Quantum physics exhibits remarkable distinguishing characteristics. For example, it gives only probabilistic predictions (non-determinism) and does not allow copying of unknown states (no-cloning). Quantum correlations may be stronger than any classical ones, nevertheless information cannot be transmitted faster than light (no-signaling). However, all these features do not single out quantum physics. A broad class of theories exist which share such traits with quantum mechanics, while they allow even stronger than quantum correlations. Here, we introduce the principle of Information Causality. It states that information that Bob can gain about a previously completely unknown to him data set of Alice, by using all his local resources (which may be correlated with her resources) and a classical communication from her, is bounded by the information volume of the communication. In other words, if Alice communicates $m$ bits to Bob, the total information access that Bob gains to her data is not greater than $m$. For $m = 0$, Information Causality reduces to the standard no-signaling principle. We show that this new principle is respected both in classical and quantum physics, whereas it is violated by all the no-signaling correlations which are stronger that the strongest quantum correlations. Maximally strong no-signalling correlations would allow Bob access to any $m$ bit subset of the whole data set held by Alice. If only one bit is sent by Alice ($m = 1$), this is tantamount to Bob being able to access the value of any single bit of Alice’s data (but of course not all of them). We suggest that Information Causality, a generalization of no-signaling, might be one of the foundational properties of Nature.

Classical (as opposed to quantum) physics rests on the assumption that all physical quantities have well defined values simultaneously. Relativity is based on clear-cut physical statements: the speed of light and the electric charge are the same for all observers. In contradistinction, the definition of quantum physics is still rather a description of its formalism: the theory in which systems are described by Hilbert spaces and dynamics is reversible. This situation is all the more unexpected as quantum physics is the most successful physical theory and quite a lot is known about it. Some of its counterintuitive features are almost a popular knowledge: all scientists, and many laymen as well, know that quantum physics predicts only probabilities, that some physical quantities (such as position and momentum) cannot be simultaneously well defined and that the act of measurement generically modifies the state of the system. Entanglement and no-cloning are rapidly claiming their place in the list of well-known quantum features: coming next in the queue are the feats of quantum information such as the possibility of secure cryptography or the teleportation of unknown states.

These features are so striking, that one could hope some of them provide the physical ground behind the formalism. Is quantum physics, for instance, the most general theory that allows violations of Bell inequalities, while satisfying no-signaling? The question was first asked by Popescu and Rohrlich and the answer was found to be negative: impossibility of being represented in terms of local variables is a property shared by a broad class of no-signaling theories. Such theories predict intrinsic randomness, no-cloning, an information-disturbance trade-off and allow for secure cryptography. As for teleportation and entanglement swapping, after a first negative attempt, it seems that they can actually be defined as well within the general no-signaling framework. In summary, most of the features that have been highlighted as “typically quantum” are actually shared by all possible no-signaling theories. Only a few discrepancies have been noticed: some no-signaling theories would lead to an implausible simplification of distributed computational tasks and would exhibit very limited dynamics. This state of affairs highlights the importance of the no-signaling principle but leaves us still rather in the dark about the specificity of quantum theory.

In the present paper we define and study a previously unnoticed feature, which we call Information Causality. Information Causality generalizes no-signaling and is respected by both classical and quantum physics. However, as we shall show, it is violated by all no-signaling theories that are endowed with correlations which are stronger than the strongest quantum correlations. It can therefore be used as a principle to distinguish physical theories from nonphysical ones and is a good candidate to be one of the foundational assumptions which are at the very root of quantum theory.
Formulated as a principle, Information Causality states that information gain that Bob can reach about previously unknown to him data set of Alice, by using all his local resources and m classical bits communicated by Alice, is at most m bits. The standard no-signaling condition is just Information Causality for m = 0. It is important to keep in mind that the principle assumes classical communication: if quantum bits were allowed to be transmitted the information gain could be higher as demonstrated in the quantum super-dense coding protocol[22]. The efficiency of this protocol is based on the use of quantum entanglement and Information Causality holds true even if the quantum bits are transmitted provided they are disentangled from the systems of the receiver. This follows from the Holevo bound, which limits information gain after transmission of m such qubits to m classical bits.

We demonstrate that in a world in which certain tasks are “too simple” (compare with Refs.[17][18]), and there exists implausible accessibility of remote data, Information Causality is violated. Consider a generic situation in which Alice has a database of N bits described by a string \( \vec{a} \). She would like to grant Bob access to as big portion of information as possible within fixed amount of classical communication. If there were no pre-established correlations between them, communication of m bits would open access to at most m bits of the database. With pre-shared correlations they could expect to do better (but, as we shall show, in the real world they would be mistaken). For concreteness, consider a generic task illustrated in Figure 1. It is a distributed version of random access coding[23, 24], oblivious transfer[14, 25] and related communication complexity problems[26]. Alice receives a string of N random and independent bits, \( \vec{a} = (a_0, a_1, ..., a_{N-1}) \). Bob receives a random value of \( b = 0, ..., N-1 \), and is asked to give a value of the \( b \thinspace \)th bit of Alice after receiving from her a message of m classical bits. The restrictions are only on the communication that can take place after the inputs have been provided. The resources that Alice and Bob may have shared in advance are assumed to be no-signaling because allowing signaling resources would open other communication channels. In a classical world, these additional resources would be correlated lists of bits; in a quantum world, Alice and Bob may share an arbitrary quantum state. But the task itself is open to accommodate any hypothetical resource producing no-signaling correlations, even such that go beyond the possibilities of quantum physics. We shall call these imaginary resources no-signaling boxes, in short NS-boxes. The impact of stronger-than-quantum correlations on the efficiency of random access coding has been studied recently from a different angle[24].

