Abstract. We prove a conjecture of Crapo and Penne which characterizes isotopy classes of skew configurations with spindle-structure. We use this result in order to define an invariant, spindle-genus, for spindle-configurations.

We also slightly simplify the exposition of some known invariants for configurations of skew lines and use them to define a natural partition of the lines in a skew configuration.

Finally, we describe an algorithm which constructs a spindle in a given switching class, or proves non-existence of such a spindle.

1. Introduction

A configuration of $n$ skew lines in $\mathbb{R}^3$ or a skew configuration is a set of $n$ non-intersecting lines in $\mathbb{R}^3$ containing no pair of parallel lines.

Two skew configurations $C_1$ and $C_2$ are isotopic if there exists an isotopy (continuous deformation of skew configurations) from $C_1$ to $C_2$.

The study and classification of configurations of skew lines (up to isotopy) was started by Viro [19] and continued for example in [1], [2], [3], [6], [10], [13], [14], [15], [16] and [20].

A spindle (or isotopy join or horizontal configuration) is a particularly nice configuration of skew lines in which all lines intersect an oriented additional line $A$, called the axis of the spindle. Its isotopy class is completely described by a spindle-permutation $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ encoding the order in which an open half-plane revolving around its boundary $A$ intersects the lines during a half-turn (see Section 7 for the precise definition). A spindle-configuration is a skew configuration isotopic to a spindle.

Consider the spindle-equivalence relation on permutations of $\{1, \ldots, n\}$ generated by transformations of the following three types:

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(1) \( \sigma \sim \mu \) if \( \mu(i) = s + \sigma(i + t \mod n) \mod n \) for some integers \( 0 \leq s, t < n \) (all sums are modulo \( n \)).
(2) \( \sigma \sim \mu \) if \( \sigma(i) \leq k \) for \( i \leq k \) and
\[
\mu(i) = \begin{cases} 
  k + 1 - \sigma(k + 1 - i) & i \leq k \\
  \sigma(i) & i > k 
\end{cases}
\]
for some integer \( k \leq n \).
(3) \( \sigma \sim \mu \) if \( \sigma(i) \leq k \) for \( i \leq k \) and
\[
\mu(i) = \begin{cases} 
  \sigma^{-1}(i) & i \leq k \\
  \sigma(i) & i > k 
\end{cases}
\]
for some integer \( k \leq n \).

Conjecture 59 in [3] states that two spindle-configurations are isotopic if and only if they are described by spindle-equivalent permutations. Its proof is the main result of this paper:

**Theorem 1.1.** Two spindle-permutations \( \sigma, \sigma' \) give rise to isotopic spindle-configurations if and only if \( \sigma \) and \( \sigma' \) are spindle-equivalent.

Orienting and labeling all lines of a skew configuration, one gets a linking matrix by considering the signs of crossing lines. The associated switching class or homology equivalence class is independent of labelings and orientations. A result of Khashin and Mazurovskii, Theorem 3.2 in [7], states that homology-equivalent spindles (spindles defining the same switching class) are isotopic. We have thus:

**Corollary 1.2.** Two spindle permutations define the same switching class if and only if they are spindle-equivalent.

Isotopy classes of spindle-configurations have thus an easy combinatorial description and can be considered as “understood”, either in terms of spindle-equivalence classes or in terms of switching classes, in contrast to the general case where no (provenly) complete invariants are available.

An invariant is a map
\[
\{ \text{Configurations of skew lines up to isotopy} \} \longrightarrow \mathcal{R}
\]
where \( \mathcal{R} \) is an algebraic (or combinatorial) structure, usually a group, vector-space or ring (or a finite set). An invariant is complete if the corresponding map is injective.

A few useful invariants for configurations of skew lines are:

1. *Equivalence classes of skew pseudoline diagrams* (see Section 2): Completeness unknown (this is a major problem since the obvious planar representation of skew configurations is perhaps
not faithful). A powerful combinatorial invariant somewhat tedious to handle. Switching classes and Kauffman polynomials factorize through it.

(2) **Switching classes or two-graphs**, see page 7 of [21], also called homological equivalence classes, see [2], are equivalent to the description of the sets of linking numbers or linking coefficients, see [2] or [20]. The definition of this invariant uses a linking matrix encoding the signs of oriented crossing (not intersecting in the compactification $\mathbb{RP}^3 \supset \mathbb{R}^3$) lines. The switching class is a complete invariant for configurations of up to 5 skew lines and is not complete for more than 5 lines. Theorem 3.2 of [7] states that switching-classes are a complete invariant for spindle-configurations.

Many weaker invariants can be derived from switching classes: Section 6 describes the Euler partition of a switching class which yields a natural set-partition on the lines of a skew configuration. The definition of the Euler partition is more natural (and better known) for an odd number of lines.

Slightly weaker (but more elementary to handle) than the switching class is the characteristic polynomial of a linking matrix

$$P_X(t) = \sum_{i=0}^{n} \alpha_i t^i = \det(tI - X).$$

The spectrum $\text{spec}(X) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\}$ of $X$ or the traces $\text{tr}(X) = \sum_{i=1}^{n} \lambda_i^k$, $k = 1, \ldots, n$ of its first powers yield of course the same information.

The coefficient $\alpha_{n-3}$ of $P_X(t)$ conveys the same information as chirality, a fairly weak invariant considered for instance in [3, Section 3].

(3) **Kauffman polynomials**: Completeness unknown. A powerful invariant which is unfortunately difficult to compute, see [4] or [3] Section 14 and Appendix] for the definition and examples.

(4) **Link invariants** for links in the $3$–sphere $S^3$ applied to the preimage $\pi^{-1}(C) \subset S^3$ (called a Temari model by some authors, see for instance Section 11 of [3]) of a skew configuration $C \subset \mathbb{R}^3 \subset \mathbb{RP}^3$ under the double covering $\pi : S^3 \to \mathbb{RP}^3$.

(5) Existence of a spindle structure. A generally very weak invariant since spindle structures are rare among configurations with many lines. Theorem 1.1 shows however that the spindle-equivalence class provides a complete invariant for the very small subset of skew configurations with a spindle structure.
(6) One should also mention the \textit{shellability order}, a generalization of the notion of spindle structure, used as a classification-tool in \cite{2}.

The sequel of the paper is organized as follows:
Section 2 introduces skew pseudoline diagrams.
Sections 3-6 are devoted to (various aspects of) switching classes.
Section 7 describes spindle-configurations and contains a proof of the easy (and known) direction in Theorem 1.1: Spindle-equivalent permutations yield isotopic configurations.
Section 8 proves the difficult direction: Isotopic spindles have spindle-equivalent permutations. This completes the proof of Theorem 1.1.
Section 9 recalls (for self-containedness) the proof of Theorem 3.2 in \cite{7}: Spindle configurations with switching-equivalent linking matrices are isotopic. Corollary 1.2 follows.
Section 10 describes a somewhat curious invariant for spindle-equivalence classes of permutations (or spindle-configurations) which involves \(2\)-dimensional topology.
In Section 11 we describe a fast algorithm which computes (or proves non-existence of) a spindle-permutation (which is unique up to spindle-equivalence by Corollary 1.2) having a linking matrix of given switching class.
Section 12 contains a few computational data.

2. Skew pseudoline diagrams

Skew pseudoline diagrams are combinatorial objects providing a convenient tool for studying configurations of skew lines.

\textbf{Definition 2.1.} \textbf{A pseudoline in} \(\mathbb{R}^2\) \textbf{is a smooth simple curve representing a non-trivial cycle in} \(\mathbb{RP}^2\). \textbf{An arrangement of} \(n\) \textbf{pseudolines in} \(\mathbb{R}^2\) \textbf{is a set of} \(n\) \textbf{pseudolines with pairs of pseudolines intersecting transversally exactly once}. \textbf{An arrangement is generic} if no triple intersections occur.

\textbf{Definition 2.2.} \textbf{A skew pseudoline diagram of} \(n\) \textbf{pseudolines in} \(\mathbb{R}^2\) \textbf{is a generic arrangement of} \(n\) \textbf{pseudolines with crossing data at intersections}. \textbf{The crossing data selects at each intersection the overcrossing pseudoline}.

We draw skew pseudoline diagrams with the conventions used for knots and links: under-crossing curves are slightly interrupted at crossings.

Skew pseudoline diagrams are equivalent if they are related by a finite sequence of the following moves (see \cite{3} Section 9):
(1) Reidemeister-3 (or $*$-move), the most interesting of the three classical moves for knots and links (see Figure 1).

![Figure 1. Local description of a Reidemeister-3 move](image1)

(2) Projective move (or $||$-move): pushing a crossing through infinity (see Figure 2).

![Figure 2. Projective move](image2)

Generic projections of isotopic skew configurations yield equivalent skew pseudoline diagrams (cf. for instance [3, Theorem 48]).

Not every (equivalence class of a) skew pseudoline diagram arises by projecting of a suitable skew configuration: a configuration involving at least 4 skew lines never projects on a skew pseudoline diagram which is alternating (see [12]). There are even generic arrangements of $n \geq 9$ pseudolines which are not stretchable, i.e. which cannot be realized as an arrangement of straight lines, see [5] for an example with 9 pseudolines.

The existence of non-isotopic skew configurations inducing equivalent skew pseudoline diagrams is unknown, cf. [3, Section 17, Problem 2]. This is a major difficulty for classifications: All known invariants for skew configurations factor through skew pseudoline diagrams.

3. Linking matrices and switching classes

One assigns signs to pairs of oriented under- or over-crossing curves (as arising for instance from oriented knots and links) which are drawn in the oriented plane. Figure 3 shows a positive and a negative crossing.

The *sign* or *linking number* $\text{lk}(L_A, L_B)$ between two oriented skew lines $L_A, L_B \subset \mathbb{R}^3$ can be computed as follows: choose ordered pairs of points $(A_\alpha, A_\omega)$ on $L_A$ (resp. $(B_\alpha, B_\omega)$ on $L_B$) which induce the
orientations. The sign of the crossing determined by $L_A$ and $L_B$ is then given by

$$lk(L_A, L_B) = \text{sign} \left( \det \begin{pmatrix} A_\omega - A_\alpha \\ B_\alpha - A_\omega \\ B_\omega - B_\alpha \end{pmatrix} \right) \in \{\pm 1\}$$

where $\text{sign}(x) = \frac{x}{|x|}$ for $x \neq 0$.

Signs of crossings are also defined in skew pseudoline diagrams.

The \textit{linking matrix} of a diagram of $n$ oriented and labeled skew pseudolines $L_1, \ldots, L_n$ is the symmetric $n \times n$ matrix $X$ with diagonal coefficients $x_{i,i} = 0$ and $x_{i,j} = lk(L_i, L_j)$ for $i \neq j$. 

Figure 3. Positive and negative crossings

Figure 4. A configuration of 6 labeled and oriented skew lines
Figure 4 shows a labeled and oriented configuration of six skew lines with linking matrix

\[
X = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & -1 & -1 & -1 \\
1 & -1 & 0 & 1 & 1 & -1 \\
1 & -1 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & -1 & 0 & -1 \\
1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}.
\]

Two symmetric matrices \(X\) and \(Y\) are switching-equivalent if

\[Y = D P^t X P D\]

where \(P\) is a permutation matrix and \(D\) is a diagonal matrix with diagonal coefficients in \(\{\pm 1\}\). Since \(PD \in O(n)\) is orthogonal, we have \((PD)^{-1} = DP^t\). Switching-equivalent matrices are thus conjugate and share a common characteristic polynomial.

**Proposition 3.1.** All linking matrices of a fixed skew pseudoline diagram are switching-equivalent.

**Proof.** Relabeling the lines conjugates a linking matrix \(X\) by a permutation matrix. Reversing the orientation of some lines amounts to conjugation by a diagonal \(\pm 1\) matrix. \(\Box\)

**Remark 3.2.** The terminology “switching classes” (many authors use also “two-graphs”) is motivated by the following combinatorial interpretation and definition of switching classes:

Two finite simple (loopless and no multiple edges) graphs \(\Gamma_1\) and \(\Gamma_2\) with vertices \(V\) and (unoriented) edges \(E_1, E_2 \subset V \times V\) are switching-related with respect to a subset \(V_- \subset V\) of vertices if their edge-sets \(E_1, E_2\) coincide on \((V_- \times V_-) \cup ((V \setminus V_-) \times (V \setminus V_-))\) and are complementary on \((V_- \times (V \setminus V_-)) \cup ((V \setminus V_-) \times V_-)\). A switching class of graphs is an equivalence class of switching-related graphs.

**Figure 5.** \(\Gamma_1\) and \(\Gamma_2\) are switching-related with respect to \(\{1, 2\} \subset \{1, 2, 3, 4\}\)

Encoding adjacency, resp. non-adjacency, of distinct vertices by \(\pm 1\) yields a bijection between switching classes of graphs and switching
classes of matrices. Conjugation by permutation-matrices corresponds to relabeling the vertices of a graph $\Gamma$ and conjugation by a diagonal $\pm 1$–matrix corresponds to the substitution of $\Gamma$ by a switching-related graph.

**Remark 3.3.** The characteristic polynomial of a linking matrix of a skew pseudoline diagram (or of a configuration of skew lines) is in general weaker than its switching class: The linking matrices

$$
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 0 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}
$$

are in different switching classes (see Example 6.3), but have the same characteristic polynomial

$$(t - 3)(t - 1)^2(t + 1)(t + 3)^2(t^2 - 2t - 11).$$

This example is minimal: Distinct switching classes of order less than 8 have distinct characteristic polynomials.

**Remark 3.4.** Isotopy classes of skew configurations with $\leq 5$ lines are characterized by their switching class.

However, Figure D shows two non-isotopic skew configurations of 6 lines (their Kauffman polynomials, see Section 14 and Appendix],

$$
5A^{12}B^3 + 10A^{11}B^4 - 10A^9B^6 + A^8B^7 + 16A^7B^8 + 10A^6B^9
$$

$$
-6A^5B^{10} - 5A^4B^{11} + 6A^3B^{12} + 6A^2B^{13} - B^{15}
$$

and

$$
-A^{15} + 6A^{13}B^2 + 6A^{12}B^3 - 5A^{11}B^4 - 6A^{10}B^5 + 10A^9B^6 + 16A^8B^7 + A^7B^8 - 10A^6B^9 + 10A^4B^{11} + 5A^3B^{12}
$$
are different) having a common switching class represented by

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & -1 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & 1 & 0
\end{pmatrix}
\]

(a) (b)

Figure 6. Non-isotopic mirror configurations with switching-equivalent linking matrices

All \(2^{\binom{n}{2}}\) possible linking matrices can be obtained from a fixed skew pseudoline diagram by an appropriate choice of crossing data. The number of equivalence classes of skew pseudoline diagrams with \(n\) pseudolines equals thus at least the number of switching classes of order \(n\). Remark 3.4 shows that this inequality is in general strict.

The mirror configuration \(\overline{C}\) obtained by reflecting a skew configuration \(C\) through the \(z = 0\) hyperplane has opposite crossing data. We have thus \(\overline{X} = -X\) for the associated linking matrices. An example of two such configurations \(C, \overline{C}\) is also given by Figure 6.

A configuration \(C\) is amphicheiral if it is isotopic to its mirror \(\overline{C}\).

**Proposition 3.5.**

(i) The linking matrix \(X\) of an amphicheiral configuration of skew lines is switching-equivalent to \(-X\). In particular, amphicheiral configurations containing an odd number of skew lines have non-invertible linking matrices.

(ii) Amphicheiral configurations with \(n \equiv 3 \pmod{4}\) lines do not exist.

**Proof.** Assertion (i) is obvious.
Assertion (ii) is [20, Theorem 1]. We rephrase the proof using properties of linking matrices.

Let \( \sum_{i=0}^{n} \alpha_i t^i = \det(tI - X) \) be the characteristic polynomial of the linking matrix \( X \) for an amphicheiral configuration with \( n \) skew lines. Assertion (i) shows that we have \( \alpha_{n-1} = \alpha_{n-3} = \alpha_{n-5} = \cdots = 0 \). This implies

\[
0 = \alpha_{n-3} = - \sum_{1 \leq i,j,k \leq n} (x_{i,j}x_{j,k}x_{k,i} + x_{i,k}x_{k,j}x_{j,i}) = -2 \left( \sum_{1 \leq i < j < k \leq n} x_{i,j}x_{j,k}x_{k,i} \right).
\]

For \( n \equiv 3 \pmod{4} \) the number \( \binom{n}{3} \) of summands in \( \sum_{1 \leq i < j < k \leq n} x_{i,j}x_{j,k}x_{k,i} \) is odd. Since all these summands are \( \pm 1 \), we get a contradiction. \( \square \)

4. Switching classes and linking numbers

Linking numbers (also called homological equivalence classes or chiral signatures) are a classical and well-known invariant for skew pseudoline diagrams. We sketch below briefly the well-known proof that they correspond to the switching class of a linking matrix.

In this paper we work with switching classes mainly because they are easier to handle.

The linking number \( \operatorname{lk}(L_i, L_j, L_k) \) of three lines ([2], [20]) is defined as the product \( x_{i,j}x_{j,k}x_{k,i} \in \{ \pm 1 \} \) of the signs for the corresponding three crossings. The result is independent of the chosen orientations for \( L_i, L_j \) and \( L_k \) and yields an invariant

\[ \{ \text{triplets of lines in skew pseudoline diagrams} \} \longrightarrow \{ \pm 1 \} \. \]

The set of linking numbers is the list of the numbers \( \operatorname{lk}(L_i, L_j, L_k) \) for all triplets \( \{ L_i, L_j, L_k \} \) of lines in a skew pseudoline diagram.

Linking numbers (defining a two-graph, see [21]) and switching classes are equivalent. Indeed, linking numbers of a diagram \( D \) of skew lines can easily be retrieved from a linking matrix for \( D \). Conversely, given all linking numbers \( \operatorname{lk}(L_i, L_j, L_k) \) of a diagram \( D \), choose an orientation of the first line \( L_1 \). Orient the remaining lines \( L_2, \ldots, L_n \) such that they cross \( L_1 \) positively. A linking matrix \( X \) for \( D \) is then given by \( x_{i,1} = x_{i,1} = 1 \), \( 2 \leq i \leq n \) and \( x_{a,b} = \operatorname{lk}(L_a, L_b) \) for \( 2 \leq a \neq b \leq n \).

Two skew pseudoline diagrams are homologically equivalent if there exists a bijection between their lines, which preserves all linking numbers. Two diagrams are homologically equivalent if and only if they have switching-equivalent linking matrices.

