Non-Stabilizer Quantum Codes from Abelian Subgroups of the Error Group

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Abstract
This paper is motivated by the computer-generated nonadditive code described in Rains et al [RHSN97]. We describe a theory of non-stabilizer codes of which the nonadditive code of Rains et al is an example. Furthermore, we give a general strategy of constructing good nonstabilizer codes from good stabilizer codes and give some explicit constructions and asymptotically good nonstabilizer codes. Like in the case of stabilizer codes, we can design fairly efficient encoding and decoding procedures.

1 Introduction
Let $A$ be a finite abelian group with operation denoted by $+$ and identity $0$. We identify $A$ with the alphabet of symbols transmitted on a classical communication channel. Consider the $n$-fold cartesian product $A^n$ of copies of $A$. Elements of $A^n$ are called words of length $n$. A commonly used group is $\{0, 1\}$ with addition modulo 2. Let $\hat{A}$ denote the character group of $A$, the multiplicative group of all homomorphisms from $A$ into the multiplicative group of complex numbers of modulus unity. For $a = (a_1, a_2, \ldots, a_n)^T \in A^n$ we define its weight $w(a)$ to be $\# \{i \mid a_i \neq 0 \}$. We say that a subgroup $C_n$ of $A^n$ is a $t$-error correcting group code if for every non-zero element $x = (x_1, x_2, \ldots, x_n)^T$ in $C_n$, $w(x) \geq 2t + 1$. In other words, if messages transmitted through a noisy channel are encoded into words from $C_n$ and during transmission of a word errors at the output occur in at most $t$ positions, then the message can be decoded without any error. There is a vast literature on the construction of $t$-error correcting group codes and the reader may find an introduction to this subject and pointers to literature in [MS78, vL98].

A broad class of quantum error correcting codes known as stabilizer codes was introduced by Gottesman [Got96] and Calderbank et al [CRSS98] (also see [CMP98, RHSN97, Rai99]). To the best of our knowledge, apart from one computer-generated example of a code proposed by Rains et al [RHSN97], all known quantum error-correcting codes are stabilizer codes.

In this paper we develop a theory of nonstabilizer codes based on the Weyl commutation relations. The nonadditive code of Rains et al [RHSN97] is an instance of our theory and we
derive it directly from the theory. Furthermore, we give a general strategy of constructing
good nonstabilizer codes from good stabilizer codes and give some explicit constructions
and asymptotically good nonstabilizer codes. For a rich family of nonstabilizer codes, we
also give elegant and efficient encoding circuits. We also give a simple effective
and asymptotically good nonstabilizer codes. For a rich family of nons tabilizer codes, we
derive it directly from the theory. Furthermore, we give a general strategy of constructing
good nonstabilizer codes from good stabilizer codes and give some explicit constructions
and asymptotically good nonstabilizer codes. For a rich family of nonstabilizer codes, we
also give elegant and efficient encoding circuits. We also give a simple effective
decoding procedure for these nonstabilizer codes.

First we introduce some definitions. We choose and fix an \( M \)-dimensional complex Hilbert
space \( \mathcal{H} \) and consider the unit vectors of \( \mathcal{H} \) as pure states of a finite level quantum system.
If \( A \) is a finite abelian group with \( M \) elements and \( \{ e_x \mid x \in A \} \) is an orthonormal basis
of \( \mathcal{H} \) indexed by elements of \( A \) we express it in the Dirac notation as \( |x\rangle = e_x \). If \( x = (x_1, x_2, \ldots, x_n)^T \in A^n \) is a word of length \( n \), we write
\[
|\mathbf{x}\rangle = |x_1x_2\ldots x_n\rangle = e_{x_1} \otimes e_{x_2} \otimes \ldots \otimes e_{x_n}
\]
where the right-hand side is a product vector in the \( n \)-fold tensor product \( \mathcal{H}^\otimes n \) of \( n \) copies of
\( \mathcal{H} \). Thus, with the chosen orthonormal basis, every word \( \mathbf{x} \) in \( A^n \) is translated into a basis state \( |\mathbf{x}\rangle \) of \( \mathcal{H}^\otimes n \).

A \textit{quantum code} is a subspace \( \mathcal{C}_n \) of \( \mathcal{H}^\otimes n \). Note that a pure state in \( \mathcal{H}^\otimes n \) described by
a unit vector \( |\psi\rangle \) in \( \mathcal{H}^\otimes n \) has density matrix \( |\psi\rangle \langle \psi| \). A density matrix \( \rho \) in \( \mathcal{H}^\otimes n \) is a non-negative operator of unit trace. In quantum probability, a projection operator \( E \) in \( \mathcal{H}^\otimes n \) is interpreted as an event concerning the quantum system and a density matrix \( \rho \) as a state of
the quantum system. The probability of the event \( E \) in the state \( \rho \) is given by \( \text{Tr} \rho E \).
Messages to be transmitted through a quantum channel are encoded into pure states in \( \mathcal{H}^\otimes n \).
When a pure state \( |\psi\rangle \), or equivalently, a density matrix \( |\psi\rangle \langle \psi| \) is transmitted the channel output is hypothesized to be a state of the form
\[
\rho = \sum_i L_i |\psi\rangle \langle \psi| L_i^\dagger
\]
where the operators \( \{L_i\} \) belong to a linear subspace \( \mathcal{A} \) of the algebra of all operators on
\( \mathcal{H}^\otimes n \). The operators \( \{L_i\} \) may depend on \( \rho \), but in order to ensure that \( \rho \) is a density matrix
it is assumed that \( \langle \psi| \sum_i L_i^\dagger L_i |\psi\rangle = 1 \). By the spectral theorem \( \rho \) can be expressed as
\[
\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|
\]
where \( |\psi_j\rangle \) is an orthonormal set in \( \mathcal{H}^\otimes n \) and \( \{p_j\} \) is a probability distribution with \( p_j > 0 \)
for each \( j \). In other words, the output state \( \rho \) is not necessarily pure even though the input
state is pure. The operators \( L_i \) are called \textit{error operators} and the linear space \( \mathcal{A} \) from which
they come is called the \textit{error space}.

Let \( P \) be the projection operator corresponding to a quantum code \( \mathcal{C}_n \). The subspace
\( \mathcal{D}(P) \) of \textit{error operators detected} by \( P \) is defined as
\[
\mathcal{D}(P) = \{ L \in \mathcal{B}(\mathcal{H}^\otimes n) \mid P L P = c P \text{ for some } c \in \mathbb{C} \}.
\]

It is evident that there is a complex-valued functional \( \phi : \mathcal{D}(P) \to \mathbb{C} \) so that we can write
\( P L P = \phi(L) P \) for all \( L \in \mathcal{D}(P) \).

A finite family \( \{ M_j \} \subseteq \mathcal{B}(\mathcal{H}^\otimes n) \) constitutes a set of \textit{decoding operators} for the code \( \mathcal{C}_n \)
and error space \( \mathcal{A} \) if the following conditions are satisfied.
(a) \( \sum_j M_j^\dagger M_j = I \).

(b) For any pure state \( |\psi\rangle \in C_n \) let \( \rho \) be the output corresponding to a subset \( \{ L_i \} \subseteq A \),
\[
\sum_j M_j \rho M_j^\dagger = \sum_{i,j} M_j L_i |\psi\rangle \langle L_i | M_i^\dagger = |\psi\rangle \langle \psi |.
\]

In this case we say that \( C_n \) is an \( A \)-error correcting quantum code.

We have the following fundamental theorem of Knill and Laflamme [KL00] which characterizes the errors that a quantum code can correct. It essentially states that errors coming from a family \( A \) of operators can be corrected for a quantum code with projection \( P \) if and only if
\[
\{ L_1^\dagger L_2 \mid L_1, L_2, \in A \} \subseteq \mathcal{D}(P).
\]

**Theorem 1.1** [KL00] Let \( A \) be a family of operators in \( \mathcal{H}^{\otimes n} \) and let \( C_n \subset \mathcal{H}^{\otimes n} \) be a quantum code with an orthonormal basis \( \psi_1, \psi_2, \ldots, \psi_d \). Let \( P \) be the projection corresponding to the code \( C_n \). Then \( C_n \) is an \( A \)-error correcting quantum code if and only if
\[
A^\dagger A = \{ L_1^\dagger L_2 \mid L_1, L_2, \in A \} \subseteq \mathcal{D}(P).
\]

**Remark 1.2** The proof of the above theorem is constructive and yields the decoding operators in terms of \( A \) and the basis \( \psi_1, \ldots, \psi_d \) of \( C_n \).

Now we specialize the choice of \( A \). Consider all unitary operators in \( \mathcal{H}^{\otimes n} \) of the form \( U = U_1 \otimes U_2 \otimes \ldots \otimes U_n \) where each \( U_i \) is a unitary operator on \( \mathcal{H} \) and all but \( t \) of the \( U_i \)'s are equal to \( I \). Such a \( U \) when operating on \( \psi = \psi_1 \otimes \ldots \otimes \psi_n \in \mathcal{H}^{\otimes n} \) produces \( U |\psi\rangle \) which is an \( n \)-fold tensor product that differs from \( \psi \) in at most \( t \) places. Denote by \( A_t \) the linear span of all such unitary operators \( U \). A quantum code \( C_n \) is called a \( t \)-error correcting quantum code if \( C_n \) is an \( A_t \)-correcting quantum code.

