A note on one-sided recognizable morphisms

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Abstract

We revisit the notion of one-sided recognizability of morphisms and its relation to two-sided recognizability.

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1 Introduction

The notion of recognizability for morphisms is an important one with a long history (see [15] for an account of it).

The first attempts used a one-sided notion adapted to one-sided infinite sequences. The main progress realized with Mossé’s Theorem (see Theorem 3.9 below) was made possible by turning to a two-sided version of recognizability. Since then, several generalizations of Mossé’s theorem have been obtained (see [6], [5] and [2]).

In this note, we come back to the one-sided version of recognizability. It was studied in [8] for alphabets with two letters and more recently in [1]. The authors of [8] prove that although a primitive aperiodic endomorphism on two letters is not always one-sided recognizable, it is almost so, in the sense that the one-sided sequences for which this fails have a special form. We generalize their result to arbitrary primitive aperiodic endomorphisms.
More precisely, given a shift space $X$ on the alphabet $A$, we define the recognizability of a morphism $\sigma: A^* \to B^*$ on $X$ by a uniqueness desubstitution property and we relate this definition with the original definition of Mossé (Proposition 3.3). We state without proof the theorem of Mossé asserting that a primitive aperiodic morphism $\sigma$ is recognizable on the shift $X(\sigma)$ defined by $\sigma$ (Theorem 3.9).

We next define the one-sided recognizability of a morphism $\sigma$ on a one-sided shift. We first relate the notion of (two-sided) recognizability on a shift $X$ with the one-sided recognizability on the one-sided shift $X^+$ associated to $X$ (Proposition 4.1). Next, we relate it with the original definition of Mossé (Proposition 4.3). The main results are

1. Theorem 4.6 which characterizes the endomorphisms $\sigma$ which are not one-sided recognizable on $X(\sigma)^+$.

2. Theorem 4.7 which states that every endomorphism $\sigma$ is almost one-sided recognizable on the shift $X(\sigma)$, in the sense that it is one-sided recognizable except at a finite number of points.

We end the paper with a mention of the point which has motivated us for this note and concerns continuous eigenvalues of shift spaces. Indeed, a result of Host [14] on the eigenvalues of substitution shifts is formulated using one-sided shifts and one-sided recognizability (and was recently extended to $S$-adic systems in [11]). Its proof can be however be read without change using two-sided shifts (and two-sided recognizability). We contribute to the clarification of the situation, giving a simple proof that a recurrent shift space and its associated one-sided shift have the same spectrum (Proposition 5.3 see also [4, Proposition 2.1]).

## 2 Shift spaces

Let $A$ be a finite alphabet. We denote by $A^*$ the set of words on $A$, by $\varepsilon$ the empty word, and by $A^+$ the set of nonempty words.

We consider the set $A^\mathbb{Z}$ of two-sided sequences of elements of $A$ and the corresponding set $A^\mathbb{N}$ of one-sided sequences. For $x \in A^\mathbb{Z}$ and $i \leq j$, we denote $x_{[i,j]} = x_ix_{i+1}\cdots x_j$ and $x_{[i,j)} = x_ix_{i+1}\cdots x_{j-1}$.

For a nonempty word $w \in A^+$, we denote $w^0 = www\cdots$ and $w^\infty = \cdots ww$.

We denote by $S$ the shift transformation defined for $x \in A^\mathbb{Z}$ (resp. $x \in A^\mathbb{N}$) by $S(x) = y$ if $y_n = x_{n+1}$ for $n \in \mathbb{Z}$ (resp. $n \in \mathbb{N}$). A subset $X$ of $A^\mathbb{Z}$ (resp. $A^\mathbb{N}$) is shift invariant if $S(X) = X$.

The orbit of a point $x \in A^\mathbb{Z}$ is the set of all $S^n(x)$ for $n \in \mathbb{Z}$.

The set $A^\mathbb{Z}$ of two-sided infinite sequences of elements of $A$ is a compact metric space for the distance $d(x,y) = 2^{-r(x,y)}$ with

$$r(x,y) = \max\{n \geq 0 \mid x_{\lfloor -n, n \rfloor} = y_{\lfloor -n, n \rfloor}\}.$$
Similarly, the set $A^n$ of one-sided sequences of elements of $A$ is a compact metric space for the distance defined using $r(x, y) = \max\{n \geq 0 \mid x_{[0,n]} = y_{[0,n]}\}$.

A shift space (resp. one-sided shift space) on the alphabet $A$ is a closed and shift invariant subset of $A^\mathbb{Z}$ (resp. $A^n$). The set $A^n$ (resp. $A^\mathbb{Z}$) itself is a a shift space (resp. a one-sided shift space) called the full shift (resp. the full one-sided shift).

The shift space generated by a sequence $x \in A^\mathbb{Z}$ (resp. $x \in A^n$) is the topological closure of the set $\cup_{n \in \mathbb{Z}} S^n x$ (resp. $\cup_{n \in \mathbb{N}} S^n x$). It is the smallest shift space (resp. one-sided shift space) containing $x$.

A shift space is a particular case of a (topological) dynamical system, which is by definition a pair $(X, T)$ of a compact metric space $X$ and a continuous map $T$ from $X$ into itself. It is invertible if $T$ is invertible (and thus a homeomorphism).

A morphism from a dynamical system $(X, T)$ to a dynamical system $(X', T')$ is a continuous map $\varphi : X \to X'$ which interleaves with $T, T'$, that is, such that $T' \circ \varphi = \varphi \circ T$.

Given an invertible system $(X, T)$, the orbit of $x \in X$ is the set $\{T^n(x) \mid n \in \mathbb{Z}\}$. Its forward orbit is the set $\{T^n(x) \mid n \geq 0\}$.

The language of a shift space (resp. a one-sided shift space) $X$, denoted $\mathcal{L}(X)$ is the set of factors of the sequences in $X$. We denote by $\mathcal{L}_n(X)$ the set of words of length $n$ in $\mathcal{L}(X)$.

Let $X, Y$ be shift spaces on alphabets $A, B$ respectively. Given an integer $N$, a block map of window size $N$ is a map $f : \mathcal{L}_{2N+1}(X) \to B$. The sliding block code defined by $f$ is the map $\varphi : X \to B^\mathbb{Z}$ defined by $\varphi(x) = y$ if

$$y_n = f(x_{[n-N,n+N]}) \quad (n \in \mathbb{Z})$$

By a classical result, a map $\varphi : X \to Y$ is a morphism if and only if it is a sliding block code from $X$ into $Y$ [10, Theorem 6.2.9].

