HYPERGEOMETRIC FUNCTIONS ON REDUCTIVE GROUPS

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An essential feature of the classical theory of hypergeometric functions is the interplay between various ways of representing such functions: by integrals, solutions of differential equations, or series. The theory of \( A \)-hypergeometric functions \([12-17]\) unifies and generalizes a considerable part of the classical theory by putting into the forefront the concept of a torus action. From a purely algebraic point of view, considering the torus actions means that we are interested in how a polynomial breaks down into a sum of monomials. Thus, an \( A \)-hypergeometric function is a function in the coefficients of an indeterminate polynomial \( f(x) = \sum_{\omega \in A} a_\omega x^\omega \) and \( A \subset \mathbb{Z}^m \) is a finite set of characters of the algebraic torus \((\mathbb{C}^*)^m\), i.e., of Laurent monomials. For example, a typical Euler integral \([16]\) is the function

\[
J(f) = \int f(x)^{\alpha_0}x_1^{\alpha_1}...x_m^{\alpha_m}dx_1...dx_m.
\]

In the same vein, the \( A \)-hypergeometric series (or \( \Gamma \)-series) \([13-15]\) \([17]\) are explicit power series in the coefficients \( a_\omega \). On the geometric side, the theory becomes allied with the algebro-geometric theory of toric varieties \([10]\) \([18]\), the integral (0.1) being a period of a hypersurface in such a variety.

The purpose of this paper is to give a generalization of the theory of \( A \)-hypergeometric functions to the case when the torus is replaced by an arbitrary reductive group \( H \). The set \( A \) is then a finite set of irreducible representations of \( H \). The space of polynomials on a given set of monomials is replaced by the space \( M_A \) of functions on \( H \) obtained as linear combinations of matrix elements of representations from \( A \). Thus a hypergeometric function is a function on the space \( M_A \); for instance, the Euler integral is written, similarly to (0.1), by \( J(f) = \int f(x)^{\alpha}x^\sigma dx \) where \( x^\sigma \) is a multivalued character of \( H \). The related algebro-geometric theory is that of spherical varieties, i.e., of equivariant compactifications of reductive groups and their homogeneous spaces coming from Gelfand pairs \([3-5]\) \([7-8]\) \([32]\). In fact, one of the reasons for the author’s thinking about this subject was to begin understanding mirror symmetry for hypersurfaces in spherical varieties and its relation to the Langlands duality for reductive groups.

The space \( M_A \), being naturally a product of matrix spaces, can be regarded as a partial compactification of another reductive group \( G = \prod GL_{d(\omega)} \) where \( d(\omega), \omega \in A \) is the dimension of the representation. So the analogs of hypergeometric series in this new situation are, again, series in matrix elements of irreducible representations of \( G \). Their construction can be obtained by generalizing the idea implicit in \([14]\), namely, expanding the delta-function along a subgroup \( H \subset G \subset M_A \) into a series in matrix elements of
representations of \( G \) and then taking the termwise Fourier transform. This leads to a very general class of generalized power series. Among their coefficients one finds the Clebsch-Gordan coefficients and other classical quantities of representation theory. Given that in the toric case the combinatorics of the coefficients of hypergeometric series is at the basis of numerical predictions for the number of curves in mirror symmetry [1] [25-27] [46], we give in §6 an example where the coefficients involve the \( 3j \)-symbols for the group \( SU_2 \).

The paper is organized as follows. In section 1 we introduce notation for decomposing functions on a reductive group into matrix elements of irreducible representations, so as to emphasize the analogy with ordinary monomials and polynomials and make writing formulas easier. Section 2 studies the spherical varieties \( X_A, Y_A \) associated with a given set \( A \) of irreducible representations. These are not the most general spherical varieties, being compactifications of reductive groups, of the kind studied by De Concini and Procesi [7-8]. The varieties \( Y_A \) are in fact, reductive algebraic semigroups studied in [41] [49]. In Section 3 we discuss “power series” on a group, i.e., series in matrix elements of irreducible representations and their use in representing analytic functions and distributions. Section 4 studies the Fourier transform of matrix elements of irreducible representations of \( GL(n) \) and the corresponding analog of the gamma function. These questions have been investigated in [21-23]. Next, in Section 5 we introduce the \( A \)-hypergeometric system and develop the analogs of all three approaches known for the toric case: differential systems, Euler integrals, hypergeometric series. Finally, Section 6 is devoted to some examples.

I was recently informed by I.M. Gelfand that a part of the constructions and results of this paper has been found by M.I. Graev (unpublished).

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§1. Monomials and polynomials on a group.

In this section we recall some well known properties of reductive groups.

(1.1) Monomials. Let $G$ be a reductive algebraic group over $\mathbb{C}$. By $\text{Irr}(G)$ we denote the set of isomorphism classes of (finite-dimensional, algebraic) irreducible representations of $G$. For $\alpha \in \text{Irr}(G)$ we choose a representation $V_\alpha$ in the isomorphism class of $\alpha$, and denote by $\rho_\alpha : G \to \text{Aut}(V_\alpha)$ the $G$-action on $V_\alpha$. The value of $\rho_\alpha$ on $x \in G$ will be simply denoted by $x_\alpha$. We will denote by $\alpha^-$ the label of the dual representation: $V_\alpha^- = V_\alpha^*$. We also set $d(\alpha) = \dim(V_\alpha)$.

Let $\mathbb{C}[G]$ be the ring of regular functions on $G$, and $M_\alpha \subset \mathbb{C}[G]$ be the subspace spanned by the matrix elements of $x_\alpha$, i.e., the space of functions of the form

$$f(x) = (C, x_\alpha) = \text{tr}(C \cdot x_\alpha), \quad C \in \text{End}(V_\alpha)^* \simeq \text{End}(V_\alpha).$$

Here we identify $\text{End}(V_\alpha)$ with its dual by means of the form $\text{tr}(ab)$. The space $M_\alpha$ will be called the space of monomials of type $\alpha$ on $G$. The group $G \times G$ acts on the left on $\mathbb{C}[G]$ by the formula $((g_1, g_2)f)(g) = f(g_1^{-1}gg_2)$, and it is well known that we have a decomposition into irreducible subspaces

$$\mathbb{C}[G] = \bigoplus_{\alpha \in \text{Irr}(G)} M_\alpha, \quad M_\alpha \simeq V_\alpha \otimes V_\alpha^*.$$

In other words, this means that any $f \in \mathbb{C}[G]$ can be uniquely written as a finite sum of monomials:

$$f(x) = \sum_\alpha (C_\alpha, x_\alpha), \quad C_\alpha \in \text{End}(V_\alpha).$$

(1.1.4) Proposition. A function $f \in \mathbb{C}[G]$ is conjugacy invariant if and only if all the coefficients in its expansion (1.1.3) are scalar matrices: $C_\alpha = c_\alpha \cdot \text{Id}$.

The monomial corresponding to $\text{Id}_{V_\alpha}$ is just the character of the representation, which will be denoted by

$$s_\alpha(x) = (\text{Id}, x_\alpha) = \text{tr}(x_\alpha).$$

(1.1.6) Examples. (a) Let $G = (\mathbb{C}^*)^n$ be the algebraic torus of dimension $n$. Then $\text{Irr}(G)$ consists of 1-dimensional representations forming a lattice $\mathbb{Z}^n$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, the corresponding monomial is just the usual Laurent monomial

$$x_\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n.$$
Let $G = GL_n(\mathbb{C})$. In this case, as well known, irreducible representations of $G$ are labelled by sequences $\alpha = (\alpha_1 \geq \ldots \geq \alpha_n)$ with $\alpha_i \in \mathbb{Z}$. We will sometimes call such sequences Young diagrams. We will write $\alpha \geq 0$ if $\alpha_n \geq 0$. The case $\alpha = (1, 0, \ldots, 0)$ corresponds to the standard representation $V = \mathbb{C}^n$ of $GL_n(\mathbb{C})$. For arbitrary $\alpha$ the representation space $V_\alpha$ will be denoted $\Sigma^\alpha(V)$, so that $\Sigma^\alpha$ is the Schur functor. The dimension of this space is given by the formula

$$\dim(\Sigma^\alpha(\mathbb{C}^n)) = d_n(\alpha) := \frac{\prod_{1 \leq i < j \leq n}(\alpha_i - \alpha_j + j - i)}{\prod_{1 \leq i < j \leq n}(j - i)}.$$ 

The character $s_\alpha(x)$ is the Schur symmetric function of the eigenvalues of $x$. Note that

$$\Sigma^{(\alpha_1 + 1, \ldots, \alpha_n + 1)}(V) = \Sigma^{(\alpha_1, \ldots, \alpha_n)}(V \otimes \det(V), \quad x^{(\alpha_1 + 1, \ldots, \alpha_n + 1)} = x^\alpha \cdot \det(x).$$

We set $|\alpha| = \sum \alpha_i$.

The entries of the matrix monomial $x^\alpha$ are homogeneous functions in the matrix elements of $x$ of degree $|\alpha|$. They are polynomials if and only if $\alpha \geq 0$. The dual representation corresponds to $\alpha^- = (-\alpha_n, \ldots, -\alpha_1)$.

### (1.2) Multiplication of monomials.

The problem of finding the product of two monomials

$$(C_\alpha, x^\alpha) : (C_\beta, x^\beta), \quad C_\alpha \in \text{End}(V_\alpha), C_\beta \in \text{End}(V_\beta),$$

is reduced to the problem of decomposing $V_\alpha \otimes V_\beta$ into irreducibles. Namely, let us write:

$$V_\alpha \otimes V_\beta \simeq \bigoplus_\gamma N^\gamma_{\alpha\beta} \otimes V_\gamma, \quad N^\gamma_{\alpha\beta} = \text{Hom}_G(V_\gamma, V_\alpha \otimes V_\beta),$$

so that $N^\gamma_{\alpha\beta}$ is the multiplicity space. Then we have the tensor product of the coefficients

$$C_\alpha \otimes C_\beta \in \text{End}(V_\alpha) \otimes \text{End}(V_\beta) = \text{End}(V_\alpha \otimes V_\beta).$$

Let

$$p = p_{\alpha\beta} : \text{End}(V_\alpha \otimes V_\beta) = \text{End}\left(\bigoplus_\gamma N^\gamma_{\alpha\beta} \otimes V_\gamma\right) \rightarrow \bigoplus_\gamma \text{End}(N^\gamma_{\alpha\beta} \otimes V_\gamma)$$

be the projection onto the diagonal blocks, and

$$p^\gamma_{\alpha\beta} : \text{End}(V_\alpha \otimes V_\beta) \rightarrow \text{End}(N^\gamma_{\alpha\beta} \otimes \text{End}(V_\gamma)$$

be the $\gamma$th component of $p_{\alpha\beta}$. The following is a well known general fact.
**Proposition.** We have

\[(C_{\alpha}, x^\alpha) \cdot (C_{\beta}, x^\beta) = \sum_\gamma (\text{tr}_1(p_{\alpha \beta}^\gamma(C_{\alpha} \otimes C_{\beta})), x^\gamma),\]

where

\[\text{tr}_1 : \text{End}(N_{\alpha \beta}^\gamma) \otimes \text{End}(V_\gamma) \to \text{End}(V_\gamma)\]

is the trace with respect to the second argument.

**Multiplication and differentiation of monomials on \(GL_n\).** Consider the case \(G = GL_n\). Then \(x = \|x_{ij}\|\) is itself a monomial corresponding to \(\alpha = (1) := (1, 0, \ldots, 0)\), so \(V_{(1)} = V\). The decomposition of \(V \otimes V_\alpha\) is given by the Pieri formula (see, e.g., [11], Proposition 15.25):

\[(1.3.1) \quad V \otimes V_{\alpha} = \bigoplus_{\beta = \alpha + e_i} V_{\beta},\]

where the sum is over \(\beta = (\beta_1 \geq \ldots \geq \beta_n)\) such that \(\beta - \alpha\) is equal to the standard basis vector \(e_i\) for some \(i\). In particular, the decomposition is multiplicity-free, so each occurrence of the trace (1.2.4) is taken over a 1-dimensional space. Thus we can eliminate it from the explicit formulas, saying that we have canonical maps \(p_{(1), \alpha}^\beta : \text{End}(V) \otimes \text{End}(V_{\alpha}) \to \text{End}(V_{\beta})\), even though there is no canonical element in the 1-dimensional space of maps \(V \otimes V_{\alpha} \to V_{\beta}\).

We can write therefore:

\[(1.3.2) \quad (a, x) \cdot (b_{\alpha}, x^\alpha) = \sum_{\beta = \alpha + e_i} \left( p_{(1), \alpha}^\beta (a \otimes b_{\alpha}), x^\beta \right).\]

Let now \(\partial = \|\partial x_{ij}\|\). This is a matrix-valued vector field on \(G\); more precisely, \(\partial \in \text{Vect}(G) \otimes \text{End}(V^*)\). If \(a\) is an \(n\) by \(n\) matrix, we denote by \((a, \partial)\) the scalar differential operator \(\sum a_{ij} \partial / \partial x_{ij}\). The following fact is the generalization of the formula \(d(x^\alpha)/dx = \alpha x^{\alpha - 1}\) for scalar monomials.

**Proposition.** For \(a \in \text{End}(V^*)\) and \(b_{\alpha} \in \text{End}(V_{\alpha})\) we have

\[(a, \partial) \cdot (b_{\alpha}, x^\alpha) = \sum_{\beta = \alpha - e_i} (\alpha_i + n - i) \cdot \left( (p_{(1), \alpha}^\beta)^t (a \otimes b_{\alpha}), x^\beta \right),\]

where

\[(p_{(1), \alpha}^\beta)^t : \text{End}(V^* \otimes V_{\alpha}) \to \text{End}(V_{\beta})\]

is the transpose of

\[p_{(1), \alpha}^\beta : \text{End}(V) \otimes \text{End}(V_{\beta}) \to \text{End}(V_{\alpha}).\]
Proof: Let $L$ be the vector space of translation invariant vector fields on $\text{Mat}_n(\mathbb{C})$. It is an irreducible representation of $G \times G$, isomorphic to $V^* \otimes V$ while $M_\alpha \simeq V_\alpha \otimes V_\alpha^*$. The LHS and the RHS of the proposed equality are $G \times G$-equivariant maps

$$l, r : L \otimes M_\alpha \to \bigoplus_{\beta = \alpha - e_i} M_\beta.$$ 

The source of $l, r$ is the tensor product of two irreducible representations. The target is a direct sum of several distinct irreducibles. Further, in the decomposition of the source into irreducibles each irreducible enters with multiplicity $\leq 1$ by Pieri's formula. Thus we have the following fact.

**Lemma.** Suppose there is a vector $v \in L \otimes M_\alpha$ whose projection to each $M_\beta$-isotypic component from the target of $l, r$, is nonzero and such that $l(v) = r(v)$. Then $l = r$ on the entire $L \otimes M_\alpha$.

Take now $v = \text{Id} \otimes \text{Id} \in L \otimes M_\alpha$. We claim that $l(v) = r(v)$. Indeed, this statement reduces to a computation involving only the characters $s_\alpha$. Consider $s_\alpha$ as a symmetric function of $n$ variables $t_1, ..., t_n$, and let $D = \sum \partial / \partial t_i$. Our statement that $l(v) = r(v)$ thus reduces to the identity

$$(Ds_\alpha)(t) = \sum_{\beta = \alpha - e_i} (\alpha_i + n - i)s_\beta(t).$$

This identity follows at once from the Weyl character formula [11] once we notice the denominator of that formula is annihilated by $D$ and thus can be treated as a constant.

Further, in the case when $\alpha_i + n - i \neq 0$ for all $i$, the equality (1.3.5) shows also that the projection of $v$ to any $M_\beta$-isotypic component we are interested in, is nonzero, so Proposition 1.3.3 is true for such an $\alpha$. Let us now identify $M_\alpha$ with $M_{\alpha_1+s, ..., \alpha_n+s}$ for any $s \in \mathbb{Z}$ in the standard way. Then, we get two families of maps

$$l(s), r(s) : L \otimes M_\alpha \to \bigoplus_{\beta = \alpha - e_i} M_\beta, \quad s \in \mathbb{Z}$$

which obviously depend on $s$ in a polynomial way. By the above, $l(s) = r(s)$ for all except possibly finitely many $s$. Therefore $l(s) = r(s)$ for any $s$ and thus $l = r$ for any $\alpha$. Proposition is proved.
§2. Some algebraic geometry related to monomials and polynomials.

(2.1) The space $M_A$. The (in)homogeneity condition. We start with a reductive group $H$ and a finite set $A \subset \text{Irr}(H)$ of irreducible representations. Let

$$(2.1.1) \quad M_A = \bigoplus_{\omega \in A} M_\omega = \left\{ \sum_{\omega \in A} (a_\omega, x_\omega), \ a_\omega \in \text{End}(V_\omega) \right\} \subset \mathbb{C}[H]$$

be the space of polynomials on monomials from $A$. So $M_A$ is a finite-dimensional subspace in $\mathbb{C}[H]$ invariant with respect to both left and right $H$-actions. The coefficients $a_\omega$ serve as matrix coordinates in $M_A$, identifying it with $\bigoplus_{\omega \in A} \text{End}(V_\omega)$. We can thus identify $M_A$ with $M^*_A$ via the form $\sum_{\omega \in A} \text{tr}(a_\omega b_\omega)$ and view both $M_A, M^*_A$ as partial compactifications of the group $G = \prod_{\omega \in A} \text{GL}(V_\omega)$. We will be interested in generic behavior of functions from $M_A$.