Clearly, there exists a protocol which allows Bob to give the correct value of at least m bits. If Alice sends him an \( m \)-bit message \( \vec{x} = (x_0, ..., x_{m-1}) \) Bob will guess \( a_b \) perfectly whenever \( b \in \{0, ..., m-1\} \). The price to pay is that he is bound to make a sheer random guess for \( b \in \{m, ..., N-1\} \). Since the pre-shared correlations contain no information about \( \vec{a} \), for every strategy there will be a tradeoff between the probabilities to guess different bits of \( \vec{a} \). Let us denote Bob’s output by \( \beta \). The efficiency of Alice’s and Bob’s strategy can be quantified by

\[
I(\vec{a} : \beta|b = K) = \sum_{K=0}^{N} I(a_K : \beta|b = K)
\]

where \( I(a_K : \beta|b = K) \) is the Shannon mutual information between \( a_K \) and \( \beta \), computed under the condition that Bob has received \( b = K \). One can also show that

\[
I \geq N - \sum_{K=0}^{N} h(P_K),
\]

where \( h(x) = -x \log_2 x - (1-x) \log_2(1-x) \) is the binary entropy of \( x \), and \( P_K \) is the probability that \( a_K = \beta \), again for the case of \( b = K \). To get the inequality, the \( a_K \) have been assumed to be unbiased and independently distributed (see details in the Supplementary Information).

Ideally, we would like to define that Information Causality holds if after transferring the \( m \)-bit message, the mutual information between Alice’s data \( \vec{a} \) and everything that Bob has, i.e. the message \( \vec{x} \) and his part \( B \) of the pre-shared correlation, is bounded by \( m \). As intuitively appealing such a definition is, it has the severe issue that it is not theory-independent. Specifically, a mutual information expression “\( I(\vec{a} : \vec{x}, B) \)” has to be defined for a state involving objects from the underlying nonlocal theory (the possibilities include classical correlation, a shared quantum state, NS-boxes, etc.). It is far from clear whether mutual information can be defined consistently for all nonlocal correlations, nor whether such a definition would be unique.
Instead, we shall show that if a mutual information can be defined that obeys three elementary properties, then (a) Information Causality holds and (b) \( I(\tilde{a} : \tilde{x}, B) \geq I \). Thus we obtain the following necessary condition for Information Causality:

\[ I \leq m. \] (3)

We stress that the parameter \( I \) is independent of any underlying physical theory: \( I \) does not involve any details of a particular physical model but is fully determined by Alice’s and Bob’s input bits and Bob’s output. In this sense it resembles Bell’s parameter\(^2\), which also involves only random variables and can be used to test different physical theories.

For a system composed of parts \( A, B, C \), prepared in a state allowed by the theory, we need to assign symmetric and non-negative mutual informations \( I(A : B) \), etc. The elementary properties mentioned above are the following.

1. **Consistency:** If the subsystems \( A \) and \( B \) are both classical, then \( I(A : B) \) should coincide with Shannon’s mutual information.

2. **Data processing inequality:** Acting on one of the parts locally by any state transformation allowed in the theory cannot increase the mutual information. I.e., if \( B \rightarrow B’ \) is a permissible map between systems, then \( I(A : B) \geq I(A : B’) \). This says that any local manipulation of data can only decay information.

3. **Chain rule:** There exists a conditional mutual information \( I(A : B|C) \) such that the following identity is satisfied for all states and triples of parts: \( I(A : B, C) = I(A : C) + I(A : B|C) \). Note that this implies an identity between ordinary mutual informations:

\[
I(A : B, C) - I(A : B) = I(A : B|C) = I(A, C : B) - I(B : C).
\]

Information Causality holds both in classical and quantum physics; we may focus on the latter because the former is a special case of it. This is because one can define quantum mutual information in formal extension of Shannon’s quantity, using von Neumann entropy\(^27\), and all three of the above properties are fulfilled\(^28\). Details can be found in the Supplementary Information, but in a nutshell one argues as follows:

