Twisted complexes and simplicial homotopies

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Abstract

We consider the dg-category of twisted complexes over simplicial ringed spaces. It is clear that a simplicial map $f : (U, R) \rightarrow (V, S)$ between simplicial ringed spaces induces a dg-functor $f^* : \text{Tw}(V, S) \rightarrow \text{Tw}(U, R)$ where $\text{Tw}(U, R)$ denotes the dg-category of twisted complexes on $(U, R)$. We prove that for simplicial homotopic maps $f$ and $g$, there exists an $A_\infty$-natural transformation $\Phi : f^* \Rightarrow g^*$ between induced dg-functors. Moreover, the 0th component of $\Phi$ is an objectwise weak equivalence. If we restrict ourselves to the full dg-subcategory of twisted perfect complexes, then we prove that $\Phi$ admits an $A_\infty$-quasi-inverse when $(U, R)$ satisfies some additional conditions.

Keywords Twisted complexes · Differential graded categories · $A_\infty$-natural transformation · Simplicial spaces · Simplicial homotopy

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1 Introduction

In the late 1970s Toledo and Tong [11] introduced twisted complexes as a way to get their hands on perfect complexes of sheaves on a complex manifold. Twisted complexes, which consist of locally defined complexes together with higher transition functions, soon started to play an important role in the study of complex geometry, algebraic geometry, as well as dg-categories and $A_\infty$-categories, see [1–3,7–9,12,13].

In particular, in [1,2] it was proved that for a simplicial ringed space $(U, R)$, the dg-category of twisted complexes $\text{Tw}(U, R)$ (see Definition 2.9 below) gives the homo-
topy limit of the cosimplicial diagram of dg-categories

\[ \text{Cpx}(U_0, R_0) \longrightarrow \text{Cpx}(U_1, R_1) \longrightarrow \text{Cpx}(U_2, R_2) \cdots \]

where \( \text{Cpx}(U_i, R_i) \) denotes the dg-category of complexes of sheaves of \( R_i \)-modules on \( U_i \). See Proposition 2.13 below.

**Remark 1.1** The definition of twisted complexes in this paper is slightly different to twisted complexes introduced in [3]. See Definition 2.9 below and [3, Definition 1]. Therefore it is natural to expect that the dg-category \( \text{Tw}(U, R) \) has some kind of homotopy invariance. In particular, let \( f \) and \( g : (U, R) \rightarrow (V, S) \) be two simplicial maps which are simplicial homotopic, i.e., there exists a simplicial map \( H : U \times I \rightarrow V \) such that \( f = H \circ \varepsilon_0 \) and \( g = H \circ \varepsilon_1 \), we expect that the induced dg-functors \( f^* \) and \( g^* : \text{Tw}(V, S) \rightarrow \text{Tw}(U, R) \) can be identified.

Using \( H \) we can construct, for each object \( E \in \text{Tw}(V, S) \), a degree 0 morphism

\[ \Phi_0(E) : f^*(E) \rightarrow g^*(E). \]

In Proposition 4.6 we prove that \( \Phi_0(E) \) is closed and in addition a weak equivalence for each \( E \).

Unfortunately, for a morphism \( \phi : E \rightarrow F \) we notice that

\[ g^*(\phi) \cdot \Phi_0(E) - (-1)^{|\phi|} \Phi_0(F) \cdot f^*(\phi) \neq 0. \]

Therefore \( \Phi_0 \) does not give a dg-natural transformation from \( f^* \) to \( g^* \). Nevertheless, in this paper we extend \( \Phi_0 \) to an \( A_\infty \)-natural transformation \( \Phi : f^* \Rightarrow g^* \), see Theorem 4.12 below. In addition, if we restrict to \( \text{Tw}_{\text{perf}}(V, S) \), the full dg-subcategory of twisted perfect complexes, then we can show that \( \Phi \) has an \( A_\infty \)-quasi-inverse.

This paper is organized as follows: in Sect. 2 we review the concept of twisted complexes and in Sect. 3 we review \( A_\infty \)-natural transformations between dg-functors. In Sect. 4 we first study simplicial homotopies between simplicial maps and then construct the \( A_\infty \)-natural transformation \( \Phi \). In Sect. 5 we consider twisted perfect complexes and show that in this case the \( A_\infty \)-natural transformation \( \Phi \) admits an the \( A_\infty \)-quasi-inverse if \( (U, R) \) satisfies some additional conditions.

### 2 A review of twisted complexes

#### 2.1 A review of simplicial and cosimplicial objects

Recall that the simplicial category \( \Delta \) is the category with objects

\[ [n] = \{0, \ldots, n\} \quad \text{for} \quad n \geq 0 \]
and morphisms order preserving functions between objects.

Let \(\mathcal{C}\) be a category. A **simplicial object** \(X\) in \(\mathcal{C}\) is a contravariant functor

\[ X: \Delta^{\text{op}} \to \mathcal{C} \]

and a morphism \(f: X \to Y\) between two simplicial objects in \(\mathcal{C}\) is a natural transformation between contravariant functors.

More explicitly, a simplicial object \(X\) in \(\mathcal{C}\) consists of a collection of objects \(X_n \in \text{obj} \mathcal{C}\) for \(n \geq 0\) and a collection of face morphisms \(\partial_i: X_n \to X_{n-1}\) and degeneracy morphisms \(s_i: X_n \to X_{n+1}, 0 \leq i \leq n\), which satisfy the **simplicial identities**

\[
\begin{align*}
\partial_i \partial_j &= \partial_{j-1} \partial_i & \text{if } i < j; \\
\partial_i s_j &= s_{j-1} \partial_i & \text{if } i < j; \\
\partial_i s_j &= \text{id} & \text{if } i = j \text{ or } i = j + 1; \\
\partial_i s_j &= s_{j-i-1} & \text{if } i > j + 1; \\
s_i s_j &= s_{j+1} s_i & \text{if } i \leq j.
\end{align*}
\]

A morphism \(f: X \to Y\) between two simplicial objects consists of a collection of morphisms \(f_n: X_n \to Y_n\) for \(n \geq 0\) in \(\mathcal{C}\) such that \(f_n\) is compatible with all \(\partial_i\)s and \(s_j\)s.

Dually a **cosimplicial object** \(X\) in \(\mathcal{C}\) is a covariant functor

\[ X: \Delta \to \mathcal{C} \]

and a morphism \(f: X \to Y\) between two cosimplicial objects in \(\mathcal{C}\) is a natural transformation between covariant functors. We also have an explicit description of cosimplicial objects and morphisms which is dual to the simplicial case.

**Example 2.1** (Classifying space of an open cover) Let \(X\) be a topological space and let \(l = \{U_i\}\) be an open cover of \(X\). Let \(U_{i_0 \ldots i_n}\) denote the intersection \(U_{i_0} \cap \cdots \cap U_{i_n}\) where repetitions of indices are allowed. Then we get a simplicial space \(N\) where

\[ N_n = \coprod_{i_0, \ldots, i_n} U_{i_0 \ldots i_n}. \]

The face map \(\partial_k: N_n \to N_{n-1}\) is induced by the inclusion map

\[ U_{i_0 \ldots i_n} \hookrightarrow U_{i_0 \ldots \hat{i}_k \ldots i_n} \]

and the degeneracy map \(s_k: N_n \to N_{n+1}\) is induced by the identity map

\[ U_{i_0 \ldots i_n} \twoheadrightarrow U_{i_0 \ldots i_k i_k \ldots i_n}. \]
2.2 Notations of bicomplexes and sign conventions

In this section by a ringed space we mean a topological space $X$ together with a sheaf of (not necessarily commutative) rings $\mathcal{R}$ on $X$. Examples include:

- A scheme $X$ with the structure sheaf $\mathcal{O}_X$;
- A complex manifold $X$ with the sheaf of analytic functions $\mathcal{O}_X$;
- A topological space $X$ with the constant sheaf of rings $\mathbb{C}$;
- A scheme $X$ with the sheaf of rings of differential operators $\mathcal{D}_X$.

