THE DEGREE OF THE ALGEBRA OF COVARIANTS OF A BINARY FORM

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ABSTRACT. We calculate the degree of the algebra of covariants $C_d$ for binary $d$-forms. We obtain the integral representation and asymptotic behavior of the degree.

1. Introduction. Let $R = R_0 \oplus R_1 \oplus \cdots$, $R_0 = \mathbb{C}$, be a finitely generated graded commutative $\mathbb{C}$-algebra without zero divisors. Denote by

$$P(R, z) = \sum_{j=0}^{\infty} \dim R_j z^j,$$

its Poincaré series. Letting $r$ be the transcendence degree of the quotient field of $R$ over $\mathbb{C}$, the number

$$\deg(R) := \lim_{z \to 1} (1 - z)^r P(R, z),$$

is called the degree of the algebra $R$. The first two terms of the Laurent series expansion of $P(R, z)$ at the point $z = 1$ have the following form

$$P(R, z) = \frac{\deg(R)}{(1 - z)^r} + \frac{\psi(R)}{(1 - z)^{r-1}} + \cdots.$$

The numbers $\deg(R)$ and $\psi(R)$ are important characteristics of the algebra $R$. For instance, if $R$ is an algebra of invariants of a finite group $G$, then $\deg(R)^{-1}$ is an order of the group $G$ and $2 \frac{\psi(R)}{\deg(R)}$ is the number of pseudo-reflections in $G$, see [2].

Let $V_d$ be the standard $(d+1)$-dimensional complex representation of $SL_2$, and let $\mathcal{I}_d := \mathbb{C}[V_d]^{SL_2}$ be the corresponding algebra of invariants. In the language of classical invariant theory the algebra $\mathcal{I}_d$ is called the
algebra of invariants for binary forms of degree \(d\). The following explicit formula for the degree \(\deg(I_d)\) was derived by Hilbert in [4]:

\[
\deg(I_d) = \begin{cases} 
-\frac{1}{4d!} \sum_{0 \leq e < d/2} (-1)^e \binom{d}{e} \left( \frac{d}{2} - e \right)^{d-3}, & \text{if } d \text{ is odd,} \\
-\frac{1}{2d!} \sum_{0 \leq e < d/2} (-1)^e \binom{d}{e} \left( \frac{d}{2} - e \right)^{d-3}, & \text{if } d \text{ is even.}
\end{cases}
\]

In [6, 7], Springer obtained two different proofs of this result. Also, he found an integral representation and the asymptotic behavior for Hilbert’s constants. For this purpose, Springer [7] derived an explicit formula for the Poincaré series \(\mathcal{P}(I_d, z)\).

Let \(\mathcal{C}_d\) be the algebra of the covariants of binary \(d\)-forms, i.e., \(\mathcal{C}_d \cong \mathbb{C}[V_1 \oplus V_d]^{SL_2}\). In the present paper, acting in the spirit of Springer’s papers, we calculate \(\deg(\mathcal{C}_d)\) and \(\psi(\mathcal{C}_d)\). The following formulas hold:

\[
\deg(\mathcal{C}_d) = \frac{1}{d!} \sum_{0 \leq j < d/2} (-1)^j \binom{d}{j} \left( \frac{d}{2} - j \right)^{d-1}
\]

and

\[
\psi(\mathcal{C}_d) = \frac{1}{2} \deg(\mathcal{C}_d).
\]

Also, we calculate both an integral representation and the asymptotic behavior of the constants. For this purpose we use the explicit formula for the Poincaré series \(\mathcal{P}(\mathcal{C}_d, z)\) derived by the first author in [1].

2. Computation of \(\deg(\mathcal{C}_d)\). The algebra of covariants \(\mathcal{C}_d\) is a finitely generated graded algebra

\[
\mathcal{C}_d = (\mathcal{C}_d)_0 \oplus (\mathcal{C}_d)_1 \oplus \cdots \oplus (\mathcal{C}_d)_i \oplus \cdots,
\]

where the subspaces \((\mathcal{C}_d)_i\) of covariants of degree \(i\) are each finite-dimensional, and \((\mathcal{C}_d)_0 \cong \mathbb{C}\). The formal power series

\[
\mathcal{P}(\mathcal{C}_d, z) = \sum_{i=0}^{\infty} \dim((\mathcal{C}_d)_i) z^i,
\]
is called the Poincaré series of the algebra of covariants $C_d$. The finite
generation of $C_d$ implies that its Poincaré series is the power series
expansion of a rational function.