For a two-sided sequence $x \in A^\mathbb{Z}$, we denote $x^+ = x_0 x_1 \cdots$. If $X$ is a shift space, we denote by $X^+$ the set of $x^+$ for $x \in X$. It is a one-sided shift space. Note that $X$ is determined by $X^+$ since for every shift space $X$, one has the equality

$$X = \{x \in A^\mathbb{Z} \mid x_n x_{n+1} \cdots \in X^+, \text{for all } n \in \mathbb{Z}\}.$$ 

Thus, the map $X \to X^+$ is a bijection from the family of shift spaces on $A$ onto the family of one-sided shift spaces on $A$.

A sequence $x \in A^\mathbb{Z}$ (resp. $x \in A^n$) is periodic if $S^n(x) = x$ for some $n \geq 1$. Otherwise, it is aperiodic. A shift space (resp. a one-sided shift space) is periodic if all its elements are periodic. It is aperiodic if all its elements are aperiodic.

A topological dynamical system is recurrent if there is a point with a dense forward orbit.

A nonempty topological dynamical system is minimal if, for every closed subset $Y$ of $X$ such that $T(Y) \subseteq Y$, one has $Y = \emptyset$ or $Y = X$. Equivalently, $X$ is minimal if and only if the orbit of every point $x$ is dense.

A shift space $X$ is recurrent if for every $u, v \in \mathcal{L}(X)$ there is a word $w$ such that $uvw \in \mathcal{L}(X)$. 

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A shift space $X$ is uniformly recurrent if for every $w \in \mathcal{L}(X)$ there is an $n \geq 1$ such that $w$ is a factor of every word in $\mathcal{L}_n(X)$.

Given a word $u \in \mathcal{L}(X)$, a right return word to $u$ is a nonempty word $w$ such that $uw \in \mathcal{L}(X)$ and that $uw$ has exactly two occurrences of $u$, one as a prefix and one as a suffix.

Similarly, a left return word to $u$ is a nonempty word $w$ such that $wu \in \mathcal{L}(X)$ and that $wu$ has exactly two occurrences of $u$, one as a prefix and one as a suffix.

A shift space is uniformly recurrent if and only if it is recurrent and for every $u \in \mathcal{L}(X)$ the set of return words to $u$ is finite.

The following is well known (see [13] for example).

**Proposition 2.1** A shift space is minimal if and only if it is uniformly recurrent.

Let $X$ be a shift space on $A$. For $w \in \mathcal{L}(X)$, we denote

$$\ell(w) = \text{Card}\{a \in A \mid aw \in \mathcal{L}(X)\}, \quad r(w) = \text{Card}\{a \in A \mid wa \in \mathcal{L}(X)\}. $$

A word $w \in \mathcal{L}(X)$ is left-special (resp. right-special) if $\ell(w) \geq 2$ (resp. $r(w) \geq 2$).

Let $X$ be a one-sided shift. For a one-sided sequence $x \in X$, we denote

$$\ell(x) = \text{Card}\{a \in A \mid ax \in X\}. $$

A one-sided sequence $x \in A^\mathbb{N}$ is left-special if $\ell(x) \geq 2$.

Two points $x, y$ of a two-sided shift space $X$ are right asymptotic if there is an $n \geq 0$ such that $S^n(x)^+ = S^n(y)^+$. They are asymptotically equivalent if there are $n, m \in \mathbb{Z}$ such that $S^n(x)^+ = S^m(y)^+$. The classes of this equivalence are called the asymptotic classes. Every class is a union of orbits. An asymptotic class is non-trivial if it is not reduced to one orbit.

The complexity of a shift space $X$ is the sequence $p_n(X) = \text{Card}(\mathcal{L}_n(X))$. We denote $s_n(X) = p_{n+1}(X) - p_n(X)$. It is classical that

$$s_n(X) = \sum_{w \in \mathcal{L}_n(X)} (\ell(w) - 1) = \sum_{w \in \mathcal{L}_n(X)} (r(w) - 1). \quad (2.1)$$

Indeed, one has

$$s_n(X) = p_{n+1}(X) - p_n(X) = \text{Card}(\mathcal{L}_{n+1}(X)) - \text{Card}(\mathcal{L}_n(X))$$

$$= \sum_{w \in \mathcal{L}_n(X)} (\ell(w) - 1).$$

If the sequence $s_n(X)$ is bounded, the complexity $p_n(X)$ is at most linear, that is $p_n(X) \leq kn$ for some $k \geq 1$. The converse is true by an important result due to Cassaigne [7].

**Proposition 2.2** If the complexity of a shift $X$ is at most linear, then $s_n(X)$ is bounded.
A shift space $X$ is linearly recurrent if there is a constant $K$ such that for every $w \in \mathcal{L}(X)$, the length of every return word to $w$ is bounded by $K|w|$. The following result is from [12].

**Proposition 2.3** Every linearly recurrent shift has at most linear complexity.

Note that this implies that, in a linearly recurrent shift, the sequence $s_n(X)$ is bounded and thus, by [2.1], the number of left-special words of length $n$ is bounded, and finally that the number of left-special one-sided infinite sequences is finite. In this case, every one-sided sequence in $X^+$ but a countable number of them, has a unique left extension in $X$. In particular, the shifts $X$ and $X^+$ are measurably isomorphic (see [13]).

### 3 Morphisms and recognizability

Let $\sigma: A^* \to B^*$ be a morphism. Then $\sigma$ extends to a map from $A^\mathbb{Z}$ (resp. $A^\mathbb{N}$) to $A^* \cup A^\mathbb{Z}$ (resp. $A^* \cup A^\mathbb{N}$).

Let $\sigma: A^* \to B^*$ be a morphism. A letter $a \in A$ is erasable if $\sigma^n(a) = \varepsilon$ for some $n \geq 1$. The morphism is non-erasing if there is no erasable letter.

Let $\sigma: A^* \to B^*$ be a morphism. A $\sigma$-representation of a point $y \in B^\mathbb{Z}$ is a pair $(x, k)$ with $x \in A^\mathbb{Z}$ and $0 \leq k < |\sigma(x_0)|$ such that $y = S^k(\sigma(x))$.

