Families of stationary modes in complex potentials

Vladimir V. Konotop and Dmitry A. Zezyulin

Centro de Física Teórica e Computacional, Faculdade de Ciências, Universidade de Lisboa, Avenida Professor Gama Pinto 2, Lisboa 1649-003, Portugal
Departamento de Física, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, Ed. C8, Lisboa 1749-016, Portugal

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It is shown that a general class of complex asymmetric potentials of the form \( w^2(x) - iw_x(x) \), where \( w(x) \) is a real function, allows for the existence of one-parametric continuous families of the stationary nonlinear modes bifurcating from the linear spectrum and propagating in Kerr media. As an example, we introduce an asymmetric double-hump complex potential and show that it supports continuous families of stable nonlinear modes. © 2014 Optical Society of America

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There are several fundamental differences in properties of nonlinear conservative and dissipative systems. One of them is related to the structure of stationary modes. In conservative systems such modes constitute one- (or several-) parametric families, while in the presence of gain and dissipation stationary solutions represent isolated fixed points (see e.g. [1]). This difference stems from the fact that, besides the balance between the nonlinearity and dispersion necessary for existence of any conservative mode, stationary modes in a dissipative system must satisfy an additional constraint of the balance between the dissipation and gain. The latter constraint, generally speaking, eliminates one (or even several) free parameters and does not allow for existence of continuous parametric families which could exist in the conservative case. The described difference has particular importance for nonlinear optics, where the propagation of nonlinear modes is defined by the input signal in the conservative case and by the system parameters in the dissipative case. A possibility of simultaneous manipulating nonlinear modes by both means would greatly enhance versatility of an optical system.

Recently, this issue attracted special attention in the context of nonlinear extensions of parity-time- (\( \mathcal{PT} \)-) symmetric [2] systems, whose optical applications were introduced theoretically [3-5] and implemented experimentally [6, 7]. It is now established that the \( \mathcal{PT} \)-symmetric optical potentials, although being complex, i.e. involving gain and dissipation, still can possess continuous parametric families of the localized modes (resembling in this way conservative systems). In particular, continuous families of nonlinear modes were obtained for discrete optical systems [8, 9], for parabolic [10, 11], periodic [12], localized [13] and double-well [14-16] \( \mathcal{PT} \)-symmetric potentials with Kerr nonlinearity. The continuous families of solutions were also obtained for nonlinear \( \mathcal{PT} \)-symmetric potentials [17] and for mixed linear and nonlinear potentials [18], as well as for optical \( \mathcal{PT} \)-symmetric systems with \( \chi^{(2)} \) nonlinearity, in both continuous [19] and discrete [20] settings. Models allowing for families of exact solutions were reported for continuous [21] and discrete [22] media with \( \mathcal{PT} \)-symmetric defects. It was also conjectured that the \( \mathcal{PT} \) symmetry is a necessary condition for the existence of continuous families of nonlinear modes in one-dimensional potentials [23].

In this context, an interesting numerical result was reported in the recent work [24]. Within the framework of the parabolic approximation

\[
i\Psi_z = -\Psi_{xx} - V(x)\Psi - g|\Psi|^2\Psi,
\]

with \( g = 1 \) \( (g = -1) \) corresponding to the focusing (defocusing) nonlinearity, and with the complex potential

\[
V(x) = w^2(x) - iw_x(x),
\]

the authors found that for \( w(x) = \eta/\cosh(\alpha x) \), where \( \eta \) is a constant, the system supports continuous families of nonlinear modes even if \( \alpha = \alpha_- \) for \( x < 0 \) and \( \alpha = \alpha_+ \) for \( x > 0 \) with \( \alpha_- \neq \alpha_+ \). The importance of this finding stems from the fact that for \( \alpha_- \neq \alpha_+ \) the potential (2) is not \( \mathcal{PT} \) symmetric, i.e., \( V(x) \neq V^*(x) \). Notice that if \( \alpha \) is constant in the whole space \( \text{i.e.}, \alpha_- = \alpha_+ \), then potential (2) becomes \( \mathcal{PT} \) symmetric and is usually referred to as Scarf II potential. Its linear spectrum is well known [25], and its nonlinear modes at specific values of the parameters were reported in [5].

