SPECTRAL ZETA FUNCTION ON DISCRETE TORI AND EPSTEIN-RIEMANN CONJECTURE

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Abstract. We consider the combinatorial Laplacian on a sequence of discrete tori which approximate the $\alpha$-dimensional torus. In the special case $\alpha = 1$, Friedli and Karlsson derived an asymptotic expansion of the corresponding spectral zeta function in the critical strip, as the approximation parameter goes to infinity. There, the authors have also formulated a conjecture on this asymptotics, that is equivalent to the Riemann conjecture. In this paper, inspired by the work of Friedli and Karlsson, we prove that a similar asymptotic expansion holds for $\alpha = 2$. Similar argument applies to higher dimensions as well. A conjecture on this asymptotics gives an equivalent formulation of the Epstein-Riemann conjecture, if we replace the standard discrete Laplacian with the 9-point star discrete Laplacian.

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1. Introduction and statement of the main results

In this paper we focus on two main objectives. First, we study asymptotics of the spectral zeta function on finite torus graphs as they grow to infinity, or equivalently approximate the $m$-dimensional torus after rescaling. Such sequences of graphs play an important role in mathematical physics and statistical mechanics, see e.g. [DuDa88, Lov12, ReVe15] to name a few.

Second objective is to provide a new perspective to the higher-dimensional analogue of the Riemann conjecture — the Epstein-Riemann conjecture, see [Eps03]. Hereby, one intriguing feature is that our reformulation of the Epstein-Riemann conjecture requires a refinement of the standard discrete Laplacian — the 9-point star operator that is widely used in numerical analysis, see e.g. [Bra13].

Our paper is a generalization of the previous work by [FrKa17]. In fact, aspects of spectral analysis on discrete graphs has been the focus of intensive research by several groups. We shall name a few, without attempting to provide a complete list.

In the case of rectilinear polygonal domains, [Ken00] derived a partial asymptotic expansion for the determinant of combinatorial Laplacian. A similar problem in a different discrete setting of half-translation surfaces endowed with a unitary flat vector bundle, has been studied recently in [Fin20]. In the special case of a discrete torus, [CJK10] as well as the second named author in [Ver17] identified the constant term in that expansion in terms of the zeta-regularized determinant of the Laplace-Beltrami operator. This relation to the zeta-regularized determinant was shown to be true in a much more general setting in [IzKh20]. Let us also mention [Sri15], which studied the asymptotic determinant for variations of the Riemannian metric, and a recent result in [TrSa19], where this problem was discussed on a symmetric discretization of surfaces glued together by a finite number of equilateral triangles. In this case, the constant term is given by the zeta regularized determinant plus some lattice depending summands.

While the references above are concerned with the asymptotic behaviour of the discrete determinant, we study asymptotics of the discrete zeta function in the critical strip and its relation to the (Epstein-) Riemann conjecture. In that respect, [FrKa17] and our current work here, stand alone. Our main results are in Theorems 1.10 and 1.12. Informally, they can be formulated as follows.

**Theorem 1.1.** Consider the 9-point star combinatorial Laplacian $\tilde{\Delta}_n$ on a finite torus graph that approximates the 2-dimensional torus $\mathbb{T}^2$ as $n \to \infty$. Then the spectral zeta function of $\tilde{\Delta}_n$ admits an asymptotic expansion as $n \to \infty$ for $s \in \mathbb{C}$ with $\text{Re}(s) \in (0, 1)$

$$
\zeta(\tilde{\Delta}_n, s) \sim A(s)n^{s-2} + B(s) + \sum_{j=1}^{\infty} C_j(s)n^{-2j},
$$

(1.1)
where the coefficients are explicitly computable and $B(s)$ is the spectral zeta function of the Laplace Beltrami operator on $\mathbb{T}^2$. Moreover, if we define
\[
H_n(s) := \pi^{-s} \Gamma(s) \left( \zeta(\tilde{\Delta}_n, s) - A(s)n^{s-2s} \right),
\]
then the Epstein-Riemann conjecture in 2 dimensions is equivalent\(^{\text{1}}\) to
\[
\lim_{n \to \infty} \left| \frac{H_n(1-s)}{H_n(s)} \right| = 1. \tag{1.2}
\]

In the remainder of this section we introduce the setting and formulate our main results explicitly.

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1.1. Combinatorial Laplacians on finite torus graphs. For each integer $n \in \mathbb{N}$ we consider the modular quotient ring $\mathbb{S}_n^1 := \mathbb{Z}/n\mathbb{Z}$. As the notation suggests, this is viewed as a discretization of the smooth submanifold $\mathbb{S}^1 \subset \mathbb{R}^2$.

**Definition 1.2.** Consider any function $u : \mathbb{S}_n^1 \to \mathbb{R}$.

1. The Dirac operator $D_n$ on $\mathbb{S}_n^1$ is defined by
\[
D_n u(k) := \frac{n}{2\pi} (u(k) - u(k-1)).
\]

2. The combinatorial Laplacian $L_n$ on $\mathbb{S}_n^1$ is defined by
\[
L_n u(k) := (D_n^\dagger D_n) u(k) = \frac{n^2}{4\pi^2} (2u(k) - u(k+1) - u(k-1)),
\]

where the adjoint is defined with respect to the usual scalar product on functions $u : \mathbb{S}_n^1 \to \mathbb{R}$ identified with elements of $\mathbb{R}^n$.

Let $\alpha \in \mathbb{N}$ be a positive integer. The $\alpha$-dimensional discrete torus $\mathbb{T}_n^\alpha$ is defined as the Cartesian product of $\alpha$ discrete circles $\times_{i=1}^\alpha \mathbb{S}_n^1$. The corresponding combinatorial Laplacian on $\mathbb{T}_n^\alpha$ acting on functions $u : \mathbb{T}_n^\alpha \to \mathbb{R}$ is defined by
\[
\Delta_n u(k) := \sum_{j=1}^\alpha L_{n,j} u(k), \quad L_{n,j} := \text{id} \otimes \cdots \otimes \text{id} \otimes L_n \otimes \text{id} \otimes \cdots \otimes \text{id}, \tag{1.3}
\]

where $k = (k_1, \ldots, k_\alpha)$ and $L_n$ appears at the $j$-the place of the tensor product, i.e. $L_{n,j}$ is the combinatorial Laplacian on $\mathbb{S}_n^1$ acting on $k_j$. In the case $\alpha = 2$ the combinatorial Laplacian is explicitly given by
\[
\Delta_n u(k) = (L_n \otimes \text{id} + \text{id} \otimes L_n) u(k) = \frac{n^2}{4\pi^2} (4u(k_1, k_2) - u(k_1 + 1, k_2) - u(k_1 - 1, k_2) - u(k_1, k_2 + 1) - u(k_1, k_2 - 1)) \tag{1.4}
\]

\(^{\text{1}}\)This equivalence fails if we instead of $\tilde{\Delta}_n$ we consider the classical combinatorial Laplacian.
This corresponds up to the normalization of \( \frac{n^2}{4\pi^2} \) to the 5-point star Laplacian, known from finite difference methods in numerical analysis. Despite the presence of the normalizing factor, we will still refer to (1.4) as the 5-point star Laplacian. In Figure 1 one sees a schematic representation of the 5-point star Laplacian. Here the number on each node of the figure denotes weighting of this node, as in (1.4).

**Figure 1.** Schematic representation of the 5-point star Laplacian

In our considerations, we also work with a slightly different operator, that can be viewed as a refinement of \( \Delta_n \). The specific reason for its use are much better properties of the coefficients in the asymptotics of the associated spectral zeta function as \( n \to \infty \).

**Definition 1.3.** The 9-point star Laplacian acting on \( u : \mathbb{T}_n^2 \to \mathbb{R} \) is defined by

\[
\tilde{\Delta}_n u(k) := (\mathcal{L}_n \otimes \text{id}) u(k) + (\text{id} \otimes \mathcal{L}_n) u(k) - \frac{2}{3} \frac{n^2}{4\pi^2} (\mathcal{L}_n \otimes \mathcal{L}_n) u(k) \quad (1.5)
\]

Note that the first two summands are the usual combinatorial Laplacian, see (1.4) in two dimensions. A straightforward computation yields the following explicit form

\[
\tilde{\Delta}_n u(j_1, j_2) := \frac{n^2}{4\pi^2} \left( \frac{10}{3} u(j_1, j_2) - \frac{2}{3} u(j_1, j_2) + u(j_1, j_2 - 1) + u(j_1, j_2 - 1) - \frac{1}{6} \left( u(j_1, j_2 - 1) + u(j_1, j_2 - 1) + u(j_1, j_2 + 1) \right) \right) \quad (1.6)
\]

Up to the \( \frac{n^2}{4\pi^2} \) factor, the operator corresponds to the compact 9-point star Laplacian, known from numerical analysis. This operator can be visualized as in Figure 2. Here, exactly as in Figure 1, the number on each node denotes the weighting of this node, see (1.6).
1.2. Spectrum of combinatorial Laplacians on finite torus graphs. The spectrum of the operator $L_n$, represented by an $n \times n$ real-valued matrix, can be computed explicitly and is given by (eigenvalues appear multiple times according to their multiplicity)

$$ \sigma(L_n) = \left\{ \frac{n^2 \pi^2 \sin^2 \left( \frac{\pi k}{n} \right)}{\pi^2} \mid k = 0, \ldots, n-1 \right\}. \quad (1.7) $$

Note that the eigenvalue $\frac{n^2 \pi^2 \sin^2 \left( \frac{\pi k}{n} \right)}{\pi^2}$ with $k \neq 0$ has multiplicity two, i.e. appears twice in the enumeration above: once for $k$ and once again for $n - k$. The eigenvalue $0$ has multiplicity one, i.e. it appears only once in the enumeration above: only for $k = 0$.

The spectrum of $\Delta_n$ is then given in view of (1.7) and (1.3) by

$$ \sigma(\Delta_n) = \left\{ \frac{n^2 \pi^2}{\pi^2} \sum_{j=1}^\alpha \sin^2 \left( \frac{\pi k_j}{n} \right) \mid \forall_{j=1,\ldots,\alpha} : k_j = 0, \ldots, n-1 \right\} \quad (1.8) $$

where each eigenvalue appears multiple times according to its multiplicity. In the case $\alpha = 2$, we can similarly compute the spectrum of the 9-point star Laplacian from (1.7) and (1.5). This proves the formulae in the next proposition.

**Proposition 1.4.** (1) The spectrum of the 5-point star Laplacian $\Delta_n$ is given by

$$ \sigma(\tilde{\Delta}_n) = \left\{ \frac{n^2 \pi^2}{\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) + \frac{n^2 \pi^2}{\pi^2} \sin^2 \left( \frac{\pi k_2}{n} \right) \mid \forall_{j=1,2} : k_j = 0, \ldots, n-1 \right\}. $$
The spectrum of the 9-point star Laplacian $\Delta_n$ is given by
\[
\sigma(\Delta_n) = \left\{ \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_2}{n} \right) - \frac{2n^2}{3\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) \sin^2 \left( \frac{\pi k_2}{n} \right) \left| \forall j = 1,2 : k_j = 0, \ldots, n - 1 \right. \right\}.
\]

In both expressions, eigenvalues appear multiple times according to their multiplicity.

**Remark 1.5.** Note that despite a negative summand in Proposition 1.4 (2), the 9-point star Laplacian $\Delta_n$ is still a non-negative operator. Indeed, its eigenvalues are of the form $\frac{n^2}{\pi^2}(a^2 + b^2 - \frac{2}{3}a^2b^2)$ with $|a|, |b| < 1$. These are non-negative, since
\[
a^2 + b^2 - \frac{2}{3}a^2b^2 \geq a^2 + b^2 - \frac{2}{3}a^2 = \frac{1}{3}a^2 + b^2 \geq 0.
\]

### 1.3. Spectral zeta functions of combinatorial Laplacians.

The spectral zeta function $\zeta(s, T)$ of a non-negative symmetric linear operator $T$ acting on a finite dimensional real vector space is defined by summing over all its non-zero eigenvalues taken to the $s$-th power, with $s \in \mathbb{C}$. Hereby, non-negativity of eigenvalues allows to use the principal branch of logarithm in the definition of the complex $s$ power. For $\Delta_n$ and $\tilde{\Delta}_n$ we find in view of Proposition 1.4

\[
\zeta(s, \Delta_n) := \sum_{k_1, k_2 = 0}^{n-1} \left( \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_2}{n} \right) \right)^{-s},
\]
\[
\zeta(s, \tilde{\Delta}_n) := \sum_{k_1, k_2 = 0}^{n-1} \left( \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_2}{n} \right) - \frac{2n^2}{3\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) \sin^2 \left( \frac{\pi k_2}{n} \right) \right)^{-s},
\]

where we exclude $(k_1, k_2) = (0, 0)$ from summation.

### 1.4. Hadamard regularized limits and integrals.

In order to define spectral zeta function of the Laplace Beltrami operator, that is the analytic counterpart of the combinatorial 5- and 9-point star Laplacians in the limit as $n \to \infty$, we need to introduce Hadamard regularized integrals.

Consider function $u \in C^\infty(0, \infty)$ such that for $x \to \infty$
\[
u(x) = \sum_{j=1}^{N} \sum_{k=0}^{n_j} a_{jk} x^{a_j} \log^{k}(x) + \sum_{k=0}^{n_0} a_{0k} \log^{k}(x) + o(x^{a_N} \log^{n_N}(x))
\]
for some $N \in \mathbb{N}$, with $(a_j)_{j=1,\ldots,N} \subset \mathbb{C}$ with $\text{Re}(a_j)$ monotonously decreasing and $\text{Re}(a_N) < 0$. We define the regularized limit of $u$ as $x \to \infty$ by the constant term in the asymptotic
\[
\text{LIM}_{x \to \infty} u(x) := a_{00}.
\]
If $\text{Re}(\alpha_N) < -1$, the integral of $u$ over $[1, R]$ has an asymptotic expansion of the same form as in (1.11) for $R \to \infty$. In that case we define the regularized integral as follows

$$
\int_1^\infty u(x)\,dx := \text{LIM}_{R \to \infty} \int_1^R u(x)\,dx
$$

Similarly, if $u(x)$ as an asymptotic expansion for $x \to 0$ of the form

$$
u(x) = \sum_{j=1}^{N'} \sum_{k=0}^{n_j} b_{jk} x^{a_j} \log^k(x) + \sum_{k=0}^{n_0} b_{0k} \log^k(x) + o(x^{\alpha_{N'}} \log^{n_{N'}}(x))
$$

for some $N' \in \mathbb{N}$, with $(\alpha_j)_{j=1}^{N'} \subset \mathbb{C}$ with $\text{Re}(\alpha_j)$ monotonously increasing and $\text{Re}(\alpha_{N'}) > 0$, then we can define the regularized limit of $u$ at zero. In the same way, if $\text{Re}(\alpha_{N'}) > -1$, we also can define the regularized integral on $[0, 1]$. If both regularized integrals are well-defined, we may write

$$
\int_0^\infty u(x)\,dx := \int_0^1 u(x)\,dx + \int_1^\infty u(x)\,dx.
$$

The regularized integral satisfies a peculiar change of variables rule.

