INVERSE MEAN CURVATURE FLOW WITH A FREE BOUNDARY IN HYPERBOLIC SPACE

XIAOXIANG CHAI

Abstract. We study the inverse mean curvature flow with a free boundary supported on geodesic spheres in hyperbolic space. Starting from any convex hypersurface inside a geodesic ball with a free boundary, the flow converges to a totally geodesic disk in finite time. Using the convergence result, we show a Willmore type inequality.

1. Introduction

Let $B_0$ be a geodesic ball of radius $\rho_0 > 0$ in hyperbolic space $\mathbb{H}^{n+1}$ and $M_0$ be a convex hypersurface sitting inside $B_0$ with free boundary supported on $\partial B_0$. We study the inverse mean curvature flow $M_t$ starting from $M_0$. The flow $M_t$ is a family of free boundary immersions $F : [0, T^*) \times \mathbb{D} \rightarrow B_0$ of $n$ dimensional disks $\mathbb{D}$ into $B_0$ with a choice of normal vector field $\nu$ and $M_t = F(t, \cdot)$. The immersions $F$ satisfies the following evolution:

$$\begin{cases}
\frac{\partial F}{\partial t} = \frac{1}{H} \nu & \text{in } \mathbb{D} \\
\langle \nu, \eta \rangle = 0 & \text{on } \partial \mathbb{D},
\end{cases}$$

where $F(t, \partial B_0) \subset \partial B_0$ and $\eta$ is the outward unit normal to $\partial B_0$ in $B_0$.

The inverse mean curvature flow is an expanding flow first studied by [Ger11, Urb90]. Starting from a star-shaped mean convex hypersurface in Euclidean space, after rescaling the hypersurfaces converges to a standard sphere.

The work of Huisken and Ilmanen [HI01] is closely related to our study. Besides a weak theory, they utilized a Geroch monotonicity of the Hawking mass [Haw68] to give a proof of the Penrose inequality under the assumption of nonnegative scalar curvature. Similar monotonicity were observed in other expanding flows and geometric quantity preserving flows leading to rich results in convex geometry, isoperimetric problems and general relativity. Hawking mass has a natural generalization to the free boundary case. A similar monotonicity of Hawking mass with boundary was observed under the free boundary inverse mean curvature flow by [Mar17]. Lambert and Scheuer [LS16] developed a convergence result which says that a free boundary closed convex hypersurface in the unit ball converges to a totally geodesic disk under the inverse mean curvature flow. Later, they [LS17] showed a Willmore type inequality in higher dimensions. See the works [SWX18, WX20] for some further interesting developments.

Unlike in Euclidean space, the inverse mean curvature flow starting from a closed hypersurface in hyperbolic space does not always converge to a round sphere after rescaling [Ger11]. In the free boundary case, Fraser-Schoen [FS15] showed that minimal immersions of disks in hyperbolic geodesic ball of dimension 3 can only
be totally geodesic. We expect that the free boundary inverse mean curvature flow in hyperbolic geodesic ball converges to a totally geodesic disk and it is hopeful to achieve a similar theory as in [LS16]. This is the goal of our work. We have showed the following.

**Theorem 1.1.** Let $F_0$ the embedding of a smooth and strictly convex free boundary hypersurface in a geodesic ball of radius $\rho_0$ with unit normal vector field $\nu$. Then there exists a finite time $T^*$, $0 < \alpha < 1$ and a unique solution such that

$$F \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T^*) \times \mathbb{D}) \cap C^\infty((0, T^*) \times \mathbb{D})$$

of free boundary inverse mean curvature flow (1.1) with initial hypersurface $M_0$. The flow $F(t, \cdot)$ converges to a unique totally geodesic disk as $t \to T^*$.

Using this convergence result, we establish the Willmore type inequality.

**Theorem 1.2.** Let

$$\lambda = \omega_{n-1} \int_0^{\rho_0} \sinh^{n-1} s ds$$

be the volume of $n$-dimensional hyperbolic geodesic ball of radius $\rho_0$ and $\Lambda = 2 \coth \rho_0 \lambda^{-\frac{n}{2}}$. Any weakly convex free boundary hypersurface $M$ in $B_0$ satisfies the Willmore type inequality

$$\frac{2-n}{n} \int_M (H^2 - n^2) + \Lambda |\partial M| \geq -n^2 \lambda^{\frac{2}{n}} + \Lambda \omega_{n-1} \sinh^{n-1} \rho_0,$$

where $\omega_{n-1}$ is the volume of standard $(n-1)$-sphere. Equality occurs if and only if when $M$ is totally geodesic.

It is worth mentioning that Volkmann [Vol16] established a Willmore type inequality for free boundary surfaces or more generally two dimensional free boundary varifolds in the Euclidean unit ball. Volkmann used a Simon type monotonicity argument. It seems an interesting problem to generalize Volkmann’s result to hyperbolic space.

The article is organized as follows:

In Section 2 we collect basic facts on the geometry of hypersurfaces in hyperbolic space. In Section 3 we calculate the evolutions and boundary derivatives of various quantities under inverse mean curvature flow. In Section 4 we study the geometry of a strictly convex free boundary hypersurface from the point of view of convexity, and these results would be used in Section 5 to show that a convex free boundary hypersurface is graphical over some hyperbolic subspace. In Section 6 we show important estimates vital to the proof of Theorem 1.1. In Section 7 we prove the Willmore inequality Theorem 1.2 and characterize the equality case.

**Acknowledgments** Research of the author is supported by KIAS Grants under the research code MG074402. The author would also like to thank Yizhi Wang of Chinese University of Hong Kong for some useful discussions and Julian Scheuer of Cardiff University for pointing out some inaccuracies in an earlier version of the paper.
2. Geometry of Hypersurfaces

We make heavy use of the hyperboloidal model of the hyperbolic space. The book [BP92] is a good reference for basics of hyperbolic spaces. The hyperbolic space $\mathbb{H}^{n+1}$ can be realized as the upper sheet of the two-sheeted hyperboloid in Minkowski space $\mathbb{R}^{n+1,1}$, that is

$$\mathbb{H}^{n+1} = \{z \in \mathbb{R}^{n+1,1} : \langle z, z \rangle = -1, z^0 > 0\},$$

where

$$\langle z, w \rangle = -z^0 w^0 + z^1 w^1 + \cdots + z^{n+1} w^{n+1}.$$

The single-sheeted hyperboloid in Minkowski space known as the de Sitter sphere is a timelike hypersurface and given by

$$dS^{n,1} = \{z \in \mathbb{R}^{n+1,1} : \langle z, z \rangle = 1\}.$$

The Poincaré ball model of hyperbolic space is good for visualization of some concepts. A point $z$ of the hyperboloid model is sent to the hyperboloidal coordinates via the stereographic projection

$$x^i = \frac{z^i}{1 + z^0}, i \geq 1$$

of the Poincaré ball model. The metric $b$ of the ball model is then given by

$$b = \frac{4}{(1 - |x|^2)^2} |dx|^2, |x| < 1.$$

On the other hand, a point $x$ in the Poincaré ball model is sent to

$$z^0 = \frac{1 + |x|^2}{1 - |x|^2}, z^i = \frac{2 x^i}{1 - |x|^2}, i \geq 1$$

of the hyperboloid. This is the inverse of the stereographic projection.

The distance $\rho(z, w)$ between two points $z, w \in \mathbb{H}^{n+1}$ in hyperboloid model is

$$\cosh \rho(z, w) = -\langle z, w \rangle.$$  \hspace{1cm} (2.1)

In particular, the distance $\rho(z)$ from $z$ to $e_0 = (1, 0)$ satisfies

$$\cosh \rho(z) = z^0.$$  \hspace{1cm} (2.2)

In the ball model, the distance $\rho_x$ from $x$ to $o = (0, \ldots, 0)$ is

$$\rho_x = \int_0^{|x|} \frac{2}{1 - |s|^2} ds = \log \left(\frac{1 + |x|}{1 - |x|}\right), \sinh \rho_x = \frac{2|x|}{1 - |x|^2}.$$

A geodesic sphere $\partial B_0$ centered at $(1, 0)$ is given by

$$\{z \in \mathbb{H}^{n+1} : z^0 = \cosh \rho_0\}.$$

By the distance formula (2.2), the outward normal to $\partial B_0$ is

$$\nabla \rho = \frac{1}{\sinh \rho} \nabla z^0 = \frac{1}{\sinh \rho} (-e_0 + z^0 z).$$

A typical example of a free boundary convex hypersurface inside the ball $B_0$ is the spherical cap which is part of a geodesic sphere.
2.1. **Differential geometry of hypersurfaces.** An immersion into the hyperbolic space which is given by a map

\[ z : \mathbb{D} \to \mathbb{R}^{n+1,1} \]

from an \( n \)-dimensional disk \( \mathbb{D} \) in \( \mathbb{H}^{n+1} \) is a codimension two submanifold with boundary in \( \mathbb{R}^{n+1,1} \). We denote this hypersurface by \( M \). Let \( \xi^i \) be the coordinates on \( \mathbb{D} \), then \( z_i := \frac{\partial z}{\partial \xi^i} \) is a tangent vector at \( z(\xi) \), satisfying the relation \( \langle z, z_i \rangle = 0 \).

We reserve the superscript notation \( z_i \) and the alike to denote coordinate components and subscript to denote taking derivatives. The induced metric on \( M \) is then

\[ g_{ij} = \langle z_i, z_j \rangle. \]

Let \( \nu \) be the normal vector of \( M \) in \( \mathbb{H}^{n+1} \) at \( z \), the second fundamental form \( h_{ij} \) of \( M \) in \( \mathbb{H}^{n+1} \) is

\[ h_{ij} = -\langle z_{ij}, \nu \rangle. \]

We have the Weingarten relation

\[ z_{ij} = g_{ij} z - h_{ij} \nu. \]

Taking the trace of the above, and written in components,

\[ \Delta z^\alpha = nz^\alpha - H \nu^\alpha, \quad \alpha = 0, 1, \ldots, n + 1. \]

Replacing the Hessian \( \nabla_i \nabla_j \) in (2.3) by the Hessian of the hyperbolic space, we use the letter \( D \) to denote connection on \( \mathbb{H}^{n+1} \), instead we get

\[ D_\alpha D_\beta z = b_{\alpha \beta} z. \]

The functions \( z^0 \) and \( z^i, \quad i = 1, \ldots, n + 1 \) are called static potentials in general relativity literature \[CH03\]. The components \( \nu^\alpha \) of the normal \( \nu \) are also

\[ \nu^\alpha = \langle Dz^\alpha, \nu \rangle. \]

The vector \( \nu \) is spacelike in \( \mathbb{R}^{n+1,1} \), hence an element of the de Sitter sphere \( \text{dS}^{n,1} \). We call the map \( \nu : M \to \text{dS}^{n,1} \) the **Gauss map**, and to emphasize the dependence on \( z \), we use the notation \( \tilde{z} := \nu \) as well.

2.2. **Christoffel symbols in the Poincaré ball model.** Let \( u = \frac{2}{1-|x|^2} \). Then

\[ D_i \log u = \frac{2x^i}{1-|x|^2} = z^i. \]

The metric \( b \) is conformal to Euclidean metric, we use \( \partial_i \) to denote the \( i \)-th unit vector in Euclidean space, so the Christoffel symbol defined by \( D_j \partial_i = \Gamma^k_{ij} \partial_k \) is given in the following

\[ \Gamma^k_{ij} = \delta^k_i D_j \log u + \delta^k_j D_i \log u - \delta^k_{ij} D^k \log u \]

\[ = \delta^k_i z^j + \delta^k_j z^i - \delta^k_{ij} z_k. \]

So

\[ D_j \partial_i = z^j \partial_i + z^i \partial_j - \delta_{ij} \hat{z}, \]

where we use \( \hat{z} = (z^1, \ldots, z^{n+1}) \in \mathbb{B}^{n+1} \).

The formula

\[ D_j(x^l \partial_l) = \partial_j + x^l D_j \partial_l \]

is useful for later use.
2.3. Useful computations. The vector $\partial_i$ in the Poincaré ball model is (by stereoprojection map $\pi$) under the hyperboloidal model is

$$\pi_*(\partial_i) = \frac{4x^i}{1 - |x|^2}e_0 + \frac{2}{1 - |x|^2}e_i + \frac{4x^i x^j}{1 - |x|^2}e_j,$$

Note that $1 - |x|^2 = \frac{2}{1 + \rho^2}$, so we have that

$$\pi_*(\partial_i) = (1 + \rho)z^i e_0 + (1 + \rho^0) e_i + z^i z^j e_j.$$

From the above,

$$\pi_*(x) = (z^0)^2 e_0 + z^0 z^i e_i.$$

And

$$\langle \pi_*(x), \nu \rangle = -((z^0)^2 - 1)\nu^i + z^0 z^i \nu^i = \nu^0,$$

where we have used $\langle z, \nu \rangle = 0$.

