Regularity of some invariant distributions on nice symmetric pairs

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Abstract

J. Sekiguchi determined the semisimple symmetric pairs \((\mathfrak{g}, \mathfrak{h})\), called nice symmetric pairs, on which there is no non-zero invariant eigendistribution with singular support. On such pairs, we study regularity of invariant distributions annihilated by a polynomial of the Casimir operator. We deduce that invariant eigendistributions on \((\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))\) are locally integrable functions.

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Introduction

Let \(G\) be a reductive group such that \(\text{Ad}(G)\) is connected. Let \(\sigma\) be an involutive automorphism of \(G\). We denote by the same letter \(\sigma\) the corresponding involution on the Lie algebra \(\mathfrak{g}\) of \(G\). Let \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}\) be the decomposition into \(+1\) and \(-1\) eigenspaces with respect to \(\sigma\). Then \((\mathfrak{g}, \mathfrak{h})\) is called a reductive symmetric pair (or semisimple when \(\mathfrak{g}\) is semisimple). Let \(H\) be the group of fixed points of \(\sigma\) in \(G\).

In \([7]\), J. Sekiguchi describes semisimple symmetric pairs on which there is no non-zero invariant eigendistribution with support in \(\mathfrak{q} - \mathfrak{q}^{reg}\) where \(\mathfrak{q}^{reg}\) is the set of semisimple regular elements of \(\mathfrak{q}\). These pairs, called nice symmetric pairs, are characterized by a property on distinguished nilpotent elements and we can generalize this notion to reductive pairs (Definition 4.1). Our main result is the following. Let \(\omega\) be the Casimir polynomial of \(\mathfrak{q}\) and \(\partial(\omega)\) the corresponding differential operator on \(\mathfrak{q}\).

**Theorem 0.1.** Let \((\mathfrak{g}, \mathfrak{h})\) be a nice reductive symmetric pair. Let \(\mathcal{V}\) be an \(H\)-invariant open subset of \(\mathfrak{q}\). Let \(\Theta\) be an \(H\)-invariant distribution on \(\mathcal{V}\) such that

1. There exists \(P \in \mathbb{C}[X]\) such that \(P(\partial(\omega))\Theta = 0\),
2. There exists \(F \in L^1_{loc}(\mathcal{V})^H\) such that \(\Theta = F\) on \(\mathcal{V} \cap \mathfrak{q}^{reg}\).

Then \(\Theta = F\) as distribution on \(\mathcal{V}\).

In \([2]\), E. Galina and Y. Laurent obtained stronger results on invariant distributions on nice symmetric pairs by different methods based on algebraic properties of \(D\)-modules. They proved...
that any invariant distribution on a nice pair which is annihilated by a finite codimensional ideal of the algebra of $H$-invariant differential operators with constant coefficients on $\mathfrak{q}$ is a locally integrable function ([2] Corollary 1.7.6).

Our approach uses properties of distributions. Assuming that $S = \Theta - F$ is non-zero, we are led to a contradiction. By the work of G. van Dijk ([8]) and J. Sekiguchi ([7]), we can adapt the descent method of Harish-Chandra. Thus, we construct a non-zero distribution $\tilde{\Theta}$ leading to a contradiction. By the work of G. van Dijk ([8]) and J. Sekiguchi ([7]), we can adapt the method developed by M. Atiyah in [1], one studies the degree of singularity along $\{0\} \times \mathbb{R}^m$ of different distributions in this equation. One deduces that $\tilde{\Theta}$ is integrable function $\tilde{\Theta}$ and a differential operator $\tilde{D}$ of $\mathfrak{g}$ to the space of differential operators with complex constant coefficients on $\mathfrak{g}$.

In the last section, we complete the results of [3] on the nice symmetric pair $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$ and deduce that any invariant eigendistribution for a regular character on this pair is given by a locally integrable function.

1 Notation

Let $M$ be a smooth variety. Let $C^\infty(M)$ be the space of smooth functions on $M$, $\mathcal{D}(M)$ the subspace of compactly supported smooth functions, $L^1_{\text{loc}}(M)$ the space of locally integrable functions on $M$, endowed with their standard topology and $\mathcal{D}'(M)$ the space of distributions on $M$.

For a group $G$ acting on $M$, one denotes by $\mathcal{F}^G$ the points of $\mathcal{F}$ fixed by $G$ for each space $\mathcal{F}$ defined as above.

If $N \subset M$ and if $f$ is a function defined on $M$, one denotes by $f_{/N}$ its restriction to $N$.

If $V$ is a finite dimensional real vector space then $V^*$ is its algebraic dual and $V_C$ is its complexified vector space. The symmetric algebra $S[V]$ of $V$ can be identified to the space $\mathbb{R}[V^*]$ of polynomial functions on $V^*$ with real coefficients and to the space of differential operators with real constant coefficients on $V$. Similarly, one has $S[V_C] = \mathbb{C}[V^*]$ and this algebra can be identified to the space of differential operators with complex constant coefficients on $V_C$. If $u \in S[V]$ (resp. $S[V_C]$), then $\partial(u)$ will denote the corresponding differential operator.

Let $G$ be a reductive group such that $\text{Ad}(G)$ is connected, and $\sigma$ an involution on $G$. This defines an involution, denoted by the same letter $\sigma$ on the Lie algebra $\mathfrak{g}$ of $G$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the direct decomposition of $\mathfrak{g}$ into the $+1$ and $-1$ eigenspaces of $\sigma$. Then $(\mathfrak{g}, \mathfrak{h})$ is called a reductive symmetric pair. Let $H$ be the subgroup of fixed points of $\sigma$ in $G$.

Let $\mathfrak{c}_\mathfrak{g}$ be the center of $\mathfrak{g}$ and $\mathfrak{g}_s$ its derived algebra. We set

$$\mathfrak{c}_\mathfrak{g} = \mathfrak{c}_\mathfrak{g} \cap \mathfrak{q} \quad \text{and} \quad \mathfrak{g}_s = \mathfrak{g}_s \cap \mathfrak{q}.$$  

If $x$ is an element of $\mathfrak{g}$ and $\mathfrak{r}$ is a subspace of $\mathfrak{g}$, we denote by $\mathfrak{r}_x$ the centralizer of $x$ in $\mathfrak{r}$.

We fix a non-degenerate bilinear form $B$ on $\mathfrak{g}$ which is equal to the Killing form on $\mathfrak{g}_s$. Then $\omega(X) = B(X, X)$ is the Casimir polynomial of $\mathfrak{g}$.
2 Transfer of distributions and differential operators

We recall results of \([8]\) sections 2 and 3) and \([\mathcal{I}]\) section (3.2)) on restriction of distributions and radial parts of differential operators. Their proofs are similar to (\([\mathcal{II}]\) or \([\mathcal{II}]\) Part I, chapter 2).

Let \(x_0 \in \mathfrak{q}_a\). Let \(U\) be a linear subspace of \(\mathfrak{q}\) such that \(\mathfrak{q} = U \oplus [x_0, \mathfrak{h}]\) and \(V\) be a linear subspace of \(\mathfrak{h}\) such that \(\mathfrak{h} = V \oplus \mathfrak{h}_{x_0}\). Consider the open subset \(U = \{Z \in U; U + [x_0 + Z, \mathfrak{h}] = \mathfrak{q}\}\) containing 0. Then the map \(\Psi\) from \(H \times U\) to \(\mathfrak{q}\) defined by \(\Psi(h, u) = h \cdot (x_0 + u)\) is a submersion. In particular, \(\Omega = \Psi(H \times U)\) is an open \(H\)-invariant subset of \(\mathfrak{q}\) containing \(x_0\). We fix an Haar measure \(dh\) on \(H\) and we denote by \(du\) (respectively \(dx\)) the Lebesgue measure on \(U\) (respectively \(\mathfrak{q}\)). The submersion \(\Psi\) induces a continuous surjective map \(\Psi_*\) from \(D(H \times U)\) onto \(D(\Omega)\) such that, for any \(F \in L^1_{\text{loc}}(\mathfrak{q})\) and any \(f \in D(H \times U)\), one has

\[
\int_{H \times U} F \circ \Psi(h, u) f(h, u) du = \int_{\mathfrak{q}} F(x) \Psi_*(f)(x) dx.
\]

**Theorem 2.1.** For \(T \in D'(\Omega)^H\) there exists a unique distribution \(\text{Res}_U T\) defined on \(\mathfrak{q}\), called the restriction of \(T\) to \(\mathfrak{q}\) with respect to \(\Psi\), such that for any \(f \in D(H \times U)\), one has

\[
<T, \Psi_*(f) > = <\text{Res}_U T, p_*(f) >
\]

where \(p_*(f) \in D(U)\) is defined by \(p_*(f)(u) = \int_{H} f(h, u) dh\).

