Reduced $\mathcal{N} = 2$ Quantum Mechanics: Descendants of the Kähler Geometries

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Abstract

We discuss an $\mathcal{N} = 2$ quantum mechanics with or without a central charge. A representation is constructed with the number of bosonic degrees of freedom less that one half of the fermionic degrees of freedom. We suggest a systematic method of reducing the bosonic degrees of freedom called “dynamical reduction.” Our consideration opens a problem of a general classification of nonstandard representations of $\mathcal{N} = 2$ superalgebra.

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1 Introduction

Solution of many physical problems lead to supersymmetric quantum mechanics (SQM) \[1\]. Suffice it to mention quantization of moduli of supersymmetric solitons. On the other hand many geometrical concepts (such as Kähler and hyper-Kähler structures) are in one-to-one correspondence with extended-SUSY quantum mechanics. Therefore, it is natural to expect that, studying novel examples of extended-SUSY quantum mechanics, one can reveal new geometries of interest.

Quantum mechanics with extended supersymmetry was considered previously both in the physical and mathematical literature \[2, 3, 4, 5, 6, 7, 8\]. The most common strategy for obtaining such systems is dimensional reduction from \(D = 2\) or \(D = 4\). In four dimensions minimal supersymmetry has four supercharges. Upon reduction to \(D = 1\) one obtains a quantal system with four supercharges, i.e. \(\mathcal{N} = 2\). The situation most extensively studied in the literature corresponds to

\[
\nu_F = 2\nu_B, \tag{1}
\]

where \(\nu_{B,F}\) stands for the number of bosonic (fermionic) degrees of freedom, respectively. We will refer to this pattern as standard. A few exotic examples with \(\nu_B > \nu_F/2\) were analyzed too. In this work we address the problem of constructing and analyzing systems with

\[
\nu_B < \nu_F/2. \tag{2}
\]

We will suggest a regular method which may be called a dynamical reduction of bosonic variables.

The general representation of the superalgebra with four supercharges (SQM\(_2\))

\[
\{Q_i, Q_j\} = 2\delta_{ij} H, \quad i, j = 1, 2, 3, 4, \tag{2}
\]

to be investigated below, is

\[
Q_i = \psi^a \epsilon_{i; a}^\mu \frac{\partial}{\partial X^\mu} + \eta_{i; abc} \psi^b \psi^c, \tag{3}
\]

where

\[
\{\psi^a, \psi^b\} = 2\delta^{ab}, \tag{4}
\]

\(X^\mu\) and \(\psi^a\) are bosonic and fermionic coordinates, respectively. As well-known, extended supersymmetry implies \(R\) symmetry. Let \(G_A\) be a generator of \(R\) symmetry, i.e.

\[
[G_A, Q_i] = U_{A; i}^j Q_j, \tag{5}
\]

where \(U_{A; i}^j\) are constant matrices. The generators \(G_A\) form the \(R\) algebra. In the standard situation it is \(SU(2)\).
Our main results are:

(I) Obtaining representation (3) of SQM2 with
   (i) \( \mu = 1, \ a = 1, 2, 3, 4 \), i.e. \( \nu_F = 4\nu_B \);
   (ii) overextended \( R \) symmetry so(4) = su(2)+su(2);
   (iii) a non-trivial Hamiltonian;

(II) Modifying representations obtained in (I) to incorporate central charges (see Eq. (20)):
   (i) the central charge has a geometrical meaning;
   (ii) we found a mirror-like symmetry;

(III) Our procedure of dynamical reduction explains, in part, points (I), (II).

There is a certain overlap between the results presented here and those obtained previously within different approaches. In particular, the issue of constructing superalgebras of a more general form was addressed in Ref. [4]. Moreover, the overextended \( R \) symmetry naturally appears in a superfield approach of Ref. [7] which explains also why it disappears upon inclusion of the central charge\(^1\). The inclusion of the central charge was studied in Ref. [8]. Our approach of dynamical reduction of bosonic coordinates has its own merits; it is transparent, has a clear-cut geometrical interpretation and reveals the geometric connection of the central charges.

The organization of the paper is as follows. In Sec. 2 we present a new solution of the SQM2 equations, and explain that its \( R \) symmetry is larger than for the standard solutions. In Sec. 3 we review some aspects of centrally extended superalgebra with four supercharges, SQM2,Z. The solution presented in Sec. 2 is generalized to include the central charge. It is noted that the quantal system thus obtained, SQM2,Z possesses a surprising mirror-like symmetry. We investigate the nonrelativistic limit of this system. In Sec. 4 we explain how such systems can be obtained through a dynamical reduction of the standard \( N = 2 \) quantum mechanics on the Kähler manifolds. We present a clear-cut geometrical formulation. Finally, in Sec. 5 we outline some issues to be studied in the future.

