Portfolio optimization in the case of an exponential utility function and in the presence of an illiquid asset

Ljudmila A. Bordag

*Faculty of Natural and Environmental Sciences, University of Applied Sciences Zittau/Görlitz, Theodor-Körner-Allee 16, D-02763 Zittau, Germany

Abstract

We study an optimization problem for a portfolio with an illiquid, a liquid risky and a risk-free asset in the framework of continuous time. Problems of such type lead to a three dimensional nonlinear Hamilton-Jacobi-Bellman (HJB) equation on the value function. The corresponding solution of the three dimensional partial differential equation (PDE) describes the value function and the investment - consumption strategies. In this framework we suppose that the illiquid asset is sold in an exogenous random moment of time $T$ with a prescribed liquidation time distribution. Earlier we studied similar optimization problems with a HARA (hyperbolic absolute risk aversion) utility function and with a logarithmic utility function.

In this paper we study the optimization problem with two types of exponential utility functions, with negative and positive ones (denoted as EXPn and EXPp), which are equivalent from an economical point of view. Exponential utility functions belong to the type of CARA (constant absolute risk aversion) utility functions. It is well known that both the logarithmical (LOG) and the negative exponential (EXPn) utility functions are connected with the HARA utility function by means of a limiting procedure: in the first case the parameter of a HARA utility function is going to zero and in the second case to infinity. In our previous papers devoted to the optimization problem with a general HARA and LOG utility functions we proved that also the corresponding analytical and Lie algebraic structures are connected with the same limiting procedure. In this paper we show that the case of EXPn utility function differs from the case of the HARA utility function and has its own special Lie algebraic structure which is not connected to the HARA case by the limiting procedure. We carry out the Lie group analysis of the three dimensional PDEs for the cases positive and negative exponential utility functions and we are able to obtain the admitted symmetry algebras.

Here we prove also explicitly that both optimization problems with negative and positive exponential utility functions are connected by a one-to-one analytical substitution and are identical from the economical, analytical or Lie algebraic point of view. We use admitted Lie algebras to prove equivalence of the both problems as well as to obtain reductions of the studied PDEs equations. We provide the complete set of nonequivalent group invariant reductions of the three dimensional PDE corresponding to the optimization problem with the EXPn utility function to two dimensional PDEs in accordance with an optimal system of sub algebras of the admitted Lie algebra. We prove that in one case the invariant reduction is consistent with the boundary condition. The two dimensional PDE is more convenient for
applications of numerical methods as the original three dimensional PDE. Because of the
uniqueness of the solution of the HJB equation we can use the reduced two dimensional PDE
to study the properties of the optimal solution and the investment - consumption strategies.

Keywords: portfolio optimization, illiquidity, Lie group analysis, invariant reductions,
2010 MSC: 22E60, 35Q93, 35K55, 91G10

1. Introduction

We study an optimization problem for a portfolio with an illiquid, a liquid risky and a
risk-free asset in the framework of continuous time. We suppose that the illiquid asset is sold
in an exogenous random moment of time \( T \) with a prescribed liquidation time distribution.

Study of optimization problems with three assets including an illiquid asset leads to three
dimensional nonlinear Hamilton-Jacobi-Bellman (HJB) equations. The corresponding non-
linear three dimensional PDEs include a lot of parameters describing the behavior of assets
and are challenging for analytical and numerical methods. To simplify the investigated prob-
lem one try to find an inner symmetry of such equation and reduce the number of variables at
least to two or if possible to one. Usually low dimensional problems better studied and are,
therefore, easier to handle.

The influence of the risk tolerance preffered by the investor on the solution of an op-
timization problem was studied before for different portfolio steeting with and without an
illiquid asset. In the paper \[11\] the authors started with the classical Merton’s optimization
problem used in \[9\]. The portfolio contains one liquid risky asset and a risk free money mar-
ket account, the trading take place within the fixed time horizon. The authors explore the
question of the risk management under different risk preferences. They study the optimal
wealth process and portfolio process across different utilities and provide transformations be-
tween two such professeses corresponding to two arbitrary utilities. It is possible to find a
deterministic transformation using the lokal absolute risk tolerance function associated with
the corresponding utility. This transformation is defined by a solution of a linear heat equa-
tion with risk tolerance function as a coefficient by the second spatial derivative. Because of
the classical problem setting it is possible study the influence of the chosen utility and risk
tolerance on the wealth process and on the different characteristics of the optimal portfolio
in detail. They prove that the main role play the curvature of the risk tolerance function of
the preferred utility function. Certainly we cannot expect such tractability from a problem
setting with an illiquid asset, but we can use this model as a benchmark for the case if the
volume of the illiquid position of the studied portfolio vanish.

The dependence of optimal liquidation strategies from the risk aversion of investors were
studied in the work of Schied and Schöneborn \[16\]. The authors consider the infinite-horizon
optimal portfolio liquidation problem and use a stochastic control approach. In this model a
large investor trades one risky and one risk free asset. There by due to insufficient liquidity of
the risky asset the investor’s trading rate move the market price for the risky asset. Authors
obtain for the value function and the optimal strategy nonlinear parabolic PDEs. They have to
determine the adaptive trading strategy that maximizes the expected utility of the proceeds of
an large asset sale. Where by authors studied different types of the investor’s utility functions.

They found that the optimal strategy is aggressive or passive in-the-money in dependence
on the investor’s risk tolerance, i.e. if the utility function displays increasing or decreasing
risk aversion. The authors proved that such strategies are rational for investors with different absolute risk aversion profiles.

Another approach to liquidation problem of an illiquid asset provided in paper [10]. It devoted to the problem of how efficiently liquidate large assets positions up to an exogenous fixed terminal time. The author suppose that the investor prefer the negative exponential utility function and seeks to maximize the expected utility of the terminal value of his wealth. The portfolio contains the illiquid asset called a primary risky asset, a liquid asset which is imperfectly correlated with the primary asset and is called a proxy risky asset as well as a riskless money market account that pays zero interest rate. In practice the investor try to reduce the price impact by the trading a large amount of assets and to hedge market risk of the liquidated portfolio. As a common strategy one choose splitting of the given order into smaller pieces and to trade these pieces sequentially over time. The author is able to find the optimal strategies explicitly and study their properties. The strategies depend on time and parameters of the model only and are solutions of a linear ordinary differential equation of the second order. The author proves that this case is a generalization of the original Merton’s model studied in [9]. He also noticed that the explicit and simple results for optimal strategies was possible to obtain just by the using finite terminal time and because of the investor used the negative exponential utility function. A more realistic setting, for instance where the investor receives multiply orders at random times or the liquidation time is not fixed in the beginning leads to an essential more complicate model. In comparison to our case the illiquid asset in [10] do not pay any dividends and the investor can also split the illiquid asset and sell them piece by piece as well as the investor has no consumption during the life time of the portfolio.

The paper is organized as follows. We introduce the economical problem setting in detail in Section 2. In Section 3 we provide the Lie group analyses of the optimization problems with a general liquidation time distribution and different utility functions. In Section 4 for the optimization problem with the negative exponential utility function we chose an optimal system of sub algebras of the admitted Lie algebra and provide the complete set of all invariant reductions of the corresponding three dimensional PDE. In each case we prove if the invariant substitutions are compatible with the boundary condition. In Section 5 we discuss the connection between different results.

2. Economical problem setting

In [4], [5], [3] and in [2] the authors described in detail an optimization problem for a portfolio with three assets. An investor has an illiquid asset that has some paper value and can not be sold till some moment of time $T$ that is random with a prescribed liquidation time distribution. The investor tries to maximize her average consumption investing into a liquid risky asset that is partly correlated with the illiquid one and into a risk free asset, that has a constant dividend rate. The investor is free to chose an utility function in correspondence with her risk tolerance.

We notice that a risk tolerance $R(c)$ of an investor is defined as $R(c) = -\frac{U''(c)}{U''(c)}$ for any utility function $U(c)$. For the HARA and LOG utility functions the risk tolerance $R(c)$ is a linear function of $c$. We obtain for the HARA utility function $R(c) = \frac{c}{1-\gamma}$, $0 < \gamma < 1$ where $\gamma$ is a parameter of the HARA utility function (given in the form introduced later in (2.10)), and $R(c) = c$ for the LOG utility function. It means also that the absolute risk tolerance is increasing with the consumption in these two cases. For the positive EXPp or negative exponential
(EXPn) utility function the risk tolerance $R(c)$ is a constant. In other words all the time the absolute risk tolerance stay unaltered in the frame work of these optimization problems. We see that despite the fact that both the LOG and EXP utility functions can be regarded as a limit case of the HARA utility function they describes quite different economical situations: in the first case the risk tolerance growing up with the increasing consumption $c$ and in the case of the EXP utility the risk tolerance do not depend on the level of the consumption at all.

In previous papers we studied the optimization problems with HARA and LOG utility functions, i.e. the investor prefers a growing up risk tolerance, now we suppose that the investor chose an EXP utility function. It means that now the investor has a constant risk tolerance. May be it explains that both optimization problems studied before and presented now have quite different algebraic structures as we show it later.

