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Generalized Equivalence Principle in Extended New General Relativity

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In extended new general relativity, which is formulated as a reduction of a Poincaré gauge theory of gravity whose gauge group is the covering group of the Poincaré group, we study the problem of whether the total energy-momentum, total angular momentum and total charge are equal to the corresponding quantities of the gravitational source. We examine this problem for charged axi-symmetric solutions of gravitational field equations. Our main concern is the restriction on the asymptotic form of the gravitational field variables imposed by the requirement that physical quantities of the total system are equivalent to the corresponding quantities of the charged rotating source body. This requirement can be regarded as an equivalence principle in a generalized sense.

§1. Introduction

Energy-momentum, angular momentum and electric charge play central roles in modern physics. The conservation properties of the first two are related to the homogeneity and isotropy of space-time, respectively, and charge conservation corresponds to the invariance of the action integral under internal $U(1)$ transformations. Also, local quantities such as energy-momentum density, angular momentum density and charge density are well defined if gravitational fields are not present in the system in question.

In general relativity (G. R.), however, well-behaved energy-momentum and angular momentum densities have not yet been defined, although total energy-momentum and total angular momentum can be defined for an asymptotically flat space-time surrounding an isolated finite system. The total energy of the system is regarded as the inertial mass multiplied by the square of the velocity of light, and there arises the following question: Is the active gravitational mass of the isolated system equal to its inertial mass? This question regards an aspect of the equivalence principle, and it is usually considered to be affirmatively answered within G. R. However, the equality of the active gravitational mass and the inertial mass is violated for the Schwarzschild metric when it is expressed with a certain coordinate system.

New general relativity (N. G. R.), which is formulated by gauging coordinate translations and is constructed within the Weitzenböck space-time, is a possible alternative to G. R. The most general gravitational Lagrangian density, which is quadratic in the torsion tensor and is invariant under global Lorentz transformations including also space inversion and general coordinate transformations, has three free

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parameters, c₁, c₂ and c₃. Solar system experiments show that c₁ and −c₂ are very likely to be equal to −1/(3κ). In Ref. 5), Shirafuji, Nashed and Hayashi give, for the case with c₁ = −1/(3κ) = −c₂, the most general spherically symmetric solution, and they clarified the restriction on the behavior of the vierbeins at spatial infinity imposed by the requirement that the inertial mass is equal to the active gravitational mass. Such analysis has been extended to the case in which c₁ ≠ −1/(3κ) ≠ −c₂ in Ref. 2).

Extended new general relativity (E. N. G. R.), 6) is obtained as a reduction of the Poincaré gauge theory (P. G. T.) of gravity, 7) which is formulated on the basis of the principal fiber bundle over the space-time, possessing the covering group P₀ of the Poincaré group as the structure group, by following the lines of the standard geometric formulation of Yang-Mills theories as closely as possible. E. N. G. R. 6) is also constructed within the Weitzenböck space-time and has many points in common with N. G. R. The field equations for the vierbeins in E. N. G. R., for example, are identical to those in N. G. R., if fields with non-vanishing “intrinsic” energy-momentum do not exist.

In this paper, considering charged axi-symmetric solutions in E. N. G. R., we examine the condition imposed on the asymptotic behavior of the field variables by the requirement∗ that the total energy-momentum, the total angular momentum and the total charge of the system are all equal to the corresponding quantities of the central gravitating body.

This paper is organized as follows. In §2, we give the basic formulation of E. N. G. R. In §3, we calculate conserved quantities for a charged axi-symmetric solution. In §4, we study solutions obtainable from the solution discussed in §3 and examine restrictions on the field variables imposed by the requirement mentioned above. In the final section, we give a summary and discussion.

§2. Basic formulation

2.1. Poincaré gauge theory

P. G. T. 7) is formulated on the basis of the principal fiber bundle P over the space-time M possessing the covering group P₀ of the proper orthochronous Poincaré group as the structure group. The space-time is assumed to be a noncompact four-dimensional differentiable manifold having a countable base. The bundle P admits a connection Γ. The translational and rotational parts of the coefficients of Γ will be written as Aₖμ and Aᵏᵢμ, respectively. The fundamental field variables are Aₖμ, Aᵏᵢμ, the Higgs-type field ψ = {ψᵏ}, and the matter field φ = {φᴬ|A = 1, 2, 3, ⋯, N}. **

∗ This is a generalization of the requirement that the inertial mass is equal to the active gravitational mass.

** Unless otherwise stated, we use the following conventions for indices: The middle part of the Greek alphabet, μ, ν, λ, ⋯, denotes 0, 1, 2 and 3, while the initial part, α, β, γ, ⋯, denotes 1, 2 and 3. In a similar way, the middle part of the Latin alphabet, i, j, k, ⋯, denotes 0, 1, 2 and 3, unless otherwise stated, while the initial part, a, b, c, ⋯, denotes 1, 2 and 3. The capital letters A and B are used for indices of components of the field φ, and N denotes the dimension of the representation ρ.
These fields transform as\(^1\)
\[
\psi^k = (A(a^{-1}))^k_l(\psi^l - t^l), \\
A^k_{\mu} = (A(a^{-1}))^k_l(A^l_{\mu} + t^l_{\mu} + A^l_{\mu} t^m_m), \\
A^k_{\mu} = (A(a^{-1}))^k_m A^m n_{\mu}(A(a))^{n}_{l} + (A(a^{-1}))^k_m (A(a))^{m}_{l\mu}, \\
\phi^A = (\rho(t, a^{-1}))^A_B\phi^B,
\]
\[
(2.1)
\]
under the Poincaré gauge transformation
\[
\sigma'(x) = \sigma(x) \cdot (t(x), a(x)), \quad t(x) \in T^4, \quad a(x) \in SL(2, C).
\]
\[
(2.2)
\]
Here, \(A\) is the covering map from \(SL(2, C)\) to the proper orthochronous Lorentz group, and \(\rho\) stands for the representation of the Poincaré group to which the field \(\phi\) belongs. Also, \(\sigma\) and \(\sigma'\) represent the local cross sections of \(\mathcal{P}\). The dual components \(b^k_{\mu}\) of the vierbeins \(b^{\alpha}_{\mu} \partial/\partial x^\alpha\) are related to the field \(\psi\) and the gauge potentials \(A^k_{\mu}\) and \(A^k_{\mu} \psi^l\) through the relation
\[
b^k_{\mu} = \psi^k_{\mu} + A^k_{\mu} \psi^l + A^k_{\mu},
\]
\[
(2.3)
\]
and these transform according to
\[
b^k_{\mu} = (A(a^{-1}))^k_l b^l_{\mu},
\]
\[
(2.4)
\]
under the transformation \((2.2)\). Also, they are related to the metric \(g_{\mu\nu} dx^\mu \otimes dx^\nu\) of \(M\) through the relation
\[
g_{\mu\nu} = \eta_{kl} b^k_{\mu} b^l_{\nu},
\]
\[
(2.5)
\]
with \((\eta_{kl}) \overset{\text{def}}{=} \text{diag}(-1,1,1,1)\).

There is a 2 to 1 bundle homomorphism \(F\) from \(\mathcal{P}\) to the affine frame bundle \(\mathcal{A}(M)\) over \(M\), and an extended spinor structure and a spinor structure exist in association with it. \(^3\) The space-time \(M\) is orientable, which follows from its assumed noncompactness and from the fact that \(M\) has a spinor structure.

The affine frame bundle \(\mathcal{A}(M)\) admits a connection \(\Gamma_A\). The \(T^4\)-part, \(\Gamma^\mu_{\nu,\lambda}\), and \(GL(4, R)\)-part, \(\Gamma^\mu_{\nu}\), of its connection coefficients are related to \(A^k_{\mu}\) and \(b^k_{\mu}\) through the relations
\[
\Gamma^\mu_{\nu} = \delta^\mu_{\nu}, \quad A^k_{\mu} = b^k_{\nu} b^\lambda_{\mu} \Gamma^\nu_{\lambda,\mu} + b^k_{\nu} b^\nu_{\mu},
\]
\[
(2.6)
\]
by the requirement that \(F\) maps the connection \(\Gamma\) into \(\Gamma_A\), and the space-time \(M\) is of the Riemann-Cartan type.

The field strengths \(R^k_{l\mu\nu}, R^k_{\mu\nu}\) and \(T^k_{\mu\nu}\) of \(A^k_{\mu}\) and \(b^k_{\mu}\) are given by\(^**\)
\[
R^k_{l\mu\nu} \overset{\text{def}}{=} 2(A^k_{[\mu|\nu]} + A^k_{m[|\mu} A^m_{\nu]}),
\]
\[
(2.7)
\]
\(^1\) For the function \(f\) on \(M\), we define \(f_{,\mu} \overset{\text{def}}{=} \partial f/\partial x^\mu\).

\(^**\) We define
\[
A_{\ldots[\mu|\nu|\ldots]} \overset{\text{def}}{=} \frac{1}{2}(A_{\ldots\mu\nu|\ldots} - A_{\ldots\nu\mu|\ldots}),
\]
\[
A_{\ldots(\mu|\nu|\ldots)} \overset{\text{def}}{=} \frac{1}{2}(A_{\ldots\mu\nu|\ldots} + A_{\ldots\nu\mu|\ldots}).
\]
\[ R^k_{\mu\nu} \overset{\text{def}}{=} 2(A^k_{[\nu,\mu]} + A^k_{[\mu,\nu]}), \]
\[ T^k_{\mu\nu} \overset{\text{def}}{=} 2(b^k_{[\nu,\mu]} + A^k_{[\mu,\nu]}b^l_{\nu}), \]
\[ (2.7) \]

and we have the relation
\[ T^k_{\mu\nu} = R^k_{\mu\nu} + R^k_{l\mu\nu}\psi^l. \]
\[ (2.8) \]

The field strengths \( T^k_{\mu\nu} \) and \( R^{kl}_{\mu\nu} \) are both invariant under internal translations. The torsion is given by
\[ T^\lambda_{\mu\nu} = 2\Gamma^\lambda_{[\nu,\mu]}, \]
\[ (2.9) \]

and the \( T^4 \)- and \( GL(4,R) \)-parts of the curvature are given by
\[ R^\lambda_{\mu\nu} = 2(\Gamma^\lambda_{[\nu,\mu]} + \Gamma^\lambda_{[\rho,\mu} \Gamma^\rho_{\nu]}), \]
\[ R^\rho_{\mu\nu} = 2(\Gamma^\rho_{[\nu,\mu]} + \Gamma^\rho_{[\tau,\mu} \Gamma^\tau_{\rho\nu]}), \]
\[ (2.10, 2.11) \]

respectively. Also, we have
\[ T^k_{\mu\nu} = b^k_\lambda T^\lambda_{\mu\nu} = b^k_\lambda R^\lambda_{\mu\nu}, \]
\[ R^k_{l\mu\nu} = b^k_\lambda b^l_\rho R^\lambda_{\rho\mu\nu}, \]
\[ (2.12, 2.13) \]

which follow from Eq. (2.6).

