Chiral-Anomaly-Induced Nonlinear Hall Effect in Weyl Semimetals

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We predict a nonlinear Hall effect in certain Weyl semimetals with broken inversion symmetry. When the energy dispersions about pairs of Weyl nodes are skewed – the Weyl cones are “tilted” – the concerted actions of the anomalous velocity and the chiral anomaly gives rise to the nonlinear Hall effect. This Hall conductivity is linear in both electric and magnetic fields, and depends critically on the tilting of the Weyl cones. We also show that this effect does not rely on a finite Berry curvature dipole, in contrast to the intrinsic quantum nonlinear Hall effect that was recently observed in type-II Weyl semimetals.

Introduction. – Weyl semimetals (WSMs) \cite{1–10} are a newly discovered class of quantum materials which can host a number of exotic massless quasiparticles called Weyl fermions with a well-defined chirality near the band-crossing points (Weyl nodes). One of the most unique features of Weyl fermions is the chiral anomaly \cite{11, 12} – breaking of the chiral symmetry at the quantum level leading to the nonconservation of chiral charges. The manifestation in WSMs is that a pair of Weyl nodes of opposite chiralities act as source and drain of electrons in the presence of non-perpendicular electric and magnetic fields, resulting in a density difference between the two nodes, while preserving the total electron density \cite{13, 14}.

To date, the most remarkable phenomenon induced by the chiral anomaly is the longitudinal negative magnetoresistance \cite{14–16}, which was observed experimentally in WSMs such as TaAs \cite{17, 18}. Intuitively, this phenomenon can be understood via the chiral magnetic effect \cite{19, 20} in WSMs: In the absence of an electric field, there are equal numbers of Weyl fermions with opposite chiralities moving in opposite directions (collinear with the external magnetic field), which results in zero net charge current; when an electric field is applied along the magnetic field direction, an effective chemical potential difference between Weyl fermions with opposite chiralities is created due to the chiral anomaly \cite{14}, giving rise to an imbalance between the two fluxes of Weyl fermions and consequently a net charge current $j \propto (E \cdot B)B$.

More recently, another related transport phenomenon induced by the chiral anomaly in WSMs called the planar Hall effect was proposed \cite{21–23} and experimentally detected \cite{24–27}, wherein the Hall current, the electric and magnetic fields are all coplanar. It is worth noting that both the negative magnetoresistance and the planar Hall effect in WSMs are linear responses to the external electric field.

In this work, we predict another transport signature of the chiral anomaly in WSMs – a nonlinear Hall effect with the Hall conductivity proportional to $E \cdot B$. The physical mechanism of the chiral-anomaly-induced nonlinear Hall effect is illustrated in Fig. 1, which shows a combined effect of the anomalous velocity and the chiral anomaly. It is well established that in the presence of non-perpendicular electric and magnetic fields, the chiral anomaly results in a chiral-dependent modification of the electron density in the vicinity of each Weyl node \cite{14}, i.e., $\delta n_k^s \sim sE \cdot B$ with $s = \pm 1$ denoting the chirality. Moreover, due to the finite Berry curvature $\Omega_k^s$ of the Bloch states, the conduction electrons on the Fermi surface acquire an additional anomalous velocity $v_{s}^a = \frac{e}{s}E \times \Omega_k^s$ \cite{28, 29}, which is perpendicular to the applied electric field. Note that the direction of the anomalous velocity depends also on the chirality of the Weyl nodes. These two effects conspire to produce a nonlinear Hall current density $j_{\text{CNH}} = -e \sum_{k,s} \delta n_k^s v_{s}^a$, which is finite as long as the whole Fermi surface is asymmetric. One way to achieve this is via tilting in WSMs, as demonstrated in Fig. 1. Here we consider three special cases for a pair of Weyl cones, which are untilted, tilted in opposite directions, and tilted in the same direction. As shown in the lower panels of Fig. 1, in the first two cases, the whole Fermi surface for a pair of Weyl nodes is symmetric about $k = 0$, resulting in a vanishing $j_{\text{CNH}}$, whereas in the third case, the asymmetric Fermi surface leads to a finite $j_{\text{CNH}}$. The underlying reason will be discussed in detail later. In what follows, we evaluate the chiral-anomaly-induced nonlinear Hall effect in tilted WSMs.

