Vector Bundles, Linear Representations, and Spectral Problems

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Abstract

This paper is based on my talk at ICM on recent progress in a number of classical problems of linear algebra and representation theory, based on new approach, originated from geometry of stable bundles and geometric invariant theory.

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1. Introduction

Theory of vector bundles brings a new meaning and adds a delicate geometric flavour to classical spectral problems of linear algebra, relating them to geometric invariant theory, representation theory, Schubert calculus, quantum cohomology, and various moduli spaces. The talk may be considered as a supplement to that of Hermann Weyl [35] from which I borrow the following quotation

“In preparing this lecture, the speaker has assumed that he is expected to talk on a subject in which he had some first-hand experience through his own work. And glancing back over the years he found that the one topic to which he has returned again and again is the problem of eigenvalues and eigenfunctions in its various ramifications.”

2. Spectra and representations

Let’s start with two classical and apparently independent problems.

**Hermitian spectral problem.** Find all possible spectra \( \lambda(A + B) \) of sum of Hermitian operators \( A, B \) with given spectra

\[
\lambda(A) : \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A),
\]

\[
\lambda(B) : \lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B).
\]

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Among commonly known restrictions on spectra are *trace identity*

\[ \sum \lambda_i(A + B) = \sum \lambda_j(A) + \sum \lambda_k(B) \]

and a number of classical inequalities, like that of Weyl [34]

\[ \lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B). \quad (2.0) \]

**Tensor product problem.** Find all components \( V_\gamma \subset V_\alpha \otimes V_\beta \) of tensor product of two irreducible representations of \( \text{GL}_n \) with highest weights (=Young diagrams)

\[
\begin{align*}
\alpha &: a_1 \geq a_2 \geq \cdots \geq a_n \\
\beta &: b_1 \geq b_2 \geq \cdots \geq b_n.
\end{align*}
\]

In contrast to the spectral problem (2.1) the coefficients of tensor product decomposition

\[ V_\alpha \otimes V_\beta = \sum_\gamma c^\gamma_{\alpha\beta} V_\gamma \quad (2.1) \]

can be evaluated algorithmically by *Littlewood–Richardson rule*, which may be described as follows. Fill \( i \)-th row of diagram \( \beta \) by symbol \( i \). Then \( c^\gamma_{\alpha\beta} \) is equal to number of ways to produce diagram \( \gamma \) by adding cells from \( \beta \) to \( \alpha \) in such a way that the symbols

i) weakly increase in rows,

ii) strictly increase in columns,

iii) reading all the symbols from right to left, and from top to bottom produces a *lattice permutation*, i.e. in every initial interval symbol \( i \) appears at least as many times as \( i + 1 \).

It turns out that these two problems are essentially equivalent and have the same answer. To give it, let’s associate with a subset \( I \subset \{1, 2, \ldots, n\} \) of cardinality \( p = |I| \) Young diagram \( \sigma_I \) in a rectangular of format \( p \times q \), \( p + q = n \), cut out by polygonal line \( \Gamma_I \), connecting SW and NE corners of the rectangular, with \( i \)-th unit edge running to the North, for \( i \in I \), and to the East otherwise. One can formally multiply the diagrams by L-R rule

\[ \sigma_I \sigma_J = \sum_K c^K_{IJ} \sigma_K \quad (2.2) \]

where \( c^K_{IJ} := c^K_{\sigma_I \sigma_J} \) are L-R coefficients. Geometrically (2.2) is decomposition of product of two Schubert cycles in cohomology ring of Grassmannian \( G^p_q \) of linear subspaces of dimension \( p \) and codimension \( q \).
Theorem 2.1. The following conditions are equivalent

i) There exist Hermitian operators $A$, $B$, $C = A + B$ with spectra $\lambda(A)$, $\lambda(B)$, $\lambda(C)$.

ii) Inequality

$$\lambda_K(C) \leq \lambda_I(A) + \lambda_J(B),$$

holds each time L-R coefficient $c^K_{IJ} \neq 0$. Here $I, J, K \subset \{1, 2, \ldots, n\}$ are subsets of the same cardinality $p < n$, and $\lambda_I(A) = \sum_{i \in I} \lambda_i(A)$.

iii) For integer spectra $\alpha = \lambda(A)$, $\beta = \lambda(B)$, $\gamma = \lambda(C)$ the above conditions are equivalent to

$$V_\gamma \subset V_\alpha \otimes V_\beta.$$ (2.3)

Remarks 2.2. (1) The last claim iii) implies a recurrence procedure to generate all $\alpha, \beta, \gamma$ with $c^\gamma_{\alpha\beta} \neq 0$:

$$c^\gamma_{\alpha\beta} \neq 0 \iff V_\gamma \subset V_\alpha \otimes V_\beta \iff \gamma_K \leq \alpha_I + \beta_J \text{ each time } c^K_{IJ} \neq 0.$$ Here $c^\gamma_{\alpha\beta}$ are Littlewood-Richardson coefficients for group $GL_n$, while $c^K_{IJ}$ are L-R coefficients for group $GL_p$ of smaller rank $p < n$. An explicit form of this recurrence has been conjectured by A. Horn [13] in the framework of Hermitian spectral problem.

