On the CP-violating phase $\delta_{\text{CP}}$ in fermion mixing matrices

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The recent established large $\theta_{13}$ in neutrino mixing provides an optimistic possibility for the investigation of the CP violation, therefore it is necessary to study the CP-violating phase $\delta_{\text{CP}}$ in detail. Based on the maximal CP violation hypothesis in the original Kobayashi-Maskawa (KM) scheme of neutrino mixing matrix, i.e., $\delta_{\text{KM}} = 90^\circ$, we calculate $\delta_{\text{CK}}$ for both quarks and leptons in the Chau-Keung (CK) scheme of the standard parametrization and find that $\delta_{\text{CK}}^{\text{quark}} = (68.62^{+0.89}_{-0.81}, 1.82)$ and $\delta_{\text{CK}}^{\text{lepton}} = (85.39^{+4.76}_{-4.82})$, provided with three mixing angles to be given. We also examine the sensitivity of $|V_{13}|$ and $|U_{ij}|$ to $\delta_{\text{CK}}$ and $\delta_{\text{KM}}$. As a convention-independent investigation, we discuss the $\Phi$ matrix, which has elements correspond to angles of the unitarity triangles. We demonstrate the $\Phi$ matrices for both quark and lepton sectors and discuss the implications as well as the variations of the $\Phi$ matrix elements with $\delta_{\text{CP}}$.

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I. INTRODUCTION

The oscillation of neutrinos have been verified for more than a decade, and such a phenomenon can be described by the standard model of particle physics (SM) as the misalignment of the flavor eigenstates with the mass eigenstates, as shown explicitly in the lagrangian of the charged current (CC) interaction

\[ L = \frac{g}{\sqrt{2}} U_L^\dagger \gamma^\mu V_{\text{CKM}} D_L W_{\mu}^+ - \frac{g}{\sqrt{2}} E_L^\dagger \gamma^\mu U_{\text{PMNS}} N_L W_{\mu}^- + \text{h.c.}, \]

(1)

where

\[ U_L = (u_L, c_L, t_L)^T; \quad D_L = (d_L, s_L, b_L)^T; \]
\[ E_L = (e_L, \mu_L, \tau_L)^T; \quad N_L = (\nu_1, \nu_2, \nu_3)^T. \]

(2)

We discuss the situation with only three generations, while additional generation can also be collaborated in the lagrangian. In Eq. (1), $V_{\text{CKM}}$, namely the Cabibbo-Kobayashi-Maskawa (KM) matrix \cite{1,2}, is the mixing matrix describing the mixing between different generations of quarks, and correspondingly $U_{\text{PMNS}}$, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix \cite{3}, describes the misalignment of the flavor eigenstates with the mass eigenstates of leptons. By choosing the mass matrix of charged leptons to be diagonal, the PMNS matrix represents the neutrino mixing, therefore we can also call it the neutrino mixing matrix.

Both the CKM matrix and the PMNS matrix are unitary matrices that can be parameterized by three Euler angles representing three rotations in certain planes and one Dirac phase angle representing the CP violation. If the neutrinos are of Majorana type, two additional Majorana phases are needed to fully determine the mixing matrix. While the Majorana phases do not manifest themselves in the oscillation, we discuss the Dirac phase only. This kind of parametrization, which can be referred to as the angle-phase parametrization, has the freedom of arranging the orders of these three rotations. Of the twelve ways to do the product, only nine are independent and the standard parametrization, i.e., the Chau-Keung (CK) scheme \cite{4} adopted by Particle Data Group \cite{5,6}, is one of the nine \cite{8,10}.

Recent progress on the measurement of the smallest neutrino mixing angle $\theta_{13}$ by the Daya-Bay \cite{11} and RENO collaborations \cite{12} has established a non-zero and relatively large value. This progress might be considered as a signal of the era of precise measurement of neutrino oscillation as well as an optimistic possibility for future measurements of the CP violating phase $\delta_{\text{CP}}$ in the neutrino mixing. In fact, a non-zero $\theta_{13}$ was indicated by various experiments, i.e., the T2K, MINOS and Double-Chooz collaborations since last year \cite{13,15}.