The covariant derivative of the matter field \( \phi \) takes the form
\[ D_k\phi^A = b^k_\mu D_\mu\phi^A, \]
\[ D_\mu\phi^A \overset{\text{def}}{=} \partial_\mu\phi^A + \frac{i}{2} A^{lm}_\mu (M_{lm})^A + iA^l_\mu (P_l\phi)^A. \]
\[ (2.14) \]

Here, \( M_{kl} \) and \( P_k \) are representation matrices of the standard basis of the group \( P_0 : M_{kl} = -i\rho_*(\bar{M}_{kl}), P_k = -i\rho_*(\bar{P}_k) \). The matrix \( P_k \) represents the intrinsic energy-momentum of the field \( \phi^A \), and it is vanishing for all the observed fields.\(^*\)

From the requirement of invariance of the action integral under internal \( P_0 \) gauge transformations, it follows that the gravitational Lagrangian density is a function of \( T_{klm} \) and of \( R_{klmn} \). The gravitational Lagrangian density is identical to that in Poincaré gauge theory, and hence gravitational field equations take the same forms in these theories. The field equation for the field \( \psi^k \) is automatically satisfied if those for \( A^k_\mu \) and \( \phi^A \) are both satisfied, and \( \psi^k \) is a non-dynamical field in this sense.

2.2. Extended new general relativity

2.2.1. Reduction of Poincaré gauge theory to an extended new general relativity

In P. G. T., we consider the case in which the field strength \( R^{kl}_{\mu\nu} \) vanishes identically,
\[ R^{kl}_{\mu\nu} \equiv 0, \]
\[ (2.15) \]

\(^*\) In what follows, the field components \( b^k_\mu \) and \( b^l_\mu \) are used to convert Latin and Greek indices, in analogy to the case of \( D_k\phi^A \) and \( D_\mu\phi^A \). Also, raising and lowering the indices \( k, l, m, \cdots \) are accomplished with the aid of \( (\eta^{kl}) \overset{\text{def}}{=} (\eta_{kl})^{-1} \) and \( (\eta_{kl}) \).
and we choose $SL(2,C)$-gauge such that

$$A^{kl}_{\mu} \equiv 0 .$$  \hfill (2.16)

Then, the curvature vanishes, and the space-time reduces to the Weitzenböck space-time, which means that we have a teleparallel theory.

Also, the vierbeins $b^k_\mu$, the affine connection coefficients $\Gamma^\lambda_{\mu\nu}$ and the torsion tensor $T^\lambda_{\mu\nu}$ are given by

$$b^k_\mu = \psi^k_{\mu\nu} + A^k_\mu ,$$  \hfill (2.17)

$$\Gamma^\lambda_{\mu\nu} = b^\lambda_{l\mu\nu} ,$$  \hfill (2.18)

and

$$T^\lambda_{\mu\nu} = 2 \Gamma^\lambda_{[\mu\nu]} = 2b^\lambda_k b^k_{[\mu\nu]} = b^\lambda_k T^k_{\mu\nu} ,$$  \hfill (2.19)

respectively. Introducing the volume element $dv$ by

$$dv \equiv \sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 ,$$  \hfill (2.20)

with $g \equiv \det(g_{\mu\nu}) = -\{\det(b^k_\mu)\}^2$, we consider the action integral

$$I = \int_D L(\psi^k_{\mu\nu}, \psi^k, A^k_{\mu\nu}, A^k_\mu, A_{\mu\nu}, A^A_{\mu\nu}, A^A_\mu, A^A_{\mu\nu}, A^*A_{\mu\nu}, A^*A_\mu, A^*A_{\mu\nu}, \phi^A, \phi^*A, \phi^*A) dv ,$$  \hfill (2.21)

where $A_\mu$ and $D$ denote the electromagnetic vector potential and a compact region in $M$, respectively. Also, the symbol $*$ represents the operation of complex conjugation, and thus $\phi^*A$ denotes the complex conjugate of $\phi^A$.

We impose the following requirement:

(R.i) The action $I$ is invariant under local internal translations and global $SL(2, C)$-transformations. From this, the identities \hfill (6)

$$\frac{\delta L}{\delta \psi^k} + \partial_\mu \left( \frac{\delta L}{\delta A^k_\mu} \right) + \frac{i}{2} \frac{\delta L}{\delta \phi^A} (P_k \phi)^A - i \frac{\delta L}{\delta \phi^*A} (P_k \phi)^*A \equiv 0 ,$$  \hfill (2.22)

$$F_k^{(\mu\nu)} \equiv 0 ,$$  \hfill (2.23)

$$\partial_\mu T^k_{\mu\nu} - \partial_\nu T^k_{\mu\nu} - \frac{\delta L}{\delta A^k_\mu} \equiv 0 ,$$  \hfill (2.24)

$$\partial_\mu S^{kl}_{\mu\nu} - 2\frac{\delta L}{\delta \psi^k \psi^l} - 2\frac{\delta L}{\delta A^k_{\mu\nu}} A^l_{\mu\nu} - i \frac{\delta L}{\delta \phi^A} (M_{kl} \phi)^A + i \frac{\delta L}{\delta \phi^*A} (M_{kl} \phi)^*A \equiv 0$$  \hfill (2.25)

follow, where we have defined

$$L \equiv \sqrt{-g} L ,$$  \hfill (2.26)
\[ F^\mu_\nu = \frac{\partial L}{\partial A^\mu_\nu} , \]  
\[ \text{tot} T^\mu_\nu \overset{\text{def}}{=} \frac{\partial L}{\partial \psi^{[k}_\mu} + i \frac{\partial L}{\partial \phi^{A}_\mu} (P_k \phi)^A - i \frac{\partial L}{\partial \phi^{*}_\mu} (P_k \phi)^* A , \]  
\[ \text{tot} S^{kl}_\mu \overset{\text{def}}{=} -2 \frac{\partial L}{\partial \psi^{[k}_\mu} \psi^{l]} - 2 F^{[k}_\nu \mu \overset{\text{def}}{=} A^{[\nu} \mu \]  
(2.28)

\[ \text{tot} T^\mu_\nu \overset{\text{def}}{=} \frac{\partial L}{\partial A^\mu_\nu} . \]  
(2.30)

By virtue of the identity (2.24), the density \( \text{tot} T^\mu_\nu \) can be expressed in the usual form of the current in Yang-Mills theories, \( \text{tot} T^\mu_\nu = \frac{\partial L}{\partial A^\mu_\nu} . \) (2.31)

When the field equations \( \delta L/\delta A^k_\mu \overset{\text{def}}{=} \partial L/\partial A^k_\mu - \partial_\nu (\partial L/\partial A^k_\mu_\nu) = 0 \) and \( \delta L/\delta \phi^A = 0 \) are both satisfied, we have the following:

(i) The field equation \( \delta L/\delta \psi^{k} = 0 \) is automatically satisfied, and hence \( \psi^k \) is not an independent dynamical variable.

(ii) \( \partial_\mu \text{tot} T^\mu_\nu = 0 , \) \( \partial_\mu \text{tot} S^{kl}_\mu = 0 , \) (2.32)

which are the differential conservation laws of the dynamical energy-momentum and the “spin” angular momentum, respectively. The assertions (i) and (ii) follow from Eqs. (2.22) – (2.25).

Also, we require the following:

(R.ii) The Lagrangian density \( L \) is a scalar field on \( M \).

Then, we have

\[ \tilde{T}^{\mu}_\nu - \partial_\lambda \Psi^{\mu}_\nu - \frac{\delta L}{\delta A^k_\nu} A^k_\mu - \frac{\delta L}{\delta A_\mu} A^k_\nu \equiv 0 \]  
(2.33)

and

\[ \Psi^{(\mu_\nu)} \equiv 0 , \]  
(2.34)

with

\[ \tilde{T}^{\mu}_\nu \overset{\text{def}}{=} \delta^{\mu}_\nu L - F^{\lambda}_k A^k_\nu A^\lambda_\mu - F^{\lambda}_\nu A^\lambda_\mu - \frac{\partial L}{\partial \phi^{A}_\nu} \phi^A_\mu - \frac{\partial L}{\partial \phi^{*}_\nu} \phi^{* A}_\mu - \frac{\partial L}{\partial \psi^{k}_\nu} \psi^k_\mu , \]  
(2.35)

and

\[ \Psi^{\mu_\nu} \overset{\text{def}}{=} F^{\mu_\nu}_k A^k_\lambda + F^{\mu_\nu} A^\lambda_\mu , \]  
(2.36)

\[ F^{\mu_\nu} \overset{\text{def}}{=} \frac{\partial L}{\partial A^\mu_\nu} . \]  
(2.37)
The identities (2.33) and (2.34) lead to
\begin{align}
\partial_\mu T^\mu_{\nu} &= 0, \\
\partial_\mu M^{\lambda\mu\nu} &= 0,
\end{align}
when \( \delta L/\delta A^k_\mu = 0 \) and \( \delta L/\delta A_\mu = 0 \), where \( M^{\lambda\mu\nu} \equiv 2(\Psi^{\lambda\mu\nu} - x^\mu \tilde{T}_\lambda^{\nu}) \). Equations (2.38) and (2.39) are the differential conservation laws of the canonical energy-momentum and “extended orbital angular momentum”, respectively. We also require invariance of the action under the \( U(1) \) gauge transformation:
\begin{align}
A'_\mu &= A_\mu + \lambda_\mu , \\
\phi'^A &= \exp(iqA)\phi^A , \\
\psi'^k &= \psi^k , \\
A'^{k}_\mu &= A^{k}_\mu ,
\end{align}
with \( \lambda \) being an arbitrary function on \( M \), from which we can obtain
\begin{align}
F^{(\mu\nu)} &\equiv 0 , \\
J^\mu + iq \left( \frac{\partial L}{\partial \phi^A_{\mu}} \phi^A - \frac{\partial L}{\partial \phi^{*A}_{\mu}} \phi^{*A} \right) &= 0 , \\
\partial_\mu \left( \frac{\delta L}{\delta A^*_\mu} - J^\mu \right) &\equiv 0 ,
\end{align}
with
\begin{align}
J^\mu &\equiv \frac{\partial L}{\partial A^*_\mu} .
\end{align}
From the identity (2.43), the differential conservation law of the electric charge,
\begin{align}
\partial_\mu J^\mu &= 0 ,
\end{align}
follows when the field equation \( \delta L/\delta A^\mu = 0 \) is satisfied. The functional dependence of \( L \) is restricted as
\begin{align}
L &= \mathcal{L}(\psi^k, T_{klm}, F_{\mu\nu}, \nabla_k \phi^A, \phi^A, \nabla^*_{k} \phi^{*A}, \phi^{*A}) ,
\end{align}
with \( \mathcal{L} \) satisfying certain identities, where
\begin{align}
F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu , \\
\nabla_k \phi^A &\equiv b^A_k \nabla_{\mu} \phi^A \equiv b^A_k (\phi^A_{\mu} + iA^k_{\mu}(P_k \phi)^A - iqA_{\mu} \phi^{*A}) , \\
\nabla^*_{k} \phi^{*A} &\equiv b^A_k \nabla^*_{\mu} \phi^{*A} \equiv b^A_k (\phi^{*A}_{\mu} - iA^k_{\mu}(P_k \phi)^A + iqA_{\mu} \phi^{*A}) .
\end{align}
The gravitational action 4)
\begin{align}
\mathcal{I}_G = \int_D (c_1 t_{klm} t_{klm} + c_2 v^k v_k + c_3 a^k a_k) dv ,
\end{align}
with \( c_i (i = 1, 2, 3) \) being real constants, satisfies the requirements (R.i) and (R.ii). Here \( t_{klm} \), \( v_k \) and \( a_k \) are irreducible components of the torsion tensor:
\begin{align}
t_{klm} &\equiv \frac{1}{2}(T_{klm} + T_{lkm}) + \frac{1}{6}(\eta_{mk} v_l + \eta_{ml} v_k) - \frac{1}{3} \eta_{kl} v_m , \\
v_k &\equiv T^l_{lk} , \\
a_k &\equiv \frac{1}{6} \varepsilon_{klm} T^{lmn} ,
\end{align}
\footnote{4) This action is identical to the gravitational action in N. G. R. 4.}
where $\varepsilon_{klmn}$ is a completely anti-symmetric Lorentz tensor with $\varepsilon_{(0)(1)(2)(3)} = -1.\)  

