LACUNARY SERIES, RESONANCES, AND AUTOMORPHISMS OF $\mathbb{C}^2$ WITH A ROUND SIEGEL DOMAIN

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Abstract. We construct transcendental automorphisms of $\mathbb{C}^2$ having an unbounded and regular Siegel domain.

1. Introduction

As shown by Cremer in 1927, a holomorphic germ $f(z) = e^{2\pi i \theta}z + a_2 z^2 + \cdots$, with irrational rotation number $\theta$, is not always linearisable at the origin [Cre38]. However, Siegel proved in 1940 that linearisability occurs when $\theta$ is Diophantine [Sie42].

Thirty years later Brjuno extended Siegel’s result showing that the arithmetic condition $\sum q_n^{-1} \log q_{n+1} < +\infty$ on the best rational approximants $(p_n/q_n)_{n \geq 0}$ of $\theta$ is a sufficient condition for linearisability. In 1995, Yoccoz proved that if the Brjuno condition does not hold for some $\theta$, then the quadratic map $e^{2\pi i \theta}z + z^2$ is not linearisable [Yoc95].

When $f$ is a rational function or entire, any such linearisable fixed point gives rise to a Siegel domain, i.e. a Fatou component of $f$ on which $f$ is holomorphically conjugate to the irrational rotation $z \mapsto e^{2\pi i \theta}z$. Boundaries of Siegel discs are usually very irregular [McM98, PZ04] but, Perez-Marco (unpublished manuscript [PM97]) and Avila-Buff-Cheritat [ABC04] proved that Siegel domains with smooth boundaries do actually exist.

Siegel domains also exist in higher dimension and the canonical class of biholomorphic models for them is that of Reinhardt domains. This is a rather large class where one does not expect a Siegel domain to be holomorphically equivalent to a familiar Reinhardt domain, such as the Euclidean ball or the poly-disc. It is very likely that, generically, two distinct Siegel domains in $\mathbb{C}^k$ are not bi-holomorphic equivalent. These questions remain widely open and very few explicit examples are known [Fes92, Bed18].

Here, we exhibit examples of shear automorphisms of $\mathbb{C}^2$ for which the product of the complex line with the Euclidean unit disc $\mathbb{C} \times \Delta$ is a Siegel domain. Indeed, the Siegel domain is the unique Fatou component of the map. These automorphisms are tangent to a rotation by an irrational number $R_\theta := (e^{2\pi i \theta}w, e^{2\pi i \theta}z)$, where $\theta$ does not satisfy the Brjuno’s arithmetic condition. This is in strong contrast with Yoccoz’s theorem in dimension one, and is related to the resonance in the tangent map $R_\theta$.

Assume that $\theta \in \mathbb{R} \setminus \mathbb{Q}$ with best rational approximants $(p_n/q_n)_{n \geq 0}$. Let $(q_n')_{n \geq 0}$ be a subsequence of $(q_n)_{n \geq 0}$, $\mu \in \mathbb{R} \setminus \mathbb{Q}$, and $u = (u_n)_{n \geq 0}$ belong to $l^\infty$, the space of bounded sequences in $\mathbb{C}$. Consider the formal power series

$$\varphi_{\mu,q',u}(z) := z e^{2\pi i \mu} \sum_{n=0}^{\infty} u_{q'_n} (1 - e^{2\pi i q'_n \mu}) z^{q'_n},$$

and let

$$A_{\mu,q',u}(w, z) := (e^{2\pi i \mu}w + \varphi_{\mu,q',u}(z), e^{2\pi i \mu}z).$$
Theorem 1.1. Assume that $\theta$ is an irrational number with best rational approximants $(p_n/q_n)_{n \geq 0}$ satisfying $\lim_{n \to \infty} q_n^{-1} \log q_{n+1} = \infty$. Let $S$ be a countable dense subset of $\mathbb{C} \times \{|z| > 1\}$. Then, there exist a subsequence $q'$ of $(q_n)_{n \geq 0}$ and a dense $G_\delta$-subset $E_2 \subseteq l^\infty$ such that for any $u \in E_2$, $A_{\mu,q',w}(w,z)$ is an automorphism of $\mathbb{C}^2$ satisfying the following properties:

i) $A_{\theta,q',u}$ is analytically conjugate to the rotation $R_\theta$ on $\mathbb{C} \times \Delta$;

ii) the orbit of any $(w, z) \in \mathbb{C} \times \{|z| > 1\}$ by $A_{\theta,q',u}$ is unbounded and recurrent;

iii) the sequence of derivatives $(\|DA_{\theta,q',u}(w,z)\|)_{N \geq 0}$ is unbounded, for any $(w, z) \in S$.

