High order tensor moments of random vectors

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Abstract

A random vector $\mathbf{x} \in \mathbb{R}^n$ is a vector whose coordinates are all random variables. A random vector is called a Gaussian vector if it follows Gaussian distribution. These terminology can also be extended to a random (Gaussian) matrix and random (Gaussian) tensor. The classical form of an $k$-order moment (for any positive integer $k$) of a random vector $\mathbf{x} \in \mathbb{R}^n$ is usually expressed in a matrix form of size $n \times n^{k-1}$ generated from the $k$th derivative of the characteristic function or the moment generating function of $\mathbf{x}$, and the expression of an $k$-order moment is very complicated even for a standard normal distributed vector. With the tensor form, we can simplify all the expressions related to high order moments. The main purpose of this paper is to introduce the high order moments of a random vector in tensor forms and the high order moments of a standard normal distributed vector. Finally we present an expression of high order moments of a random vector that follows a Gaussian distribution.

Keywords: Tensor; random matrix; high order moment; Gaussian distribution;

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1 Introduction

Higher order moments are important in statistics. The concept of covariance matrix gives rise to that of co-skewness and co-kurtosis when it is extended to the higher order moments, say, the third and fourth moments like skewness and kurtosis. This follows from the generalization of the concept of mean and variance to moments and central moments. Higher-order moments of a normal distribution can be used to derive the recursive relationship of Hermite polynomials[3]. They are also widely used in the insurance industry[7], color transmission[8], fault diagnosis[9], large reflector antenna simulation[10] and

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other aspects also play an important role. The higher order moments are also useful in risk management. An example would be when the fund performance of four different fund managers are analyzed separately and they are then combined together so that in the end only 2 sets of results are compared. In both cases the moments i.e. the mean, standard deviation, skewness and kurtosis for each manager remains the same.

The covariance, i.e., the second order centralized moment of a random variable, determines the holistic divergence from its location (centroid) in one dimension. The shape and the other features of the distributions of a multidimensional random vector are not so obvious and hard to describe and illustrated by traditional approach. Note that the variance of $u$ is $D[u] = E[(u - E(u))^2] = E[u^2] - (E[u])^2 = m_2 - m_1^2$. In statistical analyses, the fundamental tasks include the characterization of the location and variability of the distribution of a data set or a population. Further characterization of the data includes the skewness and kurtosis, which involves the computation of the third and fourth order moments respectively. The skewness is a measurement of symmetry (or lack of symmetry) of the data distribution. A data set is said to be symmetrically distributed if it looks the same to the left and right of the center point. Kurtosis is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution, that is, data sets with high kurtosis tend to have heavy tails, or outliers, and these with low kurtosis tend to have light tails, or lack of outliers.

There are many ways to express the moments, one of the commonly used approach is to use derivatives to the characteristic function or the moment generating function\[^3\] \[^4\]. For the standardized 2-dimensional normal distribution, Kendall and Stuart (1963) gave the recurrence relation of the second order moment\[^5\]. Johnson (2000) and others have given analytical formulas for the same problem. Holmquist (1988) proved the general form of higher-order moments\[^6\] and extended the result to include the derivation of normal distribution quadratic higher moments\[^1\]. The problem of moments and cumulants of normal random matrices is considered by Ghazal and Neudecker\[^2\]. Using the Kronecker product, a simple formula for the special case of the second and fourth moments of the random matrix is derived\[^2\].

In this paper, we mainly introduce the higher-order tensor moments, present some tensor expressions of the higher order moments, and investigate their properties.

Recall that a tensor $\mathcal{A}$ is a multi-way array which can be regarded as a hypermatrix. An $m$-order tensor $\mathcal{A}$ can be of size $I_1 \times I_2 \times \ldots \times I_m$. $\mathcal{A}$ is called a $m$th order $n$-dimensional real tensor if $n := I_1 = I_2 = \ldots = I_m$. The set of all $m$th order $n$-dimensional real tensors is denoted as $\mathcal{T}_{m,n}$. For any positive integers $m, n > 1$, we usually denote $[n] := \{1, 2, \ldots, n\}, [n]_0 :=$
\{0, 1, 2, \ldots, n\}\), and
\[ S(m, n) = \{(i_1, i_2, \ldots, i_m) : i_k \in [n], \forall k \in [m]\} \]
and
\[ S(k; m, n) = \{\sigma := (i_1, i_2, \ldots, i_m) \in S(m, n) : i_1 + i_2 + \ldots + i_m = m + k\} \]
where \(k \in [N]_0\) with \(N = m(n - 1)\). For any \(\tau \in S(m, n)\), it is easy to see that \(\tau \in S(0; m, n)\) if and only if \(\tau = (1, 1, \ldots, 1)\), the smallest element in set \(S(m, n)\) according to the lexical order, and \(\tau \in S(N; m, n)\) if and only if \(\tau = (n, n, \ldots, n)\), the largest element in \(S(m, n)\).

An \(m\)th order \(n\)-dimensional real tensor \(A\) with size \(n \times n \times \ldots \times n\) is an \(m\)-array whose entries are indexed by indices \((i_1, i_2, \ldots, i_m) \in S(m, n)\). As element \(A_{i_1i_2\ldots i_m}\) is also denoted by \(A_\sigma\) where \(\sigma = (i_1, i_2, \ldots, i_m)\). We denote the set of all \(m\)th order \(n\)-dimensional real tensors by \(T_{m;n}\). A tensor \(A = (A_\sigma) \in T_{m:n}\) is called a symmetric tensor if each entry \(A_{i_1i_2\ldots i_m}\) is invariant under any permutation of its indices, that is,

\[ A_\sigma = A_{\tau(\sigma)} \forall \tau \in \text{Sym}_m, \forall \sigma \in S(m, n). \]

where \(\text{Sym}_m\) is the set of all permutations on \([m]\). We denote the set of all \(m\)th order \(n\)-dimensional symmetric tensors by \(\text{ST}_{m;n}\).

In the next section, we will introduce some notations related to the multiplications of tensors, which will be used to characterize higher order moments. Also we will define the tensor form of high order derivatives (HOD) of a multivariate function. Some interesting results of 4-order tensors will also be addressed in order to prepare for the description of the covariance tensor of a random matrix.

