Genus-one correction to asymptotically free
Seiberg-Witten prepotential from
Dijkgraaf-Vafa matrix model

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Abstract

We find perfect agreements on the genus-one correction to the prepotential of $SU(2)$
Seiberg-Witten theory with $N_f = 2, 3$ between field theoretical and Dijkgraaf-Vafa-
Penner type matrix model results.

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1 Introduction

Recently, owing to a milestone discovery made by Alday, Gaiotto and Tachikawa \[1\], there have been lots of publications and research related to their work \[2\]-\[30\]. In particular, Dijkgraaf and Vafa \[31\] proposed a Penner type matrix model whose classical spectral curve can reproduce the so-called Gaiotto curve \(\mathcal{G}\) \[32\]. Note that \(\mathcal{G}\) consists of a punctured Riemann surface \(C_{g,n}\) whose moduli space \(\mathcal{M}_{g,n}\) (\(g\): genus, \(n\): puncture) is referred to as a Teichmuller space. Surprisingly, \(\mathcal{M}_{g,n}\) boils down to the space of exactly marginal gauge couplings of a large family of 4D \(\mathcal{N} = 2\) superconformal gauge theories whose weakly-coupled \textit{cusps} correspond to various patterns of colliding punctures on \(C_{g,n}\). In addition, when \((g,n) = (0,6)\) there appear \textit{generalized} quiver SCFTs in contrast to known linear quiver SCFTs. Further studies towards this newly proposed matrix model can be found in \([33, 34, 35, 36]\). Because \(\mathcal{G}\) is a rewritten Seiberg-Witten curve which emerges by taking a thermodynamic limit of Nekrasov’s partition function \(Z_{\text{Nekrasov}} = Z_{\text{classical}}Z_{\text{1-loop}}Z_{\text{inst}}\) \[37, 38, 39\], attempts towards proving an equivalence between both sides are naturally expected.

At the level of \(\mathcal{F}_0\) (tree-level free energy), Eguchi and Maruyoshi \[34\] showed that \(\mathcal{F}_0\) (including asymptotically free cases) coincides with the original Seiberg-Witten prepotential \[40\]. Moreover, in \[35, 36\] all-genus proofs in certain restricted cases are presented by executing exact matrix integrals and comparing them with \(Z_{\text{Nekrasov}}\). Motivated by these works, in this letter we would like to show agreements between matrix model and field theoretical results on the genus-one free energy \(\mathcal{F}_1\) of \(\mathcal{N} = 2\ SU(2)\) Seiberg-Witten theory with \(N_f = 2, 3\). As a matter of fact, we have closely followed previous approaches in \[41, 42\].

In Section 2, we begin with a topologically twisted theory living on a hyperKähler manifold and extract a physical \(\mathcal{F}_1\). In Section 3, a matrix model proposed by \[34\] is used to compute \(\mathcal{F}_1\). We summarize our result in Section 4.

2 Field theory

Gravitational couplings of the form \(\int d^4x \mathcal{F}_g R_g^2 F_+^{2g-2}\) \((g \geq 1)\) due to a curved four-manifold \(\mathcal{M}_4\) give rise to a corrected Seiberg-Witten prepotential in terms of a \textit{genus}
expansion:

\[ \mathcal{F} = \sum_{g \geq 0} \hbar^{2g-2} \mathcal{F}_g(a, m) = - \log Z_{\text{Nekrasov}}, \]

\( a \) : Coulomb branch parameters, \( m \) : hypermultiplet masses. \hspace{1cm} (2.1)

Here, \( R_+ \) and \( F_+ = \hbar \) are the self-dual part of the Riemann curvature and the graviphoton field strength respectively. In particular, when \( \mathcal{M}_4 \) is Euclidean, the genus-one correction is given by

\[
\int d^4 x \mathcal{F}_1 \text{Tr} R^2_+ = \frac{1}{2} \mathcal{F}_1(\chi - \frac{3}{2} \sigma), \quad R_\pm = \frac{1}{2} (R \pm R^*)
\]

\[
\chi = \frac{1}{32\pi^2} \int R \wedge R^*, \quad \sigma = \frac{1}{24\pi^2} \int R \wedge R \hspace{1cm} (2.2)
\]

where \( \chi(\mathcal{M}_4) \) and \( \sigma(\mathcal{M}_4) \) denote the Euler number and the Hirzebruch signature respectively.

