The statistical properties of the q-deformed Dirac oscillator in one and two-dimensions

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In this paper, we study the behavior of the eigenvalues of the one and two dimensions of q-deformed Dirac oscillator. The eigensolutions have been obtained by using a method based on the q-deformed creation and annihilation operators in both dimensions. For a two-dimensional case, we have used the complex formalism which reduced the problem to the problem of one dimensional case. The influence of the q-numbers on the eigenvalues has been well analyzed. Also, the connection between the q-oscillator and a quantum optics is well established. Finally, for very small deformation $\eta$, we have mentioned to existence of well-known q-deformed version of Zitterbewegung in relativistic quantum dynamics, and calculated the partition function and all thermal quantities such as the free energy, total energy, entropy and specific heat: here we consider only the case of a pure phase ($q = e^{i\eta}$). The extension to the case of graphene has been discussed.

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I. INTRODUCTION

Quantum groups and quantum algebras have attracted much attention of physicists and mathematicians during the last eight years. There had been a great deal of interest in this field, especially after the introduction of the q-deformed harmonic oscillator. Quantum groups and quantum algebras have found unexpected applications in theoretical physics. From the mathematical point of view they are q-deformations of the universal enveloping algebras of the corresponding Lie algebras, being also concrete examples of Hopf algebras. When the deformation parameter q is set equal to 1, the usual Lie algebras are obtained. The realization of the quantum algebra SU(2) in terms of the q-analogue of the quantum harmonic oscillator has initiated much work on this topic. Biedenharn and Macfarlane have studied the q-deformed harmonic oscillator based on an algebra of q-deformed creation and annihilation operators. They have found the spectrum and eigenvalues of such a harmonic oscillator under the assumption that there is a state with a lowest energy eigenvalue. Recently, the theory of the q-deformed has become a topic of great interest in the last few years, and it has been finding applications in several branches of physics because of its possible applications in a wide range of areas, such as a q-deformation of the harmonic oscillator, a q-deformed Morse oscillator, a classical and quantum q-deformed physical systems, Jaynes-Cummings model and the deformed-oscillator algebra, q-deformed super-symmetric quantum mechanics, Morse oscillator, a classical and quantum q-deformed systems. The relativistic harmonic oscillator is one of the most important quantum system, as it is one of the very few that can be solved exactly. The Dirac relativistic oscillator interaction is an important potential both for theory and application. It was for the first time studied by Ito et al. They considered a Dirac equation in which the momentum $\vec{p}$ is replaced by $\vec{p} - im\beta\vec{r}$, with $\vec{r}$ being the position vector, $m$ the mass of particle, and $\omega$ the frequency of the oscillator. The interest in the problem was revived by Moshinsky and Szczepaniak, who gave it the name of Dirac oscillator (DO) because, in the non-relativistic limit, it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Physically, it can be shown that the (DO) interaction is a physical system, which can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field. The electromagnetic potential associated with the DO has been found by Benitez et al. The Dirac oscillator has attracted a lot of interest both because it provides one of the examples of the Dirac’s equation exact solvability and because of its numerous physical applications. We can note here that Franco-Villafane et al. have exposed the proposal of the first experimental microwave realization of the one-dimensional (DO).

The q-deformed oscillator systems have attracted much attention and have been considered in many papers (see Ref. and references therein). The representation theory of the quantum algebras has led to the development of q-deformed oscillator algebra. Since, there have been an increasing interest in the study of physical systems using q-oscillator algebra. It has found applications in several branches of physics such as vibrational spectroscopy, nuclear physics, many body theory and quantum optics. The q-analogue of the one-dimensional non-relativistic harmonic oscillator has been studied by several authors. Realizations of the quantum algebra $SU_q(1,1)$ via the one-dimensional q-harmonic oscillator were suggested by Chaichian et al. (see also Ref.). The representation theory of quantum algebras with a single deformation parameter $q$, has led to the development of the so-called q-deformed harmonic oscillator algebra.

The extension of the non-relativistic q-harmonic oscillator to the relativistic case, in the best of our knowledge, is not available in the literature. In this context, and in order the overcome this lack in the literature, the principal aim of this paper will be the studied of q-deformed Dirac oscillator in one and two dimensions. The concept of q-deformation is also applied to investigation of the connection of q-deformed Dirac oscillator with quantum optics, and the existence of the well-known Zitterbewegung in relativistic quantum dynamics of the problem in question. In addition, we have evaluated various thermodynamic quantities such as partition function, entropy and internal energy. Such studies are expected to be relevant when we want to extended them to the case of Graphene.

The structure of this paper is as follows: Sec. II is devoted to the case of the standard q-harmonic oscillator. The extension to the q-deformed Dirac oscillator will be treated in Sec. III. Different numerical results about the thermal properties of q-deformed Dirac oscillator are discussed in Sec. IV. Finally, Sec. V will be a conclusion.

