THE EXPECTED NUMBER OF ELEMENTS TO GENERATE A FINITE GROUP WITH $d$-GENERATED SYLOW SUBGROUPS

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Abstract. Given a finite group $G$, let $e(G)$ be expected number of elements of $G$ which have to be drawn at random, with replacement, before a set of generators is found. If all the Sylow subgroups of $G$ can be generated by $d$ elements, then $e(G) \leq d + \kappa$ with $\kappa \sim 2.75239495$. The number $\kappa$ is explicitly described in terms of the Riemann zeta function and is best possible. If $G$ is a permutation group of degree $n$, then either $G = \text{Sym}(3)$ and $e(G) = 2$ or $e(G) \leq \lfloor n/2 \rfloor + \kappa^*$ with $\kappa^* \sim 1.606695$. These results improved the weaker ones obtained in [13].

1. Introduction

In 1989, R. Guralnick [5] and the first author [11] independently proved that if all the Sylow subgroups of a finite group $G$ can be generated by $d$ elements, then the group $G$ itself can be generated by $d+1$ elements. A probabilistic version of this result was obtained in [13]. Let $G$ be a nontrivial finite group and let $x = (x_n)_{n \in \mathbb{N}}$ be a sequence of independent, uniformly distributed $G$-valued random variables. We may define a random variable $\tau_G$ by $\tau_G = \min\{n \geq 1 \mid \langle x_1, \ldots, x_n \rangle = G\}$. We denote by $e(G)$ the expectation $E(\tau_G)$ of this random variable: $e(G)$ is the expected number of elements of $G$ which have to be drawn at random, with replacement, before a set of generators is found. In [13] it was proved that if all the Sylow subgroups of $G$ can be generated by $d$ elements, then $e(G) \leq d + \eta$ with $\eta \sim 2.875065$. This bound is not too far from being best possible. Indeed in [13], Pomerance proved that if $\Omega_d$ is the set of all the $d$-generated finite abelian groups, then

$$\sup_{G \in \Omega_d} e(G) = d + \sigma, \text{ where } \sigma \sim 2.11846.$$ 

However the bound $e(G) \leq d + \eta$ is approximative, and one could be interest in finding a best possible estimation for $e(G)$. We give an exhaustive answer to this question, proving the following result.

Theorem 1. Let $G$ be a finite group. If all the Sylow subgroups of $G$ can be generated by $d$ elements, then $e(G) \leq d + \kappa$ with $\kappa \sim 2.75239495$. The number $\kappa$ is explicitly described in terms of the Riemann zeta function and is best possible.

This bound can be further improved under some additional assumptions on $G$. For example we prove that if all the Sylow subgroups of $G$ can be generated by $d$ elements and $G$ is not soluble, then $e(G) \leq d + 2.7501$ (Proposition [10]). A stronger result holds if $|G|$ is odd.

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Theorem 2. Let $G$ be a finite group of odd order. If all the Sylow subgroups of $G$ can be generated by $d$ elements, then $e(G) \leq d + \kappa$ with $\kappa \sim 2.148668$.

If $G$ is a $p$-subgroup of $\text{Sym}(n)$, then $G$ can be generated by $\lfloor n/p \rfloor$ elements (see [7]), so Theorem 11 has the following consequence: if $G$ is a permutation group of degree $n$, then $e(G) \leq \lfloor n/2 \rfloor + \kappa$. However this bound is not best possible and a better result can be obtained:

Corollary 3. If $G$ is a permutation group of degree $n$, then either $G = \text{Sym}(3)$ and $e(G) = 2.9$ or $e(G) \leq \lfloor n/2 \rfloor + \kappa^*$ with $\kappa^* \sim 1.606695$.

The number $\kappa^*$ is best possible. Let $m = \lfloor n/2 \rfloor$ and set $G_n = \text{Sym}(2)^m$ if $m$ is even, $G_n = \text{Sym}(2)^{m-1} \times \text{Sym}(3)$ if $m$ is odd. If $n \geq 8$, then $e(G_n) - m$ increase with $n$ and $\lim_{n \to \infty} e(G) - m = 1.606695$.

2. Preliminary results

Let $G$ be a finite group and use the following notations:

- For a given prime $p$, $d_p(G)$ is the smallest cardinality of a generating set of a Sylow $p$-subgroup of $G$.
- For a given prime $p$ and a positive integer $t$, $\alpha_{p,t}(G)$ is the number of complemented factors of order $p^t$ in a chief series of $G$.
- For a given prime $p$, $\alpha_p(G) = \sum_t \alpha_{p,t}(G)$ is the number of complemented factors of $p$-power order in a chief series of $G$.
- $\beta(G)$ is the number of nonabelian factors in a chief series of $G$.

Lemma 4. For every finite group $G$, we have:

1. $\alpha_p(G) \leq d_p(G)$.
2. $\alpha_2(G) + \beta(G) \leq d_2(G)$.
3. If $\beta(G) \neq 0$, then $\beta(G) \leq d_2(G) - 1$.
4. If $\alpha_{2,1}(G) = 0$, then $\alpha_2(G) + \beta(G) \leq d_2(G) - 1$.
5. If $\alpha_{p,1}(G) = 0$, then $\alpha_p(G) \leq d_p(G) - 1$.

