Note On The Catalan Constant And Prime Triples

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Abstract

The existence of infinitely many consecutive prime triples \( p_n, p_{n+1}, \) and \( p_{n+2} \) as \( n \to \infty \), is sufficient to prove that the Catalan constant \( \beta(2) = 0.9159655941\ldots \) is an irrational number. This note provides the detailed analysis. Moreover, the numerical data suggests that the irrationality measure is \( \mu(\beta(2)) = 2 \), the same as almost every irrational real numbers.

1 Introduction and the Result

The Dirichlet beta function is defined by the series

\[
\beta(s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_{p \geq 3} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},
\]

where \( \chi(n) \) is the quadratic symbol, and \( s \in \mathbb{C} \) is a complex number. A beta constant \( \beta(s) \) at an odd integer argument \( s = 2n + 1 \) has an exact evaluation

\[
\beta(2n + 1) = \frac{n^{2n+1}E_n}{4^{n+1}(2n)!}
\]

in terms of the Euler numbers

\[
E_1 = 1, \quad E_2 = 5, \quad E_4 = 1385, \quad E_n, \ldots,
\]

for \( n \geq 1 \). This formula expresses each Dirichlet beta constant \( \beta(2n + 1) \) as a rational multiple of \( \pi^{2n+1} \), see [1] and related references. In contrast, the evaluation of a beta constant at an even integer argument can involves the zeta function and a power series, and other complicated formulas, [7], [8], et cetera. One of the simplest of these formulas is

\[
\beta(s) = \frac{3}{4} \zeta(s) - 2 \sum_{n \geq 1} \frac{1}{(4n + 3)^s},
\]

where \( \zeta(s) \) is the zeta function and \( s \geq 2 \). These expressions are summarized in a compact formula.

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Definition 1.1. Let $s \geq 2$ be an integer. The $\pi$-representation of the Dirichlet beta constant $\beta(s)$ is defined by the formula

$$
\beta(s) = \begin{cases} 
  r_n \pi^s & \text{if } s = 2n + 1, \\
  r_n \pi^s - u_n & \text{if } s = 2n,
\end{cases}
$$

(5)

where $r_n \in \mathbb{Q}$ is a rational number and $u_n \in \mathbb{R}$ is a real number.

The arithmetic nature of the first even constant, called the Catalan constant, is unknown. A proof based on a result for the existence of infinitely many consecutive prime triples

$$
p_n = 8n_1 + 1, \quad p_{n+1} = 8n_2 + 5, \quad \text{and} \quad p_{n+2} = 8n_3 + 7
$$

(6)

as the integers variables $n_1, n_2, n_3, n \to \infty$, and other results is given here.

Theorem 1.1. The Catalan constant

$$
\beta(2) = \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)^2} = 0.9159655941 \ldots,
$$

is an irrational number.

The essential basic foundation and background are presented in Section 2, and the simple proof of Theorem 1.1 is presented in Section 3.

Conjecture 1.1. The irrationality measure of the Catalan constant is $\mu(\beta(2)) = 2$.

The experimental data compiled in Section 5 confirms the prediction very accurately.

Last, but not least, it should be observed that the same proof seamlessly generalizes to all the beta constants $\beta(s)$, where $s \geq 2$ is an integer. For example, the same argument given in Section 3 provides a new proof of the irrationality of the number $\beta(3) = \prod_{p \geq 3} (1 - \chi(p)p^{-3})^{-1} = \pi^3/32$.

Now, recall that an elementary argument based on the irrationality of the number $\pi^2/6 = \prod_{p \geq 2} (1 - p^{-2})^{-1}$ implies the existence of infinitely many primes. Similarly, if the converse of the Brun irrationality criterion holds, see Theorem 2.1, then the known irrationality of the number $\pi^3/32 = \prod_{p \geq 3} (1 - \chi(p)p^{-3})^{-1}$ implies the existence of infinitely many prime triples, see (6). Assuming the consecutive prime triples are on three dependent arithmetic progressions, for example $n_1 = n_2 = n_3$ in (6), this is a stronger result than the twin primes conjecture. However, this is open problem closely related to the Bateman-Horn conjecture, see [5] and [6] for more information.

2 Foundation

The conditional result for the Catalan constant, and more generally, the Dirichlet beta constants, is based on a proven irrationality criterion, and the conjectured distribution of prime triples.
2.1 Irrationality Criterion

**Theorem 2.1.** (Brun irrationality criterion) Let \( x_n \geq 1 \) and \( y_n \geq 1 \) be a pair of monotonic increasing increasing integers sequences. Suppose that the following properties are true.

