became the subject of intensive studies (both analytical \[5\] and numerical \[6\]).

Another limiting case is quasi-one-dimensional systems (transverse dimensions smaller than $\xi$ and $\lambda$) such as narrow microstrips and channels. Their resistance is caused by phase-slippage centers (PSC), which nucleate spontaneously in some narrow region of currents even without a magnetic field \[7,8,9\].

We consider here systems with transverse geometry intermediate between the two cases described above; i.e., for which the radius $R$ of 3D wires satisfies the condition: $\xi \ll R \ll \lambda$. Likewise for strips one assumes that $hd \ll \lambda^2$ together with $h \gg \xi$, where $h$ is the thickness and $d$ is the width of the strip. We assume that the homogeneous external magnetic field is applied normal to the strip and parallel to the wire’s axis. The current is applied lengthwise to the strip and parallel to the wire’s axis. Our aim is to elucidate the onset and penetration of vortices into a sample.

The problems of the surface instabilities and consequent nucleation of the vortices in bulk superconductors was widely discussed (see, e.g. \[10\]). The critical magnetic field for bulk semiinfinite superconducting sample, known as the ”superheating field”, was obtained by several authors (see, e.g. \[11\]). However, to our knowledge, for thin superconducting films and wires with intermediate geometry ($hd \ll \lambda^2$ or $\xi \ll R \ll \lambda$) such problems were not studied.

One can expect that the stationary, spatially inhomogeneous current-carrying superconducting state becomes unstable when the applied current $j$ and/or magnetic field $B$ exceed some critical value(s). Because of the translation invariance along the longitudinal direction, the perturbations arising for the order parameter have the form of a standing wave with finite modulation wavenumber $q$. In virtue of the inhomogeneity of the stationary state, the
perturbative eigenfunctions should be mostly localized near the edges. Exponential growth in time of the perturbations finally results in the formation of single zeroes (lines of zeroes for wires) of the order parameters on a set of points placed at the edges of a strip at a distance $2\pi/q$ apart. Such zeroes serve as the ”seeds” for vortices which will be driven away later on from the edges by the current and fields. In this article we present the analytic treatment of this problem. Our results are supported by the numerical simulations of the TDGLE in the region where the analytic approach fails.

Although the vortices in thin strips and wires are not topologically different from the usual vortices in massive superconductors, the intermediate geometry results in significant differences. In contrast to massive superconductors the magnetic field remains practically homogeneous on the scale of the strip or wire. As a result, only the vortex phase rather than the magnetic flux carried by the vortex is quantized [3]. Moreover, the values of critical currents and fields obtained happens to be strongly dependent on the transverse dimension. The vortices in thin strips and wires are different also from the PSC nucleating in a very thin wire ($R \sim \xi$) where one needs the potential difference across a sample, and instability typically appears only in a narrow band of currents [1]. Note that still another type of vortices occurs in superfluid jets above the critical velocity [12], which formally corresponds to zero magnetic field with the superconducting current greater than the depairing current.

We start from the celebrated time-dependent Ginzburg - Landau Equation (TDGLE) in the form [3]

$$u(\partial_t + i\mu)\Psi = (\nabla - 2iA)^2\Psi + \left(1 - |\Psi|^2\right)\Psi, \quad (1)$$

$$j = |\Psi|^2(\nabla\varphi - 2A) - (\nabla\mu + 2\partial_t A), \quad (2)$$

$$\nabla \cdot j = 0, \quad \nabla \cdot A = 0, \quad (3)$$

where $\Psi$ is the (complex) order parameter, $\varphi = arg\Psi$, $A$ and $\mu$ are vector and scalar potentials, and $j$ is the current density. The value of the parameter $u$ has to be obtained from microscopic theory [3]. The usual dimensionless units are used [3].

Eqs. (1-3) apply for thin strips and wires. Indeed, the condition $dh \ll \lambda^2$ for strips and
$R \ll \lambda$ for wires enable us to neglect the magnetic field created by currents \cite{13,16} and omit thereby the appropriate Maxwell equation.

