A GENERALIZATION OF THE 3D DISTANCE THEOREM

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Abstract. Let \( P \) be a positive rational number. Call a function \( f : \mathbb{R} \to \mathbb{R} \) to have finite gaps property mod \( P \) if the following holds: for any positive irrational \( \alpha \) and positive integer \( M \), when the values of \( f(m\alpha) \), \( 1 \leq m \leq M \), are inserted mod \( P \) into the interval \( [0, P) \) and arranged in increasing order, the number of distinct gaps between successive terms is bounded by a constant \( k_f \) which depends only on \( f \). In this note, we prove a generalization of the 3d distance theorem of Chung and Graham. As a consequence, we show that a piecewise linear map with rational slopes and having only finitely many non-differentiable points has finite gaps property mod \( P \). We also show that if \( f \) is distance to the nearest integer function, then it has finite gaps property mod 1 with \( k_f \leq 6 \).

1. Introduction

The well known three gaps theorem was first observed by H. Steinhaus and proved independently by V. T. Sós [6,7] and others [8,9] (see [1] for a nice summary and recent generalization). The three gaps theorem is a special case \((d = 1)\) of the following more general theorem of Chung and Graham [2].

Theorem (3d distance theorem). Let \( \alpha > 0 \) be an irrational number and \( N_1, \ldots, N_d \) be positive integers. When the fractional parts of \( d \) arithmetic sequences \( n\alpha + k_i \), \( k_i \in \mathbb{Z}, 1 \leq i \leq d \), are inserted into a circle of unit circumference, the gaps between successive terms takes at most \( 3d \) distinct values.

What makes this theorem surprising is the fact that the fractional parts of the sequence \( n\alpha \) are known to be uniformly distributed in the interval \( [0, 1) \).

For a positive rational \( \lambda \), let \( I_\lambda \) be the discrete set \( \{n\lambda \mid n \in \mathbb{Z}\} \). For \( x \in \mathbb{R} \), define \( \lambda \)-floor \( [x]_\lambda \) and \( \lambda \)-roof \( \{x\}_\lambda \) as:

\[
[x]_\lambda = \max\{r \in I_\lambda \mid r \leq x\}, \quad [x]_\lambda = \min\{r \in I_\lambda \mid r \geq x\}.
\]

Define \( \lambda \)-fractional part functions \( \{-\}^\lambda_1 \) and \( \{-\}^\lambda_2 \) as:

\[
\{x\}_\lambda = \begin{cases} \frac{x - [x]_\lambda}{\lambda} & \text{if } x \geq 0 \\ \frac{x - [x]_\lambda}{\lambda} & \text{if } x < 0, \end{cases}
\]

\[
[x]_\lambda = x - \{x\}_\lambda.
\]

Define \( \{x\}_\infty = x \). We write \( \{x\}_{1\lambda} \) as \( \{x\} \).

Choose a \( \lambda \)-fractional part function \( \{-\}^\lambda_1 \) or \( \{-\}^\lambda_2 \) and denote it by \( \{-\}^\lambda_1 \). We prove the following generalization of the 3d distance theorem.

Theorem 1.1. Let \( \alpha > 0 \) be an irrational number, \( N_1, \ldots, N_d \) be positive integers and \( n_1, \ldots, n_d \) be non-negative integers such that \( n_i \leq N_i, 1 \leq i \leq d \). Write \( N = \sum_{i=1}^d (N_i - n_i) \). Consider the linear maps \( f_i : x \in \mathbb{R} \mapsto \frac{p_i}{q} x + k_i \in \mathbb{R}, 1 \leq i \leq d \), where \( 0 \neq p_i \in \mathbb{Z}, q \in \mathbb{Z}_{>0} \) and \( k_i \in \mathbb{R} \). Fix \( P \) to be a positive rational and let \( \lambda \) be
any positive integer multiple of $Pq$. Define $f_i : \mathbb{R} \to \mathbb{R}$ by $f_i(x) = \hat{f}_i(\{x\} \chi)$. Insert mod $P$, the values of $f_i(m\alpha)$, $n_i < m \leq N_i$, $1 \leq i \leq d$, in the interval $[0, P)$ to form an increasing sequence $(b_n)_{1 \leq n \leq N}$. Write $\ell = \text{lcm}(p_1, \ldots, p_d) > 0$, $c_i := \ell/p_i$, and $c = \sum_{i=1}^d |c_i|$. Then there are at most $3c$ distinct values in the set of gaps $g_m$ defined by

$$g_1 = P + b_1 - b_N, \quad g_m = b_m - b_{m-1}, \quad m = 2, \ldots, N.$$

Note that Theorem 1.1 allows the possibility of some points to coincide. The ordering of coincidental points is defined in Section 2. Let $\| \cdot \| : \mathbb{R} \to [0, 1/2]$ denote the distance to the nearest integer function. By definition

$$\|x\| = \min(\{|x|\}, 1 - \{|x|\}).$$

As a special case of Theorem 1.1, we obtain the following result which was proved in [3] using different methods.

