A K-CONTACT SIMPLY CONNECTED 5-MANIFOLD WITHOUT SASAKIAN STRUCTURE

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Abstract. We construct the first example of a 5-dimensional simply connected compact manifold that admits a K-contact structure but does not admit a semi-regular Sasakian structure. For this, we need two ingredients: (a) to construct a suitable simply connected symplectic 4-manifold with disjoint symplectic surfaces spanning the homology, all of them but one of genus 1 and the other of genus \( g > 1 \), (b) to prove a bound on the second Betti number \( b_2 \) of an algebraic surface with \( b_1 = 0 \) and having disjoint complex curves spanning the homology when all of them but one are of genus 1 and the other of genus \( g > 1 \).

1. Introduction

In geometry, it is a central question to determine when a given manifold admits an specific geometric structure. Complex geometry provides with numerous examples of compact manifolds with rich topology, and there is a number of topological properties that have to be satisfied by complex manifolds. For instance, compact Kähler manifolds satisfy strong topological properties like the hard Lefschetz property, the formality of its rational homotopy type [8], or restrictions on the fundamental group [1]. A natural approach is to weaken the given structure and to ask to what extent a manifold having the weaker structure may admit the stronger one. In the case of Kähler manifolds, if we forget about the integrability of the complex structure, then we are dealing with symplectic manifolds. There has been enormous interest in the construction of (compact) symplectic manifolds that do not admit Kähler structures, and in determining its topological properties [16]. In dimension 4, when we deal with complex surfaces, we have the Enriques-Kodaira classification [3] that helps in the understanding of this question.

In odd dimension, Sasakian and K-contact manifolds are analogues of Kähler and symplectic manifolds, respectively. Sasakian geometry has become an important and active subject, especially after the appearance of the fundamental treatise of Boyer and Galicki [5]. Chapter 7 of this book contains an extended discussion of topological problems of Sasakian and K-contact manifolds.

The precise definition of the structures that we are dealing with in this paper is as follows. Let \((M, \eta)\) be a co-oriented contact manifold with a contact form \( \eta \in \Omega^1(M) \), that is \( \eta \wedge (d\eta)^n > 0 \) everywhere, with \( \dim M = 2n + 1 \). We say that \((M, \eta)\) is K-contact if there is an endomorphism \( J \) of \( TM \) such that:

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\[ J^2 = -\text{Id} + \xi \otimes \eta, \]
where \( \xi \) is the Reeb vector field of \( \eta \), \( i_\xi \eta = 1, \]
\( i_\xi (d\eta) = 0 \),

\[ d\eta(JX, JY) = d\eta(X, Y), \]
for all vector fields \( X, Y \), and \( d\eta(JX, X) > 0 \) for all nonzero \( X \in \ker \eta \),

the Reeb field \( \xi \) is Killing with respect to the Riemannian metric \( g(X, Y) = d\eta(JX, Y) + \eta(X)\eta(Y) \).

In other words, the endomorphism \( J \) defines a complex structure on \( D = \ker \eta \) compatible with \( d\eta \), hence \( J \) is orthogonal with respect to the metric \( g|_D \). By definition, the Reeb vector field \( \xi \) is orthogonal to \( D \).

Just as in the case of an almost Hermitian structure, there is the notion of integrability of an almost contact metric structure. More precisely, an almost contact metric structure \((\eta, \xi, J, g)\) is called normal if the Nijenhuis tensor \( N_J \) associated to the tensor field \( J \), defined by

\[ N_J(X, Y) := J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \]

satisfies the equation \( N_J = -d\eta \otimes \xi \). A Sasakian structure is a normal contact metric structure. If \((\eta, \xi, J, g)\) is a Sasakian structure on \( M \), then \((M, \eta, \xi, J, g)\) is called a Sasakian manifold.

Let \((M, \eta, g, J)\) be a K-contact manifold. Consider the contact cone as the Riemannian manifold \( C(M) = (M \times \mathbb{R}_+, t^2 g + dt^2) \). One defines the almost complex structure \( I \) on \( C(M) \) by:

\[ I(X) = J(X) \text{ on } \ker \eta, \]
\[ I(\xi) = t \frac{\partial}{\partial t}, \]
\[ I\left(t \frac{\partial}{\partial t}\right) = -\xi, \]
for the Killing vector field \( \xi \) of \( \eta \).

Then \( M \) is Sasakian if \( I \) is integrable, that is, \( C(M) \) is a Kähler manifold. Accordingly, \( M \) is K-contact if \( C(M) \) is almost-Kähler, that is, symplectic with a compatible almost complex structure.

There is much interest on constructing K-contact manifolds which do not admit Sasakian structures. The odd Betti numbers up to degree \( n \) of Sasakian \((2n + 1)\)-manifolds must be even. The parity of \( b_1 \) was used to produce the first examples of K-contact manifolds with no Sasakian structure [5, example 7.4.16]. More refined tools are needed in the case of even Betti numbers. The cohomology algebra of a Sasakian manifold satisfies a hard Lefschetz property [6]. Using it examples of K-contact non-Sasakian manifolds are produced in [7] in dimensions 5 and 7. These examples are nilmanifolds with even Betti numbers, so in particular they are not simply connected.

When one moves to simply connected manifolds, K-contact non-Sasakian examples of any dimension \( \geq 9 \) were constructed in [9] using the evenness of \( b_3 \) of a compact Sasakian manifold. Alternatively, using the hard Lefschetz property for Sasakian manifolds there are examples [13] of simply connected K-contact non-Sasakian manifolds of any dimension \( \geq 9 \). In [4, 18] the rational homotopy type of Sasakian manifolds is studied. All higher order Massey products for simply connected Sasakian manifolds vanish, although there are Sasakian manifolds with non-vanishing triple Massey products [4]. This yields examples of simply connected
K-contact non-Sasakian manifolds in dimensions $\geq 17$. However, Massey products are not suitable for the analysis of lower dimensional manifolds.

The problem of the existence of simply connected K-contact non-Sasakian compact manifolds (open problem 7.4.1 in [5]) is still open in dimension 5. It was solved for dimensions $\geq 9$ in [6, 7, 9] and for dimension 7 in [14] by a combination of various techniques based on the homotopy theory and symplectic geometry. In the least possible dimension the problem appears to be much more difficult. A simply connected compact oriented 5-manifold is called a Smale-Barden manifold. These manifolds are classified topologically by $H_2(M, \mathbb{Z})$ and the second Stiefel-Whitney class. Chapter 10 of the book by Boyer and Galicki is devoted to a description of some Smale-Barden manifolds which carry Sasakian structures. The following problem is still open [5, open problem 10.2.1].

Do there exist Smale-Barden manifolds which carry K-contact but do not carry Sasakian structures?

In the pioneering work [15], it is done a first step towards a positive answer to the question. A homology Smale-Barden manifold is a compact 5-dimensional manifold with $H_1(M, \mathbb{Z}) = 0$. A Sasakian structure is regular if the leaves of the Reeb flow are a foliation by circles with the structure of a circle bundle over a smooth manifold. The Sasakian structure is quasi-regular if the foliation is a Seifert circle bundle over a (cyclic) orbifold. It is semi-regular if this foliation has only locus of non-trivial isotropy of codimension 2, that is, if the base orbifold is a topological manifold. It is a remarkable result, although not difficult, that any manifold admitting a Sasakian structure has also a quasi-regular Sasakian structure. Therefore, a Sasakian manifold is a Seifert bundle over a cyclic Kähler orbifold [15].

Correspondingly, for K-contact manifolds we also define regular, quasi-regular and semi-regular K-contact structures with the same conditions. Any K-contact manifold admits a quasi-regular K-contact structure [14], and hence a K-contact manifold is a Seifert bundle over a cyclic symplectic orbifold. Such orbifold has isotropy locus which are a (stratified) collection of symplectic sub-orbifolds. The K-contact structure is semi-regular if the symplectic orbifold has isotropy locus of codimension 2. The main result of [15] is:

**Theorem 1** ([15]). There exists a homology Smale-Barden manifold which admits a semi-regular K-contact structure but which does not carry any semi-regular Sasakian structure.

The construction of [15] relies upon subtle obstructions to admit Sasakian structures in dimension 5 found by Kollár [11]. If a 5-dimensional manifold $M$ has a Sasakian structure, then it is a Seifert bundle structure over a Kähler orbifold $X$ with isotropy locus a collection of complex curves $D_i$ with isotropy (multiplicity) $m_i$. We have the following topological characterization of the homology of $M$ in terms of that of $X$. 
Theorem 2 ([15, Theorem 16]). Suppose that \( \pi : M \to X \) is a semi-regular Seifert bundle with isotropy surfaces \( D_i \) with multiplicities \( m_i \). Then \( H_1(M, \mathbb{Z}) = 0 \) if and only if

1. \( H_1(X, \mathbb{Z}) = 0 \),
2. \( H^2(X, \mathbb{Z}) \to \sum H^2(D_i, \mathbb{Z}/m_i) \) is surjective,
3. The Chern class of the circle bundle \( c_1(M/e^{2\pi i/\mu}) \in H^2(X, \mathbb{Z}) \) is primitive,

where \( \mu \) is the lcm of all \( m_i \).

Moreover, \( H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/m_i)^{2g_i} \), \( g_i = \text{genus of } D_i \), \( k + 1 = b_2(X) \).

Corollary 3 ([15, Corollary 18]). Suppose that \( M \) is a 5-manifold with \( H_1(M, \mathbb{Z}) = 0 \) and \( H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p_i)^{2g_i} \), \( k \geq 0 \), \( p_i \) a prime, and \( g_i \geq 1 \). If \( M \to X \) is a semi-regular Seifert bundle, then \( H_1(X, \mathbb{Z}) = 0 \), \( H_2(X, \mathbb{Z}) = \mathbb{Z}^{k+1} \), and the ramification locus has \( k + 1 \) disjoint surfaces \( D_i \) linearly independent in rational homology, and of genus \( g(D_i) = g_i \).

In [15, Theorem 23], the authors construct a symplectic 4-dimensional orbifold with disjoint symplectic surfaces spanning the second homology. This is the first example of such phenomenon and has \( b_2 = 36 \). The genus of the isotropy surfaces satisfy \( 1 \leq g_i \leq 3 \), with several of them with genus 3. Using this symplectic orbifold \( X \), we obtain a semi-regular K-contact 5-manifold \( M \) with

\[
H_1(M, \mathbb{Z}) = 0, \quad H_2(M, \mathbb{Z}) = \mathbb{Z}^{35} \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p_i)^{2g_i}. \tag{1}
\]

For understanding the Sasakian side, in [15] it is proved the following result

Theorem 4 ([15, Theorem 32]). Let \( S \) be a smooth Kähler surface with \( H_1(S, \mathbb{Q}) = 0 \) and containing \( D_1, \ldots, D_b \), \( b = b_2(S) \), smooth disjoint complex curves with \( g(D_i) = g_i > 0 \), and spanning \( H_2(S, \mathbb{Q}) \). Assume that:

- at least two \( g_i \) are \( > 1 \),
- \( 1 \leq g_i \leq 3 \).

Then \( b \leq 2 \max\{g_i\} + 3 \).

As a corollary [15, Proposition 31], there is no Sasakian semi-regular 5-dimensional manifold with homology given by (1). So \( M \) is K-contact, but does not admit a semi-regular Sasakian structure, proving Theorem 1.

Theorem 4 is an instance of the following conjecture:

Conjecture 5. There does not exist a Kähler manifold or a Kähler orbifold \( X \) with \( b_1 = 0 \) and with \( b_2 \geq 2 \) having disjoint complex curves spanning \( H_2(X, \mathbb{Q}) \), all of genus \( g \geq 1 \).

The present work enhances the previous work to achieve a 5-manifold that it is furthermore simply connected. Our main result is the following:
Theorem 6. There exists a (simply connected) Smale-Barden manifold which admits a semi-regular K-contact structure but which does not carry any semi-regular Sasakian structure.

On the one hand, we provide a new construction of a symplectic 4-manifold $X$ with $b_1 = 0$ and $b_2 = b > 1$, having a collection of disjoint symplectic surfaces $C_1, \ldots, C_b$ spanning $H_2(X, \mathbb{Q})$, and all with genus $g_i \geq 1$. This is based on the following phenomenon which can be performed in the symplectic setting but not in the algebro-geometric situation.

Start with the complex projective plane $\mathbb{C}P^2$ and two generic complex cubic curves $C_1, C_2$. They intersect in nine points $P_1, \ldots, P_9$. A third complex cubic curve passing through $P_1, \ldots, P_8$ has to go necessarily through $P_9$. This is a purely algebraic phenomenon. However, it is possible to construct a third symplectic cubic $C_3$ going through $P_1, \ldots, P_8$, but intersecting $C_1$ at another point $P_{10}$, and $C_2$ at a different point $P_{11}$. Note that each $C_i$ misses exactly one of the eleven points $P_1, \ldots, P_{11}$. Looking at this more symmetrically, we aim to have a collection of 11 points $\Delta = \{P_1, \ldots, P_{11}\}$ and 11 cubic complex curves $C_1, \ldots, C_{11}$ such that $C_i$ passes through the points of $\Delta - \{P_i\}$, $i = 1, \ldots, 11$. In this way, the intersections are $C_i \cap C_j = \Delta - \{P_i, P_j\}$ and no more points. Blowing up at all points of $\Delta$, we get the (symplectic) 4-manifold $X = \mathbb{C}P^2 \# 11 \mathbb{C}P^2$, with 11 complex curves of genus 1 and disjoint. An extra (complex) curve can be obtained by taking a singular complex curve $G$ of degree 10 with ordinary triple points at the points of $\Delta$. As $G \cdot C_i = 30$ equals the geometric intersection, that is, 3 times for each of the 10 triple points in $G \cap C_i = \Delta - \{P_i\}$, we would not have more intersections. This curve is of genus $g_G = 3$, and it becomes a smooth genus 3 curve in the blow-up, that is disjoint from the others. This heuristic argument has to be carried out in a slightly different guise, by making a symplectic construction in a tubular neighbourhood of a cubic curve and a complex line and gluing it in symplectically (see section 2).