To show (a), denote by \( B \) Bob’s quantum system holding the shared quantum state \( \rho_{AB} \), Alice’s data \( \tilde{a} = (a_0, \ldots, a_{N-1}) \), and the \( m \)-bit message \( \tilde{x} \); our objective is to prove \( I(\tilde{a} : \tilde{x}, B) \leq m \). First, the chain rule for mutual information yields \( I(\tilde{a} : \tilde{x}, B) = I(\tilde{a} : B) + I(\tilde{a} : \tilde{x}|B) \). Second, \( I(\tilde{a} : B) = 0 \) because without the message Alice’s data and Bob’s quantum state are independent (expressing the no-signaling condition). Third, we use chain rule again to express the conditional mutual information as \( I(\tilde{a} : \tilde{x}|B) = I(\tilde{x} : \tilde{a}, B) - I(\tilde{x} : \tilde{a}) \leq I(\tilde{x} : \tilde{a}, B) \). Finally, the latter can be upper bounded by \( I(\tilde{x} : \tilde{a}) \leq m \), invoking data processing. Similarly, (b) is obtained by repeated application of the chain rule, data processing inequality and non-negativity of mutual information (see the Supplementary Information for details).

In order to study how other no-signaling theories can violate Information Causality, we focus on the necessary condition \([3]\). First consider the simplest example of two-bit input of Alice, \((a_0, a_1)\); it is described in Figure 2. The probability that Bob correctly gives the value of the bit \( a_0 \) is

\[
P_1 = \frac{1}{2} \left[ P(A \oplus B = 0|0,0) + P(A \oplus B = 0|1,0) \right], \quad (4)
\]

and the analogous probability for the bit \( a_1 \) reads

\[
P_{11} = \frac{1}{2} \left[ P(A \oplus B = 0|0,1) + P(A \oplus B = 1|1,1) \right], \quad (5)
\]

where the symbol \( \oplus \) denotes summation modulo 2.

One can recognize that these probabilities are intimately linked with the Clauser–Horne–Shimony–Holt parameter\(^{29}\) \( S \), which can be used to quantify the strength of correlations. Indeed,

\[
S = \sum_{a=0}^{1} \sum_{b=0}^{1} P(A \oplus B = ab|a,b) = 2 \left( P_1 + P_{11} \right). \quad (6)
\]

The classical correlations are bounded by \( S \leq S_C = 3 \) (the equivalent form of Bell inequality\(^2,29\)). Quantum correlations exceed this limit up to \( S \leq S_Q = 2 + \sqrt{2} \) (the
The protocol works just as well for any Boolean function of the inputs, \( f(\bar{a}, b) \). It is sufficient that Alice inserts to her PR-box the sum of \( f(\bar{a}, 0) \oplus f(\bar{a}, 1) \). If Information Causality is maximally violated, Bob can learn the value of \( f(\bar{a}, b) \) for any of his inputs, irrespectively of Alice’s input data. Even more surprisingly, this is so also if he does not know the function to be computed.

We shall now demonstrate that Information Causality is violated as soon as the quantum Tsirelson limit for the CHSH inequality is exceeded. This result of ours can be also seen as an information-theoretic proof of the Tsirelson bound, independent of the formalism of Hilbert spaces, relying instead only on the existence of a consistent information calculus for certain correlations.

First we note that, using a suitable local randomization procedure that does not change the value of the parameter \( S \), any NS-box can be brought to a simple form\[^3\]: the local outcomes are uniformly random and the correlations are given by

\[
P(A \oplus B = ab | a, b) = \frac{1}{2} (1 + E),
\]

with \( 0 \leq E \leq 1 \). The case \( E = 1 \) corresponds to the PR-box; \( E = 0 \) describes uncorrelated random bits. The classical bound \( S \leq S_C \) is violated as soon as \( E > \frac{2}{\sqrt{3}} \); the Tsirelson bound of quantum physics becomes \( E \leq E_Q = \frac{1}{\sqrt{2}} \), attained by performing suitable measurements on the singlet state of two two-level systems\[^2\]\[^30\].

The bound that Information Causality imposes on correlations can be identified using a pyramid of NS-boxes and nesting the simple protocol described above (see Figure 3). Now Alice receives \( N = 2^n \) input bits and the probability that Bob guesses \( a_K \) correctly is given by

\[
P_K = \frac{1}{2} [1 + E^n].
\]

Inserting this expression into \[^4\], one finds that the Information Causality condition \( I \leq 1 \) is violated as soon as \( 2E^2 > 1 \) and \( n \) large enough, i.e. \( E > E_Q \). Since all NS-boxes can be brought to the form \[^7\] without changing the value of \( S \), we conclude indeed that every NS-box with stronger than quantum correlations violates the Information Causality condition. In Supplementary Information the more general result is proved, that for any

\[
\frac{1}{2} (E_1^2 + E_2^2) > E_Q^2
\]

where \( E_J = 2P_J - 1 \) — see eqs. \[^1\] and \[^5\]. Information Causality is violated, and conversely if it is fulfilled, that there exists a quantum correlation with these probabilities.

It is fulfilled, that there exists a quantum correlation with these probabilities.

In conclusion, we have identified the principle of Information Causality, which precisely distinguishes physically realized correlations from nonphysical ones (in the sense that quantum mechanics cannot reach them). It is phrased in operational terms and in a theory-independent way and therefore we suggest it is at the same foundational level as the no-signaling condition itself, of which it is a generalization.

