Anyonic Liquids in Nearly Saturated Spin Chains

Armin Rahmani,1 Adrian E. Feiguin,2 and Cristian D. Batista1

1 Theoretical Division, T-4 and CNLS, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA
2 Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA

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Most spin chains in a magnetic field flow to a universal free-fermion fixed point near saturation. Here we show that an exotic fixed point, characterized by two species of low-energy excitations with mutual anyonic statistics and no inter-species interactions, may also emerge in such spin chains if the dispersion relation has two minima. By using bosonization, two-magnon exact calculations, and numerical density-matrix-renormalization-group, we demonstrate the existence of this anyonic liquid fixed point in an XXZ spin chain with up to second neighbor interactions, and identify a range of microscopic parameters, which support this phase.

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Magnetic-field induced saturation of quantum magnets is one of the most widely studied quantum critical points (QCP) of nature: magnets with axial symmetry along the field axis become fully polarized at a critical field value. In two and three spatial dimensions, the corresponding QCP (between fully and partially polarized states) belongs to the “Bose-Einstein condensate” (BEC) universality class:1–5 the magnets can be mapped onto a dilute gas of bosons. In contrast, in most models studied thus far, the weakly interacting quasiparticles at the QCP of one-dimensional systems have fermionic statistics.6 Here we demonstrate that a much richer spectrum of QCPs, including novel anyonic liquids, may emerge in nearly saturated axially symmetric spin chains.

The essential ingredient for our results is magnetic frustration, which can provide natural realizations of single-particle dispersions with degenerate minima at multiple wave vectors Q>1, such single-particle dispersions do not change the universality class of the BEC QCP, but they can give rise to multi-Q condensates,9–11 such as long-range ordered magnetic vortex crystals.12–13 In contrast, long-range order is forbidden in d = 1 due to strong quantum fluctuations.14 However, the fate of the QCP is easy to understand if the dispersion has a single minimum: the Jordan-Wigner transformation allows us to describe the magnet as a dilute gas of interacting fermions near the fully polarized state. The Pauli exclusion principle then renders all fermion-fermion interactions irrelevant (in a renormalization-group sense), resulting in a free-fermion fixed point.8 The central question addressed in this paper is the fate of the QCP in d = 1 when the dispersion relation has two minima.

We show that magnetic frustration can lead to a novel anyonic-liquid universality class in such one-dimensional systems. This extends the classification of QCPs for saturated quantum magnets from simple theories of free bosons (d > 1) and free fermions (d = 1) to an exotic line of QCPs that continuously interpolate between these two fixed points, where the elementary excitations have emergent Abelian anyonic statistics. There are two species of quasiparticles (originating from the two degenerate minima) in our anyonic-liquid universality class. There is no inter-species interactions between quasiparticles. However, these quasiparticles do not have simple bosonic or fermionic commutation relations: they are Abelian anyons. In fact, similar theories of d = 1 Abelian anyons have been envisioned in the field-theory literature through abstract flux attachment to free bosonic theories.18–26 However, no experimentally relevant microscopic models have been shown to support such anyonic liquids. By combining different techniques, including bosonization, renormalization-group arguments and numerical density-matrix renormalization group (DMRG) computations34,35, we provide an experimentally relevant realization for these elusive exotic liquids in the context of frustrated magnetism. Moreover, we propose experimental signatures of these anyonic liquids, which could facilitate their observation.

We focus on an S=1/2 XXZ chain with second-neighbor interactions, which is the simplest extension of the nearest-neighbor model, with doubly degenerate minima in the single-particle dispersion. The model has been studied in several limits with various numerical and analytical techniques. Several phases, such as two-component Luttinger liquids (LL) and vector chiral states, have been found in its phase diagram27–33. The corresponding XXZ Hamiltonian is illustrated in Fig. (1a) and it is given by:

$$H = \sum_{j,a=1,2} \left[ J_a \left( S_j^a S_{j+a}^a + S_j^a S_{j+a}^a S_j^a S_{j+a}^a \right) + \Delta_a J_a \left( S_j^a S_{j+a}^a - \frac{1}{4} \right) \right],$$

(1)

with up to second neighbor interactions (for a possible physi-
cal realization in a zigzag ladder, see Refs. \[36, 37\]). A magnetic field \(B_z\), which couples to \(S_j^z = \sum_j S_j^z\), can be used to tune \(S_j\) (\(S_j^z\) is conserved because \([H, S_j^z] = 0\)).