Let $X$ be a shift space on $A$. A morphism $\sigma: A^* \to B^*$ is recognizable on $X$ at $y \in B^\mathbb{Z}$ if $y$ has at most one $\sigma$-representation $(x, k)$ with $x \in X$. It is recognizable on $X$ if it is recognizable on $X$ at every $y \in B^\mathbb{Z}$.

For a shift space $X$ on $A$ and a word $w$ of length $n$, we denote $[w]_X = \{x \in X \mid x_{[0,n)} = w\}$.

**Proposition 3.1** Let $\sigma: A^* \to B^*$ be a morphism, let $X$ be a shift space on $A$ and let $Y$ be the closure under the shift of $\sigma(X)$. The morphism $\sigma$ is recognizable on $X$ if and only if the family $\mathcal{P}$ of sets $S^k \sigma([a]_X)$ for $a \in \mathcal{L}_1(X)$ and $0 \leq k < |\sigma(a)|$ forms a partition of $Y$.

**Proof.** Assume first that $\sigma$ is recognizable on $X$. Every element of $Y$ has a $\sigma$-representation and thus the union of the elements of $\mathcal{P}$ is $Y$. Next if two of them intersect, then some element of $Y$ has two distinct $\sigma$-representations, which is impossible.

Conversely, if $\sigma$ is not recognizable on $X$, there is some $y$ with two distinct $\sigma$ representations $(x, k)$ and $(x', k')$. If $x = x'$ then $S^k \sigma([x_0])$ and $S^{k'} \sigma([x_0])$ are two distinct elements of the family $\mathcal{P}$ with nonempty intersection. Otherwise, shifting $y$ if necessary, we may assume that $x_0 \neq x'_0$, whence the conclusion that $\mathcal{P}$ is not a partition again.

The partition $\mathcal{P}$ above, called a partition in towers, plays an important role in the definition of a Bratteli diagram associated to a substitution shift (see [13]).
Example 3.2 The morphism $\sigma: a \mapsto ab, b \mapsto a$ is called the Fibonacci morphism. It is recognizable on $X = A^Z$ since the family 

$$[ab]_Y, [b]_Y, [aa]_Y$$

forms a partition of $Y$. One has $\sigma([b]_X) = [aa]_Y$ because $\sigma(a)$ and $\sigma(b)$ begin with $a$.

The definition of recognizability given above is a dynamical one and the one in current use now (see [5] for example) but it was given in a different form (and only for endomorphisms) in the articles of Mossé [17, 18].

Let $\sigma: A^* \rightarrow B^*$ be a morphism, let $x \in A^Z$ be such that $y = \sigma(x)$ is two-sided infinite. Define the set of cutting points of $x$ as

$$C(x) = \{ |\sigma(x|_{[0,n]})| \mid n \geq 0 \} \cup \{ -|\sigma(x|_{[n,0]})| \mid n < 0 \}.$$ 

Given $N \geq 1$, let us say that $\sigma$ is recognizable in the sense of Mossé for $x$ with scope $N$ if for $i, j \in \mathbb{Z}$, whenever $y_{[i-N,i+N]} = y_{[j-N,j+N]}$, then $i \in C(x) \iff j \in C(x)$.

The following result connecting the two notions of recognizability is proved in [5, Theorem 2.5]. All morphisms are supposed to be non-erasing in [5], but the proof of (i) remains the same in the general case.

Proposition 3.3 Let $\sigma: A^* \rightarrow B^*$ be morphism, let $x \in A^Z$ be such that $y = \sigma(x)$ is two-sided infinite and let $X$ be the shift generated by $x$. The following assertions hold.

(i) If $\sigma$ is recognizable on $X$ then it is recognizable in the sense of Mossé for $x$.

(ii) If the shift $X$ is minimal, if the morphism $\sigma$ is non-erasing, injective on $A$ and recognizable in the sense of Mossé for $x$, then $\sigma$ is recognizable on $X$.

Assertion (ii) is not true without its restrictive hypotheses on $X$ and $\sigma$. Indeed, for example, if $\sigma: a \mapsto a, b \mapsto \varepsilon$, then $\sigma$ is recognizable in the sense of Mossé for every $x \in A^Z$ with a finite number of $b$, but it is not recognizable on the shift $X$ generated by $x$ since $\sigma(X) = a^\infty$.

The notion of recognizability is closely related to the notion of tower construction that we recall now. Given a morphism $\sigma: A^* \rightarrow B^*$ and a shift space $X$ on $A$, let $(X^\sigma, T)$ be the dynamical system defined by

$$X^\sigma = \{(x, k) \mid x \in X, 0 \leq k < |\sigma(x_0)| \}$$

and

$$T(x, k) = \begin{cases} (x, k + 1) & \text{if } k + 1 < |\sigma(x_0)| \\ (S(x), 0) & \text{otherwise} \end{cases}$$

(3.1)
Then the map $\hat{\sigma}: (x, i) \mapsto S^i \sigma(x)$ is a morphism of dynamical systems from $(X\sigma, T)$ onto the shift $Y$ which is the closure under the shift of $\sigma(X)$. The morphism $\sigma$ is recognizable on $X$ if and only if $\hat{\sigma}$ is a homeomorphism.

Note that we may consider $(X\sigma, T)$ as a shift space on the alphabet $A\sigma = \{(a, k) \mid a \in A, 0 \leq k < |\sigma(a)|\}$. (3.2)

Indeed, there is a unique morphism $\alpha$ from $(X\sigma, T)$ into $(A\sigma)^\mathbb{Z}$ such that

$$\alpha(x, k)_0 = (x_0, k)$$

(3.3)

Endomorphisms  A morphism $\sigma: A^* \to A^*$ is called an endomorphism. Let $\sigma: A^* \to A^*$ be an endomorphism. The language $L(\sigma)$ is the set of factors of the words $\sigma^n(a)$ for some $n \geq 0$ and some $a \in A$. The shift $X(\sigma)$ is the set of $x \in A^\mathbb{Z}$ with all their factors in $L(\sigma)$. Such a shift is called a substitution shift.

Endomorphisms are often called substitutions (in general with additional requirements, such as being non erasing and with $L(\sigma) = L(X(\sigma))$ as in [13]).

The troubles arising with erasable letters are simplified for substitution shifts since by [3, Lemma 3.13], for every $x \in X(\sigma)$, the sequence $\sigma(x)$ is in $X(\sigma)$ (in particular, $\sigma(x)$ is a two-sided infinite sequence).

