Research Article

Quantitative Uncertainty Principles Associated with the Deformed Gabor Transform

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Abstract

The nonstationary signals constitute a wider class of signals arising in natural or artificial communication systems. As such, the mathematical representation of such signals is one of the core areas of interest among researchers working in diverse aspects of harmonic analysis. Indeed, the nonstationary signals require frequency analysis that is local in time, resulting in the notion of time-frequency analysis. The utmost development in the context of time-frequency analysis came in the form of the well-known Gabor transform [1], which deals with the decomposition of nontransient signals in terms of time- and frequency-shifted basis functions, known as Gabor window functions. With the aid of these window functions, one can analyze the spectral contents of nontransient signals in localized neighbourhoods of time [2].

Mathematically, the Gabor transformation of any \( f \in L^2(\mathbb{R}) \) evaluated at the location \((y,v)\) in the time-frequency plane is defined by [3]

\[
G_h(f)(y,v) = \int_{\mathbb{R}} f(t)h(t-y)e^{-iyv} \, dt,
\]

where \( h \in L^2(\mathbb{R}) \) is an arbitrary window function. Keeping in view the fact that Gabor transform (1) relies on the family of analyzing functions \( h(t-y)e^{iyv} \) determined by the translation and modulation operators acting on the window function \( h \), Mejjaoli [4] introduced the notion of deformed Gabor transform by revamping the classical family of analyzing elements as

\[
h_{v,y}(x) = t^{kn}_{(v,y)} h_{y}(x), \quad x \in \mathbb{R}, \quad (2)
\]

where \( t^{kn}_{v,y} \) and \( h_y \) denote the generalized translation and modulation operators acting on the window function \( h \) as

\[
h_y = F_{k,n}(\sqrt{t^{kn}_{v,y}(|h|^2)}),
\]

\[
F_{k,n}(t^{kn}_{v,y} h) = B_{k,n}(\cdot,x) F_{k,n}(h),
\]

where \( F_{k,n}, n \in \mathbb{N}, k \geq (n-1)/n \), denotes the well-known deformed Hankel transform. For any \( f \in L^2_{k,n}(\mathbb{R}) \), the deformed Gabor transform with respect to \( h \in L^2_{k,n,c}(\mathbb{R}) \) is given as

\[
G^{kn}_{h}(f)(y,v) = \int_{\mathbb{R}} f(x)h_{v,y}(x)dy_{k,n}(x),
\]

where \( h_{v,y}(x) \) is given by (2).

On the flip side, the notion of uncertainty principles is central in harmonic analysis and with the advent time-frequency analysis, the investigation of the uncertainty inequalities received considerable heed and such inequalities have already been extensively studied for diverse integral transforms ranging from the classical Fourier to the recently introduced quadratic-phase Fourier transforms [5]. The pioneering Heisenberg’s uncertainty principle asserts that it is impossible for any ideal function to attain compact support in both the time and frequency domains. In literature, many amendments of the usual Heisenberg’s uncertainty principle have been carried out, with the most notable ones being the Beckner-type uncertainty principles.

1. Introduction

The nonstationary signals constitute a wider class of signals arising in natural or artificial communication systems. As such, the mathematical representation of such signals is one of the core areas of interest among researchers working in diverse aspects of harmonic analysis. Indeed, the nonstationary signals require frequency analysis that is local in time, resulting in the notion of time-frequency analysis. The utmost development in the context of time-frequency analysis came in the form of the well-known Gabor transform [1], which deals with the decomposition of nontransient signals in terms of time- and frequency-shifted basis functions, known as Gabor window functions. With the aid of these window functions, one can analyze the spectral contents of nontransient signals in localized neighbourhoods of time [2].

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2. Deformed Hankel and Gabor Transforms

The aim of this section is to present the prerequisites concerning the deformed Hankel and Gabor transforms which shall be frequently used in formulating the main results. The main references are [12–15].

2.1. The Deformed Hankel Transform. Here, we shall take a survey of the deformed Hankel transform together with the fundamental properties. To facilitate the narrative, we set some notations as follows:

1. $C_b(R)$ the space of bounded continuous functions on $R$.
2. The space $C_{b,e}(R)$ of even bounded continuous functions on $R$.
3. For $p \in [1, \infty]$, the conjugate exponent shall be denoted by $p'$.
4. $M_k,n := n^{(2k-1)/2}2^{n(2k-1)+2}1/(n(2k-1) + 2/2),
   d_{k,n}(x) := M_k,n|x|^{(2nk-2)/2n}dx, \ k \geq n - 1/n, \ n \in \mathbb{N}$.
5. $L^p_{k,n} (R), 1 \leq p \leq \infty$, denotes the space of measurable functions $f$ on $R$ satisfying
   $$\|f\|_{L^p_{k,n} (R)} := \left( \int_R |f(x)|^p d_{k,n}(x) \right)^{1/p} < \infty, \text{ if } 1 \leq p < \infty,$$
   $$\|f\|_{L^\infty_{k,n} (R)} := \sup_{x \in R} |f(x)| < \infty.$$ (5)

In case $p = 2$, the inner product on the space $L^2_{k,n} (R)$ is given by
   $$\langle f, g \rangle_{L^2_{k,n} (R)} := \int_R f(x)g(x)d_{k,n}(x).$$ (6)

We are now in a position to recall the notion of deformed Hankel transform. In this direction, we have the following definition.

Definition 2.1. For $k \geq n - 1/n, \text{ and } f \in L^1_{k,n} (R)$, the integral operator
   $$\mathcal{F}_{k,n} (f)(\lambda) = \int_R f(x)B_{k,n}(\lambda, x)d_{k,n}(x), \text{ for all } \lambda \in R,$$
   is termed as the deformed Hankel transform, where $B_{k,n}(\lambda, x)$ denotes the kernel
   $$B_{k,n}(\lambda, x) = J_{nk-n/2}(n|x|^{|1/|n|}) + (-i)^n(n/2)^n
   \Gamma(nk - n/2 + 1)\Gamma(nk + n/2 + 1)\lambda x J_{nk-n/2}(n|x|^{|1/|n|}),$$ (8)

and $J_{\alpha}(u)$ are the normalized Bessel function of index $\alpha$:
   $$J_{\alpha}(u) := \Gamma(\alpha + 1)(\frac{u}{2})^{-\alpha} \int_{-\alpha}^{\alpha} J_{\alpha}(u) = \Gamma(\alpha + 1)$$
   \frac{(-1)^m}{m!\Gamma(\alpha + m + 1)}(\frac{u}{2})^m.$$ (9)

Deformed Hankel kernel (8) satisfies the following properties:

(i) For $z, t \in R$, we have
   $$B_{k,n}(z, t) = B_{k,n}(t, z),$$
   $$B_{k,n}(z, 0) = 1,$$
   $$B_{k,n}(z, t) = B_{k,n}(-1)^nz, t.$$ (10)

Moreover, for all $\lambda \in R$, we have $B_{k,n}(\lambda z, t) = B_{k,n}(z, \lambda t)$.

(ii) There exists a finite positive constant $C$ depending on $n$ and $k$ such that
Theorem 2.1. For any pair of functions \( f, g \in L^2_{k,n}(\mathbb{R}) \), the following assertions are true:

(i) The deformed Hankel transform \( \mathcal{T}_{k,n} \) is bounded on the space \( L^1_{k,n}(\mathbb{R}) \) and

\[
\| \mathcal{T}_{k,n}(f) \|_{L^1_{k,n}(\mathbb{R})} \leq \| f \|_{L^1_{k,n}(\mathbb{R})}, \quad \forall f \in L^1_{k,n}(\mathbb{R}).
\]

(ii) The deformed Hankel transform provides a natural generalization of the classical Hankel transform. Indeed, if we set

\[
B_{k,n}(x, y) = \frac{1}{2} (B_{k,n}(x + y) + B_{k,n}(x - y)) = \frac{\lambda}{2} (|xy|^{1/n}).
\]

(iii) For all \( n \in \mathbb{N} \), we have

\[
\mathcal{T}_{k,n}(f)(\xi) = \frac{(n!)^{2k-2n}}{2k+2-n!} \int_0^\infty \left[ \int_0^\infty f(x) dx \right] x^{1/2} (\xi^2 x^{n/2} - 1) dx, \quad \forall \xi \in \mathbb{R}.
\]

Proposition 2.2. Let \( f \) be in \( L^p_{k,n}(\mathbb{R}) \) and \( p \in [1, 2] \). Then, \( \mathcal{T}_{k,n}(f) \) belongs to \( L^p_{k,n}(\mathbb{R}) \) and

\[
\| \mathcal{T}_{k,n}(f) \|_{L^p_{k,n}(\mathbb{R})} \leq \| f \|_{L^p_{k,n}(\mathbb{R})}.
\]

Proposition 2.2. If \( \tau_{x}^{k,n} f \) is the generalized translation operator on \( L^2_{k,n}(\mathbb{R}) \), then the following statements are true:

(i) For any \( f \in L^2_{k,n}(\mathbb{R}) \), we have

\[
\| \tau_{x}^{k,n} f \|_{L^2_{k,n}(\mathbb{R})} \leq \| f \|_{L^2_{k,n}(\mathbb{R})}, \quad \forall x \in \mathbb{R}.
\]

(ii) If \( f \in \mathcal{W}_{k,n}(\mathbb{R}) \), then

\[
\tau_{x}^{k,n} f(y) = \int_\mathbb{R} B_{k,n}(\xi) \chi_{k,n}(\xi) d\gamma_{k,n}(\xi), \quad \forall x, y \in \mathbb{R}.
\]

(iii) For all \( f \in \mathcal{W}_{k,n}(\mathbb{R}) \) and \( g \in \mathcal{W}_{k,n}(\mathbb{R}) \), we have

\[
\tau_{x}^{k,n} f(y) = \tau_{y}^{k,n} f(x).
\]

(iv) For all \( f \in \mathcal{W}_{k,n}(\mathbb{R}) \) and \( g \in L^1_{k,n}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R}) \), we have

\[
\int_\mathbb{R} \tau_{x}^{k,n} f(y) g(y) \gamma_{k,n}(y) = \int_\mathbb{R} f(y) \tau_{-1}^{k,n} \chi_{k,n}(y) g(y) \gamma_{k,n}(y), \quad \forall x \in \mathbb{R}.
\]
\[ d\zeta_{x,y}(z) = \begin{cases} \mathcal{H}_{k,n}(x,y,z)dy_{k,n}(z), & \text{if } xy \neq 0, \\ d\delta_x(z), & \text{if } y = 0, \\ d\delta_y(z), & \text{if } x = 0, \end{cases} \]

where \( \mathcal{H}_{k,n}(x,y,z) \) denotes the kernel, which is supported on the set \( \{ z \in \mathbb{R} : \|x\|^{1/n} - |y|^{1/n} < |z|^{1/n} < |x|^{1/n} + |y|^{1/n} \} \). Moreover, operator (26) satisfies the following norm inequality:

\[ \|T^k_x f\|_{L^p_k(\mathbb{R})} \leq 4\|f\|_{L^p(\mathbb{R})}, \quad \forall x \in \mathbb{R}. \]  

Apart from the integral representation of the generalized translation operator given in Theorem 2.2, another useful “trigonometric” form has been obtained in [14, 15]. In continuation to this, we present the next theorem.

