Goldfish geodesics and Hamiltonian reduction of matrix dynamics

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Abstract

We relate free vector dynamics to the eigenvalue motion of a time-dependent real-symmetric \( N \times N \) matrix, and give a geodesic interpretation to Ruijsenaars Schneider models.

Despite of more than two decades of extensive work on Ruijsenaars-Schneider models [1, 2], their interpretation as describing geodesic motions seems to have gone unnoticed. Apart from wishing to fill this gap, the (related) second topic of this paper is the Hamiltonian reduction of free matrix dynamics (to, as we will show, free vector dynamics).

To start with the latter, let \( \ddot{X}(t) = 0, X(t) \) a real symmetric \( N \times N \) matrix (depending on "time"), be described by

\[
H[X, P] := \frac{1}{2} \text{Tr} P^2. \tag{1}
\]

Writing

\[
X(t) = R(t)D(t)R^{-1}(t) \\
\dot{X}(t) = R\left(\dot{D} + [M, D]\right)R^{-1} = RLR^{-1}, \tag{2}
\]

the symplectic form \(-\text{Tr} dX \wedge dP\) becomes (cp. [3]), with \(dA := R^{-1}dR\),

\[
\begin{aligned}
-dq_i \wedge dp_i + 2 \sum_{i<j} df_{ij} \wedge da_{ij} \\
- 2 \sum_{i<j<k} \left( f_{ij}da_{jk} \wedge da_{ik} + f_{ik}da_{ij} \wedge da_{jk} + f_{jk}da_{ik} \wedge da_{ij} \right)
\end{aligned} \tag{3}
\]

which (inverting (3)) gives the non-trivial Poisson-brackets

\[
\{q_i, p_j\} = \delta_{ij} \tag{4}
\]

\[
\{f_{ij}, f_{kl}\} = -\frac{1}{2} \delta_{jk}f_{il} + \frac{1}{2} \delta_{ik}f_{jl} + \frac{1}{2} \delta_{jl}f_{ik} - \frac{1}{2} \delta_{il}f_{jk} \tag{5}
\]

\[
\{f_{i<j}, a_{k<l}\} = -\frac{1}{2} \delta_{ik} \delta_{jl} \tag{6}
\]

resp. \( \{r_{ij}, f_{kl}\} = -\frac{1}{2} \left( \delta_{jk}r_{il} - \delta_{il}r_{jk} \right) \) \( \tag{7}\)
for the eigenvalues of $X$, their time derivatives $\dot{q}_i$, and $f_{ij} = -f_{ji} := \left( R^{-1} \dot{R} \right)_{ij} (q_i - q_j)^2$. (1) becomes

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} \sum_{i \neq j} f_{ij}^2 (q_i - q_j)^2,$$

(8)

which, as $H = H[q, \tilde{p}; f_{ij}]$, is known under the name "Euler Calogero-Moser Hamiltonian" ([4, 5]), with equations of motion

$$\ddot{q}_i = 2 \sum_{k \neq i} f_{ik} f_{kj} \left( \frac{1}{q_{ik}} - \frac{1}{q_{kj}} \right),$$

$$\dot{f}_{ij} = -\sum_{k \neq i, j} f_{ik} f_{kj} \left( \frac{1}{q_{ik}} - \frac{1}{q_{kj}} \right).$$

(9)

While it is well known (see e.g. [6, 7]) that both (types of) equations in (9) consistently reduce to

$$\ddot{q}_i = 2 \sum_{j \neq i} \frac{\dot{q}_i \dot{q}_j}{q_i - q_j}$$

(10)

upon setting

$$f_{ij} = -(q_i - q_j) \sqrt{\dot{q}_i \dot{q}_j},$$

(11)

the Hamiltonian reduction of (1) (resp. (8)) to (10) has remained open for many years. Simply counting the degrees of freedom ($H[q, \tilde{p}; f_{i<j}]$ has at most $2N + N(N-1)/2$ variables) together with

$$\{G_{ij}, G_{kl}\} = -\delta_{jk} G_{il} + \delta_{ik} G_{jl} + \delta_{il} G_{jk} - \delta_{ij} G_{lk}$$

$$G_{ij} := 2 \left( f_{ij} + (q_i - q_j) \sqrt{p_i p_j} \right)$$

(12)

