Some Properties of the Topological Entropy of a Family of Dynamical Systems Defined on an Arbitrary Metric Space

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Abstract—We consider a family of dynamical systems defined on a noncompact metric space and continuously depending on a parameter varying in some metric space. For any such family, the topological entropy of the dynamical systems in the family is studied as a function of the parameter from the viewpoint of the Baire classification of functions.

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The topological entropy of autonomous dynamical systems on an invariant compact metric space was defined in [1]. Later, this concept was extended in [2] to dynamical systems defined on an arbitrary metric space.

1. BAIRE CLASS OF THE TOPOLOGICAL ENTROPY OF A FAMILY OF DYNAMICAL SYSTEMS IN THE CASE OF A NONINVARIANT COMPACT SET

Following [2], let us give a definition needed in what follows. Let $(X,d)$ be a metric space, let $K(X)$ be the set of compact subsets of $X$, and let $f : X \to X$ be a continuous mapping. Along with the original metric $d$, we define an additional system of metrics $d^n_f$, $n \in \mathbb{N}$, on $X$ by the formula

$$d^n_f(x,y) = \max_{0 \leq i \leq n-1} d(f^{\circ i}(x), f^{\circ i}(y)), \quad x, y \in X, \; n \in \mathbb{N},$$

where $f^{\circ i}$, $i \in \mathbb{N}$, is the $i$th iteration of $f$ and $f^0 \equiv \text{id}_X$. Given a $K \in K(X)$, for any $n \in \mathbb{N}$ and $\varepsilon > 0$ we denote by $N_d(K,f,\varepsilon,n)$ the maximum number of points in $K$ with pairwise $d^n_f$-distances greater than $\varepsilon$. The numbers

$$\overline{h}_{\text{top}}(K,f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln N_d(K,f,\varepsilon,n) \quad \text{and} \quad \underline{h}_{\text{top}}(K,f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln N_d(K,f,\varepsilon,n) \quad (1)$$

are called the upper and lower topological entropies, respectively, of $f$ on $K$.

Note that if the metric $d$ is replaced by a metric generating the same topology as $d$, then the numbers (1) do not change [3, p. 121 of the Russian translation].

Let us also recall formulas for the upper and lower topological entropies, to be used below. For any $x \in X$, $\varepsilon > 0$, and $n \in \mathbb{N}$, let $B_f(x,\varepsilon,n)$ be the open ball $\{y \in K : d^n_f(x,y) < \varepsilon\}$ of radius $\varepsilon$ centered at $x$ in the space $(X,d^n_f)$. A set $D \subset K$ is called an $(f,\varepsilon,n)$-cover of $K$ if

$$K \subset \bigcup_{x \in D} B_f(x,\varepsilon,n).$$

Let $S_d(K,f,\varepsilon,n)$ be the minimum possible number of elements in an $(f,\varepsilon,n)$-cover of $K$. Then the upper and lower topological entropies of $f$ on $K$ can be calculated by the formulas [3, p. 122 of the Russian translation]

$$\overline{h}_{\text{top}}(K,f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln S_d(K,f,\varepsilon,n), \quad \underline{h}_{\text{top}}(K,f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln S_d(K,f,\varepsilon,n). \quad (2)$$
Formulas (1) (or (2)) obviously imply the inequality $\bar{h}_{\text{top}}(K, f) \geq h_{\text{top}}(K, f)$. If $K$ is $f$-invariant, i.e., $f(K) \subset K$, then the upper and lower topological entropies of $f$ on $K$ coincide [3, p. 122 of the Russian translation]. The following example shows that the numbers (1) do not necessarily coincide in the general case. Consider the set $\Omega_2$ of sequences $x = (x_1, x_2, x_3, \ldots)$, where $x_i \in \{0, 1\}$, with the metric

$$d_{\Omega_2}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/\min\{i : x_i \neq y_i\} & \text{if } x \neq y. \end{cases}$$

Note that the space $(\Omega_2, d_{\Omega_2})$ is homeomorphic to the Cantor set on the interval $[0,1]$ with the metric induced by the standard metric of the real line. Let $K_0 \subset \Omega_2$ be the compact set defined by the condition

$$(x_1, x_2, x_3, \ldots) \in K_0 \iff x_i = 0, \quad i \in \bigcup_{k \in \mathbb{N}\setminus\{0\}} \{(2k)!, \ldots,(2k+1)!\},$$

and let $\sigma : \Omega_2 \to \Omega_2$ be the left shift mapping, $\sigma((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots)$. Then

$$\bar{h}_{\text{top}}(K_0, \sigma) = \ln 2, \quad h_{\text{top}}(K_0, \sigma) = 0.$$

