Monotone bargaining is Nash-solvable

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Abstract

Given two finite ordered sets \( A = \{a_1, \ldots, a_m \} \) and \( B = \{b_1, \ldots, b_n \} \), introduce the set of \( mn \) outcomes of the game \( O = \{(a, b) \mid a \in A, b \in B \} \). Two players, Alice and Bob, have the sets of strategies \( X \) and \( Y \) that consist of all monotone non-decreasing mappings \( x : A \to B \) and \( y : B \to A \), respectively. It is easily seen that each pair \((x, y) \in X \times Y\) produces at least one \( \text{deal} \), that is, an outcome \((a, b) \in O \) such that \( x(a) \geq b \) and \( y(b) \geq a \). Denote by \( G(x, y) \subseteq O \) the set of all such deals related to \((x, y)\). The obtained a mapping \( G = G_{m,n} : X \times Y \to 2^O \) is a \( \text{game correspondence} \). Choose an arbitrary deal \((x, y) \in G(x, y)\) to obtain a mapping \( g : X \times Y \to O \), which is a \( \text{game form} \). We use notation \( g \in G \) and show that each such game form is tight and, hence, Nash-solvable, that is, for any pair \( u = (u_A, u_B) \) of utility functions \( u_A : O \to \mathbb{R} \) of Alice and \( u_B : O \to \mathbb{R} \) of Bob, the obtained monotone bargaining game \((g, u)\) has at least one Nash equilibrium in pure strategies. Moreover, the same equilibrium can be selected for all \( g \in G \). We also obtain an efficient algorithm that determines such an equilibrium in time linear in \( mn \), although the numbers of strategies \(|X| = \binom{m+n-1}{m} \) and \(|Y| = \binom{m+n-1}{n} \) are exponential in \( mn \). Our results show that, somewhat surprising, the players have no need to hide or randomize their bargaining strategies, even in the zero-sum case.

Key words: Monotone bargaining game, deal, game form, game correspondence, Nash equilibrium, Nash-solvability, tightness.

1 Introduction

1.1 Main results

Two players, Alice and Bob, possess items \( A = \{a_1, \ldots, a_m \} \) and \( B = \{b_1, \ldots, b_n \} \), respectively. Both sets are ordered: the smaller is the index, the less valuable is the corresponding item. Both players know both orders. A pair \((a, b)\) with \( a \in A \) and \( b \in B \) is called an outcome.

Alice chooses her strategy, which is a mappings \( x : A \to B \), showing that she is ready to exchange \( a \) for \( x(a) \) for any \( a \in A \). Naturally, such a mapping \( x \) is assumed to be \( \text{monotone non-decreasing}: x(a) \geq x(a') \) whenever \( a > a' \). Similarly, Bob chooses his strategy, which is a monotone non-decreasing mapping \( y : B \to A \).

It is not difficult to compute the cardinalities of the sets of strategies and outcomes:

\[
|X| = \binom{m+n-1}{m}, \quad |Y| = \binom{m+n-1}{n}, \quad |O| = mn.
\]

Let us fix a pair of strategies \((x, y)\). An outcome \((a, b) \in O\) is called a \( \text{deal} \) if \( x(a) = b \) and \( y(b) = a \). Denote by \( G(x, y) \subseteq O \) the set of all deals. We will show that set \( G(x, y) \) is not empty; yet, it may contain several deals. The obtained mapping \( G : X \times Y \to 2^O \) is called a \( \text{(monotone bargaining) game correspondence} \). Note that \( G = G_{m,n} \) is uniquely defined by \( m \) and \( n \). For each \((x, y)\), let us fix an arbitrary deal \((g(x, y))\) from \( G(x, y) \). The obtained a mapping \( g : X \times Y \to O \), is called a \( \text{(monotone bargaining) game form} \).

An arbitrary (not necessarily monotone bargaining) two-person game form \( g \) is called \( \text{Nash-solvable} \) (NS) if for any pair \( u = (u_A, u_B) \) of utility functions \( u_A : O \to \mathbb{R} \) of Alice and \( u_B : O \to \mathbb{R} \) of Bob the obtained game \((g, u)\) has at least one Nash equilibrium (NE) in pure strategies. A game correspondence \( G \) is called NS if every its game form \( g \in G \) is NS.

We will apply some general criteria of Nash-solvability (based on the concept of Boolean duality and developed for the two-person game forms and correspondences in [13, 15, 27]) to the monotone bargaining case considered in this paper. These criteria imply that \( G = G_{m,n} \) is NS for all \( m \) and \( n \). Moreover, given \( m \) and \( n \), the same NE \((x, y)\) exists for all \( g \in G \), and this equilibrium is simple, that is, \( G(x, y) \) contains a unique

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deal. We also construct an efficient algorithm that determines such an equilibrium in time polynomial in \(mn\), although, by (1), the sizes of \(X\) and \(Y\) are exponential in \(mn\).

In our model, Alice and Bob are restricted to their monotone non-decreasing strategies. Let us replace them by the monotone non-increasing ones. Obviously, this case is equivalent with the original one: it is enough just to inverse the enumeration of either \(A\), or \(B\), but not both. However, if we restrict Alice by her monotone non-decreasing and Bob by his monotone non-increasing) strategies, or vice versa, then \(G(x, y)\) may become empty for some \(x\) and \(y\).

In particular, our results imply that the players have no need to keep their bargaining strategies secret and choose them randomly, even in the zero-sum case. This is unusual, since in most bargaining models players do have to hide and randomize their strategies [5, 22, 23, 25, 26, 7].

The considered monotone bargaining games can be viewed as positional games on complete bipartite digraphs, in which, however, both players are restricted to their monotone non-decreasing (or equivalently, non-increasing) strategies. Restricting the set of strategies can be viewed as a generalization of the concept of imperfect information [14, Section 6]. Nevertheless, we show that the monotone bargaining games can be solved in pure strategies. It would be interesting to find other classes of positional games with imperfect information NS in pure strategies.

The paper is organized as follows. In Section 2 we derive some combinatorial properties of a monotone bargaining game correspondence \(G = G_{m,n}\) and then explain their role and meaning, after recalling some general definitions and results related to the two-person normal form games in Section 3. In particular, we show that \(G\), as well as all game forms \(g \in G\), are tight. Moreover, they have the same pair of dual hypergraphs (or the same self-dual effectivity functions, in terminology of [16] and [17]). It is known that tightness and Nash-solvability are equivalent properties of the two-person game forms; see [13, 15], and also [27]. In Section 3.3 we recall a constructive proof of this fact that is based on an algorithm finding a NE in any game \((g, u)\) in time linear in the size of the set \(O\) of its outcomes, whenever \(g\) is tight. Recall that \(|O| = mn\) for the monotone bargaining game forms. In section 4 we recall other classes of two-person tight game forms that appear from: positional structures, Section 4.1 self-dual hypergraphs, Section 4.2); symmetric dual hypergraphs and veto voting, Section 4.3 topological structure, where tightness results from Jordan’s curve theorem, Section 4.4. Finally, in Section 5 we show that a natural extension of our results to the case of three players fails.

### 1.2 Examples

Three game correspondences \(G = G_{m,n}\) are shown in Figures 1, 2, and 3 for \((m = n = 2)\), \((m = 3, n = 2)\), and \((m = n = 3)\) respectively. The corresponding tables are of sizes \(3 \times 3\), \(4 \times 6\), and \(10 \times 10\), respectively, and contain 4, 6, and 9 outcomes, in accordance with (1).

