Slopes of 2-adic overconvergent modular forms with small level

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1 Abstract

Let \( \tau \) be the primitive Dirichlet character of conductor 4, let \( \chi \) be the primitive even Dirichlet character of conductor 8 and let \( k \) be an integer. Then the \( U_2 \) operator acting on cuspidal overconvergent modular forms of weight \( 2k + 1 \) and character \( \tau \) has slopes in the arithmetic progression \( \{ 2, 4, \ldots, 2n, \ldots \} \), and the \( U_2 \) operator acting on cuspidal overconvergent modular forms of weight \( k \) and character \( \chi \cdot \tau^k \) has slopes in the arithmetic progression \( \{ 1, 2, \ldots, n, \ldots \} \).

We then show that the characteristic polynomials of the Hecke operators \( U_2 \) and \( T_p \) acting on the space of classical cusp forms of weight \( k \) and character either \( \tau \) or \( \chi \cdot \tau^k \) split completely over \( \mathbb{Q}_2 \).

2 Introduction

Definition 1 Let \( f \) be a cuspidal modular eigenform with \( q \)-expansion at \( \infty \) given by \( \sum_{n=1}^{\infty} a_n q^n \). Let \( f \) be normalised; that is, \( a_1 = 1 \). The \( (p) \)-slope of \( f \) is defined to be the \( p \)-valuation of \( a_p \); we normalise the \( p \)-valuation of \( p \) to be 1. If we do not specify \( p \), then we mean the 2-slope.

In this paper, we prove the following theorem on the slopes of classical modular cusp forms:

Theorem 2 Let \( \tau \) be the nontrivial character of conductor 4, and let \( k \) be an integer greater than 2. The slopes of the \( U_2 \) operator acting on \( S_{2k-1}(\Gamma_0(4), \tau) \) are

\[ 2, 4, 6, \ldots, 2k - 4. \]

Let \( \chi \) be the even primitive Dirichlet character of conductor 8. The slopes of the \( U_2 \) operator acting on \( S_k(\Gamma_0(8), \chi \cdot \tau^k) \) are

\[ 1, 2, 3, \ldots, k - 2. \]

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As a corollary of this theorem, we also prove the following result about the field over which cusp forms of weight $k$ and character $\chi \cdot \tau^k$ or $\tau$ are defined:

**Corollary 3** Let $k$ be an integer greater than 2, and let $S$ be either $S_{2k-1}(\Gamma_0(4), \tau)$ or $S_{k}(\Gamma_0(8), \chi \cdot \tau^k)$.

Let $f \in S$ be a normalised eigenform. Then the coefficients of the Fourier expansion of $f$ are elements of $\mathbb{Q}_2$.

This corollary gives a partial answer to an extension of Questions 4.3 and 4.4 of Buzzard [2], which give a conjectural bound on the degree of the field of definition of certain spaces of modular forms over $\mathbb{Q}_2$.

### 3 Previous work

Matthew Emerton determines in his thesis [11] the smallest slope for the spaces of modular cuspforms $S_k(\Gamma_0(2^n), \chi)$, where $\chi$ is a primitive Dirichlet character of conductor $2^n$.

**Theorem 4** (Emerton [11], Proposition 5.1) Let $m$ be a positive integer greater than 1, and let $\chi$ be a primitive Dirichlet character of conductor $2^m$.

The smallest slope of the $U_2$ operator acting on cuspforms of weight $k$ and character $\chi$ is $2^{3-m}$.

If we look at the character of conductor 4 and the odd character of conductor 8, there is a CM modular form which is defined over the field $\mathbb{Q}$. We quote a result of Schoeneberg, proved in Ogg [16]:

**Theorem 5** (Ogg [16], Theorem VI.22) Let $i$ be the square root of $-1$, and let $k$ be a positive odd integer greater than 3 and congruent to 1 mod 4. Then there is a normalised cuspidal modular eigenform in $S_k(\Gamma_0(4), \tau)$ with $q$-expansion

$$f_k(q) = \frac{1}{4} \sum_{m,n \in \mathbb{Z}} (m + n \cdot i)^{k-1} \cdot q^{m^2 + n^2}.$$ 

Let $l$ be a positive odd integer greater than 1. Then there is a normalised cuspidal modular eigenform in $S_l(\Gamma_0(8), \tau \cdot \chi)$ with $q$-expansion

$$g_l(q) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}} (m + 2n \cdot i)^{l-1} \cdot q^{m^2 + 2n^2}.$$ 

We see by inspection that the slope of $f_k$ is $(k - 1)/2$, and that the slope of $g_l$ is $l - 1$.

