PFAFFIANS AND SHUFFLING RELATIONS FOR THE SPIN MODULE

R. CHIRIVÌ AND A. MAFFEI

Abstract. We present explicit formulas for a set of generators of the ideal of relations among the pfaffians of the principal minors of the antisymmetric matrices of fixed dimension. These formulas have an interpretation in terms of the standard monomial theory for the spin module of orthogonal groups.

Introduction

Let $\mathfrak{X}_{n+1}$ be the set of $(n+1) \times (n+1)$ antisymmetric matrices over the complex number. It is well known that the determinant $\mathfrak{X}_{n+1} \ni X \mapsto \det X \in \mathbb{C}$ is the square of a polynomial function $\mathfrak{X}_{n+1} \ni X \mapsto \text{Pf}(X) \in \mathbb{C}$ called the pfaffian of a matrix. In particular only the even dimensional principal minors of $X \in \mathfrak{X}_{n+1}$ have non zero pfaffian.

Let $\mathbb{B}$ be the ring $\mathbb{C}[x_{ij} \mid 1 \leqslant i < j \leqslant n+1]$. As a set of generators of $\mathbb{B}$ we choose the pfaffians of the (even dimensional) principal minors of matrices in $\mathfrak{X}_{n+1}$. In this paper we want to describe the relations among these generators. This is a classical problem and may be seen as analogous to the Plücker relations for determinants. Indeed the formulas we present are very similar to Plücker formulas. In order to state such formulas we introduce some notations.

The pfaffians of the principal minors are indexed by lists of even length of integers in $\{1, 2, \ldots, n+1\}$, or, equivalently, by row, i.e. list $I = i_1 i_2 \cdots i_r$ (of any length) of integers in $\{1, 2, \ldots, n\}$ in the following way: define $I^0$ be the either the same sequence $I$ if $r$ is even or the sequence $i_1 i_2 \cdots i_r, n + 1$ if $r$ is odd and let $\text{pf}_I$ be the polynomial function on $\mathfrak{X}_{n+1}$ given by the pfaffian of the principal minor with rows and columns in $I^0$. For convenience we set also $\text{pf}_\varnothing$ to be equal to the constant polynomial 1. This indexing procedure may look unusual but it makes easier to write the relations among pfaffians.

Consider the action $\sigma \cdot (i_1 \cdots i_r) = \sigma(i_1) \cdots \sigma(i_r)$ of the symmetric group $S_n$ on the set of rows. Given a row $I$ containing distinct entries let $\tau_I \in S_n$ be the permutation reordering the entries of $I$ in increasing order and fixing all integers not appearing in $I$ and let $\varepsilon(I) = (-1)^{\tau_I}$. We have $\text{pf}_\sigma \cdot I = \varepsilon(I) \text{pf}_I$ and $\text{pf}_I = 0$ if there are repetitions in $I$.

If $I$ and $J$ are rows we set $IJ$ to be the row obtained by listing the elements of $J$ after the elements of $I$. For a row $I = i_1 i_2 \cdots i_r$, let $|I| = r$ be its length.

Let us define an order on rows as follows: $R = i_1 i_2 \cdots i_r \leqslant S = j_1 j_2 \cdots j_s$ if $r \geqslant s$ and $i_h \leqslant j_h$ for $h = 1, 2, \ldots, s$. For each pair of rows $R, S$ with increasing entries which are not comparable with respect to this order we construct a relation among pfaffians in the following way. Assume $R$ is not shorter than $S$ and let $R = IJ$ and $S = HK$ be such that $|I| = r$, $|H| = r + 1$, any entry of $I$ is less or equal to the corresponding entry of $H$ and the first entry of $J$ is greater than the last entry of $H$; so we have the first violation of the order condition in the $(r+1)$-th column.

Theorem. For each pair of rows as above, we have the following relation among pfaffians:

$$
\sum (-1)^{h(h+1)/2} \left[ J \setminus K + h \atop h \right] \varepsilon(H' J') \varepsilon(Z_2 K'_1) \text{pf}_{IJ'K'_1} \text{pf}_{H'K'_2} = 0
$$

(1)
with \( h = |H| - |H'| \), \( Z_1 \cong I \cap H \) and \( Z_2 \cong J \cap K \) as ordered rows and the sum running over the set of all quadruples \((J', H', K_1', K_2')\) of rows with increasing entries such that

1. \(|H'| \leq |H|\),
2. \(H' \cup J' = H \cup J\),
3. \(K_1' \cap K_2' = K\),
4. \(Z_1 \subset H'\),
5. \(Z_2 \subset J' \cap K_2'\).

Moreover these relations generate the ideal of relations among pfaffians.

In the formula above we have denoted, for \(0 \leq k \leq m\), by \([m]_k\) the gaussian binomials defined as the element \(\frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}\) of \(\mathbb{C}[q]\) and \([m]_{k-1}\) is this polynomial evaluated in \(-1\).

Our interest for this topic stems from the standard monomial theory. Consider \(V = \mathbb{C}^{2n+2}\) equipped with a non degenerate symmetric bilinear form such that the subspaces \(V_1\) and \(V_2\), generated respectively by the first and the last \(n+1\) vectors of the canonical basis of \(V\), are totally isotropic. The variety of the \((n+1)\)-dimensional totally isotropic subspaces of \(V\) has two connected components; let the positive lagrangian grassmannian \(\mathcal{G}^r\) be the component of those subspaces whose intersection with \(V_2\) is even dimensional. The special orthogonal group of \(V\) acts on this variety, let \(G = \text{Spin}(2n+2)\) be its simply connected cover and \(\mathbb{S}\) the Spin module of \(G\).

Then \(\mathcal{G}^r\) embeds into \(\mathbb{P}(\mathbb{S})\) and the study of relations among pfaffians is equivalent to the study of the equations defining the cone over this embedding of \(\mathcal{G}^r\) (see section 3). Notice that on the grassmannian side we have a natural action of a bigger group of symmetries which is not apparent on the pfaffian picture of the problem. Moreover in the study of the coordinate ring of this embedding we can make use of the standard monomial theory.

The standard monomial theory is a very general theory which construct a basis of the projective coordinate ring given by the immersion of a generalized grassmannian or a flag variety. The prototypical example of the theory is given by the work of Hodge on the Plücker immersion of a grassmannian \(2\). The idea of Hodge was generalized to a projective immersion of partial flag varieties for a semisimple group \(G\) (and even to a Kac Moody one) between the seventies and the nineties by the work of many peoples, Seshadri, Lakshmibai, Musili, De Concini, Eisenbud, Procesi and Littlemann. The results of standard monomial theory have many consequences both in representation theory and in the study of singularities of Schubert varieties and other related varieties. However the theory does not give an explicit description of the equations beyond the original case of the Plücker immersion of a grassmannian and some other very simple cases.

In our case the standard monomial theory takes the following form. Let \(A\) be the coordinate ring of the embedding \(\mathcal{G}^r \hookrightarrow \mathbb{P}(\mathbb{S})\) and let \(A_m\) be the subspace of homogeneous polynomials of degree \(m\). There exists a basis \(x(I)\) of \(A_1\) indexed by increasing rows \(I\) as above. A tableau \(T = (I_1, I_2, \cdots, I_m)\) is simply a sequence of rows; it is standard if all rows have increasing entries and \(I_1 \leq I_2 \leq \cdots \leq I_m\). The set of standard monomials \(x(T) = x(I_1)x(I_2)\cdots x(I_m)\) for \(T\) standard are a basis for \(A_m\). So each section \(x(T)\) with \(T\) non standard may be written as a linear combination of standard ones, these relations are called straightening relations for \(A\). Since the ideal of relations in the generators \(x(I)\) is generated by quadratic elements it is enough to consider only non standard tableaux with two rows. An important point is that in the straightening relation for the tableau \(T = (I, J)\) only standard tableaux \((H, K)\) with \(H \leq I, J \geq K \geq I\) do appear.
This is the core idea of the standard monomial theory: to replace the knowledge of the (very) complicated explicit straightening relations by that of the order condition stated above. Indeed this condition is sufficient to deduce quite a lot about the geometry of the flag variety $Gr$.

The equations we write down for the generators $x(I)$ are given by formula (1) replacing $pf_I$ with $x(I)$. These are not straightening relations but they are what we call *shuffling relations*: given a non standard tableau $T$ with two rows we say that the element $f = \sum a_h x(I_h) x(J_h)$, with the $a_h \in \mathbb{C}$, is a shuffling relation for $T$ if $f = 0$, $T$ appears with coefficient 1 and all other tableaux do fulfill the order condition of a straightening relation for $T$. In particular we are not asking that all tableaux but $T$ must be standard. It is clear that by a finite number of steps we may deduce the straightening relations from the shuffling relations; so these weaker relations are still a set of generators for the ideal of relations. (The name ‘shuffling’ is the same name used for the analogous relations for the determinants of minors of a matrix of indeterminates.) Notice that also the classical Plücker equations are shuffling relations and not straightening relations.

From the point of view of standard monomial theory, the easiest cases, which were also the first ones to be analyzed [5], are those in which the embedding is in the projective space of a minuscule $G$–module. Recall that an irreducible $G$–module is said to be minuscule if its weights are a single orbit under the Weyl group.

The condition for a module (for a general semisimple group) to be minuscule is very strong and, so, very few modules are minuscule. Moreover for all minuscule modules for classical groups but the spin module explicit shuffling relations are known. Indeed all such modules have an extremely simple order structure except the fundamental representations of type $A$ which corresponds to the Plücker immersion of a grassmannian and the Spin modules. In particular our module $S$ is a minuscule one, and the other are either twisted form of $S$ or restriction of $S$ to Spin($2n+1$), so the same result for these other cases may be easily deduced from our result.

For the exceptional groups only two modules for $E_6$ (one dual to the other) and one module for $E_7$ are minuscule. (For $E_7$ one may see [1] for some hints to explicit formulas for straightening relations.)

After completing this paper we came to known that formulas very similar to ours have already been found by Kustin in [4]. However the proof in that paper is very different from our: while being much elementary, since it uses only multilinear algebra, it does not exploit the role played by the representation theory of the spin module. So we still think our proof of the shuffling relations may have some interest; at least we hope this paper may bring some attention to the paper of Kustin which, in our opinion, is not very known.