A last invariant considered by some authors (see [3, Section 3 and Appendix]) is the chirality \((\gamma_+, \gamma_-)\) of a skew pseudoline diagram. It is
defined as
\[
\gamma_+ = \sharp \{ 1 \leq i < j < k \leq n \mid \text{lk}(L_i, L_j, L_k) = 1 \} \\
\gamma_- = \sharp \{ 1 \leq i < j < k \leq n \mid \text{lk}(L_i, L_j, L_k) = -1 \}
\]
One has of course
\[
\gamma_+ = \binom{n}{3} + c \\
\gamma_- = \binom{n}{3} - c
\]
where
\[
c = \sum_{1 \leq i < j < k \leq n} x_{i,j} x_{j,k} x_{k,i} = \frac{1}{6} \text{trace}(X^3) = -\frac{\alpha_{n-3}}{2}
\]
is proportional to the coefficient of \(t^{n-3}\) in the characteristic polynomial
\[
\det(tI - X) = \sum_{i=0}^{n} \alpha_i t^i
\]
of a linking matrix \(X\).

The sign indeterminacy in linking matrices representing switching classes makes their use tedious. For switching classes of odd order, a satisfactory answer addressing this problem will be given in the next section. For even orders, there seems to be no completely satisfactory way to get rid of all sign-indeterminancies.

One possible normalization consists of choosing a given (generally the first) row of a representing matrix and to make all entries in this row positive by conjugation with a suitable diagonal \(\pm 1\) matrix. The resulting matrix describes a simple graph on \(n-1\) vertices encoding all entries outside the chosen row (and column). The choice of the suppressed row leads to the notion of graphs which are called “cousins” in [3] where this point of view is adopted instead of switching-equivalence.

5. Signature for configurations of \(2n-1\) lines

Consider the signature
\[
\epsilon(X) = \prod_{1 \leq i < j \leq 2n-1} x_{i,j} \in \{\pm 1\}
\]
of a linking matrix \(X\) of odd order \(2n-1\). It is easy to check that \(\epsilon(X)\) depends only on the switching-class of \(X\). (This follows also from Proposition 5.1 below.)

**Proposition 5.1.** We have
\[
\epsilon(X) = (-1)^{\gamma_-}
\]
where \(\gamma_- = \sharp \{ 1 \leq i < j < k \leq 2n + 1 \mid \text{lk}(L_i, L_j, L_k) = -1 \}\) counts the number of triplets of pseudolines with linking number \(-1\) in a pseudoline diagram with linking matrix \(X\).
Proof. The identity is correct if $x_{i,j} = 1$ for all $1 \leq i < j \leq 2n - 1$. Reversing exactly one pair of coefficients $x_{i,j}, x_{j,i}$ reverses the sign of the $2n - 3$ linking numbers $lk(L_i, L_j, L_k, k \neq i, j)$ and thus changes the parity of $\gamma_-$. This implies the result by induction on $\gamma_-$. □

Remark 5.2. If the matrix $D$ corresponds to a configuration of $2n - 1$ pseudolines arising from a spindle-configuration with spindle permutation $\sigma$, then $\epsilon(D)$ corresponds to the signature of the permutation $\sigma$ (which is well-defined for the spindle-equivalence class of $\sigma$). This fact provides another proof of assertion (ii) in Proposition 3.5 for spindle-configurations: Given a spindle-permutation $\sigma$ of $\{1, \ldots, 2n - 1\}$, a spindle-permutation of the mirror configuration is for instance given by $\sigma\tau$ where $\tau$ is the involution defined by $\tau(i) = 2n - i$, $1 \leq i < 2n$ and $\tau$ has signature $-1$ if $n \equiv 0 \pmod{4}$.

Remark 5.3. Defining invariants of switching classes is fairly easy. A few examples are:

1. The set of numbers $|\alpha_{i,j}|$, $i \neq j$ where $\alpha_{i,j} = \sum_{k=1}^{n} x_{i,k}x_{j,k}$ or the set of all triplets $\alpha_{i,j}\alpha_{i,k}\alpha_{j,k}$.
2. The set of all $4-$tuplets $|\sum_{k=1}^{n} x_{i_1,k}x_{i_2,k}x_{i_3,k}x_{i_4,k}|$ with $i_1 < i_2 < i_3 < i_4$.
3. Properties of the (non-Euclidean) Lattice with Gram matrix (scalar products between generators) $X$ representing a given switching class.
4. Properties of the Euclidean lattice spanned by the rows of $X$, considered as integral vectors in standard Euclidean space.

6. Euler partitions

In this section we study invariants of switching classes which have a computational cost of $O(n^2)$ operations. For skew configurations involving a huge number of lines they are thus easier to compute and to compare than chirality-invariants (with computational cost $O(n^3)$).

The behaviour of switching classes depends on the parity of their order $n$.

Switching classes of odd order $2n - 1$ are in bijection with Eulerian graphs. This endows pseudoline diagrams consisting of an odd number of pseudolines with a canonical orientation (up to a global change). We get a partition of the pseudolines into equivalence classes according to the number of positive crossings in which they are involved for an Eulerian orientation.

We consider the case of odd order in Subsection 6.1.

The situation is more complicated for switching classes of even order $2n$. We replace Euler graphs appearing for odd orders by a suitable
kind of planar rooted binary trees which we call Euler trees. The leaves of the Euler tree induce again a natural partition, which we call the Euler partition, of the set of pseudolines into equivalence classes of even cardinalities. Subsection 6.2 addresses the even case.

6.1. Switching classes of odd order - Euler orientations. A simple finite graph $\Gamma$ is Eulerian if all its vertices are of even degree. The following well-known result goes back to Seidel [17].

**Proposition 6.1.** Eulerian graphs with an odd number $2n - 1$ of vertices are in bijection with switching classes of order $2n - 1$.

We recall here the simple proof since it yields a fast algorithm for computing Eulerian orientations on configurations with an odd number of skew lines.

**Proof.** Choose a representing matrix $X$ in a given switching class. For $1 \leq i \leq 2n - 1$ define the number

$$v_i = \# \{ j \mid x_{i,j} = 1 \} = \frac{1}{2} \sum_{j=1, j \neq i}^{2n-1} x_{i,j} + 1$$

counting all entries equal to 1 in the $i$–th row of $X$. Since $X$ is symmetric, the vector $(v_1, \ldots, v_{2n-1})$ has an even number of odd coefficients and conjugation of the matrix $X$ with the diagonal matrix having diagonal entries $(-1)^{v_i}$ turns $X$ into a matrix $X_E$ with an even number of 1’s in each row and column. The matrix $X_E$ is well defined up to conjugation by a permutation matrix and defines an Eulerian graph with vertices $\{1, \ldots, 2n - 1\}$ and edges $\{i, j\}$ if $(X_E)_{i,j} = 1$. This construction can easily be reversed. \hfill \Box

Let $D$ be a skew diagram having an odd number of pseudolines. Label and orient its pseudolines arbitrarily thus getting a linking matrix $X$. Reverse the orientations of all lines having an odd number of crossings with positive sign. Call the resulting orientation Eulerian. It is unique up to reversing the orientation of all lines.

An Eulerian linking matrix $X_E$ associated to an Eulerian orientation of $D$ is uniquely defined up to conjugation by a permutation matrix. Its invariants coincide with those of the switching class of $X_E$ but are slightly easier to compute since there are no sign ambiguities. In particular, some of them can be computed using only $O(n^2)$ operations.

An Eulerian partition of the set of pseudolines of a diagram consisting of an odd number of pseudolines is by definition the partition of the pseudolines into subsets $L_k$ consisting of all pseudolines involved in exactly $2k$ positive crossings for an Eulerian orientation.
A few more invariants of Eulerian matrices are:

1. The total sum $\sum_{i,j} x_{i,j}$ of all entries in an Eulerian linking matrix $X_E$ (this is of course equivalent to the computation of the number of entries 1 in $X_E$). The computation of this invariant needs only $O(n^2)$ operations.

2. Its signature $\epsilon = \prod_{i<j} x_{i,j}$. The easy identity

$$\epsilon = (-1)^{\binom{n}{2} + \sum_{i<j} x_{i,j}}$$

relates the signature to the total sum $\sum_{i,j} x_{i,j}$ of all entries in an Eulerian linking matrix.

3. The number of rows of $X_E$ with given row-sum (this can also be computed using $O(n^2)$ operations). These numbers yield of course the cardinalities of the sets $L_0, L_1, \ldots$

4. All invariants of the associated Eulerian graph (having edges associated to entries $x_{i,j} = 1$) defined by $X_E$, e.g. the number of triangles or of other fixed subgraphs.

For example, for 7 vertices, there are 54 different Eulerian graphs, 36 different sequences (up to a permutation of the vertices) of vertex degrees, and 18 different numbers for the cardinality of 1’s in $X_E$.

Figure 7 shows all seven Eulerian graphs on 5 vertices.

6.2. Switching classes of even order - Euler partitions. The situation in this case is slightly more complicated and less satisfactory.

There exists a natural partition of the $2n$ rows $R$ of $X$ into two subsets $R_+$ and $R_-$ according to the sign

$$\epsilon_i = \prod_{j \neq i} x_{i,j} \prod_{s<t} x_{s,t}$$
associated to the \(i\)-th row of \(X\). This sign is indeed well-defined since switching (conjugation by a diagonal \(\pm 1\) matrix \(D\)) multiplies both factors \(\prod_{j \neq i} x_{i,j}\) and \(\prod_{s < t} x_{s,t}\) by \(\det(D) \in \{\pm 1\}\).

Since \(\prod_{i} \epsilon_{i} = \prod_{i \neq j} x_{i,j} (\prod_{s < t} x_{s,t})^{2n} = 1\), both subsets \(R_{+}, R_{-}\) have even cardinalities.

If \(R_{+}\) (or equivalently, \(R_{-}\)) is non-empty, it defines a symmetric submatrix \(X_{+}\) of even size \(\#(R_{+})\) corresponding to all rows and columns with indices in \(R_{+}\). Iterating the above construction we get thus a partition \(R = R_{+} \cup R_{-}\). This construction is most conveniently encoded by a planar rooted binary tree which we call the Euler-tree of \(X\): Draw a root \(R\) corresponding to the row-set \(R\) of \(X\). If the partition \(R = R_{+} \cup R_{-}\) is non-trivial join the root \(R\) to a left successor called \(R_{-}\) and a right successor called \(R_{+}\). The Euler tree of \(X\) is now constructed recursively by gluing the root \(R_{\pm}\) of the Euler tree associated to \(X_{\pm}\) onto the corresponding successor \(R_{\pm}\) of the root \(R\).

The leaves of the Euler tree \(T(X)\) of \(X\) correspond to subsets \(R_{w}\) (with \(w \in \{\pm\}^{*}\)) of even cardinality \(2n_{w}\) summing up to \(2n\). The leaves of \(T(X)\) define symmetric submatrices in \(X\) which we call “Eulerian”: All their row-sums are identical modulo 2 and can be chosen to be even perhaps after switching an odd number of vertices (corresponding to conjugation by a diagonal \(\pm 1\) matrix of determinant \(-1\)). The row partition \(R = R_{+} \cup R_{-}\) of an Eulerian matrix \(X\) of even size is by definition trivial. The sign \(\epsilon \in \{\pm\}\) such that \(R = R_{\epsilon}\) is called the the “signature” of \(X\). An Eulerian matrix of size 2 has always signature 1. For Eulerian matrices of size \(2n \geq 4\) both signs can occur as signatures since changing the signs of the entries \(x_{i,j}\), \(1 \leq i \neq j \leq 3\) reverses the signature of an Eulerian matrix. The signature of an Eulerian matrix encodes the parity of the number of edges in an Eulerian graph (having only vertices of even degrees) in the switching class of \(X\).

**Enumerative digression.** Associating a weight \(n \in \{1, 2, \ldots\}\) to a (non-empty) leaf corresponding to an Euler matrix of order \(2n\), the generating function \(F(z) = \sum_{n=0}^{\infty} \alpha_{n} z^{n}\) enumerating the number \(\alpha_{n}\) of Euler trees with total weight \(n\) (associated to even switching classes of order \(2n\)) satisfies the equation

\[
F(z) = \frac{1}{1-z} + (F(z) - 1)^2
\]

(with \(\alpha_{0} = 1\) corresponding to the empty tree). Indeed, Euler trees reduced to a leaf contribute \(1/(1-z)\) to \(F(z)\). All other Euler trees are obtained by gluing two Euler trees of strictly positive weights below a root and are enumerated by the factor \((F(z) - 1)^2\).
Solving for $F(z)$ we get the closed form

$$F(z) = \sum_{n=0}^{\infty} \alpha_n z^n = \frac{3(1-z) - \sqrt{(1-z)(1-5z)}}{2(1-z)}.$$

showing that

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = 5.$$

The first terms $\alpha_0, \alpha_1, \ldots$ are given by

$$1, 1, 2, 5, 15, 51, 188, 731, 2950, 12235, \ldots$$

(see also Sequence A7317 in [18]).

Similarly, the generating function $F_s(z) = \sum_{n=0}^{\infty} \beta_n z^n$ enumerating the number $\beta_n$ of signed weighted Euler trees (keeping track of the signature of all leaves with weight $\geq 2$) with total weight $n$ satisfies the equation

$$F_s(z) = \frac{1 + z^2}{1 - z} + (F(z) - 1)^2.$$

We get thus

$$F_s(z) = \sum_{n=0}^{\infty} \beta_n z^n = \frac{3(1-z) - \sqrt{(1-z)(1-5z-4z^2)}}{2(1-z)}.$$

and

$$\lim_{n \to \infty} \frac{\beta_{n+1}}{\beta_n} = \frac{5 + \sqrt{41}}{2} \sim 5.7016.$$

The first terms $\beta_0, \beta_1, \ldots$ are given by

$$1, 1, 3, 8, 27, 104, 436, 1930, 8871, 41916, \ldots$$

The leaves of the Euler tree define a natural partition of the set of rows of $X$ into subsets. We call this partition the Euler partition.

**Example 6.2.** The symmetric matrix

$$\begin{pmatrix}
0 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\
-1 & 0 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 0 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 \\
-1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 0
\end{pmatrix}$$
yields the Euler partition
\[ R_+ = \{1, 2, 4, 7, 8, 9\} \cup R_- = \{6, 10\} \cup R_{--} = \{3, 5\} \]
(where the Eulerian submatrix associated to \(R_+\) has signature 1) with Euler tree presented in Figure 8.

\[ \text{Figure 8. The Euler tree of Example 6.2} \]

Example 6.3. The two matrices mentioned in Remark 3.3 are indeed not switching-equivalent: The row-partition \(R = R_+ \cup R_-\) of the first matrix is given by \(R_+ = \{1, 3, 4, 6\}\) and \(R_- = \{2, 5, 7, 8\}\). The associated Euler matrices \(X_+\) and \(X_-\) are both of signature 1. On the other hand, the second matrix is Eulerian and has also signature 1.

Let us mention a last invariant related to the Euler tree for a switching class \(X\) having even order \(2n\). Let \(R_1, \ldots, R_r \subset R\) be the Euler partition of \(X\). For \(1 \leq i, j \leq r\) define numbers \(a_{i,j} \in \{\pm 1\}\) by
\[
a_{i,j} = \left\{ \begin{array}{ll}
\prod_{t \neq s \in R_i} x_{s_0, t} \prod_{s, t \in R_i, s < t} x_{s, t} = \text{signature}(X_i) & i = j \\
\prod_{s \in R_i, t \in R_j} x_{s, t} & i \neq j
\end{array} \right.
\]
where \(s_0 \in R_i\) is a fixed element (and where \(X_i\) denotes the Eulerian submatrix defined by \(R_i\)). One can easily check that the numbers \(a_{i,j}\) are well-defined.

This invariant has an even stronger analogue for switching classes of odd order: Given an Eulerian matrix of order \(2n + 1\) with Euler partition \(A_0, \ldots, A_r\) (where \(A_i\) corresponds to the set of vertices of degree \(2i\) in the Euler graph \(\Gamma\)) one can consider the numbers
\[
a_{i,j} = \sum_{s \in A_i, t \in A_j} x_{s, t}, \quad 0 \leq i, j
\]
related to the number of edges joining vertices of given degree in \(\Gamma\).
7. Spindles

7.1. Spindle-configuration. Recall that a spindle is a configuration of skew lines intersecting an auxiliary line $A$, called its axis. A spindle-configuration (or a spindle structure) is a configuration of skew lines isotopic to a spindle.

The orientation of the axis $A$ induces a linear order $L_1 < \cdots < L_n$ on the $n$ lines of a spindle $C$. Each line $L_i \in C$ defines a hyperplane $\Pi_i$ containing $L_i$ and the axis $A$.

A second directed auxiliary line $B$ (called a directrix) in general position with respect to $A, \Pi_1, \ldots, \Pi_n$ and crossing $A$ negatively, intersects the hyperplanes $\Pi_1, \ldots, \Pi_n$ in points $\sigma(L_i) = B \cap \Pi_i$. One can assume $\sigma(L_i) \in B$ by a suitable rotation of $L_i \subset \Pi_i$ around $A \cap \Pi_i$. Since the orientation of $B$ induces a linear order on the points $\sigma(L_i)$, we get a spindle-permutation (still denoted) $i \mapsto \sigma(i)$ of the set $\{1, \ldots, n\}$ by identifying the two linearly ordered sets $L_1, \ldots, L_n$ and $\sigma(L_1), \ldots, \sigma(L_n)$ in the obvious way with $\{1, \ldots, n\}$. Figure 9 displays an example corresponding to $\sigma(1) = 1$, $\sigma(2) = 4$, $\sigma(3) = 2$, $\sigma(4) = 5$, $\sigma(5) = 3$.

![Figure 9. A spindle](image_url)

Spindles can also be represented by (the isotopy classes of) configurations of skew lines with all lines contained in distinct affine horizontal planes of $\mathbb{R}^3$ (horizontal configurations). An associated spindle permutation encodes then the two orders on the set of lines given by considering the heights of the horizontal planes containing them and by their slopes after vertical projection onto such a horizontal plane.

A linking matrix $X$ of a spindle $C$ is easily computed as follows. Transform $C$ isotopically into a spindle with oriented axis $A$ and directrix $B$ as above. Orient a line $L_i$ from $L_i \cap A$ to $\sigma(L_i) = L_i \cap B$. A straightforward computation shows that the linking matrix $X$ of this
labeled and oriented skew configuration has coefficients
\[ x_{i,j} = \text{sign}\left((i-j)(\sigma(i) - \sigma(j))\right) \]
where \(\text{sign}(0) = 0\) and \(\text{sign}(x) = \frac{x}{|x|}\) for \(x \neq 0\) and where \(\sigma\) is the corresponding spindle-permutation.