**Theorem 2.3** (see [14, 15]). Suppose \( f \in C_0(\mathbb{R}) \) is such that \( f = f_x + f_\psi \) where \( f_x \) is an even function and \( f_\psi \) is an odd function. Then, the generalized translation operator \( T^k_x \) can be expressed as

\[ T^k_x f(y) = M_{k,n} \int_0^\pi f\left( \langle x, y \rangle, \phi_n \right) \left\{ 1 + (-1)^n n! \text{sgn}(xy) C_{n}^{nk-n/2} (\cos \phi) \right\} (\sin \phi)^{2nk-n} d\phi \]

\[ + \int_0^\pi f\left( \langle x, y \rangle, \phi_n \right) \left\{ n! \text{sgn}(y) C_{n}^{nk-n/2} \left( \frac{|x|^{1/n} - |y|^{1/n} \cos \phi}{\langle x, y \rangle^{1/n} \phi_n} \right) \right\} (\sin \phi)^{2nk-n} d\phi, \]  

where \( C_{n}^{nk-n/2} \) are the degenerate polynomials and

\[ \langle x, y \rangle_{\phi_n} = \left( |x|^{2/n} + |y|^{2/n} - 2|xy|^{1/n} \cos \phi \right)^{n/2}. \]  

**Corollary 2.1** (see [14, 15]). For all \( f \in C_0(\mathbb{R}) \), we have

\[ T^k_x f(y) = \frac{M_{k,n}}{2n} \int_0^\pi f\left( \langle x, y \rangle, \phi_n \right) \left\{ 1 + (-1)^n n! \text{sgn}(xy) C_{n}^{nk-n/2} (\cos \phi) \right\} (\sin \phi)^{2nk-n} d\phi. \]

Besides the aforementioned trigonometric form, the authors in [14, 15] have also studied certain important properties of the generalized translation operator, which play a crucial role in the subsequent developments on the subject. Here, we recall some of the needful properties, whose proof can be found in [14, 15].

**Proposition 2.3.** Suppose that \( f \) is nonnegative, is even, and belongs to generalized Wigner space \( \mathcal{W}_{k,n}(\mathbb{R}) \). Then, for any \( x, y \in \mathbb{R} \), we have

(i) \( T^k_x f \geq 0 \).

(ii) \( T^k_x f \in L^1_{k,n}(\mathbb{R}) \) and

\[ \int_\mathbb{R} T^k_x f(y)dy_{k,n}(y) = \int_\mathbb{R} f(y)dy_{k,n}(y). \]  

2.2. The Deformed Gabor Transform. In this section, we shall take a tour of the deformed Gabor transform introduced in [4]. Primarily, we fix some notations which shall be frequently used while formulating the main results. For \( 1 \leq p \leq \infty \), we denote \( L^p_{k,n}(\mathbb{R}^2) \) as the space of measurable functions \( f \) on \( \mathbb{R}^2 \) satisfying

\[ \|f\|_{L^p_{k,n}(\mathbb{R}^2)} = \left( \int_{\mathbb{R}^2} |f(x,y)|^p d\mu_{k,n}(x,y) \right)^{1/p}, \quad 1 \leq p < \infty, \]

\[ \text{ess sup}_{(x,y) \in \mathbb{R}^2} |f(x,y)| < \infty, \quad p = \infty, \]

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where $d\mu_{k,n}(x, y) := dy_{k,n}(x)dy_{k,n}(y)$.

**Definition 2.3.** For $v \in \mathbb{R}$, the generalized modulation acting on $h \in L^2_{k,n,e}(\mathbb{R})$ is given by

$$h_y := \mathcal{F}_{k,n}( \sqrt{r_{k,n}(|h|^2)} ),$$

(37)

where $r_{k,n}$ is the usual generalized translation operator.

Remark: by virtue of positivity of the generalized translation operator as mentioned in Theorem 2.4, we infer that (37) is well defined. Moreover, by virtue of (17) and (32), we have

$$\|h\|_{L^2_{k,n}(\mathbb{R})} = \|h\|_{L^2_{k,n}(\mathbb{R})}, \quad \forall h \in L^2_{k,n,e}(\mathbb{R}).$$

(38)

Based on the generalized translation and modulation operations defined in (20) and (37), respectively, we consider the family of functions $h_{v,y}, v, y \in \mathbb{R}$ as

$$h_{v,y}(x) = h_{k,n}^{-1}(\gamma y)h_{k,n}(x), x \in \mathbb{R}.$$  

(39)

Then, it can be easily verified that

$$\|h_{v,y}\|_{L^2_{k,n}(\mathbb{R})} \leq \|h\|_{L^2_{k,n}(\mathbb{R})}, \quad \forall y, v \in \mathbb{R}.$$  

(40)

With the aid of (39), we are in a position to present the formal definition of the deformed Gabor transform.

**Definition 2.4.** Given $f \in L^2_{k,n}(\mathbb{R})$, the deformed Gabor transform with respect to the window $h \in L^2_{k,n,e}(\mathbb{R})$ is denoted by $g_{h}^{k,n}(f)$ and is defined as

$$g_{h}^{k,n}(f)(v, y) := \int_{\mathbb{R}} f(x) \overline{h_{v,y}(x)} dy_{k,n}(x),$$

(41)

where $h_{v,y}$ is given by (39).

For any $\lambda > 0$ and $(v, y) \in \mathbb{R}^2$, it can be easily verified that

$$g_{h}^{k,n}(f_{\lambda})(v, y) = g_{h}^{k,n}(f)(\frac{v}{\lambda}, \frac{y}{\lambda}),$$

(42)

with

$$g_{\lambda}(x) := \frac{1}{\lambda^{(2k+1)n+2m}} g_{\frac{x}{\lambda}}, \quad \forall x \in \mathbb{R}. $$

(43)

In the next theorem, some basic properties of deformed Gabor Fourier transform (41) are assembled.

**Theorem 2.5 (see [4]).** For any $f \in L^2_{k,n}(\mathbb{R})$ and $h \in L^2_{k,n,e}(\mathbb{R})$, the following statements are true:

(i) The deformed Gabor Fourier transform $g_{h}^{k,n}(f)$ is a bounded operator and

$$\|g_{h}^{k,n}(f)\|_{L^2_{k,n}(\mathbb{R})} \leq \|f\|_{L^2_{k,n}(\mathbb{R})} \|h\|_{L^2_{k,n}(\mathbb{R})}. $$

(44)

(ii) The deformed Gabor transform (41) satisfies the following energy preserving relation:

$$\|g_{h}^{k,n}(f)\|_{L^2_{k,n}(\mathbb{R})} = \|h\|_{L^2_{k,n}(\mathbb{R})} \|f\|_{L^2_{k,n}(\mathbb{R})}. $$

(45)

(iii) For any given pair of functions $f, g \in L^2_{k,n}(\mathbb{R})$, the following orthogonality relation holds:

$$\int_{\mathbb{R}^2} g_{h}^{k,n}(f)(v, y) \overline{g_{h}^{k,n}(g)(v, y)} dy_{k,n}(y) dv_{k,n}(v, y) \leq \|f\|_{L^2_{k,n}(\mathbb{R})} \|g\|_{L^2_{k,n}(\mathbb{R})}. $$

(46)

(iv) For $p \in [2, \infty)$, we have

$$\|g_{h}^{k,n}(f)\|_{L^p_{k,n}(\mathbb{R})} \leq \|f\|_{L^p_{k,n}(\mathbb{R})} \|h\|_{L^2_{k,n}(\mathbb{R})}. $$

(47)

The following lemma follows by a straightforward calculation.

**Lemma 2.1.** Let $h \in L^2_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n,e}(\mathbb{R})$; then, for any $f \in L^2_{k,n}(\mathbb{R})$, we have

$$\mathcal{F}_{k,n}(g_{h}^{k,n}(f)(., y))(\xi) = \mathcal{F}_{k,n}(f)(\xi) \sqrt{\lambda^{2m}|h|^2(-1)^0\xi}. $$

(48)

**Theorem 2.6.** Suppose that a function $h \in L^2_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n,e}(\mathbb{R})$ satisfies $\|h\|_{L^2_{k,n}(\mathbb{R})} = 1$. Then, the function

$$f_{j}(x) = \int_{-j}^{j} \int_{\mathbb{R}} g_{h}^{k,n}(f)(v, y)h_{v,y}(x) dy_{k,n}(y) dv_{k,n}(v, y), $$

(49)

belongs to $L^2_{k,n}(\mathbb{R})$ and satisfies

$$\lim_{j \to \infty} \|f_{j} - f\|_{L^2_{k,n}(\mathbb{R})} = 0, \quad \text{whenever} \quad f \in L^2_{k,n}(\mathbb{R}). $$

(50)

3. Heisenberg-Type Inequalities for the Deformed Gabor Transform

Heisenberg’s uncertainty principle is surely the stepping stone for harmonic analysis, which is in fact an analogy of the prominent Heisenberg’s uncertainty principle in quantum mechanics asserting that it is impossible to ascertain both the position and momentum of particles simultaneously [5]. The harmonic analysis variant of the uncertainty principle is also referred to as the duration-bandwidth theorem, due to the fact that the principle states that the widths of a signal in the time domain (duration) and in the frequency domain (band-width) are constrained and cannot be made arbitrarily narrow. In this section, we shall establish certain Heisenberg-type uncertainty inequalities in the context of the deformed Gabor transform $g_{h}^{k,n}$ by choosing the window function $h$ as a nontrivial even function in the space $L^2_{k,n}(\mathbb{R})$.

3.1. Generalized Heisenberg’s Uncertainty Principle. In order to facilitate the formulation of new variants of Heisenberg’s principle for the deformed Gabor transform (41), we ought to recall a fundamental uncertainty inequality in the context of deformed Hankel transform $\mathcal{F}_{k,n}$.


Proposition 3.1 (see [12, 10]). For $s, t > 0$, there exists a positive constant $C_{k,n}(s, t)$, such that for every $f$ in $L^2_{k,n}(\mathbb{R})$, we have
\[
\|\xi^t \mathcal{F}_{k,n}(f)(\xi)\|_{L^2_{k,n}(\mathbb{R})}^{t+s} \geq C_{k,n}(s, t) \|f\|_{L^2_{k,n}(\mathbb{R})}^{s+t},
\]
where $C_{k,n}(s, t) = (2(k-1)n + 2/2n)^{s+t}$, $s, t \geq 1/n$.

Theorem 3.1. Let $\mathcal{G}_{k,n}(f)$ be the deformed Gabor transform of any $f \in L^2_{k,n}(\mathbb{R})$. Then, for $s, t > 0$, we have
\[
\left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(\mathcal{G}_{k,n}(f))(\xi)|^2 d\mu_{k,n}(\xi) \right)^{s+t} \geq \left( C_{k,n}(s, t) \right)^2 \left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(f)(\xi)|^2 d\mu_{k,n}(\xi) \right)^{s+t}. \tag{52}
\]
where $C_{k,n}(s, t)$ is the same constant as in Proposition 3.1.

