The Dynamics of Multi-agent Multi-option Decision Making

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Abstract

Decision making is a dynamical phenomenon and thus the transition from indecision to decision can be described well by bifurcation theory. Agreement, or consensus, is only one of many ways in which collective decisions can be made. However, theories of multi-agent multi-option decision making either focus on agreement decisions or average at the agent level to achieve a mean-field description of the decision dynamics. Introducing the agent level back into collective decision-making models uncovers a plethora of novel collective behaviors beyond agreement and mean-field descriptions of disagreement and polarization. These include uniform and moderate-extremist disagreement, and switchy-and-fast versus continuous-and-slow transitions from indecision to decision. Which of these behaviors emerge depends on the model parameters, in particular, the number of agents and the number of options. Our study is grounded in Equivariant Bifurcation Theory, which allows us to formulate model-independent predictions and to develop a constructive sensitivity analysis of the decision dynamics at the organizing equivariant singularity. The localized sensitivity analysis reveals how collective decision making can be both flexible and robust in response to subtle changes in the environment or in the deciding agent interactions. Equivariant
Bifurcation Theory also guides the construction of new multi-agent multi-option decision-making dynamics, for an arbitrary number of agents and an arbitrary number of options, and with fully controllable dynamical behaviors.

Keywords: Consensus, Polarization, Nonlinear Dynamics, Equivariant Bifurcation Theory, Sensitivity Analysis, Computational Modeling

Contents

1 Introduction 3

2 A dynamic model of decision making 6

3 Model-independent predictions 10

3.1 Agreement and disagreement decision making 10

3.2 Moderate and extremist decision makers 12

3.3 The balance between cooperation and competition selects between agreement and disagreement 14

3.3.1 Mutual exclusion of agreement and disagreement 16

3.3.2 Co-existence of agreement and disagreement 17

3.4 Continuity of the decision process depends on the number of agents and the number of options 18

3.4.1 Agreement decision making is always switchy for $N_o > 3$ 20

3.4.2 Disagreement decision making is always switchy for $N_a, N_o > 2$ 21

4 Flexibility and robustness of nonlinear decision making 22

5 Equivariant analysis of collective decision making 24

5.1 Preliminaries 24

5.2 Examples 30
1 Introduction

Decision making among options can be viewed as a nonlinear process that depends on features of the decision-making agents, the options, and the context, all of which are subject to change. For example, suppose that on your way home at the end of the day you consider which among a set of alternative routes you could take. Perhaps you need to pick something up on the way and so you decide to follow the route where you expect to find what you need. However, when you get outside, you see that it is starting to rain and you determine that you would do better by taking a less exposed route.

When there are multiple decision-making agents who can communicate or observe the opinions of one another, the nonlinear decision-making process will then also depend on features of the exchange, for instance, the level of attention that each agent pays to the opinions of others. Returning to your commute home, suppose some office friends
Choose A

Options are equally valued

Choose A

Option A is higher valued

Choose A

Option B is higher valued

Choose B

Figure 1: Decision making as a bifurcation phenomenon. Three bifurcation diagrams are shown with a collective decision variable on the vertical axis and bifurcation parameter $\lambda$ on the horizontal axis. Solid (dashed) lines are stable (unstable) solutions. Arrows show flow of the quasi steady-state as $\lambda$ increases. $\lambda$ models a feature of the agents, options, context, or exchange, which can be varied. When options are equally valued, deadlock is broken by traversing a symmetric bifurcation. When one of the alternatives is more highly valued, the bifurcation unfolds in the direction of the favored option.

spontaneously decide to visit you after work. Your friends express their choice of route and you then decide to follow suit and travel with them. However, when you get outside, you remember that you need to stop at the store and so you decide ultimately to make your way home along yet a different route.

We can view the multi-agent, multi-option decision-making process as a transition in state of a continuous time and continuous state-space nonlinear dynamical system driven by parameters and time-dependent external inputs. The state represents the opinions of the agents about the options, and the parameters and external inputs represent the features of the agents, options, context, or exchange. The prediction of state transitions as a function of quasi-static variation of parameters and inputs is the subject of bifurcation theory: a bifurcation diagram expresses the set of solutions as a bifurcation parameter (or input) varies. The theory also classifies how a bifurcation diagram itself can evolve as other parameters (or inputs or imperfections) in the system vary, for instance, accumulated evidence in favor of one of the options. Recent works have used bifurcation theory to model decision making in a variety of biological systems, from animal groups to gene regulatory networks to neurons [22, 28, 3, 15, 17, 5, 21, 14, 27, 20, 23]. Figure 1 summarizes the general idea behind those works.

A bifurcation point in a bifurcation diagram corresponds to a singularity in the dy-
dynamics where the existence and/or the stability of solutions can change. Near the singularity the system has a heightened sensitivity, which explains how a decision-making system responds when there are changes in parameters and external inputs, such as individual biases or option values. Away from the singularity, the bifurcation diagram is highly robust, which explains how a decision-making system will reject small disturbances. In this way, bifurcation theory reveals how decision-making systems can tune the balance between flexibility and robustness.

The most challenging decision-making problems correspond to those in which the options have near-equal value or the agents have no marked biases but where making no decision is costly. These problems are also compelling because of the flexibility they afford: an agent without bias can more quickly be won over to another decision as compared to an agent who starts with a bias. So, for our nominal decision-making system we assume that the multiple options are interchangeable (equal value) and the multiple agents are also interchangeable (no biases). We can then use bifurcation theory to infer what can happen when there are small differences in options and/or agents.

Interchangeability of options and agents corresponds to (permutation) symmetry in the dynamics; that is, the dynamics are equivariant to permutation of options and permutation of agents. As a result, the powerful tools of equivariant bifurcation theory apply. The (linearized) state space decomposes into a direct sum of irreducible representations of the symmetry group. The theory predicts that generically at a singularity all bifurcation branches are tangent to one irreducible representation. For the decision-making system there are two irreducible representations, which are two complementary spaces with a clear interpretation: the agreement space, also known as the consensus space, where all the agents have the same opinion, and the disagreement space, sometimes called the balanced space, where the agents have polarized opinions that are spread out over the options in such a way that the average opinion is neutral.
The nonlinear dynamics of multi-agent, multi-option decision making are complex and their complexity grows with numbers of agents and options. However, equivariant bifurcation theory provides the means to understand this richness: the theory allows us to classify properties of the likely decision-making solutions and the transitions among them, as a function of the number of options $N_o$ and the number of agents $N_a$. This yields a large menu of possible dynamic scenarios and outcomes, which can be used, for example, to derive testable predictions of behavior, to help explain experimental and numerical data, and to design mechanisms for flexible and robust decision making in systems where design or intervention is possible.

When restricted to agreement decision making, our theory recovers existing modeling results, for example [22, 28, 3, 4, 17, 5, 21, 14]. Moreover, our general theory summarizes these results in a single, model-independent theoretical framework, and generalizes them to a wider spectrum of decision-making behavior. The proposed framework might also support novel mechanistic insights on recent computational and statistical studies that highlight the appearance of disagreement and polarization in large-scale networks [26, 24, 2, 16, 19, 18, 24]. Finally, the theory guides the construction of a three-agent three-option computational model, provided in Appendix A.1 that we use to illustrate the theoretical, model-independent predictions and that can be used as a starting point to develop a new generation of computational multi-agent multi-option decision-making models.

2 A dynamic model of decision making

We consider a network of $N_a \geq 1$ agents that have to make a decision between $N_o \geq 2$ options. The state of agent $i$ is denoted by the vector $X_i = (x_{i1}, \ldots, x_{iN_o}) \in \mathbb{R}_{\geq 0}^{N_o}$. The positive number $x_{ij}$ represents the opinion of agent $i$ about option $j$. A larger $x_{ij}$ means a greater preference by agent $i$ for option $j$. The entries of the vector $X_i$ satisfy the
\( N_o - 1 \) dimensional simplex conditions \( x_j^i \geq 0 \) and

\[
x_1^i + \cdots + x_{N_o}^i = 1.
\] (1)

Constraint (1) is based on the assumption that each agent has the same number of votes to distribute among the various options and this total has been normalized to 1. Hence, each agent state space is the \( N_o - 1 \) dimensional (unit) simplex \( \Delta \) and the state space of the decision-making network is

\[
\mathcal{V} = \Delta \times \cdots \times \Delta_{N_a-times}
\] (2)

of dimension \( d = N_a(N_o - 1) \). A generic point in the state space is denoted by the vector \( \mathbf{X} = (X_1, \ldots, X_{N_a}) \). The neutral point \( \mathbf{O} \) is the unique state where each agent assigns the same proportional vote to each option; that is, \( \mathbf{O} = (O_1, \ldots, O_{N_a}) \) where

\[
O_i = \left( \frac{1}{N_o}, \ldots, \frac{1}{N_o} \right) \in \Delta.
\] (3)

At the neutral point, all options are equally preferred by all agents. Mathematical analysis will be performed in the translated variables defined by state variations around the neutral point

\[
\mathbf{Z} = \mathbf{X} - \mathbf{O}.
\] (4)

This is equivalent to doing the analysis on \( \mathcal{V} = T_\mathbf{O}\mathcal{V} \), the tangent space to \( \mathcal{V} \) at \( \mathbf{O} \). The new variables \( \mathbf{Z}_i \) satisfy the linear constraint

\[
z_1^i + \cdots + z_{N_o}^i = 0,
\] (5)
instead of (1); that is, \( Z \) belongs to the \( d \)-dimensional linear subspace \( V \subset \mathbb{R}^{N_0N_a} \) defined by (5). Let

\[
V_i = T_{O_i} \Delta = \mathbb{R}^{N_0-1}
\]

be the tangent space to the simplex of options of agent \( i \) at the neutral point. Then the linearized state space \( V \) decomposes as \( V = V_1 \times \cdots \times V_{N_a} = \mathbb{R}^d \).

In this model, an agent makes a decision when its proportional vote for one option is greater than its proportional vote for any other option. We illustrate this concept in Figure 2 for the case \( N_0 = 3 \). If two or more options are equally favored and have a maximum proportional vote, then the agent is indecisive between those options. In this sense, a decision is a qualitative, binary event: any point in the agent state space is either a decision point or an indecision point. However, because the model state space is continuous, decisions possess a quantitative nature too. If the agent state lies close to the neutral point or to decision boundaries, its decision is weak, soft, or moderate. If the agent state lies far from the neutral point and from decision boundaries, its decision is strong, hard, or extremist. In practice, state thresholds or time deadlines can help in determining when an agent makes a decision. Based on their decisions, multiple agents can either agree or disagree.

We aim at generalizing and unifying existing decision-making models to answer the following questions: How does a group of identical agents make decisions about a set of \( a \) priori equally valued options? What are the possible decision transitions that can generically occur for decision-making networks as defined above? How do the dynamics of these transitions depend on system parameters and inputs, such as numbers of agents and options, interconnection topology, and external signals? To answer these questions, we use equivariant bifurcation theory [12, 11] and model the time evolution of agent
Figure 2: A) State space and decision states of one agent $i$ for $N_o = 3$. The 2-simplex is the triangle $\Delta \in \mathbb{R}^3$ with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. The center of the triangle is the neutral point $O_i$. The dashed lines in the triangle indicate points of indecision, or decision boundaries, where two options are equally favored over the third. B) Agreement and disagreement states of two agents $i$ and $j$ for $N_o = 3$.

opinions as a smooth dynamical system

$$\dot{X} = G(X, \lambda),$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter. The distinguished bifurcation parameter $\lambda$ can model a variety of features and parameters depending on the context. For instance, the bifurcation parameter could be the strength of exchange between the agents, the pressure exerted by an upcoming deadline and by other environmental factors, or a combination of all of them. When any of these factors is sufficiently strong, we expect the neutral, undecided state to become unstable. Indecision is no longer sustainable and the system transitions, or bifurcates, to a decision state.