**Proposition 1.6** ([Les96] Lemma 2.1.4). Assume $u$ satisfies (1.11) and (1.12) with $\text{Re}(\alpha_N) < -1$ and $\text{Re}(\alpha_{N'}) > -1$. Then for $\lambda > 0$ the following holds

$$
\int_0^\infty u(\lambda x)\,dx = \lambda^{-1}\left(\int_0^\infty u(x)\,dx + \sum_{k=0}^{m_j} b_{jk} \frac{\log^{k+1}(\lambda)}{k+1} - \sum_{k=0}^{n_{N'}} a_{j/k} \frac{\log^{k+1}(\lambda)}{k+1}\right)
$$

where $a_{j/k}$ is the coefficient to the term $x^{-1} \log^k(x)$ in the asymptotic expansion of $u$ as $x \to \infty$ and $b_{jk}$ is the coefficient to the term $x^{-1} \log^k(x)$ in the asymptotic expansion of $u$ as $x \to 0$.

**Remark 1.7.** If in both asymptotic expansions (1.11) and (1.12) no $x^{-1} \log^k x$ terms exist, then it is just the usual change of variables rule.

1.5. **Spectral zeta function of the Laplace Beltrami operator on a torus.** Now let us consider the Laplace Beltrami operator $\Delta$ on the $\alpha$-dimensional torus $T^\alpha = \times_{i=1}^\alpha S^1$. Similar to (1.3), we can write $\Delta$ in terms of the Laplace-Beltrami operator $\Delta_{S^1}$, acting on the individual components $S^1$

$$
\Delta u(x_1, \cdots, x_\alpha) = \sum_{j=1}^\alpha \Delta_{S^1,j} u(x_1, \cdots, x_\alpha),
$$

where $\Delta_{S^1,j}$ is the operator $\Delta_{S^1}$, acting on $x_j$. The functions $\left(\frac{1}{\sqrt{2\pi}} e^{ikx}\right)_{k \in \mathbb{Z}}$ form an orthonormal basis of $L^2(S^1)$ of eigenfunctions of $\Delta_{S^1}$ to the eigenvalues $k^2, k \in \mathbb{Z}$. The eigenvalues $k^2 \neq 0$ have multiplicity two, the eigenvalue $k^2 = 0$ has multiplicity one. Thus the spectrum of $\Delta$ is given by

$$
\sigma(\Delta) = \left\{ k_1^2 + \cdots + k_\alpha^2 : k = (k_1, \ldots, k_\alpha) \in \mathbb{Z}^\alpha \right\},
$$

where the eigenvalues appear multiple times according to their multiplicity.
The spectral zeta function $\zeta(\Delta, s)$ of $\Delta$ is defined by summing over non-zero eigenvalues, according to their multiplicity, with $(-s)$-exponent. That sum converges and is holomorphic for $\Re(s) > \frac{\alpha}{2}$. In this work we will use the following integral representation for $\Re(s) > \frac{\alpha}{2}$, cf. [LeVe13, (1.24), (1.26)]

$$\zeta(\Delta, s) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} (k_1^2 + \cdots + k_n^2)^{-s}$$

$$= \frac{2}{\pi} \frac{\sin(\pi s)}{\Gamma(1 - s)\Gamma(\alpha)} \int_0^\infty z^{2\alpha - 2s - 1} \text{Tr}(\Delta + z^2)^{-\alpha} \, dz,$$

where in the first sum we used the multi-index notation $k = (k_1, \ldots, k_n)$. In the second expression, $(\Delta + z^2)^{-\alpha}$ denotes $\alpha$-th power of the resolvent of $\Delta$, $\text{Tr}$ the trace, and $\Gamma$ the Gamma function. The front factor will be abbreviated by

$$V_\alpha(s) = \frac{2}{\pi} \frac{\sin(\pi s)}{\Gamma(1 - s)\Gamma(\alpha)}\frac{\Gamma(1 - s)\Gamma(\alpha)}{\Gamma(\alpha - s)}.$$  

The standard asymptotic expansion of the resolvent trace $\text{Tr}(\Delta + z^2)^{-\alpha}$ gives a meromorphic continuation of (1.14) to the whole complex plane. Classical references on the spectral zeta function of $\Delta$ and its applications are e.g. [MiPl49, See67, Haw77].

Exactly the same integral representation (1.14) holds for the spectral zeta functions $\zeta(s, \Delta_n)$ and $\zeta(s, \tilde{\Delta}_n)$ of the combinatorial 5- and 9-point star Laplacians, where $\alpha = 2$ and $\text{Tr}(\Delta + z^2)^{-\alpha}$ is replaced by

$$\text{Tr}(\Delta_n + z^2)^{-2} = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \left( \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_2}{n} \right) + z^2 \right)^{-2},$$

$$\text{Tr}(\tilde{\Delta}_n + z^2)^{-2} = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \left( \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k_2}{n} \right) \right) - \frac{2n^2}{3\pi^2} \sin^2 \left( \frac{\pi k_1}{n} \right) \sin^2 \left( \frac{\pi k_2}{n} \right) + z^2 \right)^{-2},$$

(1.16)

respectively. The explicit structure of these combinatorial resolvent traces follows directly from the explicit form of the eigenvalues in Proposition 1.4.

1.6. **Riemann- and Epstein zeta function.** The spectral zeta function on the torus $\mathbb{T}^n$ is a special case of a wide class of zeta functions, namely the Epstein zeta functions. Those functions are defined for any real-valued positive definite $\alpha \times \alpha$-matrix $Q$ by

$$\zeta_Q(s) := \sum_{k \in \mathbb{Z}^n \setminus \{0\}} (k^T Q k)^{-s}, \quad \Re(s) > \frac{\alpha}{2}.$$  

(1.17)

If $Q = \text{id}$ is the identity matrix, we recover the spectral zeta function of the torus

$$\zeta_{\text{id}}(s) = \zeta(\Delta, s),$$  

(1.18)

for all $\alpha \in \mathbb{N}$. Epstein zeta functions generalize many concepts of the Riemann zeta function. First of all, $\zeta_Q(s)$ can be continued to a meromorphic function to
the whole complex plane. The poles are simple poles at \( s = 0 \) and \( s = \alpha/2 \). The Epstein zeta function also has trivial zeros, located at \( s = -z \) for \( z \in \mathbb{N} \).

A very common technical tool in working with the Epstein zeta function is the complete Epstein zeta function, also called the Epstein xi function. It is defined by

\[
\xi_Q(s) := \pi^{-s}\Gamma(s)\zeta_Q(s).
\]

(1.19)

The complete Epstein zeta function satisfies the functional equation

\[
\xi_Q(s) = (\det Q)^{-1/2}\xi_{Q^{-1}}(\alpha/2 - s),
\]

(1.20)

which is due to [Eps03]. This relation indicates some interesting behaviour of \( \zeta_Q(s) \) in the so-called critical strip \( s \in \mathbb{C} : 0 < \Re(s) < \alpha/2 \). Zeros located in the critical strip are called critical zeros. Similar to the Riemann conjecture, we can formulate a conjecture for the Epstein zeta function

**Conjecture 1.1** (Epstein Riemann (abbreviated as E.R.) conjecture).

All critical zeros \( s \in \mathbb{C} \) (i.e. zeros with \( 0 < \Re(s) < \alpha/2 \)) of the Epstein zeta function \( \zeta_{id}(s) \) have real part \( \Re(s) = \alpha/4 \).

In dimensions \( \alpha = 2, 4, 6 \) and \( 8 \), the Epstein zeta function \( \zeta_{id}(s) \) (and hence by (1.18) also \( \zeta(\Delta, s) \) — the spectral zeta function on a torus) can be expressed in terms of the Riemann zeta function \( \zeta_R(s) \), see [Zuc74]. For example, in case \( \alpha = 2 \) we have the beautiful formula due to Glasser, see [Zuc74, (1.1)]

\[
\zeta_{id}(s) \equiv \zeta(\Delta, s) = 4\zeta_R(s)\beta(s),
\]

(1.21)

where \( \beta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^s} \) the Dirichlet beta function, that can be viewed as a Dirichlet \( L \)-function of a Dirichlet-character \( \chi \). It is conjectured by numerical evidence that the non-trivial zeros of \( \beta(s) \) are located on the critical line \( \Re(s) = 1/2 \), cf. e.g. [Kaw20]. Provided this were true, for \( \alpha = 2 \), the Epstein-Riemann conjecture and the Riemann conjectures were equivalent.

For \( \alpha \geq 3 \) the Epstein-Riemann (E.R.) conjecture fails, as is demonstrated by the formulae in [TrSa19, (1.4) – (1.6)], as well as other numerical computations in [TrSa19, Section 5]. We summarize this as the answer to the Conjecture (1.1).

**Answer 1.8.**

1. For \( \alpha = 1 \) the E.R. conjecture is the Riemann conjecture.
2. For \( \alpha = 2 \) the E.R. conjecture is equivalent to the Riemann conjecture, if zeros of Dirichlet beta function \( \beta(s) \) are located on the critical line \( \Re(s) = 1/2 \).
3. For \( \alpha \geq 3 \) the E.R. conjecture is wrong. However it may still hold for imaginary part of critical zeros being sufficiently large.

The main contribution of this work is the relation between the E.R. conjecture and the asymptotics of \( \zeta(\tilde{\Delta}_n, s) \) in dimension \( \alpha = 2 \), with a clear path for generalization to higher dimensions.

\footnote{In fact [McPhe13] claims this is equivalent to the Riemann conjecture.}
1.7. **Statement of the main results.** For an arbitrary $\alpha \in \mathbb{N}$, the eigenvalues of the combinatorial Laplacians $\Delta_n$ and $\tilde{\Delta}_n$ converge to the eigenvalues of the Laplace Beltrami operator $\Delta$ as $n \to \infty$. Hence for any $s \in \mathbb{C}$ with $\text{Re}(s) > \alpha/2$

$$
\lim_{n \to \infty} \zeta(\Delta_n, s) = \lim_{n \to \infty} \zeta(\tilde{\Delta}_n, s) = \zeta(\Delta, s).
$$

However, for $s$ in the critical strip $0 < \text{Re}(s) < \alpha/2$, convergence fails and is replaced by an intricate asymptotic expansion. For $\alpha = 1$, in [FrKA17, Theorem 3] the authors have shown a partial asymptotics of the following form\(^3\)

**Theorem 1.9** ([FrKA17] Theorem 3). For $\text{Re}(s) \in (0, 1/2)$ we have

$$
\zeta(\mathcal{L}_n, s) = \pi^{2s-1/2} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)} n^{1-2s} + 2\zeta(2s) + \frac{2s}{3} \pi^2 \zeta(2s - 2)n^{-2} + o(n^{-2})
$$

$$
= \pi^{2s-1/2} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)} n^{1-2s} + \zeta(\Delta_S, s) + \frac{s}{3} \pi^2 \zeta(\Delta_S, s - 1)n^{-2} + o(n^{-2}).
$$

For an arbitrary $\alpha \in \mathbb{N}$ in [FrKA17, Theorem 1] the authors established a one term shorter partial asymptotic expansion, with the first two terms explicitly identified and existence & structure of the next term left as an open question. Our first main result addresses this open question for $\alpha = 2$.

**Theorem 1.10.** For $\text{Re}(s) \in (0, \alpha/2)$ we have for any integer $M \in \mathbb{N}$ as $n \to \infty$

$$
\zeta(\Delta_n, s) = V_\alpha(s) \left( a(s) n^{\alpha - 2s} + \sum_{m=0}^{M-1} b_m(s) n^{-2m} \right) + O(n^{-2M-2s+2}),
$$

$$
\zeta(\tilde{\Delta}_n, s) = V_\alpha(s) \left( \tilde{a}(s) n^{\alpha - 2s} + \sum_{m=0}^{M-1} \tilde{b}_m(s) n^{-2m} \right) + O(n^{-2M-2s+2}).
$$

The leading coefficients $a(s)$ and $\tilde{a}(s)$ are explicitly given by

$$
a(s) = \int_0^\infty z^{2\alpha - 2s - 1} \int_0^1 \int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2 \right)^{-\alpha} \text{d}x \text{d}y \text{d}z,
$$

$$
\tilde{a}(s) = \int_0^\infty z^{2\alpha - 2s - 1} \int_0^1 \int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2 \right)^{-\alpha} \text{d}x \text{d}y \text{d}z - \frac{2n^2}{3\pi^2} \sin^2\left(\frac{\pi x}{n}\right) \sin^2\left(\frac{\pi y}{n}\right) \text{d}x \text{d}y \text{d}z.
$$

The first two higher order coefficients are explicitly given by

$$
b_0(s) = \tilde{b}_0(s) = V_\alpha(s)^{-1} \zeta(\Delta, s), \quad \tilde{b}_1(s) = \frac{s\pi^2}{3} V_\alpha(s)^{-1} \zeta(\Delta, s - 1),
$$

$$
b_1(s) = \frac{s\pi^2}{3} V_\alpha(s)^{-1} \zeta(\Delta, s - 1) + \frac{4\pi^2 V_2(s)}{\alpha - s} \int_0^\infty z^{4 - 2s + 1} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{k_1^2 k_2^2}{(k_1^2 + k_2^2 + z^2)^2}.
$$

\(^3\)The formula in [FrKA17, Theorem 3] differs from the presentation here by normalization.
Remark 1.11. The computations in Section 2 can be adapted in a completely analogous way to higher dimensions. For this reason, we will continue writing $\alpha$ instead of the explicit 2, as above. Naturally, our techniques may be applied to the one-dimensional case as well and would reprove the result from [FrKA17].

