MONOTONICITY OF THE OVER-ROTATION INTERVAL FOR BIMODAL MAPS

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ABSTRACT. We show that the over-rotation interval is a monotone function of a bimodal interval map.

1. INTRODUCTION

One-dimensional combinatorial dynamics started when O. M. Sharkovsky proved his theorem on the coexistence of periods of cycles for interval maps (in what follows by cycles we mean periodic orbits). To state it, we recall the (transitive) Sharkovsky order of the set natural numbers:

\[ 3 \succ 5 \succ 7 \succ \ldots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \]

\[ \ldots \succ 2^2 \cdot 3 \succ 2^2 \cdot 5 \succ 2^2 \cdot 7 \succ \ldots \succ 2^2 \succ 2 \succ 1. \]

In what follows by the period we mean the minimal period. Below \( I \) always denotes a closed interval.

Theorem 1.1 ([12]; [13] for English translation). If \( g : I \to I \) is continuous, \( m \succ n \) and \( m \) is the period of a cycle of \( g \) then \( n \) is also the period of a cycle of \( g \).

Theorem 1.1 inspired a lot of developments. One of them is the discovery of over-rotation numbers [8], a combinatorial tool used for classifying interval cycles and invariant under topological conjugacy of interval maps. It can be defined as follows.

Let \( I \) be the unit interval, let \( f : I \to I \) be a continuous interval map, let \( P \) be a cycle of \( f \) of period \( q > 1 \), and let \( m \) be the number of points \( x \in P \) such that \( f(x) - x \) and \( f^2(x) - f(x) \) have different signs. Call the pair \( (\frac{m}{q}, q) \) the over-rotation pair of \( P \) and denote it by \( \text{orp}(P) \); call \( \frac{m}{q} = \rho(P) \) the over-rotation number of \( P \). An over-rotation pair \( (p, q) \) is coprime if \( p \) and \( q \) are coprime. The set of over-rotation pairs of all cycles of \( f \) is denoted by \( \text{ORP}(f) \). Since the number \( 0 < m \leq q \) is even, then, in an over-rotation pair \( (p, q) \), \( p \) and \( q \) are integers and \( 0 < \frac{p}{q} \leq \frac{1}{2} \). The number \( p \) can be interpreted as the number of times \( f(x) \) goes around \( x \) as we move along the orbit of the “vector” \( xf(x) \). If \( f \) has a unique fixed point \( a \), then \( f(x) > x \) if \( x < a \) and \( f(x) < x \) if \( x > a \), and \( p \) is the number of points to the right of \( a \) which are mapped to the left of \( a \) (or vice versa).

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Theorem 1.2 ([8]). Suppose that \((p, q)\) and \((r, s)\) are over-rotation pairs. Moreover, suppose that one of the following holds.

1. \(\frac{p}{q} < \frac{r}{s}\),
2. \(\frac{p}{q} = \frac{r}{s}\) so that for a coprime over-rotation pair \((k, l)\) we have \(p/k = q/l = u\) and \(r/k = s/l = v\) are integers, and \(u \succ v\).

Then any interval map with a cycle of over-rotation pair \((p, q)\) has a cycle of over-rotation pair \((r, s)\).

Definition 1.3. Given an interval map \(f\), denote by \(I_f\) the closure of the union of over-rotation numbers of \(f\)-periodic points, and call \(I_f\) the over-rotation interval of \(f\).

By Theorem 1.1, any map \(f\) with non-fixed periodic points has a cycle of period 2 with over-rotation number \(\frac{1}{2}\); by Theorem 1.2 if \(\rho(P) = \frac{p}{q}\) for a cycle \(P\) then \([\frac{p}{q}, \frac{1}{2}] \subset I_f\); hence for any interval map \(f\) there exists a number \(r_f, 0 \leq r_f < \frac{1}{2}\), such that \(I_f = [r_f, \frac{1}{2}]\).

A continuous self-mapping \(f\) of an interval is said to have a horseshoe if there exist subintervals \(I\) and \(J\) of the domain of \(f\) disjoint except perhaps a common endpoint, such that \(f(I) \cap f(J) \supset I \cup J\). By [8], if \(f\) has a horseshoe then \(I_f = [0, \frac{1}{2}]\).

A map \(f\) is piecewise-monotone if the domain of \(I\) can be partitioned into finitely many intervals (laps) on which \(f\) is strictly monotone. If the smallest number of such intervals is 2, then \(f\) is said to be unimodal; in this case \(f\) has one turning point. If the smallest number of such intervals is 3, then \(f\) is said to be bimodal; in this case \(f\) has two turning point. In this paper we always consider only piecewise-monotone maps that are increasing on their leftmost laps.

Every cycle \(P\) of a map \(f\) induces a cyclic permutation \(\Pi\) obtained by looking at how the map \(f\) acts on the points of \(P\) ordered from the left to the right. One can introduce a relation \(\sim\) on the family of all cycles such that for two cycles \(P\) and \(Q\), \(P \sim Q\) iff \(P\) and \(Q\) induce the same permutation; \(\sim\) is an equivalence relation whose equivalence classes are called patterns. If an interval map \(f\) has a cycle \(P\) from a pattern \(\pi\) associated with permutation \(\Pi\), say that \(P\) is a representative of \(\pi\) \((\Pi)\) (in \(f\)) and \(f\) exhibits \(\pi\) \((\Pi)\) (on \(P\)). A pattern \(\pi\) (a permutation \(\Pi\)) forces a pattern \(\theta\) (a permutation \(\Theta\)) if any continuous interval map \(f\) which exhibits \(\pi\) also exhibits \(\theta\). By [2], forcing is a partial ordering. In what follows we will interchangeably talk about patterns and permutations.

A useful algorithm allows one to describe all patterns forced by a pattern \(\pi\). Consider a cycle \(P\) of pattern \(\pi\), and denote the leftmost point of \(P\) by \(a\) and the rightmost point of \(P\) by \(b\). Every component of \([a, b] - P\) is called a \(P\)-basic interval. Extend this map from \(P\) to \([a, b]\) by defining it linearly on each \(P\)-basic interval and call the resultant map \(f_P\) the \(P\)-linear map. Then, the patterns of all cycles of \(f_P\) are exactly the patterns forced by the pattern of \(P\) (see [1] and [2]).
Over-rotation pairs and numbers for patterns are defined just like for cycles. Denote the over-rotation pair and over-rotation number of a pattern $\pi$ by $\text{orp}(\pi)$ and $\rho(\pi)$, resp. If $P$ is a cycle of the pattern $\pi$, call the over-rotation interval $I_\pi = [r_\pi, \frac{1}{2}]$ of the $P$-linear map $f_P$ the over-rotation interval of $\pi$.

**Definition 1.4.** A pattern $\pi$ is called over-twist if it does not force any other pattern of the same over-rotation number.

