EXTREMAL DISCS AND THE HOLOMORPHIC EXTENSION FROM CONVEX HYPERSURFACES

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1. Introduction

Let $D \subset \mathbb{C}^n$ be a convex domain with smooth boundary $\partial D$ and let $f$ be a continuous function on $\partial D$. Suppose for every complex line $L$ the restriction $f|_{L \cap \partial D}$ holomorphically extends into $L \cap D$. Then $f$ extends to $D$ as a holomorphic function of $n$ variables (Stout [10]). The conclusion is still true if instead of the holomorphic extendibility of $f$ into the sections $L \cap D$, we assume the weaker Morera condition

\[(1.1) \quad \int_{L \cap \partial D} f\alpha = 0\]

for every $(1,0)$-form $\alpha$ with constant coefficients and every complex line $L$ (Globevnik and Stout [4]).

The condition of holomorphic extendibility into sections $L \cap D$ and even the Morera condition (1.1) for all lines $L$ seem excessively strong because it suffices to use only the lines close to the tangent lines to $\partial D$. Indeed, for simplicity assume $f \in C^1(\partial D)$. Then the Morera condition for $L$ as $L$ approaches a tangent line $L_0$ at $z_0 \in \partial D$ implies that the $\bar{\partial}$ derivative of $f$ at $z_0$ along $L_0$ equals zero. Then $f$ holomorphically extends to $D$ by the classical Hartogs-Bochner theorem. Therefore of great interest are “small” families of lines, for which the result is still true. In particular, the family of lines should not contain the lines close to the tangent lines to $\partial D$.

Reducing the family of lines, Agranovsky and Semenov [1] show that if $D_2 \subset D_1$ are domains in $\mathbb{C}^n$ and $f \in C(\partial D_1)$ holomorphically extends into sections $L \cap D_1$ by the lines that meet $D_2$, then $f$ holomorphically extends to $D_1$. In the case of two concentric balls $D_2 \subset D_1$, Rudin [9] proves that the same conclusion is valid if one only assumes the extendibility into sections by the lines tangent to $\partial D_2$. Globevnik [5] observes that in Rudin’s result one only needs the lines tangent to a sufficiently large open set in $\partial D_2$.

Globevnik and Stout [4] conjecture that Rudin’s result is valid for every convex domains $D_2 \subset \subset D_1$, that is if $f \in C(\partial D_1)$ holomorphically extends into sections $L \cap D_1$ by the lines tangent to $\partial D_2$, then $f$ holomorphically extends to $D_1$. They also observe in [4] (see also [2]) that for $n = 2$ in Rudin’s result one generally cannot replace the extendibility into the sections $L \cap D_1$ by the Morera condition (1.1), that is the latter suffices unless the ratio $r_1/r_2$ of the radii of the balls belongs to an exceptional countable set. For a counterexample
in $\mathbb{C}^2$, take $r_1 = 1$, $r_2 = \sqrt{1/3}$, and $f = z_1 \bar{z}_2$. However if $n > 2$, then Berenstein, Chang, Pascuas, and Zalcman \cite{2} show that for the concentric balls the Morera condition for the tangent lines suffices without exceptions.

Further reduction of the family of lines is possible. Globevnik \cite{6} shows that for the unit ball $D \subset \mathbb{C}^2$, the holomorphic extension property into sections by lines of certain two parameter family suffices for the holomorphic extendibility into $D$. The set of lines in his result consists of two disjoint tori. The second author shows \cite{12} that for every generating CR manifold $M \subset \mathbb{C}^n$ of dimension $d$ there exists a $(d-1)$-parameter family of analytic discs attached to $M$ so that if $f \in C(M)$ holomorphically extends to those discs, then $f$ is a CR function on $M$.

Despite the large amount of work done on the subject, the conjecture of Globevnik and Stout has been open so far. In this paper we prove a version of the conjecture in which the complex lines are replaced by the complex geodesics of the Kobayashi metric for $D_1$ also known as extremal or stationary discs, whose theory was developed by Lempert \cite{8}. We believe that the extremal discs for a general convex domain $D_1$ are more appropriate in the problem than the lines because they are intrinsically defined, invariant under biholomorphisms, and coincide with the lines for the ball. Hence, if $D_1$ is the ball and $D_2$ is an arbitrary strictly convex subdomain, then our result proves the conjecture of Globevnik and Stout for the lines as stated. As in Globevnik’s result \cite{5} cited above, we only need the extendibility into the extremal discs tangent to a sufficiently large open set in $\partial D_2$ (cf. Remark \ref{remark3}.

The authors of the results for the concentric balls use the Fourier analysis and decomposition into spherical harmonics. This method does not seem to work for general convex domains. We employ the method of \cite{13} according to which we add an extra variable, the fiber coordinate in the projectivized cotangent bundle. Then using the lifts of the extremal discs we lift the given function $f$ to a CR function on the boundary of a wedge $W$ whose edge is the projectivized conormal bundle of $\partial D_1$. Then using the theory of CR functions we extend it to a bounded holomorphic function in $W$. Finally since $W$ contains “large” discs, we prove that the lifted function actually does not depend on the extra variable, which proves the result.

We feel that the method developed here has a wider scope, and we plan to use it on other occasions.