(2.1.2) $\rho_A = \bigoplus_{\omega \in A} \rho_\omega : H \to \prod_{\omega \in A} \text{GL}(V_\omega) = G \subset M^*_A - \{0\}$

be the direct sum of representations from $A$. We will assume that $\rho_A$ is injective so that we can regard $H$ as a subgroup of $G$.

In the sequel we will impose one of the following conditions on $\rho_A(H) \subset G$ (or, equivalently, on $A \subset \text{Irr}(H)$). Clearly, one of these conditions is always satisfied.

(2.1.3) Homogeneity condition. The image $\rho_A(H)$ contains the subgroup of scalars $\mathbb{C}^* = \{(\lambda, ..., \lambda) \in \prod GL(V_\omega), \ \lambda \in \mathbb{C}^*\}$.

(2.1.4) Inhomogeneity condition. The intersection $\rho_A(H) \cap \mathbb{C}^*$ is finite.

We shall say that the pair $(H, A)$ is homogeneous, resp. inhomogeneous, if it satisfies (2.1.3), resp. (2.1.4).

Given $(H, A)$ satisfying (2.1.4), we can construct $(\tilde{H}, \tilde{A})$ satisfying (2.1.3) so that $M_{\tilde{A}} = M_A$. namely, let $\tilde{H}$ be the image of $\mathbb{C}^* \times H$ under the multiplication map

$$(2.1.5) \quad p : \mathbb{C}^* \times H \to G = \prod GL(V_\omega), \ (\lambda, x) \mapsto \lambda x.$$ 

Then each $V_\omega, \omega \in A$, is an irreducible $\tilde{H}$-module. Denoting $\tilde{\omega}$ its isomorphism class, we get a bijection

$$A \to \tilde{A}, \ \omega \mapsto \tilde{\omega}, \ V_\omega = V_{\tilde{\omega}}.$$ 

(2.1.6) Examples. (a) Let $H = \text{GL}_n$. A set $A \subset \text{Irr}(\text{GL}_n)$ satisfies the homogeneity condition if and only if for any $\omega = (\omega_1 \geq ... \geq \omega_n) \in A$ the sum $|\omega| = \sum \omega_i$ is the same.

(b) let $H = \text{SL}_n$. Then any set $A \subset \text{Irr}(\text{SL}_n)$ satisfies the inhomogeneity condition.

(2.2) The varieties $Y_A, X_A$. We assume that $A$ satisfies the homogeneity condition. Let $Y_A$ be the Zariski closure of $\rho_A(H)$ in $M^*_A$. This is a conic variety, and we denote by $X_A = P(Y_A) \subset P(M^*_A)$ its projectivization. The following properties of $X_A, Y_A$ are obvious from the construction.
(2.2.1) Proposition. (a) $Y_A$ is invariant under the $H \times H$-action in $M_A^*$ and contains $\rho_A(H)$ as an open orbit.
(b) $X_A$ is projective, invariant under the action of $P(H) \times P(H)$ and contains $P(\rho_A(H))$ as an open orbit.
(c) Functions from $M_A$ are precisely the restrictions to $\rho_A(H)$ of linear functions on $Y_A \subset M_A^*$.
(d) $\mathbb{C}[Y_A]$, the algebra of regular functions on $Y_A$, is the subalgebra of $\mathbb{C}[H]$ generated by $M_A$.

Call a subset $S \subset \text{Irr}(H)$ monoidal if 0 (the label for the trivial representation) is in $S$ and for each $\alpha, \beta \in S$ and each embedding $V_\gamma \subset V_\alpha \otimes V_\beta$ we have $\gamma \in S$. Let $\langle A \rangle$ be the minimal monoidal subset generated by $A$.

(2.2.2) Proposition. The subalgebra $\mathbb{C}[Y_A]$ has the form

$$\mathbb{C}[Y_A] = \bigoplus_{\alpha \in \langle A \rangle} M_\alpha \subset \bigoplus_{\alpha \in \text{Irr}(H)} M_\alpha = \mathbb{C}[H], \quad M_\alpha = V_\alpha \otimes V_\alpha^*.$$ 

Proof: $\mathbb{C}[H]$, as an $H \times H$-module, splits into a direct sum of distinct irreducible representations $M_\alpha$ for all $\alpha \in \text{Irr}(H)$. Thus $\mathbb{C}[Y_A]$, being an $H \times H$-submodule, should have the form $\bigoplus_{\alpha \in S} M_\alpha$ where $S \subset \text{Irr}(H)$ is some subset. Since $\mathbb{C}[Y_A]$ is also an algebra generated by the $M_\omega, \omega \in A$, we find that $S = \langle A \rangle$.

(2.2.3) Corollary. The variety $Y_A$ has a natural structure of an algebraic semigroup containing the group $H$.

(2.2.4) Proposition. The number of $H \times H$-orbits on $Y_A$ is finite.

Proof: This is a consequence of the fact that each irreducible representation of $H \times H$ enters $\mathbb{C}[Y_A]$ no more than once, see [44].

(2.2.5) Definition. If $(H, A)$ does not satisfy the homogeneity condition, then we define $X_A := X_{\bar{A}}, Y_A = Y_{\bar{A}}$, where $(\bar{H}, \bar{A})$ is constructed in (2.1.5).

(2.3) Tannakian point of view on $Y_A$. We can generalize the construction of $Y_A$ as follows. Let $S$ be any monoidal subset in $\text{Irr}(H)$ generated by a finite subset. Then the subspace $M[S] := \bigoplus_{\alpha \in S} M_\alpha$ is a finitely generated subalgebra in $\mathbb{C}[H]$. The corresponding affine algebraic variety will be denoted by $Y[S] = \text{Spec} M[S]$. Thus $Y_A$ is obtained when $S = \langle A \rangle$.

To the set $S$ we associate the category $\mathcal{R} = \mathcal{R}_S$ all regular representations $V$ of $H$ such that each irreducible component of $V$ is isomorphic to some $V_\alpha, \alpha \in S$. This is a monoidal category, i.e., if $V, W \in \mathcal{R}$, then $V \otimes W \in \mathcal{R}$. Denote by $f : \mathcal{R} \to \text{Vect}$ the forgetful functor to the monoidal category of vector spaces (with the usual tensor product). Thus $f(V)$ is $V$ regarded as a vector space. The following fact is a version of the well known Tannaka-Krein theorem reconstructing a group from the category of its representations.
(2.3.1) Proposition. (a) \( Y_S \) is an affine algebraic semigroup equipped with a semigroup homomorphism \( i_S : H \to Y_S \) with an open dense image. The map \( i_S \) is injective if and only if every representation \( V_\alpha, \alpha \in \text{Irr}(H) \) can be embedded into \( V_\beta \otimes V_\gamma \) for \( \beta, \gamma \in S \).

(b) The correspondence \( S \mapsto Y_S \) establishes a bijection between finitely generated monoidal subsets \( S \subset \text{Irr}(H) \) and isomorphism classes of homomorphisms \( i : H \to Y \) where \( Y \) is an affine semigroup and \( \text{Im}(H) \) is open dense.

(c) Points of \( Y_S \) are in bijection with monoidal natural transformations \( a \) of functors from \( f \) to itself, i.e., with systems of endomorphisms \( a_V : V \to V \) given for all \( V \in \mathcal{R} \) and satisfying the following properties:

(i) If \( \phi : V \to W \) is a morphism of representations, then \( \phi a_V = a_W \phi \);
(ii) \( a_V \oplus W = a_V \oplus a_W, a_V \otimes W = a_V \otimes a_W \).

The semigroup structure on \( Y_S \) is given by the composition of the operators: \( (ab)_V = a_V \circ b_V \).

The varieties \( Y_S \) are precisely the reductive algebraic semigroups studied in [41] [49].

(2.3.2) Example. Suppose that \( H \) is a torus. Then \( \text{Irr}(H) \) is the lattice of characters of \( H \), a monoidal subset in it is just a sub-semigroup, \( M[S] = \mathbf{C}[S] \) is the semigroup ring and Proposition 2.3.1 is the classification of affine toric varieties.

(2.4) The Newton polytope and the orbit structure of \( X_A \). The varieties \( X_A, Y_A \) belong to the following general class of varieties with group action [3] [33].

(2.4.1) Definition. (a) Let \( K \) be a reductive group. An algebraic subgroup \( \Delta \subset K \) is called spherical if some Borel subgroup in \( K \) has a dense orbit in \( K/\Delta \).

(b) An algebraic variety \( M \) with \( K \)-action is called spherical if it contains an open orbit isomorphic to \( K/\Delta \), where \( \Delta \) is a spherical subgroup.

(c) For any affine algebraic group \( G \) its rank \( \text{rk}(G) \) is defined as the dimension of any maximal torus. The rank of a spherical \( K \)-variety \( M \) is defined by \( \text{rk}(M) = \text{rk}(K) - \text{rk}(\Delta) \) where \( \Delta \) is as in (b).

An equivalent formulation of the condition in (a) is that each irreducible representation of \( K \) occurs in the space \( \mathbf{C}[K/\Delta] \) not more than once. Pairs \( (K, \Delta) \) with this property are known as Gelfand pairs.

The example relevant for us is \( K = H \times H \), and \( \Delta = \{(x, x)\} \simeq H \) being the diagonal. Clearly, \( Y_A \) is spherical with this choice of \( K, \Delta \). Similarly, \( X_A \) is spherical.

Usually, one includes normality in the definition of spherical varieties. Since this is unnatural in our context, we will not do this and will take care in applying results which are stated in the literature for normal varieties.

Choose a maximal torus \( T \subset H \). Let \( \Lambda = \text{Ch}(T) \) be its character lattice (i.e., the lattice of weights of \( H \)), and set \( \Lambda_\mathbf{R} = \Lambda \otimes \mathbf{R} \). Let also \( W \) be the Weyl group of \( H \) (with respect to \( T \)). We will identify \( \text{Irr}(H) \) with the set of dominant weights in \( \Lambda \).

Given a representation \( \omega \in \text{Irr}(H) \), we denote by \( Q_\omega \subset \Lambda_\mathbf{R} \) the convex hull of all the weights of \( T \) on \( V_\omega \). As well known, \( Q_\omega = \text{Conv}(W \cdot \omega) \). For a finite subset \( A \subset \text{Irr}(H) \) as
before, we denote by $Q = Q_A$ the convex hull of the union of the $Q_\omega$ for $\omega \in A$. This is a $W$-invariant convex polytope in $\Lambda_\mathbb{R}$. It will be called the Newton polytope of $A$.

(2.4.2) **Theorem.** (a) $H \times H$-orbits on $Y_A$ are in order-preserving bijection with $W$-orbits on faces of $Q_A$ (including the empty face). Denote by $Y(\Gamma)$ the orbit corresponding to a face $\Gamma$.

(b) Therefore the orbits on $X_A$ are the projectivizations $X(\Gamma) := P(Y(\Gamma))$ for all nonempty faces $\Gamma \subset Q_A$ (considered up to $W$-action).

(c) Explicitly, $Y(\Gamma)$ is the orbit of the vector $e(\Gamma) \in \prod_{\omega \in A} \text{End}(V_\omega)$ whose components $e(\Gamma)_\omega$ are defined as follows. Write the weight decomposition $V_\omega = \bigoplus_{\lambda \in \Lambda} V_{\omega}^\lambda$ with respect to the $T$-action. Then $e(\Gamma)_\omega$ is the block-diagonal operator with respect to this decomposition whose diagonal block on $V_{\omega}^\lambda$ is zero if $\lambda \notin \Gamma$ and is the identity if $\lambda \in \Gamma$.

(d) Each $Y(\Gamma)$ is a spherical variety of rank equal to $\text{rk}(H) - \text{codim}(\Gamma)$.

(e) In particular, minimal (closed) orbits correspond to $W$-orbits on the vertices of $Q_A$. For such a vertex $\lambda$ the orbit $X(\lambda)$ is isomorphic to $(H/P_\lambda) \times (H/P_\lambda)$, where $P_\lambda \subset H$ is the parabolic subgroup whose relative Weyl group is the stablizer $W_\lambda \subset W$.

(2.4.3) **Examples.** (a) Let $H = GL_n = GL(V)$ and let $A$ consist of one element $\omega = (1)$, so that $V_\omega = V$ is the tautological representation. Then $Y_A = \text{Mat}_n(\mathbb{C})$. Orbits of $H \times H$ on $\text{Mat}_n(\mathbb{C})$ are just sets of matrices with fixed rank. Thus there are $n + 1$ orbits. The polytope $Q_A$ is the $(n - 1)$-dimensional simplex, the convex hull of the $n$ basis vectors in $\Lambda_\mathbb{R} = \mathbb{R}^n$, and the Weyl group acts by permutations of the vertices. Thus there is only one orbit on faces of each dimension, and there are $n + 1$ orbits.

(b) Let $H$ be a semisimple group of adjoint type and let $A = \{ \omega \}$ consist of one element which is strictly dominant. The variety $X_A$ corresponding to the homogenized pair $(\tilde{H}, \tilde{A})$ from (2.1), is the minimal compactification of $H$ constructed by De Concini and Procesi [7]. In this case $Q$ is the “permutohedron” $\text{Conv}(W \cdot \omega)$, with $|W|$ vertices forming one orbit. As shown in [7], $X_A$ has in this case one closed orbit, isomorphic to $(H/B) \times (H/B)$, where $B$ is a Borel subgroup in $H$.

(2.5) **Proof of Theorem 2.4.2.** The statement is covered by the existing general theory of spherical varieties [3] [8] [33] and especially that of reductive algebraic semigroups [41] [49]. Because of various notational differences, however, it seems the easiest to give a more self-contained proof, using the ideas of the cited papers.

(2.5.1) **Lemma.** Let

$$K = H \times H, \quad L = T \times T, \quad V = \bigoplus_{\omega \in A} M_\omega, \quad v = (1, \ldots, 1).$$

Then any $K$-orbit in $Y_A = K \cdot v$ intersects the closure of $L \cdot v$.

**Proof:** (cf. [41], §3, Lemma 3.) Let $H((t))$, resp. $H[[t]]$ be the group of $\mathbb{C}((t))$, resp. $\mathbb{C}[[t]]$-points of $H$, and similarly for $T$. The Iwahori theorem (also known as the Cartan
decomposition) for reductive groups over local fields such as \( C((t)) \) gives

\[
(2.5.2) \quad H((t)) = H[[t]] \cdot T((t)) \cdot H[[t]].
\]

If \( x \in Y_A \), we have \( x = \lim_{t \to 0} g(t)v \) for some meromorphic analytic curve \( g(t), 0 < |t| < \epsilon \) in \( H \). Such a curve can be viewed as an element of \( H((t)) \). Factoring now \( g(t) = f_1(t)\gamma(t)f_2(t) \) with \( f_i(t) \in H[[t]] \) and \( \gamma(t) \in T((t)) \), we get, taking into account that \( v \) is the unit element of \( H \),

\[
x = \lim_{t \to 0} g(t)v = \lim_{t \to 0} (f_1(t)\gamma(t)f_2(t))v = f_1(0)\left(\lim_{t \to 0} \gamma(t)v\right)f_2(0),
\]

which means that \( \lim_{t \to 0} \gamma(t)v \) exists in \( Y_A \) and that its \( H \times H \)-orbit contains \( x \), proving the lemma.

Let \( \text{Ch}(L) \) be the lattice of characters of \( L \) and \( \text{Ch}_R(L) = \text{Ch}(L) \otimes \mathbb{R} \). Let now \( V \) be any algebraic representation of the torus \( L \) and \( V = \bigoplus_{\lambda \in \text{Ch}(L)} V^\lambda \) be its weight decomposition. For a vector \( v \in V \) denote by \( v_\lambda \in V^\lambda \) its component with respect to this decomposition. Set

\[
(2.5.3) \quad A_L(v) = \{ \lambda : v_\lambda \neq 0 \} \subset \text{Ch}(L), \quad \text{Wt}_L(v) = \text{Conv}(A_L(v)) \subset \text{Ch}_R(L).
\]

Thus \( \text{Wt}_L(v) \) is a convex polytope known as the weight polytope of \( v \), see [18] Ch.5, §1. For any nonzero vector \( w \in V \) let \( P(w) \in P(V) \) be the corresponding point of the projective space. The following fact is well known, see, e.g., loc. cit. Proposition 1.9:

(2.5.4) Proposition. Let \( \alpha : \mathbb{C} \to L \) be any rational map and let \( v' \in V \) be any nonzero vector representing the point \( \lim_{\tau \to 0} P(\alpha(\tau) \cdot v) \in P(V) \). Then there is a face \( \Gamma \subset \text{Wt}_L(v) \) such that \( v_\lambda' = 0 \) for \( \lambda \notin A_L(v) \cap \Gamma \) while \( v_\lambda' \) is a nonzero multiple of \( v_\lambda \neq 0 \) for \( \lambda \in A_L(v) \cap \Gamma \).

Setting now as before \( V = \bigoplus_{\omega \in \Lambda} M_\omega \), we see that \( A_L(v) \) is the set of all the weights in all the representations \( V_\omega \), embedded diagonally into \( \Lambda \oplus \Lambda = \text{Ch}(L) \). So \( \text{Wt}_L(v) = Q_A \).

