Quantum phase transitions in algebraic models of mesoscopic systems

F Iachello
Center for Theoretical Physics, Sloane Physics Laboratory, Yale University, New Haven, CT 06520-8120, USA

Abstract. The theory of quantum phase transitions (QPTs) and excited states quantum phase transitions (ESQPTs) is briefly reviewed. The phase structure and phase diagram of the interacting boson model of nuclei are discussed. Experimental evidence for QPTs in nuclei is presented.

1. Algebraic models of mesoscopic systems
Mesoscopic systems are many-body systems with a finite number of constituents, \( N \). The constituents can be either bosonic or fermionic or mixtures of bosons and fermions. In the last 40 years, it has become apparent that a convenient way to treat these systems is by algebraic methods [1]. In algebraic models of many-body systems, the Hamiltonian, \( H \), and other operators are expanded in elements \( G_{\alpha\alpha'} \) of an algebra \( g \),

\[
H = E_0 + \sum_{\alpha\alpha'} \varepsilon_{\alpha\alpha'} G_{\alpha\alpha'} + \sum_{\alpha\alpha'\beta\beta'} u_{\alpha\alpha'\beta\beta'} G_{\alpha\alpha'} G_{\beta\beta'} + \ldots
\]  

(1)

States are characterized by the irreducible representations of \( g \), labelled by the total number of particles, \( N \). The representations are totally symmetric for bosons and totally antisymmetric for fermions.

For fermionic systems the algebra \( g \) is composed by the bilinear products of fermion creation and annihilation operators, satisfying \( \{ a_i, a_{i'}^\dagger \} = \delta_{ii'} \); \( \{ a_i, a_{i'} \} = \{ a_{i'}^\dagger, a_i^\dagger \} = 0 \),

\[
G_{ii'} = a_{i}^\dagger a_{i'} , \quad i, i' = 1, \ldots, n ; \quad G_{ii'} \in g \equiv U(n).
\]  

(2)

The Hamiltonian of any (number-conserving) fermionic system can be written in the form

\[
H = E_0 + \sum_{ii'} \eta_{ii'} a_{i'}^\dagger a_{i'} + \frac{1}{2} \sum_{ii'jj'} u_{ii'jj'} a_{i'}^\dagger a_{j'}^\dagger a_{j} a_{i} + \ldots
\]  

(3)

which, upon rearrangement of the fermion operators, can be cast in the form given above. For rotationally invariant systems, it is convenient to label the single-particle states by \( i \equiv \ell_i, m_{\ell_i}, s_i, m_{s_i} \), and \( n = \sum_i (2\ell_i + 1) (2s_i + 1) \), with \( s_i \) = half-integer. Examples are the atomic shell model and the nuclear shell model.

Published under licence by IOP Publishing Ltd
For bosonic systems, the algebra \( g \) is composed of bilinear products of creation and annihilation operators for bosons, satisfying

\[
G_{\alpha \alpha'} = b_{\alpha}^\dagger b_{\alpha'}^\dagger, \quad \alpha, \alpha' = 1, \ldots, n; \quad G_{\alpha \alpha'} \in g \equiv U(n). \tag{4}
\]

[Bohonic and fermionic systems have the same \( g \). Their differ by its representations.] The Hamiltonian of any (number-conserving) bosonic system can be written in the form

\[
H = E_0 + \sum_{\alpha \alpha'} \varepsilon_{\alpha \alpha'} b_{\alpha}^\dagger b_{\alpha'} + \frac{1}{2} \sum_{\alpha \beta \beta'} \nu_{\alpha \beta \beta'} b_{\alpha}^\dagger b_{\alpha}^\dagger b_{\beta} b_{\beta'} + \ldots, \tag{5}
\]

which, upon rearrangement of the boson operators, reduces to the Hamiltonian given above. [For non-number conserving bosonic systems (for example, ensembles of photons or phonons) bilinear products of creation and annihilation operators are added, \( G'_{\alpha \alpha'} = \hat{b}_{\alpha}^\dagger \hat{b}_{\alpha}^\dagger, G''_{\alpha \alpha'} = b_{\alpha} b_{\alpha}, \) and the algebra \( g \) is \( \text{Sp}(2n, \mathbb{R}) \).] Examples of number conserving bosonic systems are the Interacting Boson Model of nuclei [2], for which \( n = 6 \), and the Vibron Model of molecules [3], for which \( n = 2, 3, 4 \).