The paper is organized as follows. In the first section we collect some combinatorial definitions about rows, tableaux and standardness and the definition and some properties of the gaussian binomials. The second section introduce the spin group, the spin module and the related grassmannian. Then the standard monomial theory for this module is shortly discussed with a remark about a general invariance of the relations defining a flag variety. In the third section we see the definition of a pfaffian and the relations of such polynomial functions with the spin module. Using this links we are able to prove some invariance properties of the ideal of relations. In Section 4 we introduce our formulas and prove some combinatorial properties of such formulas. In Section 5 we prove that the formulas are indeed shuffling relations. Finally in the last section we extend these results to an arbitrary field.
1. Combinatorics

1.1. Rows and tableaux. We call a finite sequence (maybe empty) of integers a \textit{row}. Let $\mathcal{R}_n \supset \mathcal{R}_n^+ \supset \mathcal{S}\mathcal{R}_n$ be respectively the set of all rows in the alphabet $\{1, 2, \ldots, n\}$, the subset of rows containing distinct elements and the subset of \textit{standard} rows, i.e. of rows $I = i_1i_2\cdots i_r$ with $i_1 < i_2 < \cdots < i_r$. We define the length of the row $I = i_1i_2\cdots i_r$ as $|I| = r$, moreover the (standard) empty row is denoted by $\emptyset$. If $I \in \mathcal{R}_n^+$ we define $\bar{I}$ as the \textit{complementary} row $j_1 \ldots j_{n-r}$ of $I$ where $\{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_{n-r}\} = \{1, 2, \ldots, n\}$ and $j_1 < j_2 < \cdots < j_{n-r}$. If $I = i_1 \ldots i_r$ and $J = j_1 \ldots j_s$ are two rows we set $IJ = i_1 \ldots i_r j_1 \ldots j_s$.

Given an integer $h$ we define $I \ast h$ as the row obtained by adding $h$ to the end of the row $I$, futhur if $k$ is another integer then $I(h \rightarrow k)$ is the row obtained by replacing all occurrences (if any) of $h$ by $k$ in $I$. Hereafter we write $h \in I$ to say that $h$ appears as an element in the row $I$ and, in general, we use the language of sets with rows when this does not create any ambiguity.

The symmetric group $S_n$ acts on the set of rows: for $I = i_1i_2\cdots i_r$ and $\sigma \in S_n$ we set $\sigma \cdot I = \sigma(i_1)\sigma(i_2)\cdots \sigma(i_r)$. Given a row $I = i_1i_2\cdots i_r$ in $\mathcal{R}_n^+$ let $I^\xi$ be the row with the same entries of $I$ rearranged in ascending order, moreover let $\tau_I \in S_n$ be such that $\tau_I \cdot I = I^\xi$ and $\tau_I$ fixes each integer $h \notin I$, let also $\varepsilon(I)$ be the sign of $\tau_I$.

We define the partial order for the standard rows: if $I = i_1 \ldots i_r$, $J = j_1 \ldots j_s$ are standard rows we set $I \leq J$ if $|I| \geq |J|$ and $i_m \leq j_m$ for all $1 \leq m \leq |J|$. A \textit{tableau} is a sequence of rows $T = (I_1, I_2, \ldots, I_d)$ with $|I_1| \geq |I_2| \geq \cdots \geq |I_d|$; it is standard if all rows are standard and $I_1 \leq I_2 \leq \cdots \leq I_d$, the \textit{degree} of $T$ is the number $d$ of rows. If $I_m = i_{m,1}i_{m,2}\cdots i_{m,r_m}$ for $m = 1, \ldots, d$ then, as customary, we write $T$ arranged vertically and left justified

$$
\left( \begin{array}{cccc}
  i_{1,1} & i_{1,2} & \cdots & i_{1,r_1} \\
  i_{2,1} & i_{2,2} & \cdots & i_{2,r_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  i_{d,1} & i_{d,2} & \cdots & i_{d,r_d}
\end{array} \right)
$$

so that $T$ is standard if and only if its entries increase along the rows and do not decrease along the columns.

Finally let $\mathcal{R}_n^0$ be the subset of rows of even length of $\mathcal{R}_n$ and define analogously $\mathcal{R}_n^{+0}$ and $\mathcal{S}\mathcal{R}_n^0$. Given a row $I \in \mathcal{R}_n$ let $I^0 \in \mathcal{R}_{n+1}^0$ be either the same row if $|I|$ is even or the row $I \ast (n+1)$ otherwise; then the map $I \mapsto I^0$ is clearly a bijection between $\mathcal{S}\mathcal{R}_n$ and $\mathcal{S}\mathcal{R}_n^0$.

1.2. Gaussian binomials. Let $q$ be an indeterminate and define the \textit{gaussian binomials} as the following elements of $\mathbb{C}[q]$; for all $m, k \in \mathbb{Z}$

$$
\left[ \begin{array}{c}
m \\
k \end{array} \right]_q = \begin{cases} 
(1-q^m)(1-q^{m-1})\cdots(1-q^{m-k+1}) \\
(1-q^k)(1-q^{k-1})\cdots(1-q) 
\end{cases} \begin{cases} 
\text{if } 0 \leq k \leq m \\
\text{otherwise.}
\end{cases}
$$

The evaluation of $q$ to $-1$ is of particular interest for our aims; the following result is easily proved by induction.
Lemma 1. For all $0 \leq k \leq m$ we have

$$\left[ \begin{array}{c} m \\ k \end{array} \right]_{-1} = \left\{ \begin{array}{ll} \binom{m}{k} & \text{if } m \text{ is odd or } k \text{ is even} \\ 0 & \text{otherwise.} \end{array} \right.$$

In the next Lemma we see a $q$–analogous of a well known binomial formula; it is needed in the proof of our main result.

Lemma 2. For all $1 \leq s \leq m$ we have

$$\sum_{h=0}^{s} (-1)^h q^{h(k-1)/2} \left[ \begin{array}{c} m + 1 \\ s - h \end{array} \right] q^{[m+1-s+h]} = 0.$$

Proof. Let $\psi_{m,s}$ be the left hand side of the above identity. We use induction on $m$. By the definition of the gaussian binomials we see that for all $n \geq 0$ and $0 \leq k \leq n + 1$

$$\left[ \begin{array}{c} n + 1 \\ k \end{array} \right] = \frac{1 - q^{n+1}}{1 - q^{n+1-k}} \left[ \begin{array}{c} n \\ k \end{array} \right].$$

Hence for all $1 \leq s \leq m$ we have

$$\psi_{m+1,s} = \sum_{h=0}^{s} (-1)^h q^{h(k-1)/2} \left[ \begin{array}{c} m+1 \\ s - h \end{array} \right] q^{[m+1-s+h]} = \sum_{h=0}^{s} (-1)^h q^{h(k-1)/2} \frac{1 - q^{m+1}}{1 - q^{m+1+s+h}} \left[ \begin{array}{c} m \\ s - h \end{array} \right] q^{[m-s+h]} = \frac{1 - q^{m+1}}{1 - q^{m+1}} \psi_{m,s} = 0$$

by induction on $m$.

So we need to show that $\psi_{m,m} = 0$ for all $m \geq 1$ to complete the proof. But

$$\psi_{m,m} = \sum_{h=0}^{m} (-1)^h q^{h(k-1)/2} \left[ \begin{array}{c} m \\ h \end{array} \right]$$

and our claim follows by evaluating in $t = -1$ the Newton binomial formula

$$\prod_{k=0}^{m-1} (1 + q^k t) = \sum_{h=0}^{m} q^{h(k-1)/2} \left[ \begin{array}{c} m \\ h \end{array} \right] t^h.$$

\[ \square \]

Remark 3. There is a certain link between gaussian binomials and Coxeter groups of type A. Recall that if $W = \langle s_1, s_2, \ldots, s_r \rangle$ is a Coxeter group and $I \subset \{s_1, s_2, \ldots, s_r\}$, then one may define the set $W^I \hookrightarrow W$ of representatives of minimal length of the quotient $W/W_I$, where $W_I$ is the subgroup generated by $I$ in $W$. Moreover for any subset $S$ of $W$ the Poincaré polynomial of $S$ is defined as $P_S(q) = \sum_{\tau \in S} q^{d(\tau)}$, where $d : W \rightarrow \mathbb{N}$ is the length function.

We have $P_{W^I}(q) = P_W(q)/P_{W_I}(q)$, the quotient of the Poincaré polynomials of the two Coxeter groups $W$ and $W_I$. Moreover $P_W(q) = \prod_{i=1}^{r} \frac{1 - q^{d_i}}{1 - q}$ where $d_1, d_2, \ldots$ are the degrees of $W$ (see [3]).

Now let $W$ be the symmetric group $S_m$ on $m$ elements, which is a Coxeter group of type $A_{m-1}$ with respect to the set of generators $s_i = (i, i+1)$ for $i = 1, 2, \ldots, m-1$. Since the degrees of a Coxeter group of type $A_{m-1}$ are $2, 3, \ldots, m$, it follows at once that the gaussian binomial $\left[ \begin{array}{c} m \\ k \end{array} \right]_q$ is the Poincaré polynomial of the set of minimal representatives of the quotient $S_m/S_k \times S_{m-k}$.  

5
2. The spin module

We fix once and for all the notation $\varepsilon_1, \ldots, \varepsilon_n$ for the standard basis of $\mathbb{C}^n$ and we denote by $E_{ij}$ the matrix associated to the linear map sending $\varepsilon_j$ to $\varepsilon_i$ and all other elements of this basis to zero.

2.1. The spin group. On $V \cong \mathbb{C}^{2n+2}$ fix the symmetric bilinear form whose associated matrix is $(\frac{1}{2} I_n)$. Let $\mathfrak{g} = \mathfrak{so}(2n+2)$ and $G = \text{Spin}(2n+2)$ the associated simply connected group. In this basis $\mathfrak{g}$ is the set of all matrices $(A \ B \ C \ D)$ with $A, B, C \ (n+1) \times (n+1)$ matrices and $B, C$ antisymmetric.

