The Cauchy Problem of the
Schrödinger-Korteweg-de Vries System

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Abstract We study the Cauchy problem of the Schrödinger-Korteweg-de Vries system. First, we establish the local well-posedness results, which improve the results of Corcho, Linares (2007). Moreover, we obtain some ill-posedness results, which show that they are sharp in some well-posedness thresholds. Particularly, we obtain the local well-posedness for the initial data in $H^{-\frac{1}{16}}(\mathbb{R}) \times H^{-\frac{2}{4}}(\mathbb{R})$ in the resonant case, it is almost the optimal except the endpoint. At last we establish the global well-posedness results in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ when $s > \frac{1}{2}$ no matter in the resonant case or in the non-resonant case, which improve the results of Pecher (2005).

Keywords: Schrödinger-Korteweg-de Vries system, local well-posedness, ill-posedness, global well-posedness, Bourgain space, $I$-method

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1 Introduction

The Cauchy problem of the Schrödinger-Korteweg-de Vries equations

\[
\begin{cases}
    i\partial_t u + \partial_x^2 u = \alpha uv + \beta |u|^2 u, & x, t \in \mathbb{R}, \\
    \partial_t v + \partial_x^2 v + \frac{1}{2} \partial_x v^2 = \gamma \partial_x (|u|^2), \\
    u(x, 0) = u_0(x) \in H^s(\mathbb{R}), v(x, 0) = v_0(x) \in H^l(\mathbb{R}),
\end{cases}
\]  

(1.1)

where $\alpha, \beta, \gamma \in \mathbb{R}$. The system governs the interactions between the short-wave and the long-wave, which appears in several fields of physics and fluid dynamics. The case $\beta = 0$ describes the resonant interactions, while the case $\beta \neq 0$ describes the non-resonant interactions. See [13], [17], [18], [23] for the applications.

The Cauchy problem for the system (1.1) was considered by several authors. The local well-posedness was studied in [3], [4], [12], etc., where the last paper [12] obtained the local well-posedness for $(u_0, v_0) \in H^s(\mathbb{R}) \times H^l(\mathbb{R})$ when $s \geq 0$, $l > -\frac{3}{4}$, and

- $s - 1 \leq l \leq 2s - 1/2$, if $s \leq 1/2$;
- $s - 1 \leq l < s + 1/2$, if $s > 1/2$,

by the Bourgain argument (see [1], [19] for instances). Moreover, when $\alpha \gamma > 0$, Tsutsumi [24] proved by some conservation laws that for $(u_0, v_0) \in H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ with $s \in \mathbb{Z}^+$, (1.1) was global well-posedness; Guo and Miao [15] showed that the system in the resonant case was globally well-posed for $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s \in \mathbb{Z}^+$; In [22], the author improved the results and obtained the global well-posedness for $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ when $s < 1$ and

- $s > 3/5$ in the case of $\beta = 0$;
- $s > 2/3$ in the case of $\beta \neq 0$,

by using the $I-$method of Colliander, Keel, Staffilani, Takaoka and Tao (see [9], [10] for examples).
In [12], the authors have obtained the local well-posedness for the initial data belongs to $L^2(\mathbb{R}) \times H^{-\frac{1}{4}+}(\mathbb{R})$, but it seems not the natural one as the best result (except the endpoint) and also exists some room if the system [11] has no power type nonlinearity (that is, $\beta = 0$). As what studied in the first part of this paper, we establish the local well-posedness results at a relatively wide region of the indices $(s, l)$, compared to the results in [12]. Especially, we obtain the local well-posedness for $(u_0, v_0) \in H^{-\frac{3}{16}+}(\mathbb{R}) \times H^{-\frac{3}{4}+}(\mathbb{R})$ in the case of $\beta = 0$. The second aim here is to establish some ill-posedness results, by the breakage of continuity in Picard iterative scheme (see [7], [21], [5], etc.), which show that some thresholds are sharp except the boundary in the well-posedness region. By these results, we will see that the index $(-\frac{3}{16}+, -\frac{3}{4}+)$ is almost the best in the resonant case. Further, the third aim in this article is to obtain the global well-posedness by the $I$-method. As we know, the thresholds of the global well-posedness in the Sobolev spaces, studied by the $I$-method, are decided by two ingredients: the almost conserved quantities and the lifetime in the local theory, particularly if the solutions of the equations lack of the scale invariance. Our motivation here is to lengthen the lifetime of the local existence. Our argument is establishing some special type multilinear estimates, as an available technique we obtain a uniformly control of the Bourgain $X_{0,b}$—norm of $u$ by the $L^2$-mass conservation. But unfortunately, these special type estimates break the framework of the fixed point theorem, we finally use the iterate technical to overcome this problem. However, we believe that our global results may not be the best and might be improved especially by some more sophisticated estimates on the almost conserved quantities.

**Some basic notations.** We use $A \lesssim B$ or $B \gtrsim A$ to denote the statement that $A \leq CB$ for some large constant $C$ which may vary from line to line. We use $A \ll B$ to denote the statement $A \leq C^{-1}B$, and use $A \sim B$ to mean $A \lesssim B \lesssim A$. The notation $a+$ denotes $a+\epsilon$ for any small $\epsilon$, and $a-$ for $a-\epsilon$. $\langle \cdot \rangle = (1+|\cdot|^2)^{\frac{1}{2}}$ and $D_x = (-\partial_x^2)^{\frac{1}{2}}$. We use $\|f\|_{L^p_xL^q_t}^\beta$ to denote the mixed norm $\left(\int \|f(x,\cdot)\|_{L^p_t}^\beta dx\right)^{\frac{1}{\beta}}$. Moreover, we denote $\hat{u}$ to be the spatial or spacetime Fourier transform of $u$, and use $\hat{f}$ or $\mathcal{F}^{-1}$ (such as $\mathcal{F}_\xi^{-1}$, $\mathcal{F}_\tau^{-1}$, $\mathcal{F}_{\xi\tau}^{-1}$ etc.) to denote the inverse Fourier transform of $f$ (on the corresponding variables).
Now we introduce some definitions before presenting our main results. We use $U_\phi(t) = \exp(-it\phi(-i\partial_x))$ to denote the unitary group generated by the linear equation

$$iu_t - \phi(-i\partial_x)u = 0,$$

and define the Bourgain spaces $X_{s,b}(\phi)$ to be the closure of the Schwartz class under the norms

$$\|f\|_{X_{s,b}(\phi)} \equiv \left( \int \int \langle \xi \rangle^{2s} \langle \tau + \phi(\xi) \rangle^{2b} |\hat{f}(\xi, \tau)|^2 d\xi d\tau \right)^{\frac{1}{2}},$$

for $s, b \in \mathbb{R}$. We write $X_{s,b}^{\pm}$, $Y_{s,b}$ to be $X_{s,b}(\phi)$ when $\phi = \pm \xi^2, -\xi^3$, which is corresponding to the Schrödinger and KdV respectively, and we write $X_{s,b} \equiv X_{s,b}^+$ in default. For an interval $\Omega$, we define $X_{\Omega, s,b}(\phi)$ to be the restriction of $X_{s,b}(\phi)$ on $\mathbb{R} \times \Omega$ with the norms

$$\|f\|_{X_{\Omega, s,b}(\phi)} = \inf \{ \|F\|_{X_{s,b}(\phi)} : F|_{t \in \Omega} = f|_{t \in \Omega} \}.$$

When $\Omega = [-\delta, \delta]$, we write $X_{\Omega, s,b}(\phi)$ as $X_{\delta, s,b}(\phi)$ ($X_{s,b}^\delta$ as $X_{s,b}^\delta$, $Y_{s,b}^\Omega$ as $Y_{s,b}^\delta$).

Let $s < 0$ and $N \gg 1$ be fixed, the Fourier multiplier operator $I_{N,s}$ is defined as

$$\widehat{I_{N,s}u}(\xi) = m_{N,s}(\xi)\hat{u}(\xi),$$

(1.2)

where the multiplier $m_{N,s}(\xi)$ is a smooth, monotone function satisfying $0 < m_{N,s}(\xi) \leq 1$ and

$$m_{N,s}(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ N^{1-s}|\xi|^{s-1}, & |\xi| > 2N. \end{cases}$$

(1.3)

Sometimes we denote $I_{N,s}$ and $m_{N,s}$ as $I$ and $m$ respectively for short if there is no confusion. It is obvious that the operator $I_{N,s}$ maps $H^s(\mathbb{R})$ into $H^1(\mathbb{R})$ with equivalent norm such that

$$\|f\|_{H^s} \lesssim \|I_{N,s}f\|_{H^1} \lesssim N^{1-s}\|f\|_{H^s}.$$

(1.4)

Moreover, $I_{N,s}$ can be extended to a map (still denoted by $I_{N,s}$) from $X_{s,b}$ to $X_{1,b}$ which satisfies

$$\|f\|_{X_{s,b}} \lesssim \|I_{N,s}f\|_{X_{1,b}} \lesssim N^{1-s}\|f\|_{X_{s,b}}$$

for any $s < 1, b \in \mathbb{R}$.

Our main results in this paper are given as follows.
Theorem 1.1 The Cauchy problem of the system (1.1) is locally well-posed on some time interval \([-\delta, \delta]\) for the initial data \((u_0, v_0) \in H^s(\mathbb{R}) \times H^l(\mathbb{R})\) when

- \(l > -3/4, l < 4s, s - 2 \leq l < s + 1, \) if \(\beta = 0;\)
- \(l > -3/4, s \geq 0, l < 4s, s - 2 \leq l < s + 1, \) if \(\beta \neq 0.\)

The solutions satisfy

\[(u, v) \in C^0_t([-\delta, \delta]; H^s(\mathbb{R}) \times H^l(\mathbb{R})).\]

Remark. The best result obtained in Theorem 1.1 is local well-posedness in \(L^2(\mathbb{R}) \times H^{-\frac{3}{16} +} (\mathbb{R})\) in the non-resonant case (\(\beta \neq 0\)), which has been contained in [12]. However, in the resonant case (\(\beta = 0\)), the best result we obtained is local well-posedness in \(H^{-\frac{3}{16} +} (\mathbb{R}) \times H^{-\frac{3}{4} +} (\mathbb{R})\), which improves the one in [12]. As we see, it follows from the assumptions of \(l > -3/4\) and \(l < 4s\). The next result tells that it is almost the best in the following sense except the endpoint case.

Theorem 1.2 Let \(l > 4s\), and let \(\delta\) and the solution map \((u_0, v_0) \mapsto (u, v)\) is defined in Theorem 1.1 from \(H^{s_0}(\mathbb{R}) \times H^{l_0}(\mathbb{R})\) to \(C^0_t([0, \delta]; H^{s_0}(\mathbb{R}) \times H^{l_0}(\mathbb{R}))\) for some well-posed index \((s_0, l_0)\). Then the map is not \(C^2\)-differentiable at zero from \(H^s(\mathbb{R}) \times H^l(\mathbb{R})\) to \(C^0_t([0, \delta]; H^s(\mathbb{R}) \times H^l(\mathbb{R})).\)

If we take the initial data \((0, v_0)\) with \(v_0 \in H^l(\mathbb{R})\) in the system (1.1), then by the uniqueness, \(u \equiv 0\) and the system (1.1) can be deduced to the single KdV equation which is well known to ill-posedness when \(l < -\frac{3}{4}\) (see [6], [20]). Thus we say that the system (1.1) is ill-posedness in \(H^s(\mathbb{R}) \times H^l(\mathbb{R})\) for \(s \in \mathbb{R}\) and \(l < -\frac{3}{4}\). To sum up, we draw the following figure for the corresponding regions.
Theorem 1.3 Let $\alpha \gamma > 0$, then the Cauchy problem of the system (1.1) is globally well-posed for the initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ when $1 > s > \frac{1}{2}$, no matter in the resonant case or in the non-resonant case.

The rest of this article is organized as follows. In Section 2, we derive some preliminary estimates. In Section 3, we establish some multilinear estimates and prove Theorem 1.1. In Section 4, we establish some multilinear estimates of special types, give a variant local result, and prove Theorem 1.3. In Section 5, we prove Theorem 1.2. In Section 6, as an appendix, we prove some auxiliary lemmas about the spaces $X_{s,b}^\Omega(\phi)$. 
2 Some Preliminary Estimates

First, we present two Stricharz estimates in the Bourgain spaces.

