On the Local Well-posedness of a 3D Model for Incompressible Navier-Stokes Equations with Partial Viscosity

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Abstract
In this short note, we study the local well-posedness of a 3D model for incompressible Navier-Stokes equations with partial viscosity. This model was originally proposed by Hou-Lei in [4]. In a recent paper, we prove that this 3D model with partial viscosity will develop a finite time singularity for a class of initial condition using a mixed Dirichlet Robin boundary condition. The local well-posedness analysis of this initial boundary value problem is more subtle than the corresponding well-posedness analysis using a standard boundary condition because the Robin boundary condition we consider is non-dissipative. We establish the local well-posedness of this initial boundary value problem by designing a Picard iteration in a Banach space and proving the convergence of the Picard iteration by studying the well-posedness property of the heat equation with the same Dirichlet Robin boundary condition.

1 Introduction
In this short note, we prove the local well-posedness of the 3D model with partial viscosity. The 3D model with partial viscosity has the following form:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= 2u\psi_z, \\
\frac{\partial \omega}{\partial t} &= (u^2)_z + \nu \Delta \omega, \quad (x, z) \in \Omega = \Omega_x \times (0, \infty), \\
-\Delta \psi &= \omega
\end{aligned}
\]

where \(\Omega_x = (0, a) \times (0, a)\). Let \(\Gamma = \{(x, z) \mid x \in \Omega_x, \, z = 0\}\). The initial and boundary conditions for (1) are given as following:

\[
\begin{aligned}
\omega|_{\partial \Omega \setminus \Gamma} &= 0, \quad (\omega_z + \gamma \omega)|_{\Gamma} = 0, \\
\psi|_{\partial \Omega \setminus \Gamma} &= 0, \quad (\psi_z + \beta \psi)|_{\Gamma} = 0, \\
\omega|_{t=0} &= \omega_0(x, z), \quad u|_{t=0} = u_0(x, z).
\end{aligned}
\]
This 3D model with viscosity in both $u$ and $\omega$ components was first proposed by Hou and Lei in [4]. The only difference between this 3D model and the reformulated Navier-Stokes equations is that convection term is neglected in the model. If one adds the convection term back to the 3D model, one would recover the full Navier-Stokes equations. This model preserves almost all the properties of the full 3D Navier-Stokes equations. Despite the striking similarity at the theoretical level between the 3D model and the Navier-Stokes equations, the former seems to have a very different behavior from the full Navier-Stokes equations. In a recent paper [5], we prove that the above 3D model with partial viscosity develops a finite time singularity for a class of initial condition using a mixed Dirichlet Robin boundary condition.

The analysis of finite time singularity formation of the 3D model [5] uses the local well-posedness result of the 3D model. The local well-posedness of the 3D model can be proved by using a standard energy estimate and a mollifier if there is no boundary or if the boundary condition is a standard one, see e.g. [6]. For the mixed Dirichlet Robin boundary condition we consider here, the analysis is a bit more complicated since the mixed Dirichlet Robin condition gives rise to a growing eigenmode.

There are two key ingredients in our local well-posedness analysis. The first one is to design a Picard iteration for the 3D model. The second one is to show that the mapping that generates the Picard iteration is a contraction mapping and the Picard iteration converges to a fixed point of the Picard mapping by using the Contraction Mapping Theorem. To establish the contraction property of the Picard mapping, we need to use the well-posedness property of the heat equation with the same Dirichlet Robin boundary condition as $\omega$. The well-posedness analysis of the heat equation with a mixed Dirichelet Robin boundary has been studied in the literature. The case of $\gamma > 0$ is more subtle because there is a growing eigenmode. Nonetheless, we prove that all the essential regularity properties of the heat equation are still valid for the mixed Dirichlet Robin boundary condition with $\gamma > 0$.

2 The main result

The local existence result of our 3D model with partial viscosity is stated in the following theorem.

**Theorem 2.1** Assume that $u_0 \in H^{s+1}(\Omega)$, $\omega_0 \in H^s(\Omega)$ for some $s > 3/2$, $u_0|_{\partial \Omega} = u_{0z}|_{\partial \Omega} = 0$ and $\omega_0$ satisfies [2]. Moreover, we assume that $\beta \in S_\infty$ (or $S_b$) as defined in Lemma 2.1. Then there exists a finite time $T = T\left(\|u_0\|_{H^{s+1}(\Omega)}, \|\omega_0\|_{H^s(\Omega)}\right) > 0$ such that the system [7] with boundary condition [2], [3] and initial data [4] has a unique solution, $u \in C([0,T], H^{s+1}(\Omega))$, $\omega \in C([0,T], H^s(\Omega))$ and $\psi \in C([0,T], H^{s+2}(\Omega))$.