Our second main result relates the asymptotics of Theorem 1.10 to the Epstein-Riemann conjecture. For this it is crucial for the $n^{-2}$ coefficient in (1.22) to satisfy some functional equation. The coefficient $b_1(s)$ contains an additional term, known as an angular lattice sum, cf. [BGM13]. Due to this term, $b_1(s)$ does not admit a functional relation. This is why we have introduced a "corrected" discrete Laplacian $\tilde{\Delta}_n$ with coefficient $\tilde{b}_1(s)$, which no longer has any angular lattice sum term and satisfies a functional relation. Therefore we use $\tilde{\Delta}_n$ to obtain an equivalent reformulation of the E.R. conjecture.

Theorem 1.12. Consider (1.22) and define for $s \in \mathbb{C}$ with $\text{Re}(s) \in (0, 1)$

$$H_n(s) := \pi^{-3} \Gamma(s) \left( \zeta(\tilde{\Delta}_n, s) - V_\alpha(s) \tilde{a}(s)n^{\alpha-2s} \right)$$

Then, the Epstein-Riemann conjecture 1.1 for $\alpha = 2$ is equivalent to

$$\lim_{n \to \infty} \left| \frac{H_n(1-s)}{H_n(s)} \right| = 1. \quad (1.25)$$

By the Answer 1.8 the Epstein-Riemann conjecture for $\alpha = 2$ is related to the Riemann conjecture and the generalized Riemann conjecture for the Dirichlet beta function. We hope that our arguments apply to derive similar reformulations for $\alpha \geq 3$. We expect that in higher dimensions further refinements of the discrete Laplacian will be necessary.

Our result fits the work [FrKA17] and [Fri16], where similar statements are proved for the Riemann zeta function and certain Dirichlet L-functions. The case for $\alpha = 2$ shows the most similarities with the statements in these previous works. As discussed above, other Epstein zeta functions do not satisfy an Epstein-Riemann conjecture. It might be interesting to consider discrete spectral zeta functions in higher dimensions to examine how the E.R. conjecture has to be adapted.

2. Euler-Maclaurin formula and the discrete resolvent trace

2.1. Generalities on the Euler-Maclaurin-formula. In this section, we derive integral representations of the discrete resolvent traces in (1.16), using the classical Euler-Maclaurin-formula iteratively. Let us recall the latter here for convenience.

\footnote{The authors were surprised to see that the necessary correction that annihilates the angular lattice term, in fact leads to the well-known 9-point star Laplacian.}
Theorem 2.1. For any $M, n \in \mathbb{N}$ and $u \in C^{2M+1}([0, n])$ we have the identity
\[
\sum_{i=0}^{n} u(i) = \int_{0}^{n} u(x)dx + \frac{1}{2}(u(0) + u(n)) + \sum_{j=1}^{M} \frac{B_{2j}}{(2j)!} \left( u^{(2j-1)}(n) - u^{(2j-1)}(0) \right) + \frac{1}{(2M+1)!} \int_{0}^{n} B_{2M+1}(x-[x])u^{(2M+1)}(x)dx,
\]
where the $B_k$'s are the $k$-th Bernoulli-numbers and $B_k(x)$ denotes the Bernoulli-polynomial of order $k$.

There are various generalizations of the Euler-Maclaurin-formula. We mention those that are most closely related to the problem studied here. First, [LyMc69] derived an Euler-Maclaurin-formula for summation in multiple variables. Application of that formula would provide an alternative argument for our problem here. However, one encounters a problem to control the decay of the error-terms. Moreover, for an arbitrary number of summations, the formulas become cumbersome.

Another generalization of the Euler-Maclaurin-formula was formulated in [Sto04] and [MoLy98]. In [Sto04] functions with algebraic singularities at the endpoints of the interval are allowed. In this generalization, the integral part of the expansion is allowed to exist in the Hadamard finite part sense. Those algebraic singularities are precisely those that appear here as well. However, application of [Sto04] then becomes problematic in the second iteration. This is why we only use the classical statement as in Theorem 2.1 iteratively.

There are also combinations of these two generalizations e.g. [VPL97], and many more. We do not attempt to compile a complete list.

Theorem 2.1 implies by subtracting $u(n)$ the following formula
\[
\sum_{i=0}^{n-1} u(i) = \int_{0}^{n} u(x)dx + \frac{1}{2}(u(0) - u(n)) + \sum_{j=1}^{M} \frac{B_{2j}}{(2j)!} \left( u^{(2j-1)}(n) - u^{(2j-1)}(0) \right) + \frac{1}{(2M+1)!} \int_{0}^{n} B_{2M+1}(x-[x])u^{(2M+1)}(x)dx.
\]
We introduce an auxiliary notation for the terms in (2.2)
\[
I_{x_{i}}^{n}u := \int_{0}^{n} u(x_{i})dx_{i}, \quad A_{x_{i}}^{n} := \frac{1}{2} \left( u(x_{i} = n) - u(x_{i} = 0) \right),
\]
\[
D_{x_{i},M}^{n}u := \sum_{j=1}^{M} \frac{B_{2j}}{(2j)!} \left( \partial_{x_{i}}^{2j+1}u(x_{i} = n) - \partial_{x_{i}}^{2j+1}u(x_{i} = 0) \right),
\]
\[
E_{x_{i},M}^{n}u := \frac{1}{(2M+1)!} \int_{0}^{n} B_{2M+1}(x_{i} - [x_{i}]) \partial_{x_{i}}^{2M+1}u(x_{i})dx.
\]
So (2.2), with summation over $x_{i}$, can be written in the short form as
\[
\sum_{j=0}^{n-1} u(x_{i} = j) = I_{x_{i}}^{n}u + A_{x_{i}}^{n}u + D_{x_{i},M}^{n}u + E_{x_{i},M}^{n}u.
\]
2.2. Estimates of the resolvent trace of 5-point star Laplacian $\Delta_n$.

In view of (1.16), we simplify notation below by introducing

$$f(x, y, n, z) := \frac{n^2}{\pi^2} \sin^2 \left( \frac{n \pi x}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{n \pi y}{n} \right) + z^2. \quad (2.5)$$

We want to apply (2.2) to the resolvent trace in (1.16) for the classical (5-point star) combinatorial Laplacian $\Delta_n$ on $\mathbb{T}_n^2$.

$$\text{Tr}(\Delta_n + z^2)^{-\alpha} = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} f(k_1, k_2, n, z)^{-\alpha}. \quad (2.6)$$

**Proposition 2.2.** For any $z, \alpha > 0$ and positive integers $M$ and $n$, the resolvent trace $\text{Tr}(\Delta_n + z^2)^{-\alpha}$ has the representation

$$\text{Tr}(\Delta_n + z^2)^{-\alpha} = n^{-2\alpha+2} \int_0^1 \int_0^1 f(x, y, 1, z/n)^{-\alpha} \, dx \, dy \quad (2.6)$$

$$+ 2 I^n_y \circ E^n_{x,M} f^{-\alpha} + E^n_{y,M} \circ E^n_{x,M} f^{-\alpha},$$

where for some constant $C > 0$, depending only on $M$

$$\left| I^n_y \circ E^n_{x,M} f^{-\alpha} \right| \leq C n^{-2M-2\alpha+1} \left( \frac{z}{n} \right)^{-2\alpha-1} \left( \sqrt{\left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2}} \right)^{-1}, \quad (2.7)$$

$$\left| E^n_{y,M} \circ E^n_{x,M} f^{-\alpha} \right| \leq C n^{-4M-2\alpha} \left( \frac{z}{n} \right)^{-2\alpha-2} \left( \sqrt{\left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2}} \right)^{-2}.$$

**Proof.** We compute the individual operators in (2.3) applied to $f(x, y, n, z)^{-\alpha}$. Since $f(0, y, n, z) = f(n, y, n, z)$, we find

$$A^n_y f(x, y, n, z)^{-\alpha} = 0.$$

For every odd derivative of $f(x, y, n, z)^{-\alpha}$ we get a factor $\sin(\frac{m\pi}{n})$ in all summands. These vanish at $x = 0$ and $x = n$. Thus

$$D^n_{x,M} f(x, y, n, z)^{-\alpha} = 0.$$

Applying now (2.2), we conclude

$$\text{Tr}(\Delta_n + z^2)^{-\alpha} = \sum_{k_1=0}^{n-1} I^n_x f(k_2, n, z)^{-\alpha} + \sum_{k_2=0}^{n-1} E^n_{x,M} f(k_2, n, z)^{-\alpha}.$$

Repeat the same argument, applying (2.4) to both remaining sums. As before we have that $D^n_{y,M} f(x, y, n, z)^{-\alpha} = 0$ and $A^n_y f(x, y, n, z)^{-\alpha} = 0$. Hence we arrive at

$$\text{Tr}(\Delta_n + z^2)^{-\alpha} = I^n_y \circ I^n_x f^{-\alpha} + 2 \cdot I^n_y \circ E^n_{x,M} f^{-\alpha} + E^n_{y,M} \circ E^n_{x,M} f^{-\alpha} \quad (2.8)$$

By a change of variables $\tilde{x} = x/n$ and $\tilde{y} = y/n$, the individual summands in (2.8) can be written as follows (note that $\partial_{\tilde{x}} = n^{-1} \partial_x$ and $\partial_{\tilde{y}} = n^{-1} \partial_y$)
It remains to estimate the last two terms. We write $x, y$ for $\tilde{x}, \tilde{y}$, respectively and note for the partial derivatives of $f(x, y, 1, z/n)^{-\alpha}$

$$
\partial^{-2\alpha+1}_x f(x, y, 1, z/n)^{-\alpha} = \sum_{i=0}^{2M} A_i(x, y) f(x, y, 1, z/n)^{-\alpha-i-1},
$$

$$
\partial^{-2\alpha+1}_y f(x, y, 1, z/n)^{-\alpha} = \sum_{i=0}^{4M} B_i(x, y) f(x, y, 1, z/n)^{-\alpha-i-2},
$$

(2.9)

where $A_i, B_i \in C^\infty([0, 1]^2)$ are independent of $z$ and $n$. We estimate $A_i, B_i$ and the Bernoulli polynomials $B_{2M+1}$ against a constant, $f(x, y, 1, z/n)^{-\alpha-i}$ against $(z/n)^{-2\alpha}$, and obtain for some constant $C > 0$, depending only on $M$

$$
\left| \int_{y,M}^n \int_0^n \partial^{-2\alpha+1}_y f(x, y, 1, z/n)^{-\alpha} \right| \leq C \left( \frac{n^{-2\alpha+1}}{(2M+1)!} \right) \left( \frac{z}{n} \right)^{-2\alpha} \int_0^1 f\left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{z^2}{n^2} \right)^{-1} dx,
$$

(2.10)

$$
\left| \int_{x,M}^n \int_0^n \partial^{-2\alpha+1}_x f(x, y, 1, z/n)^{-\alpha} \right| \leq C \left( \frac{n^{-4\alpha+2}}{(2M+1)!} \right) \left( \frac{z}{n} \right)^{-2\alpha} \int_0^1 f\left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{z^2}{n^2} \right)^{-1} dx \times \int_0^1 f\left( \frac{\sin^2(\pi y)}{\pi^2} + \frac{z^2}{n^2} \right)^{-1} dy.
$$

By [ZMGR14] pp. 177, 2.562 we know that for $b/a > -1$

$$
\int_{a+b}^{a+b} \frac{dx}{a + b \sin^2(x)} = \frac{\text{sign}(a)}{\sqrt{a(a+b)}} \arctan\left( \frac{\sqrt{a+b} \tan(x)}{a} \right).
$$

(2.11)

Since $\tan(\pm \pi/2) = \pm \infty$ and $\arctan(\pm \infty) = \pm \pi/2$ we get using $\sin^2(\pi x) \geq 0$ for all $x \in [0, 1]$ and symmetry

$$
\int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{z^2}{n^2} \right)^{-1} dx = \frac{1}{\frac{z}{n} \sqrt{\left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2}}}. 
$$

(2.12)
Summarizing, we have shown for some constant $C > 0$, depending only on $M$

$$I^n_y \circ I^n_x f^{-\alpha} = n^{-2\alpha + 2} \int_0^1 \int_0^1 f(x, y, 1, z/n)^{-\alpha} \, dx \, dy,$$

$$| I^n_y \circ E^n_{x,M} f^{-\alpha} | \leq C \, n^{-2M - 2\alpha + 1} \left( \frac{z}{n} \right)^{-2\alpha - 1} \left( \sqrt{ \left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2} } \right)^{-1},$$

$$| E^n_{y,M} \circ E^n_{x,M} f^{-\alpha} | \leq C \, n^{-4M - 2\alpha} \left( \frac{z}{n} \right)^{-2\alpha - 2} \left( \sqrt{ \left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2} } \right)^{-2}.$$

This proves in view of (2.8) the statement. □

2.3. Estimates of the resolvent trace of $9$-point star Laplacian $\tilde{\Delta}_n$.

In view of (1.16), we simplify notation below by introducing

$$g(x, y, n, z) := \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi x}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi y}{n} \right) - \frac{2n^2}{3\pi^2} \sin^2 \left( \frac{\pi x}{n} \right) \sin^2 \left( \frac{\pi y}{n} \right) + z^2. \quad (2.13)$$

By (1.16), we have

$$\operatorname{Tr}(\tilde{\Delta}_n + z^2)^{-\alpha} = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} g(k_1, k_2, n, z)^{-\alpha}. \quad (2.14)$$

Now exactly the same argument as in Proposition 2.2 carries over with $f$ replaced by $g$, and the only difference lies in the estimate (2.10). There, the integral is replaced by (cf. the inequality in (1.9))

$$\int_0^1 \left( \frac{\sin^2(\pi x)}{3\pi^2} + \frac{z^2}{n^2} \right)^{-1} \, dx = \frac{\sqrt{3}}{\frac{n}{\sqrt{3}} \left( \frac{z}{n} \right) + \frac{1}{\pi^2}},$$

where we used (2.11) again. Thus, we obtain the following counterpart to Proposition 2.2.