For any weight \( \lambda \in A_L(v) \) the component \( v_\lambda \) is the identity map of the corresponding weight space. Thus Proposition 2.5.4 implies that any point in the closure of \( P(L \cdot v) \subset X_A \) is in the \( L \)-orbit of the point described in Theorem 2.4.2 (c). Together with Lemma 2.5.1, this proves parts (a)-(c) of Theorem 2.4.2.

Let us show (e). Let \( \lambda \) be a vertex of \( Q_A \); by applying the Weyl group transformations, if necessary, we can assume that \( \lambda \) is dominant. Since \( \lambda \) is a vertex, the corresponding weight subspace of \( \bigoplus_{\omega \in \Lambda} V_\omega \) is 1-dimensional, and thus the vector \( e(\lambda) \) of operators has only one nonzero component, namely \( e(\lambda)_\lambda : V_\lambda \to V_\lambda \). Further, this component is just the identity operator on the 1-dimension weight \( \lambda \) subspace extended by 0 along the other weight spaces. In other words, \( e(\lambda)_\lambda = \xi_\lambda \otimes \xi_\lambda^* \in V_\lambda \otimes V_\lambda^* \), where \( \xi_\lambda \) is the highest vector in \( V_\lambda \) and \( \xi_\lambda^* \) is the highest vector in \( V_\lambda^* \). This implies (e).

Finally, part (d) follows from Theorem 7.3 of [33] applied to the normalization of \( Y_A \) once we translate the contravariant description of spherical varieties (via fans) used there into the dual covariant description (via polytopes) used here.
(2.6) Orbit structure of semigroups. For future reference we note a slight general-
ization of Theorem 2.4.2. Let \( S \subset \mathsf{Irr}(H) \) be any finitely generated monoidal subset and \( Y_S = \mathsf{Spec}(M[S]) \) be the corresponding affine semigroup from (2.3). The group \( H \times H \) acts on \( Y_S \) via the homomorphism \( i_S : H \to Y_S \) and has an open orbit whose stabilizer contains \( H \). So \( Y_S \) is an affine spherical variety. Let \( T, \Lambda \) be as in (2.4) and let \( C_S \subset \Lambda_R \) be the convex hull of all the \( T \)-weights of all the representations \( V_\alpha, \alpha \in S \). This is a polyhedral cone. The Weyl group \( W \) naturally acts on \( C_S \). The proof of the next theorem is similar to that of 2.4.2.

(2.6.1) Theorem. \( H \times H \)-orbits on \( Y_S \) are in bijection with \( W \)-orbits on nonempty faces of the cone \( C_S \). The orbit \( Y(\Gamma) \) corresponding to a face \( \Gamma \) is a spherical variety of rank equal to \( \dim(\Gamma) \).

We can also exhibit a distinguished point \( e(\Gamma) \in Y(\Gamma) \). For this, we use the description of points of \( Y_S \) from Proposition 2.3.1(c). Let \( R \) be the monoidal category of representations corresponding to \( S \). For every \( V \in R \) we have the decomposition \( V = V'_\Gamma \oplus V''_\Gamma \) where \( V'_\Gamma \) is the direct sum of all weight subspaces whose weights lie in \( \Gamma \) and \( V''_\Gamma \) is the direct sum of all the other weight subspaces. To describe a point \( e(\Gamma) \) we will describe the corresponding system of operators \( e(\Gamma)_V : V \to V, V \in R \).

(2.6.2) Proposition. The collection of operators \( e(\Gamma)_V = \text{Id}_{V'_\Gamma} \oplus 0_{V''_\Gamma} \) satisfies the conditions of Proposition 2.3.1 (c) and thus defines a point \( e(\Gamma) \in Y_S \). The orbit \( Y(\Gamma) \) contains \( e(\Gamma) \).

(2.7) The degree of \( X_A \). In the notation of (2.4), let \( \Lambda^\vee = \mathsf{OP}(T) \) be the lattice of 1-parameter subgroups in \( T \), i.e., the dual lattice to \( \Lambda \). Denote by \( R \subset \Lambda \) the root system of \( H \) and choose a system of positive roots \( R_+ \subset R \). Let also \( R^\vee \subset \Lambda^\vee \) be the system of coroots [45] and \( R_+^\vee \) be the system of positive coroots. Recall that \( R \) and \( R^\vee \) are in bijection; for \( \alpha \in R \) let \( \alpha^\vee \in R^\vee \) be the corresponding coroot. Each \( \alpha^\vee \) is a linear function on the real space \( \Lambda_R \).

(2.7.1) Theorem. The degree of the projective variety \( X_A \subset P(M_A^*) \) is equal to
\[
\frac{1}{\prod(d_i - 1)!} \int_{Q_A} \prod_{\alpha \in \Lambda_R^\vee} \langle \alpha^\vee, \lambda \rangle d\lambda,
\]
where \( d\lambda \) is the Lebesgue measure on \( \Lambda_R \) normalized so that \( \text{Vol}(\Lambda_R/\Lambda) = 1 \), and the \( d_i \) are the characteristic exponents of \( H \), i.e., the degrees of the polynomial generators of the algebra of \( W \)-invariant polynomials on \( \Lambda_R \).

This theorem was proved by B. Kazarnovski [31] in 1987. A more general result for arbitrary spherical varieties was proved by M. Brion [4] in 1989. Note that the function \( \prod_{\alpha \in R_+} \langle \alpha^\vee, \lambda \rangle \) is the famous Weyl volume element for the Langlands dual group \( G^L \), see [45]. The maximal torus \( T^L \) of \( G^L \) is dual to \( T \), so the quotient of \( T \) by its maximal
compact subgroup is canonically identified with $\Lambda_{\mathbf{R}}$. Thus the integral in (2.7.1) can be viewed as the volume of a conjugacy invariant domain in $G^L$ whose intersection with $T^L$ is the preimage of $Q_A$ under the map $T^L \rightarrow T^L/T^L_c = \Lambda_{\mathbf{R}}$.

**(2.8) The $A$-discriminant.** Let $A \subset \text{Irr}(H)$ be an arbitrary finite subset (satisfying or not the homogeneity condition). As in the toric case [18], we set

$$\nabla_A^0 = \left\{ f \in M_A \mid \exists x_0 \in H : f(x_0) = d_{x_0}f = 0 \right\}$$

and let $\nabla_A \subset M_A$ be the Zariski closure of $\nabla_A^0$. We call $\nabla_A$ the $A$-discriminantal variety; in the case when it has codimension 1, the equation of $\nabla_A$ is called the $A$-discriminant and denoted by $\Delta_A$. Thus $\Delta_A$ is a polynomial function on $M_A = \bigoplus_{\omega \in A} \text{End}(V_\omega)$.

**(2.8.1) Proposition.** (a) $\nabla_A$ is conic and is invariant under both left and right actions of $H$ in $M_A$.

(b) The projectivization of $\nabla_A$ is projectively dual to $X_A$.

Extending this construction, for any nonempty face $\Gamma \subset Q_A$ we denote by $\nabla_{A, \Gamma} \subset M_A$ the conic variety whose projectivization is projectively dual to the orbit $X(\Gamma) \subset X_A$. This variety is irreducible, since $X(\Gamma)$ is. It is clear that $\nabla_{A, \Gamma}$ is also invariant under the left and right actions of $H$ on $M_A$. In the case when $\nabla_{A, \Gamma}$ is a hypersurface, we will denote by $\Delta_{A, \Gamma}$ the irreducible equation of $\nabla_{A, \Gamma}$ (defined up to a constant factor).

In the case when $H$ is a torus, the discriminants have been studied in [18] from the point of view of monomials entering into their expansion. In the present case it is natural to view the space $M_A$ on which the $\Delta_{A, \Gamma}$ are defined, as a partial compactification of the group $G = \prod GL(V_\omega)$ and expand the discriminants not in ordinary monomials but in the matrix elements of irreducible representations of $G$:

$$(2.8.2) \quad \Delta_{A, \Gamma}\left( \sum_{\omega \in A} (a_\omega, x^\omega) \right) = \sum_{\alpha \in \text{Irr}(G)} (c_\alpha, a^\alpha).$$

Here $a = (a(\omega))_{\omega \in A}$, thus $\alpha = (\alpha(\omega))_{\omega \in A}$ with each $\alpha(\omega)$ being an element of $\text{Irr}(GL(V_\omega))$, i.e., a Young diagram. The coefficient $c_\alpha$ lies in $\text{End}\left( \bigotimes_{\omega \in A} \Sigma^{\alpha(\omega)} V_\omega \right)$. The condition of left and right $H$-invariance of $\nabla_{A, \Gamma}$ implies the following quasi-homogeneity property of its equation.

**(2.8.3) Corollary.** Suppose that $\Delta_{A, \Gamma}$ is defined. There is a 1-dimensional character $\chi = \chi_\Gamma$ of $H$ such that each coefficient $c_\alpha$ of $\Delta_{A, \Gamma}$ satisfies

$$c_\alpha \in \text{End}\left( \left( \bigotimes_{\omega \in A} \Sigma^{\alpha(\omega)} V_\omega \right)^{\chi} \right),$$

13
where the superscript $\chi$ means the subspace of $\chi$-quasiinvariants.

§3. Power series and distributions on a group.

(3.0) Orthogonality relations. Let $G$ be a reductive group, as in §1, and $G_c \subset G$ a compact form. Denote by $d^*x$ the regular invariant volume form on $G$ normalized so that $\int_{G_c} d^*x = 1$. The orthogonality relations for the matrix elements of irreducible representations have the form:

\[ \int_{G_c} (a_\alpha, x^\alpha) \cdot (b_\beta, x^\beta) d^*x = \begin{cases} 0, & \text{if } \alpha \neq \beta^-; \\ d(\alpha)^{-1} \text{tr}(a_\alpha b_\beta^t), & \text{if } \alpha = \beta^-. \end{cases} \]

Here $d(\alpha) = \dim(V_\alpha)$ and $V_\alpha^- = V_\alpha^*$. In particular, for the characters we have:

\[ \int_{G_c} s_\alpha(x) s_\beta(x) d^*x = \begin{cases} 0 & \text{if } \alpha \neq \beta^-; \\ 1 & \text{if } \alpha = \beta^-. \end{cases} \]

(3.1) Power series and analytic functions. By a power series on $G$ we mean a series of the form

\[ f(x) = \sum_{\alpha \in \text{Irr}(G)} (a_\alpha, x^\alpha), \quad a_\alpha \in \text{End}(V_\alpha). \]

We are interested in representing analytic functions on (some domains in) $G$ in this way. Representing of functions on $G_c$ in this way is the content of the Peter-Weyl theorem which, together with the orthogonality relations (3.0.1) gives the following.

(3.1.2) Proposition. Let $f(x)$ be an analytic function in some neighborhood of $G_c$ in $G$. Then $f$ can be written as the sum of an absolutely convergent series (3.1.1) with

\[ a_\alpha = d(\alpha) \int_{G_c} f(x)x^{\alpha^-} d^*x. \]

(3.2) Example. The case of $GL_n(C)$. For $G = GL_n$ we take $G_c = U_n$, the unitary group. In this case, as well known [29]

\[ d^*x = \frac{1}{C \det(x)^n} \frac{dx}{\det(x)^n}, \quad dx = \prod dx_{ij}, \quad C = \int_{U_n} \frac{dx}{\det(x)^n} = \frac{(2\pi)^{n(n+1)/2}}{\prod_{i=1}^{n-1} i!}. \]
Also, if \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a Young diagram, then \( \alpha^- = (-\alpha_n, \ldots, -\alpha_1) \). So the formula for the coefficients becomes

\[
a_\alpha = \frac{d(\alpha)}{C} \int_{U_n} f(x)x^{(-n-\alpha_n, \ldots, -n-\alpha_1)}dx.
\]

Recall that \( U_n \) is the Shilov boundary of the domain

\[
D_n = \{ x \in \text{Mat}_n(C) : I - xx^* > 0 \}.
\]

Here, as usual, the notation \( > 0 \) means that the (Hermitian) matrix in question is positive definite. So we have the following fact.

**Proposition.** Let \( f \) be a function analytic on a neighborhood of \( U_n \) in \( GL_n(C) \). If \( f \) can be continued to an analytic function in a connected domain in \( \text{Mat}_n(C) \) containing 0, then \( f \) can be analytically continued to the entire \( D_n \) and, in addition, the coefficients \( a_\alpha \) in (3.2.2) are zero unless \( \alpha \geq 0 \).

The reader can consult [35] for the general study of domains of convergence of power series on reductive groups.

**Rings of formal series and convergence.** Denote by \( C[[G]] \) the vector space of all formal power series of the form (3.1.1). This space is not a ring. However, we have the following fact.

**Proposition.** Let \( D(G) \) be the ring of all regular differential operators on \( G \). Then \( C[[G]] \) is naturally a \( D(G) \)-module, containing the space \( C[G] \) of regular functions.

**Proof:** The structure of \( C[G] \)-module on \( C[[G]] \) follows from Proposition 1.2.4 describing the product of two monomials. If \( L \) is a left-invariant vector field on \( G \), then \( L \) preserves each \( M_\alpha \) and so acts on \( C[[G]] \). The ring \( D(G) \) is generated by \( C[G] \) and invariant vector fields. The check that the relations holding in \( D(G) \) are satisfied, is standard.

Let now \( S \subset \text{Irr}(G) \) be a finitely generated monoidal subset. Let also \( T \subset G \) be a maximal torus and \( \Lambda \) its character lattice, and \( C_S \subset \Lambda_R \) be the cone defined in (2.6). We shall say that \( S \) is strictly convex if \( C_S \) is a strictly convex cone, i.e., it does not contain linear subspaces and this \( 0 \in \Lambda_R \) is a vertex of \( C_S \). Let \( Y[S] \) be the affine semigroup associated to \( S \). Theorem 2.6.1 implies that \( Y[S] \) contains a unique \( G \times G \)-fixed point which we also denote 0.

Let \( M[[S]] \) be the completion of the ring \( M[S] = C[Y[S]] \) at 0. This completion can be viewed as the ring of formal series

\[
f(x) = \sum_{\alpha \in S} (a_\alpha, x^\alpha), \quad a_\alpha \in \text{End}(V_\alpha).
\]

Because of the strict convexity, the multiplication of two such series is now a well defined, purely algebraic operation accomplished via Proposition 1.2.4. Here is a useful sufficient condition for convergence of a series from \( M[[S]] \).
**Theorem.** Let \( f \in M[[S]] \) be a formal series. Suppose that there is a holonomic system of linear PDE on \( G \) with regular singularities (i.e., a holonomic regular \( \mathcal{D}(G) \)-module \([2]\)) of which \( f \) is a solution. Then \( f \) converges to an analytic function defined on a neighborhood of 0 in \( Y \). In particular, the domain of convergence of \( f \) in \( G \) is nonempty.

**Proof:** For a smooth algebraic variety \( Z \) let \( \mathcal{D}_Z \) be the sheaf of rings of differential operators on \( Z \). Let \( \mathcal{L} \) be the sheaf of \( \mathcal{D}_G \)-modules corresponding to the holonomic system in question. Let also \( G' \subset Y \) be the open \( G \times G \)-orbit, i.e., the set of all invertible elements of \( Y \) as a semigroup. Then we have a surjective morphism of reductive groups \( \sigma : G \to G' \). By Hironaka’s theorem, there exists a resolution of singularities \( p : Y \to Y \) bijective over \( G' \) and such that \( p^{-1}(0) \) is a divisor with normal crossings. Then \( p^*f \) is a function on the formal neighborhood of \( p^{-1}(0) \) in \( Y \). In particular, for any \( y \in p^{-1}(0) \) we have the germ \( (p^*f)_y \) lying in \( \hat{\mathcal{O}}_{Y,y} \approx \mathbb{C}[[y_1, \ldots, y_n]] \), the completion of the local ring of \( Y \) at \( y \). Let \( \phi \) be the composition

\[
G \xrightarrow{\sigma} G' \xrightarrow{p^{-1}} p^{-1}(G') \to Y.
\]

By the general theory, see \([2]\), we have that \( \phi_* \mathcal{L} \) is a holonomic regular \( \mathcal{D}_Y \)-module. It is also clear that each \( (p^*f)_y \), \( y \in p^{-1}(0) \), is a formal solution of this module, i.e.,

\[
(p^*f)_y \in \text{Hom}_{\mathcal{D}_Y}(\phi_* \mathcal{L}, \hat{\mathcal{O}}_{Y,y}).
\]

Now, it is the main property of holonomic regular systems, see, e.g., \([2]\), Prop. 14.8, that any formal solution is convergent. Thus each \( (p^*f)_y \) is convergent in some neighborhood of \( y \). This means that there is a neighborhood \( U \supset p^{-1}(0) \) in \( Y \) such that \( p^*(f) \) converges to an analytic function in \( U \). But \( U \cap p^{-1}(G') \) is identified with an open set in \( G' \), so we get that \( f \) converges to an analytic function on a nonempty open subset in \( G' \) (and hence to a function on a domain of \( G \)). Theorem is proved.

**Distributions on a group.** As with usual power series, series of the form (3.1.1) can be used to represent distributions on the group as well. We need the following examples which follow at once from the Peter-Weyl theorem.