Now, let us focus on a topologically twisted \( \mathcal{N} = 2 \) \( SU(2) \) theory with hypermultiplets living on \( \mathcal{M}_4 \). The low-energy partition function looks like

\[
Z = \int [du] A^x B^x \exp (-S),
\]

\[
A = \alpha \sqrt{\frac{\partial u}{\partial a}}, \quad B = \beta \Delta_{SW}^{\frac{1}{2}}, \quad \alpha, \beta : \text{constants}
\]

where \( u \) stands for the gauge- and monodromy-invariant coordinate of the complex one-dimensional Coulomb branch. Forms of \( A \) and \( B \) appearing above are required to ensure the modular invariance of \( Z \) and necessarily cancel the modular anomaly caused by \([du]\) \[43,44\]. These considerations then define a field theoretical version of the coupling to gravity, i.e.

\[
A^x B^x = \exp \left( b(u) \chi + c(u) \sigma \right), \quad b(u) = \frac{1}{2} \log \left( \frac{du}{da} \right), \quad c(u) = \frac{1}{8} \log \left( \Delta_{SW} \right). \hspace{1cm} (2.3)
\]

Here, \( a \) is the electric period integral of the corresponding Seiberg-Witten curve and \( \Delta_{SW} \) denotes its discriminant. When \( \mathcal{M}_4 \) is hyperKähler (\( \sigma = -2\chi/3 \)) or a \( K3 \) manifold (\( \chi = 24 \) and \( \sigma = -16 \)), the effect of twist\(^1\) is not visible and (2.3) of a twisted theory becomes

\(^1\)The topological twist is performed through replacing \( SU(2)_+ \subset SO(4) \cong SU(2)_+ \times SU(2)_- \) by the diagonal part of \( SU(2)_+ \times SU(2)_R \) where \( SU(2)_R \) represents the \( R \)-symmetry. For hyperKähler manifolds, that no holonomy is involved in \( SU(2)_+ \) implies that to twist will not be visible.
compatible with that of a physical theory. Therefore, equating \( b(u)\chi + c(u)\sigma \) of a hyperKähler \( M_4 \) in \( 2.3 \) we see that

\[
\mathcal{F}_1 = b(u) - \frac{2}{3} c(u). \tag{2.4}
\]

In order to determine \( b(u) \) and \( c(u) \), one needs an explicit Seiberg-Witten curve \( \Sigma \)

\[
\Sigma : \prod_{l=0}^{n} (t - t_l) v^2 = M_{n+1}(t) v + U_{n+1}(t), \quad (t, v) \in (\mathbb{C}^* - \{t_0, \ldots, t_n\}) \times \mathbb{C} \tag{2.5}
\]

and notices that (\( \lambda_{SW} \): Seiberg-Witten one-form)

\[
\frac{da}{du} = \frac{d}{du} \int_A \lambda_{SW}.
\]

The subscript of polynomials \( M \) and \( U \) denotes their degree. According to \[15\], \( \Sigma \) arises from an M-theory lift of Type IIA D4- and NS5-branes engineering \( \mathcal{N} = 2 \) \( SU(n+1) \) Yang-Mills theory with fundamental matters which are encoded at two asymptotic ends (\( t = 0, \infty \)) of \( \Sigma \). The gauge coupling \( \tau_I \) of \( I \)-th gauge factor of a conventional linear quiver is expressed in terms of \( t = \exp \left( x^6 + i x^{10} / R_M \right) \) (\( R_M \): M-circle radius) parameterizing a cylinder along \((x^6, x^{10})\):

\[
i \pi \tau_I = \log \frac{t_{I-1}}{t_I}, \quad \tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}.
\]

As shown in \[32\], through performing a change of variables \( v = xt \) and certain proper Möbius transformation on \( t \), one finally obtains a so-called Gaiotto curve:

\[
\mathcal{G} : x^2 = \phi^\text{SW}_2(z), \quad xdz \equiv \lambda_{SW}. \tag{2.6}
\]