(2) Inequalities (IJK) for $c^K_{IJ} \neq 0$ define a cone in the space of triplets of spectra, and the facets of this cone correspond to $c^K_{IJ} = 1$. P. Belkale [3] was first to note that all inequalities (IJK) follow from those with $c^K_{IJ} = 1$, and in recent preprint A. Knutson, T. Tao, and Ch. Woodward [23] proved their independence. In my original paper [19] condition (2.3) appears in a weaker form

$$V_N \gamma \subset V_N \alpha \otimes V_N \beta \quad \text{for some } N > 0,$$ (2.3′)

and its equivalence to (2.3), known as saturation conjecture, was later proved by A. Knutson and T. Tao [22], and in more general quiver context by H. Derksen and J. Weyman [6].

Note that inequalities (IJK), although complete, are too numerous to be practical for large $n$. That is why L-R rule, in its different incarnations [22, 11], often provides a more intuitive way to see possible spectra for sum of Hermitian operators.

Example 2.3. Let $A$ be Hermitian matrix with integer spectrum $\lambda(A) : a_1 \geq a_2 \geq \ldots \geq a_n$ and $B \geq 0$ be a nonnegative matrix of rank one with spectrum $\lambda(B) : b \geq 0 \geq \cdots \geq 0$. Viewing the spectra as Young diagrams, and applying L-R rule we find out that $\lambda(A) \otimes \lambda(B)$ is a sum of diagrams $\gamma : c_1 \geq c_2 \geq \cdots \geq c_n$ satisfying the following intracing inequalities

$$c_1 \geq a_1 \geq c_2 \geq a_2 \geq \cdots \geq c_n \geq a_n.$$
By Theorem 2.3 this implies Cauchy interlacing theorem for spectra
\[ \lambda_i(A) \leq \lambda_i(A + B) \leq \lambda_{i-1}(A), \quad \text{rk}B = 1, \quad B \geq 0, \]
known in mechanics as Rayleigh-Courant-Fisher principle: Let mechanical system \( S' \) is obtained from another one \( S \), by imposing a linear constraint, e.g. by fixing a point of a drum. Then spectrum of \( S \) separates spectrum of \( S' \).

3. Toric bundles

Historically Theorem 2.3 first appears as a byproduct of theory of toric vector bundles and sheaves, originated in [15, 17]. See other expositions of the theory in [21, 30], and further applications in [16, 33]. Vector bundles form a cross point at which the diverse subjects of this paper meet together.

3.1. Filtrations

To avoid technicalities let’s consider the simplest case of projective plane
\[ \mathbb{P}^2 = \{(x^\alpha : x^\beta : x^\gamma)| x \in \mathbb{C}\} \]
on which diagonal torus
\[ T = \{(t_\alpha : t_\beta : t_\gamma)| t \in \mathbb{C}^*\} \]
acts by the formula
\[ t \cdot x = (t_\alpha x^\alpha : t_\beta x^\beta : t_\gamma x^\gamma). \]
Orbits of this action are vertices, sides and complement of the coordinate triangle. In particular there is unique dense orbit, consisting of points with nonzero coordinates.

The objects of our interest are \( T \)-equivariant (or toric for short) vector bundles \( E \) over \( \mathbb{P}^2 \). This means that \( E \) is endowed with an action \( T : E \) which is linear on fibers and makes the following diagram commutative
\[
\begin{array}{cccccc}
E & @> t >> & E \\
\mathbb{P}^2 & @VVV & \mathbb{P}^2 \\
@V\pi VV & t & >> & \mathbb{P}^2 \\
\end{array}
\]
Let us fix a generic point \( p_0 \in \mathbb{P}^2 \) not in a coordinate line, and denote by
\[ E := E(p_0) \]
the corresponding generic fiber. There is no action of torus \( T \) on the fiber \( E \). Instead the equivariant structure produces some distinguished subspaces in \( E \) by the following construction. Let us choose a generic point \( p_0 \in X^\alpha \) in coordinate line \( X^\alpha : x^\alpha = 0 \). Since \( T \)-orbit of \( p_0 \) is dense in \( \mathbb{P}^2 \), we can vary \( t \in T \) so that \( tp_0 \)
tends to \( p_\alpha \). Then for any vector \( e \in E = E(p_0) \), we have \( te \in E(tp_0) \) and can try the limit

\[
\lim_{tp_0 \to p_\alpha} (te)
\]

which either exists or not. Let us denote by \( E^\alpha(0) \) the set of vectors \( e \in E \) for which the limit exists:

\[
E^\alpha(0) := \{ e \in E \mid \lim_{tp_0 \to p_\alpha} (te) \text{ exists} \}.
\]

Evidently \( E^\alpha(0) \) is a vector subspace of \( E \), independent of \( p_0 \) and \( p_\alpha \).