II. ONE-DIMENSIONAL Q-DEFORMED STANDARD HARMONIC OSCILLATOR: A REVIEW

In the case of q-deformed harmonic oscillator, the creation and annihilation operators $a^+$ and $a$ satisfy the commutation relation

$$[a, a^+]_q = aa^+ - q^{-1}a^+a = q^N$$  \hspace{1cm} (1)
where \( N \) is the number operator, satisfying
\[
[N, a^+] = a^+, [N, a] = -a.
\] (2)

The relevant Fock space is defined as
\[
a |0\rangle = 0, |n\rangle = \frac{(a^+)^n}{\sqrt{|n|!}} |0\rangle,
\] (3)
where the q-factorial is defined as
\[
[n]! = [n][n-1] \cdots [1],
\] (4)
and the q-numbers are defined by
\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.
\] (5)

When \( q \) is real (\( q = e^\eta \)), the q-numbers take the form
\[
[k] = \frac{\sinh(\eta k)}{\sinh(\eta)}
\] (6)
and when \( q \) is imaginary (\( q = e^{i\eta} \)), the q-numbers take the form
\[
[k] = \frac{\sin(\eta k)}{\sin(\eta)}
\] (7)

It is clear that in both cases \([k] \to k\) in the limit \( q \to 1 \).

The Hamiltonian of the q-deformed harmonic oscillator is
\[
H = \frac{P_q^2}{2m} + \frac{1}{2}m\omega^2 Q_q^2
\] (8)
where the q-momentum \((P_q)\) and q-position \((Q_q)\) operators are directly written in terms of the q-boson operators \( a \) and \( a^+ \) introduced above with
\[
P_q = i\sqrt{\frac{m\omega\hbar}{2}} (a - a^+) ,
\] (9)
\[
Q_q = \sqrt{\frac{\hbar \omega}{2m}} (a + a^+) ,
\] (10)

Following this, we obtain
\[
H = \frac{\hbar \omega}{2} (aa^+ + a^+ a) .
\] (11)

The eigenvalues, in the Fock space defined above, are
\[
E_n = \frac{\hbar \omega}{2} ([n] + [n + 1]) .
\] (12)

Hence the energy levels are no longer uniformly spaced as \( q \) is not equal to one. From the above Eq. (12), we find that the q-deformed harmonic oscillator has a spectrum, given by
\[
E_n = \frac{\hbar \omega}{2} \frac{\sinh (q (n + \frac{1}{2}))}{\sinh (\frac{q}{2})} ,
\] (13)
when \( q \) is real, and by
\[
E_n = \frac{\hbar \omega}{2} \frac{\sin (q (n + \frac{1}{2}))}{\sin (\frac{q}{2})}
\] (14)
when $q$ is complex. In both cases, when $q \to 1$ ($\eta \to 0$) the well-known relation

$$E_n = \hbar \omega \left( n + \frac{1}{2} \right)$$  (15)

is recover. According to the Eq. [15], one can see that for $q$ real the energy eigenvalues increase more rapidly than the ordinary case, in which the spectrum is equidistant, i.e. the spectrum gets "expanded". In contrast, when $q$ is a pure phase, the eigenvalues of the energy increase less rapidly than the ordinary (equidistant) case, i.e. the spectrum is "compressed" or squeezed.

In what follow, we treat the case of the one and two dimensional $q$-deformed Dirac oscillator.

### III. SOLUTIONS OF A Q-DEFORMED DIRAC OSCILLATOR

#### A. One-dimensional $q$-deformed Dirac oscillator

The one-dimensional Dirac oscillator is

$$\{ c \alpha_x (P_q - i m \omega \beta Q_q) + \beta m c^2 \} \psi_D = \epsilon \psi_D, $$  (16)

with $\psi_D = (\psi_1 \; \psi_2)^T$, $\alpha_x = \sigma_x$ and $\beta = \sigma_z$. In this case, Eq. (16) becomes :

$$H_D \psi_D = \epsilon \psi_D, $$  (17)

with

$$H_D = \begin{pmatrix} mc^2 & \frac{m c^2}{2} \frac{c (p_x + im \omega x)}{-mc^2} \\ \frac{m c^2}{2} \frac{c (p_x - im \omega x)}{mc^2} & \frac{mc^2}{2} \end{pmatrix} $$  (18)

by introducing the usual annihilation and creation operators of the $q$-deformed harmonic oscillator

$$P_q = i \frac{\sqrt{m \omega \hbar}}{2} (a - a^+),$$  (19)

$$Q_q = \frac{\sqrt{\hbar \omega}}{2m} (a + a^+),$$  (20)

this Hamiltonian transforms into

$$H_D = \begin{pmatrix} mc^2 & ga^1 \\ g^* a & -mc^2 \end{pmatrix} $$  (21)

with $q = imc^2 \sqrt{2r}$, is the coupling strength between orbital and spin degrees of freedom, and $r = \frac{\hbar \omega}{mc^2} = 1$ is a parameter which controls the non-relativistic limit. It is an important parameter that specifies the importance of relativistic effects in the Dirac oscillator.