The following theorem shows an explicit form for this rational
function. Let $\varphi_n$, $n \in \mathbb{N}$, be the linear operator that transforms a
rational function $f$ in $z$ to a rational function $\varphi_n(f)$ which is defined
on the power $z^n$ by

$$(\varphi_n(f))(z^n) = \frac{1}{n} \sum_{j=0}^{n-1} f(\zeta_n z), \quad \zeta_n = e^{2\pi i/n}.$$ 

**Theorem 2.1** ([1]). The Poincaré series $P(C_d, z)$ has the following
form:

$$P(C_d, z) = \sum_{0 \leq j < d/2} \varphi_{d-2j} \left( \frac{(-1)^j z^{j+1}(1+z)}{(z^2, z^2)_j (z^2, z^2)_{d-j}} \right),$$

where $(a, q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$ denotes the $q$-shifted
factorial.

It is well known that the transcendence degree of the quotient field
for the algebra of covariants $C_d$ over $\mathbb{C}$ coincides with the order of pole
$z = 1$ for the rational function $P(C_d, z)$ and equals $d$. Therefore, the
first terms of the Laurent series for $P(C_d, z)$ at the point $z = 1$ are

$$P(C_d, z) = \frac{\deg(C_d)}{(1-z)^d} + \frac{\psi(C_d)}{(1-z)^{d-1}} + \cdots.$$ 

In order to calculate the rational coefficients $\deg(C_d)$ and $\psi(C_d)$ we
shall prove several auxiliary facts.

**Lemma 2.2.** The following statements hold:

(i) the first terms of the Taylor series for the function $(z^2, z^2)_j$ at
$z = 1$ are $(z^2, z^2)_j = 2^j j!(1-z)^j - 2^{j-1} j! j^2 (1-z)^{j+1} + \cdots$;
(ii) the first terms of the Laurent series for the function

$$\frac{(-1)^j z^{j+1}(1+z)}{(z^2, z^2)_j (z^2, z^2)_{d-j}}$$

at $z = 1$
are:
\[
\frac{(-1)^j}{2^{d-1}j!(d-j)!} \cdot \frac{1}{(1-z)^d} + \frac{(-1)^j(d+1)}{2^d j!(d-j)!} (d-2j-1) \cdot \frac{1}{(1-z)^{d-1}} + \cdots.
\]

Proof.

(i) We have
\[
(z^2, z^2)_j = (1 - z^2)(1 - z^4) \cdots (1 - z^{2j}).
\]
Let us expand the polynomial \(1 - z^n\) in a Taylor series about \(z = 1\). We have
\[
1 - z^n = -n(z - 1) - \frac{n(n-1)}{2!} (z-1)^2 + \cdots
= n(1-z) - \frac{n(n-1)}{2!} (1 - z)^2 + O((1-z)^3).
\]
Therefore,
\[
(z^2, z^2)_j = (1 - z^2)(1 - z^4) \cdots (1 - z^{2j})
= (2(1-z) - \frac{2}{2!} (1-z)^2 + \cdots)
\times \left(4(1-z) - \frac{4 \cdot 3}{2!} (1-z)^2 + \cdots\right)
\times \left(2j(1-z) - \frac{2j(2j-1)}{2!} (1-z)^2 + \cdots\right)
= (2 \cdot 4 \cdots 2j(1-z)^j + (1 + 3 + 5 \cdots + 2j-1)
\times 2^{j-1}j!(1-z)^{j+1} + \cdots)
= 2^j j!(1-z)^j - 2^{j-1}j! j^2(1-z)^{j+1} + \cdots.
\]
It follows that
\[
(z^2, z^2)_j(z^2, z^2)_{d-j} = (2^j j!(1-z)^j - 2^{j-1}j! j^2(1-z)^{j+1} + \cdots)
\times ((d-j)^2 + j^2)(1-z)^{d+1} + \cdots.
\]
(ii) To find the first terms of the Laurent series for the function