2. Proofs

Let $\theta$ be an irrational real number, with continued fraction expansion

$$\theta = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}$$

with integers $a_i \geq 1$, for all $i \geq 1$, and $a_0 \in \mathbb{Z}$. The best rational approximants of $\theta$ are defined as the sequence of rational numbers

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}, \quad n \geq 0.$$

We shall assume that $p_n$ and $q_n$ are relatively prime, and $q_n \geq 0$, for all $n \geq 0$. It is known that the sequence of integers $p_n$ and $q_n$, for $n \geq 0$, may be defined according to the following recursive relations

$$p_{-2} = q_{-1} = 0, \quad p_{-1} = q_{-2} = 1,$$

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}.$$ The sequence $(q_n)_{n \geq 0}$ has exponential growth. For instance, since $a_i \geq 1$, the sequence $q_n$ grows at least as fast as the Fibonacci sequence. Indeed, by an inductive argument, one may see that

$$q_n \geq 2^{(n-1)/2}, \quad \forall n \geq 0.$$ (1)

Moreover, one has the following estimate on the size of approximation

$$|\theta - \frac{p_n}{q_n}| \leq \frac{1}{q_n q_{n+1}}, \quad \forall n \geq 0.$$ (2)

One may consult [Kh64] or [Sch50] for basic properties of continued fractions.

Given $z_0 \in \mathbb{C}$ and $r > 0$, we define

$$\Delta(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}, \quad \Delta := \Delta(0, 1).$$

Let us consider the (complex) Banach space $(l^\infty, \| \|_\infty)$, where $\|(u_n)_{n \geq 0}\|_\infty$ is defined as $\sup_{n \geq 0} |u_n|$. We assume $u_n \in \mathbb{C}$, for $n \geq 0$.

Proposition 2.1. Let $\theta$ be an irrational number with best rational approximants $(p_n/q_n)_{n \geq 0}$. For every subsequence $q' := (q'_n)_{n \geq 0}$ of $(q_n)_{n \geq 0}$, the following hold:

(i) the Taylor series

$$h_{q'}(z) := z \sum_{n=0}^\infty z^n$$

is holomorphic on $\Delta$, but does not have holomorphic extension across any

$z \in \partial \Delta$;
(ii) for any \( u = (u_n)_{n \geq 0} \in l^\infty \) the Taylor series

\[
h_{q', u}(z) := \sum_{n=0}^{\infty} u_{q'_n} z^{q'_n}
\]

is holomorphic on \( \Delta \);

(iii) there exists a dense \( G_\delta \) subset \( E_1 \) of \( l^\infty \) (which only depends on \( q' \)) such that for every \( u \in E_1 \), \( h_{q', u}(z) \) does not have holomorphic extension across any \( z \in \partial \Delta \).

The series in the above proposition are special cases of Lacunary series, studied first by Weierstrass; see [Kah64] or [EM54].

**Proof.** By Equation (1), the series \( h_{q', u} \) satisfies Hadamard’s lacunarity condition [Had92], that is, \( \liminf_{n \to 0} (q'_{n+1}/q'_n) > 1 \). This implies that, if the radius of convergence of the series \( h_{q', u}(z) \) about 0 is equal to some \( r > 0 \) then \( h_{q', u}(z) \) does not extend holomorphically across any \( z \in \partial \Delta \).

Evidently, the radius of convergence of \( h_{q'} \) is equal to +1. The radius of convergence of \( h_{q', u}(z) \) is at least +1. Below we show that the radius of convergence is indeed equal to +1 for many series of the form \( h_{q', u}(z) \).