**Delta-function at 1.** It follows from Proposition 3.1.2 that the series

\[
\delta_1(x) = \sum_{\alpha \in \text{Irr}(G)} (d(\alpha) \cdot \text{Id}, x^\alpha) = \sum_{\alpha \in \text{Irr}(G)} d(\alpha) \text{tr}(x^\alpha)
\]

(sum over all the representations) represents the Dirac delta-function situated at the unity of \( G \). This is an equality of distributions on \( G_c \).

Accordingly, the delta-function situated at any \( x_0 \in G \) can be written as

\[
\delta_1(x_0^{-1} x) = \delta_1(x x_0^{-1}) = \sum_{\alpha \in \text{Irr}(G)} d(\alpha)(x_0^{-1})^\alpha, x^\alpha).
\]
(3.4.2) **Delta-function along a subgroup.** Let $H \subset G$ be a reductive subgroup. For any $\alpha \in \text{Irr}(G)$ the space $V_\alpha$ splits canonically into direct sum

$$V_\alpha = V^H_\alpha \oplus V'_\alpha,$$

where $V^H_\alpha$ is the space of $H$-invariants, and $V'_\alpha$ is the sum of the other isotypical components. Thus we have the embedding:

$$\text{End}(V^H_\alpha) \subset \text{End}(V_\alpha).$$

Let $I_\alpha(H)$ be the image of the identity operator under this embedding. Then:

$$\delta_H(x) = \sum_{\alpha \in \text{Irr}(G)} d(\alpha) \cdot (I_\alpha(H), x^\alpha)$$

is the delta-function along $H$. Again, this is an equality of distributions on $G_c$. Note that

$$\delta_H(x) = \int_{y \in H_c} \delta_1(xy^{-1})d^*y,$$

where $H_c = H \cap G_c \subset H$ is a compact form. This follows from the fact that the operator $\int_{y \in H_c} \rho_\alpha(y)d^*y$ is equal to the projection onto $V^H_\alpha$ along $V'_\alpha$.

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§4. The matrix gamma function.

(4.1) **Motivation.** In the one-dimensional Fourier analysis it is well known that the Fourier transform of a monomial $x^\alpha$, $\alpha \in \mathbb{C}$, is again a monomial $x^{-1-\alpha}$ but taken with the coefficient $\Gamma(1 + \alpha)$, the Euler gamma function. For application to hypergeometric series we need a generalization of this to the case when $x$ is an $n \times n$ matrix and $\alpha$ is a Young diagram, i.e., we need to study the Fourier transform of the matrix elements of irreducible representations of $GL_n(\mathbb{C})$. This generalization was essentially done in [21-23], and we just formulate it in the form needed for applications.

(4.1) **Multivalued monomials.** Throughout this section we set:

$$V = \mathbb{C}^n, \ G = GL_n(\mathbb{C}) = \text{Aut}(V), \ E = \text{Mat}_n(\mathbb{C}) = \text{End}(V).$$

We denote $\mathfrak{g} = gl_n(\mathbb{C})$ the Lie algebra of $G$. Let also $j : G \hookrightarrow E$ be the embedding. We also set

$$\text{Irr} = \text{Irr}(G) = \{\alpha = (\alpha_1 \geq \ldots \geq \alpha_n) \in \mathbb{Z}^n\}.$$
For $\alpha \in \text{Irr}$ we denote $V_\alpha = \Sigma^\alpha V$ the space of the corresponding representation.

By means of the standard basis in $V = \mathbb{C}^n$ we identify $\Lambda^n V \simeq \mathbb{C}$. Accordingly, for any $s \in \mathbb{Z}$ and $\beta \in \text{Irr}$ we identify $V(\beta_1 + s, ..., \beta_n + s)$ with $V_\beta$ as a vector space, so that the $G$-action differs by $\rho(\beta_1 + s, ..., \beta_n + s)(x) \cdot \det(x)^s$.

Let now

$$I = \left\{ \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n \mid \alpha_i - \alpha_{i+1} \in \mathbb{Z}_+ \right\}.$$  

(4.1.1)

Every $\alpha \in I$ has the form

(4.1.2) $\alpha = \beta + (s, ..., s), \quad s \in \mathbb{C}, \beta \in \text{Irr}.$

The set $I$ serves to label multivalued holomorphic representations of $G$. Explicitly, if $\alpha \in I$ has the form (4.1.2), we set $V_\alpha = V_\beta$ and define the $G$-action by $\rho_\alpha(x) = \rho_\beta(x) \cdot \det(x)^s$. As usual, we write $x^\alpha$ for $\rho_\alpha(x)$. This is a multivalued representation, but the corresponding representation of the Lie algebra $g$ is well-defined and irreducible.

Let $M_\alpha$ be the local system on $G$ spanned by all the determinations (branches) of all the matrix elements of $x^\alpha$. More precisely, if $\alpha$ is as in (4.1.2), and $L_s$ is the 1-dimensional local system spanned by all the branches of $\det(x)^s$, then $M_\alpha = M_\beta \otimes L_s$, with $M_\beta = \text{End}(V_\beta)$. Note that by construction $M_\alpha$ is a subsheaf in $\mathcal{O}_G$, the sheaf of holomorphic functions.

(4.1.3) **Proposition.** (a) $M_\alpha$ is a $G \times G$-equivariant local system on $G$. As such, it is irreducible.

(b) We have an isomorphism of equivariant local systems $M_\alpha^* \simeq M_{(-\alpha_n, ..., -\alpha_1)}$.

(4.1.4) **Corollary.** There is a natural action of the Lie algebra $g \oplus g$ on each stalk of $M_\alpha$ and thus on any space $\Gamma(U, M_\alpha)$ where $U$ is a connected and simply connected open set in $G$. This action is irreducible, of type $V_\alpha^* \otimes V_\alpha$.

(4.2) **Fourier integrals over Hermitian matrices.** Denote by $\text{Herm}$ the subspace of Hermitian matrices in $E$. So $\text{Herm}$ is an $n^2$-dimensional totally real subspace in $E$. Let $\text{Herm}_+$ be the cone of positive definite matrices in $\text{Herm}$, and let $\mathcal{H} \subset E$ be the Siegel upperhalf plane

$$\mathcal{H} = \text{Herm} + i\text{Herm}_+ = \{ x : i(x^* - x) \geq 0 \}.$$ 

Let also $\mathcal{P} = -i\mathcal{H}$ be the generalized right half-plane, so it is given by the condition $x + x^* \geq 0$. The action of the group $G \times G$ on $E$ does not preserve $\text{Herm}$, but the action of the real Lie subgroup $\Delta = \{ (g^{-1}, g^*) \in G \times G \}$ does. As a real Lie group, $\Delta \simeq G$ but its embedding into $G \times G$ is not holomorphic. Let $d$ be the (real) Lie algebra of $\Delta$. Thus $d$ is a real Lie subalgebra in the complex Lie algebra $g \oplus g$ and $d \otimes \mathbb{C} = g \oplus g$. 

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For a subset $U \subset G$ we denote by $\mathcal{M}_\alpha(U)$ the space of sections of $\mathcal{M}_\alpha$ over $U$. Since $\text{Herm}_+$ and $\mathcal{P}$ are, topologically, cells containing 1, we have identifications

\begin{equation}
\mathcal{M}_\alpha(\mathcal{P}) \simeq \mathcal{M}_\alpha(\text{Herm}_+) \simeq \mathcal{M}_{\alpha,1}.
\end{equation}

The power function $t^z, t \in \mathbb{R}_+, z \in \mathbb{C}$, will be always normalized by putting $t^z = e^{z \log(t)}$ with the standard choice of branch of logarithm on $\mathbb{R}_+$. We will understand the expression $|x|^z$ (power of the determinant) for $x \in \text{Herm}_+$ accordingly. This defines an identification $\mathcal{M}_{\alpha,1} \simeq M_\beta$, where $\beta$ is related to $\alpha$ as in (4.1.2). Similarly, we define an identification

\begin{equation}
\mathcal{M}_\alpha(\mathcal{H}) \simeq \mathcal{M}_\alpha(\text{Herm}_+) \simeq \mathcal{M}_{\alpha,1} \simeq M_\beta
\end{equation}

via the inclusion of $\text{Herm}_+$ into the closure of $\mathcal{H}$.

Let $\alpha \in \text{I}$ be such that $\Re(\alpha) > 0$. Define the linear operator (Fourier integral)

\begin{equation}
F_\alpha : \mathcal{M}_\alpha(\text{Herm}_+) \to \mathcal{O}(\mathcal{H}), \quad (F_\alpha f)(y) = \int_{\text{Herm}_+} f(x) e^{\text{tr}(xy)} dx,
\end{equation}

the integral being absolutely convergent. By identifying $\mathcal{H}$ with $\mathcal{P}$ via multiplication by $i$, we can write the same map in the form of a Laplace integral:

\begin{equation}
\Lambda_\alpha : \mathcal{M}_\alpha(\text{Herm}_+) \to \mathcal{O}(\mathcal{P}), \quad (\Lambda_\alpha f)(y) = \int_{\text{Herm}_+} f(x) e^{-\text{tr}(xy)} dx,
\end{equation}

Recall that the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ acts on $\mathcal{M}_\alpha(\text{Herm}_+)$. We will also use the following twisted action of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{M}_\alpha(\text{Herm}_+)$:

\[(\gamma_1, \gamma_2) * f = n(\text{tr}(\gamma_2) - \text{tr}(\gamma_1)) \cdot f + (\gamma_2, \gamma_1)f,
\]

where the last summand on the right is the standard action.

(4.2.6) Proposition. The map $\Lambda_\alpha$ is $\mathfrak{g} \oplus \mathfrak{g}$-equivariant, if we consider the standard action on $\mathcal{M}_\alpha(\text{Herm}_+)$ and the twisted action on $\mathcal{O}(\mathcal{P})$.

Proof: Since $\Lambda_\alpha$ is $\mathbb{C}$-linear, it is enough to prove $\mathbb{d}$-equivariance. This follows via a linear change of variables on $\text{Herm}_+$.

(4.2.7) Theorem. The map $\Lambda_\alpha$ defines an isomorphism of $\mathcal{M}_\alpha(\text{Herm}_+)$ with $\mathcal{M}_{-n-\alpha_n,\ldots,-n-\alpha_1}(\mathcal{P}) \subset \mathcal{O}(\mathcal{P})$.

Proof: The space $\mathcal{M}_\alpha(\text{Herm}_+)$ being an irreducible $\mathfrak{g} \oplus \mathfrak{g}$-module, it is enough to find the effect of $\Lambda_\alpha$ on a generator of it, for example on the function $s_\alpha(x) = \text{tr}(x^\alpha)$. This has been done in [23], Th. 5.9, which, after being translated to our notation, reads:

\[(\Lambda_\alpha s_\alpha)(y) = \int_{\text{Herm}_+} e^{-\text{tr}(xy)} s_\alpha(x) dx =
\]
\[(4.2.8) \quad \pi^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \Gamma(\alpha_j + n - j + 1)s_{-n-\alpha_n, \ldots, -n-\alpha_1}(y).\]

(See also [32], formula (12.6) for an evaluation of an equivalent integral.) So the generator is mapped into a nonzero scalar multiple of a generator and our statement follows.

\textbf{(4.3) The matrix gamma function.} For \(\alpha \in I\) set

\[(4.3.1) \quad \Gamma_n(\alpha) = \prod_{j=1}^{n} \Gamma(\alpha_j + n - j)\]

and call this meromorphic function on \(I\) the \textit{matrix gamma function}. For \(s \in C\) we will write \(\Gamma_n(\alpha + s)\) for \(\Gamma_n(\alpha_1 + s, \ldots, \alpha_n + s)\). The map \(\Lambda_\alpha\), as it follows from (4.2.8) and equivariance, is essentially scalar:

\[(4.3.2) \quad \Lambda_\alpha[(a, x^\alpha)] = \pi^{\frac{n(n-1)}{2}} \Gamma_n(\alpha + 1)(a^t, x^{-n-\alpha_n, \ldots, -n-\alpha_1}).\]

Here \(a \in \text{End}(V_\alpha)\) and \(a^t \in \text{End}(V^*_\alpha)\) is its transpose. We can also express this as follows:

\[(4.3.3) \quad \Lambda_\alpha(x^\alpha) = \pi^{\frac{n(n-1)}{2}} \Gamma_n(\alpha + 1)(x^{-n-\alpha_n, \ldots, -n-\alpha_1})^t,\]

where we regard \(x^\alpha\) as a \(V^*_\alpha \otimes V_\alpha\)-valued function, while \(x^{-n-\alpha_n, \ldots, -n-\alpha_1}\) is a \(V_\alpha \otimes V^*_\alpha\)-valued function.

\textbf{(4.4) Distributional version.} Since the Fourier transform is involutive, a different setting will lead to the appearance of \(\Gamma_n(\alpha+1)^{-1}\) and not of \(\Gamma_n(\alpha+1)\) as a proportionality factor in the Fourier transform of \(x^\alpha\). Tooverview both settings, it is convenient to work with distributions on Herm, directly generalizing [20], §II.2.

First, define a \(\text{End}(V_\alpha)\)-valued distribution \(x^\alpha_+\) on Herm given by

\[(4.4.1) \quad x^\alpha_+ = \begin{cases} x^\alpha, & \text{if } x \in \text{Herm}_+; \\ 0, & \text{otherwise}. \end{cases}\]

Second, since Herm is the Shilov boundary of the Siegel upperhalf plane \(H\), any analytic function \(f\) in \(H\) has a boundary value on Herm, which is a hyperfunction denoted by

\[(4.4.2) \quad \bar{f}(x) = f(x + i0) = \lim_{\tau \to 0} f(x + i\tau).\]

The identification \(M_\alpha(H) \simeq M_{\alpha,1}\) from (4.2.2), gives an \(\text{End}(V_\alpha)\)-valued analytic function \(x^\alpha\) in \(H\). Its boundary value \((x + i0)^\alpha\) on Herm is in fact a distribution.

Now, the Fourier transform of distributions on Herm is induced by the transform of test functions given by

\[(4.4.3) \quad F[\phi](y) = \int_{\text{Herm}} \phi(x)e^{i\mathrm{tr}(xy)}dx.\]
It is also denoted by $F$ and satisfies
\begin{equation}
FF[f(x)] = (2\pi)^n f(-x).
\end{equation}

Now, (4.3.3) implies, similarly to [20], the following equality of matrix-valued distributions on Herm:
\begin{equation}
F[x_+^\alpha] = e^{i\pi (|\alpha|+n)} \pi^{\frac{n(n+1)}{2}} \Gamma_n(\alpha + 1)(y^t + i0)^{(-n-\alpha_n, \ldots, -n-\alpha_1)},
\end{equation}
and therefore
\begin{equation}
F[(x + i0)^\alpha] = 2^{n^2} \pi^{\frac{n(n+1)}{2}} \left(\frac{y^{(-n-\alpha_n, \ldots, -n-\alpha_1)}}{\Gamma_n(-n-\alpha_n + 1, \ldots, -n-\alpha_1 + 1)}\right)^t.
\end{equation}

(4.5) **Divided powers.** Let $\alpha \in I$. We denote
\begin{equation}
x^{((\alpha))} = \frac{x^\alpha}{\Gamma_n(\alpha + 1)},
\end{equation}
regarding this as a matrix-valued function in the right half-plane $P$. Here we use the normalization of $x^\alpha$ in $P$ described in (4.2.1). The function $x^{((\alpha))}$ will be called the divided power.

It follows from Proposition 1.3.3 that the derivatives of $x^{((\alpha))}$ have a particularly simple form. In the next proposition we conserve the notation of (1.3).

(4.5.2) **Proposition.** For $a \in \text{End}(V^*)$, $b_\alpha \in \text{End}(V_\alpha)$ we have
\begin{equation}
(a, \partial) \cdot (b_\alpha, x^{((\alpha))}) = \sum_{\beta = \alpha - e_i} (p_\alpha^{\beta})^t (a \otimes b_\alpha, x^{((\beta))}).
\end{equation}

(4.6) **Example: exponential series.** If $\alpha \in \text{Irr}$, $\alpha \geq 0$ is a positive Young diagram with $|\alpha| = m$, then let $W_\alpha$ be the corresponding representation of the symmetric group $S_m$. Let $w_\alpha = \text{dim}(W_\alpha)$. By the Weyl reciprocity theorem,
\begin{equation}
V^\otimes m \simeq \bigoplus_{|\alpha|=m} W_\alpha \otimes V_\alpha
\end{equation}
as a $G$-module, which implies that
\begin{equation}
\text{tr}(x)^m = \sum_{|\alpha|=m} w_\alpha \cdot s_\alpha(x).
\end{equation}
For any Young diagram \( \alpha = (\alpha_1 \geq \ldots \geq \alpha_m) \) with \( m \) cells the number \( w_\alpha \) is given by the well known formula (see [11], formula (4.11)):

\[
(4.6.3) \quad w_\alpha = \frac{m!}{\prod_{j=1}^{m} (\alpha_j + m - j)!} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j + j - i).
\]

Note that as long as \( \alpha \) has no more than \( n \) non-zero parts (which is true for \( \alpha \) entering into (4.6.2)),

\[
\prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j + j - i) = \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j + j - i) / \prod_{j=1}^{m} (\alpha_j + m - j)!.
\]

Therefore, using the formula (1.1.7) for \( d(\alpha) \) we get:

\[
(4.6.4) \quad \frac{w_\alpha s_\alpha(x)}{|\alpha|!} = c_n \frac{d(\alpha)s_\alpha(x)}{\Gamma_n(\alpha + 1)}, \quad c_n = \prod_{j=1}^{n-1} j!
\]

or, equivalently,

\[
(4.6.5) \quad e^{\text{tr}(x)} = \sum_{m=0}^{\infty} \frac{\text{tr}(x)^m}{m!} = c_n \sum_{\alpha \in \text{Irr}} d(\alpha) \frac{s_\alpha(x)}{\Gamma_n(\alpha + 1)}.
\]

Here we extended the sum over the entire set \( \text{Irr} \), since \( 1/\Gamma_n(\alpha + 1) = 0 \) unless \( \alpha \geq 0 \). The equality (4.6.4) can be seen as the termwise Fourier transform of the formula

\[
(4.6.6) \quad \delta_1(x) = \sum_{\alpha \in \text{Irr}} d(\alpha) s_\alpha(x),
\]

discussed in (3.4.1). Note that (4.6.6) is an equality of distributions on the unitary group, so it is not possible to formally deduce (4.6.5) from it, if in the Fourier transform we integrate over the space of Hermitian matrices.