Formulation. – We consider a pair of tilted Weyl nodes of opposite chiralities ($s = \pm 1$) separately and sum their contributions to obtain the total response function corresponding to the nonlinear Hall effect. Without loss of generality, we assume that the Weyl cones are tilted along the $z$-axis. The simplest low-energy Hamiltonian for each Weyl node is given by \cite{30}

$$H^s(k) = \hbar v_F (sk \cdot \sigma + R_k^s \sigma_0),$$

where $\sigma_0$ is the $2 \times 2$ identity matrix, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the three Pauli matrices, $v_F$ is the Fermi velocity, $s = \pm 1$ specifies the chirality of the Weyl node,
FIG. 1. Schematics illustrating the physical mechanism of the chiral-anomaly-induced nonlinear Hall (CNH) effect for a pair of Weyl nodes of opposite chiralities, denoted by the blue ($s = 1$) and red ($s = −1$) dots. In the linear model used in our analysis, for a fixed value of $k_y$, each Weyl node has a linear dispersion along the $k_x$ and $k_z$ axes, forming a Weyl cone, as shown in the upper panel in each subfigure. For simplicity, the two Weyl nodes are separated along the $z$-axis and the external electric and magnetic fields are assumed to be in the $x$-direction. We show the scenarios where the pair of Weyl cones are (a) untilted, (b) tilted in opposite directions, and (c) tilted in the same direction. The gray horizontal planes cut through the energy dispersions at the equilibrium chemical potential $\mu$ with the corresponding Fermi surface cross-sections shown in the lower panels. The nonlinear Hall current arises as a consequence of the chiral anomaly and the anomalous velocity. On one hand, the chiral anomaly effectively leads to unequal electron densities between the two Weyl cones, as shown by the orange-filled parts of the cones. On the other hand, the anomalous velocity, whose direction and magnitude depend on the chirality of the Weyl cone as well as the location on the Fermi surface, is indicated by the blue arrows in the lower panels. In scenarios (a) and (b), the whole Fermi surface is symmetric about $k = 0$, leading to a vanishing CNH current, whereas in scenario (c), an asymmetric Fermi surface leads to a finite CNH current.

and the parameter $R_s$ characterizes the tilting of the Weyl cone. For small tilting $|R_s| < 1$, the Fermi surface encloses only an electron pocket (assuming that the chemical potential lies in the conduction band). In this case, the Hamiltonian describes a type-I Weyl node. When $|R_s| > 1$, which corresponds to a type-II Weyl node, unbounded electron and hole pockets are present at the Fermi energy [31]. Consequently, both the conduction and valence bands will contribute to the nonlinear Hall current in type-II WSMs. Moreover, due to the unbounded nature of the Fermi surface in a linear model, to calculate any physical quantity related to the Fermi surface, such as the current response, we must introduce a ultraviolet momentum cutoff ($\Lambda$), beyond which the linear model (1) can no longer be taken as a valid description of the WSM [32, 33]. The energy dispersion of this two-band model can be written as

$$\varepsilon^s(k) = \hbar v_F(R_s k_z \pm k),$$

where $+$ ($-$) sign corresponds to the conduction (valence) band and $k = |k|$. The Berry curvature is unaffected by tilting and is given by

$$\Omega^s(k) = -\frac{s \pm \mathbf{k}}{2k^3}. \tag{3}$$

The semiclassical equations of motion for a Weyl fermion wavepacket can be written as [29]:

$$D^s \dot{\mathbf{r}}^s = \mathbf{v}^s + \frac{e}{\hbar} \mathbf{E} \times \Omega^s + \frac{e}{\hbar} (\mathbf{v}^s \cdot \Omega^s) \mathbf{B}, \tag{4a}$$

$$D^s \dot{\mathbf{k}}^s = -\frac{e}{\hbar} \mathbf{E} - \frac{e}{\hbar} \mathbf{v}^s \times \mathbf{B} - \frac{e^2}{\hbar^2} (\mathbf{E} \cdot \mathbf{B}) \Omega^s, \tag{4b}$$

where $\mathbf{v}^s \equiv \partial \varepsilon^s / \partial \mathbf{k}^s$ and $D^s \equiv 1 + \frac{e}{\hbar} (\mathbf{B} \cdot \Omega^s)$. To compute the current density, we substitute Eq. (4b) into the homogeneous steady-state Boltzmann equation with the relaxation-time approximation