By Theorem 1.2, an over-twist pattern has a coprime over-rotation pair (i.e., an over-rotation pair $(p, q)$ where $p$ and $q$ are coprime); in particular, there is a unique over-twist pattern of over-rotation number $\frac{1}{2}$ associated with a unique cyclic permutation of period 2, so from now on we consider over-twists of over-rotation number distinct from $\frac{1}{2}$. By [8] and by properties of forcing relation, for any $\frac{p}{q} \in (r_f, \frac{1}{2})$, an over-twist pattern of over-rotation number $\frac{p}{q}$ is exhibited by a cycle of $f$; over-twists are patterns that are guaranteed to be exhibited by a map $f$ if the interior of $I_f$ contains the appropriate over-rotation number. This can be sharpened if $f$ is piecewise-monotone.

**Theorem 1.5** ([5], [6], [7]). If $f : [0, 1] \rightarrow [0, 1]$ is a piecewise monotone continuous map with over-rotation interval $[r_f, \frac{1}{2}]$, then for any $\frac{p}{q} \in [r_f, \frac{1}{2}]$, there exists a cycle $P$ which exhibits over-twist pattern of over-rotation number $\frac{p}{q}$.

The over-rotation interval of an over-twist pattern $\pi$ is $[\rho(\pi), \frac{1}{2}]$ so that $r_\pi = \rho(\pi)$ holds. In [3] the following version of the opposite statement is proven: a pattern $\pi$ with coprime over-rotation pair is over-twist if and only if $r_\pi = \rho(\pi)$.

In [9], it was proven that for a given rational number $\frac{p}{q}$ there exists a unique unimodal over-twist pattern $\gamma_{\frac{p}{q}}$ of over-rotation number $\frac{p}{q}$; moreover, the dynamics of $\gamma_{\frac{p}{q}}$ was described. It was also shown that the over-rotation interval is a monotone function of a map considered on a wide variety of one-parameter families of unimodal maps.

The goal of the present paper is to prove a similar result for the family of bimodal interval maps. To this end let us recall the following result from [11].

**Theorem 1.6** ([11]). To any bimodal map $f : [0, 1] \rightarrow [0, 1]$ there is associated a canonical truncation of bimodal horseshoe map $H_P : [0, 1] \rightarrow [0, 1]$ which has exactly the same kneading data. Furthermore, there exists a monotone but discontinuous correspondence $\theta : [0, 1] \rightarrow [0, 1]$ which semi-conjugates $f$ to $H_P$.

From Theorem 1.6 it follows that any bimodal interval map can be “modeled” by a truncation of a bimodal horseshoe map. So, we parameterize the family of all truncations of a bimodal horseshoe map and then show that the set of the parameters which correspond to a fixed over-rotation interval is a connected subset of the parameter space. For this we will use the explicit description of all bimodal over-twist patterns obtained in a recent paper [4]. Observe that our results do not
imply monotonicity of the over-rotation interval in any given family of bimodal maps increasing on the leftmost lap, nor do they imply its monotonicity in any natural space of smooth bimodal maps. This issue is non-trivial even in the unimodal case (we are not aware of the relevant results).

It is natural to find an efficient measure of dynamical complexity of maps which could be effectively computed. Hopefully, over-rotation numbers can serve this end. Not only can they be easily computed, our work provides a way to partition the parameter space of bimodal maps into areas associated with a fixed over-rotation interval which in turn encapsulates the limiting dynamical behavior of points. The generalization of the monotonicity of over-rotation intervals from unimodal to bimodal maps suggests that future generalizations to maps of higher modality are also possible.

We discuss our plans for this paper in detail below.

**Figure 1.** The graphs of the maps $H_2$ and $H_{\alpha,\beta}$ with the latter shown in bolder lines

A bimodal saw map $H_2 : [0, 1] \to [0, 1]$ has cycles of all bimodal patterns; it is defined by

$$H_2(x) = \begin{cases} 
3x & \text{if } 0 \leq x \leq \frac{1}{3} \\
2 - 3x & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\
3x - 2 & \text{if } \frac{2}{3} \leq x \leq 1
\end{cases}$$

For $\alpha, \beta \in [0, 1]$ with $\alpha \geq \beta$, we call the interval $\xi_{\alpha}^{\max} = \left[\frac{\alpha}{3}, \frac{2}{3} - \frac{\alpha}{3}\right]$ where $H_2(x) \geq \alpha$ the level $\alpha$ max flat spot and the interval $\xi_{\beta}^{\min} = \left[\frac{2}{3} - \frac{\beta}{3}, \frac{2}{3} + \frac{\beta}{3}\right]$ where $H_2(x) \leq \beta$ the level $\beta$ min flat spot. The map $H_{\alpha,\beta} : [0, 1] \to [0, 1]$ defined by:
\[ H_{\alpha,\beta}(x) = \begin{cases} 
H(x) & \text{if } x \notin \xi_{\alpha}^{\text{max}} \cup \xi_{\beta}^{\text{min}} \\
\alpha & \text{if } x \in \xi_{\alpha}^{\text{max}} \\
\beta & \text{if } x \in \xi_{\beta}^{\text{min}} 
\end{cases} \]

is called the \textit{bimodal truncation} of \( H \) with parameters \( \alpha \) and \( \beta \). Figure 1 shows the graphs of \( H_{2} \) and \( H_{\alpha,\beta} \).

Recall that by [10] any bimodal map can be modeled by some \( H_{\alpha,\beta} \). Moreover, there is an obvious hierarchy among truncations which is used in our paper. Namely, suppose that \( \alpha \leq \alpha' \) and \( \beta \geq \beta' \) and compare two truncations, \( H_{\alpha,\beta} \) and \( H_{\alpha',\beta'} \). Then any orbit of \( H_{\alpha,\beta} \) that does not enter the interior of its flat spots is also an orbit of \( H_{\alpha',\beta'} \). In other words, \textit{the dynamics of the map in the parameter space of truncations becomes richer as the first coordinate in the parameter space increases and the second coordinate decreases.} This relates to all orbits that do not enter the interior of the flat spots of \( H_{\alpha,\beta} \), but this does not complicate the situation since a lot of characteristics of the map do not depend on just one orbit.

Let \( \mathcal{H} = \{ H_{\alpha,\beta} : \alpha, \beta \in [0, 1] & \alpha \geq \beta \} \) be the family of all truncations of \( H_{2} \). Denote the \textit{over-rotation interval} of \( H_{\alpha,\beta} \) by \( I_{\alpha,\beta} \). It is easy to see that if \( \alpha = \beta \), \( \alpha \leq \frac{1}{2} \), or \( \beta \geq \frac{1}{2} \), then all periodic points of \( H_{\alpha,\beta} \) are fixed. Hence, the \( \omega \)-limit set \( \omega(x) \) is a fixed point for every \( x \), and we are not considering such maps. Excluding these parameters, we define the \textit{parameter space} as \( \mathcal{P} = \{ (\alpha, \beta) \in [0, 1] \times [0, 1] : \alpha \geq \frac{1}{2} \geq \beta \}; \) \( \mathcal{P} \) is a \textit{square} with vertices \((1, 0), (1, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \) and \((\frac{1}{2}, 0) \).