The asymmetric Scarf II potential considered in [24] was the first example of an asymmetric complex one-dimensional potential allowing for continuous families of nonlinear modes. Naturally, the results of [24] raised several important issues. The first one is the elaboration of analytical arguments which would corroborate the numerical results of [24] on the existence of continuous families. Second, it is highly relevant to understand the nature of such modes and to examine possibilities for generalization of the reported numerical findings. Third, it is of practical importance to establish whether continuous families persist if instead of an asymmetric one-hump function \( w(x) \) (this was an important constraint in [24]) one considers more complex (say one- or multi-hump) asymmetric functions \( w(x) \). Finally, it is important to understand if the class of families \( V(x) \) having...
the form (2) can be extended to a more general class that preserves the existence of the continuous families.

The main goal of this Letter is to propose answers to the above questions. More specifically, we show that continuous families in the problem (1)–(2) are explained by a “hidden” symmetry, which is expressed in the form of a conserved quantity of the nonlinear dynamical system describing profiles of the nonlinear modes. This conserved quantity exists for any real differentiable function \( w(x) \), and therefore, the continuous families exist in complex potentials of a rather general form. Our finding allows us to develop a demonstrative computation approach which can be used as a tool for a simple and reliable numerical calculation of nonlinear modes of this novel type. In order to illustrate our findings and to emphasize possibilities for generalization of the previous results, we introduce an asymmetric complex double-hump potential and show that it features real linear spectrum and possesses continuous families of nonlinear modes. The main properties of the found nonlinear modes are also described.

Let us briefly discuss the properties of the spectrum of potential (2). The idea of using (2) for constructing PT-symmetric potentials with entirely real spectra was suggested in [26]. It is based on the existence of a connection between the Zakharov-Shabat (ZS) spectral problem

\[
\phi_{1,x} + i \zeta \phi_1 = w(x, \tau) \phi_2, \quad \phi_{2,x} - i \zeta \phi_2 = -w(x, \tau) \phi_1, \tag{3}
\]

associated with the modified Korteweg-de Vries (mKdV) equation, \( w_x + 6w^2w_x + w_{xxx} = 0 \), and the Schrödinger eigenvalue problem \( \beta \phi = -\phi_{xx} - V(x, \tau) \phi \) with the potential \( V(x) \) given by (2). The spectral parameters of the two problems are related as \( \beta = -\zeta^2 \) (see e.g. [26, 27] for more details). Presence of this connection has several important consequences. To formulate them, we recall (see e.g. [27]) that discrete eigenvalues of the ZS problem (3), if any, are either purely imaginary or situated symmetrically with respect to the imaginary axis (i.e. if \( i \zeta \) is an eigenvalue, then \( -\zeta^2 \) is an eigenvalue, as well), and the continuous spectrum is the real axis. Thus from any solution \( w(x, \tau) \) of the mKdV equation that provides pure imaginary discrete eigenvalues of (3), one can obtain a complex potential \( V(x, \tau) \), defined by (2) with purely real spectrum \( \beta \). Next, we notice that \( w(x, \tau) \) depends on \( \tau \) which is time in the mKdV equation [in this paragraph we explicitly indicate this dependence by writing down \( w(x, \tau) \) instead of \( w(x) \)]. The spectrum of the ZS problem does not depend on \( \tau \). This means that \( \tau \) can be considered as a “deformation” parameter, and each solution \( w(x, \tau) \) represents a family of deformable potentials \( V(x, \tau) \) which can be either PT symmetric, as in [26], or not, as in [24], but in any case having real spectrum. Finally, the constraint \( \zeta = -\zeta^2 \), imposed on discrete eigenvalues, makes the real spectrum of \( V(x, \tau) \) sufficiently robust, i.e. starting with a purely solitonic solutions, like in [26], one can change their shape creating potentials of more sophisticated forms, without violating reality of the spectrum of \( V(x) \). In this way, one can construct asymmetric one- and multi-hump potentials \( V(x) \) of the form (2) with purely real spectrum.