Remark 2.2 Papers [8,9,11] focus on the special case that $X$ is a complex manifold and $\mathcal{R} = \mathcal{O}_X$ is the sheaf of holomorphic functions on $X$. In this paper we consider more general $(X, \mathcal{R})$.

Remark 2.3 In this section by $\mathcal{R}$-modules we always mean left $\mathcal{R}$-modules, unless it is explicitly pointed out otherwise.

A simplicial ringed space is a simplicial object in the category of ringed spaces, and a simplicial map is a morphism between two simplicial objects in the category of ringed spaces.

In this section we introduce some notations which are necessary in the definition of twisted complexes. Let $(U, \mathcal{R})$ be a simplicial ringed space. Let $\partial_i : (U_n, \mathcal{R}_n) \to (U_{n-1}, \mathcal{R}_{n-1})$ and $s_i : (U_n, \mathcal{R}_n) \to (U_{n+1}, \mathcal{R}_{n+1})$ be the face and degeneracy maps, respectively. Moreover for $k \geq p$, we define $\rho_{k,p} : (U_k, \mathcal{R}_k) \to (U_p, \mathcal{R}_p)$ to be the front face map, i.e.,

$$\rho_{k,p} := \partial_{p+1} \circ \partial_{p+2} \circ \cdots \circ \partial_k.$$

Similarly we define $\tau_{k,p} : (U_k, \mathcal{R}_k) \to (U_p, \mathcal{R}_p)$ to be the back face map, i.e.,

$$\tau_{k,p} := \partial_0 \circ \partial_0 \circ \cdots \circ \partial_0.$$

We have the following identities.

Lemma 2.4 For $k \geq p \geq r$ we have

$$\rho_{p,r} \circ \rho_{k,p} = \rho_{k,r},$$

$$\tau_{p,r} \circ \tau_{k,p} = \tau_{k,r},$$

$$\rho_{p,r} \circ \tau_{k,p} = \tau_{k+r-p,r} \circ \rho_{k,k+r-p}.$$

Moreover, for a morphism $f : (X, \mathcal{R}) \to (Y, \mathcal{I})$ between simplicial ringed spaces, we have

$$f_p \circ \rho_{k,p} = \rho_{k,p} \circ f_k, \quad f_p \circ \tau_{k,p} = \tau_{k,p} \circ f_k.$$

Proof It follows from the simplicial identities (1).}

Let $E^\bullet$ be a graded sheaf of $\mathcal{R}_0$-modules on $U_0$. Let

$$C^\bullet(U, \mathcal{R}, E^\bullet) = \prod_{p,q} \Gamma(U_p, \rho_{p,0}^* E^q)$$
be the bigraded complexes of $E^\bullet$.

Now, if another graded sheaf $F^\bullet$ of $\mathcal{R}_0$-modules is given on $U_0$, then we can consider the bigraded complex

$$C^\bullet(\mathcal{U}, \mathcal{R}, \text{Hom}^\bullet(E, F)) = \prod_{p,q} \text{Hom}^q_{\mathcal{R}_p\text{-Mod}}(\tau^*_p E, \rho^*_q F).$$

**Remark 2.5** In this paper when we talk about degree $(p, q)$, the first index always indicates the simplicial degree while the second index always indicates the graded sheaf degree. We use $|u|$ to denote the total degree of $u$.

We need to study the compositions of $C^\bullet(\mathcal{U}, \mathcal{R}, \text{Hom}^\bullet(E, F))$. Let $G^\bullet$ be a third graded sheaf of $\mathcal{R}_0$-modules, then there is a composition map

$$C^\bullet(\mathcal{U}, \mathcal{R}, \text{Hom}^\bullet(F, G)) \times C^\bullet(\mathcal{U}, \mathcal{R}, \text{Hom}^\bullet(E, F)) \to C^\bullet(\mathcal{U}, \mathcal{R}, \text{Hom}^\bullet(E, G)).$$

In fact, for $u^{p,q} \in C^p(\mathcal{U}, \mathcal{R}, \text{Hom}^q(F, G))$ and $v^{r,s} \in C^r(\mathcal{U}, \mathcal{R}, \text{Hom}^s(E, F))$, their composition $(u \cdot v)^{p+r, q+s}$ is given by

$$(u \cdot v)^{p+r, q+s} = (-1)^{qr}(\rho^*_{p+r, p} u^{p,q}) \circ (\tau^*_{p+r, r} v^{r,s})$$

where the right-hand side is the naïve composition of sheaf maps.

In particular, $C^\bullet(\mathcal{U}, \mathcal{R}, \text{Hom}^\bullet(E, E))$ becomes an associative algebra under this composition. (It is easy but tedious to check the associativity.) We also notice that $C^\bullet(\mathcal{U}, \mathcal{R}, E^\bullet)$ becomes a left module over this algebra. In fact the action

$$C^\bullet(\mathcal{U}, \mathcal{R}, \text{Hom}^\bullet(E, E)) \times C^\bullet(\mathcal{U}, \mathcal{R}, E^\bullet) \to C^\bullet(\mathcal{U}, \mathcal{R}, E^\bullet)$$

is given by

$$(u \cdot c)^{p+r, q+s} = (-1)^{qr}(\rho^*_{p+r, p} u^{p,q}) \circ (\tau^*_{p+r, r} c^{r,s})$$

where the right-hand side is given by evaluation.

**Remark 2.6** The definition of compositions and actions makes sense because we have Lemma 2.4.

There is also a Čech-style differential operator $\delta$ on $C^\bullet(\mathcal{U}, \mathcal{R}, \text{Hom}^\bullet(E, F))$ and $C^\bullet(\mathcal{U}, \mathcal{R}, E^\bullet)$ of bidegree $(1, 0)$ given by the formula

$$(\delta u)^{p+1, q} = \sum_{k=1}^{p} (-1)^k \partial^*_k u^{p,q}$$

for $u^{p,q} \in C^p(\mathcal{U}, \mathcal{R}, \text{Hom}^q(E, F))$ \hspace{1cm} (4)

and

$$(\delta c)^{p+1, q} = \sum_{k=1}^{p+1} (-1)^k \partial^*_k c^{p,q}$$

for $c^{p,q} \in C^p(\mathcal{U}, \mathcal{R}, E^q)$. \hspace{1cm} (5)
Caution 2.7 Notice that the map $\delta$ defined above is different from the usual Čech differential. In equation (4) we do not include the 0th and the $(p+1)$th indices and in equation (5) we do not include the 0th index.

**Proposition 2.8** The differential satisfies the Leibniz rule. More precisely, we have

$$
\delta (u \cdot v) = (\delta u) \cdot v + (-1)^{|u|} u \cdot (\delta v),
\delta (u \cdot c) = (\delta u) \cdot c + (-1)^{|u|} u \cdot (\delta c)
$$

where $|u|$ is the total degree of $u$.

**Proof** This is a routine check. \qed

Now we consider a ringed space $(X, \mathcal{R})$ and an open cover $\{U\}$ of $X$. The classifying space $N$ of $U$ as in Example 2.1 is a simplicial ringed space with structure sheaves inherited from the $\mathcal{R}$ on $X$ and we denote this simplicial ringed space by $(N, \mathcal{R})$. In this case we have the following observations. Actually they are exactly the conventions in [8, Section 1].