\[
\frac{(-1)^j z^{j(j+1)}(1 + z)}{(z^2, z^2)_j (z^2, z^2)_{d-j}},
\]

we expand the numerator in the Taylor series expressed in terms of powers of \((1 - z)\). We have:

\[
1 + z = 2 - (1 - z),
\]

\[
z^{j(j+1)} = 1 - j(j + 1)(1 - z) + \cdots,
\]

\[(1 + z)z^{j(j+1)} = 2 - (2j(j + 1) + 1)(1 - z) + \cdots.
\]

It is easy to check that the following decomposition holds:

\[
\frac{a_0 + a_1 x + \cdots}{b_0 + b_1 x + \cdots} = \frac{a_0}{b_0} + \frac{a_1 b_0 - a_0 b_1}{b_0^2} x + \cdots, b_0 \neq 0.
\]

Then

\[
\frac{(-1)^j z^{j(j+1)}(1 + z)}{(z^2, z^2)_j (z^2, z^2)_{d-j}} = \frac{2 - (2j(j+1)+1)(1-z) + \cdots}{2^d j! (d-j)! (1-z)^{d-j} - 2^d j! (d-j)! ((d-j)^2 + j^2)(1-z)^{d+1} + \cdots}
\]

\[
= \frac{1}{(1-z)^d} \left( \frac{1}{2^d j! (d-j)!} + \frac{(-1)^j (d+1)}{2^d j! (d-j)!} (d-2j-1)(1-z) + \cdots \right).
\]

\[\square\]

The following lemma shows how the function \(\varphi_n\) acts on the negative powers of \(1 - z\).

**Lemma 2.3.** For \(h \in \mathbb{N}\),

\[
\varphi_n \left( \frac{1}{(1-z)^h} \right) = \sum_{i=0}^{h} \frac{\alpha_{n,i}}{(1-z)^i},
\]

where \(\alpha_{nh} = n^{h-1}\) and \(\alpha_{n,h-1} = -n^{h-2}(n-1)h/2\).
Proof. Using article [1, Lemma 4], we get
\[ \varphi_n \left( \frac{1}{(1 - z)^h} \right) = \frac{\varphi_n \left( (1 + z + z^2 + \ldots + z^{n-1})^h \right)}{(1 - z)^h}. \]

Obviously, \( \alpha_{nh} \) is the remainder after the division of \( \varphi_n \left( (1 + z + z^2 + \ldots + z^{n-1})^h \right) \) by \( (1 - z) \).

Using the definition of the function \( \varphi_n \) we get
\[ \varphi_n \left( (1 + z + z^2 + \ldots + z^{n-1})^h \right) = \frac{1}{n} \sum_{j=0}^{n-1} \left( 1 + \zeta_n^j z + (\zeta_n^j)^2 z^2 + \ldots + (\zeta_n^j)^{n-1} z^{(n-1)} \right)^h \bigg|_{z^n = z}. \]

The remainder of division of this polynomial by \( (1 - z) \) is equal to its value at the point \( z = 1 \). Thus,
\[ \varphi_n \left( (1 + z + z^2 + \ldots + z^{n-1})^h \right) \bigg|_{z=1} = \frac{1}{n} \sum_{j=0}^{n-1} \left( 1 + \zeta_n^j + (\zeta_n^j)^2 + \ldots + (\zeta_n^j)^{n-1} \right)^h = n^{h-1}. \]

Obviously, \( \alpha_{n,h-1} \) is the coefficient of \( (1 - z) \) in the Taylor series expansion for
\[ \varphi_n \left( (1 + z + z^2 + \ldots + z^{n-1})^h \right) \]
at the point \( z = 1 \). Therefore,
\[ \alpha_{n,h-1} = -\lim_{z \to 1} (\varphi_n \left( (1 + z + z^2 + \ldots + z^{n-1})^h \right))^\prime. \]

