Interpolating by Functions from Model Subspaces in $H^1$

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Abstract. Given an interpolating Blaschke product $B$ with zeros $\{a_j\}$, we seek to characterize the sequences of values $\{w_j\}$ for which the interpolation problem
\[ f(a_j) = w_j \quad (j = 1, 2, \ldots) \]
can be solved with a function $f$ from the model subspace $H^1 \cap B\overline{H_0}$ of the Hardy space $H^1$.

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1. Introduction

Let $\mathbb{D}$ stand for the disk $\{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T}$ for its boundary, and $m$ for the normalized Lebesgue measure on $\mathbb{T}$. For $0 < p \leq \infty$, we denote by $H^p$ the classical Hardy $p$-space of holomorphic functions on $\mathbb{D}$ (see [4, Chapter II]), viewed also as a subspace of $L^p(\mathbb{T}, m)$.

Now suppose $\theta$ is an inner function on $\mathbb{D}$, that is, $\theta \in H^\infty$ and $|\theta| = 1$ almost everywhere on $\mathbb{T}$. The associated model subspace $K^p_\theta$ is then defined, this time for $1 \leq p \leq \infty$, by putting
\[ K^p_\theta := H^p \cap \theta \overline{H_0^p}, \tag{1} \]
where $H_0^p := z H^p$ and the bar denotes complex conjugation. In other words, $K^p_\theta$ is formed by those $f \in H^p$ for which the function
\[ \tilde{f} := \overline{z f \theta} \]
(living a.e. on $\mathbb{T}$) is in $H^p$. Moreover, the map $f \mapsto \tilde{f}$ leaves $K^p_\theta$ invariant.

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We may also rephrase the above definition by saying that $K_\theta^p$ is the kernel (in $H^p$) of the Toeplitz operator with symbol $\theta$. For further – and deeper – operator theoretic connections, we refer to [6]. Finally, we mention the direct sum decomposition formula

$$H^p = K_\theta^p \oplus \theta H^p, \quad 1 < p < \infty$$  \hspace{1cm} (2)

(with orthogonality for $p = 2$), which is a consequence of the M. Riesz projection theorem; cf. [4, Chapter III].

From now on, the role of $\theta$ will be played by an interpolating Blaschke product, i.e., by a function of the form

$$B(z) = B_{\{a_k\}}(z) := \prod_k \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z},$$

with $\{a_k\} \subset \mathbb{D}$ and $\sum_k (1 - |a_k|) < \infty$, satisfying

$$\inf_k |B'(a_k)| (1 - |a_k|) > 0.$$  \hspace{1cm} (4)

The zero sequences $\{a_k\}$ of such products are precisely the interpolating sequences for $H^\infty$ (see [4, Chapter VII]), meaning that

$$H^\infty_{\{a_k\}} = \ell^\infty.$$  \hspace{1cm} (3)

Here and throughout, we use the notation $X_{\{a_k\}}$ (where $X$ is a given function space on $\mathbb{D}$) for the set of all sequences $\{w_k\} \subset \mathbb{C}$ such that the interpolation problem

$$f(a_k) = w_k \quad (k = 1, 2, \ldots)$$  \hspace{1cm} (5)

has a solution $f \in X$. Equivalently, $X_{\{a_k\}}$ consists of the traces $f_{\{a_k\}} = \{f(a_k)\}$ that arise as $f$ ranges over $X$.

Now, for $0 < p < \infty$, the appropriate $H^p$ analogue of (3) is known to be

$$H^p_{\{a_k\}} = \ell^p_1(\{a_k\}),$$  \hspace{1cm} (5)

where $\ell^p_1(\{a_k\})$ stands for the set of sequences $\{w_k\} \subset \mathbb{C}$ with

$$\sum_k |w_k|^p (1 - |a_k|) < \infty.$$  \hspace{1cm} (6)

Indeed, a theorem of Shapiro and Shields (see [7]) tells us that, for $1 \leq p < \infty$, (5) holds if and only if $\{a_k\}$ is an interpolating sequence for $H^\infty$; the case $0 < p < 1$ is, in fact, no different. Letting $B = B_{\{a_k\}}$ be the associated (interpolating) Blaschke product, we may then combine (5) with the observation that

$$H^p_{\{a_k\}} = K^p_{B_{\{a_k\}}}, \quad 1 < p < \infty$$  \hspace{1cm} (2)

