Sum of K-frames in Hilbert C*-Modules

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Abstract. In this paper, we investigate some conditions under which the action of an operator on a K-frame, remain again a K-frame for Hilbert module E. We also give a generalization of Douglas theorem to prove that the sum of two K-frames under certain condition is again a K-frame. Finally, we characterize the K-frame generators in terms of operators.

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [6]. Frames can be viewed as redundant bases which are generalization of orthonormal bases. Many generalizations of frames were introduced, e.g., frames of subspaces [4], Pseudo-frames [1], G-frames [17], and fusion frames [3]. Recently, L. Gavruta introduced the concept of K-frame for a given bounded operator K on Hilbert space in [10]. Hilbert C*-modules arose as generalizations of the notion of Hilbert space. The basic idea was to consider modules over C*-algebras instead of linear spaces and to allow the inner product to take values in the C*-algebra of coefficients being C-(anti-)linear in its arguments [13]. In [8] authors generalized frame concept for operators in Hilbert C*-modules. The paper is organized as follows. In Section 2, some notations and preliminary results of Hilbert Modules, their frames and K-frames are given. In Section 3, we study the action of operators on K-frames and under certain conditions, we shall show that it is again a K-frame. The next section is devoted to sum of K-frames. In fact, to show that the sum of two K-frames under certain conditions is again a K-frame we need to say a generalization of the Douglas Theorem [18], which may interest by its own. Finally, in the last section, we consider a unitary system of operators and characterize the K-frame generators in terms of operators. We also look forward to sum of two K-frame generators to be a K-frame generator.

2. Preliminaries

In this section we give some preliminaries about frames, K-frames in Hilbert spaces and Hilbert modules and related operators which we need in the following sections. A finite or countable sequence \(\{f_k\}_{k \in J}\) is
called a frame for a separable Hilbert space $H$ if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{k \in J} |(f, f_k)|^2 \leq B \|f\|^2, \quad \forall f \in H.$$  

Frank and Larson [8] introduced the notion of frames in Hilbert $C^*$-modules as a generalization of frames in Hilbert spaces. A (left) Hilbert $C^*$-module over the $C^*$-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module $E$ equipped with an $\mathcal{A}$-valued inner product satisfy the following conditions:

1. $\langle x, x \rangle \geq 0$ for every $x \in E$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
2. $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in E$,
3. $\langle \cdot, \cdot \rangle$ is $\mathcal{A}$-linear in the first argument,
4. $E$ is complete with respect to the norm $\|x\|^2 = \|\langle x, x \rangle\|_\mathcal{A}$.

Given Hilbert $C^*$-modules $E$ and $F$, we denote by $L_\mathcal{A}(E, F)$ or $L(E, F)$ the set of all adjointable operators from $E$ to $F$ i.e. the set of all maps $T : E \to F$ such that there exists $T^* : F \to E$ with the property

$$\langle Tx, y \rangle = \langle x, T^*y \rangle,$$

for all $x \in E, y \in F$. It is well-known that each adjointable operator is necessarily bounded $\mathcal{A}$-linear in the sense $T(ax) = aT(x)$, for all $a \in \mathcal{A}, x \in E$. We denote $L(E)$ for $L(E, E)$. In fact $L(E)$ is a $C^*$-algebra.

Let $\mathcal{A}$ be a $C^*$-algebra and consider

$$\ell^2(\mathcal{A}) := \{|a_j|_\mathcal{A} \subseteq \mathcal{A} : \sum_j a_j a_j^* \text{ converges in norm in } \mathcal{A}\}.$$  

It is easy to see that $\ell^2(\mathcal{A})$ with pointwise operations and the inner product

$$\langle |a_j|, |b_j| \rangle = \sum_j a_j b_j^*,$$

becomes a Hilbert $C^*$-module which is called the standard Hilbert $C^*$-module over $\mathcal{A}$. Throughout this paper, we suppose $E$ is a Hilbert $\mathcal{A}$-module and $J$ a countable index set. Also, we denote the range of $T \in L(E)$ by $R(T)$, and the kernel of $T$ by $N(T)$. A Hilbert $\mathcal{A}$-module $E$ is called finitely generated (countably generated) if there exists a finite subset $\{|x_1, \ldots, x_j|\}$ (countable set $\{|x_j|\}_{j \in \mathbb{N}}$) of $E$ such that $E$ equals the closed $\mathcal{A}$-linear hull of this set. The basic theory of Hilbert $C^*$-modules can be found in [13].

