López de Medrano and Verjovsky discovered in 1997 a way to construct many compact complex manifolds (cf. [LdM-V]). They start with a \( \mathbb{C} \)-action on \( \mathbb{P}^n \) induced by a diagonal linear vector field (satisfying certain properties), and find an open dense subset \( U \subset \mathbb{P}^n \) where the action is free, proper and cocompact, so the quotient \( N = U/\mathbb{C} \) is a compact complex manifold. Their construction was extended to \( \mathbb{C}^m \)-actions by Meersseman in [M], yielding a vast family of non-Kähler compact manifolds, called LVM-manifolds. These manifolds lend themselves very well to various computations, and a thorough study of their properties is conducted in [M]. Furthermore, they are (deformations of) a very natural generalization of Calabi-Eckmann manifolds. Finally, the topology of LVM-manifolds can be extraordinarily complicated: We refer to [B-M] for the most recent results about a study started off in [W] and [LdM-V].

It was also remarked that each LVM-manifold carries a transversely Kähler foliation \( F \) ([L-N], [M]).

There were two main developments on LVM-manifolds:

(a) In [Bo], Bosio showed that Meersseman’s construction could actually be generalized to more general actions of \( \mathbb{C}^m \). He produced a family of quotient manifolds, containing the family of LVM-manifolds, that we will call LVMB-manifolds. He showed that many properties of LVM-manifolds carry over to the case of LVMB-manifolds, and also pointed out non-trivial combinatorics relevant to these quotients. However, he did not mention the foliation \( F \) (that still exists).

(b) In 2004, Meersseman and Verjovsky investigated in [M-V] a link between LVM-manifolds and Mumford’s Geometric Invariant Theory (Mumford’s GIT) that was soon discovered after [M] was completed. Their main results are that each LVM-manifold satisfying condition \( (K) \) (cf. Sect. 2) admits a Seifert fibration over a projective simplicial toric variety \( X \), and moreover any such \( X \) can appear this way.

Building on both (a) and (b), we establish a link between LVMB-manifolds and GIT. We show that the extension from LVM-manifolds to LVMB-manifolds parallels exactly the extension from Mumford’s GIT to the generalized GIT of Białynicki-Birula and Święcicka, which allows for non-projective quotients (cf. [BB-Św2]).

We describe a construction of LVMB-manifolds from a GIT point of view, from which it will be clear that some of them are Seifert-fibered over a complete simplicial toric variety \( X \). Using a result of Hamm, we show that

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any such $X$ can appear that way, i.e., below some LVMB-manifold. This generalizes and simplifies the proof of the main result of [M-V].

In most cases $X$ is an algebraic reduction of $N$, and if $X$ is projective then $N$ is actually an LVM-manifold. Using this and the almost-homogeneous structure of these manifolds, we produce examples of LVMB-manifolds that are not biholomorphic to any LVM-manifold.

Now consider an LVM-manifold $N$ satisfying condition $(K)$. Meersseman and Verjovsky showed that the foliation $\mathcal{F}$ is given by a Seifert fibration $N \to X$, with $X$ projective. The fact that $X$ is Kähler is the “reason” for $\mathcal{F}$ to be transversely Kähler.

By our results, an LVMB-manifold that is not an LVM-manifold, and that satisfies condition $(K)$, also admits a map $N \to X$, although this time $X$ is not projective. We prove that $\mathcal{F}$ on such an $N$ is not transversely Kähler. The difficulty here is to deal with the singularities of $X$ as an orbifold.

Finally, our GIT point of view on LVMB-manifolds leads naturally to a further extension of the LVMB family, in the context of the Bialynicki-Birula and Sommese conjecture (cf. [BB-So]). Our results show that the construction of LVMB-manifolds actually follows from the solution of this conjecture for linear algebraic ($\mathbb{C}^*$) actions on the projective space (cf. [BB-Sw2]).

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### 1. LVMB-manifolds

Let $m$ and $n \geq 2m$ be positive integers. Let $\mathcal{L} = (\ell_0, \ldots, \ell_n)$ be an ordered $(n+1)$-tuple of linear forms on $\mathbb{C}^m$. Fixing an isomorphism between $\mathbb{C}^m$ and its dual vector space, we will look at each $\ell_i$ as a row vector in $\mathbb{C}^m$. We will also look at $\ell_i$ as an element in $\mathbb{R}^{2m}$ via the identification $\ell_i \mapsto (\Re \ell_i, \Im \ell_i) \in \mathbb{R}^m \times \mathbb{R}^m \cong \mathbb{R}^{2m}$.

Let $\mathcal{E} = \{\mathcal{E}_\alpha\}_\alpha$ be a family of subsets of $\{0, \ldots, n\}$, each of these subsets having cardinality $2m + 1$.

We denote these data by $(\mathcal{L}, \mathcal{E}, m, n)$. To any given $(\mathcal{L}, \mathcal{E}, m, n)$, we associate the following objects:

- a Zariski open subset $U = U(\mathcal{E}) \subset \mathbb{P}^n$ defined by
  $$U = \{[x_0 : \cdots : x_n] \mid \text{there exists } \mathcal{E}_\alpha \in \mathcal{E} \text{ such that for all } i \in \mathcal{E}_\alpha, x_i \neq 0\};$$

- a $\mathbb{C}^m$-action on $U$ defined by
  $$(\mathbb{C}^m \times U) \longrightarrow U$$
  $$(z, [x_0 : \cdots : x_n]) \longmapsto [e^{\ell_0(z)}x_0 : \cdots : e^{\ell_n(z)}x_n];$$

- for all $\alpha$, we define $P_\alpha$ as the convex hull of $\{\ell_i, i \in \mathcal{E}_\alpha\}$. Notice that $2m + 1$ is precisely the number of vertices of a simplex in $\mathbb{R}^{2m}$.

**Definition.** Let $G$ be a complex Lie group acting holomorphically and effectively on $\mathbb{P}^n$. A $G$-stable open subset $U$ of $\mathbb{P}^n$ will be called **good with**
respect to the $G$-action when the restricted action on $U$ is proper (therefore the topological quotient $U/G$ is Hausdorff), and $U/G$ is compact.