Lemma 2.1 (1) For any $u \in X_{0, \theta^+}^\pm, \theta \geq \frac{3}{2} \left(\frac{1}{2} - \frac{1}{p}\right), p \in [2, 6]$, we have

$$\|u\|_{L^p_{xt}} \lesssim \|u\|_{X_{0, \theta^+}^\pm}. \quad (2.1)$$

(2) For any $v \in Y_{0, \rho^+}, \rho \geq \frac{4}{3} \left(\frac{1}{2} - \frac{1}{q}\right). q \in [2, 8]$, we have

$$\|v\|_{L^q_{xt}} \lesssim \|v\|_{Y_{0, \rho^+}}. \quad (2.2)$$

Proof. They are easily obtained by the interpolation between the following well-known inequalities

$$\|u\|_{L^6_{xt}} \lesssim \|u\|_{X_{0, \frac{1}{2}^+}}; \quad \|v\|_{L^8_{xt}} \lesssim \|v\|_{Y_{0, \frac{1}{2}^+}}$$

and the equalities

$$\|u\|_{L^2_{xt}} = \|u\|_{X_{0, 0}^\pm}; \quad \|v\|_{L^2_{xt}} = \|v\|_{Y_{0, 0}}.$$

Now, we introduce some multiplier operators and the estimates on them (similar results are appeared in [8] and [14]). They are the important tools in the estimations. For the nonnegative functions $\hat{f}, \hat{g}, \hat{h}$, we define

$$I_k^\pm(\hat{f}, \hat{g}, \hat{h}) = \int_D m_k(\xi, \xi_1, \xi_2)^\pm \hat{f}(\xi, \tau)\hat{g}(\xi_1, \tau_1)\hat{h}(\xi_2, \tau_2) \quad (2.3)$$

for $k = 1, \cdots, 7$, where the set $D = \{(\xi_1, \xi_2, \tau_1, \tau_2) : \xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2\}$, and the multipliers $m_k$ are defined as

$$m_1 = |3\xi_2^2 + 2\xi_1|; \quad m_2 = |3\xi_2^2 + 2\xi_2|; \quad m_3 = |\xi_2|;$$

$$m_4 = |\xi|; \quad m_5 = |3\xi_2^2 - 2\xi_2|; \quad m_6 = |3\xi_2^2 + 2\xi_1|;$$

$$m_7 = |\xi_1 - \xi_2||\xi_1 + \xi_2|.$$
Lemma 2.2 Let \( f, g, h \) are reasonable functions, then

\[
\begin{align*}
(1) \quad I_{1}^{2}(\hat{f}, \hat{g}, \hat{h}) & \lesssim \|f\|L^{2} \|g\|_{X_{0,\frac{1}{2}+}} \|h\|_{Y_{0,\frac{1}{2}+}}; \quad I_{2}^{2}(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{X_{0,\frac{1}{2}+}} \|h\|L^{2} \|g\|_{Y_{0,\frac{1}{2}+}}; \\
I_{3}^{2}(\hat{f}, \hat{g}, \hat{h}) & \lesssim \|f\|_{X_{0,\frac{1}{2}+}} \|g\|L^{2} \|h\|_{X_{0,\frac{1}{2}+}}; \\
I_{4}^{2}(\hat{f}, \hat{g}, \hat{h}) & \lesssim \|f\|_{X_{0,\frac{1}{2}+}} \|g\|_{Y_{0,\frac{1}{2}+}} \|h\|_{Y_{0,\frac{1}{2}+}}; \\
I_{5}^{2}(\hat{f}, \hat{g}, \hat{h}) & \lesssim \|f\|_{X_{0,\frac{1}{2}+}} \|g\|_{X_{0,\frac{1}{2}+}} \|h\|_{Y_{0,\frac{1}{2}+}}; \\
I_{6}^{2}(\hat{f}, \hat{g}, \hat{h}) & \lesssim \|f\|_{X_{0,\frac{1}{2}+}} \|g\|_{L^{2}} \|h\|_{X_{0,\frac{1}{2}+}}; \\
I_{7}^{2}(\hat{f}, \hat{g}, \hat{h}) & \lesssim \|f\|_{X_{0,\frac{1}{2}+}} \|g\|_{L^{2}} \|h\|_{X_{0,\frac{1}{2}+}}; \\
I_{8}^{2}(\hat{f}, \hat{g}, \hat{h}) & \lesssim \|f\|_{L^{2}} \|g\|_{Y_{0,\frac{1}{2}+}} \|h\|_{Y_{0,\frac{1}{2}+}}.
\end{align*}
\]

Proof. We use the argument in [8] to prove the lemma. For \( I_{1}^{2} \), we change variables by

\[
\begin{align*}
\tau & = \lambda - \xi^{2}, \quad \tau_{1} = \lambda_{1} - \xi_{1}^{2}, \quad \tau_{2} = \lambda_{2} + \xi_{2}^{3},
\end{align*}
\]
then, \( I_{1}^{2}(f, g, h) \) is changed into

\[
\int m_{1}^{2} f(\xi_{1} + \xi_{2}, \lambda_{1} + \lambda_{2} - \xi_{1}^{2} + \xi_{2}^{2}) \hat{g}(\xi_{1}, \lambda_{1} - \xi_{1}^{2}) \hat{h}(\xi_{2}, \lambda_{2} + \xi_{2}^{3}) \, d\xi_{1} d\xi_{2} d\lambda_{1} d\lambda_{2}. \tag{2.4}
\]

We change variables again as follows. Let

\[
(\eta, \omega) = T(\xi_{1}, \xi_{2}), \tag{2.5}
\]
where

\[
\eta = T_{1}(\xi_{1}, \xi_{2}) = \xi_{1} + \xi_{2}, \quad \omega = T_{2}(\xi_{1}, \xi_{2}) = \lambda_{1} + \lambda_{2} - \xi_{1}^{2} + \xi_{2}^{3}.
\]
Then the Jacobian \( J \) of this transform satisfies

\[
|J| = |3\xi_{2}^{2} + 2\xi_{1}|.
\]

Define

\[
H(\eta, \omega, \lambda_{1}, \lambda_{2}) = \hat{g} \hat{h} \circ T^{-1}(\eta, \omega, \lambda_{1}, \lambda_{2}),
\]
then, by using \(|J|^{\frac{1}{2}}\) to eliminate \( m_{1}^{2} \), \( I_{1}^{2} \) has a bound of

\[
\int \hat{f}(\eta, \omega) \cdot \frac{H(\eta, \omega, \lambda_{1}, \lambda_{2})}{|J|^{\frac{1}{2}}} \, d\eta d\omega d\lambda_{1} d\lambda_{2}. \tag{2.6}
\]
By Hölder’ inequality, we have

\[
\begin{align*}
\langle 2.6 \rangle \quad & \leq \left\| f \right\|_{L^2_{\eta\omega}} \cdot \int \left( \int \frac{|H(\eta, \omega, \lambda_1, \lambda_2)|^2}{|J|} \, d\eta \omega \right)^{\frac{1}{2}} \, d\lambda_1 \, d\lambda_2 \\
&= \left\| f \right\|_{L^2_{\eta\omega}} \cdot \int \left\| \hat{g}(\xi_1, \lambda_1 - \xi_1^2) \right\|_{L^2_{\xi_1}} \, d\lambda_1 \cdot \int \left\| \hat{h}(\xi_2, \lambda_2 + \xi_2^3) \right\|_{L^2_{\xi_2}} \, d\lambda_2 \\
&\lesssim \| f \|_{L^2} \| g \|_{X_{0, \frac{1}{2}}} \| h \|_{Y_{0, \frac{1}{2}}},
\end{align*}
\]

where we employed the inverse transform of \langle 2.5 \rangle in the second step and Hölder’ inequality in the third step.

For \( I_1^2 \), the modification of the proof is replacing the variable transform \((\eta, \omega)\) by

\[
\eta = T_1(\xi, \xi_2) = \xi - \xi_2,
\]

\[
\omega = T_2(\xi, \xi_2) = \lambda - \lambda_2 - \xi_2^2 - \xi_2^3.
\]

Then the Jacobian \( J \) in this situation satisfies

\[
|J| = |3\xi_2^2 + 2\xi|.
\]

Therefore, we have the claim by the same argument as above but separating \( \hat{g} \) from the integration in this time.

For \( I_2^3 \), we take \((\eta, \omega)\) in this time that

\[
\eta = T_1(\xi, \xi_1) = \xi - \xi_1,
\]

\[
\omega = T_2(\xi, \xi_1) = \lambda - \lambda_1 - \xi_1^2 + \xi_1^2.
\]

Then the Jacobian \( J \) in this situation satisfies

\[
|J| = 2|\xi_2|.
\]

So the claim follows again.

For \( I_4^\frac{1}{2}, I_5^\frac{1}{2}, I_6^\frac{1}{2} \), we change variables first by setting

\[
\tau = \lambda + \xi^3, \quad \tau_1 = \lambda_1 - \xi_1^2, \quad \tau_2 = \lambda_2 + \xi_2^2, \tag{2.7}
\]

then for \( I_4^\frac{1}{2} \), we change variables again as

\[
\eta = T_1(\xi_1, \xi_2) = \xi_1 + \xi_2, \quad \omega = T_2(\xi_1, \xi_2) = \lambda_1 + \lambda_2 - \xi_1^2 + \xi_2^2.
\]
Then the Jacobian \( J \) satisfies
\[ |J| = 2|\xi|. \]

For \( I_{5}^{\frac{1}{2}} \), we change variables as
\[ \eta = T_{1}(\xi, \xi_{2}) = \xi - \xi_{2}, \]
\[ \omega = T_{2}(\xi, \xi_{2}) = \lambda - \lambda_{2} + \xi^{3} - \xi_{2}^{2}. \]
Then the Jacobian \( J \) satisfies
\[ |J| = |3\xi^{2} - 2\xi_{2}|. \]

For \( I_{6}^{\frac{1}{2}} \), we change variables as
\[ \eta = T_{1}(\xi, \xi_{1}) = \xi - \xi_{1}, \]
\[ \omega = T_{2}(\xi, \xi_{1}) = \lambda - \lambda_{1} + \xi^{3} + \xi_{1}^{2}. \]
Then the Jacobian \( J \) satisfies
\[ |J| = |3\xi^{2} + 2\xi_{1}|. \]

Therefore, we have the conclusions in the second term.

For \( I_{7}^{\frac{1}{2}} \), we change variables first by setting
\[ \tau = \lambda + \xi^{3}, \quad \tau_{1} = \lambda_{1} + \xi_{1}^{3}, \quad \tau_{2} = \lambda_{2} + \xi_{2}^{3}, \] (2.8)
then we change variables again as
\[ \eta = T_{1}(\xi_{1}, \xi_{2}) = \xi_{1} + \xi_{2}, \]
\[ \omega = T_{2}(\xi_{1}, \xi_{2}) = \lambda_{1} + \lambda_{2} + \xi_{1}^{3} + \xi_{2}^{3}. \]
Then the Jacobian \( J \) satisfies
\[ |J| = 3|\xi_{1}^{2} - \xi_{2}^{2}|. \]

Thus the claim follows by the same argument. \( \square \)

When \( s = 0 \), by Lemma 2.1 we have
\[ I_{2}^{0}(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{L_{\text{a}x}^{6/3}} \|g\|_{L_{\text{a}x}^{2}} \|h\|_{L_{\text{a}x}^{6}} \lesssim \|f\|_{X_{0,\frac{1}{4}+}} \|g\|_{L_{\text{a}x}^{2}} \|h\|_{Y_{0,\frac{1}{4}+}}; \]
\[ I_{3}^{0}(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{L_{\text{a}x}^{3}} \|g\|_{L_{\text{a}x}^{6}} \|h\|_{L_{\text{a}x}^{3}} \lesssim \|f\|_{X_{0,\frac{1}{4}+}} \|g\|_{X_{0,\frac{1}{4}+}} \|h\|_{L_{\text{a}x}^{2}}; \]
\[ I_{5}^{0}(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{L_{\text{a}x}^{3}} \|g\|_{L_{\text{a}x}^{6}} \|h\|_{L_{\text{a}x}^{3}} \lesssim \|f\|_{Y_{0,\frac{1}{4}+}} \|g\|_{L_{\text{a}x}^{2}} \|h\|_{X_{0,\frac{1}{4}+}}; \]
\[ I_{6}^{0}(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{L_{\text{a}x}^{3}} \|g\|_{L_{\text{a}x}^{6}} \|h\|_{L_{\text{a}x}^{3}} \lesssim \|f\|_{Y_{0,\frac{1}{4}+}} \|g\|_{X_{0,\frac{1}{4}+}} \|h\|_{L_{\text{a}x}^{2}}. \]

Interpolation between them and the results in Lemma 2.2, we have the following lemma.
Lemma 2.3 For any \( s \in \left[0, \frac{1}{2}\right] \), the following estimates hold:

\[
I_s^2(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{X_{0,\rho_2}} \|g\|_{L^2} \|h\|_{Y_{0,\frac{3}{4}+}};
\]

\[
I_s^3(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{X_{0,\rho_3}} \|g\|_{X_{0,\frac{3}{4}+}} \|h\|_{L^2};
\]

\[
I_s^5(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{Y_{0,\rho_5}} \|g\|_{L^2} \|h\|_{X_{0,-\frac{1}{4}+}};
\]

\[
I_s^6(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{Y_{0,\rho_6}} \|g\|_{X_{0,\frac{3}{4}+}} \|h\|_{L^2};
\]

where \( \rho_2 \geq \frac{3}{16} + \frac{5}{8}s, \rho_3 \geq \frac{1}{4} + \frac{1}{2}s \) and \( \rho_5, \rho_6 \geq \frac{2}{9} + \frac{5}{9}s \).

Further, we denote \( \psi(t) \) to be an even smooth characteristic function of the interval \([-1, 1]\), then we have the following estimates.