The following lemma found the relation between the range of an operator and the kernel of its adjoint operator.

**Lemma 2.1.** ([19], Lemma 15.3.5; [13], Theorem 3.2) Let $T \in L(E, F)$. Then

1. $N(T) = N(|T|), N(T^*) = R(T)^+, N(T^*)^+ = R(T)^+ \supseteq R(T)$;
2. $R(T)$ is closed if and only if $R(T^*)$ is closed, and in this case $R(T)$ and $R(T^*)$ are orthogonally complemented with $R(T) = N(T^*)^+$ and $R(T^*) = N(T)^+$.

The following theorem is extended Douglas theorem [7] for Hilbert modules.

**Theorem 2.2.** [18] Let $T^* \in L(G, F)$ and $T \in L(E, F)$ with $\overline{R(T^*)}$ orthogonally complemented. The following statements are equivalent:

1. $T^*T^* \leq \lambda TT^*$ for some $\lambda > 0$;
2. There exists $\mu > 0$ such that $\|T^*z\| \leq \mu \|Tz\|$ for all $z \in F$;
3. There exists $D \in L(G, E)$ such that $T^* = TD$, i.e. the equation $TX = T^*$ has a solution;
4. $R(T^*) \subseteq R(T)$. 
Here, we recall the concept of frame in Hilbert $C^*$-modules which is defined in [8]. Let $E$ be a countably generated Hilbert module over a unital $C^*$-algebra $\mathcal{A}$. A sequence $\{x_j\}_{j \in J} \subset E$ is said to be a frame if there exist two constant $C,D > 0$ such that

$$C\langle x, x \rangle \leq \sum_{j} \langle x, x_j \rangle \langle x_j, x \rangle \leq D\langle x, x \rangle, \text{ for all } x \in E. \quad (1)$$

The optimal constants (i.e. maximal for $C$ and minimal for $D$) are called frame bounds. If the sum in (1) converges in norm, the frame is called standard frame. In this paper all frames consider standard frames. The sequence $\{x_j\}_{j \in J}$ is called a Bessel sequence with bound $D$ if the upper inequality in (1) holds for every $x \in E$.

Let $\{x_j\}_{j \in J}$ be a Bessel sequence for Hilbert module $E$ over $\mathcal{A}$. The operator $T : E \to \ell^2(\mathcal{A})$ defined by $Tx = \{(x,x_j)\}_{j \in J}$ is called the analysis operator. The adjoint operator $T^* : \ell^2(\mathcal{A}) \to E$ which is given by

$$T^*[c_j]_{j \in J} = \sum_{j \in J} c_j x_j,$$

is called the pre-frame operator or the synthesis operator. By composing $T$ and $T^*$, we obtain the frame operator $S : E \to E$ given by

$$Sx = T^*Tx = \sum_{j \in J} \langle x, x_j \rangle x_j, \quad (x \in E).$$

By [8], if $\{x_j\}_{j \in J}$ is a frame, the frame operator is positive and invertible. Also it is the unique operator in $L(E)$ such that the reconstruction formula

$$x = \sum_{j \in J} \langle x, S^{-1}x_j \rangle x_j = \sum_{j \in J} \langle x, x_j \rangle S^{-1}x_j, \quad x \in E,$$

holds. It is easy to see that the sequence $\{S^{-1}x_j\}_{j \in J}$ is a frame for $E$, and it is called the canonical dual frame of $\{x_j\}_{j \in J}$.

**Theorem 2.3.** [14], Proposition 2.2] Let $\{x_j\}_{j \in J}$ be a sequence in $E$ such that $\sum_{j \in J} c_j x_j$ converges for all $c = \{c_j\}_{j \in J} \in \ell^2(\mathcal{A})$. Then $\{x_j\}_{j \in J}$ is a Bessel sequence in $E$.

**Theorem 2.4.** [12] Let $E$ be a finitely or countably generated Hilbert module over a unital $C^*$-algebra $\mathcal{A}$, and $\{x_j\}_{j \in J}$ be a sequence in $E$. Then $\{x_j\}_{j \in J}$ is a frame for $E$ with bounds $C$ and $D$ if and only if

$$C\|x\|^2 \leq \|\sum_{j \in J} \langle x, x_j \rangle \langle x_j, x \rangle\| \leq D\|x\|^2, \quad (x \in E).$$

Najati in [14] extended the concept of atomic system and a $K$-frame to Hilbert modules.