Hence we can, in certain cases, determine the smallest slope and another classical slope of $U_2$ acting on modular newforms of level $\Gamma_1(4)$ or $\Gamma_1(8)$ using previously known results.

Lawren Smithline has also proved results about the slopes of classical modular forms, and the techniques used in [19] are similar to those in this paper.
Theorem 6 (Smithline [19], Corollary 6.1.3.3) Let $v$ be a non-negative integer and let $k = 2 \cdot 3^{v+1}$. Then there are exactly $3^v$ classical modular eigenforms of weight $k$ and level 3 with 3-slope $3^{v+1} - 1$.

Buzzard and Calegari [4] have proved the following theorem on the slopes of modular forms:

Theorem 7 (Buzzard-Calegari [4], Corollary 1) The slopes of the $U_2$ operator on the space of cusp forms of weight 0 are given by

$$1 + 2v_2\left(\frac{(3n)!}{n!}\right).$$

Jacobs [13] has proved, using similar techniques to those in this paper, that the slopes of the operator $U_3$ acting on the space of automorphic forms over a definite quaternion algebra are in the arithmetic progression $\{1/2, 3/2, 5/2, \ldots\}$.

4 Overconvergent modular forms

A famous quote of Jacques Hadamard [12] says that “the shortest and best way between two truths of the real domain often passes through the imaginary one.” It seems that often the best way to prove results like Theorem 2 about classical modular forms is to prove a theorem for the overconvergent modular forms and then derive the theorem for classical modular forms as a consequence.

We therefore recall the definition of the 2-adic overconvergent modular forms, first by defining overconvergent modular forms of weight 0, and then by deriving the definition for forms with weight and character.

Following Katz [14], section 2.1, we recall that, for $E$ be an elliptic curve over an $\mathbb{F}_2$-algebra $R$, there is a mod 2 modular form $A(E)$ called the Hasse invariant, which has the $q$-expansion over $\mathbb{F}_2$ equal to 1.

We consider the Eisenstein series of weight 4 and tame level 1 defined over $\mathbb{Z}$, with $q$-expansion

$$E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \left( \sum_{0<d|n} d^3 \right) \cdot q^n.$$

We see that $E_4$ is a lifting of $A(E)^4$ to characteristic 0, as the reduction of $E_4$ to characteristic 2 has the same $q$-expansion as $A(E)^4$, and therefore $E_4$ mod 2 and $A(E)^4$ are both modular forms of level 1 and weight 4 defined over $\mathbb{F}_2$, with the same $q$-expansion.

Now the value of $E_4(E)$ may not be well-defined, but it can be shown that the valuation $v_2(E_4(E))$ is well-defined. This will allow us to define the ordinary locus of $X_0(2^m)$ and certain neighbourhoods of it. We follow the work of Coleman [9], and first define structures on the modular curve $X_1(2^m)$. 
Definition 8 (Coleman [9], page 448) Consider $X_1(2^m)/\mathbb{Q}_2$ as a rigid analytic space, and let $t$ be a point of $X_1(2^m)$.

If $t$ is a point of $X_1(2^m)$ which corresponds to a cusp, then we define $v(E_4(t)) = 0$, following [8], section 4.

We define the ordinary locus of $X_1(2^m)$ to be the set of points $t$ of $X_1(2^m)$ such that $v_2(E_4(t)) = 0$, and define $Z_1(2^m)$ to be the rigid connected component of the ordinary locus in $X_1(2^m)$ which contains the cusp $\infty$. This is a rigid analytic space.

In [10], page 36, it is shown that $Z_1(2^m)$ is an affinoid subdomain of the rigid space $X_1(2^m)/\mathbb{Q}_2$.

We will perform calculations in later sections on the modular curve $X_0(2^m)$, which we will now define.

Definition 9 Consider $X_1(2^m)$ as a modular curve. We see that the group $G := (\mathbb{Z}/2^m\mathbb{Z})^\times$ acts upon the non-cuspidal points of $X_1(2^m)$, by the following action: if $a \in (\mathbb{Z}/2^m\mathbb{Z})^\times$, then the action of $a$ sends the pair $(E, P)$ to $(E, aP)$. This action extends to the cuspidal points of $X_1(2^m)$, and it sends cusps to cusps.

We will define the modular curve $X_0(2^m)/\mathbb{Q}_2$ to be the quotient of $X_1(2^m)$ by $(\mathbb{Z}/2^m\mathbb{Z})^\times$.

We note that the action of the group $G$ does not change the valuation of a given elliptic curve $E$. We define $Z_0(2^m)$ to be the rigid connected component of the ordinary locus in $X_0(2^m)$ which contains the cusp $\infty$. It is a rigid analytic space.