2 A Nonstandard Example with Overextended \( R \) Symmetry

2.1 The example

The realization of the algebra (2) to be considered here, is built on one bosonic variable \( X \), and 4 fermionic,

\[
\{\psi_i, \psi_j\} = 2\delta_{ij}, \quad i, j = 1, \ldots, 4.
\]  

\(^1\)We have learned of the existence of this illuminating work only after the submission of our paper to hep-th. We are grateful to E. Ivanov, S. Krivonos and A. Pashnev for drawing our attention to Ref. [7].
For what follows we will introduce also $\psi_5$,

$$\psi_5 = \psi_1 \psi_2 \psi_3 \psi_4, \quad \{\psi_i, \psi_5\} = 0, \quad (i = 1, \ldots, 4), \quad (\psi_5)^2 = 1. \quad (7)$$

The representation we want to construct depends on one function of one real variable $f(X)$, and is given by

$$Q_j = i\psi_j \frac{\partial}{\partial X} + if(X)\psi_j \psi_5, \quad j = 1, \ldots, 4,$$

$$H = -\frac{d^2}{dX^2} + (f(X))^2 - \psi_5 \frac{df}{dX}. \quad (8)$$

Since the algebra (3) is nothing but the Clifford algebra, it is realized by four-by-four (Euclidean) $\gamma$ matrices. Let $S$ be the space of the four-component spinors, and $S_{\pm}$ the spaces of chiral and antichiral spinors,

$$\psi_5 S_{\pm} = \pm S_{\pm}.$$

If we look only at the Hamiltonian, and, for a short while, forget about the supercharges, we will immediately see that this system is a combination of two identical decoupled Witten’s Hamiltonians, each of them presenting $\mathcal{N} = 1$ supersymmetric quantum mechanics. Therefore, the system (8) has the following obvious properties.

(i) Any excited (nonvacuum) state has degeneracy equal to 4, i.e. the dimension of the supermultiplet is four.

As for the ground state, there are several possibilities.

(ii) The bosonic target space is compact (i.e. the coordinate $X$ lives on a circle $S_1$); and the period

$$\Pi = \int_{S_1} f(X) dX$$

vanishes. Then SUSY is unbroken, there are four zero-energy states. Two are in the space of the chiral spinor fields, and two in the space of the antichiral spinor fields,

$$\Psi_{a,\pm}(X) = s_{a,\pm} \exp \left( \pm \int_{T}^X f(T) dT \right), \quad a = 1, 2, \quad (10)$$

where $s_{+,a}$ ($s_{-,a}$) denote the basis in the space of the (anti)chiral spinors.

(iii) The bosonic target space is compact and the period (9) does not vanish. Then SUSY is broken, there are no zero-energy states. The degeneracy of the (nonsupersymmetric) ground state is four.

If the bosonic target space is noncompact, then there are two possibilities:

(iv) neither $\Psi_+$, nor $\Psi_-$ (see Eq. (10)) are normalizable; SUSY is broken, there are no zero-energy states.

(v) either $\Psi_+$, or $\Psi_-$ is normalizable; SUSY is unbroken, the degeneracy of the zero-energy state is two.
2.2 The overextended \( R \) symmetry

As well-known, in the standard (Kähler) case the \( R \) symmetry of \( \mathcal{N} = 2 \) quantum mechanics is SU(2). In the Kähler sigma models (without superpotential) this symmetry is known as the Lefshetz SL(2) symmetry.

In the example constructed above the \( R \) symmetry is larger, it is SO(4). The supercharges \( Q_i \) (\( i = 1, 2, 3, 4 \)) form a representation \( \mathbf{4} \) of this SO(4). Indeed, the generators of SO(4) are

\[
G_{ab} = \frac{1}{2} [\psi^a, \psi^b].
\]

(11)

Note that in the standard realization, even on the flat metric complex plane, the four supercharges form two doublets of SU(2), which cannot be transformed one into another by \( R \) symmetry, since one pair of the supercharges contain \( \partial/\partial z \) and another pair \( \partial/\partial \bar{z} \). (Here \( z \) is the complex coordinate on the plane).

Our SO(4) \( R \) symmetry should not be confused with SU(4) that acts on nonvacuum states in the standard realization of \( \mathcal{N} = 2 \) SUSY, with the generators

\[
[Q_i, Q_j] \frac{1}{2H}.
\]

(12)

3 Centrally Extended \( \mathcal{N} = 2 \) Superalgebra

The general \( \mathcal{N} = 2 \) algebra with the central charges can be written as follows (in the real notations)

\[
(Q_1)^2 = (Q_2)^2 = H - Z, \quad (Q_3)^2 = (Q_4)^2 = H + Z,
\]

\[
\{Q_1, Q_3\} = \{Q_2, Q_4\} = -2P,
\]

(13)

with all other anticommutators vanishing. We assumed that the central charge \( Z \) in Eq. (13) is real (and positive). This can be always achieved by an appropriate phase rotation in the definition of the supercharges. Here \( H \) is the Hamiltonian, \( P \) is the momentum operator. Restricting ourselves to the sector with the vanishing spatial momentum \( P = 0 \) we obtain the following algebra

\[
\{Q_i, Q_j\} = 2 \begin{pmatrix}
H - Z & 0 & 0 & 0 \\
0 & H - Z & 0 & 0 \\
0 & 0 & H + Z & 0 \\
0 & 0 & 0 & H + Z
\end{pmatrix}_{ij}, \quad i, j = 1, ..., 4.
\]