To accommodate the experimental data of neutrino mixing, a certain parametrization should be adopted. As mentioned before, the Chau-Keung (CK) scheme is adopted as the standard one, and the mixing angles in this scheme are directly related to the observed oscillation probabilities. Since all the parameterizations are equivalent to each other mathematically and other schemes may still have some advantages in phenomenological analysis as well as model building, it is meaningful to explore schemes other than the standard one for the possibility to find some clues towards a better understanding of fermion properties. For example, the original Kobayashi-Maskawa (KM) scheme \cite{2} allows for almost a perfect maximal CP violation of the quark mixing, i.e., the CP violating phase $\delta_{\text{CK}}^{\text{quark}} = 90^\circ$ \cite{14,21}, whereas in the standard parametrization $\delta_{\text{CK}}^{\text{quark}} = 68.8^\circ$ \cite{5}, which deviates from the maximal CP violation. This arises naturally the following questions: Is there a maximal CP violation in the lepton sector? What happens if $\delta_{\text{KM}} = 90^\circ$? Is there any reason behind the hypothesis? As there

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is no experimental information on $\delta_{\text{CP}}^{\text{lepton}}$ yet, we take $\delta_{\text{KM}}^{\text{lepton}} = 90^\circ$ as an Ansatz, to see what can we get for neutrino mixing.

Cautions should be taken when talking about the maximal CP violation, because historically, the notion refers to a maximized $J$ [22], i.e., the Jarlskog invariant [23]. The conception we adopt here refers to the case that the CP-violating phase equals to $90^\circ$, which guarantees the term $\sin \delta$ in its maximal value as such term always shows up in the Jarlskog invariant [17].

We make our Ansatz of a maximal CP violation $\delta_{\text{KM}}^{\text{lepton}} = 90^\circ$ in the lepton sector based on all the similarities shared by quarks and leptons and the fact that there is no information on the CP-violating phase experimentally now. Based on the Ansatz, we can work out all the elements of the neutrino mixing matrix together with a prediction of the CP-violating phase $\delta_{\text{lepton}}^{\text{CP}}$ as has been shown in Ref. [24] briefly. In Section III, we firstly perform a replaying procedure in the quark sector as a test for $\delta_{\text{KM}}^{\text{lepton}} = 90^\circ$ as well as an exercise for the method. Then we take $\delta_{\text{KM}}^{\text{lepton}} = 90^\circ$ as an Ansatz to provide a prediction of $\delta_{\text{lepton}}^{\text{CP}}$. In Section IV, we examine the dependence of the mixing matrix on the CP-violating phase. As there is a rephasing freedom in the mixing matrix, we also discuss the convention-independent $\Phi$ matrix in Section V where we provide predictions for all of the unitarity triangles of neutrinos. Section V serves for some discussions and conclusions.

II. THE CP-VIOLATING PHASES $\delta_{\text{CK}}$ IN QUARK AND LEPTON SECTORS

A. Reproduction of $\delta_{\text{CK}}$ in the quark sector

The four parameters needed to determine the CKM matrix have been measured to high precision. Using the global fit result of the four Wolfenstein parameters [7], we can get three mixing angles together with the CP-violating phase in any angle-phase parametrization. In the following we perform a replay to obtain the CP-violating phase $\delta_{\text{CK}}^{\text{quark}}$ in the standard parametrization (i.e., the CK scheme) provided with three mixing angles to be given together with a maximal CP violation hypothesis in the KM scheme. Our purpose is to check the procedure and then to make prediction of the CP-violating phase $\delta_{\text{CK}}^{\text{lepton}}$ for the lepton sector. Our calculation is to obtain $\delta_{\text{CK}}^{\text{lepton}}$ from $\delta_{\text{CK}}^{\text{quark}} = 90^\circ$, i.e., we assume that $\delta_{\text{CK}}^{\text{quark}}$ is unknown. The mixing angles in the standard parametrization are deduced from the Wolfenstein parameters,

\begin{equation}
\begin{align*}
    s_{12} &= \lambda = 0.2253 \pm 0.0007; \\
    s_{23} &= A \lambda^2 = 0.0410^{+0.0011}_{-0.0008}; \\
    s_{13} &= |A \lambda^3 (\rho + i \eta)| = 0.0035^{+0.0002}_{-0.0001};
\end{align*}
\end{equation}

in which $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$ ($i, j = 1, 2, 3$). From the expression of the Chau-Keung (CK) scheme

$$
V_{\text{CK}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13} e^{-i \delta_{\text{CK}}} \\
0 & 1 & 0 \\
-s_{13} e^{i \delta_{\text{CK}}} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

we can get the moduli of five CKM matrix elements

\begin{align*}
    |V_{ud}| &= c_{12} c_{13} = 0.9743 \pm 0.0002; \\
    |V_{us}| &= s_{12} c_{13} = 0.2253 \pm 0.0007; \\
    |V_{ub}| &= s_{13} = 0.0034^{+0.0002}_{-0.0001}; \\
    |V_{cb}| &= s_{23} c_{13} = 0.0410^{+0.0011}_{-0.0008}; \\
    |V_{tb}| &= c_{23} c_{13} = 0.9992^{+0.0005}_{-0.0003}.
\end{align*}