An easy modification of the previous construction allows to define for integer \( m \in \mathbb{Z} \), the subspace

\[
E^\alpha(m) := \left\{ e \in E \mid \lim_{tp_0 \to p_\alpha} \left( \frac{t_\alpha}{t_\beta} \right)^{-m} (te) \text{ exists} \right\}.
\]

Roughly speaking \( E^\alpha(m) \) consists of vectors \( e \in E \) for which \( te \) vanishes up to order \( m \) as \( tp_0 \) tends to coordinate line \( X^\alpha \). The subspaces \( E^\alpha(m) \) form a non-increasing exhaustive \( \mathbb{Z} \)-filtration:

\[
E^\alpha : \cdots \supset E^\alpha(m - 1) \supset E^\alpha(m) \supset E^\alpha(m + 1) \supset \cdots,
\]

\[
E^\alpha(m) = 0, \text{ for } m \gg 0,
\]

\[
E^\alpha(m) = E, \text{ for } m \ll 0.
\]

Applying this construction to other coordinate lines, we get a triple of filtrations \( E^\alpha, E^\beta, E^\gamma \) in generic fiber \( E = E(p_0) \), associated with toric bundle \( \mathcal{E} \).

**Theorem 3.1.** The correspondence

\[
\mathcal{E} \mapsto (E^\alpha, E^\beta, E^\gamma)
\]

establishes an equivalence between category of toric vector bundles on \( \mathbb{P}^2 \) and category of triply filtered vector spaces.

We’ll use notation \( \mathcal{E}(E^\alpha, E^\beta, E^\gamma) \) for toric bundle corresponding to triplet of filtrations \( E^\alpha, E^\beta, E^\gamma \).

### 3.2. Stability

The previous theorem tells that every property or invariant of a vector bundle has its counterpart on the level of filtrations. For application to spectral problems the notion of stability of a vector bundle \( \mathcal{E} \) is crucial. Recall that \( \mathcal{E} \to \mathbb{P}^2 \) is said to be Mumford–Takemoto stable iff

\[
\frac{c_1(\mathcal{F})}{\deg(\mathcal{F})} < \frac{c_1(\mathcal{E})}{\deg(\mathcal{E})}
\]

for every proper subsheaf \( \mathcal{F} \subset \mathcal{E} \), and semistable if weak inequalities hold. Here \( c_1(\mathcal{E}) = \deg(\det \mathcal{E}) \) is the first Chern class. Donaldson theorem [7] brings a deep geometrical meaning to this seemingly artificial definition: Every stable bundle carries unique Hermit-Einstein metric (with Ricci curvature proportional to metric).
Theorem 3.2. Toric bundle $\mathcal{E} = \mathcal{E}(E^\alpha, E^\beta, E^\gamma)$ is stable iff for every proper subspace $F \subset E$ the following inequality holds
\[
\frac{1}{\dim F} \sum_{\nu=\alpha,\beta,\gamma} \sum_{i \in \mathbb{Z}} i \dim F^{[\nu]}(i) < \frac{1}{\dim E} \sum_{\nu=\alpha,\beta,\gamma} \sum_{i \in \mathbb{Z}} i \dim E^{[\nu]}(i) \tag{3.5}
\]
where $F^{\nu}(i) = F \cap E^{\nu}(i)$ is induces filtration with composition factors $F^{[\nu]}(i) = F^{\nu}(i)/F^{\nu}(i+1)$.

There is nothing surprising in this theorem since the sums in (3.5) are just Chern classes of the corresponding toric bundles and sheaves.

Remark 3.3. Inequality (3.5) depends only on relative positions of subspace $F \subset E$ with respect to filtrations $E^\alpha, E^\beta, E^\gamma$, which are given by three Schubert cells $s_\alpha, s_\beta, s_\gamma$. Hence we have one inequality each time.
\[
s_\alpha \cap s_\beta \cap s_\gamma \neq \emptyset. \tag{3.6}
\]
For filtrations in general position (3.6) is equivalent to nonvanishing of the product of Schubert cycles $\sigma_\alpha \cdot \sigma_\beta \cdot \sigma_\gamma \neq 0$ in cohomolgy ring of Grassmannian, and in this case stability inequalities (3.5) amount to inequalities (IJK) of Theorem 2.1.

3.3. Back to Hermitian operators

Let now $E$ be Hermitian space and $H : E \to E$ be Hermitian operator with spectral filtration
\[
E^H(x) = \left(\text{sum of eigenspaces of } H \text{ with eigenvalues at least } x\right). \tag{3.7}
\]
The operator can be recovered from the filtration using spectral decomposition
\[
H = \int_{-\infty}^{\infty} xdP_H(x)
\]
where $P_H(x)$ is orthogonal projector with kernel $E^H(x)$. So in Hermitian space we have equivalence
\[
\text{Hermitian operators } = \mathbb{R}\text{-filtrations.}
\]
Let $H^\alpha$ be Hermitian operator with spectral filtration $E^\alpha$. Its spectrum depends only on filtration $E^\alpha$, and we define $\text{Spec } E^\alpha := \text{Spec } H^\alpha$.

Theorem 3.3. Indecomposable triplet of $\mathbb{R}$-filtrations $E^\alpha, E^\beta, E^\gamma$ is stable iff there exists a Hermitian metric in $E$ such that the sum of the corresponding Hermitian operators is a scalar
\[
H^\alpha + H^\beta + H^\gamma = \text{scalar}. \tag{3.8}
\]

This is a toric version of Donaldson theorem on existence of Hermit–Einstein metric in stable bundles. Together with Theorem 3.2 it reduces solution of Hermitian spectral problem to stability inequalities (3.5), which by remark 3.3 amounts to inequalities (IJK) of Theorem 2.1.