(14)

We will refer to it as (SQM\(_{2,Z}\) for short).
The simplest realization of the algebra (14), which presents a straightforward generalization of Witten’s quantum mechanics with two supercharges [1], was suggested in [9],

\[
Q_1 = \psi_1 p + W'(x)\psi_2 , \quad Q_2 = \psi_2 p - W'(x)\psi_1 \\
Q_3 = \psi_4 \left(2Z + p^2 + (W')^2 - i\psi_1\psi_2 W''\right)^{1/2} , \\
Q_4 = -\psi_3 \left(2Z + p^2 + (W')^2 - i\psi_1\psi_2 W''\right)^{1/2} , \tag{15}
\]

where \(W\) is a superpotential depending on one bosonic variable \(x\), and \(p = -id/dx\). Moreover,

\[\{\psi_i, \psi_j\} = 2\delta_{ij}.\]  \tag{16}

This realization is natural from the standpoint of the nonrelativistic expansion. In the nonrelativistic limit the first two supercharges are small (\(p \sim W' \sim \beta\)), and so is \(H - Z\) (which is proportional to \(\beta^2\)). The last two supercharges are large, to the leading order in \(\beta\)

\[
Q_3 = \psi_4 \sqrt{2Z} , \quad Q_4 = -\psi_3 \sqrt{2Z} . \tag{17}
\]

The Hamiltonian can be obtained by squaring \(Q_1\) or \(Q_2\),

\[H - Z = p^2 + (W')^2 - i\psi_1\psi_2 W'' . \tag{18}\]

The realization (15) is nonlinear in \(p\), however (see [9] for further details). It is evident that the fermionic variables \(\psi_3,\psi_4\) are factored out; \(H - Z\) does not depend on them; the supercharges \(Q_3, Q_4\) play no dynamical role apart from ensuring the proper dimension of the supermultiplets. In essence, the system described by (15) reduces to Witten’s quantum mechanics with two supercharges.

Our task is to explore other realizations of \(\mathcal{N} = 2\) centrally extended SUSY algebra – linear in \(p\).

\section{3.1 The Centrally Extended Reduced Quantum Mechanics}

So far we considered a superalgebra with four supercharges and no central extension. In the presence of central charges the general representation of the supercharges takes the form

\[
Q_1 = \psi^a \xi^\mu_i \frac{\partial}{\partial X^\mu} + \eta_{iabc} \psi^a \psi^b \psi^c + \zeta_{i;a} \psi^a , \tag{19}
\]
i.e. one adds terms linear in \(\psi\) to the supercharges, and obtain an SQM\(_{2,Z}\) algebra.

More concretely,

\[
Q_1 = i\psi_1 \frac{\partial}{\partial X} + if(X)\psi_1 \psi_5 + \psi_2 \left(-vg(X) + \frac{\omega}{g(X)}\right) ,
\]
\[ Q_2 = \frac{i}{2} \psi_2 \frac{\partial}{\partial X} + i f(X) \psi_2 \psi_5 + \psi_1 \left( +v g(X) - \frac{\omega}{g(X)} \right), \]
\[ Q_3 = \frac{i}{2} \psi_3 \frac{\partial}{\partial X} + i f(X) \psi_3 \psi_5 + \psi_4 \left( +v g(X) + \frac{\omega}{g(X)} \right), \]
\[ Q_4 = \frac{i}{2} \psi_4 \frac{\partial}{\partial X} + i f(X) \psi_4 \psi_5 + \psi_3 \left( -v g(X) - \frac{\omega}{g(X)} \right), \]

(20)

where \( v \) and \( \omega \) are constants, while the functions \( f(X) \) and \( g(X) \) are related as follows:
\[- \frac{1}{2} \frac{d}{dX} \log g = f(X). \]

(21)

The central charge is equal to
\[ Z = 2 v \omega. \]

(22)

Equation (21) implies, in particular, that if the target space is compact and the period \( \Pi \neq 0 \) (see Eq. (9)) the central extension of the type (20) is impossible, additional terms linear in \( \psi \)'s cannot be introduced.

As we will see below, the realization (20) is obtained by a certain reduction from the standard Kählerian formulation; therefore, we will refer to it as to reduced Kähler-related.

The Hamiltonian corresponding to the given realization of SQM is
\[ H = -\frac{d^2}{dX^2} + \left( -v g(X) + \frac{\omega}{g(X)} + i \psi_3 \psi_4 f(X) \right)^2 \]
\[ + \frac{i}{2} \psi_1 \psi_2 \frac{d}{dX} \left( -v g(X) + \frac{\omega}{g(X)} + i \psi_3 \psi_4 f(X) \right) + 2v \omega. \]

(23)

Alternatively this Hamiltonian can be rewritten as
\[ H = -\frac{d^2}{dX^2} + v^2 g^2 + \frac{\omega^2}{g^2} + f^2 \]
\[ + 2i \psi_3 \psi_4 f \left( -v g + \frac{\omega}{g} \right) + 2i \psi_1 \psi_2 f \left( v g + \frac{\omega}{g} \right) - \psi_5 \frac{df}{dX}, \]

(24)

where the first and the second line represent the boson and fermion terms, respectively. Putting \( v = \omega = 0 \) returns us to Eq. (3).