For example, the configuration of 5 skew lines depicted in Figure 9 corresponds to the linking matrix
\[
X = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & -1 \\
1 & -1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & -1 \\
1 & -1 & 1 & -1 & 0
\end{pmatrix}.
\]

**Remark 7.1.** A configuration \(C\) of \(n\) skew lines has a spindle structure if and only if its mirror configuration \(\overline{C}\) has a spindle structure. A spindle permutation \(\sigma\) for \(C\) is then for instance given by \(\sigma(i) = n + 1 - \sigma(i), 1 \leq i \leq n\), where \(\sigma\) is a spindle-permutation for \(C\).

### 7.2. Spindle-equivalent permutations.

The aim of this subsection is to show that three types of transformations of spindle-permutations, defined by Crapo and Penne in [3, Section 15] preserve the associated spindle-configuration, up to isotopy. This is the easy direction in Theorem 1.1 and follows for instance from Theorem 3.2 in [7].

A (linear) block of size \(k\) or a \(k\)-block in a permutation \(\sigma\) of \(\{1, \ldots, n\}\) is a subset \(\{i+1, \ldots, i+k\}\) of \(k\) consecutive integers in \(\{1, \ldots, n\}\) such that
\[
\sigma(\{i+1, \ldots, i+k\}) = \{j+1, \ldots, j+k\}
\]
(i.e. the image under \(\sigma\) of a set \(\{i+1, \ldots, i+k\}\) of \(k\) consecutive integers is again a set of \(k\) consecutive integers). In the sequel, we denote by \([\alpha, \beta]\) = \(\{\alpha, \alpha+1, \ldots, \beta-1, \beta\} \subset \{1, \ldots, n\}\) a subset of consecutive integers and by \(\sigma([i+1,i+k]) = [j+1,j+k]\) a \(k\)-block as above.

Recall that two spindle-permutations are equivalent (see [3, Section 15]) if they are equivalent under the equivalence relation generated by

1. **Circular move**
   \[ \sigma \sim \mu \text{ if } \mu(i) = (s + \sigma((i+t) \mod n)) \mod n \]
   for some integers \(0 \leq s, t < n\) (all integers are modulo \(n\)).

2. **Vertical reflection of a block or local reversal**
   \[ \sigma \sim \mu \text{ if } \sigma([1,k]) = [1,k] \text{ and } \mu(i) = \begin{cases} k + 1 - \sigma(k+1-i) & i \leq k \\ \sigma(i) & i > k \end{cases} \]
for some integer $k \leq n$ (see Figure 10).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Vertical reflection of a block}
\end{figure}

(3) (Horizontal reflection of a block or local inversion) $\sigma \sim \mu$ if there exists an integer $1 < k \leq n$ such that $\sigma([1, k]) = [1, k]$ and

$$
\mu(i) = \begin{cases} 
    \sigma^{-1}(i) & i \leq k \\
    \sigma(i) & i > k
\end{cases}
$$

(see Figure 11).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{Horizontal reflection of a block}
\end{figure}

Circular moves suggest to extend the linear order on $\{1, \ldots, n\}$ to the cyclic order induced by the compactification $\mathbb{R}P^2 \supset \mathbb{R}$. We have an obvious notion of cyclic blocks and can consider linear horizontal and vertical reflections of cyclic blocks related to vertical and horizontal reflections as above by conjugations involving circular moves. Such (more general) moves lead to the same equivalence relation as the three moves considered above and we allow them for the sake of concision.

We restate and reprove the easy direction (see also [7]) of Theorem 1.1:

**Proposition 7.2.** Equivalent spindle-permutations yield isotopic spindle-configurations.

**Proof.** A transformation of type (1) amounts to pushing the last few lines of the spindle on the axis and directrix through infinity. This can be done isotopically (continuously without leaving the set of skew configurations).
An isotopy inducing a vertical reflection (type (2) above) can be described as follows: Consider the two complementary subblocks $\sigma([1, k]) = [1, k]$ and $\sigma([k + 1, n]) = [k + 1, n]$ in $\sigma$. All lines of the first block $\sigma([1, k]) = [1, k]$ can be squeezed isotopically into the interior of a small one-sheeted hyperboloid $H$ whose axis of revolution intersects orthogonally the axis $A$ and the directrix $B$ of $\sigma$. Moreover, we may assume that no line of the complementary block $\sigma([k + 1, n]) = [k + 1, n]$ intersects the interior of $H$. The isotopy of skew lines given by rotating the interior (containing the block $\sigma([1, k]) = [1, k]$) of $H$ by a half-turn around its axis of revolution induces a vertical reflection (see Figure 12).

**Figure 12.** Isotopy for a transformation of type (2)

For constructing an isotopy inducing a horizontal reflection (type (3)) we start as above by pushing the $k$ lines of the first block $\sigma([1, k]) = [1, k]$ into a small hyperboloid $H_1$ with revolution axis $C_1$ intersecting the axis $A$ and the directrix $B$ orthogonally. Denote by $I_B$ the open segment of $B$ contained in the interior of $H_1$. Moreover, suppose that the directions of the axis $A$ and the directrix $B$ are orthogonal. Push the lines of the complementary subblock $\sigma([k + 1, n]) = [k + 1, n]$ in the positive sense along the directrix $B$ until they can be squeezed into the interior of a small revolution hyperboloid $H_2$ not intersecting $H_1$ with revolution axis $C_2$ parallel to the directrix $B$ of $\sigma$.

Rotate the interior of the first hyperboloid $H_1$ containing the subblock $\sigma([1, k]) = [1, k]$ by a half-turn around the revolution axis $C_2$ of $H_2$ (see Figure 13) and call the resulting hyperboloid $H'_1$. Finally, rotate
the hyperboloid $H'_1$ and the lines inside it by $\pi/2$ around its revolution axis $C'_1$ and translate it along $C'_1$ until the image of $I_B$ is contained in the directrix $A$ of $\sigma$. This yields a spindle-configuration whose spindle-permutation is related by a circular move and a horizontal reflection (and perhaps a vertical reflection of the first subblock, depending on the sense of the half-turn around $C'_1$) to the initial spindle-permutation.

Figure 13. Isotopy for realizing a transformation of type (3)
Proposition 8.1. Let $\sigma$ be a spindle-permutation with linking matrix $X$. Let $I$ be a subset of indices such that

$$x_{i_1,i_2}x_{i_1,j}x_{i_2,j} \in \{\pm 1\}$$

depends only on $i_1 \neq i_2 \in I$ and is independent of $j \notin I$.

Then, up to spindle moves of $\sigma$, the lines $\{L_i\}_{i \in I}$ corresponding to $I$ form a block of $\sigma$.

Corollary 8.2. Let $B \subset \sigma$ be a block of a spindle-permutation $\sigma$. Let $i$ be an isotopy between $\sigma$ and a second spindle-permutation $\sigma'$. Then, up to spindle moves of $\sigma'$, the set of lines $i(B) \subset \sigma'$ is a block.

Proof of Proposition 8.1. We suppose the rows and columns of $X$ indexed by $\{1, \ldots, n\}$.

We consider first the case $|I| = 2$. Up to circular moves, we can assume that $I = \{1, \alpha\}$ corresponds to the lines $(1,1), (\alpha, \beta) \in \sigma$. Up to replacing $\sigma$ by its mirror spindle $\overline{\sigma}$, we can also assume $x_{1,\alpha}x_{1,\gamma}x_{\alpha,\gamma} = 1$ for $\gamma \in \{2, \ldots, n\} \setminus \{\alpha\}$. This implies $(\alpha - \gamma)(\beta - \sigma(\gamma)) > 0$ and shows $\alpha = \beta$. Moreover, the lines $(2, \sigma(2)), (3, \sigma(3)), \ldots, (\alpha - 1, \sigma(\alpha - 1)), (\alpha, \alpha)$ form a block $B \subset \sigma$ and a vertical reflection with respect to $B$ sends $(\alpha, \alpha)$ onto $(2,2)$.

Suppose now that the subset $I$ containing $k > 2$ lines is minimal in the sense that it contains no strict subset $I' \subset I$ of $k' \geq 2$ indices satisfying the condition of Proposition 8.1. We claim that the corresponding subset of lines $\{(i, \sigma(i))\}_{i \in I}$ is a block of $\sigma$. Indeed, suppose that our claim does not hold. Up to a horizontal reflection and a circular move, there exist $1 < j_1 < i < j_2 \leq n$ with $1, i \in I$ and $j_1, j_2 \notin I$. Up to replacing $\sigma$ by its mirror and up to circular moves we can assume $1 = \sigma(1) < \sigma(j_1) < \sigma(i)$. This implies $\sigma(i) < \sigma(j_2)$ and shows that the two non-empty sets $I_1 = I \cap (\{1, 2, \ldots, j_1 - 1\} \cup \{j_2 + 1, \ldots, n\})$ and $I_2 = I \cap \{j_1 + 1, \ldots, j_2 - 1\}$ yield a partition of $I$ into subsets which satisfy both the condition of Proposition 8.1. Since at least one of the sets $I_1, I_2$ contains $\geq 2$ elements, we get a contradiction with the assumed minimality of $I$.

In the general case we consider a subset $I' \subset I$ which satisfies the condition of the Proposition 8.1 and which contains either only two lines or is minimal as defined previously. Since Proposition 8.1 holds for $I'$ we can suppose (up to spindle moves if $|I'| = 2$) that the lines of $I'$ form a block $B' \subset \sigma$. Considering the lines of $B'$ as rigidly linked...
(and thus allowing only spindle-moves of \( \sigma \) transforming all lines of \( B' \) similarly) we can consider \( B' \) as being represented by a single line \( L' \in B' \). This yields a smaller spindle-permutation \( \tilde{\sigma} \) and a subset of indices \( \tilde{I} \subset \) satisfying the condition of Proposition 8.1 for the corresponding linking matrix \( \tilde{X} \) obtained by removing from \( X \) all rows and columns corresponding to \( B' \setminus L' \). Proposition 8.1 holds now for \( \tilde{\sigma} \) by induction on the number of lines and gluing back the rigid block \( B' \) (which is well-defined up to isotopy and a vertical and horizontal reflection of \( B' \)) onto \( L' \in \tilde{\sigma} \) yields the result. \( \square \)

Proof of Corollary 8.2. The set of indices corresponding to a subblock \( B \subset \sigma \) satisfies the condition of Proposition 8.1 for a common linking matrix \( X \) of the isotopic spindles \( \sigma \) and \( \sigma' \). \( \square \)

Theorem 8.3. Let \( \iota \) be an isotopy relating two spindle-configurations (associated to spindle-permutations) \( \sigma \) and \( \sigma' \). Then the bijection from the lines of \( \sigma \) onto the lines of \( \sigma' \) induced by \( \iota \) can be given by spindle-moves.

Proof of Theorem 1.1. Follows from Proposition 7.2 and Theorem 8.3. \( \square \)

Remark 8.4. There might exist “exotic” isotopies between spindle-equivalent spindle-configurations which do not arise from (a continuous deformation of a sequence of) spindle-moves.

A few notations: We generalize the notions of spindle-permutations, spindle-configurations etc. as follows: A spindle-permutation is a bijection \( \sigma : E \rightarrow F \) between two finite subsets \( E, F \subset \mathbb{R} \) which we consider either linearly ordered or cyclically ordered by the cyclic order induced on \( \mathbb{R} \) from the compactification \( \mathbb{R} \subset \mathbb{R} \cup \{\infty\} = \mathbb{R}P^1 \sim S^1 \). For \( e \in E \) we denote by \((e, \sigma(e))\) the line of the spindle-configuration associated in the obvious way to \( \sigma \) and we identify \( \sigma \) with the set \( \{(e, \sigma(e))\}_{e \in E} \) of its lines. For \( e \in E \), the notation \( \sigma \setminus (e, \sigma(e)) \) denotes the spindle or spindle-permutation obtained by restricting \( \sigma \) to \( E \setminus \{e\} \). For \( e_1 < e_2 \in E \) and \( f_1 < f_2 \in F \) we denote by \([e_1, e_2]\) the subsets \([e_1, e_2]\) \( \cap E \) and \([f_1, f_2]\) \( \cap F \) of \( E \) and \( F \). For subsets \( E' \subset E, F' \subset F \) of the same cardinality such that \( \sigma(e) \in F' \) for all \( e \in E' \) we denote by \((E', \sigma(E')) = F' \) the spindle-permutation obtained by restricting \( \sigma \) to \( E' \). A subblock of \( \sigma \) can thus be written as \(([e_1, e_2], \sigma([e_1, e_2]) = [f_1, f_2]) \subset \sigma \) and we will also use the shorthand notation \( \sigma([e_1, e_2]) = [f_1, f_2] \). In the sequel, a \( k \)-block (of a spindle-permutation \( \sigma \)) will almost always denote a cyclic block consisting of \( k \) lines, i.e. a subset \( E' \subset E \) of \( k \) cyclically consecutive
elements with $\sigma(E')$ cyclically consecutive in $F$. Let us also remark that given a spindle-permutation $\sigma : E \rightarrow F$, its mirror configuration is for instance associated to the spindle-permutation $\bar{\sigma} : E \rightarrow \bar{F}$ where $\bar{F} = F$ as a set but equipped with the opposite (cyclic) order. The application $\sigma \mapsto \bar{\sigma}$ which replaces a spindle-permutation by its mirror enjoys good properties (preserves the spindle-equivalence relation, the isotopy relation, yields a bijection between subblocks etc.) and will often be used to reduce the number of possible cases.

The proof of Theorem 8.3 is by induction on the number $n$ of lines involved in $\sigma$ and $\sigma'$. The result holds clearly for configurations of $\leq 3$ lines (in this case, spindle-moves generate the complete permutation group of all lines).

Call a permutation irreducible if it contains no non-trivial block (consisting of $2 \leq k \leq n - 2$ cyclically consecutive lines).

Call a block minimal if it consists of $2 \leq k \leq n - 2$ lines and if it contains no subblock of strictly smaller cardinality $k' \geq 2$.

Proposition 8.1 shows that the set of possible subblocks of a spindle-permutation $\sigma$ is encoded in its linking matrix. Thus, either both or none of the spindle permutations $\sigma, \sigma'$ are irreducible.

The proof of Theorem 8.3 splits into two cases, depending on the reducibility of $\sigma$ and $\sigma'$.

8.1. The reducible case. Consider a non-trivial $k-$subblock $B \subset \sigma$. Corollary 8.2 shows that, up to spindle moves, $B' = \iota(B)$ is a non-trivial $k-$subblock of $\sigma'$.

Up to circular spindle moves we can assume $B$ and $B'$ be given by $\sigma([1, k]) = [1, k]$ and $\sigma'([1, k]) = [1, k]$. We denote by $\overline{B} = \sigma \setminus B$ and $\overline{B}' = \sigma' \setminus B'$ the complementary blocks.

By induction on the number of lines, the bijection of lines obtained by restricting $\iota$ to the subspindles $(1, \sigma(1)) \cup \overline{B}$ and $(\iota(1), \sigma'(\iota(1))) \cup \overline{B}'$ can be obtained by an isotopy $\mu$ given by a composition of spindle-moves. Up to a vertical and/or horizontal reflection of $B$, the isotopy $\mu$ can be extended uniquely to all lines of $\sigma$ by considering the subblock $B \subset \sigma$ as rigid (and thus by allowing only spindle moves having the same effect on all lines of $B$). Replacing $\iota$ with the isotopy $\mu^{-1} \circ \iota$ we can thus suppose that the permutation induced by $\iota$ fixes all lines of $\overline{B} = \sigma \setminus B$. Applying the above argument to the complementary block $\overline{B}$ we get the result. \qed

8.2. The irreducible case. The case where $\sigma$ and $\sigma'$ are irreducible is more involved. It splits into the three following subcases:

Subcase (1): There exists a line $L \in \sigma$ with $\sigma \setminus L$ irreducible.
Subcase (2): There exists a line $L$ such that $\sigma \setminus L$ contains a minimal block $B$ consisting of $3 \leq k < n - 3$ lines.

Subcase (3): $\sigma \setminus L$ contains a $2-$block for every line $L \in \sigma$.

Subcases (1) and (2) are dealt with by induction on the number of lines. We call a spindle-permutation giving rise to subcase (3) exceptional. Subcase (3) is then handled by classifying all exceptional spindle-permutations.

8.2.1. **Subcase (1).** Choose a line $L \in \sigma$ with $\sigma \setminus L$ irreducible. By Corollary 8.2, the spindle-permutation $\sigma' \setminus \iota(L)$ is also irreducible. Up to circular moves of $\sigma$ and $\sigma'$ we can assume $L = (n, n) \in \sigma, \iota(L) = (n, n) \in \sigma'$.

By induction on $n$, the restriction of $\iota$ to the spindle configurations $\sigma([1, n-1]) = [1, n-1], \sigma'([1, n-1]) = [1, n-1]$ yields a bijection between their lines which can be realized by spindle moves. Up to applying horizontal and vertical reflections to $\sigma'$, there exist (by irreducibility of $\sigma \setminus L$ and $\sigma'\iota(L)$) integers $0 \leq \alpha, \beta < n - 1$ such that

$$\sigma'(i) = \sigma\left(i - \alpha \pmod{(n - 1)}\right) + \beta \pmod{(n - 1)}$$

for $i = 1, \ldots, n - 1$ where $x \pmod{(n - 1)} \in \{1, \ldots, n - 1\}$. Say that line $(i, \sigma(i))$ (with $i < n$) is moved through infinity if either $i \geq n - \alpha$ or $\sigma(i) \geq n - \beta$. If both inequalities hold, we say that $(i, \sigma(i))$ is moved twice through infinity.

If there exists a line $(i, \sigma(i))$ which is moved exactly once through infinity, then every line $(j, \sigma(j)), 1 \leq j \leq n - 1$ is moved exactly once through infinity: Otherwise, consider a line $(i, \sigma(i))$ which is moved once and a line $(j, \sigma(j))$ which is not moved through infinity or moved twice. This implies that the $3-$subspindles

$$\{(i, \sigma(i)), (j, \sigma(j)), (n, n)\} \text{ and } \{\iota(i, \sigma(i)), \iota(j, \sigma(j)), \iota(n, n)\}$$

are mirrors (and thus not isotopic) which is impossible. Every line of $\sigma \setminus (n, n)$ is thus moved through infinity exactly once which implies $\alpha + \beta = n - 1$. If $2 \leq \alpha \leq n - 3$, we get a contradiction with irreducibility of $\sigma \setminus (n, n)$. If $\alpha \in \{1, n - 2\}$, we get a contradiction with irreducibility of $\sigma$.