3. Evolution equations and boundary derivatives

In this section, we derive evolution equations and boundary derivatives of various quantities.

Lemma 3.1. Let $X$ be a tangent vector field on $\partial M$, then

$$(3.1) \quad A(X, \eta) = 0.$$  

Proof. We differentiate the relation $\langle \eta, \nu \rangle = 0$ along $X$ on $\partial M$ and get

$$\langle D_X \eta, \nu \rangle + \langle D_X \nu, \eta \rangle = 0.$$  

So

$$A(X, \eta) = \langle D_X \nu, \eta \rangle = -\langle D_X \eta, \nu \rangle.$$  

Recall that the vector $\eta$ is also outward normal of the geodesic sphere $\partial B_0$, $X$ and $\nu$ are tangent to $\partial B_0$, so $\langle D_X \eta, \nu \rangle = 0$. Hence $A(X, \eta) = 0$. \hfill \Box

Lemma 3.2. Let $X$ and $Y$ be tangent vector fields on $\partial M$,

$$(3.2) \quad (\nabla_\eta A)(X, Y) = \coth \rho_0 \langle A(\eta, \eta)(X, Y) - A(X, Y) \rangle.$$  

Proof. We differentiate (3.1) along $Y$,

$$\nabla_Y (A(X, \eta)) = 0.$$  

By product rule,

$$\langle \nabla_Y A(X, \eta) \rangle = -A(\nabla_Y X, \eta) - A(X, \nabla_Y \eta).$$  

In $TM$, the vector field is decomposed into components normal to $\eta$ and parallel to $\eta$, but because of (3.1),

$$A(\nabla_Y X, \eta) = A(\eta, \eta)(\nabla_Y X, \eta) = -A(\eta, \eta)(\nabla_Y \eta, X),$$  

and that $\langle \nabla_Y \eta, X \rangle = \coth \rho_0 (X, Y)$, so

$$-A(\nabla_Y X, \eta) = \coth \rho_0 A(\eta, \eta)(X, Y).$$  

The vector field $\eta$ is unit normal to $\partial M$ in $M$, so $\langle \nabla_Y \eta, \eta \rangle = 0$ and $\nabla_Y \eta$ coincides with the second fundamental form of $\partial B_0$ in $B_0$, so $\nabla_Y \eta = \coth \rho_0 Y$. So

$$-A(X, \nabla_Y \eta) = -\coth \rho_0 A(X, Y).$$
And by Codazzi equation,

$$(\nabla_Y A)(\eta, X) = (\nabla_\eta A)(X, Y).$$

Collecting all the above leads to (3.2).

**Lemma 3.3.** We have

$$(\nabla_\eta A)(\eta, \eta) = -n \coth \rho_0 A(\eta, \eta).$$

**Proof.** We decompose $H$, so

$$-\coth \rho_0 H = \nabla_\eta H = [(g^{ij} - \eta^i \eta^j) + \eta^i \eta^j] \nabla_\eta h_{ij}.$$

Note that $(g^{ij} - \eta^i \eta^j) \nabla_\eta h_{ij}$ is just trace of the tensor $\nabla_\eta h_{ij}$ restricted to tangent space of $\partial M$, from (3.2),

$$(g^{ij} - \eta^i \eta^j) \nabla_\eta h_{ij} = \coth \rho_0 [A(\eta, \eta)(n - 1) - \text{tr}_{\partial M} A] = \coth \rho_0 [nA(\eta, \eta) - H],$$

leads to (3.3).

3.1. **Curvatures.** We are interested in the evolution of the second fundamental form and the mean curvature. They are given by the following.

**Lemma 3.4.** The Weingarten tensor $\tilde{h}_{ij} := g^{ik} h_{ik}$ evolves by

$$\partial_t \tilde{h}_{ij} = -\nabla_i \nabla_j \frac{1}{H} - (-\delta_i^j + h^k_i h^j_k) \frac{1}{H}.$$

The quantity $\log H$ evolves by the equation

$$(\partial_t - \frac{1}{H^2} \Delta) \log H = -\frac{1}{H^2} |\nabla H|^2 - (-n + |A|^2) \frac{1}{H^2},$$

and on the boundary $\partial M$,

$$\nabla_\eta \log H = -\coth \rho_0.$$

**Proof.** For the proof, see for example [And94]. Taking the trace of (3.4),

$$\partial_t H = -\Delta \frac{1}{H} - (-n + |A|^2) \frac{1}{H^2}.$$

This implies

$$(\partial_t - \frac{1}{H^2} \Delta) H = -\frac{2}{H^2} |\nabla H|^2 - (-n + |A|^2) \frac{1}{H^2},$$

and

$$(\partial_t - \frac{1}{H^2} \Delta) \frac{1}{H} = (-n + |A|^2) \frac{1}{H^2}.$$

We obtain the evolution for $\log H$, and $\nabla_\eta \log H = -\coth \rho_0$ follows from [Sta96a].

**Lemma 3.5.** Let $\tilde{h}_{ij}$ be the inverse matrix of $h_{ij}$, then the evolution of $\tilde{H} := g_{ij} \tilde{h}^{ij}$ is given by

$$\partial_t \log \tilde{H} - \frac{1}{\tilde{H}^2} \Delta \log \tilde{H}$$

$$= -\frac{2n}{\tilde{H}} \frac{1}{H^2} - \frac{|A|}{H^2} + \frac{1}{H^2} |\nabla \tilde{H}|^2$$

$$+ \frac{2}{H^2} \left( \frac{1}{\tilde{H}} g_{ij} \tilde{h}^{ik} \tilde{h}^{jq} \nabla_k H \nabla_l H - g_{ij} \tilde{h}^{wpq} \nabla h_{pq} \cdot \nabla \tilde{h}_{kl} \right).$$

(3.7)
Proof. We write in short $\phi = \frac{1}{H}$ for some convenience. First, $\partial_t g_{ij} = 2\phi h_{ij}$ and

$$\partial_t h_{ij} = -\nabla_i \nabla_j \phi - \phi (h_{ik}^k h_{kj}^l - \delta_i^l).$$

So

$$\partial_t h_{ij} = \partial_t (g_{ik} h_{kj}^k)$$
$$= h_{ij}^k \partial_t g_{ik} + g_{ik} \partial_t h_{kj}^k$$
$$= 2\phi h_{ik}^k h_{kj}^l - \nabla_i \nabla_j \phi - \phi (h_{ik}^k h_{kj}^l - g_{ij})$$
$$= -\nabla_i \nabla_j \phi + \phi g_{ik}^k h_{kj}^l + \phi g_{ij}.$$

Now

$$\partial_t \tilde{H} = \partial_t (g_{ij} \tilde{h}^{ij})$$
$$= \tilde{h}_{ij} \partial_t g_{ij} + g_{ij} \partial_t \tilde{h}^{ij}$$
$$= 2\phi \tilde{h}_{ij} - g_{ij} \tilde{h}_{ik}^k \tilde{h}_{kj}^l \partial_t h_{kl}$$
$$= 2n \phi - g_{ij} \tilde{h}_{ik}^k \tilde{h}_{kj}^l (-\nabla_k \nabla_l \phi + \phi g_{ps} h_{ks} h_{lp} + \phi g_{kl})$$
$$= n \phi + g_{ij} \tilde{h}_{ik}^k \tilde{h}_{kj}^l \nabla_k \nabla_l \phi - \phi g_{ij} \tilde{h}_{ik}^k \tilde{h}_{kj}^l g_{kl}$$
$$= \nabla_k \nabla_l H + 2g_{ij} \tilde{h}_{ik}^k \tilde{h}_{kj}^l \nabla_k H \nabla_l H - g_{ij} \tilde{h}_{ik}^k \tilde{h}_{kj}^l g_{kl}.$$

We compute now $\Delta H$ and we have

$$\Delta \tilde{H} = g_{ij} \Delta \tilde{h}^{ij}$$
$$= - g_{ij} \nabla (\tilde{h}_{ik}^k \tilde{h}_{kj}^l \nabla h_{kl})$$
$$= - g_{ij} \tilde{h}_{ik}^k \Delta h_{kl} - g_{ij} \tilde{h}_{ik}^l \nabla h_{kl} \cdot \nabla h_{kl} - g_{ij} \tilde{h}_{ij}^l \nabla h_{ik} \cdot \nabla h_{kl}.$$

We give a quick derivation of the Simons identity:

$$\Delta h_{ij} = \nabla_k \nabla_k h_{ij}$$
$$= \nabla_k \nabla_k h_{kj}$$
$$= \nabla_k \nabla_k h_{kj} - R_{ik}^l h_{lj} - R_{kj}^l h_{kl}$$
$$= \nabla_k \nabla_k h_{kj} - (\bar{R}_{ik}^l + h_{ik}^l - h_{kk} h_{lj}^l) h_{lj}$$
$$- (\bar{R}_{kj}^l + h_{kj}^l - h_{kk} h_{lj}^l) h_{kl}$$
$$= \nabla_k \nabla_k H - (g_{kk} \bar{g}_{ij}^l + \bar{g}_{ik}^l g_{kj} + h_{ik}^l h_{kj}^l - h_{kk} h_{lj}^l) h_{lj}$$
$$- (g_{kj}^l \bar{g}_{ik}^l + \bar{g}_{kj}^l g_{ij} + h_{kj}^l h_{ij}^l - h_{kk} h_{lj}^l) h_{kl}$$

(3.8) $$= \nabla_k \nabla_k H - nh_{ij} + H h_{lj}^l h_{jl} + H g_{ij} - h_{ij} |A|^2.$$
\[\Delta \tilde{H} = -g_{ij}\tilde{h}^{ik}\tilde{h}^{jl}(\nabla_k \nabla_l H - nh_{kl} + H\tilde{h}^s_{kl}s + Hg_{kl} - h_{kl}|A|^2) \]

\[= -g_{ij}\tilde{h}^{ik}\nabla_l \tilde{h}^{jl} \cdot \nabla h_{kl} - g_{ij}\tilde{h}^{i junk}\cdot \nabla h_{kl} \]

\[= -g_{ij}\tilde{h}^{ik}\tilde{h}^{jl}\nabla_k \nabla_l H + n\tilde{H} - nH - Hg_{ij}g_{kl}\tilde{h}^{ik}\tilde{h}^{jl} + \tilde{H}|A|^2 \]

\[-g_{ij}\tilde{h}^{ik}\nabla^j \tilde{h}^{jl} \cdot \nabla h_{kl} - g_{ij}\tilde{h}^{i junk}\cdot \nabla h_{kl}.\]

Now

\[\left(\partial_t - \frac{1}{H^2}\Delta\right) \log \tilde{H} = \frac{1}{H}\partial_t \tilde{H} - \frac{1}{H^2}\tilde{H} + \frac{1}{H^2}\tilde{H}^2 \]

\[= \frac{2\eta}{H} + 2g_{ij}\tilde{h}^{ik}\tilde{h}^{jl}\frac{1}{H^2}\nabla_k H\nabla_l H \]

\[-\frac{n}{H^2} - \frac{|A|^2}{H^2} + \frac{1}{H^2}g_{ij}(\tilde{h}^{ik}\nabla^j \tilde{h}^{jl} + \tilde{h}^{jl}\nabla^i \tilde{h}^{ik}) \cdot \nabla h_{kl} + \frac{1}{H^2}\tilde{H}^2.\]

Considering that \(\tilde{h}^{ij}\) is the inverse matrix of \(h_{ij}\), \(\nabla h^{ij} = -\tilde{h}^{ik}\tilde{h}^{jl}\nabla h_{kl}\) and symmetry of indices leads to (3.7). \(\square\)

**Lemma 3.6.** If \(M_t\) is convex, then boundary derivatives \(\nabla_\eta \tilde{H}\) satisfies the estimate

\[(3.9) \quad \nabla_\eta \tilde{H} \leq n \coth \rho_0 \tilde{H}.\]

**Proof.** We fix coordinates on \(M_t\) such that \(\partial_t = \eta\), so (3.3) and (3.2) implies that

\[\nabla_\eta \tilde{H} = -\eta^k g_{ij}\tilde{h}^{ij}\tilde{h}^{k\nu}\nabla_k h_{ls}\]

\[= -g_{ij}\tilde{h}^{ij}\tilde{h}^{k\nu}\nabla_k h_{ls} - \sum_{l,s \neq 1} g_{ij}\tilde{h}^{ij}\tilde{h}^{l\nu}\nabla_l h_{ls}\]

\[= n\coth \rho_0 g_{ij}\tilde{h}^{ij}\tilde{h}^{k\nu}(h_{11}g_{ls} - h_{ls}).\]

Since the cross term (3.1), \(\tilde{h}^{ij} = 0\) for \(i \neq 1\) and we can assume that \(g_{ij}\) for \(i, j \geq 2\) is unit matrix, then \(h_{ij}\) is diagonal matrix with diagonal entries \((\kappa_1, \ldots, \kappa_n)\). So \(\tilde{h}^{ij}\) is diagonal with diagonal entries \((\kappa_1, \ldots, \kappa^{-1})\). We see then

\[\nabla_\eta \tilde{H} = \coth \rho_0\left(\frac{\rho_0}{\kappa_1} - \kappa_1 \sum_{i \neq 1} \kappa_i^{-2} + \sum_{i \neq 1} \kappa_i^{-1}\right)\]

\[\leq \coth \rho_0\left(\frac{n - 1}{\kappa_1} + \tilde{H}\right).\]

Since \(\tilde{H} = \sum_i \kappa_i^{-1} \) obviously, so \(\nabla_\eta \tilde{H} \leq n \coth \rho_0 \tilde{H}\). \(\square\)

### 3.2. Hyperboloidal coordinates.

The functions \(z^0\) and \(z^i\) are used in later sections to construct auxiliary functions.