For \( k \geq 1 \), consider the set

\[
E^k := \left\{ (u_n)_{n \geq 0} \in l^\infty : \exists N \in \mathbb{N}, \sum_{n=0}^{N} u_{q'_n} > k \right\}.
\]

For every \( k \in \mathbb{N}, E^k \) is open and dense in \( l^\infty \). By Baire’s theorem, \( E_1 = \bigcap_{k \in \mathbb{N}} E^k \) is a dense subset of \( l^\infty \). For every \( u \in E_1 \), the series \( \sum_{n \geq 0} u_{q'_n} \) is divergent. □

Let \( h_{q', u} : \Delta \to \mathbb{C} \) be a map as in Proposition 2.1 with \( u \in E_1 \). We may consider the shear map

\[
S_{h_{q', u}}(w, z) = (w + h_{q', u}(z), z).
\]

This is an automorphism of \( \mathbb{C} \times \Delta \), with inverse \( S_{h_{q', u}}^{-1} = S_{-h_{q', u}} \). For \( \mu \in \mathbb{R} \), let us consider the two dimensional rotation \( R_\mu : \mathbb{C}^2 \to \mathbb{C}^2 \),

\[
R_\mu(w, z) = (e^{2\pi i \mu} w, e^{2\pi i \mu} z).
\]

We define the automorphism \( A_{\mu, q', u} : \mathbb{C} \times \Delta \to \mathbb{C} \times \Delta \) as

\[
A_{\mu, q', u} = S_{h_{q', u}}^{-1} \circ R_\mu \circ S_{h_{q', u}} = S_{-h_{q', u}} \circ R_\mu \circ S_{h_{q', u}}.
\]

We note that

\[
A_{\mu, q', u}(w, z) = S_{-h_{q', u}} \circ R_\mu \circ S_{h_{q', u}}(w, z) = S_{-h_{q', u}} \circ R_\mu(w + h_{q', u}(z), z) = S_{-h_{q', u}}(e^{2\pi i \mu} w + e^{2\pi i \mu} h_{q', u}(z), e^{2\pi i \mu} z) = e^{2\pi i \mu} w + e^{2\pi i \mu} h_{q', u}(z) - h_{q', u}(e^{2\pi i \mu} z) = (e^{2\pi i \mu} w + e^{2\pi i \mu} h_{q', u}(z) - h_{q', u}(e^{2\pi i \mu} z), e^{2\pi i \mu} z).
\]

Let us define the map \( \varphi_{\mu, q', u} : \Delta \to \mathbb{C} \) as

\[
\varphi_{\mu, q', u}(z) = e^{2\pi i \mu} h_{q', u}(z) - h_{q', u}(e^{2\pi i \mu} z),
\]

so that

\[
A_{\mu, q', u}(w, z) = (e^{2\pi i \mu} w + \varphi_{\mu, q', u}(z), e^{2\pi i \mu} z).
\]

By Equation (3), for all \( N \in \mathbb{N} \),

\[
A_{\mu, q', u}^N(w, z) = A_{N\mu, q', u}(w, z) = (e^{2\pi i N \mu} w + \varphi_{N\mu, q', u}(z), e^{2\pi i N \mu} z).
\]

When the sequence \( u \in l^\infty \) is the constant sequence \( u_n = 1 \) we shall simply denote the above maps by \( A_{\mu, q'} \) and \( \varphi_{\mu, q'} \), that is,

\[
\varphi_{\mu, q'}(z) = e^{2\pi i \mu} h_{q'}(z) - h_{q'}(e^{2\pi i \mu} z),
\]
Proposition 2.2. Assume that the best rational approximants \((p_n/q_n)_{n \geq 0}\) of \(\theta \in \mathbb{R} \setminus \mathbb{Q}\) satisfy \(q_n^{-1} \log q_{n+1} \to \infty\), as \(n \to \infty\). Then, for any subsequence \(q'\) of \((q_n)_{n \geq 0}\) and any \(u \in l^\infty\), the Taylor series of \(\varphi_{\theta, q', u}\) defines an entire holomorphic map on \(\mathbb{C}\).

In particular, \(A_{\theta, q', u}\) is an automorphism of \(\mathbb{C}^2\).

By choosing \(a_n\) large enough, one may identify values of \(\theta\) such that \(q_n\) grows arbitrarily fast. In particular, there are irrational numbers \(\theta\) for which \(q_n^{-1} \log q_{n+1} \to +\infty\) as \(n \to +\infty\).

Proof. By the estimate in Equation (2), we have \(|q_n \theta - p_n| \leq 1/q_{n+1}\). On the other hand, for any \(\beta \in [-1/2, 1/2]\), \(1 - e^{2\pi i \beta} \leq 2\pi |\beta|\). In particular, \(1 - e^{2\pi i q_n \theta} \leq 2\pi/q_{n+1}\). By the hypothesis of the proposition on the growth of \(q_n\), we conclude that the radius of convergence of the Taylor series for \(\varphi_{\theta, q', u}\) is \(\infty\). The Taylor series of \(\varphi_{\theta, q', u}\) is a subseries of the Taylor series of \(\varphi_{\theta, q, u}\), and hence, its radius of convergence is also \(\infty\).