**Corollary 1.2.** Let $\alpha > 0$ be an irrational number and $M > 1$ be an integer. When the values $|n\alpha|$, $1 \leq n \leq M$, are arranged in ascending order in the interval $[0, 1/2]$, the gaps between successive terms may take at most 6 distinct values.

Our proof of Corollary 1.2 is significantly shorter than the proof in [3]. However, the bound obtained in loc. cit. is effective.

**Corollary 1.3.** Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise linear map with rational slopes and having only finitely many non-differentiable points. Let $\alpha > 0$ be an irrational number and $M > 1$ be an integer. For any positive rational $P$, when the values $f(m\alpha)$, $1 \leq m \leq M$, are inserted mod $P$ in $[0, P)$ and arranged in ascending order, the gaps between successive terms may take at most $k_f$ distinct values, where $k_f$ is a constant which depends only on $f$.

Our proof of Theorem 1.3 is an adaptation of the elegant proof of the 3d distance Theorem by Liang [4].

2. **Proof of Theorem 1.1**

**Proof.** For $1 \leq i \leq d$, let $B_i$ be the set of all triples $\beta_{im} = (\gamma_{im}, i, m) \in [0, P) \times \mathbb{Z} \times \mathbb{Z}$ where $\gamma_{im} \cong f_i(m\alpha)$ mod $P$, $n_i \leq m \leq N_i$. Write $B = \bigcup_{i=1}^d B_i$. We give a strict ordering $\prec$ on $B$ by declaring $\beta_{im} \prec \beta_{jn}$, iff

$$\gamma_{im} < \gamma_{jn} \quad \text{or} \quad \gamma_{im} = \gamma_{jn} \text{ and } [i < j \text{ or } (i = j \text{ and } m < n)].$$

Arrange the elements of $B$ in a strictly increasing sequence $(b_n)_{1 \leq n \leq N}$ with this ordering. Applying arithmetic modulo $P$, we identify $P$ with 0 and consider $\{\gamma_{im} | n_i \leq m \leq N_i, 1 \leq i \leq d\}$ as living in this circle $[0, P)$. This makes the ordering on $B$ a cyclic ordering, which we again denote by $\prec$. Thus $b_N$ and $b_1$ are consecutive in this cyclic ordering. To simplify notation, will often abuse notation and write $\beta_{im}$ when we mean $\gamma_{im}$.

A gap interval is an interval in the circle $[0, P]$ of the form $[\beta_{in}, \beta_{jm}]$ where $\beta_{im}$, $\beta_{jm}$ are consecutive points of $B$ in the cyclic ordering $\prec$. Write $\ell_0 = \ell/q$. A gap interval is rigid if translating a gap interval by $\ell_0\alpha$ does not produce a gap interval. Observe that gap intervals cannot loop upon successive translations by $\ell_0\alpha$. To see this, suppose $s$ is a positive integer such that translation by $s\ell_0\alpha$ maps $[\beta_{in}, \beta_{jm}]$ to itself. Then either $\beta_{in} + s\ell_0\alpha = \beta_{in}$ and $\beta_{jm} + s\ell_0\alpha = \beta_{jm}$, or $\beta_{in} + s\ell_0\alpha = \beta_{jm}$ and $\beta_{jm} + s\ell_0\alpha = \beta_{in}$. Either of these cases contradicts the irrationality of $\alpha$. 


Now, a gap interval $I$ is rigid if upon translation by $t_0\alpha$ it produces an interval $J$ for which one of the following holds:

(i) At least one of the end points of $J$ is not in $\mathbb{B}$.

(ii) The translated interval $J$ has endpoints in $\mathbb{B}$ but they are not consecutive.