Many-body systems composed of a mixture of bosons and fermions can also be treated by algebraic methods by mapping onto an algebraic superstructure, \( g^* \), composed of elements

\[
G_{\alpha \alpha'} = b_{\alpha}^\dagger b_{\alpha'}, \quad G_{i i'} = a_{i}^\dagger a_{i'}, \quad F_{\alpha i} = b_{\alpha}^\dagger a_{i}, \quad F_{i \alpha} = a_{i}^\dagger b_{\alpha}; \quad g^* \equiv U(n/m). \tag{6}
\]

States are characterized by the totally supersymmetric representations of \( U(n/m) \) labelled by the total number of bosons plus fermions \( N = N_B + N_F \). Examples are the Interacting Boson-Fermion Model of nuclei [4] and the Electron-Vibron Model of molecules [5].

In this contribution only bosonic systems will be considered.

2. Quantum phase transitions
Quantum phase transitions (QPT) are qualitative changes in the structure of a physical system that occur as a function of one (or more) parameters, \( \xi_1, \xi_2, \ldots \), the so-called control parameters that appear in the quantum Hamiltonian describing the system

\[
H = \varepsilon (H_1 + \xi_1 H_2 + \xi_2 H_3 + \ldots). \tag{7}
\]

Originally introduced within the context of nuclear physics [6], they have had many applications to various areas of physics [7]. In recent years, the concept of QPT has been extended to excited states quantum phase transitions (ESQPT), that is qualitative changes in the state of a system that occur as a function of excitation energy [8]. In this contribution, after a review of the theory of QPT and ESQPT, applications of these concepts to mesoscopic systems will be discussed. An important aspect of QPTs in mesoscopic systems is the study of finite size scaling and its large \( N \) limit. As a specific example, QPTs and ESQPTs in atomic nuclei (\( N \sim 2 \sim 250 \)) will be discussed. The advantages of using atomic nuclei as examples are: (i) there are in this case specific models that can be studied in depth and (ii) the theory can be tested with experiments.

2.1. QPTs in algebraic models
Algebraic models are particularly well-suited to study QPTs, because they possess exact solutions, called dynamic symmetries (for purely bosonic or fermionic systems) or dynamical super-symmetries (for mixed Bose-Fermi systems) which describe properties of the system in explicit analytic form. These symmetries can be used to define the phases of the system. The
possible phases of a system described by the algebra \( g \), are obtained by breaking \( g \) into its subalgebras

\[
\begin{align*}
g & \supset g_1 \supset g'_1 \supset ... \\
g & \supset g_2 \supset g'_2 \supset ... \\
& \vdots \\
g & \supset g_\phi \supset g'_\phi \supset ...
\end{align*}
\]

(8)

Dynamical symmetries are situations in which the Hamiltonian \( H \) does not contain all elements of \( g \), but only invariant (Casimir) operators, \( C \), of a subalgebra chain

\[
H = aC(g) + a_1C(g_1) + a'_1C(g'_1) + ...
\]

(9)

In this case, the energy eigenvalues can be written in explicit analytic form in terms of quantum phases. Taking out a scale factor, \( \varepsilon = a_1 \), and introducing the control parameters \( \xi_1 = \frac{a_2}{a_1}, \xi_2 = \frac{a_3}{a_1}, ... \), the Hamiltonian can be written in the standard form

\[
H = \varepsilon (H_1 + \xi_1 H_2 + \xi_2 H_3 + ...).
\]

Note that the number of control parameters is equal to the number of phases \( \phi \) minus 1. It is sufficient to take one Casimir operator for each \( g_i \). The corresponding Hamiltonian is often referred to as the "essential" Hamiltonian. For systems with only two phases, it is convenient to parametrize \( H \) as

\[
H = \varepsilon [(1 - \xi) H_1 + \xi H_2]
\]

(12)

in such a way that the control parameter varies in the interval 0-1 instead of 0-\( \infty \).