We can now assume that every line of $\sigma \setminus (n, n)$ is moved an even number of times through infinity. This implies $\alpha = \beta$ and the existence of a non-trivial subblock $\sigma([n-\alpha, n-1]) = [n-\alpha, n-1]$ if $2 \leq \alpha \leq n - 3$ which contradicts irreducibility of $\sigma \setminus (n, n)$. The case $\alpha = \beta \in \{1, n - 2\}$ leads to a contradiction with irreducibility of $\sigma$. We get thus $\alpha = \beta = 0$ which shows $\sigma = \sigma'$ and proves the result.
8.2.2. **Subcase (2).** Up to circular moves, we can assume that $\sigma \setminus (n, n)$ contains a block $B$ of size $3 \leq k \leq n - 4$. Irreducibility of $\sigma$ shows now the existence (up to horizontal and vertical reflections of $\sigma$) of integers $\alpha, \beta \geq 1$ with $3 \leq \alpha + \beta = \sharp(B) \leq n - 4$ and of an integer $\gamma$ with $1 \leq \gamma < n - 2 - \alpha - \beta$ such that

$$B : \sigma([1, \alpha]) \cup [n - \beta, n - 1]) = [\gamma + 1, \gamma + \alpha + \beta].$$

Up to considering mirror-configurations, we can suppose $\gamma < \sigma(\alpha) < \sigma(n - \beta) \leq \alpha + \beta + \gamma$. There exist now $\alpha < \nu_1 < \nu_2 < n - \beta$ such that $\sigma(\nu_1) > \alpha + \beta + \gamma$ and $\sigma(\nu_2) \leq \gamma$. Indeed, otherwise $\sigma([\alpha + 1, \alpha + \gamma]) = [1, \gamma]$ and $\sigma([\alpha + \gamma + 1, n - \beta - 1]) = [\alpha + \beta + \gamma + 1, n - 1]$ which contradicts the irreducibility of $\sigma$ since this yields at least one non-trivial block in $\sigma$.

![Figure 14. A schematical picture of the subspindle $\tilde{\sigma} \subset \sigma$.](image)

**Lemma 8.5.** The subspindle $\tilde{\sigma} \subset \sigma$ defined by

$$B \cup (\nu_1, \sigma(\nu_1)) \cup (\nu_2, \sigma(\nu_2)) \cup (n, n)$$

is irreducible.

**Proof.** Consider the 5–subspindle $\tau \subset \tilde{\sigma}$ containing the five lines

$$(\alpha, \sigma(\alpha)), (\nu_1, \sigma(\nu_1)), (\nu_2, \sigma(\nu_2)), (n - \beta, \sigma(n - \beta)), (n, n).$$

The inequalities $\alpha < \nu_1 < \nu_2 < n - \beta < n$ and $\sigma(\nu_2) < \sigma(\alpha) < \sigma(n - \beta) < \sigma(\nu_1) < n$ imply easily that $\tau$ is irreducible (it is enough to check that $\tau$ contains no 2–block). Thus any block $\tilde{B} \subset \tilde{\sigma}$ intersects $\tau$ in a subset with cardinality $\sharp(\tilde{B} \cap \tau) \in \{0, 1, 4, 5\}$. Up to replacing $\tilde{B}$ by its complementary block $\tilde{\sigma} \setminus \tilde{B} \subset \tilde{\sigma}$, we can assume $\sharp(\tilde{B} \cap \tau) \leq 1$.

If $\tilde{B} \cap \tau \subset \{(\alpha, \sigma(\alpha)), (n - \beta, \sigma(n - \beta))\}$ then $\tilde{B} \subset B$ is also a nontrivial subblock in $\sigma$. This contradicts the irreducibility of $\sigma$.

If $\tilde{B} \cap \tau = \{(n, n)\}$, then a non-trivial subblock $\tilde{B}$ of of $\tilde{\sigma}$ contains also at least one line of the set $\{((\nu_1, \sigma(\nu_1)), (\nu_2, \sigma(\nu_2))\}$. This contradicts our assumption $\sharp(\tilde{B} \cap \tau) \leq 1$. 
If $\tilde{B} \cap \tau = \{(\nu_1, \sigma(\nu_1))\}$, then a non-trivial subblock $\tilde{B}$ contains also at least one line of $\{(\alpha, \sigma(\alpha)), (\nu_2, \sigma(\nu_2))\}$ contradicting again $\sharp(\tilde{B} \cap \tau) \leq 1$.

The case $\tilde{B} \cap \tau = \{(\nu_2, \sigma(\nu_2))\}$ is analogous.

The case $\tilde{B} \cap \tau = \emptyset$ implies that $\tilde{B}$ is a block of $\sigma$ which is impossible.

□

Apply now Theorem 8.3 to $\iota$ restricted to the subspindles $\tilde{\sigma}$ and $\tilde{\sigma}' = \iota(\tilde{\sigma}) \subset \sigma'$ (which contain at most $n - 1$ lines). Assuming minimality of $B$, the subset $B' = \iota(B)$ is already a subblock in $\sigma' \setminus \iota(n, n)$ (see the proof of Proposition 8.1). Moreover, up to spindle moves, the relative position of the subblock $B' \subset \sigma' \setminus (n, n)$ (with $(n, n) = \iota(n, n)$) inside $\sigma'$ is described by the integers $\alpha, \beta, \gamma$ considered above: One can indeed compute these integers by counting isotopy classes of suitable triplets $L_B \in B, L_{\tilde{\sigma}} \in \sigma \setminus (B \cup (n, n)), (n, n)$ which are in bijection with the corresponding triplets in $\sigma' = \iota(\sigma)$. Induction on $n$ implies now, that (perhaps up to a vertical reflection of $\sigma'$) the permutation induced by $\iota$ fixes the lines of $B \cup (n, n)$ which can be assumed to be common to $\sigma$ and $\sigma'$. Replacing $B$ by its complement $\overline{B}$ in $\sigma \setminus (n, n)$ we can find an irreducible subspindle $\overline{\sigma} \subset \sigma$ containing $\overline{B}, (n, n)$ and two suitable lines of $B$. Theorem 8.3 holds by induction on $n$ for the isotopy $\tau$ obtained by restricting $\iota$ to the subspindles $\overline{\sigma} \subset \sigma, \iota(\overline{\sigma}) \subset \sigma'$ and implies that $\iota$ fixes also all lines of $\overline{\sigma}$. (In this case, we have no longer to care about minimality of $\overline{B}$: the corresponding parameters $\overline{\sigma}, \overline{\beta}, \overline{\gamma}$ are fixed by the relative position of the already coinciding subblocks $B = B'$ inside $\sigma$ and $\sigma'$.)

This shows $\sigma = \sigma'$ and ends the proof of subcase (2).

8.2.3. Exceptional irreducible spindles. Call a spindle-permutation $\sigma$ of $n \geq 4$ lines exceptional if $\sigma$ is irreducible and $\sigma \setminus L$ contains a (cyclic) 2−block for every line $L \subset \sigma$.

**Proposition 8.6.** (i) For $n \geq 5$ odd, the spindle-permutation $\tau = \tau_n$ of $\{0, \ldots, n-1\}$ defined by

\[ \tau : i \mapsto \tau(i) = 2i \pmod{n}, \quad 0 \leq i \leq n-1 \]

and its mirror $\overline{\tau} = \overline{\tau}_n$ given (up to a circular move) by

\[ \overline{\tau} : i \mapsto \tau(i) = -2i \pmod{n}, \quad 0 \leq i \leq n-1 \]

are exceptional. The spindle-permutations $\tau_5$ and $\overline{\tau}_5$ are spindle-equivalent. For $n > 5$ odd, the spindle-permutations $\tau_n$ and $\overline{\tau}_n$ are not spindle-equivalent and have linking matrices which are not in the same switching class. In particular, the associated spindle-configurations are non-isotopic.
(ii) The spindle-permutations \( \tau_n, \; \tilde{\tau}_n, \; n \geq 5 \) odd, are the only exceptional spindle-permutations having \( \geq 4 \) lines, up to spindle-equivalence.

(iii) If \( \iota \) is an isotopy of the exceptional spindle-configuration \( \tau_n \) onto itself, then the line permutation \( L \mapsto \iota(L) \) induced by \( \iota \) can be realized by spindle moves.

Proof. We write \( \tau = \tau_n \) for \( n \geq 5 \) odd. We have \( \tau = \tilde{\tau}^k \) with

\[
\tilde{\tau}^k : i \mapsto \tilde{\tau}^k(i) = 2k + \tau(i - k \pmod{n}) \pmod{n}
\]

showing that \( \tau \) has a group of automorphisms acting transitively on its lines. Since \( \tau \setminus (0, 0) \) contains the (cyclic) 2–block \( \{(\frac{n-1}{2}, n - 1), (\frac{n+1}{2}, 1)\} \), we get thus a cyclic 2–block in \( \tau \setminus L \) for any line \( L \in \tau \).

Suppose now that \( \tau \) is reducible and consider a non-trivial subblock \( B \subset \tau \). Up to replacing \( B \) by the complementary block \( \tau \setminus B \), we can assume that \( B \) contains fewer than \( n/2 \) lines. Up to cyclic moves we can assume that \((0, 0), (1, 2) \in B \). This implies either \( B = \tau \setminus \{(\frac{n+1}{2}, 1)\} \) which contradicts non-triviality of \( B \) or \( B \) contains the line \( (\frac{n+1}{2}, 1) \).

But then \( B \) contains either all lines \((k, 2k), \; 1 \leq k \leq \frac{n-1}{2} \) or all lines \((k, 2k - n), \; \frac{n+1}{2} \leq k \leq n - 1 \). Since \((0, 0) \in B \), we have in both cases \( \sharp(B) \geq \frac{n+1}{2} \) which contradicts the assumption \( \sharp(B) < \frac{n}{2} \).

This shows that \( \tau = \tau_n \) and its mirror \( \overline{\tau}_n \) are exceptional.

A vertical reflection transforms \( \tau_n \) into \( \overline{\tau}_5 \) which proves their equivalence under spindle moves. For \( n > 5 \) odd, \( \tau_n \setminus L \) and \( \overline{\tau}_n \setminus \overline{L} \) contain both a unique 2–block \( B \), resp. \( \overline{B} \) (the choice of the lines \( L, \overline{L} \) is irrelevant since they are transitively permuted by automorphisms). We get thus 3–subspindles \( L \cup B \subset \tau_n \) and \( \overline{L} \cup \overline{B} \subset \overline{\tau}_n \) which are not isomorphic since each is the mirror of the other. Since such a 3–spindle and its isotopy class can be recovered from the linking matrix, the associated crossing matrices are not switching equivalent and the corresponding spindle configurations are non-isotopic. This proves assertion (i).

Notice that this argument fails for \( n = 5 \): In this case, \( \tau \setminus L \) gives rise to two complementary 2–blocks \( B, \overline{B} \) such that \( L \cup B \) and \( \overline{L} \cup \overline{B} \) are non-isotopic 3–spindles.

An inspection shows that no irreducible 4–spindle exists. This proves assertion (ii) for \( n = 4 \). Hence, we can suppose \( n \geq 5 \). Indexing the \( n \) lines of an exceptional spindle-permutation \( \sigma \) by \( \{0, 1, \ldots, n-1\} \) and using circular moves, we can assume \( \sigma(0) = 0 \). Denote by \( B_0 \) a 2–block contained in \( \sigma \setminus (0, 0) \). Up to a horizontal reflection, there exists \( 1 \leq \alpha \leq n - 1 \) such that \( B_0 = \{(\alpha, \sigma(\alpha)), (\alpha + 1, \sigma(\alpha + 1))\} \) with \( \{\sigma(\alpha), \sigma(\alpha + 1)\} \setminus \{1, n - 1\} \). Up to replacing \( \sigma \) by its mirror spindle, we can suppose \( \sigma(\alpha) = n - 1 \) and \( \sigma(\alpha + 1) = 1 \). Irreducibility of \( \sigma \) and \( n \geq 5 \) imply then \( 1 < \alpha < n - 2 \).
For a 2–block $B_{\alpha+1}$ contained in $\sigma \setminus (\alpha + 1, 1)$, we have now the following three possibilities:

$$B_{\alpha+1} = \begin{cases} 
\{(\alpha, n-1), (\alpha+2, n-2)\} & \text{case (a)} \\
\{(0,0), (n-1, 2)\} & \text{case (b)} \\
\{(0,0), (1, 2)\} & \text{case (c)}
\end{cases}$$

Indeed, such a 2–block is either given by the two lines $(\alpha, x), (\alpha+2, y)$ (case (a)) with $x, y$ cyclically adjacent, or it is given by $(x, 0), (y, 2)$ (cases (b) and (c)) with $x, y$ cyclically adjacent, see Figure 15 (with dots indicating the extremities and fatter lines indicating the 2–block $B_{\alpha+1}$).

**Figure 15.** The subcases (a), (b) and (c).

The following chart clarifies the steps of the proof:

```
\[
\begin{array}{c|c|c}
\text{a} & \text{aa} & \tau_5 \text{ or contradiction} \\
& \text{ab} & \tau_5 \text{ or contradiction} \\
& \text{ba} & \text{contradiction} \\
\hline
\text{b} & \text{bb} & \text{contradiction} \\
& \text{ca} & \tau_5 \text{ or contradiction} \\
\hline
\text{c} & \text{cb} & \tau_7 \\
& \phantom{aa} & \phantom{\tau_5} \\
\end{array}
\]
```

Case (a): Consider a 2–subblock $B_\alpha$ contained in $\sigma \setminus (\alpha, n-1)$. There are two possibilities for $B_\alpha$:

$$B_\alpha = \begin{cases} 
\{(0,0), (\alpha+2, n-2)\} & \text{subcase (aa)} \\
\{(\alpha+1, 2), (\alpha+1, 1)\} & \text{subcase (ab)}
\end{cases}$$

Subcase (aa) leads to $\alpha = n - 3$ and the lines $\{(0,0), (n-3, n-1), (n-2, 1), (n-1, n-2)\}$ form a non-trivial subblock of $\sigma$ for $n > 5$ which is a contradiction. For $n = 5$ we get $\sigma = \tau_5$.

Subcase (ab): The 2–block $B_{\alpha-1}$ contained in $\sigma \setminus (\alpha-1, 2)$ is either given by $B_{\alpha-1} = \{(\alpha+1, 1), (\alpha+2, n-2)\}$ which implies $1+2 = 3 = n-2$ and $\sigma = \tau_5$ or we have $B_{\alpha-1} = \{(\alpha, n-1), (0, 0)\}$ which implies $\alpha = 2$. 

For $n = 5$ we get again $\sigma = \tau_5$ and for $n > 5$ the spindle permutation $\sigma$ contains the non-trivial subblock $\{(0, 0), (1, 2), (2, n - 1), (3, 1)\}$. This contradicts the irreducibility of $\sigma$ and finishes the discussion of case (a).

Case (b): For the subblock $B_\alpha$ contained in $\sigma \setminus (\alpha, n - 1)$ we have

$$B_\alpha = \begin{cases} 
\{(0, 0), (\alpha + 1, 1)\} & \text{subcase (ba)} \\
\{(0, 0), (1, n - 2)\} & \text{subcase (bb)}
\end{cases}$$

Subcase (ba) implies $\alpha = 1$ which contradicts the irreducibility of $\sigma$.

Subcase (bb): The $2$–subblock $B_1$ associated to $\sigma \setminus (1, n - 2)$ is either given by $B_1 = \{(0, 0), (\alpha, n - 1)\}$ which implies $\alpha = 2$ contradicts the irreducibility (consider the $2$–block $\{(1, n - 2), (2, n - 1)\} \subset \sigma$) or $B_1$ is given by $B_1 = \{(\alpha, n - 1), (\alpha - 1, n - 3)\}$. Considering the $2$–block of $\sigma \setminus (\alpha - 1, n - 3)$ we get the existence of the line $(2, n - 4)$ and iterating this argument, we get the existence of the lines $(i, n - 2i), (\alpha + 1 - i, n + 1 - 2i), 1 \leq i \leq \left\lfloor \frac{\alpha - 1}{2} \right\rfloor$ forming a non-trivial subblock $([1, \alpha], \sigma([1, \alpha]) = [n - \alpha, n - 1]) \subset \sigma$ and contradicting the irreducibility of $\sigma$. This rules out case (b).

Case (c): For the $2$–block $B_\alpha$ of $\sigma \setminus (\alpha, n - 1)$ we have

$$B_\alpha = \begin{cases} 
\{(1, 2), (\alpha + 1, 1)\} & \text{subcase (ca)} \\
\{(0, 0), (n - 1, n - 2)\} & \text{subcase (cb)}
\end{cases}$$

Subcase (ca) yields $\alpha = 2$ and the four lines $(0, 0), (1, 2), (2, n - 1), (3, 1)$ form a non-trivial subblock of $\sigma$ if $n > 5$ which is a contradiction. For $n = 5$ we get $\sigma = \tau_5$.

Subcase (cb): Considering $\sigma \setminus (1, 2)$ we have either $n = 5, \alpha = 2$ and $\sigma = \tau_5$ or we obtain the existence of the line $(\alpha + 2, 3)$. A symmetric argument (involving $\sigma \setminus (n - 1, n - 2)$) yields the line $(\alpha - 1, n - 3)$.

We have now either $n = 7, \alpha = 3$ and $\sigma = \tau_7$ or we get the existence of two new lines $(2, 4)$ and $(n - 2, n - 4)$ by considering the $2$–blocks of $\sigma \setminus (\alpha + 2, 2)$, respectively $\sigma \setminus (\alpha - 1, n - 2)$. More generally, we get by iteration of this construction either a $(4k + 1)$–subspindle $\widetilde{\sigma}_{4k+1}$ containing the lines

$$(\alpha + 1 - k, n + 1 - 2k), (\alpha + k, 2k - 1), (k, 2k), (n - k, n - 2k)$$

or a $(4k - 1)$–subspindle $\widetilde{\sigma}_{4k-1}$ containing the lines

$$(k - 1, 2k - 2), (n + 1 - k, n + 2 - 2k), (\alpha + 1 - k, n + 1 - 2k), (\alpha + k, 2k - 1).$$

First consider $\widetilde{\sigma}_{4k+1}$. If $\widetilde{\sigma}_{4k+1} = \sigma$, we get $\alpha = 2k$ implying $\sigma = \tau_{4k+1}$. Otherwise, a consideration of the $2$–blocks contained in $\sigma \setminus (k, 2k)$ and $\sigma \setminus (n - k, n - 2k)$ implies the existence of the lines $(\alpha + 1 + k, 2k + 1), (\alpha - k, n - 1 - 2k)$ showing that $\widetilde{\sigma}_{4k+1}$ can be extended to $\widetilde{\sigma}_{4(k+1)-1}$. 
Now consider $\tilde{\sigma}_{4k-1}$. If $\tilde{\sigma}_{4k-1} = \sigma$ we have $\alpha = 2k - 1$ and $\sigma = \tau_{4k-1}$. Otherwise, considering the $2$-blocks contained in $\sigma \setminus (\alpha + 1 - k, n + 1 - 2k)$ and $\sigma \setminus (\alpha + k, 2k - 1)$, we get the existence of the lines $(n - k, n - 2k)$ and $(k, 2k)$. This shows that $\tilde{\sigma}_{4k-1}$ can be extended to $\tilde{\sigma}_{4k+1}$. Iteration of this construction stops by finiteness of $\sigma$ and ends always with $\sigma = \tau_n$ for odd $n$. This proves assertion (ii).