Let us use the following terminology to describe \( \mathcal{P} \) throughout the paper. The \textit{points} \((1, 0) \) and \((\frac{1}{2}, \frac{1}{2}) \) in the \textit{parameter plane} \( \mathcal{P} \) are called the \textit{focal point} \( \mathcal{F} \) and the \textit{vertex} \( \mathcal{V} \) respectively. Let us call the set \( \mathcal{B} = \{ (x, \frac{1}{2}) \mid \frac{1}{2} \leq x \leq 1 \} \cup \{ (\frac{1}{2}, y) \mid 0 \leq y \leq \frac{1}{2} \} \) the \textit{base set}. Similarly, call the set \( \mathcal{S} = \{ (x, 0) \mid \frac{1}{2} \leq x \leq 1 \} \cup \{ (1, y) \mid 0 \leq y \leq \frac{1}{2} \} \) the \textit{side arm}.

**Figure 2.** The Parameter space \( \mathcal{P} \)
Define a map $\psi : \mathcal{P} \to [0, \frac{1}{2}]$ with $\psi(\alpha, \beta) = \rho(\alpha, \beta)$ where $I_{\alpha, \beta} = [\rho(\alpha, \beta), \frac{1}{2}]$. Observe that at the focal point the value of $\psi$ is the least (namely, 0), while at any point in the base $\mathcal{B}$, the value of $\psi$ is the greatest (namely, $\frac{1}{2}$). We will call the set $\psi^{-1}(\nu) = \psi_{\nu} = \{(\alpha, \beta) \in \mathcal{P} : \psi((\alpha, \beta) = \nu)\}$ the Bimodal Iso-over-rotation-tract corresponding to $\nu$ or the $\nu$-Bimodal Iso-over-rotation-tract. The main objective of our paper is to show that for any $\nu \in [0, \frac{1}{2}]$ the $\nu$-Bimodal Iso-over-rotation-tract is a connected set. In other words the map $\psi : \mathcal{P} \to [0, \frac{1}{2}]$ is a monotone map.

We divide our paper into three sections. Section 1 is our Introduction. Section 2 contains some preliminary ideas. Section 3 is the main section of the paper where we prove that Bimodal Iso-over-rotation-tracts are connected.

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2. PRELIMINARIES

2.1. Kneading sequences. Let $f : [0, 1] \to [0, 1]$ be a bimodal map with turning points $0 < c_1 < c_2 < 1$. Call $\mathcal{S} = \{I_0, C_1, I_1, C_2, I_2\}$ where $I_0 = [0, c_1)$, $C_1 = \{c_1\}$, $I_1 = (c_1, c_2)$, $C_2 = \{c_2\}$ and $I_2 = (c_2, 1]$ the symbolic set of $f$. Order its elements as $I_0 < C_1 < I_1 < C_2 < I_2$; we will write $\mathcal{J}' \leq \mathcal{J}''$ if $\mathcal{J}' = \mathcal{J}''$ or $\mathcal{J}' < \mathcal{J}''$. Define a location function $\mathcal{J} : [0, 1] \to \mathcal{S}$ which assigns to each point $x \in [0, 1]$, its unique location $\mathcal{J}(x) \in \mathcal{S}$ defined by $x \in \mathcal{J}(x)$. Observe that if $x < y$, then $\mathcal{J}(x) \leq \mathcal{J}(y)$. Let $\mathcal{S}^\mathbb{N}$ be the set of all infinite sequences of symbols $(\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \ldots)$ where $\mathcal{J}_i \in \mathcal{S}$ for every $i$.

Define a function $\mathcal{I} : [0, 1] \to \mathcal{S}^\mathbb{N}$ which associates with each point $x \in [0, 1]$, the unique sequence $\mathcal{I}(x) = (\mathcal{J}(x), \mathcal{J}(f(x)), \mathcal{J}(f^2(x)), \ldots) \in \mathcal{S}^\mathbb{N}$. We call $\mathcal{I}(x)$, the itinerary of the point $x$. Let $\sigma : \mathcal{S}^\mathbb{N} \to \mathcal{S}^\mathbb{N}$ be the shift map defined by $\sigma(\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \ldots) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4, \ldots)$. It follows that $\mathcal{I} : [0, 1] \to \mathcal{S}^\mathbb{N}$ conjugates the map $f : [0, 1] \to [0, 1]$ to the shift map $\sigma : \mathcal{S}^\mathbb{N} \to \mathcal{S}^\mathbb{N}$. This allows one to model the dynamics of the map $f$ using the properties of the itineraries.

Define a sign function $\Theta : \mathcal{S} \to \{-1, 0, 1\}$ as follows: for any $j$, $\Theta(I_j) = +1$ if $f$ is increasing on $I_j$, $\Theta(I_j) = -1$ if $f$ is decreasing on $I_j$, and $\Theta(C_j) = 0$ for any $j$.

We define a partial order $\succ$ on all itineraries as follows. Let $\mathcal{J} = (\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \ldots)$ and $\mathcal{K} = (\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \ldots)$ be two itineraries such that $\mathcal{J} \neq \mathcal{K}$. Let $k$ be the first $i \in \mathbb{N}$ such that $\mathcal{J}_i \neq \mathcal{K}_i$ and assume without loss of generality that $\mathcal{J}_k > \mathcal{K}_k$. We set $\Lambda_k = 1^k = \Pi_{i=0}^k \Theta(\mathcal{J}_i) = \Pi_{i=0}^k \Theta(\mathcal{K}_i)$. Clearly $\Lambda_k \in \{0, 1, -1\}$. It is easy to see that if the itineraries are associated to points under the action of the same map, say, $f$, then $\Lambda_k \neq 0$. Indeed, if $\Lambda_k = 0$, then $\mathcal{J}_i = C_j$ for some $i \in \{0, 1, \ldots k - 1\}$ and $j \in \{1, 2\}$. Since we are considering itineraries
of points under the same map $f$, then $\mathcal{J} = \mathcal{K}$, a contradiction with the assumption that $\mathcal{J} \neq \mathcal{K}$. Hence there are two cases (in general if $\Lambda_k = 0$, then the ordering is not defined).

(1) If $\Lambda_k = 1$, then we say that $\mathcal{J}$ is stronger than $\mathcal{K}$ and write $\mathcal{J} \succ \mathcal{K}$.

(2) If $\Lambda_k = -1$, then we say that $\mathcal{K}$ is stronger than $\mathcal{J}$ and write $\mathcal{K} \succ \mathcal{J}$.

Also, we will write $\mathcal{J} \succeq \mathcal{K}$ if $\mathcal{J} \succ \mathcal{K}$ or $\mathcal{J} = \mathcal{K}$.

**Theorem 2.1** ([10]). Let $x, y \in [0, 1]$ with $x > y$. Then $\mathcal{I}(x) \succeq \mathcal{I}(y)$.

Conversely, if for $x, y \in [0, 1]$, we have $\mathcal{I}(x) \succ \mathcal{I}(y)$, then $x > y$.

