ABELIAN CARTER SUBGROUPS IN FINITE PERMUTATION GROUPS

ENRICO JABARA AND PABLO SPIGA

Abstract. We show that a finite permutation group containing a regular abelian self-normalizing subgroup is soluble.

1. Introduction

Let $G$ and $A$ be finite groups with $A$ acting on $G$ as a group of automorphisms. When $C_G(A) = 1$, it is customary to say that $A$ acts fixed-point-freely on $G$, that is, $1$ is the only element of $G$ invariant by every element of $A$. From the seminal work of Thompson [11], fixed-point-free groups of automorphisms have attracted considerable interest and have shown to be remarkably important within finite group theory. It is well-established that in many cases the condition $C_G(A) = 1$ forces the group $G$ to be soluble. One of the main contributions is the result of Rowley [9], showing that groups admitting a fixed-point-free automorphism are soluble. This result was then generalized by Belyaev and Hartley [2], showing that if a nilpotent group acts fixed-point-freely on $G$, then $G$ is soluble. In this paper, as an application of this remarkable result, we prove the following.

Theorem 1.1. If $G$ is a finite transitive permutation group with a regular abelian self-normalizing subgroup, then $G$ is soluble.

(A self-normalizing nilpotent subgroup of a finite group is called a Carter subgroup.) Theorem[1.1] is provoked by some recent investigations [4] on Cayley graphs over abelian groups. In fact, Dobson, Verret and the second author have recently proved a conjecture of Babai and Godsil [1, Conjecture 2.1] concerning the enumeration of Cayley graphs over abelian groups. In the approach in [5], at a critical juncture [6, proof of Theorem 1.5], it is necessary to have structural information on finite permutation groups admitting a regular abelian Carter subgroup. In view of [5], any improvement in our understanding of the structure of the automorphism group of a Cayley graph over an abelian group requires a detailed description of the finite permutation groups containing a regular abelian Carter subgroup. Theorem[1.1] is a first step in this direction.

The hypothesis of $A$ being abelian is rather crucial in Theorem[1.1]. For instance, for each Mersenne prime $p = 2^ℓ - 1$, the group $\text{PSL}_2(p)$ acts primitively on the points of the projective line and contains a regular Carter subgroup isomorphic to the dihedral group of order $2^ℓ$. 

2010 Mathematics Subject Classification. Primary 20B25; Secondary 05E18.

Key words and phrases. fixed-point-free, regular group, Carter subgroup, permutation group.

Address correspondence to Pablo Spiga: pablo.spiga@unimi.it.
A well-known theorem of Carter \cite{3} shows that every finite soluble group contains exactly one conjugacy class of Carter subgroups. On the other hand, a finite non-soluble group may have no Carter subgroup, as witnessed by the alternating group $\text{Aut}(5)$. However, using the Classification of the Finite Simple Groups, Vdovin \cite{12} has extended the result of Carter by showing that, if a finite group $G$ contains a Carter subgroup, then every two distinct Carter subgroups of $G$ are conjugate.

The main ingredients in our proof of Theorem 1.1 are two results of fundamental importance in finite group theory. The first is the classification of Li \cite{7} of the finite primitive groups containing an abelian regular subgroup. This classification will allow us to prove Theorem 1.1 for primitive groups by a simple direct inspection (see Proposition 2.1). The second is the remarkable theorem of Vdovin \cite{12} showing that any two Carter subgroups of a finite group are conjugate. This result will allow us to use induction for proving Theorem 1.1. In particular, as \cite{2, 7, 12} depend upon the Classification of the Finite Simple Groups, so does Theorem 1.1.

2. Proof of Theorem 1.1

Before proving Theorem 1.1 we need an auxiliary result.

**Proposition 2.1.** Let $G$ be a finite primitive permutation group with a regular abelian subgroup $A$. Then either $\mathbf{N}_G(A) \neq A$, or $G = A$ has prime order.

**Proof.** The finite primitive groups containing an abelian regular subgroup are classified by Li in \cite{7}. In fact, from \cite{7} Theorem 1.1, we see that either the socle $N$ of $G$ is abelian, or $N$ is the direct product of $\ell \geq 1$ pairwise isomorphic non-abelian simple groups and $G$ is endowed of the primitive product action. We consider these two cases separately.