**Proposition 2.3.** For any $z, \alpha > 0$ and positive integers $M$ and $n$, the resolvent trace $\operatorname{Tr}(\tilde{\Delta}_n + z^2)^{-\alpha}$ has the representation

$$\operatorname{Tr}(\tilde{\Delta}_n + z^2)^{-\alpha} = n^{-2\alpha + 2} \int_0^1 \frac{f(x, y, 1, z/n)^{-\alpha} \, dx \, dy}{g(x, y, n, z)^{-\alpha} + 2 I^n_y \circ E^n_{x,M} g^{-\alpha} + E^n_{y,M} \circ E^n_{x,M} g^{-\alpha},} \quad (2.15)$$

where for some constant $C > 0$, depending only on $M$ and $\alpha$

$$| I^n_y \circ E^n_{x,M} f^{-\alpha} | \leq C \, n^{-2M - 2\alpha + 1} \left( \frac{z}{n} \right)^{-2\alpha - 1} \left( \sqrt{3 \left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2} } \right)^{-1},$$

$$| E^n_{y,M} \circ E^n_{x,M} f^{-\alpha} | \leq C \, n^{-4M - 2\alpha} \left( \frac{z}{n} \right)^{-2\alpha - 2} \left( \sqrt{3 \left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2} } \right)^{-2}. \quad (2.16)$$
3. Asymptotic expansion of the discrete spectral zeta function

In this section we use Propositions 2.2 and 2.3 to derive a complete asymptotic expansion of the spectral zeta functions \( \zeta(\Delta_n, s) \) and \( \zeta(\Delta_n, s) \) as \( n \to \infty \) in the critical strip \( \text{Re}(s) \in (0, \alpha/2) \). Recall the auxiliary functions (2.5) and (2.13)

\[
\begin{align*}
  f(x, y, n, z) &:= \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi x}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi y}{n} \right) + z^2, \\
g(x, y, n, z) &:= \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi x}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi y}{n} \right) + z^2 \\
  &\quad - \frac{2n^2}{3\pi^2} \sin^2 \left( \frac{\pi x}{n} \right) \sin^2 \left( \frac{\pi y}{n} \right).
\end{align*}
\]

(3.1)

3.1. Uniform series representation for \( f^{-\alpha} \) and \( g^{-\alpha} \). Recall the notation for \( f(x, y, n, z) \) from (2.5). We will use the series representation, derived here, decisively in the next subsection.

**Lemma 3.1.** We have the following series representation for any \( N \in \mathbb{N} \)

\[
\begin{align*}
f(x, y, n, z)^{-\alpha} &= \sum_{m=0}^{\infty} n^{-2m} \sum_{j=0}^{m} \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \\
  &=: \sum_{m=0}^{N-1} n^{-2m} \sum_{j=0}^{m} \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} + n^{-2N} \frac{G \cdot (x^2 + y^2)^N}{(x^2 + y^2 + z^2)^\alpha}
\end{align*}
\]

(3.2)

\[
\begin{align*}
g(x, y, n, z)^{-\alpha} &= \sum_{m=0}^{\infty} n^{-2m} \sum_{j=0}^{m} \frac{\tilde{F}_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \\
  &=: \sum_{m=0}^{N-1} n^{-2m} \sum_{j=0}^{m} \frac{\tilde{F}_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} + n^{-2N} \frac{\tilde{G} \cdot (x^2 + y^2)^N}{(x^2 + y^2 + z^2)^\alpha}
\end{align*}
\]

where, \( F_{m,j} \) and \( \tilde{F}_{m,j} \) are symmetric polynomials in \( (x, y) \), homogeneous of order \( 2m + 2j \). Moreover, if \( n \geq n_0 \) sufficiently large, \( G \) and \( \tilde{G} \) are bounded uniformly in \( x, y, z \geq 0 \).

**Proof.** We prove the statement for \( f \). The statement for \( g \) follows along the same lines. Using \( \sin^2 \theta = (1 - \cos 2\theta)/2 \) and the cosinus series, we obtain for \( u \in \mathbb{C} \)

\[
\frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi u}{n} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k-2}}{(2k)!} \frac{u^{2k}}{n^{2k-2}}.
\]

(3.3)

Plugging this in \( f^{-\alpha}(x, y, n, z) \) for \( x \) and \( y \), we obtain

\[
\begin{align*}
f(x, y, n, z)^{-\alpha} &= \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k-2}}{(2k)!} \frac{u^{2k}}{n^{2k-2}} (x^{2k} + y^{2k}) \right)^{-\alpha} \\
  &= (x^2 + y^2 + z^2)^{-\alpha} \left( 1 - \sum_{k=2}^{\infty} \frac{(-1)^{k+1} 2^{2k-1} \pi^{2k-2}}{(2k)!} \frac{2^{2k} \pi^{2k}}{n^{2k-2}} (x^{2k} + y^{2k}) \right)^{-\alpha} \\
  &= (x^2 + y^2 + z^2)^{-\alpha} \left( 1 - \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2^{2k+3} \pi^{2k+2}}{(2k+4)!} \frac{2^{2k+1} \pi^{2k+2}}{n^{2k+2}} (x^{2k+4} + y^{2k+4}) \right)^{-\alpha}.
\end{align*}
\]
We simplify notation by setting
\[ d_k := \frac{(-1)^{k+1}}{(2k+4)!} 2^{2k+3} \pi^{2k+2}. \]
Then we find by the generalized binomial formula
\[ f(x, y, n, z)^{-\alpha} = (x^2 + y^2 + z^2)^{-\alpha} \sum_{j=0}^{\infty} \binom{\alpha + j - 1}{j} \left( \sum_{k=0}^{\infty} d_k \frac{(x^{2k+4} + y^{2k+4})}{(x^2 + y^2 + z^2)} n^{-2k-2} \right)^j. \]
In order to evaluate the \( j \)-th power of the infinite sum above, we recall the general rule for the \( j \)-th Cauchy product of an absolutely convergent series
\[ \left( \sum_{k=0}^{\infty} a_k \right)^j = \sum_{k=0}^{\infty} \sum_{k_1=0}^{k} \cdots \sum_{k_j=0}^{k_{j-1}} a_{k_1} \cdots a_{k_j} \prod_{\ell=1}^{j-1} a_{k_{\ell} - k_{\ell+1}}. \] (3.4)
This leads to the following expression
\[ \left( \alpha + j - 1 \right) \binom{\alpha + j - 1}{j} \left( \sum_{k=0}^{\infty} d_k \frac{(x^{2k+4} + y^{2k+4})}{(x^2 + y^2 + z^2)} n^{-2k-2} \right)^j = \sum_{k=0}^{\infty} \frac{F'_{k,j}(x, y)}{(x^2 + y^2 + z^2)} n^{-2(k+j)}, \]
where functions \( F'_{k,j} \) are given in view of (3.4) by
\[ F'_{k,j}(x, y) = \left( \alpha + j - 1 \right) \sum_{k_2=0}^{k} \cdots \sum_{k_j=0}^{k_{j-1}} d_{k_2} \cdots d_{k_j} \prod_{\ell=1}^{j-1} d_{k_{\ell} - k_{\ell+1}} (x^{2k_2+4} + y^{2k_j+4}) \prod_{\ell=1}^{j-1} (x^{2(k_{\ell} - k_{\ell+1})+4} + y^{2(k_{\ell} - k_{\ell+1})+4}). \] (3.5)
Note that each \( F'_{k,j}(x, y) \) is homogeneous of order \( 2k + 4j \) in \( x, y \).
We also need an estimate of \( F'_{k,j} \) uniformly in \( x, y \geq 0 \) and \( k, j \in \mathbb{N}_0 \). Note that the binomial factor in (3.5) can be estimated against \( j^{\alpha-1} \) up to a uniform constant. Also, the number of summands in the Volterra series in (3.5) can be estimated against \( \frac{k_{j-1}^{j-1}}{(j-1)!} \), up to a uniform constant. This yields
\[
\left| F'_{k,j}(x, y) \right| \leq C_j j^{\alpha-1} \frac{k_{j-1}^{j-1} 2^{2k+3j} \pi^{2k+2j} \left( x^2 + y^2 \right)^{k+2j}}{(j-1)! \Gamma(2k_j + 5) \prod_{\ell=1}^{j-1} \Gamma(2(k_{\ell} - k_{\ell+1}) + 5)} \\
\leq C_j j^{\alpha-1} \frac{k_{j-1}^{j-1} 2^{2k+3j} \pi^{2k+2j} \left( x^2 + y^2 \right)^{k+2j}}{(j-1)! \left( (2k/j + 5) \right)^j},
\]
for some uniform constant \( C_1 > 0 \). Here, we have used in the second step the logarithmic convexity and Jensen’s inequality to estimate the product of Gamma functions in the denominator. Studying the behaviour of the individual terms...
as \( k \leq j \to \infty \), we conclude that for yet another uniform constant \( C_2 > 0 \), sufficiently large

\[
|F'_{k,j}(x, y)| \leq C_2 \left( x^2 + y^2 \right)^{k+2j}.
\] (3.6)

Summarizing, we find after some reshuffling for any \( N \in \mathbb{N} \)

\[
f(x, y, n, z)^{-\alpha} = \sum_{m=0}^{\infty} n^{-2m} \sum_{k+j=m} \frac{F'_{k,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} = \sum_{m=0}^{N-1} n^{-2m} \sum_{j=0}^{m} \frac{F'_{m-j,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} + n^{-2N} G \cdot (x^2 + y^2)^N \] (3.7)

where, provided \( n > C_2 \), the term \( G \) in the nominator is estimated by (3.6)

\[
|G| \leq \sum_{m=N}^{\infty} C_2^m n^{-2(m-N)} \sum_{j=0}^{m} \frac{(x^2 + y^2)^j}{(x^2 + y^2 + z^2)^j} \leq C,
\] (3.8)

for some uniform constant \( C > 0 \), independent of \( n \) and \( z \). This proves the statement if we set \( F_{m,j} := F'_{m-j,j} \). \( \square \)

**Remark 3.2.** The coefficients \( F_{m,j} \) and \( F'_{m,j} \) arise from the Taylor expansion (3.3) and are explicit from (3.5). For instance we have for any \( x, y \geq 0 \) and \( \alpha = 2 \)

\[
F_{0,0}(x, y) = \tilde{F}_{0,0}(x, y) = 1,
\]

\[
F_{1,0}(x, y) = 0, \quad F_{1,1}(x, y) = \frac{2}{3} \pi^2 \left( x^4 + y^4 \right),
\]

\[
\tilde{F}_{1,0}(x, y) = 0, \quad \tilde{F}_{1,1}(x, y) = \frac{2}{3} \pi^2 \left( x^2 + y^2 \right)^2.
\] (3.9)

### 3.2. Asymptotic expansion of the spectral zeta functions.

As in (1.14) we write

\[
\zeta(\Delta_n, s) = V_\alpha(s) \int_0^\infty z^{2\alpha-2s-1} \text{Tr}(\Delta_n + z^2)^{-\alpha} dz,
\]

\[
\tilde{\zeta}(\tilde{\Delta}_n, s) = V_\alpha(s) \int_0^\infty z^{2\alpha-2s-1} \text{Tr}(\tilde{\Delta}_n + z^2)^{-\alpha} dz.
\] (3.10)

We proceed below with studying the 5-point star Laplacian \( \Delta_n \). The estimates for the 9-point star Laplacian \( \tilde{\Delta}_n \) follows along the same lines. We split the regularized integral in two parts, namely

\[
\int_0^\infty z^{2\alpha-2s-1} \text{Tr}(\Delta_n + z^2)^{-\alpha} dz = \int_0^n + \int_n^\infty = : J_n^\alpha(s) + J_n^\infty(s).
\]

Below, Proposition 3.3 derives the asymptotics of \( J_n^\infty \). Proposition 3.4 discusses the asymptotics of \( J_n^0 \). Both results use the (reduced) discrete resolvent trace asymptotics in Proposition 2.2. The corresponding result for the 9-point star Laplacian is discussed in the next subsection.

**Proposition 3.3.** For \( \text{Re}(s) > 0 \) we have as \( n \to \infty \)

\[
J_n^\infty(s) \sim a_\infty(s) n^{\alpha-2s} + O(n^{-\infty}),
\] (3.11)
with the coefficient given by

\[ a_\infty(s) = \int_1^\infty z^{2\alpha-2s-1} \int_0^1 \int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2 \right)^{-\alpha} \, dx \, dy \, dz. \]  

(3.12)

**Proof.** Let \( M \in \mathbb{N} \). We plug in (2.6) and (2.7) into \( J_\alpha(s) \), which gives

\[
J_\alpha(s) = n^{-2\alpha+2} \int_n^\infty \int_0^1 \int_0^1 z^{2\alpha-2s-1} \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + \frac{z^2}{n^2} \right)^{-\alpha} \, dx \, dy \, dz \\
+ n^{-2M-2\alpha+1} \int_n^\infty z^{2\alpha-2s-1} O \left( \left( \frac{z}{n} \right)^{-2\alpha-1} \left( \sqrt{\left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2}} \right)^{-1} \right) \, dz \\
+ n^{-4M-2\alpha} \int_n^\infty z^{2\alpha-2s-1} O \left( \left( \frac{z}{n} \right)^{-2\alpha-2} \left( \sqrt{\left( \frac{z}{n} \right)^2 + \frac{1}{\pi^2}} \right)^{-2} \right) \, dz.
\]

Here, the absolute value of each \( O \)-term may be estimated against the term in its brackets, up to a constant, depending only on \( M \). For a fixed \( n \in \mathbb{N} \), each integrand is \( O \left( z^{-2s-1} \right) \) as \( z \to \infty \). Hence, for \( \text{Re}(s) > 0 \), the regularized integrals exist in the usual sense and we obtain after a change of coordinates \( z \mapsto zn \)

\[
J_\alpha(s) = n^{2-2s} \int_1^\infty z^{2\alpha-2s-1} \int_0^1 \int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2 \right)^{-\alpha} \, dx \, dy \, dz \\
+ n^{-2M-2s+1} O \left( \int_1^\infty z^{2\alpha-2s-1} z^{-2\alpha-1} \left( \sqrt{z^2 + \frac{1}{\pi^2}} \right)^{-1} \, dz \right) \\
+ n^{-4M-2s} O \left( \int_1^\infty z^{2\alpha-2s-1} z^{-2\alpha-2} \left( \sqrt{z^2 + \frac{1}{\pi^2}} \right)^{-2} \, dz \right),
\]

Since \( M \in \mathbb{N} \) was arbitrary, taking \( M \to \infty \), proves the statement. \( \square \)

In contrast to the discussion of \( J_\alpha(s) \) in Proposition 3.3 above, the asymptotic analysis of \( J_0(s) \) is much more intricate, since the integrals exist only in the Hadamard regularized sense.

