The total energy–momentum tensor for electromagnetic fields in a dielectric

Michael E. Crenshaw

*US Army Aviation and Missile Research, Development, and Engineering Center, Redstone Arsenal, AL 35898, USA*

There are various formulations of energy–momentum tensors for an electromagnetic field in a linear dielectric. The total energy–momentum tensor, comprised of electromagnetic and material components, must be unique. We discuss the construction of the total energy–momentum tensor and the associated conservation laws.

I. INTRODUCTION

Radiation pressure is an observable consequence of optically induced forces on materials. On cosmic scales, radiation pressure is responsible for the bending of the tails of comets as they pass near the sun. At a much smaller scale, optically induced forces are being investigated as part of a toolkit for micromanipulation and nanofabrication technology \[1\]. A number of practical applications of the mechanical effects of light–matter interaction are discussed by Qiu, et al. \[2\]. The promise of the nascent nanophotonic technology for manufacturing small, low-power, high-sensitivity sensors and other devices has likely motivated the substantial current interest in optical manipulation of materials at the nanoscale, see, for example, Ref. \[2\] and the references therein. While substantial progress toward optical micromanipulation has been achieved, *e.g.* optical tweezers \[1\], in this report we limit our consideration to the particular issue of optically induced forces on a transparent dielectric material. As a matter of electromagnetic theory, these forces remain indeterminate and controversial. Due to the potential applications in nanotechnology, the century-old debate regarding these forces, and the associated momentums, has ramped up considerably in the physics community.

The energy–momentum tensor is the centerpiece of conservation laws for the unimpeded, inviscid, incompressible flow of non-interacting particles in the continuum limit in an otherwise empty volume. The foundations of the energy–momentum tensor and the associated tensor conservation theory come to electrodynamics from classical continuum dynamics by applying the divergence theorem to a Taylor series expansion of a property density field of a continuous flow in an otherwise empty volume. The dust tensor is a particularly simple example of an energy–momentum tensor that deals with particles of matter in the continuum limit in terms of the mass density \(\rho_m\), energy density \(\rho_m c^2\), and momentum density \(\rho_m v\). Newtonian fluids can behave very much like dust with the same energy–momentum tensor. The energy and momentum conservation properties of light propagating in the vacuum were long-ago cast in the energy–momentum tensor formalism in terms of the electromagnetic energy density and electromagnetic momentum density. However, extrapolating the tensor theory of energy–momentum conservation for propagation of light in the vacuum to propagation of light in a simple linear dielectric medium has proven to be problematic and controversial. A dielectric medium is not "otherwise empty" and it is typically assumed that optically induced forces accelerate and decelerate nanoscopic material constituents of the dielectric. The corresponding material energy–momentum tensor is added to the electromagnetic energy–momentum tensor to form the total energy–momentum tensor, thereby ensuring that the total energy and the total momentum of the thermodynamically closed system remain constant in time.

II. THE TOTAL ENERGY–MOMENTUM TENSOR

The total energy–momentum tensor for the flow of light in a linear medium is the sum of the electromagnetic energy–momentum tensor and a material energy–momentum tensor. The typical development \[3–5\], reviewed by Pfeifer, Nieminen, Heckenberg, and Rubinsztein-Dunlop \[2\], shows

\[
T^{\alpha\beta}_{\text{EM,Abr}} = \begin{pmatrix}
\frac{1}{2} (D \cdot E + H \cdot B) \\
E \times H \\
- E \wedge D - H \wedge B + \frac{1}{2} (E \cdot D + H \cdot B) I
\end{pmatrix},
\]

(2.1)
in Heaviside–Lorentz units and

\[
T_{\text{mat, Abr}}^{\alpha\beta} = \begin{bmatrix}
\rho_m c^2 & \rho_m c \mathbf{v} \\
\rho_m c \mathbf{v} & \rho_m \mathbf{v} \wedge \mathbf{v}
\end{bmatrix}.
\] (2.2)