2 The multiplications of tensors and the 4-order tensors

An \(m\)th order \(n\)-dimensional real tensor \(A \in T_{m;n}\) can be associated with an \(m\)-order \(n\)-variate homogeneous polynomial in form

\[ f_A(x) = A x^m := \sum_{i_1, i_2, \ldots, i_m} A_{i_1i_2\ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m} \]

A symmetric tensor \(A \in \text{ST}_{m;n}\) is called positive semidefinite or simply PSD if \(f_A(x) \geq 0\) for all \(x \in \mathbb{R}^n\) and is called positive definite (PD) if \(f_A(x) > 0\) for all nonzero \(x \in \mathbb{R}^n\). Let \(A, B\) be any tensors of order \(p\) and \(q\) respectively. Now we denote \(\lfloor p+q \rfloor := \{1, 2, \ldots, p + q\}\) and let \(\lfloor p+q \rfloor = S \cup T\) be a proper partition of set \(\lfloor p+q \rfloor\) where the carnalities of \(S\) and \(T\) are respectively \(p\) and \(q\). For convenience, we write \(S = \{s_1, s_2, \ldots, s_p\}\) and \(T = \{t_1, t_2, \ldots, t_q\}\),
both in increasing order. Then we denote $C := A \times_T B$ for the outer-product of $A$ and $B$, defined by

$$C_{i_1 \ldots i_p i_{p+1} \ldots i_{p+q}} = A_{i_S} B_{i_T}$$

(2.1)

where $i_S := (i_{s_1}, i_{s_2}, \ldots, i_{s_p}), i_T := (i_{t_1}, i_{t_2}, \ldots, i_{t_q})$. $C$ is called the outer-product of $A$ with $B$ along mode-$T$, which is a tensor of order $p + q$. Note that the out-product of $m$ (column) vectors produces a tensor of order $m$.

We denote $x^n := x \times x \times \ldots \times x$ for any $x \in \mathbb{R}^n$ for our convenience. Thus $x^m$ is a rank-1 $m$th order $n$-dimensional symmetric tensor. In the following example, we consider the outer-product of two $n \times n$ real matrices.

Example 2.1. Let $A, B \in \mathbb{R}^{n \times n}$. There are six different out-products for $(A, B)$, each product $A \times_\theta B$ is a 4-order $n$-dimensional tensor where $\theta$ is any 2-set of $[4]$, i.e., $\theta \in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$. Furthermore, there are three different products when $B = A$, i.e.,

$$A \times_{(1, 2)} A, \quad A \times_{(1, 3)} A, \quad A \times_{(1, 4)} A$$

since $A \times_\theta A = A \times_{\theta'} A$ for any 2-set $\theta \subset [4]$.

Note that generally these three tensors are different. For example, let $A = I_n$, the identity matrix. Then we have

$$(I_n \times_{(1, 2)} I_n)_{i_1 i_2 i_3 i_4} = \delta_{i_1 i_2}$$

and

$$(I_n \times_{(1, 3)} I_n)_{i_1 i_2 i_3 i_4} = \delta_{i_2 i_4}$$

for any index $(i_1, i_2, i_3, i_4) \in S(4, n)$, where $\delta_{ij}$ is the Kronecker constant ($\delta_{ij} \in \{0, 1\}$ and $\delta_{ij} = 1 \iff j = i$).

Sometimes we need to reduce or preserve the order of tensors by multiplication. For this purpose, we introduce the contractive multiplications of tensors, which may be regarded as the extension of the Einstein multiplications of tensors. Let $A \in \mathcal{T}_{p,n}, B \in \mathcal{T}_{q,n}$ and let $S \subset [p], T \subset [q]$ with $r = |S| = |T|$ ($1 \leq r \leq \min(p, q)$). Here $|S|$ denotes the cardinality of a set $S$. Denote $m = p + q - 2r$. The Einstein product of $A$ with $B$ along mode-$(S, T)$ as an $m$-order tensor $C := A \times_{(S, T)} B$ which is defined by

$$C_\theta = \sum_{i_S} A_{\theta_S} B_{\tau_T}$$

where $\theta_S \in S(p, n), \tau_T \in S(q, n)$. For example, if $A, B \in \mathcal{T}_{4,n}$, and $S = \{3, 4\}, T = \{1, 2\}$. Then we have $C := A \times_{(S, T)} B \in \mathcal{T}_{4,n}$ whose entries are

$$C_{i_{12} i_{34}} = \sum_{j_{12}} A_{i_{12} j_{12}} B_{j_{12} i_{34}}$$
Lemma 2.2. For any tensor $A \in T_{m,n}$, $B \in \mathbb{R}^{n \times p}$. Then an $k$-mode multiplication of $A$ by $B$ from the right side, denoted $A \times_k B$, is defined by

$$(A \times_k B)_{i_1 \ldots i_k i_{k+1} \ldots i_n} = \sum_{j=1}^{n} A_{i_1 \ldots i_{k-1} j i_{k+1} \ldots i_n} B_{j i_k}$$

(2.2)

Sometimes we briefly denote it by $AB$ when $k = n$. Similarly, the $k$-mode multiplication of $A$ by $B$ from the left, denoted $B \times_k A$, is defined by

$$(B \times_k A)_{i_1 \ldots i_k i_{k+1} \ldots i_m} = \sum_{j} A_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_m} B_{k j}$$

(2.3)

We denote $[B]A = B \times_1 A \times_2 \ldots \times_k A$ when $A \in T_{m,n}, B \in \mathbb{R}^{n \times n}$.

The contractive product of an $m$th order $n$-dimensional symmetric tensor $A$ with an $n$-dimensional vector $x$ in all modes yields an $m$-degree $n$-variate homogeneous polynomial $f(x) := A x^{m}$, and $y := A x^{m-1}$, which is defined as a vector $y = (y_1, y_2, \ldots, y_n)^\top$ with

$$y_i = \sum_{i_2, i_3, \ldots, i_m} A_{i_2 i_3 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}, \quad i = 1, 2, \ldots, n$$

where the summation is over all $j_1, j_2 \in [n]$. Moreover, if $A \in T_{m,n}, B \in \mathbb{R}^{n \times p}$. Then an $k$-mode multiplication of $A$ by $B$ from the right side, denoted $A \times_k B$, is defined by

$$(A \times_k B)_{i_1 \ldots i_k i_{k+1} \ldots i_n} = \sum_{j=1}^{n} A_{i_1 \ldots i_{k-1} j i_{k+1} \ldots i_n} B_{j i_k}$$