Lemma 2.4 Let \( \delta \in (0, 1), s \in \mathbb{R} \), then the following estimates hold:

(i) \( \|f\|_{C^0_t(\mathbb{R};H^s)} \lesssim \|f\|_{X_{s,b}(\phi)}, \forall \ b \in (\frac{1}{2}, 1]\);  

(ii) \( \|\psi(t)U_{\phi}(t)u_0\|_{X_{s,b}(\phi)} \lesssim \|u_0\|_{H^s}, \forall \ b \in (\frac{1}{2}, 1]\);  

(iii) \( \left\| \psi(t) \int_0^t U_{\phi}(t-s)F(s) \, ds \right\|_{X_{s,b}(\phi)} \lesssim \|F\|_{X_{s,b-1}(\phi)}, \forall \ b \in (\frac{1}{2}, 1]\);  

(iv) \( \|\psi(t/\delta)f\|_{X_{s,b}(\phi)} \lesssim \delta^{s-b} \|f\|_{X_{s,b'(\phi)}}, \forall \ 0 \leq b \leq b' < \frac{1}{2} \).

Proof. See cf. [16], [22], [8] for the proofs. \( \square \)

3 Multilinear Estimates and Local Well-posedness

3.1 Bi- and Trilinear Estimates

We begin with two well-known estimates.

Lemma 3.1 ([4]) Let \( s \geq 0, b = \frac{1}{2}+ \), then for any \( u_1, u_2, u_3 \in X_{s,b} \),

\[
\|u_1u_2u_3\|_{L^2_t} \lesssim \|u_1\|_{X_{s,b}} \|u_2\|_{X_{s,b}} \|u_3\|_{X_{s,b}}.
\]

Lemma 3.2 ([19]) Let \( l > \frac{3}{4}, c, c' = \frac{1}{2}+ \), then for any \( v_1, v_2 \in Y_{l,c} \),

\[
\|\partial_x(v_1v_2)\|_{Y_{l,c'-1}} \lesssim \|v_1\|_{Y_{l,c}} \|v_2\|_{Y_{l,c}}.
\]
Now we turn to prove other two bilinear estimates which improve the results in [3], [12]. Before stating the next lemma, we note an arithmetic fact that

\[(\tau + \xi^2) - (\tau_1 + \xi_1^2) - (\tau_2 - \xi_2^3) = \xi^2 - \xi_1^2 + \xi_2^2 = \xi_2(\xi_2^3 + \xi_2 + 2\xi_1),\]

if \(\xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2\). It implies that one of the following three cases always occur:

\[(a) |\tau + \xi^2| \gtrsim |\xi_2||\xi_2^2 + \xi_2 + 2\xi_1|; \quad (b) |\tau_1 + \xi_1^2| \gtrsim |\xi_2||\xi_2^2 + \xi_2 + 2\xi_1|;\]

\[(c) |\tau_2 - \xi_2^3| \gtrsim |\xi_2||\xi_2^2 + \xi_2 + 2\xi_1|. \tag{3.1}\]

**Lemma 3.3** Let \(l \geq -1, s - l \leq 2\) when \(s \geq 0, s + l \geq -2\) when \(s < 0\), and \(b, b', c = \frac{1}{2} + \),

then for any \(u \in X_{s,b}, v \in Y_{l,c}\),

\[\|uv\|_{X_{s,b',-1}} \lesssim \|u\|_{X_{s,b}} \|v\|_{Y_{l,c}}.\]

**Proof.** By duality and Plancherel’s identity, it suffices to show that for any \(f \in X_{0,1-b'}\),

\[\int_D \frac{(\xi)^s}{(\xi_1)^s(\xi_2)^s} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \|f\|_{X_{0,1-b'}} \|g\|_{X_{0,b}} \|h\|_{Y_{0,c}} \equiv RHS,\]

where the set \(D = \{(\xi_1, \xi_2, \tau_1, \tau_2) : \xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2\}\). We may assume that \(\hat{f}, \hat{g}, \hat{h}\) are nonnegative, otherwise we can replace them by their absolute value without lose of generality. This point of view will also be used at the following multilinear estimates without any mentioned. We divide the integral domain \(D\) into three parts by writing

\[\int_D = \int_{D_1} + \int_{D_2} + \int_{D_3},\]

where

\[D_1 = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi|, |\xi_1|, |\xi_2| \lesssim 1\},\]

\[D_2 = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi| \lesssim |\xi_1|, |\xi_1| \gg 1\},\]

\[D_3 = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi| \gg |\xi_1|, |\xi| \gg 1\}.\]

**Estimate in \(D_1\).** By Lemma 2.1 we have

\[\int_{D_1} \sim \int_{D_1} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \|f\|_{L_{2t}^2} \|g\|_{L_{2t}^2} \|h\|_{L_{2t}^2} \lesssim RHS.\]
**Estimate in** $D_2$. We shall split it into two cases,

\[(1) : s \geq 0; \quad (2) : s < 0.\]

For (1): $s \geq 0$, then

\[
\int_{D_2} \lesssim \int_{D_2} (\xi_2)^{-l} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2). \tag{3.2}
\]

We may assume that $|\xi_2| \geq 1$, otherwise it can be gotten as $\int_{D_1}$. Thus,

\[
(3.2) \sim \int_{D_2} |\xi_2|^{-l} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2).
\]

We divide $D_2$ again into two subregions:

\[D_{21} = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D_2 : |\xi_2^2 + \xi_2 + 2\xi_1| \ll |\xi_2|^2 \}, \]

\[D_{22} = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D_2 : |\xi_2^2 + \xi_2 + 2\xi_1| \gg |\xi_2|^2 \}. \]

**Estimate in** $D_{21}$. Note that $|3\xi_2^2 + 2\xi_1| \sim |\xi_2|^2$ in $D_{21}$, therefore, $\int_{D_{21}}$ is equivalent to

\[
\int_{D_{21}} |\xi_2|^{-l-1} \cdot \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2)
\lesssim \|f\|_{L^2} \|g\|_{X_{0,\frac{1}{2}}} \|h\|_{Y_{0,\frac{1}{2}}} \lesssim \text{RHS},
\]

where we note that $l \geq -1$ in the second step.

**Estimate in** $D_{22}$. By (3.1), we can divide the integral domain into three parts again. But they are similar to each other, we just take (a): $|\tau + \xi^2| \gtrsim |\xi_2||\xi_2^2 + \xi_2 + 2\xi_1|$ for example, then since $l \geq -1$ and by Lemma 2.1, $\int_{D_{22}}$ is equivalent to

\[
\int_{D_{22}} (\tau + \xi^2)^{\frac{1}{2}} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \|f\|_{X_{0,\frac{1}{2}}} \|g\|_{L^4_{2t}} \|h\|_{L^4_{2t}} \lesssim \text{RHS}.
\]

Now we turn to consider the case (2): $s < 0$. We also further split $D_2$ into following two parts $D'_{21}$ and $D'_{22}$,

\[D'_{21} = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D_2 : |\xi| \sim |\xi_1| \}, \]

\[D'_{22} = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D_2 : |\xi| \ll |\xi_1| \sim |\xi_2| \}. \]
**Estimate in** $D_{21}'$. We can give the claim as $s \geq 0$.

**Estimate in** $D_{22}'$. We note that $|\xi_2^2 + \xi_2 + 2\xi_1| \sim |\xi_2|^2$, $m_1 \sim m_2 \sim |\xi_2|^2$. By (3.2), we split $D_{22}'$ into three parts again, but similarly we only consider (c): $|\tau_2 - \xi_2^3| \gtrsim |\xi_2||\xi_2^2 + \xi_2 + 2\xi_1|$, then by Lemma 2.3,

\[
\int_{D_{22}'} \sim \int_{D_{22}'} \langle \xi \rangle^s |\xi_2|^{-s-l} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \\
\lesssim \int_{D_{22}'} |\xi_2|^{-s-l} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \\
\lesssim \int_{D_{22}'} |\xi_2|^{-s-l-3c-s_1} m_3^{s_1} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \langle \tau_2 - \xi_2^3 \rangle c \hat{h}(\xi_2, \tau_2) \\
\lesssim I_3^{s_1} (\hat{f}, \hat{g}, \langle \tau - \xi^3 \rangle \hat{h}) \\
\lesssim RHS,
\]

where $s_1 = \frac{1}{2} -$, such that $1 - l > \frac{1}{4} + \frac{1}{2} s_1$ and $s + l \geq -3c - s_1$ (ensured by $s + l \geq -2$).

**Estimate in** $D_3$. We have,

\[
\int_{D_3} \sim \int_{D_3} \langle \xi_1 \rangle^{-s} |\xi_2|^{s-l} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2).
\]

It is much similar to $\int_{D_{22}'}$. Therefore, when $s \geq 0$, then

\[
\int_{D_3} \lesssim |\xi_2|^{-s-l} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2),
\]

thus we have the claim by noting that $s - l \leq 2$; when $s < 0$, since $|\xi_1| \lesssim |\xi_2|$, we have

\[
\int_{D_3} \lesssim \int_{D_3} |\xi_2|^{-l} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2),
\]

thus we have the claim again by noting $l \geq -2$.

In the next proof of lemma we will use the following algebraic relation

\[
(\tau - \xi^3) - (\tau_1 + \xi_1^2) - (\tau_2 - \xi_2^2) = -\xi^3 - \xi_1^2 + \xi_2^2 = -\xi(\xi^2 + \xi - 2\xi_2),
\]

if $\xi = \xi_1 + \xi_2$, $\tau = \tau_1 + \tau_2$. It implies that one of the following three cases must occur:

\[
(a) |\tau - \xi^3| \gtrsim |\xi||\xi^2 + \xi - 2\xi_2|; \quad (b) |\tau_1 + \xi_1^2| \gtrsim |\xi||\xi^2 + \xi - 2\xi_2|; \\
(c) |\tau_2 - \xi_2^3| \gtrsim |\xi||\xi^2 + \xi - 2\xi_2|.
\]

(3.3)
Lemma 3.4 Let $s > -\frac{1}{4}$, $l < 4s$, $l - s < 1$ when $s \geq 0$, $l - 2s < 1$ when $s < 0$, and $b, c' = \frac{1}{2} +$, then for any $u_1, u_2 \in X_{s,b}$, 

$$\|\partial_x(u_1 \overline{u_2})\|_{Y_{s,c'-1}} \lesssim \|u_1\|_{X_{s,b}} \|u_2\|_{X_{s,b}}.$$  

Proof. By duality and Plancherel’s identity and the fact $\hat{u}(\xi, \tau) = \overline{u}(-\xi, -\tau)$, it suffices to show that 

$$\int_D \frac{|\xi|}{(|\xi| s + |\xi_2|)^s} f(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \|f\|_{L^4_{\xi}} \|g\|_{L^4_{\xi}} \|h\|_{L^2_{\xi}} \equiv RHS,$$  

where the set $D = \{ (\xi_1, \xi_2, \tau_1, \tau_2) : \xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2 \}$. We may assume that $|\xi_1| \geq |\xi_2|$ (the other is similar). We divide the integration domain $D$ into three parts: 

$$D_1 = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi|, |\xi_1|, |\xi_2| \lesssim 1 \},$$  

$$D_2 = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi| \ll |\xi_1| \sim |\xi_2|, |\xi_1| \gg 1 \},$$  

$$D_3 = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi| \sim |\xi_1|, |\xi_1| \gg 1 \}.$$  

Estimate in $D_1$. By Lemma 2.1 we have 

$$\int_{D_1} \sim \int_{D_1} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \|f\|_{L^4_{\xi}} \|g\|_{L^4_{\xi}} \|h\|_{L^2_{\xi}} \lesssim RHS.$$  

Estimate in $D_2$. We have 

$$\int_{D_2} \sim \int_{D_2} |\xi|^{|\xi_1|}|\xi_2|^{-2s} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2)$$  

We divide $D_2$ again into two subparts: 

$$D_{21} = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D_2 : |\xi^2 + \xi - 2\xi_2| \ll |\xi_2| \},$$  

$$D_{22} = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D_2 : |\xi^2 + \xi - 2\xi_2| \gg |\xi_2| \}.$$  

Estimate in $D_{21}$. We note that $|\xi|^2 \sim |\xi_2|$ and $|3\xi^2 - 2\xi_2| \sim |\xi_2|$ in $D_{21}$, therefore, 

$$\int_{D_{21}} \sim \int_{D_{21}} |\xi_2|^{-2s + \frac{1}{2} + \frac{1}{2}} f(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \int_{D_{21}} |\xi_2|^{-2s + \frac{1}{2} + \frac{1}{2} + s} \cdot m_5^{s_2} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim I_5^{s_2}(\hat{f}, \hat{g}, \hat{h}) \lesssim \|f\|_{Y_{0,1-c'}} \|g\|_{L^2} \|h\|_{X_{s,b}^-} \lesssim RHS.$$
where we use Lemma 2.3 in the fourth step and choose $s_2 = \frac{1}{2} - c^\prime$, such that $1 - c^\prime > \frac{2}{9} + \frac{5}{9} s_2$ and $-2s + \frac{l+1}{2} - s_2 \geq 0$ (which is ensured by $l < 4s$).