Theorem 7. Let $P_1, \ldots, P_{11}$ be 11 points in $\mathbb{C}P^2$. Then there exist symplectic surfaces

$$C_1, C_2, \ldots, C_{11}, G \subset \mathbb{C}P^2$$

such that:

- $C_i$ is a genus 1 smooth surface and $P_j \in C_i$ for $j \neq i$, $P_i \notin C_i$.
- The surfaces $C_i, C_j$, $i \neq j$ intersect exactly at $\{P_1, \ldots, P_{11}\} - \{P_i, P_j\}$, positively and transversely.
- $G$ is a genus 3 singular symplectic surface whose only singularities are 11 triple points at $P_i$ (with different branches intersecting positively). Moreover $G$ intersects each $C_i$ only at the points $P_j$, $j \neq i$, and all the intersections of $C_i$ with the branches of $G$ are positive and transverse.

Using this, we construct our K-contact 5-manifold. First we blow up $\mathbb{C}P^2$ at the 11 points $P_1, \ldots, P_{11}$, to obtain a symplectic manifold, which topologically is $X = \mathbb{C}P^2 \# 11 \mathbb{C}P^2$. The proper transforms of $C_1, \ldots, C_{11}, G$ are symplectic surfaces
in $X$, via the method in [15, section 5.2]. The proper transform of $G$ becomes a smooth genus 3 symplectic surface. Therefore $b_2(X) = 12$ and it has 12 disjoint symplectic surfaces, 11 of them of genus $g_i = 1$ and one of genus $g_{12} = 3$. Take numbers $m_i$. Using [15, Proposition 7], we make $X$ into an orbifold $X'$ whose isotropy locus is $C_i$ with multiplicity $m_i$ and $G$ with multiplicity $m_{12}$. Then we can take a Seifert bundle $M \to X'$ with primitive Chern class $c_1(M/e^{2\pi i/\mu}) = [\omega]$ as in [15, Lemma 20]. The manifold $M$ is K-contact and has

$$H_1(M, \mathbb{Z}) = 0, \quad H_2(M, \mathbb{Z}) = \mathbb{Z}^{11} \oplus \bigoplus_{i=1}^{12} (\mathbb{Z}/m_i)^{2g_i}. \quad (2)$$

We choose a prime $p$ and $m_i = p^i$, so that all $m_i$ are distinct and pairwise non-coprime.

Given a Seifert bundle $M \to X'$, the fundamental group of $M$ is directly related to the orbifold fundamental group of $X'$ by the long exact sequence

$$\ldots \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(M) \to \pi^\text{orb}_1(X') \to 1$$

When $\pi^\text{orb}_1(X') = 1$, we have that $\pi_1(M)$ is abelian, and hence if $H_1(M, \mathbb{Z}) = 0$ then $M$ is simply connected. We prove in section 4 the following

**Theorem 8.** For the orbifold $X'$ constructed above, $\pi^\text{orb}_1(X') = 1$. Hence $M$ is a Smale-Barden manifold.

On the second hand, we have to prove that $M$ cannot admit a semi-regular Sasakian structure. It this were the case, then there would be a Seifert orbifold $M \to Y$, where $Y$ is a Kähler orbifold. By [15, Proposition 10], this manifold $Y$ is a complex manifold, and as the Sasakian structure is semi-regular, $Y$ is smooth. As the homology of $M$ is given by (2), then Corollary 3 guarantees that $Y$ has $b_1 = 0$, $b_2 = 12$ and contains 12 disjoint smooth complex curves $C_1', \ldots, C_{11}', G'$, where $g(C_i') = 1$ and $g(G') = 3$. We prove the corresponding instance of Conjecture 5.

Note that this is not covered by Theorem 4.

**Theorem 9.** Let $S$ be a smooth complex surface with $H_1(S, \mathbb{Q}) = 0$ and containing $D_1, \ldots, D_b$, $b = b_2(S)$, smooth disjoint complex curves with genus $g(D_i) = g_i > 0$, and spanning $H_2(S, \mathbb{Q})$. Assume that $g_i = 1$, for $1 \leq i \leq b - 1$. Then $b \leq 2g_b^2 - 4g_b + 3$.

In particular, the case $b_2 = 12$, $g_i = 1$, for $1 \leq i \leq 11$ and $g_{12} = 3$ cannot happen.

**Corollary 10.** Let $M$ be a 5-dimensional manifold with $H_1(M, \mathbb{Z}) = 0$ and

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^{11} \oplus \bigoplus_{i=1}^{12} (\mathbb{Z}/p^i)^{2g_i},$$

where $g_i = 1$ for $1 \leq i \leq 11$, $g_{12} = 3$, and $p$ is a prime number. Then $M$ does not admit a semi-regular Sasakian structure.
This proves Theorem 6. It remains to see Theorems 7, 8 and 9. We prove Theorem 7 in Section 3, Theorem 8 in Section 4 and Theorem 9 in Section 5.

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2. Symplectic plumbing

2.1. Definition of symplectic plumbing. Let \((S, \omega)\) be a compact symplectic surface and \(\pi : E \to S\) be a complex line bundle. Topologically, \(E\) is determined by the Chern class \(d = c_1(E)\) which is the self-intersection of \(S\) inside \(E\), \(d = [S]^2\). We put a hermitian structure in \(E\), so we can define a neighbourhood via a disc bundle of some fixed radius \(c > 0\), denoted by \(B_c(S) \subset E\). We construct a symplectic form on \(B_c(S)\) next. First, we write \(V' \subset V\) if \(V'\) is an open subset such that its closure \(\overline{V'} \subset V\).

Lemma 11. For small enough \(c > 0\), \(B_c(S)\) admits a symplectic form \(\omega_E\) which is compatible with the complex structure of the fibers of the complex line bundle, and such that the inclusion \((S, \omega) \hookrightarrow (B_c(S), \omega_E)\) is symplectic. If \(V \subset S\) is a trivializing open set, \(E|V \cong V \times \mathbb{C}\), and \(V' \subset V\), we can arrange so that \(\omega_E|_{B_c(S) \cap E|V'}\) is the symplectic product structure on \(B_c(S) \cap E|V' \cong V' \times B_c(0)\).

Proof. Take \(S = \bigcup \alpha U_\alpha\) a cover of \(S\), with each \(U_\alpha\) symplectomorphic to a ball, and trivializations \(E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}\). In the fiber \(\mathbb{C}\) we put coordinates \(u + iv\) and consider the standard symplectic form \(\omega_0 = du \wedge dv = d(u dv) = d\eta\). Denote \(\varpi_\alpha : U_\alpha \times \mathbb{C} \to \mathbb{C}\) the projection over the second factor, and take \(\rho_\alpha\) a smooth partition of unity subordinated to the cover \(U_\alpha\) of \(S\). Define

\[
\omega_E = \pi^* \omega_S + \sum_\alpha d((\pi^* \rho_\alpha) \cdot (\varpi_\alpha^* \eta)).
\]

For \(x \in S\), we have \(\omega_E|_{E_x} = \sum \rho_\alpha(x) \omega_0 = \omega_0\), using that the changes of trivializations preserve \(\omega_0\). Then using the decomposition \(T_x E = T_x S \oplus E_x\), we have that \((\omega_E)^2(x) = \omega_S(x) \wedge \omega_0 > 0\). Therefore \(\omega_E\) is symplectic on the zero section \(S \subset E\). Since this is an open condition, it holds in some neighborhood \(B_c(S)\) of the zero section.

For the last part, just take an open cover \(U_\alpha\) of \(S - V'\) together with \(V\) in the construction above. \(\square\)

The submanifold \(S \subset E\) and any fiber \(E_x \subset E\) are symplectic, and they are symplectically orthogonal.

Now we move to the definition of plumbing as a symplectic neighbourhood of the union of two intersecting symplectic surfaces \(S_1, S_2\). Take points \(P_1, \ldots, P_m \in S_1\) and \(Q_1, \ldots, Q_m \in S_2\). We define

\[
S = S_1 \cup S_2/P_i \sim Q_i, i = 1, \ldots, m
\]
and we can write \( S = S_1 \cup S_2 \). Let now \( E_1 \to S_1 \) and \( E_2 \to S_2 \) be two complex line bundles, where \( d_i = c_i(E_i) \) which is the self-intersection of \( S_i \) inside \( E_i \), \( d_i = S_i^2 \).

Take hermitian metrics on the line bundles, so that \( B_c(S_1) \subset E_1 \) and \( B_c(S_2) \subset E_2 \) are symplectic manifolds for \( c > 0 \) by using Lemma 11.

For each \( i = 1, \ldots, m \), take small neighbourhoods \( B(P_i) \subset S_1 \), symplectomorphic to the ball \( B_c(0) \) via \( f_{1i} : B(P_i) \to B_c(0) \). Take a trivialization \( \varphi_{1i} : E_1|_{B(P_i)} \xrightarrow{\cong} B(P_i) \times \mathbb{C} \), therefore we have

\[
(f_{1i} \times \text{Id}) \circ \varphi_{1i} : B_c(S_1) \cap E_1|_{B(P_i)} \xrightarrow{\cong} B_c(0) \times B_c(0)
\]

(3)

Using Lemma 11, we endow \( E_1 \) with a 2-form \( \omega_{E_1} \) such that \( (B_c(S_1), \omega_{E_1}) \) is symplectic and the symplectic form is a product on \( B_c(S_1) \cap E_1|_{B(P_i)} \). This means that (3) is a symplectomorphism. We do the same for \( Q_i \in S_2 \), obtaining a symplectomorphism \( f_{2i} : B(Q_i) \to B_c(0) \), a trivialization \( \varphi_{2i} : E_2|_{B(Q_i)} \xrightarrow{\cong} \mathbb{C} \times \mathbb{C} \), a symplectic form \( \omega_{E_2} \) on \( B_c(S_2) \), and a symplectomorphism

\[
(f_{2i} \times \text{Id}) \circ \varphi_{2i} : B_c(S_2) \cap E_2|_{B(Q_i)} \xrightarrow{\cong} B_c(0) \times B_c(0)
\]

Let \( R : B_c(0) \times B_c(0) \to B_c(0) \times B_c(0) \), \( R(z_1, z_2) = (z_2, z_1) \), be the map reversal of coordinates, which is a symplectomorphism swapping horizontal and vertical directions. Then we take the gluing map

\[
\Phi_i = ((f_{2i} \times \text{Id}) \circ \varphi_{2i})^{-1} \circ R \circ ((f_{2i} \times \text{Id}) \circ \varphi_{1i}) : B_c(S_1) \cap E_1|_{B(P_i)} \to B_c(S_2) \cap E_2|_{B(Q_i)}
\]

**Definition 12.** We define the symplectic plumbing \( P_c(S_1 \cup S_2) \) of \( S = S_1 \cup S_2 \) as the symplectic manifold

\[
X = (B_c(S_1) \cup B_c(S_2))/x \sim \Phi_i(x), x \in B_c(S_1) \cap E_1|_{B(P_i)}, i = 1, \ldots, m
\]

Note that \( S_1 \cup S_2 \subset P_c(S_1 \cup S_2) \) are symplectic submanifolds and they intersect transversely.

### 2.2. Symplectic tubular neighbourhood

We need a symplectic tubular neighbourhood theorem for two intersecting surfaces \( S_1 \cup S_2 \). We start with the case of a single submanifold. We include the proof since our result is a minor modification of the one appearing in the literature.

**Proposition 13** (Symplectic tubular neighborhood). Suppose that \( (X, \omega) \) and \( (X', \omega') \) are two symplectic 4-manifolds (maybe open) with compact symplectic surfaces \( S \subset X \) and \( S' \subset X' \). Suppose that \( S \) and \( S' \) are symplectomorphic as symplectic manifolds via \( f : S \to S' \), and assume also that their normal bundles are smoothly isomorphic.