The action $\tilde{I}_G$ with\)

$$c_1 = -c_2 = -\frac{1}{3\kappa}, \quad (2.53)$$

with $\kappa$ being the Einstein gravitational constant agrees quite well with experimental results. In what follows, we assume that the condition (2.53) is satisfied. Thus, our gravitational action is

$$I_G = \int_D L_G dv, \quad (2.54)$$

with

$$L_G = -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} . \quad (2.55)$$

2.2.2. The gravitational and electromagnetic field equations in vacuum

The electromagnetic Lagrangian density $L_{em}$ is given by\)

$$L_{em} = -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} . \quad (2.56)$$

We consider a system described by the Lagrangian density $L \equiv L_G + L_{em}$. The gravitational and electromagnetic field equations for this system are the following:

$$G_{\mu\nu}({}\{\}) + K_{\mu\nu} = \kappa T_{em}^{\mu\nu}, \quad (2.57)$$

$$\partial_\mu (\sqrt{-g} J_{kl\mu}^a) = 0 , \quad (2.58)$$

$$\partial_\nu (\sqrt{-g} F_{\mu\nu}) = 0 . \quad (2.59)$$

Here we have defined

$$G_{\mu\nu}({}\{\}) = R_{\mu\nu}({}\{\}) - \frac{1}{2} g_{\mu\nu} R({}\{\}) , \quad (2.60)$$

and

$$R_{\mu\nu}({}\{\}) = R^\lambda_{\mu\lambda\nu}({}\{\}) , \quad (2.61)$$

with the Riemann-Christoffel curvature tensor

$$R^\lambda_{\rho\mu\nu}({}\{\}) = 2 \left( \partial_\mu \left\{ \begin{array}{c} \lambda \\ \rho \\ \nu \end{array} \right\} + \left\{ \begin{array}{c} \lambda \\ \sigma \\ \mu \end{array} \right\} \left\{ \begin{array}{c} \sigma \\ \rho \\ \nu \end{array} \right\} \right) . \quad (2.62)$$

Also, $T_{em}^{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field,

$$T_{em}^{\mu\nu} = F^{\mu\rho} F_{\nu\rho} g_{\rho\sigma} + g^{\mu\nu} L_{em} , \quad (2.63)$$

and the tensors $K^{\mu\nu}$ and $J^{k\mu\nu}$ are defined by

$$K^{\mu\nu} = \frac{k}{\lambda} \left[ \frac{1}{2} \left( \varepsilon^{\mu\rho\sigma\lambda} (T^{\rho\sigma} - T_{\rho\sigma}) + \varepsilon^{\nu\rho\sigma\lambda} (T^{\rho\sigma} - T_{\rho\sigma}) \right) a_\lambda - \frac{3}{2} a_\mu a_\nu - \frac{3}{4} g^{\mu\nu} a_\lambda a_\lambda \right] . \quad (2.64)$$

\(a\) Latin indices are put in parentheses to discriminate them from Greek indices.

\(\ast\) We will use natural units in which $\hbar = c = 1$.

\(\ast\ast\) Here we use Heaviside-Lorentz rationalized unit.
and
\[ J^{k l \mu} \overset{\text{def}}{=} -\frac{3}{2} b^k \rho b^l \sigma \varepsilon^{\rho \sigma \mu \nu} a_{\nu}, \tag{2.65} \]
respectively, where we have used
\[ \lambda \overset{\text{def}}{=} \frac{9 \kappa}{4 \kappa c^3 + 3}. \tag{2.66} \]

2.2.3. An exact solution of gravitational and electromagnetic field equations with a charged rotating source

An exact solution of the field equations (2.57) – (2.59), which represents the gravitational and electromagnetic fields surrounding a charged rotating source, is given in Ref. 10). This will be described below. The vector fields \( b^k \mu \) and the electromagnetic potential \( A_{\mu} \) have the expressions
\[ b^k \mu = (0)^k \mu + \frac{a}{2} l^k l_{\mu} - \frac{Q^2}{2} m^k m_{\mu}, \tag{2.67} \]
\[ A_{\mu} = -\frac{q}{4 \pi} \sqrt{Z} l_{\mu}, \tag{2.68} \]
where \((0)^k \mu\) are the dual components of constant vierbeins and are defined by \((0)^k \mu \overset{\text{def}}{=} \delta^k \mu\). The functions \( Z, l_{\mu}, m_{\mu}, l^k \) and \( m^k \) are given by
\[ Z = \frac{N}{D}, \]
\[ l_0 = \sqrt{Z}, \quad l_\alpha = \frac{2\sqrt{Z}}{D + r^2 + h^2} \left[ N x_\alpha + \frac{h^2 x^3 \varepsilon_{\alpha \beta \gamma}}{N} - \varepsilon_{\alpha \beta \gamma} x^3 \right], \]
\[ m_{\mu} = \frac{l_\mu}{\sqrt{N}}, \quad l^k = \delta^k \mu l^{\mu \nu} l_\nu, \quad m^k = \delta^k \mu l^{\mu \nu} m_\nu, \tag{2.69} \]
where \( D \) and \( N \) are given by
\[ D = \sqrt{(r^2 - h^2)^2 + 4h^2 x^3 \right)^2}, \quad N = \frac{\sqrt{r^2 - h^2} + D}{\sqrt{2}}, \tag{2.70} \]
with \( r \overset{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \). In Eq. (2.69), the \( \varepsilon_{\alpha \beta \gamma} \) are three-dimensional totally anti-symmetric tensor with \( \varepsilon_{123} = 1 \). Also, we have defined
\[ a \overset{\text{def}}{=} \frac{km}{4 \pi}, \quad Q \overset{\text{def}}{=} \frac{q}{4 \pi} \sqrt{\kappa} \frac{2}{2}, \quad h \overset{\text{def}}{=} -\frac{J}{m}, \tag{2.71} \]
with \( m, J \) and \( q \) being the active gravitational mass, the absolute value of the angular momentum, and the electromagnetic charge of the source body, respectively.

For the solution given by Eqs. (2.67) and (2.68), the axial vector part \( a_{\mu} \) of the torsion tensor vanishes,
\[ a_{\mu} = 0, \tag{2.72} \]
and the metric is identical to the charged Kerr metric in G. R.
The asymptotic forms of the vierbeins and the electromagnetic vector potential for large $r$ are given by

\[ b^a_{\alpha} = \delta^a_{\alpha} + \frac{1}{2} \left( a - \frac{Q^2}{r} \right) n^\alpha n_a - \frac{ah}{2r^2} (\varepsilon_{\alpha\beta\gamma} n^\alpha + \varepsilon^a_{\beta\gamma} n_a) + O_a \left( \frac{1}{r^3} \right), \]

\[ b^{(0)}_{\alpha} = -\frac{n_\alpha}{2r} \left( a - \frac{Q^2}{r} \right) + \frac{ah}{2r^2} \varepsilon_{\alpha\beta} n^a + O_{\alpha} \left( \frac{1}{r^3} \right), \]

\[ b^n = \frac{n^\alpha}{2r} \left( a - \frac{Q^2}{r} \right) - \frac{ah}{2r^2} \varepsilon_{\alpha\beta} n^a + O^a \left( \frac{1}{r^3} \right), \]

\[ b^{(0)}_0 = 1 - \frac{1}{2r} \left( a - \frac{Q^2}{r} \right) + O \left( \frac{1}{r^3} \right) \]

and

\[ A_0 = -\frac{q}{4\pi r} + O \left( \frac{1}{r^3} \right), \quad A_{\alpha} = -\frac{q}{4\pi r^2} x^\alpha + O_{\alpha} \left( \frac{1}{r^3} \right), \]

respectively, which follow from Eqs. (2.67) and (2.68). Here, we have defined $n^\alpha = x^\alpha / r$, and the expression $O \left( \frac{1}{r^w} \right)$ with positive $w$ denotes a term such that $\lim_{r \to \infty} r^w O \left( \frac{1}{r^w} \right) = \text{constant.} \quad \ast$

2.2.4. Asymptotic form of $\psi^k$

The space-time in this theory has vanishing curvature tensor. When, additionally, the torsion tensor vanishes identically, the space-time is the Minkowski space-time, for which the translational gauge potentials $A^k_\mu$ can be chosen to be zero and Eq. (2.3) is reduced to

\[ b^k_\mu = \psi^k,\mu. \quad (2.75) \]

For the solution given by Eqs. (2.67) and (2.68), we have\textsuperscript{**}

\[ T^k_{\mu\nu} = O^k_{[\mu\nu]} \left( \frac{1}{r^2} \right), \quad (2.76) \]

and the space-time asymptotically approaches the Minkowski space-time for large $r$.

The above discussion and the consideration given in Ref. 11) to introduce vierbeins suggest that $\psi^k$ can be regarded as a Minkowskian coordinate at spatial infinity, and we are naturally led to employ the following form of $\psi^k$:

\[ \psi^k = (0) b^k_\mu x^\mu + (0) \psi^k + O^k \left( \frac{1}{r^3} \right), \quad (\beta > 0) \]

\[ \psi^k,_{\mu} = (0) b^k_\mu + O^k_\mu \left( \frac{1}{r^{1+\beta}} \right), \]

\[ \psi^k,_{\mu\nu} = O^k_{(\mu\nu)} \left( \frac{1}{r^2} \right), \quad (2.77) \]

where $^{(0)} \psi^k$ and $\beta$ are constants.