(i) \( \frac{y_n}{x_n} \) converges to a real number \( \alpha \neq 0 \) as \( n \to \infty \).

(ii) \( \frac{y_n}{x_n} < \frac{y_{n+1}}{x_{n+1}} \) is monotonically increasing as \( n \to \infty \).

(iii) \( \frac{y_{n+1} - y_n}{x_{n+1} - x_n} > \frac{y_{n+2} - y_{n+1}}{x_{n+2} - x_{n+1}} \) is monotonically decreasing as \( n \to \infty \).

Then, the number \( \alpha \) is irrational.

The details of the proof are discussed in [3], and the most recent application of this result is given in [2].

2.2 Sequences of Consecutive Primes

Let \( p_1, p_2, p_3, \ldots, p_n, \ldots \) be the sequence of primes in increasing order. Let \( q \geq 1 \) be an integer, and let \( a = (a_1, a_2, \ldots, a_k) \) be an \( k \)-tuple of congruences \( p_n \equiv a_n \mod q \), with \( \gcd(a_n, q) = 1 \) for \( n \in \{1, 2, \ldots, k\} \). Define the primes counting function

\[
\pi(x, q, a) = \#\{p \leq x : p_n \equiv a_n \mod q\}.
\]

A few results have been proved for the constant case

\[
a = (a_1 = a, a_2 = a, \ldots, a_k = a),
\]

where \( \gcd(a, q) = 1 \), see [4], [6], et cetera, for the most recent literature. The nonconstant case of (8) for \( k \) dependent arithmetic progressions appears to be a difficult problem. But, for \( k \) independent arithmetic progression, this problem is manageable.

**Theorem 2.2.** Let \( x \geq 1 \) and \( q \leq \log x \) be an integer. If \( a = (a_1, a_2, \ldots, a_k) \) is a congruence vector such that \( \gcd(a_n, q) = 1 \) for \( n = 1, 2, \ldots, k \), then,

\[
\pi(x, q, a) = \frac{\text{li}(x)}{\varphi(q)^k} \left(1 + O\left(\frac{1}{\log x}\right)\right).
\]

**Proof.** Let \( x \geq 1 \) be a large number, and let \( c \geq 0 \) be a constant. Fix a modulo \( q \ll (\log x)^c \), and an admissible \( k \)-tuple \( a_1, a_2, \ldots, a_k \) such that \( \gcd(q, a_n) = 1 \) for \( n \leq k \), take the cross product of \( k \) independent arithmetic progressions

\[
qn_1 + a_1, \quad qn_2 + a_2, \quad \ldots, \quad qn_k + a_k.
\]

By Dirichlet theorem (or Siegel-Walfisz theorem), the corresponding prime \( k \)-tuples has the natural density

\[
\delta(a) = \lim_{x \to \infty} \frac{\#\{p \leq x : p_n \equiv a_n \mod q\}}{x} = \frac{1}{\varphi(q)^k}.
\]

Each prime in the consecutive prime \( k \)-tuple \( p_n, p_{n+1}, \ldots, p_{n+k} \) is independently generated, but satisfies the specified congruence \( p_{n+i} \equiv a_{n+i} \mod q \). ■
As a new application, the sequence of prime triples
\[(97, 101, 103), \ (193, 197, 199), \ (457, 459, 463), \ldots \] defined in (6), is used here to develop an argument for the irrationality of the beta constant \(\beta(s)\).

## 3 An Irrationality Result

The simple argument in support of Theorem 1.1 is the following.

**Proof.** Theorem 1.1 Suppose that
\[p_n \equiv 1 \mod 8, \ \ p_{n+1} \equiv 5 \mod 8, \ \text{and} \ \ p_{n+2} \equiv 7 \mod 8. \] This hypothesis immediately implies that
\[\chi(p_n) = 1, \ \chi(p_{n+1}) = 1, \ \text{and} \ \chi(p_{n+2}) = -1. \] (13)

By Theorem 2.2, there are infinitely many consecutive prime triples (12) that satisfy the triple character values (13) as \(n \to \infty\). The rest of the proof verifies the three steps specified in Theorem 2.1 to prove the irrationality of the number \(G = \beta(2)\).

