Theorem 1. [4] Let $n$ be an arbitrary positive integer. Given any sequence $S$ of $2n - 1$ integers, there is a subsequence $T$ of $S$ with length $n$, the sum of whose terms is divisible by $n$.

The natural generalization of this result is the study of sequences over finite abelian groups which are guaranteed to have zero-sum subsequences of some prescribed length.

We will operate extensively with sets of integers. Write $\mathbb{N}$ for the set of all positive integers and $\mathbb{Z}$ for the set of all integers. We use the notation $[a, b]$ to denote the set of all integers $\{a, a + 1, \ldots, b\}$ between $a$ and $b$ inclusive. For a set $K \subset \mathbb{Z}$ and an integer $q$, write $Kq = \{kq : k \in K\}$, $K + q = \{k + q : k \in K\}$, and $q - K = \{q - k : k \in K\}$.

Let $(G, +)$ be a finite abelian group written additively. Then, we write $|G|$ for the size of $G$ and $\exp(G)$ for the exponent of $G$, i.e. the largest order of any element of $G$. A sequence $S$ over $G$ will be written multiplicatively in the form

$$S = g_1 g_2 \cdots g_\ell = \prod_{g \in G} g^{v_g(S)},$$

with $v_g(S) \geq 0$ being the number of times that $g$ appears in $S$. If $S_1, S_2$ are two sequences then we write

$$S_1 S_2 = \prod_{g \in G} g^{v_g(S_1) + v_g(S_2)}$$

for their product, or concatenation.
With these definitions, we call

$$|S| = \sum_{g \in G} v_g(S)$$

the length of $S$ and

$$\sigma(S) = \sum_{g \in G} v_g(S) \cdot g$$

the sum of $S$ (which is an element of $G$). A sequence $S$ is zero-sum if $\sigma(S) = 0$.

Throughout, we write $C_m$ for the cyclic group of order $m$ and identify it with $\mathbb{Z}/m\mathbb{Z}$.

We say that $T$ is a subsequence of $S$, written $T|S$, if $v_g(T) \leq v_g(S)$ for every $g \in G$. If $T$ is a subsequence of $S$ then we write $ST^{-1}$ for the sequence

$$ST^{-1} = \prod_{g \in G} g^{v_g(S) - v_g(T)}.$$ 

Following Gao and Thangadurai [11], we define $s_k(G)$ to be the smallest positive integer $\ell$ for which any sequence $S$ of length $\ell$ over $G$ contains a zero-sum subsequence of length $k$. We will only be concerned with the case where $\exp(G)|k$; it is easy to check that if $\exp(G) \nmid k$ then $s_k(G) = \infty$. Theorem 1 proved that $s_n(C_n) = 2n - 1$, where $C_n$ is the cyclic group of order $n$. The case $G = C_n^d$ and $k = n$ was first studied by Harborth [13].

We generalize this definition to sets of lengths, following notation from Geroldinger, Grynkiewicz and Schmid [9]. For any set $K$ of positive integers, let $s_K(G)$ be the shortest length $\ell$ for which any sequence $S$ of length $\ell$ over $G$ contains a zero-sum subsequence with length in $K$. Geroldinger, Grynkiewicz and Schmid were interested in the case that $K$ is an infinite progression $a\mathbb{N}$, but we will primarily work with finite sets $K$.

As an important special case, define $D(G) = s_2(G)$ to be the Davenport constant of $G$, which is the shortest length $\ell$ for which any sequence $S$ of length $\ell$ has some nontrivial zero-sum subsequence. For a $p$-group of the form

$$G = \bigoplus_{i=1}^d C_{p^{\alpha_i}},$$

the Davenport constant was determined by Olson [16] to be

$$D(G) = 1 + \sum_{i=1}^d (p^{\alpha_i} - 1).$$

It will henceforth be implicitly understood that tight lower bounds on all of the quantities $s_K(G)$ can be proved by construction and that it suffices to prove upper bounds.

In the rank 2 case it was first conjectured by Kemnitz [14] that $s_n(C_n^2) = 4n - 3$. Alon and Dubiner [1, 2] obtained the first linear bound $s_n(C_n^2) \leq 6n - 5$. Later Rónyai [18] showed for primes $p$, $s_p(C_p^2) \leq 4p - 2$, and the full Kemnitz conjecture was resolved by Reiher [17]. All of these results follow from algebraic considerations related to the Chevalley-Warning theorem.

