ASYMPTOTICS OF THE CRITICAL NON-LINEAR WAVE EQUATION FOR
A CLASS OF NON STAR-SHAPED OBSTACLES

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Abstract. Scattering for the energy critical non-linear wave equation for domains exterior to non trapping obstacles in 3+1 dimension is known for the star-shaped case. In this paper, we extend the scattering for a class of non star-shaped obstacles called illuminated from exterior. The main tool we use is the method of multipliers with weights that generalize the Morawetz multiplier to suit the geometry of the obstacle.

1. Introduction

In this paper we are working on the energy critical nonlinear wave equation in 3+1 dimension in a domain $\Omega = \mathbb{R}^3 \setminus V$ where $V$ is a non-trapping obstacle with smooth boundary

\begin{equation}
\Box u = (\partial_t^2 - \Delta) u = -u^5 \quad \text{in } \mathbb{R} \times \Omega \n\end{equation}

\begin{align}
|u|_{\mathbb{R} \times \partial \Omega} = 0 \\
(\nabla u(t, \cdot), \partial_t u(t, \cdot)) \in L^2(\Omega) & \quad t \in \mathbb{R}
\end{align}

which enjoys the conservation of energy

$$E = E(t) = \int_{\Omega} \left(\frac{\partial_t u|^2}{2} + \frac{\nabla u|^2}{2} + \frac{u^6}{6}\right) dx.$$ 

In the boundaryless case ($\Omega = \mathbb{R}^3$), the first results for the global existence were obtained by Grillakis [8, 9]. He showed that there are global smooth solutions of the critical wave equation, if the data is smooth. Shatah and Struwe [17, 18] extended this theorem by showing that there are global solutions for the data lying in the energy space $H^1 \times L^2$. They also obtained results for critical wave equation in higher dimensions.

For the case of obstacles, the first results were due to Smith and Sogge [19]. They showed that Grillakis theorem extends to the case where $\Omega$ is the complement of a smooth, compact, strictly convex obstacle with Dirichlet boundary conditions. This result was later extended to the case of arbitrary domains in $\mathbb{R}^3$ and data in the energy space by Burq, Lebeau and Planchon [6]. The case of the nonlinear critical Neumann wave equation in 3-dimensions was subsequently handled by Burq and Planchon [7].

More specifically, in this paper we are interested in asymptotics, i.e. how solutions to the nonlinear equation scatter to a solution to the homogeneous linear equation. In the boundaryless case ($\Omega = \mathbb{R}^3$), first results were obtained by Bahouri-Gérard in [1]; in their paper they used the following decay estimate proved by Bahouri-Shatah [2]

$$\lim_{|t| \to +\infty} \frac{1}{6} \int_{\Omega} |u(t, x)|^6 dx = 0.$$ 

to get

$$\|u\|_{L^5(\mathbb{R}; L^{10}(\Omega))} + \|u\|_{L^4(\mathbb{R}; L^{12}(\Omega))} < \infty,$$

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and thus scattering. Moreover, in [1] Bahouri-Gérard used profile decomposition to show that 
\[ \|u\|_{L^5(\mathbb{R}; L^{10}(\Omega))} \] is also controlled by a universal function of the energy \( f(E) \).
Then the scattering result was extended to the case of star-shaped obstacles (\( x \cdot n \geq 0 \) for \( x \in \partial V \) with \( n \) the outward pointing unit normal vector to \( \partial V \)) by Blair, Smith, and Sogge in [3]. They used the same \( L^6 \) decay estimate proved by Bahouri-Shatah in [2] after they extended it to their case of obstacles making slight modifications on the proof to handle the boundary term.

In the papers by Bahouri-Shatah [2] and Blair, Smith, and Sogge [3], the \( L^6 \) decay estimate which is the main key to prove scattering was proved using the method of multipliers. The method of multipliers is also called Friedrichs' ABC method as it dates back to Kurt O. Friedrichs in the 1950's. The idea of this method is to multiply the equation with a factor \( Nu \), with \( N \) is a linear first-order differential operator, defined as
\[
Nu = Au + B \cdot \nabla u + C \partial_t u
\]
and then to express the product as a divergence or energy identity of the form
\[
\text{div}_{t,x}(\cdots) + \text{Remaining terms} = 0
\]
and finally to integrate this divergence identity over a domain in \( \mathbb{R}^{n+1} \) and subsequently derive the required estimates. The only case where the differential multiplier is adapted to both the wave equation in terms of commutation (avoiding remaining terms) and the geometry of the obstacle in terms of the sign of the boundary term, is the star-shaped case. The method of multipliers was used in the 1960's and 1970's to prove uniform decay results for the homogeneous linear wave equation (\( \Box u = 0 \)) outside obstacles. Cathleen S. Morawetz was the first to succeed in proving uniform local energy decay for star-shaped obstacles with Dirichlet boundary condition using this method ([13] and [15]). Since then, the results of Morawetz have been considerably improved. Better decay rates have been achieved (as in odd dimensions \( n \geq 3 \), Huygen's principle has been shown to imply an exponential rate of decay whenever there is some sort of decay [11], [14]). Moreover, the class of obstacles under consideration has been enlarged; decay results have been derived for a special case of non trapping obstacles referred to as “almost star-shaped regions” (Ivrii [10]) and for non trapping obstacles with simple and direct geometrical generalizations to the star-shaped such as the “illuminated from interior” (Bloom and Kazarinoff [4]) and the “illuminated from exterior” (Bloom and Kazarinoff [5], Liu [12]). For these cases, decay results have been proved using the method of multipliers after generalizing the multipliers to suit the geometry of the obstacle, and although these generalized multipliers lead to volume integrals that were avoided before, it turned out that these integrals were actually useful in the estimates. Other wider generalizations that include all the above geometries later followed, Strauss [20] proved uniform local energy decay for the homogeneous linear wave equation in exterior domains in \( \mathbb{R}^n \) \( n \geq 3 \), provided a strictly expansive vector field (now called the Straussian vector field) exists, that leaves \( \Omega \) strictly invariant, then these Straussian vector fields were generalized by Morawetz, Ralston, and Strauss [16] by introducing escape functions, using them to construct a pseudo-differential operator \( P(x,D) \), and finally setting \( Pu \) as a multiplier.

Though the microlocal methods of Morawetz, Ralston, and Strauss provided general results for the linear case, they cannot be easily extended to the nonlinear wave equation for which the star-shapedness has been so far a restriction to obtain decay results. Thus, we consider in this paper obstacles with explicit geometry that is a direct generalization of the star-shapedness, and we prove the \( L^6 \) decay estimate and thus the scattering for such obstacles using Friedrichs' ABC method after generalizing the multipliers to suit our case. The price to pay for such a generalization is that the our multiplier, which is adapted to the geometry of the obstacle, no
longer has exact commutation properties with the wave operator. Therefore, unlike the star-shaped case we do get volume integrals that we deal with using a Gronwall-like argument.

Our main results are:

**Theorem 1.1.** Suppose \( u \) solves the nonlinear wave equation \([1.1]\) and that \( V \) is a non-trapping obstacle with regular boundary that can be illuminated from its exterior by a strictly convex body \( C \) satisfying the geometric condition:

\[\min_{\partial V} (s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0 \]

where

- \( s_0 \) is the algebraic distance from \( \partial C \) to \( x \in \partial V \) along the exterior normal to \( \partial C \).
- \( \rho_{2M} = \max_{(\sigma_1, \sigma_1)} \rho_2 \) where \( \rho_i \) (\( i = 1, 2 \)) are the radii of curvature of \( \partial C \) (\( \rho_2 \geq \rho_1 \)).

then

\[\lim_{t \to -\infty} \int_\Omega |u(t, x)|^6 dx = 0\]

**Remark 1.2.** We can construct obstacles that are illuminated from the exterior or from the interior that satisfy the condition \([1.2]\). In particular, it would be easy to see this for illuminated from interior obstacles, where the illuminating body is inside the obstacle and thus \( s_0 > 0 \), by considering a dog bone like obstacle (Figure 7) that is a small perturbation of the star-shaped.

**Remark 1.3.** Remark that if the data has compact support, the computation that proves the above result provides an explicit decay rate for the local energy. In particular, it recovers Bloom-Kazarinoff for the linear equation \([4]\) without using the fact that \( \Box \partial_t u = 0 \).