Now we turn to stationary nonlinear modes which are written down in the form \( \Psi(x, z) = e^{i\beta z} \psi(x) \), where \( \beta \) is the real propagation constant and \( \psi(x) \) is the complex field. It is convenient to employ the “hydrodynamic” representation \( \psi(x) = \rho(x)e^{i\theta(x)} \), where \( \rho(x) \) is the amplitude and \( \theta(x) \) is the phase of the field. Then, introducing the phase gradient \( v(x) \equiv \theta_{x}(x) \), we reduce Eq. (1) to the system

\[
\rho_{xx} - \beta \rho + w^2 \rho + gp^2 - v^2 \rho = 0, \tag{4a}
\]

\[
2\rho_x v + \rho v_x - w_x \rho = 0. \tag{4b}
\]

It is convenient to look at these equations as at a nonlinear dynamical system with respect to evolution variable \( x \), i.e. if a solution \( (\rho, v) \) is given at some \( x_0 \), then its “evolution” towards growing or decaying \( x \) will describe the transverse profile of the stationary nonlinear mode.

An important finding of our study is that system (4) has a “conserved”, i.e. \( x \)-independent, quantity

\[
I = \rho_x^2 + \rho^2(v-w)^2 - \beta \rho^2 + gp^4/2, \quad dI/dx = 0. \tag{5}
\]

The specific form of the potential (2) is crucial for the existence of this conserved quantity: it does not exist in a system with a generic complex potential \( V(x) \). The conserved quantity \( I \) appears to be crucial for understanding the existence of continuous families of nonlinear modes. Focusing on spatially localized modes \( \psi(x) \) which satisfy the boundary conditions \( \psi(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \), we conclude that for any localized mode \( I = 0 \), i.e.

\[
\rho_x^2 + \rho^2(v-w)^2 - \beta \rho^2 + gp^4/2 = 0. \tag{6}
\]

This constraint imposes an additional algebraic relation among the dependent variables \( \rho, p_x \), and \( v \), which allows one to elaborate simple arguments for explanation of the existence of continuous families of nonlinear modes.

Indeed, for a function \( w(x) \) vanishing as \( |x| \rightarrow \infty \) sufficiently rapidly, the respective asymptotic of the modes reads \( \psi \sim r_+ e^{-\sqrt{\beta x+i\theta_+}} \) as \( x \rightarrow \infty \) and \( \psi \sim r_- e^{\sqrt{\beta x+i\theta_-}}, \) as \( x \rightarrow -\infty \), where \( r_\pm \) and \( \theta_\pm \) are real constants. Any mode must satisfy the continuity condition for the field and for its derivative at any intermediate point \( x_0 \) [28], i.e. there are three matching conditions which must be satisfied at \( x_0 \): \( r_+(x_0) = r_+(x_0), \quad r_+(x_0) = r_+(x_0), \) and \( v_-(x_0) = v_+(x_0), \) where the subscripts “−” and “+” stay for the solutions at \( x \leq x_0 \) and \( x \geq x_0 \), respectively. Since the constant phases \( \theta_\pm \) do not affect either the amplitude or the phase gradient, for a generic complex potential these matching conditions must be satisfied using only two constants \( r_\pm \), which is generally not possible unless the propagation constant \( \beta \) is adjusted to acquire some particular value. This explains the absence of the continuous families in complex potentials of a generic type. However, in our case, due to the integral \( I \), there effectively exist only two matching conditions: if any two conditions are satisfied then the third
one is satisfied automatically due to (6). For example, if \( \rho_-(x_0) = \rho_+(x_0) \) and \( v_-(x_0) = v_+(x_0) \) then Eq. (6) implies \( [\rho_{-,x}(x_0)]^2 = [\rho_{+,x}(x_0)]^2 \). Notice, that in this case Eq. (6) is satisfied also by \( \rho_{-,x}(x_0) = -\rho_{+,x}(x_0) \), which is a “phantom” solution not corresponding to a differentiable nonlinear mode (i.e. it must be discarded).