We shall also need to identify some \(\mu \in \mathbb{R}\) for which the convergence radius of the Taylor series of \(\varphi_{\mu, q', u}\) is precisely equal to 1.

Lemma 2.3. Let \(\theta \in \mathbb{R} \setminus \mathbb{Q}\) with best rational approximants \((p_n/q_n)_{n \geq 0}\) and let \(q' := (q''_n)_{n \geq 0}\) be a subsequence of \((q_n)_{n \geq 0}\). There exist \(\mu \in \mathbb{R}\) and a dense \(G_\delta\)-subset \(E_1\) of \(l^\infty\) such that for any \(u \in E_1\), the convergence radius of the Taylor series of \(\varphi_{\mu, q', u}\) and \(\varphi_{\mu, q'}\) are equal to 1.

Proof. Let \((q''_n)_{n \geq 0}\) be a subsequence of \(q'\) such that for all \(n \geq 0\) we have \(q''_{n+1} \geq 4q''_n\). We may inductively choose a nest of closed connected intervals \(I_n\) on \([0, 1]\), for \(n \geq 0\), such that each interval \(I_n\) has length \(1/q''_n\) and each set \(q''_n I_{n+1}\) is contained in \([1/4, 3/4] + \mathbb{Z}\). We define \(\mu\) as the unique element in the nest \(I_n\). It follows that for all \(n \geq 0\), \(1 - e^{2\pi i q''_n \mu} \geq \sqrt{2}\).

According to Proposition 2.2, there exists a dense \(G_\delta\)-subset \(E_1\) of \(l^\infty\) such that for any \(u \in E_1\) the radius of convergence of \(h_{q'', u}(z) = e^{\sum_{n=0}^{\infty} u_{q''_n} z^{q''_n}}\) is equal to 1. By Cauchy-Hadamard’s formula, we must have \(\limsup_n |u_{q''_n}|^{1/q''_n} = 1\). Since \(2 \geq |1 - e^{2\pi i q''_n \mu}| \geq \sqrt{2}\), it follows that for \(u \in E_1\),

\[
\limsup_{n} |u_{q''_n}(1 - e^{2\pi i q''_n \mu})|^{1/q''_n} = 1.
\]

As \(q''\) is a subsequence of \(q'\), and \(\limsup_n |u_{q''_n}(1 - e^{2\pi i q''_n \mu})|^{1/q''_n} \leq 1\), we conclude that

\[
\limsup_{n} |u_{q''_n}(1 - e^{2\pi i q''_n \mu})|^{1/q''_n} = 1.
\]

Therefore, for \(u \in E_1\), the radius of convergence of \(\varphi_{\mu, q', u}(z)\) is equal to 1.

The above arguments also work when the sequence \(u\) is constant equal to 1 and thus the convergence radius of \(\varphi_{\mu, q'}(z)\) equals 1 as well.
Assume that \( \theta \) is an irrational number which satisfies the hypothesis of Proposition \ref{prop:existence}. For any subsequence \( q' \) of \( q \) and any \( u \in l^\infty \),
\[
A_{\theta,q'}(w, z) = (e^{2\pi i \theta} w + \varphi_{\theta,q',u}(z), e^{2\pi i \theta} z)
\]
is an automorphism of \( \mathbb{C}^2 \). The map \( A_{\theta,q',u} \) has a constant Jacobian of size 1. So, it is area preserving. Moreover, \( A_{\theta,q',u} \) preserves the origin, with derivative
\[
DA_{\theta,q',u}(0, 0) = \begin{pmatrix}
e^{2\pi i \theta} & 0 \\
0 & e^{2\pi i \theta}
\end{pmatrix}.
\]

Any irrational number \( \theta \) satisfying \( \lim_{n \to \infty} q_n^{-1} \log q_{n+1} = \infty \) does not satisfy the Brjuno’s arithmetic condition for the linearisability \cite{Sie42, Brj71, Her87, Yoc95}. However, here \( DA_{\theta,q',u}(0, 0) \) enjoys a resonance condition. By virtue of the commutative relation in Equation \ref{eq:commutative}, \( A_{\theta,q',u} \) is conjugate to the rotation \( R_\theta \) on the round domain \( \mathbb{C} \times \Delta \), via the map \( S_{h,q',u} \). We would like to understand the global dynamics of \( A_{\theta,q',u} \). In particular, if the Siegel disk of \( A_{\theta,q',u} \) centred at \((0,0)\) extends beyond \( \mathbb{C} \times \Delta \).