For the simplest \( SU(2) \) Seiberg-Witten theory with \( N_f = 2 \) and 3, the period integral \( a(u) \) had been computed by Ohta \[46\] in terms of a large-\( u \) expansion (weak coupling expansion). Therefore, it is straightforward to evaluate \( b(u) \) for \( N_f = 2 \):

\[
b(u) = -\frac{1}{2} \log \left( \frac{da}{du} \right) = \frac{3}{4} \log 2 - \frac{1}{4} \log \zeta - \frac{3\Lambda^2}{2048}(\Lambda^2 + 64m_1m_2)\zeta^2 + \frac{15\Lambda^4}{2048}(m_1^2 + m_2^2)\zeta^3 + \mathcal{O}(\zeta^4) \tag{2.7}
\]

where \( \zeta = 1/u \). Similarly, for \( N_f = 3 \),

\[
b(u) = \frac{1}{2} \log 2 - \frac{1}{4} \log \zeta - \frac{\Lambda^2}{2048}\zeta - \frac{\Lambda}{8388608} \left( 7\Lambda^3 + 12288(m_1^2 + m_2^2 + m_3^2)\Lambda + 786432m_1m_2m_3 \right)\zeta^2 + \mathcal{O}(\zeta^3). \tag{2.8}
\]
We have denoted flavor bare masses and the dynamical scale by $m_i$’s and $\Lambda$ respectively. In Section 3, we will find perfect agreements with these results in carrying out a computation via the matrix model proposed by Eguchi and Maruyoshi [34].

### 3 Matrix model

Before computing the genus-one free energy $F_1$, we first give a brief introduction about the newly proposed Dijkgraaf-Vafa matrix model. Without the background charge $Q = b + b^{-1}$ ($b = i$), in computing correlators of vertex operators $\langle \prod_i V_i(\xi_i) \rangle$ in Liouville theory, Dijkgraaf and Vafa [31] have replaced the usual Liouville wall by a chiral one $\int d\xi e^{\sqrt{2} b \phi(\xi)}$. This results in a hermitian matrix model with an usual Vandermonde, and inserted operators $V_i(\xi_i) = e^{i \sqrt{2} \nu_i \phi(\xi)}$ as a whole consequently lead to a logarithmic potential of Penner type, i.e.

$$Z_{DV} = \left\langle \prod_i V_i(\xi_i) \right\rangle_{\text{chiral Liouville}} = \int_{N \times N} dM \exp \left( \frac{1}{g_s} \text{Tr} W(M) \right) = \exp \left( - \sum_{g \geq 0} g_s^{2g-2} F_g \right),$$

$$W(M) = \sum_i \mu_i \log(M - \xi_i), \quad \mu_i = 2g_s \nu_i, \quad \sum_i \mu_i + \mu_0 + 2g_s N = 0,$$

$p_i, \ N \to \infty, \ g_s \to 0, \ \mu_i, \ g_sN = \text{fixed}. \quad (3.1)$

The charge conservation is respected by placing $\mu_0$ units at infinity.

Interpreting the above chiral free boson $\phi$ as a Kodaira-Spencer (collective) field which is especially powerful in dealing with quantizing the Riemann surface complex moduli, one can express the matrix model quantum spectral curve as

$$-i \langle \partial \phi(z) \rangle = \left( -\nu_0 z^{-1} + \sum_{n>0} n \nu_n z^{n-1} + g_s^2 \sum_{n \geq 0} z^{-n-1} \frac{\partial}{\partial \nu_n} \right) Z_{DV}$$

with $z$ parameterizing it. $\nu_n$ and its conjugate are referred to as symplectic coordinates of the moduli space. Eventually, $\langle \partial \phi(z) \rangle_{g_s \to 0}$ just reduces to a Gaiotto curve $\mathcal{G}$ of $\mathcal{N} = 2$ SU(2) SCFTs as will be explained more below. Dijkgraaf and Vafa’s intuition seems due to the marvelous discovery of Alday, Gaiotto and Tachikawa [1] relating correlators in Liouville theory to Nekrasov’s partition function. Recall that $\mathcal{G}$ was yielded by reorganizing a Seiberg-Witten curve which emerges via taking a thermodynamic limit ($\hbar \to 0$) of $Z_{\text{Nekrasov}}$. It is thus very tempting to recognize a full equivalence
\[ Z_{DV} = Z_{\text{chiral Liouville correlator}} = Z_{\text{Nekrasov}} \text{ with } g_s = \hbar. \] This line has been pursuit in \[34, 35, 36\].

