Consistent $\sigma$-models in N=1 supergravity

S. Groot Nibbelink and J.W. van Holten

NIKHEF, P.O. Box 41882,
1009 DB, Amsterdam NL

Revised version
August 8, 1999

Abstract
A consistent $N = 1$ supersymmetric $\sigma$-model can be constructed, given a Kähler manifold, by adding chiral matter multiplets. Their scalar components are covariant tensors on the underlying Kähler manifold. The Kähler $U(1)$-charges can be adjusted such that the anomalies cancel, using the holomorphic functions in which the Kähler potential transforms. The arbitrariness of the $U(1)$-charges of matter multiplets is related to their Weyl-weights in superconformal gravity, before it is reduced to supergravity. The covariance of the Kähler potential forces the superpotential to be covariant as well. This relates the cut-off, the Planck scale and the matter charges to each other. A non-vanishing VEV of the covariant superpotential breaks the Kähler $U(1)$ spontaneously. If this VEV vanishes, the gravitino is massless and depending on the above mentioned parameters there may be additional internal symmetry breaking. The separation of the different representations of chiral multiplets can be achieved by covariantizations of derivatives and fermions. Using non-holomorphic transformations, the full Kähler metric can be block-diagonalized and the necessary covariantizations come out naturally. Various aspects are illustrated by applying them to Grassmannian coset models. As an example the coset $SU(5)/SU(2) \times U(1) \times SU(3)$ with the field content of the standard model is constructed. Phenomenological aspects of this model are analyzed.
1 Introduction

Like gravity, non-linear $\sigma$-models in four dimensions are not renormalizable. They involve a natural scale $\Lambda$ above which new dynamics or new physical degrees of freedom can come into play. The existence of such a scale requires the introduction of a parameter $f \propto \Lambda^{-1}$ of inverse mass dimension. Below this scale the $\sigma$-models are useful as effective field theories, for example to describe the low-energy dynamics of bound states in strongly interacting gauge theories like QCD. Effective field theories also arise in the low-energy description of quantum string theory, in which case the string scale sets the limit of applicability. This scale is connected with the Planck scale, as classical gravity described in a general relativistic formulation is part of the effective long-distance physics which comes out of string theory.

Because realistic string theories require supersymmetry for their consistency, the most obvious candidates for effective low-energy theories arising from string models are four-dimensional $N = 1$ supergravity theories, subject to phenomenological constraints as well. These theories describe gravity and the other interactions in the context of a locally supersymmetric field theory, which is not renormalizable but well-behaved below the Planck scale. With well-behaved we mean, that by taking into account the presence of a cut-off, these theories give unambiguous and consistent answers to questions related to phenomena at distance scales large compared to the cut-off. In particular one requires the proper incorporation of symmetries and the absence of anomalies in local gauge-invariances like those of the electro-weak or grand-unified interactions.

For such reasons it is important to be able to construct the most general locally supersymmetric field theories including local chiral and non-chiral gauge-interactions free of anomalies at the quantum level. In this paper we address this question within the context of conventional representations of $N = 1$ supersymmetry, which besides the supergravity multiplet include complex chiral and real vector multiplets.

The complex scalars of the chiral-multiplet sector of an $N = 1$ supersymmetric theory parameterize a Kähler manifold. In particular, the low-energy lagrangean of this sector is determined by a single real function of the chiral superfields $\Sigma^A$, the Kähler potential $K(\bar{\Sigma}, \Sigma)$, from which objects like the metric and curvature of the manifold can be computed. Interesting examples of Kähler manifolds include cosets of the Grassmannian type \cite{1,2,3,4}, and those based on exceptional groups like $E_6/\text{SO}(10) \times U(1)$, $E_7/\text{SU}(5) \times \text{SU}(3) \times U(1)$ or $E_8/\text{SO}(10) \times \text{SU}(3) \times U(1)$ \cite{5,6,7,8,9}

The pure $N = 1$ supersymmetric $\sigma$-models can be extended in several ways: by adding superpotentials, by gauging some or all of the isometries and by adding additional matter superfields in appropriate representations of the isometry group. For model-building purposes it is necessary to analyze what kind of low-energy physics then emerges from these models and their various extensions. This is
a highly non-trivial question, as supersymmetry requires the inclusion of many rather special scalar potential terms and Yukawa couplings for its consistency. The patterns of internal and supersymmetry breaking, and of boson and fermion masses emerging from these models can become quite intricate.

Further constraints come from requiring the light particles in the model to have assigned charges and other quantum numbers to them, consistent with standard-model phenomenology, and to be free of chiral anomalies in the σ-model or gauge interactions. This has been the subject of various earlier investigations \[10, 11, 12, 13, 14\]. In these studies it was concluded that many models based on mathematically interesting coset spaces like the symmetric spaces, appear to suffer from problems in these respects. However, in a recent paper \[16\] we showed that the \(U(1)\) charge assignments of the chiral superfields allow more freedom then was previously thought. As a result anomaly cancellation can be realized in phenomenologically interesting ways by combining appropriate representations of isometry groups, even in models based on these coset spaces. It is then of clear importance to study which consistent effective supergravity models can be constructed incorporating these ideas.

The present paper is a step in this direction. Building on the results of \[17, 18, 24\], we discuss the extension of the supergravity lagrangeans necessary to include non-linear internal gauge symmetries. We describe how the non-linear transformations can be modified to assign arbitrary \(U(1)\) charges to supermultiplets in any representation of the internal gauge group, and we discuss the cancellation of anomalies.

This paper is composed as follows. The main aspects of gauged supersymmetric σ-models on Kähler manifolds are reviewed in section 2. We discuss their extension to local supersymmetry in section 3. This includes a description of the role of the non-linear compensating scalar multiplet introduced in \[16\]. Section 4 analyses the phenomenology in supergravity of models where the Kähler potential transforms covariantly. In section 5 the various constructions are applied in the context of the Grassmannian coset spaces \(U(n + m)/U(n) \times U(m)\) and its non-compact analogs. An anomaly-free supersymmetric model of a family of quarks and leptons in representations of \(SU(5)/SU(3) \times SU(2) \times U(1)\) is presented in sect.6. Diagonalization of the metric and propagators is discussed in a general geometrical setting in section 7, using some of the geometrical constructions presented in the appendix. Its implementation in the case of Grassmannian models is given in section 8. The vacuum configuration of the Grassmannian model based on \(SU(5)\) with the standard model particle spectrum is analysed. The conclusions and lessons drawn from these investigations are summarized in section 10. Their applications to other models are described in a separate publication \[19\].
2 Supersymmetric lagrangeans

In this section we present the machinery to describe $N = 1$ supersymmetric lagrangeans [23, 1, 2, 3]. The geometrical objects we use in this section are just short-hand to cast the formulae in a more systematic form. The geometrical aspects are of use later on in this article. All the supersymmetric field theories which are developed in this section have to be interpreted as effective field theories involving a cut-off scale $f^{-1}$. This cut-off scale is used explicitly only when needed to give a certain object its canonical dimension.

Let $\Sigma^A = (Z^A, \psi^A_L, H^A)$ be a set of chiral multiplets, where $Z^A$ is a physical complex scalar, $\psi^A_L$ a chiral fermion and $H^A$ is an auxiliary complex scalar. The index $A$ enumerates the multiplets in the set. The kinetic part of the lagrangian for such chiral multiplets is given in terms of a real composite superfield $K(\bar{\Sigma}, \Sigma)$ by the following supersymmetric expression

$$L_K = K(\bar{\Sigma}, \Sigma) |_{\mathcal{D} = -g^A_A \partial^\mu \bar{Z}^A \partial_\mu Z^A + \bar{\psi}^A_L \Gamma^A_{LR} \psi^A_R - \hat{H}^A \hat{H}^A)$$

(2.1)

$$+ R_{AB \bar{B}} (\bar{\psi}^A_{\bar{B}} \psi^B_R) (\bar{\psi}^A_L \psi^B_R).$$

The complex hermitean metric $g_{AB}$ can be derived from the Kähler potential $K(\bar{Z}, Z)$ by

$$g_{AB} = K_{,AB}. \quad (2.2)$$

The auxiliary fields $H^A$ are redefined as $\hat{H}^A = H^A - \Gamma^A_{BC} \bar{\psi}^B_R \psi^C_L$ and the Kähler covariant derivative $D^A = \partial^A + \Gamma^A_{BC} \psi^B_L \partial^\mu Z^C_C$ is introduced with

$$\Gamma^A_{BC} = g^{A \bar{A}} g_{AB, \bar{C}}, \quad \bar{\Gamma}^A_{BC} = g^{A \bar{A}} g_{\bar{B} \bar{C}, A} \quad (2.3)$$

the complex connections. The four-fermion terms have been rewritten with the help of the curvature tensor

$$R_{AB \bar{B}} = g_{AB, \bar{B}} - g_{AB, A} g^{C \bar{C}} g_{\bar{C} \bar{B}, \bar{B}}. \quad (2.4)$$

In addition one can write down a lagrangean determined by a holomorphic function $W$, called the superpotential, of chiral superfields

$$L_W = [W(\Sigma)]_F = \frac{1}{2} W_{,A} H^A - \frac{1}{2} W_{,AB} \bar{\psi}^A_R \psi^B_R + \text{h.c.}. \quad (2.5)$$

It follows from the lagrangean (2.1) that the symmetries of this supersymmetric model are given by the isometries of the metric $g_{AB}$ which leave (2.5) invariant.\footnote{In fact $W$ may transform with a phase factor which does not depend on the fields, which can be compensated by a chiral rotation of the fermions to leave the Yukawa couplings invariant. This so-called $R$-symmetry is broken if the superpotential does not transform homogeneously.}
The isometries are generated by a complete set of the Killing vectors $\mathcal{R}_i^A(Z)$ which determine the transformation rules for the chiral multiplet completely

$$
\delta_i \Sigma^A = \mathcal{R}_i^A(\Sigma) = \begin{cases} 
\delta_i Z^A & = \mathcal{R}_i^A(Z), \\
\delta_i \psi^A_L & = \mathcal{R}_{i,B}^A(Z)\psi^B_L, \\
\delta_i H^A & = \mathcal{R}_{i,B}^A(Z)H^B - \mathcal{R}_{i,BC}^A(Z)\bar{\psi}^B_L\psi^C_L.
\end{cases}
$$

(2.6)

The Killing vectors satisfy the Killing conditions

$$
\left(g_{AB}^A, g_{BC}^B\right) = 0,
$$

therefore they obey

$$
\mathcal{R}^B_{[i} \mathcal{R}^A_{j]} = f_{ij}^k \mathcal{R}^A_k.
$$

(2.7)

This defines a representation of the abstract algebra $\delta_i \delta_j = f_{ij}^k \delta_k$ of a group with the structure coefficients $f_{ij}^k$, satisfying the Jacobi identities for consistency.

The Kähler potential $\mathcal{K}$ may transform under the isometries (2.6) as

$$
\delta_i \mathcal{K}(\bar{Z}, Z) = \mathcal{K}_{,A} \mathcal{R}^A_i + \mathcal{K}_{,\bar{A}} \bar{\mathcal{R}}^A_i = \mathcal{F}_i(Z) + \bar{\mathcal{F}}_i(\bar{Z}).
$$

(2.8)

The functions $\mathcal{F}_i$ (\bar{\mathcal{F}}_i) are (anti-)holomorphic functions, as the metric is defined by (2.2). By using the group property of the isometries and the fact that $\mathcal{F}_i$ and $\mathcal{R}_i$ are both holomorphic, it follows that the algebra of the functions $\mathcal{F}_i$ is determined by the structure constants, up to an imaginary constant part: $\delta_i \mathcal{F}_j = f_{ij}^k \mathcal{F}_k + a_{ij}$ where the constants $a_{ij}$ are real and anti-symmetric. By an appropriate shift of the functions $\mathcal{F}_i$ these constants can be absorbed into the definition of $\mathcal{F}_i$, so as to give

$$
\delta_i \mathcal{F}_j = f_{ij}^k \mathcal{F}_k.
$$

(2.9)

Thus the holomorphic functions $\mathcal{F}_i$ transform as a 1-cycle. In case the Ricci tensor $R_{\bar{A}A}$ is proportional to the metric: $R_{\bar{A}A} = f^2 g_{\bar{A}A}$ (Einstein spaces), the Kähler potential can be written as $\mathcal{K} = f^{-2} \ln \det g$ and the holomorphic functions are given by $\mathcal{F}_i = 1/(2f^2) \mathcal{R}^A_i$. Defining the Killing potentials $\mathcal{M}_i(Z, \bar{Z})$ as

$$
i \mathcal{M}_i = \mathcal{K}_{,A} \mathcal{R}^A_i - \mathcal{F}_i = -\mathcal{K}_{,\bar{A}} \bar{\mathcal{R}}^A_i + \bar{\mathcal{F}}_i,
$$

(2.10)

with the second identity following from eq.(2.8), one observes that the Killing potentials $\mathcal{M}_i$ are real functions. The Killing vectors $\mathcal{R}^A_i$ can be obtained from them by

$$
i \mathcal{M}_{i,\bar{A}} = g_{\bar{A}A} \mathcal{R}^A_i, \quad i \mathcal{M}_{i,A} = -g_{\bar{A}A} \bar{\mathcal{R}}^A_i.
$$

(2.11)

They transform under the isometries in the adjoint representation

$$
\delta_i \mathcal{M}_j = -\delta_j \mathcal{M}_i = f_{ij}^k \mathcal{M}_k.
$$

(2.12)
When the isometry group is semi-simple, all geometrical objects of the Kähler manifold can be expressed in terms of Killing potentials [20, 21].

If part of the internal symmetries are local, the partial derivatives in eq.(2.1) and in the Kähler covariant derivative $D_{\mu}$ have to be replaced by gauge covariant ones given by

$$
\partial_{\mu} Z^A \rightarrow D_{\mu} Z^A = \partial_{\mu} Z^A - A_{\mu}^i R_i^A,
$$

$$
D_{\mu} \psi^A_L \rightarrow D_{\mu} \psi^A_L = \partial_{\mu} \psi^A_L - A_{\mu}^i R_i^A \psi^B_L + D_{\mu} Z^C \Gamma_{CDE}^A \psi^B_L, \quad (2.13)
$$

where $A_{\mu}^i$ are the gauge fields corresponding to the local symmetries. They are components of the vector multiplets $V^i = (A_{\mu}^i, \lambda^i, D^i)$, with $\lambda^i$ representing the gauginos and $D^i$ the real auxiliary fields.

After the introduction of the gauge fields in the lagrangean (2.1) via the covariant derivatives (2.13), the $\sigma$-model itself is not invariant under supersymmetry transformations anymore. This is resolved by adding the terms

$$
\Delta L_K = 2 g_A \left( \bar{R}^A_{iR} \lambda^i_R \psi^A_L + \bar{R}^A_{iL} \lambda^i_L \psi^A_R \right) - D^i (\mathcal{M}_i + \xi_i) \quad (2.14)
$$

to the lagrangean (2.1), including Fayet-Illiopoulos terms if applicable.