Assume that $N$ is abelian. Then $N$ is an elementary abelian $p$-group, for some prime $p$. Since $N$ and $A$ both act regularly, we have $|N| = |A|$ and hence $A$ is a $p$-group. Therefore $AN$ is also a $p$-group. Suppose $AN > A$. Then (from basic properties of $p$-groups) we have $\mathbf{N}_{AN}(A) > A$ and hence $\mathbf{N}_G(A) \neq A$. Suppose $AN = A$. Then $A = N < G$ and, if $G > A$, then $\mathbf{N}_G(A) = G \neq A$, and if $G = A$, then $A$ must have prime order.

Assume that $N$ is non-abelian. Let $T_1, \ldots, T_\ell$ be the simple direct factors of $N$. So $N = T_1 \times \cdots \times T_\ell$. From \cite{7} Theorem 1.1 (2)], we see that $G$ contains a normal subgroup $\overline{N} = \overline{T_1} \times \cdots \times \overline{T_\ell}$ with $\overline{T_i}$ having socle $T_i$ for each $i \in \{1, \ldots, \ell\}$, with $A \leq \overline{N}$ and with

$$A = (A \cap \overline{T_1}) \times \cdots \times (A \cap \overline{T_\ell}).$$

Furthermore, for $i \in \{1, \ldots, \ell\}$, each of the possible pairs $(\overline{T_i}, A \cap \overline{T_i})$ is listed in \cite{7} Theorem 1.1 (2) (i),\ldots,(iv)]. An easy inspection on each of these four cases reveals that $\mathbf{N}_{\overline{T_i}}(A \cap \overline{T_i}) > (A \cap \overline{T_i})$, for each $i \in \{1, \ldots, \ell\}$. Therefore

$$\mathbf{N}_G(A) \supseteq \mathbf{N}_{\overline{N}}(A) = \mathbf{N}_{\overline{T_1}}(A \cap \overline{T_1}) \times \cdots \times \mathbf{N}_{\overline{T_\ell}}(A \cap \overline{T_\ell}) > (A \cap \overline{T_1}) \times \cdots \times (A \cap \overline{T_\ell}) = A.$$

\[\square\]

**Proof of Theorem 1.1.** We argue by contradiction and among all possible counterexamples we choose $G$ and $A$ with $|G| + |A|$ as small as possible. We let $\Omega$ be the transitive set acted upon by $G$.

As $G$ is insoluble and $A$ is self-normalizing, from Proposition 2.1 we deduce that $G$ is imprimitive. Among all non-trivial systems of imprimitivity for $G$ choose $\mathcal{B}$
with blocks of minimal size. We denote by $G_{(B)}$ the pointwise stabilizer of $B$, that is, $G_{(B)} = \{ g \in G \mid B^g = B \}$, for each $B \in \mathcal{B}$. In particular, $G_{(B)}$ is the kernel of the action of $G$ on $B$. Also, for $B \in \mathcal{B}$, we write $G_{(B)} = \{ g \in G \mid B^g = B \}$ for the setwise stabilizer of $B$ and $G_{(B)} = \{ g \in G \mid \alpha^g = \alpha, \text{ for each } \alpha \in B \}$ for the pointwise stabilizer of $B$. Moreover, given a subgroup $X$ of $G$, we let $X^B$ denote the permutation group induced by $X$ on $B$. In particular, $X^B \cong XG_{(B)}/G_{(B)}$.

Observe that $A^B$ is a transitive abelian subgroup of $G^B$, and hence $A^B$ acts regularly on $B$. We show that $A^B$ is a Carter subgroup of $G^B$. As $A^B \cong AG_{(B)}/B$, it suffices to show that $N_G(A^B) = AG_{(B)}$. Let $g \in N_G(A^B)$. Now $(AG_{(B)})^g = AG_{(B)}$ and hence $A, A^g \leq AG_{(B)}$. As $A$ and $A^g$ are Carter subgroups of $G$, they are also Carter subgroups of $AG_{(B)}$. In particular, by [12], $A$ and $A^g$ are conjugate in $AG_{(B)}$. Therefore there exists $h \in G_{(B)}$ with $A^h = A^g$. Thus $gh^{-1} \in N_G(A) = A$ and $g \in AG_{(B)}$.

Since $A^B$ is a Carter subgroup of $G^B$, we see that $G^B$ and $A^B$ satisfy the hypothesis of Theorem 1.3. As $\mathcal{B}$ is a non-trivial system of imprimitivity, $|A^B| < |A|$ and $|G^B| \leq |G|$ and hence, by minimality, $G^B \cong G/G_{(B)}$ is soluble. In particular, $G_{(B)}$ is insoluble.