Writing that $\psi_D = (|n\rangle, |n - 1\rangle)^T$, this equation can be solved algebraically. Following the above section, when $q$ is real, the spectrum of energy is

$$\epsilon_n = \pm mc^2 \sqrt{1 + 2 \frac{\sinh (\eta n)}{\sinh (\eta)}}, $$  (22)

Now, if $q$ is complex, its becomes

$$\tilde{\epsilon}_n = \pm mc^2 \sqrt{1 + 2 \frac{\sin (\eta n)}{\sin (\eta)}}, $$  (23)

In both cases, when $q \to 1$ ($\eta \to 0$) the well-known relation

$$\tilde{\epsilon}_n = \pm mc^2 \sqrt{1 + 2n} $$  (24)
is recover \[25\]. The eigensolutions of a two-dimensional Dirac oscillator, in both cases, can be written as

\[
|\psi\rangle = \begin{cases} 
\sqrt{\frac{E_n + mc^2}{2Emn}} |n\rangle \\
\mp i \sqrt{\frac{E_n + mc^2}{2Emn}} |n-1\rangle
\end{cases}
\]

Using the creation and annihilation operators and the raising and lowering operators \(\sigma^\pm = \frac{1}{2} (\sigma^x \pm i\sigma^y)\) Dirac Spinor one can rewrite the previous equation as

\[
H_{1D} = g (\sigma^+ a + \sigma^- a^\dagger) + \Delta \sigma_z
\]

This Hamiltonian is exactly the JCM Hamiltonian in quantum optics \[36\]. Thus the one-dimensional Dirac oscillator maps exactly onto the Jaynes-Cummings (JC), provided that one identifies the isospin with the atomic system and the spatial degrees of freedom with the cavity mode. Thus, as a result, we would like mentioned the relativistic Hamiltonian of a q-deformed one-dimensional Dirac oscillator can be mapped onto a q-deformed Jaynes-Cummings(JC).

**B. Two-dimensional q-deformed Dirac oscillator**

1. **Complex formalism**

In terms of complex coordinates and its complex conjugate \[37–40\], we have

\[
z = x + iy, \quad \bar{z} = x - iy,
\]

and

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

The operators momentum \(p_x\) and \(p_y\), in the Cartesian coordinates, are defined by

\[
p_x = -i\hbar \frac{\partial}{\partial x}, \quad p_y = -i\hbar \frac{\partial}{\partial y}.
\]

When we use \(p_z = -i\hbar \frac{\partial}{\partial z}\), we get

\[
p_z = -i\hbar \frac{d}{dz} = \frac{1}{2} \left( p_x - ip_y \right),
\]

\[
\bar{p}_z = -i\hbar \frac{d}{d\bar{z}} = \frac{1}{2} \left( p_x + ip_y \right),
\]

with \(p_z = -\bar{p}_z\). These operators obey the basic commutation relations

\[
[z, p_z] = [\bar{z}, p_z] = i\hbar, \quad [z, \bar{z}] = [\bar{z}, p_z] = 0.
\]

The usual creation and annihilation operators, \(a_x\) and \(a_y\) with

\[
a_x = \sqrt{\frac{m\omega}{2\hbar}} x + \frac{1}{\sqrt{2m\omega\hbar}} p_x, \quad a_y = \sqrt{\frac{m\omega}{2\hbar}} y + \frac{1}{\sqrt{2m\omega\hbar}} p_y,
\]

can be reformulated, in the formalism complex, as follows

\[
a_z = i \left( \frac{1}{\sqrt{m\omega\hbar}} \bar{p}_z - \frac{i}{2} \sqrt{\frac{m\omega}{\hbar}} \bar{z} \right),
\]

\[
\bar{a}_z = -i \left( \frac{1}{\sqrt{m\omega\hbar}} p_z + \frac{i}{2} \sqrt{\frac{m\omega}{\hbar}} z \right).
\]
These operators, also, satisfy the habitual commutation relations
\[ [a_z, a_{\bar{z}}] = 1, [a_z, a_{\bar{z}}] = 0, [a_{\bar{z}}, a_z] = 0. \] (36)

Now, in the case of q-deformed Dirac oscillator, the creation and annihilation operators \( \bar{a}_z \) and \( a_z \) satisfy the commutation relation
\[ [a_z, \bar{a}_{\bar{z}}]_q = a_z \bar{a}_{\bar{z}} - q^{-1} \bar{a}_{\bar{z}} a_z = q^N \] (37)
where \( N \) is the number operator, satisfying
\[ [N, \bar{a}_{\bar{z}}] = \bar{a}_{\bar{z}}, [N, a_z] = -a_z. \] (38)