The kinetic terms for these vector multiplets take the form [17, 18]

\[
\mathcal{L}_f = \left[ f_{ij} W^i(V) W^j(V) \right]_F = \frac{1}{2} f_{ij} \left( -\bar{\lambda}^i_R \bar{\psi}^A_R \lambda^j_L - \frac{1}{2} F^{i-} \cdot F^{j-} + \frac{1}{2} D^i D^j \right)
\]

\[
+ \frac{1}{2} f_{ij} \mathcal{A} \left( -\sigma \cdot F^{i-} + i D^i \right) \bar{\psi}^A_R \lambda^j_L - \frac{1}{4} f_{ij} \mathcal{A} H^A \lambda^i_R \lambda^j_L \quad (2.15)
\]

+ \frac{1}{4} f_{ij} \mathcal{A} \mathcal{B} \left( \bar{\psi}^A_R \psi^B_L \right) \left( \bar{\lambda}^i_R \lambda^j_L \right) + \text{h.c.},
\]

where the $f_{ij}$ are chiral superfields transforming covariantly under the group of isometries; for example, they can be holomorphic functions of the chiral superfields $\Sigma$. The anti-selfdual field strength is defined as $F_{\mu \nu}^{i-} = \frac{1}{2} \left( F_{\mu \nu}^{i} - \tilde{F}_{\mu \nu}^{i} \right)$. The covariant derivative acting on the gauginos is defined in the adjoint representation. The standard form of the function $f_{ij}$ is $f_{ij}(Z) = \sigma(Z) \eta_{ij}$ where $\eta_{ij}$ is the Killing metric defined from the structure coefficients

$$
-2 \eta_{ij} = -2 C_A \delta_{ij} = f_{ik} f_{jl} \quad (2.16)
$$

and $\sigma(Z)$ is an invariant holomorphic scalar coefficient. The indices $i, j, \ldots$ run over the gauged part of the isometries and $\delta_{ij}$ is the Killing metric normalized by the Casimir $C_A$ of the adjoint. When a direct product group of subgroups is gauged, there are as many different coefficients $\sigma^{(i)}$ as there are subgroups. The
real parts of the coefficients: \( \text{Re} \left( C_A \sigma^{(i)} \right) \), can be interpreted as coupling constants \( 1/(g^{(i)})^2 \).

From the covariance of the Killing potentials, eq.(2.12), it is obvious that in non-supersymmetric models one can write down a general class of non-minimal kinetic terms for the gauge fields, with \( f_{ij} \) of the form

\[
f_{ij} = \sigma \eta_{ij} + \rho \mathcal{M}_i \mathcal{M}_j.
\]

(2.17)

Here the coefficients \( \sigma \) and \( \rho \) must be scalars under the internal symmetries. The inverse of \( f_{ij} \) can be obtained if its determinant is non-zero; this happens in particular if the Killing metric \( \eta_{ij} \) is invertable and \( \sigma, \sigma + \rho (\mathcal{M}_k)^2 \neq 0 \). The non-minimal kinetic terms of this type are more complicated than the ones usually discussed in the context of \( N = 1 \) supersymmetry, as for \( \rho \neq 0 \) the \( f_{ij} \) are non-holomorphic functions of the chiral goldstone-boson superfields. Supersymmetrization of these terms is possible, but only at the expense of introducing higher-derivative terms for the chiral multiplets.

To see this, we recall that in terms of components the \( W^i \) form a particular kind of chiral spinor-tensor multiplet, obtained from the vector multiplet \( V \) by

\[
W_L(V) = \left( \lambda^i_L, \frac{1}{2} \left( -\sigma \cdot F^i - iD^i \right), D \lambda_R^i \right).
\]

(2.18)

Now the \( \sigma \)-model action itself is derived from a composite abelian vector multiplet \( \mathcal{K} = (B_\mu, \Lambda, D) \), with components defined as

\[
B_\mu = iK_\alpha \partial_\mu z^\alpha - iK_\alpha \partial_\mu z^\alpha + 2gV^i M_i - 2ig\bar{\omega}_\alpha \bar{\psi}_L \gamma_\mu \psi_L^\alpha,
\]

\[
\Lambda_R = 2ig\bar{\omega}_\alpha D \bar{\psi}_L^\alpha - 2ig\bar{\omega}_\alpha \hat{H}^\alpha \psi_R^\alpha - 2g\lambda_R^i M_i,
\]

\[
D = \mathcal{L} \equiv \mathcal{L}_\mathcal{K} + \Delta \mathcal{L}_\mathcal{K}.
\]

(2.19)

Here \( \mathcal{L} = \mathcal{L}_\mathcal{K} + \Delta \mathcal{L}_\mathcal{K} \) is the gauged extension of the \( \sigma \)-model lagrangean, as discussed previously; by construction it is invariant under supersymmetry modulo a total derivative.

To the vector multiplet \( \mathcal{K} \) corresponds a similar spinor-tensor multiplet, with components

\[
W_L(\mathcal{K}) = \left( \Lambda_L, \frac{1}{2} \left( -\sigma \cdot T^i - i\mathcal{L} \right), \partial \Lambda_R \right),
\]

(2.20)

where

\[
T_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu
\]

\[
= g_{A A} \left( D_\mu \bar{Z} \mathcal{A} D_\nu Z^A - D_\nu \bar{Z} \mathcal{A} D_\mu Z^A \right)
\]

\[
+ D_\mu \left( g_{A A} \bar{\psi}_L^A \gamma_\nu \psi_L^A \right) - D_\nu \left( g_{A A} \bar{\psi}_L^A \gamma_\mu \psi_L^A \right) - ig F^i_{\mu \nu} \mathcal{M}_i.
\]

(2.21)
Then the supersymmetric extension of the terms (2.17) is constructed by taking the direct analogue of eq.(2.15):

$$\Delta \mathcal{L}_{\text{non-min}} = \left[ \sigma \eta_{ij} W^i(V) W^j(V) \right]_F + \left[ \rho W(\mathcal{K})^2 \right]_F.$$  \hfill (2.22)

Clearly, this involves the square of the contracted field strength $[F^i_{\mu
u} M_i]^2$, but in addition there are higher-derivative terms for the components of the chiral superfields $\Sigma^A$.

Until now we treated all chiral multiplets $\Sigma^A \equiv (\Phi^\alpha, \Psi^A)$ on the same footing; we now classify the chiral multiplets $\Phi^\alpha$ and $\Psi^A$ by their transformation properties under the isometries. The chiral multiplets $\Phi^\alpha = (z^\alpha, \psi^\alpha_L, h^\alpha)$ transforming non-linearly into themselves under a part of the isometries are called $\sigma$-model multiplets. The chiral multiplets $\Psi^A = (x^A, \chi^A_L, f^A)$ transforming linearly into themselves under all isometries, but possibly with $\sigma$-model-field dependent parameters, are called matter multiplets. The transformations eq.(2.6) of $\sigma$-model and matter multiplets take the form

$$\delta_i \Phi^\alpha = R^\alpha_i (\Phi), \quad \delta_i \Psi^A = R^A_i (\Phi) \Psi^B.$$  \hfill (2.23)

according to the definitions above. The Killing vectors (2.23) for the $\sigma$-model and matter multiplets satisfy

$$R^{\beta}_{\ [i} R^\alpha_{j], \beta} = f_{ij}^k R^\alpha_k, \quad R^B_{[i \ C} R^A_{j] B} + R^B_{[i \ J} R^A_{j] C, \beta} = f_{ij}^k R^A_{k \ C}.$$  \hfill (2.24)

The components of the $\sigma$-model multiplets $\Phi^\alpha$ transform according to (2.6) but with $Z^A$ replaced by $z^\alpha$, etc. The transformation rules for components of the matter multiplets $\Psi^A$ are more involved

$$\delta_i x^A = R^A_i x^B,$$

$$\delta_i \chi^A_L = R^A_i \chi^B_L + R^A_{i B, \beta} x^B \psi^\beta_L,$$

$$\delta_i f^A = R^A_i f^B + R^A_{i B, \beta} (x^B h^\beta - 2 \bar{\chi}^B_R \psi^\beta_L) - R^A_{i B, \beta \gamma} x^B \bar{\psi}^\beta_R \psi^\gamma_L.$$  \hfill (2.25)

Notice that the chiral matter fermions $\chi^A_L$ do not transform into themselves if the transformations $R^A_i$ depend on the $\sigma$-model fields. In section 7 the chiral matter fermions are redefined such that they transform covariantly, see eq. (7.14) in that section.

Below we give a number of examples of matter multiplets and construct their Kähler potentials. Because the transformation rules for the complex matter scalars $x^A$ are linear in themselves, it follows that Kähler potentials for the matter multiplets are invariant unless the Kähler potential is a sum of holomorphic functions of these matter fields and their complex conjugates already.

\footnote{Of this trivial fact we make use later, see eq.(2.29)}
Given the Killing vectors $R_i^\alpha$, the Kähler potential $K_\sigma$ and hence the metric $g_{\sigma \bar{\alpha} \alpha}$ for the $\sigma$-model multiplets, it is straightforward to give explicit examples of the transformations of matter multiplets. By noticing that the metric $g_\sigma$ defines an invariant line element on the Kähler manifold by $ds^2 = dz^\alpha g_\sigma \bar{\alpha}\alpha dz^\alpha$, it follows that scalar fields $x^\alpha$ which transform as differentials

$$\delta_i x^\alpha = R_i^\alpha(z) x^\beta,$$

have an invariant Kähler potential given by

$$K_1(\bar{x}, x; \bar{z}, z) = i \bar{\alpha} g_{\sigma \alpha \bar{\alpha}} x^\alpha.$$

With the subscript 1 we indicate that this is the coupling of a rank one tensor (a vector) to the $\sigma$-model. The complex scalar $x^\alpha$ can be part of a chiral multiplet $\Psi^\alpha = (x^\alpha, \chi^\alpha_L, f^\alpha)$. Its transformation rules are given by equations (2.25) when $A, B$ are replaced by $\alpha, \beta$. By taking tensor products of $n$ such vectors one can built a rank $n$ tensor chiral multiplet which is coupled to the $\sigma$-model. It is possible to construct irreducible representations of the linear isometries by (anti)-symmetrizations and by taking traces. This construction is called covariant matter coupling [15, 22, 16].

It is also possible to couple a singlet chiral superfield $\Omega = (s, \chi_L, f)$ non-trivially to a $\sigma$-model [16]. The singlet chiral multiplet $\Omega$ transforms as

$$\delta_i \Omega = - f^2 F_i(\Phi) \Omega,$$

which forms a representation, see eq.(2.9). Note, that as the matter fields transform linearly into themselves, we have $F_i(\Phi, \Psi) = F_i(\Phi)$. We have taken the scalar component $\Omega$ dimensionless, as is convenient for the applications of $\Omega$ later. A covariant Kähler potential for this singlet $\Omega$ is given by

$$K_\Omega = f^{-2} \ln \left( \bar{\Omega} \Omega \right),$$

which transforms opposite to the Kähler potential $K_\sigma$ of the $\sigma$-model fields. The components of the multiplet $\Omega$ are non-propagating, as $K_\Omega$ is a sum of a holomorphic and an anti-holomorphic function. One can use this compensating multiplet $\Omega$ to rescale other matter multiplets so as to assign arbitrary $U(1)$ charges $q^{(A)}$ to them [16]. Indeed, let $\Psi^A$ be a set of matter multiplets described by a Kähler potential $\bar{\Psi}^A g_{A A} \Psi^A$. Define the rescaled multiplets $\Psi'^A$ by $\Psi'^A = \Omega^{-q^{(A)}} \Psi^A$. Their transformation rules become

$$\delta_i \Psi'^A = R_i^B(\Phi) \Psi'^B + q^{(A)} f^2 F_i(\Phi) \Psi'^A,$$

and their Kähler potential has to be modified to

$$\bar{\Psi}'^A g'_{A A} \Psi'^A = \bar{\Psi}^A g_{A A} \Psi^A e^{-q^{(A)} f^2 K_{\text{cov}}}.$$
The Kähler potential $K_{\text{cov}}$ denotes any Kähler potential transforming in the same way as $K_\sigma$. The case where $K_{\text{cov}} = K_\sigma$ was discussed in ref. [16]. The numbers $q^{(A)}$ are arbitrary real numbers and may be fixed by dynamical considerations, like anomaly cancellation [16].

We finish this section by fixing the notation for the general considerations below. In the following we denote the Kähler potential for all physical $\sigma$-model multiplets $\Phi$ by $K_\sigma(\bar{\Phi}, \Phi)$, and the Kähler potential for all physical matter multiplets $\Psi^A$ by $K^m(\bar{\Psi}, \Psi; \bar{\Phi}, \Phi)$. The matter fields $\Psi^A$ residing in $K^m$ may be rescaled by some power of $\Omega$. $K = K_\sigma + K^m$ is the sum of these two Kähler potentials.

In the discussion of superpotentials (2.5), it is often convenient to introduce a compensating superpotential $w(\Sigma)$: a dimensionless composite chiral superfield which transforms as $\delta_i w = q f^2 F_i w$ under the internal symmetries, with $q$ a real number. With such a holomorphic function $w$, an invariant Kähler potential can be defined in terms of the physical fields only

$$K(\bar{\Sigma}, \Sigma) = K(\bar{\Sigma}, \Sigma) - \frac{1}{q f^2} \ln |W(\Sigma)|^2.$$  

(2.32)

Here the covariant superpotential $W$ is defined by

$$W(\Sigma) = f^3 w(\Sigma) W(\Sigma),$$  

(2.33)

combining the invariant superpotential $W$, as in eq. (2.5), with the compensating superpotential $w$ introduced above. Observe, that with a compensating singlet $\Omega$ one can not make more general superpotentials than with physical multiplets alone, as $\Omega$ can always be integrated out.

3 Non-linear chiral multiplet coupled to supergravity

We now turn to the coupling of gauged chiral multiplets to supergravity, as discussed for example in [17], [18], generalized to include (holomorphic) non-linear gauge transformations. This coupling to supergravity has also been discussed in [24] using the superspace formalism [25]. A related approach using Kähler superspace [26] can be found in [27]. Besides presenting a review of this coupling, the main purpose of this section is to relate it to the rescaling of the matter multiplets we discussed in the previous section. We make the same distinction between $\sigma$-model multiplets $\Phi^\alpha$ and matter multiplets $\Psi^A$ as in section 2. Of the latter

---

3 This works because one can define covariant matter fermions $\hat{\chi}^A_L$ using (7.14) transforming as $\delta_i \hat{\chi}^A_L = R^A_{iB} \hat{\chi}^A_L$, as we show in section 6.

4 Such a compensating superpotential $w$ can always be constructed: for instance add two physical singlets $S_+$ and $S_-$ with opposite charge to cancel any anomalies, and consider $w = f S_+$.
the compensating singlet $\Omega$ plays a crucial role in the locally supersymmetric case as well. The matter fields $\Psi^A$ are initially assumed to transform covariantly; rescalings by powers of $\Omega$ come out naturally, as we show below.

