Instanton interaction in de Sitter spacetime

Dimitrios Metaxas

Department of Physics,
National Technical University of Athens,
Zografou Campus, 15780 Athens, Greece
metaxas@central.ntua.gr

Abstract

Because of the presence of a cosmological horizon the dilute instanton gas approximation used for the derivation of the Coleman-De Luccia tunneling rate in de Sitter spacetime receives additional contributions due to the finite instanton separation. Here I calculate the first corrections to the vacuum decay rate that arise from this effect and depend on the parameters of the theory and the cosmological constant of the background spacetime.
1 Introduction

The calculation of the decay rate of metastable vacua in flat and curved space-time [1, 2] and zero or finite temperature [3] is of great interest due to various applications to problems at the interface of particle physics and cosmology. Especially, the recent discovery of a multitude of string vacua suggests that a detailed understanding of these rates for different parameters of the models may be relevant to the investigation of the various cosmological phase transitions that took place in the evolution of the universe as we know it [4]. In particular, it is important to have quantitative results for the dependence of the decay rate on physical parameters such as the temperature [5] and the Hubble expansion rate or the curvature of the background space-time [6].

To this effect, the usual semiclassical calculation at zero temperature in flat space-time, which I review in Sec. 2, needs various modifications and admits additional contributions depending on the physical situation at hand. In Sec. 3 I proceed to examine the corrections that arise in de Sitter space-time because of the presence of a cosmological horizon. The dilute instanton gas approximation then receives corrections, the first of which arises from the two-instanton interaction, and I calculate it in the thin wall and fixed background approximation. The fact that the instanton interaction in de Sitter space-time turns out to be repulsive, essentially because of the cosmological expansion, and unlike most instanton and soliton interactions in scalar field theories, makes the calculation feasible, without the need of additional cutoffs.

The corrections obtained here are, of course, subleading with respect to the semiclassical exponential factor, they are of interest, however, because of their dependence on the various parameters of the theory, especially the cosmological constant of the background space-time. Finally, in Sec. 4 I conclude with some comments regarding possible extensions of this work.

2 Review of the flat space-time results

In order to calculate the false vacuum decay rate for a scalar field theory in flat, 3 + 1-dimensional space-time and zero temperature one evaluates the imaginary part of the energy of the false vacuum, obtained from the path
integral

\[ Z(T) = \int [d\phi] \exp(-S_E(\phi)), \]  

(1)

with the Euclidean action

\[ S_E(\phi) = \int d^3x \int_{-T/2}^{T/2} dt \, L_E, \]  

(2)

and

\[ L_E = \frac{1}{2} (\partial_\mu \phi)^2 + U(\phi). \]  

(3)

where the potential \( U(\phi) \) has a metastable minimum at \( \phi = \phi_f \) and an absolute minimum at \( \phi = \phi_t \), and the path integral is restricted to paths satisfying \( \phi(-T/2) = \phi(T/2) = \phi_f \).

Then, in the limit \( T \to \infty \),

\[ Z(T) \approx \exp(-E_f T \Omega), \]  

(4)

where \( \Omega \) is the volume of space and \( E_f \) the energy density of the false vacuum and the imaginary part of this quantity will give the decay rate

\[ \Gamma = -2 \frac{\text{Im} \ln Z(T)}{T \Omega}, \]  

(5)

that is, in this case, the bubble nucleation rate, or the number of true vacuum bubbles produced per unit volume per unit time.

The path integral can be evaluated in the saddle-point approximation by obtaining an instanton (bounce) solution, \( \phi_b \), of the Euclidean field equation

\[ \Box_E \phi = \frac{dU}{d\phi}. \]  

(6)

Then

\[ Z(T) \approx Z(\phi_f) + Z(\phi_b) + Z_2 + \ldots \]  

(7)

where the first contribution comes from a homogeneous false vacuum configuration,

\[ Z(\phi_f) = [\det S''_E(\phi_f)]^{-1/2} \exp(-S_E(\phi_f)) \]  

(8)

(\( S''_E \) denotes the second variation of the Euclidean action), the second contribution to (7) comes from the bounce configuration,

\[ Z(\phi_b) = \frac{i}{2} \Omega T J \, | \det' S''_E(\phi_b) |^{-1/2} \exp(-S_E(\phi_b)) \]  

(9)
and higher contributions come from multi-bounce configurations. In the evaluation of (9) one takes into account that around the bounce configuration $S_E^{\prime\prime}$ has one negative mode and four zero modes (which are excluded from the determinant). The treatment of the negative mode gives the factor of $i/2$, the zero modes can be treated by changing to collective coordinates, which gives a Jacobean $J$, and the integration over these coordinates gives a factor of $\Omega T$, the total physical volume of the system.

