Abstract. We characterize genera of even unimodular \( \mathbb{Z} \)-lattices admitting an isometry of a given odd prime order. If the lattice is moreover indefinite, this leads to a classification of conjugacy classes of prime order isometries. This is applied to automorphisms of irreducible holomorphic symplectic manifolds.

1. Introduction

Let \( l_+, l_- \) be non-negative integers and \( L \) an even unimodular \( \mathbb{Z} \)-lattice of signature \((l_+, l_-)\). Such a lattice exists if and only if \( l_+ \equiv l_- \pmod{8} \). If \( l_+ \) and \( l_- \) are both non-zero, then the lattice is unique up to isometry. We treat the following problem: does \( L \) admit an isometry of odd prime order \( p \)? If yes, classify their conjugacy classes.

In order to state our main results, we define some invariants of a prime order isometry \( f \) in the orthogonal group \( O(L) \). The fixed and cofixed lattices (of \( f \)) are defined as

\[
L^f = \{ x \in L \mid f(x) = x \} \quad \text{and} \quad L_f = (L^f)^\perp.
\]

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Theorem 1.1. Let $p$ be an odd prime number. There exists an even unimodular lattice $L$ of signature $(l_+, l_-)$ admitting an isometry $f \in O(L)$ of order $p$ if and only if there exist $n, m, s_+, s_- \in \mathbb{Z}_{\geq 0}$ such that

1. $l_+ \equiv l_- \pmod{8}$
2. $s_+ + s_- = (2m + n)(p - 1) > 0$;
3. $s_+, s_- \in 2\mathbb{Z}$;
4. if $n = 0$, then $s_+ \equiv s_- \pmod{8}$;
5. $s_+ \leq l_+$, $s_- \leq l_-;
6. s_+ + s_- + n \leq l_+ + l_-;
7. if equality in (6) holds, then $s_+ - s_- \equiv (p - 1)(l_+ - l_- + s_+ + s_-) \pmod{8}$.

If $f$ exists, then $L^f$ and $L_f$ are $p$-elementary of determinant $p^n$ and the signature of $L_f$ equals $(s_+, s_-)$.

The invariants of the theorem determine the genus of the fixed and cofixed lattice of $f$. But in order to distinguish conjugacy classes, we now define additional invariants. Let $\zeta_p$ be a primitive $p$-th root of unity. The cofixed lattice $L_f$ has the structure of a $\mathbb{Z}[\zeta_p]$-module via $\zeta_p \cdot x = f(x)$. Its Steinitz class $I_f$ is given by the top exterior product $\bigwedge^{rk_{\mathbb{R}}(L_f)} L_f$ in the class group of $\mathbb{Q}(\zeta_p)$. It measures the deviation of $L_f$ from being a free $\mathbb{Z}[\zeta_p]$-module. Finally, the signatures of $(L, f)$ are given by the signatures $(k^+_i, k^-_i) \in (2\mathbb{N})^2$ of the real quadratic spaces

$K_i = \ker(f^i_R + f^{-i}_R - \zeta_p^i + \zeta_p^{-i})$, $i \in \{1, \ldots, (p - 1)/2\}$.

They satisfy $(s_+, s_-) = \sum_{i=0}^{(p-1)/2} (k^+_i, k^-_i)$.

Theorem 1.2. Let $L$ be an even unimodular lattice, $p$ an odd prime number, $E = \mathbb{Q}[\zeta_p]$ and $K = \mathbb{Q}[\zeta_p + \zeta_p^{-1}]$. Then conjugacy classes of isometries $f \in O(L)$ of order $p$ with $L_f$ indefinite are determined by the following complete set of 3 invariants

1. the isometry class of the fixed lattice $L^f$,
2. the Steinitz class of $L_f$ in the relative class group $C(E)/C(K)$,
3. the signatures of $(L, f)$.

Remark 1.3. In the positive definite case the classification is typically of algorithmic nature. This has been carried out for instance in [28, 18] where the authors classify extremal unimodular lattices of rank $48$ admitting certain prime order isometries.

Remark 1.4. The analogous results for $p = 2$ are easily derived from [29] Thms. 3.6.2, 3.6.3 due to Nikulin. They boil down to a classification of primitive $2$-elementary sublattices of even unimodular lattices up to the action of the orthogonal group. Combining Proposition 2.10 and [29, §16], one can derive analogous statements for odd unimodular lattices.

In Section 3 we present some geometric applications of these classification results, concerning the study of automorphisms of odd prime order of K3 surfaces and, more generally, of irreducible holomorphic symplectic (IHS) manifolds. The cohomology group $H^2(X, \mathbb{Z})$ of an IHS manifold $X$ admits an integral $\mathbb{Z}$-lattice structure (by use of the Beauville–Bogomolov–Fujiki quadratic form). The global Torelli theorem explains how to recover automorphisms of $X$ from their pull-back action on this
lattice, which is an isometry. We consider non-symplectic automorphisms (i.e. automorphisms whose action on $H^{2,0}(X)$ is not trivial), since their cofixed lattice inside $H^2(X,\mathbb{Z})$ is indefinite. In the case of K3 surfaces, whose second cohomology lattice is unimodular, Theorems 1.1 and 1.2 can be combined to give the following classification.

**Theorem 1.5.** Let $p$ be an odd prime number. There exists a K3 surface $S$ and a non-symplectic automorphism $\sigma \in \text{Aut}(S)$ of order $p$ if and only if $p \leq 19$ and there are non-negative integers $n, m$ such that

1. $(n + 2m)(p - 1) \leq \min\{21, 22 - n\}$;
2. if $n = 0$, then $m(p - 1) \equiv 2 \pmod{4}$;
3. if $(n + 2m)(p - 1) = 22 - n$, then $(3 + m)(p - 1) \equiv 2 \pmod{4}$.

Moreover, up to the choice of a marking $\eta : H^2(S,\mathbb{Z}) \to L := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, the action of the group $\langle \sigma \rangle$ on $L$ is determined by the triple $(p, n, m)$.

**Remark 1.6.** A study of non-symplectic automorphisms of odd prime order on K3 surfaces was already conducted in [1] and [2], where all possible isometry classes for the fixed and cofixed lattices inside the second cohomology lattice are listed and related to the topology of the fixed locus of the automorphism.

In higher dimensions, we consider IHS manifolds of $K3^{[n]}$-type (i.e. deformation equivalent to Hilbert schemes of $n$ points on a K3 surface) and of $\text{Kum}_{\sigma}$-type (i.e. deformation equivalent to the $2n$-dimensional generalized Kummer variety of an abelian surface). Here we do not obtain a classification as simple as the one for K3 surfaces. The reason is twofold. Firstly, the second cohomology lattice is no longer unimodular. Secondly, instead of the orthogonal group one has to consider the monodromy group for geometric purposes. In Theorem 3.7 we show that conjugacy classes of non-symplectic monodromies of odd prime order are determined by conjugacy classes of isometries (of the same order) of a suitable unimodular lattice $M \supseteq H^2(X,\mathbb{Z})$ together with the class of a primitive embedding of a “Mukai-Vector” in the fixed lattice. Moreover, we obtain the following generalization of the classification results of [8] and [26].

**Theorem 1.7.** Let $n \geq 2$ and $S$ be a primitive sublattice of $L = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n - 1) \rangle$ (respectively, $L = U^{\oplus 3} \oplus \langle -2(n + 1) \rangle$). Then there exist an IHS manifold $X$ of $K3^{[n]}$-type (respectively, of $\text{Kum}_{\sigma}$-type), a marking $\eta : H^2(X,\mathbb{Z}) \to L$ and a non-symplectic automorphism $\sigma \in \text{Aut}(X)$ of odd prime order $p$ such that $\eta(H^2(X,\mathbb{Z})_{\sigma}) = S$ and $\eta(H^2(X,\mathbb{Z})_{\sigma^*}) = S^\perp$ if and only if $S$ is $p$-elementary of determinant $p^n$ and signature $(2, (n + 2m)(p - 1) - 2)$ for some non-negative integer $m$.