Each of these three figures consists of two tables. In the first one, items \(a \in A\) and \(b \in B\) are represented as the vertices of a bipartite digraph, denoted by white and black discs that are always located on the left and on the right, respectively. The values of items increase with their indices \(i \in I\) and \(j \in J\), the higher is a disc, the smaller is its index and the cheaper is the corresponding item. The strategies (monotone non-decreasing mappings) \(x\) and \(y\) are denoted by directed arcs, which may have common ends, but no other intersections. Indeed, they cannot have a common beginning, because \(x : A \to B\) and \(y : B \to A\) are mappings, and since both mappings are monotone non-decreasing, the arcs cannot cross. The strategies and deals are shown on the corresponding bipartite graphs; To save space, for every \(i \in I\) and \(j \in J\) we denote the deal \(a_{ij} = (a_i, b_j)\) by a non-directed arc between \(a_i\) and \(b_j\). In the second table the strategies and deals are specified simply by the corresponding pairs of indices \((i, j)\).

Rows and columns of both tables are labeled by the strategies \(x \in X\) and \(y \in Y\), respectively. The set of deals \(G(x, y)\) is shown for each entry \((x, y)\). These sets are always non-empty (see Proposition 2 below) but may contain several deals. For example, in Figure 3, three deals \(\{a_i, b_i\} \mid i = 1, 2, 3\) are assigned to the pair \((x, y)\) with \(x(a_i) = b_i\) and \(y(b_i) = a_i\), for \(i = 1, 2, 3\); and one or two deals are assigned to any other pair \((x, y)\).
Figure 1. Monotone bargaining game correspondence $G(2, 2)$. 
Figure 2. Monotone bargaining game correspondence $G(2, 3)$. 

|     | 1 1 | 1 1 | 1 1 | 2 1 | 2 1 | 3 1 | 3 1 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 1 |     |     |     |     |     |     |     |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 1 |     |     |     |     | 2 1 | 2 1 | 3 1 |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 1 |     |     |     | 2 1 | 1 1 | 2 1 | 3 1 |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 1 |     |     |     | 2 1 | 1 1 | 2 1 | 3 1 |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 1 |     |     |     | 2 1 | 1 1 | 2 1 | 3 1 |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 1 |     |     |     | 2 1 | 1 1 | 2 1 | 3 1 |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 1 |     |     |     | 2 1 | 1 1 | 2 1 | 3 1 |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 1 |     |     |     | 2 1 | 1 1 | 2 1 | 3 1 |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 1 |     |     |     | 2 1 | 1 1 | 2 1 | 3 1 |
| 2 1 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |

|     | 1 2 | 1 2 | 2 2 | 3 2 | 2 2 | 3 2 | 3 2 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 2 |     |     |     |     |     |     |     |
| 2 2 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 2 |     |     |     |     |     |     |     |
| 2 2 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
| 1 2 |     |     |     |     |     |     |     |
| 2 2 |     |     |     |     |     |     |     |
| 3 2 |     |     |     |     |     |     |     |
Figure 3. Monotone bargaining game correspondence \( G(3,3) \).

For each of the three examples let us verify also the following combinatorial properties, which will play an important role too:

1. For any deal \((a, b) \in G(x, y)\) there exist \(x' \in X\) and \(y' \in Y\) such that sets \(G(x, y')\) and \(G(x', y)\) contain only this deal \((a, b)\).

2. In each row \(x \in X\) (respectively, column \(y \in Y\)) choose an arbitrary deal \((a, b) \in G(x, y)\) for some \(y \in Y\) (respectively, for some \(x \in X\)). Then,

3. The obtained two selections always have a deal in common.
• (t’ t'') There exists a column y₀ (respectively, a row x₀) such that the set of deals in it is a subset of the above row-selection (respectively, column-selection).

Let us consider, for example, the 3 × 3 table in Figure 1. The central entry \( G(x_2, y_2) \) contains two deals \((a_1, b_1)\) and \((a_2, b_2)\), but the second row and the second column each contains \((a_1, b_1)\) and \((a_2, b_2)\), separately, in \( G(x_1, y_2), G(x_2, y_1) \) and in \( G(x_2, y_3), G(x_3, y_2) \). Thus, \( (r) \) holds for this table. In the same table, the side diagonal is a row selection that provides the set of deals \((a_2, b_1), (a_1, b_1), (a_1, b_2)\), but the first column provides \((a_1, b_1), (a_1, b_2)\), which is its subset. Similarly, in the 10 × 10 table in Figure 3 the entry \( G(x_5, y_5) \) contains three deals \((a_1, b_1), (a_2, b_2)\), and \((a_3, b_3)\), but each of these three deals appears in the fifth row and column of this table separately, in agreement with \( (r) \).

In Sections 3 and 4 we will recall that properties \((t'), (t'')\) and \((t)\) are equivalent and imply \( (r) \). Due to \( (r) \), all game forms \( g \in G \) have the same pair of hypergraphs (or in other words, the same effectivity functions). Furthermore, \((t'), (t'')\) and \((t)\) mean that these hypergraphs are dual (respectively, the effectivity function is self-dual) implying that all game forms \( g \in G \) are tight and, hence, Nash-solvable, in accordance with the criteria of [13] and [15].

1.3 Interpretation

Let Alice and Bob be dealers possessing the sets of objects \( A \) and \( B \), respectively, and a deal \((a, b) \in A \times B\) means that they exchange \( a \) and \( b \). They may be art-dealers, car dealers; or one of them may be just a buyer with a discrete budget.

It would be natural to order sets \( A \) and \( B \) in accordance with the values of the objects. This fully justifies the requirement of monotonicity for the strategies \( x \in X \) and \( y \in Y \). Yet, then, it is not natural to assume that the utility functions \( u_A \) and \( u_B \) are arbitrary. Indeed, for a pair of deals \((a, b)\) and \((a', b')\) such that \( a \leq a', b' \leq b \) and at least one of these inequalities is strict, it would be natural to assume that Alice strictly prefers \((a, b)\) to \((a', b')\), while Bob vice versa. Under such assumption, two constant strategies \( x : a \rightarrow b_n \) and \( y : b \rightarrow a_m \) form a trivial NE with the outcome \((a_m, b_n)\). Although, we prove that an equilibrium exists for any \( u_A \) and \( u_B \) (which is non-trivial) but why to consider any?

However, it becomes natural if the values of the deals can be corrected by side-payments. Then, let us also require that the total value of each collection, \( A \) and \( B \), should not be reduced. These total values are estimated subjectively by Alice and Bob (or by some experts) and the side-payments are not taken into account. Such a requirement would suffice to eliminate all strategies except for the monotone non-decreasing ones.

Alternatively, \( A \) and \( B \) are ordered in accordance with some other attribute of the objects, distinct from their commercial value. Such an attribute must be important enough to justify the requirement of monotonicity of the strategies. For example, it could be the shelf life of commodities, or the age of pieces of art. (The age is also a reasonable attribute in case of the matching problems, when \( A \) and \( B \) are brides and grooms, and a deal is a marriage.) We leave to the reader suggesting more practical cases when \( x \) and \( y \) must be monotone non-decreasing, while \( u \) can be arbitrary.