**Lemma 3.7.** The functions \(z^0\) and \(z^i\) evolves by

\[(3.10) \quad \partial_t z^i - \frac{1}{M^2}\Delta z^i = -\frac{2}{M^2}z^i + \frac{2}{M^2}v^i.\]
Proof. First, by the equation of the inverse mean curvature flow (1.1),

\[ \partial_t z^i = \frac{1}{H} \nu^i. \]

Combining with (2.4), we obtain (3.10). □

We are also interested in the evolution of \( f(z) \).

Lemma 3.8. Evolution of \( f(z) \) is given by

\[ (\partial_t - \frac{1}{H^2} \Delta) f = -n \frac{\partial f}{\partial z^\alpha} z^\alpha + 2 \frac{\partial f}{\partial \nu^\alpha} \nu^\alpha - \frac{\partial^2 f}{\partial z^\alpha \partial z^\beta} \langle \nabla z^\alpha, \nabla z^\beta \rangle. \]

In particular, when \( f = \log z^\alpha \)

\[ (\partial_t - \frac{1}{H^2} \Delta) \log z^\alpha = -n \frac{\partial}{\partial z^\alpha} + 2 \frac{\partial}{\partial \nu^\alpha} \nu^\alpha + \frac{1}{H^2} \frac{\partial^2}{\partial z^\alpha \partial z^\beta} |\nabla z^\alpha|^2. \]

Proof. First, by (3.10),

\[ \partial_t f(z) = \frac{\partial f}{\partial z^\alpha} \partial_t z^\alpha = \frac{\partial f}{\partial z^\alpha} (\frac{1}{H^2} \Delta z^\alpha - \frac{n}{H^2} z^\alpha + \frac{\nu^\alpha}{H}). \]

By Leibniz rule,

\[ \Delta f(z) = \nabla^i (\frac{\partial f}{\partial z^\alpha} \nabla_i z^\alpha) = \frac{\partial f}{\partial z^\alpha} \Delta z^\alpha + \frac{\partial^2 f}{\partial z^\alpha \partial z^\beta} \langle \nabla z^\alpha, \nabla z^\beta \rangle. \]

Subtraction the above from \( \partial_t f(z) \) gives (3.11). □

Lemma 3.9. The boundary derivatives of \( \nabla_\eta z^i \) and \( \nabla_\eta z^i \) are given by

\[ \nabla_\eta \log z^0 = \tanh \rho_0, \nabla_\eta z^i = \coth \rho_0 z^i. \]

Proof. We use the hyperboloidal coordinates. The normal \( N \) to the geodesic sphere \( \partial B(\rho_0) \) is just \( \nabla \rho \), where \( \rho \) is the distance function to \( e_0 \). By (5.2),

\[ \eta = \nabla \rho = \frac{1}{\sinh \rho} \nabla z^0 = \frac{1}{\sinh \rho} (-e_0 + z^0 z). \]

It readily leads to

\[ \nabla_\eta \log z^0 = \frac{\nabla z^0}{z^0} = \frac{(z^0)^2 - 1}{\sinh \rho \cosh \rho} = \tanh \rho, \]

and

\[ \nabla_\eta z^i = \frac{1}{\sinh \rho} \frac{z^0 z^i}{z^0} = \coth \rho z^i. \]

We can also calculate the normal derivatives \( \nabla_\eta z \) by using the Poincaré ball model. The calculation is slightly longer. □

4. Geometry from convexity

First, we show that \( \partial M \) is a convex hypersurface in \( \partial B_0 \).

Lemma 4.1. Let \( M \) be a convex free boundary hypersurface in \( B_0 \), then \( \partial M \) is a closed convex hypersurface in \( \partial B_0 \).

Proof. The convexity of \( \partial M \) in \( \partial B_0 \) readily follows from the free boundary condition and that the normal of \( M \) in \( B_0 \) is also the normal of \( \partial M \) in \( \partial B_0 \). □

The following lemma states that \( \langle z, \tilde{y} \rangle \) is geometric and does not depend on the choice of embedding of \( \mathbb{H}^{n+1} \) into \( \mathbb{R}^{n+1,1} \). The quantity \( \langle z, \tilde{y} \rangle \) is from the work [Ger06] and closely related to convexity.
Lemma 4.2. Let \( y \) a point in \( \mathbb{H}^{n+1} \), \( \tilde{y} \) be a vector in the tangent space \( T_y \mathbb{H}^{n+1} \) and \( P \) be the subspace \( \mathbb{H}^{n+1} \cap \{ x : \langle x, \tilde{y} \rangle = 0 \} \). Let \( d \) be the distance from a point \( z' \in \mathbb{H}^{n+1} \) to \( P \). By assigning \( d \) to be positive if \( z' \) lies in the same side with which \( \tilde{y} \) points into and negative if \( z' \) lies in the opposite side, we can view \( d \) as a signed distance, and moreover
\[
\sinh d = \langle z', \tilde{y} \rangle.
\]

Proof. The vectors \( y \) and \( \tilde{y} \) are orthogonal unit vectors in \( \mathbb{R}^{n+1} \), we extend \( \{ y, \tilde{y} \} \) to an orthonormal basis of \( \mathbb{R}^{n+1} \). Any point \( z' \in \mathbb{H}^{n+1} \) can then be written as
\[
z' = y \cosh \rho' + \xi' \sinh \rho', \quad \rho', \rho' \geq 0,
\]
where \( \xi' \) is a unit vector orthogonal to \( y \) and \( \rho' \) is the distance in \( \mathbb{H}^{n+1} \) from \( y \) to \( z' \). Any point \( z \) in \( P \) can be written as \( z = y \cosh \rho + \xi \sinh \rho \) where \( \rho \) is the distance in \( \mathbb{H}^{n+1} \) from \( y \) to \( z \), and \( \xi \) is a unit vector in \( \mathbb{R}^{n+1} \) with \( \langle \xi, \tilde{y} \rangle = \langle \xi, y \rangle = 0 \). We see from (2.1) we just have to find out the maximum of \( \langle z, z' \rangle \). Since
\[
- \cosh \text{dist}(z, z') = \langle z, z' \rangle = - \cosh \rho \cosh \rho' + \langle \xi, \xi' \rangle \sinh \rho' \sinh \rho
\]
\[
\leq - \cosh \rho \cosh \rho' + \sinh \rho' \sinh \rho \sqrt{1 - \langle \xi', \tilde{y} \rangle^2} = F(\rho).
\]
So \( F(\rho) \) achieve its maximum when
\[
- \sinh \rho \cosh \rho' + \sinh \rho' \cosh \rho \sqrt{1 - \langle \xi', \tilde{y} \rangle^2} = 0.
\]
This gives
\[
\tanh^2 \rho = \tanh^2 \rho' (1 - \langle \xi', \tilde{y} \rangle^2).
\]
Now we calculate the maximum of \( F(\rho) \) using the condition above,
\[
F(\rho) = - \cosh \rho' (\cosh \rho - \sinh \rho \tanh \rho' \sqrt{1 - \langle \xi', \tilde{y} \rangle^2})
\]
\[
= - \cosh \rho' (\cosh \rho - \sinh \rho \tanh \rho)
\]
\[
= - \frac{\cosh \rho'}{\cosh \rho}
\]
\[
= - \cosh \rho' \sqrt{1 - \tanh^2 \rho}
\]
\[
= - \cosh \rho' \sqrt{1 - \tanh^2 \rho' + \tanh^2 \rho' \langle \xi', \tilde{y} \rangle^2}
\]
\[
= - \sqrt{\cosh^2 \rho' - \sinh^2 \rho' + \sinh^2 \rho' \langle \xi', \tilde{y} \rangle^2}
\]
\[
= - \sqrt{1 + \sinh^2 \rho' \langle \xi', \tilde{y} \rangle}
\]
\[
= - \cosh d,
\]
\[
= - \sqrt{1 + \langle z', \tilde{y} \rangle^2}.
\]
where \( d \) is the distance from \( z' \) to the hyperbolic subspace determined by \( \tilde{y} \). Note that \( \langle z', \tilde{y} \rangle = \langle \tilde{y}, \xi' \rangle \sinh \rho' \). \( \square \)

The corollary below immediately follows from (2.3) and convexity of \( M \).

Corollary 1. For each point \( z \) of \( M \), there exists a small neighborhood \( U_z \) in \( \mathbb{H}^{n+1} \) of \( z \) such that all points in \( (M \cap U_z) \setminus \{ z \} \) lie strictly on the opposite side of \( \tilde{z} \) in the hyperbolic subspace orthogonal to \( \tilde{z} \).
Note that it allows more choices of \( \tilde{z} \) if \( z \) is a boundary point of \( M \). We show in the following that through a point in the free boundary \( \partial M \) there is a 2-subspace \( P \) such that points in a tiny neighborhood of \( z \) in \( M \cap P \) lies on one side of a geodesic in \( P \).

**Lemma 4.3.** Let \( z \in \partial M \), \( U_z \) be a neighborhood as in Corollary \( \ref{cor1} \) and \( P \) be a 2-subspace such that \( P \cap M \) contains \( z \) and at least one more point from \( U_z \cap M \). Then there exists a geodesic line \( L \) such that \( (P \cap M \cap U_z) \setminus \{z\} \) lie on one side of \( L \).

**Proof.** Let \( \tilde{z} \) be a vector as in Corollary \( \ref{cor1} \) then for any point \( w \in U \cap M \), we have
\[
\langle w, \tilde{z} \rangle < 0.
\]

Let \( e_i, i = 1, 2 \) be orthonormal tangent vectors at \( P \). We extend \( e_0 := z, e_1, \) and \( e_2 \) to a set of orthonormal basis of \( \mathbb{R}^{n+1,1} \). We write \( \tilde{z} = \sum_{i=0}^{n+1} \tilde{z}^i e_i \) and any point \( w = \sum_{i=0}^{n} w^i e_i \) in \( P \). We define \( v = \sum_{i=0}^{n} \tilde{z}^i e_i \). Obviously \( v \) is a tangent vector in \( P \), and
\[
\langle w, v \rangle = \langle \tilde{z}, w \rangle < 0.
\]

Geometrically, \( v \) is just the projection of \( \tilde{z} \) to \( P \) and it determines a geodesic line \( L \) such that \( (U \cap P \cap M) \setminus \{z\} \) lies on one side of \( L \). \( \square \)

**Lemma 4.4.** Let \( \Omega' \) be the region bounded by \( \partial B_0 \) and \( M \) such that \( v \) points outward of \( \Omega' \), and \( \Sigma = \partial \Omega' \), and \( y \) be a point in \( \Sigma \), then
\[
(z, \hat{y}) \leq 0
\]
for all \( z \in \Sigma \) and equality occurring only when \( z = y \).

**Proof.** Note that Corollary \( \ref{cor1} \) and Lemma \( \ref{lem1.3} \) are valid at \( \Sigma \setminus M \) as well. First, we assume that \( y \) is a smooth point of \( \Sigma \). Note that by Corollary \( \ref{cor1} \) for all points \( z \in \Sigma \) near \( y \), \( \langle z, \hat{y} \rangle < 0 \). Assume that for some \( z \in \Sigma \) that \( \langle z, \hat{y} \rangle > 0 \). Then by continuity, there is at least another point not \( y \) itself in \( M \) which we still \( w' \) satisfies \( \langle w', \hat{y} \rangle = 0 \). By Lemma \( \ref{lem1.2} \), \( w' \) lies in the subspace normal to \( \hat{y} \). There is a geodesic in \( \mathbb{H}^{n+1} \) from \( y \) to \( w' \), there is a unique hyperbolic 2-subspace \( P \) linearly spanned by \( y, \hat{y} \) and the unit tangent vector of this geodesic at \( y \). Obviously, \( w' \in P \).