shows that one can not simply "fix the gauge". Viewing (8) as originating from (1), however, one has the degrees of freedom corresponding to the orthogonal matrix $R$ (resp. the antisymmetric matrix $A$, which – due to (6) – naturally provides $N(N-1)/2$ variables that are canonically conjugate to the $G_{ij}$). Note that (8), when written in terms of the variables $G_{ij}$, reads

$$H = \frac{1}{2} \left( \sum_{i=1}^{N} p_i \right)^2 + \frac{1}{8} \sum_{i \neq j} G_{ij}^2 (q_i - q_j)^2 - \frac{1}{4} \sum_{i \neq j} \frac{G_{ij} \sqrt{p_i p_j}}{q_i - q_j},$$

(13)

clarifying (in the Hamiltonian framework) that it is consistent to put the $G_{ij} = 0$, due to $\dot{G}_{ij} = \{G_{ij}, H\}$ being "weakly zero" (i.e. using $G_{kl} = 0$ after computing the Poisson-brackets, according to (12) and (4)/(5); note that while the $q$’s and $p$’s do not commute with the $G_{ij}$, the total momentum, $P := \sum p_i$, does, as the $G_{ij}$ contain only the differences of the $q$’s). In order to find out what the reduced phase space is (consisting of functions that Poisson-commute with all the $G_{ij}$), the reduced Hamiltonian (cp. (13)) being

$$H = \frac{1}{2} \left( \sum p_i \right)^2 = \frac{1}{2} P^2,$$

(14)
it is useful to bring the "orbital angular momentum",

\[ L_{ij} := 2(q_i - q_j)\sqrt{p_ip_j} \tag{15} \]

(satisfying (12), with \( G \) replaced by \( L \)) into canonical form \((Q_iP_j - Q_jP_i)\), by making the canonical transformation

\[ P_i = \sqrt{p_i}, \quad Q_i = 2q_i\sqrt{p_i}. \tag{16} \]

While the (first order form, \( \dot{q}_i = p_i, \dot{p}_i = \ldots \), of the) goldfish-equations (10) do(es) not simplify at all,

\[ \dot{Q}_i = 2P_i^3 + 2Q_iP_j \sum_j \frac{P_j^3}{Q_jP_j - Q_jP_i} \]

\[ \dot{P}_i = 2P^2_i \sum_j \frac{P_j^3}{Q_jP_j - Q_jP_i} \tag{17} \]

(with \( \dot{Q}_iP_i - Q_i\dot{P}_i = 2P_i^4 \)) \( \dot{Q} \) and \( \dot{P} \) now transform as ordinary vectors \((\dot{Q} \rightarrow S\dot{Q}, \dot{P} \rightarrow S\dot{P})\) under the rotations generated by the \( G_{ij} \):

\[ \{G_{ij}, Q_k\} = \delta_{ik}Q_j - \delta_{jk}Q_i \]

\[ \{G_{ij}, P_k\} = \delta_{ik}P_j - \delta_{jk}P_i. \tag{18} \]

Together with (cp. (7))

\[ \{G_{ij}, r_{kl}\} = \delta_{il}r_{kj} - \delta_{jl}r_{ki}, \tag{19} \]

corresponding to \( R \rightarrow RS \), the 2\( N \) independent variables

\[ \vec{u} := R\vec{Q}, \quad \vec{v} := R\vec{P} \tag{20} \]

are therefore invariant, i.e. natural coordinates for the reduced phase-space. The original free matrix-dynamics (with the initial condition that \( \dot{X}(0) \) has 1 positive eigenvalue, and \( N - 1 \) zero) is thereby reduced to free vector dynamics,

\[ \ddot{u} = 2\vec{v}(\vec{v}^2), \quad \ddot{v} = 0. \tag{21} \]

governed by

\[ H[\vec{u}, \vec{v}] = \frac{1}{2} (\vec{v}^2)^2 = \frac{1}{2} \left( (R\vec{P})^2 \right)^2 = \frac{1}{2} (\vec{P}^2)^2 = \frac{1}{2} P^2, \tag{22} \]