Substituting the five elements values into the expression of the Kobayashi-Maskawa (KM) scheme,
The corresponding trigonometric functions are

\[ V_{\text{KM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 - s_2 & 0 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & e^{i\delta_{\text{KM}}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & s_3 & -c_3 \end{pmatrix} \]

\[ = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta_{\text{KM}}} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta_{\text{KM}}} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta_{\text{KM}}} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta_{\text{KM}}} \end{pmatrix}, \] (6)

and we can make a prediction of the CK phase in the Chau-Keung (CK) scheme by solving the equation

\[ J_{\text{CK}}^\text{quark} = J_{\text{CK}}^\text{lepton} \]

where

\[ J_{\text{KM}}^\text{quark} = \frac{1}{8} \sin \theta_1 \sin 2 \theta_2 \sin 2 \theta_3 \sin \delta_{\text{KM} \text{quark}} \]

\[ = (2.90^{+0.19}_{-0.15}) \times 10^{-5}. \] (10)

Thus we can make a reproduction of the CP phase in the Chau-Keung (CK) scheme by solving the equation

\[ J_{\text{CK}}^\text{quark} = J_{\text{CK}}^\text{lepton} \]

where

\[ J_{\text{CK}}^\text{quark} = \frac{1}{8} \cos \theta_{13} \sin 2 \theta_{12} \sin 2 \theta_{23} \sin 2 \theta_{13} \sin \delta_{\text{CK} \text{quark}} \] (11)

We get

\[ \delta_{\text{CK} \text{quark}} = (68.62^{+0.89}_{-0.81})^\circ, \] (12)

which is in accordance with the result extracted from the Wolfenstein parameters [1], which indicate

\[ \delta_{\text{CK} \text{quark}} = (68.81^{+3.30}_{-2.17})^\circ. \] (13)

B. Prediction of \( \delta_{\text{CK}} \) in the lepton sector

We have seen that the hypothesis of the maximal CP violation can reproduce a reasonable \( \delta_{\text{CK} \text{quark}} \). Under the same procedure, together with the Ansatz of a maximal CP violation in the lepton sector, i.e., \( \delta_{\text{KM}} = 90^\circ \) for the KM-scheme of mixing matrix, we can make a prediction of \( \delta_{\text{CK} \text{lepton}} \).

We adopt the global fit of neutrino mixing angles based on previous experimental data including T2K and MINOS experiments (1\( \sigma (3\sigma) \)) [23]

\[ \sin^2 \theta_{12} = 0.312^{+0.017}_{-0.016} (0.052); \]
\[ \sin^2 \theta_{23} = 0.43^{+0.08}_{-0.07} (0.22); \] (14)
for our input of $\theta_{12}$ and $\theta_{23}$. While for $\theta_{13}$, it is reasonable to make use of all the recent data showed in Table III. The resulting $\theta_{13}$ is

$$\sin^2 2\theta_{13} = 0.097 \pm 0.013.$$  \hspace{1cm} (15)

Explicitly, the three mixing angles in our input are

$$\theta_{12} = (33.96^{+1.03}_{-0.99}(-2.91))^{{\circ}};$$
$$\theta_{23} = (40.40^{+4.64}_{-1.74}(+12.77))^{{\circ}};$$
$$\theta_{13} = (9.07 \pm 0.63(1.89))^{{\circ}}. \hspace{1cm} (16)$$

The moduli of five matrix elements are

$$|U_{e1}| = c_{12}c_{13} = 0.819 \pm 0.010; \hspace{1cm} (17)$$
$$|U_{e2}| = s_{12}c_{13} = 0.552^{+0.015}_{-0.014}; \hspace{1cm} (18)$$
$$|U_{e3}| = s_{13} = 0.158 \pm 0.011; \hspace{1cm} (19)$$
$$|U_{\mu 3}| = s_{23}c_{13} = 0.640^{+0.061}_{-0.023}; \hspace{1cm} (20)$$
$$|U_{\tau 3}| = c_{23}c_{13} = 0.752^{+0.052}_{-0.019}; \hspace{1cm} (21)$$

