In recent years, realistic unquenched QCD simulations have been carried out with various lattice actions. In this report, I explain the progress in theory and algorithms and some of the physics results.

1. INTRODUCTION

The hadronic states in QCD have various energy scales: the smallest scales are the light pseudoscalar meson masses $m_\pi$ and $m_K$, and the medium energy scales other light hadronic masses of order $\Lambda_{\text{QCD}}$, and the larger energy scales are heavy hadron masses $m_D$ and $m_B$. These mass scales range from 100 MeV to 5 GeV, giving a hierarchy of order 50. If one wants to fully describe such a system with completely controlled systematic errors, one should ideally carry out lattice simulations 1) with sufficiently large volume $V = L^4$ and small pseudoscalar meson masses $m_{\text{PS}}$ to cover the chiral dynamics of the $\pi$ meson and 2) with sufficiently large heavy quark mass $m_Q$ and large cutoff scale $a^{-1}$ in order to cover the heavy meson mass scale. However, in reality, due to the limitation of the present computer power our lattice simulation can marginally cover the energy scales as

$$L^{-1} < m_\pi < m_{\text{PS}} < m_K < \Lambda_{\text{QCD}} < m_c < m_Q < a^{-1} < m_b. \quad (1)$$

Even if one cannot totally cover the energy scale, one can extrapolate the simulation result to the physical result with the help of the chiral perturbation theory and the heavy meson effective theory. Therefore, the real practical problem is to achieve lattice computations with larger volume, smaller light quark mass, and finer lattice as much as possible so that the kinematic parameters lie within the range where the effective theory descriptions are valid.

In addition, in order to obtain the hadronic matrix element of the operator $O$ with high precision, we need to determine the renormalization factor $Z_O$ very accurately since the matrix element in the continuum theory can be related to that on the lattice as

$$\langle O_{\text{cont}}(\mu) \rangle = \lim_{\alpha \to 0} Z(a\mu, g_0(a)^2)\langle O_{\text{lat}}(a) \rangle. \quad (2)$$

In recent years, there have been several new developments both in theoretical formulations and algorithms. With these developments, recent lattice simulations can cover much wider energy scale so that the extrapolation of the quark masses to their physical values can be carried out reliably with the help of the low energy effective theories.

1.1. Lattice fermion actions

Let us now comment on various lattice fermion actions. The staggered fermion or improved staggered fermions (AsqTad, HISQ) have been extensively used for large scale unquenched simulations. The biggest advantages are the small simulation cost due to the small degrees of freedom and the exact partial chiral symmetry at finite lattice spacing. Since it has extra 'taste' degrees of freedom, square or quartic rooted trick are needed. The Wilson fermion or O(a)-improved Wilson fermion (=Clover fermion) are now widely used in recent unquenched simulations. The biggest advantage is that this action is theoretically most simple and the computational cost is reasonable. Although the chiral symmetry is violated at finite lattice spacing, it can be recovered in the continuum limit. The Domain-Wall fermion is a Wilson fermion in 5 dimensions with a mass term which has kinks in the 5-th dimensions. The chiral symmetry violations are exponentially suppressed as a function of the lattice extent in the 5-th dimension $N_5$. It is $N_5$ times more costly than the conventional Wilson fermion. The overlap fermion has an exact chiral symmetry on the lattice. This action is very costly but still feasible by the new method to fix topology during the hybrid
Monte Carlo simulation. The twisted mass Wilson (tmWilson) is the Wilson fermion with chirally rotated mass. It has a partial exact chiral symmetry but partial vector symmetry is violated, instead. At maximal twist, this action is automatically O(a)-improved.