We fix the Chevalley generators, a Cartan subalgebra $\mathfrak{t}$ and a Borel subalgebra $\mathfrak{b}$ in the following way:

$$e_i \doteq \begin{pmatrix} E_{i,i+1} & 0 \\ 0 & -E_{i+1,i} \end{pmatrix} \quad \text{for } i = 1, \ldots, n, \quad e_{n+1} \doteq \begin{pmatrix} 0 & E_{n,n+1} - E_{n+1,n} \\ 0 & 0 \end{pmatrix},$$

$$f_i \doteq e_i^t \quad \text{for } i = 1, \ldots, n + 1,$$

$$h_i \doteq \begin{pmatrix} E_{i,i} - E_{i+1,i+1} & 0 \\ 0 & -E_{i,i} + E_{i+1,i+1} \end{pmatrix} \quad \text{for } i = 1, \ldots, n,$$

$$h_{n+1} \doteq \begin{pmatrix} E_{n,n} + E_{n+1,n+1} & 0 \\ 0 & -E_{n,n} - E_{n+1,n+1} \end{pmatrix},$$

$$t \doteq \{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \in \mathfrak{g} \mid A \text{ is diagonal} \},$$

$$b \doteq \{ \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} \in \mathfrak{g} \mid A \text{ is upper triangular and } B \text{ is antisymmetric} \}.$$

We denote with $T$ (resp. $B$) the maximal torus (resp. Borel subgroup) of $G$ whose Lie algebra is given by $\mathfrak{t}$ (resp. $\mathfrak{b}$) and let $B^-$ be the Borel opposite to $B$.

We identify the Cartan subalgebra with $\mathbb{C}^{n+1}$ mapping $\varepsilon_i$ to $(\frac{e_{i,i}}{0} - \frac{E_{i,i}}{})$ for $i = 1, \ldots, n+1$ and, further, we identify the Cartan subalgebra with its dual using the standard form $\varepsilon_i \varepsilon_j = \delta_{ij}$ for $1 \leq i, j \leq n+1$. In particular if we set $\alpha_i = h_i$, for $i = 1, \ldots, n+1$, the set $\{ \alpha_1, \ldots, \alpha_{n+1} \}$ is a simple basis for the roots of $\mathfrak{g}$. Let $\Lambda \doteq \text{Hom}(T, \mathbb{C}^*) \subset \mathfrak{t}^*$ be the set of integral weights and $\Lambda^+$ be the subset of $\Lambda$ of dominant weights and let $\omega_1, \ldots, \omega_{n+1}$ be the fundamental weights. For $\lambda \in \Lambda^+$ let $V_\lambda$ be the irreducible representation of $G$ of highest weight $\lambda$.

2.2. The spin module. We want now to give an explicit description of the dual of the irreducible module of highest weight $\omega_{n+1}$. This is called the positive spin module and will be denoted with $S$.

Define the vector space $S \doteq \bigoplus I \mathbb{C} \cdot x(I)$ with basis elements $x(I)$, $I \in \mathcal{S}R_n$. Next we define the weight of a row $I \in \mathcal{R}^+_n$ as

$$\text{wt}(I) \doteq \frac{1}{2} (\sum_{i \in I^0} \varepsilon_i - \sum_{i \notin I^0} \varepsilon_i) \in \Lambda.$$

Now $V$ has a $\mathfrak{g}$–module structure defined as follows:

$$e_i(x(I)) = \begin{cases} x(I(i+1 \rightarrow i)) & \text{if } i + 1 \in I, \ i \notin I, \\ 0 & \text{otherwise} \end{cases}$$
\[ e_n(x(I)) = \begin{cases} x(I \ast n) & \text{if } n \not\in I \text{ and } |I| \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \]
\[ e_{n+1}(x(I)) = \begin{cases} x(I \ast n) & \text{if } n \not\in I \text{ and } |I| \text{ is even} \\ 0 & \text{otherwise} \end{cases}. \]
\[ h_i(x(I)) = \text{wt}(I)x(I) \text{ for } i = 1, 2, \ldots, n+1. \]
\[ f_i(x(I)) = \begin{cases} x(I(i \rightarrow i + 1)) & \text{if } i \in I, i + 1 \not\in I, \\ 0 & \text{otherwise} \end{cases} \]
\[ f_n(x(I)) = \begin{cases} x(I \setminus \{n\}) & \text{if } n \in I \text{ and } |I| \text{ is even} \\ 0 & \text{otherwise} \end{cases}. \]
\[ f_{n+1}(x(I)) = \begin{cases} x(I \setminus \{n\}) & \text{if } n \in I \text{ and } |I| \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \]

Here is an example of this action for \( n = 3 \): for each basis vector of \( S \) we have drawn all operators \( e_1, e_2, e_3 \) which does not send that vector to 0.

\[
\begin{align*}
x(123) & \xleftarrow{e_4} x(12) \\
& \xleftarrow{e_2} x(13) \\
& \xleftarrow{e_3} x(23) \xrightarrow{e_1} x(1) \\
& \xrightarrow{e_2} x(2) \\
& \xrightarrow{e_4} x(3) \xrightarrow{e_1} x(\emptyset)
\end{align*}
\]

Notice that \( S \) is irreducible, and its lowest weight is \( -\omega_{n+1} \) so it is the dual of \( V_{-\omega_{n+1}} \). Notice also that all the weights are in the orbit of \( -\omega_{n+1} \), in particular the spin module is miniscule.

2.3. The lagrangian grassmannian. Let \( V_1 \) and \( V_2 \) be the span, respectively, of the first, and the last, \( n+1 \) vector of the canonical basis of \( V \). Define the positive lagrangian grassmannian \( Gr \) as the variety of \( (n + 1) \)–dimensional subspaces of \( V \) which have even dimensional intersection with \( V_2 \). This is an homogeneous space for the special orthogonal group of \( V \) and for \( G \).

The Picard group of \( Gr \) has a unique ample generator that we denote by \( L \) which is \( G \)–linearizable and the \( G \)–module \( H^0(Gr, L) = \{ \eta : G \rightarrow \mathbb{C} \mid \eta \text{ is holomorphic and } \eta(gp^{-1}) = \omega_{n+1}(p)\eta(g) \text{ for all } g \in G \text{ and } p \in B \} \) is isomorphic to \( S \).

The main object of study of this paper is the graded ring \( A = \bigoplus_m H^0(Gr, L^m) \). For \( m \geq 0 \) let \( A_m = H^0(Gr, L^m) \) be the space of its homogeneous components of degree \( m \).
In particular $A_1 = V_{\omega_n}^\ast$. The ring $A$ is generated in degree one with quadratic relations. We denote by $K \subset S^2(A_1) \rightarrow A_2$ the kernel of the multiplication map; our aim here is to find explicitly generators for this kernel. When we need to stress the rank $n+1$ of the spin group we add a subscript $n$, for example $K_n$.

2.4. Standard monomial theory. On the module $S$ we have defined the basis $x(I)$, $I \in S\mathcal{R}_n$, now we want to extend the symbol $x(I)$ to any row $I \in \mathcal{R}_n$: let $x(I) = (-1)^{r(I)} x(I_\prec)$ if $I \in \mathcal{R}^+_n$ and $x(I) = 0$ for all $I \notin \mathcal{R}^+_n$.

Next we extend $x : \mathcal{R}_n \rightarrow A_1$ to tableaux as

$$x \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_r \end{pmatrix} = x(I_1)x(I_2) \cdots x(I_r) \in S^rA_1$$

We will call such a monomial standard if the tableau is standard. Notice that if this monomial is non zero (i.e. if and only if $I_h \in \mathcal{R}^+_n$ for all $h = 1, \ldots, r$) then we may always consider all of its rows as standard up to a sign change.

Let $y(T)$ be the image of $x(T)$ in $A$, and recall that, by a well known result of Standard Monomial Theory, the set of monomials $y(T)$ with $T$ a standard tableau of degree $r$ is a $\mathbb{C}$-basis of $A_r$. Moreover for each non standard tableau $\begin{pmatrix} I \\ J \end{pmatrix}$ we have a straightening relation

$$y \begin{pmatrix} I \\ J \end{pmatrix} = \sum a_{H,K} \left( \begin{pmatrix} H \\ K \end{pmatrix} \right), \quad a_{H,K} \in \mathbb{C}$$

where the sum runs over all standard tableaux $\begin{pmatrix} H \\ K \end{pmatrix}$ with $H \leq I$, $J$ and $K \geq I$, $J$ (see Corollary 1 of [5]). In the case of a minuscule module these properties are not difficult to prove and were the starting point of the standard monomial theory.

Notice that for all $\begin{pmatrix} H \\ K \end{pmatrix}$ we have $I \cup J = H \cup K$ with multiplicities, by $t$-homogeneity. We may write the relation above also as $x(\begin{pmatrix} I \\ J \end{pmatrix}) = \sum a_{H,K} x(\begin{pmatrix} H \\ K \end{pmatrix}) \in K$; in particular these elements (that we still call straithening relations) generate the space $K$ of quadratic relations, hence they generate the ideal defining the ring $A$.

Later we will see direct formulas for what we call shuffling relation: given a non standard tableau $T = \begin{pmatrix} I \\ J \end{pmatrix}$ as above, a shuffling relation for $T$ is any element

$$\sum m a_m x \left( \begin{pmatrix} H_m \\ K_m \end{pmatrix} \right), \quad a_m \in \mathbb{C}$$

do $K$ such that $x(\begin{pmatrix} I \\ J \end{pmatrix})$ appears with coefficient 1 and, for all $m$, $H_m \leq I$, $J$ and $K_m \geq I$, $J$. Notice that we are not requiring $\begin{pmatrix} H_m \\ K_m \end{pmatrix}$ to be standard. Clearly the straightening relation for $\begin{pmatrix} I \\ J \end{pmatrix}$ can be obtained in a finite number of steps using the shuffling relations, so also the shuffling relations generates $K$.

2.5. Complementary invariance of relations. In this section we want to prove a symmetry property of the ideal defining a (general) flag variety, so here we allow $g$ to be any semisimple Lie algebra.