- An element $cp_{p, q}$ of $C^p (N, \mathcal{R}, E^q)$ consists of a section $cp_{i_0 ... i_p}$ of $E^q_{i_0}$ over each non-empty intersection $U_{i_0 ... i_p}$. If $U_{i_0 ... i_p} = \emptyset$, then the component on it is zero.
- An element $up_{p, q}$ of $C^p (N, \mathcal{R}, Hom^q (E, F))$ gives a section $up_{i_0 ... i_p}$ of $Hom^q_{\mathcal{R}, \text{Mod}} (E^*_{i_p}, F^*_{i_0})$, i.e., a degree $q$ map from $E^*_{i_p}$ to $F^*_{i_0}$ over the non-empty intersection $U_{i_0 ... i_p}$. Notice that we require $up_{p, q}$ to be a map from the $F^*$ on the last subscript of $U_{i_0 ... i_p}$ to the $E^*$ on the first subscript of $U_{i_0 ... i_p}$. Again, if $U_{i_0 ... i_p} = \emptyset$, then the component on it is zero.

The compositions and actions are given in the following formula (see [8, Equations (1.1) and (1.2))]:

$$
(u \cdot v)_{i_0 ... i_p + r} = (-1)^{q r} up_{i_0 ... i_p} v_{i_p ... i_p + r}
$$

and

$$
(u \cdot c)_{i_0 ... i_p + r} = (-1)^{q r} up_{i_0 ... i_p} c_{i_p ... i_p + r}.
$$

Moreover, the differentials are given by

$$
(\delta u)_{i_0 ... i_p + 1} = \sum_{k=1}^{p} (-1)^{k} up_{i_0 ... i_k ... i_{p+1}} |_{U_{i_0 ... i_p + 1}} \text{ for } up_{p, q} \in C^p (N, \mathcal{R}, Hom^q (E, F)),
$$

$$
(\delta c)_{i_0 ... i_p + 1} = \sum_{k=1}^{p+1} (-1)^{k} cp_{i_0 ... i_k ... i_{p+1}} |_{U_{i_0 ... i_p + 1}} \text{ for } cp_{p, q} \in C^p (N, \mathcal{R}, E).
$$
2.3 Twisted complexes

With the notations in Sect. 2.2 we can define twisted complexes on simplicial ringed spaces.

**Definition 2.9** Let \((\mathcal{U}, \mathcal{R})\) be a simplicial ringed space. A twisted complex on \((\mathcal{U}, \mathcal{R})\) consists of a graded sheaf of \(\mathcal{R}_0\)-modules \(E^*\) on \(U_0\) together with

\[
a = \prod_{k \geq 0} a^{k,1-k} \in \prod_{k \geq 0} \text{Hom}^{1-k}_{\mathcal{R}_k\text{-Mod}}(\tau^*_{k,0}(E), \rho^*_{k,0}(E))
\]

where

\[
a^{k,1-k} \in \text{Hom}^{1-k}_{\mathcal{R}_k\text{-Mod}}(\tau^*_{k,0}(E), \rho^*_{k,0}(E))
\]

and they satisfy the following two conditions:

- The Maurer–Cartan equation
  \[
  \delta a + a \cdot a = 0,
  \]
  or more explicitly
  \[
  \sum_{j=1}^{k-1} (-1)^j \partial_j^* (a^{k-1,2-k}) + \sum_{j=0}^{k} (-1)^{(1-j)(k-j)} \rho_{k,j}^* (a^{j,1-j}) \tau^*_{k,j} (a^{k-j,1-k+j}) = 0.
  \]

- The non-degenerate condition: \(a^{1,0} \in \text{Hom}^0_{\Omega^{1-1}_{\mathcal{R}_1\text{-Mod}}}(\tau^*_{1,0}(E), \rho^*_{1,0}(E))\) is invertible up to homotopy.

A morphism \(\theta\) of degree \(m\) from \((E, a)\) to \((F, b)\) is given by a collection

\[
\theta^{k,m-k} \in \text{Hom}^{m-k}_{\mathcal{R}_k\text{-Mod}}(\tau^*_{k,0}(E), \rho^*_{k,0}(F)) \text{ for all } k \geq 0
\]

and the differential is given by

\[
d \theta = \delta \theta + b \cdot \theta - (-1)^m \theta \cdot a
\]
or more explicitly

\[(d\theta)^{k,m+1-k} = \sum_{j=1}^{k-1} (-1)^j \partial_j^* \theta^{k-1,m+1-k} \]

\[+ \sum_{l=0}^{k} (-1)^{(l-k)(k-l)} \rho^*_{k,l} b^l_{k,k-l} \theta^{k-l,m-k+1} \]

\[+ \sum_{l=0}^{k} (-1)^{(m-l)(k-l)} \rho^*_{k,l} a^l_{k,k-l} \tau^*_{k,k-l} \theta^{k-l,1-k+l} \).

We denote the dg-category of twisted complexes on a simplicial ringed space \((\mathcal{U}, \mathcal{R})\) by \(\text{Tw}(\mathcal{U}, \mathcal{R})\).

**Remark 2.10** People who are familiar with \(A_\infty\)-categories may find that the definition of twisted complexes is similar to the construction of \(A_\infty\)-functors. Actually this is the approach taken by [1,12]. In this paper we satisfy ourselves with Definition 2.9 and refer interested readers to [12, Section 16] and [1, Section 4] for the \(A_\infty\)-approach.

**Definition 2.11** Let \(f : (\mathcal{U}, \mathcal{R}) \to (\mathcal{V}, \mathcal{S})\) be a simplicial map between simplicial ringed spaces. Then \(f\) naturally induces a dg-functor \(f^* : \text{Tw}(\mathcal{V}, \mathcal{S}) \to \text{Tw}(\mathcal{U}, \mathcal{R})\). More precisely, for \(E = (E, a) \in \text{Tw}(\mathcal{V}, \mathcal{S})\), \(f^* E\) is given by \((f_0^* E, f^* a)\) where

\[(f^* a)^{k,1-k} = f_k^* a_{k,1-k}^k \in \text{Hom}_{\mathcal{R}_k^1}(\tau_{k,0}^*(f_0^* E), \rho_{k,0}^*(f_0^* E)).\]

For a degree \(m\) morphism \(\phi : E \to F\) we define \(f^* \phi : f^* E \to f^* F\) as

\[(f^* \phi)^{k,m-k} = f_k^* \phi_{k,m-k} \in \text{Hom}_{\mathcal{R}_k^m}(\tau_{k,0}^*(f_0^* E), \rho_{k,0}^*(f_0^* F)).\]

By Lemma 2.4 this definition makes sense. It is clear that \(\delta(f^* \phi) = f^* (\delta \phi)\) and \(f^* (\phi \cdot \psi) = f^* \phi \cdot f^* \psi\).

In the case that the simplicial space is the classifying space \(\mathcal{N}\) of an open cover \(\mathcal{U}\) as in Example 2.1 we have the following more concrete description of twisted complexes.

**Definition 2.12** ([8, Definition 1.3] or [13, Definition 5]) Let \((X, \mathcal{R})\) be a ringed space and \(\mathcal{U} = \{U_i\}\) be a locally finite open cover of \(X\). A twisted complex consists of graded sheaves \(E_i^*\) of \(\mathcal{R}\)-modules on each \(U_i\) together with a collection of morphisms for \(k \geq 0\) and every multi-index \((i_0 \ldots i_k)\)

\[a_{i_0 \ldots i_k}^{k,1-k} \in \text{Hom}_{\mathcal{U}_{i_0 \ldots i_k}^1}(E_{i_k}, E_{i_0})\]

which satisfies the Maurer–Cartan equation

\[\sum_{j=1}^{k-1} (-1)^j a_{i_0 \ldots i_j i_k}^{k-1,2-k} + \sum_{l=0}^{k} (-1)^{(l-k)(k-l)} a_{i_0 \ldots i_l i_{l+1} \ldots i_k}^{l,k-l,1-k+l} = 0.\]
Moreover we impose the following non-degenerate condition: for each $i$, the chain map
\[ a_{ij}^{1,0} : (E_i^\bullet, a_i^{0,1}) \to (E_i^\bullet, a_i^{0,1}) \]
is invertible up to homotopy.

Morphisms and differentials are defined similarly.