We will now define strict affinoid neighbourhoods of $Z_0(2^m)$.

Definition 10 (Coleman [9], Section B2) We think of $X_0(2^m)$ as a rigid space over $\mathbb{Q}_2$, and we let $t \in X_0(2^m)(\overline{\mathbb{Q}_2})$ be a point, corresponding either to an elliptic curve defined over a finite extension of $\mathbb{Q}_2$, or to a cusp. Let $w$ be a rational number, such that $0 < w < \min(2^{-m}/3, 1/4)$.

We define $Z_0(2^m)(w)$ to be the connected component of the affinoid

\[ \{ t \in X_0(2^m) : v_2(E_4(t)) \leq 4w \} \]

which contains the cusp $\infty$.

The condition involves $4w$ rather than $w$ because we are working with a lifting of the fourth power of the Hasse invariant. Also, note that as $E_4$ is a lifting of the mod 2 modular form $A^4$, and that any another lifting of $A^4$ would be of the form $E_4 + 2F$, where $F$ is a modular form, then this valuation is well-defined if $0 \leq v(E_4(t)) < 1$. This corresponds to the condition $0 \leq w < 1/4$ in Definition 10.

We can now define overconvergent modular forms of weight 0.

Definition 11 (Coleman, [8], page 397) Let $w$ be a rational number, such that $0 < w < \min(2^{-m}/3, 1/4)$. Let $\mathcal{O}$ be the structure sheaf of $Z_0(2^m)(w)$.

We call sections of $\mathcal{O}$ on $Z_0(2^m)(w)$ $w$-overconvergent 2-adic modular forms of weight 0 and level $\Gamma_0(2^m)$. 
If a section \( f \) of \( \mathcal{O} \) is a \( w \)-overconvergent modular form, then we say that \( f \) is an overconvergent 2-adic modular form.

Let \( K \) be a complete subfield of \( \mathbb{C} \), and define \( Z_0(2^m)(w)/K \) to be the affinoid over \( K \) induced from \( Z_0(2^m)(w) \) by base change from \( \mathbb{Q}_2 \). The space \( M_0(2^m, w; K) := \mathcal{O}(Z_0(2^m)(w)/K) \) of \( w \)-overconvergent modular forms of weight 0 and level \( \Gamma_0(2^m) \) is a \( K \)-Banach space.

We now use non-cuspidal modular forms of the desired weight and character to define overconvergent modular forms with non-zero weight.

**Definition 12** Let \( w \) be a real number such that \( 0 < w < \min(2^{2-m}/3, 1/4) \). Let \( k \) be an integer and let \( \chi \) be a character such that \( \chi(-1) = (-1)^k \), and let \( K \) be a complete subfield of \( \mathbb{C} \). Let \( E_{k, \chi} \) be the Eisenstein series of weight \( k \) and character \( \chi \).

The space of overconvergent 2-adic modular forms of weight \( k \) and character \( \chi \) is given by

\[
\mathcal{M}_{k, \chi}(2^m, w; K) := E_{k, \chi}^* \cdot M_0(2^m, w; K).
\]

This is a Banach space over \( K \).

There are continuous Hecke operators \( T_p \) which act on \( \mathcal{M}_{k, \chi}(2^m, w; K) \); the operator \( U := U_2 \), which is defined on \( q \)-expansions as

\[
U \left( \sum_{n=0}^{\infty} a_n q^n \right) = \sum_{n=0}^{\infty} a_{2n} q^n
\]

is also compact and therefore has a spectral theory.

As a consequence of results of Coleman, we have the following theorem:

**Theorem 13 (Coleman) [9], Theorem B3.2** Let \( w \) be a real number such that \( 0 < w < \min(2^{2-m}/3, 1/4) \), let \( k \) be an integer and let \( \chi \) be a character such that \( \chi(-1) = (-1)^k \).

The characteristic polynomial of \( U_2 \) acting on overconvergent 2-adic modular forms of weight \( k \) and character \( \chi \) is independent of the choice of \( w \).

This theorem allows us to choose a convenient value of \( w \) and prove results for that \( w \), and guarantees that these results will hold for any \( w \).

The connected component in Definition 10 is hard to work with. We will therefore rewrite it in terms of modular functions of level greater than 1, to prove the following theorem:

**Theorem 14** The spaces of overconvergent modular forms of weight 0 and level \( N = 4 \) or 8 are Tate algebras in one variable over \( \mathbb{Q}_2(2^{4/N}) \).