**Condition (i): Convergence Property.** Define the sequence of rational approximations
\[\frac{y_n}{x_n} = \prod_{k \leq n} \left(1 - \frac{\chi(p_k)}{p_k^2}\right)^{-1} = \prod_{k \leq n} \frac{p_k^2}{p_k^2 - \chi(p_k)}. \] (14)

Since \(\chi(p_n) = 1\), the sequence \(\{y_n/x_n : n \geq 1\}\) is composed of the two sequences of monotonically increasing integers
\[x_n = \prod_{k \leq n} (p_k^2 - \chi(p_k)), \ \text{and} \ \ y_n = \prod_{k \leq n} p_k^2. \] (15)

This shows that the sequence of rational approximations \(y_n/x_n\) converges to \(\beta(2) = 0.9159655941 \ldots\) as \(n \to \infty\), see (1). This step verifies Theorem 2.1-i.

**Condition (ii): Monotonically Increasing Ratios** \(y_n/x_n\). Utilizing the hypothesis (13), a pair of consecutive ratios have the forms
\[\frac{y_n}{x_n} = \prod_{k \leq n} \frac{p_k^2}{p_k^2 - \chi(p_k)} = \left(\frac{p_n^2}{p_n^2 - 1}\right) \prod_{k \leq n-1} \frac{p_k^2}{p_k^2 - 1}, \] (16)

and
\[\frac{y_{n+1}}{x_{n+1}} = \prod_{k \leq n+1} \frac{p_k^2}{p_k^2 - \chi(p_k)} = \left(\frac{p_{n+1}^2}{p_{n+1}^2 - 1}\right) \left(\frac{p_n^2}{p_n^2 - 1}\right) \prod_{k \leq n-1} \frac{p_k^2}{p_k^2 - 1}. \] (17)

Clearly, this is a monotonically increasing sequence:
\[\frac{y_n}{x_n} = \left(\frac{p_n^2}{p_n^2 - 1}\right) \prod_{k \leq n-1} \frac{p_k^2}{p_k^2 - 1} < \frac{y_{n+1}}{x_{n+1}} = \left(\frac{p_{n+1}^2}{p_{n+1}^2 - 1}\right) \left(\frac{p_n^2}{p_n^2 - 1}\right) \prod_{k \leq n-1} \frac{p_k^2}{p_k^2 - 1}. \] (18)
This step verifies Theorem 2.1-ii.

**Condition (iii): Monotonically Decreasing Slopes.** A pair of consecutive slopes are computed using the hypothesis (13).

The first ratio of shifted differences (slope, see [3, Figure 1] for a graphical description of this sequence) is

\[
\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{(p_{n+1}^2 - 1) y_n}{(p_{n+1}^2 - 1 - 1) x_n} = \frac{(p_{n+1}^2 - 1)}{(p_{n+1}^2 - 2)} x_n, \tag{19}
\]

where \(\chi(p_{n+1}) = 1\).

The next ratio of shifted differences is

\[
\frac{x_{n+2} - x_{n+1}}{y_{n+2} - y_{n+1}} = \frac{(p_{n+2}^2 p_{n+1}^2 - p_{n+1}^2)}{(p_{n+2}^2 + 1)(p_{n+1}^2 - 1) - (p_{n+1}^2 - 1)} \frac{y_n}{x_n} \tag{20}
\]

\[
= \frac{(p_{n+2}^2 - 1)p_{n+1}^2 y_n}{(p_{n+2}^2 + 1 - 1)(p_{n+1}^2 - 1)} x_n,
\]

where \(\chi(p_{n+2}) = -1\).

Comparing a pair of consecutive ratios yields

\[
\frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \frac{(p_{n+1}^2 - 1)}{(p_{n+1}^2 - 2)} x_n \tag{21}
\]

\[
> \frac{x_{n+2} - x_{n+1}}{y_{n+2} - y_{n+1}} = \frac{(p_{n+2}^2 - 1)p_{n+1}^2}{p_{n+2}^2(p_{n+1}^2 - 1)} x_n.
\]

Equivalently,

\[
\frac{(p_{n+1}^2 - 1)}{(p_{n+1}^2 - 2)} > \frac{(p_{n+2}^2 - 1)p_{n+1}^2}{p_{n+2}^2(p_{n+1}^2 - 1)} \tag{22}
\]

Expanding and simplifying it return

\[
p_{n+1}^4 > p_{n+1}^2. \tag{23}
\]

Therefore, the slope (19) is a strictly monotonically decreasing sequence. This step verifies Theorem 2.1-iii.

Therefore, since all the conditions of Theorem 2.1 are satisfied, the number \(\beta(2)\) is irrational.

