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WHEN IS A REDUCTIVE GROUP SCHEME LINEAR?

PHILIPPE GILLE

Abstract. We show that a reductive group scheme over a base scheme $S$ admits a faithful linear representation if and only if its radical torus is isotrivial; that is, it splits after a finite étale cover.

to the 75-th anniversary of Gopal Prasad

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1. Introduction

Let $S$ be a scheme. Let $G$ be an $S$–group scheme. It is natural to ask whether $G$ is linear; that is, there exists a group monomorphism $G \to \text{GL}(E)$ where $E$ is a locally free $O_S$-module of finite rank. In particular, $G$ admits a faithful representation on $E$. This holds for affine algebraic groups over a field [4, II, §2.3.3].

In the case $S$ is locally noetherian and $G$ is of multiplicative type of constant type and of finite type, Grothendieck has shown that $G$ is linear if and if $G$ is isotrivial, i.e. $G$ is split by a finite étale extension of $S$ [11, XI.4.6]. In particular there exist non linear tori of rank 2 over the local ring (at a node) of a nodal algebraic curve (ibid, X.1.6). Firstly we extend that criterion over an arbitrary base by using Azumaya and separable $O_S$–algebras (Theorem 3.3).

Secondly we deal with the case $G$ reductive; that is, $G$ is smooth affine with reductive (connected) geometric fibers. In this case a faithful representation is necessarily a closed immersion [11, XVI.1.5]. Positive results on the linearity question are due to M. Raynaud [11, VI.B] and R. Thomason [13, 3.1] which is essentially the implication $(i) \implies (ii)$ in the theorem below.

We can restrict our attention to the case when $G$ is of constant type (recall that the type is a locally constant function on $S$); this implies that there exists a Chevalley $\mathbb{Z}$–group scheme $G$ such that $G$ is locally isomorphic to $G_S$ for the étale topology [11, XXII.2.3, 2.5]. A short version of our main result is the following.

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Theorem 1.1. The following are equivalent:

(i) The radical torus rad(\mathfrak{G}) is isotrivial;

(ii) \mathfrak{G} is linear.

Furthermore if \(S\) is affine, the above are equivalent to

(iii') there exists a closed immersion \(i: \mathfrak{G} \to \text{GL}_n\) with \(n \geq 1\) which is a homomorphism.

We recall that \(\text{rad}(\mathfrak{G})\) is the maximal central subtorus of \(\mathfrak{G}\) [11, XXIV.4.3.6] and that (i) means that \(\text{rad}(\mathfrak{G})\) splits after passing to a finite étale cover \(S' \to S\). In the noetherian setting, a variant of the implication (i) \(\implies\) (ii) has been shown by Margaux who furthermore provided an \(\text{Aut}(\mathfrak{G})\)-equivariant representation [9]. Note that condition (i) depends only on the quasi-split form of \(\mathfrak{G}\) and also that it is always satisfied if \(\mathfrak{G}\) is semisimple or if \(\text{rad}(\mathfrak{G})\) is of rank one. Furthermore, if \(S\) is a semilocal scheme, Demazure’s characterization of isotrivial group schemes [11, XXIV.3.5] permits us to deduce that \(\mathfrak{G}\) is isotrivial if and only if \(\mathfrak{G}\) is linear, see section 5.

Finally for \(S = \text{Spec}(R)\) with \(R\) noetherian, we complete Thomason’s approach by showing that linearity for \(\mathfrak{G}\) is equivalent to the resolution property (Th. 6.2).

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2. Definitions and basic facts

2.1. Notation. We use mainly the terminology and notation of Grothendieck-Dieudonné [5, §9.4 and 9.6] which agrees with that of Demazure-Grothendieck used in [11, Exp. I.4].

(a) Let \(S\) be a scheme and let \(\mathcal{E}\) be a quasi-coherent sheaf over \(S\). For each morphism \(f: T \to S\), we denote by \(\mathcal{E}_T = f^*(\mathcal{E})\) the inverse image of \(\mathcal{E}\) by the morphism \(f\). We denote by \(V(\mathcal{E})\) the affine \(S\)-scheme defined by \(V(\mathcal{E}) = \text{Spec}(\text{Sym}^*\mathcal{E})\); it represents the \(S\)-functor \(Y \mapsto \text{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{O}_Y)\) [5, 9.4.9].