For more details on twisted complexes see [13]. In this paper we just mention the relation between twisted complexes and homotopy limits. Let
\[ \text{Cpx} : \text{Ringed Space}^{\text{op}} \to \text{dgCat} \]
be the contravariant functor which assigns to each ringed space $(X, \mathcal{R})$ the dg-category of complexes of left $\mathcal{R}$-modules on $X$. This is a presheaf of dg-categories. For a simplicial ringed space $(U, \mathcal{R})$ we get a cosimplicial diagram of dg-categories
\[ \text{Cpx}(U_0, \mathcal{R}_0) \to \text{Cpx}(U_1, \mathcal{R}_1) \to \text{Cpx}(U_2, \mathcal{R}_2) \cdots \]
Then we have the following result.

**Proposition 2.13** ([2, Corollary 4.8], [1, Proposition 4.0.2]) Let $U$ be a simplicial ringed space. Then the dg-category of twisted complexes $\text{Tw}(U, \mathcal{R})$ gives an explicit construction of $\text{holim} \text{Cpx}(U, \mathcal{R})$.

Proposition 2.13 shows the importance of twisted complexes in descent theory, see [1, Introduction] for some discussions and [14] for an application.

**Remark 2.14** In practice we are often less interested in the category of all complexes of $\mathcal{R}$-modules than in some well-behaved subcategory, say complexes with quasi-coherent cohomology on a scheme, or $D_X$-modules which are quasi-coherent as $\mathcal{O}_X$-modules. As long as the condition we impose is local the theory works equally well in those cases. We will explicitly consider the case of perfect complexes in Sect. 5.

For later purpose we need the following concept. See [13, Definition 2.27].

**Definition 2.15** Let $(U, \mathcal{R})$ be a simplicial ringed space. Let $E = (E^\bullet, a)$ and $F = (F^\bullet, b)$ be two objects in $\text{Tw}(U, \mathcal{R})$. A morphism $\phi : E \to F$ is called a weak equivalence if it satisfies the following two conditions:

- $\phi$ is closed and of degree zero.
- Its $(0,0)$ component $\phi^{0,0} : (E^\bullet, a^{0,1}) \to (F^\bullet, b^{0,1})$ is a quasi-isomorphism of complexes of $\mathcal{R}_0$-modules on $U_0$.

### 3 $A_\infty$-natural transformations

In this section we review $A_\infty$-natural transformations between dg-functors. For more details see [15]. See for example [5] or [1] for an introduction of more general $A_\infty$-categories, $A_\infty$-functors and $A_\infty$-natural transformations. Since we restrict ourselves to $A_\infty$-natural transformations between dg-functors, the notations and $\pm$ sign conventions of $A_\infty$-natural transformations can be dramatically simplified.
Definition 3.1 (\(A_\infty\)-prenatural transformation) Let \(F, G : \mathcal{C} \to \mathcal{D}\) be two dg-functors between dg-categories. An \(A_\infty\)-prenatural transformation \(\Phi : F \Rightarrow G\) of degree \(n\) consists of the following data:

- For any object \(X \in \text{obj} (\mathcal{C})\), a morphism \(\Phi^0_X \in \mathcal{D}^n(FX, GX)\).
- For any \(l \geq 1\) and any objects \(X_0, \ldots, X_l \in \text{obj} (\mathcal{C})\), a morphism

\[
\Phi^l_{X_0, \ldots, X_l} \in \text{Hom}^{n-l}_k(\mathcal{C}(X_{l-1}, X_l) \otimes \cdots \otimes \mathcal{C}(X_0, X_1), \mathcal{D}(FX_0, GX_l)).
\]

Definition 3.2 (Differential of \(A_\infty\)-prenatural transformation) Let \(F, G : \mathcal{C} \to \mathcal{D}\) be two dg-functors between dg-categories. Let \(\Phi : F \Rightarrow G\) be an \(A_\infty\)-prenatural transformation of degree \(n\) as in Definition 3.1. Then the differential \(d\Phi : F \Rightarrow G\) is an \(A_\infty\)-natural transformation of degree \(n+1\) whose components are given as follows:

- For any object \(X \in \text{obj} \mathcal{C}\), \((d^{\infty}\Phi)^0_X = d(\Phi^0_X) \in \mathcal{D}^{n+1}(FX, GX)\).
- For any \(l \geq 1\) and a collection of morphisms \(u_i \in \mathcal{C}(X_{i-1}, X_i), i = 1, \ldots, l\),

\[
(d^{\infty}\Phi)^l(u_1 \otimes \cdots \otimes u_l) = d(\Phi^l(u_1 \otimes \cdots \otimes u_l)) + (-1)^{|u_1|-1}G(u_1)\Phi^{l-1}(u_{l-1} \otimes \cdots \otimes u_1) + (-1)^{|u_1|+\cdots+|u_l|+l-1}\Phi^{l-1}(u_1 \otimes \cdots \otimes u_2)F(u_1)
\]

\[
+ \sum_{i=1}^{l} (-1)^{|u_1|+\cdots+|u_{i+1}|+l-i+1}\Phi^i(u_1 \otimes \cdots \otimes du_i \otimes \cdots \otimes u_1)
\]

\[
+ \sum_{i=1}^{l-1} (-1)^{|u_1|+\cdots+|u_{i+1}|+l-i+1}\Phi^{l-1}(u_1 \otimes \cdots \otimes u_{i+1}u_i \otimes \cdots \otimes u_1).
\]

Remark 3.3 The last term in (6) exists only if \(l \geq 2\).

Remark 3.4 The \(d^{\infty}\) above is differed from the \(\mu^{1}_Q\) in [10, Section I.1 (d)] by

\[
(-1)^{|u_1|+\cdots+|u_l|} \text{ on each term, which does not affect the properties of } d^{\infty}.
\]

We can check that \(d^{\infty} \circ d^{\infty} = 0\) on \(A_\infty\)-prenatural transformations.

Definition 3.5 (\(A_\infty\)-natural transformation) Let \(F, G : \mathcal{C} \to \mathcal{D}\) be two dg-functors between dg-categories. Let \(\Phi : F \Rightarrow G\) be an \(A_\infty\)-prenatural transformation. We call \(\Phi\) an \(A_\infty\)-natural transformation if \(\Phi\) is of degree 0 and closed under the differential \(d^{\infty}\) in Definition 3.2.

For an \(A_\infty\)-natural transformation \(\Phi : F \Rightarrow G\), the \(l = 0\) component of (6) is simply

\[
d(\Phi^0_X) = 0\]

for any object \(X\). The \(l = 1\) condition is that for any \(u \in \mathcal{C}(X_0, X_1)\) we have

\[
d(\Phi^1(u)) - \Phi^1(d(u)) + (-1)^{|u|}\Phi^0_{X_1}F(u) + (-1)^{|u|+1}G(u)\Phi^0_{X_0} = 0.
\]

The \(l = 2\) condition is that for any \(u_1 \in \mathcal{C}(X_0, X_1)\) and \(u_2 \in \mathcal{C}(X_1, X_2)\) we have

\[
d(\Phi^2(u_2 \otimes u_1)) - (-1)^{|u_1|+|u_2|}\Phi^1(u_2)F(u_1) - (-1)^{|u_2|}G(u_2)\Phi^1(u_1)
\]

\[
+ (-1)^{|u_2|}\Phi^2(u_2 \otimes du_1) - \Phi^2(du_2 \otimes u_1) + (-1)^{|u_2|}\Phi^1(u_2u_1) = 0.
\]
It is clear that a closed degree 0 dg-natural transformation $\Phi$ can be considered as an $A_{\infty}$-natural transformation with $\Phi^l = 0$ for all $l \geq 1$.