Proof. We shall take into consideration the ideal case, assuming that the integrals appearing in (52) are finite. For an arbitrary $\nu$, inequality (51) yields
\[
\left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(\mathcal{G}_{k,n}(f))(\xi)|^2 d\mu_{k,n}(\xi) \right)^{s+t} \geq \left( C_{k,n}(s, t) \right)^2 \left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(f)(\xi)|^2 d\mu_{k,n}(\xi) \right)^{s+t}. \tag{53}
\]
Integrating under the measure $d\mu_{k,n}(\xi)$ followed by the implication of Cauchy-Schwarz’s inequality yields
\[
\left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(\mathcal{G}_{k,n}(f))(\xi)|^2 d\mu_{k,n}(\xi, \nu) \right)^{s+t} \geq \left( C_{k,n}(s, t) \right)^2 \left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(f)(\xi, \nu)|^2 d\mu_{k,n}(\xi, \nu) \right)^{s+t}. \tag{54}
\]
Using the fact that
\[
\left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(f)(\xi, \nu)|^2 d\mu_{k,n}(\xi, \nu) \right)^{s+t} = \|h\|_{L^2_{k,n}(\mathbb{R})}^2 \left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(f)(\xi)|^2 d\mu_{k,n}(\xi) \right), \tag{55}
\]
we deduce that
\[
\|h\|_{L^2_{k,n}(\mathbb{R})}^2 \left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(f)(\xi)|^2 d\mu_{k,n}(\xi) \right)^{s+t} \geq \left( C_{k,n}(s, t) \right)^2 \left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(f)(\xi)|^2 d\mu_{k,n}(\xi, \nu) \right)^{s+t} = \left( C_{k,n}(s, t) \right)^2 \left( \int_{\mathbb{R}} |\xi|^t |\mathcal{F}_{k,n}(f)(\xi)|^2 d\mu_{k,n}(\xi) \right). \tag{56}
\]
Hence, the proof of Theorem 3.1 is complete.

Proposition 3.2 (Nash’s inequality for $\mathcal{G}_{k,n}^m$). Given $s > 0$, we can always find a constant $C'(k, n, s) > 0$ such that
\[
\|h\|_{L^2_{k,n}(\mathbb{R})}^2 \|f\|_{L^2_{k,n}(\mathbb{R})}^2 \leq C'(k, n, s) \|(y, \nu)\|^2 \|\mathcal{G}_{k,n}^m(f)\|_{L^2_{k,n}(\mathbb{R})}^2 \tag{57}
\]
for all $f \in L^2_{k,n}(\mathbb{R})$.

Proof. Clearly inequality (57) is true in case $f = 0$. For a given $R > 0$, we consider $0 \neq f \in L^2_{k,n}(\mathbb{R})$; then, it follows from Plancherel’s formula (45) that
\[
\|h\|_{L^2_{k,n}(\mathbb{R})}^2 \|f\|_{L^2_{k,n}(\mathbb{R})}^2 = \|\mathcal{G}_{k,n}^m(f)\|_{L^2_{k,n}(\mathbb{R})}^2 \tag{58}
\]
Moreover, we have
\[
\left( \|1 - 1_{B_2(0, R)} \mathcal{G}_{k,n}^m(f)\|_{L^2_{k,n}(\mathbb{R})}^2 \right) \leq R^{-2s} \left( 1 - 1_{B_2(0, R)} \|(y, \nu)\|^2 \|\mathcal{G}_{k,n}^m(f)\|_{L^2_{k,n}(\mathbb{R})}^2 \right) \leq R^{-2s} \left( 1 - 1_{B_2(0, R)} \|(y, \nu)\|^2 \|\mathcal{G}_{k,n}^m(f)\|_{L^2_{k,n}(\mathbb{R})}^2 \right). \tag{61}
\]
Therefore, it follows that
\[
\|h\|_{L^2_{k,n}(\mathbb{R})}^2 \|f\|_{L^2_{k,n}(\mathbb{R})}^2 \leq C' R^{2((k-1)n+2)ns} + R^{-2s} \left( 1 - 1_{B_2(0, R)} \|(y, \nu)\|^2 \|\mathcal{G}_{k,n}^m(f)\|_{L^2_{k,n}(\mathbb{R})}^2 \right) \tag{62}
\]
After minimizing over $R > 0$, the RHS of the above inequality implies that
\[
\|h\|_{L^2_{k,n}(\mathbb{R})}^2 \|f\|_{L^2_{k,n}(\mathbb{R})}^2 \leq C(k, n, s) \|h\|_{L^2_{k,n}(\mathbb{R})}^2 \|f\|_{L^2_{k,n}(\mathbb{R})}^2 \tag{63}
\]
where $B_2(0, R) = \{(y, \nu) \in \mathbb{R}^2 : \|y, \nu\| \leq R \}$. The intended outcome occurs fairly from (63).
3.2. Heisenberg’s Uncertainty Principle via \((k,n)\)-Entropy. The \((k,n)\)-entropy pertaining to a given probability density function \(p\) over \(\mathbb{R}^2\) is given via the integral expression [17]:

\[ E_{k,n}(p) := -\int_{\mathbb{R}^2} \log(p(y,v))p(y,v)d\mu_{k,n}(y,v), \]  

where

\[ \int_{\mathbb{R}^2} p(y,v)d\mu_{k,n}(y,v) = 1. \]

The main goal of the present section is to investigate upon the local characteristics of the \((k,n)\)-entropy associated with deformed Gabor transform (41).

**Proposition 3.3.** For all \(f \in L^2_{k,n}(\mathbb{R})\), we have

\[ E_{k,n}\left( \left| \mathcal{G}_h^{k,n}(f) \right|^2 \right) \geq -2 \| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})} \ln \left( \| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})} \right). \]  

**Proof.** Assume that \( \| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})} = 1 \). Then, by virtue of (44), we obtain

\[ \left| \mathcal{G}_h^{k,n}(f)(y,v) \right| \leq \left( \| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})} \right) = 1. \]  

In particular, \( E_{k,n}(\left| \mathcal{G}_h^{k,n}(f) \right|^2) \geq 0 \). We now relax the above assumption and consider

\[ \phi := \frac{f}{\| f \|^2_{L^2_{k,n}(\mathbb{R})}}, \]

\[ \psi := \frac{h}{\| h \|^2_{L^2_{k,n}(\mathbb{R})}}, \]  

then, it is quite evident that \( \phi, \psi \in L^2_{k,n}(\mathbb{R}) \) and \( \| \phi \|^2_{L^2_{k,n}(\mathbb{R})} \| \psi \|^2_{L^2_{k,n}(\mathbb{R})} = 1 \).

Hence, \( E_{k,n}(\left| \mathcal{G}_h^{k,n}(\phi) \right|^2) = 0 \). Indeed, we have

\[ \mathcal{G}_h^{k,n}(\phi) = \frac{1}{\| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})}} \mathcal{G}_h^{k,n}(f), \]

which further implies that

\[ E_{k,n}\left( \left| \mathcal{G}_h^{k,n}(\phi) \right|^2 \right) = \frac{1}{\| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})}} E_{k,n}\left( \left| \mathcal{G}_h^{k,n}(f) \right|^2 \right) \]

\[ + 2 \ln \left( \| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})} \right). \]

Invoking \( E_{k,n}(\left| \mathcal{G}_h^{k,n}(\phi) \right|^2) \geq 0 \), it follows that

\[ E_{k,n}\left( \left| \mathcal{G}_h^{k,n}(f) \right|^2 \right) \geq -2 \| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})} \ln \left( \| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})} \right). \]  

Hence, the proof of Proposition 3.3 is complete. Employing the \((k,n)\)-entropy of deformed Gabor transform (41), we can have another variant of Heisenberg’s principle for \( \mathcal{G}_h^{k,n} \).

**Theorem 3.2.** For every \( f \in L^2_{k,n}(\mathbb{R}) \) and \( p,q > 0 \), we have

\[ \left( \int_{\mathbb{R}^2} |y|^p |\mathcal{G}_h^{k,n}(f)(y,v)|^q \right) \geq \frac{\left( \int_{\mathbb{R}^2} |y|^p |\mathcal{G}_h^{k,n}(f)(y,v)|^q \right)^{p/q}}{\left( \int_{\mathbb{R}^2} |y|^p |\mathcal{G}_h^{k,n}(f)(y,v)|^q \right)^{p/q}}. \]  

\[ \geq C_{p,q}(k,n) \| f \|^2_{L^2_{k,n}(\mathbb{R})} \| h \|^2_{L^2_{k,n}(\mathbb{R})}, \]

where

\[ C_{p,q}(k,n) = (2k - 1)n + 2/n \frac{pq}{p^q + p^q} \exp \left\{ \frac{pq}{(2k - 1)n + 2/n} \right\}. \]  

**Proof.** For every \( t \in \mathbb{R}^+ \), we consider

\[ \eta_{t,p,q}^{k,n}(y,v) := \frac{pq}{4(M_{k,n})^2 \Gamma((2k - 1)n + 2/n) \Gamma((2k - 1)n + 2/n)} \exp \left\{ \frac{|y|^p + |v|^q}{t^{(2k - 1)n + 2/n}} \right\} \cdot \frac{1}{t^{(2k - 1)n + 2/n}}. \]

After doing some elementary computations, we observe that

\[ \int_{\mathbb{R}^2} \eta_{t,p,q}^{k,n}(y,v)d\mu_{k,n}(y,v) = 1, \]  

where \( d\sigma_{t,p,q}(y,v) = \eta_{t,p,q}^{k,n}(y,v)d\mu_{k,n}(y,v) \) is the probability measure on \( \mathbb{R}^2 \). Since \( \varphi(t) = t \log(t) \) is convex on \( (0, \infty) \), Jensen’s inequality implies that

\[ \int_{\mathbb{R}^2} \frac{\left| \mathcal{G}_h^{k,n}(f)(y,v) \right|^2}{\eta_{t,p,q}^{k,n}(y,v)} \ln \left( \frac{\left| \mathcal{G}_h^{k,n}(f)(y,v) \right|^2}{\eta_{t,p,q}^{k,n}(y,v)} \right) d\sigma_{t,p,q}(y,v) \geq 0, \]  

which further implies that
\[ E_{k,n}\left(\|\mathcal{S}_h^{k,n}(f)\|^2\right) + \ln\left(\frac{p^q}{4(M_{k,n})^2}\Gamma((2k-1)n+2np)\Gamma((2k-1)n+2np)\right) \|f\|^2_{L^2_{k,n}(R)} \|h\|^2_{L^2_{k,n}(R)} \]

\[
\leq \ln\left(t^{(2k-1)n}/2np\right) \|f\|^2_{L^2_{k,n}(R)} \|h\|^2_{L^2_{k,n}(R)} + \frac{1}{t} \int_{\mathbb{R}^1} |y|^p + |v|^q \left|\mathcal{S}_h^{k,n}(f)(y,v)\right|^2 d\mu_{k,n}(y,v). \tag{77}
\]

