ANALYSIS OF TIME-DOMAIN ELECTROMAGNETIC SCATTERING PROBLEM BY MULTIPLE CAVITIES

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ABSTRACT. Consider the time-domain multiple cavity scattering problem, which arises in diverse scientific areas and has significant industrial and military applications. The multiple cavity embedded in an infinite ground plane, is filled with inhomogeneous media characterized by variable dielectric permittivities and magnetic permeabilities. Corresponding to the transverse electric or magnetic polarization, the scattering problem can be studied for the Helmholtz equation in frequency domain and wave equation in time-domain, respectively. A novel transparent boundary condition in time-domain is developed to reformulate the cavity scattering problem into an initial-boundary value problem in a bounded domain. The well-posedness and stability are established for the reduced problem. Moreover, a priori estimates for the electric field is obtained with a minimum requirement for the data by directly studying the wave equation.

1. INTRODUCTION

This paper is concerned with the mathematical analysis of the time-domain electromagnetic scattering problem of multiple cavities, which is embedded in a conducting ground planes. The cavity scattering problem arises in diverse scientific areas and has significant industrial and military applications, including the design of cavity-backed conformal antennas for civil and military use, and the characterization of radar cross-section (RCS) of vehicles with grooves, especially to design RCS. It is used to detect airplanes in a wide variation of ranges. For instance, a stealth aircraft will have design features that give it a low RCS, as opposed to a passenger airliner that will have a high RCS. RCS is integral to the development of radar stealth technology, particularly in applications involving aircraft and ballistic missiles. The cavity RCS caused by jet engine inlet ducts or cavity-backed antennas can dominate the total RCS. A thorough understanding of the electromagnetic scattering characteristic of a target, particularly a cavity, is necessary for successful implementation of any desired control of its RCS.

The descriptions of cavity scattering problem were centered on methods developed in the time-harmonic and time-domain. For the time-harmonic problems were introduced firstly by engineers [16,18,20,29]. The mathematical analysis of the cavity scattering problem was given by three fundamental papers [1–3], where the existence and uniqueness of the solutions were obtained based on a non-local transparent boundary condition on the cavity opening. A large amount of information was available regarding their solutions for both the two-dimensional Helmholtz and the three-dimensional Maxwell equations [4,5,7,8,22,25,26,28]. A good survey to the problem of cavity scattering can be found in [23]. The time-domain scattering problems have recently attracted considerable attention due to their capability of capturing wide-band signals and modeling more general material and nonlinearity [9,19,21,30], which motivates us to tune our focus from seeking the best possible conditions for those physical parameters to the time-domain problem. Comparing with the time-harmonic problems, the time-domain problems are less studied due to the additional challenge of the temporal dependence. The analysis can be found in [6,12,15,31] for the time-domain acoustic, elastic and electromagnetic scattering problems in different structures including bounded obstacles, periodic surfaces, and unbounded rough surfaces. Inspired by the one open cavity structure in [24], we extends the results to the multiple cavity scattering problem. It appears more complicated because of the unbounded nature of the domain and the novel transparent boundary condition on multiple apertures. Utilizing the

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Laplace transform as a bridge between the time-domain and the frequency domain, we develop an exact time-domain transparent boundary condition (TBC) and reduce the scattering problem equivalently into an initial boundary value problem in a bounded domain. Using the energy method with new energy functions, we can show the well-posedness and stability of the time-domain multiple cavity scattering problem.

The paper is organized as follow. In section 2, we introduce the model problem of one cavity scattering problem and establish a time-domain TBC. Section 3 is concentrated on the analysis of two cavities scattering problem, while the well-posedness and stability are addressed in both the frequency and time-domain. The multiple cavity problem is proposed in section 4, while a priori estimates with explicit time dependence for the quantities of electric filed is obtained with a minimum requirement for the data by directly studying the wave equation. We conclude the paper with some remarks in section 5.

2. ONE CAVITY SCATTERING PROBLEM

In this section, we shall introduce the mathematical model for a single cavity scattering problem and develop an exact TBC to reduce the scattering problem from an unbounded domain into a bounded domain.

2.1. Problem formulation. Consider a simpler model for the open cavity scattering problem by assuming that the medium and material are invariant along the $z$-axis. Let $\Omega \subset \mathbb{R}^2$ be the cross section of a $z$-invariant cavity with a Lipschitz continuous boundary $\partial \Omega = S \cup \Gamma$, as seen in Figure 1. The cavity is filled with some inhomogeneous medium, characterized by the variable dielectric permittivity $\varepsilon(x, y)$ and magnetic permeability $\mu(x, y)$. The exterior region $\Omega^e$ is filled with some homogeneous material with a constant permittivity $\varepsilon_0$ and a constant permeability $\mu_0$. Here the cavity wall $S$ is assumed to be a perfect electric conductor and the cavity opening $\Gamma$ is aligned with the perfectly electrically conducting infinite ground surface $\Gamma^c$. An open cavity $\Omega$, enclosed by the aperture $\Gamma$ and the wall $S$, is placed on a perfectly conducting ground plane $\Gamma^c$.

The electromagnetic wave propagation is governed by the time-domain Maxwell equations

\begin{equation}
\begin{aligned}
\nabla \times E(r, t) + \mu \partial_t H(r, t) &= 0, \\
\nabla \times H(r, t) - \varepsilon \partial_t E(r, t) &= 0,
\end{aligned}
\end{equation}

where $r = (x, y, z) \in \mathbb{R}^3$, $E$ is the electric field, $H$ is the magnetic field, $\varepsilon$ and $\mu$ are the dielectric permittivity and magnetic permeability, respectively, and satisfy

$$0 < \varepsilon_{\text{min}} \leq \varepsilon \leq \varepsilon_{\text{max}} < \infty, \quad 0 < \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}} < \infty,$$

while $\varepsilon_{\text{min}}, \varepsilon_{\text{max}}, \mu_{\text{min}}, \mu_{\text{max}}$ are constants. The system is constrained by the initial conditions

$$E|_{t=0} = 0, \quad H|_{t=0} = 0. \quad (2.2)$$

Since the structure is invariant in the $z$-axis, the problem can be decomposed into two fundamental polarizations: transverse electric (TE) and transverse magnetic (TM). The three-dimensional Maxwell equations can be reduced to the two-dimensional wave equation.

(i) TE polarization: the magnetic field is transverse to the $z$-axis, the electric and magnetic fields are

$$E(r, t) = [0, 0, u(\rho, t)]^T, \quad H(r, t) = [H_1(\rho, t), H_2(\rho, t), 0]^T,$$
where \( \rho = (x, y) \in \mathbb{R}^2 \). Eliminating the magnetic field from (2.1), we get the wave equation for the electric field

\[ \varepsilon \partial_t^2 u - \nabla \cdot (\mu^{-1} \nabla u) = 0 \quad \text{in} \quad \Omega^e \cup \Omega, \quad t > 0. \quad (2.3) \]

By the perfectly conducting boundary condition on the ground plane and cavity wall we can get

\[ u = 0 \quad \text{on} \quad S \cup \Gamma^c, \quad t > 0. \]

It follows from the initial condition (2.2) that \( u(\rho, t) \) satisfies the homogeneous initial conditions

\[ u(\rho, t) \big|_{t=0} = 0, \quad \partial_t u(\rho, t) \big|_{t=0} = 0 \quad \text{in} \quad \Omega^e \cup \Omega. \]

(ii) TM polarization: the electric field is transverse to the \( z \)-axis, the electric and magnetic fields are

\[ E(r, t) = [E_1(\rho, t), E_2(\rho, t), 0]^\top, \quad H(r, t) = [0, 0, u(\rho, t)]^\top. \]

We may eliminate the electric field from (2.1) and obtain the wave equation for the magnetic field

\[ \mu \partial_t^2 u - \nabla \cdot (\varepsilon^{-1} \nabla u) = 0 \quad \text{in} \quad \Omega^e \cup \Omega, \quad t > 0. \quad (2.4) \]

It also follows from the perfectly conducting boundary condition on the ground plane and cavity wall that

\[ \partial_n u = 0 \quad \text{on} \quad S \cup \Gamma^c, \quad t > 0, \]

where \( \nu \) is the unit outward normal vector on \( S \cup \Gamma^c \). The initial conditions for the TM is

\[ u(\rho, t) \big|_{t=0} = 0, \quad \partial_t u(\rho, t) \big|_{t=0} = 0 \quad \text{in} \quad \Omega^e \cup \Omega. \]

It is clear to note from (2.3) and (2.4) that TE and TM polarizations can be handled in a unified way by formally exchanging the roles of \( \varepsilon \) and \( \mu \). We will just discuss the results in detail by using (2.3) (TE case) as the model equation in the rest of the paper. The method can be extended to the TM polarization with obvious modifications.

Let an incoming plane wave \( u^{\text{inc}} = f(-t - c_1 x - c_2 y) \) be incident on the cavity from above, where \( f \) is a smooth function and its regularity will be specified later, and \( c_1 = \cos \theta/\sqrt{\varepsilon_0 \mu_0}, c_2 = \sin \theta/\sqrt{\varepsilon_0 \mu_0}, 0 < \theta < \pi \). Clearly, the incident field satisfies the wave equation (2.3) with \( \varepsilon = \varepsilon_0, \mu = \mu_0 \). The total field can be split into the incident field, the reflected field and the scattered field:

\[ u = u^{\text{inc}} + u^r + u^{sc}, \]

where \( u^r = -f(-t - c_1 x + c_2 y) \) (or \( u^r = f(-t - c_1 x + c_2 y) \)) is the reflected field in TE (or TM) case. To impose the initial conditions, we assume that the total field, the incident field and the reflected field vanish for \( t < 0 \), so that the scattered field \( u^{sc} = 0 \) for \( t < 0 \). Moreover, the scattered field is required to satisfies the Sommerfeld radiation condition:

\[ \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \partial_r u^{sc} + \partial_t u^{sc} = o(r^{-1/2}) \quad \text{as} \quad r = |\rho| \to \infty, \quad t > 0. \quad (2.5) \]

To analyze the problem, the open domain needs to be truncated into a bounded domain. Therefore, a suitable boundary condition has to be imposed on the boundary of the bounded domain so that no artificial wave reflection occurs. We shall present a transparent boundary condition on the open domain enclosing the inhomogeneous cavity.

2.1.1. Laplace transform and some notation. We first introduce the Laplace transform and present some identities for the transform. For any \( s = s_1 + is_2 \) with \( s_1, s_2 \in \mathbb{R}, \ s_1 > 0, \ i = \sqrt{-1} \), define by \( \hat{u}(s) \) the Laplace transform of the function \( u(t) \), i.e.,

\[ \hat{u}(s) = \mathcal{L}(u)(s) = \int_0^\infty e^{-st} u(t)dt. \]

Using the integration by parts yields

\[ \int_0^t u(\tau)d\tau = \mathcal{L}^{-1}(s^{-1}\hat{u}(s)). \]
More precisely, we will address the reduced initial-boundary value problem

\[ \text{Dirichlet–to–Neumann (DtN) operator which maps the Dirichlet data to the Neumann data of the wave field.} \]

2.1.2. Transparent boundary condition.

Let \( \partial \Omega \) be a bounded Lipschitz domain with boundary \( \partial \Omega \). Denote the Sobolev space: \( H^1(\Omega) = \{ u : D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leq 1 \} \). To describe the boundary operator and transparent boundary condition in the formulation of the boundary value problem, we define the trace functional space

\[ H^\nu(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + \xi^2)^\nu |\hat{u}|^2 d\xi < \infty \right\}, \]

whose norm is defined by

\[ ||u||_{H^\nu(\mathbb{R})} = \left( \int_{\mathbb{R}} (1 + \xi^2)^\nu |\hat{u}|^2 d\xi \right)^{1/2}, \]

where \( \hat{u} \) is the Fourier transform of \( u \) defined as

\[ \hat{u}(\xi) = \int_{\mathbb{R}} u(x)e^{ix\xi} dx. \]

It is clear to note that the dual space of \( H^{1/2}(\mathbb{R}) \) is \( H^{-1/2}(\mathbb{R}) \) under the \( L^2(\mathbb{R}) \) inner produce

\[ \langle u, v \rangle = \int_{\mathbb{R}} \hat{u}\hat{v} dx = \int_{\mathbb{R}} \hat{u}\hat{v} d\xi. \]

2.1.2. Transparent boundary condition. We introduce a time-domain TBC to formulate the cavity scattering problem into an equivalent initial-boundary value problem in a bounded domain. The idea is to design a Dirichlet–to–Neumann (DtN) operator which maps the Dirichlet data to the Neumann data of the wave field. More precisely, we will address the reduced initial-boundary value problem

\[
\begin{aligned}
\varepsilon \partial_t^2 u - \nabla \cdot (\mu^{-1} \nabla u) &= 0 &\text{in } \Omega, \ t > 0, \\
|u|_{t=0} &= 0, \quad \partial_t u|_{t=0} &= 0 &\text{in } \Omega, \\
|u| &= 0 &\text{on } S, \ t > 0, \\
\partial_n u &= \mathcal{F} u + g &\text{on } \Gamma, \ t > 0,
\end{aligned}
\]
where $\mathcal{T}$ is a time-domain boundary operator and $g$ will be given later. In what follows, we derive the formulation of the operator $\mathcal{T}$ and analyze its important properties.

Since $\varepsilon = \varepsilon_0, \mu = \mu_0$ in $\Omega^e$, the equations (2.3) and (2.4) together with the radiation condition (2.5) implies the scattered field $u^{sc}$ satisfies

$$\begin{cases}
\Delta u^{sc} - \varepsilon_0\mu_0\partial_t^2 u^{sc} = 0 & \text{in } \Omega^e, \ t > 0, \\
u^{sc}\big|_{t=0} = 0, \ \partial_t u^{sc}\big|_{t=0} = 0 & \text{in } \Omega^e, \\
u^{sc} = -\left(\hat{u}^{inc} + \hat{u}^t\right) & \text{on } \Gamma^c, \ t > 0, \\
(\varepsilon_0\mu_0)^{-1/2}\partial_t u^s + \partial_r u^s = o(r^{-1/2}) & \text{as } r = |\rho| \to \infty, \ t > 0.
\end{cases}$$

(2.8)

Let $\bar{u}(\rho, s) = \mathcal{L}(u)(\rho, t)$ be the Laplace transform of $u(\rho, t)$ with respect to $t$. Recalling that $\mathcal{L}(\partial_t u) = s\hat{u}(\cdot, s) - u(\cdot, 0)$, $\mathcal{L}(\partial_t^2 u) = s^2\hat{u}(\cdot, s) - s\hat{u}(\cdot, 0) - \partial_t u(\cdot, 0)$.

Taking the Laplace transform of (2.8) with the initial conditions, we can get the time-harmonic Helmholtz equation for the scattered field with the complex wave number $c$.

$$\begin{cases}
\Delta \hat{u}^{sc} - \frac{s^2}{c^2}\hat{u}^{sc} = 0 & \text{in } \Omega^e, \\
\hat{u}^{sc} = -\left(\hat{\hat{u}}^{inc} + \hat{u}^t\right) & \text{on } \Gamma^c, \\
\frac{s}{c}\hat{u}^{sc} + \partial_r \hat{u}^{sc} = o(r^{-1/2}) & \text{as } r = |\rho| \to \infty,
\end{cases}$$

(2.9)

where $c := \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ is the light speed in the free space.