Let us start with strips. We choose the origin of the coordinate frame at the mid-point of the strip with the $x$ axis lengthwise and the $y$ axis in the lateral direction, so that the edges are located at $(x,-d/2)$ and $(x,d/2)$. The normal to the strip magnetic field $B$ is described by the vector potential $\mathbf{A} = (-By,0,0)$ (see Fig. 1).

The TDGLE has a stationary current-carrying superconducting solution of the form

$$\Psi(x,y) = F(y) \exp(ikx), \quad \mu = 0$$
(4)

$$j(y) = F(y)^2(k + 2By)$$
(5)

where $F(y)$ satisfies the following nonlinear equation

$$\partial_y^2 F + \left(1 - F^2 - (k + 2By)^2\right) F = 0$$
(6)

subject to the no-flux (superconductor-vacuum) boundary conditions $\partial_y F(d/2) = \partial_y F(-d/2) = 0$. This solution is the only stable superconducting state for $B < B_c$.

For the case of weak magnetic fields ($B \ll k/d$) one can find a perturbative solution of Eq. (6). In lowest order the solutions for $F(y)$ and the mean current $j_0 = \frac{1}{d} \int_{-d/2}^{d/2} j(y) dy$ have the form

$$F = F_0 - \frac{2kB y}{F_0} + \sqrt{2kB} \sinh(\sqrt{2}F_0 y) \frac{1}{F_0^2} \cosh(\sqrt{2}F_0 d/2) + O(B^2)$$
(7)

where $j_0 = k(1 - k^2) - kd^2(B^2 + O(1/d^2))$ and $F_0 = \sqrt{1-k^2}$ (note that $F(y) - F_0$ is antisymmetric only for $Bd \ll 1$). For arbitrary values of the magnetic field $B$ Eq. (6) can be solved only numerically. For the numerical solution of the nonlinear boundary problem we used the deferred-difference method \cite{14}. The characteristic form of $F(y)$ is shown in Fig. 2.

We seek the perturbative solution of the TDGLE in the form $\Psi = \left(F(y) + \hat{\zeta}(x,y,t)\right) \exp(ikx)$. Linearizing TDGLE with respect to $\hat{\zeta}, \mu$, splitting real and imaginary parts of $\hat{\zeta} = \hat{a} + i\hat{b}$, and representing the solution in
the form \((\hat{a}, \hat{b}, \mu) \sim (a(y), b(y), \mu(y)) \exp(iqx + \omega t)\), we obtain the following equations for the functions \(a(y), b(y), \mu(y)\)

\[
\begin{align*}
\omega u_a &= \partial_y^2 a + (Z(y, q) - 2 F^2) a - 2iq(k + 2By)b \\
\omega u_b &= \partial_y^2 b + Z(y, q)b + 2iq(k + 2By)a - uF\mu \\
F\omega ub &= \partial_y^2 \mu - (q^2 + uF^2)\mu
\end{align*}
\]

(8)

where \(Z(y, q) = 1 - (k + 2By)^2 - F^2 - q^2\). For convenience (in order to deal only with real coefficients in the Eqs. (8) we redefine \(b \rightarrow ib\). Then the Eqs. (8) are of the form

\[
\begin{align*}
\omega u_a &= \partial_y^2 a + (Z(y, q) - 2 F^2) a - 2q(k + 2By)b \\
\omega u_b &= \partial_y^2 b + Z(y, q)b - 2q(k + 2By)a - uF\mu \\
F\omega ub &= \partial_y^2 \mu - (q^2 + uF^2)\mu
\end{align*}
\]

(9)