The itineraries $\mathcal{K}_j = \mathcal{I}(f(c_j)) \in \mathcal{S}^N$ of the critical points $c_j$ for $j = 1, 2$ are called the kneading sequences of the map $f$. The vector $\mathcal{K}(f) = (\mathcal{K}_1, \mathcal{K}_2)$ is called the kneading vector of the map $f$.

In this paper we will use Theorem 2.1 to pinpoint the distribution of the cycles corresponding to different bimodal over-twist patterns under the action of a bimodal horseshoe map $H_2$ by comparing the itineraries of the points of absolute maxima and minima of these orbits.

### 2.2. Some results on degree one circle maps

Recall some results on circle maps that we need (see [1] for detail). Consider the unit circle $\mathbb{S}$ normalized so that the circumference equals 1 and identify $[0, 1)$ with it. Define the natural projection $\pi : \mathbb{R} \to \mathbb{S}$ associating with $t \in \mathbb{R}$ its fractional part $\pi(t)$ considered as a point of $\mathbb{S}$. If $f : \mathbb{S} \to \mathbb{S}$ is continuous, then there is a continuous map $F : \mathbb{R} \to \mathbb{R}$ such that $F \circ \pi = \pi \circ f$. Such a map $F$ is called a lifting of $f$. It is unique up to translation by an integer. An integer $d$ with $F(x+1) = F(x) + d$ for all $x \in \mathbb{R}$ is called the degree of the map $f$ and is independent of the choice of $F$. Conversely, maps $G : \mathbb{R} \to \mathbb{R}$ such that $G(x+1) = G(x) + d$ for every real $x$ are said to be maps of the real line of degree $d$, can be defined independently, and are semiconjugate by the same map $\pi$ to the circle maps of degree $d$. In this paper we consider both maps of the circle and maps of the real line of degree one.

Denote by $\mathcal{L}_1$ the set of all liftings of continuous degree one self-mappings of $\mathbb{S}$ endowed with the sup norm. Let $F \in \mathcal{L}_1$. Define upper and lower rotation numbers of $x \in \mathbb{R}$ for $F \in \mathcal{L}_1$ as $\overline{\rho_F}(x) = \limsup_{n \to \infty} \frac{F^n(x) - x}{n}$ and $\rho_F(x) = \liminf_{n \to \infty} \frac{F^n(x) - x}{n}$ respectively. If $\overline{\rho_F}(x) = \rho_F(x)$, this number is called the rotation number of $x$ for $F$ and is denoted by $\rho_F(x)$. Let $\mathcal{L}'_1$ be the space of all non-decreasing elements of $\mathcal{L}_1$.

**Theorem 2.2** ([1]). If $F \in \mathcal{L}'_1$ is a lifting of a circle map $f$ then $\rho_F(x)$ exists for all $x \in \mathbb{R}$ and is independent of $x$. Moreover, it is rational if and only if $f$ has a periodic point.

In the situation of Theorem 2.2, set $\rho(F) = \rho_F(x)$ and call it the rotation number of $F$. The following lemma is left to the reader.

**Lemma 2.3.** The function $\rho : \mathcal{L}'_1 \to \mathbb{R}$ is continuous.
For $F \in \mathcal{L}_1$ we define maps $F_l, F_u \in \mathcal{L}'_1$ as follows: $F_l(x) = \inf \{ F(y) : y \geq x \}$ and $F_u(x) = \sup \{ F(y) : y \leq x \}$.

**Theorem 2.4 ([1]).** The following statements are true.

1. Let $F \in \mathcal{L}_1$. Then, $F_l(x) \leq F(x) \leq F_u(x)$ for every $x \in \mathbb{R}$.
2. If $F, G \in \mathcal{L}_1$ with $F \leq G$, then $F_l \leq G_l$ and $F_u \leq G_u$.
3. Let $F \in \mathcal{L}'_1$. Then, $F_l = F_u = F$.
4. The maps $F \mapsto F_l$ and $F \mapsto F_u$ are Lipschitz continuous with constant $1$ in the sup norm.
5. The maps $F \mapsto \rho(F_l)$ and $F \mapsto \rho(F_u)$ are continuous.
6. If $F_l(x) \neq F(x)$, then $x \in \text{Const}(F_l)$ where $\text{Const}(F_l)$ denotes the union of all open intervals on which $F_l$ is constant.
7. Let $F \in \mathcal{L}_1$ be a lifting of a circle map $f$. Then the set of all rotation numbers of points is equal to the interval $[\rho(F_l), \rho(F_u)]$.
   Moreover, for each rational $a$ from this interval there is a point $x$ such that $\pi(x)$ is periodic for $f$ and $\rho_F(x) = a$.
8. The points $\rho(F_l), \rho(F_u)$ depend continuously on $F \in \mathcal{L}_1$.

The interval $[\rho(F_l), \rho(F_u)]$ from Theorem 2.4(7) is called the rotation interval of $F$ and is denoted by $\text{Rot}(F)$.

2.3. Results on over-rotation numbers of bimodal maps. In what follows when talking about (over-)rotation numbers $\frac{p}{q}$ and the like we assume (unless specified otherwise) that the numerator and the denominator are coprime. The article [4] relates the over-rotation numbers of bimodal interval maps and the classical rotation numbers of degree one maps of the circle and of the real line. This yields a description of bimodal over-twist patterns. Namely, given an over-rotation number $\frac{p}{q}$ there are $q - 2p + 1$ bimodal over-twist patterns $\Gamma_{r, \frac{p}{q}}$ with $r \in \{0, 1, 2, \ldots, q - 2p\}$ of over-rotation number $\frac{p}{q}$. The patterns with $r = 0$ and $r = q - 2p$ are unimodal while the remaining $q - 2p - 1$ patterns are strictly bimodal. Denote the permutation associated to the over-twist pattern $\Gamma_{r, \frac{p}{q}}$ by $\Pi_{r, \frac{p}{q}}$. By [4] $\Pi_{r, \frac{p}{q}}$ is as follows.

$$\Pi_{r, \frac{p}{q}}(j) = \begin{cases} j + p & \text{if } 1 \leq j \leq r \\ q - j + r + 1 & \text{if } r + 1 \leq j \leq r + p \\ 2p - j + r + 1 & \text{if } r + p + 1 \leq j \leq r + 2p \\ j - p & \text{if } r + 2p + 1 \leq j \leq q \end{cases}$$  \quad (2.1)
Let $P_{r,s}^q$ be a cycle of a map $f$ which exhibits the pattern $\Gamma_{r,s}^q$. Number its $q$ points using spatial labelling as $x_1 < x_2 < \cdots < x_q$ and partition them into 4 disjoint parts. Strictly speaking, in the notation that we introduce now, the dependence upon the cycle at hand should be reflected. However we omit it to make notation lighter.

The first $r$ points $x_1, x_2, \ldots, x_r$ are called red points and are denoted by $\mathcal{R}_1^r, \mathcal{R}_2^r, \mathcal{R}_3^r, \ldots, \mathcal{R}_r^r$. Under the action of $f$ the red points are shifted to the right by $p$ points. The next $p$ points $x_{r+1}, x_{r+2}, \ldots, x_{r+p}$ are called green points and are denoted by $\mathcal{G}_1^p, \mathcal{G}_2^p, \mathcal{G}_3^p, \ldots, \mathcal{G}_p^p$. The green points map onto the last $p$ points $x_{q-p+1}, x_{q-p+2}, \ldots, x_{q-1}, x_q$ of the orbit with a flip, that is orientation is reversed but without any expansion.