With these five moduli, together with an Ansatz of maximal CP violation $\delta_{KM}^{lepton} = 90^\circ$, we can get the mixing angles in the KM parametrization

$$\theta_1 = (35.01^{+1.02}_{-0.96})^{{\circ}};$$
$$\theta_2 = (39.85^{+5.20}_{-3.95})^{{\circ}};$$
$$\theta_3 = (15.96 \pm 1.14)^{{\circ}}. \hspace{1cm} (22)$$

The corresponding trigonometric functions are

$$\sin \theta_1 = 0.574^{+0.015}_{-0.014}; \quad \cos \theta_1 = 0.819 \pm 0.010; \hspace{1cm} (23)$$
$$\sin \theta_2 = 0.641^{+0.070}_{-0.026}; \quad \cos \theta_2 = 0.768^{+0.058}_{-0.022}; \hspace{1cm} (24)$$
$$\sin \theta_3 = 0.275 \pm 0.019, \quad \cos \theta_3 = 0.961 \pm 0.005; \hspace{1cm} (25)$$

Thus we have all the parameters in Eq. (6). Then we can get all the moduli of the PMNS matrix, which is

$$|U_{PMNS}| = \begin{pmatrix}
0.819 \pm 0.010 & 0.552^{+0.015}_{-0.014} & 0.158 \pm 0.011 \\
0.440^{+0.035}_{-0.016} & 0.639^{+0.059}_{-0.016} & 0.640^{+0.061}_{-0.023} \\
0.368^{+0.041}_{-0.018} & 0.541^{+0.045}_{-0.018} & 0.752^{+0.052}_{-0.019} 
\end{pmatrix}. \hspace{1cm} (26)$$

The corresponding Jarlskog invariant is

$$J_{KM}^{lepton} = \frac{1}{8} \sin \theta_1 \sin 2\theta_1 \sin 2\theta_2 \sin 2\theta_3 \sin \delta_{KM}^{lepton} = 0.035^{+0.003}_{-0.002}. \hspace{1cm} (27)$$

As the same procedure used in the quark sector, we can give our prediction of the CP-violating phase in the Chau-Keung (CK) scheme by solving the equation $J_{CK}^{lepton} = J_{KM}^{lepton}$, where

$$J_{CK}^{lepton} = \frac{1}{8} \cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \delta_{CK}^{lepton}. \hspace{1cm} (28)$$

The resulting $\delta_{CK}^{lepton}$ is

$$\delta_{CK}^{lepton} = (85.39^{+4.76}_{-1.82})^{{\circ}}. \hspace{1cm} (29)$$

Note that corresponding to our input of the mixing angles, $J_{lepton}^{lepton}$ is in the range

$$0 \leq J_{lepton}^{lepton} \leq 0.035^{+0.003}_{-0.002}. \hspace{1cm} (30)$$

where "=" happens for $\delta_{lepton}^{lepton} = 0$ or $\delta_{lepton}^{lepton} = 90^\circ$. Though $\delta_{lepton}^{lepton} = 90^\circ$ corresponds to $\delta_{lepton}^{lepton} = 85.39^\circ$, there is a slight difference in the $J_{lepton}^{lepton}$ range if we let $\delta_{lepton}^{lepton} = 90^\circ$. However, the difference is of $\mathcal{O}(10^{-2})$ and we just neglect it.

It is interesting to notice that our procedure leads to a quasi-maximal CP violation in the standard parametrization. However, there have been some theoretical investigations indicating that a large CP-violating phase $\delta_{lepton}^{lepton}$ can be understood from some basic asymmetries. For example, the near maximal CP violation with a large $\theta_{13}$ from our analysis is in accordance with a general approach based on residual $Z_2$ symmetries [20]. The maximal or large CP violation are also predicted from some theoretical reasoning [27,29], thus our prediction of a quasi-maximal $\delta_{lepton}^{lepton}$ or a maximal $\delta_{lepton}^{lepton}$ might acquire theoretical support from basic considerations.

III. THE SENSITIVITY OF THE MIXING MATRIX ELEMENTS TO THE CP-VIOLATING PHASE $\delta_{CP}$

Previous discussions are based on the hypothesis of the maximal CP violation, while it is still helpful to check how the moduli of the mixing matrix elements vary with the CP-violating phase $\delta_{CP}$. We can get the information on the sensitivity of $|V_{ij}|$ and $|U_{ij}|$ to $\delta_{CP}$, and such information would be useful when extracting the CP-violating information from the mixing matrix elements. We make such an investigation in both the CK scheme and the KM scheme.