Here, \(T_{\text{EM, Abr}}^{\alpha\beta}\) is the well-known Abraham electromagnetic energy–momentum tensor, \(T_{\text{mat, Abr}}^{\alpha\beta}\) is the well-known dust tensor, and \(\rho_m\) is the mass density of the dust. The total energy–momentum tensor,

\[
T_{\text{total}}^{\alpha\beta} = T_{\text{EM, Abr}}^{\alpha\beta} + T_{\text{mat, Abr}}^{\alpha\beta}
\]

is

\[
T_{\text{total}}^{\alpha\beta} = \left[ \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B}) + \rho_m c^2 - \mathbf{E} \wedge \mathbf{H} + \rho_m c \mathbf{v} - \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) I + \rho_m \mathbf{v} \wedge \mathbf{v} \right].
\] (2.3)

According to the scientific literature [3, 5], the total energy–momentum tensor can also be constructed from the sum of the Minkowski electromagnetic energy–momentum tensor

\[
T_{\text{EM, Mink}}^{\alpha\beta} = \left[ \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B}) - \frac{\rho_m c^2}{c} + \mathbf{E} \times \mathbf{H} - \mathbf{D} \times \mathbf{B} - \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) I \right]
\] (2.4)

and an appropriate material tensor \(T_{\text{mat, Mink}}^{\alpha\beta}\). However, there is no physical model for this material tensor as there is for \(T_{\text{mat, Abr}}^{\alpha\beta}\). Instead, the material tensor [3, 5]

\[
T_{\text{mat, Mink}}^{\alpha\beta} = \left[ \rho_m c \mathbf{v} + \mathbf{E} \times \mathbf{H} - \mathbf{D} \times \mathbf{B} + \rho_m c \mathbf{v} \wedge \mathbf{v} \right]
\] (2.5)

that accompanies the Minkowski electromagnetic energy–momentum tensor, Eq. (2.4), is obtained phenomenologically by starting with the total energy–momentum tensor, Eq. (2.3), then subtracting the Minkowski electromagnetic energy–momentum tensor, Eq. (2.4).

The total energy and total linear momentum are constrained by the conservation law [6]

\[
\frac{\partial T_{\text{total}}^{\alpha\beta}}{\partial x^\beta} = 0,
\] (2.6)

where \(x^\beta \in \{x^0 = ct, x^1 = x, x^2 = y, x^3 = z\}\). Substituting Eq. (2.3) into Eq. (2.6), one obtains

\[
\frac{1}{c} \frac{\partial}{\partial t} \left( \rho_e c^2 + \mathbf{E} \times \mathbf{H} + \rho_m c \mathbf{v} \right) = 0
\] (2.7)

for the \(\alpha = 0\) element. Here, \(\rho_e\) denotes the electromagnetic energy density

\[
\rho_e = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}).
\] (2.8)

An additional constraint on conservation is that the total energy and the total momentum must remain constant as a wave packet transits from free-space into the material. For the total energy and total linear momentum, this constraint is

\[
P^\alpha (t) = \int_\sigma T_{\text{total}}^{\alpha\beta} dv = P^\alpha (t_0)
\] (2.9)

where the volume of integration has been extended to all-space, \(\sigma\). For the conservation of total angular momentum, the constraint

\[
T_{\text{total}}^{\alpha\beta} = T_{\text{total}}^{\beta\alpha}
\] (2.10)
is often employed although the requirement that the total angular momentum is constant in time does not necessarily require the total energy–momentum tensor to be diagonally symmetric \[5\]. Some authors take this caveat to mean that the specific relationship between \(T^{\alpha\beta}_{\text{total}}\) and its transpose is immaterial because it is what it needs to be to insure that total angular momentum is conserved in a thermodynamically closed system. There is, however, a relationship, although it is more complicated than Eq. \(2.10\). In order to avoid unnecessary complications, we posit the case of ordinary unstructured fields in the common plane-wave limit incident on the dielectric medium from the vacuum of free space. Then, then the total energy–momentum tensor of the transmitted field can be taken as diagonally symmetric if the total energy–momentum tensor of the incident field, from vacuum, is diagonally symmetric.