2. Extremal discs

We will collect here, and develop in some details, the main results of \cite{8} which are needed for our discussion. Let $D$ be a bounded domain of $\mathbb{C}^n$ with $C^k$-boundary; according to \cite{8} we assume $k \geq 6$. We also assume that $D$ is strongly convex in the sense that $D$ has a global defining function with positive real Hessian. An analytic disc in $\mathbb{C}^n$ is a holomorphic mapping $\Delta \to \mathbb{C}^n$, smooth up to $\partial \Delta$, where $\Delta$ is the standard disc in $\mathbb{C}$. We denote by $A$ the image set under $\varphi$. The disc $A$ is said to be “attached” to $\partial D$ when $\partial A \subset \partial D$. 
Definition 2.1. An analytic disc \( \varphi \) in \( D \) is said to be “stationary” when it is attached to \( \partial D \) and endowed with a meromorphic lift \( \varphi^*(\tau) \in (T^*\mathbb{C}^n)_{\varphi(\tau)} \) \( \forall \tau \in \Delta \) with one simple pole at 0 such that \( \varphi^*(\tau) \in (T^*\partial D\mathbb{C}^n)_{\varphi(\tau)} \) when \( |\tau| = 1 \). In other words, \( (\varphi, \varphi^*) \) is attached to the conormal bundle \( T^*\partial D\mathbb{C}^n \).

Definition 2.2. An analytic disc \( \varphi \) in \( D \) is said to be “extremal” when for any other disc \( \psi \) in \( D \) with \( \psi(0) = \varphi(0) \) and \( \psi'(0) = \lambda \varphi'(0) \), \( \lambda \in \mathbb{C} \), we have \( |\lambda| < 1 \).

It is shown in [8] that extremal and stationary discs coincide. Also, it is shown that they are stable under reparametrization. In particular, in Definitions 2.1 and 2.2 we can replace 0 by any other value of \( \tau \in \Delta \) which does not affect the stationarity or extremality of \( \varphi \). It follows that the extremal discs are the geodesics of the Kobayashi metric in \( D \); in particular they are embeddings of \( \bar{\Delta} \) into \( \mathbb{C}^n \). We recall basic facts about the existence, uniqueness, and smooth dependence of the extremal discs on parameters (see [8], Proposition 11’):

(2.0) For any \( z \in D \) and \( v \in \hat{\mathbb{C}}^n := \mathbb{C}^n \setminus \{0\} \), there exists a unique extremal disc \( \varphi = \varphi_{z,v} \) such that \( \varphi(0) = z \) and \( \varphi'(0) = rv \) for \( r \in \mathbb{R}^+ \). Also, the mapping
\[
D \times \hat{\mathbb{C}}^n \to C^{2, \frac{1}{2}}(\bar{\Delta}), \quad (z, v) \mapsto \varphi_{z,v},
\]
where \( C^{2, \frac{1}{2}} \) is the space of functions whose derivatives up to order 2 are \( \frac{1}{2} \)-Hölder-continuous.

If \( \varphi^* \) has its pole at \( \tau_o \), we multiply it by
\[
\nu(\tau) = \frac{(\tau - \tau_o)(1 - \tau_o \tau)}{\tau}, \quad \tau \in \Delta,
\]
so that the pole is moved to 0. Next, we multiply \( \varphi^* \) by a real constant \( \neq 0 \) so that \( \varphi^*(1) \) is the unit outward conormal to \( D \) at \( \varphi(1) \). We will assume that \( \varphi^* \) is normalized by the two above conditions. It is essential for our discussion also to clarify the dependence of \( \varphi^*_{z,v} \) on the parameters \( z \) and \( v \) which is not explicitly stated in [8].

Proposition 2.3. The mapping
\[
(2.1) \quad D \times \hat{\mathbb{C}}^n \to C^{2, \frac{1}{2}}(\bar{\Delta}), \quad (z, v) \mapsto \varphi^*_{z,v},
\]
is \( C^{k-4} \).

Proof. Our starting remark is that we can describe \( \varphi^*_{z,v} \) quite explicitly only over \( \partial \Delta \). In fact, we must have
\[
(2.2) \quad \varphi^*_{z,v}(\tau) = g_{z,v}(\tau)\partial \rho(\varphi^*_{z,v}(\tau)) \quad \forall \tau \in \partial \Delta,
\]
where \( g_{z,v} \) is real and normalized by the condition \( g_{z,v}(1) = 1 \). On the other hand, when evaluating \( \varphi^*_{z,v} \) at points \( \tau \in \Delta \), we can use the Cauchy integral over \( \partial \Delta \). Thus, if we are able to show that \( (z, v) \mapsto \varphi^*_{z,v} \) with values in \( C^{2, \frac{1}{2}}(\partial \Delta) \) is \( C^{k-4} \), the same will be true with values in \( C^{2, \frac{1}{2}}(\bar{\Delta}) \) since the Cauchy integral preserves fractional regularity. Now, the second term on the right side of (2.2) can be handled by means of (2.0) and so what is
needed is to describe $g_{z,v}$. We can suppose without loss of generality $\partial z_1 \rho \neq 0$ for any point of $\varphi_{z,v}(\partial \Delta)$. According to Proposition 9, we can further normalize our coordinates in a neighborhood of the disc $\varphi_{z,v}(\Delta)$ so that

\begin{equation}
\partial z_1 \rho(\varphi_{z,v}(\tau)) \approx \bar{\tau} \quad \text{on } \partial \Delta,
\end{equation}

where “$\approx$” means close in $C^1$-norm. In particular, the index of the curve $\{\tau \partial z_1 \rho(\varphi_{z,v}(\tau)) \mid \tau \in \partial \Delta\}$ around 0 is 0 and hence it is well defined the function $\log(\tau \partial z_1 \rho(\varphi_{z,v}(\tau)))$ that we will denote by $f$. We are thus reduced to solve the Riemann problem of finding $G = G_{z,v}$ holomorphic such that

\begin{equation}
\text{Im } G_{z,v} = \text{Im } (\log(\tau \partial z_1 \rho(\varphi_{z,v}(\tau)))).
\end{equation}