Up to considering mirror configurations, it is enough to prove assertion (iii) for $\tau_n$. Up to spindle-moves (circular moves, horizontal and vertical reflections) we can assume that the line-bijection induced by $\iota$ fixes the three lines $(0, 0), (\frac{n + 1}{2}, 1), (\frac{n - 1}{2}, n - 1)$. This implies that it has also to fix the lines $(1, 2)$ and $(n - 1, n - 2)$. Iteration shows finally that the lines of all subspindles $\tilde{\sigma}_{4k+1}$ and $\tilde{\sigma}_{4k-1}$ are fixed. This implies the result.

\begin{proof}[Proof of Theorem 1.3] Follows from subsections 8.1 and 8.2 \end{proof}

9. Theorem 3.2 of Kashin-Muzurovskii

Theorem 3.2 of [7] can be restated in our terminology as follows:

\textbf{Theorem 9.1.} Spindle-configurations with switching-equivalent linking-matrices are isotopic.

The proof is constructive and provides an explicit isotopy between the two spindle configurations.

Let $\sigma$ and $\mu$ be two spindles with switching-equivalent linking-matrices $X_\mu = P^t X_\sigma P$ where $P$ is a signed permutation matrix inducing a bijection between the lines of $\sigma$ and $\mu$. (Recall that the linking matrix $X_\tau$ of a spindle with permutation $\tau$ is defined by

$$(X_\tau)_{i,j} = \text{sign}((i - j)(\tau(i) - \tau(j))).$$

Up to circular moves, we can assume that the line-bijection

$$(i, \sigma(i)) \mapsto (i', \mu(i'))$$

induced by $P$ sends the first line $(1, \sigma(1) = 1)$ of $\sigma$ onto the first line $(1, \mu(1) = 1)$ of $\mu$. This implies that the conjugating matrix $P$ is an ordinary permutation matrix and we have thus

$$(i - j)(\sigma(i) - \sigma(j))(i' - j')(\mu(i') - \mu(j')) > 0$$

for all $1 \leq i < j \leq n$.

For $t \in \mathbb{R}$ fixed, consider the line $L_i(t) = sP_o^i(t) + (1 - s)P_o^i(t)$ parametrized by $s \in \mathbb{R}$, oriented from $P_o^i(t)$ to $P_o^i(t)$ where

$$\begin{cases}
P_o^i(t) &= (1 - t)(i, 1, 0) + t(0, 1, i') \\
P_o^i(t) &= (1 - t)(0, -1, \sigma(i)) + t(-\mu(i'), -1, 0)
\end{cases}$$
Proposition 9.2. For \( t \in \mathbb{R} \), the lines \( \mathcal{L}(t) = \{ L_1(t), \ldots, L_n(t) \} \) form a skew configuration.

Moreover, \( \mathcal{L}(0) \) realizes the spindle \( \sigma \) and \( \mathcal{L}(1) \) realizes the spindle \( \mu \).

This is essentially Theorem 3.2 of \([7]\). The following proof is a transcription of the original proof in \([7]\), made slightly more elementary in the sense that we avoid the use of Theorem 2.13 (involving configurations of subspaces in spaces of dimension higher than 3) of \([7]\) at the cost of a determinant-computation.

Proof. We have to prove that

\[
\text{lk}(L_i(t), L_j(t)) = \text{sign} \begin{pmatrix}
P_{\omega}^i(t) - P_{\omega}^j(t) \\
P_{\alpha}^i(t) - P_{\alpha}^j(t) \\
P_{\omega}^i(t) - P_{\alpha}^j(t)
\end{pmatrix}
\]

is well-defined (takes a constant value). The matrix involved is given by

\[
\begin{pmatrix}
(t-1)i - t\mu(i') & -2 & (1-t)\sigma(i) - ti' \\
(1-t)j + t\mu(i') & 2 & (t-1)\sigma(i) + tj' \\
(t-1)j - t\mu(j') & -2 & (1-t)\sigma(j) - tj'
\end{pmatrix}
\]

and its determinant \( p \) equals

\[
2((i-j)(\sigma(i) - \sigma(j)) + (i' - j')(\mu(i') - \mu(j'))t^2 - 4(i-j)(\sigma(i) - \sigma(j))t + 2(i-j)(\sigma(i) - \sigma(j))
\]

The discriminant (with respect to \( t \)) of \( p \) given by

\[-16(i-j)(\sigma(i) - \sigma(j))(i' - j')(\mu(i') - \mu(j'))\]

is strictly negative for \( i \neq j \) which shows that \( p \) is non-zero for \( t \in \mathbb{R} \).

This proves that \( \mathcal{L}(t) \) is a skew configuration for \( t \in \mathbb{R} \).

For \( t = 0 \) we get a spindle with axis \( (\mathbb{R}, 1, 0) \) and directrix \( (0, -1, \mathbb{R}) \) realizing the spindle \( \sigma \) with lines

\[
L_i = s(i, 1, 0) + (1-s)(0, -1, \sigma(i)), \ i = 1, \ldots, n, s \in \mathbb{R}
\]

For \( t = 1 \) we get a spindle with axis \( (0, 1, \mathbb{R}) \) and directrix \( (-\mathbb{R}, -1, 0) \) (where \( -\mathbb{R} \) denotes the real line endowed with the opposite order) realizing \( \mu \) with lines

\[
L'_i = s(0, 1, i') + (1-s)(-\mu(i'), -1, 0), \ i' = 1, \ldots, n, s \in \mathbb{R}
\]

The proof of Theorem 9.1 is immediate.
Proof of Corollary 1.2. Spindle-equivalent permutations give rise to isotopic spindles and their linking matrices are thus switching-equivalent.

On the other hand, given two permutations $\sigma, \sigma'$ with switching-equivalent linking matrices, Theorem 9.1 yields an isotopy between the associated spindle-configurations and Theorem 1.1 implies that $\sigma, \sigma'$ are spindle-equivalent. $\square$

10. Spindlegenus

This section describes a topological invariant of permutations up to spindle-equivalence. This yields an invariant for spindle-configurations by Theorem 1.1.

Let $P$ be a regular polygon with $n$ edges $E_1, \ldots, E_n$ in clockwise cyclical order. Reading indices modulo $n$ we orient the edge $E_i$ from $E_i \cap E_{i-1}$ and denote by $E_{-i}$ the edge of opposite orientation. Consider a second polygon $P'$ with edges $E'_1, \ldots, E'_n$ obtained from $P$ by an orientation-reversing isometry (e.g. an orthogonal symmetry with respect to a line). Given a permutation $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$, gluing the oriented edge $E_i \in P$ onto the oriented edge $E'_{\sigma(i)} \in P'$ for $1 \leq i \leq n$ yields a compact orientable surface $\Sigma(\sigma)$. We call the genus $g(\sigma) \in \mathbb{N}$ of $\Sigma(\sigma)$ the spindlegenus of $\sigma$.

This construction can be generalized as follows:

A signed permutation is a permutation of the set $\{\pm 1, \ldots, \pm n\}$ such that $\tilde{\sigma}(-i) = -\tilde{\sigma}(i)$. The group of all signed permutations is the full group of all isometries acting on the regular $n$–dimensional cube $[-1, 1]^n \subset \mathbb{R}^n$. Such a permutation can be graphically represented by segments $[(i, 0), (|\tilde{\sigma}(i)|, 1)]$ carrying signs $\epsilon_i = \frac{|\tilde{\sigma}(i)|}{\tilde{\sigma}(i)} \in \{\pm 1\}$. The notion of spindle-equivalence extends to signed permutations in the obvious way. The construction of the compact surface $\Sigma(\tilde{\sigma})$ can now be applied to a signed permutation $\tilde{\sigma}$ (glue the oriented edge $E_i \in P$ onto the oriented edge $E'_{\tilde{\sigma}(i)} \in P'$ for all $i = 1, \ldots, n$) and yields a generally non-orientable surface $\Sigma(\tilde{\sigma})$ presented as a CW–complex with two open 2–cells (corresponding to the interiors of the polygons $P$ and $P'$), $n$ open 1–cells (the edges of $P$ or $P'$) and with $v(\tilde{\sigma})$ points or 0–cells. The surface $\Sigma(\tilde{\sigma})$ is orientable if and only if $|\sum_{i=1}^{n} \epsilon_i| = n$, i.e. if $\tilde{\sigma}$ is either an ordinary permutation or the opposite of an ordinary permutation. The classification of compact surfaces (cf. for instance [1]) shows that, up to homeomorphisms, the surface $\Sigma(\tilde{\sigma})$ is completely described by $n, v(\tilde{\sigma})$ and orientability.

Proposition 10.1. If $\tilde{\sigma}, \tilde{\sigma}'$ are two signed spindle-equivalent permutations, then $v(\tilde{\sigma}) = v(\tilde{\sigma}')$ and the compact surfaces $\Sigma(\tilde{\sigma})$ and $\Sigma(\tilde{\sigma}')$ are homeomorphic.
Corollary 10.2. Two permutations $\sigma$ and $\sigma'$ which are spindle-equivalent have the same genus (i.e. $g(\sigma) = g(\sigma')$).

The following table lists the multiplicities for the spindlegenus $g(\sigma)$ (related to the number $v(\sigma)$ of vertices in the $CW-$complex considered above by the formula $\chi(\Sigma) = 2 - n + v(\sigma) = 2 - 2g(\sigma)$ for the Euler characteristic of $\Sigma(\sigma)$) of permutations (normalized by $\sigma(1) = 1$, multiply by $n$ in order to get the corresponding numbers for not necessarily normalized permutations):

| $n$ | $g = 0$ | $g = 1$ | $g = 2$ | $g = 3$ | $g = 4$ | $g = 5$ |
|-----|---------|---------|---------|---------|---------|---------|
| 1   | 1       |         |         |         |         |         |
| 2   | 1       |         |         |         |         |         |
| 3   | 1       | 1       |         |         |         |         |
| 4   | 1       | 5       |         |         |         |         |
| 5   | 1       | 15      | 8       |         |         |         |
| 6   | 1       | 35      | 84      |         |         |         |
| 7   | 1       | 70      | 469     | 180     |         |         |
| 8   | 1       | 126     | 1869    | 3044    |         |         |
| 9   | 1       | 210     | 5985    | 26060   | 8064    |         |
| 10  | 1       | 330     | 16401   | 152900  | 193248  |         |
| 11  | 1       | 495     | 39963   | 696905  | 2286636 | 604800  |
| 12  | 1       | 715     | 88803   | 2641925 | 18128396| 19056960|

(see also Sequence A60593 of [18]).

Given a permutation $\sigma$ of $\{1, \ldots, n\}$ and a sequence of signs $(\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$, we consider the signed permutation $\sigma_\epsilon : i \mapsto \epsilon_i \sigma(i)$ and set

$$V_\sigma(t, z) = \sum_{(\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n} z^{v(\sigma_\epsilon)} t^{\sum_i \epsilon_i} \in \mathbb{Z}[t, \frac{1}{t}, z].$$

Corollary 10.3. The application $\sigma \mapsto V_\sigma(t, z)$ is well-defined on spindle-equivalence classes and satisfies

$$V_{\overline{\sigma}}(t, z) = V_\sigma(\frac{1}{t}, z),$$

where $\overline{\sigma}(i) = n + 1 - \sigma(i)$ denotes a mirror spindle-permutation of $\sigma$.

Proof of Proposition 10.1. The number $v(\overline{\sigma})$ is obviously invariant under circular moves.

The compact surface $\Sigma(\overline{\sigma})$ is orientable if and only if $\sum_{i=1}^n \epsilon_i \in \{\pm n\}$ and $\sum_{i=1}^n \epsilon_i$ is preserved under spindle-equivalence. It is thus enough to show the invariance of the number $v(\sigma)$ of vertices in the $CW-$complex representing $\Sigma(\overline{\sigma})$ under horizontal and vertical reflections.
This number $v = v(\tilde{\sigma})$ can be computed graphically as follows: Represent the signed permutation $\tilde{\sigma}(i)$ of $\sigma$ by drawing $n$ segments joining the points $(i, 0), \ i = 1, \ldots, n$ to $(|\sigma(i)|, 1)$ as shown in Figure 16 for the signed permutation $\tilde{\sigma}(1) = -2, \ \tilde{\sigma}(2) = 3, \ \tilde{\sigma}(3) = 1$.  

![Figure 16. Example for the computation of $v(\tilde{\sigma})$](image)

Add $2n + 2$ additional points $(0.5, 0), (1.5, 0), \ldots, (n + 0.5, 0)$ and $(0.5, 1), (1.5, 1), \ldots, (n + 0.5, 1)$ by drawing all points at height $y = 0$ or 1 which are at distance $1/2$ from an endpoint of a segment $[(i, 0), (|\tilde{\sigma}(i)|, 1)]$. Join $(0.5, 0), (n + 0.5, 0)$ (respectively $(0.5, 1), (n + 0.5, 1)$) by dotted convex (respectively concave) arcs and join the points $(i \pm 0.5, 0), (|\tilde{\sigma}(i)| \pm \epsilon_i 0.5, 1)$ by dotted arcs with $\epsilon_i = \frac{\tilde{\sigma}(i)}{|\tilde{\sigma}(i)|}$ corresponding to the sign of the $i$-th segment. This yields a graph with vertices $(0.5, 0), (1.5, 0), \ldots, (n + 0.5, 0), (0.5, 1), (1.5, 1), \ldots, (n + 0.5, 1)$ of degree 2 by considering all dotted arcs as edges. The number of connected components of this graph (2 in Figure 16) equals $v(\tilde{\sigma})$. This can be seen as follows: The interior of the polygons $P, P'$ correspond to the half-planes $y < 0$ and $y > 1$. Vertices of $P, P'$ correspond to the points $(0.5, 0), (1.5, 0), \ldots, (n + 0.5, 0), (0.5, 1), (1.5, 1), \ldots, (n + 0.5, 1)$ where the pairs of points $(0.5, 0), (n + 0.5, 0)$ and $(0.5, 1), (n, 1.5, 1)$ have to be glued together (achieved by additional arcs joining these points). The segments $(i, 0), (|\tilde{\sigma}(i)|, 1)$ represent glued edges with dotted arcs joining vertices identified under gluing.

Let us consider the local situation around a block $B$ of $\sigma$. The (internal) dotted arcs associated to lines of $B$ connect the four boundary points adjacent to $B$ in one of the three ways depicted in Figure 17. The proof is now obvious since each of these three situations is invariant under vertical and horizontal reflections.  

\[\square\]

We leave the easy proofs of Corollaries 10.2 and 10.3 to the reader.
Remark 10.4. Many similar invariants of spindle-permutations, up to spindle-equivalence, can be defined similarly by considering a set $S$ of subsets of lines in $\sigma$ which is defined in a topological way (e.g. triplets of lines of given isotopy class, subsets of $k$ lines defining a subspindle in a set of prescribed spindle-equivalence classes, or a given subset of lines arising in the Euler partition of a spindle having $2n$ lines) and by considering all sign sequences $(\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$ such that $\{i \mid \epsilon_i = -1\} \in S$. The corresponding sum $\sum z^{v(\sigma_i)}$ is then of course well-defined for spindles and thus for spindle-permutations, up to equivalence.

We end this section with the following natural questions:

(1) By Corollary 1.22 the (complete) spindlegenus factorize through switching classes associated to spindles. Can the spindlegenus (or some related invariant) be extended to all switching classes?

(2) Can the (complete) spindlegenus be extended to skew configurations which are not spindles?

11. SPINDLE STRUCTURES FOR SWITCHING CLASSES

The existence of a linking matrix of a spindle in a given switching class is a natural question which we want to address algorithmically in this section.

The following algorithm exhibits a spindle-permutation with linking matrix in a given switching class or proves non-existence of such a permutation. By Corollary 1.22 such a spindle-permutation is unique, up to spindle-equivalence.

Algorithm 11.1.
Initial data. A natural number \( n \) and a switching class represented by a symmetric matrix \( X \) of order \( n \) with rows and columns indexed by \( \{1, \ldots, n\} \) and coefficients \( x_{i,j} \) satisfying
\[
x_{i,i} = 0, \quad 1 \leq i \leq n,
\]
\[
x_{i,j} = x_{j,i} \in \{\pm 1\}, \quad 1 \leq i \neq j \leq n.
\]

Initialization. Conjugate the symmetric matrix \( X \) by the diagonal matrix with diagonal coefficients \((1, x_{1,2}, x_{1,3}, \ldots, x_{1,n})\). Set \( \gamma(1) = \gamma(2) = 1 \), \( \sigma(1) = 1 \) and \( k = 2 \).

Main loop. Replace \( \gamma(k) \) by \( \gamma(k) + 1 \) and set
\[
\sigma(k) = 1 + \#\{ j \mid x_{\gamma(k),j} = -1 \} + \sum_{s=1}^{k-1} x_{\gamma(s),\gamma(k)}.
\]
Check the following conditions:
(1) \( \gamma(k) \neq \gamma(s) \) for \( s = 1, \ldots, k-1 \).
(2) \( x_{\gamma(k),\gamma(s)} = \text{sign}(\sigma(k) - \sigma(s)) \) for \( s = 1, \ldots, k-1 \) (where \( \text{sign}(0) = 0 \) and \( \text{sign}(x) = \frac{x}{|x|} \) for \( x \neq 0 \)).
(3) for \( j \in \{1, \ldots, n\} \setminus \{\gamma(1), \ldots, \gamma(k)\} \) and \( 1 \leq s < k \):
if \( x_{j,\gamma(s)} x_{\gamma(s),\gamma(k)} = -1 \) then \( x_{j,\gamma(k)} = x_{j,\gamma(s)} \).

If all conditions are fulfilled then:
if \( k = n \) print all the data (mainly the spindle-permutation \( i \mapsto \sigma(i) \) and perhaps also the conjugating permutation \( i \mapsto \gamma(i) \)) and stop.
if \( k < n \) then set \( \gamma(k+1) = 1 \), replace \( k \) by \( k + 1 \) and redo the main loop.

If at least one of the above conditions is not fulfilled then:
while \( \gamma(k) = n \) replace \( k \) by \( k - 1 \).
if \( k = 1 \): print “no spindle structure exists for this switching class” and stop.
if \( k > 1 \): redo the main loop.

Explanation of the algorithm. The initialization is actually a normalization: we assume that the first row of the matrix represents the first line of a spindle-permutation \( \sigma \) normalized to \( \sigma(1) = 1 \) (up to a circular move, this can always be assumed for a spindle-permutation).