We have
\[ \left( \frac{1}{n} \sum_{j=0}^{n-1} \left( 1 + \zeta_n^j z + (\zeta_n^j)^2 z^2 + \ldots + (\zeta_n^j)^{n-1} z^{(n-1)} \right)^h \right)^\prime = \frac{h}{n} \sum_{j=0}^{n-1} \left( 1 + \zeta_n^j z + (\zeta_n^j)^2 z^2 + \ldots + (\zeta_n^j)^{n-1} z^{(n-1)} \right)^{h-1} \times \left( \zeta_n^j + 2(\zeta_n^j)^2 z + \cdots + (n-1)(\zeta_n^j)^{n-1} z^{(n-2)} \right). \]
It now follows that

\[
\lim_{z \to 1} \left( \frac{1}{n} \sum_{j=0}^{n-1} \left( 1 + \zeta_n^j z + (\zeta_n^j)^2 z^2 + \cdots + (\zeta_n^j)^{n-1} z^{(n-1)} \right)^h \right)'
\]

\[
= \frac{h}{n} \sum_{j=0}^{n-1} \left( \sum_{k=0}^{h-1} (\zeta_n^j)^k \right)^{h-1} \left( \zeta_n^j + 2(\zeta_n^j)^2 + \cdots + (n-1)(\zeta_n^j)^{n-1} \right)
\]

\[
= \frac{h}{n} h^{-1}(1 + 2 + \ldots + (n-1)) = \frac{1}{2} h(n-1)n^{h-1}.
\]

By using the relation

\[
\lim_{z \to 1} (f(z^n)|_{z^n=z})' = \frac{1}{n} \lim_{z \to 1} f'(z^n),
\]

we get

\[
\alpha_{n,h-1} = - \lim_{z \to 1} (\varphi_n \left( (1 + z + z^2 + \cdots + z^{n-1})^h \right))'
\]

\[
= - \lim_{z \to 1} \left( \frac{1}{n} \sum_{j=0}^{n-1} \left( 1 + \zeta_n^j z + (\zeta_n^j)^2 z^2 + \cdots + (\zeta_n^j)^{n-1} z^{(n-1)} \right)^h \right)'
\]

\[
= - \frac{1}{2} h(n-1)n^{h-2}.
\]

Now we can compute \( \deg(C_d) \) and \( \psi(C_d) \).

**Theorem 2.4.**

\[
\deg(C_d) = \lim_{z \to 1} (1-z)^d \mathcal{P}(C_d, z)
\]

\[
= \frac{1}{d!} \sum_{0 \leq j < d/2} (-1)^j \binom{d}{j} \left( \frac{d}{2} - j \right)^{d-1}
\]

and

\[
\psi(C_d) = \lim_{z \to 1} \left( -(1-z)^d \mathcal{P}(C_d, z) \right)' = \frac{1}{2} \deg(C_d).
\]

**Proof.** Using Lemmas 1 and 2 we get

\[
\mathcal{P}(C_d, z) = \sum_{0 \leq j < d/2} \varphi_{d-2j} \left( \frac{(-1)^j z^j (j+1)}{(z^2, z^2)_{j+1}} + \frac{-(1 + z)}{(z^2, z^2)_{d-j}} \right)
\]
\[
\begin{align*}
&= \sum_{0 \leq j < d/2} \varphi_{d-2j} \left( \frac{(-1)^j}{2^{d-1}j!(d-j)!} \frac{1}{(1-z)^d} + \cdots \right) \\
&= \sum_{0 \leq j < d/2} \frac{(-1)^j}{2^{d-1}j!(d-j)!} \varphi_{d-2j} \left( \frac{1}{(1-z)^d} \right) \\
&\quad + \sum_{0 \leq j < d/2} \frac{(-1)^j (d+1)}{2^{d-1}j!(d-j)!} \left( \frac{1}{2} d - j - \frac{1}{2} \right) \\
&\quad \times \varphi_{d-2j} \left( \frac{1}{(1-z)^{d-1}} \right) + \cdots \\
&= \frac{1}{(1-z)^d} \sum_{0 \leq j < d/2} \frac{(-1)^j (d-2j)^{d-1}}{2^{d-1}j!(d-j)!} \\
&\quad - \frac{1}{(1-z)^{d-1}} \frac{1}{2} \sum_{0 \leq j < d/2} \frac{(-1)^j}{2^{d-1}j!(d-j)!} \\
&\quad \times (d-2j)^{d-2}(d-2j-1)(d-1) + \frac{1}{(1-z)^{d-1}} \frac{1}{2} \sum_{0 \leq j < d/2} \\
&\quad \times \frac{(-1)^j}{2^{d-1}j!(d-j)!} (d+1)(d-2j-1)(d-2j)^{d-2} + \cdots.
\end{align*}
\]