This restriction satisfies the following properties:

1. If \(U\) is stable under the action of a subgroup \(H_0\) of \(H\) then \(\text{Res}_U T\) is \(H_0\)-invariant.
2. \(x_0 + \text{supp} (\text{Res}_U T) \subset \text{supp} (T) \cap (x_0 + \mathfrak{q})\).
3. If \(F \in L^1_{\text{loc}}(\Omega)^H\) then \(\text{Res}_U F\) is the locally integrable function on \(\mathfrak{q}\) defined by \(\text{Res}_U F(u) = F(x_0 + u)\).
4. If \(\text{Res}_U T = 0\) then \(T = 0\) on \(\Omega\).

**Theorem 2.2.** Let \(D\) be a \(H\)-invariant differential operator on \(\mathfrak{q}\). Then there exists a differential operator \(\text{Rad}_U(D)\), called the radial part of \(D\) with respect to \(\Psi\), defined on \(\mathfrak{q}\) such that for any \(f \in D(\Omega)^H\), one has \((D \cdot f)(x_0 + u) = \text{Rad}_U(D) \cdot \text{Res}_U f(u)\) for \(u \in \mathfrak{q}\).

Moreover, for any \(T \in D'(\Omega)^H\), one has

\[
\text{Res}_U (D \cdot T) = \text{Rad}_U(D) \cdot \text{Res}_U (T).
\]

3 Semisimple elements

We recall that a Cartan subspace of \(\mathfrak{q}\) is a maximal abelian subspace of \(\mathfrak{q}\) consisting of semisimple elements.

If \(\mathfrak{r} = \mathfrak{q}\) or \(\mathfrak{q}_a\), we denote by \(S(\mathfrak{r})\) the set of semisimple elements of \(\mathfrak{r}\).

Let \(\mathfrak{a}\) be a Cartan subspace of \(\mathfrak{q}\). If \(\lambda \in \mathfrak{g}_C^\ast\), we set

\[
\mathfrak{g}_C^\lambda = \{X \in \mathfrak{g}_C; [A, X] = \lambda(A)X \text{ for any } A \in \mathfrak{a}_C\}
\]
and 
\[ \Sigma(a) = \{ \lambda \in g_C^0 : g_C^0 \neq \{0\} \}. \]

Then \( \Sigma(a) \) is the root system of \((g_C, a_C)\).

An element \( X \) of \( S(q) \) is \( q \)-regular (or regular) if its centralizer \( q_X \) in \( q \) is a Cartan subspace. If \( X \in a \) then \( X \) is regular if and only if \( \lambda(X) \neq 0 \) for all \( \lambda \in \Sigma(a) \). We denote by \( q^{reg} \) the open dense subset of semisimple regular elements of \( q \).

Let \( A_0 \in S(q) \). Its centralizer \( z = g_{A_0} \) in \( g \) is a reductive \( \sigma \)-stable Lie subalgebra of \( g \). We denote by \( c \) its center and by \( z_0 \) its derived algebra. We set 
\[ c^- = c \cap q, \quad c^+ = c \cap h, \quad z^-_0 = z_0 \cap q \quad \text{and} \quad z^- = z_0 \cap h. \]

The pair \((z_0, z^-)\) is a semisimple symmetric subpair of \((g_s, h_s)\) which is equal to \((g_s, h_s)\) if \( A_0 \in c_q \).

Let \( H^+_s \) be the analytic subgroup of \( H \) with Lie algebra \( z^+_s \).

We assume that \( A_0 \notin c_q \). We take a Cartan subspace \( a \) of \( q \) containing \( A_0 \) and consider the corresponding root system \( \Sigma = \Sigma(a) \). We fix a positive system \( \Sigma^+ \) of \( \Sigma \). For any \( \lambda \in \Sigma^+ \), we choose a \( C \)-basis \( X_{\lambda,1}, \ldots, X_{\lambda,m_\lambda} \) of \( g_C^0 \) such that \( B(X_{\lambda,i}, \sigma(X_{\lambda,j})) = -\delta_{i,j} \) for \( i, j \in \{1, \ldots, m_\lambda\} \).

Let \( \Sigma^+_1 = \{ \lambda \in \Sigma^+ : \lambda(A_0) \neq 0 \} \). We set 
\[ V_{C}^\pm = \sum_{\lambda \in \Sigma^+_1}^{m_\lambda} (X_{\lambda,j} \pm \sigma(X_{\lambda,j})), \quad V^+ = V_{C}^+ \cap h, \quad V^- = V_{C}^- \cap q. \]

We have the decompositions \( h = z^+ + V^+ \) and \( q = z^- + V^- \), with \( \dim V^+ = \dim V^- \) and \( [A_0, h] = V^- \).

If \( Z_0 \in z^- \), we define the map \( \eta_{Z_0} \) from \( V^+ \times z^- \) to \( q \) by \( \eta_{Z_0}(v, Z) = Z + [v, A_0 + Z_0] \). Then \( \eta_0 \) is a bijective map. We set \( \xi(Z_0) = det(\eta_{Z_0} \circ \eta^{-1}) \) and \( z^-_0 = \{ Z \in z^- : \xi(Z) \neq 0 \} \). Then \( z^-_0 \) is invariant under \( H^+_s \).

Thus the map \( \gamma \) from \( H \times z^-_0 \) to \( q \) defined by \( \gamma(h, Z) = h \cdot (A_0 + Z) \) is a submersion. By Theorem 2.2 for any \( H \)-invariant distribution \( \Theta \) on \( q \), there exists a unique \( H^+_s \)-invariant distribution \( Res_{z_0^-(\Theta)} \) defined on \( z^-_0 \) such that, for any \( f \in D(H \times z^-_0) \), one has \( < \Theta, \gamma_*(f) > = < Res_{z^-_0(\Theta)}, p_*(f) > \).

Let \( \omega_j^- \) be the restriction of \( \omega \) to \( z^-_0 \). Then, one has:

**Lemma 3.1.** ([?] Lemma 4.4). Let \( Rad_{z^-}(\partial(\omega)) \) be the radial part of \( \partial(\omega) \) with respect to \( \gamma \) (Theorem 2.2). Then 
\[ Rad_{z^-}(\partial(\omega)) = \xi^{-1/2} \partial(\omega_j^-) \circ \xi^{1/2} - \mu \]
where \( \mu(Z) = \xi(Z)^{-1/2}(\partial(\omega_j^-) \xi^{1/2})(Z) \) is an analytic function on \( z^-_0 \).

4 Nilpotent and distinguished elements

Let \( Z_0 \in q \). Let \( Z_0 = A_0 + X_0 \) be its Jordan decomposition ([?] Lemma 1.1). We construct the symmetric pair \((z_0, z^-_0)\) related to \( A_0 \) as in [3].

We assume that \( X_0 \) is different from zero. From ([?] Lemma 1.7), there exists a normal \( sl_2 \)-triple \((B_0, X_0, Y_0)\) of \((z_0, z^-_0)\) containing \( X_0 \), i.e. satisfying \( B_0 \in z^+_0 \) and \( Y_0 \in z^-_0 \) such that \([B_0, X_0] = 2X_0, [B_0, Y_0] = -2Y_0 \) and \([X_0, Y_0] = B_0 \).
We set $\mathfrak{z}_0 = \mathbb{R}B_0 + \mathbb{R}X_0 + \mathbb{R}Y_0$. The Cartan involution $\theta_0$ of $\mathfrak{z}_0$ defined by $\theta_0 : (B_0, X_0, Y_0) \to (-B_0, -Y_0, -X_0)$ extends to a Cartan involution of $\mathfrak{z}_s$, denoted by $\theta$, which commutes with $\sigma$. (Lemma 4.1.) The bilinear form $(X, Y) \mapsto -B(\theta(X), Y)$ defines a scalar product on $\mathfrak{z}_s$.