**4.7 Contour Fourier transform of single-valued monomials.** When considering single-valued functions on \( G = GL_n \), we can take the contour in the Fourier integrals to be the unitary group \( U_n \). Thus, we define the contour Fourier transform to be the map

\[
(4.7.1) \quad \text{FC} : \mathbb{C}[G] \to \mathbb{C}[G], \quad \text{FC}[f](y) = \int_{x \in U_n} f(x) e^{\text{tr}(xy)} dx.
\]

Here the integral can be, of course, calculated purely algebraically, in terms of coefficients of the expansion of \( f \) into monomials. The analog of the formula (4.4.6) in this algebraic situation is as follows:

\[
(4.7.2) \quad \text{FC}[x^{\alpha_1 \ldots \alpha_n} x^{-\alpha_1 \ldots \alpha_n}] = C \cdot \frac{y^\alpha}{\Gamma_n(\alpha + 1)},
\]
§5. The $A$-hypergeometric system.

(5.1) The system and its Fourier transform. We consider a pair of reductive groups

$$H \subset G = \prod_{\omega \in A} GL(V_\omega), \quad A \subset \text{Irr}(H),$$

satisfying the homogeneity property (2.1.3). We define the space $M^*_A = \bigoplus_{\omega \in A} \text{End}(V_\omega)$ and varieties $Y_A \subset M^*_A, X_A \subset P(M^*_A)$ as in §2. Thus $Y_A$ is $H$-biinvariant and contains $H$ as an open orbit. Let $I_A \subset S^*(M_A)$ be the homogeneous ideal of $Y_A$.

Let $h$ be the Lie algebra of $H$, and $\chi : h \to \mathbb{C}$ be a character. Let us identify $S^*(M_A)$ with the ring of differential operators on $M_A$ with constant coefficients. More precisely, for $f \in S^*(M_A)$ we denote by $P_f$ the corresponding differential operator.

By definition, the $A$-hypergeometric system corresponding to the character $\chi$, is the following system of linear differential equations on a function $\Phi \in \mathcal{O}(M_A)$:

$$\begin{cases}
L_h \Phi = R_h \Phi = \chi(h) \cdot \Phi, & h \in h, \\
P_f \Phi = 0, & f \in I_A.
\end{cases}$$

Here $L_h$ and $R_h$ are the infinitesimal generators of the left and right $H$-actions on $M_A$. Thus the first group of equations just expresses the quasihomogeneity of $\Phi$. Holomorphic solutions of this system (defined over some open sets in $M_A$) will be called $A$-hypergeometric functions. We denote by $\mathcal{H} = \mathcal{H}_{A,\chi}$ the sheaf of such functions.

For a finite-dimensional space $V$ let $\mathcal{D}(V)$ denote the algebra of differential operators on $V$ with polynomial coefficients. There is a natural isomorphism $\phi : \mathcal{D}(V) \to \mathcal{D}(V^*)$ called the (formal) Fourier transform. If $x_1, ..., x_N$ are linear coordinates in $V$ and $y_1, ..., y_N$ are the dual coordinates in $V^*$, then

$$\phi(x_i) = -\frac{\partial}{\partial y_i}, \quad \phi \left( \frac{\partial}{\partial x_i} \right) = y_i.$$

For $P \in \mathcal{D}(V)$ we will denote $\phi(P)$ simply by $\hat{P}$.

Now take $V = M_A$ and apply the Fourier transform to the system (5.1.1). In this way we get the following system on a function $\Psi$ on $M_A^*$:

$$\begin{cases}
\hat{L}_h \Psi = \hat{R}_h \Psi = \chi(h) \cdot \Psi, & h \in h, \\
f \cdot \Psi = 0, & f \in I_A.
\end{cases}$$

where $C$ is the same as in (3.2.1). The formula (4.7.2) follows at once from (3.2.2) and from the series expansion (4.6.5) of $e^{\text{tr}(x)}$. 

Note that $\hat{L}_h$, resp. $\hat{R}_h$ is just the infinitesimal generator of the left, resp. right $H$-action on $M^*_A$ dual to that on $M_A$. The second group of equations means that $\Psi$ is supported on $Y_A$, so solutions exist only among distributions.

(5.1.3) Theorem. (a) The system (5.1.1) is holonomic, with regular singularities. In particular, there is an open subset $M^\text{gen}_A$ in $M_A$ (“generic stratum”) such that the restriction of the sheaf $\mathcal{H}$ to $M^\text{gen}_A$ is a locally constant sheaf on $M_A$ of finite rank.

(b) More precisely, $M^\text{gen}_A$ is obtained from $M_A$ by deleting the varieties $\nabla_{A,\Gamma}$, see (2.8), for all proper faces $\Gamma$ of the polytope $Q_A$.

(c) Suppose that the variety $Y_A$ is Cohen-Macaulay. Then the rank of the sheaf $\mathcal{H}$ at any point of $M^\text{gen}_A$ is less or equal to the degree of $X_A$ (which is given by the Kazarnovskii-Brion theorem 2.7.1).

(5.1.4) Remark. In the toric case, the Cohen-Macaulay property implies that the generic rank of $\mathcal{H}$ is equal to $\deg(X_A)$, see [15]. Without this property, the generic rank of $\mathcal{H}$ can be greater, as shown by B. Sturmfels and N. Takayama [47].

(5.2) The hypergeometric $\mathcal{D}$-module. The proof of (5.1.3) is similar to that in [15] and is based on the analysis of the $\mathcal{D}$-module corresponding to the system (see [2] [28] for general background on $\mathcal{D}$-modules). If $Z$ is a smooth algebraic variety, we denote by $\mathcal{D}_Z$ the sheaf of regular differential operators on $Z$ and by $\mathcal{D}(Z)$ the ring of its global sections. For a coherent $\mathcal{D}_Z$-module $\mathcal{N}$ we denote by $SS(\mathcal{N}) \subset T^*Z$ its characteristic variety and by $\overline{SS}(\mathcal{N})$ the characteristic cycle.

We denote for short $\mathcal{D} = \mathcal{D}_{M_A}$ and $\hat{\mathcal{D}} = \mathcal{D}_{M^*_A}$. Let $\mathcal{M} = \mathcal{M}_{A,\chi}$ be the $\mathcal{D}$-module corresponding to the system (5.1.1):

$$\mathcal{M}_{A,\chi} = \mathcal{D} \left/ \sum_{f \in I_A} \mathcal{D} \cdot P_f + \sum_{h \in h} \mathcal{D} \cdot (L_h - \chi(h)) + \sum_{h \in h} \mathcal{D} (R_h - \chi(h)) \right..$$

Thus the sheaf of $A$-hypergeometric functions can be written as

$$\mathcal{H}_{A,\chi} = \overline{\text{Hom}}_{\mathcal{D}}(\mathcal{M}_{A,\chi}, \mathcal{O}_{M_A}).$$

Let also $\hat{\mathcal{M}}$ be the $\hat{\mathcal{D}}$-module corresponding to the system (5.1.2). It is a particular case of $\mathcal{D}$-modules described in the following theorem, see [2], Ch. VII, Th. 12.11 or [28], §5.

(5.2.3) Theorem. Let $K$ be an algebraic group over $\mathbb{C}$ acting on a smooth algebraic variety $Z$. Let $W \subset Z$ be a closed subvariety which is the union of finitely many $K$-orbits and let $\mathcal{I}_W \subset \mathcal{O}_Z$ be the sheaf of functions vanishing on $W$. Let $\mathfrak{k}$ be the Lie algebra of $K$ and $\beta : \mathfrak{k} \rightarrow \mathbb{C}$ be a character. Then the $\mathcal{D}_Z$-module

$$\mathcal{D}_Z \left/ \mathcal{D}_Z \cdot \mathcal{I}_W + \sum_{k \in \mathfrak{k}} \mathcal{D}_Z (L_k - \beta(k)) \right..$$

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is holonomic, with regular singularities. Its characteristic variety is contained in the union of the closures of the conormal bundles $T^*_O(Z)$, where $O$ runs over the $K$-orbits in $W$.

The module $\hat{M}$ corresponds to $K = H \times H$, $Z = M_A^*$, $W = Y_A$. The proof of Theorem 5.1.3 (a-b) is now done in the same way as in [15]: the module $\hat{M}$ being monodromic in the sense of [6], its Fourier transform, i.e., $\check{M}$, is again holonomic regular by Corollary 7.25 of [6], which gives (a). To see (b), let $\tilde{\phi} : T^*M_A \to T^*M_A^*$ be the natural identification of the cotangent spaces. The result of [6] just cited implies also that $SS(\check{M}) = \tilde{\phi}^{-1}(SS(\hat{M}))$ which gives (b).

We now prove part (c). We start with the following lemma.

(5.2.4) Lemma. Let $Y \subset \mathbb{C}^N$ be a conic ($\mathbb{C}^*$-invariant) closed algebraic variety, whose projectivization $P(Y) \subset P^{N-1}$ has degree $d$, and let $E \subset \mathbb{C}^N$ be a linear subspace such that $\dim(E) + \dim(Y) = N$ and $Y \cap E = \{0\}$. If $Y$ is Cohen-Macaulay, then

$$\dim\mathbb{C}\left(\mathbb{C}[E] \otimes \mathbb{C}[x_1, \ldots, x_N] \mathbb{C}[Y]\right) = d.$$  

Proof: Let $\mathbb{P}^N$ be the projective space containing $\mathbb{C}^N$, and $\tilde{E}, \tilde{Y}$ be the closures of $E, Y$ in $\mathbb{P}^N$, so that $\tilde{E}$ is a projective subspace and $\deg(\tilde{Y}) = d$. Then $\tilde{E} \cap \tilde{Y} = E \cap Y$ consist of one point, namely $0 \in \mathbb{C}^N$. Therefore

$$\sum_i (-1)^i \dim\mathbb{C} \text{Tor}_{i}^{\mathbb{C}[x_1, \ldots, x_n]}(\mathbb{C}[E], \mathbb{C}[Y]) = d,$$

as this is the contribution of the only intersection point of $\tilde{E}$ and $\tilde{Y}$ into their global intersection index on $\mathbb{P}^N$, which is $d = \deg(\tilde{Y})$, see [43]. On the other hand, let $l_1, \ldots, l_m$ be independent linear equations of $E$. As $\dim(E \cap Y) = 0 = \dim(Y) - m$, we conclude that $l_1, \ldots, l_m$ form a regular sequence in $E$. As $\dim(E \cap Y) = 0$, we conclude that $\text{Tor}_i = 0$ for $i \neq 0$ and the lemma follows.

Let now $h \in \mathfrak{h}$. The vector field $L_h$ on $M_A$ can be regarded as a family of linear functionals $L_h(a) : M_A^* \to \mathbb{C}$ depending on $a \in M_A$. These functionals form the highest symbol of the equation $L_h \Phi = \chi(h) \cdot \Phi$ from (5.1.1). For given $a \in M_A$ let $\lambda_h(a) \subset M_A^*$ be the kernel of $L_h(a)$. Similarly, let $R_h(a) : M_A^* \to \mathbb{C}$ be the linear functional corresponding to $R_h$, and $\rho_h(a)$ be its kernel. Now, the following fact is clear from the definitions.

(5.2.5) Lemma. Let $a \in M_A, h \in \mathfrak{h}$. A point $b \in M_A^*$ lies in $\lambda_h(a)$, resp. in $\rho_h(a)$, if and only if the vector

$$\left.\frac{d}{dt}\right|_{t=0} e^{th} \cdot b, \text{ resp. } \left.\frac{d}{dt}\right|_{t=0} b \cdot e^{th} \in M_A$$

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is orthogonal to $a$.

Let

$$
\lambda(a) = \bigcap_{h \in h} \lambda_h(a), \quad \rho(a) = \bigcap_{h \in h} \rho_h(a)
$$

be the zero loci over $a$ of the highest symbols of the first two groups of equations in (5.1.1).

(5.2.6) Corollary. A point $b \in M_A^*$ lies in $\lambda(a)$, resp. $\rho(a)$, resp. $\lambda(a) \cap \rho(a)$ if and only if the hyperplane $\ker(a) \subset M_A^*$ is tangent to the left $H$-orbit, resp. right $H$-orbit, resp. $H \times H$-orbit of $b$.

In particular, for $x_0 \in H$ the point $b = (x_0^\omega)_{\omega \in A}$ of $\rho_A(H) \subset M_A^*$ lies in $\lambda(a)$ or $\rho(a)$ if and only if the polynomial $f_a(x) = \sum(a_\omega, x^\omega)$ on $H$ has a critical point at $x = x_0$ (the critical value is automatically 0 by homogeneity).

(5.2.7) Corollary. If $a \in M_A^{gen}$, then $\lambda(a) \cap \rho(a) \cap Y_A = \{0\}$.

Now, noticing that codim $\lambda(a) = \text{codim} \rho(a) = \text{dim} Y_A$, we can find a linear subspace $E \subset M_A^*$ of codimension equal to $\text{dim}(Y_A)$, containing $\lambda(a) \cap \rho(a)$ and such that $E \cap Y_A = \{0\}$. Therefore, by Lemma 5.2.4, the quotient of $\mathcal{C}[Y_A]$ by the ideal generated by the linear equations of $E$, has dimension equal to $\text{deg}(X_A)$. Thus the quotient by the ideal generated by the linear equations of $\lambda(a) \cap \rho(a)$ has dimension less or equal to $\text{deg}(X_A)$. Finally, this latter quotient is just obtained by taking the highest symbols of the equations in (5.1.1), so the quotient by the full characteristic ideal has the rank less or equal to that. This proves part (c) of Theorem 5.1.3.

(5.3) Fourier integrals. We keep the above assumptions on $H, A, \chi$. The system of differential equations

$$(5.3.1) \quad L_h(u) = \chi(h) \cdot u, \quad h \in h,$$

together with the initial condition $u(1) = 1$, defines a multivalued “monomial” function on $H$ which we will denote $x^\sigma$, $\sigma = \sigma(\chi)$. Denote by $\mathcal{L} = \mathcal{L}_\chi$ the 1-dimensional local system on $H$ spanned by all the branches $x^\sigma$. Let $C^i_\bullet(H, \mathcal{L})$ be the singular chain complex of $H$ with coefficients in $\mathcal{L}$, see, e.g., [16]. An element of $C^i_\chi(H, \mathcal{L})$ is thus a finite linear combination of pairs $(\gamma, \phi)$ where $\gamma : \Delta^m \to H$ is a singular $m$-simplex and $\phi$ is a section of $\gamma^* \mathcal{L}$ on $\Delta^m$. The differential is defined by the usual rule. Let $C^i_{\chi}^f(H, \mathcal{L})$ be the bigger complex consisting of locally finite combinations as above. As well known, the homology of these complexes is expressed via sheaf cohomology (with or without compact support):

$$H^m_m(H, \mathcal{L}) = H^m_m(H, \mathcal{L}^*)^*, \quad H^m_{\chi}^f(H, \mathcal{L}) = H^m_{\chi}^f(H, \mathcal{L}^*)^*.$$

We regard $H$ as embedded into $M_A^*$. Let $K \subset M_A^*$ be any strictly convex closed cone. Denote by $C^i_\bullet(K)(H, \mathcal{L})$ the subcomplex in $C^i_{\chi}^f(H, \mathcal{L})$ formed by such locally finite chains which are contained in $K$ except for finitely many simplices and which are semi-algebraic at
the infinity of $M^*_A \supset H$ in the sense of [40]. Let also $G^{[K]}_*(H, \mathcal{L})$ be the bigger subcomplex consisting of such locally finite chains that all their simplices that do not belong to $K$, lie in a bounded subset of $M^*_A$ (and which satisfy the same condition at the infinity of $M^*_A$ as before). We will denote the homology of these complexes by $H_m^{[K]}(H, \mathcal{L})$ etc.