As a result of this decay estimate, we get scattering

**Corollary 1.4.** Suppose \( u \) solves the nonlinear wave equation \([1.1]\) and that \( V \) is a non-trapping obstacle with regular boundary that can be illuminated from its exterior by a strictly convex body satisfying the geometric condition:

\[\min_{\partial V} (s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0 \]

then there exists unique solutions \( v_{\pm} \) to the homogeneous linear problem

\[\begin{cases}
\Box v = 0 & \text{in } \mathbb{R} \times \Omega \\
v|_{\mathbb{R} \times \partial \Omega = 0}
\end{cases}\]

such that

\[\lim_{t \to \pm \infty} E_0(u - v_{\pm}; t) = 0.\]

Moreover, \( u \) satisfies:

\[\|u\|_{L^5(\mathbb{R}; L^{10}(\Omega))} + \|u\|_{L^4(\mathbb{R}; L^{12}(\Omega))} < \infty.\]

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2. The geometry of the obstacle

We consider in this paper illuminated from exterior obstacles, defined as such (Liu [12]):

**Definition 2.1.** We say that the boundary of an exterior domain \( \Omega = \mathbb{R}^3 \setminus V \) (or the obstacle \( V \)) can be illuminated from the exterior if and only if there exists a convex body \( C \) containing \( \partial V \) with smooth boundary \( \partial C \) such that \( \partial V \) is filled by a family of non-intersecting rays normal to \( \partial C \). Each ray is completely contained in \( \Omega \) in the following sense: for each \( x_0 \in \partial V \) there exists a unique \( x_1 \in \partial C \) and a number \( s_0(x_1) \leq 0 \) such that

\[
x_0 = s_0(x_1)\nu(x_1) + x_1,
\]

where \( \nu \) is the outward unit normal to \( \partial C \) at \( x_1 \), and

\[
x = t\nu(x_1) + x_1 \in \Omega, \quad t \in [s_0, \infty).
\]

This definition actually generalizes the following definition of illuminated from interior obstacles introduced by Bloom and Kazarinoff [4]:

**Definition 2.2.** We say that a body \( V \) can be illuminated from the interior if and only if there exists a smooth convex body \( C \) inside \( V \) such that \( \text{ext} C \) is filled by a family of non-intersecting rays normal to \( \partial C \) and such that each ray intersects \( \partial V \) exactly once.

In fact, every body that can be illuminated from the interior can also be illuminated from its exterior by enlarging the original convex body (Liu [12]); thus our results proven for illuminated from exterior obstacles also hold for illuminated from interior. Furthermore, as we mentioned in the introduction, these geometries are direct generalizations of the star-shaped. More precisely, the illuminated from interior is a generalization of the strict star-shapedness \( (x \cdot n > 0) \). The condition \( x \cdot n > 0 \) implies that each ray beginning at the origin intersects \( \partial V \) exactly once, which means that the interior of a strictly star-shaped obstacle can be illuminated by a source of light situated at the origin. A small ball centered at the origin is contained in the interior of \( V \) and the above light rays are perpendicular to the surface of this ball, hence our strictly star-shaped is illuminated from its interior by this ball.

An example of a non star-shaped body that can be illuminated from interior is a “dog bone” (Figure 1), and a “snake-shaped” body (Figure 2) is an example of an obstacle that cannot be illuminated from its interior but can be illuminated from the exterior.

![Figure 1. dog bone](image-url)
2.1. The illuminating coordinate system. In this section we will introduce the coordinate system that we are going to use which is the one in the paper by Liu [12] in which he was dealing with similar obstacle for the linear equation. We again denote by $\partial C$ the smooth and convex surface of the illuminating body. Let $x = (x_1, x_2, x_3)$ be Cartesian coordinates in $\mathbb{R}^3$ with the origin inside $C$ and $V$. If $X^0$ is on $\partial C$, then in a neighborhood of $X^0$ we choose the parametric curves to be the two principal curves on $\partial C$. If the neighborhood of $X^0$ is an all-umbilic surface, then we still can choose the parametric curves to be orthogonal to each other. Furthermore, we let the parameters be the arc-length parameters. Thus, if $X^0 \in \partial C$, then $X^0$ is given in local coordinates by

$$X^0 = X^0(\sigma_1, \sigma_2) = (X_0^1(\sigma_1, \sigma_2), X_0^2(\sigma_1, \sigma_2), X_0^3(\sigma_1, \sigma_2)),$$

where $\sigma_1 = \text{const.}$ and $\sigma_2 = \text{const.}$ are the parameterizations of the arc-length of the principal curves near $X^0$. A finite number of $(\sigma_1, \sigma_2)$ coordinate patches cover $\partial C$. Next, corresponding to each point $X^0(\sigma_1, \sigma_2)$ on $\partial C$, we make the choice $x = X(s, \sigma_1, \sigma_2)$, where

$$(2.1) \quad x = X(s, \sigma_1, \sigma_2) = s\nu(\sigma_1, \sigma_2) + X^0(\sigma_2, \sigma_2) = sX_s + X^0(\sigma_1, \sigma_2),$$

where

$$\nu(\sigma_1, \sigma_2) = \left[\frac{X_0^i}{|X_0^i|} \times \frac{X_0^j}{|X_0^j|}\right]$$

is the unit exterior normal to $\partial C$, with $X_0^i = \partial X^0/\partial \sigma_i, \ i = 1, 2$. By Definition 2.1 for each $X^1 \in \partial V$, there is a unique triple $(s_0, \sigma_1, \sigma_2)$ with $s_0 \leq 0$ such that

$$X^1 = s_0\nu(\sigma_1, \sigma_2) + X^0(\sigma_1, \sigma_2),$$

We denote by $\kappa_1$ and $\kappa_2$ the principal curvatures at $X^0(\sigma_1, \sigma_2)$ and $\rho_i = \frac{1}{\kappa_i} (i = 1, 2)$ the radii of curvature of $\partial C$. We assume $0 < \kappa_2 \leq \kappa_1$ ($\rho_2 \geq \rho_1 > 0$). Furthermore, we always assume that

$$\min_{\partial V}(s_0 + \rho_1) > 0 \quad (2.2)$$

This condition implies that for every $x \in \Omega$, we have $s + \rho_i > 0 (i = 1, 2)$ since $x = s\nu(\sigma_1, \sigma_2) + X^0(\sigma_2, \sigma_2)$ with $s_0 \leq s < \infty$, where $s_0$ corresponds to the point on $\partial V$ associated with $X^0$.

**Remark 2.3.** Generically, Definition 2.1 implies the condition (2.2) (page 26 Lemma 2.1, [12] page 316 Remark after Lemma 1). Moreover, note that for a star-shaped obstacle, which is in fact an obstacle that is illuminated from the exterior by some ball $B(0, R_0)$, $s + \rho_i$ is nothing
but \( r \). This explains the significance of this value and makes the need of such an assumption in a computation that is a generalization of the star-shaped totally logical.

Now, we state the following geometrical lemma that we will use later:

**Lemma 2.4.** There exist a constant \( a_0 > 0 \) such that \( s + \rho_{2M} \geq a_0 r \).

**Proof.** The existence of \( a_0 \) is due to the boundedness of the the obstacle \( V \) and the illuminating body \( C \). In fact, if \( s \geq 0 \) (\( x \in \text{ext} C \)) then \( r - r_0(\sigma_1, \sigma_2) \leq s < r \) where \( r_0(\sigma_1, \sigma_2) = |X^0(\sigma_1, \sigma_2)| \) and if \( s < 0 \) (\( x \in C \cap \Omega \)) then \( s + \rho_{2M} \geq c \) for some positive constant \( c \). \( \square \)

We also recall the following lemmas about the coordinate system, these lemmas were originally stated and proved by Bloom and Kazarinoff [4] for illuminated from interior obstacles, and then they were extended by Liu [12] for illuminated from exterior obstacles.

**Lemma 2.5.** The level surfaces \( s = \text{const.}, \sigma_i = \text{const.} \) \((i = 1, 2)\) define a set of local coordinate systems in \( \Omega \) with each ray \( \{x : \sigma_1 = \text{const.}, \sigma_2 = \text{const.}, \text{and } s_0 \leq s < \infty \} \) normally incident on \( \partial C \) and

\[
\frac{D(x_1, x_2, x_3)}{D(s, \sigma_1, \sigma_2)} = \Lambda(\kappa_1 s + 1)(\kappa_2 s + 1) > 0
\]

where \( \Lambda = |X^0_{\sigma_1}||X^0_{\sigma_2}|. \)

**Lemma 2.6.** \( \nu \cdot n \geq 0 \) on \( \partial V \).

**Remark 2.7.** We recall the following calculus formulas that we will use in our computation \((i = 1, 2)\):

\[
\partial_{\sigma_i} \nu = \frac{\partial \nu}{\partial \sigma_i} = \kappa_i X^0_{\sigma_i}
\]

and thus \( X_{\sigma_i} = (\kappa_i s + 1)X^0_{\sigma_i} \).