\textsuperscript{\ast} The symbols $a$ and $\alpha$ in $O^a_\alpha(1/r^3), O_\alpha(1/r^3)$ and $O^a(1/r^3)$ are to show that these terms have indices as indicated.

\textsuperscript{\ast\ast} The square brackets [ ] in the suffix of $O^k_{(\mu\nu)}(1/r^2)$ indicate that this term is anti-symmetric with respect to $\mu, \nu$. 

\textsuperscript{\ast\ast\ast} The symbols $a$ and $\alpha$ in $O^a_\alpha(1/r^3), O_\alpha(1/r^3)$ and $O^a(1/r^3)$ are to show that these terms have indices as indicated.
§3. Equivalence relations for the case of the solution represented by Eqs. (2·67) and (2·68)

In this section, on the basis of the discussion in Refs. 9) and 12), we examine the energy-momentum, the angular momentum and the charge for the solution given in the preceding section.

3.1. The case in which \( \{ \psi^k, A^k, A_\mu \} \) is employed as the set of independent field variables

We regard the Lagrangian density \( L \equiv \sqrt{-g} L \) as a function of \( \psi^k, A^k, A_\mu \) and of their derivatives. For this case, the generator \( M_k \) of internal translations and \( S_{kl} \) of internal Lorentz transformations are the dynamical energy-momentum and the total (=spin + orbital) angular momentum, respectively, and they are expressed as

\[
M_k \equiv \int_\sigma T^k_{\mu} d\sigma_\mu \quad (3·1)
\]

and

\[
S_{kl} \equiv \int_\sigma S^k_{\mu\nu} d\sigma_\mu \quad (3·2)
\]

Here, \( \sigma \) is a space-like surface, and \( d\sigma_\mu \) is the surface element on it:

\[
d\sigma_\mu = \frac{1}{3!} \epsilon_{\mu\nu\lambda\rho} dx^\nu \wedge dx^\lambda \wedge dx^\rho \quad (3·3)
\]

We have

\[
F^k_{\mu\nu} = F^{(1)}_{k\mu\nu} + F^{(2)}_{k\mu\nu} \quad (3·4)
\]

with

\[
F^{(1)}_{k\mu\nu} \equiv \frac{1}{\kappa \sqrt{-g}} b_{k\rho} \partial_\sigma \left\{ (-g)^{\rho[\mu} g^{\nu]\sigma} \right\} + Z^k_{\mu\nu} \quad (3·5)
\]

\[
F^{(2)}_{k\mu\nu} \equiv \frac{3}{2\lambda \sqrt{-g}} \epsilon_k^{lmn} b^l_{\mu} b^m_{\nu} a_{\lambda} \quad (3·6)
\]

\[
Z^k_{\mu\nu} \equiv \frac{\sqrt{-g}}{\kappa} \left\{ b^k_{\mu l}(\epsilon^{\nu]\rho\lambda b^{\rho\sigma l,\lambda}) + b^l_{k \mu} b^{\nu\rho l\sigma} \right\} \quad (3·7)
\]

as is known by using the explicit form of \( L_G \). From Eqs. (2·17), (2·24), (2·28) and (3·4), we find that \( \text{tot} S^k_{\mu\nu} \) defined by Eq. (2·29) can be rewritten as

\[
\text{tot} S^k_{\mu\nu} = 2 \partial_\nu(\psi^k_{[k} F^l_{\mu]} \mu^\nu) + \frac{2}{\kappa} \partial_\nu \left( \sqrt{-g} b^k \mu_{[k} b^\nu_{\mu]} - (0)^{b\mu}_{[k} b^{\nu]}_{\mu]l} \right) - b_{[k\nu} F^{(2)}_{l\mu]} \mu^\nu + 2 \psi^k \frac{\delta L}{\delta A^l_{[\mu}} \quad (3·8)
\]

where \( (0)^{b\mu}_{[k} \) are the components of the constant vierbeins: \( (0)^{b\mu}_{[k} \equiv \delta^\mu_{[k} \). For the vierbeins given by Eq. (2·67), we have

\[
Z^k_{\mu\nu} = 0 \quad F^{(2)}_{k\mu\nu} = 0 \quad (3·9)
\]
and hence
\[ F^\mu_\nu = F^{(1)}_\mu_\nu = \frac{1}{\kappa \sqrt{-g}} b_{k\rho} \partial_\sigma \left\{ (-g) g^{\rho \mu} g^{\nu \rho} \right\}. \] (3.10)

### 3.1.1. Energy-momentum

When the field equation \( \delta L / \delta A_k^\mu = 0 \) is satisfied, Eq. (3.1) can be rewritten as
\[ M_k = \int_\sigma \partial_\nu F^\mu_\nu d\sigma_\mu = \int_S F^{(0)}_k \epsilon^2 n_\alpha d\Omega, \] (3.11)
by using the identity (2.24). Here, \( S \) and \( d\Omega \) stand for the two-dimensional surface of \( \sigma \) and the differential solid angle, respectively. Equations (3.4), (3.9) – (3.11) and (A.1) give
\[ M^{(0)} = -m, \quad M_a = 0. \] (3.12)
The quantity \( M_k \) is the total energy-momentum vector of the system. The first relation in Eq. (3.12) expresses the equality of the active gravitational mass and the inertial mass.

### 3.1.2. Angular momentum

From Eqs. (3.2) and (3.8), the total angular momentum can be expressed as
\[ S_{kl} = 2 \int_\sigma \partial_\nu \left[ \psi [ k F^I_\mu_\nu + \frac{1}{\kappa} \left\{ \sqrt{-g} b^\mu_\nu [ k b^\nu_\nu - (^{(0)} b^\mu_\nu [ k (^{(0)} b^\nu_\nu ] \right\} \right] d\sigma_\mu, \] (3.13)
from which the expression
\[ S^{(0)}_a = (^{(0)} \psi_a) m = (^{(0)} \psi_a) M^{(0)} - (^{(0)} \psi (^{(0)}) M_a, \]
\[ S_{ab} = J_{eab3} = J_{eab3} + (^{(0)} \psi_a M_b - (^{(0)} \psi_b M_a \] (3.14)
is obtained by the use of Eqs. (2.73), (2.77), (3.1), (3.4), (3.10) and (A.1).

In the above, terms of the form \( (^{(0)} \psi_k M_l - (^{(0)} \psi l M_k \) are regarded as to represent the conserved orbital angular momentum around the origin of the internal space. \(^{12}\)
Equation (3.14) implies that the total angular momentum is equal to the angular momentum of the rotating source.

### 3.1.3. Canonical energy-momentum and “extended orbital angular momentum”

The generator \( M^e_\mu \) of coordinate translations and the generator \( L_\mu_\nu \) of \( GL(4, \mathbb{R}) \) coordinate transformations are the canonical energy-momentum and the “extended orbital angular momentum”, \(^{\ast}\) respectively. They have the expressions
\[ M^e_\mu \text{ def } = \int_\sigma \bar{T}^\mu_\nu d\sigma_\nu = \int_\sigma \partial_\nu \Psi^\mu_\nu d\sigma_\nu, \] (3.15)
\[ L^\nu_\mu \text{ def } = \int_\sigma M^\nu_\mu d\sigma_\lambda = -2 \int_\sigma \partial_\sigma (x^\nu \Psi^\lambda_\mu d\sigma_\lambda. \] (3.16)

\(^{\ast}\) Note that the anti-symmetric part \( L_{[\mu_\nu]} \text{ def } = \eta_{[\nu_\lambda]\mu_\alpha} L^\lambda_\mu \) is the orbital angular momentum.
The asymptotic behavior of the translational gauge potentials \( A^k_\mu \) at spatial infinity is given as
\[
A^{(0)}_0 = -\frac{a}{2r} + O\left(\frac{1}{r^{1+\beta}}\right), \quad A^{(0)}_\alpha = -\frac{a}{2r} n_\alpha + O\left(\frac{1}{r^{1+\beta}}\right),
\]
\[
A^a_0 = \frac{a}{2r} n^a + O\left(\frac{1}{r^{1+\beta}}\right), \quad A^a_\alpha = \frac{a}{2r} n^a n_\alpha + O\left(\frac{1}{r^{1+\beta}}\right),
\]
which are known from Eqs. (2.17), (2.73) and (2.77). Then \( M^c_\mu \) vanishes trivially,
\[
M^c_\mu = \int_S F_k^{0\alpha} A^k_\mu r^2 n_\alpha d\Omega = 0 ,
\]
while \( L^\nu_\mu \) is expressed as
\[
L^\nu_\mu = -2 \int_S x^\nu (F_k^{0\alpha} A^k_\mu + F^{0\alpha} A_\mu) r^2 n_\alpha d\Omega ,
\]
and the non-zero components are given by
\[
L^1_1 = L^2_2 = L^3_3 = \frac{q^2}{6\pi} .
\]
These can be shown by use of Eqs. (2.36), (2.74), (3.4), (3.17) and (A.1). Thus, the orbital angular momentum \( L_{[\mu\nu]} \) is vanishing: \( L_{[\mu\nu]} = 0 \).

3.1.4. Charge

The charge is defined as the generator of \( U(1) \) gauge transformations and is given by
\[
C \overset{\text{def}}{=} \int_\partial \left( \frac{\partial L}{\partial A_{\mu,\nu}} \right) d\sigma_\mu = \int_\partial \left( \partial_\nu (\sqrt{-g} F^{\mu\nu}) \right) d\sigma_\mu = - \int_S (\partial^\alpha A^0) n_\alpha r^2 d\Omega = q ,
\]
where we have used Eq. (2.74). This implies the equality of the total charge of the system and the charge of the source.