By applying the Jordan-Wigner transformation \(S_j^z = c_j \exp(-i\sum_{k<j} n_k)\) and \(S_j^z = n_j - \frac{1}{2}\) with \(n_j = c_j^\dagger c_j\), we can reinterpret \(H = H_0 + H_1\) as the Hamiltonian for an interacting chain of spinless fermions:

\[
H_0 = \sum_{x,\alpha=r,l,1,2} \left( \frac{\hbar^2}{2m} c_x^\dagger c_x + \text{H.c.} \right) = \sum_{k} \epsilon(k) c_k^\dagger c_k, \\
H_1 = \sum_{x,\alpha=r,l,1,2} \left( \Delta_a J_{\alpha} n_{x+\alpha} - J_2 \sum_{\alpha} \left( c_{x+1}^\dagger n_{x+2} + \text{H.c.} \right) \right),
\]

where the coefficients \(g\) represent the effective interactions at the fixed point, where the renormalization-group flow stops. (We have derived the bare coupling constants, which provide a good approximation to the fixed-point values for weak interactions, in terms of the microscopic parameters of the XXZ chain in the Supplemental Material \[38\].)

To write the bosonized Hamiltonian in the dilute limit, we first define the fields

\[
\varphi(x) = \frac{1}{2} \left[ \phi_1(x) + \phi_2(x) \right], \quad \bar{\varphi}(x) = \frac{1}{2} \left[ \phi_1(x) - \phi_2(x) \right],
\]

and their conjugate momenta \(\Pi(x) = -\frac{\hbar}{\pi v} \left[ \partial_x \phi_1(x) - \partial_x \phi_2(x) \right]\) and \(\bar{\Pi}(x) = \frac{\hbar}{\pi} \left[ \partial_x \phi_1(x) - \partial_x \phi_2(x) \right]\). Physically, \(\Pi(x)\) and \(\bar{\Pi}(x)\) are proportional to current operators from fermions in the vicinity of respectively the right and left minima [see Fig. [b]]. Similarly, \(\partial_x \varphi(x)\) and \(\partial_x \bar{\varphi}(x)\) are proportional to densities near these minima.

In the dilute limit, we have \(v_1 \approx v_2 = v\). Also \(g_{12}\) and \(g_{12}'\) are irrelevant for precisely the same reason as for the single-minimum case: the Pauli exclusion principle forbids interactions like \(\psi_1^\dagger \psi_1 \bar{\psi}_2 \bar{\psi}_2\), so the most relevant interactions must have two derivatives, \(\psi_1^\dagger \partial_\phi^2 \partial_\phi^2 \psi_2 \bar{\psi}_2\), making them irrelevant perturbations to the free-fermion fixed point (see Ref. [6]).

Moreover, the spatial derivative that appears in the fermionic currents \(i \left( \psi_1^\dagger \partial_\phi \psi_1 - \partial_\phi \bar{\psi}_1 \bar{\psi}_2 \right)\) makes the coefficient of \(\Pi(x)\bar{\Pi}(x)\) irrelevant (the terms proportional to \(\Pi^2\) and \(\bar{\Pi}^2\) are, however, relevant as the fermionic anticommutation relations yield relevant terms of type \(\partial_\phi^2 \partial_\phi^2 \bar{\psi}_1 \bar{\psi}_1\)). In addition, inversion symmetry requires \(g_{12}' = g_{12}\). The general form of the Hamiltonian in the dilute limit is then given by

\[
H = \left( \frac{1}{2\pi} \right)^2 \int dx \left[ 2\pi v \left( \partial_x \varphi \right)^2 + \left( \partial_x \bar{\varphi} \right)^2 \right] + 2\pi v^2 \left( \Pi^2 + \bar{\Pi}^2 \right) + g_\varphi \left( \partial_x \varphi \Pi - \bar{\partial}_\phi \bar{\Pi} \right) + g_\varphi \partial_x \varphi \bar{\partial}_\phi \bar{\Pi} + \text{g}_2 \cos \left( 2(\bar{\varphi} - \varphi) \right) \right],
\]

where \(g_\varphi \equiv g_{11} - g_{22}, g_\varphi' \equiv g_{11} + 2g_{12} + g_{22}\) and the explicit dependence of the fields on \(x\) is suppressed.