As was mentioned above, the algebra (3) can be realized by the (Euclidean) \( \gamma \) matrices. One can choose them in such a way that \( \psi_1 \psi_2, \psi_3 \psi_4 \) and \( \psi_5 \) are all diagonal, e.g.
\[ i \psi_1 \psi_2 = \text{diag}(-1,1,1,-1), \quad i \psi_3 \psi_4 = \text{diag}(-1,1,-1,1), \]
\[ \psi_5 = \text{diag}(-1,-1,1,1). \]

(25)
It is not difficult to construct the wave functions of the ground state, assuming for definiteness that $Z > 0$. To this end one multiplies the supercharge $Q_1$ by $\psi_1$ and requires

$$\psi_1 Q_1 |\Psi_0\rangle = 0. \quad (26)$$

Consider the basis in $S_\pm$ such that $\psi_1 \psi_2$ is diagonal in this basis. The formal solutions to this equation have the form

$$\Psi = S^{(a)} (g(X))^{\varepsilon_a/2} \exp \left\{ \delta_a \int \left( v g(X) - \frac{\omega}{g(X)} \right) dX \right\}, \quad (27)$$

here $\varepsilon_a$ and $\delta_a$ are $\pm 1$. They are common eigenvalues of the operators $i\psi_1 \psi_2$ and $i\psi_3 \psi_4$, while $S^{(a)}$ are the corresponding eigenvectors. For instance, if

$$S^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (28)$$

then

$$\varepsilon_1 = -1, \quad \delta_1 = 1. \quad (29)$$

Whether the formal solution is the actual ground state with $E_0 = Z$ depends on normalizability. Assume that the target space is noncompact. It is natural to start from the case when $g(X)$ neither vanishes nor becomes infinite at $-\infty < X < \infty$. Then the answer depends on the number of zeros of

$$F(X) \equiv v g(X) - \frac{\omega}{g(X)}. \quad (30)$$

The point where $F(X) = 0$ is the classical ground state. If this number is odd, two $E_0 = Z$ states of the type (27) are normalizable. They correspond to a specific choice of $\delta_a$ and $\varepsilon_a = \pm 1$. If the number of zeros is even (or no zeros at all), supersymmetry is broken, and the ground state has $E_0 > Z$. All multiplets with $E_0 > Z$ have dimension four. This is a quantum-mechanical manifestation of the multiplet shortening at $E_0 = Z$.

If $g(X) = 0$ or $g(X) = \infty$ at some finite $X = X_*$, the problem actually splits in two, one defined on the interval $(-\infty, X_*)$, another on the interval $(X_*, \infty)$. They can and must be considered separately.

Unlike the Hamiltonian (18) where two fermion variables are factored out, in the case at hand, see Eq. (23), there is no factorization – spin is entangled with the coordinate motion.
3.2 “Mirror” symmetry

Note, that the realization of SQM\(_{2,Z}\) we have constructed has a strange “mirror” symmetry. Namely, if one interchanges

\[ \omega \rightarrow v, \quad v \rightarrow \omega; \]
\[ g(X) \rightarrow \frac{1}{g(X)}, \quad \text{(then } f \rightarrow -f), \] (31)

this leaves the purely bosonic part of the Hamiltonian (the first line in Eq. (24)) invariant. Now one can promote this symmetry of the bosonic part of the Hamiltonian to the symmetry of the supercharges, and the fermionic part of the Hamiltonian. To this end one must supplement Eq. (31) by

\[ \psi_1 \rightarrow \psi_2, \quad \psi_2 \rightarrow \psi_1; \]
\[ \psi_3 \rightarrow \psi_4, \quad \psi_4 \rightarrow -\psi_3. \] (32)

Under the combined action of (31) and (32)

\[ Q_1 \leftrightarrow Q_2, \quad Q_3 \rightarrow Q_4, \quad Q_4 \rightarrow -Q_3. \] (33)

Summarizing, the “mirror” symmetry is an intrinsic property of the reduced Kähler-related SQM. A mirror symmetry was also observed by Ivanov et al. [7] within their superfield approach.