Proof. (1), (2) and (3) are proved in [13] Lemma 4. Now assume that no complemented chief factor of $G$ has order 2 and let $r = \alpha_2(G) + \beta(G)$. There exists a sequence $X_r \leq Y_r \leq \cdots \leq X_1 \leq Y_1$ of normal subgroups of $G$ such that, for every $1 \leq i \leq r$, $Y_i/X_i$ is a complemented chief factor of $G$ of even order. Notice that $\beta(G/Y_1) = \alpha_2(G/Y_1) = 0$, hence $G/Y_1$ is a finite solvable group all of whose complemented chief factors have odd order, but then $G/Y_1$ has odd order and consequently $d_2(G) = d_2(Y_1)$. Moreover, as in the proof of [13] Lemma 4, $d_2(Y_1) \geq d_2(Y_1/X_1) + r - 1$. Since $|Y_1/X_1| \neq 2$ and the Sylow 2-subgroups of a finite nonabelian simple cannot be cyclic [16] 10.1.9, we deduce $d_2(Y_1/X_1) \geq 2$ and consequently $d_2(G) = d_2(Y_1) \geq r + 1$. This proves (4). The proof of (5) is similar. \qed

Recall (see [13] (1.1) for more details) that

\begin{equation}
(2.1) \quad e(G) = \sum_{n \geq 0} (1 - P_G(n))
\end{equation}

where

$$P_G(n) = \frac{|\{(g_1, \ldots, g_n) \in G^n \mid \langle g_1, \ldots, g_n \rangle = G\}|}{|G|^n}$$
is the probability that \( n \) randomly chosen elements of \( G \) generate \( G \). Denote by \( m_n(G) \) the number of index \( n \) maximal subgroups of \( G \). We have (see [10, 11.6]):

\[
1 - P_G(k) \leq \sum_{n \geq 2} \frac{m_n(G)}{n^k}.
\]

Using the notations introduced in [8, Section 2], we say that a maximal subgroup \( M \) of \( G \) is of type A if \( \text{soc}(G/\text{Core}_G(M)) \) is abelian, of type B otherwise, and we denote by \( m^A_n(G) \) (respectively \( m^B_n(G) \)) the number of maximal subgroups of \( G \) of type A (respectively B) of index \( n \). Given \( t \in \mathbb{N} \) and \( p \in \pi(G) \), define

\[
\mu^*(G,t) = \sum_{k \geq t} \left( \sum_{n \geq 5} \frac{m^B_n(G)}{n^k} \right), \quad \mu_p(G,t) = \sum_{k \geq t} \left( \sum_{n \geq 1} \frac{m^A_n(G)}{p^{nk}} \right).
\]

**Lemma 5.** Let \( t \in \mathbb{N} \). Then \( e(G) \leq t + \mu^*(G,t) + \sum_{p \in \pi(G)} \mu_p(G,t). \)

**Proof.** By (2.1) and (2.2),

\[
e(G) \leq t + \sum_{n \geq t} (1 - P_G(n)) \leq t + \sum_{k \geq t} \left( \sum_{n \geq 2} \frac{m_n(G)}{n^k} \right). \quad \Box
\]

**Lemma 6.** Let \( t \in \mathbb{N} \). If \( \beta(G) = 0 \), then \( \mu^*(G,t) = 0 \). If \( t \geq \beta(G) + 3 \), then

\[
\mu^*(G,t) \leq \frac{\beta(G)(\beta(G) + 1)}{2 \cdot 5^{t-4}} \cdot \frac{1}{4}.
\]

**Proof.** It follows from [13, Lemma 8] and its proof. \( \square \)

**Lemma 7.** For \( t \in \mathbb{N} \) and \( p \in \pi(G) \). If \( \alpha_p(G) = 0 \), then \( \mu_p(G,t) = 0 \).

1. If \( \alpha_2(G) \leq t - 1 \) and \( \alpha_{2,u}(G) \leq t - 2 \) for every \( u > 1 \), then

\[
\mu_2(G,t) \leq \frac{1}{2^{t-\alpha_2(G)-1}}.
\]

2. Let \( p \) be an odd prime. If \( \alpha_p(G) \leq t - 2 \) then

\[
\mu_p(G,t) \leq \frac{1}{p^{t-\alpha_p(G)-2}} \cdot \frac{1}{(p-1)^2}.
\]

**Proof.** It follows from [13, Lemma 7] and its proof. \( \square \)

Let \( G \) be a finite soluble group and let \( A \) be a set of representatives for the irreducible \( G \)-module that are \( G \)-isomorphic to some complemented chief factor of \( G \). For every \( A \in A \), let \( \delta_A \) be the number of complemented factors \( G \)-isomorphic to \( A \) in a chief series of \( G \). \( q_A = |\text{End}_G(A)| \), \( r_A = \dim_{\text{End}_G(A)}(A) \), \( \zeta_A = 0 \) if \( A \) is a trivial \( G \)-module, \( \zeta_A = 1 \) otherwise. Moreover, for every \( l \in \mathbb{N} \), let \( Q_{A,l}(s) \) be the Dirichlet polynomial defined by

\[
Q_{A,l}(s) = 1 - \frac{\text{dim}_{\text{End}_G(A)}(A)^l}{q_A^{sA}}.
\]

By [4, Satz 1], for every positive integer \( k \) we have

\[
P_G(k) = \prod_{A \in A} \left( \prod_{0 \leq l \leq \delta_A - 1} Q_{A,l}(k) \right).
\]
For every prime \( p \) dividing \(|G|\), let \( \mathcal{A}_p \) be the subset of \( \mathcal{A} \) consisting of the irreducible \( G \)-modules having order a power of \( p \) and let

\[
P_{G,p}(k) = \prod_{A \in \mathcal{A}_p} \left( \prod_{0 \leq i \leq \delta_A - 1} Q_{A,i}(k) \right).
\]

**Definition 8.** For every prime \( p \) and every positive integer \( \alpha \) let

\[
C_{p,\alpha}(s) = \prod_{0 \leq i \leq \alpha - 1} \left( 1 - \frac{p^i}{p^s} \right), \quad D_{p,\alpha}(s) = \prod_{1 \leq i \leq \alpha} \left( 1 - \frac{p^i}{p^s} \right).
\]

**Lemma 9.** Let \( G \) be a finite soluble group and let \( k \) be a positive integer.

\( (1) \) If \( d_p(G) \leq d \), then \( P_{G,p}(k) \geq d_{p,d}(k) \).

\( (2) \) If \( p \) divides \(|G/G'|\), then \( P_{G,p}(k) \geq P_{G,p}(k) \).

\( (3) \) If \( \alpha_{p,1}(G) = 0 \), then \( P_{G,p}(k) \geq C_{p,d}(k) \).