See also Faltings talk [9] on arithmetical applications of stable filtrations.
### 3.4. Components of tensor product

In the previous section we explain that stability inequalities (3.5) \((\Leftrightarrow (IJK))\) via toric Donaldson-Yau theorem solve Hermitian spectral problem. To relate this with tensor product part of Theorem 2.1 we need another interpretation of the stability inequalities via Geometric Invariant Theory [26].

Recall, that point \(x \in \mathbb{P}(V)\) is said to be \textit{GIT stable} with respect to linear action \(G:V\) if \(G\)-orbit of the corresponding vector \(x \in V\) is closed and its stabilizer is finite. Let \(X = F^\alpha \times F^\beta \times F^\gamma\) be product of three flag varieties of the same types as flags of the filtrations \(E^\alpha, E^\beta, E^\gamma\), and \(\mathcal{L}^\alpha\) be line bundle on the flag variety \(F^\alpha\) induced by character \(\omega_\alpha: \text{diag}(x_1, x_2, \ldots, x_n) \mapsto x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\), where \(\alpha: a_1 \geq a_2 \geq \cdots \geq a_n\) is the spectrum of filtration \(E^\alpha\), i.e. spectrum of the corresponding operator \(H^\alpha\).

**Observation 3.4.** Vector bundle \(\mathcal{E} = \mathcal{E}(E^\alpha, E^\beta, E^\gamma)\) is stable iff the corresponding triplet of flags \(x = F^\alpha \times F^\beta \times F^\gamma \in F^\alpha \times F^\beta \times F^\gamma = X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{L}))\) is a GIT stable point w.r. to group \(\text{SL}(E)\) and polarization \(\mathcal{L} = \mathcal{L}^\alpha \boxtimes \mathcal{L}^\beta \boxtimes \mathcal{L}^\gamma\).

This observation is essentially due to Mumford [25]. Notice that by Borel-Weil-Bott theorem [5] the space of global sections \(\Gamma(F^\alpha, \mathcal{L}^\alpha) = V_\alpha\) is just an irreducible representation of \(\text{SL}(E)\) with highest weight \(\alpha\). Hence \(\Gamma(X, \mathcal{L})) = V_\alpha \otimes V_\beta \otimes V_\gamma\). Every stable vector \(x\) can be separated from zero by a \(G\)-invariant section of \(\mathcal{L}^N\).

Therefore triplet of flags in generic position is stable iff \(|V_{N_\alpha} \otimes V_{N_\beta} \otimes V_{N_\gamma}|_{\text{SL}(E)} \neq 0\) for some \(N \geq 1\). This proves the last part of Theorem 2.1, modulo the saturation conjecture.

### 4. Unitary operators and parabolic bundles

We have seen in the previous section that solution of the Hermitian spectral problem amounts to stability condition for toric bundles. A remarkable ramification of this idea was discovered by S. Angihotri and Ch. Woodward [2] for unitary spectral problem.

Let \(U \in \text{SU}(n)\) be unitary matrix with unitary spectrum \(\varepsilon(U) = (e^{2\pi i \lambda_1}, e^{2\pi i \lambda_2}, \ldots, e^{2\pi i \lambda_n})\).

Let’s normalize exponents \(\lambda_i\) as follows

\[
\lambda(U) := \begin{cases} 
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \\
\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0, \\
\lambda_1 - \lambda_n < 1,
\end{cases}
\]

(4.1)

and, admitting an abuse of language, call \(\lambda(U)\) \textit{spectrum} of \(U\).
Unitary spectral problem. Find possible spectra of product $\lambda(UV)$, when spectra of the factors $\lambda(U), \lambda(V)$ are given.

To state the result we need in quantum cohomology $H^*_q(G^r_p)$ of Grassmannian $G^r_p$ of linear subspaces of dimension $p$ and codimension $r$. This is an algebra over polynomial ring $\mathbb{C}[q]$ generated by Schubert cycles $\sigma_I, I \subset \{1, 2, \ldots, n\}, |I| = p, n = p + r$ with multiplication given by the formula

$$\sigma_I \ast \sigma_J = \sum_{K,d} c^K_{IJ}(d) q^d \sigma_K$$

where structure constants $c^K_{IJ}(d)$ are defined as follows. Let $G^r_p \hookrightarrow \mathbb{P}(\Lambda^p \mathbb{C}^n)$ be Plücker imbedding and

$$\varphi : \mathbb{P}^1 \rightarrow G^r_p$$

be a rational curve of degree $d$ in Grassmanian $G^r_p \subset \mathbb{P}(\Lambda^p \mathbb{C}^n)$. One can check that $\varphi$ depends on $\text{dim} G^r_p + nd$ parameters. For fixed point $x \in \mathbb{P}^1$ the condition $\varphi(x) \in \sigma_I$ imposes codim $\sigma_I$ constraints on $\varphi$. Hence for

$$\text{codim} \sigma_I + \text{codim} \sigma_J + \text{codim} \sigma_K = \text{dim} G^r_p + nd$$

the numbers

$$(\sigma_I, \sigma_J, \sigma_K)_d = \#\{\varphi : \mathbb{P}^1 \rightarrow G^r_p \mid \varphi(x) \in \sigma_I, \varphi(x) \in \sigma_J, \varphi(x) \in \sigma_K, \deg \varphi = d\}$$

supposed to be finite. They are known as Gromov-Witten invariants and related to the structure constants by the formula

$$c^K_{IJ}(d) = (\sigma_I, \sigma_J, \sigma_K)_d$$

where $K^* = \{n + 1 - k \mid k \in K\}$. For $d = 0$ they are just conventional Littlewood-Richardson coefficients $c^K_{IJ}$.