**Remark 3.1.** As stated in Section 2, there is a proof for the existence of infinitely many consecutive prime triples of constant congruences \(p_n \equiv p_{n+1} \equiv p_{n+2} \equiv 1 \mod 4\) and \(p_n \equiv p_{n+1} \equiv p_{n+2} \equiv -1 \mod 4\) on arithmetic progressions of a single variable. However, the same argument fails for any of these sequences of consecutive prime triples. For example, the same analysis as above using the sequence

\[
p_n \equiv 1 \mod 16, \quad p_{n+1} \equiv 5 \mod 16, \quad \text{and} \quad p_{n+2} \equiv 9 \mod 16 \tag{24}
\]
of constant quadratic symbol
\[ \chi(p_n) = 1, \quad \chi(p_{n+1}) = 1, \quad \text{and} \quad \chi(p_{n+2}) = 1, \]
(25)
or using the sequence
\[ p_n \equiv 3 \mod 16, \quad p_{n+1} \equiv 7 \mod 16, \quad \text{and} \quad p_{n+2} \equiv 11 \mod 16, \]
(26)
of constant quadratic symbol
\[ \chi(p_n) = -1, \quad \chi(p_{n+1}) = -1, \quad \text{and} \quad \chi(p_{n+2}) = -1, \]
(27)
fails to prove that \( \beta(2) \) is irrational.

4 Basic Diophantine Approximations

The basic results recorded below are standard results in the literature, see [9], [10], et alii.

4.1 Basic Continued Fractions

**Lemma 4.1.** Let \( \alpha = [a_0, a_1, \ldots, a_n, \ldots] \) be the continue fraction of the real number \( \alpha \in \mathbb{R} \). Then, the following properties hold.

(i) \( p_n = a_np_{n-1} + p_{n-2}, \) \( p_{-2} = 0, \) \( p_{-1} = 1, \) \( \text{for all } n \geq 0. \)
(ii) \( q_n = a_nq_{n-1} + q_{n-2}, \) \( q_{-2} = 1, \) \( q_{-1} = 0, \) \( \text{for all } n \geq 0. \)
(iii) \( p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}, \) \( \text{for all } n \geq 1. \)
(iv) \( \frac{p_n}{q_n} = a_0 + \sum_{0 \leq k < n} \frac{(-1)^k}{q_kq_{k+1}}, \) \( \text{for all } n \geq 1. \)

4.2 The Irrationality Measure

The irrationality measure measures the quality of the rational approximation of an irrational number. It is lower bound of all the rational approximations. Specifically,
\[ \left| \alpha - \frac{p}{q} \right| \geq \frac{1}{q^\mu} \]
(28)
for all sufficiently large \( q \geq 1. \)
5 Numerical Data

The data to support the claim in Conjecture 1.1 is compiled in this section. The continued fraction of the Catalan constant

\[
\beta(2) = 0.91596559417721901505460351493238411077414937428167\ldots,
\]

is of the form, (first 100 partial quotients \(a_n\)),

\[
\beta(2) = [0; 1, 10, 1, 8, 1, 88, 4, 1, 1, 7, 22, 1, 2, 3, 26, 1, 11, 1, 10, 1, 9, 3, 1, 1, 1, 1, 1, 1, 6, 1, 12, 1, 4, 7, 1, 1, 2, 5, 1, 5, 9, 1, 1, 1, 1, 33, 4, 1, 1, 3, 5, 3, 2, 1, 2, 1, 2, 1, 7, 6, 3, 1, 3, 3, 1, 1, 2, 1, 14, 1, 4, 4, 1, 2, 4, 1, 17, 4, 1, 14, 1, 1, 12, 1, 1, 1, 1, 3, 1, 2, 3, 1, 6, 2, 1, 2, 2, 322, 1, 1, 1, 2, 1, 108, 3, 1, 2, 82, 1, 5, 4, 1, 2, 2, 1, 1, 1, 5, 1, 12, 2, 11, 8, 2, 17, 1, 11, 1, 6, 1, 18, 1, 5, 2, 24, 4, 1, 1, 1, 8, 4, 3, 8, 3, \ldots].
\]

These are archived as sequence A006752 and sequence A014538, respectively, on the OEIS.

The sequence of convergents \(\{p_n/q_n : n \geq 1\}\), listed in Table 1, is computed via the recursive formula provided in the Lemma 4.1.

An approximation \(\mu_n(\alpha)\) of the irrationality measure satisfies the inequality

\[
\left| \alpha - \frac{p_n}{q_n} \right| \geq \frac{1}{q^{\mu_n(\alpha)}}
\]

for \(n \geq 2\). The values of the approximate irrationality measure \(\mu_n(\alpha) \geq 2\) of the irrational number \(\alpha \neq 0\) is defined by

\[
\mu_n(\alpha) = -\frac{\log |\alpha - p_n/q_n|}{\log q_n},
\]

where \(n \geq 2\).