QPTs have been studied in many bosonic systems. A class extensively investigated is the so-called s-b systems, defined in terms of a singlet boson \( s^{(0)} \) and a \((2\ell + 1)\)-fold degenerate boson \( b^{(\ell)} \). These models have had many applications in physics (see [9]). The algebraic structure of these models is \( U(n) \) with \( n = 2\ell + 2 \), with elements \( U(n) \ni s^1s, s^1b_m, b^1_m, b^1_mb_{m'}, (m, m') = -\ell, ..., +\ell \). For applications to nuclear and molecular physics it is convenient to impose (3D) rotational invariance for problems with \( 2\ell + 1 \)=odd and (2D) invariance for problems with \( 2\ell + 1 \)=even. The decomposition of the algebra of \( U(n) \) with rotational invariance has been carried out explicitly up to \( n = 6 \). QPTs in s-b models have been studied both classically and quantum mechanically [10].

The classical analysis is done by first introducing a geometry of algebraic models through the coset space \( U(n)/U(n - 1) \otimes U(1) \) and then introducing coherent (or intrinsic) states expressed in terms of complex variables \( \alpha_m \). For finite (mesoscopic) systems with a conserved number of particles it is convenient to use number-projected coherent states. The ground state of the system is written as a boson condensate

\[
|N, \alpha \rangle = \frac{1}{\sqrt{N!}} \left( s^1 + \sum_m \alpha_m b^1_m \right)^N |0\rangle.
\]

(13)

The energy expectation value in this state is \( \frac{\langle N, \alpha | H | N, \alpha \rangle}{\langle N, \alpha | N, \alpha \rangle} = E(\alpha) \). Coordinates and momenta are introduced as \( q_m = [(-)^m \alpha_{-m} + \alpha_{-m}^*]/\sqrt{2}, p_m = i[(-)^m \alpha_{-m}^* + \alpha_m]/\sqrt{2} \). A convenient set of
coordinates in this \((n-1)\)-dimensional space is the hyperspherical coordinates set \((r,\Theta)\) with \(\Theta \equiv (\theta_1, \theta_2, \ldots, \theta_{n-2})\). The classical Hamiltonian associated with the quantal Hamiltonian \(H\) is

\[
H_{cl} = T(p_r, p_\Theta, r, \Theta) + V(r, \Theta).
\]

The classical potential function, \(V(\Xi; r, \Theta)\) depends on coordinates \((r, \Theta)\) and on control parameters \(\Xi \equiv (\xi_1, \xi_2, \ldots, \xi_{(n-1)})\). Minimizing \(V\) with respect to the coordinates, one obtains the equilibrium values \((r_e, \Theta_e)\), which are the classical order parameters, and the classical ground state energy, \(V_c(\Xi; r_e, \Theta_e) \equiv E_0\). The study of \(V_c\) and its derivatives with respect to the control parameters determines the order of the phase transition. A classification often used is Erhenfest classification.