**Definition 3.6** (Compositions) Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be three dg-functors between dg-categories. Let $\Phi : F \Rightarrow G$ and $\Psi : G \Rightarrow H$ be two $A_{\infty}$-natural transformations. Then the composition $\Psi \circ \Phi$ is defined as follows: For any object $X \in \text{obj} \mathcal{C}$,

$$(\Psi \circ \Phi)^0_X = \Psi^0_X \Phi^0_X : F X \to G X \to H X$$

and for any $u_i \in \mathcal{C}(X_{i-1}, X_i), i = 1, \ldots, l$,

$$(\Psi \circ \Phi)^i(u_i \otimes \cdots \otimes u_1) = \sum_{k=1}^{l-1} \Psi^{i-k}(u_i \otimes \cdots \otimes u_{k+1}) \Phi^k(u_k \otimes \cdots \otimes u_1)$$

$$+ \Psi^i(u_i \otimes \cdots \otimes u_1) \Phi^0_{X_0} + \Phi^0_X \Psi^i(u_i \otimes \cdots \otimes u_1).$$

We can check that $\Psi \circ \Phi$ is an $A_{\infty}$-natural transformation.

**Remark 3.7** We can define compositions for general $A_{\infty}$-prenatural transformations. See [5, Section 3] or [10, Section I.1 (d)].

**Definition 3.8** ($A_{\infty}$-quasi-inverse) Let $F, G : \mathcal{C} \to \mathcal{D}$ be two dg $k$-functors between dg-categories. Let $\Phi : F \Rightarrow G$ be an $A_{\infty}$-natural transformation. We call an $A_{\infty}$-natural transformation $\Psi : G \Rightarrow F$ an $A_{\infty}$-quasi-inverse of $\Phi$ if there exist $A_{\infty}$-prenatural transformations $\eta : F \Rightarrow F$ and $\omega : G \Rightarrow G$ both of degree $-1$ such that

$$\Psi \circ \Phi - \text{id}_F = d^\infty \eta \quad \text{and} \quad \Phi \circ \Psi - \text{id}_G = d^\infty \omega.$$

In more details, this means that we have

$$\sum_{k=1}^{l-1} \Psi^{l-k}(u_l \otimes \cdots \otimes u_{k+1}) \Phi^k(u_k \otimes \cdots \otimes u_1)$$

$$+ \Psi^l(u_l \otimes \cdots \otimes u_1) \Phi^0_{X_0} + \Phi^0_X \Psi^l(u_l \otimes \cdots \otimes u_1)$$

$$= d(\eta^l(u_l \otimes \cdots \otimes u_1)) + (-1)^{|u_l|+1} G(u_l) \eta^l_{l-1}(u_{l-1} \otimes \cdots \otimes u_1)$$

$$+ (-1)^{|u_{l-1}|+\cdots+|u_i|+l-1} \eta^l_{l-1}(u_l \otimes \cdots \otimes u_2) F(u_1)$$

$$+ \sum_{i=1}^{l} (-1)^{|u_l|+\cdots+|u_{i+1}|+l-i+1} \eta^l_{l-1}(u_l \otimes \cdots \otimes d u_i \otimes \cdots \otimes u_1)$$

$$+ \sum_{i=1}^{l-1} (-1)^{|u_l|+\cdots+|u_{i+1}|+l-i+1} \eta^l_{l-1}(u_l \otimes \cdots \otimes u_{i+1} u_i \otimes \cdots \otimes u_1)$$
and

\[
\sum_{k=1}^{l-1} \Phi^{l-k}(u_l \otimes \cdots \otimes u_{k+1}) \Psi^k(u_k \otimes \cdots \otimes u_1) \\
+ \Phi^l(u_l \otimes \cdots \otimes u_1) \Psi^0_{X_0} + \Phi^0_{X_l} \Phi^l(u_l \otimes \cdots \otimes u_1)
\]

\[
= d(\omega^l(u_l \otimes \cdots \otimes u_1)) + (-1)^{|u_l|-1} G(u_l) \omega^{l-1}(u_{l-1} \otimes \cdots \otimes u_1) \\
+ (-1)^{|u_2|-\cdots-|u_l|+l-1} \omega^{l-1}(u_l \otimes \cdots \otimes u_2) F(u_1)
\]

\[
+ \sum_{i=1}^{l} (-1)^{|u_2|+\cdots+|u_{i+1}|} \omega^l(u_l \otimes \cdots \otimes u_i \otimes \cdots \otimes u_1)
\]

\[
+ \sum_{i=1}^{l-1} (-1)^{|u_2|+\cdots+|u_{i+1}|+l-i+1} \omega^{l-1}(u_l \otimes \cdots \otimes u_{i+1} u_i \otimes \cdots \otimes u_1).
\]

**Proposition 3.9** Let \( F, G : C \to D \) be two dg-functors between dg-categories and \( \Phi : F \Rightarrow G \) be an \( A_\infty \)-natural transformation. Then \( \Phi \) admits an \( A_\infty \)-quasi-inverse if and only if \( \Phi^0_X : FX \to GX \) is invertible in the homotopy category \( \text{Ho}D \) for any object \( X \in C \).

**Proof** See [5, Proposition 7.15] or [15, Theorem 4.1]. \( \square \)

**Remark 3.10** Proposition 7.15 in [5] is a more general result on \( A_\infty \)-natural transformation of \( A_\infty \)-functors between \( A_\infty \)-categories.

## 4 Simplicial homotopies

### 4.1 A review of simplicial homotopies

First we review the definition of simplicial homotopy between simplicial maps. For more details see [4, Section I.6].

**Definition 4.1** Let \( C \) be a category which admits finite colimits. For a simplicial object \( U \) in \( C \), we can construct the tensor product \( U \times I \) where \( I \) is the simplicial set \( \Delta_1 \). Two simplicial maps \( f, g : U \to V \) between simplicial objects are called **simplicial homotopic** if there is a map \( H : U \times I \to V \) such that

\[
f = H \circ \varepsilon_0 \quad \text{and} \quad g = H \circ \varepsilon_1
\]

where \( \varepsilon_\mu : U \to U \times I, \mu = 0, 1, \) are the two obvious inclusions. In this case we call \( H \) a simplicial homotopy between \( f \) and \( g \).

**Remark 4.2** In the literature a simplicial homotopy is sometimes called a strict simplicial homotopy. In Definition 4.1 we simply call it simplicial homotopy. Nevertheless we notice that simplicial homotopy is not an equivalence relation if we put no restriction on \( V \). See [4, Section I.6] for further discussions.
We have the following equivalent definition of simplicial homotopy, which is useful in the proof of Proposition 4.6 below. See [6, Definition 5.1].

**Definition 4.3** Two maps \( f, g : U \to V \) between simplicial objects are called combinatorial simplicial homotopic if for each \( p \geq 0 \), there exist morphisms

\[
h_i = h_i^p : U_p \to V_{p+1} \quad \text{for } i = 0, \ldots, p,
\]

such that the following conditions hold:

- \( \partial_0 h_0 = f_p, \partial_{p+1} h_p = g_p; \)
- \[
\partial_i h_j = \begin{cases} 
  h_{j-1} \partial_i, & i < j, \\
  \partial_i h_{j-1}, & i = j \neq 0, \\
  h_j \partial_{i-1}, & i > j + 1;
\end{cases}
\]
- \[
s_i h_j = \begin{cases} 
  h_{j+1} s_i, & i \leq j, \\
  h_j s_{i-1}, & i > j.
\end{cases}
\]

**Lemma 4.4** Let \( \mathcal{C} \) be a category which admits finite colimits. Then the two versions of simplicial homotopy in Definitions 4.1 and 4.3 are equivalent.