## 2 Quantum Codes and Projections in a Group Algebra

Let \((A, +)\) be a finite abelian group with \( M \) elements and identity denoted by \( 0 \). By the fundamental theorem of finite abelian groups, \( A \) is isomorphic to \( \bigoplus_{i=1}^k \mathbb{Z}_{n_i} \) via the isomorphism \( \tau \). For every \( m \), let \( \omega_m = e^{2\pi i/m} \). Define the **canonical bicharacter** of the group \( A \) as the following complex-valued function on \( A \times A \).
\[
\langle\langle a, b \rangle\rangle = \prod_{j=1}^k \omega_{x_j y_j}^{x_j y_j}, \text{ where } \tau(a) = (x_1, \ldots, x_k) \text{ and } \tau(b) = (y_1, \ldots, y_k).
\]

Notice that for all \( a, b, c \in A \) we have \( \langle\langle a, b \rangle\rangle = \langle\langle b, a \rangle\rangle, \langle\langle a + b, c \rangle\rangle = \langle\langle a, c \rangle\rangle \langle\langle b, c \rangle\rangle, \text{ and } \langle\langle a, b \rangle\rangle = 1 \) for all \( b \in A \) if and only if \( a = 0 \). Denote by \( \hat{A} \) the character group of \( A \). For each fixed \( a \in A \), the bicharacter \( \langle\langle a, b \rangle\rangle \), as a function of \( b \), is a distinct element \( \chi_a \) of \( \hat{A} \) and the correspondence \( a \mapsto \chi_a \) is a group isomorphism between \( A \) and the multiplicative character group \( \hat{A} \).
Denote by $\mathcal{H}$ the $M$-dimensional Hilbert space $L^2(A)$ of all complex-valued functions on $A$, spanned by $\{|x\rangle\}_{x \in A}$ (where the vector $|x\rangle$ denotes the indicator function $1_x$ of the singleton $\{x\}$). Define the unitary operators $U_a$ and $V_a$ on $\mathcal{H}$ for every $a \in A$ by

$$U_a |x\rangle = |x + a\rangle, \quad V_a |x\rangle = \langle \langle a, x \rangle |x\rangle$$

where $x \in A$. Then we have

$$U_a U_b = U_{a+b} \quad V_a V_b = V_{a+b}, \text{ and } \langle \langle a, b \rangle \rangle U_a V_b = V_b U_a \quad \forall \ a, b \in A.$$ 

These are the Weyl commutation relations between the unitary operators $U_a$ and $V_a$ on $\mathcal{H}$. The family of operators $\{U_a V_b \mid a, b \in A\}$ is irreducible.

The canonical bicharacter on $A$ gives rise to the following bicharacter on $A^n$. For two elements $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $A^n$, $\langle \langle a, b \rangle \rangle$ is defined as

$$\langle \langle a, b \rangle \rangle = \prod_{i=1}^{n} \langle \langle a_i, b_i \rangle \rangle.$$ 

Put $U_a = U_{a_1} \otimes \ldots \otimes U_{a_n}$ and $V_b = V_{b_1} \otimes \ldots \otimes V_{b_n}$. Then $\{U_a V_b \mid a, b \in A^n\}$ is again an irreducible family of unitary operators such that $U_a U_b = U_{a+b}$ and $V_a V_b = V_{a+b}$, and they satisfy the Weyl commutation relations

$$\langle \langle a, b \rangle \rangle U_a V_b = V_b U_a \forall \ a, b \in A^n.$$ 

In the Hilbert space $\mathcal{B}(\mathcal{H}^\otimes_n)$ of all linear operators on $\mathcal{H}^\otimes_n$ with the scalar product $\langle X | Y \rangle = Tr X^\dagger Y$ the set $\{M^{-n/2} U_a V_b \mid a, b \in A^n\}$ is an orthonormal basis. In particular

$$Tr U_a V_b = \begin{cases} 0 & \text{if } \langle \langle a, b \rangle \rangle \neq (0, 0), \\ M & \text{otherwise}, \end{cases}$$

The weight $wt(a, b)$ of a pair $(a, b) \in A^n \times A^n$ is defined to be $\#\{i \mid 1 \leq i \leq n, (a_i, b_i) \neq (0, 0)\}$, where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. The irreducibility of $\{U_a V_b \mid a, b \in A^n\}$ implies that $\{U_a V_b \mid a, b \in A^n, wt(a, b) \leq t\}$ spans $\mathcal{A}_t$. As a result, the Knill-Laflamme theorem for $\mathcal{A}_t$-correcting quantum codes takes the following form which is easy to derive from Theorem [7].

**Theorem 2.1** $\mathcal{C}_n \subset L^2(A)^\otimes_n$ is a $t$-error correcting quantum code if and only if

$$\{U_a V_b \mid a, b \in A^n, wt(a, b) \leq 2t\} \subseteq D(P),$$

where $P$ is the projection corresponding to $\mathcal{C}_n$.

Let $N$ be the least positive integer such that $Na = 0$ for all $a \in A$. Let $\omega = e^{2\pi i/N}$, We define the error group $\mathcal{E}$ as follows.

**Definition 2.2** The error group $\mathcal{E}$ is defined as

$$\mathcal{E} = \{\omega^i U_a V_b \mid 0 \leq i \leq N - 1, a, b \in A^n\},$$

with the group operation defined by

$$\omega^i U_a V_b \omega^j U_c V_d = \omega^{i+j} \langle \langle b, c \rangle \rangle U_{a+c} V_{b+d}.$$
Let $S \leq \mathcal{E}$ be a subgroup of the error group. In the theory of stabilizer codes we are interested in subspaces of $L^2(\mathcal{A})^{\otimes n}$ that are left invariant under the action of $S$. It turns out that the invariant subspace is nontrivial if and only if $S$ is abelian with the property that $\omega^i I \notin S$ for all $1 \leq i \leq N - 1$.

**Definition 2.3** An abelian subgroup $S$ of $\mathcal{E}$ is said to be a Gottesman subgroup of $\mathcal{E}$ if $\omega^i I \notin S$ for all $1 \leq i \leq N - 1$. The closure of a Gottesman subgroup $S$ is the abelian subgroup $\overline{S}$ of $\mathcal{E}$ defined as

$$\overline{S} = \{\omega^i g \mid g \in S, \ 0 \leq i \leq N - 1\}.$$ 

**Remark 2.4** For any Gottesman subgroup $S$ of the error group $\mathcal{E}$ the element $\omega^i U_n V_b \in S$ for at most one $i : 0 \leq i \leq N - 1$.

Let $\mathbb{C}[S]$ be the group algebra of formal sums $\sum_{s \in S} T_s s$, $T_s \in \mathbb{C}$ where the plus and product (convolution) are defined as follows

$$T + T' = \sum_{s \in \mathcal{E}} (T_s + T'_s)s,$$

$$T' * T'' = T, \text{ where } T_s = \sum_{g \in \mathcal{E}} T'_g T''_{g^{-1} s}.$$ 

Since any Gottesman subgroup $S$ is a set of linearly independent elements of $B(\mathcal{H}^{\otimes n})$, the identity map is a natural injective linear embedding from $\mathbb{C}[S]$ into $B(\mathcal{H}^{\otimes n})$. In other words, $\mathbb{C}[S]$ is a subalgebra of $B(\mathcal{H}^{\otimes n})$ under operator addition and composition, where the convolution operation “*” in $\mathbb{C}[S]$ coincides with composition operator.

Every subspace (i.e. quantum code) in $\mathcal{H}^{\otimes n}$ is defined by its corresponding projection operator in $B(\mathcal{H}^{\otimes n})$. In this paper we are interested in the projection operators in $\mathbb{C}[S]$ for a Gottesman subgroup $S$ of $\mathcal{E}$. In the following easy proposition we characterize the elements in $\mathbb{C}[S]$ which are projection operators in $\mathcal{H}^{\otimes n}$. Then our goal will be to seek for projections whose range is a good $t$-error correcting quantum code.

**Proposition 2.5** The element $T \in \mathbb{C}[S]$ is a projection operator on $\mathcal{H}^{\otimes n}$ if and only if $T_s = T_{s-1}$ and $T * T = T$.

We use the Fourier transform over $\mathbb{C}[S]$ and the above proposition to describe projection operators in $\mathbb{C}[S]$. We recall the Fourier transform and some of its properties.

Let $S$ be an abelian group and let $\hat{S}$ denote the character group of $S$. For each $s \in S$ we can associate the element $\sum \alpha_g g$ in the algebra $\mathbb{C}[S]$ where, $\alpha_s = 1$ and $\alpha_g = 0$ for $g \neq s \in S$. Similarly, to $\chi \in \hat{S}$ we associate the element $\sum \chi(g) g$ of $\mathbb{C}[S]$. Fix an isomorphism $s \mapsto \chi_s$ between the groups $S$ and $\hat{S}$. The Fourier transform over $\mathbb{C}[S]$ is now defined as follows.