**Example 5.1.** A large convergent is used here to illustrate the calculations, using 50 digits accuracy in the computer algebra system SAGE. The 100th convergent \(p_{100}/q_{100}\) is given by

(a) \(p_{100} = 24078868662746347429760476964387436156348637833\),

(b) \(q_{100} = 26287961923259336649196821919541159881600485419\).

The corresponding 100th approximation of the irrationality measure is

\[
\mu_{100}(\beta(2)) = -\frac{\log |\beta(2) - p_{100}/q_{100}|}{\log q_{100}}
\]

\[
= 2.009837567910985080940738967354842545238309309668.
\]

The range of values for \(n \leq 45\) is plotted in Figure 1.
Figure 1: Approximate Irrationality Measure $\mu_n(\beta(2))$ Of The Number $\beta(2)$.

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Table 1: Numerical Data For The Exponent $\mu(\beta(2))$ Of The Number $\beta(2)$

| $n$ | $p_n$ | $q_n$ | $\mu_n(\beta(2))$ |
|-----|-------|-------|--------------------|
| 1   | 1     | 1     | 2.07678354372      |
| 2   | 10    | 11    | 2.92280567895      |
| 3   | 11    | 12    | 2.02491750642      |
| 4   | 98    | 107   | 2.93949885584      |
| 5   | 109   | 119   | 2.16337045430      |
| 6   | 9690  | 10579 | 2.0707391493       |
| 7   | 38869 | 42435 | 2.06103252758      |
| 8   | 48559 | 53014 | 2.06103252758      |
| 9   | 87428 | 95449 | 2.1768724408       |
| 10  | 66055 | 72115 | 2.23191137105      |
| 11  | 1461963 | 1596090 | 2.02341311597 |
| 12  | 15280193 | 16682060 | 2.07153881185 |
| 13  | 45180024 | 49325023 | 2.06867467056 |
| 14  | 150820265 | 164657129 | 2.11498233047 |
| 15  | 3966506914 | 4330410377 | 2.00518561271 |
| 16  | 4117327179 | 4495067506 | 2.11498233047 |
| 17  | 49257105883 | 5376152943 | 2.00653757661 |
| 18  | 5374433062 | 58271220449 | 2.0444250285 |
| 19  | 583001436503 | 636488357433 | 2.0066821796 |
| 20  | 63675869565 | 694759577882 | 2.08514590463 |
| 21  | 6310384262588 | 6889324558371 | 2.0444250285 |
| 22  | 1956728567329 | 2136273252995 | 2.0217482851 |
| 23  | 2587912919917 | 2825205781366 | 2.0275664628 |
| 24  | 454454141577246 | 49614791064361 | 2.0249903825 |
| 25  | 7132354497163 | 7786884875727 | 2.0250051325 |
| 26  | 116768796074409 | 127481639940088 | 2.0257968126 |
| 27  | 188092150571572 | 20534888815815 | 2.0217482851 |
| 28  | 304860946465981 | 332830128755903 | 2.0324463682 |
| 29  | 797814043863534 | 871008746327621 | 2.0347350242 |
| 30  | 190048903437049 | 207484762141145 | 2.0116024732 |
| 31  | 2698303078236583 | 294585636738766 | 2.0705476506 |
| 32  | 3158182289475462 | 3447926766537571 | 2.01265439798 |
| 33  | 34280125973212045 | 3742512403276337 | 2.0263563878 |
| 34  | 65861948868187507 | 71904391700813908 | 2.0131249833 |
| 35  | 100142074841399552 | 109329515753590245 | 2.05164524679 |
| 36  | 666714397916584819 | 72788148611135378 | 2.0049959412 |
| 37  | 766856472757984371 | 837211001846445623 | 2.0633797686 |
| 38  | 9868992071012397271 | 10774413508268702854 | 2.00633061985 |
| 39  | 10635845843770381642 | 1161162451011514877 | 2.0369255654 |
| 40  | 52412386246093923839 | 57220911548729296762 | 2.0451224382 |
| 41  | 377522552266427848515 | 412158005351220225811 | 2.012651795 |
| 42  | 429934935812521772354 | 46937891689994522573 | 2.0178488368 |
| 43  | 807457490778949620869 | 88153692251169748384 | 2.0206299817 |
| 44  | 2044849920070421014092 | 2232452761402289019341 | 2.0372180851 |
| 45  | 11031707091131054691329 | 1204380072926261485089 | 2.00635311938 |