We have given a valuation on the points \( t \) of the rigid space \( X_0(2^m) \), based on the lifting of the Eisenstein series \( E_4 \). We recall that the modular \( j \)-invariant
is defined to be \( j = E_4^3 / \Delta \). Therefore, we see that, if the elliptic curve corresponding to \( t \) has good reduction, then \( \Delta(t) \) has valuation 0, and therefore that

\[
v(t) = \frac{1}{4} v(E_4(t)) = \frac{1}{12} v((E_4)^3(t)) = \frac{1}{12} v(j(t)).
\]

From Lemma 2.3 of [11], we see that there is a modular function \( j_8 \) which is a uniformiser on \( X_0(8) \). It has \( q \)-expansion at \( \infty \)

\[
j_8 = \frac{1}{q \prod_{n=1}^{\infty} (1 + q^n)(1 + q^{2n})(1 + q^{4n})^4} = \left( \frac{\Delta(q)\Delta(q^4)}{\Delta(q^2)\Delta(q^8)} \right)^{1/12}.
\]

Also, \( j_8(\infty) = \infty \).

There is another modular function \( j_{16} \) which is a uniformiser on \( X_0(16) \), with \( q \)-expansion at \( \infty \) given by

\[
j_{16} = \frac{1}{q \prod_{n=1}^{\infty} (1 + q^n)(1 + q^{2n})(1 + q^{4n})(1 + q^{8n})^2} = \left( \frac{\Delta(q^{16})\Delta(q^2)}{\Delta(q^8)\Delta(q^4)} \right)^{1/24}.
\]

We see also that \( j_{16}(\infty) = \infty \).

By an explicit calculation of \( q \)-expansions, using the formulae in Chapter 2 of [11], we see that

\[
j = \frac{(j_8^4 + 256j_8^3 + 5120j_8^2 + 32768j_8 + 65536)^3}{(j_8^2 + 16j_8 + 64) \cdot (j_8 + 4)}
\]

and

\[
\frac{1}{j_8} = \frac{1}{j_{16}} + \frac{2}{j_{16}^2}.
\]

Because we know that \( j_8(\infty) = \infty \), the connected component of \( Z_0(8) \) which contains \( \infty \) is of the form \( v(j_8) < D \) for some rational number \( D \). We see that, if \( v(j_8) < 2 \), then \( v(j_8) = v(j) \). This means that we have shown that

\[
Z_0(8)(w) = \{ x \in X_0(8) : v(j_8(x)) \leq 12w \} \text{ for } 0 < w < 1/4.
\]

Similarly, we see that the connected component of \( Z_0(16) \) which contains \( \infty \) is of the form \( v(j_{16}) < D \) for some rational number \( D \). We see that, if \( v(j_{16}) < 1 \), then \( v(j_{16}) = v(j_{16}) \), and therefore \( v(j_{16}) = v(j) \). This means that we have shown that

\[
Z_0(16)(w) = \{ x \in X_0(8) : v(j_{16}(x)) \leq 12w \} \text{, for } 0 < w < 1/6.
\]

We now define another modular function on \( X_0(2^n) \), in terms of Eisenstein series.

Let \( k \) be an integer, and let \( \theta : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times \) be a primitive Dirichlet character, such that \( \theta(-1) = (-1)^k \). Recall from Washington [21], page 30, that the extended Bernoulli numbers \( B_{k,\theta} \) are defined by

\[
\sum_{a=1}^{N} \frac{\theta(a) \cdot t \cdot \exp(at)}{\exp(Nt) - 1} = \sum_{k=0}^{\infty} B_{k,\theta} \cdot \frac{t^i}{i!}
\]
We define the Eisenstein series $E^*_{k,\theta}$ to be
\[
E^*_{k,\theta} := \frac{-B_{k,\theta}}{2k} + \sum_{n=1}^{\infty} \left( \sum_{\substack{0 < d | n \\ (d, p) = 1}} \theta(d) \cdot d^{k-1} \right) \cdot q^n,
\]
where $B_{k,\theta}$ is the extended Bernoulli number attached to $k$ and $\theta$.

There is an operator $V$ on the space of modular forms $S_k(\Gamma_0(N))$; its effect on $q$-expansions is to send $q$ to $q^2$. We define $V^*_{k}$ to be $V(E^*_{k})$.

We define modular functions on $X_0(8)$ and $X_0(16)$ by
\[
z_4 := \frac{E^*_{1,c}/V^*_{1,c} - 1}{2} = \frac{2}{j_8 + 2},
\]
and
\[
z_8 := \frac{E^*_{1,\chi r}/V^*_{1,\chi r} - 1}{\sqrt{2}} = \frac{\sqrt{2}}{j_{16} + 2},
\]
where we choose and fix a square root of 2 in $\mathbb{C}_2$. These identities can be verified by explicit calculation.