Eqs. (9) represent the eigenvalue problem for \(\omega\) which has to be solved for the no-flux boundary conditions for \(a, b\) and \(\mu\). The stationary solution becomes unstable when for the first time the negative \(\omega(q)\) achieves zero. For \(\omega = 0\) the equation for \(\mu\) has no solutions obeying the boundary condition. Therefore, we are left with two equations for \(a\) and \(b\) (\(\mu = 0\))

\[
\begin{align*}
\partial_y^2 a + (Z(y, q) - 2 F^2) a - 2q(k + 2By)b &= 0 \\
\partial_y^2 b + Z(y, q)b - 2q(k + 2By)a &= 0
\end{align*}
\]

(10)

The system (10) in general requires numerical solution. These equations contain four control parameters, namely, the magnetic field \(B\), the mean current \(j_0\) which is determined by \(k\), the width of the strip \(d\) and the modulation wavenumber \(q\). The numerical solution was implemented in the following way: 1) The stationary solution \(F(y)\) was determined (numerically) and this was then substituted into Eqs. (10); 2) For fixed \(q\), Eqs. (10) were solved for two sets of boundary conditions at the lower edge: \(\partial_y a_1(-d/2) = \partial_y b_1(-d/2) = 0, a_1(-d/2) = 1, b_1(-d/2) = 0\) and \(\partial_y a_2(-d/2) = \partial_y b_2(-d/2) = 0\).
\( \partial_y b_2(-d/2) = 0, a_2(-d/2) = 0, b_2(-d/2) = 1; \)

3) Then the solutions at the second edge were examined; 4) The eigenvalue \( q \) was obtained from the condition

\[
\partial_y a_1 \partial_y b_2 - \partial_y a_2 \partial_y b_1 = 0. \tag{11}
\]

The critical curves for the onset of instability in the \( (j, B) \) plane for several widths of the strip \( d \) are depicted on Fig. 3. The eigenfunctions \( a(y), b(y) \) are shown in Fig. 2. It can be seen that for \( B \neq 0 \) the eigenfunctions are strongly localized near the edge of the strip where the suppression of the order parameter is maximal. The critical curves are terminated at the depairing current \( j_p = 2/\sqrt{27} \approx 0.387 \), which corresponds to \( k_p = 1/\sqrt{3} \). In this region a very weak magnetic field is needed to destroy superconductivity. In the practically relevant limit \( d \gg 1 \) (well-separated edges), Eqs. (10) can be essentially simplified by an adiabatic elimination of \( a(y) \) from the second Eqs. (10) resulting in:

\[
\partial_y^2 b = q^2 \left( 1 - \frac{4(k + 2By)^2}{2 - 2(k + 2By)^2 + q^2} \right) b + O(1/d^2). \tag{12}
\]

For \( k \to k_p \) also \( q \to 0 \) and the critical curve is given by the solvability condition \( \int_{-d/2}^{d/2} \left( 1 - \frac{4(k + 2By)^2}{2 - 2(k + 2By)^2 + q^2} \right) dy = 0 \). The straightforward calculations result in the following conditions:

\[
j_p - j = k_p B^2 d^2 \quad \text{and} \quad B^2 d^2 = -3\delta/k_p \quad \text{where} \quad \delta = k - k_p \ll 1. \]

However this trivial solution is relevant only for very small \( \delta \sim 1/d^2 \). For larger \( \delta \) this curve merges with another curve which turns out to be unstable for smaller \( B \). This curve obeys the scaling \((q^2d, \delta d, Bd^2) \sim O(1)\).

For this case the Eq. (12) is reduced to the Airy equation:

\[
\partial_y^2 b = (\alpha + \beta y)b \tag{13}
\]

where \( \alpha = \frac{3}{16} q^2 \left( -16\sqrt{3}\delta(1 - q^2) + 4q^2 - 72\delta^2 - 3q^4 \right), \beta = -6\sqrt{3}q^2B \) (these expressions were obtained with the aid of the Maple program for analytical calculations). The critical curve for the Eq. (13) is given by the conditions (cf. (11))

\[
\partial_y Ai(\gamma_1) \partial_y Bi(\gamma_2) - \partial_y Bi(\gamma_1) \partial_y Ai(\gamma_2) = 0
\]

