Exact mesoscopic correlation functions of the pairing model

Alexandre Faribault\textsuperscript{1}, Pasquale Calabrese\textsuperscript{2} and Jean-Sébastien Caux\textsuperscript{1}

\textsuperscript{1}Institute for Theoretical Physics, Universiteit van Amsterdam, 1018 XE Amsterdam, The Netherlands and
\textsuperscript{2}Dipartimento di Fisica dell’Università di Pisa and INFN, 56127 Pisa, Italy

(Dated: February 2, 2008)

We study the static correlation functions of the Richardson pairing model (also known as the reduced or discrete-state BCS model) in the canonical ensemble. Making use of the Algebraic Bethe Ansatz formalism, we obtain exact expressions which are easily evaluated numerically for any value of the pairing strength up to large numbers of particles. We provide explicit results at half-filling and extensively discuss their finite-size scaling behavior.

I. INTRODUCTION

The pairing phenomenon is ubiquitous in quantum many-body systems of sizes ranging from the very small, like quarks and nuclei, to the very large, like stars\textsuperscript{11,12}. The common feature of all these seemingly unrelated systems is the instability against the formation of Cooper pairs for an arbitrarily weak attractive force, the basis of the BCS theory of superconductivity\textsuperscript{3}. Despite the diverse nature of pairing systems, many of their fundamental properties can be understood phenomenologically from a so-called reduced BCS model,

\[ H_{\text{BCS}} = \sum_{\sigma = \pm, \alpha=1}^{N} \frac{\epsilon_{\alpha}}{2} c_{\alpha\sigma}^{\dagger} c_{\alpha\sigma} - g \sum_{\alpha, \beta = 1}^{N} c_{\alpha +}^{\dagger} c_{\alpha-} - c_{\beta -} c_{\beta +}, \]

which was introduced by Richardson in the early 1960’s in the context of nuclear physics\textsuperscript{4}. The model describes (pseudo) spin-1/2 fermions (electrons, nucleons, etc.) in a shell of doubly degenerate single particle energy levels with energies $\epsilon_{\alpha}/2$, $\alpha = 1, \ldots, N$. $c_{\alpha \sigma}$ are the annihilation operators, $\sigma = +, -$ labels the degenerate time reversed states (i.e. spin or isospin) and $g$ denotes the pairing coupling constant. Despite its simplified character (all levels interact uniformly), this Hamiltonian captures the main essence of the problem: fixing (ab-initio or phenomenologically) the energy levels $\epsilon_{\alpha}$ and the coupling $g$ allows to obtain quantitative predictions, since the model remains solvable for an arbitrary choice of parameters.

In the thermodynamic limit, and within the grand-canonical ensemble, the properties of the Richardson model are correctly described via the BCS variational ansatz\textsuperscript{2}. However, for finite numbers of particles, the situation is more complex. The actual solution then depends on the ensemble chosen, and for physically relevant systems the grand-canonical ensemble is not always the appropriate one. For example, nuclei have a fixed number of nucleons; due to the typically large charging energy, experiments on ultra-small metallic grains are also performed at fixed number of electrons\textsuperscript{5}. In these cases a treatment based on the canonical ensemble would be more appropriate, precluding a BCS mean-field approach. Moreover, dealing with a system in the mesoscopic regime precludes the use of quantum statistical mechanics, and one is thus forced to rely either on uncontrolled approximations, or nonperturbative methods.

Fortunately, Richardson’s Hamiltonian\textsuperscript{12} is one of the theories for which an exact solution can be constructed in the canonical ensemble\textsuperscript{4}. This solution explains several interesting features of the mesoscopic physics of superconductors, complementing previous approximate treatments (see the review\textsuperscript{6}). In particular it allowed to give a definitive answer to Anderson’s 1959 question: What is the size limit for a metallic grain to have superconducting properties? The utility of the model is thus indisputable (see also the reviews\textsuperscript{6,7} for some non condensed-matter applications), however most of the attention has been concentrated on thermodynamical quantities. On the other hand, experiments typically give access to static or dynamical correlation functions, which are not easily obtained in this framework. Richardson himself in 1965\textsuperscript{8} derived a first exact expression for static correlation functions, which unfortunately has a degree of complexity that grows factorially with system size, and was therefore not suitable for actual calculations. In a significant development, Amico and Osterloh\textsuperscript{9} proposed a new method (based on a generalization of earlier work by Sklyanin\textsuperscript{10}) to write down such correlations explicitly.

The complexity of this method was still factorial and all the numerical results were therefore limited to system sizes of up to 16 particles. A disadvantage of these methods is that all the eigenstates of the Hamiltonian must be known to get the correlation functions.

A major simplification was then proposed by Zhou et al\textsuperscript{11} (see also\textsuperscript{14}). Using the Algebraic Bethe Ansatz (ABA) and the Slavnov formula for scalar products of states\textsuperscript{15}, they managed to write the static correlation functions as sums over $N_{r}^{2}$ determinants of $N_{r} \times N_{r}$ matrices, reducing the complexity of the problem to order $N_{r}^{5}$ (here $N_{r}$ is the number of rapidities in the eigenstates). Furthermore, in this approach only the knowledge of the ground-state wavefunction is required. Surprisingly, this approach has not been used until now to obtain quantitative numerical results for the correlation functions, with the notable exception of the calculation of ground-state entanglement properties\textsuperscript{16}.

In this paper we fill this gap. As a first step we reanalyze the results of Refs.\textsuperscript{13,14}, rewriting all static correlation functions as sums over only $N_{r}$ determinants (thereby reducing the complexity of the problem by a
further factor of \( N_r \)). We then provide analytical formulas for the physically relevant correlation functions, and evaluate them for some model Hamiltonians. We stress that having reduced the complexity of the problem by this amount, correlation functions of systems with many more particles than before can be calculated on a simple computer. In this way we can describe the crossover from mesoscopic to macroscopic physics, going beyond previous results limited to fewer particles\(^{11,17}\).

The paper is organized as follows. In Sec. \([\text{II}]\) we discuss the model and its general properties. In Sec. \([\text{III}]\) we recall how to calculate static correlation functions by means of Algebraic Bethe Ansatz, and derive their general expressions in terms of sums of \( N_r \) determinants. This section is rather technical, thus the reader interested in the physical results can skip directly to Sec. \([\text{IV}]\) where we discuss how to solve the Richardson equations for the ground state, and derive quantities that do not require knowledge of the determinant representation. In Sec. \([\text{V}]\) all the correlation functions are explicitly calculated at half-filling. The paper is closed by Sec. \([\text{VI}]\) where we also discuss open problems for future investigation.

II. THE MODEL

A simple but very important property of the system is the so-called blocking effect\(^{12,12}\), i.e. unpaired particles completely decouple from the dynamics and behave as if they were free. We will denote the total number of fermions as \( N_f \), and the total number of pairs as \( N_p \). Due to level blocking, we will only consider \( N_f = 2N_p \) paired particles in \( N \) unblocked levels. In terms of pair annihilation and creation operators

\[
\begin{align*}
  b_\alpha &= c_\alpha - c_\alpha^\dagger, \\
  b_\alpha^\dagger &= c_\alpha^\dagger + c_\alpha,
\end{align*}
\]

the Hamiltonian is

\[
H = \sum_{\alpha=1}^{N} \epsilon_\alpha b_\alpha^\dagger b_\alpha - g \sum_{\alpha,\beta=1}^{N} b_\alpha^\dagger b_\beta,
\]

and \( n_\alpha = 2b_\alpha^\dagger b_\alpha \) is the number of particles in level \( \alpha \).

The pair creation and annihilation operators satisfy the commutation relations

\[
[b_\alpha, b_\beta^\dagger] = \delta_{\alpha \beta} (1 - 2b_\alpha^\dagger b_\alpha), \quad [b_\alpha, b_\beta] = [b_\alpha^\dagger, b_\beta^\dagger] = 0.
\]

The term \( 2b_\alpha^\dagger b_\alpha \) in the first commutator makes the model different from free bosons and therefore non-trivial.

Using the pseudo-spin realization of electron pairs \( S_z^\alpha = b_\alpha^\dagger b_\alpha - 1/2, \ S_\alpha^+ = b_\alpha, \ S_\alpha^- = b_\alpha^\dagger \), the BCS Hamiltonian becomes (up to a constant)

\[
H = \sum_{\alpha=1}^{N} \epsilon_\alpha S_\alpha^z - g \sum_{\alpha,\beta=1}^{N} S_\alpha^+ S_\beta^-.
\]

The operators \( S_{\alpha}^{\pm, z} \) obey a standard spin algebra and so the Hamiltonian \([\text{III}]\) describes a spin-1/2 magnet with long-range interaction for the \( XY \) components in a site-dependent transverse magnetic field \( \epsilon_\alpha \). Such a magnetic Hamiltonian is known in the literature as a Gaudin magnet\(^{18}\). An important relation is

\[
S_\alpha^+ S_\alpha^z = S_\alpha^2 - (S_\alpha^x)^2 \pm S_\alpha^z.
\]

Since the normalization of the pairing strength in the literature is not uniform, care must be taken when comparing the results we will obtain with other papers (e.g. our \( g \) is the half of the coupling used in Refs. \([11,17])\).