**Theorem 1.1** ([Bo], 1.4). Take any $(\mathcal{L}, \mathcal{E}, m, n)$, with corresponding $U$ and $\mathbb{C}^m$-action. Then $U$ is good with respect to that action if and only if the following two conditions are satisfied:

1. **(sep)** $P_\alpha \cap P_\beta \neq \emptyset$ for every $\alpha, \beta$;
2. **(comp)** for every $\mathcal{E}_\alpha \in \mathcal{E}$ and every $i \in \{0, \ldots, n\}$ there exists $j \in \mathcal{E}_\alpha$ such that $(\mathcal{E}_\alpha - \{j\}) \cup \{i\} \in \mathcal{E}$.

When $(\mathcal{L}, \mathcal{E}, m, n)$ satisfies these conditions, we write $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB$, and denote the quotient $U/C^m$ by $N = N(\mathcal{L}, \mathcal{E}, m, n)$. By definition, the $\mathbb{C}^m$-action is proper, but $\mathbb{C}^m$ has no compact subgroups, so this action is necessarily free. It is known that a quotient by a free and proper action is a complex manifold ([cf. Hu] Proposition 2.1.13). Therefore $N$ is a compact complex manifold of dimension $n - m$, that we call an *LVMB-manifold* (Bosio calls it a *generalized LVM-manifold*).

When $\bigcap_\alpha P_\alpha \neq \emptyset$, the action on $U$ is called an *LVM-action*. We write $(\mathcal{L}, \mathcal{E}, m, n) \in LVM$, and call $N$ an *LVM-manifold*. By Proposition 1.3 in [Bo], LVM-manifolds are exactly the manifolds constructed in [M].

**Example 1.2.** Take $n = 5, m = 1$, so the $\ell_i$’s are just complex numbers, that—for later use— we take in $\mathbb{Z} + \sqrt{-1} \mathbb{Z}$ according to the diagram:

![Diagram of example 1.2](image_url)

and we take

$$\mathcal{E} = \left\{ \{024\}, \{135\}, \{025\}, \{124\}, \{034\}, \{035\}, \{134\}, \{125\} \right\}.$$

Then it is easy to check that $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB$. It is also immediate that $(\mathcal{L}, \mathcal{E}, m, n) \notin LVM$ (cf. [BB-Šw1] p.26 and [Bo] p.1291). We will show that the LVMB-manifold $N(\mathcal{L}, \mathcal{E}, m, n)$ is not isomorphic to any LVM-manifold.

**Remark.** (i) For an action on $\mathbb{P}^n$ of form (hol), there are in general many good $U$’s, and on some of them the restricted action is an *LVM*-action, whereas the restricted action on others is not an *LVM*-action. In other words, for $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB$, there may be $\mathcal{E}' \neq \mathcal{E}$ such that $(\mathcal{L}, \mathcal{E}', m, n)$ is in $LVMB$ or in $LVM$.

(ii) For $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB$ but $(\mathcal{L}, \mathcal{E}, m, n) \notin LVM$, the natural question is whether it is possible that $N(\mathcal{L}, \mathcal{E}, m, n)$ be isomorphic to an *LVM*-manifold. This question was unfortunately left open by [Bo] (cf. its review MR1860666-2002i:32015). We will give a partial answer in Sect. 3.2.
(iii) In the proof of Proposition 2.2 in \[Bo\], the author shows that if \(n > 2m\), \(N\) contains a submanifold with odd first Betti number. From this it follows that \(N\) is not Kähler. (Remark: Our \(n\) corresponds to \(n - 1\) in \[Bo\].) In the limiting case \(n = 2m\), \(N\) is a compact torus, so it is Kähler.

**Definition.** (\[ALV\]) We say that \((\mathcal{L}, \mathcal{E}, m, n)\) satisfies condition \((K)\), and write \((\mathcal{L}, \mathcal{E}, m, n) \in (K)\), when there exists a real affine automorphism of \((\mathbb{C}^m)^* \cong \mathbb{C}^m \cong \mathbb{R}^{2m}\) sending each \(\ell_i\) to a vector with integer coefficients.

For example, an action whose \(\ell_i\)'s have only rational coordinates satisfies condition \((K)\). In particular, \((K)\) is a dense condition. \(\square\)

2. **Intermediate generalized GIT-quotients**

Take integers \(m, n\) with \(n \geq 2m\). Consider an algebraic torus \((\mathbb{C}^*)^{2m}\) acting effectively and linearly on \(\mathbb{P}^n\). Such an action is given by \(n + 1\) row vectors \(\lambda_i = (\lambda_{i,1}, \ldots, \lambda_{i,2m}) \in \mathbb{Z}^{2m}\) for \(i = 0 \ldots n\). Explicitly:

\[
(\mathbb{C}^*)^{2m} \times \mathbb{P}^n \longrightarrow \mathbb{P}^n
\]

\[
(alg) \quad (t, [x_0 : \ldots : x_n]) \mapsto [t^{\lambda_0} x_0 : \ldots : t^{\lambda_n} x_n]
\]

where \(t^{\lambda_i}\) denotes the product \(t_{1}^{\lambda_{i,1}} \ldots t_{2m}^{\lambda_{i,2m}}\).

In this setting, Bia/‌łynicki-Birula and Świecieckia describe all the good open subsets in \([BB-\acute{S}w2]\). This is a generalization of Mumford’s GIT, because the quotient of such a good open subset \(U\) is not necessarily projective: In general \(X = U / (\mathbb{C}^*)^{2m}\) is a complete simplicial toric abstract-algebraic variety\(^3\). In their language, \(U \rightarrow X\) is a “strongly geometric complete quotient” (NB: their “good quotients” are not assumed to be compact). In particular, all the \((\mathbb{C}^*)^{2m}\)-isotropy subgroups are finite (cf. \([BB-\acute{S}w2]\) 1.4).

Now pick \(G\) a closed cocompact complex Lie subgroup of \((\mathbb{C}^*)^{2m}\) isomorphic to \(\mathbb{C}^m\) (there are plenty of these: see Lemma \[2.2\]). As \((\mathbb{C}^*)^{2m}\) acts properly on \(U\), so does \(G\). Thus \(U\) is good with respect to the \(G\)-action.

Moreover, \(G\) has no torsion, so it can’t intersect the \((\mathbb{C}^*)^{2m}\)-isotropy subgroups. Therefore \(G\) acts freely, and \(N = U / G\) is a compact complex manifold, that we call an intermediate (generalized) GIT-quotient:

\[
\begin{array}{ccc}
U & \rightarrow & N \cong U / G \\
\downarrow & & \downarrow \\
X = U / (\mathbb{C}^*)^{2m} & \rightarrow & N \rightarrow X
\end{array}
\]

and the map \(N \rightarrow X\) is the quotient of \(N\) by the action of \(\mathbb{T} = (\mathbb{C}^*)^{2m} / G\), which is a compact complex \(m\)-torus.

