A complex semigroup approach to group algebras of infinite dimensional Lie groups

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To K. H. Hofmann on the occasion of his 75th birthday

Abstract

A host algebra of a topological group $G$ is a $C^*$-algebra whose representations are in one-to-one correspondence with certain continuous unitary representations of $G$. In this paper we present an approach to host algebras for infinite dimensional Lie groups which is based on complex involutive semigroups. Any locally bounded absolute value $\alpha$ on such a semigroup $S$ leads in a natural way to a $C^*$-algebra $C^*(S, \alpha)$, and we describe a setting which permits us to conclude that this $C^*$-algebra is a host algebra for a Lie group $G$. We further explain how to attach to any such host algebra an invariant weak-$*$-closed convex set in the dual of the Lie algebra of $G$ enjoying certain nice convex geometric properties. If $G$ is the additive group of a locally convex space, we describe all host algebras arising this way. The general non-commutative case is left for the future.

Keywords: complex semigroup, infinite dimensional Lie group, host algebra, multiplier algebra, unitary representation.

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Introduction

If $G$ is a locally compact group, then Haar measure on $G$ leads to the convolution algebra $L^1(G)$, and we obtain a $C^*$-algebra $C^*(G)$ as the enveloping $C^*$-algebra of $L^1(G)$. This $C^*$-algebra has the universal property that each
(continuous) unitary representation $(\pi, \mathcal{H})$ of $G$ on some Hilbert space $\mathcal{H}$ defines a unique non-degenerate representation of $C^*(G)$ on $\mathcal{H}$ and, conversely, each non-degenerate representation of $C^*(G)$ arises from a unique unitary representation of $G$. This correspondence is a central tool in the harmonic analysis on $G$ because the well-developed theory of $C^*$-algebras provides a powerful machinery to study the set of all irreducible representations of $G$, to endow it with a natural topology and to understand how to decompose representations into irreducibles or factor representations.

For infinite dimensional Lie groups, i.e., Lie groups modeled on infinite dimensional locally convex spaces, there is no natural analog of the convolution algebra $L^1(G)$, so that we cannot hope to find a $C^*$-algebra whose representations are in one-to-one correspondence to all unitary representations of $G$. However, in [Gr05] H. Grundling introduces the notion of a host algebra of a topological group $G$. This is a pair $(\mathcal{A}, \eta)$, consisting of a $C^*$-algebra $\mathcal{A}$ and a morphism $\eta: G \to U(M(\mathcal{A}))$ of $G$ into the unitary group of its multiplier algebra $M(\mathcal{A})$ with the following property: For each non-degenerate representation $\pi$ of $\mathcal{A}$ and its canonical extension $\tilde{\pi}$ to $M(\mathcal{A})$, the unitary representation $\tilde{\pi} \circ \eta$ of $G$ is continuous and determines $\pi$ uniquely. In this sense, $\mathcal{A}$ is hosting a certain class of representations of $G$. A host algebra $\mathcal{A}$ is called full if it is hosting all continuous unitary representations of $G$. Now it is natural to ask to which extent infinite dimensional Lie groups, or other non-locally compact groups, possess host algebras. One cannot expect the existence of a full host algebra because, f.i., the topological dual $E'$ of an infinite dimensional locally convex space $E$ carries no natural locally compact topology. Therefore one is looking for host algebras that accommodate certain classes of continuous unitary representations.

In the present paper we discuss a construction of host algebras based on holomorphic extensions of unitary representations of a Lie group $G$ to certain complex semigroups $S$. Some of the basic ideas of our constructions appear already in [Ne95], where one finds the construction of the enveloping $C^*$-algebras $C^*(S, \alpha)$ of a complex involutive semigroup $S$, endowed with a locally bounded absolute value $\alpha$, and also in [Ne98], where this is applied to the special case where $S$ is a complex Banach–Lie group. Here we address the situation where $S$ may be an infinite dimensional semigroup which is not a group.

The structure of the paper is as follows. In Section 1 we first recall the concept of a complex involutive semigroup $S$ and associate to any locally bounded absolute value $\alpha$ on $S$ a $C^*$-algebra $C^*(S, \alpha)$ with a holomorphic
morphism $\eta : S \rightarrow C^*(S,\alpha)$ having a suitable universal property. Since our goal is to construct host algebras for infinite dimensional Lie groups, we build in Section 2 a bridge between complex involutive semigroups and Lie groups by defining the notion of a host semigroup of a Lie group. Roughly speaking, this a complex involutive semigroup $S$ on which the Lie group $G$ acts smoothly by unitary multipliers and for which there exists an open convex cone $W$ in the Lie algebra $L(G)$ of $G$, invariant under the adjoint action, for which all $\mathbb{R}$-actions on $S$ defined by the one-parameter semigroups $\gamma_x(t) = \exp_G(tx)$, $x \in W$, extend to “holomorphic” one-parameter semigroups $C_+ = \mathbb{R} + i[0,\infty] \rightarrow M(S)$ mapping the open upper halfplane $\mathbb{C}_+$ holomorphically into $S \subseteq M(S)$. The main result of Section 2 is that for each locally bounded absolute value $\alpha$ on a host semigroup $S$, the $C^*$-algebra $C^*(S,\alpha)$ is a host algebra of $G$.

This leaves us with the problem to understand the classes of representations of $G$ hosted by such $C^*$-algebras. To clarify this point, we consider in Section 3 multiplier actions $\eta : G \rightarrow U(M(A))$ of a Lie group $G$ on a $C^*$-algebra $A$ and study to which extent the action of certain one-parameter semigroups of $G$ extends holomorphically to the upper halfplane. This leads to the momentum map

$$\Psi_\eta : S(A)^\infty \rightarrow L(G)', \quad \varphi \mapsto \frac{1}{i} d(\tilde{\varphi} \circ \eta)(1).$$

Here $S(A)^\infty$ denotes the set of all states $\varphi$ of $A$ for which the canonical extension $\tilde{\varphi}$ to $M(A)$ yields a smooth function $\tilde{\varphi} \circ \eta : G \rightarrow \mathbb{C}$. The weak-$*$-closed hull $I_\eta$ of the image of $\Psi_\eta$ is a convex set invariant under the coadjoint action, called the momentum set of $(A, \eta)$. A crucial observation is that, if the multiplier action comes from a host algebra $C^*(S,\alpha)$, where $S$ is a host semigroup, the convex cone

$$B(I_\eta) := \{ x \in L(G) : \inf \langle I_\eta, x \rangle > -\infty \}$$

has non-empty interior and the support function

$$s : B(I_\eta)^0 \rightarrow \mathbb{R}, \quad x \mapsto -\inf \langle I_\eta, x \rangle$$

is locally bounded. This observation suggests that to find host algebras for $G$, one should start with an $\text{Ad}^*(G)$-invariant weak-$*$-closed convex subset $C \subseteq L(G)'$ for which the corresponding support function $s_C : B(C)^0 \rightarrow \mathbb{R}$ is locally bounded. As the function $s_C$ is convex, we take in Section 4 a closer
look at convex functions on open convex domains in locally convex spaces. In
particular, we show that whenever \( L(G) \) is barrelled, the existence of interior
points in the cone \( B(C) \) automatically implies that \( s_C \) is locally bounded
and even continuous.

In Section 5 we then show how this circle of ideas can be completed in
the abelian case. Here the Lie group \( G \) is a locally convex space \( V \) and
the semigroup is a tube \( S = V + iW \), where \( W \subseteq V \) is an open con-

vex cone. In this case it suffices to consider absolute values of the form
\( \alpha(a + ib) = e^{-\inf(C,b)} \), where \( C \subseteq V' \) is a weak-*-closed convex subset. Now \( \alpha \) is locally bounded if and only if the support function \( s_C(x) = -\inf(C,x) \)
is locally bounded on \( W \). If this is the case, then the results of Section 4
imply that \( C \) is locally compact and \( C^*(S, \alpha) \cong C_0(C) \) is a host algebra
of \( V \) hosting precisely all unitary representations of \( V \) arising from spectral
measures on the locally compact subset \( C \subseteq V' \).

Section 6 contains a brief discussion of the finite dimensional case, which
is developed in detail in [Ne99]. Here we give a short and direct proof of the
fact that any host algebra of \( G \) coming from a host semigroup is a quotient
of the group \( C^*-algebra C^*(G) \).

The next steps of this project aim at a better understanding of the
classes of representations of a Lie group \( G \) hosted by \( C^*-algebras of the
form C^*(S, \alpha) \). The first major problem one has to solve here is to find a
suitable complex involutive semigroup \( S \) whenever the invariant convex set
\( C \subseteq L(G)' \) is given. For finite dimensional groups this has been carried out in
[Ne99], but for infinite dimensional groups many key tools are still missing.
Furthermore, once the semigroup \( S \) is constructed, one has to find the class
of unitary representations of \( G \) extending to holomorphic representations of
\( S \). We leave all that to the future.

**Preliminaries**

For the sake of easier reference, we collect some of the basic definitions con-
cerning infinite dimensional manifolds and Lie groups.

Let \( X \) and \( Y \) be locally convex topological vector spaces, \( U \subseteq X \) open
and \( f: U \to Y \) a map. Then the **derivative of \( f \) at \( x \) in the direction of \( h \)** is
declared as

\[
df(x)(h) := \lim_{t \to 0} \frac{1}{t} (f(x + th) - f(x))
\]

whenever the limit exists. The function \( f \) is called **differentiable at \( x \)** if
$df(x)(h)$ exists for all $h \in X$. It is called **continuously differentiable** or $C^1$ if it is continuous and differentiable at all points of $U$ and

$$df : U \times X \rightarrow Y, \quad (x, h) \mapsto df(x)(h)$$

is a continuous map. It is called a $C^n$-map if it is $C^1$ and $df$ is a $C^{n-1}$-map, and $C^\infty$ (or **smooth**) if it is $C^n$ for all $n \in \mathbb{N}$. This is the notion of differentiability used in [Mil84], [Ha82] and [Gl02], where the latter reference deals with the modifications needed for incomplete spaces. If $X$ and $Y$ are complex, $f$ is called **holomorphic** if it is smooth and its differentials $df(x)$ are complex linear. If $Y$ is Mackey complete, it suffices that $f$ is $C^1$.

Since we have a chain rule for $C^1$-maps between locally convex spaces, we can define smooth manifolds as in the finite dimensional case. A chart $(\varphi, U)$ with respect to a given manifold structure on $M$ is an open set $U \subset M$ together with a homeomorphism $\varphi$ onto an open set of the model space.

A Lie group $G$ is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $1 \in G$ for the identity element and $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on $G$. Then each $x \in T_1(G)$ corresponds to a unique left invariant vector field $x_l$ with $x_l(1) = x$. The space of left invariant vector fields is closed under the Lie bracket, hence inherits a Lie algebra structure. We thus obtain on the tangent space $T_1(G)$ a continuous Lie bracket which is uniquely determined by $[x, y] = [x_l, y_l]$ for $x, y \in T_1(G)$. We write $L(G) := (T_1(G), [\cdot, \cdot])$ for the so-obtained topological Lie algebra. Then $L$ defines a functor from the category of locally convex Lie groups to the category of locally convex topological Lie algebras. The adjoint action of $G$ on $L(G)$ is defined by $\text{Ad}(g) := L(c_g)$, where $c_g(x) = gxg^{-1}$. This action is smooth and each $\text{Ad}(g)$ is a topological isomorphism of $L(G)$. The coadjoint action on the topological dual space $L(G)'$ is defined by $\text{Ad}^*(g).f := f \circ \text{Ad}(g)^{-1}$ and all these maps are continuous with respect to the weak-$*$-topology on $L(G)'$, but in general the coadjoint action of $G$ is not continuous with respect to this topology.

# $C^*$-algebras associated to complex semigroups

In this section we associate to each complex involutive semigroup $S$, endowed with an absolute value $\alpha$, a $C^*$-algebra $C^*(S, \alpha)$. As we shall see later on, one
can use these semigroup algebras to construct host algebras for Lie groups, and this is our main purpose for their construction.

**Definition 1.1** (a) An involutive complex semigroup is a complex manifold $S$ modeled on a locally convex space which is endowed with a holomorphic semigroup multiplication and an antiholomorphic antiautomorphism denoted $s \mapsto s^*$. 

(b) A function $\alpha : S \to \mathbb{R}^+$ is called an absolute value if 

$$\alpha(s) = \alpha(s^*) \quad \text{and} \quad \alpha(st) \leq \alpha(s) \alpha(t)$$

for all $s, t \in S$. 

(c) A holomorphic representation $(\pi, H)$ of a complex involutive semigroup $S$ on the Hilbert space $H$ is a morphism $\pi : S \to B(H)$ of involutive semigroups which is holomorphic if $B(H)$ is endowed with its natural complex Banach space structure defined by the operator norm. If $\alpha$ is an absolute value on $S$, then the representation $\pi$ is said to be $\alpha$-bounded if $\|\pi(s)\| \leq \alpha(s)$ holds for each $s \in S$. A representation is called non-degenerate if $\pi(S).v = \{0\}$ implies $v = 0$.

**Examples 1.2** (1) If $H$ is a complex Lie group and $s \mapsto s^*$ an antiholomorphic antiautomorphism $H$, then $H$ is a complex involutive (semi)group. Any open $\ast$-subsemigroup of $H$ is a complex involutive semigroup. 

(2) If $V$ is a locally convex space and $W \subseteq V$ an open convex cone, then $S := V + iW \subseteq V_C$ is an involutive subsemigroup with respect to the involution $(x + iy)^* := -x + iy$. 

(3) If $A$ is a $C^\ast$-algebra, then its multiplicative semigroup $(\mathcal{A}, \cdot)$ is a complex involutive semigroup and $\alpha(a) := \|a\|$ is an absolute value on $\mathcal{A}$. An important example is the $C^\ast$-algebra $B(H)$ of bounded operators on the Hilbert space $H$. 