Using (17), and the time-evolution of the \( r_{ij} \) (cp. (7)/(8)) given according to

\[ \left( R^{-1} \dot{R} \right)_{ij} = M_{ij} = \frac{\dot{f}_{ij}}{q_{ij}^2} = \frac{-\sqrt{p_ip_j}}{q_{ij}} = -2 = \frac{p^2_ip_j}{Q_iP_j - Q_jP_i} \tag{23} \]

one can check the time-evolution of the invariant variables \( \vec{u} = R\vec{Q}, \vec{v} = R\vec{P} \):

\[ \frac{d}{dt} \left( R\vec{Q} \right) = 2 \left( R\vec{P} \right) \vec{P}^2, \quad \frac{d}{dt} \left( R\vec{P} \right) = 0. \tag{24} \]
Let us focus on (10), independent of any previous considerations, as (coupled) second order ODEs. Their most natural, and extremely simple (though apparently unnoticed), interpretation is that of geodesic equations,

\[ \ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0 \]  
\[ \Gamma^i_{jk} = - \left( \frac{\delta_j^i (1 - \delta_k^i)}{q^j - q^k} + \frac{\delta_k^i (1 - \delta_j^i)}{q^k - q^j} \right) . \]

The simple expression (26), which is of the form

\[ \Gamma^i_{jk} = \delta_j^i w_{ik} + \delta_k^i w_{ij}, \]

with \( w_{ik} = -\frac{1}{2} w(q^i - q^k)(1 - \delta_{ik}) \), makes the calculation of the curvature-tensor

\[ R^c_{adb} = \partial_d \Gamma^c_{ab} + \Gamma^e_{ab} \Gamma^c_{de} - (b \leftrightarrow d) \]

straightforward:

\[ R^c_{adb} = \delta^c_d \delta^e_a w'_{ab} - \delta^c_d \delta^e_b w'_{ad} + w'_{ca} (\delta_{ab} \delta^e_d - \delta^e_b \delta_{da}) + \delta^c_d (w_{ab} w_{ca} + w_{ba} w_{cb} - w_{ca} w_{cb}) - \delta^c_b (w_{ad} w_{ca} + w_{da} w_{cd} - w_{ca} w_{cd}) \]

is identically zero for \( w(x) = 2/x \) (i.e. (26)), due to

\[ \frac{1}{q_c - q_a} \frac{1}{q_a - q_d} + \frac{1}{q_a - q_d} \frac{1}{q_d - q_c} + \frac{1}{q_d - q_c} \frac{1}{q_c - q_a} = 0 \quad (a \neq c \neq d \neq a), \]

and the \( w' \) term is cancelled by those \( w w \) terms, for which an additional equality of indices holds; alternatively, one could e.g. bring the last 3 \( w w \)-terms in (28), provided \( w(-x) = -w(x) \), into the form

\[ w(x)w(y) - w(x + y) (w(x) + w(y)), \]

\( x = q_c - q_a, y = q_a - q_d, \) and then note the linearity of \( 1/w \).

In order to find the change of variables that will make the Christoffel-symbols vanish, it is easiest to observe that

\[ \ddot{q}_i = 2 \sum'_j \frac{\dot{q}_i \dot{q}_j}{q_i - q_j} \]

directly implies that the functions

\[ b_n(t) := \frac{1}{(n - 1)!} \sum \hat{q}_{i_1} \hat{q}_{i_2} \cdots \hat{q}_{i_{n-1}} \hat{q}_{i_n} \quad n = 1, 2, \ldots \]

(with \( \sim \) indicating the indices to all be different) do not depend on time:

\[ \frac{1}{(n - 2)!} \sum \hat{q}_{i_1} \hat{q}_{i_2} \cdots \hat{q}_{i_{n-1}} \hat{q}_{i_n} + \frac{2}{(n - 1)!} \sum \hat{q}_{i_1} \hat{q}_{i_2} \cdots \hat{q}_{i_{n-1}} \sum_{j \neq i_n} \frac{\dot{q}_i \dot{q}_j}{q_i - q_j} = 0, \]