**Estimate in $D_{22}$.** In $D_{22}$, all the cases of $|\xi|^2 \ll |\xi_2|^2$, $|\xi|^2 \sim |\xi_2|^2$ or $|\xi|^2 \gg |\xi_2|^2$, and (a), (b) or (c) in (3.3) maybe occur. Note that $|\xi|^2 + |\xi - 2\xi_2| \sim \max\{|\xi|^2, |\xi_2|^2\}$, we see that the worst case is: $|\xi_2| \gg |\xi|^2$ and $|\tau - \xi^3| \gg \max\{|\tau_1 + \xi_2^2|, |\tau_2 - \xi_2^2|\}$. We only consider the integration under this part (the others can be treated similarly but employing the estimates on $I_{5}^s$ or $I_{6}^s$ in Lemma 2.3). Then $|\tau - \xi^3| \gtrsim |\xi||\xi_2|$, and thus,

\[
\int_{D_{22}} \sim \int_{D_{22}} |\xi|^1 \langle \xi \rangle^l \langle \xi^2 \rangle^{2s-1-c^\prime} \langle \tau - \xi^3 \rangle^{1-c^\prime} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \\
\lesssim \int_{D_{22}} \langle \tau - \xi^3 \rangle^{1-c^\prime} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \\
\lesssim \|f\|_{Y_{0,1-c^\prime}} \|g\|_{L^4_{xt}} \|h\|_{L^4_{xt}} \\
\lesssim RHS,
\]

where we note $s > -\frac{1}{4}, l < 4s$ in the second step and use Lemma 2.1 in the fourth step.

**Estimate in $D_3$.** We have

\[
\int_{D_3} \sim \int_{D_3} |\xi|^{1+l-s} \langle \xi_2 \rangle^{-s} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2). \tag{3.4}
\]

We split it into two cases to analysis,

\[
(1): s \geq 0; \quad (2): s < 0.
\]

For (1): $s \geq 0$, then

\[
\tag{3.4} \lesssim \int_{D_3} |\xi|^{1+l-s} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2). \tag{3.5}
\]

In $D_3$, we have $|\xi|^2 + |\xi - 2\xi_2| \sim |\xi|^2, m_5 \sim m_6 \sim |\xi|^2$. By (3.3), we split the domain $D_3$ into three parts, but we only take (a) for example. Therefore, by Lemma 2.2,

\[
\tag{3.5} \lesssim \int_{D_3} |\xi|^{1+l-s+3c^\prime-\frac{3}{2}} \frac{m_4^\lambda}{m_4^\lambda} \langle \tau - \xi^3 \rangle^{1-c^\prime} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \\
\lesssim I_{7}^\lambda(\langle \tau - \xi^3 \rangle^{1-c^\prime} \hat{f}, \hat{g}, \hat{h}) \\
\lesssim RHS,
\]

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where we note that \( l - s < 1 \) and \( c' = \frac{1}{2} + \), thus \( l - s + 3c' - \frac{5}{2} \leq 0 \).

For (2): \( s < 0 \), since \( |\xi_2| \lesssim |\xi| \), we have,

\[
\text{(3.4)} \lesssim \int_{D_\delta} |\xi|^{1 + l - 2s} \hat{f}(\xi, \tau) \hat{g}(\xi, \tau_1) \hat{h}(\xi_2, \tau_2).
\]

Similar to (1), we have the claim since \( l - 2s < 1 \). \( \square \)

### 3.2 Local Well-posedness

Write the sets

\[
R_\beta \equiv \{(s, l) : s \geq 0\}; \quad R_{kdc} \equiv \{(s, l) : l > -\frac{3}{4}\};
\]

\[
R_\alpha \equiv \{(s, l) : l \geq -1; s - l \leq 2 \text{ when } s \geq 0, s + l \geq -2 \text{ when } s < 0\};
\]

\[
R_\gamma \equiv \{(s, l) : s > -\frac{1}{4}; l < 4s; l - s < 1 \text{ when } s \geq 0, l - 2s < 1 \text{ when } s < 0\},
\]

and let

\[
R_{\beta=0} \equiv R_{kdc} \cap R_\alpha \cap R_\gamma = \{(s, l) : l > -\frac{3}{4}, l < 4s, s - 2 \leq l < s + 1\};
\]

\[
R_{\beta \neq 0} \equiv R_{kdc} \cap R_\alpha \cap R_\gamma \cap R_\beta = \{(s, l) : l > -\frac{3}{4}, s \geq 0, l < 4s, s - 2 \leq l < s + 1\}.
\]

We assume that \((s, l) \in R_{\beta=0}\) in the case of \( \beta = 0 \) and \((s, l) \in R_{\beta \neq 0}\) in the case of \( \beta \neq 0 \).

Define the maps

\[
\Phi_1(u, v) = \psi(t)S(t)u_0 - i\psi(t)\int_0^t S(t - t')\psi(t'/\delta) \left[ \alpha(uv)(t') + \beta(|u|^2u)(t') \right] \, dt',
\]

\[
\Phi_2(u, v) = \psi(t)W(t)v_0 + \psi(t)\int_0^t W(t - t')\psi(t'/\delta) \left[ \gamma \partial_x(|u|^2)(t') - \frac{1}{2} \partial_x(v^2)(t') \right] \, dt',
\]

where \( S(t), W(t) \) are \( U_\phi(t) \) with \( \phi = \xi^2, -\xi^3 \) respectively, then \((\Phi_1, \Phi_2)\) is locally well-posed on \([-\delta, \delta]\) only if \((\Phi_1, \Phi_2)\) has a unique fixed point. By Lemma 2.4, Lemmas 3.1–3.4, the fixed point theory and a standard process (see \cite{12} cf.), we prove Theorem 1.1 with the estimates on the lifetime and solutions that for \( \mu = \max\{b' - b, c' - c\} > 0 \),

\[
\delta \sim (\|u_0\|_{H^s} + \|v_0\|_{H^i})^{-\mu}; \quad \|u\|_{X^s_{\frac{1}{2}+}} + \|v\|_{Y^s_{\frac{1}{2}+}} \lesssim \|u_0\|_{H^s} + \|v_0\|_{H^i},
\]

or

\[
\delta \sim \min \left\{ \|u_0\|_{H^s}, \|v_0\|_{H^i}, \frac{\|v_0\|_{H^i}^2}{\|u_0\|_{H^s}} \right\}^{-\mu}; \quad \|u\|_{X^s_{\frac{1}{2}+}} \lesssim \|u_0\|_{H^s}, \|v\|_{Y^s_{\frac{1}{2}+}} \lesssim \|v_0\|_{H^i}.
\]
Remark. From the proof of Lemma 3.3 and Lemma 3.4, more general conditions for the local well-posedness region (see Figure 1) on the top-left and bottom-right areas are

\[ 3c + \frac{3}{2} - 2b' < l - s \leq \frac{5}{2} - 3c', \]

for any \( c, c', b' \) large and suitable close to \( \frac{1}{2} \), and \( c' > c \). However, one always has the restriction that

\[ \text{if } b < l - s \leq a, \quad \text{then } a + b \leq 3. \]

This implies that the well-posedness region is contained in a belt with the distance of 3.

4 The Proof of Theorem 1.3

In this section, we consider the global well-posedness of the solutions obtained in Theorem 1.1 when \( \alpha \gamma > 0 \) and \((u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})\) for some \( s > 0 \). We assume that \( v \) is real valued from now. In this paper, we pursue to lengthen the lifetime by some special techniques and a useful conservation law on \( L^2 \)-norm of \( u \). We do nothing on the almost conserved quantities but cite what obtained in [22] directly.

4.1 Some Variant Multilinear Estimates

Now we turn to establish some special multilinear estimates, which are useful in the next subsection although there is a bit cumbersome in some.

We will use the following two inequalities frequently in the multilinear estimates below, which follow from Lemma 2.4 (iv) and Lemma 6.3. They are,

\[ \|f\|_{X_{s,b}^\delta(\phi)} \lesssim \delta^{\frac{1}{2} - b} \|f\|_{X_{s-\frac{1}{2}}^\delta(\phi)}; \quad \|\psi(t/\delta)f\|_{X_{s,b}^\delta(\phi)} \lesssim \delta^{b'-b} \|f\|_{X_{s,b}^\delta(\phi)} \tag{4.1} \]

for \( b, b' \in [0, \frac{1}{2}] \) with \( b' \geq b \).

Lemma 4.1 Let \( s \geq 0, c = \frac{1}{2} +, \delta \in (0, 1) \), then for any \( v_1, v_2 \in Y_{s,c}^\delta \),

\[ \|\psi(t/\delta)\partial_x (\tilde{v}_1 \tilde{v}_2)\|_{Y_{s-1,c}} \lesssim \delta^{\frac{1}{2} -} \|v_1\|_{Y_{s,c}^\delta} \|v_2\|_{Y_{s,c}^\delta}, \]

where \( \tilde{v}_1, \tilde{v}_2 \) are the extensions of \( v_1|_{t \in [-\delta, \delta]} \) and \( v_2|_{t \in [-\delta, \delta]} \) such that \( \|v_1\|_{Y_{s,c}^\delta} = \|\tilde{v}_1\|_{Y_{s,c}}, \|v_2\|_{Y_{s,c}^\delta} = \|\tilde{v}_2\|_{Y_{s,c}}. \)
Proof. By duality and Plancherel's identity, it suffices to show that for any $f \in Y_{0,1-c}$,

$$\int_{D} \frac{|\xi|}{|\xi\rangle} \frac{\langle \xi \rangle}{|\xi\rangle} \mathcal{F}_0(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \delta^{\frac{1}{2}} \|f\|_{Y_{0,1-c}} \|g\|_{Y_{0,c}} \|h\|_{Y_{0,c}} \equiv \text{RHS},$$

where the set $D = \{ (\xi_1, \xi_2, \tau_1, \tau_2) : \xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2 \}$, $f_0 = \psi(t/\delta)f$, and

$$\hat{g}(\xi, \tau) = \langle \xi \rangle^{s} \hat{v}_1(\xi, \tau); \quad \hat{h}(\xi, \tau) = \langle \xi \rangle^{s} \hat{v}_2(\xi, \tau).$$

Further, since $s \geq 0$, it is sufficient to show

$$\int_{D} |\xi| \mathcal{F}_0(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \text{RHS}.$$  

We may assume that $|\xi_1| \geq |\xi_2|$ by symmetry and write

$$\int_{D} = \int_{D_1} + \int_{D_2},$$

where

$$D_1 = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi| \lesssim |\xi_2| \}; \quad D_2 = \{ (\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi| \ll |\xi_2| \sim |\xi_1| \}.$$  

First, by the arithmetic fact

$$(\tau - \xi^3) - (\tau_1 - \xi_1^3) - (\tau_2 - \xi_2^3) = -3\xi_1 \xi_2,$$

we can split $D_1$ into three parts:

(a) $|\tau - \xi^3| \gtrsim |\xi||\xi_1||\xi_2|$; (b) $|\tau_1 - \xi_1^3| \gtrsim |\xi||\xi_1||\xi_2|$; (c) $|\tau_2 - \xi_2^3| \gtrsim |\xi||\xi_1||\xi_2|.$

We just take (a) for example, since the other two are similar, then $\int_{D_1}$ is controlled by

$$\int_{D_1} (\tau - \xi^3)^{\frac{3}{2}} \mathcal{F}_0(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \|f_0\|_{Y_{0,\frac{3}{2}}} \|g\|_{Y_{0,1}} \|h\|_{L^2_t} \lesssim \text{RHS}$$

by Lemma 2.1 and (4.1). Second, note that $\xi_1 \cdot \xi_2 < 0$, by Lemma 2.2 and (4.1), we have

$$\int_{D_2} \lesssim \|f_0\|_{L^2_t} \|g\|_{Y_{0,\frac{3}{2}}} \|h\|_{Y_{0,1}} \lesssim \text{RHS}.$$

This completes the proof of the lemma. $\square$

By a general result in [11], Lemma 6.1 and Lemma 4.1, we have
Corollary 4.2 Let $I = I_{N,s}, s \geq 0, c = \frac{1}{2} +, \delta \in (0, 1)$, then for any $v_1, v_2 \in Y^\delta_{s,c},$

$$\|\psi(t/\delta) \partial_x I(\tilde{v}_1 \tilde{v}_2)\|_{Y_{1,c}^{-1}} \lesssim \delta^{\frac{1}{2} -} \cdot \|Iv_1\|_{Y^\delta_{1,c}} \cdot \|Iv_2\|_{Y^\delta_{1,c}}.$$ 

where $\tilde{v}_1, \tilde{v}_2$ are same with Lemma 4.1.

Lemma 4.3 Let $s \geq 0, b, c = \frac{1}{2} +, \delta \in (0, 1)$, then for any $u \in X^\delta_{s,b}, v \in Y^\delta_{s,c},$

$$\|\psi(t/\delta) \tilde{u} \tilde{v}\|_{X_{s,b-1}} \lesssim \delta^{\frac{13}{16} -} \cdot \|u\|_{X^\delta_{s,b}} \cdot \|v\|_{Y^\delta_{s,c}},$$

where $\tilde{u}, \tilde{v}$ are the extensions of $u|_{t \in [-\delta, \delta]}$ and $v|_{t \in [-\delta, \delta]}$ such that $\|u\|_{X^\delta_{s,b}}, \|v\|_{Y^\delta_{s,c}} = \|\tilde{u}\|_{Y_{s,c}},$.

