The $\eta'$ signal from partially quenched Wilson fermions
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We present new results from our ongoing study of flavor singlet pseudoscalar mesons in QCD. Our approach is based on (a) performing truncated eigenmode expansions for the hairpin diagram and (b) incorporating the ground state contribution for the connected meson propagator. First, we explain how the computations can be substantially improved by even-odd preconditioning. We extend previous results on early mass plateauing in the $\eta'$ channel of two-flavor full QCD with degenerate sea and valence quarks to the partially quenched situation. We find that early mass plateau formation persists in the partially quenched situation.

1. Introduction

Hadron spectroscopy has been the testbed for monitoring the progress of lattice QCD calculations for over more than two decades. The $\eta'$ meson has been rather evasive, however. Being a flavor singlet object, its correlation function is a combination of a connected (octet) contribution and the infamous hairpin diagram. The latter is difficult to evaluate, for it is the correlator of two trace expressions over the inverse hermitean Dirac operator, $Q^{-1}$.

So far we have been computing $D$ by means of a truncated eigenmode representation of $Q^{-1}$,

$$Q^{-1} = \sum_{i=1}^{N} \frac{1}{\lambda_i} |\psi_i \rangle \langle \psi_i|,$$

with $Q|\psi_i \rangle = \lambda_i |\psi_i \rangle$ and $N = \dim(Q)$. 'Truncated' means that we restrict the sum Eq. (1) to $i \ll N$, i.e. to some low lying eigenmodes. Another important ingredient of our approach is a ground state projection of the connected correlator (one loop ground state analysis = OLGA), prior to the combination with the data for the hairpin-operator.

For the $16^3 \times 32$ SESAM lattice at $\beta = 5.6$ we showed for 2 different quark mass values that the 300 lowest lying eigenmodes of $Q$ suffice to approximate $D$ well enough. With OLGA, we found long $\eta'$ mass plateaus with onset at the very first time slice. The results appear to be fully consistent with former stochastic estimator analyses, which provides evidence that truncation effects from Eq. (1) in the SESAM operating conditions can be safely discarded.

We observed in Ref. [4] that stochastic estimator techniques and TEA require comparable computational effort. Actually, we can reduce the costs of TEA by means of preconditioning.

2. Even-odd preconditioning of $Q$

Since the Wilson Dirac matrix $M$ connects nearest neighbor sites only, it can be readily even-odd preconditioned, i.e. cast into the form

$$\tilde{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \kappa^2 D_{ee} D_{oo} \end{pmatrix},$$

where $e(o)$ stands for even(odd). Note that $\gamma_5 (1 - \kappa^2 D_{ee} D_{oo})$ is a hermitean matrix with orthogonal eigenvectors $\phi_i$ and real eigenvalues $\sigma_i$. In terms of these eigenmodes, $\tilde{Q}^{-1} = (\gamma_5 \tilde{M})^{-1}$ reduces to

$$\tilde{Q}^{-1} = \sum_{i=1}^{N/2} \frac{1}{\sigma_i} \left| \begin{array}{c} \phi_i \\ \phi_i \end{array} \right> \left< \begin{array}{c} \phi_i \\ \phi_i \end{array} \right|.$$
Eq. (1) now becomes
\[
\text{Tr} Q^{-1} = \text{Tr} \sum_{i=1}^{N/2} \frac{1}{\sigma_i} \left\langle \kappa D_{ee} \phi_i \right| \kappa D_{ee} \phi_i \right\rangle .
\] (4)

The quark loop signals as determined from the two different sets of vectors, i.e. from Eq. (1) and Eq. (4), are practically indistinguishable. Note that \( \tilde{M} \) offers two crucial benefits: 1. it requires only half the memory and 2. the eigenmode algorithm applied on \( \gamma_5 (1 - \kappa^2 D_{oe} D_{eo}) \) is improved by a factor 4 in convergence rate. In fact the order of the Chebyshev (accelerating) polynomials can be cut down from 80 to 20, see \[4\].

3. Unquenched 2-flavor \( \eta' \) mass analysis

Let us present next an update of our recent unquenched 2-flavor \( \eta' \) mass analysis \[3\]. We have by now completed the quark loop computations at 4 different sea quark mass values, each gauge field ensemble containing \( \mathcal{O}(200) \) gauge fields.

The local \( \eta' \) masses as extracted from logarithmic derivatives of the \( \eta' \)-correlator are exhibited in Fig. (1). We find that OLGA produces long and stable mass plateaus. As expected the quality of the plateaus improves when going more chiral in sea quark masses.

In Fig. (2) we plot the chiral extrapolations for the singlet masses. Our data clearly favor the quadratic fit, which yields \( \chi^2 = 0.14 \), compared to \( \chi^2 = 5.5 \) for the linear extrapolation.

We retrieve a definitely non vanishing mass gap between singlet and nonsinglet masses after quadratic chiral extrapolation. The value of the computed mass gap is far from the experimental one (even if we scale the \( \eta' \)-mass according to the Witten-Veneziano expectation down to a pseudoexperimental value as indicated in Fig. (2)). Physically this could be due either to limitations of unimproved Wilson fermions or the omission of strange quarks. Here we address the latter.

4. Partially quenched \( \eta - \eta' \) mass analysis

Given the SESAM gauge fields with just two dynamical flavors, let us turn now to a partially quenched scenario (PQS) \[6,7,1\]: we introduce 3 quenched quarks, that will be referred to as \( u, d \) and \( s \), with \( m_u = m_d \). For \( u \) and \( d \) we will choose \( \kappa_{val} \), independently from \( \kappa_{sea} \). In this setting we can extrapolate eventually to light \( u \) and
\[ G_{ij} = \frac{\delta_{ij}}{p^2 + m_i^2} - \frac{m_0^2}{(p^2 + m_i^2)(p^2 + m_j^2)} F(p^2), \]  

(5)

with

\[ F(p^2) = 1 + \frac{2m_0^2}{p^2 + m_f^2}, \]  

(6)

and

\[ 2m_0^2 = m_\eta^2 - m_i^2, \quad \eta' \equiv \langle \bar{u}u + \bar{d}d \rangle, \]  

(7)

\[ m_0^2 = m_\eta^2 - m_i^2, \quad \eta' \equiv \langle \bar{s}s \rangle. \]  

(8)

In these formulae, the indices \( i,j \) refer to the valence quark masses, while the index \( f \) points to the sea quark mass of the QCD vacuum configurations.

For any value of time separation \( t \), the mass gap, \( m_0(t) \), can be extracted by fitting the momentum zero Fourier transform of Eq. (5) to the lattice data, once the local nonsinglet pseudoscalar meson masses \( m_i(t) \) have been determined from a standard lattice analysis. The \( \eta' \) masses in the \( u-d, s \) sectors follow subsequently from Eq. (7), Eq. (8), respectively.

It is nontrivial to trace the \( \eta' \) signals under partial quenching, since we see statistically well defined linear and constant terms in the propagators, reflecting the double pole structure. Their effect increase as we move \( m_{val} \) away from \( m_{sea} \). To illustrate the actual \( \eta' \) signal from PQS, we plotted the mass plateaus of \( \eta' \equiv \langle \bar{u}u + \bar{d}d \rangle \) for \( \kappa_{sea} = 0.1575 \) and three different \( \kappa_{val} \) values in Fig. (3). Increasing \( m_{val} \), the plateau signal remains stable. Similarly, Fig. (4) shows the plateauing in the \( s \) sector as computed from Eq. (8).

The quality of our plateaus gives us confidence that we will be able to successfully complete the partially quenched calculations and move on to the study of improved fermions.

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