\[
\frac{\partial}{\partial q} \left[ \partial_y Ai(\gamma_1) \partial_y Bi(\gamma_2) - \partial_y Bi(\gamma_1) \partial_y Ai(\gamma_2) \right] = 0 \tag{14}
\]
where $\gamma_{1,2} = \beta^{1/3}(\mp d/2 + \alpha/\beta)$. The critical curves obtained by solution of Eq. (12) are shown also on Fig. 3. The critical curves obtained from the Airy equation (13,14) coincide with those within the thickness of the line. We conclude that the "$d \gg 1$" approximation works extremely well. As can be seen from Fig. 3, the critical field $B_c$ decreases with $d$, and, finally $B_c$ vanishes for $d \to \infty$. This is consistent with well-known results [1] that the critical magnetic field in an infinite film is always zero. Figures 2-3 support the physical picture of an appearance of a chain of vortices at one edge of the strip (where the eigenfunctions are localized). It is useful to mention that the critical curves appear to be insensitive to the particular properties of the superconductor given by the material constant $u$.

In order to follow the consequent stages of the instability, which cannot be covered by the linear analysis, we performed direct numerical simulations of TDGLE (with $u = 5.79$ [3]). The initial conditions were chosen typically as a homogeneous superconducting state perturbed by small amplitude noise. We used no-flux boundary conditions in the transverse direction ($\partial_y \Psi = 0$) and normal metal- superconductor boundary conditions in the longitudinal direction ($\Psi(x,y) \to 0$ for $x \to 0, L$, where $L$ is the strip length). The numerical scheme was the generalization of the split-step method described in [15,16,17], the number of the grid points was $256 \times 128$ and the timestep was $0.05 - 0.1$. Results of the simulations, shown in Fig. 3, 4 clearly agree with the stability analysis.

The analytical solution and the numerical simulations of TDGLE clarify all stages of the process: initial development of small perturbations due to applied current and/or field (Fig. 4a), their localization at some points at the lower edge (Fig. 4b) and an appearance of the chain of single vortices and its consequent tearing off and propagation inside the strip (Fig. 4c). Then the process repeats itself. When the vortex is sufficiently far away from the edges, its motion can be described, in principle, by a simple first order equation [15,16]. This periodic nucleation of vortices at the edge was observed for sufficiently wide range of transport currents $0 < j < j_p$. Thus, one has a dynamic resistive state near the critical line. A gradual increas of the field $B$ leads to faster nucleation, also the vortex motion
itself looses the regularity and reminds of some sort of vortex turbulence \([15,6]\). In that case one has also nonperiodic oscillations of the voltage across the sample, as it was observed in numerical simulations \([15,6]\). Very close to the critical current \(j_p\) the film doesn’t exhibit dynamic resistive states and undergoes direct transition to normal state. Alternatively, for \(j \to 0\) the period of the nucleation diverges.

Our analysis can be generalized for the case of superconducting wires when \(B\) is parallel to the axis \(z\) of wire. Then the potential can be chosen in the form \(A = (A_r, A_\theta, A_z) = (0, Br/2, 0)\).

The equation for the stationary order parameter has the form (compare with Eq. (6))

\[
\partial^2_r F + \frac{1}{r} \partial_r F + \left(1 - F^2 - k^2 - (Br)^2\right) F = 0
\]

Eq. (15) is subjected to the boundary conditions \(\partial_r F(0) = \partial_r F(R) = 0\).

We seek a perturbative solution in the form \(\Psi = (F(r) + \zeta(r, \theta, z, t)) \exp(ikz)\) Notice, that unlike Eqs. (9), one now has the full 3D problem. In contrast to strips the wires also possess rotation symmetry. Therefore, the perturbative solution should break both translation and rotation symmetry, creating, thereby, helicoidal structure.