(b) We assume now that \(\mathcal{E}\) is locally free of finite rank and denote by \(\mathcal{E}^\vee\) its dual. In this case the affine \(S\)-scheme \(V(\mathcal{E})\) is of finite presentation (ibid, 9.4.11); also the \(S\)-functor \(Y \mapsto H^0(Y, \mathcal{E}_Y) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_Y)\) is representable by the affine \(S\)-scheme \(V(\mathcal{E}^\vee)\) which is also denoted by \(W(\mathcal{E})\) [11, I.4.6].

The above applies to the locally free quasi-coherent sheaf \(\mathcal{E}_{nd}(\mathcal{E}) = \mathcal{E}^\vee \otimes_{\mathcal{O}_S} \mathcal{E}\) over \(S\) so that we can consider the affine \(S\)-scheme \(V(\mathcal{E}_{nd}(\mathcal{E}))\) which is an \(S\)-functor in associative and unital algebras [5, 9.6.2]. Now we consider the \(S\)-functor \(Y \mapsto \text{Aut}_{\mathcal{O}_S}(\mathcal{E}_Y)\). It is representable by an open \(S\)-subscheme of \(V(\mathcal{E}_{nd}(\mathcal{E}))\) which is denoted by \(\text{GL}(\mathcal{E})\) (loc. cit., 9.6.4).
(c) If \( \mathcal{B} \) is a locally free \( \mathcal{O}_S \)–algebra (unital, associative) of finite rank, we recall that the functor of invertible elements of \( \mathcal{B} \) is representable by an affine \( S \)–group scheme which is denoted by \( \text{GL}_1(\mathcal{B}) \) [3, 2.4.2.1].

For separable and Azumaya algebras, we refer to [8]. Note that in [3, §2.5.1], separable algebras are supposed furthermore to be locally free of finite rank.

If \( \mathcal{B} \) is a separable \( \mathcal{O}_S \)–algebra which is a locally free \( \mathcal{O}_S \)–algebra of finite rank, then \( \text{GL}_1(\mathcal{B}) \) it is a reductive \( S \)–group scheme [3, 3.1.0.50].

(d) We use the theory and terminology of tori and multiplicative group schemes of [11]; see also Oesterlé’s survey [10].

2.2. Finite étale covers. The next lemma is a consequence of the equivalence of categories describing finite étale \( \mathcal{O}_S \)–algebra of rank \( N \) [3, §2.5.2]; it admits a simple direct proof.

**Lemma 2.1.** Let \( N \) be a positive integer and let \( \mathcal{C} \) be a finite étale \( \mathcal{O}_S \)–algebra of rank \( N \). Then there exists a finite étale cover \( T \) of \( S \) of degree \( N! \) such that \( \mathcal{C} \otimes_{\mathcal{O}_S} \mathcal{O}_T \rightarrow (\mathcal{O}_T)^N \).

**Proof.** We proceed by induction on \( N \), the case \( N = 1 \) being obvious. We put \( S' = \text{Spec}(\mathcal{O}_C) \), this is a finite étale cover of \( S \) of degree \( N \). Since the diagonal map \( S' \rightarrow S' \times_S S' \) is closed and open [6, 4.17.4.2] there exists a decomposition \( \mathcal{C} \otimes_{\mathcal{O}_S} \mathcal{O}_S' = \mathcal{O}_{S'} \times \mathcal{C}' \) where \( \mathcal{C}' \) is a finite étale \( \mathcal{O}_{S'} \)–algebra of rank \( N - 1 \). Applying the induction process to \( \mathcal{C}' \) provides a finite étale cover \( T \) of \( S' \) of degree \( (N - 1)! \) such that \( \mathcal{C}' \otimes_{\mathcal{O}_{S'}} \mathcal{O}_T \cong (\mathcal{O}_T)^{N-1} \). Thus \( \mathcal{C} \otimes_{\mathcal{O}_S} \mathcal{O}_T = \mathcal{O}_T \times (\mathcal{O}_T)^{N-1} \) and \( T \) is a finite étale cover of \( S \) of degree \( N! = N \times (N - 1)! \).