Moreover, remark that for any scalar function \( f = f(s, \sigma_1, \sigma_2) \) and every vector field written in the new coordinate system

\[
F = F^0 \nu + F^1 \frac{X^0_{\sigma_1}}{|X^0_{\sigma_1}|} + F^2 \frac{X^0_{\sigma_2}}{|X^0_{\sigma_2}|},
\]

we can express the gradient and the divergence as follows:

\[
\nabla f = \partial_s f \nu + \frac{1}{|X_{\sigma_1}|} \partial_{\sigma_1} f \frac{X^0_{\sigma_1}}{|X^0_{\sigma_1}|} + \frac{1}{|X_{\sigma_2}|} \partial_{\sigma_2} f \frac{X^0_{\sigma_2}}{|X^0_{\sigma_2}|}
\]

and

\[
\text{div} F = \frac{1}{|X_{\sigma_1}| |X_{\sigma_2}|} \left[ \partial_s \left( |X_{\sigma_1}| |X_{\sigma_2}| F^0 \right) + \partial_{\sigma_1} \left( |X_{\sigma_2}| F^1 \right) + \partial_{\sigma_2} \left( |X_{\sigma_1}| F^2 \right) \right]
\]

In particular, we have

\[
\nabla s = \nu \]

\[
\nabla f \cdot \nu = \partial_s f
\]

\[
|\nabla f|^2 = (\partial_s f)^2 + \frac{1}{(\kappa_1 s + 1)^2|X^0_{\sigma_1}|^2}(\partial_{\sigma_1} f)^2 + \frac{1}{(\kappa_2 s + 1)^2|X^0_{\sigma_2}|^2}(\partial_{\sigma_2} f)^2
\]

Denote by

\[
|\nabla^i f|^2 = \frac{1}{(\kappa_i s + 1)^2|X^0_{\sigma_i}|^2}(\partial_{\sigma_i} f)^2, \ i = 1, 2
\]

and

\[
|\nabla^i f|^2 = |\nabla^i_1 f|^2 + |\nabla^i_2 f|^2
\]
\[ |\nabla f|^2 = |\partial_x f|^2 + |\nabla^* f|^2. \]

3. **Proof of the \( L^6 \) decay estimate (Theorem 1.1)**

We must show that for any \( \epsilon_0 > 0 \), there exists \( T_0 \) such that whenever \( t \geq T_0 \),

\[ \frac{1}{6} \int_{\Omega} |u(t,x)|^6 dx \leq \epsilon_0. \]

First, we begin by multiplying the wave equation \( \square u + u^5 = 0 \) by \( \partial_t u \), we get the following divergence or energy identity

\[ \partial_t (e(u)) - \text{div}(\nabla u \partial_t u) = 0 \]

where

\[ e(u) = \frac{1}{2} (|\partial_t u|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \]

denotes the energy density. Integrating over the region \( \{ (x,t); s + \rho_{2M} > t + M, 0 \leq t \leq T \} \), where \( \rho_{2M} = \max \{ \rho_1, \rho_2 \} \) and \( M \) is a positive constant chosen such that \( M \geq \rho_{2M} \) and the illuminating body \( C \subset \{ x; s + \rho_{2M} \leq M \} \), and using the divergence theorem and the Dirichlet boundary condition, we get

\[ \int_{s+\rho_{2M} > T+M} e(u)(T,x) dx + \frac{1}{\sqrt{2}} f\text{lux}(0,T) \leq \int_{s+\rho_{2M} > M} e(u)(0,x) dx \]

where the flux on the mantle is defined by:

\[ f\text{lux}(a,b) = \int_{M_a^b} \left( \frac{1}{2} |\nu \partial_t u + \nabla u|^2 + \frac{u^6}{6} \right) d\sigma \]

with

\[ M_a^b = \{ (x,t); s + \rho_{2M} = t + M, a \leq t \leq b \} \]

Since the solution has finite energy, we may select \( M \) large so that the right hand side of (3.1) is less than \( \frac{\epsilon_0}{2} \). Hence, it will suffice to show the existence of \( T_0 \) such that whenever \( T > T_0 \) we have

\[ \frac{1}{6} \int_{s+\rho_{2M} \leq T+M} |u(T,x)|^6 dx \leq \frac{\epsilon_0}{2} \]

This is a consequence of the following proposition:

**Proposition 3.1.** Suppose \( u \) solves the nonlinear wave equation (1.1) and that \( V \) is a non-trapping obstacle with regular boundary that can be illuminated from its exterior by a strictly convex body satisfying the geometric condition:

\[ \min_{\partial V} (s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0 \]

then

\[ \sqrt{\eta_0} \int_{s+\rho_{2M} \leq t+M} \left( |\nabla^* u|^2 + \left| \frac{\partial_x ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right)^{1\over 2} dx + \int_{s+\rho_{2M} \leq T+M} \frac{u^6(T,x)}{3} dx \leq 2c_1 \beta E + \frac{1}{T} (C_0 E + C_2 E \ln(1 + T) + 2(c_2 + c_3 T) f\text{lux}(0,T)) \]

\[ + \frac{\eta_0}{T} \int_0^T \int_{s+\rho_{2M} \leq t+M} \left( |\nabla^* u|^2 + \left| \frac{\partial_x ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx dt \]

for some arbitrary \( 0 < \beta < 1 \) and where \( 0 < \eta_0, \epsilon < 1 \) and all the other constants depend on the geometry of the obstacle.
As a result of Proposition 3.1, we will show that given \( \epsilon_0 > 0 \), \( \exists T_0 \) such that \( \forall T \geq T_0 \) we have

\[
\frac{1}{T} \int_0^T \int_{s + \rho_2 M \leq cT + M} \left( |\nabla^* u|^2 + \left| \frac{\partial_u ((s + \rho_2 M)u)}{s + \rho_2 M} \right|^2 \right) dx dt < \frac{\epsilon_0}{2}.
\]

For simplicity, let:

\[
\phi(t) = \int_{s + \rho_2 M \leq cT + M} \left( |\nabla^* u|^2 + \left| \frac{\partial_u ((s + \rho_2 M)u)}{s + \rho_2 M} \right|^2 \right) dx,
\]

\[
\psi(T) = \frac{1}{T} \int_0^T \phi(t) dt,
\]

\[
\gamma(t) = \frac{2c_1 \beta E}{\eta} + \frac{1}{t \eta} \left( C_0 E + C_2 E \ln(1 + t) + 2(c_2 + c_3 T) \text{flux}(0, t) \right).
\]

with \( \eta = \sqrt{\eta_0} \).

Thus we want to show the existence of \( T_0 \) such that \( \forall T \geq T_0 \) we have \( \psi(T) < \frac{\epsilon_0}{2} \).

\[
\psi(T) + T \frac{d\psi}{dt} = \phi(T)
\]

From the differential inequality in Proposition 3.1 we have

\[
\phi(T) \leq \gamma(T) + \eta \psi(T)
\]

so

\[
\frac{d\psi}{dt} + (1 - \eta) \frac{\psi(T)}{T} \leq \frac{\gamma(T)}{T}
\]

\[
\frac{1}{T^{1-\eta}} \frac{d(T^{1-\eta} \psi)}{dt} \leq \frac{\gamma(T)}{T}
\]

\[
\frac{d(T^{1-\eta} \psi)}{dt} \leq \frac{\gamma(T)}{T^\eta}
\]

\[
T^{1-\eta} \psi(T) \leq \psi(1) + \int_1^T \frac{\gamma(t)}{t^\eta} dt
\]

Hence,

\[
(3.3) \quad \psi(T) \leq J(T) + \frac{\psi(1)}{T^{1-\eta}}
\]

with

\[
J(T) = \frac{1}{T^{1-\eta}} \int_1^T \frac{\gamma(t)}{t^\eta} dt.
\]

But, note that \( \text{flux}(0, t) \xrightarrow{t \to \infty} 0 \) by the classical energy-conservation law on the exterior of a truncated cone stated above. Thus, \( \exists t_0 \) such that \( \forall t \geq t_0 \) we have

\[
\frac{1}{t \eta} \left( C_0 E + C_2 E \ln(1 + t) + 2(c_2 + c_3 T) \text{flux}(0, t) \right) < \frac{\epsilon_0 (1 - \eta)}{12},
\]

and choose \( \beta \) such that

\[
\frac{2c_1 \beta E}{\eta} = \frac{\epsilon_0 (1 - \eta)}{12}.
\]

Hence,

\[
\exists t_0 \text{ such that } \forall t \geq t_0 \text{ we have } \gamma(t) < \frac{\epsilon_0 (1 - \eta)}{6}
\]

and \( \gamma \) is bounded:

\[
\gamma(t) \leq M, \forall t.
\]
Moreover,
\[ \exists T_0 > t_0 \text{ such that } \forall t > T_0, \text{ we have } \frac{1}{t_1^{1-\eta}} \frac{M}{1-\eta} (t_0^{1-\eta} - 1) < \frac{\epsilon_0}{6} \text{ and } \psi(1) \frac{t_1^{1-\eta}}{t_1^{1-\eta}} < \frac{\epsilon_0}{6}. \]
Thus, for all \( T > T_0 \),
\[
J(T) = \frac{1}{T^{1-\eta}} \int_0^{T_0} \frac{\gamma(t)}{t_0^{1-\eta}} dt + \frac{1}{T_1^{1-\eta}} \int_{T_0}^{T} \frac{\gamma(t)}{t_0^{1-\eta}} dt 
\leq \frac{1}{T_1^{1-\eta}} \frac{M}{1-\eta} (t_0^{1-\eta} - 1) + \frac{1}{T_1^{1-\eta}} \frac{\epsilon_0}{6} (T_1^{1-\eta} - t_1^{1-\eta}) 
< \frac{\epsilon_0}{3}
\]
and by (3.3), \( \psi(T) < \frac{\epsilon_0}{2} \) which is (3.2).