Adopting the constraint on the total linear momentum, \(P_{\text{total}} = (P_1^\text{total}, P_2^\text{total}, P_3^\text{total})\), Pfeifer, Nieminen, Heckenberg, and Rubinsztein-Dunlop \[3\] derive
\[
\rho_m c \nu = (n - 1) E \times H
\]
by subtracting the electromagnetic momentum from the total momentum, where the latter is determined from Eq. \(2.10\) taking into account the well-known change in amplitude and spatial width of electromagnetic fields in a dielectric. Taking the mass density of the material \(\rho_m\) to be constant for a quasimonochromatic/monochromatic field in the plane wave limit and substituting Eq. \(2.11\) into Eq. \(2.7\), we find that
\[
\frac{1}{c} \frac{\partial \rho_e}{\partial t} + \nabla \cdot (n E \times H) = 0.
\]

This result is manifestly false for a quasimonochromatic field because the two non-zero terms depend on different powers of the independent parameter \(n\). To see this, one can perform the calculus operations on fields written in terms of a constant field amplitude and a carrier wave, \(e^{-(i\omega t \pm n(\omega/c)z)}\), as one does when deriving the Fresnel relations, or one can simply note that Eq. \(2.12\) is incommensurate with the Poynting theorem (Eq. \(3.2\), below). Consequently, we are forced to argue that \(\rho_m = (n - 1)\rho_e\) oscillates at optical frequencies such that Eq. \(2.7\) is commensurate with the Poynting theorem, but at the expense of violating the conservation law, Eq. \(2.6\), which is violated because the timelike coordinate, \(ct\), becomes index-dependent, \(ct/n\), if one follows this line of reasoning.

III. MAXWELLIAN CONTINUUM ELECTRODYNAMICS

Having determined that the standard model of the total energy–momentum tensor produces a total energy conservation law that is false, we would like to circumscribe the problem. We start with the derivation of the electromagnetic energy–momentum tensor. The familiar Maxwell–Minkowski equations can be written as \[7\]
\[
\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = 0 \quad (3.1a)
\]
\[
\nabla \cdot \mathbf{B} = 0 \quad (3.1b)
\]
\[
\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3.1c)
\]
\[
\nabla \cdot \mathbf{D} = 0, \quad (3.1d)
\]

where the macroscopic fields, \(\mathbf{E}, \mathbf{D}, \mathbf{B},\) and \(\mathbf{H}\), are functions of position, \(r\), and time, \(t\). As is commonly done, we limit consideration to simple linear dielectric media in which the center frequency of the exciting quasimonochromatic/monochromatic field is away from material resonances. In this regime, absorption and frequency dispersion can be treated as negligible in the lowest order of approximation. Then, the refractive
index, \(n(r)\), depends on the center frequency of the exciting quasimonochromatic/monochromatic field but is otherwise a real, time-independent, single-valued function of position. In this regime, we can freely use the familiar constitutive relations, \(\mathbf{D} = n^2 \mathbf{E}\) and \(\mathbf{B} = \mathbf{H}\), for a simple linear dielectric. Poynting’s theorem

\[
\frac{1}{c} \frac{\partial \rho_e}{\partial t} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = 0
\]  

(3.2)

is derived by subtracting the scalar product of Eq. (3.1a) with \(\mathbf{E}\) from the scalar product of Eq. (3.1c) with \(\mathbf{H}\) and applying common vector identities. Likewise,

\[
\frac{1}{c} \frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B})_i + \sum_j \frac{\partial}{\partial x_i} W^{\text{Mink}}_{ij} = f^\text{Mink}_i
\]  

(3.3)

is rigorously derived from the Maxwell–Minkowski field equations, Eqs. (3.1), by adding the cross-product of Eq. (3.1a) with \(\mathbf{B}\) to the cross-product of \(\mathbf{D}\) with Eq. (3.1c) and simplifying the result using Eqs. (3.1b) and (3.1d) [7]. Here,

\[
f^\text{Mink}_i = (\mathbf{E} \cdot \nabla (n^2)) \mathbf{E} = \mathbf{E} \times (\mathbf{E} \times \nabla (n^2)) + \mathbf{E}^2 \nabla (n^2)
\]  