Fix a triangular decomposition $g = b^- \oplus h \oplus b$, a corresponding set of simple roots $\alpha_1, \ldots, \alpha_\ell$ with fundamental weights $\omega_1, \ldots, \omega_\ell$ and corresponding Chevalley generators $e_1, \ldots, e_\ell, h_1, \ldots, h_\ell$ and $f_1, \ldots, f_\ell$. For any dominant weight $\lambda$, let $v_\lambda$ be a fixed highest weight vector of the irreducible $g$-module $V_\lambda$ and let $v^-_\lambda$ be the lowest weight vector in the Weyl group orbit of $v_\lambda$. Denote by $K_\lambda$ the kernel of the $g$-module projection $S^2V_\lambda \rightarrow V_{2\lambda}$; this kernel is the direct sum of all isotypic components of $S^2V_\lambda$ of weight less than $2\lambda$. 

Let $w_0$ be the longest element of the Weyl group of $\mathfrak{g}$ and denote by $d$ the linear map of $\Lambda$ given by $-w_0$; hence, in particular, $V_\lambda^* \cong V_{d(\lambda)}$ for all dominant weights $\lambda$. We still denote by $d$ the permutation of $\{1, \ldots, \ell\}$ defined in the following way: $1 \leq h \leq \ell$ is mapped to $k$ if $d(\omega_h) = \omega_k$. Further let $a : \mathfrak{g} \to \mathfrak{g}$ be the unique automorphism defined by $e_i \mapsto f_{d(i)}, h_i \mapsto -h_{d(i)}$ and $f_i \mapsto e_{d(i)}$ for $i = 1, \ldots, \ell$ and, finally, extend it to the universal enveloping algebra $U(\mathfrak{g})$.

It is clear that any element of $\mathcal{S}^2 V_\lambda$ may be written as $\sum_{h=1}^r \varphi_h v_{\lambda} \cdot \psi_h v_{\lambda}$ where $\varphi_h$, $\psi_h$ are in the universal enveloping algebra $U(\mathfrak{g})$.

Lemma 4. If the element $\sum_{h=1}^r \varphi_h v_{\lambda} \cdot \psi_h v_{\lambda}$ of $\mathcal{S}^2 V_\lambda$ is in $\mathcal{K}_\lambda$ then also $\sum_{h=1}^r a(\varphi_h) v_{\lambda}^\perp \cdot a(\psi_h) v_{\lambda}^\perp$ is an element of $\mathcal{K}_\lambda$.

Proof. Given a $\mathfrak{g}$–module $V$, let $V^a$ be the $\mathfrak{g}$–module on the vector space $V$ with action defined by $x \cdot v = a(x)v$ for all $x \in \mathfrak{g}$ and $v \in V$. It is clear that $(\mathcal{S}^2 V)^a = \mathcal{S}^2 V^a$ and $(U \oplus V)^a = U^a \oplus V^a$ for all $\mathfrak{g}$–modules $U$ and $V$.

Notice that $v_{\lambda}^\perp$ is a highest weight vector of $V_\lambda^a$, and its weight is $a(\omega_0 \lambda) = -d\omega_0(\lambda) = \lambda$; hence there exists a $\mathfrak{g}$–module isomorphism $\alpha_\lambda : V_\lambda \to V_\lambda^a$ and we may normalize it by $\alpha_\lambda(v_{\lambda}) = v_{\lambda}^\perp$.

Now consider the map $\mathcal{S}^2 \alpha_\lambda$. Since $V_\lambda \subset \mathcal{S}^2 V_\lambda$ is sent to itself by $\mathcal{S}^2 \alpha_\lambda$, we see that $\mathcal{S}^2 \alpha_\lambda(\mathcal{K}_\lambda) \subset \mathcal{K}_\lambda$. So

$$\mathcal{S}^2 \alpha_\lambda(\sum_{h=1}^r \varphi_h v_{\lambda} \cdot \psi_h v_{\lambda}) = \sum_{h=1}^r \alpha_\lambda(\varphi_h v_{\lambda}) \cdot \alpha_\lambda(\psi_h v_{\lambda}) = \sum_{h=1}^r a(\varphi_h) v_{\lambda}^\perp \cdot a(\psi_h) v_{\lambda}^\perp$$

is an element of $\mathcal{K}_\lambda$ as claimed. $\square$

Specializing to our context, and using the notation of previous sections, we see that if $\varphi v_{\lambda}$, with $\varphi \in U(\mathfrak{b}^-)$, is the vector $x(I)$, with $I \in \mathcal{S} \mathcal{R}_n$, then $a(\varphi) v_{\lambda}^\perp$ is the vector $x(\tilde{I})$ where $\tilde{I}$ is the complementary row of the row $I$. Hence we have proved the following corollary.

Corollary 5. If the element $\sum_h a_h x(\tilde{I}_h)$, with $a_h \in \mathbb{C}$ and $I_h, J_h \in \mathcal{S} \mathcal{R}_n$, is in the kernel $\mathcal{K}$ of the multiplication map $\mathcal{S}^2 \mathcal{A}_1 \to \mathcal{A}_2$ then also $\sum_h a_h x(\tilde{I}_h)$ is an element of $\mathcal{K}$.

3. Pfaffians

3.1. Definition. Let $\mathcal{X}_n$ be the set of $n \times n$ antisymmetric matrices with complex coefficients. We identify $\mathcal{X}_n$ with $\Lambda^2 \mathbb{C}^n$ mapping $X = (x_{ij}) \in \mathcal{X}_n$ to $\omega_X = \frac{1}{2} \sum_{i,j} x_{ij} \varepsilon_i \wedge \varepsilon_j$. We recall that the pfaffian of $X \in \mathcal{X}_{2m}$, denoted by $\text{Pf}(X)$, is defined by

$$\frac{1}{m!} \omega_X^{2m} = \text{Pf}(X) \varepsilon_1 \wedge \cdots \wedge \varepsilon_{2m}.$$ 

Notice that $\text{Pf}(X)^2 = \det X$ and, in particular, the pfaffian of $X$ vanishes when the matrix $X$ is singular. Moreover if we let $S_{2m}$ act on $\mathcal{X}_{2m}$ by permuting the rows and the columns of a matrix as $\sigma \cdot (x_{i,j}) = (x_{\sigma(i), \sigma(j)})$, then $\text{Pf}(\sigma \cdot X) = (-1)^\sigma \text{Pf}(X)$.

Now we see an example; it will be used in the proof of our theorem.
For each $I$ with $\{i_1, \ldots, i_h\} \subseteq \{1, \ldots, n\}$ denote with $I[n]$ the corresponding $n \times n$ submatrix of \( B \). Then $\phi : X \mapsto B$ is an isomorphism between $X$ and $B$. More precisely it is an orbit under the unipotent radical of the stabilizer $\mathcal{R}_n$. Define $\psi : A \mapsto X$ by $\psi(A) = \phi^{-1}(A)$. Notice also that the isomorphism $\psi$ is an isomorphism of the spin module $\mathfrak{g}_\varphi$.

### Example 6

The pfaffian of the following antisymmetric matrix

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
-1 & 0 & 1 & \ldots & 0 \\
0 & -1 & 0 & 1 & \ldots & 0 \\
\vdots \\
0 & \ldots & -1 & 0 \\
\end{pmatrix}
\]

is 1; clearly the same is true for all even dimensional principal submatrices of $M$.

Let $B$ be the ring of polynomial functions on $X_{n+1}$. For each $X \in X_{n+1}$ and $I = \{i_1, \ldots, i_h\} \subseteq \{1, \ldots, n\}$ define $X_I$ to be the antisymmetric matrix given by the principal minor corresponding to the rows and the columns $i_1, \ldots, i_h$ of $X$. In this definition we allow repetitions and we consider also the order of the elements in $I$; for example if

\[
X = \begin{pmatrix}
\begin{smallmatrix}
a & b \\
-b & c \\
\end{smallmatrix}
\end{pmatrix}
\text{ then } X_{13} = \begin{pmatrix}
0 & b \\
-b & 0
\end{pmatrix} \text{ while } X_{31} = \begin{pmatrix}
0 & b \\
-b & 0
\end{pmatrix}.
\]

For each $I \in \mathcal{R}_n$ we define a (polynomial) function $\text{pf}_I$ on $X_{n+1}$ by

\[
\text{pf}_I(X) = \text{Pf}(X_I).
\]

The ordering of the elements of $I$ is not essential here since $\text{pf}_I(X) = (-1)^r \text{pf}_{I \subseteq I}(X)$ for all $I \in \mathcal{R}_n^+$; however it will be convenient for us to have this more general notation. Notice that $\text{pf}_I = 0$ for all sequences $I$ with a repetition since the pfaffian of a singular matrix is zero. It is also clear that the functions $\text{pf}_I$ with $I \in \mathcal{R}_n$ generate the ring $B$; indeed $\text{pf}_{ij}(X) = x_{ij}$ for all $1 \leq i < j \leq n$.

### 3.2. Pfaffians and the spin module

We now describe the relation between the pfaffians and the spin module. Notice that $\mathcal{G}_r$ has a unique $B^-$-stable divisor that we will denote with $S$. This is the variety of all subspaces $W \subseteq \mathcal{G}_r$ with non trivial intersection with $V_2$. As a subvariety of $\mathcal{G}_r$ it is defined by the equation $x(\varnothing) = 0$.

Let now $p \in \mathcal{G}_r$ be the point corresponding to $V_1$. The $B^-$-orbit $\mathcal{U}$ of $p$ is the complement of $S$. More precisely it is an orbit under the unipotent radical of the stabilizer $P$ of $p$. Define $u : X_{n+1} \rightarrow G$ by $u(X) = (x_{ij})$ and $\varphi : \mathfrak{X}_{n+1} \rightarrow \mathcal{G}_r$ by $\varphi(X) = u(X) \cdot p$. Then $\varphi$ is an isomorphism between $X_{n+1}$ and $\mathcal{U}$.