If the term proportional to \(g_\varphi\) becomes relevant, it can open a gap and destroy criticality. However, if it is irrelevant (to be checked a posteriori), we have a quantum liquid. Now, if \(g_\varphi'\) also flows to zero for a certain range of microscopic parameters, we can write the Hamiltonian as

\[
H = \frac{u}{2\pi} \int dx \sum_{\alpha=r,l,1,2} \left[ \frac{1}{K} \left( \partial_x \varphi_\alpha \right)^2 + K \left( \pi \Pi_{\alpha} \right)^2 \right],
\]

where the new fields (which decouple the Hamiltonian above) are related to the old ones through the following anyonic gauge transformation:

\[
\varphi_+ \equiv \varphi, \quad \Pi_+ = \Pi - \frac{\alpha}{\pi^2} \bar{\partial}_\phi \bar{\varphi}, \quad \varphi_- \equiv \bar{\varphi}, \quad \Pi_- = \bar{\Pi} + \frac{\alpha}{\pi^2} \partial_\phi \varphi,
\]

with \(\alpha \equiv \frac{\bar{\varphi}}{\varphi}, \quad K = 1/\sqrt{1 - (\bar{\varphi}/\varphi)^2}\), and \(u = v/K\). Notice that as in flux-attachment arguments, the momentum of one
species is shifted by a gauge field times the density of the other species \[18\]. The anyonic nature of the new quasiparticles corresponds to a generalized Jordan-Wigner transformation (discussed below) and can be inferred from the commutation relations given below Eq. (4) \[18\]. It is easy to check that the scaling dimension of \[2\varphi - \varphi\] is \(2K\) for the anyonic liquid and \(g_e\) indeed flows to zero.

In fact, Hamiltonian (3) is a direct generalization of the Shastry-Schulz model of noninteracting anyons \[18\], which can be understood in terms of simple flux attachment, with \(\alpha\) representing the mutual statistical phase for exchanging the two types of particles. In our system, just like the Shastry-Schulz model, there is no coupling between the two anyonic species (there is a unique statistics of quasiparticles for which the theory breaks into two decoupled sectors). The Shastry-Schulz model, however, corresponds to the special case of \(K = 1\), indicating no intra-species interactions. The \(\alpha\)-dependent \(K\) in our model results in a continuous interpolation from free bosons (a LL with Luttinger parameter \(K = 1\)). For such noninteracting LL, the two-particle state \(|c_j^\dagger c_k^\dagger 0\rangle\) is an exact eigenstate of the Hamiltonian. As soon as \(K\) moves away from unity, this state scatters into other two-particle states and will not remain an eigenstate. Thus, if the effective Hamiltonian has the general Luttinger-liquid form and \(c_k^\dagger c_k^\dagger 0\rangle\) is an exact eigenstate of the microscopic Hamiltonian, the Luttinger parameter must be equal to unity (fermion fixed point). Similarly, we require that a two-anyon state is an exact eigenstate of the Hamiltonian (1).

Going back to Eq. (1), we perform a generalized Jordan-Wigner transformation to anyons with statistical phase \(\phi\) and annihilation operator \(a_j\) on site \(j\): \(S_j^- = a_j e^{i\phi} \sum_i \delta_j i a_i\) and \(S_j^z = n_j - \frac{1}{2}\) with \(n_j = a_j^\dagger a_j\). The anyonic statistics of these particles can be observed in the relationship \(a_j^\dagger a_k^\dagger e^{-i\phi} a_k a_j\) for \(j < l\) (see Ref. \[37\]) for the physical interpretation of anyons in terms of spins). In the dilute limit, the possible momenta are \(\pm Q\). We would like to find a relationship between the microscopic parameters so that the two-particle state \(a_j^\dagger a_k^\dagger 0\rangle\), with \(\tilde{Q} \equiv -Q\), is an exact eigenstate of Eq. (1). The Hamiltonian has the same form as Eqs. (2) and (3) in terms of anyonic operators (with \(e\) replaced by \(a\)), except for the correlated hopping term (the term in \(H_j\) proportional \(J_2\)), which now reads \(\frac{J_2}{2} \sum_{j} n_{j+1} \left( e^{i\phi} - 1 \right) a_j^\dagger a_{j+2}^\dagger + (e^{-i\phi} - 1) a_j a_{j+2}\). Requiring \(H a_j^\dagger a_k^\dagger 0\rangle = e^{-i\phi} a_j^\dagger a_k^\dagger 0\rangle\) leads to the following relationship between the microscopic parameters:

\[
\Delta_1 = \cos(Q) + \frac{\sin(Q)}{2} \left[ \tan(Q) + \tan(Q + \phi/2) \right], \tag{9}
\]
\[
\Delta_2 = \cos(2Q) + \sin(2Q) \tan(2Q + \phi/2), \tag{10}
\]

with the energy given by \(\varepsilon = -2(\Delta_1 J_1 + \Delta_2 J_2) + 2J_1 \cos(Q) + 2J_2 \cos(2Q)\). Because there is only one anyon of each species in \(a_j^\dagger a_k^\dagger 0\rangle\), the intra-species interactions characterized by the parameter \(K\), play no role in the above argument.

