A new two parameter lifetime distribution: model and properties

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Abstract
In this paper a new lifetime distribution which is obtained by compounding Lindley and geometric distributions, named Lindley-geometric (LG) distribution, is introduced. Several properties of the new distribution such as density, failure rate, mean lifetime, moments, and order statistics are derived. Furthermore, estimation by maximum likelihood and inference for large sample are discussed. The paper is motivated by two applications to real data sets and we hope that this model be able to attract wider applicability in survival and reliability.

Keywords: Bathtub failure rate, EM algorithm, Geometric distribution, Lindley distribution, Maximum likelihood estimation, Unimodal failure rate.

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1. Introduction

The Lindley distribution specified by the probability density function (p.d.f.)

\[ f(x) = \frac{\theta^2}{\theta + 1} (1 + x) e^{-\theta x}, \quad x > 0, \quad \theta > 0, \]  \hspace{1cm} (1)

was introduced by Lindley [13]. The corresponding cumulative distribution function (c.d.f.) is given by

\[ F(x) = 1 - (1 + \frac{\theta x}{\theta + 1}) e^{-\theta x}, \quad x > 0, \quad \theta > 0. \]  \hspace{1cm} (2)

The Lindley distribution, in spite of little attention in the statistical literature, is important for studying stress-strength reliability modeling. Besides, some researchers have proposed new classes of distributions based on modifications of the Lindley distribution, including also their properties. Sankaran [23] introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany et al. [9] investigated most of the statistical
properties of the Lindley distribution, showing this distribution may provide a better fitting than the exponential distribution. Mahmoudi and Zakerzadeh [15] proposed an extended version of the compound Poisson distribution which was obtained by compounding the Poisson distribution with the generalized Lindley distribution which is obtained and analyzed by Zakerzadeh and Dolati [25]. Recently a new extension of the Lindley distribution, called extended Lindley (EL) distribution, which offers a more flexible model for lifetime data is introduced by Bakouch et al. [3].

Adamidis and Loukas [1] introduced a two-parameter lifetime distribution with decreasing failure rate by compounding exponential and geometric distributions, which was named exponential geometric (EG) distribution. In the same way, Kus [12] and Tahmasbi and Rezaei [24] introduced the exponential Poisson (EP) and exponential logarithmic distributions, respectively. Marshall and Olkin [19] presented a method for adding a parameter to a family of distributions with application to the exponential and Weibull families.

Recently, Chahkandi and Ganjali [8] introduced a class of distributions, named exponential power series (EPS) distributions, by compounding exponential and power series distributions, where compounding procedure follows the same way that was previously carried out by Adamidis and Loukas [1]; this class contains the distributions mentioned before. Extensions of the EG distribution was given by Adamidis et al. [2] and Barreto-Souza et al. [6], where the last was obtained by compounding Weibull and geometric distributions. A three-parameter extension of the EP distribution was obtained by Barreto-Souza and Cribari-Neto [5].

This new class of distributions has been received considerable attention over the two last years. Weibull power series (WPS), complementary exponential geometric (CEG), two-parameter Poisson-exponential, generalized exponential power series (GEP), exponentiated Weibull-Poisson (EWP) and generalized inverse Weibull-Poisson (GIWP) distributions were introduced and studied by Morais and Barreto-Souza [20], Louzada-Neto et al. [14], Cancho et al. [7], Mahmoudi and Jafari [16], Mahmoudi and Sepahdar [17] and Mahmoudi and Torki [18].

In this paper, we introduce a new lifetime distribution by compounding Lindley and geometric distributions as follows: Consider the random variable $X$ having the Lindley distribution where its pdf and cdf are given in (1) and (2).

Given $N$, let $X_1, \cdots, X_N$ be independent and identically distributed (iid) random variables from Lindley distribution. Let the random variable $N$ is distributed according to the geometric
distribution with pdf

\[ P(N = n) = (1 - p)p^{n-1}, \quad n = 1, 2, \ldots, 0 < p < 1. \]

Let \( Y = \min(X_1, \ldots, X_N) \), then the conditional cdf of \( Y|N = n \) is given by

\[ F_{Y|N}(y|n) = 1 - \left[ (1 + \frac{\theta y}{\theta + 1})e^{-\theta y} \right]^n, \tag{3} \]

The Lindley-geometric (LG) distribution, denoted by \( \text{LG}(p, \theta) \), is defined by the marginal cdf of \( Y \), i.e.,

\[ F_Y(y) = \frac{1 - (1 + \frac{\theta y}{\theta + 1})e^{-\theta y}}{1 - p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y}}, \quad y > 0, \quad \theta > 0, \quad 0 < p < 1. \tag{4} \]

The paper is organized as follows. In Section 2, the density function, survival and hazard rate functions of the LG with some of their properties are given. Section 3 provides a general expansion for the quantiles and moments of the LG distribution. Its moment generating function is derived in this section. Section 4 provides the moments of order statistics of the LG distribution. Residual life and reversed residual functions of the LG distribution is discussed in Section 5. Section 6 is devoted to the Bonferroni and Lorenz curves of the LG distribution. In Section 7 we explain the probability weighted moments. Mean deviations from the mean and median are derived in Section 8. Estimation of the parameters by maximum likelihood via an EM-algorithm and inference for large sample are presented in Section 9. Applications to two real data sets are given in Section 10 and conclusions are provided in Section 11.

2. Density function, survival and hazard rate functions

The probability density function of the LG distribution is given by

\[ f(y) = \frac{\theta^2}{\theta + 1}(1-p)(1+y)e^{-\theta y} \left[ 1 - p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y} \right]^{-2}, \quad y > 0, \tag{5} \]

where \( \theta > 0 \) and \( 0 < p < 1 \).