The new principle is respected by all correlations accessible with quantum physics while it excludes all no-signaling correlations, which violate the quantum Tsirelson bound. Among the correlations that do not violate that bound it is not known whether Information Causality singles out exactly those allowed by quantum physics. If it does, the new principle would acquire even
stronger status.

We thank Matthias Christandl, Vlatko Vedral and Stephanie Wehner for stimulating discussions. This work was supported by the National Research Foundation and the Ministry of Education in Singapore, and by the European Commission through IP “QAP”. AW acknowledges support by the U.K. EPSRC through the “QIP IRC” and an Advanced Fellowship, by a Royal Society Wolfson Merit Award, and a Philip Leverhulme Prize.

[1] Wootters, W. K., Zurek, W. H. A single quantum cannot be cloned. Nature 299, 802-803 (1982).
[2] Bell, J. S. On the Einstein Podolsky Rosen paradox. Physics 1, 195-200 (1964).
[3] Popescu, S., Rohrlich, D. Quantum nonlocality as an axiom. Found. Phys. 24, 379-385 (1994).
[4] Bennett, C.H., Brassard, G. Quantum Cryptography: Public key distribution and coin tossing. Quantum Cryptography: Public key distribution and coin tossing. Proceedings IEEE Int. Conf. on Computers, Systems and Signal Processing, Bangalore, India (IEEE, New York), p. 175-179 (1984).
[5] Ekert, A.K. Quantum cryptography based on Bells theorem. Phys. Rev. Lett. 67, 661-663 (1991).
[6] Bennett, C.H., Brassard, G., Cr´epeau, C., Jozsa, R., Peres, A., Wootters, W.K. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Phys. Rev. Lett. 70, 18951899 (1993).
[7] Masanes, Ll., Ac´in, A., Gisin, N. General properties of nonsignaling theories. Phys. Rev. A. 73, 012112 (2006).
[8] Barnum, H., Barrett, J., Leifer, M., Wilce, A. Generalized No-Broadcasting Theorem. Phys. Rev. Lett. 99, 240501 (2007).
[9] Scarani, V., Gisin, N., Brunner, N., Masanes, Ll., Pino, S., Acín, A. Secrecy extraction from no-signaling correlations. Phys. Rev. A 74, 042339 (2006).
[10] Barrett, J., Hardy, L., Kent, A. No Signaling and Quantum Key Distribution. Phys. Rev. Lett. 95, 010503 (2005).
[11] Acín, A., Gisin, N., Masanes, Ll. From Bell’s Theorem to Secure Quantum Key Distribution. Phys. Rev. Lett. 97, 120405 (2006).
[12] Masanes, Ll. Universally-composable privacy amplification from causality constraints. Phys. Rev. Lett. 102, 140501 (2009), and references therein.
[13] Żukowski, M., Zeilinger, A., Horne, M.A., Ekert, A.K. “Event-ready-detectors” Bell experiment via entanglement swapping. Phys. Rev. Lett. 71, 4287 (1993).
[14] Short, A.J., Popescu, S., Gisin, N. Entanglement swapping for generalized nonlocal correlations. Phys. Rev. A 73, 012101 (2006).
[15] Barnum, H., Barrett, J., Leifer, M., Wilce, A. Teleportation in General Probabilistic Theories. arXiv:0805.3553.
[16] Skrzypczyk, P., Brunner, N., Popescu, S. Emergence of Quantum Correlations from Nonlocality Swapping. Phys. Rev. Lett. 102, 110402 (2009).
[17] van Dam, W. Ph.D. thesis, University of Oxford (2000); also: Implausible Consequences of Superstrong Nonlocality. arXiv:quant-ph/0501159.
[18] Brassard, G., Buhrman, H., Linden, N., Méthot, A.A., Tapp, A., Unger, F. Limit on Nonlocality in Any World in Which Communication Complexity Is Not Trivial. Phys. Rev. Lett. 96, 250401 (2006).
[19] Linden, N., Popescu, S., Short, A.J., Winter, A. Quantum Nonlocality and Beyond: Limits from Nonlocal Computation. Phys. Rev. Lett. 99, 180502 (2007).
[20] Brunner, N., Skrzypczyk, P. Non-locality distillation and post-quantum theories with trivial communication complexity. Phys. Rev. Lett. 102, 160403 (2009).
[21] Barrett, J. Information processing in generalized probabilistic theories. Phys. Rev. A 75, 032304 (2007).
[22] Bennett, C.H., Wiesner, S.J. Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. Phys. Rev. Lett. 69, 2881-2884 (1992).
[23] Ambainis, A., Nayak, A., Ta-Shma, A., Vazirani, U. Quantum dense coding and quantum finite automata. Journal of the ACM 49, 496-511 (2002).
[24] Ver Steeg, G., Wehner, S. Relaxed uncertainty relations and information processing. arXiv[quant-ph]:0811.3771 (2008).
[25] Wolf, S., Wallischegler, J. Oblivious transfer and quantum non-locality. arXiv:quant-ph/0502030.
[26] Brassard, G. Quantum communication complexity. Found. Phys. 33, 1593-1616 (2003).
[27] Cerf, N.J., Adami, C. Negative Entropy and Information in Quantum Mechanics. Phys. Rev. Lett. 79, 5194-5197 (1997).
[28] Nielsen, M.A., Chuang, I.L. Quantum Computation and Quantum Information (Cambridge University Press, 2000).
[29] Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A. Proposed Experiment to Test Local Hidden-Variable Theories. Phys. Rev. Lett. 23, 880-884 (1969).
[30] Cirel’son, B.S. Quantum generalizations of Bell’s inequality. Lett. Math. Phys. 4, 93-100 (1980).