**Definition 2.6 (Fourier Transform)** The linear transformation that maps $s \in \mathbb{C}[S]$ to $\chi_s \in \mathbb{C}[S]$ is called the Fourier transform over $\mathbb{C}[S]$. The Fourier transform $\hat{T}$ of $T$ in $\mathbb{C}[S]$ is given by the formula

$$\hat{T}_u = \sum_{s \in S} \chi_u(s)T_s.$$ 

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The inverse Fourier transform is given by the following formula:

\[ T_s = \frac{1}{\#S} \sum_{u \in S} \chi_u(s) \hat{T}_u. \]

Let \( TT' \) denote the component-wise product of \( T, T' \in \mathbb{C}[S] \). I.e.

\[ TT' = \sum_{s \in S} T_s T'_s. \]

We now recall some useful properties of the Fourier transform.

\[ \hat{T}_1 \ast \hat{T}_2 = \hat{T}_1 \hat{T}_2, \quad T_1, T_2 \in \mathbb{C}[S]. \]

\[ \hat{gT} = \chi_g \hat{T}, \quad g \in S, \quad T \in \mathbb{C}[S]. \]

We can characterize projection operators in \( \mathbb{C}[S] \) using the Fourier transform.

**Theorem 2.7** An element \( T \in \mathbb{C}[S] \) is a projection (and hence a code) iff \( \hat{T} = 1_B = \sum_{g \in B} g \) for some subset \( B \) of \( S \).

**Proof.** \( T \) is a projection iff \( T \ast T = T \) and \( T^\dagger = T \). On taking Fourier transforms on both sides of the equation \( T \ast T = T \) we get

\[ \hat{T} \hat{T} = \hat{T}. \]

This implies \( \hat{T}_s^2 = \hat{T}_s \) for every \( s \in S \). Thus, \( \hat{T}_s \in \{0, 1\} \) for every \( s \in S \) which gives the desired result.

Conversely, note that if \( \hat{T} = 1_B \) then by inverse Fourier transform we get

\[ T = \frac{1}{\#S} \sum_{s \in S} \sum_{u \in B} \chi_u(s) s. \]

From the above equation it is clear that the condition \( T^\dagger = T \) is automatically satisfied.

From Theorem 2.7 it is clear that the code defined by a projection \( T \) in \( \mathbb{C}[S] \) is completely specified by the subset \( B \) of \( S \). We will call \( B \) the Fourier description of the quantum code and denote the corresponding projection by \( P(B) \).

Next, we give a formula for the dimension of a quantum code defined by a projection \( T \) in \( \mathbb{C}[S] \).

**Lemma 2.8** Let \( C_n \subseteq H^\otimes n \) be a quantum code defined by a projection \( T \) in \( \mathbb{C}[S] \) with Fourier description \( B \subseteq S \). Then \( C_n \neq 0 \) if and only if \( \omega^i I \notin S \) for \( i : 1 \leq i \leq N - 1 \) (i.e. \( S \) is a Gottesman subgroup), and if \( C_n \neq 0 \) the dimension of the code \( C_n \) is given by

\[ \dim(C_n) = \frac{\#A^n \#B}{\#S}. \]
Proof. If \( B \) is the Fourier description of the code then the projection corresponding to the code is given by
\[
P = \frac{1}{\#S} \sum_{s \in S} \sum_{u \in B} \chi_u(s)s.
\]
The dimension is given by \( \text{Tr}(P) \). Observe that \( \text{Tr}(U_a V_b) = 0 \) if \( U_a V_b \neq I \), and \( \text{Tr}(\omega^i I) = \omega^i \# A^n \), for \( \omega^i I \in S \). Now, since \( \omega \) is a nontrivial root of unity, \( \sum_{\omega^i I \in S} \omega^i \# A = 0 \) if \( \omega^i I \in S \) for some \( i \) such that \( 1 \leq i \leq N - 1 \). Thus, we have \( \text{Tr}(P) = 0 \) if \( \omega^i I \in S \) for some \( i : 1 \leq i \leq N - 1 \), and otherwise \( \text{Tr}(P) = \frac{\#B \# S \# A^n}{\# S} \). This proves the lemma.

Proposition 2.9 Let \( g = \omega^i U_a V_b \) and \( h = \omega^j U_c V_d \) be elements in the error group \( E \). Then
\[
ghg^{-1}h^{-1} = \gamma(g, h)I
\]
where \( \gamma(g, h) = \langle \langle b, c \rangle \rangle \langle \langle a, d \rangle \rangle \). In particular we have
\[
\gamma(g_1 g_2, h) = \gamma(g_1, h) \gamma(g_2, h)
\]
and
\[
\gamma(g, h_1 h_2) = \gamma(g, h_1) \gamma(g, h_2)
\]
Proof. Straightforward from the Weyl commutation relations.

Remark 2.10 When \( h \in E \) is fixed and \( s \) varies in \( S \) the map \( s \mapsto \gamma(s, h) \) is a character of \( S \) which we will denote by \( \gamma_h \).

Let \( C_n \subseteq H^\otimes n \) be a code with Fourier description \( B \). In the next theorem we derive a condition on \( B \) such that \( C_n \) is a \( t \)-error correcting quantum code. We introduce a convenient notation: For \( g = \omega^i U_a V_b \) in the error group \( E \), let \( wt(g) \) denote the number \( wt(a, b) \).

Theorem 2.11 A quantum code \( C_n \subseteq H^\otimes n \) with Fourier description \( B \) is \( t \)-error correcting iff the following two conditions hold.

1. For each \( g \in S \) such that \( wt(g) \leq 2t \)
\[
\chi_g(u_1^{-1}u_2) = 1 \text{ for all } u_1, u_2 \in B.
\]
(i.e. every \( u \in B \) is in the same coset of the kernel of \( \chi_g \).)

2. For each \( g \in E \setminus S \) such that \( wt(g) \leq 2t \), and for every \( u \in B^{-1}B \) we have
\[
\sum_{s \in S} \gamma_g(s) \chi_u(s) = 0.
\]
(i.e. the character \( \gamma_g \) is different from \( \chi_{u^{-1}} \) for every \( u \in B^{-1}B \).)
Proof.

Let $C_n$ be a $t$-error correcting code and let $T = \sum_{s \in S} T_s s$ be the corresponding projection. Recall that if $B$ is the Fourier description for the code then

$$T_s = \frac{1}{\#S} \sum_{u \in B} \bar{\chi}_u(s).$$

By Theorem [1], $C_n$ is $t$-error correcting if and only if there is a scalar-valued function $\phi$ such that for every $g \in E$ with $wt(g) \leq 2t$

$$T g T = \phi(g) T.$$

This is equivalent to the following condition.

$$\left( \sum_{s_1 \in S} T_{s_1} s_1 \right) g \left( \sum_{s_2 \in S} T_{s_2} s_2 \right) = \phi(g) \sum_{s \in S} T_s s.$$

Case 1 $g \in S$: In this case Equation 2 yields $g T \ast T = \phi(g) T$. Taking Fourier transform on both sides we get

$$\chi_g \hat{T} \hat{T} = \phi(g) \hat{T}.$$

Since $\hat{T} = 1_B$, we have $\chi_g(u) = \phi(g)$ for all $u \in B$. Thus, $\chi_g$ is constant on $B$ for every $g \in S$ such that $wt(g) \leq 2t$. This is true precisely when $B$ is contained in some coset of the kernel of $\chi_g$.

Case 2 $g \notin S$: In this case the Knill-Laflamme condition takes the following form

$$\sum_{s_1, s_2} T_{s_1} T_{s_2} \gamma(s_1, g) g s_1 s_2 = \phi(g) \sum_{s \in S} T_s s.$$

Since the operators on the two sides of the above equation have disjoint support, each side of the equation should to be 0. Consequently, $\phi(g) = 0$ and for all $s \in S$

$$\sum_{s_1, s_2 = s} T_{s_1} T_{s_2} \gamma(s_1, g) = 0.$$

This yields

$$\sum_{s_1, s_2 = s} \left( \sum_{u_1 \in B} \bar{\chi}_{u_1}(s_1) \right) \left( \sum_{u_2 \in B} \bar{\chi}_{u_2}(s_2) \right) \gamma(s_1, g) = 0.$$

On simplification we get

$$\sum_{u_1, u_2 \in B} \sum_{s_1, s_2 = s} \gamma_g(s_1, s_2) \bar{\chi}_{u_1}(s_1) \bar{\chi}_{u_2}(s_2) = 0.$$
which gives
\[
\sum_{u_1, u_2 \in B} \chi_{u_2}(s) \sum_{s_1 \in S} (\gamma_g \chi_{u_2} \chi_{u_1})(s_1) = 0.
\] (2)

Note that the inner summation in equation 2 is summing up of a character of \( S \), namely, \( \gamma_g \chi_{u_2} \chi_{u_1} \), over the whole group \( S \). Therefore, the inner summation evaluates to either 0 or \( \#S \). Hence, the necessary and sufficient condition for equation 2 to hold is
\[
\sum_{s \in S} \gamma_g(s) \chi_u(s) = 0 \quad \forall u \in B^{-1}B.
\]

\[\Box\]

Remark 2.12 If the Fourier description of a code, \( B \), is a subgroup of \( S \) then the code is actually a stabilizer code with stabilizer group \( B^\perp \), where \( B^\perp \) is the annihilator of \( B \) in \( S \) defined by
\[
B^\perp = \{ a \in S : \forall b \in B \ (\langle a, b \rangle) = 0 \}.
\] In particular if we set \( B = \{ I \} \), where \( I \) is the identity element, the code \( C_n \) with Fourier description \( B \) is the stabilizer quantum code:
\[
C_n = \{|\psi\rangle \ | s |\psi\rangle = |\psi\rangle \forall s \in S\}.
\]

Thus the stabilizer codes of \([Got90]\) are a subclass of the class of codes defined in this paper.