**Definition 2.5.** A sequence $\{x_j\}_{j \in J}$ of $E$ is called an atomic system for $K \in L(E)$ if the following statement hold:

1. The series $\sum_{j \in J} c_j x_j$ converges for all $c = \{c_j\}_{j \in J} \in \ell^2(\mathcal{A})$;
2. There exists $C > 0$ such that for every $x \in E$ there exists $\{a_j\}_{j \in J} \in \ell^2(\mathcal{A})$ such that $\sum_{j \in J} a_j a_j^* \leq C\langle x, x \rangle$ and $Kx = \sum_{j \in J} a_j x_j$.

By Theorem 2.3, the condition (1) in the above definition, actually says that $\{x_j\}_{j \in J}$ is a Bessel sequence.

**Theorem 2.6.** [14] If $K \in L(E)$, then there exists an atomic system for $K$.

**Theorem 2.7.** [14] Let $\{x_j\}_{j \in J}$ be a Bessel sequence for $E$ and $K \in L(E)$. Suppose that $T \in L(E, \ell^2(\mathcal{A}))$ is given by $T(x) = \{(x, x_j)\}_{j \in J}$ and $R(T)$ is orthogonally complemented. Then the following statements are equivalent:
1. \( \{x_j\}_{j \in \mathbb{J}} \) is an atomic system for \( K \);
2. There exist constants \( C, B > 0 \) such that
\[
B\|K^*x\|^2 \leq \| \sum_j \langle x, x_j \rangle \langle x_j, x \rangle \| \leq C\|x\|^2;
\]
3. There exists \( D \in L(E, \ell^2(\mathcal{A})) \) such that \( K = T^*D \).

**Definition 2.8.** Let \( E \) be a Hilbert \( \mathcal{A} \)-module, \( \{x_j\}_{j \in \mathbb{J}} \subset E \) and \( K \in L(E) \). The sequence \( \{x_j\}_{j \in \mathbb{J}} \) is said to be a \( K \)-frame if there exist constants \( C, D > 0 \) such that
\[
C\langle K^*x, K^*x \rangle \leq \sum_{j \in \mathbb{J}} \langle x, x_j \rangle \langle x_j, x \rangle \leq D\langle x, x \rangle, \quad x \in E.
\]

The following theorem gives a characterization of \( K \)-frames using linear adjointable operators.

**Theorem 2.9.** [14] Let \( K \in L(E) \) and \( \{x_j\}_{j \in \mathbb{J}} \) be a Bessel sequence for \( E \). Suppose that \( T \in L(E, \ell^2(\mathcal{A})) \) is given by \( T(x) = \{\langle x, x_j \rangle\}_{j \in \mathbb{J}} \) and \( R(T) \) is orthogonally complemented. Then \( \{x_j\}_{j \in \mathbb{J}} \) is a \( K \)-frame for \( E \) if and only if there exists a linear bounded operator \( L : \ell^2(\mathcal{A}) \to E \) such that \( Lx_j = x_j \) and \( R(K) \subseteq R(L) \), where \( \{e_j\}_{j \in \mathbb{J}} \) is the canonical orthonormal basis for \( \ell^2(\mathcal{A}) \).

3. Operators On \( K \)-frames

In this section we study the action of an operator on a \( K \)-frame. The following lemma shows that the action of an adjointable operator on a Bessel sequence is again a Bessel sequence.

**Lemma 3.1.** Let \( E \) be a Hilbert \( \mathcal{A} \)-module and \( \{x_j\}_{j \in \mathbb{J}} \) be a Bessel sequence. Then \( \{Mx_j\}_{j \in \mathbb{J}} \) is a Bessel sequence for every \( M \in L(E) \).

**Proof.** Since \( \{x_j\}_{j \in \mathbb{J}} \) is a Bessel sequence there exists constant \( D \) such that
\[
\sum_{j \in \mathbb{J}} \langle x, x_j \rangle \langle x_j, x \rangle \leq D\langle x, x \rangle,
\]
for every \( x \in E \). So
\[
\sum_{j \in \mathbb{J}} \langle x, Mx_j \rangle \langle Mx_j, x \rangle = \sum_{j \in \mathbb{J}} \langle M^*x, x_j \rangle \langle x_j, M^*x \rangle \leq D\langle M^*x, M^*x \rangle = D\langle MM^*x, x \rangle \leq D\|M\|^2 \langle x, x \rangle,
\]
for every \( x \in E \). \( \Box \)

**Theorem 3.2.** Let \( E \) be a Hilbert \( \mathcal{A} \)-module, \( K \in L(E) \) and \( \{x_j\}_{j \in \mathbb{J}} \) be a \( K \)-frame for \( E \). Let \( M \in L(E) \) with \( R(M) \subset R(K) \) and \( R(K^*) \) is orthogonally complemented. Then \( \{x_j\}_{j \in \mathbb{J}} \) is an \( M \)-frame for \( E \).