Now, from Proposition 3.1, we have
\[
\int_{s+\rho_2M \leq t+M} \frac{u^6(T,x)}{3} dx \leq \eta \gamma(T) + \eta^2 \psi(T) < \gamma(T) + \psi(T) < \epsilon_0
\]
Hence, for all \( T \geq T_0 \), we have
\[
\int_{s+\rho_2M \leq t+M} \frac{u^6(T,x)}{3} dx < \epsilon_0
\]
which ends the proof of Theorem 1.1.

4. Proof of the Differential Inequality (Proposition 3.1)

The method we are going to use to prove our result is the method of multipliers and we will generalize the Morawetz multipliers that were used for star-shaped obstacles in way that suits our obstacle.

4.1. Divergence Identity and Integral Equality. We multiply the wave equation by
\[ (u + \alpha \cdot \nabla u + (t + M) \partial_t u), \]
where \( M \geq \rho_2M \) is the positive constant chosen in the previous section and \( \alpha \) is a vector field defined as follows:
\[ \alpha = (s + \rho_2M) \nu \]
and we get the following divergence identity:
\[ \partial_t Q + \text{div} P + R = 0 \]
where
\[
Q = (t + M) \frac{\left| \nabla u \right|^2}{2} + (t + M) \frac{|u|^6}{6} + (t + M) \frac{\left| \partial_t u \right|^2}{2} + \partial_t u (\alpha \cdot \nabla u) + (\partial_t u) u
\]
\[
P = \left( \frac{\left| \nabla u \right|^2}{2} + \frac{|u|^6}{6} - \frac{\left| \partial_t u \right|^2}{2} \right) \alpha - ((t + M) \partial_t u + \alpha \cdot \nabla u + u) \nabla u
\]
\[
R = (\text{div} \alpha - 3) \frac{\left| \partial_t u \right|^2}{2} + (1 - \text{div} \alpha) \frac{\left| \nabla u \right|^2}{2} + (5 - \text{div} \alpha) \frac{|u|^6}{6} + H_\alpha (\nabla u, \nabla u)
\]
where \( H_α(∇u, ∇u) = \sum_{i,j=1}^{3} \partial_i α_j \partial_i u \partial_j u = ∇u \cdot ((∇u \cdot ∇) α). \)

Integrating the divergence identity over the truncated cone

\[
K_{T_1}^{T_2} = \{ (x, t); s + ρ_{2M} \leq t + M, T_1 \leq t \leq T_2 \}, \quad 0 < T_1 < T_2,
\]

and applying the divergence theorem, we get

\[
\int_{D(T_2)} Q(T_2, x) dx - \int_{D(T_1)} Q(T_1, x) dx - \frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} (Q - P \cdot ν) dσ
\]

(4.2)

\[
- \int_{T_1}^{T_2} \int_{∂V} P \cdot n dσ dt + \int_{K_{T_1}^{T_2}} Rdσ dt = 0
\]

where \( dσ \) denotes the Lebesgue measure on the corresponding surface and \( n \) is the outward pointing unit normal vector to \( ∂V \); and where

\[
D(T_1) = \{ x ∈ Ω; s + ρ_{2M} \leq T_1 + M \}
\]

and

\[
M_{T_1}^{T_2} = \{ (x, t); s + ρ_{2M} = t + M, T_1 \leq t \leq T_2 \}.
\]

4.2. The differential inequality. Now, we deal with the terms of the integral equality (4.2) in order to get the desired differential inequality.

The boundary term

By the Dirichlet boundary condition, we get:

\[
- \int_{T_1}^{T_2} \int_{∂V} P \cdot n dσ dt = - \int_{T_1}^{T_2} \int_{∂V} \frac{|∇u|^2}{2} α \cdot n - (α \cdot ∇u)(∇u \cdot n) dσ dt
\]

(4.3)

\[
= \int_{T_1}^{T_2} \int_{∂V} \frac{1}{2} |∇u \cdot n|^2 (α \cdot n) dσ dt ≥ 0
\]

since \( α \cdot n = (s + ρ_{2M})ν \cdot n \) with \( s + ρ_{2M} > 0 \) (by our assumption) and \( ν \cdot n ≥ 0 \) (Lemma 2.6).

The terms on the time slices

We have

\[
Q(T_2, x) = (T_2 + M) \frac{|∇u|^2}{2} + (T_2 + M) \frac{|u|^6}{6} + (T_2 + M) \frac{|∂_t u|^2}{2} + ∂_t u(α \cdot ∇u) + (∂_t u) u
\]

Introduce

\[
I(T_2) = \frac{1}{4} (T_2 + M + (s + ρ_{2M})) \left[ ∂_t u + \frac{∂_s((s + ρ_{2M})u)}{s + ρ_{2M}} \right]^2 + \frac{1}{4} (T_2 + M - (s + ρ_{2M})) \left[ ∂_t u - \frac{∂_s((s + ρ_{2M})u)}{s + ρ_{2M}} \right]^2
\]

\[
+ (T_2 + M) \frac{u^6}{6}
\]

\[
= (T_2 + M) \left( \frac{|∂_t u|^2}{2} + \frac{|∂_s u|^2}{2} + \frac{u^6}{6} \right) + (∂_t u)(s + ρ_{2M})∂_s u + (∂_t u) u + \frac{T_2 + M}{2} \left( \frac{u^2}{(s + ρ_{2M})^2} + \frac{2u∂_s u}{s + ρ_{2M}} \right)
\]

by Remark 2.7 we have

\[
|∇u|^2 = |∂_t u|^2 + |∇^s u|^2
\]

where

\[
|∇^s u|^2 = |∇_1^s u|^2 + |∇_2^s u|^2
\]
Thus, so

\[
I(T_2) + (T_2 + M) \frac{| \nabla^* u |^2}{2} = Q(T_2, \cdot) + \frac{T_2 + M}{2} \left( \frac{u^2}{(s + \rho_{2M})^2} + \frac{2u \partial_s u}{s + \rho_{2M}} \right)
\]

Thus,

\[
\int_{D(T_2)} Q(T_2, x) dx = \int_{D(T_2)} I(T_2) + (T_2 + M) \frac{| \nabla^* u |^2}{2} dx - \frac{T_2 + M}{2} \int_{\partial D(T_2)} \frac{2u \partial_s u}{s + \rho_{2M}} + \frac{u^2}{(s + \rho_{2M})^2} d\Sigma
\]

Now, integrating by parts and using Dirichlet boundary condition, we get

\[
- \int_{D(T_2)} \partial_s ((s + \rho_{2M})u^2) \frac{(s + \rho_{2M})^2}{(s + \rho_{2M})^2} dx = \int_{D(T_2)} (s + \rho_{2M})u^2 \partial_s \left( \frac{(s + \rho_{1})(s + \rho_{2})}{(s + \rho_{2M})^2} \right) \Lambda \rho_1 \rho_2 ds d\sigma_1 d\sigma_2 - \int_{\partial D(T_2)} \frac{u^2}{T_2 + M} dS_2
\]

where \(dS_2\) is the measure on \(\partial D(T_2)\) and

\[
\partial D(T_2) = \{ x, s + \rho_{2M} = T_2 + M \}.
\]

But

\[
\partial_s \left( \frac{(s + \rho_{1})(s + \rho_{2})}{(s + \rho_{2M})^2} \right) = \frac{(2s + \rho_1 + \rho_2)(s + \rho_{2M}) - 2(s + \rho_1)(s + \rho_2)}{(s + \rho_{2M})^3}
\]

\[
= \frac{1}{(s + \rho_{2M})^3} \left( \sum_{i,j=1, i \neq j}^2 (\rho_{2M} - \rho_i)(s + \rho_j) \right) \geq 0
\]

so

\[
\int_{D(T_2)} Q(T_2, x) dx = \int_{D(T_2)} I(T_2) + (T_2 + M) \frac{| \nabla^* u |^2}{2} dx - \frac{T_2 + M}{2} \int_{\partial D(T_2)} \frac{u^2}{T_2 + M} dS_2
\]

\[
+ \frac{T_2 + M}{2} \int_{D(T_2)} \frac{u^2}{(s + \rho_{2M})^2} \left( \sum_{i,j=1, i \neq j}^2 (\rho_{2M} - \rho_i)(s + \rho_j) \right) \Lambda \rho_1 \rho_2 ds d\sigma_1 d\sigma_2
\]

\[
= \int_{D(T_2)} I(T_2) + (T_2 + M) \frac{| \nabla^* u |^2}{2} dx - \frac{1}{2} \int_{\partial D(T_2)} u^2 dS_2
\]

\[
+ \frac{T_2 + M}{2} \int_{D(T_2)} \left( \sum_{i=1}^2 \frac{\rho_{2M} - \rho_i}{s + \rho_i} \right) \frac{u^2}{(s + \rho_{2M})^2} dx
\]