(4) Let $\mathcal{A}$ be a unital $C^\ast$-algebra and $\tau = \tau^* \in \mathcal{A}$ an involution, i.e., $\tau^2 = 1$. For $a, b \in \mathcal{A}$ we write $a < b$ if there exists an invertible element $c \in \mathcal{A}$ with $b - a = c^*c$. Then 

$$S := \{s \in A : s^*\tau s < \tau\}$$

is an open subsemigroup of $\mathcal{A}$ with respect to multiplication. To see that it is non-empty, we observe that we may write $\tau = 1 - 2p = (1 - p) - p$ for a
projection \( p = p^* = p^2 \in \mathcal{A} \). For \( \lambda \in \mathbb{C}^\times \) and \( s := \lambda(1 - p) + \lambda^{-1}p \) we then have

\[
s^* \tau s = |\lambda|^2(1 - p) - |\lambda|^{-2}p < \tau = (1 - p) - p
\]

if and only if \( |\lambda| < 1 \). The boundary of \( S \) contains the real Banach–Lie group

\[
U(\mathcal{A}, \tau) := \{ g \in \mathcal{A}^\times : g^* \tau g = \tau \}.
\]

**Definition 1.3** Let \( S \) be a complex involutive semigroup and \( \alpha \) a locally bounded absolute value on \( S \). We associate to the pair \( (S, \alpha) \) a \( C^* \)-algebra \( C^*(S, \alpha) \) as follows.

First, we endow the semigroup algebra \( C[\mathcal{S}] \), whose elements we write as finitely supported functions \( f: S \to \mathbb{C} \), with the submultiplicative seminorm

\[
\|f\|_\alpha := \sum_{s \in S} |f(s)| \alpha(s)
\]

and the involution \( f^*(s) := f(s) \). Let \( \ell^1(S, \alpha) \) be the complex involutive Banach algebra obtained by completion of this seminormed \(*\)-algebra. We define \( \eta_{\alpha}(s) \in \ell^1(S, \alpha) \) as the image of the function \( \delta_{s,t} \) in \( \ell^1(S, \alpha) \) and note that \( \|\eta_{\alpha}(s)\| = \alpha(s) \).

If \( \mathcal{A} \) is a \( C^* \)-algebra, then each homomorphism \( \beta: S \to (\mathcal{A}, \cdot) \) of involutive semigroups, which is \( \alpha \)-bounded in the sense that \( \|\beta(s)\| \leq \alpha(s) \) holds for each \( s \in S \), defines a unique contractive morphism

\[
\hat{\beta}: \ell^1(S, \alpha) \to \mathcal{A}, \quad \hat{\beta}(f) := \sum_{s \in S} f(s) \beta(s)
\]

of Banach-\( * \)-algebras satisfying \( \hat{\beta} \circ \eta_{\alpha}^1 = \beta \). Let \( I \trianglelefteq \ell^1(S, \alpha) \) denote the intersection of the kernels of all such homomorphism \( \hat{\beta} \) for which \( \beta \) is a holomorphic map. On the quotient algebra \( \ell^1(S, \alpha)/I \), we obtain a \( C^* \)-norm by

\[
\|\left[ f \right]\| := \sup_{\beta \text{ holomorphic}} \|\hat{\beta}(f)\| \leq \|f\|_\alpha.
\]

We now define \( C^*(S, \alpha) \) as the completion of \( \ell^1(S, \alpha)/I \) with respect to this norm. It follows immediately from the construction that we thus obtain a \( C^* \)-algebra.

Before we turn to the universal property of \( C^*(S, \alpha) \), we recall the following criterion for holomorphy ([Ne99], Cor. A.III.3):

**Lemma 1.4** Let \( M \) be a complex manifold, \( V \) a Banach space and \( N \subseteq V' \) a subset which is norm-determining, i.e., \( \|v\| = \sup\{|\lambda(v)| : \lambda \in N, \|\lambda\| \leq 1\} \) for all \( v \in V \). Then a locally bounded function \( f: M \to V \) is holomorphic if and only if for each \( \lambda \in N \) the function \( \lambda \circ f \) is holomorphic.
The following theorem could also be derived from Theorem IV.2.7 in [Ne99], but the construction we give here is much more direct.

**Theorem 1.5** The C*-algebra $C^*(S, \alpha)$ has the following properties:

(i) There exists a holomorphic morphism $\eta_\alpha : S \to C^*(S, \alpha)$ of involutive semigroups with total range, i.e., $\eta_\alpha(S)$ generates a dense subalgebra.

(ii) For each $\alpha$-bounded holomorphic morphism of involutive semigroups $\pi : S \to A$ to the multiplicative semigroup of a C*-algebra $A$, there exists a unique morphism of C*-algebras $\tilde{\pi} : C^*(S, \alpha) \to A$ with $\tilde{\pi} \circ \eta_\alpha = \pi$.

**Proof.** (i) We define $\eta_\alpha(s) \in C^*(S, \alpha)$ as the image of the element $\eta^1_\alpha(s) \in \ell^1(S, \alpha)$. Then $\|\eta_\alpha(s)\| \leq \|\eta^1_\alpha(s)\| = \alpha(s)$ implies that $\eta_\alpha$ is a locally bounded morphism of involutive semigroups.

To see that $\eta_\alpha(S)$ spans a dense subspace of $C^*(S, \alpha)$, we first note that the subspace $N$ of continuous linear functionals on $C^*(S, \alpha)$ spanned by the functionals of the form $\phi \circ \hat{\beta}$ on $\ell^1(S, \alpha)$, where $\beta : S \to A$ is a holomorphic morphism of involutive semigroups into a C*-algebra $A$, where $\phi \in A'$, and $\phi$ separates the points of $C^*(S, \alpha)$ and determines the norm (by definition of the norm on $C^*(S, \alpha)$). For each functional $\psi = \phi \circ \hat{\beta}$ as above the map

$$\psi \circ \eta_\alpha = \phi \circ \beta : S \to \mathbb{C}$$

is holomorphic. Therefore Lemma 1.4 implies that $\eta_\alpha$ is holomorphic.

That $\eta_\alpha(S)$ spans a dense subspace of $C^*(S, \alpha)$ follows from the construction because the image of $S$ spans a dense subspace of $\ell^1(S, \alpha)$, hence also in the quotient by the ideal $I$.

(ii) Let $\tilde{\pi} : \ell^1(S, \alpha) \to A$ denote the canonical extension of $\pi$ which is a contractive morphism of involutive Banach algebras and note that $\ker \tilde{\pi} \supseteq I$, so that $\tilde{\pi}$ factors through a morphism $\ell^1(S, \alpha)/I \to A$ of involutive Banach algebras which, by definition, extends to the completion $C^*(S, \alpha)$. }

**Remark 1.6** (a) The preceding theorem entails that for each C*-algebra $A$, we have $\text{Hom}(C^*(S, \alpha), A) \cong \text{Hom}_{\text{hol}}((S, \alpha), (A, \| \cdot \|))$, where the right hand side denote the holomorphic contractive morphisms of complex involutive semigroups with absolute value. This means that $C^*(S, \alpha)$ defines an adjoint of the forgetful functor from the category of C*-algebras to the category
of complex involutive semigroups with absolute value, assigning to a \( C^* \)-algebra \( \mathcal{A} \) the semigroup \((\mathcal{A}, \cdot, \| \cdot \|)\). It follows in particular that the universal property in Theorem 1.5(ii) determines \( C^*(S, \alpha) \) up to isomorphism.

(b) If \( \beta \) is an absolute value on \( S \) satisfying \( \| \eta_\alpha \| \leq \beta \leq \alpha \), then the natural map \( \varphi: C^*(S, \beta) \to C^*(S, \alpha) \) is an isomorphism with \( \varphi \circ \eta_\beta = \eta_\alpha \).

Examples 1.7

(a) We take a closer look at the case where \( S \) is commutative. Let \( \widehat{S} := \text{Hom}(S, (\mathbb{C}, \cdot)) \setminus \{0\} \) denote the set of non-zero holomorphic characters of \( S \), i.e., the one dimensional (=irreducible) non-degenerate representations. A holomorphic character \( \chi \) extends to a character of the \( C^* \)-algebra \( C^*(S, \alpha) \) if and only if it is \( \alpha \)-bounded. Hence the set \( \widehat{S}_\alpha \) of \( \alpha \)-bounded non-zero holomorphic characters form the spectrum of the commutative \( C^* \)-algebra \( C^*(S, \alpha) \). We conclude that \( C^*(S, \alpha) \cong C_0(\widehat{S}_\alpha) \), where \( \widehat{S}_\alpha \subseteq C^*(S, \alpha)' \) carries the weak*-topology. Moreover, the set \( \widehat{S}_\alpha \cup \{0\} \) is weak*-compact in \( C^*(S, \alpha)' \), and the canonical map \( \eta_\alpha^*: C^*(S, \alpha)' \to \mathbb{C}^* \), \( \varphi \mapsto \varphi \circ \eta \) is continuous with respect to the weak*-topology on the left and the product topology on the right. This shows that \( \widehat{S}_\alpha \cup \{0\} \) is compact in \( C^*_S \) with respect to the product topology which, therefore, coincides with the weak*-topology defined by \( C^*(S, \alpha) \). We conclude that \( \widehat{S}_\alpha \) is locally compact with respect to the product topology and that \( C^*(S, \alpha) \cong C_0(\widehat{S}_\alpha) \). We now have

\[
\| \eta_\alpha(s) \| = \sup \{ |\chi(s)| : \chi \in \widehat{S}_\alpha \},
\]

and this absolute value defines the same \( C^* \)-algebra by Remark 1.6(b).

(b) We specialize to the particular case where \( S = V + iW \subseteq V_\mathbb{C} \) holds for a real locally convex space \( V \) and an open convex cone \( W \subseteq V \). This is an open complex subsemigroup of the complex vector space \( V_\mathbb{C} \). Any non-zero holomorphic character \( \chi: S \to \mathbb{C} \) maps into \( \mathbb{C}^* \) and induces a unique continuous character \( V \to \mathbb{T} \), hence is of the form \( \chi = e^{if} \) for some continuous linear functional \( f \in V' \) (which we also extend to a complex linear functional on \( V_\mathbb{C} \)).

Now let \( \alpha \) be a locally bounded absolute value on \( S \) and

\[
C_\alpha := \{ f \in V' : e^{if} \in \widehat{S}_\alpha \}
\]

the set of linear functionals defining \( \alpha \)-bounded characters of \( S \). In view of (b), we may w.l.o.g. assume that

\[
\alpha(x + iy) = \| \eta_\alpha(x + iy) \| = \sup \{ e^{-f(y)} : f \in C_\alpha \} = e^{-\inf_{(C_\alpha, y)}}
\]
without changing $C^*(S, \alpha)$ or $C_\alpha$. A holomorphic character $e^{if}$, $f \in V'$, is $\alpha$-bounded, i.e., contained in $\hat{S}_\alpha$, if and only if $e^{-f(y)} \leq e^{-\inf(C_\alpha, y)}$ for each $y \in W$, which is equivalent to

$$f(y) \geq \inf(C_\alpha, y) \quad \text{for} \quad y \in W. \quad (1)$$

This implies that $C_\alpha$ is a weak-$\ast$-closed convex subset of $V'$, and (a) further shows that $\hat{S}_\alpha$ is locally compact as a subset of $\mathbb{C}^S$. We continue the discussion of these examples in Section 5 below.

(c) Let $S := \mathbb{C}_+^0 = \mathbb{R} + i[0, \infty] \subseteq \mathbb{C}$ be the open upper half plane and $\alpha$ as in (b). Then $V = \mathbb{R}$, $W = [0, \infty]$ and $C_\alpha \subseteq \mathbb{R}$ is a closed convex subset bounded from below. Let $m := \inf C_\alpha$. Then

$$\|\eta_\alpha(x + iy)\| = e^{-\inf(C_\alpha)\cdot y} = e^{-my},$$

and (1) implies that $C_\alpha = [m, \infty[$. Therefore $C^*(\mathbb{C}_+^0, \alpha) \cong C_0([m, \infty[)$.

### Some holomorphic representation theory

**Lemma 1.8** Let $\mathcal{H}$ be a Hilbert space and $\pi : S \to B(\mathcal{H})$ a morphism of involutive semigroups. Then $\pi$ is a holomorphic representation if and only if it satisfies the following conditions:

1. $\pi$ is locally bounded, i.e., for every $s \in S$ there exists a neighborhood $U$ such that $\pi(U)$ is a bounded subset of the Banach space $B(\mathcal{H})$.
2. There exists a dense subspace $E \subseteq \mathcal{H}$ such that the functions $\pi^v(s) := \langle \pi(s).v, v \rangle$ are holomorphic for all $v \in E$.

**Proof.** The necessity of conditions (1) and (2) is obvious, and the converse follows from [Ne99], Cor. A.III.5, which also holds for general locally convex manifolds since [He89], Prop. 2.4.9(a) applies to functions on locally convex spaces that are not necessarily complete. \hfill \(\blacksquare\)

**Proposition 1.9** Let $S$ be an involutive complex semigroup and $\alpha$ a locally bounded absolute value on $S$.

(a) If $(\pi_j, \mathcal{H}_j)_{j \in J}$ is a set of $\alpha$-bounded holomorphic representations of $S$, then the operators induced by $s \in S$ on $\bigoplus_{j \in J} \mathcal{H}_j$ are bounded, and we thus obtain an $\alpha$-bounded holomorphic representation of $S$ on the direct Hilbert space sum $\bigoplus_{j \in J} \mathcal{H}$.  

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Every non-degenerate $\alpha$-bounded holomorphic representation is a direct sum of cyclic $\alpha$-bounded holomorphic representations.

(c) Every $\alpha$-bounded cyclic holomorphic representation of $S$ is equivalent to a representation $(\pi, H_{\varphi})$ on a reproducing kernel Hilbert space $H_{\varphi} \subseteq \mathcal{O}(S)$ by $(\pi(s), f)(x) = f(xs)$, where the reproducing kernel is given by $K(s, t) = \varphi(st^*)$ for some holomorphic function $\varphi \in H_{\varphi}$.