Equivariant bifurcation theory is a model-independent theory, in the sense that pre-
cise predictions about model behavior can be based solely on empirical assumptions, without proposing a specific model. Implicitly, we have already made one assumption; it is that decision making can be modeled as a smooth dynamical system of the form (7). Explicitly, we have made a second assumption concerning model symmetries. When options are equally valued and agents have no hierarchies nor biases, the system is symmetric with respect to interchanging options and interchanging agents. In the language of group theory, this means that model (7) has symmetry group

$$\Gamma = S_{N_a} \times S_{N_o},$$

(8)

where the permutation group $S_{N_a}$ interchanges the agents and the permutation group $S_{N_o}$ interchanges the options. The precise form that these symmetries take will be discussed in Section 3.1. We stress that these symmetry assumptions will never be satisfied in practice. However, the predictions of the theory remains practically valid when the symmetry assumptions are weakly violated. The perfectly symmetric situations constitute an organizing center where the sensitivity of the perturbed, real situations can be constructively examined. We will come back to this point in Section 4.

A key observation for our development is that all agents have the same state at $O$ and all options are equally favored at $O$. Hence, $\gamma O = O$ for all $\gamma \in \Gamma$, that is, the neutral point is fixed by the symmetry group of the model.

3 Model-independent predictions

3.1 Agreement and disagreement decision making

Given the symmetry assumptions, equivariant bifurcation theory predicts that decision making follows two generic types of paths, or bifurcation branches, as sketched in Fig-
One type is tangent at $O$ to the agreement space

$$W_1 = \{(\tilde{Z}, \ldots, \tilde{Z}) : \tilde{Z} \in V_1\}$$

(9)

and the other to the disagreement space

$$W_2 = \{(Z_1, \ldots, Z_{N_a}) : Z_1 + \cdots + Z_{N_a} = 0, Z_i \in V_i\}.$$  

(10)

Observe that $ToV = W_1 \oplus W_2$. States that are close to the agreement space correspond to situations in which all the agents have agreed on the same option. Conversely, close to the disagreement space, agent opinions are evenly spread across the options in such a way that the average opinion is close to neutral, an almost perfect collective indecision.

Agreement and disagreement spaces appear as important objects in the decision-making dynamics (7) because they are the two irreducible representations [12, Section XII.2], [11, Page 14] of the symmetry group $\Gamma$. The dimensions of the two spaces are different in general, as are the ways in which the symmetry group acts on the two spaces. The agreement space is generally lower dimensional and, because all agents have the same state, the permutation group $S_{N_a}$ acts trivially on this space. Decision making along the agreement space is thus governed by $S_{N_o}$-equivariant bifurcations acting on $\mathbb{R}^{N_o-1}$, a type of symmetry breaking that has already been studied [6], [11, Sections 1.5, 2.6, 2.7]. The disagreement space is generally higher dimensional and the full symmetry group acts nontrivially on this space. Decision making along this space is governed by $S_{N_a} \times S_{N_o}$-equivariant bifurcations acting on $\mathbb{R}^{N_a-1} \otimes \mathbb{R}^{N_o-1}$. These types of bifurcations have been studied in only a handful of cases [1], [12, Section XIII.5], [12, Chapter X]. We rigorously extend these results in Section 5.3.
Figure 3: The agreement space $W_1$ and disagreement space $W_2$ define complementary orthogonal subspaces at the neutral point $O$. Equilibria of the decision-making dynamics vary according to the bifurcation parameter $\lambda$. At a bifurcation, new equilibria can emerge from the neutral point along agreement and disagreement branches. Agreement branches are tangent to $W_1$, disagreement branches are tangent to $W_2$. As external signals and/or variations in system features modulate the bifurcation parameter, the network decision state follows one of the branches, either towards an agreement state or towards a disagreement state. Along the agreement space, all agents make the same decision ($a_1,a_2$). In $a_1$ the agent group reaches consensus on one of the options, whereas in $a_2$ it remains undecided between two options. Along the disagreement space the agent group remains totally undecided, in the sense that the average opinion is neutral. The are two generic types of disagreement states: uniform ($d_1$) and moderates/extremists ($d_3$).

### 3.2 Moderate and extremist decision makers

Figure 4 illustrates 3-agent 3-option agreement and disagreement decision making in the computational model presented in Appendix A.1. In both cases, parameters are such that the neutral point (represented by the circle at the center of the 2-simplex) is unstable. In agreement decision making (Figure 4 left), the agents’ states rapidly converge to a common value and then move together away from the neutral point and toward an agreement decision equilibrium. In disagreement decision making (Figure 4 center and right) two situations can happen. In one case (Figure 4 center), called uniform disagreement, the agents split uniformly from the neutral point toward the different decision
states, with each agent preferring a different option with the same opinion strength. In other case (Figure 4 right), called moderate-extremist disagreement, two agents start the decision process in near agreement. Those two agents move away from the neutral point in a cluster toward the same decision (in this numerical example, disfavoring Option 1 and being undecided between Options 2 and 3). The third agent moves away in the opposite decision direction (in this numerical example, choosing Option 1). Although two agents (the moderates) want to make the same choice, agreement is not possible because the third agent (the extremist) disagrees in such a strong manner that the moderates cannot lead the group toward their preferred choice(s). Eventually, this moderate/extremist configuration loses stability (the associated equilibrium is a saddle point in the proposed model with the parameter values used) and the group converges toward uniform disagreement.

Figure 4: Agreement and disagreement decision in a 3-agent 3-option computational model. The evolution of Agent 1’s state on the 2-simplex is represented by the solid line. The evolution of Agent 2’s state is represented by the short dashed line. The evolution of Agent 3’s state is represented by the long dashed line. The empty circle at the center of the simplex denotes the neutral point $O$. The final agent states are denoted by the black dots. In the Moderate/Extremist Disagreement case the transient (saddle) agent state is denoted by the gray dots. The parameters used for the agreement and disagreement cases are different. The parameters used for the two disagreement cases are the same, only the initial conditions differ. Code available upon request from the first author.

The appearance of the two types of disagreement equilibria, uniform and moderate/extremist, is another general prediction of the theory. The two types of disagreement decision making are also sketched in Figure 3. The disagreement equilibria d1 sketches
the generic uniform disagreement configuration for \( N_a \geq 3 \). The disagreement equilibria \( d3 \) sketches the generic moderate/extremist disagreement configuration for \( N_a \geq 3 \). The two disagreement types appear as non-conjugate axial subgroups \([12, Section XIII.3],[11, Section 1.4]\) of the disagreement irreducible representation.

### 3.3 The balance between cooperation and competition selects between agreement and disagreement

Using symmetry properties of the generic vector field, in particular its \( \Gamma \)-equivariance, we break down its form in some detail. This allows us to reinterpret agreement and disagreement bifurcations in terms of cooperation and competition between the agent opinions. Let

\[
F^i_j = F \left( x^i_j, \{ x^i_{l \neq j} \}, \{ x^i_{l \neq j} \} \right) \tag{11}
\]

and

\[
\langle F^i \rangle = \frac{1}{N_o} \sum_{j=1}^{N_o} F^i_j \tag{12}
\]

where \( F : \mathbb{R} \times \mathbb{R}^{N_o-1} \times \mathbb{R}^{N_a-1} \times \mathbb{R}^N \rightarrow \mathbb{R} \), \( N = N_a N_o - N_a - N_o + 1 \), is a smooth function, and the notations \( \{ x^i_{l \neq j} \}, \{ x^i_{l \neq j} \}, \{ x^i_{l \neq j} \} \) mean that \( F \) is invariant with respect to permutations of the elements of those sets of arguments. It then follows by symmetry properties of \( G \) (see \([25, 13]\) for details) that

\[
\dot{x}^i_j = G^i_j = F^i_j - \langle F^i \rangle. \tag{13}
\]

Subtracting \( \langle F^i \rangle \) in (13) ensures that \( \sum_{j=1}^{N_o} x^i_j(t) \) remains constant for all \( t \).

Let \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) be defined as

\[
\alpha = \left. \frac{\partial F^i_j}{\partial x^i_j} \right|_o + 1, \quad \beta = \left. \frac{\partial F^i_j}{\partial x^i_l} \right|_o, \quad \gamma = \left. \frac{\partial F^i_j}{\partial x^i_l} \right|_o, \quad \delta = \left. \frac{\partial F^i_j}{\partial x^i_l} \right|_o, \tag{14}
\]
where in the definition of $\beta$, $j \neq l$, in the definition of $\gamma$, $i \neq k$, and in the definition of $\delta$, $i \neq k, j \neq l$. By invariance properties of $F$, the definitions above do not depend on the index choices. The number $\alpha$ characterizes the uncoupled dynamics. Observe in particular that, if $\alpha = 0$, then $\left. \frac{\partial F}{\partial x} \right|_O = -1$. The numbers $\beta, \gamma, \delta$ characterize opinion interactions, both intra- and inter-agent, at the neutral point $O$:

- $\beta$ gives the direction and strength of intra-agent opinion interaction. If $\beta > 0$ the interaction is cooperative, if $\beta < 0$, the interaction is competitive. It is natural to consider $\beta < 0$ to model the fact that options are mutually exclusive because $\sum_{j=1}^{N_0} x_j^i = 1$.

- $\gamma - \delta$ gives the direction and strength of inter-agent opinion interaction. If $\gamma - \delta > 0$, then each agent pulls every other agent opinion toward its own opinion. The interaction is cooperative. If $\gamma - \delta < 0$, then each agent pushes every other agent opinion away from its own opinion. The interaction is competitive.

A simple computation then shows that

$$J := \frac{\partial G}{\partial X} = \begin{bmatrix} A & B & \cdots & B \\ B & \ddots & \vdots & \vdots \\ \vdots & \ddots & B & \vdots \\ B & \cdots & B & A \end{bmatrix}$$ (15)

where

$$A = \begin{bmatrix} a & b & \cdots & b \\ b & \ddots & \vdots & \vdots \\ \vdots & \ddots & b & \vdots \\ b & \cdots & b & a \end{bmatrix}, \quad B = \begin{bmatrix} p & q & \cdots & q \\ q & \ddots & \vdots & \vdots \\ \vdots & \ddots & q & \vdots \\ q & \cdots & q & p \end{bmatrix}$$ (16)

and $a = \frac{N_0 - 1}{N_0} (-1 + \alpha - \beta)$, $b = -\frac{a}{N_0 - 1}$, $p = \frac{N_0 - 1}{N_0} (\gamma - \delta)$, $q = -\frac{q}{N_0 - 1}$. Observe that for $\alpha = \beta = \gamma = \delta = 0$, the Jacobian of $G$ at the neutral point has $N_a(N_0 - 1)$ identical negative eigenvalues. This reflects the natural hypothesis that in the uncoupled
dynamics, when agent opinions do not interact and the decision process has not started, all opinions must converge to the neutral point, i.e., $X(t) \to O$ exponentially.

It is easy to derive conditions on $\alpha, \beta, \gamma, \delta$ under which an agreement bifurcation or a disagreement bifurcation can happen. Let $X_{agree} \in W_1$ and $X_{disagree} \in W_2$. Then we can solve

$$JX_{agree} = 0$$

or

$$JX_{disagree} = 0$$

in $\alpha, \beta, \gamma, \delta$ to determine which type of bifurcation can occur. If (17) admits a solution then agreement bifurcations are possible. If (18) admits a solution then disagreement bifurcations are possible. In some cases agreement and disagreement are mutually exclusive and in others they can co-exist. Solving (17) leads to

$$\alpha - \beta + (N_a - 1)(\gamma - \delta) = 1.$$  

(19)

Solving (18) leads to

$$\alpha - \beta - (\gamma - \delta) = 1.$$  

(20)

3.3.1 Mutual exclusion of agreement and disagreement

To interpret (19) and (20) in terms of agent cooperation and competition let’s fix $\alpha = 0$ and $\beta < 0$. If $-1 < \beta \leq 0$, which corresponds to weak intra-agent opinion competition, then it follows from (19) and (20) that agreement and disagreement are mutually exclusive. In particular, if agents cooperate, that is, $\gamma - \delta > 0$, then only agreement decision making is possible. This is the case in Figure 4 left. Conversely, if agents compete, that is, $\gamma - \delta < 0$, then only disagreement decision making is possible. This is the case in Figure 4 right.
3.3.2 Co-existence of agreement and disagreement

Now suppose that $\alpha = 0$ and $\beta < -1$, which corresponds to strong intra-agent opinion interactions. Then agreement and disagreement decision making can co-exist. Simultaneously imposing (19) and (20) leads to

$$\gamma = \delta, \quad \alpha - \beta = 1.$$  \hspace{1cm} (21)

For such parameter combination the neutral point bifurcates simultaneously along both the agreement and the disagreement branches. Close to such a degenerate situation there exist parameter combinations where both agreement and disagreement decision equilibria are stable. Figure 5 illustrates this situation.