At this point we recall some useful facts from the theory of stabilizer codes as developed in \([AP02]\).

Let \( S \) be a Gottesman subgroup of \( E \)
\[
C(S) = \{ \psi \in L^2(A)^{\otimes n} \ | \ U \psi = \psi \ \forall \ U \in S\}.
\]
Let \( Z(S) \) denote the centralizer of \( S \) in \( E \), that is,
\[
Z(S) = \{ U \in E \ | \ UU' = U'U \ \forall \ U' \in S\}.
\]
For Gottesman subgroup \( S \) recall that the closure of \( S \) (denoted by \( \overline{S} \)) is defined as
\[
\overline{S} = \{ \omega^i U_a V_y : 0 \le i \le N - 1 \text{ and } \exists j \ 0 \le j \le N - 1 \text{ and } \omega^j U_a V_y \in Z(S) \setminus \overline{S} \}.
\]

Theorem 2.13 \([AP02]\) Let \( S \) be an Gottesman subgroup of the error group \( E \) and \( \overline{S} \) be the closure of \( S \). Then \( C(S) \) is a \( t \)-error correcting quantum code if \( wt(a, b) > 2t \) for each \( \omega^j U_a V_y \in Z(S) \setminus \overline{S} \).

We introduce a useful notation for describing quantum stabilizer codes. Let \( S \) be a Gottesman subgroup of \( E \) with centralizer \( Z(S) \). The minimum distance \( d(S) \) is defined to be the minimum of
\[
\{ wt(a, b) \ | \ \omega^i U_a V_y \in Z(S) \setminus \overline{S} \}.
\]
When \( A \) is the additive abelian group of the finite field \( \mathbb{F}_q \) we define an \( [[n, k, d]] \) quantum stabilizer code to be a \( q^k \)-dimensional subspace \( C(S) \) of \( L^2(\mathbb{F}_q)^{\otimes n} \), where \( S \) is a Gottesman subgroup of \( E \) with \( d(S) \ge d \) and cardinality \( q^{n-k} \).

By Theorem 2.13 it follows that an \( [[n, k, d]] \) quantum stabilizer code is a \( \lfloor (d - 1)/2 \rfloor \)-error correcting quantum code.
3 Nonstabilizer codes over finite fields

We focus our attention to the case when the abelian group $A$ is the field $\mathbb{F}_q$. Recall that the additive group $\mathbb{F}_q^n$ is a vector space over $\mathbb{F}_q$. If $\omega$ is a nontrivial character then the characters of the additive group $\mathbb{F}_q^n$ is the set $\{\omega_a | a \in \mathbb{F}_q^n\}$ where $\omega_a(b) = \omega(a,b), a,b = \sum a_ib_i$.

The set of operators $\{\omega_p^i U_a V_b | a,b \in \mathbb{F}_q^n\}$, $\omega_p$ is a $p^{th}$ root of unity, where $p$ is the characteristic of the field $\mathbb{F}_q$ forms an irreducible representation of associated error group. It is shown in [AP02] that any Gottesman subgroup $S$ is of the following form

$$\{\omega(\rho(a)) U_{La} V_{M}a | a \in \mathbb{F}_q^r\},$$

where $L$ and $M$ are $n \times r$ matrices over $\mathbb{F}_q$ such that $L^T M$ is symmetric and $\rho(.)$ satisfies the condition that

$$\rho(v_1 + v_2) - \rho(v_1) - \rho(v_2) = v_2^T L^T M v_1.$$

Our goal in this section is to seek for nonstabilizer codes with the help of Theorem 2.11.

**Definition 3.1** A Gottesman subgroup $S$ of the error group $E$ is said to be $d$-pure if $wt(g) \geq d$ for every $g \in Z(S)$.

By the theory of stabilizer codes, it follows that the corresponding stabilizer code $C(S) = \{\psi \in L^2(A)^{\otimes n} | U \psi = \psi \forall U \in S\}$ is a $[(d - 1)/2]$-error correcting quantum code.

More precisely, our aim is to start with the stabilizer code $C(S)$ of distance $d$, and use Theorem 2.11 to construct nonstabilizer codes of the same distance but larger dimension.

Observe that if $S$ is a $2t$-pure Gottesman subgroup of $E$, the first condition in Theorem 2.11 is vacuously true. Thus, we only need to ensure that the second condition in Theorem 2.11 is satisfied. For a $d$-pure Gottesman code we define the forbidden set as follows

**Definition 3.2** Let $S$ be a $d$-pure Gottesman subgroup of the error group $E$. We define the $d$-forbidden subset of $S$, denoted by $F_d(S)$, to be the subset

$$F_d(S) = \{u \in S : \exists g \in E \setminus S \text{ wt}(g) < d \text{ and } \sum_{s \in S} (\gamma_g \chi_u)(s) = \#S\}.$$ 

We have the following theorem that is straight forward consequence of Theorem 2.11.

**Theorem 3.3** Let $S$ be a $d$-pure Gottesman subgroup of the error group $E$ then $B \subseteq S$ is the Fourier description of a distance $d$ code iff $B^{-1} B \cap F_d(S)$ is empty.

Let $s_a$ denote $\omega_p(\rho(a)) U_{La} V_{Ma} \in S$. Observe that $\tau : s_a \mapsto a$ is a group isomorphism from $S$ to $\mathbb{F}_q^n$ and $\chi_{s_a} \mapsto \omega_a$ is an isomorphism from $S$ to $\mathbb{F}_q^n$.

Let $g = \omega_i U_x V_y \not\in S$ with $wt(g) \leq 2t$. By applying the Weyl commutation relations we get $\gamma_g(s_a) = \omega(a^T M^T x - a^T L^T y) = \omega_a(M^T x - L^T y)$. We have the following theorem.

10
Theorem 3.4 If \( S = \{ \omega(\rho(a))U_{Ia}V_{Ma} : a \in \mathbb{F}_q^r \} \) is \( d \)-pure Gottesman subgroup of the error group \( E \) over \( \mathbb{F}_q^n \). The \( d \)-forbidden subset of \( S \) is given by

\[
\mathcal{F}_d(S) = \{ s_u : \exists x, y \in \mathbb{F}_q^n \text{ wt}(x, y) < d \text{ and } u = L^T y - M^T x \}
\]

Proof.

\[
\sum_{s \in S} \gamma_g(s) \chi_u(s) = \sum_{a \in \mathbb{F}_q^r} \omega_a(u + M^T x - L^T y)
\]

Note that the right hand side of the equation is the sum over all character of \( \mathbb{F}_q^r \) and hence is nonzero iff \( u + M^T x - L^T y = 0 \). Hence

\[
\mathcal{F}_d(S) = \{ s_u : \exists x, y \in \mathbb{F}_q^n \text{ wt}(x, y) < d \text{ and } u = L^T y - M^T x \}
\]

Remark 3.5 In the above setting we will call the set

\[
F_d(S) = \{ u : \exists x, y \in \mathbb{F}_q^n \text{ wt}(x, y) < d \text{ and } u = L^T y - M^T x \}
\]

the forbidden set. Note that \( \mathcal{F}_d(S) = \{ s_u : u \in F_d(S) \} \).

4 Bounds on the dimension of codes

We now give upper and lower bounds on the dimension of nonstabilizer codes built from pure Gottesman subgroups of \( E \). Let the encoding space be \( L^2(A)^{\otimes n} \), and \( N(n, q, d) \) denote the number \( \sum_{i=0}^d \binom{n}{i} (q^2 - 1)^i \). We have the following upper bound on the dimension of the code.