(which follows upon applying (2) with $\theta = B$), to deduce that

$$K^p_{B_{\{a_k\}}} = \ell^p_1(\{a_k\}), \quad 1 < p < \infty.$$  \hspace{1cm} (7)

The trace space $K^p_{B_{\{a_k\}}}$ with $1 < p < \infty$ is thereby nicely characterized, but the endpoints $p = 1$ and $p = \infty$ present a harder problem. The latter
case was recently settled by the author in [3]. There, it was shown that a sequence \( \{w_k\} \in \ell^\infty \) belongs to \( K^\infty_B|_{\{a_k\}} \) if and only if the numbers
\[
\tilde{w}_k := \sum_j \frac{w_j}{B'(a_j) \cdot (1 - a_j \bar{a}_k)} \quad (k = 1, 2, \ldots)
\]
satisfy \( \{\tilde{w}_k\} \in \ell^\infty \). We mention in passing that a similar method, coupled with [2, Theorem 5.2], yields the identity
\[
K^*_B|_{\{a_k\}} = \{ \{w_k\} \in \ell^2(\{a_k\}) : \{\tilde{w}_k\} \in \ell^\infty \},
\]
where \( K^*_B := K^2_B \cap \text{BMO}(\mathbb{T}) \).

It is the other extreme (i.e., the case \( p = 1 \)) that puzzles us. We feel that the numbers \( \tilde{w}_k \), defined as in (8), must again play a crucial role in the solution, and we now proceed to discuss this in detail.

2. Problems and discussion

**Problem 1.** Given an interpolating Blaschke product \( B \) with zeros \( \{a_k\} \), characterize the set \( K^1_B|_{\{a_k\}} \).

An equivalent formulation is as follows.

**Problem 1'.** For \( B = B_{\{a_k\}} \) as above, describe the functions from \( K^1_B + BH^1 \) in terms of their values at the \( a_k \)’s.

The equivalence between the two versions is due to the obvious fact that, given a function \( f \in H^1 \), we have \( f \in K^1_B + BH^1 \) if and only if \( f|_{\{a_k\}} = g|_{\{a_k\}} \) for some \( g \in K^1_B \).

It should be noted that, whenever \( B = B_{\{a_k\}} \) is an infinite interpolating Blaschke product, the trace space \( K^1_B|_{\{a_k\}} \) is strictly smaller than \( H^1|_{\{a_k\}} = \ell^1(\{a_k\}) \), a result that can be deduced from [8, Theorem 3.8]. On the other hand, it was shown by Vinogradov in [9] that
\[
K^1_{B^2}|_{\{a_k\}} = \ell^1(\{a_k\})
\]
for each interpolating Blaschke product \( B = B_{\{a_k\}} \). The subspace \( K^1_{B^2} \) is, however, essentially larger than \( K^1_B \).

In light of our solution to the \( K^\infty_B \) counterpart of Problem 1, as described in Section 1 above, the following conjecture appears to be plausible.

**Conjecture 1.** Let \( B = \text{B}_{\{a_k\}} \) be an interpolating Blaschke product. In order that a sequence of complex numbers \( \{w_k\} \) belong to \( K^1_B|_{\{a_k\}} \), it is necessary and sufficient that
\[
\sum_k |w_k| (1 - |a_k|) < \infty
\]
and
\[
\sum_k |\tilde{w}_k| (1 - |a_k|) < \infty
\]
(i.e., that both \( \{w_k\} \) and \( \{\tilde{w}_k\} \) be in \( \ell_1^1(\{a_k\}) \)).

As a matter of fact, the necessity of (9) and (10) is fairly easy to verify. First of all, letting \( \delta_k \) denote the unit point mass at \( a_k \), we know that the measure \( \sum_k (1 - |a_k|) \delta_k \) is Carleson (see [4, Chapter VII]), and so

\[
\sum_k |F(a_k)| (1 - |a_k|) < \infty \tag{11}
\]

for every \( F \in H^1 \).