![Figure 5: Co-existence of agreement and disagreement decision in a 3-agent 3-option computational model. The parameters used for the agreement and disagreement case are the same. Code available upon request from the first author.](image)

We stress that Figure 4 and Figure 5 correspond to two very different situations. In Figure 4 agreement and disagreement were obtained for different parameter combinations. In Figure 5 the state reaches agreement or disagreement solely depending on initial conditions. These two situations describe two different types of sensitivity (to parameters and to initial conditions, respectively) that can occur in practice.

Close to the degenerate situation (21), where agreement and disagreement decisions bifurcate at the same time another type of decision branches can also be detected, which
is neither agreement nor disagreement in the exact sense discussed until now. These new branches arise as connecting branches between exact agreement and disagreement. The existence and characterization of the connecting branches is the topic of mode interaction analysis in equivariant singularity theory. It is in general hard to develop such analysis. The only case in which it has previously been developed is the lowest possible dimensional case of two agents and two options [10, Chapter X]. We won’t pursue this topic further in the present work.

3.4 Continuity of the decision process depends on the number of agents and the number of options

We begin this section with a short introduction on the use of equivariant bifurcation diagrams in providing more quantitative details about the evolution of the model equilibria along the agreement and disagreement branches sketched in Figure 3. We illustrate for two simple cases: agreement decision making between two and three options. Because all agent states are the same in agreement decision making, the illustration is independent of the number of agents. However, it does depends on the number of options.

When $N_o = 2$, agreement decision making is symmetric with respect to swapping the two options, and the associated symmetry group is $S_{N_o} = S_2 \cong Z_2$. When $N_o = 3$, agreement decision making is symmetric with respect to permuting the three options, and the associated symmetry group is $S_{N_o} = S_3$. Figure 5 depicts the generic $S_2$-equivariant bifurcation diagram, the pitchfork bifurcation (left), and one of the generic $S_3$-equivariant bifurcation diagrams (right). For small bifurcation parameter values, in both cases the neutral equilibrium is the only steady state and it is stable. As $\lambda$ increases new equilibria appear and the neutral point becomes unstable. The new stable equilibria correspond to agreement decision-making states to which the system converges; the associated agreement decision configurations are sketched in the insets.
In the two-option case all agents choose either of the two options. In the three-option case all agents choose one of the three options.

Figure 6: A) Left: $\mathbf{S}_2$-equivariant bifurcation diagram for agreement decision making between two options and its interpretation. Right: $\mathbf{S}_3$-equivariant bifurcation diagram for agreement decision making between three options and its interpretation. The bifurcation diagrams are plots of model equilibria versus bifurcation parameter $\lambda$. Solid bifurcation branches are made of stable equilibria, dashed bifurcation branches by unstable ones. B) Smoothness of agreement decision making depends on the number of options.

Observe that the single bifurcating branch represents two distinct agreement equilibria in the two-option case and three distinct agreement equilibria in the three-option case, one per option. For clarity only one is drawn. The other branches are symmetrically disposed in the state space with respect to the model symmetry group (like the end points of a segment centered at the $O$ in the two-option case, the vertexes of an equilateral triangle centered at $O$ in the three-option case, the vertex of an $N_o - 1$ dimensional simplex in $\mathbb{R}^{N_o}$ centered at $O$ in the $N_o$-option case). In technical terms, the
various branches are *conjugate* by the symmetry group action [12, Section XIII.1], [11, Pages 10, 11].

### 3.4.1 Agreement decision making is always switchy for $N_o > 3$

Observe that agreement decision making exhibits two sharply distinct ways to transition from indecision to decision depending on whether $N_o = 2$ or $N_o = 3$. For $N_o = 2$, the transition from indecision to one of the two agreement decision states is continuous (Figure 6B, left). The agent opinion changes slowly along the decision branch. For $N_o = 3$, the transition from indecision to one of the three decision states is discontinuous, or "switchy" (Figure 6B, right). The agent opinion changes abruptly, i.e., it switches, from the neutral point to one of the decision branches. This difference is due to the fact that in the two-option case the agreement decision branches bifurcating from the neutral point are stable, whereas in the three-option case they are all unstable. In the latter case, however, the unstable branches departing to the left fold into a stable upper branch that attracts the model state after the neutral point has lost stability. Observe that there exists a range of $\lambda$ values where both the agreement decision and the neutral point are stable. The model is bistable between decision and indecision.

The same qualitative structure as the three-option case is present in all $S_{N_o}$-equivariant, $N_o > 2$, bifurcation diagrams. This is due to the existence in the agreement space of a non-trivial *equivariant quadratic mapping* [12, Example 2.15], which implies that all bifurcating branches predicted by the theory are unstable (see [12, Section XIII.4.c]). As a consequence we can formulate the general prediction that agreement decision making between two options is in general a more continuous, but slower, process as compared to agreement decision between three or more options, where the instability of the bifurcating decision branches engenders a switchy, but fast, transition from indecision to decision.
We finally stress that the folded stable branches that attract the system solutions at a switchy transition from indecision to decision might exhibit secondary bifurcations and lose stability. This is the case, for instance, in the other two types of generic $S_3$ equivariant bifurcation diagrams \cite[Section XV.4]{[12]}, \cite[Example 2.16]{[11]}. In this case, the system state is attracted to one of the secondary branches that can either correspond to the same decision state as the primary branch or not. The switchy nature of decision making is however preserved independently of the presence of secondary bifurcations. The singularity theory needed to study the existence and structure of secondary bifurcations has been worked out only in a handful of low-dimensional cases and won’t be further pursued here.

3.4.2 Disagreement decision making is always switchy for $N_a, N_o > 2$

A similar phenomenon as for agreement decision making happens for disagreement decision making. Disagreement decision making is always switchy for $N_a N_o > 6$ with $N_a, N_o > 2$ because, when these two conditions are satisfied, there exists a non-trivial equivariant quadratic mapping in the disagreement space. We prove this fact in Section \ref{sec:5.3.1}. Conversely, when $N_a N_o \leq 6$ or when either $N_a < 3$ or $N_o < 3$, such quadratic mapping does not exist and stable bifurcating branches are allowed. For instance, using \cite[Theorem 4.1]{[1]}, in the $N_a > 3$ and $N_o = 2$ case, we generically expect decision branches corresponding to roughly half of the agents picking one option and the remaining agents picking the other option to be stable. In other words, in a disagreeing agent population that decides over two options, we expect the group to continuously split over the two options in two roughly equally-sized clusters. Conversely, using the construction in Section \ref{sec:5.3.1} below, in the $N_a > 3$ and $N_o = 3$, we expect all disagreement decisions to happen abruptly.
4 Flexibility and robustness of nonlinear decision making

The hypothesis of exactly interchangeable agents and options is of course a mathematical idealization. It however provides precise information about decision making behavior when agents are not interchangeable or when options are not exactly equally valued. It reveals in particular how nonlinear decision making can both be flexible and robust. We illustrate this point, in our 3-agent 3-option computational model in the agreement regime, by introducing noisy perturbations, which take into account uncertainty and variability in the environment and agent interactions, and biases in the options, which take into account accumulated evidence in favor of options or a priori heuristics. The equations used in the illustration are provided in Section A.1. The associated Julia code is available upon request from the first author.

When options are equally valued, the agreement equivariant bifurcation diagram is perfectly symmetric (Figure 7 left), which reflects the fact that all options are equally likely to be chosen by the agent group. When one of the options has higher value, even if infinitesimally so, the equivariant bifurcation diagram breaks, or unfolds (Figure 7 right). Its geometry is then such that the only likely path from indecision to decision is the one toward agreement on the favored option. The bifurcation branches corresponding to agreement on disfavored options are disconnected from the indecision branch, preventing the model state from converging toward those branches as in an originally undecided situation. The decision making process is thus extremely flexible, or ultra-sensitive in the sense originally introduced in [9], because arbitrary small differences in option values push the system toward the highest valued options. We stress that the structure of the perturbed bifurcation diagram on the right cannot be derived by the equivariant bifurcation theory. We formalize the prediction of the appearance
of the sketched geometry under generic affine, non-symmetric perturbations (modeling differently-valued options) in Section 5.4.

Figure 7: Left: an $S_3$-equivariant bifurcation diagram. Right: a non-symmetric unfolding in the presence of affine perturbations modeling option value differences.

The dual property to flexibility, inherent to nonlinear decision making, is robustness. Away from the bifurcation point, where the neutral point loses stability, the bifurcation diagram is barely affect by the presence of exogenous inputs, both constructive ones, like evidence about higher-valued options, and destructive ones, like uncertainties or noise. This property ensures that away from the bifurcation point, decision making is a robust phenomenon, purely determined by its intrinsic geometry. In turn, this provides good signal-to-noise filtering capabilities to nonlinear decision making, where, for instance, the signal is given by differences in option values and the noise accounts for any random perturbation to the system.

We illustrate this fact by comparing robustness to noisy disturbances in nonlinear and linear decision making obtained by linearization at the bifurcation point. Linear decision making has the same local properties of its nonlinear counterpart, in particular, they both have the same number of zero eigenvalues with the same eigenvectors. However, linear decision making lacks the global organizing structure of nonlinear decision making. For a small fixed bias toward Option 1 and increasing noise intensity, nonlinear decision
making consistently chooses the correct option much more reliably than linear decision making when a time limit on the decision is imposed (Figure 8 top). More importantly, linear decision making is not able to reach a decision in the constrained time for small noise intensities because the linear drift, proportional to the perceived option value differences, is too small (Figure 8 bottom right). Nonlinear decision making does not suffer from this limitation because convergence to the favored equilibrium is exponential with a characteristic time constant that is fixed by the system intrinsic properties and not by the perceived option value differences (Figure 8 bottom left). Observe that for large noise intensity the linear and nonlinear decision making exhibit indistinguishable performances with the favored option being chosen slightly more than one third of the time.

5 Equivariant analysis of collective decision making

5.1 Preliminaries

The decision making dynamics (7) have symmetry $\Gamma = S_{Na} \times S_{No}$ in the sense that we are now going to make rigorous. Agent permutation $S_{Na}$ acts on $\mathcal{V}$ by permuting the agent axes, that is, if $\sigma \in S_{Na}$,

$$\sigma X = (X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(Na)})$$

Options permutations $S_{No}$ acts diagonally on $\mathcal{V}$, that is, if $\tau \in S_{No}$,

$$\tau X = (\tau X_1, \ldots, \tau X_{Na}), \quad \tau X_i = (x_{\tau^{-1}(i)(1)}, \ldots, x_{\tau^{-1}(i)(No)}).$$

Representing this action in the shifted (linearized) variables $Z$, defined by (4), we obtain a representation of $\Gamma$ on the linear space $V = T_{O}\mathcal{V}$. The analysis will be developed on
Figure 8: Stochastic, option biased simulations in agreement three-agent three-option decision making. Left, full model; right, linearized model. The bifurcation parameter is chosen to be exactly at bifurcation, that is, the agreement subspace is the generalized eigenspace of the zero eigenvalue. Noisy intensity, bias in favor of Option 1, and decision threshold are the same in the linear and nonlinear cases. Upper plots: probability of choosing the correct decision (Option 1) in less than 200 time units as a function of noise intensity. Bottom plots: 30 sample trajectory of $x_1$ for noise intensity 0.002. Bold lines are timely correct decisions.

this linear representation.

The given action of $S_{N_o}$ on each $V_i = T_{O_i} \Delta$ is isomorphic to the standard action of $S_{N_o}$ on $\mathbb{R}^{N_o-1}$. In particular, $\text{Fix}_{V_i}(S_{N_o}) = \{0\}$, where the set $\text{Fix}_{V_i}(S_{N_o})$ is defined for a general compact Lie group acting on a finite-dimensional vector space as follows.