$$\frac{\partial f^s}{\partial \mathbf{k}^s} = -\frac{f^s - f^{s\mu}}{\tau}, \tag{5}$$

and solve for $f^s \approx f^s_0 + f^s_1 + f^s_2$, where $f^s_0$ is the equilibrium Fermi-Dirac distribution [at zero temperature $f^s_0 = \Theta (\mu - \varepsilon^s(\mathbf{k}))$ with $\mu$ the chemical potential], $f^s_1$ and $f^s_2$ are the corrections to the equilibrium distribution at the first- and second-order in electric field, respectively, and $\tau$ is the intranode relaxation time. We have also assumed that the internode scattering rate is much smaller than than the intranode scattering rate $1/\tau$ and hence can be neglected. The current density can then be calculated via

$$\mathbf{j} = \langle -e \rangle \sum_s \int \mathbf{D}^s \dot{\mathbf{r}}^s f^s, \tag{6}$$

where $f^s_{\mathbf{k}}$ is the shorthand notation for $\int d\mathbf{k} / (2\pi)^3$. 


Note that a nonlinear response can only arise from terms in Eq. (6) that involve $f_1^s$ and $f_2^s$. Therefore, when the chemical potential lies above the Weyl nodes, we only need to calculate the contribution from the conduction band for a type-I WSM, whereas for a type-II WSM, we need to sum the contributions from both the conduction and valence bands due to the emergence of the electron and hole pockets at the Fermi level. Details of the calculation of nonlinear responses can be found in Appendix A.

Results and Discussion. – We find, up to $O(E^2 B^1)$, that the chiral-anomaly-induced nonlinear Hall (CNH) current stems from the following integral,

$$j_{\text{CNH}} = \frac{e^4 r}{\hbar^2} \sum_s \int \frac{\partial f_0^s}{\partial \varepsilon^s} (\mathbf{E} \times \partial \varepsilon^s) \cdot (\mathbf{v}^s \times \mathbf{B}).$$

By inspecting the structure of the integral above, it is evident that if the energy dispersion of the WSM is invariant under $k \rightarrow -k$, corresponding to a Fermi surface symmetric about $k = 0$, the group velocity $\mathbf{v}^s = \partial \varepsilon^s / \partial \mathbf{k}^s$ is an odd function of $\mathbf{k}$ and hence the integral over the reciprocal space vanishes. Thus, to obtain a nonzero $j_{\text{CNH}}$, an asymmetric Fermi surface is necessary and in this case, it is provided by the tilt-dependent term in Eq. (2). Evaluating the integral given in Eq. (7) for type-I and type-II WSMs, the CNH current density can be expressed in the following general form,

$$j_{\text{CNH}} = \sum_s \kappa^s (\mathbf{E} \cdot \mathbf{B}) (\mathbf{E} \times \mathbf{i}),$$

where $\mathbf{i}$ is the unit vector in the direction of the tilt ($\mathbf{i} = \hat{e}_z$ in the present setup) and $\kappa^s$ is the nonlinear current response function for a Weyl node of chirality $s$. We find, for type-I WSMs

$$\kappa^s_1 = \frac{3 \varepsilon^s \mathbf{v}^2}{40 \pi^2 \hbar \mu^2} R_s a,$$

and for type-II WSMs,

$$\kappa^s_2 = \frac{5 \varepsilon^s}{12 |R_s|^3} \left( 6 \mu^2 - 3 \mu^4 \right),$$

where we have introduced a dimensionless ultraviolet cutoff $\tilde{\Lambda} \equiv \Lambda / k_F$ [with the Fermi wavevector defined as $k_F \equiv \mu / (\hbar v_F)$] [32, 33] to deal with the open Fermi surface in the two-band model of type-II WSMs described by Eq. (1). In real materials, the cutoff may be considered as an upper-bound of $|k_z|$ beyond which the bands cease to be linearly dispersing. It is worthy to point out that for type-II WSMs, when the momentum cutoff $\Lambda$ is much larger than the Fermi wavevector, that is, $\tilde{\Lambda} \gg 1$, the cutoff-dependent terms in the response function $\kappa^s_1$ become negligible. This regime is desirable as we are mostly interested in the physics near the Weyl nodes, that is, when the Fermi energy is close to zero. It further removes the dependence on the seemingly artificial cutoff $\Lambda$, making the result more universal.