2.1. Formulation of the optimization problem

The investor’s portfolio includes a risk free bond $B_t$, a risky asset $S_t$ and a non-traded asset $H_t$ that generates stochastic income, i.e., dividends or costs of maintaining the asset. The liquidation time of the portfolio $T$ is a randomly-distributed continuous variable. The risk free bank account $B_t$ with the interest rate $r$, follows

$$dB_t = rB_t \, dt, \quad t \leq T, \quad (2.1)$$

where $r$ is constant. The lower case index $t$ denotes the spot value of the asset at the moment $t$. The stock price $S_t$ follows the geometrical Brownian motion

$$dS_t = S_t(\alpha \, dt + \sigma \, dW^1_t), \quad t \leq T, \quad (2.2)$$

with the continuously compounded rate of return $\alpha > r$ and the standard deviation $\sigma$. The illiquid asset $H_t$, that can not be traded up to the time $T$ and its paper value is correlated with the stock index and the illiquid risky asset. The parameters $\mu, \delta, \eta, \rho$ are all assumed to be constant.

The randomly distributed time $T$ is an exogenous time and it does not depend on the Brownian motions $(W^1_t, W^2_t)$. The probability density function of the liquidation time distribution is denoted by $\phi(t)$, whereas $\Phi(t)$ denotes the cumulative distribution function, and $\Phi(t)$, the survival function, also known as a reliability function, $\Phi(t) = 1 - \Phi(t)$. We skip here the explicit notion of the possible parameters of the distribution in order to make the formulas shorter. In dependence on the rate of illiquidity the liquidation time distribution can take different forms. Typically one use the simplest one parameter exponential distribution with the reliability function $\Phi(t) = e^{-\kappa t}$, where $\kappa$ is the parameter of the distribution or a more advanced Weibull distribution with $\Phi(t) = e^{-t/\lambda^k}$ with two parameters $\lambda$ and $k$. We will take these two distributions as examples in our investigation. We notice that the exponential distribution is a special case of the Weibull distribution by $k = 1$ and $\kappa = 1/\lambda$. 4
We assume that the investor consumes at rate \( c(t) \) from the liquid wealth and the allocation-consumption plan \( (\pi, c) \) consists of the allocation of the portfolio with the cash amount \( \pi = \pi(t) \) invested in stocks, the consumption stream \( c = c(t) \) and the rest of the capital kept in bonds. The consumption stream \( c \) is admissible if and only if it is positive and there exists a strategy that finances it. Further on we sometimes omit the dependence on \( t \) in some of the equations for the sake of clarity of the formulas. All the income is derived from the capital gains and the investor must be solvent. In other words, the liquid wealth process \( L \) must cover the consumption stream. The wealth process \( L \) is the sum of cash holdings in bonds, stocks and random dividends from the non-traded asset minus the consumption stream, i.e. it must satisfy the balance equation

\[
\frac{dL}{dt} = (rL + \delta H_t + \pi_t(\alpha - r) - c_t) dt + \pi_t \sigma dW_t^1.
\]

The investor wants to maximize the overall utility consumed up to the random time of liquidation \( T \), given by

\[
\Upsilon(c) := E \left[ \int_0^T \Phi(t) U(c(t)) dt \right],
\]

as it was shown in [4]. It means we work with the problem (2.4) that corresponds to the value function \( V(l, h, t) \), which is defined as

\[
V(l, h, t) = \max_{(\pi, c)} \left\{ \int_t^\infty \Phi(t) U(c(t)) dt \mid L(t) = l, H(t) = h \right\},
\]

where \( l \) could be regarded as an initial capital and \( h \) as a paper value of the illiquid asset. The value function \( V(l, h, t) \) satisfies the Hamilton–Jacobi–Bellman (HJB) equation

\[
V_t(l, h, t) + \frac{1}{2} \eta^2 h^2 V_{hh}(l, h, t) + (rl + \delta h)V_l(l, h, t) + (\mu - \delta)hV_h(l, h, t) + \max_{\pi} G[\pi] + \max_{c \geq 0} H[c] = 0,
\]

where

\[
G[\pi] = \frac{1}{2} V_{\eta\eta}(l, h, t) \pi^2 \sigma^2 + V_{\eta\pi}(l, h, t) \eta \rho \pi \sigma h + \pi(\alpha - r)V_l(l, h, t),
\]

\[
H[c] = -cV_l(l, h, t) + \Phi(t) U(c),
\]

with the boundary condition

\[
V(l, h, t) \rightarrow 0, \quad \text{as} \quad t \rightarrow \infty.
\]

In [5] and [4] the authors have already demonstrated that the formulated problem has a unique solution under certain conditions. Namely, the following theorems were proved.

**Theorem 2.1.** [4]. There exists a unique viscosity solution of the corresponding HJB equation (2.5)-(2.9) if

1. \( U(c) \) is strictly increasing, concave and twice differentiable in \( c \),
2. \( \lim_{c \rightarrow 0} \Phi(t) E[U(c(t))] = 0, \quad \Phi(t) \sim e^{-\kappa t} \) or faster as \( t \rightarrow \infty \),
3. \( U(c) \leq M(1 + c)^{\gamma} \) with \( 0 < \gamma < 1 \) and \( M > 0 \),
4. \( \lim_{c \rightarrow 0} U'(c) = +\infty, \lim_{c \rightarrow +\infty} U'(c) = 0 \).

**Lemma 2.1.** [4]. Under the conditions (1) – (4) from Theorem 2.1, the value function \( V(t, l, h) \)

[2.3] has the following properties:

5
V(l, h, t) is concave and non-decreasing in l and in h,
(ii) V(l, h, t) is strictly increasing in l,
(iii) V(l, h, t) is strictly decreasing in t starting from some point,
(iv) 0 \leq V(l, h, t) \leq O(|l|^γ + |h|^γ) uniformly in t.

In this paper we restrict ourselves to the case of exponential utility functions that satisfy three first conditions of Theorem 2.1 by definition. The fourth condition should be replaced by the condition \( \lim_{c \to 0} U'(c) > 0, \lim_{c \to +\infty} U'(c) = 0 \) which will be satisfied by negative or positive exponential utility functions. We checked the proof of the theorem in [4] and see that the condition \( \lim_{c \to 0} U'(c) \to \infty \) can be replaced by the condition \( \lim_{c \to 0} U'(c) > 0 \) and the existence and uniqueness of the viscosity solution of HJB equation is still guaranteed.

In the previous papers [4], [5], [3] and [2] we used the HARA and LOG utility functions and studied connection between both optimization problems. We used the HARA utility function in the form

\[
U_{HARA}^1(c) = \frac{1 - \gamma}{\gamma} \left( \frac{c}{1 - \gamma} \right)^{\gamma} - 1, \quad 0 < \gamma < 1. \tag{2.10}
\]

It is easily to see that as \( \gamma \to 0 \) then the HARA utility function written as (2.10) tends to the LOG utility

\[
U_{HARA}^1(c) \to U^{LOG}(c) = \ln c. \tag{2.11}
\]

In common literature is often noticed that we obtain an exponential utility function as a limit case of a HARA utility function by \( \gamma \to \infty \). This assertion is correct just if the HARA utility function takes a special form, for instance, for the HARA utility in the form (2.10) it is not the case. It is easy to prove that if we take the HARA utility in the form

\[
U_{HARA}^2(c) = \frac{1 - \gamma}{\gamma} \left( \frac{ac}{1 - \gamma} + 1 \right)^{\gamma}, \quad 0 < \gamma < 1, \quad a > 0, \tag{2.12}
\]

then we obtain by the limiting procedure an exponential utility function

\[
U_{HARA}^2(c) \to U^{EXPn}(c) = -e^{-ac}. \tag{2.13}
\]

It is so called negative utility function (denoted as EXPn). The most common form of the the exponential utility function is

\[
U^{EXPp}(c) = \frac{1}{a} (1 - e^{-ac}), \quad a > 0. \tag{2.14}
\]

We call it positive exponential utility function (and denote it as EXPp). Both negative and positive exponential utility functions differs just by an additive and a multiplicative constant \( \frac{1}{a} \). We will study later both optimization problems, with EXPn and EXPp utility functions.

The forms \( U_{HARA}^1 \) and \( U_{HARA}^2 \) of the HARA utility function are often used and from an economical point of view both of them have properties of a HARA type utility functions. From the analytical point of view the HARA utility functions \( U_{HARA}^1 \) and \( U_{HARA}^2 \) are different.

In the first case \( U_{HARA}^1 \) we obtain by limiting transition \( \gamma \to 0 \) a logarithmical utility function (2.11). As we mentioned before the risk tolerance is in this case is equal to

\[
R_1(c) = -\frac{U'(c)}{U''(c)} = \frac{c}{1 - \gamma}. \tag{2.15}
\]
and by $\gamma \to 0$ we obtain $R_{LOG}(c) = c$ as it to expect. But we get neither a finite limit by $\gamma \to \infty$ of the function $U_1(c)$ nor a relevant value for the risk aversion $R_1(c)$ in this case.

In the second case for the utility function $U^HARA_2(2.12)$ we obtain for the risk tolerance the expression

$$R_2(c) = \frac{ac + 1 - \gamma}{a(1-\gamma)} = \frac{c}{1-\gamma} + \frac{1}{a}. \quad (2.16)$$

Here $U^HARA_2$ tends by the limiting transition $\gamma \to \infty$ to an exponential utility function $U^HARA_2(2.12)$ and the risk tolerance takes a constant value $R_2(c) = a^{-1}$ as it is to expect in the case of the exponential utility function. But here in contradiction to the first case of $U^HARA_1$ we cannot obtain any meaningful expression by the limiting procedure $\gamma \to 0$, it means we do not obtain a transition to the logarithmic utility function.