### 1.2. New algorithms in unquenched simulations

It is known that the simulation cost grows towards smaller quark mass, larger volume, and finer lattice spacing. In addition to the obvious effect from simply having more lattice points, there are effects from more iterations in Dirac matrix inversion, smaller step size in keeping the acceptance rate constant, and longer Monte Carlo trajectories to compensate longer autocorrelations. For example, the empirical formula \[ \text{cost} \approx C \left( \frac{\#\text{conf}}{1000} \right) \left( \frac{0.6}{m_{\pi}/m_\rho} \right)^6 \left( \frac{L}{3 \text{ fm}} \right)^5 \left( \frac{0.1 \text{ fm}}{a} \right)^7 \] with \( C \approx 2.8 \). From this formula, one can estimate that a typical simulation of 1000 configurations with \( a = 0.1 \text{ fm} \), \( L = 3 \text{ fm} \) would cost 25 Tflops years for \( m_{\pi} = 300 \text{ MeV} \) and more than 600 Tflops years for physical pion mass. For this reason, it has been thought that the staggered fermion is the only feasible choice for unquenched QCD simulations for \( m_{\pi} \approx 300 \text{ MeV} \) with Tflops machine.

One of the most striking developments in lattice QCD is the proposal of new algorithms to reduce the simulation cost of unquenched QCD simulation. The new algorithms are the Hybrid Monte Carlo based on the combination of the 'preconditioning' (mass preconditioning \[ \text{[3, 4]} \] or domain decomposition \[ \text{[7, 8]} \]) to reduce the cost of the Dirac operator inversion and 'multi-time scale' in Molecular dynamics to reduce the frequency of the Dirac operator inversion in time-step. It was found that these new algorithms can give rise to significant cost reductions of factor 20-30. Therefore, the study of O(a)-improved Wilson fermions has also become quite feasible with O(1-10) Tflops machines. The same algorithm also enables large scale simulations of Domain-wall or overlap fermions, although one must restrict oneself to work on a single relatively coarse lattice spacing with the present computer.

### 1.3. Nonperturbative renormalization

Another important theoretical methods is the nonperturbative renormalization. There are several different methods. The first method is a simple method using the PCAC relation \( \Delta_\mu A_\mu = 2m_q P \). If the axial current is exactly conserved on the lattice (ex. staggered, tmWilson, Overlap), one can use PCAC relation to obtain the decay constant of the pseudoscalar meson with nonperturbative accuracy as \( f_{PS} = \lim_{a \rightarrow 0} \left( \frac{0.2 m_q P(0)}{m_{PS}^2} \right) \). The second method is the nonperturbative renormalization by the Regularization Independent Momentum (RI-MOM) scheme proposed by...
Figure 1: Low energy constants $\bar{l}_3, \bar{l}_4$ from NLO SU(2) ChPT fits of the lattice data from various groups. For comparison, phenomenological estimates by Gasser and Leutwyler are also plotted.

Table II: SU(3) breaking in the pseudoscalar decay constant.

| Group          | $n_f$ | Action        | $f_K/f_\pi$ input          |
|----------------|-------|---------------|-----------------------------|
| MILC 2+1      |       | Asqtad        | $m_{\pi,K}$ 1.202(3)$^{+6}_{-13}$ |
| HPQCD/MILC 2+1|       | Asqtad+HISQ   | $m_{\pi,K}$ 1.189(7)        |
| PACS-CS 2+1   |       | Clover        | $m_{\pi,K}$ 1.19(2)         |
| BMW 2+1       |       | Clover        | $m_{\pi,K}$ 1.19(1)(1)      |
| RBC/UKQCD 2+1 |       | Domain-wall   | $m_{\pi,K}$ 1.22(2)(6)      |
| JLQCD 2+1     |       | Overlap       | $f_\pi, m_{\pi,K}$ 1.20(3) |
| ETMC 2        |       | tmWilson      | $f_\pi, m_{\pi,K}$ 1.196(13)(7)(8) |

Martinelli et al. [19]. This is a scheme defined by the quark-gluon amplitudes in Landau gauge with large off-shell momentum in Euclidean region. Schrodinger functional (SF) scheme proposed by the Alpha collaboration [20, 21, 22]. This scheme is defined the amplitudes in a box with boundaries with physical box size $L^4$ under the external gauge potential introduced by the boundary values of the gauge field.