By taking the Fourier transform of the first equation in (2.9) with respect to $x$, we have an ordinary differential equation with respect to $y$:

$$\frac{\partial^2 \hat{u}^{sc}}{\partial y^2} - \left(\xi^2 + \frac{s^2}{c^2}\right)\hat{u}^{sc} = 0, \ y > 0.$$

(2.10)

It follows from the radiation condition in (2.9), we deduce that the solution of (2.10) has the analytical form

$$\hat{u}^{sc} = \hat{u}^{sc}(\xi, 0)e^{\beta(\xi)y},$$

(2.11)

where

$$\beta(\xi) = \left(\xi^2 + \frac{s^2}{c^2}\right)^{1/2} \quad \text{with } \Re(\beta(\xi)) < 0.$$

(2.12)

Taking the inverse Fourier transform of (2.11), we find that

$$\hat{u}^{sc}(x, y) = \int_{\mathbb{R}} \hat{u}^{sc}(\xi, 0)e^{\beta(\xi)y}e^{-i\xi x}d\xi \quad \text{in } \Omega^e.$$

(2.13)

Taking the normal derivative on $\Gamma^c \cup \Gamma$ and evaluating at $y = 0$ yields

$$\partial_n \hat{u}^{sc}(x, y)|_{y=0} = \int_{\mathbb{R}} \beta(\xi)\hat{u}^{sc}(\xi, 0)e^{-i\xi x}d\xi,$$

(2.14)

where $n$ is the unit outward normal on $\Gamma^c \cup \Gamma$, i.e. $n = (0, 1)$.

For any $w \in H^{1/2}(\Gamma^c \cup \Gamma)$ with $w = \int_{\mathbb{R}} \hat{w}(\xi, 0)e^{-i\xi x}d\xi$, define the boundary operator $\mathcal{B}$

$$\mathcal{B}w := \int_{\mathbb{R}} \beta(\xi)\hat{w}(\xi, 0)e^{-i\xi x}d\xi,$$

(2.15)

which leads to a transparent boundary condition for the scattered field on $\Gamma^c \cup \Gamma$:

$$\partial_n \hat{u}^{sc} = \mathcal{B}\hat{u}^{sc}. $$

From $\hat{u}^{sc} = \hat{u} - \left(\hat{u}^{inc} + \hat{u}^t\right)$, we can get an equivalent transparent boundary condition for the total field

$$\partial_n \hat{u} = \mathcal{B}\hat{u} + \hat{g} \quad \text{on } \Gamma^c \cup \Gamma,$$

(2.15)

where $\hat{g} = \partial_n \left(\hat{u}^{inc} + \hat{u}^t\right) - \mathcal{B} \left(\hat{u}^{inc} + \hat{u}^t\right).$
Taking the inverse Laplace transform of (2.15) yields the TBC in the time-domain
\[ \partial_n u = \mathcal{T} u + g \quad \text{on } \Gamma^c \cup \Gamma, \quad t > 0, \]  
(2.16)
where \( \mathcal{T} = \mathcal{L}^{-1} \circ \mathcal{B} \circ \mathcal{L} \) and \( g = \mathcal{L}^{-1} \circ \tilde{g} \).

Since \( u \) is defined on \( \Gamma^c \cup \Gamma \) and the transparent boundary condition above is derived for \( u \in H^{1/2}(\mathbb{R}) \). In order to derive the transparent boundary condition for the total field on \( \Gamma \), we make the zero extension as follows: for any given \( u \) on \( \Gamma \), define
\[ \tilde{u}(x) = \begin{cases} 
  u & \text{for } x \in \Gamma, \\
  0 & \text{for } x \in \Gamma^c.
\end{cases} \]

Since the cavity is placed on a perfectly conducting ground plane \( \Gamma^c \), i.e. the total filed is required to be zero on \( \Gamma^c \), it is obviously that above zero extension is consistent with the problem geometry. Based on the extension and the transparent boundary conditions (2.15) and (2.16), we have the transparent boundary conditions for the total field on the opening
\[ \partial_n \tilde{u} = \mathcal{B} \tilde{u} + \tilde{g} \quad \text{on } \Gamma; \quad \partial_n u = \mathcal{T} \tilde{u} + g \quad \text{on } \Gamma, \quad t > 0. \]

Define a dual paring \( \langle \cdot, \cdot \rangle_{\Gamma} \) by
\[ \langle u, v \rangle_{\Gamma} = \int_{\Gamma} u \bar{v} d\gamma. \]
By the definition of extension, this dual paring for \( u \) and \( v \) is equivalent to the scalar product in \( L^2(\mathbb{R}) \) for their extension, i.e.,
\[ \langle u, v \rangle_{\Gamma} = \langle \tilde{u}, \tilde{v} \rangle. \]

The following lemmas are useful in the proof of the well-posedness of the reduced problem.

**Lemma 2.2.** The boundary operator \( \mathcal{B} : H^{1/2}(\mathbb{R}) \rightarrow H^{-1/2}(\mathbb{R}) \) is continuous, i.e.,
\[ \| \mathcal{B} u \|_{H^{-1/2}(\mathbb{R})} \leq C \| u \|_{H^{1/2}(\mathbb{R})}, \quad \forall u \in H^{1/2}(\mathbb{R}). \]

**Proof.** For any \( u, v \in H^{1/2}(\mathbb{R}) \), it follows from the definitions (2.14) that
\[ \langle \mathcal{B} u, v \rangle = \int_{\mathbb{R}} \mathcal{B} u \bar{v} d\xi = \int_{\mathbb{R}} \frac{\beta(\xi)}{(1 + \xi^2)^{1/4}} (1 + \xi^2)^{1/4} u \cdot (1 + \xi^2)^{1/4} \bar{v} d\xi. \]
To prove the lemma, it is required to estimate
\[ \frac{|\beta(\xi)|}{(1 + \xi^2)^{1/4}}, \quad -\infty < \xi < \infty. \]
Let
\[ \frac{s^2}{c^2} = a + ib, \quad a := \frac{s^2 - s_c^2}{c^2}, \quad b := \frac{2s_1 s_c}{c^2}. \]
Denote
\[ \beta^2(\xi) = \frac{s^2}{c^2} + \xi^2 = \phi + ib, \]
where \( \phi := \text{Re} \left( \frac{s^2}{c^2} \right) + \xi^2 = a + \xi^2 \). A simple calculation gives
\[ \frac{|\beta(\xi)|}{(1 + \xi^2)^{1/4}} = \left[ \frac{\phi^2 + b^2}{(1 + \phi - a)^2} \right]^{1/4}. \]
Define
\[ F(t) = \frac{t^2 + b^2}{(1 + t - a)^2}, \quad t \geq a. \]
It follows
\[ F'(t) = \frac{2(1 + t - a)(t(1 - a) - b^2)}{(1 + t - a)^4}. \]

We consider it in two cases:

(i) \(1 - a > 0\). It can be verified that the function \(F(t)\) decreases for \(a \leq t \leq \frac{b^2}{1-a}\) and increase for \(t > \frac{b^2}{1-a}\).

Thus
\[ F(\phi) \leq \max \left\{ F(a) = a^2 + b^2, \; F(+\infty) = 1 \right\}. \]

(ii) \(1 - a \leq 0\). It is easy to verify that \(F(t)\) decreases for \(t \geq a\). Thus, we have
\[ F(\phi) \leq F(a) = a^2 + b^2. \]

Combining above estimates and using the Cauchy–Schwarz inequality yield
\[ |\langle \mathcal{B} u, v \rangle| \leq C \|u\|_{H^{1/2}(\mathbb{R})} \|v\|_{H^{1/2}(\mathbb{R})}, \]

where
\[ C = \max\{(a^2 + b^2)^{1/4}, 1\}. \]

Thus we have
\[ \|\mathcal{B} u\|_{H^{-1/2}(\mathbb{R})} \leq \sup_{v \in H^{1/2}(\mathbb{R})} \frac{|\langle \mathcal{B} u, v \rangle|}{\|v\|_{H^{1/2}(\mathbb{R})}} \leq C \|u\|_{H^{1/2}(\mathbb{R})}. \]

\[ \square \]

It follows from Lemma 2.1 and Lemma 2.2 that the inverse Laplace transform in (2.16) is make sense.

**Lemma 2.3.** It holds that
\[ -\text{Re}(\langle (s\mu)^{-1} \mathcal{B} u, u \rangle) \geq 0, \quad u \in H^{1/2}(\mathbb{R}). \]

**Proof.** By the definition (2.14), we find
\[ -\langle (s\mu)^{-1} \mathcal{B} u, u \rangle = -\int_{\mathbb{R}} (s\mu)^{-1} \beta(\xi) |u|^2 d\xi = -\int_{\mathbb{R}} \frac{s\beta(\xi)}{\mu |s|^2} |u|^2 d\xi. \]

Let \(\beta(\xi) = \varsigma + i\eta\) with \(\varsigma < 0\). Taking the real part of the above equation gives
\[ -\text{Re}(\langle (s\mu)^{-1} \mathcal{B} u, u \rangle) = -\int_{\mathbb{R}} \frac{s_1 \varsigma + s_2 \eta}{\mu |s|^2} |u|^2 d\xi. \quad (2.17) \]

Recalling \(\beta^2(\xi) = \xi^2 + c^{-2}s^2\), we have
\[ \varsigma^2 - \eta^2 = \xi^2 + c^{-2}(s_1^2 - s_2^2), \quad \varsigma \eta = c^{-2}s_1 s_2. \quad (2.18) \]

Using (2.18), it gives
\[ s_1 \varsigma + s_2 \eta = \frac{s_1}{\varsigma} (\varsigma^2 + c^{-2}s_2^2). \quad (2.19) \]

Substituting (2.19) into (2.17), we have
\[ -\text{Re}(\langle (s\mu)^{-1} \mathcal{B} u, u \rangle) = -\int_{\mathbb{R}} \frac{1}{\mu |s|^2} \frac{s_1}{\varsigma} (\varsigma^2 + c^{-2}s_2^2) |u|^2 d\xi \geq 0, \]

which completes the proof. \(\square\)

**Lemma 2.4.** For any \(u(\cdot, t) \in L^2(0, T; H^{1/2}(\Gamma))\) with initial value \(u(\cdot, 0) = 0\), it holds that
\[ -\text{Re} \int_0^T \langle \mathcal{F} u(\cdot, t), \partial_t u(\cdot, t) \rangle_{\Gamma} dt \geq 0. \]
Proof. Let \( \tilde{u}(\cdot, t) \) be the extension of \( u(\cdot, t) \) with respect to \( t \) in \( \mathbb{R} \) such that \( \tilde{u}(\cdot, t) = 0 \) outside the interval \([0, T]\), and \( \tilde{u} = \mathcal{L}(\tilde{u}) \) be the Laplace of \( \tilde{u} \). By the Parseval identity (2.6) and Lemma 2.3, we get

\[
-\text{Re} \int_0^T e^{-2s_1t} (\mathcal{T} u, \partial_t u)_\Gamma dt = -\text{Re} \int_0^T e^{-2s_1t} (\mathcal{T} \tilde{u}, \partial_t \tilde{u})_\Gamma dt
\]

\[
= -\text{Re} \int_\Gamma \int_0^\infty e^{-2s_1t} (\mathcal{T} \tilde{u}) \partial_t \tilde{u} dt d\gamma
\]

\[
= -\frac{1}{2\pi} \int_\infty^{-\infty} \text{Re} (\mathcal{B} \tilde{u}, s\tilde{u})_\Gamma ds_2
\]

\[
= -\frac{1}{2\pi} \int_\infty^{-\infty} |s|^2 \mu \text{Re} \left( (s\mu)^{-1} \mathcal{B} \tilde{u}, \tilde{u} \right)_\Gamma ds_2 \geq 0,
\]

which completes the proof after taking \( s_1 \to 0 \). \( \Box \)

The following trace theorem are useful in the following reduced problem, the proof can be found in (cf. [11]).

Lemma 2.5. (trace theorem) Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz domain with boundary \( \partial \Omega \). For \( 1/2 < \nu < 3/2 \) the interior trace operator

\[
T_0 : H^\nu(\Omega) \to H^{\nu-1/2}(\Gamma)
\]

is bounded, \( \forall w \in H^\nu(\Omega) \), where \( T_0 w = w|_\Gamma \).

2.2. The reduced one cavity scattering problem. In this section, we will present the well-posedness of the reduced problem by a variation method, and given the stability of one cavity scattering problem.

2.2.1. well-posedness in the s-domain. Taking the Laplace transform of (2.7) and using the transparent boundary condition, we may consider the following reduced boundary value problems

\[
\begin{cases}
\text{s} \varepsilon \tilde{u} - \nabla \cdot ((s\mu)^{-1} \nabla \tilde{u}) = 0 & \text{in } \Omega, \\
\tilde{u} = 0 & \text{on } S, \\
\partial_n \tilde{u} = \mathcal{B} \tilde{u} + \tilde{g} & \text{on } \Gamma,
\end{cases}
\]

(2.20)

where \( \tilde{g} = \partial_n \left( \tilde{u}^{\text{inc}} + \tilde{u}^\ast \right) - \mathcal{B} \left( \tilde{u}^{\text{inc}} + \tilde{u}^\ast \right), s = s_1 + is_2 \) with \( s_1 > 0 \).

By multiplying a test function \( v \in H^1_S \) and using the transparent boundary condition, we may consider the following reduced boundary value problems

\[
\begin{align*}
a_1(\tilde{u}, v) &= \langle \tilde{g}, v \rangle_\Gamma, & \forall v \in H^1_S(\Omega),
\end{align*}
\]

(2.21)

where the sesquilinear form

\[
a_1(\tilde{u}, v) = \int_\Omega \left( ((s\mu)^{-1} \nabla \tilde{u} \cdot \nabla v + s\varepsilon \tilde{u}v) \right) d\rho - \left( (s\mu)^{-1} \mathcal{B} \tilde{u}, v \right)_\Gamma.
\]

(2.22)

Theorem 2.6. The variational problem (2.21) has a unique solution \( \tilde{u} \in H^1_S(\Omega) \) which satisfies

\[
\| \nabla \tilde{u} \|_{L^2(\Omega)^2} + \| s \tilde{u} \|_{L^2(\Omega)} \lesssim s_1^{-1} \| s \tilde{g} \|_{H^{-1/2}(\Gamma)}.
\]

(2.23)

Proof. It suffices to show the coercivity of the sesquilinear form of \( a_1(\tilde{u}, v) \). The continuity of sesquilinear form follows directly from the Cauchy–Schwarz inequality, Lemma 2.2 and Lemma 2.5

\[
|a_1(\tilde{u}, v)| \leq \frac{1}{|s|\mu_{\min}} \| \nabla \tilde{u} \|_{L^2(\Omega)^2} \| \nabla v \|_{L^2(\Omega)^2} + |s|\varepsilon_{\max} \| \tilde{u} \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)}
\]

\[
+ \frac{1}{|s|\mu_{\min}} \| \mathcal{B} \tilde{u} \|_{H^{-1/2}(\Gamma)} \| v \|_{H^{1/2}(\Gamma)}
\]

\[
\lesssim \| \tilde{u} \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)}.
\]
Let \( v = \hat{u} \) in (2.22), we get
\[
a_1(\hat{u}, \hat{u}) = \int_{\Omega} \left( (s\mu)^{-1}|\nabla \hat{u}|^2 + s\varepsilon |\hat{u}|^2 \right) d\rho - (s\mu)^{-1} \phi(\hat{u}, \hat{u})_\Gamma. ~ (2.24)
\]
Taking the real part of (2.24) and using Lemma 2.3 yields
\[
\text{Re} (a_1(\hat{u}, \hat{u})) \geq C_1 \frac{\mu}{2} \left( \|\nabla \hat{u}\|_{L^2(\Omega)}^2 + \|s\hat{u}\|_{L^2(\Omega)}^2 \right),
\]
where \( C_1 = \min \{ \mu_\text{max}^{-1}, 1 \} \).