If the effective theory of the system is given by Eq. (7), the above values of \(\Delta_1\) and \(\Delta_2\) guarantee that there is no scattering between the two anyonic species and the effective Hamiltonian must reduce to Eq. (8) with \(\alpha = \pi - \phi\). In other words, we have a family of Hamiltonians characterized by two parameters \(Q\) and \(\phi\), which can potentially flow to the anyonic-liquid fixed point (8). However, formation of low-energy bound states may lead to either a first-order phase transition from the saturated state (the number of particles changes discontinuously at the the saturation field) or a continuous transition into a state with dominant nematic (BEC of pairs) or higher-order multipolar fluctuations. As discussed below, by using exact two-magnon calculations, we found the range of parameters that give rise to low-energy bound states, destabilizing the anyonic liquid, and obtained the phase diagram of Fig. 2.

The bound states are most easily analyzed in the original spin representation (1). We denote the fully saturated state \(|\uparrow \uparrow \uparrow \cdots\rangle\) by \(0\rangle\). The states \(|\phi_{ij}\rangle = S_j^z S_i^z 0\rangle\) with \(i < j\) provide a basis for the the two-magnon subspace. Translation invariance implies that an arbitrary two-magnon eigenstate has a well defined center-of-mass momentum \(q\): \(|\Psi\rangle = \sum_{i,j} e^{iqr_{ij}} \phi_{ij}\rangle\) for \(r_{ij} = (r_i + r_j)/2\) and \(r_{ij} = r_j - r_i\). The eigenvalue equation \(H(\phi) = \varepsilon(\phi)\) then gives coupled equations for \(u(r)\) \[38\]. For \(r > 2\), the equation for \(u(r)\) generally admits plane-wave scattering solutions giving rise to a continuous spectrum, but it is also possible to obtain bound states, which correspond either to a single exponentially decaying solution \(u(r) = e^{-\gamma r}\) for \(r > 2\) (for a real positive \(\gamma\)) or a linear combination of two such solutions \(u(r) = e^{-\gamma r} + e^{\gamma r} e^{-r}\) with \(Re(\kappa) > 0\) for \(r > 1\) (where \(\theta\) is a phase shift) \[39\]. If such solutions exist for some center-of-mass momentum \(q\), and the corresponding energy is below the two-particle continuum, the system will form low-energy bound states in the two-magnon sector and it is vulnerable to phase separation.

Returning to the anyonic liquid, we now present analytical predictions for the correlation functions of the system, which we also numerically verify with the DMRG method. For the fermionic Green’s function \(G(x) = \langle c_i^\dagger c_{i+x} \rangle\), we find (to lead-
numerical results (fit). Fitting to Eq. (11) gives $Q_1/\pi = 0.16$, $Q_2/\pi = 0.21$ and an exponent 0.108 in excellent agreement with analytical predictions $Q_1/\pi = 0.17$, $Q_2/\pi = 0.22$ and an exponent 0.108.

FIG. 3: The fermionic Green’s function for $\phi/\pi = 0.615$ and $Q/\pi = 0.2$ at density $\rho_0 = 0.05$. The black circles (blue line) represent numerical results (fit). Fitting to Eq. (11) gives $Q_1/\pi = 0.16$, $Q_2/\pi = 0.21$ and an exponent 0.108 in excellent agreement with analytical predictions $Q_1/\pi = 0.17$, $Q_2/\pi = 0.22$ and an exponent 0.108.


g(x) \propto [\sin (Q_1 x + \omega_1) - \sin (Q_2 x + \omega_2)] x^{-1/\sqrt{1-x^2}}, \quad (11)

with $\lambda = \alpha/\pi$ in the dilute limit. In general, the ordering vectors change at finite densities (because of the string operator that relates fermions to anyons), but the change is negligible in the limit of small density considered here [18]. Moreover, the ordering vectors $Q_i$ have an uncertainty of order $1/Q$ in a finite system of length $L$. We therefore compare the above prediction with the numerical results by fitting the numerically computed correlation function to expression (11), with the ordering vectors, the overall coefficient, and the exponent as fitting parameters (using the fact that the exponents are relatively close to 1 we neglect the phase shifts in the oscillatory prefactor $\cos \pi x$ in fitting the data). An exponent close to $-1/\sqrt{1-x^2}$ and ordering vectors close to the computed (for the given density of fermions) $Q_1$ and $Q_2$ would corroborate our analytical prediction for an anyonic liquid.

