Lower bounds on the maximum dimension of a simple module in characteristic $p$

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Abstract

We obtain lower bounds for the maximum dimension of a simple $F G$-module, where $G$ is a finite group, $p$ is a prime, and $F$ is an algebraically closed field of characteristic $p$. These bounds are described in terms of properties of $p$-subgroups of $G$. It turns out that it suffices to treat the case that $O_p(G) = \Phi(G) = 1$. In that case, our main result asserts that if $p$ is an odd prime which is not Mersenne, then there is a simple $F G$ module $S$ of dimension at least $|G|_p |\text{Out}_G(X)|_p$, where $X = O^p(E(G))$ and $\text{Out}_G(X) = G/XC_G(X)$ is (isomorphic to) the group of outer automorphisms of $X$ induced by the conjugation action of $G$. Note, in particular, that this quotient is $|G|_p$ when $X = 1$, that is to say, when no component of $G$ has order divisible by $p$ (and, more generally, the quotient is $|G|_p$ when $\text{Out}_G(X)$ is a $p'$-group).

We also use simplicial complex methods to obtain a weaker general bound which still applies when $p = 2$ or when $p$ is a Mersenne prime.

1 Introduction and Statement of Theorem 1

In this note, we are interested in exhibiting simple $F G$-module $S$ whose dimension is large relative to the size of $p$-subgroups of $G$, where $F$ is an algebraically closed field of characteristic $p$. We will prove:

Theorem 1: Let $G$ be a finite group, $p$ be a prime, and $F$ be an algebraically closed field of characteristic $p$. Suppose further that $O_p(G) = \Phi(G) = 1$. Set $X = O^p(E(G))$ and $\text{Out}_G(X) = G/XC_G(X)$. Then:

i) If $p$ is an odd prime which is not Mersenne, then there is a simple $F G$ module $S$ of dimension at least $|G|_p |\text{Out}_G(X)|_p$.

ii) If $p = 2$, or if $p$ is a Mersenne prime, then there is a simple $F G$-module of dimension at least $|A|$, where $A$ is a maximal Abelian $p$-subgroup of $XC_G(X)$ which contains an Abelian $p$-subgroup of maximal order of $C_G(X)$. 

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Remark: Note, in particular, that the quotient given in part i) of Theorem 1 is \([G]_p\) when \(X = 1\), that is to say, when no component of \(G\) has order divisible by \(p\).

2 On the maximal dimension of a simple module

Let \(G\) be a finite group, \(p\) be a prime, and \(F\) be an algebraically closed field of characteristic \(p\). Let \(P\) be a Sylow \(p\)-subgroup of \(G\). We define the invariant \(m_\ast(G) = m_\ast(G, p)\) to be the maximal dimension of a simple \(FG\)-module, and we are concerned in this note with obtaining lower bounds for \(m_\ast(G)\) in terms of properties of \(P\) and its subgroups. Clearly, we have \(m_\ast(G) = m_\ast(G/O_p(G))\), so it suffices to deal with the case that \(O_p(G) = 1\). Note that we clearly have \(m_\ast(G) = m_\ast(X)m_\ast(Y)\) when \(G = X \times Y\) is a direct product of subgroups \(X\) and \(Y\).

There are many examples of finite groups \(G\) with \(O_p(G) = 1\) and \(m_\ast(G) = |P|\). For example, when \(p = 2\) and \(G = SL(2, 2^n)\), then we have \(m_\ast(G) = 2^n\), the degrees of the simple \(FG\)-modules being \(1, 2, 2^2, \ldots, 2^n\) (with \(\binom{n}{j}\) occurrences of simple modules of dimension \(2^j\)), so we can’t expect to obtain a general bound stronger than \(m_\ast(G) \geq |P|\). It is known that finite simple groups of Lie type have a \(p\)-blocks of defect zero for every prime \(p\) by theorem of Michler [5] and Willems [9], so we do have \(m_\ast(G) \geq |P|\) for such groups \(G\) (whatever the defining characteristic of \(G\) is). For most primes \(p\) (other than the defining characteristic and prime divisors of the order of the Weyl group), such groups \(G\) have Abelian Sylow \(p\)-subgroups.