Thus, the coefficient of \( \frac{1}{(1-z)^d} \) is

\[
\text{deg} (C_d) = \sum_{0 \leq j < d/2} \frac{(-1)^j (d-2j)^{d-1}}{2^{d-1}j!(d-j)!} \\
= \frac{1}{d!} \sum_{0 \leq j < d/2} (-1)^j \binom{d}{j} \left( \frac{d}{2} - j \right)^{d-1},
\]

and the coefficient of \( \frac{1}{(1-z)^{d-1}} \) is

\[
\psi (C_d) = \frac{1}{2d!} \sum_{0 \leq j < d/2} (-1)^j \binom{d}{j} \left( \frac{d}{2} - j \right)^{d-1}.
\]

\[\square\]
3. **Asymptotic behavior of** $\deg(C_d)$. Let us establish an integral representation for the degree $\deg(C_d)$. We denote by

$$c_d := \deg(C_d) \cdot d! = \sum_{0 \leq j < d/2} (-1)^j \left( \frac{d}{j} \right) \left( \frac{d}{2} - j \right)^{d-1}.$$

The following statement holds:

**Lemma 3.1.**

(i) $c_d = 2\pi^{-1}(d - 1)! \int_0^\infty \frac{\sin^d x}{x^d} dx$,

(ii) $\deg(C_d) > 0$.

**Proof.**

(i) We have:

$$2c_d = \sum_{0 \leq j < d/2} (-1)^j \left( \frac{d}{j} \right) \left( \frac{d}{2} - j \right)^{d-1}$$

$$+ \sum_{0 \leq j < d/2} (-1)^j \left( \frac{d}{j} \right) \left( \frac{d}{2} - j \right)^{d-1}$$

$$= \sum_{0 \leq j < d/2} (-1)^j \left( \frac{d}{j} \right) \left( \frac{d}{2} - j \right)^{d-1}$$

$$+ \sum_{d/2 \leq j < d} (-1)^j \left( \frac{d}{j} \right) \text{sign} \left( \frac{d}{2} - j \right) \left( \frac{d}{2} - j \right)^{d-1}$$

$$= \sum_{j=0}^{d} (-1)^j \left( \frac{d}{j} \right) \text{sign} \left( \frac{d}{2} - j \right) \left( \frac{d}{2} - j \right)^{d-1}.$$

We use that

$$\frac{\pi}{2} \text{sign} (a) = \int_0^\infty \frac{\sin ax}{x} dx.$$
Then
\[
\pi c_d = \frac{\pi}{2} \sum_{j=0}^{d} (-1)^j \binom{d}{j} \text{sign} \left( \frac{d}{2} - j \right) \left( \frac{d}{2} - j \right)^{d-1}
\]
\[
= \sum_{j=0}^{d} (-1)^j \binom{d}{j} \left( \frac{d}{2} - j \right)^{d-1} \int_{0}^{\infty} \frac{\sin(d/2-j)x}{x} \, dx
\]
\[
= \int_{0}^{\infty} \text{Im} \left( \sum_{j=0}^{d} (-1)^j \binom{d}{j} \left( \frac{d}{2} - j \right)^{d-1} e^{i(d/2-j)x} \right) \frac{dx}{x}, \quad i^2 = -1.
\]