It follows from the Lax–Milgram lemma that the variational problem (2.21) has a unique solution \( \hat{u} \in H^1(\Omega) \). Moreover, we have from (2.21) that
\[
|a_1(\hat{u}, \hat{u})| \leq |s|^{-1} \|\hat{g}\|_{H^{-1/2}(\Gamma)} \|s\hat{u}\|_{L^2(\Omega)}, ~ (2.26)
\]
Combining (2.25)–(2.26) leads to
\[
\|\nabla \hat{u}\|_{L^2(\Omega)}^2 + \|s\hat{u}\|_{L^2(\Omega)}^2 \lesssim s_1^{-1} \|s\hat{g}\|_{H^{-1/2}(\Gamma)} \|s\hat{u}\|_{L^2(\Omega)},
\]
which gives estimate of (2.23) after applying the Cauchy–Schwarz inequality.

2.2.2. well-posedness in the time-domain. Using the time-domain transparent boundary condition, we consider the reduced initial-boundary value problem
\[
\begin{align*}
\varepsilon \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\mu^{-1} \nabla u) &= 0 & \text{in } \Omega, \ t > 0, \\
\left. u \right|_{t=0} &= 0, \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = 0 & \text{in } \Omega, \\
u &= 0 & \text{on } S_t, \ t > 0, \\
\frac{\partial u}{\partial n} &= \mathcal{T} u + g & \text{on } \Gamma, \ t > 0.
\end{align*}
(2.27)
\]

**Theorem 2.7.** The initial-boundary problem (2.27) has a unique solution \( u \), which satisfies
\[
u \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)),
\]
and the stability estimate
\[
\max_{t \in [0, T]} (\|\partial_t u\|_{L^2(\Omega)} + \|\partial_t (\nabla u)\|_{L^2(\Omega)})
\lesssim \|g\|_{L^1(0, T; H^{-1/2}(\Gamma))} + \max_{t \in [0, T]} \|\partial_t g\|_{H^{-1/2}(\Gamma)} + \|\partial^2 g\|_{L^1(0, T; H^{-1/2}(\Gamma))}.
(2.28)
\]

**Proof.** First, we have
\[
\int_0^T \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 \right) dt \leq \int_0^T e^{-2s_1(t-T)} \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 \right) dt
\]
\[
eq e^{2s_1T} \int_0^T e^{-2s_1t} \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 \right) dt
\]
\[
\lesssim \int_0^\infty e^{-2s_1t} \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 \right) dt.
\]

Hence it suffices to estimate the integral
\[
\int_0^\infty e^{-2s_1t} \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 \right) dt.
\]
Let \( \tilde{u} = \mathcal{L} u \). By Theorem 2.6, we have
\[
\|\nabla \tilde{u}\|_{L^2(\Omega)}^2 + \|s\tilde{u}\|_{L^2(\Omega)}^2 \lesssim s_1^{-2} |s|^2 \|\tilde{g}\|_{H^{-1/2}(\Gamma)}^2 \lesssim s_1^{-2} |s|^2 \|\tilde{u}\|_{H^1(\Omega)}^2.
\]
It follows from (cf. [27] Lemma 44.1)) that \( \tilde{u} \) is a holomorphic function of \( s \) on the half plane \( s_1 > \sigma_0 > 0 \), where \( \sigma_0 \) is any positive constant. Hence we have from Lemma 2.1 that the inverse Laplace transform of \( \tilde{u} \) exists and is supported in \((0, \infty)\).
One may verify from the inverse Laplace transform that
\[ \ddot{u} = \mathcal{L}(u) = \mathcal{F}(e^{-s_1 t}u), \]
where \( \mathcal{F} \) is the Fourier transform in \( s_2 \). Recalling the Plancherel or Parseval identity for the Laplace transform in (2.6), it follows
\[
\int_0^\infty e^{-2s_1 t} \left( \|\nabla u\|^2_{L^2(\Omega)^2} + \|\partial_t u\|^2_{L^2(\Omega)} \right) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \|\nabla \hat{u}\|^2_{L^2(\Omega)} + s\|\hat{u}\|^2_{L^2(\Omega)} \right) ds_2 
\leq s_1^{-2} \int_{-\infty}^\infty \left( \|s(\hat{u}^{\text{inc}} + \hat{u}^t)\|^2_{L^2(\Omega)} + \|s(\nabla \hat{u}^{\text{inc}} + \nabla \hat{u}^t)\|^2_{L^2(\Omega)^2} \right) ds_2.
\]
Since \( (u^{\text{inc}} + u^t)|_{t=0} = \partial_t(u^{\text{inc}} + u^t)|_{t=0} = 0 \) in \( \Omega \), we have \( \mathcal{L}(\partial_t(u^{\text{inc}} + u^t)) = s(\hat{u}^{\text{inc}} + \hat{u}^t) \) in \( \Omega \). It is easy to note that
\[
|s|^2(\hat{u}^{\text{inc}} + \hat{u}^t) = (2s_1 - s)s(\hat{u}^{\text{inc}} + \hat{u}^t) = 2s_1\mathcal{L}(\partial_t u^{\text{inc}} + \partial_t u^t) - \mathcal{L}(\partial_t^2 u^{\text{inc}} + \partial_t^2 u^t),
\]
and
\[
|s|^2(\nabla \hat{u}^{\text{inc}} + \nabla \hat{u}^t) = 2s_1\mathcal{L}(\partial_t \nabla u^{\text{inc}} + \partial_t \nabla u^t) - \mathcal{L}(\partial_t^2 \nabla u^{\text{inc}} + \partial_t^2 \nabla u^t).
\]
Hence we have
\[
\int_0^\infty e^{-2s_1 t} \left( \|\nabla u\|^2_{L^2(\Omega)^2} + \|\partial_t u\|^2_{L^2(\Omega)} \right) dt 
\leq \int_{-\infty}^\infty \mathcal{L}(\partial_t^2 u^{\text{inc}} + \partial_t^2 u^t) ds_2 + s_1^{-2} \int_{\mathbb{R}} \mathcal{L}(\partial_t^2 u^{\text{inc}} + \partial_t^2 u^t) ds_2 
\nonumber 
+ \int_{-\infty}^\infty \mathcal{L}(\partial_t \nabla u^{\text{inc}} + \partial_t \nabla u^t) ds_2 + s_1^{-2} \int_{\mathbb{R}} \mathcal{L}(\partial_t^2 \nabla u^{\text{inc}} + \partial_t^2 \nabla u^t) ds_2.
\]
Using the Parseval identity (2.6) again gives
\[
\int_0^\infty e^{-2s_1 t} \left( \|\nabla u\|^2_{L^2(\Omega)^2} + \|\partial_t u\|^2_{L^2(\Omega)} \right) dt 
\leq \int_0^\infty e^{-2s_1 t} \|\partial_t^2 u^{\text{inc}} + \partial_t^2 u^t\|^2_{H^1(\Omega)} dt + s_1^{-2} \int_0^\infty e^{-2s_1 t} \|\partial_t^2 u^{\text{inc}} + \partial_t^2 u^t\|^2_{L^2(\Omega)^2} dt,
\]
which shows
\[
u \in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)).
\]

Next, we prove the stability. For any \( 0 < t < T \), define the energy function
\[
e_1(t) = \|e^{1/2 \partial_t} u(\cdot, t)\|^2_{L^2(\Omega)} + \|\mu^{-1/2} \nabla u(\cdot, t)\|^2_{L^2(\Omega)^2}.
\]
It follows from (2.27) and integration by parts that
\[
e_1(t) - e_1(0) = \int_0^t e_1'(\tau) d\tau = 2 \text{Re} \int_0^t \int_\Omega (\nabla \cdot (\mu^{-1} \nabla u) \partial_t \bar{u} + \mu^{-1} (\nabla \partial_t u) \cdot \nabla \bar{u}) d\rho d\tau.
\]
Since $e_1(0) = 0$, we obtain from Lemma 2.4 that
\[
e_1(t) = \int_0^t e_1'(\tau)\,d\tau = 2\text{Re} \int_0^t \int_\Omega (-\mu^{-1} \nabla u \cdot (\nabla \partial_t \bar{u}) + \mu^{-1} (\nabla \partial_t u) \cdot \nabla \bar{u}) \,d\rho \,d\tau
+ 2\text{Re} \int_0^t \int_\Gamma \mu^{-1}(\partial_t u) \partial_t \bar{u} \,d\gamma \,dt
= 2\text{Re} \int_0^t \mu^{-1}(\mathcal{S} u, \partial_t u)_\Gamma \,dt + 2\text{Re} \int_0^t \langle g, \partial_t u \rangle_\Gamma \,dt
\leq 2\text{Re} \int_0^t (\|g\|_{H^{-1/2}(\Gamma)} \|\partial_t u\|_{H^{1/2}(\Gamma)}) \,dt
\leq 2\text{Re} \int_0^t (\|g\|_{H^{-1/2}(\Gamma)} \|\partial_t u\|_{H^1(\Omega)}) \,dt
\leq 2 \left( \max_{t \in [t, T]} \|\partial_t u\|_{H^1(\Omega)} \right) \|g\|_{L^1(0,T;H^{-1/2}(\Gamma))}. \tag{2.29}
\]

Since the right-hand side of (2.29) contains the term $\partial_t \nabla u$, which cannot be controlled by the left-hand side of (2.29), hence we need to consider a new reduced system. Taking the derivative of (2.27) with respect to $t$, we know that $\partial_t u$ also satisfies the same equations with $g$ replaced by $\partial_t g$. Hence we may consider the similar energy function
\[e_2(t) = \|e^{1/2} \partial_t^2 u(\cdot, t)\|_{L^2(\Omega)}^2 + \|\mu^{-1/2} \partial_t (\nabla u(\cdot, t))\|_{L^2(\Omega)}^2,
\]
and get the estimate
\[e_2(t) \leq 2\text{Re} \int_0^t \int_\Gamma (\partial_t g) \partial_t^2 \bar{u} \,d\gamma \,dt
= 2\text{Re} \int_\Gamma (\partial_t g) \partial_t \bar{u} \,d\gamma - 2\text{Re} \int_0^t \int_\Gamma (\partial_t^2 g) \partial_t \bar{u} \,d\gamma \,dt
\leq 2\text{Re}(\max_{t \in [t, T]} \|\partial_t u\|_{H^1(\Omega)})(\max_{t \in [t, T]} \|\partial_t g\|_{H^{-1/2}(\Gamma)} + \|\partial_t^2 g\|_{L^1(0,T;H^{-1/2}(\Gamma))}),
\]
Combing above estimates, we can obtain
\[
\max_{t \in [t, T]} \|\partial_t u\|_{H^1(\Omega)} \lesssim (\|g\|_{L^1(0,T;H^{-1/2}(\Gamma))} + \max_{t \in [t, T]} \|\partial_t g\|_{H^{-1/2}(\Gamma)} + \|\partial_t^2 g\|_{L^1(0,T;H^{-1/2}(\Gamma))}) \|\partial_t u\|_{H^1(\Omega),}
\]
which give the estimate (2.28) after applying Young’s inequality inequality.

2.3. **A priori estimates of one cavity problem.** In this section, we derive a priori estimates for the total field with a minimum regularity requirement for the data and an explicit dependence on the time.

The variation problem of (2.27) in time-domain is to find $u \in H^1_{\text{S}}(\Omega)$ for all $t > 0$ such that
\[
\int_\Omega (\varepsilon \partial_t^2 u) \bar{v} \,d\rho = -\int_\Omega \mu^{-1} \nabla u \cdot \nabla \bar{v} \,d\rho + \int_\Omega \mu^{-1}(\mathcal{S} u) \bar{v} \,d\gamma + \int_\Gamma g \bar{v} \,d\gamma, \quad \forall v \in H^1_{\text{S}}(\Omega). \tag{2.30}
\]
To show the stability of its solution, we follow the argument in [27] but with careful study of the TBC.

**Theorem 2.8.** Let $u \in H^1_{\text{S}}(\Omega)$ be the solution of (2.27). Given $g \in L^1(0,T;H^{-1/2}(\Gamma))$, we have for any $T > 0$ that
\[
\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|_{L^\infty(0,T;L^2(\Omega))} \lesssim T \|g\|_{L^1(0,T;H^{-1/2}(\Gamma))} + \|\partial_t g\|_{L^1(0,T;H^{-1/2}(\Gamma))}, \tag{2.31}
\]
and
\[
\|u\|_{L^2(0,T;L^2(\Omega))} + \|\nabla u\|_{L^2(0,T;L^2(\Omega))} \lesssim T^{3/2} \|g\|_{L^1(0,T;H^{-1/2}(\Gamma))} + T^{1/2} \|\partial_t g\|_{L^1(0,T;H^{-1/2}(\Gamma))}. \tag{2.32}
\]
Proof. Let \(0 < \theta < T\) and define an auxiliary function
\[
\psi_1(\rho, t) = \int_t^\theta u(\rho, \tau) d\tau, \quad \rho \in \Omega, \ 0 \leq t \leq \theta.
\]
It is clear that
\[
\psi_1(\rho, \theta) = 0, \quad \partial_t \psi_1(\rho, t) = -u(\rho, t).
\] (2.33)
For any \(\phi(\rho, t) \in L^2 (0, \theta; L^2(\Omega))\), we have
\[
\int_0^\theta \phi(\rho, t) \psi_1(\rho, t) dt = \int_0^\theta \left( \int_0^t \phi(\rho, \tau) d\tau \right) \bar{u}(\rho, t) dt.
\] (2.34)
Indeed, using integration by parts and (2.33), we have
\[
\int_0^\theta \phi(\rho, t) \psi_1(\rho, t) dt = \int_0^\theta \int_0^\tau \bar{u}(\rho, \tau) d\tau d\phi(\rho, \tau)
\[
= \int_0^\theta \bar{u}(\rho, \tau) d\tau \int_0^\tau \phi(\rho, \tau) d\tau + \int_0^\theta \left( \int_0^\tau \phi(\rho, \tau) d\tau \right) \bar{u}(\rho, t) dt
\[
= \int_0^\theta \left( \int_0^t \phi(\rho, \tau) d\tau \right) \bar{u}(\rho, t) dt.
\]
Next, we take the test function \(v = \psi_1\) in (2.30) and get
\[
\int_\Omega \varepsilon(\partial_t^2 u) \psi_1 d\rho = -\int_\Omega \mu^{-1} \nabla u \cdot \nabla \psi_1 d\rho + \int_\Gamma \mu^{-1} (\mathcal{T} u) \psi_1 d\gamma + \int_\Gamma g \bar{\psi}_1 d\gamma.
\] (2.35)
It follows from the facts in (2.33) and the initial conditions in (2.27) that
\[
\text{Re} \int_0^\theta \int_\Omega \varepsilon(\partial_t^2 u) \psi_1 d\rho dt = \text{Re} \int_\Omega \varepsilon \left( \partial_t u \partial_t \psi_1 \right)_0^\theta + \frac{1}{2} |u|_0^\theta d\rho d\rho
\[
= \frac{1}{2} \| \varepsilon^{1/2} u(\cdot, \theta) \|_{L^2(\Omega)}^2.
\]
Integrating (2.35) from \(t = 0\) to \(t = \theta\) and taking the real parts yields
\[
\frac{1}{2} \| \varepsilon^{1/2} u(\cdot, \theta) \|_{L^2(\Omega)}^2 + \text{Re} \int_0^\theta \int_\Omega \mu^{-1} \nabla u \cdot \nabla \bar{\psi}_1 d\rho dt
\[
= \frac{1}{2} \| \varepsilon^{1/2} u(\cdot, \theta) \|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega \mu^{-1} \left| \int_0^\theta \nabla u(\cdot, t) dt \right|^2 d\rho
\]
\[
= \text{Re} \int_0^\theta \mu^{-1} (\mathcal{T} u, \psi_1)_{\Gamma} dt + \text{Re} \int_0^\theta \int_\Gamma g \bar{\psi}_1 d\gamma dt.
\] (2.36)
In what follows, we estimate the two terms on the right-hand side of (2.36) separately.
By the property (2.34), we can obtain
\[
\text{Re} \int_0^\theta \mu^{-1} (\mathcal{T} u, \psi_1)_{\Gamma} dt = \text{Re} \int_0^\theta \int_0^t \left( \int_\Gamma \mu^{-1} (\mathcal{T} u(\cdot, \tau) d\gamma \right) d\tau \bar{u}(\cdot, t) dt.
\]
Let $\tilde{u}$ be the extension of $u$ with respect to $t$ in $\mathbb{R}$ such that $\tilde{u} = 0$ outside the interval $[0, \theta]$. We obtain from the Parseval identity and Lemma 2.3 that