\( \square \)

2.3. Isotriviality. [11, XXIV.4] Let \( \mathcal{H} \) be a fppf \( S \)–sheaf in groups and let \( \mathcal{X} \) be a \( \mathcal{H} \)–torsor. We say that \( \mathcal{X} \) is **isotrivial** if there exists a finite étale cover \( S' \) of \( S \) which trivializes \( \mathcal{X} \); that is, satisfying \( \mathcal{X}(S') \neq \emptyset \).

The notion of locally isotrivial (with respect to the Zariski topology) is then clear and there is also the following variant of **semilocally isotrivial**.

We say that \( \mathcal{X} \) is **semilocally isotrivial** if for each subset \( \{s_1, \ldots, s_n\} \) of points of \( S \) contained in an affine open subset of \( S \), there exists an open subscheme \( U \) of \( S \) containing \( s_1, \ldots, s_n \) such that \( \mathcal{X} \times_S U \) is isotrivial over \( U \).

A reductive \( S \)–group scheme \( \mathfrak{G} \) is **isotrivial** if it is split by a finite étale cover \( S' \) of \( S \). An isotrivial reductive \( S \)–group scheme \( \mathfrak{G} \) is necessarily of constant type. If \( \mathfrak{G} \) is of constant type with underlying Chevalley group scheme \( G \), \( \mathfrak{G} \) is isotrivial if and only if the \( \text{Aut}(G) \)–torsor \( \text{Isom}(G_S, \mathfrak{G}) \) is isotrivial.

2.4. Rank one tori. The simplest case is that of \( G = \mathbb{G}_{m,S} \), the split \( S \)–torus of rank 1. The \( S \)–functor \( S' \mapsto \text{Hom}_{S'\text{-gp}}(\mathbb{G}_{m,S'}, \mathbb{G}_{m,S'}) \) is representable by the constant \( S \)–group scheme \( \mathbb{Z}_S \) [11, VIII.1.5]. It follows that the \( S \)–functor \( S' \mapsto \text{Isom}_{S'\text{-gp}}(\mathbb{G}_{m,S'}, \mathbb{G}_{m,S'}) \) is representable by the constant \( S \)–group scheme \( (\mathbb{Z}/2\mathbb{Z})_S = \text{Aut}_{S\text{-gp}}(\mathbb{Z}_S) \). On the other hand, \((\mathbb{Z}/2\mathbb{Z})_S \) is the automorphism group of the split
étale cover $S \sqcup S \to S$ of degree 2. By definition an $S$–torus of rank one is a form of $\mathbb{G}_m$ for the fpqc topology; in the other hand, a degree 2 étale cover is a form of $S \sqcup S$ for the finite étale topology (see for example Lemma 2.1) and a fortiori for the fpqc topology.

According to the faithfully flat descent technique (e.g. [11, XXIV.1.17]), there is then an equivalence of categories between the groupoid of rank one tori over $S$ and a fortiori for the fpqc topology.
Lemma 2.3. Let $\mathcal{G}$ be an $S$–group scheme and let $S'$ be a finite locally free cover of $S$. Then $\mathcal{G}$ is linear if and only if $\mathcal{G} \times_S S'$ is linear.