This follows by the same method as in [7, Lemma 3.4.7]. We have:
\[
\sin^d \frac{x}{2} = \left( \frac{e^{ix/2} - e^{-ix/2}}{2i} \right)^d
\]
\[
= \frac{1}{2^d d!} \sum_{j=0}^{d} \binom{d}{j} \left( e^{ix/2} \right)^{d-j} \left( e^{-ix/2} \right)^j
\]
\[
= \frac{1}{2^d d!} \sum_{j=0}^{d} (-1)^j \binom{d}{j} e^{ix(d/2-j)}.
\]

Differentiating \(d-1\) times with respect to \(x\), we obtain
\[
\left( \sin^d \frac{x}{2} \right)^{(d-1)} = \frac{i^{d-1}}{2^d d!} \sum_{0 \leq j \leq d} (-1)^j \binom{d}{j} \left( \frac{d}{2} - j \right)^{d-1} e^{ix(d/2-j)}.
\]

Hence,
\[
\text{Im} \left( \sum_{j=0}^{d} (-1)^j \binom{d}{j} \left( \frac{d}{2} - j \right)^{d-1} e^{i(d/2-j)x} \right) = 2^d \left( \sin^d \frac{x}{2} \right)^{d-1} + \frac{1}{\pi} \int_{0}^{\infty} 2^d \left( \sin^d \frac{x}{2} \right)^{d-1} \frac{dx}{x}.
\]

Thus,
\[
c_d = \frac{1}{\pi} \int_{0}^{\infty} 2^d \left( \sin^d \frac{x}{2} \right)^{d-1} \frac{dx}{x} = \frac{2}{\pi} \int_{0}^{\infty} \left( \sin^d x \right)^{d-1} \frac{dx}{x}.
\]
Integrating by parts $d - 1$ times, we obtain

$$c_d = \frac{2(d - 1)!}{\pi} \int_{0}^{\infty} \frac{\sin^d x}{x^d} \, dx.$$  

(ii) It is enough to prove that

$$\int_{0}^{\infty} \frac{\sin^d x}{x^d} \, dx > 0.$$  

First of all, we prove that the integral is absolutely convergent. Let us split the integral into two parts:

$$\int_{0}^{\infty} \frac{\sin^d x}{x^d} \, dx = \int_{0}^{1} \frac{\sin^d x}{x^d} \, dx + \int_{1}^{\infty} \frac{\sin^d x}{x^d} \, dx$$

Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, the function $(\frac{\sin x}{x})^p$ is continuous on $[0, 1]$. Thus, the first integral is convergent. Since

$$\left| \frac{\sin^d x}{x^d} \right| \leq \frac{1}{x^d},$$

then the second integral is absolutely convergent for $d > 1$.

Now the integral can be represented in the form

$$\int_{0}^{\infty} \frac{\sin^d x}{x^d} \, dx$$

$$= \sum_{j=0}^{\infty} \left( \int_{2j\pi}^{(2j+1)\pi} \frac{\sin^d x}{x^d} \, dx + \int_{(2j+1)\pi}^{(2j+1)\pi} \frac{\sin^d x}{x^d} \, dx \right)$$

$$= \sum_{j=0}^{\infty} \left( \int_{2j\pi}^{(2j+1)\pi} \frac{\sin^d x}{x^d} \, dx + \int_{2j\pi}^{(2j+1)\pi} \frac{\sin^d (x + \pi)}{(x + \pi)^d} \, dx \right)$$

$$\geq \sum_{j=0}^{\infty} \int_{2j\pi}^{(2j+1)\pi} \left( \frac{\sin^d x}{x^d} - \frac{\sin^d x}{(x + \pi)^d} \right) \, dx$$

$$= \sum_{j=0}^{\infty} \int_{2j\pi}^{(2j+1)\pi} \frac{\sin^d x}{x^d (x + \pi)^d} \left( (x + \pi)^d - x^d \right) \, dx > 0, \quad d > 1.$$
For the case $d = 1$, we have
\[
\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2} > 0. \tag*{\square}
\]

Condition $\deg(C_d) > 0$ is equivalent to the statement that the transcendence degree of the field of fractions of the algebra $C_d$ is equal to $d$.