As \(\mathbb{T}\) is compact, the structure of the map \(N \rightarrow X\) is very well understood thanks to the results of Holmann (cf. \([O]\) pp. 82-84). In some cases, \(\mathbb{T}\) acts freely, so \(X\) is a manifold and \(N \rightarrow X\) is simply a \(\mathbb{T}\)-principal bundle. In general, we have a so-called **Seifert principal bundle**: \(X\) is an orbifold whose

---

\(^3\) These authors also significantly improved Mumford’s theory of projective quotients.
singularities correspond to orbits with non-trivial isotropy. The map \( N \to X \)
 is “not locally trivial” around those orbits.

**Remark.** An intuitive (but real) picture of such an exceptional orbit in a Seifert fibration is provided by the circle action \( t.(z, w) = (t^2z, t^3w) \) on the solid torus \( T = \{|z| \leq 1\} \times \{|w| = 1\} \). This yields a foliation of \( T \) for which the core of \( T \) (i.e., \( \{|z| = 0\} \times \{|w| = 1\} \)) is a fiber which is covered 3 times by neighboring fibers, which are \((2, 3)\)-torus knots.

**Theorem 2.1.** Let \((\mathcal{L}, \mathcal{E}, m, n) \in LVMB \cap (K)\), and \( N = N(\mathcal{L}, \mathcal{E}, m, n) \). Then \( N \) can be obtained as an intermediate GIT-quotient as above, for some choice of a \((\mathbb{C}^*)^{2m}\)-action on \( \mathbb{P}^n \), a good open subset \( U \) and a subgroup \( G \). In particular, any \( LVMB \)-manifold can be obtained as a small deformation of such an intermediate GIT-quotient.

**Lemma 2.2.** Let \( G \approx \mathbb{C}^m \) be a Lie subgroup of \((\mathbb{C}^*)^{2m}\). Let \( A \in \mathbb{C}^{2\times m} \) be a matrix whose columns \( A_1, \ldots, A_m \) form a \( \mathbb{C} \)-basis of \( \text{Lie}(G) \). Then:

(i) If \( G \) is closed then \( G \) is cocompact;

(ii) \( G \) is closed if and only if the matrix \((\text{Re } A | \text{Im } A) \) \( \in \mathbb{R}^{2\times 2m} \) is invertible.

**Proof.** Denote the exponential map \( \exp : \mathbb{C}^{2m} \to (\mathbb{C}^*)^{2m} \), whose kernel is \( i\mathbb{Z}^{2m} \). As \( G \approx \mathbb{C}^m \), \( \text{Lie}(G) \cap i\mathbb{Z}^{2m} = \{0\} \). Thus \( G \) is closed if and only if \( \text{Lie}(G) \cap i\mathbb{R}^{2m} = \{0\} \). Thus, when \( G \) is closed, \( \text{Span}_{\mathbb{R}}(\alpha \cup \beta) \) is a full lattice in \( \mathbb{C}^{2m} \) for any real bases \( \alpha \) and \( \beta \) of \( \text{Lie}(G) \) and \( i\mathbb{R}^{2m} \) respectively. Therefore a closed \( G \) is cocompact.

Now: \( \text{Lie}(G) \cap i\mathbb{R}^{2m} \neq \{0\} \) if and only if there exists \( (z_1, \ldots, z_m) \in \mathbb{C}^m \setminus \{0\} \) such that

\[
(\ast) \quad \text{Re} \left( \sum_{j=1}^{m} z_j A_j \right) = 0.
\]

Writing \( z_j \) as \( x_j + iy_j \), the equation \( \ast \) is equivalent to

\[
\sum_{j=1}^{m} x_j \text{Re } A_j - y_j \text{Im } A_j = 0.
\]

Thus \( G \) is not closed if and only if \( \{\text{Re } A_1, \ldots, \text{Re } A_m, \text{Im } A_1, \ldots, \text{Im } A_m\} \)
 is \( \mathbb{R} \)-linearly dependent, that is, \( (\text{Re } A | \text{Im } A) \) is not invertible. \( \square \)

**Proposition 2.3.** An action of form \( \text{hol} \) satisfies \( (K) \) if and only if it is the restriction of a \((\mathbb{C}^*)^{2m}\)-action of form \( \text{alg} \) to a closed cocompact subgroup isomorphic to \( \mathbb{C}^m \).

**Proof.** Take a \((\mathbb{C}^*)^{2m}\)-action of form \( \text{alg} \) given by \( \lambda_0, \ldots, \lambda_n \in \mathbb{Z}^{2m} \). Let \( G \) be a closed cocompact subgroup of \((\mathbb{C}^*)^{2m}\) which is isomorphic to \( \mathbb{C}^m \). Take \( A \in \mathbb{C}^{2\times m} \) as in Lemma 2.2. We know that \( (\text{Re } A | \text{Im } A) \) is invertible. Writing \( G = \{\exp(Az), z \in \mathbb{C}^m\} \), a direct computation shows that the \((\mathbb{C}^*)^{2m}\)-action restricted to \( G \) is of form \( \text{hol} \), with \( \ell_i = \lambda_i A \) for \( i = 0 \ldots n \). That restricted action satisfies \( (K) \) because \( (\text{Re } \ell_i, \text{Im } \ell_i)(\text{Re } A | \text{Im } A)^{-1} \) \( \in \mathbb{R}^m \times \mathbb{R}^m \approx \mathbb{R}^{2m} \) equals \( \lambda_i \), so has integer coefficients.

Conversely, take a \( \mathbb{C}^m\)-action of form \( \text{hol} \) satisfying \( (K) \), given by the linear forms \( \ell_0, \ldots, \ell_n \). Notice that translating all the \( \ell_i \)'s by the same vector does not change the action. There exist \( \lambda_0, \ldots, \lambda_n \in \mathbb{Z}^{2m} \), an invertible
$M \in \mathbb{R}^{2m \times 2m}$ and $b \in \mathbb{R}^{2m}$ such that for all $i$, $(Re \ell_i, Im \ell_i) = \lambda_i M + b$. Now take $A \in \mathbb{C}^{2m \times m}$ such that $(Re A|Im A) = M$, and denote by $G$ the subgroup of $(\mathbb{C}^*)^{2m}$ such that $\text{Lie}(G) = \text{Image}(A)$. By Lemma 2.2 $G$ is closed and cocompact. Moreover the restriction to $G$ of the $(\mathbb{C}^*)^{2m}$-action given by the $\lambda_i$’s is the $\mathbb{C}^m$-action we started with.