The local well-posedness analysis relies on the following local well-posedness of the heat equation and the elliptic equation with mixed Dirichlet and Robin boundary conditions. First, the local well-posedness of the elliptic equation with the mixed Dirichlet and Robin boundary condition is given by the following lemma [5]:

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Lemma 2.1 There exists a unique solution $v \in H^s(\Omega)$ to the boundary value problem:

\begin{align*}
-\Delta v &= f, \quad (x, z) \in \Omega, \quad (5) \\
v|_{\partial \Omega \setminus \Gamma} &= 0, \quad (v_x + \beta v)|_{\Gamma} = 0, \quad (6)
\end{align*}

if $\beta \in S_\infty \equiv \{ \beta \mid \beta \neq \frac{\pi |k|}{\sqrt{a}} \text{ for all } k \in \mathbb{Z}^2 \}$, $f \in H^{s-2}(\Omega)$ with $s \geq 2$ and $f|_{\partial \Omega \setminus \Gamma} = 0$. Moreover we have

\[ \|v\|_{H^s(\Omega)} \leq C_s \|f\|_{H^{s-2}(\Omega)}, \quad (7) \]

where $C_s$ is a constant depending on $s$, $|k| = \sqrt{k_1^2 + k_2^2}$.

Definition 2.1 Let $K : H^{s-2}(\Omega) \to H^s(\Omega)$ be a linear operator defined as following:

for all $f \in H^{s-2}(\Omega)$, $K(f)$ is the solution of the boundary value problem (5)-(6).

It follows from Lemma 2.1 that for any $f \in H^{s-2}(\Omega)$, we have

\[ \|K(f)\|_{H^s(\Omega)} \leq C_s \|f\|_{H^{s-2}(\Omega)}. \quad (8) \]

For the heat equation with the mixed Dirichlet and Robin boundary condition, we have the following result.

Lemma 2.2 There exists a unique solution $\omega \in C([0, T]; H^s(\Omega))$ to the initial boundary value problem:

\begin{align*}
\omega_t &= \nu \Delta \omega, \quad (x, z) \in \Omega, \quad (9) \\
\omega|_{\partial \Omega \setminus \Gamma} &= 0, \quad (\omega_x + \gamma \omega)|_{\Gamma} = 0, \quad (10) \\
\omega|_{t=0} &= \omega_0(x, z). \quad (11)
\end{align*}

for $\omega_0 \in H^s(\Omega)$ with $s > 3/2$. Moreover we have the following estimates in the case of $\gamma > 0$

\[ \|\omega(t)\|_{H^s(\Omega)} \leq C(\gamma, s) e^{\nu \gamma t} \|\omega_0\|_{H^s(\Omega)}, \quad t \geq 0, \quad (12) \]

and

\[ \|\omega(t)\|_{H^s(\Omega)} \leq C(\gamma, s, t) \|\omega_0\|_{L^2(\Omega)}, \quad t > 0. \quad (13) \]

Remark 2.1 We remark that the growth factor $e^{\nu \gamma t}$ in (12) is absent in the case of $\gamma \leq 0$ since there is no growing eigenmode in this case.

Proof First, we prove the solution of the system (9)-(11) is unique. Let $\omega_1, \omega_2 \in H^s(\Omega)$ be two smooth solutions of the heat equation for $0 \leq t < T$ satisfying the same initial
condition and the Dirichlet Robin boundary condition. Let \( \omega = \omega_1 - \omega_2 \). We will prove that \( \omega = 0 \) by using an energy estimate and the Robin boundary condition at \( \Gamma \):

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \omega^2 \, dx \, dz = \nu \int_{\Omega} \omega \Delta \omega \, dx \, dz \\
= -\nu \int_{\Omega} |\nabla \omega|^2 \, dx \, dz - \nu \int_{\Gamma} \omega \omega_z \, dx \\
= -\nu \int_{\Omega} |\nabla \omega|^2 \, dx \, dz + \nu \gamma \int_{\Gamma} \omega^2 \, dx \\
= -\nu \int_{\Omega} |\nabla \omega|^2 \, dx \, dz - 2\nu \gamma \int_{\Gamma} \int_{z}^{\infty} (\omega^2) \, dz \, dx \\
\leq -\nu \int_{\Omega} |\nabla \omega|^2 \, dx \, dz + \nu \int_{\Omega} \omega^2 \, dx \, dz \\
\leq -\nu \int_{\Omega} |\nabla \omega|^2 \, dx \, dz - 2\nu \gamma \int_{\Gamma} \int_{z}^{\infty} \omega \omega_z \, dx \, dz,
\]

(14)

where we have used the fact that the smooth solution of the heat equation \( \omega \) decays to zero as \( z \to \infty \). Thus, we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \omega^2 \, dx \, dz \leq 2\nu \gamma^2 \int_{\Omega} \omega^2 \, dx \, dz.
\]

(15)

It follows from Gronwall’s inequality

\[
e^{-2\nu \gamma^2 t} \int_{\Omega} \omega^2 \, dx \, dz \leq \int_{\Omega} \omega_0^2 \, dx \, dz = 0,
\]

(16)

since \( \omega_0 = 0 \). Since \( \omega \in H^s(\Omega) \) with \( s > 3/2 \), this implies that \( \omega = 0 \) for \( 0 \leq t < T \) which proves the uniqueness of smooth solutions for the heat equation with the mixed Dirichlet Robin boundary condition.