In this paper we are primarily interested in finite abelian $p$-groups of higher rank, for which the following conjecture has been made. If $G$ has exponent $q$ it will be convenient to write $d = \left\lceil \frac{D(G)}{q} \right\rceil$. 
Conjecture 2. [11, 15] Let $p$ be a prime and let $G$ be a finite abelian $p$-group with $\exp(G) = q$ and $\left\lceil \frac{D(G)}{q} \right\rceil = d$. Then for any $k \geq d$, we have $s_{kq}(G) = kq + D(G) - 1$.

Conjecture 2 has been proved when $G$ has rank at most 2: see for instance Theorem 6.12 in the survey of Gao and Geroldinger [8]. For groups of higher rank, progress has been made towards the computation of $s_{kn}(C_n^r)$ for $r = 3, 4$ by Edel et al. [3], and more general inductive bounds for arbitrary rank have been obtained by Fan, Gao, and Zhong [5].

Our first main result bounds $s_{Kq}(G)$ when $p$ is prime, $G$ is a $p$-group with exponent $q$, and $|K| \geq d$. The same result was proved for the special case $G = C_q^d$ by Kubertin [15], and our argument is similar.

**Theorem 3.** Let $p$ be a prime and let $G$ be a finite abelian $p$-group with $\exp(G) = q$ and $\left\lceil \frac{D(G)}{q} \right\rceil = d$. If $K \subseteq [1, p]$ is a set satisfying $|K| \geq d$, then

$$s_{Kq}(G) \leq (\max K + 1 - |K|)q + D(G) - 1.$$ 

Gao, Han, Peng, and Sun [6, 10] have shown the following preliminary results in the most general setting, assuming nothing about $G$.

**Theorem 4.** [6, 10] Let $G$ be a finite abelian group with $\exp(G) = n$. Then for any $k \geq 1$, we have $s_{kn}(G) \geq kn + D(G) - 1$. If $kn \geq |G|$, then equality holds, whereas if $kn < D(G)$, then the inequality is strict.

From here they introduced the threshold function $\ell(G)$ defined as the smallest positive integer $\ell$ for which $s_{kn}(G) = kn + D(G) - 1$ for any $k \geq \ell$. From Theorem 4 we have

$$\frac{D(G)}{\exp(G)} \leq \ell(G) \leq \frac{|G|}{\exp(G)}.$$ 

The lower bound is conjectured to be tight; our primary goals in this paper are to bound the growth of $s_{kn}(G)$ and in turn give a much stronger upper bound on $\ell(G)$ when $G$ is a $p$-group.

Using Theorem 3 it is possible to prove the following bound on $s_{kq}(G)$ when $G$ is a $p$-group with exponent $q$. For comparison, Kubertin’s methods [15] allow one to prove $s_{kq}(C_q^d) \leq (k + Cq^2)q - d$ for some fixed constant $C > 0$.

**Theorem 5.** Let $p$ be a prime, let $G$ be a finite abelian $p$-group with $\exp(G) = q$ and $\left\lceil \frac{D(G)}{q} \right\rceil = d$. If $p \geq 2d - 3 + 3\left\lceil \frac{D(G)}{2q} \right\rceil$, and $k \geq d$, then

$$s_{kq}(G) \leq (k + 2d - 2)q + 3D(G) - 3.$$ 

When restricted to $G = C_q^d$, the Davenport constant is just $D(G) = qd - d + 1$. Thus, the bound reduces to $s_{kq}(C_q^d) \leq (k + 5d - 2)q - 3d$ when $2p \geq 7d - 3$ and $k \geq d$, achieving a bound linear in $d$ where Kubertin proved only a quadratic one. Theorem 5, together with Olson’s calculation of $D(G)$, gives a new bound for $\ell(G)$ for a $p$-group.

**Corollary 6.** Let $p$ be a prime and let $G$ be a finite abelian $p$-group with $\exp(G) = q$ and $\left\lceil \frac{D(G)}{q} \right\rceil = d$. If $p \geq 4d - 2$, then $\ell(G) \leq p + d$. That is, $s_{kq}(G) = kq + D(G) - 1$ whenever $k \geq p + d$. 
In the next section, we collect two well-known lemmas about the behavior of $s_{kq}(G)$, before proceeding to the proofs of Theorems 3 and 5. For more discussion of the implications of Theorem 5 and Corollary 6 on the general problem and on open questions, see the final section.

**Preliminary Lemmas**

First, we show a sub-additivity result on $s_k(G)$ for general $G$. Note that if $\exp(G) \nmid a$ or $\exp(G) \nmid b$, the following lemma is vacuously true.