Let \( F: \mathbb{C}^n \to \mathbb{C}^n \) be a holomorphic map. We say that \( \Omega \) is a Fatou component for the iterates of \( F \), if the family \( \{ F^n \}_{n \geq 0} \) forms a locally equi-continuous family of maps near any point in \( \Omega \). One may consult \cite{Mil06, CG93, MNTU00} for classic books in complex dynamics in dimension one, and refer to \cite{CGSY03, Shi09} for general theory of complex dynamics in higher dimensions.

**Theorem 2.4.** Assume that \( \theta \) is an irrational number whose best rational approximants \( (p_n/q_n)_{n \geq 0} \) satisfies \( \lim_{n \to \infty} q_n^{-1} \log q_{n+1} = \infty \). Let \( q' \) be a subsequence of \( (q_n)_{n \geq 0} \). There exists a dense \( \mathbb{G}_1 \)-subset \( E_1 \) of \( l^\infty \) such that for every \( u \in E_1 \), \( \mathbb{C} \times \Delta \) is a Fatou component of \( A_{\theta,q',u} \), and for every \((w,z) \in \mathbb{C}^2 \) with \(|z| > 1 \) the orbit of \((w,z) \) under \( A_{\theta,q',u} \) is unbounded.

Moreover, the same conclusions hold for \( A_{\theta,q'} \), that is, for the constant sequence \( u_n \equiv 1 \).

**Proof.** Let us fix \( q' \), and let \( \mu \in \mathbb{R} \) and \( E_1 \subset l^\infty \) be the \( \mathbb{G}_1 \) dense set obtained in Lemma \ref{lem:existence}. By Proposition \ref{prop:existence}, for every \( u \in l^\infty \), \( A_{\theta,q',u} \) is a well-defined holomorphic map on \( \mathbb{C}^2 \). From now on we assume that \( u \in E_1 \), or \( u \) is the constant sequence \( u_n \equiv 1 \).

Note that \( A_{\theta,q',u} \) preserves \( \mathbb{C} \times \Delta \). Indeed, by Equation \ref{eq:commutative} and Proposition \ref{prop:existence}, \( A_{\theta,q',u} \) is conjugate to an irrational rotation on \( \mathbb{C} \times \Delta \). Thus, \( \mathbb{C} \times \Delta \) is contained in the Fatou set of \( A_{\theta,q',u} \). Let \( \Omega \subset \mathbb{C}^2 \) be a Fatou component of \( A_{\theta,q',u} \) which contains \( \mathbb{C} \times \Delta \). Assume in the contrary that \( \Omega \neq \mathbb{C} \times \Delta \).

We may choose \( w_0 \in \mathbb{C} \), \( z_0 \in \partial \Delta \), and \( \epsilon > 0 \) such that the bidisk \( \Delta(w_0, \epsilon) \times \Delta(z_0, \epsilon) \) is contained in \( \Omega \). It follows from the formula
\[
A_{\theta,q',u}^N(w, z) = (e^{2\pi i N \theta} w + \varphi_{N\theta,q',u}(z), e^{2\pi i N \theta} z), \quad N \geq 0,
\]
that the iterates of \( A_{\theta,q',u} \) are locally equi-continuous on the tube \( \mathbb{C} \times \Delta(z_0, \epsilon) \). On the other hand, since \( \Omega \) is invariant under \( A_{\theta,q',u} \), and \( A_{\theta,q',u} \) acts as an irrational rotation in the \( z \)-coordinate, \( \Omega \) contains \( \mathbb{C} \times \{ z \in \mathbb{C} : ||z| - 1| < \epsilon \} \). Thus, the iterates of \( A_{\theta,q',u} \) are equi-continuous on the tube \( T = \mathbb{C} \times \Delta(0,1+\epsilon) \).