\( (4) \) If \( d_2(G) \leq d \), then \( P_{G,2}(k) \geq C_{p,d}(k) \).

**Proof.** Suppose that \( \mathcal{A}_p = \{ A_1, \ldots, A_l \} \) and let \( q_i = q_{A_i}, r_i = r_{A_i}, \xi_i = \xi_{A_i} \) and \( \delta_i = \delta_{A_i} \). Recall that

\[
P_{G,p}(k) = \prod_{0 \leq i \leq \delta_A - 1} Q_{A,i}(k).
\]

By Lemma 1, \( \delta_1 + \delta_2 + \cdots + \delta_l = \alpha_p(G) \leq d_p(G) \leq d \), hence the number of factors \( Q_{A_1,i}(k) \) in \( P_{G,p}(k) \) is at most \( d \). We order these factors in such a way that \( A_1 \) is the trivial \( G \)-module if \( p \) divides \(|G/G'|\).

1) Since \( d_{p,d}(k) = 0 \) if \( k \leq d \), we may take \( k > d \). To show that \( P_{G,p}(k) \geq d_{p,d}(k) \), it is sufficient to show that the \( j \)-th factor \( Q_j(k) = Q_{A_1,i}(k) \) of \( P_{G,p}(k) \) is greater than the \( j \)-th factor

\[
D_j(k) = 1 - \frac{p^j}{p^k}
\]

of \( d_{p,d}(k) \). If \( j \leq \delta_1 \) then \( Q_j(k) = Q_{A_1,l}(k) \) with \( l = j - 1 \). If \( j > \delta_1 \) then \( Q_j(k) = Q_{A_1,i}(k) \) for some \( i \in \{ 2, \ldots, t \} \) and \( l \in \{ 0, \ldots, \delta_i - 1 \} \), thus

\[
j = \delta_1 + \delta_2 + \cdots + \delta_{i-1} + l + 1 \geq l + 2.
\]

In any case,

\[
q_i^{r_i \xi_i} q_i^{l \xi_i} \leq q_i^{r_i \xi_i} q_i^{l \xi_i} \leq q_i^{r_i \xi_i}.
\]

We have \( q_i = p^{n_i} \) for some \( n_i \in \mathbb{N} \). Since \( j \leq d < k \), we deduce that

\[
q_i^{r_i \xi_i} q_i^{l \xi_i} \leq q_i^{r_i \xi_i} q_i^{l \xi_i} = \left( \frac{p^i}{p^k} \right)^{r_i n_i} \leq \frac{p^i}{p^k}.
\]

But then

\[
Q_j(k) = 1 - \frac{q_i^{r_i \xi_i} q_i^{l \xi_i}}{q_i^{r_i \xi_i} q_i^{l \xi_i}} \geq 1 - \frac{p^j}{p^k} = D_j(k).
\]

2) Since \( C_{p,d}(k) = 0 \) if \( k < d \), we may take \( k \geq d \). To show that \( P_{G,p}(k) \geq C_{p,d}(k) \), it is sufficient to show that the \( j \)-th factor \( Q_j(k) = Q_{A_1,i}(k) \) of \( P_{G,p}(k) \) is greater than the \( j \)-th factor

\[
C_j(k) = 1 - \frac{p^{j-1}}{p^k}
\]
of $C_{p,d}(k)$. If $i = 1$, then, by the way in which we ordered the elements of $A_p$, we have $Q_i(k) = C_j(k)$. Otherwise, as we have seen in the proof of (1), $l + 2 \leq j$ so $r_i\zeta_i + l \leq r_i + j - 2 \leq r_i(j - 1)$. Since $j \leq d \leq k$, we deduce that

$$\frac{q_i^{r_i\zeta_i}q_i^l}{q_i^{r_i\zeta_i}q_i^l} \leq \frac{p^{j-1}}{p^k} \quad \text{and} \quad Q_j(k) = 1 - \frac{q_i^{r_i\zeta_i}q_i^l}{q_i^{r_i\zeta_i}q_i^l} \geq 1 - \frac{p^{j-1}}{p^k} = C_j(k).$$

3) Assume that no complemented chief factor of $G$ has order $p$. By (5) of Lemma 4, $\alpha_p(G) \leq d_p(G) - 1 \leq d - 1$. But then the factors $Q_{A_i}(k)$ in (2.4) are at most $d - 1$ and, arguing as in the proof of (1), we conclude $P_{G,p}(k) \geq D_{p,d-1}(k) \geq C_{p,d}(k)$.

4) We may assume $\alpha_2(G) \neq 0$ (otherwise $P_{G,2}(k) = 1$). Since $\alpha_{2,1}(G) \neq 0$ if and only if 2 divides $|G/G'|$, the conclusion follows from (2) and (3).

3. THE MAIN RESULT

**Proposition 10.** Let $G$ be a finite group. If all the Sylow subgroups of $G$ can be generated by $d$ elements and $G$ is not soluble, then

$$e(G) \leq d + \kappa^* \quad \text{with} \quad \kappa^* \leq 2.7501.$$

**Proof.** Let $\beta = \beta(G)$. Since $G$ is not soluble, $\beta > 0$, hence by (2) and (3) of Lemma 4, we have $1 \leq \beta \leq d_2(G) - 1 \leq d - 1$ and $\alpha_2(G) \leq d_2(G) - \beta \leq d - 1$. We distinguish two cases:

a) $\beta < d - 1$. By Lemma 5 and 6 and using an accurate estimation of $\sum_p (p-1)^{-2}$ given in [1], we conclude

$$e(G) \leq d + 2 + \mu^*(G, d + 2) + \mu_2(G, d + 2) + \sum_{p>2} \mu_p(G, d + 2) \leq d + 2 + \frac{1}{20} + \frac{1}{4} + \sum_{p>2} \frac{1}{(p-1)^2} \leq d + 2.6751.$$

b) $\beta = d - 1$. By (2) and (4) of Lemma 4, either $\alpha_2(G) = 0$ or $\alpha_2(G) = \alpha_{2,1}(G) = 1$. In the first case $\mu_2(G, d + 2) = 0$, in the second case $m^2_2(G) = 1$ and consequently

$$\mu_2(G, d + 2) = \sum_{k \geq d+2} \frac{m^2_2(G)}{2^k} \leq \sum_{k \geq d+2} \frac{1}{2^k} \leq \sum_{k \geq 4} \frac{1}{2^k} \leq \frac{1}{8}.$$