However, in any practical situation, one can easily check whether the necessary condition \( \rho_{-,x}(x_0) = \rho_{+,x}(x_0) \) holds, i.e. the artificial ambiguity is easily removed.

Thus we argue that for an arbitrary propagation constant \( \beta \) a localized nonlinear mode can be found by solving two matching equations with two real parameters \( r_+ \) and \( r_- \). Hence by varying the free parameter \( \beta \) one can construct continuous families of nonlinear modes.

The described arguments hold for a general class of functions \( w(x) \) and allow for simple and illustrative computation of the nonlinear modes. As an example, we consider an asymmetric double-hump potential (cf. the single-hump potential in [24]) choosing

\[
w(x) = a_- e^{-(x+l)^2} + a_+ e^{-(x-l)^2}.
\]  

(7)

The \( \mathcal{PT} \)-symmetric version of this potential (with \( a_- = a_+ \)) was recently considered in [16]. We focus on an asymmetric case \( a_- \neq a_+ \) and for numerical illustrations set \( a_- = 2.3, a_+ = 2 \) and \( l = 1.5 \). Real and imaginary parts of the resulting asymmetric potential are shown in Fig. 1(a). Potential \( V(x) \) has two real isolated eigenvalues, \( \beta_1 \approx 1.51 \) and \( \beta_2 \approx 2.38 \), and the continuous spectrum lying on the negative half of the real axis.

To verify the existence of nonlinear modes, we scanned a certain range of parameters \( r_- \) and \( r_+ \), and computed the values \( \rho_{\pm}(0), \rho_{\pm,x}(0) \) and \( v_{\pm}(0) \), i.e. choosing \( x_0 = 0 \). The obtained dependencies are plotted in Fig. 2 on the plane \( (\rho(0), v(0)) \). Any intersection of the solid (blue) and dashed (red) curves corresponds to the moment when \( \rho_-(0) = \rho_+(0) \) and \( v_-(0) = v_+(0) \) for certain \( r_+ \) and \( r_- \). As explained above, this automatically implies \( [\rho_{-,x}(0)]^2 = [\rho_{+,x}(0)]^2 \). We also checked that for both the intersections \( \rho_{-,x}(0) = \rho_{+,x}(0) \), and therefore they correspond to the two localized modes. The values of the parameters \( r_+ \) at the crossing points can be used to compute spatial profiles of the nonlinear modes. Notice that in order to establish the existence of the solutions, significant efforts in the high numerical accuracy were necessary in the previous works (the accuracy of the simulations reported in [24] and [16] was respectively up to \( 10^{-10} \) and \( 10^{-30} \)). The “numerical proof” of the existence of the asymmetric modes presented here does not require any specific concern about the accuracy, as the existence of the modes is established simply by crossing of the two curves, as in Fig. 2.

Using the described approach for different values of \( \beta \), we have identified different families of nonlinear modes in the asymmetric complex double-hump potential, in both focusing \( (g = 1) \) and defocusing \( (g = -1) \) nonlinearities. The results can be visualized in the form of dependencies of the power flow \( P = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \) on the propagation constant \( \beta \) [Fig. 1(b)]. The families of nonlinear modes presented in Fig. 1(b) bifurcate from the isolated eigenvalues \( \beta_{1,2} \) of the complex potential \( V(x) \). In the case of the defocusing nonlinearity, the families asymptotically approach the edge of the continuous spectrum, i.e. \( \beta = 0 \).