The integration over the zero modes can be better illustrated in the case of the $0 + 1$-dimensional case of a simple quantum-mechanical system with the Euclidean action

$$S^{(1)}(\phi) = \int_{-T/2}^{T/2} dt_E \left( \frac{1}{2} (\partial_t \phi)^2 + U(\phi) \right),$$

with a similar potential that admits a (one-dimensional) bounce solution, $\phi^{(1)}_b(t)$, to the Euclidean equation $\partial_t^2 \phi = dU/d\phi$. Because of translational invariance there is a single zero mode.

$$\phi_s = \phi^{(1)}_b(t - s)$$

is a one-parameter set of solutions with the same action, and $\phi_0 = \partial_t \phi_s/\sqrt{S^{(1)}_b}$, where $S^{(1)}_b$ is the action for $\phi^{(1)}_b$, is the corresponding zero mode.

The corresponding path integral can be evaluated in the semiclassical approximation around the saddle point \[7\], after writing $\phi(t) = \phi_s + \psi$, expanding $\psi = \sum \alpha_n \psi_n$, in eigenfunctions of $Q = \delta^2 S^{(1)}/\delta^2 \phi$, such that

$Q \psi_n = \lambda_n \psi_n$, and inserting the identity

$$1 = \Delta(\phi) \int ds \delta[<\phi_0, \phi - \phi_s>],$$

where $<f, g> = \int f(t)g(t)dt$. The action to second order is $S = S^{(1)}_b + \frac{1}{2} \sum \lambda_n \alpha_n^2$, and the divergence of the integration over the zero mode is taken care of by the delta function, resulting in an overall factor of $T$, the “volume” of the system, and a Jacobean factor, $J = \Delta(\phi) \approx \Delta(\phi_s)$ in the semiclassical approximation.

Going back to the $3 + 1$-dimensional case, one continues with the higher contributions to (7), which come from multi-instanton configurations: for
example, for the two-bounce configuration, dividing the total space-time volume into two volumes \( V_1 \) and \( V_2 \), each containing one instanton, one obtains [8]

\[
Z_2 = \frac{1}{2!} \int [d\phi] V_1 \int [d\phi] V_2 e^{-\int V_1 L_E} e^{-\int V_2 L_E} = \frac{1}{2!} \int [d\phi] V_2 e^{-S_1} \int [d\phi] V_1 e^{-S_0} \int [d\phi] V_1 e^{-S_0} = \frac{1}{2!} Z_f \left( \frac{Z_b}{Z_f} \right)^2,
\]

and

\[
Z(T) = Z_f + Z_f \frac{Z_b}{Z_f} + Z_f \frac{1}{2!} \left( \frac{Z_b}{Z_f} \right)^2 + \ldots = Z_f \exp \left( \frac{Z_b}{Z_f} \right).
\]

From (8, 9) we get

\[
\frac{Z_b}{Z_f} = i \frac{\Omega T}{2} A e^{-B}
\]

where \( A \) denotes the Jacobean and determinant factors and

\[
B = S(\phi_b) - S(\phi_f),
\]

and, finally, from (5) we have

\[
\Gamma = A e^{-B}.
\]

As a final note one should mention that the bounce solution of (6) is obtained in flat space-time using the fact that it enjoys \( O(4) \) symmetry [9], and that the dilute instanton gas approximation that was used when considering widely separated instantons in (14) can be justified when \( B > 1 \).

3 The bounce and the instanton gas in de Sitter space-time

When extending the previous results to include the gravitational effects on the false vacuum decay rate one assumes again \( O(4) \) symmetry (which has
not been rigorously proven in this case) with the metric

\[ ds^2 = d\tau^2 + \rho(\tau)^2 d\Omega_3^2, \]

and then the Euclidean action becomes

\[ S_E = 2\pi^2 \int d\tau \left( \rho^3 \left[ \frac{1}{2} \dot{\phi}^2 + U(\phi) \right] + \frac{3}{8\pi G} \left( \rho \ddot{\rho} + \rho \dot{\rho}^2 - \rho \right) \right), \]

for which the Euclidean field equations are:

\[ \dot{\rho}^2 - 1 = \frac{8\pi G}{3} \rho^2 \left[ \frac{1}{2} \dot{\phi}^2 - U(\phi) \right], \]

\[ \ddot{\phi} + \frac{3}{\rho} \dot{\rho} \dot{\phi} = \frac{dU}{d\phi}. \]

I will work in the fixed background approximation where

\[ U(\phi) = U_0 + \tilde{U}(\phi) \]

with \( |\tilde{U}(\phi)| < U_0 \), and a mass scale \( \mu \) such that \( \phi_f, \phi_t \sim \mu, U(\phi_f) \sim U_0 \gg \mu^4 \). As before, \( \phi_t \) and \( \phi_f \) are, respectively, the absolute minimum (true vacuum) and the relative minimum (metastable vacuum) of the potential.