The main loop assumes that row number \( \gamma(k) \) of \( X \) contains the crossing data of the \( k \)--th line \( L_k \) (assuming that the rows representing the crossing data of \( L_1, \ldots, L_{k-1} \) are correctly chosen). The image \( \sigma(k) \) of \( k \) under a spindle-permutation is then uniquely defined and given by the formula used in the main loop.

One has to check three necessary conditions:
• The first condition checks that row number $\gamma(k)$ has not been used before.
• The second condition checks the consistency of the choice for $\gamma(k)$ with all previous choices.
• If the third condition is violated, then the given choice of rows $\gamma(1), \ldots, \gamma(k)$ cannot be extended up to $k = n$.

The algorithm runs correctly even without checking for Condition (3). However, it loses much of its interest: Condition (3) is very strong (especially in the case of non-existence of a spindle structure) and ensures a fast running time.

The algorithm, in the case of success, produces two permutations $\sigma$ and $\gamma$. The linking matrix of the spindle permutation $\sigma$ is in the switching class as $X$ and $\gamma$ yields a conjugation between these two matrices. More precisely:

$$x_{\gamma(i), \gamma(j)} = \text{sign}((i - j)(\sigma(i) - \sigma(j)))$$

under the assumption $x_{1, i} = x_{i, 1} = 1$ for $2 \leq i \leq n$.

Failure of the algorithm (the algorithm stops after printing “no spindle structure exists for this switching class”) proves non-existence of a spindle structure in the switching class of $X$.

In practice, the average running time of this algorithm should be of order $O(n^3)$ or perhaps $O(n^4)$. Indeed Condition (3) is only very rarely satisfied for a wrong choice of $\gamma(k)$ with $k > 2$. On the other hand, for a switching class containing a crossing matrix of a spindle, checking all cases of Condition (3) needs at least $O(n^3)$ operations (or more precisely $\binom{n-1}{3}$ operations after suppressing the useless comparisons involving $\gamma(1) = 1$).

**Remark 11.2.** The algorithm can be improved. Condition (3) can be made considerably stronger.

12. Computational results and lower bounds for the number of non-isotopic configurations

In this section, we describe some computational results.

The number of skew configurations, up to isotopy, having $n \leq 7$ lines are known (see the survey of Viro and Drobotukhina [20] and the results of Borobia and Mazurovskii [1, 2, 11]):
The following table enumerates the number of switching classes of order $6 - 9$. The middle row shows the number of distinct polynomials which arise as characteristic polynomials of switching classes (this is of course the same as the number of conjugacy classes under the orthogonal group $O(n)$ of matrices representing switching classes).

| Lines | Isotopy classes |
|-------|----------------|
| 2     | 1              |
| 3     | 2              |
| 4     | 3              |
| 5     | 7              |
| 6     | 19             |
| 7     | 74             |

In fact, one can use representation theory of the symmetric groups in order to derive a formula for the number of switching classes of given order (see [8] and Sequence A2854 in [18]).

The map

$$\{\text{configurations of skew lines}\} \longrightarrow \{\text{switching classes}\}$$

is perhaps not surjective in general (there seems to be an unpublished counterexample of Peter Shor for $n = 71$, see [3] Section 3). We rechecked however a claim of Crapo and Penne (Theorem 5 of Section 4) stating that all 243 switching classes of order 8 arise as linking matrices of suitable skew configurations of 8 lines (the corresponding result holds also for fewer lines). There are thus at least 243 isotopy classes of configurations containing 8 skew lines, 180 of them are spindle classes.

The following table shows the number of spindle-permutations, up to equivalence, for $n \leq 13$. We also indicate the number of amphicheiral spindle-permutations, up to equivalence.

| Lines | Characteristic polynomials | Switching classes |
|-------|----------------------------|-------------------|
| 6     | 16                         | 16                |
| 7     | 54                         | 54                |
| 8     | 235                        | 243               |
| 9     | 1824                       | 2038              |
| $n$ | spindle classes | amphicheiral classes |
|-----|-----------------|---------------------|
| 1   | 1               | 1                   |
| 2   | 1               | 1                   |
| 3   | 2               | 0                   |
| 4   | 3               | 1                   |
| 5   | 7               | 1                   |
| 6   | 15              | 3                   |
| 7   | 48              | 0                   |
| 8   | 180             | 12                  |
| 9   | 985             | 5                   |
| 10  | 6867            | 83                  |
| 11  | 60108           | 0                   |
| 12  | 609112          | 808                 |
| 13  | 6909017         | 47                  |

Assertion (ii) of Proposition 3.5 explains of course the non-existence of amphicheiral classes for $n \equiv 3 \pmod{4}$.

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SPINDLE CONFIGURATIONS OF SKEW LINES

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ABSTRACT. We simplify slightly the exposition of some known invariants for configurations of skew lines and use them to define a natural partition of the lines in a skew configuration. Finally, we describe an algorithm constructing a spindle in a given switching class, provided such a spindle exists.

1. Introduction

A configuration of $n$ skew lines in $\mathbb{R}^3$ is an arrangement of $n$ lines in general position (distinct lines are never coplanar).

Two configurations of skew lines $C_1$ and $C_2$ are isotopic if there exists an isotopy from $C_1$ to $C_2$ (a continuous deformation of configurations of skew lines starting at $C_1$ and ending at $C_2$).

The study and classification of configurations of skew lines (up to isotopy) was initiated by Viro [14] and continued for example in [1], [2], [3], [5], [8], [9], [10], [11] and [15].

A spindle (or isotopy join) is a particularly nice configuration of skew lines in which all lines of the configuration intersect an oriented extra line, called the axis of the spindle. The isotopy class of a spindle is completely described by its spindle permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ encoding the order in which a half-plane bounded by the axis $A$ hits the lines during a half-turn around its boundary $A$ (see Section 6 for the precise definition). A spindle configuration is a configuration of skew lines which is isotopic to a spindle.

Consider the spindle-equivalence relation on permutations of $\{1, \ldots, n\}$ which is generated by transformations of the following three types:

1. $\sigma \sim \mu$ if $\mu(i) = s + \sigma(i + t \mod n)$ (mod $n$) for some integers $0 \leq s, t < n$ (with integers read modulo $n$).

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(2) $\sigma \sim \mu$ if $\sigma(i) < k$ for $i < k$ and
\[
\mu(i) = \begin{cases} 
  k - \sigma(k - i) & i < k \\
  \sigma(i) & i \geq k
\end{cases}
\]
for some integer $k \leq n + 1$.

(3) $\sigma \sim \mu$ if $\sigma(i) < k$ for $i < k$ and
\[
\mu(i) = \begin{cases} 
  \sigma^{-1}(i) & i < k \\
  \sigma(i) & i \geq k
\end{cases}
\]
for some integer $k \leq n + 1$.

Conjecture 59 in [3] states that two spindle configurations are isotopic if and only if they are described by spindle-equivalent permutations.

This paper contains a slight improvement of Theorem 62 in [3]:

**Theorem 1.1.** Equivalent spindle-permutations yield isotopic spindle configurations.

Part of the information about a configuration of labeled and oriented skew lines can be encoded by the *crossing matrix* $X$ (a symmetric matrix encoding the crossing data). Its *switching class* (or *two-graph*, see page 7 of [16]) is equivalent to the *homological equivalence class* of the underlying configuration $C$ of unoriented unlabeled skew lines (see [2]). It is also equivalent to the description of the set of *linking numbers* or *linking coefficients* of $C$ (see [2] or [15]).

An *invariant* is an application

\[
\{\text{isotopy classes of configurations of skew lines}\} \longrightarrow \mathcal{R}
\]

where $\mathcal{R}$ is some set, usually a ring or vector space. An invariant is *complete* if the above map is injective.

The set of known invariants for configurations of skew lines together with information about their completeness can be resumed as follows:

1. Equivalence classes of skew pseudoline diagrams (see Section 2): Completeness unknown. A powerful combinatorial invariant somewhat tedious to describe and handle. Switching classes and Kauffman polynomials factorize through this invariant.

2. Switching classes (or homological equivalence classes, see [2], or sets of linking numbers, see [15]). Not complete: There exist two configurations of 6 skew lines which are not isotopic (they have different Kauffman polynomials) but they define the same switching class (see Remark 3.4). This example is minimal in the sense that switching classes characterize isotopy classes of configurations of at most 5 skew lines.
An interesting feature of a switching class is the definition of its Euler partition - a natural partition of the corresponding lines into equivalence classes. The definition of this partition depends on the parity of the number \( n \) of lines. For odd \( n \), the definition is fairly simple since odd switching classes are in bijection with so-called Euler graphs (see Proposition 5.1). For even \( n \) the definition is more delicate. Some features attached to these partitions will be studied in Section 5.

Slightly weaker (but more elementary to handle) than the switching class is the characteristic polynomial

\[
P_X(t) = \sum_{i=0}^{n} \alpha_i t^i = \det(tI - X)
\]

of a switching class represented by a crossing matrix \( X \). The description of \( P_X \) is of course equivalent to the description of the spectrum of \( X \) or of the traces of the first \( n \) powers of \( X \). The coefficient \( \alpha_{n-3} \) of the characteristic polynomial conveys the same information as chirality, a fairly weak invariant considered sometimes (see [3, Section 3]).

(3) Kauffman polynomials: Completeness unknown. A powerful invariant which is very difficult to compute if there are many lines (its complexity is \( O(2^n) \)).

(4) Link invariants for links in \( S^3 \) applied to the preimage \( \pi^{-1}(C) \subset S^3 \) (called a Temari model by some authors, see for instance section 11 of [3]) of a configuration \( C \subset \mathbb{R}^3 \subset \mathbb{R}P^3 \) under the double covering \( \pi : S^3 \rightarrow \mathbb{R}P^3 \).

(5) Existence (and description) of a spindle structure. A generally very weak invariant since spindle structures are rare among configurations with many lines. Section 7 contains a somewhat curious (and rather weak) invariant of permutations, up to spindle-equivalence. In Section 8 we describe an algorithm which computes a spindle structure or proves non-existence of a structure representing a given switching class.

In this context, one should mention also the so-called shellability order, a kind of generalization of the notion of spindle structure, cf. [2] where it is used as a tool for classifications.

The above invariants can be used in order to distinguish isotopy classes of configurations of skew lines. Section 9 contains some computational data.
2. Skew pseudoline diagrams

Skew pseudoline diagrams are purely combinatorial objects and are among the main tools for studying configurations of skew lines.

**Definition 2.1.** A pseudoline in \( \mathbb{R}^2 \) is a smooth simple curve representing a non-trivial cycle in \( \mathbb{R}^2 \). An arrangement of \( n \) pseudolines in \( \mathbb{R}^2 \) is a set of \( n \) pseudolines with pairs of pseudolines intersecting transversally exactly once. The arrangement is **generic** if all intersections involve only two distinct pseudolines.

**Definition 2.2.** A skew pseudoline diagram of \( n \) pseudolines in \( \mathbb{R}^2 \) is a generic arrangement of \( n \) pseudolines with crossing data at intersections. The crossing data selects at each intersection the overcrossing pseudoline.

One represents skew pseudoline diagrams graphically by drawing them with the conventions used for knots and links: under-crossing curves are slightly interrupted at the crossing.

Skew pseudoline diagrams are **equivalent** if and only if they are related by a finite sequence of the following two moves (see [3, Section 9]):

1. Reidemeister-3 (or \( * \)-move), the most interesting of the three classical moves for knots and links (see Figure 1).

   ![Figure 1. Local description of a Reidemeister-3 move](image)

2. Projective move (or \( || \)-move): pushing a crossing through infinity (see Figure 2).

   ![Figure 2. Projective move](image)
Generic projections of isotopic configurations of skew lines are easily seen to yield equivalent skew-pseudoline diagrams (cf. for instance [3, Theorem 48]).

There exist equivalence classes of skew pseudoline diagrams which are not projections of configurations of skew lines: for instance alternating skew pseudoline diagrams with more than 3 pseudolines (see [7]). There are even generic arrangements of pseudolines which are not stretchable, i.e. cannot be realized as an arrangement of straight lines, see [4] for the smallest possible example having 9 lines.

The existence of non-isotopic configurations of skew lines inducing equivalent skew pseudoline diagrams is however unknown, cf. [3, Section 17, Problem 2]. This is a major difficulty for classifications. Indeed, classifying diagrams of skew pseudolines is purely combinatorial, but lifting such diagrams into configurations of skew lines (if such configurations exist) seems not obvious and it would of course be interesting to understand the number of possible (non-isotopic) liftings for instance in purely combinatorial terms.

3. CROSSING MATRICES AND SWITCHING CLASSES

We assign signs to pairs of oriented under- or over-crossing lines. Figure 3 displays a positive and a negative crossing using the conventions of knot theory which we adopt in the sequel of this paper.

![positive crossing](image1)

**Figure 3.** Positive and negative crossings

One can compute the sign between two oriented skew lines \( L_A, L_B \subset \mathbb{R}^3 \) algebraically as follows: choose ordered pairs of points \((A_\alpha, A_\omega)\) on \( L_A \) (resp. \((B_\alpha, B_\omega)\) on \( L_B \)) which induce the orientations. The sign of the crossing determined by \( L_A \) and \( L_B \) is then given by

\[
\text{sign}\left( \det \begin{pmatrix}
A_\omega - A_\alpha \\
B_\alpha - A_\omega \\
B_\omega - B_\alpha
\end{pmatrix}\right) \in \{\pm 1\}
\]

where \( \text{sign}(x) = 1 \) if \( x > 0 \) and \( \text{sign}(x) = -1 \) if \( x < 0 \).
The signs of crossings are of course also well-defined between pairs of oriented pseudolines in skew pseudoline diagrams.

The crossing matrix of a diagram of \( n \) oriented and labeled skew pseudolines \( L_1, \ldots, L_n \) is the symmetric \( n \times n \) matrix \( X \) where \( x_{i,j} \) is the sign of the crossing between \( L_i \) and \( L_j \) for \( i \neq j \) and \( x_{i,i} = 0 \).

\[
X = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & -1 & -1 & -1 \\
1 & -1 & 0 & 1 & 1 & -1 \\
1 & -1 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & -1 & 0 & -1 \\
1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}.
\]

Two symmetric matrices \( X \) and \( Y \) are switching-equivalent if
\[
Y = D \ P^t \ X \ P \ D
\]
where \( P \) is a permutation matrix and \( D \) is a diagonal matrix with diagonal coefficients in \( \{ \pm 1 \} \). Since \( (PD)^{-1} = DP^t \), the diagonalizable matrices \( X \) and \( Y \) have identical characteristic polynomials and the same eigenvalues.

**Proposition 3.1.** The switching class of a crossing matrix \( X \) of a skew pseudoline diagram \( D \) (and hence of a configuration of skew lines) is well-defined (i.e. independent of the labeling and the orientations of the lines in \( D \)).

*Proof.* Changing the labeling of the lines in \( D \) conjugates \( X \) by a permutation matrix. Reversing the orientation of some lines amounts to conjugating \( X \) by a diagonal matrix with diagonal entries \( \pm 1 \). \( \Box \)
Remark 3.2. The terminology “switching classes” (many authors use also “two-graphs”) has its origin in the following combinatorial interpretation and definition of switching classes:

Two finite simple graphs $\Gamma_1$ and $\Gamma_2$ with vertices $V = \{v_1, \ldots, v_n\}$ are switching-equivalent if there exists a partition $I \cup J = \{1, \ldots, n\}$ such that $\{v_i, v_j\}$ is an edge in $\Gamma_2$ if and only if either $\{v_i, v_j\}$ is an edge in $\Gamma_1$ and the labels $i, j$ belong either both to $I$ or both to $J$ or there is no edge between $v_i$ and $v_j$ in $\Gamma_1$ and exactly one of the indices $i, j$ is an element of $I$ (see Figure 5 where $I = \{1, 2\}$ and $J = \{3, 4\}$).

![Figure 5](image)

Figure 5. $\Gamma_1$ and $\Gamma_2$ are switching-equivalent (switch the vertices 1 and 2)

One says then that $\Gamma_1$ and $\Gamma_2$ are related by switching the vertices $\{v_i\}_{i \in I}$. The switching class of a graph $\Gamma_1$ defines a switching class of matrices by considering the symmetric matrix $X$ with entries $x_{ii} = 0$ and $x_{ij} = \pm 1$ according to the existence (giving rise to an entry 1) of an edge between the two distinct vertices $v_i$ and $v_j$. Conjugation by permutations is of course introduced in order to get rid of the labeling of the vertices in $\Gamma_1$. In the sequel we will also use the terminology “switching” in order to denote conjugation by a diagonal matrix with entries $\pm 1$.

Remark 3.3. The characteristic polynomial of a crossing matrix of a skew pseudoline diagram (or of a configuration of skew lines) is a weaker invariant than its switching class: One can see that the crossing matrices

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 0 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & 0 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 0 & 1 & -1 \\
1 & 1 & 1 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 0 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & 0 & 0
\end{pmatrix}
\]
are in different switching classes (see Example 5.3), but have common characteristic polynomial

\[(t - 3)(t - 1)^2(t + 1)(t + 3)^2(t^2 - 2t - 11) .\]

This example is minimal: suitable (i.e. zero entries on the diagonal and ±1 entries elsewhere) symmetric matrices of order less than 8 in distinct switching classes have distinct characteristic polynomials.

**Remark 3.4.** Configurations of at most 5 lines are (up to isotopy) completely characterized by the switching class of an associated crossing matrix.

However, the two configurations of 6 lines in Figure 6 have both crossing matrices in the switching class of
\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & -1 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & -1 & 0 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & 0
\end{pmatrix}
\]

![Figure 6](image_url)

**Figure 6.** Two different configurations with switching-equivalent crossing matrices
but are not isotopic. The Kauffman polynomial (see [3, Section 14 and Appendix]) of the first configuration is

\[5A^{12}B^3 + 10A^{11}B^4 - 10A^9B^6 + A^8B^7 + 16A^7B^8 + 10A^6B^9 - 6A^5B^{10} - 5A^4B^{11} + 6A^3B^{12} + 6A^2B^{13} - B^{15}\]

whereas the second configuration has Kauffman polynomial

\[-A^{15} + 6A^{13}B^2 + 6A^{12}B^3 - 5A^{11}B^4 - 6A^{10}B^5 + 10A^9B^6 + 16A^8B^7 + A^7B^8 - 10A^6B^9 + 10A^4B^{11} + 5A^3B^{12}\]

Each skew pseudoline diagram \(D\) with oriented and labeled pseudolines gives rise to a crossing matrix. Moreover, given an arbitrary pseudoline arrangement on \(n\) labeled and oriented lines and a suitable symmetric matrix \(X\) of order \(n\), then the matrix \(X\) can be turned into the crossing matrix of \(D\) by choosing the appropriate (and uniquely defined) crossing data on \(D\). This shows for instance that the number of equivalence classes of skew pseudoline diagrams having \(n\) pseudolines equals at least the number of switching classes of order \(n\). This inequality is generally strict as shown by the example of Remark 3.4.