It is clear that \(m_\ast(G) \geq m_\ast(H)\) for each section \(H\) of \(G\). For if \(N \triangleleft G\), we certainly have \(m_\ast(G/N) \leq m_\ast(G)\), while if \(K\) is a subgroup of \(G\), and \(S\) is a simple \(FK\)-module of maximal dimension, then \(\text{Ind}_K^G(S)\) has a composition factor \(T\) such that \(S\) is a composition factor of \(\text{Res}_K^G(T)\), so that \(\dim_F(T) \geq \dim_F(S)\).

Slightly less obvious than the reduction to \(O_p(G) = 1\) is that it suffices to deal with the case \(\Phi(G) = 1\) when \(O_p(G) = 1\). For if \(O_p(G) = 1\), then \(\Phi(G)\) is a nilpotent \(p'\)-group. We reproduce the well-known proof that \(O_p(G/\Phi(G)) = 1\). For suppose otherwise, and let \(N \triangleleft G\) denote the full pre-image in \(G\) of \(O_p(G/\Phi(G))\). Let \(Q\) be a Sylow \(p\)-subgroup of \(N\). Then \(G = NN_G(Q)\) by the Frattini argument, and \(N = \Phi(G)Q\), so that \(G = \Phi(G)N_G(Q)\). Hence \(G = N_G(Q)\), contrary to \(O_p(G) = 1\).

Note that when \(\Phi(G) = 1\), we have \(\Phi(M) = 1\) whenever \(M \triangleleft \triangleleft G\). In particular, each component of \(G\) (if any exist) is a non-Abelian simple group. Also, \(F(G)\) is a direct product of minimal normal subgroups of \(G\), each of which is complemented in \(G\). In particular, \(F(G)\) is a direct product of elementary Abelian groups (for various primes). Hence \(F^*(G) = \text{soc}(G)\) is the direct product of the minimal normal subgroups of \(G\).
In determining a lower bound for $m_s(G)$ in terms of properties of $p$-subgroups of $G$ when $O_p(G) = \Phi(G) = 1$, it suffices to treat the case that $G = PF^*(G)$. For $O_p(PF^*(G))$ centralizes $F^*(G)$ in that case, and $Z(F(G))$ is a $p'$-group. Hence we assume from now on that $O_p(G) = \Phi(G) = 1$, and that $G = PF^*(G)$, with $F^*(G)$ being a direct product of (possibly Abelian) simple groups.

#### 3 When $G \neq O^p(G)$

In the previous section, we reduced to a configuration where $G = PF^*(G)$. Note that $G \neq O^p(G)$ if $G \neq F^*(G)$. It may be worthwhile to digress to make some general remarks about the relationship between $m_s(G)$ and $m_s(H)$ when $H = O^p(G)$. In one sense, the Clifford theory between simple $FH$-modules and simple $FG$-modules is transparent, yet the relationship between $m_s(G)$ and $m_s(G)$ is less clear in general.

Given a simple $FH$-module $S$, with inertial subgroup $I_G(S) = K$, say, then $S$ has a unique extension $S$ to a simple $FK$-module, and then $\text{Ind}_K^G(S)$ is the unique simple $FG$-module covering $S$. Hence the number of simple $FG$-modules is the number of orbits of $G$ on isomorphism types of simple $FH$-modules, and a simple $FH$-module $S$ in a given $G$-orbit gives rise to a simple $FG$-module of dimension $\frac{[G : I_G(S)] \dim_F(S)}{\dim_F(S)}$.

Hence we obtain $m_s(H) \leq m_s(G) \leq \frac{[G : H]m_s(H)}{\dim F(S)}$ but it usually difficult to ascertain without further knowledge which simple $FH$-module gives rise in this fashion to a simple $FG$-module of maximal dimension.

A case we will return to later is when $G = F^*(G)P$, and $G$ has no component of order divisible by $p$. Even in the case that $F^*(G) = N$ is Abelian, there are examples where $m_s(G) = |A|$, the maximal order of an Abelian $p$-subgroup of $G$.