(3.4)

is the Minkowski force density, which does not reduce to \(-(1/2)\mathbf{E}^2 \nabla (n^2)\) [8]. In fact, the Minkowski force is zero in the plane-wave limit. The elements of \(W^{\text{Mink}}\) are the elements of the matrix

\[
W^{\text{Mink}}_{ij} = -D_i E_j - H_i B_j + \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \delta_{ij}.
\]  

(3.5)

Next, we write the scalar equation, Eq. (3.2), and the three scalar components of the vector equation, Eq. (3.3), row-wise, as a differential equation

\[
\partial_\beta T^{\alpha \beta}_{\text{EM,Mink}} = f^\text{Mink}_\alpha
\]  

(3.6)

where \(f^\text{Mink}_\alpha = (0, f^\text{Mink})\) is an element of the four-force density and

\[
T^{\alpha \beta}_{\text{EM,Mink}} = \begin{bmatrix}
\frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) & \frac{1}{2} (\mathbf{E} \times \mathbf{H}) \\
\frac{1}{2} (\mathbf{D} \times \mathbf{B}) & -\mathbf{E} \wedge \mathbf{D} - \mathbf{H} \wedge \mathbf{B} + \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \mathbf{I}
\end{bmatrix},
\]  

(3.7)

is, by definition, a four-by-four square matrix that is historically known as the Minkowski energy–momentum tensor.

We can also write the matrix differential equation, Eq. (3.6), in terms of the Abraham electromagnetic momentum density. Subtracting the Abraham force density

\[
f^\text{Abr}_\alpha = \frac{\partial}{\partial t} \left(\frac{(n^2 - 1)\mathbf{E} \times \mathbf{H}}{c}\right)
\]  

(3.8)

from both sides of Eq. (3.3), we obtain

\[
\frac{1}{c} \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{H})_i + \sum_j \frac{\partial}{\partial x_i} W^{\text{Mink}}_{ij} = f^\text{Mink}_i - \left(\frac{\partial}{\partial t} \left(\frac{(n^2 - 1)\mathbf{E} \times \mathbf{H}}{c}\right)\right)_i.
\]  

(3.9)

Combining Eq. (3.2) with Eq. (3.9), row-wise, as before, we obtain a differential equation

\[
\partial_\beta T^{\alpha \beta}_{\text{EM,Abr}} = f^\text{Mink}_\alpha - \left(\frac{(n^2 - 1)\mathbf{E} \times \mathbf{H}}{c}\right)_\alpha.
\]  

(3.10)
for each \( \alpha \), where \( \partial_\beta \) is an element of the four-divergence operator

\[
\partial_\beta = \left( \frac{1}{c} \frac{\partial}{\partial t}, \partial_x, \partial_y, \partial_z \right).
\] (3.11)

Many authors claim that the Abraham force density, Eq. (3.8), that comprises the last term of Eq. (3.8) can be neglected because the time average of the fluctuating fields is essentially zero. This is a consequence of analyzing the Abraham force density, Eq. (3.8), separate from the dynamical equation, Eq. (3.9). If the Abraham force can be neglected, then

\[
\sum_j \frac{\partial}{\partial x_j} W_{ij}^{\text{Mink}} = f_i^{\text{Mink}}
\] (3.12)

because the first and last terms of Eq. (3.9) differ by a non-negligible multiplicative constant, \( n^2 - 1 \). Consequently, neglect of the Abraham force results in a manifestly false static equation for propagating electromagnetic fields.