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2. Hermitian lattices over cyclotomic fields

2.1. $\mathbb{Z}$-lattices. A $\mathbb{Z}$-lattice, or simply lattice, $(L, b)$ consists of a free $\mathbb{Z}$-module $L$ equipped with a non-degenerate symmetric bilinear form $b: L \times L \to \mathbb{Q}$. The lattice is called integral if $b(L \times L) \subseteq \mathbb{Z}$ and even if $b(x, x) \in 2\mathbb{Z}$ for all $x \in L$. If $b$ is understood, we abbreviate $(L, b)$ to $L$. Let $(L', b')$ be a further lattice. A homomorphism $f: L \to L'$ of $\mathbb{Z}$-modules is called an isometry if it satisfies $b(x, y) = b'(f(x), f(y))$ for all $x, y \in L$. We call $f$ a homomorphism of lattices. The orthogonal group $O(L, b)$ consists of the automorphisms of $(L, b)$. We assume that the reader is familiar with the standard terminology concerning lattices. We refer to [29, 10, 20] for further details and definitions.

2.2. Hermitian lattices.

**Definition 2.1.** Let $K$ be a field of characteristic zero and let $a \in K$. Set $E = K[x]/(x^2 - a)$ and denote by $\sigma: E \to E, [x] \mapsto [-x]$ the canonical involution. A hermitian space $(V, h)$ over $(E, \sigma)$ is a finitely generated free $E$-module $V$ equipped with a non-degenerate $K$-bilinear form

$$h: V \times V \to E$$

which is $E$-linear in the first argument and satisfies $h(x, y) = h(y, x)^\sigma$ for all $x, y \in V$. Let $\mathbb{Z}_E$ be the maximal order of $E$. A hermitian $\mathbb{Z}_E$-lattice $(L, h)$ consists of a finitely generated $\mathbb{Z}_E$-module $L \subseteq V$ of full rank, equipped with the hermitian form $h$.

If $E$ and $h$ are understood, we drop them from notation and simply speak of a hermitian lattice $L$. For $a_1, \ldots, a_n \in K$ we denote by $\langle a_1, \cdots, a_n \rangle$ a free $\mathbb{Z}_E$-module spanned by $e_1, \ldots, e_n$ equipped with the hermitian form defined by $h(e_i, e_j) = \delta_{ij}a_i$.

**Remark 2.2.** More generally one can define hermitian lattices over orders in algebras with involution. If we assume the algebra to be étale, then the trace construction (see below) works as well. But in this generality not too much is known.

**Notation for cyclotomic fields.** Let $p$ be an odd prime and $\zeta$ a fixed primitive $p$-th root of unity. See [33, Ch. 2] for details.

- $\Phi_p(x)$ is the $p$-th cyclotomic polynomial.
- $E = \mathbb{Q}[\zeta]$ and $K = \mathbb{Q}[\zeta + \zeta^{-1}]$.
- $C(E), C(K)$ are the respective class groups.
- $C(E/K) = C(E)/C(K)$ the relative class group.
- $\# C(E/K) = \# C(E)/\# C(K)$ is the relative class number.
- $\zeta^\sigma = \zeta^{-1}$, i.e. the involution $\sigma$ is complex conjugation.
- $\mathbb{Z}_E = \mathbb{Z}[\zeta]$ and $\mathbb{Z}_K = \mathbb{Z}[\zeta + \zeta^{-1}]$.
- $\pi = (1 - \zeta)$.
- $p = \pi \pi^\sigma \mathbb{Z}_K$.
- $\mathfrak{P} = \pi \mathbb{Z}_E$.
- $\mathfrak{N} = \mathfrak{D}_E^E = \pi^{n-2} \mathbb{Z}_E$ is the absolute different.
- $\mathfrak{R} = \mathfrak{D}_E^K = \pi \mathbb{Z}_E$ is the relative different.
- $N = N_E^K$ and $T = \text{Tr}^E_K$ are the relative norm and trace.
- $\Omega(K)$ is the set of places of $K$.
- $\mathfrak{P}(K)$ is the set of prime ideals of $K$.

**Notation concerning hermitian lattices.**
2.3. The trace lattice. If \((L, h)\) is a hermitian \(\mathbb{Z}[\zeta]\)-lattice, then \((L, b)\), where \(b = \text{Tr}_Q^E \circ h\), is a \(\mathbb{Z}\)-lattice. It is called the trace lattice. Multiplication by \(\zeta\) induces an isometry \(f\) of \((L, b)\) with minimal polynomial \(\Phi_p(x)\). Conversely, if \((L, b)\) is a lattice and \(f\) an isometry with minimal polynomial \(\Phi_p(x)\), then \(\zeta \cdot x = f(x)\) defines a \(\mathbb{Z}_E\)-module structure on \(L\) and

\[
h(x, y) = \sum_{i=0}^{p-1} b(x, f^i(y))\zeta^i \in E
\]
defines a hermitian form. We have thus shown the following well known proposition. Note that it involved the choice of a fixed root of unity \(\zeta\).

**Proposition 2.3.** The trace construction sets up an equivalence of the category of hermitian \(\mathbb{Z}_E\)-lattices \((L, h)\) and the category consisting of pairs \(((L, b), f)\) where \(f \in O(L, b)\) is an isometry of minimal polynomial \(\Phi_p(x)\) and morphisms are \(f\)-equivariant isometries.

We denote by

\[
L^\vee = (L, b)^\vee = \{ x \in L \otimes \mathbb{Q} \mid b(x, L) \subseteq \mathbb{Z} \}
\]
and

\[
L^\# = (L, h)^\# = \{ x \in L \otimes \mathbb{Q} \mid h(x, L) \subseteq \mathbb{Z}_E \}
\]
the respective dual lattices. They satisfy the following relation

\[
(L, b)^\vee = \mathfrak{A}^{-1}(L, h)^\# .
\]

If \(\mathfrak{B}L^\# = L\) for some fractional ideal \(\mathfrak{B}\) of \(E\), we call \((L, h)\) \(\mathfrak{B}\)-modular. A \(\mathbb{Z}_E\)-modular hermitian lattice is also called unimodular.

**Lemma 2.4.** Let \((L, h)\) be a hermitian lattice over a prime cyclotomic field such that its trace lattice \((L, b)\) is integral. Then \((L, b)\) is even.

**Proof.** That \((L, b)\) is integral is equivalent to \(\mathfrak{s}(L) \subseteq \mathfrak{A}\). Let \(x \in L\). Then \(h(x, x) \in \mathfrak{n}(L) \subseteq K \cap \mathfrak{s}(L) \subseteq K \cap \mathfrak{A} = \mathcal{D}_K^K\). Thus \(\text{Tr}_Q^E(h(x, x)) = 2 \text{Tr}_Q^K(h(x, x)) \in 2\mathbb{Z}\). □

**Example 2.5.** The root lattice \(A_{p-1}\) admits a fixed point free isometry of order \(p\). To see this, consider the extended Dynkin diagram \(\widetilde{A}_{p-1}\). It is a regular polygon with \(p\) vertices. It has a rotational symmetry \(f\) of order \(p\) which fixes the sum of the \(p\) vertices. This sum is precisely the kernel of \(\widetilde{A}_{p-1}\). Thus \(f\) descends to a fixed point free isometry of the quotient. But the quotient is isomorphic to \(A_{p-1}\), therefore it can be seen as a hermitian \(\mathbb{Z}[\zeta]\) lattice of rank one.

2.4. Genera of hermitian lattices. The classification of hermitian lattices largely parallels that of \(\mathbb{Z}\)-lattices. In this section we recall the parts of the classification needed for our purposes.

**Definition 2.6.** Two hermitian lattices \(L\) and \(L'\) are said to be in the same genus if the completions \(L_{\nu}\) and \(L'_{\nu}\) are isomorphic for all \(\nu \in \Omega(K)\).
Let $\nu \in \Omega(K)$ be a place of $K$. The place $\nu$ is called good, if $E_\nu$ is either isomorphic to $K_\nu \times K_\nu$ or $E_\nu/K_\nu$ is an unramified field extension of degree 2. Otherwise we call $\nu$ bad, and then $E_\nu/K_\nu$ is a ramified extension of degree 2. A hermitian lattice over $E_\nu$ can be decomposed as an orthogonal direct sum of modular hermitian lattices. In the sequel we recall the classification of modular hermitian lattices [15]. A unimodular hermitian lattice over $E_\nu$ with $\nu$ a good prime is unique up to isomorphism (cf. [16] Prop. 3.3.5)). In our case the only bad prime is $\pi$.

