A SIMPLIFIED BEAM-LIKE MODEL FOR THE DYNAMIC ANALYSIS OF MULTI-STOREY BUILDINGS

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Abstract. The analysis of multi-storey frames by means of simplified models, introduced in the last century, is still a subject of current interest since it allows drastically containing the required computational burden. Aiming at simulating the static and dynamic behaviour of entire buildings, beam-like models, equivalent to framed structures, have been formulated in recent researches. In the present study a performing beam-like model, suitable for the representation of buildings with non uniform mass and stiffness distribution along the height and within the floors, is proposed. Only shear and torsional deformability are considered in the proposed continuous beam thus leading to a simpler model, compared to those already presented in the literature taking into account also the flexural deformability. A Rayleigh-Ritz approach, based on a suitable number of modal shapes of the uniform shear-torsional beam models, allows to discretize the analysis and to evaluate the response in the generalized reduced space. Aiming at providing a numerical validation, the frequencies and modes of vibration of some reference multi-storey buildings, obtained by 3D FEM models, have been compared with those obtained considering the non-uniform beam-like model. The obtained results show a satisfactory correspondence with those provided by the detailed linear FEM simulation and proved the capability of the proposed equivalent non-uniform beam model to approximate the linear dynamic response of spatial asymmetric frames to seismic ground motion at a very low computational cost.
1 INTRODUCTION

The advancements on computational procedures and parallel processing of the last few years enhance accurate dynamic analysis and seismic assessments of multi-storey buildings. The analyses can be carried out at building or urban level and high fidelity models or sophisticated large scale simulations can be adopted respectively. In particular, the dynamic analysis of entire urban area needs optimized computational procedures and simplified but accurate numerical models. Thus, a new and renovated interest has grown recently on the beam-like models that have been introduced in the last century [1, 2]. The beam-like models, which are based on the equivalence of multi-level structures to flexural-shear coupling continuum beams, aim to simulate the dynamic behaviour of multi-level buildings by drastically reducing the computational effort. The simplified beam-like approaches firstly proposed in the literature make usually adopt homogeneous elements with uniform stiffness distribution [3]. Some contributions take into account uniform beams with eccentricity between the centers of stiffness and mass [4, 5, 6, 7]. Different mathematical models, such as Timoshenko and Euler-Bernoulli, have been considered for the analysis of the equivalent beam.

In the present paper a beam-like model suitable for a simplified modeling of buildings, which do not have uniform mass and stiffness distribution along the height and are characterized by unsymmetrical plans, is proposed. The approach takes into account only the shear and torsional deformability thus providing a simpler model with respect to the others already presented in the literature. As it will be shown in the applicative section, in spite of its simplicity, the shear torsional beam is able to provide a satisfactory dynamic behaviour of low- and mid-rise buildings with a significant reduction of the computational effort with respect to the one required by more general models.

In the model a rigid floors assumption is considered, however some corrective coefficients are introduced in order to take into account the actual flexural flexibility of the floors. The equations of motion of the proposed spatial unsymmetrical beam-like model are derived through the application of Hamilton's principle. The linear dynamic behaviour of the non-uniform beam-like element is then evaluated by discretizing the continuous model according to a Rayleigh-Ritz approach based on an appropriate number of modal shapes of the uniform shear torsional beam model [8].

Different multi-storey buildings have been studied by means of the proposed equivalent beam models and the results have been compared to those obtained through a 3D FEM modeling approach. This comparison has shown very satisfactory results both in terms of eigen-properties and dynamic response to earthquake excitations thus confirming the capability of the proposed simplified model for investigating the seismic behavior of multi-storey buildings with a strongly reduced computational demand.

2 THE CONSIDERED BEAM-LIKE MODEL

In this paper multi-storey unsymmetrical buildings are modeled by means of equivalent 3D shear-torsional beams with decreasing size of the cross section as described in Figure 1. Each k-th building interstorey is modeled by means of equivalent beam segments having the same length and constant cross section.

The beam-like model is here assumed clamped at the base and free at the top (individuated respectively through the abscissae z=0 and z=h). At each k-th storey (in the x, y plane) the centers of mass and stiffness are opportune calculated and their mutual position is taken into account generating torsional effects during the seismic response of the investigated building. The beam-like shear and torsional stiffness, at each floor, are here opportune evaluated by
means of an equivalence to the corresponding inter-storey values of the analyzed building according to the procedure previously proposed by Piccardo et al. [6], the equivalence is endorsed by means of deformation energies which are expressed in terms of displacement components of both buildings and the corresponding equivalent beams.