(4.4)

Similarly we get

\[
\int_{D(T_1)} Q(T_1, x) dx = \int_{D(T_1)} I(T_1) + (T_1 + M) \frac{| \nabla^* u |^2}{2} dx - \frac{1}{2} \int_{\partial D(T_1)} u^2 dS_1
\]

(4.5)

\[
+ \frac{T_1 + M}{2} \int_{D(T_1)} \left( \sum_{i=1}^2 \frac{\rho_{2M} - \rho_i}{s + \rho_i} \right) \frac{u^2}{(s + \rho_{2M})^2} dx
\]

The term on the mantle

\[
...\]
On the mantle $M_{T_1}^{T_2}$, we have $s + \rho_{2M} = t + M$, and recall that $\nabla u \cdot \nu = \partial_s u$ (Remark 2.7), thus we get:

$$Q - P \cdot \nu = (s + \rho_{2M})(\partial_t u + \partial_s u)^2 + u(\partial_t u + \partial_s u)$$

$$= (s + \rho_{2M}) \left( \partial_t u + \partial_s u + \frac{u}{s + \rho_{2M}} \right)^2 - u(\partial_t u + \partial_s u) - \frac{u^2}{s + \rho_{2M}}$$

so

$$-\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} (Q - P \cdot \nu) \, ds = -\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} (s + \rho_{2M}) \left( \partial_t u + \partial_s u + \frac{u}{s + \rho_{2M}} \right)^2 \, ds$$

$$+ \frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} u(\partial_t u + \partial_s u) + \frac{u^2}{s + \rho_{2M}} \, ds$$

(4.6)

Now setting $\bar{\pi}(y) = u(s + \rho_{2M} - M, y)$ on the mantle, we have

$$\nabla \bar{\pi} = \nu \partial_t u + \nabla u \text{ and } \partial_s \bar{\pi} = \partial_t u + \partial_s u,$$

the second term in (4.6) can be written as such

$$\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} u(\partial_t u + \partial_s u) + \frac{u^2}{s + \rho_{2M}} \, ds = \int_{M_{T_1}^{T_2}} \left( \bar{\pi} \partial_t \bar{\pi} + \frac{\bar{\pi}^2}{s + \rho_{2M}} \right) \, dy = \int_{M_{T_1}^{T_2}} \frac{\partial_s ((s + \rho_{2M})^2 \bar{\pi}^2)}{2(s + \rho_{2M})^2} \, dy$$

where

$$\overline{M}_{T_1}^{T_2} = \{ x, T_1 + M \leq s + \rho_{2M} \leq T_2 + M \}$$

Integrating by parts and using the Dirichlet boundary condition, we get

$$\int_{M_{T_1}^{T_2}} \frac{\partial_s ((s + \rho_{2M})^2 \bar{\pi}^2)}{2(s + \rho_{2M})^2} \, dy = \int_{M_{T_1}^{T_2}} \frac{\partial_s ((s + \rho_{2M})^2 \bar{\pi}^2)}{2(s + \rho_{2M})^2} \frac{\Lambda}{\rho_1 \rho_2} (s + \rho_1)(s + \rho_2) \, ds \, d\sigma_1 \, d\sigma_2$$

$$= -\int_{M_{T_1}^{T_2}} \frac{1}{2} (s + \rho_{2M})^2 \bar{\pi}^2 \partial_s \left( \frac{(s + \rho_1)(s + \rho_2)}{(s + \rho_{2M})^2} \right) \frac{\Lambda}{\rho_1 \rho_2} \, ds \, d\sigma_1 \, d\sigma_2$$

$$+ \frac{1}{2} \int_{\partial D(T_2)} \bar{\pi}^2 \, dS_2 - \frac{1}{2} \int_{\partial D(T_1)} \bar{\pi}^2 \, dS_1$$

$$= -\frac{1}{2} \int_{M_{T_1}^{T_2}} \left( \sum_{i,j=1, i \neq j}^2 (\rho_{2M} - \rho_i)(s + \rho_j) \right) \frac{\bar{\pi}^2}{s + \rho_{2M}} \frac{\Lambda}{\rho_1 \rho_2} \, ds \, d\sigma_1 \, d\sigma_2$$

$$+ \frac{1}{2} \int_{\partial D(T_2)} \bar{\pi}^2 \, dS_2 - \frac{1}{2} \int_{\partial D(T_1)} \bar{\pi}^2 \, dS_1$$

Thus the term on the mantle (4.6) becomes

(4.7)

$$-\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} (Q - P \cdot \nu) \, ds = -\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} (s + \rho_{2M}) \left( \partial_t u + \partial_s u + \frac{u}{s + \rho_{2M}} \right)^2 \, ds$$

$$-\frac{1}{2} \int_{M_{T_1}^{T_2}} \left( \sum_{i,j=1, i \neq j}^2 (\rho_{2M} - \rho_i)(s + \rho_j) \right) \frac{\bar{\pi}^2}{s + \rho_{2M}} \frac{\Lambda}{\rho_1 \rho_2} \, ds \, d\sigma_1 \, d\sigma_2$$

$$+ \frac{1}{2} \int_{\partial D(T_2)} u^2 \, dS_2 - \frac{1}{2} \int_{\partial D(T_1)} u^2 \, dS_1$$
The remainder term

We have $\alpha = (s + \rho_2M)\nu$ thus by Remark 2.7, we get

$$\text{div} \alpha = \frac{1}{(\kappa_1s + 1)(\kappa_2s + 1)} \partial_s((\kappa_1s + 1)(\kappa_2s + 1)(s + \rho_2M))$$

$$= 1 + \frac{\kappa_1(s + \rho_2M)}{\kappa_1s + 1} + \frac{\kappa_2(s + \rho_2M)}{\kappa_2s + 1}$$

$$= 3 + \frac{\rho_2M - \rho_1}{s + \rho_1} + \frac{\rho_2M - \rho_2}{s + \rho_2}$$

since $s + \rho_i > 0$ and $\rho_2M \geq \rho_2 \geq \rho_1$ then $\text{div} \alpha - 3 \geq 0$ and

(4.8) $$\int_{K_{T_1}^2} (\text{div} \alpha - 3) \frac{|\partial_s u|^2}{2} dx dt \geq 0$$

Now, in the following remainder term

$$\int_{K_{T_1}^2} (5 - \text{div} \alpha) \frac{|u|^6}{6} dx dt$$

we have

$$5 - \text{div} \alpha = 2 - \frac{\rho_2M - \rho_1}{s + \rho_1} - \frac{\rho_2M - \rho_2}{s + \rho_2}$$

Imposing the following geometric condition

$$\min_{\partial V}(s_0 + \rho_1 - (\rho_2M - \rho_1)) > 0$$

we get that (recall that $s \geq s_0$)

$$\frac{\rho_2M - \rho_1}{s + \rho_1} + \frac{\rho_2M - \rho_2}{s + \rho_2} < 2$$

and thus

(4.9) $$\int_{K_{T_1}^2} (5 - \text{div} \alpha) \frac{|u|^6}{6} dx dt \geq 0$$

We still have to deal with the following term in $R$:

$$H_\alpha(\nabla u, \nabla u) \cdot ((\nabla u \cdot \nabla) \alpha)$$

We have $H_\alpha(\nabla u, \nabla u) = \nabla u \cdot ((\nabla u \cdot \nabla) \alpha)$; and by Remark 2.7, we have

$$\nabla u = \partial_s u \nu + \frac{1}{(\kappa_1s + 1)} |X_\sigma| \partial_{\sigma_1} u X_\sigma^0 + \frac{1}{(\kappa_2s + 1)} |X_\tau| \partial_{\tau_1} u X_\tau^0$$

$$\nabla u \cdot \nabla = \partial_s u \partial_s + \frac{1}{(\kappa_1s + 1)^2} |X_\sigma| \partial_{\sigma_1} u \partial_{\sigma_1} + \frac{1}{(\kappa_2s + 1)^2} |X_\tau| \partial_{\tau_1} u \partial_{\tau_1}$$

$$= \partial_s u \nu + (s + \rho_2M) \left( \frac{\kappa_1}{(\kappa_1s + 1)^2} |X_\sigma| \partial_{\sigma_1} u X_\sigma^0 + \frac{\kappa_2}{(\kappa_2s + 1)^2} |X_\tau| \partial_{\tau_1} u X_\tau^0 \right)$$