Proof. (a) [Ne99], Prop. IV.2.3; (b) [Ne99], Prop. II.2.11(ii); (c) [Ne99], Lemma IV.2.6.

2 Host semigroups and host algebras

In this section we describe the connection between Lie groups and complex semigroups. The key point is that there is a Lie theoretic notion of a host semigroup $S$ of a Lie group $G$ which can be used to obtain host algebras for $G$. We start with the definition of a host semigroup of a Lie group and turn in the second subsection to host algebras of topological groups.

2.1 Host semigroups of Lie groups

Definition 2.1 Let $S$ be a complex involutive semigroup. A multiplier of $S$ is a pair $(\lambda, \rho)$ of holomorphic mappings $\lambda, \rho: S \to S$ satisfying the following conditions:

$$a\lambda(b) = \rho(a)b, \quad \lambda(ab) = \lambda(a)b, \quad \text{and} \quad \rho(ab) = a\rho(b).$$

We write $M(S)$ for the set of all multipliers of $S$ and turn it into an involutive semigroup by

$$(\lambda, \rho)(\lambda', \rho') := (\lambda \circ \lambda', \rho' \circ \rho) \quad \text{and} \quad (\lambda, \rho)^* := (\rho^*, \lambda^*),$$

where $\lambda^*(a) := \lambda(a^*)^*$ and $\rho^*(a) = \rho(a^*)^*$ (cf. [FD88], p.778).

Remark 2.2 The assignment $\eta_S: S \to M(S), a \mapsto (\lambda_a, \rho_a)$ defines a morphism of involutive semigroups which is surjective if and only if $S$ has an identity. Its image is an involutive semigroup ideal in $M(S)$. 

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Proposition 2.3 Let $\alpha$ be a locally bounded absolute value on the complex involutive semigroup $S$. Then the following assertions hold:

(1) For each non-degenerate $\alpha$-bounded holomorphic representation $(\pi, \mathcal{H})$ of $S$ there exists a unique unitary representation $(\tilde{\pi}, \mathcal{H})$ of $U(M(S))$, determined by $\tilde{\pi}(g)\pi(s) = \pi(gs)$ for $g \in U(M(S)), s \in S$.

(2) There exists a unique homomorphism $\tilde{\eta}: U(M(S)) \to U(M(C^*(S, \alpha)))$ with $\tilde{\eta}(g)\eta(s) = \eta(gs)$ for $g \in U(M(S)), s \in S$.

Proof. (a) Every $\alpha$-bounded holomorphic representation is a direct sum of cyclic ones which in turn are of the form $(\pi, \mathcal{H})$ (Proposition 1.9). We therefore may assume that $\pi = \pi_\varphi$ is realized on a reproducing kernel space $\mathcal{H}_\varphi \subseteq O(S)$ with reproducing kernel $K(s, t) := \varphi(st^*)$. Then $K$ is invariant under the right action of any $g = (\lambda_g, \rho_g) \in U(M(S))$:

$$K(g, t) = \varphi((sg)(tg)^*) = \varphi(sgg^{-1}t^*) = \varphi(st^*) = K(s, t).$$

Hence $\tilde{\pi}_\varphi(g)(f) := f \circ \rho_g$ defines a unitary operator on $\mathcal{H}_\varphi$ satisfying $\tilde{\pi}_\varphi(g)\pi_\varphi(s) = \pi_\varphi(gs)$ for $s \in S$ (cf. [Ne99], Remark II.4.5).

(b) Since there exists a faithful representation $\pi: C^*(S, \alpha) \to B(\mathcal{H})$, this follows directly from (a). $\blacksquare$

Definition 2.4 (a) For a Lie group $G$ with Lie algebra $L(G)$, we call a smooth function $\exp_G: L(G) \to G$ an exponential function if for each $x \in L(G)$ the curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. In general an exponential function need not exist, but it is unique ([GN07]).

(b) A Lie group $G$ with an exponential function is called locally exponential if there exists an open 0-neighborhood $U$ in $L(G)$ for which $\exp_G|_U$ is a diffeomorphism onto an open subset of $G$.

Definition 2.5 We say that a net $(u_i)_{i \in I}$ in a topological involutive semigroup $S$ is an approximate identity if $\lim u_is = \lim su_i = s$ holds for all $s \in S$.

Remark 2.6 For any complex involutive semigroup $S$ with an approximate identity, the natural map $S \to M(S)$ is injective. In fact, if $(u_i)$ is an approximate identity of $S$, then the assertion follows from $\eta(s)u_i = su_i \to s$. We may thus identify $S$ with a subsemigroup of $M(S)$. 12
**Definition 2.7** Let $G$ be a connected Lie group with a smooth exponential function $\exp_G: L(G) \to G$. A triple $(S, \eta, W)$, consisting of a complex involutive semigroup $S$ with an approximate identity, a group homomorphism $\eta: G \to U(M(S))$ into the group of unitary holomorphic multipliers of $S$ and an open convex $\text{Ad}(G)$-invariant cone $W \subseteq L(G)$ is called a *host semigroup* for $G$ if the following conditions are satisfied:

(HS1) The left action of $G$ on $S$ defined by $\eta$ is smooth.

(HS2) For each $x \in W$, the one-parameter group $\eta_x: \mathbb{R} \to U(M(S))$, $t \mapsto \eta(\exp_G(tx))$ extends to a morphism $\hat{\eta}_x: \mathbb{C}_+ = \mathbb{R} + i[0, \infty[ \to M(S)$ of involutive semigroups defining a continuous left action of the closed upper halfplane $\mathbb{C}_+$ on $S$, $\hat{\eta}_x(\mathbb{C}^+_+ \subseteq S$ (considered as a subsemigroup of $M(S)$), and the corresponding map $\mathbb{C}^+_+ \to S$ is holomorphic.

(HS3) If $f: S \to \mathbb{C}$ is a holomorphic function for which all functions $f \circ \gamma^S_x$ vanish on the open upper half plane, then $f = 0$.

**Remark 2.8** Suppose that $(S, \eta, W)$ is a host semigroup of $G$ and $x \in W$. Then the one-parameter subsemigroup $\hat{\eta}_x(it)$, $t > 0$, is an approximate identity for $S$ in the sense that for each $s \in S$ we have

$$\lim_{t \to 0} \hat{\eta}_x(it)s = \lim_{t \to 0} s\hat{\eta}_x(it) = s$$

because the left action of the closed halfplane $\mathbb{C}_+$ on $S$ defined by $\hat{\eta}_x$ is continuous.

**Proposition 2.9** Let $G_C$ be a connected complex locally exponential Lie group whose Lie algebra $L(G)_C$ is the complexification of the real Lie algebra $L(G)$, $\sigma$ a holomorphic involutive automorphism of $G_C$ with $L(\sigma)(x + iy) = x - iy$, and $G := (G_C)^0$ the identity component of the group $G_C^0$ of $\sigma$-fixed points in $G_C$. Let $S \subseteq G_C$ be an open connected subsemigroup invariant under the involution $s^* := \sigma(s)^{-1}$ and $W \subseteq L(G)$ an open convex invariant cone with $\exp_{G_C}(iW) \subseteq S$ and $GSG = S$.

Then we obtain for each $g \in G$ a holomorphic multiplier $\eta(g) \in U(M(S))$ by $\eta(g) = (\lambda_g, \rho_g)$ and $(S, \eta, W)$ is a host semigroup for $G$. 

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Proof. (HS1) follows from the smoothness of the multiplication in $G_C$.

(HS2): For $x \in W$ we put $\hat{\eta}_x(z) := (\lambda_{\exp(zx)}, \rho_{\exp zx})$ and note that this is the multiplier corresponding to the semigroup element $\exp(zx)$ because for $z = a + ib$ we have $\exp(zx) = \exp(ax) \exp(ibx) \in GS \subseteq S$.

(HS3): Let $f : S \to \mathbb{C}$ be a holomorphic function vanishing on the sets $\exp(C^0_0, x), x \in W$. Let $\Omega \subseteq \exp^{-1}(S) \subseteq L(G)_C$ denote the connected component containing the cone $iW$. Then the holomorphic function $f \circ \exp : \Omega \to \mathbb{C}$ (cf. [GN07] for the holomorphy of $\exp$) vanishes on $iW$, hence on a neighborhood of $iW$, and therefore on all of $\Omega$. Since $G_C$ is locally exponential, there exists a point $x_0 \in iW$ (sufficiently close to 0) and an open neighborhood $U$ of $x_0$ in $\Omega$, such that $\exp(U)$ is an open subset of $S$. Then $f$ vanishes on $\exp(U)$ and hence on all of $S$ because $S$ is connected.

Example 2.10 Let $G = V$ be a locally convex space, $W \subseteq V$ an open convex cone and $S := V + iW$. Then $\eta(v)(s) := s + v$ yields a host semigroup $(S, \eta, W)$ for $V$.

2.2 Host algebras

Definition 2.11 If $\mathcal{A}$ is a $C^*$-algebra, then we write $M(\mathcal{A})$ for the set of continuous linear multipliers on $\mathcal{A}$. Then $M(\mathcal{A})$ carries a natural structure of a $C^*$-algebra and the map $\eta_A : \mathcal{A} \to M(\mathcal{A})$ is injective (cf. [Pe79], Sect. 3.12). We write $\mathcal{A}$ for its image in $M(\mathcal{A})$. The strict topology on $M(\mathcal{A})$ is the locally convex topology defined by the seminorms

$$p_a(m) := \|ma\| + \|am\|, \quad a \in \mathcal{A}, m \in M(\mathcal{A}).$$

The involution is continuous with respect to this topology and the multiplication is continuous on bounded subsets, which implies in particular that the unitary group $U(M(\mathcal{A}))$ is a topological group (cf. [Wo95], Sect. 2).

For a complex Hilbert space $\mathcal{H}$, we write $\text{Rep}(\mathcal{A}, \mathcal{H})$ for the set of non-degenerate representations of $\mathcal{A}$ on $\mathcal{H}$. Each representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ which is non-degenerate in the sense that $\pi(\mathcal{A})v = \{0\}$ implies $v = 0$ extends to a unique representation $\tilde{\pi}$ of $M(\mathcal{A})$ satisfying $\tilde{\pi} \circ \eta_A = \pi$ which is continuous with respect to the strict topology on $M(\mathcal{A})$ and the strong operator topology on $B(\mathcal{H})$ (cf. Proposition 8.4 below).

Examples 2.12 ([Pe79], Sect. 3.12) (a) If $\mathcal{A}$ is a closed $*$-subalgebra of $B(\mathcal{H})$, then $M(\mathcal{A}) \cong \{X \in B(\mathcal{H}) : XA + AX \subseteq \mathcal{A}\}$

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(b) If $\mathcal{A} = K(\mathcal{H})$ is the ideal of compact operators in $B(\mathcal{H})$, then $M(K(\mathcal{H})) \cong B(\mathcal{H})$.

c) If $\mathcal{A} = C_0(X)$ is the $C^*$-algebra of continuous functions vanishing at infinity on the locally compact space $X$, then $M(\mathcal{A}) \cong C_b(X)$ is the $C^*$-algebra of bounded continuous functions on $X$.

**Definition 2.13** Let $G$ be a topological group. A host algebra for $G$ is a pair $(\mathcal{A}, \eta)$, where $\mathcal{A}$ is a $C^*$-algebra and $\eta: G \to U(M(\mathcal{A}))$ is a group homomorphism such that:

(H1) For each non-degenerate representation $(\pi, \mathcal{H})$ of $\mathcal{A}$, the representation $\tilde{\pi} \circ \eta$ of $G$ is continuous.

(H2) For each complex Hilbert space $\mathcal{H}$, the corresponding map $\eta^*: \text{Rep}(\mathcal{A}, \mathcal{H}) \to \text{Rep}(G, \mathcal{H})$, $\pi \mapsto \tilde{\pi} \circ \eta$ is injective.

We say that $(\mathcal{A}, \eta)$ is a full host algebra if $\eta^*$ is surjective for each Hilbert space $\mathcal{H}$.

**Remark 2.14**

(a) If $\eta: G \to U(M(\mathcal{A}))$ is strictly continuous, i.e., continuous with respect to the strict topology on $U(M(\mathcal{A}))$, then Proposition 8.3(3) implies (H1).

(b) Since the extension $\tilde{\pi}$ of a non-degenerate representation $\pi$ of $\mathcal{A}$ is strictly continuous, condition (H2) holds if $\eta(G)$ spans a strictly dense subalgebra of $M(\mathcal{A})$. In view of [Wo95], Prop. 2.2, (H2) conversely implies that $\text{span}(\eta(G))$ is strictly dense in $M(\mathcal{A})$.

(c) For any multiplier action $\eta: G \to U(M(\mathcal{A}))$ of a topological group $G$ on the $C^*$-algebra $\mathcal{A}$, the subspace $R$, consisting of all elements $a \in \mathcal{A}$ for which the map $G \mapsto \mathcal{A}, g \mapsto \eta(g)a$ is continuous is a right ideal which is closed because $G$ acts by isometries on $\mathcal{A}$. It is biinvariant under $G$. Hence $\mathcal{A}_c := R \cap R^*$ is a $C^*$-subalgebra of $\mathcal{A}$ which is $G$-biinvariant, and the corresponding homomorphism $\eta_c: G \to U(M(\mathcal{A}_c))$ is strictly continuous.

(d) If a homomorphism $\eta: G \to U(M(\mathcal{A}))$ satisfies (H2), then (b) implies that $\eta(G)$ spans a strictly dense subalgebra of $M(\mathcal{A})$. This implies in particular, that the $G$-biinvariant closed subalgebra $\mathcal{A}_c$ of $\mathcal{A}$ is a two-sided ideal of $M(\mathcal{A})$ and the corresponding morphism $\gamma: M(\mathcal{A}) \to M(\mathcal{A}_c)$ is obviously strictly continuous and satisfies $\gamma \circ \eta = \eta_c$. 