One may look at the mirror image of a configuration: Given a configuration \(C\) of skew lines with crossing matrix \(X\), the mirror configuration \(\overline{C}\) (obtained for instance by reflecting \(C\) through the \(z = 0\) plane) has opposite crossing data. The pair of configurations of 6 lines in Figure 6 is an example of two configurations which are each others mirror. The crossing matrix \(\overline{X}\) of \(\overline{C}\) is then given by \(-X\).

A configuration \(C\) is amphicheiral if it is isotopic to its mirror \(\overline{C}\).

**Proposition 3.5.**

(i) The crossing matrix \(X\) of an amphicheiral configuration of skew lines is switching-equivalent to \(-X\). In particular, a crossing matrix of an amphicheiral configuration containing an odd number of skew lines has determinant 0.

(ii) Amphicheiral configurations with \(n\) lines do not exist if \(n \equiv 3 \pmod{4}\).

**Proof.** Assertion (i) is obvious.

Assertion (ii) is actually [15, Theorem 1]. We rephrase the proof using some properties of crossing matrices.

Let \(\sum_{i=0}^{n} a_i t^i = \det(tI - X)\) be the characteristic polynomial of the crossing matrix \(X\) for an amphicheiral configuration with \(n\) skew lines. Assertion (i) shows that we have \(a_{n-1} = a_{n-3} = a_{n-5} = \cdots = 0\). This implies

\[0 = a_{n-3} = -\sum_{1 \leq i, j, k \leq n} (x_{i,j}x_{j,k}x_{k,i} + x_{i,k}x_{k,j}x_{j,i}) = -2 \left( \sum_{1 \leq i < j < k \leq n} x_{i,j}x_{j,k}x_{k,i} \right) .\]
For \( n \equiv 3 \pmod{4} \) the number \( \binom{n}{3} \) of summands in \( \sum_{1 \leq i < j < k \leq n} x_{i,j}x_{j,k}x_{k,i} \) is odd. Since all these summands are \( \pm 1 \), we get a contradiction. \( \Box \)

4. Switching classes and linking numbers

Linking numbers (also called homological equivalence classes or chiral signatures) are a classical and well-known invariant for skew pseudoline diagrams. We sketch below briefly the well-known proof that they correspond to the switching class of a crossing matrix.

In this paper we prefer to work with switching classes mainly because they are easier to handle.

The linking number \( \text{lk}(L_i, L_j, L_k) \) of three lines ([2],[15]) is defined as the product \( x_{i,j}x_{j,k}x_{k,i} \in \{ \pm 1 \} \) of the signs (after an arbitrary orientation of all lines) of the corresponding three crossings. It is elementary to check that the result is independent of the chosen orientations for \( L_i, L_j \) and \( L_k \). It yields hence an invariant

\[
\{ \text{triplets of lines in skew pseudoline diagrams} \} \longrightarrow \{ \pm 1 \}.
\]

The set of linking numbers is the list of the numbers \( \text{lk}(L_i, L_j, L_k) \) for all triplets \( \{ L_i, L_j, L_k \} \) of lines in a skew pseudoline diagram.

Linking numbers (defining a two-graph, see [16]) and switching classes are equivalent. Indeed, linking numbers of a diagram \( D \) of skew lines can easily be read of a crossing matrix for \( D \). Conversely, given all linking numbers \( \text{lk}(L_i, L_j, L_k) \) of a diagram \( D \), choose an orientation of the first line \( L_1 \). Orient lines \( L_2, \ldots, L_n \) such that they cross \( L_1 \) positively.

A crossing matrix \( X \) for \( D \) is then given by \( x_{1,i} = x_{i,1} = 1 \), \( 2 \leq i \leq n \) and \( x_{a,b} = \text{lk}(L_a, L_b) \) for \( 2 \leq a \neq b \leq n \).

Two skew pseudoline diagrams are called homologically equivalent if there exists a bijection between the two sets of lines, which preserves the set of linking numbers. Two diagrams are homologically equivalent if and only if they have switching-equivalent crossing matrices.

A last invariant considered by some authors (see [3, Section 3 and Appendix]) is the chirality \( (\gamma_+, \gamma_-) \) of a skew pseudoline diagram. It is defined as

\[
\gamma_+ = \frac{1}{2} \left\{ 1 \leq i < j < k \leq n \mid \text{lk}(L_i, L_j, L_k) = 1 \right\},
\gamma_- = \frac{1}{2} \left\{ 1 \leq i < j < k \leq n \mid \text{lk}(L_i, L_j, L_k) = -1 \right\}.
\]

One has of course

\[
\gamma_+ = \frac{\binom{n}{3} + c}{2}, \quad \gamma_- = \frac{\binom{n}{3} - c}{2}.
\]
where
\[ c = \sum_{1 \leq i < j < k \leq n} x_{i,j,k} x_{j,k,i} = \frac{1}{6} \text{trace}(X^3) = -\frac{\alpha_{n-3}}{2} \]
is proportional to the coefficient of \( t^{n-3} \) in the characteristic polynomial \( \det(tI - X) = \sum_{i=0}^{n} \alpha_i t^i \) of a crossing matrix \( X \).

Let us stress that switching classes are somewhat annoying to work with because of possible signs (\(-1\) entries) in the conjugating matrix between two representatives. For switching classes of odd order there is a very satisfactory answer for this problem which will be addressed in the next section. For even orders, no completely satisfactory way to get rid of signs seems to exist.

One possible normalization consists in choosing a given (generally the first) row of a representing matrix and to make all entries in this row positive by conjugation with a suitable diagonal \( \pm 1 \) matrix. Such a matrix can then be encoded by a graph on \( n-1 \) vertices encoding all entries outside the chosen row and the corresponding column. This leads to the notion of graphs which are “cousins” in [3]. This notion is of course equivalent to the notion of switching-equivalence as can be easily checked.

5. Euler partitions

In this section we study some properties of switching classes. They lead to invariants with computational cost of \( O(n^2) \) operations. I the number of lines is huge they are easier to compute and to compare than the so-called chirality-invariants with computational cost \( O(n^3) \). All these invariants factor of course through switching-classes and they are thus useless for improving known classification results. They seem interesting mainly because of their simplicity and low complexity.

As already mentioned, invariants of switching classes behave quite differently according to the parity of their order \( n \).

Switching classes of odd order \( 2n-1 \) are in bijection with Eulerian graphs. This endows pseudoline diagrams consisting of an odd number of pseudolines with a canonical orientation (up to a global change). We get a partition of the pseudolines into equivalence classes according to the number of positive crossings in which they are involved for an Eulerian orientation.

We consider the case of odd order in Subsection 5.1 below.

For switching classes of even order \( 2n \) the situation is more complicated. We replace the Euler graphs appearing for odd orders by a suitable kind of planar rooted binary trees which we call Euler trees. In this case we lose the injectivity which we have in the odd case.
However, the leaves of the Euler tree induce in this case again a natural partition, also called an Euler partition, of the set of pseudolines into equivalence classes of even cardinalities. Subsection 5.2 below addresses the even case.

5.1. Switching classes of odd order - Euler orientations. A simple finite graph $\Gamma$ is Eulerian if all its vertices are of even degree. Connected Eulerian graphs are good designs for zoological gardens or expositions since they can be visited by walking exactly once along every edge. The following well-known result goes back to Seidel [12].

**Proposition 5.1.** Eulerian graphs on an odd number $2n-1$ of vertices are in bijection with switching classes of order $2n-1$.

We recall here the simple proof since it yields an algorithm for computing Eulerian orientations on configurations with an odd number of skew lines.

**Proof.** Choose an arbitrary representative $X$ in a given switching class. For $1 \leq i \leq 2n-1$ count the number

$$v_i = \sharp \{ j \mid x_{i,j} = 1 \} = \sum_{j=1,j \neq i}^{2n-1} \frac{x_{i,j} + 1}{2}$$

of entries equal to 1 in the $i$-th row of $X$. Since $X$ is symmetric, the vector $(v_1, \ldots, v_{2n-1})$ has an even number of odd coefficients and conjugation of the matrix $X$ with the diagonal matrix having diagonal entries $(-1)^{v_i}$ turns $X$ into a matrix $X_E$ with an even number of 1’s in each row and column. The matrix $X_E$ is well defined up to conjugation by a permutation matrix and defines an Eulerian graph with vertices $\{1, \ldots, 2n-1\}$ and edges $\{i, j\}$ if $(X_E)_{i,j} = 1$. This construction can easily be reversed. \qed

Let $D$ be a skew diagram having an odd number of pseudolines. Label and orient its pseudolines arbitrarily thus getting a crossing matrix $X$. Reverse the orientations of all lines having an odd number of crossings with positive sign. Call the resulting orientation Eulerian. It is unique up to globally reversing the orientations of all lines.

An Eulerian crossing matrix $X_E$ associated to an Eulerian orientation of $D$ is uniquely defined up to conjugation by a permutation matrix. Its invariants coincide with those of the switching class of $X_E$ but are slightly easier to compute since there are no sign ambiguities. In particular, some of them can be computed using only $O(n^2)$ operations.
An *Eulerian partition* of the set of pseudolines of a diagram consisting of an odd number of pseudolines is by definition the partition of the pseudolines into subsets $\mathcal{L}_k$ consisting of all pseudolines involved in exactly $2k$ positive crossings for an Eulerian orientation.

A few more invariants of Eulerian matrices are:

1. The total sum $\sum_{i,j} x_{i,j}$ of all entries in an Eulerian crossing matrix $X_E$ (this is of course equivalent to the computation of the number of entries 1 in $X_E$). The computation of this invariant needs only $O(n^2)$ operations.
2. The number of rows of $X_E$ whose entries have a given sum (this can also be computed using $O(n^2)$ operations). These numbers yield of course the cardinalities of the sets $\mathcal{L}_0, \mathcal{L}_1, \ldots$.
3. All invariants of the associated Eulerian graph (having edges associated to entries $x_{i,j} = 1$) defined by $X_E$, e.g. the number of triangles or of other given finite subgraphs, etc.

For example, for 7 vertices, there are 54 different Eulerian graphs, 36 different sequences (up to a permutation of the vertices) of vertex degrees, and 18 different numbers for the cardinality of 1’s in $X_E$.

Figure 7 shows all seven Eulerian graphs on 5 vertices.

![Eulerian Graphs on 5 Vertices](image)

**Figure 7.** All Eulerian graphs on 5 vertices

5.2. **Switching classes of even order - Euler partitions.** The situation in this case is unfortunately more complicated and less satisfactory.

There exists a natural partition of the rows of $X$ into two subsets $A$ and $B$ according to the parity of $\left(1 + \sum_j x_{i,j}\right)/2$. Conjugation by a diagonal $\pm 1$ matrix $D$ preserves or exchanges these two sets according to the determinant $\prod_i d_{i,i} \in \{\pm 1\}$ of $D$. This construction can then be iterated: The sets $A$ and $B$ have cardinalities $\alpha$ and $\beta$ which are both even and they determine thus two switching classes of even order...
\( \alpha \) and \( \beta \) by erasing all rows and columns not in \( A \), respectively not in \( B \). This stops if one of the subsets \( A, B \) is empty. We get in this way a partition of all rows (or columns) of \( X \) into subsets \( A_1, \ldots, A_r \) such that the corresponding symmetric sub-matrices are either Eulerian, i.e. represent an Eulerian graph on \( |A_i| \) vertices (having an edge between two vertices \( L_s, L_t \in A_i \) if and only if \( X_{s,t} = 1 \)) or anti-Eulerian, i.e. represent the complement of an Eulerian graph. This situation can be encoded by a planar binary rooted tree with non-zero integral weights as follows: A matrix \( X \) of order \( 2n \) representing a switching class is represented by a root-vertex. If the above partition of the rows into two subsets \( A \) and \( B \) is non-trivial, then draw a left successor representing all rows containing an even number of coefficients 1 and a right successor representing all rows containing an odd number of coefficients 1. Consider the new vertices as the roots of the corresponding symmetric sub-matrices representing switching classes and iterate. If the partition of all rows of \( X \) is trivial then put a weight \( n \) on the corresponding leaf if \( X \) is an Eulerian matrix of order \( 2n \) and put a weight of \( -n \) if \( X \) is anti-Eulerian of order \( 2n \).

We have yet to understand the effect of switching (conjugation by a diagonal \( \pm 1 \) matrix) on this decorated planar tree: switching one line in a leaf \( L_i \) of the above tree changes the weight \( w \) of the leaf into \( -w \) and has the effect of transposing the sons in all predecessors of the leaf \( L_i \).

There exists thus a unique representative having only Eulerian leaves. We call this representative the Euler tree of the switching class. Its weight \( n \) is the sum of all (positive) weights of its leaves.

**Enumerative digression.** The generating function \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) enumerating the number \( a_n \) of Eulerian trees of weight \( n \) satisfies the equation

\[
F(z) = \frac{1}{1 - z} + (F(z) - 1)^2.
\]

Indeed, Eulerian trees reduced to a leaf contribute \( 1/(1 - z) \) to \( F(z) \). All other Eulerian trees are obtained by gluing two Eulerian trees of strictly positive weights below a root and are enumerated by the factor \((F(z) - 1)^2\).

Solving for \( F(z) \) we get the closed form

\[
F(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{3(1 - z) - \sqrt{(1 - z)(1 - 5z)}}{2(1 - z)}.
\]
showing that
\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 5. \]
The first terms \(a_0, a_1, \ldots\) are given by
\[ 1, 1, 2, 5, 15, 51, 188, 731, 2950, 12235, \ldots \]
(see also Sequence A7317 in [13]).

The leaves of the Euler tree define a natural partition of the set of rows of \(X\) into subsets. We call this partition the \emph{Euler partition}.

**Example 5.2.** The symmetric matrix

\[
\begin{pmatrix}
0 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\
-1 & 0 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 0 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 \\
-1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 0
\end{pmatrix}
\]
yields the left tree of Figure 8. It gives hence the Euler partition
\[ A = \{1, 2, 4, 7, 8, 9\} \quad \cup \quad B = \{3, 5\} \quad \cup \quad C = \{6, 10\} \]
associated to the Euler tree at the right side of of Figure 8.

![Figure 8. Decorated trees and the Euler tree corresponding to Example 5.2](image)

**Example 5.3.** The two matrices mentioned in Remark 3.3 are indeed not switching-equivalent: The Euler partition of the first one has four rows in each class (more precisely, the Euler tree has two leaves: the left leaf is associated to rows 2, 5, 7, 8 and the right leaf to rows 1, 3, 4, 6). The second matrix is anti-Eulerian and its Euler partition is hence trivial. It can be turned into an Eulerian matrix by switching an odd number of lines.
Euler trees can also be used to define fast invariants for skew diagrams of an even number of pseudolines: for instance, compute a crossing matrix $X$ (this needs $O(n^2)$ operations) and compute then the cardinality
\[
\alpha = \# \{ i \mid \sum_{j=1, j \neq i}^{2n} \left( \frac{x_{i,j} + 1}{2} \right) \equiv 0 \pmod{2} \}.
\]
The unordered set $\{ \alpha, \beta = 2n - \alpha \}$ is an invariant with computational cost $O(n^2)$. It yields the sum of the weights of all leaves at the left, respectively at the right, of the root. Unfortunately, at this stage, one can not decide if $\alpha$ corresponds to the left leaves or not. In order to get rid of this indetermination without computing the complete Euler tree, one can use the following trick: obviously the difference
\[
\alpha - \beta
\]
changes its sign by switching an odd number of rows. It is straightforward to check that the same holds also for the number
\[
\lambda = \prod_{1 \leq i < j \leq 2n} x_{i,j} \in \{ \pm 1 \}.
\]
The product
\[
\lambda(2\alpha - 2n) \in 2\mathbb{Z}
\]
is a well-defined invariant of the switching class of $X$. If the matrix $X$ represents an Eulerian graph $\Gamma_E$, then the sign $\lambda$ is related to the parity of the number of edges in the Eulerian graph $\Gamma_E$.

Let us finally mention a last invariant which is a kind of decoration of the Euler tree for a switching class $X$ of even order $2n$. Suppose $X$ normalized in the obvious way (all leaves of the corresponding Euler tree are Eulerian) and let $A_1, \ldots, A_r$ be the Euler partition. For $1 \leq i \leq j \leq r$ define numbers $\alpha_{i,j} \in \{ \pm 1 \}$ in the following way
\[
\alpha_{i,j} = \prod_{s,t \in A_i, s < t} x_{s,t}
\]
(this is the definition of the sign $\lambda$ considered above of the Eulerian graph $A_i$) and
\[
\alpha_{i,j} = \prod_{s \in A_i, t \in A_j} x_{s,t}
\]
if $i < j$. One checks easily that the numbers $\alpha_{i,j}$ are well-defined.

Let us also remark that this invariant has an even stronger analogue for switching classes of odd order: Given an Eulerian matrix of order $2n + 1$ with Euler partition $A_0, \ldots, A_r$ (where $A_i$ corresponds to the
set of vertices of degree $2i$ in the Euler graph) one can consider the numbers

$$a_{i,j} = \sum_{s \in \mathcal{A}_i, t \in \mathcal{A}_j} x_{s,t}, \quad 0 \leq i, j$$

which can easily be shown to be well-defined. They are of course related to the number of edges between vertices of given type.

6. Spindles

This section is devoted to the study of spindles.

6.1. The definition of a spindle configuration. Recall that a spindle is a configuration of skew lines with all lines intersecting a supplementary line, called the axis. A spindle configuration (or a spindle structure) is a configuration of skew lines isotopic to a spindle.

Orienting the axis $A$ labels the lines of a spindle $C$ according to the order in which they intersect $A$. Each line $L_i \in C$ defines then a plane $\Pi_i$ containing $L_i$ and the axis $A$. The isotopy type of $C$ is now defined by describing the circular order in which the distinct planes $\Pi_1, \ldots, \Pi_n$ are arranged around the axis $A$. A convenient way to summarize this information is to project $\mathbb{R}^3$ along the directed axis $A$ onto an oriented plane, to choose an arbitrary initial plane $\Pi_{i_0}$ and to read off clockwise the appearance of the remaining planes on the projection thus getting a spindle permutation $k \mapsto \sigma(k) = j$ if $i_j = k$ (Figure 9(a) for instance shows a spindle encoded by the spindle permutation $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(4) = 2$).

This description can also be summarized as follows. Given an extra line $B$ (called a directrix) in general position with respect to $A$ and the planes $\Pi_i$, one can rotate each line $L_i$ in the plane $\Pi_i$ around the intersection $L_i \cap A$ until the line $L_i$ hits $B$. Choose the unique orientation of the directrix $B$ such that it crosses the oriented axis $A$ negatively. This orders the intersection points $L_i \cap B$ linearly. The spindle permutation of $C$ encodes then the relation between the linear orders $L_i \cap A$ and $L_i \cap B$ (see Figure 9(b)).

