A TWO-SPECIES WEAK COMPETITION SYSTEM OF REACTION-DIFFUSION-ADVECTION WITH DOUBLE FREE BOUNDARIES

BO DUAN AND ZHENGCE ZHANG*

School of Mathematics and Statistics, Xi’an Jiaotong University
Xi’an, 710049, China

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Abstract. In this paper, we investigate a two-species weak competition system of reaction-diffusion-advection with double free boundaries that represent the expanding front in a one-dimensional habitat, where a combination of random movement and advection is adopted by two competing species. The main goal is to understand the effect of small advection environment and dynamics of the two species through double free boundaries. We provide a spreading-vanishing dichotomy, which means that both of the two species either spread to the entire space successfully and survive in the new environment as time goes to infinity, or vanish and become extinct in the long run. Furthermore, if the spreading or vanishing of the two species occurs, some sufficient conditions via the initial data are established. When spreading of the two species happens, the long time behavior of solutions and estimates of spreading speed of both free boundaries are obtained.

1. Introduction. To study the spreading of a new or invasive species in population ecology, plenty of works have been done. In recent years, it has been attracting researchers’ attention to understand the role that free boundary problem plays in the dynamics of species. In 2010, Du and Lin [10] studied a logistic model with free boundary. In their work, they elaborated the spreading of a new or invasive species, whose population density is represented by $u(t, x)$, and they used one free boundary $h(t)$ or double free boundaries $g(t)$ and $h(t)$ to represent the expanding fronts. Moreover, spreading-vanishing dichotomy and spreading speed were gained. Later on, a growing number of researchers investigated further extensions, including the model in higher dimension spaces and in spatial heterogeneous environment [7, 8] and so on. Besides the homogeneous and spatial heterogeneous environment, if the environment is heterogeneous time-periodic, plenty of results [9, 34] have been studied for the case recently. Taking the place of logistic reaction term, Du and Lou [12] investigated the general nonlinear term $f(u)$, containing general monostable, bistable and combustion types. To get more related results on general models for single species case, we refer to [2, 25, 32, 41] and the references cited therein.

For two species case, a typical model is the Lotka-Volterra type competition system. In [18], the two weak competition species shared the same free boundary.

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* Corresponding author.
We will elaborate it later. By removing the restriction of weak competition, Wang and Zhao [37] presented a more complete description for the model. Du and Lin [11] and Wang and Zhang [35] considered the diffusive competition problem, which is described as the invasion of a superior or inferior competitor. In their problem, an invasive species exists in a ball initially, and invades into the environment, while the resident species distributes in the whole space. Two different free boundaries for the two species were considered by Guo and Wu [19] and Wu [39]. By using more specific considerations, they established a spreading-vanishing trichotomy, a spreading-vanishing quartering, the notion of the minimal habitat size and so on. For more results, we refer to [6, 38] and references cited therein.

Previous works of the investigation of two species models in population ecology were assumed to be in a bounded domain, which originated from the works of Mimura, Yamada and Yotsutani [29, 30, 31]. In their works, they assumed that the two species were considered by Guo and Wu [19] and Wu [39]. By using more specific considerations, they established a spreading-vanishing trichotomy, a spreading-vanishing quartering, the notion of the minimal habitat size and so on. For more results, we refer to [6, 38] and references cited therein.

Now, we elaborate the model in [18] of Guo and Wu in 2012, which is the following problem

\[
\begin{align*}
  u_t &= u_{xx} + u(1 - u - kv), & t > 0, & 0 < x < s(t), \\
  v_t &= Dv_{xx} + rv(1 - v - hu), & t > 0, & 0 < x < s(t), \\
  u_x(t, 0) = v_x(t, 0) &= 0, & t > 0, \\
  u(t, s(t)) &= v(t, s(t)) = 0, & t > 0, \\
  s'(t) &= -\mu[u_x(t, s(t)) + \rho v_x(t, s(t))], & t > 0, \\
  u(0, x) &= u_0(x), & v(0, x) &= v_0(x), & 0 \leq x \leq s_0, \\
  s(0) &= s_0,
\end{align*}
\]

with parameters \( D, r, k, h, \mu, \rho > 0 \), and the initial data \((u_0, v_0, s_0)\) satisfies

\[
\begin{align*}
  s_0 &> 0, & u_0, v_0 &\in C^2([0, s_0]), & u_0(x), & v_0(x) > 0 &\text{ for } x \in [0, s_0), \\
  u_0(s_0) &= v_0(s_0) = u_0'(0) = v_0'(0) = 0.
\end{align*}
\]

The model describes the competition between two species with population densities \( u(t, x) \) and \( v(t, x) \) at time \( t \) and position \( x \). \( D \) is diffusion coefficient of species \( v \), \( r \) is intrinsic growth rate of species \( v \), \( k \) and \( h \) are interspecific competition coefficients, the rest parameters are 1 due to nondimensionalization.

From the perspective of biology, this model describes the way of the two competing species invade over a one-dimensional habitat with the initial region \([0, s_0] \). It is assumed that the zero Neumann boundary condition is imposed for \( x = 0 \). In addition, they supposed that both species have a expanding front \( s(t) \), which represents a trend to emigrate from the right boundary to get their new habitat. Here, \( s(t) \) satisfies the well-known Stefan type condition. For more biological background of the Stefan type condition, we can refer to [2, 20, 21, 27].

In their paper, they considered the weak competition case: \( 0 < h, k < 1 \), and gave the criteria for spreading or vanishing. Besides, in the case of spreading, they obtained a more precise asymptotic behavior and an upper bound for the \( \limsup_{t \to \infty} s(t)/t \), which shows that the asymptotic spreading speed can not be faster
than the minimal speed of travelling wavefront solutions of the problem in the whole line without a free boundary.

In addition, as a result of rich resource, appropriate climate and so on, organisms can often sense and respond to local environmental factor by moving towards one direction in the field of population ecology. For instance, some diseases spread along the wind direction. In 2009, the propagation of West Nile virus from New York City to California state was studied by Maidana and Yang in [28]. In the summer of 1999, it was observed that West Nile virus appeared for the first time in New York City. In the second year, the wave front travelled 187km to the north and 1100km to the south. Hence, they gave thought to the advection movement and showed that bird advection becomes an important factor for lower mosquito biting rates. In [1], the effect of intermediate advection on the dynamics of two-species competition system was considered by Averill, and a specific range of advection strength for the coexistence of two competing species was provided. Additionally, they illustrated three different kinds of transitions from small advection to large advection theoretically and numerically. From a mathematical point of view, in order to contain the effect of advection, it is one of the simplest but probably still realistic approaches that supposing species can move up along the gradient of the density.

For the case of fixed boundary, there are also many results involving the effect of the advection term. In [4], Cantrell, Cosner and Lou studied a Lotka-Volterra model for two competing species in a heterogeneous environment, i.e.,

\[
\begin{align*}
  u_t &= \nabla \cdot [\mu \nabla u - \alpha u \nabla m(x)] + u(m(x) - u - v), \\
  v_t &= \nabla \cdot [\nu \nabla v] + u(m(x) - u - v),
\end{align*}
\]

with no-flux boundary conditions

\[
\frac{\partial u}{\partial n} - \alpha u \frac{\partial m}{\partial n} = \frac{\partial v}{\partial n} = 0, \text{ on } \partial \Omega \times (0, \infty).
\]

This model describes that in a bounded region \( \Omega \) in \( \mathbb{R}^N \), species \( v \) disperses by random diffusion only, species \( u \) disperses by both random diffusion and advection along an environmental gradient. The boundary \( \partial \Omega \) is smooth, \( n \) denotes the unit normal vector on \( \partial \Omega \), and the no-flux boundary condition means that no individuals cross the boundary. The migration rates \( \mu \) and \( \nu \) are two positive constants, and intrinsic growth rates \( m(x) \) of species is assumed to be twice continuously differentiable in \( \Omega \). Based on this model, Chen, Lam and Lou [5] studied a model with both species disperse by random diffusion and advection. More models involving the effect of the advection term can be found in related references.

For the case of free boundary involving the effect of the advection term, some results can be found in [14, 15, 16, 17, 22]. Gu et al. [14, 15] considered the following single case

\[
\begin{align*}
  u_t &= u_{xx} - \beta u_x + u(1 - u), \\
  u(t, g(t)) &= 0, \quad g'(t) = -\mu u_x(t, g(t)), \\
  u(t, h(t)) &= 0, \quad h'(t) = -\mu u_x(t, h(t)), \\
  u(0, x) &= u_0(x), \\
  g(0) &= h(0) = h_0, \quad -h_0 \leq x \leq h_0,
\end{align*}
\]
where $\beta$ and $\mu$ are positive constants, and the initial function $(u_0, h_0)$ satisfies
\[
\begin{aligned}
& h_0 > 0, \quad u_0(x) \in C^2([-h_0, h_0]), \quad u_0(x) > 0 \text{ for } x \in (-h_0, h_0), \\
& u_0(-h_0) = u_0(h_0) = 0.
\end{aligned}
\]
This model describes the spreading of a new or invasive species under the influence of dispersal and advection (expressed by $\beta u_x$). Especially, they studied the influence of small advection coefficient $\beta \in (0, 2)$ on the long time behavior of the solutions, and got a spreading-vanishing dichotomy. Moreover, they gave a sharp threshold between spreading and vanishing. Furthermore, when spreading happens, the asymptotic spreading speed is derived. i.e., there exist two positive constant $c_1^*$ and $c_2^*$ such that
\[
-c_1^* := \lim_{t \to \infty} \frac{g(t)}{t}, \quad c_2^* := \lim_{t \to \infty} \frac{h(t)}{t},
\]
which shows how the advection term $\beta u_x$ influences the spreading speed.

When the right term is replaced by a more general nonlinear term $f(u)$, which is monostable, bistable or combustion type, such results were carried out in [16, 17, 22].

In [22], a much sharper estimate for the different spreading speeds of the fronts was obtained, i.e., when $\beta \in (0, c_0)$,
\[
\begin{aligned}
\lim_{t \to \infty} g'(t) &= -c_1^*, \quad \lim_{t \to \infty} h'(t) = c_2^*, \\
\lim_{t \to \infty} [g(t) + c_1^* t] &= G_\infty, \quad \lim_{t \to \infty} [h(t) - c_2^* t] = H_\infty,
\end{aligned}
\]
for some $G_\infty, H_\infty \in \mathbb{R}$, where $c_0$ is the minimal speed of the travelling waves of the problem
\[
\begin{aligned}
q'' - cq' + f(q) &= 0, \quad q > 0 \text{ in } \mathbb{R}, \\
q(-\infty) &= 0, \quad q(\infty) = 1.
\end{aligned}
\]
Apart from the above results, how the solution approaches a semi-wave was also described.

In [17], Gu et al. extended the small advection case to the general case of $\beta \in (0, \infty)$. They found a critical value $\beta^* > c_0$, where $c_0 = 2\sqrt{f'(0)}$ is the minimal speed of the travelling waves of (2). When $c_0 \leq \beta < \beta^*$, a trichotomy result was obtained. When $\beta \geq \beta^*$, a vanishing result was shown.

Motivated by the above works, specifically, we consider the following problem (TFB):
\[
\begin{aligned}
u_t &= D\nu_{xx} - \beta_1 \nu_x + u(1 - u - cu), \quad t > 0, \quad g(t) < x < h(t), \\
v_t &= Dv_{xx} - \beta_2 v_x + rv(1 - v - bu), \quad t > 0, \quad g(t) < x < h(t), \\
u(t, g(t)) &= u(t, h(t)) = 0, \quad t > 0, \\
v(t, g(t)) &= v(t, h(t)) = 0, \quad t > 0, \\
\begin{aligned}
g'(t) &= -\mu[u_x(t, g(t)) + rv_x(t, g(t))], \quad t > 0, \\
h'(t) &= -\mu[u_x(t, h(t)) + rv_x(t, h(t))], \quad t > 0,
\end{aligned} \\
u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), \quad -h_0 \leq x \leq h_0, \\
-g(0) &= h(0) = h_0,
\end{aligned}
\]
where $x = g(t)$ and $x = h(t)$ are free boundaries which satisfy the well-known one-phase Stefan type condition, and the parameters $\mu$ and $\mu \rho$ measure the intention for spreading into new habitats of $u$ and $v$, respectively. For both $x = g(t)$ and $x = h(t)$, we impose the Dirichlet boundary condition. Moreover, we assume $D, r,$
Besides, the initial data \((u_0(x), v_0(x), -h_0, h_0)\) satisfies
\[
\begin{aligned}
&h_0 > 0, \quad u_0(x), v_0(x) \in C^2([-h_0, h_0]), \quad u_0(x), \ v_0(x) > 0 \text{ for } x \in (-h_0, h_0), \\
u_0(-h_0) = v_0(-h_0) = u_0(h_0) = v_0(h_0) = 0.
\end{aligned}
\]

In what follows, we only consider the nonnegative solutions, and focus on the weak competition case:
\[(A1) \quad 0 < c, \ b < 1.\]

Besides, we only consider the small advection case:
\[(A2) \quad 0 < \beta_1 < 2\sqrt{1 - c}, \quad 0 < \beta_2 < 2\sqrt{(1 - b)rD}.\]

We give some notations which are often used in the sequel:

**Notation.**
\[
g_{\infty} := \lim_{t \to \infty} g(t), \quad h_{\infty} := \lim_{t \to \infty} h(t), \quad h_s = \min \left\{ \frac{\pi}{2} \sqrt{\frac{D}{r}} \frac{1}{\sqrt{1 - \frac{\beta^2_1}{4rD}}}, \frac{\pi}{2} \frac{1}{\sqrt{1 - \frac{\beta^2_2}{4rD}}} \right\},
\]
\[
h^* = \begin{cases} 
\frac{\pi}{2} \frac{1}{\sqrt{1 - b - \frac{\beta^2_2}{4rD}}}, & \text{if } \frac{D}{r} 1 - \frac{\beta^2_2}{4rD} < \frac{1}{1 - \frac{\beta^2_2}{4}}, \\
\frac{\pi}{2} \frac{1}{\sqrt{1 - c - \frac{\beta^2_1}{4rD}}}, & \text{if } \frac{D}{r} 1 - \frac{\beta^2_1}{4rD} > \frac{1}{1 - \frac{\beta^2_1}{4}}, \\
\min \left\{ \frac{\pi}{2} \frac{1}{\sqrt{1 - b - \frac{\beta^2_2}{4rD}}}, \frac{\pi}{2} \frac{1}{\sqrt{1 - c - \frac{\beta^2_1}{4rD}}} \right\}, & \text{if } \frac{D}{r} 1 - \frac{\beta^2_1}{4rD} = \frac{1}{1 - \frac{\beta^2_1}{4}}.
\end{cases}
\]

We can see that \(h_s < h^*\) easily.

We give the following definition:

**Definition 1.1. Spreading of the two species:** \(g_{\infty} = -\infty, \ h_{\infty} = \infty, \) and
\[
\liminf_{t \to \infty} u(t, x) > 0, \quad \liminf_{t \to \infty} v(t, x) > 0,
\]
uniformly on any compact subset of \((-\infty, \infty)\).