A. Grand-canonical BCS wavefunction

In the grand-canonical (GC) ensemble the ground-state wavefunction is the BCS variational ansatz

\[
|GS\rangle = \prod_\alpha (u_\alpha + e^{i\phi_\alpha} v_\alpha b_\alpha^\dagger)|0\rangle, \quad \text{with } u_\alpha^2 + v_\alpha^2 = 1,
\]

where the variational parameters \( u_\alpha \) and \( v_\alpha \) are real and \( \phi_\alpha \) is a phase which, it turns out, must be \( \alpha \)-independent. \( |GS\rangle \) is not an eigenstate of the particle number operator \( N_f \) and the average condition \( \langle N_f \rangle = N_f \) determines the GC chemical potential. Likewise, the commonly used definition

\[
\Delta_{GC} = 2g \sum_\alpha \langle b_\alpha \rangle = 2g \sum_\alpha u_\alpha v_\alpha e^{i\phi_\alpha},
\]

for the superconducting gap makes sense only in a GC ensemble, since \( \langle b_\alpha \rangle \) is zero when evaluated at fixed particle number. The variational parameters are obtained as

\[
\begin{align*}
  v_\alpha^2 &= \frac{1}{2} \left[ 1 - \frac{\epsilon_\alpha - \mu}{\sqrt{\epsilon_\alpha - \mu)^2 + |\Delta_{GC}|^2}} \right],
\end{align*}
\]

where \( \mu \) is the GC chemical potential.

It is then easy to calculate (static) correlation functions on this GS:

\[
\begin{align*}
\langle b_\alpha^\dagger b_\alpha \rangle &= v_\alpha^2, \\
\langle b_\alpha b_\beta^\dagger \rangle &= u_\alpha^2, \\
\langle b_\alpha^\dagger b_\beta \rangle &= u_\alpha v_\alpha u_\beta v_\beta.
\end{align*}
\]

B. Canonical description and Richardson solution

The exact solution of \([\text{III}]\) was worked out by Richardson.\(^{19}\) In the canonical ensemble the model is integrable\(^{20,21}\) and tractable by means of algebraic methods\(^{11,14,20,21}\). We review here only the main points of this solution.

In the ABA, eigenstates are constricted by applying raising operators on a so-called reference state (pseudovacuum). We here choose the pseudovacuum (in the spin representation) to be fully polarized along the \( \hat{z} \) axis

\[
S_\alpha^z |0\rangle = \frac{1}{2} |0\rangle, \quad \forall \alpha.
\]
In the pair representation, this state thus corresponds to having one pair in each available level. Eigenstates with \( N_p \) pairs are then characterized by \( N_r = N - N_p \) spectral parameters (rapidities) \( w_j \), and take the form of Bethe wavefunctions
\[
|\{w_j\}⟩ = \prod_{k=1}^{N_r} B(w_k)|0⟩. \tag{12}
\]

The operators \( B \), together with operators \( A, C, D \) defined as
\[
A(w_k) = -\frac{1}{g} + \frac{N}{\alpha=1} \frac{S_α^z}{w_α - \varepsilon_α}, \quad B(w_k) = \frac{N}{\alpha=1} \frac{S_α^-}{w_α - \varepsilon_α},
\]
\[
C(w_k) = \sum_{\alpha=1}^{N} \frac{S_α^+}{w_α - \varepsilon_α}, \quad D(w_k) = \frac{1}{g} - \sum_{\alpha=1}^{N} \frac{S_α^-}{w_α - \varepsilon_α} \tag{13}
\]
obeys the Gaudin algebra, which is the quasi-classical limit of the quadratic Yang-Baxter algebra associated to the \( gl(2) \) invariant \( R \)-matrix (we refer the readers to \[14\] for details).

The wavefunctions \( |\{w_j\}⟩ \) are eigenstates of the transfer matrix, and thus of the Hamiltonian \( \{1\} \), when the parameters \( w_j \) satisfy the Richardson equations
\[
\frac{1}{g} = \sum_{\alpha=1}^{N} \frac{1}{w_j - \varepsilon_α} - \sum_{k \neq j}^{N_r} \frac{2}{w_j - w_k}, \quad j = 1, \ldots, N_r. \tag{14}
\]

Throughout the paper we will refer with latin indices to the rapidities and with greek ones to the energy levels. The total energy of a Bethe state is \( E = \sum_{\alpha=1}^{N} \frac{\varepsilon_α}{\alpha} - N - \sum_j w_j + g(2N_r - N) \). For a given \( N \) and \( N_r \), the number of solutions of Richardson equations is \( \binom{N_r}{N} \), and coincides with the dimension of the Hilbert space of \( N_r \) pair vacancies distributed into \( N \) different levels, i.e. the solutions to Richardson equations give all the eigenstates of the model.

Note that the Richardson equations \( \{14\} \) have a different sign of \( g \) compared to the ones mostly considered in the literature. This is due to the particular choice of the pseudovacuum we made in Eq. \( \{1\} \), whereas the most common choice is \( S_α^z|0⟩ = -1/2|0⟩ \). With our choice of pseudovacuum, Bethe states are built by destroying pairs, as in Eq. \( \{12\} \) and not by creating them. We use this somewhat unusual pseudovacuum following Refs. \[13,14\] in order to use all the formulas there without any adaptation. At half-filling (that is the only case considered numerically here) the different choice of the pseudovacuum only matters as a global normalization and a different labeling of the states.

The connection between the canonical and grand-canonical ensembles was first pointed out by Richardon himself\[22\], who showed how in the large \( N_f \) limit one recovers the BCS gap equation as
\[
N_f = \sum_{\alpha=1}^{N} \left( 1 - \frac{\varepsilon_α - \mu}{\sqrt{(\varepsilon_α - \mu)^2 + |N_f\Delta|^2}} \right), \tag{15}
\]
where now \( \mu \) is fixed by the density and the equation can be solved to find \( \Delta \), which with this normalization is an intensive quantity and corresponds to \( \Delta_{GC}/N_f \). For the ground-state energy per pair \( E_0 \) one finds
\[
N_f E_0 = \sum_{\alpha=1}^{N} \varepsilon_α \left( 1 - \frac{\varepsilon_α - \mu}{\sqrt{(\varepsilon_α - \mu)^2 + |N_f\Delta|^2}} \right) - \frac{N_f^2 \Delta^2}{g}. \tag{16}
\]

Anderson\[1\] argued that increasing the mean energy spacing \( d \) (that is inversely proportional to the volume in a metallic grain) superconductivity should disappear when \( d \) becomes of the order of the bulk gap \( \Delta_{GC} \). Our study of correlation functions to be presented below clearly shows this crossover.

### III. Algebraic Bethe Ansatz and Correlation Functions

The starting point to calculate correlation functions with the Algebraic Bethe Ansatz (ABA) is having a representation for the scalar products of two generic states defined by \( N_r \) rapidities \((N - N_r)\) pairs
\[
\langle \{w\}|\{v\}⟩ = \langle 0 | \prod_{b=1}^{N_r} C(w_b) \prod_{a=1}^{N_r} B(v_a)|0⟩, \tag{17}
\]
when at least one set of parameters (e.g. \( w_b \) but not \( v_a \)) is a solution to the Richardson equations. Following standard notations, \( C \) is the conjugate of the operator \( B \).

Such a representation exists, and is known as the Slavnov formula\[15\], which for the case at hand specifically reads \(13\)
\[
\langle \{w\}|\{v\}⟩ = \frac{\prod_{b \neq a} (v_b - v_a)}{\prod_{b < c} (w_b - v_a) \prod_{a < b} (v_b - v_a)} \times \det N_r J(\{v_a\}, \{w_b\}), \tag{18}
\]
where the matrix elements of \( J \) are given by
\[
J_{ab} = \frac{v_b - v_a}{v_a - w_b} \left( \sum_{\alpha=1}^{N} \frac{1}{(v_a - \varepsilon_α)(w_b - \varepsilon_α)} - 2 \sum_{c \neq a} \frac{1}{(v_a - v_c)(w_b - v_c)} \right), \tag{19}
\]
from which the norms of states simply follow from \( v \rightarrow w \) as \( ||\{v\}||^2 = \det N_r G \) with a Gaudin matrix
\[
G_{ab} = \begin{cases} \frac{1}{(v_a - \varepsilon_β)^2} - 2 \sum_{c \neq a} \frac{1}{(v_a - v_c)^2} & a = b, \\ \frac{2}{(v_a - v_b)^2} & a \neq b, \end{cases} \tag{20}
\]
recovering Richardson’s old result\[10\].