**Proposition 2.7.** [16] Prop. 3.3.5] Let $\pi$ be a bad prime which is coprime to 2. Let $L$ be a $\pi^i$-modular lattice of rank $r$ over $E_\pi$. If $i$ is even, then

$$L \cong (\langle \pi^i \rangle)^i, \det L(\pi^{2i})^{(1-r)/2}.$$ If $i$ is odd, then

$$L \cong H_i^{\oplus r/2} = \begin{pmatrix} 0 & \pi^i \\ \pi^i & 0 \end{pmatrix}^{\oplus r/2}.$$ Let $v_1, \ldots, v_s$ be the real places of $K$. Denote by $n_i$ the number of negative entries in a diagonal Gram matrix of $(L_{v_i}, h_{v_i})$. They are called the signatures of $(L, h)$. Let $(s_+ , s_-)$ be the signature of the trace lattice $(L, b)$ and $k_+^\pm$ the signatures of $(L, b, f)$ as defined in the introduction. For a place $q$ of $\mathbb{Q}$, we obtain the orthogonal splitting

$$(L, h) \otimes \mathbb{Z}_q \cong \bigoplus_{v|q} (L, h)_v.$$ For $q = -1$ the infinite place, we obtain the orthogonal splitting of $\mathbb{H}$. By carrying out the trace construction for a hermitian lattice over $\mathbb{C}/\mathbb{R}$ of rank one we get that $k_1^- = 2n_i$ and thus

$$s_- = 2 \sum_{i=0}^s n_i.$$ The genus of a hermitian lattice $(L, h)$ is uniquely determined by its modular decompositions at all primes dividing its determinant and its signatures $n_i$.

**Proposition 2.8.** [16] 3.4.2 (3) and 3.5.6] Given hermitian lattices $(L_{\nu}, h_{\nu})$ at each place $\nu \in \Omega(K)$, all but finitely many of which are unimodular, there is a global hermitian lattice $(L, h)$ with $(L, h)_\nu \cong (L_{\nu}, h_{\nu})$ at all places if and only if the set $S = \{\nu \in \Omega(K) \mid \det(L_{\nu}, h_{\nu}) \notin N(E^*)\}$ is of even cardinality.

**Lemma 2.9.** Let $E = \mathbb{Q}([\zeta]$ and $K = \mathbb{Q}([\zeta + \zeta^{-1}]$ where $\zeta$ is a primitive $p$-th root of unity and $p$ an odd prime. Let $(L, h)$ be a hermitian lattice over $E/K$. Suppose that $(L, h)$ is of rank one or indefinite. Then the number of classes in the genus of $(L, h)$ is the relative class number $c(E)/c(K)$. In particular, two classes are isomorphic if and only they have the same Steinitz invariant in $C(E/K)$.

**Proof.** If $(L, h)$ is a hermitian $\mathbb{Z}_E$-lattice of rank one, see [3] Prop. 9 (ii) and p.176]. If $(L, h)$ is of rank two, we need to introduce notation concerning determinants of hermitian $\mathbb{Z}_E$-lattices before we can proceed. Let $q \in \mathbb{P}(K)$ be a prime. We define:

- $\mathcal{E}_0 = \{u \in \mathbb{Z}^*_E \mid uu^* = 1\}$,
- $\mathcal{E}_1 = \{u/u^* \mid u \in \mathbb{Z}^*_E\} \subseteq \mathcal{E}_0$,
- $\mathcal{E}(L_q) = \{\det(g) \mid g \in O(L_q, h_q)\} \subseteq \mathcal{E}_0^3$. 


Thus \( P(L) = \{ q \in \mathcal{P}(E) \mid \mathcal{E}(L_q) \neq \mathcal{E}_0 \} \) consists only of primes ramified in \( E/K \),
\[
\mathcal{E}(L) = \prod_{q \in \mathcal{P}(L)} \mathcal{E}_0^{\mathcal{E}(L_q)},
\]
\[ R(L) = \{ (e \mathcal{E}(L_q))_{q \in \mathcal{P}(L)} \in \mathcal{E}(L) \mid e \in \mathbb{Z}_E \text{ and } ee^\# = 1 \}, \]
\[ C = C(E) \text{ the class group of } E, \]
\[ C_0 = \{ [\mathfrak{a}] \in C \mid \mathfrak{a} = \mathfrak{a}^* \} \text{ is the subgroup of } C \text{ generated by the image of } C(K), \]
and the prime ideals of \( \mathbb{Z}_E \) ramified in \( E/K \).

The number of special genera in the genus is \( [C : C_0][\mathcal{E}(L) : R(L)] \) (cf. [17] Lem. 4.6). Since the genus is indefinite, each special genus consists of a single isometry class and the number above is actually the number of classes in the genus. Note that \( p \) is the unique ramified prime. Hence, \( \mathcal{E}(L) = \mathcal{E}_0^p/\mathcal{E}(L_p) \). By [17] Thm. 3.7, \( \mathcal{E}_1^p \subseteq \mathcal{E}(L) \), and further the quotient \( \mathcal{E}_0^p/\mathcal{E}_1^p \) is of order 2 and generated by \(-1 \cdot \mathcal{E}_1^p\) (cf. [17] Lem. 3.5). Since \((-1 \cdot \mathcal{E}_1^p)\) is clearly in \( R(L) \), the index \( [\mathcal{E}(L) : R(L)] = 1 \). It remains to compute the index \([C : C_0]\). Now, \( C_0 \) is the subgroup of \( C \) generated by the image of the class group \( C(K) \) of \( K \) and the prime ideals \( \mathbb{Z}_E \) ramified in \( E/K \). In our case this is the principal ideal \( \mathfrak{p} = (1 - \zeta) \). By [33] Thm. 4.14] the natural homomorphism \( C(K) \to C(E) \) is injective. This yields that \( #(C/C_0) = #C(E)/#C(K) \) is the relative class number. \( \square \)

2.5. Fixed point free isometries of prime order.

Proposition 2.10. Let \( p \) be an odd prime number and \( (L, b) \) an integral lattice of signature \((s_+, s_-)\). Then some lattice in the genus of \( (L, b) \) admits a fixed point free isometry \( f \) of order \( p \) acting trivially on the discriminant group, if and only if there are non-negative integers \( n, m \in \mathbb{Z} \) such that

(i) \( L \) is even, \( p \)-elementary of determinant \( p^n \),

(ii) \( s_+ + s_- = (n + 2m)(p - 1) \),

(iii) \( s_+ \in 2\mathbb{Z} \).

Conversely, every combination of integers \( s_+, s_- \), \( n, m \in \mathbb{Z}_{\geq 0} \) satisfying (ii), (iii) and

(iv) \( n = 0, \text{ then } 2s_- \equiv 2m(p - 1) \pmod{8}; \)

yields such a triple \((L, b, f)\) with \( L \) satisfying (i).

Suppose that \((L, b)\) is unique in its genus. Then a complete set of invariants of the conjugacy classes of fixed point free isometries \( f \in O(L, b) \) of order \( p \) acting trivially on the discriminant group is given by the signatures and the Steinitz class of \((L, b, f)\) seen in the relative class group \( C(E/K) \).

Proof. (i) Since \( f = \zeta \) acts as identity on the discriminant group and \((L, b)\) is integral, we have
\[
1 - \zeta)(L, b)^\vee \subseteq (L, b) \subseteq (L, b)^\vee.
\]
Thus \((L, b)^\vee/(L, b)\) is isomorphic to \((\mathbb{Z}[\zeta]/(1 - \zeta))^n\) for some \( n \leq \dim_E L \otimes \mathbb{Q} \). Since \( \mathbb{Z}[\zeta]/(1 - \zeta) \cong \mathbb{F}_p \) as abelian groups, we see that \((L, b)\) is \( p \)-elementary. By Lemma 2.4, it is moreover even.