**Definition 1.2. Vanishing of the two species:** \(g_{\infty} > -\infty, \ h_{\infty} < \infty, \) and
\[
\lim_{t \to \infty} \|u(t, \cdot)\|_{C([g(t), h(t)])} = 0, \quad \lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0.
\]

The purposes of our paper are to analyse the model of two species competition with the influence of advection on the criteria for spreading and vanishing, the long time behavior of the solution, and the asymptotic spreading speed when spreading occurs. In this sense, this paper can be regarded as an improvement and extension of [18] with adding the advection effect.

The difference between our problem and [18] lies in the discussion of the following problem
\[
\begin{aligned}
u_t &= Du_{xx} - \beta u_x + rv(1 - u), \quad t > 0, \quad x \in (L_1, L_2), \\
u(t, L_1) &= 0, \quad u(t, L_2) = 0, \quad t > 0, 
\end{aligned}
\tag{4}
\]
and the case in [18]

\[
\begin{cases}
    u_t = Du_{xx} + ru(1-u), & t > 0, \ x \in (0, L), \\
    u_x(t, 0) = 0, \ u(t, L) = 0, & t > 0.
\end{cases}
\]

(5)

In [18], Guo and Wu considered the case of \( \beta = 0 \), and they investigated the Neumann boundary condition on the left side \( x = 0 \) and the Dirichlet boundary condition on the right side \( x = s(t) \). In contrast to the problem (5), we consider the left boundary \( x = g(t) \), and boundary condition here is Dirichlet type, so the corresponding problems is (4). Therefore, the conclusion in [18] cannot be used directly in this paper. To overcome the difficulty induced by the advection term \( \beta \neq 0 \) and the Dirichlet boundary condition on the left side, we calculate the principal eigenvalue for the new problem, and use it to determine the corresponding sharp threshold related to the spreading or vanishing of the two species, and find that both of the advection coefficients \( \beta_1 \) and \( \beta_2 \) can influence the sharp threshold, and the techniques of dealing with the two advection coefficients \( \beta_1 \) and \( \beta_2 \) are more complicated than the single case with only one advection coefficient \( \beta \). Besides, we need to deal with the double free boundaries at the same time.

Moreover, we give a simple criterion governing spreading or vanishing of the two species, then by adding some more restrictions on the parameters, we derive a spreading-vanishing dichotomy, which was put forward by Du and Lin [10] for a case of single species initially. Last, by using the comparison principle, we give some sufficient conditions for the spreading and vanishing via the initial data \((u_0, v_0, -h_0, h_0)\). When spreading of the two species happens, we give two theorems on the long time behavior of solutions, a natural method is finding a pair of supersolution and subsolution to squeeze the solution. Nevertheless, to find them immediately seems not easy, therefore an iteration scheme with constructing better supersolutions and subsolutions step by step is adopted to derive our goal. For the spreading speeds of both \( h(t) \) and \( g(t) \), we can draw a conclusion that the advection can influence the spreading speed. For the upper bound of \( \limsup_{t \to \infty} h(t)/t \) and the lower bound of \( \liminf_{t \to \infty} g(t)/t \), we adopt the idea of [18], which indicates that \( \limsup_{t \to \infty} h(t)/t \) (\( \liminf_{t \to \infty} g(t)/t \)) can not be faster (slower) than the minimal (maximal) travelling wave speed of the related problems. For the lower bound of \( \liminf_{t \to \infty} h(t)/t \) and the upper bound of \( \limsup_{t \to \infty} g(t)/t \), we refer to [38]. Then we give a second estimate of spreading speed, which is better than the first one. The tool we here use is a modification of the single species case.

From this paper, we can see that when the interspecific competition coefficients are small, i.e., \( 0 < b, c < 1 \), the two species can coexist when spreading happens. Furthermore, when the advection coefficients are small, i.e., the advection influence is not strong, the two species share similar dynamic behavior with the case of \( \beta = 0 \), i.e., they will spreading or vanishing, and no third case exists. So natural questions arise such as: (1) When \( 0 < b < 1 \leq c \) or \( 0 < c < 1 \leq b \), i.e., one species has a more competition ability than the other, can the two species coexist, or one overcome another? (2) When the advection coefficients are large, can spreading-vanishing dichotomy still exist? Or does the third case which between spreading and vanishing happen? These interesting problems will be our future research subjects.

The organization of this paper is as follows. In Section 2, we show the problem (TFB) exists a unique global solution. In Section 3, we show several preliminaries.
including the comparison principles and so on, which will be used throughout the paper. In Section 4, we prove the long time behavior of solutions when \( g_\infty = -\infty \) and \( h_\infty = \infty \). In Section 5, we demonstrate both \( h_\infty \) and \( g_\infty \) are finite or infinite simultaneously, and give some criteria for spreading and vanishing. In Section 6, we provide estimates of spreading speed of both \( h(t) \) and \( g(t) \) when spreading happens, and give a more detailed depiction on the long time behavior of solutions. In Appendixes A, B and C, we give the brief proofs of Lemmas 2.2, 2.3 and Theorem 2.1, respectively.

2. Existence and uniqueness. In this section, we give Theorem 2.1, which is the global existence and uniqueness of the solution of the problem (TFB). The proof of Theorem 2.1 is standard by modifying the ideas of [10] and [18], and we put the brief proofs of this section in Appendixes A, B and C.

**Theorem 2.1.** The problem (TFB) admits a unique global solution
\[
(u, v, g, h) \in C^{1+\frac{1}{2}, 2+\alpha}(\Omega) \times C^{1+\frac{1}{2}, 2+\alpha}(\Omega) \times C^{1+\frac{1}{2}}([0, \infty)) \times C^{1+\frac{1}{2}}([0, \infty)),
\]
where \( \Omega := \{(t, x) : t > 0, \ g(t) \leq x \leq h(t)\} \).

In order to prove Theorem 2.1, we give the following two lemmas.

**Lemma 2.2.** (Local existence and uniqueness) The problem (TFB) admits a unique global solution
\[
(u, v, g, h) \in C^{1+\frac{1}{2}, 2+\alpha}(\Omega_T) \times C^{1+\frac{1}{2}, 2+\alpha}(\Omega_T) \times C^{1+\frac{1}{2}}([0, T]) \times C^{1+\frac{1}{2}}([0, T]),
\]
for any \( \alpha \in (0, 1) \) and some \( T > 0 \) small enough, where \( \Omega_T := \{(t, x) : 0 < t \leq T, \ g(t) \leq x \leq h(t)\} \).

In order to prove the global existence of solution, we need the following lemma.

**Lemma 2.3.** Let \((u, v, g, h)\) be a solution of the problem (TFB) for \( t \in [0, T]\) for some \( T > 0 \). Then
\[
0 < u(t, x) \leq \max\{1, \|u_0\|_{L^\infty([-h_0, h_0])}\} \text{ for } t \in [0, T], \quad x \in (g(t), h(t)),
\]
\[
0 < v(t, x) \leq \max\{1, \|v_0\|_{L^\infty([-h_0, h_0])}\} \text{ for } t \in [0, T], \quad x \in (g(t), h(t)),
\]
\[
0 < -g'(t) \leq \mu \Lambda \text{ for } t \in [0, T],
\]
\[
0 < h'(t) \leq \mu \Lambda \text{ for } t \in [0, T],
\]
where \( \Lambda > 0 \) depends only on \( D, r, \rho, \beta_1, \beta_2, \|u_0\|_{L^\infty([-h_0, h_0])}, \|v_0\|_{L^\infty([-h_0, h_0])}, \|u_0\|_{C^1([-h_0, h_0])}, \|v_0\|_{C^1([-h_0, h_0])} \).

3. Preliminaries. In this section, we will give some basic lemmas which will be used in the following part. Consider the problem \((P_0)\):
\[
\begin{align*}
\left\{ \begin{array}{l}
   u_t = Du_{xx} - \beta u_x + ru(1-u), \quad t > 0, \quad x \in (L_1, L_2), \\
   u(t, L_1) = 0, \quad u(t, L_2) = 0, \quad t > 0,
\end{array} \right.
\end{align*}
\]
where \( r > 0, \ L_1 \) and \( L_2 \) are constants. The first lemma will be used frequently in the later sections, which can be considered as a special case of Corollary 3.4 in [3].

**Lemma 3.1.** Let \((L_2 - L_1)^* = \pi \sqrt{\frac{D}{r}} \sqrt{\frac{1}{1 - \frac{r}{2D}}} \), then one of the following happens:
(i) If \( L_2 - L_1 \leq (L_2 - L_1)^* \), then all positive solutions of \((P_0)\) tend to zero in \( C([L_1, L_2]) \) as \( t \to \infty \).
(ii) If \( L_2 - L_1 > (L_2 - L_1)^* \), then there exists a unique positive stationary solution \( \phi \) of \((P_0)\) such that all positive solutions of \((P_0)\) approach \( \phi \) in \( C([L_1, L_2]) \) as \( t \to \infty \).
Lemma 3.2. Let \((u, v, g, h)\) be a solution of the problem (TFB). If \(h_\infty - g_\infty < \infty\), then
\[
\lim_{t \to \infty} g'(t) = 0, \quad \lim_{t \to \infty} h'(t) = 0.
\]

Proof. For any \(\tau \geq 1, t \in [\tau, \tau + 1]\), by the standard transformation
\[
y = \frac{2x - h(t) - g(t)}{h(t) - g(t)}, \quad \bar{u}(t, y) := u(t, x) \text{ and } \bar{v}(t, y) := v(t, x),
\]
the problem (TFB) can be replaced by a fixed boundary problem. By the standard \(L^p\) theory and the Sobolev embedding theorem, we obtain that \(\bar{u}\) and \(\bar{v}\) have a uniform \(C^{1+\alpha},1+\alpha\) bound over \(\{(t, y) : t \in [\tau, \tau + 1], -1 \leq y \leq 1\}\), where \(\alpha \in (0, 1)\). Notice that this uniform bound is independent of \(\tau\), hence by using the Stefan type condition in (3), there exists a positive constant \(C\) such that
\[
\|g'\|_{C^{\frac{n}{2}}([1, \infty))} \leq C, \quad \|h'\|_{C^{\frac{n}{2}}([1, \infty))} \leq C. \tag{10}
\]

We only prove \(\lim_{t \to \infty} g'(t) = 0\), since another is parallel. Assume there exists a sequence \(\{t_n\}\) satisfying \(t_n \to \infty, g'(t_n) \to \sigma_1\) as \(n \to \infty\) for some \(\sigma_1 < 0\). Due to (10), we can find \(\varepsilon > 0\) small enough such that \(g'(t) \leq \frac{\sigma_1}{2}\) for all \(t \in [t_n - \varepsilon, t_n + \varepsilon]\) and for all large \(n\). Then we get \(g_{\infty} = -h_0 + \int_{t_n}^{\infty} g'(t)dt \leq -h_0 + \int_{t_n}^{\infty} \sigma_1 dt = -\infty\), hence a contradiction is obtained.

Lemma 3.3. (Comparison Principle I) (see [10, 18]) Suppose that \(\bar{g}, \bar{h} \in C^1([0, \infty)), w_1, w_2 \in C(D) \cap C^{1,2}(D)\) with \(D = \{(t, x) \in \mathbb{R}^2 : t > 0, \bar{g}(t) < x < \bar{h}(t)\}\), and
\[
\begin{align*}
\begin{cases}
w_{1,t} \geq w_{1,x} - \beta_1 w_{1,x} + w_1(1 - w_1), \text{ in } D, \\
w_{2,t} \geq Dw_{2,x} - \beta_2 w_{2,x} + rw_2(1 - w_2), \text{ in } D, \\
w_1(t, \bar{g}(t)) \geq 0, \quad w_i(t, \bar{h}(t)) = 0, \quad t > 0, \quad i = 1, 2,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
h'(t) \geq -\mu(1 + \rho)w_{1,x}(t, \bar{h}(t)), \quad t > 0, \quad i = 1, 2.
\end{cases}
\end{align*}
\]
If \(w_1(0, x) \geq u_0(x), w_2(0, x) \geq v_0(x)\) for \(x \in [-h_0, h_0]\), \(\bar{h}(0) \geq h_0\), and \(\bar{g}(t) \leq g(t)\) for \(t \geq 0\), then the solution \((u, v, g, h)\) of the problem (TFB) satisfies: \(\bar{h}(t) \geq h(t)\) for \(t \geq 0, u_1(t, x) \geq u(t, x), \) and \(w_2(t, x) \geq v(t, x)\) for \(t \geq 0, g(t) \leq x \leq h(t)\).

Next we give a variant of Lemma 3.3, whose proof only requires some obvious modifications.

Lemma 3.4. (Comparison Principle II) (see [10]) Suppose that \(\bar{g}, \bar{h} \in C^1([0, \infty)), w_1, w_2 \in C(D) \cap C^{1,2}(D)\) with \(D = \{(t, x) \in \mathbb{R}^2 : t > 0, \bar{g}(t) < x < \bar{h}(t)\}\), and
\[
\begin{align*}
\begin{cases}
w_{1,t} \geq w_{1,x} - \beta_1 w_{1,x} + w_1(1 - w_1), \text{ in } D, \\
w_{2,t} \geq Dw_{2,x} - \beta_2 w_{2,x} + rw_2(1 - w_2), \text{ in } D, \\
w_1(t, \bar{g}(t)) = 0, \quad w_i(t, \bar{h}(t)) = 0, \quad t > 0, \quad i = 1, 2,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
g'(t) \leq -\mu(1 + \rho)w_{1,x}(t, \bar{g}(t)), \quad t > 0, \quad i = 1, 2.
\end{cases}
\end{align*}
\]
If \(w_1(0, x) \geq u_0(x), w_2(0, x) \geq v_0(x)\) for \(x \in [-h_0, h_0]\), \(\bar{h}(0) \geq h_0\), and \(\bar{g}(0) \leq -h_0\) for \(t \geq 0\), then the solution \((u, v, g, h)\) of the problem (TFB) satisfies: \(\bar{h}(t) \geq h(t)\), and \(\bar{g}(t) \leq g(t)\) for \(t \geq 0, u_1(t, x) \geq u(t, x), \) and \(w_2(t, x) \geq v(t, x)\) for \(t \geq 0, g(t) \leq x \leq h(t)\).
Remark 1. There is a symmetric version of Lemma 3.3, in which the condition on the left and right boundaries are interchanged. We also have a corresponding comparison principle for subsolutions in each case.

4. Long time behavior of solutions when $g_{\infty} = -\infty$ and $h_{\infty} = \infty$. In this section, we shall give the long time behavior of solutions when $g_{\infty} = -\infty$ and $h_{\infty} = \infty$.

**Theorem 4.1.** Suppose that spreading of the two species happens. Then

$$\lim_{t \to \infty} u(t, x) = \frac{1 - c}{1 - bc}, \quad \lim_{t \to \infty} v(t, x) = \frac{1 - b}{1 - bc},$$

uniformly on any compact subset of $(-\infty, \infty)$.

In order to prove Theorem 4.1, we give some results firstly.

**Lemma 4.2.** The positive solution $u(x)$ of

$$\begin{cases} u_{xx} - \beta u_x + u(a - bu) = 0, & -l < x < l, \\
 u(-l) = u(l) = 0, \end{cases}$$

satisfies $u(x) \to \frac{a}{b}$ uniformly on any compact subset of $(-\infty, \infty)$ as $l \to \infty$.