The next $p$ points $x_{r+p+1}, x_{r+p+2}, \ldots, x_{r+2p}$ are called pink points and are denoted by $\mathcal{P}_1^p, \mathcal{P}_2^p, \mathcal{P}_3^p, \ldots, \mathcal{P}_p^p$. The pink points map onto the first $p$ points $x_1, x_2, \ldots, x_p$ of the orbit with a flip but with no expansion. The last $q-2p-r$ points are called blue points and are denoted by $\mathcal{B}_1^r, \mathcal{B}_2^r, \mathcal{B}_3^r, \ldots, \mathcal{B}_{q-2p-r}^r$. They are shifted to the left by $p$ points.

If $r = 0$, we have no red points; if $r = q-2p$, we have no blue points. For all other values of $r$, we have points of all colors.

The dynamics of $P_{3, \frac{3}{11}}^q$ is depicted in the Figure 3.

Remark 2.5. We will need an interpretation of the dynamics of $P_{r,s}^q$ (or $\Gamma_{r,s}^q$, or $\Pi_{r,s}^q$) that relates this dynamics (or combinatorics) and that of the rotation of the circle by the rational angle $\frac{p}{q}$. This relation is based upon a special discontinuous lifting of a bimodal interval map to a discontinuous degree one map of the real line which yields a relation between $P_{r,s}^q$ and the rotation of the circle by the rational angle $\frac{p}{q}$. In fact, we will consider the combinatorial rotation by $\frac{p}{q}$ understood as a rotation with the combinatorial rotation number $\frac{p}{q}$ of $q$ circularly ordered points. One can visualize points $y_0 < y_1 < \cdots < y_{q-1} < y_0$ sitting on the unit circle with the order understood in terms of counterclockwise order; then the combinatorial rotation in question is defined as $\varphi_{\frac{p}{q}}(y_i) = y_{i+p} \mod q$. 

Figure 3. The Bimodal over-twist pattern $\Gamma_{3, \frac{3}{11}}$
To relate $P_{r, \frac{p}{q}}$ and $\varphi_{\frac{p}{q}}$, do the following. Place all $r$ red points on the circle in their normal order. Then place all $p$ green points on the circle right after the red points in the normal order. Then put the blue points on the circle in the reversed order. Finally, put all the pink points on the circle in the reversed order. The above given description of $P_{r, \frac{p}{q}}$ (i.e., the main result of [4]) simply means that the just described map of $P_{r, \frac{p}{q}}$ to $q$ circularly ordered points conjugates $f|_{P_{r, \frac{p}{q}}}$ and the combinatorial rotation of these $q$ circularly ordered points by $\frac{p}{q}$; we leave an easy verification of this fact to the reader.

3. CONNECTEDNESS OF BIMODAL ISO-OVER-ROTATION-TRACTS

Recall that $H = \{ H_{\alpha, \beta} : \alpha, \beta \in [0, 1] \& \alpha \geq \beta \}$ is the family of all truncations of $H_2$, $I_{\alpha, \beta}$ is the over-rotation interval of $H_{\alpha, \beta}$ and $\psi : \mathcal{P} \to [0, \frac{1}{2}]$ is defined by $\psi(\alpha, \beta) = \rho(\alpha, \beta)$ where $\rho(\alpha, \beta)$ is the left end point of $I_{\alpha, \beta}$. For $\alpha, \beta \in [0, 1]$ with $\alpha \geq \beta$, we call the interval $\xi_{\alpha}^{max} = \left[\frac{q}{3}, \frac{2}{3} - \frac{\alpha}{3}\right]$ where $H_2(x) \geq \alpha$ the level $\alpha$ max flat spot and the interval $\xi_{\beta}^{min} = \left[\frac{2}{3} - \frac{\beta}{3}, \frac{2}{3} + \frac{\beta}{3}\right]$ where $H_2(x) \leq \beta$ the level $\beta$ min flat spot. For two sets $U, V \subseteq \mathbb{R}^2$, $d(U, V) = \inf \{ d(u, v) : u \in U \& v \in V \}$.

Also, we set $I_0 = [0, \frac{1}{3}], I_1 = \left[\frac{1}{3}, \frac{2}{3}\right], I_2 = \left[\frac{2}{3}, 1\right]$.

For any rational $\frac{p}{q}$ with coprime $p$ and $q$, there are $q - 2p + 1$ bimodal over-twist patterns $\Gamma_{r, \frac{p}{q}}$ for $r \in \{0, 1, 2, \ldots, q - 2p\}$. Out of these the patterns $\Gamma_{0, \frac{p}{q}}$ and $\Gamma_{q - 2p, \frac{p}{q}}$ are unimodal and the remaining $q - 2p - 1$ patterns are strictly bimodal. Suppose that a cycle $P$ represents a pattern that forces a unique fixed point. Then we denote this fixed point $a$ and use the following colors introduced in Section 2.3: red points are to the left of $a$ and stay to the left of $a$ under the map, green points are to the left of $a$ but map to the right of $a$, pink points are to the right of $a$ but map to the left of $a$ and blue points are to the right of $a$ and stay to the right of $a$ under the action of the map. When talking about the corresponding pattern itself, we use the same colors. This coloring scheme applies to cycles exhibiting patterns $\Gamma_{r, \frac{p}{q}}$.

For a given cycle $P$ let us denote its left endpoint by $le(P)$ and right endpoint by $ri(P)$. Let $P$ be a cycle exhibiting one of the patterns $\Gamma_{r, \frac{p}{q}}$. We will need the next lemma describing the monotonicity of the $P$-linear map $f : [le(P), ri(P)] \to [le(P), ri(P)]$ on certain $P$-basic intervals.

**Lemma 3.1.** Let $P$ be a cycle exhibiting one of the patterns $\Gamma_{r, \frac{p}{q}}$. Consider the $P$-linear map $f_P = f$ with a unique fixed point $a$. Let $u \in P$ be such that $f(u) = ri(P)$; let $v \in P$ be such that $f(v) = le(P)$. Choose a small interval $I = (u, a + \varepsilon)$ and consider $f^j|_I$ for some $j$. Then $f^j|_I$ is decreasing if $f^j(u) > a$ and increasing if $f^j(u) < a$.

**Proof.** The map $f$ is decreasing on an iterated image $K$ of $I$ if and only if $K \subset (u, v)$. The structure of the pattern of $P$ implies that the orbit of $I$ can be viewed as a concatenation of a few segments each of which looks as either (1) an iterated image of $I$ contained in $(u, a)$
and then several (maybe none) images of $I$ contained in $(v, ri(P))$, or 
(2) an iterated image of $I$ contained in $(a, v)$ and then several (maybe 
none) images of $I$ contained in $(le(P), u)$. This immediately implies 
the claim of the lemma. 