2. The solutions

The two-dimensional Dirac oscillator is
\[ [\sigma_x (p_x - i m \omega x) + \sigma_y (p_y - i m \omega y)] \psi = \varepsilon \psi, \] (39)
with \( \psi_D = (\psi_1 \ \psi_2)^T \), \( \alpha_x = \sigma_x \) and \( \beta = \sigma_z \). With the following definitions of Dirac matrices,
\[ \alpha_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \alpha_y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \] (40)
Eq. (14) can be decoupled in a set of equations as follows
\[ \varepsilon |\psi_1\rangle = c (p_x + i m \omega x - i p_y + m \omega y) |\psi_2\rangle, \] (41)
\[ \varepsilon |\psi_2\rangle = c (p_x - i m \omega x + i p_y + m \omega y) |\psi_1\rangle, \] (42)
and so, Eq. (13) reads
\[ H_D = \begin{pmatrix} mc^2 & c (p_x + i m \omega x - i p_y + m \omega y) \\ c (p_x - i m \omega x + i p_y + m \omega y) & -mc^2 \end{pmatrix}. \] (43)
This last form of Hamiltonian of Dirac can be written, in the complex formalism, by
\[ H_D = \begin{pmatrix} mc^2 & 2c p_z + i m \omega c \bar{z} \\ 2c \bar{p}_z - i m \omega c z & -mc^2 \end{pmatrix} = \begin{pmatrix} mc^2 & 2g \bar{a}_z \\ 2g^* a_z & -mc^2 \end{pmatrix}. \] (44)
Thus the problem is transformed to the one-dimensional case with a complex variable \( z \).

Now, following Eqs. (15) and (16), the wave functions \( \psi_1 \) and \( \psi_2 \) can be rewritten in the language of the complex annihilation-creation operators as
\[ |\psi_1\rangle = \frac{g}{\varepsilon - mc^2 \bar{a}_z} |\psi_2\rangle, \] (45)
\[ |\psi_2\rangle = \frac{g^*}{\varepsilon + mc^2 a_z} |\psi_1\rangle. \] (46)
When we write the component \( |\psi_1\rangle \) in terms of the quanta bases, \( |n\rangle = \frac{(a^\dagger)^n}{\sqrt{|n|!}} |0\rangle \), these equations can be simultaneously diagonalized, and the energy spectrum can be described by
\[ \varepsilon_n = \pm mc^2 \sqrt{1 + 4 \frac{\sinh |\eta_n|}{\sinh (|\eta|)}}. \] (47)
when \( q \) is real, and by
\[ \bar{\varepsilon}_n = \pm mc^2 \sqrt{1 + 4 \frac{\sin |\eta_n|}{\sin (|\eta|)}}. \] (48)
if \( q \) is complex. In both cases, when \( q \to 1 \) (\( \eta \to 0 \)) the well-known relation

\[
\bar{\varepsilon}_n = \pm mc^2 \sqrt{1 + 4n}
\]  

(49)
is recovered. Our results are in a good agreement with those obtained by Hatami et al [41].

According to last equation, the Dirac Hamiltonian can be written into another form as

\[
H_{2D} = g (\sigma^1 \bar{a}_z + \sigma^2 a_z) + \Delta \sigma_z,
\]

(50)
and it correspond to the q-deformed Anti-Jaynes-Cummings (AJC) model. Here \( \sigma^z = \frac{1}{2} (\sigma_x \pm i\sigma_y) \) are the spin arising and lowering operators, and \( \Delta = mc^2 \) is a detuning parameter. Before go further, we like to mentioned the relativistic Hamiltonian of a q-deformed two-dimensional Dirac oscillator can be mapped onto a couple of q-deformed Anti-Jayne-Cummings- (AJC) which describe the interaction between the relativistic spin and bosons.

We further observe that the Zitterbewegung frequency for the q-deformed \((2 + 1)\)-dimensional Dirac oscillator depends on the parameter of deformation \( \eta \). To show this we, first, start with the following eigensolutions of a two-dimensional Dirac oscillator

\[
|\psi_{1,2}\rangle = \begin{pmatrix}
\sqrt{\frac{E_n + mc^2}{2\varepsilon_n}} |n\rangle \\
\mp i\sqrt{\frac{E_n + mc^2}{2\varepsilon_n}} |n - 1\rangle
\end{pmatrix}.
\]

(51)
Here, \( E_n \equiv \varepsilon_n \) (or \( E_n \equiv \varepsilon_n \)) for \( q \) real (or \( q \) complex) respectively. The eigenstates can be expressed transparently in terms of two-component Pauli spinors \( |\chi_{\uparrow}\rangle \) and \( |\chi_{\downarrow}\rangle \). 

\[
|\psi_1\rangle = \alpha_n |n\rangle |\chi_{\uparrow}\rangle - i\gamma_n |n - 1\rangle |\chi_{\downarrow}\rangle,
\]

(52)

\[
|\psi_2\rangle = \gamma_n |n\rangle |\chi_{\uparrow}\rangle + i\alpha_n |n - 1\rangle |\chi_{\downarrow}\rangle,
\]

(53)
where \( \alpha_n = \sqrt{\frac{\varepsilon_n + mc^2}{2\varepsilon_n}} \) (or \( \sqrt{\frac{\varepsilon_n + mc^2}{2\varepsilon_n}} \)) and \( \delta_n = \sqrt{\frac{\varepsilon_n - mc^2}{2\varepsilon_n}} \) (or \( \sqrt{\frac{\varepsilon_n - mc^2}{2\varepsilon_n}} \)) are real. We can observe that the energy eigenstates present entanglement between the orbital and spin degrees of freedom. The clarify this, we start with some initial pure state at \( t = 0 \),

\[
|\Psi (0)\rangle = |n - 1\rangle |\chi_{\uparrow}\rangle = i\alpha_n |\psi_1\rangle - i\gamma_n |\psi_2\rangle,
\]

(54)
This equation shows that the starting initial state is a superposition of both the positive and negative energy solutions, which is the fundamental ingredient that leads to Zitterbewegung in relativistic quantum dynamics.