(32)
as the sum over $j \notin (i_1, \ldots, i_{n-1})$ gives zero, while $q_{n-1}$ times the sum over $j \in (i_1, \ldots, i_{n-1})$ can be replaced by $-\frac{n-1}{2} \hat{q}_n \hat{q}_{n-1}$. The searched transformation $\vec{q} \to \vec{x}(q_1, \ldots, q_n)$ (non-singular, as long as the $q_i$ are all different) is therefore provided by

$$x_n[\vec{q}] := \frac{1}{n!} \sum_{i=1}^{\infty} q_i \cdots q_n,$$  \hspace{1cm} (33)

$$\det \left( \frac{\partial x_n}{\partial q_i} \right) = \prod_{i<j}(q_i - q_j),$$  \hspace{1cm} (34)

$$\vec{x}_n[\vec{q}(t)] = \dot{b}_n = 0.$$  \hspace{1cm} (35)

The natural induced metric (giving (26)) is

$$g_{ij} = \partial_i \vec{x} \cdot \partial_j \vec{x} = (J^T J)_{ij},$$  \hspace{1cm} (36)

with

$$J^n_j = \frac{\partial x_n}{\partial q^j} = \frac{1}{(n-1)!} \sum_{\neq j} q^j \cdots q^{n-1},$$  \hspace{1cm} (37)

(now writing the $q$ coordinates with upper indices) and the goldfish-equations can therefore be described by the (geodesic flow) Hamiltonian

$$H[\vec{q}, \vec{\pi}] := \frac{1}{2} \sum_i g^{ij} \pi_i \pi_j,$$  \hspace{1cm} (38)

$$g^{ij} = (J^{-1}(J^T)^{-1})_{ij} = \frac{1 + q^j q^i + \cdots + (q^j q^i)^{N-1}}{\prod_k(q^k - q^j) \prod_l(q^j - q^l)},$$  \hspace{1cm} (39)

$$\dot{q}^i = g^{ik} \pi_k, \hspace{0.5cm} \dot{\pi}_k = -\frac{1}{2} \pi_i g^{ij} g^{k\ell} \pi_j$$  \hspace{1cm} (40)

giving (25)/(26).

To the best of our knowledge, this geodesic interpretation (and natural quadratic Hamiltonian structure) of the Ruijsenaars-Schneider model has not been observed before.

The inverse of the Jacobian is

$$(J^{-1})^t_m = (-)^{m-1} \frac{(q^j)^{N-m}}{\prod_{j \neq i}(q^i - q^j)},$$  \hspace{1cm} (41)

and the conserved quantities

$$B_n[\vec{q}, \vec{\pi}] := \frac{1}{(n-1)!} \sum_{i=1}^{\infty} q^i \cdots q^{i\cdots i^{n-1}} g^{i n j} \pi_j$$  \hspace{1cm} (42)

(all linear in the momenta) Poisson-commute, as the transformation $(\vec{x}, \vec{p}) \leftrightarrow (\vec{q}, \vec{\pi})$, with $p_n = \dot{x}_n = B_n$ trivially gives $\{p_m, p_n\} = 0$.

Hamiltonian relations between zeroes of polynomials and their coefficients have been considered before (cp. [8, 10]), but – to our surprise – apparently not for the original goldfish equation. We also became aware of chapter 27 in [9]¹, where (just as in [10]) the time-independence of the quantities (31) is

¹thanks to E. Langmann!
stated (and proved), but with the standard exponential (cp. (43)) Hamiltonian
structure, and not the geodesic structure (with respect to which the conserved
quantities Poisson-commute). Note that, rewriting [9] the canonical Poisson-
bracket, \( \{ q_i, \tilde{p}_j \} = \delta_{ij} \), in terms of the exponential variables suggested by the
standard RS Hamiltonian

\[
\sum_{i=1}^{N} e^{P_i} \prod_{j=1}^{N} (q_i - q_j)^{-1} =: \sum_{i=1}^{N} \tilde{\pi}_i = P, \tag{43}
\]

one has

\[
\{ q_i, q_j \} = 0, \quad \{ q_i, \tilde{p}_j \} = \delta_{ij} \tilde{\pi}_i, \quad \{ \tilde{\pi}_i, \tilde{\pi}_j \} = \frac{\tilde{\pi}_i \tilde{\pi}_j}{q_i - q_j} (1 - \delta_{ij}) \tag{44}
\]

which (interpreting them as Dirac-brackets, in the context of our Hamiltonian
reduction to \( H = \frac{1}{2} \) \((\sum p_i)^2\), with \( p_i \) playing the role of \( \tilde{\pi}_i \)) tells one how to
obtain the goldfish equations directly from the reduced Hamiltonian (14): using
(44), with \( \tilde{\pi}_i \) replaced by \( p_i \), and (43) by \( H = \frac{1}{2} P^2 \), gives (10).