Denote the curve in \( \Sigma \) from \( y \) to \( w' \) by \( \gamma \) and the geodesic from \( y \) to \( w' \) in \( \Sigma \) by \( \gamma_0 \). Since \( \langle w', \hat{y} \rangle = \langle y, \hat{y} \rangle = 0 \), let \( z \) be the point \( z \in \gamma \) such that the function \( \langle \gamma(t), \hat{y} \rangle \) of \( t \) achieves its minimum.

Again, we assume that \( z \) is a smooth point of \( \Sigma \). Let \( v \) the normal to \( \gamma \) in \( P \). We know that \( \gamma_0 \) and \( \gamma \) bounds a region \( \Omega \) in \( P \), we fix the orientation of \( v \) so that it is consistent with \( \hat{y} \), that is \( v \) points into \( \Omega \).

We change coordinates so that \( e_0 = y, \hat{y} = e_1 \) and a unit tangent vector \( e_2 \) at \( y \). The set \( \{e_0, e_1, e_2\} \) is an orthonormal basis of \( P \). We can identify \( P \) with \( \mathbb{R}^{2,1} \).

The region \( \Omega \) is located in the strip
\[
\{ z \in P = \mathbb{R}^{2,1} : \langle z, \hat{y} \rangle \leq z^1 \leq 0 \}.
\]
Since \( v \) points into \( \Omega \), so \( v \) cannot point to the outward of \( \Omega \) which is \( -e_1 \) direction. Hence \( \langle v, \hat{y} \rangle = \langle v, e_1 \rangle > 0 \). Since \( \langle \gamma(t), \hat{y} \rangle \) achieves minimum at \( z \), \( \hat{y} \) is normal to \( \gamma' \) at \( z \). We can assume that
\[
\hat{y} = a_0 z + a_1 v.
\]
From \( \langle \tilde{y}, z \rangle < 0 \), \( z \) is time-like, so \( a_0 > 0 \); and from \( \langle v, \tilde{y} \rangle > 0 \), \( a_1 > 0 \). Actually \( a_1^2 = a_0^3 + 1 \), but it is not needed. Let \( w \) be a point very close to \( z \), since \( \langle \gamma(t), \tilde{y} \rangle \) achieves minimum at \( z \), so \( \langle w - z, \tilde{y} \rangle \geq 0 \). Hence,

\[
0 \leq \langle w - z, \tilde{y} \rangle = \langle w - z, a_1v \rangle + \langle w - z, a_0z \rangle = a_1\langle w - z, v \rangle + a_0(\langle w, z \rangle + 1) = a_1\langle w, v \rangle + a_0(\langle w, z \rangle + 1).
\]

From the distance formula (2.1), \( \langle w, z \rangle + 1 < 0 \). By positivity of \( a_0 \) and \( a_1 \),

\[
\langle w, v \rangle > 0.
\]

Now we center the coordinate system at \( z \), let \( z = e_0 \), unit tangent vector \( e_1 \) at \( z \) and \( v = e_2 \). Any point \( w \) in \( P \) satisfies \( \langle w, e_i \rangle = 0 \) for \( i \geq 3 \). Since \( \langle w, v \rangle > 0 \) near \( z \) and the normal \( \tilde{z} \) in \( \mathbb{H}^{n+1} \) is a linear combination of \( e_i \) for \( i \geq 2 \), so

\[
\langle w, \tilde{z} \rangle = \langle w, v \rangle \langle \tilde{z}, v \rangle.
\]

There is a curve going over \( z \) in \( P \) to meet the point \( w' \), so \( v \) must points outward of the region bounded by \( M \), so \( \langle \tilde{z}, v \rangle > 0 \). Therefore, we get \( \langle w, \tilde{z} \rangle > 0 \). However, because of convexity near \( z \) and that \( \langle w, \tilde{z} \rangle < 0 \), it leads to a contradiction.

Now the same proof works through if \( y \) is not a smooth point. If \( \Sigma \) is not smooth at \( z \), it is sufficient that we invoke Lemma 4.3 instead.

Let \( (\partial M)^* \) be the convex hull of \( \partial M \) in \( \partial B \) and \( S^* \) be the set of \( \omega \in S^n \) such that \( e_0 \cosh \rho_0 + \omega \sinh \rho_0 \in (\partial M)^* \). Define the following two sets:

\[
C_1 = \{ z : z = e_0 \cosh \rho + \omega \sinh \rho, \rho \geq 0, \omega \in S^* \},
\]

and

\[
C_2 = \cap_{y \in \partial M} \{ z : \langle z, \tilde{y} \rangle \leq 0 \}.
\]

**Lemma 4.5.** The two sets \( C_1 \) and \( C_2 \) are equivalent:

\[
C_1 = C_2.
\]

**Proof.** We only show that \( C_1 \subset C_2 \) and the reverse relation is similarly proved. We define

\[
\gamma_\omega(t) = e_0 \cosh t + \omega \sinh t, t \geq 0
\]

be the radial geodesic in \( \mathbb{H}^{n+1} \) starting from \( e_0 \). Let \( z = \gamma_\omega(t) \) be a point in \( C_1 \) and \( x = \gamma_\omega(\rho_0) \). Since \( x \) and \( y \) are on the same level set of \( z_0 \), and \( \tilde{y} \) has no \( e_0 \) component, by results from convex geometry of the sphere [FIN13], we have that \( \langle x - y, \tilde{y} \rangle \leq 0 \). So \( \langle x, \tilde{y} \rangle \leq 0 \) and

\[
\langle z, \tilde{y} \rangle = \sinh t \langle \xi, \tilde{y} \rangle = \frac{\sinh t}{\sinh \rho_0} (\sinh \rho_0, \tilde{y}) = \frac{\sinh t}{\sinh \rho_0} \langle x, \tilde{y} \rangle \leq 0.
\]

Hence \( z \in C_2 \). \( \square \)

It follows immediately that

**Corollary 2.** We have that \( M \subset C_1 = C_2 \) and \( e_0 \notin M \).

**Proof.** The fact \( M \subset C_2 \) is obvious. If \( e_0 \) is a point in \( M \), it has to be an interior point of \( M \), and \( M \) is smooth here. But \( C_1 \) is a convex radial cone and \( M \subset C_1 \) says that \( M \) is not smooth at \( e_0 \). \( \square \)
Let $\hat{M}$ be the enclosed region by $\partial B$ and $M$ with the unit normals $\nu$ pointing outside of $\hat{M}$. Let

$$C_0 = \bigcap_{z \in \partial M} \{ w \in \mathbb{H}^{n+1} : \langle w, \tilde{z} \rangle \leq 0 \}.$$ 

We see that $\hat{M}$ is contained in $C_0$.

**Lemma 4.6.** Let $\rho > 0$ and $C \subset \mathbb{R}^{n+1,1}$ be the cone

$$C = \{ z = e_0 \cosh t + \omega \sinh t : t \geq 0, \xi \in S^* \}$$

for some convex set $S^* \subset \mathbb{S}^n$. If for all $\omega \in S^*$ and some $\varepsilon \in (0, \frac{\pi}{2})$ such that $\langle \omega, e_1 \rangle \geq \cos(\frac{\pi}{2} - \varepsilon)$,

$$z^1 = (z, e_1) \geq \frac{\sinh R}{\cos \varepsilon}$$

if $B_R(z) \subset C$.

**Proof.** We define a new cone

$$C' = \{ z = e_0 \cosh t + \omega \sinh t : t \geq 0, \langle \omega, e_1 \rangle \geq \cos(\frac{\pi}{2} - \varepsilon) \}.$$ 

Obviously, $C \subset C'$ and the distance from $z$ to $\partial C'$ denoted by $R'$ is greater than $R$. Assume that $\partial B_R(z)$ touches $\partial C'$ at $D$ and let $A$ represents the point $z$. The geodesic starting from $A$ to $D$ must be orthogonal to the ray $OD$. The three points $O$, $A$ and $D$ spans a two dimensional hyperbolic subspace. By the hyperbolic sine law (see [Kat92, Chapter 1]), we have that

$$\frac{\sinh R'}{\sin \angle DOA} = \frac{\sinh OA}{\sin \frac{\pi}{2}},$$

where $OA = \frac{2|x|}{1-|x|^2}$. We have that

$$x^1 = |x| \cos \angle AOB',$$

where $B'$ is point realizing the shortest Euclidean distance from $A$ to the line $x^1$ axis. So

$$z^1 = \frac{2x^1}{1-|x|^2} = \frac{2|x| \cos \angle AOB'}{1-|x|^2} = \frac{\cos \angle AOB'}{\sin \angle DOA} \sinh R'.$$

We just have to estimate

$$\frac{\cos \angle AOB'}{\sin \angle DOA} = \frac{\cos(\frac{\pi}{2} - \varepsilon - \angle DOA)}{\sin \angle DOA} = \frac{\sin(\varepsilon + \angle DOA)}{\sin \angle DOA} = \sin \varepsilon \cot \angle DOA + \cos \varepsilon.$$

The right hand of the the above is uniformly bounded below by $\frac{1}{\cos \varepsilon}$ since $\angle DOA = \frac{\pi}{2} - \varepsilon$ and

$$\sin \varepsilon \cot \angle DOA + \cos \varepsilon = \sin \varepsilon \cot(\frac{\pi}{2} - \varepsilon) + \cos \varepsilon = \frac{1}{\cos \varepsilon},$$

This concludes our proof. \(\Box\)

We can also look at this in the Poincaré ball model. Then if $B_R(A) \subset C$, in terms of ball model,

$$z^1 := \frac{2x^1}{1-|x|^2} \geq \sinh R(1 + \delta),$$

($x$ is the point of $A$) where $\delta$ only depends only on $\varepsilon$. If $x^1/|x| = \cos \theta \in [0, \frac{\pi}{2} - \varepsilon]$, we have a lower bound on $x^1$. 


Corollary 3. Under the same conditions of Lemma 4.6, we have that there exists a \( \delta' \in (0, 1) \) such that
\[
x^1 \geq \delta' > 0.
\]

Proof. We solve \( x^1 \) in terms of \( z^1 \), we got
\[
z^1 \left( \frac{(x^1)^2}{\cos^2 \theta} - 1 \right) + 2x^1 = 0,
\]
and
\[
x^1 = \frac{-2 + \sqrt{4 + 4z^2/\cos^2 \theta}}{2 \cos \theta} = \frac{\sqrt{1 + z^2/\cos^2 \theta} - 1}{z/\cos^2 \theta} = \frac{z}{\sqrt{1 + z^2/\cos^2 \theta}}.
\]
(we have dropped the negative solution) since the function \( \sqrt{1 + x^2} \) is increasing, we have that \( x^1 \) is bounded below by the above when \( z = \sinh R(1 + \delta) \). \( \square \)

Lemma 4.7. For every interior point \( z \) of \( M \), we have that the zeroth component of the normal at \( x \) is strictly negative i.e.
\[
\tilde{z}^0 < 0.
\]

Proof. Let \( w(t) \) be the geodesic ray starting from \( e_0 \) and passing through \( z \). Note \( z \) is an interior point of \( \Sigma \). So we have that
\[
\langle \tilde{z}, w'(t_0) \rangle < 0 = \langle w(t_0), \tilde{z} \rangle.
\]
The \( \langle \tilde{z}, w'(t_0) \rangle < 0 \) follows from convexity of \( \Sigma \) and \( \langle w(t_0), \tilde{z} \rangle = 0 \) follows from the fact that \( \tilde{x} \) is in the tangent space at \( z = w(t_0) \).

We use the unit parametrization \( w(t) = e_0 \cosh t_0 + \xi \sinh t_0 \) where \( \xi \in S^n \) for the curve \( w(t) \). The above condition are then
\[
\langle w'(t_0), z \rangle = \langle e_0, \tilde{z} \rangle \sinh t_0 + \langle \xi, \tilde{z} \rangle \cosh t_0 < 0
\]
\[
\langle w(t_0), \tilde{z} \rangle = \langle e_0, \tilde{z} \rangle \cosh t_0 + \langle \xi, \tilde{z} \rangle \sinh t_0 = 0.
\]
Eliminating \( \langle \xi, \tilde{z} \rangle \), we have
\[
\langle w'(t_0), \tilde{z} \rangle
\]
\[
= \langle e_0, \tilde{z} \rangle \left( \sinh t_0 - \frac{\cosh^2 t_0}{\sinh t_0} \right)
\]
\[
= - \frac{1}{\sinh t_0} \langle e_0, \tilde{z} \rangle = \frac{\tilde{z}^0}{\sinh t_0} < 0.
\]
So the zeroth component is less than zero. \( \square \)

The following corollary shows that \( M \) lies in a half geodesic ball omitting one point in the direction of the axis. It is an important fact in writing the inverse mean curvature flow in coordinates and turning it into a parabolic partial differential equation of a scalar function with Neumann boundary conditions.