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$$y_i = \sum_{i_2, i_3, \ldots, i_m} A_{i_2 i_3 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}, \quad i = 1, 2, \ldots, n$$

Now we consider the linear space $T_{4,n}$, the set of all 4-order $n$-dimensional real tensors. Let $A, B \in T_{4,n}$. The product $C = A \times B \in T_{4,n}$ is defined as

$$C_{i_1 i_2 i_3 i_4} = \sum_{j_1, j_2} A_{i_1 i_2 j_1 j_2} B_{j_1 j_2 i_3 i_4}$$

(2.5)

for any $(i_1, i_2, i_3, i_4) \in S(4,n)$. In this case, we may define the identity tensor $I = (\epsilon_{i_1 i_2 i_3 i_4}) \in T_{4,n}$ as $\epsilon_{i_1 i_2 i_3 i_4} = \delta_{i_1 i_3} \delta_{i_2 i_4}$ where $\delta_{ij}$ is the Kronecker constant, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for all distinct $i, j$. It is easy to see that $I = I_n \times (1,3) I_n$. We can also show that

**Lemma 2.2.** For any tensor $A \in T_{4,n}$, we have

$$A \times I = I \times A = A$$

(2.6)

**Proof.** We show the equality $A \times I = A$. For any given index $(i_1, i_2, i_3, i_4) \in S(4,n)$, we have

$$(A \times I)_{i_1 i_2 i_3 i_4} = \sum_{j_1, j_2} A_{i_1 i_2 j_1 j_2} I_{j_1 j_2 i_3 i_4}$$

$$= \sum_{j_1, j_2} A_{i_1 i_2 j_1 j_2} \delta_{j_1 i_3} \delta_{j_2 i_4}$$

$$= A_{i_1 i_2 i_3 i_4}$$

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Thus we have $A \times I = A$. Similarly we can also prove the equality $I \times A = A$.

We are now ready to define the tensor form of high order derivatives (HOD) of a multivariate function. Let $f(x) = f(x_1, x_2, \ldots, x_n)$ be the function defined on $\mathbb{R}^n$ which maps $\mathbb{R}^n$ to $\mathbb{R}$. Then the first derivative of $f$ with respect to $x$, also called the gradient of $f$, denoted by $\frac{df}{dx}$, is defined as $\frac{df}{dx} := (\frac{df}{dx_1}, \frac{df}{dx_2}, \ldots, \frac{df}{dx_n})^\top \in \mathbb{R}^n$. The second derivative of $f$ is defined accordingly by $\frac{d^2f}{dx^2} = \frac{df}{dx}(\frac{df}{dx})$, which yields the definition

$$H = (h_{ij}) \in \mathbb{R}^{n \times n} \text{ is called the Hessian matrix of } f(x).$$

Note that sometimes the Hessian of $f$ refers to the determinant of the Hessian matrix. Here we only concern the Hessian matrix. The higher order derivatives of $f(x)$, i.e., $\frac{d^k f}{dx^k}$ for $k > 2$, is bit of more complicate traditionally since all $\frac{d^k f}{dx^k}$ are defined in matrix form which is achieved by recursive vectorization of the matrix $\frac{d(k-1)f}{dx(k-1)}$ before the next derivative, i.e.,

$$H^{(k+1)} := \frac{d^{(k+1)}f}{dx^{(k+1)}} = \frac{d}{dx}(\text{vec}(\frac{d^k f}{dx^k})) \\ (2.8)$$

Thus $\frac{d^k f}{dx^k} \in \mathbb{R}^{n \times n^{k-1}}$ for all $k \geq 2$. Note that $H^{(1)} \in \mathbb{R}^n$ and $H = H^{(2)} \in \mathbb{R}^{n \times n}$ is exactly the Hessian of $f$. This conventional definition of HOD of $f$ ambiguous the meaning of each element when $k > 2$. A more natural definition is the following:

$$\mathcal{H}^{(k)} := (H_{i_1i_2\ldots i_k}), \quad H_{i_1i_2\ldots i_k} = \frac{d^k f}{dx_{i_1}dx_{i_2}\ldots dx_{i_k}} \\ (2.9)$$

Thus we call $\mathcal{H}^{(k)}$ the $k$-order Hessian tensor of $f(x)$. Note that $\mathcal{H}^{(k)}$ is symmetric due to the commutivity of derivatives of $f$. Hence we have $\mathcal{H}^{(k)} \in ST_{kn}$.

Now we consider the derivatives of a matrix variable $Y \in \mathbb{R}^{m \times n}$ with respect to another matrix variable $X \in \mathbb{R}^{m \times n}$ (taking each entry of $Y$ as a function of the elements of $X$). Conventionally this is defined as a matrix

$$H = (H_{ij}) \in \mathbb{R}^{mn \times mn} \text{ with } H_{ij} = \frac{\partial y_{ij}}{\partial x_{j_1j_2}} \text{ where}$$

$$i = (j_1 - 1)m + i_1, \quad j = (j_2 - 1)m + i_2 \\ (2.10)$$

where $0 \leq i_1, i_2 < m$ and $1 \leq j_1, j_2 \leq n$. Note that

$$\frac{\partial Y}{\partial X} = \frac{\partial \text{vec}(Y)}{\partial \text{vec}(X)}^\top$$
It follows that \( \frac{\partial y}{\partial x} = (h_{ij}) \) with \( h_{ij} = \frac{\partial y_i}{\partial x_j} \) when \( x, y \) are both vectors.

The two equations in (2.10) are obtained from the division theorem with remainder properties. Note that when \( i_1 = 0 (i_2 = 0) \), we replace \( j_1(j_2) \) by \( j_1 - 1(j_2 - 1) \) and let \( i_1 = n (i_2 = n) \). The definition provokes some conveniences especially when coping with the \( k \)th derivative of \( Y \) w.r.t. \( X \), as seen from the above. Now we introduce the tensor form of the derivatives.