Even when \( p \leq 0 \), Equation (5) is a density function. We can then define the LG distribution by Equation (5) for any \( p < 1 \). Some special sub-models of the LG distribution (5) are obtained as follows. If \( p = 0 \), we have the Lindley distribution. When \( p \rightarrow 1^- \), the LG distribution tends to a distribution degenerate in zero. Hence, the parameter \( p \) can be interpreted as a concentration parameter. LG density functions are displayed in Figure 1 for selected values of \( \theta \) and \( p = -2, \ -0.05, \ 0, \ 0.05, \ 0.09. \)
Theorem 1. The density function of the LG distribution is (i) decreasing for all values \( p \) and \( \theta \) for which \( p > \frac{1-\theta^2}{1+\theta^2} \), (ii) unimodal for all values \( p \) and \( \theta \) for which \( p \leq \frac{1-\theta^2}{1+\theta^2} \).

Proof. See Appendix.

The survival function and hazard rate function of the LG distribution, are given respectively by

\[
S(y) = \frac{(1 - p)(1 + \frac{\theta y}{\theta + 1})e^{-\theta y}}{1 - p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y}},
\]

and

\[
h(y) = \frac{\theta^2(y + 1)}{\theta y + \theta + 1} \left[ 1 - p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y} \right]^{-1}.
\]

Figure 2 provides the plots of the hazard rate function of the LG distribution for different values \( p = -2, -0.5, 0, 0.5, 0.9 \) and \( \theta = 0.03, 0.08, 1, 3 \). We have the following results regarding the shapes of the hazard rate function of the LG distribution. The proof is provided in the Appendix.

Theorem 2. The hazard function of the LG distribution in (7) is (i) bathtub-shaped if \( p > \frac{1}{1+\theta^2} \), (ii) firstly increasing then bathtub-shaped if \( p \leq \frac{1}{1+\theta^2} \).

Proposition 1. The hazard rate function of the LG distribution in (7) tends to \( \frac{\theta^2}{(\theta+1)(1-p)} \) and \( \theta \) where \( y \to 0 \) and \( y \to \infty \), respectively.

Using the series expansion

\[
(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k) j!} z^j,
\]

where \( |z| < 1 \) and \( k > 0 \), the density function [5] can be demonstrated by

\[
f_{LG}(y; p, \theta) = \frac{\theta^2}{\theta+1}(1-p)(1+y)e^{-\theta y} \sum_{j=0}^{\infty} (j+1)p^j(1 + \frac{\theta y}{\theta + 1})^j e^{-j\theta y}.
\]

Various mathematical properties of the LG distribution can be obtained from [9] and the corresponding properties of the Lindely distribution.

In the following theorem, we give the stochastically ordering property of the random variable \( Y \) with LG distribution.

Theorem 3. Consider the two random variables \( Y_1 \) and \( Y_2 \) with LG\((p_1, \theta)\) and LG\((p_2, \theta)\) distributions, respectively.

(i) If \( p_1 \leq p_2 \), then \( S_{Y_1}(t) \leq S_{Y_2}(t) \) (\( Y_1 \leq Y_2 \)) and \( h_{Y_1}(t) \leq h_{Y_2}(t) \) (\( Y_1 \leq Y_2 \)).

(ii) If \( p_1 > p_2 \), then \( \frac{f_{Y_1}(t)}{f_{Y_2}(t)} \) is decreasing in \( t \), i.e., \( Y_1 \leq Y_2 \).
3. Quantiles and moments of the LG distribution

Applying the equation \( F(x_{\xi}) = \xi \), the \( \xi \)th quantile of the LG distribution is the solution of equation

\[
\frac{1 - \xi}{1 - p_{\xi}} = \left(1 + \frac{\theta x_{\xi}}{\theta + 1}\right)e^{-\theta x_{\xi}},
\]

which is used for data generation from the LG distribution.

Suppose that \( Y \sim LG(p, \theta) \), using the equation (10) and applying the binomial expression for \( (1 + \frac{\theta y}{\theta + 1})^i \), the \( r \)th moment of \( Y \) is given by

\[
E(Y^r) = \frac{\theta^2(1-p)}{\theta + 1} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i}(j + 1)p^i(\frac{\theta}{\theta + 1})^i \frac{\Gamma(r + i + 1)}{(\theta(j + 1))^{r+i+1}}(1 + \frac{r + i + 1}{\theta(j + 1)}).
\]

Using Eq. (10), the moment generating function of the LG distribution is given by

\[
M_Y(t) = \sum_{k=0}^{\infty} \frac{\theta^2(1-p)}{\theta + 1} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i}(j + 1)p^i(\frac{\theta}{\theta + 1})^i \frac{\Gamma(k+i+1)}{(\theta(j + 1))^{k+i+1}}(1 + \frac{k+i+1}{\theta(j + 1)}).
\]

Proposition 2. The mean of the LG distribution is given by

\[
E(Y) = \frac{\theta^2(1-p)}{\theta + 1} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{(j + 1)!}{(j - i)!}p^i(\frac{\theta}{\theta + 1})^i \frac{i + 1}{\theta(j + 1)} + \frac{i + 2}{\theta(j + 1)}.
\]

4. Order statistics and their moments

Order statistics make their appearance in many areas of statistical theory and practice. Order statistics are among the most fundamental tools in non-parametric statistics and inference and play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures.

