A short note on the computation of the generalised Jacobsthal function for paired progressions

Mario Ziller and John F. Morack

Abstract

Jacobsthal’s function was recently generalised for the case of paired progressions. It was proven that a specific bound of this function is sufficient for the truth of Goldbach’s conjecture and of the prime pairs conjecture as well. We extended and adapted algorithms described for the computation of the common Jacobsthal function, and computed respective function values of the paired Jacobsthal function for primorial numbers for primes up to 73. All these values fulfil the conjectured specific bound. In addition to this note, we provide a detailed review of the algorithmic approaches and the complete computational results in ancillary files.

1 Introduction

Henceforth, we denote the set of integral numbers by \( \mathbb{Z} \) and the set of natural numbers, i.e. positive integers, by \( \mathbb{N} \). Let \( \mathbb{P} = \{p_i \mid i \in \mathbb{N}\} \) be the set of prime numbers with \( p_1 = 2 \).

The commonly known Jacobsthal function \( j(n) \) is defined to be the smallest natural number \( m \), such that every sequence of \( m \) consecutive integers contains at least one integer coprime to \( n \) \[2, 1\].

Definition 1. For \( n \in \mathbb{N} \), the Jacobsthal function \( j(n) \) is defined as

\[
j(n) = \min \{m \in \mathbb{N} \mid \forall a \in \mathbb{Z} \ \exists q \in \{1, \ldots, m\} : a + q \perp n\}.
\]

Jacobshal remarked that the entire function is determined by its values for products of distinct primes \[2\]. The particular case of primorial numbers is therefore of great interest. The function values at these points characterise the maximum growth of the function.
The Jacobsthal function was recently generalised for the case of paired progressions. A paired progression \( [4] \) is defined as an ordered sequence of consecutive pairs of integers of a definite length.

**Definition 2.** Let \( a, b \in \mathbb{Z} \) and \( m \in \mathbb{N} \). A paired progression \( \langle a, b \rangle_m \) is defined as

\[
\langle a, b \rangle_m = \{(a + i, b + i) \in \mathbb{Z}^2 \mid i = 1, \ldots, m\}.
\]

The generalisation \( j_2(n) \) of the Jacobsthal function with respect to paired progressions is called paired Jacobsthal function. It is defined to be the smallest natural number \( n \) such that every paired progression of length \( m \), where the difference of its pair elements is even, contains at least one pair coprime to \( n \) [4].

**Definition 3.** For \( n \in \mathbb{N} \), the paired Jacobsthal function \( j_2(n) \) is defined as

\[
j_2(n) = \min \{m \in \mathbb{N} \mid \forall (a, b) \in \mathbb{Z}^2 \text{ with } 2 \mid (b - a) : \exists q \in \{1, \ldots, m\} : n \perp (a + q, b + q)\}.
\]

This function is also determined by its values for products of distinct primes [4], and the case of primorial numbers is most interesting. The function values at these points generate a separate function.

**Definition 4.** For \( n \in \mathbb{N} \), the primorial paired Jacobsthal function \( h_2(n) \) is defined as

\[
h_2(n) = \min \{m \in \mathbb{N} \mid \forall (a, b) \in \mathbb{Z}^2 \text{ with } 2 \mid (b - a) : \exists q \in \{1, \ldots, m\} \forall i \in \{1, \ldots, n\} : p_i \perp (a + q, b + q)\}.
\]

It was proven that the conjectured upper bound \( h_2(n) < p_n^2 - p_n, \ n \geq 3 \) of this function represents a sufficient condition for the truth of Goldbach’s conjecture and of the prime pairs conjecture as well [4].

### 2 Computation and results

Several algorithmic concepts were successfully applied to the computation of Jacobsthal’s common function for primorials [3]. For the computation of the primorial paired Jacobsthal function, we extended and adapted these approaches for the case of paired progressions. A detailed analysis of how to compute \( h_2(n) \) is provided in an ancillary file.

We computed the entire values of the function \( h_2(n) \) for \( n \leq 21 \) or \( p_n \leq 73 \), respectively. The results for \( 3 \leq n \leq 21 \) fulfil the conjectured bound \( h_2(n) < p_n^2 - p_n [4] \). The data are listed in table 1, including the maximum processed prime \( p_n \), the values of \( h_2(n) \), and the conjectured bound \( p_n^2 - p_n \).
| $n$ | $p_n$ | $h_2(n)$ | $p_n^2 - p_n$ | $n$ | $p_n$ | $h_2(n)$ | $p_n^2 - p_n$ |
|-----|-------|----------|----------------|-----|-------|----------|----------------|
| 1   | 2     | 2        | 2              | 12  | 37    | 708      | 1332           |
| 2   | 3     | 6        | 6              | 13  | 41    | 894      | 1640           |
| 3   | 5     | 18       | 20             | 14  | 43    | 1044     | 1806           |
| 4   | 7     | 30       | 42             | 15  | 47    | 1284     | 2162           |
| 5   | 11    | 66       | 110            | 16  | 53    | 1422     | 2756           |
| 6   | 13    | 150      | 156            | 17  | 59    | 1656     | 3422           |
| 7   | 17    | 192      | 272            | 18  | 61    | 1902     | 3660           |
| 8   | 19    | 258      | 342            | 19  | 67    | 2190     | 4422           |
| 9   | 23    | 366      | 506            | 20  | 71    | 2460     | 4970           |
| 10  | 29    | 450      | 812            | 21  | 73    | 2622     | 5256           |

Table 1: Results of computation.

In addition to this note, we provide several ancillary files. The file "full_details.pdf" includes a detailed description of theory, algorithms, and further results. This paper widely follows the structure of our former paper [3]. Furthermore, it outlines the content of all other auxiliary data files representing exhaustive lists of the computational results.

Contact
marioziller@arcor.de
axelmorack@live.com

References

[1] Paul Erdös, *On the Integers Relatively Prime to n and a Number-Theoretic Function Considered by Jacobsthal*, MATHEMATICA SCANDINAVICA 10 (1962), 163–170.

[2] Ernst Jacobsthal, *Über Sequenzen ganzer Zahlen, von denen keine zu n teilerfremd ist. I*, D.K.N.V.S. Forhandlinger 33 (1960), no. 24, 117–124.

[3] Mario Ziller and John F. Morack, *Algorithmic concepts for the computation of Jacobsthal’s function*, arXiv:1611.03310 [math.NT] (2016).

[4] Mario Ziller and John F. Morack, *Divisibility in paired progressions, Goldbach’s conjecture, and the infinitude of prime pairs*, arXiv:1706.00317 [math.NT] (2017).