**Definition 2.3.** Let \( X, Y \in \mathbb{R}^{m \times n} \) where each entry \( y_{ij} \) is regarded as a function of \( mn \) variables \( \{x_{ij}\} \). We define the derivative \( \frac{\partial Y}{\partial X} \) as the 4-order tensor \( A = (A_{i_1i_2j_1j_2}) \) with

\[
A_{i_1i_2j_1j_2} = \frac{\partial y_{i_1j_1}}{\partial x_{i_2j_2}} \tag{2.11}
\]

which is of size \( m \times m \times n \times n \). Now recursively we define the \( k \)th derivative as

\[
\frac{\partial^k Y}{\partial X^k} = \frac{\partial}{\partial X} \frac{\partial^{k-1} Y}{\partial X^{k-1}}
\]

If we denote \( A^{(k)} := \frac{\partial^k Y}{\partial X^k} \), then \( A^{(k)} \) is an \( 2(k + 1) \)-order tensor, with

\[
A_{i_1i_2...i_{k+1}j_1j_2...j_{k+1}} = \frac{\partial^k y_{i_1j_1}}{\partial x_{i_2j_2} \ldots \partial x_{i_{k+1}j_{k+1}}} \tag{2.12}
\]

When both \( X, Y \) are reduced to vectors, \( A^{(k)} \) reduces to an \( (k + 1) \)-order tensor.

In the next section, we will use the derivative tensors to present the higher order moments of random vectors and random matrices.

### 3 High order tensor moments

High order moments can be expressed in the form of tensors which can simplify their expressions. The covariance of a random variable \( x \) is the second central moment of \( x \), i.e., \( \text{Var}(x) = E[(x - E[x])^2] \), and the covariance matrix of a random vector \( x \in \mathbb{R}^n \), defined as

\[
\text{Var}(x) = E[(x - E[x])(x - E[x])'] = E[xx'] - E[x]E[x]^\top = m_2 - \mu^2
\]

can be regarded as a function of \( m_1 \) and \( m_2 \). Furthermore, the covariance matrix of a random matrix \( X \in \mathbb{R}^{m \times n} \) is conventionally defined as

\[
\text{Cov}(X) = E[(x - \mu) \times (x - \mu)] \quad \text{(3.1)}
\]

(note that \( x \times y = xy^\top \) for any column vectors \( x, y \) where \( x = \text{vec}(X), \mu = \text{vec}(E[X]) = E[\text{vec}(X)] \in \mathbb{R}^{mn} \). Thus \( \text{Cov}(X) \) is a PSD \( mn \times mn \) matrix. However, the definition (3.1) ruins the structure of \( X \) and thus makes the
Similarly the risk of confusion. Note that any arbitrary matrix. Then the matricization of a random vector vec \( X \) when

\[
\text{Cov}(X) = E[(X - E[X]) \times (1, 3)] (X - E[X])
\]

(3.2)

Here we use outer-product \( A \times (1, 3) A \), which is the same to \( A \times (2, 4) A \), to make the size of tensor \( C \) as \( m \times m \times n \times n \). Specifically, the 4-order tensor \( C = (C_{i1i2j1j2}) \in \mathbb{R}^{m \times m \times n \times n} \) is defined by

\[
C_{i1i2j1j2} = \text{Cov}(x_{i1j1}, x_{i2j2}) = E[(x_{i1j1} - \mu_{i1j1})(x_{i2j2} - \mu_{i2j2})]
\]

(3.3)

where \( \mu = (\mu_{ij}) = E[X] \in \mathbb{R}^{m \times n} \).

An important tool for deriving moments is the characteristic function (CF). Let \( u \in \mathbb{R} \) be a random variable. We denote by \( \varphi_u(t) \) the CF of \( u \), which is defined by

\[
\varphi_u(t) = E[e^{itu}]
\]

(3.4)

where \( i \) is the imaginary unit. Let the CF \( \varphi_X(t) \) be \( k \) times differentiable. Then the \( k \)-moment of \( x \in \mathbb{R}^n \) equals

\[
m_k[x] = \frac{1}{i^k} \frac{d^k}{dt^k} \varphi_X(t) \bigg|_{t=0}
\]

(3.5)

Similarly the \( k \)-central moment of a random vector \( x \in \mathbb{R}^n \) is given by

\[
\bar{m}_k[x] = m_k[x - E[x]] = \frac{1}{i^k} \frac{d^k}{dt^k} \varphi_{x - E[x]}(t) \bigg|_{t=0}, \quad t \in \mathbb{R}^p
\]

(3.6)

From (3.5) we have \( m_1 \in \mathbb{R}^n \) and \( m_2 \) is an \( n \times n \) matrix. The definition of \( m_k \) for \( k > 2 \) is constraint by that of the high order derivative of a multivariate function.

A matrix is called a random matrix if each of its entries is a random variable. A random matrix \( X \in \mathbb{R}^{p \times q} \) can be regarded as a consequence of matricization of a random vector \( \text{vec}(X) \in \mathbb{R}^{pq} \). We suppose the characteristic function \( \varphi_X(T) \) of \( X \) be \( k \) times differentiable with \( T \in \mathbb{R}^{p \times q} \) being any arbitrary matrix. Then the \( k \)-moment of \( X \) is defined by

\[
m_k[X] = \frac{1}{i^k} \frac{d^k}{dT^k} \varphi_X(T) \bigg|_{T=0}
\]

(3.7)

Similarly the \( k \)-central moment \( \bar{m}_k[X] \) of \( X \) is defined by

\[
\bar{m}_k[X] = \frac{1}{i^k} \frac{d^k}{dT^k} \varphi_{X - E[X]}(T) \bigg|_{T=0}
\]

(3.8)

Sometimes we denote by \( m_k(\bar{m}_k) \) instead of \( m_k[X] \) (\( \bar{m}_k[X] \)) if there is no risk of confusion. Note that \( m_k(\bar{m}_k) \) is a tensor of order 2\( k \) whose size is

\[
\underbrace{m \times m \times \ldots \times m}_{k} \times \underbrace{n \times n \times \ldots \times n}_{k}
\]

when \( X \in \mathbb{R}^{p \times q} \). We have
Theorem 3.1. Let $X = (x_{ij}) \in \mathbb{R}^{p \times q}$ be a random matrix and let $M = E[X]$ be the mean matrix of $X$. Then its $k$-moment tensor and the $k$-central moment tensor are respectively

$$m_k[X] = E[X^{\times k}]$$  \hspace{1cm} (3.9)

and

$$\bar{m}_k[X] = E[(X - M)^{\times k}]$$  \hspace{1cm} (3.10)

where $X^{\times k} := \overbrace{X \times X \times \ldots \times X}^{k}$ is the $k$th power of $X$ in the sense of outer-product.

