Quantum Dynamical Algebra $SU(1,1)$ in One-Dimensional Exactly Solvable Potentials

Ming-Guang Hu$^1$ and Jing-Ling Chen$^{1,2}$

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In this paper, we establish the underlying quantum dynamical algebra $SU(1,1)$ for some one-dimensional exactly solvable potentials by using the shift operators method. The connection between $SU(1,1)$ algebra and the radial Hamiltonian problems is also discussed.

KEY WORDS: quantum dynamical algebra; exactly solvable potentials.

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1. INTRODUCTION

Exactly solvable models are very important in physics. They are important not just from a theoretical point of view but also from the experimentalist’s perspective because in such cases theoretical results and experimental results can be compared without ambiguity. Exactly solvable potentials especially including one-dimensional or spherically symmetric ones have extensively played the indispensable roles in condensed matter, biophysics, nuclear physics, quantum optics and solid-state physics. Those familiar potentials embrace, for instance, the typical harmonic oscillator potential, conventional Coulomb potential, one dimensional Morse potential (1929), the Rosen–Morse potential (1932), the Pöschl–Teller potential (Pöschl and Teller, 1933; Grosche, 1989), the Hulthén potential, the Kratzer’s molecular potential, and the famous Yukawa potential, etc. (Flügge, 1974). Thereinto, the Morse potential and Kratzer’s molecular potential are utilized to describe the anharmonicity and bond dissociation of diatomic molecules. Another noticeable potential with short-range properties is the Pöschl–Teller potential, of which the generalized coherent states (Crawford and Vrsay, 1998), nonlinear properties (Chen et al., 1998; Quesne, 1999), and

$^1$Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, P. R. China.

$^2$To whom correspondence should be addressed; e-mail: chenjl@nankai.edu.cn.
supersymmetric extension (Díaz et al., 1999) have been widely studied. Additionally, in supersymmetric (SUSY) quantum mechanics, the shaped invariant potentials have also been mooted (Gendenstein, 1983; Dutt et al., 1986; Cooper et al., 1988; Khare and Sukhatme, 1988; Dabrowska et al., 1988) In particular, a large class of such potentials is the Natanzon class (Natanzon, 1979; Fukui and Aizawa, 1993; Dutt et al., 1993; Roychoudhury et al., 2001; Znojil et al., 2001).

The preference to deal with those potentials in modern quantum mechanics adopts the abstract formulation and stresses the special nature of wave mechanics. However, the machinery of wave mechanics such as choice of coordinate system, separation of variables, boundary conditions, single-valuedness can obscure the underlying quantum mechanical principles and complicate the analysis. In this way, the operator methods which mainly consist of noncommutative algebra and the shift operator factorization to some extent can avoid these flaws. Algebraic methods (Kamran and Olver, 1990; Celeghini et al., 1985) exploring the underlying Lie symmetry and its associated algebra have been widely used to study many of these exactly solvable potentials, for instance, Darboux transformation, Infeld–Hull transformation (Infeld and Hull, 1951; Stahlhofen and Bleuler, 1989), Mielnick facorization (1984), SUSY quantum mechanics, inverse scattering theory (Hoppe, 1992), and intertwining technique (Díaz et al., 1999). Operators methods with shift operators (i.e., raising and lowering operators) for the Hamiltonian of exactly solvable potentials have been presented in De Lange and Raab (1991). In 1993, nonlinear deformations of $SU(2)$ and $SU(1,1)$ algebras (NLDA) with two deforming functions $g(J_0)$ and $f(J_0)$ are introduced by Delbecq and Quesne (1993). Subsequently, in 1998 Chen et al. (1998) applied successfully the nonlinear deformation algebra to a physical system with Pöschl–Teller potential, and furthermore they tuned the NLDA naturally to a linear algebra $SU(1,1)$ by readjusting the generators (Chen et al., 1998; Quesne, 1999); in other words, the underlying quantum dynamical algebra (QDA) $SU(1,1)$ was revealed for the Pöschl–Teller potential, and one may note that the crucial step to establish the QDA is firstly to construct the shift operators for the potentials. In 2000, following the similar method a unified approach that emphasized on constructing the shift operators of exactly one-dimensional solvable potentials was provided (Ge et al., 2000), but without pointing out the concomitant algebraic structures. The purpose of this paper is to go beyond Ge et al. (2000) and establish systematically the corresponding underlying quantum dynamical algebra for the potentials discussed. Interestingly and also surprisingly, these exactly solvable potentials possess the same simple algebra $SU(1,1)$.

The paper is organized as follows: In Sec. 2, we first briefly review the Pöschl–Teller potential problem discussed in Chen et al. (1998) and Quesne (1999), and then present a systematic method to construct the linear algebra $SU(1, 1)$ for one-dimensional exactly solvable potentials. In Sec. 3, quantum dynamical algebra $SU(1, 1)$ is established for some one-dimensional exactly solvable potentials. In