2 Main properties of monotone bargaining

To any pair of strategies \( x : A \rightarrow B \) and \( y : B \rightarrow A \) (not necessarily monotone non-decreasing) let us assign a digraph \( \Gamma = \Gamma(x, y) \) on the vertex-set \( A \cup B \) as follows: \((a, b)\) (respectively, \((b, a)\)) is an arc of \( \Gamma(x, y) \) whenever \( x(a) = b \) (respectively, \( y(b) = a \)).

Some visualization helps. Let us consider an embedding of \( \Gamma(x, y) \) into the plane, as in Figures 1, 2, and 3, where many \((3 \times 3 + 4 \times 6 + 10 \times 10 = 133)\) examples are given. Obviously, no two arcs corresponding to \( x \) may have a common tail, but may have common heads. Furthermore, for a monotone non-decreasing \( x \) the corresponding arcs cannot cross. The same is true for \( y \). However, two arcs corresponding to \( x \) and to \( y \) may cross; also the head of one may coincide with the tail of the other.

Remark 1 In contrast, for a monotone non-increasing mapping, every two of its arcs intersect: either cross or have common heads. Indeed, one turns a monotone non-decreasing mapping \( x \) (respectively, \( y \)) into a monotone non-increasing one \( x' \) (respectively, \( y' \)) just replacing the order of \( B \) (respectively, of \( A \)) by the inverse one.

By construction, digraph \( \Gamma \) is bipartite, with parts \( A \) and \( B \). Hence, every directed cycle (dicycle) in \( \Gamma \) is even. There is an obvious one-to-one correspondence between the dicycles of length 2 (or 2-dicycles, for short) in \( \Gamma(x, y) \) and the deals of \( G(x, y) \). In the figures all 2-dicycles are replaced by the nondirected edges.
Proposition 1 Digraph $\Gamma(x, y)$ contains at least one dicycle of length 2 and no longer dicycles.

Proof. For any initial vertex $v \in A \cup B$, strategies $x$ and $y$ uniquely define an infinite walk from $v$. Since sets $A$ and $B$ are finite, this walk is a lasso, that is, it consists of an initial directed path and a dicycle $C$ repeated infinitely. Furthermore, $C$ is a 2-dicycle whenever mappings $x$ and $y$ are monotone non-decreasing. Indeed, it is easily seen that either $x$, or $y$, or both are not monotone whenever $C$ is longer, since in this case crossing arcs corresponding either to $x$ or to $y$ must appear. □

The next statement results immediately from Proposition 1.

Proposition 2 Game correspondence $G$ is well-defined, that is, $G(x, y) \neq \emptyset$ for all monotone non-decreasing $x \in X$ and $y \in Y$. □

As Figures 1, 2, and 3 show, $G(x, y)$ may contain several deals. These deals can be naturally ordered in accordance with their importance, as the following claim shows.

Proposition 3 For any two deals $(a, b), (a', b') \in G(x, y)$ we have: either $a > a'$ and $b > b'$, or $a < a'$ and $b < b'$.

Proof. Indeed, equalities $a = a'$ or $b = b'$ cannot hold, because any two arcs corresponding to a mapping cannot have a common tail. Also they cannot cross whenever the considered mapping is monotone non-decreasing. The above two observations imply the statement. □

Proposition 4 Any set of pairwise disjoint (nondirected) edges between $A$ and $B$ may form $G(x, y)$ for some monotone non-decreasing $x$ and $y$.

Proof. Consider any $k$ deals $(a_{i_1}, b_{j_1}, \ldots, a_{i_k}, b_{j_k})$ such that $a_{i_1} < \ldots < a_{i_k}$ and $b_{j_1} < \ldots < b_{j_k}$. Define $x(a) = b_{j_i}$ for all $a$ such that $a_{i_{t-1}} < a \leq a_{i_t}$ and $x(b) = a_{i_t}$ for all $b$ such that $b_{j_{t-1}} < b \leq b_{j_t}$, where the first inequalities are redundant for $t = 1$. By construction, $x$ and $y$ are monotone non-decreasing and $G(x, y) = \{(a_{i_1}, b_{j_1}, \ldots, a_{i_k}, b_{j_k})\}$. □

The remaining claims of this subsection we formulate and prove for Alice. Of course, similar statements and proofs hold for Bob as well: just replace $x, X, a, A, i, I$, "row", "Alice", and "she" with $y, Y, b, B, j, J$, "column", "Bob", and "he", respectively, and vice versa.

Proposition 5 Any set of pairwise disjoint (nondirected) edges between $A$ and $B$ corresponding to a monotone non-decreasing $x$ forms $G(x, y)$ for some $y$.

Proof. Define (a monotone non-decreasing) mapping $y$ as in the proof of Proposition 4. □

Choosing a game form $g \in G$ we select one deal for each pair $(x, y)$. The following statement implies that, in a sense, this choice does not matter.

Proposition 6 For any deal $(a, b) \in G(x, y)$ there exists a strategy $x' \in X$ such that $G(x', y)$ consists of a unique deal $\{(a, b)\}$.

Proof. Obviously, the constant mappings $x' : A \rightarrow \{b\}$ satisfies the condition and proves the statement. □

Given a subset of outcomes $W \subseteq O$ and a game correspondence $G : X \times Y \rightarrow 2^{O}$ (respectively, a game form $g : X \times Y \rightarrow O$) we say that Alice is effective for $W$ if she has a strategy $x \in X$ such that $G(x, y) \subseteq W$ (respectively, $g(x, y) \in W$) for any $y \in Y$; or in other words, if there is a row $x$ in which all outcomes are from $W$. We write $E(A, W) = 1$ in this case and $E(A, W) = 0$ otherwise.

For each $W \subseteq O$, the following simple greedy algorithm verifies in time linear in $|O| = mn$ whether Alice is effective for $W$, that is, if $E(A, W) = 1$. For each $a_i \in A$, $i \in I = \{1, \ldots, m\}$, select the minimal $b_j_i \in B$, $j_i \in J = \{1, \ldots, n\}$, if any, such that $(a_i, b_j_i) \in W$ and $b_j_i \geq b_{j_{i-1}}$. (The last condition is redundant for $i = 1$.) If such $b_j_i$ exists for each $i \in I$ then stop and output $E(B, W) = 1$. In this case $x : A \rightarrow B$ such that $x(a_i) = b_j_i$ for all $i \in I$ as a required strategy; furthermore, by construction, $x$ is the lexicographically minimal monotone non-decreasing one. If $b_j_i$ fails to exist for some $i \in I$ then stop and output $E(A, W) = 0$.

However, the last option is impossible when each column contains an outcome from $W$. 

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Given $G : X \times Y \to 2^O$ and $W \subseteq O$, if for each $y \in Y$ there exists an $x \in X$ such that $W \cap G(x, y) \neq \emptyset$ then there exists an $x \in X$ such that $G(x, y) \subseteq W$ for each $y \in Y$.

This claim can be equivalently reformulated in many ways; for example, $E(B, O \setminus W) = 0 \Rightarrow E(A, W) = 1$; see Section 3 for more options.

**Proof** of the theorem. We apply the greedy algorithm verifying that for each $i = 1, \ldots, m$ a required $b_i$, exists for $a_i$.