In other words to study connection between two optimization problems with a HARA utility function and with a logarithmic utility function we should use the HARA utility for instance in the form $U^HARA_1(c)$ to be able to provide the limiting procedure $\gamma \to 0$ in all formulas. For study the connection between two optimization problems with a HARA utility and with an exponential utility we should use other form of the HARA utility, for instance, of type $U^HARA_2(2.12)$ to be able to make the limiting transition for $\gamma \to \infty$ in corresponding formulas.

Because of the relation $(2.13)$ to correct comparison of the results for the HARA utility function $U^HARA_2$ with the results for an exponential utility function we need to study first the optimization problem with the negative exponential utility function. Other sides the positive $U^HARA_2$ form of utility function and with a logarithmic utility function we should use the HARA utility for

$$V_1(l,h,t) + \frac{1}{2} \eta^2 h^2 V_{hh}(l,h,t) + (r l + \delta h)V_l(l,h,t) + (\mu - \delta) h V_h(l,h,t) \quad (3.17)$$

$$- \frac{1}{2} \sigma^2 V_{ll}(l,h,t)$$

$$+ \frac{1}{a} V_l(l,h,t) \ln V_l(l,h,t) - \frac{1}{a} \ln \Phi(t) V_l(l,h,t) - \ln \frac{a}{c} V_l(l,h,t) = 0, \quad V \to 0, \ t \to \infty.$$
the value function \( V(l, h, t) \)

\[
\pi(l, h, t) = -\frac{\eta \rho \sigma h V_l(l, h, t) + (\alpha - r)V_l(l, h, t)}{\sigma^2 V_l(l, h, t)}, \tag{3.18}
\]

\[
c(l, h, t) = \frac{1}{a} \ln \left( \frac{\Phi(t)}{aV_l(l, h, t)} \right). \tag{3.19}
\]

Equation (3.17) is a nonlinear three dimensional PDE with three independent variables \( l, h, t \) such equations are demanding by study with analytical or numerical methods. The Lie group analysis of a nonlinear PDE is a proper tool to obtain the Lie algebra admitted by this equation. Using the generators of the symmetry algebra one can reduce the dimension of the equation (3.17) and make a problem better tractable. In detail one can find the description of this method in [12], [6] or in [1] where a short and comprehensive introduction in this method is given as well as applications to other PDEs arising in financial mathematics.

Here we formulate the main theorem of Lie group analysis for the optimization problem with the negative exponential utility function.

**Theorem 3.1.** The HJB equation (3.17) with the negative exponential utility function (2.13) and with a general liquidation time distribution \( \Phi(T) \) admits the four dimensional Lie algebra \( L_4^{Exp} \) spanned by generators \( U_1, U_2, U_3, U_4 \), i.e. \( L_4^{Exp} = \langle U_1, U_2, U_3, U_4 \rangle \), where

\[
U_1 = \frac{1}{a r} \frac{\partial}{\partial l} - V \frac{\partial}{\partial V}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad U_2 = \frac{\partial}{\partial V}, \tag{3.20}
\]

\[
U_3 = -\frac{1}{a r} \left( e^{\alpha t} \int e^{-\alpha t} d\Phi(t) \right) \frac{\partial}{\partial l} + \frac{1}{r} \frac{\partial}{\partial t}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad U_4 = e^{\alpha t} \frac{\partial}{\partial l},
\]

with following non trivial commutation relations

\[
[U_1, U_2] = U_2, \quad [U_3, U_4] = U_4. \tag{3.21}
\]

Except finite dimensional Lie algebra \( L_4^{Exp} \) (3.20) the equation (3.17) admits also an infinite dimensional algebra \( L_{\infty} = \langle \psi(h, t) \frac{\partial}{\partial h} \rangle \) where the function \( \psi(h, t) \) is any solution of the linear parabolic PDE

\[
\psi_t(h, t) + \frac{1}{2} \sigma^2 h^2 \psi_{hh}(h, t) + (\mu - \delta) \psi(h, t) = 0. \tag{3.22}
\]

**Proof of Theorem 3.1.** As in [12], [6] or [1] we introduce the second jet bundle \( j^2 \) and present the equation (3.17) in the form \( \Delta(l, h, t, V, l, l, V, l, h, V, l, h, V, h) = 0 \) as a function of these variables in the jet bundle \( j^2 \). We look for generators of the admitted Lie algebra in the form

\[
\mathbf{U} = \xi_1(l, h, t, V) \frac{\partial}{\partial l} + \xi_2(l, h, t, V) \frac{\partial}{\partial h} + \xi_3(l, h, t, V) \frac{\partial}{\partial t} + \eta_1(l, h, t, V) \frac{\partial}{\partial V}, \tag{3.23}
\]

where the functions \( \xi_1, \xi_2, \xi_3, \eta_1 \) can be found using the over determined system of determining equations

\[
\mathbf{U}^{(2)} \Delta(l, h, t, V, l, l, V, l, l, V, l, h, V, h) |_{\Delta=0} = 0, \tag{3.24}
\]

where \( \mathbf{U}^{(2)} \) is the second prolongation of \( \mathbf{U} \) in \( j^2 \). We look at the action of \( \mathbf{U}^{(2)} \) on \( \Delta(l, h, t, V, l, l, V, l, l, V, l, h, V, h) \) located on its solution subvariety \( \Delta = 0 \) and obtain an
The equations (3.27) contain four arbitrary constants \( \xi_1, \xi_2, \xi_3 \) and \( \eta_1 \) from (3.23). This system has 130 PDEs on the functions \( \xi_1, \xi_2, \xi_3, \eta_1 \). The most of them are trivial and lead to following conditions on the functions

\[
(\xi_1)_t = 0, \quad \xi_1 = \xi_1(t) \\
(\xi_2)_t = 0, \quad \xi_2 = \xi_1(t) \\
(\xi_3)_t = 0, \quad \xi_2 = \xi_2(t) \\
(\eta_1)_t = 0, \quad \eta_1 = \eta_1(h,t).
\]

Consequently the unknown functions in (3.23) have the following structure

\[
\xi_1(l,h,t,V) = \xi_1(t)l + \xi_1(t), \quad \xi_2(l,h,t,V) = \xi_1(t), \quad \xi_3(l,h,t,V) = \xi_1(t), \quad \eta_1(l,h,t,V) = \eta_1(h,t).
\]

Here \( \xi_1(t), \xi_2(t), \xi_3(t), \eta_1(h,t), \) and \( \eta_1(h,t) \) are some functions which will be defined later. To find these unknown functions we should have a closer look on the non-trivial equations of the obtained system, that are left. After all simplifications we get the system of seven PDEs

\[
\eta_{11} + \eta^2 h^2 \eta_{1h} + (\mu - \delta) h \eta_{1h} = 0, \quad (3.26)
\]

We introduce the differential operator \( L = \frac{\partial}{\partial t} + \frac{1}{2} \eta^2 h^2 \frac{\partial^2}{\partial t^2} + (\mu - \delta) h \frac{\partial}{\partial t} \) using this operator we can rewrite the first equation in the above system as conditions on the functions \( \eta_{11}(h,t) \) and \( \eta_{12}(h,t) \) which appears in the last equation of (3.25) correspondingly as \( L \eta_{11}(h,t) = 0 \) and \( L \eta_{12}(h,t) = 0 \). Other equations in the above system do not contain the function \( \eta_{12}(h,t) \) at all. If we now denote \( \eta_{12}(h,t) = \psi(h,t) \) then we see that we proved last statement of the theorem, see (3.22).

Solving the system (3.26) for an arbitrary function \( \Phi(t) \) we obtain

\[
\xi_1 = c_{11} e^{\alpha t} + \eta_{11} + \frac{1}{a} \int e^{-\alpha t} \frac{\Phi}{\Phi} dt, \quad \xi_2 = 0, \quad \xi_3 - \text{const}. \quad \eta_1 = \eta_{11} V + \eta_{122} + \psi(h,t), \quad c_{11}, \eta_{11}, \eta_{122} - \text{const.}
\]

The equations (3.27) contain four arbitrary constants \( \xi_3, \xi_1, \eta_{11}, \eta_{122} \) and a function \( \eta_{12}(h,t) = \psi(h,t) \) which is an arbitrary solution of \( L \psi(h,t) = 0 \). Formulas (3.27) define four generators of the finite dimensional Lie algebra \( L_{\exp n}^{(4)} \) and the infinitely dimensional algebra \( L_{\exp n} \) as it was described in Theorem 3.2.
Remark 3.1. The found four dimensional Lie algebra describes the symmetry property of the equation (3.17) for any function $\overline{\Phi}(t)$. In [4], [5] we have proved the theorem for existence and uniqueness of the solution of HJB equation for a liquidation time distribution for which $\overline{\Phi}(t) \sim e^{-\gamma t}$ or faster as $t \to \infty$, therefore we will regard this type of the distribution studying the analytical properties of the equation further on.

First we explain a meaning of some generators of the Lie algebra listed in Theorem [3.1]. We start with the second generator $U_2 = \frac{\partial}{\partial \psi}$. It means that the original value function $V(l,h,t)$ which is a solution of the equation (3.17), can be shifted on any constant and still be a solution of the same equation. Neither allocation $\pi$ or consumption function $c$ will change their values, because they also depend only on the derivatives of the value functions. In some sense it is a trivial symmetry, since the equation (3.17) contains just the derivatives of $V(l,h,t)$ so we certainly can add a constant to this function and it sill will be a solution of the equation. Following this symmetry does not give a rise to any reductions of the studied three dimensional PDE and this symmetry do not satisfied the boundary condition $V(l,h,t) \to 0$, $t \to \infty$ because of that it is not interesting by solution of the possed optimization problem.