A. $|V_{ij}|$ as functions of $\delta_{CK}$ and $\delta_{KM}$

Since the moduli of the CKM matrix elements are measured by various processes, discussing the sensitivity of $|V_{ij}|$ to $\delta_{CP}$ can be useful when these measurements reach a higher precision so that information on the CP violation can be extracted directly from the moduli. We try to find out which elements are more sensitive to the CP-violating phase.

The mixing angles in the angle-phase parametrizations are not observables in the case of quark mixing, so we adopt the global fit result of the Wolfenstein parameters to get the mixing angles in the standard parametrization as we do in Section II. Thus we get the CKM matrix with only one unrestrained parameter $\delta_{CK}$. Taking the range indicated by the global fit as a reasonable one, we plot $|V_{ij}|$ as a function of $\delta_{CK}$ in Fig. [1].

Using the same procedure as in the previous section, namely, calculating the mixing angles in the KM scheme
TABLE I: The recent experimental data on $\theta_{13}$

| Experimental collaboration | Data on $\theta_{13}$ |
|----------------------------|-----------------------|
| T2K                        | $0.03(0.04) < \sin^2 2\theta_{13} < 0.28(0.34)$, NH(IH) $\sin^2 2\theta_{13} = 0.11 \pm 0.17^a$ |
| MINOS                      | $2 \sin^2 \theta_{23} \sin^2 2\theta_{13} = 0.041 \pm 0.003(0.079^{+0.071}_{-0.053})$, NH(IH) $\sin^2 2\theta_{13} = 0.049 \pm 0.075^b$ |
| Double Chooz               | $\sin^2 2\theta_{13} = 0.086 \pm 0.041$ (stat) $\pm 0.030$ (syst) $\sin^2 2\theta_{13} = 0.086 \pm 0.051^c$ |
| Daya Bay                   | $\sin^2 2\theta_{13} = 0.092 \pm 0.016$ (stat) $\pm 0.005$ (syst) $\sin^2 2\theta_{13} = 0.092 \pm 0.017^c$ |
| RENO                       | $\sin^2 2\theta_{13} = 0.113 \pm 0.013$ (stat) $\pm 0.019$ (syst) $\sin^2 2\theta_{13} = 0.113 \pm 0.023^c$ |

The weighted average of $\theta_{13}$ $\langle \sin^2 2\theta_{13} \rangle = 0.097 \pm 0.013$

$^a$Taking the latest results into account, we adopt the NH case with a symmetrized error range $\pm 0.17$.

$^b$We take the NH case in accordance with the former one and extract $\theta_{13}$ with $\theta_{23}$ taking its global fit value.

$^c$The 1$\sigma$ deviation is estimated by $\sigma^2 = \sigma^2_{\text{stat}} + \sigma^2_{\text{syst}}$.

FIG. 1: $|V_{ij}|$ as a function of $\delta_{\text{CK}}$ in the CK scheme. The dashed lines denote $\delta_{\text{CK}} = 90^\circ$ and $|V_{ij}|$ (for $\delta_{\text{CK}} = 90^\circ$, 180$^\circ$).

with the matrix elements that are independent of the phase, and leaving $\delta_{\text{KM}}$ unrestrained, we plot $|V_{ij}|$ as a function of $\delta_{\text{KM}}$ in Fig. 2.

From Fig. 2 we can see that in the range indicated by the global fit, the hypothesis of the maximal CP violation works quite well.

We list the range of $|V_{ij}|$ for a $\delta_{\text{CP}}$ ranging $(0-180)^\circ$ in Table II for both the CK scheme and the KM scheme. In each scheme, there are four $|V_{ij}|$ depending on $\delta_{\text{CP}}$ and the four elements are not necessarily the same with respect to their corresponding elements in another scheme. With a certain scheme, which four elements are dependent on $\delta_{\text{CP}}$ is a question of phase convention. We adopt the phase convention as in Eq. 4 and Eq. 6.