**Proof of Theorem 2.1.** Let $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB \cap (K)$, with associated $N = U/\mathbb{C}^m$. Take a $(\mathbb{C}^*)^{2m}$-action of form $\text{alg}$ given by Proposition 2.3.

By definition, $U$ is the complement of some coordinate subspaces, so it is stable by the “big” algebraic torus $(\mathbb{C}^*)^{n+1}$ acting diagonally on $\mathbb{P}^n$, so in particular $U$ is stable by that $(\mathbb{C}^*)^{2m}$-action. Moreover the topological quotient space $U/(\mathbb{C}^*)^{2m}$ is homeomorphic to $X = (U/\mathbb{C}^m)/((\mathbb{C}^*)^{2m}/\mathbb{C}^m)$, which is the quotient of the compact manifold $N$ by a compact group, therefore it is compact and Hausdorff. Hence $U \rightarrow X$ is one of the “strongly geometric complete” quotients considered in [BB-Sw2]. □

### 3. Applications

**Notation.** For a given $(\mathcal{L}, \mathcal{E}, m, n)$, we denote by $d$ the minimal codimension of $\mathbb{P}^n - U$ (cf. [M]). The condition $d > 1$ is equivalent to $\bigcap_\alpha \mathcal{E}_\alpha = \emptyset$. □

#### 3.1. LVMB-manifolds and complete toric varieties

An action of form $\text{hol}$ (resp. $\text{alg}$) determines a subgroup $H_1$ (resp. $H_2$) of $(\mathbb{C}^*)^{n+1}$, with $H_1 \approx \mathbb{C}^m$ and $H_2 \approx (\mathbb{C}^*)^{2m}$. When $(K)$ is satisfied, the extension from a $\mathbb{C}^m$-action to a $(\mathbb{C}^*)^{2m}$-action in Proposition 2.3 is unique in the sense that $H_2$ must be the smallest algebraic subtorus of $(\mathbb{C}^*)^{n+1}$ containing $H_1$, as follows from the cocompactness of $H_1$ in $H_2$. In particular, for any given $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB \cap (K)$, there is a well-defined

$$X = X(\mathcal{L}, \mathcal{E}, m, n) := U/H_2,$$

with a map $\pi : N(\mathcal{L}, \mathcal{E}, m, n) \rightarrow X$ which is a Seifert principal fibration with compact torus $\mathbb{T} = H_2/H_1$, called a *generalized Calabi-Eckmann fibration* in [M-V]. The fibers of $\pi$ define a foliation on $N$ denoted by $\mathcal{F} = \mathcal{F}(\mathcal{L}, \mathcal{E}, m, n)$.

The next result extends one of [M-V]’s main results to a more general setting. We also give a simpler proof, based on a result of H. Hamm.

**Theorem 3.1.** Let $X$ be any complete simplicial toric variety. Then there exists an LVMB-manifold $N$ giving a generalized Calabi-Eckmann fibration over $X$.

**Proof.** Applying Theorem 6.1 in [Ha], one can realize $X$ as a geometric quotient $V/(\mathbb{C}^*)^r$ with $V$ an open subset of some $\mathbb{C}^n$ acted on linearly by $(\mathbb{C}^*)^r$ and $n > r$. This action is the restriction (to a subgroup isomorphic to $(\mathbb{C}^*)^r$) of the diagonal $(\mathbb{C}^*)^n$-action on $\mathbb{C}^n$. If $r$ is odd, we let $(\mathbb{C}^*)^{r+1} = (\mathbb{C}^*)^r \times \mathbb{C}^*$ act on $V \times \mathbb{C}^*$ by $(t_1, \ldots, t_r, t)_*(v, z) = ((t_1, \ldots, t_r), v, tz)$ then $V \times \mathbb{C}^*$ is an open subset of $\mathbb{C}^{n+1}$, and $V \times \mathbb{C}^*/((\mathbb{C}^*)^{r+1} = X$. Therefore (up to replacing $V$ with $V \times \mathbb{C}^*$ and $n$ with $n+1$) we can assume that $r$ is even, i.e., $r = 2m$.

The map $(z_1, \ldots, z_n) \mapsto [1 : z_1 : \cdots : z_n]$ defines a $(\mathbb{C}^*)^{2m}$-equivariant embedding $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$ sending $V$ to an open subset $U$ of $\mathbb{P}^n$ whose quotient
by $(\mathbb{C}^*)^{2m}$ is $X$. Taking any $G \cong \mathbb{C}^m$ closed and cocompact in $(\mathbb{C}^*)^{2m}$ gives a generalized Calabi-Eckmann fibration $N = U/\mathbb{C}^m$ above $X$. \hfill \Box

**Proposition 3.2.** Let $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB \cap (K)$, with corresponding $N$ and $X$.

(i) If $d > 1$ then $X$ is the algebraic reduction of $N$;

(ii) $X$ is projective if and only if $(\mathcal{L}, \mathcal{E}, m, n) \in LVM$.

In particular, if $X$ is projective then $N$ is an LVM-manifold.

**Proof.** (i) The proof of Theorem 4 in $[M]$ applies here and shows that any meromorphic function on $N$ is a function of some functions $M_1, \ldots, M_s$. But $M_1, \ldots, M_s$ are pull-backs of meromorphic functions on $X$ by Remark 2.13 in $[M-V]$. As $X$ is an abstract-algebraic variety, it is a (non-necessarily projective) algebraic reduction of $N$.

(ii) Take any $(\mathbb{C}^*)^{2m}$-action of form $\text{alg}$ given by Proposition 2.3. From the proof of Proposition 2.3, we know that the $\lambda_i$’s of this action are obtained from the $\ell_i$’s of the $\mathbb{C}^m$-action by an affine automorphism, that we denote $\varphi$. For each polytope $P_\alpha$ associated to $(\mathcal{L}, \mathcal{E}, m, n)$, we define $Q_\alpha = \varphi(P_\alpha)$.