A phase transition is called of 0th order if the ground state energy is discontinuous, 1st order if \(\partial E_0 / \partial \Xi\) is discontinuous, 2nd order if \(\partial^2 E_0 / \partial \Xi^2\) is discontinuous, etc.. If there is no discontinuity it is called a crossover. The behavior of the order parameters at (or around) the critical points of the control parameters also determines the order of the phase transitions and the associated critical exponents. A classification often used is Landau classification. A phase transition is called of 1st order if the order parameters \((r_e, \Theta_e)\) are discontinuous, 2nd order if \((\partial r_e / \partial \Xi, \partial \Theta_e / \partial \Xi)\) are discontinuous, etc. If there is no discontinuity, it is called a crossover. Critical exponents describe the behavior around the critical point. For example, for 2nd order transitions

\[
r_e \propto (\xi - \xi_c)^\mu \quad (\xi \geq \xi_c), \quad r_e = 0 \quad (\xi < \xi_c).
\]

For first order transitions the situation is more complex, since there are three points of importance, the spinodal, critical and antispinodal points [10]. Critical exponents have been evaluated for \(U(n)\) models with at most quadratic Hamiltonians.

The quantal analysis is done by diagonalizing the Hamiltonian \(H\) as a function of the control parameters \(\Xi\) in one of the dynamical symmetry bases \(g \supset g_1 \supset g_1' \supset \ldots\), thus evaluating numerically the ground state energy \(E_0\) and its derivatives \(\partial E_0 / \partial \Xi, \ldots\). If the diagonalization is done for sufficiently large \(N\), one can observe precursors of QPTs and determine the order of the phase transition. The quantum order parameters are the expectation values in the ground state of some suitably chosen operators. The most basic order parameter for s-b models is the expectation value of the number of b bosons \(\omega = \langle \hat{N}_b \rangle / N\). This is related to the classical order parameter by \(\omega = r^2 \xi / (1 + r^2)^2\). The behavior of the order parameter and other quantities as \(N \to \infty\) is called finite size scaling. Finite size scaling in s-b models can be studied both numerically and analytically. For second order transitions all physical quantities scale as a power law [11]. For example, \(\omega (\xi_c) = A_{c0} N^{-\Delta_{ccl}}\). [For first order transitions a complete theory of scaling is not yet available].

3. Excited state quantum phase transitions (ESQPT)

ESQPTs are qualitative changes in the state of the system as a function of excitation energy [8]. In QPTs one studies classically the potential \(V\) and quantum mechanically the ground state energy, \(E_0\). In ESQPTs one needs to do dynamics, either semi-classically or quantum mechanically.

Semi-classical dynamics has been done completely for s-b models with pairing interactions with quantum Hamiltonian

\[
H = \frac{(1 - \xi)}{N} \hat{N}_b - \frac{\xi}{N^2} \left( s^{\dagger} \hat{b} + b^{\dagger} \hat{s} \right) \cdot \left( s^{\dagger} \hat{b} + b^{\dagger} \hat{s} \right)
\]

and classical Hamiltonian

\[
H_{cl} = \frac{1 - \xi}{2N} [p_r^2 + r^{-2} T_0 (v)] + \frac{\xi}{N^2} [r^2 p_r^2 + T_0 (v)] + V(r, \xi)
\]

\[
V(r, \xi) = \frac{1 - 5\xi}{2} r^2 + \xi r^4.
\]
A study of $V$ and its derivatives shows that s-b models with pairing interactions have a 2nd order QPT at $\xi_c = 1/5$, with critical exponent $\mu = 1/2$. In order to study ESQPTs, one considers the full Hamiltonian (kinetic plus potential) and studies the classical action $S(\xi; E)$ shown in figure 1.

![Figure 1](image_url)

**Figure 1.** Contour plot of the classical action $S(\xi; E)$ for the classical Hamiltonian (17), through the different regimes determined by the shape of the potential energy function, which is shown for (a) $\xi < \xi_c$, (b) $\xi = \xi_c$, and (c) $\xi > \xi_c$. The individual contours are related semi-classically to the evolution of the level eigenvalues. Figure from Ref. [8].

At the top of the barrier in (c), the action has a logarithmic singularity, leading semi-classically to an infinite level density (ESQPT). The quantity $(\partial S/\partial E)_\xi$ diverges at $E = 0$.