Now suppose that we can solve the interpolation problem (4) with a function \( f \in K_B^1 \). An application of (11) with \( F = f \) then yields (9). Furthermore, putting \( g = \tilde{f}(:= r_f B) \) a.e. on \( \mathbb{T} \), we know that \( g \in K_B^1(\subset H^1) \); in particular, \( g \) extends to \( \mathbb{D} \) by the Cauchy integral formula. Therefore,

\[
\overline{g(a_k)} = \int_{\mathbb{T}} \frac{g(\zeta)}{1 - \zeta a_k} \, dm(\zeta)
= \int_{\mathbb{T}} \frac{\overline{\zeta f(\zeta)B(\zeta)}}{1 - \zeta a_k} \, dm(\zeta)
\]

\[
= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{B(\zeta) \cdot (1 - \zeta a_k)} \, d\zeta, \tag{12}
\]

and computing the last integral by residues we find that

\[
\overline{g(a_k)} = \sum_j \frac{w_j}{B'(a_j) \cdot (1 - a_j a_k)} = \tilde{w}_k, \tag{13}
\]

for each \( k \in \mathbb{N} \). Finally, we apply (11) with \( F = g \) and combine this with (13) to arrive at (10).

Thus, conditions (9) and (10) are indeed necessary in order that \( \{w_k\} \) be in \( K_B^1|_{\{a_k\}} \). It is the sufficiency of the two conditions that presents an open problem.

In contrast to the case \( 1 < p < \infty \), where (6) remains valid upon replacing \( \{w_k\} \) by \( \{\tilde{w}_k\} \), no such thing is true for \( p = 1 \). In other words, (9) no longer implies (10), so the two conditions together are actually stronger than (9) alone. Consider, as an example, the “radial zeros” situation where the interpolating sequence \( \{a_k\} \) satisfies

\[
0 \leq a_1 < a_2 < \cdots < 1 \tag{14}
\]

(in which case the quantities \( 1 - a_k \) must decrease exponentially, see [4, Chapter VII]). Further, let \( \{\gamma_k\} \) be a sequence of nonnegative numbers such that \( \sum_k \gamma_k < \infty \) but \( \sum_k k \gamma_k = \infty \), and let \( w_k = B'(a_k) \cdot \gamma_k \). Then

\[
(1 - |a_k| \cdot |w_k| = (1 - |a_k|) \cdot |B'(a_k)| \cdot \gamma_k \leq \gamma_k, \quad k \in \mathbb{N}
\]

(since \( B \) is a unit-norm \( H^\infty \) function), and (9) follows. On the other hand,

\[
\tilde{w}_k = \sum_j \frac{\gamma_j}{1 - a_j a_k} \geq \sum_{j=k}^\infty \frac{\gamma_j}{1 - a_j a_k} \geq \frac{1}{2(1 - a_k)} \sum_{j=k}^\infty \gamma_j,
\]
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because for $j \geq k$ we have $a_j \geq a_k$, and hence

$$1 - a_j a_k \leq 1 - a_k^2 \leq 2(1 - a_k).$$

Consequently,

$$\sum_k |\tilde{w}_k| (1 - |a_k|) = \sum_k \tilde{w}_k (1 - a_k) \geq \frac{1}{2} \sum_k \sum_{j=k}^\infty \gamma_j = \frac{1}{2} \sum_j j \gamma_j = \infty,$$

which means that (10) breaks down.

Our next problem involves the notion of an ideal sequence space, which we now recall. Suppose $S$ is a vector space consisting of sequences of complex numbers. We say that $S$ is ideal if, whenever $\{u_j\} \in S$ and $\{v_j\}(\subset \mathbb{C})$ is a sequence satisfying $|v_j| \leq |u_j|$ for all $j$, it follows that $\{v_j\} \in S$. Roughly speaking, this property means that the sequences belonging to $S$ admit a nice description in terms of a certain “size condition” on their components.

When $1 < p < \infty$, identity (7) tells us that the trace space $K_B^p|_{\{a_k\}}$ coincides with $l^p(\{a_k\})$ and is therefore ideal. However, assuming that Conjecture 1 is true and arguing by analogy with the $p = \infty$ case, as studied in \cite{3}, we strongly believe that $K_B^1|_{\{a_k\}}$ is no longer ideal in general. We would even go on to conjecture that this last trace space is never ideal, as soon as $B = B_{\{a_k\}}$ is an infinite Blaschke product. The following problem arises then naturally.