We performed the DMRG calculations for a chain of length $L = 400$ with periodic boundary conditions (implemented by constructing a system of two chains with length $L/2$ connected at the endpoint [40]). We also compared the results with a calculation for $L = 200$ and chose the range of $x$ where the two data sets overlap. An excellent convergence was obtained by keeping 1000 states in the DMRG iterations. As seen in Fig. 3 the exponent of the correlation function differs from $\delta = 1$ (free fermion fixed point) and is consistent with the exponents of an anyonic liquid. The ordering momenta are also very close to our analytical predictions (note that the agreement cannot be perfect because of the finite density $\rho_0 = 0.05$ that is needed for the numerical calculations).

Fitting the above expression with DMGRG. The bosonic correlators have a stronger finite-size dependence so in fitting the data we replaced $x$ in $x^{-1/\sqrt{1-x^2}}$ with its finite-size counterpart $\bar{x} = x + \epsilon x^2$. The agreement is excellent as shown in Fig. 4. The anyonic fixed point can be detected by comparing the above exponent with the exponent of the correlator that determines the longitudinal susceptibility $\chi_{zz}$. The oscillatory $|k = \pm (Q_2 - Q_1)|$ components of $(S_+^z S_0^z)$ decay as $x^{-1/\sqrt{1-x^2}}$ (the leading nonoscillatory component decays as $x^{-2}$ but extracting the oscillatory part is not too difficult).

In summary, by using multiple analytical techniques and state-of-the-art DMRG calculations we examined the effects of strong magnetic frustration in nearly saturated spin chains. We extended the classification of the saturation QCPs in quantum magnets from the standard paradigm of simple free fermionic (bosonic) theories in $d = 1$ ($d > 1$) [2] to an exotic continuous line of fixed points termed anyonic liquids, which are characterized by two species of anyonic quasiparticles with vanishing inter-species interactions. The emergent statistical phase of the quasiparticles in this novel phase continuously interpolates between bosons and fermions. Our results provide natural realizations of one-dimensional anyonic liquids in a very simple experimentally relevant model, which, while envisioned in the field-theory literature, had thus far remained as an abstract theoretical construction. Our findings also open a promising direction in the experimental search for anyons in frustrated magnets: as only one exchange parameter needs to be tuned in order to realize our anyonic liquids (apart from the magnetic field which can be easily brought to the vicinity of the critical point), physical or chemical pressure could drive generic highly frustrated one-dimensional magnetic materials into the anyonic-liquid phase. Relationships between the transverse and longitudinal magnetic susceptibilities serve as experimental signatures of this exotic phase.

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Other realizations of BEC’s, such as atomic gases or superconductors, rarely exhibit single-particle dispersions with more than one global minimum.

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Supplemental Material for “Anyonic Liquids in Nearly Saturated Spin Chains”

Armin Rahmani,1 Adrian Feiguin,2 and Cristian D. Batista1

1 Theoretical Division, T-4 and CNLS, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA
2 Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA

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I. BARE COUPLING CONSTANTS

If we neglect the variations of the slow fields over a distance of order a few lattice spacings, inserting Eq. (4) of the main text into the expression for \( H_f \) in Eq. (3) leads to the form given in Eq. (5) with the bare coupling constants below:

\[
\begin{align*}
\bar{g}_{11} &= 4\Delta_1J_1 \sin^2{(Q_1)} + 4\Delta_2J_2 \sin^2{(Q_2)} + 8J_2 \sin^2{(Q_2)}, \\
\bar{g}_e &= -4\Delta_1J_1 \sin{(Q_1)} \sin{(Q_2)} - 4\Delta_2J_2 \sin{(Q_1)} \sin{(Q_2)} - 8J_2 \sin{(Q_1)} \sin{(Q_2)}, \\
\bar{g}_{12} &= 4\Delta_1J_1 \sin^2{(Q_1)} + 4\Delta_2J_2 \sin^2{(Q_1)} + 2J_2 \{2\cos{(Q_1 + Q_2)} - \cos{(2Q_1)} - \cos{(2Q_2)}\}, \\
\bar{g}_{12} &= 4\Delta_1J_1 \sin^2{(Q_1)} + 4\Delta_2J_2 \sin^2{(Q_1)} + 2J_2 \{2\cos{(Q_1 - Q_2)} - \cos{(2Q_1)} - \cos{(2Q_2)}\}, \\
\bar{g}_{12} &= 4\Delta_1J_1 \sin^2{(Q_2)} + 4\Delta_2J_2 \sin^2{(Q_2)} + 2J_2 \{2\cos{(Q_1 + Q_2)} - \cos{(2Q_1)} - \cos{(2Q_2)}\}, \\
\bar{g}_{12} &= 4\Delta_1J_1 \sin^2{(Q_2)} + 4\Delta_2J_2 \sin^2{(Q_2)} + 2J_2 \{2\cos{(Q_1 - Q_2)} - \cos{(2Q_1)} - \cos{(2Q_2)}\}.
\end{align*}
\]