As was discussed in ref. [17] an elegant way of coupling chiral multiplets to supergravity goes as follows: first couple the chiral multiplets to superconformal gravity, using a compensating multiplet $\Omega$. By fixing a set of gauges involving this compensating chiral multiplet $\Omega$ the superconformal algebra is reduced to the super-Poincaré algebra. On a chiral multiplet $\Sigma = (Z, \psi_L, H) \in \{ \Phi^\alpha, \Psi^A, \Omega \}$ the local superconformal algebra with transformations $\delta = \delta_Q(\epsilon) + \delta_S(\eta) + \delta_D(\lambda) + \delta_A(\theta)$ is realized by

$$\begin{align*}
\delta Z &= \bar{\epsilon}_R \psi_L + \omega(\lambda - \frac{i}{3} \theta) Z \\
\delta \psi_L &= \frac{1}{2} \left( D Z \epsilon_R + H \epsilon_L \right) + \omega Z \eta_L + \left[ (\omega + \frac{1}{2}) \lambda + i \left( \frac{1}{2} - \frac{\omega}{3} \right) \theta \right] \psi_L, \\
\delta H &= \bar{\epsilon}_L \left( D \psi_L - \lambda^i R_i(Z) \right) + 2(1 - \omega) \bar{\eta}_R \psi_L \\
&\quad + \left[ (\omega + 1) \lambda + i \left( 1 - \frac{\omega}{3} \right) \theta \right] H. \tag{3.1}
\end{align*}$$

Here $(\lambda, \theta)$ are the parameters of local scale and chiral $U(1)$ transformations, whilst the spinors $(\epsilon, \eta)$ parameterize local $Q$- and $S$-supersymmetry transformations, respectively. Furthermore $\omega = (\omega^{(a)}, \omega^{(A)})$ denote the Weyl-weights of the chiral multiplets; the Weyl-weight of $\Omega$ is taken to be $\omega^{(\Omega)} = 1$. The special conformal boosts do not have to be considered here as their only role is to fix the Weyl gauge field $b_\mu$ to zero when we restrict to Poincaré supergravity. The covariant derivatives are superconformal derivatives with the non-linear gauge-covariantizations (2.13) included.

Under the internal symmetries the $\sigma$-model fields $\Phi^\alpha$ and the matter fields $\Psi^A$ transform according to eqs. (2.23). Generically this requires the conformal Weyl weights of the $\sigma$-model bosons to vanish; formally this is derived by requiring the internal symmetries and the space-time symmetries to commute:

$$0 = [\delta_D, \delta_i] z^\alpha = \omega^{(\beta)} R^{\alpha}_{i,\beta} z^\beta - \omega^{(\alpha)} R^{\alpha}_{i} \Rightarrow \omega^{(\alpha)} = 0, \forall \alpha. \tag{3.2}$$

By a similar argument any additional $U(1)$ symmetries entering the theory should leave the $\sigma$-boson fields inert as well. Furthermore, the Weyl weights of the matter multiplets in a single irreducible representation must all be equal. We can make them vanish as well by multiplying with an appropriate power of the compensator:

$$\Psi'^A = \Omega^{-\omega^{(A)}} \Psi^A \Rightarrow \omega'^{(A)} = 0. \tag{3.3}$$

Clearly, the dimension of the physical fields (as opposed to the Weyl weight) is kept fixed by taking $\Omega$ dimensionless. For later use in constructing invariant
actions we demand that the compensating superfield $\Omega$ transforms like a non-
trivial singlet (2.28) under the internal symmetries as

$$\delta_i \Omega = -\kappa^2 F_i(\Phi) \Omega,$$

(3.4)

with $F_i(\Phi)$ having vanishing Weyl weight: $\omega(F_i) = 0$, but —like for the Kähler
potential itself— the mass-dimension $\text{dim}[F_i] = 2$. Therefore we have introduced
the inverse Planck scale $\kappa = 1/M_P$. By eq.(2.28) this implies that the multiplet
$\Psi'^A$ transforms under the internal symmetries as

$$\delta_i \Psi'^A = R^A_{iB}(\Phi) \Psi'^B + \omega(A) \kappa^2 F_i(\Phi) \Psi'^A.$$

(3.5)

which is precisely the form of equation (2.30) with $q_f^2 = \omega(A) \kappa^2$. These charges
were introduced more or less ad hoc in ref.[16], so as to cancel anomalies. From
now on we assume that we have performed this rescaling to all the matter fields,
therefore we drop the prime on the matter fields.

With these results in mind we proceed in the standard way [17, 18] to c on-
struct invariant functions of the superfields and use the density formula for real
superfields of Weyl-weight 2 and chiral superfields of Weyl-weight 3 to obtain
superconformally invariant lagrangeans. Let $K$ be the Kähler potential for the
$\sigma$-model fields $\Phi^\alpha$ and the matter fields $\Psi^A$ which is covariant

$$\delta_i K = F_i + \bar{F}_i.$$

(3.6)

One defines a dimensionless invariant Kähler potential $\mathcal{G}$ by

$$e^\mathcal{G} = \bar{\Omega} \Omega e^{\kappa^2 K(\bar{\Phi}, \bar{\Psi}; \Phi, \Psi)}.$$ 

(3.7)

$e^\mathcal{G}$ is a real superfield with Weyl-weight 2 and is inert under all internal symme-
tries. Hence by using the density formula for a real Weyl-weight 2 superfield it
follows that the lagrangean $\left[ e^\mathcal{G} \right]_D = \left[ \bar{\Omega} \Omega e^{\kappa^2 K(\bar{\Phi}, \bar{\Psi}; \Phi, \Psi)} \right]_D$ is invariant under super-
conformal and internal symmetries. For similar reasons the only Weyl-weight 3
$F$-term lagrangean one can write down is $\left[ (\Omega)^3 \mathcal{W}^{3/\omega}(\Phi, \Psi) \right]_F$, where the covariant
superpotential $\mathcal{W}$ is a dimensionless holomorphic function of $\Phi^\alpha$ and $\Psi^A$. This la-
grangean is inert under the internal symmetries provided that the superpotential transforms as

$$\delta_i \mathcal{W}(\Phi, \Psi) = \omega \kappa^2 F_i(\Phi) \mathcal{W}(\Phi, \Psi),$$

(3.8)

again with the Weyl weight and rescaling charge $q$ of eq. (2.32) related by $q_f^2 = \omega \kappa^2$. The particular power $3/\omega$ of the superpotential $\mathcal{W}$ is required precisely to
satisfy this transformation rule. By redefining the compensating multiplet as

$$\Omega' = \Omega \mathcal{W}^{1/\omega}(\Phi, \Psi),$$

(3.9)
it is inert under all internal symmetries and the superpotential can be absorbed into the extended Kähler potential $K$ \cite{17},
\begin{equation}
e^G = \bar{\Omega}' \Omega' e^{\kappa^2 K},
\end{equation}
where $K$ is given by eq. (2.32) with the above substitution $qf^2 \to \omega \kappa^2$. Therefore a superpotential for the physical fields necessarily transforms as in eq.(3.8). From now on we use the redefined singlet $\Omega'$ of eq.(3.9), unless explicitly stated otherwise; therefore we drop the primes on $\Omega$. Although we now consider non-linear internal symmetries the Kähler potential $K$ takes the same form as given in ref.\cite{17}. But because of this non-linear nature, the gauging of part of the internal symmetries leads to some modifications of the invariant lagrangean. These modifications only come from the $D$-terms and gaugino-matter coupling which we discuss below.\footnote{All gauge couplings now involve the Killing vectors $R^A_i$ as well.} The crucial part of the lagrangean in eq. (3.14) of ref.\cite{17} generalizes to
\begin{equation}
e^{-1} \Delta L = \frac{1}{4} f_{ij} D^i D^j + \frac{i}{2\kappa^2} \left(e^G\right)_{A \bar{A}} \mathcal{R}^A_i \left(D^i + i\bar{\Psi}_R \cdot \gamma \lambda_R^i\right)
+ \frac{2}{\kappa^2} \left(e^G\right)_{A \bar{A}} \mathcal{R}^A_i \bar{\lambda}_L^i \lambda^A_R + \text{h.c.}
\end{equation}
where $\mathcal{R}^A_i$ are the Killing vectors defined for all the fields in the model. Notice that $\mathcal{R}^{\Omega}_i = 0$ since the compensating multiplet $\Omega$ does not transform under the internal symmetries. These Killing vectors can be obtained from Killing potentials $\mathcal{M}_i$ defined by $i\kappa^2 \mathcal{M}_i = \left(e^G\right)_{A \bar{A}} \mathcal{R}^A_i$, using (2.11) and the vanishing of $\mathcal{F}_i$, as $G$ is inert under the internal symmetries. This holds for the Kähler potential $K$ as well, and the Killing potential can be expressed as
\begin{equation}
\mathcal{M}_i = e^G M_i = e^G \left(M_{\sigma i} - iK_{m,\alpha} R^m_i - iK_{m,A} \left\{ R^A_{iB} + \omega^{(A)} F_{i}^a \delta^A \right\} \Psi^B \right).
\end{equation}
For example, with the matter terms of the form
\begin{equation}
K_m = \sum_{\Psi} \bar{\Psi}^A \eta_{A \bar{A}} \Psi^A e^{-\kappa^2 \omega^{(A)} K_\sigma},
\end{equation}
the Killing potentials can be expressed as \cite{10}
\begin{equation}
\mathcal{M}_i = e^G M_i = e^G \left( M_{\sigma i} + \eta \kappa^2 \sum_{\Psi} \omega^{(A)} \bar{\Psi}\eta_{A \bar{A}} \Psi^A e^{-\omega^{(A)} \kappa^2 K_\sigma} \right)
- \frac{i}{2} \sum_{\Psi} \bar{\Psi} \eta_{A \bar{A}} R^A_{\sigma i} \Psi^A e^{-\omega^{(A)} \kappa^2 K_\sigma \right).}
\end{equation}
The part of the lagrangean (3.11) can be written in terms of Killing potentials as

\[ e^{-1} \Delta \mathcal{L} = \frac{1}{4} f_{ij} D^i D^j - \frac{1}{2} \mathcal{M}_i \left( D^i + i \bar{\Psi}_R \cdot \gamma^i \lambda_R \right) + 2i \mathcal{M}_i \Delta \bar{\lambda}_L \lambda_R^A + \text{h.c.} \]  

(3.15)

The total lagrangean in superconformal gravity of gauged non-linear isometries with matter couplings is given by

\[ \mathcal{L} = \frac{1}{\kappa^2} \left[ \bar{\Omega} \Omega e^{\kappa^2 K} \right]_D + \frac{1}{\kappa^3} \left[ \Omega^3 W^3 \right]_F + \left[ f_{ij} W^i W^j \right]_F. \]  

(3.16)

To reduce the lagrangean (3.16) to Poincaré supergravity with matter coupled to it, one has to perform a number of gauge-fixings [17]. This can be done in a clever way [18] by choosing

\[ D: \quad \bar{s}se^{\kappa^2 K} = 3, \quad A: \quad \text{Im } s = 0, \]  

(3.17)

\[ S: \quad \chi_L = -\kappa^2 sK \psi^A, \quad K_m: \quad b_\mu = 0, \]

using the components of \( \Omega = (s, \chi_L) \).

We now briefly review the relationship between this setup and the formulation of refs. [24, 26]. If one does not perform the redefinition (3.9) the superconformal lagrangean reads

\[ \mathcal{L} = \frac{1}{\kappa^2} \left[ \bar{\Omega} \Omega e^{2\kappa K} \right]_D + \frac{1}{\kappa^3} \left[ \Omega^3 W^3 \right]_F + \left[ f_{ij} W^i W^j \right]_F. \]  

(3.18)

It is inert under all internal symmetries provided that the compensator \( \Omega \) transforms according to eq. (3.4). If one reduces to Poincaré supergravity by applying the gauge fixings (3.17) with \( K \) replaced by \( K \) the results of refs. [24] are obtained. The gauge fixings eq. (3.17) in this situation are not invariant under the internal symmetry transformations; this can be compensated by a chiral rotation [24]

\[ \delta_i \psi = i \frac{1}{2} \text{Im} F_i \gamma_5 \psi, \]  

(3.19)

on all spinors \( \psi \). One could also consider arbitrary holomorphic functions \( F \) in eqs. (3.6) and (3.4) instead of the functions \( F_i \), which are dictated by the isometries (2.23). This is the basis of Kähler superspace [24] where the Kähler \( U(1) \) transformations are gauged. Notice that the redefinition of the compensator (3.9) is a special case of this. It is clear that the lagrangean (3.18) is invariant under these transformations. This reflects the fact that in supergravity the Kähler potential \( K \) and the superpotential \( W \) are not independent.

In this article our primary concerns are the isometries of the \( \sigma \)-model and the holomorphic functions \( F_i \) they induce. Therefore we choose to work with the invariant Kähler potential \( K \) and the lagrangean (3.16), amounting to a specific gauge in Kähler supergravity.
4 Vacuum configuration

In the previous section the construction of (locally) supersymmetric lagrangeans for $\sigma$-models with non-linear symmetries was discussed. We now make a first step in the analysis of the phenomenology of such models. The scalar potential $V$ is given here before the auxiliary fields are eliminated. This has the advantage that breaking of supersymmetry is encoded in the vacuum expectation values of the auxiliary fields: supersymmetry is broken iff at least one auxiliary field has a non-vanishing VEV. We do not consider any fermion condensates here. Combining the results of eqs. (2.1), (2.5) and (3.15), the scalar potential for the auxiliary and physical scalars in Poincaré supergravity reads

$$V = V_D + V_F = -\frac{1}{2} \text{Re} f_{ij} D^i D^j - g_{\Delta A} \bar{H}^A H^A$$

$$+ D^i M_i - \kappa^{-1} \left( \mathcal{K}_{\Delta A} \bar{H}^A + \mathcal{K}_{\omega A} H^A \right) e^{\frac{1}{2} \kappa^2 \mathcal{K}} - 3\kappa^{-4} e^{\kappa^2 \mathcal{K}},$$

using the results of ref. together with the generalization for non-linear symmetries of section 3. In this scalar potential the Kähler potential is given by eq. (2.32). If supersymmetry is unbroken, the gravitino may still have a non-vanishing mass

$$\kappa^{-1} e^{\frac{1}{2} \kappa^2 \mathcal{K}} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu = \kappa^{-1} |\mathcal{W}|^{-\frac{1}{2}} e^{\frac{1}{2} \kappa^2 \mathcal{K}} \bar{\psi}_{R\mu} \sigma^{\mu\nu} \psi_{L\nu} + \text{h.c.},$$

as this term, together with the negative cosmological last term of eq. (4.1), is supersymmetric. A vanishing gravitino mass is only possible if $-\frac{1}{\omega} > 0$ and the covariant superpotential vanishes in the vacuum. Because of the covariance of the superpotential $\mathcal{W}$ the gravitino mass is linked to the breaking of internal symmetries, as

$$<\delta_i \mathcal{W}> = q f^2 <F_i> <\mathcal{W}>.$$

If $<\mathcal{W}>=0$, the symmetries for which $<F_i>=0$ are broken. In particular the $U(1)$-factor of the linear isometries produces a constant function $F_{U(1)} = f^{-2} a$, hence this $U(1)$ is broken as soon as $<\mathcal{W}>=0$. Observe that the inverse is not necessarily true: when the gravitino mass vanishes ($<\mathcal{W}> = 0$) symmetry breaking is not automatically ruled out.