Hence

$$H_\alpha(\nabla u, \nabla u) = \nabla u \cdot ((\nabla u \cdot \nabla) \alpha) = (\partial_s u)^2 + \frac{\kappa_1(s + \rho_2M)}{(\kappa_1s + 1)^3} |X_\sigma| \partial_{\sigma_1} u X_\sigma^0 \partial_{\sigma_1} u + \frac{\kappa_2(s + \rho_2M)}{(\kappa_2s + 1)^3} |X_\tau| \partial_{\tau_1} u X_\tau^0 \partial_{\tau_1} u$$
Using

\[ |\nabla u|^2 = (\partial_s u)^2 + \frac{1}{(\kappa_1 s + 1)^2|X_{s_1}^0|^2}(\partial_{s_1} u)^2 + \frac{1}{(\kappa_2 s + 1)^2|X_{s_2}^0|^2}(\partial_{s_2} u)^2 \]

we get

\[ H_\alpha(\nabla u, \nabla u) = |\nabla u|^2 + \frac{\rho_2 M - \rho_1}{s + \rho_1} |\nabla_1^* u|^2 + \frac{\rho_2 M - \rho_2}{s + \rho_2} |\nabla_2^* u|^2 \geq 0 \]

Moreover, we have

\[ (1 - \text{div}_0) \frac{|\nabla u|^2}{2} = -|\nabla u|^2 - \left( \frac{\rho_2 M - \rho_1}{s + \rho_1} + \frac{\rho_2 M - \rho_2}{s + \rho_2} \right) \frac{|\nabla u|^2}{2} \]

Hence

\[ (4.10) \quad \int_{K_{T_1}^T} H_\alpha(\nabla u, \nabla u) + (1 - \text{div}_0) \frac{|\nabla u|^2}{2} dx dt \]

Recall that

\[ |\nabla u|^2 = |\partial_s u|^2 + |\nabla^* u|^2 \]

and note that

\[ \frac{\partial_s ((s + \rho_2 M) u)}{s + \rho_2 M} = \partial_s u + \frac{u}{s + \rho_2 M}, \]

hence

\[ |\nabla u|^2 = |\nabla^* u|^2 + \left| \frac{\partial_s ((s + \rho_2 M) u)}{s + \rho_2 M} \right|^2 - \left| \frac{u}{s + \rho_2 M} \right|^2 - \frac{2 u \partial_s u}{s + \rho_2 M} \]

Substituting this in (4.10), we get:

\[ \int_{K_{T_1}^T} H_\alpha(\nabla u, \nabla u) + (1 - \text{div}_0) \frac{|\nabla u|^2}{2} dx dt \]

\[ = \int_{K_{T_1}^T} \left( \sum_{i=1}^{2} \frac{\rho_2 M - \rho_i}{s + \rho_i} |\nabla_1^* u|^2 \right) dx dt - \frac{1}{2} \left( \sum_{i=1}^{2} \frac{\rho_2 M - \rho_i}{s + \rho_i} \right) |\nabla^* u|^2 \]

\[ (4.11) \]

\[ = I + II + III + IV \]

\[ \frac{1}{2} \int_{K_{T_1}^T} \left( \frac{\rho_2 M - \rho_1}{s + \rho_1} - \frac{\rho_2 M - \rho_2}{s + \rho_2} \right) |\nabla_1^* u|^2 + \left( \frac{\rho_2 M - \rho_2}{s + \rho_2} - \frac{\rho_2 M - \rho_1}{s + \rho_1} \right) |\nabla^*_2 u|^2 dx dt \]
Integrating by parts the last term and using the Dirichlet boundary condition, we get:

\[
IV = \sum_{i,j=1, i \neq j}^{2} \frac{1}{2} \int_{t_{1}}^{T_{2}} \int_{s + \rho_{2M} \leq t + M} \left( \rho_{2M} - \rho_{i} \right) \frac{2 u \partial_{s} u}{s + \rho_{i}} \frac{s + \rho_{j}}{s + \rho_{2M}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

\[
= \sum_{i,j=1, i \neq j}^{2} \frac{1}{2} \int_{t_{1}}^{T_{2}} \int_{s + \rho_{2M} \leq t + M} \left( \rho_{2M} - \rho_{i} \right) \frac{u^2}{s + \rho_{2M}} \frac{1}{\rho_{1} \rho_{2}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

\[
= - \sum_{i,j=1, i \neq j}^{2} \frac{1}{2} \int_{t_{1}}^{T_{2}} \int_{s + \rho_{2M} \leq t + M} \left( \rho_{2M} - \rho_{i} \right) u^2 \frac{s + \rho_{j}}{s + \rho_{2M}} \frac{1}{\rho_{1} \rho_{2}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

\[
+ \sum_{i,j=1, i \neq j}^{2} \frac{1}{2} \int_{t_{1}}^{T_{2}} \int_{s + \rho_{2M} \leq t + M} \left( \rho_{2M} - \rho_{i} \right) u^2 \frac{s + \rho_{j}}{s + \rho_{2M}} \frac{1}{\rho_{1} \rho_{2}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

and we have

\[
III = \sum_{i,j=1, i \neq j}^{2} \frac{1}{2} \int_{t_{1}}^{T_{2}} \int_{s + \rho_{2M} \leq t + M} \left( \rho_{2M} - \rho_{i} \right) (s + \rho_{j}) \frac{u^2}{(s + \rho_{2M}) \rho_{1} \rho_{2}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

Hence,

\[
III + IV = \frac{1}{2} \int_{t_{1}}^{T_{2}} \sum_{i,j=1, i \neq j}^{2} \left( \rho_{2M} - \rho_{i} \right) (s + \rho_{j}) \frac{u^2}{s + \rho_{2M}} \frac{1}{\rho_{1} \rho_{2}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

\[
+ \frac{1}{2} \int_{t_{1}}^{T_{2}} \sum_{i,j=1, i \neq j}^{2} \left( \rho_{2M} - \rho_{i} \right) (s + \rho_{j}) \frac{u^2}{s + \rho_{2M}} \frac{1}{\rho_{1} \rho_{2}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

Now substituting in (4.11) we get

\[
\int_{K_{T_{1}}}^{T_{2}} H_{\alpha}(\nabla u, \nabla u) + (1 - \text{div} u) \frac{1}{2} \left( \frac{\nabla u}{s + \rho_{1}} - \frac{\rho_{2M} - \rho_{1}}{s + \rho_{2}} \right) \left( \frac{\rho_{2M} - \rho_{2}}{s + \rho_{2}} \right) ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

\[
\frac{1}{2} \int_{K_{T_{1}}}^{T_{2}} \left( \sum_{i=1}^{2} \frac{\rho_{2M} - \rho_{i}}{s + \rho_{i}} \left( \frac{\partial_{s}(s + \rho_{2M}) u}{s + \rho_{2M}} \right) \right) \frac{\partial_{s}(s + \rho_{2M}) u}{s + \rho_{2M}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

\[
\frac{1}{2} \int_{K_{T_{1}}}^{T_{2}} \left( \sum_{i,j=1, i \neq j}^{2} \left( \rho_{2M} - \rho_{i} \right) (s + \rho_{j}) \frac{u^2}{s + \rho_{2M}} \frac{1}{\rho_{1} \rho_{2}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

\[
+ \frac{1}{2} \int_{K_{T_{1}}}^{T_{2}} \left( \sum_{i,j=1, i \neq j}^{2} \left( \rho_{2M} - \rho_{i} \right) (s + \rho_{j}) \frac{u^2}{s + \rho_{2M}} \frac{1}{\rho_{1} \rho_{2}} ds d\sigma_{1} d\sigma_{2} \frac{dt}{dt}
\]

and note that since we imposed the geometric condition

\[
\min_{\partial V}(s_{0} + \rho_{1} - (\rho_{2M} - \rho_{1})) > 0
\]
we have \( s + 2\rho_j - \rho_{2M} > 0 \) and thus the last term in (4.13) is nonnegative.