Then one has a Euclidean de Sitter background spacetime which is topologically a four-sphere with

\[ \rho(\tau) = \frac{1}{H} \sin(H\tau) \]

where

\[ H^2 = \frac{8\pi G}{3} U_0 \]

and the coordinate \( \tau \) extends from 0, on the “south pole”, to \( \frac{\pi}{H} \), on the “north pole” of the sphere.

The bounce solution to (21), in the thin wall approximation of [2], takes values close to \( \phi_t \) at the “south pole”, \( \tau \approx 0 \), and falls rapidly to \( \phi_f \) at larger values of \( \tau \), satisfying the boundary conditions \( \dot{\phi} = 0 \) at \( \tau = 0 \) and \( \tau = \frac{\pi}{H} \).
Provided that the bounce radius, $R_b$, is much smaller than $\frac{\pi}{H}$ one can justify the dilute instanton gas approximation described in the previous section and proceed to derive the Coleman-De Luccia bubble nucleation rate, $\Gamma_{CDL} = Ae^{-B}$, where

$$B = 2\pi^2 \int d\tau \rho^3 \left[ \frac{1}{2} \dot{\phi}^2 + \tilde{U}(\phi) \right], \quad (25)$$

and $A$ is the functional determinant pre-factor around the single bounce solution in de Sitter spacetime. Since, however, the Euclidean de Sitter four-sphere has a finite extend, $\tau$ extends up to the de Sitter horizon

$$R_{dS} = \frac{\pi}{H}, \quad (26)$$

and there are always contributions from the interactions between instantons, multi-bounce configurations, as they are distributed on the four-sphere, especially when their radius grows larger.

Here I will estimate the first corrections to $\Gamma_{CDL}$ that appear because of this effect. In order to get a quantitative estimate I will assume that we are still inside the limits of the previous approximations (in particular, the condition $B \gg 1$ also has to hold, as in the flat case, and the thin wall and fixed background limits are also assumed) and obtain, therefore, a correction to the pre-exponential factor for $\Gamma$, which is of interest, however, since it depends on the parameters of the model and especially on the cosmological constant of the background de Sitter space-time.

A first estimate of the corrections involved comes from considering the two-instanton interaction, as it appears in (14) and its generalization to the Euclidean de Sitter spacetime. The method involved is similar to the calculation of soliton interactions in [10] and a basic reason that it can be used here in order to get quantitative results is that the instanton interaction in de Sitter spacetime turns out to be repulsive, essentially because of the cosmological expansion rate, and unlike most scalar soliton and instanton interactions in quantum field theory. The two-instanton configuration that gives the greatest contribution to the saddle-point evaluation of (14) will consist, therefore, of two bounce solutions symmetrically on the “north pole” and “south pole” of the de Sitter four-sphere.

In order to justify the previous statements we consider, accordingly, a two-instanton configuration of the form

$$\phi_s(\tau) = \tilde{\phi}_b(\tau) + \tilde{\phi}_b(s - \tau) + \phi_f \quad (27)$$
where
\[
\tilde{\phi} = \phi - \phi_f
\]  
(28)
and \(\phi_b\) a solution to the bounce equation
\[
\ddot{\phi} + 3H \cot(H\tau) \dot{\phi} = \frac{dU}{d\phi}.
\]  
(29)
When the two-instanton separation \(s\) is very close to \(\frac{\pi}{H}\) the previous configuration is an approximate solution of the Euclidean field equations and one can estimate their interaction as a function of \(s\). As in [10], the interaction will come from the overlap of the “tails” of the two widely separated instantons, that is around the “equator” \(\tau = \frac{\pi}{2H}\). One needs, therefore, the asymptotic behavior of the bounce solution to (29). First we write around the false vacuum
\[
U(\phi) = U_0 + \frac{\beta H^2}{2} (\phi - \phi_f)^2
\]  
(30)
with
\[
\beta = \frac{U''(\phi_f)}{H^2}.
\]  
(31)
After setting \(x = \tau - \frac{\pi}{2H}\), writing \(\cot(H\tau) \approx -Hx\) for small \(x\), changing to \(\tilde{\phi} = u(x)e^{-\frac{H^2 x^2}{2}}\) and neglecting terms of higher order in \(x\) in the final expressions, we find the asymptotic behavior for \(\tilde{\phi} = \phi - \phi_f\) around \(\tau \approx \frac{\pi}{2H}\):
\[
\tilde{\phi} \approx c e^{-H\sqrt{\beta - \frac{3}{2}}} 
\]  
(32)
with \(c\) a constant of order \(\mu\).