In the absence of fermion condensates the equations of motion of the auxiliary fields in the vacuum are

$$\text{Re} f_{ij} D^j = M_i,$$

$$\bar{H}^A g_{\Delta A} = -\kappa^{-1} \left( \mathcal{W} K_{\Delta A} - \frac{1}{q f^2} \mathcal{W}_{r A} \right) |\mathcal{W}|^{-\frac{1}{2}} e^{-\frac{1}{2} \kappa^2 \mathcal{K}}.$$
If the metrics $\text{Re} f_{ij}$ and $g_{\Delta A}$ are invertable, the scalar potential can be written in hybrid form as

$$V = \frac{1}{2} \text{Re} f_{ij} D^i D^j + g_{\Delta A} \bar{H}^A H^A - 3 \kappa^{-4} |W|^{-\frac{3}{2}} e^{\kappa^2 K},$$  \hspace{1cm} (4.5)

with $D^i$ and $H^A$ the solutions (4.4). Using eqs. (4.4) and (4.5) the following phenomenological picture emerges. First of all observe that if

$$-\frac{\kappa^2}{q f^2} = -\frac{1}{\omega} < 1 \quad (\exists W, A \neq 0)$$  \hspace{1cm} (4.6)

the scalar potential diverges for $W = 0$ unless for all $A$ the first derivative of the superpotential vanishes as well. In that case the scalar potential still diverges if

$$-\frac{1}{\omega} < 0 \quad (\forall W, A = 0).$$  \hspace{1cm} (4.7)

In this case the scalar potential can diverge to $\pm$ infinity depending on the details of the Kähler potential. (In the case where not all $W, A = 0$ the potential always diverges to $+\infty$, as $|W, A|^2$ is always positive in that situation.)

We now discuss the consequences of the analysis above. If $-\frac{1}{\omega} > 1$, the $\sigma$-model cut-off is in general bigger than the Planck-scale. In this case $< W >= 0$ but the derivatives of $W$ do not have to vanish. The condition $< W >= 0$ may give rise to additional internal symmetry breaking. In the situation $1 \geq -\frac{1}{\omega} > 0$ the Planck scale may be much bigger than the $\sigma$-scale, and not only $< W >= 0$ but also all $< W, A >= 0$. This means that there are more restrictions on the VEVs of the scalars and hence there may be more symmetry breaking and/or more parameters are fixed. In this case all the auxiliary fields (4.4) of the chiral multiplets vanish, therefore $F$-term supersymmetry breaking is not possible. The spontaneous supersymmetry breaking can therefore only occur if the auxiliary $D$-fields (4.4) acquire non-vanishing VEVs. Soft supersymmetry breaking masses can still arise because of non-renormalizable contributions. We show how this works out in practice in section (9), where the vacuum configuration of a Grassmann $\sigma$-model with a standard model-like spectrum is discussed.

## 5 Grassmannian Manifolds

In this section we illustrate the general constructions discussed above with the example of Grassmann manifolds. Considering a particular model which describes quark doublets, we show that anomaly cancellation is possible when we extend it to a non-linear version of the standard model. A Grassmann manifold is a homogeneous space which is obtained as the coset $U_q(m, n)/U(m) \times U(n) \simeq SU_q(m, n)/SU(m) \times SU(n) \times U(1)$. The parameter $\eta$ distinguishes the compact
and non-compact case: the compact group \( (S)U(m + n) \) has \( \eta = 1 \) and the non-compact group \((S)U(m, n)\) has \( \eta = -1 \). Note that the second expression \( G/H \) for the coset manifold is obtained from the first one by cancelling a \( U(1) \) factor between the numerator and the denominator. This \( U(1) \) may still act on the fields in our models, where it then represents a central charge (it commutes with all generators of \( G \)). Refs. \[1, 2, 3\] provide the Kähler potential for these models, which can be written as

\[
K_\sigma(\bar{Q}, Q) = \frac{1}{\eta f^2} \left( a \, \text{tr}_m \ln(g^{-1}) + b \, \text{tr}_n \ln(\bar{g}^{-1}) \right) \tag{5.1}
\]

with the inverse metrics \( g^{-1} \) and \( \bar{g}^{-1} \)

\[
\left(g^{-1}\right)_i^i = \left[1 + \eta f^2 Q Q\right]_i^i, \quad \left(\bar{g}^{-1}\right)_a^a = \left[1 + \eta f^2 \bar{Q} \bar{Q}\right]_a^a. \tag{5.2}
\]

Here \( f \) is the parameter with the dimension of inverse mass setting the scale; it gives the fields \( Q \) their canonical dimension. Two traces have been introduced: \( \text{tr}_m \) acts on \( m \times m \)-matrices and \( \text{tr}_n \) on \( n \times n \)-matrices. The superfield matrix \( Q = (Q^a) \) has vector indices in both \( SU(m) \) and \( SU(n) \) and \( Q = (\bar{Q}_a) \) is its conjugate. We take the indices \( i = 1, \ldots, m \) and \( a = 1, \ldots, n \). In subsection \( \square \) we interpret \( Q^a \) as a chiral multiplet containing a quark-doublet. The two real constants \( a \) and \( b \) obey \( a + b = 1 \) and hence drop out of (5.1) after evaluating the traces. The constant \( c \) defined by \( mn c = ma - nb \), which may be used to characterize the central charge, is therefore not fixed uniquely. The non-linear realization of the \( U_n(m, n) \) algebra on multiplets \( Q \) and \( \bar{Q} \) takes the form

\[
\delta Q = R(Q) = \frac{1}{f} \epsilon + \eta f Q \epsilon Q + i M Q - i Q N + i(m + n) \theta_Y Q,
\]

\[
\delta \bar{Q} = \bar{R}(\bar{Q}) = \frac{1}{f} \epsilon + \eta f \bar{Q} \epsilon \bar{Q} + i N \bar{Q} - i \bar{Q} M - i(m + n) \theta_Y \bar{Q}, \tag{5.3}
\]

where \( M (N) \) represents the matrix of infinitesimal parameters of \( SU(m) \) \( (SU(n)) \), \( \epsilon \) an \( n \times m \)-matrix and \( \theta_Y \) is a real number. We also introduce a real parameter \( \theta_C \) for the central charge, but by construction the goldstone fields \( Q, \bar{Q} \) are inert under the central \( U(1) \). The Lie algebra corresponding to the transformation rules (5.3) can be stated as

\[
[Y, X^i_a] = (m + n) X^i_a, \quad [Y, X_{ai}] = -(m + n) X_{ai},
\]

\[
[U^k_l, X^i_a] = \delta^i_l X^k a - \frac{1}{m} \delta^k_l X^i_a, \quad [U^k_l, X_{ai}] = -\delta^i_l X_{al} + \frac{1}{m} \delta^k_l X_{ai},
\]

\[
[V^c_d, X^i_a] = -\delta^a_d X^i_c + \frac{1}{n} \delta^c_d X^i_a, \quad [V^c_d, X_{ai}] = \delta^a_c X_{di} - \frac{1}{n} \delta^c_d X_{ai}, \tag{5.4}
\]

\[
[U^i_j, U^k_l] = \delta^i_l U^k_j - \delta^k_l U^i_j, \quad [V^a_b, V^c_d] = \delta^a_d V^c_b - \delta^a_b V^c_d,
\]

\[
[X_{ai}, X^{jb}] = \eta \left( \delta^b_d U^i_j - \delta^i_l V^b_d \right) + \eta \frac{1}{mn} Y \delta^i_j \delta^b_d.
\]
where $U, V, X, \bar{X}, Y$ are the generators of $SU_\eta(m, n)$. By adding the generator $C$ for the central $U(1)$ we complete this to a full set of generators for $U_\eta(m, n)$. The generators $U$ and $V$ are taken anti-hermitean and $X$ and $\bar{X}$ are each others hermitean conjugates. The $U_j$ span the subalgebra $SU(m)$ of $U_\eta(m, n)$ and similarly the generators $V^a$ span the subalgebra $SU(n)$. The two $U(1)$-factors in $U(m)$ and $U(n)$ combine to form the charges $Y$ and $C$. On $Q^a$ the generators $U$ ($V$) act via left (right) multiplication. For this reason the commutators involving $V$ differ from the commutators involving $U$ by a minus sign. (By a redefinition of $V$ this minus sign could be absorbed.) The inverse metrics (5.2) transform under these symmetries as

$$
\delta g^{-1} = H g^{-1} + g^{-1} H^\dagger, \quad \delta \tilde{g}^{-1} = \tilde{g}^{-1} H + \tilde{H} \tilde{g}^\dagger.
$$ (5.5)

Here the holomorphic matrix-valued functions

$$
H = \eta f Q \bar{\epsilon} + i M + i n \theta_Y + i \theta_C, \quad \tilde{H} = \eta f \bar{\epsilon} Q - i N + i m \theta_Y - i \theta_C.
$$ (5.6)

and their conjugates transform in the adjoint representation of $U_\eta(m, n)$. Using (5.5) it is easy to show that $K_\sigma$ in eq.(5.1) transforms as a Kähler potential

$$
\delta K_\sigma(\bar{Q}, Q) = F(Q) + \tilde{F}(\bar{Q}).
$$ (5.7)

As the functions $H$ and $\tilde{H}$ transform in the adjoint representation, so does the holomorphic function

$$
F(Q) = \frac{1}{\eta f^2} \left( \text{atr}_m H + \text{btr}_n \tilde{H} \right) = \frac{1}{\eta f^2} \left( \eta f \text{tr}_m (Q \bar{\epsilon}) + i m n \theta_Y + i m n c \theta_C \right).
$$ (5.8)

Next we discuss matter coupling to the Grassmannian model. Let $R(\bar{\Sigma}, \Sigma)$ and $\tilde{R}(\Sigma, \Sigma)$ be $m \times m$, resp. $n \times n$-matrix-valued composite real superfields. They are called left, resp. right, covariant if they transform as

$$
\delta R = H R + RH^\dagger, \quad \delta \tilde{R} = \tilde{R} \tilde{H} + \tilde{H}^\dagger \tilde{R}
$$ (5.9)

under the $U_\eta(m, n)$ isometries of the Grassmannian manifold. Invariant Kähler potentials for these real composite superfields $R$ and $\tilde{R}$ are provided by

$$
\text{tr}_m(g R), \quad \text{tr}_n(\tilde{g} \tilde{R}).
$$ (5.10)

By eqs.(5.5) it follows that $g^{-1}$, resp. $\tilde{g}^{-1}$, are left, resp. right, covariant but the construction mentioned above gives trivial results for these examples. To obtain non-trivial results, consider the chiral multiplets $L^i$ and $D^a$ which transform under $U_\eta(m, n)$ by left, resp. right, multiplication

$$
\delta L = H L = (\eta f Q \bar{\epsilon} + i M + i n \theta_Y + i \theta_C) L,
$$ (5.11)

$$
\delta D = D \tilde{H} = D(\eta f \bar{\epsilon} Q - i N + i m \theta_Y - i \theta_C).
$$
We will later interpret $L$ and $D$ as chiral superfields containing the left-handed lepton doublets and charge conjugate of the right-handed $d$-quark. It follows that $L$ has a $Y$ charge $n$ and central $C$ charge 1 and $D$ has $Y$ charge $m$ and central $C$ charge $-1$. However this interpretation does not work directly as for $m = 3, n = 2$ the $Y$ charges of $L$ and $D$ with respect to $Q$ do not reproduce the standard hypercharges $Y_w$. Notice that $(L\bar{L})^a_j$ and $(\bar{D}D)^b_a$ are left-, resp. right-, covariant composite superfields and hence from the expressions (5.10) the Kähler invariants can be constructed

$$\bar{L}gL \quad \text{and} \quad \bar{D}\tilde{g}D.$$ (5.12)

By taking tensor products of multiplets which transform like $L$ and $D$, one can obtain higher rank $U(m) \times U(n)$ tensors chiral multiplets.

As the function $F$ defines a cycle, transforming with the structure constants of the gauge group, see eq.(2.9), we can use (2.28) to couple a multiplet $\Omega$ which is a singlet under the semi-simple part of the unbroken symmetries to a Grassmannian manifold by

$$\delta \Omega = \eta f^2 F(Q)\Omega = (\eta f \text{tr}_m(Q\bar{\epsilon}) + imn\theta_Y + imn\theta_C) \Omega.$$ (5.13)

For later convenience, we take $\Omega$ dimensionless. The rescalings with this singlet changes the $Y$ charge as well as the central charge $C$. Notice that we can introduce another singlet $\Omega'$ with the same transformation rules as $\Omega$ but with a different value for the central charge $c'$, as the choice of parameters $a$ and $b$ in eq.(5.1) is not unique. Therefore we can define two independent non-trivial singlets $\Omega_Y$ and $\Omega_C$ which transform as

$$\delta \Omega_Y = (\eta f \text{tr}_m(Q\bar{\epsilon}) + imn\theta_Y) \Omega_Y, \quad \delta \Omega_C = imn\theta_C \Omega_C,$$ (5.14)

where we set the central charge $c$ of $\Omega_C$ to unity. When rescalings with $\Omega_Y$ are performed, one needs to modify the metrics (5.2) because this rescaling also generates additional non-linear transformations. For rescalings with $\Omega_C$ this is not the case; it can in principle be applied to all multiplets. In the following we discuss the effects of rescalings on matter with a general singlet $\Omega$ only, as rescalings with $\Omega_Y$ or $\Omega_C$ are just particular examples of this.