We claim that \((\mathcal{A}_c, \eta_c)\) also is a host algebra for \(G\). In fact, (H1) follows from the strict continuity of \(\eta_c\). Next we note that \(\gamma(M(\mathcal{A}))\) contains \(\mathcal{A}_c\), so that it is strictly dense in \(M(\mathcal{A}_c)\) \([\text{Wo95}, \text{Prop. 2.2}]\). Therefore \(\eta_c(G) = \gamma(\eta(G))\) spans a strictly dense subalgebra of \(M(\mathcal{A}_c)\), which implies (H2), as we have seen in (b).

**Examples 2.15** (a) Let \(G\) be a locally compact group and \(C^*(G)\) the enveloping \(C^\ast\)-algebra of the group algebra \(L^1(G)\). Then we have a natural homomorphism \(\eta: G \to U(M(C^*(G)))\) which is determined by the left action \(\eta(g)(f)(x) = f(g^{-1}x)\) on \(L^1\)-functions. Since \(G\) acts continuously from the left and the right on \(L^1(G)\) and the image of \(L^1(G)\) is dense in \(C^*(G)\), \(\eta\) is continuous with respect to the strict topology. It is well known that \((C^*(G), \eta)\) is a full host algebra of \(G\) \([\text{Dix64}, \text{Sect. 13.9}]\).

(b) Let \(G\) be an abelian topological group and \(\hat{G} := \text{Hom}(G, \mathbb{T})\) its character group. Then any host algebra \((\mathcal{A}, \eta)\) for \(G\) is commutative because of the strict density of \(\eta(G)\) in \(M(\mathcal{A})\). Hence there exists a locally compact space \(X\) with \(\mathcal{A} \cong C_0(X)\) and \(M(\mathcal{A}) \cong C_b(X)\) \((\text{cf. Example 2.12})\). Then \(U(M(\mathcal{A})) \cong C(X, \mathbb{T})\), where the strict topology on this group corresponds to the compact open topology and a \(*\)-subalgebra of \(C_b(X)\) is strictly dense if and only if it separates the points of \(X\) \([\text{Br77}, \text{Lemma 3.5}]\). Therefore the map \(\gamma: X \to \hat{G}\) defined by \(\gamma(x)(g) := \eta(g)(x)\) is injective, so that we may consider \(X\) as a subset of the character group \(\hat{G}\).

If, conversely, \(X \subseteq \hat{G}\) is a subset, endowed with a locally compact topology finer than the topology of pointwise convergence on \(G\), then the natural map \(\eta: G \to C(X, \mathbb{T}) = U(M(C_0(X)))\) defined by \(\eta(g)(x) := \chi(g)\) satisfies (H2) because \(\eta(G)\) separates the points of \(X\). If, in addition, \(\eta\) is strictly continuous, i.e., each compact subset of \(X\) is equicontinuous, then (H1) is also satisfied, so that \((C_0(X), \eta)\) is a host algebra of \(G\).

(c) If \(\mathcal{A}\) is any \(C^\ast\)-algebra and \(G := U(M(\mathcal{A}))\), endowed with the strict topology, then the fact that \(G\) spans \(M(\mathcal{A})\) implies that \(\eta = \text{id}_G\) satisfies (H1) and (H2), so that \((\mathcal{A}, \text{id}_G)\) is a host algebra for \(G\).

**Example 2.16** (Non-uniqueness of host algebras) Let \(G := \mathbb{Z}\). Then its character group is \(\hat{G} \cong \mathbb{T}\), which is a compact group with respect to the topology of pointwise convergence. Since \(G\) is locally compact, \(C^*(G) \cong C(\mathbb{T})\) is a full host algebra for \(G\). Let \(\mathcal{A} := C_0([0, 1])\) and define a homomorphism \(\eta: \mathbb{Z} \to U(M(C_0([0, 1]))) \cong C([0, 1], \mathbb{T})\) by \(\eta(n)(x) := e^{2\pi i nx}\). Then \(\eta(1): [0, 1] \to \mathbb{T}\) is a continuous bijection, which implies in particular that
η(Z) separates the points, so that (H2) holds. Further, Z is discrete, so that (H1) is trivially satisfied, and thus (A, η) is a host algebra. This host algebra is full because the representations of Z are in one-to-one correspondence with Borel spectral measures on T and η(1) is a Borel isomorphism. Note in particular that the full host algebra A is not unital, although G is a discrete group.

Remark 2.17 Let G be a Lie group, A a C*-algebra and η: G → U(M(A)) a group homomorphism. We then obtain left and right actions of G on A by isometries. Let A∞ ⊆ A denote the set of all elements for which the orbit maps of these actions are smooth. The space A∞ r of smooth vectors for the left action ηl is a right ideal, the set A∞ r of smooth vectors for the right action ηr is a left ideal and both are exchanged by the involution. Hence their intersection A∞ is a *-subalgebra on which G acts by multipliers.

Definition 2.18 Let G be a Lie group and A a C*-algebra. We say that a homomorphism γ: G → U(M(A)) is strictly smooth if the *-subalgebra A∞ of smooth vectors for the left and right action of G on A is dense.

If this is the case, then the maximal C*-subalgebra Ac of A on which G acts continuously is dense, hence coincides with A, so that η is in particular strictly continuous (cf. Remark 2.14(c)).

Proposition 2.19 Let (S, η, W) be a host semigroup of the Lie group G and α a G-invariant locally bounded absolute value on S. Then η induces a strictly smooth homomorphism ˜η: G → U(M(C*(S, α))) determined uniquely by ˜η(g)(ηα(s)) = ηα(gs) for g ∈ G, s ∈ S, and (C*(S, α), ˜η) is a host algebra for G.

Proof. The existence of ˜η follows from Proposition 2.3, since U(M(S)) acts by unitary multipliers on C*(S, α). That ˜η defines a strictly smooth multiplier action of G on C*(S, α) follows from the relation ˜η(g)ηα(s) = ηα(gs), which implies that ηα(S) consists of smooth vectors for G because ηα: S → C*(S, α) is a holomorphic map. Hence ˜η is strictly smooth.

To see that ˜η defines a host algebra, let (πi, H), i = 1, 2, be two representations of C*(S, α) with π1 ◦ ˜η = π2 ◦ ˜η. For each x ∈ W we then have

π1 ◦ ˜ηx = π2 ◦ ˜ηx: R → U(H).

Now πi ◦ ηk: C+ → B(H), i = 1, 2, are two continuous representations of the involutive semigroup C+ which are holomorphic on C+ and coincide
on \( \mathbb{R} \), so that \([\text{Ne}99], \text{Lemma XI.2.2, implies that they are equal}\). Hence \( \pi_1 \circ \eta_x = \pi_2 \circ \eta_x \) for each \( x \in W \), so that \( \pi_1 - \pi_2 : S \rightarrow B(H) \) is a holomorphic function vanishing on all sets \( \gamma_x^S(C^*_+) \), and now (HS3) leads to \( \pi_1 = \pi_2 \). 

3 Multiplier actions of Lie groups on \( C^* \)-algebras

In the preceding section we have seen that we can associate to each host semigroup \( S \) and any \( G \)-invariant locally bounded absolute value \( \alpha \) on \( S \) a host algebra \( C^*(S, \alpha) \) for \( G \). In this section we slightly change our perspective and ask for properties of a homomorphism \( \eta : G \rightarrow U(M(A)) \) which are characteristic for a host algebras of the form \( A = C^*(S, \alpha) \). This will lead us to the momentum set \( I_\eta \subset L(G)' \) of the pair \( (A, \eta) \). We shall see in particular that the weak-*closed convex Ad\(^*\)(\( G \))-invariant set \( I_\eta \) tells us for which open invariant cones \( W \subset L(G) \) there might be a corresponding host semigroup.

3.1 Strictly continuous multiplier actions

If \( G \) is a topological group and \( A \) a \( C^* \)-algebra, we also call a strictly continuous homomorphism \( \eta : G \rightarrow U(M(A)) \) a strictly continuous multiplier action of \( G \) on \( A \).

Remark 3.1 (a) If \( G \) is a locally compact group, \( A \) is a \( C^* \)-algebra and the homomorphism \( \gamma : G \rightarrow U(M(A)) \) is strictly continuous, then integration yields a morphism

\[
\gamma : L^1(G) \rightarrow M(A), \quad \gamma(f) := \int_G f(g) \gamma(g) \, dg
\]

of Banach-*-algebras, so that the universal property of the \( C^* \)-algebra \( C^*(G) \), the enveloping \( C^* \)-algebra of \( L^1(G) \), implies the existence of a corresponding morphism of \( C^* \)-algebras \( \tilde{\gamma} : C^*(G) \rightarrow M(A) \) for which \( \tilde{\gamma}(C^*(G))A \supseteq \gamma(L^1(G))A \) is dense in \( A \) (use an approximate identity in \( L^1(G) \) and the strict continuity of the action of \( G \)) and we have \( \tilde{\gamma} \circ \eta = \gamma \).

If, conversely, \( \alpha : C^*(G) \rightarrow M(A) \) is a morphism of \( C^* \)-algebras for which \( \alpha(C^*(G))A \) is dense in \( A \), then Proposition 8.3 in the appendix implies that \( \alpha \) extends to strictly continuous morphism \( \tilde{\alpha} : M(C^*(G)) \rightarrow M(A) \). Hence
\[
\gamma := \tilde{\alpha} \circ \eta : G \rightarrow U(M(\mathcal{A}))
\]
is a strictly continuous homomorphism with \( \tilde{\gamma} = \tilde{\alpha} \).

We conclude that strictly continuous multiplier actions of \( G \) on a \( \mathcal{C}^* \)-algebra \( \mathcal{A} \) are in one-to-one correspondence with morphisms \( \alpha : \mathcal{C}^*(G) \rightarrow M(\mathcal{A}) \) for which \( \alpha(\mathcal{C}^*(G))\mathcal{A} \) is dense in \( \mathcal{A} \).

**Example 3.2** Let \( \mathcal{A} := K(\mathcal{H}) \) be the \( \mathcal{C}^* \)-algebra of compact operators on the complex Hilbert space \( \mathcal{H} \). Then \( M(\mathcal{A}) \cong B(\mathcal{H}) \) is the algebra of all bounded operators on \( \mathcal{H} \). Hence \( U(M(\mathcal{A})) \cong U(\mathcal{H}) \) is the unitary group of \( \mathcal{H} \) and the strict topology on this group coincides with the strong operator topology because if \( U(\mathcal{H}) \) carries the strong operator topology, the closed subalgebra \( K(\mathcal{H})c \) contains finite rank operators, hence coincides with \( K(\mathcal{H}) \). Therefore a strictly continuous multiplier action of a topological group \( G \) on \( K(\mathcal{H}) \) is the same as a continuous unitary representation on \( \mathcal{H} \).

**Remark 3.3** If \( G \) is a finite dimensional Lie group and \( \eta : G \rightarrow U(M(\mathcal{A})) \) is strictly continuous, then it is also strictly smooth. In fact, there exists a sequence \( (\delta_n)_{n \in \mathbb{N}} \in C^\infty_c(G, \mathbb{R}) \) which is an approximate identity in \( L^1(G) \). Then each \( a \in \mathcal{A} \) is the norm limit of the elements \( \eta(\delta_n)a\eta(\delta_n) \in \mathcal{A}^\infty \).

**Momentum sets of unitary representations**

**Definition 3.4** Let \( (\pi, \mathcal{H}) \) be a continuous unitary representation of the Lie group \( G \) on the Hilbert space \( \mathcal{H} \). We write \( \mathcal{H}^\infty \subseteq \mathcal{H} \) for the subspace of smooth vectors and note that this is a linear subspace on which we have a derived representation \( d\pi \) of the Lie algebra \( \mathfrak{g} = \mathfrak{L}(G) \). We call the representation \( (\pi, \mathcal{H}) \) smooth if \( \mathcal{H}^\infty \) is dense in \( \mathcal{H} \).

(a) Let \( \mathbb{P}(\mathcal{H}^\infty) = \{ [v] := \mathbb{C}v : 0 \neq v \in \mathcal{H}^\infty \} \) denote the projective space of the subspace \( \mathcal{H}^\infty \) of smooth vectors. The map

\[
\Phi_\pi : \mathbb{P}(\mathcal{H}^\infty) \rightarrow \mathfrak{g}' \quad \text{with} \quad \Phi_\pi([v])(x) = \frac{1}{i} \frac{\langle d\pi(x).v, v \rangle}{\langle v, v \rangle}
\]
is called the **momentum map of the unitary representation** \( \pi \). The right hand side is well defined because it only depends on \( [v] = \mathbb{C}v \). The operator \( i \cdot d\pi(x) \) is symmetric so that the right hand side is real, and since \( v \) is a smooth vector, it defines a continuous linear functional on \( \mathfrak{g} \).

(b) The weak-*-closed convex hull \( I_\pi \subseteq \mathfrak{g}' \) of the image of \( \Phi_\pi \) is called the **convex momentum set of** \( \pi \).
Remark 3.5 If $\pi: \mathcal{A} \to B(\mathcal{H})$ is a non-degenerate representation of a $C^*$-algebra and $\eta: G \to U(M(\mathcal{A}))$ a strictly smooth multiplier action, then the corresponding representation $\tilde{\pi} \circ \eta$ of $G$ on $\mathcal{H}$ has a dense space $\mathcal{H}^\infty$ of smooth vectors because the dense subspace spanned by $\pi(\mathcal{A}^\infty)\mathcal{H}$ consists of smooth vectors for $G$.