3.1. Weakest Cycles. Clearly, for each $r \in \{0, 1, 2, \ldots, q - 2p\}$ there 
exist cycles of $H_2$ that exhibit $\Gamma_{r, \frac{q}{r}}$. Each such cycle $P$ gives rise to 
the corresponding truncation $T_P$ defined as follows: choose the point 
$u_P = u \in P$ such that $H_2(u)$ is the rightmost point of $P$. Clearly 
$u \in (0, \frac{1}{2})$; draw a horizontal line at the point $(u, H_2(u))$ on the graph 
of $H_2$ creating the max flat spot of $T_P$. Likewise, take $v_P = v \in P$ 
such that $H_2(v)$ is the left most point of $P$, and use it to create the 
min flat spot of $T_P$. Evidently, $P$ remains a cycle of $T_P$. We will use 
the notation $u_P$ and $v_P$ in what follows.

Definition 3.2. We say that a cycle $P$ of $H_2$ is a weakest cycle ex-
hibiting $\Gamma_{r, \frac{q}{r}}$ if $P$ is the unique cycle of $T_P$ exhibiting $\Gamma_{r, \frac{q}{r}}$.

Lemma 3.3. Let $P$ be a weakest cycle of $H_2$ which exhibits $\Gamma_{r, \frac{q}{r}}$ for 
some $r$. Then all pink and green points of $P$ lie in the decreasing lap 
$I_1$ and $P$ is unique.

Proof. We first prove that all green and pink points of $P$ lie in the 
decreasing lap $I_1$. By way of contradiction suppose that $x = u_P \notin I_1$. 
Set $f = T_P$. Consider the maximal interval $T_1 = [x, z], z \in P$, on which 
$f$ is non-strictly decreasing. Let $T_0 = [0, x]$ and $T_2 = [z, 1]$, and call 
$T_0, T_1$ and $T_2$ $T$-intervals. Let $X$ be the set of all points $s \in [0, 1]$ 
such that for any $i$ the points $f^i(s)$ and $f^i(x)$ belong to the same $T$-
interval, additionally requiring that if $f^i(x) \in T_1$ then $f^i(s) \in T_1$, too 
(the additional requirement resolves the ambiguous situation one faces 
if $f^i(x)$ is the common endpoint of $T_1$ and $T_0$, or a common endpoint 
of $T_1$ and $T_2$). It is well-known (and not hard to observe) that $X$ is a 
closed interval of the form $[x, y]$. Consider the point $y$.

The definition implies that $f^{q}(X) \subset X$ (because $f^{q}(x) = x$) and that 
$f^{q}|_X$ is (non-strictly) monotone. If $y_0 \in I_1$ is such that $f(x) = f(y_0)$, 
then $y_0 \leq y$. Moreover, by Lemma 3.1 a small interval $[y_0, y_0 + \varepsilon]$ has 
the $f^q$-image contained in $[x, y_0]$; thus, $[y_0, y_0 + \varepsilon] \subset X$ and $y_0 < y$. We 
claim that $f^{q}(y) = y$. Indeed, the fact that $y_0 < y$ and the definitions 
imply that $x < f^{q}(y)$. If now $f^{q}(y) < y$ then by continuity a small interval $[y, y + \delta]$ is contained in $X$, a contradiction with the fact that 
$X = [x, y]$.

The structure of the pattern $\Gamma_{r, \frac{q}{r}}$ implies that $X$ cannot contain 
points of $P$ other than $x$ itself (in particular, $y$ cannot be a point of 
P). Thus, all points of $X$ at every moment of time are mapped into 
the same $P$-basic interval as $x$. Hence the orbit of $y$ exhibits the same 
pattern as that of $x$, i.e. the orbit of $y$ exhibits $\Gamma_{r, \frac{q}{r}}$, a contradiction 
with the assumption that $P$ is weakest. Thus, $u_P \in I_1$; similarly one 
shows that $v_P \in I_1$.

Now, we show that $P$ is unique. Indeed otherwise there exists an-
other weakest cycle $Q$ which exhibits the pattern $\Gamma_{r, \frac{q}{r}}$. Then $u_P \neq u_Q$. 
On the other hand, by the previous paragraph and because the orbits of \(u_P\) and \(u_Q\) exhibit the same pattern it follows that for each \(k\) the points \(H_2^k(u_P)\) and \(H_2^k(u_Q)\) belong to the same lap of \(H_2\), be it \(I_0, I_1\) or \(I_2\). In other words, \(u_P\) and \(u_Q\) have the same itinerary under \(H_2\) which implies that \(u_P = u_Q\), a contradiction. \(\square\)

Observe that an alternative proof of Lemma 3.3 is based upon the techniques of so-called admissible intervals introduced in \([8]\); we choose our arguments to make the paper shorter and more self-contained.

From now on \(P_{r,q}^2\) shall denote the weakest cycle of \(H_2\) which exhibits \(\Gamma_{r,q}\). Let \(S_{r,q}^2\) be the collection of these \(q - 2p + 1\) cycles \(P_{r,q}^2\) for \(r \in \{0,1,2,\ldots,q - 2p\}\).

3.2. Order among weakest cycles. Interestingly, there is an order among these \(q - 2p + 1\) orbits on \(H_2\) which we study now. For \(P_{r,q}^2 \in S_{r,q}^2\), denote its left endpoint, right endpoint, point of absolute maxima and point of absolute minima by \(l_{e,q}^r, r_{i,q}^r, M_{r,q}^r\) and \(m_{r,q}^r\) respectively. Clearly, \(H_2(m_{r,q}^r) = l_{e,q}^r\) and \(H_2(M_{r,q}^r) = r_{i,q}^r\).

**Lemma 3.4.** For any rational \(\frac{p}{q}\) where \(p\) and \(q\) are coprime we have

\[
le_{q-2p} < le_{q-2p-1} < le_{q-2p-2} < le_{q-2p-3} < \cdots < le_0
\]

**Proof.** Our proof is based upon the interpretation of the results of \([4]\) given in Remark 2.5. According to it, let us place \(q\) points \(Y = \{y_0 < \cdots < y_{q-1} < y_0\}\) on the unit circle and consider the map \(\varphi_{r,q} : Y \to Y\). Moreover, let us associate points of \(P_{r,q}^2\) and points of \(Y\) as explained in Remark 2.5. Then in the case of \(P_{r,q}^2\) we have the partition, say, \(L\) of \(Y\) into sets \(R', G', P'\) and \(B'\) of \(r, p, p, q - 2p - r\) points respectively of colors: red, green, pink and blue. In the case of \(P_{r+1,q}^2\) we have the partition \(L''\) of \(Y\) into sets \(R'', G'', P''\) and \(B''\) of \(r + 1, p, p, q - 2p - r - 1\) points respectively of the same colors. We can view the transformation from \(L'\) to \(L''\) as follows: (i) the set \(G'\) of \(p\) green points in \(L'\) moves one click away from \(y_0\) (in other words, \(G'\) occupies points from \(y_r\) through \(y_{r+p-1}\) while \(G''\) occupies points from \(y_{r+1}\) through \(y_{r+p}\)) (ii) the set \(R'\) adds the point: \(y_r\) while the set \(B'\) loses the point: \(y_{r+p}\).