Let \( V, V' \) be tubular neighbourhoods of \( S \) and \( S' \), with projections \( \pi : V \to S \), \( \pi' : V' \to S' \), and let \( g : V \to V' \) be a diffeomorphism of tubular neighbourhoods of \( S \) and \( S' \) with \( g|_S = f \). Let \( W \subset S \), \( W' \subset S' \) such that \( g|_{\pi^{-1}(W)} : \pi^{-1}(W) \to \pi'^{-1}(W') \) is a symplectomorphism. Suppose that \( H^1(W) = 0 \), and let \( \tilde{W} \subset W \). Then there are tubular neighbourhoods \( S \subset U \subset X \) and \( S' \subset U' \subset X' \) which are symplectomorphic via \( \varphi : U \to U' \), where \( \varphi|_S = f \) and \( \varphi|_{U \cap \pi^{-1}(\tilde{W})} = g \).
Proof. This is an extension of the symplectic tubular neighbourhood theorem [17], which is the case where \( W \) is empty. Let \( g : V \to V' \) be the diffeomorphism of tubular neighbourhoods where \( g|_S = f \). We start by isotopying \( g \) so that \( d_x g : T_x X \to T_{g(x)} X' \) is a symplectomorphism, for all \( x \in S \). We do this without modifying \( g \) on \( \pi^{-1}(\hat{W}) \). Then the symplectic orthogonal to \( T_x S \subset T_x V \) is sent to the symplectic orthogonal to \( T_{f(x)} S' \subset T_{f(x)} V' \).

We take \( \omega_0 = \omega \) and \( \omega_1 = g^* \omega' \) and note that \( i^* (\omega_1 - \omega_0) = 0 \), where \( i : S \to V \) is the inclusion map. As \( \omega \) is the symplectic orthogonal to \( g \) tubular neighbourhoods where \( f \) is the case where \( T \) can suppose that \( i^* \rho = 0 \), hence there exists a 1-form \( \mu \in \Omega^1(\hat{W}) \) such that \( d \mu = \omega_1 - \omega_0 \). We can suppose that \( i^* \mu = 0 \), since otherwise we would consider the form \( \mu - \pi^* i^* \rho \).

Take an open set \( \hat{W} \) such that \( \hat{W} \in \hat{W} \in W \). We can also suppose that \( \mu|_{\pi^{-1}(\hat{W})} \) is an extension of the symplectic tubular neighborhood theorem \([17]\). Moreover \( \pi^{-1}(\hat{W}) \) is an isomorphism. We can also suppose that the restriction \( \mu|_S = 0 \). In local coordinates \( (x_1, x_2, y_1, y_2) \) where \( S = \{ (x_1, x_2, 0, 0) \} \), we have \( \mu = \sum a_j (x_1, x_2) dy_j + O(y) \). We cover \( S \) with balls \( B_\alpha \), and then \( (\mu|_S)|_{B_\alpha} = \sum a_j^\alpha dy_j^\alpha. \) The balls are chosen so that they are inside \( S \) and \( \hat{W} \). Take a partition of unity \{ \rho_\alpha \} subordinated to it. We define \( k_\alpha = \sum a_j^\alpha y_j^\alpha \) and \( k = \sum \rho_\alpha k_\alpha. \) For those \( B_\alpha \subset \hat{W} \), we can take \( k_\alpha = 0 \). Then \( d k|_S = \mu|_S \), and we can substitute \( \mu \) by \( \mu - d k \). Note that \( k = 0 \) on \( \pi^{-1}(\hat{W}) \), so we keep \( \mu|_{\pi^{-1}(\hat{W})} = 0 \).

Now consider the form \( \omega_t = t \omega_1 + (1 - t) \omega_0 = \omega_0 + t \mu \), for \( 0 \leq t \leq 1 \). Since \( d_x g \) is a symplectomorphism for all \( x \in S \), we have \( \omega_t|_S = \omega_0|_S \) and hence \( \omega_t|_S = \omega_0|_S \) is symplectic over all points of \( S \). So, reducing \( V \) if necessary, \( \omega_t \) is symplectic on some neighborhood \( V \) of \( S \). The equation \( \iota_{X_t} \omega_t = - \mu \) admits a unique solution \( X_t \) which is a vector field on \( V \). By the above, \( X_t|_S = 0 \) and \( X_t|_{\hat{W}} = 0 \). Take the flow \( \varphi_s \) of the family of vector fields \( X_t \). There is some \( U \subset V \) such that \( \varphi_t : U \to V \) for all \( t \in [0, 1] \). Moreover \( \varphi_0 = \text{Id}_U \), and \( \varphi_t|_S = \text{Id}_S \) and \( \varphi_t|_{\hat{W}} = \text{Id}_{\hat{W}}. \) We compute
\[
\frac{d}{dt}|_{t=s} \varphi_s^* \omega_t = \varphi_s^* (L_{X_s} \omega_s) + \varphi_s^* (d \mu) = \varphi_s^* (d (\iota_{X_s} \omega_s) + \iota_{X_s} d \omega_s) + \varphi_s^* d \mu
\]
\[
= - \varphi_s^* (d \mu) + \varphi_s^* (d \mu) = 0.
\]
This implies that \( \omega_0 = \varphi_0^* \omega_0 = \varphi_s^* \omega_0 \). So \( \varphi_1 : (U, \omega) \to (V, g^* \omega') \) is a symplectomorphism. The composition \( \varphi = g \circ \varphi_1 : (U, \omega) \to (V', \omega') \) is a symplectomorphism of \( U \) onto \( U' = \varphi(U) \subset V' \).

2.3. Symplectic tubular neighbourhood of two intersecting submanifolds.
Now we move to the case of the union of two intersecting symplectic submanifolds.

**Definition 14.** Let \((X, \omega)\) be a symplectic 4-manifold. We say that two symplectic surfaces \( S_1, S_2 \subset X \) intersect \( \omega \)-orthogonally if for every \( \phi \in S_1 \cap S_2 \) there are
Darboux coordinates \((z_1, z_2)\) such that \(S_1 = \{z_2 = 0\}\) and \(S_2 = \{z_1 = 0\}\) around \(p\).

By definition, \(S_1\) and \(S_2\) intersect \(\omega\)-orthogonally in the symplectic plumbing \(P_\epsilon(S_1 \cup S_2)\).

**Lemma 15** ([15, Lemma 6]). Let \((X, \omega)\) be a symplectic 4-manifold, and suppose that \(S_1, S_2 \subset X\) are symplectic surfaces intersecting transversely and positively. Then we can perturb \(S_1\) to get another surface \(S'_1\) in such a way that:

1. The perturbed surface \(S'_1\) is symplectic.
2. The perturbation is small in the \(C^0\)-sense and only changes \(S_1\) near the intersection points with \(S_2\), leaving these points fixed, i.e. \(S_1 \cap S_2 = S'_1 \cap S_2\).
3. \(S'_1\) and \(S_2\) intersect \(\omega\)-orthogonally.

**Theorem 16** (Symplectic tubular neighborhood). Suppose that \((X, \omega)\) and \((X', \omega')\) are two symplectic 4-manifolds (maybe open) with compact symplectic surfaces \(S_1, S_2 \subset X\) and \(S'_1, S'_2 \subset X'\). Assume that \(S_1\) and \(S_2\) intersect symplectically orthogonally, and similarly for \(S'_1\) and \(S'_2\). Suppose that there is a map \(f : S = S_1 \cup S_2 \to S' = S'_1 \cup S'_2\) which is a symplectomorphism \(f : S_1 \to S'_1\) and a symplectomorphism \(f : S_2 \to S'_2\). Assume also that the normal bundles \(\nu_{S_1} \cong \nu_{S'_1}\) and \(\nu_{S_2} \cong \nu_{S'_2}\). Then, there are tubular neighborhoods \(S \subset U \subset X\) and \(S' \subset U' \subset X'\) which are symplectomorphic via \(\varphi : U \to U'\), with \(\varphi|_S = f\).

**Proof.** Take a point \(P_i \in S_i \cap S_2\). Let \(\varphi_i : B_i \to B_i(0) \subset \mathbb{C}^2\) be Darboux coordinates so that \(S_1 = \{z_2 = 0\}\) and \(S_2 = \{z_1 = 0\}\), \(\varphi_i(P_i) = 0\). For \(f(P_i) \in S'_1 \cap S'_2\) we also take \(\varphi'_i : B'_i \to B'_i(0) \subset \mathbb{C}^2\) Darboux coordinates so that \(S'_1 = \{z'_2 = 0\}\) and \(S'_2 = \{z'_1 = 0\}\), \(\varphi'_i(f(P_i)) = 0\). The composite \((\varphi'_i)^{-1} \circ \varphi_i : B_i \to B'_i\) may not coincide with \(f\) on \(B_i \cap (S_1 \cup S_2)\). To arrange this, take

\[
\begin{align*}
  h_1 &= \varphi'_i \circ (f|_{B_i \cap S_1}) \circ \varphi_i^{-1} : B'_i(0) \times \{0\} \to B_i(0) \times \{0\} \\
  h_2 &= \varphi'_i \circ (f|_{B_i \cap S_2}) \circ \varphi_i^{-1} : \{0\} \times B'_i(0) \to \{0\} \times B_i(0)
\end{align*}
\]

which are symplectomorphisms onto their image. Then \(h = h_1 \times h_2\) is a symplectomorphism of \(\mathbb{C}^2\) on a neighbourhood of the origin. So consider the symplectomorphism

\[
\psi_i = (\varphi'_i)^{-1} \circ h \circ \varphi_i : W_i \to W'_i
\]

defined on a neighbourhood \(W_i \subset B_i\). It satisfies \(\psi_i|_{B_i \cap (S_1 \cup S_2)} = f|_{B_i \cap (S_1 \cup S_2)}\). Fix also \(\tilde{W}_i \Subset W_i\), and denote \(W = \bigcup W_i, \tilde{W} = \bigcup \tilde{W}_i, W'_i = \psi_i(W_i), W'' = \bigcup W'_i, \tilde{W}'_i = \psi_i(\tilde{W}_i), \tilde{W}' = \bigcup \tilde{W}'_i\), and \(\psi : W \to W''\) the map which is \(\psi_i\) on each \(W_i\).

Now take small tubular neighbourhoods \(U_1, U_2\) of \(S_1, S_2\) respectively. Then \(U_1 \cap U_2\) is a neighbourhood of the intersection \(S_1 \cap S_2\), and can be made as small as we want. We require that \(U_1 \cap U_2 \subset \tilde{W}\). We also take neighbourhoods \(U'_1, U'_2\) of \(S'_1, S'_2\) respectively such that \(U'_1 \cap U'_2 \subset \tilde{W}'\). We can define diffeomorphisms \(g_j : U_j \to U'_j\) with \(g_j|_{U_j} = f|_{U_j}\) and \(g_j|_{\tilde{W} \cap U_j} = \psi|_{\tilde{W} \cap U_j}\) for some \(\tilde{W} \Subset \tilde{W}' \subset \tilde{W}\) for \(j = 1, 2\). Apply Proposition 13 to \(g_j\), to obtain symplectomorphisms \(\varphi_j : V_j \to V'_j\), where
Remark 19. \(S_j \subset V_j \subset U_j\) and \(S_j' \subset V_j' \subset U_j'\), such that \(\varphi_j|_{S_j} = f|_{S_j}\) and \(\varphi_j|_{W \cap W_j} = \psi|_{W \cap W_j}\). As \(V_1 \cap V_2 \subset U_1 \cap U_2 \subset W\), we have that \(\varphi_1, \varphi_2\) coincide in the overlap region, defining thus a symplectomorphism
\[
\varphi : V_1 \cup V_2 \to V_1' \cup V_2'
\]
with \(\varphi|_S = f|_S\).

\[\square\]

**Corollary 17.** Let \((X, \omega)\) be a symplectic 4-manifold and \(S_1, S_2 \subset X\) two compact symplectic surfaces intersecting symplectically orthogonally. Then there is a neighbourhood \(U\) of \(S = S_1 \cup S_2\) which is symplectomorphic to a symplectic plumbing \(P_c(S)\).

**Proof.** Let \(i_j : S_j \hookrightarrow S\) be the inclusion map, and denote \(\{P_1, \ldots, P_m\} = i_1^{-1}(S_1 \cap S_2) \subset S_1\) and \(\{Q_1, \ldots, Q_m\} = i_2^{-1}(S_1 \cap S_2) \subset S_2\). Take complex line bundles \(E_j \to S_j\) with \(c_1(E_j) = d_j = |S_j|^2\), and define a symplectic plumbing \(P_c(S_1 \cup S_2)\) with these data. Now apply Theorem 16 to \(S \subset X\) and \(S \subset P_c(S)\).

**Corollary 18.** Let \((S_1, \omega_1), (S_2, \omega_2)\) and \((S_1', \omega_1'), (S_2', \omega_2')\) be compact symplectic surfaces. Consider a symplectic plumbing \(P_c(S_1 \cup S_2)\) with \#\(S_1 \cap S_2 = m\) and \(d_j = |S_j|^2\), \(j = 1, 2\), and another symplectic plumbing \(P_c(S_1' \cup S_2')\) with \#\(S_1' \cap S_2' = m'\) and \(d'_j = |S'_j|^2\), \(j = 1, 2\). If \(m = m', \langle [\omega_1], [S_j]\rangle = \langle [\omega_1'], [S'_j]\rangle\) and \(d_j = d'_j, j = 1, 2\), then there are neighbourhoods \(S_1 \cup S_2 \subset U \subset P_c(S_1 \cup S_2)\) and \(S_1' \cup S_2' \subset U' \subset P_c(S_1' \cup S_2')\) which are symplectomorphic.