Lemma 3.6 Let $G$ be a Lie group with exponential function and $(\pi, \mathcal{H})$ a unitary representation with a dense space $\mathcal{H}^\infty$ of smooth vectors. Then, for each $x \in L(G)$, the unbounded operator

$$d\pi(x): \mathcal{H}^\infty \to \mathcal{H}, \quad d\pi(x)v := \frac{d}{dt}_{t=0}\pi(\exp_G(tx)).v$$

is essentially skewadjoint and its closure $\overline{d\pi(x)}$ is the infinitesimal generator of the unitary one-parameter group $\pi_x := \pi \circ \gamma_x$, $\gamma_x(t) = \exp_G(tx)$.

Proof. Since the dense subspace $\mathcal{H}^\infty$ is invariant under $\pi(\gamma_x(\mathbb{R}))$, the assertion follows from [RS75], Thm. VIII.10.

Lemma 3.7 If $(\pi, \mathcal{H})$ is a smooth unitary representation of the Lie group $G$ with exponential function, $x \in L(G)$, $\pi_x := \pi \circ \gamma_x$ and $A_x := -i d\pi_x(1)$ the corresponding selfadjoint operator, then

$$\inf \text{Spec}(A_x) = \inf \langle I_\pi, x \rangle.$$

Proof. For $m(x) := \inf \langle I_\pi, x \rangle \in \mathbb{R} \cup \{-\infty\}$, we have

$$\langle A_x.v, v \rangle \geq m(x)\langle v, v \rangle \quad \text{for each} \quad v \in \mathcal{H}^\infty,$$

and since the graph of the operator $-i \cdot d\pi(x)$ on $\mathcal{H}^\infty$ in dense in the graph of $A_x$ (Lemma 3.6), $\langle A_x.v, v \rangle \geq m(x)\langle v, v \rangle$ holds for each $v$ in the domain of $A_x$. This shows that $\inf \text{Spec}(A_x) \geq m(x)$. The converse inequality holds trivially.

Problem 3.8 Let $(\pi, \mathcal{H})$ be a unitary representation of the Lie group $G$ on $\mathcal{H}$. If $v$ is a smooth vector, then the function $\pi^v(g) := \langle g.v, v \rangle$ is smooth. In [Ne99], Prop. X.6.4 it is shown that the converse also holds if $G$ is finite dimensional. Does this result generalize to infinite dimensional Lie groups?
Holomorphic extension of multiplier actions

**Definition 3.9** Let \( \eta: G \to U(M(A)) \) be a strictly smooth multiplier action of \( G \) on the \( C^* \)-algebra \( A \) and \( \mathfrak{g} = \mathcal{L}(G) \).

We write \( S(A) \) for the set of states of \( A \). Since each state of \( A \) is of the form \( \varphi(a) = \pi^v(a) := \langle \pi(a).v, v \rangle \) for a unit vector \( v \in \mathcal{H} \) and a non-degenerate representation \((\pi, \mathcal{H})\) of \( A \), there exists a canonical extension \( \tilde{\varphi} := \tilde{\pi}^v \) to a state of \( M(A) \).

We call a state \( \varphi \) of \( A \) \( \eta \)-smooth if \( \tilde{\varphi} \circ \eta \) is smooth and write \( S(A)^\infty \) for the set of \( \eta \)-smooth states of \( A \). We now have a momentum map

\[ \Psi_\eta: S(A)^\infty \to \mathfrak{g}', \quad \Psi_\eta(\varphi) = \frac{1}{i} d(\tilde{\varphi} \circ \eta)(1). \]

Since \( S(A)^\infty \) is a convex set, the weak-\(*\)-closure

\[ I_\eta := \overline{\Psi_\eta(S(A)^\infty)} \subseteq \mathfrak{g}' \]

also is a convex subset of \( \mathfrak{g}' \); called the momentum set of \((A, \eta)\).

**Proposition 3.10** Let \( \eta: G \to U(M(A)) \) be a strictly smooth multiplier representation, \( I_\eta \subseteq \mathfrak{g}' \) its momentum set and \( m \in \mathbb{R} \). Then the following are equivalent for \( x \in \mathfrak{g} \):

1. If \( \eta_x(t) := \eta(\exp_G(tx)) \), then the corresponding homomorphism of \( C^* \)-algebras \( \tilde{\eta}_x: C^*(\mathbb{R}) \cong C_0(\mathbb{R}) \to M(A) \) factors through the quotient algebra \( C_0([m, \infty[) \).
2. \( I_\eta(x) \geq m \).
3. \( \eta_x \) extends to a strictly continuous homomorphism \( \tilde{\eta}_x: \mathbb{C}_+ \to M(A) \) of involutive semigroups which is holomorphic on \( \mathbb{C}_0^+ \) and satisfies

\[ \|\tilde{\eta}_x(z)\| \leq e^{-m \text{Im} z}. \]

If these conditions are satisfied, then \( \|\tilde{\eta}_x(a + ib)\| = e^{-b \inf I_\eta(x)} \).

**Proof.** (1) \( \iff \) (2): Let \((\pi, \mathcal{H})\) be a universal representation of \( A \), i.e., each state of \( A \) is of the form \( \pi^v(a) = \langle \pi(a).v, v \rangle \) for some unit vector \( v \in \mathcal{H} \). Then \( \tilde{\pi} \circ \eta \) is a smooth representation of \( G \) (Remark 3.5) and \( \eta_x \) also defines a continuous unitary representation \( \pi_x := \tilde{\pi} \circ \eta_x \) of \( \mathbb{R} \) on \( \mathcal{H} \).
For any smooth unit vector \( v \in \mathcal{H}^\infty \) and the corresponding smooth state \( \pi^v \) we then obtain with \( (\pi^v \circ \eta)(\exp_G(tx)) = \pi_x(t) \):

\[
\Psi_\eta(\pi^v)(x) = -i \cdot d(\pi^v \circ \eta)(x) = -i(d\pi(x).v, v) = \Phi_x([v])(x),
\]

which leads to \( I_\pi(x) \subseteq I_\eta(x) \). In view of \cite{Ne99}, Prop. X.6.4, the smoothness of a state \( \pi^v \) implies the smoothness of \( v \) for \( \pi_x \), so that \( I_\eta(x) \subseteq I_\pi(x) \). Lemma 3.7 now implies that \( \inf I_{\pi_x} = \inf \langle \pi_x, x \rangle = \inf I_\pi(x) \), so that we arrive at

\[
\inf I_\pi(x) = \inf I_\eta(x) = \inf I_{\pi_x}.
\]

A simple argument with the spectral measure of the unitary one-parameter group \( \pi_x \) shows that the kernel of the corresponding representation \( \hat{\pi}_x \) of \( C^*(\mathbb{R}) \cong C_0(\mathbb{R}) \) contains the ideal

\[
I_m := \{ f \in C_0(\mathbb{R}) : \text{supp}(f) \subseteq ]-\infty, m[ \}
\]

if and only if \( m \leq I_{\pi_x} \), which means that it factors through the quotient algebra \( C_0(\mathbb{R})/I_m \cong C_0([m, \infty[) \). If this is the case, then the image of \( \hat{\pi}_x \) lies in the multiplier algebra \( M(A) \cong \{ T \in B(\mathcal{H}) : T.A + A.T \subseteq A \} \) (Proposition 8.3). This proves the equivalence of (1) and (2).

(1) \(\Rightarrow\) (3): First we consider the map

\[
\gamma : \mathbb{C}^+ \to C_b([m, \infty[) = M(C_0([m, \infty[)), \quad \gamma(z) := e^{izt}.
\]

Then \( \|\gamma(z)\| = e^{-m\text{Im}z} \), so that \( \gamma \) is locally bounded. Since the strict topology on bounded subsets of \( C_b([m, \infty[) \) coincides with the compact open topology (cf. \cite{Br77}, Lemma 3.5), the strict continuity of \( \gamma \) follows from the continuity of the map \( \beta : \mathbb{C} \to C_b([m, \infty[), \beta(z)(t) := e^{izt} \) with respect to the compact open topology. That \( \gamma \) is holomorphic on the open upper halfplane \( \mathbb{C}^+ \) is a consequence of Example 17(c). Now (3) follows by composing the strictly continuous extension \( M(C_0([m, \infty[)) \cong C_b([m, \infty[) \to M(A) \) with \( \gamma \).

(3) \(\Rightarrow\) (2): By definition, \( \hat{\eta}_x \) induces a morphism \( \beta : C^*(\mathbb{C}^+_0, \alpha) \to M(A) \), where \( \alpha(z) := e^{-m\text{Im}z} \). Since \( \hat{\eta}_x \) is strictly continuous, \( \beta(\mathbb{C}^+_0)A \) is dense in \( A \), so that \( \beta \) extends to a strictly continuous morphism \( \hat{\beta} : M(C^*(\mathbb{C}^+_0, \alpha)) \cong C_b([m, \infty[) \to M(A) \) with \( \hat{\beta}(\gamma(t)) = \eta_x(t) \) for \( t \in \mathbb{R} \). Therefore the homomorphism \( \hat{\eta}_x : C^*(\mathbb{R}) \to M(A) \) factors through the quotient \( C_b([m, \infty[) \).
Remark 3.11  (a) As the example \( A = C_0([m, \infty[) \) shows, the map \( \hat{\eta}_x \) need not be norm continuous because the natural map

\[
\gamma : \mathbb{C}_+ \to C_b([m, \infty[), \quad \gamma(z)(t) = e^{itz}
\]

is not norm continuous at the boundary \( \mathbb{R} = \partial \mathbb{C}_+ \).

(b) Assume that the conditions of Proposition 3.10 are for the element \( x \in \mathfrak{g} \). Let \( \mathcal{B} := M(\mathcal{A}) \) denote the \( C^\ast \)-subalgebra consisting of all elements on which \( G \) acts continuously by multipliers from the left and the right. Then \( \eta_x(\mathbb{R})\mathcal{B} + \mathcal{B}\eta_x(\mathbb{R}) \subseteq \mathcal{B} \) implies that \( \eta_x \) induces a strictly continuous morphism \( \eta_x^\ast : \mathbb{R} \to U(M(\mathcal{B})) \).

Since the induced homomorphism \( C^\ast(\mathbb{R}) \to M(\mathcal{A}) \) factors through \( C_0([m, \infty[) \), the same holds for the corresponding morphism \( C^\ast(\mathbb{R}) \to M(\mathcal{B}) \). From that we conclude that we even obtain a strictly continuous morphism \( \mathbb{C}_+ \to M(\mathcal{B}) \) which is holomorphic on \( \mathbb{C}_0^\ast \).

Proposition 3.12  Let \( \eta : G \to U(M(\mathcal{A})) \) be the strictly smooth multiplier action defined by a host algebra of \( G \) obtained from a host semigroup \( (S, \eta, W) \) for which the map

\[
\text{Exp} : W \to S, \quad x \mapsto \hat{\eta}_x(i)
\]

is continuous. Then

\[
s : W \to \mathbb{R}, \quad s(x) := -\inf\langle I_\eta, x \rangle
\]

is a locally bounded function on \( W \).

Proof.  For each \( x \in W \), the homomorphism \( \eta_x : \mathbb{R} \to U(M(C^\ast(S, \alpha))) \) extends to a homomorphism of involutive semigroups \( \hat{\eta}_x : \mathbb{C}_+ \to M(C^\ast(S, \alpha)) \) which is holomorphic on \( \mathbb{C}_0^\ast \) and comes from a smooth multiplier action of \( \mathbb{C}_+ \) on \( S \) (Definition 2.7). We also have

\[
\|\hat{\eta}_x(z)\| \leq \alpha(\gamma_x(z)) \quad \text{for} \quad z \in \mathbb{C}_0^\ast.
\]

From Example 1.7(c) we now derive that \( \|\hat{\eta}_x(z)\| = e^{-m_x \text{Im} z} \) for some \( m_x \in \mathbb{R} \) and in particular that \( \hat{\eta}_x \) is locally bounded on \( \mathbb{C}_+ \). Therefore the continuity of the corresponding multiplier action of \( \mathbb{C}_+ \) on \( S \) implies the strict continuity of the corresponding multiplier action on \( C^\ast(S, \alpha) \). Now Proposition 3.10(3) tells us that \( \inf I_\eta(x) = m_x \), so that \( s(x) = -m_x \). Further,

\[
e^{s(x)} = e^{-m_x} = \|\hat{\eta}_x(i)\| = \|\eta_\alpha(\hat{\eta}_x(i))\| \leq \alpha(\text{Exp}(x))
\]

shows that \( s \) is locally bounded on the open cone \( W \).
4 Convex functions on infinite dimensional domains

In Proposition 3.12 we have seen how host algebras of a Lie group \( G \) coming from host semigroups lead to weak-\( \ast \)-closed convex subsets \( I_\eta \) of the dual \( L(G)' \) of the locally convex Lie algebra \( L(G) \) with the property that the support function \( x \mapsto -\inf \langle I_\eta, x \rangle \) is locally bounded on some open convex cone in \( L(G) \).

In this section we therefore take a closer look at weak-\( \ast \)-closed convex subsets \( C \) of the dual \( V' \) of a locally convex space \( V \). We are in particular interested in conditions for the cone \( B(C) = \{ v \in V : \inf \langle C, v \rangle > -\infty \} \) to have non-empty interior and the corresponding support function \( s_C(v) := -\inf \langle C, v \rangle \) to be locally bounded on \( B(C) \).

Convex subsets of locally convex spaces

Proposition 4.1 Let \( \emptyset \neq C \subseteq V \) be a closed convex set in the topological vector space \( V \).

(1) \( \lim(C) := \{ v \in V : C + v \subseteq C \} \) is a closed convex cone, called the recession cone of \( C \).

(2) \( \lim(C) = \{ v \in V : v = \lim_{n \to \infty} t_n c_n, c_n \in C, t_n \to 0, t_n \geq 0 \} \).

(3) If \( c \in C \) and \( x \in V \) satisfy \( c + \mathbb{R}^+ x \subseteq C \), then \( x \in \lim(C) \).

(4) If \( C \) is bounded, then \( \lim(C) = \{0\} \).