### IV. LAGRANGIAN EQUATIONS OF MOTION

It is often claimed that Eqs. (3.2) and (3.3) are the electromagnetic energy and momentum conservation laws. However, the energy and momentum conservation laws originate in classical continuum dynamics, principally fluid dynamics, and not in continuum electrodynamics. Equations (3.2) and (3.3) are theorems of the macroscopic Maxwell–Minkowski equations, Eqs. (3.2), and have the outward appearance of the conservation law, Eq. (2.6). However, we have seen that it is impossible to construct a total energy–momentum tensor from the Maxwell–Minkowski equations that satisfies the spacetime conservation laws. Although it is common to introduce an adjustable relationship, e.g., Eq (2.11), between the electromagnetic and material components of energy and momentum in order to satisfy one of the conservation laws this action causes the violation of another conservation law that was previously satisfied. Specifically, it is not possible to satisfy both Eq. (2.6) and (2.7) simultaneously within the formalism of Maxwellian continuum electrodynamics.

The spacetime conservation law, Eq. (2.6), reflects the conservation of a scalar property in the continuum limit of an unimpeded inviscid, incompressible flow of non-interacting particles in terms of the equality of the net rate of flux out of an otherwise empty volume and the time rate of change of the property density field. This description perfectly fits the propagation of light in the vacuum. However, several of the conditions are violated if the light propagates through a simple linear dielectric medium. First, the light slows down as it enters the medium indicating some sort of impediment to free flow. Second, the volume contains a linear medium and is therefore not "otherwise empty". Third, the volume occupied by the field inside the dielectric is smaller than the volume occupied by the same field as it is incident from the vacuum due to the reduced speed of light in a dielectric, violating the condition of incompressible flow.

At this point, we turn to Lagrangian field dynamics to derive new field equations for macroscopic fields in a simple linear dielectric. The classical Lagrangian is [6, 7]

\[
L = \frac{1}{2} \int_{\sigma} (T - V) dv,
\] (4.1)

where \( T \) is the kinetic energy density, \( V \) is the potential energy density, and integration is performed over all-space \( \sigma \). We write the classical Lagrangian for macroscopic fields in a linear dielectric as

\[
L = \frac{1}{2} \int_{\sigma} \left( \left( \frac{n}{c} \frac{\partial A}{\partial t} \right)^2 - (\nabla \times A)^2 \right) dv.
\] (4.2)
The Lagrangian density,

$$L = \frac{1}{2} \left( \left( \frac{n \partial A}{c \partial t} \right)^2 - (\nabla \times A)^2 \right),$$

is the integrand of the Lagrangian, Eq. (4.2).

We consider an arbitrarily large region of space to be filled with an isotropic homogeneous transparent linear dielectric medium that is characterized by a linear refractive index $n$. We limit our attention to simple linear media and we write a new time-like variable

$$\bar{x}^0 = \frac{ct}{n}.$$ 

We take the re-parameterized Lagrangian density

$$L = \frac{1}{2} \left( \left( \frac{\partial A}{\partial \bar{x}^0} \right)^2 - (\nabla \times A)^2 \right)$$

as our starting point and apply Lagrangian field theory to systematically derive equations of motion for the macroscopic fields in an arbitrarily large isotropic homogenous block of simple linear dielectric material.

The Lagrange equation for electromagnetic fields in the vacuum is

$$\frac{d}{dt} \frac{\partial L}{\partial (\partial A_j / \partial t)} + \sum_i \frac{\partial}{\partial x_i} \frac{\partial L}{\partial (\partial A_j / \partial x_i)} = \frac{\partial L}{\partial A_j},$$

although Eq. (4.6) is also used for electromagnetic fields in linear media [9]. In terms of the re-parameterized temporal coordinate, Eq. (4.4), the preceding equation becomes

$$\frac{d}{d\bar{x}^0} \frac{\partial L}{\partial (\partial A_j / \partial \bar{x}^0)} + \sum_i \frac{\partial}{\partial x_i} \frac{\partial L}{\partial (\partial A_j / \partial x_i)} = \frac{\partial L}{\partial A_j}$$

for simple linear dielectric materials. Substituting the Lagrangian density, Eq. (4.5), into Eq. (4.7), we obtain the components

$$\frac{\partial L}{\partial (\partial A_j / \partial \bar{x}^0)} = \frac{\partial A_j}{\partial \bar{x}^0}$$

$$\frac{\partial L}{\partial A_j} = 0$$

$$\sum_i \frac{\partial}{\partial x_i} \frac{\partial L}{(\partial A_j / \partial x_i)} = [\nabla \times (\nabla \times A)]_j.$$