$$
\Re \int_0^\theta e^{-2s_1 t} \int_0^t \left( \int_\Gamma \mu^{-1} \mathcal{T} u(\cdot, \tau) d\gamma \right) d\tau \tilde{u}(\cdot, t) dt = \Re \int_0^\theta \int_0^\infty e^{-2s_1 t} \left( \int_0^t \mu^{-1} \mathcal{T} \tilde{u}(\cdot, \tau) d\tau \right) \tilde{u}(\cdot, t) dt d\gamma
$$

$$= \Re \int_0^\theta \int_0^\infty e^{-2s_1 t} \left( \int_0^t \mathcal{L}^{-1} \circ \mu^{-1} \mathcal{B} \circ \mathcal{L} \tilde{u}(\cdot, \tau) d\tau \right) \tilde{u}(\cdot, t) d\gamma dt
$$

$$= \Re \int_0^\theta \int_0^\infty e^{-2s_1 t} \left( \int_0^t \mathcal{L}^{-1} \circ (s \mu)^{-1} \mathcal{B} \circ \mathcal{L} \tilde{u}(\cdot, t) \right) \tilde{u}(\cdot, t) d\gamma dt
$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \Re \langle (s \mu)^{-1} \mathcal{B} \tilde{u}, \tilde{u} \rangle d\gamma_2 \leq 0,$$

where we have used the fact that

$$
\int_0^t u(\tau) d\tau = \mathcal{L}^{-1}(s^{-1} \tilde{u}(s)).
$$

After taking $s_1 \to 0$, we obtain

$$
\Re \int_0^\theta \mu^{-1} (\mathcal{T} u, \psi_1) d\tau \leq 0. \quad (2.37)
$$

For $0 \leq t \leq \theta \leq T$, by (2.34) we have

$$
\Re \int_0^\theta \int_\Gamma g \tilde{\psi}_1 d\gamma dt = \Re \int_0^\theta \left( \int_0^t \int_\Gamma g(\cdot, \tau) d\gamma d\tau \right) \tilde{u}(\cdot, t) dt
$$

$$\leq \int_0^\theta \left( \int_0^t \| g(\cdot, \tau) \|_{H^{-1/2}(\Gamma)} d\tau \right) \| u(\cdot, t) \|_{H^{1/2}(\Gamma)} d\tau dt
$$

$$\leq \int_0^\theta \left( \int_0^t \| g(\cdot, \tau) \|_{H^{-1/2}(\Gamma)} d\tau \right) \| u(\cdot, t) \|_{H^1(\Omega)} d\tau dt
$$

$$\leq \left( \int_0^\theta \| g(\cdot, t) \|_{H^{-1/2}(\Gamma)} d\tau \right) \left( \int_0^\theta \| u(\cdot, t) \|_{H^1(\Omega)} d\tau d\tau \right). \quad (2.38)
$$

Combining (2.36)–(2.38), we have for any $\theta \in [0, T]$ that

$$
\frac{1}{2} \| e^{1/2} u(\cdot, t) \|^2_{L^2(\Omega)} \leq \frac{1}{2} \| e^{1/2} u(\cdot, t) \|^2_{L^2(\Omega)} + \frac{1}{2} \int_\Omega \mu^{-1} \left( \int_0^\theta \nabla u(\cdot, t) d\tau \right) d\rho
$$

$$\leq \left( \int_0^\theta \| g(\cdot, t) \|_{L^2(\Omega)} d\tau \right) \left( \int_0^\theta \| u(\cdot, t) \|_{L^2(\Omega)} d\tau d\tau \right). \quad (2.39)
$$

Taking the derivative of (2.27) with respect to $t$, we know that $\partial_t u$ satisfies the same equation with $g$ replaced by $\partial_t g$. Define

$$
\psi_2(\rho, t) = \int_t^\theta \partial_t u(\rho, \tau) d\tau, \quad \rho \in \Omega, \ 0 \leq t \leq \theta.
$$

We may follow the same steps as those for $\psi_1$ to obtain

$$
\frac{1}{2} \| e^{1/2} \partial_t u(\cdot, t) \|^2_{L^2(\Omega)} \leq \frac{1}{2} \| e^{1/2} \partial_t u(\cdot, t) \|^2_{L^2(\Omega)} + \frac{1}{2} \int_\Omega \mu^{-1} \left( \int_0^\theta \partial_t (\nabla u(\cdot, t)) d\tau \right) d\rho
$$

$$\leq \left( \int_0^\theta \| \partial_t g(\cdot, t) \|_{L^2(\Omega)} d\tau \right) \left( \int_0^\theta \| \partial_t u(\cdot, t) \|_{L^2(\Omega)} d\tau d\tau \right). \quad (2.40)
$$
By (2.33) and Lemma 2.5, we get

\[ \frac{1}{2} \int_\Omega \mu^{-1} \left| \int_0^\theta \partial_t (\nabla u(\cdot, t)) \, dt \right|^2 \, d\rho = \frac{1}{2} \mu^{-1/2} \nabla u(\cdot, \theta) \| L^2(\Omega). \] (2.41)

The first term on the right-hand side of (2.40) can be discussed as above, we only consider the second term. By (2.33) and Lemma 2.5 we get

\[ \int_0^\theta (\partial_t g) \psi_2 \, d\gamma dt = \int_\Gamma \left( \int_0^t \partial_t g(\cdot, \tau) \, d\tau \right) \tilde{u}(\cdot, t) \| \theta dt - \int_\Gamma (\partial_t g(\cdot, t)) u(\cdot, t) \| \theta dt \]

\[ \lesssim \int_0^\theta \| \partial_t g(\cdot, t) \|_{H^{-1/2}(\Gamma)} \| u(\cdot, t) \|_{H^{1/2}(\Gamma)} dt \]

\[ \lesssim \int_0^\theta \| \partial_t g(\cdot, t) \|_{H^{-1/2}(\Gamma)} \| u(\cdot, t) \|_{H^{1}(\Omega)} dt. \] (2.42)

Substituting (2.41)–(2.42) into (2.40), we have for any \( \theta \in [0, T] \) that

\[ \frac{1}{2} \| \varepsilon^{1/2} \partial_t u(\cdot, \theta) \|_{L^2(\Omega)}^2 + \frac{1}{2} \mu^{-1/2} \nabla u(\cdot, \theta) \|_{L^2(\Omega)}^2 \lesssim \int_0^\theta \| \partial_t g(\cdot, t) \|_{H^{-1/2}(\Gamma)} \| u(\cdot, t) \|_{H^{1}(\Omega)} dt. \] (2.43)

Combining the estimates (2.39) and (2.43), it follows

\[ \| u(\cdot, \theta) \|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 \lesssim \left( \int_0^\theta \| g(\cdot, t) \|_{H^{-1/2}(\Gamma)} dt \right) \left( \int_0^\theta \| u(\cdot, t) \|_{H^{1}(\Omega)} dt \right) 

+ \int_0^\theta \| \partial_t g(\cdot, t) \|_{H^{-1/2}(\Gamma)} \| u(\cdot, t) \|_{H^{1}(\Omega)} dt. \] (2.44)

Taking the \( L^\infty \)-norm with respect to \( \theta \) on both sides of (2.44) yields

\[ \| u \|_{L^\infty(0,T;L^2(\Omega))}^2 + \| \nabla u \|_{L^\infty(0,T;L^2(\Omega))}^2 \lesssim T \| g \|_{L^1(0,T;L^{-1/2}(\Gamma))} \| u \|_{L^\infty(0,T;H^1(\Omega))} + \| \partial_t g \|_{L^1(0,T;H^{-1/2}(\Gamma))} \| u \|_{L^\infty(0,T;H^1(\Omega))}, \]

which gives the estimate (2.31) after applying the Young’s inequality.

Integrating (2.44) with respect to \( \theta \) from 0 to \( T \) and using the Cauchy–Schwarz inequality, we obtain

\[ \| u \|_{L^2(0,T;L^2(\Omega))}^2 + \| \nabla u \|_{L^2(0,T;L^2(\Omega))}^2 \lesssim T^{3/2} \| g \|_{L^1(0,T;L^{-1/2}(\Gamma))} \| u \|_{L^2(0,T;H^1(\Omega))} 

+ T^{1/2} \| \partial_t g \|_{L^1(0,T;H^{-1/2}(\Gamma))} \| u \|_{L^2(0,T;H^1(\Omega))}, \]

which implies the estimate (2.32) by using Young’s inequality again.

3. TWO CAVITIES SCATTERING PROBLEM

In order to address the general multiple cavity scattering problem, in this section, we first give the discussion on the two cavity scattering problem. As it shows that the two cavity scattering problem shares the same features with the general multiple cavity scattering problem, but is easier to present the major ideas in the proof of the well-posedness and stability for the multiple cavity scattering problem.
3.1. **Problem formulation.** As shown in the Figure 2, two open cavities $\Omega_1$ and $\Omega_2$, enclosed by the apertures $\Gamma_1$ and $\Gamma_2$ and the walls $S_1$ and $S_2$, are placed on a perfectly conducting ground plane $\Gamma^c$. Above the flat surface $\Gamma^c \cup \Gamma_1 \cup \Gamma_2$, the medium is assumed to be homogeneous with positive dielectric permittivity $\varepsilon_0$ and magnetic permeability $\mu_0$. The medium inside the cavity $\Omega_1$ and $\Omega_2$ is inhomogeneous with a variable dielectric permittivity $\varepsilon_j(x,y)$, respectively and the same variable magnetic permeability $\mu(x,y)$. Assume further that $\varepsilon_j(x,y) \in L^\infty(\Omega_j)$ and $\mu(x,y) \in L^\infty(\Omega_j)$ and satisfy

$$0 < \varepsilon_{j,\text{min}} \leq \varepsilon_j \leq \varepsilon_{j,\text{max}} < \infty, \quad 0 < \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}} < \infty \quad \text{for } j = 1, 2.$$  

3.1.1. **Transparent boundary condition.** In TE polarization, the three-dimensional Maxwell equations can be reduced to the two-dimensional wave equation with initial-boundary value problem

$$\begin{cases}
\varepsilon \partial_t^2 u - \nabla \cdot (\mu^{-1} \nabla u) = 0 & \text{in } \Omega^e \cup \Omega_1 \cup \Omega_2, \ t > 0, \\
u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0 & \text{in } \Omega^e \cup \Omega_1 \cup \Omega_2, \\
u = 0 & \text{on } \Gamma^c \cup S_1 \cup S_2, \ t > 0.
\end{cases} \quad (3.1)$$

Let the plane wave $u^{\text{inc}}$ be incident on the cavities from above. Due to the interaction between the incident wave and the ground plane and the two cavity, it can be shown that the total field $u$ is composed of the incident field $u^{\text{inc}}$, the reflected field $u^r$ and the scattered field $u^{sc}$. The scattered field $u^{sc}$ is also required to satisfy the radiation condition (2.5).

To reduce the scattering problem from the open domain $\Omega^e \cup \Omega_1 \cup \Omega_2$ into the bounded domain, we need to derive transparent boundary conditions on the apertures $\Gamma_1$ and $\Gamma_2$. We want to reduce (3.1) into two single cavity scattering problem: for $j = 1, 2$,

$$\begin{cases}
\varepsilon_j \partial_t^2 u_j - \nabla \cdot (\mu^{-1} \nabla u_j) = 0 & \text{in } \Omega_j, \ t > 0, \\
u_j|_{t=0} = 0, \quad \partial_t u_j|_{t=0} = 0 & \text{in } \Omega_j, \\
u_j = 0 & \text{on } S_j, \ t > 0, \\
\partial_n u_j = \mathcal{T} u_j + g & \text{on } \Gamma_j, \ t > 0,
\end{cases} \quad (3.2)$$

where $\mathcal{T}$ is transparent boundary conditions in time-domain. Obviously, if $u$ is the solution of (3.1), then $u_j$ are solutions of (3.2). Moreover, it has $u|_{\Omega_j} = u_j$.

Due to the homogeneous medium in the upper half space $\Omega^e$ and the radiation condition (2.5), after taking the Laplace transform with respect to $t$, the scattered field $u^{sc}$ still satisfies the same ordinary differential equation (2.13). Thus, in $s$-domain and in the time-domain, the transparent boundary condition can be respectively written as

$$\partial_n \tilde{u} = \mathcal{B} \tilde{u} + \tilde{g}, \quad \partial_n u = \mathcal{T} u + g \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \Gamma_2. \quad (3.3)$$

For and $u_j(x,0)(j = 1, 2)$ defined on $\Gamma_j$, define the extension to the whole $x$-axis by

$$\tilde{u}_j(x,0) = \begin{cases} u_j(x,0) & \text{for } x \in \Gamma_j, \\
0 & \text{for } x \in \mathbb{R} \setminus \Gamma_j.
\end{cases}$$

For the total field $u(x,0)$, define its extension to the whole $x$-axis by

$$\tilde{u}(x,0) = \begin{cases} u_1(x,0) & \text{for } x \in \Gamma_1, \\
u_2(x,0) & \text{for } x \in \Gamma_2, \\
0 & \text{for } x \in \Gamma^c.
\end{cases}$$

It follows from the definitions of these extensions that

$$\tilde{u} = \tilde{u}_1 + \tilde{u}_2 \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \Gamma_2.$$  

The transparent boundary conditions (3.3) can be respectively written as

$$\partial_n \tilde{u} = \mathcal{B} \tilde{u} + \tilde{g} \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \Gamma_2; \quad \partial_n \tilde{u} = \mathcal{T} \tilde{u} + g \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \Gamma_2, \ t > 0.$$
These lead to the transparent boundary conditions for $u_j$ on $\Gamma_j$ in frequency domain and time-domain, respectively:

$$
\partial_n \tilde{u}_1 = \mathcal{R} \tilde{u}_1 + \mathcal{B} \tilde{u}_2 + \tilde{g} \quad \text{on } \Gamma_1; \quad \partial_n u_1 = \mathcal{I} \tilde{u}_1 + \mathcal{I} \tilde{u}_2 + g \quad \text{on } \Gamma_1, \; t > 0, \tag{3.4}
$$

and

$$
\partial_n \tilde{u}_2 = \mathcal{R} \tilde{u}_2 + \mathcal{B} \tilde{u}_1 + \tilde{g} \quad \text{on } \Gamma_2; \quad \partial_n u_2 = \mathcal{I} \tilde{u}_2 + \mathcal{I} \tilde{u}_1 + g \quad \text{on } \Gamma_2, \; t > 0. \tag{3.5}
$$

From (3.4) and (3.5), we find the boundary conditions for $u_1$ and $u_2$ are coupled with each other, which is the major difference between the single cavity scattering problem.