**Proposition 3.4.** For \( \text{Re}(s) < \alpha/2 \) we have for any integer \( M \in \mathbb{N} \) as \( n \to \infty \)

\[
J_0(s) \sim a_0(s) n^{\alpha-2s} + \sum_{m=0}^{M-1} b_m(s) n^{-2m} + O(n^{-2M-2s+1}).
\]

(3.13)

The leading coefficient is explicitly given by

\[
a_0(s) = \int_0^1 \int_0^1 \int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2 \right)^{-\alpha} \, dx \, dy \, dz
\]

(3.14)
The higher order coefficients are given by
\[ b_j(s) = \frac{4}{(2M + 1)!} \int_0^\infty z^{2s-2s-1} \int_0^\infty \int_0^\infty B_{2M+1}(x - [x]) B_{2M+1}(y - [y]) \int_0^\infty B_{2M+1}(x - [x]) \int_0^\infty \int_0^\infty \int_0^\infty \] 
\[ \partial_x^{(2M+1)} \partial_y^{(2M+1)} \sum_{j=0}^m \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{m+j}} \] 
\[ + \frac{\delta}{(2M + 1)!} \int_0^\infty z^{2s-2s-1} \int_0^\infty \int_0^\infty B_{2M+1}(x - [x]) \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \] 
\[ \partial_x^{(2M+1)} \sum_{j=0}^m \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{m+j}} dx dy dz \] 
(3.15)

**Proof.** Recall the Euler Maclaurin formula (2.4) applied to the resolvent trace
\[ \text{Tr}(\Delta_n + z^2)^{-\alpha} = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} f(k_1, k_2, n, z)^{-\alpha} \] 
\[ = \left( I_y^n \circ I_x^n + 2 I_y^n \circ E_{x,M}^n + E_{y,M}^n \circ E_{x,M}^n \right) f(x, y, n, z)^{-\alpha}. \] 
(3.16)

For a function \( u \in C^\infty[0, n] \), such that \( u(x) = u(n - x) \), we compute
\[ \int_0^n B_{2M+1}(x - [x]) \partial_x^{2M+1} u(x) dx \] 
\[ = \int_0^n B_{2M+1}(x - [x]) \partial_x^{2M+1} u(x) dx + \int_0^n B_{2M+1}(x - [x]) \partial_x^{2M+1} u(n - x) dx \] 
\[ = \int_0^n B_{2M+1}(x - [x]) \partial_x^{2M+1} u(x) dx + \int_0^n B_{2M+1}(1 - (y - [y])) (-\partial_y)^{2M+1} u(y) dy \] 
\[ = 2 \int_0^n B_{2M+1}(x - [x]) \partial_x^{2M+1} u(x) dx, \]
where in the last step we used symmetry of the Bernoulli polynomials \( B_p(1-t) = (-1)^p B(t) \) for \( t \geq 0 \). Using symmetry of \( f(x, y, n, z) \) and the computation above, we can rewrite (3.16) as follows
\[ \text{Tr}(\Delta_n + z^2)^{-\alpha} = \left( I_y^n \circ I_x^n + 4 \left( 2 I_y^n \circ E_{x,M}^n + E_{y,M}^n \circ E_{x,M}^n \right) \right) f(x, y, n, z)^{-\alpha}. \]

Now we plug in the expansion in Lemma 3.1 with
\[ N := M + 1, \] for the expansion of \( I_y^n \circ E_{x,M}^n f^{-\alpha} \),
\[ N' := 2(M + 1), \] for the expansion of \( E_{y,M}^n \circ E_{x,M}^n f^{-\alpha} \).
We obtain
\[
\text{Tr}(\Delta_n + z^2)^{-\alpha} = \int_0^n \int_0^n \left( \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi x}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi y}{n} \right) + z^2 \right)^{-\alpha} \, dx \, dy \\
+ 8 I_{y}^{n} \circ E_{x,M}^{n} \left( \sum_{m=0}^{N-1} n^{-2m} \sum_{j=0}^{m} \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} + n^{-2N} \frac{G \cdot (x^2 + y^2)^N}{(x^2 + y^2 + z^2)^{\alpha}} \right) \\
+ 4 E_{y,M}^{n} \circ E_{x,M}^{n} \left( \sum_{m=0}^{N-1} n^{-2m} \sum_{j=0}^{m} \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} + n^{-2N'} \frac{G' \cdot (x^2 + y^2)^{N'}}{(x^2 + y^2 + z^2)^{\alpha}} \right).
\]
(3.17)

Plug this into (3.10) and simplify notation by writing for the first term in (3.17)
\[
I(s, n) := \int_0^n z^{2\alpha-2s-1} \int_0^n \int_0^n \left( \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi x}{n} \right) + \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi y}{n} \right) + z^2 \right)^{-\alpha} \, dx \, dy \, dz.
\]

For the individual summands in the second and third lines in (3.17) we set
\[
W_{m,j}^1(s, n) := \int_0^n z^{2\alpha-2s-1} \left( I_{y}^{n} \circ E_{x,M}^{n} \right) \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \, dz,
\]
\[
W_{m,j}^2(s, n) := \int_0^n z^{2\alpha-2s-1} \left( E_{y,M}^{n} \circ E_{x,M}^{n} \right) \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \, dz.
\]

For the other remaining summands in (3.17) (plugged into (3.10)) we set
\[
R_{N}^1(s, n) := \int_0^n z^{2\alpha-2s-1} \left( I_{y}^{n} \circ E_{x,M}^{n} \right) \frac{G \cdot (x^2 + y^2)^N}{(x^2 + y^2 + z^2)^{\alpha}} \, dz,
\]
\[
R_{N'}^2(s, n) := \int_0^n z^{2\alpha-2s-1} \left( E_{y,M}^{n} \circ E_{x,M}^{n} \right) \frac{G' \cdot (x^2 + y^2)^{N'}}{(x^2 + y^2 + z^2)^{\alpha}} \, dz.
\]

This yields in view of (3.17) the following expression
\[
J_\alpha^1(s) = I(s, n) + 8 \left( \sum_{m=0}^{N-1} n^{-2m} \sum_{j=0}^{m} W_{m,j}^1(s, n) + n^{-2N} R_{N}^1(s, n) \right) \\
+ 4 \left( \sum_{m=0}^{N'} n^{-2m} \sum_{j=0}^{m} W_{m,j}^2(s, n) + n^{-2N'} R_{N'}^2(s, n) \right).
\]

The rest of the proof proceeds with estimating the individual terms. For the term \(I(s, n)\) note that by (2.12) the \(dz\) integrand is \(O(z^{\alpha-2s-1})\) as \(z \to 0\). Hence for \(\text{Re}(s) < \alpha/2\), the regularized integral exists in the usual sense and we obtain after a change of variables
\[
I(s, n) = n^{2-2s} \int_0^1 z^{2\alpha-2s-1} \int_0^1 \int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2 \right)^{-\alpha} \, dx \, dy \, dz.
\]
(3.18)

This proves (3.14).
For the other terms we estimate for any \( x, y, z > 0 \) and a uniform constant \( C > 0 \), using that \( F_{m,j} \) is a polynomial and any derivative just lowers the exponent in (3.6) by one half:\(^5\)

\[
\left| \partial_x^{(2M+1)} F_{m,j}(x, y) \right| \leq C \begin{cases} x(x^2 + y^2)^{m-M} \frac{(x^2 + y^2 + z^2)^{\alpha+1}}{(x^2 + y^2 + z^2)^{\alpha+1}}, & \text{for } m \geq M, \\ x(x^2 + y^2 + z^2)^{-\alpha-M+m-1}, & \text{for } m \leq M - 1. \end{cases} \tag{3.19}
\]

\[
\left| (\partial_y \partial_x)^{(2M+1)} F_{m,j}(x, y) \right| \leq C \begin{cases} xy(x^2 + y^2)^{m-2M} \frac{(x^2 + y^2 + z^2)^{\alpha+2}}{(x^2 + y^2 + z^2)^{\alpha+2}}, & \text{for } m \geq 2(M + 1), \\ xy \frac{(x^2 + y^2 + z^2)^{\alpha+2M-m+2}}{(x^2 + y^2 + z^2)^{\alpha+2M-m+2}}, & \text{for } m \leq 2M + 1. \end{cases}
\]

For the estimate of the error term \( R_N^1(s, n) \), we note using (3.19) and \( N = M + 1 \)

\[
\partial_x^{2M+1} \frac{\mathcal{G} \cdot (x^2 + y^2)^N}{(x^2 + y^2 + z^2)^{\alpha}} = \sum_{m=N}^{\infty} n^{-2(m-N)} \sum_{j=0}^{m} \partial_x^{2M+1} \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+1}} \tag{3.20}
\]

\[
= \sum_{m=N}^{\infty} n^{-2(m-N)} \sum_{j=0}^{m} \mathcal{O} \left( \frac{x(x^2 + y^2)^{m-M}}{(x^2 + y^2 + z^2)^{\alpha+1}} \right)
\]

\[
= \mathcal{O} \left( x(x^2 + y^2 + z^2)^{-\alpha} \right),
\]

for \( x, y \in [0, n/2] \), such that the series converge absolutely and the \( O \) constant is uniform in \((n, z)\). From here we conclude for \( z \in [0, n], n \geq 2 \) and a constant \( C > 0 \), depending only on \( M \)

\[
\left| I_x^n \circ E_{x,M}^n \frac{\mathcal{G} \cdot (x^2 + y^2)^N}{(x^2 + y^2 + z^2)^{\alpha}} \right| \leq C \int_0^{1/2} (n^2 + y^2 + z^2)^{-\alpha+1} dy + C \int_0^{1/2} (y^2 + z^2)^{-\alpha+1} dy 
\]

\[
\leq C \int_0^{1/2} (n^2 + y^2 + z^2)^{-\alpha+1} dy + C \int_0^{1} (y + z^2)^{-\alpha+1} dy 
\]

\[
+ C \int_1^{1/2} (1 + z^2)^{-\alpha+1} dy \leq C \log n.
\]

Similarly, we obtain for \( R_N^2(s, n) \), using (3.19) and \( N' = 2(M + 1) \)

\[
\partial_y^{2M+1} \partial_x^{2M+1} \frac{G' \cdot (x^2 + y^2)^{N'}}{(x^2 + y^2 + z^2)^{\alpha}} = \sum_{m=N'}^{\infty} n^{-2(m-N')} \sum_{j=0}^{m} \partial_y^{2M+1} \partial_x^{2M+1} \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+1}} 
\]

\[
= \sum_{m=N'}^{\infty} n^{-2(m-N')} \sum_{j=0}^{m} \mathcal{O} \left( \frac{xy(x^2 + y^2)^{m-2M}}{(x^2 + y^2 + z^2)^{\alpha+2}} \right) = \mathcal{O} \left( xy(x^2 + y^2 + z^2)^{-\alpha} \right), \tag{3.21}
\]

\(^5\)The order of differentiation in \( x, y \) is odd, so that the additional \( x, y \) appears in the estimates.
for \( x, y \in [0, n/2] \), where the \( O \) constant is as before uniform in \((n, z)\). From here we conclude for \( z \in [0, n] \) and a constant \( C > 0 \), depending only on \( M \)
\[
\left| E_{\frac{x}{n}, M} \circ E_{\frac{y}{M}, M} \frac{G' \cdot (x^2 + y^2)^N}{(x^2 + y^2 + z^2)^\alpha} \right| \leq C \log(x^2 + y^2 + z^2) \bigg|_{x=0} \bigg|_{y=0} \leq C \log n.
\]
Plugging these estimates into the expressions for \( R_N^1(s, n) \) and \( R_{N'}^0(s, n) \), and noting that for \( \text{Re}(s) < \alpha/2 \) the \( z \)-integrals exist in the usual sense, we find
\[
|R_N^1(s, n)| \leq C \log(n) \int_0^n z^{2\alpha - 2s - 1} \leq C, \quad \text{for } N = M + 1,
\]
\[
|R_{N'}^0(s, n)| \leq C \log(n) \int_0^n z^{2\alpha - 2s - 1} \leq C, \quad \text{for } N' = 2(M + 1).
\]
(3.22)

It remains to study \( W_{m,j}^1(s, n) \) and \( W_{m,j}^2(s, n) \). We write
\[
W_{m,j}^1(s, n) = \left( \int_0^n - \int_n^\infty \right) \frac{z^{2\alpha - 2s - 1}}{(2M + 1)!} \left( \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \right) B_{2M+1}(x - [x]) \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+1}} dx dy dz,
\]
\[
W_{m,j}^2(s, n) = \left( \int_0^n - \int_n^\infty \right) \frac{z^{2\alpha - 2s - 1}}{(2M + 1)!} \left( \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \right) B_{2M+1}(y - [y]) B_{2M+1}(x - [x]) \frac{\partial^{2M+1} F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+1}} dx dy dz.
\]
(3.23)

We want to use (3.19) to estimate the individual integrals in (3.23): e.g. we find after change of variables
\[
\int_0^n \int_0^\infty \int_0^\infty \frac{z^{2\alpha - 2s - 1} x(x^2 + y^2 + z^2)^{-\alpha-M+m-1}}{4} dx dy dz
\]
\[
= \int_0^n \int_0^\infty z^{2\alpha - 2s - 1} (\alpha + M - m)^{-1} (n^2/4 + y^2 + z^2)^{-\alpha-M+m} dy dz
\]
\[
= (n/2)^{-2M+2m-2s+1} \int_0^n \int_0^\infty z^{2\alpha - 2s - 1} (\alpha + M - m)^{-1} (1 + y^2 + z^2)^{-\alpha-M+m} dy dz
\]
\[
= O(n^{-2M+2m-2s+1}), \quad \text{as } n \to \infty.
\]

Similarly, we compute
\[
\int_n^\infty \int_0^\infty \int_0^\infty \frac{z^{2\alpha - 2s - 1} x(x^2 + y^2 + z^2)^{-\alpha-M+m-1}}{4} dx dy dz
\]
\[
= \int_n^\infty \int_0^\infty z^{2\alpha - 2s - 1} (\alpha + M - m)^{-1} (y^2 + z^2)^{-\alpha-M+m} dy dz
\]
\[
= n^{-2s-2M+2m+1} \int_0^\infty \int_0^\infty z^{2\alpha - 2s - 1} (\alpha + M - m)^{-1} (y^2 + z^2)^{-\alpha-M+m} dy dz
\]
\[
= O(n^{-2M+2m-2s+1}), \quad \text{as } n \to \infty.
\]
Similar estimates hold for other integrals where at least one of the variables \((x, y, z)\) is integrated over \([n, \infty)\) or \([n/2, \infty)\). In all of those cases, the \(z\)-integral exists in the usual sense and hence (3.19) may be applied. We conclude

\[
W_{m,j}^1(s, n) = \int_0^\infty \frac{z^{2\alpha-2s-1}}{(2M+1)!} \int_0^\infty \int_0^\infty B_{2M+1}(x-[x]) \partial_x^{2M+1} \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \, dx \, dy \, dz + O\left(n^{-2M+2m-2s+1}\right), \quad \text{as } n \to \infty,
\]

\[
W_{m,j}^2(s, n) = \int_0^\infty \frac{z^{2\alpha-2s-1}}{(2M+1)!} \int_0^\infty \int_0^\infty B_{2M+1}(y-[y])B_{2M+1}(x-[x]) \partial_y^{2M+1} \partial_x^{2M+1} \frac{F_{m,j}(x, y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \, dx \, dy \, dz + O\left(n^{-4M+2m-2s+1}\right), \quad \text{as } n \to \infty.
\]

This, together with (3.22) and (3.18) proves the statement. \(\square\)

Propositions 3.3 and 3.4, as well as their analogous estimates for the 9-point star Laplacian \(\tilde{\Delta}_n\), combine to the main result of this section.