The above expressions for the coupling constants in terms of the microscopic parameters are only valid in the limit of small interactions \( |H_f| \ll |H_0| \). Generally, we can not neglect the short-distance (high-energy) physics stemming from the variations of the slow fields. Integrating them out, however, does not change the form of \( H_f \) (as it includes all allowed scattering processes); it merely renormalizes the coupling constants.

To derive the expression above, it is convenient to define \( H_{ab'c'} \equiv \int dxc_x^a c_x^a c_x^a c_x^b c_x^c c_x^d \). Now, each of the creation and annihilation operators in \( H_{ab'c'} \) can be written as a linear combination of four chiral operators as in Eq. (4) of the main text. As mentioned above, \( a, b, a', \) and \( b' \) are assumed to be of the order the lattice spacing. The chiral fields have slow variations over such distances and we have used, e.g., \( \psi_0(x + a) \approx \psi_0(x) \) and \( \psi_0(x + a) \approx \psi_0(x) \). Assuming the momenta \( Q_i \) do not take any special values that allow for Umklapp processes, all terms (out of the 4\(^2\) terms coming from expanding the product in the integrand) that have an \( x \)-dependent oscillatory factor vanish upon integration due to momentum conservation. We can then write

\[
H_{ab'c'} \approx \int dx \left[ 4 \sin{[Q_1(a - b)]} \sin{[Q_1(a' - b')]} \psi_1^1 \bar{\psi}_1 \psi_2 \right. \\
+ 4 \sin{[Q_1(a - b)]} \sin{[Q_2(a' - b')]} \psi_1^2 \bar{\psi}_2 \psi_2 \\
+ \left. e^{Q_1(a' - a)} + e^{Q_1(b' - b)} e^{-Q_1(b' - a)} + e^{Q_1(a' - b)} + e^{Q_1(b' - a)} \right] \psi_1^1 \bar{\psi}_1 \psi_2 \psi_2 \\
+ \left. e^{Q_1(a' - a)} + e^{Q_1(b' - b)} e^{-Q_1(b' - a)} + e^{Q_1(a' - b)} + e^{Q_1(b' - a)} \right] \psi_1^2 \bar{\psi}_1 \psi_2 \psi_2 \\
+ \left. e^{Q_1(a - a')} + e^{Q_1(b - b')} e^{-Q_1(b - a')} + e^{Q_1(a - b')} + e^{Q_1(b - a')} \right] \psi_1^1 \bar{\psi}_1 \psi_1 \psi_2 \\
+ \left. e^{Q_1(a - a')} + e^{Q_1(b - b')} e^{-Q_1(b - a')} + e^{Q_1(a - b')} + e^{Q_1(b - a')} \right] \psi_1^2 \bar{\psi}_2 \psi_2 \psi_2 \\
+ 4 \sin{[Q_1(a' - b')]} \sin{[Q_2(a - b)]} \psi_1^2 \bar{\psi}_1 \psi_1 \\
+ \left. 4 \sin{[Q_2(a' - b')]} \sin{[Q_2(a - b)]} \psi_1^1 \bar{\psi}_2 \psi_2 \right],
\]

where all the chiral fields are calculated at position \( x \). The dependence of \( \bar{g} \) on the microscopic parameters then readily follows from the relationship \( H_f = -\Delta_1J_1H_{0101} - \Delta_2J_2H_{2020} - J_2(H_{0112} + H_{2110}) \).