SUPPLEMENTARY INFORMATION

THE GENERIC NATURE OF THE CONSIDERED TASK

Assume that Alice has a data set, a CD or whatever, which can be encoded into a N-bit string, ⃗a. Bob may wish to have an access to that set of data. Of course, without any communication he has no access at all. However, if they share randomness, or a source of randomness, and a protocol, Alice can, by transferring m bits, allow him an access to a specific m-bit subsequence of her data. Thus N − m bits are still inaccessible to Bob. Transfer of m-bits reduces the number of inaccessible bits from N to N − m, or more if the protocol is not optimal. We have an accessibility gain of up to m bits. PR-boxes clearly violate this limit. A transfer of m-bits, due to no-signaling still allows to access m bit sequences only, however the full data set is open for such a readout. The number of inaccessible bits is reduced to 0. Accessibility gain is N.

In the most elementary case of just one bit transfer, the
Information Causality allows one in an optimal protocol to decode the value of just one specific bit. For PR-boxes one bit transfer opens access to any bit. That is, all bits are readable, with no-signaling constraining the actual readout to just one of them.

Further, note that, any Boolean function of Alice’s and Bob’s data can be put in the following way

$$ f(\vec{a}, b) = \sum_{b'} \delta_{bb'} f'_a(b'), $$

where $f'_a(b')$ is again Boolean. For example, the original data string is a simple function of this kind $A_\vec{x}(b) = a_b$. Therefore any function $f'_a(b)$ is just a preparation of a new string of data, in form of a Boolean function of the old string. It has exactly the same length. Thus our problem contains within itself a completely general problem of obtaining the value of any $f(\vec{a}, b)$. That is, it is a generic problem for dichotomic functions.

**INFORMATION CALCULUS AND INFORMATION CAUSALITY**

Here we prove bounds for the mutual information $I(\vec{a} : \vec{x}, B)$, where $\vec{a}$ is a string of Alice’s bits, $\vec{x}$ is her classical message and $B$ denotes Bob’s part of the pre-shared no-signalling resource, assuming the three abstract properties for the mutual information, as follows.

1. **Consistency:** If the subsystems $A$ and $B$ are both classical, then $I(A : B)$ should coincide with Shannon’s mutual information.

2. **Data processing inequality:** Acting on one of the parts locally by any state transformation allowed in the theory cannot increase the mutual information. I.e., if $B \rightarrow B'$ is a permissible map between systems, then $I(A : B) \geq I(A : B')$. This says that any local manipulation of data can only decay information.

3. **Chain rule:** There exists a conditional mutual information $I(A : B|C)$ such that the following identity is satisfied for all states and triples of parts: $I(A : B, C) = I(A : B|C) + I(A : B|C)$. Note that this implies an identity between ordinary mutual informations:

$$ I(A : B, C) - I(A : C) = I(A : B|C) = I(A : C : B) - I(B : C) $$

We show first $I(\vec{a} : \vec{x}, B) \geq I(\vec{a} : \vec{x}, B) = \sum_{K=0}^{N-1} I(a_K : \vec{x}, B)$ in the case of independent Alice’s input bits. Namely, by the chain rule (3), we can isolate Alice’s first bit, obtaining

$$ I(0, \ldots, a_{N-1} : \vec{x}, B) = I(0 : \vec{x}, B) + I(a_1, \ldots, a_{N-1} : \vec{x}, B | a_0). $$

The second term on the right-hand side equals, using chain rule once more, $I(a_1, \ldots, a_{N-1} : \vec{x}, B | a_0) = I(a_1, \ldots, a_{N-1} : \vec{x}, B | a_0) - I(a_1, \ldots, a_{N-1} : a_0)$, in which, due to the independence of Alice’s inputs, $I(a_1, \ldots, a_{N-1} : a_0) = 0$. Applying the data processing inequality (2) to the first term here then implies

$$ I(\vec{a} : \vec{x}, B) \geq I(0 : \vec{x}, B) + I(a_1, \ldots, a_{N-1} : \vec{x}, B). \tag{9} $$

Iterating these steps $N - 1$ times to the rightmost information gives

$$ I(\vec{a} : \vec{x}, B) \geq \sum_{K=0}^{N-1} I(a_K : \vec{x}, B). \tag{10} $$

Finally, we observe that Bob’s guess bit $\beta$ is obtained at the end from $b$, $\vec{x}$ and $B$. Hence, the data processing inequality puts a limit of $I(a_K : \beta | b = K) \leq I(a_K : \vec{x}, B)$ on Bob’s accessible information. Putting this together with eq. (10) yields the result,

$$ I(\vec{a} : \vec{x}, B) \geq \sum_{K=0}^{N-1} I(a_K : \beta | b = K) \equiv I, \tag{11} $$

which is the efficiency described in the main text. Note that implicitly we have made use of the consistency condition (1) here.