Theorem 4.1 Let \( C_n \subseteq L^2(A)^{\otimes n} \) be a \( d \)-error correcting quantum code such that its corresponding projection \( P \) has support in a \( 2d + 1 \)-pure Gottesman subgroup \( S \) of the error group \( E \). Let \( B \) be the Fourier description of \( C_n \). Then the dimension of the code \( C_n \) satisfies the inequality

\[
\dim(C_n) \leq \frac{\# A^n}{N(n, \# A, d)}.
\]

Proof. Since \( S \) is \( 2d + 1 \)-pure, \( \{ g \in E \mid \text{ wt}(g) \leq 2d \} \subseteq E \setminus S \). By Theorem 2.11 we have:

\[
P g P = 0
\]

for all \( g \in E \) such that \( \text{ wt}(g) \leq 2d \). Let \( P_g \) denote the projection \( g^{-1} P g \). The range of \( P_g \) has dimension \( \dim(C_n) \) for every \( g \in E \). Furthermore, for all \( g_1, g_2 \in E \) such that \( \text{ wt}(g_1) \leq d \) and \( \text{ wt}(g_2) \leq d \), we have

\[
P_{g_1} P_{g_2} = g_1^{-1} P g_1 g_2^{-1} P g_2 = 0,
\]
since \( wt(g_1g_2) \leq 2d \) implies \( P_{g_1}g_2P = 0 \). Thus, \( \{ P_g \mid g \in \mathcal{E}, wt(g) \leq d \} \) is a collection of mutually orthogonal projections in \( L^2(A)^{\otimes n} \). Furthermore, the range of each \( P_g \) is \( \dim(C_n) \). Since there are \( N(n, \#A, d) \) elements \( g \in \mathcal{E} \) with \( wt(g) \leq d \), it follows by adding dimensions that

\[
N(n, \#A, d) \dim(C_n) \leq \dim L^2(A)^{\otimes n} = \#A^n.
\]

We now show a lower bound for the code dimension for codes satisfying the conditions of Theorem 3.4.

**Theorem 4.2** Let \( A = \mathbb{F}_q^n \), and \( \mathcal{S} = \{ \omega(\rho(a))U_{La}V_{Ma} \mid a \in \mathbb{F}_q^n \} \) be a \( 2d + 1 \)-pure Gottesman subgroup of the error group \( \mathcal{E} \) for the encoding space \( L^2(A)^{\otimes n} \). Then there is a \( d \)-error correcting code \( C_n \) such that its corresponding projection has support in \( \mathcal{S} \) and

\[
\dim(C_n) \geq \frac{q^n}{N(n, q, 2d)}.
\]

**Proof.** For the \( 2d + 1 \)-pure Gottesman subgroup \( \mathcal{S} = \{ \omega(\rho(a))U_{La}V_{Ma} \mid a \in \mathbb{F}_q^n \} \), let \( X \) be the corresponding forbidden set. By Theorem 3.4, \( X \) is the image of the set \( \{ (x, y) \mid x, y \in \mathbb{F}_q^n, wt(x, y) \leq d - 1 \} \) under the map \( (x, y) \mapsto L^T y - M^T x \). The number of \( x, y \in \mathbb{F}_q^n \) such that \( wt(x, y) \leq 2d \) is \( N(n, q, 2d) \). Hence \( \#X \leq N(n, q, 2d) \). We prove the existence of the code \( C_n \) by constructing its Fourier description \( B \subseteq \mathcal{S} \) using the following “greedy” strategy to pick elements from \( \mathcal{S} \):

1. Initially, let \( B = \{ \} \) and let \( A = \mathcal{S} \).
2. Pick any \( u \in A \) and include in \( B \).
3. Remove from \( A \) all elements \( v \) such that difference \( u - v \) is in \( X \), where \( u \) is the element picked in the previous step.
4. If \( A = \emptyset \) stop. Otherwise, return to Step 2.

Note that this strategy will eliminate at most \( \#X \) elements from \( A \) every time we include a new element in \( B \). Thus, the number of elements picked into \( B \) will be at least \( \left\lceil \frac{\#S}{\#X} \right\rceil \geq \frac{\#S}{N(n, q, 2d)} \). Applying Lemma 2,8 yields the desired lower bound.

We can now easily argue about the existence of asymptotically good nonstabilizer codes. The following theorem is a paraphrase of a result we proved in [AP02] about the existence of \( d \)-pure maximal Gottesman subgroups of the error group \( \mathcal{E} \) for large \( d \), for encoding space \( L^2(A)^{\otimes n} \) for large \( n \) \((A = \mathbb{F}_2)\).

We first need the following technical definition.

**Definition 4.3** [AP02] An \( n \times n \) matrix \( R \) over \( \mathbb{F}_2 \) is said to be \( \alpha \)-good if the following conditions are true.
(i) The sum of every $\lfloor \alpha n \rfloor$ columns of $R$ has weight at least $\alpha n$.

(ii) The sum of every $\lfloor \alpha n \rfloor$ rows of $R$ has weight at least $\alpha n$.

(iii) The sum of every $\lfloor \alpha n \rfloor$ columns of $R$ has weight at most $(1 - \alpha)n$.

(iv) The sum of every $\lfloor \alpha n \rfloor$ rows of $R$ has weight at most $(1 - \alpha)n$.

It is shown in [AP02] that there is a constant $\alpha > 0$ and a corresponding positive integer $n_\alpha$ such that

$$\Pr[R \text{ is } \alpha\text{-good}] > 0.$$ 

**Theorem 4.4** [AP02] For $0 < \alpha < 1$, suppose $R$ is an $n \times n$ $\alpha$-good matrix over $\mathbb{F}_2$. Let $L$ be the following $2n \times 2n$ symmetric matrix over $\mathbb{F}_2$:

$$
\begin{pmatrix}
0 & R \\
R^T & 0
\end{pmatrix}
$$

If we write $L = D + D^T$, where $D$ is the upper triangular matrix with zeros on the principal diagonal, then $\mathcal{S} = \{\tilde{\omega}(a^T Da)U_aV_{La-b} \mid a \in C, b \in C^\perp\}$, is an $\lfloor \alpha n \rfloor$-pure maximal Gottesman subgroup $\mathcal{S}$ of the error group $\mathcal{E}$.

Now, applying Theorem 4.2 we immediately get the following family of asymptotically good nonstabilizer codes.

**Corollary 4.5** For $0 < \alpha < 1$, suppose $R$ is an $n \times n$ $\alpha$-good matrix over $\mathbb{F}_2$ and $\mathcal{S}$ is the $\lfloor \alpha n \rfloor$-pure maximal Gottesman subgroup $\mathcal{S}$ of the error group $\mathcal{E}$ (defined in the above theorem). Then there is an $\lfloor \alpha n - 1/2 \rfloor$-error correcting quantum code of dimension $\frac{2^n}{N(n,2,\lfloor \alpha n \rfloor)}$, whose projection has support in $\mathcal{S}$.

## 5 Explicit construction of Non-stabilizer codes

We now give an explicit construction of a family of distance 2 code. Recall that any abelian group of the error group is of the form

$$\mathcal{S} = \{\omega(\rho(a))U_aV_{M a} : a \in \mathbb{F}_q^n\}$$

where $L$ and $M$ are $r \times n$ matrices over $\mathbb{F}_q$ such that $L^T M$ is symmetric and $\rho$ satisfies the condition

$$\rho(a_1 + a_2) - \rho(a_1) - \rho(a_2) = a_1^T L^T M a_2.$$ 

Given an odd integer $n = 2m+1$, we give the explicit construction of a $((n, 1+n(q-1), 2))_q$ code. Note that if $q = 2$ and $n = 5$ we get a $((5, 6, 2))_2$ code. In [RHSN97] a $((5, 6, 2))_2$ code is given which is generated by a computer search. They have also shown that for distance 2 this is the best possible code. We also show that there is a code of dimension greater than $\left\lfloor \frac{q^n}{n(q^2-1)} \right\rfloor$. 13
Let \( x \in \mathbb{F}_q^n \) is all zeros except at positions \( m + 1 \) and \( m + 2 \) where it is 1. Define the matrices \( S \) and \( L \) as follows

\[
S = \begin{pmatrix}
    x \\
    \sigma x \\
    \vdots \\
    \sigma^{n-1} x \\
\end{pmatrix}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Theorem 5.3  The set $B$ as defined above is the Fourier description of a $((n, 1+n(q-1), 2))_q$ code.

Proof. Let $D = (B - B) \setminus \{0\}$. Since $F_2$ does not contain the zero vector it is sufficient to prove that $F_2 \cap D$ is empty. Let $u_i = \left(\sum_{j=1}^{n-1} e_j\right) - e_i, 1 \leq i \leq n-1$. Now $D = A_1 \cup A_2 \cup A_3 \cup A_4$ where

$A_1 = \{\alpha e_0, \alpha \neq 0\}$  
$A_2 = \{e_0 + \alpha u_i : \alpha \neq 0\}$  
$A_3 = \{\alpha e_0 + \beta u_i : \beta \neq 0; \alpha \neq -1; 1 \leq i \leq n-1\}$  
$A_4 = \{\alpha u_i + \beta u_j : \alpha, \beta \neq 0; 1 \leq i, j \leq n-1; \alpha \neq \beta \text{ or } i \neq j\}$

where $\alpha, \beta \in F_q$. The elements of the forbidden set $F_2$ are given by

$R_1 = \{a(e_m + e_{m+1}) + be_0 + (a - b)\mathbb{1} : (a, b) \neq (0, 0)\}$
$R_2 = \{a(e_0 + e_1) + be_{m+1} : (a, b) \neq (0, 0)\}$
$R_3 = \{a(e_0 + e_{n-1}) + be_m : (a, b) \neq (0, 0)\}$
$R_4 = \{a(e_{j+m} + e_{j+m+1}) + be_j + a\mathbb{1} : (a, b) \neq (0, 0); j \neq 0, j \neq m, j \neq m + 1\}$

Now it can be verified that $A_i \cap R_j$ is empty for every $1 \leq i, j \leq 4$. ■

6 Examples of 1-error correcting nonstabilizer codes

In this section we give explicit constructions of a $((33, 155, 3))$ code and a $((15, 8, 3))$ code. The codes we construct will be over the field $F_2$.