Suppose that $AG_{(B)} < G$. Then $A$ is a regular abelian Carter subgroup of $AG_{(B)}$ with $|AG_{(B)}| < |G|$. By minimality, $AG_{(B)}$ is soluble and hence so is $G_{(B)}$, a contradiction. Thus

$$G = AG_{(B)}.$$ (1)

From [1], we have $G^B = A^B$ and hence $G$ acts regularly on $B$. In particular, $G_\omega \leq G_{(B)}$, for every $\omega \in \Omega$, and $G_{(B)} = G_{(B)}$, for every $B \in \mathcal{B}$. Moreover, by the minimality of $|B|$, the group $G_{(B)} = G_{(B)}$ acts primitively on $B$.

We show that

$$N_G(A_0) = C_G(A_0), \quad \text{for every } A_0 \leq A.$$ (2)

Clearly the right hand side is contained in the left hand side. Let $g \in N_G(A_0)$. Now $A_0$ is centralized by $A$ and by $A^g$, and so by $\langle A, A^g \rangle$. As $A$ and $A^g$ are Carter subgroups of $\langle A, A^g \rangle$, by [12] there exists $x \in \langle A, A^g \rangle$ with $A^g = A^x$. Thus $gx^{-1} \in N_G(A) = A$. In particular, $gx^{-1}$ and $x$ centralize $A_0$, and hence so does $g$. Thus $g \in C_G(A_0)$.

Applying (2) with $A_0 = A \cap G_{(B)}$ we obtain

$$N_G(A \cap G_{(B)}) = C_G(A \cap G_{(B)}).$$ (3)

Fix $B_1 \in \mathcal{B}$ and write $H = G_{(B_1)}^{B_1}$, that is, the permutation group induced by $G_{(B_1)}$ on the block $B_1$. By the Embedding Theorem [5] Theorem 1.2.6, the group $G$ is permutation isomorphic to a subgroup of the wreath product $W = H wr A^B$ (in its natural imprimitive product action on the cartesian product $B_1 \times \mathcal{B}$). Write $\ell = |B|$. In particular, under this embedding $G_{(B)}$ is identified with a subgroup of $H^\ell$. As $G_{(B)}$ is insoluble and primitive in its action on $B_1$, it follows that

$$H \quad \text{is an insoluble primitive permutation group.}$$ (4)

Let $N$ be a minimal normal subgroup of $G$ with $N \leq G_{(B)}$. We now divide the proof in two cases: depending on whether $N$ is abelian or non-abelian.

**Case I:** $N$ is non-abelian.
Observe that $A$ is a regular abelian Carter subgroup of $AN$ and that $AN$ is not soluble, hence by minimality $G = AN$. The modular law gives $G(B) = (A \cap G(B)) N$ and thus $H = G(B)_{\ell} = (A \cap G(B))_{\ell} N^{B_1}$. It is easy to check that $(A \cap G(B))_{\ell}$ is a regular abelian subgroup of $H$ and that $N^{B_1}$ is a characteristically simple normal subgroup of $H$.

We now appeal to the classification of Li [7] of the primitive groups with a regular abelian subgroup. Using the notation in [7], we see that in all the groups $X$ in [7] Theorem 1.1(2) the regular abelian subgroup $G$ of $X$ acts trivially by conjugation on the simple direct factors of the socle of $X$. In particular, as $(A \cap G(B))_{\ell}$ acts transitively on the simple direct factors of $N^{B_1}$, we deduce that $N^{B_1}$ is a non-abelian simple group, that is, $H$ is an almost simple group. Furthermore, $(H, (A \cap G(B))_{\ell})$ is one of the pairs (denoted by $(\bar{T}, G_i)$) in [7] Theorem 1.1(2) (i), . . . , (iv)].

Let $T_1, \ldots, T_p$ be the simple direct factors of $N$ with $T_1$ acting faithfully on $B_1$ and with $N(B_1) = T_2 \times \cdots \times T_p$. Observe that $T_1, \ldots, T_p$ are pairwise isomorphic and that $A$ acts transitively by conjugation on $\{T_1, \ldots, T_p\}$. Let $\pi_1 : N \to T_1$ be the projection on the first coordinate and set $A_1 = \pi_1(N \cap A)$. In particular, replacing $G$ by a suitable conjugate in $\text{Aut}(N)$, we may suppose that $N \cap A = \{(a_1, \ldots, a_p) \mid a \in A_1\} \leq A_{\ell}$. Now, a direct inspection on each of the pairs in [7] Theorem 1.1(2) (i), . . . , (iv)] shows that there exists $h \in N_{T_1}(A_1) \setminus C_{T_1}(A_1)$. The element $g = (h, \ldots, h) \in T_1 \times \cdots \times T_p = N$ and $g \not\in N_G(A \cap N) \setminus C_G(A \cap N)$, contradicting [2] (applied with $A_0 = A \cap N$).