The evolution of this initial state can be expressed as

\[
|\Psi (t)\rangle = i\gamma_n e^{-i\omega_n t} |\psi_1\rangle - i\alpha_n e^{i\omega_n t} |\psi_2\rangle
\]

(55)
where

\[
\omega_n = \frac{\varepsilon_n}{\hbar} = \frac{mc^2}{\hbar} \sqrt{1 + 4\sinh (\eta n) \sinh (\eta)}
\]

(56)
for \( q \) real, and

\[
\omega_n = \frac{\bar{\varepsilon}_n}{\hbar} = \frac{mc^2}{\hbar} \sqrt{1 + 4\sin (\eta n) \sin (\eta)}
\]

(57)
for \( q \) complex. In both cases, \( \omega_n \) describes the frequency of oscillations: the frequency oscillation between positive and negative energy solutions.

If we consider very small deformation and neglect all terms proportional to \( \eta^4 \),

\[
\omega_n = \frac{\varepsilon_n}{\hbar} = \frac{mc^2}{\hbar} \sqrt{1 + 4\sin [\eta n] \sinh (\eta)} \approx \frac{mc^2}{\hbar} \sqrt{1 + 4n + \frac{2}{3} \eta^2 n^3} \approx \omega_{DO} \left( 1 \pm \frac{1}{3} \frac{n^3}{(1 + 4n)^2} \eta^2 \right),
\]

(58)
with $\omega_{DO} = \frac{mc}{\hbar} \sqrt{1 + 4n}$ is the frequency of a two-dimensional Dirac oscillator without deformation, and the sign $+$ denotes the case for $q$ real and the sign $-$ for the case of $q$ complex.

Expression $|\Psi(t)\rangle$ in terms of two-component $|\psi_1\rangle$ and $|\psi_2\rangle$, and in the approximation of very small $\eta$, the final form will be

\[ |\Psi(t)\rangle = |\Psi(t)\rangle_{\eta=0} + \eta^2 |\Psi(t)\rangle_{\eta\neq0}, \]

where

\[ |\Psi(t)\rangle_{\eta=0} = \left( \cos \omega_{DO}t + \frac{i \sin \omega_{DO}t}{\sqrt{1+4n}} \right) |\psi_1\rangle + \sqrt{\frac{4n}{1+4n}} \sin \omega_{DO}t |\psi_2\rangle, \]

and

\[ |\Psi(t)\rangle_{\eta\neq0} = \left( \left\{ \frac{1}{3} \frac{n^3 \omega_{DO}t}{1+4n} \sin \omega_{DO}t + \frac{i}{\sqrt{1+4n}} \left\{ \frac{1}{3} \frac{n^3 \omega_{DO}t}{1+4n} \left( \cos \omega_{DO}t + \sin \omega_{DO}t \right) \right\} \right\} |\psi_1\rangle + \right\} \sqrt{\frac{4n}{1+4n}} \frac{1}{3} \frac{n^3 \omega_{DO}t}{1+4n} \sin \omega_{DO}t |\psi_2\rangle. \]

The sign $(-)$ for the case of $q$ real, and $(+)$ for the case of pure phase. This equation shows the oscillatory behavior between the states $|n\rangle \langle \chi_\uparrow|$ and $|n-1\rangle \langle \chi_\downarrow|$ which is exactly similar to atomic Rabi oscillations occurring in the JC/AJC models. The q-deformed Rabi frequency is given by $\omega_n$ (Eq. (22)) for both cases.

C. Discussions

This section is devoted to study the influence of q-deformed algebra on the eigenvalues of the Dirac oscillator in one and two dimensions. This influence has been well established through the parameter $\eta$ with $q = e^\eta$.

In Fig. 1, we present the eigenvalues of the q-deformed Dirac oscillator in one and two dimensions versus the quantum number with different values of parameter $\eta$ in both cases of $q$ real and complex. In order to argued this figure, we use the same explication used by Neskovic and Urosevic \[47\] in their study of the statistical properties of quantum oscillator: thus, the energy levels of the q-oscillator are not uniformly spaced for $q = 1$. The behavior of the energy spectra is completely different in the cases $q = e^{\eta}$ and $q = e^{\eta}$. When $q$ is real, $q = e^\eta$, the separation between the levels increases with the value of $n$ i.e, the spectrum is extended. On the other hand, when $q$ is a pure phase, the separation between all the levels decreases with increasing $n$, i.e, the spectrum is squeeze.