The key point is that any form factor of a local spin operator between two Bethe eigenstates can be represented...
via (13) as a scalar product with one set, e.g. \{v\} not satisfying the Bethe equations, for which Slavnov’s formula is applicable. This has been explicitly worked out in Ref. [13]. For \{w\}, \{v\} containing respectively \(N_r + 1\) and \(N_r\) elements, the nonzero form factors are:
\[
\langle \{w\}|S^+_\alpha S^-_\beta \{v\} \rangle = \frac{\prod_{b=1}^{N_r+1}(w_b - \epsilon_\alpha)}{\prod_{a=1}^{N_r}(v_a - \epsilon_\alpha)} \frac{\det_{N_r+1}T(\alpha, \{w\}, \{v\})}{\prod_{b>a}(w_b - w_a)\prod_{b\leq a}(v_b - v_a)},
\]
(21)
and, for both \{w\} and \{v\} containing \(N_r\) rapidities
\[
\langle \{w\}|S^+_\alpha \{v\} \rangle = \frac{\prod_{a=1}^{N_r}(w_a - \epsilon_\alpha)}{(v_a - \epsilon_\alpha)} \times \frac{\det_{N_r}T(\alpha, \{w\}, \{v\})}{\prod_{b>a}(w_b - w_a)\prod_{b\leq a}(v_b - v_a)},
\]
(22)
with the matrix elements of \(T\) given by
\[
T_{ab}(\alpha) = \frac{N_r+1}{\prod_{c\neq a}(w_c - v_b)} \left( \sum_{\alpha=1}^{N_r} \frac{1}{v_b - \epsilon_\alpha} \frac{1}{w_a - \epsilon_\alpha} \right) - 2 \sum_{c\neq a} \frac{1}{v_b - \epsilon_\alpha} \frac{1}{w_a - \epsilon_\alpha} , \ b < N_r + 1,
\]
\[
T_{a,N_r+1}(\alpha) = \frac{1}{(w_a - \epsilon_\alpha)^2}, \quad Q_{ab}(\alpha) = \frac{\prod_{c\neq b}v_c - v_b}{(w_a - \epsilon_\alpha)^2}.
\]
(23)

Above, \(T_z\) is the \(N_r \times N_r\) matrix obtained from \(T\) by deleting the last row and column and replacing \(N_r + 1\) by \(N_r\) in the matrix elements. Here it is assumed that both \(\{v_a\}\) and \(\{w_b\}\) are solutions to Richardson’s Bethe equations. However, the results are still valid for \(S^+_\alpha\) if only \(\{w_b\}\) satisfy the Bethe equations.

A. Determinant representation of the correlation functions

In Ref. [13] it has been pointed out that due to the simplicity of the solution of the ABA not only the form factors, but any static correlation function can be written in a determinant representation. This simplicity puts the BCS model in an extremely privileged position for a detailed study of the static correlation functions.

The result for \(\langle \{w\}|S^+_\alpha S^-_\beta \{v\} \rangle\) has been explicitly worked out\(^{13}\)
\[
\langle \{w\}|S^+_\alpha S^-_\beta \{v\} \rangle = \sum_{i=1}^{N_r} \frac{1}{v_i - \epsilon_\beta} \langle \{w\}|S^-_\alpha \{v\} \rangle_i
\]
\[
- \sum_{i' \neq i} \frac{1}{(v_i - \epsilon_\beta)(v_{i'} - \epsilon_\beta)} \langle \{w\}|S^+_\alpha S^-_\beta \{v\} \rangle_{i,i'}.
\]
(24)

Here the sets indicated by \(\{v\}_i\) stands for sets where the rapidity \(i\) has been removed and similarly for \(\{v\}_{i,i'}\) both \(i\) and \(i'\) rapidities have been removed. The \(S^-S^-\) form factor is given by\(^{13}\)
\[
\langle \{w\}|S^-_\alpha S^-_\beta \{v\} \rangle = \frac{\prod_{b=1}^{N_r}(w_b - \epsilon_\alpha)(w_b - \epsilon_\beta)}{\prod_{a=1}^{N_r}(v_a - \epsilon_\alpha)(v_a - \epsilon_\beta)} \times \frac{\det_{N_r}T(\alpha, \{w\}, \{v\})}{\prod_{b>a}(w_b - w_a)\prod_{b\leq a}(v_b - v_a)},
\]
(25)
with
\[
T_{ab}(\alpha, \beta) = \frac{N_r}{\prod_{c\neq a}(w_c - v_b)} \left( \sum_{\gamma=1}^{N_r} \frac{1}{v_b - \epsilon_\gamma} \frac{1}{w_a - \epsilon_\gamma} \right) - 2 \sum_{c\neq a} \frac{1}{v_b - \epsilon_\gamma} \frac{1}{w_a - \epsilon_\gamma}, \ b < N_r - 1,
\]
\[
T_{a,N_r-1}(\alpha, \beta) = \frac{2w_a - \epsilon_\alpha - \epsilon_\beta}{(w_a - \epsilon_\alpha)^2}, \quad T_{a,N_r}(\alpha, \beta) = \frac{1}{(w_a - \epsilon_\alpha)^2}.
\]
(26)

where \(\alpha \neq \beta\) is assumed, with the convention that it vanishes when \(\alpha = \beta\). Note that \(\langle \{w\}|S^-_\alpha S^-_\beta \{v\} \rangle\) is symmetric under the exchange of \(\alpha\) and \(\beta\), although this is not manifest in the formal expression. This nontrivial property will be checked during the numerical computation.

This correlation is then written as the sum of \(N^2\) determinants, which is much less than the sum over the full Hilbert space needed in other approaches. In the following we will determine a similar expression for \(\langle S^+_\alpha S^+_\gamma \rangle\) and then we will show that it is possible to reduce these formulas to sums of only \(N\) terms.

1. Determinant representation of \(\langle S^+_\alpha S^+_\gamma \rangle\)

The operator \(A(u)\) only has simple poles at \(\epsilon_\alpha\) such that\(^{13}\)
\[
S^+_\alpha = \lim_{u \to \epsilon_\alpha} (u - \epsilon_\alpha)A(u).
\]
(27)
This allows one to write
\[
\langle \{w\}|S^+_\alpha S^+_\gamma \{v\} \rangle = \lim_{u \to \epsilon_\alpha, \beta \to \epsilon_\beta} \langle u' - \epsilon_\alpha \rangle (u - \epsilon_\beta)
\]
\[
\times \langle \{w\}|A(u')A(u) \prod_{i=1}^{N_r} B(v_i) |0\rangle,
\]
(28)
which can be easily computed by commuting the \(A\) operators until they reach the far right and act on the pseudovacuum in the following way:
\[
A(u)|0\rangle = a(u)|0\rangle = -\frac{1}{g}|0\rangle + \frac{1}{2} \sum_{\gamma=1}^{N_r} \frac{1}{u - \epsilon_\gamma}|0\rangle.
\]
(29)
Using the commutation relation \[ [A(u), B(v)] = \frac{B(u)}{u-v} - \frac{B(v)}{u-v} \] (30)
and defining \( \mathcal{G} \equiv \langle \{ w \} | A(u') A(u) \prod_{i=1}^{N_r} B(v_i) | 0 \rangle \), we find by commuting \( A(u) \) and \( B(v_1) \) that
\[
\mathcal{G} = \langle \{ w \} | A(u') B(u) (v_1) A(u) \prod_{i=2}^{N_r} B(u)(v_i) | 0 \rangle 
+ \sum_{i=1}^{N_r} \frac{1}{u-v_i} \langle \{ w \} | A(u') \{ v_i \}; u \rangle - \langle \{ w \} | A(u') \{ v \} \rangle, \tag{31}
\]
where \( \{ v \}; u \) is the (non-Bethe) state built by replacing the rapidity \( v_1 \) by \( u \). By commuting again \( N_r - 1 \) times and using Eq. (24), we find
\[
\mathcal{G} = F(u) \langle \{ w \} | A(u') \{ v \} \rangle
+ \sum_{i=1}^{N_r} \frac{1}{u-v_i} \left[ \sum_{i'=1}^{N_r} \frac{1}{u'-v'} \langle \{ w \} | A(u') \{ v_{i'} \}; u, u' \rangle 
+ F_i(u') \langle \{ w \} | \{ v \}; u \rangle + \frac{1}{u'-u} \langle \{ w \} | \{ v \}; u' \rangle 
- \frac{1}{u'-u} \langle \{ w \} | \{ v \}; u \rangle \right], \tag{32}
\]
with \( F_i(u) \equiv -\frac{1}{\beta} + \frac{1}{2} \sum_{i=1}^{N_r} 1 \frac{1}{u_i-v_i} - \sum_{i=1}^{N_r} 1 \frac{1}{u_i-v_i} \). It is then easy to take the limit as prescribed by Eq. (23) and find
\[
\langle \{ w \} | S^-_\alpha S^-_\beta \{ v \} \rangle = \frac{\langle \{ w \} | \{ v \} \rangle}{4} + \sum_{i=1}^{N_r} \frac{\langle \{ w \} | S^-_\alpha \{ v \} \rangle}{2(\epsilon_\alpha - v_i)} 
+ \sum_{i=1}^{N_r} \frac{\langle \{ w \} | S^-_\beta \{ v \} \rangle}{2(\epsilon_\beta - v_i)} + \sum_{i'=1}^{N_r} \frac{\langle \{ w \} | S^-_\alpha S^-_\beta \{ v \}; i, i' \rangle}{(\epsilon_\alpha - v_i)(\epsilon_\alpha - v_i')}. \tag{34}
\]

Note that we use the same notation as in Eq. (21) for the \( S^- \) form factor but there it is evaluated between two states with \( N_r \) and \( N_r + 1 \) rapidities, while here it is between two states with \( N_r - 1 \) and \( N_r \). This should not be a source of confusion.