Assume that \( \|f\|_{L^2_{k,n}(R)} \|h\|_{L^2_{k,n}(R)} = 1 \). Then, Proposition 3.3 implies that

\[
\int_{\mathbb{R}^1} |y|^p + |v|^q \left|\mathcal{S}_h^{k,n}(f)(y,v)\right|^2 d\mu_{k,n}(y,v) \geq C_{p,q}. \tag{78}
\]

However, the RHS of inequality (78) attains its upper bound at

\[
t_0 = \exp\left\{\frac{p^q}{((2k-1)n+2np)(p+q)} \ln\left(\frac{p^q}{4(M_{k,n})^2}\Gamma((2k-1)n+2np)\Gamma((2k-1)n+2np)\right) - 1\right\}, \tag{79}
\]

and consequently,

\[
\int_{\mathbb{R}^1} |y|^p + |v|^q \left|\mathcal{S}_h^{k,n}(f)(y,v)\right|^2 d\mu_{k,n}(y,v) \geq C_{p,q}. \tag{80}
\]

where

\[
C_{p,q} = \frac{(2k-1)n+2np}{p^q} \times \exp\left\{\frac{p^q}{((2k-1)n+2np)(p+q)} \ln\left(\frac{p^q}{4(M_{k,n})^2}\Gamma((2k-1)n+2np)\Gamma((2k-1)n+2np)\right) - 1\right\}. \tag{81}
\]

Therefore, for every \( f \in L^1_{k,n}(R) \) and \( h \in L^2_{k,n}(R) \) satisfying \( \|f\|_{L^1_{k,n}(R)} \|h\|_{L^2_{k,n}(R)} = 1 \), we have

\[
\int_{\mathbb{R}^1} |y|^p |\mathcal{S}_h^{k,n}(f)(y,v)|^2 d\mu_{k,n}(y,v) + \int_{\mathbb{R}^1} |v|^q |\mathcal{S}_h^{k,n}(f)(y,v)|^2 d\mu_{k,n}(y,v) \geq C_{p,q}. \tag{82}
\]

It is evident that for \( \lambda > 0 \), the dilates \( f_\lambda \) and \( h_{1/\lambda} \) belong to \( L^2_{k,n}(R) \). Therefore, after substituting \( f \) with \( f_\lambda \) and \( h \) with \( h_{1/\lambda} \) and noting \( \|f\|_{L^1_{k,n}(R)} \|h\|_{L^2_{k,n}(R)} = 1 \), the above inequality yields

\[
\int_{\mathbb{R}^1} |y|^p |\mathcal{S}_{\lambda}^{k,n}(f_\lambda)(y,v)|^2 d\mu_{k,n}(y,v) + \int_{\mathbb{R}^1} |v|^q |\mathcal{S}_{\lambda}^{k,n}(f_\lambda)(y,v)|^2 d\mu_{k,n}(y,v) \geq C_{p,q}. \tag{83}
\]
Using (42), we obtain
\[
\lambda^{-p} \int_{\mathbb{R}^1} | y^p | \left| g^{k,n}_h (f)(y,v) \right|^2 \, d\mu_{k,n}(y,v) \\
+ \lambda^q \int_{\mathbb{R}^1} | y^q | \left| g^{k,n}_h (f)(y,v) \right|^2 \, d\mu_{k,n}(y,v) \geq C_{p,q}.
\]
(84)
In particular, the inequality holds at the point
\[
\lambda = \left( p \int_{\mathbb{R}^1} | y^p | \left| g^{k,n}_h (f)(y,v) \right|^2 \, d\mu_{k,n}(y,v) \right)^{1/p} / \left( q \int_{\mathbb{R}^1} | y^q | \left| g^{k,n}_h (f)(y,v) \right|^2 \, d\mu_{k,n}(y,v) \right)^{1/q},
\]
(85)
where
\[
\mathcal{E}_{p,q}(k,n) = C_{p,q} \frac{p^{p/q}q^{q/p} \lambda^{p+q}}{p+q}
\]
so that
\[
\left( \int_{\mathbb{R}^1} | y^p | \left| g^{k,n}_h (f)(y,v) \right|^2 \, d\mu_{k,n}(y,v) \right)^{p/q} \\
\cdot \left( \int_{\mathbb{R}^1} | y^q | \left| g^{k,n}_h (f)(y,v) \right|^2 \, d\mu_{k,n}(y,v) \right)^{q/p} \geq \mathcal{E}_{p,q}(k,n),
\]
(86)

Hence, the desired result is obtained after replacing \( f \) with \( f/\| f \|_{L_{k,n}^p(\mathbb{R}^1)} \) and \( h \) with \( h/\| h \|_{L_{k,n}^q(\mathbb{R}^1)} \).

Remark: for \( p = q = 2 \), we have
\[
\left\| | y^2 | g^{k,n}_h (f) \right\|_{L_{k,n}^2(\mathbb{R}^1)} \left\| | y^2 | g^{k,n}_h (f) \right\|_{L_{k,n}^2(\mathbb{R}^1)} \\
\geq \frac{(2k-1)n + 2/n}{2e(M_{k,n}\Gamma^2 ((2k-1)n + 2/n))^{2/(2k-1)n+2/n}} \cdot \| f \|_{L_{k,n}^2(\mathbb{R}^1)}^2 \| h \|_{L_{k,n}^2(\mathbb{R}^1)}^2.
\]
(88)

3.3. \( L^p \)-Heisenberg’s Uncertainty Principle. In this section, we shall establish a unified form of \( L^p \)-Heisenberg’s inequality for deformed Gabor transform (41). Our strategy of the proof is motivated by [18], wherein the authors have studied the \( L^2 \)-Heisenberg’s uncertainty inequality in the context of Lie groups. To facilitate the narrative, we set the following notation:
\[
\Gamma_1(y,v) := e^{-\lambda |(y,v)|^2}, \quad (y,v) \in \mathbb{R}^2, \lambda > 0.
\]
(91)
It is quite straightforward to verify that for every \( 1 \leq q < \infty \), there exist \( C > 0 \) with
\[
\Gamma_1 g^{k,n}_h \in L_{k,n}^q(\mathbb{R}^1) \quad \Rightarrow \quad C\lambda^{-(2k-1)n+2/nq}.
\]
(90)

Lemma 3.1. Let \( g^{k,n}_h (f) \) be the deformed Gabor transform of any \( f \in L_{k,n}^p(\mathbb{R}^1) \) and \( 1 < p \leq 2, 0 < s < (2k-1)n + 2np \). Then, there exists a positive constant \( C \) such that
\[
\left\| \Gamma_1 g^{k,n}_h (1_{(-r,r)} f) \right\|_{L_{k,n}^q(\mathbb{R}^1)} \]
\[
\leq \left\| \Gamma_1 g^{k,n}_h (1_{(-r,r)} f) \right\|_{L_{k,n}^q(\mathbb{R}^1)} \\
\leq \left\| \| h \|_{L_{k,n}^q(\mathbb{R}^1)} \right\|_{L_{k,n}^q(\mathbb{R}^1)} \left\| f \right\|_{L_{k,n}^q(\mathbb{R}^1)}.
\]
(93)

On the flip side, relation (44) and Hölder’s inequality implies that
\[
\left\| \Gamma_1 g^{k,n}_h (1_{(-r,r)} f) \right\|_{L_{k,n}^q(\mathbb{R}^1)} \]
\[
\leq \left\| \Gamma_1 g^{k,n}_h (1_{(-r,r)} f) \right\|_{L_{k,n}^q(\mathbb{R}^1)} \\
\leq \left\| \| h \|_{L_{k,n}^q(\mathbb{R}^1)} \right\|_{L_{k,n}^q(\mathbb{R}^1)} \left\| f \right\|_{L_{k,n}^q(\mathbb{R}^1)}.
\]
(94)
It is quite straightforward to demonstrate that there exists a positive constant \( C \) such that
\[
\left\| \Gamma_1 g^{k,n}_h (1_{(-r,r)} f) \right\|_{L_{k,n}^q(\mathbb{R}^1)} \\
\leq \left\| \| h \|_{L_{k,n}^q(\mathbb{R}^1)} \right\|_{L_{k,n}^q(\mathbb{R}^1)} \left\| f \right\|_{L_{k,n}^q(\mathbb{R}^1)}.
\]
\[\|y\|^{-t}1_{(-r,r)}(y)\|_{L_{\mathbb{R}}^p} = C r^{-t + (2k-1)n + 2/2np}. \quad (95)\]

Consequently, we obtain
\[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} \leq \left[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p}^2 + \|G_{\lambda,y}^{k,n}(f')\|_{L_{\mathbb{R}}^p}^2\right]^{1/2}, \quad \text{for } \lambda > 0. \quad (96)\]

Choosing \( r = \lambda^2 \) and using (90), we obtain the desired inequality. \( \square \)

**Theorem 3.3.** Let \( G_{\lambda,y}^{k,n}(f) \) be the deformed Gabor transform of an arbitrary function \( f \in L_{\mathbb{R}}^2 \). Then, for \( 1 < p \leq 2, 0 < s < (2k-1)n + 2/2np \) and \( t > 0 \), the following inequality holds:
\[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} \leq C|\lambda|^{2t}\left(\|\langle y, r \rangle\|^2 \right) \|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} + \left[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p}^2 + \|G_{\lambda,y}^{k,n}(f')\|_{L_{\mathbb{R}}^p}^2\right]^{1/2}. \quad (97)\]

However, on the flip side, we have
\[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} \leq C|\lambda|^{2t}\left(\|\langle y, r \rangle\|^2 \right) \|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} + \left[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p}^2 + \|G_{\lambda,y}^{k,n}(f')\|_{L_{\mathbb{R}}^p}^2\right]^{1/2}. \quad (98)\]

Therefore, we have
\[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} \leq C|\lambda|^{2t}\left(\|\langle y, r \rangle\|^2 \right) \|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} + \left[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p}^2 + \|G_{\lambda,y}^{k,n}(f')\|_{L_{\mathbb{R}}^p}^2\right]^{1/2}. \quad (99)\]

\[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} \leq \left[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p}^2 + \|G_{\lambda,y}^{k,n}(f')\|_{L_{\mathbb{R}}^p}^2\right]^{1/2}. \quad (100)\]

as \( (1 - e^{-u})u^{-2t} \) is bounded for \( u \geq 0 \) and \( t \leq 1/2 \). By optimizing the above inequality over \( \lambda \), we can obtain inequality (97) for \( 0 < s < (2k-1)n + 2/2np \) and \( t \leq 1/2 \). Next, we shall consider the case, when \( t \leq 1/2 \). For \( u \geq 0 \) and \( t \leq 1/2 < t \), it is easy to see that \( u^{2t} \leq 1 + u^t \), which is for \( u = \|\langle y, r \rangle\|/\varepsilon \) becomes
\[\left(\frac{\|\langle y, r \rangle\|}{\varepsilon}\right)^{4t} < 1 + \left(\frac{\|\langle y, r \rangle\|}{\varepsilon}\right)^{4t}, \quad \text{for all } \varepsilon > 0. \quad (101)\]

Therefore, we have
\[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} \leq \left[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p}^2 + \|G_{\lambda,y}^{k,n}(f')\|_{L_{\mathbb{R}}^p}^2\right]^{1/2}. \quad (102)\]

Upon optimizing over \( \varepsilon \), we shall obtain a positive constant \( C \) such that
\[\|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p} \leq C \|G_{\lambda,y}^{k,n}(f)\|_{L_{\mathbb{R}}^p}^{4t}. \quad (103)\]

Combining (97) and (103), we get the desired inequality. \( \square \)

**Corollary 3.1.** For \( 0 < s < (2k-1)n + 2/2np \) and \( t > 0 \), we can always find some \( C > 0 \) satisfying
\[
\|f\|_{L^2_{k,n}(\mathbb{R})} \leq C \|h\|_{L^2_{k,n}(\mathbb{R})}^{\lambda + \epsilon} \left( \|y^2 f\|_{L^2_{k,n}(\mathbb{R})} + \|y^3 f\|_{L^2_{k,n}(\mathbb{R})} \right)^{1/3},
\]
for all \( f \in L^2_{k,n}(\mathbb{R}) \).