An endomorphism $\sigma: A^* \to A^*$ is primitive if there is an integer $n \geq 1$ such that for every $a, b \in A$ one has $|\sigma^n(a)cb| \geq 1$. The following is well known (see [13] for example).

**Proposition 3.4** If $\sigma: A^* \to A^*$ is primitive and $\text{Card}(A) \geq 1$, then $X(\sigma)$ is minimal.

The following result is from [9] (see also [13]).

**Proposition 3.5** Every minimal substitution shift is linearly recurrent.

Combining Propositions 3.4 and 2.3, we obtain that every minimal substitution shift has at most linear complexity.

Note that, by Equation (2.1), this implies that for a minimal substitution shift $X(\sigma)$, the number of left-special sequences is finite.

**Example 3.6** The Fibonacci morphism $\sigma: a \mapsto ab, b \mapsto a$ is primitive. The complexity of $X(\sigma)$ is $p_n(X) = n + 1$. As a minimal shift of complexity $n + 1$, the shift $X(\sigma)$ is, by definition, a Sturmian shift. The sequence $\sigma^\omega(a)$ is the unique one-sided sequence having all $\sigma^n(a)$ as prefixes. It is the unique fixed point of $\sigma$ in $X(\sigma)^+$ and also the unique left-special sequence in $X(\sigma)^+$. As a consequence, we have the following finiteness result.

**Proposition 3.7** Let $\sigma$ be a morphism. If $X(\sigma)$ is minimal, the number of non-trivial asymptotic classes of $X(\sigma)$ is finite and bounded by the number of left-special sequences.
Proof. If $X(\sigma)$ is periodic, there is no non-trivial asymptotic class. Thus, since $X(\sigma)$ is minimal, we may assume that $X(\sigma)$ (and also $X(\sigma)^+$) is aperiodic. Let $x$ be the map which assigns to a left-special sequence $x \in X(\sigma)^+$ the asymptotic class of the points $z \in X(\sigma)$ such that $z^+ = x$. Since $x$ is left-special and since $x$ cannot be periodic, the class $\alpha(x)$ is non-trivial. Let indeed $z, z' \in X$ be distinct such that $z^+ = z^+ = x$ (they exist since $x$ is left-special). If $z$ is a shift of $z'$, then $x$ is a proper shift of itself and thus it is periodic, a contradiction.

The map $\alpha$ is surjective from the set of left-special sequences to the set of non-trivial asymptotic classes. Indeed, let $C$ be a non-trivial asymptotic class. Let $x, y \in C$ be in distinct orbits and let $n, m \in \mathbb{Z}$ be such that $S^n x^+ = S^m y^+$. Set $z = S^n x$ and $t = S^m y$. Then $z, t$ are distinct points in $C$ such that $z^+ = t^+$ and thus there is at least one $u \in C$ such that $u^+$ is left-special.

Let $X$ be a shift space. For an asymptotic class $C$ of $X$, we denote $\omega(C) = \text{Card}(o(C)) - 1$ where $o(C)$ is the set of orbits contained in $C$. For a right infinite word $u \in X^+$, let

$$\ell_C(u) = \text{Card}\{a \in A \mid x^+ = au \text{ for some } x \in C\}.$$

We denote by $LS_\omega(C)$ the set of right infinite words $u$ such that $\ell_C(u) \geq 2$.

The following statement is proved in [10, Proposition 4.3].

**Proposition 3.8** Let $X$ be a shift space and let $C$ be a right asymptotic class. Then

$$\omega(C) = \sum_{u \in LS_\omega(C)} (\ell_C(u) - 1)$$

where both sides are simultaneously finite.

The following result is Mossé’s Theorem (see [13] for references). A morphism is aperiodic if $X(\sigma)$ is aperiodic.

**Theorem 3.9** If $\sigma: A^* \rightarrow A^*$ is a primitive aperiodic morphism, then $\sigma$ is recognizable on $X(\sigma)$.

The following generalization of Mossé’s Theorem was proved in [5] for non-erasing morphisms. A different proof holding in the more general case of morphisms with erasable letters was given in [2].

**Theorem 3.10** Any endomorphism $\sigma: A^* \rightarrow A^*$ is recognizable on $X(\sigma)$ at aperiodic points.

The following example illustrates the case of an erasing morphism.

**Example 3.11** Let $\sigma: a \mapsto ab, b \mapsto ac, c \mapsto \varepsilon$. The shift $X(\sigma)$ is the Fibonacci shift with letters $c$ inserted at the cutting points. The morphism $\sigma$ is recognizable on $X(\sigma)$.
Corollary 3.12 Let \( \sigma : A^* \to A^* \) be a morphism such that \( X(\sigma) \) is minimal and aperiodic. Every left-special sequence \( x \in X(\sigma)^{+} \) is a fixed point of a power of \( \sigma \).

Proof. Since, by Theorem 3.10, \( \sigma \) is recognizable on \( X(\sigma) \) at aperiodic points and since \( X(\sigma) \) is minimal aperiodic, the morphism \( \sigma \) is recognizable on \( X(\sigma) \). The map \( x \mapsto \sigma(x) \) induces a bijection from the set of orbits in \( X(\sigma) \) onto itself (as we have seen before, \( \sigma(x) \) is a two-sided infinite sequence for every \( x \in X(\sigma) \)). This bijection maps every non-trivial asymptotic class onto a non-trivial asymptotic class. Since there is a finite number of these classes by Proposition 3.7, and since each class is formed of a finite number of orbits by Proposition 3.8, some power of \( \sigma \) fixes each of the orbits forming each of these classes.

Example 3.13 Let \( \sigma : a \mapsto ab, b \mapsto a \) be the Fibonacci morphism. The sequence \( \sigma^\omega(a) \) is the unique one-sided sequence having all \( \sigma^n(a) \) as prefixes. It is the unique fixed point of \( \sigma \) and also the unique left-special sequence.

We say that a morphism \( \sigma : A^* \to B^* \) is almost recognizable on a shift \( X \) if \( \sigma \) is recognizable on \( X \) except at a finite number of points of \( B^\mathbb{N} \).

Theorem 3.14 Every morphism \( \sigma : A^* \to A^* \) is almost recognizable on \( X(\sigma) \).

The proof results directly from Theorem 3.10 using the following result, proved in [3].

Theorem 3.15 Let \( \sigma : A^* \to A^* \) be a morphism. The set of periodic points in \( X(\sigma) \) is finite.

4 One-sided recognizability

We now focus on one-sided shifts and the corresponding notion of one-sided recognizability.