First, let us consider the following simple case, just as an example. Suppose that $(a_1, b_2), (a_2, b_1) \in W$, while $(a_1, b_1) \notin W$ and $(a_2, b_2) \notin W$ for all $j = 2, \ldots, n$. Then, $E(A, W) = 0$, since $b_2$ does not exist. However, in this case $W$ cannot satisfy the conditions of the theorem. Indeed, let us consider a (monotone non-decreasing) strategy $y \in Y$ such that $y(b_1) = a_1$ and $y(b_2) = a_2$ for all $j = 2, \ldots, n$. Then, either $(a_1, b_1) \in W$ or $(a_2, b_2) \in W$ for at least one $j = 2, \ldots, n$. Thus, $G(x, y) \cap W = \emptyset$ for all $x \in X$, which contradicts the requirements to $W$.

The general case can be analyzed similarly. Suppose that the greedy algorithm assigns $b_j$ to $a_i$ for $i = 1, \ldots, k$, for some $k < m$, but for $a_{k+1}$ there exists no $b = b_{j+1}$ satisfying the required conditions: (l) $(a_{k+1}, b) \in W$ and (ll) $b \geq b_{j+1}$, that is, either (l), or (ll), or both fail for every $b$. In each case we will construct a strategy $y \in Y$ that violate conditions of the theorem. When (l) fails, it is enough to take the constant mapping $y$ that assigns $b$ to every $a \in A$. When (ll) fails, $y$ is chosen in a more sophisticated way. Recursively, for $j = 1, \ldots, k$, assign to each $b_j \in B$ the minimal $a_{i_j} \in A$ such that $(a_{i_j}, b_j) \notin W$ and $a_{i_j} > a_{i_{j-1}}$. In particular, $a_{i_j} = a_{k+1}$ for all $j \geq j_k$.

By construction, $y$ is a monotone non-decreasing mapping such that $W \cap G(x, y) = \emptyset$ for all $x \in X$, which contradicts conditions of the theorem. $\square$

### 3 Tightness and Nash-solvability

#### 3.1 Dual hypergraphs

Given a ground-set $O = \{o_1, \ldots, o_p\}$, consider two hypergraphs

$$C = \{C_x \mid x \in X\} \text{ and } D = \{D_y \mid 2^Y \subseteq 2^O\}.$$

Here $X, Y$, and $O$ are arbitrary finite sets. However, later they will be interpreted as strategies and outcomes, as in Sections 1 and 2. For this reason we keep the notation. Consider the following properties of $C$ and $D$:

- (i) $C_x \cap D_y \neq \emptyset$ for all $x \in X$ and $y \in Y$;
- (i') For any $D' \subseteq O$ such that $D' \cap C_x \neq \emptyset$ for each $x \in X$ there exists an $y \in Y$ such that $D_y \subseteq D'$;
- (i'') For any $C' \subseteq O$ such that $C' \cap D_y \neq \emptyset$ for each $y \in Y$ there exists an $x \in X$ such that $C_x \subseteq C'$;
- (t) $C' \cap D' \neq \emptyset$ whenever $C' \cap D_y \neq \emptyset$ for each $y \in Y$ and $D' \cap C_x \neq \emptyset$ for each $x \in X$;
- (r) For any $x \in X, y \in Y$ and $o \in C_x \cap D_y$ there exist an $x' \in X$ and $y' \in Y$ such that $C_{x'} \cap D_{y'} = C_x \cap D_y = \{o\}$.

**Theorem 2** The following implications hold:

$$((i) \& (t)) \Leftrightarrow ((i) \& (i')) \Leftrightarrow ((i) \& (i'')) \Rightarrow (r). \quad (2)$$

Hypergraphs $C$ and $D$ satisfying the first three equivalent properties are called *dual*. Note that $((i) \& (r))$ does not imply duality yet.

The last implication of Theorem 2 is obvious and all are well-known; see, for example, [1] Part I Section 4.2, where the above properties are formulated in the Boolean language, in terms of the monotone (positive) Conjunctive and Disjunctive Normal Forms; see also [13], [15]. We leave to the reader to verify all implication of Theorem 2.

It is also well-known [11] that duality of $C, D \subseteq 2^O$ is respected by any identification of the elements of $O$. Let us fix a subset $O' \subseteq O$ such that $|O'| > 1$ and replace all $o \in O'$ by one new element $o'$. This operation turns $C$ and $D$ into $C'$ and $D'$, which remain dual whenever $C$ and $D$ were dual. Clearly, such operation can be applied successively several times.
Remark 2 A hypergraph is called Sperner if its edges do not contain one another and, in particular, cannot be equal. Sperner hypergraphs are associated with the irredundant monotone Conjunctive and Disjunctive Normal Forms \([2]\). For any hypergraph \(C\) (respectively, \(D\)) properties (i) and (i') (respectively, (i) and i'') uniquely determine the so-called dual hypergraph \(C_d\) (respectively, \(D_d\)), which is Sperner, by the definition. If \(C\) (respectively, \(D\)) is Sperner itself then duality is an involution, that is, \(C_{dd} = C\) (respectively, \(D_{dd} = D\)).

Although in all our examples only dual pairs of Sperner hypergraphs appear, but we do not assume that this always holds; in particular, we allow hypergraphs to have embedded or equal edges.

### 3.2 Tight game correspondences and game forms

Now, we interpret \(X, Y\) and \(O\) as the sets strategies and outcomes. A mapping \(G : X \times Y \rightarrow 2^O\) assigning a set of outcomes to each pair of strategies is called a game correspondence. A mapping \(g : X \times Y \rightarrow O\) assigning a single outcome of \(O\) to each pair \((x, y)\) is called a game form. We will call \(G' : X \times Y \rightarrow 2^O\) a subcorrespondence of \(G\) and write \(G' \preceq G\) if \(G'(x, y) \subseteq G(x, y)\) for each \(x \in X\) and \(y \in Y\). If \(G' = g\) is a game form, we call it a subform of \(G\) and write \(g \in G\). All these concepts were already considered in Sections 1 and 2 in the special case of monotone bargaining; here we extend them to the general case.

To each game correspondence \(G : X \times Y \rightarrow 2^O\) we assign two hypergraphs

\[
C = \{C_x = \bigcup_{y \in Y} G(x, y) \mid x \in X\}, \quad D = \{D_y = \bigcup_{x \in X} G(x, y) \mid y \in Y\} \subseteq 2^O \quad \text{and}
\]

and call \(G\) tight if they are dual. Note that \(G(x, y) \subseteq C_x \cap D_y\) for every \(x \in X, y \in Y\), by definition, and hence, property (i) holds for \(C\) and \(D\) automatically, since \(G(x, y) \neq \emptyset\).