**Proof.** Note that two compact surfaces \(\Sigma, \Sigma'\) are symplectomorphic if and only if they have the same area \(\langle [\omega], [\Sigma]\rangle = \langle [\omega'], [\Sigma']\rangle\). Moreover the symplectomorphism can be chosen so that it sends some finite collection of \(m\) points of \(\Sigma\) to another collection of \(m\) points of \(\Sigma'\). Applying this to \(S_j, S_j'\), we get a symplectomorphism \(f_j : S_j \to S_j'\) with \(f_j|_{S_1 \cap S_2} : S_1 \cap S_2 \to S_1' \cap S_2'\) sending the intersection points in the required order, \(j = 1, 2\). Therefore \(f_1|_{S_1 \cap S_2} = f_2|_{S_1 \cap S_2}\), thus defining a map \(f : S_1 \cup S_2 \to S_1' \cup S_2'.\) As the intersections are symplectically orthogonal, we can apply Theorem 16 to get the stated result.

This gives uniqueness of symplectic plumbings. In particular, they do not depend on the choices of symplectomorphisms of the surfaces, or the choice of Darboux coordinates at the intersection points.

**Remark 19.** Theorem 16 holds for a symplectic manifold \(X\) of any dimension, and symplectic submanifolds \(S_1, S_2 \subset X\) of complementary dimension intersecting symplectically orthogonally.

The plumbing can be defined for symplectic manifolds \(S_1, S_2\) of any dimension \(2n\), and \(P_c(S_1 \cup S_2)\) will have dimension \(4n\).

**3. A Configuration of Symplectic Surfaces in \(\mathbb{CP}^2\# 11\overline{\mathbb{CP}^2}\).**

**3.1. Homology of \(\mathbb{CP}^2\# 11\overline{\mathbb{CP}^2}\).** Let \(X = \mathbb{CP}^2\# 11\overline{\mathbb{CP}^2}\) be the symplectic manifold obtained by blowing up the projective plane \(\mathbb{CP}^2\) at 11 points \(\Delta = \{P_1, \ldots, P_{11}\}\).
We call $h \in H_2(X)$ the homology class of the line, and $e_i$, $1 \leq i \leq 11$, the homology classes of the exceptional divisors, so that $H_2(X) = \langle h, e_1, \ldots, e_{11} \rangle$. Moreover, the intersection form of $X$ is diagonal with respect to the basis $\{h, e_1, \ldots, e_{11}\}$. Now consider the collection of homology classes in $H_2(X)$ given by:

$$c_k = 3h - \sum_{i \neq k} e_i, \quad 1 \leq k \leq 11,$$

$$d = 10h - \sum_{i=1}^{11} 3e_i.$$

**Proposition 20.** The homology classes $\{c_1, \ldots, c_{11}, d\}$ form a basis of $H_2(X)$. The intersection form is diagonal with respect to this basis, and the self-intersections are $c_k^2 = -1$, for $1 \leq k \leq 11$, and $d^2 = 1$.

**Proof.** It follows from the fact that $e_i \cdot h = 0$ for all $i$, $e_i^2 = -1$ and $h^2 = 1$. This implies that the determinant of the intersection form with respect to this basis is $-1$, hence it is a basis over $\mathbb{Z}$. \hfill $\square$

Our focus is to prove that the basis $\{c_1, \ldots, c_{11}, d\}$ of $H_2(X)$ can be realized by symplectic surfaces. For this, we need the following configuration of symplectic surfaces in $\mathbb{C}P^2$:

- Eleven symplectic surfaces $C_1, \ldots, C_{11}$ such that its homology classes are $[C_i] = 3h$ in $\mathbb{C}P^2$. These surfaces $C_i$, being cubics, must have $g = 1$ by the symplectic adjunction formula. The surface $C_i$ is required to pass through the 10 points in $\Delta - \{P_i\}$, but not through $P_i$. Therefore, the proper transform $\tilde{C}_i$ of $C_i$ in the blow-up $X = \mathbb{C}P^2 \# 11\mathbb{C}P^2$ of $\mathbb{C}P^2$ at $S$, has homology class $[\tilde{C}_i] = c_i$.
- The intersection $C_i \cap C_j$ contains the 9 points $\Delta - \{P_i, P_j\}$, for $i \neq j$. Note that the algebraic intersection is $C_i \cdot C_j = 9$. If these intersections are transverse and positive (e.g. if $C_i$ are holomorphic around the intersection points) and if there are no more intersections, then the proper transforms $\tilde{C}_i, \tilde{C}_j$ are disjoint.
- One singular symplectic surface $G$ such that $[G] = 10h$ and $G$ has 11 ordinary triple points at the points of $\Delta$. By the adjunction formula the genus of $G$ is

$$g = \frac{1}{2}(10-1)(10-2) - 11 \cdot \frac{3 \cdot 2}{2} = 36 - 33 = 3.$$

If $G$ is holomorphic at a neighbourhood of the triple points, and the branches intersect transversely (and hence also positively), then the proper transform $\tilde{G}$ of $G$ in the blow-up $X = \mathbb{C}P^2 \# 11\mathbb{C}P^2$ of $\mathbb{C}P^2$ at $S$, has homology class $[\tilde{G}] = 10h - 3(e_1 + \ldots + e_{11}) = d$. Moreover, if there are no more singularities, then $\tilde{G}$ is smooth and symplectic surface in $X$.
- The intersections $C_i \cap G$ contain the 10 points $\Delta - \{P_i\}$. Note that the algebraic intersection is $C_i \cdot G = 30$. If the intersections with each of the
three branches at each intersection point are transverse and positive (e.g. if $C_i, G$ are holomorphic around the intersection points), and if there are no more intersections, then these account for all intersections. In the blow-up $X$, the proper transforms $\tilde{C}_i, \tilde{G}$ are disjoint.

Our aim now is to construct these surfaces in $\mathbb{C}P^2$. For this, we will make the construction in a local model, and then we will transplant it to $\mathbb{C}P^2$.

3.2. Construction of a local model. Now we are going to construct the required 11 surfaces of genus 1 and the singular surface of genus 3, in a local model. The local model is as follows: take a genus 1 complex curve $C$ and a rational complex curve $L \cong \mathbb{C}P^1$. Take three points $Q_1, Q_2, Q_3 \in C$ and another three $Q'_{1}, Q'_{2}, Q'_{3} \in L$. Take a line bundle $E \rightarrow C$ of degree 9 and a line bundle $E' \rightarrow L$ of degree 1, and perform the plumbing as given in section 2.1. This produces a symplectic manifold $P_c(C \cup L)$, which contains $C \cup L$.

**Proposition 21.** Let $C' \subset \mathbb{C}P^2$ and $L' \subset \mathbb{C}P^2$ be a smooth cubic and a line in the complex plane, intersecting transversely. Then $P_c(C \cup L)$ can be symplectically embedded in a neighbourhood of $C' \cup L'$, where $C$ is sent to $C'$ and $L$ is sent to a $C^0$-small perturbation of $L'$, preserving the intersection points.

**Proof.** We start by modifying $L'$ to $L''$ using Lemma 15, so that $C'$ and $L''$ intersect symplectically orthogonally. By corollary 17, a small neighbourhood of $C' \cup L''$ is symplectomorphic to a small neighbourhood of the plumbing of $C \cup L$, that is some $P_c'(C \cup L)$, for $c > 0$ small, and the symplectomorphism sends $C$ to $C'$ and $L$ to $L''$.

Therefore, to prove Theorem 7, it is enough to prove the following:

**Theorem 22.** There are 11 points $P_1, \ldots, P_{11}$ in $P_c(C \cup L)$, and symplectic surfaces $C_1, C_2, \ldots, C_{11}, G \subset P_c(C \cup L)$ such that:

- $C_i$ is a section of a complex line bundle $E \rightarrow C$ of degree 3, and $P_j \in C_i$ for $j \neq i$, $P_i \notin C_i$. In particular, they have genus 1.
- The surfaces $C_i, C_j$, $i \neq j$ intersect exactly at $\{P_1, \ldots, P_{11}\} - \{P_i, P_j\}$, positively and transversely.
- $G$ is a genus 3 singular symplectic surface whose only singularities are 11 triple points at $P_i$ (with different branches intersecting positively). Moreover $G$ intersects each $C_i$ only at the points $P_j$, $j \neq i$, and all the intersections of $C_i$ with the branches of $G$ are positive and transverse.

To be more concrete, we do as follows. We fix a complex structure on $C$, and a degree 9 complex line bundle $E \rightarrow C$. This is going to be as follows: take a complex disc $D \subset C$, which we assume as the radius 1 disc $D = D(0, 1) \subset \mathbb{C}$. Let $V = C - \bar{D}(0, 1/2)$, and consider the change of trivialization given by the function $g(z) = z^9 \cos D(0, 1) - \bar{D}(0, 1/2)$. This means that $E$ is formed by gluing $E|_D = D \times \mathbb{C}$ with $E|_V = V \times \mathbb{C}$ via $(z, y) \sim (z, z^9 y)$. We endow $E$ with an auxiliary hermitian metric which is of the form $h(z) = 1$ on the trivialization $E|_D$. 

We will choose the points $Q_1, Q_2, Q_3 \in D \subset C$. We also take a complex line bundle $E' \to L$ of degree 1, for which we fix a hermitian structure. Fixing three points $Q_1', Q_2', Q_3' \in L$, we perform the plumbing $P_c(C \cup L)$.

### 3.3. Construction of the genus 1 surfaces.

The genus 1 symplectic surfaces will be constructed as sections of the line bundle $E \to C$. Consider the previous cover $C = V \cup D$ and trivializations $E|_V \cong V \times \mathbb{C}$ and $E|_D \cong D \times \mathbb{C}$. Fix distinct numbers $z_1, \ldots, z_{10}, z_{11} \in D$ with $z_{11} = 0$ the origin. Take $\lambda > 0$ a small positive real number to be fixed later. Take the points

$$P_j = (\lambda z_j, 0), \quad j = 1, \ldots, 10, \quad \text{and} \quad P_{11} = (0, 1)$$

in $E$, in the given trivialization $E|_D \cong D \times \mathbb{C}$. We define 11 holomorphic local sections in the chart $D \subset C$ as

$$\sigma_j(z) = \prod_{i \neq j}^{10} \left(1 - \frac{z}{\lambda z_i}\right), \quad j = 1, \ldots, 10,$$

and $\sigma_{11}(z) = 0$.

Clearly $\sigma_j(z) = \sigma_{11}(z) = 0$ at the 9 points $\lambda z_1, \ldots, \lambda z_9, \lambda z_{10}$. Also, for $1 \leq j < k \leq 10$, we have that $\sigma_j(z) = \sigma_k(z)$ at the 9 points given by

$$\lambda z_1, \ldots, \lambda z_9, \lambda z_{10}, z_{11} = 0.$$  \hspace{1cm} (5)

All the intersections of the graphs are transverse and positive since the points $\lambda z_i$ are simple roots and $\sigma_j$ are holomorphic sections. By construction, the graph $\Gamma(\sigma_j)$ of the local section $\sigma_j$ in the trivialization $E|_D \cong D \times \mathbb{C}$ contains the set of points $\{P_1, \ldots, P_{11}\} - \{P_j\}$, as desired.

Now we move to the trivialization $E|_V$. Let us see that we can extend the sections $\sigma_j$ to all of $V$ without introducing any new intersection points between their graphs. For $z \in D \cap V$, the sections $\sigma_j$ become, for $|z| \geq 1/2$, in the trivialization of $E|_V \cong V \times \mathbb{C}$,

$$\tilde{\sigma}_j = z^{-9} \prod_{i \neq j}^{10} \left(1 - \frac{z}{\lambda z_i}\right) = \lambda^{-9} A z_j \prod_{i \neq j}^{10} \left(1 - \frac{\lambda z_i}{z}\right), \quad A = -(z_1 \ldots z_{10})^{-1},$$

and $\tilde{\sigma}_{11} = 0$. Then $\tilde{\sigma}_j$ has the form

$$\tilde{\sigma}_j = A \lambda^{-9} z_j (1 + \lambda f_j(z, \lambda)),$$

where

$$f_j(z, \lambda) = \frac{1}{\lambda} \left(\prod_{i \neq j}^{10} \left(1 - \frac{\lambda z_i}{z}\right) - 1\right)$$

is a holomorphic function of $z$ depending on the parameter $\lambda$, such that

$$|f_j(z, \lambda)| \leq M_0, \quad \text{for} \quad \lambda \leq \frac{1}{4}, \quad |z| \geq \frac{1}{2},$$

being $M_0$ a constant depending only on $z_1, \ldots, z_{11}$.

Let $\rho$ be the smooth non-increasing function with $\rho(r) = 0$ for $r \geq 3/4$ and $\rho(r) = 1$ for $r \leq 2/3$. Here $r = |z|$ is the radius in the disc $D$. Now we modify the
local sections $\hat{\sigma}_j$ to sections $\hat{\sigma}_j$ that can be extended to global sections in $E \to C$. We define for $z \in U$, $|z| \geq 1/2$,

$$\hat{\sigma}_j(z) = \rho(|z|)\sigma_j(z) + (1 - \rho(|z|))\lambda^{-9}A_j = \lambda^{-9}A_j(1 + \lambda\rho(|z|)f_j(z, \lambda)).$$

(6)

We also put $\hat{\sigma}_{11} = 0$.