Proof. We denote by $p : S' \to S$ the structure map. If $\mathcal{G}$ is linear, then $\mathcal{G} \times_S S'$ is linear. Conversely we assume that there exists a monomorphism $i : \mathcal{G} \times_S S' \to \text{GL}(\mathcal{E}')$ where $\mathcal{E}'$ is a locally free $\mathcal{O}_{S'}$–module of finite rank. We put $\mathcal{E} = p_*(\mathcal{E}')$, this is a locally free $\mathcal{O}_S$–module of finite rank. We consider the sequence of $S$–functors in $S$–groups

$$\mathcal{G} \to R_{S'/S}(\mathcal{G} \times_S S') \xrightarrow{R_{S'/S}(i)} R_{S'/S}(\text{GL}(\mathcal{E}')) \to \text{GL}(\mathcal{E})$$

where $R_{S'/S}$ stands for the Weil restriction and the first map is the diagonal map which is a monomorphism. Since the Weil restriction for $S'/S$ transforms monomorphisms into monomorphisms, the map $R_{S'/S}(i)$ is also a monomorphism and so is the last map since $R_{S'/S}(\text{GL}(\mathcal{E}'))(T) \subset \text{GL}(\mathcal{E})(T)$ corresponds to automorphisms of $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ which are $\mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_T$–linear. Since all maps are monomorphisms, we conclude that $\mathcal{G}$ is linear. \hfill \Box

3. Tori and group of multiplicative type

3.1. Maximal tori of linear groups. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}_S$–algebra. We consider the reductive $S$–group scheme $\text{GL}_1(\mathcal{A})$. Let $\mathcal{B} \subset \mathcal{A}$ be a separable $\mathcal{O}_S$–subalgebra of $\mathcal{A}$ which is a locally free $\mathcal{O}_S$–module of finite rank and which is locally a direct summand of $\mathcal{A}$ as $\mathcal{O}_S$–module. We get a monomorphism of reductive $S$–group schemes $\text{GL}_1(\mathcal{B}) \to \text{GL}_1(\mathcal{A})$. In particular, if $\mathcal{B}$ is commutative, $\mathcal{B}$ is a finite étale $\mathcal{O}_S$–algebra and $\text{GL}_1(\mathcal{B})$ is a torus. We come now to Grothendieck's definition of maximal étale subalgebras.

Definition 3.1. [7, Déf. 5.6]. We say that a finite étale $\mathcal{O}_S$–subalgebra $\mathcal{C} \subset \mathcal{A}$ is maximal if $\mathcal{C}$ is locally a direct summand of $\mathcal{A}$ as $\mathcal{O}_S$–module and if the rank of $\mathcal{C} \otimes_{\mathcal{O}_S} \kappa(s)$ is the degree of $\mathcal{A}_s \otimes_{\mathcal{O}_S} \kappa(s)$ for each $s \in S$.

If $\mathcal{C} \subset \mathcal{A}$ is maximal finite étale $\mathcal{O}_S$–subalgebra of $\mathcal{A}$, then the torus $\text{GL}_1(\mathcal{C})$ is a maximal torus of $\text{GL}_1(\mathcal{A})$ since it is the case on geometric fibers. According to [7, §7.5], all maximal $S$–tori of $\text{GL}_1(\mathcal{A})$ occur in that manner; this is part (3) of the following enlarged statement.

Proposition 3.2. Let $\mathcal{G}$ be a subgroup scheme of multiplicative type of $\text{GL}_1(\mathcal{A})$ and put $\mathcal{B} = \mathcal{A}^\circ$, the centralizer subalgebra of $\mathcal{G}$.

(1) $\mathcal{B}$ is a separable $\mathcal{O}_S$–algebra which is locally a direct summand of $\mathcal{A}$ as $\mathcal{O}_S$–module.

(2) Let $\mathcal{C}$ be the center of $\mathcal{B}$; this is a finite étale $\mathcal{O}_S$–algebra of positive rank which is locally a direct summand of $\mathcal{B}$ (and $\mathcal{A}$) as $\mathcal{O}_S$–module. We have the closed immersions

$$\mathcal{G} \subset \text{GL}_1(\mathcal{C}) \subset \text{GL}_1(\mathcal{A}).$$

(3) If $\mathcal{G}$ is a maximal torus of $\text{GL}_1(\mathcal{A})$, then $\mathcal{G} = \text{GL}_1(\mathcal{C})$ and $\mathcal{C}$ is a maximal finite étale $\mathcal{O}_S$–subalgebra of $\mathcal{A}$. 

(4) If $S$ is of constant type, then $S$ is isotrivial.