In Fig. 2(a) we show $\kappa^s$ for a single Weyl node and its derivative with respect to the tilt parameter, $\partial \kappa^s / \partial R_s$, as a function of $R_s$, assuming $\tilde{\Lambda} = 10$. For both type-I and type-II WSMs, the contribution to the nonlinear Hall effect from a single Weyl node becomes more prominent as tilting of the Weyl cone gets larger due to the monotonic nature of the nonlinear current response function $\kappa^s$ (solid curve). Moreover, the phase transition from type-I to type-II WSM can be clearly seen from the derivative of $\kappa^s$ with respect to $R_s$ (dashed curve) due to the discontinuity at $R_s = \pm 1$.

In Fig. 2(b), we consider a pair of Weyl nodes of opposite chiralities ($s = \pm 1$) and the total nonlinear current response $\kappa = \kappa^+ + \kappa^-$ as a function of $R_+ = \mp R_-$. Region I (II) in the parameter space corresponds to the case where both Weyl nodes are type-I (type-II). Region III corresponds to the case where one of the Weyl nodes is type-I and the other is type-II. When $R_+$ and $R_-$ have the same sign, the magnitude of $\kappa$ increases as the magnitude of either of the tilt parameters increases. On the other hand, when they have opposite signs, the magnitude of $\kappa$ first decreases as the magnitude of one of the tilt parameters increases while the other is fixed. It then increases after the tilt parameter crosses the line $R_+ = -R_-$. It is worth noting that on the line $R_+ = -R_- = \kappa = 0$ in both type-I and -II WSMs, due to the symmetric Fermi surfaces as depicted in Fig. 1(b).

A few remarks on the CNH effect are in order. First, the nonlinear Hall effect vanishes if the system is inversion-symmetric. This can be seen from the general form of the nonlinear Hall current $j_{\text{CNH}}$ as given by Eq. (8); the whole set of the external fields, i.e., $(\mathbf{E} \cdot \mathbf{B}) \mathbf{E}$, is even under space inversion whereas $j_{\text{CNH}}$ is parity-odd, so the response function must be zero if the system is invariant under space inversion. Furthermore, we have addressed explicitly based on Eq. (7) that an asymmetric Fermi surface is also required to have a nonvanishing $j_{\text{CNH}}$, which can be realized in tilted WSMs. [34]

Secondly, the nonlinear Hall response $\kappa^s$ and hence the corresponding nonlinear Hall conductivity are proportional to $\mu^{-2}$ for both types of WSMs, at variance with the Drude conductivity which is proportional to $\mu^2$, as is clear from Eqs. (9) and (10). This implies that the nonlinear Hall effect becomes sizable when the Fermi energy approaches the Weyl node; such enhancement originates from the singularity of the Berry curvature at the Weyl node. The divergence of $\kappa^s$ as the Fermi energy falls on the Weyl nodes, however, can be evaded by the disorder-induced energy broadening, which imposes a lower bound of the Fermi energy $\mu \gtrsim \hbar / \tau$ [35]. Also, in our semiclassical treatment, we...
be expressed as

\[ j = |j| \cos \theta \mathbf{E} + |j| \sin \theta \mathbf{B} \]

The corresponding nonlinear Hall current density can be derived by Morimoto and coworkers [35] for Weyl fermions with linear and isotropic dispersion, which is finite when the electric and magnetic fields are perpendicular and hence does not emanate from the chiral anomaly. Another nonlinear Hall effect arises from the momentum-space Berry curvature dipoles [36–39].

We are now in position to compare the CNH effect with other nonlinear Hall effects that were discovered previously. A nonlinear Hall conductivity linearly proportional to both electric and magnetic fields was derived by Morimoto and coworkers [35] for Weyl fermions with linear and isotropic dispersion, which is finite when the electric and magnetic fields are perpendicular and hence does not emanate from the chiral anomaly. Another nonlinear Hall effect arises from the momentum-space Berry curvature dipoles [36–39].