**Proof.** The process is similar to Lemma 2.2 of [13], so we omit the details. \hfill \Box

**Lemma 4.3.** Let $(u, v, g, h)$ be a solution of the problem (TFB) with $g_{\infty} = -\infty$ and $h_{\infty} = \infty$. Then

(i) $\limsup_{t \to \infty} u(t, x) \leq 1$ and $\limsup_{t \to \infty} v(t, x) \leq 1$ uniformly in $x \in (-\infty, \infty)$,

(ii) $\liminf_{t \to \infty} u(t, x) \geq 1 - c$ and $\liminf_{t \to \infty} v(t, x) \geq 1 - b$ uniformly on any compact subset of $x \in (-\infty, \infty)$.

**Proof.** The process is similar to Lemma 4.1 of [18], so we omit the details. \hfill \Box

**Lemma 4.4.** Assume that $0 < b$, $c < 1$.

(i) Define two sequences $\{\bar{u}_n\}_{n \in \mathbb{N}}$ and $\{\underline{u}_n\}_{n \in \mathbb{N}}$ as follows:

$$(\bar{u}_1, \underline{u}_1) := (1, 1 - b), \quad (\bar{u}_{n+1}, \underline{u}_{n+1}) := (1 - c\underline{u}_n, 1 - b(1 - c\underline{u}_n)).$$

Then we obtain $\bar{u}_n > \bar{u}_{n+1} > 0$, and $\underline{u}_n < \underline{u}_{n+1} < 1$ for all $n \in \mathbb{N}$. In addition,

$$\lim_{n \to \infty} \bar{u}_n = \frac{1 - c}{1 - bc}, \quad \lim_{n \to \infty} \underline{u}_n = \frac{1 - b}{1 - bc}.$$

(ii) Define two sequences $\{\bar{v}_n\}_{n \in \mathbb{N}}$ and $\{\underline{v}_n\}_{n \in \mathbb{N}}$ as follows:

$$(\bar{v}_1, \underline{v}_1) := (1 - c, 1), \quad (\bar{v}_{n+1}, \underline{v}_{n+1}) := (1 - c(1 - b\underline{u}_n), 1 - b\underline{u}_n).$$

Then we obtain $\bar{v}_n < \bar{v}_{n+1} < 1$, and $\underline{v}_n > \underline{v}_{n+1} > 0$ for all $n \in \mathbb{N}$. In addition,

$$\lim_{n \to \infty} \bar{v}_n = \frac{1 - c}{1 - bc}, \quad \lim_{n \to \infty} \underline{v}_n = \frac{1 - b}{1 - bc}.$$

**Proof.** The process is similar to Lemma 4.2 of [18], so we omit the details. \hfill \Box

**Lemma 4.5.** Let $(u, v, g, h)$ be a solution of the problem (TFB) with $g_{\infty} = -\infty$, $h_{\infty} = \infty$. Then

$$\underline{u}_2 \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{u}_2,$$

$$\underline{v}_2 \leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \bar{v}_2,$$
uniformly on any compact subset of \((-\infty, \infty)\).

**Proof.** First, we prove \(\limsup_{t \to \infty} u(t,x) \leq \bar{u}_2\) uniformly on any compact subset of \((-\infty, \infty)\). For any given \(\varepsilon \in (0, \frac{1+\sqrt{5}}{2\beta_2})\), by Lemma 4.3, there exists \(T_\varepsilon \gg 1\) such that

\[
v(t,x) \geq 1 - b - 3\beta \varepsilon \geq \underline{v}_1 - 3\beta \varepsilon > 0, \quad (t,x) \in [T_\varepsilon, \infty) \times [-2S_\varepsilon, 2S_\varepsilon],
\]

\[
u(t,x) \leq 1 + \varepsilon = \underline{u}_1 + \varepsilon, \quad (t,x) \in [T_\varepsilon, \infty) \times [-2S_\varepsilon, 2S_\varepsilon],
\]

where \(S_\varepsilon\) satisfies \(\lim_{\varepsilon \to 0} S_\varepsilon = \infty\) and is to be determined.

In order to use the comparison principle for \(u\) and \(v\), constructing some suitable supersolution \(\bar{U}(t,x)\) and subsolution \(\bar{V}(t,x)\) is what we need. Let \(a_\varepsilon := \bar{u}_2 + 3\beta \varepsilon = 1 - c(\underline{v}_1 - 3\beta \varepsilon) > 0\). Define

\[
\bar{U}(t,x) := \phi(t) + \psi(x) + \varepsilon, \quad \bar{V}(t,x) := \underline{v}_1 - 3\beta \varepsilon = 1 - b - 3\beta \varepsilon,
\]

where \(\phi(t)\) is the solution of \(\phi_t = \phi(a_\varepsilon - \phi)\) with initial value \(\phi(T_\varepsilon) = 1 + \varepsilon\), and

\[
\psi(x) := \begin{cases}
\frac{b}{2\beta_2} \left[-x - S_\varepsilon + \frac{2S_\varepsilon}{\beta} \sin \left(\frac{(x+S_\varepsilon)\varepsilon}{2S_\varepsilon}\right)\right], & x \in [-2S_\varepsilon, -S_\varepsilon), \\
0, & x \in [-S_\varepsilon, S_\varepsilon], \\
\frac{b}{2\beta_2} \left[-x - S_\varepsilon - \frac{2S_\varepsilon}{\beta} \sin \left(\frac{(x-S_\varepsilon)\varepsilon}{2S_\varepsilon}\right)\right], & x \in (S_\varepsilon, 2S_\varepsilon),
\end{cases}
\]

where \(a_\varepsilon, b_\varepsilon\) are to be determined. It is easy to obtain that \(\phi(t) = a_\varepsilon - \frac{a_\varepsilon}{1+3\varepsilon + \bar{u}}\) with \(C = \frac{1+3\varepsilon}{1+3\varepsilon + \bar{u}} < 0\), so we have \(\phi(t) \downarrow a_\varepsilon\) as \(t \to \infty\). For \(\psi(x)\), we can see that \(\psi'(x) \leq 0, \psi''(x) \geq 0\) in \([-2S_\varepsilon, -S_\varepsilon]\), and \(\psi'(x) \geq 0, \psi''(x) \geq 0\) in \([S_\varepsilon, 2S_\varepsilon]\).

It is easy to see \(\bar{U}(t,x) \in C^{1,2}(\Omega_\varepsilon)\), where \(\Omega_\varepsilon := \{(t,x) : t \geq T_\varepsilon, -2S_\varepsilon \leq x \leq 2S_\varepsilon\}\). By a direct computation, we get

\[
\bar{V}_x - D\bar{V}_{xx} + \beta_2 \bar{V}_x - r\bar{V}(1 - \bar{V} - b\bar{U}) = -rb\bar{V}(1 + 3\varepsilon - \bar{U}) \leq 0, \quad \text{if} \quad \bar{U} \leq 1 + 3\varepsilon. \tag{13}
\]

The region in which (13) holds is to be determined. Divide \([-2S_\varepsilon, 2S_\varepsilon]\) into \([-2S_\varepsilon, -S_\varepsilon) \cup [-S_\varepsilon, S_\varepsilon] \cup (S_\varepsilon, 2S_\varepsilon]\), and choose \(S_\varepsilon = \beta_1 b + \sqrt{\frac{2\beta^2 + 4\alpha^2 b_\varepsilon^2}{4\alpha^2}}\).

For \((t,x) \in [T_\varepsilon, \infty) \times [-S_\varepsilon, S_\varepsilon]\), \(\bar{U}(t,x) = \phi(t) + \varepsilon\). Since \(\phi(t) > a_\varepsilon\) for all \(t \geq T_\varepsilon\),

\[
\bar{U}_t - \bar{U}_{xx} + \beta_1 \bar{U}_x - \bar{U}(1 - \bar{V} - c\bar{V}) = \phi(a_\varepsilon - \phi) + (\phi + \varepsilon)(\phi + \varepsilon - a_\varepsilon) > -\phi(a_\varepsilon - \phi) + (\phi + \varepsilon)(\phi - a_\varepsilon) = \varepsilon(\phi - a_\varepsilon) > 0. \tag{14}
\]

For \((t,x) \in [T_\varepsilon, \infty) \times (S_\varepsilon, 2S_\varepsilon]\),

\[
\bar{U}_t - \bar{U}_{xx} + \beta_1 \bar{U}_x - \bar{U}(1 - \bar{V} - c\bar{V}) = \phi'(t) - \psi''(x) + \beta_1 \psi'(x) - (\phi + \psi + \varepsilon)(1 - \phi - \psi - c(\underline{v}_1 - 3\beta \varepsilon)) \geq \phi(a_\varepsilon - \phi) - \psi''(x) - (\phi + \psi + \varepsilon)(a_\varepsilon - \phi) + \varepsilon^2 \geq \varepsilon^2 - \psi''(x) \geq \varepsilon^2 - \frac{b_\varepsilon \pi}{4\alpha S_\varepsilon} \geq 0.
\]

For \((t,x) \in [T_\varepsilon, \infty) \times [-2S_\varepsilon, -S_\varepsilon]\),

\[
\bar{U}_t - \bar{U}_{xx} + \beta_1 \bar{U}_x - \bar{U}(1 - \bar{V} - c\bar{V}) = \phi'(t) - \psi''(x) + \beta_1 \psi'(x) - (\phi + \psi + \varepsilon)(1 - \phi - \psi - c(\underline{v}_1 - 3\beta \varepsilon)) \geq \varepsilon^2 - \frac{b_\varepsilon \pi}{4\alpha S_\varepsilon} - \frac{\beta_1 b_\varepsilon^2}{2\alpha S_\varepsilon} \geq 0. \tag{16}
\]
For (13), in order to use the comparison principle, we need to adjust the region. Let \( t \geq T_\varepsilon, x = L(t) > 0 \) be the curve such that \( \bar{U}(t, L(t)) = 1 + 3\varepsilon \), then it is easy to see that \( S_\varepsilon \leq L(t) \). Otherwise, if \( L(t) < S_\varepsilon \), we have \( \psi(L(t)) = 0 \), \( \bar{U}(t, L(t)) = \phi(t) + \varepsilon = 1 + 3\varepsilon, \phi(t) = 1 + 2\varepsilon \), this is impossible. Then, setting \( \Omega_T := \{(t, x) : t \geq T_\varepsilon, -L(t) \leq x \leq L(t)\} \), we can see

\[
\{(t, x) : t \geq T_\varepsilon, -S_\varepsilon \leq x \leq S_\varepsilon\} \subset \bar{\Omega}_T \cap \Omega_T. \tag{17}
\]

In the following, we get the following initial and boundary value conditions:

\[
\begin{align*}
\bar{U}(t, L(t)) & \geq u(t, L(t)), \quad \bar{U}(t, -L(t)) \geq u(t, -L(t)), \quad \text{for } t \geq T_\varepsilon, \\
\bar{U}(t, 2S_\varepsilon) & \geq u(t, 2S_\varepsilon), \quad \bar{U}(t, -2S_\varepsilon) \geq u(t, -2S_\varepsilon), \quad \text{for } t \geq T_\varepsilon,
\end{align*}
\]

if we choose \( \alpha := \frac{\varepsilon - 2}{\varepsilon}, \beta := 1 + \varepsilon - \alpha \varepsilon > 0 \). On the other hand,

\[
\begin{align*}
\bar{V}(t, L(t)) & \leq v(t, L(t)), \quad \bar{V}(t, -L(t)) \leq v(t, -L(t)), \quad \text{for } t \geq T_\varepsilon, \\
\bar{V}(t, 2S_\varepsilon) & \leq v(t, 2S_\varepsilon), \quad \bar{V}(t, -2S_\varepsilon) \leq v(t, -2S_\varepsilon), \quad \text{for } t \geq T_\varepsilon, \\
\bar{U}(T_\varepsilon, x) & \geq u(T_\varepsilon, x), \quad \bar{V}(T_\varepsilon, x) \leq v(T_\varepsilon, x), \quad \text{for } x \in [-2S_\varepsilon, 2S_\varepsilon].
\end{align*}
\]

Together with (13)-(16), and using the comparison principle, we get \( \bar{U} \geq u \) in \( \bar{\Omega}_T \cap \Omega_T \). Particularly, by (17), \( \phi(t) + \varepsilon \geq u \) for \( (t, x) \in [T_\varepsilon, \infty) \times [-S_\varepsilon, S_\varepsilon] \). Thus

\[
\limsup_{t \to \infty} u(t, x) \leq \liminf_{t \to \infty} \phi(t) + \varepsilon = a_\varepsilon + \varepsilon = \bar{u}_2 + \varepsilon(3bc + 1), \quad \text{for } x \in [-S_\varepsilon, S_\varepsilon].
\]

Letting \( \varepsilon \to 0 \), we get that \( \limsup_{t \to \infty} u(t, x) \leq \bar{u}_2 \) uniformly on any compact subset of \( (-\infty, \infty) \).

Next, by using the proof similar to Lemma 4.3(ii), i.e., replacing \( 1+\varepsilon \) with \( \bar{u}_2 + \varepsilon \), we prove that \( \liminf_{t \to \infty} v(t, x) \geq \underline{v}_2 \) uniformly on any compact subset of \( (-\infty, \infty) \).

Last, we obtain \( \limsup_{t \to \infty} v(t, x) \leq \bar{v}_2 \) and \( \liminf_{t \to \infty} u(t, x) \geq \underline{u}_2 \) uniformly on any compact subset of \( (-\infty, \infty) \) in a similar way.

In order to prove Theorem 4.1, we continue the strategy as above to get the following corollary.

**Corollary 1.** Let \((u, v, g, h)\) be a solution of the problem (TFB) with \( g_\infty = -\infty \) and \( h_\infty = \infty \). Then for each \( n \in \mathbb{N} \),

\[
\underline{u}_n \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{u}_n,
\]

\[
\underline{v}_n \leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \bar{v}_n,
\]

uniformly on any compact subset of \( (-\infty, \infty) \).

Now, we are ready to give the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Letting \( n \to \infty \) in Corollary 1 and using Lemma 4.4, we can get the results. \( \square \)

Besides Theorem 4.1, we will give a more detailed depiction on the long time behavior of solutions when spreading of the two species happens, which is stated in Section 6.
5. The criteria governing spreading and vanishing. In this section, we first give a simple criterion for the spreading or vanishing of the two species, then a spreading-vanishing dichotomy is presented. Finally, based on the previous results, we give a corollary.

By Lemma 2.3, we get $-g_\infty, h_\infty \in (0, \infty]$, where $g_\infty := \lim_{t \to \infty} g(t)$ and $h_\infty := \lim_{t \to \infty} h(t)$. Therefore, we have four cases: (1) $g_\infty = -\infty, h_\infty = \infty$, (2) $g_\infty > -\infty, h_\infty < \infty$, (3) $g_\infty = -\infty, h_\infty < \infty$, (4) $g_\infty > -\infty, h_\infty = \infty$. Recall the notation of $h^*$ from page 5, we get Lemma 5.1, which indicates that the last two cases do not happen, i.e., both $g_\infty$ and $h_\infty$ are finite or infinite simultaneously.

**Lemma 5.1.** Let $(u, v, g, h)$ be a solution of the problem (TFB). If $h_\infty - g_\infty > 2h^*$, then $g_\infty = -\infty, h_\infty = \infty$.