By Lemma 5 and 6, we conclude

$$e(G) \leq d + 2 + \frac{1}{4} + \frac{1}{8} + \sum_{p>2} \frac{1}{(p-1)^2} \leq d + 2.7501. \quad \square$$

The previous proposition reduces the proof of Theorem 11 to the particular case when $G$ is soluble. To deal with this case, we are going to introduce, for every positive integer $d$ and every set of primes $\pi$, a supersoluble group $H_{\pi,d}$ with the property that $e(G) \leq e(H_{\pi,d})$ whenever $G$ is soluble, $\pi(G) \subseteq \pi$ and the Sylow subgroups of $G$ are $d$-generated.
Definition 11. Let $\pi$ be a finite set of prime integers with $2 \in \pi$, and let $d$ be a positive integer. We define $H_{\pi,d}$ as the semidirect product

$$H_{\pi,d} = \left( \prod_{p \in \pi \setminus \{2\}} C_p^d \right) \rtimes C_2 \rtimes C_2^{d-1}$$

where $C_p$ is the cyclic group of order $p$ and $C_2 = \langle y \rangle$ acts on $A = \prod_{p \in \pi \setminus \{2\}} C_p^d$ by setting $x^y = x^{-1}$ for all $x \in A$.

Theorem 12. Let $G$ be a finite soluble group. If all the Sylow subgroups of $G$ can be generated by $d$ elements, then $e(G) \leq e(H_{\pi,d})$, where $\pi = \pi(G) \cup \{2\}$.

Proof. Let $H = H_{\pi,d}$, $p \in \pi$ and $k \in \mathbb{N}$. By (2.3), $P_{H,p}(k) = D_{p,d}(k)$ if $p \neq 2$, while $P_{H,2}(k) = C_{2,d}(k)$. By Lemma 9, $P_{G,p}(k) \geq P_{H,p}(k)$ for every $p \in \pi(G)$. This implies

$$P_{G}(k) = \prod_{p \in \pi(G)} P_{G,p}(k) \geq \prod_{p \in \pi} P_{H,p}(k) = P_{H}(G)$$

and consequently $e(G) = \sum_{k \geq 0} (1 - P_{G}(k)) \leq \sum_{k \geq 0} (1 - P_{H}(k)) = e(H)$. □

Definition 13. Let $e_d = \sup_{\pi} e(H_{\pi,d})$ and $\kappa = \sup_{d} (e_d - d)$.

Let $2 \in \pi$ and let $\pi^* = \pi \setminus \{2\}$. Since $P_{H,2,d}(k) = 0$ for all $k \leq d$ we have

$$e(H_{\pi,d}) = \sum_{k \geq 0} (1 - P_{H,\pi,d}(k)) = d + 1 + \sum_{k \geq d+1} \left(1 - C_{2,d}(k) \prod_{p \in \pi^*} D_{p,d}(k)\right)$$

$$= d + 1 + \sum_{k \geq d+1} \left(1 - \prod_{1 \leq i \leq d} \left(1 - \frac{2^{i-1}}{2^k}\right) \prod_{p \in \pi^*} \prod_{1 \leq i \leq d} \left(1 - \frac{p^i}{p^{k+1}}\right)\right)$$

$$= d + 1 + \sum_{k \geq 0} \left(1 - \prod_{1 \leq i \leq d} \left(1 - \frac{2^{i-1}}{2^{k+(d+1)}}\right) \prod_{p \in \pi^*} \prod_{1 \leq i \leq d} \left(1 - \frac{p^i}{p^{d+1}}\right)\right).$$

We immediately deduce that $e(H_{\pi,d}) - d$ increase as $d$ increase. Moreover we have

$$e_d - d = \sup_{\pi} (e(H_{\pi,d}) - d)$$

$$= 1 + \sum_{k \geq d+1} \left(1 - \frac{(1 - \frac{1}{2^k})}{(1 - \frac{1}{2^{d+1}})} \prod_{p \leq k \leq d} \prod_{1 \leq i \leq d} \left(1 - \frac{p^i}{p^{k+1}}\right)\right).$$

For $k = d+1$ the double product goes to $0$ while for $k \geq d+2$ goes to $0$ and so we get

$$e_d - d = 2 + \sum_{k \geq d+2} \left(1 - \frac{(1 - \frac{1}{2^k})}{(1 - \frac{1}{2^{d+1}})} \prod_{1 \leq i \leq d} \zeta(k - i)^{-1}\right)$$

$$= 2 + \sum_{j \geq 1} \left(1 - \frac{(2j+1 - 2^d)}{2j+1 - 1} \prod_{1 \leq i \leq d} \zeta(j + l)^{-1}\right)$$

$$= 2 + \sum_{j \geq 1} \left(1 - \frac{2j+1 - 2^d}{2j+1 - 1} \prod_{1 \leq j \leq d+j} \zeta(n)^{-1}\right).$$
Let $c = \prod_{2 \leq n \leq \infty} \zeta(n)^{-1}$. Since $e_d - d$ increases as $d$ grows, we get
\[
\kappa = \lim_{d \to \infty} e_d - d
\]
\[
= 2 + \left(1 - \frac{2^2}{2^2 - 1}\right) c + \sum_{j \geq 2} \left(1 - \frac{2^j + 1}{2^{j+1} - 1}\right) c \prod_{2 \leq n \leq j} \zeta(n)
\]
\[
= 2 + \left(1 - \frac{4}{3} \cdot c\right) + \sum_{j \geq 2} \left(1 - \frac{1}{2^{j+1} - 1}\right) c \prod_{2 \leq n \leq j} \zeta(n)
\]
Using the computer algebra system PARI/GP \cite{14}, we get
\[
\kappa = 2 + \left(1 - \frac{4}{3} \cdot c\right) + \sum_{j \geq 2} \left(1 - \frac{1}{2^{j+1} - 1}\right) c \prod_{2 \leq n \leq j} \zeta(n) \sim 2.75239495.
\]
Combining this result with Proposition \cite{10} and Theorem \cite{12} we obtain the proof of Theorem \cite{11}