(ii) Recall that \( \pi = (1 - \zeta) \). Using equation \( (5) \) we translate \( (\mathfrak{p}) \) to
\[
\pi^{3-p}(L, h)^\# \subseteq (L, h) \subseteq \pi^{2-p}(L, h)^\#.
\]
Thus \((L, h)\) is unimodular at all primes \( \mathfrak{q} \) of \( \mathcal{O}_K \) except \( \mathfrak{p} = \pi\mathcal{O}_K \). Since the primes \( \mathfrak{q} \) are good, a unimodular hermitian lattice over \( \mathcal{O}_K \) is uniquely determined by its rank. By \( \mathfrak{q} \) the modular decomposition of \( L \) at \( \mathfrak{p} \) may only have \( \pi^i \) modular
blocks for $2 - p \leq i \leq 3 - p$. From the classification of $\pi^i$-modular hermitian lattices in Proposition 2.7, we extract that

$$(L, h)_p \cong M \oplus H_{2-p}^\oplus$$

where $M$ is $\pi^{3-p}$-modular of rank $k$ and determinant $\epsilon \in K'_p / N(E_\pi^\times)$. Since $\mathfrak{A}^{-1}H_{2-p}^\oplus$ equals $H_{2-p}$, we obtain

$$L^\vee / L = \mathfrak{A}^{-1}L^\vee / L \cong \mathfrak{A}^{-1}M^\vee / M \cong \mathbb{Z}_E / \mathfrak{P}^k \cong \mathbb{F}_p^k.$$

Giving $\det(L, b) = p^k$, and thus $k = n$.

(iii) This follows from the fact that $s_- = 2 \sum_{i=1}^n n_i$.

(iv) Recall from Proposition 2.8 that a collection $(L_v, h_v)_{v \in \Omega(K)}$ of local hermitian lattices, all but finitely many of which are unimodular, is coming from a single global hermitian lattice $(L, h)$ if and only if the set $S = \{v \in \Omega(K) | \det L_v \not\in N(E_v)\}$ is finite of even cardinality. An infinite place $v$ lies in $S$ if and only if $n_v$ is odd. Thus we obtain

$$\#S \equiv \sum_{i=1}^n n_i + \begin{cases} 0 & \text{for } \epsilon(-1)^m \in N(E_\pi^\times) \\ 1 & \text{for } \epsilon(-1)^m \not\in N(E_\pi^\times) \end{cases} \pmod{2}$$

where $\epsilon = 1$ if $n = 0$. We see that for $n \neq 0$ this condition uniquely determines the norm class of $\epsilon$. We show that for $n = 0$ condition (iv) is equivalent to $\#S \equiv 0 \pmod{2}$. To this end note that $-1$ is a local norm at $\pi$ if and only if $p \equiv 1 \pmod{4}$. Thus we can rewrite (1) with $\epsilon = 1$ as

$$2\#S \equiv 2 \sum_{i=1}^n n_i + m(p - 1) \pmod{4}.$$

With $s_- = 2 \sum_{i=1}^n n_i$ and multiplying by 2 we arrive at (iv).

Note that $n = 0$ if and only if $(L, b)$ is unimodular. Then it is well known that $s_+ - s_- \equiv 0 \pmod{8}$. With $s_+ + s_- = (p - 1)(n + 2m)$ this is clearly equivalent to (iv). We have settled the existence conditions.

The lattice $(L, b)$ determines $n$ and $m$. By the previous considerations this determines the isomorphism class of $(L, h)_v$ at all finite places $v \in \mathbb{F}(K)$. Now the signatures $n_i$ determine it at the infinite places. Thus they give the genus of $(L, h)$.

By Lemma 2.9 the classes in the genus of $(L, h)$ are given by their Steininitz class. □

Remark 2.11. The relative class number $h^{-}(\mathbb{Q}[[\zeta_p]])$ is 1 for all primes $p < 19$. For $p = 23, 29, 31, 37, 41$ it equals $3, 2^3, 3^2, 37, 11^2$ respectively (cf. [33, Tables §3]).

2.6. The action on the discriminant group. For an integral $\mathbb{Z}$-lattice $(L, b)$ and $f \in O(L, b)$ fixed point free of order $p$ we denote by $O(L, b, f)$ the centralizer of $f$ in $O(L, b)$. Note that $O(L, b, f) = O(L, h)$, where $h$ is defined as in (2). Let $\tilde{f}$ be the isometry of $L^\vee / L$ induced by $f$. In what follows, we want to compute the image of the morphism

$$O(L, f) \to O(L^\vee / L, \tilde{f})$$

in the case that $\tilde{f} = \text{id}_{L^\vee / L}$.

Let $x \in L^\vee$ and $[x] = x + L$. If $b(x, x) \neq 0 \pmod{\mathbb{Z}}$, we obtain the reflection

$$\tau_{[x]} : L^\vee / L \to L^\vee / L, \quad y \mapsto y - 2\frac{b(x, y)}{b(x, x)}.$$
This can be adapted for hermitian lattices. For any \( \delta \in \mathcal{E}_0 = \{ \delta \in \mathbb{Z}_E \mid \delta \delta^* = 1 \} \) and \( x \in V = L \otimes E \) with \( h(x, x) \neq 0 \) we obtain the quasi reflection \( \tau_{x, \delta} \in \text{O}(V, h) \) defined by
\[
\tau_{x, \delta} : V \to V, \quad y \mapsto y + (\delta - 1) \frac{h(y, x)}{h(x, x)} x.
\]
It maps \( x \) to \( \delta x \) and is trivial on \( x^\perp \).

**Lemma 2.12.** Let \( x \in L^\vee \otimes \mathbb{Z}_p = L_p^\vee \) with \( b(x, x) \neq 0 \) mod \( \mathbb{Z}_p \). Suppose that \( \mathfrak{P} L_p^\vee \subseteq L_p \). Then \( \tau_{x, -1} \) acts as the reflection \( \tau_{[x]} \) on the discriminant group \( L^\vee / L \).

**Proof.** As \( \text{Tr}_{L_p^\vee}^E(h(x, x)) = b(x, x) \neq 0 \) mod \( \mathbb{Z}_p \), we obtain that \( h(x, x) \notin \mathfrak{A}^{-1} \). On the other hand \( h(x, x) \in \mathfrak{g}(L^\vee, h)_p = \mathfrak{P}^{-1} \mathfrak{A}^{-1} \). Thus \( h(x, x) \mathcal{Z}_{E_p} = \mathfrak{g}(L^\vee) \), and we can write \( L^\vee = \mathcal{Z}_{E_p} x \oplus x^\perp \). This induces a compatible splitting on \( L^\vee / L \). Since \( \pi x \in L \), we have \( x \mathcal{Z}_{E_p} + L_p = x \mathcal{Z}_p + L_p \) and this splitting is \( \{ x \mathcal{Z}_p \oplus x \}^\perp \). We can now conclude by comparing the actions of \( \tau_{x, -1} \) and \( \tau_{[x]} \) on \( [x] \) and \( [x] \). \( \square \)

**Lemma 2.13.** The morphism
\[
\text{O}(L_p, h_p) \to \text{O}(L^\vee / L)
\]
is surjective.

**Proof.** Note that in our case \( \text{O}(L^\vee / L) \) is just an orthogonal group over a field. Then the Cartan-Dieudonné theorem [30] Ch. 1, Thm. 5.4] says that it is generated by reflections. But these reflections lie in the image. \( \square \)

**Corollary 2.14.** The special orthogonal group \( \text{SO}(L^\vee / L) \) is contained in the image of \( \text{O}(L, h) \to \text{O}(L^\vee / L) \).

**Proof.** The special orthogonal group is generated by pairs \( \tau_{[x]} \). Then the corresponding isometry \( t = \tau_{x, -1} \tau_{y, -1} \) has determinant one. By the strong approximation theorem, for any \( k \) we can find \( g \in \text{O}(L) \) with \( (g - t)(L) \subseteq p^k L \). For us \( k = 1 \) is enough. \( \square \)

Since \( [\text{O}(L^\vee / L) : \text{SO}(L^\vee / L)] \) is at most 2, we need only one more generator to get full surjectivity.

**Proposition 2.15.** Let \( (L, h) \) be a hermitian \( \mathbb{Z}_E \)-lattice with \( \mathfrak{P} L^\vee \subseteq L \). If \( (L, h) \) is indefinite or of rank one, then the natural map
\[
\text{O}(L, h) \to \text{O}(L^\vee / L)
\]
is surjective.