The corresponding nonlinear Hall current density can be expressed as

\[ j^\text{CNH} = \sum_{abcd} \chi^s_{abcd} E_a E_b B_c \]

with the nonlinear response function

\[ \chi^s_{abcd} = \epsilon^{sabcd} \frac{1}{2} \int \frac{d^3 \mathbf{k}}{V} \left( \frac{\partial}{\partial k_n} \Omega^s_+ \right) \]

Comparing Eq. (11) with (12), it is evident that the nonlinear response function \( \chi^s_{abcd} \) is different from a Berry curvature dipole. It follows that, by simple power counting, the nonlinear Hall current in WSMs originating from the Berry curvature dipole has a logarithmic divergence as the Fermi energy approaches zero since \( \Omega^s \propto k^{-2} \), in contrast to the quadratic divergence of the chiral-anomaly-induced nonlinear Hall current discussed above. Such different scalings with \( \mu \) can in principle be used to distinguish these two nonlinear Hall effects in experiments where a finite magnetic field is present.

Additionally, the CNH effect can be distinguished experimentally from linear Hall effects [21, 22, 33, 40] in WSMs as well. In a.c. measurements, this can be easily achieved by measuring the second harmonic Hall resistance [41, 42] wherein linear Hall contributions are automatically excluded. In d.c. measurements, they can also be distinguished by proper alignment of the external electric and magnetic fields.

Finally, we give an order-of-magnitude estimate of the size of the CNH effect for a pair of Weyl cones, which are assumed to tilt in the same direction. For reference, we can compare the size of the nonlinear Hall conductivity, \( \sigma_{1(II)} = \kappa_{1(II)} \mathbf{E} \cdot \mathbf{B} \), to the size of the Drude conductivity \( \sigma_D \) in tilted WSMs, which is given in [42]. For a type-I WSM such as TaAs [43, 44], using typical parameters \( v_F = 3 \times 10^5 \text{ m/s}, \mu = 20 \text{ meV} \), and assuming a tilt parameter \( R_s = 0.1 \) leads to a ratio of the nonlinear Hall conductivity to the Drude conductivity, \( \sigma_{1}/\sigma_D \approx 1\% \), for an electric field \( E = 100 \text{ V/cm} \) applied in the x-direction and a parallel magnetic field \( B = 9 \text{ T} \). Similarly, a type-II semimetal such as MoTe\(_2\) [45, 46] with \( R_s = 1.5 \) would lead to a ratio of the same order of magnitude. Note that a real WSM material typically hosts more than a pair of Weyl nodes and tilting may be different among them. To obtain a more accurate estimate of this effect, one needs to perform a first-principle calculation of the nonlinear response function given by Eq. (12) for a particular material by taking into account all the pairs in the Brillouin zone, which is beyond the scope of this study.

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**APPENDIX A: CALCULATION OF THE NONLINEAR CURRENT DENSITY WITH SPHERICAL GEOMETRY**

Assuming $\mu > 0$, we note that for a type-I WSM, we only need to calculate the contribution from the conduction band, whereas for a type-II WSM, we need to sum the contributions from both the conduction and valence bands. In general, the current density is given by

$$j = (-e) \sum_s \int \mathbf{D}^s \hat{s} \cdot \mathbf{f}^s, \quad (A1)$$

where $\mathbf{f}^s \simeq f^s_0 + f^s_1 + f^s_2$. Since a nonlinear response to an external electric field can only arise from terms in Eq. (A1) that involve $f^s_1$ and $f^s_2$, exploiting the spherical symmetry of the system described by Eq. (1), the nonlinear current density components can be written as

$$j_i = \sum_s \int_0^\infty k^2 dk \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos \theta) \mathcal{M}^s(k, \phi, \cos \theta)$$

$$\times \delta(\mu - \hbar v_F k (R_s \cos \theta \pm 1)), \quad (A2)$$

where $\mathcal{M}^s(k, \phi, \cos \theta)$ is a function of $k$, $\phi$, and $\cos \theta$ proportional to $E^2$ that comes from expanding the integrand in Eq. (A1). The root for the conduction band in the Dirac delta function is given by

$$k = \frac{k_F}{R_s \cos \theta + 1}, \quad (A3)$$

whereas for the valence band the root is

$$k = \frac{k_F}{R_s \cos \theta - 1}, \quad (A4)$$

where $k_F = \mu/\hbar v_F$. Since $k \geq 0$ and $k_F > 0$ is assumed, we must have $R_s \cos \theta - 1 > 0$ for the conduction band and $R_s \cos \theta + 1 > 0$ for the valence band. Then for type-I WSMs with $|R_s| < 1$, the integration range of $\cos \theta$ is from $-1$ to $1$.