To this end we have just to set $G = -T_1(\text{Im } f) + i\text{Im } f$ where $T_1$ is the Hilbert transform normalized by the condition $T_1 f(1) = 1$. Since $T_1$ preserves fractional regularity, then $(z, v) \mapsto G_{z,v} \in C^{1/2}$ is also $C^{k-4}$. We finally put

\begin{equation}
\nu_{z,v}(\tau) := \frac{e^{G(\tau)}}{\tau \partial z_1 \rho(\varphi_{z,v})}.
\end{equation}

We have

\begin{equation}
\nu_{z,v} = \exp(\text{Re } G + i\text{Im } \log(\tau \partial z_1 \rho(\varphi_{z,v})) - \log(\tau \partial z_1 \rho(\varphi_{z,v}))) = \exp(\text{Re } G - \text{Re } \log(\tau \partial z_1 \rho)) \text{ is real },
\end{equation}

\begin{equation}
\nu_{z,v} \tau \partial z_1 \rho(\varphi_{z,v}) \text{ extends holomorphically from } \partial \Delta \text{ to } \bar{\Delta}.
\end{equation}

Finally, since

\[ g_{\nu} \mid_{\partial \Delta} \in \mathbb{R}, \quad g_{\nu} \text{ extends holomorphically }, \quad g_{\nu}(1) = 1, \]

then $g \equiv \nu$. It follows that $g$ and hence also $\varphi^*$ depends on a $C^{k-4}$ fashion on $z$, $v$.

\[ \square \]

There is a statement perfectly analogous to (2.0) above, in which vectors $v$ are replaced by covectors $\zeta$: for any $z \in D$ and $\zeta \in \mathbb{C}^n$ there are unique the stationary disc and its lift $(\varphi, \varphi^*) = (\varphi_{z,\zeta}, \varphi^*_{z,\zeta})$ such that $\varphi(0) = z$ and $\varphi^*(0) = \zeta$ where $\varphi^*(0)$ stands for the residue $\text{Res } \varphi^*(0)$.

We recall now some basics about the Lempert Riemann mapping. For any pair of points $(z, w)$ in $D$, let $\varphi_{z,w}$ be the (unique) stationary disc through $z$ and $w$ normalized by the condition $z = \varphi_{z,w}(0)$, $w = \varphi_{z,w}(\xi)$ for some $\xi \in (0, 1)$; we define

\[ \Psi_z(w) := \xi \frac{\varphi'_{z,w}(0)}{|\varphi_{z,w}(0)|}. \]

Let $\mathbb{B}^n$ (resp. $\mathbb{D}$) denote the unit ball of $\mathbb{C}^n$ (resp. the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$), and set $\mathring{\mathbb{B}}^n = \mathbb{B}^n \setminus \{0\}$. Consider the correspondence

\begin{equation}
(D \times D) \setminus \mathbb{D} \to D \times \mathring{\mathbb{B}}^n, \quad (z, w) \mapsto (z, \Psi_z(w)).
\end{equation}
We have

- For fixed \( z \), \( \Psi_z \) is a diffeomorphism of class \( C^{k-4} \) which extends as a diffeomorphism between the boundaries \( \partial D \) and \( \partial \mathbb{B}^n \).
- \( (2.7) \) is differentiable of class \( C^{k-4} \).

Write \( v = v(z, w) \) for \( \Psi_z(w) \). By the above statements, the smoothness of (2.0) and (2.1) are equivalent to those of

\[
(z, w) \mapsto \varphi_{z,w}, \quad (z, w) \mapsto \varphi^*_{z,w}.
\]

**Remark 2.4.** Let \( z_\nu \) and \( w_\nu \) be sequences converging to \( z_0 \), and put \( v_\nu := \varphi'_{z_\nu, w_\nu}(0) \). If we define \( v := \lim \nu \frac{w_\nu - z_\nu}{|w_\nu - z_\nu|} \), then \( v = \lim \nu \frac{v_\nu}{|v_\nu|} \). Hence we have convergence (in the \( C^{2,\frac{1}{2}}(\Delta) \) space):

\[
\varphi_{z_\nu, w_\nu}(= \varphi_{z_\nu, v_\nu}) \to \varphi_{z_0, v}, \quad \varphi^*_{z_\nu, w_\nu}(= \varphi^*_{z_\nu, v_\nu}) \to \varphi^*_{z_0, v}.
\]

For our further needs it is convenient to state the following uniqueness theorem which is largely contained in former literature.

**Theorem 2.5.** Let us be given two stationary discs \( \varphi_j \), \( j = 1, 2 \) in a strongly convex domain \( D \) and assume that for \( \tau_j \in \Delta \), \( j = 1, 2 \) we have

\[
(2.10) \quad \begin{cases} \varphi_1(\tau_1) = \varphi_2(\tau_2), \\ \varphi^*_1(\tau_1) = \lambda \varphi^*_2(\tau_2) \end{cases} \text{ for some } \lambda \in \mathbb{C}.
\]

Then, after reparametrization of \( \Delta \) we have, for a complex scalar function \( \mu = \mu(\tau) \):

\[
\varphi_1 = \varphi_2, \quad \varphi^*_1 = \mu \varphi^*_2.
\]

As before, if \( \tau_j \) is a pole of \( \varphi^*_j \), then \( \varphi^*_j(\tau_j) \) stands for \( \text{Res} \ varphi^*_j(\tau_j) \).