The quantum Hamiltonian $H$ can be diagonalized numerically for finite $N$. The evolution of the eigenvalues as a function of $\xi$ is called correlation diagram, shown in figure 2. It can be seen that features (discontinuities when $N \to \infty$) appear by traversing the correlation diagram both in the horizontal direction, as a function of the control parameter $\xi$, QPT, and in the vertical direction as a function of excitation energy, ESQPT. An important property of excited states is that, at a critical value of a QPT, $\xi = \xi_c$, all excited states collapse to zero energy. For finite $N$, a gap develops. This gap scales also with a power law $\Delta = A_{\Delta0}N^{-A_{\Delta1}}$. The scaling exponent can be obtained either numerically or analytically.

4. **Quantum phase transitions in nuclei: the Interacting Boson Model (IBM)**

The interacting boson model (IBM) is a model of even-even nuclei in terms of correlated pairs of nucleons treated as bosons with $L = 0$ (s-boson) and $L = 2$ (d-boson) [2]. (S- and D-wave pairing in 3D).
4.1. Algebraic structure

The s and d bosons span a six-dimensional space with algebraic structure $U(6)$ and dynamical symmetries

$$
U(6) \supset U(5) \supset SO(5) \supset SO(3) \supset SO(2) \quad (I),
$$
$$
U(6) \supset SU(3) \supset SO(3) \supset SO(2) \quad (II),
$$
$$
U(6) \supset SO(6) \supset SO(5) \supset SO(3) \supset SO(2) \quad (III).
$$

The energy eigenvalues are

$$
E(N, n_d, v, n_\Delta, L, M_L) = E_0 + \alpha n_d(n_d + 4) + \beta v(v + 3) + \gamma L(L + 1),
$$

$$
E(N, \lambda, \mu, K, L, M_L) = E_0 + \kappa (\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) + \kappa' L(L + 1),
$$

$$
E(N, \sigma, \tau, \nu_\Delta, L, M_L) = E_0 + A\sigma(\sigma + 4) + B\tau(\tau + 3) + CL(L + 1).
$$

4.2. Phase structure and phase diagram

The essential Hamiltonian of IBM is

$$
H = \varepsilon \{ C_1[U(5)] + \xi_1 C_2[SU(3)] + \xi_2 C_2[SO(6)] \},
$$

where $C_p(g)$ is the Casimir operator of order $p$ of $g$. The IBM has three structural phases, Eq.(18), and its phase diagram is two-dimensional. A convenient (physical) parametrization of $H$ is

$$
H = \left( \frac{E_0}{N} \right) \left( 1 - \xi \right) \hat{n}_d - \frac{\xi}{4N} \hat{Q}^x \cdot \hat{Q}^y.
$$

When $\xi = 0$ one has $U(5)$ symmetry, when $\xi = 1, \chi = 0$ one has $SO(6)$ symmetry and when $\xi = 1, \chi = \pm \sqrt{7}/2$, one has $SU(3)$ symmetry.

The coset space of IBM is $U(6)/U(5)\otimes U(1)$, with five complex variables $\alpha_\mu (\mu = 0, \pm 1, \pm 2)$. The number projected coherent state is

$$
|N, \alpha_\mu \rangle = \left( s^\dagger + \sum_\mu \alpha_\mu d_\mu^\dagger \right)^N |0\rangle.
$$
Figure 3. Liquid drop model of the nucleus with ellipsoidal shape. Intrinsic axis orientations are described by Euler angles $\theta_i$, and axis lengths are determined by $\beta$ and $\gamma$.