Corollary 4. Let \( p_1 = e_0 \cosh \rho_0 + e_1 \sinh \rho_0 \), then
\[
M \subset B^+ \setminus \{ p_1 \}.
\]

Proof. If the interior of \( M \) touches the geodesic sphere \( \partial B_0 \) at \( p_1 \in M \), then \( p_1 \) is an interior maximum point of the function \( z^0 \). Since
\[
\Delta z^0 = nz^0 - H\nu^0.
\]
From Lemma 4.7 follows \( \Delta \rho > 0 \) at \( p_1 \) contradicting the fact \( p \) is maximum point of \( \rho \). \( \square \)
Lemma 4.8. The first component of $\nu$ is bounded, in particular,

$$\nu_1 \leq -c_0,$$

where $c_0$ depends only on the distance of $S^*$ to the equator

$$\mathcal{H}(e_1) = \{ e \in S^n : \langle e, e_1 \rangle = 0 \}.$$

Proof. Any point in $(\partial M)^*$ can be represented by

$$z' = e_0 \cosh \rho_0 + \xi' \sinh \rho_0, \quad t > 0, \xi' \in S^*.$$

By (4.1), $\langle \nu, z' \rangle \leq 0$. From Lemma 4.7, we can assume that

$$\nu = -e_0 \sinh s + \xi \cosh s, \quad s \geq 0, \xi \in S^n.$$

We have that

$$\langle \nu, z' \rangle = \sinh s \cosh \rho_0 + \langle \xi, \xi' \rangle \sinh \rho_0 \cosh s \leq 0.$$

So

$$\langle \xi, \xi' \rangle \leq -\frac{\tanh s}{\tanh \rho_0}.$$

Note that

$$\langle \xi, \xi' \rangle \leq 0, -1 \leq -\frac{\tanh s}{\tanh \rho_0}.$$

This gives a trivial bound $0 \leq s \leq \rho_0$ on $s$. And since $\xi'$ lies in a convex set $S^*$ of $S^n$, this gives a bound on $\langle \xi, e_1 \rangle$ which depends on the distance of $S^*$ to the equator $\mathcal{H}(e_1)$.

We now prove an estimate of $z^1$ for convex free boundary hypersurfaces.

Lemma 4.9. Let $M$ be a strictly convex hypersurfaces, then

$$z^1 \geq \delta > 0$$

for all $M$, where the constant $\delta$ depends only on $\sup_M |A|$ and the distance of $\partial M$ to the equator $\mathcal{H}(e_1)$.

Proof. Let $z \in M$ be a global minimum point of $z^1$. Let $e_1' = e_1 + \langle e_1, z \rangle z$. The vector $e_1'$ is a vector in $T_z \mathbb{H}^{n+1}$, moreover $\langle e_1', e_1' \rangle = 1 + (z^1)^2 > 0$. Let now $e = e_1' / \sqrt{\langle e_1', e_1' \rangle}$.

Since $z$ is global minimum, $e$ must be a normal vector of $M$ pointing inward of $\hat{M}$. Let $P$ be the hyperbolic subspace normal to $e$. Due to the previous lemma, it is possible to write $M$ as a graph around $z$ over $P$. The graph function satisfies

$$w_{ij} = -h_{ij} \langle \nu, e \rangle + g_{ij} \langle w, e \rangle.$$

We only have to show that $\langle \nu, e \rangle$ is bounded by a negative constant. Indeed,

$$\langle \nu, e \rangle = \frac{1}{\sqrt{\langle e_1', e_1' \rangle}} \langle \nu, e_1 + \langle e_1, z \rangle z \rangle = \frac{1}{1 + (z^1)^2} \langle \nu, e_1 \rangle + \frac{z^1}{1 + (z^1)^2} \langle \nu, z \rangle.$$

By Lemma 4.7, $\langle \nu, z \rangle \leq 0$. So

$$\langle \nu, e \rangle \leq \frac{1}{1 + (z^1)^2} \langle \nu, e_1 \rangle \leq \frac{c_0}{1 + (z^1)^2},$$

where the constant $c_0$ is from Lemma 4.8. Define

$$\hat{M}' = \cap_{\theta \in M} \{ z \in \mathbb{H}^{n+1} : \langle z, \tilde{y} \rangle \leq 0 \}.$$
By the previous consideration, we have that $\hat{M}'$ satisfies an interior sphere condition at $z$ with radius of the ball depending only on the second fundamental form and $\langle \nu, e \rangle$. Due to $B_R(z) \subset \hat{M}' \subset C_1 = C_2$, from Lemma 4.6 we have that $z^1 \geq \delta > 0$.

\[ \square \]

5. Moebius coordinates

Lambert and Scheuer [LS16] introduced the Moebius coordinates for free boundary hypersurfaces in the Euclidean unit ball.

**Definition 1.** Let $D \subset \mathbb{R}^n$ be the unit disk. Define the Moebius coordinates for the pointed half ball $B^+ = B^+ \{ \{0\} \}$

to be the diffeomorphism $f: D \times [1, \infty) \to \mathbb{B}^+$,

\[ f(\xi, \lambda) = \frac{4\lambda(1 + |\xi|^2)(\lambda^2 - 1)\partial_1}{(1 + \lambda^2) + (1 - \lambda^2)|\xi|^2}. \]

In this section, we generalized the Moebius coordinates of convex free boundary hypersurfaces by [LS16] to the hyperbolic case. To this end, we will make use of the ball model. In this way, the Moebius coordinate is just composition with one more scaling with a scaling factor $r_0$ defined by the relation $1 + r_0^2 = \cosh \rho_0$. The Moebius coordinates are then given by the diffeomorphism

\[ \psi: \mathbb{B}^n \times [1, \infty) \to \mathbb{B} \to \mathbb{B}^n(r_0), r_0 < 1 \]

sending $(\xi, \lambda)$ to

\[ \psi(\xi, \lambda) = r_0 f(\xi) = r_0 \frac{4\lambda(1 + |\xi|^2)(\lambda^2 - 1)\partial_1}{(1 + \lambda^2) + (1 - \lambda^2)|\xi|^2}. \]

The metric components of the metric $b$ under this new coordinate are $b(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \lambda})$, $b(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \lambda})$ and $b(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \lambda})$. It is easy to see that

\[ b(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \lambda}) = \frac{4}{(1 - r_0^2|f|^2)^2} \frac{\partial(f_0)}{\partial \xi} \cdot \frac{\partial(f_0)}{\partial \xi} = \frac{4r_0^2}{(1 - r_0^2|f|^2)^2} \frac{\partial f}{\partial \xi} \cdot \frac{\partial f}{\partial \xi} \]

and similar formulas hold for other components. The dot · here represents the Euclidean inner product. These components can be explicitly calculated although what is really needed is $b(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \lambda}) = 0$.

**Lemma 5.1.** Let $\phi = (1 + \lambda)^2 + (1 - \lambda)^2|\xi|^2$. The metric is

\[ b = \phi_1^2 d\lambda^2 + \phi_2^2 \delta_{ij}, \]

where

\[ \phi_1 = \frac{4(1 + |\xi|^2)}{\phi(1 - r_0^2|f|^2)}, \phi_2 = \frac{8r_0\lambda}{\phi(1 - r_0^2|f|^2)}. \]

**Proof.** By direct but tedious calculation. The components $b(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \lambda}) = 0$ also follows from [LS16]. \[ \square \]

**Proposition 5.2.** Let $y : M \to (\mathbb{B}^{n+1}, b)$ the the embedding of a strictly convex free boundary hypersurface $M$. Then $M$ can be written as a graph in Moebius coordinates around $\partial_1$ that is

\[ x = \psi(\xi, u(\xi)). \]
Before we prove this proposition, we give the following lemma as a preparation where the idea of the proof is also used in proof of Proposition 5.2.

**Lemma 5.3.** Suppose that $M$ is convex with free boundary in $B_0$, in the ball model we have

$$\langle \partial_1, N \rangle < 0,$$

where $N$ is the unit normal to $M$ in the Poincaré ball model.

**Proof.** Since $M$ is convex and that $\partial M$ is convex in $\partial B_0$ or equivalently in $\partial B(r_0)$, $\langle \partial_1, N \rangle < 0$ along $\partial M$. Assume on the contrary that $\langle \partial_1, N \rangle \geq 0$, then the maximum of $\langle \partial_1, N \rangle$ occurs in the interior of $M$. Let $y = y^t \partial_i$ be any tangent vector fields on $M$, at a maximum point $p$ of $\langle \partial_1, N \rangle$,

$$D_y (\partial_1, N) = \langle N, D_y \partial_1 \rangle + \langle \partial_1, D_y N \rangle = 0.$$

Since

$$D_j \partial_i = z^j \partial_i + z^i \partial_j - \delta_{ij} \hat{z},$$

so

$$\langle N, D_y \partial_1 \rangle = y_j z^j \langle \partial_1, N \rangle - y_1 \langle z, N \rangle =: \langle Y, y \rangle,$$

where $Y = u^{-2} \hat{z} (\partial_1, N) - u^{-2} \langle \hat{z}, N \rangle \partial_1$. So

$$\langle \partial_1, D_y N \rangle = -\langle y, Y \rangle$$

at $p$. Note that $Y$ is orthogonal to $N$, letting $\partial_1^\top$ be the projection of $\partial_1$ to the tangent space of $M$, so

$$A(\partial_1^\top) = -Y,$$

and

$$\langle A(\partial_1^\top), \partial_1^\top \rangle = -\langle Y, \partial_1 \rangle = u [\langle x, N \rangle - x^1 \langle \partial_1, N \rangle] < 0.$$

Due the to the assumption at $p$, $\langle \partial_1, N \rangle \geq 0$ and recall that $x^1 > 0$, so

$$\langle A(\partial_1^\top), \partial_1^\top \rangle < 0.$$

However $M$ is convex, $A$ is positive. The contradiction leads to $\langle \partial_1, N \rangle < 0$. \hfill $\square$

Now we turn to the proof of Proposition 5.2.

**Proof of Proposition 5.2.** First, by Lemma 4.9 Moebius coordinate is well defined on $M$. By implicit function theorem, we have to show that

$$\langle \partial_1, N \rangle < 0.$$

Note that

$$\frac{\partial \varphi}{\partial x} = r_0 \frac{\partial f}{\partial x} = r_0 \left( \frac{1 + |\xi|^2}{\lambda(1 + \lambda)^2 + (1 - \lambda)^2 |\xi|^2} \right) (f - \frac{x^2 + 1}{x^2 - 1} \partial_1).$$

Since $\lambda > 1$ by Lemma 4.9 and $x = r_0 f$, we are reduce to show

$$\zeta(x) := (x - r_0 \frac{1 + \lambda^2}{x^2 - 1} \partial_1, N) > 0.$$

Now we follow same lines of arguments as in Lemma 5.3 to show the above inequality. Suppose on the contrary that $\zeta(x) \leq 0$. Since $\langle x, N \rangle$ vanishes along $\partial M$,

$$\zeta(x) = -\langle r_0 \frac{1 + \lambda^2}{x^2 - 1} \partial_1, N \rangle > 0.$$
along $\partial M$. So the minimum of $\zeta$ occurs at an interior point $p$ of $M$ and at $p$, $\nabla \zeta = 0$. Letting $\Lambda = r_0 \ell^2$ and $X = x - \Lambda \partial_1$, then at $p$

$$\langle D_y X, N \rangle + \langle X, D_y N \rangle = 0.$$ 

Now note that $\langle D_y x, N \rangle$ vanishes. Indeed, this follows from $D_j(x^i \partial_i) = \partial_j + x^i D_j \partial_i$, and

$$D_y x = y + y^j x^i D_j \partial_i = y + y^j x^i(z^j \partial_i + z^i \partial_j - \delta_{ij} \hat{z}) = y + y^j z^i x + yx^i z^i - x^i y^j \hat{z} = y(1 + x^i z^i).$$

So

$$\langle X, D_y N \rangle = -\langle D_y X, N \rangle = \Lambda \langle D_y \partial_1, N \rangle = \Lambda \langle y, Y \rangle$$

which gives

$$A(X^\top) = \Lambda Y,$$

again here $Y = u^{-2} \hat{z} (\partial_1, N) - u^{-2} (\hat{z}, N) \partial_1$.