Proof. Since $s \geq 0$, it suffices to prove that for any $f \in X_{0,1-b},$

$$\int_D \tilde{f}_\delta(\xi, \tau) \tilde{g}(\xi_1, \tau_1) \tilde{h}(\xi_2, \tau_2) \lesssim \delta^{\frac{13}{16} -} \cdot \|f\|_{X_{0,1-b}} \cdot \|g\|_{X_{0,b}} \cdot \|h\|_{Y_{0,c}} \equiv RHS, \quad (4.2)$$

where the set $D = \{(\xi_1, \xi_2, \tau_1, \tau_2) : \xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2\}, f_\delta = \psi(t/\delta)f,$ and

$$\tilde{g}(\xi, \tau) = \langle \xi \rangle^s \tilde{u}(\xi, \tau); \quad \tilde{h}(\xi, \tau) = \langle \xi \rangle^s \tilde{v}(\xi, \tau).$$

But the left hand side of (4.2) is controlled by

$$\|f_\delta\|_{L^2_{st}} \cdot \|g\|_{L^\frac{8}{3}_{st}} \cdot \|h\|_{L^\frac{8}{5}_{st}} \lesssim \|f_\delta\|_{X_{0,0}} \cdot \|g\|_{X_{0,\frac{1}{3} +}} \cdot \|h\|_{Y_{0,\frac{1}{4} +}} \lesssim RHS,$$

by using Lemma 2.1 and (4.1).

Again, by the general result in [11], Lemma 6.1 and Lemma 4.3, we have

Corollary 4.4 Let $I = I_{N,s}, s \geq 0, b, c = \frac{1}{2} +, \delta \in (0, 1)$, then for any $u \in X^\delta_{s,b}, v \in Y^\delta_{s,c},$

$$\|\psi(t/\delta) I(\tilde{u} \tilde{v})\|_{X_{1,b-1}} \lesssim \delta^{\frac{13}{16} -} \cdot \|Iu\|_{X^\delta_{1,b}} \cdot \|Iv\|_{Y^\delta_{1,c}},$$

where $\tilde{u}, \tilde{v}$ are same with Lemma 4.3.

Lemma 4.5 Let $I = I_{N,s}, s \geq 0, b = \frac{1}{2} +, \delta \in (0, 1)$, then for any $u \in X^\delta_{s,b},$

$$\|\psi(t/\delta) I(\tilde{u})\|_{X_{1,b-1}} \lesssim \delta^{\frac{13}{16} -} \cdot \|Iu\|_{X^\delta_{1,b}} \cdot \|u\|_{X^\delta_{0,b}},$$

where $\tilde{u}$ is the extension of $u|_{t \in [-\delta, \delta]}$ such that $\|u\|_{X^\delta_{s,b}} = \|\tilde{u}\|_{X_{s,b}}.$
Proof. By Lemma 2.4 (iv), it suffices to show that

\[ \|I(|u|^2u)\|_{X_{1,0}} \lesssim \|u\|_{X_{1,b}} \|u\|_{Y_{X_{1,b}}^2}^2 \]

for any \( u \in X_{s,b} \). Further, it is equivalent to

\[ \int_D m(\xi) \langle \xi \rangle f(\xi, \tau) \hat{u}(\xi_1, \tau_1) \hat{u}(\xi_2, \tau_2) \overline{\hat{u}}(-\xi_3, -\tau_3) \lesssim \|f\|_{L^2_{xt}} \|u\|_{X_{1,b}} \|u\|_{Y_{X_{1,b}}^2}^2 \] (4.3)

for any \( f \in L^2(\mathbb{R}^2) \), where \( D = \{(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) : \xi = \xi_1 + \xi_2 + \xi_3, \tau = \tau_1 + \tau_2 + \tau_3\} \).

Note that there is at least one of \( \xi_j, j = 1, 2, 3 \) such that \( |\xi| \lesssim |\xi_j| \). Without loss of generality, we assume that \( j = 1 \), then \( m(\xi)|\xi| \lesssim m(\xi_1)|\xi_1| \). Therefore, by Lemma 2.1, (4.3) is bounded by

\[ \int_D f(\xi, \tau) \langle \xi_1 \rangle \overline{\hat{u}}(\xi_1, \tau_1) \hat{u}(\xi_2, \tau_2) \overline{\hat{u}}(-\xi_3, -\tau_3) \lesssim \|f\|_{L^2_{xt}} \|(1 + D_x)Iu\|_{L^6_{xt}} \|u\|_{L^6_{xt}}^2 \lesssim RHS. \]

This completes the proof of the lemma.

\[ \square \]

Lemma 4.6 Let \( I = I_{N,s} \), \( s > 0, b, c = \frac{1}{2} +, \delta \in (0, 1) \), then for any \( u \in X_{s,b}^\delta \),

\[ \|\psi(t/\delta) \partial_x I(|\tilde{u}|^2)\|_{Y_{1,c-1}} \lesssim \delta^a \|u\|_{X_{1,b}^\delta} \|u\|_{X_{1,b}^\delta} + (N^{-1}\delta^a + \delta^{13}\delta^{-1}) \cdot \|u\|_{X_{1,b}^\delta}^2. \]

where \( a = \left( \frac{13}{16} - \frac{1}{3} \right) \), and \( \tilde{u} \) is the extension of \( u|_{\xi \in [-\delta, \delta]} \) such that \( \|u\|_{X_{s,b}^\delta} = \|\tilde{u}\|_{X_{s,b}} \).

Proof. By duality and Plancherel’s identity, it suffices to show that for any \( f \in Y_{0,1-c} \),

\[ \int_D \langle \xi \rangle m(\xi) f(\xi, \tau) \hat{u}(\xi_1, \tau_1) \overline{\hat{u}}(-\xi_2, -\tau_2) \lesssim \|f\|_{Y_{0,1-c}} \left( \delta^{\alpha} \|u\|_{X_{1,b}^\delta} \|u\|_{X_{1,b}^\delta} + (N^{-1}\delta^a + \delta^{13}\delta^{-1}) \|u\|_{X_{1,b}^\delta}^2 \right) \equiv RHS, \]

where the set \( D = \{(\xi_1, \xi_2, \tau_1, \tau_2) : \xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2\} \), \( f_\delta = \psi(t/\delta)f \). We restrict in \( D \) that \( |\xi_1| \geq |\xi_2| \) (the other is similar) and divide it into the following four parts:

\[ D_1 = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi|, |\xi_1|, |\xi_2| \lesssim N\}; \]
\[ D_2 = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi| \lesssim N, |\xi_1| \sim |\xi_2| \gg N\}; \]
\[ D_3 = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi_2| \lesssim N, |\xi| \sim |\xi_1| \gg N\}; \]
\[ D_4 = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D : |\xi|, |\xi_1|, |\xi_2| \gg N\}. \]
**Estimate in $D_1$.** Note that $m(\xi), m(\xi_1), m(\xi_2) \sim 1$ in $D_1$, thus,

$$\int_{D_1} \sim \int_{D_1} |\xi| \langle \xi \rangle \hat{f}_\delta(\xi, \tau) \hat{u}(\xi_1, \tau_1) \overline{\hat{u}}(-\xi_2, -\tau_2).$$

We divide $D_1$ into two parts again:

$$D_{11} = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D_1 : |\xi| \lesssim |\xi_2|\};$$

$$D_{12} = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D_1 : |\xi| \gg |\xi_2|\}.$$

**Estimate in $D_{11}$.** By (2.1), (2.2) and (4.1), we have

$$\int_{D_{11}} \lesssim \int_{D_{11}} \langle \xi_1 \rangle \langle \xi_2 \rangle \hat{f}_\delta(\xi, \tau) \hat{I}\bar{u}(\xi_1, \tau_1) \overline{\hat{u}}(-\xi_2, -\tau_2)$$

$$\lesssim \|f_\delta\|_{L^{p_1}_x} \|(1 + D_x)I\bar{u}\|_{L^{p_1}_x}^2 \lesssim \delta^{13\over 16} \|f\|_{Y_{0,1-c}} \|I\bar{u}\|_{X_{1,b}}^2,$$

where $1 + {2 \over p} = 1$ such that $1 - c > {4 \over 3} \left(1 - {1 \over p} - {1 \over q}\right)$.

**Estimate in $D_{12}$.** By (3.3), we further split $D_{12}$ into three parts, but each part is similar, we only take (a) for example, then $|\tau - \xi^3| \gtrsim |\xi|^3$ and $|\xi| \sim |\xi_1|$, thus we have

$$\int_{D_{12}} \lesssim \int_{D_{12}} \langle \tau - \xi^3 \rangle \langle \xi_1 \rangle \hat{f}_\delta(\xi, \tau) \hat{I}\bar{u}(\xi_1, \tau_1) \overline{\hat{u}}(-\xi_2, -\tau_2)$$

$$\lesssim \left\| F_{x_\tau}^{-1} \right\| \langle \tau - \xi^3 \rangle \langle \xi_1 \rangle \hat{f}_\delta \| (1 + D_x)I\bar{u}\|_{L^{q_1}_x} \|\tilde{u}\|_{L^{q_1}_x}$$

$$\lesssim \delta^a \|f\|_{Y_{0,1-c}} \|I\bar{u}\|_{X_{1,b}} \|\tilde{u}\|_{X_{0,b}},$$

where $1 + {2 \over q} = 1$ such that $1 - c - {1 \over 3} > {4 \over 3} \left(1 - {1 \over 2} - {1 \over q}\right)$.

**Estimate in $D_2$.** We will show at the following that

$$\int_{D_2} \lesssim \delta^{13\over 16} \cdot \|f\|_{Y_{0,1-c}} \|I\bar{u}\|_{X_{1,b}}^2.$$

Indeed, it is sufficient if we show

$$\int_{D_2} \xi \langle \xi \rangle \langle \xi_1 \rangle \langle \xi_2 \rangle \cdot m(\xi) \cdot m(\xi_1)m(\xi_2) \cdot \hat{f}_\delta(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \delta^{13\over 16} \|f\|_{Y_{0,1-c}} \|g\|_{X_{0,b}} \|h\|_{X_{0,b}} , \quad (4.4)$$

where $\hat{g} = \hat{I}\bar{u}, \hat{h} = \hat{I}\bar{u}$. The left hand side of (4.4) is equivalent to

$$N^{2s-2} \int_{D_2} |\xi| \langle \xi \rangle \langle \xi_1 \rangle \langle \xi_2 \rangle \cdot m(\xi) \cdot m(\xi_1)m(\xi_2) \cdot \hat{f}_\delta(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \int_{D_2} \hat{f}_\delta(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2)$$

$$\lesssim \delta^{13\over 16} \|f\|_{Y_{0,1-c}} \|g\|_{X_{0,b}} \|h\|_{X_{0,b}}.$$
by the same way used in $\int_{D_{11}}$ in the last step.

**Estimate in $D_3$.** We have

$$\int_{D_3} \sim \int_{D_3} |\xi| \hat{f}_\delta(\xi, \tau) \langle \xi_1 \rangle \hat{u}(\xi_1, \tau_1) \hat{u}(\xi_2, -\tau_2).$$

By the same manner used in $\int_{D_{12}}$, we have

$$\int_{D_3} \lesssim \delta^a \|f\|_{Y_{0,1-c}} \|I\hat{u}\|_{X_{1,b}} \|\hat{u}\|_{X_{0,b}}.$$ 

**Estimate in $D_4$.** We will show in the following that

$$\int_{D_4} \lesssim N^{-1} \delta^a \cdot \|f\|_{Y_{0,1-c}}^2 \|I\hat{u}\|_{X_{1,b}}^2.$$ 

In fact, it suffices to show

$$\int_{D_4} \frac{\xi(\xi)}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \cdot \frac{m(\xi)}{m(\xi_1)m(\xi_2)} \cdot \hat{f}_\delta(\xi, \tau) \hat{g}(\xi, \tau) \hat{h}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim N^{-1} \delta^a \|f\|_{Y_{0,1-c}} \|g\|_{X_{0,b}} \|h\|_{X_{0,b}^{-}}.$$ 

The left hand side is controlled by

$$N^{s-1} \int_{D_4} |\xi|^{s+1} |\xi_1|^{-s} |\xi_2|^{-s} \hat{f}_\delta(\xi, \tau) \hat{g}(\xi, \tau) \hat{h}(\xi, \tau) \hat{h}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2).$$

We divide $D_4$ into two subregions again:

$$D_{41} = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D_4 : |\xi^2 + \xi - 2\xi_2| \ll |\xi|^2\};$$

$$D_{42} = \{(\xi_1, \xi_2, \tau_1, \tau_2) \in D_4 : |\xi^2 + \xi - 2\xi_2| \gtrsim |\xi|^2\}.$$

**Estimate in $D_{41}$.** Note that $|\xi_2| \sim |\xi|^2$ and $|3\xi^2 - \xi_2| \sim |\xi|^2$ in $D_{41}$, then by Lemma 2.3, we have

$$\int_{D_{41}} \lesssim N^{s-1} \int_{D_{41}} |\xi|^{1-3s} \hat{f}_\delta(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2)$$

$$\lesssim N^{s-1} N^{-1} \delta^a \|f\|_{Y_{0,1-c}} \|g\|_{X_{0,b}} \|h\|_{X_{0,b}^{-}}.$$ 

note that $s > 0$ in the second step. Since $N^{(2s-1)} \delta^{\frac{1}{2}} \lesssim N^{-1} \delta^a$, we obtain the claim.
Estimate in $D_{42}$. By (3.3), we can split $D_{42}$ again into three parts, as above, we only consider (a): $|\tau - \xi^3| \gtrsim |\xi|^3$. First we have (since $|\xi| \leq 2|\xi_1|$),

$$\int_{D_{42}} \lesssim N^{-1} \int_{D_{42}} |\xi| \hat{f}_3(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2).$$

(4.5)

Further,

$$\int_{D_{42}} \lesssim N^{-1} \int_{D_{42}} (\tau - \xi^3)^{\frac{1}{2}} \hat{f}_3(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2)$$

$$\lesssim N^{-1} \delta^a \|f\|_{Y_{0,1-c}} \|g\|_{X_{0,\delta}} \|h\|_{X_{0,\delta}},$$

by the same way used in $\int_{D_{12}}$ in the last step. \hfill \square

### 4.2 Some Variant Local Well-posedness

Now we turn to obtain a variant local well-posedness result. Compared with the standard local well-posedness result Theorem 1.1, it is established in order to fit the $I$-method. It gives the estimates on the lifetime and the solutions under the $X^{\delta}_{\mathcal{A},\frac{1}{2}^+}$-norm, with the operator $I_{N,s}$. Indeed, along the lines of [11] and the estimates from Lemma 3.1–lemma 3.4, we have the following result as an adaptation of Theorem 1.1 (see [10], [22] cf.).