The fourth generator $U_4 = e^{\pi t} \frac{\partial}{\partial t}$ means that the value of the independent variable $l$ can be shifted on the arbitrary value $de^{\pi t}$, i.e. the shift $l \to l + de^{\pi t}$, $d - \text{const.}$ leaves the solution unaltered. From economical point of view it means that the absolut value of the initial capital is not important for this problem. We can arbitrary shift the initial liquidity $l$ on a bank account $d$, $d > 0$ or credit $d$, $d < 0$ yet $l + de^{\pi t}$ should be positiv in the initial time moment. The value function $V(l,h,t)$ as a solution of the equation (3.17) and the allocation-consumption strategy $(\pi,c)$ will be unaltered. This symmetry is also trivial and it does not provide any reductions of the original three dimensional PDEs.

We also get an infinitely-dimensional algebra $L_\infty = \langle \psi(h,t) \frac{\partial}{\partial \psi} \rangle$ where the function $\psi(h,t)$ is any solution of the linear PDE $\psi_t + \frac{1}{2} \eta^2 h^2 \psi_{hh} + (\mu - \delta) h \psi_h = 0$, see Theorem 3.1. It has a special meaning - we can add any solution $\psi(h,t)$ of this equation to the value function $V(l,h,t)$ without any changes of the allocation-consumption strategy $(\pi,c)$. From economical point of view it means that the additional use of some financial instrument which is the solution of $\psi_t + \frac{1}{2} \eta^2 h^2 \psi_{hh} + (\mu - \delta) h \psi_h = 0$ do not change the investment-allocation strategies in this optimization problem. The boundary condition $V(l,h,t) \to 0$, $t \to \infty$ leads to the following boundary condition on the solution of this equation $\psi(h,t) \to 0$, $t \to \infty$. It means it is a financial instrument which value is defined just by the paper value of the illiquid asset and time only, can not change the allocation-consumption strategy $(\pi,c)$. We notice also that after the substitution $h = e^t$, $t = -2 \tau/\eta^2$, $\psi(t,h) = \tilde{h}\left(\mu - \delta - \frac{1}{2} \eta^2\right) e^{-\left(\mu - \delta - \frac{1}{2} \eta^2\right)^2/\eta^4} \psi(\tau,x)$ we obtain on the function $\psi(\tau,x)$ the parabolic equation of the type $v_t = v_{xx}$, which is well studied. The solution methods as well as the fundamental solution of this equation are well known.

3.1. Relation between two optimization problems with the HARA and with the negative exponential utility functions

In our previous papers we studied the optimization problem with an illiquid asset in the case if the investor used the HARA utility function (2.10) or the logarithmic utility function (2.11). It is well known that both problems are connected by limiting procedure if $\gamma \to 0$. In previous paper [3] we proved that also analytic and algebraic structures of both optimization problems also connected with the same limiting procedure.
We noticed before that also negative exponential utility function is connected to the HARA utility function $U_{2}^{\text{HARA}}(c)$ with the limiting procedure by $\gamma \to \infty$, see (2.13).

It means also that we cannot use direct the results of the Lie group analysis obtained in previous works [3] and [2] to compare the admitted Lie algebras for the optimization problem with the HARA utility function in the form $U_{1}^{\text{HARA}}(c)$ with the results in this work for an optimization problem with an exponential utility function. Because of that we should recalculate the results of the Lie group analysis for the new form of the HARA utility function. We remember that we first provide the formal maximization in the HJB equation (2.6) and correspondingly to the chosen utility function obtain a three dimensional PDE. In our previous works [3] and [2] we used the utility function in the form (2.10) and got following PDE

\[
\begin{align*}
V_t(t,l,h) + \frac{1}{2} \eta^2 h^2 V_{hh}(t,l,h) + (rl + \delta h)V_l(t,l,h) + \mu hV_h(t,l,h) + \left(\alpha - r\right)^2 V_r^2(t,l,h) + 2(\alpha - r) \eta \rho V_r V_l(t,l,h) + \left(\alpha - r\right) \eta^2 \rho^2 \sigma^2 h^2 V_{rl}(t,l,h) + \frac{2 \sigma^2 V_{rr}(t,l,h)}{2} + \frac{1 - \gamma}{\gamma} \bar{\Phi}(t) V_l(t,l,h) - \frac{1}{\gamma} - \frac{1 - \gamma}{\gamma} \bar{\Phi}(t) = 0, \quad V \to 0. \tag{3.28}
\end{align*}
\]

Now if we insert in the HJB equation (2.6) the HARA utility function $U_{2}^{\text{HARA}}(2.12)$ then we obtain the PDE in the form

\[
\begin{align*}
V_t(t,l,h) + \frac{1}{2} \eta^2 h^2 V_{hh}(t,l,h) + (rl + \delta h)V_l(t,l,h) + \mu hV_h(t,l,h) + \left(\alpha - r\right)^2 V_r^2(t,l,h) + 2(\alpha - r) \eta \rho V_r V_l(t,l,h) + \left(\alpha - r\right) \eta^2 \rho^2 \sigma^2 h^2 V_{rl}(t,l,h) + \frac{2 \sigma^2 V_{rr}(t,l,h)}{2} + \frac{1 - \gamma}{\gamma} \bar{\Phi}(t) V_l(t,l,h) - \frac{1}{\gamma} - \frac{1 - \gamma}{\gamma} \bar{\Phi}(t) = 0, \quad V \to 0. \tag{3.29}
\end{align*}
\]

The equations (3.28) and (3.29) differs analytically in the last terms, from economical point of view they describe equivalent optimization problems. The Lie group analysis of the first equation was provided in [3]. Now we use the same method and find the admitted Lie group for the second equation (3.29). We formulate the results of in the following theorem

**Theorem 3.2.** The equation (3.29) admits the three dimensional Lie algebra $L_3^{\text{HARA}}$ spanned by generators $L_3^{\text{HARA}} = \langle U_1, U_2, U_3 \rangle$, where

\[
\begin{align*}
U_1 &= \frac{\partial}{\partial V}, \quad U_2 = \epsilon r \frac{\partial}{\partial l}, \quad U_3 = \left(1 - \frac{\gamma}{ar}\right) \frac{\partial}{\partial l} + \frac{\partial}{\partial h} + \gamma V \frac{\partial}{\partial V}, \tag{3.30}
\end{align*}
\]

for any liquidation time distribution. Moreover, if and only if the liquidation time distribution has the exponential form, i.e. $\bar{\Phi}(t) = e^{-\lambda d}$, where $d, \lambda$ are constants the studied equation admits a four dimensional Lie algebra $L_4^{\text{HARA}}$ with an additional generator

\[
U_4 = \frac{\partial}{\partial t} - \kappa V \frac{\partial}{\partial V}, \tag{3.31}
\]

i.e. $L_4^{\text{HARA}} = \langle U_1, U_2, U_3, U_4 \rangle$. Except finite dimensional Lie algebras (3.35) and (3.31) correspondingly equation (3.28)
admits also an infinite dimensional algebra \( L_\infty = \langle \psi(h,t) \frac{\partial}{\partial \psi} \rangle \) where the function \( \psi(h,t) \) is any solution of the linear PDE

\[
\psi_t(h,t) + \frac{1}{2} \eta^2 h^2 \psi_{hh}(h,t) + (\mu - \delta) h \psi_t(h,t) = 0.
\] (3.32)

The Lie algebra \( L_3^{\text{HARA}_1} \) has the following non-zero commutator relations

\[
[U_1, U_3] = \gamma U_1, \quad [U_2, U_3] = U_2
\] (3.33)

The Lie algebra \( L_4^{\text{HARA}_2} \) has the following non-zero commutator relations

\[
[U_1, U_3] = \gamma U_1, \quad [U_1, U_4] = -\kappa U_1, \quad [U_2, U_3] = U_2, \quad [U_2, U_4] = -r U_2
\] (3.34)