Let \( Y_1, \cdots, Y_n \) be a random sample taken from the LG distribution and \( Y_{1:n}, \cdots, Y_{n:n} \) denote the corresponding order statistics. Then, the pdf \( f_{r:n}(y) \) of the \( r \)th order statistics \( Y_{r:n} \) is given by

\[
f_{r:n}(y) = \frac{1}{B(r, n-r+1) \theta + 1} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \binom{r}{l} \frac{\Gamma(n-r+l)}{n} \left(1 + \frac{\theta y}{\theta + 1}\right)^{n-r+i+j+l} e^{-(n-r+i+j+l)\theta y}.
\]
where $Be(a, b) = \int_0^1 w^{a-1}(1 - w)^{b-1} dw$ is the beta function. After some calculations and using the binomial expression for $(1 + \frac{\theta y}{\theta + 1})^{-r-i+j+l}$, we have

$$f_{r:n}(y) = \frac{1}{Be(r, \theta + 1)} \frac{\theta^2}{\theta + 1} (1 - p)(1 + y)e^{-\theta y} \sum_{i=0}^\infty \sum_{t=0}^r \sum_{m=0}^\infty \frac{(-1)^i (\theta y)^m}{\theta + 1} e^{-(n-r+i+j+l)\theta y}.$$  

The $k$th moment of the $r$th order statistic $Y_{r:n}$ can be obtained from the known result,

$$E[Y_{r:n}^k] = \frac{1}{Be(r, \theta + 1)} \frac{\theta^2}{\theta + 1} (1 - p) \sum_{j=0}^\infty \sum_{i=0}^r \sum_{m=0}^\infty \frac{(j+1)p^j+l(1-p)^{n-r+i} \left( \frac{\theta y}{\theta + 1} \right)^m e^{-(n-r+i+j+l)\theta y}}{\theta^{(n-r+i+j+l+1)}k+m+r} \frac{\Gamma(k+m+2)}{\theta^{(n-r+i+j+l+1)}k+m+r}.$$  

5. Residual life and reversed failure rate function of the LG distribution

Given that a component survives up to time $t > 0$, the residual life is the period beyond $t$ until the time of failure and defined by the conditional random variable $Y - t | Y > t$. In reliability, it is well known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely (Gupta and Gupta, [10]). Therefore, we obtain the $r$th-order moment of the residual life via the general formula

$$\mu_r(t) = E[(Y - t)^r | Y > t] = \frac{1}{S(t)} \int_t^\infty (y - t)^r f(y) dy,$$

where $S(t) = 1 - F(t)$, is the survival function. Applying the binomial expansion to $(y - t)^r$ into the above formula gives

$$\mu_r(t) = \frac{(1-p)p^2}{(\theta + 1)S(t)} \sum_{i=0}^r \sum_{j=0}^\infty \sum_{k=0}^j (-1)^i \binom{j}{i} (j+1)t^i p^j \left( \frac{\theta}{\theta + 1} \right)^k \frac{1}{\theta^{(j+1)}k+i+2} \Gamma(r + k - i + 2; \theta t(j+1)) + \theta(j+1) \Gamma(r + k - i + 2; \theta t(j+1)),$$

where $\Gamma(s; t) = \int_t^\infty x^{s-1} e^{-x} dx$, shows the upper incomplete gamma function.

The mean residual life (MRL) of the LG distribution is given by

$$\mu(t) = \frac{(1-p)p^2}{(\theta + 1)S(t)} \sum_{j=0}^\infty \sum_{k=0}^j \binom{j}{i} (j+1) p^j \left( \frac{\theta}{\theta + 1} \right)^k \frac{1}{\theta^{(j+1)}k+i+2} \Gamma(r + k + 3; \theta t(j+1)) + \theta(j+1) \Gamma(r + k + 2; \theta t(j+1)).$$

In particular, we obtain

$$\mu(0) = E(Y) = \frac{\theta^2(1-p)}{\theta + 1} \sum_{j=0}^\infty \sum_{i=0}^{j+1} \frac{(j+1)!}{(j-i)!} p^i \left( \frac{\theta}{\theta + 1} \right)^i \frac{1}{\theta^{(j+1)}i+2} (1 + \frac{i+2}{\theta(j+1)}),$$
Also, if \( p = 0 \), then
\[
\mu(t) = \frac{2 + \theta + \theta t}{\theta (1 + \theta + \theta t)},
\]
which is the MRL function of the original Lindley distribution. The variance of the residual life of the LG distribution can be obtained easily by using \( \mu_2(t) \) and \( \mu(t) \).

The reversed residual life can be defined as the conditional random variable \( t - Y | Y \leq t \) which denotes the time elapsed from the failure of a component given that its life is less than or equal to \( t \). This random variable may also be called the inactivity time (or time since failure); for more details one can see (Kundu and Nanda, [11]; Nanda et al., [21]). Also, in reliability, the mean reversed residual life and ratio of two consecutive moments of reversed residual life characterize the distribution uniquely. Using (4) and (5), the reversed failure (or reversed hazard) rate function is given by
\[
r(y) = \frac{f(y)}{F(y)} = \frac{\theta^2 (1 - p)(1 + y)e^{-\theta y}}{[1 - p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y}][1 - (1 + \frac{\theta y}{\theta + 1})e^{-\theta y}]}, \quad y > 0.
\]
(15)

It is noticed that \( h(0) = \infty \) and \( h(0) \) is discontinuous in the parameters of the LG distribution.

The \( r \)th-order moment of the reversed residual life can be obtained by the well known formula
\[
m_r(t) = E[(t - Y)^r | Y \leq t] = \frac{1}{F(t)} \int_0^t (t - y)^r f(y) dy,
\]
hence,
\[
m_r(t) = \frac{(1-p)\theta^2}{(\theta+1)F(t)} \sum_{i=0}^r \sum_{j=0}^\infty \sum_{k=0}^j (-1)^{r+i} \binom{j}{k} (j + 1) t^i p^j \left( \frac{\theta}{\theta + 1} \right)^k \frac{1}{(\theta(j+1))^{r+k+\sigma + 2}}
\times \left( \gamma(r + k - i + 2; \theta t(j + 1)) + \theta(j + 1) \gamma(r + k - i + 1; \theta t(j + 1)) \right), \quad r \geq 1,
\]
(16)

where \( \gamma(s; t) = \int_0^t x^{s-1} e^{-x} dx \), shows the lower incomplete gamma function. Thus, the mean of the reversed residual life of the LG distribution is given by
\[
m(t) = t - \frac{(1-p)\theta^2}{(\theta+1)S(t)} \sum_{i=0}^\infty \sum_{k=0}^j \binom{j}{k} (j + 1) t^i p^j \left( \frac{\theta}{\theta + 1} \right)^k \frac{1}{(\theta(j+1))^{k+r+\sigma + 3}}
\times \left( \gamma(k + 3; \theta t(j + 1)) + \theta(j + 1) \gamma(k + 2; \theta t(j + 1)) \right).
\]
(17)