**Proof.** We divide the discussion into three cases: (i) $\frac{D}{r} \frac{1}{1-\frac{h^*}{4+b}} < \frac{1}{2}$; (ii) $\frac{D}{r} \frac{1}{1-\frac{h^*}{4+b}} > \frac{1}{1-\frac{h^*}{2}}$; (iii) $\frac{D}{r} \frac{1}{1-\frac{h^*}{4+b}} = \frac{1}{1-\frac{h^*}{2}}$.

**Case (i).** In this case, we have
$$h^* = \frac{\pi}{2} \sqrt[4]{\frac{D}{r}} \frac{1}{\sqrt{1-b-\frac{h^*}{4+b}}}.$$

The idea of the proof comes from Lemma 3.1 of [26], whose difficulty is induced by the introducing of advection term and double free boundaries. By a contradiction, there are three cases: (1) $g_\infty > -\infty, h_\infty < \infty$, (2) $g_\infty = -\infty, h_\infty < \infty$, (3) $g_\infty > -\infty, h_\infty = \infty$. Without loss of generality, we only assume cases (1) and (2): $g_\infty \geq -\infty, h_\infty < \infty$. (Whether $g_\infty > -\infty$ or $g_\infty = -\infty$ does not make contributions to the proof), since the proof of case (3) is parallel.

As $h_\infty - g_\infty > 2h^*$, we can find $T_0 > 0$ large enough and $l > 0$ such that $h(t) - g(t) > 2l > \pi \sqrt[4]{\frac{D}{r}} \frac{1}{\sqrt{1-b-\frac{h^*}{4+b}}}$, for $t \geq T_0$. It follows that there exists $x_0 \in (g(T_0), h(T_0))$ such that
$$g(t) \leq g(T_0) < x_0 - l < x_0 < x_0 + l < h(T_0) \leq h(t), \text{ for } t \geq T_0.$$

For such a fixed $l$, there exists small $\varepsilon > 0$ and $\varepsilon_1 > 0$ such that
$$l > \pi \sqrt[4]{\frac{D}{r}} \frac{1}{\sqrt{1-b-\varepsilon - \frac{(\beta_2-\varepsilon_1)^2}{4rD}}}.$$

Therefore, by Lemma 3.1, the principal eigenvalue $\lambda_1$ of the following eigenvalue problem
\begin{equation}
\begin{cases}
-D\phi'' + (\beta_2 - \varepsilon_1)\phi' - r(1-b-\varepsilon)\phi = \lambda \phi, & x_0 - l < x < x_0 + l, \\
\phi(x_0 - l) = 0, & \phi(x_0 + l) = 0,
\end{cases}
\end{equation}

satisfies $\lambda_1 < 0$, and the following problem
\begin{equation}
\begin{cases}
-Dw'' + (\beta_2 - \varepsilon_1)w' = rw(1-b-\varepsilon - w), & x_0 - l < x < x_0 + l, \\
w(x_0 - l) = 0, & w(x_0 + l) = 0,
\end{cases}
\end{equation}

admits a unique positive solution $0 < w < 1 - b - \varepsilon$ in $(x_0 - l, x_0 + l)$. According to the maximum principle, $w'(x_0 + l) < 0 < w'(x_0 - l)$, then $w'$ has zero points in $(x_0 - l, x_0 + l)$. Let $x_1 \in (x_0 - l, x_0 + l)$ be such point which is closest to $x_0 + l$. 

Then \( w'(x) < 0 \) in \((x_1, x_0 + l)\), which is crucial in constructing a subsolution of \( v \) later.

Set \( \varepsilon_1 = 0 \) in (18), denote the corresponding principal eigenvalue and eigenfunction by \((\lambda_1^0, \phi_1^0)\). Define \( \overline{u}(t, x) := 1 + \frac{x}{2} \), we can easily show there exists \( T_1 > T_0 \) such that

\[
u(t, x) \leq \overline{u}(t, x), \quad (t, x) \in [T_1, \infty) \times [x_1, h(t)],
\]

and can easily check that \( \varepsilon_2 \phi_1^0 \) is a subsolution of \( v \) in \([T_1, \infty) \times [x_0 - l, x_0 + l] \), provided \( \varepsilon_2 \) is small (depends on \( T_1 \)). Therefore, \( v(t, x_1) \geq \varepsilon_2 \phi_1^0(x_1) > 0 \), for \( t \geq T_1 \).

Now, define \( g(t, x) := \varepsilon_3 w(x_1 + \frac{x_0 + l - x_1}{h(t) - x_1}(x - x_1)) \), \( x_1 \leq x \leq h(t) \), we need to compare \((u, v)\) with \((\overline{u}, \overline{v})\) over \( \Omega_T \), where \( \Omega_T := \{(t, x) : t > T, x_1 \leq x \leq h(t)\} \) for some \( T > 1 \). We calculate

\[
u_x - D\nu_{xx} + \beta_2 \nu_x = - \left( \frac{x_0 + l - x_1}{h(t) - x_1} \right)^2 h'(t) \varepsilon_3 w' - D \left( \frac{x_0 + l - x_1}{h(t) - x_1} \right)^2 \varepsilon_3 w'' + \beta_2 \frac{x_0 + l - x_1}{h(t) - x_1} \varepsilon_3 w'
\]

Using the fact that \( h'(t) \to 0 \) as \( t \to \infty \), we can find \( T > T_1 \) such that \( h'(t) \leq \frac{x_0 + l - x_1}{h_{\infty} - x_1} \varepsilon_1 \), together with the fact that \( w'(y) \leq 0 \) and \( \frac{h(t) - x_1}{x_0 + l - x_1} \geq 1 \) for \((t, y) \in [T, \infty) \times [x_1, x_0 + l] \), we get

\[
u_x - D\nu_{xx} + \beta_2 \nu_x \leq \varepsilon_3 r w(1 - b - \varepsilon - w) \leq r \nu(1 - b - \overline{u}).
\]

Next, choose \( \varepsilon_3 \) small enough such that \( \varepsilon_3 w(x_1) \leq \varepsilon_2 \phi_1^0(x_1) \), and \( \varepsilon_3 w(x_1 + \frac{x_0 + l - x_1}{h(t) - x_1}(x - x_1)) \leq v(T, x) \) in \([x_1, h(T)]\). Moreover, because of \( \overline{u}(t, h(t)) = v(t, h(t)) = 0 \), by the comparison principle, we get \( \nu(t, x) \leq v(t, x) \) for \((t, x) \in \Omega_T \). Therefore, \( \nu_x(t, h(t)) \geq v_x(t, h(t)) \) for \( t \geq T \). Then \( \varepsilon_3 w'(x_0 + l + \frac{x_0 + l - x_1}{h(t) - x_1}) \geq \frac{1}{\rho} h'(t) - \frac{w_u(t, h(t))}{\rho} \geq -\frac{1}{\rho} h'(t) \). Taking \( t \to \infty \), it follows that \( \varepsilon_3 w'(x_0 + l) \geq 0 \), which is in contradiction with \( w'(x_0 + l) < 0 \). Thus, we get \( h_{\infty} = \infty \).

By a similar argument as in Case(i), the proof of Case(ii) can be obtained, so we omit the details here. For Case(iii), without loss of generality, we assume

\[
D \frac{1}{r - 1 - b - \frac{\beta_2}{4D}} < \frac{1}{1 - c - \frac{\beta_2}{4}}.
\]

Then by using the same method as in Case(i), we can prove it. \( \square \)

Recall the notation of \( h_* \) from page 5, we get the following lemma.

**Lemma 5.2.** When \( \frac{D}{r - 1 - \frac{\beta_2}{4D}} = \frac{1}{1 - \frac{\beta_2}{4}} \), we have

\[
 h_{\infty} - g_\infty \notin \left( 2h_* + 2\max \left\{ \frac{\pi}{2} \sqrt{\frac{D}{r - 1 - \frac{\beta_2}{4D}}}, \frac{\pi}{2} \sqrt{1 - \frac{\beta_2}{4}} \right\} \right).
\]

**Proof.** We give an argument by contradiction. Since the proof of \( \frac{D}{r - 1 - \frac{\beta_2}{4D}} < \frac{1}{1 - \frac{\beta_2}{4}} \)
and \( \frac{D}{r - 1 - \frac{\beta_2}{4D}} > \frac{1}{1 - \frac{\beta_2}{4}} \) are similar, we only consider the assumption of \( \frac{D}{r - 1 - \frac{\beta_2}{4D}} < \frac{1}{1 - \frac{\beta_2}{4}} \). In this case, \( h_{\infty} - g_\infty \in \left( \pi \sqrt{\frac{D}{r - 1 - \frac{\beta_2}{4D}}}, \frac{\pi}{2} \sqrt{1 - \frac{\beta_2}{4}} \right) \). This means it implies
that there exists $\sigma \geq 0$ such that $h_\infty + \sigma - (g_\infty - \sigma) = \frac{\pi}{\sqrt{1 - \frac{r^2}{\beta^2}}}$. By showing
that $v(t, \cdot)$ converges to some function in $C^2((g_\infty, h_\infty))$, our goal is to obtain a contradiction. Such idea comes from [7].

Let $(u, v, g, h)$ be the solution of the problem (TFB) and $\bar{u}$ be the solution of
\begin{equation}
\begin{aligned}
\bar{u}_t &= \bar{u}_{xx} - \beta_1 \bar{u}_x + \bar{u}(1 - \bar{u}), \quad t > 0, \quad g_\infty - \sigma < x < h_\infty + \sigma, \\
\bar{u}(t, g_\infty - \sigma) &= 0, \quad \bar{u}(t, h_\infty + \sigma) = 0, \quad t > 0, \\
\bar{u}(0, x) &= \begin{cases}
u(0, x), & x \in [-h_0, h_0], \\
0, & x \in [g_\infty - \sigma, -h_0) \cup (h_0, h_\infty + \sigma].
\end{cases}
\end{aligned}
\end{equation}

Then, by Lemma 3.1,
\begin{equation}
\lim_{t \to \infty} \|\bar{u}(t, \cdot)\|_{C([g_\infty - \sigma, h_\infty + \sigma])} = 0.
\end{equation}

Comparing $(\bar{u}, 0)$ with $(u, v)$, we get
\begin{equation}
\bar{u}(t, x) \geq u(t, x) \text{ for all } t > 0, \ x \in [g(t), h(t)].
\end{equation}

On the other hand, let $\bar{v}$ be the solution of
\begin{equation}
\begin{aligned}
\bar{v}_t &= D\bar{v}_{xx} - \beta_2 \bar{v}_x + r\bar{v}(1 - \bar{v}), \quad t > 0, \quad g_\infty < x < h_\infty, \\
\bar{v}(t, g_\infty) &= 0, \quad \bar{v}(t, h_\infty) = 0, \quad t > 0, \\
\bar{v}(0, x) &= \begin{cases}v(0, x), & x \in [-h_0, h_0], \\
0, & x \in [g_\infty, -h_0) \cup (h_0, h_\infty].
\end{cases}
\end{aligned}
\end{equation}

By Lemma 3.1 again, we have
\begin{equation}
\lim_{t \to \infty} \|\bar{v}(t, \cdot) - v(\cdot)\|_{C([g_\infty, h_\infty])} = 0,
\end{equation}

where $v_\infty > 0$ satisfies
\begin{equation}
\begin{cases}
Dv_\infty'' - \beta_2 v_\infty' + rv_\infty(1 - v_\infty) = 0, \quad g_\infty < x < h_\infty, \\
v_\infty(g_\infty) = 0, \quad v_\infty(h_\infty) = 0.
\end{cases}
\end{equation}

Comparing $(0, \bar{v})$ with $(u, v)$ yields that
\begin{equation}
\bar{v}(t, x) \geq v(t, x) \text{ for all } t > 0, \ x \in [g(t), h(t)].
\end{equation}

Combining (22) with (24), we get
\begin{equation}
\limsup_{t \to \infty} v(t, x) \leq v_\infty(x) \text{ for } x \in (g_\infty, h_\infty).
\end{equation}

Next, we estimate $\liminf_{t \to \infty} v(t, x)$. Choose $h_n - g_n \in \left(\pi \sqrt{\frac{r}{\beta}} \frac{1}{\sqrt{1 - \frac{r^2}{\beta^2}}}, h_\infty - g_\infty\right)$
with $g_n \downarrow g_\infty$ and $h_n \uparrow h_\infty$ as $n \to \infty$, and fix $h_1 - g_1$ which is close enough to
$h_\infty - g_\infty$, then $\{h_n - g_n\}$ can hold the following property:
\begin{equation}
h_n - g_n > \pi \sqrt{\frac{D}{r}} \frac{1}{\sqrt{1 - (h_\infty - g_\infty - h_n + g_n)}} \frac{1}{\sqrt{1 - \frac{r^2}{\beta^2}}} > 0,
\end{equation}
for all $n \in \mathbb{N}$. Thanks to Lemma 3.1, for each fixed $n$, there exists a unique $v_n(x)$
> 0 satisfying
\begin{equation}
\begin{cases}
Dv_n'' - \beta_2 v_n' + rv_n[1 - (h_\infty - g_\infty - h_n + g_n) - v_n] = 0, \quad g_n < x < h_n, \\
v_n(g_n) = 0, \quad v_n(h_n) = 0.
\end{cases}
\end{equation}
For each \( j \in \mathbb{N} \), as \( v_n \) is bounded in \( C^{2+\alpha}([g_l, h_r]) \) for all \( n \geq j \), therefore, by using the Arzela-Ascoli Theorem and the diagonal process, we get \( v_n \to v_\infty \) in \( C^{2,\alpha}_0([g_\infty, h_\infty]) \) as \( n \to \infty \) (up to a subsequence), where \( v_\infty \) satisfies (23).

For each \( n \in \mathbb{N} \), recalling (20), we can find \( T_n > 0 \) such that
\[
b_n \leq h_\infty - g_\infty - h_n + g_n \text{ for all } t \in [T_n, \infty), \quad x \in [g(t), h(t)].
\]
(26)

Besides, \( g(t) < g_n \) and \( h(t) > h_n \) for \( t > T_n \). Last, let \( v_n(t, x) \) be the solution of
\[
\begin{cases}
(v_n)_t = D(v_n)_{xx} - \beta_2 v_n x + r v_n [1 - (h_\infty - g_\infty - h_n + g_n) - v_n], & t > T_n, \quad g_n < x < h_n, \\
v_n(t, g_n) = 0, \quad v_n(t, h_n) = 0, & t > T_n, \\
v_n(T_n, x) = \begin{cases}
v(T_n, x), & x \in [-h_0, h_0], \\
0, & x \in [g_n, -h_0) \cup (h_0, h_n].
\end{cases}
\end{cases}
\]
From (26), we also see that
\[
(v_n)_t - D(v_n)_{xx} + \beta_2 v_n x - r v_n [1 - (h_\infty - g_\infty - h_n + g_n) - v_n] = 0,
\]
for all \((t, x) \in [T_n, \infty) \times [g_n, h_n]\). Therefore, due to (19) and (21), we shall compare \((\bar{u}, \bar{v}_n)\) with \((u, v)\) over \( \{(t, x) : (t, x) \in [T_n, \infty) \times [g_n, h_n]\} \), which implies \( v \geq v_n \) for all \((t, x) \in [T_n, \infty) \times [g_n, h_n]\). Using Lemma 3.1 again, we get \( v_n \to v_\infty \) in \( C([g_n, h_n]) \) as \( t \to \infty \). Thus, for each \( n \),
\[
\lim_{t \to \infty} \inf_{x \in [g_n, h_n]} v(t, x) \geq v_n(x) \text{ for } x \in [g_n, h_n].
\]
Taking \( n \to \infty \), we get
\[
\lim_{t \to \infty} \inf_{x \in [g_\infty, h_\infty]} v(t, x) \geq v_\infty(x) \text{ for } x \in (g_\infty, h_\infty), \tag{27}
\]
where \( v_\infty \) satisfies (23).