Given a metric space $M$, a compact set $K \subset X$, and a continuous mapping

$$f : M \times X \to X,$$  (3)

consider the functions

$$\mu \mapsto \bar{h}_{\text{top}}(K, f(\mu, \cdot)), \quad (4)$$

$$\mu \mapsto h_{\text{top}}(K, f(\mu, \cdot)). \quad (5)$$

In the present paper, for each mapping (3) the functions (4) and (5) are studied from the viewpoint of the Baire classification of functions. Recall that continuous functions $M \to \mathbb{R}$ on a metric space $M$ are called functions of Baire class 0, and the functions of Baire class $p$ are defined for each positive integer $p$ as the functions that are pointwise limits of sequences of functions of Baire class $p - 1$.

We have already mentioned that the numbers (1) coincide if $K$ is $f$-invariant; in that case, their common value is called the topological entropy of $f$ and is denoted by $h(f)$. In was established in the paper [4] that for each mapping (3) the function

$$\mu \mapsto h(f(\mu, \cdot)) \quad (6)$$

is of Baire class 2 on $M$. In [5], a family of homeomorphisms (3) with $X = M = \Omega_2$ was constructed such that the function (6) is not of Baire class 1; consequently, the functions (4) and (5), generally speaking, are not of Baire class 1 either. It was shown in [4] that if the space $M$ is metrizable by a complete metric, then the set of points of lower semicontinuity of the function (6) contains a $G_δ$ set everywhere dense in $M$, while the paper [6] established that the set of points of lower semicontinuity itself is an everywhere dense $G_δ$ set in $M$. Further, let $X = \Omega_2$, and let $M$ be an arbitrary complete metric separable zero-dimensional space (for example, $\Omega_2$). In this case, for an arbitrary $G_δ$ set $G$ everywhere dense in $M$, the paper [7] presents the construction of a mapping (3) such that the set of points of lower semicontinuity of the corresponding function (6) coincides with $G$. It turns our that the following assertion holds in the case of a noninvariant compact set $K$.

**Theorem 1.** For any $K \in \mathcal{K}(X)$ and any mapping (3), the function (4) is of Baire class 3 and the function (5) is of Baire class 2 on $M$. If $M$ is metrizable by a complete metric, then the set of points of lower semicontinuity of the function (5) is an everywhere dense $G_δ$ set.

**Proof.** For any $\varepsilon > 0$ and $n \in \mathbb{N}$, the function $\mu \mapsto n^{-1} \ln N_{\bar{d}}(K, f(\mu, \cdot), \varepsilon, n)$ is lower semicontinuous [8] and the function $\mu \mapsto n^{-1} \ln S_{\bar{d}}(K, f(\mu, \cdot), \varepsilon, n)$ is upper semicontinuous [4];
consequently [9, Ch. IX, Sec. 37, XI of the Russian translation], there exist sequences of continuous functions \( \mu \mapsto \varphi_d^m(K; \mu, \varepsilon, n) \) and \( \mu \mapsto \psi_d^m(K; \mu, \varepsilon, n) \), \( m \in \mathbb{N} \), on \( \mathcal{M} \) such that

\[
\frac{1}{n} \ln N_d(K, f(\mu, \cdot), \varepsilon, n) = \sup_{m \in \mathbb{N}} \varphi_d^m(K; \mu, \varepsilon, n), \quad \mu \in \mathcal{M},
\]

(7)

\[
\frac{1}{n} \ln S_d(K, f(\mu, \cdot), \varepsilon, n) = \inf_{m \in \mathbb{N}} \psi_d^m(K; \mu, \varepsilon, n), \quad \mu \in \mathcal{M}.
\]

(8)

Hence we obtain the representations

\[
\bar{h}_{\text{top}}(K, f(\mu, \cdot)) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln N_d(K, f(\mu, \cdot), \varepsilon, n) = \sup_{k \in \mathbb{N}} \sup_{m \in \mathbb{N}} \lim_{n \to \infty} \varphi_d^m(K; \mu, 1/k, n)
\]

(9)

\[
= \sup_{k \in \mathbb{N}} \inf_{n \in \mathbb{N}} \sup_{l \geq n} \varphi_d^m(K; \mu, 1/k, l)
\]

(10)

\[
= \lim_{p \to \infty} \max_{1 \leq k \leq p} \lim_{q \to \infty} \min_{1 \leq l \leq q} \varphi_d^m(K; \mu, 1/k, l)
\]