The static correlation function can now be evaluated by setting \( \{ v \} = \{ w \} = \{ w_0 \} \); the set of rapidities corresponding to the ground state of the system. Using equations (21) and (23), we can directly express this correlation function as a sum of \( N_r^2 + N_r \) determinants.

### B. Reduction formulas

As anticipated in the introduction, the previous expressions can be reduced to sums over only \( N_r \) determinants. This is explicitly worked out in the following. We will assume that \( \alpha \neq \beta \), because for intra-level correlations from Eq. (6) we have
\[
\langle \{ w \} | S^-_\alpha S^-_\beta | \{ w \} \rangle = \frac{1}{2} + \langle \{ w \} | S^+_\alpha | \{ w \} \rangle, \tag{35}
\]
which is already a single determinant expression.

#### 1. Reduction of \( \langle S^-_\alpha S^-_\beta \rangle \)

We here need to evaluate Eq. (24) in the limit \( v \to w \). In this case, Eqs. (21) and (24) for the form factors simplify to
\[
\frac{\langle \{ w \} | S^-_\alpha | \{ w_0 \} \rangle_{N_r-1}}{w_q - w_{q-1}} \frac{w_q - \epsilon_\alpha}{w_q - \epsilon_\beta} \det_{N_r} U^{(q)}, \tag{36}
\]
\[
\frac{\langle \{ w \} | S^-_\alpha S^-_\beta | \{ w_0 \} \rangle_{N_r-2}}{(w_q - \epsilon_\beta)(w_l - \epsilon_\beta)} \frac{w_l - w_q}{w_l - w_q} \times \det_{N_r} U^{(q)} \tag{37}
\]
The matrices \( U \) are defined as follows. \( U^{(q)} \) and \( U^{(q,l)} \) are equal to the Gaudin matrix \( 24 \) except for columns \( q \) and \( q, l \) respectively, where
\[
U^{(q)}_{aq} = U^{(q)}_{aq} = \frac{1}{(w_a - \epsilon_\alpha)^2}, \tag{38}
\]
\[
U^{(q)}_{al} = \frac{2w_a - \epsilon_\alpha - \epsilon_\beta}{(w_a - \epsilon_\alpha)^2(w_a - \epsilon_\beta)^2}. \tag{39}
\]

In Eq. (37) it is explicitly assumed that \( \alpha \neq \beta \) with the convention that for \( \alpha = \beta \) it is zero. And in fact, for \( \alpha = \beta \) it is not difficult to prove that Eq. (36) reproduces the correct result given by Eq. (33).

Since everything is symmetric under exchange of \( l \) and \( q \) we only perform the sum over \( l < q \) and in the end we multiply the result by 2. Thus we need to perform the sum (neglecting for the moment the \( l \) independent factors)
\[
\sum_{l=1}^{N_r} \frac{w_l - \epsilon_\alpha}{w_l - w_q} \det U^{(q)} l \equiv \sum_{l=1}^{N_r} K_{lq} \det U^{(q)}. \tag{40}
\]
Let us write the matrix \( U^{(q)} \) as a vector of vectors
\[
U^{(q)} = | \tilde{G}_1 \ldots \tilde{G}_{l-1} \ldots \tilde{G}_{l+1} \ldots \tilde{G}_{q-1} \ldots \tilde{C} \ldots | \tag{41}
\]
where \( \tilde{G}_i \) corresponds to the columns of the matrix \( G \), \( \tilde{C} \) (at position \( q \)) corresponds to the vector given by Eq. (35) and \( \tilde{B} \) by Eq. (39).
The sum we want to calculate is (we use $| · |$ for the determinant)

\[
\sum_{l=1}^{q-1} K_{lq} \det U^{(q)} = K_{1q}|\vec{B}, \vec{G}_2, \vec{G}_3 \ldots | \\
+ K_{2q}|\vec{G}_1, \vec{B}, \vec{G}_3 \ldots | + K_{3q}|\vec{G}_1, \vec{G}_2, \vec{B} \ldots | + \ldots \quad (42)
\]

Using the fact that two determinants which differ by a single column can easily be expressed as a single determinant the two first terms of the sum can be written as $|K_{2q}\vec{G}_1 - K_{1q}\vec{G}_2, \vec{B}, \vec{G}_3 \ldots |$. Elementary column operations allow us to write the third determinant as $K_{3q}|\vec{G}_1 - \frac{\epsilon_\alpha}{\epsilon_\beta} \vec{G}_2, \vec{G}_2, \vec{B} \ldots |$. This term differs by a single column from the preceding sum. We can then write the sum of the first three terms as a single determinant $K_{3q}|\vec{G}_1 - \frac{\epsilon_\alpha}{\epsilon_\beta} \vec{G}_2, \vec{G}_2, \vec{B} \ldots | = K_{3q}|\vec{G}_1 - \frac{\epsilon_\alpha}{\epsilon_\beta} \vec{G}_2, \vec{G}_2, \vec{B} \ldots |$. This term differs by a single column from the preceding sum.

We can keep on adding terms in the same way until we reach column $q-1$ and find that Eq. (42) can be written as a single determinant

\[
K_{q-1q}|\vec{G}_1 - \frac{K_{1q}}{K_{q-1q}} \vec{G}_2, \vec{G}_2 - \frac{K_{2q}}{K_{q-1q}} \vec{G}_3, \ldots |, \quad (43)
\]

In this way we reduced the double sum to a single one. The additional terms in the correlation function (coming from $\langle S^- \rangle$) can also be incorporated to the sum in a similar fashion. The $\langle S^- \rangle$ term in Eq. (24) is given by Eq. (36) and can be simply encoded in the representation we just obtained for the sum over $l < q$ of $\langle S^- \rangle$. In this way, we finally have for the full correlation function

\[
\langle \{w\}|S^+_{\alpha}|\{w\} \rangle = \sum_{q=1}^{N_r} \frac{w_q - \epsilon_\alpha}{w_q - \epsilon_\beta} D_q^{(\alpha, \beta)}, \quad (44)
\]

where we defined the matrix

\[
D_q^{(\alpha, \beta)} = \begin{bmatrix} \vec{G}_1 & \frac{K_{1q}}{K_{q-1q}} \vec{G}_2 & \ldots & \vec{G}_q \end{bmatrix} \begin{bmatrix} D_{q,1}^{(\alpha, \beta)} & D_{q,2}^{(\alpha, \beta)} & \ldots & D_{q,N_r}^{(\alpha, \beta)} \end{bmatrix}
\]

that has the following structure

\[
D_{q,i}^{(\alpha, \beta)} = \begin{cases} \vec{G}_i - \frac{K_{q-1q}}{K_{i+1q}} \vec{G}_{i+1} & i < q - 1, \\
\vec{G}_i + \left( \frac{w_q - \epsilon_\beta}{w_{q-1} - w_q} \right) \vec{B} & i = q - 1, \\
\vec{G}_i & i = q, \\
\vec{G}_i & i > q. \end{cases}
\]

The low level of complexity of this representation as sum of $N_r$ determinants of $N_r$ by $N_r$ matrices allows us to access easily the static correlation functions for systems with large number of pairs compared to previously published results.

We stress again that these formulas are true only for $\alpha \neq \beta$.