2. LIGHT FLAVOR PHYSICS

2.1. Quark mass dependence of the pseudoscalar masses and decay constants

Next-to-Leading Order (NLO) SU(2) Chiral Perturbation Theory (ChPT) predicts the quark mass dependences of the pseudoscalar masses and decay constants as

$$m_{PS}^2 = 2m_q B \left[ 1 + \frac{2m_q B}{16\pi^2 f_\pi^2} \left( \bar{l}_3 + \ln \left( \frac{2m_q B}{m_\pi} \right) \right) \right], \quad f_{PS} = f \left[ 1 + \frac{2m_q B}{8\pi^2 f_\pi^2} \left( \bar{l}_4 - \ln \left( \frac{2m_q B}{m_\pi} \right) \right) \right],$$

where $m_\pi = 139.6$ MeV. The ratio of the decay constant is also obtained by various groups. Table III lists the ratio $f_K/f_\pi$ in 2 and 2+1 flavor QCD from various groups. A remarkable observation was made by the 2+1 flavor simulation PACS-CS collaboration [11], where the lightest quark mass is as small as the physical up and down quark masses. They find that the non-analytic behavior as functions of $m_{ud}$ arising from the chiral log can be clearly reproduced for $m_{ud}^2/m_\pi$ and $f_K/f_\pi$. RBC/UKQCD collaboration [17] found that the NLO partially quenched chiral perturbation theory (PQChPT) with the expansion parameter $x = m_\pi^2/(4\pi f_\pi)^2$ ($x$-expansion) fits the data only for $m_\pi < 450$ MeV. While JLQCD collaboration finds similar results for ChPT, they also find that the NLO ChPT
with expansion parameter $\xi = \frac{m^2_{\pi}}{4\pi f_{\pi}^2}$ ($\xi$-expansion) makes the convergence better for heavier quark masses and NNLO ChPT with $\xi$-expansion can nicely describe the lattice data up to the strange quark mass regime as shown in Fig. 2 [18].

2.2. $B_K$

Indirect CP violation in the K meson system $\epsilon_K$ is one of the most crucial quantities to test the standard model and the physics beyond. The experimental value is determined with high accuracy as

$$|\epsilon_K| = (2.233 \pm 0.015) \times 10^{-3}.$$ (5)

Theoretically this quantity is described as $|\epsilon_K| = f(\rho, \eta) \times C(\mu) \times B_K(\mu)$. Here, $f(\rho, \eta)$ is a factor which depends on the CKM matrix elements, $C(\mu)$ is the Wilson coefficient from short-distance QCD corrections and $B_K(\mu)$ is the bag parameter defined as

$$B_K(\mu) = \frac{\langle K^0 | [\bar{d} \gamma^\mu (1 - \gamma_5)s \bar{d} \gamma^\mu (1 - \gamma_5)s] (\mu) | K^0 \rangle}{4f_K m_K^2}.$$ (6)

The main problem in unquenched lattice calculations is the mixing of operators with wrong chiralities or tastes. For Wilson-type fermions there exists mixing with operators of the same dimension having wrong chirality. Since they cannot be extrapolated away in the continuum limit, the only way to remove the contamination is to determine the counter term nonperturbatively. However, due to the chiral enhancement of the operators with wrong chirality, it is quite difficult to control the systematic error. The operator mixing for staggered fermion is also quite complicated and only perturbative renormalization exists. The overlap fermion is free from operator mixing owing to the exact chiral symmetry, while the operator mixing for the domain-wall fermion is exponentially suppressed and practically under control. In fact, for both cases studies with nonperturbative renormalization in RI-MOM scheme, one cannot see visible effect of the the operator mixing. For tmWilson, due to the parity odd nature and the cu symmetry the operator mixing can be prohibited for 2-flavor QCD. Therefore, tmWilson is another promising approach to the precise determination of $B_K$. However, for 2+1 flavors it is difficult to full realize the O(a)-improvent, no mixing and non-degenerate ($m_s, m_d$) quark simultaneously.