Second, we prove that the same assumptions lead to $I(\vec{a} : \vec{x}, B) \leq m$, i.e. Information Causality. To do so we need a little preparation. Note that from consistency (1) and data processing (2), we inherit automatically an important property of Shannon mutual information, namely the fact that $I(A : B) = 0$ if two systems $A$ and $B$ are independent, i.e. the state is a (tensor) product of states on $A$ and on $B$, respectively. To prove this, observe that the state can thus be prepared by allowed local operations starting from two classical independent systems. These have zero mutual information by consistency (1), so $I(A : B) \leq 0$ by data processing (2). On the other hand, mutual information must be non-negative, hence $I(A : B) = 0$.

Now,

$$ I(\vec{a} : \vec{x}, B) = I(\vec{a} : B) + I(\vec{a} : \vec{x}|B) $$

$$ = I(\vec{a} : B) $$

$$ = I(\vec{x} : \vec{a}, B) - I(\vec{x} : B) $$

$$ \leq I(\vec{x} : \vec{a}, B), $$

where we have invoked the chain rule (3), the independence of $\vec{a}$ from $B$ (which owes itself to the no-signaling condition), chain rule once more and non-negativity of mutual information.

We are finished now once we argue that $I(\vec{x} : \vec{a}, B) \leq I(\vec{x} : \vec{x})$, because the latter is a quantity only involving classical objects, so it can be evaluated as the Shannon entropy of $\vec{x}$ by the consistency requirement (1), and the entropy is upper bounded by $m$. But this inequality follows once more from data processing (2), because the joint state of $\vec{x}$, $\vec{a}$ and $B$ is given by some distribution on $\vec{x}$, and a joint state for $\vec{a}$ and $B$ for each value $\vec{x}$ can take. In other words, there is a state preparation for each value of $\vec{x}$, hence there must exist the corresponding state transformation $\vec{x} \rightarrow \vec{a}, B$ in the theory.
SIMPLIFIED LOWER BOUND ON $I$

The conditional mutual informations can be simplified using the probability of Bob’s correct guess of $a_K$, denoted by $P_K$, i.e. the probability that $a_K \oplus \beta = 0$, given $b = K$. Since Alice’s inputs are uniformly random, the binary entropy $h(a_K) = 1$, we have $I(a_K : \beta|b = K) = 1 - H(a_K|\beta, b = K)$. Note that, $H(a_K|\beta, b = K) = H(a_K \oplus \beta|\beta, b = K)$ because knowing $\beta$ leaves the same uncertainty about $a_K$ and $a_K \oplus \beta$ (this can also be proved using the chain rule for conditional entropy). Omitting the conditioning on $\beta$ can only increase the entropy, $H(a_K \oplus \beta|\beta, b = K) \leq H(a_K \oplus \beta|b = K) = h(P_K)$. Therefore,

$$I \geq N - \sum_{K=0}^{N-1} h(P_K), \quad (12)$$

as stated in the main text. This inequality can also be seen as a special case of Fano’s inequality [28]. In a more general case, in which Alice’s inputs acquire values from an alphabet of $d$ elements, Fano’s inequality gives the bound

$$I \geq N \log_2 d - \sum_{K=0}^{N-1} h(P_K) - \sum_{K=0}^{N-1} (1 - P_K) \log_2 (d-1). \quad (13)$$

Similarly, one can write a bound for any inputs of Alice.

Since the necessary condition for Information Causality to hold reads $I \leq m$, one finds by looking at the expression [12] that Information Causality limits the probability of Bob’s correct guess, unless all information about Alice’s bits is communicated to Bob:

$$\sum_{K=0}^{N} h(P_K) \geq N - m. \quad (14)$$

INFORMATION CAUSALITY IN CLASSICAL AND QUANTUM PHYSICS

Here we show that Information Causality holds in classical and quantum physics. All we have to do, in the light of our previous reasoning, is to write down expressions for the mutual information and conditional mutual information, and confirm that they satisfy properties (1)–(3). We focus on quantum correlations because classical correlations form a subset of quantum correlations. With respect to any tripartite state $\rho_{ABC}$, denote by $\rho_A$, etc. its reduced states, and write $S(\rho) = -\text{Tr} \rho \log \rho$ for the von Neumann entropy. Then let

$$I(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

$$I(A : B|C) = S(\rho_{AC}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_C).$$

Both expressions are manifestly invariant under swapping $A$ and $B$, and non-negative by strong subadditivity [28]. Clearly, consistency (1) holds, as classical correlations are embedded as matrices diagonal in some fixed local bases and then von Neumann entropy reduces to Shannon entropy. Also the chain rule (3) is an easily verified identity. The data processing inequality (2) is equivalent once more to the data processing inequality [28].