Let \( \sigma : A^* \to B^* \) be a morphism. As in the case of two-sided sequences, a \( \sigma \)-representation of a one-sided sequence \( y \in A^\mathbb{N} \) is a pair \((x, k)\) with \( x \in A^\mathbb{N} \) and \( 0 \leq k < |\sigma(x_0)| \) such that \( y = S^k(\sigma(x)) \).

Let \( X \) be a one-sided shift space on \( A \). A morphism \( \sigma : A^* \to B^* \) is one-sided recognizable on \( X \) at \( y \in B^\mathbb{N} \) if \( y \) has at most one \( \sigma \)-representation \((x, k)\) with \( x \in X \). It is one-sided recognizable if it is recognizable on \( X \) at every \( y \in B^\mathbb{N} \).

As for two-sided recognizability, the notion of one-sided recognizability can be formulated using the map \( \hat{\sigma} : (X^\sigma, T) \to Y \) where \( X^\sigma \) is defined by \eqref{eq:3.1} and \( Y \) is the closure of \( \sigma(X) \) under the shift.

We first have the following statement describing the relation between recognizability and one-sided recognizability. A non-erasing morphism \( \sigma : A^* \to B^* \) is called right-marked if the words \( \sigma(a) \) for \( a \in A \) end with different letters.

Proposition 4.1 Let \( \sigma : A^* \to B^* \) be a morphism and let \( X \) be a shift space on \( A \).
1. If $\sigma$ is one-sided recognizable on $X^+$, it is recognizable on $X$.

2. If $\sigma$ is right-marked and is recognizable on $X$, then $\sigma$ is one-sided recognizable on $X^+$.

Proof. 1. Let $y \in B^\mathbb{Z}$ have two $\sigma$-representations $(x, k)$ and $(x', k')$ with $x, x' \in X$. Since $y^+ = S^k(\sigma(x^+)) = S^{k'}(\sigma(x'^+))$, we have $x^+ = x'^+$ and $k = k'$. Since we may apply this argument for every shift $S^{-n}(y)$, we conclude that $x = x'$.

2. Let $y \in B^\mathbb{N}$ have two $\sigma$-representations $(z, k)$ and $(z', k')$ with $z, z' \in X^+$. Let $t, t' \in X$ be such that $t^+ = z$ and $t'^+ = z'$. Consider the system $(X^\sigma, T)$ obtained by the tower construction, with $X^\sigma$ defined by Equation (3.1). Let $Y$ be the closure of $\sigma(X)$ under the shift. Since $\sigma$ is recognizable, the map $\sigma: (x, k) \mapsto S^k \sigma(x)$ is a homeomorphism from $(X^\sigma, T)$ onto $Y$. Since $X^\sigma$ may be considered as a shift space on the alphabet $A^\sigma$, the morphism $\sigma^{-1}$ is a sliding block code defined by a blockmap $f: L_{2N+1}(Y) \to A^\sigma$ of window size $N$. Consequently, we have $T^n(t, k) = T^n(t', k')$ for all $n \geq 2N$. This implies that $z = uz$, $z' = u'z$ and $y = vy'$ with $vy' = S^k ux = S^k u' x$ and $\sigma(x) = y'$. Since $\sigma$ is right marked, this implies that $k = k'$ and $u = u'$.

Example 4.2 The morphism $\sigma: a \mapsto ab, b \mapsto aa$ is called the period-doubling morphism and the shift $X(\sigma)$ the period-doubling shift (see [13]). Since it is primitive aperiodic, it is recognizable on $X(\sigma)$. Since it is right-marked, it is also one-sided recognizable on $X(\sigma)^+$.

As for (two-sided) recognizability, the definition of one-sided recognizability was given in a different form in the articles of Mossé [17, 18].

Let $\sigma: A^* \to B^*$ be a morphism, let $x \in A^\mathbb{N}$ (resp. $x \in A^{\mathbb{Z}}$) be such that $y = \sigma(x)$ is infinite (resp. two-sided infinite). Set

$$C^+(x) = \{||\sigma(x_{[i,n]})|| \mid n \geq 0\}. $$

For $N \geq 1$, let us say that $\sigma$ is one-sided recognizable in the sense of Mossé for $x$ with scope $N$ if for $i, j \geq 0$ (resp. $i, j \in \mathbb{Z}$), whenever $y_{[i,i+N]} = y_{[j,j+N]}$, then $i \in C^+(x) \iff j \in C^+(x)$ (resp. $i \in C(x) \iff j \in C(x)$).

Note that if $\sigma$ is one-sided recognizable in the sense of Mossé for $x \in A^\mathbb{Z}$, then it is one-sided recognizable in the sense of Mossé for $x^+$.

The following statement relates the two notions of recognizability in the sense of Mossé. A set $U$ of words is a suffix code if no element of $U$ is a suffix of another one. In particular, the words in $U$ are nonempty.

Proposition 4.3 Let $\sigma: A^* \to B^*$ be a morphism and let $x \in A^{\mathbb{Z}}$ be such that $\sigma(x)$ is two-sided infinite.

1. If $\sigma$ is one-sided recognizable in the sense of Mossé for $x$, it is recognizable in the sense of Mossé for $x$.

2. If $\sigma(A)$ is a suffix code and if $\sigma$ is recognizable in the sense of Mossé for $x$, then it is one-sided recognizable in the sense of Mossé for $x$.  

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Proof. Set \( y = \sigma(x) \).

1. Assume that \( \sigma \) is one-sided recognizable in the sense of Mossé for \( x \) with scope \( N \). Suppose that \( y_{i-N,i+N} = y_{j-j+N} \). Then \( y_{i,i+N} = y_{j,j+N} \) and thus \( i \in C(x) \Leftrightarrow j \in C(x) \).

2. Assume that \( y_{i,i+M} = y_{j,j+M} \) for some \( M \geq 1 \) and \( i \in C(x) \). If \( M \) is chosen large enough, there is an \( L \geq 1 \) such that \( i + L \in C(x) \) with

\[
i < i + L - N < i + L < i + L + N < i + M.
\]

Since \( y_{i+L-N,i+L+N} = y_{j+L-N,j+L+N} \), we have \( j + L \in C(x) \). But since \( \sigma(A) \) is a suffix code, it implies \( j \in C(x) \) and thus \( \sigma \) is one-sided recognizable in the sense of Mossé.