3.2. The case in which \( \{ \psi_k, b^k_\mu, A_\mu \} \) is employed as the set of independent field variables

Let us denote \( \hat{L} \) and \( L \), expressed as functions of \( \psi_k, b^k_\mu, A_\mu \) and their derivatives, by \( \hat{\hat{L}} \) and \( \hat{L} \), respectively. The action \( I \) is now written as
\[
I = \int_D \hat{L} dv = \hat{I} .
\]
Various identities can be derived from the requirements (R.i) and (R.ii), among which we have
\[
\frac{\delta \hat{L}}{\delta \psi_k} \equiv 0 ,
\]
\[
\text{tot.} \frac{\delta \hat{\hat{L}}}{\delta \psi_k} \equiv 0 ,
\]
\[ \partial_{\mu} \text{tot} \hat{S}_{kl}^{\mu} - 2 \frac{\delta \hat{L}}{\delta \psi^{[k]} \psi_{[l]}} - 2 \frac{\delta \hat{L}}{\delta b_{[k]} b_{[l]}} \equiv 0 , \]  
(3.25)

\[ \hat{T}^{\mu}_{\nu} - \partial_{\lambda} \hat{\Psi}^{\nu \lambda} - \frac{\delta \hat{L}}{\delta b_{[k]} b_{\lambda}} \equiv 0 , \]  
(3.26)

where we have defined
\[ \hat{L} \overset{\text{def}}{=} \sqrt{-g} L , \]  
(3.27)

\[ \text{tot} \hat{T}^{\mu}_{\nu} \overset{\text{def}}{=} -2 \frac{\partial \hat{L}}{\partial \psi^{[k] \nu}} - 2 \hat{F}^{[k \mu]} b_{[l] \nu} , \]  
(3.28)

\[ \text{tot} \hat{S}_{kl}^{\mu} \overset{\text{def}}{=} -2 \frac{\partial \hat{L}}{\partial \psi^{[k]} \psi_{[l]}} - 2 \hat{F}^{[k \mu]} b_{[l] \nu} - \frac{\partial \hat{L}}{\partial b_{[k]} b_{\lambda}} \]  
(3.29)

\[ \hat{T}^{\mu}_{\nu} \overset{\text{def}}{=} \delta^{\mu \nu} \hat{L} - \hat{F}^{k \lambda \nu \mu} b_{k} b_{\lambda} - \hat{F}^{\lambda \nu} A_{\lambda \mu} - \frac{\partial \hat{L}}{\partial b_{[k]} b_{\lambda} \psi^{[k] \psi_{[l]}}} , \]  
(3.30)

\[ \hat{\Psi}^{\mu \nu \lambda} \overset{\text{def}}{=} \hat{F}^{k \mu \nu} b_{k} b_{\lambda} + \hat{F}^{\mu \nu} A_{\lambda} = - \hat{\Psi}^{\lambda \nu \mu} , \]  
(3.31)

\[ \hat{F}^{k \mu \nu} \overset{\text{def}}{=} \frac{\partial \hat{L}}{\partial b_{k \mu} b_{\nu}} = F^{k \mu \nu} , \quad \hat{F}^{\mu \nu} \overset{\text{def}}{=} \frac{\partial \hat{L}}{\partial A_{\mu \nu}} = F^{\mu \nu} . \]  
(3.32)

From Eqs. (3.23) and (3.24), we see that
\[ \partial_{\mu} \text{tot} \hat{S}_{kl}^{\mu} = 0 \]  
(3.33)

when the field equations for \( b_{[k]} b_{\lambda} \) are satisfied. From Eqs. (3.26) and (3.31), it follows that
\[ \partial_{\nu} \hat{T}^{\mu}_{\nu} = 0 , \]  
(3.34)

\[ \partial_{\nu} \hat{M}^{\mu \nu} = 0 \]  
(3.35)

when the field equations \( \delta \hat{L} / \delta b_{[k]} b_{\lambda} = 0 \) and \( \delta \hat{L} / \delta A_{\mu} = 0 \) are both satisfied, where \( \hat{M}^{\mu \nu} \overset{\text{def}}{=} 2(\hat{\Psi}^{\mu \nu} - x^{\mu} \hat{T}^{\nu}) \). Equations (3.33) – (3.35) are the differential conservation laws of the "spin" angular momentum, the canonical angular momentum, and the "extended orbital angular momentum", respectively.

The density \( \text{tot} \hat{S}_{kl}^{\mu} \) defined by Eq. (3.29) can be rewritten as
\[ \text{tot} \hat{S}_{kl}^{\mu} = \frac{2}{\kappa} \partial_{\nu} \left( \sqrt{-g} b_{[k]} b_{\lambda} - b_{[k]} b_{\lambda} - b_{[l]} b_{\lambda} \right) - b_{[k]} b_{\lambda} F_{[k \mu \nu]}^{(2)} , \]  
(3.36)

by the use of Eqs. (3.4), (3.24), (3.32) and (3.28).

3.2.1. Energy-momentum

The dynamical energy-momentum \( \hat{M}_{k} \), which is the generator of internal translations, vanishes identically:
\[ \hat{M}_{k} \overset{\text{def}}{=} \int_{\sigma} \hat{T}_{k}^{\mu} d\sigma_{\mu} \equiv 0 . \]  
(3.37)

This is evident from Eq. (3.24).
3.2.2. Spin angular momentum

The generator $\hat{S}_{kl}$ of internal Lorentz transformations is expressed as

$$\hat{S}_{kl} \overset{\text{def}}{=} \int_{\sigma} S_{kl}^{\mu} d\sigma_{\mu} = \frac{2}{\kappa} \int_{S} \left( \sqrt{-g} b^{\alpha}_{\lambda} \left[ b^{\alpha}_{\lambda} \right] - (0) b^{\alpha}_{\lambda} \left[ (0) b^{\alpha}_{\lambda} \right] \right) n_{\alpha} r^2 d\Omega ,$$

(3.38)

as can be shown by using Eq. (3.36), and we obtain

$$\hat{S}_{(0)a} = 0, \quad \hat{S}_{ab} = \frac{1}{3} J_{\varepsilon ab3}$$

(3.39)

by using Eq. (2.73).

3.2.3. Canonical energy-momentum and “extended orbital angular momentum”

The generator $\hat{M}_{\mu}^{c}$ of coordinate translations and the generator $\hat{L}_{\mu\nu}$ of GL(4, $\mathbb{R}$) coordinate transformations are the canonical energy-momentum and the “extended orbital angular momentum”, respectively. They have the expressions

$$\hat{M}_{\mu}^{c} \overset{\text{def}}{=} \int_{\sigma} \hat{T}_{\mu}^{\nu} \sigma_{\nu} = \int_{\sigma} \partial_{\nu} \hat{\Psi}_{\mu}^{\nu} d\sigma_{\nu} ,$$

(3.40)

$$\hat{L}_{\mu\nu} \overset{\text{def}}{=} \int_{\sigma} \hat{M}_{\mu\nu\lambda} d\sigma_{\lambda} = -2 \int_{\sigma} \partial_{\nu} \left( x^{\nu} \hat{\Psi}_{\mu}^{\lambda} \right) d\sigma_{\lambda} .$$

(3.41)

Then we have

$$\hat{M}_{0}^{c} = -m, \quad \hat{M}_{a}^{c} = 0 .$$

(3.42)

Thus, $\hat{M}_{\mu}^{c}$ is the total energy-momentum, and the equality of the active gravitational mass and the inertial mass holds. Also, $\hat{L}_{\mu\nu}$ is given by

$$\hat{L}_{0}^{\beta} = 2 x^{0} m, \quad \hat{L}_{0}^{\alpha} = 0, \quad \hat{L}_{a}^{0} = 0, \quad \hat{L}_{a}^{\beta} = \frac{2}{3} J_{\varepsilon a33}, \quad (\alpha \neq \beta), \quad \hat{L}_{1}^{1} = \hat{L}_{2}^{2} = \hat{L}_{3}^{3} = \infty .$$

(3.43)

Equations (3.42) and (3.43) are obtained by using Eqs. (2.73), (2.74), (3.10), (3.31), (3.32) and (A.1).

The orbital angular momentum $\hat{L}_{[\mu\nu]}$ is given by

$$\hat{L}_{[0a]} = 0, \quad \hat{L}_{[a3]} = \frac{2}{3} J_{\varepsilon a33} .$$

(3.44)

If we define the total angular momentum of the system by

$$\hat{J}_{kl} \overset{\text{def}}{=} \hat{S}_{kl} + (0) b^{\mu}_{k} (0) b^{\nu}_{l} \hat{L}_{[\mu\nu]} ,$$

(3.45)

then we have

$$\hat{J}_{(0)k} = 0, \quad \hat{J}_{ab} = J_{\varepsilon ab3} ,$$

(3.46)

and the total angular momentum is equal to the angular momentum of the rotating source.
3.2.4. Charge

The generator $\hat{C}$ of $U(1)$ gauge transformations is given by

$$\hat{C} \overset{\text{def}}{=} \int_{\sigma} \partial_{\nu} \left( \frac{\partial \hat{L}}{\partial A_{\mu,\nu}} \right) \, d\sigma_{\mu} = q , \quad (3.47)$$

which implies the equality of the total charge of the system and the charge of the source.

§4. Restrictions imposed on field variables by the generalized equivalence principle

In the preceding section, we examined the solution given by Eqs. (2.67) and (2.68), and the results show that the total energy-momentum, the total angular momentum, and the total charge of the system, which are generators of transformations, are equal to the corresponding active quantities of a central gravitating body. The total mass, which is equal to the total energy divided by the square of the velocity of light, can be regarded as the inertial mass of the system. Thus, the results include the equality of the inertial mass and the active gravitational mass, which implies that the equivalence principle is satisfied by this solution.

In view of the above, we regard the total momentum, the total angular momentum and the total charge as “inertial” quantities, and we say that a generalized equivalence principle (G. E. P.) is satisfied if the total energy-momentum, total angular momentum and total charge are all equal to the corresponding quantities of the source.

The axial vector part $a_{\mu}$ vanishes for our solution, as stated above, and the field equations (2.57) – (2.59) are covariant under general coordinate transformations and under local Lorentz transformations that keep $a_{\mu}$ vanishing. Thus, new solutions can be obtained by applying the general coordinate transformations and restricted local Lorentz transformations $b_{\mu}^{\prime} = A_{\mu,\nu}^{\prime} b_{\nu}^\prime$, that satisfy the condition

$$\varepsilon_{\mu\nu\rho\sigma} b_{\mu}^{\prime} b_{\rho}^{\prime} a^{m}_{k}(x) A_{m l}(x) , \sigma = 0 \quad (4.1)$$

to the solution represented by Eqs. (2.67) and (2.68).