Linearizing now the TDGLE, and substituting the perturbations in the form \((a, b, \mu) \sim \exp(in\theta + iqz + \omega t)\) \((a, b\) have the same meaning as for a strip), we are left for the critical value \(\omega = 0\) with the following equations

\[
\Delta_n a_n + (Z_1(r, q) - 2F^2)a_n - 2qkb_n + 2Bnb_n = 0
\]

\[
\Delta_n b_n + Z_1(r, q)b_n - 2qka_n + 2Bna_n = 0
\]

where \(Z_1(r, q) = 1 - k^2 - (Br)^2 - F^2 - q^2, \Delta_n = \partial^2_r + \frac{1}{r} \partial_r - \frac{n^2}{r^2}\).

The boundary conditions read \(\partial_r a_n = \partial_r b_n = 0\) for \(r = R\) and \(a_n, b_n \sim r^n\) for \(r \to 0, n \neq 0\). For \(n = 0\) the conditions translate to \(\partial_r a_0(0) = \partial_r b_0(0) = 0\). The numerical solution of Eqs. (16) is similar to that of the strip if the fact that they contain the rotation number \(n\) is taken into account. Critical curves for each \(n\) intersect in some complicated way on the \((j, B)\) plane. In order to select the most unstable rotation mode we used the following
algorithm. We reproduced the critical curves for the 6 first modes ($n = 0..5$, other modes turned out to be unimportant for the chosen radius of the wire) as a function of $k$, and then for each given $k$ we selected the mode which is unstable for lowest value of $B$. As a result, the critical curve in the $(j,B)$ plane consists of several pieces of different $n$. As can be seen from Fig. 5 the rotation symmetry changes with $B$, namely, for small $B$ the axi-symmetric mode $n = 0$ appears while for larger $B$ the higher helicoid solutions occur.

The number of the "important" modes grows with the radius of the wire. For $R \gg 1$ the critical curves can be described by the reduction to effective Airy equation as that for the strip. For macroscopic wires the helicoid structures are well-known (see, e.g., [18]).

It is a challenging problem to generalize the above formalism to a magnetic field orthogonal to the axis of wire. However, the stationary state is then described by a 2D nonlinear partial differential equation, and the analysis is very hard. One can speculate that the critical curves for this case will not much differ from that of a strip with $d \approx 2R$.

Finally, we have described all the stages of the appearance and the propagation of a chain of vortices in thin strips and wires, in contrast to the usual approach where the existence of vortices is assumed ad hoc. All existing methods of experimental observation of fluxon dynamics, e.g., voltage-current characteristics, temporal pulses at a measured voltage, etc. may be used to check our analysis. High-temperature superconductors are most suitable for these experiments. Indeed, they have large values of $\lambda$ and, hence, samples can be prepared satisfying the conditions $dh \ll \lambda^2$ and $R \ll \lambda$. In particular, for the $YBaCuO$ system with parameters $\lambda = 1500 \text{ Å}, d = 500 \text{ Å}$ and $h = 10 \text{ Å}$, one has $dh/\lambda^2 \ll 1$. For this system the scale of physical units shown in Fig. 2-5 are: $y_{phys} \sim y \times 10\text{ Å}, B_{phys} \sim B \times 5 \cdot 10^3 \text{ Gs}$ and $j_{phys} = j \times 2.5 \cdot 10^8 \text{ A/cm}^2$. The characteristic time scale is given by $t_{phys} = t \times 10^{-9} \text{ sec}$.

The authors are grateful to L. Kramer, A.I. Larkin and N.B. Kopnin for fruitful discussions. The work of I.A. and B.Y. S. was supported in part by the Rashi Foundation. The support of the Israeli Ministry of Science and Technology is kindly acknowledged.
† Such a vortex is often called a fluxoid (see [2]).

[1] A.A. Abrikosov, *Fundamentals of the Theory of Metals*, (Elsevier, New York, 1988).

[2] P. G. de Gennes, *Superconductivity of Metals and Alloys*, (Addison-Wesley, Redwood City, 1989).

[3] L.P. Gor'kov and N.B. Kopnin, Usp. Fiz. Nauk, 116, 413 (1975).