The differential inequality

Now, summing up all the terms ((4.3), (4.4), (4.5), (4.7), (4.8), (4.9), and (4.13)) in the integral equality (4.2) and dropping the nonnegative terms, we get the following differential inequality:

\[
\int_{D(T_2)} I(T_2) + (T_2 + M) \frac{|\nabla^* u|^2}{2} dx \\
\leq \int_{D(T_1)} I(T_1) + (T_1 + M) \frac{|\nabla^* u|^2}{2} dx \\
+ \frac{T_1 + M}{2} \int_{D(T_1)} \left( \sum_{i=1}^{2} \rho_{2M} - \rho_i \right) \frac{u^2}{(s + \rho_i)^2} dx \\
+ \frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} (s + \rho_{2M}) \left( \partial_t u + \partial_s u + \frac{u}{s + \rho_{2M}} \right)^2 d\sigma \\
+ \frac{1}{2} \int_{K_{T_1}^{T_2}} \left( \frac{\rho_{2M} - \rho_2}{s + \rho_2} - \frac{\rho_{2M} - \rho_1}{s + \rho_1} \right) |\nabla^* u|^2 + \left( \frac{\rho_{2M} - \rho_1}{s + \rho_1} - \frac{\rho_{2M} - \rho_2}{s + \rho_2} \right) |\nabla^* u|^2 dx dt \\
+ \frac{1}{2} \int_{K_{T_1}^{T_2}} \left( \sum_{i=1}^{2} \rho_{2M} - \rho_i \right) \left| \partial_s ((s + \rho_{2M})u) \right|^2 dx dt
\]

Recall that for \( i = 1, 2 \) we have:

\[
I(T_i) = \frac{1}{4} (T_i + M + (s + \rho_{2M})) \left[ \partial_t u + \partial_s ((s + \rho_{2M})u) \right]^2 + \frac{1}{4} (T_i + M - (s + \rho_{2M})) \left[ \partial_t u - \partial_s ((s + \rho_{2M})u) \right]^2 \\
+ (T_i + M) \frac{u^6}{6}
\]

Thus

\[
I(T_2) \geq \frac{1}{2} (T_2 + M - (s + \rho_{2M})) \left( \partial_t u + \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right) \left( \partial_t u + \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right) + (T_2 + M) \frac{u^6}{6}
\]

we also have,

\[
I(T_1) \leq \frac{1}{2} (T_1 + M + (s + \rho_{2M})) \left( \partial_t u + \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right)^2 + (T_1 + M) \frac{u^6}{6}
\]
so we get
\[
\int_{D(T_2)} \frac{1}{2}(T_2 + M - (s + \rho_{2M})) \left( (\partial_t u)^2 + \left| \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) + (T_2 + M) \frac{u_6}{6} + (T_2 + M) \frac{|\nabla u|^2}{2} \, dx
\]
(4.14)

\[
\leq \int_{D(T_1)} \frac{1}{2}(T_1 + M + (s + \rho_{2M})) \left( (\partial_t u)^2 + \left| \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 + \frac{u_6}{3} + |\nabla u|^2 \right) \, dx
\]
\[
+ \frac{T_1 + M}{2} \int_{D(T_1)} \left( \sum_{i=1}^{2} \frac{\rho_{2M} - \rho_i}{s + \rho_i} \right) \frac{u^2}{(s + \rho_{2M})^2} \, dx + \frac{1}{\sqrt{2}} \int_{M_{T_1}^2} (s + \rho_{2M}) \left( \partial_t u + \partial_s u + \frac{u}{s + \rho_{2M}} \right)^2 \, d\sigma
\]
\[
+ \frac{1}{2} \int_{K_{T_1}^2} \left( \sum_{i=1}^{2} \frac{\rho_{2M} - \rho_i}{s + \rho_i} \right) \left| \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 \, dx dt
\]
\[
\leq A_1 + A_2 + A_3 + A_4 + A_5
\]

We have
\[
\left| \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 = \left| \frac{u}{s + \rho_{2M}} + \partial_s u \right|^2 \leq 2 \left| \frac{u}{s + \rho_{2M}} \right|^2 + 2 |\partial_s u|^2 \leq 2 \left| \frac{u}{s + \rho_{2M}} \right|^2 + 2 |\nabla u|^2
\]
and on \(D(T_1)\) we have: \(s + \rho_{2M} \leq T_1 + M\), thus
\[
A_1 \leq (T_1 + M) \int_{D(T_1)} \left( (\partial_t u)^2 + 2 \left| \frac{u}{s + \rho_{2M}} \right|^2 \right) \, dx + 3|\nabla u|^2 + \frac{u_6}{3}) \, dx
\]
Moreover, by the geometric condition we imposed
\[
\min_{\partial V} (s_0 + \rho_1 - (\rho_{2M} - \rho_1)) > 0
\]
we have
\[
\sum_{i=1}^{2} \frac{\rho_{2M} - \rho_i}{s + \rho_i} \leq 2
\]
Thus
\[
A_2 \leq (T_1 + M) \int_{D(T_1)} \left| \frac{u}{s + \rho_{2M}} \right|^2 \, dx
\]
so
\[
A_1 + A_2 \leq (T_1 + M) \int_{D(T_1)} \left( (\partial_t u)^2 + 3 \left| \frac{u}{s + \rho_{2M}} \right|^2 \right) \, dx + 3|\nabla u|^2 + \frac{u_6}{3}) \, dx
\]
Recall that there exist a constant \(a_0 > 0\) such that \(s + \rho_{2M} \geq a_0r\) (Lemma 2.4). Thus,
\[
\int \left| \frac{u}{s + \rho_{2M}} \right|^2 \, dx \leq \frac{1}{a_0^2} \int \frac{u^2}{r^2} \, dx
\]
and by Hardy’s inequality:
\[
\int \frac{u^2}{r^2} \, dx \leq C \int |\nabla u|^2 \, dx
\]
we get that
\[
(4.15) \quad \int \left| \frac{u}{s + \rho_{2M}} \right|^2 \, dx \lesssim \int |\nabla u|^2 \, dx
\]
where \( c_0 \) and \( c_1 \) are constants that depend on the geometry of the obstacle and \( E \) is the conserved energy. Now, the term on the mantle \( A_3 \) can be written as follows:

\[
A_3 = \int_{\Omega_{T_1}} (s + \rho_{2M}) \left( \partial_s \pi + \frac{\pi}{s + \rho_{2M}} \right)^2 dy
\]

\[
\leq (T_2 + M) \int_{\Omega_{T_1}} \left( \partial_s \pi + \frac{\pi}{s + \rho_{2M}} \right)^2 dy
\]

\[
\leq 2(T_2 + M) \int_{\Omega_{T_1}} \left( |\partial_s \pi|^2 + \frac{|\pi|^2}{s + \rho_{2M}} \right) dy
\]

Similarly to (4.15), we have

\[
\int |\frac{\pi}{s + \rho_{2M}}|^2 dy \lesssim \int |\nabla \pi|^2 dy
\]

hence

\[
A_3 \lesssim (T_2 + M) \int_{\Omega_{T_1}} |\nabla \pi|^2 dy \leq (c_2 + c_3 T_2) \text{flux}(T_1, T_2)
\]

where \( c_2 \) and \( c_3 \) are constants that depend on the geometry of the obstacle, and where

\[
\text{flux}(T_1, T_2) = \int_{\Omega_{T_1}} \left( \frac{1}{2} |\nu \partial_t u + \nabla u|^2 + \frac{u^6}{6} \right) d\sigma = \sqrt{2} \int_{\Omega_{T_1}} \left( \frac{|\nabla u|^2}{2} + \frac{u^6}{6} \right) dy.
\]

\[
A_4 \leq \frac{1}{2} \int_{\Omega_{T_1}} \left( \frac{\rho_{2M} - \rho_2}{s + \rho_2} - \frac{\rho_{2M} - \rho_1}{s + \rho_1} \right) |\nabla^* u|^2 dx dt
\]

\[
\leq \frac{1}{2} \int_{\Omega_{T_1}} \left( \frac{\rho_{2M} - \rho_2}{s + \rho_2} + \frac{\rho_{2M} - \rho_1}{s + \rho_1} \right) |\nabla^* u|^2 dx dt
\]

thus,

\[
A_4 + A_5 \leq \frac{1}{2} \int_{\Omega_{T_1}} \left( \frac{\rho_{2M} - \rho_2}{s + \rho_2} + \frac{\rho_{2M} - \rho_1}{s + \rho_1} \right) \left( |\nabla^* u|^2 + \left| \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx dt
\]

Thus (4.14) becomes

(4.16)

\[
\int_{D(T_2)} (T_2 + M - (s + \rho_{2M})) \left( |\nabla^* u|^2 + \left| \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) + T_2 \frac{u^6}{3} dx
\]

\[
\leq 2(c_0 + c_1 T_1) E + 2(c_2 + c_3 T_2) \text{flux}(T_1, T_2) + \sum_{i=1}^{2} \int_{T_1} \int_{\Omega} \frac{\rho_{2M} - \rho_i}{s + \rho_i} \left( |\nabla^* u|^2 + \left| \frac{\partial_s ((s + \rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx dt
\]