Finally, let us make some remarks about the hyperbolic case: Starting, as
was done for the Calogero-Moser case in [11],

\[
\frac{d}{dt} \left( \dot{X} X^{-1} + X^{-1} \dot{X} \right) = 0, \tag{45}
\]

\[X\] a positive definite matrix – which we take to be real, with eigenvalues \( e^{2a \lambda_j(t)} \)
(a real),

\[
X(t) = R(t)e^{2a \Lambda(t)} R^{-1}(t), \tag{46}
\]

\( R^T = R^{-1} \), one obtains (with, as before, \( M := R^{-1} \dot{R} \))

\[
L_{ij} = \delta_{ij} \dot{\lambda}_i - \frac{\sinh (2a (\lambda_i - \lambda_j))}{2a} M_{ij}, \tag{47}
\]

\[
\dot{L} = [L, M] \tag{48}
\]

from (45), when defining \( L \) via

\[
R(t)L(t)R^{-1}(t) = \frac{1}{4a} \left( \dot{X} X^{-1} + X^{-1} \dot{X} \right). \tag{49}
\]

The solution of (45), on the other hand, can be written as

\[
X(t) = e^{a \Lambda_0} e^{2a V_0} e^{a \Lambda_0} \tag{50}
\]

(with \( \Lambda_0 = \Lambda(0) \) when choosing \( R(0) = 1 \)). The crucial point is that (48)
consistently reduces to

\[
\dot{\lambda}_i = 2 \sum_{j \neq i} \prime \frac{2a \dot{\lambda}_i \dot{\lambda}_j}{\sinh (2a (\lambda_i - \lambda_j))} \tag{51}
\]

when setting

\[
M_{ij} = -\frac{2a}{\sinh (2a (\lambda_i - \lambda_j))} \sqrt{\dot{\lambda}_i \dot{\lambda}_j} \tag{52}
\]
– corresponding to the initial conditions

\[ (V_0)_{ij} = a \frac{\sqrt{c_i c_j}}{\cosh a(a_i - a_j)}, \quad (53) \]

where \( a_i = \lambda_i(0), \ c_i = \hat{\lambda}_i(0) \) (geometrically speaking, the matrix geodesics (46) project down to geodesics in the space of positive diagonal matrices, if the initial conditions are chosen “perpendicular to the action of the rotation-group”). Except for their intrinsically geodesic interpretation (made explicit below) the equations (51), and variants thereof, are well-studied (see e.g. [10, 12]). In particular one can show, that the solutions of

\[ \ddot{q}_i = 2 \sum_j q_i \dot{q}_j \coth(q_i - q_j) \quad (54) \]

are the \( N \) roots of

\[ f(q) := \sum_{i=1}^{N} \frac{c_i \tanh(Pt)}{\tanh(q - a_i)} - 1 = 0, \quad (55) \]

resp. (see e.g. [9]) the eigenvalues of \( Z(t) = e^{2\Lambda_0} e^{2t L_0}, \) with \( P = \sum \dot{q}_i = \sum c_i, \)

\( (L_0)_{ij} = c_j \) (avoiding the somewhat artificial positivity restrictions in (53)). Rewriting (55) as

\[ \frac{d}{dt} \left( e^{-2tP} \frac{d}{dt} \det(Z(t) - e^{2q_1}) \right) = 0, \quad (56) \]

the crucial step then (not taken in [9]) is to note that the symmetric functions of the eigenvalues \( e^{2q_i(t)}, \)

\[ s_n(t) = \frac{1}{n!} \sum_{1 \leq i_1 < \ldots < i_n} e^{2(q_{i_1} + \ldots + q_{i_n})} \quad n = 1, \ldots, N, \quad (57) \]

in analogy with (33), evolve in time (almost uncoupled as \( 2P = \dot{s}_N/s_N = \text{const.} \)) as

\[ \ddot{s}_n - 2P \dot{s}_n = 0 \quad n = 1, \ldots, N. \quad (58) \]

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