Any given chiral multiplet, for example $L$, can be rescaled by a (non-physical) singlet $\Omega$ to $L' = \Omega L$, which transform as

$$\delta L' = \left(l \eta f^2 F + H\right) L'$$ (5.15)

using the transformation (5.13) of the singlet $\Omega$. In this way the right charges can be assigned to multiplets allowing for specific physical applications. The additional terms in the transformation rule for $L'$ have to be compensated in the Kähler potential. Again let $\mathfrak{R}(\Sigma, \bar{\Sigma})$ and $\mathfrak{R}(\bar{\Sigma}, \Sigma)$ be left and right covariant real

18
composite multiplets. Using eqs. (5.8), (5.9) and (5.15) a left covariant composite
real superfield is constructed for \(L'\) by

\[
\det^{-l\mathfrak{R}} \det^{-\bar{l}\mathfrak{R}} \quad L' \bar{L}' = e^{-l(\text{atr}_m \ln \mathfrak{R} + \text{btr}_n \ln \bar{\mathfrak{R}})} \quad L' \bar{L}'
\]  

and hence using (5.10) a Kähler invariant for \(L'\) is obtained. Notice that this is an
eexample of eq. (2.31) with \(K_{\text{cov}} = \text{atr}_m \ln \mathfrak{R} + \text{btr}_n \ln \bar{\mathfrak{R}}.\) If one takes \(g^{-1}\) and \(\bar{g}^{-1}\)
for the composite superfields \(\mathfrak{R}\) and \(\bar{\mathfrak{R}}\) then one obtains from this construction
the invariant

\[
\bar{L}' g L' e^{-l f^2 K_{\sigma}}
\]  

by eq.(5.1). Of course a similar construction works for \(D\) as well. After rescaling
\(L\) by \(l\) and \(D\) by \(d\) such that:

\[
\delta L = (H + lnf^2 F) L, \quad \delta D = D(\bar{H} + dnf^2 F),
\]

the generalizations of the Kähler invariants (5.12) are given by

\[
K_L = \bar{L} g^{(L)} L, \quad K_D = D \bar{g}^{(D)} \bar{D},
\]  

with the modified metrics

\[
g^{(L)} = e^{-lnf^2 K_{\sigma}} g, \quad \bar{g}^{(D)} = e^{-dnf^2 K_{\sigma}} \bar{g}.
\]

We next turn to discuss the Killing potentials. We denote all Killing potentials
\(\mathcal{M}_i\) collectively as \(\mathcal{M} = \theta^i \mathcal{M}_i\), where \(\theta^i\) stands for the parameters of the isome-
tries. We first focus on the part \(\mathcal{M}_\sigma\) of the Killing potential depending on the
\(\sigma\)-model fields \(Q\) and \(\bar{Q}\) only; afterwards the matter contribution \(\mathcal{M}_m\) is exam-
ined. The complete Killing potential is given by \(\mathcal{M} = \mathcal{M}_\sigma + \mathcal{M}_m\). Both the
\(\sigma\)-model and matter Killing potentials can be written conveniently in terms of
the matrices

\[
\Delta = R^{(ia)}_{\ (ia)} (g^{-1})_{\ (ia)} g - H
\]

\[
= - i\theta_C + i\theta_Y \left( (m + n)nf^2 QQg - n \right) - iMg
- inf^2 QN\bar{Q}g + nf \left( \bar{Q} - Q\bar{\epsilon} \right) g,
\]  

\[
\tilde{\Delta} = \tilde{g}^{(ia)}_{\ (ia)} R^{(ia)} - \tilde{H}
\]

\[
= + i\theta_C + i\theta_Y \left( (m + n)nf^2 \bar{Q}Q - m \right) + inf^2 \bar{g} \bar{Q} M Q
+ \tilde{g} N + nf \tilde{g} \left( \bar{Q} \epsilon - \bar{\epsilon} Q \right),
\]

under some mild assumptions as we see below. Here we have used the index
notation \((ia)\) to emphasize that this index refers to the superfield \(Q^{ia}\). Using
eq.(2.10) for the σ-model fields $Q$ and $\bar{Q}$ we find that their Killing potential can be written as

$$iM_\sigma = K_{\sigma,(ia)}R^{(ia)} - \frac{1}{q\eta f^2}W^{-1}\delta W = \frac{a}{\eta f^2}\text{tr}_m\Delta + \frac{b}{\eta f^2}\text{tr}_n\tilde{\Delta}.$$ (5.22)

Notice that $W^{-1}\delta W$ plays the role of $F$ and that the covariant superpotential $W$ in combination with the σ-model Kähler potential forms an invariant.

To discuss the Killing potential due to matter fields in some generality we introduce some further notation. We discuss only the rescaled matter field $L$ here, as it is easy to generalize our discussion to the matter field $D$ and tensor products. Define the $m \times m$ matrix real composite superfield $[L\bar{L}]^j_i = (g(L)L)^j_i \bar{L}$ where $g^{(L)}$ is the rescaled metric defined in eq.(5.20). Notice that $[L\bar{L}]^{-1}g^{-1}$ is a left covariant real composite superfield, hence by (5.10) we obtain the Kähler invariant: $\text{tr}_m[L\bar{L}] = K_L$. From now on we assume that the matter Kähler potential $K_m$ can be written entirely in terms of matrices like $[L\bar{L}]$. As $K_m$ is an invariant Kähler function, one can define the Killing potential for the matter field $L$ as

$$iM_L = \text{tr}_m\left[K_m,[LL] \left((\delta Q^{ia}(g^{(L)}),(ia)L\bar{L}) + g^{(L)}\delta L\bar{L}\right)\right].$$ (5.23)

where $K_m,[LL]$ denotes the derivative of $K_m$ with respect to the matrix $[L\bar{L}]$. This can be expressed in terms of $\Delta$ and $M_\sigma$ as

$$iM_L = -\bar{L}K_m,[LL]^{-1}g^{(L)}(l\eta f^2 iM_\sigma + \Delta) L.$$ (5.24)

The Killing potential $M_m$ due to all the different matter fields is a sum of Killing potentials like $M_L$. As the Killing potentials $M_\sigma, M_m$ for the σ-model fields and the matter fields are linear in $\Delta$ and $\tilde{\Delta}$, cf. eq.(5.22), we can always express the full Killing potential as:

$$iM = \text{tr}_m\Delta P + \text{tr}_n\tilde{\Delta}\tilde{P}.$$ (5.25)

where the field dependent matrices $P$ and $\tilde{P}$ encode the details of the full Kähler potential. Using these matrices one can state the Killing potentials for the different symmetries of $U_\eta(m,n)$ as

$$M_C = -\text{tr}_mP + \text{tr}_n\tilde{P},$$

$$M_Y = \text{tr}_mP \left((m+n)\eta f^2\bar{Q} \bar{Q} - n\right) + \text{tr}_n\tilde{P} \left((m+n)\eta f^2\bar{Q} \bar{Q} - m\right),$$

$$M_U = -gP + \eta f^2\bar{Q} \bar{P} \bar{g} \bar{Q},$$

$$M_V = \tilde{P} \bar{g} - \eta f^2\bar{Q} \bar{P} \bar{g} \bar{Q},$$

$$iM_X = \eta f \left(\bar{Q} P \bar{P} \bar{Q} \bar{g} \right),$$

$$iM_{\bar{X}} = -\eta f \left(gP \bar{Q} + \bar{P} \bar{g} \right).$$ (5.26)
In combination with the vector field strengths, also transforming according to the adjoint representation, we can now construct the invariant \( M_{\mu\nu} = f^2 M_i F_{i\mu\nu} \).

For example, if we gauge the full \( SU_\eta(m, n) \) (but without the central charge), the Killing potentials \( M_{\sigma i} \) of the pure \( \sigma \)-model give

\[
M_{\mu\nu} = -if \mathrm{tr}_m \left\{ g \left( F_{i\mu\nu}(Z) \bar{Q} + i f Q \bar{Q} (F_{i\mu\nu}(V) + n F_{i\mu\nu}(A)) \right) \right\} 
+ if \mathrm{tr}_n \left\{ \tilde{g} \left( F_{i\mu\nu}(\bar{Z}) Q + i f \bar{Q} \bar{Q} (F_{i\mu\nu}(W) - m F_{i\mu\nu}(A)) \right) \right\}.
\]

(5.27)

Here \((V_\mu, W_\mu, Z_\mu, \bar{Z}_\mu, A_\mu)\) are the vector fields for \( SU(m) \), \( SU(n) \), the off-diagonal generators of \( SU_\eta(m, n) \), and \( U(1) \), respectively. The kinetic terms for the gauge fields then can be constructed as

\[
e^{-1} \mathcal{L}_{gk} = -\sigma \eta_{ij} F_{i\mu\nu} F_{j\mu\nu} + \rho \left[ M_{\mu\nu} \right]^2 + ..., \quad (5.28)
\]

where the dots denote the supersymmetric completion. We observe, that in the case that all isometries are gauged (for \( \eta = +1 \)), the unitary gauge is \( Q = \bar{Q} = 0 \), and therefore the higher-order scalar derivative terms are absent. As \( M_\sigma \) acquires a vacuum expectation value, it becomes constant and the non-minimal gauge-kinetic terms become of minimal type, but with a renormalized value of the \( U(1) \) gauge coupling w.r.t. the gauge coupling of the other \( SU(n+m) \) fields; after some rescaling we find

\[
e^{-1} \mathcal{L}_{gk} = -\frac{\sigma}{4} \left( \mathrm{tr}_m F_{i\mu\nu}(V)^2 + \mathrm{tr}_n F_{i\mu\nu}(W)^2 + 2\mathrm{tr}_m F_{i\mu\nu}(Z) F_{i\mu\nu}(\bar{Z}) \right) 
- (m + n)mn \frac{\sigma}{4} \left( 1 + \frac{\rho}{\sigma m + n} \right) F_{\mu\nu}(A)^2.
\]

(5.29)

Note that the \( D \)-potential which accompanies the gauging induces mass terms for the vector bosons \((Z, \bar{Z})\). In the present normalization of the lagrangean the mass-term for the heavy gauge bosons is just \( m_Z = 1/f \), but the physical masses then become to lowest order \( M_{\mu\nu}^{\text{phys}} = 1/f \sqrt{\sigma} \).

### 6 Grassmannian standard model

We now turn to an example illustrating how one can cancel anomalies by adding rescaled matter multiplets. If we consider the case with \( m = 2 \) and \( n = 3 \) then the Grassmannian manifold may be the basis of an \( SU(5) \) unification model with the standard model group \( SU(2) \times U(1) \times SU(3) \) as the unbroken subgroup. We do not require the \( U_\eta(2, 3) \) to be compact nor do we disregard the central charge \( C \). In the standard model the field content is such that all possible anomalies cancel in each
Table 1: Grassmannian (matter) multiplets and their chiral fermion content. $Y$ is the canonical charge of the $\sigma$ model and $Y_w$ denotes the hypercharge needed for anomaly cancelation within the standard model. These charges can be identified if $Y = 15Y_w$. The number $k$ gives the rescalings with a singlet $\Omega_Y$. $C$ is the central charge, which can be chosen differently for each (matter) multiplet, using the singlet $\Omega_C$.

| Multiplet | Fermion | $Y$ | $C$ | $Y_w$ | $k$ |
|-----------|---------|-----|-----|-------|-----|
| $Q^a$    | $q^a_L$ | $n + m$ | $c_q$ | $+1/3$ | 0   |
| $Q'^a$   | $q'^a_L$ | $n + m$ | $c_q$ | $+1/3$ | 0   |
| $L_i$    | $l_i$   | 3   | $c_l$ | -1    | $l = -3$ |
| $H^{-i}$ | $h^-_{i,L}$ | 3 | $c_-$ | -1    | $h^- = -3$ |
| $H^{+i}$ | $h^+_{i,L}$ | 3 | $c_+$ | +1    | $h^+ = 4/2$ |
| $D^a$    | $d^a_L$ | 2   | $c_d$ | +2/3  | $d = 4/3$ |
| $U^a$    | $u^a_L$ | 2   | $c_u$ | -4/3  | $u = -11/3$ |
| $E$      | $e^c_L$ | 0   | $c_e$ | 2     | $e = 5$ |
| $\Omega$ | -       | $mn$ | $c$  | -     | -   |

The hypercharges in the standard model are assigned so as to produce anomaly cancelation. In the supersymmetric models the chiral fermion representations have to be completed to the chiral supermultiplets $Q^a, L^i, D^a, U^a, H^\pm$, and $E$. However if we use the standard coupling of matter multiplets to the Grassmann $\sigma$-model we do not obtain the correct charge assignment.

In table 1 the hypercharge $Y_w$ for the chiral multiplets containing the quarks and leptons is compared to the canonical charge $Y$ defined on the Grassmannian manifold, as e.g. obtained from the couplings in eqs. (5.3) and (5.11). In the third column we have evaluated these $U(1)$ charges in the case of $U_\eta(2,3)$ ($SU(5)$). In the fourth column we have given the central charges $C$ of the multiplets. As is obvious from this table the hypercharges $Y_w$ required in the standard model do not match the charges $Y$. (For this to happen, we should have $Y = 15Y_w$ for all fields.) However from eq. (5.13) we see that the singlet chiral multiplet $\Omega_Y$ has $U(1)$ charge $mn = 6$ in the $U_\eta(2,3)$ model. By employing the rescaling: 

$Y_w = 15Y_w$.


any chiral multiplet \( \Psi \) can be given an additional charge \( kmn \). In the last column we have given the powers \((l, d, u, e, h\pm)\) to which the singlet has to be raised in order the find the right hypercharge assignment for the standard model. In a similar way the central charges \( C \) may be adjusted to coincide with the \( B - L \) quantum numbers.

In the following we assume that we have performed the rescaling to the chiral multiplets as given in this table and hence we can state the Kähler potential.

\[
K = K_{\sigma} + K_{E} + K_{L} + K_{D} + K_{U} + K_{H^+} + K_{H^-},
\]

(6.1)

where \( K_{L} \) and \( K_{D} \) are defined in eqs.(5.19) and \( K_{E}, K_{H^\pm} \) and \( K_{U} \) are defined in a similar fashion.

As fundamental compensating superpotentials we may take

\[
\begin{align*}
W_E &= fE \quad (q_E = e = 5), \\
W_{L^-} &= f^2\varepsilon_{ij}L^iH^{-j} \quad (q_{L^-} = 1 + l + h^- = -5).
\end{align*}
\]

Using the \( SU(2) \) invariant \( \varepsilon \)-tensor, it follows that \( W_{L^-} \) is a \( SU(2) \) singlet. These compensating superpotentials transform as \( \delta w = q\eta f^2 Fw \) where the numbers \( q \) is given in the brackets in (6.2). The central charge \( c_w \) of the compensating superpotential determines the central charge of the holomorphic functions into which the Kähler potential transforms: \( qmnC = c_w \). Combining these compensating superpotentials to another superpotential puts restrictions on the choice of \( C \)-charges of the superpotentials (6.2). For example

\[
W = aw_E + b(w_{L^-})^{-1},
\]

(6.3)

where \( a \) and \( b \) are complex constants, demands that \( c_{wE} = -c_{wL^-} \). This in turn puts restrictions on the \( C \)-charges of the matter fields.

For the invariant superpotential \( W \) we take a part of the standard model superpotential:

\[
W = \alpha + \beta E\varepsilon_{ij}H^{-i}L^j - \mu \varepsilon_{ij}H'^{+i}H^{-j}.
\]

(6.4)

The first term \( \alpha \) is a constant with dimension of \((\text{mass})^3\); the second term is the usual Yukawa coupling in supersymmetric models and the third term is the Higgs interaction. Notice that in this model there are no Yukawa interactions for the quarks, as the quark doublet superfield \( Q \) does not transform covariantly. Notice that the superpotential \( W \) has a homogeneously vanishing central charge if \( c_e + c_l + c_- = 0 \) and \( c_+ + c_- = 0 \). Also notice that these central charges can be choosen in accordance with the lepton number \( L \). In that case we can take \( c_+ = c_- = 0 \) and \( -c_e = c_l = 1 \). The central charges of the quark multiplets \( (D, U) \) can be chosen to match the baryon number. However, this method does not apply to the left-handed quarks described by the \( \sigma \)-model superfields \( Q \)


7 Separation of submanifolds

If one considers the combined system of a non-linear $\sigma$-model with additional matter coupling to it, the metric of that total system is in general not diagonal: one can have mixing between different representations of the symmetry algebra in the quadratic kinetic terms of the scalars and chiral fermions. This is carried over to the definition of propagators. If one knows that the theory is constructed out of several sectors, one would like to be able to assign to each sector a separate metric, without mixing between different sectors. This requires the metric to be block diagonal, with each block representing the metric of a different representation of the isometry group. With the machinery developed in the appendix this can be done elegantly without too much computational difficulty.