Proof. We denote by $\phi = \phi_X(T)$ for simplicity. By the tensor form of the higher order derivatives defined in (2.12), we have

$$\frac{d^k \phi}{dT^k} = \frac{1}{k!} E[\exp i \langle T, X \rangle X^{\times k}]$$  \hspace{1cm} (3.11)

Thus by (3.7), we get (3.9). Similarly we can prove (3.10).

For any given positive integers $n, s$ where $1 \leq s \leq n$, we denote

$$\pi_s(n) := \{\theta_s := (i_1, i_2, \ldots, i_s) : 1 \leq i_1 < i_2 < \ldots < i_s \leq n\}$$

and $\pi_0 := \emptyset$ (the empty set). Now let $\mathcal{A}, \mathcal{B}$ be tensors of order $p$ and $q$ respectively, and let $p + q = n$. The outer-product $\mathcal{A} \times_{\theta_s} \mathcal{B}$ is a tensor of order $n$ as defined by (2.1). For $s = 0$, $\pi_s(n) = \emptyset$, and we denote $\mathcal{A} \times_0 \mathcal{B} = \mathcal{A}$.

On the other hand, we denote $\mathcal{A} \times_n \mathcal{B} = \mathcal{B}$ if $s = n = q$.

From Theorem 3.1 we have

Corollary 3.2. Let $x = (x_j) \in \mathbb{R}^n$ be a random vector with mean vector $\mu = E[x]$. Then its $k$-moment (central moment) is the $k$-order $n$-dimensional tensor $m_k[x] = E[x^{\times k}]$ (or $\bar{m}_k[X] = E[(x - \mu)^{\times k}]$). Furthermore, we have

$$\bar{m}_k = \sum_{s=0}^{k} (-1)^s \sum_{\theta_s \in \pi_s} m_{k-s} \times_{\theta_s} \mu^s$$  \hspace{1cm} (3.12)

Proof. The first part of the corollary is immediate from Theorem 3.1 and (3.12) can be obtained by

$$\bar{m}_k = E[(x - \mu)^{\times k}]$$

$$= E[\sum_{s=0}^{k} (-1)^s \sum_{\theta_s \in \pi_s(k)} x^{k-s} \times_{\theta_s} \mu^s]$$

$$= \sum_{s=0}^{k} (-1)^s \sum_{\theta_s \in \pi_s(k)} E[x^{k-s}] \times_{\theta_s} \mu^s$$

$$= \sum_{s=0}^{k} (-1)^s \sum_{\theta_s \in \pi_s(k)} m_{k-s} \times_{\theta_s} \mu^s$$
By Corollary 3.2, we have $\bar{m}_2 = m_2 - \mu^2 \in \mathbb{R}^{n \times n}$ which is the covariance of $x$, and

$$\bar{m}_3 = m_3 - \sum_{k=1}^{3} m_2 \times_k \mu + 2\mu^3 \quad (3.13)$$

We note that the entries of the $k$-moment of a random vector $x \in \mathbb{R}^n$, by Corollary 3.2, is

$$(m_k)_{i_1 i_2 \ldots i_k} = E[x_{i_1} x_{i_2} \ldots x_{i_k}] \quad (3.14)$$

which conforms to the traditional definition when $m_k$ is flattened to a matrix.

### 4 Higher order moments of multivariate Gaussian distribution

Gaussian distribution is the most basic and important distribution in statistics. The density function of a Gaussian vector $x \in \mathbb{R}^n$ with mean vector $\mu \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n \times n}$ (usually assumed to be nonsingular), is

$$f_x(t) = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left\{-\frac{1}{2} \text{Tr}\left\{\Sigma^{-1}(t - \mu)(t - \mu)^\top\right\}\right\} \quad (4.1)$$

where $t \in \mathbb{R}^n$ is arbitrary. A random matrix $U \in \mathbb{R}^{m \times n}$ is called a Standard Normal matrix or a SN-matrix if

$$\text{vec}(U) \sim N_{mn}(0, I_{mn}) \quad (4.2)$$

where $I_k$ stands for the identity matrix. A SN-matrix $U \in \mathbb{R}^{m \times n}$ is denoted by $U \sim N_{m,n}(0, I_m, I_n)$, meaning that all the columns $u_j (1 \leq j \leq n)$ of $U$ are i.i.d. with $u_j \sim N_m(0, I_m)$ and all rows $w_i (1 \leq i \leq m)$ of $U$ are i.i.d. with $w_i \sim N_n(0, I_n)$. A random matrix $X \in \mathbb{R}^{m \times n}$ is called a Gaussian matrix if there exist constant matrices $\mu \in \mathbb{R}^{m \times n}, A \in \mathbb{R}^{m \times p}, B \in \mathbb{R}^{n \times q}$ such that

$$X \overset{\text{d}}{=} \mu + AB\top, \quad U \sim N_{p,q}(0, I_p, I_q) \quad (4.3)$$

where $U \overset{\text{d}}{=} V$ means that two random variables (vectors, matrices) $U, V$ have the same distribution. This is denoted by $X \sim N_{m,n}(\mu, \Sigma_1, \Sigma_2)$. In this situation, we call $X$ is a Gaussian matrix with parameters $\mu, \Sigma_1, \Sigma_2$ where $\Sigma_1 := AA\top, \Sigma_2 := BB\top$.

Let $\mu \in \mathbb{R}^{m \times n}, \Sigma_1 \in \mathbb{R}^{m \times m}, \Sigma_2 \in \mathbb{R}^{n \times n}$ with $\Sigma_k$ being PSD. It is shown\[8] that

**Lemma 4.1.** Let $X \in \mathbb{R}^{m \times n}$ be a random matrix. Then $X \sim N_{m,n}(\mu, \Sigma_1, \Sigma_2)$ if and only if it satisfies
where \( T \) for each \( j \) we have three 2-partitions of set \( [4] := [a] \) where we define \( \sum_k \). Furthermore, if \( k \) is called a 2-partition of set \( [2 \times m] \), which satisfies the recurrence \( \gamma_i = \begin{cases} s_i, & \text{for } k = 1, 2; \\
 s_i, & \text{for } k = 3, 4. \end{cases} \)

An immediate corollary from Lemma 4.1 is

**Corollary 4.2.** Let \( \mu \in \mathbb{R}^{m \times n}, \Sigma_1 \in \mathbb{R}^{m \times m}, \Sigma_2 \in \mathbb{R}^{n \times n} \) with \( \Sigma_k (k = 1, 2) \) both are positive definite, then the density function of \( X \) is

\[
f_X(T) = (2\pi)^{-mn/2} \det(\Sigma_1)^{-n/2} \det(\Sigma_2)^{-m/2} \exp \{ \psi(T) \}
\]

where \( T \in \mathbb{R}^{m \times n} \) is arbitrary and \( \psi(T) := -\frac{1}{2} \text{Tr} \Sigma_1^{-1}(T - \mu) \Sigma_2^{-1}(T - \mu)\top \).