Substituting components, Eqs. (4.8), into Eq. (2.2), the Lagrange equations of motion for the electromagnetic field in a dielectric are the three orthogonal components of the vector wave equation

$$\nabla \times (\nabla \times A) + \frac{\partial^2 A}{\partial (\bar{x}^0)^2} = 0.$$ 

The second-order equation, Eq. (4.9), can be written as a set of first-order differential equations and we introduce macroscopic field variables

$$\Pi = \frac{\partial A}{\partial \bar{x}^0}$$

(4.10)
\[ \mathbf{B} = \nabla \times \mathbf{A}. \] (4.11)

The macroscopic field variable \( \Pi \), Eq. (4.10), is selected by the canonical momentum field density whose components were defined in Eq. (4.8a). Note that both of the terms in the Lagrangian density, \( \mathcal{L} = \Pi^2 - \mathbf{B}^2 \), and the Hamiltonian density, \( \mathcal{H} = \Pi^2 + \mathbf{B}^2 \), are quadratic.

We substitute the definition of the canonical momentum field \( \Pi \), Eq. (4.10), and the definition of the magnetic field \( \mathbf{B} \), Eq. (4.11), into the wave equation, Eq. (4.9), to obtain

\[ \nabla \times \mathbf{B} + \frac{\partial \Pi}{\partial \bar{x}^0} = 0, \] (4.12)

which is similar to the Maxwell–Ampère law. Taking the divergence of Eq. (4.11), we obtain

\[ \nabla \cdot \mathbf{B} = 0. \] (4.13)

Applying the operator Eq. (4.10) produces a version of Faraday’s Law,

\[ \nabla \times \Pi - \frac{\partial \mathbf{B}}{\partial \bar{x}^0} = \frac{\nabla n}{n} \times \Pi. \] (4.14)

Finally,

\[ \nabla \cdot \Pi = -\frac{\nabla n}{n} \cdot \Pi \] (4.15)

is a modified version of Gauss’s law that is obtained by integrating the divergence of Eq. (4.12) with respect to the temporal coordinate. This completes the set of first-order equations of motion for the macroscopic fields, Eqs. (4.12)–(4.15).

The unusual appearance of Eqs. (4.12)–(4.15) may cause some readers to question their validity or range of applicability. Various reviewers of Ref. [10] have said that the superficially equivalent equations derived as identities of the Maxwell–Minkowski equations under a re-parameterization by the direct substitution of \( \Pi = -n \mathbf{E} \) and \( \bar{x}^0 = ct/n \) into Eqs. (3.1) are wrong because: 1) they violate Einstein’s special theory of relativity, 2) they violate spacetime conservation laws, 3) they violate conditions of holonomy, 4) they are valid only in some pathological limit, such as a spatially infinite medium, 5) they were derived using (unspecified) implicit or hidden axioms, 6) there are (unspecified) manifest errors in the derivation. Other reviewers contend that Eqs. (4.12)–(4.15) cannot possibly be false because they are identities of the Maxwell–Minkowski equations, Eqs. (3.1), under a re-parameterization by the direct substitution of \( \Pi = -n \mathbf{E} \) and \( \bar{x}^0 = ct/n \). However, both sets of careful readers are only partially correct.