We consider a Kähler manifold parametrized by the coordinates $Z^A = (z^\alpha, x^A)$ and their conjugates. The method we follow generalizes the result of ref. [28] where only quadratically coupled rank 1 matter was considered:

$$K_m = \bar{x}^\alpha g_{\bar{x}x} x^\alpha,$$

with the metric $g_{\bar{x}x}$ depending on $z^\alpha$ and $\bar{z}^\alpha$. However, the starting point of this section is more general, allowing any Kähler potential $K$ of the $\sigma$-model and matter fields. First we identify the $\sigma$-model Kähler potential $K_\sigma(\bar{z}, z)$ and the matter Kähler potential $K_m(\bar{x}, x; \bar{z}, z)$ by

$$K_\sigma = K |_{x=\bar{x}=0}, \quad K_m \equiv g_{\bar{x}x} = K - K_\sigma. \quad (7.1)$$

The notation $g_{\bar{x}x}$ for the matter Kähler potential is very suggestive as it reduces to $g_{\bar{x}x} = \bar{x}^\alpha g_{\bar{x}x} x^\alpha$ when matter is quadratically coupled. To take this analogy to the case of quadratic matter coupling a bit further, we define

$$g_{\bar{x}A} = K_{\bar{x}A}, \quad g_{\bar{x}A} = K_{\bar{x}A}, \quad (7.2)$$

whilst the metrics for the matter and $\sigma$-model fields are

$$g_{\bar{x}A} \equiv K_{\bar{x}A}, \quad g_{\bar{x}A} \equiv K_{\bar{x}A}. \quad (7.3)$$

To be able to use the method explained in the appendix, we first need to define the non-holomorphic transformation matrices $X_A^A'$ and $\bar{X}_A^A$. We do this by demanding that the transformations (A.5) block-diagonalize the metric of the combined system of $\sigma$-model fields and matter and leave the metric for the matter fields unchanged. The metric of the combined system is

$$g_{\bar{x}A} = 
\begin{pmatrix}
    g_{\bar{x}x} + g_{\bar{x}x,\bar{x}x} & g_{\bar{x}A,\bar{x}} \\
g_{\bar{x}x,\bar{x}} & g_{\bar{x}A}
\end{pmatrix}, \quad (7.4)$$

where $g_{\bar{x}x,\bar{x}}$ is the metric of the $\sigma$-model without matter coupling. Then the appropriate transformation is given by the matrices

$$X_A^A = \begin{pmatrix}
    \delta^\alpha_{\alpha'} & 0 \\
    -\Gamma^A_{\bar{x}A'} & \delta^\alpha_{A'}
\end{pmatrix}, \quad \bar{X}_A^A = \begin{pmatrix}
    \delta^\alpha_{\bar{x}} & -\bar{\Gamma}^A_{\bar{x}A'} \\
    0 & \delta^\alpha_{A'}
\end{pmatrix}. \quad (7.5)$$

24
In analogy to the quadratically coupled case \([28, 16]\) we have introduced generalizations of the connections

\[
\Gamma^\alpha_{\beta\gamma} \equiv g^\alpha_{\sigma} g^{\sigma \beta, \gamma}, \quad \Gamma^A_{B\gamma} \equiv g^{A\Delta}g_{\Delta B, \gamma},
\]

\[
(7.6)
\]

\[
\Gamma^A_{BC} \equiv g^{A\Delta}g_{\Delta B, C}, \quad \Gamma^A_{x\gamma} \equiv g^{A\Delta}g_{\Delta x, \gamma}
\]

and their conjugates. (There is no object \(\Gamma^A_{x C}\) as a similar definition as in eqs. (7.6) just gives \(\Gamma^A_{x C} = \delta^A_C\).) Indeed, the metric of the full system after this transformation is

\[
g_{A'A'} = \begin{pmatrix} g^\alpha_{\alpha'} & 0 \\ 0 & g^{A'A'} \end{pmatrix},
\]

\[
(7.7)
\]

with the effective metric for the \(z^\alpha, \bar{z}^\alpha\) scalars given by

\[
g_{\alpha \alpha} = g_{\sigma \alpha} + R_{\bar{x} x \alpha}.
\]

(7.8)

In this derivation we have assumed that the metric \(g_{\sigma \alpha}\) is invertable, and we have used the generalized curvature \(R_{\bar{x} x \alpha}\) defined by

\[
R_{\bar{x} x \alpha} \equiv g_{\bar{x} x \alpha} - g_{\bar{x} B \alpha} g^{B B}g_{B x, \alpha} = g_{\bar{x} x \alpha} - \bar{\Gamma}_{\bar{x} \bar{x}}^{\alpha}g_{\bar{x} B}g^{B}.
\]

(7.9)

In the following we also assume that the metric (7.7) is invertable. Notice that the inverse of this transformation is given by the same matrices (7.5) but the primed-indices now are downstairs and there is an additional minus-sign in front of the off-diagonal parts.

We could also have chosen the matrices (7.5) differently to block diagonalize the total metric (for example use an upper triangle matrix for the first one) but as the metric \(g_{\sigma \alpha}\) was already modified (see eq. (7.4)), it is most convenient to include all the other modifications in there as well. They can all be combined simply in the curvature (7.9).

Using the connections (7.6) one can define quite a number of generalized curvature components

\[
R_{\alpha \beta \gamma} \equiv g_{\sigma \alpha} \left( \Gamma^\gamma_{\alpha \beta} \right)_{\beta} = g_{\sigma \beta} \left( \Gamma^\gamma_{\beta \gamma} \right)_{\alpha},
\]

\[
R_{AABB} \equiv g_{\Delta C} \left( \Gamma^C_{AB} \right)_{B} = g_{C A} \left( \Gamma^C_{A B} \right)_{B},
\]

\[
R_{\alpha \alpha} \equiv g_{\Delta C} \left( \Gamma^C_{\alpha \alpha} \right)_{\alpha} = g_{C A} \left( \Gamma^C_{\alpha A} \right)_{\alpha},
\]

\[
R_{\bar{x} x \alpha} \equiv g_{\bar{C} A} \left( \Gamma_{\bar{C} x} \alpha \right), \quad R_{\bar{x} \alpha A} \equiv g_{\bar{C} A} \left( \Gamma_{\bar{C} \alpha x} \right),
\]

\[
R_{\bar{x} A \bar{B}} \equiv g_{\bar{C} A} \left( \Gamma_{\bar{C} \bar{B} x} \right), \quad R_{\bar{x} B \alpha} \equiv g_{\bar{C} A} \left( \Gamma_{\bar{C} B x} \alpha \right).
\]

(7.10)

Other generalized curvature components either vanish or are irrelevant in the following.
Let us mention an important application of the transformation diagonalizing the metric to eq. (7.7). For several physical applications, like determining whether there is soft supersymmetry breaking, one needs to know the contracted connection \( \Gamma^A = \Gamma^B_{BA} \) and the Ricci-tensor \( R_{AA} = g^{BB}R_{BBAA} \) of the full model. In particular the calculation of the curvature can be very tedious even in the setup presented here, and it is hard to obtain the Ricci tensor in this way. However it is well known that the contracted connection and the Ricci tensor can be obtained from the determinant \( \det g \) of the metric

\[
\Gamma^A = (\ln \det g)_{,A}, \quad R_{AA} = (\ln \det g)_{,AA}.
\]

As the transformation matrices (7.5) are upper- or lower-triangular matrices their determinants are unity. Therefore we may use the block-diagonal metric (7.7) to calculate the determinant of the full metric: \( \det g' = \det g \).

There are some further applications of the method discussed in the appendix. If the transformations described by the matrices (7.5) are applied to the derivative of the coordinates \( z^A, x^A \) we find that (A.4)

\[
(\partial_\mu z)^A_{\mu'} = \left( \partial_\mu x^A + \Gamma^A_{x\beta} \partial_\mu z^\beta \right).
\]

The derivative \( D_\mu x^A \) is covariant under holomorphic transformations. From now on we drop the primes on the indices if no confusion is possible. Using these definitions the kinetic energy of the boson fields \( z^\alpha \) and \( x^A \) can be written as

\[
-L_B = g_{\alpha \alpha} \partial_\mu \bar{z}^\alpha \partial^\mu z^\alpha + g_{AA} D_\mu \bar{x}^A D^\mu x^A.
\]

The fermion \( \chi^L_A \), the fermionic partner of \( x^A \), is turned into a covariant vector by the same transformation

\[
\hat{\chi}^A_L = \chi^A_L + \Gamma^A_{x\beta} \psi^\beta \equiv \hat{\chi}^A_L
\]

where the hat denotes covariantization.

So far we have only discussed how the metric and covariant vectors behave under the transformations described by the matrices (7.5). This is sufficient to write the kinetic lagrangean for the complex scalars in a convenient form. We now turn to the calculation of the kinetic lagrangean of the chiral fermions. These terms in eq.(2.1) involve the covariant derivative on the chiral fermions of the full system, so we have known the form of the covariant derivative on a covariant vector \( V^A \). To calculate this we use eq.(A.8) of the appendix:

\[
(\mathcal{D}_\mu V^A)_{\mu'} = (\mathcal{D}_\mu V^A)_{\mu'} - \bar{\psi}^{E} g^{E}_{EB} \partial_\mu z^{EB} V^B + g^{A}_{A'} U^{E}_{EB} \partial_\mu \bar{z}^{E} V^{B'}.
\]

This means we have to calculate the non-vanishing contributions to the connection

\[
\Gamma^{A'}_{B'} g^{A' c'} = \Gamma^{A' A}_{B' C} g^{A' B'} g^{C' C'}
\]

26
of the full system

\begin{equation}
\begin{align*}
g'_{\alpha'\alpha',C'} &= \left( g_{\alpha'\beta'} \Gamma^{\beta'}_{\alpha'\gamma'} + R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} + R_{3B2a'C} \Gamma^{B}_{2a'} \Gamma^{C}_{\gamma'}, \ R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} \right) \\
g'_{\alpha'\alpha',C'} &= g_{\alpha'\alpha'} \left( \bar{\Gamma}^{\beta'}_{A'\gamma'}, \ \Gamma^{B'}_{A'\gamma'} C' \right) \equiv \left( g_{\alpha'\alpha'} \left( \Gamma^{B'}_{A'\gamma'} \right)_{A'}, \ g_{\alpha'\alpha'} \Gamma^{B'}_{A'\gamma'} \right) \quad (7.16)
\end{align*}
\end{equation}

which involves the metric \( g_{\alpha'\beta'} \) of the transformed system. On the r.h.s. the index \( C' = (\gamma', C') \) is written out explicitly using a row-vector notation. The non-vanishing components of \( U_{\alpha'\alpha',C'} \) are:

\begin{equation}
U_{\alpha'\alpha',C'} = -g_{\alpha'\alpha'} \left( R_{\alpha'\gamma'} - R_{A'\gamma'} \bar{\Gamma}^{\gamma'}_{x\gamma'} \right) \quad (7.17)
\end{equation}

In these expressions we have made use of a covariant derivative in \( R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} \) which is defined in the usual way using the connections given in equations (7.6), whilst we have used the identities

\begin{equation}
\begin{align*}
&\left( \Gamma^{A}_{x\gamma} \right)_{C} = \Gamma^{A}_{C\gamma} - \Gamma^{A}_{BC} \Gamma^{B}_{x\gamma} \equiv \bar{\Gamma}^{A}_{C\gamma}, \\
&= R_{\tilde{\alpha} \tilde{\beta} \alpha} - R_{\tilde{\alpha} \tilde{\beta} 2a} \Gamma^{C}_{x\gamma}. \quad (7.18)
\end{align*}
\end{equation}

With this it is easy to give the rewritten covariant derivative explicitly. As an application we give here the kinetic terms of the supersymmetric lagrangean (2.1) for the chiral fermions including covariantizations:

\[ -\mathcal{L}_{F} = g_{\alpha'\alpha'} \bar{\psi}^{\alpha'}_{L} D\bar{\psi}^{\alpha'}_{L} + g_{\alpha'\alpha'} \bar{\psi}^{\alpha'}_{L} D\bar{\psi}^{\alpha'}_{L} \]

\begin{equation}
\begin{align*}
&+ \left( \left( R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} \right)^{\gamma}_{\alpha'} + 2R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} - 2R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} \bar{\Gamma}^{\gamma'}_{x\gamma'} \right) \partial_{\mu} \bar{z}^{\mu} \\
&+ 2R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} D_{\mu} \bar{z}^{\mu} \bar{\psi}^{\alpha'}_{L} \\
&+ \left( \left( R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} \right)^{\gamma}_{\alpha'} + 2R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} - 2R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} \bar{\Gamma}^{\gamma'}_{x\gamma'} \right) \partial_{\mu} z^{\gamma'} \\
&+ 2R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} D_{\mu} z^{\gamma'} \psi^{\alpha'}_{L} \\
&+ \left( R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} + R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} \Gamma^{\gamma'}_{x\alpha'} \Gamma^{C'}_{x\gamma'} \right) \partial_{\mu} \bar{z}^{\gamma'}_{\mu} \psi^{\alpha'}_{L} \\
&+ \left( R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} + R_{\tilde{\alpha} \tilde{\beta} \alpha' \gamma'} \Gamma^{\gamma'}_{x\alpha'} \Gamma^{C'}_{x\gamma'} \right) \partial_{\mu} z^{\gamma'}_{\mu} \bar{\psi}^{\alpha'}_{L} \quad (7.19)
\end{align*}
\end{equation}
with the covariant derivatives defined in eq. (7.12) and
\[ D_\mu \psi^\alpha_L \equiv \partial_\mu \psi^\alpha_L + \Gamma^\alpha_{\beta\gamma} \partial_\mu z^\gamma \psi^\beta_L \]
\[ D_\mu \hat{\chi}_A^L \equiv \partial_\mu \hat{\chi}_A^L + \hat{\Gamma}_B^A \partial_\mu z^B \hat{\chi}_L^B + \Gamma^A_{BC} \partial_\mu x_C \hat{\chi}_L^B. \]  
(7.20)

The four-fermion terms can be calculated by using eq. (A.11) in the appendix.