We have that $\hat{\sigma}_j = \sigma_j$ in $\{1/2 \leq |z| \leq 2/3\}$, so $\hat{\sigma}_j$ extends to the trivialization $E|_D$ as $\sigma_j$ in $\{|z| \leq 1/2\} \subset D$. Moreover, $\hat{\sigma}_j(z) = A_j$ is constant for $|z| \geq 3/4$, so $\hat{\sigma}_j$ extends to all $V$, hence they give global sections in the line bundle $E \to C$. We call $\sigma_j$ these global sections, and $\Gamma(\hat{\sigma}_j)$ its graphs.

Now let us check that no undesired intersection points are introduced between any pair of surfaces $C_j$, $1 \leq j \leq 11$. On $|z| \leq 1/2$, $\hat{\sigma}_j = \sigma_j$, so $\hat{\sigma}_j, \sigma_k, j \neq k$, have 9 intersection points given by (5), which are the set $\{P_1, \ldots, P_{11}\} - \{P_j, P_k\}$. As $\hat{\sigma}_j$ and $\sigma_k$ are holomorphic there, and the roots are simple, the intersections are positive and transverse.

For $|z| \geq 3/4$, $\hat{\sigma}_j = \lambda^{-9}A_j$, $j = 1, \ldots, 10$ and $\hat{\sigma}_{11} = 0$. Therefore the sections do not intersect since the $\{z_j\}$ are distinct points. Now assume that $1/2 \leq |z| \leq 3/4$. If $\hat{\sigma}_j(z) = \hat{\sigma}_k(z)$ with $k \neq j \leq 10$ then

$$z_j + z_j\lambda\rho(|z|)f_j(z, \lambda) = z_k + z_k\lambda\rho(|z|)f_k(z, \lambda).$$

Taking $\lambda > 0$ small enough, the discs $B(z_j, M_0|z_j|\lambda)$ and $B(z_k, M_0|z_k|\lambda)$ are all pairwise disjoint, so the above equality does not happen. Analogously, if $\hat{\sigma}_j(z) = \hat{\sigma}_{11}(z) = 0$ for $1/2 \leq |z| \leq 3/4$ we have a contradiction as long as $\lambda$ is small enough so that the discs $B(z_j, M_0|z_j|\lambda)$ do not contain the origin.

Finally considering $\hat{\sigma}_j = \epsilon\hat{\sigma}_j$, being $\epsilon > 0$ small enough, the intersections of the graphs remains the same except that $P_{11}$ is changed to the point $(0, \epsilon)$. This ensures that the graphs are all contained in the given neighbourhood $B_c(C)$, for any $c > 0$ given beforehand. Moreover the graphs become $C^1$-close to the zero section $C \subset E$, in particular the graphs are symplectic surfaces of $B_c(E)$.

### 3.4. Construction of the genus 3 surface.

The genus 3 surface will be constructed inside the neighbourhood $P_c(C \cup L)$ of $C \cup L$, where $C$ is the genus 1 surface and $L$ the genus 0 surface, both intersecting at three points. We will take 3 sections of the bundle $E \to C$, all of them passing through the eleven points $P_1, \ldots, P_{11}$. In this way we get the 11 triple points. Then we add the line $L$, and glue the three sections with $L$ around the intersection points of $L$ and $C$. Let us carry out the details.

As before, take the previous cover $C = V \cup D$, $D = D(0, 1)$, $V = C - D(0, 1/2)$, and trivializations $E|_D \cong D \times \mathbb{C}$ and $L|_V \cong V \times \mathbb{C}$, with change of trivialization $g(z) = z^9$. We have fixed $z_1, \ldots, z_{10}, z_{11} = 0 \in D$ and the points

$$P_j = (\lambda z_j, 0), j = 1, \ldots, 10, \text{ and } P_{11} = (0, 1)$$

in $E|_D = D \times \mathbb{C}$, where $0 < \lambda \leq 1/4$ is some small number as arranged in section 3.3.
We choose another three distinct values \( w_1, w_2, w_3 \in D \), different to \( z_1, \ldots, z_{11} \). We take the points
\[
Q_1 = (\lambda w_1, 0), Q_2 = (\lambda w_2, 0), Q_3 = (\lambda w_3, 0),
\]
in the trivialization \( E|_D = D \times \mathbb{C} \). Consider (meromorphic) sections \( \tau_k \), defined in \( D - \{Q_k\} \) by the formula
\[
\tau_k(z) = \frac{\prod_{i=1}^{10} \left( 1 - \frac{z}{\lambda z_i} \right)}{1 - \frac{z}{\lambda w_k}},
\]
for \( k = 1, 2, 3 \). The graph of \( \tau_k \) passes through all 11 points (4).

Let us see that we can extend the sections \( \tau_k \) to the trivialization \( E|_V \), giving thus sections over \( C - \{Q_k\} \). For \( z \in D \cap V \), i.e. \( |z| \geq 1/2 \), we express \( \tau_k \) in the trivialization \( L|_V \), which is given by \( \tilde{\tau}_k(z) = z^{-9} \tau_k(z) \).

\[
\tilde{\tau}_k(z) = z^{-9} \frac{\prod_{i=1}^{10} \left( 1 - \frac{z}{\lambda z_i} \right)}{1 - \frac{z}{\lambda w_k}} = \lambda^{-9} A w_k \frac{\prod_{i=1}^{10} \left( 1 - \frac{z}{\lambda z_i} \right)}{1 - \frac{z}{\lambda z_k}} = \lambda^{-9} A w_k (1 + \lambda g_k(z, \lambda)),
\]
where \( A = -(z_1 \cdots z_{10})^{-1} \) as before, and
\[
g_k(z, \lambda) = \frac{1}{\lambda} \left( \frac{\prod_{i=1}^{10} \left( 1 - \frac{z}{\lambda z_i} \right)}{1 - \frac{z}{\lambda w_k}} - 1 \right)
\]
are bounded functions for \( |z| \geq 1/2 \) and \( 0 < \lambda \leq 1/4 \), say \( |g_k(\lambda, z)| \leq M \), for \( M > 0 \) a constant.

Let \( \rho : [0, \infty) \to \mathbb{R} \) be a non-increasing smooth function such that \( \rho(r) = 1 \) for \( r \leq 1/2 \) and \( \rho(r) = 0 \) for \( r \geq 3/4 \). Now we modify \( \tilde{\tau}_k(z) \) for \( z \in D \cap V \), i.e. \( |z| \geq 1/2 \). Consider
\[
\hat{\tau}_k(z) = \lambda^{-9} A w_k (1 + \rho(|z|) \lambda g_k(\lambda, z)).
\]
Clearly, \( \hat{\tau}_k(z) = \tilde{\tau}_k(z) \) for \( |z| \leq 1/2 \), so \( \hat{\tau}_k \) extends to the trivialization \( E|_D \). Also, for \( |z| \geq 3/4 \) we have \( \hat{\tau}_k(z) = \lambda^{-9} A w_k \) is constant so \( \hat{\tau}_k \) extends to all the trivialization \( L|_V \). This yields a global section defined in \( C - \{Q_k\} \), given by \( \tau_k \) in \( \{ |z| \leq 1/2 \} \subset D \), and by \( \hat{\tau}_k \) in \( V \). We call from now on \( \hat{\tau}_k \) this global section. Let us denote
\[
\Theta_k = \Gamma(\hat{\tau}_k) = \{(z, \hat{\tau}_k(z)) \mid z \in C - \{Q_k\}\}
\]
the graph of \( \hat{\tau}_k \).

Let us see that the graphs \( \Theta_1, \Theta_2, \Theta_3 \) only intersect at the points \( P_1, \ldots, P_{10}, P_{11} \), i.e. that the sections only coincide for the values \( \lambda z_1, \ldots, \lambda z_{10}, z_{11} = 0 \). Let \( j \neq k \). On \( |z| \leq 1/2, z \neq \lambda w_j, \lambda w_k \), if \( \hat{\tau}_j(z) = \hat{\tau}_k(z) \), then
\[
\frac{\prod_{i=1}^{10} \left( 1 - \frac{z}{\lambda z_i} \right)}{1 - \frac{z}{\lambda w_j}} = \frac{\prod_{i=1}^{10} \left( 1 - \frac{z}{\lambda z_i} \right)}{1 - \frac{z}{\lambda w_k}}.
\]
Hence either \( z = \lambda z_i \) for some \( 1 \leq i \leq 10 \) or \( \frac{z}{\lambda w_j} = \frac{z}{\lambda w_k} \). The latter implies \( z = 0 = z_{11} \).
For $|z| \geq 3/4$, if $\tilde{\tau}_j(z) = \tilde{\tau}_k(z)$ then $\lambda^{-9} Aw_j = \lambda^{-9} Aw_k$, which is false since $w_j \neq w_k$. Finally, for $1/2 \leq |z| \leq 3/4$, if $\tilde{\tau}_j(z) = \tilde{\tau}_k(z)$ then

$$w_j + w_j \rho(|z|) \lambda g_j(\lambda, z) = w_k + w_k \rho(|z|) \lambda g_k(\lambda, z).$$

Choosing $\lambda$ small enough, we have that the discs $D(w_j, \lambda|w_j|M)$ and $D(w_k, \lambda|w_k|M)$ do not intersect. So the above equality does not happen.

Finally, let us check the intersections of $\Theta_k$ with $\Gamma(\hat{\sigma}_j)$. Take $|z| \leq 1/2$. Suppose that $\tau_k(z) = \sigma_j(z)$. This implies that

$$\sigma_j(z) \frac{1 - \frac{z}{x_{\lambda z_j}}}{1 - \frac{z}{x_{\lambda w_k}}} = \sigma_j(z),$$

hence either $\sigma_j(z) = 0$ or $\frac{z}{x_{\lambda z_j}} = \frac{z}{x_{\lambda w_k}}$. In the first case we have that $z = \lambda z_j$ for some $i \neq j$. In the second case we have that either $z = 0 = z_{11}$, or $\lambda z_j = \lambda w_k$ which is not possible.

Suppose now that $1/2 \leq |z| \leq 3/4$ and $\sigma_j(z) = \tau_k(z)$. Then

$$z_j + z_j \rho(|z|) \lambda f_j(z, \lambda) = w_k + w_k \rho(|z|) \lambda g_k(\lambda, z).$$

if we take $\lambda$ small, the discs $D(z_j, M_0|z_j|\lambda)$ and $D(w_k, M|w_k|\lambda)$ are disjoint, so the above equality is impossible. Finally, if $|z| \geq 3/4$ and $\tau_k(z) = \sigma_j(z)$ then $\lambda^{-9} A z_j = \lambda^{-9} A w_k$, and this is false.

3.5. **Gluing the transversal in the plumbing.** The plumbing $P_c(C \cup L)$ is defined only for $c > 0$ small enough. Let us arrange that our sections lie inside it suitably. For this let $N > 0$ be an upper bound of all $|\hat{\sigma}_j|, j = 1, \ldots, 11$ such that $(|\hat{\tau}_k|)^{-1}([N, \infty)) \subset B(Q_k)$, where $B(Q_k) \subset D_{1/2}$ are small balls around $Q_k$, $k = 1, 2, 3$. Recall that $\hat{\tau}_k = \tau_k$ is holomorphic on $B(Q_k) - \{Q_k\}$. As $\tau_k$ has a simple pole at $Q_k$, we have that

$$z' = z_k' = h_k(z) = \frac{1}{\tau_k(z)}$$

is a biholomorphism from a neighbourhood of $Q_k$ (that we keep calling $B(Q_k)$) to a ball $B_c(0)$. We take the coordinate $z_k'$ on $B(Q_k)$. We need to modify the symplectic form so that $z_k'$ is also a Darboux coordinate.

**Lemma 23.** Consider the disc $D = D(0, 1)$. We can perturb the standard symplectic form $\omega_D$ to a nearby symplectic form $\omega_D'$ such that, maybe after reducing the balls $B(Q_k)$, the coordinate $z_k'$ are Darboux. The perturbation is made only on a (slightly larger) ball around $Q_k$, and keeping the total area.

**Proof.** We write $z' = z_k' = x' + iy'$. The standard symplectic form $\omega_D$ on the coordinates $z$ is clearly Kähler, therefore it is also Kähler for the holomorphic coordinate $z'$. In particular, it has a Kähler potential $\phi(x', y')$, with $\omega_D = \partial \bar{\partial} \phi(x', y')$. We can assume that $\phi$ has no linear part, so $\phi(x', y') = \phi_2(x', y') + \phi_3(x', y')$, where $\phi_2(x', y')$ is quadratic and $|\phi_3| = O((x', y')^3)$. Then take some bump function $\rho$ that vanishes on a neighbourhood $B_{\eta}(Q_k)$ of $Q_k$ (the size measured with respect
to the radial coordinate $r' = |z'|$, and $\rho \equiv 1$ on a slightly larger neighbourhood $B_{2\eta}(Q_k)$, $|d\rho| = O(\eta^{-1})$ and $|\nabla d\rho| = O(\eta^{-2})$. Set $\omega'_D = \partial\bar{\partial}((\rho - 1)\phi_3)$. Then $|\omega'_D - \omega_D| = O(\eta)$, and $\omega'_D$ is standard on $(B_{\eta}(Q_k), z')$. As the difference $\omega'_D - \omega_D = \partial\bar{\partial}((\rho - 1)\phi_3) = d(\partial(\rho - 1)\phi_3)$ is exact and compactly supported, the total area remains the same.

\[ \square \]

**Remark 24.** Lemma 23 also holds in higher dimension. More concretely, let $(Z, \omega, J)$ be a Kähler manifold of real dimension $2n$, $p \in Z$ and $\varphi : U \to B \subset \mathbb{C}^n$ holomorphic coordinates around $p$. Then there exists a symplectic form $\omega'$ on $Z$ so that $(Z, \omega', J)$ is Kähler, $\omega' = \omega$ in $Z - U$, and $\omega'$ is a linear symplectic form near $p$ on the coordinates $\varphi$, in some smaller neighborhood $V \subset U$. Moreover, the cohomology classes $[\omega] = [\omega']$.