The pull back $\varphi^* \mathcal{L}$ of $\mathcal{L}$ on $X_{n+1}$ is isomorphic to the trivial line bundle so it induces a ring homomorphism $\psi : A \rightarrow B$. Notice that $\psi(x(\varnothing))$ is a nowhere vanishing function so it is a non zero constant that we can normalize to be 1. Hence $\psi$ induces an isomorphism

\[
\psi : A \rightarrow B;
\]

\[
(x(\varnothing) = 1) \cong B.
\]

In particular, since $\mathcal{U}$ is open in $\mathcal{G}_r$, the restriction $\psi_m = \psi|_{\mathcal{A}_m}$ to the homogeneous component is injective and we define $B_m = \psi(\mathcal{A}_m)$. Since $\psi(x(\varnothing)) = 1$ we have $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$. Notice also that the isomorphism $\psi$ does not define a $G$-structure on $B$ however it defines a $G$-structure on $B_m$ such that the multiplication maps $B_m \times B_{m'}$ in $B_{m+m'}$. In particular we have the following commutative diagram

\[
\begin{array}{ccc}
A_1 \otimes A_1 & \rightarrow & A_2 \\
\downarrow \psi_1 \otimes \psi_1 & & \downarrow \psi_2 \\
B_1 \otimes B_1 & \rightarrow & B_2
\end{array}
\]

where vertical maps are isomorphisms and horizontal ones are induced by multiplication. Hence the homogenous relations between elements of $A_1$ and $B_1$ are the same.
We want to identify in a more explicit way $B_1$; in order to do this we need to make explicit the trivialization of the line bundle $\mathcal{L}$. If $\sigma : G \to \mathbb{C}$ is a meromorphic function such that $\sigma(gp^{-1}) = \omega_n^{-1}(p)\sigma(g)$ for all $p \in P$ then it defines naturally a meromorphic section of $\mathcal{L}$. We can associate to $\sigma$ also a function on $\mathcal{X}_{n+1}$ by $f_\sigma(X) = \sigma(u(X))$.

On the other hand every meromorphic function on $\mathcal{X}_{n+1}$ can be constructed in this way. So we obtain an action of the Lie algebra $\mathfrak{g}$ on the space of meromorphic functions on $\mathcal{X}_{n+1}$. Explicitly this action is given as follows:

\[
\begin{align*}
&\left( \begin{array}{cc}
A & 0 \\
0 & -^t A
\end{array} \right) \cdot f(X) = \left. \frac{d}{ds}f(X + s(A^t X + X A)) \right|_{s=0} - \frac{1}{2} \text{Tr}(A) f(X) \\
&\left( \begin{array}{cc}
0 & B \\
0 & 0
\end{array} \right) \cdot f(X) = \left. \frac{d}{ds}f(X + sBX) \right|_{s=0} - \frac{1}{2} \text{Tr}(BX) f(X) \\
&\left( \begin{array}{cc}
0 & 0 \\
C & 0
\end{array} \right) \cdot f(X) = \left. \frac{d}{ds}f(X - sC) \right|_{s=0}
\end{align*}
\]

for all $A \in \text{End}(\mathbb{C}^{n+1})$ and $B,C \in \mathcal{X}_{n+1}$.

In particular with simple computations we get that if $I \in \mathcal{SR}_n$:

\[
e_i(\text{pf}_I) = \begin{cases} 
\text{pf}_{e_i(I)} & \text{if } e_i(I) \neq 0, \\
0 & \text{otherwise};
\end{cases}
\]

\[
h_i(\text{pf}_I) = \langle h_i, \text{wt}(I) \rangle \text{pf}_I.
\]

Since we have normalized $\psi$ in such a way that $\psi(x(\emptyset)) = 1 = \text{pf}_\emptyset$ we have

\[
\psi(\text{pf}_I) = x(I)
\]

for all $I \in \mathcal{SR}_n$. So, by our conventions, $\psi(\text{pf}_I) = x(I)$ for all rows $I$ and not only for standard rows. Notice that, in particular, $B_1$ is the vector space spanned by the functions $\text{pf}_I$ with $I \in \mathcal{SR}_n$.

So we have proved the following result.

**Proposition 7.** The element $\sum_h a_h x^{(I_h)}_h$ of $S^2 A_1$ is in $\mathcal{K}$ if and only if $\sum_h a_h \text{pf}_{I_h} \text{pf}_{J_h} = 0$ as a polynomial function on $\mathcal{X}_{n+1}$.

3.3. The action of the symmetric group and submatrices. The previous identification of the weight vectors of the positive spin module and the pfaffians allows us to prove the following invariance properties. These will be used in the proof of our main result.

**Lemma 8.** If $\sum_h a_h x^{(I_h)}_h$ is an element of $\mathcal{K}$, with $a_h \in \mathbb{C}$, and $\sigma \in S_n$ then $\sum_h a_h x^{(\sigma I_h)}_{\sigma J_h}$ is in $\mathcal{K}$ too.

**Proof.** Indeed $\sum_h a_h x^{(I_h)}_h$ is an element of $\mathcal{K}$ if and only if $\sum a_h \text{pf}_{I_h} \text{pf}_{J_h} = 0$ by Proposition 7. If we change the enumeration of the rows and columns of an antisymmetric $(n+1) \times (n+1)$ matrix from $1, 2, \ldots, n+1$ to $\sigma(1), \sigma(2), \ldots, \sigma(n), n+1$ the same relation holds. But in terms of the original enumeration this relation is $\sum a_h \text{pf}_{I_h} \text{pf}_{J_h} = 0$; hence our claim follows using again Proposition 7.

**Lemma 9.** If $f = \sum_h a_h x^{(I_h)}_h$ is an element of $\mathcal{K}_n$, with $a_h \in \mathbb{C}$, and $I_h, J_h \subset \{1, \ldots, n-1\}$ for all $h$, then we may consider $f$ as an element of $\mathcal{K}_{n-1}$. On the converse any relation in $\mathcal{K}_{n-1}$ may be considered as an element of $\mathcal{K}_n$. 

11
Proof. The element $f \in K_n$ corresponds to the relation $\sum a_h \text{pf}_h \text{pf}_{j_h} = 0$ in terms of pfaffians. These pfaffians involve the rows and columns $1, 2, \ldots, n-1, n+1$ by hypothesis. Now we change the enumeration from $1, 2, \ldots, n-1, n+1$ to $1, 2, \ldots, n-1, n+1, n$. The same relation is still true with the new enumeration; but it is obtained by completing each odd length row $I$ by adding $n$ and not $n+1$ before computing the pfaffian $\text{pf}_I$. Hence this last relation is the same we obtain if we consider $f$ has an element of the ring $A$ for $n-1$ instead of $n$.

The second claim is analogously proved using again Proposition. □

If $I = i_1 \ldots i_r$ is a row and $1 \leq s \leq n$ an integer, let $j_s(I)$ be the row $i_1 \cdots i_{i_s} s i_{h+1} \cdots i_r$ where $h$ is the maximal index such that $i_t \leq s$ for all $t = 1, \ldots, h$. Notice that $j_s(I)$ is standard if $I$ is standard and $s \notin I$.

Let $d_s(I)$ be the row obtained by deleting any occurrence of $s$ in $I$. We have that $d_s(I)$ is standard if $I$ is standard.

Further we define $j_s, d_s$ on weight vectors by $j_s(x(I)) = x(j_s(I)), d_s(x(I)) = x(d_s(I))$ respectively, for all standard rows $I$; clearly $j_s(x(I)) = 0$ if and only if $s \in I$. We extend $j_s, d_s$ from $A_1$ to $S^*A$ as algebra homomorphisms.

Lemma 10. Let $f = \sum_a a_h x(I_h^h)$ be an element of $K$ with $I_h, J_h$ standard rows for all $h$. If $s \notin I_h \cup J_h$ for all $h$, then $j_s(f) \in K$. If $s \in I_h \cap J_h$ for all $h$, then $d_s(f) \in K$.

Proof. Given a subset $\Delta$ of $\{1, 2, \ldots, n\}$ and a standard row $I \subset \Delta$ let $c_{\Delta}(I)$ be the standard row complementary to $I$ in $\Delta$. We define $c_{\Delta}$ on weight vectors by $c_{\Delta}(x(I)) = x(c_{\Delta}(I))$ for all $x(I)$ such that $I$ is a standard row contained in $\Delta$; further we extend it as an algebra homomorphism from the subalgebra $A_{\Delta}$ of $A$ generated by $x(I)$ with $I \subset \Delta$ to $A$.

By Corollary, Lemma and Lemma we have: if $f \in A_{\Delta} \cap K$ then $c_{\Delta}(f) \in K$.

Now let $\Delta_1 = \{1, 2, \ldots, s-1, s+1, \ldots, n\}, \Delta_2 = \{1, 2, \ldots, n\}$. If $I$ is a standard row and $s \notin I$ then $j_s(I) = c_{\Delta_2}c_{\Delta_1}(I)$ and hence our first claim follows.

The second claim is analogous; indeed if $s \in I$ then $d_s(I) = c_{\Delta_1}c_{\Delta_2}(I)$. □

4. Shuffling relations

4.1. Shuffling polynomial. We say that a tableau is $r$-standard if all rows are standard, the entries of $T$ do not decrease along the first $r$ columns but the same is not true in the $(r+1)$-th column. If moreover $T$ has two rows then we may write it as $T = \left(\frac{IJ}{HK}\right)$ where $I = i_1i_2\cdots i_r, J = j_1j_2\cdots j_s, H = h_1h_2\cdots h_{r+1}$ and $K = k_1k_2\cdots k_t$ with $s > t$ and with the following inequalities

$$T = \left( \begin{array}{c} i_1 < i_2 < \cdots < i_r < j_1 < j_2 < \cdots < j_s \\ \wedge \\ h_1 < h_2 < \cdots < h_r < h_{r+1} < k_1 < \cdots < k_t \end{array} \right)$$

We call this subdivision of the two rows of $T$ its canonical form. Notice in particular that we have a chain of strict inequalities from $h_1$ to $j_s$, hence there is no repetition in $HJ$.

However, despite the name, a standard tableau is not $r$-standard for any $r$.

Definition 11. Given an $r$-standard tableau $T = \left(\frac{IJ}{HK}\right)$ in canonical form, we define the shuffling polynomial of $T$ as the following element of $S^2A_1$:

$$\Theta(T) = \sum (-1)^{\frac{h |J| + h}{h}} \binom{|J \setminus K| + h}{h} \varepsilon(H', J') \in (Z_2K_1') x \left( \frac{IJ'}{H'K_2'} \right)$$

with $h = |H| - |H'|$, $Z_1 = Z_1(T) \equiv I \cap H$ and $Z_2 = Z_2(T) \equiv J \cap K$ as ordered rows and the sum running over the set $\mathcal{I}(T)$ of all $(J', H', K_1', K_2') \in S^R_n$ such that
Hence under this bijection. Clearly we have
\[ K'_i \cup K'_2 = K, \]
(4) \( Z_1 \subseteq H', \)
(5) \( Z_2 \subseteq J' \cap K'_2. \)

For short let
\[ \alpha(J', H', K'_1, K'_2) \doteq (-1)^{s(h+1)/2} \left[ \frac{|J \setminus K| + h}{h} \right] \varepsilon(H', J') \varepsilon(Z_2 K'_1). \]

**Remark 12.** Notice that \( T \) appears in \( \Theta(T) \) with coefficient 1 and each tableau in \( \Theta(T) \) but \( T \) fulfills the conditions of a straightening relation for \( T \).