Let $w$ be a rational number such that $0 < w < 1/6$. Then using the formulae above, we see that
\[
Z_0(8)(w) = \{ x \in X_0(8) : v(z_4(x)) \geq 1 - 12w \}.
\]
We now choose $w = 1/12$, to obtain
\[
Z_0(8)(1/12) = \{ x \in X_0(8) : v(z_4(x)) \geq 0 \}.
\]
Now, the rigid functions on the closed disc over $\mathbb{Q}_2$ with centre 0 and radius 1 are defined to be power series of the form
\[
\sum_{n \in \mathbb{N}} a_n z^n : a_n \in \mathbb{Q}_2, \ a_n \to 0.
\]
Therefore, the $1/12$-overconvergent modular forms of level $\Gamma_0(4)$ and weight 0 are
\[
\mathbb{Q}_2(z_4),
\]
which is what we wanted to show. We now follow the same procedure for $X_0(16)$.

Let $w$ be a rational number such that $0 < w < 1/12$. Then using the formulae above, we see that
\[
Z_0(16)(w) = \{ x \in X_0(16) : v(z_8(x)) \geq 1/2 - 12w \}.
\]
We now choose $w = 1/24$, to obtain
\[
Z_0(16)(1/24) = \{ x \in X_0(16) : v(z_8(x)) \geq 0 \}.
\]
Now, the rigid functions on the closed disc over $\mathbb{Q}_2$ with centre 0 and radius 1 are defined to be power series of the form
\[
\sum_{n \in \mathbb{N}} a_n z^n : a_n \in \mathbb{Q}_2(\sqrt{2}), \ a_n \to 0.
\]

Therefore, the $1/24$-overconvergent modular forms of level $\Gamma_0(8)$ and weight 0 are
\[
\mathbb{Q}_2(\sqrt{2})(z_8),
\]
so we have shown that these spaces of modular forms are Tate algebras in one variable.

We now show that the odd powers of $z_N$ are sent to 0 under the $U_2$ operator.

**Lemma 15** Let $N = 4$ or 8 and let $i$ be a positive integer. Then
\[
U_2(z_2^{2i+1}) = 0, \text{ and } U(z_2^{2i}) = (U(z_2^2))^i.
\]

Recall that the Eisenstein series $E_k^*$ is an eigenvector with eigenvalue 1 for $U_2$, and that $U_2(V(E_k^*)) = E_k^*$. Then we see that we have (for $\mu = 2$ or $\sqrt{2}$):
\[
U_2(z) = U_2\left(\frac{E_k^*/V_k^* - 1}{\mu}\right) = \frac{1}{\mu} \cdot U_2\left(\frac{E_k^* - V_k^*}{V_k^*}\right)
\]
\[
= \frac{1}{\mu E_k^*} \cdot U_2(E_k^* - V_k^*) = \frac{1}{\mu E_k^*} \cdot (E_k^* - E_k^*) = 0.
\]

Hence we see that $z_N$ has only odd $q$-coefficients, and that therefore $z_N^2$ has only even $q$-coefficients.

Therefore $z_N^{2i+1}$ has only odd $q$-coefficients. Hence for all non-negative integers $t$, we see that
\[
U_2(z_N^{2i+1}) = 0.
\]

Because we have just shown that $z_N$ has only odd $q$-coefficients, we see that
\[
z_N = qF(q^2) = qV(F(q)),
\]
for some power series $F(q)$.

Therefore we have
\[
U_2(z_N^2) = U_2(q^{2i}V(F(q)^{2i})) = U_2(V(q^iF(q)^{2i})),
\]
and hence we see that
\[
U_2(z_N^2) = q^iF(q)^{2i} = (qF(q)^2)^i = U_2(z_N^2)^i,
\]
which proves the Lemma.

Because we have written down the overconvergent modular forms as an explicit Banach space, we can write down its Banach basis: the set \{ $z_4, z_4^2, z_4^3, \ldots$ \}.
forms a Banach basis for the overconvergent modular forms of weight 0 and level \( \Gamma_0(4) \) and the functions \( \{ z_8, z_2^8, z_3^8, \ldots \} \) form a Banach basis for the overconvergent modular forms of weight 0 and level \( \Gamma_0(8) \).

These Banach bases are composed of weight 0 modular functions — we want to be able to consider the action of the \( U \) operator on overconvergent modular forms with non-zero weight-character \( k \). Using an observation from the work of Coleman, we will be able to move between weight-character 0 and weight-character \( k \) via multiplication by a suitable quotient of modular forms.