The most general Hamiltonian before invoking inversion symmetry and the irrelevance of several coupling constants in the
The eigenvalue equation can be written as

\[ H = \left( \frac{1}{2\pi} \right)^2 \int dx \left\{ (g_{12} + \pi v_1 + \pi v_2) [\partial_x \varphi(x)]^2 + (g_{12} + \pi v_1 + \pi v_2) [\partial_x \bar{\varphi}(x)]^2 \right. \]

\[ + \pi^2 (g_{12} + \pi v_1 + \pi v_2) [\Pi(x)]^2 + \pi^2 (g_{12} + \pi v_1 + \pi v_2) [\bar{\Pi}(x)]^2 \]

\[ + \pi (g_{12} + \pi v_1 - \pi v_2) \Pi(x) \partial_x \varphi(x) + \pi (g_{12} + \pi v_1 - \pi v_2) \bar{\Pi}(x) \partial_x \bar{\varphi}(x) \]

\[ - \pi (g_{12} + \pi v_1 - \pi v_2) \bar{\Pi}(x) \partial_x \varphi(x) - \pi (g_{12} + \pi v_1 - \pi v_2) \bar{\Pi}(x) \partial_x \bar{\varphi}(x) \]

\[ + (g_{11} + g_{12} + g_{12} + g_{22}) [\partial_x \varphi(x)] [\partial_x \bar{\varphi}(x)] \]

\[ + \pi^2 (g_{11} + g_{12} + g_{12} - g_{22}) \Pi(x) \bar{\Pi}(x) \]

\[ + \pi (g_{11} + g_{12} - g_{12} - g_{22}) \partial_x \varphi(x) \bar{\Pi}(x) \]

\[ + \pi (g_{11} + g_{12} - g_{12} + g_{22}) \partial_x \bar{\varphi}(x) \Pi(x) \]

\[ + 2g \cos \left( \tilde{\varphi}(x) - \varphi(x) \right) \right\} \]

(9)

Notice that the bare values of the coupling constants, which are irrelevant in the dilute limit, vanish as \((Q_1 - Q_2)^2\).

II. MAGNON BOUND STATES

We denote the vacuum \(|\uparrow\uparrow\uparrow\ldots\rangle\) by \(\mid 0 \rangle\) and represent the two-magnon states as

\[ |\phi_{i,j} \rangle = S_i S_j |0 \rangle, \quad i < j. \]

(10)

Due to translation invariance of the Hamiltonian, two-magnon eigenstates have a well-defined center-of-mass momentum \(q\):

\[ |\psi \rangle = \sum_{i,j \geq 1} e^{iq_R \cdot u(r_{i,j})} |\phi_{i,j} \rangle, \quad R_{i,j} = (r_i + r_j)/2, \quad r_{i,j} = r_j - r_i. \]

(11)

The eigenvalue equation \(H|\psi \rangle = \epsilon|\psi \rangle\) in this sector [with \(H\) given by Eq. (1) of the main text] then reduces to

\[ \left( \epsilon + J_1 + 2J_2 \right) u(1) = J_2 \cos (q) u(1) + J_1 \cos \left( \frac{q}{2} \right) u(2) + J_2 \cos (q) u(3), \]

(12)

\[ \left( \epsilon + 2J_1 + J_2 \right) u(2) = J_1 \cos \left( \frac{q}{2} \right) u(1) + J_1 \cos \left( \frac{q}{2} \right) u(3) + J_2 \cos (q) u(4), \]

(13)

\[ \left( \epsilon + 2J_1 + J_2 \right) u(r) = J_1 \cos \left( \frac{q}{2} \right) u(r-1) + u(r+1) \]

\[ + J_2 \cos (q) [u(r-2) + u(r+2)], \quad r > 2. \]

(14)

The last relationship above [Eq. (14)] (in the bulk) has exponential solutions \(u(r) = e^{-\kappa r}\), where \(z = e^{-\kappa}\), for a \(\kappa\) on the complex plane, satisfies the characteristic polynomial equation \(\epsilon + 2J_1 + 2J_2 = J_1 \cos \left( \frac{q}{2} \right) \left( z + \frac{1}{z} \right) + J_2 \cos (q) \left( z^2 + \frac{1}{z^2} \right)\). For any energy, there are four solutions for \(z\) but solutions with \(\text{Re}(\kappa) < 0\) \((|z| > 1)\) are unphysical as they cannot be normalized. Wave functions with \(|z| = 1\) are extended scattering states, while wave functions with \(|z| < 1\) are bound states. Since for a given solution \(z, z^*\) and \(z^*\) are also solutions to the characteristic equation, there are at most two independent bound-state solutions with \(|z| < 1\). An ansatz bound-state solution satisfying the boundary conditions (12) and (13) is then given by a linear combination of these normalizable wave functions: \(u(r) = e^{-\kappa r} + se^{-\kappa r}\) for all \(r\), where \(\text{Im}(\kappa) \neq 0\). To solve Eqs. (12) and (13), we then need \(s = \gamma'/\gamma'' = -\Xi/\Xi''\) (therefore \(|s| = 1\), where

\[ \gamma' \equiv - \left( \epsilon + J_1 + 2J_2 \right) e^{-\chi} + J_2 \cos (q) e^{-\chi} + J_1 \cos \left( \frac{q}{2} \right) e^{-2\chi} + J_2 \cos (q) e^{-3\chi}, \]