Denote $\Pi = \{Q_\alpha\}_\alpha$, and define $\hat{U} \subset \mathbb{P}^n$ by: $[x_0 : \cdots : x_n] \in \hat{U}$ if and only if the convex hull of $\{\lambda_i | x_i \neq 0\}$ contains a polytope of $\Pi$. We claim that $\hat{U} = U$. The inclusion $U \subset \hat{U}$ follows from the definitions. On the other hand, it follows from Theorem 1.1(sep) that

$$\bigcap_{Q, Q' \in \hat{\Pi}} \hat{Q} \cap \hat{Q}' \neq \emptyset.$$  

In particular, any polytope of $\Pi$ has dimension $2m$, so the quotient of $\hat{U}$ by $(\mathbb{C}^*)^{2m}$ is a geometric quotient (cf. Theorem 7.8 in $[BB-\text{Sw2}]$), i.e., the orbit space $\hat{U}/(\mathbb{C}^*)^{2m}$ is Hausdorff. But this space also contains the compact set $U/(\mathbb{C}^*)^{2m}$ as an open subset, so $\hat{U}/(\mathbb{C}^*)^{2m} = U/(\mathbb{C}^*)^{2m}$. This proves the claim, so $U$ is given by $\Pi$ in the sense of $[BB-\text{Sw2}]$.

From Example 7.12 and Corollary 7.16 in $[BB-\text{Sw2}]$, it follows that a necessary and sufficient condition for $X$ to be projective is that $\bigcap_{Q \in \Pi} Q \neq \emptyset$.

By this is equivalent to

$$\bigcap_{Q \in \Pi} \hat{Q} \neq \emptyset,$$

which is in turn equivalent to $\bigcap_{\alpha} \hat{P}_\alpha \neq \emptyset$, i.e., $(\mathcal{L}, \mathcal{E}, m, n) \in LVM$. \hfill \Box

### 3.2. Existence of LVMB-manifolds that are not LVM-manifolds.

#### 3.2.1. **Standard submanifolds.** Let $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB$, with associated $N = N(\mathcal{L}, \mathcal{E}, m, n)$. Let $U^1$ be a subset of $U \subset \mathbb{P}^n$ defined by the vanishing of some homogeneous coordinates. Then the $\mathbb{C}^m$-action on $U$ restricts on $U^1$ to an action of form $[ho]$, and when it satisfies the conditions of Theorem 1.1 it gives a submanifold $N^1 \subset N$ called a standard submanifold of $N$ with respect to $(\mathcal{L}, \mathcal{E}, m, n)$ (cf. $[M]$ p.100).

Examples of standard submanifolds are obtained as follows: Take any element $\mathcal{E}_\alpha$ of $\mathcal{E}$, and define a $2m$-dimensional $U^1$ by letting $z_i = 0$ for $i \notin \mathcal{E}_\alpha$. By doing this for each element of $\mathcal{E}$, one gets a finite family of standard $m$-submanifolds. They have minimal dimension among standard submanifolds.
of $N$ w.r.t. $(\mathcal{L}, \mathcal{E}, m, n)$. We call them \textit{minimal} standard submanifolds of $N$ w.r.t. $(\mathcal{L}, \mathcal{E}, m, n)$.

\textbf{Lemma 3.3.} A standard submanifold of $N$ is minimal if and only if it is a compact complex torus.

\textit{Proof.} Each standard submanifold is itself an $LVMB$-manifold, so the statement follows from Proposition 2.1 in \cite{Bo}, and the fact that tori are symplectic (NB: our $n$ corresponds to $n - 1$ in \cite{Bo}). \hfill $\square$

3.2.2. \textit{Action of $Aut^0(N)$}. We state a few consequences of \cite{Bo} Proposition 2.4 and of \cite{M} Part V, whose proofs apply also to $LVMB$-manifolds.

Let $N = N(\mathcal{L}, \mathcal{E}, m, n)$ be an $LVMB$-manifold. The linear diagonal action of $(\mathbb{C}^*)^{n + 1}$ on $\mathbb{P}^n$ descends to an action on $N$, which corresponds to a subgroup of $Aut(N)$ that we call $G$. Each $G$-orbit is a standard submanifold $N^1$ minus all standard submanifolds strictly contained in $N^1$. In particular, minimal standard submanifolds are $G$-orbits.

In general, $G$ is contained in $Aut^0(N)$, the connected component of the identity. Therefore the closure of any $Aut^0(N)$-orbit is a union of standard submanifolds.

Finally, if $d > 1$ and $\ell_i \neq \ell_j$ when $i \neq j$, then $G = Aut^0(N)$. This follows from \cite{M} Theorem 6, which states the result at the $-\text{commutative-}$Lie algebra level.

3.2.3. \textit{A family of $LVMB$-manifolds that are not $LVM$-manifolds.}

\textbf{Proposition 3.4.} Let $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB$, with associated $d$ and $\ell_i$'s such that $d > 1$ and $\ell_i \neq \ell_j$ when $i \neq j$. Let $N = N(\mathcal{L}, \mathcal{E}, m, n)$.

If there exists $(\mathcal{L}', \mathcal{E}', m', n')$ such that $N(\mathcal{L}', \mathcal{E}', m', n') = N$, then:

(i) $m' = m$, $n' = n$, $d' > 1$;
(ii) if $(\mathcal{L}, \mathcal{E}, m, n) \in (K)$ then $(\mathcal{L}', \mathcal{E}', m', n') \in (K)$.

\textit{Proof.} We use the facts stated in \cite{222}.

(i) The $Aut^0(N)$-orbits of minimal dimension are exactly the minimal standard submanifolds of $N$ w.r.t. $(\mathcal{L}, \mathcal{E}, m, n)$. Therefore they are tori of dimension $m$.

Let $p$ belong to one of these tori. We know \textit{a priori} that the closure of the orbit of $p$ under $Aut^0(N)$ is a union of standard submanifolds w.r.t. $(\mathcal{L}', \mathcal{E}', m', n')$. That orbit being closed and smooth, it must equal exactly one such submanifold $N'_i$. By Lemma 3.3 it is a minimal standard submanifold w.r.t. $(\mathcal{L}', \mathcal{E}', m', n')$, so $m' = m$. Then $n' = n$ because $n' - m' = \dim N = n - m$.

From the proof of Proposition 2.1 in \cite{Bo}, $d > 1$ implies that

$$H^2(N, \mathbb{Z}) \cong \mathbb{Z} \text{ and } \pi_1(N) = 0$$

($\pi_1(N) = 0$ because there is a principal bundle over $N$ with fiber a circle and a 2-connected total space).