The following two lemmas are analogous to Lemmas 2.3,2.4 which will be used to analysis the uniqueness and existence for the solution of the two cavity scattering problem.

**Lemma 3.1.** It holds that

$$
-\Re \left( \left( (s \mu)^{-1} \mathcal{B} u, v \right)_{\Gamma_1} + \left( (s \mu)^{-1} \mathcal{B} v, u \right)_{\Gamma_1} + \left( (s \mu)^{-1} \mathcal{B} v, v \right)_{\Gamma_2} + \left( (s \mu)^{-1} \mathcal{B} u, v \right)_{\Gamma_2} \right) \geq 0, \quad u, v \in H^{1/2}(\mathbb{R}).
$$

**Proof.** Recalling $\beta^2(\xi) = \xi^2 + c^{-2} s^2$ and using (2.18), we get

$$
\begin{align*}
\Re \left( \left( (s \mu)^{-1} \mathcal{B} u, u \right)_{\Gamma_1} + \left( (s \mu)^{-1} \mathcal{B} v, u \right)_{\Gamma_1} + \left( (s \mu)^{-1} \mathcal{B} v, v \right)_{\Gamma_2} + \left( (s \mu)^{-1} \mathcal{B} u, v \right)_{\Gamma_2} \right) \\
= -\Re \int_{\mathbb{R}} \frac{1}{\mu |s|^2} \left( \beta(\xi)(|u|^2 + |v|^2 + \bar{v}u + u\bar{v}) \right) d\xi \\
= -\int_{\mathbb{R}} \frac{1}{\mu |s|^2} \left( \beta(\xi)(|u|^2 + |v|^2) \right) d\xi \leq 0,
\end{align*}
$$

where $\beta(\xi) = \xi + i\eta$ with $\eta < 0$. \hfill $\Box$

**Lemma 3.2.** For any $u(\cdot, t) \in L^2(0, T; H^{1/2}(\Gamma_1))$, $v(\cdot, t) \in L^2(0, T; H^{1/2}(\Gamma_2))$ with initial values $u(\cdot, 0) = 0, v(\cdot, 0) = 0$, denote their zero extension on $L^2(0, T; H^{1/2}(\mathbb{R}))$ by $\tilde{u}(\cdot, t)$ and $\tilde{v}(\cdot, t)$, respectively. Then, it holds that

$$
-\Re \int_0^T \left( \left( \mathcal{J} \tilde{u}, \partial_t \tilde{u} \right)_{\Gamma_1} + \left( \mathcal{J} \tilde{v}, \partial_t \tilde{v} \right)_{\Gamma_1} + \left( \mathcal{J} \tilde{v}, \partial_t \tilde{v} \right)_{\Gamma_2} + \left( \mathcal{J} \tilde{u}, \partial_t \tilde{v} \right)_{\Gamma_2} \right) dt \geq 0.
$$

**Proof.** Let $\tilde{u}(\cdot, t), \tilde{v}(\cdot, t)$ be the extension of $u(\cdot, t), v(\cdot, t)$ with respect to $t$ in $\mathbb{R}$ such that $\tilde{u}(\cdot, t) = 0, \tilde{v}(\cdot, t) = 0$ outside the interval $[0, T]$, and $\tilde{\tilde{u}} = \mathcal{L}(\tilde{u}), \tilde{\tilde{v}} = \mathcal{L}(\tilde{v})$ be the Laplace transform of $\tilde{u}, \tilde{v}$. By the Parseval identity (2.6), we get

$$
\begin{align*}
&-\Re \int_0^T e^{-2s_1 t} \left( \left( \mathcal{J} \tilde{u}, \partial_t \tilde{u} \right)_{\Gamma_1} + \left( \mathcal{J} \tilde{v}, \partial_t \tilde{v} \right)_{\Gamma_1} + \left( \mathcal{J} \tilde{v}, \partial_t \tilde{v} \right)_{\Gamma_2} + \left( \mathcal{J} \tilde{u}, \partial_t \tilde{v} \right)_{\Gamma_2} \right) dt \\
&= -\Re \sum_{j=1}^{\infty} \left( \int_{\Gamma_j} \int_0^\infty e^{-2s_1 t} \left( \mathcal{J} \tilde{u}, \partial_t \tilde{u} \right)_{\Gamma_j} dt \mathcal{J}_j + \int_{\Gamma_j} \int_0^\infty e^{-2s_1 t} \left( \mathcal{J} \tilde{v}, \partial_t \tilde{v} \right)_{\Gamma_j} dt \mathcal{J}_j \right) \\
&= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left( \left( \mathcal{B} \tilde{\tilde{u}}, \bar{s} \tilde{\tilde{u}} \right)_{\Gamma_1} + \left( \mathcal{B} \tilde{\tilde{v}}, \bar{s} \tilde{\tilde{v}} \right)_{\Gamma_1} + \left( \mathcal{B} \tilde{\tilde{v}}, \bar{s} \tilde{\tilde{v}} \right)_{\Gamma_2} + \left( \mathcal{B} \tilde{\tilde{u}}, \bar{s} \tilde{\tilde{v}} \right)_{\Gamma_2} \right) ds_2 \\
&= -\frac{1}{2\pi} \int_{-\infty}^{\infty} |s|^2 \Re \left( \left( s^{-1} \mathcal{B} \tilde{\tilde{u}}, \bar{s} \tilde{\tilde{u}} \right)_{\Gamma_1} + \left( s^{-1} \mathcal{B} \tilde{\tilde{v}}, \bar{s} \tilde{\tilde{v}} \right)_{\Gamma_1} + \left( s^{-1} \mathcal{B} \tilde{\tilde{v}}, \bar{s} \tilde{\tilde{v}} \right)_{\Gamma_2} + \left( s^{-1} \mathcal{B} \tilde{\tilde{u}}, \bar{s} \tilde{\tilde{v}} \right)_{\Gamma_2} \right) ds_2.
\end{align*}
$$

It follows from Lemma 3.1 and $\beta(\xi) = \xi + i\eta$ with $\eta < 0$ that

$$
-\Re \int_0^T e^{-2s_1 t} \left( \left( \mathcal{J} \tilde{u}, \partial_t \tilde{u} \right)_{\Gamma_1} + \left( \mathcal{J} \tilde{v}, \partial_t \tilde{v} \right)_{\Gamma_1} + \left( \mathcal{J} \tilde{v}, \partial_t \tilde{v} \right)_{\Gamma_2} + \left( \mathcal{J} \tilde{u}, \partial_t \tilde{v} \right)_{\Gamma_2} \right) dt \geq 0
$$

which completes the proof after taking $s_1 \to 0$. \hfill $\Box$
3.2. The reduced two cavity scattering problem. In this section, we will discuss the well-posedness and stability for the reduced problem of the two cavity scattering problem. Firstly, we denote \( \Omega = \Omega_1 \cup \Omega_2 \), \( \Gamma = \Gamma_1 \cup \Gamma_2 \), and \( S = S_1 \cup S_2 \). Let

\[
  u = \begin{cases} 
  u_1 & \text{in } \Omega_1, \\
  u_2 & \text{in } \Omega_2.
\end{cases}
\]

Define a trace functional space

\[
  \tilde{H}^{1/2}(\Gamma) = \tilde{H}^{1/2}(\Gamma_1) \times \tilde{H}^{1/2}(\Gamma_2),
\]

whose norm is characterized by \( \| u \|^2_{\tilde{H}^{1/2}(\Gamma)} = \| u_1 \|^2_{\tilde{H}^{1/2}(\Gamma_1)} + \| u_2 \|^2_{\tilde{H}^{1/2}(\Gamma_2)} \). Denote by

\[
  H^{-1/2}(\Gamma) = H^{-1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_2),
\]

which is the dual space of \( \tilde{H}^{1/2}(\Gamma) \). The norm on the space \( H^{-1/2}(\Gamma) \) is characterized by

\[
  \| u \|^2_{H^{-1/2}(\Gamma)} = \| u_1 \|^2_{H^{-1/2}(\Gamma_1)} + \| u_2 \|^2_{H^{-1/2}(\Gamma_2)}.
\]

Define the space

\[
  H^1_S(\Omega) = H^1_{S_1}(\Omega_1) \times H^1_{S_2}(\Omega_2),
\]

which is a Hilbert space with norm characterized by \( \| u \|^2_{H^1_S(\Omega)} = \| u_1 \|^2_{H^1_{S_1}(\Omega_1)} + \| u_2 \|^2_{H^1_{S_2}(\Omega_2)} \).

3.2.1. well-posedness in the s-domain. Now we present a variational formulation for the two cavity scattering problem. For \( j = 1, 2 \), taking the Laplace transform of (3.2), we get

\[
  \begin{cases} 
  \varepsilon_j s \tilde{u}_j - \nabla \cdot \left( s^{-1} \mu^{-1} \nabla \tilde{u}_j \right) = 0 & \text{in } \Omega_j, \\
  \tilde{u}_j = 0 & \text{on } S_j, \\
  \partial_n \tilde{u}_j = \mathcal{B} \tilde{u}_j + \tilde{g} & \text{on } \Gamma_j. 
\end{cases}
\]  

(3.6)

Multiplying the complex conjugate of test function \( v_j \in H^1_{S_j}(\Omega_j) \), \( j = 1, 2 \) on both sides of the first equation of (3.6), integrating over \( \Omega_j \), we have

\[
  \int_{\Omega_j} (s\mu)^{-1} \nabla \tilde{u}_j \nabla \tilde{v}_j + s\varepsilon_j \tilde{u}_j \tilde{v}_j \, d\rho - \sum_{i=1}^{2} \langle (s\mu)^{-1} \mathcal{B} \tilde{u}_i, \tilde{v}_j \rangle_{\Gamma_j} = \langle \tilde{g}, v_j \rangle_{\Gamma_j}.
\]

We deduce the variational formulation for the two cavity scattering problem: find \( \tilde{u} \in H^1_S(\Omega) \) with \( \tilde{u}|_{\Omega_j} = \tilde{u}_1 \in H^1_{S_1}(\Omega_j) \), such that for all \( v \in H^1_S(\Omega) \) with \( v_j = v|_{\Omega_j} \in H^1_{S_1}(\Omega_j) \), it holds

\[
  a_2(\tilde{u}, v) = \langle \tilde{g}, v_1 \rangle_{\Gamma_1} + \langle \tilde{g}, v_2 \rangle_{\Gamma_2},
\]

(3.7)

where the sesquilinear form

\[
  a_2(\tilde{u}, v) = \sum_{j=1}^{2} \int_{\Omega_j} (s\mu)^{-1} \nabla \tilde{u}_j \nabla \tilde{v}_j + s\varepsilon_j \tilde{u}_j \tilde{v}_j \, d\rho - \sum_{j=1}^{2} \sum_{i=1}^{2} \langle (s\mu)^{-1} \mathcal{B} \tilde{u}_i, \tilde{v}_j \rangle_{\Gamma_j}.
\]

Theorem 3.3. The variational problem (3.7) has a unique solution \( \tilde{u} \in H^1_S(\Omega) \) which satisfies

\[
  \| \nabla \tilde{u} \|_{L^2(\Omega)^2} + \| s\tilde{u} \|_{L^2(\Omega)} \lesssim s_1^{-1} \| s\tilde{g} \|_{H^{-1/2}(\Gamma)}.
\]

(3.8)
Proof. The continuity of the sesquilinear follows directly from the Cauchy–Schwarz inequality, Lemma 2.2 and Lemma 2.5

\[ |a_2(\tilde{u}, v)| \leq \sum_{j=1}^{2} \left( \frac{1}{s|\mu_{\text{min}}|} \|\nabla \tilde{u}_j\|_{L^2(\Omega_j)^2} \|\nabla v_j\|_{L^2(\Omega_j)^2} + \frac{s|\varepsilon_{\text{max}}|}{2} \|\tilde{u}_j\|_{L^2(\Omega_j)} \|v_j\|_{L^2(\Omega_j)} \right) \]

\[ + \frac{1}{s|\mu_{\text{min}}|} \sum_{j=1}^{2} \sum_{i=1}^{2} \|\mathcal{B} \tilde{u}_i\|_{H^{-1/2}(\Gamma_j)} \|v_j\|_{H^{1/2}(\Gamma_j)} \]

\[ \leq \|\nabla \tilde{u}\|_{L^2(\Omega)^2} \|\nabla v\|_{L^2(\Omega)^2} + \|\tilde{u}\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|\tilde{u}\|_{H^{1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} \]

\[ \lesssim \|\tilde{u}\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \]

where \( \varepsilon_{\text{max}} = \max\{\varepsilon_1, \varepsilon_2\} \). It suffices to show the coercivity of \( a_2(\tilde{u}, v) \). A simple calculation yields

\[ a_2(\tilde{u}, \tilde{u}) = \sum_{j=1}^{2} \int_{\Omega_j} (s\mu)^{-1} |\nabla \tilde{u}_j|^2 + s\varepsilon_j |\tilde{u}_j|^2 d\rho_j - \sum_{j=1}^{2} \sum_{i=1}^{2} \langle (s\mu)^{-1} \mathcal{B} \tilde{u}_i, \tilde{u}_j \rangle_{\Gamma_j}. \tag{3.9} \]

Taking the real part of (3.9) and using Lemma 3.1, we get

\[ \text{Re}(a_2(\tilde{u}, \tilde{u})) \geq C_1 \frac{\tilde{s}_1}{|s|^2} \left( \|\nabla \tilde{u}\|_{L^2(\Omega)^2}^2 + \|\tilde{s}\tilde{u}\|_{L^2(\Omega)}^2 \right), \tag{3.10} \]

where \( C_1 = \min\{\mu_{\text{max}}^{-1}, 1\} \).

It follows from the Lax–Milgram lemma that the variational problem (3.7) has a unique solution \( \tilde{u} \in H^1_S(\Omega) \) and satisfies \( u|_{\Omega_j} = u_i \). Moreover, we have from (3.7) that

\[ |a_2(\tilde{u}, \tilde{u})| \leq |s|^{-1} \|\tilde{g}\|_{H^{-1/2}(\Gamma)} \|\tilde{s}\tilde{u}\|_{L^2(\Omega)}. \tag{3.11} \]

Combining (3.10)–(3.11) leads to

\[ \|\nabla \tilde{u}\|_{L^2(\Omega)^2}^2 + \|\tilde{s}\tilde{u}\|_{L^2(\Omega)}^2 \lesssim s_1^{-1} \|\tilde{g}\|_{H^{-1/2}(\Gamma)} \|\tilde{s}\tilde{u}\|_{L^2(\Omega)}, \]

which completes the proof of estimates of (3.8) after applying the Cauchy–Schwarz inequality.