Given an even integer \( k = 2m, m \geq 1 \). A partition \( \gamma := \{ \gamma_1, \gamma_2, \ldots, \gamma_m \} \) of set \( [k] \) is called a 2-partition if \( [k] = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_m \) with \( |\gamma_j| = 2 \) for each \( j \in [m] \). Denote by \( \Gamma_2[k] \) the set of all 2-partitions of \([k]\) and let \( a_m := |\Gamma_2[k]| \) denote the cardinality of \( \Gamma_2[k] \). Then

\[
a_m = (2m - 1)!!
\]

where we define \( a_m = 1 \) when \( m \leq 1 \). There are many methods (e.g. the graph theory) to prove (4.6). Since a 2-partition of set \([2m]\) corresponds to a 1-factor of a complete graph \( K_{2m} \), \( a_m \) is exactly the number of the 1-factors of \( K_{2m} \), which satisfies the recurrence \( a_m = (2m - 1)a_{m-1} \), by which \([4.6]\) follows.

Now we extend the 2-partitions of a set \([k]\) for any positive integer \( k \). For any integer \( s \) with \( k \geq 2s \geq 0 \), we let \( W \) be a subset of \([k]\) with \(|W| = k - 2s \), and \( \gamma := \{ \gamma_1, \ldots, \gamma_s \} \) be a 2-partition of \( W^c := [k] \setminus W \) \((W = \emptyset \text{ if } s = \lfloor k/2 \rfloor)\). Define

\[
\tilde{\gamma} := \{ \gamma_1, \gamma_2, \ldots, \gamma_s, \gamma_{s+1} \}
\]

with \( \gamma_{s+1} = W \). Then \( \tilde{\gamma} \) is a partition of \([k]\). We call \( \tilde{\gamma} \) a \([s, 2]-\text{partition} \) of \([k]\), and denote \( \Pi(s, k) \) the set of all \([s, 2]-\text{partitions of } [k]\).

A 2-partition \( \gamma \) uniquely determines the pattern of the \( m \)th power of a matrix \( A \) in terms of the outer product. For example, when \( m = 2(k = 4) \), we have three 2-partitions of set \([4] := \{1, 2, 3, 4\} \), that is,

\[
\{1, 2\} \cup \{3, 4\}, \quad \{1, 3\} \cup \{2, 4\}, \quad \{1, 4\} \cup \{2, 3\}.
\]

\[\text{1}\gamma_{s+1} \text{ may be an empty set.}\]
Thus we have three different patterns of $I_n \times I_n \times I_n$, i.e.,

$$I_n \times (1,2) I_n, \quad I_n \times (1,3) I_n, \quad I_n \times (1,4) I_n.$$ 

For any matrices $A_1, A_2, \ldots, A_m$ and any $\gamma \in \Gamma_2[m]$, we denote

$$A^\gamma := A_1 \times_{\gamma_2} A_2 \times_{\gamma_3} A_3 \times \ldots \times_{\gamma_m} A_m \quad (4.7)$$

Note that (4.7) is well-defined since the outer-product satisfies the associativity law. Moreover, (4.7) is denoted by $I_n^\gamma$ when $A_1 = A_2 = \ldots = A_m = I_n$. Obviously $I_n^\gamma \in \mathcal{T}_{k,n}$. For any given index $\sigma := (i_1, i_2, \ldots, i_k)$ $(k = 2m)$, we have

$$(I_n^\gamma)_\sigma = \delta_{i_1} \delta_{i_2} \ldots \delta_{i_m} \quad (4.8)$$

where $\delta_{i_\alpha} := \delta_{i_\alpha} \gamma_\alpha$ if $\gamma_\alpha \in \{s,t\}$.

The following result gives the expressions for $k$-order moment of a SND vector.

**Theorem 4.3.** Let $u \in \mathbb{R}^n$ be a SND random vector and let $m_k$ denotes the $k$-order moment of $\mathbf{u}$. Then

(1). $m_k = 0 \in \mathcal{T}_{k,n}$ for all odd integer $k = 1, 3, 5, \ldots$

(2). For all even integers $k = 2m$, we have

$$m_k = \sum_{\gamma \in \Gamma_2} \mathcal{T}_{n}^\gamma \quad (4.9)$$

**Proof.** Let $u = (u_1, u_2, \ldots, u_n)^\top$ where $n > 1$. For any given positive integers $k > 1$, we denote

$$R[n,k] := \{ \alpha = (r_1, r_2, \ldots, r_n) : r_1 + r_2 + \ldots + r_n = k, r_s \in [k], \forall s \in [n] \}$$

For any $\phi = (i_1, i_2, \ldots, i_k) \in S(m,n)$, we call $\phi$ an $\alpha$-type where $\alpha = (r_1, r_2, \ldots, r_n) \in R[n,k]$, if for each $s \in [k]$,

$$r_s = |\{ t : i_t = s, t = 1, 2, \ldots, k \}|$$

that is equivalent to the condition $x_{i_1} x_{i_2} \ldots x_{i_k} = x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}$ for any vector $x = (x_1, x_2, \ldots, x_n)^\top$. For any index $\tau := (i_1, i_2, \ldots, i_k) \in S(k,n)$, suppose $\tau$ is $\alpha$-type where $\alpha = (r_1, r_2, \ldots, r_n) \in R[n,k]$. Then

$$(m_k)_{i_1 i_2 \ldots i_k} = E[u_{i_1} u_{i_2} \ldots u_{i_k}] = E[u_1^{r_1} u_2^{r_2} \ldots u_n^{r_n}] = E[u_1^{r_1}] E[u_2^{r_2}] \ldots E[u_n^{r_n}]$$

The last equality follows from the fact that $u_1, u_2, \ldots, u_n$ are independent since $u \sim \mathcal{N}_n(0, I_n)$. Note that $E[u_j^{r_j}] = 1$ if $r_j = 0$. 