Now we use that $N = N(\mathcal{L}', \mathcal{E}', m', n')$. Denote by $k'$ the cardinality of $\bigcap_{j} E_{\alpha_j}$. Assume that $d' = 1$. Then $k' \geq 1$, and Bosio's description of the homotopy type of $N$ implies that $H^2(N, \mathbb{Z})$ has rank $0$ or $(k' - 1)(k' - 2)/2$.
and $\pi_1(N) = \mathbb{Z}^{k-1}$. Thus, by $(*)$, $(k'-1)(k'-2)/2 = 1$, so $k' = 3$. Then $\pi_1(N) = \mathbb{Z}^2$, which is a contradiction, so $d' > 1$.

(ii) For $N = N(\mathcal{L}, \mathcal{E}, m, n)$ with $d > 1$, Theorem 4 (ii) in [M] equates the algebraic dimension of $N$ to an integer $a$ (the proof given for LVM-manifolds applies also in the LVMB case). It follows from the definitions of $a$ and $(K)$ that in general $a \leq \dim N - m$, with equality if and only if $(\mathcal{L}, \mathcal{E}, m, n) \in (K)$. Therefore (ii) follows from (i). \qed

**Proposition 3.5.** Let $N = N(\mathcal{L}, \mathcal{E}, m, n)$ be an LVMB-manifold. Assume that $(\mathcal{L}, \mathcal{E}, m, n) \in (K)$, $d > 1$, and $\ell_i \neq \ell_j$ for $i \neq j$.

Then there exists an open, dense and $\mathcal{F}$-saturated subset $N_* \subset N$ such that the following are equivalent:

(i) $p, q \in N_*$ belong to the same leaf of $\mathcal{F}$.

(ii) for all $f \in \mathcal{M}(N)$ holomorphic at $p$ and $q$, $f(p) = f(q)$.

*Proof.* The foliation $\mathcal{F}$ is given by the fibers of the map $\pi : N \to X(\mathcal{L}, \mathcal{E}, m, n)$ and $X$ is an abstract-algebraic variety, so in particular $X$ is a Moishezon space. By Hironaka’s and Moishezon’s theorems, there exists $\tilde{X}$ smooth and projective, with a sequence of blow-downs $\varphi : \tilde{X} \to X$.

Then $f \mapsto f \varphi$ is an isomorphism between the fields of meromorphic functions $\mathcal{M}(X)$ and $\mathcal{M}(\tilde{X})$, and there exist $X_s$ and $\tilde{X}_s$ open dense subsets in $X$ and $\tilde{X}$ between which $\varphi$ induces a biholomorphism. Define $N_s = \pi^{-1}(X_s)$.

(ii) implies (i):

Take $p, q \in N_*$ not on the same leaf, i.e., $\pi(p)$ and $\pi(q)$ are distinct points in $X_s$, so $p_s = \varphi^{-1}\pi(p)$ and $q_s = \varphi^{-1}\pi(q)$ are distinct in $\tilde{X}_s$. Then there exists $f \in \mathcal{M}(\tilde{X})$ holomorphic at $p_s$ and $q_s$ such that $f(p_s) \neq f(q_s)$ (cf. [G-F], V, Theorem 3.14). Then $f \varphi^{-1}\pi \in \mathcal{M}(N)$ is holomorphic at $p$ and $q$, and $f(p) \neq f(q)$.

(i) implies (ii):

Take $p, q \in N_*$ on the same leaf $L$, i.e., $\pi(p) = \pi(q)$. Let $f \in \mathcal{M}(N)$, holomorphic at $p$ and $q$. Then there exists $f_1 \in \mathcal{M}(X)$ such that $f = f_1 \pi$ (cf. proof of Proposition 3.2).

As $f$ is holomorphic at $p$, it is bounded on a neighborhood of $p$. As $\pi$ is an open map, $f_1$ is bounded on a neighborhood $A$ of $\pi(p) = \pi(q)$. So $f_1$ is holomorphic on $A$, and $f$ is holomorphic on $\pi^{-1}(A)$, and in particular on $L$, which is compact. So $f$ is constant on $L$, and in particular $f(p) = f(q)$. \qed

**Theorem 3.6.** Let $(\mathcal{L}, \mathcal{E}, m, n) \in LVMB \cap (K)$, with $d > 1$ and $\ell_i \neq \ell_j$ for $i \neq j$. If $(\mathcal{L}, \mathcal{E}, m, n) \notin LVM$ then $N$ is not biholomorphic to any LVM-manifold.
Proof. Assume on the contrary that $N$ can also be written as $N(L', E', m', n')$ with $(L', E', m', n') \in LVM$. By Proposition 3.4 (ii), $(L', E', m', n') \in (K)$. Now denote by $F$ and $F'$ the foliations of $N$ with respect to $(L, E, m, n)$ and $(L', E', m', n')$. By Proposition 3.5, $F$ and $F'$ agree on a dense open subset, so $F = F'$ everywhere. But by Proposition 3.2, $X' = N/F'$ is projective, whereas $X = N/F$ is not. Contradiction. \hfill $\square$

**Example.** In Ex. 3.2 it is immediate to check that $(L, E, m, n) \in (K)$ and that $\cap_{a} E_{a} = \emptyset$, so $d > 1$ (actually $d = 2$). Also $(L, E, m, n) \notin LVM$, so by Theorem 3.6, $N(L, E, m, n)$ is not isomorphic to any LVM-manifold.

### 3.3. Foliations

Let $(L, E, m, n) \in LVM$, and $N = N(L, E, m, n)$. Generalizing the results of Loeb and Nicolau, Meersseman shows (cf. [M]) the existence of a foliation $F = F(L, E, m, n)$ on $N$, and proves it is transversely Kähler. This means that $N$ admits a closed, real and $J$-invariant two-form $\omega$, positive on the normal bundle of $F$, and such that $\ker(\omega) = F$. This is a strong property, that has interesting geometric consequences on $N$.

Now assume that $(L, E, m, n) \in (K)$. Then the foliation’s leaves are just the fibers of $\pi : N \to X(L, E, m, n)$. Assume moreover that $(L, E, m, n) \notin LVM$. By Proposition 3.5 (ii), we know that $X$ is not projective. Using this fact, we prove below that $F$ is not transversely Kähler. This is an unexpected difference with the case of LVM-manifolds.