**Lemma 7.** If $G$ is a finite abelian group and $a, b \in \mathbb{N}$, then
$$s_{a+b}(G) \leq \max\{s_a(G) + b, s_b(G)\}.$$  

**Proof.** Suppose that $S$ is a sequence over $G$ with $|S| \geq s_a(G) + b$ and $|S| \geq s_b(G)$. By the latter inequality, $S$ has a zero-sum subsequence $S'_1$ of length $b$. By the former inequality, $SS'_1^{-1}$ has a zero-sum subsequence $S'_2$ of length $a$. Thus, $S$ contains a zero-sum subsequence $S'_1S'_2$ of length $a + b$ as desired. \hfill $\Box$

Also we will need an easy special case of Conjecture 2, which follows directly from a result of Geroldinger, Grynkiewicz and Schmid [9]. We include a quick proof for convenience, using the result of Olson [16] which determines the value of $D(G)$ for all $p$-groups.

**Lemma 8.** Let $p$ be a prime and let $G$ be a finite abelian $p$-group with $\exp(G) = q$ and $\frac{D(G)}{q} = d$. If $p \geq d$, then $s_{kpq}(G) = kpq + D(G) - 1$ for any integer $k \geq 1$.

**Proof.** It suffices by Theorem 4 and Lemma 7 to prove that $s_{pq}(G) \leq pq + D(G) - 1$. Let $S$ be a sequence of this latter length over $G$.

Let $e$ be a generator of $C_{pq}$. Construct a sequence $S'$ over $G \oplus C_{pq}$ such that if $S = \prod_{g \in G} g^{\nu_g(S)}$, then $S' = \prod_{g \in G} (g, e)^{\nu_g(S)}$. Then, a zero-sum subsequence of $S'$ corresponds exactly to a zero-sum subsequence of $S$ with length divisible by $pq$. Since $|S| = pq + D(G) - 1 < 2pq$, it follows that any such subsequence would have length exactly $pq$.

Since $D(G \oplus C_{pq}) = pq + D(G) - 1$ by Olson’s theorem [16], it follows that $S$ itself had a length $pq$ zero-sum subsequence, as desired. \hfill $\Box$

Combining Lemmas 7 and 8, it remains to control the behavior of $s_{kq}(G)$ in the finite interval $k \in [d, p - 1]$.

**The Algebraic Method of Rónyai and Kubertin**

In this section we generalize the algebraic method of Rónyai [18], who showed that $s_p(C_p^2) \leq 4p - 2$ for all primes $p$, to prove Theorem 3. Theorem 3 was proved for $C_q^d$ in the paper of Kubertin [15].

We require the following elementary result. Rónyai used the same statement for fields [18], but we will also need the case $R = \mathbb{Z}$.

**Lemma 9.** Let $R$ be an integral domain and $m$ a positive integer. Then the (multilinear) monomials $\prod_{i \in I} x_i, I \subseteq [1, m]$, constitute a basis for the free $R$-module $M$ of all functions from $\{0, 1\}^m$ to $R$. (Here 0 and 1 are viewed as elements of $R$.)
Proof. For a given point \( \mathbf{a} = (a_1, a_2, \ldots, a_m) \in \{0, 1\}^m \), we define the indicator function

\[
p_{\mathbf{a}}(\mathbf{x}) = (-1)^{m - \sum_{i=1}^{m} a_i} \prod_{i=1}^{m} (x_i - 1 + a_i),
\]

where \( \mathbf{x} = (x_1, \ldots, x_m) \in \{0, 1\}^m \). Thus \( (p_{\mathbf{a}}(\mathbf{x}), \mathbf{a} \in \{0, 1\}^m) \) is a basis for \( M \).

Since \( p_{\mathbf{a}}(\mathbf{x}) \) is a \( R \)-linear combination of the given monomials \( \prod_{i \in I} x_i, I \subseteq [1, m] \), these monomials generate \( M \). There are \( 2^m \) monomials in this set of generators and \( M \) has rank \( 2^m \), so we conclude that the given monomials \( \prod_{i \in I} x_i, I \subseteq [1, m] \) form a basis for \( M \). □

Using this lemma, we can prove Theorem 3.

Proof. (of Theorem 3) Let \( G = C_{q_1} \oplus \cdots \oplus C_{q_e} \) with each \( q_i = p^{m_i} \) for some \( m_i > 0 \). Write \( q = \exp(G) = \max(q_i) \). Let \( S = \prod_{i=1}^{m_i} g_i \) be a sequence over \( G \) with length

\[
m = (\max K + 1 - |K|)q + D(G) - 1
\]

\[
= (\max K + 1 - |K|)q + \sum_{i=1}^{e} q_i - e.
\]

We show that, given any set \( K \) of positive integers in \([1, p]\) with cardinality at least \( d \), some zero-sum subsequence of \( S \) has length in \( Kq \). Suppose otherwise.