A crossing matrix $X$ of a spindle $C$ is easily computed as follows. Transform your spindle into a spindle with oriented axis $A$ and directrix $B$ as above. Orient a line $L_i$ from $L_i \cap A$ to $L_i \cap B$. A straightforward computation shows that the crossing matrix $X$ of this labeled oriented configuration has entries

$$x_{i,j} = \text{sign}((i - j)(\sigma(i) - \sigma(j)))$$

where $\text{sign}(0) = 0$ and $\text{sign}(x) = \frac{x}{|x|}$ for $x \neq 0$ and where $\sigma$ is the associated spindle permutation.
One can easily see:

**Remark 6.1.** A configuration $C$ of $n$ skew lines has a spindle structure if and only if its mirror configuration $\overline{C}$ has a spindle structure. A spindle permutation $\overline{\sigma}$ for $C$ is then for instance given by setting $\overline{\sigma}(i) = n + 1 - \sigma(i)$, $1 \leq i \leq n$, where $\sigma$ is a spindle permutation for $C$.

For example, the configuration of 5 skew lines depicted in Figure 10 corresponds to the spindle permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$$

and to the crossing matrix

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

6.2. **Spindle-equivalent permutations.** The aim of this subsection is to show that three types of transformations of spindle permutations, defined by Crapo and Penne in [3, Section 15], preserve the isotopy type of the associated spindle configuration.

A block of width $w$ ($w \geq 1$) in a permutation $\sigma$ of $\{1, \ldots, n\}$ is a subset $\{i + 1, \ldots, i + w\}$ of $w$ consecutive integers in $\{1, \ldots, n\}$ such that

$$\sigma(\{i + 1, \ldots, i + w\}) = \{j + 1, \ldots, j + w\}$$

(i.e. the image under $\sigma$ of a set $\{i + 1, \ldots, i + w\}$ of $w$ consecutive integers is again a set of $w$ consecutive integers).
A block containing \( n \) lines is \textit{minimal} if it has no sub-block of width \( 2 \leq w \leq n - 1 \).

For example, Figure 11 is a graphical representation of the cyclic permutation \( \sigma = (1 \ 2 \ 4 \ 3 \ 5) \) (i.e., \( \sigma(1) = 2, \sigma(2) = 4, \) etc.). The three lines contained in the shaded area of Figure 11 form a block of width 3 which is not minimal since it contains \( \sigma(\{2, 3\}) = \{4, 5\} \) as a proper sub-block of width 2.

Now, recall that two spindle-permutations are \textit{equivalent} (see [3, Section 15]) if they are equivalent for the equivalence relation generated by

1. (Circular move) \( \sigma \sim \mu \) if \( \mu(i) = (s + \sigma((i + t) \; \text{mod} \; n)) \; \text{mod} \; n \) for some integers \( 0 \leq s, t < n \) (all integers are modulo \( n \)).
2. (Vertical reflection of a block or local reversal) \( \sigma \sim \mu \) if \( \sigma(i) < k \) for \( i < k \) and

\[
\mu(i) = \begin{cases} 
    k - \sigma(k - i) & i < k \\
    \sigma(i) & i \geq k 
\end{cases}
\]

for some integer \( k \leq n + 1 \) (see Figure 12).
(3) (Horizontal reflection of a block or local exchange) $\sigma \sim \mu$ if $\sigma(i) < k$ for $i < k$ and

$$\mu(i) = \begin{cases} 
\sigma^{-1}(i) & i < k \\
\sigma(i) & i \geq k 
\end{cases}$$

for some integer $k \leq n + 1$ (see Figure 13).

One should notice that circular moves behave in a different manner from vertical or horizontal reflections in the following sense: Crossing matrices associated to permutations related by vertical or horizontal reflections are conjugated by a permutation matrix. This is generally no longer true for circular moves: Crossing matrices of spindle permutations related by circular moves are conjugated by signed permutation matrices.

We give now the proof of Theorem 1.1: Equivalent spindle-permutations yield isotopic spindle configurations.

Proof. A transformation of type (1) amounts to pushing the last few lines of the spindle on the axis and/or on the directrix through infinity. This obviously does not change the isotopy type of a spindle.

A transformation of type (2) can be achieved by an isotopy as follows. First, observe that the condition that $\sigma(i) < k$ for $i < k$ implies the existence of two blocks: $\sigma(\{1, \cdots, k - 1\}) = \{1, \cdots, k - 1\}$ and $\sigma(\{k, \cdots, n\}) = \{k, \cdots, n\}$. All the lines of the first block can then be moved by an isotopy into the interior of a small one-sheeted
revolution-hyperboloid whose axis of revolution intersects the axis \( A \) and the directrix \( B \) at right angles. We may moreover assume that this hyperboloid separates the lines of the first block from all the remaining lines. Rotating the interior (containing all lines of the first block) of this hyperboloid by a half-turn around its axis of revolution accomplishes the requested vertical reflection (see Figure 14).

![Figure 14. Isotopy for a transformation of type (2)](image)

For a transformation of type (3) we start as for type (2) by pushing the \( k \) lines of the first block into a small revolution hyperboloid with revolution axis intersecting the axis \( A \) and the directrix \( B \) orthogonally. Suppose moreover that the axis \( A = P_A + \mathbb{R}D_A \) and the directrix \( B = P_B + \mathbb{R}D_B \) have orthogonal directions \( D_A \perp D_B \). Push the remaining lines into the interior of a second small revolution hyperboloid such that the two hyperboloids are disjoint and the axis \( H_2 \) of the second hyperboloid intersects \( A \) orthogonally and is parallel to the directrix \( B \).

Rotate now the first hyperboloid together with its interior containing the lines \( L_1, \ldots, L_{k-1} \) of the first block by \( 180^\circ \) around the revolution axis \( H_2 \) of the second hyperboloid (see Figure 15). Finally, rotate the first hyperboloid and the lines inside it by \( 90^\circ \) around its revolution axis and translate it along its axis until the image of the original directrix \( B \) coincides with the axis \( A \). This yields a spindle configuration whose spindle permutation is related by a horizontal reflection and a circular move to the original spindle permutation.
7. SPINDLEGENDUS

This short section is devoted to a strange invariant of permutations, up to spindle-equivalence. We call this invariant the *spindlegenus* of a permutation. We do not know if this invariant gives rise to an invariant of spindle configurations up to isotopy.

Given a permutation $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ one can construct a compact oriented surface by gluing “together along $\sigma$” two polygons

\begin{figure}[h]
\centering
\includegraphics{figure15}
\caption{Isotopy for realizing a transformation of type (3)}
\end{figure}
having \( n \) sides. More precisely, consider two oriented polygons \( P, P' \) of the oriented plane with clockwise read edges \( E_1, \ldots, E_n, E'_1, \ldots, E'_n \). Glue the edge \( E_i \) of \( P \) onto in the edge \( E'_{\sigma(i)} \) of \( P' \) in the unique way (up to isotopy) which respects all orientations. This produces an oriented compact surface \( \Sigma(\sigma) \) of genus \( g(\sigma) \in \mathbb{N} \).

**Proposition 7.1.** Two permutations \( \sigma \) and \( \sigma' \) which are spindle-equivalent have the same genus (i.e. \( g(\sigma) = g(\sigma') \)).

Proposition 7.1 yields in fact a second invariant: the mirror-genus defined as the genus of the mirror spindle-permutation \( \psi = \sigma \circ \mu \) where \( \mu(i) = n + 1 - i \) of a permutation \( \sigma \) of \( \{1, \ldots, n\} \). The mirror-genus is generally different from the genus (consider for instance the case of the identity permutation). The genus and mirror-genus of amphicheiral permutations (giving rise to amphicheiral spindles) are equal.

The following questions are natural:

1. Is spindle genus the restriction of an invariant of switching classes?
2. If no, is the spindle genus an invariant of spindle configurations?
   This is a weaker statement than the Crapo-Penne conjecture.
3. If the answer to the second question is yes, is there a natural extension of the invariant \( g(C) \) to non-spindle configurations?

**Proof of Proposition 7.1.** The proposition obviously holds if \( \sigma \) and \( \sigma' \) are related by a circular move.

In order to prove the invariance of \( g(\sigma) \) under vertical or horizontal reflections it is useful to recall how \( g(\sigma) \) can be computed graphically. By the definition of the Euler characteristic

\[
\chi(\Sigma) = 2 - n + S(\sigma) = 2 - 2g(\sigma)
\]

where \( S(\sigma) \) counts the number of points in \( \Sigma(\sigma) \) originating from vertices of \( P \) and \( P' \), the proposition boils down to the equality \( S(\sigma) = S(\sigma') \).

Draw the permutation \( \overline{\sigma} : i \mapsto n+1-\sigma(i) \) graphically with segments joining the points \((i, 0)\) to \((\sigma(i), 1)\) as in Figure 16.

Add the points \((0.5, 0), (1.5, 0), \ldots, (n+0.5, 0)\) and \((0.5, 1), (1.5, 1), \ldots, (n+0.5, 1)\) and join \((0.5, 0), (n+0.5, 0)\) (respectively \((0.5, 1), (n+0.5, 1)\)) by dotted convex (respectively concave) arcs. Join the points \((i + 0.5, 0), (\overline{\sigma}(i) + 0.5, 1)\) by dotted segments. Do the same with the points \((i - 0.5, 0), (\overline{\sigma}(i) - 0.5, 1)\). One gets in this way a graph with vertices of degree 2 by considering all dotted segments. The number of connected components (3 in Figure 16) of this graph is easily seen to be \( S(\sigma) \).
Figure 16. Example for the computation of $S(\sigma)$

Let us consider the local situation around a block $B$ of $\sigma$. The edges of $B$ connect the four boundary points adjacent $B$ in one of the three ways depicted in Figure 17. The proof is now obvious since each of these three situations is invariant under vertical or horizontal reflections.

Figure 17. Three local situations around a block

The following table lists the multiplicities for the spindlegenus of spindle-permutations (normalized by $\sigma(1) = 1$, multiply by $n$ in order to get the corresponding numbers for not necessarily normalized
permutations):

| $n$ | $g = 0$ | $g = 1$ | $g = 2$ | $g = 3$ | $g = 4$ | $g = 5$ |
|-----|---------|---------|---------|---------|---------|---------|
| 1   | 1       |         |         |         |         |         |
| 2   | 1       |         |         |         |         |         |
| 3   | 1       |         | 1       |         |         |         |
| 4   | 1       |         | 5       |         |         |         |
| 5   | 1       |         | 15      |         | 8       |         |
| 6   | 1       |         | 35      |         | 84      |         |
| 7   | 1       |         | 70      |         | 469     |         |
| 8   | 1       |         | 126     |         | 1869    | 3044    |
| 9   | 1       |         | 210     | 5985    | 3115    | 28060   |
| 10  | 1       |         | 330     | 16401   | 152900  | 193248  |
| 11  | 1       |         | 495     | 39963   | 696905  | 2286636 |
| 12  | 1       |         | 715     | 88803   | 2641925 | 18128396| 19056960|

(see also Sequence A60593 of [13]).

8. **Spindle structures for switching classes**

The existence of a crossing matrix of a spindle in a given switching class is a natural question which we want to address algorithmically in this section.

The following algorithm exhibits a spindle permutation with crossing matrix in a given switching class or proves non-existence of such a permutation.

**Algorithm 8.1.**

**Initial data.** A natural number $n$ and a switching class represented by a symmetric matrix $X$ of order $n$ with rows and columns indexed by $\{1, \ldots, n\}$ and coefficients $x_{i,j}$ satisfying

\[
x_{i,i} = 0, \quad 1 \leq i \leq n, \\
x_{i,j} = x_{j,i} \in \{\pm 1\}, \quad 1 \leq i \neq j \leq n.
\]

**Initialization.** Conjugate the symmetric matrix $X$ by the diagonal matrix with diagonal coefficients $(1, x_{1,2}, x_{1,3}, \ldots, x_{1,n})$. Set $\gamma(1) = \gamma(2) = 1$, $\sigma(1) = 1$ and $k = 2$.

**Main loop.** Replace $\gamma(k)$ by $\gamma(k) + 1$ and set

\[
\sigma(k) = 1 + \sum_{j} \{j \mid x_{\gamma(k), j} = -1\} + \sum_{s=1}^{k-1} x_{\gamma(s), \gamma(k)}.
\]

Check the following conditions:

1. $\gamma(k) \neq \gamma(s)$ for $s = 1, \ldots, k - 1$. 
(2) \( x_{\gamma(k), \gamma(s)} = \text{sign}(\sigma(k) - \sigma(s)) \) for \( s = 1, \ldots, k - 1 \) (where \( \text{sign}(0) = 0 \) and \( \text{sign}(x) = \frac{x}{|x|} \) for \( x \neq 0 \)).

(3) for \( j \in \{1, \ldots, n\} \setminus \{\gamma(1), \ldots, \gamma(k)\} \) and \( 1 \leq s < k \):
\( \quad \) if \( x_{j, \gamma(s)} x_{\gamma(s), \gamma(k)} = -1 \) then \( x_{j, \gamma(k)} = x_{j, \gamma(s)} \).

If all conditions are fulfilled then:
\( \quad \) if \( k = n \) print all the data (mainly the spindle permutation \( i \mapsto \sigma(i) \)) and perhaps also the conjugating permutation \( i \mapsto \gamma(i) \) and stop.
\( \quad \) if \( k < n \) then set \( \gamma(k + 1) = 1 \), replace \( k \) by \( k + 1 \) and redo the main loop.

If at least one of the above conditions is not fulfilled then:
\( \quad \) while \( \gamma(k) = n \) replace \( k \) by \( k - 1 \).
\( \quad \) if \( k = 1 \): print “no spindle structure exists for this switching class” and stop.
\( \quad \) if \( k > 1 \): redo the main loop.

Explanation of the algorithm. The initialization is actually a normalization: we assume that the first row of the matrix represents the first line of a spindle permutation \( \sigma \) normalized to \( \sigma(1) = 1 \) (this can always be assumed after a suitable circular move).

The main loop assumes that row number \( \gamma(k) \) of \( X \) contains the crossing data of the \( k \)-th line (assuming that the rows representing the crossing data of the lines labeled \( 1, \ldots, k - 1 \) have already been correctly chosen). The image \( \sigma(k) \) of \( k \) under a spindle permutation is then uniquely defined and given by the formula used in the main loop.

One has to check three necessary conditions:
\( \quad \) The first condition checks that row number \( \gamma(k) \) has not been used before.
\( \quad \) The second condition checks the consistency of the choice for \( \gamma(k) \) with all previous choices.
\( \quad \) If the third condition is violated, then the given choice of rows \( \gamma(1), \ldots, \gamma(k) \) cannot be extended up to \( k = n \).

The algorithm runs correctly even without checking for Condition (3). However, it loses much of its interest: Condition (3) is very strong (especially in the case of non-existence of a spindle structure) and ensures a fast running time.

The algorithm, in the case of success, produces two permutations \( \sigma \) and \( \gamma \). The crossing matrix of the spindle permutation \( \sigma \) is in the same switching class as \( X \) and \( \gamma \) yields a conjugation between these
two matrices. More precisely:

\[ x_{\gamma(i),\gamma(j)} = \text{sign}\left((i - j)(\sigma(i) - \sigma(j))\right) \]

under the assumption \(x_{1,1} = x_{i,1} = 1\) for \(2 \leq i \leq n\).

Failure of the algorithm (the algorithm stops after printing “no spindle structure exists for this switching class”) proves non-existence of a spindle structure in the switching class of \(X\).

In practice, the average running time of this algorithm should be of order \(O(n^3)\) or perhaps \(O(n^4)\). Indeed Condition (3) is only very rarely satisfied for a wrong choice of \(\gamma(k)\) with \(k > 2\). On the other hand, for a switching class containing a crossing matrix of a spindle, checking all cases of Condition (3) needs at least \(O(n^3)\) operations (or more precisely \(\binom{n-1}{3}\) operations after suppressing the useless comparisons involving \(\gamma(1) = 1\)).

**Remark 8.2.** The algorithm can be improved. Condition (3) can be made considerably stronger.

9. **Computational results and lower bounds for the number of non-isotopic configurations**

In this section, we describe some computational results.

The numbers of non-isotopic classes of configurations having at most 7 lines are known (see the survey of Viro and Drobotukhina [15] and the results of Borobia and Mazurovskii [1], [2]):

| Lines | Isotopy classes |
|-------|----------------|
| 2     | 1              |
| 3     | 2              |
| 4     | 3              |
| 5     | 7              |
| 6     | 19             |
| 7     | 74             |

The following table enumerates the number of switching classes of order 6 – 9. The middle row shows the number of distinct polynomials which arise as characteristic polynomials of switching classes (this is of course the same as the number of conjugacy classes under the orthogonal group \(O(n)\) of matrices representing switching classes).
| Lines | Characteristic polynomials | Switching classes |
|-------|-----------------------------|-------------------|
| 6     | 16                          | 16                |
| 7     | 54                          | 54                |
| 8     | 235                         | 243               |
| 9     | 1824                        | 2038              |

In fact, one can use representation theory of the symmetric groups in order to derive a formula for the number of switching classes of given order (see [6] and Sequence A2854 in [13]).

The map

\[
\{ \text{isotopy classes of configurations of skew lines} \} \longrightarrow \{ \text{switching classes} \}
\]

is perhaps not surjective for all \( n \) (there seems to be an unpublished counterexample of Peter Shor for \( n = 71 \), see [3, Section 3]). We rechecked however a claim of Crapo and Penne (Theorem 5 of Section 4) stating that all 243 switching classes of order 8 arise as crossing matrices of suitable skew configurations of 8 lines (the corresponding result holds also for fewer lines). There are thus at least 243 isotopy classes of configurations containing 8 skew lines, 180 of them are spindle classes.

The following table shows the number of spindle permutations, up to equivalence, for \( n \leq 13 \). We also indicate the number of amphicheiral spindle permutations, up to equivalence.

| \( n \) | spindle classes | amphicheiral classes |
|--------|-----------------|----------------------|
| 1      | 1               | 1                    |
| 2      | 1               | 1                    |
| 3      | 2               | 0                    |
| 4      | 3               | 1                    |
| 5      | 7               | 1                    |
| 6      | 15              | 3                    |
| 7      | 48              | 0                    |
| 8      | 180             | 12                   |
| 9      | 985             | 5                    |
| 10     | 6867            | 83                   |
| 11     | 60108           | 0                    |
| 12     | 609112          | 808                  |
| 13     | 6909017         | 47                   |
Assertion (ii) of Proposition 3.5 explains of course the non-existence of amphicheiral classes for $n \equiv 3 \pmod{4}$.

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