Using \( m(t) \) and \( m_2(t) \) one can obtain the variance and the coefficient of variation of the reversed residual life of the LG distribution.
6. Bonferroni and Lorenz curves of the LG distribution

The Bonferroni and Lorenz curves and Gini index have many applications not only in economics to study income and poverty, but also in other fields like reliability, medicine and insurance. The Bonferroni curve \( B_{F}[F(y)] \) is given by

\[
B_{F}[F(y)] = \frac{1}{\mu F(y)} \int_{0}^{y} u f(u) du.
\]

Using this fact that

\[
I(y) = \int_{0}^{y} u f(u) du = \frac{(1-p)\theta^2}{\mu(\theta+1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( \begin{array}{c} j + 1 \\ k \end{array} \right) p^j \left( \frac{\theta}{\theta+1} \right)^k \left( 1 - \frac{1}{(\theta(j+1))^{k+\gamma}} \right) \left( \gamma(k + 3; \theta y(j + 1)) + \theta(j + 1)\gamma(k + 2; \theta y(j + 1)) \right),
\]

the Bonferroni curve of the distribution function \( F \) of LG distribution is given by

\[
B_{F}[F(y)] = \frac{(1-p)\theta^2[1-p(1-\frac{\theta y}{\theta+1})e^{-\theta y}]}{\mu(\theta+1)[1-(1+\frac{\theta y}{\theta+1})e^{-\theta y}]} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( \begin{array}{c} j + 1 \\ k \end{array} \right) p^j \left( \frac{\theta}{\theta+1} \right)^k \left( 1 - \frac{1}{(\theta(j+1))^{k+\gamma}} \right) \left( \gamma(k + 3; \theta y(j + 1)) + \theta(j + 1)\gamma(k + 2; \theta y(j + 1)) \right),
\]

where \( \mu \) (the mean of LG distribution) is given in (11).

Also, the Lorenz curve of \( F \) that follows the LG distribution can be obtained via the expression \( L_{F}[F(y)] = B_{F}[F(y)]F(y) \). The scaled total time and cumulative total time on test transform of a distribution function \( F \) (Pundir et al., 22) are defined by

\[
S_{F}[F(t)] = \frac{1}{\mu} \int_{0}^{t} S(u) du,
\]

and

\[
C_{F} = \int_{0}^{1} S_{F}[F(t)] f(t) dt,
\]

respectively, where \( S(.) \) denotes the survival function. If \( F(t) \) denotes the LG distribution function specified by (11) then we have,

\[
S_{F}[F(t)] = \frac{1-p}{\mu} \sum_{j=0}^{\infty} \sum_{k=0}^{j+1} \left( j + 1 \atop k \right) p^j \left( \frac{\theta}{\theta+1} \right)^k (\theta(j + 1))^{-(k+1)} \gamma(k + 1; \theta t(j + 1)).
\]

The Gini index can be obtained from the relationship \( G = 1 - C_{F} \).
7. Probability weighted moments

The probability weighted moments (PWMs) method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. We calculate the PWMs of the LG distribution since they can be used to obtain the moments of the LG distribution. The PWMs of a random variable \( Y \) are formally defined by

\[
\tau_{s,r} = E[Y^s F(Y)^r] = \int_0^\infty y^s F(y)^r f(y) dy,
\]

where \( r \) and \( s \) are positive integers and \( F(.) \) and \( f(.) \) are the cdf and pdf of the random variable \( Y \). The PWMs of the LG distribution are given in the following proposition.

**Proposition 3.** The PWMs of the LG distribution with cdf (4) and pdf (5), are given by

\[
\tau_{s,r} = \theta_m + 2(1-p)(\theta + 1)^{m+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{r} \sum_{m=0}^{\infty} (-1)^l \binom{k+j+l}{l} \binom{r+k-1}{k} (j+1)p^{j+k} 
\]

\[
\times \left[ \frac{\Gamma(m+s+1)}{(\theta(k+j+l))^{m+s+1}} - \frac{\Gamma(m+s+2)}{(\theta(k+j+l))^{m+s+2}} \right].
\]

**Proposition 4.** The \( s \)th moment of the LG distribution can be obtained putting \( r = 0 \) in Eq. (21). Also, the mean and variance of the LG distribution can be obtained.

8. Mean deviations

The amount of scatter in a population can be measured by the totality of deviations from the mean and median. For a random variable \( X \) with pdf \( f(.) \), cdf \( F(.) \), mean \( \mu = E(X) \) and \( M = Median(X) \), the mean deviation about the mean and the mean deviation about the median, are defined respectively by

\[
\delta_1(X) = \int_0^\infty |x - \mu| f(x) dx = 2\mu F(\mu) - 2I(\mu),
\]

and

\[
\delta_2(X) = \int_0^\infty |x - M| f(x) dx = \mu - 2I(M),
\]

where \( I(b) = \int_0^b xf(x) dx \).