Proof. According to [11, IX.2.5], the map $\mathcal{G} \to \text{GL}_1(\mathcal{A})$ is a closed immersion so that $S$ is of finite type over $S$. As a preliminary observation we notice that (1), (2), (3) are local for the étale topology. We can assume that $S = \text{Spec}(R)$, $\mathcal{A} = M_n(R)$ and that $\mathcal{G} = D(M)$ for $M$ an abelian group. Note that $M$ is finitely generated since $S$ is of finite type (ibid, VII.2.1.b).

(1) We consider the $M$–grading $R^n = \bigoplus_{m \in M} R^n_m$. The $R$–modules $(R^n_m)_{m \in M}$ are finitely generated projective so locally free of finite rank. There is a finite subset $M' \subset M$ such that $R^n = \bigoplus_{m \in M'} R^n_m$. Then $B = \prod_{m \in M'} \text{End}_R(R^n_m)$ and each $\text{End}_R(R^n_m)$ is a separable $R$–algebra which is locally free of finite rank [8, III, example 2.8]. Since a product of separable algebras is a separable algebra (ibid, III, proposition 1.7), it follows that $B$ is a separable $R$–algebra. Furthermore $B$ is locally a direct summand of $M_n(R)$ as $R$–module.

(2) Let $C$ be the center of $B$. We have $R$–monomorphisms of groups $S \subset \text{GL}_1(C) \subset \text{GL}_1(\mathcal{A})$.

(3) We assume that $\mathcal{G}$ is a maximal torus of $\text{GL}_1(\mathcal{A})$ and want to establish that $\mathcal{G} = \text{GL}_1(C)$. So $\text{GL}_1(C)$ is an $R$–torus containing $\mathcal{T}$ and since maximality holds also in the naive sense [11, XII.1.4], we conclude that $\mathcal{T} = \text{GL}_1(C)$.

(4) We come back to the initial setting (i.e. without localizing). We want to show that $\mathcal{G}$ is isotrivial. According to [5, ch.0, 5.4.1], for each integer $l \geq 0$, $S_l = \{ s \in S \mid \text{rank}(C_{\kappa(s)}) = l \}$ is an open subset of $S$ so that we have a decomposition in clopen subschemes $S = \bigsqcup_{l \geq 0} S_l$. Without loss of generality we can assume that $C$ is locally free of rank $l$. Lemma 2.1 provides a finite étale cover $S'$ of $S$ such that $C \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \cong (\mathcal{O}_{S'})^l$. Hence $\text{GL}_1(C) \times_S S \cong (\mathbb{G}_m)^l \times_S S'$. It follows that $\mathcal{G} \times_S S'$ is a subgroup $S'$–scheme of $(\mathbb{G}_m)^l \times_S S'$ of multiplicative type. According to [11, IX.2.11.(i)] there exists a partition in clopen subschemes $S' = \bigsqcup_{i \in I} S'_i$ such that each $\mathcal{G} \times_S S'_i$ is diagonalizable. Since $\mathcal{G} \times_S S'$ is of constant type we conclude that $\mathcal{G} \times_S S'$ is diagonalizable. \hfill \Box

3.2. Characterization of isotrivial groups of multiplicative type.

Theorem 3.3. Let $\mathcal{G}$ be an $S$–group scheme of multiplicative type of finite type and of constant type. Then the following are equivalent:

(i) $\mathcal{G}$ is isotrivial;

(ii) $\mathcal{G}$ is linear;

(iii) $\mathcal{G}$ is an $S$–subgroup scheme of an $S$–group scheme $\text{GL}_1(\mathcal{A})$ where $\mathcal{A}$ is an Azumaya $\mathcal{O}_S$–algebra.

Proof. (i) $\implies$ (ii). This follows from Lemma 2.3.
(ii) $\implies$ (iii). By definition we have a monomorphism $\mathcal{G} \to \text{GL}(\mathcal{E})$ where $\mathcal{E}$ is a locally free $O_S$–module of finite rank. Since $\text{End}_{O_S}(\mathcal{E})$ is an Azumaya $O_S$–algebra, we get (iii).