Next, we will prove the existence of the solution by constructing a solution explicitly. Let \( \eta(x, z, t) \) be the solution of the following initial boundary value problem:

\[
\eta_t = \nu \Delta \eta, \quad (x, z) \in \Omega, \\
\eta|_{\partial \Omega} = 0, \quad \eta|_{t=0} = \eta_0(x, z),
\]

(17)

and let \( \xi(x, t) \) be the solution of the following PDE in \( \Omega_x \):

\[
\xi_t = \nu \Delta_x \xi + \nu \gamma^2 \xi, \quad x \in \Omega_x, \\
\xi|_{\partial \Omega_x} = 0, \quad \xi|_{t=0} = \overline{\omega}_0(x),
\]

(19)

where \( \Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \) and \( \overline{\omega}_0(x) = 2\gamma \int_0^\infty \omega_0(x, z) e^{-\gamma z} \, dz \). From the standard theory of the heat equation, we know that \( \eta \) and \( \xi \) both exist globally in time.
We are interested in the case when the initial value \( \eta_0(x, z) \) is related to \( \omega_0 \) by solving the following ODE as a function of \( z \) with \( x \) being fixed as a parameter:

\[
-\frac{1}{\gamma} \eta_{zz} + \eta = \omega_0(x, z) - \overline{\omega}_0(x) e^{-\gamma z}, \quad \eta_0(x, 0) = 0.
\]

Define

\[
\omega(x, z, t) \equiv -\frac{1}{\gamma} \eta_z + \eta + \xi(x, t) e^{-\gamma z}, \quad (x, z) \in \Omega.
\]

It is easy to check that \( \omega \) satisfies the heat equation for \( t > 0 \) and the initial condition. Obviously, \( \omega \) also satisfies the boundary condition on \( \partial \Omega \setminus \Gamma \). To verify the boundary condition on \( \Gamma \), we observe by a direct calculation that \( \omega_z + \gamma \omega \) is zero on \( \Gamma \). Since \( \eta(x, z) \) on \( \Gamma \), we obtain by using \( \eta_t = \nu \Delta \eta \) and taking the limit as \( z \to 0^+ \) that \( \Delta \eta |_{\Gamma} = 0 \), which implies that \( \eta_{zz} |_{\Gamma} = 0 \). Therefore, \( \omega \) also satisfies the Dirichlet Robin boundary condition at \( \Gamma \). This shows that \( \omega \) is a solution of the system (21)-(24). By the uniqueness result that we proved earlier, the solution of the heat equation must be given by (22).

Since \( \eta \) and \( \xi \) are solutions of the heat equation with a standard Dirichlet boundary condition, the classical theory of the heat equation \([1]\) gives the following regularity estimates:

\[
\|\eta\|_{H^s(\Omega)} \leq C \|\eta_0\|_{H^s(\Omega)}, \quad \|\xi(x)\|_{H^s(\Omega_k)} \leq C e^{\gamma \gamma^2 t} ||\overline{\omega}_0(x)||_{H^s(\Omega_k)}. \tag{23}
\]

Recall that \( \eta_{zz} |_{\Gamma} = 0 \). Therefore, \( \eta_z \) also solves the heat equation with the same Dirichlet Robin boundary condition:

\[
(\eta_z)_z = \nu \Delta \eta_z, \quad (x, z) \in \Omega, \\
(\eta_z)_z |_{\Gamma} = 0, \quad (\eta_z)|_{\partial \Omega \setminus \Gamma} = 0, \quad (\eta_z)|_{t=0} = \eta_{0z}(x, z),
\]

which implies that

\[
\|\eta_z\|_{H^s(\Omega)} \leq C \|\eta_{0z}\|_{H^s(\Omega)}. \tag{26}
\]

Putting all the above estimates for \( \eta, \eta_z \) and \( \xi \) together and using (22), we obtain the following estimate:

\[
\|\omega\|_{H^s(\Omega)} = \left\| -\frac{1}{\gamma} \eta_z + \eta + \xi(x, t) e^{-\gamma z} \right\|_{H^s(\Omega)} \leq \frac{1}{\gamma} \|\eta_z\|_{H^s(\Omega)} + \|\eta\|_{H^s(\Omega)} + \|\xi(x, t) e^{-\gamma z}\|_{H^s(\Omega)} \leq C(\gamma, s) \left( \|\eta_0\|_{H^s(\Omega)} + \|\eta_0\|_{H^s(\Omega)} + e^{\gamma \gamma^2 t} ||\overline{\omega}_0(x)||_{H^s(\Omega_k)} \right). \tag{27}
\]

