Mean Field Approach to the Giant Wormhole Problem

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Abstract
We introduce a gaussian probability density for the space-time distribution of wormholes, thus taking effectively into account wormhole interaction. Using a mean-field approximation for the free energy, we show that giant wormholes are probabilistically suppressed in a homogenous isotropic “large” universe.
1 Introduction

Some years ago it was observed \cite{1} that topological fluctuations of space-time provide a mechanism which can in principle fix the values of some fundamental constants of nature. More precisely, it appeared that taking into account the possibility of creation of tiny wormholes connecting distant regions of space-time has the effect of modifying the coupling constants contained in the model and to provide a probability distribution for them. In particular, it was shown by Coleman that, under certain assumptions, the probability distribution for the cosmological constant was infinitely peaked around zero, thus providing a candidate solution, at least neglecting one-loop effects \cite{2}, to the long-standing cosmological constant problem.

The main characteristic of wormholes is that the probability of their creation does not depend on the positions of the endpoints \(x_0\) and \(x_1\), at least when the two points are distant. In a first approximation, one can think that the wormholes are dilute, so that \(|x_1 - x_0|\) is in the average great with respect to the wormhole size \(a\), and one can forget about the wormhole interaction which is likely to appear when the endpoints become close to each other. This approximation was assumed in all previous analysis of this subject. However, it was evident from the very beginning \cite{3} that this simplified model ran over some inconsistencies, like the so-called giant wormhole paradox, that is, the non-physical prediction that wormholes of macroscopic sizes should be generated.

In this paper we introduce a finer description of the process of wormhole creation and destruction, and show that in this case giant wormholes are probabilistically disfavoured. More precisely, we assume that the wormhole space-time distribution be described by the gaussian probability density \(2.3\), which corresponds to a suppression of the probability of creation of a wormhole when the two endpoints come close to each other. We then evaluate the functional integral for the theory using methods which were previously introduced by one of us \cite{4} to deal with similar problems in solid state physics. Using the mean-field approximation \(3.12\) for the free energy, we show that the free energy for the theory is proportional to \(\Lambda^4 a^2/\ell_P^2\) (where \(\Lambda\) is an energy scale \(\leq 1/\ell_P^2\) and \(a\) is the effective size of the wormholes), and therefore has a minimum for small values of \(a\). This is our main physical result, which implies that giant wormholes are probabilistically suppressed in a homogenous isotropic “large” universe. This phenomenon is a large-scale, low-energy “cooperative effect” between the wormhole density \(2.1\) and space-time distribution \(2.3\). Previous analysis of the problem neglected a self-consistent determination of the wormhole size \(a\) because they did not take into account the wormhole space-time distribution, arriving this way to the cited giant wormhole paradox. Furthermore, our analysis shows that
the result is quite independent from the “microscopic details” of the gravity-matter interaction, at least in the domain of validity \(1/a < \Lambda\) of our model.

2 Definition of the model

As a starting point, let us assume that we have performed integration over short-range topology fluctuations. It is natural to suppose that a reasonable trial wave function, after eliminating such degrees of freedom, will give us some effective theory of geometrical and matter degrees of freedom on the wormholes background. Moreover, it seems reasonable that the only trial parameter left in the theory should be the effective size of the wormholes. Thus, let us assume that the probability density for the creation of a wormhole of size \(a\) be of the form

\[
w_0 = e^{-ca^2}
\]

where \(c = \text{const} = \mathcal{O}(1)\) and the length \(a\) is measured in units of the Plank length \(\ell_P\).

Now, it is reasonable to think that the above wormhole density distribution for finite values of \(a\) affects also the space-time distribution of the wormhole itself.

Of course, it is difficult to build a complete theory of the microscopic dynamics of wormholes, but it seems reasonable as a first step to describe this dynamics self-consistently by taking into account the size \(a\) both in the density and in the space-time correlations. Let us in the following formulate a simple model of universe with wormholes in which one may calculate explicitly the effect of the size \(a\), and the main result shall be that giant wormholes are probabilistically suppressed.