**Definition 5.1.** Let $\Gamma$ be a compact Lie group acting on $\mathbb{R}^m$ and $\Sigma \in \Gamma$ be a subgroup. The fixed point subspace of $\Sigma$ is

$$\text{Fix}(\Sigma) = \{v \in \mathbb{R}^m : \gamma v = v, \ \forall \gamma \in \Sigma\}$$

Let $L \subset \mathbb{R}^m$ be a $\Sigma$-invariant subspace, that is, $\sigma L \subset L$ for all $\sigma \in \Sigma$. The fixed point
subspace of $\Sigma$ restricted to $L$ is

$$\text{Fix}_L(\Sigma) = \{ v \in L : \gamma v = v, \ \forall \gamma \in \Sigma \}.$$  

In the original coordinates on the simplex, this means that the fixed point set of the option symmetries in the option space $\Delta$ of each agent $i$ is exactly the neutral point $[3]$. Because $S_{N_0}$ acts diagonally, the fixed-point subset of $1 \times S_{N_0} \subset \Gamma$ in $V$ is $\text{Fix}_V(1 \times S_{N_0}) = \{ 0 \}$. It follows that

$$\text{Fix}_V(\Gamma) = \{ 0 \}. \quad (22)$$

Our decision-making model has symmetry $\Gamma$ in the following sense. The decision-making dynamics (7) map to decision-making dynamics on $V$ defined by

$$\dot{Z} = G(Z, \lambda), \quad (23)$$

where $G : V \times \mathbb{R} \rightarrow V$. Our symmetry assumption is formalized by requiring that $G$ is $\Gamma$-equivariant, that is, $\Gamma G(\cdot, \lambda) = G(\Gamma \cdot, \lambda)$, for all $\lambda$. A necessary and sufficient condition for $\Gamma$-equivariance is that, if $Z(t)$ is a solution of (7), then so is $\gamma Z(t)$ for all $\gamma \in \Gamma$. We study the associated equivariant bifurcation problem

$$G(Z, \lambda) = 0. \quad (24)$$

Invoking (22) and [11, Theorem 1.17], it follows that the origin is a solution of the bifurcation problem (24) for all $\lambda$, that is,

$$G(0, \lambda) \equiv 0.$$ 

\^{1}Observe that, with a small abuse of notation, we are denoting the vector field on the tangent space $V$ with the same letter as the vector field on the base space $V$. 

26
We call the origin the *trivial equilibrium*. Observe that in the original decision-making coordinates $X$ the trivial equilibrium is exactly the neutral point $O$. Our goal is to study symmetry-breaking from the trivial equilibrium. In doing so, we will mainly rely on the Equivariant Branching Lemma [11, Lemma 1.31], [12, Theorem 3.3].

The first step in applying the Equivariant Branching Lemma is to decompose the state space into the direct sum of *irreducible representations*. A group representation on a given vector space is irreducible if the only invariant subspace with respect to the group action is the origin (see [12, Section XII.2], [11, Definition 1.21]). In our case, we can write

$$V = W_1 \oplus W_2$$

where $W_1$ and $W_2$ are the agreement and disagreement space defined in (9) and (10), respectively. It is easy to verify that both $W_1$ and $W_2$ are irreducible representations of $\Gamma$. Moreover $S_{N_a}$ acts trivially on $W_1$ but faithfully on $W_2$, whereas $S_{N_o}$ acts faithfully on both spaces. It follows that the actions of $\Gamma$ on $W_1$ and $W_2$ are not isomorphic. Because $W_1 \oplus W_2 = V$, [12, Corollary XII.2.6(a)] implies that no other $\Gamma$-irreducible representation exist.

Irreducible representations are important because of the following facts. We call a point $(0, \lambda^*)$ a *steady-state bifurcation point* from the trivial equilibrium if zero is an eigenvalue of $A_0 = (dG)_0, \lambda^*$. Without loss of generality, we assume $\lambda^* = 0$.

**Definition 5.2.** A group $\bar{\Gamma}$ acts absolutely irreducibly on $\mathbb{R}^m$ if the only linear maps that commute with $\bar{\Gamma}$ are multiples of the identity.

In particular, an absolutely irreducibly action is irreducible, because would an invariant subspace exist, the element of $\bar{\Gamma}$ would permute with the projection on that subspace. If $G$ is $\bar{\Gamma}$-equivariant, then $(dG)_{0, \lambda}$ commutes with $\bar{\Gamma}$ for all $\lambda$ (see [11, page 15]). It follows that if $\bar{\Gamma}$ acts absolutely irreducibly in some subspace $L$, $G$ is $\bar{\Gamma}$-equivariant, and $(0, 0)$ a steady-state bifurcation point, then $((dG)|_L)_{0, \lambda} = c(\lambda)I$, with $c(0) = 0$. The
following genericity result holds [11, Lemma 1.31].

**Theorem 5.3.** Let $\bar{\Gamma}$ be a compact Lie group. At a steady-state bifurcation point from the trivial equilibrium of a $\Gamma$-equivariant bifurcation problem the following are generically true.

(a) $0$ is the only eigenvalue of $A_0$ on the imaginary axis.

(b) The generalized eigenspace corresponding to $0$ is $\ker A_0$.

(c) $\bar{\Gamma}$ acts absolutely irreducibly on $\ker A_0$.

In other words, at a steady-state bifurcation point of a $\bar{\Gamma}$-equivariant bifurcation problem, (a) the only purely imaginary eigenvalue of $A_0$ is $0$, (b) $0$ has the same algebraic and geometric multiplicity, and (c) $\ker A_0$ is $\bar{\Gamma}$ invariant and the restriction of $(dG)_{0,\lambda}$ to $\ker A_0$ has the form $c(\lambda)I$ with $c(0) = 0$. The Lyapunov-Schmidt reduction [11, Section 1.3] ensures that we can restrict the $\bar{\Gamma}$-equivariant bifurcation problem to $\ker A_0$ preserving $\bar{\Gamma}$ equivariance and Theorem 5.3 ensures that, in doing so, the action of $\bar{\Gamma}$ on $\ker A_0$ is absolutely irreducible.

Going back to symmetry breaking in decision making, we can conclude that, generically, there are two types of symmetry breaking bifurcations of (7). Symmetry breaking along the agreement space, which corresponds to $\ker A_0 = W_1$, and symmetry breaking along the disagreement space, which corresponds to $\ker A_0 = W_2$. Note that, generically, the two types do not occur together, that is, either one type of bifurcation is observed or the other. The analysis in Section 3.3 determines which type is going to occur. To determine the structure of the bifurcating branches appearing inside either mode, we can apply the Equivariant Branching Lemma.

**Definition 5.4.** Let $v \in \mathbb{R}^m$ and $\bar{\Gamma}$ a compact Lie group acting on $\mathbb{R}^m$. The isotropy subgroup of $v$ is

$$\Sigma_v = \{ \gamma \in \bar{\Gamma} : \gamma v = v \}.$$
An isotropy subgroup $\Sigma$ is axial if $\dim \text{Fix}(\Sigma) = 1$.

**Lemma 5.5. (Equivariant Branching Lemma)**

(a) Assume $\bar{\Gamma}$ acts absolutely irreducibly on $\mathbb{R}^m$.

(b) Let $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ be $\bar{\Gamma}$-equivariant.

Assumptions (a) and (b) imply that

\[ f(0, \lambda) \equiv 0 \]
\[ (df)(0,\lambda) = c(\lambda) \]

Assume a bifurcation occurs at $\lambda = 0$, that is, $c(0) = 0$.

(c) Assume $c'(0) \neq 0$ (non-degeneracy condition).

(d) Assume $\Sigma \subset \bar{\Gamma}$ is an axial subgroup.

Then there exists a unique branch of solutions to $f(x, \lambda) = 0$ emanating from $(0,0)$ where the symmetry of the solutions is $\Sigma$.

The Equivariant Branching Lemma completely characterizes the types of branches that occur inside each irreducible representation in terms of the axial subgroups of $\Gamma$. Axial subgroups are relatively easy to compute and their fixed point spaces are the one-dimensional subspaces where the associated bifurcating branches lie. In general, submaximal, that is, non-axial, bifurcation branches might occur \[8\]. The appearance of submaximal branches is not a model-independent result, in the sense that they appear on open, but not dense, set of vector fields. We won’t pursue this discussion further here.
5.2 Examples

5.2.1 \( N_a = 3 \) and \( N_o = 2 \)

When \( N_a = 3 \) and \( N_o = 2 \) the state space \( V \) is isomorphic to \( \mathbb{R}^3 \). Each agent state space is isomorphic to the real line with positive and negative values associated to preferring one or the other of the options. The agreement space is thus one-dimensional and inside this space the symmetry group action reduces to the standard action of \( \mathbb{Z}_2 \) in \( \mathbb{R} \). Agreement decision making is therefore organized by the generic \( \mathbb{Z}_2 \)-equivariant bifurcation, the pitchfork. The bifurcating branches correspond to the three agents agreeing on the same option (Figure 9A).

The disagreement space is isomorphic to the subspace of \( \mathbb{R}^3 \) where the coordinate sum is zero, that is, it is isomorphic to \( \mathbb{R}^2 \). The action of the symmetry group in this subspace is the standard action of \( S_3 \times Z_2 \cong D_6 \) on \( \mathbb{R}^2 \), generated in our representation by option swapping \( \tau Z = -Z \) and by agent permutations, \( \sigma \in S_3 \),

\[
\sigma(Z_1, Z_2, Z_3) = (Z_{\sigma(1)}, Z_{\sigma(2)}, Z_{\sigma(3)}).
\]
In this representation, there are two conjugacy classes of axial subgroups [12, Section XIII.5], $Z_2(\kappa)$ and $Z_2(\tau\kappa)$, where $\kappa \in S_3$ swaps agents 1 and 2. The associated conjugacy classes of one-dimensional fixed point subspaces are

$$\text{Fix}(Z_2(\kappa)) = \{Z_1 = Z_2 = z, \ Z_3 = -2z, \ z \in \mathbb{R}\}$$

and

$$\text{Fix}(Z_2(\tau\kappa)) = \{Z_3 = 0, \ Z_1 = -Z_2 = z, \ z \in \mathbb{R}\}.$$

The class $Z_2(\kappa)$ describes the situation in which two agents agree on one of the two options and the third agent strongly disagrees by preferring the other option. It is a moderate/extremist situation. The class $Z_2(\tau\kappa)$ describes the situation in which one agent is neutral, while the other two agents disagree by preferring different options. Depending on second-order terms, the bifurcating branch of either type of conjugacy class can be stable, but not both at the same time, as illustrated in Figure 9B.

5.2.2 $N_a = 2$ and $N_o = 3$

When $N_a = 2$ and $N_o = 3$ the state space is isomorphic to $\mathbb{R}^4$. Each agent space is the two-simplex in $\mathbb{R}^3$, which is locally isomorphic to $\mathbb{R}^2$. The agreement space is thus isomorphic to $\mathbb{R}^2$ and inside this space the symmetry group action reduces to the standard action of $S_3 \cong D_3$ on $\mathbb{R}^2$, generated in our representation by option permutations. Figure 10A reproduces a generic $D_3$ equivariant diagram from [12, Figure XV.4.2] and its two interpretations in terms of agreement decision making. A similar interpretation can be worked out for the other generic $D_3$ equivariant diagram in [12, Figure XV.4.2]. There is one conjugacy class of axial subgroups, that is, $Z_2(\kappa)$, where $\kappa \in S_3$ swaps options 1 and 2. The associated conjugacy class of fixed point subspaces is

$$\text{Fix}(Z_2(\kappa)) = \{z_3^1 = z_3^2 = 0, \ z_1^1 = z_2^2 = -z_1^2 = -z_2^2 = x, \ x \in \mathbb{R}\}.$$
The two bifurcating $Z_2(\kappa)$ branches are associated to two different situations. In one, the two agents agree on one option, in the other, they remain undecided between the other two options. There are two ways of making this assignment and there are thus two possible interpretations of this bifurcation diagram in terms of decision making. Which of the two is actually observed depends on second-order terms. In this $D_3$-equivariant bifurcation diagram there exist two regions of bistability. The first region happens for small values of the bifurcation parameter and involves bistability between the neutral state and one of the two branch types. This bistability region happens also in the other two generic $D_3$ equivariant diagrams in [12] Figure XV.4.2. The second bistability region happens for larger values of the bifurcation parameter and involves bistability between the two types of bifurcating axial branches. Both bistability regions can lead to hysteretic and switching decisions.