Now one can estimate the two-instanton interaction \(\delta B\) as the difference between the value for \(B\) of the configuration \(\phi_s\) and twice the value for a single bounce,
\[
B(\phi_s) = 2B + \delta B,
\]  
(33)
and, using the previous expressions, we find
\[
\delta B = 2\pi^2 \int d\tau \rho^3 \left[ \dot{\phi}_b(\tau) \ddot{\phi}_b(s - \tau) + \beta H^2 \dot{\phi}_b(\tau) \dddot{\phi}_b(s - \tau) \right]
\]  
(34)
\[
= \left(2\pi^2 c^2 \int d\tau \frac{1}{H^3} \sin^3(H\tau) \right) \frac{3}{2} H^2 e^{-H\sqrt{\beta - \frac{3}{2}}} 
\]
\[
= 2\pi^2 c^2 \frac{4}{3H^4} \frac{3}{2} H^2 e^{-H\sqrt{\beta - \frac{3}{2}}}. 
\]
Here, the integration inside the parenthesis in the second line of the previous expression is done around the “equator” \( \tau \approx \frac{\pi}{2H} \), since, however, the integrand falls to zero away from this value, I have extended the region of integration to the entire range, the difference being a numerical factor of order unity that can, in any case, be absorbed in \( c \).

One can see immediately from this expression that the effective instanton interaction in the background de Sitter spacetime is repulsive, and, consequently, that the path integral is dominated by growing values of \( s \approx \frac{\pi}{H} \).

If this instanton interaction term did not exist, one would proceed from expressions like (14) to derive \( \Gamma_{CDL} \) as before, the integration over \( s \), from \( s = 0 \) to \( s = \frac{\pi}{H} \), corresponding to a collective coordinate contributing to the total volume factor in (15). Now, however, this integration will give a different factor, leading to an additional contribution to the pre-exponential factor for the false vacuum decay rate. In the two-instanton interaction, the factor \( Z_2 \) in (14), after treating (27) in the manner of (12), gets modified to

\[
Z_2 = \frac{1}{2!} \alpha \left( \frac{Z_b}{Z_f} \right)^2,
\]

with

\[
\alpha = \frac{\int_0^{\pi/H} ds \ e^{-\delta B}}{\pi/H} \tag{36}
\]

and in the general \( n \)-instanton contribution to (14) one may consider the nearest neighbor interaction and integrate out each instanton to get the general factor

\[
Z_n = \frac{1}{n!} \left( \frac{Z_1}{Z_0} \right)^n \alpha^{n-1}. \tag{37}
\]

Overall this leads to a correction to the pre-exponential factor of the Coleman-De Luccia result

\[
\Gamma = \alpha \Gamma_{CDL} \tag{38}
\]

with

\[
\alpha = \frac{1}{\pi} \int_0^\pi dx \exp(-a_1 e^{-a_2 x}) \tag{39}
\]

where \( a_1 = \frac{4\pi^2 c^2}{H^2} \) and \( a_2 = \sqrt{\beta - \frac{3}{2}} \). When \( \frac{H}{\mu} \ll 1 \) this has the asymptotic expansion \( \alpha \approx 1 - \frac{\ln a_1}{a_2} - \frac{\gamma}{\pi a_2} \), where \( \gamma = 0.57721... \) is the Euler-Mascheroni constant.
4 Comments

In summary, using the thin-wall, dilute instanton gas approximations of [2] and the fixed-background approximation as described before, I derived the first corrections to the pre-exponential factor of the Coleman-De Luccia vacuum decay rate that originate from the instanton interactions inside the finite horizon of the background de Sitter spacetime.

It should be stressed again that the general configuration considered in (27) is only an approximate solution to the Euclidean field equations, when the instanton separation is very close to the horizon value $s = \pi/H$; because, however, of the repulsive nature of the instanton interaction, the integrals considered are dominated by these values (this is reflected by the fact that the leading term of the above expressions for the corrections is unity). I should also mention that the analysis presented here concerns essentially what would be one of the several zero modes, not considering the rest or the important question of the negative mode(s) [11]. Possible additional contributions arising from the treatment of these modes will be considered in future investigations.

It would also be interesting to see whether some of the approximations used can be relaxed; it is possible that this may be true for the thin wall approximation, since the instanton interaction comes mainly from the overlap of the “tails” of two bounce configurations. It would also be useful, from the phenomenological point of view, to investigate the limits of the fixed-background approximation. Hopefully these issues, as well as the application of these results to models of the cosmological landscape will be the subject of future work.

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