**Problem 2.** Given an interpolating Blaschke product $B$ with zeros $\{a_k\}$, determine the maximal (largest possible) ideal sequence space contained in $K_B^1|_{\{a_k\}}$.

In particular, the calculations above pertaining to the special case (14) show that, in this “radial zeros” situation, the maximal ideal subspace of $K_B^1|_{\{a_k\}}$ must be

$$\mathcal{M} := \left\{w_k : \sum_k k |w_k| (1 - a_k) < \infty \right\},$$

provided that the validity of Conjecture 1 is established. Strictly speaking, by saying that $\mathcal{M}$ is maximal we mean that $\mathcal{M} \subset K_B^1|_{\{a_k\}}$, while every ideal sequence space $S$ with $S \subset K_B^1|_{\{a_k\}}$ is actually contained in $\mathcal{M}$.

Our last problem (posed in a somewhat vaguer form) deals with the map, to be denoted by $T_{\{a_k\}}$, that takes $\{w_k\}$ to $\{\tilde{w}_k\}$; here $\tilde{w}_k$ is again defined by (8), with $B = B_{\{a_k\}}$.

**Problem 3.** Given an interpolating Blaschke product $B$ with zeros $\{a_k\}$, study the action of the linear operator

$$T_{\{a_k\}} : \{w_k\} \mapsto \{\tilde{w}_k\}$$

on various (natural) sequence spaces.
We know from (13) that, whenever the $w_k$'s satisfy (4) for some $f \in K_B^1$, the conjugates of the $\tilde{w}_k$'s play a similar role for the “partner” $\tilde{f} := \overline{\sum f B} \in K_B^1$. Recalling (7), we deduce that $T_{\{a_k\}}$ maps $\ell^p_1(\{a_k\})$ with $1 < p < \infty$ isomorphically onto itself. Its inverse, $T_{\{a_k\}}^{-1}$, is then given by

$$T_{\{a_k\}}^{-1} : \{\tilde{w}_k\} \mapsto \left\{ \sum_j \frac{\tilde{w}_j}{B'(a_j) \cdot (1 - \overline{a_j a_k})} \right\} = \{w_k\},$$

which follows upon interchanging the roles of $f$ and $g(= \tilde{f})$ in (12) and (13). It might be interesting to determine further sequence spaces that are preserved by $T_{\{a_k\}}$. Anyhow, this $T_{\{a_k\}}$-invariance property can no longer be guaranteed for the endpoint spaces $\ell^1_1(\{a_k\})$ and $\ell^\infty$. In fact, we have seen in [3] that the set of those $\ell^\infty$ sequences that are mapped by $T_{\{a_k\}}$ into $\ell^\infty$ coincides with the trace space $K_B^{\infty}_{\{a_k\}}$ (which is, typically, smaller than $\ell^\infty$); and Conjecture 1 says that at the other extreme, when $p = 1$, the situation should be similar.

Furthermore, the suggested analysis of the $T_{\{a_k\}}$ operator could perhaps be extended, at least in part, to the case of a generic (non-interpolating) Blaschke sequence $\{a_k\}$.

Finally, going back to Problem 1, we remark that the range $0 < p < 1$ is also worth looking at, in addition to the $p = 1$ case. It seems reasonable, though, to define the corresponding $K^p_B$ space as the closure of $K^{\infty}_B$ in $H^p$, rather than use the original definition (1) for $p < 1$. The problem would then consist in describing the trace space $K^p_B|_{\{a_k\}}$ for an interpolating Blaschke product $B = B_{\{a_k\}}$. More generally, one might wish to characterize $K^p_\theta|_{\{\lambda_k\}}$, say with $0 < p \leq 1$ and $\theta$ inner, for an arbitrary interpolating sequence $\{\lambda_k\}$ in $D$. This time, however, the chances for a neat solution appear to be slim. Indeed, even the “good” case $1 < p < \infty$ is far from being completely understood; see, e.g., [1, 5] for a discussion of such interpolation problems in $K^2_\theta$.

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