We compute now

$$\langle A(X^\top), X^\top \rangle = \langle \Lambda Y, X \rangle = \Lambda \sum_{i,j} \langle x(\partial_1, N) - \langle x, N \rangle \partial_1, x - \Lambda \partial_1 \rangle = \Lambda \sum_{i,j} \langle x, N \rangle (\Lambda - x^1) - \Lambda x^1 (\partial_1, N)].$$

Since at $p$, $\zeta(x) \leq 0$ implies that $\langle x, N \rangle \leq \Lambda (\partial_1, N)$ and for all of $M$, $\Lambda - x^1 > 0$, we have

$$\langle A(X^\top), X^\top \rangle \leq \Lambda \left[ \sum_{i,j} \langle x, N \rangle (\Lambda - x^1) - \Lambda x^1 (\partial_1, N) \right] = \Lambda (\partial_1, N) \Lambda \left( \sum_{i,j} |x|^2 + (\Lambda - x^1)^2 \right).$$

The factor in the big bracket on the left is obviously positive, so $\langle A(X^\top), X^\top \rangle < 0$ by the previous lemma. This again contradicts the convexity of $M$. \hfill \Box

Now we can reduce the inverse mean curvature flow to a scalar parabolic equation.

**Theorem 5.4.** Let $F$ be a solution of (1.1) for some $[0, \varepsilon)$ such that for all $t \in [0, \varepsilon)$ the flow $M_t$ are graphs in Moebius coordinates in the pointed half ball $B_0^+ \setminus p_1$, that is

$$M_t = \{(x(t, \omega), u(t, \xi)) : (t, \omega) \in [0, \varepsilon) \times \mathbb{D} \}.$$

Adopting the notations in Lemma 5.1, then $u$ solves a parabolic partial differential equation with Neumann boundary value condition:

$$\begin{cases}
\frac{\partial}{\partial t} u - \sum_{i} \xi_i \frac{\partial}{\partial \xi_i} u = 0 & \text{in } (0, \varepsilon) \times \mathbb{D} \\
u \frac{\partial u}{\partial n} = u_0 & \text{on } (0, \varepsilon) \times \partial \mathbb{D}
\end{cases} \tag{5.2}$$
where \( v \) is defined to be
\[
v = \frac{1}{4} \phi (1 - \nu_0^2 |f|^2) \sqrt{\frac{1}{(1 + |f|^2)} + \frac{1}{4} \sum_i (\frac{\partial u}{\partial \xi_i})^2}.
\]

Proof. Let \( X = (\xi, \lambda) \), because of Proposition 5.2, the inverse mean curvature flow is equivalent to
\[
(\partial_t X) \perp = \frac{1}{H} \nu
\]
up to tangential diffeomorphisms where \( \nu \) points downward of \( \lambda \)-direction, so
\[
\frac{1}{H} = \langle \partial_t X, \nu \rangle = \langle \frac{\partial}{\partial \lambda}, \nu \rangle \frac{\partial}{\partial \lambda}.
\]

We write \( b_{\lambda \lambda} = \phi_1^2 \) and \( b_{ij} = \phi_2^2 \delta_{ij} \), the vector \( \tilde{\nu} = \phi_1^2 \partial u - \phi_2^2 \partial \lambda \) is normal to the graph, where \( \partial u = \sum_i \frac{\partial u}{\partial \xi_i} \). So
\[
\langle \frac{\partial}{\partial \lambda}, \nu \rangle = \frac{1}{|\nu|} \langle \frac{\partial}{\partial \lambda}, \tilde{\nu} \rangle = -\frac{1}{|\nu|} \phi_1^2 \phi_2^2.
\]

We see that \( v = \frac{|\nu|}{\phi_1^2 \phi_2^2} \), and the equation (5.2) now easily follows from \( \partial_t \xi^i \equiv 0 \). \( \square \)

We partially calculate the mean curvature \( H \) in terms of \( u \) and the metric in (5.1).

**Lemma 5.5.** Let \( \tilde{\nu}, u \) be as in the previous theorem, the mean curvature \( H \) of a graph of some function \( u \) in Moebius coordinates is given by
\[
(5.3) \quad H = \frac{1}{g} g^{ij} \partial_i \partial_j u + F',
\]
where \( F' \) contains lower order terms and depends on \( x, u \) and \( \partial u \).

Proof. Let the inverse metric of the graph be \( g^{ij} \), so the mean curvature of a graph of the function \( u \) is given by
\[
H = \frac{1}{|\nu|} \langle D \partial_{\xi^i} \partial_j u, \partial_{\xi^i} \nu \rangle.
\]
The terms contains \( \partial_i \partial_j u \) is \( \frac{1}{|\nu|} \langle D \partial_{\xi^i} \phi_1^2, \partial_j \nu \rangle \), it is easy to see (5.3) holds. \( \square \)

6. CONVERGENCE TO TOTALLY GEODESIC DISKS

We define \( \bar{T} \) to be the largest time such that for all \( t \in [0, \bar{T}) \) the hypersurface \( M_t \) is strictly convex and \( T^* \) be the largest time such that the flow exists. The for all \( t \in [0, T^*) \), \( M_t \) is mean convex. Obviously, \( \bar{T} \leq T^* \).

Proposition 6.1. Let \( F(t, \xi) \) be a solution to the inverse mean curvature flow on the interval \([0, \bar{T})\). Then the principal curvature are bounded, that is
\[
(6.1) \quad \sup_{M_t} |A| \leq C,
\]
for all \( t \in [0, \bar{T}) \) where \( C \) depends only on the radius \( \rho_0 \), initial position of \( M_0 \) and the mean curvature of the initial hypersurface \( M_0 \).

Proof. We consider the auxiliary function \( \zeta = \log H + \log z^0 \). From (3.6) and (3.12),
\[
\nabla_\eta \zeta = \nabla_\eta \log H + \nabla_\eta \log z^0 = -\coth \rho_0 + \tanh \rho_0 < 0.
\]
It follows that the maximum of ζ is not attained at the boundary ∂M. Using the evolutions (3.5) and (3.11),
\[ \partial_t \zeta - \frac{1}{H^2} \Delta \zeta = (\partial_t - \frac{1}{H^2} \Delta) (\log H + \log z^0) \]
\[ = - \frac{\nabla H^2}{H^2} - \frac{1}{H^2} |A|^2 + \frac{2}{H^2 z^0} \nu_0 + \frac{1}{H^2 z^0} |\nabla z^0|^2. \]
We rewrite the above evolution equation as
\[ (\partial_t - \frac{1}{H^2} \Delta) \zeta + \frac{1}{H^2} (\nabla H - \frac{\nabla z^0}{z^0}, \nabla \zeta) = - \frac{|A|^2}{H^2} + \frac{2}{H^2 z^0} \nu^i. \]
The right hand side is less than zero by (4.7). From maximum principle,
\[ \zeta = \log H + \log z^0 \leq \sup_{M_0} \zeta = \sup_{M_0} (\log H + \log z^0). \]
Since for each \( t \in [0, T] \), \( M_t \) is convex, each principal curvature is less than the mean curvature, so we obtain the curvature bound (5.11).

**Lemma 6.2.** There exists a \( C^{1, \alpha} \) limiting hypersurface \( \partial M_F \) arising as the limit of \( \partial M_t \). The boundary \( \partial M_t \) is either an equator of the sphere \( \partial B_0 \) or is contained in an open hemisphere of \( \partial B_0 \).

**Proof.** The proof is the same with [LS16] Corollary 5 relying on [MS16]. \( \square \)

**Lemma 6.3.** Let \( M_t \) be a solution to the inverse mean curvature flow, if \( \partial M_F \) is positive distance away from the equator, then
\[ \sup_{[0, T] \times D} \frac{1}{H} \leq c, \]
where \( c \) depends only on \( M_0 \) and the distance of \( \partial M_F \) to the equator.

**Proof.** Let \( f(q) = -\log(\Lambda - q), q = \lambda z^1 + z^0 \). We require that \( 0 < \Lambda < \frac{1}{\cosh \rho_0} \) and \( \lambda < -1 - \frac{\cosh \rho_0}{\delta} \), where \( \delta \) is the number in Lemma 4.9. We have that \( f''(q) = \frac{1}{(\Lambda - q)^2}, \) \( f''(q) = \frac{1}{(\Lambda - q)^2} \).

Now
\[ (\partial_t - \frac{1}{H^2} \Delta) \log \frac{1}{H} = \frac{\nabla H^2}{H^2} + \left( -n + |A|^2 \right) \frac{1}{H^2}. \]

**Evolution of \( F \):**
\[ (\partial_t - \frac{1}{H^2} \Delta) F = -n \frac{\partial F}{\partial z^2} z^\alpha + \frac{2}{H} \frac{\partial F}{\partial z^2} \nu^\alpha - \frac{\partial^2 F}{H^2 \partial z^2 z^2} (\nabla z^\alpha, \nabla z^\beta). \]
Let \( F = f \circ q(z) \), note that
\[ \frac{\partial (\log) }{\partial z^2} = \frac{\partial }{\partial z^2} (f' \frac{\partial q}{\partial z^2}) = f'' \frac{\partial q}{\partial z^2} \frac{\partial q}{\partial z^2} + f' \frac{\partial^2 q}{\partial z^2 \partial z^2}. \]
So we have that
\[ (\partial_t - \frac{1}{H^2} \Delta) F = - \frac{1}{H^2} f' \frac{\partial q}{\partial z^2} z^\alpha + \frac{2}{H} f' \frac{\partial q}{\partial z^2} \nu^\alpha - \frac{1}{H^2} f' \frac{\partial^2 q}{\partial z^2 \partial z^2} (\nabla z^\alpha, \nabla z^\beta) - \frac{1}{H^2} f' f' \frac{\partial q}{\partial z^2} \frac{\partial q}{\partial z^2} (\nabla z^\alpha, \nabla z^\beta) - f'' \frac{\partial q}{\partial z^2} \frac{\partial q}{\partial z^2} \frac{\partial q}{\partial z^2} (\nabla z^\alpha, \nabla z^\beta). \]
Since \( q \) is linear in \( z^1 \) and \( z^0 \), we have that the evolution of \( \xi \)
\[ \xi := \log \frac{1}{H} + f \circ q(z) \]
is given by the following

\[ \partial_t - \frac{1}{H^2} \Delta \xi = \frac{|\nabla H|^2}{H^2} - \frac{1}{H^2} f''|\nabla q|^2 + (-n + |A|^2 - n f' q) \frac{1}{H^2} + \frac{2}{H} f' \frac{\partial q}{\partial z} \nu^\alpha. \]

Now we compute the boundary derivatives of \( \xi \):

\[ \nabla_n \xi = \coth \rho_0 + \coth \rho_0 f' \frac{\partial q}{\partial z} z^1 + f' \frac{\partial q}{\partial z} z^0 \tanh \rho_0 \]

\[ = \coth \rho_0 \left(1 + \frac{1}{\Lambda - q} z^1 + \frac{1}{\Lambda - q} z^0 \tanh \rho_0 \right) \]

\[ = \coth \rho_0 (\Lambda - z^0 + z^0 \tanh^2 \rho_0) \]

\[ = \coth \rho_0 (\Lambda - \frac{1}{\tanh \rho_0}) < 0; \]

where in the last line we have used along the boundary \( z^0 = \cosh \rho_0 \) and \( 1 - \tanh^2 \rho_0 = \frac{1}{\cosh^2 \rho_0} \).