Let $n = 2m + 1$ be any odd integer. Let $x$ be the vector in $F_2^n$ with zeros at all positions except $m + 1$ and $m + 2$. As in the previous section, let

$$S = \begin{pmatrix} x \\ \sigma x \\ \vdots \\ \sigma^{n-1} x \end{pmatrix}$$

$$L = \begin{pmatrix} I_{n-1} & 0 \\ \vdots \\ 1 \ldots 1 & 0 \end{pmatrix},$$

where $\sigma$ is the cyclic shift. Let $J$ be the $n \times n$ matrix all of whose entries are 1’s. Recall that the generalized Laflamme code is the stabilizer code associated with stabilizer group given by

$S = \{U_{La}V_{Ma} : a \in F_2^n\}$,

where $M = SL + J$. The corresponding 1-forbidden set is given by

$F_1 = \{aM^Te_i + bL^Te_i : a, b \in F_2\}$.

It can be easily verified that $F_{d+1} = F_d + F_1$. Now, let $W_d = \{wt(a) : a \in F_d\}$. It can be easily checked that $W_1 = \{1, 2, 3, n - 3, n - 2, n - 1\}$ and $W_2 = \{1, 2, 3, 4, 5, 6, n - 6, n - 5, n - 4, n - 3, n - 2, n - 1\}$.
If \( B \) is a subset of \( \mathbb{F}_2^3 \) such that for \( u \in B - B \) we have \( \text{wt}(u) \notin W_2 \) then \( B \) is the Fourier description of a 1-error correcting quantum code. A natural approach to finding large Fourier descriptions \( B \) is to solve the following combinatorial problem.

**Problem 6.1** Construct a family of subsets \( \mathcal{F} \) of \( \{1, 2, \ldots, n\} \) such that for all \( S_1, S_2 \in \mathcal{F} \), \( S_1 \neq S_2 \) we have
\[
\#(S_1 \setminus S_2) + \#(S_2 \setminus S_1) \notin W_2.
\]

Given such a collection of subsets \( \mathcal{F} \), it is clear that the set \( B \) defined as
\[
B = \left\{ \sum_{i \in S} e_i : S \in \mathcal{F} \right\}
\]  
will yield the Fourier description of a 1-error correcting quantum code by Theorem 2.11.

For, the condition on the family of subsets \( \mathcal{F} \) will ensure that the weight of any element in \( B - B \) does not lie in the set \( \{1, 2, 3, 4, 5, 6, n - 6, n - 5, n - 4, n - 3, n - 2, n - 1\} \) and hence \( (B - B) \cap F_2 = \emptyset \).

As our first example we describe a \(((15, 8, 3))\) code. For \( n = 15 \) it suffices to construct a family of 8 subsets \( \mathcal{F} \) such that for any two distinct subsets \( S_1, S_2 \in \mathcal{F} \) we have \( \#(S_1 \setminus S_2) + \#(S_2 \setminus S_1) \in \{7, 8\} \). Then, \( B \) defined by Equation \( 6 \) will be the Fourier description of a \(((15, 8, 3))\) code. The eight subsets of \( \{1, 2, \ldots, 15\} \) that we pick are as follows:

\[
S_1 = \{1, 2, 3, 4, 13\}, \quad S_2 = \{5, 6, 7, 8, 13\}, \quad S_3 = \{9, 10, 11, 12, 13\},
\]

\[
S_4 = \{1, 2, 5, 6, 9, 10\}, \quad S_5 = \{1, 2, 7, 8, 11, 12\}, \quad S_6 = \{3, 4, 7, 8, 9, 10\},
\]

\[
S_7 = \{3, 4, 5, 6, 11, 12\}, \quad S_8 = \{14, 15\}.
\]

In order to construct such nonstabilizer codes for general \( n \) we need to construct explicit set families \( \mathcal{F} \) as a solution to Problem 6.1. To this we describe a general method and use it to construct a \(((33, 155, 3))\) code. More precisely, we will seek a special solution of Problem 6.1 in which all the sets in \( \mathcal{F} \) are of the same cardinality.

Consider the case \( n = 33 \). In our explicit construction we consider only subsets of \( \{1, 2, \ldots, 32\} \). Consider the vector space \( \mathbb{F}_2^3 \). As sets, \( \mathbb{F}_2^3 \) and \( \{1, 2, \ldots, 32\} \) are of the same size and can be identified using any 1-1 correspondence. Our goal is essentially to find a family of subsets of \( \mathbb{F}_2^3 \) satisfying the above conditions. Let \( \mathcal{F} \) be the family of all 3 dimensional subspaces of \( \mathbb{F}_2^3 \). Since any \( S \in \mathcal{F} \) is a vector space over \( \mathbb{F}_2 \) of dimension three, we have \( \#S = 2^3 = 8 \). Moreover any two distinct subspaces can have at most 4 vectors in common. Hence for every pair of distinct sets \( S_1, S_2 \in \mathcal{F} \) we have \( \#S_1 \cap S_2 \leq 4 \). Consequently, for distinct sets \( S_1, S_2 \in \mathcal{F} \) we have
\[
8 \leq \#(S_1 \setminus S_2) + \#(S_2 \setminus S_1) \leq 14.
\]

Thus, for distinct sets \( S_1, S_2 \in \mathcal{F} \) \( \#(S_1 \setminus S_2) + \#(S_2 \setminus S_1) \notin W_2 \) and hence the corresponding Fourier description \( B \) gives rise to a 1-error correcting code.

Now, to find the size of the set \( B \) which is the dimension of the code we have to find the size of \( \mathcal{F} \). The following general theorem gives the exact size.
**Theorem 6.2** Consider the vector space $\mathbb{F}_q^m$. The number of subspace of dimension $r$ is given by

$$N(n,q,r) = \frac{(q^m - 1)(q^{m-1} - 1) \ldots (q - 1)}{(q^r - 1)(q^{r-1} - 1) \ldots (q - 1)(q^{m-r} - 1)(q^{m-r-1} - 1) \ldots (q - 1)}$$

**Proof.** Let $e_1, e_2, \ldots, e_m$ be the standard basis for $\mathbb{F}_q^m$. Let $\mathcal{T}$ be the family of all $r$ dimensional subspaces of $\mathbb{F}_q^m$. We want to find $\# \mathcal{T}$. Let $R$ be the subspace of $\mathbb{F}_q^m$ spanned by the vectors $e_1, e_2, \ldots, e_r$. Consider the group $G = GL_m(\mathbb{F}_q)$. $G$ acts on $\mathcal{T}$ transitively and hence the orbit of $R$ under the $G$ action is the whole of $\mathcal{T}$. Hence the number of elements in $\mathcal{T}$ is given by

$$\# \mathcal{T} = \frac{\# G}{\# G_R}$$

where $G_R$ is the subgroup of $G$ that leaves $R$ invariant.

Any element of $G_R$ is of the form

$$\left( \begin{array}{cc} A & * \\ 0 & B \end{array} \right)$$

where $A$ and $B$ are $r \times r$ and $(m - r) \times (m - r)$ nonsingular matrices respectively and $*$ is any $r \times (m - r)$ matrix. If $g(m) = \# GL_m(\mathbb{F}_q)$ then we have

$$\# G_R = g(r)g(m - r)q^{(m-r)r}.$$ 

So the problem reduces to finding $g(m)$. Let $A$ be any matrix in $GL_m(\mathbb{F}_q)$. The first column of $A$ can be any one of the nonzero vectors in $\mathbb{F}_q^m$. The are $q^m - 1$ nonzero vectors. Having fixed the first column $a_1$, we have $q^m - q$ choices for the second column $a_2$. Similarly there are $q^m - q^2$ choices for the third column and so on. Therefore the number of elements in $GL_m(\mathbb{F}_q)$ is given by

$$g(m) = \prod_{i=0}^{m-1} (q^m - q^i).$$

Therefore the size of $\mathcal{T}$ is given by

$$\# \mathcal{T} = \frac{g(m)}{g(r)g(m - r)q^{(m-r)r}} = \frac{(q^m - 1)(q^{m-1} - 1) \ldots (q - 1)}{(q^r - 1)(q^{r-1} - 1) \ldots (q - 1)(q^{m-r} - 1)(q^{m-r-1} - 1) \ldots (q - 1)}$$

From Theorem 6.2 we have

$$\# \mathcal{T} = N(5,2,3) = 155$$

and hence the set $B$ defined as

$$B = \left\{ \sum_{i \in S} e_i : S \in \mathfrak{F} \right\}$$

gives a $((33,155,3))$ code.
Remark 6.3 We can actually obtain a \((31, 155, 3)\) code as follows: construct the same family of 3-dimensional subspaces of \(\mathbb{F}_2^5\) which are 155 in number. Now, drop the extra coordinate, which will still result in 155 distinct subsets such that the symmetric difference of any pair of these has weight in the range 7 to 13. Thus, we have a \((31, 155, 3)\) nonstabilizer code. We can easily extend this puncturing argument to other nonstabilizer codes.