We can decompose $\mathfrak{z}_s$ in an orthogonal sum $\mathfrak{z}_s = \sum_i \mathfrak{z}_i$ of irreducible representations $\mathfrak{z}_i$ under the adjoint action of $\mathfrak{z}_0$. One can choose a suitable ordering of the $\mathfrak{z}_i$ such that $(\mathfrak{z}_s^-)_{X_0} = \sum_{i=1}^r \mathfrak{z}_i \cap (\mathfrak{z}_s^-)_{Y_0} = \theta((\mathfrak{z}_s^-)_{X_0})$ with $\mathfrak{z}_1 = \mathfrak{z}_0$ and $\dim \mathfrak{z}_i \cap (\mathfrak{z}_s^-)_{Y_0} = 1$. We set $n_i + 1 = \dim \mathfrak{z}_i$. Hence, there exists an orthonormal basis $(w_1, \ldots, w_r)$ of $(\mathfrak{z}_s^-)_{Y_0}$ such that $w_1 = \frac{Y_0}{\|Y_0\|}$ and $[B_0, w_i] = -n_i w_i$ for $i \in \{1, \ldots, r\}$. In particular, one has $n_1 = 2$.

We set \[ \delta_q(Z_0) = \delta_{\mathfrak{z}_s^-}(X_0) = \sum_{i=1}^r (n_i + 2) - \dim (\mathfrak{z}_s^-). \]

Let $\mathcal{N}(\mathfrak{z}_s^-)$ be the set of nilpotent elements of $\mathfrak{z}_s$.

**Definition 4.1.** ([Lemma 1.11 and 1.13])

1. An element $X_0$ of $\mathcal{N}(\mathfrak{z}_s^-)$ is a $\mathfrak{z}_s$-distinguished nilpotent element if $(\mathfrak{z}_s^-)_{X_0}$ contains no non-zero semisimple element.

2. An element $Z_0$ of $\mathfrak{g}$ with Jordan decomposition $Z_0 = A_0 + X_0$ is called $\mathfrak{q}$-distinguished if $X_0$ is a $\mathfrak{z}_s$-distinguished nilpotent element of $\mathfrak{z}_s$.

**Definition 4.2.** The symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is nice if for any $\mathfrak{q}$-distinguished element $Z$, one has $\delta_{\mathfrak{q}}(Z) > 0$.

Let $\omega_s$ be the restriction of $\omega$ to $\mathfrak{z}_s^-$. Though $\omega_s$ is not the Casimir polynomial on $\mathfrak{z}_s^-$, one has the following result:

**Lemma 4.3.** ([Lemma 4]) The following assertions are equivalent:

1. $X_0$ is a $\mathfrak{z}_s^-$-distinguished nilpotent element.

2. $\omega_s(X) = 0$ for all $X \in (\mathfrak{z}_s^-)_{X_0}$.

3. $\omega_s(X) = 0$ for all $X \in (\mathfrak{z}_s^-)_{Y_0}$.

4. $n_i > 0$.

5. $(\mathfrak{z}_s^-)_{X_0} \cap (\mathfrak{z}_s^-)_{Y_0} = \{0\}$.

Thus, if $X_0$ is a $\mathfrak{z}_s^-$-distinguished nilpotent element then one has $\omega(X_0 + X) = 2B(X_0, X) = 2\|Y_0\| \cdot x_1$ for all $X \in (\mathfrak{z}_s^-)_{Y_0}$, where $x_1$ is the first coordinate of $X$ in the basis $(w_1, \ldots, w_r)$ of $(\mathfrak{z}_s^-)_{Y_0}$.

For any $X_0 \in \mathcal{N}(\mathfrak{z}_s^-)$, one has $\mathfrak{z}_s^- = (\mathfrak{z}_s^-)_{X_0} \oplus [\mathfrak{z}_s^+, X_0]$ and $\mathfrak{z}_s^+ = (\mathfrak{z}_s^+)_{X_0} \oplus [\mathfrak{z}_s^-, Y_0]$. From now on, we set $U = (\mathfrak{z}_s^-)_{Y_0}$.

For $X \in U$, we consider the map $\psi_X$ from $[\mathfrak{z}_s^-, Y_0] \times U$ to $\mathfrak{z}_s^-$ defined by $\psi_X(v, z) = z + [v, X_0 + X]$. The map $\psi_0$ is bijective.
Lemma 4.5. ([8] Lemma 13) The homogeneous part of degree 2 of $\omega$ $R$ by only if ([8] Theorem 14) Let Theorem 4.6.

Lemma 4.4. ([8] Lemma 17 and 18). There exists a neighborhood $U_0$ of 0 in $U$ such that

1. $\pi$ is a submersion on $H_s^+ \times U_0$,
2. $\Omega_0 = \pi(H_s^+ \times U_0)$ is an open neighborhood of $X_0$ in $\mathfrak{s}^-$ and $\Omega_0 \cap N_j = \mathcal{O}_j$,
3. $\mathcal{O}_j \cap (X_0 + U_0) = \{X_0\}$
4. Let $\Theta$ be an $H_s^+$-invariant distribution on $\Omega_0$. Let $\text{Res}_U \Theta$ be its restriction to $U$ with respect to $\pi$.

If $\text{supp} (\Theta) \subset N_j$ then $\text{supp} (\text{Res}_U \Theta) \subset \{0\}$.

We denote by $\omega_2$ and $\omega_s$ the restrictions of $\omega$ to $c^-$ and $\mathfrak{s}^-$ respectively. One has $\omega_j = \omega_{j_-} + \omega_s$. We precise now the radial part $\text{Rad}_U (\partial(\omega_s))$ of $\partial(\omega_s)$ with respect to $\pi$. We denote by $\text{Rad}_{U,X} (\partial(\omega_s))$ its local expression at $X = U_0$.

Lemma 4.5. ([8] Lemma 13) The homogeneous part of degree 2 of $\text{Rad}_{U,0} (\partial(\omega_s))$ is zero if and only if $X_0$ is $\mathfrak{s}^-$-distinguished.

Theorem 4.6. ([8] Theorem 14) Let $X_0$ be a $\mathfrak{s}^-$-distinguished nilpotent element and $c_0 = \|X_0\|$. Then, there exist analytic functions $a_{i,j}$ ($2 \leq i, j \leq r$) and $a_i$ ($2 \leq i \leq r$) on $U_0$ satisfying $a_{i,j}(0) = 0$ such that, for any $H_s^+$-invariant distribution $T$ on $\Omega_0$, one has

$$\text{Res}_U (\partial(\omega_s)T) = \text{Rad}_U ((\partial(\omega_s))\text{Res}_U (T) = \frac{1}{c_0} \left( 2x_1 \frac{\partial^2}{\partial x_1^2} + (\dim \mathfrak{s}^-) \frac{\partial}{\partial x_1} + \sum_{i=2}^r (n_i + 2)x_i \frac{\partial^2}{\partial x_1 \partial x_i} \right)$$

$$+ \sum_{2 \leq i \leq j \leq r} a_{i,j}(X) \frac{\partial}{\partial x_j \partial x_i} + \sum_{i=2}^r a_i(X) \frac{\partial}{\partial x_i}) \text{Res}_U (T)$$

where $x_1, \ldots, x_r$ are the coordinates of $X$ in the basis $(w_1, \ldots, w_r)$.

5 The main Theorem

Our goal is to prove the following Theorem:

Theorem 5.1. Let $(\mathfrak{g}, \mathfrak{h})$ be a nice reductive symmetric pair. Let $\mathcal{V}$ an $H$-invariant open subset of $\mathfrak{q}$. Let $\Theta$ be an $H$-invariant distribution on $\mathcal{V}$ such that

1. There exists $P \in \mathbb{C}[X]$ such that $P(\partial(\omega)) \Theta = 0$
2. There exists \( F \in L^1_{\text{loc}}(\mathcal{V})^H \) such that \( \Theta = F \) on \( \mathcal{V} \cap q^{reg} \).