It remains to bound \( \|\eta_0\|_{H^s(\Omega)}, \|\eta_0\|_{H^s(\Omega)} \) and \( ||\overline{\omega}_0(x)||_{H^s(\Omega_k)} \) in terms of \( \|\omega_0\|_{H^s(\Omega)} \). By solving the ODE (21) directly, we can express \( \eta \) in terms of \( \omega_0 \) explicitly

\[
\eta_0(x, z) = -\gamma e^{\gamma z} \int_0^z e^{-\gamma z'} f(x, z')dz' = \gamma \int_z^\infty e^{-\gamma(z'-z)} f(x, z')dz', \tag{28}
\]
where \( f(x, z) = \omega_0(x, z) - \overline{\omega}_0(x) e^{-\gamma z} \) and we have used the property that
\[
\int_0^\infty f(x, z) e^{-\gamma z} \, dz = 0.
\]

By using integration by parts, we have
\[
\eta_0(x, z) = -\gamma f(x, z) + \gamma^2 \int_z^\infty e^{-\gamma(z' - z)} f(x, z') \, dz' = \gamma \int_z^\infty e^{-\gamma(z' - z)} f_z(x, z') \, dz'.
\] (29)

By induction we can show that for any \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \geq 0 \)
\[
D^\alpha \eta_0 = \gamma \int_z^\infty e^{-\gamma(z' - z)} D^\alpha f(x, z') \, dz'.
\] (30)

Let \( K(z) = \gamma e^{-\gamma z} \chi(z) \) and \( \chi(z) \) be the characteristic function
\[
\chi(z) = \begin{cases} 
0, & z \leq 0, \\
1, & z > 0.
\end{cases}
\] (31)

Then \( D^\alpha \eta_0 \) can be written in the following convolution form:
\[
D^\alpha \eta_0(x, z) = \int_0^\infty K(z' - z) D^\alpha f(x, z') \, dz'.
\] (32)

Using Young’s inequality (see e.g. page 232 of [2]), we obtain:
\[
\| D^\alpha \eta_0 \|_{L^2(\Omega)} \leq \| K(z) \|_{L^1(\mathbb{R}^+)} \| D^\alpha f \|_{L^2(\Omega)}
\leq C(\gamma) \left( \| D^\alpha \omega_0 - (-\gamma)^{\alpha_1} e^{-\gamma z} D^{(\alpha_1, \alpha_2)} \overline{\omega}_0(x) \|_{L^2(\Omega)} \right)
\leq C(\gamma, \alpha) \left( \| D^\alpha \omega_0 \|_{L^2(\Omega)} + \| D^{(\alpha_1, \alpha_2)} \overline{\omega}_0(x) \|_{L^2(\Omega_\delta)} \right). \] (33)

Moreover, we obtain by using the Hölder inequality that
\[
\left\| D^{(\alpha_1, \alpha_2)} \overline{\omega}_0(x) \right\|_{L^2(\Omega_\delta)} = \left( \int_{\Omega_\delta} \left( \int_0^\infty e^{-\gamma z} D^{(\alpha_1, \alpha_2)} \omega_0(x, z) \, dz \right)^2 \, dx \right)^{1/2}
\leq \left( \frac{1}{2\gamma} \int_{\Omega_\delta} \int_0^\infty \left( D^{(\alpha_1, \alpha_2)} \omega_0(x, z) \right)^2 \, dz \, dx \right)^{1/2}
= \frac{1}{\sqrt{2\gamma}} \left\| D^{(\alpha_1, \alpha_2)} \omega_0(x, z) \right\|_{L^2(\Omega)}. \] (34)

Substituting (34) to (33) yields
\[
\| D^\alpha \eta_0 \|_{L^2(\Omega)} \leq C(\gamma, \alpha) \left( \| D^\alpha \omega_0 \|_{L^2(\Omega)} + \| D^{(\alpha_1, \alpha_2)} \omega_0 \|_{L^2(\Omega)} \right), \] (35)

which implies that
\[
\| \eta_0 \|_{H^{s}(\Omega)} \leq C(\gamma, s) \| \omega_0 \|_{H^{s}(\Omega)}, \quad \forall \, s \geq 0.
\] (36)
It follows from (34) that
\[ \| \omega_0(x) \|_{H^s(\Omega_x)} \leq C(\gamma) \| \omega_0 \|_{H^s(\Omega)}, \quad \forall \ s \geq 0. \] (37)

On the other hand, we obtain from the equation for \( \eta_0 \) (21) that
\[ \| \eta_0z \|_{H^s(\Omega)} = \gamma \| f + \eta_0 \|_{H^s(\Omega)} \leq C(\gamma, s) \| \omega_0 \|_{H^s(\Omega)}, \quad \forall \ s \geq 0. \] (38)

Upon substituting (36)-(38) to (27), we obtain
\[ \| \omega \|_{H^s(\Omega)} \leq C(\gamma, s) e^{\nu \gamma^2 t} \| \omega_0 \|_{H^s(\Omega)}, \] (39)
where \( C(\gamma, s) \) is a constant depending on \( \gamma \) and \( s \) only. This proves (12).