Proof. (1) The closedness of \( \lim(C) \) is an immediate consequence of the closedness of \( C \).

(2) If \( c \in C \) and \( x \in \lim(C) \), then \( c+nx \in C \) for \( n \in \mathbb{N} \) and \( \frac{1}{n}(c+nx) \to x \).

If, conversely, \( x = \lim_{n \to \infty} t_n c_n \) with \( t_n \to 0, t_n \geq 0 \) and \( c,c_n \in C \), then \( (1 - t_n)c + t_n c_n \to c + x \in \overline{C} = C \) implies that \( C + x \subseteq C \), i.e. \( x \in \lim(C) \).

(3) In view of (2), this follows from \( \frac{1}{n}(c+nx) \to x \).

(4) If \( C \) is bounded, each continuous linear functional \( f : V \to \mathbb{R} \) is bounded on \( C \). For each \( x \in \lim(C) \) the relation \( C + \mathbb{N} x \subseteq C \) then leads to \( f(x) = 0 \). Since \( V' \) separates the points of \( V \), we obtain \( x = 0 \).
Definition 4.2 Let $V$ be a locally convex space and $C \subseteq V'$ a subset. We put

$$B(C) := \{v \in V : \inf \langle C, v \rangle > -\infty\} \quad \text{and} \quad C^* := \{v \in V : \langle C, v \rangle \subseteq \mathbb{R}_+\}.$$ 

Then $C^* \subseteq B(C)$ are convex cones and $C^*$ is called the dual cone of $C$. If $C$ is a cone, then $B(C) = C^*$.

Let $C \subseteq V'$ be a weak-$*$-closed convex subset. As a consequence of the Hahn–Banach Separation Theorem, there exists for each element $\alpha \in V' \setminus C$ some $x \in V$ (the dual of $V'$ endowed with the weak-$*$-topology) with $\alpha(v) < \inf \langle C, v \rangle$. Then $v \in B(C)$, and we thus obtain

$$C = \{\alpha \in V' : (\forall v \in B(C)) \alpha(v) \geq \inf \langle C, v \rangle\},$$

which permits us to reconstruct $C$ from its support function $s_C(v) = -\inf \langle C, v \rangle$ on $V$.

Lemma 4.3 For a non-empty weak-$*$-closed convex subset $C \subseteq V'$, the following assertions hold:

(i) $B(C)$ is a convex cone satisfying $B(C)^* = \lim(C)$.

(ii) If $B(C)$ has non-empty interior, then $B(C)$ has the same interior as $\lim(C)^*$.

(iii) If $C$ is a cone, then $B(C) = C^*$ has non-empty interior if and only if $C$ has a weak-$*$-compact equicontinuous base.

Proof. (i) The relation $C + \lim(C) = C$ implies that every element in $B(C)$ is non-negative on $\lim(C)$, i.e. $\lim(C) \subseteq B(C)^*$.

Using (2), we see that for $x \in B(C)^*$, $c \in C$ and $f \in B(C)$, we have $f(x + c) \geq f(c) \geq \inf f(C)$, so that $x + C \subseteq C$. This proves $B(C)^* \subseteq \lim(C)$ and hence equality.

(ii) From the Hahn–Banach–Separation Theorem, we further derive $B(C) = (B(C)^*)^* = \lim(C)^*$. If $B(C)$ has non-empty interior, it coincides with the interior of its closure ([Bou07], Cor. II.2.6.1).

(iii) If $C$ has an equicontinuous weak-$*$-compact base $K$ and $x \in V$ satisfies $\langle K, x \rangle > \varepsilon$, then $\varepsilon \hat{K}$ (where $\hat{K}$ denotes the polar of $K$) is a 0-neighborhood in $V$ with $x + \varepsilon \hat{K} \subseteq C^*$, showing that $C^*$ has interior points.
If, conversely, $C^*$ has an interior point $x_0$ and $U$ is a convex symmetric 0-neighborhood with $x_0 + U \subseteq C^*$, then the polar set $\hat{U}$ is a weak-$*$-compact equicontinuous subset containing $K := \{ \alpha \in C : \alpha(x_0) = 1 \}$. Therefore $K$ is weak-$*$-compact and equicontinuous with $C = \mathbb{R}_+ K$ and $0 \notin K$, i.e., $K$ is a base of $C$.

If $B(C)$ has interior points, the following proposition shows that we can reconstruct $C$ from the values of $s_C$ on the open set $B(C)^0$.

**Proposition 4.4** Let $V$ be a locally convex space and $C \subseteq V'$ be a weak-$*$-closed convex subset for which the cone $B(C)^0$ has interior points. Then

$$C = \{ \alpha \in V' : (\forall x \in B(C)^0) \alpha(x) \geq \inf \langle C, x \rangle \}.$$

**Proof.** Let $D := \{ \alpha \in V' : (\forall x \in B(C)^0) \alpha(x) \geq \inf \langle C, x \rangle \}$. Then we have $C \subseteq D$ and both are weak-$*$-closed convex sets. If $\alpha \in D \setminus C$, then this implies the existence of some $x \in B(C)$ with $\alpha(x) < \inf \langle C, x \rangle$.

Let $x_0 \in B(C)^0$. Then, for each $t \in [0, 1]$, the element $x_t := (1-t)x_0 + tx$ is contained in $B(C)^0$, so that $\alpha(x_t) \geq \inf \langle C, x_t \rangle$. Now $F : [0, 1] \to \mathbb{R}, F(t) := -\inf \langle C, x_t \rangle$ is a lower semicontinuous convex function on a real interval, hence continuous ([Ne99], Cor. V.3.3). Therefore

$$\inf \langle C, x \rangle = \lim_{t \to 1} \inf \langle C, x_t \rangle \leq \lim_{t \to 1} \alpha(x_t) = \alpha(x),$$

and we get $\alpha(x) \geq \inf \langle C, x \rangle$. This contradiction implies that $C = D$. 

The following lemma is obvious:

**Lemma 4.5** For a weak-$*$-closed convex subset $C \subseteq V'$ the following are equivalent

1. $C$ is weak-$*$-bounded.
2. $B(C) = V$.
3. The polar set $\hat{C} := \{ v \in V : |\langle C, v \rangle| \leq 1 \}$ is absorbing.
Convex functions on domains in locally convex spaces

Definition 4.6 Let $X$ be a topological space. We say that a real-valued function $f: X \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is lower semicontinuous if for each $x_0 \in X$ and $c < f(x_0)$ there exists a neighborhood $U$ of $x_0$ with $\inf f(U) > c$. We call $f$ upper semicontinuous if for each $x_0 \in X$ and $d > f(x_0)$ there exists a neighborhood $U$ of $x_0$ with $\sup f(U) < d$.

Remark 4.7 A function $f: X \to \mathbb{R}_\infty$ is lower semicontinuous if and only if its epigraph $\text{epi}(f) := \{(x, t) \in X \times \mathbb{R}: f(x) \leq t\}$ is a closed subset of $X \times \mathbb{R}$.

Proposition 4.8 (cf. [Bou07], Prop. II.2.21) Let $\Omega \subseteq V$ be an open convex subset and $f: \Omega \to \mathbb{R}$ a lower semicontinuous convex function. Then the following are equivalent

(1) $f$ is continuous.

(2) $f$ is locally bounded.

(3) $f$ is bounded in a neighborhood of one point.

(4) $f$ is upper semicontinuous.

Proof. Since $f$ is assumed to be lower semicontinuous, (1) and (4) are equivalent. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2): Suppose that $f \leq M$ holds on the open convex neighborhood $U$ of $c_0 \in \Omega$. Let $c \in \Omega$. Then there exists an element $c_1 \in \Omega$ and $0 < t < 1$ with $c = (1-t)c_1 + tc_0$. Then $(1-t)c_1 + tU$ is an open subset of $\Omega$ containing $c$, and on this subset we have

$$\sup f((1-t)c_1 + tU) \leq (1-t)f(c_1) + t \sup f(U) < \infty.$$ 

Therefore $f$ is locally bounded.

(2) $\Rightarrow$ (4): Let $c \in \Omega$ and $U$ a closed convex 0-neighborhood in $V$ with $c + U \subseteq \Omega$ on which $f$ is bounded and $d > f(c)$. Then for each $t \in [0, 1]$, we have

$$\sup f(c + tU) = \sup f((1-t)c + t(c + U)) \leq (1-t)f(c) + t \sup f(c + U),$$

so that for some $t > 0$ close to 0, we have $\sup f(c + tU) \leq d$. Therefore $f$ is upper semicontinuous in $c$. $\square$
Remark 4.9  Under the circumstances of Lemma 4.8, \( s_C(x) := -\inf \langle C, x \rangle \) defines a convex function on all of \( V \). In view of the preceding proposition, this function is locally bounded if and only if it is continuous, and this is equivalent to the polar set \( \hat{C} \) being a 0-neighborhood. The polar set \( \hat{C} \) is a \textit{barrel}, i.e., a closed absolutely convex absorbing set. According to the Bipolar Theorem, each barrel \( B \) coincides with the polar \( \hat{C} \) of its polar \( C := \hat{B} \).

A locally convex space \( V \) is said to be \textit{barrelled} if all barrels in \( V \) are 0-neighborhoods. In view of the preceding remarks, this means that all functions \( s_C \) are continuous. The functions \( s_C, \hat{C} \) a barrel, are precisely the lower semicontinuous seminorms on \( V \), so that \( V \) is barrelled if and only if all lower semicontinuous seminorms are continuous (cf. [Bou07], §III.4.1).

The following theorem extends this remark to general convex functions.

Theorem 4.10  Let \( V \) be a barrelled space, \( \Omega \subseteq V \) an open convex set and \( f : \Omega \to \mathbb{R} \) a lower semicontinuous function. Then \( f \) is continuous.

Proof.  Pick \( x_0 \in \Omega \) and let \( U \) be a closed absolutely convex 0-neighborhood with \( x_0 + U \subseteq \Omega \). We consider the set
\[
B := \{ v \in U : f(x_0 \pm v) \leq f(x_0) + 1 \}
\]
and claim that \( B \) is a barrel. Since \( f \) is lower semicontinuous, \( B \) is a closed convex subset of \( U \). Moreover, \( v \in B \) and \( \lambda \in \mathbb{R} \) with \( |\lambda| \leq 1 \) implies \( \lambda v \in B \) because
\[
f(x_0 \pm \lambda v) \leq \text{conv}\{ f(x_0 \pm v) \} \leq f(x_0) + 1.
\]
We conclude that \( B \) is absolutely convex. To see that \( B \) is absorbing, let \( v \in U \) and observe that the lower semicontinuous function \( h(t) := f(x_0 + tv) \) on \( [-1, 1] \) is continuous ([Ne99], Cor. V.3.3). Hence there exists a \( \mu > 0 \) with \( \mu v \in B \). This proves that \( B \) is a closed absolutely convex absorbing set, hence a barrel. Since \( V \) is barrelled, \( B \) is a 0-neighborhood, and thus \( f \) is bounded on a neighborhood of \( x_0 \). The continuity of \( f \) now follows from Proposition 4.8. \( \blacksquare \)

Remark 4.11  For a locally convex space we have the following implications

\[ \text{Banach} \Rightarrow \text{Fréchet} \Rightarrow \text{Baire} \Rightarrow \text{barrelled}, \]

so that in particular all Frechet spaces are barrelled.
Furthermore, each locally convex direct limit of barrelled spaces is barrelled, which implies that there are barrelled spaces which are not Baire, f.i. $V := \mathbb{R}^{(N)}$, endowed with the finest locally convex topology, is such a space.

**Example 4.12** (A barrel which is not a 0-neighborhood) Let $X = c_0$, which is a non-reflexive Banach space and $V := \ell^1$ its topological dual, endowed with the weak-$*$-topology. Then the closed unit ball $B \subseteq V$ is a barrel which is not a zero neighborhood because each 0-neighborhood contains a subspace of finite codimension.

**Proposition 4.13** Let $C \subseteq V'$ be a non-empty weak-$*$-closed convex subset and $x \in B(C)$ such that the support function $s_C$ is bounded on some neighborhood of $x$. Then the following assertions hold:

1. For each $m \in \mathbb{R}$ the subset $C_m := \{ \alpha \in C : \alpha(x) \leq m \}$ is equicontinuous and weak-$*$-compact.

2. The function $\eta(x) : C \to \mathbb{R}, \eta(x)(\alpha) := \alpha(x)$ is proper.

3. There exists an extreme point $\alpha \in C$ with $\alpha(x) = \min \langle C, x \rangle$.

4. $C$ is weak-$*$-locally compact.

**Proof.**

1. Pick a 0-neighborhood $U \subseteq V$ for which $s_C$ is bounded on $x + U$ by some constant $M$. Then $\langle C, x + U \rangle \geq -M$, and hence

$$\langle C_m, U \rangle \geq \langle C_m, x + U \rangle - m \geq -M - m.$$ 

This implies that $s_{C_m}$ is bounded from below on $U$, and hence $s_{C_m}$ is bounded from above on $-U$. Therefore the polar $\hat{C}_m$ contains a multiple of $U \cap -U$, hence is a neighborhood of 0. This is equivalent to $C_m$ being equicontinuous. Now the Banach–Alaoglu–Bourbaki Theorem implies that $C_m$ is weak-$*$-compact because it is a closed subset of the polar set of a 0-neighborhood in $U$.

2. Follows immediately from (1).

3. Pick $M > \inf\langle C, x \rangle$. Then the weak-$*$-compactness of $C_M$ implies the existence of a minimal value $m = \min \eta(x)(C)$. Then $C_m := \eta(x)^{-1}(m) \cap C$ is a weak-$*$-compact convex set, so that the Krein–Milman Theorem implies the existence of an extreme point $e$ of $C_m$. Since $C_m$ is a face of $C$, $e$ also is an extreme point of $C$.

4. For any $\alpha \in C$, (1) implies that the set $C_{\alpha(x)+1}$ is a compact neighborhood of $\alpha$ in $C$. Hence $C$ is weak-$*$-locally compact. 