We will not provide the proof of the Theorem 3.2 because it is quite similar to the proof of the previous Theorem 3.1 for the equation (3.17). In the paper \([3]\) we obtained for the equation (3.28) with a general liquidation time distribution also three dimensional Lie algebra \( L_3^{\text{HARA}_1} \) with generators of the following form

\[
U_1 = \frac{\partial}{\partial \psi}, \quad U_2 = e^{rt} \frac{\partial}{\partial t}, \quad U_3 = l \frac{\partial}{\partial l} + h \frac{\partial}{\partial h} + \left( \mathcal{V} - (1 - \gamma) \int \Phi(t) dt \right) \frac{\partial}{\partial \mathcal{V}},
\] (3.35)

as well as an infinite dimensional algebra \( L_\infty = \langle \psi(h,t) \frac{\partial}{\partial \psi} \rangle \) where the function \( \psi(h,t) \) is any solution of the linear PDE \( \psi_t(h,t) + \frac{1}{2} \eta^2 h^2 \psi_{hh}(h,t) + (\mu - \delta) h \psi_t(h,t) = 0 \). It is easy to see that both algebras \( L_3^{\text{HARA}_1} \) and \( L_4^{\text{HARA}_2} \) have the same commutation relations and are isomorph. We prove that the admitted algebras are also similar. Indeed if we take the substitutions

\[
\beta = l - \frac{1 - \gamma}{ar}, \quad \tilde{h} = h, \quad \tilde{t} = t, \quad \tilde{V} = V + \frac{1 - \gamma}{\gamma} \int \Phi(t) dt
\] (3.36)

then the equation (3.28) on the function \( \tilde{V}(\tilde{l}, \tilde{h}, \tilde{t}) \) will be replaced by the equation (3.29) on the value function \( V(l,h,t) \). Correspondingly the generators of the algebra (3.35) will take form of the generators of (3.30). It means that the Lie algebras \( L_3^{\text{HARA}_1} \) and \( L_4^{\text{HARA}_2} \) are isomorph and similar and the optimization problems with the utility functions \( U_3^{\text{HARA}_1} \) and with \( U_4^{\text{HARA}_2} \) are equivalent from an economical and analytical point of view. But now we have a correct form of generators of Lie algebra to study a limiting procedure by \( \gamma \to \infty \). Indeed using the properties of the generators of a Lie algebra we obtain from (3.30)

\[
U_1^\infty = \frac{\partial}{\partial \psi}, \quad U_2^\infty = e^{rt} \frac{\partial}{\partial t}, \quad U_3^\infty = \frac{1}{ar} \frac{\partial}{\partial l} - V \frac{\partial}{\partial \mathcal{V}}.
\] (3.37)

We apply now the limiting transition to the main equation (3.29) and see that we do not get the corresponding PDE in the form (3.17), despite the fact that the \( U_2^{\text{HARA}_1} \) and EXPn utility are connected with this limiting procedure. We obtain on this step different analytical structures.

Now we compare this algebraic structure with the described in Theorem 3.1. First we see that both three dimensional PDEs have the same infinite dimensional algebra \( L_\infty = \langle \psi(h,t) \frac{\partial}{\partial \psi} \rangle \). Then we compare the finite dimensional algebras (3.37) and (3.20) and see that the finite dimensional algebras in both cases are essentially different. In the case of HARA
utility function $U_{2}^{\text{HARA}}$ and general liquidation time distribution we have after limiting transition $\gamma \to \infty$ the three dimensional algebra (3.37) and in the case of the negative exponential utility function we got the four dimensional algebra (3.20). These algebras do not connected with the limiting procedure by $\gamma \to \infty$ as well as the both three dimensional PDEs are not connected with this limiting procedure. Other sides it is easy to see that the all three generators (3.37) coincide with the three of the four generators of (3.20). The algebra (3.20) is in some way extension of the algebra (3.37). It means that using the exponential utility functions makes the corresponding optimization problem smoother from Lie algebraic point of view.

We see that by limiting procedure $\gamma \to \infty$ neither the analytic nor the algebraic structure of the optimization problem will be preserved. If in the previous cases of HARA - and LOG - utility functions it was sufficient to study the case of HARA utility and then just take a limit by $\gamma \to 0$ to obtain the corresponding results for the optimization problem with LOG utility function now we should study the optimization problem with the exponential utility function in own rights step-by-step independently from the case of HARA utility function.

In the next section we study the optimization problem with the positive utility function (EXPp). Because the utility functions are defined up to additive and multiplicative constants the results should be in some sense equivalent. We will prove this equivalence for the optimizations problems with EXPn and EXPp utility functions.

### 3.2. Relation between two optimization problems with negative and positive exponential utility functions

A positive exponential utility function EXPp (2.14) is very close to the negative exponential function (2.13), yet this particular case rather popular therefore we analyze it separately.

The whole approach is very similar to the method described in the beginning of this section therefore we omit some details here. In the case of the positive exponential utility function (2.14) the HJB equation after the formal maximization procedure will take the following form

$$
V_t + \frac{1}{2} \frac{\eta^2}{h} V_{hh} + (r l + \delta h) V_l + (\mu - \delta) h V_h
$$

$$
- (\alpha - r)^2 V_l^2 + 2(\alpha - r) \eta \rho h V_l V_h + \eta^2 \rho^2 \sigma^2 h^2 V_{lh}^2
$$

$$
+ \frac{1}{a} V_l \ln V_l - \frac{1}{a} \left(1 + \ln \Phi(t) \right) V_l + \frac{1}{a} \Phi(t) = 0 , \quad V \to 0 , \ t \to \infty .
$$

The main PDE (3.17) for the negative exponential utility function differs from this one for positive exponential utility function by one term only. If in (3.17) the last term was $-\left(\frac{1}{a} \ln a \right) V_l$ now it is $\frac{1}{a} \Phi(t)$, i.e. now equation (3.38) has a free term $\frac{1}{a} \Phi(t)$ without the dependent variable $V(l, h, t)$ as it was before.

Analogously to the previous chapter we can formulate and prove the main theorem of Lie group analysis for the HJB optimization problem with the positive exponential utility function.

**Theorem 3.3.** The equation (3.38) admits the four dimensional Lie algebra $L_4^{\text{EXPp}}$ spanned
by generators \(U_1, U_2, U_3, U_4\), i.e. \(L_4^{\text{EXPp}} = \langle U_1, U_2, U_3, U_4 \rangle\), where

\[
U_1 = \frac{1}{ar} \frac{\partial}{\partial l} - \left( V + \frac{1}{a} \int \Phi(t) dt \right) \frac{\partial}{\partial V}, \quad U_2 = \frac{\partial}{\partial l}, \quad (3.39)
\]

\[
U_3 = -\frac{1}{ar} e^{-rt} \left( \int e^{-rt} d\ln \Phi(t) \right) \frac{\partial}{\partial l} + \frac{1}{r} \frac{\partial}{\partial t} - \frac{1}{ar} \Phi(t) \frac{\partial}{\partial V}, \quad U_4 = e^{rt} \frac{\partial}{\partial l}.
\]

for any liquidation time distribution \(\Phi(t)\). Except finite dimensional Lie algebra \(L_4^{\text{EXPp}}\) the equation (3.38) admits also an infinite dimensional algebra \(L_\infty = \langle \psi(h,t) \frac{\partial}{\partial V} \rangle\) where the function \(\psi(h,t)\) is any solution of the linear PDE

\[
\psi_t(h,t) + \frac{1}{2} \eta^2 h^2 \psi_{hh}(h,t) + (\mu - \delta) h \psi_h(h,t) = 0. \quad (3.40)
\]

The Lie algebra \(L_4^{\text{EXPp}}\) has the following two non-zero commutator relations

\[
[U_1, U_2] = U_2, \quad [U_3, U_4] = U_4. \quad (3.41)
\]

If we compare non trivial commutation relations (3.21) for \(L_4^{\text{EXPn}}\) and (3.41) for \(L_4^{\text{EXPp}}\) we see that they coincide. It means that both algebras \(L_4^{\text{EXPn}}\) and \(L_4^{\text{EXPp}}\) are isomorphic because they have the same set of the structure constants. It means also that both Lie symmetry algebras \(L_4^{\text{EXPn}}\) and \(L_4^{\text{EXPp}}\) correspond to \(A_2 \oplus A_2\) [15].

If we are able to prove that the both algebras are also similar, then the both optimization problems with the positive and with the negative exponential utility functions are equivalent and we provide in the same time the equivalence substitution. The similarity of two algebras means that they are not just isomorphic but also that there exist an analytical substitution which provided analytical identity between corresponding generators. We can then use this substitution to transform the equation (3.38) to (3.17) or both equations to one and the same equation.

It is easy to see that if we make following transformations of the variables \(l, h, t, V\) in equation (3.38)

\[
l = \tilde{l} - \ln a \frac{\alpha}{ar}, \quad h = \tilde{h}, \quad t = \tilde{t}, \quad (3.42)
\]

\[
V(l, h, t) = \tilde{V}(\tilde{l}, \tilde{h}, \tilde{t}) - \frac{1}{a} \int \tilde{\Phi}(t) dt.
\]

then the final equation in variables \(\tilde{l}, \tilde{h}, \tilde{t}, \tilde{V}\) coincide with (3.17). Because the substitution (3.42) is an invertible analytical one-to-one substitution we have to do with two identical optimization problems. The analytical and algebraic structures of optimization problems with negative and positive utility functions are equivalent and it is enough to study one of these problems in details.

4. Optimal system of subalgebras of \(L_4^{\text{EXPN}}\) and related invariant reductions

In order to find all reductions and in this way to find all classes of non equivalent group invariant solutions of a differential equation Ovsiannikov [13] has introduced the idea of an optimal system of subalgebras for a given symmetry algebra of this differential equation. This
idea is now widely used for PDEs and systems of ODEs arising in different areas of sciences \([14, 7, 8]\).

Now we will study a complete set of possible reductions of the three dimensional PDE \((3.17)\) to two dimensional PDEs. For this purpose we need an optimal system of subalgebras of \(L_4^{\text{EXP}_n}\). As before in \([3]\) we use the optimal system developed in \([15]\) for real four dimensional Lie algebras of this type. To make the comparison of the results transparenter we introduce in this Section the same notations for the generators of \(L_4^{\text{EXP}_n}\) as in \([15]\) and in \([3]\).