To prove (13), we use the classical regularity result for the heat equation with the homogeneous Dirichlet boundary condition to obtain the following estimates for \( t > 0 \):
\[ \| \eta \|_{H^s(\Omega)} \leq C(t) \| \eta_0 \|_{L^2(\Omega)}, \] (40)
\[ \| \eta_z \|_{H^s(\Omega)} \leq C(s, t) \| \eta_0z \|_{L^2(\Omega)}, \] (41)
\[ \| \bar{\omega}(x) \|_{H^s(\Omega_x)} \leq C(s, t) e^{\nu \gamma^2 t} \| \omega_0(x) \|_{L^2(\Omega_x)}, \] (42)
where \( C(s, t) \) is a constant depending on \( s \) and \( t \). By combining (40)-(42) with estimates (36)-(38), we obtain for any \( t > 0 \) that
\[ \| \omega \|_{H^s(\Omega)} \leq C(\gamma, s, t) \left( \| \eta_0z \|_{L^2(\Omega)} + \| \eta_0 \|_{L^2(\Omega)} + e^{\nu \gamma^2 t} \| \omega_0(x) \|_{L^2(\Omega_x)} \right) \]
\[ \leq C(\gamma, s, t) \| \omega_0 \|_{L^2(\Omega)}, \] (43)
where \( C(\gamma, s, t) < \infty \) is a constant depending on \( \gamma, s \) and \( t \). This proves (13) and completes the proof of the Lemma.

We also need the following well-known Sobolev inequality [3].

**Lemma 2.3** Let \( u, v \in H^s(\Omega) \) with \( s > 3/2 \). We have
\[ \| uv \|_{H^s(\Omega)} \leq c \| u \|_{H^s(\Omega)} \| v \|_{H^s(\Omega)}. \] (44)

Now, we are ready to give the proof of Theorem 2.1

**Proof of Theorem 2.1** Let \( v = u^2 \). First, using the definition of the operator \( \mathcal{K} \) (see Definition 2.1), we can rewrite the 3D model with partial viscosity in the following equivalent form:
\[ \begin{cases} v_t &= 4v \mathcal{K}(\omega)_z, \\
\omega_t &= v_z + \nu \Delta \omega, \end{cases} \quad (x, z) \in \Omega = \Omega_x \times (0, \infty), \] (45)

with the initial and boundary conditions given as follows:
\[ \omega|_{\partial \Omega \cap \Gamma} = 0, \quad (\omega_z + \gamma \omega)|_{\Gamma} = 0, \] (46)
\[ \omega|_{t=0} = \omega_0(x, z) \in W^s, \quad v|_{t=0} = v_0(x, z) \in V^{s+1}, \] (47)
where \( V^{s+1} = \{ v \in H^{s+1} : v|_{\partial \Omega} = 0, v_2|_{\partial \Omega} = 0, v_{zz}|_{\partial \Omega} = 0 \} \) and \( W^s = \{ w \in H^s : w|_{\partial \Omega\backslash \Gamma} = 0, (w_z + \gamma w)|_{\Gamma} = 0 \} \).

We note that the condition \( u_0|_{\partial \Omega} = u_0|_{\partial \Omega} = 0 \) implies that \( v_0|_{\partial \Omega} = v_0|_{\partial \Omega} = v_{0zz}|_{\partial \Omega} = 0 \) by using the relation \( v_0 = v_0^2 \). Thus we have \( v_0 \in V^{s+1} \). It is easy to show by using the \( v \)-equation that the property \( u_0|_{\partial \Omega} = u_0|_{\partial \Omega} = 0 \) is preserved dynamically. Thus we have \( v \in V^{s+1} \).

Define \( U = (U_1, U_2) = (v, \omega) \) and \( X = C([0, T]; V^{s+1}) \times C([0, T]; W^s) \) with the norm

\[
\|U\|_X = \sup_{t \in [0,T]} \|U_1\|_{H^{s+1}(\Omega)} + \sup_{t \in [0,T]} \|U_2\|_{H^s(\Omega)}, \quad \forall U \in X
\]

and let \( S = \{ U \in X : \|U\|_X \leq M \} \).