3.2.2. well-posedness in the time-domain. Using the time-domain transparent boundary conditions (3.4)–(3.5), problem (3.1) can be equivalently reduced to the initial-boundary value problem

\[ \begin{cases}
\varepsilon \partial_t^2 u - \nabla \cdot (\mu^{-1} \nabla u) = 0 & \text{in } \Omega, \ t > 0, \\
u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0 & \text{in } \Omega, \\
u = 0 & \text{on } S, \ t > 0, \\
\partial_n u = \mathcal{F} u + g & \text{on } \Gamma, \ t > 0.
\end{cases} \tag{3.12} \]

**Theorem 3.4.** The initial-boundary problem (3.12) has a unique solution \( u \), which satisfies

\[ u \in L^2(0, T; H^1_S(\Omega)) \cap H^1(0, T; L^2(\Omega)), \]

and the stability estimate

\[ \max_{t \in [0, T]} (\|\partial_t u\|_{L^2(\Omega)} + \|\partial_t (\nabla u)\|_{L^2(\Omega)^2}) \]

\[ \lesssim \left( \|g\|_{L^1(0, T; H^{-1/2}(\Gamma))} + \max_{t \in [0, T]} \|\partial_t g\|_{H^{-1/2}(\Gamma)} + \|\partial_t^2 g\|_{L^1(0, T; H^{-1/2}(\Gamma))} \right). \tag{3.13} \]

**Proof.** Using the similar way as one cavity scattering problem, we can get

\[ u \in L^2(0, T; H^1_S(\Omega)) \cap H^1(0, T; L^2(\Omega)). \]

Next, we prove the stability. For any \( 0 < t < T \), define the energy function

\[ e_3(t) = \|\varepsilon^{1/2} \partial_t u(\cdot, t)\|_{L^2(\Omega)}^2 + \|\mu^{-1/2} \nabla u(\cdot, t)\|_{L^2(\Omega)^2}^2. \tag{3.14} \]
Integrating by parts, it follows from (3.12) that
\[ \int_0^t e_3'(\tau) d\tau = e_3(t) - e_3(0) \]
\[ = 2\text{Re} \int_0^t \int_{\Omega_1} (\varepsilon(\partial_t^2 u_1) \partial_t \bar{u}_1 + \mu^{-1}(\nabla \partial_t u_1) \cdot \nabla \bar{u}_1) d\rho d\tau \]
\[ + 2\text{Re} \int_0^t \int_{\Omega_2} (\varepsilon(\partial_t^2 u_2) \partial_t \bar{u}_2 + \mu^{-1}(\nabla \partial_t u_2) \cdot \nabla \bar{u}_2) d\rho d\tau \]
\[ = \sum_{j=1}^2 2\text{Re} \int_0^t \int_{\Omega_j} (\nabla \cdot (\mu^{-1} \nabla u_j)) \partial_t \bar{u}_j + \mu^{-1}(\nabla \partial_t u_j) \cdot \nabla \bar{u}_j) d\rho d\tau. \]
Since $e_3(0) = 0$, we obtain from Lemma 3.2 that
\[ e_3(t) = \int_0^t e_3'(\tau) d\tau \]
\[ = 2\text{Re} \int_0^t \int_{\Omega_1} (-\mu^{-1} \nabla u_1 \cdot (\nabla \partial_t \bar{u}_1) + \mu^{-1}(\nabla \partial_t u_1) \cdot \nabla \bar{u}_1) d\rho d\tau + 2\text{Re} \int_0^t \int_{\Gamma_1} \mu^{-1}(\partial_t u_1) \partial_t \bar{u}_1 d\gamma_1 dt \]
\[ + 2\text{Re} \int_0^t \int_{\Omega_2} (-\mu^{-1} \nabla u_2 \cdot (\nabla \partial_t \bar{u}_2) + \mu^{-1}(\nabla \partial_t u_2) \cdot \nabla \bar{u}_2) d\rho d\tau + 2\text{Re} \int_0^t \int_{\Gamma_2} \mu^{-1}(\partial_t u_2) \partial_t \bar{u}_2 d\gamma_2 dt \]
\[ = 2\text{Re} \int_0^t \int_{\Gamma_1} (\langle F u_1, \partial_t u_1 \rangle_{\Gamma_1} + \langle F u_2, \partial_t u_1 \rangle_{\Gamma_1}) dt + 2\text{Re} \int_0^t \int_{\Gamma_2} (\langle g, \partial_t u_1 \rangle_{\Gamma_1} + \langle g, \partial_t u_1 \rangle_{\Gamma_2}) dt \]
\[ + 2\text{Re} \int_0^t \int_{\Gamma_1} \langle g \parallel H^{-1/2}(\Gamma_1) \parallel_{\Gamma_1} \parallel \partial_t u_1 \parallel_{H^1(\Omega)} \rangle dt \]
\[ + 2\text{Re} \int_0^t \int_{\Gamma_2} \langle g \parallel H^{-1/2}(\Gamma_2) \parallel_{\Gamma_1} \parallel \partial_t u_1 \parallel_{H^1(\Omega)} \rangle dt \]
\[ \leq 2(\max_{t \in [t,T]} \parallel \partial_t u \parallel_{H^1(\Omega)}) \parallel g \parallel L^1(0,T;H^{-1/2}(\Gamma)). \quad (3.15) \]
In order to give the estimate of $\parallel \partial_t (\nabla u) \parallel_{L^2(\Omega)^2}$, taking the derivative of (2.27) with respect to $t$. We find that \( \partial_t u \) also satisfies the same equations with $g$ replaced by $\partial_t g$. Hence consider
\[ e_4(t) = \parallel \varepsilon^{1/2} \partial_t^2 u(\cdot,t) \parallel_{L^2(\Omega)}^2 + \parallel \mu^{-1/2} \partial_t (\nabla u(\cdot,t)) \parallel_{L^2(\Omega)^2}^2, \quad (3.16) \]
and get the estimate
\[ e_4(t) \leq 2\text{Re} \left( \int_0^t \int_{\Gamma_1} (\partial_t g) \partial_t^2 \bar{u}_1 d\gamma_1 dt + \int_0^t \int_{\Gamma_2} (\partial_t g) \partial_t^2 \bar{u}_2 d\gamma_2 dt \right) \]
\[ = 2\text{Re} \int_{\Gamma_1} (\partial_t g) \partial_t \bar{u}_1 \parallel_0^t d\gamma_1 - 2\text{Re} \int_{\Gamma_1} (\partial_t^2 g) \partial_t \bar{u}_1 d\gamma_1 dt \]
\[ + 2\text{Re} \int_{\Gamma_2} (\partial_t g) \partial_t \bar{u}_2 \parallel_0^t d\gamma_2 - 2\text{Re} \int_{\Gamma_2} (\partial_t^2 g) \partial_t \bar{u}_2 d\gamma_2 dt \]
\[ \leq 2 \max_{t \in [t,T]} \parallel \partial_t u \parallel_{H^1(\Omega)} \left( \max_{t \in [t,T]} \parallel \partial_t g \parallel_{H^{-1/2}(\Gamma)} + \parallel \partial_t^2 g \parallel_{L^1(0,T;H^{-1/2}(\Gamma))} \right). \quad (3.17) \]
Combining the above estimates (3.14)–(3.17), we can obtain
\[ \max_{t \in [t,T]} \parallel \partial_t u \parallel_{H^1(\Omega)} \leq \left( \parallel g \parallel_{L^1(0,T;H^{-1/2}(\Gamma))} + \max_{t \in [t,T]} \parallel \partial_t g \parallel_{H^{-1/2}(\Gamma)} + \parallel \partial_t^2 g \parallel_{L^1(0,T;H^{-1/2}(\Gamma))} \right) \parallel \partial_t u \parallel_{H^1(\Omega)}, \]
which give the estimate \((3.13)\) after applying Young’s inequality.

\[\square\]

### 3.3. A priori estimates of the two cavity problem.

In this section, for the two cavity scattering problem, we derive a priori estimates for the total field with a minimum regularity requirement for the data and an explicit dependence on the time.

The variation problem of \((3.2)\) in time-domain is to find \(u_j \in H^1_{S_j}(\Omega_j), j = 1, 2\) for all \(t > 0\) such that for all \(v_j \in H^1_{S_j}(\Omega_j)\)

\[
\sum_{j=1}^2 \int_{\Omega_j} \varepsilon_j (\partial_t^2 u_j) \overline{v}_j \, d\rho = - \int_{\Omega_j} \mu^{-1} \nabla u_j \cdot \nabla \overline{v}_j \, d\rho + \sum_{i=1}^2 \int_{\Gamma_j} \mu^{-1}(\mathcal{F} u_i) \overline{v}_j \, d\gamma_j + \int_{\Gamma_j} g \overline{v}_j \, d\gamma_j. \tag{3.18}
\]

This is equivalent to find \(u \in H^1_S(\Omega)\) with \(u|_{\Omega_j} = u_j \in H^1_{S_j}(\Omega_j)\), such that for all \(v \in H^1_S(\Omega)\) with \(v_j = v|_{\Omega_j} \in H^1_{S_j}(\Omega_j)\), it holds

\[c_1(u, v) = (g, v_1)_\Gamma + (g, v_2)_\Gamma,\]

where the sesquilinear form

\[c_1(u, v) = \sum_{j=1}^2 \left( \int_{\Omega_j} \varepsilon_j (\partial_t^2 u_j) \overline{v}_j \, d\rho + \int_{\Omega_j} \mu^{-1} \nabla u_j \cdot \nabla \overline{v}_j \, d\rho \right) - \sum_{j=1}^2 \int_{\Gamma_j} \mu^{-1}(\mathcal{F} u_i) \overline{v}_j \, d\gamma_j.\]

**Theorem 3.5.** Let \(u \in H^1_S(\Omega)\) be the solution of \((3.12)\). Given \(g \in L^1(0, T; H^{-1/2}(\Gamma))\), we have for any \(T > 0\) that

\[
\|u\|_{L^\infty(0, T; L^2(\Omega))} + \|\nabla u\|_{L^\infty(0, T; L^2(\Omega)^2)} \leq T \|g\|_{L^1(0, T; H^{-1/2}(\Gamma))} + \|\partial_t g\|_{L^1(0, T; H^{-1/2}(\Gamma))}, \tag{3.19}
\]

and

\[
\|u\|_{L^2(0, T; L^2(\Omega))} + \|\nabla u\|_{L^2(0, T; L^2(\Omega)^2)} \leq T^{3/2} \|g\|_{L^1(0, T; H^{-1/2}(\Gamma))} + T^{1/2} \|\partial_t g\|_{L^1(0, T; H^{-1/2}(\Gamma))}. \tag{3.20}
\]

**Proof.** Define the test function \(\psi_1\) as in the proof of Theorem 2.8. Denote by \(\psi_1^{(1)} := \psi_1|_{\Omega_1}\) and \(\psi_1^{(2)} := \psi_1|_{\Omega_2}\). Taking the test functions \(v_j = \psi_1^{(j)}\) in \((3.18)\), we can obtain

\[
\sum_{j=1}^2 \int_{\Omega_j} \varepsilon_j (\partial_t^2 u_j) \overline{\psi}_1^{(j)} \, d\rho = - \sum_{j=1}^2 \int_{\Omega_j} \mu^{-1} \nabla u_j \cdot \nabla \overline{\psi}_1^{(j)} \, d\rho + \sum_{j=1}^2 \int_{\Gamma_j} \mu^{-1}(\mathcal{F} u_i) \overline{\psi}_1^{(j)} \, d\gamma_j + \sum_{j=1}^2 \int_{\Gamma_j} g \overline{\psi}_1^{(j)} \, d\gamma_j. \tag{3.21}
\]

It follows from the facts in \((2.33)\) and the initial conditions in \((3.2)\) that

\[
\text{Re} \sum_{j=1}^2 \int_0^\theta \int_{\Omega_j} \varepsilon_j (\partial_t^2 u_j(\cdot, \theta)) \overline{\psi}_1^{(j)} \, d\rho \, dt = \text{Re} \sum_{j=1}^2 \int_0^\theta \int_{\Omega_j} \left( (\partial_t u_j) \overline{\psi}_1^{(j)} |_0^\theta + \frac{1}{2} |u_j|_0^\theta \right) \, d\rho \, dt = \sum_{j=1}^2 \int_0^\theta \int_{\Omega_j} (\partial_t u_j(\cdot, \theta))^2 \, d\rho \, dt.
\]

Integrating \((3.21)\) from \(t = 0\) to \(t = \theta\) and taking the real parts yields