In this basis there are only two non-zero commutation relations \((3.21)\) on \(L_4^{\text{EXP}_n}\). If we introduce notations like in the paper \([15]\), i.e. we denote \(U_i = e_i\) where \(i = 1, \ldots, 4\) then we can rewrite the relations \((3.21)\) as

\[
[e_1, e_2] = e_2, \quad [e_3, e_4] = e_4.
\]

Now we can see that \(L_4^{\text{EXP}_n}\) corresponds to \(A_2 \oplus A_2\), in the classification of \([15]\) where also optimal systems of subalgebras for three and four dimensional solvable Lie algebras are provided. The corresponding system of optimal subalgebras of \(L_4^{\text{EXP}_n}\) is listed in Table 1. We will use it to obtain all non equivalent reductions of the original equation \((3.17)\).

| Dimension of the subalgebra | System of optimal subalgebras of algebra \(L_4^{\text{EXP}_n}\) |
|----------------------------|--------------------------------------------------|
| 1                         | \(h_1 = \langle e_2 \rangle, h_2 = \langle e_3 \rangle, h_3 = \langle e_4 \rangle, h_4 = \langle e_1 + \omega e_3 \rangle, h_5 = \langle e_1 \pm e_4 \rangle, h_6 = \langle e_2 \pm e_4 \rangle, h_7 = \langle e_2 \pm e_3 \rangle\) |
| 2                         | \(h_8 = \langle e_1, e_3 \rangle, h_9 = \langle e_1, e_4 \rangle, h_{10} = \langle e_2, e_3 \rangle, h_{11} = \langle e_2, e_4 \rangle, h_{12} = \langle e_1 + \omega e_3, e_2 \rangle, h_{13} = \langle e_3 + \omega e_1, e_4 \rangle, h_{14} = \langle e_1 \pm e_4, e_2 \rangle, h_{15} = \langle e_3 \pm e_2, e_4 \rangle, h_{16} = \langle e_1 + e_3, e_2 \pm e_4 \rangle\) |
| 3                         | \(h_{17} = \langle e_1, e_3, e_2 \rangle, h_{18} = \langle e_1, e_4, e_2 \rangle, h_{19} = \langle e_1, e_3, e_4 \rangle, h_{20} = \langle e_2, e_3, e_4 \rangle, h_{21} = \langle e_1 \pm e_3, e_2, e_4 \rangle, h_{22} = \langle e_1 + \omega e_3, e_2, e_4 \rangle\) |

Table 1: The optimal system of one-, two- and three- dimensional subalgebras of \(L_4^{\text{EXP}_n}\), where \(\omega\) is a parameter such that \(-\infty < \omega < \infty\).

Now we are going to study all possible invariant reductions of the problem \((3.17)\). Let us first note that the subgroups \(H_1, H_3\) and \(H_5\) generated by sub algebras \(h_1 = \left\langle \frac{\partial}{\partial \eta} \right\rangle, h_3 = \left\langle e^{\eta} \frac{\partial}{\partial \eta} \right\rangle\) and \(h_5 = \left\langle \frac{\partial}{\partial \eta} \pm e^{\eta} \frac{\partial}{\partial \eta} \right\rangle\) correspondingly, do not give us any interesting reductions so we omit the detailed study of these cases here. We start with a first interesting and non-trivial case.

**Case \(H_2(h_2)\).** The sub algebra \(h_2\) is spanned by the generator \(e_3\)

\[
h_2 = \langle e_3 \rangle = \left\langle -\frac{1}{ar} \left( e^{\eta} \int e^{-\eta} d\ln \Phi(t) \right) \frac{\partial}{\partial \tilde{t}} + \frac{1}{r} \frac{\partial}{\partial \Phi(t)} \right\rangle.
\]

To find all invariants of the subgroup \(H_2\) we solve the related characteristic system of equations

\[
\frac{dl}{-\frac{1}{ar} \left( e^{\eta} \int e^{-\eta} d\ln \Phi(t) \right)} = \frac{dt}{\frac{1}{r}} = \frac{dV}{0} = \frac{dh}{0}.
\]

\[\text{Page 15}\]
where the last two equations of the system present a formal notation that shows that the independent variable \( h \) and the dependent variable \( V \) are actually invariants under the action of the sub group \( H_2 \). We can obtain another independent invariants solving the system above. So we obtain a set of independent invariants

\[
\begin{align*}
inv_1 &= z = l + \frac{1}{ar} e^{rt} \int e^{-rt} d\ln \Phi(t) - \frac{1}{ar} \ln \Phi(t) - \frac{1}{ar} (1 + \log a), \quad inv_2 = h, \\
inv_3 &= W(z,h) = V(l,h,t).
\end{align*}
\]

The invariants (4.43) can be used as the new independent variables \( z, h \) and the invariant (4.44) as the new dependent variable \( W(t,z) \) to reduce the three dimensional PDE (3.17) to a two dimensional one

\[
\frac{1}{2} \eta^2 h^2 W_{hh} + (\mu - \delta) h W_t + (rz + \delta h) W_z + \frac{1}{a} W_z \log W_z - \frac{(\alpha - r)^2 W_z^2 + 2(\alpha - r) \eta \rho h W_t W_{zh} + \eta^2 \rho^2 \sigma^2 h^2 W_{zh}^2}{2 \sigma^2 W_{zz}} = 0.
\]

In (2.9) formulate the boundary condition and in Lemma 2.1 we formulate the main properties of the value function. Now we have to reformulate the boundary condition on the function \( W(z,h) \) after the substitution (4.43). To make further remarks transparent we take first as an example the simples form of the liquidation time distribution and suppose that \( \Phi(t) = e^{-\kappa t} \), i.e. we have to do with exponential liquidation time distribution. The new variable \( z \) will take the form

\[
z = l + \frac{\kappa}{ar} t + \frac{1}{ar} \left( \frac{\kappa}{r} - 1 - \log a \right).
\]

It means that \( z \) is increasing if \( l \) or \( t \) are growing up. But it leads to contradiction between the properties of the function \( W(z,h) = V(l,h,t) \). One sides the boundary condition demands that the value function tends to zero for \( t \to \infty \), other sides that the same function is strictly increasing by \( l \to \infty \). Because after the invariant substitution the new variable \( z \) is the sum of these two old variables \( l \) and \( t \) we are not able to solve this contradiction. The similar inconsistency problem arising if we use other form of the function \( \Phi(t) \). Following this reduction cannot be used to solve of the optimization problem.

**Case \( H_4(h_4) \).** Now we look for invariants of the sub group \( H_4 \). The corresponding subalgebra \( h_4 \) is spanned by the generator \( e_1 + \omega e_3 \), i.e.

\[
h_4 = \left\langle \frac{1}{ar} \left( 1 - \omega e^r \int e^{-rt} d\ln \Phi(t) \right) \frac{\partial}{\partial l} + \omega \frac{\partial}{\partial t} - V \frac{\partial}{\partial V} \right\rangle.
\]

We need to regard two special cases \( \omega = 0 \) and \( \omega \neq 0 \). Here if \( \omega = 0 \) then

\[
h_4 =< e_1 > = \left\langle \frac{1}{ar} \frac{\partial}{\partial l} - V \frac{\partial}{\partial V} \right\rangle.
\]

The invariants of the group \( H_4 \) are

\[
inv_1 = h, \quad inv_2 = t, \quad inv_3 = W(h,t) = V(l,h,t) e^{arl}.
\]

From the last relation follows that \( V(l,h,t) = W(h,t) e^{-arl} \). We see that in this case the value function has the form \( V(l,h,t) = e^{-arl} W(h,t) \) and the complete dependence on \( l \) is described
just by the factor $e^{-at}$. It means that we obtain a decreasing function $V(l,h,t)$ in the variable $l$ in contradiction to the properties of a value function (see Lemma 2.1). It means that this reduction do not provide any meaningful solutions for our problem.

Now we can move according to a standard procedure to find the invariants of $H_4$ when $\omega \neq 0$. We obtain three independent invariants using a corresponding characteristic system

\begin{align}
\text{inv}_1 &= z = l + \frac{1}{ar}e^{rt} \int e^{-rt}d\ln \Phi(t) - \frac{1}{ar} \ln \Phi(t) - \frac{t}{a\omega}, \quad \text{inv}_2 = h, \quad (4.46) \\
\text{inv}_3 &= W(z,h) = V(t,l,h)e^{z't}.
\end{align}

Analogously substituting expressions for the invariants $z$ as the new independent and $W(z,h)$ as the new dependent variables into (3.17) we get

\begin{align}
\frac{1}{2} \eta^2 h^2 W_{hh} + (\mu - \delta) h W_h + (r z + \delta h) W_z + \frac{1}{a} w_z \log W_z - \frac{1}{a} \left( \frac{1}{\omega} + (1 + \ln a) \right) W_z \\
- \frac{(\alpha - r)^2 W_z^2 + 2(\alpha - r)\eta pH_z W_z + \eta^2 \rho^2 \sigma^2 h^2 W_z^2}{2\sigma^2 W_z} - \frac{r}{\omega} W = 0. \quad (4.48)
\end{align}

We prove now a compatibility of the invariant substitutions (4.46) and the boundary condition (2.9). As before we look for the new invariant variables (4.46)-(4.47) in the case of exponential liquidation time with $\Phi(t) = e^{-at}$ then these formulas take the form

\begin{align}
z &= l + \frac{k\omega - r}{ar\omega} t + \frac{\kappa}{ar^2}, \quad \text{inv}_2 = h, \quad (4.49) \\
V(t,l,h) &= W(z,h)e^{-z't}, \quad \omega \neq 0, \quad (4.50)
\end{align}

From (4.49) follows that if we chose an arbitrary parameter $\omega = r/\kappa$ then the variable $z$ up to a constant shift coincides with the old variable $l$. The relation (4.50) shows that the boundary condition (2.9) will be satisfied for any solution of (4.48). We see also for other positive values of the parameter $\omega$ the invariant variables (4.49)-(4.50) are compatible with the boundary condition (2.9).