The situation is different for type-II WSMs. As mentioned in the main text, to do sensible calculations for type-II WSMs one needs to impose a radial momentum cutoff $\Lambda$ such that $k \leq \Lambda$. This cutoff then translates into a change in the integration limits of $\cos \theta$. For $R_s > 1$, requiring $k \leq \Lambda$ leads to the following integration limits for the conduction band:

$$\left( \frac{k_F}{\Lambda} - 1 \right) \frac{1}{R_s} \leq \cos \theta \leq 1, \quad (A5)$$

while for the valence band the limits are

$$\left( \frac{k_F}{\Lambda} + 1 \right) \frac{1}{R_s} \leq \cos \theta \leq 1. \quad (A6)$$

On the other hand, for $R_s < -1$, we get the following limits for the conduction band:

$$-1 \leq \cos \theta \leq \left( \frac{k_F}{\Lambda} - 1 \right) \frac{1}{R_s}, \quad (A7)$$

and for the valence band:

$$-1 \leq \cos \theta \leq \left( \frac{k_F}{\Lambda} + 1 \right) \frac{1}{R_s}. \quad (A8)$$

**APPENDIX B: Q-DEPENDENCE OF THE NONLINEAR RESPONSE FUNCTIONS**

Here we consider two Weyl nodes of opposite chiralities lying on the $z$-axis, separated by $2Q$ in momentum space. Again, the Weyl cones are assumed to be tilted along the $z$-axis. The low-energy Hamiltonian for a single Weyl node that takes into account the separation is given by

$$H^s(k) = \hbar v_F (s(k - sQ\hat{e}_z) \cdot \sigma + R_s(k_z - sQ)\sigma_0). \quad (B1)$$

The Berry curvature is modified to

$$\Omega^s(k) = -s \frac{\pm (k - sQ\hat{e}_z)}{2|k - sQ\hat{e}_z|^3}. \quad (B2)$$

Similar to Eq. (A2), in this case, the current density components are computed in the cylindrical coordinates:

$$j_i = \sum_s \int_{k_{\perp} = 0}^{k_F} \int_{k_{\perp} = 0}^{2\pi} d\phi \int_{-\Lambda}^{\Lambda} dk_z \mathcal{M}^s(k_{\perp}, \phi, \tilde{k}_{\perp z}, z)$$

$$\times \delta(\mu - \hbar v_F \left( R_s \tilde{k}_{\perp z} z, \pm \sqrt{k_{\perp}^2 + \tilde{k}_{\perp z}^2} \right)), \quad (B3)$$

where $k_{\perp} = \sqrt{k_{\perp}^2 + k_{\perp z}^2}$ and $\tilde{k}_{\perp z} = k_z - sQ$. Note that with the cylindrical geometry, the momentum cutoff $\Lambda_z$ for type-II WSMs is imposed on $k_z$. Similar to what is discussed in Appendix A, analyzing the roots of the Dirac delta function determines the integration limits for type-I and type-II WSMs. The details can be found in [33]. We find that the nonlinear response function for type-I WSMs is the same as Eq. (9). For
type-II WSMs, it is given by
\[
\kappa_{11} = \frac{e^4 v_F^2 \tau}{64 \pi^2 \hbar \mu^2} \frac{\text{sgn}(R_s)}{R_s^2} \left[ 2(10R_s^6 + 15R_s^4 - 1) - 10(R_s^2 - 1)(\delta_+^2 + \delta_-^2) + 20(R_s^2 - 1)(\delta_+^4 + \delta_-^4) - 5(2R_s^4 - 3)\left(\delta_+^2 + \delta_-^2\right) \right], \quad (B4)
\]
where
\[
\delta_+ = \left[ 1 + R_s(s\tilde{Q} + \tilde{\Lambda}_s) \right]^{-1}, \quad (B5a)
\]
\[
\delta_- = \left[ 1 + R_s(s\tilde{Q} - \tilde{\Lambda}_s) \right]^{-1}, \quad (B5b)
\]
with \(\tilde{Q} \equiv Q/k_F\) and \(\tilde{\Lambda}_s \equiv \Lambda_s/k_F\). It is not hard to see that the cutoff-independent terms in Eq. (B4) agree with the ones in Eq. (10), and they dominate when \(\tilde{\Lambda}_s \gg 1\). In this case, the result is independent of both \(Q\) and \(\Lambda_s\). The \(Q\)-dependence of the nonlinear response function is fairly weak except for values of \(Q\) close to the momentum cutoff \(\tilde{\Lambda}_s\). For brevity, we adopt a simpler Hamiltonian (1) for our calculations presented in the main text.