**Proof.** We assume that the poles of the \( \varphi^*_j \)'s are placed at 0. We compose each \( (\varphi_j, \varphi^*_j) \) with an automorphism of \( \Delta \) which brings \( \tau_j \) to 0. We are therefore reduced to the following:

\[
(2.11) \quad \begin{cases} \varphi_1(0) = \varphi_2(0), \\ \text{Res} \ \varphi^*_1(0) = \lambda \text{Res} \ \varphi^*_2(0), \end{cases}
\]

for a new constant \( \lambda \). We put \( \lambda = re^{i\theta} \) and replace \( (\varphi_2(\tau), \varphi^*_2(\tau)) \) by \( (\varphi_2(e^{-i\theta} \tau), r \varphi^*_2(e^{-i\theta} \tau)) \). This transformation reduces (2.11) to \( \lambda = 1 \). At this point we can prove that \( \varphi_1 = \varphi_2 \). We reason by contradiction and suppose \( \varphi_1 \neq \varphi_2 \). It follows

\[
(2.12) \quad \int_0^{2\pi} \text{Re} \ (\varphi^*_1(\tau) - \varphi^*_2(\tau), \varphi_2(\tau) - \varphi_1(\tau)) \, d\theta > 0,
\]

since the integrand is almost everywhere \( > 0 \) on \( \partial \Delta \) due to the strong convexity of the domain. On the other hand \( d\theta = -i \frac{d\tau}{\tau} \); also, \( \frac{\varphi_2 - \varphi_1}{\tau} \) and \( \varphi^*_1 - \varphi^*_2 \) are holomorphic. Hence the integrand in (2.12) is a \( (1, 0) \) form whose coefficient is the real part of a holomorphic function. Hence the integral (2.12) is 0, a contradiction. \( \square \)
In particular in the situation of Theorem 2.5 we have coincidence of the image sets \( \varphi_1(\Delta) = \varphi_2(\Delta) \).

**Remark 2.6.** Let \( \dot{T}^*\mathbb{C}^n \) be the cotangent bundle to \( \mathbb{C}^n \) with the 0-section removed, and let \( \dot{T}^*\mathbb{C}^n/\dot{\mathbb{C}} \simeq \mathbb{C}^n \times \mathbb{P}^{n-1}_\mathbb{C} \) be the projectivization of its fibers. We denote by \( (z, [\zeta]) \) the variable in \( \dot{T}^*\mathbb{C}^n/\dot{\mathbb{C}} \). We can rephrase Theorem 2.5 by saying that if two discs \( (\varphi_j, [\varphi_j^*]) \) \( j = 1, 2 \) have a common point, then, after reparametrization, they need to coincide. Also, it is useful to point out that, given a stationary disc \( \varphi(\Delta) \), its lift \( [\varphi^*(\Delta)] \) is unique. In fact, the different choices of \( \varphi \) obtained by reparametrization, do not affect the class of \( \varphi^* \) in the projectivization of the cotangent bundle.

### 3. The main result

Let \( D_1 \) and \( D_2 \) be bounded domains of \( \mathbb{C}^n \) with \( D_2 \subset\subset D_1 \). We assume that \( D_1 \) is strongly convex and with \( C^k \) boundary for \( k \geq 6 \) as is the setting of [8]. Let \( D_2 \) be defined by \( \rho < 0 \) for a real function \( \rho \) of class \( C^2 \) with \( \partial \rho(z) \neq 0 \) when \( \rho(z) = 0 \).

**Definition 3.1.** The domain \( D_2 \) is said to be strongly convex with respect to the extremal discs of \( D_1 \), if every such disc \( \varphi \) tangent to \( D_2 \) at \( z_0 = \varphi(0) \in \partial D_2 \) has tangency of order 2, that is for some \( c > 0 \) we have \( \rho(\varphi(\tau)) \geq c|\tau|^2 \forall \tau \in \Delta \), in particular \( D_2 \cap \varphi(\Delta) = \{z_0\} \).

Here is the main result of our paper.

**Theorem 3.2.** Let \( D_2 \subset\subset D_1 \) be bounded domains of \( \mathbb{C}^n \) with \( D_1 \) strongly convex and \( C^k \) for \( k \geq 6 \), and \( D_2 \) strongly convex with respect to the extremal discs of \( D_1 \) and of class \( C^2 \). Let \( f \) be a continuous function which extends holomorphically along each extremal disc \( \varphi(\Delta) \) of \( D_1 \) which is tangent to \( \partial D_2 \). Then \( f \) extends holomorphically to \( D_1 \), continuous up to \( \partial D_1 \).

**Remark 3.3.** We do not think that the assumption that \( D_2 \) is strongly convex with respect to the extremal discs is essential. We add it for the sake of simplicity and convenience of the proof.