4.2. **Combinatorial properties of shuffling polynomials.** In the next purely combinatorial lemma we prove that the shuffling polynomial of a tableau does not change if we insert or remove a common entry in the rows.

**Lemma 13.** Let \( T \) be a non standard tableau with two standard rows.

1. If \( T \) does not contain \( s \), then \( j_s(T) \) is not standard and \( j_s(\Theta(T)) = \Theta(j_s(T)). \)
2. If \( s \) is contained in both rows of \( T \), then \( d_s(T) \) is not standard and \( d_s(\Theta(T)) = \Theta(d_s(T)). \)

**Proof.** We prove the first claim. Assume that \( T = (t_{H,K}) \) is \( r \)-standard in canonical form. Consider first the case \( s < h_{r+1} \). Then \( j_s(T) \) is \((r+1)\)-standard, \( j_s(T) = (j_{s(H,K)}) \) in canonical form and the map
\[
(J', H', K'_1, K'_2) \mapsto (J', j_s(H'), K'_1, K'_2)
\]
gives a bijection from \( \mathcal{I}(T) \) to \( \mathcal{I}(j_s(T)) \).

Let \( f \) be an addend in \( \Theta(T) \), we compare \( j_s(f) \) with the corresponding term in \( \Theta(j_s(T)) \) under this bijection. Clearly we have

- \((i)\) \( |j_s(H)| - |j_s(H')| = |H| - |H'|, \)
- \((ii)\) \( J \setminus K \) is the same set for \( T \) and for \( j_s(T) \) and \( Z_2 \) and \( K'_1 \) do not change in this bijection.

Let \( k \) be the number of elements of \( J' \) smaller than \( s \). Then we have
\[
\varepsilon(j_s(H'), J') = (-1)^k \varepsilon(H', J') \]
\[
\varepsilon(j_s(I), J' K'_1) = (-1)^k \varepsilon(I J' K'_1) \]
\[
\varepsilon(j_s(H'), K'_2) = \varepsilon(H' K'_2)
\]

Hence
\[
\varepsilon(j_s(H'), J') x \left( \frac{j_s(I, J' K'_1)}{j_s(H'), K'_2} \right) = \varepsilon(j_s(H'), J') \varepsilon(j_s(I), J' K'_1) j_s(H') K'_2 x \left( \frac{j_s(I, J' K'_1)}{j_s(H') K'_2} \right) = \varepsilon(H', J') \varepsilon(I, J' K'_1) \varepsilon(H', K'_2) x \left( \frac{j_s(I, J' K'_1)}{j_s(H', K'_2)} \right) = \varepsilon(H', J') x \left( \frac{I J' K'_1}{H', K'_2} \right)
\]
which implies that the two terms we are considering are equal. This proves \( j_s(\Theta(T)) = \Theta(j_s(T)) \).
Assume now that \( s > h_{r+1} \). Then \( j_s(T) \) is \( r \)-standard, \( j_s(T) = (I_{j_s(K)})_J \) in canonical form and the map

\[
(J', H', K'_1, K'_2) \mapsto (j_s(J'), H', K'_1, j_s(K'_2))
\]
gives a bijection from \( \mathcal{I}(T) \) to \( \mathcal{I}(j_s(T)) \). As above we compare the corresponding terms in \( \Theta(T) \) and \( \Theta(j_s(T)) \) under this bijection. We have: \( j_s(J) \sim j_s(K) = J \sim K \) while \( |H| - |H'| \) does not change in this bijection. Let \( k \) be the number of elements of \( H' \) bigger than \( s \) and \( m \) be the number of elements of \( K'_1 \) smaller than \( s \). Then we have

\[
\varepsilon(H'j_s(J')) = (-1)^k \varepsilon(H'J')
\]

\[
\varepsilon(I_{j_s(J')}K'_1) = (-1)^m \varepsilon(IJ'K'_1)
\]

\[
\varepsilon(H'j_s(K'_2)) = (-1)^k \varepsilon(H'K'_2)
\]

\[
\varepsilon(j_s(Z_2)K'_1) = (-1)^m \varepsilon(Z_2K'_1)
\]

which, as above, implies that the two corresponding terms are equal and hence \( j_s(\Theta(T)) = \Theta(j_s(T)) \).

Now we see how the second claim follows from the first one. Indeed let \( T \) be a non standard tableau containing \( s \) in both rows as in the second claim and let \( \overline{T} = d_s(T) \). Notice that \( j_s(\overline{T}) = j_sd_s(T) = T \) since \( T \) has standard rows, hence \( \overline{T} \) cannot be standard otherwise also \( T = j_s(\overline{T}) \) should be standard. So we may apply the first claim to \( \overline{T} \) and find \( j_s(\Theta(\overline{T})) = \Theta(j_s(\overline{T})) = \Theta(T) \). Apply \( d_s \) to both sides of this identity and notice that \( d_sj_s = \text{Id} \) to conclude \( \Theta(d_sT) = d_sj_s\Theta(\overline{T}) = d_s\Theta(T) \) as claimed.

Now we see another combinatorial property of the shuffling polynomials. We want to prove that if we permute the entries of a tableau with a permutation satisfying certain assumptions then the shuffling polynomial may change only by the sign.

Let us start by stating these assumptions of compatibilities between an \( r \)-standard tableau \( T = (J_{H\sigma}^J) \) in canonical form and a permutation \( \sigma \in S_n \). Given a row \( I \) let \( I^\sigma \) be the row \( (\sigma \cdot I)^\circ \) and let \( T^\sigma = (J_{H\sigma}^J)^\sigma \). We say that \( \sigma \) is compatible with \( T \) if \( T^\sigma \) is again \( r \)-standard with canonical form \( (J_{H\sigma}^J)^\sigma \) and \( K^\sigma = \sigma \cdot K \). In particular notice that in this case we have \( Z_1(T^\sigma) = Z_1(T^\sigma)^\sigma \) and \( Z_2(T^\sigma) = Z_2(T^\sigma)^\sigma \).

**Lemma 14.** If \( T \) is \( r \)-standard and \( \sigma \) is \( T \)-compatible then \( \sigma \cdot \Theta(T) = \pm \Theta(T^\sigma) \).

**Proof.** First notice that the map

\[
(J', H', K'_1, K'_2) \mapsto (J^\sigma, H^\sigma, K^\sigma_1, K^\sigma_2)
\]
gives a bijection from \( \mathcal{I}(T) \) to \( \mathcal{I}(T^\sigma) \).

If \( (H', J', K'_1, K'_2) \) is an element of \( \mathcal{I}(T) \), consider the following sequence of transformations which reorder the row \( H^\sigma J^\sigma \)

\[
H^\sigma J^\sigma \xrightarrow{\tau_{\sigma H', J'}^{-1}} (\sigma \cdot H')(\sigma \cdot J') \xrightarrow{\tau_{H', J'}^{-1}} H' J' \xrightarrow{\tau_{H', J'}^{-1}} H J \xrightarrow{\sigma} (\sigma \cdot H)(\sigma \cdot J) \xrightarrow{\tau_{\sigma H', J'}^{-1}} H^\sigma J^\sigma \xrightarrow{\tau_{\sigma H', J'}^{-1}} (H^\sigma J^\sigma)^\circ.
\]

So, using that the sign of \( \sigma \tau_{H', J'}^{-1} \) is that of \( \tau_{H', J'} \), we obtain

\[
\varepsilon(H^\sigma J^\sigma) = \varepsilon(\sigma \cdot H')\varepsilon(\sigma \cdot J')\varepsilon(\sigma \cdot H)\varepsilon(\sigma \cdot J).
\]

Moreover by \( \sigma \cdot K = K^\sigma \) we have \( \sigma \cdot K'_1 = K_1^\sigma \), \( \sigma \cdot K'_2 = K_2^\sigma \) and also \( \varepsilon(Z_2K'_1) = \varepsilon(Z_2K'_1)^\sigma \), hence

\[
\varepsilon(H'J')\varepsilon(Z_2K'_1)(\sigma \cdot x)\left(I_{J'K'_1}^J\right) = \varepsilon(H'J')\varepsilon(Z_2K'_1)\varepsilon(\sigma \cdot J')\varepsilon(\sigma \cdot H')x\left(I_{J'K'_1}^J\right) = \varepsilon(\sigma \cdot I)\varepsilon(\sigma \cdot J)\varepsilon(\sigma \cdot H)\varepsilon(H^\sigma J^\sigma)\varepsilon(Z_2K'_1)^x\left(I_{J'K'_1}^J\right) = \varepsilon(\sigma \cdot I)\varepsilon(\sigma \cdot J)\varepsilon(\sigma \cdot H)\varepsilon(H^\sigma J^\sigma)\varepsilon(Z_2K'_1)^x\left(I_{J'K'_1}^J\right).
\]
Set \( \varepsilon = \varepsilon(\sigma \cdot I) \varepsilon(\sigma \cdot J) \varepsilon(\sigma \cdot H) \) and notice that \( |J'| = |J^\sigma| \) and \( |H| - |H'| = |H^\sigma| - |H'^\sigma| \).

From the above identities we conclude

\[
\sigma \cdot \Theta(T) = \sum_{(H', J', K'_1, K'_2) \in \mathcal{I}(T)} \alpha(H', J', K'_1, K'_2) \sigma \cdot x \left( \frac{I \cdot J' \cdot K'_1}{H'} \cdot K'_2 \right)
\]

\[
= \sum_{(H', J', K'_1, K'_2) \in \mathcal{I}(T)} \varepsilon \alpha(H', J', K'_1, K'_2) x \left( \frac{I^\sigma \cdot J'^\sigma \cdot K'_1^\sigma}{H'^\sigma \cdot K'_2^\sigma} \right)
\]

proving our claim. \( \square \)

5. PROOF OF THE SHUFFLING RELATIONS

In this section we prove our main result.