4. Finite groups of odd order

**Theorem 14.** Let $G$ be a finite soluble group. There exists a finite supersoluble group $H$ such that

(1) $\pi(H) = \pi(G)$,

(2) $P_G(k) \geq P_H(k)$ for all $k \in \mathbb{N}$,

(3) $d_p(G) \geq d_p(H)$ for all $p \in \pi(G)$,

(4) $\pi(G/G') \leq \pi(H/H')$.

**Proof.** Let $\pi(G) = \{p_1, \ldots, p_n\}$ with $p_1 \leq \cdots \leq p_n$. For $i \in \{1, \ldots, n\}$, set $\pi_i = \{p_1, \ldots, p_i\}$. We will prove, by induction on $i$, that for every $i \in \{1, \ldots, n\}$ there exists a supersoluble group $H_i$ such that $\pi(H_i) = \pi_i$ and, for every $j \leq i$,

(1) $P_{H_i,p_j}(k) \leq P_{G,p_j}(k)$ for all $k \in \mathbb{N}$,

(2) $d_{p_j}(H_i) \leq d_{p_j}(G)$,

(3) if $C_{p_j}$ is an epimorphic image of $G$, then $C_{p_j}$ is an epimorphic image of $H_i$.

Assume that $H_i$ has been constructed and set $p_{i+1} = p$ and $d_{p_i}(G) = d_p$. We distinguish two different cases:

1) Either $p$ divides $|G/G'|$ or $G$ contains no complemented chief factor of order $p$. We consider the direct product $H_{i+1} = H_i \times C_p^{d_p}$. Clearly $P_{H_{i+1},p_j}(k) = P_{H_i,p_j}(k) \leq P_{G,p_j}(k)$ if $j \leq i$. Moreover, by (2) and (3) of Lemma \cite{3} $P_{H_{i+1},p}(k) = C_{p,d_p}(k) \leq P_{G,p}(k)$.

2) $p$ does not divide $|G/G'|$ but $G$ contains a complemented chief factor which is isomorphic to a nontrivial $G$-module, say $A$, of order $p$. In this case $G/C_G(A)$ is a non-trivial cyclic group whose order divides $p - 1$. Let $q$ be a prime divisor of $|G/C_G(A)|$ (it must be $q = p_j$ for some $j \leq i$). Since $q$ divides $|G/G'|$, we have that $q$ divides also $|H_i/H'_i|$, hence there exists a normal subgroup $N$ of $H_i$ with $H_i/N \cong C_q$ and a nontrivial action of $H_i$ on $C_p$ with kernel $N$. We use this action to construct the supersoluble group $H_{i+1} = C_p^{d_p} \times H_i$. Clearly $P_{H_{i+1},p_j}(k) = P_{H_i,p_j}(k) \leq P_{G,p_j}(k)$ if $j \leq i$. Moreover, by (1) of Lemma \cite{3} $P_{H_{i+1},p}(k) = D_{p,d_p}(k) \leq P_{G,p}(k)$.

We conclude the proof, noticing that $H = H_n$ satisfies the requests in our statement.\qed
**Proof of Theorem 7.3** Let $\pi(G) = \pi$. By Theorem 14 there exists a supersoluble group $H$ such that $\pi(H) = \pi$, $d_p(H) \leq d$ for every $p \in \pi$ and $P_G(k) \geq P_H(k)$ for every $k \in \mathbb{N}$. In particular $e(G) = \sum_{k \geq 0} (1 - P_G(k)) \leq \sum_{k \geq 0} (1 - P_H(k)) = e(H)$.

Since $H$ is supersoluble, if $A$ is $H$-isomorphic to a chief factor of $H$, then $|A| = p$ for some $p \in \pi$ and $H/C_H(A)$ is a cyclic group of order dividing $p-1$. If $p$ is a Fermat prime, then $H/C_H(A)$ is a 2-group and, since $|H|$ is odd, we must have $H = C_H(A)$. This implies that if $p \in \pi$ is a Fermat prime, then $P_{H,p}(k) = C_{p,d_p(H)}(k) \geq C_{p,d}(k)$.

For all the other primes in $\pi$, by (1) of Lemma 9 we have $P_{H,p}(k) \geq D_{p,d}(k)$. Therefore, denoting by $\Lambda$ the set of the Fermat primes and by $\Delta$ the set of the remaining odd primes, we get

$$P_H(k) = \prod_{p \in \pi} P_{H,p}(k) \geq \prod_{p \in \Lambda} C_{p,d}(k) \prod_{p \in \Delta} D_{p,d}(k).$$

It follows that

$$e(H) = \sum_{k \geq 0} (1 - P_H(k))$$

$$\leq \sum_{k \geq 0} \left( 1 - \prod_{p \in \Lambda} \prod_{1 \leq i \leq d} \left(1 - \frac{p^{i-1}}{p^k}\right) \prod_{p \in \Delta} \prod_{1 \leq i \leq d} \left(1 - \frac{p^i}{p^k}\right) \right)$$

$$= d + 1 + \sum_{k \geq 2d+1} \left( 1 - \prod_{p \in \Lambda} \prod_{1 \leq i \leq d} \left(1 - \frac{p^{i-1}}{p^k}\right) \prod_{p \in \Delta} \prod_{1 \leq i \leq d} \left(1 - \frac{p^i}{p^k}\right) \right)$$

$$= d + 1 + \sum_{\ell \geq 0} \left( 1 - \prod_{p \in \Lambda} \prod_{1 \leq i \leq d} \left(1 - \frac{p^{i-1}}{p^\ell p^{(d+1)}}\right) \prod_{p \in \Delta} \prod_{1 \leq i \leq d} \left(1 - \frac{p^i}{p^\ell p^{(d+1)}}\right) \right) + 1.$$