[4] G. Blatter, M.V. Feigelman, V.B. Geshkenbein, A.I. Larkin, and V.M. Vinokur, Vortices in High Temperature Superconductors, Rev. Mod. Phys, 1994 (to be published).

[5] F. Liu, M. Mondello, and N. Goldenfeld, Phys. Rev. Lett., 66, 3071 (1991); H. Frahm, S. Ullah, and A. Dorsey, Phys. Rev. Lett., 66, 3067 (1991).

[6] A. Shinozaki and Y. Oono, Phys. Rev. Lett, 66, 173 (1991); R. Kato, Y. Enomoto, and S. Maekawa, Phys. Rev. B, 44, 6916 (1991); 47, 8016 (1993); M. Machida and Kaburaki, Phys. Rev. Lett. 71, 3206 (1993).

[7] J.S. Langer, Rev. Mod. Phys., 52, 1 (1980).

[8] B.I. Ivlev and N.B. Kopnin, Adv. Phys, 33, 47 (1984).

[9] R.J. Watts-Tobin, Y. Krähenbühl, and L. Kramer, J. of Low. Temp. Phys., 42, 459 (1981); L. Kramer and R. J. Watts-Tobin, Phys. Rev. Lett, 40, 1041 (1978).

[10] Y.V. Sharvin, Zh. E.T.F. -Pi’sma, 2, 287 (1965) [translation: JETP Letters 2, 183 (1965)]; B. S. Chandrasekhar, D. E. Farrell, and S. Huang, Phys. Rev. Lett. , 18, 43 (1967); B. L. Brandt and R. D. Parks, Phys. Rev. Lett. , 19, 163 (1967).

[11] V.L. Ginzburg, Zh. E.T.F., 34, 113 (1958) [translation: JETP 7, 78 (1958)]; L. Kramer, Phys. Rev. 170, 475 (1968); Z. Physik, 259, 333 (1973).

[12] P.I. Soininen and N.B. Kopnin, Phys. Rev. B , 49, 12087 (1994).
[13] J. Pearl, Appl. Phys. Lett, 5, 65 (1966).

[14] NAG Fortran Library, Mark 15.

[15] I. Aranson, M. Gitterman, and B. Y. Shapiro, Motion of vortices in thin superconducting films, to appear in J. of Low Temp. Phys., 1994.

[16] I. Aranson, L. Kramer, and A. Weber, J. Low. Temp. Phys, 89, 859 (1992).

[17] A. Weber and L. Kramer, J. of Low Temp. Phys., 84, 289 (1991).

[18] E.H. Brandt, Phys. Rev. Lett., 69, 1105 (1992).
FIGURES

FIG. 1. Coordinate frame for the superconducting strip.

FIG. 2. The amplitude of order parameter $F(y)$ and the perturbative solutions $a(y), b(y)$ as functions of $y$ obtained from numerical solutions of Eq. (1) and Eqs. (2) for $q = 0.2804, d = 30, k = 0.3, B = 0.014$ and $j = 0.22$.

FIG. 3. The critical $q_c$ and $B_c$ versus $j$ for two different values of $d$. The diamonds and heavy dots show the loci of stable and unstable stationary solutions of TDGLE obtained by numerical simulations. Dashed lines show the critical curves are given by Eq. (12).

FIG. 4. Results of numerical simulations of the TDGLE. The time runs from the top to the bottom: a) $t = 90$, b) $t = 150$ and c) $t = 210$. The parameters used are $d = 30, B = 0.018, j = 0.2$ and the length of the strip $L = 70$. The current is applied along $x$-axis and the magnetic field is perpendicular to the strip. On the gray coded snapshots of $|\Psi(x, y)|$ ($|\Psi(x, y)| = 0$ is shown in black and $|\Psi(x, y)| = 1$ is shown in white) one clearly sees the onset and propagation of vortices.

FIG. 5. The $(j, B)$ critical curve for a wire with $R = 10$