In this section, we examine restrictions imposed on solutions by the requirement that this G. E. P. is satisfied. We look for new solutions having suitable asymptotic behavior by considering the following Poincaré gauge transformations:

(1) Local $SL(2,C)$ transformation

$$H_{\mu}^{k} \overset{\text{def}}{=} \left( A(a^{-1}) \right)_{k}^{l} = A^{k}_{\mu} + A^{k}_{\mu} \omega^{m}_{l}(x) . \quad (4.2)$$

(2) Local internal translation

$$t^{k} = (0) t^{k} + b^{k}(x) . \quad (4.3)$$

*) Note that $a_{\mu}$ is invariant under the local Lorentz transformation $A^{k}_{\mu}(x)$ if and only if this condition is satisfied.
Here, $A^k_l$ and $(0)^l_k$ denote a constant internal Lorentz transformation and the constant internal translation, respectively, and $\omega^k_l$ and $b^k_l$ are functions satisfying the following conditions:

$$
\omega^k_l(x) = O^k_l\left(\frac{1}{r^p}\right), \quad \omega^k_{l,\mu}(x) = O^k_{l,\mu}\left(\frac{1}{r^{p+1}}\right), \quad (p > 0)
$$

$$
b^k_l(x) = O^k\left(\frac{1}{r^\beta}\right), \quad b^k_{\mu}(x) = O^k_{\mu}\left(\frac{1}{r^{\gamma+1}}\right), \quad (\gamma > 0)
$$

The transformation $H^k_l \equiv (A(a^{-1}))^k_l$ is a Lorentz transformation satisfying Eq. (4.1), if and only if

$$
\omega_{kl} + \omega_{lk} + \omega_{mk}\omega^m_l = 0, \quad (4.6)
$$

$$
\varepsilon^{\mu\nu\lambda\rho}(\omega_{kl,\rho} + \omega_{mk}\omega^m_{l,\rho})b^k_{\nu}b^l_{\lambda} = 0 \quad (4.7)
$$

are both satisfied. The conditions (4.6) and (4.7) are equivalent to

$$
\omega_{\mu\nu} + \omega_{\nu\mu} + \omega_{\lambda\mu}\omega^\lambda_{\nu} = 0 \quad (4.8)
$$

and

$$
X_{\mu\nu\lambda} + X_{\lambda\mu\nu} + X_{\nu\lambda\mu} = 0, \quad (4.9)
$$

respectively, where we have defined

$$
\omega_{\mu\nu} \equiv b^k_{\mu}b^l_{\nu}\omega_{kl}, \quad (4.10)
$$

$$
X_{\mu\nu\lambda} \equiv \omega_{\mu\nu,\lambda} + \omega_\tau^\mu \omega_{\tau\nu,\lambda} - b^k_{\mu,\lambda}\omega_\tau^\nu - b^k_{\nu,\mu}\omega_\tau^\lambda + b^k_{\nu,\lambda}\omega_\rho^\mu\omega_{\rho\nu} \quad (4.11)
$$

The function $X_{\mu\nu\lambda}$ is anti-symmetric with respect to the first two indices:

$$
X_{\mu\nu\lambda} = -X_{\nu\mu\lambda}. \quad (4.12)
$$

From Eqs. (4.4), (4.8) and (4.9), $\omega_{\mu\nu}$ is known to have the expression

$$
\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + f_{\mu\nu}(x), \quad (4.13)
$$

with

$$
\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = O_{[\mu\nu]}\left(\frac{1}{r^p}\right), \quad \partial_\mu \omega_\nu - \partial_\nu \omega_\mu = O_{[\mu\nu]}\left(\frac{1}{r^s}\right), \quad (p < s)
$$

$$
f_{\mu\nu}(x) = O_{[\mu\nu]}\left(\frac{1}{r^s}\right). \quad (p < s)
$$

In addition, we require the leading term of $\omega_{kl}$ at spatial infinity to be spherically symmetric. Then we can write

$$
\omega^0 = A(r, x^0), \quad \omega^\alpha = n^\alpha B(r, x^0), \quad (4.15)
$$

with certain functions $A$ and $B$ of $r$ and $x^0$. 
Also, we consider the following coordinate transformation:

\[ x'_{\mu} = C_{\mu\nu}x_{\nu} + D_{\mu}(x), \]
\[ \frac{\partial x'_{\mu}}{\partial x_{\nu}} = C_{\mu\nu} + a_{\mu\nu}(x), \]
\[ a_{\mu\nu} \equiv D_{\mu\nu}, \]
\[ a_{\mu\nu}(x) = O_{\mu\nu} \left( \frac{1}{r^u} \right), \quad a_{\mu\nu,\lambda} = O_{\mu\nu\lambda} \left( \frac{1}{r^{u+1}} \right), \quad (u > 0) \]  \hspace{1cm} (4.16)

where \( C_{\mu\nu} \) denotes a constant Lorentz transformation, and \( D_{\mu}(x) \) satisfies the condition

\[ \lim_{r \to \infty} \frac{D_{\mu}(x)}{r} = 0. \]  \hspace{1cm} (4.17)

We write \( \partial x_{\mu} / \partial x_{\nu} \) as

\[ \frac{\partial x_{\mu}}{\partial x_{\nu}} = (C^{-1})_{\mu\nu} + d_{\mu\nu}(x), \]  \hspace{1cm} (4.18)

with \( (C^{-1})_{\mu\nu} \) being constants satisfying \( (C^{-1})_{\mu\lambda}C_{\lambda\nu} = \delta_{\mu\nu} \).

The vierbeins and vector potentials given by

\[ b_{k\mu} \equiv H_{kl} \frac{\partial x_{\nu}}{\partial x_{\mu}} b_{l\nu}, \quad A_{\mu}' \equiv \frac{\partial x_{\nu}}{\partial x'_{\mu}} A_{\nu}, \]  \hspace{1cm} (4.19)

with \( b_{k\mu} \) and \( A_{\mu} \) given by Eqs. (2.67) and (2.68), are solutions of the gravitational and electromagnetic field equations. This is true irrespective of the values of the parameters \( p, s, \beta, \gamma \) and \( u \). The G. E. P. is considered to be satisfied if the energy-momentum, the angular momentum and the charge all have correct transformation properties as their indices indicate. But, this is not necessarily the case for arbitrary values of these parameters; i.e. there are solutions which do not satisfy the G. E. P. We examine restrictions imposed on these parameters by the requirement that the G. E. P. is satisfied.

4.1. The case in which \( \{\psi^k, A_{k\mu}, A_{\mu}\} \) is employed as the set of independent field variables

Under the combined transformation of the Poincaré gauge transformation given by Eqs. (4.2) and (4.3) and satisfying the conditions (4.6) and (4.7) and of the coordinate transformation (4.16), \( F^{(1)}_{k\mu\nu}, F^{(2)}_{k\mu\nu} \) and \( F^{\mu\nu} \) transform according to

\[ F^{(1)}_{k\mu\nu} = \Delta \frac{\partial x_{\mu}'}{\partial x_{\rho}} \frac{\partial x_{\nu}'}{\partial x_{\sigma}} H_{k}^{l} F^{(1)}_{l\rho\sigma}, \]
\[ F^{(2)}_{k\mu\nu} = 0 = \Delta \frac{\partial x_{\mu}'}{\partial x_{\rho}} \frac{\partial x_{\nu}'}{\partial x_{\sigma}} H_{k}^{l} F^{(2)}_{l\rho\sigma}, \]  \hspace{1cm} (4.20)
\[ F^{\mu\nu} = \frac{\partial x_{\mu}'}{\partial x_{\rho}} \frac{\partial x_{\nu}'}{\partial x_{\sigma}} F^{\rho\sigma}, \]  \hspace{1cm} (4.21)
\[ F^{(1)}_{k\mu\nu} = \frac{\partial x_{\mu}'}{\partial x_{\rho}} \frac{\partial x_{\nu}'}{\partial x_{\sigma}} H_{k}^{l} F^{(1)}_{l\rho\sigma}, \]  \hspace{1cm} (4.22)
where we have defined

\[ U_{kmn\lambda} \overset{\text{def}}{=} H_{kl}^m V_{mn\lambda}, \quad \text{(4.23)} \]

\[ V_{mn\lambda} \overset{\text{def}}{=} H^m H_{n\lambda} = -V_{mn\lambda}, \quad \text{(4.24)} \]

\[ W^{\mu
u\lambda}_{klm} \overset{\text{def}}{=} b^{[\mu}_{k} b^{\nu]}_{m} b_{l}^{\lambda} + b^{[\mu}_{m} b^{\nu]}_{k} b_{l}^{\lambda} + b^{[\mu}_{l} b^{\nu]}_{m} b_{k}^{\lambda}, \quad \text{(4.25)} \]

\[ \Delta \overset{\text{def}}{=} \text{det} \left( \frac{\partial x^\mu}{\partial x'^\nu} \right). \quad \text{(4.26)} \]

Equations (4.20) – (4.22) show that \( F^{(1)}_{k\mu\nu} \), \( F^{(2)}_{k\mu\nu} \) and \( F^{\mu\nu} \) transform as tensor densities under coordinate transformations. The function \( W^{\mu\nu\lambda}_{klm} \) is totally antisymmetric, both in upper indices and in lower indices.

4.1.1. Energy-momentum

From Eqs. (3.11) and (4.20), \( M_{k} \) is found to transform as \(^{**} \)

\[ M'_{k} \overset{\text{def}}{=} \int_{\sigma} \text{tot} \text{T'}_{k\mu} d\sigma'_{\mu} = \int_{\sigma} \partial'_{\mu} F'_{k\mu\nu} d\sigma'_{\mu} = A_{k} M_{l} + \frac{3}{\kappa} \int U_{k[0\alpha\beta]} b_{\alpha\beta} r^2 d\Omega, \quad \text{(4.27)} \]

where we represent \( \partial'_{\mu} = \partial/\partial x'^{\mu} \). The energy-momentum \( M_{k} \) obeys the correct transformation rule

\[ M'_{k} = A_{k} M_{l} \quad \text{(4.28)} \]

if the condition

\[ p > \frac{1}{2}, \quad s > 1 \quad \text{(4.29)} \]

is satisfied. \(^{***}\)

4.1.2. Angular momentum

From Eqs. (3.13) and (4.20), we find that \( S_{kl} \) transforms as \(^{**} \)

\begin{align*}
S'_{kl} & \overset{\text{def}}{=} \int_{\sigma} \text{tot} S'_{kl\mu} d\sigma'_{\mu} \\
& = 2 \int_{\sigma} \partial'_{\nu} \left[ \psi_{[k} F'_{l]}^{\mu\nu} + \frac{1}{\kappa} \left( \sqrt{-g} b^{[\mu}_{k} b^{\nu]}_{l} - b^{(0)[\mu}_{k} b^{(0)\nu]}_{l} \right) \right] d\sigma'_{\mu} \\
& = A_{k} A_{l}^{-1} \left( S_{mn} - 2 S_{0} M_{n} \right) \\
& \quad + 2 A_{k} A_{l}^{-1} \left( \omega_{m}^{n} \left( \psi_{n} - t_{n} \right) - b_{m}(x) \right) H_{l}^{n} F_{n}^{0\alpha\beta} n_{\alpha} r^{2} d\Omega \\
& \quad + \frac{2}{\kappa} \int H_{[k}^{m} H_{l]}^{n} \left( \psi_{m} - t_{m} \right) H_{i}^{j} H_{i}^{h} \beta W^{0\alpha\beta} n_{h} n_{\alpha} r^{2} d\Omega
\end{align*}

\(^{*}\) For simplicity, we restrict our consideration to the case in which \( \Delta > 0 \). An extension to the case of arbitrary non-vanishing \( \Delta \) can be made without difficulty.

\(^{**} \) \( A_{[\mu\lambda]} \overset{\text{def}}{=} \frac{1}{2} \left( A_{\mu[\nu\lambda]} + A_{\nu[\mu\lambda]} + A^{\lambda[\mu\nu]} \right) \).