HPQCD collaboration studied $B_K$ on 2+1 flavor MILC configuration with Aqstad (staggered) sea quark with lattice spacing $a = 0.125$ fm [23]. They employ two different improved staggered actions (Asqtad and HYP) for the valence quark. The calculation was carried out by degenerate valence quarks for the K meson $(m_1, m_2) = m_s/2$ while
the sea quark is chosen to be 0.2m_{sea}, 0.4m_s. Chiral extrapolation is performed by linear fit. The renormalization of ΔS = 2 operator is obtained by perturbation at 1-loop. They find \( \hat{B} = 0.83(18) \), where the dominant error comes from the unknown higher order perturbative correction of the renormalization factor of \( O(\alpha^2) \simeq 20\% \). RBC/UKQCD collaborations computed \( B_K \) with domain-wall fermion in 2+1 flavor QCD at lattice spacing \( a = 0.11 \text{ fm} \) \cite{17}. They have 3 points for the sea quark and 7 combinations of valence quark masses \( (m_1, m_2) \). They fit the data with next-to-leading order partially quenched chiral perturbation theory (NLO PQChPT). The renormalization factor is determined nonperturbatively by RI-MOM scheme. They obtain \( \hat{B} = 0.720(13)_{\text{stat.}}(17)_{\text{sys.}} \), where the dominant error comes from the discretization error of \( O(a^2) \simeq 4\% \). JLQCD collaborations study \( B_K \) with overlap fermion in 2 flavor QCD at lattice spacing \( a = 0.12 \text{ fm} \) on physical volume with \( L = 2 \text{ fm} \) \cite{24}. They have 4 points for the sea quark and 10 combinations of valence quark masses \( (m_1, m_2) \). They fit the data with NLO PQChPT. The renormalization factor is determined nonperturbatively by RI-MOM scheme. They obtain \( \hat{B} = 0.734(5)_{\text{stat.}}(50)_{\text{sys.}} \), where the dominant error comes from the finite size effect of order 5%.

It should be noted that the long standing operator mixing problem is solved with the advent of Ginsparg-Wilson fermions (domain-wall, overlap) with (almost) exact chiral symmetry. Thus the above studies are the real start of the precision study of \( B_K \) for which significant progress will be expected near future. RBC/UKQCD collaboration are planning to work at finer lattice spacing \( a = 0.09 \text{ fm} \) to reduce the discretization error. JLQCD collaborations are studying 2+1 flavor QCD for physical volumes of 2 fm and 3 fm. ETMC and other groups are studying on mixed action approach where they employ the Ginsparg-Wilson valence quarks on gauge configuration with tmWilson or staggered dynamical quarks. This can also be an economical and promising approach, with the help of mixed action PQChPT so that one can even take the continuum limit with the existing 2+1 flavor gauge configurations.

More precise determination of \( B_K \) are important in view of the recent tension in \( \sin(2\beta_1) \) versus the determination of unitarity triangle by \( \epsilon_K \) and \( \Delta m_{B_s}/\Delta m_{B_d} \) as pointed out by Lunghi and Soni \cite{25} or Buras and Guadagnoli \cite{26}.

### 2.3. Nucleon sigma term

Nucleon sigma term is the finite quark mass effect of the nucleon mass and is defined by the following matrix element

\[
\sigma_{\pi N} = m_{ud} \langle N | \bar{u}u + \bar{d}d | N \rangle,
\]

which contains both the valence and the sea quark contributions. The sea quark contribution also arise from the strange quark and it is often parameterized by the ratio \( y \) defined as

\[
y = \frac{2 \langle N | \bar{s}s | N \rangle}{\langle N | \bar{u}u + \bar{d}d | N \rangle}.
\]

The sigma term and the strange quark content can be determined using the quark mass dependence of the nucleon mass. The Feynman-Hellman theorem tells that the quark mass dependence can be related to the sigma term as

\[
\frac{\partial m_N}{\partial m_{val}} = \langle N | \bar{u}u + \bar{d}d | N \rangle_{\text{conn}}, \quad \frac{\partial m_N}{\partial m_{sea}} = \langle N | \bar{u}u + \bar{d}d | N \rangle_{\text{disc}},
\]

where “conn” and “disc” denotes contributions to the sigma terms. The disconnected contribution is nothing but 2\( \langle N | \bar{s}s | N \rangle \) when the sea quark mass is equal to strange quark mass.