**Corollary 4.7** Let $s > 0, I = I_{N,s}$, then the solutions obtained in Theorem 1.1 for the initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ exist on $[-\delta_0, \delta_0]$ with

$$\delta_0 \sim (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^{-\mu}; \quad \|Iu\|_{X^{s_0}_{1, \frac{1}{2}^+}} + \|Iv\|_{Y^{s_0}_{1, \frac{1}{2}^+}} \lesssim \|Iu_0\|_{H^1} + \|Iv_0\|_{H^1},$$

for some $\mu > 0$.

But we have no intention of exploiting it as our basic of the iteration to establish the global well-posedness results. In order to extend the lifetime, we shall reconstruction it and ultimately establish the refined local result as follows.

**Proposition 4.8** Let $s > 0, I = I_{N,s}$, then the solutions obtained in Corollary 4.7 exists on $[-\delta, \delta]$ with

$$\delta \sim (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^{-2},$$

(4.6)
when $N \gg N_0$ for some large number $N_0$ such that

$$\gamma N_0^{-2}(\|I_{N_0,s}u_0\|_{H^1} + \|I_{N_0,s}v_0\|_{H^1}) \sim 1.$$  \hspace{1cm} (4.7)

Moreover, the solutions satisfy

$$\|Iu\|_{X^s_{1,\frac{1}{2}+}} + \|Iv\|_{Y^s_{1,\frac{1}{2}+}} \lesssim \|u_0\|_{H^1} + \|v_0\|_{H^1}. \hspace{1cm} (4.8)$$

In the following text, we may assume that $\|u(t)\|_{L^2_t} \sim 1$ by fixing $u_0$ and the $L^2$-mass conservation: $\|u(t)\|_{L^2_t} = \|u_0\|_{L^2}$. Further, by local result in Corollary 4.7 and the iteration, one can conclude the existence of solutions on $[-\delta, \delta]$ in Proposition 4.8 by the estimate in (4.8), which implies a priori estimate of the solutions in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. Therefore, to prove Proposition 4.8, we may assume at the beginning that the solutions exist on the time interval $[-\delta, \delta]$ with the $\delta$ defined in (4.6), and turn to prove (4.8).

**Lemma 4.9** Let $s > 0$, assume that $(u, v)$ are the solutions of (1.1) on $[-\delta, \delta]$ for small $\delta > 0$ with the initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, then if it satisfies that

$$\alpha \delta^{\frac{11}{12}} \cdot \|v\|_{Y^s_{1,\frac{1}{2}+}}^{\frac{3}{2}} \leq \epsilon_0$$

for some small $\epsilon_0$, we have

$$\|u\|_{X^s_{0,\frac{1}{2}+}} \lesssim \|u_0\|_{L^2}. \hspace{1cm} (4.10)$$

**Proof.** Define the operator

$$\Phi(u)(x, t) = \psi(t)S(t)u_0 - i\psi(t) \int_0^t S(t-t') \psi(t'/\delta) \left[ \alpha(uv)(t') + \beta(|u|^2u)(t') \right] dt'. \hspace{1cm} (4.11)$$

Taking $X^s_{0,\frac{1}{2}+}$ on the two sides of (4.11), and by Lemma 2.4, Lemma 3.1, Lemma 4.3 (when $s = 0$), we have

$$\|\Phi(u)\|_{X^s_{0,\frac{1}{2}+}} \lesssim \|u_0\|_{L^2} + \alpha \|\psi(t/\delta) \tilde{u}v\|_{X_{0,-\frac{1}{2}+}} + \beta \|\psi(t/\delta) \, \tilde{u} |u|^2u\|_{X_{0,-\frac{1}{2}+}}$$

$$\lesssim \|u_0\|_{L^2} + \alpha \frac{5\delta^{13}}{12} \|u\|_{X^s_{0,\frac{1}{2}+}} \|v\|_{Y^s_{0,\frac{1}{2}+}} + \beta \frac{\delta^{13}}{2} \|u\|_{X^s_{0,\frac{1}{2}+}}^3,$$

where $\|u\|_{X^s_{0,0}} = \|\tilde{u}\|_{X_{0,0}}$, $\|v\|_{Y^s_{0,0}} = \|\tilde{v}\|_{Y_{0,0}}$. Thus, by (4.9), we have

$$\|\Phi(u)\|_{X^s_{0,\frac{1}{2}+}} \leq c \|u_0\|_{L^2} + C \left( \epsilon_0 \|u\|_{X^s_{0,\frac{1}{2}+}} + \beta \delta^{\frac{13}{14}} \|u\|_{X^s_{0,\frac{1}{2}+}}^3 \right) \hspace{1cm} (4.12)$$
for some large constants $c, C > 0$. Let the ball $B \in X^0_{\delta, \frac{1}{2}^+}$ be defined as

$$B = \left\{ u \in X^0_{\delta, \frac{1}{2}^+} : \| u \|_{X^0_{\delta, \frac{1}{2}^+}} \leq 2c\| u_0 \|_{L^2} \right\},$$

then by (4.12), we have $\Phi$ maps $B$ into itself. We also have the contraction of $\Phi$ by a similar way. Thus we complete the proof of the lemma by the fixed point theory.

**Proof of Proposition 4.8.** By Duhamel’s formula and acting the operator $I = I_{N,s}$, we have, for $t \in [-\varrho, \varrho]$,

$$iu(x, t) = \psi(t) e^{i\varrho(t)} + \int_0^t \psi(t') e^{i\varrho(t')} \left[ \alpha I(\varrho)(t') + \beta I(\varrho^2)(v(t')) \right] dt',$$

$$iv(x, t) = \psi(t) e^{i\varrho(t)} + \int_0^t \psi(t') e^{i\varrho(t')} \left[ \gamma \partial_x I(\varrho^2)(t') - \frac{1}{2} \partial_x I(\varrho^2)(t') \right] dt'.$$

Therefore, by Lemma 2.4, Corollarys 4.2, 4.4 and Lemmas 4.5, 4.6, we have

$$\|Iu\|_{X^\varrho_{1, \frac{1}{2}^+}} \leq \|Iu_0\|_{H^1} + \alpha \|\psi(t/\varrho)I(\varrho)\|_{X^\varrho_{1, \frac{1}{2}^+}} + \beta \|\psi(t/\varrho)I(\varrho^2)(\varrho)\|_{X^\varrho_{1, \frac{1}{2}^+}}$$

$$\leq c\|Iu_0\|_{H^1} + C\varrho^{\frac{3}{16}} \|Iu\|_{X^\varrho_{1, \frac{1}{2}^+}} \|iv\|_{Y^\varrho_{1, \frac{1}{2}^+}}$$

$$+ C\beta \varrho^{\frac{1}{2}} \|Iu\|_{X^\varrho_{1, \frac{1}{2}^+}} \|iv\|_{X^\varrho_{0, \frac{1}{2}^+}};$$

$$\|iv\|_{Y^\varrho_{1, \frac{1}{2}^+}} \leq \|iv_0\|_{H^1} + \gamma \|\psi(t/\varrho)\partial_x I(\varrho^2)(\varrho)\|_{Y^\varrho_{1, \frac{1}{2}^+}} + \|\psi(t/\varrho)\partial_x I(\varrho^2)(\varrho)\|_{Y^\varrho_{1, \frac{1}{2}^+}}$$

$$\leq c\|iv_0\|_{H^1} + C\gamma \left( \varrho^a \|Iu\|_{X^\varrho_{1, \frac{1}{2}^+}} \|uv\|_{X^\varrho_{0, \frac{1}{2}^+}} \right) + (N^{-1} \varrho^a + \varrho^{\frac{13}{16} -}) \|Iu\|_{X^\varrho_{1, \frac{1}{2}^+}} + C\varrho^{\frac{1}{2}} \|iv\|_{Y^\varrho_{0, \frac{1}{2}^+}}$$

(4.13)

(4.14)

for some constants $c, C > 0$. Let $\delta$ be the quantity satisfying

$$\alpha \delta^{\frac{13}{16} -} R(N) \leq \epsilon_0; \quad \beta \delta^{\frac{1}{2}} \|u_0\|_{L^2}^2 \leq \epsilon_0;$$

$$\gamma \delta^a \|u_0\|_{L^2} \leq \epsilon_0; \quad \gamma N^{-1} \delta^a R(N) \leq \epsilon_0; \quad \gamma \delta^{\frac{13}{16} -} R(N) \leq \epsilon_0; \quad \delta^{\frac{1}{2}} - R(N) \leq \epsilon_0$$

(4.15)

for some small $\epsilon_0$ and $R(N) = \|Iu_0\|_{H^1} + \|iv_0\|_{H^1}$. We claim that for any $\varrho \in [0, \delta]$,

$$\|Iu\|_{X^\varrho_{1, \frac{1}{2}^+}} + \|iv\|_{Y^\varrho_{1, \frac{1}{2}^+}} \leq 2cR(N).$$

(4.16)

Indeed, it can be shown by the iteration which we present as follows. We only consider the positive time, since it is similar to the negative time. First, by Corollary 4.7, we have

$$\|Iu\|_{X^\varrho_{0, \frac{1}{2}^+}} + \|iv\|_{Y^\varrho_{0, \frac{1}{2}^+}} \leq 2cR(N),$$

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then by Lemma 2.4 (i), we obtain

\[ R_1(N) \equiv \|I u(\delta_0)\|_{H^1} + \|I v(\delta_0)\|_{H^1} \leq 2c\bar{c}R(N) \]

for some constant \( \bar{c} > 0 \). Now we take \((u(\delta_0), v(\delta_0))\) for the new initial data, and employ Corollary 4.7 again, then we obtain that for some \( \delta_1 \sim \delta_0 \),

\[ \|I u\|_{X^{1,1/2}} + \|I v\|_{Y^{1,1/2}} \leq 2c(1 + 2c\bar{c})R(N). \quad (4.17) \]

By (4.15) and (4.17), we have (4.9), and thus we obtain (4.10) by Lemma 4.9. Therefore, inserting (4.10) and (4.17) into (4.13) and (4.14), we have exactly

\[ \|I u\|_{X^{1,1/2}} + \|I v\|_{Y^{1,1/2}} \leq 2cR(N). \]

The process above can always be repeated under (4.15), and ultimately we prove the claim (4.16) and obtain the proposition.

\[ \Box \]

**4.3 The Global Well-posedness**

In this section, we establish the global result by combining Proposition 4.8 and the results in [22]. We only consider the positive time in the following. Compared to the process in Section 6 in [22], it just needs to modify the estimate on the lifetime. Define

\[ M_I(t) = \|I u\|_{L^2}; \]
\[ L_I(t) = \alpha\|I v\|_{L^2}^2 + 2\gamma \int \text{Im}(I u I\bar{u}_x) \, dx; \]
\[ E_I(t) = \alpha\gamma \int I v \, |I u|^2 \, dx + \gamma\|I u_x\|_{L^2}^2 + \frac{\alpha}{2}\|I v_x\|_{L^2}^2 - \frac{\alpha}{6}\|I v\|_{L^3}^3 + \frac{\beta\gamma}{2}\|I v\|_{L^4}^4, \]

then by the Sobolev interpolation inequalities, we have (see [22] for the details)

\[ \|I u\|_{H^1}^2 + \|I v\|_{H^1}^2 \lesssim |E_I| + |L_I|^\frac{3}{2} + |M_I|^8 + 1. \quad (4.18) \]

Moreover, we have the following estimates, which are proved in [22].
Lemma 4.10 Let $I = I_{N,s}$, $s > \frac{1}{2}$, $(u, v)$ is the solution of (1.1), then

$$|E_I(\delta) - E_I(0)| \lesssim \left( N^{-1+\frac{1}{2}} + N^{-\frac{7}{4}} \right) \left( \|I u\|_{H^1}^3 + \|I v\|_{Y^1}^3 \right) + N^{-2+}.$$

$$\left( \|I u\|_{X^1}^{\frac{4}{1}} + \|I v\|_{Y^1}^{\frac{4}{1}} \right) + N^{-3+} \|I u\|_{X^1}^{\frac{4}{1}} \left( \|I u\|_{Y^1}^{\frac{4}{1}} + \|I v\|_{Y^1}^{\frac{4}{1}} \right).$$

Lemma 4.11 Let $I, s, (u, v)$ be the same with Lemma 4.10, then

$$|L_I(\delta) - L_I(0)| \lesssim N^{-2+\frac{1}{2}} \left( \|I u\|_{H^1}^{\frac{3}{1}} + \|I v\|_{Y^1}^{\frac{3}{1}} \right) + N^{-3+} \|I u\|_{X^1}^{\frac{4}{1}}.$$

Further, we have the trivial estimate of $M_I(t)$ that

$$M_I(t) \lesssim \|u(t)\|_{L^2} \sim 1,$$

which follows from the $L^2$-mass conservation and $m(\xi) \leq 1$.