**Proof.** We consider the cotangent (respectively tangent) bundle \( \dot{T}^*\mathbb{C}^n/\dot{\mathbb{C}} \), resp. \( \dot{T}\mathbb{C}^n/\dot{\mathbb{C}} \), with projectivized fibers \( \mathbb{P}^{n-1}_\mathbb{C} \) and with coordinates \( (z, [\zeta]) \) and \( (z, [v]) \) respectively. The prefix \( T^c \) will be used to denote the complex tangent bundle. We fix a rule for selecting a “distinguished” representative \( v \) of \([v]\) and define a mapping

\begin{align*}
(3.1) & \quad (\dot{T}^c \partial D_2/\dot{\mathbb{C}}) \times \Delta \xrightarrow{\Phi} (\dot{T}^*\mathbb{C}^n/\dot{\mathbb{C}})|_{D_1 \setminus D_2}, \\
(3.2) & \quad (z, [v], \tau) \xrightarrow{\Phi} (\varphi_{z,v}(\tau), [\varphi_{z,v}^*(\tau)]) ,
\end{align*}

where \( \varphi_{z,v} \) is the unique stationary disc such that \( (\varphi(0) = z, \varphi'(0) = rv) \) for some \( r \in \mathbb{R}^+ \) and \( \varphi_{z,v}^* \) is its “lift” according to §2, (2.0). Note that by multiplying \( \varphi_{z,v}^* \) by \( \nu(\tau) \), real on \( \partial \Delta \), we can move the pole to \( \tau = 0 \).
We denote by $S$ the image-set of $\Phi$. We show that $\Phi$ is an injective smooth local parametrization of $S$. First, it is injective: in fact, if $(z_1, [v_1], \tau_1)$ and $(z_2, [v_2], \tau_2)$ go to the same image, then by Theorem 2.4 in §2, $\varphi_{z_1,v_1}$ and $\varphi_{z_2,v_2}$ coincide up to reparametrization. On the other hand, by the strong convexity of $D_2$ with respect to the stationary discs of $D_1$, we must have $z_2 = z_1$. Then we also have $v_1 = v_2$ by our rule of taking representatives and therefore the discs coincide (without need of reparametrization). Finally $\tau_2 = \tau_1$ because they are injective (cf. §2). As for the smoothness, we make a choice of our representative $v$ smoothly depending on $z$, and point our attention to (2.0) and (2.1) of §2. If we then take evaluation of the discs and their lifts at $\tau \in \Delta$ we get the $C^{k-4}$-smoothness of $z$, $2$. In the lines of what was remarked after Proposition 2.3, for any $(z,[\zeta]) \in (\hat{T}\ast\mathbb{C}^n/\hat{\mathbb{C}})|_{D_1 \setminus D_2}$ there is a unique $(\varphi, \varphi^*)$, up to reparametrization, such that $\varphi(\tau) = z$, $[\varphi^*(\tau)] = [\zeta]$ for some $\tau \in \Delta$. On the other hand, the class of stationary discs which are tangent to $\partial D_2$ divides the set of all stationary discs into two sets, the ones which are transversal to (resp. disjoint from) $D_2$. Accordingly, $S$ divides $(\hat{T}\ast\mathbb{C}^n/\hat{\mathbb{C}})|_{D_1 \setminus D_2}$ into two sets. We denote by $\mathcal{W}$ the first set and refer to it as to the “finite” side of $S$ the complement being called a neighborhood of the “plane at infinity”. The set $\mathcal{W}$ is a wedge type domain with boundary $S$ and edge $\mathcal{E} := T_{\partial D_1} \mathbb{C}^n/\hat{\mathbb{C}}$.

We now describe the fibers $S_{z_0} = \pi^{-1}(z_0) \cap S$ where $\pi : \hat{T}\ast\mathbb{C}^n/\hat{\mathbb{C}} \to \mathbb{C}^n$ is the projection $\pi(z,[\zeta]) = z$. Our plan is to use $\Psi_{z_0}$, interchange $D_1$ with $\mathbb{B}^n$ and $z_0$ with $0$, analyze the situation in this new setting, and then bring back the conclusions to the former by $\Psi_{z_0}^{-1}$. Recall that $\Psi_{z_0}$ interchanges the stationary discs through $z_0$ with the complex lines (the stationary discs of the ball) through $0$. We first describe the set

\[(3.3) \quad \gamma_0 = \{z \in \partial(\Psi_{z_0} D_2) : \text{ for some } v \in T_z^C \partial(\Psi_{z_0} D_2), \ \varphi_{z,v} \text{ passes through } 0\} .\]

If $\rho = 0$ is an equation for $\partial(\Psi_{z_0} D_2)$, $\gamma_0$ is defined by

\[\rho(z) = 0, \ \partial \rho(z) \cdot z = 0.\]

This is a system of three real equations that we denote by $r = 0$. We normalize our coordinates so that

\[\partial \rho(z) = (1,0,\ldots), \quad z = (0,c,0,\ldots).\]

We then have for the partial Jacobian $J_{z_1,z_2}\gamma_0(z)$:

\[(3.4) \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ \ast & \ast & c\partial^2_{z_2,z_2} \rho & c\partial^2_{\bar{z}_2,z_2} \rho \\ \ast & \ast & c\partial^2_{z_2,\bar{z}_2} \rho & c\partial^2_{\bar{z}_2,\bar{z}_2} \rho \end{bmatrix} .\]
where the asterisks denote unimportant matrix coefficients. Let $A$ be the $3 \times 3$ minor obtained by discarding the first column. We have

$$
\begin{align*}
\det A &= c^2 \det \begin{bmatrix}
\partial_{z_2,z_2}^2 \rho & \partial_{z_2,z_3}^2 \rho \\
\partial_{z_3,z_2}^2 \rho & \partial_{z_3,z_3}^2 \rho
\end{bmatrix} \\
&= -c^2 \det (Hess(\rho)|_{z_2}) < 0,
\end{align*}
$$