7 Encoding circuits for a class of nonstabilizer codes

In this section we discuss the encoding algorithm for the class of non-stabilizer codes defined in Section 5 and the asymptotically good codes of Corollary 4.5. Recall that given a Gottesman subgroup, a code can be specified by giving its Fourier description. We fix our encoding space to be \(L^2(\mathbb{F}_n^q)\). Let \(C = \{a| \sum a_i = 0\}\) and \(C^\perp\) the set \(\{b| b^T a = 0\text{ for all }a\in C\}\) We restrict attention to maximal Gottesman subgroups of the form

\[
S = \{s_{a,b} = \omega(a^T D a) U_a V_{la+b}| a \in C, b \in C^\perp\}
\]

where \(D\) is an upper triangular matrix and \(L = D + D^T\). Consider a code \(C_n\) with Fourier description \(B \subseteq S\). Recall that the dimension of the code is \(#B\). Due to the isomorphism \(S \cong C \times C^\perp \rightarrow \mathbb{F}_q^n \times \mathbb{F}_q^n\) we have the character group of \(S\) as

\[
\chi_{s_{c,d}}(s_{a,b}) = \chi_{c,d}(a, b) = \omega(a^T c + b^T c).
\]

For \(u \in S\) define \(S_u\) to be the abelian group \(\{\chi_u(s)|s \in S\}\). It is easy to see that \(S_u\) is also a maximal Gottesman subgroup. In this notation we have \(S_1 = S\). Let \(C_{S_u}\) denote the stabilizer (one dimensional) code corresponding to the Gottesman subgroup \(S_u\). Let \(\{|\varphi_u\rangle\}\) denote a (singleton) orthonormal basis for \(C_{S_u}\) for each \(u \in S\).

Theorem 7.1 The vectors \(|\varphi_u\rangle, u \in B\) forms an orthonormal basis for the code with Fourier description \(B\).

Proof. The projection operators for the code with Fourier description \(B\) is given by

\[
P = \frac{1}{\#S} \sum_{u \in B} \sum_{s \in S} \chi_u(s)s = \sum_{u \in B} P_u
\]

where \(P_u = \frac{1}{\#S} \sum_{s \in S} \chi_u(s)s\). Note that \(P_u\) is nothing but the projection operator corresponding to the stabilizer code \(C_{S_u}\). To prove that \(\{|\varphi_u\rangle | u \in B\}\) forms an orthonormal basis for the code given by \(B\) it suffices to show that

\[
P_u P_v = \begin{cases} 0 & \text{if } u \neq v, \\ P_u & \text{otherwise,} \end{cases}
\]

which is an immediate consequence of the following:

\[
P_u P_v = P_{\chi_u \chi_v} = \begin{cases} 0 & \text{if } u \neq v, \\ P_u & \text{otherwise.} \end{cases}
\]
It follows that \( \{ |\varphi_u \rangle \mid u \in B \} \) is an orthonormal basis for \( C_n \).

For \( s_{a,b} \in S \) instead of writing \( S_{s_{ab}} \) we will write \( S_{a,b} \). Similarly \( |\varphi_{a,b} \rangle \) will be used to denote \( |\varphi_{s_{a,b}} \rangle \). It is easy to see that

\[
|\varphi_{c,d} \rangle = \sqrt{\frac{1}{#C}} \sum_{x \in C} \omega((x + d)^T D(x + d) \bar{\omega}(x^T c) |x + d \rangle.
\]

Since the code has dimension \( #B \), we will assume that the encoding message space is a Hilbert space of dimension \( #B \) with basis are indexed by elements of \( B \) viz. \( \{ |c,d \rangle \mid s_{c,d} \in B \} \).

To summarize we have the following observation.

**Proposition 7.2** For a code with Fourier description \( B \) then the encoding procedure is given by linear map with the following property \( |c,d \rangle \mapsto |\varphi_{c,d} \rangle \) for all \( s_{c,d} \in B \).

We give encoding circuits for the codes over \( \mathbb{F}_q \). We assume that some basic gates over \( L^2(\mathbb{F}_q) \), which we will define in a moment, are given as black boxes. It is to be noted that for a fixed \( q \) these gates are easily implementable using the standard set of gates. The most basic of the gates which we require is the inverter gate defined as follows

\[
\mathcal{I} |a \rangle = |-a \rangle ; \ a \in \mathbb{F}_q.
\]

Let \( C-U \) and \( C-V \) stand for the unitary transformation define as follows

\[
C-U |a \rangle |b \rangle = |a \rangle |a + b \rangle.
\]

and

\[
C-V |a \rangle |b \rangle = \omega(ab) |a \rangle |b \rangle ; \ a, b \in \mathbb{F}_q.
\]

Note that for \( q = 2 \) the gate \( C-U \) is just the \( C-NOT \) gate and the gate \( C-V \) is the \( Z \) gate. We also assume that the operator \( CC-U \) defined as is available:

\[
CC-U |a,b,c \rangle = |a,b,c + ab \rangle.
\]

For \( \mathbb{F}_2 \) it is the \( CC-NOT \) gate. Note that \( CC-V \) defined as

\[
CC-V |a,b,c \rangle = \omega(c + ab) |a,b,c \rangle
\]

can be defined using \( CC-U, \mathcal{I} \) and \( C-V \). See Figure [ ] for a pictorial description.

We need one more operator which is the Fourier transform operator over \( \mathbb{F}_q \) which we will denote by \( F \). It is defined as follows

\[
F |a \rangle = \frac{1}{\sqrt{q}} \sum_{x \in \mathbb{F}_q} \omega(ax) |x \rangle.
\]

Note that if \( q = 2 \) then \( F \) is nothing but the Hadamard operator. Consider any gate \( C-f \) of two arguments \( a,b \in \mathbb{F}_q \) defined as
\[ C-f |a\rangle |b\rangle = |a\rangle |f(a,b)\rangle. \]

We extend it to an operator \(C-f_n\) acting on two arguments \(a, b \in \mathbb{F}_q^n\) in a natural way as follows (see Figure 2):

\[ C-f_n |a\rangle |b\rangle = C-f_n |a_1 a_2 \ldots a_n \rangle |b_1 b_2 \ldots b_n \rangle = |a\rangle |f(a_1, b_1), f(a_2, b_2), \ldots, f(a_n, b_n)\rangle. \]

In a similar fashion we extend the gate \(CC-f\) acting on \(a, b, c \in \mathbb{F}_q\) to \(CC-f_n\) acting on \(a, b, c \in \mathbb{F}_q^n\). In the circuits we draw a \(C-f\) gate with thick wires to indicate that it takes a tuple from \(\mathbb{F}_q^n\).

From these gates we can construct a circuit that computes for an \(y\) an \(a, b \in \mathbb{F}_q^n\) the dot product \(\sum_i a_i b_i\) (see Figure 3). The second circuit in the figure is a symbolic representation of the circuit (note the thick lines). Using this inner product circuit we can also define the circuit that takes a \(n \times n\) matrix \(D\) and a vector \(a \in \mathbb{F}_q^n\) and computes the vector \(Da\) (see Figure ?? note the thick lines and the cut).

We also need a circuit which will take the vector \(|0^n\rangle\) and generate the uniform superposition \(\frac{1}{\sqrt{q^n-1}} \sum_{x \in C} |x\rangle\). This circuit is given in Figure ??.

Given the message \(c, d\), we can describe the main steps of the encoding algorithm as follows.

1. Initialize \(|R\rangle := |0^n\rangle\)

2. Apply \(C\) on \(R\) so that

\[ |R\rangle = \sqrt{\frac{1}{q^n-1}} \sum_{x \in C} |x\rangle. \]

3. Apply \(C-V_n\) on \(|c\rangle |R\rangle\) so that

\[ |c\rangle |R\rangle \mapsto |c\rangle \otimes (\frac{1}{\sqrt{q^n-1}} \sum_{x \in C} \omega(c^T x) |x\rangle). \]

4. Apply \(C-U_n\) on \(|d\rangle |R\rangle\)

\[ |d\rangle |R\rangle \mapsto |d\rangle \otimes (\frac{1}{\sqrt{q^n-1}} \sum_{x \in C} \omega c^T x |x + d\rangle). \]

5. Apply the circuit in Figure ?? on \(|R\rangle |D\rangle |0^n\rangle\)

\[ |R\rangle |D\rangle |0^n\rangle \mapsto \frac{1}{\sqrt{q^n-1}} \sum_{x \in C} \omega(c^T x) |x + d, D(x + d)\rangle |D\rangle. \]

6. Apply \(C-V_n\) on \(|R_1\rangle = \frac{1}{\sqrt{q^n-1}} \sum_{x \in C} \omega(c^T x) |x + d, D(x + d)\rangle\) to get

\[ |R_1\rangle \mapsto \frac{1}{\sqrt{q^n-1}} \sum_{x \in C} \omega((x + d)^T D(x + d) + c^T x) |x + d, D(x + d)\rangle. \]
8 Decoding for a class of nonstabilizer codes

Let $\mathcal{C} \subseteq \mathcal{H}^\otimes n$ be a $t$-error correcting quantum code satisfying the conditions of Theorem 2.11 with Fourier description $B$, and such that its projection $P$ has support in the Gottesman subgroup $S$ of the error group $E$. The two conditions that $\mathcal{C}$ satisfies are:

1. For each $g \in S$ such that $wt(g) \leq 2t$
   \[ \chi_g(u_1^{-1}u_2) = 1 \text{ for all } u_1, u_2 \in B. \]

2. For each $g \in E \setminus S$ such that $wt(g) \leq 2t$, and for $s \in S$ we have
   \[ \sum_{u_1, u_2 \in B} \chi_{u_2}(s) \sum_s \gamma(g, s_1) \chi_{s_1}(u_1^{-1}u_2) = 0, \]
   where $\gamma(g, s)$ is the scalar such that $\gamma(g, s)sg = gs$.