Fix the large number $N$, $s > \frac{1}{2}$. By Proposition 4.8, the solution $(u, v)$ of the system (1.1) exists on $[0, \delta]$, with

$$\delta \sim (\|I u_0\|_{H^1} + \|I v_0\|_{H^1})^{-2-} \gtrsim N^{-2(1-s)-}$$

by (1.4). We repeat this local existence results by iteration. In order to ensure the same length of the lifespan, we need to get the uniform control of $H^1$-norm of the solution at $t = k\delta$ for $k = 1, 2, \cdots$, which follows from the uniform control of $|E_I|$ and $|L_I|$. More precisely, we shall obtain that

$$|E_I(k\delta)| \leq cN^{2(1-s)}; \quad |L_I(k\delta)| \leq cN^{1-s}, \quad (4.19)$$

which imply by (1.18) that $\|I u(k\delta)\|_{H^1}^2 + \|I v(k\delta)\|_{H^1}^2 \leq cN^{2(1-s)}$ for the constants $c$, $\tilde{c}$ independent of $k, N$. We note that the condition (4.7) is valid in every step. By Lemmas 4.10, 4.11, and the estimate of (1.8) in each step, we have

$$|E_I(k\delta) - E_I(0)| \leq \tilde{c}k \left( \left( N^{-1+\frac{1}{2}} + N^{-\frac{7}{4}} \right) N^{3(1-s)} + N^{-2+}N^{4(1-s)} + N^{-3+}N^{6(1-s)} \right);$$

$$|L_I(k\delta) - L_I(0)| \leq \tilde{c}k \left( N^{-2+}N^{3(1-s)} + N^{-3+}N^{4(1-s)} \right)$$

for the constant $\tilde{c}$ independent of $k, N$. 28
Set $T = k\delta$, for (4.19), we only need to show

$$T\delta^{-1} \left( \left( N^{-1+\frac{1}{2}} N^{-\frac{7}{2}} + N^{-2+4(1-s)} + N^{-3+6(1-s)} \right) \right) \lesssim N^{2(1-s)}; \quad (4.20)$$

$$T\delta^{-1} \left( \left( N^{-2+\frac{1}{2}} N^{-3(1-s)} + N^{-3+4(1-s)} \right) \right) \lesssim N^{1-s}. \quad (4.21)$$

In order to $T \sim N^{0+}$, and note that $\delta \sim N^{-2(1-s)-}$, (4.20) is fulfilled if

$$N^{-1+N^{-1-s}}+N^{3(1-s)}+N^{-\frac{7}{2}+3(1-s)}+N^{-3+6(1-s)} \lesssim N^{0-},$$

which is valid if

$$-1 - (1 - s) + 3(1 - s) < 0; -\frac{7}{4} + 3(1 - s) < 0; -2 + 4(1 - s) < 0; -3 + 6(1 - s) < 0,$$

which hold when $s > \frac{1}{2}$. Similarly, (4.21) is fulfilled if

$$N^{-2+N^{-1-s}}+N^{3(1-s)}+N^{-3+4(1-s)} \lesssim N^{(s-1)-}.$$ 

It is valid if

$$-2 - (1 - s) + 3(1 - s) < s - 1; -3 + 4(1 - s) < s - 1,$$

they hold when $s > \frac{2}{5}$. Therefore, we prove the global well-posedness in $H^{s}(\mathbb{R}) \times H^{s}(\mathbb{R})$ when $s > \frac{1}{2}$ and thus finish the proof of Theorem 1.3.

## 5 The Proof of Theorem 1.2

Suppose for the contradiction that the system (1.1) is locally well-posed on $[0, \delta]$ for $\delta \in (0,1)$, and the solution map $(u_0, v_0) \mapsto (u, v)$ is $C^2$ from $H^{s}(\mathbb{R}) \times H^1(\mathbb{R})$ to $C^0([0, \delta]; H^{s}(\mathbb{R}) \times H^1(\mathbb{R}))$. Then, by the Picard iterative scheme, so is the operator $A = (A_1, A_2) : H^{s}(\mathbb{R}) \times H^1(\mathbb{R}) \to C^0([0, \delta]; H^{s}(\mathbb{R}) \times H^1(\mathbb{R}))$ defined as

$$A_1(u_0, v_0) = -i \int_0^t S(t-t') \left[ \alpha(S(t') u_0 \cdot W(t') v_0) + \beta(|S(t') u_0|^2 S(t') u_0) \right] dt';$$

$$A_2(u_0, v_0) = \int_0^t W(t-t') \left[ \gamma \partial_x(|S(t') u_0|^2)(t') - \frac{1}{2} \partial_x(W(t') v_0)^2 \right] dt'. $$

In particular, $A_2$ is $C^2$-differentiable from $H^s(\mathbb{R}) \times H^l(\mathbb{R})$ to $C^0([0, \delta]; H^l(\mathbb{R}))$. 

\[29\]
Fix a large number $N \gg 1$ such that $(N - \frac{1}{2})^3 = 2k\pi$ for some $k \in \mathbb{N}$, and let the sets
\[
\mathcal{Y} = \left\{ \xi \in \mathbb{R} : \left| (\xi + \frac{1}{2}) - N \right| \leq \frac{1}{100N^2} \right\};
\mathcal{Y}_1 = \left\{ \xi \in \mathbb{R} : \left| \xi - \left( N - \frac{1}{2}N^2 - \frac{3}{8} \right) \right| \leq \frac{1}{N} \right\};
\mathcal{Y}_2 = \left\{ \xi \in \mathbb{R} : \left| (2\xi + \frac{1}{4}) - N^2 \right| \leq \frac{1}{100N} \right\};
\Lambda = \left\{ \xi \in \mathbb{R} : |\xi - 1| \leq \frac{1}{N^n} \right\}
\]
for some $n \in \mathbb{N}$. Note that
\[
\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi = \xi_1 + \xi_2, \xi \in \mathcal{Y}, \xi_2 \in \mathcal{Y}_2 \} \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi = \xi_1 + \xi_2, \xi_1 \in \mathcal{Y}_1 \}.
\]
(5.1)

Put the initial data $(u_0, v_0)$ such that
\[
\widehat{u}_0(\xi) = \epsilon_0 N^{-2s+\frac{1}{2}} \chi_{\mathcal{Y}_1}(\xi); \quad \widehat{v}_0(\xi) = \epsilon_0 N^2 \chi_{\Lambda}(\xi),
\]
then $\|u_0\|_{H^s}, \|v_0\|_{H^s} \sim \epsilon_0$. We may set $\delta = 1$ by choosing $\epsilon_0$ small enough.

Further, $\|A_2\|_{C^0([0,1];H^s)}$ is equal to
\[
\begin{align*}
&\sup_{0 \leq t \leq 1} \left\| \int_0^t W(t-t') \left[ \gamma \partial_x(|S(t')u_0|^2) - \frac{1}{2} \partial_x(W(t')v_0)^2 \right] dt' \right\|_{H^s} \\
= &\sup_{0 \leq t \leq 1} \left\| \xi \langle \xi \rangle \int_0^t \int_0^{t'} \exp \left\{ i(t-t')\xi^3 \right\} \left[ \gamma \exp \left\{ -it' (\xi - \xi_2)^2 \right\} \exp \left\{ it' \xi_2^2 \right\} \widehat{u}_0(\xi - \xi_2) \widehat{u}_0(\xi_2) \\
&\quad - \frac{1}{2} \exp \left\{ it' (\xi - \xi_2)^3 \right\} \exp \left\{ it' \xi_2^3 \right\} \widehat{v}_0(\xi - \xi_2) \widehat{v}_0(\xi_2) \right\|_{L^2} \\
\geq &\gamma \left\| \xi \langle \xi \rangle \int_0^1 \int \exp \left\{ i\xi^3 \right\} \exp \left\{ -it\xi (\xi^2 + \xi - 2\xi_2) \right\} \widehat{u}_0(\xi - \xi_2) \widehat{u}_0(\xi_2) d\xi_2 dt \right\|_{L^2} \\
&\quad - \left\| \xi \langle \xi \rangle \int_0^1 \int \exp \left\{ i\xi^3 \right\} \exp \left\{ -3it\xi (\xi - 2\xi_2) \right\} \widehat{v}_0(\xi - \xi_2) \widehat{v}_0(\xi_2) d\xi_2 dt' \right\|_{L^2}.
\end{align*}
\]
(5.2)
The first term of (5.2) has a lower bound of
\[
\gamma N^{t+1} \left\| \int_0^1 \exp \left\{ i\xi^3 \right\} \exp \left\{ -it\xi (\xi^2 + \xi - 2\xi_2) \right\} \widehat{u}_0(\xi - \xi_2) \widehat{u}_0(\xi_2) d\xi_2 dt' \right\|_{L^2}[\mathcal{Y}].
\]
(5.3)

When $\xi \in \mathcal{Y}$, set $\xi = N - \frac{1}{2} + p(\xi, N)$, then $|p| \leq \frac{1}{100N^2}$. Therefore,
\[
\left| \xi^3 - \left( N - \frac{1}{2} \right)^3 \right| \leq 2N^2 \cdot |p| \leq \frac{1}{50}.
\]
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Besides, when $\xi \in \Upsilon, \xi_2 \in \Upsilon_2$, we have
\[
|\xi(\xi^2 + \xi - 2\xi_2)| \leq \frac{1}{10}.
\]
Note that $(N - \frac{1}{2})^3 = 2k\pi$, therefore, for any $0 \leq t' \leq 1$, we have,
\[
\text{Re} \left( \exp \left\{ i\xi^3 \right\} \exp \left\{ -it'\xi(\xi^2 + \xi - 2\xi_2) \right\} \right) > \frac{1}{2}. \tag{5.4}
\]
Thus, by (5.1) and (5.4), we obtain
\[
(5.3) \gtrsim \gamma \epsilon_0^2 N^{-l+1} N^{-4s+1} m(\Upsilon_2) m(\Upsilon)^{\frac{1}{2}} \sim \gamma \epsilon_0^2 N^{l-4s},
\]
where $m(\cdot)$ is the Lebesgue measure.

Second term of (5.2), by the support of $v_0$, have a bound of
\[
\left\| \int_0^1 \hat{v}_0(\xi - \xi_2) \hat{v}_0(\xi_2) d\xi_2 dt' \right\|_{L^2_\xi((\xi-\frac{2}{\pi\tau})} \lesssim \epsilon_0^2 N^{-\frac{4}{7}}.
\]
Therefore, by choosing $n$ large enough, we have
\[
\| A_2(u_0, v_0) \|_{C^l([0,1] ; H^s)} \gtrsim \gamma \epsilon_0^2 N^{l-4s}. \tag{5.5}
\]
Since $A_2$ is $C^2$-differentiable, we must have
\[
\| A_2(u_0, v_0) \|_{C^l([0,1] ; H^s)} \lesssim \| u_0 \|_{H^s}^2 + \| v_0 \|_{H^l}^2,
\]
but it fails to hold when $l > 4s$ by (5.5). This completes the proof of Theorem 1.2.

Remark. More facts related to the condition $l < 4s$ may be interesting to the readers. As we see, the condition appears in the bilinear estimate in Lemma 3.4, which is necessary in the framework of Bourgain method. But on the other hand, it is optimal. More precisely, if $l > 4s$, then for any $b_1, b_2, b_3 \in \mathbb{R}$, the estimates
\[
\| \partial_x(u_1 \overline{u}_2) \|_{Y_{b_1}} \lesssim \| u_1 \|_{X_{s,b_2}} \| u_2 \|_{X_{s,b_3}}
\]
fail to hold. The proof is based on the counterexample of $u_1, u_2$ such that
\[
\hat{u}_1(\xi, \tau) = \chi_{\Upsilon_1} \cdot \chi_{\{\tau - \xi^2 \leq 100\}}(\xi, \tau); \quad \hat{u}_2(\xi, \tau) = \chi_{\Upsilon_2} \cdot \chi_{\{\tau - \xi^2 \leq 10\}}(\xi, \tau),
\]
and specially take the integration in the left hand side over
\[
\{(\xi, \tau) : \xi \in \Upsilon, |\tau - \xi^2| \leq 10\}
\]
in the spacetime-frequency space. The detailed computation is omitted here.
6 Appendix: Some Auxiliary Lemmas

As some handy tools, we give some properties and estimates on the restricted spaces $X_{s,b}(\phi)$ in the following.