**APPENDIX C: TWO-NODE HAMILTONIAN AND ASYMMETRIC FERMI SURFACES**

As illustrated in Fig. 1, tilting of a Weyl cone leads to a Fermi surface that is asymmetric about the Weyl node, so that the anomalous velocities at a pair of nodes symmetric about the center of the Fermi surface do not cancel out each other, giving rise to a finite contribution to the nonlinear Hall current. A natural question to ask is whether such asymmetric Fermi surface can be produced in ways other than tilting. One way to achieve this is to go beyond the linearized model (1) or (B1). One may start with a more general two-node Hamiltonian of WSMs:

\[
H = A(k_x\sigma_x + k_y\sigma_y) + M(Q^2 - k_z^2)\sigma_z, \quad (C1)
\]
whose dispersions of the two energy bands are
\[
\varepsilon_{\pm}(k) = \pm \sqrt{A^2(k_x^2 + k_y^2) + M^2(Q^2 - k_z^2)^2}. \quad (C2)
\]
This Hamiltonian describes two Weyl nodes situated at \((0, 0, \pm Q)\). Examining the dispersion of the conduction band, it is clear that it is an even function of \(k\), i.e. \(\varepsilon_+(-k) = \varepsilon_+(k)\). Hence, based on our discussion about Eq. (7), the chiral-anomaly-induced nonlinear Hall current is zero for the given Hamiltonian. To look at the matter more closely and make a better connection with the low-energy one-node Hamiltonian used in our calculations, one may do an expansion around the nodes \(k_z^2 \approx Q^2 + 2sQ(k_z - sQ) + (k_z - sQ)^2\). Plugging it into Eq. (C1), one arrives at the following low-energy Hamiltonian for each of the Weyl nodes,

\[
H^\ast(k) = -M \left[ 2sQ(k_z - sQ) + (k_z - sQ)^2 \right]\sigma_z + A(k_x\sigma_x + k_y\sigma_y). \quad (C3)
\]
Note that if one keeps only terms in the expansion up to the first order in \(k_z\), the low-energy Hamiltonian matches exactly Eq. (B1) without tilting \((R_s = 0)\), with
\[
M = \frac{\hbar v_F}{2Q}, \quad A = s\hbar v_F. \quad (C4)
\]
Therefore, the second-order correction in the low-energy Hamiltonian gives rise to an asymmetric Fermi surface about the Weyl node. However, after taking a closer look at its energy dispersion of e.g. the conduction band,

\[
\varepsilon_+^s = \hbar v_F \sqrt{s(k_z - sQ) + \frac{1}{2Q}(k_z - sQ)^2} + k_x^2 + k_y^2, \quad (C5)
\]
one finds \(\varepsilon_+^s(k) = \epsilon_-^{s^*}(-k)\), that is, the energy at point \(k\) on one Weyl cone is the same as the energy at point \(-k\) on the other cone. Therefore, even though the distribution for a single Weyl cone is asymmetric about the Weyl node, the overall distribution for the pair of cones is, however, symmetric about \(k = 0\). This is analogous to the case of two oppositely tilted Weyl nodes, as illustrated in Fig. 1(b). In this case, the chiral-anomaly-induced nonlinear Hall current is expected to vanish, which is confirmed by explicit calculations.

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Alternatively, one may use a low-energy Hamiltonian which takes into account the separation of the two Weyl nodes in momentum space.\[ H(k) = \hbar v_p [s(k - sQ) \cdot \sigma + B(k_z - sQ) \sigma_0]. \] But the calculation using this Hamiltonian shows that Q dependence of the conductivity is very weak. Thus for our analysis we may use a simpler Hamiltonian given by Eq. (1). See details in Appendix B.

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