(15)

\[ \Xi \equiv - \left( \epsilon + 2J_1 + J_2 \right) e^{-2\chi} + J_1 \cos \left( \frac{q}{2} \right) e^{-3\chi} + J_1 \cos \left( \frac{q}{2} \right) e^{-4\chi}. \]

(16)

The condition for this ansatz is then \(\text{Im}(\gamma' \Xi) = 0\). We then scan all energies below the minimum of the two-particle continuum and above exact lower bounds for the two-magnon energy, find \(\kappa\) by solving the characteristic equation, and check the condition \(\text{Im}(\gamma' \Xi) = 0\).
Another possibility is that there are real solutions for \( z \) and a single exponential satisfies the equations. In this case, we can not require \( u(1) \) to have the same form \( u(r) = e^{-\gamma r} \) for \( r \gg 2 \). However, we can simply eliminate \( u(1) \) and obtain the condition

\[
2J_1 \cos \left( \frac{q}{2} \right) \cosh(\gamma) + 2J_2 \cos(q) \cosh(2\gamma) - J_1^2 - J_2^2 \cos(q) = J_1 \cos \left( \frac{q}{2} \right) \left( J_1 \cos \left( \frac{q}{2} \right) + J_2 \cos(q) \right) e^{-\gamma}.
\]

Checking for the two types of bound states above, we obtain the phase diagram shown in Fig. 2 of the main text. We have also checked this phase diagram by direct numerical calculation of the ground-state energy with Lanczos diagonalization in the two-particle subspace in a finite system of \( L = 100 \), which showed excellent agreement.

### III. SCALING DIMENSIONS

The correlation functions presented in the main text can be computed easily from Eq. (4) of the main text using the mapping of the chiral modes to new chiral modes that give rise to two noninteracting Luttinger liquids:

\[
\phi_1' + \phi_2' = \frac{1}{\sqrt{K}} (\phi_1 + \phi_2), \quad \phi_1' - \phi_2' = \sqrt{K} (\phi_1 - \phi_2 - \lambda \phi_1 - \lambda \phi_2),
\]

\[
\bar{\phi}_1' + \bar{\phi}_2' = \frac{1}{\sqrt{K}} (\bar{\phi}_1 + \bar{\phi}_2), \quad \bar{\phi}_1' - \bar{\phi}_2' = \sqrt{K} (\bar{\phi}_1 - \bar{\phi}_2 + \lambda \phi_1 + \lambda \phi_2),
\]

which gives

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\bar{\phi}_1 \\
\bar{\phi}_2
\end{pmatrix}
= \frac{1}{2} \begin{pmatrix}
\frac{k+1}{\sqrt{K}} & -\lambda \sqrt{K} & \frac{k+1}{\sqrt{K}} & -\lambda \sqrt{K} \\
-\lambda \sqrt{K} & \frac{k-1}{\sqrt{K}} & -\lambda \sqrt{K} & \frac{k-1}{\sqrt{K}} \\
\frac{k-1}{\sqrt{K}} & \lambda \sqrt{K} & \frac{k+1}{\sqrt{K}} & \lambda \sqrt{K} \\
\lambda \sqrt{K} & \frac{k+1}{\sqrt{K}} & \lambda \sqrt{K} & \frac{k-1}{\sqrt{K}}
\end{pmatrix}
\begin{pmatrix}
\phi_1' \\
\phi_2' \\
\bar{\phi}_1' \\
\bar{\phi}_2'
\end{pmatrix}.
\]

As the free-fermion chiral correlation functions are known, all correlators of vertex operators can be easily computed from the above expression. For example, \( \langle e^{-i\phi_1(t)} e^{i\phi_1(x)} \rangle \), which appears in \( G(x) \) is given by

\[
\langle e^{-i\phi_1(t)} e^{i\phi_1(x)} \rangle = \frac{i}{x} \left( \frac{1}{x} \right)^{\frac{k+1}{2}} \left( \frac{k+1}{2} \right)^{\frac{k-1}{2}} x^{\frac{k+1}{2}} \left( \frac{k-1}{2} \right)^{\frac{k-1}{2}} x^{\frac{k-1}{2}},
\]

which leads to Eq. (11) of the main text. In case of the spin-spin correlation functions, the exponents are far from unity and the phase shifts \( \omega \) cannot be neglected.