(11)

by (7) and

\[
\underline{h}_{\text{top}}(K, f(\mu, \cdot)) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln S_d(K, f(\mu, \cdot), \varepsilon, n) = \sup_{k \in \mathbb{N}} \inf_{m \in \mathbb{N}} \lim_{n \to \infty} \psi_d^m(K; \mu, 1/k, n)
\]

(12)

\[
= \sup_{k \in \mathbb{N}} \sup_{n \in \mathbb{N}} \inf_{m \in \mathbb{N}} \psi_d^m(K; \mu, 1/k, l)
\]

(13)

\[
= \lim_{p \to \infty} \max_{1 \leq k \leq p} \lim_{q \to \infty} \min_{1 \leq l \leq q} \psi_d^m(K; \mu, 1/k, l)
\]

(14)

by (8).

Since the maximum and minimum of finitely many functions in a Baire class belong to the same class [9, Ch. IX, Sec. 37, III of the Russian translation], it obviously follows from these representations that the function \( \mu \mapsto \bar{h}_{\text{top}}(K, f(\mu, \cdot)) \) is of Baire class 3 and the function \( \mu \mapsto \underline{h}_{\text{top}}(K, f(\mu, \cdot)) \) is of Baire class 2 on \( \mathcal{M} \).

Since the function \( \mu \mapsto \underline{h}_{\text{top}}(K, f(\mu, \cdot)) \) can be represented as the limit of a nondecreasing sequence of functions of Baire class 1, we see that its set of points of lower semicontinuity is an everywhere dense \( G_\delta \) set [10, Lemma 2]. The proof of the theorem is complete.

Note that if \( \mathcal{M} \) is a complete metric space, then, by Baire’s theorem [9, Ch. IX, Sec. 39, VI of the Russian translation], Theorem 1 implies that for each mapping (3) there exists an everywhere dense \( G_\delta \) set \( G \subset \mathcal{M} \) such that the restrictions of the functions \( \mu \mapsto \underline{h}_{\text{top}}(K, f(\mu, \cdot)) \) and \( \mu \mapsto \bar{h}_{\text{top}}(K, f(\mu, \cdot)) \) to \( G \) are continuous.

There arises a natural question about the least Baire class containing the function (4). To answer this question, we construct metric spaces \( \mathcal{B} \) and \( \mathcal{C} \). By definition, the points of \( \mathcal{B} \) are all possible (countable) sequences \( \mu = (\mu_k)_{k=1}^\infty \) of positive integers. The distance between two points \( \mu \) and \( \nu \) is defined by the formula

\[
d_{\mathcal{B}}(\mu, \nu) = \begin{cases} 0 & \text{if } \mu = \nu \\ 1/\min\{k : \mu_k \neq \nu_k\} & \text{if } \mu \neq \nu. \end{cases}
\]

Note that the space \((\mathcal{B}, d_{\mathcal{B}})\) is homeomorphic to the set of irrational numbers on the interval \([0,1]\) with the metric induced by the natural metric of the real line. The points of \( \mathcal{C} \) are all possible pairs \((x, i)\), where \( x \in [0,1] \) and \( i \in \mathbb{N} \). The distance between points \((x, i)\) and \((y, j)\) is defined by the formula

\[
d_{\mathcal{C}}((x, i), (y, j)) = \begin{cases} |x - y| & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases}
\]

For each \( r \in \mathbb{N} \), let \( K_r \in \mathcal{K}(\mathcal{C}) \) be the compact set \( K_r = [0,1] \times \{1, \ldots, r\} \) in \( \mathcal{C} \).
Theorem 2. Let $\mathcal{M} = \mathcal{B}$, $X = \mathcal{C}$, and $K = K_r$; then there exists a mapping (3) such that the function (4) is everywhere discontinuous and does not belong to the second Baire class on $\mathcal{M}$.

Proof. Given a sequence $\mu = (\mu_k)_{k=1}^{\infty} \in \mathcal{B}$, we construct the sequence $\alpha(\mu)$ with elements $\alpha_k(\mu) = \mu[\log_2(k+1)]$ (where $\lfloor \cdot \rfloor$ is the integer part of a number). Consider the sequence $(f_k)$ of mappings of $\mathcal{B} \times [0,1]$ into $[0,1]$ given by

$$f_k(\mu, x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 - 1/\alpha_k(\mu) \\ 2x - 1 + 1/\alpha_k(\mu) & \text{if } 1 - 1/\alpha_k(\mu) < x \leq 1 - 1/(2\alpha_k(\mu)) \\ -2x + 3 - 1/\alpha_k(\mu) & \text{if } 1 - 1/(2\alpha_k(\mu)) < x \leq 1. \end{cases}$$

We use this sequence to construct a mapping $f : \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ by setting

$$f(\mu, (x,k)) = (f_k(\mu, x), k+1).$$

(9)

By definition, the function $f$ is continuous on $\mathcal{B} \times \mathcal{C}$.