3.3 Nonrelativistic limit

It is instructive to examine the nonrelativistic limit of the mirror-symmetric realization (20). In our convention the particle mass is 1/2; we will adjust the central charge correspondingly, putting \( Z = 1/2 \). The simplest way to proceed to the nonrelativistic limit is as follows

\[ v \rightarrow \frac{1}{2}, \quad \omega \rightarrow \frac{1}{2}; \]
\[ g(X) \rightarrow 1 + \beta(X), \] (34)

where it will be assumed that

\[ |\beta(X)| \ll 1, \quad \beta(X) \sim \beta \text{ (velocity)} \]

is the expansion parameter. Then

\[ d/dX \sim \beta, \quad \text{while } \frac{d\beta(X)}{dX} \sim \beta^2, \]
and $H - Z \sim \beta^2$. The expansion of the first two supercharges in Eq. (20) starts from $O(\beta)$, while that of the last two supercharges starts from $O(\beta^0)$. To the leading order

$$
Q_1 = i\psi_1 \frac{\partial}{\partial X} - \psi_2 \beta(X), \quad Q_2 = i\psi_2 \frac{\partial}{\partial X} + \psi_1 \beta(X),
$$

$$
Q_3 = \psi_4, \quad Q_4 = -\psi_3,
$$

which identically coincides with the nonrelativistic limit of (15), provided $\beta(X)$ is identified with $W'(x)$ and $Z$ with $1/2$, see also (17). The relativistic corrections are different, however. In particular, in the mirror-symmetric realization $Q_3$ and $Q_4$ have corrections $O(\beta)$ while in the realization (13) the first correction is $O(\beta^2)$.

4 The Geometrical Meaning of Reduced Kähler-Related Realizations – a Local Reduction

4.1 Generalities

Above we presented the SQM$_{2,Z}$ algebra (20) without explaining how we arrived at it. In fact, it has a geometric origin. Some mysterious features of the mirror-symmetric realization (20) could be easily understood since (locally) it is obtained as a reduction of the standard representation of SQM$_2$, which is obtained, in turn, by dimensional reduction of the Wess-Zumino model. Moreover, the very defining property of the “mirror” symmetry finds a geometrical interpretation.

Let us remind that to obtain the standard quantum mechanics with four supercharges one takes the Wess-Zumino model in four dimensions (where the minimal number of supercharges is four) and reduces it to one (time) dimension. In this way one arrives at a supersymmetric quantal system with the target space parametrized by one complex variable. The existence of four supercharges implies that it must be the Kähler space. Thus, in the standard case the number of the (real) boson variables is two, while the number of (real) fermion variables is four, so that Eq. (1) is satisfied.

Suppose we have the algebra (13), and the operator $P$ commutes with the supercharges. Then one can restrict oneself to the states that are the eigenfunctions of the operator $P$, for example, one can restrict to the eigenfunctions with the zero eigenvalue of $P$. If the operator $P$ is a derivative along the bosonic coordinate, the reduced theory will have more fermionic degrees of freedom than twice the number of the bosonic degrees of freedom. To the best of our knowledge, this is the only way to construct realizations of the centrally extended $SQM$ with $\nu_B < \nu_F / 2$ – by a reduction of the bosonic coordinates in the algebra (13), through restriction to the states which are annihilated by $P$.

Physically, this procedure can be interpreted as follows [9]. Consider a supersymmetric $\mathcal{N} = 2$ system built of two (interacting) components. This system can be
described by two bosonic coordinates, corresponding to the motion of the center of mass of the system, plus the dynamics of an internal coordinate. The motion of the center of mass can be factored out in the general form. What remains is a reduced system with $\nu_B = 1$ and $\nu_F = 4$.

4.2 The geometric formulation

The standard solutions of the SQM$_2$ equations are constructed from the following geometrical data: a complex manifold $\mathcal{M}$ equipped with the Kähler metric $G$ and a closed holomorphic 1-differential on $\mathcal{M}$ (the latter is needed for SQM$_{2Z}$),

$$\Omega = \Omega_n dz^n, \quad \partial \Omega = 0, \quad \bar{\partial} \Omega = 0. \quad (36)$$

Now we will outline the standard solutions (without central charges) mainly with the purpose of introducing our notation. We then show how the central charges appear.

As well-known, the standard $N = 2$ superalgebra is generated by four supercharges

$$Q_\psi = i\partial + \bar{\Omega} = i\psi^m \frac{\partial}{\partial z^m} + \bar{\psi}^m \bar{\Omega}_m,$$

$$\bar{Q}_\psi = i\bar{\partial} + \Omega = i\bar{\psi}^m \frac{\partial}{\partial \bar{z}^m} - \psi^m \Omega_m,$$

$$Q_\chi = (i\partial + \bar{\Omega}) = iG^{mn} \chi_n \left( \frac{\partial}{\partial z^m} - \Gamma_{\bar{m}\bar{n}}^p \bar{\psi}^\bar{p} \bar{\chi}_n \right) + G^{\bar{m}n} \bar{\chi}_m \Omega_n,$$

$$\bar{Q}_\bar{\chi} = (i\bar{\partial} + \Omega) = iG^{\bar{m}n} \bar{\chi}_n \left( \frac{\partial}{\partial \bar{z}^m} - \Gamma_{m\bar{n}}^p \psi^p \chi_p \right) - G^{\bar{m}n} \chi_n \bar{\Omega}_m, \quad (37)$$

where

$$\psi^m = dz^m, \quad \bar{\psi}^m = d\bar{z}^m, \quad \chi_m = \iota \frac{\partial}{\partial z^m}, \quad \bar{\chi}_m = \iota \frac{\partial}{\partial \bar{z}^m}.$$