Working over the field \( \mathbb{Q} \), we define the following polynomial on \( m \) variables \( x_1, x_2, \ldots, x_m \). Write \( \mathbf{x} = (x_1, \ldots, x_m) \). By identifying \( C_{q_i} \) with \( \mathbb{Z}/q_j \mathbb{Z} \) and picking representatives \([0, q_j - 1] \subseteq \mathbb{Z} \) for this quotient, we let \( a_{i,j} \) denote the representative in \([0, q_j - 1] \) for the \( j \)-th component of \( g_i \in S \), where \( i \in [1, m] \) and \( j \in [1, e] \). Also, given a polynomial \( Q(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}] \) and an integer \( n \geq 0 \), we define

\[
\binom{Q(\mathbf{x})}{n} = \frac{Q(\mathbf{x})(Q(\mathbf{x}) - 1) \cdots (Q(\mathbf{x}) - (n-1))}{n!} \in \mathbb{Q}[\mathbf{x}],
\]

with an empty product taken to be 1. Then, define \( P \) to be the integer-valued polynomial

\[
P(\mathbf{x}) = P_L(\mathbf{x})P_S(\mathbf{x})P_K(\mathbf{x}),
\]

where

\[
P_L(\mathbf{x}) = \binom{\sum_{i=1}^{m} x_i - 1}{q - 1}
\]

\[
P_S(\mathbf{x}) = \prod_{j=1}^{e} \binom{\sum_{i=1}^{m} a_{i,j} x_i - 1}{q_j - 1}
\]

\[
P_K(\mathbf{x}) = \prod_{\ell \in [1, \max K] \setminus K} \left( \binom{\sum_{i=1}^{m} x_i}{q} - \ell \right).
\]

Each \( \mathbf{x} \in \{0, 1\}^m \) uniquely indexes a subsequence \( T_\mathbf{x}|S \) with the terms of \( T_\mathbf{x} \) precisely those \( g_i \) for which \( x_i = 1 \). Now \( P \) vanishes modulo \( p \) except when \( \mathbf{x} = \mathbf{0} \) (the null vector). Suppose \( \mathbf{x} \neq \mathbf{0} \) and indexes a sequence \( T_\mathbf{x}|S \). The classical theorem of Lucas, of which Granville gave an excellent exposition [12], implies that for any power \( q_j \) of \( p \),

\[
\binom{y - 1}{q_j - 1} \equiv \begin{cases} 1 \pmod{p} & \text{if } q_j|y \\ 0 \pmod{p} & \text{otherwise}, \end{cases}
\]

"
and furthermore
\[
\left( \frac{\ell q_j}{q_j} \right) \equiv \ell \pmod{p},
\]
for any integers \( y, \ell \). Thus the polynomial \( P_L \) vanishes modulo \( p \) whenever \( |T_x| \) is not a multiple of \( q \), and the polynomial \( P_S \) vanishes modulo \( p \) whenever \( \sigma(T_x) \neq 0 \). Given that \( |T_x| = \ell q \) for some \( \ell > 0 \), the polynomial \( P_K \) vanishes modulo \( p \) whenever \( \ell \in [1, \max K] \setminus K \).

It follows that the only possibility for \( Q(x) \) not to vanish modulo \( p \) is if \( T_x \) is a zero-sum sequence with length congruent to \( \ell q \) for some \( \ell \in K \cup [\max K + 1, \ell] \pmod{p} \). But since \( |T_x| = \max |S| = m \) and \( m \) is constructed to be less than \( (\max K + 1)q \), it follows that \( |T_x| \in Kq \), a contradiction.

We see that \( P \) vanishes modulo \( p \) on \( \{0, 1\}^m \) with the sole exception of the all-0’s vector. On that vector we can explicitly compute
\[
P(0) = P_L(0)P_S(0)P_K(0) = (-1)^{q-1} \prod_{i=1}^{\ell} (-1)^{q_{ij}-1} \prod_{\ell \in [1, \max K] \setminus K} (-\ell),
\]
which is nonzero mod \( p \) as \( [1, \max K] \setminus K \) is a subset of \( [1, p-1] \). Now \( P \) is an integer-valued polynomial which vanishes everywhere modulo \( p \) except at \( 0 \). Applying Lemma 9 with \( R = \mathbb{Z}/p\mathbb{Z} \), we can write
\[
P(x) \equiv C \prod_{i=1}^{m} (1 - x_i) \pmod{p},
\]
as functions \( \{0, 1\}^m \to \mathbb{Z}/p\mathbb{Z} \) for some \( C \in \mathbb{Z} \) not divisible by \( p \). Pulling back to the integers, we have
\[
P(x) = C \prod_{i=1}^{m} (1 - x_i) + pQ_1(x)
\]
as functions \( \{0, 1\}^m \to \mathbb{Z} \) for some integer-valued function \( Q_1(x) \). We can then apply Lemma 9 to \( Q_1 \) with \( R = \mathbb{Z} \). Thus \( Q_1(x) \) is equal as a function to some integer linear combination \( Q_2 \) of monomials \( \prod_{i \in I} x_i, I \subseteq [1, m] \) satisfying \( Q_2 \in \mathbb{Z}[x] \) and \( \deg Q_2 \leq m \). Writing
\[
Q(x) = C \prod_{i=1}^{m} (1 - x_i) + pQ_2(x),
\]
we see that \( P \) and \( Q \) agree as functions \( \{0, 1\}^m \to \mathbb{Z} \). Since \( p \nmid C \), the top-degree term \( \prod_{i=1}^{m} x_i \) in \( Q(x) \) has a nonzero coefficient, so \( \deg Q = m \).