**Proof.** Since \( \{x_j\}_{j \in \mathbb{J}} \) is a \( K \)-frame then there exist positive numbers \( \mu \) and \( \lambda \) such that
\[
\lambda\langle K^*x, K^*x \rangle \leq \sum_{j \in \mathbb{J}} \langle x, x_j \rangle \langle x_j, x \rangle \leq \mu\langle x, x \rangle.
\]
Using Theorem 2.2, the fact that \( R(M) \subset R(K) \) shows that \( MM^* \leq \lambda^* KK^* \) for some \( \lambda > 0 \). So

\[
\langle MM^* x, x \rangle \leq \lambda^* \langle KK^* x, x \rangle,
\]

and hence,

\[
\frac{\lambda}{\lambda^*} \langle MM^* x, x \rangle \leq \lambda \langle K^* x, K^* x \rangle.
\]

From (3), we have

\[
\frac{\lambda}{\lambda^*} \langle MM^* x, x \rangle \leq \sum_{j \in \mathbb{J}} \langle x, x_j \rangle \langle x_j, x \rangle \leq \mu(x, x).
\]

Therefore, \( \{x_j\}_{j \in \mathbb{J}} \) is an \( M \)-frame with the bounds \( \frac{\lambda}{\lambda^*} \) and \( \mu \) for \( E \). \[\Box\]

In the following theorem, we obtain the result of the last theorem by different conditions.

**Theorem 3.3.** Let \( \{x_j\}_{j \in \mathbb{J}} \) be a \( K \)-frame for Hilbert \( \mathfrak{A} \)-module \( E \). Suppose that \( T \in L(E, \ell^2(\mathfrak{A})) \) with \( T(x) = \{(x, x_j)\}_{j \in \mathbb{J}} \) for every \( x \in E \), \( \mathbb{R}(T) \) is orthogonally complemented and \( M \in L(E) \) such that \( R(M) \subset R(K) \). Then \( \{x_j\}_{j \in \mathbb{J}} \) is an \( M \)-frame for \( E \).

**Proof.** By Theorem 2.9, there exists \( L : \ell^2(\mathfrak{A}) \to E \) such that \( Le_j = f_j \), \( j \in \mathbb{J} \) and \( R(K) \subset R(L) \). So \( R(M) \subset R(L) \). Now again by Theorem 2.9, we conclude that \( \{x_j\}_{j \in \mathbb{J}} \) is an \( M \)-frame for \( E \). \[\Box\]

**Theorem 3.4.** Let \( E \) be a Hilbert \( \mathfrak{A} \)-module and \( K \in L(E) \) with the dense range. Let \( \{x_j\}_{j \in \mathbb{J}} \) be a \( K \)-frame for \( E \) and \( T \in L(E) \) has closed range. If \( \{Tx_j\}_{j \in \mathbb{J}} \) is a \( K \)-frame for \( E \) with \( T \) is surjective.

**Proof.** Suppose that \( K^* x = 0 \) for \( x \in E \). Then for each \( y \in E \), \( \langle Ky, x \rangle = \langle y, K^* x \rangle = 0 \) and \( \langle z, x \rangle = 0 \) for each \( z \in E \). Since \( R(K) \) is dense in \( E \), hence \( x = 0 \) and so \( K^* \) is injective. Now, we show that \( T^* \) is injective too. Note that if \( \{Tx_j\}_{j \in \mathbb{J}} \) is a \( K \)-frame for \( E \) with bounds \( \lambda \) and \( \mu \), then

\[
\lambda \|K^* x\|^2 \leq \| \sum_{j \in \mathbb{J}} \langle x, Tx_j \rangle \langle Tx_j, x \rangle \| \leq \mu \|x\|^2,
\]

and therefore,

\[
\lambda \|K^* x\|^2 \leq \| \sum_{j \in \mathbb{J}} \langle T^* x, x_j \rangle \langle x_j, x \rangle \| \leq \mu \|x\|^2.
\]

If \( x \in N(T^*) \) then \( T^* x = 0 \). Hence \( \langle T^* x, x_j \rangle = 0 \) for each \( j \in \mathbb{J} \), and so \( K^* x = 0 \), by the last inequality. Since \( K^* \) is injective, it follows that \( x = 0 \), and so \( T^* \) is injective. Therefore

\[
E = N(T^*) + \mathbb{R}(T) = \mathbb{R}(T) = R(T),
\]

and this completes the proof. \[\Box\]