\[
= \sum_{j=1}^2 \int_0^\theta \int_{\Omega_j} (\mathcal{F} u_i, \psi_1^{(j)})_\Gamma \, dt + \int_0^\theta \int_{\Gamma_j} g \overline{\psi}_1^{(j)} \, d\gamma_j \, dt. \tag{3.22}
\]
In the following, we estimate the two terms on the right-hand side of (3.22) separately. It follows from Lemma 3.2 that
\[
\text{Re} \left( \int_0^\theta \mu^{-1} (\mathcal{L} u_1, \psi_1^{(1)})_{\Gamma_1} + \mu^{-1} (\mathcal{L} u_2, \psi_1^{(1)})_{\Gamma_1} \, dt \right) \\
+ \int_0^\theta \mu^{-1} (\mathcal{L} u_2, \psi_1^{(2)})_{\Gamma_2} + \mu^{-1} (\mathcal{L} u_1, \psi_1^{(2)})_{\Gamma_2} \, dt \leq 0. \tag{3.23}
\]
For \(0 \leq t \leq \theta \leq T\), by the fact in (2.34), we have
\[
\text{Re} \sum_{j=1}^2 \int_0^\theta \int_{\Gamma_j} g \psi_j^{(j)} d\gamma_j \, dt \leq \sum_{j=1}^2 \left( \int_0^\theta \|g(\cdot, t)\|_{H^{-1/2}(\Gamma_j)} \, dt \right) \left( \int_0^\theta \|u_j(\cdot, t)\|_{H^{1/2}(\Gamma_j)} \, dt \right) \\
\leq \left( \int_0^\theta \|g(\cdot, t)\|_{H^{-1/2}(\Gamma)} \, dt \right) \left( \int_0^\theta \|u(\cdot, t)\|_{H^{1/2}(\Gamma)} \, dt \right). \tag{3.24}
\]
Combining (3.23)–(3.24), we have for any \(\theta \in [0, T]\) that
\[
\frac{1}{2} \sum_{j=1}^2 \|e_{j/2} u_j(\cdot, \theta)\|_{L^2(\Omega_j)}^2 \leq \left( \int_0^\theta \|g(\cdot, t)\|_{L^2(\Omega)} \, dt \right) \left( \int_0^\theta \|u(\cdot, t)\|_{L^2(\Omega)} \, dt \right). \tag{3.25}
\]
Taking the derivative of (3.12) with respect to \(t\), we know that \(\partial_t u\) satisfies the same equation with \(g\) replaced by \(\partial_t g\). In similar way, define
\[
\psi_2(\rho, t) = \int_t^\theta \partial_t u_i(\rho, \tau) \, d\tau, \quad \rho \in \Omega, \ 0 \leq t \leq \theta, \ i = 1, 2,
\]
and denote by \(\psi_2^{(1)} = \psi_2|_{\Omega_1}\) and \(\psi_2^{(2)} = \psi_2|_{\Omega_2}\). It follows the same step as above
\[
\sum_{j=1}^2 \frac{1}{2} \left( \int_{\Omega_j} \|e_{j/2} \partial_t u_j(\cdot, \theta)\|_{L^2(\Omega_j)}^2 \, d\rho \right) + \int_{\Omega_j} \mu^{-1} \left| \int_0^\theta \partial_t (\nabla u_j(\cdot, t)) \, dt \right|^2 \, d\rho \\
= \text{Re} \sum_{j=1}^2 \left( \int_0^\theta \mu^{-1} (\mathcal{L} \partial_t u_i, \psi_2^{(j)})_{\Gamma_j} \, dt \right) + \int_0^\theta \int_{\Gamma_j} g \psi_2^{(j)} d\gamma_j \, dt. \tag{3.26}
\]
Integrating by parts yields
\[
\frac{1}{2} \int_{\Omega_j} \mu^{-1} \left| \int_0^\theta \partial_t (\nabla u_j(\cdot, t)) \, dt \right|^2 \, d\rho = \frac{1}{2} \|\mu^{-1/2} \nabla u_j(\cdot, \theta)\|_{L^2(\Omega_j)}^2, \ j = 1, 2. \tag{3.27}
\]
The estimate of the first term on the right-hand side of (3.26) can be discussed similarly as above, we only consider the second term. By the fact in (2.33) and Lemma 2.5, we get
\[
\sum_{j=1}^2 \int_0^\theta \int_{\Gamma_j} g \psi_2^{(j)} d\gamma_j \, dt = \sum_{j=1}^2 \left( \int_0^\theta \int_{\Omega_j} \left( \int_0^t \partial_t g(\cdot, \tau) \, d\tau \right) \tilde{u}_j(\cdot, \tau) \, d\gamma_j - \int_0^\theta \int_{\Gamma_j} \partial_t g(\cdot, t) u_j(\cdot, t) \, d\gamma_j \, dt \right) \\
\leq \int_0^\theta \|\partial_t g(\cdot, t)\|_{H^{-1/2}(\Gamma)} \|u(\cdot, t)\|_{H^{1}(\Omega)} \, dt. \tag{3.28}
\]
Substituting (3.27)–(3.28) into (3.26), we have for any \(\theta \in [0, T]\) that
\[
\frac{1}{2} \sum_{j=1}^2 \left( \|e_{j/2} \partial_t u_j(\cdot, \theta)\|_{L^2(\Omega_j)} + \|\mu^{-1/2} \nabla u_j(\cdot, \theta)\|_{L^2(\Omega_j)^2} \right) \lesssim \int_0^\theta \|\partial_t g(\cdot, t)\|_{H^{-1/2}(\Gamma)} \|u(\cdot, t)\|_{H^{1}(\Omega_j)} \, dt. \tag{3.29}
\]
Consider the similar model of the wave equation for the total field:

\[ \Delta u - \text{grad} \cdot (\mu^{-1} \text{grad} u) = 0 \]

Above the flat surface \( \Gamma \), the total field \( u \) is assumed to consist of the incident field \( u^{\text{inc}} \), the reflected field \( u^{r} \), and the scattered field \( u^{\text{sc}} \), where the scattered field is required to satisfy the radiation condition \( u^{\text{sc}} \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow \infty \).

Figure 3. The problem geometry of the multiple cavity.

Combining the estimates (3.25) and (3.29), we obtain

\[
\| u(x, t) \|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 \lesssim \left( \int_0^\theta \| g(\cdot, t) \|_{H^{-1/2}(\Gamma)} dt \right) \left( \int_0^\theta \| u(\cdot, t) \|_{H^1(\Omega)} dt \right) + \int_0^\theta \| \partial_t g(\cdot, t) \|_{H^{-1/2}(\Gamma)} \| u(\cdot, t) \|_{H^1(\Omega)} dt.
\]

Taking the \( L^\infty \)-norm with respect to \( \theta \) on both sides of (3.30) yields

\[
\| u \|_{L^\infty(0, T; L^2(\Omega))}^2 + \| \nabla u \|_{L^2(0, T; L^2(\Omega))}^2 \lesssim T \| g \|_{L^1(0, T; H^{-1/2}(\Gamma))} \| u \|_{L^\infty(0, T; H^1(\Omega))} + \int_0^\theta \| \partial_t g \|_{L^1(0, T; H^{-1/2}(\Gamma))} \| u \|_{L^\infty(0, T; H^1(\Omega))},
\]

which gives the estimate (3.19) after applying the Young’s inequality.

Integrating (3.30) with respect to \( \theta \) from 0 to \( T \) and using the Cauchy–Schwarz inequality, we obtain

\[
\| u \|_{L^2(0, T; L^2(\Omega))}^2 + \| \nabla u \|_{L^2(0, T; L^2(\Omega))}^2 \lesssim T^{3/2} \| g \|_{L^1(0, T; H^{-1/2}(\Gamma))} \| u \|_{L^2(0, T; H^1(\Omega))} + \int_0^\theta \| \partial_t g \|_{L^1(0, T; H^{-1/2}(\Gamma))} \| u \|_{L^2(0, T; H^1(\Omega))},
\]

which implies the estimate (3.20) by using Young’s inequality again.

4. MULTIPLE CAVITIES SCATTERING PROBLEM

In this section, we generalize the model problem and techniques to the case of multiple cavity scattering. The proofs and results are analogous to those for the two cavity scattering problem. For completes, we briefly state the results and give the results.

4.1. Problem formulation. As shown in the Figure 3, the \( n \)-multiple open cavities \( \Omega_1, \Omega_2, \ldots, \Omega_n \) are placed on a perfectly conducting ground plane \( \Gamma_c \), with apertures \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \) and walls \( S_1, S_2, \ldots, S_n \). Above the flat surface \( \{ y = 0 \} = \Gamma_c \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n \), the medium is assumed to be homogeneous with the positive dielectric permittivity \( \varepsilon_0 \) and magnetic permeability \( \mu_0 \). The medium inside the cavity \( \Omega_i \) is inhomogenous with the variable dielectric permittivity \( \varepsilon_j(x, y) \) and the same magnetic permeability \( \mu(x, y) \).

Assume further that \( \varepsilon_j(x, y), \mu(x, y) \in L^\infty(\Omega_j) \) are positive for \( j = 1, 2, \ldots, n \), and satisfy

\[ 0 < \varepsilon_{j, \text{min}} \leq \varepsilon_j \leq \varepsilon_{j, \text{max}} < \infty, \quad 0 < \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}} < \infty. \]

Consider the similar model of the wave equation for the total field:

\[
\begin{cases}
\varepsilon \partial_t^2 u - \nabla \cdot (\mu^{-1} \nabla u) = 0 & \text{in } \Omega^c \cup \Omega_1 \cup \cdots \cup \Omega_n, \ t > 0, \\
u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0 & \text{in } \Omega^c \cup \Omega_1 \cup \cdots \cup \Omega_n, \\
u = 0 & \text{on } \Gamma_c \cup S_1 \cup \cdots \cup S_n, \ t > 0.
\end{cases}
\]

The total field \( u \) is assumed to consist of the incident field \( u^{\text{inc}} \), the reflected field \( u^{r} \), and the scattered field \( u^{\text{sc}} \), where the scattered field is required to satisfy the radiation condition (2.5).
4.1.1. **Transparent boundary condition.** As the two cavity situation, to reduce the scattering problem from the open domain $\Omega^0 \cup \Omega_1 \cup \ldots \cup \Omega_n$ into the bounded domain, we need to derive transparent boundary conditions on the aperture $\Gamma_j$, $j = 1, \ldots, n$. Reformulating the multiple cavity scattering problem (4.1) into $n$ single cavity scattering problem with the coupled boundary conditions

\[
\begin{aligned}
\varepsilon_j \partial_t^2 u_j - \nabla \cdot \left( \mu_j^{-1} \nabla u_j \right) &= 0 \quad \text{in } \Omega_j, \ t > 0, \\
u_j|_{t=0} &= 0, \quad \partial_t u_j|_{t=0} = 0 \quad \text{in } \Omega_j, \\
u_j &= 0 \quad \text{on } S_j, \ t > 0, \\
\partial_n u_j &= \mathcal{T} u_j + g \quad \text{on } \Gamma_j, \ t > 0, 
\end{aligned}
\]

(4.2)

where the transparent boundary operator $\mathcal{T}$ will be given later and $u_j = u|_{\Omega_j}, \ j = 1, \ldots, n$.

Due to the homogeneous medium in the upper half space $\mathbb{R}^2_+$ and the radiation condition (2.5), the scattered field $u^{sc}$ still satisfies the same ordinary differential equation (2.13) after taking the Laplace transform with respect to $t$. Thus the total field $\tilde{u}$ and $u$ satisfy the transparent boundary conditions in frequency domain and time-domain, respectively:

\[
\partial_n \tilde{u} = \mathcal{B} \tilde{u} + \tilde{g}, \quad \partial_n u = \mathcal{T} u + g \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \ldots \cup \Gamma_n.
\]

(4.3)

Next, we derive the transparent boundary condition for each $u_j$ on $\Gamma_j$.

For $u_j(x, 0), j = 1, \ldots, n$ defined on $\Gamma_j$, we extend them to the whole $x$-axis by

\[
\tilde{u}_j(x, 0) = \begin{cases} u_j(x, 0) & \text{for } x \in \Gamma_j, \\ 0 & \text{for } x \in \mathbb{R} \setminus \Gamma_j. \end{cases}
\]

For the total field $u(x, 0)$, define its extension to the whole $x$-axis by

\[
\tilde{u}(x, 0) = \begin{cases} u_j(x, 0) & \text{for } x \in \Gamma_j, \\ 0 & \text{for } x \in \Gamma^c. \end{cases}
\]

By the definitions above, it is obvious that

\[
\tilde{u} = \sum_{j=1}^{n} \tilde{u}_j \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \ldots \cup \Gamma_n.
\]

The transparent boundary condition (4.3) can be written as

\[
\partial_n \tilde{u} = \mathcal{B} \tilde{u} + \tilde{g} \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \ldots \cup \Gamma_n; \quad \partial_n \tilde{u} = \mathcal{T} \tilde{u} + g \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \ldots \cup \Gamma_n,
\]

which leads to the transparent boundary conditions for $u_j$ on $\Gamma_j$:

\[
\partial_n \tilde{u}_j = \mathcal{B} \tilde{u}_j + \sum_{i \neq j}^{n} \mathcal{B} \tilde{u}_i + \tilde{g} \quad \text{on } \Gamma_j; \quad \partial_n u_j = \mathcal{T} \tilde{u}_j + \sum_{i \neq j}^{n} \mathcal{T} \tilde{u}_i + g \quad \text{on } \Gamma_j, \ t > 0.
\]

(4.4)

From (4.4), it is obvious that the boundary conditions for $u_1, \ldots, u_n$ are coupled with each other, which is the major difference between the single cavity scattering problem and the multiple cavity scattering problem.

The following lemma is analogous to Lemma 3.1 which is used for analysis the uniqueness and existence for the multiple cavity scattering problem.

**Lemma 4.1.** It holds that

\[
- \text{Re} \sum_{j=1}^{n} \sum_{i=1}^{n} \langle (s\mu)^{-1} \mathcal{B} u_i, u_j \rangle_{\Gamma_j} \geq 0, \quad u_j \in H^{1/2}(\mathbb{R}), \ j = 1, 2, \ldots, n.
\]
Proof. By definition \(4.4\), recalling \(\beta^2(\xi) = \xi^2 + c^{-2}s_2^2\), it gives
\[
\operatorname{Re} \left( \sum_{j=1}^{n} \sum_{i=1}^{n} \langle (s\mu)^{-1} \mathcal{B} u_i, u_j \rangle_{\Gamma_j} \right) = \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{\mathbb{R}} \frac{1}{s|\xi|^2} s_1 (\xi^2 + c^{-2}s_2^2) u_i \tilde{u}_j \, d\xi
\]
\[
\leq \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{\mathbb{R}} \frac{1}{s|\xi|^2} s_1 (\xi^2 + c^{-2}s_2^2) u_i \tilde{u}_j \, d\xi
\]
\[
\leq \int_{\mathbb{R}} \frac{1}{s|\xi|^2} s_1 (\xi^2 + c^{-2}s_2^2) \sum_{j=1}^{n} u_j \, d\xi \leq 0.
\]
\[\square\]

Lemma 4.2. For any \(u_j(\cdot, t) \in L^2(0, T; H^{1/2}(\Omega_j))\) with initial value \(u_j(\cdot, 0) = 0\), denote their zero extension on \(L^2(0, T; H^{1/2}(\mathbb{R}))\) by \(\tilde{u}_j(\cdot, t), j = 1, \ldots, n\). Then, it holds
\[-\operatorname{Re} \int_{0}^{T} \sum_{j=1}^{n} \sum_{i=1}^{n} \langle (\mathcal{F} \tilde{u}_i, \partial_t \tilde{u}_j) \rangle_{\Gamma_j} dt \geq 0.\]

Proof. Extending \(\tilde{u}_j(\cdot, t)\) with respect to \(t\) in \(\mathbb{R}\) such that \(\tilde{u}_j(\cdot, 0) = 0\) outside the interval \([0, T]\), for convenience, we still denote it by \(\tilde{u}_j(\cdot, t)\). Let \(\tilde{u}_j = \mathcal{L}(\tilde{u}_j)\) be the Laplace of \(\tilde{u}_j\). By the Parseval identity (2.6) and Lemma 4.1, we get
\[-\operatorname{Re} \int_{0}^{T} e^{-2s_1t} \sum_{j=1}^{n} \sum_{i=1}^{n} \langle (\mathcal{F} u_i, \partial_t u_j) \rangle_{\Gamma_j} dt = -\operatorname{Re} \int_{0}^{T} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{\Gamma_j} e^{-2s_1t} (\mathcal{F} \tilde{u}_i, \partial_t \tilde{u}_j) \, dt \, d\gamma
\]
\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{n} \sum_{i=1}^{n} \langle \mathcal{F} \tilde{u}_i, s \tilde{u}_j \rangle_{\Gamma_j} \, ds_2
\]
\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} |s|^2 \sum_{j=1}^{n} \sum_{i=1}^{n} \left| s^{-1}\beta(\xi) \tilde{u}_i \tilde{u}_j \right| \, ds_2
\]
\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| s_2 \beta(\xi) \tilde{u}_i \tilde{u}_j \right| \, ds_2 \geq 0,
\]
which completes the proof after taking \(s_1 \to 0\). \[\square\]