From Table II we see that, for $|V_{ij}|$ that are dependent on different $\delta_{\text{CP}}$, the dependence might differ in orders of magnitude if we compare the dependence in terms of $\Delta|V_{ij}|$ range, e.g., $\Delta|V_{ts}|$ is of $O(10^{-4})$ on dependence of $\delta_{\text{CK}}$ and of $O(10^{-5})$ on dependence of $\delta_{\text{KM}}$. More explicitly, the dependence can be classified into,

1. $O(10^{-2})$: $\Delta|V_{cb}|$, $\Delta|V_{ts}|$ versus $\delta_{\text{KM}}$;

2. $O(10^{-3})$: $\Delta|V_{td}|$, $\Delta|V_{ts}|$ versus $\delta_{\text{CK}}$, and $\Delta|V_{cb}|$, $\Delta|V_{tb}|$ versus $\delta_{\text{KM}}$;

3. $O(10^{-4})$: $\Delta|V_{td}|$, $\Delta|V_{ts}|$ versus $\delta_{\text{CK}}$.

From these results, we can observe that the KM scheme is more sensitive when the CP-violating information is extracted from the measured CKM matrix elements. The results also indicate that $|V_{cb}|$ and/or $|V_{ts}|$ are good candidates for extracting $\delta_{\text{KM}}$.

B. $|U_{ij}|$ as functions of $\delta_{\text{CK}}$ and $\delta_{\text{CP}}$

The mixing angles in the neutrino mixing matrix are related to the observed oscillation probabilities and are consequently observables. We adopt the global fit values of $\theta_{12}$ and $\theta_{23}$, and the averaged $\theta_{13}$ in Table I as our input. Since $|U_{ij}|$ are not well determined and there is no experimental information on $\delta_{\text{lepton}}$, we plot $|U_{ij}|$ in a range where $\delta_{\text{CP}}$ takes all its possible values in Fig. 3 and Fig. 4.
FIG. 2: $|V_{ij}|$ as a function of $\delta_{KM}$ in the KM scheme. The dashed lines denote $\delta_{CK} = 90^\circ$ and $|V_{ij}|$ (for $\delta_{KM} = 90^\circ, 180^\circ$).

FIG. 3: $|U_{ij}|$ as a function of $\delta_{CK}$ in the CK scheme. The dashed lines denote $\delta_{CK} = 90^\circ$ and $|U_{ij}|$ (for $\delta_{CK} = 90^\circ, 180^\circ$).

### TABLE II: The $\Delta|V_{ij}|$ range indicated by $\delta$ of $(0 - 180)^\circ$

| $\Delta|V_{ij}|$   | The CK scheme | The KM scheme |
|-------------------|---------------|---------------|
| $\Delta|V_{ud}|$   | $-a$          | -             |
| $\Delta|V_{us}|$   | -             | -             |
| $\Delta|V_{ub}|$   | -             | -             |
| $\Delta|V_{cd}|$   | 0.0003        | -             |
| $\Delta|V_{cs}|$   | 0.0001        | 0.0011        |
| $\Delta|V_{cb}|$   | -             | 0.0292        |
| $\Delta|V_{td}|$   | 0.0066        | -             |
| $\Delta|V_{ts}|$   | 0.0015        | 0.0300        |
| $\Delta|V_{tb}|$   | -             | 0.0011        |

$^a$ $-a$ denotes this matrix element being independent of $\delta$. 
We also calculate the $\Delta |U_{ij}|$ range and list the result in Table 11. From Table 11 we see that $\Delta |U_{ij}|$ are all of the same order, which is different from the situation as in quarks. Such a result indicates that all $|U_{ij}|$ within a certain scheme are of similar sensitivity when they are used to extract the CP-violating information. Besides, $\Delta |U_{ij}|$ is larger for a variation of $\delta_{KM}$. This indicates that $|U_{ij}|$ are more sensitive to $\delta_{KM}$.

It should be emphasized that the discussions here are only purposed for illustration because of the ambiguity caused by the rephasing transformation. Our discussions are based on the phase convention in Eq. [4] and Eq. [6]. It is natural to seek for phase-convention independent ways as we show in the following section.

IV. THE MATRIX OF UNITARITY TRIANGLE ANGLES

The physical observables are not affected by rephasing the corresponding fields so it is better to find rephasing-invariant descriptions. There have been several successful attempts. For example, the Jarlskog invariant $J$, which was proposed by Jarlskog in seeking for the commutator of the mass matrices [23], is invariant under rephasing transformation. All CP violation effects can be expressed as functions of $J$. It was also pointed out by Wu in Ref. [30] that special quartet forms like $V_{ij}V_{kj}V_{ik}$, ($i,j,k$ cyclic and $\alpha,\beta,\gamma$ cyclic) are convention-invariant quantities. For three generation case, there are nine such quantities, which can form a matrix like,

$$
\Pi = \begin{pmatrix}
V_{tb}V_{ts}V_{cs}V_{cb}^{*} & V_{td}V_{ts}V_{cs}V_{cb}^{*} & V_{ts}V_{td}V_{cd}V_{cs}^{*} \\
V_{ub}V_{us}V_{ts}V_{ub}^{*} & V_{ud}V_{us}V_{ts}V_{ub}^{*} & V_{us}V_{ud}V_{td}V_{ts}^{*} \\
V_{cb}V_{cs}V_{us}V_{ub}^{*} & V_{cd}V_{cs}V_{us}V_{ub}^{*} & V_{cs}V_{cd}V_{us}V_{ub}^{*}
\end{pmatrix}.
$$