On the other hand, \( P \) can be written as a linear combination of basis monomials over \( \mathbb{Q} \) in another way, simply by expanding the product \( P = P_LP_SP_K \) and applying the relation \( x_i^2 = x_i \) for functions on \( \{0, 1\}^m \). Both expansions represent \( P \) in terms of the basis \( \{\prod_{i \in I} x_i, I \subseteq [1, m]\} \) for functions \( \{0, 1\}^m \to \mathbb{Q} \). By Lemma 9 with \( R = \mathbb{Q} \), they to be equal, so their degrees must equate. On the other hand, the second expansion has degree at most \( \deg P_L + \deg P_S + \deg P_K \), which we easily compute to be
\[
\deg P_L + \deg P_S + \deg P_K = (q - 1) + (D(G) - 1) + (\max K - |K|)q = m - 1
\]
by the definition of \( m \). This cannot agree with the degree of \( Q \), so we have a contradiction and the theorem is proved. \( \square \)
Bounds on Small Lengths

We now prove Theorem 5. Theorem 3 gives us
\[ s_{Kq}(G) \leq (\max K + 1 - |K|)q + D(G) - 1 \]
whenever \(|K| \geq d\) and \(\max(K) \leq p\). For the next step, we obtain a bound for \(s_{Kq}(G)\) when
\(|K| \geq d/2\), allowing for \(K\) half as large.

**Lemma 10.** Let \(p\) be a prime and let \(G\) be a finite abelian \(p\)-group with \(\exp(G) = q\) and
\[ \left\lceil \frac{D(G)}{q} \right\rceil = d. \]
If \(K \subset \mathbb{N}\) is a finite set with \(|K| \geq d/2\) and \(2\max K + |K| \leq p\), then
\[ s_{Kq}(G) \leq (2\max K + 1 - |K|)q + D(G) - 1. \]

**Proof.** For any zero-sum sequence \(T\) with \(|T| = nq\) and \(2\max K + 1 \leq n \leq p + 1\), we can
define \(L = K \cup (n - K) \subseteq [1, p]\) having \(|L| = 2|K| \geq d\) and \(\max L \leq n - 1\). Since
\[ |T| = nq \geq (n - 2|K|)q + D(G) \geq (\max L + 1 - |L|)q + D(G) - 1, \]
we can apply Theorem 3 to \(T\) with length set \(L\). Thus \(T\) has a zero-sum subsequence with
length in \(Lq = Kq \cup (n - K)q\). However, if it had a zero-sum subsequence \(T_1\) with length
\((n - k)q\) and \(k \in K\), then \(TT_1^{-1}\) has length \(kq\) with \(k \in K\), and is also zero-sum since \(T\)
itself is zero-sum. It follows that \(T\) has a zero-sum subsequence with length in \(Kq\).

Let \(S\) be a sequence over \(G\) of length \((2\max K + 1 - |K|)q + D(G) - 1\). Let \(K' = K \cup \{2\max(K) + i : i \in [1, |K|]\}\). We have \(|K'| = 2|K| \geq d\) and \(\max(K') = 2\max(K) + |K| \leq p\), by hypothesis. Also,
\[ |S| = (2\max K + 1 - |K|)q + D(G) - 1 \geq (\max K' + 1 - |K'|)q + D(G) - 1, \]
so by Theorem 3 again, this time applied to \(K'\) and \(S\), we see that \(S\) has a zero-sum subsequence \(T\) with length in \(K'q\). If \(|T| \in Kq\) we’re done. Otherwise, \(|T| = nq\) with
\[ 2\max K + 1 \leq n \leq 2\max K + |K| \leq p. \]
But then \(T\) itself has a zero-sum subsequence with length in \(Kq\) by the previous argument, and
so \(S\) does as well. \(\square\)

Next we prove a much stronger bound than Theorem 5 on the interval \(k \in [2d - 1, p]\).