**Proof** It is an easy but complicated combinatorial check. See [6, Proposition 6.2]. The proof there is for \( \mathcal{C} = \text{Sets} \) but it also works for general \( \mathcal{C} \). \( \square \)

**Lemma 4.5** For any \( k \geq p \) let \( \rho_{k,p} \) and \( \tau_{k,p} \) be the front and back face maps as in (2) and (3). We have the following identities:

\[
\begin{align*}
  h_i \partial_j &= \begin{cases} 
  \partial_{j+1} h_i, & i < j, \\
  \partial_j h_{i+1}, & i \geq j;
\end{cases} \\
  h_i \circ \tau_{k,p} &= \tau_{k+1,p+1} \circ h_{i+k-p} \quad \text{for all } 0 \leq i \leq p; \\
  h_i \circ \rho_{k,p} &= \rho_{k+1,p+1} \circ h_i \quad \text{for all } 0 \leq i \leq p; \\
  f \circ \tau_{k,p} &= \tau_{k+1,p} \circ h_i \quad \text{for all } 0 \leq i \leq k - p; \\
  g \circ \rho_{k,p} &= \rho_{k+1,p} \circ h_i \quad \text{for all } p \leq i \leq k.
\end{align*}
\]

**Proof** It is a routine check of Definition 4.3 and the simplicial identities. \( \square \)

### 4.2 Simplicial homotopical maps and twisted complexes

In the sequel we compare \( f^* \) and \( g^* \) for simplicial homotopic maps \( f \) and \( g \).

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Proposition 4.6  Let \( f \) and \( g \) be two simplicial maps between simplicial ringed spaces \((\mathcal{U}, \mathcal{R}) \) and \((\mathcal{V}, \mathcal{S})\). Let \( h \) be a simplicial homotopy between \( f \) and \( g \) as in Definition 4.3. Then for any twisted complex \( \mathcal{E} = (E^\bullet, a) \) on \((\mathcal{V}, \mathcal{S})\), the homotopy \( h \) induces a weak equivalence

\[
\Phi_0(\mathcal{E}): f^*(\mathcal{E}) \sim g^*(\mathcal{E}).
\]

Proof  For any \( k \geq 0 \) we have \( d^{k+1,-k} \in \text{Hom}^{-k}_{S_{k+1}}(\tau^*_{k+1,0}(E), \rho^*_{k+1,0}(E)) \). Using \( h_i: (U_k, \mathcal{R}_k) \to (V_{k+1}, S_{k+1}) \) we obtain

\[
h_i^*(d^{k+1,-k}) \in \text{Hom}^{-k}_{\mathcal{R}_k}(h_i^*\tau^*_{k+1,0}(E), h_i^*\rho^*_{k+1,0}(E)) \quad \text{for} \quad 0 \leq i \leq k.
\]

By (9) and (10) we have

\[
\tau_{k+1,0} \circ h_i = f_0 \circ \tau_{k,0} \quad \text{and} \quad \rho_{k+1,0} \circ h_i = g_0 \circ \rho_{k,0}
\]

hence we get

\[
h_i^*(d^{k+1,-k}) \in \text{Hom}^{-k}_{\mathcal{R}_k}(\tau^*_0 f_0^*(E), \rho^*_0 g_0^*(E)).
\]

Then we define \( \Phi_0(\mathcal{E}) \) as follows:

\[
\Phi_0^{k,-k}(\mathcal{E}) = \sum_{i=0}^{k} (-1)^i h_i^*(d^{k+1,-k}) \in \text{Hom}^{-k}_{\mathcal{R}_k}(\tau^*_0 f_0^*(E), \rho^*_0 g_0^*(E)).
\]

Lemma 4.7  For any \( k \geq 0 \) we have

\[
\sum_{i=1}^{k-1} \sum_{j=0}^{k-1} (-1)^{i+j} \partial_j^* h_i^* = \sum_{i=1}^{k} \sum_{j=0}^{k} (-1)^{i+j-1} h_j^* \partial_i^*. \tag{11}
\]

Moreover, for two morphisms \( \phi: \mathcal{F} \to \mathcal{G} \) and \( \psi: \mathcal{E} \to \mathcal{F} \) with degree \( m \) and \( n \), respectively, we have

\[
\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^{(m-i)(k-i)+j} (\rho^*_{k,i} \phi^* i, m-i) (\tau^*_{k, k-i} \psi^* k-i+1, n-1+i-k)
\]

\[
= \sum_{j=0}^{k} (-1)^{j+m} h_j^* \sum_{i=0}^{j} (-1)^{(m-i)(k-i+1)} (\rho^*_{k+1, i} \phi^* i, m-i) (\tau^*_{k+1, k-i+1} \psi^* k-i+1, n-1+i-k). \tag{12}
\]
Then we prove that the morphism $\Phi_1$ and $\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{(m-i-1)(k-i)+j} \left( \rho_{k,i}^* \phi^i \Phi_{i+1,m-i-1} \left( \tau_{k,k-i}^* f^* y_{k-i,n+i-k} \right) \right)$

$$\Phi_1 = \sum_{j=0}^{k} (-1)^j h_j^* \left[ \sum_{i=j+1}^{k+1} (-1)^{(m-i)(k-i+1)} \left( \rho_{k+1,i}^* \phi^i \right) \tau_{k+1,k-i+1}^* y_{k-i+1,n-1+i-k} \right].$$

(13)

**Proof of Lemma 4.7** These identities follow from Lemma 4.5 and re-indexing. □

Then we prove that the morphism $\Phi_0(\mathcal{E})$ is closed, i.e., for any $k \geq 0$ we have

$$\delta \Phi_0(\mathcal{E}) + g^* (a) \cdot \Phi_0(\mathcal{E}) - \Phi_0(\mathcal{E}) \cdot f^* (a) = 0. \quad (14)$$

First we have

$$(\delta \Phi_0(\mathcal{E}))^{k,1-k} = \sum_{i=1}^{k-1} (-1)^i \partial_i^* \Phi_{0,\mathcal{E}}^{k-1,1-k}$$

$$= \sum_{i=1}^{k-1} (-1)^i \partial_i^* \sum_{j=0}^{i} (-1)^j h_j^* a^{k,1-k} = \sum_{j=0}^{k-1} \sum_{i=1}^{k-1} (-1)^{i+j} \partial_i^* h_j^* a^{k,1-k}.$$  

By (11) we have

$$(\delta \Phi_0(\mathcal{E}))^{k,1-k} = \sum_{j=0}^{k} \sum_{i=1}^{j} (-1)^i \partial_i^* \Phi_{0,\mathcal{E}}^{k-1,1-k}$$

$$= \sum_{j=0}^{k} (-1)^j h_j^* \sum_{i=1}^{k} (-1)^i \partial_i^* a^{k,1-k}.$$  

Similarly, by (12) we have

$$(g^* (a) \cdot \Phi_0(\mathcal{E}))^{k,1-k}$$

$$= \sum_{j=0}^{k} (-1)^j h_j^* \sum_{i=0}^{j} (-1)^{(1-i)(k+1-i)} \rho_{k+1,i}^* a^{i,1-i} \tau_{k+1,k+1-i}^* a^{k+1-i,i-k},$$

and by (13) we have

$$(\Phi_0(\mathcal{E}) \cdot f^* (a))^{k,1-k}$$

$$= \sum_{j=0}^{k} (-1)^j h_j^* \sum_{i=j+1}^{k+1} (-1)^{(1-i)(k+1-i)} \rho_{k+1,i}^* a^{i,1-i} \tau_{k+1,k+1-i}^* a^{k+1-i,i-k}.$$
Summing up these three identities and using $\delta a + a \cdot a = 0$ we get (14).

Finally we notice that $\Phi_0(\mathcal{E}) = h_0^*(a^{1,0}) \in \text{Hom}^0_\mathcal{S}_0(\tau_{0,0}^*(E), \rho_{0,0}^*g_0^*(E))$ is a quasi-isomorphism, since $a^{1,0} \in \text{Hom}^0_\mathcal{S}_1(\tau_{1,0}^*(E), \rho_{1,0}^*(E))$ is invertible up to homotopy. Therefore we know that $\Phi_0(\mathcal{E})$ is a weak equivalence. □

**Remark 4.8** In general for a morphism $\phi: \mathcal{E} \to \mathcal{F}$,

$$g^*(\phi) \cdot \Phi_0(\mathcal{E}) - (-1)^{|\phi|} \Phi_0(\mathcal{F}) \cdot f^*(\phi) \neq 0.$$ 

Therefore $\Phi_0(-)$ does not give a dg-natural transformation from $f^*$ to $g^*$. Nevertheless we can extend $\Phi_0(-)$ to an $A_{\infty}$-natural transformation.