Combining (25) with (27), we obtain \( \lim_{t \to \infty} v(t, x) = v_\infty(x) \) for \( x \in (g_\infty, h_\infty) \).
Finally, following the process of the proof of Lemma 2.2 in [7], we derive
\[
\lim_{t \to \infty} \|v(t, \cdot) - v_\infty(\cdot)\|_{C^2([g(t), h(t)])} = 0.
\]
It yields that \( v_\infty(g(t)) \to v_\infty'(g_\infty) > 0 \) and \( v_\infty(h(t)) \to v_\infty'(h_\infty) < 0 \) as \( t \to \infty \). Therefore, by using (3), we find \( \beta > 0 \) such that \( g'(t) \leq -\beta \) and \( h'(t) \geq \beta \) for all large \( t \). But this is in contradiction with Lemma 3.2. Then the proof is completed. \( \square \)

Based on the above two lemmas, we have the following two theorems.

**Theorem 5.3.** Let \((u, v, g, h)\) be a solution of the problem (TFB). Then we have

(i) If \( h_\infty - g_\infty \leq 2h_*, \) then vanishing of the two species happens.
(ii) If \( h_\infty - g_\infty > 2h^* \), then spreading of the two species happens.

**Proof.** (i) Choose \( l_1 \leq g_\infty, l_2 \geq h_\infty \), and \( l_2 - l_1 \in [h_\infty - g_\infty, 2h_*] \). Let \( \bar{u} \) be the solution of
\[
\begin{cases}
\bar{u}_t = \bar{u}_{xx} - \beta_1 \bar{u}_x + \bar{u}(1 - \bar{u}), & t > 0, \quad l_1 < x < l_2, \\
\bar{u}(t, l_1) = 0, \quad \bar{u}(t, l_2) = 0, & t > 0, \\
\bar{u}(0, x) = \begin{cases}
u(0, x), & x \in [-h_0, h_0], \\
0, & x \in [l_1, -h_0) \cup (h_0, l_2].
\end{cases}
\end{cases}
\]
At the same time, let \( \bar{v} \) be the solution of

\[
\begin{align*}
\bar{v} &= D\bar{v}_{xx} - \beta_2\bar{v}_x + r\bar{v}(1 - \bar{v}), \\
\bar{v}(t, l_1) &= 0, \quad \bar{v}(t, l_2) = 0, \\
\bar{v}(0, x) &= \begin{cases} v(0, x), & x \in [-h_0, h_0], \\
0, & x \in [l_1, -h_0) \cup (h_0, l_2].
\end{cases}
\end{align*}
\]

Then, using Lemma 3.1,

\[
\lim_{t \to \infty} \|\bar{v}(t, \cdot)\|_{C([l_1, l_2])} = 0, \quad \lim_{t \to \infty} \|\bar{v}(t, \cdot)\|_{C([l_1, l_2])} = 0. \tag{28}
\]

Comparing \((\bar{u}, 0)\) with \((u, v)\) and \((0, \bar{v})\) with \((u, v)\) over \( \Omega := \{(t, x) \in \mathbb{R}^2 : t > 0, \ g(t) < x < h(t)\} \) respectively, we get \( 0 \leq u \leq \bar{u} \) and \( 0 \leq v \leq \bar{v} \) over \( \Omega \).
Combining with (28), we finish the proof.

(ii) The proof is a direct result of Lemmas 4.3 and 5.1. \( \square \)

As we can see, when \( 2h_* < h_\infty - g_\infty \leq 2h^* \), Theorem 5.3 does not give any information for spreading or vanishing. If some more restrictions on the parameters for the problem (TFB) are added, that is, the following sets are introduced:

\[
A := \left\{ \frac{D}{r} \frac{1}{1 - \frac{\beta_2^2}{4rD}} < \frac{1}{1 - \frac{\beta_1^2}{4}}, \quad 0 < b \leq 1 - \frac{\beta_2^2}{4rD} - \frac{D}{r}(1 - \frac{\beta_1^2}{4}), \quad 0 < c < 1, \quad \mu, \rho > 0 \right\},
\]

\[
B := \left\{ \frac{D}{r} \frac{1}{1 - \frac{\beta_2^2}{4rD}} > \frac{1}{1 - \frac{\beta_1^2}{4}}, \quad 0 < c \leq 1 - \frac{\beta_2^2}{4rD} - \frac{r}{D}(1 - \frac{\beta_1^2}{4}), \quad 0 < b < 1, \quad \mu, \rho > 0 \right\},
\]

then we can get a spreading-vanishing dichotomy, and it was introduced by Du and Lin [10] for a single species case for the first time.

**Theorem 5.4.** Let \((u, v, g, h)\) be a solution of the problem (TFB) with \((D, r, \beta_1, \beta_2, b, c, \mu, \rho) \in A \cup B\). Then either \( h_\infty - g_\infty \leq 2h_* \) (and so vanishing of the two species happens), or spreading of the two species happens.

**Proof.** To prove the result, by Theorem 5.3, it is enough to show that \( g_\infty = -\infty \) and \( h_\infty = \infty \) when \( h_\infty - g_\infty > 2h_* \). When \((D, r, \beta_1, \beta_2, b, c, \mu, \rho) \in A \cup B\), we get

\[
h^* \in \left( h_*, \max \left\{ \frac{\pi}{2} \sqrt{\frac{D}{r} \frac{1}{\sqrt{1 - \frac{\beta_2^2}{4rD}}}}, \frac{\pi}{2} \sqrt{\frac{1}{\sqrt{1 - \frac{\beta_1^2}{4}}} \frac{1}{\sqrt{1 - \frac{\beta_2^2}{4rD}}}} \right\} \right).
\]

Then by Lemma 5.2, we obtain

\[
h_\infty - g_\infty > 2\max \left\{ \frac{\pi}{2} \sqrt{\frac{D}{r} \frac{1}{\sqrt{1 - \frac{\beta_2^2}{4rD}}}}, \frac{\pi}{2} \sqrt{\frac{1}{\sqrt{1 - \frac{\beta_1^2}{4}}} \frac{1}{\sqrt{1 - \frac{\beta_2^2}{4rD}}}} \right\} \geq 2h^*.
\]

Thus by Theorem 5.3, we have the result. \( \square \)

**Remark 2.** As we can see, roughly speaking, this theorem says that if \( D, r, \beta_1, \beta_2, \mu \) and \( \rho \) are given and \( \frac{D}{r} \frac{1}{1 - \frac{\beta_1^2}{4rD}} \neq \frac{1}{1 - \frac{\beta_2^2}{4rD}} \), then either \( b \) or \( c \) is small, a spreading-vanishing dichotomy can be obtained.

In the last part of this section, as a corollary, for the spreading or vanishing of the two species happens, we give some sufficient conditions via the initial data.
Corollary 2. Let \((u, v, g, h)\) be a solution of the problem (TFB). Then we have

(i) If \(h_0 \geq h^\ast\), then spreading of the two species happens.

(ii) Assume \((D, r, \beta_1, \beta_2, b, c, \mu, \rho) \in A \cup B\). If \(h_0 \geq h^\ast\), then spreading of the two species happens.

(iii) If \(h_0 < h^\ast\), and

\[
\|u_0\|_{L^\infty([-h_0, h_0])} \leq \frac{h_0^2 \delta^2 (2 + \delta) \cos\left(\frac{\pi}{2 + \delta}\right)}{2 \pi \mu (1 + \rho) e^{\frac{\pi h_0}{2 + \delta} (2 + \frac{\delta}{2})}},
\]

\[
\|v_0\|_{L^\infty([-h_0, h_0])} \leq \frac{h_0^2 \delta^2 (2 + \delta) \cos\left(\frac{\pi}{2 + \delta}\right)}{2 \pi \mu (1 + \rho) e^{\frac{\pi h_0}{2 + \delta} (2 + \frac{\delta}{2})}},
\]

where \(\delta < \min\{\delta_1, \delta_2, \delta_3\}\), and \(\delta_1, \delta_2, \delta_3\) satisfy

\[
\begin{align*}
\left(\frac{\pi}{2}\right)^2 \frac{1}{h_0^2 (1 + \delta_1)^2} & \geq 1 - \frac{\beta_1^2}{4} + \frac{\beta_1 h_0 \delta^2}{4}, \\
D \left(\frac{\pi}{2}\right)^2 \frac{1}{h_0^2 (1 + \delta_2)^2} & \geq r - \frac{\beta_2^2}{4D} + \delta_2 + \frac{\beta_2 h_0 \delta_2^2}{4D}, \\
h_0 (1 + \delta_3) & \leq h^\ast,
\end{align*}
\]

respectively. Then vanishing of the two species happens.

Proof. (i) Since \(h'(t) > 0\) and \(g'(t) < 0\) for all \(t > 0\), we have \(h_\infty - g_\infty > 2h^\ast\). Then the result is obtained immediately from Theorem 5.3.

(ii) The same as (i), we have \(h_\infty - g_\infty > 2h^\ast\), and together with the condition of \((D, r, \beta_1, \beta_2, b, c, \mu, \rho) \in A \cup B\), the result is obtained from Theorem 5.4.

(iii) We adopt Lemma 3.4 to complete our proof. Inspired by the arguments from [10], we use the following functions

\[
s(t) := h_0 (1 + \delta - \frac{\delta}{2} e^{-\delta t}), \quad t \geq 0,
\]

\[
w_1(t, x) := M_1 e^{-\delta t - \frac{\delta}{2} (s(t) - x)} V\left(\frac{x}{s(t)}\right), \quad t \geq 0, \quad -s(t) \leq x \leq s(t),
\]

\[
w_2(t, x) := M_2 e^{-\delta t - \frac{\delta}{2} (s(t) - x)} V\left(\frac{x}{s(t)}\right), \quad t \geq 0, \quad -s(t) \leq x \leq s(t),
\]

where \(V(y) := \cos\left(\frac{\pi}{2} y\right), y = \frac{x}{s(t)}, -1 \leq y \leq 1\). \(\delta, M_1\) and \(M_2\) are positive constants which are to be determined. To apply Lemma 3.4, we shall confirm (12). By \(h_0 < h^\ast\), (31) and (32), we have

\[
w_{1,t} - w_{1,xx} + \beta_1 w_{1,x} - w_{1} (1 - w_1)
\]

\[
\geq M_1 e^{-\delta t - \frac{\delta}{2} (s(t) - x)} V\left(\frac{\pi}{2}\right)^2 \frac{1}{h_0^2 (1 + \delta)^2} - 1 - \delta + \frac{\beta_1^2}{4} - \frac{\beta_1 h_0 \delta^2}{4}
\]

\[
w_{2,t} - D w_{2,xx} + \beta_2 w_{2,x} - rw_{2} (1 - w_2)
\]

\[
\geq M_2 e^{-\delta t - \frac{\delta}{2} (s(t) - x)} V\left(D \left(\frac{\pi}{2}\right)^2 \frac{1}{h_0^2 (1 + \delta)^2} - r - \delta + \frac{\beta_2^2}{4D} - \frac{\beta_2 h_0 \delta^2}{4D}\right) \geq 0,
\]

in \(\{t, x\} : t > 0, -s(t) < x < s(t)\}. By choosing

\[
M_1 := \frac{\|u_0\|_{L^\infty([-h_0, h_0])} e^{\frac{\pi h_0}{2 + \delta} (2 + \frac{\delta}{2})}}{\cos\left(\frac{\pi}{2 + \delta}\right)}, \quad M_2 := \frac{\|v_0\|_{L^\infty([-h_0, h_0])} e^{\frac{\pi h_0}{2 + \delta} (2 + \frac{\delta}{2})}}{\cos\left(\frac{\pi}{2 + \delta}\right)},
\]
Lemma 6.1. Assume we call detailed depiction on the long time behavior of solutions. 

Moreover, \( w_1(t, -\sigma(t)) = 0, w_2(t, -\sigma(t)) = 0, w_1(t, \sigma(t)) = 0, w_2(t, \sigma(t)) = 0, \sigma(0) \geq h_0, -\sigma(0) \leq -h_0. \) By Lemma 3.4, we have \( -\sigma(t) \leq g(t) \) and \( \sigma(t) \geq h(t) \) for \( t \geq 0. \) Using (33) and letting \( t \to \infty \), we can get \( h_\infty - g_\infty \leq 2h_0(1 + \delta) \leq 2h_*. \) Then the result follows from Theorem 5.3(i).

6. Spreading speed. In this section, we provide upper bound and lower bound for spreading speeds of both \( h(t) \) and \( g(t) \) in the spreading case, and give a more detailed depiction on the long time behavior of solutions.

In order to give Theorem 6.4, we give three lemmas first.

Lemma 6.1. Assume \( 0 < \beta < 2\sqrt{aD} \). Let \( (u, g, h) \) be a solution of the equation

\[
\begin{aligned}
\begin{cases}
u_t = Du_{xx} + \beta u_x + u(a - bu), & t > 0, \ g(t) < x < h(t), \\
u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\
g'(t) = -\mu u_x(t, g(t)), & t > 0, \\
h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
u(0, x) = u_0(x), & -h_0 \leq x \leq h_0, \\
-g(0) = h(0) = h_0,
\end{cases}
\end{aligned}
\]

which spreading happens. Then the leftward and rightward asymptotic spreading speeds exist:

\[
-c^*_l := \lim_{t \to \infty} \frac{h(t)}{t}, \quad c^*_r := \lim_{t \to \infty} \frac{g(t)}{t}.
\]

Moreover, \( 0 < c^*_l < c^* < c^*_r \), where \( c^* \) is the spreading speed of the solution with the case of \( \beta = 0 \). We write \( c^*_l = c^*_l(a, b, D, \mu, \beta) \) and \( c^*_r = c^*_r(a, b, D, \mu, \beta) \) to emphasize the dependence on the parameters of the equation.

Proof. The proof is a slight modification of Theorem 1.1 of [15], whose right term \( f(u) = u(1 - u) \) is replaced by \( f(u) = u(a - bu) \). So we omit the process.

Lemma 6.2. We call \( q(z) \) a semi-wave with speed \( c \) if \((c, q(z))\) satisfies

\[
\begin{aligned}
\begin{cases}
Dq'' - (c - \beta)q' + q(a - bq) = 0, & z \in (0, \infty), \\
q(0) = 0, \ q(\infty) = \frac{a}{b}, q'(z) > 0, & z \in (0, \infty).
\end{cases}
\end{aligned}
\]

For each \( \mu > 0 \), problem (35) has exactly one solution \((c, q) = (c^*_r, q^*_r)\) such that

\[
\mu((q^*_r)'(0)) = c^*_r,
\]

where \( c^*_r \) is defined in (34). Moreover, \( c^*_r \in (0, 2\sqrt{aD} + \beta) \).
Proof. The proof is a slight modification of Proposition 2.2 of [15].

Similarly, we also have the following lemma, which is similar to Proposition 2.4 of [15].