Then \( \Theta = F \) as distribution on \( \mathcal{V} \).

We will use the method developed by M. Atiyah in [1]. First we recall some facts about distributions on \( \mathbb{R}^r \times \mathbb{R}^m \). Let \( \mathbb{N} \) be the set of non-negative integers. For \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \), we set \( |\alpha| = \alpha_1 + \ldots + \alpha_r \) and

\[
x^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r}, \quad \partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_r^{\alpha_r}}.
\]

For \( \varphi \in \mathcal{D}(\mathbb{R}^r \times \mathbb{R}^m) \) and \( \varepsilon > 0 \), we set \( \varphi_{\varepsilon}(x,y) = \varphi(\frac{x}{\varepsilon}, y) \) for \( (x,y) \in \mathbb{R}^r \times \mathbb{R}^m \). For \( T \in \mathcal{D}'(\mathbb{R}^r \times \mathbb{R}^m) \) we denote by \( T_\varepsilon \) the distribution defined by \( \langle T_\varepsilon, \varphi \rangle = \langle T, \varphi_{\varepsilon} \rangle \).

**Definition 5.2.** Let \( V = \{0\} \times \mathbb{R}^m \subset \mathbb{R}^r \times \mathbb{R}^m \) and \( T \in \mathcal{D}'(\mathbb{R}^r \times \mathbb{R}^m) \).

1. The distribution \( T \) is regular along \( V \) if \( \lim_{\varepsilon \to 0} T_\varepsilon = 0 \).

2. The distribution \( T \) has a degree of singularity along \( V \) smaller than \( k \) if for all \( \alpha \in \mathbb{N}^r \) with \( |\alpha| = k \), the distribution \( x^\alpha T \) is regular.

We denote by \( d^T \) the degree of singularity of \( T \) along \( V \) and we omit in what follows to precise "along \( V \)". Regularity corresponds to a degree of singularity equal to 0.

3. The degree of singularity of \( T \) is equal to \( k \) if \( d^T k \leq k \) and \( d^T k \neq k - 1 \).

**Lemma 5.3.**

1. If \( F \in L^1_{\text{loc}}(\mathbb{R}^r + m) \) then \( d^F = 0 \).

2. If \( d^F_i = k \geq 1 \) then \( d^F_i (x_i T) = k - 1 \) for \( i \in \{1, \ldots r\} \).

3. If \( d^F_i \leq k \) then \( \frac{\partial}{\partial x_i} T \leq k + 1 \) for \( i \in \{1, \ldots r\} \).

4. Let \( \delta_0 \) be the Dirac measure at 0 in \( \mathbb{R}^r \) and \( \delta_0^{(\alpha)} = \delta^\alpha \delta_0 \). If \( S \in \mathcal{D}'(\mathbb{R}^m) \) then the degree of singularity of \( \delta^{(\alpha)}_0 \otimes S \) is equal to \( |\alpha| + 1 \).

**Proof.**

1. Let \( F \in L^1_{\text{loc}}(\mathbb{R}^r + m) \) and \( \phi \in \mathcal{D}(\mathbb{R}^r + m) \) with \( \text{supp}(\phi) \subset K_1 \times K_2 \) where \( K_1 \) (resp., \( K_2 \)) is a compact subset of \( \mathbb{R}^r \) (resp., \( \mathbb{R}^m \)). One has

\[
|\int_{\mathbb{R}^r \times \mathbb{R}^m} F(x,y)\phi(x,y)dx dy| \leq \sup_{(x,y)\in \mathbb{R}^r + m} |\phi(x,y)| \int_{(\varepsilon K_1) \times K_2} |F(x,y)| dx dy
\]

and the first assertion follows.

2. is clear.

3. Let \( \alpha \in \mathbb{N}^r \) such that \( |\alpha| = k + 1 \). If \( \alpha_j \geq 1 \) for some \( j \in \{1, \ldots, r\} \), we set \( \bar{\alpha}^j = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_r) \). Let \( \varphi \in \mathcal{D}(\mathbb{R}^r + m) \).

If \( \alpha_j \geq 1 \), one has

\[
\langle x^\alpha \frac{\partial}{\partial x_i} T, \varphi_{\varepsilon} \rangle = \langle - \alpha_i + x^\alpha x^i \varphi_{\varepsilon} + \frac{\partial}{\partial x_i} (\frac{\partial}{\partial x_i} \varphi)_{\varepsilon} \rangle > 0.
\]

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thus \((x^\alpha T)_\varepsilon\) converges to 0 since \(d_x^\alpha T \leq k\).

If \(\alpha_i = 0\), we choose \(j\) such that \(\alpha_j \geq 1\). One has \(x^\alpha \frac{\partial}{\partial x_i} T, \phi_\varepsilon \geq -x^{\alpha j} T, (x_j \frac{\partial}{\partial x_i} \phi)_\varepsilon\) which tends to 0 as before.

4. We recall that for \(i \in \{1, \ldots, r\}\), one has

\[
x_i^* \delta^{(s_0)} = \begin{cases} (-1)^l \frac{(\alpha_1)!}{(\alpha_i-l)!} \delta^{(s_1, \ldots, s_l)} & \text{if } \alpha_i \geq l \\
0 & \text{if } \alpha_i < l.
\end{cases}
\]

Hence, one has \(x^\alpha \delta_0^{(s_0)} = (-1)^{\alpha \lambda} \alpha! \delta_0\) and for all \(\beta \in \mathbb{N}^r\) with \(|\beta| = |\alpha| + 1\), one has \(x^\beta \delta_0^{(s_0)} = 0\). The assertion follows.

**Definition 5.4.** Let \(\Gamma = x^\beta \partial^\gamma D\) where \(D\) is a differential operator on \(\mathbb{R}^m\). Then \(\Gamma\) increases the degree of singularity at most \(|\alpha| - |\beta|\). The integer \(|\alpha| - |\beta|\) is called the total degree of \(\Gamma\) in \(x\).

We can define the homogeneous part of highest total degree (in \(x\)) of an analytic differential operator developing its coefficients in Taylor series.

**Proof of the Theorem.** Let \(\Theta \in D'(\mathcal{V})^H\) and \(F \in L^1_{\text{loc}}(\mathcal{V})^H\) such that \(P(\partial(\omega))\Theta = 0\) for a unitary polynomial \(P \in \mathbb{C}[X]\) and \(\Theta = F\) on \(\mathcal{V}^{reg} = \mathcal{V} \cap q^{reg}\). We write \(\Theta = F + S\) where \(S\) is an \(H\)-invariant distribution with support contained in \(\mathcal{V} \setminus \mathcal{V}^{reg}\). We want to prove that \(S = 0\), which is equivalent to \(\text{supp } (S) = \emptyset\).

Assuming \(S\) is non-zero, we are led to a contradiction. We will study \(S\) near an element \(Z_0 \in \text{supp } (S)\) chosen as follows:

For \(Z_0 \in \text{supp } (S)\) with Jordan decomposition \(Z_0 = A_0 + X_0\), we construct the symmetric subpair \((\mathfrak{s}_-^+, \mathfrak{s}_-^+)\) related to \(A_0\) and we set \(q_{A_0} = \mathfrak{s}_-^+ = \mathfrak{c}^- \oplus \mathfrak{z}_-^+\) as in section 3. Let \(S_k\) be the set of \(Z_0\) in the support of \(S\) such that \(\text{rank} (\mathfrak{z}_-^+) = k\). Since \(\text{supp } (S) \subset \mathcal{V} \setminus \mathcal{V}^{reg}\), if \(Z_0 = A_0 + X_0\) belongs to \(\text{supp } (S)\) then \(A_0\) is not \(q\)-regular. One deduces that \(S_0 = \emptyset\). Let \(k_0 > 0\) such that \(S_0 = S_1 = \ldots = S_{k_0-1} = \emptyset\) and \(S_{k_0} \neq \emptyset\).