Since \( A_{\theta,q',u} \) is conjugate to the irrational rotation \( R_\theta \) on \( \mathbb{C} \times \Delta \), there is an increasing sequence of positive integers \( m_i \) such that
\[
A_{\theta,q',u}^{m_i} \to S_{-h,q',u} \circ R_\mu \circ S_{h,q',u},
\]
with the convergence uniform on compact subsets of \( \mathbb{C} \times \Delta \). Since \( \Omega \) is a Fatou set for \( A_{\theta,q',u} \), we may extract an increasing subsequence \( k_i \) of \( m_i \) such that the iterates \( A_{\theta,q',u}^{k_i} \) converges uniformly on compact subsets of \( T \) to a holomorphic map
ψ : T → C^2. Thus, we must have

ψ = S_{-h_{q',u}} ∘ R_μ ∘ S_{h_{q',u}} on C ∝ Δ.

However, the map S_{-h_{q',u}} ∘ R_μ ∘ S_{h_{q',u}} is of the form

(w, z) ↦ (e^{2πiμ}w + ϕ_{μ, q', u}(z), e^{2πiμ}z),

which, by our choice of μ and u and according to Lemma 2.3, diverges at some point on C ∝ Δ. This contradicts ψ being defined on T.

We have just proved that C ∝ Δ is a Fatou component of A_{θ,q',u}. As we shall show below, this implies that for any (w, z) ∈ C^2 with |z| > 1, the orbit of (w, z) by A_{θ,q',u} is unbounded.

Let us fix (w_0, z_0) ∈ C^2 with |z_0| > 1. Assume in the contrary that there is M > 0 such that |A_{θ,q',u}^n(w_0, z_0)| ≤ M, for all N ≥ 0. Then |A_{θ,q',u}^{N+n}(w_0, z_0)| ≤ M, for all N ≥ 0 and all n ≥ 0. Writing A_{θ,q',u}^n(w_0, z_0) =: (w_n, e^{2πiθ_n}z_0), we thus have simultaneously |w_n| ≤ M and |A_{θ,q',u}^n(w_n, e^{2πiθ_n}z_0)| ≤ M, for all n ≥ 0 and all N ≥ 0. Then, since

A_{θ,q',u}^{N+n}(w_0, z_0) = A_{θ,q',u}^N(w_n, e^{2πiθ_n}z_0)

= (e^{2πiθ_n}w_n + ϕ_{θ,q', u}(e^{2πiθ_n}z_0), e^{2πi(N+n)θ_n}z_0),

we get that |ϕ_{θ,q', u}| ≤ 2M on {e^{2πiθ_n}z_0 : n ≥ 0}, for any N ≥ 0. Thus, the family of maps ϕ_{θ,q', u}, for N ≥ 0, is uniformly bounded on the circle |z| = |z_0| and, by the maximum principle, on the disk Δ(0, |z_0|).

As A_{θ,q', u}^n(w, z) = (e^{2πiθ_n}w + ϕ_{θ,q', u}(z), e^{2πiθ_n}z), by the above paragraph, the iterates of A_{θ,q', u} are locally uniformly bounded on C ∝ Δ(0, |z_0|). Therefore, C ∝ Δ(0, |z_0|) must be a Fatou component of A_{θ,q', u}, which strictly contains the Siegel tube C ∝ Δ. This contradicts the earlier statement that C ∝ Δ itself is a Fatou component of A_{θ,q', u}.

We now aim to show that for a certain choice of the subsequence q' and for a generic u ∈ t∞ the Julia set of A_{θ,q',u} coincides with the complement of the Siegel tube C ∝ Δ. This will prove Theorem 1.3.

The proof is based on the following two lemmas.

Lemma 2.5. Assume that θ is an irrational number whose best rational approximants \((p_n/q_n)_{n≥0}\) satisfies \(q_n^{-1} \log q_{n+1} \rightarrow ∞\), as n → ∞. For any sequence of positive numbers (ε_p)_{p≥1} converging to zero, there exist a subsequence (q'_n)_{n≥0} of (q_n)_{n≥0} and an increasing sequence of integers (N_p)_{p≥1} such that for every p ≥ 1, we have

\[ |1 - e^{2πiθ_p}| + \sum_{n≥0} |1 - e^{2πiθ_pq'_n}|p^{q'_n} ≤ ε_p. \]

Proof. Let us set N_0 = 1 and q'_0 = q_0. We inductively identify the pair of integers q'_p and N_p so that the following properties are satisfied for all p ≥ 1:

(i) N_p > N_{p-1},
(ii) |1 - e^{2πiθ_p}| + ∑_{n=0}^{p-1} |1 - e^{2πiθ_pq'_n}|p^{q'_n} ≤ ε_p/2, 
(iii) q'_p ∈ \{q_n : n ≥ 0, q_n > q_{p-1}, q_n ≥ 2N_p\},
(iv) ∑_{n,q_n ≥ q'_p, q_n/q_{n+1} ≥ ε_p/(4πN_p)} p^{q'_n} ≤ ε_p.