**Case II:** $N$ is an elementary abelian $p$-group for some prime $p$.

Now $N^{B_1}$ is a normal elementary abelian $p$-subgroup of $G^{B_1}_{\ell} = H$. As $H$ is primitive, $N^{B_1}$ is the socle of $H$ and $H$ is a primitive group of affine type, that is, $H$ is isomorphic to a primitive subgroup of the affine general linear group $\text{AGL}_d(p)$ for some $d \geq 1$.

Let $\mathcal{V}$ be the socle of $H$, let $\mathcal{V}_{\ell}$ be the socle of $H_{\ell}$ (as a subgroup of $W$) and let $U = \mathcal{V}_{\ell} \cap G$. Let $\pi_1 : G_{(B)} \to H$ be the projection onto the first coordinate and set $A_1 = \pi_1(A \cap G(B))$. Since $A_1 = (A \cap G(B))_{\ell}$, we see that $A_1$ is a regular abelian subgroup of $H$. Moreover, as $A$ acts transitively by conjugation on the $\ell$ coordinates of $H_{\ell}$, replacing $G$ by a suitable conjugate, under the embedding $G \leq W$ we have

\begin{equation}
A \cap G(B) = \{(a, \ldots, a) \mid a \in A_1\}. \tag{5}
\end{equation}

Suppose that $A_1 = V$. Then $A \cap G(B) \leq V_{\ell} \cap G = U$. Moreover, $G_{(B)}/U$ is isomorphic to a subgroup of the direct product $\text{GL}_d(p)^\ell$. Since $G_{(B)}$ is insoluble, so is $X = G_{(B)}/U$. Furthermore, $A$ acts as a group of automorphisms on $X$. From [2] Theorem 0.11, we have $C_X(A) \neq 1$ and hence there exists $g \in G_{(B)} \setminus U$ with $[g, A] \leq U$. As $A$ acts transitively on the elements of $B$, $g = (v_1h, v_2h, \ldots, v_\ell h)$ for some $h \in H$ and some $v_1, \ldots, v_\ell \in V$. As $g \not\in U$, we have $h \not\in V$ (if $h \in V$, then $g \in V_{\ell} \cap G = U$, a contradiction). As $V = C_H(V)$, we get $h \not\in C_H(V)$ and $h \in H = N_H(V)$. From [5] it is clear that $g \not\in C_G(A \cap G(B)) \setminus C_G(A \cap G(B))_{\ell}$, which contradicts [3].

Suppose that $A_1 \neq V$. We show that there exists $u \in N_V(A \cap G(B)) \setminus C_V(A \cap G(B))$, from which the theorem will immediately follow by contradicting [3]. Since $A_1 V > A_1$ and $C_H(A_1) = A_1$, there exists $\bar{v} \in N_V(A_1) \setminus C_H(A_1)$. We label the elements of $B$ by $B_1, B_2, \ldots, B_\ell$, where the action of $g = (g_1, \ldots, g_\ell) \in G(B)$ on $B_i$ is given by the element $g_{i, h}$ on the $i$th coordinate of $g$. 

Assume that $G_{(B_1)} = 1$. Let $u \in U$ with $u^{B_1} = \bar{v}$, that is, $u = (\bar{v}, v_2, \ldots, v_\ell)$, for some $v_2, \ldots, v_\ell \in V$. By construction $u$ normalizes $A \cap G_{(B)}$ modulo $G_{(B_1)}$ and hence $u$ normalizes $A \cap G_{(B)}$ because $G_{(B_1)} = 1$.

Assume that $G_{(B_1)} \neq 1$. Let $B \in \mathcal{B} \setminus \{B_1\}$ with $(G_{(B_1)})^B \neq 1$. Relabelling the set $\mathcal{B}$ if necessary, we may assume that $B = B_2$. As $G_{(B_1)} \trianglelefteq G_{(B)}$ and $G_{(B)}$ acts primitively on $B_2$, we get $V \leq (G_{(B_1)})^{B_2}$. We show that

$$U = U_{(B_1)}U_{(B_2)}.$$  

Let $z = (v_1, v_2, v_3, \ldots, v_\ell) \in U$. As $V = (U_{(B_1)})^{B_2}$, there exists $x = (1, v_2, v'_3, \ldots, v'_\ell) \in U_{(B_1)}$, for some $v'_3, \ldots, v'_\ell \in V$. Clearly, $zx^{-1} = (v_1, 1, v_3v'^{-1}_3, \ldots, v_\ell v'^{-1}_\ell) \in U_{(B_2)}$.