For case (i), let $\beta_{in}$ be an end point of $I$ such that $\beta_{in} + t_0\alpha = \beta_{i(n+c_i)} \notin \mathbb{B}$. Then in particular, $\beta_{i(n+c_i)} \notin B_i$. If $c_i > 0$, then $\beta_{i(n+c_i)} \notin B_i$ if $n + c_i > N_i$. Then, $\beta_{in}$ will be in the set

$$S_i = \{\beta_{im} \mid N_i - c_i + 1 \leq m \leq N_i\}.$$  

If $c_i < 0$, then $\beta_{i(n+c_i)} \notin B_i$ if $n + c_i \leq n_i$. Then $\beta_{in}$ will be in the set

$$T_i = \{\beta_{im} \mid 1 + n_i \leq m \leq -c_i + n_i\}.$$  

Call the elements of $S_i$ and $T_i$ to be starting points. Then for each $i$, the starting points have cardinality $|c_i|$. Since each starting point is the boundary of at most two gap intervals, case (i) contributes at most $2 \sum_{i=1}^{d} |c_i| = 2c$ rigid intervals.

For case (ii), let $\beta_{kp} \in \mathbb{B}$ be an internal point of $I$. Then $\beta_{k(p-c_k)}$ is an internal point of $I$. Since $I$ is a gap interval, this implies that $\beta_{k(p-c_k)} \notin \mathbb{B}$. In particular $\beta_{k(p-c_k)} \notin B_k$. This implies that $\beta_{kp}$ belongs to the set

$$T'_k = \{\beta_{km} \mid 1 + n_k \leq m \leq c_k + n_k\}$$  

or

$$S'_k = \{\beta_{km} \mid N_k + c_k + 1 \leq m \leq N_k\}$$  

according as $c_k > 0$ or $c_k < 0$. Call the elements of $S'_k$ and $T'_k$ to be finish points. Then for each $i$, the finish points have cardinality at most $|c_i|$. Thus case (ii) contributes at most $\sum_{i=1}^{d} |c_k| = c$ rigid intervals.

We have shown that there can be at most $3c$ distinct rigid intervals and consequently at most $3c$ gap interval sizes. This completes the proof.

\section{Proof of Corollaries 1.2 and 1.3}

Proof of Corollary 1.2. We retain the notations of Theorem 1.1 and its proof in Section 2. Put $d = 2$, $q = 1$, $p_1 = 1$, $p_2 = -1$, $k_1 = 0$, $k_2 = 1$, $N_1 = N_2 = M$, $P = 1 = \lambda$ and $\{-\}^\lambda = \{-\}^\lambda$. Then $c = 2$, the starting points are $\{Ma\}$ and $1 - \{\alpha\}$, and the finish points are $1 - \{Ma\}$ and $\{\alpha\}$. Now write $D \subset \mathbb{B}$ for the set of points $\{||ma|| \mid 1 \leq m \leq N\}$. Since $\alpha$ is irrational, the points of $\mathbb{B}$ are all distinct. Arrange the points in $D$ in usual increasing order. Since the ordering $\prec$ on $\mathbb{B}$ is the usual order on the circle $[0, 1]$, and since $\mathbb{B} \setminus D \subset \left(\frac{1}{2}, 1\right)$, it follows that if $u, v$ are consecutive points of $D$, then it they are also consecutive points of $\mathbb{B}$. Consequently, it follows from Theorem 1.1 that the number of distinct gap values in $D$ is at most $3c = 6$.

Remark 3.1. When $\alpha$ is a positive cube root of 15, we get four distinct gap sizes: $0.000612999, 0.006205886, 0.006818885, 0.007125385$. Henk Don\cite{3} has shown that the bound is precisely 4.

Proof of Corollary 1.3. Let $I_i$ be a partition of the interval $[0, M\alpha]$ into smallest possible number of connected parts such that $f |I_i = f_i |I_i$ for some linear functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $x + k_i \in \mathbb{R}$, $1 \leq i \leq d$, where $0 \neq p_i \in \mathbb{Z}$, $q \in \mathbb{Z}_{>0}$ and $k_i \in \mathbb{R}$. Let $n_i \leq N_i$ be uniquely defined integers such that $m\alpha \in I_i$ iff $n_i < m \leq N_i$,
\[ 1 \leq i \leq d. \] The result then follows from Theorem 1.1 by putting \( N = M \) and \( \{ - \}^\lambda = \{ - \}^\infty. \) □

4. Acknoledgement

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