The variables $\alpha_{\mu}$ can be transformed to Bohr variables $(\beta, \gamma)$ and Euler angles $(\theta_1, \theta_2, \theta_3)$ by rotation. This is motivated by the liquid drop model of nuclei, shown in figure 3. The intrinsic state can be written in terms of Bohr variables as

$$|N; \beta, \gamma\rangle = \left(s^\dagger + \beta \left[ \cos \gamma d_{0}^\dagger + \frac{\sin \gamma}{\sqrt{2}} \left( d_{+2}^\dagger + d_{-2}^\dagger \right) \right] \right)^N |0\rangle.$$

and the potential function is

$$V(\xi, \chi; \beta, \gamma) = \frac{(1 - \xi) \beta^2}{1 + \beta^2} - \xi \left[ 4\beta^2 - 4\sqrt{2/7} \chi\beta^3 \cos 3\gamma + 2\chi^2 \beta^4/7 \right] \frac{1}{(1 + \beta^2)^2}.$$

The classical order parameters are the equilibrium deformations, $\beta_e, \gamma_e$. A study of $V_e$ and its derivatives shows that there is in the phase diagram a line of 1st order transitions ending in a point of second order, as shown in figure 4 [12].

Figure 4. Phase diagram of the IBM. A curve of first-order QPT $[\xi_e(\chi) = (5 + \frac{1}{7}\chi^2)^{-1}]$ and point of second-order QPT separate spherical and deformed phases.

A quantal analysis of QPTs in IBM has been done, showing the smoothing out of critical properties. The order parameter $\omega$ is shown in figure 5 [13]. The ground state energy, $E_0$, and its derivatives are shown in figure 6 [14].
Figure 5. Quantum calculations of the order parameter $\omega$, for the interacting boson model. (Right) The U(5)-SO(6) transition. (Left) The U(5)-SU(3) transition. Calculations are shown for various finite $N$.

Figure 6. The ground state energy $E(0^n)$ and its derivatives for the IBM, across first-order U(5)-SU(3) (left) and second-order U(5)-SO(6) (right) QPTs, at finite $N$. The lines shown are for $N=10, 100, 200, 300,$ and $400$. The control parameter used in these plots is $\zeta \equiv 4\xi/(1 + 3\xi)$. Thus, the critical point is at $\zeta_c \approx 0.47$ at left or $\zeta_c = 1/2$ at right. Figure from Ref. [14].

Scaling exponents for 2nd order transitions can be obtained either analytically or numerically [15]. Writing for any quantity $\Phi(\zeta_c) = A_0 N^{-A_1}$ one obtains in the U(5)-SO(6) transition, for the ground state energy $E_0$, $A_1 = 4/3$, for the order parameter $\langle n_d \rangle$, $A_1 = -1/3$, for the gap $\Delta$, $A_1 = 1/3$, and for the operator $T = B(E2)/5N$, $A_1 = -4/3$. These analytic values have been verified numerically as shown in figure 7. [For first order transitions a complete theory of scaling is not yet available].
4.3. Experimental evidence

In the quantal Hamiltonian of the IBM, $H = \varepsilon \hat{n}_d - \kappa \hat{Q}^x \cdot \hat{Q}^x$, the parameters $\varepsilon$ and $\kappa$ depend on proton and neutron number. It turns out that the control parameter depends (almost) linearly on proton and neutron number. Three quantities have been experimentally investigated: (a) The ground state energy, $E_0$. This can be measured easily. It is useful to plot the differences $S_2 n(N) = E_0(N + 1) - E_0(N) \propto \frac{\partial E_0}{\partial \xi}$. (b) The order parameter $\omega$. This is difficult to measure, but its square, $B(E2; 2_1^+ \rightarrow 0_1^+)$, can be measured easily. (c) The gap $\Delta = E(0_2^+) - E(0_1^+)$ easily measured. These three quantities are shown in figure 8. It is seen that critical behavior (precursors of QPT) is observed even for small values of $N$.