2. Reduction of $\langle S^+_{\alpha} S^+_{\beta} \rangle$

According to Eq. (31), when $v \to w$

\[
\langle \{w\}|S^+_{\alpha} S^+_{\beta}|\{w\} \rangle = \frac{1}{4} \langle \{w\}|\{w\} \rangle \\
- \frac{1}{2} \left[ \sum_{i=1}^{N_r} \langle \{w\}|S^+_{\alpha}|\{w\}\rangle_i - \sum_{i' \neq i} \langle \{w\}|S^-_{\alpha} S^-_{\beta}|\{w\}\rangle_i \right] \\
- \frac{1}{2} \left[ \sum_{i=1}^{N_r} \langle \{w\}|S^-_{\alpha} S^-_{\beta}|\{w\}\rangle_i - \sum_{i' \neq i} \langle \{w\}|S^+_{\alpha} S^+_{\beta}|\{w\}\rangle_i \right].
\]

From Eqs. (13) and (14) it is straightforward to show that when $\alpha \neq \beta$

\[
\langle \{w\}|S^+_{\alpha} S^+_{\beta}|\{w\} \rangle = \frac{1}{4} - \frac{1}{2} \sum_{q=1}^{N_r} (\det D_q^{(\beta, \alpha)} + \det D_q^{(\alpha, \beta)}).
\]

For $\alpha = \beta$ the result is trivially $\langle \{w\}|(S^+_{\alpha})^2|\{w\} \rangle = 1/4$.

This completes the representation of the static correlation functions in terms of determinants. To make further progress, we need explicit results for the ground-state rapidities $\{w\}$, i.e. the lowest-energy solutions to the Richardson equations. The following section is devoted to this.

IV. THE SOLUTION OF THE RICHARDSON EQUATIONS FOR THE GROUND STATE

A. General properties

At $g = 0$, for $N_r$ pair vacancies in $N$ energy levels $\epsilon_\alpha$ with only double degeneracy, the $(N_r)$ solutions to the Richardson equations are trivial. They are given by Eq. (12) with the $N_r$ rapidities set to be strictly equal to one of the energies $\epsilon_\alpha$. Clearly, the GS in that limit is built by choosing the $N_r$ highest energy levels, i.e. $w_1 = \epsilon_N, w_2 = \epsilon_{N-1}, \ldots, w_{N_r} = \epsilon_{N-N_r+1}$.

Apart from a few particular cases with a small number of particles, the Richardson equations are not solvable analytically when $g \neq 0$. The perturbative expansions for small$^{22,23}$ and large$^{24,25}$ coupling are not predictive for all values of the pairing strength and so the most accurate results come from the numerical solution. The solutions are such that every $w_j$ is either a real quantity or forms, with another parameter $w_j^*$, a complex conjugate pair (CCP), i.e. $w_j^* = w_j$. The mechanism for the CCPs formation is very easy: as interactions are turned on, all $w_j$ are real quantities for small enough $g$, but at a certain critical value of the coupling $g_j^*$ two rapidities will be exactly equal to one of the energy levels $(w_j = w_j^* = \epsilon_j)$ and for $g > g_j^*$, the two parameters that
FIG. 1: $g$ dependence of the real (left) and imaginary parts (right) of the ground state rapidities. From top to bottom they correspond to $N_r = 8, 15, 64$ always at half-filling ($N = 2N_r$).

collapsed will form a CCP at least for a finite interval in $g$. The situation is in fact rather intricate: the values $g_j^*$ are implicit functions of all other rapidities, and can only be read off a full solution of the Richardson equations for a specific choice of state. Moreover, CCPs can split back into real pairs, whose components can then re-pair with neighbouring rapidities. Finding complex solutions to the Richardson equations is thus difficult in general, since there is no equivalent to the 'string hypothesis' as for e.g. integrable spin chains.

The solutions for the ground state have a particularly simple structure. In fact, the set of critical points is such that the smaller a rapidity is at $g = 0$ the smaller the $g$ at which it forms a CCP will be. As we raise $g$ from zero there will come a point at which $w_{N_r}$ will form a CCP with $w_{N_r - 1}$ when they are both equal to $\epsilon_{N - N_r + 2}$. As $g$ is raised some more, $w_{N_r - 2}$ and $w_{N_r - 3}$ will form a CCP at $\epsilon_{N - N_r + 4}$ and this will go on until every rapidity has formed a CCP in the case of even $N$. Oppositely, with an odd number of rapidities in the system, $w_1$ (the largest one at $g = 0$) will always remain a real quantity no matter how large the coupling strength is. After the CCPs are formed no further collapse happens in the case of the ground state, while for excited states further collapses...
can take place and complex solutions can become real again.

Different choices of the parameters $\epsilon_\alpha$ and of their eventual degenerations specify different models. In all the preceding sections everything was completely general (modulo having to take some extra precautions in the case of coinciding levels $\epsilon_\alpha$), but from now on we specialize to the case of equally spaced doubly degenerate levels. We make the choice to use $\epsilon_\alpha = \alpha$ which sets the zero of energy and implies that every energy will be given in units of the (pair) inter-level spacing. Furthermore we consider only half-filling of the energy levels ($N = 2N_r = 2N_p = N_f$). In this case, as $g \to \infty$, the real part of every rapidity will go to $+\infty$ whereas the CCP's imaginary parts will go to $\pm \infty$.

B. Numerical procedure and results

At the precise value of $g$ at which a pair of rapidities $(w_j, w_{j'})$ collapse into a CCP $(w_j = w_{j'} = \epsilon_\gamma(j))$, the Richardson equations (Eq. (14)) labelled $j$ and $j'$ will include two diverging terms whose sum remains finite. In order to be able to treat these points numerically, one can define the following real variables,

\begin{align}
  w_{1,j} &\equiv w_j + w_{j'} \quad (49) \\
  w_{2,j} &\equiv \frac{2\epsilon_\gamma(j) - w_j - w_{j'}}{(w_j - w_{j'})^2} \quad (50)
\end{align}

whose inverse transformation reads

\begin{align}
  w_j &= \frac{1}{2} \left[ w_{1,j} + \sqrt{\frac{2\epsilon_{j-1} - w_{1,j}}{w_{2,j}}} \right] \quad (51) \\
  w_j &= \frac{1}{2} \left[ w_{1,j} - \sqrt{\frac{2\epsilon_{j-1} - w_{1,j}}{w_{2,j}}} \right] \quad (52)
\end{align}

As discussed in Ref. [23], we need to know beforehand which rapidities will form a CCP and at which $\epsilon_\gamma(j)$ it will happen in order to use this type of change of variables. Since in this article we only need ground state solutions, this requirement is easily met.

At the critical point $w_{2,j}$ goes to a well defined (though a priori unknown) finite 0/0 form. Using it as a variable in the system of equations therefore avoids some potential numerical complications when close to a critical point.

By multiplying the $j$ and $j'$ Richardson equations respectively by $\epsilon_\gamma(j) - w_j$ and $\epsilon_\gamma(j) - w_{j'}$, we can get rid of the divergences that show up at critical points. Adding the resulting equations (giving $F_{1,j}$) and subtracting them and then dividing it by $w_j - w_{j'}$ (giving $F_{2,j}$), it is simple to obtain the following two real equations

\begin{align}
  F_{1,j} &= \sum_{i \neq j-1}^N \frac{(\epsilon_{j-1} - \epsilon_i)(2\epsilon_i - w_{1,j})}{(\epsilon_i - \frac{w_{1,j}}{4})w_{1,j} - \frac{2\epsilon_{j-1} - w_{1,j}}{4w_{2,j}}} - \sum_{j' \neq j-1}^N 2 \frac{(\epsilon_{j-1} - w_{j'})(2w_{j'} - w_{1,j})}{(w_{j'})^2 - (w_{j'} - \frac{w_{1,j}}{4})w_{1,j} - \frac{2\epsilon_{j-1} - w_{1,j}}{4w_{2,j}}} + \frac{2\epsilon_{j-1} - w_{1,j}}{g} + 2(N - 1) - 4(N_r - 2) = 0 \quad (53) \\
  F_{2,j} &= \sum_{i \neq j-1}^N \frac{(\epsilon_{j-1} - \epsilon_i)(2\epsilon_i - w_{1,j})}{(\epsilon_i - \frac{w_{1,j}}{4})w_{1,j} - \frac{2\epsilon_{j-1} - w_{1,j}}{4w_{2,j}}} - \sum_{j' \neq j-1}^N 2 \frac{(\epsilon_{j-1} - w_{j'})(2w_{j'} - w_{1,j})}{(w_{j'})^2 - (w_{j'} - \frac{w_{1,j}}{4})w_{1,j} - \frac{2\epsilon_{j-1} - w_{1,j}}{4w_{2,j}}} - \frac{1}{g} + 2w_{2,j} = 0 \quad (54)
\end{align}

The resulting system of non-linear equations can then easily be solved using Newton’s method. Notice that every element of the Jacobian matrix has an analytical expression that is easy to obtain and therefore is explicitly written here. Of course, for Newton’s procedure to converge to the correct solution at a given $g$, we need a good approximation to it. It is simple to do so by slowly incrementing $g$ starting from $g = 0$, where the GS is known. One can then use a simple linear regression on $w_{1,j}, w_{2,j}$ to obtain an educated guess to the ground state at $g' = g + \Delta g$. Despite its simplicity this method, very similar to other ones in the literature [23,27,28], is sufficient for obtaining the ground state solutions. For general states, for which the formation of and splitting apart of CCPs can be highly non-trivial, a more refined algorithm (see Refs. [29,30] for example) is needed to find the solutions.
N lapses into a CCP. The latter is limited at large \(N\) of 
\(g\) stand qualitatively the different behaviors as function of 
shown in the figure. 