**Theorem 3.5.** Let \( K \in L(E) \) and \( \{x_j\}_{j \in \mathbb{J}} \) be a \( K \)-frame for \( E \). If \( T \in L(E) \) has closed range, \( R(K^*) \subset R(T) \), \( \mathbb{R}(TK) \) is orthogonal complemented and \( KT = TK \), then \( \{Tx_j\}_{j \in \mathbb{J}} \) is a \( K \)-frame for \( R(T) \).

**Proof.** It was proved in [20] that if \( T \) has closed range, then \( T \) has the Moore-Penrose inverse operator \( T^+ \) such that \( TT^+ T = T \) and \( T^+ T T^+ = T^+ \). So \( TT^+ |_{\mathbb{R}(T)} = I_{\mathbb{R}(T)} \) and \( (TT^+)^* = I = T^+ \). For every \( x \in R(T) \) we have

\[
\langle K^* x, K^* x \rangle = \langle (TT^+)K^* x, (TT^+)K^* x \rangle
\]

\[
= \langle T^+ TK^* x, T^+ TK^* x \rangle
\]

\[
\leq \|T^+\|^2 \langle T^* K^* x, T^* K^* x \rangle,
\]

}\[\Box\]
and so
\[ \|(T^*)^*\|^2 \langle K^*x, K^*x \rangle \leq \langle T^*K^*x, T^*K^*x \rangle. \]

Since \( \{x_j\}_{j \in J} \) is a K-frame, with lower frame bound \( \lambda \) and \( R(T^*K^*) \subseteq R(K^*T^*) \), then by Theorem 2.2, there exists some \( \lambda' > 0 \) such that
\[
\sum_{j \in J} \langle x, T x_n \rangle \langle T x_n, x \rangle = \sum_{j \in J} \langle T^* x, x_n \rangle \langle x_n, T^* x \rangle
\geq \lambda \langle K^*T^* x, K^*T^* x \rangle
\geq \lambda' \lambda' \langle T^*K^* x, T^*K^* x \rangle.
\]

This implies that \( \{Tx_j\}_{j \in J} \) satisfies in lower frame condition. On the other hand, by Lemma 3.1, \( \{Tx_j\}_{j \in J} \) is a Bessel sequence and therefore \( \{Tx_j\}_{j \in J} \) is a K-frame for Hilbert module \( R(T) \).

**Theorem 3.6.** Let \( E \) be a Hilbert \( \mathcal{A} \)-module, \( K \in L(E) \) and \( \{x_j\}_{j \in J} \) be a K-frame for \( E \). Moreover, let \( T \in L(E) \) be a co-isometry such that \( R(T^*K^*) \subseteq R(K^*T^*) \) and \( R(TK) \) is orthogonal complemented. Then \( \{Tx_j\}_{j \in J} \) is a K-frame for \( E \).

**Proof.** Using Lemma 3.1, \( \{Tx_j\}_{j \in J} \) is a Bessel sequence. Also, by Theorem 2.2, there exists \( \lambda' > 0 \) such that
\[ \|(T^*K^*)x\|^2 \leq \lambda' \|K^*T^*x\|^2 \]
for each \( x \in E \). Suppose \( \lambda \) is a lower bound for the K-frame \( \{x_j\}_{j \in J} \). Since \( T \) is a co-isometry, then
\[
\frac{\lambda}{\lambda'} \|K^*x\|^2 = \frac{\lambda}{\lambda'} \|T^*K^*x\|^2
\leq \lambda \|K^*T^*x\|^2
\leq \sum_{j \in J} \langle T^*x, x_n \rangle \langle x_n, T^*x \rangle
= \sum_{j \in J} \langle x, T x_n \rangle \langle T x_n, x \rangle,
\]
which implies that \( \{Tx_j\}_{j \in J} \) is a K-frame for \( E \).

**Remark 3.7.** Consider \( K \in L(E) \) with dense range, \( T \in L(E) \) with closed range such that \( TK = KT \) and \( \{x_j\}_{j \in J} \) is a K-frame for \( E \). Then \( \{Tx_j\}_{j \in J} \) is a K-frame for \( E \) if and only if \( T \) is surjective.