As pointed out by AGT, one can yield a quadratic Seiberg-Witten differential from Ward identities in Liouville theory:\[3\]:
\[
\phi_2(z)dz^2 = \frac{\left< T(z) \prod_i O_i(z_i) \right>}{\left< \prod_i O_i(z_i) \right>}, \quad T(z) : \text{stress tensor},
\]
\[
\phi_2(z)dz^2 \rightarrow \phi_2^{SW}(z)dz^2 = \lambda_{SW}^2, \quad \text{when } 1 \gg \epsilon_{1,2}. \quad (3.2)
\]

Note that \(O\)'s are inserted at the level of conformal blocks in Liouville theory, while \(\epsilon_i\) denotes the non-self-dual graviphoton field strength appearing in Nekrasov’s formula. Through \(x^2 = \phi_2^{SW}(z)\) one obtains a Gaiotto curve which is a double cover of a punctured sphere with cuts. From \[3.2\], a reasonable analogy is strongly recommended in the aforementioned \(Z_{DV}\). Because a stress tensor on the hermitian matrix model side can be defined through a Kodaira-Spencer field, i.e.
\[
T(z) = -\frac{1}{2}(\partial \phi)^2 = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},
\]
a classical spectral curve emerging in large-\(N\) limit is written as
\[
\langle T(z) \rangle = -\frac{1}{2}\langle \partial \phi(z)^2 \rangle \rightarrow \frac{1}{2} \mathcal{W}'(z)^2 + 2f(z) \quad (3.3)
\]
where the average is w.r.t. \(Z_{DV}\). Equivalently,
\[
-x = i\langle \partial \phi(z) \rangle = -\mathcal{W}'(z) - 2\omega(z), \quad \omega(z) = g_s \text{Tr} \left( \frac{1}{z-M} \right)
\]
with which an \(SU(2)\) Gaiotto curve is identified by Dijkgraaf and Vafa. The arrow in \[3.3\] is completely owing to a factorization of the resolvent operator at large-\(N\) limit:
\[
g_s^2 \left< \text{Tr} \frac{1}{z-M} \text{Tr} \frac{1}{z-M} \right> = \omega(z)^2
\]

\[2\text{An early attempt towards interpreting Nekrasov’s partition function as a kind of tachyon’s scattering amplitude in the self-dual } c = 1 \text{ string theory can be found in } [47]. \text{ There, vertex operators made of a collective field of a Fermi fluid are inserted at } q \text{-numbered positions on two asymptotic regions of a sphere.}
\]

\[3\text{We must apologize for using } \phi \text{ in expressing quadratic differentials and Kodaira-Spencer fields simultaneously.}\]
such that the all-genus loop equation becomes

$$\omega(z)^2 + \omega(z)W'(z) = g_s \left( \frac{W'(z) - W'(M)}{z - M} \right) = f(z).$$

In [34], $x^2 = W'(z)^2 + 4f(z)$ was shown to coincide with $x^2 = \phi_{SW}^2(z)$ in (3.2) by fully exploiting known properties of standard Seiberg-Witten curves.

Let us pause to see a canonical example $SU(2)$ $N_f = 4$. Four insertions $(V_0, V_1, V_2, V_3)$ are prescribed to be located at $(\infty, q, 1, 0)$ in order and $V_0$ at $\infty$ will never show up in $W$ though. It is evident that residues of $x_{DV}(z)$ at $(\infty, q, 1, 0)$ correspond to momenta of $V_i$'s which are identified with flavor bare masses according to AGT dictionary. Also, $q$ stands for the cross-ratio of four distinct punctures on a sphere and hence lives on $\mathbb{CP}^1 \{0, 1, \infty\}$.