**Proof.** Set \( V = L \otimes E \). If \( (L, h) \) is of rank one, then \( \text{O}(L^\vee / L) \subseteq \{ \pm 1 \} \) and the natural map is surjective. If \( (L, b) \) is unimodular, the proposition is certainly true. Otherwise, by the proof of Proposition 2.10 \( \mathfrak{s}(L) = \mathfrak{n}(L) \mathcal{Z}_E \). Let \( x \in L \) be a local norm generator at \( p \), that is, \( h(x, x) \mathcal{Z}_{K_p} = \mathfrak{n}(L)_p \). Since \( \mathfrak{n}(L_p) \mathcal{Z}_{E_p} = \mathfrak{s}(L_p) \), the reflection \( \tau = \tau_{x, -1} \in \text{O}(V, h) \) satisfies \( \tau(L_p) = L_p \). However, it has denominators at the finite set of primes
\[
Q = \{ q \in \mathbb{P}(K) \mid \tau(L_q) \neq L_q \}.
\]
We shall use the strong approximation theorem ([31] 5.12), ([19]) in the formulation of ([18] Thm. 5.1.3) to compensate the denominators. Take \( S = \mathbb{P}(K) \). Since \( V \) is indefinite \( \Omega(K) \setminus S \) contains an isotropic place. We set \( T = Q \cup \{ p \} \), and define
\[ \sigma_q = \tau^{-1} \circ \phi_q \] for \( q \in Q \) where \( \phi_q \in O(L_q) \) is of determinant \(-1\) (which is possible by [17 Cor. 3.6]). Finally set \( \sigma_q = \text{id}_{L_q} \). By the strong approximation theorem for any \( k \in \mathbb{N} \) we can find \( \sigma \in O(V) \) with

- \( \sigma(L_v) = L_v \) for \( v \in S \setminus T \) and
- \( (\sigma - \sigma_q)(L_q) \subseteq q^kL_q \) for \( q \in Q \).

Choose \( k \geq 1 \) large enough such that \( q^k\tau(L_q) \subseteq L_q \) for all \( q \in Q \). Then

\[
\tau \circ \sigma(L_q) \subseteq \tau \circ (\sigma - \sigma_q)(L_q) + \tau \circ \sigma_q(L_q)
\]

\[
\subseteq \tau(q^kL_q) + \tau \circ \sigma_q(L_q)
\]

\[
\subseteq q^k\tau(L_q) + \phi_q(L_q)
\]

\[
\subseteq L_q.
\]

Hence \( \tau \circ \sigma \) preserves \( L_q \) for all \( q \in \mathbb{P}(K) \). As it is moreover an element of \( O(V,h) \), it must be in \( O(L,h) \). Since \( k \geq 1 \), both \( \tau \) and \( \tau \circ \sigma \) induce the reflection \( \tau_{[x]} \) on the discriminant group. This reflection generates \( O(L^\vee/L)/SO(L^\vee/L) \).

For later use we prove the following Lemma.

**Lemma 2.16.** Let \((L,h)\) be a hermitian \( \mathbb{Z}_E \)-lattice with trace lattice \((L,b,f)\). Then \( O(L,b,f) = SO(L,b,f) \).

**Proof.** Let \( f \in O(L,b,f) = O(L,h) \). When we view \( f \) as an \( E \)-linear map its determinant \( d = \det_E(f) \in E \) satisfies \( dd^* = 1 \). Viewed as a \( \mathbb{Q} \)-linear map one obtains \( \det_Q(f) = N_{E/Q}(\det_E(f)) = N_{E/K}(N_{E/Q}(\det_E(f))) = N_{E/K}(1) = 1 \). \( \square \)

### 2.7. Conjugacy classes of finite order in unimodular lattices.

Here we use the results of the previous section to obtain existence and uniqueness results on prime order isometries of unimodular lattices. To make notation lighter, in the following we will denote by \( A_L = L^\vee/L \) the discriminant group of an even \( \mathbb{Z} \)-lattice \((L,b)\) and by \( q_L \) its discriminant quadratic form. The length \( l(A_L) \) is defined as the minimal number of generators of the group \( A_L \).

**Proof of Theorem 1.1.** If \( L \) is unimodular, there exists an isometry of order \( p \) if and only if there exists a primitive sublattice \( S \subset L \) and a fixed point free isometry \( f \in O(S) \) of order \( p \) which acts trivially on the discriminant group of \( S \). Indeed, such an isometry \( f \in O(S) \) glues with \( \text{id}_{S^L} \) (see [29 Cor. 1.5.2]) to give an isometry of \( L \) whose cofixed lattice is \( S \). By Proposition 2.11 the lattice \( S \) is \( p \)-elementary of rank \( (n + 2m)(p - 1) \) and discriminant \( p^n \), for some non-negative integers \( n,m \). If we denote its signature by \((s_+, s_-)\), this gives conditions (2), (3) and (4). By [29 Thm. 1.12.2], such a lattice \( S \) embeds primitively into some even unimodular lattice \( L \in \Pi_{(l_+, l_-)} \) if and only if (5) and (6) hold (note that \( l(A_S) = n \), and further

\[
\text{if } (n + 2m)(p - 1) = l_+ + l_- - n, \text{ then } (-1)^{l_++l_-}p^n \equiv \text{disc}(K(q_S)) \mod (\mathbb{Z}_p^*)^2
\]

where \( K(q_S) \) is the unique \( p \)-adic lattice of rank \( n \) and discriminant form \( q_S \), i.e. the \( p \)-modular Jordan component of \( S \otimes \mathbb{Z}_p \). Its discriminant is computed in [10 Ch. 15, Thm. 13] via \( s_+ - s_- \equiv 2\epsilon - 2 - (p - 1)n \mod 8 \) where \( \epsilon \in \{ \pm 1 \} \) indicates the unit-square class of the determinant. Noting that \( s_+ \) is even, the left side is computed by \( (-1)^{l_++s_+} \equiv (p - 1)l_+ + 1 \mod 4 \). Inserting this for \( \epsilon \), we arrive at

\[
s_+ - s_- \equiv 2(p - 1)l_+ - (p - 1)n \mod 8
\]

which gives (7). \( \square \)
Proof of Theorem 2.19. Let \( L \) be an even unimodular lattice and \( f, g \in O(L) \) prime order isometries with \( L_f, L_g \) indefinite. Suppose that \( L_f \cong L_g \), and the signatures and Steinitz class of \( L_f \) and \( L_g \) agree. We prove that \( f \) and \( g \) are conjugated as isometries of \( L \). By assumption there is an isometry \( u : L_f \to L_g \), and by Proposition 2.10 an isometry \( v : (L_f,f|_{L_f}) \to (L_g,g|_{L_g}) \). Let \( \epsilon_f : A_{L_f} \to A_{L_g} \) and \( \epsilon_g : A_{L_g} \to A_{L_s} \) be the standard isomorphisms between the discriminant groups of orthogonal primitive sublattices inside a unimodular lattice. By Proposition 2.15 there exists an isometry \( w \in O(L_f) \) centralizing \( f|_{L_f} \) whose action on the discriminant group \( A_{L_f} \) is \( \bar{w} = \bar{v}^{-1} \circ \epsilon_g^{-1} \circ \bar{u} \circ \epsilon_f \). It follows from \cite{29} Cor. 1.5.2 that \( u \circ (w \circ v) : L_f \oplus L_f \to L_g \oplus L_g \) extends to an isometry of \( O(L) \) conjugating \( f \) and \( g \).

Of special interest for geometric applications are isometries with \( s_+ = 2 \). In the following we give a closer study.

Corollary 2.17. In the situation of Theorem 2.14 suppose that \( s_+ = 2 \) and \( n = 0 \). Then \( p \equiv 3 \pmod{4} \) and \( m \equiv 1 \pmod{2} \).

Corollary 2.18. Let \( L \) be an even unimodular lattice. Then conjugacy classes of subgroups \( G \leq O(L) \) of odd prime order \( p \leq 19 \) with \( L_G \) of signature \( (2,*) \) are determined by the isometry class of their fixed lattice \( L^G \). If \( p = 23 \), then there are exactly two such subgroups up to conjugacy. They are distinguished by the Steinitz class of \( L_G \) being trivial or not.