Let us briefly examine these misconceptions: 1). Einstein derived special relativity using coordinate transformations between inertial reference frame in the vacuum of free space. If a light pulse is emitted from the origin at a time \( t = 0 \) into the vacuum then spherical wavefronts are defined by

\[ x^2 + y^2 + z^2 - (x^0)^2 = 0 \] (4.16)

in a flat four-dimensional Minkowski spacetime \( S_v(x^0 = ct, x, y, z) \). Von-Laue used the velocity sum rule to derive a theory of special relativity for a dielectric and Eqs. (4.12)–(4.15) are proven false by Einstein–von Laue dielectric special relativity. This should be “case-closed” except, in the case of a pulse emitted into a simple isotropic homogeneous dielectric linear medium, spherical wavefronts are defined by

\[ x^2 + y^2 + z^2 - (\bar{x}^0)^2 = 0 \] (4.17)

in a flat four-dimensional non-Minkowski material spacetime \( S_d(\bar{x}^0, x, y, z) \). Consequently, a different version of continuum electrodynamics that is defined on a corresponding material spacetime is associated with each isotropic homogeneous linear medium. This result is correlated with Rosen’s special relativities in which a
different index-dependent version of special relativity is associated with each isotropic homogeneous linear dielectric [11]. Although Rosen’s theory of dielectric special relativities is phenomenological in nature, it was derived rigorously in Ref. [10] using coordinate transformations between inertial frames of reference in an arbitrarily large isotropic homogeneous simple linear dielectric medium in which the speed of light is $c/n$. Because each value of the refractive index is identified with a different material spacetime, the rigorous theory is limited to arbitrarily large isotropic homogeneous simple dielectric linear media, although piece-wise homogeneous materials can also be treated using the appropriate boundary conditions.

2). The spacetime conservation laws were derived for an unimpeded inviscid, incompressible flow of non-interacting particles in the continuum limit through an otherwise empty volume. The resulting spacetime conservation law, Eq. (2.6), is clearly violated by Eqs. (4.12)–(4.15). However, the premises of the conservation law, Eq. (2.6), are violated for light propagating into a simple linear dielectric, as described in the introduction to Sec. IV. Combining Eqs. (4.12)–(4.15) as in Sec. 3, we obtain the energy conservation law,

$$\frac{\partial \rho_e}{\partial \bar{x}^0} + \nabla \cdot (B \times \Pi) = 0$$  

and the momentum conservation law,

$$\frac{\partial}{\partial \bar{x}^0} (B \times \Pi)_i + \sum_j \frac{\partial}{\partial x_j} W_{ij}^{\text{new}} = f_i^{\text{new}},$$  

where

$$W_{ij}^{\text{new}} = -\Pi_j B_i - B_i B_j + \frac{1}{2} (\Pi \cdot \Pi + B \cdot B) \delta_{ij}$$

and

$$f_i^{\text{new}} = \left(0, \Pi \times \left(\nabla \frac{n}{n} \times \Pi\right)\right).$$

Combining Eq. (4.18) with Eq. (4.19), row-wise, as before, these two conservation laws can be written as a single matrix differential equation

$$\bar{\partial}_\beta T_{\alpha\beta}^{\text{total}} = f_{\alpha}^{\text{new}}$$

for each $\alpha$, where $\bar{\partial}_\beta$ is an element of the material four-divergence operator

$$\bar{\partial}_\beta = \left(\frac{\partial}{\partial \bar{x}^0}, \partial x, \partial y, \partial z\right).$$

and

$$T_{\alpha\beta}^{\text{total}} = \begin{bmatrix} \frac{1}{2} (\Pi \cdot \Pi + B \cdot B) & -\Pi \times \Pi - H \times B + \frac{1}{2} (\Pi \cdot \Pi + B \cdot B) \mathbf{I} \\ B \times \Pi & \mathbf{I} \end{bmatrix},$$

is the diagonally symmetric total energy–momentum tensor. Then the conservation law that was derived for a continuous flow in the vacuum, Eq. (2.6), becomes

$$\frac{\partial T_{\alpha\beta}^{\text{total}}}{\partial \bar{x}^\beta} = 0,$$

where $\bar{x}^\beta \in \{\bar{x}^0, x^1, x^2, x^3\}$ in a dielectric. Like the relativity theory, there is a different conservation law for each linear dielectric.