For \(Z_0 = A_0 + X_0\) in \(S_{k_0}\), we denote by \(\mathcal{N}(\mathfrak{z}_-^+) = \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_\nu\) the set of nilpotent elements in \(\mathfrak{z}_-^+\) as in section 3. Since \(\text{supp } (S) \cap (A_0 + \mathcal{N}(\mathfrak{z}_-^+)) \neq \emptyset\), one can choose \(j_0 \in \{1, \ldots, \nu\}\) such that \(\text{supp } (S) \cap (A_0 + \mathcal{O}_i) = \emptyset\) for \(i \in \{1, \ldots, j_0 - 1\}\) and \(\text{supp } (S) \cap (A_0 + \mathcal{O}_{j_0}) \neq \emptyset\).

From now on, we fix \(Z_0 = A_0 + X_0\) in \(S_{k_0}\) such that \(X_0 \in \mathcal{O}_{j_0}\).

For \(\varepsilon > 0\), we denote by \(W_\varepsilon\) the set of \(x\) in \(\mathfrak{z}_-^+\) such that, for any eigenvalue \(\lambda\) of \(\text{ad}_x\), one has \(|\lambda| < \varepsilon\). The choice of \(k_0\) implies that there exists \(\varepsilon > 0\) such that \(\text{supp } (S) \subset (Z_0 + \mathcal{N}(\mathfrak{z}_-^+))^\varepsilon \subset \text{supp } (S) \cap (Z_0 + \mathcal{N}(\mathfrak{z}_-^+))\). Hence, we can choose an open neighborhood \(W_\varepsilon\) of 0 in \(\mathfrak{c}^-\) and an open neighborhood \(W_\varepsilon\) of 0 in \(\mathfrak{z}_-^+\) such that

\[
\text{supp } (S) \cap (A_0 + W_\varepsilon + W_\varepsilon) \subset \text{supp } (S) \cap (A_0 + W_\varepsilon + \mathcal{N}(\mathfrak{z}_-^+)).
\]

**First case.** \(A_0 \notin \mathfrak{o}_2\) and \(X_0 \neq 0\).

We keep the notation of section 3. We fix a normal \(sl_2\)-triple \((B_0, Y_0, X_0)\) in \((\mathfrak{z}_-^+, \mathfrak{z}_+^+)\). We choose an open neighborhood \(U_0\) of 0 in \(U\), the centralizer of \(Y_0\) in \(\mathfrak{z}_-^+\), as in Lemma 4. We keep the notation of this lemma. We recall that the map \(\gamma\) from \(H \times \mathfrak{z}_-^+\) to \(\mathfrak{q}\) defined by \(\gamma(h, Z) = h \cdot (A_0 + Z)\) is a submersion. Reducing \(U_0\), \(W_\varepsilon\) and \(W_\varepsilon\) if necessary, we may assume
that $\mathcal{W}_c + \Omega_0 \subset \mathcal{W}_c + W_s \subset \mathfrak{g}^-$ and that $V_0 = \gamma(H \times (\mathcal{W}_c + \Omega_0))$ is an open neighborhood of $Z_0$ contained in $\mathcal{V}$.

If $T$ is an $H$-invariant distribution on $\mathcal{V}$, we denote by $T_0$ its restriction to $V_0$. By theorem 2.1, one can consider its restriction $T_1 = R_{\gamma^{-1}}^{-1}T_0$ to $\mathcal{W}_c + \Omega_0$ with respect to $\gamma$. One has $A_0 + \text{supp}(T_1) \subset \text{supp}(T) \cap (A_0 + \mathcal{W}_c + \Omega_0)$.

We set $T_2 = \xi^{1/2}T_1$ where $\xi^{1/2}$ is the analytic function on $\mathcal{W}_c + \Omega_0$ defined in section 3.

Now, we consider the submersion $\pi_0$ from $H^+_S \times U_0 \times \mathcal{W}_c$ to $\mathfrak{g}^-$ defined by $\pi_0(h, X, C) = h \cdot (X_0 + X) + C$. One denotes by $T_3$ the restriction on $U_0 \times \mathcal{W}_c$ of $T_2$ with respect to $\pi_0$. We have $X_0 + \text{supp}(T_3) \subset \text{supp}(T_2) \cap (X_0 + U_0)$.

Since $F$ is a locally integrable function, the distribution $F_3$ is the locally integrable function on $U_0 \times \mathcal{W}_c$ defined by $F_3(X, C) = \xi^{1/2}(C + X)F(C + X)$.

By assumption, the distribution $S_3$ is non-zero. By (5.3) and Lemma 4.4 (2), one has $\text{supp}(S_2) = \text{supp}(S_1) \subset \mathcal{W}_c + \Omega_0 \cap \mathcal{N}_j = \mathcal{W}_c + \mathcal{O}_j$. We deduce from Lemma 4.1 (3) that $\text{supp}(S_3) \subset \{0\} \times \mathcal{W}_c$. By (6), Lemma 3, there exists a family $(S_\alpha)_\alpha$ of $\mathcal{D}(\mathcal{W}_c)$ such that $S_3 = \sum_{\alpha \in \mathbb{N}^r; |\alpha| \leq l} \delta_0^{(\alpha)} \otimes S_\alpha$ where $\delta_0$ is the Dirac measure at 0 of $U_0$ and for $\alpha \in \mathbb{N}^r$, the $S_\alpha$ with $|\alpha| = l$ are not all zero.

By assumption, the distribution $\Theta$ satisfies $P(\partial(\omega))\Theta = 0$. By Lemma 3.1 one has

$$P\left(\left(\partial(\omega_s) + \partial(\omega_c)\right) - \mu(Z)\right) \Theta_2 = 0 \text{ on } \mathcal{W}_c + \Omega_0.$$

Using the restriction with respect to $\pi_0$, one obtains

$$P\left(\left(\text{Rad}_U(\partial(\omega_s)) + \partial(\omega_c) - \tilde{\mu}\right)\Theta_3 = 0 \text{ on } U_0 \times \mathcal{W}_c$$

where $\tilde{\mu}(X, C) = \mu(C + X)$ for $X \in U_0$ and $C \in \mathcal{W}_c$.

Let $D_0$ be the homogeneous part of highest total degree $d$ of $\text{Rad}_U(\partial(\omega_s))$. We set

$$P\left(\left(\text{Rad}_U(\partial(\omega_s)) + \partial(\omega_c) - \tilde{\mu}\right)\right) = D_0^N + D_1$$

where $N$ is the degree of $P$ and $D_1$ is a differential operator with total degree in $X$ strictly smaller than $Nd$. Since $\Theta_3 = F_3 + S_3$ with $S_3 = \sum_{\alpha \in \mathbb{N}^r; |\alpha| \leq l} \delta_0^{(\alpha)} \otimes S_\alpha$, we obtain the following relation on $U_0 \times \mathcal{W}_c$:

$$(D_0^N + D_1)S_3 = (D_0^N + D_1)(\sum_{\alpha \in \mathbb{N}^r; |\alpha| \leq l} \delta_0^{(\alpha)} \otimes S_\alpha) = -(D_0^N + D_1)F_3 \quad (5.2)$$

We study now the degree of singularity along $\{0\} \times \mathcal{W}_c$ of the two members of (5.2).

If $X_0$ is not a $\mathfrak{g}^-$-distinguished nilpotent element then by Lemma 4.5, the homogeneous part of degree 2 of $\text{Rad}_l(\partial(\omega_s))$ does not vanish and is a differential operator with constant coefficients of degree 2. Hence the total degree of $D_0$ is equal to $d = 2$. Since $F_3$ is a locally integrable function, it follows from Lemma 5.3 that one has $d_s^0F_3 = 0$ and $d_s^0((D_0^N + D_1)F_3) \leq 2N$. By the same Lemma, one has $d_s^0((D_0^N + D_1)S_3) = l + 1 + 2N$. Hence, we have a contradiction.
Assume that $X_0$ is a $3_-$-distinguished nilpotent element. Lemma [4.4] gives $c_0D_0 = 2x \frac{\partial^2}{\partial x_1^2} + (\dim 3_-) \frac{\partial}{\partial x_1} + \sum_{i=2}^r (n_i + 2)x_i \frac{\partial^2}{\partial x_1 \partial x_i} + \sum_{2 \leq i \leq j \leq r} a_{i,j}(X) \frac{\partial^2}{\partial x_j \partial x_i} + \sum_{i=2}^r a_i(X) \frac{\partial}{\partial x_i}$ where $c_0 = \|X_0\|$.