(3.5)

where the real Hessian of $\rho$ at $z$ along the $z_2$-plane is positive because $\Psi_{z_0} D_2$ is strongly convex with respect to $\Psi_{z_0} (\varphi_{z_0,z}(\Delta))$. In conclusion, rank $J(r) = 3$ and hence $\gamma_0$ is a regular real manifold of dimension $2n - 3$ compact and without boundary. We now use the fact that $\Psi_{z_0}$ is a diffeomorphism and conclude that $\gamma_{z_0} := \Psi_{z_0}^{-1}(\gamma_0)$ is also a regular manifold of dimension $2n - 3$ in $\partial D_2$, which enjoys the same properties as $\gamma_0$. It represents the set of points where the geodesics of $D_1$ through $z_0$ are tangent to $\partial D_2$. Let $\tilde{\gamma}_{z_0}$ be the section $(z, [v(z)])$ of $(\hat{T}_c^0 \partial D_2 / \hat{\mathcal{C}})|_{\gamma_{z_0}}$, where $[v(z)]$ is the direction tangent to $z$ to the stationary disc connecting $z_0$ and $z$. We can parametrize the fiber $S_{z_0}$ over $\tilde{\gamma}_{z_0} \times \Delta$ by the same parametrization $\Phi$ as in (3.4). This being bijective, we conclude that $S_{z_0}$ is a finite family of regular closed manifolds of dimension $2n - 3$ without boundary, which do not intersect. We move now $z$ from the fixed $z_0$ and describe the behavior of the fibers $S_z$; they depend in a $(C^{k-4})$ fashion on $z$ since the mapping in (2.7) is also $C^{k-4}$. As for their behavior at $z_0 \in \partial D_2$, we consider the set $\Pi_{z_0}$ defined by the diagram

$$
\begin{align*}
\hat{T}_c^0 \mathbb{C}^n / \hat{\mathcal{C}} &\sim \hat{T}_0^0 / \hat{\mathcal{C}} \sim \mathbb{P}^{n-1} \\
\cup &
\cup \\
\hat{T}_c^0 \partial D_2 / \hat{\mathcal{C}} &\sim \Pi_{z_0}
\end{align*}
$$

where the two horizontal arrows are given by the smooth injective mapping $v \mapsto [\varphi_{z_0,v}^*(0)]$. Thus $\Pi_{z_0} := \{[\varphi_{z_0,v}^*(0)] : \varphi$ is tangent to $\partial D_2$ at $z_0\}$ is a 2-codimensional real submanifold of $\mathbb{P}^{n-1}$ which reduces to a single point when $n = 2$.

**Lemma 3.4.** The sets $S_{z_0}$ shrink to $\Pi_{z_0}$ as $z_0 \to z_0 \in \partial D_2$; in particular, $S_{z_0}$ consists of just one component when $z_0$ is close to $z_0$.

**Proof.** By the strong convexity of $\partial D_2$, the manifolds $\gamma_{z_0}$ shrink to $\{z_0\}$ as $z_0 \to z_0$. If we pick up any sequence $w_\nu \in \gamma_{z_0}$, we have

$$
\begin{align*}
\frac{w_\nu - z_0}{|w_\nu - z_0|} \to v \in T_{z_0}^c \partial D_2.
\end{align*}
$$

Let $\varphi_{z_0,w_\nu}$ (resp. $\varphi_{z_0,v}$) be the geodesic through $z_0$ and $w_\nu$ (resp. through $z_0$ with tangent $v$), normalized by the condition $z_0 = \varphi_{z_0,w_\nu}(0)$, $w_\nu = \varphi_{z_0,w_\nu}(\xi)$ for $\xi \in (0, 1)$, (resp. $z_0 = \varphi_{z_0,v}(0)$, $rv = \varphi_{z_0,v}^*(0)$ for $r \in \mathbb{R}^+\)$. Then

$$
\varphi_{z_0,w_\nu} \to \varphi_{z_0,\nu}, \quad \varphi_{z_0,w_\nu}^* \to \varphi_{z_0,v}^*.
$$
with convergence in the $C^{2,\frac{1}{2}}(\Delta)$ norm. In particular, since $S_{z_\nu} = \bigcup_{w_\nu \in z_{z_\nu}} [\varphi_{z_\nu,w_\nu}^\ast(0)]$, then

$$S_{z_\nu} \to \bigcup_{v \in T^\nu_{z_\nu} \partial \mathbb{D}_2} [\varphi_{z_\nu,v}(0)].$$

It follows that for the fibers $W_{z_\nu}$, which are open domains of $\mathbb{P}^{n-1}_C$ with boundary $S_{z_\nu}$, we have merely by definition:

$$W_{z_\nu} \to \mathbb{P}^{n-1}_C \setminus \Pi_{z_\nu} \text{ as } z_\nu \to z_0 \in \partial \mathbb{D}_2.$$ If, instead, we move $z_\nu$ towards $z \in \partial \mathbb{D}_1$, then each $S_{z_\nu}$ as well as their “finite” sides $W_{z_\nu}$, shrink to the single point $T^\ast_{\partial \mathbb{D}_1} \mathbb{C}^n/\mathbb{C}_z$.

Now we move $z_\nu$ all over $\mathbb{D}_1 \setminus \mathbb{D}_2$. If we take a closer look to (3.4), (3.5) we see that the set $\Psi_{z_\nu}(\mathbb{D}_2)$ as well as its equation $\rho_{z_\nu} = 0$, moves smoothly with respect to $z_\nu$ by the regularity properties of $\Psi$. It follows that the set defined by $\{(z_\nu, z) : z_\nu \in D_1 \setminus \mathbb{D}_2, z \in \gamma_{z_\nu}\}$ is a $(4n-3)$-dimensional manifold. In particular, the set $\gamma_{z_\nu}$ is a $(2n-3)$-dimensional manifold and it cannot turn from one to several components without passing through a singular point $z_\nu$. It follows that the set $S_{z_\nu}$ also consists of one component.