Game correspondences \(G_1, G_2, G_3, G_4\) in Figures 1–4 have the following four pairs of hypergraphs:

- \(C_1 = \{(o_{11}, o_{21}), (o_{11}, o_{22}), (o_{12}, o_{22})\}\)
- \(D_1 = \{(o_{11}, o_{12}), (o_{11}, o_{22}), (o_{21}, o_{22})\}\)
- \(C_2 = \{(o_{11}, o_{21}, o_{31}), (o_{11}, o_{21}, o_{32}), (o_{11}, o_{22}, o_{32}), (o_{12}, o_{22}, o_{32})\}\)
- \(D_2 = \{(o_{11}, o_{12}), (o_{11}, o_{22}), (o_{21}, o_{22}), (o_{21}, o_{32}), (o_{31}, o_{32})\}\)
- \(C_3 = \{(o_{11}, o_{21}, o_{31}), (o_{11}, o_{21}, o_{32}), (o_{11}, o_{22}, o_{32}), (o_{12}, o_{22}, o_{32}), (o_{11}, o_{22}, o_{33}), (o_{11}, o_{23}, o_{33}), (o_{12}, o_{22}, o_{33}), (o_{12}, o_{23}, o_{33}), (o_{13}, o_{23}, o_{33})\}\)
- \(D_3 = \{(o_{11}, o_{12}, o_{13}), (o_{11}, o_{12}, o_{23}), (o_{11}, o_{22}, o_{23}), (o_{11}, o_{22}, o_{33}), (o_{11}, o_{23}, o_{33}), (o_{11}, o_{32}, o_{33}), (o_{21}, o_{22}, o_{33}), (o_{21}, o_{32}, o_{33}), (o_{31}, o_{32}, o_{33})\}\)
- \(C_6 = \{\{o_1, \ldots, o_k\} \subseteq \{\{0, 0, 0\}, \{0, 0, 1\}, \ldots, \{0, k\}\} \forall k = 1, 2, 3, 4\}\)
- \(D_6 = \{\{o_1, \ldots, o_k\} \subseteq \{\{0, 0, 0\}, \{0, 0, 1\}, \ldots, \{0, k\}\} \forall k = 1, 2, 3, 4\}\)

We leave to the careful reader to

- (j) verify that each pair contains two dual (Sperner) hypergraphs, or in other words, that the eight game correspondences in Figures 1–8 are tight;
- (jj) reformulate all statements of Section 3.1 related to tightness for game correspondences and forms;
- (jii) check that some of them were proven in Section 2 for the case of monotone bargaining.

**Proposition 7** If \(G' \preceq G\) then \(G'\) is tight whenever \(G\) is tight; moreover, in this case \(G\) and \(G'\) have the same pair of dual hypergraphs.

Note that the inverse implication fails. For example, \(G(x, y) = O\) may hold for all \(x \in X\) and \(y \in Y\), in which case \(G\) is not tight unless \(|X| = 1\) or \(|Y| = 1\) or \(|O| = 1\), while \(G'\) may be tight. Note also that \(G'\) may be a game form.
Proof of Proposition \[\mathbb{I}\] If \(G\) is tight then property \((r)\) holds, implying that \(G\) and \(G'\) generate the same pairs of hypergraphs. \(\square\)

Any two hypergraphs \(C\) and \(D\) on the ground-set \(O\) uniquely define the game correspondence \(G = G_{C,D}\) by formula \(G(x,y) = C_x \cap D_y\). By definition, \(G\) is tight if and only if \(C\) and \(D\) are dual. By Proposition \[\mathbb{I}\] in this case every subcorrespondence \(G' \leq G\) and, in particular, each game form \(g \in G\) is tight too; also tightness is preserved by any identifications of the outcomes, as we already mentioned. Note finally that hypergraphs \(C\) and \(D\) need not to be Sperner: each of them may contain embedded and multiple edges.

### 3.3 Nash-solvability

To a game form \(g : X \times Y \to O\), we assign a (normal form) game \((g,u)\) introducing the utility function \(u = (u_A, u_B)\) for the players: \(u_A : O \to \mathbb{R}\) for Alice and \(u_B : O \to \mathbb{R}\) for Bob. A strategy profile \((x,y) \in X \times Y\) of a game \((g,u)\) is called a Nash equilibrium (NE) if

\[u_A(g(x',y)) \leq u_A(g(x,y))\] for all \(x' \in X\); \(u_B(g(x,y')) \leq u_A(g(x,y))\) for all \(y' \in Y\).

A game \((g,u)\) and its utility function \(u = (u_A, u_B)\) are called zero-sum if \(u_A(o) + u_B(o) = 0\) for each \(o \in O\). In this case a Nash equilibrium is usually referred to as a saddle point. A game form \(g\) is called

- Nash-solvable (NS) if game \((g,u)\) has a Nash equilibrium (NE) for any \(u\);
- zero-sum-solvable if \((g,u)\) has a saddle point for every zero-sum \(u\);
- \(\pm 1\)-solvable if \((g,u)\) has a saddle point for every zero-sum \(u\) taking values \(+1\) and \(-1\) only.

**Theorem 3** (\(\mathbb{I}\mathbb{I}\mathbb{I}, \mathbb{I}\mathbb{I}\mathbb{II}\)). The following properties of game forms are equivalent: (S1) NS, (S2) zero-sum-solvability, (S3) \(\pm 1\)-solvability, (T) tightness.

Implications \((S1) \Rightarrow (S2) \Rightarrow (S3)\) are immediate from the definitions.

For the zero-sum case, the equivalence \((T) \iff (S2) \iff (S3)\) was proven by Edmonds and Fulkerson (who called it the “Bottle-Neck Extrema Theorem”) in \(\mathbb{I}\mathbb{I}\mathbb{I}\) and later, independently, in \(\mathbb{I}\mathbb{I}\mathbb{II}\). Then, NS (and the implication \((T) \Rightarrow (S1)\)) was added to the list in \(\mathbb{I}\mathbb{X}\). The same proof was repeated in a more focused paper \(\mathbb{I}\mathbb{I}\mathbb{II}\). This proof is constructive: a NE in the game \((g,u)\) is obtained by an explicit algorithm whenever the game form \(g\) is tight. Unfortunately, this algorithm is exponential in the size of \(O\).

Another proof, based on a polynomial algorithm, appeared later; see \(\mathbb{I}\mathbb{I}\mathbb{I}\) Section 4.3 and \(\mathbb{I}\) Section 3. Some new ideas from \(\mathbb{I}\) were used in this proof. Here, for reader’s convenience, we reproduce it in a stronger and slightly simplified form.

**Proof** of the implication \((T) \Rightarrow (S1)\). We will prove a stronger claim. Given a tight game correspondence \(G : X \times Y \to 2^O\) and an arbitrary utility function \(u = (u_A, u_B)\), we suggest an algorithm that finds a simple NE \((x,y)\) such that \(|G(x,y)| = 1\) in time linear in \(|O|\).

Let us recall notation. Given a subset of outcomes \(W \subseteq O\), we say that Alice is effective for \(W\) if she has a strategy \(x \in X\) such that \(G(x,y) \subseteq W\) for any \(y \in Y\); or in other words, if there is a row \(x\) in which all outcomes are from \(W\), that is, if \(C_x \subseteq W\). We write \(E(A,W) = 1\) in this case and \(E(A,W) = 0\) otherwise. The effectivity function \(E(B,W)\) of Bob is introduced similarly.

Obviously, the equalities \(E(A,W) = E(B,O \setminus W) = 1\) for no \(W \subseteq O\) can hold simultaneously. Indeed, if Alice is effective for \(W\) and Bob for \(O \setminus W\) then \(\emptyset = G(x,y) \subseteq C_x \cap D_y\) for the corresponding strategies \(x \in X\) and \(y \in Y\), but \(G(x,y)\) cannot be empty. Thus, implication \(E(A,W) = 1 \Rightarrow E(B,O \setminus W) = 0\) holds for any \(G\) and, by definition, \(G\) is tight if and only if the inverse implication holds: \(E(A,W) = 0 \Rightarrow E(B,O \setminus W) = 1\).