The disagreement space is also isomorphic to $\mathbb{R}^2$ because each agent state space is isomorphic to $\mathbb{R}^2$ and $X_1 + X_2 = O$ inside the disagreement space. The symmetry group action is the standard action of $Z_2 \times S_3 \cong D_6$ on $\mathbb{R}^2$, generated in our representation by agent swapping $\tau(Z_1, Z_2) = (Z_2, Z_1) = -(Z_1, Z_2)$ and by option permutations.
\[ \sigma \in S_3, \]
\[ \sigma Z = (\sigma Z_1, \sigma Z_2). \]

Note that the symmetry group action is isomorphic to the case \( N_a = 3, N_o = 2 \). The conjugacy classes of axial subgroups are therefore the same. However, the conjugacy classes of fixed point subspaces and their interpretation are different. They are now given by
\[ \text{Fix}(Z_2(\kappa)) = \{ z_1^1 = z_2^1 = -z_1^2 = -z_2^2, z_3^1 = -z_3^2 = z, z \in \mathbb{R} \} \]
and
\[ \text{Fix}(Z_2(\tau\kappa)) = \{ z_3^3 = z_3^2 = 0, z_3^1 = -z_2^1 = -z_1^2 = z_2^2 = z, z \in \mathbb{R} \}. \]

The conjugacy class \( \text{Fix}(Z_2(\kappa)) \) describes the situation in which one agent chooses one of the three options and the other agent remains undecided between the other two options. The conjugacy class \( \text{Fix}(Z_2(\tau\kappa)) \) describes the situation in which both agents remain neutral about one of the three options and each of them makes a different choice between the remaining two options. Depending on second-order terms, both types of branches can be stable, but not both of them at the same time as illustrated in Figure 10B.

### 5.2.3 Disagreement axials for \( N_a = N_o = 3 \)

The main difference between the \( N_a = N_o = 3 \) case considered here and the \( N_a \cdot N_o = 6 \) cases considered in Sections [5.2.1] and [5.2.2] is that the group \( S_3 \times S_3 \) is not isomorphic to any \( S_k \) or \( D_k \) for any \( k \). Novel types of axial conjugacy classes appear. Moreover, the existence of a quadratic equivariant, proved in Section [5.3.1], implies that all disagreement bifurcation branches are unstable. Agreement decision-making analysis is the same as for the two-agent three-option case. We hence focus on disagreement decision making.

The following theorem classifies the axial subgroups of \( \Gamma = S_3 \times S_3 \) acting on the
disagreement space

\[ W_2 = \{ (Z_1, Z_2, Z_3) : Z_1 + Z_2 + Z_3 = 0, \ Z_i \in V_i \} \cong \mathbb{R}^2 \otimes \mathbb{R}^2, \]

where the first factor of \( \Gamma \) permutes the agent index and the second factor permutes the options. It is a corollary of the more general Theorem 5.8. We first need some notation.

Let \( \kappa^a \in S_3 \) be the order-two element that swaps the first two agents and \( \kappa^o \in S_3 \) be the order-two element that swaps the first two options. Let \( \theta^a \in S_3 \) be the order-three element that cycles forward the agents and \( \theta^o \in S_3 \) be the order-three element that cycles forward the options. Let \( \rho \in S_3 \times S_3 \) be the order-two element defined by \( \rho = (\kappa^a, \kappa^o) \). Let \( \nu \in S_3 \times S_3 \) be the order-three element defined by \( \nu = (\theta^a, \theta^o) \).

**Theorem 5.6.** There are two conjugacy classes of axial subgroups for the action of \( \Gamma = S_3 \times S_3 \) on \( W_2 \cong \mathbb{R}^2 \otimes \mathbb{R}^2 \) defined above. They are

I) \( \mathbf{Z}_2(\kappa^a) \times \mathbf{Z}_2(\kappa^o) \) with fixed point subspace

\[
\text{Fix}(\mathbf{Z}_2(\kappa^a) \times \mathbf{Z}_2(\kappa^o)) = \{ Z_1 = Z_2 : z_1^1 = z_2^1 \}. \tag{25}
\]

II) \( \mathbf{Z}_3(\nu) \times \mathbf{Z}_2(\rho) \) with fixed point subspace

\[
\text{Fix}(\mathbf{Z}_3(\nu) \times \mathbf{Z}_2(\rho)) = \{ Z_3 = \theta^o Z_2 = \theta^o Z_1 : z_2^1 = z_1^2 \}. \tag{26}
\]

Moreover, all the bifurcation branches are unstable.

The interpretation of the two subgroups is provided in Figure 11. The axial \( \mathbf{Z}_2(\kappa^a) \times \mathbf{Z}_2(\kappa^o) \) describes the situation in which the disagreeing agents form two clusters (Figure 11A). One is made of two agents, the moderates, that develop a weak opinion either toward one of the options (right) or remain undecided between two options (left).
second cluster is made of one agent, the extremist, that develops a stronger opinion exactly opposing the moderate cluster. The axial $\mathbb{Z}_3(\nu) \times \mathbb{Z}_2(\rho)$ describes the situation in which the disagreeing agents develop symmetrically opposed decisions. They either favor different options, or remain undecided between different pairs of options.

We stress that because all bifurcation branches are unstable the transition from indecision to disagreement is discontinuous. In general, other types of branches, not predicted by the Equivariant Branching Lemma, can appear at and away from the singularity. In those cases, the two types of disagreement branches described here might be globally unstable and thus never asymptotically reached by the system. However, such a situation is not generic [8], as opposed to the Equivariant Branching Lemma, which holds generically. We can therefore expect that, similarly to the three-option agreement decision making in Figure 6A(right) and Figure 10A, the disagreement decision branches predicted here bend in a fold and become stable away from the singularity. The simulations in Figures 4 and 5 show, for the developed three-agent three-option model, that the $\mathbb{Z}_3(\nu) \times \mathbb{Z}_2(\rho)$ axial branches indeed undergo such stabilization bending.
5.2.4 Disagreement decision making for \( N_a = 2 \) or \( N_o = 2 \)

When either \( N_a = 2 \) and \( N_o = n \geq 2 \), or \( N_o = 2 \) and \( N_a = n \geq 2 \), the action of \( \Gamma \) on the disagreement space is the action of \( S_n \times Z_2 \) on \( \mathbb{R}^{n-1} \). In both cases, the non-trivial element of \( Z_2 \), corresponding to agent swapping if \( N_a = 2 \) or to option swapping if \( N_o = 2 \), can be represented as minus the identity. The abstract equivariant analysis is thus the same for the two cases and can be found in [1, Corollary 3.2]. We summarize here the relevant results. Of course, their interpretation will depend on whether \( N_a = 2 \) or \( N_o = 2 \).

To state the theoretical results we need some notation and definitions. Let \( \tilde{V} \cong \mathbb{R}^{n-1} \) be the subspace of \( \mathbb{R}^n \) where coordinates sum to zero. Let \( T_l \in S_n \times Z_2 \) be defined as follows. Partition \( n \) into 3 blocks, in such a way that the first two blocks posses \( 1 \leq l \leq \frac{n}{2} \) elements each and the last block possesses \( n - 2l \) elements. Let

\[
T_l = S_l \times S_l \times S_{n-2l} \times Z_2(\rho_l)
\]

where \( \rho_l = (\tilde{\rho}_l, -I) \in S_n \times Z_2 \) and \( \tilde{\rho}_l \) swaps the first two blocks.

**Theorem 5.7.** The conjugacy classes of axial subgroups of \( S_n \times Z_2 \) acting on \( \tilde{V} \) and associated conjugacy classes of one-dimensional fixed point subspaces are:

1. \( \Sigma_k = S_k \times S_{n-k}, \ 1 \leq k < \frac{n}{2} \), with conjugacy class of fixed point subspaces

\[
\text{Fix}(\Sigma_k) = \{(y_1, y_2) \in \tilde{V} : y_1 \in \mathbb{R}^k, \ y_2 \in \mathbb{R}^{n-k}, \\
y_i = c_i(1, \ldots, 1), \ kc_1 + (n-k)c_2 = 0\}.
\]

2. \( T_l = S_l \times S_l \times S_{n-2l} \times Z_2(\rho_l), \ 1 \leq l \leq \frac{n}{2} \), with conjugacy class of fixed point subspaces

\[
\text{Fix}(T_l) = \{(z, -z, 0) \in \tilde{V} : z = c(1, \ldots, 1) \in \mathbb{R}^l, \ c \in \mathbb{R}\}.
\]
Let us interpret Theorem 5.7 in terms of decision making. When \(N_a = 2\) and \(N_o = n\), the fixed point subspace \(\text{Fix}(\Sigma_k)\) describes the situation in which one agent has opinion \(1/N_o + c_1\) about the first \(k\) options and opinion \(1/N_o + c_2\) about the last \(N_o - k\) options, while the other agent has opinion \(1/N_o - c_1\) about first \(k\) options and opinion \(1/N_o - c_2\) about the last \(N_o - k\) options. See Figure 12a. Because \(k < \frac{n}{2}\), it follows that \(|c_1| > |c_2|\). Suppose that \(c_1 > 0\) (the reader can easily work out the interpretation for \(c_1 < 0\)). Then the first agent has a strong preference for the first \(k\) options while the second agent has a weak preference for the last \(N_o - k\) options. The first agent behaves with sureness in situations in which agent decisions are mutually exclusive by giving all its votes to a small number of options and strongly securing them. The second agent behaves more insecurely, by giving all its votes to the remaining options, which are more numerous since \(k < \frac{n}{2}\), but with weaker preference.

When \(N_o = 2\) and \(N_a = n\), the fixed point subspace \(\text{Fix}(\Sigma_k)\) describes the situation in which a small group of \(k\) agents strongly prefers the first option (with opinion \(1/2 + c_1\)) and a larger group of \(N_a - k\) agents weakly prefer the second option (with opinion \(1/2 + c_2\)). See Figure 12b. The members of the small group can be considered as the extremists, who manage to avoid agreement on the option favored by a larger number of agents, by developing a strong preference for their favorite option. The members of the large group can be considered as the moderates, who do not develop a strong preference but rely on their large number to achieve agreement. A similar situation was studied in schooling fish [4].

A similar interpretation can be worked out for the other axial as follows. When \(N_a = 2\) and \(N_o = n\), the fixed point subspace \(\text{Fix}(T_l)\) describes the situation in which the two agents are neutral about the last \(N_o - 2l\) options and undecided, with exactly opposite opinions, about the first \(2l\) options. In particular, the first agent favors the first \(l\) options with opinion \(1/N_o + c\) and disfavors the second \(l\) options with opinion \(1/N_o - c\),
and vice versa for the second agent. When $N_o = 2$ and $N_a = n$, the first $l$ agents equally favor the first option with opinion $1/2 + c$, the second $l$ agents equally favor the second option with opinion $1/2 - c$, and the last $N_a - 2l$ agents form an undecided group.

Invoking [1, Theorems 4.1 and 4.2] we conclude that, generically, only the bifurcation branches corresponding to the axial $\Sigma_k$, with $n/3 < k < n/2$ are stable. This fact does not imply that the other types of disagreement decision making won’t be observed in practice. The associated equilibria will indeed be saddle points and, depending on initial conditions, the system can transiently converge to these points along their stable mani-
Figure 13: Interpretations of the $S_n \times Z_2$ axial $T_1$ in terms of disagreement decision making.

fold before diverging toward a stable disagreement equilibrium. A similar phenomenon happens in the three-agent three-option case simulated in Figure 4.
5.3 Main results

5.3.1 Existence of a disagreement quadratic equivariant for \( N_a \geq 3 \) and \( N_o \geq 3 \)

Let \( F_i : V_i \to V_i \) be the quadratic map defined as

\[
F_i(Z_i) = \begin{bmatrix}
N_o(z_i^1)^2 - ((z_i^1)^2 + \cdots + (z_{N_o}^i)^2) \\
\vdots \\
N_o(z_{N_o}^i)^2 - ((z_i^1)^2 + \cdots + (z_{N_o}^i)^2)
\end{bmatrix}
\]

If \( N_o \geq 3 \), \( F_i \) is not identically zero and \( S_{N_o} \)-equivariant.