(31)

It was pointed out by Harrison, Dallison and Scott in Ref. [31] that the matrix $-\Pi^{*}$ can be decomposed into two matrices like,

$$
-\Pi^{*} = \begin{pmatrix}
K_{ud} & K_{us} & K_{ub} \\
K_{cd} & K_{cs} & K_{cb} \\
K_{td} & K_{ts} & K_{tb}
\end{pmatrix} + i \begin{pmatrix}
J & J & J \\
J & J & J \\
J & J & J
\end{pmatrix},
$$

(32)

and we call the first one $K$ matrix. The orthogonality condition aroused from the unitarity of the mixing matrix can be translated into a geometrical language, i.e., the unitarity triangle. There are six unitarity triangles for a 3 by 3 unitarity matrix. It is also pointed out in Ref. [31] that the matrix of unitarity triangle angles $\Phi$, can be constructed from the matrix in Eq. [31],

$$
\Phi_{\text{quark}} = \begin{pmatrix}
\arg(-\Pi_{ud}^{*}) & \arg(-\Pi_{us}^{*}) & \arg(-\Pi_{ub}^{*}) \\
\arg(-\Pi_{cd}^{*}) & \arg(-\Pi_{cs}^{*}) & \arg(-\Pi_{cb}^{*}) \\
\arg(-\Pi_{td}^{*}) & \arg(-\Pi_{ts}^{*}) & \arg(-\Pi_{tb}^{*})
\end{pmatrix}.
$$

(33)

Each element in the $\Phi$ matrix corresponds to an inner angle of a unitarity triangle, and 3 elements in each row or column correspond to the three angles of a unitarity triangle. The $\Phi$ matrix is rephasing-invariant, real and related to the geometrical image, i.e., the unitarity triangles directly. The $K$ matrix is correlated to the $\Phi$ matrix by $K = J\cot\Phi$. Similar $\Phi$ matrix can also be constructed in the lepton sector.

Using the mixing angles and the phase indicated by Wolfenstein parameters, we work out the complex CKM matrix in the CK scheme and then get the following $\Phi_{\text{quark}}$ matrix,

$$
\Phi_{\text{CKM}} = \begin{pmatrix}
101^{\circ} & 20.88^{\circ} & 158.11^{\circ} \\
67.83^{\circ} & 90.31^{\circ} & 21.86^{\circ} \\
111.16^{\circ} & 68.81^{\circ} & 0.03^{\circ}
\end{pmatrix}.
$$

(34)

Similarly, we can construct the $\Phi_{\text{quark}}$ matrix in the KM.
values in the Φ
the lepton sector, we can get the following Φ matrix,
well in an explicit way for quarks.




Moreover, in the Φ
quark
scheme, these five elements are almost symmetric versus δ
len,
and




Note that the maximal CP violation hypothesis is used to get the above result rather than deducing δ
KM
directly from the Wolfenstein parameters. Comparing these angles in Eq. (34) and Eq. (35), we see that there are only slight difference in the obtained Φ
quark
matrix elements between the two schemes. Besides, in the Φ
quark
matrix, F
us,
F
ts,
and
F
ts
correspond to β,
α,
and
γ
in the db unitarity triangle separately. We see that the predicted
beta,
alpha,
and
gamma
in the Φ
quark
matrix are very close to the fit values in the Φ
CK
matrix. Both values for
beta,
alpha,
and
gamma
are compatible with the measured ones. This demonstrates that the maximal CP violation hypothesis works well in an explicit way for quarks.

Taking the hypothesis of the maximal CP violation in the lepton sector, we can get the following Φ matrix,



where we use the predicted δ
CK
= 85.38° as an input. Notice that this δ
CK
is deduced from a maximal CP phase in the KM scheme, therefore we do not expect anything new from the Φ
lepton
matrix except for very small differences caused by precision.