In Fig. 2 we present the frequency $\omega_n$ versus a quantum number $n$ for both $q$ real and complex in one and two-dimensions: this frequency describes the oscillations between positive and negative energy solutions. As consequence, we are in the case of well-known Zitterbewegung in relativistic quantum dynamics. This phenomenon, due to the interference of positive and negative energies, has never been observed experimentally. The reason is that the amplitude of these rapid oscillations lies below the Compton wavelength. The influence of the deformation on this frequency is well established also.

Finally, the exact connection of the q-deformed Dirac oscillator with both Jaynes-Cummings (JC) and anti-Jaynes-Cummings (AJC) models has been established.

IV. THERMAL PROPERTIES OF Q-DEFORMED DIRAC OSCILLATOR

The theory of q-deformed statistics has become a topic of great interest in the last few years because of its possible applications in a wide range of areas, such as anyon physics, vertex models, quantum mechanics in discontinuous space-time, quantum oscillator, and vibration of polyatomic molecules, etc \[48\]. In recent years, many researchers have studied the q-deformed physical systems and have obtained a lot of research progress. Among them, the statistical mechanics of q-deformed quantum oscillator studied by Neskovic and Urosevic \[49\]. Using the boson realization of q-oscillator algebra and taking q to be real, they have calculated the partition function $Z$ and thermodynamic potentials such as free energy $F$, entropy $S$ and internal energy $U$ for a Slightly Deformed Oscillator.
Figure 1: Spectrum of energy of one and two dimensions versus quantum number for both q cases.

Figure 2: Reduced frequency versus quantum number: here $\hbar = m = c = 1$. 

(a) q-deformed one-dimensional Dirac oscillator

(b) q-deformed two-dimensional Dirac oscillator
The probability of finding a system in a state with energy \( E_n \) is given by

\[
P_n = e^{-\frac{E_n}{k_B T}} Z, \tag{62}
\]

here \( Z = \text{tr} e^{-\frac{H}{k_B T}} = \sum e^{-\beta E_n} \) is the partition function. Our main object is to obtain this partition function for small deformation \( \beta \): in this case the summation appears in \( Z \) can be easily performed. Before doing so, let’s rewrite the form of energy in a more convenient form. Starting with the following equation

\[
\epsilon_n = \pm mc^2 \sqrt{1 + a \sinh (\eta n) / \sinh (\eta)}, \tag{63}
\]

for \( q \) real and

\[
\bar{\epsilon}_n = \pm mc^2 \sqrt{1 + a \sin (\eta n) / \sin (\eta)} \tag{64}
\]

for \( q \) complex. Here \( a = 2 \) for one-dimensional (4 for a two-dimensional case). Now, in order to extract these properties of our \( q \)-oscillator, we will only restrict ourselves to stationary states of positive energy. The Dirac oscillator possesses an exact Foldy–Wouthuysen transformation (FWT): so, the positive- and negative-energy solutions never mix. Following this, we only consider the positive part of energy.

As \( \frac{\sinh (\eta n)}{\sinh (\eta)} \) is even as a function of \( \eta \), so that the same property has the energy and all quantities derived from it. Now, we will consider very small deformation and neglect all terms proportional to \( \eta^4 \). In this case, we have

\[
\epsilon_n \simeq \sqrt{1 + an} \left( 1 + \frac{an^3}{12 (1 + an)} \eta^2 \right). \tag{65}
\]

With the same argument, the energy spectrum of the \( q \) complex case can be written as

\[
\bar{\epsilon} = \simeq \sqrt{1 + an} \left( 1 - \frac{an^3}{12 (1 + an)} \eta^2 \right). \tag{66}
\]

A both equations can be written in a compact form as

\[
\xi_n = \sqrt{1 + an} \left( 1 \pm \frac{an^3}{12 (1 + an)} \eta^2 \right), \tag{67}
\]

or

\[
\xi_n = \xi_{n0} + \xi_{n\pm}, \tag{68}
\]

with

\[
\xi_{n0} = \sqrt{1 + an}, \tag{69}
\]

is the reduced spectrum of energy of the ordinary Dirac oscillator, and

\[
\xi_{n\pm} = \pm \frac{an^3}{12 \sqrt{1 + an}} \eta^2 \tag{70}
\]

is the correction on the energy when the deformation exists.

In this case, the partition function is

\[
Z = \sum_{n=0}^{\infty} e^{-\frac{\xi_{n\pm}}{\tau}} \simeq \sum_{n=0}^{\infty} e^{-\frac{\sqrt{1 + an}}{\tau}} \left( 1 \pm \frac{an^3}{12 \tau \sqrt{1 + an}} \eta^2 \right) = Z_0 + Z_1 \tag{71}
\]