4.2. The reduced multiple cavity scattering problem. We now present the well-posedness and stability of the reduced problem. For simplicity, we shall use the same notation as those adopted in Section 3 for the two cavity scattering problem. Denote by \(\Omega = \Omega_1 \cup \cdots \cup \Omega_n, \Gamma = \Gamma_1 \cup \cdots \Gamma_n, \) and \(S = S_1 \cup \cdots \cup S_n\). Define the trace functional space
\[\tilde{H}^{1/2}(\Gamma) = \tilde{H}^{1/2}(\Gamma_1) \times \cdots \times \tilde{H}^{1/2}(\Gamma_n),\]
whose norm is characterized by \(\|u\|_{\tilde{H}^{1/2}(\Gamma)} = \sum_{j=1}^{n} \|u_j\|^2_{\tilde{H}^{1/2}(\Gamma_j)}\). Denote by
\[H^{-1/2}(\Gamma) = H^{-1/2}(\Gamma_1) \times \cdots \times H^{-1/2}(\Gamma_n),\]
which is the dual space of \(\tilde{H}^{1/2}(\Gamma)\). The norm on the space \(H^{-1/2}(\Gamma)\) is characterized by
\[\|u\|^2_{H^{-1/2}(\Gamma)} = \sum_{j=1}^{n} \|u_j\|^2_{H^{-1/2}(\Gamma_j)}.\]
Denote the space 

$$H^1_S(\Omega) = H^1_{S_1}(\Omega_1) \times \cdots \times H^1_{S_n}(\Omega_n),$$

which is a Hilbert space with norm characterized by $$\|u\|_{H^1_S(\Omega)}^2 = \sum_{j=1}^n \|u_j\|_{H^1_{S_j}(\Omega_j)}^2.$$ 

### 4.2.1. well-posedness in the s-domain.

Taking the Laplace transform of (4.2), we can get for $$j = 1, \cdots, n,$$

$$\begin{cases}
\varepsilon_j s \tilde{u}_j - \nabla \cdot (s^{-1} \mu^{-1} \nabla \tilde{u}_j) = 0 & \text{in } \Omega_j, \\
\tilde{u}_j = 0 & \text{on } S_j, \\
\partial_n \tilde{u}_j = \mathcal{B} \tilde{u}_j + \tilde{g} & \text{on } \Gamma_j.
\end{cases} \tag{4.5}$$

Multiplying the complex conjugate of test function $$v_j \in H^1_{S_j}(\Omega_j)$$ on both sides of the first equality of (4.5) and integrating over $$\Omega_j$$, we have

$$\int_{\Omega_j} (s\mu)^{-1} \nabla \tilde{u}_j \nabla \tilde{v}_j + s \varepsilon_j \tilde{u}_j \tilde{v}_j \, d\rho - \sum_{i=1}^n \langle (s\mu)^{-1} \mathcal{B} \tilde{u}_i, \tilde{v}_j \rangle_{\Gamma_j} = \langle \tilde{g}, v_j \rangle_{\Gamma_j}.$$ 

The variational formulation for the multiple cavity scattering problem $$\mathbf{(4.5)}$$ is: find $$\tilde{u} \in H^1_S(\Omega)$$ with $$\tilde{u}|_{\Omega_j} = u_j \in H^1_{S_j}(\Omega_j)$$, such that for all $$v \in H^1_S(\Omega)$$ with $$v|_{\Omega_j} = v_j \in H^1_{S_j}(\Omega_j)$$, it holds

$$a_3(\tilde{u}, v) = \sum_{j=1}^n \langle \tilde{g}, v_j \rangle_{\Gamma_j}, \tag{4.6}$$

where the sesquilinear form

$$a_3(\tilde{u}, v) = \sum_{j=1}^n \int_{\Omega_j} ((s\mu)^{-1} \nabla \tilde{u}_j \nabla \tilde{v} + s \varepsilon_j \tilde{u}_j \tilde{v}_j) \, d\rho - \sum_{j=1}^n \sum_{i=1}^n \langle (s\mu)^{-1} \mathcal{B} \tilde{u}_i, \tilde{v}_j \rangle_{\Gamma_j}.$$ 

**Theorem 4.3.** The variational problem $$\mathbf{(4.6)}$$ has a unique solution $$\tilde{u} \in H^1_S(\Omega)$$ which satisfies

$$\|\nabla \tilde{u}\|_{L^2(\Omega)^2} + \|s \tilde{u}\|_{L^2(\Omega)} \lesssim s_1^{-1} \|s \tilde{g}\|_{H^{-1/2}(\Gamma)}.$$ \tag{4.7}

**Proof.** The continuity of the sesquilinear form $$a_3(\tilde{u}, v)$$ follows

$$|a_3(\tilde{u}, v)| \leq \frac{1}{|s|\mu} \sum_{j=1}^n \|\nabla \tilde{u}_j\|_{L^2(\Omega)^2} \|\nabla v_j\|_{L^2(\Omega_j)^2} + |s| \varepsilon_{\max}(\sum_{j=1}^n \|\tilde{u}_j\|_{L^2(\Omega_j)})^2 \\
+ \frac{1}{|s|\mu} \sum_{j=1}^n \sum_{i=1}^n \|\mathcal{B} \tilde{u}_i\|_{H^{-1/2}(\Gamma_j)} \|v_j\|_{H^{1/2}(\Gamma_j)} \\
\lesssim \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 \|\nabla v\|_{L^2(\Omega)}^2 + \|\tilde{u}\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|\tilde{u}\|_{H^{1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} \\
\lesssim \|\tilde{u}\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)},$$

where $$\varepsilon_{\max} = \max \{\varepsilon_j, \, j = 1, \cdots, n\}$$. A simple calculation yields

$$a_3(\tilde{u}, \tilde{u}) = \sum_{j=1}^n \int_{\Omega_j} (s\mu)^{-1} \|\nabla \tilde{u}_j\|^2 + s \varepsilon_j \|\tilde{u}_j\|^2 \, d\rho - \sum_{j=1}^n \sum_{i=1}^n \langle (s\mu)^{-1} \mathcal{B} \tilde{u}_i, \tilde{u}_j \rangle_{\Gamma_j}. \tag{4.8}$$

Taking the real part of (4.8) and using Lemma 4.1, we get

$$\text{Re}(a_3(\tilde{u}, \tilde{u})) \geq C_1 \frac{s_1}{|s|^2} \left( \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 + \|s \tilde{u}\|_{L^2(\Omega)}^2 \right),$$

where $$C_1 = \min\{\mu^{-1}, 1\}$$. It follows from the Lax–Milgram lemma that the variational problem $$\mathbf{(4.6)}$$ has a unique solution $$\tilde{u} \in H^1_S(\Omega)$$. Moreover, we have from (4.6) that

$$|a_3(\tilde{u}, \tilde{u})| \leq |s|^{-1} \|\tilde{g}\|_{H^{-1/2}(\Gamma)} \|s \tilde{u}\|_{L^2(\Omega)}. \tag{4.10}$$
Combining (4.9)–(4.10) leads to
\[ \| \nabla \bar{u} \|_{L^2(\Omega)^2}^2 + \| s \bar{u} \|_{L^2(\Omega)}^2 \lesssim s_1^{-1} \| s \bar{g} \|_{H^{-1/2}(\Gamma)} \| s \bar{u} \|_{L^2(\Omega)}, \]
which implies the estimate of (4.7) after applying the Young’s inequality.

4.2.2. well-posedness in the time-domain. Using the time-domain transparent boundary condition, we consider the reduced initial-boundary value problem:
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
\varepsilon \partial_t^2 u - \nabla \cdot (\mu^{-1} \nabla u) = 0 & \text{in } \Omega, \ t > 0, \\
|u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0 & \text{in } \Omega, \\
u = 0 & \text{on } S, \ t > 0, \\
\partial_r u = \mathcal{F} u + g & \text{on } \Gamma, \ t > 0.
\end{array} \right.
\]
\tag{4.11}
\]

**Theorem 4.4.** The initial-boundary problem (4.11) has a unique solution \(u\), which satisfies
\[ u \in L^2(0, T; H^1_\delta(\Omega)) \cap H^1(0, T; L^2(\Omega)), \]
and the stability estimate
\[
\max_{t \in [0, T]} \left( \| \partial_t u \|_{L^2(\Omega)} + \| \partial_t (\nabla u) \|_{L^2(\Omega)^2} \right) \lesssim (\| g \|_{L^1(0, T; H^{-1/2}(\Gamma))} + \max_{t \in [0, T]} \| \partial_t g \|_{H^{-1/2}(\Gamma)} + \| \partial_t^2 g \|_{L^1(0, T; H^{-1/2}(\Gamma))}).
\]
\tag{4.12}

**Proof.** Using the same way of the two cavity scattering problem, we can get that
\[ u \in L^2(0, T; H^1_\delta(\Omega)) \cap H^1(0, T; L^2(\Omega)) \].

Next, we prove the stability. For any \(0 < t < T\), define the energy function
\[ e_5(t) = \| \varepsilon^{1/2} \partial_t u(\cdot, t) \|_{L^2(\Omega)}^2 + \| \mu^{-1/2} \nabla u(\cdot, t) \|_{L^2(\Omega)^2}. \]

It follows from (4.11) and integration by parts that
\[
\int_0^t e_5'(\tau) d\tau = 2 \Re \sum_{j=1}^n \int_0^t \int_{\Omega_j} \left( \nabla \cdot (\mu^{-1} \nabla u_j) \right) \partial_t \bar{u}_j + \mu^{-1} \nabla \partial_t u_j \cdot \nabla \bar{u}_j \right) d\rho d\tau.
\]

Since \(e_5(0) = 0\), we obtain from Lemma 4.2 that
\[
e_5(t) = 2 \Re \sum_{j=1}^n \int_0^t \mu^{-1} \langle \mathcal{F} u_j, \partial_t u_j \rangle_{T_j} dt + 2 \Re \sum_{j=1}^n \int_0^t \langle g, \partial_t u_j \rangle_{T_j} dt \]
\[
\leq 2 \Re \sum_{j=1}^n \int_0^t \| g \|_{H^{-1/2}(T_j)} \| \partial_t u_j \|_{H^{1/2}(T_j)} dt \]
\[
\lesssim 2 \Re \int_0^t \| g \|_{H^{-1/2}(T)} \| \partial_t u \|_{H^1(\Omega)} dt \]
\[
\leq 2 \left( \max_{t \in [0, T]} \| \partial_t u \|_{H^1(\Omega)} \right) \| g \|_{L^1(0, T; H^{-1/2}(\Gamma))}. \]

Taking the derivative of (2.27) with respect to \(t\), we know that \(\partial_t u\) also satisfies the same equations with \(g\) replaced by \(\partial_t g\). In order to control \(\| \partial_t (\nabla u(\cdot, t)) \|_{L^2(\Omega)^2}\), consider the energy function
\[ e_6(t) = \| \varepsilon^{1/2} \partial_t^2 u(\cdot, t) \|_{L^2(\Omega)}^2 + \| \mu^{-1/2} \partial_t (\nabla u(\cdot, t)) \|_{L^2(\Omega)^2}. \]
Similarly, we get the estimate
\[ e_6(t) \leq 2\text{Re} \sum_{j=1}^{n} \int_{0}^{t} (\partial_t g) \partial^2_t \bar{u}_j d\gamma_j dt \]
\[ = 2\text{Re} \sum_{j=1}^{n} \int_{\Gamma_j} (\partial_t g) \partial_t \bar{u}_j d\gamma_j - 2\text{Re} \sum_{j=1}^{n} \int_{0}^{t} (\partial^2_t g) \partial_t \bar{u}_j d\gamma_j dt \]
\[ \leq 2 \left( \max_{t \in [t,T]} \| \partial_t u \|_{H^1(\Omega)} \right) \left( \max_{t \in [t,T]} \| \partial_t g \|_{H^{-1/2}(\Gamma)} + \| \partial^2_t g \|_{L^1(0,T;H^{-1/2}(\Gamma))} \right). \]

Combing the above estimates, we can obtain
\[ \max_{t \in [t,T]} \| \partial_t u \|^2_{H^1(\Omega)} \lesssim (\| g \|_{L^1(0,T;H^{-1/2}(\Gamma))} ) \| \partial_t g \|_{H^{-1/2}(\Gamma)} + \| \partial^2_t g \|_{L^1(0,T;H^{-1/2}(\Gamma))} \| \partial_t u \|_{H^1(\Omega)}, \]
which give the estimate (4.12) after applying the Young’s inequality. \( \square \)

4.3. A priori estimates of the multiple cavity problem. In this section, for the multiple cavity scattering problem, we also derive a priori estimates for the total field with a minimum regularity requirement for the data and an explicit dependence on the time.

The variation formulation of (4.2) is to find \( u_j \in H^1_S(\Omega_j) \) for all \( t > 0 \) such that
\[ \int_{\Omega_j} \varepsilon_j (\partial^2_t u_j) \bar{v}_j d\rho = - \int_{\Omega_j} \mu^{-1} \nabla u_j \cdot \nabla \bar{v}_j d\rho + \sum_{i=1}^{n} \int_{\Gamma_j} \mu^{-1} (\mathcal{F}_j u_j) \bar{v}_j d\gamma_j + \int_{\Gamma_j} g \bar{v}_j d\gamma_j, \quad j = 1, \cdots, n. \]
This is equivalent to: find \( u \in H^1_S(\Omega) \) with \( u|_{\Omega_j} = u_j \in H^1_S(\Omega_j) \), such that for all \( v \in H^1_S(\Omega) \) with \( v_j = v|_{\Omega_j} \in H^1_S(\Omega_j) \), it holds
\[ c_2(u, v) = \sum_{j=1}^{n} \langle g, v_j \rangle_{\Gamma_j}, \]
where the sesquilinear form
\[ c_2(u, v) = \sum_{j=1}^{n} \left( \int_{\Omega_j} \varepsilon_j (\partial^2_t u_j) \bar{v}_j d\rho + \int_{\Omega_j} \mu^{-1} \nabla u_j \cdot \nabla \bar{v}_j d\rho \right) - \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{\Gamma_j} \mu^{-1} (\mathcal{F}_j u_j) \bar{v}_j d\gamma_j. \]

Theorem 4.5. Let \( u \in H^1_S(\Omega) \) be the solution of (4.11). Given \( g \in L^1(0,T;H^{-1/2}(\Gamma)) \), we have for any \( T > 0 \) that
\[ \| u \|_{L^\infty(0,T;L^2(\Omega))} + \| \nabla u \|_{L^\infty(0,T;L^2(\Omega))} \lesssim T \| g \|_{L^1(0,T;H^{-1/2}(\Gamma))} + \| \partial_t g \|_{L^1(0,T;H^{-1/2}(\Gamma))}, \]
and
\[ \| u \|_{L^2(0,T;L^2(\Omega))} + \| \nabla u \|_{L^2(0,T;L^2(\Omega))} \lesssim T^{3/2} \| g \|_{L^1(0,T;H^{-1/2}(\Gamma))} + T^{1/2} \| \partial_t g \|_{L^1(0,T;H^{-1/2}(\Gamma))}. \]
The proof is similar in nature as that of the two cavity model problem and is omitted here for brevity.

5. Conclusion

The problem of electromagnetic scattering by cavities embedded in the infinite two-dimensional ground plane is an important area of research. In this paper, we present the multiple cavity scattering problem in time-domain. We reduce the overall scattering problem to coupled scattering problem in bounded domain via the introduction of a novel transparent boundary condition over the cavity aperture in time-domain. The uniqueness, existence and stability of the reduced problem are obtained in frequency domain and time-domain, respectively. The main ingredients of the proofs are the Laplace transform, the Lax-Milgram lemma, and the Parseval identity. Moreover, by directly considering the variational problem of the time-domain wave equation, we obtain a priori estimates with an explicit dependence on the time.
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