Split \( \Omega \) into \( \{ s + \rho_{2M} \leq \epsilon t + M \} \) and \( \{ s + \rho_{2M} > \epsilon t + M \} \), where \( 0 < \epsilon < 1 \) is a constant that depends on the geometry of the obstacle to be later specified, then the last term in (4.16)
becomes

\[
J = \sum_{i=1}^{2} \int_{T_{1}}^{T_{2}} \int_{\Omega} \frac{\rho_{2M} - \rho_{i}}{s + \rho_{i}} \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx dt
\]

\[
\leq \sum_{i=1}^{2} \int_{T_{1}}^{T_{2}} \int_{s+\rho_{2M} > t+M \atop \rho_{2M} - \rho_{i}} \frac{\rho_{2M} - \rho_{i}}{s + \rho_{2M} - (\rho_{2M} - \rho_{i})} \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx dt
\]

\[
+ \max_{\partial V} \left( \frac{\rho_{2M} - \rho_{1}}{s_{0} + \rho_{1}} + \frac{\rho_{2M} - \rho_{2}}{s_{0} + \rho_{2}} \right) \int_{T_{1}}^{T_{2}} \int_{s+\rho_{2M} \leq t+M \atop s + \rho_{2M}} \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx dt
\]

\[
= J_{1} + J_{2}
\]

\[
J_{1} \leq \sum_{i=1}^{2} \int_{T_{1}}^{T_{2}} \frac{\rho_{2M} - \rho_{i}}{\epsilon t + N} \left( \int_{s+\rho_{2M} > t+M} \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx \right) dt
\]

\[
\leq 2 \int_{T_{1}}^{T_{2}} \frac{\rho_{2M} - \rho_{1m}}{\epsilon t + N} \left( \int_{s+\rho_{2M} > t+M} \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx \right) dt
\]

with \( N = M - \rho_{2M} \geq 0 \). Since \( s + \rho_{2M} \geq a_{0} r \) and using Hardy’s inequality, we get

\[
\int_{s+\rho_{2M} > t+M} \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx \lesssim E
\]

hence,

\[
J_{1} \lesssim \frac{2(\rho_{2M} - \rho_{1m})}{\epsilon} \ln \left( \frac{t_{2} + N}{t_{1} + N} \right) E \leq C_{1} E + C_{2} E \ln(1 + T_{2})
\]

where \( C_{1} \) and \( C_{2} \) are constants that depend on the geometry of the obstacle. Thus

(4.17) \( J \leq C_{1} E + C_{2} E \ln(1 + T_{2}) + \eta_{0} \int_{T_{1}}^{T_{2}} \int_{s+\rho_{2M} \leq t+M} \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx dt \)

with

\[
\eta_{0} = \max_{\partial V} \left( \frac{\rho_{2M} - \rho_{1}}{s_{0} + \rho_{1}} + \frac{\rho_{2M} - \rho_{2}}{s_{0} + \rho_{2}} \right)
\]

The geometric condition

\[
\min_{\partial V}(s_{0} + \rho_{1} - (\rho_{2M} - \rho_{1})) > 0
\]

which we assumed so far implies that \( \eta_{0} < 2 \) which is not enough as we want \( 0 < \eta_{0} < 1 \) for the proof of the \( L^{6} \) decay estimate (Theorem 1.1). Thus, we impose a stronger condition

\[
\min_{\partial V}(s_{0} + \rho_{1} - 2(\rho_{2M} - \rho_{1})) > 0
\]

On the other hand,

(4.18) \( \int_{D(T_{2})} (T_{2} + M - (s + \rho_{2M})) \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) + \frac{T_{2} u^{6}}{3} dx \)

\[
\geq T_{2}(1 - \epsilon) \int_{s+\rho_{2M} \leq t_{2} + M} \left( |\nabla * u|^2 + \left| \frac{\partial_{s}((s+\rho_{2M})u)}{s + \rho_{2M}} \right|^2 \right) dx + T_{2} \int_{D(T_{2})} \frac{u^{6}}{3} dx
\]
Using (4.17) and (4.18), (4.16) becomes

\[ T_2(1 - \epsilon) \int_{s + \rho_2M \leq s + T + M} \left( |\nabla^s u|^2 + \left| \frac{\partial_s ((s + \rho_2M)u)}{s + \rho_2M} \right|^2 \right) \, dx + T_2 \int_{D(T_2)} \frac{u^6}{3} \, dx \]

(4.19)

\[ \leq C_0 + 2c_1 T_1 E + C_2 E \ln(1 + T_2) + 2(c_2 + c_3 T_2) \text{flux}(T_1, T_2) \]
\[ + \eta_0 \int_{T_1}^{T_2} \int_{s + \rho_2M \leq t + M} \left( |\nabla^s u|^2 + \left| \frac{\partial_s ((s + \rho_2M)u)}{s + \rho_2M} \right|^2 \right) \, dx \, dt \]

Now, setting \( T_2 = T \) and \( T_1 = \beta T \) for some \( 0 < \beta < 1 \) and choosing \( 0 < \epsilon < 1 \) such that \( \epsilon = 1 - \sqrt{\eta_0} \), (4.19) yields

\[ T \sqrt{\eta_0} \int_{s + \rho_2M \leq t + M} \left( |\nabla^s u|^2 + \left| \frac{\partial_s ((s + \rho_2M)u)}{s + \rho_2M} \right|^2 \right) \, dx + T \int_{s + \rho_2M \leq T + M} \frac{u^6(T, x)}{3} \, dx \]

(4.20)

\[ \leq C_0 + 2c_1 \beta T E + C_2 E \ln(1 + T) + 2(c_2 + c_3 T) \text{flux}(0, T) \]
\[ + \eta_0 \int_{0}^{T} \int_{s + \rho_2M \leq t + M} \left( |\nabla^s u|^2 + \left| \frac{\partial_s ((s + \rho_2M)u)}{s + \rho_2M} \right|^2 \right) \, dx \, dt \]

which ends the proof of Proposition 3.1.

5. Proof of the scattering (Corollary 1.4)

As a result of the \( L^6 \) decay estimate we get the scattering result of Corollary 1.4. The proof of this corollary was done in the paper of Blair, Smith, and Sogge [3] and we replicate it here for the sake of completeness.

We have the following Strichartz estimate on functions \( w(t, x) \) satisfying homogeneous Dirichlet boundary condition on non-trapping obstacles

\[ \|w\|_{L^6([\mathbb{R}, L^1(\Omega)])} + \|w\|_{L^6([\mathbb{R}, L^2(\Omega)])} \leq c \left( \|\nabla w(0, \cdot), \partial_t w(0, \cdot)\|_{L^2(\Omega)} + \|\Box w\|_{L^6([\mathbb{R}, L^2(\Omega)])} \right) \]

and we define the conserved energy of the linear equation (1.3)

\[ E_0(v; t) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 + |\partial_t v|^2 \, dx \]

Attention was restricted to the \( v_+ \) function, as symmetric arguments will yield the existence of a \( v_+ \) asymptotic to \( u \) at \(-\infty\). As remarked in [3], it is enough to prove (1.5) as (1.4) follows as a consequence. They first established the existence of the wave operator, that is for any solution \( v \) to the linear equation (1.3), there exists a unique solution \( u \) to the non linear equation (1.1) such that

\[ \lim_{t \to \infty} E_0(u - v; t) = 0 \]

Given (5.1), for any \( \delta > 0 \) one may select \( T \) large so that \( \|v\|_{L^6([T, \infty); L^1(\Omega))} \leq \delta \). Given any \( w(t, x) \) satisfying \( \|w\|_{L^6([T, \infty); L^1(\Omega))} \leq \delta \), we have a unique solution to the linear problem

\[ \Box \hat{w} = -(v + \hat{w})^5 \]
\[ \lim_{t \to \infty} E_0(\hat{w}; t) = 0 \]

as the right hand side is in \( L^1([T, \infty); L^2(\Omega)) \). The estimate (5.1) then also ensures that

\[ \|\hat{w}\|_{L^6([T, \infty); L^1(\Omega))} \leq c \|v + \hat{w}\|_{L^6([T, \infty); L^1(\Omega))}^5 \leq 32c_5^5 \]

Hence for \( \delta \) sufficiently small, the map \( w \to \hat{w} \) is seen to be a contraction on the ball of radius \( \delta \) in \( L^5 ([T, \infty); L^{10}(\Omega)) \). The unique fixed point \( w \) can be uniquely extended over all of \( \mathbb{R} \times \Omega \). Hence taking \( u = v + \hat{w} \) shows the existence of the wave operator.
To see that the wave operator is surjective, they used the $L^6$ decay estimate which we proved in Theorem 1.1 for our obstacle. This decay estimate establishes that the non linear effects of the solution map for the non linear equation (1.1) diminish as time evolves.

By the result of Theorem 1.1, given any $\epsilon > 0$, there exists $T$ sufficiently large such that

$$\sup_{t \geq T} \| u(t, \cdot) \|_{L^6} < \epsilon$$

Hence for any $S > T$ we obtain the following for any solution $u$ to (1.1)

$$\| u \|_{L^5([T,S]; L^{10}(\Omega))} + \| u \|_{L^4([T,S]; L^{12}(\Omega))} \leq c \left( E + \| u^5 \|_{L^1([T,S]; L^2(\Omega))} \right)$$

$$\leq cE + \epsilon \| u \|_{L^4([T,S]; L^{12}(\Omega))}^4$$

A continuity argument now yields

$$\square w = -u^5$$

$$\lim_{t \to \infty} E_0(w; t) = 0$$

admits a solution, showing that the wave operator is indeed surjective as $v = u - w$ is the desired solution to (1.3).

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