Theorem 4.1. The following conditions are equivalent

i) There exist unitary matrices $W = UV$ with given spectra $\lambda(U), \lambda(V), \lambda(W)$.

ii) The inequality

$$\lambda_I(U) + \lambda_J(V) \leq d + \lambda_K(W)$$

holds each time $c^K_{IJ}(d) \neq 0$.

4.1. Parabolic bundles

As in the Hermitian case solution of the unitary problem comes from its holomorphic interpretation in terms of vector bundles. To explain the idea let’s start with vector bundle $\mathcal{E}$ over compact Riemann surface $\overline{X}$ of genus $g \geq 2$. It has unique topological invariant $c_1(\mathcal{E}) = \deg \det \mathcal{E}$, which for simplicity we suppose to be zero,
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i.e. $\mathcal{E}$ be topologically trivial. Narasimhan-Seshadri theorem [27] claims that every stable bundle carries unique flat metric, and hence defines unitary monodromy representation

$$\rho_\mathcal{E} : \pi_1(\overline{X}, x_0) \to \text{SU}(E), \quad E = \mathcal{E}(x_0).$$

This gives rise to equivalence

$$\mathcal{M}_g := \left\{ \text{stable bundles of degree zero} \right\} = \left\{ \text{irreducible unitary representations } \rho : \pi_1 \to \text{SU}(E) \right\}. \quad (4.2)$$

This theorem is an ancestor of the Donaldson-Yau generalization [7] to higher dimensions, and may be seen as a geometric version of Langlands correspondence.

In algebraic terms the theorem describes stable bundles in terms of solutions of equation

$$[U_1, V_1][U_2, V_2] \cdots [U_g, V_g] = 1$$

in unitary matrices $U_i, V_j \in \text{SU}(E)$. This is not the matrix problem we are currently interested in. To modify it let’s consider punctured Riemann surface $X = \overline{X} \setminus \{p_1, p_2, \ldots, p_\ell\}$. It has distinguished classes

$$\gamma_\alpha = \text{(small circle around } p_\alpha\text{)}$$

in fundamental group $\pi_1(X)$, and we can readily define an analogue of RHS of (4.2):

$$\mathcal{M}_g(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell)}) = \left\{ \rho : \pi_1(X) \to \text{SU}(E) \mid \lambda(\rho(\gamma_\alpha)) = \lambda^{(\alpha)} \right\}, \quad (4.3)$$

where $\lambda^{(\alpha)}$ is a given spectrum of monodromy around puncture $p_\alpha$. C. S. Seshadri [31] manages to find an analogue of more subtle holomorphic LHS of (4.2) in terms of so called parabolic bundles.

Parabolic bundle $\mathcal{E}$ on $X$ is actually a bundle on compactification $\overline{X}$ together with $\mathbb{R}$-filtration in every special fiber $E^\alpha = \mathcal{E}(p_\alpha)$ with support in an interval of length $\leq 1$. The filtration is a substitution for spectral decomposition of $\rho(\gamma_\alpha)$, cf. (4.1). Seshadri also defines (semi)stability of parabolic bundle $\mathcal{E}$ by inequalities

$$\frac{\text{Par deg } \mathcal{F}}{\text{rk } \mathcal{F}} \leq \frac{\text{Par deg } \mathcal{E}}{\text{rk } \mathcal{E}}, \quad \forall \mathcal{F} \subset \mathcal{E}, \quad (4.4)$$

where the parabolic degree is given by equation $\text{Par deg } \mathcal{E} = \deg \mathcal{E} + \sum_{\alpha} \lambda^{(\alpha)}_i$. Metha-Seshadri theorem [24] claims that every stable parabolic bundle $\mathcal{E}$ on $X$ carries unique flat metric with given spectra of monodromies $\lambda(\gamma_\alpha) = \lambda^{(\alpha)}$. This gives a holomorphic interpretation of the space (4.3)

$$\mathcal{M}_g(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell)}) = \left\{ \text{stable parabolic bundles of degree zero} \right\}, \quad (4.5)$$

In the simplest case of projective line with three punctures (4.3) amounts to space of solutions of equation $UVW = 1$ in unitary matrices $U, V, W \in \text{SU}(n)$ with given
spectra. By Metha-Seshadry theorem solvability of this equation is equivalent to stability inequalities (4.4). In the case under consideration holomorphic vector bundle $E$ on $\mathbb{P}^1$ is trivial, $E = E \times \mathbb{P}^1$, and hence its subbundle $F \subset E$ of rank $p$ is nothing but a rational curve $\varphi : \mathbb{P}^1 \to G_p(E)$ in Grassmannian. This allows to write down stability condition (4.4) in terms of quantum cohomology, and eventually arrive at Theorem 4.1.

5. Further ramifications

The progress in Hermitian and unitary spectral problems open way for solution of a variety of others classical, and not so classical, problems. Most of them, however, have no holomorphic interpretation, and require different methods, borrowed from harmonic analysis on homogeneous spaces, symplectic geometry, and geometric invariant theory.