## 8 Geometry of matter in Grassmannian Models

The previous section was devoted to the question of how one could make the metric of the combined system of matter and \( \sigma \)-model fields block diagonal. In doing so we noticed, that these techniques can be applied to make all kinds of functions of fields covariant under the isometries of the underlying \( \sigma \)-model. In this section we show how these methods may work in practice with the example discussed in section 3 of consistent Grassmann \( \sigma \)-models with the field content of the standard model with one generation. Our starting point is the quadratically coupled matter Kähler potential (6.1). Using the results of section 7 we have computed the connections (7.6)
\[ \Gamma^{(kc)}_{(ia)(jb)} = - f^2 \left( \delta^c_b \delta^k_j (\tilde{g}Q)_{aj} + \delta^c_a (\tilde{g}Q)_b \right), \]
\[ \Gamma^{L}_{E(jb)} = - e f^2 (\bar{Q}g)_{jb} E, \]
\[ \Gamma^{k}_{L(jb)} = - f^2 \left( l(\bar{Q}g)_{bj} L^k + \delta^k_j (\tilde{Q}gL)_b \right), \]
\[ \Gamma^{c}_{D(jb)} = - f^2 \left( d(\bar{Q}g)_{bj} D^c + \delta^c_b (\tilde{D}gQ)_j \right). \]  
(8.1)

The connection for \( U \) is similar to the one for \( D \), and the connections for the Higgses \( H^\pm \) are similar to the one for \( L \). (In models with more generations, the quark doublets \( Q^i \) have the same connection as the \( \sigma \)-model field \( Q \).) To make a distinction between indices referring the original \( \sigma \)-model fields \( Q^i_a \) and matter indices \( a \) and \( i \), we write \((ai)\) for the former ones. Notice that the normal gauge, in which all connections vanish, coincides with the unitary gauge \( Q = 0 \). Because of the global \( U_{\eta}(2, 3) \) invariance, the vacuum can always be studied in the normal gauge by setting \(< Q > = 0\). Using these connections, one obtains the covariant chiral fermions by eq. (7.14), for example
\[ \hat{e}_L^c \equiv e_L^c = e_L^c - e f^2 E \text{ tr}_m \tilde{Q}gq_L, \]
\[ \hat{l}_L^i \equiv l_L^i = l_L^i - f^2 \left( lL^i \text{ tr}_m \tilde{Q}gq_L + (qL\tilde{Q}gL)^i \right). \]  
(8.2)
Because we only consider quadratically coupled matter here, we have $\Gamma^A_{x\alpha} = x^B \Gamma^A_{B\alpha}$ and $\Gamma^A_{BC} = 0$. For the same reason most of the curvatures of eq.(7.10) are related; we find

$$R^{(ia)}_{(bj)} (kc)_{(dl)} = -\eta f^2 (g^{(ic)}_{\sigma (bj)} g^{(ka)}_{\sigma (dl)} + g^{(ka)}_{\sigma (bj)} g^{(ic)}_{\sigma (dl)}) ,$$

$$R^{(bj)}_{EE} (ia) = -\eta f^2 e K_E g^j_b \tilde{g}^i_a = -\eta f^2 K_E g^{(bj)}_{\sigma (ia)}$$

$$R^{(bj)}_{LL} (ia) = -\eta f^2 (l K_L g^j_i + ([LL] g)^j_i) \tilde{g}^i_a ,$$

$$R^{(bj)}_{DD} (ia) = -\eta f^2 (d K_D \tilde{g}^b_a + (\tilde{g} [DD])^b_a) g^i_i ,$$

using the notation $[LL]$, etc., of section 5. The metric $G^{(bj)}_{\sigma (ia)}$ of the $\sigma$-model fields $Q$ and $\bar{Q}$ in the presence of matter multiplets $E, L, D, U$ becomes

$$G^{(bj)}_{\sigma (ia)} \equiv g^{(bj)}_{\sigma (ia)} + \sum_x R^{(bj)}_{xx} (ia) = \alpha (g \otimes \bar{g} + g A \otimes \bar{g} + g \otimes B \bar{g})^{(bj)}_{(ia)} ,$$

using eq.(7.8) as well as the curvatures (8.3) with the short-hand notation

$$\alpha = 1 - \eta f^2 \sum_x q_x K_x ,$$

$$A = -\eta f^2 \alpha^{-1} \left([LL] + [H^+ \bar{H}^+] + [H^- \bar{H}^-]\right) ,$$

$$B = -\eta f^2 \alpha^{-1} \left([DD] + [UU]\right) .$$

Notice that in the unitary gauge $Q = 0$ the metric $G_{\sigma}$ does not reduce to the metric without matter coupling $g_{\sigma}$ evaluated at $Q = 0$. The inverse of this metric can be written as infinite sum of tensor products

$$G^{-1} = \alpha^{-1} \sum_{n=0}^{\infty} (1 + A)^{-n-1} A^n g^{-1} \otimes \bar{g}^{-1} B^n (1 + B)^{-n-1} .$$

It turns out that it is very profitable to express other quantities using the covariant objects defined in section 4 as well. First of all we find that the first order derivative of the Kähler potential simplifies to

$$K_{A'} = \left((\tilde{g}Q)_{(ia)}, E g^{(E)}_{\sigma}, (\bar{L} g^{(L)}_{\sigma})_i, (\bar{H}^+ g^{(H^+)}_{\sigma})_i, (\bar{g}^{(D)} D)_{\alpha}, (\bar{g}^{(U)} U)_{\alpha}\right) .$$

The full metric in the transformed system is given by

$$G' = \text{diag} \left(G_{\sigma}, g^{(E)}_{\sigma}, g^{(L)}_{\sigma}, g^{(H^+)}_{\sigma}, g^{(H^-)}_{\sigma}, g^{(D)}_{\sigma}, g^{(U)}_{\sigma}\right) ,$$

where $G_{\sigma}$ is given by (8.4).
9 Vacua of the Grassmannian standard model

Section 6 discussed a chiral anomaly-free Grassmannian model with the fermion particle spectrum of the standard model. We now discuss the possible vacuum configurations of this model. Grassmannian models with doubling have been studied in a supergravity background [29], but the authors did not include superpotential terms which can alter their claim that the fermion masses are of the order of the gravitino mass. Using the geometrical results of section 8 we can discuss the vacuum solutions of this model in an elegant and straightforward fashion.

Before going into the details of the model we first observe that barring non-trivial topological effects [19] the vacuum can always be chosen such that \( \langle Q \rangle = 0 \). As the vacuum expectation values of \( Q \) and \( \bar{Q} \) are constants, they can be set to zero by a global gauge transformation. Notice that \( \langle Q \rangle = 0 \) is indeed a vacuum solution, because in the scalar potential \( Q \) and \( \bar{Q} \) always appear together.

In the supergravity background the consistent model of \( U_\eta(2,3)/U(2) \times U(3) \) with the chiral fermion content of the standard model should satisfy at least the following requirements in order not contradict the standard model phenomenology: the gauge group \( SU(3) \times U_{\text{em}}(1) \) is unbroken, and the gauginos and the complex scalar bosons should acquire masses above the scale of the gauge bosons and the chiral fermions.

Here we analyze the restrictions resulting from the electroweak symmetry breaking. The subgroup \( SU(3) \times U_Y(1) \times SU(2) \) is gauged and the generator \( Q_{\text{em}} = \frac{1}{2} Y_w + I_3 \) is unbroken. Therefore all singlets under \( SU(2) \) should vanish in the vacuum; this holds in particular for the covariant superpotential \( \mathcal{W} \). Furthermore we demand that \( B - L \) is a good symmetry, also below the electroweak symmetry breaking scale. Only neutral parts of the Higgs \( SU(2) \) doublets may acquire a vacuum expectation value

\[
< H^+ > = \begin{pmatrix} 0 \\ H_0^+ \end{pmatrix}, \quad < H^- > = \begin{pmatrix} H_0^- \\ 0 \end{pmatrix}.
\]  

(9.1)

The Killing potentials of the \( Y \)-charge and the weak-isospin

\[
\mathcal{M}_Y = -\frac{6}{\eta f^2} + 15 \left( |H_0^+|^2 - |H_0^-|^2 \right),
\]

\[
\mathcal{M}_{I_3} = \frac{1}{2} \left( |H_0^-|^2 - |H_0^+|^2 \right),
\]

(9.2)

are the only Killing potentials which do not necessarily vanish. The non-vanishing part of the scalar potential due to the \( D \)-terms is given by

\[
V_D = \frac{1}{2} g_Y^2 \mathcal{M}_Y^2 + \frac{1}{2} g_3^2 \mathcal{M}_{I_3}^2.
\]

(9.3)
When only the standard model gauge group is gauged, the gauge couplings are independent. We denote the $U(1)$ gauge coupling constant by $g_Y$, the gauge coupling constants for $SU(2)$ and $SU(3)$ by $g_2$, $g_3$. We observe, that there always is a $D$-term supersymmetry and internal symmetry breaking, and the minimum of the potential occurs at

$$|H^+_0|^2 - |H^-_0|^2 = -2M_{I_3} = \frac{15g_Y^2}{(15g_Y)^2 + \frac{1}{7}g_2^2} \left( \frac{6}{\eta f^2} \right)$$  \hspace{1cm} (9.4)$$

in both cases ($\eta = \pm 1$). The other Killing potential takes the value

$$\mathcal{M}_Y = \frac{-\frac{1}{7}g_2^2}{(15g_Y)^2 + \frac{1}{7}g_2^2} \left( \frac{6}{\eta f^2} \right).$$  \hspace{1cm} (9.5)$$

In section [section number] we discussed special requirements which have to be fulfilled in order for $\langle \mathcal{W} \rangle = 0$ to be allowed. We investigate the consequences of these conditions. We assume that the covariant superpotential can be written as $\mathcal{W} = wf^3\mathcal{W}$, eq. (2.32). (In general this may be a sum of such products.) For the invariant superpotential $\mathcal{W}$ one can take (6.4). For the compensating superpotential $w$ we can choose between two compensating superpotentials, see (6.2) where also their charges can be found. Because of the strong restriction that the covariant superpotential has to vanish in the vacuum, it follows that the gravitino mass vanishes. Therefore soft supersymmetry breaking masses can only arise due to the non-linear nature of the model. The minimal condition for which the covariant superpotential may vanish is that

$$0 < -\frac{1}{\omega} = -\frac{\kappa^2}{q_f f^2}.$$  \hspace{1cm} (9.6)$$

This requirement specifies which version of $U_\eta(2, 3)$ one should use. The compensating superpotential $w_E$ is relevant in the non-compact version ($\eta = -1$) as the charge $q_E$ is positive, to incorporate proper electroweak symmetry breaking. However the charge of $w_L$ is negative, so it should be used in the compact version ($\eta = 1$). When $0 < -\frac{1}{\omega} \leq 1$ the derivatives of the covariant superpotential should vanish as well.

We will now analyse the different cases in more detail, starting with $w_E$. In the case that $-\frac{1}{\omega} > 1$ the scalar potential is always at its minimum if eq. (9.4) is satisfied, but $\tan \beta$ is arbitrary. When $0 < -\frac{1}{\omega} \leq 1$ we find that in addition

$$\alpha - \mu |H^+_0||H^-_0| = 0.$$  \hspace{1cm} (9.7)$$

From this equation together with (9.4) we get a prediction for the ratio of the two VEVs of the Higgses

$$\tan^2 \beta \equiv \frac{|H^+_0|^2}{|H^-_0|^2} = \frac{\sqrt{\mathcal{M}^2_{I_3} + (\alpha/\mu)^2} - \mathcal{M}_{I_3}}{\sqrt{\mathcal{M}^2_{I_3} + (\alpha/\mu)^2} + \mathcal{M}_{I_3}}.$$  \hspace{1cm} (9.8)$$

31
where $\mathcal{M}_{I_3}$ is given by eq. (9.4).

Finally we consider the case of $w = w_L$. When $1 < -\frac{1}{\eta}$, the only restriction on $H_0^\pm$ is eq. (9.4); $\tan \beta$ remains undetermined. However when $0 < -\frac{1}{\eta} \leq 1$, the vanishing of the derivatives of the covariant superpotential demands that either $H_0^0 = 0$ or eq. (9.7) is satisfied. There are two inequivalent vacua which both break the electroweak symmetry. First of all

$$H_0^- = 0, \quad |H_0^+|^2 = -2\mathcal{M}_{I_3} = \frac{15g_Y^2}{(15g_Y)^2 + \frac{1}{4}g_2^2} \left(\frac{6}{\eta f^2}\right),$$

(9.9)

which gives the unacceptable result $\tan \beta = \infty$. The other vacuum solution leads to a $\tan \beta$ as given in eq. (9.8).

10 Conclusions

Effective field theory may serve as a powerful tool in the study of physics beyond the standard model up to the intrinsic cut-off scale, which could be as large as the Planck scale. If the theory is realized in a broken phase, the symmetries are non-linear. For $N = 1$ supersymmetric theories this involves the study of Kähler manifolds. Kählerian coset models provide a class of interesting examples, but unfortunately in their simplest version these models are inconsistent. Until recently the only known method to make these models consistent in a supersymmetric way, was by doubling the spectrum by adding mirror chiral superfields. The phenomenology of these doubling models is unsatisfactory as the fermions can easily get masses of the cut-off scale. If a renormalizable supersymmetric field theory is plagued by anomalies, one adds extra matter representations with the appropriate quantum numbers. When matter is coupled to Kähler models, this can be done similarly if the charges of the matter superfields can be manipulated freely. In ref. [16] we showed that it was possible to do this, and construct consistent supersymmetric $\sigma$-models without resorting to mirror chiral superfields. The crucial step is that one can couple a singlet to the Kähler manifold using the holomorphic functions $F_i$ in which the Kähler potential transforms. Once it was understood how to couple a singlet with an arbitrary $U(1)$ charge to the $\sigma$-model, the door was open to change the charges of other matter representations as well using rescaling of these matter fields by a non-trivial singlet.

In this article we have reviewed and extended these ideas. The Killing potentials were used also to give a non-trivial example of a non-standard, non-minimal gauge kinetic function. We have discussed in detail the coupling of Kähler models with additional matter to supergravity. We showed that in supergravity the rescaling of the matter fields is a consequence of their Weyl-weights and the covariance of the Kähler potential. The compensating singlet of superconformal gravity can be used to cancel the transformation of the Kähler potential, before reducing it to supergravity. This singlet is used to set the Weyl-weights of the
matter fields to zero. Doing this introduces the same additional transformation rules for the matter fields. Because of the transformation properties of the compensating singlet in supergravity, the superpotential has to be covariant as well. Using this covariant superpotential one can construct an invariant Kähler potential. With the auxiliary fields of the gauge multiplets coupling to the scalars via the Killing potentials we obtain additional contributions to the scalar potential.

The study of the vacuum configurations of these models implies that either the Kähler $U(1)$ isometry is broken or additional requirements have to be satisfied. Either there is a relation between the cut-off scale $f^{-1}$, the Planck scale and the transformation properties of the covariant superpotential or there are more requirements on the VEVs of the scalar fields. Another consequence is that the gravitino mass vanishes in the case of an unbroken Kähler $U(1)$.

The consistent system of $\sigma$-model and matter superfields can become quite complicated. In particular the various irreducible representations have mixed kinetic terms as the metric is not block diagonal. By applying a non-holomorphic transformation on covariant objects, like chiral fermions of the full model and derivatives of the scalars, it is possible to block-diagonalize the metric. This transformation also turns non-covariant objects, like the chiral fermions belonging to the matter sectors, into covariant ones. This method is explained in section \[ \text{and the geometrical background can be found in the appendix. The automatic covarantizations are convenient consequences of this method, but other calculations are simplified as well.} \]

All these different aspects are illustrated by the example of Grassmannian coset models $U_\eta(m, n)/U(m) \times U(n)$. The properties of matter coupling to Grassmannian Kähler manifolds are described by using left- and right-covariant real composite superfields. This offers many different ways to construct non-equivalent invariant Kähler potentials for the matter fields. The algebra of isometries of the $\sigma$-models is discussed in detail, identifying the Kähler $U(1)$ charge and a central charge.