Since $a_{i,j}(0) = 0$, the total degree of $D_0$ is equal to 1.

For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, we set $\tilde{\alpha}^i = (\alpha_1, \ldots, \alpha_i-1, \alpha_i+1, \alpha_i+1, \ldots, \alpha_r)$ and $\tilde{\alpha}^i = (\alpha_1, \ldots, \alpha_i-1, \alpha_i-1, \alpha_i+1, \ldots, \alpha_r)$. The relation $x_1d_0^{(\alpha)} = -\alpha_0d_0^{(\tilde{\alpha}^i)}$ and the above expression of $D_0$ give

$$c_0D_0 \cdot d_0^{(\alpha)} \otimes S_\alpha = \lambda_\alpha \delta^{(\tilde{\alpha}^i)} \otimes S_\alpha + \sum_{2 \leq i \leq j \leq r} a_{i,j}(X) \delta^{(\tilde{\alpha}^i)} \otimes S_\alpha + \sum_{i=2}^r a_i(X) \delta^{(\tilde{\alpha}^i)} \otimes S_\alpha$$

where

$$\lambda_\alpha = -2(\alpha_1 + 2) + \dim 3_- - \sum_{i=2}^r (n_i + 2)(\alpha_i + 1).$$

Since $n_1$ is equal to 2 and $(\mathfrak{g}, \mathfrak{h})$ is a nice pair, we obtain

$$\lambda_\alpha = -\delta_q(Z_0) - \left[2\alpha_1 + \sum_{i=2}^r (n_i + 2)\alpha_i\right] < 0 \text{ for all } \alpha \in \mathbb{N}^r.$$

Consider $\alpha_0 = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ such that $|\alpha_0| = l$, $S_{\alpha_0} \neq 0$ and $\alpha_1$ is maximal for these properties. One deduces that the coefficient of $\delta^{(\tilde{\alpha}^i)} \otimes S_{\alpha_0}$ in $D_0 \cdot (\sum_{\alpha \in \mathbb{N}^r;|\alpha| = l} \delta^{(\alpha)} \otimes S_\alpha)$ is non-zero. Thus, the degree of singularity of $(D_0^N + D_1)S_3$ is equal to $1 + l + N$. Since $F_3$ is locally integrable and the total degree of $D_0$ is equal to 1, we have $d_0^2(D_0^N + D_1)F_3 \leq N$. This gives a contradiction in [5.2].

**Second case.** $A_0 \in \mathfrak{c}_q$ and $X_0 \neq 0$.

The symmetric pair $(3_-, 3_+)$ is equal to $(\mathfrak{g}_s, \mathfrak{h}_s)$. We just consider the submersion $\pi_0$ from $H \times U_0 \times \mathcal{W}_c$ to $\mathfrak{q}$ defined by $\pi_0(h, X, C) = h \cdot (X_0 + X) + A_0 + C$ where $U_0$ is defined as in Lemma [4.4] for the symmetric pair $(\mathfrak{g}_s, \mathfrak{h}_s)$.

For $T \in \mathcal{D}'(\mathfrak{q})^H$, we denote by $T_1$ the restriction of $T$ to $U_0 \times \mathcal{W}_c$ with respect to $\pi_0$. As in the first case, we have $\Theta_1 = F_1 + S_1$ where $F_1$ is a locally integrable function on $U_0 \times \mathcal{W}_c$ and $S_1$ is a non-zero distribution such that $\text{supp} \ (S_1) \subset \{0\} \times \mathcal{W}_c$. Moreover the distribution $\Theta_1$ satisfies the relation

$$P\left(\text{Rad}_c(\partial(\omega_s)) + \partial(\omega_s)\right) \Theta_1 = 0 \text{ on } U_0 \times \mathcal{W}_c.$$

The same arguments as in the first case lead to the contradiction $S_1 = 0$.

**Third case.** $X_0 = 0$.

The open sets $\mathcal{W}_c$ and $\mathcal{W}_s$ satisfy $\text{supp} \ (S) \cap (A_0 + \mathcal{W}_c + \mathcal{W}_s) \subset \text{supp} \ (S) \cap (A_0 + \mathcal{W}_c + \mathcal{N}(3_+))$. By the choice of $j_0$, we deduce that $\text{supp} \ (S) \cap (A_0 + \mathcal{W}_c + \mathcal{W}_s) \subset \text{supp} \ (S) \cap (A_0 + \mathcal{W}_c)$.

If $A_0 \in \mathfrak{c}_q$, then $V_0 = A_0 + \mathcal{W}_c + \mathcal{W}_s$ is an open neighborhood of $A_0$ in $\mathfrak{q}$. We identify $\mathfrak{q}$ with $\mathfrak{q}_s \times \mathfrak{q}_s$. Thus, the restriction $S_0$ of $S$ to $V_0$ is different from zero and satisfies $\text{supp}(S_0) \subset \{0\} \times (A_0 + \mathcal{W}_c)$. On the other hand, one has $P(\partial(\omega))S_0 = -P(\partial(\omega))F|V_0$. Since $\partial(\omega)$ is a second order operator with constant coefficients, we obtain a contradiction as above.
If $A_0 \notin \mathfrak{q}$, we may assume that $\mathcal{W}_c + \mathcal{W}_s \subset \mathfrak{g}^-$. We denote by $T_1$ the restriction of an $H$-invariant distribution $T$ to $\mathcal{W}_c + \mathcal{W}_s$ with respect to the submersion $\gamma$ from $H \times \mathfrak{g}^-$ to $\mathfrak{q}$ and we consider $T_2 = \xi^{1/2}T_1$ as distribution on $\mathcal{W}_s \times \mathcal{W}_c$. Thus, we have $S_2 \neq 0$ and $\text{supp} (S_2) = \{0\} \times \mathcal{W}_c$. Moreover, the distribution $\Theta_2 = F_2 + S_2$ satisfies $P \left( (\partial(\omega_s) + \partial(\omega_t)) - \mu(Z) \right) \Theta_2 = 0$ on $\mathcal{W}_s \times \mathcal{W}_c$ by Lemma 3.1. This is equivalent to

$$P \left( (\partial(\omega_s) + \partial(\omega_t)) - \mu(Z) \right) S_2 = -P \left( (\partial(\omega_s) + \partial(\omega_t)) - \mu(Z) \right) F_2.$$ 

Since $\partial(\omega_s)$ is a second order operator with constant coefficients, we obtain a contradiction as above.

This achieves the proof of the Theorem. □

6 Application to $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$

On $G = GL(4, \mathbb{R})$ and its Lie algebra $\mathfrak{g} = \mathfrak{gl}(4, \mathbb{R})$, we consider the involution $\sigma$ defined by $\sigma(X) = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} X \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ where $I_2$ is the $2 \times 2$ identity matrix. We have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ with

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} ; A, B \in \mathfrak{gl}(2, \mathbb{R}) \right\} \quad \text{and} \quad \mathfrak{q} = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} ; Y, Z \in \mathfrak{gl}(2, \mathbb{R}) \right\}.$$

By [7] Theorem 6.3, the symmetric pair $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$ is a nice pair.

We first recall some results of [3]. Let $\kappa(X, X') = \frac{1}{2} \text{tr}(XX')$. The restriction of $\kappa$ to the derived algebra of $\mathfrak{q}$ is a multiple of the Killing form. Let $S(\mathfrak{q}_C)^{HC}$ be subalgebra of $S(\mathfrak{q}_C)$ of all elements invariant under $H_C$. We identify $S(\mathfrak{q}_C)^{HC}$ with the algebra of $H_C$-invariant differential operators on $\mathfrak{q}_C$ with constant coefficients. Using $\kappa$, we identify $S(\mathfrak{q}_C)^{HC}$ with the algebra $\mathbb{C}[\mathfrak{q}_C]^{HC}$ of $H_C$-invariant polynomials on $\mathfrak{q}_C$. A basis of $\mathbb{C}[\mathfrak{q}_C]^{HC}$ is given by $Q(X) = \frac{1}{2} \text{tr}(X^2)$ and $S(X) = \det(X)$. The Casimir polynomial is just a multiple of $Q$.