Consider partition \(O = W \cup W_A \cup W_B\) satisfying the following properties:

- \((a1)\) any outcome of \(W_A\) is not better for Alice than any outcome of \(W\), that is, \(u_A(o') \leq u_A(o)\) for any \(o \in W\) and \(o' \in W_A\);
- \((b1)\) any outcome of \(W_B\) is not better for Bob than any outcome of \(W\), that is, \(u_B(o') \leq u_B(o)\) for any \(o \in W\) and \(o' \in W_B\);
- \((a2)\) \(E(A,W_B) = 0\), that is, Alice cannot punish Bob;
- \((b2)\) \(E(B,W_A) = 0\), that is, Bob cannot punish Allice.
Our arguments will be based on a dynamical process. We begin with \( W = O \) and then reduce \( W \) sending outcomes to \( W_A \) and to \( W_B \). First, let us show that \( W \) cannot be eliminated completely. Indeed, suppose that \( W = \emptyset \) and, hence, \( O = W_A \cup W_B \) is a partition. Then, \((a2), (b2), \) and tightness of \( G \) imply that \( E(A, W_B) = E(B, W_A) = 1 \), which is impossible, as we already mentioned. Thus, the dynamical process of reducing \( W \) will get stuck at some point, say, at a partition \( O = W \cup W_A \cup W_B \).

Let \( o^* \in O \) be the worst outcome in \( W \) for Alice, that is, \( u_A(o^*) \leq u_A(o) \) for each \( o \in W \). We would like to move \( o^* \) from \( W \) to \( W_A \) but cannot. The only possible reason is that \((b2)\) will be violated, that is, \( E(B, W_A \cup \{o^*\}) = 1 \). Denote by \( y^* \) a strategy of Bob that is effective for \((W_A \cup \{o^*\})\).

Moving \( o^* \) alone from \( W \) to \( W_B \) may violate \((b1)\). To keep it, we have to add to \( o^* \) all outcomes \( W_B(o^*) \subseteq W \) that are not better than \( o^* \) for Bob, that is, \( u_B(o^*) \geq u_B(o) \) for all \( o \in W_B(o^*) \); in particular, \( o^* \in W_B(o^*) \). Then, moving \( W_B(o^*) \) from \( W \) to \( W_B \) is OK with \((b1)\), but still cannot be done. The only possible reason is that \((a2)\) is violated, that is, \( E(A, W_B \cup W_B(o^*)) = 1 \). Denote by \( x^* \) a strategy of Alice that is effective for \((W_B \cup W_B(o^*))\).

By construction, \( G(x^*, y^*) = (W_A \cup \{o^*\}) \cap (W_B \cup W_B(o^*)) = \{o^*\} \). \((x^*, y^*)\) is a NE, moreover, it is simple, since \( G(x^*, y^*) \) consists of the single outcome \( o^* \). \( \square \)

## 4 More classes of tight game forms

A criterion of tightness for the game forms associated with veto voting schemes can be found in the textbooks [29] Chapter 6 or [24] Chapter 6; see also [8, 19]. It can be reformulated as a criterion of duality for the monotone (positive) threshold Boolean functions [4, Part II Chapter 9].

Here we will recall five more classes of tight game forms, whose NS follows in all cases from Theorem 3, but sometimes can be shown simpler.

### 4.1 Positional game structures

Let \( \Gamma = (V, E) \) be a digraph whose vertices and arcs are interpreted as positions and moves, respectively. Furthermore, let \( V_A \) and \( V_B \) be positions of positive out-degree, controlled by Alice and Bob, respectively, while \( V_T \) be the set of terminal positions, of out-degree zero. We assume that \( V = V_A \cup V_B \cup V_T \) is a partition. A strategy \( x \in X \) of Alice (resp., \( y \in Y \) of Bob) is a mapping that assigns to each position \( v \in V_A \) (resp., \( v \in V_B \)) an arbitrary move from this position. An initial position \( v_0 \in V_A \cup V_B \) and a pair \((x, y)\) of strategies uniquely define a walk that begins at \( v_0 \) and then follows the decisions made by \( x \) and \( y \). In theory of positional games this walk is called a play. Each play either terminates in \( V_T \) or it is infinite; in the latter case, it must form a lasso: first, an initial path and then a dicycle repeated infinitely.

Positional structures defined above can also be represent in normal form. We will introduce a game form \( g: X \times Y \to O \), where standardly \( O \) denotes a set of outcomes. There are several ways to introduce this set. One is to identify all infinite plays and consider them as a single outcome \( c \); then \( O = V_T \cup \{c\} \). Such games were introduced by Washburn [31] who called them deterministic graphical (DG) games. In this case, for any positional structure the associated game form is tight ([2] Section 3; see also [8, Section 12]) and, hence, it is also NS, as we know.

The following generalization was suggested recently in [18]. Digraph \( \Gamma \) is called strongly connected if for any \( v, v' \in V \) there is a directed path from \( v \) to \( v' \) (and, hence, from \( v' \) to \( v \), as well). By this definition, the union of two strongly connected digraphs is strongly connected whenever they have a common vertex. A vertex-inclusion-maximal strongly connected induced subgraph of \( \Gamma \) is called its strongly connected component. Obviously, any digraph \( \Gamma = (V, E) \) admits a unique decomposition into such components: \( \Gamma^j = \Gamma[V_j] = (V_j, E_j) \) for \( j \in J \), where \( J \) is a set of indices; furthermore \( V = \cup_{j \in J} V_j \) is a partition of \( V \). It can be determined in time linear in the size of \( \Gamma \) (that is, in \( (|V| + |E|) \)) and has numerous applications; see [30] and [29] for more details.

One more application, to positional games was suggested in [18]. For each \( j \in J \), let us contract \( \Gamma^j \) into a single vertex \( v^j \). Then, all edges of \( E_j \) (in particular, the loops) disappear and we obtain an acyclic digraph \( \Gamma^* = (V^*, E^*) \).

In the DG games, all infinite plays (lassos) represent a unique outcome \( c \). In [18], the following weaker assumption was considered. Two plays are equivalent, and represent the same outcome, when the directed cycles of their lassos are contained in the same strongly connected component of \( \Gamma \). Thus, \( J \) becomes the set of outcomes. Such games were called deterministic graphical multi-stage (DGMS). In this case too, for any positional structure, the associated game form is tight [18]. This result DGMS games strengthens the similar one for the DG games, because merging outcomes, obviously, respects tightness and NS.
Every strongly connected component $\Gamma^j$ can be interpreted as a diplomatic issue. (The level may be not very high; see examples in [18]). If this issue is resolved, the play leaves $\Gamma^j$, enters another component, and the parties (players) proceed with the negotiation process. Otherwise, the play cycles in $\Gamma^j$ and negotiations terminate resulting in the outcome $c_j$. It only matters at what issue $\Gamma^j$ the parties have been forced into an impasse, while details (a particular lasso, or a directed cycle in which it ends in $\Gamma^j$) are irrelevant. Thus, a DGMS game is interpreted as a multistage diplomatic negotiations. The main result shows that the corresponding DGMS game form is NS; furthermore, for any payoff a NE in the obtained game can be found in linear time, in other words, any bilateral multistage diplomatic conflict can be efficiently resolved and the resolving strategies can be efficiently determined.