When $G = NP$, with $N$ an Abelian normal $p$-complement acted on faithfully by the finite $p$-group $P$, the discussion at the beginning of this section shows that each simple $FG$-module has dimension a power of $p$ dividing $|P|$. Also, each projective indecomposable $FG$-module has dimension $|P|$. In general, there need not be a projective simple $FG$-module (indeed, by dimension considerations, there certainly can be no such simple module if $|P| > |N|$, a situation which can occur).

Examples of the last phenomenon occur as familiar types of counterexamples to “Burnside’s other $p^nq^h$-theorem”, or to various types of regular orbit theorems. One standard type is when $p$ is a Merseenne prime, $P \cong C_p \wr C_p$, and $N$ is an elementary Abelian group of order $(p + 1)P$. In this case, we do have $|P| > |N|$, so that $m_s(G) < |P|$, and we find that $m_s(G) = \frac{|P|}{p}$, which is the maximal order of an Abelian subgroup of $P$.

Another standard type of counterexample is when $p = 2$ and $N$ is an elementary Abelian $q$-group of order $q^2$, where $q > 3$ is a Fermat prime. Then the
2-group \( P = C_{q-1} \wr C_2 \) acts faithfully on \( N \), and we have \( |P| > |N| \), the semidirect product \( G = NP \) has \( m_s(G) < |P| \), and we again find that \( m_s(G) = \frac{|P|}{p} \), which is the maximum order of an Abelian subgroup of \( P \). Even in the case \( p = 2, q = 3 \), we find that \( m_s(G) = 4 \), the maximum order of an Abelian 2-subgroup of \( G = NP \), although we do have \( |P| = 8 < 9 = |N| \) in this case.

When \( p = 2 \) or \( p \) is a Mersenne prime, we may use direct products of some of the groups above to construct arbitrarily large examples of solvable \( p \)-nilpotent groups \( G \) with \( O_p(G) = 1 \) such that \( m_s(G) \) is the maximum order of an Abelian \( p \)-subgroup \( A \) of \( P \), yet \( G \) has a non-Abelian Sylow \( p \)-subgroup \( P \). In fact, it is clear that \( [P : A] \) may be made as large as desired.

In the next section, we use \( p \)-subgroup complexes to obtain a weaker general bound which suffices to cover these exceptional cases, but which will be significantly improved in the last section for odd primes \( p \) which are not Mersenne.

4 Using \( p \)-subgroup complexes

When \( G \) is a finite group, we let \( \mathcal{P} \) denote the simplicial complex associated to the poset of non-trivial \( p \)-subgroups of \( G \). This was introduced by D. Quillen in [7], and has been extensively studied by K.S. Brown, S. Bouc, P.J. Webb, and J. Thévenaz, among others. We let \( \sigma \) denote a strictly increasing chain of non-trivial \( p \)-subgroups of \( G \). We let \( |\sigma| \) denote the number of inclusions in the chain. The group \( G \) acts by conjugation on chains, and the reduced Euler characteristic of this complex is (up to a sign depending on conventions chosen) \( \sum_{\sigma \in \mathcal{P}/G} (-1)^{|\sigma|} [G : G_{\sigma}] \), where we include the empty chain, and consider it to have length 0.

In the Green Ring for \( FG \), there is defined what P.J. Webb described as the Steinberg (virtual) module for \( \mathcal{P} \), which is

\[
\sum_{\sigma \in \mathcal{P}/G} (-1)^{|\sigma|} \text{Ind}_{G_{\sigma}}^G(F),
\]

and which we denote by \( St_p(G) \). This was proved in (Webb,[8]), to be a virtual projective module, that is to say, a difference of projective modules (allowing the possibility of the zero module).

The virtual module \( St_p(G) \) may instead be calculated with respect to other \( G \)-homotopy equivalent complexes (with the same result). One of these, favoured by Quillen, is the complex \( \mathcal{E} \) associated to the poset of elementary Abelian \( p \)-subgroups of \( G \). Another, favoured by S. Bouc, is the complex \( \mathcal{U} \) associated to the poset of non-trivial \( p \)-subgroups \( U \) of \( G \) which satisfy \( U = O_p(N_G(U)) \).

Since \( St_p(G) \) is a virtual projective module, it is a difference of projective \( FG \)-modules, each of which is liftable to characteristic zero. Hence \( St_p(G) \) yields
a well-defined complex virtual character. Furthermore, $\text{St}_p(G)$ is uniquely determined (as a linear combination of indecomposable projectives) by this virtual character, using the non-singularity of the Cartan matrix.