### 4.3 Simplicial homotopies and $A_{\infty}$-natural transformations

In this section we introduce higher $\Phi_l$'s. Consider a degree $m$ morphism $\phi: \mathcal{E} \to \mathcal{F}$ in $\text{Tw}(\mathcal{V}, \mathcal{S})$. For any $k \geq 0$ we have

$$\phi^{k+1, m-k-1} \in \text{Hom}^{m-k-1}_{\mathcal{S}_{k+1}}(\tau_{k+1,0}^*(E), \rho_{k+1,0}^*(F)).$$

Hence for $0 \leq i \leq k$ we have

$$h_i^* \phi^{k+1, m-k-1} \in \text{Hom}^{m-k-1}_{\mathcal{S}_k}(h_i^* \tau_{k+1,0}^*(E), h_i^* \rho_{k+1,0}^*(F))$$

and by (9) and (10) we have

$$h_i^* \phi^{k+1, m-k-1} \in \text{Hom}^{m-k-1}_{\mathcal{S}_k}(\tau_{k,0}^*(F), \rho_{k,0}^*(F))\right).$$

Now we are ready for the following definition.

**Definition 4.9** For a degree $m$ morphism $\phi: \mathcal{E} \to \mathcal{F}$ in $\text{Tw}(\mathcal{V}, \mathcal{S})$, we define $\Phi_1(\phi): f^* \mathcal{E} \to g^* \mathcal{F}$ as

$$[\Phi_1(\phi)]^{k, m-k-1} := (-1)^{m-k-1} \sum_{i=0}^{k} (-1)^i h_i^* \phi^{k+1, m-k-1}.$$ 

For $l \geq 2$ we simply define $\Phi_l = 0$.

We need to prove that $\Phi_0$ and $\Phi_1$ together form an $A_{\infty}$-natural transformation from $f$ to $g$. First we prove the following proposition.

**Proposition 4.10** For a degree $m$ morphism $\phi: \mathcal{E} \to \mathcal{F}$ in $\text{Tw}(\mathcal{V}, \mathcal{S})$, we have

$$d[\Phi_1(\phi)] - \Phi_1(d\phi) + (-1)^{m-1} g^*(\phi) \Phi_0(\mathcal{E}) + (-1)^{m} \Phi_0(\mathcal{F}) f^*(\phi) = 0.$$
Proof First we have

$$[d \Phi_1(\phi)]^{k, m-k} = [\delta \Phi_1(\phi)]^{k, m-k} + [g^*(b) \cdot \Phi_1(\phi)]^{k, m-k} - (-1)^{m-1} [\Phi_1(\phi) \cdot f^*(a)]^{k, m-k}.$$ 

By definition,

$$1(\phi) t^{k, m-k} = \sum_{i=1}^{k-1} (-1)^i \sum_{j=0}^{k-1} (-1)^{j+m-1} h^*_j \phi^{k, m-k}$$

and by (11) we have

$$[\delta \Phi_1(\phi)]^{k, m-k} = \sum_{i=0}^{k} (-1)^i m \sum_{j=1}^{k} (-1)^j \partial^*_j \phi^{k, m-k}. \quad (15)$$

Next

$$[g^*(b) \cdot \Phi_1(\phi)]^{k, m-k} = \sum_{i=0}^{k} (-1)^{i-k} \sum_{j=0}^{k-1} (-1)^{j+m-1} \rho_{k,i} g^* b^* \Phi_1(\phi)^{k-i, m-1-k+i} \tau_{k-1, k-i} h^*_j \phi^{k-i, m-1-k+i}$$

and by (12) we have

$$[g^*(b) \cdot \Phi_1(\phi)]^{k, m-k} = \sum_{i=0}^{k} (-1)^i m \sum_{j=0}^{i} (-1)^{j-k} \sum_{k=1}^{i+k-1} (-1)^{j+k-1} \rho_{k,j} g^* b^* \Phi_1(\phi)^{k-j, m-1-k+j} \tau_{k+k-1, k-i} h^*_j \phi^{k-j, m-1-k+j}.$$ 

Similarly, by (13) we have

$$[\Phi_1(\phi) \cdot f^*(a)]^{k, m-k} = \sum_{i=0}^{k} (-1)^i m \sum_{j=0}^{i} (-1)^{j-k} \sum_{k=1}^{i+k-1} (-1)^{j-k} \rho_{k,j} \Phi_1(\phi)^{k-j, m-1-k+j} \tau_{k+k-1, k-i} a^{k-j, m-1-k+j}.$$ 

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Again by (12) we get

\[
\left[ g^*(\phi) \Phi_0(\mathcal{E}) \right]^{k, m-k} = \sum_{i=0}^{k} (-1)^i m h_i^* \sum_{j=0}^{i} (-1)^{m-j} (k-j+1) \rho_{k+1, j}^* \phi^j a^{k-j+1, j-k}
\]

and by (13) we get

\[
\left[ \Phi_0(\mathcal{F}) f^*(\phi) \right]^{k, m-k} = \sum_{i=0}^{k} (-1)^i h_i^* \sum_{j=i+1}^{k+1} (-1)^{1-j} (k-j+1) \rho_{k+1, j}^* b^{j, 1-j} a^{k-j+1, m-1-k+j}.
\]

Adding up equations (15) through (16) we get

\[
[d[\Phi_1(\phi)]]^{k, m-k} + (-1)^m [g^*(\phi) \Phi_0(\mathcal{E})]^{k, m-k} - (-1)^m [\Phi_0(\mathcal{F}) f^*(\phi)]^{k, m-k} = (-1)^m \sum_{i=0}^{k} (-1)^i h_i^* \left[ \sum_{j=1}^{k} (-1)^j \rho_{j, k}^* \phi^{j, m-k} \right.

\]

\[
+ \sum_{j=0}^{k+1} (-1)^{1-j} (k-j+1) \rho_{k+1, j}^* b^{j, 1-j} \phi^{k-j+1, m-1-k+j} - (-1)^m \sum_{j=0}^{k+1} (-1)^{m-j} (k-j+1) \rho_{k+1, j}^* j \phi^{j, m-j} \phi^{k-j+1, m-1-k+j} \right].
\]

We observe that the right-hand side of (17) is exactly \([\Phi_1(d\phi)]^{k, m-k}\), hence we complete the proof. \(\square\)

**Proposition 4.11** For two morphisms \(\phi: \mathcal{F} \to \mathcal{G}\) and \(\psi: \mathcal{E} \to \mathcal{F}\) in \(\text{Tw} (\mathcal{V}, S)\) with degree \(m\) and \(n\), respectively, we have

\[
(-1)^m g^*(\phi) \cdot \Phi_1(\psi) + (-1)^{m-n+1} \Phi_1(\phi) \cdot f^*(\psi) + (-1)^m \Phi_1(\phi \cdot \psi) = 0.
\]