3) and 4). There is a different set of field equations, Eqs. (4.12)–(4.15), with a different timelike coordinate, $\bar{x}^0(n)$ for each material. Consequently, the boundary conditions are indeed non-holonomic. As we saw above,
there is a different principle of dielectric special relativity and a different conservation law corresponding to, and valid for, a specific linear isotropic homogeneous material with $x^0(n)$. Then, without loss of generality, we can drop the right-hand-sides of Eqs. (4.14) and (4.14), leaving us with homogeneous equations of motion

$$\nabla \times \mathbf{B} + \frac{\partial \Pi}{\partial x^0} = 0$$

(4.26a)

$$\nabla \cdot \mathbf{B} = 0$$

(4.26b)

$$\nabla \times \Pi - \frac{\partial \mathbf{B}}{\partial x^0} = 0$$

(4.26c)

$$\nabla \cdot \Pi = 0$$

(4.26d)

with fields connected by boundary conditions. The boundary conditions are equivalent to the Fresnel relations [12]

Finally, there is no evidence for 5) and 6) because the equations are carefully derived using small, carefully documented steps from explicit axioms.

We note that Eqs. (4.12)–(4.15) are not identities of the Maxwell–Minkowski equations, Eqs. (3.1), under a re-parameterization by the direct substitution of $\Pi = -nE$ and $x^0 = ct/n$. Although that superficially appears to be the case, the refractive index forms a part of the timelike coordinate, $x^0$, and defines the flat non-Minkowski material spacetime. Specifically, $n$ is not a free parameter as it is in Maxwellian continuum electrodynamics.

V. CONCLUSIONS

The macroscopic Maxwell equations lead to a contradiction with energy–momentum conservation laws. In this report, we resolved these contradictions by deriving new equations of motion for macroscopic fields in a simple linear dielectric from Lagrangian field theory and by deriving new conservation laws based on the properties of light in the dielectric.

[1] Nieminen, T. A., Loke, V. L. Y., Stilgoe, A. B., Knöner, G., Brańczyk, A. M., Heckenberg, N. R., and Rubinsztein-Dunlop, H., “Optical tweezers computational toolbox,” J. Optics, 9, S196–S203 (2007).
[2] Qiu, C.-W., Palima, D., Novitsky, A., Gao, D., Ding, W., Zhukovsky, S. V., and Gluckstad, J., “Engineering light–matter interaction for emerging optical manipulation applications,” Nanophotonics, 3, 181–201 (2014).
[3] For a recent review, see: Pfeifer, R. N. C., Nieminen, T. A., Heckenberg, N. R., and Rubinsztein-Dunlop, H., “Colloquium: Momentum of an electromagnetic wave in dielectric media,” Rev. Mod. Phys. 79, 1197–1216 (2007).
[4] Griffiths, D. J., “Resource Letter EM-1: Electromagnetic Momentum,” Am. J. Phys. 80, 7–18 (2012).
[5] Ramos, T., Rubilar, G. F., and Obukhov, Y. N., “Relativistic analysis of the dielectric Einstein box: Abraham, Minkowski and total energy–momentum tensors, Phys. Lett. A. 375, 1703–1709, (2011).
[6] Goldstein, H. [Classical Mechanics], (Addison-Wesley, 1980). Chap. 12, particularly Eq. (12-33).
[7] Jackson, J. D., [Classical Electrodynamics] 2nd ed., Wiley, (1975).
[8] Milonni, P. W., and Boyd, R. W., “Momentum of light in a dielectric medium,” Adv. Opt. Photonics, 2, 519–553 (2010).
[9] Cohen-Tannoudji, C., Dupont-Roc, J., and Grynberg, T., [Photons and Atoms], Wiley, New York (1989).
[10] Crenshaw, M. E., “Application of axiomatic formal theory to the Abraham–Minkowski momentum controversy”, [arXiv:1509.05052]
[11] Rosen, N., “Special theories of relativity,” Am. J. Phys, 20, 161–164 (1952).
[12] Crenshaw, M. E., unpublished.