More explicitly, one is allowed to choose certain Möbius transformation $f$:

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad a, b, c, d \in \mathbb{C}$$

which brings three points $(z_1, z_2, z_3)$ on a sphere to the triple $(0, 1, \infty)$, while $z_4$ is mapped to $f(z_4) = q$. Ultimately, $q$ is just $q_{UV} = e^{i\pi\tau_{UV}}$ because this interpretation is totally supported by the known space of the exactly marginal (ultra-violet) gauge coupling constant $\tau_{UV}$.

### 3.1 Genus-one correction

For four insertions at $(\infty, q, 1, 0)$ in (3.1), it is obvious that there will be two critical points (zeros) for $W'(z) = 0$ of $Z_{DV}$ if one recalls that $V_0(\infty)$ does not show up. When quantum effects introduced by the resolvent are incorporated, they blow up into two cuts whose filling fractions $N_1$ and $N_2$ subject to the constraint $N_1 + N_2 = N$ ($N$: rank of $M$). The classical spectral curve is a double cover of a punctured sphere with two cuts and this kind of two-cut model has been fully investigated [48, 49]. Borrowing Akemann’s analysis, we are able to have the genus-one free energy expressed in an universal form$^4$

$$\mathcal{F}_1 = -\frac{1}{24} \sum_{i=1}^{4} \log M_i - \frac{1}{12} \log \Delta - \frac{1}{2} \log |K(\ell)| + \frac{1}{4} \log |(x_1 - x_3)(x_2 - x_4)|$$

$^4$Strictly speaking, this form was prescribed for a polynomial potential $W(z)$. 

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where
\[ M_i = \oint_{C} \frac{dz}{2\pi i (z-x_i)} \sqrt\prod_{i=1}^{4} (z-x_i), \quad C : \text{contour encircling both cuts} \]
\[ \ell^2 = \frac{(x_1-x_4)(x_2-x_3)}{(x_1-x_3)(x_2-x_4)}, \quad \Delta = \prod_{i<j} (x_i-x_j)^2. \quad (3.5) \]

\([x_1, x_2]\) and \([x_3, x_4]\) stand for branch points of these two cuts with their cross-ratio denoted by \(\ell^2\), while \(K(\ell)\) is the complete elliptic integral of the first kind. One can soon realize that \(M_i = 0\) when the contour \(C\) is deformed to enclose \(\infty\). Divergent terms like \(\log M_i\) will then be omitted. To deal with subsequent terms without knowing explicitly four branch points, we can appeal to a very helpful formula suggested by Masuda and Suzuki [50]. That is, noting the equality between a hypergeometric function and a complete elliptic integral \(\frac{\pi}{2} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \ell^2\right) = 2K(\ell)\), one is able to rewrite the last two terms in (3.4) as

\[-\frac{1}{2} \log \left(\frac{\pi}{2} (-D)^{-\frac{1}{4}} 2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta_g}{4\Delta_g^3}\right)\right).\]

Here, \(2F_1(\alpha, \beta; \gamma; \delta)\) is the hypergeometric function, \(\Delta_g\) is the discriminant of certain quartic polynomial \(g(y) = y^4 + ay^3 + by^2 + cy + d\) whose four roots are previous \((x_1, x_2, x_3, x_4)\) with

\[ \Delta_g = -\left(27a^4d^2 + a^3c(4c^2 - 18bd) + ac(-18bc^2 + 80b^2d + 192d^2) \right. \]
\[ + a^2(-b^2c^2 + 4b^3d + 6c^2d - 144bd^2) + 4b^3c^2 + 27c^4 \]
\[ - 16b^4d - 144bc^2d + 128b^2d^2 - 256d^3 \]

and \(D \equiv -b^2 + 3ac - 12d\).