Let \( F : V \to V \) be the quadratic map defined as

\[
F(Z) = \begin{bmatrix}
N_aF_1(Z_1) - (F_1(Z_1) + \cdots + F_{N_a}(Z_{N_a})) \\
\vdots \\
N_aF_{N_a}(Z_{N_a}) - (F_1(Z_1) + \cdots + F_{N_a}(Z_{N_a}))
\end{bmatrix}
\]

\( F \) is \( \Gamma \)-equivariant. Moreover, if \( N_a \geq 3 \), \( F|_{W_2} : W_2 \to W_2 \) is not identically zero.

It follows that, if \( N_a \geq 3 \) and \( N_o \geq 3 \), there exists a non-zero quadratic equivariant in the disagreement space. Invoking [11, Theorem 2.14] (see also [12, page 90]), we expect all disagreement branches predicted by the Equivariant Branching Lemma to be unstable.

5.3.2 Disagreement axials for \( N_a = 3 \) or \( N_o = 3 \)

When either \( N_a = 3 \) and \( N_o = n \geq 3 \), or \( N_o = 3 \) and \( N_a = n \geq 3 \), the action of \( \Gamma \) on the disagreement space is the action of \( S_n \times S_3 \cong S_n \times D_3 \) on \( \mathbb{R}^{2(n-1)} \cong \mathbb{C}^{n-1} \).

In the complex representation, \( S_n \) permutes the coordinate axis and \( D_3 \) is generated by the complex conjugation element \( \kappa z = \bar{z} \), corresponding to two agent swapping if
\( N_a = 3 \) or two option swapping if \( N_o = 3 \), and by the multiplicative element \( \theta z = e^{i2\pi z} \), corresponding to agent forward cycling if \( N_a = 3 \) or option forward cycling if \( N_o = 3 \). The complex representation leads to more compact notation, but the construction and proof worked out in this section can easily be translated to the standard representation of \( S_3 \) on \( \mathbb{R}^2 \).

Let \( \sigma_m \in S_n \) be the order-two element defined as follows. Partition \( n \) into \( r \geq 2 \) blocks such that the first and second blocks have each the same number \( m \) of elements. Let \( a_1, \ldots, a_m \) and \( b_1, \ldots, b_m \) be the elements of the first and second block, respectively. Define \( \sigma_m \) as the permutation that swaps the elements of the first and second block,

\[
\sigma_m = (a_1, b_1) \cdots (a_m, b_m). \tag{27}
\]

Let also \( \rho_m \in S_n \times D_3 \) be the order-two element defined as

\[
\rho_m = (\sigma_m, \kappa). \tag{28}
\]

Let \( \mu_m \in S_n \) be the order-three elements defined as follows. Partition \( n \) into \( r \geq 3 \) blocks such that the first three blocks have each the same number \( m \) of elements. Let \( a_1, \ldots, a_m, b_1, \ldots, b_m, \) and \( c_1, \ldots, c_m \) be the element of the first, second, and third block, respectively. Define \( \mu_m \) as the permutation that cycles forward the elements of the first three blocks,

\[
\mu_m = (a_1, b_1, c_1) \cdots (a_m, b_m, c_m). \tag{29}
\]

Let finally \( \nu_m \in S_n \times D_3 \) be the order-three element defined as

\[
\nu_m = (\mu_m, \theta). \tag{30}
\]

The following theorem fully characterizes axial subgroups of \( S_n \times D_3 \) acting on \( \mathbb{C}^{n-1} \).
Theorem 5.8. The conjugacy classes of axial subgroups of $S_n \times D_3$ acting on $\mathbb{C}^{n-1}$ as specified above are given by:

I)

$$\Sigma_m^\times \equiv S_m \times S_{n-m} \times Z_2(\kappa),$$  \hspace{1cm} (31)

where $1 \leq m \leq n/2$.

II)

$$\Sigma_{Z_2} = S_{n/2} \times S_{n/2} \times Z_2(\rho_{n/2}),$$  \hspace{1cm} (32)

with fixed point subspace

$$\text{Fix}(\Sigma_{Z_2}) = \mathbb{R}\{(i, \ldots, i, -i, \ldots, -i)\}. \hspace{1cm} (33)$$

III)

$$\Sigma_{D_3} = S_m \times S_m \times S_m \times Z_2(\rho_m) \times Z_3(\nu_m),$$  \hspace{1cm} (34)

where $1 \leq m \leq n/3$, with fixed point subspace

$$\text{Fix}(\Sigma_{D_3}) = \mathbb{R}\{(e^{i2\pi/3}, \ldots, e^{i2\pi/3}, e^{i4\pi/3}, \ldots, e^{i4\pi/3}), \hspace{1cm} (35)$$

$$1, \ldots, 1, 0, \ldots, 0)\}. \hspace{1cm} (35)$$

IV) If $n \geq 3$, then all branches are unstable.

Observe that for $n = 2$, we recover the axial classification of $Z_2 \times S_3 \cong D_6$ with its standard action on $\mathbb{C}$.

Proof. Notice that all the subgroups listed in the statement of the theorem are axial subgroups for the given action, as evident from the associated fixed point subspace. Item
IV is a direct consequence of the existence of a disagreement quadratic equivariant. The rest of the proof aims at showing that no other axial subgroups exist.

1) Product type axials.

We begin by proving that, modulo conjugacy, if an axial subgroup $\Sigma$ satisfies $\Sigma = A \times B$ with $A \subset S_n$ and $B \subset D_3$, then necessarily $\Sigma = \Sigma^h_m$ for some $m$.

Up to conjugacy, $B$ is either $1$, $Z_2(\kappa)$, $Z_3(\theta)$, or $D_3$. If $B = 1$, then $\text{Fix}(\Sigma) = \text{Fix}(A)$ is even dimensional; and $\Sigma$ is not axial. If $B = Z_3(\theta)$ or $B = D_3$, then $(1, \theta) \in \Sigma$ and

$$\text{Fix}(\Sigma) \subset \text{Fix}(Z_3(\theta)) = \{0\}. \quad (36)$$

Hence, $\Sigma$ is not axial. If $B = Z_2(\kappa)$, then $\text{Fix}(B) = \mathbb{R}^{n-1}$ and $\text{Fix}(\Sigma) = \text{Fix}(A|_{\mathbb{R}^{n-1}})$, $A \subset S_n$. So $\Sigma$ is axial if $A$ is an axial subgroup of $S_n$ acting on $\mathbb{R}^{n-1}$. Axial subgroups of $S_n$ acting on $\mathbb{R}^{n-1}$ are known [8, 1] and are exactly of the form $A = S_m \times S_{n-m}$, for $1 \leq m \leq n/2$.

II-III) Non product type axials.

Let $\Pi : S_n \times D_3 \to S_n$ be the projection homomorphism. Then $H = \Pi(\Sigma)$ is a subgroup of $S_n$ for any axial subgroup $\Sigma$.

**Lemma 5.9.** Suppose $\Sigma \subset S_n \times D_3$ is axial with respect to the given action and that, up to conjugacy, $\Sigma \neq \Sigma^h_m$, for any $1 \leq m \leq n/2$. Then $\ker(\Pi|_\Sigma) = (1, 1)$, and

$$\Sigma = \{(\tau, \varphi(\tau)) : \tau \in H\} \quad (37)$$

where $\varphi : H \to D_3$ is a group homomorphism. Moreover, up to conjugacy, $\varphi(H)$ can neither be $1$ nor $Z_3(\theta)$.

**Proof.** The elements of $\ker(\Pi|_\Sigma)$ are of the form $(1, \alpha)$ with $\alpha \in D_3$. Using the fact that conjugation leaves the kernel invariant, to prove the first part of the statement it
suffices to exclude that \((1,\theta) \in \Sigma\) and \((1,\kappa) \in \Sigma\).

If \((1,\theta) \in \Sigma\) then it follows by (36) that \(\Sigma\) is not axial.

If \((1,\kappa) \in \Sigma\) two cases are possible: either \((\tau,\theta) \not\in \Sigma\) for any \(\tau \in S_a\) or \((\tau,\theta) \in \Sigma\) for some \(\tau \in S_n\). The former can be excluded because in that case \(\Sigma = A \times \mathbb{Z}_2(\kappa)\) for some \(A \subset S_a\), and it follows as in \(I\) that \(\Sigma = \Sigma_m^\infty\) for some \(1 \leq m \leq a/2\). The second case can also be excluded because, if \((1,\kappa) \in \Sigma\), then \(\text{Fix}(\Sigma) \subset \text{Fix}(\mathbb{Z}_2(\kappa)) = \mathbb{R}^{n-1}\), and the isotropy condition imposed by the element \((\tau,\theta)\) reads \(z_{\tau(j)} = e^{i2\pi/3}z_j, z_j, z_{\tau(j)} \in \mathbb{R}\), which implies \(z_j = 0\) for all \(j\) and therefore \(\Sigma\) is not axial.

It follows by the First Isomorphism Theorem that \(\Sigma/\ker(\Pi|_{\Sigma}) = \Sigma\) is isomorphic to \(H\), and therefore (37) holds. Up to conjugacy, the image \(\varphi(H)\) can be either \(1, \mathbb{Z}_2(\kappa), \mathbb{Z}_3(\theta)\), or \(D_3\). If \(\varphi(H) = 1\), then \(\Sigma = A \times 1\), with \(A \subset S_n\), a case that was already excluded. If \(\varphi(H) = \mathbb{Z}_3(\theta)\), then \(\Sigma\) only contains elements of \(S_n\) and elements of the form \((\tau,\theta)\), for some \(\tau \in S_n\). The associated isotropy conditions take either the form \(z_i = z_j\) or \(z_i = e^{i2\pi/3}z_j\). The solution space of both equations is even dimensional and therefore \(\Sigma\) cannot be axial.

Let \(L = \ker \varphi \subset H \subset S_n\). Then

\[
\Sigma/(L,1) \cong H/L \cong \varphi(H).
\] (38)

**Lemma 5.10.** If \(\Sigma\) is an isotropy subgroup then \(L\) is an isotropy subgroup.

**Proof.** Let \(x\) be a vector which is fixed by \(\Sigma\). Then, in particular, \(x\) is fixed by \(L\), because \((L,1) \subset \Sigma\). Let \(\gamma\) be an element of \(S_n\) which also fixes \(x\). Then \((\gamma,1)\) must be in \(\Sigma\) because \(\Sigma\) is an isotropy subgroup, and therefore \(\gamma\) is in \(\ker \varphi = L\). \(\square\)
It follows from \([8, 1]\) and Lemma 5.10 that 
\[ L = S_{k_1} \times \cdots \times S_{k_s} \] 
and
\[ \text{Fix}(L) = \{(c_1, \ldots, c_1, \ldots, c_s, \ldots, c_s), c_i \in \mathbb{C}\}, \quad (39) \]
where the \(i\)-th block has dimension \(k_i\). Notice also that \(L\) is a normal subgroup of \(H\), that is, \(hLh^{-1} = L\) for all \(h \in H\), because kernels of group homomorphisms are always normal.\(^2\)

We now conclude the proof for case II, i.e., \(\varphi(H) = \mathbb{Z}_2(\kappa)\), which will provide the axial conjugacy class \(\Sigma^{\mathbb{Z}_2}\). If \(\varphi(H) = \mathbb{Z}_2(\kappa)\), then \((38)\) implies that \(L\) has index two (i.e., the order of \(\varphi(H)\)) in \(H\), and \((L, 1)\) has index two in \(\Sigma\). It follows that there must exist \(h \in H\), with \(h^2 \in L\), such that
\[ \Sigma = (L, 1) \cup (hL, \kappa). \]

The condition \(h^2 \in L\) follows by the fact that \(\kappa^2 = 1\) and thus, if \(a\) is in the coset \(hL\), then \((a, \kappa)^2 = (a^2, 1)\) and \(a^2\) must be in the coset \(L\). Because \(h\) commutes with \(L\), it must swap equally sized pairs of blocks in \((39)\) and up to conjugacy by elements of \(L\) we can take \(h\) to be of order two.\(^3\)

The resulting isotropy conditions read \(\bar{c}_i = c_i\) for each block that is not swapped by \(h\) and \(c_{j_1} = \bar{c}_{j_2}\) for each pair of blocks that are swapped by \(h\). Suppose there are \(r \leq s/2\) pairs of blocks which are swapped by \(h\) and \((s-2r)\) blocks which are fixed by \(h\).