If $O_p(G) \neq 1$, it is well-known (see, eg, Quillen [7]), that $\text{St}_p(G) = 0$. We know of no example of a finite group with $O_p(G) = 1$ and $\text{St}_p(G) = 0$, but it seems to be an open question at present whether $O_p(G) = 1$ implies $\text{St}_p(G) \neq 0$.

The virtual character afforded by (the lift to characteristic zero) of $\text{St}_p(G)$ is the alternating sum of characters afforded by homology modules associated to the complex $\mathcal{P}$. In particular, if $\text{St}_p(G) \neq 0$, then the homology of the complex $\mathcal{P}$ is non-zero, and the complex $\mathcal{P}$ is not contractible.

In cases where we are able to establish that $\text{St}_p(G) \neq 0$, we obtain a lower bound for $m_s(G)$:

**Lemma 1**: Let $G$ be a finite group such that $\text{St}_p(G) \neq 0$. Then there is simple $FG$-module $S$ and a chain $\sigma$ of non-trivial $p$-subgroups of $G$, such that $\text{Res}^G_{G_{\sigma}}(S)$ contains the projective cover of the trivial module as a summand. In particular, $\dim_F(S) \geq |Q|$, where $Q \in \text{Syl}_p(G_{\sigma})$. The chain $\sigma$ may be chosen to consist of elementary Abelian $p$-subgroups of $G$, in which case we have $|Q| \geq |A|$ for some maximal Abelian $p$-subgroup $A$ of $G$.

**Proof**: Let $S$ be a simple module such that the projective cover of $S$ occurs with non-zero multiplicity in $\text{St}_p(G)$. Then there is a chain $\sigma$ such that the projective cover of $S$ occurs as a summand of $\text{Ind}^G_{G_{\sigma}}(F)$, and $\sigma$ may be chosen to consist of elementary Abelian $p$-subgroups.

Then the projective cover of the trivial module occurs as a summand of $\text{Res}^G_{G_{\sigma}}(S)$, since $\text{Ind}^G_{G_{\sigma}}(F \otimes \text{Res}^G_{G_{\sigma}}(S^*))$ has the projective cover of the trivial module as a direct summand, and the multiplicity of the projective cover of the trivial module as a summand is preserved by induction of modules.

Hence the first claim follows. If we choose $\sigma$ to consist of elementary Abelian subgroups, and we choose a maximal Abelian $p$-subgroup $A$ of $G$ containing the largest subgroup of the chain, then $A \leq G_\sigma$ and $|Q| \geq |A|$.

Next, we observe that when $F^*(G)$ is a $p'$-group, (in particular, when $G$ is $p$-solvable with $O_p(G) = 1$), then we can do somewhat better.

**Corollary 2**: Let $G$ be a finite group such that $F^*(G)$ is a $p'$-group, and let $A$ be an Abelian $p$-subgroup of $G$ of maximal order. Then $m_s(G) \geq |A|$.

**Proof**: It suffices to prove that $m_s(O_{p'}(G)A) \geq |A|$, so we may suppose that $G = O_{p'}(G)A$, since $O_p(O_{p'}(G)A) = 1$.

But in this case, a Theorem of Hawkes and Isaacs ([4]) applies, and allows us to conclude that the virtual module $\text{St}_p(G) \neq 0$. In fact, Hawkes and Isaacs even prove that the reduced Euler characteristic of $\mathcal{P}$ is non-zero, and the reduced Euler characteristic of $\mathcal{P}$ is the virtual dimension of $\text{St}_p(G)$. Hence Lemma 1 applies to $G$ (note that, under current assumptions, every maximal Abelian $p$-subgroup of $G$ is conjugate to $A$).
In the paper [1] of Aschbacher-Kleidman, they establish a condition which, using a previous result of the present author, ensures that for any finite almost simple group $G$ with $F^*(G) \not\cong \text{PSL}(3, 4)$, we have $\text{St}_p(G) \neq 0$. They obtain the same conclusion when $G$ is the simple group $\text{PSL}_3(4)$, but the condition may fail for certain subgroups of $\text{Aut}(\text{PSL}(3, 4))$. This yields:

**Corollary 3:** Let $G$ be a finite non-Abelian simple group. Then $m_s(G) \geq |A|$ for some maximal Abelian $p$-subgroup $A$ of $G$.