**Proof.** The result follows immediately by applying Theorem 3.3 with \( p = 2 \) and Plancherel's formula (45).

\[\square\]

### 4. Weighted-Type Inequalities for the Deformed Gabor Transform

Pitt's inequality has a fundamental importance in the deformed Hankel setting because it describes the variance between a sufficiently smooth function and the corresponding deformed Hankel transform. Recently, Gorbachev et al. [9] have proposed a sharp form of Pitt's and Beckner-type inequalities for the deformed Hankel transform. Explicitly, for any \( f \in \mathcal{S}(\mathbb{R}) \subseteq L^2_{k,n}(\mathbb{R}) \), they formulated that

\[
\int_{\mathbb{R}} |\xi|^{2\lambda} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\mu_{k,n}(\xi)
\leq C_{k,n}(\lambda) \int_{\mathbb{R}} |x|^{2\lambda} |f(x)|^2 \, d\mu_{k,n}(x),
\]

where

\[
C_{k,n}(\lambda) = \left( \frac{n - 2\lambda}{2} \right) \left[ \Gamma((2k - 1 - \lambda)n + 2/4n) \right]^2 /
\left[ \Gamma((2k - 1 + \lambda)n + 2/4n) \right],
\]

0 \leq \lambda < (2k - 1)n + 2/2n.

Here, our primary goal is to formulate a new variant of Pitt's inequality (105) pertaining to the DGT given in (41).

**Theorem 4.1.** Let \( \mathcal{G}_{k,n}(f) \) be the deformed Gabor transform corresponding to \( \mathcal{G}_{k,n}(\mathcal{A}) \) for \( \mathcal{A} \subseteq \mathcal{S}(\mathbb{R}) \); then,

\[
\|h\|^2_{L^2_{k,n}(\mathbb{R})} \int_{\mathbb{R}^2} |\xi|^{-\lambda} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\mu_{k,n}(\xi)
\leq C_{k,n}(\lambda) \int_{\mathbb{R}^2} |y|^{2\lambda} |\mathcal{G}_{k,n}(f)(y, \nu)|^2 \, d\mu_{k,n}(y, \nu),
\]

where \( C_{k,n}(\lambda) \) is given by (106) and 0 \leq \lambda < (2k - 1)n + 2/2n.

**Proof.** By virtue of (105), we obtain

\[
\int_{\mathbb{R}} |\xi|^{-\lambda} |\mathcal{F}_{k,n}(\mathcal{A})(\xi)|^2 \, d\mu_{k,n}(\xi)
\leq C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^{2\lambda} |\mathcal{G}_{k,n}(\mathcal{A})(y, \nu)|^2 \, d\mu_{k,n}(y, \nu), \quad \text{for all } \nu \in \mathbb{R},
\]

which upon integration under Haar measure \( d\mu_{k,n}(\nu) \) implies that

\[
\int_{\mathbb{R}^2} |\xi|^{-2\lambda} |\mathcal{F}_{k,n}(\mathcal{A})(\xi)|^2 \, d\mu_{k,n}(\xi, \nu)
\leq C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^{2\lambda} |\mathcal{G}_{k,n}(\mathcal{A})(y, \nu)|^2 \, d\mu_{k,n}(y, \nu).
\]

Implementing Lemma 2.1, expression (109) takes the following form:

\[
\int_{\mathbb{R}^2} |\xi|^{-2\lambda} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\mu_{k,n}(\xi, \nu)
\leq C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^{2\lambda} |\mathcal{G}_{k,n}(f)(y, \nu)|^2 \, d\mu_{k,n}(y, \nu).
\]

Or,

\[
\int_{\mathbb{R}} |\xi|^{-2\lambda} |\mathcal{F}_{k,n}(f)(\xi)|^2 \left( \int_{\mathbb{R}} |h|^{2\lambda} \, d\mu_{k,n}(\nu) \right) \, d\mu_{k,n}(\xi)
\leq C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^{2\lambda} |\mathcal{G}_{k,n}(f)(y, \nu)|^2 \, d\mu_{k,n}(y, \nu).
\]

Using the hypothesis on \( h \), relation (32) becomes

\[
\|h\|^2_{L^2_{k,n}(\mathbb{R})} \int_{\mathbb{R}} |\xi|^{-2\lambda} |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\mu_{k,n}(\xi)
\leq C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^{2\lambda} |\mathcal{G}_{k,n}(f)(y, \nu)|^2 \, d\mu_{k,n}(y, \nu),
\]

which establishes Pitt's inequality for deformed Gabor transform (41).

**Remark.** Plugging \( \lambda = 0 \), we find that in (107), equality holds, which is exactly in accordance with (48). Next, we present the deformed Hankel–Beckner's inequality, which asserts that for \( f \in \mathcal{S}(\mathbb{R}) \) [9],

\[
\int_{\mathbb{R}} \log |y| |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\mu_{k,n}(\xi) + \int_{\mathbb{R}} \log |\xi| |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\mu_{k,n}(\xi)
\geq n \left[ \frac{\Gamma'((2k - 1)n + 2/4n)}{\Gamma((2k - 1)n + 2/4n)} + \log \left( \frac{2}{n} \right) \right] \int_{\mathbb{R}} |f(t)|^2 \, d\mu_{k,n}(t).
\]

The above inequality is intimately intertwined with Heisenberg's inequality, owing to that it is sometimes called the logarithmic variant of the uncertainty inequality. In the recent literature, many novel ramifications of such an inequality have been witnessed from time to time [10]. Our next motive is to obtain an associate of Beckner-type inequality (113) in the context of DGT defined in (41).

**Theorem 4.2.** Let \( f \in \mathcal{S}(\mathbb{R}) \); then, we have

\[

\]
Proof. By replacing $f$ in (113) with $\mathcal{G}_{h}^{k,n}(f)(\cdot,\cdot)$, we obtain

\[
\int_{\mathbb{R}} \log|y|\mathcal{G}_{h}^{k,n}(f)(y,v)|^2 dy_{k,n}(t) + \int_{\mathbb{R}} \log|\xi|\mathcal{F}_{k,n}[\mathcal{G}_{h}^{k,n}(f)(\cdot,\cdot)](\xi)|^2 dy_{k,n}(\xi) \geq n \left[ \Gamma\left( (2k-1)n + 2/4 \right) + \log\left( \frac{2}{n} \right) \right] \cdot \int_{\mathbb{R}} |\mathcal{G}_{h}^{k,n}(f)(y,v)|^2 dy_{k,n}(t) \quad \text{for all } y \in \mathbb{R}.
\]

Integrating (115) under $dy_{k,n}(\xi)$ implies that

\[
\int_{\mathbb{R}^2} \log|y|\mathcal{G}_{h}^{k,n}(f)(y,v)|^2 dy_{k,n}(y,v) + \int_{\mathbb{R}} \log|\xi|\mathcal{F}_{k,n}[\mathcal{G}_{h}^{k,n}(f)(\cdot,\cdot)](\xi)|^2 dy_{k,n}(\xi) \geq n \left[ \Gamma\left( (2k-1)n + 2/4 \right) + \log\left( \frac{2}{n} \right) \right] \cdot \int_{\mathbb{R}} |\mathcal{G}_{h}^{k,n}(f)(y,v)|^2 dy_{k,n}(y,v).
\]

By virtue of (45), we obtain

\[
\int_{\mathbb{R}} \log|y|\mathcal{G}_{h}^{k,n}(f)(y,v)|^2 dy_{k,n}(y,v) + \int_{\mathbb{R}} \log|\xi|\mathcal{F}_{k,n}[\mathcal{G}_{h}^{k,n}(f)(\cdot,\cdot)](\xi)|^2 dy_{k,n}(\xi) \geq n \left[ \Gamma\left( (2k-1)n + 2/4 \right) + \log\left( \frac{2}{n} \right) \right] \|h\|_{L^2_x(R)}^2 \|f\|_{L^2_x(R)}^2.
\]

In order to derive a useful computation for the later integral in (117), we invoke Lemma 2.1 together with (32), so that

\[
\int_{\mathbb{R}^2} \log|\xi|\mathcal{F}_{k,n}[\mathcal{G}_{h}^{k,n}(f)(\cdot,\cdot)](\xi)|^2 dy_{k,n}(\xi)
\]

\[
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \log|\xi|\mathcal{F}_{k,n}[\mathcal{G}_{h}^{k,n}(f)(\cdot,\cdot)](\xi)|^2 dy_{k,n}(\xi) \right) dy_{k,n}(v)
\]

\[
= \left( \int_{\mathbb{R}} \log|\xi|\mathcal{F}_{k,n}(f)(\xi)|^2 dy_{k,n}(\xi) \right) \|h\|_{L^2_x(R)}^2.
\]

Substituting (118) in (117), we obtain the result for DGT (41).

Next, we shall give another proof of Theorem 4.2, which primarily relies upon (107).

Proof (alternate proof of Theorem 4.2). Given $0 \leq \lambda < (2k-1)n + 2/2n$, consider

\[
S(\lambda) = \|h\|_{L^2_x(R)}^2 \int_{\mathbb{R}} |\xi|^{-2\lambda|\mathcal{F}_{k,n}(f)(\xi)|^2 dy_{k,n}(\xi) - C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^2 |\mathcal{G}_{h}^{k,n}(f)(y,v)|^2 dy_{k,n}(y,v).
\]

Differentiating (119) with respect to $\lambda$, we obtain

\[
S'(\lambda) = -2\|h\|_{L^2_x(R)}^2 \int_{\mathbb{R}} |\xi|^{-2\lambda} \log|\xi| |\mathcal{F}_{k,n}(f)(\xi)|^2 dy_{k,n}(\xi)
\]

\[
- 2C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^{2\lambda} |y| |\mathcal{G}_{h}^{k,n}(f)(y,v)|^2 dy_{k,n}(y,v)
\]

\[
- C_{k,n}(\lambda) \int_{\mathbb{R}} |y|^{2\lambda} |\mathcal{G}_{h}^{k,n}(f)(y,v)|^2 dy_{k,n}(y,v),
\]

where

\[
C_{k,n}(\lambda) = -nC_{k,n}(\lambda) \left( \frac{\Gamma\left( (2k-1-2\lambda)n + 2/4 \right)}{\Gamma\left( (2k-1-2\lambda)n + 2/4 \right)} + \frac{\Gamma\left( (2k-1+2\lambda)n + 2/4 \right) + 2 \log\left( \frac{2}{n} \right) \right}. \]

For $\lambda = 0$, relation (121) yields

\[
C_{k,n}(0) = -2n \left( \frac{\Gamma\left( (2k-1)n + 2/4 \right)}{\Gamma\left( (2k-1)n + 2/4 \right)} + \log\left( \frac{2}{n} \right) \right).
\]

Using (107), we get

\[
S(\lambda) \leq 0, \quad \forall \lambda \in \left[ 0, \frac{(2k-1)n + 2/2n + 2\lambda}{2} \right],
\]

and

\[
S(0') = \lim_{\lambda \to 0} \frac{S(\lambda)}{\lambda} \leq 0.
\]
which establishes the result.