JLQCD collaboration studied the quark mass dependence of the nucleon mass in 2-flavor QCD simulation with dynamical overlap fermion \cite{27}. Fitting the nucleon mass for the combination of 6 sea quark masses and 9 valence quark masses by partially quenched baryon chiral perturbation theory, They found that the by fitting the data with BChPT in finite volume finite size effect for the baryon mass in (2 fm)^4 box is significant. After correcting the finite size effect, they obtain

\[
\sigma_{\pi N} = 52(2)(^{+20}_{-2})(^{+5}_{-0}) \text{ MeV}.
\]
Fitting the nucleon mass data with PQChPT and differentiating the nucleon mass with valence and sea quark masses. They found that the disconnected contribution is much smaller than connected one. As a result the semi-quenched estimate of the $y$-parameter is

$$y = 0.030(16)(^{+6}_{-8}^{+1}_{-2}).$$  \hfill (11)

Previous lattice results $y$-parameters using Wilson-type fermions are 0.66(15), 0.36(3) for quenched QCD and 0.59(13) for 2-flavor QCD, which are significantly larger than the above result. It was pointed out that the the additive quark mass shift with Wilson-type fermion can give a significant contamination to the disconnected contribution. After subtracting this artifact, UKQCD obtained $y = -0.28(33)$ in 2-flavor QCD. Since they subtract the power-divergent contribution numerically, the result suffer from large statistical error. In contrast, the study with overlap fermion by JLQCD do not suffer such artifact owing to the exact chiral symmetry.

The exact chiral symmetry on the lattice enables novel studies of chiral dynamics. It can also open up new directions which has not been possible in other approaches. Examples of such studies are given by JLQCD collaboration [28, 29, 30, 31, 32, 33, 34, 35].

### 3. CHARM QUARK PHYSICS

#### 3.1. $D$ meson decay constants

Since the charm quark mass is of order 1 GeV, the discretization errors are larger than those for up, down, and strange quark if one take the same approach used for the light quark. In order to make a reliable extrapolation should make the lattice one has to employ either finer lattices or improved action.

The first choice is made by the Alpha collaboration. They carry out a quenched study the $D_s$ meson decay constant on lattices with lattice spacings $a = 0.04, 0.06, 0.08, 0.1$ fm, which correspond to the discretization errors of $(a m_c)^2 \simeq 0.04, 0.09, 0.16, 0.25$ [36]. Using the $O(a)$-improved Wilson fermion and nonperturbative renormalization in SF-scheme, they obtain

$$f_{D_s}^{n_f=0} = 252(2) \text{ MeV}. \hfill (12)$$

They same strategy is in progress for $n_f = 2$ QCD by the Coordinated Lattice Simulation(CLS)/Alpha collaboration. This is a quite promising approach but requires dedicated effort.

Improved fermion actions are also effective in reducing the discretization error. One of the popular approaches is the fermilab formalism and other related formalism in which one make a mass-dependent re-interpretation of $O(a)$-improved Wilson fermion. While it is possible to achieve non-perturbative improvement, mass-dependent errors of
the leading operators are removed at tree and 1-loop in practice. The dominant short distance error $\Delta$ for the charm quark system is

$$\Delta = c_0 a_s^2 + c_1 a_s (a_{\Lambda_{QCD}}) (am_c) + c_2 (a_{\Lambda_{QCD}})^2 + \cdots$$

(13)

Highly Improved Staggered Quark (HISQ) is another improved action [37]. Since it is based on the standard Symanzik improvement which treats the discretization errors in powers of $(ap)^2$ and $(am_q)^2$, standard nonperturbative renormalization technique also applies. The advantage of this approach is that the discretization errors are restricted due to the exact chiral symmetry of the staggered fermion. In general, after carrying out nonperturbative renormalization the dominant short distance error $\Delta$ for the charm quark system can be expanded in powers of $(am_c)^2$. In HISQ action, they remove the leading $O(a^2)$ error and the taste changing interaction at tree level so that the dominant short distance then becomes

$$\Delta = c_1 (am_c)^4 + c_2 a_s (am_c)^2 + \cdots$$

(14)

In principle, HISQ has nonperturbative accuracy in the continuum limit. Whether typical lattice spacings $a \simeq 0.10$ fm are small enough to have a good scaling for charm quark system as expected from naive order counting should be explicitly examined.