For the LG distribution we have

\[
I(b) = \frac{(1-p)p^2}{(\theta+1)^2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{j}{k} (j+1)p^j \left( \frac{\theta}{\theta+1} \right)^k \left[ \frac{1}{(\theta(j+1))^{k+3}} \right] 
\]

\[
\times \left( \gamma(k+3; \theta b(j+1)) + \theta(j+1)\gamma(k+2; \theta b(j+1)) \right).
\]

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Theorem 4. The Mean deviations of the LG distribution are given by

\[ \delta_1 = 2\mu \frac{1 - (1 + \frac{\theta\mu}{\theta+1})e^{-\theta\mu}}{1 - p(1 + \frac{\theta\mu}{\theta+1})e^{-\theta\mu}} - 2I(\mu), \]

and

\[ \delta_2 = \mu - 2I(M), \]

respectively, where \( \mu \) is the mean of LG in Eq. (11), \( I(\mu) \) and \( I(M) \) are obtained by substituting \( \mu \) and \( M \) in Eq. (22).

9. Estimation and inference

The estimation of the parameters of the LG distribution using the maximum likelihood estimation is studied in this section. Let \( Y_1, Y_2, \ldots, Y_n \) be a random sample with observed values \( y_1, y_2, \ldots, y_n \) from LG distribution with parameters \( p \) and \( \theta \). The total log-likelihood function is given by

\[ l_n(y; p, \theta) = 2n \log(\theta) - n \log(1 + \theta) + n \log(1 - p) + \sum_{i=1}^{n} \log(1 + y_i) - \theta \sum_{i=1}^{n} y_i \\
- 2 \sum_{i=1}^{n} \log \left(1 - p(1 + \frac{\theta y_i}{\theta+1})e^{-\theta y_i}\right). \]

The associated score function is given by \( U_n = (\partial l_n/\partial p, \partial l_n/\partial \theta)^T \), where

\[ \frac{\partial l_n}{\partial p} = \frac{-n}{1-p} + 2 \sum_{i=1}^{n} \frac{(1 + \frac{\theta y_i}{\theta+1})e^{-\theta y_i}}{1 - p(1 + \frac{\theta y_i}{\theta+1})e^{-\theta y_i}}, \]

\[ \frac{\partial l_n}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} y_i - 2p \sum_{i=1}^{n} y_i e^{-\theta y_i} \left\{ \frac{1 - \frac{\theta y_i}{\theta+1}}{1 - p(1 + \frac{\theta y_i}{\theta+1})e^{-\theta y_i}} \right\}^2. \]

The maximum likelihood estimation (MLE) of \( p \) and \( \theta \) is obtained by solving the nonlinear system \( U_n = 0 \). The solution of this nonlinear system of equation has not a closed form. For interval estimation and hypothesis tests on the model parameters, we require the information matrix.

9.1. Asymptotic variances and covariances of the MLEs

Applying the usual large sample approximation, MLE of \( \Theta \) i.e. \( \hat{\Theta} = (\hat{p}, \hat{\theta}) \), can be treated as being approximately bivariate normal with mean \( \Theta \) and variance-covariance matrix, which is the inverse of the expected information matrix \( J(\Theta) = E[I(\Theta)] \), i.e., \( N_2(\Theta, J(\Theta)^{-1}) \), where \( I(\Theta; y_{obs}) \) is the observed information matrix with elements \( I_{ij} = -\frac{\partial^2 l}{\partial \theta_i \partial \theta_j} \) with \( i, j = 1, 2 \).
and the expectation is to be taken with respect to the distribution of $Y$. Differentiating $\partial l/\partial p$ and $\partial l/\partial \theta$, the elements of the symmetric, second-order observed information matrix are found to be

$$I_{11} = \frac{n}{(1-p)^2} - 2 \sum_{i=1}^{n} \left( \frac{[1+\frac{\theta y_i}{(\theta+1)}]e^{-\theta y_i}}{1-p(1+\frac{\theta y_i}{(\theta+1)})e^{-\theta y_i}} \right)^2,$$

$$I_{12} = 2 \sum_{i=1}^{n} \frac{y_i e^{-\theta y_i} \left[ (1+\frac{\theta y_i}{(\theta+1)}) - \frac{1}{(\theta+1)^2} \right]}{1-p(1+\frac{\theta y_i}{(\theta+1)})e^{-\theta y_i}} + 4p \sum_{i=1}^{n} \left[ \frac{y_i e^{-\theta y_i} \left( (1+\frac{\theta y_i}{(\theta+1)}) - \frac{1}{(\theta+1)^2} \right)}{1-p(1+\frac{\theta y_i}{(\theta+1)})e^{-\theta y_i}} \right]^2,$$

$$I_{22} = \frac{2n}{\theta^2} - \frac{n}{(1+\theta)^2} + 2p \sum_{i=1}^{n} \left[ \frac{y_i e^{-\theta y_i} \left( (1+\frac{\theta y_i}{(\theta+1)}) - \frac{1}{(\theta+1)^2} \right)}{1-p(1+\frac{\theta y_i}{(\theta+1)})e^{-\theta y_i}} \right]^2 - \frac{2y_i e^{-\theta y_i} \left( 1 - \frac{1}{(\theta+1)^2} \frac{\theta y_i}{(\theta+1)} \right)}{1-p(1+\frac{\theta y_i}{(\theta+1)})e^{-\theta y_i}} \left[ \frac{y_i e^{-\theta y_i} \left( (1+\frac{\theta y_i}{(\theta+1)}) - \frac{1}{(\theta+1)^2} \right)}{1-p(1+\frac{\theta y_i}{(\theta+1)})e^{-\theta y_i}} \right].$$

The elements of the expected information matrix, $J(\Theta)$, are calculated by taking the expectations of $I_{ij}, i, j = 1, 2$, with respect to the distribution of $Y$. When the expectations of $I_{ij}, i, j = 1, 2$ is obtained, we would have the matrix $J(\Theta)$, the inverse of $J(\Theta)$, evaluated at $\hat{\Theta}$ provides the asymptotic variance-covariance matrix of MLEs. Alternative estimates can be obtained from the inverse of the observed information matrix since it is a consistent estimator of $J^{-1}(\Theta)$.