**Theorem 3.5.** For \(\text{Re}(s) \in (0, \alpha/2)\) we have for any integer \(M \in \mathbb{N}\) as \(n \to \infty\)

\[
\zeta(\Delta_n, s) = V_\alpha(s) \left(a(s) n^{\alpha-2s} + \sum_{m=0}^{M-1} b_m(s) n^{-2m}\right) + O(n^{-2M-2s+2}),
\]

\[
\zeta(\tilde{\Delta}_n, s) = V_\alpha(s) \left(\tilde{a}(s) n^{\alpha-2s} + \sum_{m=0}^{M-1} \tilde{b}_m(s) n^{-2m}\right) + O(n^{-2M-2s+2}).
\]

(3.24)

The leading coefficients \(a(s)\) and \(\tilde{a}(s)\) are explicitly given by

\[
a(s) = \int_0^\infty \frac{z^{2\alpha-2s-1}}{\pi^2} \int_0^1 \int_0^1 \left(\frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2\right)^{-\alpha} \, dx \, dy \, dz,
\]

\[
\tilde{a}(s) = \int_0^\infty \frac{z^{2\alpha-2s-1}}{\pi^2} \int_0^1 \int_0^1 \left(\frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2\right)^{-\alpha} \, dx \, dy \, dz
\]

\[-2n^2 \sin^2\left(\frac{\pi x}{n}\right) \sin^2\left(\frac{\pi y}{n}\right) \left(\frac{n}{\pi}\right)^{-\alpha} \, dx \, dy \, dz.
\]

(3.25)
The higher order coefficients are given by

\[ b_m(s) = \int_0^\infty z^{2\alpha-2s-1} 4 \left( E_{y,M}^\infty \circ E_{x,M}^\infty + 2 \cdot I_y^\infty \circ E_{x,M}^\infty \right) \sum_{j=0}^m \frac{F_{m,j}(x,y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \, dz \]

\[ = \frac{4}{(2M + 1)!} \int_0^\infty z^{2\alpha-2s-1} \int_0^\infty \int_0^\infty B_{2M+1}(x-[x])B_{2M+1}(y-[y]) \, dx \, dy \, dz \]

\[ \delta_x^{2M+1} \delta_y^{2M+1} \sum_{j=0}^m \frac{F_{m,j}(x,y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \, dx \, dy \, dz, \]  

\[ \delta_x^{2M+1} \delta_y^{2M+1} \sum_{j=0}^m \frac{F_{m,j}(x,y)}{(x^2 + y^2 + z^2)^{\alpha+j}} \, dx \, dy \, dz, \]

where \( F_{m,j}(x,y) \) is defined in Lemma 3.1. The expression for \( \tilde{b}_m(s) \) is exactly the same, with \( F_{m,j}(x,y) \) replaced by \( \bar{F}_{m,j}(x,y) \), as defined in Lemma 3.1.

4. Computing the coefficients in the asymptotic expansion

In this section, we will express the coefficients \( b_0(s), b_1(s) \) as well as \( \tilde{b}_0(s), \tilde{b}_1(s) \) in Theorem 3.5 explicitly in terms of the spectral zeta function on the torus, \( \zeta(\Delta, s) \) in (1.14), and related functions.

**Proposition 4.1.** Consider the Laplace Beltrami operator \( \Delta \) on the two-dimensional torus \( \mathbb{T}^2 \) and its spectral zeta function \( \zeta(\Delta, s) \) The coefficients \( b_0(s), \bar{b}_0(s) \) in Theorem 3.5 are given by

\[ b_0(s) = \tilde{b}_0(s) = V_\alpha(s)^{-1} \zeta(\Delta, s). \]  

\[ \text{(4.1)} \]

**Proof.** As asserted in (3.9), we have

\[ F_{0,0}(x,y) = \bar{F}_{0,0}(x,y) = 1. \]

Let us write \( I_{x_i}^R \) and \( E_{x_i,M}^R \) for the operators in (2.3), where the integration region is replaced by \( \mathbb{R} \). Then, in view of (3.26) and the symmetry of the integrands around \( x = 0 \) and \( y = 0 \), we obtain

\[ b_0(s) = \tilde{b}_0(s) \]

\[ = \int_0^\infty z^{2\alpha-2s-1} \left( E_{y,M}^\infty \circ E_{x,M}^\infty + 2 \cdot I_y^\infty \circ E_{x,M}^\infty \right) (x^2 + y^2 + z^2)^{-\alpha} \, dz \]

\[ = \int_0^\infty z^{2\alpha-2s-1} (E_{y,M}^R \circ E_{x,M}^R + 2 \cdot I_y^R \circ E_{x,M}^R) (x^2 + y^2 + z^2)^{-\alpha} \, dz. \]

\[ \text{(4.2)} \]
By the Euler-Maclaurin formula (2.4) (with \( n \to \infty \)) we conclude
\[
b_0(s) = \tilde{b}_0(s) = \int_0^\infty z^{2\alpha - 2s - 1} \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \frac{k_1^2 + k_2^2 + z^2}{z} \alpha \, dz
- \int_0^\infty z^{2\alpha - 2s - 1} \int_\mathbb{R} \int_\mathbb{R} (x^2 + y^2 + z^2)^{-\alpha} \, dx \, dy \, dz
= \int_0^\infty z^{2\alpha - 2s - 1} \int_\mathbb{R} \int_\mathbb{R} (x^2 + y^2 + z^2)^{-\alpha} \, dx \, dy \, dz,
\]
where \( \Delta \) is the Laplace Beltrami operator on the two-dimensional torus \( \mathbb{T}^2 \). Note that \((x^2 + y^2 + z^2)^{-\alpha}\) is integrable on \( \mathbb{R}^2 \). As computed e.g. in [Les98, (1.13c)], the regularized integral over \([0, \infty)\) for any power of \( z \) vanishes and hence we conclude
\[
\int_0^\infty z^{2\alpha - 2s - 1} \int_\mathbb{R} \int_\mathbb{R} (x^2 + y^2 + z^2)^{-\alpha} \, dx \, dy \, dz = 0.
\]
Thus we arrive at the following expression
\[
V_\alpha(s)b_0(s) = V_\alpha(s)\tilde{b}_0(s) = V_\alpha(s)\int_0^\infty z^{2\alpha - 2s - 1} \text{Tr} \left( \Delta + z^2 \right)^{-\alpha} \, dz = \zeta(\Delta, s),
\]
where we used the representation (1.14) and (1.15). \( \square \)

We proceed with the calculation of the coefficients \( b_1(s) \) and \( \tilde{b}_1(s) \) explicitly.

**Proposition 4.2.** Consider the Laplace Beltrami operator \( \Delta \) on the two-dimensional torus \( \mathbb{T}^2 \) and its spectral zeta function \( \zeta(\Delta, s) \) The coefficients \( b_0(s), \tilde{b}_0(s), \text{Re}(s) \in (0, \alpha) \) in Theorem 3.5 are given by\(^6\)
\[
V_0(s)b_0(s) = \frac{s\pi^2}{3} \zeta(\Delta, s - 1),
V_0(s)b_1(s) = \frac{s\pi^2}{3} \zeta(\Delta, s - 1) + \frac{4\pi^2 V_0(s)}{(\alpha - s)} \int_0^\infty z^{2s - 2s + 1} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{k_1^2 k_2^2}{(k_1^2 + k_2^2 + z^2)^2}.
\]

**Proof.** As asserted in (3.9), we have
\[
F_{0,0}(x, y) = 0, \quad F_{1,1}(x, y) = \frac{2}{3} \pi^2 \left( x^4 + y^4 \right),
\]
\[
\tilde{F}_{0,0}(x, y) = 0, \quad \tilde{F}_{1,1}(x, y) = \frac{2}{3} \pi^2 \left( x^2 + y^2 \right)^2.
\]
As in the proof of the previous result, let us write \( I^R \) and \( E^R_{x_0, M} \) for the operators in (2.3), where the integration region is replaced by \( \mathbb{R} \). Then, in view of (3.26),

\(^6\)The extra summand in the expression for \( V_2(s)b_1(s) \) is not directly a zeta function but is closely related. Such functions are known as angular lattice sums, see [BGM13].
symmetry of the integrands around \( x = 0, y = 0 \) yields as in (4.2)

\[
\begin{align*}
    b_1(s) &= \frac{2}{3} \pi^2 \int_0^\infty z^{2\alpha-2s-1} \left( E_{y,M}^R \circ E_{x,M}^R + 2 \cdot I_y^R \circ E_{x,M}^R \right) \frac{x^4 + y^4}{(x^2 + y^2 + z^2)^3} \, dz, \\
    \tilde{b}_1(s) &= \frac{2}{3} \pi^2 \int_0^\infty z^{2\alpha-2s-1} \left( E_{y,M}^R \circ E_{x,M}^R + 2 \cdot I_y^R \circ E_{x,M}^R \right) \frac{(x^2 + y^2)^2}{(x^2 + y^2 + z^2)^3} \, dz.
\end{align*}
\]

We would like to apply the Euler-Maclaurin formula (2.4) (with \( n \to \infty \)) and argue precisely as in Proposition 4.1 above. However

\[
\frac{F_{1,1}(x, y)}{(x^2 + y^2 + z^2)^3} = \frac{2}{3} \pi^2 \frac{x^4 + y^4}{(x^2 + y^2 + z^2)^3},
\]

\[
\frac{\tilde{F}_{1,1}(x, y)}{(x^2 + y^2 + z^2)^3} = \frac{2}{3} \pi^2 \frac{(x^2 + y^2)^2}{(x^2 + y^2 + z^2)^3},
\]

are not \( \mathbb{R}^2 \) integrable and thus the error terms \( A_x^n, A_y^n \) as well as \( D_{x,M}^n, D_{y,M}^n \) (applied to the terms above) do not vanish in the limit \( n \to \infty \). Therefore we first perform an integration by parts trick (for \( b_1(s) \), the coefficient \( \tilde{b}_1(s) \) studied verbatim with \( F_{1,1}(x, y) \) replaced by \( \tilde{F}_{1,1}(x, y) \))

\[
\begin{align*}
    b_1(s) &= \frac{2}{3} \pi^2 \int_0^\infty \left( E_{y,M}^R \circ E_{x,M}^R + 2 \cdot I_y^R \circ E_{x,M}^R \right) \partial_z z^{2\alpha-2s} \left[ \frac{x^4 + y^4}{(x^2 + y^2 + z^2)^3} \right] \, dz \\
    &\quad + \frac{2}{3} \pi^2 \lim_{R \to \infty} \lim_{\epsilon \to 0} \left. \frac{z^{2\alpha-2s}}{2\alpha - 2s} \left( E_{y,M}^R \circ E_{x,M}^R + 2 \cdot I_y^R \circ E_{x,M}^R \right) \frac{x^4 + y^4}{(x^2 + y^2 + z^2)^3} \right|_{z=\epsilon} \, dz.
\end{align*}
\]

For \( \text{Re}(s) \in (0, \alpha/2) \), the regularized limit as \( \epsilon \to 0 \) vanishes. For \( M \in \mathbb{N} \) sufficiently large, the regularized limit as \( R \to \infty \) vanishes as well. We conclude, performing the same computation for \( \tilde{b}_1(s) \)

\[
\begin{align*}
    b_1(s) &= \frac{2}{(\alpha - s)} \pi^2 \int_0^\infty z^{2\alpha-2s+1} \left( E_{y,M}^R \circ E_{x,M}^R + 2 \cdot I_y^R \circ E_{x,M}^R \right) \frac{x^4 + y^4}{(x^2 + y^2 + z^2)^4} \, dz, \\
    \tilde{b}_1(s) &= \frac{2}{(\alpha - s)} \pi^2 \int_0^\infty z^{2\alpha-2s+1} \left( E_{y,M}^R \circ E_{x,M}^R + 2 \cdot I_y^R \circ E_{x,M}^R \right) \frac{(x^2 + y^2)^2}{(x^2 + y^2 + z^2)^4} \, dz.
\end{align*}
\]

We can now apply the Euler-Maclaurin formula (2.4) (with \( n \to \infty \)) and conclude

\[
\begin{align*}
    b_1(s) &= \frac{2}{(\alpha - s)} \pi^2 \int_0^\infty z^{2\alpha-2s+1} \sum_{k_1=\infty}^{\infty} \sum_{k_1=\infty}^{\infty} \frac{k_1^4 + k_2^4}{(k_1^2 + k_2^2 + z^2)^4} \, dz \\
    &\quad - \frac{2}{3} \pi^2 \int_0^\infty z^{2\alpha-2s-1} \int_R^\infty \frac{x^4 + y^4}{(x^2 + y^2 + z^2)^4} \, dx \, dy \, dz \\
    \tilde{b}_1(s) &= \frac{2}{(\alpha - s)} \pi^2 \int_0^\infty z^{2\alpha-2s+1} \sum_{k_1=\infty}^{\infty} \sum_{k_1=\infty}^{\infty} \frac{(k_1^2 + k_2^2)^2}{(k_1^2 + k_2^2 + z^2)^4} \, dz \\
    &\quad - \frac{2}{3} \pi^2 \int_0^\infty z^{2\alpha-2s-1} \int_R^\infty \frac{(x^2 + y^2)^2}{(x^2 + y^2 + z^2)^4} \, dx \, dy \, dz.
\end{align*}
\]
The three-fold integrals vanish exactly as in (4.3) and hence we arrive at the following intermediate formulae

\[
\begin{align*}
\tilde{b}_1(s) &= \frac{2}{(\alpha - s)} \pi^2 \int_0^\infty z^{2\alpha - 2s + 1} \sum_{k_1 = -\infty}^\infty \sum_{k_1 = -\infty}^\infty \frac{k_1^4 + k_2^4}{(k_1^2 + k_2^2 + z^2)^4} \, dz \\
\tilde{b}_1(s) &= \frac{2}{(\alpha - s)} \pi^2 \int_0^\infty z^{2\alpha - 2s + 1} \sum_{k_1 = -\infty}^\infty \sum_{k_1 = -\infty}^\infty \frac{(k_1^2 + k_2^2)^2}{(k_1^2 + k_2^2 + z^2)^4} \, dz.
\end{align*}
\]