We checked the linear stability of the modes, performing the standard substitution \( \Psi = e^{i\beta z}[|\psi(x)| + A(x)e^{i\lambda z} + B^*(x)e^{i\Lambda^* z}], |A|, |B| \ll |\psi|, \) and computing instability increment \( \text{Re} \lambda \) from the linearized problem. This analysis shows that the families presented in Fig. 1(b) are stable, at least in the explored range of the propagation constant \( \beta \). This result complies with the Vakhitov-Kolokolov (VK) criterion [29] for the focusing and anti-VK criterion [30] for the defocusing media. We also notice that the families shown in Fig. 1(b) do not exhaust all possible nonlinear modes. For example, in the focusing medium, we also found a family of nonlinear modes bifurcating from the edge of the continuous spectrum, i.e. from \( \beta = 0 \). However, this family is completely unstable and is not presented.

Specific spatial distributions of the field intensity of the modes are illustrated in Fig. 1(c,d). The modes feature highly asymmetric intensity profiles \( \rho^2 = |\psi(x)|^2 \), as well as asymmetric currents \( j = v\rho^2 \), which reflects
results on generic potentials allowing the representation of nonlinear modes in non-homogeneous complex potentials. We have also generalized the previous families of nonlinear modes in non-symmetric com-

plex potentials. While our results are obtained for focusing and defocusing cubic nonlineari-

ties, generalizations for nonlinearities of other types (say quintic or cubic-quintic) are straightforward.

Generically, the nonlinearity (either focusing or defocusing) results in deformation of the linear modes without appreciable qualitative changes of their structure. We also computed the propagation of several nonlinear modes subject to a relatively small (5% of the am-

plitude) perturbation. The results (not presented here) indicate that the nonlinear modes are robust against the introduced perturbation, which corroborates stability of the found solutions.

To conclude, we have explained and generalized recent numerical findings of Ref. [24]. We have shown that the continuous families of nonlinear modes in a complex asymmetric potential can be understood by the peculiar property of the underlying dynamical system which possesses a “conserved” (i.e., reality of the spectrum and existence of the continuous families) persist if instead of a single-hump function $w(x)$ one considers more complex asymmetric double- (or multi-hump) functions. We expect that the families of localized nonlinear modes should also exist in asymmetric complex periodic potentials. While our results are obtained for focusing and defocusing cubic nonlinearities, generalizations for nonlinearities of other types (say quintic or cubic-quintic) are straightforward.