Consequently Anderson’s criterion for the presence of superconductivity for large \(N\) is 
\[
\Delta \gtrsim N \Rightarrow g \gtrsim \frac{1}{2 \ln 2 N_r},
\]
showing the typical logarithmic behavior of the small \(g\) expansion.

Fig. 3 shows as a function of \(N_r\), the values of the coupling constant \(g_{N_r}^\ast (N_r)\) at which the first two rapidities form a CCP. We also plot, for even \(N_r\), the values of \(g = g_0^\ast (N_r)\) at which the last couple of rapidities collapses into a CCP. The latter is limited at large \(N_r\) by 
\[
g_0^\ast (N_r) = (2 \text{arcsinh}(1))^{-1} = 0.567296 \text{ a constant which is also shown in the figure.}\]

These two numbers are particularly relevant to understand qualitatively the different behaviors as function of \(g\) and \(N_r\). In fact, when \(g\) is larger than \(g_0^\ast (N_r)\) all the particles are paired and the system has entered its asymptotic superconducting regime. Oppositely when no pair has still been formed, i.e. for \(g < g_{N_r}^\ast (N_r)\) superconductivity is absent. In fact, \(g_{N_r}^\ast (N_r)\) coincides with the critical value of the coupling given by Anderson’s criterion Eq. (50) for large \(N\). The curve resulting from Eq. (50) is plotted in Fig. 3 and the agreement with \(g_{N_r}^\ast (N_r)\) is excellent even for relatively small value of \(N_r\). 

Note also that \(g_{N_r}^\ast (N_r)\) vanishes in the thermodynamic limit, which can simply be interpreted as the Cooper instability. A quantitative understanding of these phenomena and of the crossover between small and large \(g\) at fixed finite \(N_r\) requires an accurate study of the correlation functions, which we present in the next section.

C. Ground state energy

In Fig. 4 we plot the value of the ground state energy per pair (in units of the inter-level spacing) at half-filling for a set of different number of pairs as given by 
\[
N_p E_0 = \sum_{j=1}^N \epsilon_j - \sum_j w_j.
\]

One interesting feature is the presence of a size invariant point at which every curve cross (see the inset of Fig. 4 for a zoom close to this point). Indeed at \(g_{inv} \approx 0.910\) the ground state energy seems to be independent of the number of pairs in the system \(E(g_{inv}) \approx -0.45\). However, the presence of this “fixed point” does not carry any deep meaning and can be easily understood in terms of the \(1/N_p\) expansion developed in Refs. 22,31. In fact, according to these references for large \(N_p\) the ground-state energy per particle can be written as 
\[
E_0 = N_p E_0^{(0)} + E_0^{(1)} + O(1/N_p),
\]
with \( E_0^{(0)} \) is nothing but Eq. (10)

\[
E_0^{(0)} = 1 - \frac{1}{2} \coth \frac{1}{2g},
\]

\[
E_0^{(1)} = \frac{1}{2} \left( 1 - \phi(2g) \coth 1/(2g) \right),
\]

\[
\phi(2g) = \frac{2}{\pi} \int_0^\infty \frac{dx}{1 + x^2} \frac{\cosh \pi x/2}{\sqrt{\cosh^2 (\pi x/2) + \sinh^2 (1/(2g))}},
\]

where we adapt the results to our normalization (i.e. the quantities of Ref. [32] reads \( D = 2N_p, \lambda = 2g \) and there is a global shift of the energy levels). The scale invariant point just corresponds to the value of \( g \) for which the order \( N_p \) term \( E_0^{(0)} \) vanishes, i.e. \( g_{\text{inv}} = (2\arccoth 2)^{-1} = 0.910239 \ldots \). The energy at this point, apart from \( O(1/N_p) \) corrections, is independent of \( N_p \) and given by \( E_0^{(0)}(g_{\text{inv}}) = -0.45276 \ldots \). Eq. (57) is thus practically a perfect approximation of the actual value of the ground-state energy for large enough \( N_p \), say \( N_p \geq 16 \).

V. CALCULATION OF THE CORRELATION FUNCTIONS

The formulas we obtained for the correlation functions are completely general and are valid for any choice of the Hamiltonian parameters \( \epsilon_\alpha \) and \( g \) (some care would have to be taken in the limit of coinciding energy levels, however). To obtain a physical result we still have to perform the sum over the \( N \) terms, introducing in the determinants for the form factors the solution to the Richardson equations. This cannot be done analytically, so we need to make a choice of the model to study. As we already mentioned, we only consider the most-studied case in the condensed matter literature, which consists of \( N \) equidistant levels at half-filling, i.e. \( N = 2N_r = 2N_f = N_f \). We normalize the levels as

\[
\epsilon_\alpha = \alpha \quad \text{with} \quad \alpha = 1 \ldots N,
\]

i.e. we measure the energy scale in terms of the interlevel spacing and we fix the Debye frequency (the largest energy level) to \( N \).

A. Correlations among the same level and “canonical” order parameter

Among the various correlation functions a central role is played by the ones on the same level. We consider the correlation

\[
u_\alpha v_\alpha = \sqrt{\langle S_\alpha^- S_\alpha^+ \rangle \langle S_\alpha^- S_\alpha^- \rangle} = \sqrt{1/4 - \langle S_\alpha^- \rangle^2},
\]

that can be easily obtained by the previous representation of \( \langle S_\alpha^- \rangle \) and does not require the reduction formulas because it is written in terms of a single form factor. This correlation is important because it is one of the building blocks of the BCS theory and because it allows to define a “canonical” BCS order parameter. In fact, as already discussed, Eq. (8) defining the grand-canonical gap, is always zero in the canonical ensemble. Thus following Ref. [33] we use as a canonical order parameter

\[
\Psi = \sum_{\alpha=1}^N u_\alpha v_\alpha.
\]

Note that \( \Psi \) is just half of the concurrence (which is a local entanglement measure, see as a review [34]) which has been already calculated with the present method[16].

FIG. 5: (Color online) \( u_\alpha v_\alpha \) as function of \( \alpha \). Each plot is at fixed \( g \) and for several \( N_p = N/2 \).

FIG. 6: (Color online) \( u_\alpha v_\alpha \) as function of \( \alpha \). Each plot is at fixed \( N_p \) but for several different \( g \) going from 0.1 (always the smallest) to 1 (always the largest) increasing by steps of 0.1.
In the large $N$ limit all these correlators must reduce to the value in the grand-canonical ensemble, which from Eq. (9) specialized to $\epsilon_\alpha = \alpha$ is

$$u_\alpha v_\alpha = \frac{1}{2} \frac{\Delta}{\sqrt{\Delta^2 + (\alpha - N_p)^2/N^2}},$$

where we fixed the chemical potential to $\mu = N/2 = N_p$ and we recall that $\Delta$ is given by Eq. (55). Consequently, in the same limit, the canonical order parameter is

$$\Psi_{N \to \infty} = \lim_{N_p \to \infty} \sum_{\alpha = 1}^{2N_p} u_\alpha v_\alpha = \frac{N\Delta}{4} \int_{-1}^{1} \frac{dx}{\sqrt{\Delta^2 + (x/2)^2}} = \frac{N\Delta}{2} \log \frac{\sqrt{1 + \Delta^2 + 1}}{\sqrt{1 + \Delta^2 - 1}} = \frac{N\Delta}{2g} \frac{N}{4g \sinh 1/2g}. \quad (64)$$

It is evident that $\Psi$ vanishes when the gap $\Delta$ is zero, confirming that in the thermodynamic limit it is a good order parameter.

Our results for $u_\alpha v_\alpha$ are reported in Figs. 5 and 6. In the former each plot consists of the various curves at fixed $g (=0.1, 0.2, 0.4, 0.7)$ with varying $N_p$. The latter instead shows the $g$ dependence at fixed $N_p$. In Fig. 6 also the BCS results for any $g$ are reported for comparison. It is evident that for all $g$ the results tend to converge to the BCS ones, as they must. However this convergence is slower as $g$ is smaller, for example for $g = 0.1$ the maximum at $N = 256$ is only 90% of the asymptotic result and conversely at $g = 0.7$ the $N_p = 16$ result is already 99.8%. These finite $N_p$ correlations are symmetric with respect to $(N + 1)/2$ by construction. However we point out that this will not be true for different level correlations, while in the grand-canonical ensemble they are both symmetric.