Note that in condition 2 in Proposition 4.1 the condition that \( \sigma \) is right-marked could not be replaced by the weaker condition 2 in Proposition 4.3 that \( \sigma(A) \) is a suffix code (see Example 4.4).

We now prove the following statement, in part analogous to Proposition 3.3. We say that \( \sigma: A^* \to B^* \) is weakly one-sided recognizable on a one-sided shift \( X \) if for every \( x \in X \), the sequence \( y = \sigma(x) \) has no other \( \sigma \)-representation than \((x,0)\).

**Proposition 4.4** Let \( \sigma: A^* \to B^* \) be a morphism, \( x \in A^N \) and let \( X \) be the one-sided shift generated by \( x \). The following assertions hold.

(i) If \( \sigma \) is one-sided recognizable on \( X \), it is one-sided recognizable in the sense of Mossé for \( x \).

(ii) If \( X \) is minimal, \( \sigma \) is injective on \( A \) and \( \sigma \) is one-sided recognizable in the sense of Mossé for \( x \), then \( \sigma \) is weakly one-sided recognizable on \( X \).

**Proof.** (i) Set \( y = \sigma(x) \). Since \( \sigma \) is one-sided recognizable on \( X \), its restriction to \( X \) is a homeomorphism from \((X^*,T)\) onto the closure \( Y \) under the shift of \( \sigma(X) \).

Let \( \tilde{\sigma} \) be the morphism from \((X^*,T)\) onto \( Y \) defined by \( \tilde{\sigma}(x,i) = S^i\sigma(x) \) where \((X^*,T)\) is defined by \( \{x\} \). Since \( \tilde{\sigma} \) is a homeomorphism, there is an integer \( N \) such that for every \( z \), \( z' \) \( Y \) with \( z = \tilde{\sigma}(t,i) \) and \( z' = \tilde{\sigma}(t',i') \) such that \( z_{0,N} = z'_{0,N} \), we have \( (t_0,i) = (t_0',i') \). In particular, \( z \in \sigma(X) \) if and only if \( z' \in \sigma(X) \). Suppose that \( i \in C^+(x) \) and \( y_{i,i+N} = y_{j,j+N} \). Let \( m \geq 0 \) be such that \( \sigma(x_{0,m}) = y_{0,j-k} \) with \( 0 \leq k < |\sigma(x_m)| \). Since \( S^j(y) \) is in \( \sigma(X) \), we have also \( S^j(y) \in \sigma(X) \) and thus \( S^j(y) \) has a \( \sigma \)-representation \((x',0) \) with \( x' \in X \).

Since \( \sigma \) is one-sided recognizable on \( X \), \( S^j(y) \) cannot have a \( \sigma \)-representation \((S^m(x),k) \) with \( k > 0 \). This implies that \( k = 0 \) and thus \( j \) is in \( C^+(x) \).

(ii) We follow the same steps as in [5]. The proof for the first ones (1 to 4) is the same and we don’t reproduce it.

Claim 1. For every \( z \in B^N \), one has \( z \in \sigma(X) \) if and only if for every sequence \( m_i \) such that \( S^{m_i}(y) \) converges to \( z \), one has \( m_i \in C^+(x) \) for all large enough \( i \).

Claim 2. The set \( \sigma(X) \) is clopen.
Claim 3. For every $x' \in X$ and $m \geq 0$, one has $S^m(x') \in \sigma(X)$ if and only if $m \in C^+(x')$.

Claim 4. $\sigma$ is a homeomorphism from $X$ onto $\sigma(X)$.

Assume now that $y' = \sigma(x')$ with $x' \in X$ and that $y' = S^k \sigma(x'')$ with $x'' \in X$ and $0 \leq k < |\sigma(x'')|$. By Claim 3, we have $k \in C^+(x'')$, which forces $k = 0$. Finally, by Claim 4, we obtain $x' = x''$.

Observe that Assertion (ii) is weaker than the corresponding assertion in Proposition 3.3 since the conclusion is not that $\sigma$ is one-sided recognizable on $X$. This is actually not true, as shown in the following example.

Example 4.5 Let $\sigma: a \mapsto ba, b \mapsto aa$ (note that $X(\sigma)$ is the period-doubling shift of Example 4.2). Since $\sigma$ is primitive and aperiodic, it is recognizable on $X(\sigma)$. Thus, by Proposition 3.3, it is recognizable in the sense of Mossé for every $x \in X$. Since $\sigma(A)$ is a suffix code, this implies, by Proposition 4.3, that $\sigma$ is one-sided recognizable in the sense of Mossé for every $x \in X$. This implies in turn by Proposition 4.4 that $\sigma$ is weakly one-sided recognizable on $X(\sigma)^+$. It is however not one-sided recognizable on $X(\sigma)^+$ because the sequence $y = aa(\sigma(a)^2(a) \cdots$ is such that $y = a\sigma(y)$ and consequently has the two $\sigma$-representations $(ay, 1)$ and $(by, 1)$.

There are primitive morphisms which are not one-sided recognizable on $X^+$ (see Examples 4.5 and 4.8). The following result characterizes morphisms which are not one-sided recognizable. It is closely related with the main result of [1] (see the comment after the proof).

Theorem 4.6 Let $\sigma: A^* \rightarrow B^*$ be a morphism and let $X$ be a shift space on $A$ with at most a finite number of periodic points and such that $\sigma$ is recognizable on $X$ at aperiodic points. Let $Y$ be the closure of $\sigma(X)$ under the shift and let $y \in Y^+$ be an aperiodic point.

The morphism $\sigma$ is not one-sided recognizable on $X^+$ at $y$ if and only if there are words $u, u' \in A^+$ and $v \in B^+$, and a one-sided sequence $x \in X^+$ such that (see Figure 4.1)

(i) $ux, u'x \in X^+$,

(ii) $y = v\sigma(x)$ with $v$ a suffix of $\sigma(u)$ and $\sigma(u')$,

(iii) the last letters of $u, u'$ are distinct.

In particular, $y$ is a shift of a left-special sequence in $Y^+$.

Proof. Let $y \in B^\infty$ be an aperiodic point with two distinct $\sigma$-representations $(z, k)$ and $(z', k')$ with $z, z' \in X^+$. Note that, since $y$ is aperiodic, we cannot have $z = z'$.