Let us iterate \(\varphi_{r,q}\) applying it to \(y_0\). Then for some time the foreword iterates of the point \(y_0\) will be located in the equally named sets of the partitions (\(R'\) and \(R''\), \(G'\) and \(G''\), etc). However, at some moment \(y_0\) is mapped back to itself, for the first time the sets of the two partitions containing, say, \(y^k\) will be named differently. A priori, this can happen in one of the following two ways:

(1) \(y^k = y_r\) belongs to \(G'\) and to \(R''\);
(2) \(y^k = y_{r+p}\) belongs to \(B'\) and to \(G''\).

However it immediately follows that if (2) takes places then already on the previous step the sets in the two partitions that contains the point will have different names, a contradiction. Thus, the first time the point is in two sets of partitions \(L', L''\) that have different names is when \(y^k = y_r\) belongs to \(G'\) and to \(R''\). It follows that the first
difference between itineraries of \( le_r \) and \( le_{r+1} \) is such that the entry in the itinerary of \( le_r \) is \( I_1 \) (because all green points belong to \( I_1 \)) while the simultaneous entry in the itinerary of \( le_{r+1} \) is \( I_0 \) (because all red points belong to \( I_0 \)). Let us now figure out the sign associated with this according to the definition of the order among itineraries. To do so notice that on its way to \( y_r \) the point \( y_0 \) enters green and pink points in pairs because the length of the segment with green points and the segment with pink points is the same and equals the step of the map, i.e. \( p \). Hence at the moment when we are at \( y_r \) the product of signs associated with laps of \( H_2 \) and defining the order among itineraries is positive implying that \( le_{r+1} < le_r \) as desired. \( \square \)

We can interpret the result of Lemma 3.4 as follows: the more red points our weakest cycle has, the more to the left its left endpoint is. Because of the symmetry of the map \( H_2 \) (formally this is based upon the fact that \( H_2 \) is conjugate to itself by a map that symmetrically flips \([0, 1]\)) we can state a similar claim for the rightmost points of weakest cycles: the more blue points our weakest cycle has, the more to the right its right endpoint is. This can be summarized in the next theorem stated with proof.

**Theorem 3.5.** For any rational number \( \frac{p}{q} \) where \( p \) and \( q \) are coprime we have

\[
le_{q-2p} < le_{q-2p-1} < \cdots < le_0 < ri_{q-2p} < ri_{q-2p-1} < \cdots < ri_0.
\]

**Example 3.6.** Let us illustrate Theorem 3.4 using the particular case: \( \frac{5}{3} = \frac{3}{11} \). In this case \( S_{11} \) has \( q - 2p + 1 = 6 \) elements: \( P_{r,11} \), \( r = 0, 1, 2, \ldots 5 \). The itineraries of \( H_2(m_r) \) are:

1. \( \mathcal{I}(H_2(m_0)) = \{ I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, \ldots \} \) (Fig 4).
2. \( \mathcal{I}(H_2(m_1)) = \{ I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, \ldots \} \) (Fig 5).
3. \( \mathcal{I}(H_2(m_2)) = \{ I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, \ldots \} \) (Fig 6).
4. \( \mathcal{I}(H_2(m_3)) = \{ I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, \ldots \} \) (Fig 7).
5. \( \mathcal{I}(H_2(m_4)) = \{ I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, \ldots \} \) (Fig 8).
6. \( \mathcal{I}(H_2(m_5)) = \{ I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, \ldots \} \) (Fig 9).

Comparing the itineraries it is easy to see that

\[
\mathcal{I}(H_2(m_0)) > \mathcal{I}(H_2(m_1)) > \cdots > \mathcal{I}(H_2(m_4)) > \mathcal{I}(H_2(m_5));
\]

thus, by Theorem 2.1, we have

\[
H_2(m_0) > H_2(m_1) > H_2(m_2) > H_2(m_3) > H_2(m_4) > H_2(m_5)
\]

and hence

\[
le_0^\frac{3}{11} < le_1^\frac{3}{11} < le_2^\frac{3}{11} < le_3^\frac{3}{11} < le_4^\frac{3}{11} < le_0^\frac{3}{11}.
\]
Figure 4. $P_{0,\frac{3}{11}}$

Figure 5. $P_{1,\frac{3}{11}}$

Figure 6. $P_{2,\frac{3}{11}}$

Figure 7. $P_{3,\frac{3}{11}}$
3.3. Leading Sets. Let us fix an over-rotation number $\frac{p}{q}$. Set $le_i = \beta_i$, $ri_i = \alpha_i$, $0 \leq i \leq q - 2p$. By Theorem 3.5 $\beta_{i+1} < \beta_i$ and $\alpha_{i+1} < \alpha_i$ for $i = 0, 1, \ldots, q - 2p - 1$. Plot points $C_i = (\alpha_i, \beta_i)$ that characterize the weakest cycles of over-rotation number $\frac{p}{q}$ in the parameter square.

It follows that as $r$ increases from 0 to $(q - 2p)$, the points $C_i$ line up in the “south-west” direction, i.e. in the direction in which $\alpha$ and $\beta$ coordinates decrease. The leading set of over-rotation number $\frac{p}{q}$ is a staircase that connects $C_i$’s as follows. Draw a vertical segment down from $C_i$ until we reach the level $\beta_{i+1}$; then draw a horizontal segment until we reach $C_{i+1}$. Also, at $C_0$ draw a horizontal segment until we meet the right (the vertical) side arm, and at $C_{q-2p}$ draw a vertical segment until we meet the bottom (the horizontal) side arm. Acting in this fashion, we will connect two side arms of the parameter square.

Lemma 3.7. For any $(\alpha, \beta) \in Z_{\frac{p}{q}}$ we have $\psi((\alpha, \beta)) = \frac{p}{q}$.