In Fig. 7 we report the order parameter $\Psi/N$ as a function of $g$ for several values of $N_p$, and compare it with the BCS result. This figure is exactly the same as the one for the concurrence obtained by Dunning et al.16, with the important difference that they considered only $N_p \leq 34$ while we pushed the calculation up to $N_p = 128$. We could have calculated these correlations for larger $N_p$, but the ones considered are already enough to describe the crossover from the mesoscopic to the macroscopic regime. In fact, Fig. 7 shows that for $N_p = 128$ $\Psi$ is almost indistinguishable from the BCS one Eq. (61), except for very small $g$ that are characterized by the scaling (65).

From the figure, it is evident that for $g < g^* \sim 0.18$ the BCS limit is reached from above, whereas for $g > g^*$ it is approached from below. Exactly at $g = g^*$ all the curves seem to cross in the same point. This is slightly different from what was observed before for a small number of particles11,17, where to get a similar crossing the order parameter was multiplied by $N^{-\eta}$ with $\eta \simeq 0.94$. To clarify this point we zoom in on the crossing point in Fig. 8. It is evident that for $N_p \geq 8$ all the curves approximately cross in $g^*$, but this is not the case for smaller sizes. It is then direct to interpret $g^*$ as the value of $g$ where the leading finite-size correction of order $1/N$ vanishes (in fact these are clearly negative for large $g$ and positive for very small ones). The differences for smaller size are due to higher order corrections $\sim 1/N^2$. This fixed point is thus completely analogous to results discussed in the previous section for the ground-state energy. A very interesting problem would be to calculate $g^*$ directly from the finite-size form in an analytical manner using the $1/N_p$ expansion previously discussed12,15.

The finite-size scaling $\Psi \sim N^\eta$ found in Refs. 11,17 can clearly not be true for large sizes, since $\Psi$ is an extensive quantity. To check for which sizes it stops working, in the inset of Fig. 8 we plot $\Psi N^\eta$. All the systems with sizes $N_p \leq 16$ cross indeed at the value of Refs. 11,17 $g_{cr} \sim 0.157$, but larger systems clearly deviate from this fixed point. We can then safely conclude that this scaling ansatz is effective only for $N_p \leq 16$. In Ref. 11 a second crossing point has been also found for a larger value of the pairing constant. According to our analysis also this
fixed point is present only for relatively small number of pairs.

For large \( g \) the BCS result is the leading term for large \( N \) and easily gives \( \Psi/N = 1/2 - 1/(48g^2) + O(g^{-3}, N^{-1}) \) whereas for small coupling we have:

\[
\Psi = g \frac{\ln(3 + \sqrt{8})}{\sqrt{N_p}} + O(1/\ln N_p), \text{ for } g \ll 1. \tag{65}
\]

Note that for large \( g \) we have an \( N_p \) independent result while for small \( g \) there is a square-root singularity in \( N_p \). The latter is again a manifestation of the non-perturbative nature of superconductivity. Both these analytical results are perfectly reproduced by our numerics.

### B. Static correlation functions among different levels

Despite several interesting features of the correlation functions among the same levels that we have just discussed, these are qualitatively very similar to the grand-canonical ones. On the other hand, correlation functions between different levels (known as off-diagonal ones) are a strong signature of the canonical BCS-like pairing correlations and should be relevant for the interpretation of tunneling experiments. In fact, within the grand-canonical ensemble (and so for \( N = \infty \)) these four-point correlations factorize to the product of two point ones (i.e. in this approximation the Cooper pairs are free). Oppositely, in the canonical ensemble they are non-trivial functions of both the pairs as a consequence of quantum fluctuations. Following Ref. [11], we concentrate here on the two correlation functions

\[
\langle S_i^{-} S_{i+1}^{+} \rangle, \text{ and } \langle S_i^{+} S_{i+1}^{-} \rangle. \tag{66}
\]

Our results for different values of \( g \) and \( N \) are reported in Fig. 9 and 11 respectively.

For \( N \to \infty \), as a consequence of factorization, we have

\[
\langle S_i^{-} S_{i+1}^{+} \rangle = u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} = \frac{1}{\sqrt{\Delta^2 + (\alpha - N_p)^2/N^2}} \frac{1}{\sqrt{\Delta^2 + (\beta - N_p)^2/N^2}}. \tag{67}
\]

In particular at fixed \( \beta \) these correlations are symmetric with respect to \( \alpha = N_p \). Including some trivial finite size effect, in the grand-canonical ensemble one would expect a symmetry at \((N + 1)/2\) as for the on-level correlations. However, as evident from the figures this is not the case in the canonical description. Furthermore, the smaller \( g \) is the more asymmetrical are the correlations. Such asymmetries are very pronounced for all the \( S^{-} S^{+} \) correlators, as for example showed in Fig. 10 where we report as the other extreme (compared to \( \langle S_i^{-} S_{i+1}^{+} \rangle \)), the correlator \( \langle S_{N_p+1}^{-} S_{N_p+2}^{+} \rangle \). Thus the asymmetries can be used to understand the degree of “canonicality” of a system. Note in particular the very different scales in Figs. 9 and 10 off-diagonal correlation functions are much more important when one of the levels is close to the Fermi point, a fact that is not surprising being true also in the grand-canonical ensemble.

A last property that is not apparent from the plots but that is true (even if not evident from the determinant representation) is that

\[
\langle S_{\alpha}^{-} S_{\beta}^{+} \rangle = \langle S_{\beta}^{-} S_{\alpha}^{+} \rangle, \tag{68}
\]

that checked for all the values we calculated.

The correlation function \( S_{\alpha}^{+} S_{\beta}^{-} \) in the grand-canonical ensemble also factorizes into the product of two point
that as a difference with $\Psi$ takes into account the effect of non-diagonal correlations. $\Psi_{OD}$ is clearly accessible from the direct computation of the off-diagonal correlation functions (as already proposed\textsuperscript{37}), but this is not needed. In fact, it can be obtained without the use of the determinant representation, using the Hellmann-Feynman theorem (an alternative method of calculation has been also proposed\textsuperscript{36}). The derivative of the ground-state energy with respect to the coupling strength, allows us to directly compute the double sum over all levels of the static $S^+ S^-$ correlation function, i.e.

$$\Psi_{OD} = \frac{1}{N_p} \sum_{\alpha, \beta=1}^{N} \langle S^+_{\alpha} S^-_{\beta} \rangle = -\frac{1}{N_p} \frac{\partial E_0(g)}{\partial g}. \quad (71)$$

The small $g$ behavior can be obtained analytically from the known result for the energy\textsuperscript{24}.

$$1 - E_0 = g + 2g^2 \ln 2 + O(g^3, (\ln N)^{-1})$$

$$\Rightarrow \Psi_{OD} = 1 + g4 \ln 2 + O(g^3). \quad (72)$$

This curve is shown in the inset of Fig. 12 and perfectly agrees with the numerical results. Again the deviations from this behavior start to occur at a value of $g$ given by the usual logarithmic scaling of Eq. (69).

In the thermodynamic limit, as a consequence of the factorization, $\Psi_{OD}$ is trivially related to $\Psi$ as

$$\Psi_{OD}^{N_p=\infty} = \frac{\Psi_0^{N_p=\infty}}{N_p}, \quad (73)$$

signaling that one is extensive if and only if the other one is. However, as evident from the figure, for fixed finite $N_p$ this is not true and the two quantities are independent. Furthermore for small $g$, in the regime that is not accessible by the BCS ansatz, they are both linear in $g$ and cannot be in a quadratic relation as for large $N_p$. Actually, it has been proven\textsuperscript{37} that $\Psi$ and $\Psi_{OD}$ satisfy the following relations for any value of $g$ and $N_p$.

$$\frac{1}{N_p} \Psi (\Psi - 1) \leq \Psi_{OD} \leq 1 + \frac{N}{N_p} \Psi, \quad (74)$$

that for $N_p \to \infty$ are trivial bounds, but not for finite $N_p$. We checked that our calculations satisfy these bounds.
The direct knowledge of $\Psi_{OD}$ allows for a last, very important consistency check. In fact the value found from Hellmann-Feynman theorem must equal, at half-filling, the sum $\frac{1}{N_p} \sum_{\alpha,\beta=1}^{N} \left(S^{-}_\alpha S^{+}_\beta\right)$ calculated from the determinant representation. We checked for all $N_p \leq 64$ that this is indeed the case.