Assume that q'_0, ..., q'_{p-1} and N_0, ..., N_{p-1} are already constructed. First we choose an integer N_p so that (i) holds, and e^{2πiθ_p} is close enough to 1 in order to satisfy property (ii). Then we choose q'_p according to property (iii), while large enough to guarantee property (iv). The latter choice is possible due to our assumption on the growth of (q_n)_{n≥0}, that is, the radius of convergence of the series ∑_{n≥0} q'_n/q_{n+1} is infinity.
Lemma 2.6. Using the estimate in property (ii), and the above estimates, we get

$$\left|1 - e^{2\pi i q_n N_{\theta}}\right| \leq 2\pi N_p / q_{n+1}'.$$

Also note that since \((q_n)_{n \geq 0}\) is a subsequence of \((q_n)_{n \geq 0}\), by property (iv) in the above list we have

$$\sum_{n \geq p} \frac{p_n'}{q_{n+1}} \leq \sum_{q_n \geq q_p} \frac{p_n'}{q_n+1} \leq \frac{\varepsilon_p}{4\pi N_p}.$$  

We may now check that the conclusion of the lemma is fulfilled. Let \(p \geq 1\). Using the estimate in property (ii), and the above estimates, we get

$$\left|1 - e^{2\pi i q_n N_{\theta}}\right| + \sum_{n=0}^{p-1} \left|1 - e^{2\pi i q_n' N_{\theta}}\right| p_n' + \sum_{n \geq p} \left|1 - e^{2\pi i q_n N_{\theta}}\right| p_n' \leq \varepsilon_p/2 + 2\pi N_p \sum_{n \geq p} \frac{p_n'}{q_{n+1}} \leq \varepsilon_p/2 + \varepsilon_p/2. \quad \square$$

Assume that \(\theta\) is an irrational number which satisfies the hypothesis of Proposition 2.2 so that for any subsequence \(q'\) of \(q\) and any \(u \in l^\infty\), \(\varphi_{N\theta,q',u}\) is an entire function. It follows from the formula in Equation 1 that for any integer \(N \in \mathbb{N}\), \(\varphi_{N\theta,q',u}\) is an entire function.

Lemma 2.6. Assume that \(\theta\) is an irrational number whose best rational approximants \((p_n/q_n)_{n \geq 0}\) satisfy \(q_n^{-1} \log q_{n+1} \to \infty\), as \(n \to \infty\). For any subsequence \(q'\) of \((q_n)_{n \geq 0}\) and any \(z \in \mathbb{C}\) with \(|z| > 1\), there exists a dense \(G_\delta\)-subset \(E(z)\) of \(l^\infty\) such that for every \(u \in E(z)\),

$$\sup_{N} |\varphi_{N\theta,q',u}(z)| = +\infty.$$  

Proof. Let us fix an arbitrary subsequence \(q'\) and a point \(z \in \mathbb{C}\) with \(|z| > 1\). Recall that

$$\varphi_{N\theta,q',u}(z) = z e^{2\pi i N\theta} \sum_{n=0}^{\infty} u_{q_n'} (1 - e^{2\pi i q_n' N_{\theta}}) z^{q_n'}.$$  

For each \(N \in \mathbb{N}\), we consider the bounded linear operator \(\Phi_{N,z} : l^\infty \to \mathbb{C}\), defined as

$$\Phi_{N,z}(u) := \varphi_{N\theta,q',u}(z) = e^{2\pi i N\theta} \sum_{n=0}^{\infty} u_{q_n'} (1 + q_n')(1 - e^{2\pi i q_n' N_{\theta}}) z^{q_n'}.$$  

The operator norm of \(\Phi_{N,z}\) satisfies

$$\|\Phi_{N,z}\| = \sum_{n=0}^{\infty} (1 + q_n')(1 - e^{2\pi i q_n' N_{\theta}}) |z|^{q_n'}.$$  