For asymptotically free \(SU(2)\) \(N_f = 2\) and 3, the classical spectral curve can be derived from the original \(N_f = 4\) one via scaling limits which amount to decoupling extremely massive flavors. By adhering to [34] and adopting their convention, adequate candidates responsible for the aforementioned quartic \(g(y)\) extracted from the classical spectral curve are then

\[ R_4(y) = y^4 + \frac{4M_+}{\Lambda_2^2} y^3 + \frac{4v}{\Lambda_2^3} y^2 + \frac{4\tilde{M}_+}{\Lambda_2^4} y + 1 \quad (3.6) \]
and
\[ Q_4(y) = y^4 + \frac{1}{M_0^2} \left( -v - M_0^2 + M_2^2 + \frac{1}{2} \tilde{M}_+ \Lambda_3 \right) y^3 \]
\[ + \frac{1}{M_0^2} \left( v + \frac{\Lambda_3^2}{4} - \frac{3}{2} \tilde{M}_+ \Lambda_3 \right) y^2 + \frac{1}{M_0^2} \left( -\frac{\Lambda_3^2}{2} + \tilde{M}_+ \Lambda_3 \right) y + \frac{\Lambda_3^2}{4M_0^2} \] (3.7)
for \( N_f = 2 \) and 3 respectively. Through the following identification in (3.6):
\[ v = 4u, \quad M_+ = 2m_1, \quad \tilde{M}_+ = 2m_2, \quad \Lambda_2 = \Lambda, \] (3.8)
the last two terms in (3.4) are thus found to be \((\zeta = 1/u)\)
\[ -\frac{1}{2} \log \frac{\pi}{8} - \frac{1}{2} \log \Lambda - \frac{1}{4} \log \zeta - \frac{3\Lambda^2}{2048}(\Lambda^2 + 64m_1m_2)\zeta^2 + \frac{15\Lambda^4}{2048}(m_1^2 + m_2^2)\zeta^3 + \mathcal{O}(\zeta^4) \] (3.9)
expressed in terms of a large-\( u \) expansion. Similarly, through
\[ v = 4u, \quad M_+ = 2m_1, \quad M_- = 2m_2, \quad \tilde{M}_+ = 2m_3, \quad \Lambda_3 = \frac{\Lambda}{2} \] (3.10)
in (3.7), the last two terms in (3.4) are found to be
\[ -\frac{1}{2} \log \frac{\pi}{4} - \frac{1}{2} \log |m_1 - m_2| - \frac{1}{4} \log \zeta - \frac{\Lambda^2}{2048} \zeta \]
\[ - \frac{\Lambda}{8388608} \left( 7\Lambda^3 + 12288(m_1^2 + m_2^2 + m_3^2)\Lambda + 786432m_1m_2m_3 \right) \zeta^2 + \mathcal{O}(\zeta^3). \] (3.11)

As stressed before, the matrix model classical spectral curve is just the same as the corresponding Gaiotto curve (rearranged Seiberg-Witten curve), henceforth we still have same discriminant \( \Delta_{SW} = \Delta \) in (2.3) and (3.4) even after decoupling massive flavors.\(^5\)

Equipped with these facts, we conclude that computations on both field theory and matrix model sides give perfectly the same \( \mathcal{F}_1 \) up to some irrelevant constant terms by looking at (2.7), (2.8), (3.9) and (3.11).

### 4 Summary

We have provided further evidence on the equivalence between a recently proposed Dijkgraaf-Vafa matrix model and low-energy dynamics of \( \mathcal{N} = 2 \) asymptotically free \( SU(2) \) Yang-Mills theory with \( N_f = 2, 3 \) at the level of \( \mathcal{F}_1 \). We utilized the matrix model technique

\(^5\)In fact, this can be easily checked by comparing our above \( \Delta_g \) with the known \( \Delta_{SW} \).
which prescribes an universal form of $\mathcal{F}_1$. Ingredients for computing the asymptotically free $\mathcal{F}_1$ can be gathered just from a classical spectral curve found in [34] by decoupling very massive flavors from an $N_f = 4$ one. Showing perfect agreements with the field theoretical result, we thus extend the equivalence of $Z_{DV}$ and $Z_{Nekrasov}$ at next-to-leading order non-trivially.

It will also be interesting to examine whether this check gets possible in the super-conformal $N_f = 4$ case. As shown by Eguchi and Maruyoshi in this situation $da/du = K(q_{UV})$, so it is quite tempting to consider relations between $q_{UV}$ and the cross-ratio $\ell^2$ of four branch points given a complete elliptic integral of the first kind in the universal expression of $\mathcal{F}_1$ in (3.4).

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