Over $E|_{B(Q_k)} \cong B_\epsilon(0) \times \mathbb{C}$, the section $\tau_k$ is given by $v = 1/|z'|$ (making $c > 0$ smaller if needed), writing $z' = z'_k$ for brevity. For $Q'_1, Q'_2, Q'_3 \subset L$, take holomorphic balls $B(Q'_j) \cong B_\epsilon(0)$, and arrange the symplectic structure on $L$ to be standard over them. Finally, take symplectic structures on the total spaces of the complex line bundles $\pi : E \to C$ and $\pi' : E' \to L$ so that they are product symplectic structures on $B_\epsilon(C) \cap E|_{B(Q_k)} \cong B(Q_k) \times B_\epsilon(0)$ and $B_\epsilon(L) \cap E'|_{B(Q_k)} \cong B(Q_k) \times B_\epsilon(0)$, respectively. The plumbing $P_c(C \cup L)$ is done by gluing $B_\epsilon(C)$ and $B_\epsilon(L)$ along $R : B(Q_k) \times B_\epsilon(0) \to B(Q_k) \times B_\epsilon(0)$, the map reversal of coordinates. Note that the uniqueness result of Corollary 18 allows to do the plumbing with these choices. We only have to take care of keeping the total areas $\langle [C], [\omega_E] \rangle$ and $\langle [L], [\omega_{E'}] \rangle$ fixed.

Now take $\epsilon > 0$ small enough so that:

- The graphs of the sections $\sigma_j^\epsilon = \epsilon \sigma_j$ are inside $B_\epsilon(C)$. For this $\epsilon N < c$ is enough.
- The graphs of the sections $\sigma_j^\epsilon$ are $C^1$-close to the zero section. This implies that these graphs are symplectic (a submanifold $C^1$-close to a symplectic one is symplectic).
- All sections $\tau_k^\epsilon = \epsilon \tau_k$ satisfy $|\tau_k^\epsilon| < c$ on $C - B(Q_k)$, so the graph $\Theta^\epsilon_k$ of $\tau_k^\epsilon$ satisfies that $\Theta^\epsilon_k \cap \pi^{-1}(C - B(Q_k)) \subset B_\epsilon(C)$. For this it is enough that $\epsilon N < c$ again.
- The graphs of the sections $\tau_k^\epsilon$ are $C^1$-close to the zero section on $C - B(Q_k)$, so they are symplectic.

Now we look at the graph $\Theta^\epsilon_k \cap (E|_{B(Q_k)})$. We have

\[
\Theta^\epsilon_k \cap (E|_{B(Q_k)}) \cong \left\{ (z', v) \in B_\epsilon(0) \times \mathbb{C} \mid v = \frac{\epsilon}{z'} \right\} = \left\{ (z', v) \in B_\epsilon(0) \times \mathbb{C} \mid |v| \geq \epsilon e^{-1}, z' = \frac{\epsilon}{|v|} \right\}.
\]
Make $\epsilon > 0$ smaller if necessary, so that $\epsilon c^{-1} \leq c/2$. Take $\rho(r)$ a smooth non-increasing function so that $\rho(r) = 1$ for $r \leq 1/2$ and $\rho(r) = 0$ for $r \geq 3/4$. Define

$$\hat{\Theta}_k^\epsilon = \{(z', v) \in B_c(0) \times \mathbb{C} \mid \epsilon c^{-1} \leq |v| \leq c, z' = \epsilon \rho(|v|/c) z \}$$

This can be smoothly glued to $\hat{\Theta}_k^\epsilon = \Theta_k^\epsilon \cap \pi^{-1}(C - B(Q_k))$. On the part of the plumbing corresponding to $E' \to L$, this has the form $z' = \epsilon \rho(|v|/c) z$ on $B(Q_k') \times B_c(0)$, where $v$ is the coordinate for $B(Q_k')$ and $z'$ is the vertical coordinate. Note that this can be extended as $z' = 0$ in the bundle $E' \to L$, over $L - (B(Q_1') \cup B(Q_2') \cup B(Q_3'))$. The resulting smooth manifold is

$$G = \bigcup_{k=1,2,3} \left( \hat{\Theta}_k^\epsilon \cup \Theta_k^\epsilon \right) \cup \left( L - (B(Q_1') \cup B(Q_2') \cup B(Q_3')) \right).$$

(8)

Clearly, as $|v| \geq \epsilon c^{-1}$ for the points of $\hat{\Theta}_k^\epsilon$, there are no new intersections with the graphs $\Gamma(\sigma_j^l)$ or $\Theta_k^\epsilon$, $l \neq k$, since they are bounded by $\epsilon N$, and we can take $c < N^{-1}$ to start with.

The graphs $\hat{\Theta}_k^\epsilon$ are symplectic, since the graphs of $z' = \epsilon \rho(|v|/c) z$ are symplectic over $|v| \geq c/2$, by taking $\epsilon > 0$ small enough so that it is $C^1$-close to the zero section $z' = 0$ of the bundle $E' \to L$. On $\epsilon c^{-1} \leq |v| \leq c/2$, the graph coincides with $z' = \epsilon z$, which is holomorphic hence symplectic.

Remark 25. The homology class of the graph $\Gamma(\sigma_j^l)$ in $P_c(C \cup L)$ is equal to $[C]$, since they are sections of $E \to C$. The manifold $G$ of (8) can be retracted to $3[C] + [L]$ in $P_c(C \cup L)$, by making $\epsilon \to 0$.

When we embed $P_c(C \cup L) \hookrightarrow \mathbb{C}P^2$, the class $[C] \mapsto 3h$, and $[L] \mapsto h$, where $h$ is the class of the line in $\mathbb{C}P^2$. Hence $[G] \mapsto 10h$, so $G$ has degree 10.

The genus of $G$ is 3 since topologically it is the gluing of three surfaces of genus 1, the sections $\Theta_k$, $k = 1, 2, 3$, with a sphere with 3 holes, $L - (B(Q_1') \cup B(Q_2') \cup B(Q_3'))$.

4. Fundamental group of the $K$-contact 5-manifold

Let $X$ be the symplectic manifold constructed as the symplectic blow-up of $\mathbb{C}P^2$ at the eleven points $P_1, \ldots, P_{11}$. The underlying smooth manifold is $X = \mathbb{C}P^2\#11\overline{\mathbb{C}P^2}$, with $b_2 = 12$. It has 11 surfaces of genus 1, named $\tilde{C}_1, \ldots, \tilde{C}_{11}$, and a genus 3 surface $\tilde{G}$ all of them disjoint. We set the isotropy of $\tilde{C}_i$ to be $p_i$, $i = 1, \ldots, 11$, and that of $\tilde{G}$ to be $p_{12}$, for a fixed prime $p$. This determines a symplectic orbifold $X'$ uniquely by [15, Proposition 7].

We start by computing the orbifold fundamental group $\pi_1^{orb}(X')$ of $X'$. The reader can find alternative definitions in [19, Chapter 13] and in [5, Definition 4.3.6]. We only need a presentation of $\pi_1^{orb}(X')$, which follows from [10, Théorème A.1.4]. For this, fix a base point $p_0 \in X'$. Take loops from $p_0$ to a point near $\tilde{C}_i$, followed by a loop $\delta_i$ around $\tilde{C}_i$, and going back to $p_0$, $i = 1, \ldots, 11$. In the same
vein, we add another loop $\delta_{12}$ around $\tilde{G}$. Then

$$\pi_1^{\text{orb}}(X') = \pi_1(X - (\tilde{C}_1 \cup \ldots \cup \tilde{C}_{11} \cup \tilde{G})) / \langle \delta_1^{10}, \ldots, \delta_{11}^{11}, \delta_{12}^{12} \rangle.$$ 

Let us see that $\pi_1^{\text{orb}}(X')$ is trivial. It suffices to see that $\pi_1(X - (\tilde{C}_1 \cup \ldots \cup \tilde{C}_{11} \cup \tilde{G}))$ is trivial. We start with a lemma.

**Lemma 26.** We can arrange a complex cubic curve and a complex line $C', L' \subset \mathbb{CP}^2$ intersecting transversally, such that a small neighborhood $B_\epsilon(C' \cup L')$ of $C' \cup L'$ satisfies the following: there are generators of $\pi_1(C')$ represented by loops $\alpha, \beta$ away from $B_\epsilon(L')$, that can be homotoped (outside $B_\epsilon(L')$) to loops $\hat{\alpha}, \hat{\beta}$ in $\partial B_\epsilon(C')$. The loops $\hat{\alpha}, \hat{\beta}$ are contractible in $\mathbb{CP}^2 - (B_\epsilon(C' \cup L'))$.

**Proof.** We consider a particular family of complex cubics in $\mathbb{CP}^2$ given by the affine equations $C_r = \{ y^2 = x^3 - r^2 x \}$, with $r > 0$ small. As $r \to 0$, the cubic $C_r$ collapses to a cuspidal rational curve $C_0 = \{ y^2 = x^3 \}$, which has trivial homology. It is known [12] that the vanishing cycles generate the homology $H_1(C_r)$. Here we give an explicit description, as the loops

$$\alpha_r = \{(x, y) | x \in [-r, 0], y \in \mathbb{R}, y^2 = x^3 - r^2 x \},$$

$$\beta_r = \{(x, y) | x = -x' \in [0, r], y = iy' \in i\mathbb{R}, (y')^2 = (x')^3 - r^2 x' \}.$$ 

Note that $\alpha_r, \beta_r$ intersect transversally at one point, hence they generate $\pi_1(C_r) \cong \mathbb{Z}^2$. The homotopies given by $\alpha_t, t \in [0, r]$, and $\beta_t, t \in [0, r]$ (with base-point at $(0,0)$) produce discs that contract $\alpha_r, \beta_r$. These discs do not intersect $C_r$. Now fix some $C' = C_r$ and take a tubular neighbourhood $B_\epsilon(C')$ by considering all $C_s$ with $|s - r| < \epsilon$. Then we can homotop the loops $\alpha = \alpha_r, \beta = \beta_r$ to $\hat{\alpha} = \alpha_{r-\epsilon}, \hat{\beta} = \beta_{r-\epsilon}$ which lie at the boundary, and can be contracted outside $B_\epsilon(C')$.

Finally, take a complex line $L' \subset \mathbb{CP}^2$ intersecting transversally $C'$, but well away from the loops $\alpha_r, \beta_r$ and the homotopies above (e.g. a small perturbation of the line at infinity). Therefore all previous statement happen outside $B_\epsilon(L')$.

**Proposition 27.** We have that the fundamental group $\pi_1(X - (\tilde{C}_1 \cup \ldots \cup \tilde{C}_{11} \cup \tilde{G})) = 1$. In particular, $\pi_1^{\text{orb}}(X') = 1$.

**Proof.** We constructed $C_1, \ldots, C_{11}, G$ inside a plumbing $P = P_c(C \cup L)$, and then we have transferred it to a neighbourhood $P' = P_c(C' \cup L'')$ of a cubic $C'$ and a perturbation $L''$ of a line $L'$ in $\mathbb{CP}^2$. Note that $C' \cup L''$ is smoothly isotopic to $C' \cup L'$. Then we blow-up at the eleven points $P_1, \ldots, P_{11}$ which lie inside $P'$, and took the proper transforms $\tilde{C}_1, \ldots, \tilde{C}_{11}, \tilde{G} \subset P'$, where $P'$ is the blow-up of $P'$. Let $B_\epsilon(\tilde{C}_i), B_\epsilon(\tilde{G}) \subset \tilde{P}'$ be small and disjoint tubular neighbourhoods of $\tilde{C}_i, \tilde{G}$, $i = 1, \ldots, 11$, respectively.

Put $X = W \cup W'$, with

$$W = \bigcup_i B_{2\epsilon}(\tilde{C}_i) \cup B_{2\epsilon}(\tilde{G}) \cup T_0, \quad W' = X - \left( \bigcup_i B_\epsilon(\tilde{C}_i) \cup B_\epsilon(\tilde{G}) \right)$$

Note that $H_1(W) = 0$, and $\tilde{G}$ is carried by $\tilde{C}_1 \cup \ldots \cup \tilde{C}_{11} \cup T_0$. Therefore $\pi_1(W) = 1$. Further, the Loeb's result [12] tells us that $\pi_1(W') = 1$. Therefore $\pi_1(X) = 1$.