**Lemma 6.3.** For each \( \mu > 0 \), the following problem
\[
\begin{cases}
Dq'' - (c + \beta)q' + q(a - bq) = 0, & z \in (0, \infty), \\
q(0) = 0, & q(\infty) = \frac{a}{b}, \quad q'(z) > 0, & z \in (0, \infty),
\end{cases}
\]
has exactly one solution \((c, q) = (c^*_1, q^*_1)\) such that
\[
\mu[(q^*_1)'(0)] = c^*_1,
\]
where \(c^*_1\) is defined in (34). Moreover, \(c^*_1 \in (0, 2\sqrt{aD} - \beta)\).

Now, we are ready to give Theorem 6.4.

**Theorem 6.4.** Let \((u, v, q, h)\) be a solution of the problem (TFB) with \(g_\infty = -\infty\) and \(h_\infty = \infty\). Then we have
\[
\max\{c^*_{r,u}, c^*_{r,v}\} \leq \liminf_{t \to \infty} \frac{h(t)}{t} \leq \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_{\min} := \max\{2 + \beta_1, 2\sqrt{rD} + \beta_2\},
\]
\[
- c_{\min} := \min\{\beta_1 - 2, \beta_2 - 2\sqrt{rD}\} \leq \liminf_{t \to \infty} \frac{g(t)}{t}
\]
\[
\leq \limsup_{t \to \infty} \frac{g(t)}{t} \leq \min\{-c^*_{r,u}, -c^*_{r,v}\},
\]
where \(c^*_{r,u} = c^*_r(1 - c, 1, 1, \mu, \beta_1), c^*_{r,v} = c^*_r(r(1 - b), r, D, \mu \rho, \beta_2), c^*_{1,u} = c^*_1(1 - c, 1, 1, \mu, \beta_1), c^*_{1,v} = c^*_1(r(1 - b), r, D, \mu \rho, \beta_2)\).

**Proof.** First, we prove \(\limsup_{t \to \infty} \frac{h(t)}{t} \leq c_{\min} := \max\{2 + \beta_1, 2\sqrt{rD} + \beta_2\}\). We shall construct a supersolution to prove Lemma 3.3. Let \(\sigma(t) := \sigma_0 + c_{\min} t\) for \(t > 0\), where \(\sigma_0 \gg 1\) is to be determined.

Let \((\bar{U}(\xi), \bar{V}(\xi))\), \(\xi := x - c_{\min} t\) with \(\bar{U}(0) = \bar{V}(0) = \frac{1}{2}\) be the solution of
\[
\begin{cases}
(c_{\min} - \beta_1)\bar{U}' + \bar{U}'' + \bar{U}(1 - \bar{U}) = 0 & \text{in } \mathbb{R}, \\
(c_{\min} - \beta_2)\bar{V}' + D\bar{V}'' + r\bar{V}(1 - \bar{V}) = 0 & \text{in } \mathbb{R}, \\
\bar{U}(-\infty) = 1, & \bar{V}(-\infty) = 1, \\
\bar{U}(\infty) = 0, & \bar{V}(\infty) = 0, \\
\bar{U}' < 0, & \bar{V}' < 0 & \text{in } \mathbb{R}.
\end{cases}
\]

Because \(c_{\min} - \beta_1 \geq 2\) and \(c_{\min} - \beta_2 \geq 2\sqrt{rD}\), such \((\bar{U}, \bar{V})\) exists ([23]). Now, we let
\[
\bar{w}_1(t, x) := k\bar{U}(x - c_{\min} t) - k\bar{U}(\sigma_0), \quad \bar{w}_2(t, x) := k\bar{V}(x - c_{\min} t) - k\bar{V}(\sigma_0),
\]
and choose \(k > 1\) such that \(k\bar{U}(\xi) > ||u_0||_{L^\infty([-h_0, h_0])}\) and \(k\bar{V}(\xi) > ||v_0||_{L^\infty([-h_0, h_0])}\) are satisfied for \(\xi \in [-h_0, h_0]\). Choose \(\sigma_0 > h_0\) which satisfies
\[
\bar{U}(\sigma_0) < \min_{x \in [-h_0, h_0]} \left[ \bar{U}(x) - \frac{u_0(x)}{k} \right], \quad \bar{V}(\sigma_0) < \min_{x \in [-h_0, h_0]} \left[ \bar{V}(x) - \frac{v_0(x)}{k} \right],
\]
\[
\bar{U}(\sigma_0) \leq 1 - \frac{1}{k}, \quad \bar{V}(\sigma_0) \leq 1 - \frac{1}{k},
\]
\[
-k\mu(1 + \rho)\min\{\bar{U}'(\sigma_0), \bar{V}'(\sigma_0)\} < c_{\min}.
\]
Using (38), we can get

\[ \bar{w}_{1,t} - \bar{w}_{1,xx} + \beta_1 \bar{w}_{1,x} - \bar{w}_1 (1 - \bar{w}_1) = k \left[ (k - 1) \left( \bar{U} - \frac{k \bar{U}(\sigma_0)}{k - 1} \right)^2 + \bar{U}(\sigma_0) - \frac{k}{k - 1} \bar{U}^2(\sigma_0) \right] \geq 0, \]

\[ \bar{w}_{2,t} - D \bar{w}_{2,xx} + \beta_2 \bar{w}_{2,x} - r \bar{w}_2 (1 - \bar{w}_2) = k \left[ (k - 1) \left( \bar{V} - \frac{k \bar{V}(\sigma_0)}{k - 1} \right)^2 + \bar{V}(\sigma_0) - \frac{k}{k - 1} \bar{V}^2(\sigma_0) \right] \geq 0. \]

Obviously, \( \bar{w}_1(t, \sigma(t)) = 0 \) and \( \bar{w}_2(t, \sigma(t)) = 0 \). Using the monotonicity of \( \bar{U} \) and \( \bar{V} \), we see

\[ \bar{w}_1(t, g(t)) = k \bar{U}(g(t) - c_{\text{min}} t) - k \bar{U}(\sigma_0) \geq k \bar{U}(0) - k \bar{U}(\sigma_0) > 0, \]

\[ \bar{w}_2(t, g(t)) = k \bar{V}(g(t) - c_{\text{min}} t) - k \bar{V}(\sigma_0) \geq k \bar{V}(0) - k \bar{V}(\sigma_0) > 0. \]

Note that (39) implies the last inequality of (11). Besides, \( \bar{w}_1(0, x) \geq u_0(x) \) and \( \bar{w}_2(0, x) \geq v_0(x) \) for \( x \in [-h_0, h_0] \) follow from (37). Recall \( \sigma(0) = \sigma_0 > h_0 \), so we can use Lemma 3.3 to get \( h(t) \leq \sigma(t) \) for \( t > 0 \). Therefore,

\[ \lim\sup_{t \to \infty} \frac{h(t)}{t} \leq \lim_{t \to \infty} \frac{\sigma(t)}{t} = c_{\text{min}}. \]

The proof of

\[ \lim\inf_{t \to \infty} \frac{g(t)}{t} \geq -c_{\text{max}} := \min \{ \beta_1 - 2, \beta_2 - 2\sqrt{rD} \} \]

is parallel, and what we need is also to construct a supersolution. In this case, we define \( \eta(t) := \eta_0 - c_{\text{max}} t \) for \( t > 0 \), where \( \eta_0 < -1 \) is to be determined. Let \( \bar{U}(\zeta) \) and \( \bar{V}(\zeta) \), \( \zeta := x + c_{\text{max}} t \) with \( \bar{U}(0) = \bar{V}(0) = \frac{1}{2} \) be the solution of

\[
\begin{cases}
(-c_{\text{max}} - \beta_1) U' + U'' + U(1 - U) = 0 \text{ in } \mathbb{R}, \\
(-c_{\text{max}} - \beta_2) V' + D V'' + r V(1 - V) = 0 \text{ in } \mathbb{R}, \\
U(-\infty) = 0, \quad V(-\infty) = 0, \quad U(\infty) = 1, \quad V(\infty) = 1, \\
U' > 0, \quad V' > 0 \text{ in } \mathbb{R}.
\end{cases}
\]

Let

\[ \bar{w}_1(t, x) := \bar{k} \bar{U}(x + c_{\text{max}} t) - \bar{k} \bar{U}(\eta_0), \quad \bar{w}_2(t, x) := \bar{k} \bar{V}(x + c_{\text{max}} t) - \bar{k} \bar{V}(\eta_0), \]

and we omit the subsequent process of proof.

Next, we prove that \( \lim\inf_{t \to \infty} \frac{h(t)}{t} \geq \max \{ c_{r,u}^*, c_{r,v}^* \} \). We first deal with \( \lim\inf_{t \to \infty} \frac{h(t)}{t} \geq c_{r,v}^* \). For any fixed \( \varepsilon \in (0, 1 - b) \), by Lemma 6.2, problem

\[
\begin{cases}
D q'' - (c - \beta_2) q' + q(r(1 - b - \varepsilon) - r q) = 0, \quad z \in (0, \infty), \\
q(0) = 0, \quad q(\infty) = 1 - b - \varepsilon, \quad q'(z) > 0, \quad z \in (0, \infty), \\
\mu q'(0) = c, 
\end{cases}
\]

admits a unique solution \((c, q) = (c_{r,v}^*(1 - b - \varepsilon), r, D, \mu, \beta_2, q) := \tilde{c}_{r,v}^*, \tilde{q}_{r,v}^*\).

Next, by Theorem 4.1 and Lemma 4.3, we need to choose \( T > \frac{h_0}{\tilde{c}_{r,v}^*} \) sufficiently large, which satisfies

\[ u(t, x) \leq 1 + \frac{\varepsilon}{b}, \quad t \geq T, \quad 0 \leq x \leq h(t), \]

\[ v(t, x) \geq \frac{1 - b}{1 - bc} - \varepsilon, \quad t \geq T, \quad 0 \leq x \leq h_0. \]
Follow the ideas of [38], define
\[ v_1(t, x) := \tilde{q}_{r,v}^*(\eta_v(t) - x), \quad \eta_v(t) := \tilde{c}_{r,v}^*(t - T) + \frac{h_0}{2}. \]
Using (40), \((v_1, \eta_v(t))\) satisfies
\[
\begin{cases}
v_1(t, x) = Dv_{1,x} - \beta_2 v_1 + rv_1(1 - b - \varepsilon - v), & t > T, \quad 0 < x < \eta_v(t), \\
v_1(t, \eta_v(t)) = 0, & t > T, \\
\eta_v'(t) = -\mu rv_1(t, \eta_v(t)), & t > T, \\
\eta_v(T) = \frac{h_0}{2}, & v_1(T, x) = \tilde{q}_{r,v}^*(\frac{h_0}{2} - x), \quad 0 \leq x \leq \frac{h_0}{2}.
\end{cases}
\]
Recall that \((u, v, g, h)\) is a solution of the problem (TFB), by (41) and \(u_x(t, h(t)) < 0\) for \(t > 0\), we have
\[
\begin{cases}
v_i \geq Dv_{xx} - \beta_2 v_x + r v(1 - b - \varepsilon - v), & t > T, \quad 0 < x < h(t), \\
h'(t) = -\mu u_x(t, h(t)) + \rho v_x(t, h(t)) \geq -\mu rv_x(t, h(t)), & t > T.
\end{cases}
\]
Besides, by (42) and \(\tilde{q}_{r,v}^*(z) > 0\) for all \(z > 0\), and \(\tilde{q}_{r,v}^*(\infty) = 1 - b - \varepsilon\), and noticing that \(\eta_v(T) = \frac{h_0}{2} < h_0 < h(T)\), we have
\[
\begin{align*}
v_1(T, x) &< 1 - b - \varepsilon \leq \frac{1 - b}{1 - bc} - \varepsilon \leq v(T, x), \quad x \in [0, \eta_v(T)], \\
v_1(t, 0) &< 1 - b - \varepsilon \leq \frac{1 - b}{1 - bc} - \varepsilon \leq v(t, 0), \quad t \geq T.
\end{align*}
\]
Using the results above and Proposition 1 and Remark 3 of [38], we see that \((v, h)\) forms a supersolution compared with \((v_1, \eta_v)\), and therefore, \(\eta_v(t) \leq h(t)\) for all \(t \geq T\). It follows that
\[
\tilde{c}_{r,v}^* = \lim_{t \to \infty} \frac{\eta_v(t)}{t} \leq \liminf_{t \to \infty} \frac{h(t)}{t}.
\]
Taking \(\varepsilon \to 0\), and using the continuous dependence on parameters of \(c_{r,v}^*(r(1 - b - \varepsilon), r, D, \mu, \beta_2)\), we can obtain that \(c_{r,v}^* = c_{r,v}^*(r(1 - b), r, D, \mu, \beta_2) \leq \liminf_{t \to \infty} \frac{h(t)}{t}\).

Similarly, for any fixed \(\varepsilon \in (0, 1 - c)\), consider the problem
\[
\begin{align*}
q'' - (\tilde{c} - \beta_1)q' + q(1 - c - \varepsilon - q) &= 0, \quad z \in (0, \infty), \\
q(0) &= 0, \quad q(\infty) = 1 - c - \varepsilon, \quad q'(z) > 0, \quad z \in (0, \infty), \\
\mu q'(0) &= \tilde{c},
\end{align*}
\]
whose solution is \((\tilde{c}, q) = (c_{r,v}^*(1 - c - \varepsilon), 1, 1, \mu, \beta_1), q) := (\tilde{c}_{r,v}^*, \tilde{q}_{r,v}^*).\) Define
\[
u_1(t, x) := \tilde{q}_{r,v}^*(\eta_u(t) - x), \quad \eta_u(t) := \tilde{c}_{r,v}^*(t - T) + \frac{h_0}{2},
\]
and apply the same proof as above to the equation of \(u\), we can finally derive
\[
\tilde{c}_{r,u}^* = c_{r,u}^*(1 - c, 1, \mu, \beta_1) \leq \liminf_{t \to \infty} \frac{h(t)}{t}.
\]
The proof of \(\limsup_{t \to \infty} \frac{g(t)}{t} \leq \min\{-c_{r,u}^*, -c_{r,v}^*\}\) is parallel, in which case we use Lemma 6.3 to complete our proof. \(\square\)

In order to obtain a better estimate of the spreading speed, we impose some more restrictions to overcome the difficulty coming from the Stefan condition. The idea comes from Theorem 1.2 of [38].
Theorem 6.5. Assume spreading of the two species happens, and \( D = r = 1, \beta_1 = \beta_2 = \beta \). If the initial data satisfies
\[
(1 - b)u_0 \geq (1 - c)v_0, \quad \text{for} \ x \in [-h_0, h_0],
\]
then
\[
\max \left\{ c_x^+ \left( 1 - b, 1, 1, \mu \left( 1 - \frac{c}{1 - b} + \rho \right), \beta \right), c_t^+ \left( 1, \frac{1 - bc}{1 - c}, 1, \mu, \beta \right) \right\} \leq \liminf_{t \to \infty} \frac{h(t)}{t}
\]
\[
\leq \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_x^+ \left( 1, 1, 1, \mu \left( 1 + \frac{b}{1 - c} \rho \right), \beta \right),
\]
\[
- c_t^+ \left( 1, 1, 1, \mu \left( 1 + \frac{b}{1 - c} \rho \right), \beta \right) \leq \liminf_{t \to \infty} \frac{g(t)}{t}
\]
\[
\leq \limsup_{t \to \infty} \frac{g(t)}{t} \leq \min \left\{ - c_t^+ \left( 1 - b, 1, 1, \mu \left( 1 - \frac{c}{1 - b} + \rho \right), \beta \right), - c_t^+ \left( 1, \frac{1 - bc}{1 - c}, 1, \mu, \beta \right) \right\},
\]
where \( c_x^+ \) and \( c_t^+ \) are defined in Lemma 6.1.