Let $t, t' \in X$ be such that $t^+ = z$ and $t'^+ = z'$. Since $X$ contains a finite number of periodic points, their complement is a shift-invariant open set. Thus
there is a clopen set $U \subset X$ containing the orbits of $t, t'$ such that $\sigma$ is recognizable on $U$. Let $V$ be the closure of $\sigma(U)$ under the shift. Then $\hat{\sigma}$ is a homeomorphism from $U^* = \{(u, k) \mid u \in U, 0 \leq k < |\sigma(u_0)|\}$ onto $V$. Let $N$ be the window size of the block map defining the restriction of $\hat{\sigma}$ to $U^*$.

Since $\sigma(t)_{k+i} = \sigma(t')_{k'+i} = y_i$ for every $i \geq 0$, we have $t_{k+j} = t'_{k'+j}$ for all $j \geq N$. Since $z \neq z'$, there is an index $n$ with $0 \leq n < N$ such that $z_{k+n} \neq z'_{k'+n}$. We choose $n$ maximal. Set

$$x = S^{k+n}z = S^{k'+n}z', \quad u = z_{[0,k+n]}, \quad u' = z'_{[0,k'+n]},$$
$$v = S^k\sigma(u) = S^{k'}\sigma(u').$$

It is then easy to verify that conditions (i), (ii) and (iii) are satisfied. Moreover, since $t, t'$ are right asymptotic, $y$ is a shift of a left-special sequence in $Y^+^+$. Conversely, set $\sigma(u) = pv$ and $\sigma(u') = p'v$. We may assume that $p, p'$ are proper prefixes of the image by $\sigma$ of the first letters of $u$ and $u'$ respectively (otherwise we can shorten $u$ or $u'$ by one letter). Let $k = |\sigma(p)|$ and $k' = |\sigma(p')|$. Set $z = ux$ and $z' = u'x$. Then $z, z' \in X^+, 0 \leq k < |\sigma(z_0)|$ and $0 \leq k' < |\sigma(z'_0)|$. Then

$$y = S^k\sigma(z) = S^{k'}\sigma(z')$$

and thus $(z, k)$ and $(z', k')$ are two distinct $\sigma$-representations of $y$. 

Note first that, by Theorem 4.10, the hypotheses of Theorem 4.6 are satisfied when $\sigma: A^* \to A^*$ is an endomorphism and $X = X(\sigma)$.

Note also the connection with the main result of [1]. By [1, Theorem 1.1], a primitive morphism $\sigma: A^* \to A^*$ with a fixed point $x$ is not one-sided recognizable in the sense of Mossé if and only if for every $N \geq 0$ there are $i, j \geq 0$ such that $\sigma(x_{[i+1,i+N]}) = \sigma(x_{[j+1,j+N]})$ with $\sigma(x_i)$ a proper suffix of $\sigma(x_j)$.

Note that the number of shifts of a left-special sequence at which a morphism fails to be one-sided recognizable can be arbitrary large (see Example 4.8).

We say that a morphism $\sigma: A^* \to B^*$ is almost one-sided recognizable on a one-sided shift $X$ if $\sigma$ is one-sided recognizable on $X$ except at a finite number of points of $B^X$.

In the case of an endomorphism, we have the following more precise statement.

**Theorem 4.7** Let $\sigma: A^* \to A^*$ be a morphism such that $X(\sigma)$ is minimal aperiodic. Then $\sigma$ is almost one-sided recognizable on $X(\sigma)^+$. 

![Figure 4.1: The morphism $\sigma$ is not one-sided recognizable at $y$.](image)
**Proof.** Since $\sigma$ is minimal and aperiodic, it is recognizable on $X(\sigma)$ by Theorem 3.10. Then the map $x \in X(\sigma) \mapsto \sigma(x)$ is injective and thus the restriction of $\sigma$ to $X(\sigma)$ is a homeomorphism from $X(\sigma)$ onto $\sigma(X(\sigma))$. This implies that there is an integer $N \geq 1$ such that for every $x$, $x' \in X(\sigma)$, if $\sigma(x)_{[-N,N]} = \sigma(x')_{[-N,N]}$ then $x_0 = x'_0$.

We will show that if $y \in X(\sigma)^+$ has several $\sigma$-representations, then $y = S^i(t)$ with $t \in X(\sigma)^+$ left-special and $i \leq N$. By the remark following Proposition 2.3 this implies our conclusion.

Let $y \in X(\sigma)^+$ have two distinct $\sigma$-representations $(x, k)$ and $(x', k')$. Note that, since $X(\sigma)$ is aperiodic, we cannot have $x = x'$. Then $y = S^k(\sigma(x)) = S^{k'}(\sigma(x'))$ with $x, x' \in X(\sigma)^+$. Let $z, z' \in X(\sigma)$ be such that $z^+ = x$ and $z'^+ = x'$.

If $S^k\sigma(z)_{[-N,N]} = S^{k'}\sigma(z')_{[-N,N]}$, then $S^k\sigma(z)_{[-N+N+i]} = S^{k'}\sigma(z')_{[-N+N+i]}$ for all $i \geq 0$ (because $S^k(\sigma(z))_j = S^{k'}(\sigma(z'))_j = y_j$ for all $j \geq 0$) and thus $x = x'$, a contradiction. This implies that for some $i \leq N$, we have $S^k\sigma(z)_{-i} \neq S^{k'}\sigma(z')_{-i}$. We choose $i$ minimal. Then $t = S^{k-i}\sigma(z)^+ = S^{k-i}\sigma(z')^+$ is left special and such that $S^i(t) = y$. 

**Example 4.8** Let $\sigma : a \mapsto ab, b \mapsto a$ be the Fibonacci morphism. The morphism $\tilde{\sigma} : a \mapsto ba, b \mapsto a$ is not one-sided recognizable. Since $a\tilde{\sigma}(x) = \sigma(x)a$ for every $x \in A^*$, the shift $X(\tilde{\sigma})$ is equal to the Fibonacci shift. Since $X(\sigma)$ is a Sturmian shift, there is a unique left-special sequence, which is $t = \sigma^\omega(a)$. Since $\sigma(t) = t$, we have also $a\tilde{\sigma}(t) = t$.