□

**Corollary 4.1.** Let \( \mathcal{G}_{k,n}^h(f) \) be the deformed Gabor transform of any arbitrary function \( f \in \mathcal{S}(\mathbb{R}) \) with respect to the window function \( h \in L^2_{k,n}(\mathbb{R}) \cap L^1_{k,n}(\mathbb{R}) \) with \( \|h\|^2_{L^1_{k,n}(\mathbb{R})} = 1 \). Then, we have

\[
\begin{align*}
\left\{ \int_{\mathbb{R}^2} |y|^2 |\mathcal{G}_{k,n}^h(f)(y,v)|^2 \, d\mu_{k,n}(y,v) \right\}^{1/2} \\
\cdot \left\{ \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_{k,n}(f)(\xi)|^2 \, d\nu_{k,n}(\xi) \right\}^{1/2} \\
\geq \exp \left\{ n \left[ \frac{\Gamma'((2k-1)n + 2/4)}{\Gamma((2k-1)n + 2/4)} + \log \left( \frac{2}{n} \right) \right] \|f\|^2_{L^1_{k,n}(\mathbb{R})} \right\}.
\end{align*}
\]

(129)

**Proof.** Noting \( \|h\|^2_{L^1_{k,n}(\mathbb{R})} = 1 \) and then using the well-known Jensen’s inequality in (114), we get

\[
\log \left\{ \int_{\mathbb{R}^2} \left| \frac{|\mathcal{G}_{k,n}^h(f)(y,v)|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} \right|^2 d\mu_{k,n}(y,v) \right\}^{1/2} \\
= \log \left\{ \int_{\mathbb{R}^2} \left| \frac{|\mathcal{G}_{k,n}^h(f)(y,v)|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} \right|^2 d\mu_{k,n}(y,v) \right\}^{1/2} + \log \left\{ \int_{\mathbb{R}} \left| \frac{|\mathcal{F}_{k,n}(f)(\xi)|^2}{\|f\|^2_{L^2_{k,n}(\mathbb{R})}} \right|^2 d\nu_{k,n}(\xi) \right\}^{1/2}
\]

(130)

which upon simplification yields desired result (129).

Remarks are as follows:

(i) By virtue of the identity [19]

\[
\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - 2 \int_0^\infty \frac{t}{(t^2 + z^2)(e^{2mt} - 1)} \, dt,
\]

(131)

we infer that

\[
\frac{\Gamma'(z)}{\Gamma(z)} \approx \left( \frac{2k-1}{2n} \right)^z, \quad \text{for} \quad (2k-1)n + 2 \gg 1,
\]

(132)

that is precisely the constant as appearing in Theorem 3.1.

(ii) In a manner similar to (113), we get
\[
\left\{ \int_{\mathbb{R}} |t|^2 |f(t)|^2 d\gamma_{k,n}(t) \right\}^{1/2} \\
\cdot \left\{ \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi) \right\}^{1/2} \\
\geq \exp\left\{ \frac{\Gamma'(2k-1)n + 2/4}{\Gamma(2k-1)n + 2/4} + \log\left(\frac{2}{n}\right) \right\} \\
\cdot \int_{\mathbb{R}} |f(t)|^2 d\gamma_{k,n}(t),
\]

where \( C_{k,n}(E_1, E_2) \) is given by (135).

**Proof.** For any \( v \in \mathbb{R} \), it follows that \( \mathcal{G}_{k,n}^\nu(f)(\cdot, v) \in L^2_{k,n}(\mathbb{R}) \), provided \( f \in L^2_{k,n}(\mathbb{R}) \). Therefore, changing \( f \) to \( \mathcal{G}_{k,n}^\nu(f)(\cdot, v) \) in (135) implies that

\[
\int_{\mathbb{R}} |\mathcal{G}_{k,n}^\nu(f)(y, v)|^2 d\gamma_{k,n}(t) \\
\leq C_{k,n}(E_1, E_2) \left\{ \int_{\mathbb{R}/E_1} |\mathcal{G}_{k,n}^\nu(f)(y, v)|^2 d\gamma_{k,n}(t) \\
+ \int_{\mathbb{R}/E_2} |\mathcal{F}_{k,n}[\mathcal{G}_{k,n}^\nu(f)(\cdot, v)](\xi)|^2 d\gamma_{k,n}(\xi) \right\},
\]

which after integration under \( d\gamma_{k,n}(v) \) yields the following inequality:

\[
\int_{\mathbb{R}/E_1} \int_{\mathbb{R}} |\mathcal{G}_{k,n}^\nu(f)(y, v)|^2 d\mu_{k,n}(y, v) \leq C_{k,n}(E_1, E_2) \\
\cdot \left\{ \int_{\mathbb{R}/E_1} \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f)(y, v)|^2 d\mu_{k,n}(y, v) \\
+ \int_{\mathbb{R}/E_2} |\mathcal{F}_{k,n}[\mathcal{G}_{k,n}^\nu(f)(\cdot, v)](\xi)|^2 d\mu_{k,n}(\xi, v) \right\}.
\]

Invoking Lemma 2.1 in association with Plancherel’s formula (45), we get

\[
\int_{\mathbb{R}/E_1} \int_{\mathbb{R}} |\mathcal{G}_{k,n}^\nu(f)(y, v)|^2 d\mu_{k,n}(y, v) \\
+ \int_{\mathbb{R}/E_2} |\mathcal{F}_{k,n}(f)(\xi)|^2 \sqrt{\frac{\nu}{\tau}} |h| \sqrt{(-1)^n \xi} |^2 d\mu_{k,n}(\xi, v) \\
\geq \frac{\|h\|_{L^2_{k,n}(\mathbb{R})}^2 \|f\|_{L^2_{k,n}(\mathbb{R})}^2}{C_{k,n}(E_1, E_2)},
\]

so that

\[
\int_{\mathbb{R}/E_1} \int_{\mathbb{R}} |\mathcal{G}_{k,n}^\nu(f)(y, v)|^2 d\mu_{k,n}(y, v) + \int_{\mathbb{R}/E_2} |\mathcal{F}_{k,n}(f)(\xi)|^2 \\
\cdot \left\{ \int_{\mathbb{R}} \sqrt{\frac{\nu}{\tau}} |h| \sqrt{(-1)^n \xi} d\gamma_{k,n}(v) \right\} d\gamma_{k,n}(\xi) \\
\geq \frac{\|h\|_{L^2_{k,n}(\mathbb{R})}^2 \|f\|_{L^2_{k,n}(\mathbb{R})}^2}{C_{k,n}(E_1, E_2)}.
\]

Using Lemma 2.1 and relation (32) and keeping in view that \( h \in L^2_{k,n}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R}) \), we obtain

\[
\int_{\mathbb{R}/E_1} \int_{\mathbb{R}} |\mathcal{G}_{k,n}^\nu(f)(y, v)|^2 d\mu_{k,n}(y, v) + \|h\|_{L^2_{k,n}(\mathbb{R})}^2 \\
\cdot \int_{\mathbb{R}/E_2} |\mathcal{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi) \geq \frac{\|h\|_{L^2_{k,n}(\mathbb{R})}^2 \|f\|_{L^2_{k,n}(\mathbb{R})}^2}{C_{k,n}(E_1, E_2)},
\]
which is the desired Benedick–Amrein–Berthier’s uncertainty principle for deformed Gabor transform (41).

As a consequence of Theorem 5.1, we obtain the following generalized Heisenberg-type uncertainty inequality for DGT (41).

**Corollary 5.1.** Let \( G_{k,n}^p(f) \) be the deformed Gabor transform of any arbitrary function \( f \in L^2_{k,n}(\mathbb{R}) \). Then, for \( p, q > 0 \), there exist \( \mathcal{C}_{k,n}(p, q) > 0 \) satisfying

\[
\left\{ \begin{array}{l}
\int_{\mathbb{R}} |y|^{2p} |\mathcal{F}_{k,n}(f) (y, \xi)|^2 \, d\mu_{k,n}(y, \xi) \\
\cdot \left\{ \int_{\mathbb{R}} |\xi|^{2q} |\mathcal{F}_{k,n}(f) (\xi)|^2 \, d\mu_{k,n}(\xi) \right\}^{p/2} \\
\geq \mathcal{C}_{k,n}(p, q) \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2}
\end{array} \right. \tag{142}
\]

**Proof.** Take \( E_1 = E_2 = (-1, 1) \). Then, for any \( f \in L^2_{k,n}(\mathbb{R}) \) and \( p, q > 0 \), (136) implies that

\[
\int_{\mathbb{R}^2} \left| \mathcal{F}_{k,n}(f) (y, \xi) \right|^2 \, d\mu_{k,n}(y, \xi) + \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2} 
\]

\[
\cdot \left\{ \int_{\mathbb{R}^2} |\xi|^{2q} |\mathcal{F}_{k,n}(f) (\xi)|^2 \, d\mu_{k,n}(\xi) \right\}^{p/2} \geq \mathcal{C}_{k,n}(p, q) \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2}, \tag{143}
\]

where \( C(k, n) = C_{k,n}(E_1, E_2) \). Hence, it follows that

\[
\int_{\mathbb{R}^2} |y|^{2p} |\mathcal{F}_{k,n}(f) (y, \xi)|^2 \, d\mu_{k,n}(y, \xi) + \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2} 
\]

\[
\cdot \left\{ \int_{\mathbb{R}^2} |\xi|^{2q} |\mathcal{F}_{k,n}(f) (\xi)|^2 \, d\mu_{k,n}(\xi) \right\}^{p/2} \geq \frac{\mathcal{C}_{k,n}(p, q) \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2}}{C(k, n)}, \tag{144}
\]

Replacing \( f \) by \( f_{\lambda} \) and \( h \) by \( h_{\lambda} \), relation (42) implies that

\[
\int_{\mathbb{R}^2} |y|^{2p} |\mathcal{F}_{k,n}(f_{\lambda} (y, \xi)|^2 \, d\mu_{k,n}(y, \xi) + \lambda^{(2k-1)n+2m} \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2} 
\]

\[
\cdot \left\{ \int_{\mathbb{R}^2} |\xi|^{2q} |\mathcal{F}_{k,n}(f_{\lambda}) (\lambda \xi)|^2 \, d\mu_{k,n}(\lambda \xi) \right\}^{p/2} \geq \frac{\mathcal{C}_{k,n}(p, q) \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2}}{C(k, n)}, \tag{145}
\]

Therefore, we have

\[
\lambda^{2p} \int_{\mathbb{R}^2} |y|^{2p} |\mathcal{F}_{k,n}(f) (y, \xi)|^2 \, d\mu_{k,n}(y, \xi) + \lambda^{-2q} \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2} 
\]

\[
\cdot \left\{ \int_{\mathbb{R}^2} |\xi|^{2q} |\mathcal{F}_{k,n}(f) (\lambda \xi)|^2 \, d\mu_{k,n}(\lambda \xi) \right\}^{p/2} \geq \frac{\mathcal{C}_{k,n}(p, q) \|h\|_{L^q_{k,n}(\mathbb{R})}^q \|f\|_{L^p_{k,n}(\mathbb{R})}^{p/2}}{C(k, n)}, \tag{146}
\]

As a result, optimizing the right side of the aforementioned inequality for \( \lambda > 0 \) yields desired inequality (142). \( \square \)

### 5.2. Local-Type Uncertainty Principles

In this section, we shall formulate certain local uncertainty inequalities pertaining to the deformed Gabor transform (41) by employing the following inequality of the deformed Hankel transforms [8].