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References

1. N. Akhmediev and A. Ankiewicz in *Dissipative Solitons*, edited by N. Akhmediev and A. Ankiewicz (Springer-Verlag, Berlin, 2005).
2. C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
3. A. Ruschhaupt, F. Delgado and J. G. Muga, J. Phys. A 38, L171-L176 (2005).
4. R. El-Ganainy, K. G. Makris, D. N. Christodoulides and Z. H. Musslimani, Opt. Lett. 32, 26322634 (2007).
5. Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, Phys. Rev. Lett. 100, 030402 (2008).
6. A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou and D. N. Christodoulides, Phys. Rev. Lett. 103, 093902 (2009).
7. C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev and D. Kip, Nature Phys. 6, 192–195 (2010).
8. K. Li and P. G. Kevrekidis, Phys. Rev. E 83, 066608 (2011).
9. D. A. Zezyulin and V.V. Konotop, Phys. Rev. Lett. 108, 213906 (2012).
10. D. A. Zezyulin and V. V. Konotop, Phys. Rev. A 85, 043840 (2012).
11. V. Achilleos, P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, Phys. Rev. A 86, 013805 (2012).
12. S. Nixon, L. Ge and J. Yang, Phys. Rev. A 85, 023822 (2012).
13. S. Hu, X. Ma, D. Lu, Y. Zheng and W. Hu, Phys. Rev. A 85, 043826 (2012).
14. D. Dast, D. Haag, H. Cartarius, G. Wunner, R. Eichler, and J. Main, Fortschr. Phys. 61, 124–139 (2013).
15. A. S. Rodrigues, K. Li, V. Achilleos, P. G. Kevrekidis, D. J. Frantzeskakis, C. M. Bender, Rom. Rep. Phys. 65, 5–26 (2013).
16. J. Yang arXiv:1408.0687 [physics.optics]
17. F. K. Abdullaev, Y. V. Kartashov, V. V. Konotop and D. A. Zezyulin, Phys. Rev. A 83, 041805(R) (2011)
18. Y. He, X. Zhu, D. Mihalache, J. Liu and Z. Chen, Phys. Rev. A 85, 013831 (2012).
19. F. C. Moreira, V. V. Konotop, and B. A. Malomed, Phys. Rev. A 87, 013832 (2013).
20. K. Li, D. A. Zezyulin, P. G. Kevrekidis, V. V. Konotop, and F. Kh. Abdullaev, Phys. Rev. A 88, 053820 (2013).
21. T. Mayteevarunyoo, B. A. Malomed, and A. Reoksabutr, Phys. Rev. E 88, 022919 (2013).
22. X. Zhang, J. Chai, J. Huang, Z. Chen, Y. Li, and B. A. Malomed, Opt. Expr. 22, 13927 (2014).
23. J. Yang, Phys. Lett. A 378, 367 (2014).
24. E. N. Tsoy, I. M. Allayarov and F. Kh. Abdullaev, Opt. Lett. 39, 4215 (2014).
25. Z. Akhmed, Phys. Lett. A 282, 343 (2001); ibidem 287, 295 (2001).
26. M. Wadati, J. Phys. Soc. Jpn. 77, 074005 (2008).
27. G. L. Lamb, *Elements of Soliton Theory* (Wiley, 1980).
28. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1977).
29. M. Vakhitov and A. Kolokolov, Radiophysics. Quantum. Electron. 16, 783 (1973).
30. H. Sakaguchi and B. A. Malomed, Phys. Rev. A 81, 013624 (2010).
List of references with titles