Now we show that

$$(\bar{v}, \bar{v}, v_3, \ldots, v_\ell) \in U, \quad \text{for some } v_3, \ldots, v_\ell \in V.$$  

As $U = U_{(B_1)}U_{(B_2)}$, we get $V = (U_{(B_1)})^{B_2}$ and hence there exists $y = (1, \bar{v}, v'_3, \ldots, v'_\ell) \in U_{(B_1)}$. Similarly, as $V = (U_{(B_2)})^{B_1}$, there exists $x = (\bar{v}, 1, v'^{-1}_3, \ldots, v'^{-1}_\ell) \in U_{(B_2)}$. Now, $xy \in U$ and the first two coordinates of $xy$ are $\bar{v}, \bar{v}$.

Assume that $G_{(B_1)} \cap G_{(B_2)} = 1$, that is, $G_{(B_1 \cup B_2)} = 1$. Let $u = (\bar{v}, \bar{v}, v_3, \ldots, v_\ell) \in U$. Now, from [5], we see that $u$ normalizes $A \cap G_{(B)}$ modulo $G_{(B_1 \cup B_2)}$ and hence $u$ normalizes $A \cap G_{(B)}$ because $G_{(B_1 \cup B_2)} = 1$.

Assume that $G_{(B_1)} \cap G_{(B_2)} \neq 1$. Let $B \in \mathcal{B} \setminus \{B_1, B_2\}$ with $(G_{(B_1 \cup B_2)})^B \neq 1$. Relabelling the set $\mathcal{B}$ if necessary, we may assume that $B = B_3$. As $G_{(B_1 \cup B_2)} \trianglelefteq G_{(B)}$ and $G_{(B)}$ acts primitively on $B_3$, we get $V \leq (G_{(B_1 \cup B_2)})^{B_3}$. We show that

$$U = U_{(B_1 \cup B_2)}U_{(B_3)}.$$  

Let $z = (v_1, v_2, v_3, \ldots, v_\ell) \in U$. As $V = (U_{(B_1 \cup B_2)})^{B_3}$, there exists $x = (1, v_3, v'_4, \ldots, v'_\ell) \in U_{(B_1 \cup B_2)}$, for some $v'_4, \ldots, v'_\ell \in V$. Clearly, $zx^{-1} \in U_{(B_3)}$.

Now we show that

$$(\bar{v}, \bar{v}, \bar{v}, v_4, \ldots, v_\ell) \in U, \quad \text{for some } v_4, \ldots, v_\ell \in V.$$  

Indeed, from above there exists $z = (\bar{v}, \bar{v}, w_3, \ldots, w_\ell) \in U$, for some $w_3, \ldots, w_\ell \in V$. Moreover, as $U = U_{(B_1 \cup B_2)}U_{(B_3)}$, we get $V = U^{B_3} = (U_{(B_1 \cup B_2)})^{B_3}$ and hence there exists $y = (1, 1, w'^{-1}_3, w'_4, \ldots, w'_\ell) \in U_{(B_1 \cup B_2)}$. Now, $zy \in U$ and the first three coordinates of $zy$ are $\bar{v}, \bar{v}, \bar{v}$.

Assume that $G_{(B_1 \cup B_2)} \cap G_{(B_3)} = 1$, that is, $G_{(B_1 \cup B_2 \cup B_3)} = 1$. Let $u = (\bar{v}, \bar{v}, \bar{v}, v_4, \ldots, v_\ell) \in U$. Now, from [5], we see that $u$ normalizes $A \cap G_{(B)}$ modulo $G_{(B_1 \cup B_2 \cup B_3)}$ and hence $u$ normalizes $A \cap G_{(B)}$ because $G_{(B_1 \cup B_2 \cup B_3)} = 1$.

Finally, the case $G_{(B_1 \cup B_2 \cup B_3)} \neq 1$ follows easily by induction applying the same two-step argument as in the previous paragraphs. \hfill \Box

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Enrico Jabara, Dipartimento di Filosofia e Beni Culturali, University Cà Foscari, Dorsoduro 3484/D 30123 Venezia, I-30100 Venezia, Italy. E-mail address: jabara@unive.it

Pablo Spiga, Dipartimento di Matematica Pura e Applicata, University Milano-Bicocca, Via Cozzi 53, 20126 Milano, Italy. E-mail address: pablo.spiga@unimib.it