Moreover, $\iota$ stands for the operation of the contraction with the vector field, $\Gamma^{p}_{mn}$ is the Christoffel symbol corresponding to the metric $G$,

$$\Gamma^{p}_{mn} = G^{\bar{p}\bar{\ell}} \partial_{\bar{m}} G_{\bar{n} \bar{\ell}}, \quad \Gamma^{\bar{p}}_{m\bar{n}} = G^{\bar{p}\bar{\ell}} \partial_{\bar{n}} G_{\bar{m} \bar{\ell}},$$

and $\psi, \chi$ satisfy the following anticommutation relations:

$$\{\psi^m, \chi_n\} = \delta^m_n, \quad \{\bar{\psi}^m, \bar{\chi}_n\} = \delta^m_n. \quad (38)$$

All other anticommutators are trivial.

Our basic idea, from which everything else derives, is as follows. We suppose that the manifold $\mathcal{M}$ has the $U(1)$ Lie group of diffeomorphisms that preserves metric,
complex structure, and $\Omega$. The Lie algebra of this group is represented by the vector field $V$,\[
V = V^m \frac{\partial}{\partial z^m} + \bar{V}^m \frac{\partial}{\partial \bar{z}^m}.\]

The above conditions imply that \[
\Omega_m V^m = \text{const}. \tag{40}\]

In fact, below we will use a specific \textit{ansatz} for the metric,\footnote{In the case at hand a generic metric is reducible to this \textit{ansatz} by a coordinate transformation.} in which the metric $G_{\bar{m}n}$ will depend only on the combinations $z + \bar{z}$. Moreover, $V^m$ will be a purely imaginary constant.

With the vector field switched on, one can obtain the centrally extended algebra (provided the standard supercharges (37) are modified, see below),\[
\{Q^V_{\bar{\psi}}, Q^V_\chi\} = H + P_V, \quad \{\bar{Q}^V_{\bar{\psi}}, \bar{Q}^V_\bar{\chi}\} = H - P_V, \\
\{Q^V_{\bar{\psi}}, \bar{Q}^V_\bar{\chi}\} = \bar{Z}, \quad \{\bar{Q}^V_{\bar{\psi}}, Q^V_\chi\} = Z, \tag{41}\]

where $H$ is the Hamiltonian,
\[
H = -G^{\bar{m}n} \bar{\partial}_{\bar{n}} \partial_n + G^{\bar{m}n} \bar{\Omega}_n \Omega_n + G_{\bar{m}n} \bar{V}^n V^m \\
- (\partial_m G^{\bar{m}n}) \bar{\psi}^m \chi_n \bar{\partial}_{\bar{n}} - (\bar{\partial}_{\bar{m}} G^{\bar{m}n}) \bar{\psi}^\bar{m} \bar{\chi}_n \partial_n \\
+ i \Gamma^\bar{n}_{\bar{m}n} \bar{V}^\bar{p} \bar{\psi}^{\bar{m}} \bar{\chi}_n - i \Gamma^n_{mq} V^m \psi^q \chi_p \\
- (\partial_m \bar{\partial}_{\bar{n}} G^{\bar{m}n}) \bar{\psi}^\bar{m} \chi_n \bar{\psi}^\bar{m} \bar{\chi}_n \\
+ i [\partial_m (G^{\bar{m}n} \Omega_n)] \psi^m \chi_n - i [\bar{\partial}_{\bar{m}} (G^{\bar{m}n} \bar{\Omega}_n)] \bar{\psi}^\bar{m} \bar{\chi}_n, \tag{42}\]

$P_V$ is the Lie derivative along $V$,
\[
P_V = -i V^n \partial_n - i \bar{V}^\bar{m} \bar{\partial}_{\bar{n}} - i (\partial_n V^m \psi^n \chi_m + \bar{\partial}_{\bar{n}} \bar{V}^\bar{m} \bar{\psi}^\bar{m} \bar{\chi}_m), \tag{43}\]

and
\[
\partial_m \equiv \frac{\partial}{\partial z^m}, \quad \bar{\partial}_{\bar{m}} \equiv \frac{\partial}{\partial \bar{z}^\bar{m}}.
\]

Finally,
\[
Z = 2 \Omega_m V^m. \tag{44}\]

Note that the last term in Eq. (43) is irrelevant since our \textit{ansatz} implies that $\partial_n V^m = 0$. 
To obtain the Hamiltonian (42) of the centrally extended problem one must modify the above supercharges by adding vector-field-dependent terms, namely,

\[ Q^V_\psi = i\bar{\psi}^m \frac{\partial}{\partial \bar{z}^m} + \bar{\psi}^m \bar{\Omega}_m - G_{mn} \psi^m \bar{V}^n, \]
\[ \bar{Q}^V_\bar{\psi} = i\bar{\psi}^m \frac{\partial}{\partial z^m} - \psi^m \Omega_m + G_{mn} \bar{\psi}^m V^n, \]
\[ Q^V_\chi = iG^{\bar{m}n} \chi_n \left( \frac{\partial}{\partial \bar{z}^m} - \Gamma^p_{\bar{m}n} \bar{\psi}^p \bar{\chi}_p \right) + G^{\bar{m}n} \bar{\chi}_n \Omega_m - \chi_m V^n, \]
\[ \bar{Q}^V_\bar{\chi} = iG^{m \bar{n}} \chi_n \left( \frac{\partial}{\partial z^m} - \Gamma^p_{m \bar{n}} \psi^p \chi_p \right) - G^{mn} \chi_n \bar{\Omega}_m + \bar{\chi}_m \bar{V}^m. \] (45)