Let $\mathcal{E} \subset \mathcal{B}$ be the set of sequences tending to infinity. Let us find the upper topological entropy of the mapping (9) for $\mu \in \mathcal{E}$.

Lemma 1. If $\mu \in \mathcal{E}$, then

$$\overline{h}_{exp}(K_r, f(\mu, \cdot)) = 0$$

for the mapping (9) for any $r \in \mathbb{N}$.

Proof. Fix an $\varepsilon \in (0,1)$ and a $\mu \in \mathcal{E}$. Then there exists a number $k_0(\varepsilon) > r$ such that $1/\alpha_k(\mu) < \varepsilon/2$ for each $k \geq k_0(\varepsilon)$.

Let $A_{k_0(\varepsilon)}$ be an $(f(\mu, \cdot), \varepsilon/2, k_0(\varepsilon))$-cover of $K_r$ with minimum number of elements. Let us prove that $A_{k_0(\varepsilon)}$ is an $(f(\mu, \cdot), \varepsilon, k_0(\varepsilon) + i)$-cover of $K_r$ for each $i \in \mathbb{N} \cup \{0\}$.

By the definition of $A_{k_0(\varepsilon)}$, for each point $(x,l) \in K_r$ there exists an element $(x_0,l) \in A_{k_0(\varepsilon)}$ such that $(x,l) \in B_{f(\mu, \cdot)}((x_0,l), \varepsilon/2, k_0(\varepsilon))$.

If $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x,l))$, $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0,l)) \in [0,1-\varepsilon/2] \times \mathbb{N}$, then

$$d_C(f^{\circ(k_0(\varepsilon)+i)}(\mu, (x,l)), f^{\circ(k_0(\varepsilon)+i)}(\mu, (x_0,l))) = d_C(f^{\circ(k_0(\varepsilon)-1)}(\mu, (x,l)), f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0,l))) < \varepsilon/2$$

(10)

for each $i \in \mathbb{N} \cup \{0\}$, because the interval $[0,1-\varepsilon/2]$ is invariant with respect to the mapping $f_{k_0(\varepsilon)+i}(\mu, \cdot)$ for all $i \in \mathbb{N} \cup \{0\}$.

If $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x,l))$, $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0,l)) \in [1-\varepsilon/2,1] \times \mathbb{N}$, then

$$d_C(f^{\circ(k_0(\varepsilon)+i)}(\mu, (x,l)), f^{\circ(k_0(\varepsilon)+i)}(\mu, (x_0,l))) \leq \varepsilon/2$$

(11)

for each $i \in \mathbb{N} \cup \{0\}$, because the interval $[1-\varepsilon/2,1]$ is invariant with respect to the mappings $f_{k_0(\varepsilon)+i}(\mu, \cdot)$ for all $i \in \mathbb{N} \cup \{0\}$.

If either $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x,l)) \in [0,1-\varepsilon/2] \times \mathbb{N}$ and $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0,l)) \in [1-\varepsilon/2,1] \times \mathbb{N}$ or $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0,l)) \in [0,1-\varepsilon/2] \times \mathbb{N}$ and $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x,l)) \in [1-\varepsilon/2,1] \times \mathbb{N}$, then

$$d_C(f^{\circ(k_0(\varepsilon)+i)}(\mu, (x,l)), f^{\circ(k_0(\varepsilon)+i)}(\mu, (x_0,l))) \leq d_C(f^{\circ(k_0(\varepsilon)-1)}(\mu, (x,l)), f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0,l)))+\varepsilon/2 < \varepsilon$$

(12)

for each $i \in \mathbb{N} \cup \{0\}$.