To verify the steps in our abstract derivation of Information Causality in the quantum case, denote the initial state shared between Alice and Bob by $\rho_{AB}$. Including Alice’s data as orthogonal states of a reference system $R$, the situation before the communication can be described by the state

$$\frac{1}{2^N} \sum_{\bar{a}\in\{0,1\}^N} |\bar{a}\rangle_R \otimes \rho_{AB}. \quad (15)$$

For each value of $\bar{a}$ Alice has to perform local operations to obtain the message $\bar{x}$ she wants to send to Bob. Whatever her algorithm to do so, it can be condensed into a quantum measurement (POVM) $(M^{(\bar{a})}_{\bar{x}})_{\bar{x}\in\{0,1\}^m}$, and so the joint state of Alice’s data, the message (represented by orthogonal states of a “message” system $X$) and Bob’s system is given by

$$\frac{1}{2^N} \sum_{\bar{a}\in\{0,1\}^N} |\bar{a}\rangle_R \otimes \sum_{\bar{x}\in\{0,1\}^m} \text{Tr}_A (\rho_{AB} (M^{(\bar{a})}_{\bar{x}}) \otimes \mathbb{I}). \quad (16)$$

TSIRELSON BOUND FROM INFORMATION CAUSALITY

We present a proof that Information Causality is violated by all stronger than quantum correlations. Our protocol of the main proof uses NS-boxes, which produce uniformly random local outcomes and correlations described by the probabilities $P(A \oplus B = ab|a,b)$, where $a,b = 0,1$ are the inputs to the boxes and $A,B = 0,1$ are the outputs for Alice and Bob respectively. It will be sufficient to consider the situation where Alice communicates only one bit to Bob, i.e. $m = 1$. The number of Alice’s input bits is chosen as $N = 2^n$, where $n$ is an integer parameterizing the task. Correspondingly, Bob receives $n$ input bits to encode the index $b$ as a binary string $(b_0, b_1, \ldots, b_{n-1})$, i.e. $b = \sum_{k=0}^{n-1} b_k 2^k$.

We generalize the procedure given in the main text to $N$ bits, recursively, using the insight of [17] that any function, which can be written as a Boolean formula with ANDs, XORs and NOTs, can be computed in a distributed manner using the same number of PR-boxes [3] as ANDs and one bit of communication. The function we are considering is $f_n(\bar{a}, b) \equiv a_b$, with $\bar{a} = (a_0, a_1, \ldots, a_{N-1})$. In the simplest case, $n = 1$, the function of the task reads

$$f_1((a_0, a_1), b) = a_0 \oplus b_0 (a_0 \oplus a_1). \quad (17)$$
It involves a single AND. Alice inputs $a_0 \oplus a_1$ into the PR-box, Bob $b_0$; with her output $A$ Alice forms the message $x = a_0 \oplus A$, so that Bob can obtain $x \oplus B = a_0$.

Moving to $n > 1$, write $\vec{a} = \vec{a}' \vec{a}''$ with two bit-strings $\vec{a}'$ and $\vec{a}''$ of length $N/2 = 2^{n-1}$ each. Then it is a straightforward exercise to verify that

$$f_n(\vec{a}, b) = f_{n-1}(\vec{a}', b') \oplus b_{n-1} \left[ f_{n-1}(\vec{a}'', b') \oplus f_{n-1}(\vec{a}'', b') \right],$$

where $b'$ is the string of $n - 1$ Bob’s bits ($b_0, \ldots, b_{n-2}$). Thus, if $f_{n-1}$ could be written using $N/2 - 1$ ANDs, this formula expresses $f_n$ using $N - 1$ AND operations. For instance,

$$f_2((a_0, a_1, a_2, a_3), (b_0, b_1)) = a_0 \oplus b_0(a_0 \oplus a_1) \oplus b_1(a_0 \oplus b_0(a_0 \oplus a_1) \oplus a_2 \oplus b_0(a_2 \oplus a_3)).$$

To convert this to a distributed protocol, Alice and Bob use three PR-boxes. To the first one Alice inputs $a_0 \oplus a_1$, to the second one $a_2 \oplus a_3$, and to the third one $a_0 \oplus a_1 \oplus a_2 \oplus A$, where $A_1$ and $A_2$ are her outputs from the first and second box, respectively. She transmits $x = a_0 \oplus A_1 \oplus A_3$, where $A_3$ is her output from the third box. Depending on his inputs, Bob will use two different boxes to decode $a_b$. He inserts the bit $b_1$ distinguishing groups $(a_0, a_1)$ and $(a_2, a_3)$ to the third box, obtaining output $B_3$. Due to the correlations of the boxes, the sum $x \oplus B_3$ gives the value of $a_0 \oplus A_1$ or $a_2 \oplus A_2$, depending on $b_1 = 0$ or $b_1 = 1$. These are exactly the messages he would obtain from Alice in the scenario with the single pair of boxes, and the protocol is now reduced to the previous one. For example, if $b_1 = 1$ he will input $b_0$ to the second PR-box, giving him an output $B_2$, so that he can form $x \oplus B_3 \oplus B_2 = a_2 \oplus a_1 \oplus a_2 \oplus a_2$, which is either $a_2$ or $a_3$. Note that the other PR-box is ignored by Bob, or he may as well input $b_0$ to it, too – it is not important because he doesn’t need its output.