As previously mentioned, the adopted procedure for evaluating the equivalence between the building inter-storey shear and torsional stiffness and the equivalent beam segment is based on the simplified assumption that the floors are rigid. The additional flexibility due to the flexural deformability of the floor has been taken into account by introducing stiffness reduction coefficients calculated through an optimization procedure based on the minimization of an objective function that compares the modal frequencies of the beam-like model to the correspondent ones obtained through the reference FEM models.

2.1 The Rayleigh Ritz discretization of the non-uniform equivalent beam

The masses \( M_{sk} \) and \( M_{yk} \) of the \( k \)-th storey and its second order moment \( I_{ok} \) are assumed to be concentrated at the floor levels, while the correspondent beam values \( m_n, m_v \) and \( I_o \) are assumed to be constant within each inter-storey length of the equivalent beam (but different along the height). Displacements in the \( x \) and \( y \) directions \( u_x(z,t) \) and \( u_y(z,t) \) and torsional rotations \( \theta_z(z,t) \) of the beam are continuous functions of the abscissa \( z \). The seismic excitations is related to the displacements \( u_{gx}(t), u_{gy}(t) \) at the base of the beam, while the vertical component of the ground motion is not taken into account being the beam axially rigid.

In order to evaluate the response of the equivalent non-uniform beam, with a reduced computational burden, a Rayleigh-Ritz discretization is performed. The latter is based on the use of \( m \) modal shapes of an homogeneous shear beam of total length \( h \) clamped at the base, these can be expressed as:

\[
\psi_n(\zeta) = \sin\left(\frac{\pi}{2}(2n-1)\zeta\right) \quad n = 1, 2, \ldots, \infty
\]

where \( \zeta = z/h \) is the dimensionless abscissa of the beam.

The equations of motion of the undamped system in the generalized space can be written in the following form:
being $M^*$ and $K^*$ the following generalized mass and stiffness matrices:

$$
\begin{align*}
M^* &= \begin{bmatrix} M^x & M^y \\ M^y & M^z \end{bmatrix} \\
K^* &= \begin{bmatrix} K^x & K^y \\ K^y & K^z \end{bmatrix}
\end{align*}
$$

where all the sub-matrices of $M^*$ and $K^*$ have $m \times m$ dimension (being $m$ the number of used modal shapes) and their terms have the following expressions:

$$
\begin{align*}
K^*_{y_j} &= \frac{1}{h} \int GA_y \psi'_{j} \psi'_{j} d\zeta \\
M^*_{y_j} &= h \int m_j \psi'_{j} d\zeta + \sum_{k=1}^{N} M_{x,k} \psi_{i,k} \psi_{j,k} \\
K^*_{y_j} &= \frac{1}{h} \left( GJ_z + e^2_x GA_x + e^2_y GA_y \right) \psi'_{j} \psi'_{j} d\zeta \\
M^*_{y_j} &= h \int m_j \psi'_{j} d\zeta + \sum_{k=1}^{N} M_{y,k} \psi_{i,k} \psi_{j,k} \\
K^*_{y_j,\omega} &= K^*_{y_j,\omega} = \frac{1}{h} \int e_x GA_y \psi'_{j} \psi'_{j} d\zeta \\
M^*_{y_j,\omega} &= h \int I_{z,j} \psi'_{j} d\zeta + \sum_{k=1}^{N} I_{x,k} \psi_{i,k} \psi_{j,k} \\
K^*_{y_j,\omega} &= K^*_{y_j,\omega} = -\frac{1}{h} \int e_y GA_y \psi'_{j} \psi'_{j} d\zeta \\
M^*_{y_j,\omega} &= h \int I_{y,j} \psi'_{j} d\zeta + \sum_{k=1}^{N} I_{y,k} \psi_{i,k} \psi_{j,k}
\end{align*}
$$

In equations (4) the primes denote differentiation with respect to the dimensionless abscissas, $GA_x$ and $GA_y$ are the shear stiffnesses in the $x$ and $y$ directions, respectively, and $GJ_z$ is the torsional stiffness (constant within each inter-storey length of the equivalent beam). Furthermore $e_x$ and $e_y$ are the coordinates of the center of stiffness [6].