**Theorem 15.** If \( T \) is a non standard tableau with two standard rows, then \( \Theta(T) \) is an element of \( \mathcal{K} \).

**Proof.** Assume that \( T \) is \( r \)-standard, \( T = \left( \frac{I}{H\cdot K} \right) \) in canonical form and let \( R_1 = IJ \) and \( R_2 = HK \) be its first and second row respectively.

**Step 1.** If \( R_1 \cap R_2 \neq \{1, 2, \ldots, n\} \) then there exists \( \sigma \in S_n \) such that \( n \notin \sigma \cdot T \) and \( \sigma(i) < \sigma(j) \) for any pair \( 1 \leq i < j \leq n \) in \( T \). The permutation \( \sigma \) is clearly \( T \)-compatible and, in particular, \( T^\sigma = \sigma \cdot T \), so \( \Theta(\sigma \cdot T) = \pm \sigma \cdot \Theta(T) \) by Lemma 13.

Using induction on \( n \) we may suppose that \( \Theta(\sigma \cdot T) \in \mathcal{K}_{n-1} \), but then \( \Theta(\sigma \cdot T) \in \mathcal{K}_n \) by Lemma 8. Hence \( \Theta(T) = \pm \sigma^{-1} \cdot \Theta(\sigma \cdot T) \in \mathcal{K}_n \) by Lemma 8 and our claim is proved.

The inductive base step is automatically true since for \( n = 1 \) there are no non standard tableaux.

**Step 2.** Now we proceed by induction on \( |R_1 \cap R_2| \). Suppose \( s \in R_1 \cap R_2 \) and let \( T = d_s(T) \). Since \( s \notin T \) we have \( \Theta(T) \in \mathcal{K} \) by Step 1. Hence \( j_s(\Theta(T)) \in \mathcal{K} \) by Lemma 10. But \( j_s(\Theta(T)) = \Theta(j_s(T)) \) by Lemma 13 and moreover \( j_s(T) = T \) since \( T \) has standard rows. This proves our claim.

In the following steps we assume that \( R_1 \) and \( R_2 \) do not intersect proving the induction base.

**Step 3.** Now we show that it suffices to prove our claim for a particular tableau. Indeed let \( I = i_1 \cdot i_2 \cdot \ldots \cdot i_r \), \( J = j_1 \cdot j_2 \cdot \ldots \cdot j_s \), \( H = h_1 \cdot h_2 \cdot \ldots \cdot h_{r+1} \), \( K = k_1 \cdot k_2 \cdot \ldots \cdot k_t \), with \( 2r + s + t + 1 \) since we are assuming \( R_1 \cup R_2 = \{1, \ldots, n\} \) and \( R_1 \cap R_2 = \emptyset \).

By \( R_1 \cap R_2 = \emptyset \) we deduce that there exists a unique \( \sigma \in S_n \) such that \( \sigma(i_u) = u \) for \( u = 1, \ldots, r \), \( \sigma(h_u) = u + r \) for \( u = 1, \ldots, r + 1 \), \( \sigma(j_u) = u + 2r + 1 \) for \( u = 1, \ldots, s \) and \( \sigma(k_u) = u + 2r + s + 1 \) for \( u = 1, \ldots, t \). It is clear that

\[
T^\sigma = \sigma \cdot T = T^0 = \begin{pmatrix}
1 & 2 & \cdots & r & 2r + 2 & 2r + 3 & \cdots & 2r + s + 1 \\
0 & 1 & r + 2 & \cdots & 2r & 2r + 1 & \cdots & n
\end{pmatrix}.
\]

In particular \( T^\sigma \) is \( r \)-standard and \( \sigma \) is \( T \)-compatible. Hence \( \Theta(T) = \sigma^{-1} \cdot \Theta(T^0) \) by Lemma 14 and we see that if we prove \( \Theta(T^0) \in \mathcal{K} \) then \( \Theta(T) \in \mathcal{K} \) by Lemma 8.

So in the sequel we assume \( T = T^0 \).

**Step 4.** In this step we prove our claim for \( K = \emptyset \). For short we write \( (J', H') \) instead of \( (J', H', \emptyset, \emptyset) \).

Notice that in

\[
\Theta(T) = \sum_{(J', H') \in \mathcal{I}(T)} \alpha(J', H') x \left( \frac{I \cdot J'}{H'} \right)
\]
only $T$ is non standard. Let $\Delta(T)$ be the unique element of $S^2A_1$ corresponding to the straightening relation for $T$. Each tableau $(R'_1, R'_2)$ in $\Delta(T)$ verifies $R'_1 \leq R_1, R_2$ and $R'_2 \geq R_1, R_2$; using this it is easy to see that there exists $(J', H')$ in $I(T)$ such that $R'_1 = IJ'$ and $R'_2 = H'$. So we may write
\[
\Delta(T) = \sum_{(J', H') \in I(T)} \delta(J', H') x(J', H')
\]
for some coefficients $\delta(J', H') \in \mathbb{C}$. If we show that $\alpha(J', H') = \delta(J', H')$ for all $(J', H') \in I(T)$ then we have $\Theta(T) = \Delta(T) \in \mathcal{K}$ and our claim is proved.

Step 4.1. Now we want to prove that $\alpha(J', H') = \delta(J', H')$ for all $(J', H') \in I(T)$ with $r + 1 \in H'$. We compare $d_r e_r \Theta(T)$ with $d_r e_r \Delta(T)$. Let
\[
T = \begin{pmatrix}
\delta_r(I) J & \cdots & \delta_r(I) J
\end{pmatrix}
\begin{pmatrix}
1 & 2 & \cdots & r - 1 & 2r + 2 & 2r + 3 & \cdots & n
\end{pmatrix}
\begin{pmatrix}
r + 2 & r + 3 & \cdots & 2r & 2r + 1
\end{pmatrix}
\]
Using the definition of $d_r$ and $e_r$ we have
\[
d_r e_r \Theta(T) = \sum_{(J', H') \in I(T), r + 1 \in H'} \alpha(J', H') x(J', H')
\]
The map $(J', H') \mapsto (J', d_{r+1}(H'))$ is a bijection from $I(T)$ and $I(\overline{T})$ (with inverse $(J', H') \mapsto (J', j_{r+1}(H'))$). Moreover, adding the corresponding tableau as superscript for clarity, we have
\[
\alpha^T(J', d_{r+1}(H')) = (-1)^{\frac{n r + 1}{2}} \left[ n - 2r - 1 + \frac{1}{h} \right] \sum_{-1} (H' \setminus r + 1) J'
\]
with $\overline{h} = |d_{r+1}(H')| - |d_{r+1}(H')| = |H| - |H'|$ and also $\varepsilon((H' \setminus r + 1) J') = \varepsilon(H' J')$ since $r + 1 \in H' J'$ is the minimum element. Hence $\alpha^T(J', d_{r+1}(H')) = \alpha^T(J', H')$; this proves that $d_r e_r \Theta(T) = \Delta(T)$.

But then $d_r e_r \Theta(T) = \Theta(T) \in \mathcal{K}$ by Step 1 since $r \not\in T$. Notice that $\overline{T}$ is the unique non standard tableau in $\Theta(T)$, so $d_r e_r \Theta(T)$ is the element $\Delta(\overline{T})$ of $S^2A_1$ corresponding to the straightening relation of $\overline{T}$ since it is an element of $\mathcal{K}$ and the coefficient of $\overline{T}$ is 1 in both elements.

Consider now $d_r e_r \Delta(T)$ of the definitions.
\[
d_r e_r \Delta(T) = \sum_{(J', H') \in I(T), r + 1 \in H'} \delta(J', H') x(J', H')
\]
Hence also in $d_r e_r \Delta(T)$ the unique non standard tableau is $\overline{T}$ and it appears with coefficient 1. But $\mathcal{K}$ is closed by $e_r$ since it is a $g$-module and it is closed by $d_r$ by Lemma [10] so $d_r e_r \Delta(T) = \Delta(\overline{T})$ being the straightening relation for $\overline{T}$ unique.

Hence we have showed that $d_r e_r \Theta(T) = \Delta(\overline{T}) = d_r e_r \Delta(T)$; in particular $\alpha(J', H') = \delta(J', H')$ for all $(J', H') \in I(T)$ with $r + 1 \in H'$ that our claim for this step.

Step 4.2. Our next claim is now $\alpha(J', H') = \delta(J', H')$ if $H' \neq \emptyset$. Indeed let $r + 1 \leq i \leq n$ and consider $e_i \Delta(T)$. We find at once
\[
e_i \Delta(T) = \sum_{(J', H') \in I(T), i \in H', i + 1 \in J'} \delta(J', H') x(J', H') + \sum_{(J', H') \in I(T), i \in H', i + 1 \in J'} \delta(J', H') x(J', H')
\]
In particular it is easy to see that all tableau in $e_i \Delta(T)$ are standard; but this is an element of $\mathcal{K}$, hence $e_i \Delta(T) = 0$. Each tableau in $e_i \Delta(T)$ is obtained in exactly two ways: replacing $i + 1$ by $i$ in the first row or in the second row. So we have proved that
δ(J" , H") = −δ(J', H') if i and i + 1 appear in different row in \((I_{J'}^{r})\) and \((I_{H'}^{r})\) is obtained from \((I_{J'}^{r})\) by swapping i and i + 1.

Since each tableau \((I_{J'}^{r})\) with \(H' \neq \emptyset\) is reached by a finite number of swaps of \(i, i + 1\) with \(r + 1 \leq i \leq n\) from a tableau containing \(r + 1\) in the second row, we have \(\alpha(J', H') = \delta(J', H')\) for all tableaux with \(H' \neq \emptyset\) using the result of the previous step.