5.1. Multiplicative singular value problem

The problem in question is about possible singular spectrum $\sigma(AB)$ of product of complex matrices with given singular spectra $\sigma(A)$ and $\sigma(B)$. Recall, that singular spectrum of complex matrix $A$ is spectrum of its radial part $\sigma(A) := \lambda(\sqrt{A^*A})$.

For a long time it was observed that every inequality for Hermitian problem has a multiplicative counterpart for the singular one. For example multiplicative version of Weyl’s inequality $\lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B)$ is $\sigma_{i+j-1}(AB) \leq \sigma_i(A)\sigma_j(B)$. The equivalence between these two problems was conjectured by R. C. Thompson, and first proved by the author [20] using harmonic analysis on symmetric spaces. Later on A. Alekseev, E. Menreken, and Ch. Woodward [1] gave an elegant conceptual solution based on Drinfeld’s Poisson-Lie groups [8]. Here is a precise statement for classical groups.

**Theorem 5.1.** Let $G$ be one of the classical groups $\text{SL}(n, \mathbb{C})$, $\text{SO}(n, \mathbb{C})$, or $\text{Sp}(2n, \mathbb{C})$ and $L$ be the corresponding compact Lie algebra of traceless skew Hermitian complex, real, or quaternionic $n \times n$ matrices respectively. Then the following conditions are equivalent

1. There exist $A_i \in G$ with given singular spectra $\sigma(A_i) = \sigma_i$ and $A_1A_2\cdots A_N = 1$.

2. There exist $H_i \in L$ with spectra $\lambda(H_i) = \sqrt{-1}\log \sigma_i$ and $H_1 + H_2 + \cdots + H_N = 0$.

Note, however, that neither of the above approaches solve the singular problem per se, but reduces it to Hermitian one. Both of them suggest that all three problems
must be treated in one package. More precisely, every compact simply connected group $G$ give birth to three symmetric spaces

- The group $G$ itself,
- Its Lie algebra $L_G$,
- The dual symmetric space $H_G = G^c/G$,

of positive, zero, and negative curvature, and to three “spectral problems” concerned with support of convolution of $G$ orbits in these spaces, see [20] for details. For $G = \text{SU}(n)$ we return to the package of unitary, Hermitian, and singular problems.

The first two problems may be effectively treated in framework of vector bundles with structure group $G$, as explained in sections 2–4. Many flat, i.e. additive “spectral problem” has been solved by A. Berenstein and R. Sjamaar in a very general setting [4].

5.2. Other symmetric spaces

As an example of unresolved problem let’s consider symmetric spaces associated with different incarnations of Grassmannian

- Compact $U(p+q)/U(p) \times U(q)$,
- Flat $\text{Mat}(p,q) = \text{complex } p \times q$ matrices,
- Hyperbolic $U(p,q)/U(p) \times U(q)$.

In compact case the corresponding spectral problem is about possible angles between three $p$-subspaces $U, V, W \subset \mathbb{H}^n$ in Hermitian space $\mathbb{H}^n$ of dimension $n = p + q$, $p \leq q$. The Jordan angles

$$\hat{U}V = (\varphi_1, \varphi_2, \ldots, \varphi_p), \quad 0 \leq \varphi \leq \frac{\pi}{2}$$

between subspaces $U, V$ are defined via spectrum of product of orthogonal projectors $\pi_{UV} : U \to V$ and $\pi_{VU} : V \to U$

$$\lambda(\sqrt{\pi_{UV}} \pi_{VU}) : \cos \varphi_1 \geq \cos \varphi_2 \geq \cdots \geq \cos \varphi_p \geq 0.$$

Yu. Neretin [28] proved Lidskii type inequalities\(^1\) for angles $\hat{U}V, \hat{V}W, \hat{W}U$, and conjectured that other inequalities are the same as in the Hermitian case. Note, however, that the unitary triplet suggests existence of nonhomogeneous “quantum” inequalities, e.g. sum of angles of a spherical triangle is $\leq \pi$.

In flat case the problem is about relation between singular spectra of $p \times q$ matrices $\sigma(A-B)$, $\sigma(B-C)$, $\sigma(C-A)$. This additive singular problem was resolved by O’Shea and Sjamaar [29].

In hyperbolic case the question is about angles between maximal positive subspaces $U, V, W \subset \mathbb{H}^{pq}$ in Hermitian space of signature $(p,q)$. They are defined by equation

$$\lambda(\sqrt{\pi_{UV}} \pi_{VU}) : \cosh \varphi_1 \geq \cosh \varphi_2 \geq \cdots \geq \cosh \varphi_p \geq 1.$$

\(^1\)He actually deals with real Grassmannian.
Again our experience with the unitary triplet suggests that the exponential map establishes a Thompson’s type correspondence between O’Shea-Sjamaar inequalities for additive singular problem and that of for hyperbolic angles.