Proof. Set \( P(t,x) := (1 - b)u(t,x) - (1 - c)v(t,x) \), we have \( P_t = P_{xx} - \beta P_x + [1 - (a + v^2)]P \), for all \( t > 0 \), \( g(t) < x < h(t) \). Due to (43) and the fact of \( P(t,g(t)) = P(t,h(t)) = 0 \) for all \( t \geq 0 \), we can get \( P \geq 0 \) by the standard comparison principle, i.e.,
\[
v(t,x) \leq \frac{1 - b}{1 - c} u(t,x), \ t \geq 0, \ g(t) \leq x \leq h(t).
\]
Moreover, we have \( P_x(t,g(t)) \geq 0 \) and \( P_x(t,h(t)) \leq 0 \) for all \( t > 0 \), i.e.,
\[
(1 - b)u_x(t,g(t)) \geq (1 - c)v_x(t,g(t)), \ t > 0.
\]
\[
(1 - b)u_x(t,h(t)) \leq (1 - c)v_x(t,h(t)), \ t > 0.
\]
Therefore,
\[
g'(t) = \mu [ u_x(t,g(t)) + \rho v_x(t,g(t))] \geq - \mu \left( 1 + \frac{1 - b}{1 - c} \rho \right) u_x(t,g(t)), \ t > 0.
\]
\[
h'(t) = - \mu [ u_x(t,h(t)) + \rho v_x(t,h(t))] \leq - \mu \left( 1 + \frac{1 - b}{1 - c} \rho \right) u_x(t,h(t)), \ t > 0.
\]
To prove the upper bound of \( h(t) \), for \( \varepsilon > 0 \) small, consider the following problem
\[
\begin{align*}
U'' - (c - \beta) U' + U(1 + \varepsilon - U) &= 0, \quad z \in (0, \infty), \\
U(0) &= 0, \ U(\infty) = 1 + \varepsilon, \ U'(z) > 0, \quad z \in (0, \infty), \\
\bar{\mu} U'(0) &= c,
\end{align*}
\]
where \( \bar{\mu} := \mu \left( 1 + \frac{1 - b}{1 - c} \rho \right) \). By Lemma 6.2, there exists a unique \( c = c_x^+ (1 + \varepsilon, 1, 1, \bar{\mu}, \beta) := \tilde{c}_x^+ \) such that (49) has a unique positive solution \( U \). Now introduce
\[
R(t,x) := U(q(t) - x), \ q(t) := \tilde{c}_x^+ t + M,
\]
where \( M > 0 \) is to be determined. Direct calculation shows that \( R \) and \( q \) satisfy
\[
\begin{align*}
R_t &= R_{xx} - \beta R_x + R(1 + \varepsilon - R), \quad t > 0, \ 0 < x < q(t), \\
R(t,q(t)) &= 0, \ q'(t) = - \bar{\mu} R_x(t,q(t)), \ t > 0, \\
R(0,x) &= U(M - x), \ q(0) = M.
\end{align*}
\]
Since \( U(z) \uparrow U(\infty) = 1 + \varepsilon \) as \( z \to \infty \), and by Lemma 4.3, we can choose \( M > 0 \) and \( T > 0 \) large enough such that
\[
M > h(T), \ R(T,x) \geq 1 + \frac{\varepsilon}{2} > u(T,x), \ 0 \leq x \leq h(T).
\]
According to (48), (50), (51), (52), and the fact of $u_t \leq u_{xx} - \beta u_x + u(1 + \varepsilon - u)$ for $t > 0, 0 < x < h(t)$, we get that $(u, h)$ forms a subsolution of $(R, q)$ for $t > T$, $0 < x < h(t)$. Thus,

$$
\lim_{t \to \infty} \frac{h(t)}{t} \leq \lim_{t \to \infty} \frac{q(t)}{t} = c^*_r(1 + \varepsilon, 1, 1, \mu, \beta).
$$

By taking $\varepsilon \to 0$, we get the upper bound of $\frac{h(t)}{t}$.

The proof of the lower bound of $\frac{h(t)}{t}$ is parallel, so we omit the details.

Next we prove the lower bound of $\frac{h(t)}{t}$, which can be done by using the similar proof of Theorem 6.4, and by using (46) and (44) respectively. By (46), we get

$$
V'' - (c - \beta)V' + V(1 - b - \varepsilon - V) = 0, \quad z \in (0, \infty),
$$

$$
V(0) = 0, \quad V(\infty) = 1 - b - \varepsilon,
$$

$$
\mu \left( \frac{1 - c}{1 - b} + \rho \right) V'(0) = c.
$$

We can obtain

$$
\lim_{t \to \infty} \frac{h(t)}{t} \geq c^*_r \left( 1 - b, 1, 1, \mu \left( \frac{1 - c}{1 - b} + \rho \right), \beta \right).
$$

On the other hand, by (44) and $v_x(t, h(t)) < 0$ for $t > 0$, we have

$$
\begin{align*}
&u_t - u_{xx} + \beta u_x \geq u \left( 1 - \frac{1 - bc}{1 - c} \right) u, \\
h'(t) = -\mu[u_x(t, h(t)) + \rho v_x(t, h(t))] > -\mu u_x(t, h(t)),
\end{align*}
$$

$t > 0, \quad 0 < x < h(t)$.

It is not hard to obtain

$$
\lim_{t \to \infty} \frac{h(t)}{t} \geq c^*_r \left( 1, \frac{1 - bc}{1 - c}, 1, \mu, \beta \right).
$$

The proof of the upper bound of $\frac{h(t)}{t}$ is parallel, so we omit the details.

**Remark 3.** (i) If the condition (43) is replaced by $(1 - b)u_0(x) \leq (1 - c)v_0(x)$ for all $x \in [-h_0, h_0]$, then by exchanging the role of $u$ and $v$, we can get similar estimates.

(ii) In order to show Theorem 6.5 is a better estimate, we put $r = D = 1, \beta_1 = \beta_2 = \beta$ in Theorem 6.4. From the ideas of [10] and [15], we see that both of $c^*_r$ and $c^*_l$ are strictly increasing with respect to their first and fourth arguments, and are strictly decreasing with respect to their second arguments. But $c^*_r$ is strictly increasing with respect to its fifth argument, whereas $c^*_l$ is strictly decreasing with respect to it. Therefore, it is easy to obtain

$$
c^*_r(1 - b, 1, 1, \mu \rho, \beta) < c^*_r \left( 1 - b, 1, 1, \mu \left( \frac{1 - c}{1 - b} + \rho \right), \beta \right),
$$

$$
c^*_r \left( 1, 1, 1, \mu \left( \frac{1 - b}{1 - c} \right), \beta \right) < 2 + \beta,
$$

$$
-c^*_l \left( 1, 1, 1, \mu \left( \frac{1 - b}{1 - c} \right), \beta \right) > \beta - 2,
$$

$$
-c^*_l(1 - b, 1, 1, \mu \left( \frac{1 - c}{1 - b} + \rho \right), \beta) < -c^*_l(1 - b, 1, 1, \mu \rho, \beta).
$$
For \( -c^*_t(1, \frac{1-bc}{c}, 1, \mu, \beta) < -c^*_t(1-c, 1, 1, \mu, \beta) \), rescale the space variable, i.e., let 
y = \sqrt{\frac{x}{1-c}} \text{ in (36)}, we see that 
\( c^*_t(1, \frac{1-bc}{c}, 1, \mu, \beta) = \sqrt{\frac{x}{1-c}} c^*_t(1-c, 1, 1, \mu, \sqrt{1-c}) \)
> \( \frac{1}{\sqrt{1-c}} c^*_t(1-c, 1, 1, \mu, \beta) > c^*_t(1-c, 1, 1, \mu, \beta) \), but we can not get 
\( c^*_t(1, \frac{1-bc}{c}, 1, \mu, \beta) > c^*_t(1-c, 1, 1, \mu, \beta) \) by using the same technique, since the monotonicity on \( \beta \) of 
c^*_t and \( c^*_t \) are opposite. Therefore, we need to give more restrictions, i.e., let \( b \) be 
close to 1 sufficiently, then according to the continuous dependence of \( c^*_t \) on its 
parameters, we can get the desired result.

Inspired by Theorem 7 of [36], we have the following theorem, which is an improvement of Theorem 4.1. From this theorem, we can know that when spreading occurs, the interval range of spreading is related to the spreading speed.

**Theorem 6.6.** Let \( D, r, \beta_1, \beta_2, \mu, \rho, b, c \) be fixed and \( 0 < b, c < 1 \). If \( g(\infty) = -\infty, h(\infty) = \infty \), then for each \( 0 < c_0 < \min\{c_{r,u}, c_{r,v}, c_{t,u}, c_{t,v}\} \),
\[
\lim_{t \to \infty} \max_{-c_0 \leq c \leq c_0} \left| u(t, \cdot) - \frac{1-c}{1-bc} \right| = 0, \quad \lim_{t \to \infty} \max_{-c_0 \leq c \leq c_0} \left| v(t, \cdot) - \frac{1-b}{1-bc} \right| = 0,
\]
where \( c_{r,u}, c_{r,v}, c_{t,u}, c_{t,v} \) are defined in Theorem 6.4.

**Proof.** According to the choice of \( c_0 \), and the fact of \( \max\{c_{r,u}, c_{r,v}, c_{t,u}, c_{t,v}\} \leq \inf_{t \to \infty} \frac{h(t)}{1} \), \( \limsup_{t \to \infty} 2(t) \leq \min\{-c_{r,u}, -c_{r,v}\} \), there exist \( 0 < \sigma_0 < 1 \) and \( t_\sigma > 1 \) such that 
\( c_\sigma := c_0 + \sigma \min\{c_{r,u}, c_{r,v}, c_{t,u}, c_{t,v}\}, \forall 0 < \sigma < \sigma_0, h(t) > c_\sigma t, g(t) < -c_\sigma t, \forall t \geq t_\sigma \).

**Step 1.** For any given \( 0 < \varepsilon \ll 1 \), recall Lemma 4.3, there exists \( t_1 > 0 \) such that \( \varepsilon < 1 + \varepsilon \in [t_1, \infty) \times (-\infty, \infty) \). We may enlarge \( t_1 \) if necessary, and consider 
\( h(t_1) - g(t_1) > \frac{\pi}{\sqrt{a_\varepsilon - \beta_1^2/4}} \), where \( a_\varepsilon = 1 - c(1 + \varepsilon) \). Let \( (w_1, r_1, s_1) \) be the unique solution of the following problem:
\[
\begin{align*}
w_{1,t} - w_{1,xx} + \beta_1 w_{1,x} + w_1(a_\varepsilon - w_1), & \quad t > t_1, r_1(t) < x < s_1(t) , \\
w_1(t, r_1(t)) = 0, & \quad w_1(t, s_1(t)) = 0, \\
(-\mu) w_{1,x}(t, r_1(t)), & \quad s_1(t) = -\mu w_{1,x}(t, s_1(t)), \\
r_1(t) = g(t_1), & \quad s_1(t_1) = h(t_1), w_1(t_1, x) = u(t_1, x), r_1(t_1) \leq x \leq s_1(t_1),
\end{align*}
\]
where \( \mu \) is defined in (3). Then by comparison principle, we get \( u \geq w_1 \) in \( D_{1,\varepsilon} := \{(t, x) : t \geq t_1, t_1(t) \leq x \leq s_1(t) \} \), and \( g(t) \leq r_1(t), h(t) \geq s_1(t) \) for \( t \geq t_1 \). According to Theorem B of [22],
\[
\begin{align*}
\lim_{t \to \infty} (s_1(t) - c_{\varepsilon,t}) & = \rho_1 \in \mathbb{R}, \lim_{t \to \infty} \|w_1(t, \cdot) - \bar{q}_c(c_\varepsilon t + \rho_1 - \cdot)\|_{C^2([\rho_1, s_1(t)])} = 0, \quad (53) \\
\lim_{t \to \infty} (r_1(t) + \bar{c}_\varepsilon t) & = \rho_1 \in \mathbb{R}, \lim_{t \to \infty} \|w_1(t, \cdot) - \bar{q}_c(\cdot + \bar{c}_\varepsilon t - \rho_1)\|_{C^2([r_1(t), 0])} = 0, \quad (54)
\end{align*}
\]
where \( (c_{\varepsilon}, \bar{q}_c) \) and \( (\bar{c}_\varepsilon, \bar{q}_c) \) are the unique solutions of (35) and (36) respectively, with \( D = 1, \beta = \beta_1, a = a_\varepsilon, b = 1, \) and \( \mu \) is defined in (3). Notice that \( 0 < c_\sigma < c_{r,u} = c^*_t(1-c, 1, 1, \mu, \beta_1), c_\sigma = c^*_t(1-c(1+\varepsilon)), 1, 1, \mu, \beta_1) \), we have \( c_\varepsilon > c_\sigma \) due to \( 0 < \varepsilon \ll 1 \). Similarly, \( \bar{c}_\varepsilon > c_\sigma \) is also obtained. Thus, we have \( s_1(t) - c_\varepsilon t \to \infty, r_1(t) + c_\varepsilon t \to -\infty, \) and \( \min_{x \in [c_\varepsilon t + \rho_1 - \cdot]_{t \to \infty}} (c_\varepsilon t + \rho_1 - x) \to -\infty, \min_{x \in \bar{c}_\varepsilon t - \rho_1 \to \infty} (x + \bar{c}_\varepsilon t - \rho_1) \to \infty \) as \( t \to \infty \). So \( \lim_{t \to \infty} \min_{c_{r,u}, s, c_\varepsilon t, t} \{\min_{t \to \infty} w_1(t, x) = a_\varepsilon \) is obtained according to (53) and (54). Therefore, we get \( \liminf_{t \to \infty} \min_{c_{r,u}, s, c_\varepsilon t, t} \{u(t, x) \geq 1 - c, \) since \( \varepsilon \) is arbitrary. Then, there
exists $t_2 > t_1$ such that $h(t) > c_\sigma t$, $g(t) < -c_\sigma t$, $u(t, x) \geq 1 - c - \varepsilon := b_\varepsilon$, $\forall t \geq t_2$, $-c_\sigma t \leq x \leq c_\sigma t$.