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It follows from inequalities (10)–(12) that \((x,l) \in B_{f(\mu, \cdot)}((x_0, l), \varepsilon,k_0(\varepsilon) + i)\) for each \(i \in \mathbb{N} \cup \{0\}\), and hence the set \(A_{k_0(\varepsilon)}\) is an \((f(\mu, \cdot), \varepsilon,k_0(\varepsilon) + i)\)-cover of \(K_r\) for each \(i \in \mathbb{N} \cup \{0\}\). Thus, the estimate

\[
S_{dc}(K_r, f(\mu, \cdot), \varepsilon,n) \leq S_{dc}(K_r, f(\mu, \cdot), \varepsilon/2, k_0(\varepsilon))
\]

holds for \(n \geq k_0(\varepsilon)\), which implies that

\[
\overline{h}_{top}(K_r, f(\mu, \cdot)) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln S_{dc}(K_r, f(\mu, \cdot), \varepsilon,n) \\
\leq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln S_{dc}(K_r, f(\mu, \cdot), \varepsilon/2, k_0(\varepsilon)) = 0.
\]

The proof of the lemma is complete.

Now let us estimate the upper topological entropy of the mapping (9) for \(\mu \notin \mathcal{E}\).

**Lemma 2.** If \(\mu \notin \mathcal{E}\), then

\[
\overline{h}_{top}(K_1, f(\mu, \cdot)) \geq \frac{1}{4} \ln 2
\]

for the mapping (9).

**Proof.** Let \(\mu \notin \mathcal{E}\). Then there exists a subsequence \((\mu_k)_\infty \subseteq (\mu_k)_\infty \) and a positive integer \(q\) such that \(\mu_{k_j} = q\) for all \(j \in \mathbb{N}\).

Further, \(f_k(\mu,x) = f_{2^{k_j} - 1}(\mu,x) = t_q(x)\) for all \(j \in \mathbb{N}, \ k \in \{2^{k_j} - 1, \ldots, 2^{k_j+1} - 2\}\), and \(x \in [0,1]\), where

\[
t_q(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq 1 - 1/q \\
  2x - 1 + 1/q & \text{if } 1 - 1/q < x \leq 1 - 1/(2q) \\
  -2x + 3 - 1/q & \text{if } 1 - 1/(2q) < x \leq 1.
\end{cases}
\]

The affine order-preserving transformation \(\varphi\) mapping the interval \(I_q = [1 - 1/q, 1]\) onto the interval \([0, 1]\) takes the mapping \(t_q|_q : I_q \to I_q\) to the mapping \(g = \varphi \circ t_q|_q \circ \varphi^{-1} : [0, 1] \to [0, 1]\) given by the formula

\[
g(x) = \begin{cases} 
  2x & \text{if } 0 \leq x \leq 1/2 \\
  2 - 2x & \text{if } 1/2 < x \leq 1.
\end{cases}
\]

It was established in the monograph [3, p. 502 of the Russian translation] that the topological entropy of \(g\) is \(\ln 2\); consequently, there exists an \(\varepsilon_0 < 1/q\) such that

\[
\lim_{n \to \infty} \frac{1}{n} \ln N_d([0,1], g, \varepsilon, n) \geq \frac{1}{2} \ln 2, \quad d(x, y) = |x - y|,
\]

for every \(\varepsilon < \varepsilon_0\). For each \(n \in \mathbb{N}\), consider a set \(\{a_1, \ldots, a_{\mathcal{N}_d([0,1], g, \varepsilon, n)}\}\) of points on the interval \([0, 1]\) with pairwise \(d^{\mu}_{\infty}\)-distances greater than \(\varepsilon > 0\).

Let \(\varepsilon < \varepsilon_0\) and \(n = 2^{k_j+1} - 2^{k_j} - 1\). Then the \(d^{f(\mu, \cdot)}_{2^{k_j+1} - 2^{k_j}}\)-distance between any preimages of two arbitrary points \((\varphi^{-1}(a_i), 2^{k_j} - 1)\) and \((\varphi^{-1}(a_m), 2^{k_j} - 1)\), \(i \neq m\), under the mapping \(f^{(2^k - 2)(\mu, \cdot)}\) is greater than \(\varepsilon/q > 0\), and consequently,

\[
N_{dc}(K_1, f(\mu, \cdot), \varepsilon/q, 2^{k_j+1}) \geq N_d([0,1], g, \varepsilon, 2^{k_j+1} - 2^{k_j}).
\]

Hence we obtain the estimates

\[
\overline{h}_{top}(K_1, f(\mu, \cdot)) \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{(2^{k_j+1} - 2^{k_j})}{2^{k_j+1}} \frac{1}{(2^{k_j+1} - 2^{k_j})} \ln N_d([0,1], g, \varepsilon, 2^{k_j+1} - 2^{k_j}) \geq \frac{1}{4} \ln 2.
\]