Interestingly, in the general case, the Wolstenholme formula holds:
\[
\int_0^\infty \frac{\sin^p x}{x^s} \, dx = \frac{(-1)^{p-s/2} \pi}{(s-1)!} \frac{\pi}{2^p} \sum_{p-2j>0} (-1)^j \binom{p}{j} (p - 2j)^{s-1},
\]
if $p - s$ is even, see [3, Problem 1033].

Finally, we deal with the asymptotic behavior of $\deg(C_d)$ as $d$ tends to infinity. By the previous lemma, it is enough to determine the asymptotic behavior of
\[
\int_0^\infty \frac{\sin^d x}{x^d} \, dx.
\]

**Theorem 3.2.**
\[
\lim_{d \to \infty} d^{1/2} \int_0^\infty \frac{\sin^d x}{x^d} \, dx = \frac{(6\pi)^{1/2}}{2}.
\]

**Proof.** Write
\[
I = \lim_{d \to \infty} d^{1/2} \int_0^\infty \frac{\sin^d x}{x^d} \, dx,
\]
and split the limit into two parts:
\[
I = \lim_{d \to \infty} d^{1/2} \int_0^\infty \frac{\sin^d x}{x^d} \, dx
= \lim_{d \to \infty} d^{1/2} \int_0^{\pi/2} \frac{\sin^d x}{x^d} \, dx
+ \lim_{d \to \infty} d^{1/2} \int_{\pi/2}^\infty \frac{\sin^d x}{x^d} \, dx.
\]
Since
\[
\left| \int_{\pi/2}^{\infty} \frac{\sin^d x}{x^d} dx \right| \leq \int_{\pi/2}^{\infty} \frac{x^{-d}dx}{x^{d-1}} = \lim_{b \to \infty} \frac{x^{-d+1}}{1-d} \bigg|_{\pi/2}^{b} = \frac{1}{(d-1)x^{d-1}} \longrightarrow 0,
\]

it follows that
\[
I = \lim_{d \to \infty} d^{1/2} \int_{0}^{\pi/2} \frac{\sin^d x}{x^d} dx.
\]

Fix \( \varepsilon > 0 \) sufficiently small. Since \( \sin x/x \) is monotonically decreasing as \( 0 \leq x \leq \pi/2 \), it follows that
\[
\frac{\sin x}{x} \leq \frac{\sin \varepsilon}{\varepsilon} = 1 - \frac{\varepsilon^2}{3!} + \frac{\varepsilon^4}{5!} - \cdots,
\]
as \( \varepsilon \leq x \leq \pi/2 \). It readily follows that there exists a strictly positive constant \( a \) such that
\[
\int_{\varepsilon}^{\pi/2} \frac{\sin^d x}{x^d} dx = O \left( e^{-a \cdot d^2} \right).
\]

For \( 0 \leq x \leq \varepsilon \), we have
\[
\left( \frac{\sin x}{x} \right)^d = \left( 1 - \frac{1}{6} x^2 + O(\varepsilon^4) \right)^d = e^{-1/6dx^2 + O(d\varepsilon^4)}.
\]

Hence,
\[
\int_{0}^{\varepsilon} e^{-1/6dx^2} dx = \frac{e^{O(d\varepsilon^4)}}{d^{1/2}} \int_{0}^{\varepsilon d^{1/2}} e^{-1/6\varepsilon^2} d\varepsilon.
\]

Now choose \( \varepsilon = \ln d/\sqrt{d} \). Then the limit reduces to the Euler-Poisson integral:
\[
I = \lim_{d \to \infty} d^{1/2} \int_{0}^{\pi/2} \frac{\sin^d x}{x^d} dx = \sqrt{6} \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{6\pi}}{2}.
\]

Thus, the asymptotic behavior of \( \deg(C_d) \) as \( d \to \infty \) is as follows:
\[
\deg(C_d) = \frac{c_d}{d!} \sim \sqrt{\frac{6}{\pi} \frac{1}{d^{3/2}}}.
\]
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