HPQCD collaboration studied charm quark system on MILC lattices with 2+1 dynamical Asqtad quark using HISQ action for the valence quark with the lattice spacings $a = 0.09, 0.12, 0.15$ fm [38]. The chiral and continuum extrapolation for the pseudoscalar and masses and decay constants is carried out by the global fit of the data using the ChPT and heavy meson effective theory formula including $O(\alpha_s a^2)$ and $O(a^4)$ errors. The scale is set by the upsilon spectrum and $m_{ud}, m_s$ and $m_c$ are determined from $m_\pi, m_K$ and $m_{\eta_c}$. Since there are no more free parameter, while $f_{D_s}$ and $f_{D_s^*}$ are pure predictions of theory, it should be noted that $f_\pi, f_K, m_{D_s}, m_{D_s^*}$, and $m_{D_s}$ can also give non-trivial consistency check. It turns out mixed action approach with HISQ valence and Asqtad sea quark can control the discretization error in the charm quark system nicely. They showed that the $D$ and $D_s$ meson masses are reproduced in the continuum limit. The pseudoscalar decay constant are computed with nonperturbative renormalization by AWTI, the decay constants $f_\pi$ and $f_K$ agree with experiment very well. Using the same technique they obtain $f_{D_s}$ and $f_{D_s^*}$ with 1-2% error which is dominated by the continuum and chiral extrapolation.

$$f_{D_s} = 207(4) \text{ MeV}, \quad f_{D_s^*} = 241(3) \text{ MeV}, \quad f_{D_s}/f_{D_s^*} = 1.164(11).$$

(15)

Fermilab/MILC collaboration also studied the D meson decay constants on the same 2+1 flavor MILC configurations [39, 49]. They used AsqTad action for the light valence quark and fermilab action for the charm quark. The data are fitted by the PQstaggered ChPT which include full NLO and analytic part of the NNLO contributions and $O(a^2)$ errors. The heavy-light axial vector current is renormalized by partially nonperturbative renormalization in which they use the renormalization factors of the heavy-heavy and light-light vector currents $Z_{V_A}^{QQ}, Z_{V_A}^{qq}$ and evaluate the remaining perturbative correction $\rho_{A_V}^{QQ}$

$$Z_{A_V}^{QQ} = \rho_{A_V}^{QQ} \sqrt{Z_{V_A}^{QQ} Z_{V_A}^{qq}}$$

(16)

They expect that vector current renormalization factors capture the dominant contribution from the wavefunction renormalization. Their preliminary result is

$$f_{D_s} = 207(11) \text{ MeV}, \quad f_{D_s^*} = 249(11) \text{ MeV}, \quad f_{D_s}/f_{D_s^*} = 1.20(3).$$

(17)

ETMC studied the D meson decay constants for $n_f = 2$ QCD using dynamical tmWilson with lattice spacing $a = 0.67, 0.86, 0.10$ fm [40]. Since they work at maximal twist which guarantees automatic $O(a)$ improvement, the leading discretization error is $O(a^2)$. They take degenerate ud quark masses for the sea and valence quark in the range of $m_{ud} = (0.2 - 0.4)m_s$. The data are fitted with SU(2) heavy meson ChPT with $O(a^2)$ term and nonperturbative renormalization by AWTI is used for the heavy-light axial vector current. They obtain the preliminary result as

$$f_{D_s} = 197(16) \text{ MeV}, \quad f_{D_s^*} = 244(12) \text{ MeV}, \quad f_{D_s}/f_{D_s^*} = 1.24(5).$$

(18)
Fig. 4 shows the comparison of the $D$ meson decay constants. It can be seen that $f_{D_d}$ from HPQCD is in agreement with experiment, while $f_{D_s}$ has $3 \sigma$ deviation from CLEO result. Two recent preliminary results from Fermilab/MILC and ETMC give consistent result. It was pointed out by Becirevic et al. [41] that possible experiment error by misidentifying $D \rightarrow l\nu\gamma$ decay could only give an effect smaller than a few percent. As was pointed out by the lattice talk by Fermilab group, the $3 \sigma$ deviation is dictated by the error from CLEO even if we enlarge the error from HPQCD by factor 3, the deviation still $2.8 \sigma$. Therefore the $f_D$ puzzle still remains. Further theoretical and experimental studies are required. Combining HPQCD($n_f = 2 + 1$), FNAL/MILC($n_f = 2 + 1$), and ETMC($n_f = 2$)