Similar to the case of the exponential time distribution we can study other types of liquidation time distributions. For instance we look on the frequently used Weibull distribution $\Phi(t) = e^{-(t/h)^k}$ where the invariant variables will take the form

\begin{align}
z &= l + \frac{k}{ar^{k+1} \lambda^k} e^{rt}\Gamma(k,rt) + \frac{1}{ar^k t^k} - \frac{1}{a\omega} t, \quad \text{inv}_2 = h, \quad (4.51) \\
V(t,l,h) &= W(z,h)e^{-z't}, \quad \omega \neq 0, \quad (4.52)
\end{align}

here $\Gamma(k,rt)$ is the upper incomplete gamma function.

For the studied optimization problem the most interesting case appears if the liquidation time distribution has a local minimum like we expect it in the real world. The Weibull distribution has a local minimum for the parameter $k > 1$. Because of the asymptotic behavior of the expression $e^{rt}\Gamma(k,rt) \rightarrow r^{k-1} t^{k-1}$ as $t \rightarrow \infty$ we obtain that for $k > 1$ the variable $z \rightarrow l + \frac{1}{ar^k t^k}$ as $t \rightarrow \infty$. It means that also for a Weibull distribution we have compatibility of the invariant substitutions (4.51)-(4.52) with the boundary condition (2.9).

We notice that the investment $\pi(z,h)$ and consumption $c(z,t,h)$ in the case $H_4$ look as

\begin{align}
\pi(z,h) &= \left( -\frac{\eta\sigma h W_z + (\alpha - r) W_z}{\sigma^2 W_{zz}} \right), \quad c(z,t,h) = \frac{1}{a} \ln \left( \frac{\Phi(t)}{W_z} \right) + \frac{r}{a\omega} t, \quad \omega > 0,
\end{align}
where $W(z, h)$ is a solution of the equation (4.48).

Case $H_5(h_5)$. According to the first line of Table $1$, the sub algebra corresponding to the subgroup $H_5$ algebra is spanned by

$$h_5 = <e_1 \pm e_4> = \left( \frac{1}{ar} \pm e^rt \right) \frac{\partial}{\partial l} - V \frac{\partial}{\partial V}.$$

Using a standard procedure to determine the invariants of the subgroup $H_5$ we obtain three independent invariants as a solution of the characteristic system, they have a form

$$inv_1 = h, \quad inv_2 = t, \quad inv_3 = v(h, t) = e^{\frac{ar}{r-\alpha r^a}} V(l, h, t). \quad (4.53)$$

It means also that the complete dependence of the value function $V(l, h, t)$ on the variable $l$ is described just by the factor $e^{\frac{ar}{r-\alpha r^a}}$. If $t > \frac{1}{2} \ln(ar)$ then the value function will be decreasing function in $l$ by choosing plus sign in the denominator of the fraction $-\frac{ar}{r-\alpha r^a}$ and increasing if we choose minus in the denominator of the fraction. Because the value function for the optimization problem should be increasing function in $l$ so we need to study just this one case. Therefore we choose as a new dependent variable the function $v(h, t) = e^{\frac{ar}{r-\alpha r^a}} V(l, h, t)$.

Substituting the new dependent variable $v(h, t)$ into (3.17) we get a two dimensional PDE

$$v_t + \frac{3}{2} \eta_2^2 \eta_{hh} + (\mu - \delta) v_h + \frac{r}{ar} \left( \left( a \delta h - 1 + \ln \left( \frac{r}{ar} \right) \right) v + v \ln v \right)$$

$$- \frac{(r-\alpha r^a)^2}{2 \sigma^2} v - \frac{(r-\alpha r^a) \eta \eta h}{\sigma^2} v_h - \frac{(r-\alpha r^a)^2 r^2}{2 \sigma^2} \frac{\eta \eta h}{\sigma^2} \frac{v^2}{v} = 0, \quad v(h, t) \rightarrow 0.$$

According to Lemma 2.1 the value function $V(l, h, t)$ cannot have an exponential growth in $l$ like we obtain now. It means that the invariant substitution (4.53) is inconsistent with the possed optimization problem.

Case $H_7(h_7)$). The last one dimensional subalgebra in the list of the optimal system of subalgebras in Table $1$ is spanned by $e_2 \pm e_3$

$$h_7 = <e_2 \pm e_3> = \left\{ \pm \left( -\frac{1}{ar} \left( e^r \int e^{-rt} \ln \Phi(t) \, dt \right) \frac{\partial}{\partial l} + \frac{1}{ar} \frac{\partial}{\partial V} \right) \right\}.$$

According to a standard procedure we obtain following invariants of the subgroup $H_7$

$$inv_1 = z = l + \frac{1}{ar} e^r \int e^{-rt} \ln \Phi(t) - \frac{1}{ar} \ln \Phi(t), \quad inv_2 = h, \quad (4.54)$$

$$inv_3 = W(z, h) = V(t, l, h) \mp rt. \quad (4.55)$$

Using these invariants (4.54), (4.55) as the new variables $z, h, W(z, h)$ and substituting them into (3.17) we obtain a two dimensional PDE on $W(z, h)$

$$\frac{1}{2} \eta_2^2 W_{hh} + (\mu - \delta) W_h + (rz + \delta h) W_z + \frac{1}{a} W_z \log W_z$$

$$- \frac{(r-\alpha r^a)^2}{2 \sigma^2} W_{zz} - 2(\alpha - r)^2 + 2(\alpha - r) \eta \eta h W_z W_{zh} + \eta^2 \rho^2 \sigma^2 h^2 W_{hh}^2 + \frac{1}{a} (1 + \ln a) W_z \pm r = 0.$$

In this case we see the inconsistence between the boundary condition (2.9) which demands that $V(t, l, h) \rightarrow 0$ as $t \rightarrow \infty$ and the invariant substitutions (4.54), (4.55) which say that the expression $V(t, l, h) \mp rt$ depends just on $z, h$ and not from the variable $t$. 

18
Totally there are four meaningful reductions of the three dimensional PDE (3.17) for the case of the negative exponential utility function and the general liquidation time distribution $\Phi(t)$ by using one dimensional subalgebras of the algebra $L^1_{\text{EXP}n}$. Just one of these reductions which corresponds to the case $H_4$ with $\omega \neq 0$, i.e. the substitutions (4.46), (4.47) are consistent with the boundary condition (2.9) and the two dimensional PDE (4.48) is a corresponding reduction. This equation can be studied further with numerical methods.

In Table 1 are listed also two- and three- dimensional subalgebras of $L^1_{\text{EXP}n}$. Using these subalgebras may be we can find the deeper reductions of the PDE (3.17) for instance to ordinary differential equations.

**Case $H_8(h_8)$**. We take the first two dimensional subalgebra listed in Table 1 i.e. in the subalgebra $h_8 = < e_1, e_3 >$. We rewrite the characteristic systems to the first generator $e_1$ in terms of the invariants of $e_1$ (4.43), (4.44) then $e_1$ takes the form $e_1 = \frac{\partial}{\partial t} - W \frac{\partial}{\partial W}$. Solving a corresponding characteristic system we obtain a new invariant
\[ \text{inv}_{e_1} = v(h) = W(z,h)e^{ar}, \]
which we use now as a new dependent variable to reduce the equation (4.45) to an ODE
\[
\frac{1}{2} \eta^2 h^2 \left(1 - \frac{\sigma^2}{\sigma^2 + \eta^2} \right) v'' + \left( \frac{\mu - \delta}{\sigma^2} (\alpha - r) \eta \right)hv' \\
- \left( ar \delta h + \frac{(\alpha - r)^2}{2\sigma^2} \right) v - rv \ln(-arv) = 0. \tag{4.57}
\]

In terms of original variables $l, h, t$ and $V(t,l,h)$ the substitution looks as follows
\[
V(l,h,t) = v(h)e^{-ar}, \quad z = l + \frac{1}{ar} \int e^{-rt} d\ln \Phi(t) - \frac{1}{ar} \ln \Phi(t) - \frac{1}{ar}(1 + \log a). \tag{4.58}
\]

Now we obtain a reduction of the three dimensional PDE (3.17) to an ODE. But we cannot use this reduction, because it is inconsistent with the properties of the value function $V(l,h,t)$ listed in the Lemma 2.1. The value function is an increasing function in variable $l$ and $V(l,h,t) > 0$, it means also $v(h)$ should be a positive function. From the first expression in (4.58) follows that $V(l,h,t)$ is decreasing in $z$ and following in the variable $l$ and from the equation (4.57) follows that the expression $\ln(-arv)$ is well defined just for negative functions $v(h)$.

All other two- and three- dimensional subalgebras listed in Table 1 do not give any meaningful reductions of the original equation (3.17), so we will not regard them in detail.