By [3] Lemma 1.3.1, the $H$-orbit of a semisimple element $X = \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix}$ of $\mathfrak{q}$ is characterized by $(Q(X), S(X))$ or by the set $\{\nu_1(X), \nu_2(X)\}$ of eigenvalues of $YZ$, where the functions $\nu_1$ and $\nu_2$ are defined as follows: let $Y$ be the Heaviside function. Let $S_0 = Q^2 - 4S$ and $\delta = t^{Y(-S_0)} \sqrt{|S_0|}$. We set

$$\nu_1 = (Q + \delta)/2 \quad \text{and} \quad \nu_2 = (Q - \delta)/2.$$

Regular elements of $\mathfrak{q}$ are semisimple elements with 2 by 2 distinct eigenvalues or equivalently, semisimple elements $X$ of $\mathfrak{q}$ such that $\nu_1(X)^2 - \nu_2(X)^2 = 0$ [3 Remarque 1.3.1].

Let $\chi$ be the character of $\mathbb{C}[\mathfrak{q}_C]^{HC}$ defined by $\chi(Q) = \lambda_1 + \lambda_2$ and $\chi(S) = \lambda_1 \lambda_2$ where $\lambda_1$ and $\lambda_2$ are two complex numbers satisfying $\lambda_1 \lambda_2 (\lambda_1 - \lambda_2) \neq 0$.

For an open $H$-invariant subset $\mathcal{V}$ in $\mathfrak{q}$, we denote by $\mathcal{D}'(\mathcal{V})^H$ the set of $H$-invariant distributions $T$ with support in $\mathcal{V}$ such that $\partial(P(T) = \chi(P))T$ for all $P \in \mathbb{C}[\mathfrak{q}_C]^{HC}$. Let $\mathcal{N}$ be the set of nilpotent elements of $\mathfrak{q}$ and $\mathcal{U} = \mathfrak{q} - \mathcal{N}$ its complement. In [3], we describe a basis of
the subspace of $\mathcal{D}'(\mathcal{U})^H_x$ consisting of locally integrable functions. More precisely, we obtain the following result.

We consider the Bessel operator $L_c = 4 \left( z \frac{d^2}{dz^2} + \frac{d}{dz} \right)$ on $\mathbb{C}$ and its analogous $L = 4 \left( t \frac{d^2}{dt^2} + \frac{4}{dt} \right)$ on $\mathbb{R}$. Let $\text{Sol}(L_c, \lambda)$ (resp., $\text{Sol}(L, \lambda)$) be the set of holomorphic (resp., real analytic) functions $f$ on $\mathbb{C} - \mathbb{R}$ (resp., $\mathbb{R}^+$) such that $L_c f = \lambda f$ (resp., $L f = \lambda f$). For $\lambda \in \mathbb{C}^*$, we set

$$\Phi_\lambda(z) = \sum_{n \geq 0} \frac{(\lambda z)^n}{4^n (n!)^2} \quad \text{and} \quad w_\lambda(z) = \sum_{n \geq 0} \frac{a(n)(\lambda z)^n}{4^n (n!)^2},$$

where $a(x) = -2\Gamma(x+1) \frac{\Gamma(x+1)}{1(x+1)}$. Then $(\Phi_\lambda, W_\lambda = w_\lambda + \log(\cdot) \Phi_\lambda)$ form a basis of $\text{Sol}(L_c, \lambda)$, where log is the principal determination of the logarithm function on $\mathbb{C} - \mathbb{R}_-$ and $(\Phi_\lambda, W_\lambda = w_\lambda + \log(\cdot) \Phi_\lambda)$ form a basis of $\text{Sol}(L, \lambda)$.

For two functions $f$ and $g$ defined over $\mathbb{C}$, we set

$$S^+(f, g)(X) = f(\nu_1(X))g(\nu_2(X)) + f(\nu_2(X))g(\nu_1(X))$$

and

$$[f, g](X) = f(\nu_1(X))g(\nu_2(X)) - f(\nu_2(X))g(\nu_1(X)).$$

We define the following functions on $q^{reg}$:

1. 

$$F_{ana} = \frac{[\Phi_{\lambda_1}, \Phi_{\lambda_2}]}{\nu_1 - \nu_2}$$

2. 

$$F_{sing} = \frac{[\Phi_{\lambda_1}, W_{\lambda_2}] + [W_{\lambda_1}, \Phi_{\lambda_2}] + \log |\nu_1 \nu_2|[\Phi_{\lambda_1}, \Phi_{\lambda_2}]}{\nu_1 - \nu_2}$$

3. For $(A, B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}), (\Phi_{\lambda_1}, W_{\lambda_2}^r), (W_{\lambda_1}^r, \Phi_{\lambda_2}), (W_{\lambda_1}^r, W_{\lambda_2}^r)\}$, we set

$$F_{A,B}^+ = Y(S_0) \frac{S^+(A, B)}{|\nu_1 - \nu_2|}$$

where $S_0 = Q^2 - 4S \in \mathbb{C}[q]\mathcal{U}^H_x$ and $Y$ is the Heaviside function.

**Theorem 6.1.** (33 Theorem 5.2.2 and Corollary 5.3.1).

1. The functions $F_{ana}$ and $F_{sing}$ are locally integrable on $q$.

2. For $(A, B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}), (\Phi_{\lambda_1}, W_{\lambda_2}^r), (W_{\lambda_1}^r, \Phi_{\lambda_2}), (W_{\lambda_1}^r, W_{\lambda_2}^r)\}$, the functions $F_{A,B}^+$ are locally integrable on $\mathcal{U}$.

3. The family $F_{ana}, F_{sing}$ and $F_{A,B}^+$, with $(A, B)$ as above form a basis $\mathcal{B}$ of the subspace of $\mathcal{D}'(\mathcal{U})^H_x$ consisting of distributions given by a locally integrable function.

**Corollary 6.2.** Any invariant distribution of $\mathcal{D}'(\mathcal{U})^H_x$ is given by a locally integrable function on $\mathcal{U}$. In particular, the family $\mathcal{B}$ defined in the previous Theorem is a basis of $\mathcal{D}'(\mathcal{U})^H_x$. 

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Proof. Let \( T \in \mathcal{D}'(\mathcal{U}_\lambda^H) \). We denote by \( F \) its restriction to \( \mathcal{U}^{reg} \). By \((\mathbb{F})\) Theorem 5.3 (i), \( F \) is an analytic function on \( \mathcal{U}^{reg} \) satisfying \( \partial(P)F = \chi(P)F \) on \( \mathcal{U}^{reg} \) for all \( P \in \mathbb{C}[q]^{H^c} \).

In \((\mathbb{B})\) section 4., we describe the analytic solutions of (*) in terms of \( \Phi_\lambda, W_\lambda \) and \( W_\lambda^\ast \) for \( \lambda = \lambda_1 \) and \( \lambda_2 \). By the asymptotic behaviour of orbital integrals near non-zero semisimple elements \((\mathbb{B})\) Theorems 3.3.1 and 3.4.1, and the Weyl integration formula \((\mathbb{B})\) Lemma 3.1.2), one deduces that \( F \in L^1_{loc}(\mathcal{U}_\lambda^H) \). Theorem 5.1 gives the result. \( \square \)

**Corollary 6.3.** Any invariant distribution of \( \mathcal{D}'(q)^H \) is given by a locally integrable function on \( q \).