![Figure 4: $D$ meson decay constants](image)

results where I include additional 5% error for $n_f = 2$ results my world average is

$$f_{D_d} = 206(4) \text{ MeV}, \quad f_{D_s} = 243(3) \text{ MeV},$$  

(19)

### 3.2. Charm quark mass

Another important progress is the precise determination of the charm quark mass. HPQCD and Chetyrkin, Kuhn, Steinhauser and Sturm [42, 43] determined $m_c$ and $\alpha_s$ from the moments the pseudoscalar correlators

$$G_n \equiv \sum_t \left( \frac{t}{a} \right)^n G(t), \quad G(t) \equiv a^6 \sum_x (am_c^0)^2 \langle 0 | j_5(x,t) j_5(0,0) | 0 \rangle, \quad j_5 \equiv \bar{c} \gamma_5 \psi.$$  

(20)

They match the continuum limit of the lattice calculation of $G_n$ with the continuum perturbation theory of $G_n$ at 4-loop. The lattice calculation was performed with HISQ valence charm based MILC lattice for $n_f = 2 + 1$ QCD at lattice spacings $a = 0.09, 0.12, 0.15$ fm.

In order to reduce the discretization error, modified ratios $R_n$ are defined

$$R_4 \equiv G_4/G_4(0), \quad R_n \equiv \frac{am_c}{2am_c^0} \left( \frac{G_n}{G_n(0)} \right)^{1/(n-4)} \text{ for } n \geq 6,$$  

(21)

where $G_n(0)$ are the n-th moments on the lattice at the tree level. The lattice spacing dependence of the moments can be fitted nicely with quadratic functions in $a^2$. Averaging for $n = 6, 8, 10$ one can determine the charm quark mass and the strong coupling as

$$m_c^{MS}(3\text{GeV}) = 0.986(10), \quad \alpha^{MS}(M_Z, n_f = 5) = 0.1174(12).$$  

(22)

The key ingredients are the $O(a^2)$ improvement and the exact chiral symmetry, which can control the discretization error and also allow nonperturbative renormalization by AWTI.
### 3.3. B PHYSICS

Even with the finest lattice spacing for the present unquenched lattice calculation $a = 0.04$ fm, the bottom quark mass in lattice unit is in the range of $am_b \approx 1$. Therefore, HISQ action completely loses control over the discretization error. Therefore, the only practically available methods to study B physics are either to use the effective action such as NRQCD or fermilab action or to make a very precise computation in both the static limit and in the charm quark mass regime and make an interpolation. While the former approach are taken by HPQCD and FNAL/MILC collaborations and has already produced many results in the past few years, there has also been remarkable progress in the latter approach, where serious feasibility tests of the theoretical method have been carried out in quenched QCD.

Alpha collaboration worked on the lattice HQET with nonperturbative accuracy. Their idea is to make a nonperturbative matching of the HQET and QCD at very fine lattices in small volume and then evolve the lattice HQET to coarse and larger lattice by step scaling. They could also work on $1/m_b$ correction in HQET.

Rome II group used step scaling using $O(a)$-improved Wilson fermion with nonperturbative renormalization. They make nonperturbative calculations of B meson directly at bottom quark mass in a small volume $L = 0.4$ fm at very fine lattice spacing. Then evaluate the finite volume correction with coarser lattice by extrapolations from smaller quark mass regime.

Guazzini, Sommer and Tantalo combined the above two methods and used the static results to evaluate the finite volume correction by interpolation.

The quenched QCD results are summarized in Table III. It is remarkable that one can control the systematic error from the bottom quark and achieve 2-3% accuracy in the matrix element. $n_f = 2$ unquenched QCD studies are in progress by Alpha collaboration.