Thus, the loops $\hat{\alpha}, \hat{\beta}$ are contractible in $\mathbb{CP}^2 - (B_\epsilon(C' \cup L'))$. □
where $T_0$ denotes an open contractible set constructed by fattening paths joining the base point with the tubular neighbourhoods $B_{2i}(\tilde{C}_i), B_{2i}(\tilde{G})$. As $\pi_1(X)$ is trivial, Seifert-Van-Kampen theorem shows that the map

$$\pi_1(W \cap W') \longrightarrow \pi_1(W') \cong \pi_1(X - (\tilde{C}_1 \cup \ldots \cup \tilde{C}_{11} \cup \tilde{G}))$$

is surjective. Note that $W \cap W'$ is homotopy equivalent to the wedge sum $Y_1 \vee \cdots \vee Y_{11} \vee Y_{12}$, where $Y_i = \partial B_i(\tilde{C}_i)$ is the boundary of a small tubular neighbourhood of $C_i$, and $Y_{12} = \partial B_i(\tilde{G})$. Hence it is enough to see that every loop in $Y_i$ for $1 \leq i \leq 11$ and every loop in $Y_{12}$ are contractible in $\pi_1(X - (\tilde{C}_1 \cup \ldots \cup \tilde{C}_{11} \cup \tilde{G}))$.

Take the plumbing $P = P_0(C \cup L)$ and the curves $C_1, \ldots, C_{11}, G$. We have decomposed $C = D \cup V$, where $D$ is a disc, so we may take $\alpha, \beta$ inside $C - D$. For each of the cubics $C_i, \pi_1(C_i)$ is generated by loops $\alpha_i, \beta_i$ which can be taken by lifting $\alpha, \beta$ via the sections $\tilde{\sigma}_i, i = 1, \ldots, 11$, of the complex line bundle $E \to C$.

For the curve $G \subset P$ of genus 3, we have generators $\alpha^{(1)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(2)}, \alpha^{(3)}, \beta^{(3)}$ of the fundamental group $\pi_1(G)$ with $\prod_{j=1}^3[\alpha^{(j)}, \beta^{(j)}] = 1$. These can be taken by lifting the loops $\alpha, \beta$ via the sections $\tilde{\tau}_j, j = 1, 2, 3$. The base point is also chosen outside the disc $D$. In $P - (C_1 \cup \ldots \cup C_{11} \cup G)$, we can move vertically (along the fiberwise directions of the bundle $E \to C$) all the loops $\alpha_i, \beta_i, \alpha^{(j)}, \beta^{(j)}$ without touching the other curves. Once we reach the boundary of $P \cong P'$, these can be contracted in the complement $\mathbb{C}P^2 - P'$ by Lemma 26 above.

Now we blow-up inside $P$ the eleven points $P_1, \ldots, P_{11}$ to obtain $\tilde{P}$ and the proper transforms $\tilde{C}_1, \ldots, \tilde{C}_{11}, \tilde{G}$. Consider $Y_i = \partial B_i(\tilde{C}_i)$ as before. This is a circle bundle $S^1 \to Y_i = \partial B_i(\tilde{C}_i) \to \tilde{C}_i$ with Chern class $c_1(Y_i) = [\tilde{C}_i]^2 = -1$. We have a short exact sequence

$$0 \to \pi_1(S^1) \to \pi_1(Y_i) \to \pi_1(\tilde{C}_i) \to 0.$$

Since we are away from the blow-up locus we call the generators of $\pi_1(\tilde{C}_i)$ again $\alpha_i, \beta_i$. The loop $[\alpha_i, \beta_i]$ can be homotoped in $B_i(\tilde{C}_i)$ to the base point through an homotopy transversal to $\tilde{C}_i$. This homotopy intersects $\tilde{C}_i$ in $\tilde{C}_i^2 = -1$ points counted with signs. Via the retraction $B_i(\tilde{C}_i) - \tilde{C}_i \to Y_i$, this gives a homotopy in $Y_i$ between the lifting of $[\alpha_i, \beta_i]$ and $\gamma_i^{-1}$, where $\gamma_i$ is the loop going along the fiber $S^1$. We conclude that

$$\pi_1(Y_i) = \langle \alpha_i, \beta_i, \gamma_i \mid [\alpha_i, \beta_i] = \gamma_i^{-1}, \gamma_i \text{ central} \rangle.$$

Note that $\alpha_i, \beta_i$ can be moved to $Y_i$ without touching the other cubics $\tilde{C}_j$ and then contracted in $\mathbb{C}P^2 - P$ via the blow-up map. The conclusion is that $\alpha_i$ and $\beta_i$ can be contracted to a point through an homotopy in $X - (\tilde{C}_1 \cup \ldots \cup \tilde{C}_{11} \cup \tilde{G})$. Therefore the same happens to $\gamma_i$.

Analogously, $Y_{12} = \partial B_i(\tilde{G})$ is a circle bundle $S^1 \to Y_{12} = \partial B_i(\tilde{G}) \to \tilde{G}$ with Chern class $c_1(Y_{12}) = [\tilde{G}]^2 = 1$. Denoting by $\gamma_{12}$ the loop along the fiber $S^1$, we have that

$$\pi_1(Y_{12}) = \langle \alpha^{(1)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(2)}, \alpha^{(3)}, \beta^{(3)}, \gamma_{12} \rangle \prod_{j=1}^3[\alpha^{(j)}, \beta^{(j)}] = \gamma_{12}, \gamma_{12} \text{ central} \rangle.$$
The loops \( \alpha^{(j)}, \beta^{(j)} \) can be moved to the boundary \( Y_{12} \) and then contracted in \( \mathbb{CP}^2 - \mathbb{P} \) via the blow-up map. Thus the same happens to \( \gamma_{12} \). So all generators of \( \pi_1(\partial B_2(\mathcal{C}_1)), i = 1, \ldots, 11 \), and of \( \pi_1(\partial B_2(\mathcal{G})) \), become trivial in \( \pi_1(X - (\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_{11} \cup \mathcal{G})) \). This concludes the proof. \( \square \)

Once we have the symplectic orbifold \( X' \), we construct a Seifert bundle \( M \to X' \) with primitive Chern class \( c_1(M/e^{2\pi i/\mu}) = [\omega] \). This is a K-contact manifold, which is simply-connected.

**Theorem 28.** The 5-manifold \( M \) is simply-connected, hence it is a Smale-Barden manifold.

**Proof.** By [5, Theorem 4.3.18], we have an exact sequence \( \pi_1(S^1) = \mathbb{Z} \to \pi_1(M) \to \pi_\text{forb}(X') = 1 \). In particular, \( \pi_1(M) \) is abelian. Therefore \( \pi_1(M) = H_1(M, \mathbb{Z}) = 0 \), by Theorem 2. \( \square \)

5. Non-existence of an algebraic surface with the given pattern of curves

In this section we show that it is not possible to construct an algebraic surface with the same configuration of complex curves as the manifold we constructed in section 3, that is 12 disjoint complex curves spanning \( H_2(S, \mathbb{Q}) \), one of genus 3 and all the others of genus 1. More concretely, we prove Theorem 9.

**Theorem 29.** Suppose \( S \) is a complex surface with \( b_1 = 0 \) and disjoint smooth complex curves spanning \( H_2(S, \mathbb{Q}) \), one of them of genus \( g \geq 1 \) and all the others elliptic. Then \( b_2 \leq 2g^2 - 4g + 3 \).

**Proof.** Let \( S \) be a complex surface with \( b_1 = 0 \), containing disjoint complex curves spanning \( H_2(S, \mathbb{Q}) \), one of them, say \( D_1 \), of genus \( g \) and the other curves \( D_2, \ldots, D_{b_2} \) all of genus 1.

As \( \{D_1, \ldots, D_{b_2}\} \) is a basis of \( H_2(S, \mathbb{Q}) \), the Poincaré duals \( [D_1], \ldots, [D_{b_2}] \) are a basis of \( H^2(S, \mathbb{Q}) \). Furthermore, these classes are all of type \( (1, 1) \), so we have that \( h^{1,1} = b_2 \) and the geometric genus is \( p_g = h^{2,0} = 0 \). The irregularity is \( q = h^{1,0} = 0 \) since \( b_1 = 0 \). In particular, \( S \) is an algebraic surface [3]. The holomorphic Euler characteristic is

\[
\chi(O_S) = 1 - g + p_g = 1. \tag{9}
\]

By the Riemann-Hodge relations, the signature of \( H^{1,1}(S) \) is \( (1, b_2 - 1) \). Therefore, the self-intersection of one of the \( D_i \)'s is positive and it is negative for the others.

**Case 1.** Assume for the moment that \( g = g(D_1) \geq 2 \). We show first that \( D_1^2 > 0 \). Suppose otherwise that \( D_1^2 > 0 \) for one of the genus 1 curves. After reordering, we can suppose this is true for \( D_2 \). By the adjunction formula, \( K_S \cdot D_2 + D_2^2 = 2g(D_2) - 2 = 0 \), so \( K_S \cdot D_2 = -D_2^2 \). And, by Riemann-Roch, we have,

\[
\chi(D_2) = \chi(O_S) + \frac{D_2^2 - K_S \cdot D_2}{2} = 1 + D_2^2.
\]
Hence using Serre duality,
\[
h^0(D_2) + h^0(K_S - D_2) = h^0(D_2) + h^2(D_2) \geq \chi(D_2) = 1 + D_2^2 \geq 2.
\]
Also, from the exact short sequence \( 0 \to \mathcal{O}_S(K_S - D_2) \to \mathcal{O}_S(K_S) \to \mathcal{O}_{D_2}(K_S|_{D_2}) \to 0 \) we deduce that \( h^0(K_S - D_2) = 0 \), since \( h^0(K_S) = h^{2,0}(S) = 0 \). Thus \( h^0(D_2) \geq 2 \) and we can consider a rational map \( S \dasharrow \mathbb{C}P^1 \) and, after blowing-up the base points of the pencil, an elliptic fibration \( \tilde{S} \to \mathbb{C}P^1 \), with the proper transform of \( D_2 \) as a smooth fiber. However, since they are disjoint, all the proper transforms of \( D_i, i \neq 2 \) have to lie in fibers. In particular, since all the fibers are connected, the arithmetic genus of each irreducible component of a fiber has to be at most 1, which gives rise to a contradiction as \( g(D_1) = g > 1 \). Therefore, \( D_2^2 > 0 \) and \( D_2^2, \ldots, D_{b_2}^2 < 0 \).

Denote \( m_1 = D_1^2 \) and \( m_i = -D_i^2, i = 2, \ldots, b_2 \). All of the \( m_i \)'s are positive integers. And write \( K_S \equiv \sum_{i=1}^{b_2} \lambda_i D_i \) its homology class in \( H_2(S, \mathbb{Q}) \), with \( \lambda_i \in \mathbb{Q} \). Notice that \( K_S \cdot D_i = \lambda_i D_i^2 \), from where \( \lambda_i = \frac{K_S \cdot D_i}{D_i^2} \). And, by the adjunction formula,
\[
\begin{align*}
K_S \cdot D_1 &= 2g(D_1) - 2 - D_2^2 = 2g - 2 - m_1, \\
K_S \cdot D_i &= 2g(D_i) - 2 - D_i^2 = m_i, \quad i \geq 2.
\end{align*}
\]
Therefore
\[
K_S \equiv \frac{2g - 2 - m_1}{m_1} D_1 - \sum_{i=2}^{b_2} D_i,
\]
and we get
\[
K_S^2 = \left(\frac{2g - 2 - m_1}{m_1}\right)^2 - \sum_{i=2}^{b_2} m_i. \tag{11}
\]

Consider the following short exact sequence of sheaves,
\[
0 \to \mathcal{O}(K_S) \to \mathcal{O}(K_S + D_1) \to \mathcal{O}_{D_1}(K_{D_1}) \to 0,
\]
where \( K_{D_1} = (K_S + D_1)|_{D_1} \), by adjunction. This gives a long exact sequence in cohomology,
\[
0 \to H^0(K_S) \to H^0(K_S + D_1) \to H^0(K_{D_1}) \to H^1(K_S) \to \ldots
\]
where \( H^0(K_S) = H^{2,0}(S) = 0 \) and \( H^1(K_S) = H^1(\mathcal{O}_S) = H^{0,1}(S) = 0 \). So we have an isomorphism \( H^0(K_S + D_1) \cong H^0(K_{D_1}) \), and we deduce that \( h^0(K_S + D_1) = h^0(K_{D_1}) = g \). In particular, the linear system \( |K_S + D_1| \) is not empty, and it has dimension \( g - 1 \geq 1 \). Let \( Z = Z(|K_S + D_1|) \) be the fixed part of \( |K_S + D_1| \) (that is, the largest effective divisor such that \( D \geq Z \) for all \( D \in |K_S + D_1| \)). Notice that \( Z \cdot D_1 = 0 \), since the restriction of the linear system to \( D_1, |(K_S + D_1)|_{D_1} = |K_{D_1}|, \) has no fixed points, as \( g \geq 2 \).

Write now \( Z \) as an effective divisor \( Z = \sum_{i=1}^{b_2} \alpha_i D_i + T \) where \( \alpha_i \) are non-negative integers and \( T \) is an effective divisor not containing any of the \( D_i \)'s. Notice that the latter implies \( T \cdot D_i \geq 0 \), for all \( i \). Since \( 0 = Z \cdot D_1 = \alpha_1 m_1 + T \cdot D_1 \), we have
\(\alpha_1 = 0\) and \(T \cdot D_1 = 0\). So we can write \(Z = \sum_{i=2}^{b_2} \alpha_i D_i + T\), and \(T\) does not intersect \(D_1\).