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Remark 4.14 If $C \subseteq V'$ is a weak-$*$-closed convex subset which is locally compact with respect to the weak-$*$-topology, then its recession cone $\lim(C)$ is also locally compact because, for each $\alpha \in C$, the subset $\alpha + \lim(C)$ of $C$ is closed. Therefore Exercise II.7.21(a) in [Bou07] implies that the cone $\lim(C)$ has a weak-$*$-compact base $K$, but if $K$ is not equicontinuous, this does not imply that the dual cone $\lim(C)^* = B(C)$ has interior points (Lemma 4.3).

Remark 4.15 If $C \subseteq V'$ is weak-$*$-locally compact, then we consider the commutative $C^*$-algebra $A := C_0(C)$. Clearly, the map

$$\eta: V \to M(C_0(C)) \cong C_b(C), \quad \eta(x)(\alpha) := e^{i\alpha(x)}$$

is a multiplier action of the abelian topological group $V$ on $C_0(C)$. This action is strictly continuous if and only if $\eta$ is continuous with respect to the compact open topology on $C_b(C)$ (cf. [Br77], Lemma 3.5). If this is the case, then the additive map $\tilde{\eta}: V \to C_b(C), \tilde{\eta}(x)(\alpha) = \alpha(x)$ is also continuous with respect to the compact open topology, and this is equivalent to the equicontinuity of each compact subset of $C$. Then $\eta: V \to C_b(C)$ defines a strictly continuous multiplier action and Example 2.15(b) implies that $(C_0(C), \eta)$ is a host algebra for $V$.

Problem 4.16 (a) Suppose that for some $x \in V$ all sets

$$C_m := \{\alpha \in C: \alpha(x) \leq m\}$$

are equicontinuous, hence in particular weak-$*$-compact. Then $\eta(x)(\alpha) := \alpha(x)$ defines a proper function on $C$, showing that $C$ is locally compact. The equicontinuity of the sets $C_m$ implies that the cone $\lim(C)$ has an equicontinuous weak-$*$-compact base, so that $\lim(C)^*$ has interior points (Lemma 4.3(iii)). Does this imply that $B(C)$ has interior points and that the support function $s_C$ is bounded on some neighborhood of $x$?

(b) If, in addition, some function $e^{-\tilde{\eta}(x)}, \tilde{\eta}(x)(\alpha) = \alpha(x)$, is contained in $C_0(C)$, then $\tilde{\eta}(x)$ is proper and bounded from below. Does the requirement that

$$\eta^{-1}(C_0(C)) \subseteq B(C)$$

has interior points imply that $s_C$ is bounded on some open set? According to Theorem 4.10, this is the case if $V$ is barrelled.
5 Host $C^*$-algebras coming from tubes

In this section we briefly take a closer look at the host algebras of a locally convex space $V$, defined by a complex involutive semigroup of the type $S = V + iW$ for an open convex cone $W \subseteq V$ (Proposition 2.9). In view of Remark 1.6(b) and the discussion in Example 1.7, any locally bounded absolute value $\alpha$ on such a semigroup leads to the same $C^*$-algebra as an absolute value of the form

$$\alpha_C(x + iy) := e^{-\inf\langle C,y \rangle},$$

where $C \subseteq V'$ is a weak-$\ast$-closed convex subset with $W \subseteq B(C)$ whose support function is locally bounded on $W$. The following theorem provides a converse:

**Theorem 5.1** Let $C \subseteq V'$ be a weak-$\ast$-closed convex subset for which the cone $B(C)$ has interior point and the support function $s_C$ is locally bounded on $B(C)^0$. Then the following assertions hold:

1. $S := V + iB(C)^0$ is an open complex involutive subsemigroup of $V_C$ and $\alpha_C(x + iy) := e^{-\inf\langle C,y \rangle}$ is a locally bounded absolute value on $S$.

2. $C$ is weak-$\ast$-locally compact.

3. The map $\gamma: S \to C_0(C)$, $\gamma(s)(f) := e^{if(s)}$ induces an isomorphism of $C^*$-algebras $C^*(S, \alpha_C) \to C_0(C)$.

4. The homomorphism $\eta: V \to C(C, \mathbb{T}) = U(M(C_0(C)))$, $\eta(x)(f) := e^{if(x)}$ defines a host algebra of $V$ with a strictly smooth multiplier action. The corresponding momentum set is $I_\eta = C$.

**Proof.** (1) is an immediate consequence of the definition.

(2) This follows from Proposition 4.13.

(3) For each element $s = x + iy \in S$, the continuous function $\gamma(s)$ on $C$ satisfies

$$\|\gamma(s)\| = \sup_{f \in C} |e^{if(s)}| = \sup_{f \in C} e^{-f(y)} = e^{-\inf\langle C,y \rangle} = \alpha_C(s).$$
Moreover, for each $\varepsilon > 0$, the subset
\[
\{ f \in C : e^{-f(y)} = |\gamma(s)(f)| \geq \varepsilon \} = \{ f \in C : f(y) \leq -\log \varepsilon \}
\]
is weak-$*$-compact (Proposition \textit{[1.13]}, so that $\gamma(s) \in C_0(C)$.

We thus obtain a morphism $\gamma : S \to C_0(C)$ of involutive semigroups with $\|\gamma(s)\| \leq \alpha_C(s)$. Hence $\gamma$ is locally bounded, and to see that it is holomorphic, it suffices to verify its holomorphy on the intersection of $S$ with each complex subspace $E_C \subseteq V_C$, where $E \subseteq V$ is finite dimensional.

Using that the $C^*$-algebra $C_0(C)$ has a realization as a closed subalgebra of some algebra of the form $B(\mathcal{H})$, we first use [Ne99], Thm. VI.2.3, to see that $\rho := \gamma|_{iB(C)^0 \cap E_C}$ is a norm continuous morphisms of semigroups. Further, Proposition VI.3.2 in [Ne99] implies that $\rho$ extends to a unique holomorphic homomorphism $\hat{\rho} : S \cap E_C \to B(\mathcal{H})$, but since $iB(C)^0 \cap E$ is totally real in $S \cap E_C$, the values of the unique holomorphic extension $\hat{\rho}$ also lie in the closed subspace $C_0(C)$ of $B(\mathcal{H})$. For each $f \in C$, the function $\hat{\rho}(s)(f)$ on $S \cap E_C$ is the unique holomorphic extension of the function $\eta(s)(f) = \rho(s)(f)$, and from the holomorphy of $S \to C$, $s \mapsto \gamma(s)(f)$ we derive that $\hat{\eta} = \gamma|_{S \cap E_C}$. This proves that $\gamma$ is holomorphic on $S \cap E_C$, and hence that $\gamma$ is holomorphic.

Now the universal property of the $C^*$-algebra $C^*(S, \alpha_C)$ leads to a unique morphism $\hat{\gamma} : C^*(S, \alpha_C) \to C_0(C)$ of $C^*$-algebras with $\hat{\gamma} \circ \eta = \gamma$ (Theorem \textit{[1.5]}).

To see that $\hat{\gamma}$ is injective, we recall that the characters of $C^*(S, \alpha_C)$ separate the points, so that it suffices to show that they are all of the form $f \mapsto \hat{\gamma}(f)(f)$ for some $f \in C$. Any non-zero character $\chi$ of $C^*(S, \alpha_C)$ is uniquely determined by the holomorphic character $\chi_S := \chi \circ \eta : S \to \mathbb{C}$. We claim that $\chi_S(s) \subseteq \mathbb{C}^\times$. Indeed, $\chi_S(s) = 0$ implies $\chi_S(s + S) = \{0\}$, so that $\chi_S = 0$ would follow by analytic continuation. Now $\chi_S(s) \subseteq \mathbb{C}^\times$ shows that we have a corresponding smooth character $\chi_V : V \to \mathbb{T}$, obtained from the smooth multiplier action of $V$ on $S$. We further derive that there exists some $\beta \in V'$ with
\[
\chi_S(s) = e^{i\beta(s)} \quad \text{for} \quad s \in S = V + iB(C)^0.
\]
Since any morphism of $C^*$-algebras is contractive, we get
\[
|\chi_S(x + iy)| = e^{-\beta(y)} \leq \alpha_C(s) = e^{-\inf(C,y)},
\]
i.e., $\beta(y) \geq \inf(C, y)$ for $x \in B(C)^0$. Now we apply Proposition \textit{[1.4]} to obtain $\beta \in C$. If, conversely, $\beta \in C$, then $e^{i\beta}$ defines an $\alpha_C$-bounded holomorphic
character of \( S \), and the universal property of \( C^*(S, \alpha_C) \) implies that this character extends to a character of \( C^*(S, \alpha_C) \). These arguments show that the characters of \( C^*(S, \alpha_C) \), resp., the \( \alpha_C \)-bounded holomorphic characters of \( S \), are of the form \( s \mapsto \gamma(s)(\beta) \) for some \( \beta \in C \). As we have already observed above, this implies that \( \hat{\gamma} \) is injective, hence an isometric embedding (\cite{Dix64}, Cor. I.8.3).

In view of the Stone–Weierstraß Theorem, the fact that the functions in \( \gamma(S) \) have no zeros and separate the points of \( C \) implies that \( \hat{\gamma} \) has dense image. We know already that \( \hat{\gamma} \) is isometric, so that its range is closed. Therefore \( \hat{\gamma} \) is an isomorphism of \( C^* \)-algebras.

(4) First we combine Proposition 2.19 with Example 2.10 to see that \((C_0(C), \eta)\) is a host algebra of \( V \) with strictly smooth multiplier action.

To calculate the corresponding momentum set, we first recall from (3) that \( S_\alpha = C \), so that the character \( \chi_f(\xi) := \xi(f) \) defined by \( f \in C \) defines a smooth state of \( C_0(C) \) with \( \widetilde{\chi}_f(\eta(x)) = e^{i\langle f(x) \rangle} \), which leads to \( \Psi_\eta(\chi_f) = f \), and thus \( C \subseteq I_\eta \). On the other hand, Proposition 3.10 shows that for each \( y \in B(C)_0 \), we have

\[
e^{-\inf I_\eta(y)} = \|\eta_y(i)\| = \alpha(iy) = e^{-\inf \langle C, y \rangle},
\]

so that \( \inf \langle C, y \rangle = \inf I_\eta(y) \), which leads to \( I_\eta \subseteq C \) (Proposition 4.4). \( \blacksquare \)

The preceding theorem implies in particular that each weak-\( * \)-closed convex subset \( C \subseteq V' \) whose support function is bounded on some open subset actually occurs as the momentum set of some host algebra of \( V \). Conversely, we have seen in Example 1.7(c) that all host algebras defined by complex semigroups of the form \( V + iW \), \( W \) an open convex cone in \( V \), are of this form. We thus obtain a complete picture for the case where \( G = (V, +) \) is the additive group of a locally convex space.

Remark 5.2 One can also develop a holomorphic representation theory of tubes of the form \( V + iW \) by starting with representations of the cone \( iW \cong W \) by selfadjoint operators. This program has been carried out in great generality by H. Glöckner in \cite{Gl03} (cf. also \cite{Gl00}).

6 The finite dimensional case

In the preceding section we have described all host algebras of locally convex spaces defined by complex host semigroups. In the non-commutative
case this turns out to be much harder. However, using [Ne99], we also obtain a complete picture for finite dimensional groups. We shall take up the investigation of the non-abelian infinite dimensional case in the future.

We write $S^\infty(G)$ for the set of smooth states of $G$, i.e., the set of smooth positive definite functions normalized by $\varphi(1) = 1$. We consider $S^\infty(G)$ as a convex subset of the set $S(G)$ of continuous states of $G$, which in turn can be identified with the state space $S(C^*(G))$ of the group algebra $C^*(G)$. We recall the following result from [Ne99], Prop. X.6.17.

**Proposition 6.1** Let $C \subseteq L(G)^*$ be a closed convex invariant subset and

$$\text{ev}: S^\infty(G) \to L(G)^*, \quad \varphi \mapsto \frac{1}{i} d\varphi(1).$$

Then the annihilator $I_C := ev^{-1}(C)^\perp$ is an ideal of $C^*(G)$. The non-degenerate representations of the quotient algebra $C^*(G)_C := C^*(G)/I_C$ correspond to those continuous unitary representations $(\pi, \mathcal{H})$ of $G$ satisfying $I_\pi \subseteq C$.

**Theorem 6.2** Let $G$ be a connected finite dimensional Lie group, $(S, \eta S, W)$ a host semigroup of $G$ and $\alpha$ a locally bounded absolute value on $S$. Then the following assertions hold:

(a) The host algebra $(C^*(S, \alpha), \eta)$ is a quotient of $C^*(G)$.

(b) If, in addition, $G$ acts on $C^*(S, \alpha)$ with discrete kernel and the polar map $G \times W \to S, (g, x) \mapsto g \exp x$ is a diffeomorphism, then $C^*(S, \alpha) \cong C^*(G)_{I_\eta}$.

**Proof.** (a) Let $(\pi, \mathcal{H})$ be the universal representation of $C^*(S, \alpha)$. Then we have a holomorphic representation $\hat{\pi}: S \to B(\mathcal{H})$ whose image generates the $C^*$-algebra $\mathcal{B} := \pi(C^*(S, \alpha)) \cong C^*(S, \alpha)$ (Theorem 1.5).

Let $\pi_G: G \to U(\mathcal{H})$ denote the corresponding unitary representation of $G$ and $\hat{\pi}_G$ the associated representation of $C^*(G)$. Since $G$ acts smoothly by multipliers on $S$, we obtain a continuous multiplier action of $G$ on $\mathcal{B}$, and this leads to $\hat{\pi}_G(C^*(G))\mathcal{B} \subseteq \mathcal{B}$, where the left hand side is dense in $\mathcal{B}$.