By the preceding discussion and Sard’s theorem, we can also say that $S$ is a smooth regular manifold except possibly a closed subset of measure zero. Along with its natural foliation by the discs $(\varphi_{z_\nu,v}, [\varphi_{z_\nu,v}^\ast])$, we need to endow $S$ with another foliation, locally on a neighborhood of each of its points, by CR manifolds $\mathcal{M}$ of dimension $2n$ and CR dimension 1 each one being still a union of discs. For this, we fix $z \in \mathbb{D}_1 \setminus \mathbb{D}_2$, consider the submanifold $\gamma_z$ of $\partial \mathbb{D}_2$ with dimension $2n - 3$ of points of tangency for the stationary discs through $z$, and denote by $w$ the point which moves in $\gamma_z$. As above, we denote by $\varphi_{w,z}$ the stationary disc through $w$ and $z$, normalized by $\varphi_{w,z}(0) = w$, $\varphi_{w,z}(\xi) = z$ for $\xi \in (0, 1)$; we also write $\xi = \xi(w, z)$ and define $v(w, z) := \varphi_{w,z}(0)$. We set $\Gamma_z := \{(w, [v(w, z)], \xi(w, z)) : w \in \gamma_z\}$; then $\dim \Gamma_z = \dim \gamma_z = 2n - 3$. Since $\Phi_1$ sends all points of $\Gamma_z$ to the fixed $z$, then we have an inclusion $T\Gamma_z \subset \ker \Phi_1|_{\Gamma_z}$. But since the dimensions are the same, then $T\Gamma_z = \ker \Phi_1|_{\Gamma_z}$. In particular, if $p$ is the projection $(w, [v], \tau) \mapsto w$, then

$$p' \left(\ker \Phi_1|_{\Gamma_z}\right) \subset T\gamma_z.$$ We define $\mathcal{M}$ locally at a point $(z, [\xi]) \in \mathcal{S} \cup \mathcal{E}$; if $(z, [\xi]) \in \mathcal{E}$, $\mathcal{M}$ will be in fact a manifold with boundary $\mathcal{E}$. Let $(w, [v], \tau)$ be the value of the parameter in $(T^\mathbb{C}\partial \mathbb{D}_2) \times \Delta$ which corresponds to $(z, [\xi])$ via $\Phi$. Choose a germ of submanifold $\delta_z \subset \partial \mathbb{D}_2$ transversal to $\gamma_z$ at $w$ with complementary dimension 2. By (3.6), we have

$$\ker \left(\Phi_1'(w, [v], \tau)|_{T_w \delta_z \times \mathbb{P}^{n-1}_C \times T\tau, \Delta}\right) = \{0\}.$$
Thus $\Phi_1$ induces a diffeomorphism between a neighborhood $\Sigma = \Sigma_1 \times \Sigma_2$ of $(w, [v], \tau)$ in $(T^C \partial D_2 / \mathbb{C})|_{\delta_z} \times \Delta$ and a neighborhood of $z$ in $D_1$. We define $\mathcal{M} = \Phi(\Sigma)$ that is

$$\mathcal{M} = \bigcup_{(\varphi, \varphi^*)} (\varphi, [\varphi^*])(\Sigma_2),$$

for $(\varphi(0), [\varphi'(0)]) \in \Sigma_1$. $\Phi$ is a diffeomorphic parametrization of $\mathcal{M}$ over $\Sigma$ and hence $\mathcal{M}$ is a smooth manifold, in fact a graph over a neighborhood of $z$ in $\bar{D}_1$. This was not necessarily the case of $S$ since $\Phi$ is a smooth and bijective parametrization of $S$ but it might occur that $\Phi'$ is degenerate at some point. We define a function $F$ on $S$ by collecting all extensions $f_{\varphi(\Delta)}$ of the given $f$ from $\varphi(\partial \Delta)$ to $\varphi(\bar{\Delta})$. For $(z, [\xi]) \in S$ we put

$$F(z, [\xi]) = f_{\varphi(\Delta)}(z) \text{ if } (\varphi(\tau), [\varphi^*(\tau)]) = (z, [\xi]) \text{ for some } \tau.$$

According to Theorem 2.5, $F$ is well defined. We have the following

**Proposition 3.5.** At every point of $S \setminus \mathcal{E}$, the function $F$ holomorphically extends to a one-sided neighborhood on the $W$-side of $S$.

**Proof.** The ingredients of the proof are the foliation of $S$ by manifolds with boundary $\mathcal{M}$, which are themselves union of discs, and the additional transversal foliation of $\mathcal{W}$ by the fibers $\mathcal{W}_z$. The starting remark is that $F$ is holomorphic along each disc and therefore it is CR on each $\mathcal{M}$ since $\dim_{CR}\mathcal{M} = 1$.