In the general case, Alice and Bob share $N - 1 = 2^n - 1$ PR-boxes, and for every set of inputs Bob uses $n$ of them. The protocol can be explained recursively, based on eq. [18]. Alice and Bob use the protocol for $n - 1$ Bob’s bits on the input pairs $(\vec{a}', b')$ and $(\vec{a}'', b')$, involving $N/2 - 1$ PR-boxes in each one, resulting in two single-bit messages $x'$ and $x''$ that Alice would send to Bob if their objective were to compute $f_{n-1}$. Instead, she inputs $x' \oplus x''$ into the last PR-box, while Bob inputs $b_{n-1}$: they obtain an output bits $A$ and $B$, respectively, so when Alice finally sends the message $x = x' \oplus A$, this allows Bob to obtain $x' \oplus b_{n-1}(x' \oplus x'')$, which is either $x'$ or $x''$ depending on $b_{n-1} = 0$ or $b_{n-1} = 1$. Then, the protocol for $n - 1$ bits tells him the outputs of which PR-boxes he should use in order to arrive at $a_b$. Since in the protocol for $n - 1$ Bob’s bits he only needs the outputs of $n - 1$ boxes, he reads $n$ outputs for $n$ bits; likewise, Alice uses $2(N/2 - 1) + 1 = N - 1$ PR-boxes in total.

Now, we simply substitute PR-boxes with NS-boxes, having their probabilities of guessing first and second bit of Alice’s input sum given by $P_1$ and $P_{11}$, respectively. By looking at his input bits, Bob finds that the final guess of the value of $a_0$ involves aiming $(n - k)$ times at the left bit and $k$ times at the right, where $k = b_0 + \cdots + b_{n-1}$ is the number of 1s in the binary decomposition of $b$. Since Bob’s answer is computed as the sum of the message $x$ and suitable outputs of $n$ boxes, whenever even number of boxes produce “wrong” outputs, i.e. such that $A \oplus B \neq ab$, Bob still arrives at the correct final answer. Therefore, Bob’s guess is correct whenever he has made an even number of errors in the intermediate steps.

Let us denote by

$$Q^{(k)}_{\text{even}}(P) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} P^{k-2j} (1 - P)^{2j} = \frac{1}{2} [1 + (2P - 1)^k]$$

the probability to make an even number of errors when using $k$ pairs of boxes, each producing a correct value with probability $P$. Similarly, the probability to make an odd number of errors reads

$$Q^{(k)}_{\text{odd}}(P) = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2j+1} P^{k-2j-1} (1 - P)^{2j+1} = \frac{1}{2} [1 - (2P - 1)^k].$$

With this notation, the probability that Bob’s final guess of the value of $a_K$ is correct is given by

$$P_K = Q^{(n-k)}_{\text{even}}(P_1) Q^{(k)}_{\text{even}}(P_{11}) + Q^{(n-k)}_{\text{odd}}(P_1) Q^{(k)}_{\text{odd}}(P_{11}) = \frac{1}{2} [1 + E_1^{n-k} E_{11}^{k}]$$

with $E_j = 2P_j - 1$.

We are ready to compute the Information Causality quantity (1) of the main text:

$$I = \sum_{k=1}^{N} [1 - h(P_K)]$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[ 1 - h \left( \frac{1 + E_1^{n-k} E_{11}^{k}}{2} \right) \right]$$

$$\geq \frac{1}{2 \ln 2} \sum_{k=0}^{n} \binom{n}{k} (E_1^{2})^{n-k}(E_{11}^{2})^{k}$$

$$= \frac{1}{2 \ln 2} \left( E_1^{2} + E_{11}^{2} \right)^n$$

where we have used $1 - h \left( \frac{1+\eta}{2} \right) \geq \frac{\eta^2}{2 \ln 2}$. Therefore, if

$$E_1^{2} + E_{11}^{2} > 1,$$

there exist $n$ such that $I > 1$.

It is also possible, and not difficult, to show that whenever $E_1$ and $E_{11}$ do not violate Information Causality then
there exists a quantum protocol that gives such correlations.

For the isotropic correlations (6) of the main text, $E_I = E_H = E$, hence eq. (22) becomes $P_K = \frac{1}{2} [1 + E^n]$ and eq. (24) becomes $2E^2 > 1$ as stated in the main text.