For this purpose, we describe the space-time distribution of the “ends” of the wormholes by the local field

\[
\alpha(x) = \alpha \delta(x - x_0)
\]

and we postulate the following weight function for the field \(\alpha(x)\):

\[
\mathcal{P}[\alpha(x)] = \exp\left(-\frac{1}{2} \int dx dx' \alpha(x)\alpha(x')K(x - x')\right),
\]

Here the correlation function \(K(x)\), which is the amplitude for a wormhole insertion, is assumed to be gaussian:

\[
K(x) = K_0 \exp(-ax^2),
\]

where \(K_0 \sim \exp(-S_w)\) and \(S_w\) is the (constant) wormhole action. The ansatz (2.3) allows us to take effectively into account the finite size of wormholes in the space-time distribution. As a matter of fact, if we substitute in (2.3) the distribution of
two point-like wormholes

\[ \alpha(x) = \alpha_0 \delta(x-x_0) + \alpha_1 \delta(x-x_1) \]  

(2.5)

we see that the distribution \((2.3)\) describes a suppression of probability when \(|x_1-x_0| < a\), and we get factorization when \(|x_0-x_1| \to \infty\).

As a first order of approximation one may think that the field \(\alpha\), which describes the large-scale structure of the space-time distribution of wormholes, is coupled to gravity by volume effects only. In other words, we may assume that the microscopic structure of the gravitational field is not relevant to describe the size distribution of the wormhole dynamics. Therefore the gravitational degrees of freedom associated with the spin-2 part, i.e. \(h_{\mu\nu} = g_{\mu\nu} - \langle g_{\mu\nu} \rangle\), which control the small-scale structure, can be neglected at this level. As a consequence of this picture, we impose the following coupling of \(\alpha(x)\) with gravity:

\[ S_{\text{int}} = \int d^4x \sqrt{|g(x)|} \alpha(x), \]  

(2.6)

where the “volume field” \(\sqrt{|g|}\) is expanded as

\[ \sqrt{|g(x)|} = 1 + \varphi(x), \quad |\varphi| \ll 1. \]  

(2.7)

The model that we have in mind is given by the path integral

\[ \int D\alpha(x) \exp \left( -\frac{1}{2} \int d^4x d^4x' \alpha(x)K(x-x')\alpha(x') + S_{\text{int}} \right) \]  

\[ \cong \frac{1}{\sqrt{\det K}} \exp \left( \frac{1}{2} \int d^4x d^4x' \varphi(x)K^{-1}(x-x')\varphi(x') \right) \]  

(2.8)

where \((2.8)\) is defined up to a normalization and we have excluded the term linear in \(\varphi\). Indeed this contribution may be neglected if we integrate \((2.8)\) over \(\varphi\), since it represents an arbitrary fluctuation field around the vacuum \(\langle \sqrt{|g|} \rangle = 1\). Of course, looking in this way, we are taking into account quantum gravity volume effects with a functional measure given by \(\exp(-S(\varphi))\), where \(S(\varphi)\) is the effective action

\[ S(\varphi) = \frac{1}{2} \int dxdx' \varphi(x)K^{-1}(x-x')\varphi(x'), \]  

(2.9)

coming from the above functional integration over \(\alpha\). We get Coleman’s picture when we consider only the global degrees of freedom of the field \(\alpha(x)\). For instance, if we substitute into the \(D\alpha\)-integral the factor

\[ \delta(\alpha - \int d^4x \alpha(x)) \]  

(2.10)

and perform the \(D\alpha\)-integration, we would have a theory with the global variable \(\alpha\) playing the role of an effective wormhole collective coordinate in Coleman’s spirit.
In any case, the effective model based on (2.9) is only ultralocal for $\varphi$. In a realistic scenario of quantum gravity coupled with matter fields we must investigate also the effect of the density and space-time wormhole distribution over the coupled system matter-gravity. So, a less crude model is obtained if we add to the non-local action (2.9) a matter field contribution $S_{\text{mat}}$ represented by a massless Klein-Gordon field coupled to the geometry $g_{\mu\nu}$. Assuming again that only volume effects are relevant, we may replace the covariant Laplacian in the $S_{\text{mat}}$ with the flat one, and using (2.7), $S_{\text{mat}}$ looks as

$$S_{\text{mat}} = \int d^4x \rho(x)(1 + \varphi(x))\Delta \rho(x) \quad (2.11)$$

where $\Delta$ is the flat, Euclidean 4-dimensional Laplacian.