**Proposition 5.1.** (see [8]). For \( E \subset \mathbb{R} \) satisfying \( 0 < y_{k,n}(E) = \int_{E} d y_{k,n}(x) < \infty \), there exists \( \mathcal{C}(k, n, s) > 0 \) with \( 0 < s < (2k-1)n + 2/2n \), so that

\[
\int_{E} \left| \mathcal{F}_{k,n}(f) (\xi) \right|^2 \, dy_{k,n}(\xi) \leq \mathcal{C}(k, n, s) \left( y_{k,n}(E) \right)^{2n/(2k-1)n+2}, \tag{147}
\]

for every \( f \in L^2_{k,n}(\mathbb{R}) \).

**Theorem 5.2.** Let \( 0 < s < (2k-1)n + 2/2n \); then, for \( E \subset \mathbb{R} \) satisfying \( 0 < y_{k,n}(E) < \infty \) and any \( f \in L^2_{k,n}(\mathbb{R}) \), the following is true:

\[
\int_{E} \left| \mathcal{F}_{k,n}(f) (\xi) \right|^2 \, dy_{k,n}(\xi) \leq \frac{\mathcal{C}(k, n, s) \left( y_{k,n}(E) \right)^{2n/(2k-1)n+2}}{\|h\|_{L^q_{k,n}(\mathbb{R})}^q}, \tag{148}
\]

where \( \mathcal{C}(k, n, s) \) is as mentioned in Proposition 5.1.

**Proof.** Since \( G_{k,n}^p(f) (\cdot, \cdot) \in L^2_{k,n}(\mathbb{R}) \), whenever \( f \in L^2_{k,n}(\mathbb{R}) \), we can change \( f \) to \( G_{k,n}^p(f) (\cdot, \cdot) \) in (147), so that

\[
\int_{E} \left| \mathcal{F}_{k,n}(G_{k,n}^p(f) (\cdot, \cdot)) (\xi) \right|^2 \, dy_{k,n}(\xi) \leq \mathcal{C}(k, n, s) \left( y_{k,n}(E) \right)^{2n/(2k-1)n+2}, \tag{149}
\]

Integrating the above inequality under the measure \( dy_{k,n}(\cdot) \), we get

\[
\int_{E} \int_{\mathbb{R}^2} \left| \mathcal{F}_{k,n}(G_{k,n}^p(f) (\cdot, \cdot)) (\xi) \right|^2 \, d\mu_{k,n}(\xi, \tau), \tag{150}
\]

Furthermore, using Lemma 2.1 yields

\[
\int_{E} \int_{\mathbb{R}^2} \left| \mathcal{F}_{k,n} (f) (\xi)^2 \tau, \right| h^2 ((-1)^n \xi) dy_{k,n}(\xi) \, d\mu_{k,n}(\xi, \tau) \leq \mathcal{C}(k, n, s) \left( y_{k,n}(E) \right)^{2n/(2k-1)n+2}, \tag{151}
\]

As a result, optimizing the right side of the aforementioned inequality for \( \lambda > 0 \) yields desired inequality (142). \( \square \)
As such, inequality (151) becomes
\[
\|h\|^2_{L^2_{\xi,n}(\mathbb{R})} \int_{\mathbb{R}^2} |\mathcal{F}_{h,k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi) \\
\leq \mathcal{C}(k,n,s)(\gamma_{k,n}(E))^\frac{2n/[(2k-1)n+2]}{2}
\cdot \int_{\mathbb{R}^2} |y|^2 |\mathcal{F}_{h,k,n}(f)(y,v)|^2 d\mu_{k,n}(y,v).
\] (152)

Or equivalently, for any \( s \in (0, (2k-1)n+2/2n) \), we have
\[
\int_{\mathbb{R}^2} |y|^2 |\mathcal{F}_{h,k,n}(f)(y,v)|^2 d\mu_{k,n}(y,v) \\
\leq \frac{\|h\|^2_{L^2_{\xi,n}(\mathbb{R})}}{\mathcal{C}(k,n,s)(\gamma_{k,n}(E))^\frac{2n/[(2k-1)n+2]}{2}}
\cdot \int_{\mathbb{R}^2} |\mathcal{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi).
\] (153)

Hence, the proof of Theorem 5.2 is complete. \(\Box\)

Given a subset \( E \) of \( \mathbb{R} \), the Paley–Wiener space \( PW_{k,n}(E) \) is defined by
\[
PW_{k,n}(E) = \{ f \in L^2_{\xi,n}(\mathbb{R}) : \text{supp} \mathcal{F}_{k,n}(f) \subset E \}. \] (154)

In view of Plancherel's formula (17), we are led to the following inequality.

**Corollary 5.2.** For \( E \subset \mathbb{R} \) with \( 0 < \gamma_{k,n}(E) < \infty \) and \( 0 < s < (2k-1)n+2/2n \), the following is true:
\[
\|f\|^2_{L^2_{\xi,n}(\mathbb{R}^n)} \leq \frac{\mathcal{C}(k,n,s)(\gamma_{k,n}(E))^\frac{2n/[(2k-1)n+2]}{2}}{\|h\|^2_{L^2_{\xi,n}(\mathbb{R})}}
\cdot \int_{\mathbb{R}^2} |y|^2 |\mathcal{F}_{h,k,n}(f)(y,v)|^2 d\mu_{k,n}(y,v),
\] (155)

where \( f \in PW_{k,n}(E) \) and the constant \( \mathcal{C}(k,n,s) \) is the same as given in Proposition 5.1.

Swapping the functions \( f \) and \( \mathcal{F}_{k,n}(f) \) in Proposition 5.1 yields the below mentioned inequality.

**Corollary 5.3.** For \( E \subset \mathbb{R} \) with \( 0 < \gamma_{k,n}(F) < \infty \) and \( 0 < t < (2k-1)n+2/2n \), we have
\[
\int_{\mathbb{R}^2} |f(y)|^2 d\gamma_{k,n}(y) \leq \mathcal{C}(k,n,t)(\gamma_{k,n}(F))^\frac{2n/[(2k-1)n+2]}{2}
\cdot \|\xi|^2 \mathcal{F}_{k,n}(f)\|^2_{L^2_{\xi,n}(\mathbb{R})}, \quad \forall f \in L^2_{\xi,n}(\mathbb{R}),
\] (156)

where \( \mathcal{C}(k,n,t) \) is the same constant as mentioned in Proposition 5.1.

Adopting the strategy as in Theorem 5.2 and implementing Corollary 5.3, the following inequality is obtained.

**Corollary 5.4.** For \( E \subset \mathbb{R} \) with \( 0 < \gamma_{k,n}(F) < \infty \) and \( 0 < t < (2k-1)n+2/2n \), for all \( f \in L^2_{\xi,n}(\mathbb{R}) \), we have
\[
\int_{\mathbb{R}^2} |\mathcal{F}_{h,k,n}(f)(y,v)|^2 d\mu_{k,n}(y,v) \\
\leq \mathcal{C}(k,n,t)(\gamma_{k,n}(F))^\frac{2n/[(2k-1)n+2]}{2}
\cdot \int_{\mathbb{R}^2} |\mathcal{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi),
\] (157)

where \( \mathcal{C}(k,n,t) \) is the same constant as mentioned in Proposition 5.1.

For each subset \( F \) of \( \mathbb{R} \), we consider the following generalization of Paley–Wiener spaces:
\[
GPW_{k,n}(F) = \{ f \in L^2_{\xi,n}(\mathbb{R}) : \forall \gamma \in \mathbb{R}, \text{supp} \mathcal{F}_{h,k,n}(f)(\gamma) \subset F \}. \] (158)

Applications of (45), Corollary 5.4, and the definition of generalized Paley–Wiener spaces \( GPW_{k,n}(F) \) yield the following inequality.

**Corollary 5.5.** Given are two subsets \( E, F \subset \mathbb{R} \) satisfying \( 0 < \gamma_{k,n}(E), \gamma_{k,n}(F) < \infty \). Also, let \( 0 < s, t < (2k-1)n+2/2n \); then,

(i) If \( f \in GPW_{k,n}(F) \), we have
\[
\|f\|^2_{L^2_{\xi,n}(\mathbb{R}^n)} \leq \mathcal{C}(k,n,t)(\gamma_{k,n}(F))^\frac{2n/[(2k-1)n+2]}{2}
\cdot \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi).
\] (159)

(ii) For any \( f \in PW_{k,n}(E) \cap GPW_{k,n}(F) \), we have
\[
\|f\|^2_{L^2_{\xi,n}(\mathbb{R}^n)} \leq (\mathcal{C}(k,n,t))^\frac{1}{2}(\gamma_{k,n}(E))^\frac{2n/[(2k-1)n+2]}{2}
\cdot \left( \int_{\mathbb{R}^2} |\mathcal{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi) \right)^{t/2}
\cdot \left( \int_{\mathbb{R}^2} |\mathcal{F}_{h,k,n}(f)(y,v)|^2 d\mu_{k,n}(y,v) \right)^{t/2}.
\] (160)

We now formulate yet another variant of Heisenberg-type uncertainty principle pertaining to the DGT defined in (41).