(iii) $\implies$ (i). This follows from Proposition 3.2.(4).

Examples 3.4.
(a) Grothendieck constructed a scheme $S$ and an $S$–torus $\mathcal{T}$ which is locally trivial of rank 2 but which is not isotrivial [11, §X.1.6] (e.g. $S$ consists of two copies of the projective line over a field pinched at 0 and $\infty$). Theorem 3.3 shows that such an $S$–torus is not linear.

(b) Also there exists a local ring $R$ and an $R$–torus $\mathcal{T}$ of rank 2 which is not isotrivial [11, §1.6]; the ring $R$ can be taken as the local ring of an algebraic curve at a double point. Theorem 3.3 shows that such an $R$–torus is not linear.

4. Reductive case

For stating the complete version of our main result, we need more notation. As in the introduction, $\mathcal{G}$ is a reductive $S$–group scheme of constant type and $G$ is the underlying Chevalley $\mathbb{Z}$–group scheme. We denote by $\text{Aut}(G)$ the automorphism group scheme of $G$ and we have an exact sequence of $\mathbb{Z}$–group schemes [11, th. XXIV.1.3]

$$1 \to G_{ad} \to \text{Aut}(G) \to \text{Out}(G) \to 1.$$ 

We remind the reader of the representability of the fppf sheaf $\text{Isom}(G_S, \mathcal{G})$ by a $\text{Aut}(G)_S$–torsor $\text{Isom}(G_S, \mathcal{G})$ defined in [11, XXIV.1.8]. The contracted product

$$\text{Isomext}(G_S, \mathcal{G}) := \text{Isom}(G_S, \mathcal{G}) \wedge^{\text{Aut}(G)_S} \text{Out}(G)_S$$

is an $\text{Out}(G)_S$–torsor (ibid, 1.10) which encodes the isomorphism class of the quasi-split form of $\mathcal{G}$.

Theorem 4.1. The following are equivalent:

(i) The torus $\text{rad}(\mathcal{G})$ is isotrivial;

(ii) the $\text{Out}(G)_S$–torsor $\text{Isomext}(G_S, \mathcal{G})$ is isotrivial;

(iii) $\mathcal{G}$ is linear;

(iv) $\text{rad}(\mathcal{G})$ is linear.

Furthermore if $S$ is affine, we can take a faithful linear representation in some $\text{GL}_n$ for (iii) and (iv).

Proof. (i) $\implies$ (ii). We assume that $\text{rad}(\mathcal{G})$ is isotrivial and want to show that the $\text{Out}(G)_S$–torsor $\mathcal{F} = \text{Isomext}(G_S, \mathcal{G})$ is isotrivial. In other words, we want to show that there exists a finite étale cover $S'$ of $S$ such that $\mathcal{G} \times_S S'$ is an inner form of $G$.

Without loss of generality we can assume that $\text{rad}(\mathcal{G})$ is a split torus. We quote now [11, XXIV.2.16] for the Chevalley group $G$ over $\mathbb{Z}$ which introduces the $\mathbb{Z}$–group scheme

$$H = \ker(\text{Aut}(G) \to \text{Aut}(\text{rad}(G))).$$
furthermore there is an equivalence of categories between the category of
$H$-torsors over $S$ and the category of pairs $(\mathfrak{M}, \phi)$ where $\mathfrak{M}$ is an $S$–form of $G$ and
$\phi : \text{rad}(G)_S \xrightarrow{\sim} \text{rad}(\mathfrak{M})$.

Since $\text{rad}(\mathfrak{G})$ is split, we choose an isomorphism $\phi : \text{rad}(G)_S \xrightarrow{\sim} \text{rad}(\mathfrak{G})$ and
consider an $H$-torsor $\mathfrak{P}$ mapping to an object isomorphic to $(\mathfrak{G}, \phi)$. Furthermore the quoted reference provides an exact sequence of $\mathbb{Z}$–group schemes

$$1 \to G_{\text{ad}} \to H \xrightarrow{p} F \to 1$$

where $F$ is finite étale over $\mathbb{Z}$, so is constant. We denote by $S' = \mathfrak{M} \wedge^{H_S} F_S$ the contracted product of $\mathfrak{M}$ and $F_S$ with respect to $H_S$; this is an $F$–torsor over $S$, hence is a finite étale cover of $S$.