For $u \in S$ let $S_u$ be the Gottesman group $\{ \chi_u(s) | s \in S \}$.

The projection operator for the code $\mathcal{C}_u$

\[ P = \frac{1}{\#S} \sum_{u \in B} \sum_s \chi_u(s)s = \sum_{u \in B} P_u, \]

where $P_u = \frac{1}{\#S} \sum_{s \in S} \chi_u(s)s$. Note that $P_u$ is the projection operator corresponding to the stabilizer code $\mathcal{C}_u$ with $S_u$ as stabilizer group for each $u \in B$. As argued in Theorem 7.1, $P_uP_v = 0$ for $u \neq v \in B$. Let $D_u$ denote an orthonormal basis for $\mathcal{C}_u$, $u \in B$. Then $D = \bigcup_{u \in B} D_u$ is an orthonormal basis for $\mathcal{C}$.

It suffices to describe the decoding procedure for the encoded message as a basis element from $D$ and error $g \in E$ such that $wt(g) \leq t$. W.l.o.g. let $|\psi\rangle \in D_u$ for some $u \in B$ be the encoded message, and let $g \in E$ of weight at most $t$ be the error operator. The decoding procedure takes $g |\psi\rangle = |\psi'\rangle$ as input and outputs $|\psi\rangle$. Let $\{s_1, s_2, \ldots, s_k\}$ be an independent generator set for $S$. Notice that for $1 \leq i \leq k$

\[ s_i(g |\psi\rangle) = \gamma(s_i, g) \chi_u(s_i)g |\psi\rangle. \]

Thus, $g |\psi\rangle$ is an eigen vector for operator $s_i$ with eigen value $\gamma(s_i, g) \chi_u(s_i)$.

The decoding procedure will carry out the following steps. It uses as subroutine the phase estimation algorithm of Kitaev (c.f [NC99]).

1. Let $|\psi'\rangle$ be the received state.

2. Apply $s_i$ successively, for each $1 \leq i \leq k$, and when $s_i$ is applied run Kitaev’s phase estimation algorithm to compute the eigen value $\alpha_i = \gamma(s_i, g) \chi_u(s_i)$. Let $|\rho\rangle$ be the resulting state.
3. If \( g = I \) and some \( u \in B \) constitute a solution to the system of \( k \) group equations (using a classical algorithm that searches through \( B \)):
\[
\alpha_i = \gamma(s_i, g)\chi_u(s_i), 1 \leq i \leq k,
\]
then apply \( s_i^{-1} \) for each \( 1 \leq i \leq k \) to the state \( |\rho\rangle \) and output that as the decoded state.

4. If \( g = I \) does not give a solution to the \( k \) equations, find (using a classical algorithm that searches through \( B \)) a \( g \neq I \in \mathcal{E} \) and the corresponding unique \( u \in B \) which are a solution to the \( k \) equations. Apply \( g^{-1} \) to the current state. Then apply \( s_i^{-1} \) for each \( 1 \leq i \leq k \), and output that as the decoded state.

We now argue the correctness of the procedure. Firstly, notice that if the error operator is \( g \in \mathcal{S} \) such that \( wt(g) \leq t \), then by assumption \( \chi_g(u_2) = \chi_g(u_1) \) for all \( u_1, u_2 \in B \). Denote this scalar by \( \lambda \). Notice that for any state \( |\varphi\rangle \in \mathcal{C} \), \( g |\varphi\rangle = \lambda |\varphi\rangle \). Which means that \( g \) introduces only an overall phase. We establish the following claim from which the correctness of the procedure follows.

**Claim 8.1**

1. If \( g = I \) and \( u \in B \) is a solution to the \( k \) group equations given above, then \( u \in B \) is the unique solution, and there is no \( g \notin \mathcal{S} \) which is a solution to the equations.

2. If \( g = I \) is not part of a solution to the \( k \) equations, then there is a unique \( u \in B \) and some \( g \notin \mathcal{S} \) that form a solution such that \( wt(g) \leq t \).

**Proof of Claim.** For the first part, assume that \( g = I \) and \( u \in B \) is a solution to the \( k \) group equations, and some \( g \notin \mathcal{S} \) and \( u' \in B \) is another solution. Then we have \( \gamma(g, s_i)\chi_u(s_i)\chi'_u(s_i) = 1, 1 \leq i \leq k \). Since \( s_1, s_2, \ldots, s_k \) generate \( \mathcal{S} \), it implies \( \gamma(g, s_i)\chi_u(s)\chi'_u(s) = 1 \) for all \( s \in \mathcal{S} \). It is easy to see that this contradict the second condition of Theorem 2.11 for the element \( g \notin \mathcal{S} \) of \( wt(g) \leq t \).

For the second part, notice that any solution \( g \) of \( wt(g) \leq t \) to the equations is not in \( \mathcal{S} \). For, if \( g' \in \mathcal{S} \) and \( u \in B \) are a solution then so is \( g = I \) and \( u \in B \), because \( \gamma(g', s) = 1 \) for all \( s \in \mathcal{S} \). Assume to the contrary that there are two distinct solutions \( g_1 \notin \mathcal{S} \) and \( u \in B \) and \( g_2 \notin \mathcal{S} \) and \( u' \in B \), where \( wt(g_1) \leq t \) and \( wt(g_2) \leq t \). Then, as before, the \( k \) equations will yield
\[
\gamma(g_1, s)\chi_u(s) = \gamma(g_2, s)\chi'_u(s), \quad \forall s \in \mathcal{S}.
\]
By rearranging terms we get \( \gamma(g_1^{-1}g_2, s)\chi_u(u'\chi_u^{-1}) = 1 \) for all \( s \in \mathcal{S} \).

Now, if \( g_1^{-1}g_2 \notin \mathcal{S} \), then this again contradicts the second condition of Theorem 2.11 for the element \( g_1^{-1}g_2 \notin \mathcal{S} \) of \( wt(g_1^{-1}g_2) \leq 2t \).

Next, suppose that \( g_1^{-1}g_2 \notin \mathcal{S} \). Then we get \( \chi_u(u'\chi_u^{-1}) = 1 \) for all \( s \in \mathcal{S} \) implying that \( u = u' \). Thus \( u \) is unique. Notice that by the first condition of Theorem 2.11, since \( wt(g_1^{-1}g_2) \leq 2t \), the effect of applying the error \( g_1^{-1}g_2 \) to a state in \( \mathcal{C} \) only introduces an
overall phase. Thus, decoding with either $g_1^{-1}$ or $g_2^{-1}$ will coincide up to an overall phase. This completes the proof of the claim and correctness of the decoding procedure.

To analyze the efficiency of the decoding procedure, we recall from [NC99] that the phase estimation quantum circuit is efficient (polynomial size in $n$). However, solving the $k$ group equations involves exhaustive enumeration. This takes time $O(n^{O(d)}\#B)$, which is also the dominant term in the entire time bound.

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See Figure 1 for a complete circuit. The other figures are the building blocks. Note that the extra $|D(x + d)\rangle$ can be removed by inverting it and then applying the circuit in fig 4 on $|x + d, D(x + d)) |D\rangle$.

$|a\rangle$ \hspace{1cm} \hspace{1cm} |−a\rangle$

Invertor gate

$|a\rangle$ \hspace{1cm} \hspace{1cm} |a\rangle$

$|b\rangle$ \hspace{1cm} \hspace{1cm} $|a + b\rangle$

$C − U$ gate

$|a\rangle$ \hspace{1cm} \hspace{1cm} $F|a\rangle$

$F$ \hspace{1cm} \hspace{1cm} $F|a\rangle$

Fourier Transform gate

Figure 1: Basic Gates
Figure 2: Circuit computing $C - f_n$ from $C - f$
Figure 3: Circuit computing the dot product $a^T b$
Figure 4: Circuit to compute $Dx$
Figure 5: Circuit generating $\sqrt{\frac{1}{q^{n-1}} \sum_{x \in C} |x\rangle}$
Figure 6: Complete encoding circuit