**Theorem 5.3.** Let \( \mathcal{F}_{h,k,n}(f) \) be the deformed Gabor transform of any \( f \in L^2_{\xi,n}(\mathbb{R}) \) and \( 0 < p < (2k-1)n+2/2n \) and \( q > 0 \). Then,
\[
\|f\|_{L^p_{\xi,n}(\mathbb{R}^n)}^p \leq \mathcal{C}(k,n,p,q)\|\xi|^q \mathcal{F}_{h,k,n}(f)\|_{L^q_{\xi,n}(\mathbb{R}^n)}^q
\cdot \|\xi|^q \mathcal{F}_{k,n}(f)\|_{L^q_{\xi,n}(\mathbb{R}^n)}^q
\] (161)

where
\[
\mathcal{C}(k,n,p,q) = \left( \frac{2nM_{k,n}}{(2k-1)n+2} \gamma_{k,n}(E)^{\frac{2n/[(2k-1)n+2]}{2}} \right)^{\frac{q}{p}+\frac{q}{p}}
\] (162)
Proof. Let \( q > 0 \) and \( r > 0 \). Then, for \( 0 < p < (2k - 1)n + 2/2n \), we have
\[
\|f\|_{L^q_{\mu_k,n}(\mathbb{R}^n)}^2 = \left\| \mathcal{F}_{k,n}(f) \right\|_{L^q_{\mu_k,n}(\mathbb{R}^n)}^2
\]
\[
\text{Using Theorem 5.2 and some simple calculation, we obtain}
\]
\[
\int_{-r}^{r} \left| \mathcal{F}_{k,n}(f)(\xi) \right|^2 d\mu_{k,n}(\xi)
\]
\[
\leq \left( \frac{2nM_{k,n}}{(2k - 1)n + 2} \right)^{2np/(2k - 1)n + 2} \mathcal{G}(k,n,p) r^{-2p}
\]
\[
\cdot \int_{\mathbb{R}^n} |\xi|^2 \left| \mathcal{F}_{k,n}(f)(y,v) \right|^2 d\mu_{k,n}(y,v).
\]
\[
\text{Besides, it also follows that}
\]
\[
\int_{\mathbb{R}^n} \left| \mathcal{F}_{k,n}(f)(\xi) \right|^2 d\mu_{k,n}(\xi)
\]
\[
\leq r^{-2q} \int_{\mathbb{R}^n} |\xi|^2 \left| \mathcal{F}_{k,n}(f)(\xi) \right|^2 d\mu_{k,n}(\xi).
\]
\[
\text{Combining relations (163)–(165), we get}
\]
\[
\|f\|_{L^q_{\mu_k,n}(\mathbb{R}^n)} \leq \left( \frac{2nM_{k,n}}{(2k - 1)n + 2} \right)^{2np/(2k - 1)n + 2} \mathcal{G}(k,n,p) r^{-2p}
\]
\[
\cdot \left[ \left\| \left| \xi \right|^\eta \mathcal{F}_{k,n}(f) \right\|_{L^q_{\mu_k,n}(\mathbb{R}^n)}^{1/p} \right] \left[ \left\| \left| \xi \right|^\eta \mathcal{F}_{k,n}(f) \right\|_{L^q_{\mu_k,n}(\mathbb{R}^n)}^{1/q} \right]^{1/p+q},
\]
\[
\text{We choose}
\]
\[
r = \left[ \left( \frac{q\|f\|_{L^q_{\mu_k,n}(\mathbb{R}^n)}}{p(2nM_{k,n})/(2k - 1)n + 2} \mathcal{G}(k,n,p) \right)^{1/p+q} \right]^{1/2p+2q}
\]
\[
\text{yielding the inequality as desired.}
\]

To wind up the ongoing discourse, we have the following local uncertainty inequality for deformed Gabor transform (41).

**Theorem 5.4** (Faris–Price’s inequality). Let \( \eta, p \in \mathbb{R} \) satisfy \( 0 < \eta < (2k - 1)n + 2/n \) and \( p > 1 \). For every measurable subset \( T \subset \mathbb{R}^2 \) with \( 0 < \mu_k,n(T) < \infty \) and \( f \in L^2_{\mu_k,n}(\mathbb{R}^d) \), there exists \( C_{k,n}(\eta, p) > 0 \) with
\[
\left( \int_{\mathbb{R}^2} \left| \mathcal{G}_{k,n}^\eta(f)(y,v) \right|^p d\mu_{k,n}(y,v) \right)^{1/p} \leq C_{k,n}(\eta, p) \mu_k,n(T)^{1/p+1} \left\| \left| \xi \right|^\eta \mathcal{F}_{k,n}(f) \right\|_{L^2_{\mu_k,n}(\mathbb{R}^n)}^{2((2k - 1)n + 2)/((2k - 1)n + 2 + \eta(n + 2))+(p+1)}
\]
\[
\text{where } B_{\eta}(0, r) = \{(y, v) \in \mathbb{R}^2 : \| (y,v) \| \leq r \}\] denotes the ball on \( \mathbb{R}^2 \) with radius \( r \). However, for every \( \eta \in (0,(2k - 1)n + 2/n) \), relation (43) and Hölder’s inequality implies that
\[
\|f\|_{L^q_{\mu_k,n}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^d} \left| \mathcal{G}_{k,n}^\eta(f)(y,v) \right|^p d\mu_{k,n}(y,v) \right)^{1/p}
\]
\[
\leq \left( \mu_k,n(T)^{1/p+1} \left\| \left| \xi \right|^\eta \mathcal{F}_{k,n}(f) \right\|_{L^q_{\mu_k,n}(\mathbb{R}^n)}^{1/p+1} \right) \left\| \left| \xi \right|^\eta \mathcal{F}_{k,n}(f) \right\|_{L^q_{\mu_k,n}(\mathbb{R}^n)}^{1/q} \left\| \left| \xi \right|^\eta \mathcal{F}_{k,n}(f) \right\|_{L^q_{\mu_k,n}(\mathbb{R}^n)}^{1/q}.
\]
On the flip side, we also have
\[\|\langle y, v \rangle\|_{B_r(0, \tau)}^{\kappa/n} L^{2}_{\kappa,n}(R^n)\]
\[\leq \sqrt{\frac{n \left( \frac{M_{\kappa,n}}{\tau} \right) \left( (2k-1)n + 2/2n \right)^{2}}{(2k-1-\eta)n + 2/2n)}}^{\kappa/(2k-1-\eta)n + 2/n}. \]  
(171)

Therefore,
\[\|\varphi_{h}^{\kappa,n}(f)1_{B_r(0, \tau)}\| L^{2}_{\kappa,n}(R^n) \leq \left( \mu_{\kappa,n}(T) \right)^{\kappa/(p+1)} \left\{ \left( \frac{n \left( \frac{M_{\kappa,n}}{\tau} \right) \left( (2k-1)n + 2/2n \right)^{2}}{(2k-1-\eta)n + 2/2n)}}^{\kappa/(2k-1-\eta)n + 2/n} \right\}^{\kappa/(p+1)} \]
(172)

Using the inequality due to Hölder together with (44), it follows that
\[\|\varphi_{h}^{\kappa,n}(f)1_{B_r(0, \tau)}\| L^{2}_{\kappa,n}(R^n) \leq \left( \mu_{\kappa,n}(T) \right)^{\kappa/(p+1)} \left\{ \left( \frac{n \left( \frac{M_{\kappa,n}}{\tau} \right) \left( (2k-1)n + 2/2n \right)^{2}}{(2k-1-\eta)n + 2/2n)}}^{\kappa/(2k-1-\eta)n + 2/n} \right\}^{\kappa/(p+1)} \]
(173)

Therefore, for every \( \eta \in (0, (2k-1)n + 2/n) \), we have
\[\left( \int_{T} \left| \varphi_{h}^{\kappa,n}(f)(y, v) \right|^{p} d\mu_{\kappa,n}(y, v) \right)^{1/p} \leq \left( \mu_{\kappa,n}(T) \right)^{1/(p+1)} \left\{ \left( \frac{n \left( \frac{M_{\kappa,n}}{\tau} \right) \left( (2k-1)n + 2/2n \right)^{2}}{(2k-1-\eta)n + 2/2n)}}^{\kappa/(2k-1-\eta)n + 2/n} \right\}^{1/(p+1)} \]
(174)

In particular, the inequality holds for
\[r_0 = \left( \frac{(2k-1-\eta)n + 2/n)}{n \left( \frac{M_{\kappa,n}}{\tau} \right) \left( (2k-1)n + 2/2n \right)^{2}} \right) \left( \frac{2\eta}{(2k-1-\eta)n + 2/n} \right) \]  
and hence,
\[
\left( \left\| \int_{T} |\mathcal{F}_{h}^{k,n}(f)(y,v)|^{p} \, d\mu_{k,n}(y,v) \right\|^{1/p} \right) \leq \left( \mu_{k,n}(T) \right)^{1/p(p+1)} \left( \frac{(2k - 1 - \eta)n + 2}{n} \left( \frac{(2k - 1)n + 2}{2n} \right) \right)^{2m/(2k-1+\eta)(n+2)} \left( \frac{(2k - 1 - \eta)n + 2}{2m} \right) \left( \frac{(2k - 1 - \eta)n + 2}{2m} \right) \left( \frac{(2k - 1 - \eta)n + 2}{2m} \right) \left( \frac{(2k - 1 - \eta)n + 2}{2m} \right). \]

Or equivalently,
\[
\left( \left\| \int_{T} |\mathcal{F}_{h}^{k,n}(f)(y,v)|^{p} \, d\mu_{k,n}(y,v) \right\|^{1/p} \right) \leq C_{k,n}(\eta,p)(\mu_{k,n}(T))^{1/p(p+1)} \cdot \left\| \|y, v\|^{p} \mathcal{F}_{h}^{k,n}(f) \right\|_{L^{2}_{k,n}(\mathbb{R})}^{2(2k-1+\eta)(n+2)} \cdot \left( \frac{(2k - 1 - \eta)n + 2}{2m} \right) \cdot \left( \frac{(2k - 1 - \eta)n + 2}{2m} \right). \]

Remark: for \( h \in L^{2}_{k,n}(\mathbb{R}) \), we define the modulation of \( h \) by \( v \) as follows [20]:
\[
\mathcal{M}_{v}(h) := F_{k,n}^{*}\left( \sqrt{r_{v}^{k,n}(|F_{k,n}(h)|^{2})} \right). \]

Also, the generalized Gabor transform \( \mathcal{F}_{k,n}^{h} \) is given by
\[
\forall (y, v) \in \mathbb{R}^{2}, \mathcal{F}_{k,n}^{h}(f)(y,v) := \int_{\mathbb{R}} f(x) r_{-1}^{k,n}y(x)(\mathcal{M}_{y}(h))(y) \, dy(x). \]

It is clear that
\[
\mathcal{F}_{k,n}^{h} = \mathcal{F}_{k,n}^{\mathcal{F}_{k,n}(h)} \quad \text{(178)}
\]

Therefore, by virtue of Plancherel’s formula (17), we obtain that the two integral transforms are equivalent. As such, all results proved for one are valuable for the second. Hence, we reclaim that all results proved in [4] and in this paper for the deformed Gabor transform \( \mathcal{F}_{k,n}^{h} \) are valuable for the integral transform \( \mathcal{F}_{k,n}^{h} \) and it is sufficient to replace \( h \) by \( \mathcal{F}_{k,n}(h) \) to derive the analogue results.

6. Conclusion

In the present article, we have accomplished two major objectives regarding the uncertainty inequalities pertaining to the deformed Gabor transform (DGT). Firstly, we obtained Heisenberg’s and Beckner’s uncertainty principles for the deformed Gabor transform. Besides, we also obtained certain weighted uncertainty inequalities for DGT. Secondly, we formulated a few concentration-based inequalities, such as the Benedick–Amrein–Berthier and the local-type uncertainty principles for the deformed Gabor transform.

Data Availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally to this article and approved the final manuscript.

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The first author dedicates this paper to the Emeritus Professor Khalifa Trimèche.

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