\(^{***} \) For derivations of the conditions (4.20), (4.32) – (4.34), (4.40), (4.48) and (4.52), elementary but rather tedious calculations are needed. In Appendix A, we give lists of asymptotic forms of \( F^{(1)}_{k\mu\nu}, V_{mn\lambda} \) and \( W^{\mu\nu\lambda}_{klm} \) for large \( r \), which are useful in calculations.
\[ + \frac{2}{\kappa} \int H_{[k}^{m} (H_{l]}^{n} + A_{l}^{m} \omega_{m}^{n}) (b_{[m}^{0} b_{n]}^{0} - (0)) b_{[m}^{0} b_{n]}^{0}) n_{\alpha} r^{2} d\Omega , \]

(4.30)

where we have defined \( (0) b_{k}^{\mu} \overset{\text{def}}{=} \lim_{r \to \infty} H_{k}^{i} (\partial x^{\mu} / \partial x^{\nu}) b_{i}^{\nu} \). The angular momentum obeys the correct transformation rule

\[ S'_{kl} = A_{k}^{m} A_{l}^{n} \left( S_{mn} - 2(0) t_{[m} M_{n]} \right) \]

(4.31)

if the following conditions are satisfied:

\[ p > \frac{1}{2}, \quad s > 2 \]

(4.32)

and

\[ \left\{ p + \beta > 1, \quad p + \gamma > 1 \right\} \quad \text{or} \quad \left\{ O^{(0)} \left( \frac{1}{r^{\beta}} \right) = f(r, x^{0}), \quad O^{(0)} \left( \frac{1}{r^{\gamma}} \right) = g(r, x^{0}) \right\} \]

(4.33)

\[ \left\{ p + \beta > 1, \quad p + \gamma > 1 \right\} \quad \text{or} \quad \left\{ O^{a} \left( \frac{1}{r^{\beta}} \right) = n^{a} h(r, x^{0}), \quad O^{a} \left( \frac{1}{r^{\gamma}} \right) = n^{a} k(r, x^{0}) \right\} \]

(4.34)

with \( f, g, h \) and \( k \) being some functions of \( r \) and \( x^{0} \), where the terms \( O^{k}(1/r^{\beta}) \) and \( O^{k}(1/r^{\gamma}) \) are those in Eq. (2.77) and Eq. (4.5), respectively.

4.1.3. Canonical energy-momentum and “extended orbital angular momentum”

The transformed canonical energy-momentum \( M'_{c}^{\mu} \) and the “extended orbital angular momentum” \( L'_{\mu}^{\nu} \) are given by

\[ M'_{c}^{\mu} \overset{\text{def}}{=} \int_{\sigma} \bar{T}'_{\mu}^{\nu} ds'_{\nu} = \int_{\sigma} \partial'_{\lambda} \Psi'_{\mu}^{\nu} \lambda ds'_{\nu} \]

\[ = \int_{\sigma} \Psi'_{\mu}^{\nu} \lambda \mathcal{J} \frac{\partial x^{0}}{\partial x'^{\nu}} \frac{\partial x^{\alpha}}{\partial x'^{\lambda}} n_{\alpha} r^{2} d\Omega , \]

(4.35)

\[ L'_{\mu}^{\nu} \overset{\text{def}}{=} \int_{\sigma} M'_{\mu}^{\nu} ds'_{\nu} = -2 \int_{\sigma} \partial'_{\tau}(x'^{\nu} \Psi'_{\mu}^{\lambda} \tau) ds'_{\lambda} \]

\[ = -2 \int x'^{\nu} \Psi'_{\mu}^{\lambda} \tau \mathcal{J} \frac{\partial x^{0}}{\partial x'^{\nu}} \frac{\partial x^{\alpha}}{\partial x'^{\lambda}} n_{\alpha} r^{2} d\Omega , \]

(4.36)

with \( \mathcal{J} \overset{\text{def}}{=} \det(\partial x'^{\mu} / \partial x^{\nu}) \), in which the transformed \( \Psi'_{\mu}^{\nu} \lambda \) can be obtained from Eqs. (2.1), (2.36), (4.20) – (4.22).

The relation

\[ M'_{c}^{\mu} = 0 = (C^{-1})'_{\mu} M^{c}^{\nu} \]

(4.37)
Generalized Equivalence Principle in Extended New General Relativity

holds without any additional condition on the positive parameters $p$, $s$, $\beta$, $\gamma$ and $u$, and $L_{\mu \nu}$ and $L_{[\mu \nu]}$ transform according to

$$L'_{\mu \nu} = (C^{-1})_{\rho}^\mu C^\nu_\sigma L_{\rho \sigma}, \quad (4.38)$$
$$L'_{[\mu \nu]} = 0 = (C^{-1})_{\rho}^\lambda (C^{-1})_{\mu}^\rho L_{[\lambda \rho]} \quad (4.39)$$

under the condition

$$p > 1. \quad (4.40)$$

Equations (4.37) – (4.39) are the transformation rules which we would like $M_{\mu}^c$, $L_{\mu \nu}$ and $L_{[\mu \nu]}$ to obey.

4.1.4. Charge

For the transformed charge

$$C' \stackrel{\text{def}}{=} \int_\sigma \partial'_{\nu} \left( \frac{\partial L'}{\partial A'_{\mu \nu}} \right) d\sigma'_{\mu}, \quad (4.41)$$

with the Lagrangian density $L'$ defined with transformed field variables, we can obtain

$$C' = C = q \quad (4.42)$$

without imposing any additional condition.

To summarize, the G. E. P. is satisfied if

$$p > 1, \quad s > 2. \quad (4.43)$$

The asymptotic behavior of the transformed dual components of $b^k_{\mu} = H^k_{li}(\partial x^{\nu}/\partial x^{\mu'})b^{l}_{\nu}$ and of the transformed vector potential $A'_{\mu} = (\partial x^{\nu}/\partial x^{\mu'})A_{\nu}$ can be easily obtained from Eqs. (4.2), (4.4), (4.18) and (4.43) as

$$b^k_{\mu} = A^k_{li}(C^{-1})_{\nu}^\mu b^l_{\nu} + O^k_{\mu}(1/r^u) + O^k_{\mu}(1/r), \quad (u > 0)$$
$$A'_{\mu} = (C^{-1})_{\nu}^\mu A_{\nu} + \partial'_{\nu}A_{\nu} = O_{\mu}(1/r). \quad (4.44)$$

4.2. The case in which $\{\psi^k, b^k_{\mu}, A_{\mu}\}$ is employed as the set of independent field variables

4.2.1. Energy-momentum and angular momentum

For the generator of Poincaré gauge transformations, we have

$$\hat{M}'^i_k \stackrel{\text{def}}{=} \int_\sigma \partial'_{\nu} T'_{k \mu}^{i} d\sigma'_{\mu} \equiv 0 = A_k^l \hat{M}_l, \quad (4.45)$$
$$\hat{S}'_{kl} = \int_\sigma \partial'_{\nu} S'_{k l}^{m} d\sigma'_{\mu} = A_k^m A^{l}_m \hat{S}_{mn}, \quad (4.46)$$

which hold without imposing any additional condition.
4.2.2. Canonical energy-momentum and “extended orbital angular momentum”

For $\hat{M}_\mu^c$, we have

$$\hat{M}_\mu^c \overset{\text{def}}{=} \int_\sigma \tilde{T}_\mu^\rho \rho^
u \sigma'_\nu = (C^{-1})_{\rho \mu} \hat{M}_\nu^c$$  \hspace{1cm} (4.47)

if the conditions

$$p > \frac{1}{2}, \quad s > 1, \quad p + u > 1$$  \hspace{1cm} (4.48)

are satisfied. The transformed “extended orbital angular momentum”

$$\hat{L}'_\mu^\nu \overset{\text{def}}{=} \int_\sigma \hat{M}_\mu^c \nu \sigma'_\nu$$  \hspace{1cm} (4.49)

is divergent in general, as is obvious from Eq. (3.43). However, the transformed orbital angular momentum $\hat{L}'_{[\mu \nu]}$, and hence the transformed total angular momentum $\hat{J}'_{kl} \overset{\text{def}}{=} \hat{S}'_{kl} + (0)b'^{k \rho}_{(0)} b'^{\rho \nu}_{(0)} \hat{L}'_{[\mu \nu]}$, are well defined, and they obey the rules

$$\hat{L}'_{[\mu \nu]} = (C^{-1})_{\nu \mu} (C^{-1})^\rho_{\lambda} \hat{L}'_{[\lambda \rho]},$$  \hspace{1cm} (4.50)

$$\hat{J}'_{kl} = A'^m_{k \nu} A'^n_{l \mu} \hat{J}_{mn},$$  \hspace{1cm} (4.51)

if the conditions

$$p > \frac{1}{2}, \quad s > 2, \quad u > 1, \quad p + u > 2$$  \hspace{1cm} (4.52)

are satisfied.

4.2.3. Charge

The transformed charge

$$\hat{C}' \overset{\text{def}}{=} \int_\sigma \partial'_{\nu} \left( \frac{\partial \hat{L}'}{\partial A'_{\mu, \nu}} \right) d\sigma'_\mu,$$  \hspace{1cm} (4.53)

with the Lagrangian density $\hat{L}'$ defined with transformed field variables, is evaluated as

$$\hat{C}' = \hat{C} = q,$$  \hspace{1cm} (4.54)

without imposing any additional condition.

To summarize, the G. E. P. is satisfied if the conditions in Eq. (4.52) are satisfied. The asymptotic behavior of the transformed dual components of $\hat{b}'^k_{\mu} \overset{\text{def}}{=} H^k_i (\partial x^i / \partial x'^{\mu}) b'_{\mu}$ and of the transformed vector potential $A'_{\mu} \overset{\text{def}}{=} (\partial x^\nu / \partial x'^{\mu}) A_{\nu}$ are easily determined from Eqs. (4.2), (4.4), (4.18) and (4.52) as

$$\hat{b}'^k_{\mu} = A'^k_{l (C^{-1})_{\nu \mu}^{\rho}} b'^\rho_{l \nu} + O^k_{\mu (1/x)}, \quad \left( p > \frac{1}{2} \right)$$

$$\hat{A}'_{\mu} = (C^{-1})_{\nu \mu} A_{\nu} + d'^\nu_{\mu} A_{\nu} = O_{\mu (1/r)}.$$  \hspace{1cm} (4.55)
§5. Summary and discussion

In extended new general relativity (E. N. G. R.), we have examined exact charged axi-symmetric solutions of the gravitational and electromagnetic field equations in vacuum from the point of view of the equivalence principle.