The proof of the lemma is complete.
To complete the proof of Theorem 2, we use the following assertion established in [11]: if a function \( \mu \mapsto \overline{h}_{\text{top}}(K_r, f(\mu, \cdot)) \) is of Baire class 2, then the intersection of closures of the sets \( \overline{h}_{\text{top}}(K_r, f(\mathcal{E}, \cdot)) \) and \( \overline{h}_{\text{top}}(K_r, f(B \setminus \mathcal{E}, \cdot)) \) is nonempty. By Lemmas 1 and 2,

\[
\overline{h}_{\text{top}}(K_r, f(\mathcal{E}, \cdot)) = 0 < \frac{1}{4} \ln 2 \leq \overline{h}_{\text{top}}(K_1, f(B \setminus \mathcal{E}, \cdot)) \leq \overline{h}_{\text{top}}(K_r, f(B \setminus \mathcal{E}, \cdot)),
\]

and consequently, the function \( \mu \mapsto \overline{h}_{\text{top}}(K_r, f(\mu, \cdot)) \) is not of Baire class 2. Since both \( \mathcal{E} \) and \( B \setminus \mathcal{E} \) are everywhere dense in \( B \), it follows that this function is everywhere discontinuous on \( B \). The proof of the theorem is complete.

2. Baire Class of the Topological Entropy of a Family of Dynamical Systems on a Noncompact Metric Space

Following [2], we define the upper and lower topological entropies of a mapping \( f : X \to X \) as the numbers

\[
\overline{h}_{\text{top}}(f) = \sup_{K \in \mathcal{K}(X)} \overline{h}_{\text{top}}(K, f), \quad \underline{h}_{\text{top}}(f) = \sup_{K \in \mathcal{K}(X)} h_{\text{top}}(K, f),
\]

respectively. The following example shows that the two numbers (13) may be distinct. Let us construct a space \( \mathcal{A} \) as follows. The points of \( \mathcal{A} \) are all possible pairs \((x, i)\), where \( x \in \Omega_2 \) and \( i \in \mathbb{N} \), and the metric is defined by the formula

\[
d_{\mathcal{A}}((x, i), (y, j)) = \begin{cases} d_{\Omega_2}(x, y) & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases}
\]

Consider the sequence

\[
f_n = \begin{cases} \text{id}_{\Omega_2} & \text{if } t_{2k} \leq n \leq t_{2k+1} - 1 \\ \sigma & \text{if } t_{2k+1} \leq n \leq t_{2k+2} - 1, \end{cases} \quad t_s = \sum_{m=0}^{s} m!, \quad k = 0, 1, 2, \ldots,
\]

of continuous self-mappings of \( \Omega_2 \) and define a continuous mapping \( f_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \) by setting

\[
f_{\mathcal{A}}(x, n) = (f_n(x), n + 1).
\]

**Lemma 3.** \( \underline{h}_{\text{top}}(f_{\mathcal{A}}) < \overline{h}_{\text{top}}(f_{\mathcal{A}}) \).

**Proof.** For an arbitrary \( r \in \mathbb{N} \), let \( H_r \in \mathcal{K}(\mathcal{A}) \) be the compact set \( H_r = \Omega_2 \times \{1, \ldots, r\} \). Each compact set \( K \in \mathcal{K}(\mathcal{A}) \) is contained in \( H_r \) for some \( r \); consequently,

\[
\underline{h}_{\text{top}}(f_{\mathcal{A}}) = \lim_{r \to \infty} \underline{h}_{\text{top}}(H_r, f_{\mathcal{A}}), \quad \overline{h}_{\text{top}}(f_{\mathcal{A}}) = \lim_{r \to \infty} \overline{h}_{\text{top}}(H_r, f_{\mathcal{A}}).
\]

Let \( p \in \mathbb{N} \), and let \( Q \) be an \((f_{\mathcal{A}}, 1/p, t_{2k})\)-cover of \( H_r \) with minimum number of elements. Then \( Q \) is an \((f_{\mathcal{A}}, 1/p, t_{2k+1})\)-cover of \( H_r \) by the definition of \( f_{\mathcal{A}} \). Since the points \((x, i) \in \mathcal{A}\), where \( x = (x_1, \ldots, x_{t_{2k+p}}, 0, 0, \ldots) \) and \( i \in \{1, 2, \ldots, r\} \), form an \((f_{\mathcal{A}}, 1/p, t_{2k})\)-cover of \( H_r \), it follows that the number of elements in \( Q \) does not exceed \( r2^{t_{2k+p}} \). Therefore,

\[
\underline{h}_{\text{top}}(H_r, f_{\mathcal{A}}) \leq \lim_{p \to \infty} \lim_{k \to \infty} \left( \frac{t_{2k} + p}{(2k + 1)!} \ln 2 + \frac{\ln r}{(2k + 1)!} \right) \leq \lim_{k \to \infty} \frac{\ln 2}{2k+1} \left( 2 + \frac{p}{(2k)!} \right) = 0,
\]

and hence \( \underline{h}_{\text{top}}(f_{\mathcal{A}}) = 0 \).