The coupled model is now described by $S(\varphi) + S_{\text{mat}}(\rho, \varphi)$. Performing firstly the $\rho$-functional integration, we get a free energy of the form (in Euclidean signature)

$$F[\varphi] \simeq \frac{1}{2} \log \det[(1 + \varphi(x)) \Delta]. \quad (2.12)$$

Of course we must understand the functional determinant in (2.12) as a renormalized one. For this aim in the following section we shall adopt a Schwinger-DeWitt proper-time scheme, together with an “$\frac{1}{D}$ expansion”.

### 3 Mean Field Approximation

The complete quantum partition function of our model is given by

$$Z = \int \mathcal{D}\varphi \exp \left( -\frac{1}{2} \int d^4x d^4x' \varphi(x) K^{-1}(x' - x) \varphi(x') - F[\varphi] \right)$$

$$= \langle \exp(-F[\varphi]) \rangle_{\varphi} \quad (3.1)$$

where the average symbol means

$$\langle O \rangle_{\varphi} \equiv \int \mathcal{D}\varphi \mathcal{O} \exp \left( -\frac{1}{2} \int \varphi K^{-1} \varphi \right) \quad (3.2)$$

At this point we use the mean-field approximation

$$Z \simeq \exp \left( -\langle F[\varphi] \rangle_{\varphi} \right) \quad (3.3)$$

Let us remind that, due to the convexity of the exponential function, one has the Peierls inequality

$$\langle e^A \rangle \geq e^{\langle A \rangle}. \quad (3.4)$$

Here we assume to take the lower bound of $\langle e^A \rangle$, since in our case $A$ is given by a functional $-F[\varphi]$ of $\varphi$, and the kernel (2.4), goes very rapidly to zero for $|x-y| \to \infty$; this is just the large-range approximation underlying all our model.
In other words, the true free energy $F \equiv - \log(Z)$ is approximated by

$$F \simeq \langle F[\varphi] \rangle_{\varphi} = \left\langle \frac{1}{2} \log \det[(1 + \varphi)\Delta] \right\rangle_{\varphi}. \quad (3.5)$$

By using now the Schwinger-De Witt proper-time formalism, we may write

$$\log \det[(1 + \varphi)\Delta] = - \int_{\infty}^{\epsilon} \frac{dt}{t} \text{Tr} \exp(-t[(1 + \varphi)\Delta]), \quad \epsilon \to 0^+ \quad (3.6)$$

where $\text{Tr}$ stands for the space time trace of the heat-kernel operator in (3.6) and can be represented as a Schrödinger-like path integral over closed paths according to

$$\text{Tr} \exp(-t[(1 + \varphi)\Delta]) = \int d^4x \int_{x(0)=x(t)=x} Dx(\tau) \exp\left(- \int_0^t \frac{1}{4}(1 + \varphi[x(\tau)]) \dot{x}(\tau)^2 \right). \quad (3.7)$$

Putting (3.6) and (3.7) in (3.5), we get

$$F \simeq \lim_{\epsilon \to 0^+} \int D\varphi \exp\left(- \frac{1}{2} \int dx dx' \varphi(x)K(x-x')\varphi(x') \right) \cdot \int d^4x \int_{\infty}^{\epsilon} \frac{dt}{t} \int_{x(0)=x(t)=x} Dx(\tau) \exp\left\{ - \int_0^t \left( \frac{1}{4}(1 + \varphi[x]) \dot{x}(\tau)^2 \right) \right\}, \quad (3.8)$$