Now, define the map \( \Phi : X \rightarrow X \) in the following way: let \( \Phi(\tilde{v}, \tilde{w}) = (v, \omega) \), then for any \( t \in [0, T] \),

\[
v(x, z, t) = v_0(x, z, t) + 4 \int_0^t \tilde{v}(x, z, t') K(\tilde{w})_z(x, z, t') dt', \quad (48)
\]

\[
\omega(x, z, t) = \mathcal{L}(\tilde{v}, \omega_0; x, z, t), \quad (49)
\]

where \( \omega(x, z, t) = \mathcal{L}(\tilde{v}, \omega_0; x, z, t) \) is the solution of the following equation:

\[
\omega_t = \tilde{v}_z + \nu \Delta \omega, \quad (x, z) \in \Omega = \Omega_x \times (0, \infty),
\]

with the initial and boundary conditions:

\[
\omega|_{\partial \Omega\backslash \Gamma} = 0, \quad (\omega_z + \gamma \omega)|_{\Gamma} = 0, \quad \omega|_{t=0} = \omega_0(x, z).
\]

We use the map \( \Phi \) to define a Picard iteration: \( U^{k+1} = \Phi(U^k) \) with \( U^0 = (v_0, \omega_0) \). In the following, we will prove that there exist \( T > 0 \) and \( M > 0 \) such that

1. \( U^k \in S \), for all \( k \).
2. \( \|U^{k+1} - U^k\|_X \leq \frac{1}{2} \|U^k - U^{k-1}\|_X \), for all \( k \).

Then by the contraction mapping theorem, there exists \( U = (v, \omega) \in S \) such that \( \Phi(U) = U \) which implies that \( U \) is a local solution of the system \( \text{[20]} \) in \( X \).

First, by Duhamel's principle, we have for any \( g \in C([0, T]; V^s) \) that

\[
\mathcal{L}(g, \omega_0; x, z, t) = \mathcal{P}(\omega_0; 0, t) + \int_0^t \mathcal{P}(g; t', t) dt',
\]

where \( \mathcal{P}(g; t', t) = \tilde{g}(x, z, t) \) is defined as the solution of the following initial boundary value problem at time \( t \):

\[
\tilde{g}_t = \nu \Delta \tilde{g}, \quad (x, z) \in \Omega = \Omega_x \times (0, \infty),
\]

with the initial and boundary conditions:

\[
\tilde{g}|_{\partial \Omega\backslash \Gamma} = 0, \quad (\tilde{g}_z + \gamma \tilde{g})|_{\Gamma} = 0, \quad \tilde{g}(x, z, t') = g(x, z, t').
\]
We observe that \( g(x, z, t') \) also satisfies the same boundary condition as \( \omega \) for any \( 0 \leq t' \leq t \) since \( g = v_k^2 \) and \( v_k \in V^{s+1} \).

Now we can apply Lemma 2.1 to conclude that for any \( t' < T \) and \( t \in [t', T] \) we have

\[
\| \mathcal{L}(g, \omega_0; x, z, t) \|_{H^s(\Omega)} \leq C(\gamma, s)e^{\nu \gamma (t-t')} \| g(x, z, t') \|_{H^s(\Omega)},
\]

which implies the following estimate for \( \mathcal{L} \): for all \( t \in [0, T] \),

\[
\| \mathcal{L}(g, \omega_0; x, z, t) \|_{H^s(\Omega)} \leq C(\gamma, s)e^{\nu \gamma t} \left( \| \omega_0 \|_{H^s(\Omega)} + t \sup_{t' \in [0,t]} \| g(x, z, t') \|_{H^s(\Omega)} \right).
\]

Further, by using Lemma 2.1 and the above estimate (55) for the sequence \( U^k = (v_k, \omega_k) \), we get the following estimate:

\[
\| v^{k+1} \|_{H^{s+1}(\Omega)} \leq \| v_0 \|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0, T]} \| v^k(x, z, t) \|_{H^{s+1}(\Omega)} \sup_{t \in [0, T]} \| \mathcal{K}(\omega^k, z)(x, z, t) \|_{H^{s+1}(\Omega)},
\]

\[
\| v^{k+1} \|_{H^{s+1}(\Omega)} \leq \| v_0 \|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0, T]} \| v^k(x, z, t) \|_{H^{s+1}(\Omega)} \sup_{t \in [0, T]} \| \omega^k(x, z, t) \|_{H^s(\Omega)}, \quad \forall t \in [0, T].
\]

Next, we will use mathematical induction to prove that if \( T \) satisfies the following inequality:

\[
8C(\gamma, s)Te^{\nu \gamma T} \left( \| \omega_0 \|_{H^s(\Omega)} + 2T \| v_0 \|_{H^{s+1}(\Omega)} \right) \leq 1
\]

then for all \( k \geq 0 \) and \( t \in [0, T] \), we have that

\[
\| v^k \|_{H^{s+1}(\Omega)} \leq 2 \| v_0 \|_{H^{s+1}(\Omega)},
\]

\[
\| \omega^k \|_{H^s(\Omega)} \leq C(\gamma, s)e^{\nu \gamma T} \left( \| \omega_0 \|_{H^s(\Omega)} + 2T \| v_0 \|_{H^{s+1}(\Omega)} \right).
\]