Let us establish the inequality \( \overline{h}_{\text{top}}(f_{\mathcal{A}}) \geq 0.5 \ln 2 \), whence Lemma 3 will follow. In the space \( \Omega_2 \), consider the set \( R_k, k \in \mathbb{N} \cup \{0\} \), of points of the form

\[
(x_1, \ldots, x_{(2k+2)!}, 0, 0, \ldots).
\]
Take one point \((y_{x'}, 1)\) ∈ \(H_1\) in the preimage of every point \((x, t_{2k+1})\), \(x \in R_k\), under the mapping \(f_A^{(t_{2k+1}-1)}\). If \(x' \neq x''\), \(x', x'' \in R_k\), then
\[
d_{A_{t_{2k+2}}}((y_{x'}, 1), (y_{x''}, 1)) \geq \max_{0 \leq i \leq (2k+2)!-1} d_A(f_A^i(x', t_{2k+1}), f_A^i(x'', t_{2k+1})) = 1.
\]

Thus, \(N_{A\epsilon}(H_1, f_A, \epsilon, t_{2k+2})\) is not less than the cardinality of \(R_k\), which is \(2^{(2k+2)!}\), for each \(\epsilon < 1\), and hence
\[
\overline{h}_{\text{top}}(f_A) \geq \overline{h}_{\text{top}}(H_1, f_A) \geq \lim_{\epsilon \to 0} \lim_{k \to \infty} \frac{1}{t_{2k+2}} \ln N_{A\epsilon}(H_1, f_A, \epsilon, t_{2k+2}) \geq \lim_{k \to \infty} \frac{(2k + 2)!}{t_{2k+2}} \ln 2 \geq \frac{\ln 2}{2}.
\]

The proof of the lemma is complete.

For the mapping (3), consider the functions
\[
\mu \mapsto \overline{h}_{\text{top}}(f(\mu, \cdot)), \quad (14)
\]
\[
\mu \mapsto \underline{h}_{\text{top}}(f(\mu, \cdot)). \quad (15)
\]

If \(X\) is a compact metric space, then the numbers (13) are equal to the topological entropy of \(f\). Therefore, the functions (14) and (15) are of Baire class 2 by [4] but in general not of Baire class 1 by [5].

**Theorem 3.** If \(\mathcal{M} = \mathcal{B}\) and \(X = \mathcal{C}\), then there exists a mapping (3) such that the function (14) is everywhere discontinuous and does not belong to the second Baire class on \(\mathcal{M}\).

**Proof.** Each compact set \(K \in \mathcal{K}(\mathcal{C})\) is contained in \(K_r\) for some \(r\), and consequently, the topological entropy of an arbitrary continuous mapping \(f : \mathcal{C} \to \mathcal{C}\) satisfies the relation
\[
\overline{h}_{\text{top}}(f) = \sup_{K \in \mathcal{K}(\mathcal{C})} \overline{h}_{\text{top}}(K, f) = \sup_{r \in \mathbb{N}} \overline{h}_{\text{top}}(K_r, f).
\]

By Lemmas 1 and 2, we obtain the chain of inequalities
\[
\overline{h}_{\text{top}}(f(\mathcal{E}, \cdot)) = 0 < \frac{1}{4} \ln 2 \leq \overline{h}_{\text{top}}(f(\mathcal{B} \setminus \mathcal{E}, \cdot))
\]
for the family (9). Consequently, the function \(\mu \mapsto \overline{h}_{\text{top}}(f(\mu, \cdot))\) is not of Baire class 2 [11], and since the sets \(\mathcal{E}\) and \(\mathcal{B} \setminus \mathcal{E}\) are everywhere dense in \(\mathcal{B}\), it follows that this function is everywhere discontinuous on \(\mathcal{B}\). The proof of the theorem is complete.