The undamped modes of vibration $q$ in the generalized space and the vibration frequencies $\omega$ of the equivalent beam are obtained from the resolution of the eigen-problem, equation (5).

$$
\left[ K^* - \omega^2 M^* \right] q = 0
$$

The presence of a ground motion provides the following generalized load vector:

$$
P^*_j = -\bar{u}_y h \int m_j \psi'_{j} d\zeta - \bar{u}_y \sum_{k=1}^{N} M_{x,k} \psi_{j,k}$$

whose components are:

$$
P^*_j = -\bar{u}_y h \int m_j \psi'_{j} d\zeta - \bar{u}_y \sum_{k=1}^{N} M_{x,k} \psi_{j,k}$$

$$
P^*_y = 0$$
The dynamic response in the generalized space can be expressed as the following combination of \( N \) modes of vibration:

\[
q(t) = \sum_{n=1}^{N} q_n \cdot z_n(t)
\]  
(7)

Substituting equation (7) in the equations of motion (2) and adopting the orthogonality properties of the modes of vibration, the equations of motions are simplified as follows:

\[
M_{\text{mod},n} \cdot \ddot{z}_n(t) + K_{\text{mod},n} \cdot z_n(t) = P_{\text{mod},n}
\]  
(8)

where:

\[
M_{\text{mod},n} = q_n^T M q_n \\
K_{\text{mod},n} = q_n^T K q_n \\
P_{\text{mod},n} = q_n^T P
\]  
(9)

The dynamic response in the geometric space can be obtained by the following expressions:

\[
\begin{align*}
 u_x(z,t) &= \sum_{n=1}^{N} \sum_{i=1}^{m} \psi_i(z)q_{ni} \cdot z_n(t) \\
 u_y(z,t) &= \sum_{n=1}^{N} \sum_{i=1}^{m} \psi_i(z)q_{ni} \cdot z_n(t) \\
 \mathcal{G}(z,t) &= \sum_{n=1}^{N} \sum_{i=1}^{m} \psi(z)q_{ni} \cdot z_n(t)
\end{align*}
\]  
(10)

3 VALIDATION OF THE PROPOSED MODEL

In this section the proposed beam-like model is adopted for simulating the dynamic behaviour of spatial multi-storey frames. In particular, four different structural configurations have been analyzed aiming at investigating the influence of horizontal and vertical mass distribution on the dynamic response. The results, in terms of vibration modes and seismic response, have been compared to those obtained by means of a conventional FEM model developed with SAP2000 [9]. An eight storey building has been considered in four different structural configurations. In Case A a symmetric plan at each floor and a uniform vertical distribution of mass and stiffness have been assigned. In Case B an eccentricity between the centers of mass and stiffness has been introduced. Case C refers to a building with symmetric plan and vertical mass and stiffness variation. Finally, in Case D, all the above mentioned variations have been considered on the same benchmark. The seismic response to the \( x \) component of the El Centro’s earthquake accelerogram has been evaluated in terms of the displacements of the control point \( P \) located at the top of the building (whose position in plan is shown in Figure 2).

The cross sections of all the beams are assumed to be 30x50 cm\(^2\) and the floor thickness at each level is 25 cm. The inter-storey height is 3 m and the cross sections of all the columns for the four buildings are reported in Table 1.
Figure 2: Plans of the case studies buildings (a) symmetric (b) unsymmetric with eccentricity between centers of mass and stiffness. The control point P is highlighted on the plans.

Table 1: Assignment of the concrete column sections in the four configurations

| Floor | Case A | Case B | Case C | Case D |
|-------|--------|--------|--------|--------|
|       | all    | 1, 2, 4, 5 | 3, 6, 7, 8, 9 | all    | 1, 2, 4, 5 | 3, 6, 7, 8, 9 |
| 8     | 40x40  | 40x40  | 60x60  | 30x30  | 30x30  | 60x60  |
| 7     | 40x40  | 40x40  | 60x60  | 30x30  | 30x30  | 60x60  |
| 6     | 40x40  | 40x40  | 60x60  | 35x35  | 35x35  | 60x60  |
| 5     | 40x40  | 40x40  | 60x60  | 40x40  | 35x35  | 60x60  |
| 4     | 40x40  | 40x40  | 60x60  | 45x45  | 40x40  | 60x60  |
| 3     | 40x40  | 40x40  | 60x60  | 50x50  | 45x45  | 60x60  |
| 2     | 40x40  | 40x40  | 60x60  | 55x55  | 50x50  | 60x60  |
| 1     | 40x40  | 40x40  | 60x60  | 60x60  | 50x50  | 60x60  |

As Table 1 shows in Case A all the columns have the same cross section, namely 40x40cm; in Case B two different cross sections have been assigned (40x40 and 60x60) but they are uniform along the column heights; in Case C a unique cross section has been assigned at each floor but it decreases with the height of the building; in Case D five columns have the same cross section area at each floor and all the others reduce their cross section from the base to the top level.