The estimated asymptotic multivariate normal $N_2(\Theta, I(\hat{\Theta})^{-1})$ distribution of $\hat{\Theta}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An $100(1 - \gamma)$ asymptotic confidence interval for parameters $p$ and $\theta$ is given by

$$ACI_p = (\hat{p} - Z_\gamma \sqrt{\hat{I}_{pp}}, \hat{p} + Z_\gamma \sqrt{\hat{I}_{pp}}),$$

and

$$ACI_\theta = (\hat{\theta} - Z_\gamma \sqrt{\hat{I}_{\theta\theta}}, \hat{\theta} + Z_\gamma \sqrt{\hat{I}_{\theta\theta}}),$$

where $\hat{I}_{pp}$ and $\hat{I}_{\theta\theta}$ are the diagonal element of $I(\hat{\Theta})^{-1}$ and $Z_\gamma$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

We can compute the maximized unrestricted and restricted log-likelihoods to construct likelihood ratio (LR) statistics for testing some LG sub-models. For example, we can use LR statistics to check whether the fitted LG distribution for a given data set is statistically "superior" to the fitted Lindely distribution. In any case, hypothesis tests of the type $H_0 : \Theta = \Theta_0$ versus $H_1 : \Theta = \Theta_0$ can be performed using LR statistics. In this case, the LR statistic for testing $H_0$ versus $H_1$ is $w = 2\{l(\hat{\Theta}) - l(\hat{\Theta}_0)\}$, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under $H_1$ and $H_0$. The
statistic \( w \) is asymptotically (as \( n \to \infty \)) distributed as \( \chi^2_k \), where \( k \) is the dimension of the subset \( \Theta \) of interest.

### 9.2. An EM algorithm

Let the complete-data be \( Y_1, \cdots, Y_n \) with observed values \( y_1, \cdots, y_n \) and the hypothetical random variable \( Z_1, \cdots, Z_n \). The joint probability density function is such that the marginal density of \( Y_1, \cdots, Y_n \) is the likelihood of interest. Then, we define a hypothetical complete-data distribution for each \((Y_i, Z_i)\) \( i = 1, \cdots, n \) with a joint probability density function in the form

\[
g(y, z; \Theta) = (1 - p)\frac{z\theta^2}{\theta + 1}(1 + y)e^{-\theta y}[p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y}]^{z-1},
\]

where \( \Theta = (p, \theta) \), \( y > 0 \) and \( z \in \mathbb{N} \).

Under the formulation, the E-step of an EM cycle requires the expectation of \((Z|Y; \Theta^{(r)})\) where \( \Theta^{(r)} = (\alpha^{(r)}, \beta^{(r)}, \gamma^{(r)}, \theta^{(r)}) \) is the current estimate (in the \( r \)th iteration) of \( \Theta \).

The pdf of \( Z \) given \( Y \), say \( g(z|y) \) is given by

\[
g(z|y) = z[p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y}]^{z-1}(1 - p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y})^2.
\]

Thus, its expected value is given by

\[
E[Z|Y = y] = \frac{(1 + p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y})}{(1 - p(1 + \frac{\theta y}{\theta + 1})e^{-\theta y})}.
\]

The EM cycle is completed with the M-step by using the maximum likelihood estimation over \( \Theta \), with the missing \( Z \)'s replaced by their conditional expectations given above.

The log-likelihood for the complete-data is

\[
l_n^*(y, z; \Theta) \propto \sum_{i=1}^n \log z_i + 2n \log(\theta) - n \log(1 + \theta) + n \log(1 - p) - \theta \sum_{i=1}^n y_i
\]
\[+ \sum_{i=1}^n \log(1 + y_i) + \sum_{i=1}^n (z_i - 1) \log \left( p(1 + \frac{\theta y_i}{\theta + 1})e^{-\theta y_i} \right).\]

The components of the score function \( U_n^*(\Theta) = \left( \frac{\partial l_n^*}{\partial p}, \frac{\partial l_n^*}{\partial \theta} \right)^T \) are given by

\[
\frac{\partial l_n^*}{\partial p} = \sum_{i=1}^n \frac{z_i - 1}{p} - \frac{n}{1 - p},
\]
\[
\frac{\partial l_n^*}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta + 1} - \sum_{i=1}^n y_i + \sum_{i=1}^n (z_i - 1) \frac{y_i e^{-\theta y_i} \left( \frac{1}{(\theta + 1)^2} \frac{\theta y_i}{(\theta + 1)} - 1 \right)}{(1 + \frac{\theta y_i}{\theta + 1})e^{-\theta y_i}}.
\]
From a nonlinear system of equations $U_n^*(\Theta) = 0$, we obtain the iterative procedure of the EM algorithm as

$$
\hat{p}^{(r+1)} = 1 - \frac{n}{\sum_{i=1}^{n} z_i^{(r)}},
$$

$$
\frac{2n}{\hat{q}^{(r+1)}} - \frac{n}{\hat{q}^{(r+1)}+1} - \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} (z_i^{(r)} - 1) \left\{ \frac{y_i e^{-\hat{q}^{(r+1)} y_i}}{(1 + \hat{q}^{(r+1)} y_i)^2} - \frac{\hat{q}^{(r+1)} y_i - 1}{1 + \hat{q}^{(r+1)} y_i} \right\},
$$

where $\hat{q}^{(r+1)}$ is found numerically. Hence, for $i = 1, \cdots, n$, we have that

$$
z_i^{(r)} = \frac{(1 + \hat{p}^{(r)}(1 + \hat{q}^{(r)} y_i) e^{-\hat{q}^{(r)} y_i})}{(1 - \hat{p}^{(r)}(1 + \hat{q}^{(r)} y_i) e^{-\hat{q}^{(r)} y_i})}.
$$