It follows that $\mathfrak{P} \times_S S'$ admits a reduction to a $G_{\text{ad},S'}$–torsor $\mathfrak{Q}'$. Since the map $G_{\text{ad}} \to H \to \text{Aut}(G)$ is the canonical map, we conclude that $\mathfrak{G}_{S'} \cong ^G G$ is an inner form of $G$.

$(ii) \implies (iii)$. Our assumption is that there exists a finite étale cover $S'/S$ which splits the Out$(G)$–torsor Isomext$(G_S, \mathfrak{G})$. Lemma 2.3 permits us to replace $S$ by $S'$, so we can assume that the Out$(G)$–torsor Isomext$(G_S, \mathfrak{G})$ is trivial; that is, $\mathfrak{G}$ is an inner form of $G$. There exists a $G_{\text{ad}}$–torsor $\mathfrak{Q}$ over $S$ such that $\mathfrak{G} \cong ^G G$. Since $G \times G_{\text{ad}}$ is defined over $\mathbb{Z}$, it admits a faithful representation $\rho : G \times G_{\text{ad}} \to \text{GL}_n$ over $\mathbb{Z}$ [2, §1.4.5]. The map $\rho$ is then $G_{\text{ad}}$–equivariant and can be twisted by the $G_{\text{ad}}$–torsor $\mathfrak{Q}$. We obtain a faithful representation $^G G \times ^G G_{\text{ad}} \to ^G \text{GL}_n = \text{GL}(\mathcal{E})$ where $\mathcal{E}$ is the locally free $\mathcal{O}_S$–module of rank $n$ which is the twist of $(\mathcal{O}_S)^n$ by the $\text{GL}_n$–torsor $\mathfrak{Q} \wedge^{G_{\text{ad}}} \text{GL}_n$. Thus $^G G$ is linear.

$(iii) \implies (iv)$. Obvious.

$(iv) \implies (i)$. Since $\text{rad}(\mathfrak{G})$ is a form of $\text{rad}(G)$, it is of constant rank and Theorem 3.3 shows that $\text{rad}(G)$ is isotrivial.

Finally the refinement for $S$ affine follows from Lemma 2.2.

\textbf{Corollary 4.2.} Under the assumptions of $\mathfrak{G}$, let $\mathfrak{G}^{qs}$ be the quasi-split form of $\mathfrak{G}$. Then $\mathfrak{G}$ is linear if and only if $\mathfrak{G}^{qs}$ is linear.

The next corollary slightly generalizes a result by Thomason [13, cor. 3.2].

\textbf{Corollary 4.3.} Assume that either

$(i)$ $S$ is locally noetherian and geometrically unibranch (e.g. normal);

$(ii)$ $\text{rad}(\mathfrak{G})$ is of rank $\leq 1$ (in particular if $G$ is semisimple).

Then $\mathfrak{G}$ is linear.

\textbf{Proof.} In case (i), the torus $\text{rad}(\mathfrak{G})$ is isotrivial [11, X.5.16]. In case (ii), we have $\text{rad}(G) = 1$ or $\mathbb{G}_m$ (since $G$ is split), so that $\text{rad}(\mathfrak{G})$ is split by a quadratic étale cover of $S$ (§2.4), hence is isotrivial. Hence Theorem 1.1 implies that $\mathfrak{G}$ is linear. \hfill $\Box$

The next corollary extends Demazure’s characterization of locally isotrivial reductive group schemes [11, XXIV.4.1.5].
Corollary 4.4. The following are equivalent:

(i) $\mathfrak{G}$ is locally (resp. semilocally) isotrivial;
(ii) The torus $\text{rad}(\mathfrak{G})$ is locally (resp. semilocally) isotrivial;
(iii) the $\text{Out}(G)_S$-torsor $\text{Isomext}(G_S, \mathfrak{G})$ is locally (resp. semilocally) isotrivial;
(iv) $\mathfrak{G}$ is locally (resp. semilocally) linear;
(v) $\text{rad}(\mathfrak{G})$ is locally (resp. semilocally) linear.