5. Conclusion

In this paper we study a portfolio optimization problem for a basket consisting of a risk free liquid, risky liquid and risky illiquid assets where the investor prefer to use an exponential utility function. The illiquid asset is sold in a random moment of time $T$ with known distribution of the liquidation time. It is a distribution with a survival function $\Phi(t)$, satisfying very general conditions $\lim_{t \to \infty} \Phi(t)E[U(c(t))] = 0$ and $\Phi(t) \sim e^{-\kappa t}$ or faster as $t \to \infty$. Typically one suppose that the liquidation time distribution is an exponential one, i.e. $\Phi(t) = e^{-\kappa t}$, $t \geq 0$, $\kappa > 0$, or of the Weibull type with $\Phi(t) = e^{-t^\beta}$, where $t \geq 0$, $\beta > 0$, $\lambda > 0$. The Weibull distribution turns to the exponential distribution by $k = 1$ and it can be understand
as a generalization of the exponential distribution. Based on the economical motivation we choose $k > 1$ because in this case the Weibull probability density function has a local maximum.

Before in papers [3], [5] we studied similar portfolio optimization problems just in that problem settings the investor used a HARA and LOG utility functions correspondingly instead the exponential utility function as in this paper. Both of the utility functions HARA as well as LOG utility function were widely used before for the problems of a random income and for the portfolio optimization problems. Usually it is going on the optimization problems with a portfolio that includes an illiquid asset sold in a deterministic moment of time or with the infinite time horizon. In previous papers [3], [5] we demonstrated the connection between these two problems and shown that for $\gamma \to 0$ we obtain $U^\text{HARA} \to U^\text{LOG}$ as well as a three-dimensional HJB equation (2.6) corresponding to the HARA utility formally transforms into an the HJB-equation with the logarithmic utility function. Then we proved independently from the form of the survival function $\Phi(t)$ that the algebraic structure of the PDE with logarithmic utility can be seen as a limit of the algebraic structure of the PDE with HARA-utility as $\gamma \to 0$.

Now we carry out a complete Lie group symmetry analysis for two different exponential utility functions, a negative and positive ones, i.e. for two three dimensional PDEs (3.17) and (3.38) which contain an arbitrary function $\Phi(t)$. The main results are formulated in Theorem 3.1 and Theorem 3.3. We obtained that each of these PDEs admitted four dimensional Lie algebras $L^\text{EXPn}_4$ and $L^\text{EXPp}_4$ correspondingly. These algebras are isomorph and similar, it means that the studied PDEs (3.17) and (3.38) are equivalent up to the one-to-one analytical substitution. In other words the optimization problems are identical from any point of view: an economical, analytical or algebraic one.

We also investigated a connection between the optimization problem with the HARA utility function (2.12) and with the negative exponential utility function in Section 3.1. Despite the fact that the HARA utility function is connected to the negative exponential utility function by $\gamma \to \infty$ as we mentioned in (2.13) we do not got the expected connection between the corresponding optimization problems. Instead that we obtain the quite different structures of the invariant variables by study of the symmetry reductions of the main equation (3.17). In the case of HARA utility function a typical invariant variable was the fraction $l/h$. It means that in the case of the HARA or LOG utility function the value function depends in the first place from the relation between the values of the liquid and illiquid assets. Here in the case of the exponential utility function the main invariant variable is presented as a sum of the value of the liquid asset and time, see for example (4.46). May be this difference in behavior of the main variable is to explain by the fact that the risk tolerance in the case of HARA utility function is an increasing function of the consumption $c$ and in the case of EXP utility the risk tolerance is just a constant $R(c) = \frac{1}{c}$.

Further difference between the optimization problems with HARA - and EXP - utility functions is related to the structure of the admitted Lie algebras. In the cases of the HARA and LOG utility functions the corresponding three dimensional PDEs admitted three dimensional main Lie algebras, just by the special choice of a liquidation time distribution, i.e. only for the exponential function $\Phi(t) = e^{-\kappa t}$ we got an extension of these Lie algebras to the four dimensional ones. Here in the cases of the exponential utility functions, independently EXPn or EXPP, we obtain from very beginning four dimensional Lie algebras as the symmetry algebras of the corresponding PDEs. It is remarkable that in these cases the four dimensional
Lie algebras do not allow any extension independently from the form of $Φ(t)$ as we can see it solving the system of equation (3.26) in the proof of Theorem 3.1. In the previous paper [3] we proved that the second algebra $L^{LOG}_4$ can be obtained as a limit case of $L^{HARA}_4$ by $γ → 0$. Here we see that $L^{HARA}_2$, $L^{HARA}_4$ and $L^{EXP}_4$ are quite different and they do not connected by $γ → ∞$ as well as they do not have any connections between analytical structures of their generators independently on the form of the liquidation time distribution.

In our paper we pay attention to the internal structure of the admitted algebra $L^{EXP}_4$ to obtain convenient and useful reductions of the main equation (3.17). Further on we use the system of optimal subalgebras provided in [15] and get corresponding non equivalent invariant reductions of the three dimensional PDEs (3.17) to two dimensional PDEs which describe the complete set of solutions which can not be transformed to each other with a help of the transformations of the admitted symmetry group. We show that the three dimensional PDE can be reduced to a corresponding two dimensional ones in Section 4. The low dimensional PDEs are much more convenient for further analytical or numerical studies. We also provide the formulas for optimal policies in invariant variables using solutions of the reduced equations. We demonstrate that between meaningful reductions there exists one (4.48) which is consistent with the boundary condition (2.9) and with the expected properties of the value function.

We remark also a different level of influence of the parameters on the HJB equation and the admitted Lie algebraic structure. The HJB equation contains seven parameters $r, α, σ, μ, δ, η, ρ$ which define the behavior of liquid and illiquid asset, one parameter $a$ which is fixed by the exponential utility function. There are also some parameters which define the liquidation time distribution, for instance, it is the parameter $κ$ if we take the exponential distribution with $Φ(t) = e^{-κt}$ or two parameters $λ$ and $k$ if we take the Weibull distribution with $Φ(t) = e^{-(t/λ)^k}$. If we look on the structure of the Lie algebras provided in Theorem 3.1 and Theorem 3.3 we see that the generators of the algebras are defined by the parameters $r, a$ and parameters of the liquidation time distribution only. The algebras change their structure if one or some of these parameters vanishing. Roughly said the most influence on the form of solution of this optimization problem has interest rate $r$, the type of the investor’s utility function and a marked defined liquidation time distribution for the illiquid asset.

Summing up, we carry a complete Lie group analysis for the optimization problems with negative and positive exponential utility functions and for a general liquidation time distribution. We list reduced equations and corresponding optimal policies.

Acknowledgements 5.1. The author is thankful to prof. L. Vostrikova-Jacod for interesting discussions and for organizing a very successful conference Advanced Methods in Mathematical Finance, Angers, 2018, France, where the author got the idea to write this paper.

References

References

[1] L. A. Bordag. Geometrical properties of differential equations. Applications of Lie group analysis in Financial Mathematics. World Scientific Publishing, Singapore, 2015.

[2] L. A. Bordag and I. P. Yamshchikov. Lie group analysis of nonlinear black-scholes models. In M. Günther E. Jan W. ter Maten (Eds.) Ehrhardt, M., editor, Novel Methods in Computational Finance, pages 109 – 128. Springer, Switzerland, 2017.
[3] L. A. Bordag and I. P. Yamshchikov. Optimization problem for a portfolio with an illiquid asset: Lie group analysis. *Journal of Mathematical Analysis and Applications*, 453:668 – 699, 2017.

[4] L. A. Bordag, I. P. Yamshchikov, and D. Zhelezov. Portfolio optimization in the case of an asset with a given liquidation time distribution. *International Journal of Engineering and Mathematical Modelling*, 2(2):31 – 50, 2015.

[5] L. A. Bordag, I. P. Yamshchikov, and D. Zhelezov. Optimal allocation-consumption problem for a portfolio with an illiquid asset. *International Journal of Computer Mathematics*, 93(5):749–760, 2016. DOI: 10.1080/00207160.2013.877584.

[6] N. H. Ibragimov. *Lie group analysis of differential equations*. CRS Press, Boca Raton, 1994.

[7] S. V. Meleshko. *Methods for Constructing Exact Solutions of Partial Differential Equations: Mathematical and Analytical Techniques with Applications to Engineering*. Springer, New York, 2005.

[8] S. V. Meleshko and S. Moyo. On the complete group classification of the reaction-diffusion equation with a delay. *Journal of Mathematical Analysis and Applications*, 338(1):448–466, 2008.

[9] R. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, 1969.

[10] Phillip Monin. Hedging market risk in optimal liquidation. *SSRN Electronic Journal*, 01, 2014.

[11] Phillip Monin and Thaleia Zariphopoulou. On the optimal wealth process in a log-normal market: Applications to risk management. *Journal of Financial Engineering*, 01(02):1450013, 2014.

[12] P. J. Olver. *Applications of Lie groups to differential equations*. Springer Science & Business Media, New York, 2000.

[13] L. V. Ovsiannikov. *Group Analysis of Differential Equations*. Academic Press, New York, 1982.

[14] L. V. Ovsiannikov. The programm "podmodeli". gas dynamics. *Journal of Applied Mathematics and Mechanics*, 58(4):601–627, 1994.

[15] J. Patera and P. Winternitz. Subalgebras of real three- and four-dimensional Lie algebras. *Journal of Mathematical Physics*, 18(7):1449–1455, 1977.

[16] A. Schied and T. Schöneborn. Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13(2):181–204, 2009.