Let

$$\tilde{k}_d = \sum_{\ell \geq 0} \left( 1 - \prod_{p \in \Lambda} \prod_{1 \leq i \leq d} \left(1 - \frac{p^{i-1}}{p^\ell p^{(d+1)}}\right) \prod_{p \in \Delta} \prod_{1 \leq i \leq d} \left(1 - \frac{p^i}{p^\ell p^{(d+1)}}\right) \right) + 1.$$

It can be easily check that $\tilde{k}_d$ increase as $d$ increases. Let

$$b = \prod_{1 \leq n \leq \infty} \left( 1 - \frac{1}{2^n}\right)^{-1}, \quad c = \prod_{2 \leq n \leq \infty} \zeta(n)^{-1}$$

and let $\Lambda^* = \{3, 5, 17, 257, 65537\}$ be the set of the known Fermat primes. Similar computations to the ones in the final part of Section 3 lead to the conclusion

$$\tilde{k}_d \leq 3 - \frac{b \cdot c}{2} \prod_{p \in \Lambda} \frac{p^2}{p^2 - 1} + \sum_{j \geq 2} \left( 1 - b \prod_{1 \leq n \leq j} \left( 1 - \frac{1}{2^n}\right) \prod_{p \in \Lambda} \left( 1 + \frac{1}{p^{j+1} - 1}\right) \prod_{2 \leq n \leq j} \zeta(n) \right)$$

$$\leq 3 - \frac{b \cdot c}{2} \prod_{p \in \Lambda} \frac{p^2}{p^2 - 1} + \sum_{j \geq 2} \left( 1 - b \prod_{1 \leq n \leq j} \left( 1 - \frac{1}{2^n}\right) \prod_{p \in \Lambda} \left( 1 + \frac{1}{p^{j+1} - 1}\right) \prod_{2 \leq n \leq j} \zeta(n) \right).$$
Let
\[ \kappa = 3 - \frac{b \cdot c}{2} \prod_{p \in \mathcal{P}} \frac{p^2}{p^2 - 1} + \sum_{j \geq 2} \left( 1 - b \prod_{1 \leq n \leq j} \left( 1 - \frac{1}{2n} \right) \prod_{p \in \mathcal{P}} \left( 1 + \frac{1}{p^{j+1} - 1} \right) \right) c \prod_{2 \leq n \leq j} \zeta(n). \]

With the help of PARI/GP, we get that \( \kappa \approx 2.148668. \)

5. Permutation groups

Theorem 15. \([2]\) Corollary] If \( G \) is a \( p \)-subgroup of \( \text{Sym}(n) \), then \( G \) can be generated by \( \lfloor n/p \rfloor \) elements.

Theorem 16. \([3]\) Theorem 10.0.5] The chief length of a permutation group of degree \( n \) is at most \( n - 1 \).

Lemma 17. If \( G \leq \text{Sym}(n) \) and \( n \geq 8 \), then \( \beta(G) \leq \lfloor n/2 \rfloor - 3 \).

Proof. Let \( R(G) \) be the soluble radical of \( G \). By \([3]\) Theorem 2] \( G/R(G) \) has a faithful permutation representation of degree at most \( n \), so we may assume \( R(G) = 1 \). In particular \( \text{soc}(G) = S_1 \times \cdots \times S_r \) where \( S_1, \ldots, S_r \) are nonabelian simple groups and, by \([2]\) Theorem 3.1], \( n \geq 5r \). Let \( K = N_G(S_1) \cap \cdots \cap N_G(S_r) \). We have that \( K/\text{soc}(G) \) is soluble and that \( G/K \leq \text{Sym}(r) \), so by Theorem 16, \( \beta(G/K) \leq r - 1 \) (and indeed \( \beta(G/K) = 0 \) if \( r \leq 4 \)). But then \( \beta(G) \leq 2r - 1 \leq 2 \lfloor n/5 \rfloor - 1 \) if \( r \geq 5 \), \( \beta(G) \leq r \leq \lfloor n/5 \rfloor \) otherwise.

Lemma 18. Suppose that \( G \leq \text{Sym}(n) \) with \( n \geq 8 \). If \( G \) is not soluble, then
\[ e(G) \leq \lfloor n/2 \rfloor + 1.533823. \]

Proof. Let \( m = \lfloor n/2 \rfloor \). By Theorem 15, \( d_2(G) \leq m \). Since \( G \) is not soluble, we must have \( \beta(G) \geq 1 \). By Lemma 17, \( \beta(G) \leq m - 3 \), hence, by Lemma 5, \( \mu^*(G, m) \leq 1/4 \).

By (2) and (4) of Lemma 4, \( \alpha_u(G) \leq m - 1 \) and \( \alpha_{u,1}(G) \leq m - 2 \) for every \( u > 1 \), hence, by Lemma 7, \( \mu_2(G, m) \leq 1 \). If \( p \geq 5 \), then, by Theorem 16, \( m - \alpha_p(G) \geq m - d_p(G) \geq m - [n/5] \geq 3 \) so, by Lemma 7, \( \mu_p(G, m) \leq (p(p - 1)^2)^{-1} \). Since \( n \geq 8 \) we have \( m - \alpha_3(G) \geq m - [n/3] \geq 2 \) if \( n \neq 9 \). On the other hand, it can be easily checked that \( \alpha_3(G) \leq 2 \) for every unsoluble subgroup \( G \) of \( \text{Sym}(9) \), so \( m - \alpha_3(G) \geq 2 \) also when \( n = 9 \). But then, again by Lemma 7, \( \mu_3(G, m) \leq 1/4 \). It follows
\[ e(G) \leq m + \mu^*(G, m) + \mu_2(G, m) + \mu_3(G, m) + \sum_{p > 3} \mu_p(G, m) \leq m + 1 + \frac{1}{4} + \frac{1}{4} + \sum_{p \geq 5} \frac{1}{p(p - 1)^2} \leq m + \frac{3}{2} + \sum_{n \geq 5} \frac{1}{n(n - 1)^2} \leq m + 1.533823. \]