**Lemma 11.** Let \(p\) be a prime, let \(G\) be a finite abelian \(p\)-group with \(\exp(G) = q\) and
\[ \left\lceil \frac{D(G)}{q} \right\rceil = d, \]
and let \(k \in [2d - 1, p]\) be an integer. Then,
\[ s_{kq}(G) \leq kq + 2D(G) - 2. \]

**Proof.** Let \(S\) be a sequence over \(G\) satisfying \(|S| = kq + 2D(G) - 2\). Factor \(S = S_1S_2\) where
\(|S_1| = (k + 1 - d)q + D(G) - 1\) and \(|S_2| = (d - 1)q + D(G) - 1\).

If \(d = 1\), then \(G\) is cyclic and the result is a consequence of the Erdős-Ginzburg-Ziv
Theorem, so assume \(d \geq 2\). Let \(K\) be any \(d\)-subset of \([1, 2d - 2]\), and apply Theorem 3 to \(S_2\)
with length set \(Kq\). By ranging \(K\) through all possible \(d\)-subsets of \([1, 2d - 2]\), we see that
at least \(d - 1\) of the lengths in \([1, 2d - 2]q\) appear as the lengths of zero-sum subsequences
of \(S_2\). Together with the empty subsequence, these lengths form a \(d\)-subset \(L \subset [0, 2d - 2]\)
such that every length in \(Lq\) is the length of some zero-sum subsequence \(T_2|S_2\).

It remains to show that some length in \(kq - Lq\) is the length of a zero-sum subsequence
\(T_2|S_1\). But \(kq - Lq\) has cardinality \(d\) and maximum at most \(kq\). Since \(|S_1| \geq (k + 1 - d)q + D(G) - 1\) we can apply Theorem 3 to conclude that \(S_1\) indeed contains a zero-sum
subsequence $T_1$ with length in $kq - Lq$. Combining $T_1$ with the corresponding $T_2 | S_2$ with $|T_2| \in Lq$, we find that $S$ has a zero-sum subsequence $T_1 T_2$ with the desired length $kq$. \hfill \square

Using Lemma 10 and Lemma 11 together we can prove Theorem 5.

**Proof.** (of Theorem 5) For $k \in [2d -1, p]$, the result follows by Lemma 11, and since $p \geq 2d -1$ this interval is nonempty.

Suppose $d \geq 2$ and $k \in [d, 2d-2]$, so that $k \leq p -1$. Let $m = \lceil \frac{D(G)}{2q} \rceil$ and

$$t = \left\lfloor \frac{k}{2} \right\rfloor + m - 1.$$ 

Note that since $k \geq d$ we have $t \geq 2m - 2$. Factor $S = S_1 S_2$ with lengths satisfying

$$|S_1| \geq (2t - m + 1)q + D(G) - 1 \quad \text{and} \quad |S_2| \geq (2k - 2t + 3m - 3)q + D(G) - 1. \quad (1)$$

We can apply Lemma 10 to $S_1$ with all possible $m$-subsets $K$ of $[t - 2m + 2, t]$. This is possible because for such a set $K$ we have $2 \max K + |K| \leq 2t + m \leq p$ by the hypothesis of the theorem. Thus there is a set $L \subset [t - 2m + 2, t]$ of cardinality $m$ such that every element of $Lq$ appears as the length of some zero-sum subsequence $T_1 | S_1$. In the case that $t = 2m - 2$ exactly, we first apply Lemma 10 to $m$-subsets of $[1, t]$ to get an $|L| = m - 1$, and then include the empty sequence afterwards, so that $L$ contains 0 and has the correct size $m$.

Now, we simply apply Lemma 10 to $S_2$ with the set $k - L$. This set has cardinality $m$ and maximum value at most $k - t + 2m - 2$, and $p$ satisfies

$$2(k - t + 2m - 2) + m \leq 2d + 3m - 3 \leq p,$$

so the conditions are satisfied and some $T_2 | S_2$ has sum zero and length $|T_2| \in kq - Lq$. Concatenating it with the corresponding subsequence of $S_1$, the theorem is proved for $k \in [d, 2d - 2]$.

Finally it is easy to apply Lemma 7 to prove the theorem inductively on all $k \geq 2d$. If $k \geq 2d$, then we can write $k = d + k'$ with $k' \geq d$ and Lemma 7 gives

$$s_{kq}(G) \leq \begin{cases} s_{dq}(G) + k'q, s_{k'q}(G) \end{cases} \leq (k + 2d - 2)q + 3D(G) - 3,$$

and by induction the bound is proved for all $k \geq 2d$. \hfill \square

We can now show Corollary 6.