We are very close to our final goal. In order to obtain \( \mathcal{N} = 2 \) centrally extended superalgebra, we have to make three steps. First, we restrict ourselves to the sector of states on which \( P_V = 0 \). This is always possible since \( P_V \) commutes with the supercharges (provided that \( \Omega \) is a constant). This is just the procedure of dynamical reduction we want to suggest. It reduces the number of the bosonic degrees of freedom leaving that of the fermionic degrees of freedom intact. The second step is a similarity (quasigauge) transformation. Finally, the third step is obvious: it is the passage from the complex representation of the supercharges and \( \psi \)'s and \( \chi \)'s to real supercharges and \( \bar{\psi} \)'s. In the next subsection we will demonstrate how the procedure works in a particular example.

### 4.3 SQM\( _2 \) as a U(1) Reduction of the Standard (Kähler) SQM\( _2 \)

Let us consider a concrete example. Assume \( M \) to be a manifold of the cylinder topology, with the noncompact coordinate \( X \), the compact coordinate \( Y \) and the complex structure \( z = X + iY \). Take the Kähler metric in the following form:

\[ ds^2 = g^2(X) \left( dX^2 + dY^2 \right), \] (46)

and

\[ \Omega = -i \omega dz, \quad V = v \frac{\partial}{\partial Y} = i v \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right), \] (47)

where \( \omega \) and \( v \) are real constants.

Substituting this particular metric and the corresponding Christoffel symbols in the general formulae, and setting \( P_V = 0 \) (i.e. \( \partial/\partial Y = 0 \)) we get

\[ Q^V_\psi = \frac{i}{2} \psi \frac{\partial}{\partial X} + i \bar{\psi} \omega + i g^2(X) \psi v, \]
\[ Q^V_{\bar{\psi}} = \frac{i}{2} \bar{\psi} \frac{\partial}{\partial X} + i \psi \omega + i g^2(X) \bar{\psi} v, \]

\[ Q^V_{\chi} = \frac{i}{2} \frac{1}{g^2(X)} \chi \left( \frac{\partial}{\partial X} - 2 \frac{\partial \ln g(X)}{\partial X} \psi \bar{\chi} \right) - \frac{i}{g^2(X)} \bar{\chi} \omega - i \chi v, \]

\[ \bar{Q}^V_{\bar{\chi}} = \frac{i}{2} \frac{1}{g^2(X)} \bar{\chi} \left( \frac{\partial}{\partial X} - 2 \frac{\partial \ln g(X)}{\partial X} \psi \chi \right) + \frac{i}{g^2(X)} \chi \omega + i \bar{\chi} v. \quad (48) \]

Our task is to arrive at a symmetric expression for the first and the second pair of the supercharges. To this end we make a trick of a similarity (gauge) transformation. We conjugate all operators by \( \exp\{- (\psi \chi + \bar{\psi} \bar{\chi}) \ln g \} \), namely,

\[ \psi \rightarrow e^{- (\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} \psi e^{(\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} = \frac{1}{g(X)} \psi, \]

\[ \bar{\psi} \rightarrow e^{- (\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} \bar{\psi} e^{(\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} = \frac{1}{g(X)} \bar{\psi}, \]

\[ \chi \rightarrow e^{- (\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} \chi e^{(\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} = g(X) \chi, \]

\[ \bar{\chi} \rightarrow e^{- (\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} \bar{\chi} e^{(\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} = g(X) \bar{\chi}, \quad (49) \]

and

\[ \frac{\partial}{\partial X} \rightarrow e^{- (\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} \frac{\partial}{\partial X} e^{(\psi \chi + \bar{\psi} \bar{\chi}) \ln g(X)} = \frac{\partial}{\partial X} + (\psi \chi + \bar{\psi} \bar{\chi}) \frac{\partial \ln g(X)}{\partial X}. \quad (50) \]