Proof. The claim is immediate for points $C_i = (\alpha_i, \beta_i)$. Indeed, if $\psi((\alpha_i, \beta_i)) < \frac{p}{q}$ then the map $H_{\alpha_i, \beta_i}$ has a cycle of rational over-rotation number $\rho < \frac{p}{q}$ that in turn forces a pattern of over-rotation number $\frac{p}{q}$. In other words, $H_{\alpha_i, \beta_i}$ has a cycle of over-rotation number $\frac{p}{q}$ which is distinct from the cycle $P_i$ containing $\alpha_i$ and $\beta_i$, a contradiction with the fact that the cycle $P_i$ is a weakest cycle.
Let us now consider the horizontal segment in $Z_{\gamma}$ with the left endpoint $C_i$. Then its horizontal coordinate varies from $\alpha_i$ to $\alpha_{i-1}$ while its vertical coordinate equals $\beta_i$. Suppose that the map $H_{\alpha,\beta_i}$ with $\alpha_i \leq \alpha \leq \alpha_{i-1}$ is such that $\psi((\alpha, \beta_i)) < \frac{p}{q}$. Then it must have a cycle $Q$ of over-rotation number $\rho < \frac{p}{q}$. The cycle $Q$ has its left endpoint $le(Q)$ and the point $m(Q) \in Q$ such that $H_{\alpha,\beta_i}(m(Q)) = le(Q)$. Evidently, $m(Q)$ does not belong to the level $\beta_i$ min flat spot as its orbit does not coincide with $P_i$. Hence the truncation of $H_2$ defined by $Q$ does not have $P_i$ as its cycle. On the other hand, the max value of $Q$ is less than $\alpha_{i-1}$ by construction. Hence the next (in the “north-east” direction) weakest cycle of over-rotation number $\frac{p}{q}$ (i.e., the cycle that passes through points $\alpha_{i-1}$ and $\beta_{i-1}$) is not forced by $Q$ either. It follows that no weakest cycle of over-rotation number $\frac{p}{q}$ is forced by $Q$, a contradiction with properties of over-rotation numbers (see Theorem 1.2). The arguments dealing with vertical segments in $Z_{\gamma}$ are completely analogous. This implies the claim of the lemma. \hfill $\Box$

We see that there entire parameter square is cut by countably many pairwise disjoint leading sets $Z_{\gamma}$ associated to the rational numbers $\nu$ from $(0, \frac{1}{2})$. To describe their mutual location we need the next lemma.

**Lemma 3.8.** If $(\alpha, \beta)$ and $(\gamma, \delta)$ are points in $P$ co-linear with the focal point $F$ such that $d(F, (\alpha, \beta)) \geq d(F, (\gamma, \delta))$ then $\psi((\alpha, \beta)) \geq \psi((\gamma, \delta))$.

**Proof.** Since, $(\alpha, \beta)$ and $(\gamma, \delta)$ are points in $P$ co-linear with the focal point $F$ with $d(F, (\alpha, \beta)) \geq d(F, (\gamma, \delta))$, it follows that $\gamma \geq \alpha$ and $\delta \leq \beta$. Thus, as we move from $(\alpha, \beta)$ to $(\gamma, \delta)$, the first coordinate increases and the second coordinate decreases which implies that $I_{(\alpha,\beta)} \subseteq I_{(\gamma,\delta)}$ and $\psi((\alpha, \beta)) \geq \psi((\gamma, \delta))$ as desired. \hfill $\Box$

This lemma immediately implies the next corollary.

**Corollary 3.9.** Consider rational numbers $0 < \nu < \mu \leq \frac{1}{2}$. Then $Z_{\nu}$ separates $Z_{\mu}$ from the focal point inside the parameter square $P$.

Corollary 3.9 allows us to define the notion of a leading strip $S(\mu, \nu)$, $0 < \nu < \mu \leq \frac{1}{2}$. Namely the leading strip $S(\mu, \nu)$, $0 < \nu < \mu \leq \frac{1}{2}$ is the closed set squeezed in the parameter square between $Z_{\mu}$ and $Z_{\nu}$.

**Theorem 3.10.** For any number $\gamma, 0 < \gamma \leq \frac{1}{2}$ the $\gamma$-bimodal iso-over-rotation-tract $T_{\gamma}$ is the intersection of all leading strips $S(\mu, \nu)$ with $0 < \nu < \gamma < \mu \leq \frac{1}{2}$. The set $T_{\gamma}$ is a simply connected continuum. The map $\psi : P \to [0, \frac{1}{2}]$ is continuous and monotone.

**Proof.** By Corollary 3.9, the containment $[\mu, \nu] \subset [\mu', \nu']$ implies the containment $S(\mu, \nu) \subset S(\mu', \nu')$. Since all leading strips are simply connected continua, then so is the intersection of their nested family. Hence the intersection $X$ of all leading strips $S(\mu, \nu)$ with $0 < \nu < \gamma < \mu \leq \frac{1}{2}$ is a simply connected continuum. Let us show that in fact $X$ coincides with the $\gamma$-bimodal iso-over-rotation-tract $T_{\gamma}$. Indeed, choose a point $(\alpha, \beta) \notin X$. Then by construction there is a leading set $Z_{\nu}$ separating $(\alpha, \beta)$ from $X$ such that $\nu \neq \gamma$. Assume for the sake
of definiteness that $\nu < \gamma$. Then by Lemma 3.8 $\psi((\alpha, \beta)) \leq \nu < \gamma$. On the other hand, suppose that a point $(\alpha, \beta) \in X$. Suppose that, contrary to the desired, $\psi((\alpha, \beta)) < \gamma$ (the case of $\psi((\alpha, \beta)) > \gamma$ can be considered similarly). Choose the number $\nu$ so that $\psi((\alpha, \beta)) < \nu < \gamma$. Then, again by Lemma 3.8, the leading set $Z_\nu$ separates $(\alpha, \beta)$ from $X$, a contradiction. \hfill \Box

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References

[1] Ll. Alsedà, J. Llibre and M. Misiurewicz, Combinatorial Dynamics and Entropy in Dimension One, Advanced Series in Nonlinear Dynamics (2nd edition) 5 (2000), World Scientific Singapore (2000)
[2] S. Baldwin, Generalisation of a theorem of Sharkovsky on orbits of continuous real valued functions, Discrete Math. 67 (1987), 111–127.
[3] S. Bhattacharya, A. Blokh, Very badly ordered cycles of interval maps, Journal of Difference Equations and Applications 26 (2020), 1067-1084
[4] S. Bhattacharya, A. Blokh, Over-rotation intervals of bimodal interval maps, Journal of Difference Equations and Applications 26 (2020), 1085-1113
[5] A. Blokh, On Rotation Intervals for Interval Maps, Nonlinearity 7(1994), 1395-1417.
[6] A. Blokh, The Spectral Decomposition for One-Dimensional Maps, Dynamics Reported 4 (1995), 1–59.
[7] A. Blokh, Rotation Numbers, Twists and a Sharkovsky-Misiurewicz-type Ordering for Patterns on the Interval, Ergodic Theory and Dynamical Systems 15(1995), 1–14.
[8] A. Blokh, M. Misiurewicz, A new order for periodic orbits of interval maps, Ergodic Theory and Dynamical Sys. 17(1997), 565-574
[9] A. Blokh, K. Snider, Over-rotation numbers for unimodal maps, Journal of Difference Equations and Applications 19(2013), 1108–1132.
[10] J.Milnor and W.Thurston, On Iterated Maps on the Interval, Lecture Notes in Mathematics, Springer, Berlini 1342(1988), 465–520.
[11] J. Milnor, C.Tresser On Entropy and Monotonicity for Real Cubic Maps, Communications in Mathematical Physics 209(2000), 123-178
[12] A. N. Sharkovsky, Coexistence of the cycles of a continuous mapping of the line into itself, Ukraine Mat. Zh. 16(1964), 61–71 (Russian).
[13] A. N. Sharkovsky, Coexistence of the cycles of a continuous mapping of the line into itself, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 5 (1995), 1263–1273.

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