First of all, \( U^0 = (v_0, \omega_0) \) satisfies (59) and (60). Assume \( U^k = (v^k, \omega^k) \) has this property, then for \( U^{k+1} = (v^{k+1}, \omega^{k+1}) \), using (56) and (57), we have

\[
\| v^{k+1} \|_{H^{s+1}(\Omega)} \leq \| v_0 \|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0, T]} \| v^k(x, z, t) \|_{H^{s+1}(\Omega)} \sup_{t \in [0, T]} \| \omega^k(x, z, t) \|_{H^s(\Omega)}
\]

\[
\| v^{k+1} \|_{H^{s+1}(\Omega)} \leq \| v_0 \|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0, T]} \| v^k(x, z, t) \|_{H^{s+1}(\Omega)} \left( 1 + 8C(\gamma, s)Te^{\nu \gamma T} \left( \| \omega_0 \|_{H^s(\Omega)} + 2T \| v_0 \|_{H^{s+1}(\Omega)} \right) \right)
\]

\[
\| v^{k+1} \|_{H^{s+1}(\Omega)} \leq 2 \| v_0 \|_{H^{s+1}(\Omega)}, \quad \forall t \in [0, T].
\]

\[
\| \omega^{k+1} \|_{H^s(\Omega)} \leq C(\gamma, s)e^{\nu \gamma T} \left( \| \omega_0 \|_{H^s(\Omega)} + 2T \| v_0 \|_{H^{s+1}(\Omega)} \right), \quad \forall t \in [0, T].
\]
Then, by induction, we prove that for any \( k \geq 0 \), \( U^k = (v^k, \omega^k) \) is bounded by (59) and (60).

We want to point that there exists \( T > 0 \) such that the inequality (58) is satisfied. One choice of \( T \) is given as following:

\[
T_1 = \min \left\{ \left[ 8C(\gamma, s)e^{\nu_2} \left( \| \omega_0 \|_{H^*(\Omega)} + 2\| v_0 \|_{H^{s+1}(\Omega)} \right) \right]^{-1}, 1 \right\}. \tag{63}
\]

Using the choice of \( T \) in (63), we can choose \( M = 2\| v_0 \|_{H^{s+1}(\Omega)} + C(\gamma, s)e^{\nu_2} \left( \| \omega_0 \|_{H^*(\Omega)} + 2\| v_0 \|_{H^{s+1}(\Omega)} \right) \), then we have \( U^k \in S \), for all \( k \).

Next, we will prove that \( \Phi \) is a contraction mapping for some small \( 0 < T \leq T_1 \).

First of all, by using Lemmas 2.1 and 2.3 we have

\[
\left\| v^{k+1} - v^k \right\|_{H^{s+1}(\Omega)} = \left\| \int_0^t v^k(x, t')K(\omega^k)z(x, t')dt' - \int_0^t v^{k-1}(x, t')K(\omega^{k-1})z(x, t')dt' \right\|_{H^{s+1}(\Omega)} \\
\leq \left\| \int_0^t (v^k - v^{k-1})(x, t')K(\omega^k)z(x, t')dt' \right\|_{H^{s+1}(\Omega)} \\
+ \left\| \int_0^t v^{k-1}(x, t') (K(\omega^k)z - K(\omega^{k-1})z)(x, t')dt' \right\|_{H^{s+1}(\Omega)} \\
\leq T \sup_{t \in [0, T]} \left\| v^k - v^{k-1} \right\|_{H^{s+1}(\Omega)} \sup_{t \in [0, T]} \left\| K(\omega^k)z \right\|_{H^{s+1}(\Omega)} \\
+ T \sup_{t \in [0, T]} \left\| v^{k-1} \right\|_{H^{s+1}(\Omega)} \sup_{t \in [0, T]} \left\| K(\omega^k - \omega^{k-1})z \right\|_{H^{s+1}(\Omega)} \\
\leq MT \left( \sup_{t \in [0, T]} \left\| v^k - v^{k-1} \right\|_{H^{s+1}(\Omega)} + \sup_{t \in [0, T]} \left\| \omega^k - \omega^{k-1} \right\|_{H^*(\Omega)} \right). \tag{64}
\]

On the other hand, Lemma 2.2 and (51) imply

\[
\left\| \omega^{k+1} - \omega^k \right\|_{H^*(\Omega)} = \left\| \mathcal{L}(v^k, \omega_0; x, t) - \mathcal{L}(v^{k-1}, \omega_0; x, t) \right\|_{H^*(\Omega)} \\
\leq \left\| \int_0^t \mathcal{P}(v^k - v^{k-1}; t', t)dt' \right\|_{H^*(\Omega)} \\
\leq TC(\gamma, s)e^{\nu_2} \sup_{t \in [0, T]} \left\| v^k - v^{k-1} \right\|_{H^*(\Omega)} \\
\leq TC(\gamma, s)e^{\nu_2}T \sup_{t \in [0, T]} \left\| v^k - v^{k-1} \right\|_{H^{s+1}(\Omega)}. \tag{65}
\]