(a) We approximate $F|_{\mathcal{E}}$ by a sequence of entire functions $F_{\nu}$ (cf. e.g. [3]). To this end it is important to notice, as it was first pointed out by Webster, that since $\partial D_1$ is strongly convex, then $\mathcal{E}$ is totally real. Then, in an identification $\mathcal{E} \simeq \mathbb{R}^{2n-1}$, these are defined by

$$\hat{F}_{\nu}(\xi) = \left(\frac{\nu}{\pi}\right)^{\frac{2n-1}{2}} \int_{\mathbb{R}^{2n-1}} F(\eta) e^{-\nu(\eta - \xi)^2} dV,$$

($dV$ being the element of volume in $\mathbb{R}^{2n-1}$). It is well known that $\hat{F}_{\nu} \to F$ uniformly on compact subsets of $\mathcal{E}$. Also, $F$ being CR on each $\mathcal{M}$, it is possible to deform the integration chain from $\mathcal{E}$ to another chain entering inside $\mathcal{M}$ and reaching any point of $\mathcal{M}$ in a neighborhood of $\mathcal{E}$. In other terms, the function $F$ is approximated, over each $\mathcal{M}$ near $\mathcal{E}$, by the same sequence (3.10) of entire function. Since the $\mathcal{M}$’s give a foliation of $S$, it follows that the uniform approximation of $F$ by the $F_{\nu}$’s holds on the whole $S$ in a neighborhood of $\mathcal{E}$.

(b) By using now the foliation of $\mathcal{W}$ by the fibers $\mathcal{W}_z$, we can bring the approximation by entire functions from $S$ to $\mathcal{W}$ in a neighborhood of $\mathcal{E}$: in fact, by maximum principle, the sequence $\hat{F}_{\nu}$ which is Cauchy over $S_z$ will be Cauchy on the whole $\mathcal{W}_z$.

(c) We use now the theory of propagation of wedge extendibility along discs for each CR function $F|_{\mathcal{M}}$ by [11] which develops [7]. We put a suffix $s$ in the notation of the disc $\Delta_s$ to specify its radius, and define

$$I = \{r \in (0, 1) : \text{ F extends to the side } \mathcal{W} \text{ of } S \text{ in } \cup_{(\varphi, \varphi^*)} (\varphi, [\varphi^*])(\Delta_1 \setminus \bar{\Delta}_{1-r})\},$$
for all stationary discs $\varphi$ tangent to $\partial D_2$ at $\tau = 0$. (The last requirement is just a choice of the parametrization.) We have $I \neq \emptyset$ due to (b) above. We show now that we have indeed $I = (0, 1)$ from which the proposition follows. We reason by contradiction, suppose $I \neq (0, 1)$, and denote by $r_o$ the supremum of $I$; thus $r_o < 1$. By propagation of wedge extendibility of $F|_{\mathcal{M}}$ for each $\mathcal{M}$, and since the wedge evolves continuously with the base point, then, on account also of a compactness argument, $F$ would extend to the side $\mathcal{W}$ for a value of $r$ bigger than $r_o$, a contradiction. □

End of proof of Theorem 3.2.

- First, recall that for $z$ moving from $\partial D_1$ to $z_o \in \partial D_2$, the fibers $\mathcal{W}_z$ grow from a single point to $\mathcal{W}_{z_o} = \mathbb{P}^{n-1} \setminus \Pi_{z_o}$. Also, recall that by approximation, $F$ extends holomorphically from $\mathcal{S}_z$ to $\mathcal{W}_z$ when $z$ is close to $\partial D_1$, and, by propagation, to a neighborhood of $\mathcal{S}_z$ in $\mathcal{W}$ when $z$ is no longer close to $\partial D_1$. Then $F$ extends to the whole set $\mathcal{W}$ by the Hartogs continuity principle. For $n > 2$ the same conclusion also follows by the Hartogs extension theorem.

- The boundary values of $F$ on $\pi^{-1}(\partial D_2) \cap \mathcal{W} \subset \partial \mathcal{W}$ are constant on the fibers $\mathcal{W}_{z_o}$, $z_o \in \partial D_2$. Indeed, $F|_{\mathcal{W}_{z_o}}$ holomorphically extends to the whole projective space $\mathbb{P}^{n-1}$ because the set $\Pi_{z_o}$ of codimension 2 is removable, hence it is constant. Now since $F$ is constant on the fibers of $\mathcal{W}$ on an open set of the boundary of $\mathcal{W}$, then $F$ is constant on the fibers of $\mathcal{W}$ everywhere in $\mathcal{W}$. Then $\hat{f}(z) := F(z, [\varsigma])$, $(z, [\varsigma]) \in \mathcal{W}$ is a well defined holomorphic extension of the original function $f$ to $D_1 \setminus \bar{D}_2$. Then $f$ further extends to $D_2$ by the Hartogs theorem. The proof is now complete. □

Remark 3.6. Take a line segment $I$ connecting a pair of points $z_1$ and $z_2$ of $\partial D_1$ and $\partial D_2$ resp., fix a neighborhood $U \supset I$ in $\mathbb{C}^n$, denote by $\mathcal{I}$ the family of discs tangent to $\partial D_2$ and passing through $U \cap D_1$, and set $V := \bigcup_{\varphi \in \mathcal{I}} \varphi(\partial \Delta)$. Assume that $f$ is defined and continuous in $V$ and extends holomorphically to the discs which belong to $\mathcal{I}$; then $f$ extends holomorphically to a one-sided neighborhood of $V$ in $D_1$. In fact, by moving $z$ from $z_1$ to $z_2$ along $I$ we will have the same conclusions for the fibers $\mathcal{S}_z$ and $\mathcal{W}_z$ as in the proof of Theorem 3.2. In particular we will conclude that $F$ is independent of $[\varsigma]$ in a neighborhood of $z_1$. But then $F$ is independent of $[\varsigma]$ wherever it is defined, in particular in a one-sided neighborhood of $V$.

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