Each block that is not swapped count for one degree of freedom. Each pair of swapped

---

\(^2\)Suppose \(G\) is a group and \(\varphi : G \to G'\) is a group homomorphism. Suppose \(x \in \ker \varphi\), that is \(\varphi(x) = e\). Then, given \(y \in G\), we have \(\varphi(yxy^{-1}) = \varphi(y)\varphi(x)\varphi(y)^{-1} = \varphi(y)\varphi(y)^{-1} = e\), that is, \(yxy^{-1} \in \ker \varphi\).

\(^3\)Indeed, let \(B_i\) be the set of coordinate indexes in the \(i\)-th block. Then \(L(B_i) = B_i\) for all \(i\), and because the permutation action is transitive no other \(L\)-invariant coordinate index set exists except the \(B_i\)'s. It follows that \(L(h(B_i)) = h(L(B_i)) = h(B_i)\), that is \(h(B_i) = B_j\) for some \(j\). Moreover \(h^2(B_i) = B_i\), because otherwise \(h^2 \notin L\).
blocks count for two degrees of freedom. Furthermore, we must impose the disagreement condition $k_1c_1 + \cdots + k_sc_s = 0$, which only eliminates one degree of freedom because the imaginary part is automatically zero due to the isotropy conditions. It follows that $\dim \text{Fix}(\Sigma) = (s - 2r) + 2r - 2 = s - 1$ and thus $s = 2$, and $r = 0$ or $r = 1$. If $r = 0$, then $h = 1$ and we go back to the conjugacy class of product axials $\Sigma \times \mathbb{Z}$. If $r = 1$ we obtain the conjugacy class $\Sigma \mathbb{Z}$.

To conclude the proof for case $III$, i.e., $\varphi(H) = D_3$, we proceed similarly. By (38), there exist $h_1, h_2 \in H$ such that

$$\Sigma = (L, 1) \cup (h_1L, \kappa) \cup (h_2L, \theta),$$

with $h_1^2 \in L$, $h_2^2 \in H - L$, and $h_3^2 \in L$. That is, $h_1$ swaps pairs of blocks and $h_2$ cycles triplets of blocks. Up to conjugacy by elements of $L$ we can take $h_1$ to be of order two and $h_2$ to be of order three.

Any block which is not cycled by $h_2$ is zero because the isotropy condition $\theta(c) = ce^{i2\pi/3} = c$ implies $c = 0$. Imposing the isotropy condition associated to the element $(h_2, \theta)$ on a block triplet $(c_1, c_2, c_3)$ that is cycled by $h_2$ implies $(c_1, c_2, c_3) = (c, ce^{i2\pi/3}, ce^{i4\pi/3})$, where the three blocks have the same size. If $h_1$ does not swap at least two of the three blocks cycled by $h_2$ (up to conjugacy we can take them to be the first two), then the isotropy condition $\kappa(c, ce^{i2\pi/3}) = (\bar{c}, ce^{-i2\pi/3}) = (c, ce^{i2\pi/3})$ implies $c = 0$ and the whole block triplet is zero. Any block triplet which is cycled by $h_2$ and in which $h_1$ swaps two blocks (up to conjugacy we can take them to be the first two), contribute one degree of freedom to the fixed point subspace. Indeed, the isotropy condition $(h_1, \kappa)(c, ce^{i2\pi/3}, ce^{i4\pi/3}) = (\bar{c}e^{-i2\pi/3}, \bar{c}e^{-i4\pi/3}) = (c, ce^{i2\pi/3}, ce^{i4\pi/3})$ implies $c \in \mathbb{R}\{e^{i2\pi/3}\}$. If $\Sigma$ is axial, then there exists only one such block triplet and the conjugacy class $\Sigma^{D_3}$ follows.
We now interpret Theorem 5.8 in terms of decision making. Each axial subgroup constructed in Theorem 5.8 admits two interpretations, depending on whether \( N_a = 3 \) and \( N_o = n \), or \( N_o = 3 \) and \( N_a = n \). When \( N_a = 3 \) and \( N_o = n \), the interpretation of the axial subgroup \( \Sigma_m \) is sketched in Figure 14A. In this representation, \( \kappa \in D_3 \) acts by swapping agents 1 and 2. The factor \( Z_2(\kappa) \) therefore imposes that agent 1 and 2 opinions are equal. The fixed point subspace of this subgroup describe the situation in which agent 3 strongly favors the first \( m \) options and weakly disfavors the last \( N_o - m \) options, if \( u > 0 \), or strongly disfavors the first \( m \) options and weakly favors the last \( N_o - m \) options, if \( u < 0 \). Both agents 1 and 2 are more moderate. Each of their opinions is opposite to agent 3 opinion, but with the half of the strength. When \( N_o = 3 \) and \( N_a = n \), \( \kappa \in D_3 \) acts by swapping options 1 and 2 and the resulting interpretation is sketched in Figure 14B. If \( u > 0 \), a small group of agents (the extremists) strongly favor option 3, while a larger group of agents (the moderates) weakly disfavor option 3 and are undecided between options 1 and 2. If \( u < 0 \) the situation is reversed. The moderates weakly favor option 3, whereas the extremists strongly disfavor option 3 and are undecided between options 1 and 2.

For the axial \( \Sigma Z_2 \), when \( N_a = 3 \) and \( N_o = n \), the factor \( Z_2(\rho_{n/2}) \) implies that agent 1 and agent 2 takes exactly opposite opinions about the \( N_o \) options. The axial fixed point subspace can then be interpreted as follows (Figure 15A). Agent 1 favors the first \( n/2 \) options and disfavors the last \( n/2 \) options, while agent 2 disfavors the first \( n/2 \) options and favors the last \( n/2 \) options. Agent 3 remains completely undecided about all options. When \( N_o = 3 \) and \( N_a = n \), the factor \( Z_2(\rho_{n/2}) \) implies that the agent group splits in two equally sized groups with exactly the opposite opinion. By analyzing the axial fixed point subspace, we conclude that one group favors option 1, strongly disfavors option 2, weakly disfavors option 3, while the other group strongly favors option 2, strongly disfavor option 1, and weakly disfavors option 3 (Figure 15B).
We finally interpret the axial $\Sigma^D_m$. When $N_a = 3$ and $N_o = n$, the factor $Z_2(\rho_m)$ implies that the opinion that agent 1 has about the first $m$ options is the same as the opinion that agent 2 has about the second $m$ options, and vice-versa. The factor $Z_3(\nu_m)$ implies that the opinion that agent 1 has about the first (resp. second, third) $m$ options is the same as the opinion that the second agent has about the third (resp. first, second) $m$ options and the same as the opinion that the third agent has about the second (resp. third, first) $m$ options. All agents are undecided about the last $N_o - 3m$ options. This interpretation of the resulting fixed point subspace is sketched in Figure 16A. It shows that this axial corresponds to each agent possessing an ensemble.
of $m$ strongly favored (resp. disfavored - not shown in the figure) options, while equally disfavoring (resp. favoring) $2m$ of the remaining options and remaining neutral about the remaining $N_o - 3m$ options. When $N_o = 3$ and $N_a = n$ the interpretation is straightforward. $N_a - 3m$ agent are undecided, the remaining $3m$ agent opinions are distributed with $D_3$ symmetry (Figure 16B). There are two possible configurations. In one, each group of $m$ agent has its favorite option. In the other, each group has its disfavored option and is undecided between the remaining two options.
5.3.3 A first result for general $N_a \geq 3$ and $N_o \geq 3$ disagreement

In the sequel we provide a list of axial subgroups for the action of $\Gamma = S_n \times S_k$ on the disagreement space

$$W_2 = \{(Z_1, \ldots, Z_n) : Z_1 + \cdots + Z_n = 0, \; Z_i \in V_i\},$$

(40)
where $S_n$ permutes the $Z_i$'s and $S_k$ acts diagonally on the $Z_i$'s by permuting the $z_i^k$'s.

**Remark 5.11.** Note that the disagreement space is a tensor product; that is,

$$W_2 = \mathbb{R}^{n-1} \otimes \mathbb{R}^{k-1}$$

with $\Gamma = S^n \times S^k$ acting on the tensor product in the natural way. It follows from \[41\] that the symmetry-breaking results for $(N_a, N_o)$ are identical to those of $(N_o, N_a)$. Their interpretation in terms of decision making will of course differ.

We need some preliminary notation and definitions. The normalizer of a subgroup $\Sigma \subset \Gamma$ is

$$N_\Gamma(\Sigma) = \{\gamma \in \Gamma : \gamma \Sigma = \Sigma \gamma\}.$$ 

A well known result is $\gamma \in N_\Gamma(\Sigma)$ if and only if $\gamma(\text{Fix}(\Sigma)) = \text{Fix}(\Sigma)$. Define

$$A \hat{\times} B = \{\gamma \in N_\Gamma(A \times B) : \gamma|_{\text{Fix}(A \times B)} = 1_{\text{Fix}(A \times B)}\}.$$

Note that $A \hat{\times} B$ is axial if and only if $\dim \text{Fix}(A \times B) = 1$. Indeed, by definition, if $\gamma \in A \hat{\times} B$ and $z \in \text{Fix}(A \times B)$ then $\gamma z = z$. It follows that $\text{Fix}(A \times B) \subset \text{Fix}(A \hat{\times} B)$ and $\dim \text{Fix}(A \times B) \leq \dim \text{Fix}(A \hat{\times} B)$. However, $A \times B \subset A \hat{\times} B$ and therefore $\dim \text{Fix}(A \times B) \geq \dim \text{Fix}(A \hat{\times} B)$. It follows that, in general, $\dim \text{Fix}(A \times B) = \dim \text{Fix}(A \hat{\times} B)$. If $A \hat{\times} B$ is axial, then $\dim \text{Fix}(A \times B) = \dim \text{Fix}(A \hat{\times} B) = 1$. Conversely, if $\dim \text{Fix}(A \times B) = 1$, then $\dim \text{Fix}(A \hat{\times} B) = 1$ and it remains to prove that $A \hat{\times} B$ is an isotropy subgroup. Let $\text{Fix}(A \times B) = \mathbb{R}\{z\}$ and suppose $\gamma \in \Sigma_z$. Then, $\gamma : \text{Fix}(A \times B) \to \text{Fix}(A \times B)$ and therefore $\gamma \in N_\Gamma(A \times B)$. Moreover, $\gamma$ fixes $z$ and so, by definition, $\gamma \in A \hat{\times} B$. It follows that $\Sigma_z$ is a subset of $A \hat{\times} B$ and equality follows because the reverse inclusion is straightforward.

Partition $\{1, \ldots, n\}$ into $r \geq 2$ blocks such that the first $s \geq 2$ blocks have each the
same number $m$ of elements. Let the order 2 element $\rho_m \in S_n \times S_k$ be defined as

$$ \rho_m = (\sigma_m, \sigma_{(12)}) $$

where $\sigma_m \in S_n$ swaps the first two blocks of vectors and $\sigma_{(12)} \in S_k$ swaps the first two elements of each vector. In other words, the element $\rho_m$ simultaneously swaps the first two blocks of vectors, $(Z_1, \ldots, Z_m) \leftrightarrow (Z_{m+1}, \ldots, Z_{2m})$, and the first and second element of each vector. Let also $\nu_m^{(s)} \in S_n \times S_k$ be the order $s$ element defined as

$$ \nu_m^{(s)} = (\mu_m^{(s)}, \sigma_{(12 \cdots s)}) $$

where $\mu_m^{(s)} \in S_n$ cycles forward the first $s$ blocks of vectors and $\sigma_{(12 \cdots s)} \in S_k$ cycles forward the first $s$ components of each vector. In other words, the element $\nu_m^{(s)}$ simultaneously cycles forward the first $s$ blocks of vectors and the first $s$ components of each vector.