From Eq. (34), Eq. (35), and Eq. (36), we see that ∑
κ
Φ
κ
= 180° and ∑
j
Φ
ij
= 180°, i.e., the sum of every row and every column of the Φ matrix equals to 180° as they correspond to three inner angles of a triangle.

Leaving δ
CP
unrestrained, we plot Φ
ij
as functions of δ
CK
and δ
KM
in Fig. 5 and Fig. 6.

From Fig. 5 and Fig. 6, we find that,
1. the trends of Φ
ij
with the variations of δ
CK
and δ
KM
are similar, which indicates a weak dependence on parametrizations when other parameters are fixed;
2. given same values for δ
CP,
the lepton unitarity triangles are sizable, i.e., there is no very small angles compared with those in the quark case;
3. the four elements in the left-bottom are sensitive to δ
CP
both in the quark and lepton mixing; these four elements can be in four unitarity triangles and the db unitarity triangle in the quark mixing is one of the four;
4. Φ
11
lepton,
Φ
12
lepton,
Φ
13
lepton,
Φ
23
lepton,
and Φ
33
lepton
are parabolic curves so two values of δ
are indistinguishable for only one element of Φ
ij
; in the CK scheme, these five elements are almost symmetric versus δ
len,
and 180° − δ
CK
are hard to be distinguished in these cases for each Φ
ij
.

V. DISCUSSIONS AND CONCLUSIONS

We have performed some investigations on the CP-violating phase δ
CP
from several aspects.

In Section II under the Ansatz of a maximal CP violation δ
CK
len
= 90° in the lepton sector, we provide a prediction of the CP-violating phase δ
CK
len
= (85.39±4.76)° as well as a prediction of the PMNS matrix in Eq. (26). A replaying procedure is used firstly in the quark sector as an exercise. We see that the hypothesis of the maximal CP violation in the quark sector offers δ
CP
= 90°, which means that δ
CK
len
and 180° − δ
CP
are close to the measured value δ
CK
= (68.62±0.89)°. Besides, we may mention that the priori definition of maximal CP violation, i.e., maximized J under all four mixing parameters, has been ruled out experimentally in the quark sector. In every angle-phase parametrization δ
CP
shows up in sin δ
CP,
so δ
CP
= 90° contributes most to J with respect of this parameter. We take this as our definition of a maximal CP violation.

In Section III the variations of the moduli of the mixing matrix elements with the CP-violating phase δ
CP
are demonstrated graphically. For |V
ij
| and |U
ij
| that are dependent on δ
CP
under a certain scheme, we find that ∆|V
ij
| differ in orders of magnitude while ∆|U
ij
| are all of the same order. The KM scheme stands out as more sensitive if the CP-violating information is extracted from
FIG. 5: $\Phi_{ij}$ as a function of $\delta_{CP}$ in the CK scheme. The solid curve denotes $\Phi_{ij}^{\text{quark}}$; the dashed curve stands for $\Phi_{ij}^{\text{lepton}}$; the points correspond to $\delta_{\text{CK}}^{\text{quark}} = 68.8^\circ$ and $\delta_{\text{CK}}^{\text{lepton}} = 85.4^\circ$.

FIG. 6: $\Phi_{ij}$ as a function of $\delta_{CP}$ in the KM scheme. The solid curve denotes $\Phi_{ij}^{\text{quark}}$; the dashed curve stands for $\Phi_{ij}^{\text{lepton}}$; the points correspond to $\delta_{\text{KM}}^{\text{quark}} = 90^\circ$ and $\delta_{\text{KM}}^{\text{lepton}} = 90^\circ$. 
the measured elements of the CKM matrix or the PMNS matrix. We also find that $|V_{cb}|$ and $|V_{ts}|$ are good candidates for extracting $\delta_{\text{KM}}^{\text{quark}}$. All the discussions in this section are constrained to the phase convention in Eq. [4] and Eq. [6].

The dependence of the matrix elements on $\delta_{\text{CP}}$ is convention-dependent, whereas physical observables are independent of the phase convention. So we adopt the $\Phi$ matrix description and make some discussions on it in Section II. In this section, by using the convention-independent $\Phi$ matrix, we continue our discussion by giving the $\Phi$ matrices for both quarks and leptons under the Ansatz of a maximal CP violation. We also demonstrate the variations of $\Phi_{ij}$ with $\delta_{\text{CP}}$ in both CK and KM schemes. We provide predictions of all of the unitary triangles which are directly relevant to the CP violation effect in neutrino oscillation for future experiments.

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