**Proof.** Let \( T \in \mathcal{D}'(q)^H \). By Corollary 5.2 the restriction of \( T \) to \( \mathcal{U} \) is a linear combination of elements of \( B \). By Theorem 5.1 and Theorem 6.1, it is enough to prove that the functions \( F_{\lambda,A,B}^+ \), with \( (A,B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}), (\Phi_{\lambda_1}, W_{\lambda_2}), (W_{\lambda_1}, \Phi_{\lambda_2}), (W_{\lambda_1}, W_{\lambda_2})\} \) are locally integrable on \( q \) or equivalently, that the integral \( \int_q |F_{\lambda,A,B}^+(X)f(X)|dX \) is finite for all positive function \( f \in \mathcal{D}(q) \).

For this, we will use the Weyl integration formula \((\mathbb{B})\) Proposition 1.8 and Theorem 1.27).

For \( \varepsilon = (\varepsilon_1, \varepsilon_2) \) with \( \varepsilon_j = \pm \), we define

\[
\alpha_\varepsilon = \left\{ X_\varepsilon(u_1, u_2) = \begin{pmatrix} 0 & u_1 & 0 \\ \varepsilon_1 u_1 & 0 & u_2 \\ 0 & \varepsilon_2 u_2 & 0 \end{pmatrix} ; (u_1, u_2) \in \mathbb{R}^2 \right\}.
\]

and

\[
\alpha_2 = \left\{ \begin{pmatrix} 0 & \tau - \theta & \tau \\ \tau & \theta & \tau \\ \tau & -\theta & \tau \end{pmatrix} ; (\theta, \tau) \in \mathbb{R}^2 \right\}
\]

By \((\mathbb{B})\), Lemma 1.2.1), the subspaces \( \alpha_{++}, \alpha_{+-}, \alpha_{-+} \) and \( \alpha_2 \) form a system of representatives of \( H \)-conjugaison classes of Cartan subspaces in \( q \). By \((\mathbb{B})\) Remark 1.3.1), an element \( X \in q \) satisfies \( S_0(X) \geq 0 \) if and only if \( X \) is \( H \)-conjugate to an element of \( \alpha_\varepsilon \) for some \( \varepsilon \). Furthermore, one has \( \{\nu_1(X_\varepsilon(u_1, u_2)), \nu_2(X_\varepsilon(u_1, u_2))\} = \{\varepsilon_1 u_1^2, \varepsilon_2 u_2^2\} \).

Let \( f \) be a positive function in \( \mathcal{D}(q) \). We define the orbital integral of \( f \) on \( q^{reg} \) by

\[
\mathcal{M}(f)(X) = |\nu_1(X) - \nu_2(X)| \int_{H/Z_H(X)} f(h.X)dX
\]

where \( Z_H(X) \) is the centralizer of \( X \) in \( H \) and \( dh \) is an invariant measure on \( H/Z_H(X) \).

By \((\mathbb{B})\) Theorem 1.23), the orbital integral \( \mathcal{M}(f) \) is a smooth function on \( q^{reg} \) and there exists a compact subset \( \Omega \) of \( q \) such that \( \mathcal{M}(f)(X) = 0 \) for all regular element \( X \) in the complement of \( \Omega \).

Since \( F_{\lambda,A,B}^+ \) is zero on \( \alpha^+_2 \), one deduces from the Weyl integration formula that there exist positive constants \( C_\varepsilon \) (only depending of the choice of measures), such that one has...
\[ \int_\mathbb{H} F^+_u(X) f(X) dX = \sum_{\epsilon \in \{(\pm1,\pm1,\pm1)\}} C_\epsilon \int_\mathbb{R^2} F^+_u(X\varepsilon(u_1, u_2)) \times \mathcal{M}(f)(X\varepsilon(u_1, u_2)) u_1 u_2 (\varepsilon_1 u_1^2 - \varepsilon_2 u_2^2) |du_1 du_2. \]

By definition of \( F^+_u \), there exist positive constants \( C, C_1 \) and \( C_2 \) such that, for all \( X\varepsilon(u_1, u_2) \in \Omega^{reg} \), one has

\[ |(\varepsilon_1 u_1^2 - \varepsilon_2 u_2^2) F^+_u(X\varepsilon(u_1, u_2))| \leq C(C_1 + |\log |u_1|)|(C_2 + |\log |u_2||). \]

One deduces easily the corollary from the following Lemma.

**Lemma 6.4.** Let \( f \in \mathcal{D}(q) \). Then there exist positive constants \( C', C'_1, C'_2 \) such that, for all \( X\varepsilon(u_1, u_2) \in q^{reg} \) one has

\[ |\mathcal{M}(f)(X\varepsilon(u_1, u_2))| \leq C'(C'_1 + |\log |u_1|)|(C'_2 + |\log |u_2||). \]

**Proof.** Let \( H = K N A \) be the Iwasawa decomposition of \( H \) with \( K = O(2) \times O(2) \), \( N = N_0 \times N_0 \) where \( N_0 \) consists of \( 2 \) by \( 2 \) unipotent upper triangular matrices and \( A \) is the set of diagonal matrices in \( H \). It is easy to see that the centralizer of \( X \) in \( H \) is the set of diagonal matrices \( diag((\alpha, \beta, \alpha, \beta)) \) with \( (\alpha, \beta) \in (\mathbb{R}^*)^2 \). Hence \( H/Z_H(X) \) is isomorphic to \( K \times N \times \{diag(e^x, e^y, 1, 1); x, y \in \mathbb{R}\} \).

For \( \xi \in \mathbb{R} \), we set \( n_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \). We define the function \( \tilde{f} \) by \( \tilde{f}(X) = \int_K f(k \cdot X) dk \). Then, one has

\[ \mathcal{M}(f)(X\varepsilon(u_1, u_2)) = |\varepsilon_1 u_1^2 - \varepsilon_2 u_2^2| \int_\mathbb{R^2} \left( \int_\mathbb{R^2} \tilde{f}(Y(u, \varepsilon, x, y, \xi, \eta)) d\xi d\eta \right) dxdy \]

with

\[ Y(u, \varepsilon, x, y, \xi, \eta) = \begin{pmatrix} n_\xi & 0 \\ 0 & n_\eta \end{pmatrix} \cdot diag(e^x, e^y, 1, 1) \cdot X\varepsilon,u. \]

Writing \( Y(u, \varepsilon, x, y, \xi, \eta) = \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \), one has

\[ Y = \begin{pmatrix} u_1 e^x & -\eta u_1 e^x + e^y \xi u_2 \\ 0 & u_2 e^y \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} \varepsilon_1 u_1 e^{-x} & -\xi \varepsilon_1 u_1 e^{-x} + \eta \varepsilon_2 u_2 e^{-y} \\ 0 & \varepsilon_2 u_2 e^{-y} \end{pmatrix}. \]

Since \( f \in \mathcal{D}(q) \), the function \( \tilde{f} \) has compact support in \( q \). Identify \( q \) with \( \mathbb{R}^8 \), there exists \( T > 0 \) such that supp(\( \tilde{f} \)) \( \subset [-T, T]^8 \). If \( \tilde{f}(Y(u, \varepsilon, x, y, \xi, \eta)) \neq 0 \) then we have the following inequalities:

1. \( |u_1 e^{\pm x}| \leq T \) and \( |u_2 e^{\pm y}| \leq T \),
2. \( | -\eta u_1 e^x + e^y \xi u_2| \leq T \),
3. \( | -\xi \varepsilon_1 u_1 e^{-x} + \eta \varepsilon_2 u_2 e^{-y}| \leq T \).
Changing the variables $(\xi, \eta)$ in $(r, s) = (\xi u_2 e^y - \eta u_1 e^x, -\xi \xi_1 u_2 e^{-x} + \eta \xi_2 u_2 e^{-y})$, we obtain the result.

**Remark.** By (3) Corollary 5.3.1, the function $F_{\text{ana}}$ defines an invariant eigendistribution on $\mathfrak{q}$. At this stage, we don’t know if it is the case for the functions $F_{\text{sing}}$ and $F^{+}_{A,B}$. Indeed, the proof of Theorem 6.1 of (3) is based on integration by parts using estimates of orbital integrals and some of their derivatives near non-zero semisimple elements of $\mathfrak{q}$. To determine if $F_{\text{sing}}$ and $F^{+}_{A,B}$ are eigendistributions using the same method, we have to know the behavior of derivatives of orbital integrals near 0.

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