Putting (3.7) in (3.5), we get

$$F \simeq \lim_{\epsilon \to 0^+} - \frac{1}{2} \int d^4x \int_0^t \frac{dt}{t} \int_{x(0)=x(t)=x} Dx(\tau) \cdot \exp\left(- \int_0^t d\tau \frac{\dot{x}(\tau)^2}{4} + \frac{1}{4} \int_0^t \int_0^t d\tau d\tau' \dot{x}(\tau)^2 K[x(\tau) - x(\tau')] \dot{x}(\tau')^2 \right) \quad (3.9)$$

At this point, the functional integration can be done [4], giving the expression

$$F \simeq - \frac{1}{2} V f, \quad f = \lim_{\epsilon \to 0^+} \int_0^{+\infty} \frac{dt}{t} \int_{x(0)=x(t)=0} Dx(\tau) \cdot \exp\left\{ - \int_0^t d\tau \frac{\dot{x}(\tau)^2}{4} + \frac{1}{4} \int_0^t \int_0^t d\tau d\tau' \dot{x}(\tau)^2 K[x(\tau) - x(\tau')] \dot{x}(\tau')^2 \right\}. \quad (3.10)$$

Path integrals of this kind can be found in the theory of disordered systems. Unfortunately, it is impossible to calculate exactly even this very simplified path
integral. However, we can estimate it in some sense non-perturbatively and obtain rough numerical estimates and an enough good qualitative picture.

Really, our $x(\tau)$ lies in 4-dimensional space, here assumed isotropic and homogeneous, and it seems to be reasonable that the main contribution to the path integral (3.11) is given by trajectories having high degree of isotropy. For such trajectories:

$$< x(\tau)x(\tau') > = 0,$$

$$< (x(\tau)x(\tau'))^2 > = \frac{1}{D} < x^2(\tau) >^2$$

(3.11)

where $D$ is the dimension of the space. In our case $D = 4$, and we can formulate the mean-field approximation for the path integral (3.11) by requiring that:

$$K(x(\tau) - x(\tau')) = ca^2 \exp \left( -\frac{1}{a^2}(x(\tau) - x(\tau'))^2 \right)$$

$$\approx ca^2 \exp \left( -\frac{1}{a^2}x^2(\tau) \right) \exp \left( -\frac{1}{a^2}x^2(\tau') \right).$$

(3.12)

(It is important to notice that this decomposition becomes exact in the limit $a \to \infty$; see below). In this approximation the calculation of (3.11) can be reduced to solving the quantum mechanical problem

$$f \simeq \lim_{\epsilon \to 0^+} \frac{1}{\sqrt{\pi ca^2}} \int_{t_0}^{+\infty} dt \int_{-\infty}^{+\infty} dse^{-s^2/\alpha a^2} \cdot$$

$$\int_{x(0)=x(\tau)=0} Dx(\tau) \exp \left\{ -\int_{t_0}^{t} d\tau \left[ \frac{\dot{x}^2}{4} + s\dot{x}^2 e^{-x^2/\alpha^2} \right] \right\}$$

$$= \lim_{\epsilon \to 0^+} \int_{t_0}^{+\infty} dt \int_{-\infty}^{+\infty} dse^{-s^2/\alpha a^2} G_s(t|0,0),$$

(3.13)

where $G_s(t|x,x')$ is the Green function

$$\partial_t G_s(t|x,x') = \frac{1}{4}(1 + 4se^{-x^2/\alpha^2}) \Delta G_s(t|x,x'),$$

$$G_s(0, x, x') = \delta^{(4)}(x - x').$$

(3.14)

Some remarks are in order. In our mean-field approximation the fluctuations of gravity are described by the global variable $s$, which has the meaning of a sort of order parameter. Indeed the parameter $s$ can be considered as an amplitude of the volume-field $\sqrt{g}$ fluctuations. It seems that the very existence of an order parameter of this kind has more general meaning and more deep origin than our approximation. We see from (3.15) that at $s = -1/4$ we will have $G_s(t|x,x')$ very singular at $x \to x'$. Such fluctuations can be described as bags in space-time, where $\sqrt{g} \to 0$.