FNAL/MILC collaboration recently update their $n_f = 2 + 1$ calculation of $f_{B_s}$ and $f_{B_d}$. Fig. 5 shows the $B_s$ meson decay constants in quenched QCD and $n_f = 2, 2 + 1$ unquenched QCD simulations. Using the input such as $m_\rho$ or $r_0 = 0.5$ fm for $n_f = 0, 2$, one finds that $f_{B_s}$ increase from quenched QCD to $n_f = 2 + 1$ in contrast to $f_{D_s}$. The SU(3) breaking of the $f_B$ in 2+1 flavor QCD is obtained by HPQCD and FNAL/MILC collaboration also reported their result at Lattice 2008. Since FNAL/MILC results of $f_B$ are still preliminary, I just quote their numbers and do not take the world average.

$$f_{B_s}/f_{B_d} = 1.20(3) \text{ HPQCD, } f_{B_s}/f_{B_d} = 1.25(4) \text{ HPQCD.}$$ (23)

### 3.4. $B \to D^* l \nu$

FNAL/MILC collaboration recently made an update of $B \to D^* l \nu$ form factor determination based on 2+1 MILC lattices at lattice spacings $a = 0.09, 0.12, 0.15$ fm. They employ the Asqtad action for the light sea and valence quarks and the fermilab action for the heavy quark. They developed a new double ratio method to determined the $B \to D^* l \nu$ form factor as

$$|F(\infty)|^2 = \frac{\langle D^* | \bar{c} \gamma_j \gamma_5 b | \bar{B} \rangle \langle \bar{B} | \bar{b} \gamma_j \gamma_5 c | D^* \rangle}{\langle D^* | \bar{c} \gamma_4 c | D^* \rangle \langle \bar{B} | \bar{b} \gamma_4 b | \bar{B} \rangle}.$$ (24)
They carry out the chiral extrapolation and continuum extrapolation using the NLO ChPT and analytic NNLO staggered ChPT formula. They obtain

\[ F_{n_f=2+1}(1) = 0.921(11)(19) \]  

as a preliminary result.

Rome II group studied \( B \to D^* l \nu \) form factor in quenched QCD using finite size step scaling method and combined with the HQET calculation \[51, 52\]. They used twisted boundary condition to get small continuous momentum recoil. As a result they could obtain the slope of the \( w \) dependence. They obtain

\[ F_{n_f=0}(1) = 0.917(8)(5) \]  

This proves that the finite size step scaling method is very promising also for the form factor calculation. Applications to unquenched QCD simulation are awaited.

### 3.5. \( B \to \pi l \nu \)

FNAL/MILC collaboration made a new analysis of \( B \to \pi l \nu \) form factor using MILC configuration of 2+1 flavor QCD at \( a = 0.09, 0.12, 0.15 \) fm \[53\]. Parameterizing the the form factors as

\[ f_{\perp}(E_\pi) \equiv \langle \pi | V_\| | B \rangle / \sqrt{2m_B}, \quad f_{\|}(E_\pi) \equiv \langle \pi | V_0 | B \rangle / \sqrt{2m_B} \]

and using NLO staggered ChPT

\[ f_{\perp,\|} = c_0 \left[ 1 + \text{chiral logs} + m_l + c_2(2m_u + m_s) + c_3 E_\pi + c_4 E_\pi^2 + c_5 a^2 \right] + O(m_q^2, E_\pi^3), \]

they carried out the continuum and chiral extrapolation. They then used the combined \( z \) fit of Babar and lattice data based on the dispersion relation formula and extracted \( |V_{ub}| \) as

\[ |V_{ub}| = 2.94(35) \times 10^{-3}. \]

### 3.6. New applications

The nonperturbative renormalization developed for QCD simulation for precise test of the standard model is also useful for exploring the beyond standard model physics. Recently, the running coupling constant in Schrodinger
functional scheme is applied to QCD with many flavors. Appelquist et al. [54] found an evidence of an infrared fixed point in strong coupling regime in QCD with twelve flavors. This may serve to construct an example for the walking technicolor. Application to more realistic models are now in progress.

4. CONCLUSIONS

In conclusion, the recent developments of unquenched lattice simulation allows us to study the realistic 2+1 flavor QCD. A lot of nonperturbative computation in light flavor physics is now well under control. Highly improved fermion actions are beginning to control the charm physics. The b quark still suffers from large systematic errors but several promising methods have been tested and giving precise determinations of weak matrix elements of the B mesons in quenched QCD. It was found that the exact chiral symmetry is quite useful in some cases.

Coordinated work to combine advance techniques will lead to few percent accuracy within few years, including cross-checks with different lattice actions.

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