**Proof:** By a Theorem of Aschbacher-Kleidman ([1]), we have $\text{St}_p(G) \neq 0$. Hence Lemma 1 may be applied to $G$, and $m_s(G) \geq |A|$ for some maximal Abelian $p$-subgroup of $G$.

5 Proof of Theorem 1

By the results of Granville and Ono [2], Michler [5], and Willems [9], whenever $G$ is a finite non-Abelian simple group and $p$ is a prime greater than 3, the group $G$ has a $p$-block of defect zero. However, for both $p = 2$ and $p = 3$, there are alternating groups and sporadic simple groups which have no $p$-block of defect zero.

By the proof of Lemma 2.3 of Guralnick and Robinson [3], whenever $p$ is an odd prime which is not Merseenne, and $q \neq p$ is a prime, then a Sylow $p$-subgroup $R$ of $G = \text{GL}(n, q)$ has at least two regular orbits on the natural module for $G$. When $q = 2$ and $p$ is not Merseenne, we need to adapt the latter proof as follows: In the case that $R$ is Abelian of order $p^r$ and the action of $R$ on the natural module for $G$ is irreducible, all orbits of $R$ on non-zero vectors are regular, but we don’t have $2^n - 1 = p^r$, since that equality would force $r$ to be odd and $p + 1$ to be a power of 2, contrary to the fact that $p$ is not a Merseenne prime. Hence there are at least two regular orbits of $R$ on the non-zero vectors of the natural module for $G$). Then the inductive argument proceeds in a manner similar to that of Lemma 2.3 of [3].

The proof of our main result subdivides naturally according to whether $F^*(G)$ is a $p'$-group or not.

**Lemma 2 :** Let $G$ be a finite group such that $F^*(G)$ is a $p'$-group, where $p$ is an odd prime which is not Merseenne. Then $m_s(G) \geq |P|$, where $P \in \text{Syl}_p(G)$.

**Proof:** As in earlier arguments, we may assume that $\Phi(G) = 1$ and that $G = PF^*(G)$. We may also assume that $P$ does not act faithfully on any proper $P$-invariant subgroup of $F^*(G)$. By Theorem 1.2 of Moretó-Navarro [6], we see that $F^*(G)$ is nilpotent (and even Abelian of squarefree exponent, since $\Phi(G) = 1$).

Now it follows that $P$ has a regular orbit on the Abelian group $F^*(G) = F(G)$, and on the dual group, which is the group of irreducible characters of $F(G)$. Hence $PF(G)$ has a $p$-block of defect zero and there is a simple
FG-module of dimension $|P|$. Thus $m_s(G) \geq |P|$ (in fact, equality holds in this residual configuration).

**Conclusion of Proof of Theorem 1:** i) Suppose that $p$ is neither 2 nor a Merseenne prime, and that $O_p(G) = \Phi(G) = 1$. Let $X = O^{p'}(E(G))$. Then $X$ is the normal subgroup of $E(G)$ generated the $p$-element of $E(G)$, and is the direct product of the components of $G$ of order divisible by $p$ (or is the identity subgroup if there is no such component)

It follows from remarks at the beginning of this section that $X$ has a $p$-block of defect zero, since $p \geq 5$. Now $C \cap X = 1$, where $C = C_G(X)$, and we note that $F^*(C)$ is a $p'$-group. By the previous Lemma, we have $m_s(C) \geq |C|_p$ so that we have $m_s(G) \geq m_s(XC) \geq |XC|_p$. Note that $C = G$ if $X = 1$.

Now $G/XC$ is isomorphic to a subgroup of Out$(X)$, and we denote this subgroup by Out$_G(X)$. Now we have $m_s(G) \geq |XC|_p = \frac{|G|_p}{|\text{Out}_G(X)|_p}$.

ii) If $p = 2$ or a Merseenne prime, then we may argue in a fashion similar to i), but making use instead of Corollaries 2 and 3.

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