**Theorem 3.8.** Let \( K \in L(E) \) whose range is dense and \( \{x_j\}_{j \in J} \) is a K-frame for \( E \). Moreover, let \( T \in L(E) \) has closed the range. If \( \{Tx_j\}_{j \in J} \) and \( \{T^*x_j\}_{j \in J} \) are K-frames for \( E \), then \( T \) is invertible.

**Proof.** By Theorem 3.4, \( T \) is surjective. Since \( \{T^*x_j\}_{j \in J} \) is a K-frame for \( E \) then there exist positive numbers \( \mu \) and \( \lambda \) such that for every \( x \in E \)
\[
\lambda \|K^*x\|^2 \leq \sum_{j \in J} \langle x, T^*x_j \rangle \langle T^*x_j, x \rangle \leq \mu \|x\|^2.
\]
So for \( x \in N(T) \) we have
\[
\lambda \|K^*x\|^2 \leq \sum_{j \in J} \langle x, T^*x_j \rangle \langle T^*x_j, x \rangle = 0.
\]
Then \( \|K^*x\|^2 = 0 \) and so \( x \in N(K^*) \). On the other hand, \( K \in L(E) \) has dense range. Hence \( K^* \) is injective and so \( T \) is also injective.
4. Sums of $K$-frames

In this section we show that the sum of two $K$-frames in a Hilbert $C^*$-module under certain conditions is again a $K$-frame, it was proved in Hilbert space case by Ramu and Johnson [15]. In the proof of Theorem 3.2 of [13], it was indicated that if $T$ has closed range then $R(T^*T)$ is closed and $R(T) = R(T^*T)$. The following theorem says that this result still holds for adjointable operators between Hilbert $C^*$-modules (even though $\overline{R(T^*)}$ may not be complemented).

**Theorem 4.1.** [13] For $T$ in $L(E,F)$, the sub-spaces $R(T^*)$ and $R(T^*T)$ have the same closure.

In [16], Sharifi proved that the converse of above theorem is also true.

**Theorem 4.2 (Lemma 1.1, [16]).** Suppose $T \in L(E)$. Then the operator $T$ has closed range if and only if $R(TT^*)$ has closed range. In this case, $R(T) = R(TT^*)$.

**Corollary 4.3.** Suppose $T \in L(E)^*$. Then $R(T)$ is closed if and only if $R(T^{1/2})$ is closed. In this case, $R(T) = R(T^{1/2})$.

**Proof.** The proof is immediately consequence of replacement $T$ by $T^{1/2}$ in the above theorem. □

**Theorem 4.4.** Let $E$ be a Hilbert module and $A, B \in L(E)$ such that $R(A) + R(B)$ is closed. Then

$$R(A) + R(B) = R((AA^* + BB^*)^{1/2}).$$

**Proof.** Define $T \in L(E \oplus E)$ by $T := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$. Then $T^* = \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}$, and

$$TT^* = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} = \begin{bmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{bmatrix}.$$ 

So we have

$$(TT^*)^{1/2} = \begin{bmatrix} (AA^* + BB^*)^{1/2} & 0 \\ 0 & 0 \end{bmatrix}.$$ 

On the other hand

$$T \begin{bmatrix} E \\ E \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E \\ E \end{bmatrix},$$ 

thus

$$R(T) = R(A) + R(B) \oplus \{0\}.$$ 

Since $R(T) = (R(A) + R(B))$ is closed then by Theorem 4.2, $R(T) = R(TT^*)$. But by Corollary 4.3, $R(TT^*) = R((TT^*)^{1/2})$. So we have

$$R(A) + R(B) = R((AA^* + BB^*)^{1/2}).$$

□

The following theorem is a generalization of Douglas theorem [Theorem 1.1, [18]], for Hilbert modules.

**Theorem 4.5.** Let $A, B_1, B_2 \in L(E)$ and $R(B_1) + R(B_2)$ is closed. The following statements are equivalent.

1. $R(A) \subseteq R(B_1) + R(B_2)$;
2. $AA^* \leq \lambda(B_1 B_1^* + B_2 B_2^*)$ for some $\lambda > 0$;
3. There exist $X, Y \in L(E)$ such that $A = B_1X + B_2Y$. 

Proof. (1) \(\implies\) (2): Suppose \(R(A) \subset R(B_1) + R(B_2)\). Then by Theorem 4.4, we have

\[
R(A) \subset R(B_1) + R(B_2) = R((B_1B_1^* + B_2B_2^*)^{1/2}),
\]

thus Theorem 2.2, implies \(AA^* \leq \lambda(B_1B_1^* + B_2B_2^*)\) for some \(\lambda > 0\).