One can also treat each dicycle of $\Gamma$ as a separate outcome; then, $O = V_T \cup C(\Gamma)$. In this case, not all but only some special game forms are tight. Their characterization is not known, in general, but the problem is solved for the symmetric game structures, that is, assuming that $(v, v') \in E$ whenever $(v, v') \in E$. First, tight game forms of the symmetric bipartite game structures were characterized in [9], [10], and [11]. Then, in [3] this result was strengthened by waving the assumption that $\Gamma$ is bipartite.

4.2 Selfdual hypergraphs

A hypergraph $C$ is called selfdual if it is dual to itself. Obviously, each such hypergraph generates a symmetric tight game correspondence. The Fano projective plane provides an example:

$$C = \{(0_0, 0_1, o_6), (0_0, o_2, o_5), (0_0, o_3, o_4), (0_1, o_2, o_4), (o_1, o_3, o_5), (o_2, o_3, o_6), (o_4, o_5, o_6)\}.$$ 

| $\null$ | $0_0$ | $0_1$ | $0_2$ | $0_3$ | $0_4$ | $0_5$ | $0_6$ |
|-------|------|------|------|------|------|------|------|
| $0_0$ | 0_0  | 0_1  | 0_2  | 0_3  | 0_4  | 0_5  | 0_6  |
| $0_1$ | 0_0  | 0_1  | 0_2  | 0_3  | 0_4  | 0_5  | 0_6  |
| $0_2$ | 0_0  | 0_1  | 0_2  | 0_3  | 0_4  | 0_5  | 0_6  |
| $0_3$ | 0_0  | 0_1  | 0_2  | 0_3  | 0_4  | 0_5  | 0_6  |
| $0_4$ | 0_0  | 0_1  | 0_2  | 0_3  | 0_4  | 0_5  | 0_6  |
| $0_5$ | 0_0  | 0_1  | 0_2  | 0_3  | 0_4  | 0_5  | 0_6  |
| $0_6$ | 0_0  | 0_1  | 0_2  | 0_3  | 0_4  | 0_5  | 0_6  |

Figure 4. Game correspondence of the Fano plane.

To every pair of hypergraphs $C$ and $D$ defined on the same ground-set $O$, assign the hypergraph $E = \{(C, D, (c, d))\}$, where $c$ and $d$ are two new objects, $c, d \not\in O$. To get $E$, we add $c$ (resp., $d$) to each edge of $C$ (resp., of $D$) and also add one extra edge $(c, d)$. Seymour noticed that $C$ and $D$ are dual if and only if $E$ is selfdual [28].

Let us define a large class of selfdual hypergraphs. Given $p = k + 1 \geq 3$ and the ground-set $\{o_0, o_1, \ldots, o_k\}$, the $k$-wheel is defined as the hypergraph

$$W_k = \{(o_0, o_1), (o_0, o_2), \ldots, (o_0, o_k), (o_1, o_2), \ldots, (o_k)\}.$$ 

It is easily seen that $W_k$ is selfdual for any $k \geq 2$.

Consider the following interpretation. Two players are choosing among $k + 1$ options $\{o_0, o_1, \ldots, o_k\}$, where $o_0$ is a special one, which means that players did not come to an agreement. Each player has $k + 1$ strategies, either to suggest an option, except $o_0$, or to say “any”. The game correspondence is defined as follows:

- If they suggest $o_i$ and $o_j$ (recall that $i, j \in \{1, \ldots, k\}$) then $o_0$ is chosen when $i \neq j$ and either $o = o_i = o_j$ or $o_0$ is chosen when $i = j$.
- If one suggest $o_i$, $i = 1, \ldots, k$, while the other “any”, then $o_i$ is chosen.
- If both say “any” then each option, except $o_0$, can be chosen.
For \( k = 3 \) the game correspondence is given on the following table.

| \( o_0o_1 \) | \( o_0o_2 \) | \( o_0o_3 \) | \( o_1o_2o_3 \) |
|-----------------|-----------------|-----------------|-----------------|
| \( o_0o_1 \)    | \( o_0o_1 \)    | \( o_0 \)       | \( o_0 \)       |
| \( o_0o_2 \)    | \( o_0 \)       | \( o_0o_2 \)    | \( o_0 \)       |
| \( o_0o_3 \)    | \( o_0o_3 \)    | \( o_0o_3 \)    | \( o_0 \)       |
| \( o_1o_2o_3 \) | \( o_1 \)       | \( o_2 \)       | \( o_3 \)       |

Figure 5. Game correspondence of the wheel, \( C = D = W_3 \).

For each \( k \geq 2 \), by Theorem 3 such game correspondence is NS. Moreover, we proved that a simple NE exists for every payoff. This proof can be significantly simplified in the considered case.

Let player 1 (resp., 2) chooses his (her) best option \( o_i \) (resp., \( o_j \)) among \( o_1, \ldots, o_k \), while player 2 (resp., 1) says “any”. Then \( o_i \) (resp., \( o_j \)) in the last column (resp., row) will be a simple NE unless \( o_0 \) is better for the opponent. We can assume that the latter holds for both players, since in all other cases we are done. If \( i \neq j \) then the intersection of the row \( i \) and column \( j \) form a simple NE with the outcome \( o_0 \). Interestingly, the most difficult case is \( i = j \), when the same option is the best for both. The corresponding diagonal entry is always a NE with the outcome either \( o = o_i = o_j \) or \( o_0 \), but this NE is not simple. Yet, the case is still simple. Indeed, since \( o_0 \) is the best option for both players, any simple strategy profile with the outcome \( o_0 \) (and there exist \( k(k - 1) \) of them) is a NE.

4.3 Symmetric dual hypergraphs

Let \( k \) and \( \ell \) be two positive integers and \( \mathcal{C} \) and \( \mathcal{D} \) be symmetric hypergraphs of dimension \( k \) and \( \ell \) on the ground-set \( O = \{ o_1, \ldots, o_p \} \) such that \( p = k + \ell - 1 \). In other words, \( \mathcal{C} \) (resp., \( \mathcal{D} \)) consists of all \( k \)-subsets (resp., \( \ell \)-subsets) of \( O \). It is well-known that \( \mathcal{C} \) and \( \mathcal{D} \) are dual; see, for example, [1, Part I Chapter 4].

A natural interpretation can be given in terms of veto voting. There are \( p \) candidates and two voters with veto power \( p - k \) and \( p - \ell \); see [21, Chapter 6], [24, Chapter 6], or [8]. For \( k = 3 \) and \( \ell = 2 \) the game correspondence is given on the following table.

| \( 12 \) | \( 13 \) | \( 14 \) | \( 23 \) | \( 24 \) | \( 34 \) |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 123                 | 12                  | 13                  | 14                  | 23                  | 24                  |
| 124                 | 12                  | 13                  | 14                  | 23                  | 24                  |
| 134                 | 13                  | 13                  | 14                  | 34                  | 34                  |
| 234                 | 23                  | 23                  | 34                  | 34                  | 34                  |

Figure 6. Game correspondence of the dual symmetric hypergraphs of dimensions \( k = 3 \) and \( \ell = 2 \) on the ground-set of size \( p = 4 \) represents veto voting with \( p = 4 \) candidates and \( n = 2 \) voters of the veto powers \( p - k = 1 \) and \( p - \ell = 2 \), respectively.