**Lemma 6.1** For the time interval $\Omega$ and the function $f \in X_{s,b}(\phi)$, there exists an extension $\tilde{f} \in X_{s,b}(\phi)$ such that $\tilde{f} = f$ on $\Omega$ and

$$\|f\|_{X_{s,b}(\phi)} = \|\tilde{f}\|_{X_{s,b}(\phi)}.$$  

Moreover, it holds that for any $s' \leq s$,

$$\|I_{N,s}f\|_{X_{1,b}(\phi)} = \|I_{N,s,\tilde{f}}\|_{X_{1,b}(\phi)}; \quad \|f\|_{X_{s',b}(\phi)} = \|\tilde{f}\|_{X_{s',b}(\phi)}.$$  \hspace{1cm} (6.1)

**Proof.** Fix the function $f \in X_{s,b}(\phi)$, and set

$$M_{s,b} = \left\{ F \in X_{s,b}(\phi) : F|_{t \in \Omega} = f|_{t \in \Omega} \right\},$$

then $\|f\|_{X_{s,b}^b(\phi)} = \inf_{F \in M_{s,b}} \|F\|_{X_{s,b}(\phi)}$. Since $X_{s,b}$ is a Hilbert space and $M_{s,b}$ is a closed convex subset of $X_{s,b}$, there exists a minimum $\tilde{f}$ in $M_{s,b}$. Move precisely, note that

$$M_{s,b} = \check{M}_{s,b} + \{ F \} \equiv \{ g + F : g \in \check{M}_{s,b} \}$$

for any $F \in M_{s,b}$, where $\check{M}_{s,b}$ is the closed linear subspace of $X_{s,b}$ defined as

$$\check{M}_{s,b} = \{ g \in X_{s,b}(\phi) : g|_{\Omega} = 0 \},$$

therefore, $\tilde{f}$ is exactly the function that

$$\tilde{f} \perp \check{M}_{s,b} \text{ in } X_{s,b}(\phi).$$

Therefore, for (6.1), we only need to show that

$$I_{N,s}f \perp \check{M}_{1,b} \text{ in } X_{1,b}(\phi); \quad \tilde{f} \perp \check{M}_{s',b} \text{ in } X_{s',b}(\phi)$$

for $s' \leq s$, but it is obvious. \hfill \Box

Now we show that the spaces $X_{s,b}(\phi)$ have the property of the norm-subadditivity about the restricted domain.

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Lemma 6.2  For every $f \in X^{\Omega}_{s,b}(\phi)$, it holds for any $\Omega_0 \subset \Omega$ that

$$\|f\|_{X^{\Omega_0}_{s,b}(\phi)} \leq \|f\|_{X^{\Omega_0}_{s,b}(\phi)} + \|f\|_{X^{\Omega_0}_{s,b}(\phi)}.$$

Proof. By Lemma 6.1, there exists a function $\tilde{f} \in X_{s,b}(\phi)$ such that $\tilde{f} = f$ on $\Omega$ and

$$\|f\|_{X^{\Omega_0}_{s,b}(\phi)} = \|\tilde{f}\|_{X^{\Omega_0}_{s,b}(\phi)}.$$

Define the operator $P_b$ and its inverse operator $P_b^{-1}$ as

$$\widehat{P_b f}(\xi, \tau) = \langle \tau + \phi(\xi) \rangle^{2b} \tilde{f}(\xi, \tau), \quad \widehat{P_b^{-1} f}(\xi, \tau) = \langle \tau + \phi(\xi) \rangle^{-2b} \tilde{f}(\xi, \tau).$$

We claim that

$$\|\tilde{f}\|_{X_{s,b}(\phi)} = \|P_b^{-1} (P_b \tilde{f} \cdot j)\|_{X_{s,b}(\phi)} = \|P_b^{-1} (P_b \tilde{f} \cdot j)\|_{L^2 H_x^s}$$

for any $j(t) \in C_c^\infty(\mathbb{R})$, such that

$$j(t) \equiv 1 \text{ on } \Omega' \supset \Omega.$$

Indeed, by the proof of Lemma 6.1, one only needs to show that

$$P_b^{-1} \left( P_b \tilde{f} \cdot j \right) \in M_{s,b}; \quad P_b^{-1} \left( P_b \tilde{f} \cdot j \right) \perp \tilde{M}_{s,b} \text{ in } X_{s,b}(\phi).$$

The first term is easy to check and we omit the details. On the other hand, for any $g \in \tilde{M}_{s,b}$, we have

$$\left\langle P_b^{-1} \left( P_b \tilde{f} \cdot j \right), g \right\rangle = \int \langle \xi \rangle^{2s} \langle \tau + \phi(\xi) \rangle^{2b} \mathcal{F} \left( P_b^{-1} \left( P_b \tilde{f} \cdot j \right) \right)(\xi, \tau) \cdot \overline{\mathcal{F} g(\xi, \tau)} d\xi d\tau$$

$$= \int \langle \xi \rangle^{2s} \mathcal{F} \left( P_b \tilde{f} \right) \ast \mathcal{F} j \right)(\xi, \tau) \cdot \overline{\mathcal{F} g(\xi, \tau)} d\xi d\tau$$

$$= \int \langle \xi \rangle^{2s} \mathcal{F} \left( P_b \tilde{f} \right) \right)(\xi, \tau) \cdot \overline{\mathcal{F} (gj(-\cdot))}(\xi, \tau) d\xi d\tau$$

$$= \left\langle \tilde{f}, gj(-\cdot) \right\rangle$$

$$= 0,$$

since $gj(-\cdot) \in \tilde{M}_{s,b}$, and $\tilde{f} \perp \tilde{M}_{s,b}$ in $X_{s,b}(\phi)$, where $\langle \cdot, \cdot \rangle$ is the inner product in $X_{s,b}(\phi)$. Therefore, we have the claim.
Furthermore, we keep in mind that

\[
P_b^{-\frac{1}{2}} \left( P_b \tilde{f} \cdot j \right) \bigg|_{t \in \Omega} = P_b^{-\frac{1}{2}} f \bigg|_{t \in \Omega} \quad ; \quad P_b^{-\frac{1}{2}} \left( P_b \tilde{f} \cdot j \right) \bigg|_{t \in (\text{supp}j)^c} = 0. \tag{6.2}
\]

For the simplicity, we only prove the special result that

\[
\| f \|_{X_s^{[0,\delta]}(\phi)} \leq \| f \|_{X_s^{[0,\delta]}(\phi)} + \| f \|_{X_s^{[0,\delta]}(\phi)}
\]

for \(0 \leq \delta_0 \leq \delta\), which follows from

\[
\| f \|_{X_s^{[0,\delta]}(\phi)} \leq \liminf_{\epsilon \to 0} \| f \|_{X_s^{[0,\delta]/E_{\epsilon}}(\phi)} \leq \| f \|_{X_s^{[0,\delta]}(\phi)} + \| f \|_{X_s^{[0,\delta]}(\phi)}, \tag{6.3}
\]

where \(E_{\epsilon} = (\delta_0 - 4\epsilon, \delta_0 + 4\epsilon)\).

We first show the second inequality of (6.3). Indeed, for \(f_1, f_2\) such that \(\| f \|_{X_s^{[0,\delta]}(\phi)} = \| \tilde{f}_1 \|_{X_s, \phi} \cdot \| f \|_{X_s^{[0,\delta]}(\phi)} = \| \tilde{f}_2 \|_{X_s, \phi}\) and the function \(j_1(t), j_2(t)\) such that

\[
\text{supp} j_1 \subset (-2\epsilon, \delta_0 + 2\epsilon), \quad j_1(t) \equiv 1 \text{ on } (-\epsilon, \delta_0 + \epsilon);
\]

\[
\text{supp} j_2 \subset (\delta_0 - 2\epsilon, \delta + 2\epsilon), \quad j_1(t) \equiv 1 \text{ on } (\delta_0 - \epsilon, \delta + \epsilon),
\]

we have for any small \(\epsilon > 0\),

\[
P_b^{-1} \left( P_b \tilde{f}_1 \cdot j_1 + P_b \tilde{f}_2 \cdot j_2 \right) \bigg|_{t \in [0,\delta]/E_{\epsilon}} = f \bigg|_{t \in [0,\delta]/E_{\epsilon}}.
\]

Therefore,

\[
\| f \|_{X_s^{[0,\delta]/E_{\epsilon}}(\phi)} \leq \| P_b^{-1} \left( P_b \tilde{f}_1 \cdot j_1 + P_b \tilde{f}_2 \cdot j_2 \right) \|_{X_s, \phi} \leq \| f \|_{X_s^{[0,\delta]}(\phi)} + \| f \|_{X_s^{[0,\delta]}(\phi)},
\]

since

\[
\| P_b^{-1} \left( P_b \tilde{f}_1 \cdot j_1 \right) \|_{X_s, \phi} = \| f \|_{X_s^{[0,\delta]}(\phi)} , \quad \| P_b^{-1} \left( P_b \tilde{f}_2 \cdot j_2 \right) \|_{X_s, \phi} = \| f \|_{X_s^{[0,\delta]}(\phi)}.
\]

Now we turn to the first term of (6.3). We choose the function \(j(t)\) such that

\[
\text{supp} j \subset (-2\epsilon, \delta + 2\epsilon), \quad j(t) \equiv 1 \text{ on } (-\epsilon, \delta + \epsilon),
\]

then for \(f, f_{\epsilon}\) such that \(\| f \|_{X_s^{[0,\delta]}(\phi)} = \| \tilde{f} \|_{X_s, \phi}, \| f \|_{X_s^{[0,\delta]/E_{\epsilon}}(\phi)} = \| \tilde{f}_{\epsilon} \|_{X_s, \phi}\), we have

\[
\| P_b^{-\frac{1}{2}} \left( P_b \tilde{f} \cdot j \right) \|_{L^2_t H^s_x} = \| f \|_{X_s^{[0,\delta]}(\phi)} , \quad \| P_b^{-\frac{1}{2}} \left( P_b \tilde{f}_{\epsilon} \cdot j \right) \|_{L^2_t H^s_x} = \| f \|_{X_s^{[0,\delta]/E_{\epsilon}}(\phi)}.
\]
Moreover, by (6.2) we have,
\[ m \left\{ t \in \mathbb{R} : \left\| P_b^{-\frac{1}{2}} \left( P_b (\tilde{f} - \tilde{f}_\epsilon) \cdot j \right)(t) \right\|_{H^s_x} > \varepsilon \right\} \lesssim \varepsilon \]
for any \( \varepsilon > 0 \), where \( m \) is the Lebesgue measure, which implies that
\[ \left\| P_b^{-\frac{1}{2}} \left( P_b \tilde{f} \cdot j \right)(t) \right\|_{H^s_x} \rightarrow \left\| P_b^{-\frac{1}{2}} \left( P_b \tilde{f}_\epsilon \cdot j \right)(t) \right\|_{H^s_x} \text{ in measure.} \]
Thus by Fatou’s lemma, we have
\[ \left\| P_b^{-\frac{1}{2}} \left( P_b \tilde{f}_\epsilon \cdot j \right) \right\|_{L^2_t H^s_x} \leq \liminf_{\varepsilon \to 0} \left\| P_b^{-\frac{1}{2}} \left( P_b \tilde{f}_\epsilon \cdot j \right) \right\|_{L^2_t H^s_x}, \]
and thus complete the proof of the lemma.

**Remark.** By the similar argument as above, we can prove that \( y(\delta) \equiv \|f\|_{X^s_{\delta, b}(\phi)} \) is left continuous for every fixed function \( f \).

**Lemma 6.3** Let \( \delta \in (0, 1) \), \( b, b' \in \left[ 0, \frac{1}{2} \right] \) with \( b' \geq b \), then
\[ \|f\|_{X^s_{\delta, b}(\phi)} \lesssim \delta^{b'-b} \|f\|_{X^s_{\delta, b'}(\phi)}. \]

**Proof.** Let \( \tilde{f} \) be the extension of \( f \in X^s_{\delta, b'}(\phi) \) such that \( \tilde{f} = f \) on \( [-\delta, \delta] \) and \( \|f\|_{X^s_{\delta, b'}(\phi)} = \left\| \tilde{f} \right\|_{X^s_{\delta, b'}(\phi)} \). By Lemma 2.4 (iv),
\[ \|f\|_{X^s_{\delta, b}(\phi)} \leq \left\| \tilde{f} \right\|_{X^s_{\delta, b}(\phi)} \lesssim \delta^{b'-b} \left\| \tilde{f} \right\|_{X^s_{\delta, b'}(\phi)} = \delta^{b'-b} \|f\|_{X^s_{\delta, b'}(\phi)}. \]
This completes the proof of the lemma.

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