Recall that a metric space \(X\) is said to be **locally compact** if each of its points has a compact neighborhood [12, p. 315 of the Russian translation]. A locally compact space \(X\) is said to be **countable at infinity** [12, p. 316 of the Russian translation] if it is a union of a countably many compact sets. The space \(\mathbb{R}^n\), the above-defined space \(\mathcal{C}\), and, in general, any locally compact space with countable base are examples of such spaces [12, p. 316; 13, p. 254].

**Theorem 4.** Let \(X\) be a locally compact space countable at infinity. Then, for any space \(\mathcal{M}\) and any mapping (3), the function (14) is of Baire class 3 on \(\mathcal{M}\) and the function (15) is of Baire class 2 on \(\mathcal{M}\). If \(\mathcal{M}\) is metrizable by a complete metric, then, for each mapping (3), the set of points of lower semicontinuity of the function (15) is a G\(_s\) set everywhere dense in \(\mathcal{M}\).

**Proof.** Since \(X\) is countable at infinity, it follows that there exists an increasing sequence \(\{U_s\}_{s=1}^\infty\) of relatively compact open sets that form a cover of \(X\) and satisfy \(\overline{U}_s \subset U_{s+1}\) for all \(s \in \mathbb{N}\) [12, p. 316 of the Russian translation]. Every compact set \(K \subset X\) is contained in \(U_{s_0}\) for some \(s_0\), because otherwise the cover of \(K\) by the sequence \(\{U_s\}_{s=1}^\infty\) would not contain a finite subcover, which contradicts the compactness of \(K\). Thus,
\[
\overline{h}_{\text{top}}(f) = \sup_{K \in \mathcal{K}(X)} \overline{h}_{\text{top}}(K, f) = \sup_{s \in \mathbb{N}} \overline{h}_{\text{top}}(U_s, f),
\]
\[
\underline{h}_{\text{top}}(f) = \sup_{K \in \mathcal{K}(X)} \underline{h}_{\text{top}}(K, f) = \sup_{s \in \mathbb{N}} \underline{h}_{\text{top}}(U_s, f)
\]

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for each continuous mapping \( f : X \to X \). Using formula (7), we obtain

\[
\bar{h}_{\text{top}}(f(\mu, \cdot)) = \sup_{s \in \mathbb{N}} \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln N_d(\mathcal{U}_s, f(\mu, \cdot), \epsilon, n)
\]

\[
= \sup_{s \in \mathbb{N}} \sup_{k \in \mathbb{N}} \lim_{n \to \infty} \sup_{m \in \mathbb{N}} \varphi^{(1)}_d(\mathcal{U}_s; \mu, 1/k, n) = \sup_{s \in \mathbb{N}} \sup_{k \in \mathbb{N}} \lim_{n \to \infty} \inf_{m \in \mathbb{N}} \varphi^{(1)}_d(\mathcal{U}_s; \mu, 1/k, l)
\]

\[
= \lim \max \max \lim \min \max \varphi^{(1)}_d(\mathcal{U}_s; \mu, 1/k, l).
\]

By (8), we have

\[
\underline{h}_{\text{top}}(f(\mu, \cdot)) = \sup_{s \in \mathbb{N}} \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln S_d(\mathcal{U}_s, f(\mu, \cdot), \epsilon, n)
\]

\[
= \sup_{s \in \mathbb{N}} \lim_{k \in \mathbb{N}} \inf_{\psi^{(1)}_d(\mathcal{U}_s; \mu, 1/k, n)} = \sup_{s \in \mathbb{N}} \lim_{k \in \mathbb{N}} \inf_{\psi^{(1)}_d(\mathcal{U}_s; \mu, 1/k, l)}
\]

\[
= \lim \max \max \lim \min \psi^{(1)}_d(\mathcal{U}_s; \mu, 1/k, l).
\]

Since the maximum and minimum of finitely many functions in some Baire class belong to the same class [9, Ch. IX, Sec. 37, III of the Russian translation], it follows that the function \( \mu \mapsto \bar{h}_{\text{top}}(f(\mu, \cdot)) \) is of Baire class 3 and the function \( \mu \mapsto \underline{h}_{\text{top}}(f(\mu, \cdot)) \) is of Baire class 2 on \( \mathcal{M} \).

Since the function \( \mu \mapsto \bar{h}_{\text{top}}(f(\mu, \cdot)) \) can be represented as the limit of a nondecreasing sequence of functions of Baire class 1, it follows that its set of points of lower semicontinuity is an everywhere dense \( G_\delta \) set provided that \( \mathcal{M} \) is metrizable by a complete metric [10, Lemma 2]. The proof of the theorem is complete.

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