Proof. Let \( f \) be a generator of \( G \). That \( s_+ = 2 \) means that there is a single real place \( \nu \) such that the signatures of \( f|_{L_f} \) are given by \( n_\nu = n + 2m - 1 \) and for \( \mu \neq \nu \) we have \( n_\mu = n + 2m \). This gives \((p-1)/2\) possible signatures. Replacing \( f \) by \( f^k \) corresponds to the simultaneous action of the Galois group of \( \mathbb{Q}([\zeta]) \) on the signatures and the relative class group. The signatures are attained by \( f^i \) for \( i \in \{1, \ldots, (p-1)/2\} \). Note that \( f \) and \( f^{-1} \) share the same signatures, but the Steinitz class \( I_f \) of \( (L_f,f|_{L_f}) \) is conjugate to that of \( f^{-1} \). If \( p \leq 19 \), the relative class number is one. Thus the signatures alone give the conjugacy class of \( f^k \) and the \( f^k \)'s exhaust the conjugacy classes. If \( p = 23 \), then the relative class group is isomorphic to \( \mathbb{Z}/3\mathbb{Z} \) with complex conjugation acting as \(-1\). Let \( g, h \in O(L) \) have fixed lattice isomorphic to \( L_f \). Suppose that \( I_g \) is non trivial and \( I_h \) is trivial, then \( f \) is conjugate to exactly one of \( g^k \), \( k \in \{1, \ldots, p-1\} \) or \( h^j \), \( j \in \{1, \ldots, (p-1)/2\} \). Consequently \( G \) is conjugate to \( \langle g \rangle \) or \( \langle h \rangle \).

By using Theorem 2.11 for a given genus \( \Pi_{(p,\{\pm\})} \) we can list all triples \((p,n,m)\) such that there exists a unimodular lattice \( L \in \Pi_{(p,\{\pm\})} \) and a subgroup \( G < O(L) \) of order \( p \) with \( L_G \) of signature \((2,(n+2m)(p-1)-2)\) and discriminant \( p^n \). Notice that, while the lattice \( L_G \) is uniquely determined (up to isometries) by the triple \((p,n,m)\), a priori there can be several distinct isometry classes in the genus of \( L^G \). For geometric applications, we study the uniqueness of the invariant lattice for some selected unimodular genera.

Lemma 2.19. Let \( L \) be a unimodular lattice in one of the genera \( \Pi_{(3,3)}, \Pi_{(4,4)} \), \( \Pi_{(5,5)}, \Pi_{(3,19)}, \Pi_{(4,20)} \), \( \Pi_{(5,21)} \). Let \( G < O(L) \) be a subgroup of odd prime order \( p \) with \( L_G \) of signature \((2,(n+2m)(p-1)-2)\) and discriminant \( p^n \), for some \( n,m \in \mathbb{Z}_{\geq 0} \). Then the lattice \( L^G \) is unique in its genus unless \( L \in \Pi_{(4,20)} \) and \((p,n,m) = (23,1,0)\), where \( L^G \) is isometric to either \(
\begin{pmatrix}
2 & 1 \\
1 & 12
\end{pmatrix}
\) or \(
\begin{pmatrix}
4 & 1 \\
1 & 6
\end{pmatrix}
\)
Proof: Since $L$ is unimodular, $q_{L^G} \cong -q_{L^G}$, hence $L^G$ is also $p$-elementary. Denote by $(l_+, l_-)$ the signature of $L$. If $L^G$ is indefinite and $\text{rk}(L^G) \geq 3$, then $L^G$ is unique in its genus by [10] Ch. 15, Thm. 14]. For $L$ as in the statement, the only cases where one of these two conditions fails are the following: $(l_+, l_-) = (3, 3, 3, 0, 1)$, $(3, 3, 3, 2, 0)$, $(3, 3, 5, 1, 0)$, $(4, 4, 3, 1, 1)$, $(4, 4, 7, 1, 0)$, $(4, 20, 3, 1, 5)$, $(4, 20, 23, 1, 0)$. The genera of $L^G$ are respectively $\Pi(1,1)$, $\Pi(1,1)(3^{-2})$, $\Pi(1,1)(5^{-1})$, $\Pi(2,0)(3^{-1})$, $\Pi(2,0)(7^1)$, $\Pi(2,0)(3^{-1})$, $\Pi(2,0)(23^1)$. By using [10] Ch. 15, Tables 15.1 and 15.2a we check that there is only one isometry class for each of these genera, except for $\Pi(2,0)(23^1)$, which contains the two distinct isometry classes given in the statement.

\[\square\]

3. Automorphisms of IHS manifolds

In this section we apply our results on isometries of unimodular lattices to obtain a classification of prime order automorphisms of irreducible holomorphic symplectic manifolds of a given deformation type.

Definition 3.1. An irreducible holomorphic symplectic (IHS) manifold is a compact complex Kähler simply connected manifold $X$ such that $H^0(X, \Omega^2_X) = \mathbb{C}\omega_X$, where $\omega_X$ is an everywhere non-degenerate holomorphic 2-form.

If $X$ is IHS, then $H^2(X, \mathbb{Z})$ is torsion-free and it is equipped with a non-degenerate symmetric bilinear form of topological origin, which gives it the structure of an integral lattice (see [11] Thm. 4.7]). For all known examples of IHS manifolds this lattice has been computed explicitly: it is even and only depends on the deformation type of the manifold. Let $H^2(X, \mathbb{Z}) \cong L$ for some fixed lattice $L$; a marking of $X$ is the choice of an isometry $\eta : H^2(X, \mathbb{Z}) \to L$.

Let $\sigma \in \text{Aut}(X)$ be a (biholomorphic) automorphism of an IHS manifold $X$ and let $\sigma^* \in O(H^2(X, \mathbb{Z}))$ be the isometry of the second cohomology lattice induced by pull-back. The automorphism $\sigma$ is said to be symplectic if $\sigma^*(\omega_X) = \omega_X$, non-symplectic otherwise. Conventionally, we say that $\sigma$ has order $d$ if $d$ is the order of the isometry $\sigma^*$. Denote by $\text{Mon}^2(X)$ the group of monodromy operators of $X$ (see [22] §1.1 for the definition). The natural homomorphism

$$\text{Aut}(X) \to \text{Mon}^2(X) \subseteq O(H^2(X, \mathbb{Z})), \quad \sigma \mapsto (\sigma^*)^{-1}$$

has finite kernel (see [13] Prop. 9.1]). For K3 surfaces and IHS manifolds of $K3^{[\eta]}$-type it is injective (by [4] Prop. 10] and [24] Lem. 1.2]), hence $d$ is also the order of $\sigma$ on $X$.

It can be readily checked (cf. [7] §5]) that the cofixed lattice $H^2(X, \mathbb{Z})_{\sigma^*}$ of a symplectic automorphism $\sigma \in \text{Aut}(X)$ is negative definite and contained in $\text{NS}(X)$. On the other hand, if $\sigma$ is non-symplectic then $H^2(X, \mathbb{Z})_{\sigma^*}$ has signature $(2,*$ and it contains the transcendental lattice of $X$.

3.1. K3 surfaces. Let $L = U^{\otimes 3} \oplus E_8(-1)^{\otimes 2}$. It is the unique lattice in the genus $\Pi(3,19)$, up to isometry. The lattice $L$ is isomorphic to $H^2(S, \mathbb{Z})$ for any K3 surface $S$. Let $\sigma \in \text{Aut}(S)$ be a non-symplectic automorphism of odd prime order $p$ and $\eta : H^2(S, \mathbb{Z}) \to L$ a marking. We denote by $G = \langle \eta \circ \sigma^* \circ \eta^{-1} \rangle \leq O(L)$ the group induced by the action on the lattice $L$. As recalled above, the signature $(s_+, s_-)$ of $L_G$ has $s_+ = 2$. By applying Theorem [11] we can easily determine the possible signatures and lengths of the lattices $L^G$ and $L_G$; in particular, we recover
the classification of the pairs \((L^G, L_G)\) given in [1] Table 2 (order \(p = 3\)) and [2] Tables 2–7 (prime orders \(5 \leq p \leq 19\)).

**Proof of Theorem 1.5.** From Theorem 1.1 we obtain the conditions for the existence of a group \(G = (f) \subset O(L)\) of isometries of order \(p\) when \(l_+ = 3, l_- = 19, s_+ = 2\). The lattice \(L^G\) is hyperbolic and \(\text{rk}(L^G) \leq 20\), therefore by the surjectivity of the period map there exists a marked K3 surface \((S, \eta)\) such that \(\eta(\text{NS}(S)) = L^G\) (see for instance [3, §3]). This implies that \(\eta^{-1} \circ f \circ \eta\) is an isometry of \(H^2(S, \mathbb{Z})\) which preserves a Kähler class and the Hodge decomposition, hence by the global Torelli theorem [14, Thm. 7.5.3] there exists \(\sigma \in \text{Aut}(S)\) such that \(\sigma^* = \eta^{-1} \circ f \circ \eta\).