Lemma 19. Suppose that \( G \leq \text{Sym}(n) \) with \( n \geq 8 \). If \( G \) is soluble and \( \alpha_{2,1}(G) < \lfloor n/2 \rfloor \), then
\[ e(G) \leq \lfloor n/2 \rfloor + 1.533823. \]

Proof. Let \( \alpha = \alpha_{2,1}(G) \), \( \alpha^* = \sum_{i > 1} \alpha_{2,i}(G) \) and \( m = \lfloor n/2 \rfloor \). Notice that \( \alpha^* \leq m - 1 \) by Lemma 4. Set
\[ \mu_{2,1}(G, t) = \sum_{k \geq t} \frac{m^2(G)}{2k}, \quad \mu_{2,2}(G, t) = \sum_{k \geq t} \left( \sum_{n \geq 2} \frac{m^2(G)}{2nk} \right). \]

We distinguish two cases:
a) \( \alpha_2, u(G) < m - 1 \) for every \( u \geq 2 \). Since \( m^A_2(G) = 2^\alpha - 1 \), we have
\[
\mu_{2,1}(G, m) \leq \sum_{k \geq m} \frac{2^\alpha}{2^k} = \frac{1}{2^{m-\alpha-1}} \leq 1.
\]
Moreover, arguing as in the proof of [13, Lemma 7], we deduce
\[
\mu_{2,2}(G, m) \leq \frac{1}{2^{m-\alpha-1}} \leq 1.
\]
Notice that if \( \alpha = m - 1 \), then \( \alpha^* \leq 1 \) and consequently \( \mu_{2,2}(G, m) \leq 2^{2-m} \leq 1/4 \).
Similarly, if \( \alpha^* = m - 1 \), then \( \alpha \leq 1 \) and \( \mu_{2,1}(G, m) \leq 2^{2-m} \leq 1/4 \). If follows that \( \mu_2(G, m) = \mu_{2,1}(G, m) + \mu_{2,2}(G, m) \leq 5/4 \). Except in the case when \( n = 9 \) and \( \alpha_3(G) = 3 \), arguing as in the end of Lemma [13] we conclude
\[
e(G) \leq m + \mu_2(G, m) + \mu_3(G, m) + \sum_{p > 3} \mu_p(G, m)
\leq m + \frac{5}{4} + \frac{1}{4} + \sum_{p \geq 5} \frac{1}{p(p-1)^2} \leq m + 1.533823.
\]
We remain with the case when \( G \) is a soluble subgroup of \( \text{Sym}(9) \) with \( \alpha_3(G) = 3 \). This occurs only if \( G \) is contained in the wreath product \( \text{Sym}(3) \wr \text{Sym}(3) \).
In particular \( \alpha_2(G) \leq 3 \). If \( \alpha_2(G) \leq 2 \), then, by Lemma [7]
\[
e(G) \leq 5 + \mu_2(G, 5) + \mu_3(G, 5) \leq 5 + \frac{1}{4} + \frac{1}{4} = 5.5.
\]
We have \( \alpha_2(G) = \alpha_3(G) = 3 \) only in two cases: \( \text{Sym}(3) \wr \text{Sym}(3) \times \text{Sym}(3), \langle (1, 2, 3), (4, 5, 6), (1, 4)(2, 5)(3, 6), (1, 2)(4, 5) \rangle \times \text{Sym}(3) \). In this two cases, \( G \) contains exactly 16 maximal subgroups, 7 with index 2 and 9 of index 3. But then
\[
e(G) \leq 4 + \sum_{k \geq 4} \frac{m_2(G)}{2^k} + \sum_{k \geq 4} \frac{m_3(G)}{3^k} = 4 + \sum_{k \geq 4} \frac{7}{2^k} + \sum_{k \geq 4} \frac{9}{3^k} = 4 + \frac{7}{8} + \frac{1}{6} \sim 5.0417.
\]
b) \( \alpha_2, u(G) = m - 1 \) for some \( u \geq 2 \). In this case \( m^A_2(G) \leq 1 \), so
\[
\mu_{2,1}(G, m + 1) \leq \sum_{k \geq m+1} \frac{1}{2^k} \leq \frac{1}{2m} \leq \frac{1}{16}.
\]
Moreover, by [13, Lemma 5], \( m^A_2(G) \leq 2^{u_2, (G)+u} \), hence
\[
\mu_{2,2}(G, m + 1) = \sum_{k \geq m+1} \left( \sum_{n \geq 2} \frac{m^A_2(G)}{2^{nk}} \right) \leq \sum_{k \geq m+1} \frac{m^A_2(G)}{2^k} \leq \sum_{k \geq m+1} \frac{2^{u_2, (G)+u} - 1}{2^k} \leq \frac{1}{2^m - 1} \leq \frac{1}{3}.
\]
If \( p \geq 5 \), then \( m - \alpha_p(G) \geq 3 \) so, by Lemma [7], \( \mu_p(G, m + 1) \leq (p(p-1))^{-2} \). Moreover \( m - \alpha_3(G) \geq 2 \) (notice that there is no subgroup of \( \text{Sym}(9) \) with \( \alpha_3(G) = 3 \) and \( \alpha_2, u(G) = 3 \) for some \( u \geq 2 \)), so, again by Lemma [7], \( \mu_3(G, m + 1) \leq 1/12 \). It follows
\[
e(G) \leq m + 1 + \mu_{2,1}(G, m + 1) + \mu_{2,2}(G, m + 1) + \mu_3(G, m + 1) + \sum_{p > 3} \mu_p(G, m + 1)
\leq m + 1 + \frac{1}{16} + \frac{1}{3} + \frac{1}{12} + \sum_{p \geq 5} \frac{1}{p^2(p-1)^2} \leq \frac{71}{48} + \sum_{n \geq 5} \frac{1}{n^2(n-1)^2} \leq m + 1.4843. \]
When $G \leq \text{Sym}(n)$ and $n \leq 7$, the precise value of $e(G)$ can be computed by GAP [3] using the formula

$$e(G) = -\sum_{H \leq G} \frac{\mu_G(H)|G|}{|G| - |H|},$$

where $\mu_G$ is the Möbius function defined on the subgroup lattice of $G$ (see [12, Theorem 1]). The crucial information are summarized in the following lemma.