Next we consider that the maximum of \( \xi \). From the condition \( \nabla_n \xi < 0 \), we have that

\[ \max_{(0,T) \times \mathbb{D}} \xi = \xi(t_0, \omega_0), \]

where \( \omega_0 \) is an interior point of \( \mathbb{D} \). Note that \( \partial_t \xi - \frac{1}{H^2} \Delta \xi \geq 0 \) at \((t_0, \omega_0)\), by (6.2), we have

\[ \frac{|\nabla H|^2}{H^2} - \frac{1}{H^2} f''|\nabla q|^2 + (-n + |A|^2 - n f' q) \frac{1}{H^2} + \frac{2}{H} f' \frac{\partial q}{\partial z} \nu^\alpha \]

\[ = \frac{1}{H^2} \left((f')^2 - f''\right)|\nabla q|^2 - \frac{n}{H^2} (1 + f' q) + \frac{|A|^2}{H^2} + \frac{2}{H} f' \frac{\partial q}{\partial z} \nu^\alpha \]

\[ = 0 - \frac{n}{H^2} (1 + f' q) + \frac{|A|^2}{H^2} + \frac{2}{H} f' \frac{\partial q}{\partial z} \nu^\alpha \]

\[ \geq 0, \]

where we have used the relation

\[ 0 = \nabla \xi = -\nabla H + f' \nabla q \text{ at } (t_0, \omega_0). \]

Note also that

\[ 1 + f' q = \frac{1}{\Lambda - q} > 0. \]

Now we have an inequality for \( \frac{1}{H^2} \) in the form

\[ -n(1 + f' q) \left(\frac{1}{H^2}\right)^2 + 2 \frac{f' \frac{\partial q}{\partial z} \nu^\alpha}{H^2} \left(\frac{1}{H^2}\right) + \frac{|A|^2}{H^2} \geq 0. \]

The coefficients on \( \frac{1}{H^2} \) are bounded, so this would imply a bound on \( \frac{1}{H^2} \).

\[ \square \]

**Theorem 6.4.** The strict convexity of \( M \) is preserved up to time \( T^* \) and \( \bar{T} = T^* \).

**Proof.** If \( \partial M_{\bar{T}} \) is the equator, we conclude from the Lemma 6.3 that \( M_{\bar{T}} \) is a totally geodesic disk and thus a singularity of the flow. This would yield \( \bar{T} = T^* \). If \( \partial M_{\bar{T}} \) is not the equator, then

\[ \frac{1}{H} \leq c \text{ for all } t \in (0, \bar{T}) \]

and by the estimate Lemma 4.10

\[ z^1 \geq \delta > 0. \]

We consider the \( \bar{H} \) and define

\[ \phi = \log \bar{H} - (n + 1) \log z^1 - \alpha t, t < \bar{T}, \]
where $\alpha$ will be chosen in dependence of $\delta$ and the initial data. Observe that from (3.9) and (3.12),
\[
\nabla_\eta \phi = \frac{1}{H} \nabla_\eta \tilde{H} - \frac{n+1}{z^2} \nabla_\eta z^1 \leq - \coth \rho_0 < 0,
\]
and for $0 < T < \bar{T}$, we have
\[
\sup_{[0,T] \times D} \phi = \phi(t_0, \xi_0), t_0 > 0.
\]
So $\xi_0$ is not at the boundary $\partial D$. At $(t_0, \xi_0)$, $\nabla \phi = 0$ implies that
\[
\nabla \tilde{H} = \nabla z^1 = \frac{1}{H} \nabla \tilde{H} - \frac{n+1}{z^2} \nabla z^1 z^1.
\]
From (3.7) and (3.11),
\[
\partial_t \phi - \frac{1}{H^2} \Delta \phi = - \frac{|A|^2}{H^2} + \frac{2n}{H^2} + \frac{n^2}{H^2} |\nabla \tilde{H}|^2 + \frac{1}{H^2} \left( \frac{1}{H^2} g_{ij} \tilde{h}^i \tilde{h}^j \nabla_k H \nabla_l H - g_{ij} \tilde{h}^i \tilde{h}^j \tilde{h}^l \nabla H \nabla \tilde{h} \right)
\]
\[
- \frac{2(n+1)}{H^2} |\nabla z^1|^2 - \alpha.
\]
The term in the big bracket is negative (see the proof of [Urb91, Lemma 3.8]), we have at $(t_0, \xi_0)$,
\[
0 \leq c + \frac{(n+1)^2}{H^2} |\nabla z^1|^2 - \alpha,
\]
where $c$ in the above depends on $\delta$ and the bound on $\frac{1}{H}$.
Choosing a sufficiently large $\alpha$, this is a contradiction. Then we obtain that under the assumption that $\partial M$ is not an equator, the supremum of $\phi$ would be decreasing, hence $\phi$ would be bounded up to $\bar{T}$. But then
\[
\log \tilde{H} = \phi + (n+1) \log z^1 + \alpha t \leq c + \alpha \bar{T},
\]
which contradicts the definition of $\bar{T}$ at which $\tilde{H}$ would have to blow up provided $\bar{T} < T^*$.

The estimates in this section are now sufficient to show that the inverse mean curvature flow converges to a totally geodesic disk. The proof follows the same lines in [LS16]. For completeness, we outline the proofs.

**Lemma 6.5.** If an embedded hypersurface $M \subset \mathbb{H}^{n+1}$ is mean convex with free boundary in $B_0$ such that $\partial M$ is an equator, then $M$ is totally geodesic.

**Proof.** If $\partial M$ is an equator, $\partial M$ determines a totally geodesic disk $P$ and a hyperbolic $n$-subspace with $P \subset \mathbb{H}^n$ which we denote by $\mathbb{H}^n$. We denote the center of the disk by $o$. Then $\mathbb{H}^{n+1}$ is a warped product $\mathbb{H}^n \times V \mathbb{R}$ with metric
\[
b = b' + \cosh^2 \text{dist}_{\mathbb{H}^n}(o, p') ds^2 =: b' + V^2 ds^2,
\]
where $(p', s) \in \mathbb{H}^n \times \mathbb{R}$ and $b'$ is the standard metric of $\mathbb{H}^n$. If a piece of $M$ is given by a function $u$, the mean curvature is then
\[
H = \text{div} \left( \frac{V Du}{\sqrt{1 + V^2 |Du|^2}} \right)
\]
with respect to the downward unit normal $\nu$. Here $D$ is the connection on $\mathbb{H}^n$.
Note that $H$ is translation invariant in the $s$-direction. Write
\[ L_s = \{ (p', s) \in \mathbb{H}^{n+1} : p' \in \mathbb{H}^n \} . \]
Let $s_0$ be the maximum of $s$ such that $M \cap L_s$ is nonempty. Since $\partial M \subset L_0$, $s_0 \geq 0$. Let $p \in L_s \cap M$, the tangent space of $T_pM$ and $T_pL_s$ is the same, if $s_0 > 0$, then by interior maximum principle of [Sch83, Lemma 1], there is a small neighborhood of $p$ in $M$ are also in $L_s \cap M$. By repeating this argument at points different from $p$, $L_s \cap M$ actually contains all the interior points of $M$. However, $\partial M \subset L_0$ and this would violate the continuity at $\partial M$. So $s_0 = 0$.

Now assume that $p$ is a point of $\partial M$. By the free boundary condition, we have that the two tangent space $T_pM$ and $T_pL_0$ agrees. We can invoke the boundary point lemma of [Sch83], we have that $L_0 \cap M$ contains a neighborhood of $p$ in $M$. Repeating the argument, we have that $M$ lies entirely in $L_0$ which says that $M$ is actually the geodesic disk $P$. \[ \square \]

**Remark 6.6.** Note the lemma only uses the mean-convexity of $M$. 

**Theorem 6.7.** At the maximal existence time $T^*$, $\partial M_{T^*}$ is an equator of the sphere $\partial B_0$ and $M_{T^*}$ is a totally geodesic disk. 

**Proof.** We assume that $\partial M_{T^*}$ is not an equator, then due to Lemma [6.3] we have that $\frac{1}{H} \geq c$ at $\partial M_{T^*}$. Let 
\[ F(x, u, \partial u, \partial^2 u) = -\frac{u}{H}, \]
so 
\[ \frac{\partial F}{\partial u_{ij}} = \frac{v}{H^2} \frac{\partial H}{\partial u_{ij}} = \frac{v}{H^2} (\frac{1}{v} g^{ij}) = \frac{1}{H^2} g^{ij}, \]
follows from [5.3]. The evolution is then uniformly parabolic and by regularity theory, we can extend a solution $u$ of [5.2] on $[0, T)$ where $T \in (0, T^*)$ to a solution $u$ of [5.2] on $[0, T + \varepsilon)$. The constant $\varepsilon > 0$ and depends only on the data. We can choose $T$ such that $T^* - T < \varepsilon$ and therefore we extend the solution beyond $T^*$. This contradicts the assumption $\partial M_{T^*}$ is not an equator.

Hence we have shown that $\partial M_{T^*}$ is an equator. It is easy to see that $M_{T^*}$ must be mean convex with a free boundary, by Lemma [6.5] we have that $M_{T^*}$ must be a totally geodesic disk. \[ \square \]

### 7. Willmore type inequality

We prove a version of Theorem 1.2 under the assumption that $M$ is strictly convex which is a hyperbolic analog of [LS17]. We basically follow their presentation adapting to the hyperbolic case and with a slight difference on proving the equality case of (1.2). See Lemma [6.5]. First, we control the $L^2$ norm of $H$.

**Lemma 7.1.** Let $M_t$ be the convex solution to (1.1). Then for all $1 \leq p < \infty$, there holds
\[ \lim_{t \to T^*} \int_{M_t} H^p(\cdot, t) = 0. \]

**Proof.** Since 
\[ -H(Dz^1, \nu) = \Delta z^1 - nz^1, \]
and Lemma \[\text{4.8}\] we have

\[0 \leq \int_{M_t} H(\cdot, t) \leq \frac{1}{c_0} \int_{M_t} \Delta z \cdot n \int_{M_t} z \rightarrow 0\]
as \(t \to T^*\) where the convergence follows from the fact that \(M_t\) converges to a totally geodesic disk. The result then follows from the boundness of \(H\) in \([6.1]\) and interpolation.

**Theorem 7.2.** We use the notations in Theorem \[\text{1.2}\]. Any strictly convex free boundary hypersurface \(M\) in \(B_0\) satisfies the Willmore inequality

\[|M| \frac{2-n}{n} \int_M (H^2 - n^2) + \Lambda |\partial M| > -n^2 \lambda_{n-1} + \Lambda \omega_{n-1} \sinh^{n-1} \rho_0.\]

**Proof.** We start by calculating the change rate of \(\int H^2 - n^2\), we have

\[
\begin{aligned}
\partial_t \int_M (H^2 - n^2) &\, \text{dv} \\
&= \int_M (H^2 - n^2) \, \text{dv} + \int_M 2H(-\Delta \frac{1}{H} - (-n + |A|^2) \frac{1}{H}) \, \text{dv} \\
&= \int_M (H^2 - n^2) \, \text{dv} - 2 \int_M H(\nabla \frac{1}{H}, \eta) \, \text{dv} + 2 \int_M (\nabla H, \nabla \frac{1}{H}) \, \text{dv} \\
&\quad - 2 \int_M (|A|^2 - n) \, \text{dv} \\
&\leq \int_M (H^2 - n^2) \, \text{dv} - 2 \coth \rho_0 \, |\partial M| - 2 \int_M (\frac{H^2}{n} - n) \, \text{dv} \\
&= \frac{n-2}{n} \int_M (H^2 - n^2) \, \text{dv} - 2 \coth \rho_0 \, |\partial M|.
\end{aligned}
\]

Now

\[\partial_t |\partial M| = \frac{H\partial M \cdot \partial B_0}{H} |\partial M| < |\partial M|,
\]

where the inequality is due to convexity of \(M\). The volume upper bound

\[|M| < \lambda = \omega_{n-1} \int_0^{\rho_0} \sinh^{n-1} \rho \, d\rho.
\]

Let \(q(t) = |M| \frac{2-n}{n} \int_M (H^2 - n^2) + \Lambda |\partial M|\), so

\[q'(t) < \Lambda |\partial M| - 2 \coth \rho_0 |M| \frac{2-n}{n} |\partial M|.
\]

Recall that \(\Lambda = 2 \coth \rho_0 \lambda \frac{2-n}{n}\), with this \(\Lambda\), we conclude that \(q'(t) < 0\). So \(q(t)\) is decreasing and

\[q(0) > q(T^*).
\]

Now we calculate the number

\[q(T^*) = -n^2 |M| \frac{2}{n} + \Lambda |\partial M| = -n^2 \lambda \frac{2}{n} + \omega_{n-1} \Lambda \sinh^{n-1} \rho_0.
\]

This concludes our proof. \(\square\)

We need the exact existence time for the inverse mean curvature flow.
Lemma 7.3. Suppose that $M_0$ is strictly convex. Then the maximal existence time $T^*$ is

$$T^* = \log \left( \frac{\lambda}{|M_0|} \right),$$

where $\lambda$ is given in Theorem [1.1].

Proof. Using that the evolution of $\frac{\partial}{\partial t}g_{ij} = 2H^{-1}h_{ij}$ under inverse mean curvature flow, we see that

$$\frac{d}{dt}|M_t| = |M_t|,$$

so

$$|M_t| = e^t|M_0|.$$  

Since we know from Theorem [1.1] that the flow converges to a totally geodesic disk in $C^{1,\alpha}$, we know that $\lambda = e^{T^*}|M_0|$, and the maximal existence time follows. \qed

We are using the mean curvature flow with free boundary to approximate a weakly convex free boundary hypersurface.