(2) \(\implies\) (1): By Theorems 2.2, and 4.5, it is clear.

(3) \(\implies\) (1): It is obvious.

(1) \(\implies\) (3): Define \(S, T \in L(E \oplus E)\) by

\[
S = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}.
\]

Then \(R(S) \subset R(T)\) by Theorem 2.2. Suppose

\[
X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix},
\]

is the solution of \(S = TX\), so we have \(A = B_1X_1 + B_2X_2\). This completes the proof. \(\square\)

Following lemma shows that the sum of two Bessel sequences is a Bessel sequence too.

**Lemma 4.6.** Suppose that \(\{x_j\}_{j \in J}\) and \(\{y_j\}_{j \in J}\) are two Bessel sequences in Hilbert module \(E\). Then, by the Minkowski’s inequality, \(\{x_j + y_j\}_{j \in J}\) is also a Bessel sequence for \(E\).

Now we are going to show that under certain conditions the sum of two \(K\)-frame, is a \(K\)-frame.

**Theorem 4.7.** Let \(\{x_j\}_{j \in J}\) and \(\{y_j\}_{j \in J}\) be two \(K\)-frames for \(E\) and also let the corresponding operators in Theorem 2.9, be \(L_1\) and \(L_2\) respectively. If \(L_1L_2^*\) and \(L_2L_1^*\) are positive operators and \(R(L_1) + R(L_2)\) is closed, then \(\{x_j + y_j\}_{j \in J}\) is a \(K\)-frame for \(E\).

Proof. By the hypothesis we have

\[
L_1e_j = x_j, L_2e_j = y_j, R(K) \subset R(L_1), R(K) \subset R(L_2),
\]

where \(\{e_j\}_{j \in J}\) is the canonical orthonormal basis of \(\ell^2(A)\). So \(R(K) \subset R(L_1) + R(L_2)\), by Theorem 4.5, and \(KK^* \leq \lambda(L_1L_1^* + L_2L_2^*)\) for some \(\lambda > 0\). On the other hand for each \(x \in E\),

\[
\sum_{j \in J} \langle x, x_j + y_j \rangle \langle x_j + y_j, x \rangle = \sum_{j \in J} \langle (L_1 + L_2)^*x, e_j \rangle \langle e_j, (L_1 + L_2)^*x \rangle \\
= \langle (L_1 + L_2)^*x, e_j \rangle \langle e_j, (L_1 + L_2)^*x \rangle \\
= \langle (L_1 + L_2)^*x, (L_1 + L_2)^*x \rangle \\
= \langle L_1^*x, L_1^*x \rangle + \langle L_2^*x, L_2^*x \rangle + \langle L_1^*x, L_2^*x \rangle + \langle L_2^*x, L_1^*x \rangle \\
\geq \langle (L_1^*L_1 + L_2^*L_2)x, x \rangle \\
\geq \frac{1}{\lambda} \langle KK^*x, x \rangle \\
\geq \frac{1}{\lambda} \langle K^*x, K^*x \rangle.
\]

Thus \(\{x_j + y_j\}_{j \in J}\) is a \(K\)-frame. \(\square\)
5. K-frame vectors for unitary systems

A unitary system is a set of unitary operators which contains the identity operator. A vector \( \psi \) in \( E \) is called a complete K-frame vector for a unitary system \( \mathcal{U} \) on \( E \) if \( \mathcal{U}\psi = \{ U\psi \mid U \in \mathcal{U} \} \) is a K-frame for \( E \). If \( \mathcal{U}\psi \) is an orthonormal basis for \( E \), then \( \psi \) is called a complete wandering vector for \( \mathcal{U} \). The set of all complete K-frame vectors and complete wandering vectors for \( \mathcal{U} \) is denoted by \( \mathcal{F}_K(\mathcal{U}) \) and \( \omega(\mathcal{U}) \), respectively. In this section we characterize \( \mathcal{F}_K(\mathcal{U}) \) in terms of operators and elements of \( \omega(\mathcal{U}) \).

**Definition 5.1.** For a unitary system \( \mathcal{U} \) on a Hilbert module \( E \) and \( \psi \in \mathcal{U} \), the local commutant \( C_\psi(\mathcal{U}) \) of \( \mathcal{U} \) at \( \psi \) is defined by

\[
C_\psi(\mathcal{U}) = \{ T \in L(E) \mid TU\psi = UT\psi, \quad U \in \mathcal{U} \}.
\]

Also, let \( \ell_2^\mathcal{U}(\mathcal{A}) \) be the Hilbert \( \mathcal{A} \)-module defined by

\[
\ell_2^\mathcal{U}(\mathcal{A}) = \{ \langle a_U \rangle \in \mathcal{A} : \sum a_U a_U^* \text{ converges in } \| \cdot \| \}.
\]

The following theorem characterizes complete K-frame vectors in terms of operators on complete wandering vectors.