For any positive integer \( k \) and \( \ell \), a simple NE exists; see the proof of Theorem 3. It can be simplified in the considered special case. Without loss of generality, we may assume that voter 1 prefers the candidates from \( O \) in the order \( u_1(o_1) \leq \ldots \leq u_1(o_p) \) and, moreover, that all inequalities here are strict. He can veto \( k \) his worst candidates and guarantee \( o_{k+1} \) or better. If the game is zero-sum, voter 2 just veto her \( \ell \) worst candidates \( O_2 = \{ o_{k+2}, \ldots, o_p \} \). Obviously, these two strategies form a simple NE, electing the centrist \( o_{k+1} \).

If the game is not zero-sum then among \( \ell \) candidates of \( O_2 \) may exist \( r \) that are better than \( o_{k+1} \) for voter 2 (and for voter 1 as well). Note that \( r = 0 \) in the zero-sum case). Let us choose among these \( r \) candidates one with the largest index, say, \( o_m \); in other words, the best one for both voters. By construction, there are two sets \( C, D \subseteq O \) such that \( |C| = k, |D| = \ell, C \cap D = \{ o_m \}, \) and \( o_m \) is better than any \( o \in C \) for voter 1 and any \( o \in D \) for voter 2. Thus, strategy profile \( (C, D) \) forms a simple NE.

4.4 Jordan dual hypergraphs. Choosing Battlefields in Wonderland

Wonderland is a subset of the plane homeomorphic to the closed disc. Without loss of generality, one can choose a square \( Q \) with the sides \( N, E, S, W \), as in Figure 7. Let us partition \( Q \) into areas \( O = \{ o_1, \ldots, o_p \} \) each of which is homeomorphic to the closed disc, too. Any two distinct areas \( o_i, o_j \in O \) are either disjoint or intersect in a set homeomorphic to a closed interval that contains more than one point. Equivalently, we can require that the borders of the areas of \( O \) form a regular graph of degree 3, as in Figure 7.
Two players, Tweedledee and Tweedledum, agreed to have a battle. The next thing to do is to choose a battlefield, which should be an area \( o \in O \). The strategies \( x \in X \) of Tweedledee (resp., \( y \in Y \) of Tweedledum) are all inclusion-minimal subsets \( x \subseteq O \) (resp., \( y \subseteq O \) connecting \( N \) and \( S \) (resp., \( E \) and \( W \)). The game correspondence \( G : X \times Y \to O \) defined by the equality \( G(x, y) = x \cap y \subseteq O \) will be called \textit{Jordan}; see again Figure 7 for an example. The following “topological” result is well-known.

**Theorem 4** Any Jordan game correspondence \( G \) is well defined (that is, \( G(x, y) \neq \emptyset \)) and tight.

**Proof.** It is easily seen that any two pathes in \( Q \), one connecting \( N \) with \( S \) and another connecting \( E \) with \( W \), intersect. This is one of the versions of the Jordan curve theorem. (Note that the claim fails when vertices of degree larger than 3 are allowed in \( Q \).) Hence, \( G(x, y) = x \cap y \neq \emptyset \) for any \( x \) and \( y \). Moreover, it follows from the same theorem that a subset of \( Q \) blocking all pathes between \( N \) and \( S \) must contain a path between \( E \) and \( W \), and vice versa. As we know, each of these two properties is equivalent with tightness of \( G \). □

Thus, all results of Sections 2 and 3 related to NS follow. Let us note that in this case we do not know any simpler way than to derive the existence of a simple NE from Theorem 4.

**Remark 3** We required inclusion-minimality for the subsets \( x, y \in O \) just to reduce the number of strategies. This requirement can be waved, but Theorem 4 with all its corollaries related to NS still hold.

## 5 Three-person monotone bargaining may be not tight and not NS

Let us introduce Claire to Alice and Bob and consider three sets of objects

\[
A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_n\}, C = \{c_1, \ldots, c_k\},
\]

and the set of outcomes (or possible deals) \( O = A \times B \times C \).

The strategies of Alice, Bob, and Clair are all monotone non-decreasing mappings \( x : A \to B, y : B \to C, \) and \( z : C \to A \). Given a triplet of strategies \( (x, y, z) \in X \times Y \times Z \), naturally, a triplet of objects \( (a, b, c) \in A \times B \times C \) is called a deal if \( x(a) = b, y(b) = c, \) and \( z(c) = a \).

**Proposition 8** Every triplet \( (x, y, z) \) generates at least one deal.

**Proof.** We naturally assign to \( (x, y, z) \) a tripartite graph \( \Gamma = \Gamma(x, y, z) \) with the vertex-set \( V = A \cup B \cup C \) and all arcs \( (a, b), (b, c), \) and \( (c, a) \) such that \( x(a) = b, y(b) = c, \) and \( z(c) = a \). Then, any initial vertex \( v_0 \in V \) uniquely defines a walk. Clearly, this walk is a lasso ending in a 3t-dicycle; moreover, \( t = 1 \) follows from the fact that \( x, y, \) and \( z \) are monotone non-decreasing he mappings. □

Thus, \( G(x, y, z) \neq \emptyset \) for all triplets \( (x, y, z) \) and we obtain a game correspondence \( G : X \times Y \times Z \to 2^O \). Yet, already for \( m = n = k = 2 \) the correspondence \( G_{2,2,2} \) in Figure 8 is not tight and not NS. Indeed, it is not difficult to see that Alice’s and Bob-Claire’s hypergraphs
\[ \mathcal{H}_A = \{(0_{111}, 0_{112}, 0_{121}, 0_{122}), (0_{111}, 0_{121}, 0_{212}, 0_{222}), (0_{211}, 0_{212}, 0_{221}, 0_{222})\}; \\
\mathcal{H}_{BC} = \{(0_{111}, 0_{211}), (0_{112}, 0_{212}), (0_{111}, 0_{221}), (0_{112}, 0_{222}), (0_{121}, 0_{221}), (0_{121}, 0_{222})\}. \\
\]

are not dual. For example, \((0_{121}, 0_{212})\) and \((0_{121}, 0_{212})\) are transversal to \(\mathcal{H}_A\) that do not appear in \(\mathcal{H}_{BC}\).

NS fails too, for example, for any utility \(u = (u_A, u_B, u_C)\) such that
\[
\begin{align*}
&u_A(212) > u_A(222) > u_A(111), u_A(121) > u_A(111), u_A(122) > u_A(112); \\
&u_B(112) > u_B(111), u_B(121) > u_B(122), u_B(222) > u_B(221), u_B(212) > u_B(211); \\
&u_C(111) > u_C(222), u_C(122) > u_C(122), u_C(112) > u_C(212), u_C(221) > u_C(121).
\end{align*}
\]
Figure 8. A monotone bargaining three person game game correspondence which is not tight and not NS,

In this figure 8, an upper index A, B, or C over a strategy profile \((x, y, z)\), shows which player, Alice, Bob, or Claire, could improve \((x, y, z)\) by changing his/her strategy. Since each profile has an upper index, we conclude that the game has no NE for any realization of the considered partially defined utility function \(u = (u_A, u_B, u_C)\).

Let us note that in the three-person case tightness and NS are no longer related. Tightness is neither necessary nor sufficient for NS; see examples in [13] and [15].

**Remark 4** In [13, Remark 3] it was mistakenly claimed that tightness is necessary for NS even in the \(n\)-person case. This mistake was noticed by Danilov and corrected in [15].

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