Lemma 7.4. Suppose that $F : \mathbb{D} \times [0,T) \to \mathbb{R}^{n+1}$ is a solution to the mean curvature flow

(7.2)

$$\begin{cases}
\frac{\partial}{\partial t}F = -Hu & \text{in } \mathbb{D} \\
\langle \nu, \eta \rangle = 0 & \text{on } \partial \mathbb{D},
\end{cases}$$

with initial hypersurface $M_0$ being weakly convex and perpendicular to the sphere from the inside. Then either $\partial M_0$ is an equator of the sphere or $h_{ij}$ is positive definite for $t > 0$.

Proof. Under the mean curvature flow,

$$\partial_t h_{ij} = \nabla_i \nabla_j H - H h^k_i h_{jk} - H g_{ij},$$

Using Simons identity (3.8) for $\Delta h_{ij}$, we have the evolution of the second fundamental form,

$$(\partial_t - \Delta)h_{ij} = nh_{ij} + |A|^2 h_{ij} - 2H h^k_i h_{jk} - 2H g_{ij}.$$  

If $\partial M_0$ is not an equator, then there exists a strictly convex point due to the same reasoning as [LS17] Lemma 3.1. Let

$$\chi(\xi, t) = \min_{|v| = 1} h_{ij}v^iv^j.$$  

Since $h_{ij}$ is smooth, $\chi$ is Lipschitz in space and by a simple cut-off argument, we can find a smooth function $\phi_0 : M \to \mathbb{R}$ so that $0 \leq \phi_0 \leq \chi(\xi, 0)$ and there exists $\xi' \in M$ so that $\phi_0(\xi') > 0$. We extend $\phi_0$ to $\phi : \mathbb{D}^n \times [0, \tau) \to \mathbb{R}$ by a heat flow,

$$\begin{cases}
\frac{\partial}{\partial t}\phi - \Delta g(t)\phi = 0 & \text{in } \text{int}(\mathbb{D}) \times [0, \tau) \\
\nabla_{\eta} \phi = 0 & \text{on } \partial \mathbb{D} \times [0, \tau) \\
\phi(\cdot, 0) = \phi_0,
\end{cases}$$

where $\Delta$ is the Laplace-Beltrami operator of the metrics $g(t)$ induced by the solution $F$ of (7.2). This is only a linear parabolic PDE so we can find a short time solution on $[0, \tau)$ for some small $\tau > 0$ by standard theory. By the strong maximum principle,

$$\phi(\xi, t) > 0 \text{ for } \xi \in \mathbb{D}, t > 0.$$
Let $M_{ij} = h_{ij} - \phi g_{ij}$, then
\[
(\partial_t - \Delta)M_{ij} = nh_{ij} + |A|^2 h_{ij} - 2H h^k_{ij} h_{jk} - 2H g_{ij} + 2\phi H h_{ij} =: N_{ij}.
\]
Let $v$ be a unit vector such that $M_{ij}v^j = h_{ij}v^j - \phi g_{ij}v^j = 0$.

Then tensor $N_{ij}$ applies to $v \otimes v$ is
\[
N_{ij}v^i v^j = n\phi + |A|^2 \phi - 2H\phi \geq 0,
\]
where the inequality $n + |A|^2 \geq 2H$ is simply proved by diagonalizing the matrix $h_{ij}$. We have verified that evolution of $M_{ij}$ satisfies the null eigenvector condition.

Now we wish to apply Stahl \cite{Sta96a, Sta96b} to conclude that $M_{ij} \geq 0$ on $(0, \delta)$. Because $\phi > 0$ for $t > 0$, $h_{ij} > 0$ for $0 < t < \tau$ and we finish our proof by applying \cite{Sta96a} Proposition 4.5 to the flow $F(x, t - \frac{\tau}{2})$. The rest of the proof is devoted to the assertion that $M_{ij} \geq 0$.

We use Stahl’s notation for comparability, for $p \in \partial B_0$, write $\mu \in T_p M$ for the outward pointing normal to $\partial B_0$ and some basis tangent vectors $\partial I$ of $\partial M$ so that $(\mu, \partial I)$ with $2 \leq I \leq n$ induces some coordinates near $p$. Now we show that the conditions in \cite{Sta96b} Lemma 3.4 hold.

Observe that Lemma 3.2 holds for any hypersurface perpendicular to the geodesic sphere $\partial B_0$, this implies that
\[
(7.3) \quad \nabla_{\mu} M_{ij} = \coth \rho_0 (h_{\mu\mu} \delta_{IJ} - h_{IJ}).
\]
Replacing the inverse mean curvature flow of Lemma 3.3 by mean curvature flow, we have that $\nabla_{\mu} h_{\mu\mu} = \coth \rho_0 (2H - nh_{\mu\mu})$. See also \cite{Sta96a} Theorem 4.3. So
\[
\nabla_{\mu} M_{\mu\mu} = \coth \rho_0 (2H - nh_{\mu\mu}).
\]
Suppose first that $V \in T_p \partial M$ is a minimal eigenvector with eigenvalue $\lambda \in (-\delta, 0]$, that is
\[
M_{ij}V^i = \lambda g_{ij} V^i.
\]
We see also that $V$ is also a minimal eigenvector of $h_{ij}$, and therefore $h_{ij} V^i V^j \leq h_{\mu\mu}$. So \cite{Sta96b} (7.3) implies that
\[
\nabla_{\mu} M_{IJ} V^I V^J \geq 0.
\]
Now suppose that $\mu$ is a minimal eigenvector with eigenvalue $\lambda \in (-\delta, 0]$. Again minimality of $\mu$ implies that for all $W \in T_p \partial M$, $h_{ij} W^i W^j \geq h_{\mu\mu}$, in particular, $H \geq nh_{\mu\mu}$ and so
\[
\nabla_{\mu} M_{\mu\mu} \geq H \geq 0.
\]
Now we apply the tensor maximum principle (Theorem 3.3 and Lemma 3.4 of \cite{Sta96b}) to conclude that $M_{ij} \geq 0$. \hfill \qed

**Corollary 5.** Suppose that $M$ is weakly convex free boundary hypersurface and $\partial M$ is not an equator. Then there exists an $\varepsilon > 0$ such for $0 < t < \varepsilon$ there are smooth and strictly convex free boundary hypersurfaces that satisfy
\[
\int_{M_t} H^2 \rightarrow \int_M H^2, |M_t| \rightarrow |M|, |\partial M_t| \rightarrow |\partial M|
\]
as $t \rightarrow 0$. 

Proof. It follows from \cite[Theorem 2.1]{Sta96b} and regularity at initial time $t = 0$ of (7.2).

We now prove Theorem 1.2.

Proof of Theorem 1.2. If $\partial M$ is an equator, then $M$ has to be totally geodesic by Lemma 6.5 and hence the Willmore inequality (1.2) is valid for $M$ with equality.

Due to Theorem 7.2 and 7.4 the Willmore inequality (1.2) holds for weakly convex hypersurface $M$ with non-equatorial $\partial M$.

We characterize the equality case of (1.2). If $\partial M$ is an equator, by previous argument, $M$ has to be totally geodesic. It follows the same lines from \cite[Lemma 3.4]{LS17} that

\begin{equation}
\tag{7.4}
|M| \leq |C_M| < \lambda
\end{equation}

for some constant $C_M > 0$ if we assume $\partial M$ is not an equator.

Due to Lemma 7.4 for every $\varepsilon > 0$ there exists a strictly convex hypersurface perpendicular to the sphere $\partial B_0$ from the inside $M^\varepsilon$ such that

$q(M^\varepsilon) \leq q(M) + \varepsilon$.

We know that the maximal existence time

\[ T^*_\varepsilon = \log \left( \frac{1}{|M^\varepsilon|} \right). \]

By (7.1), the quantity $q^\varepsilon(t)$ satisfy

\[ \frac{d}{dt} q^\varepsilon(t) < |\partial M^\varepsilon| \left( \lambda - 2 \coth \rho_0 |M^\varepsilon|^{\frac{2-n}{n}} \right) \]

\[ = 2 \coth \rho_0 \lambda \frac{2-n}{n} |\partial M^\varepsilon| \left( 1 - e^{-\frac{2-n}{n}(T^*_\varepsilon - t)} \right). \]

Due to (7.4) and Corollary 5 there exists a positive time $T$ which only depends on $|M|$ and is independent of $\varepsilon$ such that

\[ T^*_\varepsilon \geq 2T > 0. \]

Hence for all $\varepsilon$ and all $0 \leq t \leq T$ there holds

\[ \frac{d}{dt} q^\varepsilon(t) \leq c(1 - e^{-\frac{2-n}{n}T}) =: -c, \]

where $c$ only depends on $n$, $|M|$ and $|\partial M|$. Using the strict convexity of $M^\varepsilon$ and Lemma 7.2, we obtain that

\[ q(P) < q^\varepsilon(T) = q(M^\varepsilon) + \int_0^T \frac{d}{ds} q^\varepsilon(s) ds \]

\[ \leq q(M) + \varepsilon - cT \]

\[ = q(P) + \varepsilon - cT. \]

Choosing a sufficiently small $\varepsilon$, we obtain a contradiction and complete the proof. \[\Box\]
References

[And94] Ben Andrews. Contraction of convex hypersurfaces in Riemannian spaces. J. Differential Geom., 39(2):407–431, 1994.

[BP92] Riccardo Benedetti and Carlo Petronio. Lectures on Hyperbolic Geometry. Universitext. Springer Berlin Heidelberg, Berlin, Heidelberg, 1992.

[CH03] Piotr T. Chruściel and Marc Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. Pacific Journal of Mathematics, 212(2):231–264, 2003.

[FIN13] Ailana Fraser and Richard Schoen. Uniqueness theorems for free boundary minimal disks in space forms. Int. Math. Res. Not. IMRN, (17):8268–8274, 2015.

[Ger06] Claus Gerhardt. Minkowski type problems for convex hypersurfaces in hyperbolic space. ArXiv:math/0602597, 2006.

[Ger11] Claus Gerhardt. Inverse curvature flows in hyperbolic space. Journal of Differential Geometry, 89(3):487–527, 2011.

[Haw68] S. W. Hawking. Gravitational Radiation in an Expanding Universe. Journal of Mathematical Physics, 9(4):598–604, 1968.

[HI01] Gerhard Huisken and Tom Ilmanen. The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality. Journal of Differential Geometry, 59(3):353–437, 2001.

[Kat92] Svetlana Katok. Fuchsian Groups. University of Chicago Press, 1992.

[LS16] Ben Lambert and Julian Scheuer. The inverse mean curvature flow perpendicular to the sphere. Mathematische Annalen, 364(3-4):1069–1093, apr 2016.

[LS17] Ben Lambert and Julian Scheuer. A geometric inequality for convex free boundary hypersurfaces in the unit ball. Proc. Amer. Math. Soc., 145(9):4009–4020, 2017.

[Mar17] Thomas Marquardt. Weak solutions of inverse mean curvature flow for hypersurfaces with boundary. Journal für die reine und angewandte Mathematik (Crelles Journal), 2017(728):237–261, 2017.

[MS16] Matthias Makowski and Julian Scheuer. Rigidity results, inverse curvature flows and Alexandrov-Fenchel type inequalities in the sphere. Asian Journal of Mathematics, 20(5):869–892, 2016.

[Sch83] Richard M. Schoen. Uniqueness, symmetry, and embeddedness of minimal surfaces. Journal of Differential Geometry, 18(4):791–809, 1983.

[Sta96a] Axel Stahl. Convergence of solutions to the mean curvature flow with a Neumann boundary condition. Calc. Var. Partial Differential Equations, 4(5):421–441, 1996.

[Sta96b] Axel Stahl. Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition. Calc. Var. Partial Differential Equations, 4(4):385–407, 1996.

[SWX18] Julian Scheuer, Guofang Wang, and Chao Xia. Alexandrov-Fenchel inequalities for convex hypersurfaces with free boundary in a ball. ArXiv:1811.05776 [math], 2018. ArXiv: 1811.05776.

[Urb90] John I. E. Urbas. On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z., 205(3):355–372, 1990.

[Urb91] John I. E. Urbas. An expansion of convex hypersurfaces. J. Differential Geom., 33(1):91–125, 1991.

[Vol16] Alexander Volkmann. A monotonicity formula for free boundary surfaces with respect to the unit ball. Comm. Anal. Geom., 24(1):195–221, 2016.

[WX20] Guofang Wang and Chao Xia. Guan-Li type mean curvature flow for free boundary hypersurfaces in a ball. ArXiv:1910.07253 [math], 2020. ArXiv: 1910.07253.
INVERSE MEAN CURVATURE FLOW WITH A FREE BOUNDARY IN HYPERBOLIC SPACE

Korea Institute for Advanced Study, Seoul 02455, South Korea
Email address: xxchai@kias.re.kr