Let $N = \dim(H)$. We define the complexes of sheaves $\mathcal{F}, \mathcal{F}^!$ on $M_A$ by their stalks:

\begin{equation}
\mathcal{F}_a = \lim_{K \subset \Im{\{b, a\}} > 0} H^{(K)}_N(H, \mathcal{L}), \quad \mathcal{F}_a^! = \lim_{K \subset \Im{\{b, a\}} > 0} H^{[K]}_N(H, \mathcal{L}), \quad a = (a_\omega) \in M_A.
\end{equation}

Here $\Im{}$ means the imaginary part of a complex number. Define the morphism of sheaves (Fourier integral)

\begin{equation}
\mathcal{I} : \mathcal{F} \to \mathcal{O}_{M_A}, \quad \zeta = \sum_\nu (\gamma_\nu, \phi_\nu) \mapsto \sum_\nu \int_{\gamma_\nu} \phi_\nu(b) \exp\left(i \sum \text{tr}(a_\omega b_\omega)\right) db.
\end{equation}

Here the sum (i.e., the improper integral) always converges because of the fast decay of the exponent in any cone $K$ in (5.3.2). Note that in (5.3.3) we regarded $H$ as a subvariety in $M^*_A$ given by the parametric equations $b_\omega = x_\omega, x \in H$. Thus, regarding the linear function $\sum \text{tr}(a_\omega b_\omega)$ as a polynomial $f(x) = \sum (a_\omega, x_\omega)$ on $H$, we can rewrite the map $\mathcal{I}$ as follows:

\begin{equation}
\mathcal{I}\left(\sum_\nu (\tau_\nu, \phi_\nu)\right)(f) = \sum_\nu \int_{\tau_\nu} \phi_\nu e^{if(x)} dx.
\end{equation}

(5.3.5) Proposition. The image under $\mathcal{I}$ of any section of $\mathcal{F}$ satisfies the $A$-hypergeometric system.

Proof: Straightforward.

(5.3.6) Remark. The above proposition is just a manifestation of the general principle that the Fourier transform of a solution of a linear differential system, whenever it makes sense, is a solution of the (formally) Fourier transformed system. More precisely, since $H = \rho_A(H)$ is an open orbit in $Y_A$, the space of solutions of (5.1.2) near the generic point of $Y_A$ is 1-dimensional and generated by the following “holomorphic distribution”:

\begin{equation}
(\rho_A)_*(x^\sigma) = \delta_H(b) \cdot \rho_A^{-1}(b)^\sigma, \quad b \in G \subset M^*_A.
\end{equation}

This is just the delta function along $H \subset G$ multiplied by a power function. The sheaf $\mathcal{F}$ is formed by all possible contours which can be used to define the Fourier transform of this delta function. A more suggestive and classical way to rewrite a function from the image of $\mathcal{I}$ is:

\begin{equation}
\Phi(f) = \int_{\zeta \in H} x^\sigma e^{if(x)} dx = \int_{\zeta \in Y_A \subset M_A} \exp\left(i \sum \text{tr}(a_\omega b_\omega)\right) \delta_H(b) \rho_A^{-1}(b)^\sigma db.
\end{equation}

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Here $\zeta$ is an “appropriate contour of integration” (the exact meaning of this is provided by the sheaf $\mathcal{F}$). For the toric case Fourier integral representations of hypergeometric functions were studied in [12-13] where the case of an arbitrary locally compact ground field $k$ is considered. For $k = \mathbb{R}$ (and $H$ a torus) the framework of [12-13] gives the restrictions of $A$-hypergeometric functions to real values of variables while for $k = \mathbb{C}$ it is slightly different from the one presented here in that we allow more complicated domains of integration.

(5.4) Euler integrals. Suppose now that we have a pair $(H_0, A_0)$ satisfying the inhomogeneity condition (2.1.4) and let $(H, A)$ be the group and the set of representations obtained in (2.1), so that they satisfy the homogeneity condition and the $A$-hypergeometric system is defined. We have an exact sequence

\[(5.4.1) \quad 1 \to H_0 \cap \mathbb{C}^* \to H_0 \times \mathbb{C}^* \overset{p}{\to} H \to 1, \quad H \subset G = \prod_{\omega \in A_0} GL(V_\omega)\]

with $p$ being the multiplication map. Recall also that we have the identification $A_0 \to A$, $\omega \mapsto \bar{\omega}$ with $V_\omega = V_{\bar{\omega}}$ as a vector space, and thus $M_{A_0} = M_A$. We want to represent the solutions of the $A$-hypergeometric system (which are functions on $M_A = M_{A_0}$) as certain integrals of $f \in M_{A_0}$ over $H_0$.

More precisely, let $\chi$ be a character of $h = \text{Lie}(H)$. Since the map $p$ in (5.4.1) is finite, we have an identification $A_0 \to A$, $\omega \mapsto \bar{\omega}$ with $V_\omega = V_{\bar{\omega}}$ as a vector space, and thus $M_{A_0} = M_A$. We want to represent the solutions of the $A$-hypergeometric system (which are functions on $M_A = M_{A_0}$) as certain integrals of $f \in M_{A_0}$ over $H_0$.

For $f \in M_A$ we denote by $U_f$ the open set $\{f \neq 0\} \subset H$. Let $\mathcal{N}_\chi$ be the 1-dimensional local system on $U_f$ spanned by all the branches of the multivalued function $x^{\sigma_0} f(x)^\tau$. Let $N_0 = \dim(H_0)$. Consider the constructible sheaf $\mathcal{E} = \mathcal{E}(\chi)$ on $M_A$ defined by its stalks as follows:

\[(5.4.2) \quad \mathcal{E}_f = H_{N_0}(U_f, \mathcal{N}_\chi).\]

We have the morphism of sheaves on $M_A$

\[(5.4.3) \quad J : \mathcal{E} \to \mathcal{O}_{M_A}, \quad \zeta_0 = \sum_\nu (\gamma_\nu, \phi_\nu) \mapsto \sum_\nu \int_{\gamma_\nu} \phi_\nu(x) dx.\]

Images of sections of $\mathcal{E}$ under this map will be called Euler integrals.

(5.4.4) Proposition. Every Euler integral satisfies the $A$-hypergeometric system.

The proof is straightforward.

(5.4.5) Remark. It is more suggestive to write an Euler integral in the more classical form:

\[(5.4.6) \quad J(f) = \int_{\zeta_0} x^{\sigma_0} f(x)^\tau dx,\]
where \( \zeta_0 \subset H_0 \) is “an appropriate contour of integration”. On the heuristic level, the Euler integral over \( \zeta_0 \subset H_0 \) is obtained from a Fourier integral over the cycle \( \zeta = \zeta_0 \times (i\mathbb{R}_+) \) in \( H_0 \times \mathbb{C}^* \) by integrating away the \( \mathbb{R}_+ \)-variable:

\[
\int_{\zeta} e^{i\lambda f(x)} \lambda^\tau x^{\sigma_0} \, dx \, d\lambda = \int_{\zeta_0} \left( \int_{\mathbb{R}_+} e^{-\lambda f(x)} \lambda^\tau \, d\lambda \right) x^{\sigma_0} \, dx = \Gamma(\tau + 1) \int_{\zeta_0} f(x)^{-\tau - 1} x^{\sigma_0} \, dx.
\]

On the other hand, the first integral is equal to

\[
\int_{p(\zeta) \subset H} \exp(i\bar{f}(\bar{x})) \bar{x}^\sigma \, d\bar{x},
\]

where \( \bar{f} \in M_A \) corresponds to \( f \in M_{A_0} \) under the canonical identification.

**Nonresonance.** As in (2.4), choose a maximal torus \( T \subset H \), let \( \Lambda \) be its lattice of characters and \( \Lambda_\mathbb{R} = \Lambda \otimes \mathbb{R} \). We assume other notations of (2.4) as well. Let \( C_A \subset \Lambda_\mathbb{R} \) be the convex cone with apex 0 and base \( Q_A \). Thus nonempty faces of \( C_A \) are in bijection with faces of \( Q_A \).

Let \( \Lambda^0 \subset \Lambda \) be the lattice of characters of the full group \( H \), and \( \Lambda_\mathbb{C}^0 = \Lambda^0 \otimes \mathbb{C} \). Thus the character \( \chi \) defining the \( A \)-hypergeometric system, lies in \( \Lambda_\mathbb{C}^0 \). For a face \( \Gamma \subset C_A \) we denote \( \text{Lin}_C(\Gamma) \subset \Lambda_\mathbb{C} \) its \( \mathbb{C} \)-linear span. We will say that \( \chi \in \Lambda_\mathbb{C}^0 \) is nonresonant, if for any face \( \Gamma \subset C_A \) of codimension 1 we have \( \chi \not\in \Lambda + \text{Lin}_C(\Gamma) \).

Note that unlike the toric case [16], it may be no longer the case that a generic \( \chi \in \Lambda_\mathbb{C}^0 \) is nonresonant. This will be true, however, if the set \( A \) is chosen generic enough (so that \( \Lambda^0 \) does not lie in any \( \text{Lin}_C(\Gamma), \Gamma \subset C_A \)).

The meaning of the nonresonance property is as follows.

**Proposition.** If \( \chi \) is nonresonant, then the natural morphism \( j! \mathcal{L}_\chi \to Rj_* \mathcal{L}_\chi \) is an isomorphism in the derived category of sheaves on \( M^*_A \).

**Proof:** The closure of \( H \) in \( M^*_A \) is \( Y_A \), and \( H \times H \)-orbits \( Y(\Gamma) \) on \( Y_A \) are in bijection with nonempty faces \( \Gamma \subset C_A \). We have to prove the following statement: for each proper face \( \Gamma \), each point \( y \in Y(\Gamma) \) and small neighborhood \( U \) of \( y \) in \( Y_A \) we have \( H^i(U \cap H, \mathcal{L}_\chi) = 0 \) for all \( i \).

Consider first the case \( \text{codim}(\Gamma) = 1 \). In this case \( \text{codim}(Y(\Gamma)) = 1 \), since the open orbit \( H \subset Y_A \) is affine. Applying the slice theorem ([3], n. 1.5) to the normalization of \( Y_A \), we find that for \( y, U \) as above, \( U \cap H \) is homotopy equivalent to the disjoint union of several copies of the circle \( S^1 \). The condition \( \chi \not\in \Lambda + \text{Lin}_C(\Gamma) \) means that the monodromy of \( \mathcal{L}_\chi \) on each of these circles is nontrivial, so all the cohomology vanishes.

The case \( \text{codim}(\Gamma) > 1 \) follows from this, since for such \( \Gamma \) and \( U, y \) as above, \( U \cap H \) is fibered into unions of circles corresponding to any \( \Gamma' \supset \Gamma \), \( \text{codim}(\Gamma') = 1 \).

It seems that for Cohen-Macaulay \( Y_A \), the nonresonance condition should be sufficient to ensure that the systems of Euler and Fourier integrals are complete, i.e., the maps
\( \mathcal{I} : \mathcal{F} \to \mathcal{H} \) and \( \mathcal{J} : \mathcal{E} \to \mathcal{H} \) are isomorphisms of sheaves. A possible approach to this (generalizing that of [16]) would be to find the characteristic cycle of the perverse sheaf \( j_! \mathcal{L}_\chi \). This is a purely topological problem. For example, the multiplicity of \( T^*_0 M^*_A \) in the characteristic cycle is just the topological Euler characteristic of the hypersurface \( \{ f = \epsilon \} \subset H \) for generic \( f \in M_A, \epsilon \in \mathbb{C} \). When \( H \) is a torus, this Euler characteristic is equal to \( \text{deg}(X_A) \), as follows from [34]. It is likely that a similar answer is possible for an arbitrary reductive \( H \).

**Matrix \( \Gamma \)-series.** Another approach to solving the hypergeometric system is to expand the holomorphic distribution \((5.3.7)\) into a formal power series and perform the termwise Fourier transform, using the formulas of \((4.4)\). Since we will work with power series on \( G = \prod_{\omega \in A} \text{GL}(V_\omega) \), let us introduce some notation.

An element of \( G \) will be written as \( a = (a_\omega) \). Let \( J_\omega \simeq (\mathbb{C}^*)^{d(\omega)} \) be a maximal torus in \( \text{GL}(V_\omega) \), and \( \Xi_\omega \simeq \mathbb{Z}^{d(\omega)} \) be its lattice of characters. An element of \( \Xi_\omega \) will be written as \( \alpha(\omega) = (\alpha(\omega)_1, ..., \alpha(\omega)_{d(\omega)}) \). Let us abbreviate \( \text{Irr}_\omega := \text{Irr}(\text{GL}(V_\omega)) \) and identify this set with the set of dominant weights in \( \Xi_\omega \). A representation of \( G \) will be written as \( V[\alpha] = \bigoplus_{\omega \in A} \Sigma^{\alpha(\omega)}(V_\omega) \). Let \( \text{Irr}_\omega^+ \) be the set of positive dominant weights, i.e., those with all \( \alpha(\omega)_i \geq 0 \).

Set \( J = \prod_{\omega \in A} J_\omega, \Xi = \prod \Xi_\omega, \text{Irr} = \prod \text{Irr}_\omega \). Let also \( \Xi^0 = \mathbb{Z}^A \subset \Xi \) be the lattice of characters of \( G \). An element of \( \Xi^0 \) will be written as \( s = (s_\omega) \) and the corresponding character is \( a \mapsto \prod \det(a_\omega)^{s_\omega} \).

We denote by \( \Xi_R = \Xi \otimes R \) etc. Let also \( \text{Irr}_\omega^+ \cap \Xi_R \) be the convex hull of \( \text{Irr}_\omega^+ \) in \( \Xi_R \).

Denote by \( \mathbf{I}_\omega \) the set of \( \alpha(\omega) = (\alpha(\omega)_1, ..., \alpha(\omega)_{d(\omega)}) \in \mathbb{C}^{d(\omega)} \) with \( \alpha(\omega)_i - \alpha(\omega)_{i+1} \in \mathbb{Z}_+ \) and then set \( \mathbf{I} = \prod_{\omega \in A} \mathbf{I}_\omega \). Recall that the hypergeometric system depends on a character \( \chi \) of \( \mathfrak{h} \). Let \( d\rho(\omega) \) be the representation of \( \mathfrak{h} \) corresponding to the representation \( \rho(\omega) \) of \( H \). The set \( \mathbf{I} \), on the other hand, parametrizes irreducible representations of \( \mathfrak{g} = \text{Lie}(G) \).

We introduce the space

\[
(5.6.1) \quad L_\chi = \left\{ s = (s_\omega) \in \Xi^0 \subset \mathbb{C}^A \mid \text{tr} \sum_{\omega \in A} s_\omega d\rho(\omega) = \chi \right\}.
\]

In terms of the multivalued function \( x^\sigma \) this condition is expressed just as \( \prod |x^{s_\omega}|^{s_\omega} = x^\sigma \).

For any \( s \in L_\chi \) the formal series

\[
(5.6.2) \quad \sum_{\alpha \in \text{Irr}} d(\alpha) \cdot (I_\alpha(H), a^{\alpha+s}), \quad a^{\alpha+s} = \bigotimes a_\omega^{\alpha(\omega)} \cdot |a_\omega|^{s_\omega},
\]

represents the distribution \((5.3.7)\) on the compact form \( G_\subset G \). However, each term of this series also gives a distribution on the space \( M^*_A \text{Herm} = \prod_{\omega \in A} \text{Herm}(V_\omega) \), where \( \text{Herm}(V_\omega) \subset \text{End}(V_\omega) = M_\omega \) is the space of Hermitian operators, see \((4.4)\). The corresponding series of distributions on \( M^*_A \text{Herm} \) does not converge but we still can form its term-by-term Fourier transform, using \((4.4.6)\):

\[
(5.6.3) \quad \Phi_s(a) = \sum_{\alpha \in \text{Irr}} \left( \frac{d(\alpha) \cdot I_\alpha(H)}{\prod_{\omega} \Gamma(d(\omega)(\alpha_\omega + s_\omega + 1), a^{\alpha+s})} \right), \quad d(\alpha) = \prod d(\alpha_\omega).
\]
We get in this way a formal series $\Phi_s$ called the matrix $\Gamma$-series. It has the following meaning. Let $U \subset G$ be an open simply connected domain, $a^s$ a branch of the function $\prod \det(a_\omega)^{s_\omega}$ in $U$, and $\mathcal{M}_s$ the 1-dimensional space of functions in $U$ spanned by $a^s$. Then $\Phi_s(a) \in \mathbb{C}[[G]] \otimes \mathbb{C} \mathcal{M}_s$, where $\mathbb{C}[[G]]$ is the space of formal series on $G$ defined in (3.3). The action of differential operators on elements of $\mathbb{C}[[G]] \otimes \mathcal{M}_s$ is defined similarly to Proposition 3.3.1.

**(5.6.4) Proposition.** (a) If $s' \in L_0 \cap \mathbb{Z}^A$, then $\Phi_{s+s'}(a) = \Phi_s(a)$ as formal series.
(b) The series $\Phi_s(a)$ satisfies the $\mathcal{A}$-hypergeometric system in the sense described above.

**Proof:** Part (a) is obvious, part (b) follows from the properties of divided powers (Proposition 4.5.2).

**(5.7) Convexity and convergence of matrix $\Gamma$-series.** Let $K(G, H) \subset \Xi_\mathbb{R}$ be the convex hull of all the weights of all those representations $V_{[\alpha]} = \bigotimes \Sigma^{\alpha(\omega)}(V_\omega)$ of $G$ for which $V_{[\alpha]} \neq 0$. This is a convex but not necessarily strictly convex, cone in $\Xi_\mathbb{R}$. For a subset $B \subset A$ we denote $p_B : \Xi_\mathbb{R} \to \prod_{\omega \in B} \Xi_{\omega, \mathbb{R}}$ the coordinate projection.