To express the integrand in terms of resolvent traces we notice that the individual summands admit a partial fraction decomposition

\[
\begin{align*}
\frac{k_1^4 + k_2^4}{(k_1^2 + k_2^2 + z^2)^4} &= \frac{1}{(k_1^2 + k_2^2 + z^2)^2} \left( \frac{2z^2}{(k_1^2 + k_2^2 + z^2)^2} \right) + \frac{z^4}{(k_1^2 + k_2^2 + z^2)^4} - \frac{2k_1^2 k_2^2}{(k_1^2 + k_2^2 + z^2)^4}, \\
\frac{(k_1^2 + k_2^2)^2}{(k_1^2 + k_2^2 + z^2)^4} &= \frac{1}{(k_1^2 + k_2^2 + z^2)^2} \left( \frac{2z^2}{(k_1^2 + k_2^2 + z^2)^2} \right) + \frac{z^4}{(k_1^2 + k_2^2 + z^2)^4} + \frac{z^4}{(k_1^2 + k_2^2 + z^2)^4}.
\end{align*}
\]

Note the following identities that are special cases of (1.14)

\[
\begin{align*}
\int_0^\infty z^{4-2(s-1)-1} \sum_{k_1 = -\infty}^\infty \sum_{k_1 = -\infty}^\infty \frac{1}{(k_1^2 + k_2^2 + z^2)^2} &= V_2^{-1}(s - 1) \zeta(\Delta, s - 1), \\
\int_0^\infty z^{6-2(s-1)-1} \sum_{k_1 = -\infty}^\infty \sum_{k_1 = -\infty}^\infty \frac{1}{(k_1^2 + k_2^2 + z^2)^3} &= V_3^{-1}(s - 1) \zeta(\Delta, s - 1), \\
\int_0^\infty z^{8-2(s-1)-1} \sum_{k_1 = -\infty}^\infty \sum_{k_1 = -\infty}^\infty \frac{1}{(k_1^2 + k_2^2 + z^2)^4} &= V_4^{-1}(s - 1) \zeta(\Delta, s - 1).
\end{align*}
\]

Combining (4.5) and (4.6), we arrive at an expression of \(b_1(s)\) and \(\tilde{b}_1(s)\)

\[
\begin{align*}
\tilde{b}_1(s) &= \frac{2\pi^2}{(\alpha - s)} \zeta(\Delta, s - 1) \left( V_2^{-1}(s - 1) - 2V_3^{-1}(s - 1) + V_4^{-1}(s - 1) \right), \\
b_1(s) &= \tilde{b}_1(s) - \frac{4\pi^2}{(\alpha - s)} \int_0^\infty z^{4-2(s-1)-1} \sum_{k_1 = -\infty}^\infty \sum_{k_1 = -\infty}^\infty \frac{k_1^2 k_2^2}{(k_1^2 + k_2^2 + z^2)^2}.
\end{align*}
\]

The extra factor in the expression for \(\tilde{b}_1(s)\) amounts using (1.15) to

\[
V_2^{-1}(s - 1) - 2V_3^{-1}(s - 1) + V_4^{-1}(s - 1) = \frac{\pi(2 - s)}{2 \sin(\pi s)} \frac{s(1 - s)}{6} = \frac{s(2 - s)}{6} V_2^{-1}(s).
\]
From here the statement follows

\[ \tilde{b}_1(s) = \frac{s\pi^2}{3} V_2^{-1}(s) \zeta(\Delta, s - 1), \]
\[ b_1(s) = \frac{s\pi^2}{3} V_2^{-1}(s) \zeta(\Delta, s - 1) \]
\[ + \frac{4\pi^2}{(\alpha - s)} \int_0^\infty z^{4-2s+1} \sum_{k_1 = -\infty}^\infty \sum_{k_2 = -\infty}^\infty \frac{k_1^2 k_2^2}{(k_1^2 + k_2^2 + z^2)^2}. \]

\[ \square \]

**Remark 4.3.** This result makes the ultimate reason apparent, why we have introduced the 9-point star Laplacian \( \tilde{\Delta} \) and did not contend ourselves with the usual 5-point star Laplace operator \( \Delta \). The reason is that for the former, the coefficient \( b_1(s) \) can be expression completely in terms of the spectral zeta function.

We combine the results of Theorem 3.5, Propositions 4.1 and 4.2, and arrive at our first main result (see Theorem 1.10)

**Theorem 4.4.** For \( \text{Re}(s) \in (0, \alpha/2) \) we have for any integer \( M \in \mathbb{N} \) as \( n \to \infty \)

\[ \zeta(\Delta_n, s) = V_\alpha(s) \left( a(s) n^{\alpha - 2s} + \sum_{m=0}^{M-1} b_m(s) n^{-2m} \right) + O(n^{-2M-2s+2}), \]
\[ \zeta(\tilde{\Delta}_n, s) = V_\alpha(s) \left( \tilde{a}(s) n^{\alpha - 2s} + \sum_{m=0}^{M-1} \tilde{b}_m(s) n^{-2m} \right) + O(n^{-2M-2s+2}). \] (4.7)

The leading coefficients \( a(s) \) and \( \tilde{a}(s) \) are explicitly given by

\[ a(s) = \int_0^\infty z^{2\alpha - 2s - 1} \int_0^1 \int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2 \right)^{-\alpha} \, dx \, dy \, dz, \]
\[ \tilde{a}(s) = \int_0^\infty z^{2\alpha - 2s - 1} \int_0^1 \int_0^1 \left( \frac{\sin^2(\pi x)}{\pi^2} + \frac{\sin^2(\pi y)}{\pi^2} + z^2 \right)^{-\alpha} \, dx \, dy \, dz, \] (4.8)

\[ - \frac{2n^2}{3\pi^2} \sin^2 \left( \frac{\pi x}{n} \right) \sin^2 \left( \frac{\pi y}{n} \right) \right)^{-\alpha} \, dx \, dy \, dz. \]

The first two higher order coefficients are explicitly given by

\[ b_0(s) = \tilde{b}_0(s) = V_\alpha(s)^{-1} \zeta(\Delta, s), \quad \tilde{b}_1(s) = \frac{s\pi^2}{3} V_\alpha(s)^{-1} \zeta(\Delta, s - 1), \]
\[ b_1(s) = \frac{s\pi^2}{3} V_\alpha(s)^{-1} \zeta(\Delta, s - 1) + \frac{4\pi^2}{(\alpha - s)} \int_0^\infty z^{4-2s+1} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{k_1^2 k_2^2}{(k_1^2 + k_2^2 + z^2)^2}. \] (4.9)

5. Epstein-Riemann conjecture and discrete zeta function

In [FrKA17] the authors showed that the discrete zeta function \( \zeta(\mathcal{L}_n, s) \) on \( S_n^1 \) has a connection to the Riemann conjecture, due to its asymptotic expansion as \( n \to \infty \). Our goal is to prove an analogous result for our considerations of the
Epstein zeta-function on a two-dimensional discrete torus. We use techniques of [FrKa17] and [Fri16] and define the function (using notation in Theorem 3.5)

\[ H_n(s) := \pi^{-s} \Gamma(s) \left( \zeta(\Delta_n, s) - V_\alpha(s) \tilde{a}(s)n^{1-2s} \right) \]

\[ = \pi^{-s} \Gamma(s) \left( \zeta(\Delta, s) + \frac{s\pi^2}{3} \zeta(\Delta, s - 1) + \mathcal{O}(n^{-4}) \right). \]  

(5.1)

where in the second equality we used the asymptotic expansion in Theorem 3.5 as \( n \to \infty \) for \( \Re(s) \in (0, \alpha/2) \).

Let us introduce the following notation

\[ \xi_2(s) := \pi^{-s} \Gamma(s) \zeta(\Delta, s), \quad \Omega(s) := \frac{1}{3} s \pi^2 \zeta(s) \zeta(\Delta, s - 1). \]

The function \( \xi_2(s) \) is also known as the complete Epstein-zeta-function. Using the new notation, we can write \( (5.1) \) as

\[ H_n(s) = \xi_2(s) + \Omega(s)n^{-2} + \mathcal{O}(n^{-4}). \]

(5.3)

From here we directly obtain an asymptotic functional relation for \( H_n(s) \).

**Proposition 5.1.** For \( \Re(s) \in (0, \alpha/2) \) and \( \zeta(\Delta, s) \neq 0 \)

\[ \lim_{n \to \infty} \frac{H_n(1-s)}{H_n(s)} = 1. \]

(5.4)

**Proof.** The statement follows directly from \( (5.3) \), since (see e.g. [Epso3, (7)] and [Ter85, p.59]) the complete Epstein zeta function \( \xi_2(s) \) admits the functional equation\( \footnote{This is also true for general dimensions \( \alpha \) with 1 replaced by \( \frac{\alpha}{2} \).}

\[ \xi_2(s) = \xi_2(1-s), \]

(5.5)

We conjecture, in full analogy to [FrKa17, Sec. 9], that the same asymptotic functional relation holds (with absolute value) even at zeros of \( \zeta(\Delta, s) \).

**Conjecture 5.1.** For \( s \in \mathbb{C} \) with \( \Re(s) \in (0, 1) \) the following equality holds:

\[ \lim_{n \to \infty} \left| \frac{H_n(1-s)}{H_n(s)} \right| = 1. \]

(5.6)

5.1. **Proof of Theorem 1.12.** This section is devoted to the proof of our second main result, Theorem 1.12, which we state here once again.

**Theorem 5.2.** Conjecture 5.1 is equivalent to the Epstein-Riemann Conjecture 1.1 for \( \alpha = 2 \), i.e. to the claim that all non-trivial zeros of \( \zeta(\Delta, s) \) have \( \Re(s) = \alpha/4 = 1/2 \).

The proof follows by a sequence of lemmata.

**Lemma 5.3.** For \( \Re(s) = \frac{1}{2} \) we have

\[ \left| \frac{\Omega(1-s)}{\Omega(s)} \right| = 1. \]

(5.7)
Proof. This is the same argument as in [FrKα17, Lemma 15]. Since \( \zeta(\Delta, s) \) and \( \Gamma(s) \) are real-valued on the real axis (away from poles), we conclude by the Schwarz reflection principle that \( \zeta(\Delta, \overline{s}) = \overline{\zeta(\Delta, s)} \) and \( \Gamma(\overline{s}) = \overline{\Gamma(s)} \). Thus the same holds for \( \Omega(s) \) and we compute for \( s = \frac{1}{2} + ib, b \in \mathbb{R} \):

\[
\Omega(1 - s) = \Omega(1/2 - ib) = \overline{\Omega(1/2 + ib)} = \overline{\Omega(s)}.
\]

This proves the statement. \( \square \)

Lemma 5.3 shows that (5.7) is true if \( \text{Re}(s) = \frac{1}{2} \). We want to sharpen the statement and prove that (5.7) is satisfied only at \( \text{Re}(s) = \frac{1}{2} \).

**Proposition 5.4.** For \( \text{Re}(s) \in (0, 1) \) with\(^8\) \( \text{Im}(s) > 65 \), (5.7) holds only at \( \text{Re}(s) = \frac{1}{2} \).

Proof. In view of (5.2), we can represent \( \Omega(s) \) in terms of \( \xi_2(s) \)

\[
\Omega(s) = \frac{1}{3}s(s - 1)\pi \xi_2(s - 1).
\]

From there we arrive at the following expression

\[
\frac{\Omega(1 - s)}{\Omega(s)} = \frac{\xi_2(-s)}{\xi_2(s - 1)} = \frac{\xi_2(s + 1)}{\xi_2(s - 1)} = \frac{s(s - 1)}{\pi^2} \cdot \zeta(\Delta, s + 1) / \zeta(\Delta, s - 1),
\]

where we used the functional relation \( \xi_2(s) = \xi_2(1 - s) \) (recall (5.5)) in the numerator second equality, and plugged in the definition of \( \xi_2(s) \) in the third equality. Similarly, we compute, using (5.5) this time in the denominator

\[
\frac{\Omega(1 - s)}{\Omega(s)} = \frac{\xi_2(-s)}{\xi_2(s - 1)} = \frac{\xi_2(-s)}{\xi_2(2 - s)} = \frac{\pi^2}{s(s - 1)} \cdot \frac{\zeta(\Delta, -s)}{\zeta(\Delta, 2 - s)}.
\]

Let us introduce the following notation

\[
q(s) := \left| \frac{\pi^2}{s(s - 1)} \right|, \quad \eta(s) := \left| \frac{\zeta(\Delta, s + 1)}{\zeta(\Delta, s - 1)} \right|, \quad \rho(s) := \left| \frac{\zeta(\Delta, 2 - s)}{\zeta(\Delta, -s)} \right|.
\]

The proof idea is as follows. We assume \( \text{Im}(s) > 65 \) and prove that

- \( \frac{\Omega(1 - s)}{\Omega(s)} = \frac{q(s)}{\rho(s)} \) is strictly increasing in \( \text{Re}(s) \in (0, 1/2) \),
- \( \frac{\Omega(1 - s)}{\Omega(s)} = \frac{\eta(s)}{\rho(s)} \) is strictly increasing in \( \text{Re}(s) \in (1/2, 1) \).

Thus for any fixed \( \text{Im}(s) > 65 \), \( \left| \frac{\Omega(1 - s)}{\Omega(s)} \right| \) is injective in \( \text{Re}(s) \in (0, 1) \). By Lemma 5.3 it attains its value 1 at \( \text{Re}(s) = \frac{1}{2} \). By injectivity it attains this value only once. This proves the statement. We establish the monotonicity claims in 2 steps.

**Step 1: Monotonicity of \( q(s) \)**

**Claim:** Provided \( \text{Im}(s) > 1/4 \),

- \( q(s) \) is strictly increasing for \( \text{Re}(s) \in (0, 1/2) \),
- \( q(s) \) is strictly decreasing for \( \text{Re}(s) \in (1/2, 1) \).