### 10. Applications to real data sets

In this Section we fit LG distribution to two real data sets and compare the fitness with the extended Lindely (Bakoucha et al., [3]), Lindely, Weibull and exponential distributions, whose densities are given by

$$
f_{EL}(x; \alpha, \theta, \gamma) = \frac{1}{\Gamma(1+\theta+\theta x)} \left[ \gamma(1+\theta+\theta x)(\theta x)^{\gamma-1} - \alpha \right] e^{-(\theta x)^\gamma}, \ x, \theta, \gamma > 0, \ \alpha \in R^- \cup \{0, 1\},
$$

$$
f_L(x; \theta) = \frac{\theta^2}{\theta+1} (1+x) e^{-\theta x}, \ x, \theta > 0,
$$

$$
f_{WE}(x; \theta, \gamma) = \theta \gamma x^{\gamma-1} e^{-\theta x^\gamma}, \ x, \theta, \gamma > 0,
$$

$$
f_E(x; \theta) = \theta e^{-\theta x}, \ x, \theta > 0,
$$

respectively. The first data set represents the waiting times (in minutes) before service of 100 bank customers. This data is examined and analyzed by Ghitany et al. [9] in fitting the Lindely distribution.

In the second data set, we consider vinyl chloride data obtained from clean upgradient monitoring wells in mg/L; this data set is used by Bhaumik et al. [4] in fitting the gamma distribution for small samples.

In order to compare distributions, we consider the K-S (Kolmogorov-Smirnov) statistic with its respective $p$-value, $-2 \log(L)$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion), AD (Anderson-Darling) and CM (Cramer-von Mises) statistics for the two real data sets.

The best distribution corresponds to lower $-2 \log(L)$, AIC, AICC, BIC, AD and CM statistics values. Table 1 shows parameter MLEs with the standard errors according to each one of the
Table 1: MLEs (STDs) of the fitted distributions corresponds to data 1 and 2

| Model      | \( \hat{\alpha} \) | \( \hat{\gamma} \) | \( \hat{\theta} \) | \( \hat{p} \)          |
|------------|----------------------|----------------------|----------------------|------------------------|
| **Data 1** |                      |                      |                      |                        |
| LG         | –                    | –                    | 0.2027(0.0346)       | -0.2427(0.5270)        |
| EL         | -1e-08(4.3212)       | 1.4585(0.1098)       | 0.9128(0.0066)       | –                      |
| Lindley    | –                    | –                    | 0.1866(0.0133)       | –                      |
| Weibull    | –                    | 1.4585(0.1098)       | 0.0305(0.0096)       | –                      |
| Exp        | –                    | –                    | 0.1012(0.0101)       | –                      |

| **Data 2** |                      |                      |                      |                        |
| LG         | –                    | –                    | 0.5458(0.2305)       | 0.6346(0.3079)         |
| EL         | -1.4435(3.9990)      | 1.1380(0.4395)       | 0.2937(0.4138)       | –                      |
| Lindley    | –                    | –                    | 0.8238(0.1054)       | –                      |
| Weibull    | –                    | 1.0102(0.1327)       | 0.5202(0.1177)       | –                      |
| Exp        | –                    | –                    | 0.5321(0.0913)       | –                      |

five fitted distributions for the two real data sets. Also, Table 2 shows the values of \(-2 \log(L)\), K-S statistic with its respective \(p\)-value, AIC, AICC, BIC, AD and CM statistics values.

The values in Table 2, indicate that the LG distribution is a strong competitor to other distributions commonly used in literature for fitting lifetime data. These conclusions are corroborated by the fitted pdf, cdf and survival functions of the LG, EL, Lindley, Weibull and exponential distributions in Fig. 3. We observed a difference between the fitted curves, which is a strong motivation for choosing the most suitable distribution for fitting these two sets of data. From the above results, it is evident that the LG distribution is the best distribution for fitting these data sets compared to other distributions considered here.
Table 2: K-S, p-values, \(-2\log(L)\), AIC, AICC, BIC, AD and CM corresponds to data 1 and 2

| Model  | K-S   | p-value | \(-2\log(L)\) | AIC  | AICC | BIC  | AD    | CM    |
|--------|-------|---------|----------------|------|------|------|-------|-------|
| Data 1 |       |         |                |      |      |      |       |       |
| LG     | 0.0567| 0.9048  | 637.8          | 641.8| 642  | 647.1| 0.3984| 0.1312|
| EL     | 0.0578| 0.8926  | 637.5          | 643.5| 643.7| 651.3| 0.4056| 0.1435|
| Lindley| 0.0677| 0.7486  | 638.1          | 640.1| 640.1| 642.7| 0.4865| 0.1407|
| Weibull| 0.0573| 0.8977  | 637.5          | 641.5| 641.6| 646.7| 0.4022| 0.1425|
| Exp    | 0.1729| 0.0051  | 658            | 660  | 660.1| 662.6| 4.2237| 0.7966|
| Data 2 |       |         |                |      |      |      |       |       |
| LG     | 0.0800| 0.9814  | 110.6          | 114.6| 115  | 117.6| 0.2203| 0.1108|
| EL     | 0.0813| 0.9780  | 110.6          | 116.6| 117.4| 121.2| 0.2402| 0.1159|
| Lindley| 0.1326| 0.5880  | 112.6          | 114.6| 114.7| 116.1| 0.6873| 0.1993|
| Weibull| 0.0919| 0.9364  | 110.9          | 114.9| 115.3| 118  | 0.2826| 0.1242|
| Exp    | 0.0889| 0.9508  | 110.9          | 112.9| 113  | 114.4| 0.2719| 0.1214|