with \( \tau = \frac{mc^2}{\hbar} \) and

\[
Z_0 = \sum_{n=0}^{\infty} e^{-\frac{\sqrt{1 + an}}{\tau}}, \tag{72}
\]
In order to evaluate the first term, \( Z_0 \), we use a method based on Zeta function: so, by using the formula \[ e^{-x} = \frac{1}{2\pi i} \int_C ds x^{-s} \Gamma(s), \] the sum in Eq. (18) is transformed into
\[
\sum_n e^{-\sqrt{\alpha + n} x} = \frac{1}{2\pi i} \int_C ds \left( \frac{\sqrt{s}}{s} \right)^{-s} \sum_n (n + \alpha)^{-\frac{s}{2}} \Gamma(s) = \frac{1}{2\pi i} \int_C ds \left( \frac{\sqrt{s}}{s} \right)^{-s} \zeta_H \left( \frac{s}{2}, \alpha \right) \Gamma(s),
\]
with \( x = \frac{\sqrt{\alpha + n}}{\sqrt{\alpha}}, \gamma = \sqrt{\alpha}, \alpha = \frac{1}{a} \). Here, \( \Gamma(s) \) and \( \zeta_H \left( \frac{s}{2}, \alpha \right) \) are respectively the Euler and Hurwitz zeta function.

Applying the residues theorem, for the two poles \( s = 0 \) and \( s = 2 \), the desired partition function is written down in terms of the Hurwitz zeta function as follows:
\[
Z_0(\tau) = \frac{\tau^2}{a} + \zeta_H \left( 0, \frac{1}{a} \right).
\]

Now, the second term can be evaluated by using the Euler–MacLaurin formula; starting by the following equation
\[
Z_1 = \pm a\eta^2 \sum_{n=0}^{\infty} \frac{n^3}{\sqrt{1 + an}} e^{-\sqrt{\frac{\tau}{1 + \tau}}}.
\]
and according this approach, the sum transforms to the integral as follows
\[
\sum_{x=0}^{\infty} f(x) = \frac{1}{2} f(0) + \int_0^{\infty} f(x) dx - \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} f^{(2p-1)}(0),
\]
Here \( f(x) = \frac{x^3}{\sqrt{1 + ax}} e^{-\frac{x}{\sqrt{1 + ax}}}, \) \( B_{2p} \) are the Bernoulli numbers, \( f^{(2p-1)} \) is the derivative of order \( (2p - 1) \). Up to \( p = 1 \), the final form of \( Z_1 \) term is:
\[
Z_1(\tau, \eta) = \int_0^{\infty} f(x) dx = b\eta^2 e^{-\frac{1}{\tau}} \left( 15\tau^6 + 15\tau^5 + 6\tau^4 + \tau^3 \right),
\]
with \( b = 6e^{-\frac{1}{\tau}} \) when \( a = 2 \), and \( b = \frac{e^{-\frac{1}{\tau}}}{8} \) when \( a = 4 \). Finally, the compact final form of the q-deformed partition function of the Dirac oscillator in one and two-dimensions for both cases of q is
\[
Z_q(\tau, \eta) = \frac{\tau^2}{2} + \zeta_H \left( 0, \frac{1}{a} \right) \pm \eta^2 b \left( 15\tau^6 + 15\tau^5 + 6\tau^4 + \tau^3 \right).
\]
Here, the sign (-) describes the partition function in the case where q is real, and (+) the case of q complex.

Via Eq. (40), the determination of all thermal properties, such as the free energy, the entropy, total energy and the specific heat, can be obtained through the numerical partition function \( Z(\tau) \) via the following relations
\[
F = -\tau \ln(Z), \quad U = \tau^2 \frac{\partial \ln(Z)}{\partial \tau},
\]
\[
\frac{S}{k_B} = \ln(Z) + \tau \frac{\partial \ln(Z)}{\partial \tau}, \quad C_{\text{KB}} = 2\tau \frac{\partial \ln(Z)}{\partial \tau} + \tau^2 \frac{\partial^2 \ln(Z)}{\partial \tau^2}.
\]
Figure 3: Thermal properties of a q-deformed Dirac oscillator in one and two dimensions

A. Numerical results

The Figure shows the thermal properties of a one and two-dimensional q-deformed Dirac oscillator in both cases of q. From this figure, we can confirm that the deformation plays a significant role on these properties, and the effect of the parameters is very important on the thermodynamic properties. Also, the behavior of these quantities are completely different in the cases $q = e^{i\eta}$ and $q = e^{i\eta}$. When q is a pure phase, the behavior of different thermal quantities have a similar comportment as in the case of non-deformed Dirac oscillator in both one and two dimensions [48, 49]. On the other hand if q is real, these quantities show a strange behavior. This situation is closely related to the nature of spectrum: if q is real, the spectrum is extended which is the cause of the strange behavior of the thermodynamics quantities, contrary to the case where q is complex where the spectrum is squeeze: in this case we obtain the same form of all curves in these quantities. In what follow, in order to compare our results with those obtained in literature, we focus on the case of pure phase.

We should mention that, in all figures, we have used dimensionless quantities, and the temperature range is taken from $10^8$K to $10^{14}$K. These values give an order of the oscillator frequency about $10^{20}$Hz similar to that of Zitterbewegung frequency in the DO, which has so far been experimentally inaccessible. For the asymptotic limits, as are shown by the figures of the specific heat in the presence of deformation, all curves coincide, and reach the fixed value $C = 6k_B$ three times greater compared to the case of non-deformed Dirac oscillator in one and two dimensions.