From the discussion in Coleman [9], page 450, we see that the \( U^2 \) operator acting on overconvergent modular forms of weight-character \( (k \cdot t, \theta_t) \) where \( t \) is odd has the same characteristic power series as the composition of the \( U \) operator acting on overconvergent modular forms of weight-character 0 with multiplication by the \( z \)-expansion of \( (E^*_k,\theta/V^*_k,\theta) \).

For level 4 and weight \( 2t + 1 \) and level 8 and weight either of the form \( 2t + 1 \) or \( 4k + 2 \), we can write the weight \( k \) multiplier as a power of \( 1 + 2z^4 \) or \( 1 + \sqrt{2}z^8 \).

If \( N = 8 \) and the weight \( k \) divides 4, then we can write \( E^*_k,\theta/V^*_k,\theta \) as a rational function of \( z_8 \), because \( z_8 \) is a rational function of the uniformiser \( j_{16} \) of the genus 0 modular curve \( X_0(16) \). For instance, we see that

\[
\frac{E^*_4,\chi}{V^*_4,\chi} = \frac{11 + 2z_8 + 24z_8^2 - 48z_8^3 - 16z_8^4 - 352z_8^5}{11 + 24z_8^2 - 16z_8^4}.
\]

We can consider the action of the \( U^2 \) operator on these spaces of overconvergent modular forms.

**Definition 16** Let \( M = (m_{i,j}) \) be the infinite compact matrix of the operator \( U^2 \circ (E^*_k/V^*_k)^t \) acting on the Banach basis \( \{ z_N, z_2^N, \ldots \} \), where \( m_{i,j} \) is defined to be the coefficient of \( z_N^j \) in the \( z_N \)-expansion of \( U^2(z_N^i) \cdot (E^*_k/V^*_k)^t \).

Because we know that \( \mathcal{M}_0 \) is a Banach space, we can show that the matrix \( M \) is a compact matrix; in other words, the trace, determinant and characteristic power series of \( M \) are all well-defined.

We will use a theorem of Serre to prove our theorem on the slopes of \( U \) acting on \( \mathcal{M}_k \).

**Theorem 17 (Serre [18], Proposition 7)** 1. Let \( M \) be an \( n \times n \) matrix defined over a finite extension of \( \mathbb{Q}_2 \), and let \( 0 \neq r \in \mathbb{Q} \). Let \( \det(1-tM) = \sum_{i=0}^{n} c_it^i \). Let \( M_m \) be the matrix formed by the first \( m \) rows and columns of \( M \).

Assume that

(a) For all positive integers \( m \) such that \( 1 \leq m \leq n \), the valuation of \( \det(M_m) \) is \( r \cdot \frac{m(m+1)}{2} \).

(b) The valuation of elements in column \( j \) is at least \( r \cdot j \).

Then we have that, for all positive integers \( m \) such that \( 1 \leq m \leq n \), \( v_2(c_m) = r \cdot \frac{m(m+1)}{2} \).
2. Let $M_\infty$ be a compact infinite matrix. If there is a sequence of finite matrices $M_m$ which tend to $M_\infty$, then the finite characteristic power series $\det(1-tM_m)$ converge coefficientwise to $\det(1-tM_\infty)$, as $m \to \infty$.

We now quote a result of Coleman that tells us that overconvergent modular forms of small slope are in fact classical modular forms:

**Theorem 18 (Coleman [8], Theorem 1.1)** Let $k$ be a non-negative integer. Every 2-adic overconvergent modular eigenform of weight $k$ with slope strictly less than $k-1$ is a classical modular form.

We will now state the theorem on the slopes of overconvergent modular forms of weight-character $(2k+1, \tau)$ and weight-character $(k, \chi \cdot \tau^k)$. This, combined with Theorem 18 and a knowledge of the dimensions of spaces of classical cusp forms, will suffice to prove Theorem 17 and Corollary 3.

**Theorem 19** Let $k$ be an integer, let $\tau$ be the primitive Dirichlet character of conductor 4 and let $\chi$ be the even primitive Dirichlet character of conductor 8.

1. The slopes of overconvergent 2-adic cuspidal eigenforms of weight $2k+1$ and character $\tau$ are $\{2i\}_{i \in \mathbb{N}}$.

2. The slopes of overconvergent 2-adic cuspidal eigenforms of weight $k$ and character $\chi \cdot \tau^k$ are $\{i\}_{i \in \mathbb{N}}$.

We now recall a theorem of Cohen and Oesterlé:

**Theorem 20 (Cohen-Oesterlé [7], Théorème 1)** Let $\chi$ be a primitive Dirichlet character of conductor $2^m > 2$, and let $k$ be a positive integer. Let $k = (k, \chi)$ be an integral weight-character.