5.3. **P-adic spectral problems**

There is also a nonarchimedian counterpart of this theory, which deals with classical Chevalley groups $G_p = SL(n, \mathbb{Q}_p)$, $SO(n, \mathbb{Q}_p)$, or $Sp(2n, \mathbb{Q}_p)$ over p-adic field $\mathbb{Q}_p$ and their maximal compact subgroups $K_p = SL(n, \mathbb{Z}_p)$, $SO(n, \mathbb{Z}_p)$, or $Sp(2n, \mathbb{Z}_p)$ respectively. Double coset $K_p g K_p$ may be treated as a complete invariant of lattice $L = g L_0$, $L_0 = \mathbb{Z}_p^{\oplus n}$ with respect to $K_p$. We call lattice $L = g L_0$ **unimodular, orthogonal or symplectic** if respectively $g \in SL(n, \mathbb{Q}_p)$, $g \in SO(n, \mathbb{Q}_p)$ or $g \in Sp(2n, \mathbb{Q}_p)$.

It is commonly known that in the unimodular case there exists a basis $e_i$ of $L_0$ such that $\tilde{e}_i = p^{a_i} e_i$ form a basis of $L$ for some $a_i \in \mathbb{Z}$. We define index $(L : L_0)$ by

$$ (L : L_0) = (p^{a_1}, p^{a_2}, \ldots, p^{a_n}), \quad a_1 \geq a_2 \geq \cdots \geq a_n. \quad (5.1) $$

Notice that unimodularity $g \in SL(n, \mathbb{Q}_p)$ implies $a_1 + a_2 + \cdots + a_n = 0$.

The index $(L : L_0)$ of an orthogonal or a symplectic lattices has extra symmetries. In orthogonal case we may choose the above basis $e_i$ of $L_0$ to be **neutral**, in which case the quadratic form becomes

$$ \sum_{1 \leq i \leq n} x_i x_{-i}, \quad i \equiv n - 1 \mod 2. $$

Then the index takes the form

$$ (L : L_0) = (p^{a_{n-1}}, p^{a_{n-3}}, \ldots, p^{a_{3-n}}, p^{a_1-n}), \quad (5.2) $$

where $a_{n-1} \geq a_{n-3} \geq \ldots \geq a_{3-n} \geq a_{1-n}$, and $a_{-i} = -a_i$.

Similarly, for **symplectic** lattice $L$ we can choose symplectic basis $e_i, f_j$ of $L_0$ such that $\tilde{e}_i = p^{a_i} e_i$ and $\tilde{f}_j = p^{-a_j} f_j$ form a basis of $L$. In this case we have

$$ (L : L_0) = (p^{a_n}, p^{a_{n-1}}, \ldots, p^{a_1}, p^{-a_1}, \ldots, p^{-a_n}, p^{-a_n}), \quad (5.3) $$

with $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq 0$.

Notice that the spectra (5.1)-(5.3) have the same symmetry, as singular spectrum $\sigma(A)$ of a matrix $A \in G$ in the corresponding classical complex group.

**Theorem 5.2.** The following conditions are equivalent

1. There exists a sequence of (unimodular, orthogonal, symplectic) lattices $L_0, L_1, \ldots, L_{N-1}, L_N = L_0$

   of given indices $\sigma_i = (L_i : L_{i-1})$.

2. The indices $\sigma_i$ satisfy the equivalent conditions of Theorem 5.1 for the corresponding complex group $G$.

We’ll give proof elsewhere. The theorem is known for the unimodular lattices, see [10].
5.4. Final remarks

In the talk I try to trace the flaw of ideas from the theory of vector bundles to spectral problems. It seems C. Simpson [32] was the first to note that vector bundles technic has nontrivial implications in linear algebra. He proved that product $C_1C_2 \cdots C_N$ of conjugacy classes $C_i \subset \text{SL}(n, \mathbb{C})$ is dense in $\text{SL}(n, \mathbb{C})$ iff

$$\dim C_1 + \dim C_2 + \cdots + \dim C_N \geq (n + 1)(n - 2),$$

$$r_1 + r_2 + \cdots + r_N \geq n,$$

(5.4)

where $r_i$ is maximal codimension of root space of a matrix $A_i \in C_i$. This problem was suggested by P. Deligne, who noted that under condition

$$\dim C_1 + \dim C_2 + \cdots + \dim C_N = 2n^2 - 2$$

an irreducible solution of equation $A_1A_2 \cdots A_N = 1$, $a_i \in C_i$ is unique up to conjugacy, see book of N. Katz [14] on this rigidity phenomenon.

I think that inverse applications to moduli spaces of vector bundles are still ahead. One may consider polygon spaces [18, 12] as a toy example of this feedback, corresponding to toric 2-bundles. A similar space of spherical polygons in $\mathbb{S}^3$ with given sides is a model for moduli space of flat connections in punctured Riemann sphere. Its description is a challenge problem.

There are many interesting results, e.g. infinite dimensional spectral problems, which fall out of this survey. I refer to Fulton’s paper [10] for missing details.

References

[1] A. Alekseev, E. Meinrenken, & C. Woodward, Linearization of Poisson actions and singular values of matrix product, *Ann. Inst. Fourier (Grenoble)*, 51 (2001), no. 6, 1691–1717.

[2] S. Angihotri & C. Woodward, Eigenvalues of products of unitary matrices and quantum Schubert calculus, *Math. Res. Letters*, 5 (1998), 817–836.

[3] P. Belkale, Local systems on $\mathbb{P}^1 - S$ for $S$ a finite set, *Compositio Math.*, 129 (2001), no. 1, 67–86.