**(5.7.1) Definition.** A subset $B \subset A$ is called a cobase, if

$$K(G, H) \cap p_B^{-1}\left(\prod_{\omega \in B} \text{Irr}^+_{\omega, \mathbb{R}}\right)$$

is a strictly convex cone.

**(5.7.2) Example.** Let $H$ be a torus, so that $\text{Irr}(H) = \Lambda$ is a lattice. A subset $B \subset A$ is a cobase if and only if the complement $A - B$ is an affinely independent subset, i.e., forms the set of vertices of a simplex, see [15].

**(5.7.3) Theorem.** If $B$ is a cobase and $s \in \Xi_\mathbb{C}^0$ is such that $s_\omega \in \mathbb{Z}$ for $\omega \in B$, then the series $\Phi_s(a)$ converges in some open domain in $G$.

If $s$ satisfies the conditions of the theorem, we shall say that $\Phi_s(a)$ is adapted to $B$.

**Proof:** Because of the poles of the gamma-functions entering the factors $\Gamma(d(\omega))\alpha(\omega) + s_{\omega} + 1$ in the denominators, we find that $\Phi_s(a)$ is the product of a scalar monomial and a series $\Phi'_s(a)$ whose support lies in the cone described in Definition 5.7.1, which is, by assumption, strictly convex. Denote by $S$ the set of all irreducible representations $V$ of $G$ such that all the weights of $V$ lie in this cone. Then $\Phi'(a)$ belongs to the ring $M[[S]]$ of formal power series from (3.3) and satisfies a holonomic system with regular singularities. Thus it has a nonempty domain of convergence, by Theorem 3.3.2.

**(5.8) Terminating series.** The set $B = A$ is always a cobase. Unlike the torus case, it is not possible to guarantee the existence of any other cobases for general $(H, A)$ (see §6 for examples where nontrivial cobases exist). For $B = A$ the condition of Theorem 5.7.2 is simply that all the $s_\omega$ are integers. A matrix $\Gamma$-series with this property will be called totally resonant.
Let us identify $\mathbb{Z}^A$ with $\text{Hom}(G,C^*)$, associating to $s = (s_\omega)$ the character $|b|^s = \prod |b_\omega|^{|s_\omega|}$. Let

\[ \mathbb{Z}^A \xrightarrow{r^*} \Lambda^0 = \text{Hom}(H,C^*) \xrightarrow{q} \text{Hom}(C^*,C^*) = \mathbb{Z} \]

be the restriction maps. We think of $\Lambda^0$ as a lattice (inside $\text{Irr}(H)$) and for an element $\sigma$ of this lattice denote by $x^\sigma$ the corresponding character of $H$. Thus, the following is obvious.

(5.8.2) Proposition. (a) If $\chi$ is a character of $h$ and $s \in L_\chi \cap \mathbb{Z}^A$, $\sigma = r(s)$, then $x^\sigma$ is the same as the function defined by (5.3.1).

(b) For $s \in \mathbb{Z}^A$ we have $q(r(s)) = \sum \omega d(\omega) \cdot s_\omega$.

Now, our first remark about totally resonant series is;

(5.8.3) Proposition. (a) Every totally resonant $\Gamma$-series is terminating, i.e., it contains only finitely many nonzero terms and is therefore a polynomial on $G$ in the sense of (1.1).

(b) This polynomial is, moreover, identically zero unless $q(r(s)) \leq 0$.

For the torus case, terminating $\Gamma$-series were studied in [42].

Proof: Every representation $\rho_\omega : H \to GL(V_\omega)$ is homogeneous of degree 1 with respect to $C^* \subset H$. Thus, in order that $R_{[\alpha]} = \bigotimes \Sigma^{\alpha(\omega)}(V_\omega)$ satisfy $R^{H}_{[\alpha]} \neq 0$, it is necessary that $\sum |\alpha(\omega)| = 0$. Let $s \in \mathbb{Z}^A$. Then $|\alpha(\omega) + s_\omega| = |\alpha(\omega)| + d(\omega)s_\omega$. Notice also that the denominator in the coefficients of the series $\Phi_s(a)$ has a pole unless $\alpha(\omega) + s_\omega \geq 0$ for all $\omega$. So our statement follows from the next obvious fact.

(5.8.4) Lemma. The set of $\alpha = (\alpha(\omega))_{\omega \in A}$ such that all $\alpha(\omega) \geq 0$ and $\sum |\alpha(\omega)| = d$, is finite for all $d$ and empty for $d < 0$.

Let now $(H_0,A_0)$ be an inhomogeneous pair, and $(H,A)$ be its homogeneization, so that we have a surjective homomorphism $p : H_0 \times C^* \to H$ with finite kernel. Let $s \in \mathbb{Z}^A$ and $\sigma = r(s) \in \Lambda^0$ be as before. Let

\[ (\sigma_0, -\tau) \in \text{Hom}(H_0,C^*) \times \text{Hom}(C^*,C^*) = \text{Hom}(H_0,C^*) \times \mathbb{Z} \]

be the pullback of $\sigma$ via $p$. Clearly, $\tau = -q(r(s)) \geq 0$ if $\Phi_s$ does not vanish identically.

(5.8.5) Proposition. The terminating series $\Phi_s(a)$ is equal to the following Euler integral of $f(x) = \sum_{\omega \in A_0} (a_\omega, x^\omega)$:

\[ \Phi_s(a) = \text{const} \cdot \int_{H_{0,c}} f(x)^\tau x^{\sigma_0} d^* x, \]

where $H_{0,c} \subset H_0$ is a maximal compact subgroup.

Of course, this integral can be calculated purely algebraically, since $\tau \in \mathbb{Z}_+$. 

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Proof: We repeat the arguments of Remark 5.4.5 with $\zeta_0 = H_{0,c}$ but with $\mathbb{R}_+$ replaced by the unit circle $U_1$. Accordingly, $p(\zeta)$ is replaced by the compact form $H_c$:

$$
\int_{H_{0,c}} f(x)^{\tau} x^{s_0} dx = \frac{\tau!}{2\pi i} \int_{H_{0,c}} \left( \int_{|\lambda|=1} e^{\lambda f(x)} \lambda^{-\frac{d\lambda}{\lambda}} \right) x^{s_0} d^* x =
$$

$$
= \tau! \int_{H_{0,c} \times U_1} e^{\lambda f(x)} \lambda^{-\tau} x^{s_0} d^* x d^* \lambda = \tau! D \cdot \int_{H_c} e^{f(x)} x^{s_0} d^* \bar{x},
$$

where $D$ is the degree of the map $p$, and $\bar{\tilde{f}} \in M_A$ corresponds to $f \in M_{A_0}$ under the canonical identification. On the other hand,

$$
\int_{H_c} e^{\bar{\tilde{f}}(x)} x^{s_0} d^* \bar{x} = \int_{G_c} \exp \left( \sum_{\omega} \text{tr}(a_{\omega} b_{\omega}) \right) \delta_H(b) b^* db.
$$

Expanding now $\delta_H$ into a series of distributions on $G_c$, as in (3.4.2), we see that this time it is legitimate to integrate it termwise, since the integration is taken over $G_c$. The proof is finished by applying the formula (4.7.2) to each factor of $G_c = \prod_{\omega \in A} U_d(\omega)$.

(5.9) Deformation of vanishing $\Gamma$-series. Let $s \in \mathbb{Z}^A$ be such that $\tau = -\sum d(\omega)s_\omega < 0$. Then $\Phi_s(a)$ vanishes identically. On the Euler integral side, however, the corresponding $A$-hypergeometric system is satisfied by functions $J(f) = \int f(x)^{\tau} x^{s_0} d^* x$, which are periods of rational differential forms on the spherical variety $X_A$, or on the hypersurface $\{ f = 0 \} \subset X_A$. A way to obtain solutions from the general power series construction is by taking iterated derivatives of the $\Phi_s(a)$ in $s$. For the toric case this method was used by J. Stienstra [46] (see also [26]) to construct all the solutions in the resonant situation. Here we will consider the simplest deformation $s \mapsto (s_\omega + t), t \in \mathbb{C}$.

The function $1/\Gamma(z)$ has a first order zero at $z = -d, d = 1, 2, \ldots$ with the derivative $(-1)^d d!$. Thus, using the definition of the matrix $\Gamma$-function as a product, we find

$$
(5.9.1) \quad \frac{d}{dt} \bigg|_{t=0} \Phi_{s_\omega + t}(a) = \sum_{\gamma \in A} \Phi_s^{(\gamma)}(a), \text{ where}
$$

$$
(5.9.1') \quad \Phi_s^{(\gamma)}(a) = \sum_{\alpha \in \text{Irr} : \begin{array}{l}
\alpha(\omega) + s_\omega \geq 0, \omega \neq \gamma,
\alpha(\gamma)d(\gamma) - 1 + s_\gamma + 2 \geq 0
\end{array}} (d(\alpha) C_s^{(\gamma)}(a) I_\alpha(H), a^{\alpha + s}),
$$

where $C_s^{(\gamma)}(a)$ is the following number:

$$
(5.9.1'') \quad C_s^{(\gamma)}(a) = \frac{(-1)^{\alpha(\gamma)d(\gamma) - s_\gamma - 1} \Gamma(-\alpha(\gamma)d(\gamma) - s_\gamma) \prod_{\omega \neq \gamma} \Gamma_d(\omega)(\alpha(\omega) + s_\omega + 1) \prod_{j=1}^{d(\gamma)-1} \Gamma(\alpha(\gamma)j + s_\gamma + d(\gamma) - j + 1)}{\prod_{\gamma} \Gamma_d(\omega)(\alpha(\omega) + s_\omega + 1) \prod_{j=1}^{d(\gamma)-1} \Gamma(\alpha(\gamma)j + s_\gamma + d(\gamma) - j + 1)}.
$$

(I am grateful to the referee for correcting this formula.) As before, we prove:
\(5.9.2\) Proposition. Each \(\Phi^{(\gamma)}_s\) formally satisfies the \(A\)-hypergeometric system.

We now consider one particular case, generalizing an observation of Batyrev ([1], Proposition 14.6). Namely, let \((H_0, A_0)\) be an inhomogeneous pair such that \(0 \in A_0\) and, moreover, \(0\) is a vertex of \(Q_{A_0}\). Thus any \(f \in M_{A_0}\) is written as

\[
f(x) = a_0 + \sum_{\omega \in A'_0} (a_\omega, x^\omega), \quad A'_0 = A_0 - \{0\}.
\]

Let \(\text{Irr}'_+ = \prod_{\omega \in A'_0} \text{Irr}_+(GL(V_\omega))\). For \(\alpha \in \text{Irr}'_+\) let \(\hat{\alpha} \in \text{Irr}\) have the following components:

\[
\hat{\alpha}(0) = -\sum_{\omega \in A'_0} d(\omega) \cdot |\alpha(\omega)|, \quad \text{while} \quad \hat{\alpha}(\omega) = \alpha(\omega) \quad \text{for} \quad \omega \in A'_0.
\]

Let \((H, A)\) be the homogenization of \((H_0, A_0)\) with the standard bijection \(\omega \mapsto \bar{\omega}, \quad A_0 \rightarrow A\). Let \(A' \subset A\) be the image of \(A'_0\). Taking \(s_{\bar{0}} = 1\) and \(s_\omega = 0\) for \(\omega \in A'\), we get \(s \in \mathbb{Z}^A\) such that \(\Phi_s(a)\) vanishes identically, while

\[
\Phi^{(0)}_s(a) = \sum_{\alpha \in \text{Irr}'_+} \left(d(\alpha)I_{\hat{\alpha}}(H) \frac{\Gamma\left(1 + \sum_{\omega \in A'_0} |\alpha(\omega)|d(\omega)\right)}{\prod_{\omega \in A'_0} \Gamma_d(\alpha(\omega) + 1)}\right)^{\hat{\alpha}}.
\]

\(5.9.4\) Proposition. There is a domain \(U \subset M_A\) such that for \(f \in U\) the hypersurface \(\{f = 0\}\) does not meet the subgroup \(H_{0,c} \subset H_0\), the series \(\Phi^{(0)}_s(a)\) converges in \(U\), and its sum is equal to the following Euler integral:

\[
\Phi^{(0)}_s(a) = \text{const} \cdot \int_{H_{0,c}} \frac{d^*x}{f(x)}, \quad f = \sum_{\omega \in A_0} (a_\omega, x^\omega).
\]

Proof: This is achieved, similarly to [1], by expanding

\[
a_0 \quad \frac{f(x)}{f(x)} = \frac{1}{1 + \sum_{\omega \in A'_0} \left((a_\omega/a_0), x^\omega\right)}
\]

into the geometric series. Terms of this geometric series are labelled by integer vectors \(m = (m_\omega) \in (\mathbb{Z}_+)^{A_0}\), the term corresponding to \(m\) being

\[
(-1)^{\sum m_\omega} \frac{\left(\sum m_\omega\right)!}{\prod m_\omega!} \prod_{\omega \in A_0} \left(\text{tr}(a_\omega x^{\omega}/a_0)\right)^{m_\omega}.
\]

When \(a_0\) dominates all the other \(a_\omega\), the geometric series converges on a domain containing \(H_{0,c}\) and we can integrate it termwise. By applying the formulas (4.6.2-4) to each factor in (5.9.5) and using the orthogonality relations (3.0.1), we get the identification of the integrated series with \(\Phi^{(0)}_s(a)\).
§6. Examples.

(6.1) **A generalization of the Gauss function.** We take the group \( H_0 = \mathbb{C}^* \times GL_n = GL(L) \times GL(V), \dim(L) = 1, \dim(V) = n. \) Take \( A_0 = \{ \mathbb{C}, L, V, L \otimes V \}. \) The pair \((H_0, A_0)\) is inhomogeneous (2.1.4). A natural homogeneization of \((H_0, A_0)\) is:

\[
H = \mathbb{C}^* \times \mathbb{C}^* \times GL_n = GL(N) \times GL(L) \times GL(V), \quad A = \{ N^{\otimes 2}, N \otimes L, N \otimes V, L \otimes V \},
\]

where \( \dim(N) = 1. \) Thus

\[
(6.1.1) \quad H = \mathbb{C}^* \times \mathbb{C}^* \times GL_n = GL(N) \times GL(L) \times GL(V), \quad A = \{ N^{\otimes 2}, N \otimes L, N \otimes V, L \otimes V \},
\]

A typical point in \( M_A \) will be denoted by \((a, b, C, D)\) where \( a, b \in \mathbb{C} \) and \( C, D \in \text{Mat}_n(\mathbb{C}) \).

(6.1.3) \( Y_A = \{(u, v, x, y): uy = vx\} \),

and \( X_A \) is the projectivization of this variety. Thus

\[
\dim(X_A) = n^2 + 1, \quad \deg(X_A) = 2^{n^2}.
\]

The lattice of weights of \( H \) is \( \Lambda = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^n \), with the Weyl group \( W = S_n \) acting by permutations on the last summand only. We denote the basis vectors of \( \Lambda \) by \( e_{-1}, e_0, e_1, \ldots, e_n \), with \( e_{-1} \) and \( e_0 \) spanning the first two summands. The polytope \( Q_A \) is the cylinder over a simplex:

\[
(6.1.4) \quad Q_A = \text{Conv}\left\{2e_{-1}, e_{-1} + e_0, e_{-1} + e_i, e_0 + e_i, \quad i = 1, \ldots, n\right\} \approx \Delta^1 \times \Delta^n.
\]

The \( A \)-hypergeometric system is a system on a function \( \Phi = \Phi(f) = \Phi(a, b, C, D) \) on \( M_A \). Writing the quasihomogeneity conditions in the integrated form, we can represent the system as follows:

\[
(6.1.5) \quad \left\{ \begin{array}{l}
\frac{\partial}{\partial a} \frac{\partial}{\partial D_{ij}} \Phi = \frac{\partial}{\partial b} \frac{\partial}{\partial C_{ij}} \Phi \\
\Phi(\lambda^2 a, \lambda b, C, D) = \lambda^{x_1} \Phi(a, b, C, D) \\
\Phi(a, \lambda b, C, \lambda D) = \lambda^{x_2} \Phi(a, b, C, D) \\
\Phi(a, b, \lambda C, \lambda D) = \lambda^{x_3} \Phi(a, b, C, D) \\
\Phi(a, b, UC, UD) = \Phi(a, b, CU, DU) = |U|^x \Phi(a, b, C, D).
\end{array} \right.
\]

Here \( x_i, i = 1, 2, 3, \) are complex parameters labelling a character \( \chi \) of \( \bar{\mathfrak{h}} = \mathbb{C} \oplus \mathbb{C} \oplus gl_n. \) If \( \Phi \) is a solution, then by setting \( \phi(D) = \Phi(1, 1, 1, D) \) we have:

\[
(6.1.6) \quad \Phi(a, b, C, D) = a^{x_1} b^{x_2} - a^{x_3} b^{x_2} \cdot |C|^x \phi(ab^{-1}C^{-1}D).
\]
(6.1.7) \( \phi(UDU^{-1}) = \phi(U) \),

so, in particular, \( \phi \) is conjugacy invariant. The Euler integral solutions are:

(6.1.7) \[ \Phi(a, b, C, D) = \int_{(u, y) \in \mathbb{C}^* \times GL_n} (a + bu + (C, y) + u(D, y))^{p} t^q \det(y)^r \, du \, dy, \]

where

(6.1.8) \[ p = \frac{1}{2} \left( \chi_1 + \chi_2 + \frac{\chi_3}{n} \right), \quad q = \chi_2 - 1, \quad r = \chi_3 - n. \]

(6.1.9) Proposition. The matrix \( \Gamma \)-series corresponding to \( s = (s_1, ..., s_4) \) has the form

\[ \Phi_s(a, b, C, D) = \sum_{\nu \in \text{Irr}(GL_n)} \frac{a^{\ln + s_1} b^{-\ln + s_2} |C|^{s_3} |D|^{s_4} d(\nu) s_\nu(CD^{-1})}{\Gamma(|\nu| + s_1 + 1) \Gamma(-|\nu| + s_2 + 1) \Gamma_n(\nu + s_3 + 1) \Gamma_n(\nu + s_4 + 1)}. \]

Proof: The set \( \text{Irr} := \text{Irr}(G) \) consists of \( \sigma = (m_1, m_2, \mu, \nu) \) where \( m_i \in \mathbb{Z} \) and \( \mu, \nu \in \text{Irr}(GL_n) \). The representation of \( G \) corresponding to such a \( \sigma \), has the form

\[ R_\sigma = (N \otimes 2)^{m_1} \otimes (N \otimes L)^{m_2} \otimes \Sigma^\mu(N \otimes V) \otimes \Sigma^\nu(L \otimes V) = N^\otimes (2m_1 + m_2 + |\mu|) \otimes L^\otimes (m_2 + |\nu|) \otimes \Sigma^\mu(V) \otimes \Sigma^\nu(V). \]

(6.1.10) Proposition. (a) The space \( R_\sigma \) possesses nonzero \( H \)-invariants if and only if \( \mu = \nu \), \( m_2 = -|\nu|, m_1 = |\nu| \). In this case the space of invariants is 1-dimensional.
(b) If the conditions of (a) are satisfied, then for \( x = (a, b, C, D) \in G \) the function \( (I_\sigma(H), x) \) is equal to

\[ (I_\sigma(H), x) = \frac{1}{d(\nu)} a^{\ln |\nu|} b^{-|\nu|} s_\nu(CD^{-1}). \]

Proof: (a) is clear. To see (b), consider the following less cumbersome case to which everything is easily reduced. Let \( H' = GL(V), G' = GL(V) \times GL(V) \), so that \( H' \) is the diagonal in \( G' \). Let \( \sigma = (\nu, \nu^-) \in \text{Irr}(G') \) and let \( l \in R_\sigma = \Sigma^\nu(V) \otimes \Sigma^{\nu^-}(V) = \Sigma^\nu(V) \otimes (\Sigma^\nu(V))^* \) be the canonical pairing which generates the 1-dimensional space of \( H' \)-invariants. Let \( p \) be the projection onto this 1-dimensional space along other isotypical components, so that for \( x = (C, D) \in G_0 \) we have \( (I_\sigma(H'), x) = tr(p \circ x^\sigma) \). This last trace can be just seen as the matrix element \( \langle l | x^{\sigma} | l \rangle \) with respect to any \( H' \)-invariant scalar product \( \langle -| - \rangle \) on \( \Sigma^\nu(V) \otimes \Sigma^\nu(V^*) \) such that \( \langle l | l \rangle = 1 \).

(6.1.11) Lemma. In the above assumptions we have, for \( x = (C, D) \in G' \):

\[ \langle l | x^{\sigma} | l \rangle := \langle l | \Sigma^\nu(C) \otimes \Sigma^{\nu^-}(D) | l \rangle = \frac{1}{d(\alpha)} tr \Sigma^\nu(CD^{-1}) = \frac{1}{d(\alpha)} s_\nu(CD^{-1}). \]

Proof: It is enough to consider the case \( \nu = (1) \): in the general case we just consider the space \( \Sigma^\nu(V) \) as the new \( V \) and \( \Sigma^\nu(C) \) as the new \( C \) etc. Assuming that \( \alpha = (1) \),
we find that the element \( l \) can be viewed as the identity matrix \( 1 \in \text{End}(V) = V \otimes V^* \). The \( GL(V) \)-invariant scalar product on \( \text{End}(V) \) such that \( \langle 1 | 1 \rangle = 1 \) is given by \( \langle X | Y \rangle = \text{tr}(XY) / \dim(V) \). The operator \( \rho_\nu(C) \otimes 1 \) on \( V \otimes V^* \) corresponds, after the identification \( V \otimes V^* \simeq \text{End}(V) \), to the left multiplication by \( C \); the operator \( 1 \otimes \rho_{\nu^-}(D) \) corresponds to the right multiplication by \( D^{-1} \). Thus our matrix element is equal to

\[
\frac{1}{\dim(V)} \text{tr}((C1D^{-1}) \cdot 1) = \frac{1}{\dim(V)} \text{tr}(CD^{-1}).
\]

This establishes the particular case \( \nu = (1) \) and the general case of the lemma follows from that. This implies, in its turn, Propositions 6.1.10 and 6.1.9.

The set \( A \) possesses two one-element cobases (5.7.1), namely \( B_1 = \{ L \otimes V \} \) and \( B_2 = \{ N \otimes V \} \). The matrix \( \Gamma \) series adapted to \( B_1 \) can be obtained by putting \( s_3 = 0 \).

The series \( \phi_{s_1, s_2, s_4}(D) = \Phi_{s_1, s_2, 0, s_4}(1, 1, 1, D) \) has the form

\[
\phi_{s_1, s_2, s_4}(D) = |D|^{s_4} \sum_{\mu \in \text{Irr}^+(GL_n)} \frac{d(\mu)s_\mu(D)}{\Gamma(-|\mu| + s_1 + 1)\Gamma(|\mu| + s_2 + 1)\Gamma_n(\mu + 1)\Gamma_n(\mu^- + s_4 + 1)}.
\]

It converges near \( D = 0 \). Introducing the standard Pochhammer symbol

\[
(a)_m = \frac{\Gamma(a + m)}{\Gamma(a)} = a(a + 1) \cdots (a + m - 1), \quad a \in \mathbb{C}, m \in \mathbb{Z}_+,
\]

and the generalized, or matrix, Pochhammer symbol (cf. [23])

\[
[a]_\mu = \prod_{j=1}^{n} (a + n - j)^{\mu_j} = \frac{\Gamma_n(\mu_1 + a, \ldots, \mu_n + a)}{\Gamma_n(a, \ldots, a)}, \quad a \in \mathbb{C}, \mu \in \text{Irr}^+(GL_n)
\]

we can write, by using (4.6.4):

\[
\phi_{s_1, s_2, s_4}(D) = \text{const} \cdot |D|^{s_4} \cdot 2 F_1(-s_1, -s_4, s_2 + 1; D),
\]

where

\[
2 F_1(\alpha, \beta, \gamma; x) = \sum_{\mu \in \text{Irr}^+(GL_n)} \frac{(\alpha)[\mu][\beta]_\mu}{(\gamma)[\mu][\mu]^!} w_\mu s_\mu(x), \quad x \in \text{Mat}_n(\mathbb{C}).
\]

This function is similar to but not identical with the James-Biedenharn-Louck matrix generalization of the Gauss function [24] [30] [36], which is

\[
2 F_1(\alpha, \beta, \gamma; x) = \sum_{\mu \in \text{Irr}^+(GL_n)} \frac{[\alpha]_\mu \cdot [\beta]_\mu}{[\gamma]_\mu \cdot [\mu]^!} w_\mu s_\mu(x), \quad x \in \text{Mat}_n(\mathbb{C}), \|x\| < 1.
\]
(6.2) A generalization of the Pochhammer function. Set \( G = (\mathbb{C}^*)^{2p} \times GL_n \times GL_n \) and let

\[ H = \left\{ (u_1, \ldots, u_p, v_1, \ldots, v_p, x, y) \in G \mid u_1 \cdots u_p \cdot y = v_1 \cdots v_p \cdot x \right\}. \]

Here \( x, y \in GL_n \). Thus \( H \simeq (\mathbb{C}^*)^{2p} \times GL_n \). Let \( A \) be the set of the tautological representations of the factors of \( G \) considered as representations of \( H \). Thus \( Y_A \subset M^*_A = C^{2p} \times \text{Mat}_n(\mathbb{C}) \times \text{Mat}_n(\mathbb{C}) \) is given by the same equation as in (6.2.1). A point of \( M_A \) will be denoted by \((a_1, \ldots, a_p, b_1, \ldots, b_p, C, D)\). The subgroup \( H_0 \subset H \) given by \( u_1 = 1 \) satisfies the inhomogeneity condition. The \( A \)-hypergeometric system depends on \( 2p + 1 \) complex parameters \( \chi_1, \ldots, \chi_{2p+1} \) and consists of the equations

\[
\frac{\partial^{p+1}}{\partial a_1 \cdots \partial a_p \partial d_{ij}} \Phi = \frac{\partial^{p+1}}{\partial b_1 \cdots \partial b_p \partial c_{ij}} \Phi
\]

together with the quasihomogeneity conditions involving the \( \chi_i \) which we leave to the reader. The Euler integral solutions have the form

\[
\Phi(a_1, \ldots, a_p, b_1, \ldots, b_p, C, D) =
\]

\[
= \int_\zeta \left( a_0 + \sum_{i=2}^p a_i u_i + \sum_{i=1}^p b_i v_i + (C, y) + \frac{v_1 \cdots v_n}{u_2, \ldots, u_n} (D, y) \right)^{\lambda_1} \times
\]

\[
\prod_{i=2}^p u_i^{\lambda_i} \prod_{i=1}^p v_i^{\lambda_{p+i}} \det(y)^{\lambda_{2p+1}} dy \prod_{i=2}^p du_i \prod_{i=1}^p dv_i,
\]

where \( \zeta \) is an appropriate \( 2p - 1 + n^2 \)-dimensional contour in \( H_0 \). Any of the two \( n \)-dimensional representations in \( A \) forms a cobase with one element. The corresponding convergent \( \Gamma \)-series are easily expressed through the function

\[
p_{p+1} \mathcal{F}_p(\alpha_1, \ldots, \alpha_{p+1}, \beta_1, \ldots, \beta_p; x) = \sum_{\mu \in \text{Irr}^+(GL_n)} \frac{(\alpha_1)_{|\mu|} \cdots (\alpha_p)_{|\mu|} [\alpha_{p+1}]_{|\mu|} \mu w_\mu s_\mu(x)}{(\beta_1)_{|\mu|} \cdots (\beta_p)_{|\mu|} |\mu|!}.
\]

(6.3) A generalization of the Appell’s functions via 3j-symbols. We consider the same groups \( H_0 \) and \( H \) as in (6.1) but take a bigger set of representations of \( H_0 \), namely

\[ A_0 = \{ C, L, V, L \otimes V, L^{\otimes 2} \otimes V \} \subset \text{Irr}(H_0) \]

which entails for the homogeneization:

\[ A = \left\{ N^{\otimes 3}, N^{\otimes 2} \otimes L, N^{\otimes 2} \otimes V, N \otimes L \otimes V, L^{\otimes 2} \otimes V \right\}. \]
The space $M_A$ consists of $(a, b, x, y, z)$ with $a, b \in \mathbb{C}$ and $x, y, z \in \text{Mat}_n(\mathbb{C})$.

We will concentrate here on the form of matrix $\Gamma$-series. The group $G$ is $\mathbb{C}^* \times \mathbb{C}^* \times GL_n \times GL_n \times GL_n$. Irreducible representations of $G$ are parametrized by tuples $\sigma = (m, r, \lambda, \mu, \nu)$ with $m, r \in \mathbb{Z}$ and $\lambda, \mu, \nu \in \text{Irr}(GL_n)$. The corresponding representation is

$$R_{\sigma} = N^{\otimes (3m+2n+2|\lambda|+|\mu|)} \otimes L^{\otimes (r+|\mu|+2|\nu|)} \otimes \Sigma^\lambda(V) \otimes \Sigma^\mu(V) \otimes \Sigma^\nu(V).$$

The problem of finding $H$-invariants in $R_{\sigma}$ is thus identical with the Clebsch-Gordan problem for the group $GL_n$. When $n = 1$, the problem is trivial, and among the hypergeometric series one finds Appell’s functions $G_1, H_3, H_6$, as explained in [15], §3.2. So we consider the case $n = 2$. In this case the elementary representation theory of $SL_2$ implies the following.

(6.3.3) Proposition. (a) The necessary and sufficient conditions for non-vanishing of $R_{\sigma}^H$ are:

(a1) $3m + 2r + 2|\lambda| + |\mu| = r + |\mu| + 2|\nu| = 0$,

(a2) $|\lambda + |\mu| + |\nu| = 0$ and the nonnegative integers $\lambda_1 - \lambda_2$, $\mu_1 - \mu_2$, $\nu_1 - \nu_2$ satisfy the triangle inequalities, i.e., each of them does not exceed the sum of the two others.

(b) If the conditions of (a) are satisfied, then $R_{\sigma}^H$ is one-dimensional.

To actually write down the series, we need to fix a basis in $\Sigma^\lambda V$, $V = \mathbb{C}^2$. We choose the Gelfand-Cetlin convention, so that the basis vectors are denoted by $e_k^{(\lambda)}$ with $k$ an integer, $\lambda_1 \geq k \geq \lambda_2$. In this basis, the matrix elements are the following functions of $x = \|x_{ij}\| \in GL_2$ (see [48], p. 116):

$$t_{km}^{(\lambda)}(x) = |x|^\lambda_2 \sqrt{\frac{(k - \lambda_2)!(\lambda_1 - k)!}{(m - \lambda_2)!(\lambda_1 - m)!}} \sum_{i+j=m-k_2, i,j \geq 0} \binom{k - \lambda_2}{i} \binom{\lambda_1 - k}{j} x_{11}^i x_{12}^{k-\lambda_1+i} x_{21}^j x_{22}^{\lambda_1-k-j}.$$

We will use the hermitian scalar product in which the $e_k^{(\lambda)}$ form an orthonormal basis. It has the property that the action of $U_2 \subset GL_2$ is unitary. If $\lambda, \mu, \nu$ satisfy the conditions of (6.3.3)(a2), then the normalized generator of the 1-dimensional $GL(V)$-invariant subspace in $\Sigma^\lambda V \otimes \Sigma^\mu V \otimes \Sigma^\nu V$ is

$$v_{\lambda \mu \nu} = \sum_{i,j,k} c_{i,j,k}^{\lambda \mu \nu} e_i^{(\lambda)} \otimes e_j^{(\mu)} \otimes e_k^{(\nu)}, \quad \|v_{\lambda \mu \nu}\| = 1,$$

where the coefficients, called the 3j-symbols, are nonzero only for $i + j + k = 0$. They are easily reduced to the Clebsch-Gordan coefficients for the group $SU_2$, see [19] [37] [48] and are, explicitly, as follows ([37], §54):

$$\begin{pmatrix} \lambda \\ i \\ j \\ k \end{pmatrix} \begin{pmatrix} \mu \\ j \\ k \end{pmatrix} \begin{pmatrix} \nu \\ k \end{pmatrix} = \sqrt{\frac{(\lambda_2 + \mu_1 + \nu_1)!(\lambda_1 + \mu_2 + \nu_1)!}{(2\lambda_1 + 2\mu_1 + 2\nu_1 + 1)!}} \times \sqrt{(j - \mu_2)!(\mu_1 - j)!(k - \nu_2)!(\nu_1 - k)!}\sqrt{(\lambda_1 + \mu_1 + \nu_2)!(i - \lambda_1)!(\lambda_1 - i)!} \times$$

$$\times \sqrt{(\lambda_2 + \mu_1 + \nu_1)!(\lambda_1 + \mu_2 + \nu_1)!}.$$
\[
\times \sum_{z} z^!(\lambda_1 + \mu_1 + \nu_2 - z)!(z - j - \nu_2 - \lambda_1)!(j - z - \mu_2)!(z + i + \lambda_1 + \mu_2 + \nu_1)!(\lambda_1 - z - i)!
\]

The matrix \(\Gamma\)-series corresponding to \(s = (s_1, \ldots, s_5)\) has the form:

\[
\Phi_s(a, b, x, y, z) = a^{s_1} b^{s_2} |x|^{s_3} |y|^{s_4} |z|^{s_5} \sum_{\lambda, \mu, \nu \in \text{Irr}(GL_2)} (d(\lambda) d(\mu) d(\nu) (\frac{b}{a})^{\mu+2\nu}) \times
\]

\[
\times \frac{\sum_{i, j, k, i', j', k'} \left( \binom{\lambda}{i, j, k} \binom{\mu}{i, j', k'} \binom{\nu}{i', j', k'} t^{(\lambda)}_{i, i'}(x) t^{(\mu)}_{j, j'}(y) t^{(\nu)}_{k, k'}(z) \right)}{\Gamma(-|\mu| - 2|\nu| + s_1 + 1) \Gamma(|\mu| + 2|\nu| + s_2 + 1) \Gamma_2(\mu + s_3 + 1) \Gamma_2(\nu + s_4 + 1) \Gamma_2(\nu + s_5 + 1)}
\]

Any two 2-dimensional representations from \(A\) form a cobase. The three types of convergent \(\Gamma\) series adapted to these cobases, are natural \(GL_2\)-generalizations of Appell’s series \(G_1, H_3, H_6\).

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