**Theorem 5.2.** Suppose \( \mathcal{U} \) is a unitary system of \( E \), \( K \in L(E) \), \( \psi \in \omega(\mathcal{U}) \) and \( \eta \in E \). Moreover, suppose that \( \psi, \eta \in L(E, \ell_2^\mathcal{U}(\mathcal{A})) \) is given by \( T_\psi(x) = \langle x, U\eta \rangle \) and \( R(T_\psi^*) \) is orthogonal complemented. Then \( \eta \in \mathcal{F}_K(\mathcal{U}) \) if and only if there exists an operator \( A \in C_\psi(\mathcal{U}) \) with \( R(K) \subseteq R(A) \) such that \( \eta = A\psi \).

**Proof.** (\( \Longleftarrow \)) Suppose \( \{ e_U \} \) denote the standard orthonormal basis of \( \ell_2^\mathcal{U}(\mathcal{A}) \), where \( e_U \) takes value \( 1_A \) at \( U \) and \( 0_A \) at every other \( U \). Now suppose \( \eta \in \mathcal{F}_K(\mathcal{U}) \). Define operator \( T_\psi \) from \( E \) to \( \ell_2^\mathcal{U}(\mathcal{A}) \) by \( T_\psi x = \sum \langle x, U\eta \rangle \langle U, U\eta \rangle e_U \). It is easy to check that \( T_\psi \) is well defined, adjointable and invertible. Let \( A = T_\psi \). Then for any \( x \in E \), we have

\[
\langle A^*x, A^*x \rangle = \sum \langle x, U\eta \rangle \langle U, U\eta \rangle \langle U\eta, U\eta \rangle U^*
\]

\[
= \sum \langle x, U\eta \rangle \langle U, U\eta \rangle \langle U\eta, x \rangle
\]

\[
\geq c\langle Kx, Kx \rangle,
\]

where \( c > 0 \) is the lower bound for K-frame \( \{ U\eta \mid U \in \mathcal{U} \} \). On the other hand \( R(A) = R(T_\psi^*) \) and so by Theorem 2.2, we have \( R(K) \subseteq R(A) \). To complete the proof, it remains to prove that \( \eta = A\psi \) and \( A \in C_\psi(\mathcal{U}) \). For any \( U \) and \( V \) in \( \mathcal{U} \)

\[
\langle V\eta, AU\psi \rangle = \langle V\eta, \sum \langle U, W\eta \rangle W\psi \rangle
\]

\[
= \sum \langle V\eta, W\eta \rangle \langle W\psi, U\psi \rangle
\]

\[
= \langle V\psi, U\psi \rangle.
\]

This implies that \( AU\psi = U\eta \), so \( A\psi = \eta \). Also \( AU\psi = U\eta = UA\psi \), hence \( A \in C_\psi(\mathcal{U}) \) and this completes the proof of this part.

(\( \Longleftrightarrow \)): Suppose that there exists an operator \( A \in C_\psi(\mathcal{U}) \) with \( R(K) \subseteq R(A) \) such that \( \eta = A\psi \). Then for any \( x \in E \), we have

\[
\sum \langle x, U\eta \rangle \langle U\eta, x \rangle = \sum \langle x, UA\psi \rangle \langle UA\psi, x \rangle
\]

\[
= \sum \langle A^*x, U\psi \rangle \langle U\psi, A^*x \rangle
\]

\[
= \langle A^*x, A^*x \rangle
\]

\[
\leq \| A^* \|^2 \| x \|^2.
\]
So \(|U\eta| \in \mathcal{U}U\) is a Bessel sequence for \(E\). Now let \(T_\eta\) and \(T_\psi\) be the operators as we defined in the first part of the proof, since \(\eta = A\psi\) so we have \(T_\eta = T_\psi A^*\). Since \(\psi \in \mathcal{w}(U)\), it is easy to see that \(T_\psi^*\) is invertible and hence \(R(T_\psi^*) = R(A)\). So \(R(K) \subset R(T_\psi^*)\). Therefore, by using Theorem 3.2 of [8] it is concluded that \(\eta \in \mathcal{U}_K(U)\).

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