11. Conclusion

We propose a new two-parameter distribution, referred to as the LG distribution which contains as special the Lindley distribution. The hazard function of the LG distribution can be decreasing, increasing and bathtub-shaped. Several properties of the LG distribution such as moments, maximum likelihood estimation procedure via an EM-algorithm, moments of order statistics, residual life function and probability weighted moments are studied. Finally, we fitted LG model to two real data sets to show the potential of the new proposed distribution.
Appendix

Proof of Theorem 1:
The behavior of $f(y)$ is completely similar to the behavior of $\log(f(y))$. For simplicity we consider the behavior of $\log(f(y))$. The derivation of $\log(f(y))$ with respect to $y$ is given by

$$\frac{\partial}{\partial y}[\log(f(y))] = \frac{e^{\theta y}(1 + \theta)(1 - \theta - y\theta) - p(1 + (1 + y)^2\theta^2)}{(1 + y)(e^{\theta y}(1 + \theta) - p(1 + \theta + \theta y))}.$$ 

The dominator of $\frac{\partial}{\partial y}[\log(f(y))]$ is positive for each value $y, p$ and $\theta$, therefore we only consider its nominator. Suppose that $g_1(y) = e^{\theta y}(1 + \theta)(1 - \theta - y\theta),$

and

$g_2(y) = p(1 + (1 + y)^2\theta^2).$

Note that $g_1(0) = 1 - \theta^2$ and $g_2(0) = p(1 + \theta^2)$. $g_2(y)$ is an increasing function of $y$ for $p \geq 0$, $\theta > 0$ and a decreasing function for all values $p < 0$, $\theta > 0$. Also, $g_1(y)$ is a decreasing function of $y$ for all values $p$ and $\theta$. Consider the following comparisons between $g_1(y)$ and $g_2(y)$:

(i) For all values of $p$ and $\theta$ for which $p > \frac{1 - \theta^2}{1 + \theta^2}$, then $g_1(y) - g_2(y) > 0$ and hence $\frac{\partial}{\partial y}[\log(f(y))] < 0$ which implies the decreasing behavior of $f(y)$.

(ii) For all values of $p$ and $\theta$ for which $p \leq \frac{1 - \theta^2}{1 + \theta^2}$, then there exist a point $y^* > 0$ such that (a) for each $y < y^*$; $g_1(y) - g_2(y) \leq 0$ and hence $\frac{\partial}{\partial y}[\log(f(y))] > 0$ which implies the increasing behavior of $f(y)$; (b) for each $y \geq y^*$; $g_1(y) - g_2(y) \leq 0$ and hence $\frac{\partial}{\partial y}[\log(f(y))] < 0$ which implies the decreasing behavior of $f(y)$. Thus in this case the pdf $f(y)$ has a unique mode at point $y^*$ and also is unimodal.

Proof of Theorem 2:
The hazard rate function of the LG distribution in (7) is given by

$$h(y) = \frac{\theta^2(\theta + 1)(1 + y)}{(1 + \theta + \theta y)(\theta + 1 - p(1 + \theta + \theta y)e^{-\theta y}).$$

The derivation of $\log(h(y))$ with respect to $y$ is given by

$$\frac{\partial}{\partial y}[\log(h(y))] = \frac{e^{\theta y}(1 + \theta) - p(1 + \theta + \theta y)(1 + (1 + y)^2\theta^2)}{(1 + y)(1 + \theta + \theta y)[e^{\theta y}(1 + \theta) - p(1 + \theta + \theta y)].$$
The dominator of \( \frac{\partial}{\partial y} \left[ \log(h(y)) \right] \) is positive for each values \( y, p \) and \( \theta \), therefore we only consider its nominator. Suppose that

\[
h_1(y) = e^{\theta y}(1 + \theta),
\]

and

\[
h_2(y) = p(1 + \theta + \theta y)(1 + (1 + y)^2\theta^2).
\]

\( h_1(y) \) is an increasing function of \( y \) where \( h_1(0) = 1 + \theta > 0 \) and \( h_1(y) > 0 \) for each \( y > 0 \) and \( \theta > 0 \).

\( h_2(y) \) is a polynomial function of \( y \) which is an increasing function of \( y \) for \( 0 \leq p < 1 \) and decreasing for \( p < 0 \) and \( h_2(0) = p(1 + \theta)(1 + \theta^2) \). Note that

\[
h'_2(y) = p\theta(3(\theta(1 + y))^2 + 2\theta(1 + y) + 1),
\]

and

\[
h''_2(y) = 2p\theta^2(3(\theta(1 + y)) + 1).
\]

Comparison of functions \( h_1(y) \) and \( h_2(y) \) implies the following results:

(i) If \( 1 + \theta < p(1 + \theta)(1 + \theta^2) \) or \( p > \frac{1}{1+\theta^2} \), then there exist a unique point \( y^* > 0 \) such that

\( h_1(y) - h_2(y) < 0 \) for \( 0 < y < y^* \) and \( h_1(y) - h_2(y) > 0 \) for \( y > y^* \). In this case \( h(y) \) is first decreasing for \( 0 < y < y^* \) and then increasing function of \( y \) for \( y^* < y \), therefore \( h(y) \) is bathtub-shaped.

(ii) If \( p \leq \frac{1}{1+\theta^2} \), then there exist two points \( y^*_1 > 0 \) and \( y^*_2 > 0 \) such that \( h_1(y) - h_2(y) > 0 \) for \( 0 < y < y^*_1 \), \( h_1(y) - h_2(y) < 0 \) for \( y^*_1 < y < y^*_2 \) and \( h_1(y) - h_2(y) > 0 \) for \( y > y^*_2 \). In this case \( h(y) \) is increasing for \( 0 < y < y^*_1 \), decreasing for \( y^*_1 < y < y^*_2 \) and then increasing function of \( y \) for \( y > y^*_2 \), therefore \( h(y) \) is firstly increasing and then bathtub-shaped, in this case.
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