Proof. In view of Theorem 4.1, it remains to establish the equivalence $(i) \iff (ii)$. Since this is precisely the quoted result [11, XXIV.3.5], the proof is complete. □

Corollary 4.5. Let $\mathfrak{H}$ be a reductive $S$-subgroup scheme of $\mathfrak{G}$. If $\mathfrak{G}$ is locally (resp. semilocally) isotrivial, then $\mathfrak{H}$ is locally (resp. semilocally) isotrivial.

Proof. Corollary 4.4 shows that $\mathfrak{G}$ is locally (resp. semilocally) linear and so is $\mathfrak{H}$. Therefore $\mathfrak{H}$ is locally (resp. semilocally) isotrivial. □

5. The semilocal case

We assume that $S = \text{Spec}(R)$ where $R$ is a semilocal ring and continue to assume that the reductive $S$-group scheme $\mathfrak{G}$ is of constant type. We remind the reader that $\mathfrak{G}$ admits a maximal torus (Grothendieck, [11, XIV.3.20 and footnote]).

Corollary 5.1. Let $\Sigma$ be a maximal torus of $\mathfrak{G}$. The following are equivalent:

(i) $\mathfrak{G}$ is isotrivial;
(ii) The torus $\text{rad}(\mathfrak{G})$ is isotrivial;
(iii) the $\text{Out}(G)_S$-torsor $\text{Isomext}(G_S, \mathfrak{G})$ is isotrivial;
(iv) $\mathfrak{G}$ is linear;
(v) $\text{rad}(\mathfrak{G})$ is linear;
(vi) $\Sigma$ is linear;
(vii) $\Sigma$ is isotrivial.

Proof. From Corollary 4.4, we have the equivalences $(i) \iff (ii) \iff (iii) \iff (iv) \iff (v)$. On the other hand, the equivalence $(vi) \iff (vii)$ holds according to Theorem 3.3. Now we observe that the implications $(iv) \implies (vi)$ and $(vi) \implies (v)$ are obvious so the proof is complete. □

6. Equivariant resolution property

Definition 6.1. Let $\mathfrak{G}$ be a flat group scheme over $S$ acting on a locally noetherian $S$-scheme $\mathfrak{X}$. One says that $(\mathfrak{G}, S, \mathfrak{X})$ has the resolution property $(\text{RE})$ for short) if for every coherent $\mathfrak{G}$-module $\mathcal{F}$ on $\mathfrak{X}$, there is a locally free coherent $\mathfrak{G}$-module (i.e. a $\mathfrak{G}$-vector bundle $\mathcal{E}$) and a $\mathfrak{G}$-equivariant epimorphism $\mathcal{E} \to \mathcal{F} \to 0$.

We strengthen Thomason’s results.
Theorem 6.2. Let $G$ be a reductive $S$–group scheme. We assume that $S$ is separated noetherian and that $(1, S, S)$ satisfies the resolution property, e.g. $S$ is affine or regular or admits an ample family of line bundles. Then the following are equivalent:

(i) $G$ is linear;
(ii) $\text{rad}(G)$ is isotrivial;
(iii) $G$ satisfies $(RE)$.

Proof. $(i) \iff (ii)$. This is a special case of Theorem 4.1.

$(ii) \implies (iii)$. This is Thomason’s result [13, Theorem 2.18].

$(iii) \implies (i)$. This is Thomason’s result [13, Theorem 3.1], see also [11, VI.B.13.5]. □

Remark 6.3. Example 3.4.(b) is an example of a local noetherian ring $R$ and of a rank two non-isotrivial torus $\mathcal{T}$. Theorem 6.2 shows that $\mathcal{T}$ does not satisfy (RE). This answers a question of Thomason [13, §2.3].