The proof is straightforward when $T$ acts freely on $N$, i.e., when $N \to X = N/T$ is a genuine principal bundle: Assume that $F$ is transversely Kähler with respect to a 2-form $\omega$. Then:

(a) Make $\omega$ a $T$-invariant form by averaging it over the $T$-action, and push it forward on $X$, where it gives a Kähler form.

(b) As $X$ is smooth, Kähler and Moishezon, it is projective by a theorem of B. Moishezon.

In the general case of $N \to X$ being a Seifert principal bundle (and $X$ being singular), both steps (a) and (b) become non-trivial:

For (a): The problem is that the push forward of a $T$-invariant $\omega$ is not smooth in general (it is not even continuous). To fix this problem, we use some results and methods of D. Barlet and J. Varouchas. We take local potentials of $\omega$ on slices of the Seifert bundle, and push them forward to $X$. Then we can apply the following theorem of Varouchas: *If a complex space $X$ has an open cover $\{U_i\}_{i \in I}$, with for all $i$ a continuous strictly plurisubharmonic function $\psi_i$, such that for all $i$ and $j$, $\psi_i - \psi_j$ is plurisubharmonic, then $X$ is a Kähler space in the sense of Grauert.* (Grauert’s definition of a Kähler space is exactly the above sentence with “continuous” replaced with “smooth”.)

For (b): It is not true in general that a Moishezon Kähler space is projective. However, as $X$ has only rational singularities, a theorem of Y. Namikawa implies that $X$ is projective.

**Theorem 3.7.** Let $(L, E, m, n) \in LVM \cap (K)$, with corresponding $N$ and $F$. If $(L, E, m, n) \notin LVM$ then $F$ is not transversely Kähler.

**Proof.** Assume on the contrary that there exists a 2-form $\omega_0$ with respect to which $F$ is transversely Kähler. We have to show that $(L, E, m, n) \in LVM$. 


First step: “make \( \omega_0 \) \( \mathbb{T} \)-invariant”. Let \( A \) be an automorphism of \( N \) induced by some element of \( \mathbb{T} \). Remark that for any \( p \in N \) and any vector \( v \in T_p N \), \( v \) is tangent to the \( \mathbb{T} \)-orbit of \( p \) if and only if \( A_s v \) is.

For \( u, v \in T_p N \), define

\[
\omega(u, v) = \int_{t \in \mathbb{T}} \omega_0(tu, tv)
\]

for \( \omega_0 \) the normalized Haar measure on \( \mathbb{T} \). This defines a \( \mathbb{T} \)-invariant, closed, real and \( J \)-invariant 2-form \( \omega \). Moreover, by the above remark, \( \ker \omega = \mathcal{F} \) and \( \omega \) is positive on the normal bundle of \( \mathcal{F} \) (because an integral of positive numbers is positive). So \( \mathcal{F} \) is transversely \( \mathbb{K} \)ähler with respect to \( \omega \).

Second step: “push forward local potentials on slices”. It follows from the results of Holmann (cf. [V2] pp. 82–84) that we can find a family of local holomorphic slices \( \{S_i\}_{i \in I} \) such that for each \( i \):

- \( S_i \subset N \) is transverse to \( \mathcal{F} \) and biholomorphic to a ball of same dimension as \( X \);
- \( \pi_i = \pi|_{S_i} \) is a quotient by a finite subgroup of \( \mathbb{T} \) denoted by \( \Gamma_i \) (\( \Gamma_i \) is some isotropy subgroup);
- \( \{V_i = \pi_i(S_i)\}_{i \in I} \) is an open cover of \( X \).

For each \( i \), \( \pi_i : S_i \rightarrow V_i \) is a ramified covering. We denote by \( \pi_i : S_i^{\text{reg}} \rightarrow V_i^{\text{reg}} \) the associated regular covering (off the ramification locus). We define \( \omega_i = \omega|_{S_i} \), which is a \( \mathbb{K} \)ähler form on the ball \( S_i \). So it admits a potential \( \varphi_i \), i.e., \( \varphi_i \in C^\infty(S_i, \mathbb{R}) \), \( \varphi_i \) is strictly plurisubharmonic (p.s.h.) and \( \omega_i = \sqrt{-1}\partial\bar{\partial}\varphi_i \).

Now define for all \( i \) a map \( \psi_i : V_i \rightarrow \mathbb{R} \) by

\[
\psi_i(x) = \frac{1}{n_i} \sum_{\gamma \in \Gamma_i} \varphi_i(\gamma, p),
\]

where \( p \in \pi_i^{-1}(x) \) and \( n_i \) is the order of \( \Gamma_i \). Then for all \( i \), \( \psi \) is continuous and strictly p.s.h. by a result of Barlet (cf. [V2] Proposition 3.4.1).

Third step: we prove that for all \( i, j \), \( \psi_i - \psi_j \) is pluriharmonic on \( V_i \cap V_j \).

We use the definition of pluriharmonic (p.h.) of [V2]. For a real function, this means that the function is locally the real part of a holomorphic function.

We will first prove that \( \psi_i - \psi_j \) is p.h. on \( V_i^{\text{reg}} \cap V_j^{\text{reg}} \). Let \( V \) be any small open subset of \( V_i^{\text{reg}} \cap V_j^{\text{reg}} \) isomorphic to an open ball. Then \( \pi_i^{-1}(V) \) is a union of balls \( B_{i,1}, \ldots, B_{i,n_i} \) and \( \pi_j^{-1}(V) \) is a union of balls \( B_{j,1}, \ldots, B_{j,n_j} \). Pick any two of them \( B_\alpha \) and \( B_\beta \). Denote \( \pi_\alpha = \pi|_{B_\alpha} \) \( \pi_\beta = \pi|_{B_\beta} \). Then \( \pi_\alpha \) (resp. \( \pi_\beta \)) sends \( B_\alpha \) (resp. \( B_\beta \)) isomorphically onto \( V \).