Accordingly, the morphism $\tilde{\sigma}$ is not one-sided recognizable at $t$, since

$$t = \tilde{\sigma}(bt) = S\tilde{\sigma}(at).$$

The first equality comes from

$$\tilde{\sigma}(bt) = a\tilde{\sigma}(t) = t,$$

and the second one from

$$\tilde{\sigma}(at) = ba\tilde{\sigma}(t) = bt.$$

Consider now the morphism $\tilde{\sigma}^n$. We have

$$\tilde{\sigma}^n(at) = \tilde{\sigma}^{n-1}(b) \tilde{\sigma}^{n-1}(a) \tilde{\sigma}(t) = \tilde{\sigma}^{n-1}(b) \tilde{\sigma}^{n-1}(t)$$

$$\tilde{\sigma}^n(bt) = \tilde{\sigma}^{n-1}(a) \tilde{\sigma}^n(t).$$

Set $F_n = |\tilde{\sigma}^{n-1}(b)|$. Then, for $0 \leq i < F_{n+1}$, the sequence $S^i\tilde{\sigma}^n(bt)$ has the two $\tilde{\sigma}^n$-representations

$$(\tilde{\sigma}^n(bt), i) \text{ and } (\tilde{\sigma}^n(at), F_n + i).$$

This shows that the number of shifts of a left-special sequence at which a morphism fails to be one-sided recognizable can be arbitrary large.
Figure 4.2: The set $\mathcal{L}(X(\sigma))$ and the tree of left-special words.

The following example (due to Fabien Durand [11]) shows that, for a non-minimal morphism, the set of left-special right-infinite sequences may be infinite and that Theorem 4.7 may be false.

**Example 4.9** Let $\sigma: a \mapsto abac, b \mapsto ab, c \mapsto c$. The left-special right-infinite sequences are the $\sigma^\omega(c^n a)$ for $n \neq 1$ (see Figure 4.2).

Indeed, $a$ is left-special since $ba, ca \in \mathcal{L}(X(\sigma))$ and thus $\sigma^\omega(a)$ is left-special. Next $c^n a$ is left-special since $c^{n+1} a, ac^n a \in \mathcal{L}(X(\sigma))$ and thus $\sigma^\omega(c^{n+1} a)$ is left-special.

The morphism $\sigma$ is not one-sided recognizable on $X(\sigma)^+$ at each point $c^n \sigma^\omega(a)$ for $n \geq 1$ since.

$$c^n \sigma^\omega(a) = \sigma(c^n \sigma^\omega(a)) = S^3(\sigma(ac^{n+1} \sigma^\omega(a))).$$

The following picture summarizes the relations between the various notions of recognizability.

## 5 Continuous eigenvalues

Let $(X, T)$ be a topological dynamical system. A complex number $\lambda$ is a **continuous eigenvalue** of $(X, T)$ if there exists a continuous function $f : X \to \mathbb{C}$ with $f \neq 0$ such that $f \circ T = \lambda f$. Such a function is called a **continuous eigenfunction**. The **continuous spectrum** of $(X, T)$ is the set of continuous eigenvalues of $(X, T)$. It is invariant under conjugacy and contains always the value $\lambda = 1$ since a constant function is continuous.

**Proposition 5.1** If $(X, T)$ is recurrent, every continuous eigenvalue is of modulus $1$ and every continuous eigenfunction has constant modulus.
**Proposition 3.3** Let $X$ be a minimal, $\sigma$ injective on $A$ recognizable one-sided Mossé recognizable.

**Proposition 4.1** $\sigma$ injective on $A$.

**Proposition 4.3** $\sigma$ right-marked.

**Proposition 4.4** $\sigma$ (A) suffix.

**Figure 4.3:** The various notions of recognizability.

**Proof.** Let $f$ be a continuous eigenfunction corresponding to $\lambda$. Since $(X,T)$ is recurrent, there is a point $x$ such that the set of $T^n(x)$ for $n \geq 0$ is dense in $X$. Thus for every $\varepsilon > 0$ there is an infinity of $n$ such that $d(T^n(x), x) \leq \varepsilon$. Since $f$ is continuous, this forces $|\lambda| = 1$. Next, since $|f(T(x))| = |\lambda||f(x)| = |f(x)|$ we obtain that $|f(x)|$ is constant since $f$ is continuous.

**Example 5.2** Let $\sigma: a \mapsto ab$, $b \mapsto ba$ be the Thue-Morse morphism. The map $f: X(\sigma) \rightarrow \{-1, 1\}$ defined by $f(x) = (-1)^i$ if $x$ has a $\sigma$-representation $(y, i)$ with $y \in X(\sigma)$ is a continuous map such that $f(S(x)) = -f(x)$. Thus $-1$ is a continuous eigenvalue of $(X(\sigma), S)$.

The following statement also appears as [4 Proposition 2.1].

**Proposition 5.3** For every recurrent shift space $X$, the continuous spectrum of $X$ and $X^+$ are equal.

**Proof.** Suppose first that $f$ is a continuous eigenfunction of $X^+$ for the eigenvalue $\lambda$. Then the map $g: X \rightarrow \mathbb{C}$ defined by $g(x) = f(x^+)$ is a continuous eigenfunction of $X$ for the same eigenvalue.

Conversely, let $g$ be a continuous eigenfunction of $X$ for the eigenvalue $\lambda$. If $x, x' \in X$ are right-asymptotic, then $g(x) = g(x')$. Indeed, since $g$ is continuous, for every $\varepsilon > 0$ there is an $N \geq 1$ such that $y_{[-N,N]} = y'_{[-N,N]}$ implies $|g(y) - g(y')| < \varepsilon$. If $S^n(x)^+ = S^n(x'^+)$, we have $S^m(x)_{[-N,N]} = S^m(x')_{[-N,N]}$ for $m \geq n + N$. This implies that $|g(S^m(x)) - g(S^m(x'))| < \varepsilon$, whence $g(S^m(x)) = g(S^m(x'))$ and thus that $g(x) = g(x^+)$. We can then define $f: X^+ \rightarrow \mathbb{C}$ by $f(y) = g(x)$ for some $x$ such that $x^+ = y$. The map $f$ is clearly an eigenfunction of $X^+$ for the eigenvalue $\lambda$.

**Example 5.4** Consider again the morphism $\sigma: a \mapsto ba$, $b \mapsto aa$ of Example 4.5. As for the Thue-Morse morphism, $-1$ is an eigenvalue of $X(\sigma)$ and a corresponding eigenfunction is $f(x) = (-1)^i$ if $x$ has a $\sigma$-representation $(y, i)$ with
$y \in X(\sigma)$ and $0 \leq i < 2$. Note that, since $\sigma$ is weakly one-sided recognizable, the restriction of $f$ to $X(\sigma)^+$ is also an eigenfunction.

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