**Proof.** (of Corollary 6.) From Lemma 8, we have $s_{pq}(G) = pq + D(G) - 1$, and from Theorem 5, we have $s_{kq}(G) \leq (k + 2d - 2)q + 3D(G) - 3$ if $k \geq d$. Combining these via Lemma 7, we get for any $k \geq p + d$,

$$s_{kq}(G) \leq \max \left\{ s_{pq}(G) + (k - p)q, s_{(k-p)q}(G) \right\} = \max \left\{ kq + D(G) - 1, (k - p + 2d - 2)q + 3D(G) - 3 \right\}.$$ 

It suffices to show that under the additional assumption $p \geq 4d - 2$, the first term is larger. In fact, we have

$$p \geq 4d - 2 \quad \text{and} \quad pq \geq (4d - 2)q.$$

If $kq + D(G) - 1 \geq (k - p + 2d - 2)q + 3D(G) - 3$,

$$p \geq 4d - 2 \quad \text{and} \quad pq \geq (4d - 2)q.$$
as desired. □

CLOSING REMARKS AND OPEN PROBLEMS

We first make a few observations regarding the problem of Gao on the threshold \( \ell(G) \) after which \( s_{kn}(G) = kn + D(G) - 1 \) for all \( k \geq \ell(G) \), where \( n = \exp(G) \). Gao et al. [10] proved Theorem 4 which shows in general that

\[
D(G) \leq n \ell(G) \leq |G|.
\]

It is conjectured by Gao et al. [10] that the lower bound is tight, i.e.

\[
\ell(G) = \left\lceil \frac{D(G)}{n} \right\rceil
\]

for all \( G \). Thus we are mainly interested in improving the upper bound \( |G|/n \). In the case that \( G \) is a \( p \)-group we can do much better than \( \ell(G) \leq |G|/n \) using Theorem 5, getting \( \ell(G) \leq p + d \) when \( p \) and \( d \) satisfy the conditions of Theorem 5.

For comparison, the results of Kubertin give \( \ell(C_q^d) \leq q + Cd^2 \) for a constant \( C > 0 \), while conjectural value is \( \ell(C_q^d) = d \), so any bound independent of \( p \) would be a significant improvement on Theorem 6. The only available method for proving bounds on \( \ell(G) \) is combining bounds of the sort in Theorem 5 with Lemma 8, which depends on \( p \).

Problem. Can we remove the dependence on \( p \) in Corollary 6?

Just as in the special case \( k = 1 \), bounds on \( s_{kq}(C_q^d) \) give rise to bounds on \( s_{kn}(C_q^d) \), although in general the dependence is weaker. As an easy consequence of Theorems 5 and 6 we prove the following multiplicativity lemma.

Lemma 12. Let \( p \) be a prime, let \( q \) be a power of \( p \), and let \( G \) be a finite abelian group with \( \exp(G) = q^n \) such that the quotient group \( H = G/qG \) satisfies \( \left\lceil \frac{D(H)}{q} \right\rceil = d \) and \( p \geq 2d + 3 \left\lceil \frac{D(H)}{2q} \right\rceil - 3 \). If \( a, b \in N \) and \( b \geq d \), then

\[
s_{abq^n}(G) \leq s_{an}(qG)bq + (2d - 2)q + 3D(G) - 3.
\]

Furthermore, if \( p \geq 4d - 2 \) and \( b \geq p + d \), then

\[
s_{abq^n}(G) \leq s_{an}(qG)bq + D(G) - 1.
\]

Proof. Let \( S \) be a sequence over \( G \) with length at least \( s_{an}(qG)bq + (2d - 2)q + 3D(G) - 3 \). From \( S \) we can repeatedly remove, using Theorem 5 on \( G/qG \), length \( bq \) subsequences of \( S \) whose sums lie in \( qG \). Repeating until \( |S| \) falls below \((b + 2d - 2)q + 3D(G) - 3\), we extract a total of \( s_{an}(qG) \) disjoint zero-sum subsequences. The same can be done using Corollary 6 instead if \( b \geq p + d \).