As an application, we can extend our calculations to the case of graphene: Graphene is a two-dimensional configuration of carbon atoms organized in a hexagonal honeycomb structure. The electronic properties of graphene are exceptionally novel. For instance, the low-energy quasi-particles in Graphene behave as massless chiral Dirac fermions, which has led to the experimental observation of many interesting effects similar to those predicted in the relativistic regime. In the recent study [49], the author has shown, by using an approach based on the effective mass, that the model of a two-dimensional Dirac oscillator can be used to describe the thermodynamic properties of Graphene under an uniform magnetic field. By using the formalism of the creation and annihilation operators in complex formalism, he arrives at the following spectrum of energy

$$\varsigma_{n}^{\pm} = \pm \sqrt{2\frac{\hbar c}{l_B}} \sqrt{n},$$  \hspace{1cm} (83)

with $l_B = \sqrt{\frac{\hbar}{eB}}$ is the so-called magnetic length and $\hat{c}$ is the Fermi velocity of electrons in the graphene. The form of this spectrum of energy is in good agreement with the form of the case of graphene (See Ref [50]). This form of spectrum of energy in the presence of a deformation q can be written in a pure phase by

$$\tilde{\varsigma}_{n}^{\pm} = \pm \sqrt{2\frac{\hbar c}{l_B}} \sqrt{\frac{\sin (\eta n)}{\sin (\eta)}}.$$  \hspace{1cm} (84)

In the approximation of very small deformation, we have

$$\frac{\tilde{\varsigma}_{n}}{a_1} = \varsigma_{n}^{\pm} \left(1 + \frac{n^2}{12\eta^2}\right),$$  \hspace{1cm} (85)
with $a1 = \sqrt{2\hbar c}/k_B$, and consequently, the final partition function of the q-deformed version of graphene is

$$Z_q'(\tau, \eta) = Z'_0 + Z'_1,$$

with

$$Z'_0 = \tau^2 + \frac{1}{2},$$

$$Z'_1 = \frac{\eta^2}{12\tau} \sum_{n=0}^{\infty} n^2 e^{-\sqrt{n}\tau}.$$

Here $\tau = \frac{k_B T}{\sqrt{2\hbar c}}$. In order to evaluate $Z_1$, we use the Euler–MacLaurin formula: so we obtain

$$Z'_1 = 20\eta^2 \tau^5.$$

Thus, the final form of partition function is

$$Z_q'(\tau, \eta) = \tau^2 + \frac{1}{2} + 20\tau^5\eta^2.$$

Via equation (41), the determination of all thermal properties, such as the free energy, the entropy, total energy and the specific heat, can be obtained through the numerical partition function $Z_q'(\tau, \eta)$ via the following relations

$$F = -\frac{1}{\beta} \ln(Z) = -\tau \ln(Z), \quad U = -\frac{\partial \ln(Z)}{\partial \beta} = \tau^2 \frac{\partial \ln(Z)}{\partial \tau},$$

$$S = \frac{k_B}{\beta^2} \frac{\partial (\frac{U}{a^1})}{\partial \beta} = \ln(Z) + \tau \frac{\partial \ln(Z)}{\partial \tau}, \quad C = -\beta^2 \frac{\partial (\frac{U}{a^1})}{\partial \beta} = 2\tau \frac{\partial \ln(Z)}{\partial \tau} + \tau^2 \frac{\partial^2 \ln(Z)}{\partial \tau^2}.$$

The thermodynamic quantities are, respectively, plotted in Figure. From this figure, we observe that the behavior of the specific heat in the asymptotic regions is greater than that in the case of graphene: here the limit is $5k_B$. In addition, and as in the non-deformed case of graphene, we can argue this situation by saying that these limits follow the Dulong–Petit law for an ultra-relativistic ideal gas.

V. CONCLUSION

In this paper, after a brief preliminary around q-deformed oscillator, we studied the Dirac oscillator in this deformation formalism. We have found the eigenvalues and eigenfunctions by introducing q-deformed creation and annihilation operators using the complex formalism. It was shown that how energy eigenvalues of these oscillators in considered deformation formalism can be derived and also especially we tested them in limit cases that in both cases ordinary results were recovered. As well as treatments of energy eigenvalues for real and complex values of the $q$ parameter were depicted that difference between real and complex cases were shown clearly. It was seen that for real values of $q$, in the energy eigenvalues we faced with rapid rise so that the spectrum got expanded. On the contrary, when we set complex values for $q$, we were witness that the eigenvalues increased less rapidly than real case and also the spectrum got compressed that this treatments resembled us a periodic behaviors.

The connection between our q-deformed Dirac oscillator with quantum optics is well established via (JC) and (AJC) models, and the existence of well-known q-deformed version of Zitterbewegung in relativistic quantum dynamics has been discussed. In the case of very small deformation, we have calculated the in the pure phase case ($q = e^{i\eta}$) the partition function and all thermal quantities such as the free energy, total energy, entropy and specific heat. As application, we have extended our results to the case of graphene.

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Figure 4: Thermal properties of graphene in its q-deformed version
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