Let us see that \(T = 0\). Write \(T \equiv \sum_{i=1}^{b_2} \mu_i D_i\) its homology class in \(H_2(S, \mathbb{Q})\), with \(\mu_i \in \mathbb{Q}\). First note that, since \(T \cdot D_1 = 0\), it is \(\mu_1 = 0\), so \(T \equiv \sum_{i=2}^{b_2} \mu_i D_i\).

For \(i \geq 2\), \(0 \leq T \cdot D_i = -\mu_i m_i\), hence \(\mu_i \leq 0\). Let \(n \geq 1\) be an integer such that \(n\mu_i \in \mathbb{Z}\) for all \(i\). Hence \(nT\) is effective and \(-nT = \sum (-n\mu_i) D_i\) is also effective. This implies that \(nT = 0\) and thus \(T = 0\). This means that the fixed part is \(Z = \sum_{i=2}^{b_2} \alpha_i D_i\).

Write \(|K_S + D_1| = Z + |F|\), where \(F\) is the free part, which is a fully movable divisor. We look now at the self-intersection \(F^2 = (K_S + D_1 - Z)^2 \geq 0\). Recall that the self-intersection of a fully movable divisor is \(F^2 \geq 0\), since taking a different \(F' \equiv F\), such that \(F\) and \(F'\) do not share components, then \(F^2 = F \cdot F' \geq 0\).

Let \(j \geq 2\) and suppose both that \(m_j = 1\) and \(D_j \not\subseteq Z\). In this case, the restriction of an effective divisor \(C \in |K_S + D_1|\) to \(D_j\) is an effective degree 1 divisor, since \((K_S + D_1) \cdot D_j = K_S \cdot D_j = m_j = 1\), by (10). So \(C \cap D_j\) is a point \(P_j\). Furthermore, since \(D_j\) is not a rational curve, any pair of linearly equivalent points are actually equal. Therefore \(P_j \in D_j\) is a fixed point of \(|K_S + D_1|\) and, since \(Z \cap D_j = \emptyset\), \(P_j \in F\). So \(P_j\) is counted in the self-intersection \((K_S + D_1 - Z)^2\).

There are at most \(\sum_{i=2}^{b_2} m_i - (b_2 - 1)\) curves among \(D_2, \ldots, D_{b_2}\) with \(m_i > 1\). So there are at least \(2(b_2 - 1) - \sum_{i=2}^{b_2} m_i\) curves with self-intersection \(-1\). Hence there are at least \(2(b_2 - 1) - \sum_{i=2}^{b_2} m_i - r\) fixed points \(P_j \in D_j\) of some \(|(K_S + D_1 - Z)|_{D_j}|\), where \(r = \#\{\alpha_i > 0\}\), and thus

\[(K_S + D_1 - Z)^2 \geq 2(b_2 - 1) - \sum_{i=2}^{b_2} m_i - r.\] (12)

Note that in case we have \(2(b_2 - 1) - \sum_{i=2}^{b_2} m_i - r \leq 0\) we cannot assure the existence of any fixed point but the inequality still holds since \((K_S + D_1 - Z)^2 \geq 0\).

We now compute \((K_S + D_1 - Z)^2\),

\[
(K_S + D_1)^2 = K_S^2 + 2K_S \cdot D_1 + D_1^2 = K_S^2 + 2(2g - 2 - m_1) + m_1 =
\]

\[
= K_S^2 + 4g - 4 - m_1,
\]

\[
(K_S + D_1 - Z)^2 = (K_S + D_1)^2 - 2(K_S + D_1) \cdot Z + Z^2 =
\]

\[
= K_S^2 + 4g - 4 - m_1 - 2 \sum_{i=2}^{b_2} \alpha_i m_i - \sum_{i=2}^{b_2} \alpha_i^2 m_i.
\]

Thus (12) gives

\[
K_S^2 + 4g - 4 - m_1 - 2 \sum_{i=2}^{b_2} \alpha_i m_i - \sum_{i=2}^{b_2} \alpha_i^2 m_i \geq 2(b_2 - 1) - \sum_{i=2}^{b_2} m_i - r,
\]
from where
\[
2b_2 \leq 4g - 2 - m_1 + K_S^2 + \sum_{i=2}^{b_2} m_i + r - 2 \sum_{i=2}^{b_2} \alpha_i m_i - \sum_{i=2}^{b_2} \alpha_i^2 m_i
\]
\[
\leq 4g - 2 - m_1 + K_S^2 + \sum_{i=2}^{b_2} m_i + r - 3r \leq 4g - 2 - m_1 + K_S^2 + \sum_{i=2}^{b_2} m_i.
\]

Using (11), we have
\[
2b_2 \leq 4g - 2 - m_1 + \frac{(2g - 2 - m_1)^2}{m_1}.
\]
The expression on the right is a decreasing function on \(m_1\). Therefore, we can bound it by its value in \(m_1 = 1\), that is
\[
2b_2 \leq 4g - 3 + (2g - 3)^2 = 4g^2 - 8g + 6.
\]
Hence \(b_2 \leq 2g^2 - 4g + 3\), as required.

**Case 2.** Suppose now \(g = 1\). Using the adjunction formula, we get \(K_S \equiv -\sum_{i=1}^{b_2} D_i\). And using the same argument as above, we have that \(h^0(K_S + D_1) = h^0(K_{D_1}) = g = 1\). Thus, there is an effective divisor in \(S\) linearly equivalent to \(K_S + D_1 = -\sum_{i=2}^{b_2} D_i\) which is clearly anti-effective if \(b_2 \geq 2\). Therefore, \(b_2 \leq 1\). \(\Box\)

Let us end up by giving a different proof of the non-existence of a Kähler surface \(S\) with \(b_1 = 0\) and \(b_2 = 12\), containing disjoint smooth complex curves spanning \(H_2(S, \mathbb{Q})\), one of them of genus \(g = 3\), all the others of genus \(g_i = 1\). It makes very specific use of the numbers at hand.

We follow the notations in the proof of Theorem 29. We have the curves \(D_1, D_2, \ldots, D_b, b = 12\), with \(D_1^2 = m_1, D_i^2 = -m_i, 2 \leq i \leq b\), and all \(m_i\)’s are positive integers. The curve \(D_1\) has genus \(g = 3\) and \(D_i\) have genus 1, \(2 \leq i \leq b\). By (9) and Noether’s formula [3] we have that
\[
\frac{1}{12}(K_S^2 + c_2(S)) = \chi(\mathcal{O}_S) = 1 - q + p_g = 1.
\]
Note that \(c_2(S) = \chi(S) = 2 + b\), where \(b = b_2 = 12\) and \(b_1 = b_3 = 0\). Therefore \(K_S^2 = 10 - b = -2\). Now (11) says that
\[-2 = K_S^2 = \frac{(4 - m_1)^2}{m_1} - m_2 - \ldots - m_b \leq \frac{(4 - m_1)^2}{m_1} - 11,
\]
using that \(g = 3\). Therefore \((4 - m_1)^2 \geq 9m_1\), which is rewritten as \((m_1 - 16)(m_1 - 1) \geq 0\).

If \(m_1 \geq 16\), then the curve \(D_1\) of genus \(g = 3\) has self-intersection \(D_1^2 \geq 2g + 1\). The argument of [15, Theorem 32] concludes that \(b \leq 2g + 3\). This is a contradiction since \(g = 3\) and \(b = 12\).
Therefore we have that \( m_1 = 1 \). So
\[
K_S = 3D_1 - D_2 - \ldots - D_b
\]
and \( K_S^2 = -2 = 9 - m_2 - \ldots - m_b \leq 9 - 11 = -2 \). Therefore there must be equality and \( m_2 = \ldots = m_b = 1 \). The basis \( \{D_1, D_2, \ldots, D_b\} \) is a diagonal basis of \( H_2(S, \mathbb{Z}) \). Now we try to reconstruct \( S \) in “reverse”. Let \( H, E_2, \ldots, E_b \in H_2(S, \mathbb{Z}) \) be defined by the equalities:
\[
D_1 = 10H - 3E_2 - \ldots - 3E_b, \\
D_j = (3H - E_2 - \ldots - E_b) + E_j, \quad j = 2, \ldots, b.
\]
This is solved as:
\[
H = 10D_1 - 3D_2 - \ldots - 3D_b, \\
E_j = 3D_1 + D_j - \sum_{k=2}^b D_k, \quad j = 2, \ldots, b, \\
K_S = -3H + \sum_{k=2}^b E_k.
\]
The following self-intersections are easily computed:
\[
H^2 = 1, \quad H \cdot E_j = 0, \quad j = 2, \ldots, b, \\
E_j^2 = -1, \quad E_j \cdot E_k = 0, \quad j \neq k, \\
K_S \cdot H = -3, \quad K_S \cdot E_j = -1, \quad j = 2, \ldots, b, \\
D_j \cdot E_j = 0, \quad D_j \cdot E_k = 1, \quad j \neq k.
\]
Now let us prove that the classes \( H, E_2, \ldots, E_b \) are defined by effective divisors. First \( \chi(H) = 1 + \frac{H^2 - K_S \cdot H}{2} = 3 \). Also \( h^0(K_S - H) = 0 \) since \( K_S - H \leq K_S \) and \( h^0(K_S) = 0 \). Thus it must be \( h^0(H) \geq 3 \) and \( H \) is effective. Next \( \chi(E_j) = 1 + \frac{1}{2}(E_j^2 - K_S \cdot E_j) = 1 \). Also \( h^0(K_S - E_j) = 0 \) since \( (K_S - E_j) = -D_j < 0 \). Therefore \( h^0(E_j) \geq 1 \) and \( E_j \) is also effective.

Next note that \( K_S + D_j = E_j \). Consider the long exact sequence in cohomology associated to the exact sequence
\[
0 \rightarrow \mathcal{O}(K_S) \rightarrow \mathcal{O}(K_S + D_j) \rightarrow \mathcal{O}_{D_j}(K_{D_j}) \rightarrow 0.
\]
As \( H^0(K_S) = H^1(K_S) = 0 \), we have that \( h^0(E_j) = h^0(K_S + D_j) = h^0(\mathcal{O}_{D_j}(K_{D_j})) = 1 \), since \( D_j \) is an elliptic curve. Also \( h^2(E_j) = h^0(K_S - E_j) = 0 \), and hence \( h^1(E_j) = 0 \) since \( \chi(E_j) = 1 \).

Consider now the exact sequence
\[
0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(D_2 + \ldots + D_b) \rightarrow \bigoplus_{j=2}^b \mathcal{O}_{D_j}(D_j) \rightarrow 0, \tag{13}
\]
which holds since \( D_j \) are disjoint. As \( D_j^2 = -m_j = -1 \), and \( D_j \) is an elliptic curve, we have \( h^0(\mathcal{O}_{D_j}(D_j)) = 0 \) and \( h^1(\mathcal{O}_{D_j}(D_j)) = 1 \). Therefore (13) and the fact that
$h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$ implies that $h^0(D_2 + \ldots + D_b) = 1$ and $h^1(D_2 + \ldots + D_b) = b - 1 = 11$. Using that $3D_1 \equiv D_2 + \ldots + D_{b-1} + E_b$, we have an exact sequence

$$0 \to \mathcal{O}(E_b) \to \mathcal{O}(3D_1) \to \bigoplus_{j=2}^{b-1} \mathcal{O}_{D_j}(E_b) \to 0.$$ 

As $D_j \cdot E_b = 1$ and $D_j$ is elliptic, we have $h^0(\mathcal{O}_{D_j}(E_b)) = 1$. Then $h^0(3D_1) = b - 1 = 11$, using $h^0(E_b) = 1$ and $h^1(E_b) = 0$ computed before.

Now we compute $h^0(3D_1)$ in a different way. We have exact sequences:

$$0 \to \mathcal{O} \to \mathcal{O}(D_1) \to \mathcal{O}_{D_1}(D_1) \to 0$$
$$0 \to \mathcal{O}(D_1) \to \mathcal{O}(2D_1) \to \mathcal{O}_{D_1}(2D_1) \to 0$$
$$0 \to \mathcal{O}(2D_1) \to \mathcal{O}(3D_1) \to \mathcal{O}_{D_1}(3D_1) \to 0$$

so

$$h^0(3D_1) \leq h^0(2D_1) + h^0(\mathcal{O}_{D_1}(3D_1))$$
$$\leq h^0(D_1) + h^0(\mathcal{O}_{D_1}(2D_1)) + h^0(\mathcal{O}_{D_1}(3D_1))$$
$$\leq 1 + h^0(\mathcal{O}_{D_1}(D_1)) + h^0(\mathcal{O}_{D_1}(2D_1)) + h^0(\mathcal{O}_{D_1}(3D_1)).$$

(14)

We use Clifford’s theorem [2, p. 107] that says that for a curve of genus $g \geq 1$ and a divisor $D$ of degree $0 \leq d \leq 2g - 2$, we have $h^0(D) \leq \left\lfloor \frac{d^2}{2g} \right\rfloor + 1$. Applying this to the curve $D_1$, we have $h^0(\mathcal{O}_{D_1}(D_1)) \leq 1$, $h^0(\mathcal{O}_{D_1}(2D_1)) \leq 2$, and $h^0(\mathcal{O}_{D_1}(3D_1)) \leq 2$, recalling that $D_1^2 = 1$. Therefore (14) says that it $h^0(3D_1) \leq 1 + 1 + 2 + 2 = 6$. This is a contradiction with the previous computation of $h^0(3D_1)$.

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