Now we prove the first item. Let \( k > 1 \) be any odd integer and \( \tau := (i_1, i_2, \ldots, i_k) \in S(k, n) \) be an \( \alpha \)-type. Then there exists \( s \in [n] \) such that \( r_s \) is odd, it follows that \( E[u_{rs}^s] = 0 \) since \( u_s \sim N(0, 1) \). By (4.10) we immediately get \( (m_k)_{i_1i_2\ldots i_k} = 0 \). Consequently we have \( m_k = 0 \) for all odd integer \( k \).

To prove the second item, we let \( k = 2m \) \((m = 1, 2, \ldots)\) and denote the right hand side of (4.9) by \( A \). For any given \( \sigma \in S(k, n) \), let \( \sigma \) be a \((r_1, r_2, \ldots, r_n)\)-type. We want to show that \( A_\sigma = (m_k)_\sigma = \lambda_\sigma \) where \( \lambda_\sigma \) is defined as

\[
\lambda_\sigma = \prod_{i=1}^{n} (r_i - 1)!!
\]

(4.10)

For this purpose, we write

\[
P(\sigma) := \{ j \in [n] : r_j > 0 \} = \{ j_1, j_2, \ldots, j_T \},
\]

and let \(|P(\sigma)| = T\), i.e., the number of positive \( r_i \)'s, which is related to \( \sigma \). We call \( \sigma \) an essentially \( r[P(\sigma)] \)-type index. Then

\[
(m_k)_\sigma = E[x_{i_1} x_{i_2} \ldots x_{i_k}]
\]

\[
= E[x_{i_1}^r x_{i_2}^r \ldots x_{i_k}^r]
\]

\[
= E[x_{j_1}^r x_{j_2}^r \ldots x_{j_T}^r]
\]

It follows that

\[
(m_k)_\sigma = E[x_{j_1}^r] E[x_{j_2}^r] \ldots E[x_{j_T}^r]
\]

(4.11)

If there is a \( t \in [T] \) such that \( r_j \) is an odd number, then \( E[x_{j_T}^r] = 0 \) by (1) and thus \((m_k)_\sigma = 0\) by (4.11). It follows that each nonzero entry of \( m_k \) is associated with a \( \sigma \in S(k, n) \) of a \((r_1, r_2, \ldots, r_n)\)-type where each \( r_i \) is even (including 0). This fact is coincident with that of \( A \) as we can verify by simple deduction. Furthermore, we have

\[
A_\sigma = \sum_{\gamma} \delta_{i_{r_1}} \delta_{i_{r_2}} \ldots \delta_{i_{r_m}}
\]

by (4.8). Since \( \sigma \) is \((r_1, r_2, \ldots, r_n)\)-type or essentially \( r[P(\sigma)] \)-type where each \( r_j \) is even, there are

\[
(r_{j_1} - 1)!!(r_{j_2} - 1)!! \ldots (r_{j_T} - 1)!!
\]

(4.12)

2-partitions \( \gamma \) of \( \{i_1, i_2, \ldots, i_k\} \) such that \( \delta_{i_{r_1}} \delta_{i_{r_2}} \ldots \delta_{i_{r_m}} = 1 \). It turns out that \( A_\sigma = \lambda_\sigma \). On the other hand, we have

\[
(m_k)_\sigma = E[x_{j_1}^r x_{j_2}^r \ldots x_{j_T}^r]
\]

\[
= E[x_{j_1}^r] E[x_{j_2}^r] \ldots x_{j_T}^r
\]

\[
= (r_{j_1} - 1)!!(r_{j_2} - 1)!! \ldots (r_{j_T} - 1)!!
\]

\[
= \lambda_\sigma
\]

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Consequently we have $A_{\sigma} = (m_k)_{\sigma}$ for all $\sigma = (i_1, i_2, \ldots, i_k) \in S(k, n)$ for $k = 2m$. The proof is completed. 

It is obvious from the proof of Theorem 4.3 that

**Corollary 4.4.** Let $x \sim N(0, 1)$ be a SND random variable. Then its 2n-order moment $m_{2n} = (2n - 1)!!$.

**Corollary 4.5.** Let $u \in \mathbb{R}^n$ be a SND random vector. Then its 4-order moment $m_4$ is

$$m_4 = I_n \times \{1, 2\} I_n + I_n \times \{1, 3\} I_n + I_n \times \{1, 4\} I_n \quad (4.13)$$

Corollary 4.5 can be deduced directly by Theorem 4.3 and Example 4.1.

Here we present an alternative proof to double check the result from the different aspects.

**Proof.** For convenience, we denote by $A$ the right hand side of (4.13). Then $A = (A_{ijkl}) \in T_{4; n}$. We need to show that $M_{ijkl} = A_{ijkl}$ for all $\{i, j, k, l\} \in S(4, n)$ where $M_{ijkl}$ is the element of $m_4$ indexed by $(i, j, k, l)$. Denote by $t$ the number of distinct elements in $\{i, j, k, l\}$. Then $t \in [4]$. We need only to consider the following five cases in terms of $t$ based on the symmetry of $m_4$ and $A$.

1. $t = 1$, i.e., $i = j = k = l \in [n]$. Then $A_{iiii} = 3$ by the definition of $A$. On the other hand,

$$M_{iiii} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u_i^4 \exp \left\{ -\frac{1}{2} u_i^\top u_i \right\} du_i \nabla = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} u_i^3 \exp \left\{ -\frac{1}{2} u_i^2 \right\} du_i = 3$$

Thus we have $M_{iiii} = A_{iiii}$ for all $i \in [n]$.