The dimension of the space of cuspidal modular forms of weight-character $k$ is

$$2^{m-3}(k-1) - 1.$$

We see that the slopes of the first $2^{m-3}(k-1) - 1$ overconvergent modular forms of level $\Gamma_0(2^m)$ and primitive Dirichlet character are

$$2^{3-m}, 2^{2(3-m)}, \ldots, k-1 - 2^{3-m}.$$

Therefore, using Theorem 18, we see that all of these slopes are classical, because $k-1 - 2^{3-m} < k-1$. Hence we have proved Corollary 3.

## 5 Necessary conditions on matrices

We will show that we can apply Theorem 17 by proving the following theorem:
Theorem 21. Let $N = 4$ or $8$, and let $M$ be the matrix of the $U$-operator acting on the Banach basis $\{(z_N)^i\}$. Define the set $a_i$ by $U(z_N^2) := \sum_{i=1}^{\infty} a_i(z_N)^i$. Assume that

\[ \diamond : v(a_i) = i \cdot 4/N, \text{ for } i \text{ odd, and } v(a_i) > i \cdot 4/N, \text{ for } i \text{ even.} \]

Then the valuation of the determinant of the $i \times i$ matrix $M_n$ is $8i/N$.

We can show the precondition of this theorem directly, by considering the two identities of modular functions

\[ U(z_4^2) = \frac{2z_4}{(1 + 2z_4)^2} \text{ and } U(z_8^2) = \frac{\sqrt{2}z_8}{1 + 2z_8^2}, \]

where we have chosen a square root of 2 in the extension $Q_2(\sqrt{2})$. These can be verified by multiplying out both sides and transforming them into an identity of modular forms; we then use a theorem of Sturm [20] to show that both sides of the equation are the same by checking the $q$-expansions at $\infty$.

We then view these rational functions of $z_N$ as generating functions for power series in $z_N$.

We will also prove a result about the ring over which the $q$-expansions at $\infty$ of the cusp forms of weight-character $k$ are defined. This will allow us to prove Corollary 3.

Corollary 22. Let $k = (k, \chi)$ be an integral weight-character, where $\chi$ is a primitive Dirichlet character of conductor 4 or 8 and $k$ is a positive integer such that $\chi(-1) = (-1)^k$. Let $K$ be the field $Q_2$ if $N = 4$ or $Q_2(\sqrt{2})$ if $N = 8$.

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalised classical cuspidal modular eigenform of weight-character $k$.

Then $a_n \in K$ for all $n$.

Theorem 19 tells us that the slopes of overconvergent modular forms of weight-character $k$ are distinct. Hence by applying Theorem 20 and Theorem 18 we see that the slopes of classical modular forms of weight-character $k$ are also distinct.

We will now recall a fact from Ribet [17], page 21, to prove this Corollary. Let $\sigma$ be an element of $Gal(\overline{K}/K)$. Then we have that

\[ \sigma(f) := \sum_{n=1}^{\infty} \sigma(a_n) q^n \]

is a classical cuspidal modular eigenform of weight-character $k$.

We see that the valuation of $\sigma(a_2)$ is the same as that of $a_2$, because the characteristic polynomial of $a_2$ is stable under conjugation by $\sigma$. Therefore, $\sigma(f)$ is an eigenform of weight-character $k$ with the same slope as $f$. Hence $\sigma(f) = f$, because there is only one classical eigenform of weight-character $k$ which has any given slope, by Theorem 18 and Theorem 19.
This means that $\sigma(f) = f$ for all $\sigma$. Therefore $a_n \in K$ for all positive integers $n$. □

We will perform a series of transformations to obtain a matrix which we can apply Theorem 17 to.

We first note that the odd-numbered columns of the matrix $M_n$ are identically zero, because we have shown that $U(z_{2^a+1}^2) = 0$.

We consider the matrix $O_n$, defined by

$$(O_n)_{i,j} := (M'_{2n})_{2i,2j}, \text{ where } 1 \leq i, j \leq n.$$ 

This has the same characteristic power series as $M_{2n}$.

We now show that the matrices $O_n$ have determinant of valuation $8 \cdot n/N$, in order to be able to use Theorem 17.

To do this we will pre- and post-multiply the matrix $O_n$ by diagonal matrices to obtain a matrix $O'$ which has elements of valuation at least $8i/N$ in column $i$.

Let $D(\alpha)$ be the diagonal matrix with $(i, i)$th coefficient $\alpha^i$. We let $\alpha$ be 2 if $N = 4$ and a square root of 2 if $N = 8$, and we define

$$O' = D(\alpha^{-1}) \cdot O_n \cdot D(\alpha).$$

It can now be checked that the valuation of the elements in the $i^{th}$ column of $O'$ is at least $8i/N$.