**Step 4.3.** So \(\Theta(T) - \Delta(T) = c \cdot x(I_{J'}^{r})\) and we want to show \(c = 0\). Let \(M\) be the \((n + 1) \times (n + 1)\) antisymmetric matrix of Example 6 whose all pfaffians are 1 and identify the elements of \(S' \mathbb{A}_1\) with the corresponding polynomials as in the Proposition 7. So \(c = \Theta(T)(M)\) since \(\Delta(T)(M) = 0\) by \(\Delta(T) \in \mathcal{K}\) and \(x(I_{J'}^{r})(M) = 1\). We want to show that \(\Theta(T)(M) = 0\) proving \(c = 0\) and \(\Theta(T) = \Delta(T)\).

We have
\[
\Theta(T)(M) = \sum_{h=0}^{r+1} (-1)^{\frac{h(h+1)}{2}} \left[ \begin{array}{c} n - 2r - 1 - h \\ h \end{array} \right] \left[ \begin{array}{c} n - r \\ r + 1 - h \end{array} \right] - 1
\]
and this is zero by Lemma 2 with \(s = |H| = r + 1\), \(m = |H| + |J| = n - r\) and \(q = -1\). This finishes the proof that \(\Theta(T) \in \mathcal{K}\) for all non standard tableau \(T\) with \(K = \emptyset\).

**Step 5.** In this final step we prove that our claim for generic \(K\) follows by the case \(K = \emptyset\) of the previous steps. Let \(T = \left( I_{J'}^{r} \right)\) be as in the conclusion of Step 3; \(I = 12 \cdots r, J = 2r + 2 \cdots 2r + s + 1, H = r \cdots 2r + 1\) and \(K = 2r + s + 2 \cdots n\).

We want to proceed by induction on \(|K|\). Indeed, for \(u = 0, \ldots, |K|\) let \(T_u = \left( I_{J,K_u}^{r} \right)\), where \(K_u\) is the standard row containing the first \(u\) entries of \(K\), and notice that the base inductive step, i.e. \(\Theta(T_u) \in \mathcal{K}\), has already been proved.

Now suppose that \(\Theta(T_u) \in \mathcal{K}\) for \(0 < u < |K|\) and let \(k = 2r + s + 1 + u\) be the last entry of \(K_u\). By Lemma 9 we have \(\Theta(T_u) \in \mathcal{K}_k \subset \mathcal{K}_{k+1}\), so we apply \(e = e_{k+1} + e_{k+2}\) to \(\Theta(T_u)\). By the definition of \(e_{k+1}\) and \(e_{k+2}\) (for \(n = k + 1\)) we see that the operator \(e\) produce two tableaux from each tableau in \(\Theta(T_u)\) by adding the entry \(k + 1\) in the first and the second row. It is clear that
\[
\alpha^{T_{u+1}}(J', H', J_{k+1}(K_1'), K_2') = \alpha^{T_{u+1}}(J', H', K_1', J_{k+1}(K_2')) = \alpha^{T_u}(J', H', K_1', K_2')
\]
by the definition. Moreover \(\mathcal{I}(T_{u+1})\) is the set of tableaux obtained from \(\mathcal{I}(T_u)\) by applying \(e\). So we conclude \(\Theta(T_{u+1}) = e \cdot \Theta(T_u) \in \mathcal{K}_{k+1} \subset \mathcal{K}\) and this finishes the proof of the theorem.

So we may state our result in terms of shuffling relations.

**Corollary 16.** For any non standard tableau \(T\) with two standard rows, \(\Theta(T)\) is a shuffling relation for \(T\); the set of all such shuffling relations generate the kernel of the multiplication map \(S^2 \mathbb{A}_1 \rightarrow \mathbb{A}_2\) and the ideal of relations defining \(\mathcal{G}\) in \(\mathbb{P}(V^*_{\omega_{n+1}})\).

One may hope to simplify \(\Theta(T)\) by considering only the tableaux with \(K_1 = \emptyset\), in analogy with the shuffling relations for determinants. This is not possible as the following example for \(n = 4\) shows.
If \( T = \binom{23}{14} \) we have:

\[
\Theta(T) = x\binom{23}{14} - x\binom{13}{24} + x\binom{12}{34} - x\binom{123}{4} + x\binom{234}{1} - x\binom{134}{2} + x\binom{124}{3} - x\binom{1234}{\emptyset},
\]

\[
\Theta\binom{234}{1} = x\binom{234}{1} - x\binom{134}{2} + x\binom{124}{3} - x\binom{1234}{4}.
\]

So, if the sum \( \overline{\Theta}(T) \) of all elements with \( K'_1 = \emptyset \) in \( \Theta(T) \) (i.e. of those with 4 in the bottom row) were an element of \( \mathcal{K} \), then we have also

\[
\Theta(T) - \overline{\Theta}(T) - \Theta\binom{234}{1} = -x\binom{1234}{\emptyset} \in \mathcal{K}
\]

that is impossible since \( \binom{1234}{\emptyset} \) is standard.

6. Conclusion and relations in arbitrary characteristic

We begin by restating the main result in terms of pfaffians and we slightly generalize it to a commutative unitary base ring \( R \). Let \( B_R = R[x_{ij}]_{1 \leq i < j \leq n+1} \). It is easy to check that the pfaffians of an antisymmetric matrix as defined in Section 3 is a polynomial in the variables \( x_{ij} \) with integer coefficients so we can consider the elements \( \text{pf}_I \) as elements of \( B_R \). For a tableau \( T \), let \( \text{pf}_T \in B \) be the product of the pfaffians which appear in \( T \). Since, as we have already noticed \( \text{pf}_{ij} = x_{ij} \), the ring \( B_R \) is spanned as an \( R \)-module by the elements \( \text{pf}_T \). Notice that \( \Theta(T) \) has integral coefficients hence we may define \( \Theta_{\text{pf}}(T) \) as the element of \( B_R \) obtained by mapping \( x(I) \mapsto \text{pf}_I \) for \( I \in \mathcal{R}_n \) and \( I \neq \emptyset \), and mapping \( x(\emptyset) \) to 1.

**Theorem 17.** Let \( R \) be a commutative unitary ring.

1. The set of \( \text{pf}_T \) with \( T \) standard and \( T \) not containing the empty row is a \( R \)-basis of \( B_R \);
2. the ring \( B_R \) is defined by quadratic (but not necessarily homogenous) relations in the generators \( \text{pf}_I \);
3. for all non standard \( T \) with two rows we have \( \Theta_{\text{pf}}(T) = 0 \) and these equations generate (as a \( R \)-module) the set of quadratic relations.

**Proof.** For \( R = \mathbb{C} \) the result follows by Corollary 16 and Section 3.2. For \( R = \mathbb{Z} \) notice that \( B_\mathbb{Z} \) is a subring of \( B_\mathbb{C} \); in particular the polynomial \( \Theta_{\text{pf}}(T) \) vanish. Since \( \text{pf}_T \) has coefficient 1 in \( \Theta_{\text{pf}}(T) \), this allows to write any element \( \text{pf}_T \) as a \( \mathbb{Z} \)-linear combination of pfaffians of standard tableaux, clearly we may assume also that the empty row does not appear. Since the polynomials \( \text{pf}_T \) with \( T \) as in (1) are linear independent over \( \mathbb{C} \) we have proved that they are a \( \mathbb{Z} \)-basis, proving (1) for \( \mathbb{Z} \).

Now (2) and (3) for \( \mathbb{Z} \) follows by a standard argument. Indeed consider the ring \( \mathbb{C} \) generated over \( \mathbb{Z} \) by indeterminates \( t_I \) with \( I \in \mathcal{S}\mathcal{R}_n, I \neq \emptyset \) modulo the ideal generated by the polynomials \( \Theta^C(T) \) where \( T \) runs in the set of non standard tableau with two standard rows obtained by mapping \( x(I) \) to \( t_I \) and \( x(\emptyset) \) to 1 in \( \Theta(T) \).

Arguing as above we see that also \( C \) has a basis given by the set of \( t_T \) with \( T \) running in the set of standard tableaux without the empty row. So the map \( t_I \mapsto \text{pf}_I \) is an isomorphism and (2) and (3) follows.

The general case follows by \( B_R = B_\mathbb{Z} \otimes_\mathbb{Z} R \).

\( \square \)

In a similar way we generalize Corollary 16 from \( \mathbb{C} \) to \( R \) with \( R \) a field or the integers. The group \( G \), the variety \( \mathcal{G}_R \), and the line bundle \( \mathcal{L} \) may be defined in a flat way over the integers, hence they may be defined over \( R \) and we denote by \( G_R, \mathcal{G}_R, R \) and \( \mathcal{L}_R \) the
associated schemes and line bundles. Let us define $A_R$ as $\bigoplus_{m \geq 0} H^0(G_R, L^m)$. As proved in Remark 7 in [6], $A_Z$ is generated in degree 1 and if $R$ is a field then $A_R$ is isomorphic to $A_Z \otimes Z R$. So arguing as in the proof of the previous theorem we have the following result.

**Theorem 18.** Let $R$ be the set of integers or a field.

1. The sections $y(T)$ with $T$ standard are an $R$-basis for the ring $A_R$;
2. the ring $A_R$ is defined by quadratic (but not necessarily homogenous) relations in the generators $x(I)$;
3. for all non standard $T$ with two rows we have $\Theta(T) = 0$ and these equations generate (as a $R$-module) the set of quadratic relations.

**References**

[1] R. Chirivi, P. Littelmann, A. Maffei. *Equations defining symmetric varieties and affine Grassmannians*. I.M.R.N., vol. 2009, n. 2, 291–347.
[2] W. V. D. Hodge. Some enumerative results in the theory of forms. *Proc. Cambridge Philos. Soc.* 39::22–30, 1943.
[3] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge studies in advanced mathematics, vol. 29, Cambridge University Press, Cambridge, 2009.
[4] A. R. Kustin. Pfaffian identities, with applications to free resolutions, DG–algebras, and algebras with straightening law. *Commutative algebra: syzygies, multiplicities and birational algebra*. Contemporary mathematics, vol. 159, 269–292, 1994.
[5] C. S. Seshadri. Geometry of $G/P$. I. Theory of standard monomials for minuscule representations. in *C. P. Ramanujam—a tribute*. Tata Inst. Fund. Res. Studies in Math., vol. 8, 207–239, 1978.
[6] C. S. Seshadri. Standard monomial theory and the work of Demazure. in *Algebraic varieties and analytic varieties*. Adv. Stud. Pure Math., vol. 1, 355-384, 1983.