The material is assumed to be linear elastic and characterized by Young’s modulus of 31250 MPa, a Poisson ratio equal to 0.2 and a specific weight of 25 kN/m\(^3\).

The observation of the first three periods and modes of vibration of the beam like and the FEM models of the vertically uniform building with symmetric (Case A) and asymmetric plan (Case B), shown in Figure 3, allows to highlight the reliability of the proposed model to predict accurate eigenproperties. The nature of the first two modes can be recognized as translational while the third is torsional; furthermore it is interesting to point out that the eccentricity between the centers of mass and stiffness induces a torsional behavior not only in the third but also in the first mode.

In Figures 4 and 5 the displacement components of the control point in the seismic response of the buildings considered in cases A and B are reported. As expected the displacement has only the x direction for the uniform symmetric building while, for the asymmetric one, both displacement components can be observed. In the same figures a zoom of the first
peaks of the response functions are also reported. Once again the comparison with the results obtained for the FEM model shows a very good agreement.

| MODE | CASE A | CASE B |
|------|--------|--------|
| 1    | BEAM LIKE | FEM | BEAM LIKE | FEM |
|      | T = 0.853 s | T = 0.853 s | T = 0.742 s | T = 0.738 s |
| 2    | BEAM LIKE | FEM | BEAM LIKE | FEM |
|      | T = 0.853 s | T = 0.853 s | T = 0.684 s | T = 0.687 s |
| 3    | BEAM LIKE | FEM | BEAM LIKE | FEM |
|      | T = 0.617 s | T = 0.617 s | T = 0.396 s | T = 0.397 s |

Figure 3 First three periods and modes of vibration of Case A and Case B

Figure 4 Displacement components in $x$ and $y$ directions of the control point in Case A.
The results obtained for the vertically non uniform buildings, denoted as cases C and D, are reported in Figure 6 in terms of periods and modes of vibration, while Figures 7 and 8 show their seismic responses. It is interesting to point out, from the observation of Figures 3 and 6,
that the vertical non-uniform distribution of mass and stiffness induces a modification in the modes of vibration of the buildings with symmetric plan. In particular, it can be observed that the first floors, being stiffer than the upper ones, have smaller translations thus modifying the whole modal shape. Also for the vertically non-uniform buildings the considered asymmetry in plan has a torsional effect on the first mode.

Figure 7 Displacement components in x and y directions of the control point in Case C

Figure 8 Displacement components in x and y directions of the control point in Case D

Once again, the seismic responses obtained for the beam-like model, shown in figures 7 and 8, are very close to the correspondent ones of the FEM model thus confirming the reliability of the proposed approach. Anyway, it must be pointed out that the main differences in the response curves can be observed for the displacement component in the y direction of the building with non-uniform vertical distribution of mass and stiffness and unsymmetrical plan. In this case, the eccentricity between the centers of mass and stiffness is variable with the height of the building and leads to a small underestimation of the seismic response.

4 CONCLUSIONS

The paper introduces a methodology for the dynamic analyses of multi-storey buildings by means of an equivalent beam-like model able to take into account both non-uniform mass and stiffness distribution along the height and unsymmetrical plans. The modal shapes of a uniform shear-torsional beam have been used to perform a Rayleigh-Ritz discretization of the
continuous beam model and then the dynamic response has been evaluated in the generalized space. Aiming at investigating the influence of horizontal and vertical mass distribution on the dynamic response of multi-storey frames, different structural configurations have been analyzed. The validation of the proposed procedure has been obtained through a comparison of frequencies, modes of vibration and seismic response of multi-storey buildings modeled either by means of the beam-like approach or through 3D FEM models. The applications showed a satisfactory correspondence between the two approaches thus confirming the capability of the presented beam-like model in providing reliable results with a very limited computational effort.

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