In either case, we end up with \( s_{an}(qG) \) disjoint subsequences of \( G \), each of length \( bq \) and having sum in \( qG \). Thus there is a zero-sum subsequence of the sequence of their sums, with length \( an \), corresponding to a zero-sum subsequence of \( S \) with length \( abq^n \) as desired. □

We can bound \( s_{kn}(G) \) directly from Lemma 12 by induction. The empty product is taken to be 1.
Proposition 13. Let $G$ be a finite abelian group with $\exp(G) = n$, decomposed as

$$G = \bigoplus_{i=1}^{r} G_{p_i},$$

a direct sum of $p_i$-groups $G_{p_i}$ with $\exp(G_{p_i}) = q_i$ and $\left\lfloor \frac{\exp(G_{p_i})}{q_i} \right\rfloor = d_i$, satisfying $p_i \geq 2d_i + 3 \left\lfloor \frac{\exp(G_{p_i})}{2q_i} \right\rfloor - 3$, where $p_i$ are pairwise distinct primes, $i \in [1, r]$. Then,

$$s_{kn}(G) \leq kn + \sum_{i=0}^{r-1} \left( \prod_{j=1}^{i} a_j q_j \right) ((2d_{i+1} - 2)q_{i+1} + 3\exp(G_{p_{i+1}}) - 3),$$

where $k$ is any positive integer that can be written as a product $k = a_1 \cdots a_r$ of positive integers $a_i \geq d_i$. Also,

$$s_{kn}(G) \leq kn + \sum_{i=0}^{r-1} \left( \prod_{j=1}^{i} a_j q_j \right) (\exp(G_{p_{i+1}}) - 1)$$

if each $p_i$ satisfies $p_i \geq 4d_i - 2$ and each $a_i$ satisfies $a_i \geq p_i + d_i$.

Proof. Apply Lemma 12 with the filtration $G_j = \bigoplus_{i=1}^{j} G_{p_i}, j = 0, \ldots, r$ of $G$. Each subquotient $G_j/G_{j-1} \cong G_{p_j}, j \in [1, r]$, is a $p_j$-group so the lemma applies.

As a corollary, we have the following inequality by bounding the error term crudely by a geometric series. For clarity, we state it in terms of groups of the form $C_n^d$ though bounds on any finite abelian group can be made in the same way.

Corollary 14. For $d > 0$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with distinct prime factors $p_1, \ldots, p_r \geq \frac{7}{2}d - 3$, and $k = a_1 a_2 \cdots a_r$ a product of positive integers $a_1, a_2, \ldots, a_r \geq d$,

$$s_{kn}(C_n^d) \leq 6kn.$$

If furthermore each $p_i$ satisfies $p_i \geq 4d - 2$ and each $a_i$ satisfies $a_i \geq p_i + d$, $i \in [1, r]$, then

$$s_{kn}(C_n^d) \leq 3kn.$$

Proof. Applying the first part of Proposition 13, the first inequality reduces to showing

$$\sum_{i=0}^{r-1} \left( \prod_{j=1}^{i} a_j p_j^{\alpha_j} \right) ((2d - 2)p_{i+1}^{\alpha_{i+1}} + 3\exp(C_n^d_{p_{i+1}}) - 3) \leq 5kn.$$

Let $T_i$ be the $i$-th term of this sum. First, we have the explicit formula [16]

$$\exp(C_n^d_{p_{i+1}}) - 1 = (p_{i+1}^{\alpha_{i+1}} - 1)d,$$

giving

$$T_i \leq 5d \left( \prod_{j=1}^{i} a_j p_j^{\alpha_j} \right) p_{i+1}^{\alpha_{i+1}} \leq 5 \prod_{j=1}^{i+1} a_j p_j^{\alpha_j}.$$
If \( d = 1 \) the result follows from Erdős-Ginzburg-Ziv, so we may assume \( d \geq 2 \) and \( p_i \geq \frac{d}{2} - 3 = 4 \), whence \( p_i \geq 5 \) since \( p_i \) is prime, \( i \in [1, r] \). As a result, we have

\[
T_i \leq 5kn \left( \prod_{j=i+2}^{r} a_j p_j^{a_j} \right)^{-1} \leq 5 \cdot 10^{-r-i-1}kn.
\]

Upon summing, it follows that

\[
\sum_{i=0}^{r-1} T_i \leq \frac{50}{11}kn,
\]

as desired. The second inequality follows similarly from the second part of Proposition 13.

This is stronger than can be obtained by the iterative application of Alon and Dubiner’s general bounds [1, 2] on \( s_n(C_d^n) \), but only holds for a thin set of pairs \( (k, n) \). Of course, for any given \( n \) satisfying the conditions of Proposition 13, the inequality can be extended to all values of \( k \) in the semigroup generated additively by the \( k \) satisfying the stated condition, by Lemma 7, giving \( s_{kn}(C_d^n) \leq 6kn \) for all \( k \geq d^n (d+1)^r \), where \( r \) is the number of distinct prime factors of \( n \). Any technique achieving a bound of a strength similar to that of Corollary 14 but with the threshold of \( k \) independent of \( n \) would be significant.

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