We now check that

\[
(\pi_\alpha)_* \omega = (\pi_\beta)_* \omega.
\]

Take \( p \in V \) and \( u, v \in T_p V \). Denote: \( p_\alpha = \pi_\alpha^{-1}(p) \), \( u_\alpha = (\pi_\alpha^{-1})_* u \), \( v_\alpha = (\pi_\alpha^{-1})_* v \), and \( p_\beta \), \( u_\beta \), \( v_\beta \) in the analogous way. We need to show:

\[
\omega(u_\alpha, v_\alpha) = \omega(u_\beta, v_\beta).
\]

As \( p_\alpha \) and \( p_\beta \) are on the same fiber of \( \pi \), there exists \( A \in \text{Aut}(N) \) induced by some \( t \in \mathbb{T} \) such that \( A(p_\alpha) = p_\beta \). By the \( \mathbb{T} \)-invariance of \( \omega \),

\[
\omega(u_\alpha, v_\alpha) = \omega(A_s u_\alpha, A_s v_\alpha).
\]
On the other hand, \( \pi A = \pi \), so \( \pi_* A_* u_\alpha = \pi_* u_\alpha = u = \pi_* u_\beta \). Therefore \( u_\beta - A_* u_\alpha \in Ker \pi_* = Ker \omega \). Thus \( \omega(u_\beta - A_* u_\alpha, v_\beta) = 0 \), i.e., \( \omega(A_* u_\alpha, v_\beta) = \omega(u_\beta, v_\beta) \). Repeating this for the second entry we get

\[
(3) \quad \omega(A_* u_\alpha, A_* v_\alpha) = \omega(u_\beta, v_\beta).
\]

By (2) and (3), we get (1). Therefore (e) holds, which means that pushing forward \( \omega \) by \( \pi \) from any ball among \( B_{i,1}, \ldots, B_{i,n_i}, B_{j,1}, \ldots, B_{j,n_j} \) gives on \( V \) the same 2-form, that we denote by \( \omega_V \).

By the definition of \( \psi_i \) and \( \psi_j \) we have

\[
\psi_{j|V} = \frac{1}{n_i} \sum_{\nu=1...n_i} \pi_*(\varphi_{i,B_{i,\nu}}) \quad \text{and} \quad \psi_{j|V} = \frac{1}{n_j} \sum_{\nu=1...n_j} \pi_*(\varphi_{i,B_{j,\nu}}).
\]

So

\[
\sqrt{-1} \partial \bar{\partial} \psi_{j|V} = \frac{1}{n_i} \sum_{\nu=1...n_i} \sqrt{-1} \partial \bar{\partial} \pi_*(\varphi_{i,B_{i,\nu}}) = \frac{1}{n_i} \sum_{\nu=1...n_i} \pi_*(\sqrt{-1} \partial \bar{\partial} \varphi_{i,B_{i,\nu}}) = \omega_V,
\]

and, similarly, \( \sqrt{-1} \partial \bar{\partial} \psi_{j|V} = \omega_V \). Therefore \( \partial \bar{\partial}(\psi_i - \psi_j)|V = 0 \), so \( \psi_i - \psi_j \) is on \( V \) the real part of some holomorphic function. This proves that \( \psi_i - \psi_j \) is p.h. on \( V_i^{reg} \cap V_j^{reg} \), and we also know that it is continuous on \( V_i \cap V_j \).

Now take any small open subset \( V \subset V_i \cap V_j \) such that \( \pi^{-1}(V) \) has a simply connected locus of \( V \) as an orbifold, and denote by \( Q = \pi^{-1}(Q) \cap U \). Then \( R \) is the ramification locus of \( \pi_i : U \to V \). In particular, \( R \) is an analytic subset of \( U \). Now \( \pi_i^*(\psi_i - \psi_j|V) \) is continuous on \( U \), and it is p.h. on \( U - R \), so in particular it is p.s.h. on \( U - R \). By a result of Grauert and Remmert (in [G-R]), that function extends as a p.s.h. function on \( U \). We can show similarly that its opposite is p.s.h. on \( U \). Therefore it is p.h., so it is the real part of some holomorphic function. Now pushing forward that holomorphic function gives on \( V \) a holomorphic function whose real part is \( \psi_i - \psi_j \).

Fourth step: We can now apply Theorem 1 of [V2] (remark that our \( X \) is reduced, so “(ii) follows from (i)”), to get that \( X \) is Kähler in the sense of Grauert (cf. [G]). As \( X \) is a toric variety, we know it has only rational singularities by Theorem 5.2 in [C]. Also \( X \) is an abstract-algebraic complex variety, so it is a Moishezon complex space. By Corollary 1.7 in [N], we get that \( X \) is projective. By Proposition 3.2 (\( \mathcal{L}, \mathcal{E}, m, n \) \( \in LVM \)).

Remark. For an \( LVMB \)-manifold obtained by a non-\( LVM \) action that does not satisfy \( (K) \), one can still construct a foliation \( \mathcal{F} \), as in [M]. We expect \( \mathcal{F} \) to be non transversely Kähler, but are not able to prove it. There is no general deformation argument to get this result: Whereas a small deformation of a compact Kähler manifold is still Kähler, the analogous statement is not
true in general for transversely Kähler foliations, unless the differentiable type of the foliation is preserved (cf. [EKA-G]).

3.4. Further generalizations of the LVMB family. As a concluding remark, we give two ways of generalizing the family of LVMB-manifolds that are very natural from our GIT point of view. These two ways are independent, and can be combined.

3.4.1. Other cocompact subgroups. Start with a $(\mathbb{C}^*)^k$-action of form $[\text{alg}]$, with $k$ not necessarily even. Now take any closed cocompact complex Lie subgroup $G \subset (\mathbb{C}^*)^k$. Such a $G$ is isomorphic to $\mathbb{C}^m \times (\mathbb{Z})^l$, with $k = 2m + l$. Now take a good open subset $U$ for the $(\mathbb{C}^*)^k$-action. The quotient $N = U/G$ is a compact complex manifold. One can still deform the parameters of the action to get other manifolds.

Note that $N$ is topologically a fiber bundle over an LVMB-manifold with fiber a real torus $(S^1)^l$.

3.4.2. Quotients à la Białynicki-Birula and Sommese. Białynicki-Birula and Sommese have conjectured that part of the algebraic theory of GIT-quotients can be extended to the more general case of meromorphic actions of $(\mathbb{C}^*)^k$ on a reduced compact normal complex analytic space $Y$, with similar combinatorial properties (cf. [BB-So]). Unfortunately, only the cases of $k = 1$, and $k = 2$ with $Y$ smooth and Kähler are fully understood so far, and there are few non-toric examples worked out in the literature.

It is likely that these quotients yield many new generalized Calabi-Eckmann fibrations, possibly giving new examples over simplicial toric varieties as well.

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