After the similarity transformations, the supercharges \([48]\) become

\[ Q^V_{\psi} = \frac{i}{2g(X)} \psi \left[ \frac{\partial}{\partial X} + \bar{\psi} \bar{X} \frac{\partial g(X)}{\partial X} \right] + \frac{i}{g(X)} \bar{\psi} \omega + i g(X) \psi v, \]

\[ \bar{Q}^V_{\bar{\psi}} = \frac{i}{2g(X)} \bar{\psi} \left[ \frac{\partial}{\partial X} + \psi X \frac{\partial g(X)}{\partial X} \right] + \frac{i}{g(X)} \psi \omega + i g(X) \bar{\psi} v, \]

\[ Q^V_{\chi} = \frac{i}{2g(X)} \chi \left[ \frac{\partial}{\partial X} + \bar{X} \bar{\psi} \frac{\partial \ln g(X)}{\partial X} \right] - \frac{i}{g(X)} \bar{\chi} \omega - i g(X) \chi v, \]

\[ \bar{Q}^V_{\bar{\chi}} = \frac{i}{2g(X)} \bar{\chi} \left[ \frac{\partial}{\partial X} + \chi \bar{\psi} \frac{\partial \ln g(X)}{\partial X} \right] - \frac{i}{g(X)} \chi \omega - i g(X) \bar{\chi} v. \quad (51) \]

It is not difficult to check that the four supercharges \([51]\), and those of \([20]\), are in one-to-one correspondence. To this end one forms the following combinations:

\[ Q_1 = \frac{1}{2} \left( Q^V_{\psi} + Q^V_{\chi} - \bar{Q}^V_{\bar{\psi}} - \bar{Q}^V_{\bar{\chi}} \right), \]

\[ Q_2 = \frac{1}{2i} \left( Q^V_{\psi} - Q^V_{\chi} - \bar{Q}^V_{\bar{\psi}} + \bar{Q}^V_{\bar{\chi}} \right), \]

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\[ Q_3 = \frac{1}{2} \left( Q^V_\psi + Q^V_\chi + \bar{Q}^V_\psi + \bar{Q}^V_\chi \right), \]
\[ Q_4 = \frac{1}{2i} \left( Q^V_\psi - Q^V_\chi + \bar{Q}^V_\psi - \bar{Q}^V_\chi \right), \]  
(52)

and

\[ \psi_1 = \frac{1}{\sqrt{2}} \left( \psi - \bar{\psi} + \chi - \bar{\chi} \right), \]
\[ \psi_2 = \frac{1}{i\sqrt{2}} \left( \psi - \bar{\psi} - \chi + \bar{\chi} \right), \]
\[ \psi_3 = \frac{1}{\sqrt{2}} \left( \psi + \bar{\psi} + \chi + \bar{\chi} \right), \]
\[ \psi_4 = \frac{1}{i\sqrt{2}} \left( \psi + \bar{\psi} - \chi - \bar{\chi} \right). \]  
(53)

Then one expresses \( Q_{1,2,3,4} \) in terms of \( \psi_{1,2,3,4} \), performs an additional similarity transformation

\[ \frac{\partial}{\partial X} \rightarrow e^{(1/2)\ln g(X)} \frac{\partial}{\partial X} e^{-(1/2)\ln g(X)} = \frac{\partial}{\partial X} - \frac{1}{2} \frac{\partial \ln g(X)}{\partial X}, \]
and passes from the variable \( X \) to a new variable \( X' \),

\[ X \rightarrow X' = 2\sqrt{2} \int g(X) dX. \]  
(54)

In this way one arrives at four supercharges equivalent to (20). As was already mentioned, at \( v = \omega = 0 \) we observe an unexpected (from the geometric standpoint) overextension of the \( R \) symmetry, see Sec. 2.2.

In light of our geometric understanding, the mirror symmetry discussed in Sec. 3.2, presents the equivalence of quantum mechanics obtained by the reduction from two perfectly different manifolds (46), with distinct scale factors, see Fig. 1.

4.4 Global vs. local

So far it was assumed that the coordinate \( X \) is noncompact. Let us now compactify it, i.e. proceed to the toroidal topology. Equation (21) implies then that the period \( \Pi \) of the function \( f(X) \) defined in Eq. (9) vanishes if we start from any well-defined two-dimensional Kähler manifold. However, for the existence of SQM_{2,Z} it is sufficient to assume that (i) \( f(X) \) is single-valued on \( M \); (ii) Equation (21) is valid locally, i.e. the integral \( \int f(X) dX \) need not be single-valued. Thus, if we take the \( X \) space to be \( S_1 \) and the function \( f \) with a nonvanishing period, we arrive at SQM_{2,Z} which cannot be globally obtained by reduction from any two-dimensional Kähler manifold. (At the moment we do not know whether this solution can be obtained by reduction from a higher-dimensional Kähler manifold.)
Figure 1: Quantum mechanics obtained through reduction from these two manifolds are identical.

5 Conclusions

We have constructed, via dynamical reduction, a rather strange looking $\mathcal{N} = 2$ centrally extended quantum mechanics with $\nu_B < \nu_F/2$. Our geometrical construction partially explains the properties of the system obtained. However, not everything is explained. In particular, the origin of the mirror symmetry and the overextended $R$ symmetry are not understood.

As follows from our work, the problem of the general classification of extended superalgebras (with the standard or even overextended $R$ symmetries) in the form (19) is an open question. The solution of this problem may well lead to new interesting geometrical structures encoded in the coefficients $\epsilon, \omega, \zeta$, see Eq. (19), which could be considered as a generalization of the Kähler geometry.

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