2. $t = 2$. There are two subcases for this situation.

   i. $i = j = k \neq l$. Then

   $$M_{iiil} = \int_{\mathbb{R}^n} u_i^3 u_l f(u) du$$

   $$= ((2\pi)^{-1/2} \int_{-\infty}^{+\infty} u_i^3 \exp \left\{ -\frac{1}{2} u_i^2 \right\} du_i) ((2\pi)^{-1/2} \int_{-\infty}^{+\infty} u_l \exp \left\{ -\frac{1}{2} u_l^2 \right\} du_l)$$

   $$= 0$$

   where $f(u) = (2\pi)^{-n/2} \exp \left\{ -\frac{1}{2} u^\top u \right\}$ is the pdf of $u$. On the other hand, we have

   $$A_{iiil} = \delta_{ii} \delta_{il} + \delta_{il} \delta_{ii} + \delta_{ii} \delta_{il} = 0$$

   since $\delta_{il} = 0$ ($\delta_{ij}$ is the Kronecker constant with $\delta_{ij} = 1$ iff $i = j$). This confirms $M_{iiil} = A_{iiil}$ in this subcase.
• $i = j \neq k = l$. Then

\[
M_{iikk} = \int u_i^2 u_k^2 f(u) du = \int u_i^2 \exp \left\{-\frac{1}{2} u_i^2 \right\} du_i \int u_k^2 \exp \left\{-\frac{1}{2} u_k^2 \right\} du_l = E[u_i^2] E[u_k^2] = \sigma_i^2 \sigma_k^2 = 1
\]

On the other hand, we have

\[
A_{iikk} = \delta_{ii} \delta_{kk} + \delta_{ik} \delta_{ik} + \delta_{ik} \delta_{ik} = 1
\]

This confirms \( M_{iikk} = A_{iikk} \) in this subcase.

(3) \( t = 3 \). We need to consider the case when \( i = j \) and \( i, k, l \) are distinct. Then by the above argument we have

\[
M_{iikl} = 0
\]

for all distinct \( i, k, l \in [n] \). On the other hand, we have

\[
A_{iikl} = \delta_{ii} \delta_{kl} + \delta_{ik} \delta_{il} + \delta_{il} \delta_{ik} = 0
\]

Thus \( M_{iikl} = A_{iikl} \).

(4) \( t = 4 \), i.e., \( i, j, k, l \) are all distinct. Then similar argument as above yields \( M_{ijkl} = A_{ijkl} \) for all distinct \((i, j, k, l) \in S(4, n)\). This concludes the proof that \( m_4 = A \).

The next two lemmas will be used to prove our main result.

**Lemma 4.6.** Let \( u \in \mathbb{R}^n \) be a random vector and \( v = Au \) where \( A \in \mathbb{R}^{m \times n} \) is a constant matrix. Then

\[
m_k(v) = [A] m_k(u)
\]

where \([A] m_k(u) = A \times_1 \times_2 \ldots \times_k m_k(u)\).

**Proof.** (4.14) can be deduced by

\[
m_k(v) = E[(Au)^k] = E[[A]u^k] = [A] E[u^k] = [A] m_k(u).
\]

**Lemma 4.7.** Let \( u \sim N_n(0, I_n) \) and \( v = Au \) where \( A \in \mathbb{R}^{m \times n} \) is constant. Then we have

(1). For each odd \( k \), \( m_k(v) = 0 \).
(2). For each even \( k \),

\[
m_k(v) = \sum_{\gamma \in \pi_k} \Sigma^\gamma
\]

where \( \Sigma = AA^\top \) and \( \pi_k \) is the set of 2-partitions of set \([k]\).

**Proof.** The first item can be shown by Lemma 4.6 and \( m_k(u) = 0 \) for odd \( k \) due to (1) of Theorem 4.3. To prove the second item, we note from Lemma 4.6 that

\[
m_k(v) = [A][E[u^k]] = [A] \sum_{\gamma \in \pi_k} I_\gamma
\]  

The last equality is due to (2) of Theorem 4.3. Furthermore,

\[
[A]I_\gamma = \Sigma^\gamma
\]

Consequently we get (4.15) by combining (4.16) and (4.17).

Now we are ready to express the high order moments and central moments of a general Gaussian vector in terms of its mean vector and covariance matrix.

**Theorem 4.8.** Let \( x \in \mathbb{R}^n \) be a Gaussian vector with \( x \sim N_n(\mu, \Sigma) \) where \( \mu \in \mathbb{R}^n \) and \( \Sigma \in \mathbb{R}^{n \times n} \) is a positive semidefinite matrix. Then

\[
m_k[x] = \sum_{s=0}^{[k/2]} \sum_{\gamma \in \Pi(s,k)} (\Sigma \times_{\gamma_1} \Sigma \times_{\gamma_2} \ldots \Sigma \times_{\gamma_s} \Lambda^{1/2})
\]

**Proof.** Since \( x \sim N_n(0, \Sigma) \). We have \( x \overset{d}{=} \mu + Au \), where \( u \sim N_n(0, I_n) \) and \( A \in \mathbb{R}^{n \times n} \) is symmetric with \( A^2 = \Sigma \) (\( A \) is a square root of \( \Sigma \)). Then we have

\[
m_k[x] = E[x^k] = E[(\mu + Au)^k]
\]

\[
= E\left[\sum_{m=0}^{k} \sum_{\theta \in \Phi(k-m, m)} (Au)^m \times_\theta \mu^{k-m}\right]
\]

\[
= \sum_{m=0}^{k} \sum_{\theta \in \Phi(k-m, m)} (E[Au]^m) \times_\theta \mu^{k-m}
\]
where \( \Phi(p, m) := \{ \theta := (\theta_1, \theta_2, \ldots, \theta_p) : 1 \leq \theta_1 < \theta_2 < \ldots < \theta_p \leq m \} \). Since \( E[(Au)^m] = 0 \) for each odd \( m \) by Lemma 4.7 we have, by the last equality of (4.19)

\[
m_k[x] = \sum_{s=0}^{\lfloor k/2 \rfloor} \sum_{\theta \in \Phi(k-2s, 2s)} (E[(Au)^{2s}]) \times \theta \mu_k^{-2s}
\]

\[
= \sum_{s=0}^{\lfloor k/2 \rfloor} \sum_{\theta \in \Phi(k-2s, 2s)} ([A]E[u]^{2s}) \times \theta \mu_k^{-2s}
\]

\[
= \sum_{s=0}^{\lfloor k/2 \rfloor} \sum_{\theta \in \Phi(k-2s, 2s)} \sum_{\gamma \in \pi_{2s}} \gamma \times \theta \mu_k^{-2s}
\]

where \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s) \) is a 2-partition of \([m] = [2s]\). Set \( \gamma_{s+1} := \theta \in \Phi(k-2s, 2s) \) and denote \( \bar{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_s, \gamma_{s+1}) \). Then \( \bar{\gamma} \) is a pseudo 2-partition of \([k] \). (4.18) is immediate. The proof is completed. \( \square \)

We end this paper by pointing out that the higher order moments of a Gaussian matrix can also be expressed similarly in tensor form, which may be investigated in our future work.

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