On the other hand, the domain $s < -1/4$ corresponds to fluctuations with $\sqrt{g} < 0$, i.e., we have a first-order phase transition, since also the covariant volume
factor \( V \equiv \int d^4x \sqrt{g} \) in (3.11) changes sign, and hence we pass from a regime, say, with \( \mathcal{F} > 0 \) to one entropy dominated, i.e. \( \mathcal{F} < 0 \). From a quantum-mechanical point of view, a negative free energy should imply vacuum decay, so that in order to avoid this possibility we restrict ourselves to the domain of integration \( s > -1/4 \) (i.e., \( \sqrt{g} > 0 \)).

We come thus to the final formulation of our model: the volume density of free energy has the form:

\[
\begin{align*}
f & \simeq \lim_{\epsilon \to 0^+} \int^+_{\epsilon} dt \int_{-1/4}^{+\infty} ds e^{-s^2/\epsilon a^2} \cdot \sqrt{\frac{2}{\pi \epsilon a^2}} \cdot \\
& \quad \cdot \int_{x(0)=x(\tau)=0} Dx(\tau) \exp \left\{ - \int_0^t \left( \frac{\dot{x}^2}{4} + s\dot{x}^2 e^{-x^2/a^2} \right) \right\}
\end{align*}
\]

In (3.10) is understood the usual zero-point energy subtraction of \( \text{Tr} \log \Delta \equiv f_0 \).

4 General Estimates

A limiting case in which (3.16) can be computed is obtained by setting \( a \to +\infty \). In this limit we can neglect the \( x \)-dependence of the correlator, i.e. we can put:

\[
e^{-x^2/a^2} \simeq 1
\]

Then, one gets

\[
f \simeq \lim_{\epsilon \to 0^+} \int^+_{\epsilon} dt \sqrt{\frac{2}{\pi \epsilon a^2}} \int_{-1/4}^{+\infty} ds e^{-s^2/\epsilon a^2} \cdot <0|e^{-t(1+4s)\Delta}|0>-f_0
\]

This quantity requires an ultraviolet regularization. Let us use a momentum cut-off regularization by defining

\[
<0|e^{-t(1+4s)\Delta}|0>= \frac{2\pi^2}{(2\pi)^4} \int_0^\Lambda k^3 dk e^{-tk^2(1+4s)},
\]

where \( \Lambda \) is the UV-cutoff. By (4.3), the leading contribution to (4.2) in \( \Lambda \) is approximately equal to (see Appendix):

\[
f \approx \frac{1}{\pi} \Lambda^4 a^2 8(1 - \frac{2}{3} \sqrt{\frac{2}{\pi}}) \approx \frac{4}{\pi^2} \Lambda^4 a^2.
\]

Since in our model we have neglected systematically all quantum gravity short-range effects, we may assume that the energy-scale \( \Lambda \) is upper-bounded by the mass Planck scale \( 1/\ell_P \) (in \( c = \hbar = 1 \)).
The General Case

For arbitrary \( a \) the quantum mechanical problem (3.16) cannot be solved exactly, but we see from the previous computation that the main contribution is given by high-momentum fluctuations and that we can use semiclassical formulas. It is convenient to use Laplace (or Fourier) transform.

The Laplace transform of the Green function (3.15) in the \( t \)-variable has a cut along the imaginary semiaxis; the inverse Laplace transform can be reduced to an integral of the jump of \( G \) on this cut over this semiaxis. The resulting expression, after \( dt \)-integration, has the form:

\[
f \simeq \int_0^\infty \frac{d\omega}{\pi} \log \omega \int_{-1/4}^{+1/4} ds e^{-s^2/2a^2} \sqrt{\frac{2}{\pi c^2 a^2}} \cdot \text{Im} \, G_s(\omega + i0|0,0) - f_0,
\]

where \( G_s(\omega + i0|x,x') \) obeys the equation

\[
(V(x)\Delta + \omega)G_s(\omega|x,x') = \delta^{(4)}(x-x');
\]

\[
V(x) \equiv 1 + \varphi(x)
\]