VI. CONCLUSIONS

We have studied the static correlation functions of the reduced BCS model in the canonical ensemble. From the theoretical point of view we simplified the results of Ref. [13] giving the correlations as sums over only $N_r$ determinants of $N_r \times N_r$ matrices. This allowed us to calculate the correlation functions for very large numbers of particles, describing the crossover from mesoscopic to the thermodynamic limit, and going beyond previous exact or approximate studies. In particular with such accurate calculations we were able to discuss critically some conjectured scaling forms for the canonical order parameter. For example we rule out the idea of any phase transition as a function of the (positive) pairing strength and number of particles, in agreement with other analyses based on thermodynamical quantities.\[3-8,39\]. We also calculate the off-diagonal long-range order parameter by using the Hellmann-Feynman theorem.

A first interesting step to go beyond what has been done here would be to find a single determinant representation for the correlation functions. We made several attempts in this direction, but so far unsuccessfully.

We only analyzed the case of $N$ non-degenerate equidistant energy levels at half-filling. This is the most interesting model from the condensed matter point of view. However for the description of pairing in nuclei other choices of the parameters $e_\alpha$ are more natural\[10,20,41,44\]. These could be treated by a simple adaptation of our results.

Furthermore the method presented here allows in principle to obtain dynamical correlation functions as sums of form factors over the excited states. This is a more numerically demanding problem, but can be tackled in the same way as other Bethe Ansatz solvable models\[42,43\]. Also, by summing over the excited states, one can access the finite temperature thermodynamics and this can help in understanding some open questions\[44,45\] for ultra-small metallic grains.

The Hamiltonian (1) is the simplest one with pairing terms. More general models with several couplings have been proposed to explain complicated pairing in condensed matter\[46\] and nuclear physics\[47\]. Some have been also shown to be integrable\[44,45\]. As a consequence it would be extremely interesting to tackle these models with methods similar to those presented here.

Acknowledgments

We thank Luigi Amico for extremely fruitful discussions and correspondence. All the authors are thankful for support from the Stichting voor Fundamenteel Onderzoek der Materie (FOM) in the Netherlands. This work has been done mainly when PC was a guest of the Institute for Theoretical Physics of the Universiteit van Amsterdam. This stay was supported by the ESF Exchange Grant 1311 of the INSTANS activity.

1 K. Rajagopal and F. Wilczek, \textit{hep-ph/0011333}.
2 M. Alford, Ann. Rev. Nucl. Part. Sci. 51, 131 (2001).
3 J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 106, 162 (1957); \textit{ibid.} 108, 1175 (1957).
4 R. W. Richardson, Phys. Lett. 3, 277 (1963); \textit{ibid.} 5, 82 (1963); R. W. Richardson and N. Sherman, Nucl. Phys. 52, 221 (1964); 52, 253 (1964).
5 D. C. Ralph, C. T. Black, and M. Tinkham, Phys. Rev. Lett. 74, 3241 (1995); and \textit{ibid.} 76, 688 (1996); and \textit{ibid.} 78, 4087 (1997).
6 J. von Delft and D. C. Ralph, Phys. Rep. 345, 61 (2001).
7 P. W. Anderson, J. Phys. Chem. Solids 11, 28 (1959).
8 J. Dukelsky, S. Pittel, and G. Sierra, Rev. Mod. Phys. 76, 643 (2004).
9 D. J. Dean and M. Hjorth-Jensen, Rev. Mod. Phys. 75, 607 (2003).
10 R. W. Richardson, J. Math. Phys. 6, 1034 (1965).
11 L. Amico and A. Osterloh, Phys. Rev. Lett. 88, 127003 (2002).
12 E. K. Sklyanin, Lett. Math. Phys. 47, 275 (1999).
13 H.-Q. Zhou, J. Links, R.H. McKenzie, and M.D. Gould, Phys. Rev. B 65, 060502(R) (2002).
14 J. Links, H.-Q. Zhou, R.H. McKenzie, and M.D. Gould, J. Phys. A 36, R63 (2003).
15 N. A. Slavnov, Teor. Mat. Fiz. 79, 232 (1989).
16 C. Dunning, J. Links, and H.-Q. Zhou, Phys. Rev. Lett. 94, 227002 (2005).
17 A. Mastellone, G. Falc(i, and R. Fazio, Phys. Rev. Lett. 80, 4542 (1998).
18 M. Gaudin, \textit{Mod` eles Exactement R´ esolus} (Les ´Editions de Physique, Les Ulis, France, 1995).
19 M. C. Cambiaggio, A. M. F. Rivas, and M. Saraceno, Nucl. Phys. A 624, 157 (1997).
20 L. Amico, G. Falc(i, and R. Fazio, J. Phys. A 34, 6425, (2001).
21 J. von Delft and R. Poghossian, Phys. Rev. B 66, 134502 (2002).
22 R.W. Richardson, J. Math. Phys. 18, 1802 (1977).
23 R. W. Richardson, Phys. Rev. 141, 949 (1966).
24 M. Schechter, Y. Imry, Y. Levinson, and J. von Delft, Phys. Rev. B 63, 214518 (2001).
25 E. A. Yuzbashyan, A. A. Baytin, and B. L. Altshuler, Phys. Rev. B 68, 214509 (2003).
26 I. Snyman and H. B. Geyer, Phys. Rev. B 73, 144516 (2006).
27 J. M. Roman, G. Sierra, and J. Dukelsky, Phys. Rev. B 67, 064510 (2003).
28 M. Sambataro, Phys. Rev. C 75, 054314 (2007).
29 S. Rombouts, D. Van Neck and J. Dukelsky, Phys. Rev. C 69, 061303 (2004).
30 F. Dominguez, C. Esebbag, and J. Dukelsky, J. Phys. A 39, 11349 (2006).
31 J.M. Roman, G. Sierra, and J. Dukelsky, Nucl. Phys. B 634, 483 (2002).
32 E. A. Yuzbashyan, A. A. Baytin, and B. L. Altshuler, Phys. Rev. B 71, 094505 (2005).
33 J. von Delft, A. D. Zaikin, D. S. Golubev, and W. Tichy, Phys. Rev. Lett. 77, 3189 (1996).
34 L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys., to appear [quant-ph/0703044].
35 O. Penrose and L. Onsager, Phys. Rev. 104, 576 (1956); C. N. Yang, Rev. Mod. Phys. 34, 691 (1962).
36 G. Ortiz and J. Dukelsky, Phys. Rev. A 72, 043611 (2005).
37 G.-S. Tian, L.-H. Tang, and Q.-H. Chen, Europhys. Lett. 50, 361 (2000); Phys. Rev. B 63, 054511 (2001).
38 J. Dukelsky and G. Sierra, Phys. Rev. Lett. 83, 172 (1999).
39 G. Sierra, J. Dukelsky, G. G. Dussel, J. von Delft, F. Braun, Phys. Rev. B 61, 11890 (2000).
40 A. Belic, D.J. Dean, and M. Hjorth-Jensen, Nucl. Phys. A 731, 381 (2004).
41 V. Zelevinsky and A. Volya, Nucl. Phys. A 752, 325 (2005).
42 J.-S. Caux and J. M. Maillet, Phys. Rev. Lett. 95, 072701 (2005).
43 J.-S. Caux and P. Calabrese, Phys. Rev. A 74, 031605R (2006); J.-S. Caux, P. Calabrese, and N. A. Slavinov, J. Stat. Mech. P01008 (2007).
44 J.-S. Caux and P. Calabrese, Phys. Rev. A 74, 031605R (2006); J.-S. Caux, P. Calabrese, and N. A. Slavinov, J. Stat. Mech. P01008 (2007).
45 T. Sumaryada and A. Voyla, Phys. Rev. C 76, 024319 (2007).
46 A. Di Lorenzo, Rosario Fazio, F.W.J. Hekking, G. Falci, A. Mastellone, and G. Giaquinta, Phys. Rev. Lett. 84, 550 (2000).
47 I. L. Kurland, I. L. Aleiner, and B. L. Altshuler, Phys. Rev. B 62, 14886 (2000); J. Dukelsky, G. G. Dussel, C. Esebbag, and S. Pittel, Phys. Rev. Lett. 93, 050403 (2004); J. Dukelsky, G. Ortiz, S.M.A. Rombouts, and K. Van Houcke Phys. Rev. Lett. 96, 180404 (2006); A. M. García-García, J. D. Urbina, E. A. Yuzbashyan, K. Richter and B. L. Altshuler, arXiv:0710.2286.
48 J. Dukelsky, V. G. Gueorguiev, P. Van Isacker, S. Dimitrova, B. Errea, and S. Lerma H, Phys. Rev. Lett. 96, 072503 (2006); S. Lerma H., B. Errea, J. Dukelsky, and W. Satula, Phys. Rev. Lett. 99, 032501 (2007).
49 L. Amico, A. Di Lorenzo, and A. Osterloh, Phys. Rev. Lett. 86, 5759 (2001); J. Links, H.-Q. Zhou, R.H. McKenzie, and M.D. Gould Int. J. Mod. Phys. B 16, 3429 (2002).