By Corollary 2.18 the action of the group \(G\) is determined (up to conjugacy on \(L\)) by the isometry class of \(L^G\) (see also [2, Prop. 9.3]). Moreover, by Lemma 2.19 the lattice \(L^G\) is unique in its genus. Since \(L^G\) is \(p\)-elementary, its genus is determined by \(\text{sig}(L^G) = (1, 21 - (n + 2m)(p - 1))\) and \(l(A_{L_G}) = n\). □

**Remark 3.2.** If the action of the group \(G\) is symplectic, then \(L_G\) is negative definite and we have the additional condition that \(L_G\) has maximum at most \(-4\). Thus our results do not apply to recover the classification of symplectic automorphisms. See [21] for a survey.

### 3.2. IHSMs of \(K3\)\(^{[\alpha]}\)-type and \(Kum_3\)-type.

In order to obtain a classification of automorphisms of IHS manifolds up to deformation, one has to classify their action on cohomology up to conjugation by monodromy operators, i.e. as elements of the monodromy group. We do this by reduction to the unimodular case.

Let

\[
M = U^2 \oplus E_8(-1)^{\oplus 2} \in \Pi_{(4,20)}
\]

be the Mukai lattice of IHS manifolds of \(K3\)\(^{[\alpha]}\)-type, i.e. the full cohomology lattice of a K3 surface, and \(V \subseteq M\) a fixed sublattice of \(M\) isomorphic to \((2(n-1))\). Then

\[
(M_n, b) = V^\perp \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle
\]

is isomorphic to \(H^2(X, \mathbb{Z})\) for any IHS manifold \(X\) of \(K3\)\(^{[\alpha]}\)-type. The real spinor norm \(\text{spin}_\mathbb{R}: O(M_n) \to \mathbb{R}^*/(\mathbb{R}^*)^2 \cong \{\pm 1\}\) is defined as

\[
\text{spin}_\mathbb{R}(g) = \left(-\frac{b(v_1, v_1)}{2}\right) \cdots \left(-\frac{b(v_r, v_r)}{2}\right) \mod (\mathbb{R}^*)^2
\]

if \(g_\mathbb{R} \in O(M_n)\) factors as a product of reflections \(g_\mathbb{R} = \tau_{v_1} \circ \cdots \circ \tau_{v_r}\) with respect to elements \(v_i \in (M_n)_\mathbb{R}\) (in particular, \(r \leq \text{rk}(M_n) = 23\) by the Cartan-Dieudonné theorem). Note that this is the real spinor norm corresponding conventionally to the quadratic form \(-b\).

**Example 3.3.** Let \((L, b)\) be a lattice of signature \((s_+, s_-)\). We can diagonalize \(b \otimes \mathbb{R}\). That is, we can find \(e_1, \ldots, e_n \in L \otimes \mathbb{R}\) giving a diagonal Gram matrix with \(s_+\) ones and \(s_-\) minus ones on the diagonal. Then \(-\text{id} = \tau_{e_1} \circ \cdots \circ \tau_{e_n}\). We see that the spinor norm of \(-\text{id}\) is \((-1)^{s_+}\).

By [22, Lem. 9.2], the group of monodromy operators for manifolds of \(K3\)\(^{[\alpha]}\)-type is

\[
\text{Mon}(M_n) = \{ g \in O(M_n) \mid \tilde{g} = \pm \text{id}_{A_{M_n}}, \text{spin}_\mathbb{R}(g) = 1 \}.
\]

For IHS manifolds of \(Kum_3\)-type, the second cohomology lattice \(Y_n = U^\oplus 3 \oplus \langle -2(n+1) \rangle\) can be realized as the orthogonal complement of \(V = \langle 2(n+1) \rangle\) inside
the lattice \( Y = U^{\oplus 4} \). By [24 Cor. 4.8] and [25 Thm. 4.3], the monodromy group is now
\[
\text{Mon}(Y_n) = \{ g \in O(Y_n) \mid \tilde{g} = \det(g) \text{id}_{A_{Y_n}}, \text{spin}_\mathbb{R}(g) = 1 \}.
\]
In order to treat the \( K^3[n] \) and \( \text{Kum}_n \) case uniformly, let \( L = M \) (resp. \( Y \)), \( L_n = M_n \) (resp. \( Y_n \)) and \( S = O(M) \) (resp. \( SO(Y) \)). Let \( \chi : \text{Mon}(L_n) \to \{ \pm 1 \} \) be the character defined by \( \tilde{g} = \chi(g) \text{id}_{A_{L_n}} \) for \( g \in \text{Mon}(L_n) \). Notice that the isometry \( \chi(g) \text{id}_V \oplus g \in O(V \oplus L_n) \) extends to an element of \( S \).

**Remark 3.4.** If \( g \in O(L_n) \) has odd order, then necessarily \( \text{spin}_\mathbb{R}(g) = 1 \) and \( \det(g) = 1 \). Moreover, \( A_{L_n} = \mathbb{Z}x \cong \frac{z}{2(n+1)\mathbb{Z}} \) with discriminant form given by \( q_{L_n}(x) = -\frac{1}{2(n+1)} \): any \( \varphi \in O(A_{L_n}) \) satisfies \( \varphi^2 = \text{id} \), hence \( \tilde{g} = \text{id} \). Thus, all isometries of odd order of the lattice \( L_n \) are monodromies.

Set \( \mathcal{W}_n = \{ W \subseteq L \text{ primitive}, W \cong V \} \) where \( L_n^+ = V \cong \langle 2(n - 1) \rangle \) (resp. \( \langle 2(n + 1) \rangle \)). For any subgroup \( S \subseteq O(L) \) we define:
\[
\mathcal{C}_n(S) = \{(g, W) \mid g \in S, W \in \mathcal{W}_n, gW = W \};
\]
\[
\mathcal{C}^+_n(S) = \{(g, W) \in \mathcal{C}_n(S) \mid \text{spin}_\mathbb{R}(g|_{V^\perp}) = 1 \}.
\]

We say that \( g, g' \in S \) are conjugate (and we write \( g \sim g' \)) if and only if there exists \( f \in S \) with \( g' = f g f^{-1} \). Similarly, for \( (g, W), (g', W') \in \mathcal{C}_n(S) \), we write \( (g, W) \sim (g', W') \) if and only if there is an isometry \( f \in S \) with \( g' = f g \) and \( W' = f(W) \). In this case we also say that \( (g, W) \) and \( (g', W') \) are conjugate. We denote the respective conjugacy classes by \([g]\) and \([g, W]\). The next proposition, inspired by Markman’s characterization of monodromies [22 §9.1.2], allows us to pass to the unimodular lattice \( L \).

**Proposition 3.5.** Let \( L = M \) (resp. \( Y \)), \( L_n = M_n \) (resp. \( Y_n \)) and \( S = O(M) \) (resp. \( SO(Y) \)). Recall that \( V = L_n^+ \subseteq L \). The map
\[
\phi : \text{Mon}(L_n)/\sim \longrightarrow \mathcal{C}^+_n(S)/\sim
\]
defined by \( \phi([g]) = [\tilde{g}, V] \), where \( \tilde{g} = \chi(g) \text{id}_V \oplus g \in S \), is bijective.

**Proof.** We first show that \( \phi \) is well defined. Let \( f, g \in \text{Mon}(L_n) \) and consider the conjugated monodromies \( f \circ g \). Since \( f \) is a monodromy, \( \chi(f) \text{id}_V \oplus f \) extends to an element of \( S \) preserving \( V \). This provides the conjugacy \( (\tilde{g}, V) \sim (\tilde{f}g, V) \). To show that \( \phi \) is injective let \( g, g' \in \text{Mon}(L_n) \) with \( \phi([g]) = \phi([g']) \). i.e. \( (\tilde{g}, V) \sim (\tilde{g}', V) \). Hence there is \( f \in S \) with \( g' = f(g) \) and \( fV = V \). Then \( f \) restricts to an isometry of \( L_n = V^\perp \) which acts as \( \pm 1 \) on the discriminant group and conjugates \( g \) and \( g' \). If \( f|_{L_n} \) has real spinor norm \( -1 \), we take \( -f \) in order to have a monodromy (notice that \( \det(\chi(f|_{L_n})) = \det(-\chi(-f|_{L_n})) \)).