We will show that the matrix $O'$ has determinant with valuation $8/N \cdot n(n+1)$ by showing that it is the product of two matrices, one of which is the diagonal matrix $D(\alpha)$, and one of which is a matrix with determinant a unit.

We define the matrix $P$ to be $D(\alpha)^{-1} \cdot O'$. The entries of $P$ are given by $P_{i,j} = \alpha^{-i} \cdot O'_{i,j}$ and are therefore elements of the ring of integers of $\mathbb{Q}_2(\sqrt{2})$, because the valuation of elements in the $i^{th}$ column of $O'$ is at least $i$. Therefore, we can define the matrix $P'$ by reducing the entries of $P$ modulo 2; if $P'$ has determinant 1 in $\mathbb{F}_2$, then $O'$ has determinant $8/N \cdot n(n+1)$.

By tracing the definition of the columns through all of this, we see that the odd-indexed columns of the mod 2 matrix in weight $(2i+1) \cdot k$ have generating functions

$$\text{col}(2i+1) = \frac{x^{i+1} y^{2i+1} \cdot (1+x)^i}{(1+x)^{2i+1}},$$

and the even-indexed columns of the matrix have generating functions

$$\text{col}(2i) = \frac{x^i y^{2i} \cdot (1+x)^i}{(1+x)^{2i}}.$$ 

This is clear from the discussion earlier if $N = 4$ or $N = 8$ and 4 does not divide the weight.

If 4 does divide the weight, then we consider the quotient $E'_{k, \chi}/V'_{k, \chi}$; notice that the $q$-expansions of the Eisenstein series $E_{2k}^* \chi$ and $E_{k}^* \chi$ are congruent modulo 2, by Fermat’s little theorem. Let $z_k := (E_{k}^* \chi/V_{k}^* \chi - 1)/\lambda_k$, where $\lambda_k$ is a square root of $-2k/B_{k, \chi}$. This is also an overconvergent modular form, so it is an element of $\mathbb{Q}_2(\sqrt{2})(z_8)$. Then the effect of the changes above on the
columns is to send the multiplier $(1 + \lambda_k z_k)$ to $1 + z_k$. We see that $z_k \equiv z_8$ mod 8 because

$$z_k = \frac{E_{k,\chi}^* - V_{k,\chi}^*}{\lambda_k V_{k,\chi}^*} \equiv E_{k,\chi}^* - V_{k,\chi}^* \equiv \frac{E_{2,\chi}^* - V_{2,\chi}^*}{\sqrt{2} V_{2,\chi}^*} \equiv z_8.$$ 

Therefore the multiplier $E_{k,\chi}^*/V_{k,\chi}^*$ when expanded as a rational function of $z_8$ and then altered by the procedure above is congruent modulo 2 to $1 + z_8$.

For example, we consider the $z_8$ expansion of $E_{4,\chi}^*/V_{4,\chi}^*$. After sending $2z_8$ to $z_8$, we see that we have

$$\frac{11 + z_8 + 6z_8^2 - 6z_8^3 - z_8^4 - 11z_8^5}{1 + z_8} \equiv 1 + z_8 \mod 2.$$ 

I would like to thank Robin Chapman [6] for the idea behind the following proof. To show that the $n \times n$ mod 2 matrix has determinant 1, we will show that the elements $C := \{\text{col}'(1) = \text{col}1 \cdot (1 + x)^N, \ldots, \text{col}'(N) = \text{col}N \cdot (1 + x)^N\}$ are a basis of the ring $\mathbb{F}_2[x]/(x^{N+1})$. Because $(1 + x)$ is a unit in the ring, we may multiply each $\text{col}(i)$ by $(1 + x)^{N-1}$ to make the calculations easier. We see that there are exactly the right number of elements, so we must check that they are linearly independent.

We write $\sum_{i=1}^{N} \lambda_i \text{col}'(i) = 0$. We will show that all of the $\lambda_i = 0$, so that the columns are linearly independent.

We see that the element $\text{col}'(1) = x(1+x)^{N-1}$ is the only element of the set $C$ which has an $x^N$ term. Therefore $\lambda_1 = 0$. Also, we see that $\text{col}'(2) = x(1+x)^{N-2}$ is now the only nonzero term with an $x$ term, so therefore $\lambda_2 = 0$.

By continuing this process, we show that each $\lambda_i$ must be zero. Therefore the set $C$ is composed of linearly independent elements and hence it is a basis. So the determinant of the mod 2 matrix is 1. Hence we have shown Theorem 21 and therefore Theorem 19.

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