At the classical level the isometries can be gauged by a straightforward procedure. Non-standard non-minimal kinetic terms for the gauge fields were constructed, but supersymmetry requires them to be accompanied by higher-derivative terms involving the components of the physical scalar supermultiplets. Some of these terms disappear in the broken phase of local gauge symmetries, in which case the $U(1)$-coupling constant is renormalized w.r.t. the remaining part of the gauge group.

As a practical illustration of the cancellation of the anomalies in a Grassmannian coset models, we discussed a model of the standard model where the superpartner of the quark-doublet is interpreted as the coordinates of the coset $U_\eta(2, 3)/U(2) \times U(3)$. We showed how on this Kähler manifold matter representations could be added in such a way that the chiral fermion sector of the model coincides with the standard model. As the covariant superpotential plays an important role in supergravity, the construction of the superpotential was discussed.
in some detail. The power of the non-holomorphic transformation on covariant objects was illustrated for the calculation of the metric and first derivative of the Kähler potential; we obtained the expressions for the covariant chiral matter fermions in this way. We showed that in supergravity the Grassmannian version of the standard model leads to phenomenologically acceptable results only in very specific cases. The compensating superpotential is either $w_L$ or $w_E$, see eqs. (6.2). The former leads to two inequivalent vacua, of which only one is acceptable, as in the other case one of the two Higgses has a vanishing VEV. The covariant superpotential $w_E$ also gives rise to an acceptable vacuum, but with a different value for $\tan \beta$.

Acknowledgment

We would like to thank B.J.W. van Tent for reading the manuscript and for helpful comments.

A Appendix: Geometry of Kähler manifolds

In section $\natural$ we review the structure of $N = 1$ supersymmetric models in 4 dimensions and come across various geometrical objects like the metric, connection and curvature. These geometrical objects are used there as convenient short-hand to write the lagrangean in a compact form. They have very specific functions in the supersymmetric lagrangean: the kinetic energy of the scalars and the fermions are described by the metric. The Dirac operator that acts on the chiral fermions involved the connection and the four-fermion interactions couple via the curvature after the auxiliary fields are removed, see eq.(2.1). Section $\natural$ discusses a few of these applications like how to define fermions in chiral matter multiplets in order that they transform covariantly and how to make the metric block diagonal. In this appendix we look at these objects from a geometrical point of view but keeping physical applications in mind. We consider a Kähler manifold described (locally) by a Kähler potential $K(Z, \bar{Z})$ and treat the superpartners $\psi^A_L$ of $Z^A$ in exactly the same way as covariant fields that live on this Kähler manifold. Various transformations that can act on covariant fields are studied in this appendix. In section $\natural$ these transformations are used to cast the supersymmetric lagrangean involving matter multiplets into a form depending only on physical covariant fields.

Since a Kähler manifold is complex, the coordinate transformations preserving the complex structure are holomorphic

$$Z^A \rightarrow Z'^A = R^A(Z), \quad \bar{Z}^\dagger \rightarrow \bar{Z}'^\dagger = \bar{R}^\dagger(\bar{Z}).$$  \hspace{1cm} (A.1)

Any object $V^A$ (and its conjugate $\bar{V}^\dagger$) transforming as

$$V^A \rightarrow V'^{A'} = X^{A'}_{A}(Z)V^A, \quad \bar{V}^\dagger \rightarrow \bar{V}'^{\dagger A'} = \bar{X}^{\dagger A'}_{\dagger A}(\bar{Z})\bar{V}^\dagger$$  \hspace{1cm} (A.2)
under the holomorphic coordinate transformations with

\[ X^A_A'(Z) = \mathcal{R}^A_A(Z), \quad \bar{X}^\bar{A}_\bar{A}'(Z) = \mathcal{R}^\bar{A}_\bar{A}(Z) \]  

(A.3)

is called a covariant vector of the Kähler manifold. In the context of supersymmetric \( \sigma \)-models many covariant vectors are encountered, to name a few: the derivatives \( \partial_\mu Z^A \), the differentials \( dZ^A \) and the superpartners \( \psi^A \) of \( Z^A \).

The coordinate transformations (A.1) generally do not leave the metric of the Kähler manifold invariant, only the \( S \)-matrix of the field theory described by these coordinates. The coordinate transformations that do leave the metric invariant are called isometries.

On the covariant vectors the transformation rules (A.2) we can consider more general transformations

\[ V^A \rightarrow V'^A = X^A_A(Z, Z)V^A, \]  

(A.4)

where \( X^A_A(Z, Z) \) are possibly non-holomorphic functions. This type of transformations can be used to make the physical content of a field theory more transparent, as is illustrated in section [17]. The first thing to note is that these transformations can not be generated by non-holomorphic coordinates transformations because they would introduce terms involving \( \bar{V}^A \) in eq.(A.4) too. Therefore the transformations (A.4) can only be defined on the level of covariant vectors and geometrical objects like the metric: \( V'^A \) is nothing but a short-hand for the expression \( X^A_A(Z, Z)V^A \) for the covariant vector \( V^A \). In the following we study how the transformations (A.4) change the appearance of formulae involving the metric, connection and curvature.

If we demand that the metric defines an invariant inner product for covariant vectors, it must transform as

\[ g_{A\bar{A}} \rightarrow g'_{A'\bar{A}'} = \bar{X}^\bar{A}_{\bar{A}'}X^{A'}_A g_{A\bar{A}} \]  

(A.5)

where \( \bar{X}^\bar{A}_{\bar{A}'}(\bar{Z}, Z) \) is the inverse of \( X^A_A(\bar{Z}, Z) \).

A word about our notation is in order here: let \( A_A \) be any object with one index down, not necessarily a vector; it may be a function of covariant vectors and derivatives. Applying (A.4) to all covariant vectors transforms \( A_A \) into \( A'_A \). One can also just contract \( A_A \) with the matrix \( X^A_A \) this is denoted by \( A_{A'} = X^A_A A_A \). In the case of covariant vectors and the metric \( g_{A'\bar{A}'} = g_{A'\bar{A}'} \) these two definitions coincide but this is not true in general. (When there is no confusion possible, like with covariant vectors or the metric, we drop the prime on the symbol itself.)

The prime example where this is not the case is the connection

\[ \Gamma^A_{B'C} \rightarrow \Gamma'^A_{B'C'} = \Gamma^A_{B'C'} + U^A'_{B'C'} + g^{A'B'} \bar{U}^A_{B'C'} g_{A'B'} \]  

\[ \bar{\Gamma}^\bar{A}_{\bar{B}'\bar{C}'} \rightarrow \bar{\Gamma}'^{\bar{A}'}_{\bar{B}'\bar{C}'} = \bar{\Gamma}^{\bar{A}'}_{\bar{B}'\bar{C}'} + \bar{U}^{\bar{A}'}_{\bar{B}'\bar{C}'} + g^{\bar{B}'\bar{A}'} \bar{U}^{\bar{B}'}_{\bar{B}'\bar{C}'} g_{\bar{B}'\bar{A}'} \]  

(A.6)
with

\[ U_{B'C'}^{A'} = X_A^{A'} X_B^{B'} X_C^{C'}, \quad \bar{U}_{E'F'}^{A'} = \bar{X}_A^{A'} X_B^{B'} X_C^{C'}, \quad (A.7) \]

\[ U_{B'C'}^{A'} = X_A^{A'} X_B^{B'} \bar{X}_C^{C'}, \quad \bar{U}_{E'F'}^{A'} = \bar{X}_A^{A'} X_B^{B'} \bar{X}_C^{C'}. \]

Notice that the third term in equations (A.6) vanishes if the transformations are holomorphic. Here we see clearly that the connection is not a tensor even in the case of holomorphic transformations. But this exactly enables us to define a covariant derivative \( D_\mu \) for covariant vectors \( D_\mu V^A \equiv \partial_\mu V^A + \Gamma_\mu^{A'B'} \partial_\mu Z^{B'} V^C \).

However it is only covariant under holomorphic transformations but not under eq.(A.4); indeed

\[ (D_\mu V)^{A'} = D_\mu V^{A'} + g^{A'B'} U_{E'B'}^{A'} g^{A'C'} \partial_\mu Z^{B'} V^{C'} - U_{E'B'}^{A'} \partial_\mu Z^{B'} V^{C'}. \]

(A.8)

The second term on the r.h.s. follows from eq.(A.6) and the third compensates for the fact that the ordinary derivative \( \partial_\mu \) within \( D_\mu \) can hit the transformation matrix \( X_A^{A'} \) which may also depend on \( \bar{Z}_A \). The first term on the r.h.s. is of the same form as what one would get if the transformations (A.4) are holomorphic. The last two terms involve \( U \) and \( \bar{U} \)'s with mixed indices indicating the non-holomorphic nature of (A.4).

Finally we investigate how the transformations (A.4) influence the curvature. The calculation follows the same line as above, but now it is really convenient to separate terms which do not have mixed transformations involving \( U \) and \( \bar{U} \). With this separation one can identify which terms behave as if the transformations (A.4) are holomorphic. We call these terms holomorphic and indicate them with a superscript \( H \). The remaining terms have \( U \)'s and \( \bar{U} \)'s with mixed indices. They are called non-holomorphic and are indicated by a superscript \( N \).

As the curvature is a tensor under holomorphic transformations, the holomorphic part \( R^H \) also transforms as a tensor under (A.4). By identifying the holomorphic and non-holomorphic parts we find

\[ R'_{A'B'B'} = R^H_{A'B'B'} + g^N_{A'B'B'} - g^H_{A'C'B'} g^{C'C'} g^N_{C'A'B'} + g^N_{A'C'B'} g^{C'C'} g^N_{C'A'B'}. \]

(A.9)

As we already know how the holomorphic part of the curvature transforms, we only have to consider the terms with non-holomorphic transformations. In these terms replace the remaining holomorphic like parts \( g^H_{A'C'B'} \) by \( (g' - g^N)_{A'C'B'} \).

In section 7 we are not so much interested in the transformed curvature itself, but more in having a simple way to write expressions involving the curvature, like the four-fermion terms. Therefore we write

\[ R'_{A'B'B'} = \hat{R}'_{A'B'B'} + g'_{A'C'B'} \bar{U}_{E'B'}^{C'} \]

(A.10)
and notice that the second term depend on the order of the indices $\mathcal{B}'$ and $\mathcal{B}'$ and where the first is given by

\begin{equation}
\hat{R}_{\mathcal{A}'\mathcal{B}'\mathcal{B}'} = R_{\mathcal{A}'\mathcal{B}'\mathcal{B}'} + g_{\mathcal{A}'\mathcal{B}'} W_{\mathcal{A}'\mathcal{B}'} W_{\mathcal{A}'\mathcal{B}'} + g_{\mathcal{A}'\mathcal{B}'} W_{\mathcal{A}'\mathcal{B}'} W_{\mathcal{A}'\mathcal{B}'}
\end{equation}

\begin{equation}
+ g_{\mathcal{A}'\mathcal{D}'} U_{\mathcal{C}'\mathcal{B}'} g_{\mathcal{C}'\mathcal{A}'} - U_{\mathcal{A}'\mathcal{B}'} g_{\mathcal{A}'\mathcal{D}'} U_{\mathcal{A}'\mathcal{B}'} \tag{A.11}
\end{equation}

\begin{equation}
- g_{\mathcal{A}'\mathcal{D}'} \left( -\Gamma_{\mathcal{E}'\mathcal{B}'} U_{\mathcal{A}'\mathcal{B}'} + U_{\mathcal{E}'\mathcal{B}'} \Gamma_{\mathcal{A}'\mathcal{B}'} + U_{\mathcal{E}'\mathcal{B}'} \right)
\end{equation}

\begin{equation}
- g_{\mathcal{A}'\mathcal{D}'} \left( -\Gamma_{\mathcal{E}'\mathcal{B}'} U_{\mathcal{A}'\mathcal{B}'} + U_{\mathcal{E}'\mathcal{B}'} \Gamma_{\mathcal{A}'\mathcal{B}'} + U_{\mathcal{E}'\mathcal{B}'} \right).
\end{equation}

Here $W$ is defined as

\begin{equation}
W_{\mathcal{A}'\mathcal{B}'} = X_{\mathcal{D}'} X_{\mathcal{A}'} \bar{X}_{\mathcal{B}'} X_{\mathcal{B}'} \tag{A.12}
\end{equation}

and similarly for $\bar{W}$.

**References**

[1] C.-L. Ong *Phys. Rev.* **D27** (1983) 911.

[2] M. P. Mattis *Phys. Rev.* **D28** (1983) 2649.

[3] J. W. van Holten *Z. Phys.* **C27** (1985) 57.

[4] Y. Achiman, S. Aoyama, and J. W. van Holten *Phys. Lett.* **141B** (1984) 64.

[5] Y. Achiman, S. Aoyama, and J. W. van Holten *Nucl. Phys.* **B258** (1985) 179.

[6] M. Bando, T. Kugo, and K. Yamawaki *Phys. Rept.* **164** (1988) 217–314.

[7] T. Yanagida and Y. Yasui *Nucl. Phys.* **B269** (1986) 575.

[8] T. Kugo and T. Yanagida *Phys. Lett.* **134B** (1984) 313.

[9] C.-L. Ong *Phys. Rev.* **D27** (1983) 3044.

[10] G. Moore and P. Nelson *Phys. Rev. Lett.* **53** (1984) 1519.

[11] E. Cohen and C. Gomez *Nucl. Phys.* **B254** (1985) 235.

[12] G. M. Shore *Nucl. Phys.* **B320** (1989) 202.
[13] A. C. W. Kotcheff and G. M. Shore *Nucl. Phys.* **B333** (1990) 701.

[14] W. Buchmuller and W. Lerche *Ann. Phys.* **175** (1987) 159.

[15] J. W. van Holten *Nucl. Phys.* **B260** (1985) 125.

[16] S. Groot Nibbelink and J. W. van Holten *Phys. Lett.* **B442** (1998) 185, [hep-th/9808147](http://www.arXiv.org/abs/hep-th/9808147).

[17] E. Cremmer, S. Ferrara, L. Girardello, and A. Van Proeyen *Nucl. Phys.* **B212** (1983) 413.

[18] T. Kugo and S. Uehara *Nucl. Phys.* **B222** (1983) 125.

[19] S. Groot Nibbelink and J.W. van Holten, in preparation.

[20] S. Aoyama *Nuovo Cim.* **92A** (1986) 282.

[21] S. Aoyama *Z. Phys.* **C32** (1986) 113.

[22] T. E. Clark and S. T. Love *Nucl. Phys.* **B254** (1985) 569.

[23] J.A. Bagger and E. Witten, *Phys. Lett.* **B118** (1982) 103

[24] J. A. Bagger *Nucl. Phys.* **B211** (1983) 302.

[25] J. Bagger and J. Wess, *Supersymmetry and supergravity*. Princeton University Press, 1990,1983. JHU-TIPAC-9009.

[26] P. Binetruy, G. Girardi, and R. Grimm. LAPP-TH-275-90.

[27] R. Grimm *Phys. Lett.* **B242** (1990) 64.

[28] Y. Achiman, S. Aoyama, and J. W. van Holten *Phys. Lett.* **150B** (1985) 153.

[29] T. Goto and Y. Okada *Phys. Rev.* **D45** (1992) 3636–3640.