Let

\[
T = \min \left\{ \left[ 8C(\gamma, s)e^{\nu_2} \left( \| \omega_0 \|_{H^*(\Omega)} + 2\| v_0 \|_{H^{s+1}(\Omega)} \right) \right]^{-1}, \left[ 2C(\gamma, s)e^{\nu_2} \right]^{-1}, \frac{1}{2M} \right\}. \tag{66}
\]

Then, we have

\[
\left\| U^{k+1} - U^k \right\|_X \leq \frac{1}{2} \left\| U^k - U^{k-1} \right\|_X.
\]
This proves that the sequence $U_k$ converges to a fixed point of the map $\Phi : X \to X$, and
the limiting fixed point $U = (v, \omega)$ is a solution of the 3D model with partial viscosity.
Moreover, by passing the limit in (59)-(60), we obtain the following \textit{a priori} estimate for the solution $v$ and $\omega$:

$$
\|v\|_{H^{s+1}(\Omega)} \leq 2\|v_0\|_{H^{s+1}(\Omega)},
$$

(67)

$$
\|\omega\|_{H^s(\Omega)} \leq C(\gamma, s) e^{\gamma^2 T} \left( \|\omega_0\|_{H^s(\Omega)} + 2T \|v_0\|_{H^{s+1}(\Omega)} \right),
$$

(68)

for $0 \leq t \leq T$ with $T$ defined in (66).

It remains to show that the smooth solution of the 3D model with partial viscosity is unique. Let $(v_1, \omega_1)$ and $(v_2, \omega_2)$ be two smooth solutions of the 3D model with the same initial data and satisfying $\|v_i\|_{H^{s+1}(\Omega)} \leq M$ and $\|\omega_i\|_{H^s(\Omega)} \leq M$ for $i = 1, 2$ and $0 \leq t \leq T$, where $M$ is a positive constant depending on the initial data, $\gamma$, $s$, and $T$. Since $s > 3/2$, the Sobolev embedding theorem [1] implies that

$$
\|v_i\|_{L^\infty(\Omega)} \leq \|v_i\|_{H^{s+1}(\Omega)} \leq M, \quad i = 1, 2,
$$

(69)

$$
\|K(\omega_i)_z\|_{L^\infty(\Omega)} \leq \|K(\omega_i)_z\|_{H^s(\Omega)} \leq C_s \|\omega_i\|_{H^s(\Omega)} \leq C_s M, \quad i = 1, 2.
$$

(70)

Let $v = v_1 - v_2$ and $\omega = \omega_1 - \omega_2$. Then $(v, \omega)$ satisfies

$$
\begin{cases}
  v_t = 4vK(\omega_1)_z + 4v_2K(\omega)_z \\
  \omega_t = v_z + \nu \Delta \omega
\end{cases}, \quad (x, z) \in \Omega = \Omega_x \times (0, \infty),
$$

(71)

with $\omega|_{\partial \Omega_1} = 0$, $\omega_z|_{\partial \Omega_1} = 0$, and $\omega|_{t=0} = 0$, $v|_{t=0} = 0$. By using (69)-(70), and proceeding as the uniqueness estimate for the heat equation in [11], we can derive the following estimate for $v$ and $\omega$:

$$
\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \leq C_1 (\|v\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}),
$$

(72)

$$
\frac{d}{dt}\|\omega\|^2_{L^2(\Omega)} \leq C_2 (\|v\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}),
$$

(73)

where $C_i$ ($i = 1, 2, 3$) are positive constants depending on $M$, $\nu$, $\gamma$, $C_s$. In obtaining the estimate for (73), we have performed integration by parts in the estimate of the $v_z$-term in the $\omega$-equation and absorbing the contribution from $\omega_z$ by the diffusion term. There is no contribution from the boundary term since $v_z|_{z=0} = 0$. We have also used the property $\|K(\omega)_z\|_{L^2(\Omega)} \leq C_s \|\omega\|_{L^2(\Omega)}$, which can be proved directly by following the argument in the Appendix of [5]. Since $v_0 = 0$ and $\omega_0 = 0$, the Gronwall inequality implies that $\|v\|_{L^2(\Omega)} = \|\omega\|_{L^2(\Omega)} = 0$ for $0 \leq t \leq T$. Furthermore, since $v \in H^{s+1}$ and $\omega \in H^s$ with $s > 3/2$, $v$ and $\omega$ are continuous. Thus we must have $v = \omega = 0$ for $0 \leq t \leq T$. This proves the uniqueness of the smooth solution for the 3D model. \qed
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