**Lemma 20.** Suppose that $G \leq \text{Sym}(n)$ with $n \leq 7$. Either $e(G) \leq \lfloor n/2 \rfloor + 1$ or one of the following cases occurs:

1. $G \cong \text{Sym}(3)$, $n = 3$, $e(G) = 29/10$;
2. $G \cong C_2 \times C_2$, $n = 4$, $e(G) = 10/3$;
3. $G \cong D_8$, $n = 4$, $e(G) = 10/3$;
4. $G \cong C_2 \times \text{Sym}(3)$, $n = 5$, $e(G) = 1181/330$;
5. $G \cong C_2 \times C_2 \times C_2$, $n = 6$, $e(G) = 94/21$;
6. $G \cong C_2 \times D_8$, $n = 6$, $e(G) = 94/21$;
7. $G \cong C_2 \times C_2 \times \text{Sym}(3)$, $n = 7$, $e(G) = 241789/53130$;
8. $G \cong D_8 \times \text{Sym}(3)$, $n = 7$, $e(G) = 241789/53130$.

**Theorem 21.** Let $G$ be a permutation group of degree $n \neq 3$. If $\alpha_{2,1}(G) = \lfloor n/2 \rfloor$, then $e(G) \leq \lfloor n/2 \rfloor + \nu$, with $\nu \sim 1.606695$.

**Proof.** Let $m = \lfloor n/2 \rfloor$. We have that $\alpha_{2,1}(G) = m$ if and only if $C_2^m$ is an epimorphic image of $G$. By [2] if $C_2^m$ is an epimorphic image of $G$ then $G$ is the direct product of its transitive constituents and each constituent is one of the following: $\text{Sym}(2)$, of degree 2, $\text{Sym}(3)$, of degree 3, $C_2 \times C_2$, $D_8$, of degree 4, and the central product $D_8 \circ D_8$, of degree 8. Consequently:

$$G/\text{Frat}(G) \simeq \begin{cases} C_2^m & \text{if } n = 2m, \\ C_2^{m-1} \times \text{Sym}(3) & \text{if } n = 2m + 1. \end{cases}$$

And so, by [2,3],

$$P_G(k) = P_{G/\text{Frat}(G)}(k) = \prod_{0 \leq i \leq m-1} \left(1 - \frac{2^i}{3^k}\right) \left(1 - \frac{3}{3^k}\right)^{n-2m}.$$

Setting $\eta = 0$ if $n$ is even, $\eta = 1$ otherwise, we have

$$e(G) = \sum_{k \geq 0} (1 - P_G(k)) \leq \sum_{k \geq 0} \left(1 - \prod_{0 \leq i \leq m-1} \left(1 - \frac{2^i}{2^k}\right) \left(1 - \frac{3}{3^k}\right)^{\eta}\right) = m + \sum_{k \geq m} \left(1 - \prod_{0 \leq i \leq m-1} \left(1 - \frac{2^i}{2^k}\right) \left(1 - \frac{3}{3^k}\right)^{\eta}\right)$$

$$= m + \sum_{j \geq 0} \left(1 - \prod_{1 \leq i \leq m} \left(1 - \frac{1}{2^{j+l}}\right) \left(1 - \frac{3}{3^j+m}\right)^{\eta}\right).$$

Set

$$\omega_{m,\eta} = \sum_{j \geq 0} \left(1 - \prod_{1 \leq i \leq m} \left(1 - \frac{1}{2^{j+l}}\right) \left(1 - \frac{3}{3^j+m}\right)^{\eta}\right).$$
Clearly $\omega_{m,0}$ increase with $m$. On the other hand, if $m \geq 4$ and $j \geq 0$ then
\[
\left(1 - \frac{1}{2^{j+m+1}}\right) \left(1 - \frac{3}{3^{j+m+1}}\right) \leq \left(1 - \frac{3}{3^{j+m}}\right)
\]
and so $\omega_{m,1} \leq \omega_{m+1,1}$ if $m \geq 4$. Moreover
\[
\lim_{m \to \infty} \omega_{m,1} = \lim_{m \to \infty} \omega_{m,0} \sim 1.606695.
\]
But then $e(G) \leq m + 1.606695$ whenever $m \geq 4$. The value of $e(G)$ when $n$ is small is given by the following table (that indicates also how fast $e(G) - m$ tends to $1.606695$).

| $n$ | $e(G)$ |
|-----|--------|
| 2   | 2      |
| 3   | $\frac{20}{10} \approx 2.900$ |
| 4   | $\frac{10}{3} \approx 3.334$ |
| 5   | $\frac{181}{50} \approx 3.579$ |
| 6   | $\frac{94}{25} \approx 4.476$ |
| 7   | $\frac{241789}{33130} \approx 4.551$ |
| 8   | $\frac{194}{35} \approx 5.5429$ |
| 9   | $\frac{4633553}{812379} \approx 5.5667$ |
| 10  | $\frac{7134}{1085} \approx 6.5751$ |
| 11  | $\frac{3227369181}{490265930} \approx 6.5828$ |
| 12  | $\frac{10663922}{1240155} \approx 8.59886$ |
| 13  | $\frac{70505670417749503}{8198607229768494} \approx 8.59971$ |

From the information contained in this table, we deduce that $e(G) \leq m + 1.606695$, except when $G = \text{Sym}(3)$.

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