**Theorem 5.12.** Consider the action of $\Gamma = S_n \times S_k$ on $W_2 = \mathbb{R}^n \otimes \mathbb{R}^k$ as specified above. Then the following hold.

**I)** $A \hat{\times} B \subset S_n \times S_k$ is axial if and only if $A \subset S_n$ is an axial subgroup of the action on $\mathbb{R}^{n-1}$ and $B \subset S_k$ is an axial subgroup of the action on $\mathbb{R}^{k-1}$.

Suppose that $\Sigma \in S_n \times S_k$ is axial, $\Sigma \neq A \hat{\times} B$, and that

$$ \Sigma = \{(\tau, \varphi(\tau)), \tau \in H\} \quad (42) $$

where $H \subset S_n$ is a subgroup and $\varphi : S_n \to S_k$ is a group homomorphism.

**II)** If $\varphi(H) = S_s$, $2 \leq s < k$, then necessarily $\frac{n}{s} = m \in \mathbb{N}$ and

$$ \Sigma = \underbrace{S_m \times \cdots \times S_m}_s \times Z_2(\rho_m) \times Z_s(\nu_m^{(s)}) $$

52
and its fixed point subspace is

\[
\text{Fix}(\Sigma) = \left\{ (\mathbf{Z}, \ldots, \mathbf{Z}, \sigma(12\cdots s)(\mathbf{Z}, \ldots, \mathbf{Z}), \ldots, (\sigma(12\cdots s))^{s-1}(\mathbf{Z}, \ldots, \mathbf{Z})) \right\}
\]

where \( \mathbf{Z} = (- (s-1)c, c, \ldots, c, 0, \ldots, 0)^T \), \( c \in \mathbb{R} \).

III) If \( \varphi(H) = S_k \), then

\[
\Sigma = S_m \times \cdots \times S_m \times \mathbb{Z}_2(\rho_m) \times \mathbb{Z}_k(\nu_m^{(k)})
\]

and its fixed point subspace is

\[
\text{Fix}(\Sigma) = \left\{ (\mathbf{Z}, \ldots, \mathbf{Z}), \sigma(12\cdots k)(\mathbf{Z}, \ldots, \mathbf{Z}), \ldots, (\sigma(12\cdots s))^{k-1}(\mathbf{Z}, \ldots, \mathbf{Z}), 0, \ldots, 0 \right\}
\]

where \( \mathbf{Z} = (- (k-1)c, c, \ldots, c)^T \), \( c \in \mathbb{R} \).

Theorem 5.12 classifies axials of \( \Gamma = S_n \times S_k \) acting on \( W_2 \) in the following cases: I) the axial is of product type; the axial can be written as the graph of a group homomorphism from \( S_n \) to \( S_k \). It remains an open question whether this lists all the axials of this action. Note that all axials listed in Theorem 5.8 are special cases of those listed in Theorem 5.12, with the difference that in the \( k = 3 \) the list of axial was proved to be exhaustive.

Proof. We first prove Item I) of the statement and then sketch the proof for Items II) and III).

I) Product axials.
Lemma 5.13. Let $A \subset S_n$ and $B \subset S_k$ be subgroups. Then

$$\dim_W(\text{Fix}(A \times B)) = 1$$

if and only if

$$\dim_{R^{n-1}}(\text{Fix}(A)) = 1 \quad \text{and} \quad \dim_{R^{k-1}}(\text{Fix}(B)) = 1$$

Proof. The result follows from (41) and Lemma 3.1 in Dionne et al. . \hfill \Box

If $A$ and $B$ are axial subgroups, then $\dim \text{Fix}_W(A \times B) = 1$ and Proposition 5.13 implies that $A \times B$ is axial. Conversely, if $A \times B$ is axial, then $\dim \text{Fix}_W(A \times B) = \dim \text{Fix}_W(A \times B)$ and Lemma 5.13 implies that $A$ and $B$ are axial.

II,II) Graph axials.

Under the hypothesis that an axial subgroup is not a product subgroup and that it can be written as in the form (42), we can find its general form by following the same ideas as the $k = 3$ case above.

If $\varphi(H)$ is a cyclic or product subgroup of $S_k$, then the isotropy conditions cannot lead to a one-dimensional fixed-point subspace. It follows that necessarily $\varphi(H) = S_s$ with $2 \leq s \leq k$.

If $\varphi(H) = S_s$, $2 \leq s \leq k$, then $L = \ker \varphi$ has index $s$! (i.e., the order of $\varphi(H)$) in $H$ and $(L, 1)$ has index $s$! in $\Sigma$. It follows that, up to conjugacy, there must exist an order-two element $h_1 \in H$ and an order-$s$ element $h_2 \in H$ such that

$$\Sigma = (L, 1) \cup (h_1L, \sigma_{(12)}) \cup (h_2L, \sigma_{(12...s)}).$$
Moreover, invoking Lemma 5.10, \( L = S_{k_1} \times \cdots \times S_{k_r} \) with

\[
\text{Fix}(L) = \{(c_1, \ldots, c_1, \ldots, c_r, \ldots, c_r), c_i \in \mathbb{R}^k, (c_i)_1 + \cdots + (c_i)_k = 0\},
\]

and \( h_1 \) and \( h_2 \) must permute (equally-sized) blocks of vectors in \( \text{Fix}(L) \) because, since \( L \) is normal, they both commute with \( L \). By counting dimensions in the resulting isotropy conditions the result follows.

\[ \square \]

## 5.4 A conjecture about breaking symmetry in \( S_3 \)-equivariant system

**Theorem 5.14.** Consider the normal form of the unfolding of a codimension-1 \( S_3 \)-equivariant problem

\[
\dot{x} = -(x^2 + y^2 - \lambda)x - (x^2 + y^2 + \mu(x^3 - 2xy^2) - \alpha)(x^2 - y^2) \quad (43a)
\]

\[
\dot{y} = -(x^2 + y^2 - \lambda)y + (x^2 + y^2 + \mu(x^3 - 2xy^2) - \alpha)2xy. \quad (43b)
\]

If symmetry is broken by sufficiently small affine parameters \( \varepsilon_1, \varepsilon_2 \), leading to the perturbed normal form

\[
\dot{x} = -(x^2 + y^2 - \lambda)x - (x^2 + y^2 + \mu(x^3 - 2xy^2) - \alpha)(x^2 - y^2) + \varepsilon_1 \quad (44a)
\]

\[
\dot{y} = -(x^2 + y^2 - \lambda)y + (x^2 + y^2 + \mu(x^3 - 2xy^2) - \alpha)2xy + \varepsilon_2 \quad (44b)
\]

and \( \alpha > 0 \), then the following holds:

- **If** \( -\tan\left(\frac{\pi}{3}\right) \varepsilon_1 < \varepsilon_2 < \tan\left(\frac{\pi}{3}\right) \varepsilon_1 \), **then** the trajectory of \( (44) \) for \( \lambda = 0 \) and \( x(0) = y(0) = 0 \) converges to a stable equilibria at \( (x^*, 0) \), \( x^* > 0 \).

- **If** \( \varepsilon_2 < \min \{-\tan\left(\frac{\pi}{3}\right) \varepsilon_1, 0\} \), **then** the trajectory of \( (44) \) for \( \lambda = 0 \) and \( x(0) = \)
\( y(0) = 0 \) converges to a stable equilibria at \((- \cos \left( \frac{\pi}{3} \right) x^*, -\sin \left( \frac{\pi}{3} \right) x^* \)).

- If \( \varepsilon_2 > \max \{ \tan \left( \frac{\pi}{3} \right) \varepsilon_1, 0 \} \), then the trajectory of (44) for \( \lambda = 0 \) and \( x(0) = y(0) = 0 \) converges to a stable equilibria at \((- \cos \left( \frac{\pi}{3} \right) x^*, \sin \left( \frac{\pi}{3} \right) x^* \)).

Moreover, in the perturbed normal form (44) there exists a smooth branch of equilibria connecting the stable part of the neutral equilibrium branch and the winning stable branch.

Figure 17 illustrates Theorem 5.14. A formal proof goes beyond the scope of this work. To interpret this result in the decision making setting, observe that the unitary transformation from \( W_1 = \{ x_1 + x_2 + x_3 = 0 \} \) to \( (x, y) \in \mathbb{R}^2 \) is \( x = \frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{6}} x_2 \) and \( y = -\frac{1}{\sqrt{6}} x_1 - \frac{1}{\sqrt{6}} x_2 + \frac{1}{\sqrt{6}} x_3 \), and that \( \frac{\sqrt{6}}{\sqrt{2}} = \tan \left( \frac{\pi}{3} \right) \).

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Figure 17: A) Left: Phase portrait of the unperturbed normal form (43) for $\lambda = 0$ (bifurcation point), $\alpha = 0.1$, $\mu = 0.0$. Center: Phase portrait of the perturbed normal form (44) for $\lambda = 0$, $\alpha = 0.1$, $\mu = 0.0$, $\varepsilon_1 = 0.00001$ and $\varepsilon_2 = \frac{3}{2} \tan \left( \frac{\pi}{3} \right) \varepsilon_1$. As predicted by the theorem, the origin belongs to the basin of attraction of the right equilibrium. Right: Phase portrait of the perturbed normal form (44) for $\lambda = 0$, $\alpha = 0.1$, $\mu = 0.0$, $\varepsilon_1 = 0.00001$ and $\varepsilon_2 = \frac{3}{2} \tan \left( \frac{\pi}{3} \right) \varepsilon_1$. As predicted by the theorem, the origin belongs to the basin of attraction of the top left equilibrium.
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## A Numerical model equations

The used computational models are defined on the tangent space $T_O \mathcal{V}$. If needed, they can be lifted to $\mathcal{V}$ via a simple change of coordinates. The simplex appearing in the figures is just sketched.
A.1 Three-agent three-option model

The three-agent three-option model is defined by

\[
\dot{x}_i = F_{x_i} - \frac{1}{3} (F_{x_i} + F_{y_i} + F_{z_i}) \tag{45a}
\]

\[
\dot{y}_i = F_{y_i} - \frac{1}{3} (F_{x_i} + F_{y_i} + F_{z_i}) \tag{45b}
\]

\[
\dot{z}_i = F_{z_i} - \frac{1}{3} (F_{x_i} + F_{y_i} + F_{z_i}) \tag{45c}
\]

where

\[
F_{x_i} = -x_i + \beta \sigma_2 (y_i, z_i) + \gamma \sigma_2 (x_j, x_k) + \delta \sigma_2 (\sigma_2 (y_j, y_k), \sigma_2 (z_j, z_k)) + \sigma (x_i^2) + b_1 \tag{46a}
\]

\[
F_{y_i} = -y_i + \beta \sigma_2 (x_i, z_i) + \gamma \sigma_2 (y_j, y_k) + \delta \sigma_2 (\sigma_2 (x_j, x_k), \sigma_2 (z_j, z_k)) + \sigma (y_i^2) \tag{46b}
\]

\[
F_{z_i} = -z_i + \beta \sigma_2 (x_i, y_i) + \gamma \sigma_2 (z_j, z_k) + \delta \sigma_2 (\sigma_2 (x_j, x_k), \sigma_2 (y_j, y_k)) + \sigma (y_i^2) \tag{46c}
\]

and where \(i, j, k \in \{1, 2, 3\}\) are different indexes and \(\sigma (x) = e^{-e^{-2.74x}} - e^{-1}\), \(\sigma_2 (a, b) = \sigma (\sigma (a) + \sigma (b))\). The small number \(b_1\) is used to bias the agreement decision toward option 1. In Figure 8, \(b_1 = 0.0005\). Observe that \(\dot{x}_i + \dot{y}_i + \dot{z}_i = 0\), that is, the vector field is tangent to \(T_O V\). Non-diagonal noise was added to the three components in such a way that the noisy input is also tangent to \(T_O V\).