To see that \( \phi \) is surjective let \( (g, W) \in \mathcal{C}^+_n(S) \). Since \( W \) is primitive, Eichler’s criterion [12 Prop. 3.3(i)] applies and we can find \( f \in O(L) \) with \( fW = V \). If \( f \not\in S \), we can take a hyperbolic plane \( U \subseteq V^\perp \) and replace \( f \) by \( (t \oplus \text{id}_{U^\perp})f \) where \( t \in O(U) \setminus SO(U) \). Then \( f\tilde{g} \in S \) and \( (f\tilde{g})V = V \). The (conjugacy class of) the restriction of \( f\tilde{g} \) to \( L_n \) is a preimage of \( [f \tilde{g}, V] = [g, W] \).

**Proposition 3.6.** Let \( L = M \) (resp. \( Y \)), \( V \cong \langle 2(n - 1) \rangle \subseteq M \) (resp. \( \langle 2(n + 1) \rangle \subseteq Y \)), \( L_n = V^\perp = M_n \) (resp. \( Y_n \)), \( S = O(M) \) (resp. \( SO(Y) \)), \( g \in O(L) \), and \( S(L^9) = O(M^9) \) (resp. \( SO(Y^9) \)). Define
\[
\psi : \mathcal{C}^+_n(S)/\sim \longrightarrow S/\sim, \quad [g, W] \rightarrow [g].
\]
If $g$ is of odd prime order and $M_g$ of signature $(2, *)$, then 
\[
\varphi: \psi^{-1}([g]) \rightarrow \{W \mid W \subseteq L^g \text{ primitive, } W \cong V\} / S(L^g) \\
[h, W] \mapsto S(L^g)fW \text{ where } f' h = g
\]
is a bijection.

Proof. We first show that $\varphi$ is well defined. Let $[h', W'] \in \psi^{-1}([g])$, with $f' h' = g$ for some $f' \in S$. Suppose that $[h, W] = [h', W']$, i.e. $h' = h$ for some $s \in S$ such that $sW' = W$. We need to show that the left cosets $S(L^g)fW$ and $S(L^g)f'W'$ coincide. Let $t := fs(f')^{-1}$ and notice that $f' h = f s s^{-1} h = f'h = g$. Hence, the restriction $t|_{L^g}$ belongs to $O(L^g)$, and $t f' W' = f s W' = f W$. If $L = Y$, we have to show in addition that $t|_{L^g} \in SO(L^g)$. This follows from $\det t = 1$ and $\det t|_{L^g} = 1$ (cf. Lemma 2.16).

The map $\varphi$ is surjective. If $W \subseteq L^g$ is a primitive sublattice isomorphic to $V$, then $[g, W]$ is in the fiber $\psi^{-1}([g])$. Indeed, $g|_W = id_W$ and the restriction $g|_{W^\perp}$ has spinor norm $+1$ since $g$ has odd order.

To prove injectivity, let $[h, W]$ and $[h', W']$ be in the fiber $\psi^{-1}([g])$, with $f' h = f' h' = g$ for $f', f' \in S$, and assume that there exists $t \in S(L^g)$ such that $t f W = f' W'$. Since the natural map
\[
SO(L_g, g) = O(L_g, g) \rightarrow O(L_g^g/L_g)
\]
is surjective (cf. Proposition 2.15) and $L$ is unimodular, we can extend $t$ to an isometry $\tilde{t} \in S$ commuting with $g$. Then $(f')^{-1} \circ \tilde{t} \circ f \in S$ conjugates $h$ to $h'$ and maps $W$ to $W'$, hence $[h, W] = [h', W']$. \hfill $\square$

Together Theorems 1.1, 1.2 and Propositions 3.5, 3.6 give the classification of monodromies of odd prime order up to conjugation.

Theorem 3.7. Let $L_n = M_n$ (resp. $Y_n$) and fix an embedding $L_n \hookrightarrow L$ with $L = M$ (resp. $Y$). Set $V = L_n^+ \subset L$. Let $f, g \in \text{Mon}(L_n) \subset O(L_n)$ be monodromies of manifolds of $K3^{[n]}$-type (resp. of $\text{Kum}_n$-type) and let $f, g \in O(L)$ be the extensions of $\chi(f) id_V \oplus f, \chi(g) id_V \oplus g$ respectively. Then $f$ and $g$ are conjugated in $\text{Mon}(L)$ if and only if $\tilde{f}, \tilde{g}$ are conjugated by an element of $O(M)$ (resp. $SO(Y)$) which preserves $V$.

Moreover, if $f \in \text{Mon}(L_n)$ has odd prime order and $L_f$ has signature $(2,*)$, the number of conjugacy classes of monodromies $g \in \text{Mon}(L_n)$ such that $\tilde{f} \tilde{g}$ are conjugated coincides with the number of orbits of primitive sublattices $W \subseteq L_f$ with $W \cong V$, up to $O(L_f)$ (resp. $SO(L_f^f)$).

We conclude with the proof of Theorem 1.7, which characterizes the fixed and cofixed sublattices of non-symplectic automorphisms of odd prime order for IHS manifolds of $K3^{[n]}$-type and $\text{Kum}_n$-type.

Proof of Theorem 1.7. In order for $S$ to be the cofixed lattice of a non-symplectic automorphism, its signature needs to be $(2, \text{rk}(S) - 2)$. Any $p$-elementary lattice of determinant $p$ and signature $(2, (n + 2m)(p - 1) - 2)$ is unique in its genus, since it is either indefinite or of signature $(2, 0)$ (and therefore isomorphic to $A_2$; see [10] Table 15.1). Hence, by Proposition 2.10 the lattice $S$ admits a fixed-point free isometry $f$ of order $p$ with $\tilde{f} = id_{A_2}$ if and only if it is $p$-elementary and $\text{rk}(S) = (l(A_2) + 2m)(p - 1)$ for some $m \in \mathbb{Z}_{\geq 0}$. Let $T := S^\perp \subset L$. By [29] Cor.
1.5.2], we have that \( \text{id}_T \oplus f \in O(T \oplus S) \) extends to an isometry \( \Phi \in O(L) \) such that \( L^\Phi = T \) and \( L_f = S \). In both the \( K3^n \)-type and \( \text{Kum}_n \)-type case, \( \Phi \in \text{Mon}(L) \) since the order \( p \) of \( \Phi \) is odd (see Remark 6.4). Then, there exists an IHS manifold \( X \) of the corresponding deformation type and a marking \( \eta : H^2(X, \mathbb{Z}) \to L \) such that \( \eta(\text{NS}(X)) = T \) and \( \eta(H^{2,0}(X)) \) is an eigenvector of \( \Phi \) (see for instance [32, Thm. 3.9]). Now \( \eta^{-1} \circ \Phi \circ \eta \) is a monodromy operator of \( H^2(X, \mathbb{Z}) \) which preserves a Kähler class (since \( T = L^\Phi \)) and the Hodge decomposition. By the Hodge-theoretic global Torelli theorem [22, Thm. 1.3], \( X \) is endowed with a biregular automorphism \( \sigma \) whose action on \( H^2(X, \mathbb{Z}) \) is \( \sigma^* = \eta^{-1} \circ \Phi \circ \eta \). The automorphism is non-symplectic because \( H^{2,0}(X) \subset \text{NS}(X)^\perp \) inside \( H^2(X, \mathbb{C}) \).

\[ \square \]

Remark 3.8. For manifolds of \( K3^n \)-type, a complete classification of the pairs of fixed and cofixed lattices of non-symplectic automorphisms of odd prime order is already known in the case \( n = 2 \), by work of Boissière, Camere and Sarti [6, Thm. 7.1] and Boissière, Camere, Mongardi and Sarti [5, Thm. 6.1]. A similar classification has been achieved for \( n = 3, 4 \), and in all dimensions when \( \text{rk}(S) = 22 \), by Camere and the second author [8, Thms. 1.2 and 4.4]. For all \( n \geq 2 \), an analogous statement of Theorem 1.7 in the case of involutions has been obtained by Camere, the second author and Andrea Cattaneo in [9, Thm. 2.3]. Moreover, for \( n = 2, 3, 4 \) and \( p \neq 23 \) odd, explicit examples of automorphisms realizing all possible pairs of lattices have been exhibited in [8] and [10]. For manifolds of \( \text{Kum}_n \)-type, Mongardi, Tari and Wandel [27] have classified the pairs of fixed and cofixed lattices when \( n = 2 \) and presented examples of automorphisms for all cases where \( \text{rk}(S) \leq 6 \).

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