On the other hand, $G$ also acts continuously by unitary multipliers on $C^*(G)$, hence on $\hat{\pi}_G(C^*(G))$. For each $x \in W$, we now obtain a morphism of $C^*$-algebras $\hat{\pi}_x: C^*(\mathbb{R}) \to M(\hat{\pi}_G(C^*(G)))$ which factors through some quotient $C_0([m, \infty[)$. This implies that $\hat{\pi}(\exp(x)) \in M(\hat{\pi}_G(C^*(G)))$, and from that we obtain $\hat{\pi}(S) \subseteq M(\hat{\pi}_G(C^*(G)))$ by analytic continuation.
and (HS3) in the definition of a host semigroup. This in turn leads to
\( B \hat{\pi}_G(C^*(G)) \subseteq \hat{\pi}_G(C^*(G)) \). Since \( \tilde{\pi}_x(C_0([m, \infty[)) \hat{\pi}_G(C^*(G)) \) is dense in \( \hat{\pi}_G(C^*(G)) \), we see that \( B \hat{\pi}_G(C^*(G)) \) spans a dense subspace of \( \hat{\pi}_G(C^*(G)) \).

We now arrive that
\[
B = \text{span}(\tilde{\pi}_G(C^*(G))) = \hat{\pi}_G(C^*(G)),
\]
and this proves (a).

(b) From the proof of (1) \( \Leftrightarrow \) (2) in Proposition 3.10 we recall that \( I_\pi = I_\eta \).

Let \( C := I_\eta \). Then the ideal \( I_C \) of \( C^*(G) \) annihilates all states of the form \( \pi_v, v \in H \), so that \( I_C \subseteq \ker \hat{\pi}_G \).

By assumption, \( d\pi \) is a faithful representation of \( L(G) \), so that \( \{0\} = \ker d\pi \) implies that \( I_\pi \) spans the dual space \( L(G)' \). Therefore the open cone \( W \subseteq B(I_x) \) satisfies \( \text{Spec}(ad x) \subseteq i\mathbb{R} \) for each \( x \in W \) ([Ne99, Prop. VII.3.4(b)]), i.e., \( W \) is a weakly elliptic cone ([Ne99, Def. XI.1.11]). The construction in Section XI.1 in [Ne99] now leads to a complex involutive semigroup \( \Gamma_G(W) \) for which the polar map
\[
G \times W \to \Gamma_G(W), \quad (g, x) \mapsto g \exp(x)
\]
is a diffeomorphism. According to the Holomorphic Extension Theorem ([Ne99, XI.2.3]), each unitary representation \( (\rho, K) \) of \( G \) with \( I_\rho \subseteq C = I_\eta \) extends via
\[
\rho: \Gamma_G(W) \to \hat{\pi}(S) \subseteq B \subseteq B(H), \quad g \exp x \mapsto \pi(g)e^{i\rho(x)}
\]
to a holomorphic representation \( \hat{\rho} \) of \( \Gamma_G(W) \) with
\[
\|\hat{\rho}(g \exp x)\| = e^{-\inf(I_\rho, x)} \leq e^{-\inf(C, x)} = e^{-\inf(I_\eta, x)} \leq \alpha(g \exp x),
\]
hence to a representation of \( B \). In view of (a), these representations separate the points of \( C^*(G)_C \), which implies that \( I_C = \ker \hat{\pi}_G \). We finally obtain \( C^*(S, \alpha) \cong B \cong C^*(G)_C \).

**Remark 6.3** If \( G = V \) is a finite dimensional vector space, \( W \subseteq V \) an open convex cone, and \( C \subseteq V' \) a closed convex subset, then \( C^*(G) \cong C_0(\hat{G}) \cong C_0(V') \), and the definition of \( C^*(G)_C \), implies that \( C^*(G)_C \cong C_0(C) \). We have already seen in Section 5 that this is \( C^*(S, \alpha) \) for \( S = V + iW \) and \( \alpha(x + iy) = e^{-\inf(C, y)} \).
7 Open Problems

Problem 7.1 (Invariant convex geometry of Lie algebras) Study open invariant convex cones in the Lie algebra $L(G)$ of an infinite dimensional Lie group $G$. Here are some concrete problems:

- Does it have any consequence for the spectrum of $ad\, x$ if $x$ is contained in an open invariant convex cone $W$ not containing affine lines? What can be said about the stabilizer of $x$ in $G$ and its action on the Lie algebra $L(G)$? In the finite-dimensional case it acts like a compact group and, consequently, $ad\, x$ is semisimple with $\text{Spec}(ad\, x) \subseteq i\mathbb{R}$ (cf. [Ne99]).

- Develop a structure theory for coadjoint orbits $O_{f} := Ad^{\ast}(G).f \subseteq L(G)'$ for which the weak-$\ast$-closed convex hull $C_{f}$ has the property that $B(C_{f})$ has interior points and the support function $s_{C_{f}}$ is locally bounded on the interior. For any such orbit which separates the points of $L(G)$ (which can always be arranged after factorization of a closed ideal), the open cone $B(C_{f})^0$ does not contain affine lines. It is a natural question under which circumstances the coadjoint orbit is closed. The geometric setup leads to the alternative that either $O_{f}$ consists of extreme points of its weak-$\ast$-closed convex hull or not, where the latter case does not arise for closed orbits in the finite-dimensional case (cf. Section VIII.1 in [Ne99]).

Remark 7.2 If $x \in B(C)^0$ and $f \in C$ is a unique minimum of the function $\tilde{\eta}(x)(\alpha) = \alpha(x)$ on $C$, then the stabilizer of $x$ in $G$ is contained in the stabilizer of $G_{f}$ and it also preserves all weak-$\ast$-compact subsets

$$C_{m} = \{ \alpha \in C : \alpha(x) \leq m\}$$

of $L(G)'$. This situation should lead to interesting geometric structures on the coadjoint orbit $O_{f}$, such as weak Kähler structures.

Example 7.3 Let $A$ be a unital $C^\ast$-algebra and $G = U(A)$ its unitary group, considered as a Banach–Lie group. Then for each state $\varphi \in S(A)$ the functional

$$-i\varphi : L(G) = \{ a \in A : a^* = -a \} = u(A) \rightarrow \mathbb{R}$$

is real-valued. If $\varphi$ is a pure state, i.e., an extreme point of $S(A)$, then the coadjoint orbit $O_{-i\varphi}$ consists of extreme points of its weak-$\ast$-closed convex
hull, which is the weak-$\ast$-closed convex face of $-iS(A)$, generated by $-i\varphi$ ([Ne02], Thm. III.1).

**Problem 7.4** (Holomorphic extensions of unitary representations)

1. Suppose that $(\pi, \mathcal{H})$ is a unitary representation of the infinite dimensional Lie group $G$ for which the subspace $\mathcal{H}^\infty$ of smooth vectors is dense and $B(I_\pi)$ has interior points. Let $x \in B(I_\pi)^0$. Is the selfadjoint operator $e^{i\cdot d\pi(x)} \in B(\mathcal{H})$ a smooth vectors for the multiplier action of $G$ on $B(\mathcal{H})$?

2. Is $\pi(G)e^{i\cdot d\pi(B(I_\pi)^0)}$ a subsemigroup of $B(\mathcal{H})$?

3. Suppose that there exists a host semigroup $(S, \eta, W)$ for $G$ for which the polar map $G \times W \to S, (g, x) \mapsto g\text{Exp}(x)$ is a diffeomorphism. Assume that $W \subseteq B(I_\pi)$ for some smooth unitary representation $(\pi, \mathcal{H})$ of $G$. Is the map $\hat{\pi}: S \to B(\mathcal{H}), g\text{Exp}(x) \mapsto \pi(g)e^{i\cdot d\pi(x)}$ a holomorphic representation?

4. If $L(G)$ contains a dense locally finite subalgebra, many of the arguments seem to be reducible to the finite dimensional situation.

**Problem 7.5** (Existence of complex semigroups) Let $G$ be a Banach–Lie group (or locally exponential) and $W \subseteq L(G)$ an open convex cone satisfying $\text{Spec(ad}x) \subseteq i\mathbb{R}$ for each $x \in W$. We assume that $G$ has a faithful universal complexification $\eta: G \to GC$ (which is locally exponential if $G$ is not Banach). Is it true that $G\text{exp}(iW)$ a subsemigroup of $GC$? For some interesting examples of such semigroups we refer to [Ne01].

8 Appendix: Some useful facts on multiplier algebras

The following results are used in our discussion of general host algebras of topological groups.

**Theorem 8.1** ([Pa94], Th. 5.2.2) Let $\mathcal{A}$ be a Banach algebra with bounded left approximate identity and $T: \mathcal{A} \to B(X)$ a continuous representation of $\mathcal{A}$ on the Banach space $X$. Then for each $y \in \text{span}(T(\mathcal{A})X)$ there are elements $a \in \mathcal{A}$ and $x \in X$ with $y = T(a)x$.  

Corollary 8.2 If $A$ and $B$ are $C^*$-algebras and $\pi: A \to M(B)$ is a homomorphism for which $\pi(A)B$ is dense in $B$, then each element $y \in B$ can be written as $\pi(a)b$ for some $a \in A$ and $b \in B$.

Proposition 8.3 Let $A$ and $B$ be $C^*$-algebras. For each morphism $\alpha: A \to M(B)$ of $C^*$-algebras for which $\alpha(A)B$ is dense in $B$, there exists a unique morphism of $C^*$-algebras $\tilde{\alpha}: M(A) \to M(B)$ extending $\alpha$, and $\tilde{\alpha}$ is strictly continuous.

Proof. The uniqueness of $\tilde{\alpha}$ follows from the density of $\alpha(A)B$ in $B$ and $\tilde{\alpha}(m)\alpha(a)b = \alpha(ma)b$ for $m \in M(A)$, $a \in A$, $b \in B$.

For the existence, we realize $B$ as a closed $*$-subalgebra of some $B(H)$ for which the representation on $H$ is non-degenerate. Then $M(B) \cong \{T \in B(H) : TB + BT \subseteq B\}$ (Examples 2.12) and we interprete $\alpha$ as a representation of $A$ on $H$.

We claim that $\alpha$ is non-degenerate. Indeed, if $\alpha(A)v = \{0\}$, then

$$\{0\} = \langle \alpha(A)v, BH \rangle = \langle B\alpha(A)v, H \rangle$$

implies $B\alpha(A)v = \{0\}$, and since $B\alpha(A) = (\alpha(A)B)^*$ is dense in $B$, we obtain $v = 0$.

As $\alpha$ is non-degenerate, there exists a unique extension $\tilde{\alpha}: M(A) \to B(H)$ with $\tilde{\alpha}(m)\alpha(a) = \alpha(ma)$ for $m \in M(A)$ and $a \in A$. Then

$$\tilde{\alpha}(m)\alpha(A)B \subseteq \alpha(A)B \subseteq B \quad \text{and} \quad B\alpha(A)\tilde{\alpha}(m) \subseteq B\alpha(A) \subseteq B,$$

and the density of $\alpha(A)B$, resp., $B\alpha(A)$ in $B$ implies that $\tilde{\alpha}(M(A)) \subseteq M(B)$.

Suppose that $c_i \to c$ strictly in $M(A)$. Then we have for $a \in A$ and $b \in B$ the relation $\tilde{\alpha}(c_i)\alpha(a)b = \alpha(c_i)a)b \to \alpha(ca)b$, and since $B = \alpha(A)B$ (Corollary 8.2), we get $\tilde{\alpha}(c_i)b \to \tilde{\alpha}(c)b$ for each $b \in B$, showing that $\tilde{\alpha}(c_i) \to \tilde{\alpha}(c)$ in the strict topology on $M(B)$.

Proposition 8.4 Let $A$ be a $C^*$-algebra and $M(A)$ its multiplier algebra. Then the following properties of a representation $(\rho, H)$ of $M(A)$ are equivalent

1. $\rho|_A$ is non-degenerate.
(2) \( \rho = \tilde{\pi} \) for some non-degenerate representation of \( \mathcal{A} \).

(3) The representation \( \rho \) is continuous with respect to the strict topology on \( B \) and the strong operator topology on \( B(\mathcal{H}) \).

Proof. (1) \( \Rightarrow \) (2): Let \( \pi := \rho|_\mathcal{A} \) and assume that this representation is non-degenerate, so that it has an extension to a representation \( \tilde{\pi} \) of \( M(\mathcal{A}) \) which is uniquely determined by \( \tilde{\pi}(m)\pi(a) = \pi(ma) \) for \( m \in M(\mathcal{A}) \) and \( a \in \mathcal{A} \). Since \( \rho \) also satisfies \( \rho(m)\pi(a) = \rho(m)\rho(a) = \rho(ma) = \pi(ma) \), we obtain \( \rho = \tilde{\pi} \).

(2) \( \Rightarrow \) (3): Let \( (b_i)_{i \in I} \) be a net in \( B \) converging to some \( b \in B \) with respect to the strict topology, i.e., \( b_i a \to ba \) and \( ab_i \to ab \) for each \( a \in \mathcal{A} \). For each \( v \in \mathcal{H} \) and \( a \in \mathcal{A} \) we then have \( b_i.(a.v) = (b_i.a).v \to (ba).v = b.(a.v) \) and

\[
 b_i^*.v = (a^*b_i)^*.v \to (a^*b)^*.v = b^*.v.
\]

We conclude that \( b_i.v \to b.v \) and \( b_i^*.v \to b^*.v \) hold for all vectors \( v \in \text{span}(\mathcal{A}\mathcal{H}) \), but Theorem 8.1 implies that \( \mathcal{H} = \mathcal{A}\mathcal{H} \), proving (3).

(3) \( \Rightarrow \) (1): Let \( (u_i)_{i \in I} \) be an approximate identity in \( \mathcal{A} \). Then \( u_i \to 1 \) in the strict topology on \( M(\mathcal{A}) \). Hence we get for each \( v \in \mathcal{H} \) the relation \( u_i.v \to v \), showing that \( \rho|_\mathcal{A} \) is non-degenerate. ■

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