The Green function \( G_s(\omega|x,0) \) is spherically symmetric:

\[
G_s(\omega|x,0) = G_s(\omega, r),
\]

and taking into account the small imaginary part of \( \omega \) we obtain its semiclassical limit in the form:

\[
G_s(\omega, r) \simeq -\frac{1}{8\pi V_0 r} H_1^{(1)} \left( \sqrt{\frac{2\omega}{1 + 4s e^{-r^2/a^2}}} \right),
\]

Here we have set

\[
k(r) = \sqrt{\frac{2\omega}{1 + 4s e^{-r^2/a^2}}}, \quad V_0 \equiv 1 + 4s
\]

and \( H_1^{(1)} \) is the Hankel function of first kind. It is easy to see that the value of \( \text{Im} \, G_s(\omega, r \to 0) \) is the same as in the case \( a \to \infty \) and that we obtain the same result (4.4). As a consequence the main contribution would be given by the domain \( k(0) < \Lambda \), i.e. \( \tilde{s}t > 1/\Lambda \) in the notations of the Appendix. It should be stressed that the mean field approximation becomes exact in the limit \( a \to \infty \). On the other hand, \( a \) is large for the dominating fluctuations. Thus, we can hope that the main contribution to the free energy in the giant wormhole limit is taken into account by (4.4).

The cutoff \( \Lambda \) has physical meaning and is defined by quantum gravity effects. As we have noticed above it is natural to think that \( \Lambda \sim 1/\ell_P \). Our estimations have
relation to the case that $1/a < \Lambda < 1/\ell_P$ only. In this limit we see that the volume density of free energy increases with $a$ as $\Lambda^4a^2$, and the energy minimum is realized at small $a$.

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### 6 Appendix

We estimate the integral (4.2), (4.3). Let $s = \frac{1}{4}(\tilde{s} - 1)$, then:

$$f \approx \frac{1}{4\pi^2} \sqrt{\frac{2}{\pi ca^2}} \int_{\epsilon}^{+\infty} \frac{dt}{t} \int_{0}^{\infty} d\tilde{s} \exp\left(-\frac{1}{16ca^2}(\tilde{s} - 1)^2\right) \cdot \frac{1}{\tilde{s}^2 t^2} \left\{\left(1 - e^{-\frac{16\Lambda^2}{\pi} \tilde{s}}\right) - \frac{t \tilde{s}}{4} \Lambda^2 e^{-\frac{16\Lambda^2}{\pi} \tilde{s}}\right\} - f_0. \quad (6.1)$$

We divide the domain of $dt \, d\tilde{s}$-integration in two domains, $\tilde{s}t < 1/\Lambda^2$ and $\tilde{s}t > 1/\Lambda^2$. The first domain gives a contribution which cancels the $f_0$, while the second gives

$$f \approx \frac{1}{4\pi^2} \sqrt{\frac{2}{\pi ca^2}} \int_{\epsilon}^{+\infty} \frac{dt}{t} \int_{1/\Lambda^2 t}^{+\infty} d\tilde{s} \exp\left(-\frac{1}{16ca^2}(\tilde{s} - 1)^2\right) \cdot \frac{1}{\tilde{s}^2 t^2} \approx \int_{1/(4\Lambda^2 \sqrt{ca^2})}^{1/\Lambda^2} \frac{dt}{t^3} \left(1 - \frac{1}{4\Lambda^2 \sqrt{\frac{2}{\pi ca^2}}}\right) \approx \frac{4}{\pi^2 \Lambda^4 ca^2}, \quad a^2 \gg 1. \quad (6.2)$$

The domains of $dt$-integration $1/\Lambda^2 < t$ and $t < 1/(4\Lambda^2 \sqrt{ca^2})$ give a small contribution.
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