[4] A. Berenstein & R. Sjamaar, Coadjoint orbits, moment polytopes, and the Hilbert–Mumford criterion, *J. Amer. Math. Soc.*, 13 (2000), no. 2, 433–466.

[5] R. Bott, Homogeneous vector bundles, *Ann. of Math.*, 66 (1957), 203–248.

[6] H. Derksen & J. Weyman, Semi-invariants of quivers and saturation for Littlewood-Richardson theorem, *J. Amer. Math. Soc.*, 13 (2000), no. 3, 467–479.

[7] S. K. Donaldson, Infinite determinants, stable bundles and curvature, *Duke Math. J.*, 54 (1987), 231–247.

[8] V. G. Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians*, vol. 1,2 (Berkeley, 1986), Amer. Math. Soc., Providence, RI, 1987, 798–820.
[9] G. Faltings, Mumford-Stabilität in der algebraischen Geometrie, Proceedings of the International Congress of Mathematiciens, vol. 1,2, (Zürich, 1994), Birkhauser, Basel, 1995, 648–655.

[10] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bull. Amer. Math. Soc., 37 (2000), no. 3, 209–249.

[11] O. Gleizer & A. Postnikov, Littlewood-Richardson coefficients via Yang-Baxter equation, Internat. Math. Res. Notices (2000), no. 14, 741–774.

[12] J.-C. Hausmann & A. Knutson, The cohomology ring of polygon spaces, Ann. Inst. Fourier (Grenoble), 48 (1998), no. 1, 281–321.

[13] A. Horn, Eigenvalues of sum of Hermitian matrices, Pacific J. Math., 12 (1962), 225–241.

[14] N. M. Katz, Rigid local systems, Princeton University Press, Princeton, 1996.

[15] A. A. Klyachko, Equivariant bundles on toric varieties, Izv. Akad. Nauk SSSR Ser. Mat., 53 (1989), no. 5, 1001–1039 (Russian); Math. USSR-Izv., 35 (1990), no. 2, 63–64.

[16] A. A. Klyachko, Moduli of vector bundles and class numbers, Functional. i Priozheh., 25 (1991), 81–83 (Russian); Funct. Anal. Appl., 25 (1991), no. 1, 67–69.

[17] A. A. Klyachko, Vector bundles and torsion free sheaves on the projective plane, Preprint Max-Planck-Institute fur Mathematik MPI/91-59, (1991).

[18] A. A. Klyachko, Spatial polygons and stable configurations of points in the projective line, Algebraic geometry and its applications (Yaroslavl, 1992), Vieweg, Braunschweig, 1994, 67–84.

[19] A. A. Klyachko, Stable bundles, repesentation theory and Hermitian operators, Selecta Mathematica, 4 (1998), 419–445.

[20] A. A. Klyachko, Random walks on symmetric spaces and and inequalities for matrix spectra, Linear Algebra Appl., 319 (2000), no. 2-3, 37–59.

[21] A. Knutson & E. Sharp, Sheaves on toric varieties for physics, Adv. Theor. Math. Phys., 2 (1998), no. 4, 873–961.

[22] A. Knutson & T. Tao, The honeycomb model of $GL(n,\mathbb{C})$ tensor products. I. Proof of the saturation conjecture, J. Amer. Math. Soc., 12 (1999), no. 2, 1055–1090.

[23] A. Knutson, T. Tao & Ch. Woodward, The honecomb model for $GL(n,\mathbb{C})$ tensor products II: Facets of Littlewood–Richardson cone, Preprint (2001).

[24] V. B. Metha & C. S. Seshadri, Moduli of vector bundles on curves with parabolic structure, Math. Ann., 258 (1980), 205-239.

[25] D. Mumford, Projective invariants of projective structures, Proc. Int. Congress of Math. Stockholm, 1963, Almquist & Wiksells, Uppsala, 1963, 526–530.

[26] D. Mumford, J. Fogarty, & F. Kirwan, Geometric invariant theory, Springer, Berlin, 1994.

[27] M. S. Narasimhan & C. S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, Ann. Math., 82 (1965), 540–567.
[28] Yu. Neretin, On Jordan angles and triangle inequality in Grassmannian, 
*Geom. Dedicata*, 86 (2001), no. 1-3, 81–92.

[29] L. O'Shea & R. Sjamaar, Moments maps and Riemannian symmetric pairs, 
*Math. Ann.*, 317 (2000), no. 3, 415–457.

[30] M. Perling, Graded rings and equivariant sheaves on toric varieties, Preprint 
Univ. Kaiserslauten, (2001).

[31] C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, 
*Bull. Amer. Math. Soc.*, (1977), 124–126.

[32] C. T. Sympson, Product of Matrices, *Differential geometry, global analysis, 
and topology*, Canadian Math. Soc. Conf. Proc., vol. 12, AMS, Providence 
RI, 1992, 157–185.

[33] C. Vafa & E. Witten, A strong coupling test of S-duality, *Nuclear Phys. B*, 
431 (1994), no. 1-2, 3–77.

[34] H. Weyl, Das asymptotischer Verteilungsgesetz der Eigenwerte lineare parti-
tialer Differentialgleichungen, *Math. Ann.*, 71 (1912), 441–479.

[35] H. Weyl, Ramifications, old and new, of the eigenvalue problem, *Bull. Amer. 
Math. Soc.*, 56 (1950), 115–139.