References

1. N. Akhmediev and A. Ankiewicz, Dissipative Solitons in Complex Ginzburg-Landau and Swift-Hohenberg Equations, in Dissipative Solitons, edited by N. Akhmediev and A. Ankiewicz (Springer-Verlag, Berlin, 2005).
2. C. M. Bender, Making sense of non-Hermitian Hamiltonians, Rep. Prog. Phys. 70, 947 (2007).
3. A. Ruschhaupt, F. Delgado and J. G. Muga, Physical realization of PT-symmetric potential scattering in a planar slab waveguide, J. Phys. A 38, L171-L176 (2005).
4. R. El-Ganainy, K. G. Makris, D. N. Christodoulides and Z. H. Musslimani, Theory of coupled optical PT-symmetric structures, Opt. Lett. 32, 2632–2634 (2007).
5. Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, Optical Solitons in PT Periodic Potentials, Phys. Rev. Lett. 100, 030402 (2008).
6. A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou and D. N. Christodoulides, Observation of PT-Symmetry Breaking in Complex Optical Potentials, Phys. Rev. Lett. 103, 093902 (2009).
7. C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev and D. Kip, Observation of parity-time symmetry in optics, Nature Phys. 6, 192–195 (2010).
8. K. Li and P. G. Kevrekidis, PT-symmetric oligomers: Analytical solutions, linear stability, and nonlinear dynamics, Phys. Rev. E 83, 066608 (2011).
9. D. A. Zezyulin and V. V. Konotop, Nonlinear Modes in Finite-Dimensional PT-Symmetric Systems, Phys. Rev. Lett. 108, 213906, 2012.
10. D. A. Zezyulin and V. V. Konotop, Nonlinear modes in the harmonic PT-symmetric potential, Phys. Rev. A 85, 043840 (2012).
11. V. Achilleos, P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, Dark solitons and vortices in PT-symmetric nonlinear media: From spontaneous symmetry breaking to nonlinear PT phase transitions, Phys. Rev. A 86, 013808 (2012).
12. S. Nixon, L. Ge and J. Yang, Stability analysis for solitons in PT-symmetric optical lattices, Phys. Rev. A 85, 023822 (2012).
13. S. Hu, X. Ma, D. Lu, Y. Zheng and W. Hu, Defect solitons in parity-time symmetric superlattices, Phys. Rev. A 85, 043826 (2012).
14. D. Dast, D. Haag, H. Cartarius, G. Wunner, R. Eichler, and J. Main, A Bose-Einstein condensate in a PT symmetric double well, Fortschr. Phys. 61, 124–139 (2013).
15. A. S. Rodrigues, K. Li, V. Achilleos, P. G. Kevrekidis, D. J. Frantzeskakis, C. M. Bender, PT-Symmetric Double-Well Potentials Revisited: Bifurcations, Stability And Dynamics, Rom. Rep. Phys. 65, 5–26 (2013).
16. J. Yang, Symmetry breaking of solitons in one-dimensional parity-time-symmetric optical potentials, arXiv:1408.0687 [physics.optics]
17. F. K. Abdullaev, Y. V. Kartashov, V. V. Konotop and D. A. Zezyulin, Solitons in PT-symmetric nonlinear lattices, Phys. Rev. A 83, 041805(R) (2011).
18. Y. He, X. Zhu, D. Mihalache, J. Liu and Z. Chen, Lattice solitons in PT-symmetric mixed linear-nonlinear optical lattices, Phys. Rev. A 85, 013831 (2012).
19. F. C. Moreira, V. V. Konotop, and B. A. Malomed, Solitons in PT-symmetric periodic systems with the quadratic nonlinearity, Phys. Rev. A 87, 013832 (2013).
20. K. Li, D. A. Zezyulin, P. G. Kevrekidis, V. V. Konotop, and F. Kh. Abdullaev, PT-symmetric coupler with \( \chi(2) \) nonlinearity, Phys. Rev. A 88, 053820 (2013).
21. T. Mayteevarunyoo, B. A. Malomed, and A. Reoksabutr, Solvable model for solitons pinned to a parity-time-symmetric dipole, Phys. Rev. E 88, 022919 (2013).
22. X. Zhang, J. Chai, J. Huang, Z. Chen, Y. Li, and B. A. Malomed Discrete solitons and scattering of lattice waves in guiding arrays with a nonlinear PT-symmetric defect Opt. Expr. 22, 13927 (2014).
23. J. Yang, Necessity of PT symmetry for soliton families in one-dimensional complex potentials, Phys. Lett. A 378, 367 (2014).
24. E. N. Tsoy, I. M. Allayarov, and F. Kh. Abdullaev, Stable localized modes in asymmetric waveguides with gain and loss, Opt. Lett. 39, 4215 (2014).
25. Z. Akhmed, Real and complex discrete eigenvalues in an exactly solvable one-dimensional complex PT-invariant potential, Phys. Lett. A 282, 343 (2001); Addendum to: “Real and complex discrete eigenvalues in an exactly solvable one-dimensional complex PT-invariant potential”, ibidem 287, 295 (2001).
26. M. Wadati, Construction of Parity-Time Symmetric Potential through the Soliton Theory, J. Phys. Soc. Jpn. 77, 074005 (2008).
27. G. L. Lamb, Elements of Soliton Theory (Wiley, 1980).
28. L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Pergamon, New York, 1977).
29. M. Vakhitov and A. Kolokolov, Stationary solutions of the wave equation in the medium with nonlinearity saturation, Radiophys. Quantum. Electron. 16, 783 (1973).
30. H. Sakaguchi and B. A. Malomed, Solitons in combined linear and nonlinear lattice potentials Phys. Rev. A 81, 013624 (2010).