Recall that \(\varphi_{N\theta,q'}\) is the entire function corresponding to the constant sequence \(u_n = 1\), and observe that

$$|\varphi_{N\theta,q'}(z)| \leq \|\Phi_{N,z}\| = \|\Phi_{N,z}||.$$  

Let us consider the quantity \(K_z := \sup_N \|\Phi_{N,z}\|\). We claim that for every \(z \in \mathbb{C}\) with \(|z| > 1\), \(K_z = +\infty\). Otherwise, by Equation 6 and the maximum modulus principle, we must have \(|\varphi_{N\theta,q'}(z')| \leq K_n\), for every integer \(N\) and every \(z' \in \mathbb{C}\) with \(|z'| \leq |z|\). This implies that the family of maps \(\{\varphi_{N\theta,q'}(z') : N \geq 0\}\) is uniformly bounded on the disc \(\Delta(0, (1 + |z|)/2)\). Then, since \(A_{N\theta,q'}(w,z) = \)
We may employ the Banach-Steinhaus theorem [Rud87, page 98] to the family of functions for every $|N,ε|$.

Lemma 2.6 to obtain the dense $G$-set of $E(z)$ of $l^{∞}$, such that for every $u ∈ E(z)$, we have $sup_{N} |φ_{N,ε}| = +∞$. We may employ the Banach-Steinhaus theorem [Rud87, page 98] to the family of linear operators $[φ_{N,ε}]_{N≥1}$. That gives us a dense $G_δ$-subset $E(z)$ of $l^{∞}$.

Proof of Theorem 1.1. Let $(ε_p)^{p≥1}$ be a sequence of positive numbers converging to zero. With Lemma 2.5 we generate a subsequence $q'$ of $(q_n)^{n≥0}$ and an increasing sequence of integers $(N_p)^{p≥1}$. We also employ Theorem 2.3 with $q'$ to obtain the dense $G_δ$-subset of $l^{∞}$ denoted by $E_1$. Similarly, for $q'$ and each $z ∈ S$, we use Lemma 2.6 to obtain the dense $G_δ$-set $E(z)$ of $l^{∞}$. By Baire’s theorem, $E_2 := \cap_{z ∈ S} E(z) ∩ E_1$ is a dense $G_δ$-subset of $l^{∞}$.

Fix an arbitrary $u ∈ E_2$. According to Theorem 2.3, the map $A_{q',u}$ is analytically conjugate to the rotation $Rθ$ on $C × Δ$. Moreover, the orbit of any $(w, z) ∈ C^2$ with $|z| > 1$ by the map $A_{q',u}$ is unbounded.

Let $(N_p)^{p≥1}$ be the sequence of integers given by Lemma 2.5. Recall that $A_{q',u}^{N_p}(w, z) = (e^{2πiN_pθ}w + \varphi_{N_pθ,q',u}(z), e^{2πiN_pθ}z)$, where $\varphi_{N_pθ,q',u}(z) = ze^{2πiN_pθ} \sum_{n=0}^{∞} u_{N_p}^{n}(1 - e^{2πiN_pθ})w_{n}^{q'}$.

Fix an arbitrary $(w, z) ∈ C^2$. For integers $p ≥ |z|$, we have

$$\left\|A_{q',u}^{N_p}(w, z) - (w, z)\right\|
\leq \left\|(w, z)\right\| \cdot \left|1 - e^{2πiN_pθ}\right| + \left|z\right| \left\|u\right\|_{∞} \sum_{n≥0} \left|1 - e^{2πiN_pθ}\right| |w_{n}^{q'}|
\leq \left\|(w, z)\right\| \cdot \left|1 - e^{2πiN_pθ}\right| + \left|z\right| \left\|u\right\|_{∞} \sum_{n≥0} \left|1 - e^{2πiN_pθ}\right| |p_{n}^{q'}|
\leq \left(\left\|(w, z)\right\| + \left|z\right| \left\|u\right\|_{∞}\right) \left(\left|1 - e^{2πiN_pθ}\right| + \sum_{n≥0} \left|1 - e^{2πiN_pθ}\right| |p_{n}^{q'}\right)
\leq \left(\left\|(w, z)\right\| + \left|z\right| \left\|u\right\|_{∞}\right) \varepsilon_p.$$

In the last line of the above equation we have used the estimate in Lemma 2.5. The above bound shows that the orbit of $(w, z)$ by $A_{q',u}$ is recurrent.

Evidently, $\left\|DA_{q',u}^{N_p}(w, z)\right\| ≥ \left|\varphi_{N_pθ,q',u}(z)\right|$. Therefore, by Lemma 2.6 for every $(w, z) ∈ S$, we have $sup_{N} \left\|DA_{q',u}^{N_p}(w, z)\right\| = +∞$. □

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