\(^8\)The restriction to \( \text{Im}(s) > 65 \) is technical, not conceptual. We simply did not manage to prove the various monotonicity statements in Lemma 5.6 below without that restriction.
For $s \in \mathbb{C}$ we set $a = \text{Re}(s)$ and $b = \text{Im}(s)$ and obtain
\[ q(a, t) = \frac{\pi^2}{|a + ib||a + ib - 1|} \]
For $|b| > 0$ fixed and $a \in (0, 1)$ we calculate that
\[ \partial_a q(a, b)^2 = -\frac{\pi^4 (4a^3 - 6a^2 + 2a + 4b^2a - 2b^2)}{(a^4 - 2a^3 + 2b^2a^2 + a^2 - 2b^2a + b^4 + b^2)^2}. \]

First we have that $\partial_a q(1/2, b)^2 = 0$. Now for $|b| > 1/4$ the numerator of $\partial_a q(a, b)^2$ is strictly positive in $a \in (0, 1/2)$. Also for $|b| > 1/4$ the the numerator is strictly negative for $a \in (1/2, 1)$. Thus, for $|b| > 1/4$, $q(a, b)^2$ is strictly increasing for $a \in (0, 1/2)$, strictly decreasing for $a \in (1/2, 1)$ and has a maximum at $a = 1/2$.

**Step 2: Monotonicity of $\eta(s)$ and $\rho(s)$**

**Claim:** Provided $\text{Im}(s) > 64$,
- $\rho(s)$ is strictly decreasing for $\text{Re}(s) \in (0, 1/2)$,
- $\eta(s)$ is strictly increasing for $\text{Re}(s) \in (1/2, 1)$.

This is in fact precisely the statement of Lemma 5.6 below.

**Concluding the proof**

As a consequence, $|\Omega(1-s)/\Omega(s)|$ is strictly increasing (and thus injective) over the entire critical strip $\text{Re}(s) \in (0, 1)$. Thus, in view of Lemma 5.3 we conclude for $\text{Im}(s) > 65$ that $|\Omega(1-s)/\Omega(s)| \neq 1$ for $\text{Re}(s) \neq \frac{1}{2}$. This proves the statement. \(\square\)

Now we are able to prove the Theorem 5.2.

**Proof of Theorem 5.2.** First, we assume the E.R. Conjecture 1.1 for $\alpha = 2$ is true. We want to conclude that Conjecture 5.1 holds. By Proposition 5.1 we only need to consider $\text{Re}(s) \in (0, 1)$ where $\xi_2(s) = 0$. For such $s$ we find by (5.3)
\[ \lim_{n \to \infty} \left| \frac{H_n(1-s)}{H_n(s)} \right| = \left| \frac{\Omega(1-s)}{\Omega(s)} \right|. \]
By assumption, all non-trivial zeros of $\xi_2(s)$ have $\text{Re}(s) = \frac{1}{2}$. In that case the right hand side above equals 1 by Lemma 5.3. This proves the Conjecture 5.1.

Conversely, assume that Conjecture 5.1 holds. We want to conclude that the E.R. Conjecture 1.1 for $\alpha = 2$ is true. It is known since [LRW86] and many more, that the E.R. conjecture 1.1 holds for $\text{Im}(s) \leq 65$. Now consider a zero $s$ of $\zeta(\Delta, s)$ with $\text{Im}(s) > 65$. We compute by assumption
\[ 1 = \lim_{n \to \infty} \left| \frac{H_n(1-s)}{H_n(s)} \right| = \left| \frac{\Omega(1-s)}{\Omega(s)} \right|. \]
Thus (5.7) holds and hence by Proposition 5.4 we conclude $\text{Re}(s) = \frac{1}{2}$. This proves the E.R. Conjecture 1.1 for $\alpha = 2$. \(\square\)
5.2. Proof of auxiliary estimates. Note first the following identity, cf. [SrZv11].

**Lemma 5.5.** Let $G \subset \mathbb{C}$ be any domain and $f : G \rightarrow \mathbb{C}$ a holomorphic function with $f(z) \neq 0$. Then for any $z \in G$ with $z = a + ib$ we have

$$\text{Re} \left( \frac{f'(z)}{f(z)} \right) = \frac{1}{|f(z)|} \frac{\partial |f(z)|}{\partial a}.$$  \hfill (5.11)

This Lemma will be a main tool in the proof of the following statement.

**Lemma 5.6.** Let $s \in \mathbb{C}$ with $\text{Im}(s) > 65$. Then

- $\rho(s) = \frac{|\zeta(\Delta, 2 - s)|}{|\zeta(\Delta, -s)|}$ is strictly decreasing for $\text{Re}(s) \in (0, 1/2)$,
- $\eta(s) = \frac{|\zeta(\Delta, s + 1)|}{|\zeta(\Delta, s - 1)|}$ is strictly increasing for $\text{Re}(s) \in (1/2, 1)$.

**Proof.** Let us first show the second statement. Let $s = a + ib \in \mathbb{C}$ with $a \in (1/2, 1)$. The restriction $b = \text{Im}(s) > 65$ comes only at the very last step of the proof. By Lemma 5.5, showing that

$$\eta(s) = \frac{|\zeta(\Delta, s + 1)|}{|\zeta(\Delta, s - 1)|}$$

is strictly increasing in $a$, is equivalent to

$$\text{Re} \left( \frac{\zeta'(\Delta, s + 1)}{\zeta(\Delta, s + 1)} \right) - \text{Re} \left( \frac{\zeta'(\Delta, s - 1)}{\zeta(\Delta, s - 1)} \right) > 0.$$  \hfill (5.12)

By (1.21), this inequality is equivalent to

$$\text{Re} \left( \frac{\zeta_k(s + 1)}{\zeta_k(s + 1)} \right) - \text{Re} \left( \frac{\zeta_k(s - 1)}{\zeta_k(s - 1)} \right) + \text{Re} \left( \frac{\beta'(s + 1)}{\beta(s + 1)} \right) - \text{Re} \left( \frac{\beta'(s - 1)}{\beta(s - 1)} \right) > 0.$$

Let us consider the individual terms on the left hand side. First we note for $\Lambda(n)$ being the Mangoldt function$^9$ and $a > 1/2$

$$\text{Re} \left( \frac{\zeta_k(s + 1)}{\zeta_k(s + 1)} \right) = -\text{Re} \left( \sum_{n \in \mathbb{N}} \Lambda(n)n^{-s-1} \right) = -\sum_{n \in \mathbb{N}} \Lambda(n)n^{-a-1}\cos(b \log(n)).$$

Then the absolute value may be estimated in the following way

$$\left| \text{Re} \left( \frac{\zeta_k(s + 1)}{\zeta_k(s + 1)} \right) \right| \leq \sum_{n \in \mathbb{N}} \Lambda(n)n^{-\frac{3}{2}} \leq 1.51$$

A similar estimate holds for the Dirichlet beta function for $a > 1/2$

$$\text{Re} \left( \frac{\beta'(s + 1)}{\beta(s + 1)} \right) > -1.51.$$

$^9$We use the relation between Dirichlet series $D(s) = \sum_{n=1}^{\infty} \frac{t(n)}{n^s}$ for a completely multiplicative function $t(n)$ and the Mangoldt function $\Lambda(n)$, whenever the series converge:

$$D'(s)/D(s) = -\sum_{n=1}^{\infty} \frac{\ell(n)\Lambda(n)}{n^s}.$$
So the second statement (equivalently (5.12)) follows if (for $|b| > 65$)
\begin{equation}
- \text{Re}\left(\frac{\zeta'(s - 1)}{\zeta_R(s - 1)}\right) - \text{Re}\left(\frac{\beta'(s - 1)}{\beta(s - 1)}\right) > 3.02. \tag{5.13}
\end{equation}

To obtain such an estimate, we use techniques from [MSZ14]. First we consider the complete Riemann zeta function $\tilde{\xi}(s)$ and the completed Dirichlet beta function $\xi(s, \chi)$ introduced by [Fri16] for any Dirichlet L-function associated to a Dirichlet character $\chi$
\begin{align*}
\tilde{\xi}(s) &= (s - 1)\Gamma\left(\frac{s}{2} + 1\right) \pi^{s/2} \zeta(s), \\
\xi(s, \chi) &= \left(\frac{\pi}{4}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right) \beta(s),
\end{align*}
where the factor in front of $\zeta_R(s)$ cancels the pole at $s = 1$. We obtain by straightforward computations
\begin{align*}
0 > \text{Re}\left(\frac{\tilde{\xi}'(s)}{\tilde{\xi}(s)}\right) &= \text{Re}\left(\frac{1}{s - 1}\right) + \frac{1}{2} \text{Re}\left(\psi\left(\frac{s + 1}{2}\right)\right) - \frac{\log(\pi)}{2} + \text{Re}\left(\frac{\tilde{\xi}_R'(s)}{\tilde{\xi}_R(s)}\right), \\
0 > \text{Re}\left(\frac{\xi'(s, \chi)}{\xi(s, \chi)}\right) &= \frac{1}{2} \log\left(\frac{4}{\pi}\right) + \frac{1}{2} \text{Re}\left(\psi\left(\frac{s - 1}{2}\right)\right) + \text{Re}\left(\frac{\xi'_R(s, \chi)}{\xi_R(s, \chi)}\right),
\end{align*}
where the inequalities hold for $a < 0$ by [MSZ14, Corollary 2.6 and Corollary 2.6 L]. Here, we have introduced the digamma-function $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$. Since for $\text{Re}(s) < 1$, the real part of $(s - 1)$ is negative, we conclude
\begin{align}
- \text{Re}\left(\frac{\tilde{\xi}_R'(s - 1)}{\tilde{\xi}_R(s - 1)}\right) &> \text{Re}\left(\frac{1}{s - 1}\right) + \frac{1}{2} \text{Re}\left(\psi\left(\frac{s + 1}{2}\right)\right) - \frac{\log(\pi)}{2}, \\
- \text{Re}\left(\frac{\beta'(s - 1)}{\beta(s - 1)}\right) &> \frac{1}{2} \log\left(\frac{4}{\pi}\right) + \frac{1}{2} \text{Re}\left(\psi\left(\frac{s - 1}{2}\right)\right). \tag{5.14}
\end{align}

Let us look at the individual terms on the right hand sides of the inequalities above. According to [MSZ14, Lemma 3.2 (iii)] (a special case of the Stirling series for the digamma function) for a $0 < \theta < \pi$ and $s \in \mathbb{C}$ in the sector $-\theta < \arg(s) < \theta$ we have that
\begin{equation}
\text{Re}\left(\psi(s)\right) = \log\left(|s|\right) - \frac{\text{Re}(s)}{2|s|^2} + \text{Re}\left(R'_0(s)\right), \tag{5.15}
\end{equation}
where $R'_0(s)$ is a function, satisfying the estimate $|R'_0\left(\frac{a(s)}{2}\right)| \leq \frac{\sec^3(\theta/2)}{12|s|^2}$.

We choose the angle $\theta = 1.02452 \pi/2$ such that $(s - 1)/2$ and $(s + 1)/2$, for $a \in (1/2, 1)$ and $|b| = |\text{Im}(s)| > 65$, are in the sector. Then for this specific choice of $\theta$ we have that $\sec^3(\theta/2) < 3$. Thus (5.15) yields for $s = a + ib \in \mathbb{C}$ with $a \in (1/2, 1)$ and $|b| > 65$
\begin{align}
\text{Re}\left(\psi\left(\frac{s + 1}{2}\right)\right) &> \log\left(|b|/2\right) - \frac{2}{b^2}, \\
\text{Re}\left(\psi\left(\frac{s - 1}{2}\right)\right) &> \log\left(|b|/2\right) - \frac{2}{b^2}. \tag{5.16}
\end{align}
For the remaining term in (5.14) we find
\[
\text{Re}\left(\frac{1}{s-2}\right) = \frac{a-2}{(a-2)^2 + b^2} > -\frac{2}{1+b^2}.
\] (5.17)

Plugging (5.16) and (5.17) into (5.14), we find
\[
-\text{Re}\left(\frac{\zeta_k'(s-1)}{\zeta_k(s-1)}\right) - \text{Re}\left(\frac{\beta'(s-1)}{\beta(s-1)}\right)
\geq \log\left(\frac{|b|}{2}\right) + \frac{1}{2}\log\left(\frac{4}{\pi}\right) - \frac{\log(\pi)}{2} - \frac{2}{1+b^2} - \frac{2}{b^2} > 3.02,
\]
where only in the last inequality we assumed $|b| > 65$ to obtain 3.02 as a lower bound. This proves (5.13) and thus the second statement.

To prove the first statement, we consider
\[
\rho(s) = \left|\frac{\zeta(\Delta, 2-s)}{\zeta(\Delta, -s)}\right| : \{\text{Re}(s) \in (0, 1/2) : |b| > 65\} \to \mathbb{R}
\]
as a composition of $\eta(s)$ and
\[
g(s) = 1 - s : \{\text{Re}(s) \in (0, 1/2) : |b| > 65\} \to \{\text{Re}(s) \in (1/2, 1) : |b| > 65\},
\]
such that $\rho(s) = (\eta \circ g)(s)$. Let us rewrite this as
\[
|\rho(s)| = |\eta|\left(\text{Re}(g(s)), \text{Im}(g(s))\right).
\]
By the proof of the second statement, $|\eta(s)| = |\eta|(a, b)$ is increasing in the first variable $a \in (1/2, 1)$ for any fixed $|b| > 65$. Since $\text{Re}(g(s))$ is strictly decreasing in $\text{Re}(s) \in (0, 1/2)$, it follows immediately that $|\eta|\left(\text{Re}(g(s)), \text{Im}(g(s))\right)$ is too. This proves the first statement.

\section{Open problems and future research directions}

Our arguments apply to tori of higher dimension $\alpha \geq 3$ as well. By choosing an appropriate refinement of the discrete Laplacian (as we have done in two dimensions by studying the 9-point star Laplacian instead of the classical 5-point star operator), we may obtain a similar functional relation as in Conjecture 5.1 in higher dimensions as well.

On the other hand, as pointed out in Answer 1.8, the Epstein-Riemann conjecture does not hold for dimensions $\alpha \geq 3$. We hope that by studying the discrete functional relation, we may find a discrete geometric reason why the Epstein-Riemann conjecture fails in higher dimensions.

Finally, an interesting question is if our main result, Theorem 1.10, can be generalized to other geometries in the spirit of [IzK20] beyond tori.
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