**Proof** Again it is a consequence of (12) and (13) and details are left to the readers. \(\square\)

**Theorem 4.12** Let \(f\) and \(g\) be two simplicial maps between simplicial ringed spaces \((\mathcal{U}, \mathcal{R})\) and \((\mathcal{V}, \mathcal{S})\). Let \(h\) be a simplicial homotopy between \(f\) and \(g\) as in Definition 4.3. Let \(\Phi_0\) be as in Proposition 4.6 and \(\Phi_1\) be as in Definition 4.9. Then the collection \(\Phi = \{\Phi_0, \Phi_1, 0, 0, \ldots\}\) is an \(A_\infty\)-natural transformation from \(f^*\) to \(g^*\).
Proof According to Definitions 3.2 and 3.5, all we need to prove is that $\Phi_1$ is closed under $d_\infty$, i.e.,

$$
\begin{align*}
\sum_{i=1}^{l} (-1)^{|u_i|+\cdots+|u_{i+1}|+|u_i|+l+i+1} \Phi_{l-1}(u_i \otimes \cdots \otimes u_1) &= 0
\end{align*}
$$

for $l \geq 0$. According to (7) and (8), for $l = 0, 1, 2$ these are consequences of Propositions 4.6, 4.10, and 4.11, respectively. For $l \geq 3$ it is trivial since $\Phi_l = 0$ for $l \geq 2$. \hfill \Box

Remark 4.13 It seems surprising why we can stop at $\Phi_1$. Actually in the definition of twisted complexes, we have only differential $d^{k,1-k}$ and maps $\phi^{k,m-k}$, whose pullback under $h$ give $\Phi^0$ and $\Phi^1$ respectively. Since the compositions of morphisms between twisted complexes are strictly associative, we can stop at $\Phi^1$. If the compositions were weakly associative and we had higher associators, then we would have higher terms $\Phi^l, l \geq 2$, in the $A_\infty$-natural transformation.

5 Simplicial homotopies and twisted perfect complexes

In this section we refine Theorem 4.12 for twisted perfect complexes. First we review the concept of twisted perfect complexes.

5.1 A review of twisted perfect complexes

We are often not interested in all complexes of $\mathcal{R}$-modules but only in some more convenient subcategory. In this section we consider the contravariant functor

$$\text{StrPerf} : \text{Ringed Space}^{\text{op}} \to \text{dgCat}$$

which assigns to each ringed space $X$ the dg-category of strictly perfect complexes of $\mathcal{R}$-modules on $X$, i.e., bounded complexes of locally free finitely generated $\mathcal{R}$-modules on $X$. As before let $(\mathcal{U}, \mathcal{R})$ be a simplicial ringed space then we have a cosimplicial diagram of dg-categories.

$$
\text{StrPerf}(U_0, \mathcal{R}_0) \longrightarrow \text{StrPerf}(U_1, \mathcal{R}_1) \longrightarrow \text{StrPerf}(U_2, \mathcal{R}_2) \cdots
$$

We have the following variant of twisted complexes.
Definition 5.1 A twisted perfect complex $E = (E^i, a)$ on a simplicial ringed space $(U, R)$ is the same as twisted complex in Definition 2.9 except that each $E^i$ is a strictly perfect complex on $(U_0, R_0)$.

The twisted perfect complexes also form a dg-category and we denote it by $\text{TwPerf}(U, R)$. Obviously $\text{Tw Perf}(U, R)$ is a full dg-subcategory of $\text{Tw}(U, R)$.

Lemma 5.2 Let $f : (U, R) \to (V, S)$ be a simplicial map between simplicial ringed spaces. Then the dg-functor $f^* : \text{Tw}(V, S) \to \text{Tw}(U, R)$ restricts to the full dg-subcategory of twisted perfect complexes and gives a dg-functor $f^* : \text{TwPerf}(V, S) \to \text{TwPerf}(U, R)$.

Proof It is obvious since $f^*$ pulls back finitely generated locally free sheaves to finitely generated locally free sheaves. \qed

We have the following result for twisted perfect complexes which is similar to Proposition 2.13.

Proposition 5.3 Let $U$ be a simplicial ringed space. Then the dg-category of twisted complexes $\text{TwPerf}(U, R)$ gives an explicit construction of $\text{holim StrPerf}(U)$.

The significance of twisted perfect complexes in geometry is given by the construction in [7]. Moreover, we have the following result:

Theorem 5.4 ([13, Theorem 3.32]) Let $X$ be a quasi-compact and separated or Noetherian scheme and $U = \{U_i\}$ be an affine cover, then $\text{TwPerf}(U, O_X)$ gives a dg-enhancement of $D_{\text{Perf}}(X)$, the derived category of perfect complexes on $X$.

The following proposition is about weak equivalences between twisted perfect complexes.

Proposition 5.5 ([13, Proposition 2.31]) Suppose the simplicial space $U$ satisfies $H^k(U_i, S) = 0$ for any $i \geq 0$, any $k \geq 1$ and any locally free finitely generated sheaf of $R_i$-modules $S$. Let $E$ and $F$ be two objects in $\text{TwPerf}(U, R)$ and $\phi : E \to F$ be a degree 0 closed morphism. Then $\phi$ is a weak equivalence if and only if $\phi$ is invertible in the homotopy category $\text{HoTwPerf}(U, R)$.

Remark 5.6 The following simplicial spaces satisfy the condition in Proposition 5.5:

- $X$ is a separated scheme and $U = \{U_i\}$ is an affine cover of $X$;
- $X$ is a complex manifold and $U = \{U_i\}$ is a good cover of $X$ by Stein manifolds, i.e., all finite non-empty intersections of the cover are Stein manifolds.

Remark 5.7 In [13, Proposition 2.31] it is required that $H^k(U_i, S) = 0$ for any $i \geq 0$, any $k \geq 1$ and any quasi-coherent sheaf of $R_i$-modules $S$ because it is based on [13, Lemma 2.30] which requires a stronger condition. However, by a careful study we can see that the same proof of [13, Proposition 2.31] works if we only assume that $H^k(U_i, S) = 0$ for any $i \geq 0$, any $k \geq 1$ and any locally free finitely generated sheaf of $R_i$-modules $S$. Nevertheless, the examples in Remark 5.6 satisfy the stronger condition too.
Remark 5.8  The result in Proposition 5.5 only applies to twisted perfect complexes because we need to use the fact that quasi-isomorphisms between bounded complexes of finitely generated projective modules have quasi-inverses, which fails for general complexes of modules.

5.2 Simplicial homotopies and twisted perfect complexes

Let \( f \) and \( g \) be two simplicial maps between simplicial ringed spaces \((U, R)\) and \((V, S)\). Let \( h \) be a simplicial homotopy between \( f \) and \( g \) as in Definition 4.3. By Lemma 5.2 we have dg-functors \( f^*, g^*: \text{TwPerf}(V, S) \to \text{TwPerf}(U, R) \).

It is clear that the \( A_\infty \)-natural transformation \( \Phi: f^* \Rightarrow g^* \) in Theorem 4.12 also restricts to twisted perfect complexes. Moreover we have the following result.

**Proposition 5.9**  Let \( f \) and \( g \) be two simplicial maps between simplicial ringed spaces \((U, R)\) and \((V, S)\). Let \( h \) be a simplicial homotopy between \( f \) and \( g \) as in Definition 4.3. Let \( \Phi: f^* \Rightarrow g^* \) be the \( A_\infty \)-natural transformation as in Theorem 4.12. In addition, assume \((U, R)\) satisfies \( H^k(U_i, S) = 0 \) for any \( i \geq 0 \), any \( k \geq 1 \) and any locally free finitely generated sheaf of \( R_i \)-modules \( S \). Then \( \Phi \) admits an \( A_\infty \)-quasi-inverse.

**Proof**  By Proposition 4.6, \( \Phi_0(\mathcal{E}): f^*(\mathcal{E}) \to g^*(\mathcal{E}) \) is a weak equivalence for each \( \mathcal{E} \). By Proposition 5.5, \( \Phi_0(\mathcal{E}) \) is invertible in the homotopy category for \( \mathcal{E} \in \text{TwPerf}(V, S) \). The claim then follows from Proposition 3.9. \( \square \)

**Remark 5.10**  Although the \( A_\infty \)-natural transformation \( \Phi \) consists of only two components \( \Phi_0 \) and \( \Phi_1 \), its \( A_\infty \)-quasi-inverse may contain higher components.

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