**Step 2.** Our goal is to show that

$$\limsup_{t \to \infty} \max_{[-c_\sigma t, c_\sigma t]} v(t, \cdot) \leq 1 - b(1 - c) := v_2. \quad (55)$$

To do this, choose $0 < \delta \ll 1$, $t_3 > t_2$. Define $\varphi(t, x) = 1 - bb_\varepsilon + \varepsilon + bb_\varepsilon \cos(\frac{\pi}{2\sigma}) (t_3 - t)\delta$, and satisfy $\varphi(t, -c_\sigma t) = 1 + \varepsilon \geq v(t, c_\sigma t)$, $\varphi(t, c_\sigma t) = 1 + \varepsilon \geq v(t, c_\sigma t)$ for $t \geq t_3$, $\varphi(t_3, x) = 1 + \varepsilon \geq v(t_3, x)$ for $-c_\sigma t_3 \leq x \leq c_\sigma t_3$, and $v_1 - Dv_{xx} + \beta_2 v_x \leq rv(1 - bb_\varepsilon - v)$. Then, it suffices to show

$$\varphi_1 - D\varphi_{xx} + \beta_2 \varphi_x \geq r \varphi(1 - bb_\varepsilon - \varphi). \quad (56)$$

Due to $0 < \varepsilon < 1$, we can think of $1 - bb_\varepsilon + \varepsilon > \frac{c_\varepsilon}{2}$, and $(56)$ is guaranteed, provided $\delta(1 + \varepsilon + \frac{D\varepsilon^2}{2\sigma} + \frac{\beta_2 \varepsilon^2}{2\sigma} + \frac{\beta_2 D\varepsilon^2}{4\sigma^2}) \leq \frac{c_\varepsilon}{2}$. Then the comparison principle gives $v(t, x) \leq \varphi(t, x)$ for all $t \geq t_3$, $-c_\sigma t \leq x \leq c_\sigma t$. Therefore, $\limsup_{t \to \infty} \max_{[-c_\sigma t, c_\sigma t]} v(t, \cdot) \leq \frac{v_2}{2} + \frac{c_\varepsilon}{2}$. Then there exists $t_4 > t_3$ such that $h(t) > c_\varepsilon t$, $g(t) < -c_\varepsilon t$, $v(t, x) \leq \frac{v_2}{2} + \varepsilon =: v_2^\varepsilon < 1$, $\forall t \geq t_4$, $-c_\varepsilon t \leq x \leq c_\varepsilon t$.

**Step 3.** Due to $v_2^\varepsilon < 1$, we have $c_\varepsilon^\varepsilon(1 - c_\varepsilon^\varepsilon, 1, 1, \mu, \beta_1) > c_\varepsilon^\varepsilon(1 - c\varepsilon^\varepsilon, 1, 1, \mu, \beta_1) = c_{\varepsilon, u} > c_\varepsilon^\varepsilon$. Take $0 < \mu^* < \mu$ such that $c_\varepsilon^\varepsilon(1 - c_\varepsilon^\varepsilon, 1, \mu, \beta_1) = c_\varepsilon^\varepsilon$, we get a function $q_\varepsilon^\varepsilon$, where $(c_\varepsilon^\varepsilon, q_\varepsilon^\varepsilon)$ is the unique solution of $(36)$ with $D = 1$, $\beta = \beta_1$, $a = 1 - c_\varepsilon^\varepsilon$, $b = 1$, and $\mu = \mu^*$. Let $(w_2, r_2, s_2)$ be the unique solution of

\[
\begin{cases}
  w_{2, t} - w_{2, xx} + \beta_1 w_{2, x} = w_2(1 - c_\varepsilon^\varepsilon - w_2), & t \geq t_4, \quad r_2(t) < t < s_2(t), \\
  w_2(t, r_2(t)) = 0, & t \geq t_4, \\
  v'_2(t) = -\mu^* w_2, & t \geq t_4, \\
  s_2(t) = -\mu^* w_2, & t \geq t_4, \\
  r_2(t) = g(t), & s_2(t) = h(t), \quad w_2(t, x) = u(t, x), \quad r_2(t) \leq x \leq s_2(t).
\end{cases}
\]

Thus $u \geq w_2$ in $D_{t_4, s_2} := \{(t, x) : t \geq t_4, r_2(t) \leq x \leq s_2(t)\}$, and $g(t) \leq r_2(t)$, $h(t) \geq s_2(t)$ for $t \geq t_4$. By using Theorem B of [22] again, we get

\[
\begin{align*}
  \lim_{t \to \infty} (r_2(t) + c_\varepsilon^\varepsilon t) = \rho_2 & \in \mathbb{R}, \quad \lim_{t \to \infty} \|w_2(t, \cdot) - q_\varepsilon^\varepsilon(\cdot + c_\varepsilon^\varepsilon t - \rho_2)\|_{C^2([r_2(t), 0])} = 0, \quad (57) \\
  \lim_{t \to \infty} (s_2(t) - c_\varepsilon^\varepsilon t) = \rho_2 & \in \mathbb{R}, \quad \lim_{t \to \infty} \|w_2(t, \cdot) - q_\varepsilon^\varepsilon(\cdot + \rho_2 - \cdot)\|_{C^2([0, s_2(t)])} = 0, \quad (58)
\end{align*}
\]

where $(c_\varepsilon^\varepsilon, q_\varepsilon^\varepsilon)$ is the unique solution of $(35)$ with $D = 1$, $\beta = \beta_1$, $a = 1 - c_\varepsilon^\varepsilon$, $b = 1$ and $\mu = \mu^*$. By Lemma 6.1, $c_\varepsilon^\varepsilon = c_{\varepsilon}^\varepsilon(1 - c_\varepsilon^\varepsilon, 1, 1, \mu^*, \beta_1) < c_{\varepsilon}^\varepsilon(1 - c_\varepsilon^\varepsilon, 1, 1, \mu^*, \beta_1) = c_\varepsilon^\varepsilon$. As $c_\varepsilon^\varepsilon = c_\varepsilon^\varepsilon + \frac{c_\varepsilon}{4} > \frac{c_\varepsilon^\varepsilon + c_\varepsilon}{2} > \frac{c_\varepsilon^\varepsilon + c_\varepsilon}{2} = c_\varepsilon^\varepsilon$, then we get $s_2(t) - \frac{c_\varepsilon^\varepsilon}{2} \to \infty$, $r_2(t) + c_\varepsilon^\varepsilon t \to -\infty$, and $\min_{[-c_\varepsilon^\varepsilon t, 0]} (\tilde{c}_\varepsilon^\varepsilon t + \tilde{\rho}_2 - x) \to \infty$, $\min_{[-c_\varepsilon^\varepsilon t, 0]} (x + c_\varepsilon^\varepsilon - \rho_2) \to \infty$ as $t \to \infty$. Then $\lim_{t \to \infty} \min_{[-c_\varepsilon^\varepsilon t, 0]} w_2(t, x) = 1 - c_\varepsilon^\varepsilon$ is obtained, according to $(57)$ and $(58)$. Then we obtain $\liminf_{t \to \infty} \min_{[-c_\varepsilon^\varepsilon t, 0]} u(t, x) \geq \min_{t \to \infty} \min_{[-c_\varepsilon^\varepsilon t, 0]} u(t, x) \geq 1 - c_\varepsilon^\varepsilon := u_2$, since $\varepsilon$ is arbitrary.
Step 4. Define \((u_1, \bar{v}_1) := (1 - c, 1), (u_{n+1}, \bar{v}_{n+1}) := (1 - c\bar{v}_{n+1}, 1 - b\bar{v}_n), n \geq 1,\)
then the result of \(\lim_{n \to \infty} u_n = \frac{1-c}{1-bc}\), \(\lim_{n \to \infty} \bar{v}_n = \frac{1-b}{1-bc}\) are clear. Repeating the above
process, we can prove
\[
\liminf_{t \to \infty} \min_{[-c_0 t, c_0 t]} u(t, x) \geq u_n, \quad \limsup_{t \to \infty} \max_{[-c_0 t, c_0 t]} v(t, x) \leq \bar{v}_n, \quad \forall n \geq 1.
\]
Consequently,
\[
\liminf_{t \to \infty} \min_{[-c_0 t, c_0 t]} u(t, x) \geq \frac{1-c}{1-bc}, \quad \limsup_{t \to \infty} \max_{[-c_0 t, c_0 t]} v(t, x) \leq \frac{1-b}{1-bc}.
\]
Similarly,
\[
\limsup_{t \to \infty} \max_{[-c_0 t, c_0 t]} u(t, x) \leq \frac{1-c}{1-bc}, \quad \liminf_{t \to \infty} \min_{[-c_0 t, c_0 t]} v(t, x) \geq \frac{1-b}{1-bc}.
\]
Thus we complete the proof.

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Appendix A. Proof of Lemma 2.2.

Proof of Lemma 2.2. We make a transformation to straighten the double free boundaries, and then by using the \(L^p\) theory, the Sobolev embedding theorem (see [24]), and the contraction mapping theorem to get the conclusion. The proof is quite standard by modifying the ideas of [10] and [18], so we omit the details.

Appendix B. Proof of Lemma 2.3.

Proof of Lemma 2.3. By the strong maximum principle, \(u > 0\) and \(v > 0\) for \(t \in [0, T], x \in (g(t), h(t))\). Thus from (3), we see \(u_x(t, g(t)) > 0, v_x(t, g(t)) > 0, u_x(t, h(t)) < 0\) and \(v_x(t, h(t)) < 0\) for \(t \in [0, T]\). Then we get \(g'(t) < 0, h'(t) > 0\) for \(t \in [0, T]\). Let \(\bar{u} = \bar{u}(t)\) be the solution of \(u' = u(1-u)\) with \(\bar{u}(0) = \|u_0\|_{L^\infty([-h_0, h_0])}\).
Then the comparison principle yields that \(u(t, x) \leq \bar{u}(t) \leq \max\{1, \|u_0\|_{L^\infty([-h_0, h_0])}\}\)
for \(t \in [0, T], x \in (g(t), h(t))\). Similarly, \(v(t, x) \leq \bar{v}(t) \leq \max\{1, \|v_0\|_{L^\infty([-h_0, h_0])}\}\)
for \(t \in [0, T], x \in (g(t), h(t))\) can be obtained. Thus we have proved (6) and (7).
To prove (9), we shall construct auxiliary functions (cf. [10]) to compare \(u\) and \(v\). Define
\[
w(t, x) = R_1[2M_1(h(t) - x) - M_1^2(h(t) - x)^2],
\]
where
\[
R_1 := \max\{1, \|u_0\|_{L^\infty([-h_0, h_0])}\}.
\]
We will choose \(M_1\) so that \(w(t, x) \geq u(t, x)\) over \(\Omega_{M_1}\), where
\[
\Omega_{M_1} := \{(t, x) : 0 < t < T, h(t) - \frac{1}{M_1} < x < h(t)\}.
\]
Direct calculations indicate that, for \((t, x) \in \Omega_{M_1},\)
\[
w_t - w_{xx} + \beta_1 w_x
\]
\[\geq 0 + 2R_1 M_1^2 + \beta_1[2R_1 M_1^2(h(t) - x) - 2R_1 M_1]\]
\[\geq R_1 \geq u(1-u-cv) \quad \text{in} \ \Omega_{M_1},\]
provided
\[ M_1 \geq \sqrt{\frac{\beta_1^2}{4} + \frac{1}{2} + \frac{\beta_1}{2}}. \]

It is obvious that
\[ w(t, h(t) - \frac{1}{M_1}) = R_1 \geq u(t, h(t) - \frac{1}{M_1}), \quad w(t, h(t)) = u(t, h(t)) = 0. \]

Divide \([h_0 - \frac{1}{M_1}, h_0]\) into \([h_0 - \frac{1}{M_1}, h_0 - \frac{1}{2M_1}] \cup [h_0 - \frac{1}{2M_1}, h_0]\) to compare \(w(0, x)\) with \(u_0(x)\). For \(x \in [h_0 - \frac{1}{M_1}, h_0 - \frac{1}{2M_1}]\), \(M_1 \geq \frac{4\|u_0\|_{C^1([-h_0, h_0])}}{3R_1}\), we have
\[ w(0, x) \geq \frac{3}{4}R_1, \quad u_0(x) \leq \|u_0\|_{C^1([-h_0, h_0])} \frac{1}{M_1} \leq \frac{3}{4}R_1. \]

For \(x \in [h_0 - \frac{1}{2M_1}, h_0]\), we calculate
\[ w_x(0, x) = 2R_1M_1^2(h_0 - x) - 2R_1M_1 = -2R_1M_1[1 - M_1(h_0 - x)] \leq -R_1M_1. \]

In order to make sure \(h_0 - \frac{1}{M_1} \geq -h_0\), we need \(M_1 \geq \frac{1}{2h_0}\). Therefore, by choosing
\[ M_1 := \max\left\{ \sqrt{\frac{\beta_1^2}{4} + \frac{1}{2} + \frac{\beta_1}{2}}, \frac{4\|u_0\|_{C^1([-h_0, h_0])}}{3R_1}, \frac{1}{2h_0} \right\}, \]
we will have
\[ w_x(0, x) \leq u'_0(x). \]

As \(w(0, h_0) = u_0(h_0) = 0\), the above inequality implies
\[ w(0, x) \geq u_0(x). \]

We apply the maximum principle to get \(u(t, x) \leq w(t, x)\) in \(\Omega_{M_1}\). Then, it would follow that
\[ u_x(t, h(t)) \geq w_x(t, h(t)) = -2R_1M_1. \]

Similarly, we can prove that
\[ u_x(t, h(t)) \geq -2R_2M_2, \]
where
\[ R_2 := \max\{1, \|v_0\|_{L^\infty([-h_0, h_0])}\}, \quad M_2 := \max\left\{ \sqrt{\frac{\beta_1^2}{4D} + \frac{r}{2D} + \frac{\beta_2}{2D}}, \frac{4\|v_0\|_{C^1([-h_0, h_0])}}{3R_2}, \frac{1}{2h_0} \right\}. \]

And hence \(h'(t) \leq \mu_\Lambda\), where \(\Lambda := 2R_1M_1 + 2\rho R_2M_2\). The proof for (8) is parallel, so we omit the process.

**Appendix C. Proof of Theorem 2.1.**

**Proof of Theorem 2.1.** By Lemma 2.2, we write \(T_{\text{max}} > 0\) as the maximal existence time of the solution. On the contrary, we assume \(T_{\text{max}} < \infty\). By Lemma 2.3, there exists a positive constant \(K\) which does not depend on \(T_{\text{max}}\) such that
\[ 0 \leq u(t, x), \quad v(t, x), \quad -g'(t), \quad h'(t) \leq K \text{ for all } t \in [0, T_{\text{max}}), \quad x \in [g(t), h(t)]. \]

In particular,
\[ -h_0 - Kt \leq g(t) \leq -h_0, \quad h_0 \leq h(t) \leq h_0 + Kt \text{ for all } t \in [0, T_{\text{max}}). \]

Fix \(\varepsilon \in (0, T_{\text{max}})\) and \(\Lambda > T_{\text{max}}\), by the standard regularity theory, there exists \(K' > 0\) depending only on \(\varepsilon, \Lambda\) and \(K\) such that
\[ \|u(t, \cdot)\|_{C^2([g(t), h(t)])}, \quad \|v(t, \cdot)\|_{C^2([g(t), h(t)])} \leq K', \quad \forall t \in [\varepsilon, T_{\text{max}}]. \]
According to the proof of Lemma 2.2, there exists a constant \( \tau > 0 \) which depends only on \( K \) and \( K' \) such that the solution of the problem (TFB) with any initial time \( t \in [\varepsilon, T_{\text{max}}] \) can be uniquely extended to the interval \( [t, t + \tau) \). Therefore, the solution with the initial time \( T_{\text{max}} - \frac{\tau}{2} \) can be uniquely extended to the time \( T_{\text{max}} + \frac{\tau}{2} \), which contradicts the definition of \( T_{\text{max}} \). Thus, \( T_{\text{max}} = \infty \). This completes the proof of Theorem 2.1.

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E-mail address: duanbo2011@stu.xjtu.edu.cn
E-mail address: zhangzc@mail.xjtu.edu.cn