5. Complex situations

In recent years, QPTs in systems more complex that a single bosonic system have been investigated, especially for coupled systems. Complete results are available for: (A) Quantum
phase transitions in two-fluid bosonic systems [16], [17]. (B) Quantum phase transitions in two-level pairing models (bosonic and fermionic) [18]. The algebra $g$ is in these cases, $g = U(n_1) \otimes U(n_2)$. The phase diagrams for these systems are more complicated, as there are more phases and more control parameters. Investigations have started for (C) Quantum phase transitions in mixed Bose-Fermi systems [19], [20], [21]. Algebraic methods are particularly useful in these cases. The characterization of the phases by means of dynamic symmetries, the branching of algebras, and the geometry of these algebraic structures are of such a complexity that require full exploitation of the mathematical techniques discussed above.

6. Conclusions

The study of QPTs and ESQPTs in s-b boson models is essentially completed. The only remaining areas for further work are: (i) the study of finite-size scaling and of (ii) level densities in 1st order transitions.

The class of models discussed here is similar to models in condensed matter physics, for example the quantum Ising model, with Hamiltonian, $H = J \left( -g \sum_i \sigma_i^x - \sum_{ij} \sigma_i^z \sigma_j^z \right)$. This model has the same structure of s-b models with algebra SU(2). The difference is that in condensed matter systems the spins are on a lattice, while in nuclear and molecular systems the bosons are in free 3D or 2D space.

Important aspects of algebraic models are: (i) a rigorous mathematical definition of phases; (ii) the possibility to do semi-classical dynamics; (iii) easy computation of critical values of the control parameters, critical exponents, etc.; and (iv) simple study of finite size scaling. Future work in QPT/ESQPT includes a generalization to multi-level mesoscopic systems and a generalization to infinite systems.

Acknowledgments

This work was supported in part by US Department of Energy Grant DE-FG02-91ER-40608.

References

[1] Iachello F 2006 Lie algebras and applications (Lecture Notes in Physics Vol 708) (Berlin: Springer)
[2] Iachello F and Arima A 1987 The Interacting Boson Model (Cambridge: Cambridge University Press)
[3] Iachello F and Levine R D 1995 Algebraic Theory of Molecules (Oxford: Oxford University Press)
[4] Iachello F and van Isacker P 1991 The Interacting Boson-Fermion Model (Cambridge: Cambridge University Press)
[5] Frank A, Lemus R and Iachello F 1983 J. Chem. Phys. 91 29
[6] Gilmore R 1979 J. Math. Phys. 20 891
[7] Carr L D ed Understanding Quantum Phase Transitions (Boca Raton, FL: CRC Press)
[8] Caprio M A, Cejnar P and Iachello F 2008 Ann. Phys. 323 1106
[9] Cejnar P and Iachello F 2007 J. Phys. A: Math. Theor. 40 581
[10] Iachello F and Caprio M A 2010 in Understanding Quantum Phase Transitions (Boca Raton, FL: CRC Press) chapter 27
[11] Fisher M A and Barber M N 1972 Phys. Rev. Lett. 28 1516
[12] Feng D S, Gilmore R and Deans S R 1981 Phys. Rev. C 23 1254
[13] Iachello F and Zamfir N V 2004 Phys. Rev. Lett. 92 21501
[14] Williams E T 2009 Ph D Thesis Yale University
[15] Dusuel S, Vidal J, Arias J M, Dukelsky J and Garcia-Ramos J E 2005 Phys. Rev. C 72 011301 (R)
[16] Caprio M A and Iachello F 2004 Phys. Rev. Lett. 93 242502
[17] Arias J M, Dukelsky J and Garcia-Ramos J E 2004 Phys. Rev. Lett. 93 212501
[18] Caprio M A, Skrabacz J H and Iachello F 2011 J. Phys. A: Math. Theor. 44 075303
[19] Jolie J, Heinze S, Van Isacker P and Casten R F 2004 Phys. Rev. C 70 011305 (R)
[20] Alonso C E, Arias J M, Fortunato L and Vitturi A 2005 Phys. Rev. C 72 061302
[21] Petrellis D, Leviatan A and Iachello F 2011 Ann. Phys. 326 926