In this theory, the generators depend on the choice of the set of independent field variables. In §3, we examined the solution represented by Eqs. (2.67) and (2.68) for the case in which \( \{ \psi^k, A^k_{\mu}, A_{\mu} \} \) is employed as the set of independent field variables and for the case in which \( \{ \psi^k, b^k_{\mu}, A_{\mu} \} \) is employed as the set of independent variables. We have shown the following:

(A) For the case in which \( \{ \psi^k, b^k_{\mu}, A_{\mu} \} \) is employed as the set of independent field variables, the total energy-momentum, the total angular momentum and the total electric charge of the system are all given by generators of internal transformations. The canonical energy-momentum and the orbital angular momentum vanish trivially.

(B) For the case in which \( \{ \psi^k, b^k_{\mu}, A_{\mu} \} \) is employed as the set of independent field variables, we have following: (1) The total energy-momentum is given by the generator \( M^c_{\mu} \) of the coordinate translations, and the generator \( \hat{M}_k \) of the internal translations vanishes identically. (2) The total angular momentum is given by the sum \( \hat{J}_{kl} \) of the generator \( \hat{S}_{kl} \) of the internal Lorentz transformations and of the generator \( \hat{L}_{[\mu \nu]} \) of the coordinate Lorentz transformations. (3) The total charge is given by the generators of the internal \( U(1) \) transformations.

\( (A \cap B) \) For both cases, the total energy-momentum, the total angular momentum and the total charge of the system are identical to the corresponding active quantities of a central gravitating body.

The total mass, which is equal to the total energy divided by the square of the velocity of light, can be regarded as the inertial mass of the system. Thus, the results mentioned above include the equality of the inertial mass and the active gravitational mass, which implies that the equivalence principle is satisfied by the solution represented by Eqs. (2.67) and (2.68). In consideration of this, we have introduced the notion of a generalized equivalence principle (G. E. P.), as stated at the beginning of §4. Solutions obtained by applying \{the transformations (4.2) and (4.3) satisfying the conditions (4.1) and (4.15) followed by \} to the original solution have been examined from the point of view of the G. E. P. The following results have been obtained.

\( (A') \) For the case in which \( \{ \psi^k, A^k_{\mu}, A_{\mu} \} \) is employed as the set of independent field variables, the G. E. P. is satisfied by solutions if the conditions in Eq. (4.43) are satisfied.

\( (B') \) For the case in which \( \{ \psi^k, b^k_{\mu}, A_{\mu} \} \) is employed as the set of independent field variables, the G. E. P. is satisfied if the conditions in Eq. (4.52) are satisfied.
We would like to add several comments:

[1] For the internal transformation (4.2), the condition (4.43) gives a stronger condition than does the condition (4.52). For the coordinate transformation (4.16), no constraint is imposed by the former, while the latter gives the restrictions $u > 1$, $p + u > 2$.

[2] For the case in which $\{\psi^k, A^k_\mu, A_\mu\}$ is employed as the set of independent field variables, the G. E. P. is satisfied even by solutions which approach constant values very slowly at spatial infinity, as is seen from Eq. (4.44).

This is a direct consequence of the following:

(a) The functions $F_k^{\mu\nu}$ and $F^{\mu\nu} = \sqrt{-g} F_k^{\mu\nu}$ both describe tensor densities, and hence the total energy-momentum $M_k$, total angular momentum $S_{kl}$ and the total charge $C$ are all independent of the coordinate systems employed.

(b) The canonical energy-momentum $M^c_\mu$ and the “extended orbital angular momentum” $L^c_\mu$, which are the generators of coordinate transformations, obey the regular transformation rules (4.37) and (4.38) under the coordinate transformation (4.16) with arbitrary positive $u$ if the condition (4.40) is satisfied.

Note that $b^k_\mu$ as given by Eq. (4.44) approaches a constant value much more slowly than the vierbein components required in the general situation to give reasonable forms of energy-momentum and angular momentum.

[3] In the case in which $\{\psi^k, b^k_\mu, A_\mu\}$ is employed as the set of independent field variables, the G.E.P. for the total energy-momentum and for the total angular momentum is established when the vierbeins have asymptotic property as indicated by the first of Eq. (4.55). This is consistent with the results on the equivalence principle for the energy in new general relativity (N. G. R.).

The preceding results, together with those in Ref. 6), show that the choice $\{\psi^k, A^k_\mu, A_\mu\}$ as the set of independent field variables is preferable to the choice $\{\psi^k, b^k_\mu, A_\mu\}$. This is quite natural, because the fields $\psi^k, A^k_\mu$ and $A_\mu$ are the fundamental objects and $b^k_\mu$ is a composite of $\psi^k$ and $A^k_\mu$.

Appendix A

Asymptotic Forms of $F^{(1)}_k^{\mu\nu}, V^{mn}_\lambda$ and $W^{\mu\nu\lambda\kappa\lambda m}$ for Large $r$

In this appendix, we give the asymptotic forms of $F^{(1)}_k^{\mu\nu}, V^{mn}_\lambda$ and $W^{\mu\nu\lambda\kappa\lambda m}$ for large $r$, which are useful for the calculations in §§3 and 4.

$F^{(1)}_k^{\mu\nu}$:

$$
F^{(1)}_k^{\mu\nu}(0) = -\frac{1}{\kappa} \left( a - \frac{Q^2}{r} \right) \frac{n^\alpha}{r^2} + \frac{a h}{2\kappa r^3} \varepsilon^\alpha_{\beta\gamma} n^\beta + O\left( \frac{1}{r^4} \right),
$$

Note that E. N. G. R. is reduced to N. G. R. if fields with nonvanishing $P_k$ are not present and if the set $\{\psi^k, A^k_\mu, A_\mu, \phi^A, \phi^{*A}\}$ is employed as the set of independent field variables.
\[ F^{(1)}_{a} = -\frac{1}{kr^2} \left( a - \frac{3Q^2}{2r} \right) n_a - Q \frac{\kappa}{2} \delta^a \]
\[ - \frac{ah}{2kr^3} \left( \delta a - 3n a n \beta \right) \epsilon_{a \beta 3} + O_a \left( \frac{1}{r^4} \right) , \]
\[ F^{(1)}_{a \beta} = \frac{3ah}{2kr^3} \left( n_a \epsilon a _\gamma 3 - n a \epsilon g 3 \right) n \gamma + \frac{ah}{kr^4} \epsilon a \alpha 3 + O[\epsilon a \beta] \left( \frac{1}{r^4} \right) , \]
\[ F^{(1)}_{a \alpha} = - \frac{ah}{2kr^3} \left( \epsilon a ^\alpha 3 n a - \epsilon a ^\alpha 3 n a \right) + 2ah \left( n_a \epsilon a _\gamma 3 - n a \epsilon a _\gamma 3 \right) n_a n \gamma \]
\[ + \frac{ah}{kr^4} \epsilon a \beta 3 n a - \frac{Q}{2kr^3} \left( \delta a a \beta - \delta a a \beta n \alpha \right) + O_a \left( \frac{1}{r^4} \right) . \] (A.1)

**V mm** λ:

\[ V^{(0) a} = -n a \frac{\hat{\Xi}}{r} - a n \frac{\hat{\Xi}}{r} \hat{H} - n a \frac{\hat{\Xi}}{2} \left( \frac{\hat{\Xi}}{r} - \frac{a}{r} \hat{H} \right) + n a \frac{\hat{\Xi}}{r} H \hat{\Xi} \]
\[ + O a \left( \frac{1}{r^{s+1}} \right) + O a \left( \frac{1}{r^q} \right) + o a \left( \frac{1}{r^q} \right) , \]
\[ V^{(0) a} = -\frac{1}{r} \left( \delta a - n a n \alpha \right) \Xi - n a n a \Xi - \frac{a}{r^2} \left( \delta a - 2n a n \alpha \right) H \]
\[ - \frac{an a n a}{r} II' + \frac{n a n a}{2} \Xi^2 II' + O a \left( \frac{1}{r^{s+1}} \right) \]
\[ + O a \left( \frac{1}{r^q} \right) + o a \left( \frac{1}{r^q} \right) , \]
\[ V^{ab} = -\frac{a}{r} n a n b \left( \Xi H + \frac{\hat{\Xi}}{r} H \right) - a \frac{n a n b}{r^2} - H II + \frac{1}{2} n a n b \frac{\hat{\Xi}}{r} \]
\[ + O^{ab} \left( \frac{1}{r^{s+1}} \right) + O^{ab} \left( \frac{1}{r^q} \right) + o^{ab} \left( \frac{1}{r^q} \right) , \]
\[ V^{ab} = 1 \frac{1}{r} \left( n a d a - n b d a \right) \left( 1 - \frac{1}{4} \Xi^2 + 2 + O^{ab} \left( \frac{1}{r^{s+1}} \right) \right) \]
\[ + O^{ab} \left( \frac{1}{r^q} \right) + o^{ab} \left( \frac{1}{r^q} \right) \] (A.2)

with

\[ \Xi = A + B , \quad II = A + B - A' - B' + \frac{1}{r} (A + B) , \] (A.3)

and \( o a (1/r^3) \), for example, denotes a term such that \( \lim_{r \to \infty} r^3 o a (1/r^3) = 0. \)

**W mm** kl mn:

\[ W^{0 a} = \frac{1}{2} \left( \delta a d a - \delta a b d a \right) + \frac{a}{4r} \left( \delta a d a + \delta a b d a \right) + \left( n a d a - n b d a \right) n a \]

\( ^* \) Here, the prime and dot represent derivatives with respect to \( r \) and \( x^0 \), respectively. For example, \( A' = \partial A / \partial r \) and \( \dot{A} = \partial A / \partial x^0 \).
\[ W^{\alpha\beta\gamma}_{abc} = \frac{a}{4r} \left\{ \left( \delta^\alpha_a \delta^\beta_b - \delta^\beta_a \delta^\alpha_b \right) \delta^\gamma_c + \left( \delta^\beta_b \delta^\gamma_c - \delta^\gamma_b \delta^\beta_c \right) \delta^\alpha_a \right\} + O\left[\alpha\beta\gamma\right]_{[abc]} \left( \frac{1}{r^2} \right), \]

\[ W^{\alpha\beta\gamma}_{(0)ab} = -\frac{a}{4r} \left\{ \left( \delta^\alpha_a \delta^\beta_b - \delta^\beta_a \delta^\alpha_b \right) n^\gamma + \left( \delta^\beta_a \delta^\gamma_b - \delta^\gamma_b \delta^\beta_a \right) n^\alpha \right\} + O\left[\alpha\beta\gamma\right]_{[ab]} \left( \frac{1}{r^2} \right), \]

\[ W^{\alpha\beta\gamma}_{abc} = \frac{1}{2} \left\{ \left( \delta^\alpha_a \delta^\beta_b - \delta^\beta_a \delta^\alpha_b \right) \delta^\gamma_c + \left( \delta^\beta_b \delta^\gamma_c - \delta^\gamma_b \delta^\beta_c \right) \delta^\alpha_a \right\} + O\left[\alpha\beta\gamma\right]_{[abc]} \left( \frac{1}{r^2} \right). \]