Positivity of Energy in Einstein-Maxwell Axion-dilaton Gravity

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Abstract

Avoiding the problem of the existence of asymptotically constant spinors satisfying certain differential equations on a non-compact hypersurface we presented the proof of positivity of the ADM and Bondi energy in Einstein-Maxwell axion-dilaton gravity. In our attitude spinor fields defining the energy need only be defined near infinity and there satisfying propagation equations.

1 Introduction

In general relativity there are two distinct notions of energy-momentum for an isolated system. The Arnowitt-Deser-Misner (ADM) four momentum \[^1^] defined at spatial infinity and the Bondi one \[^2^] measured at null infinity. During last years there were heavy attempts to prove positivity of energy in general relativity. The first complete proof was presented by Schoen and Yau \[^3^] and then afterwards by Witten \[^4^]. Parker and Taubes \[^5^] gave mathematically rigorous proof of the positive energy theorem analyzing the falloff behaviour of spinors near infinity. Then Nester \[^6^] treated the problem of the positive mass theorem in a fully covariant way to avoid difficulties concerning three-dimensional truncation of the four-dimensional divergence theorem. In several works \[^7^] the issue of extending the Witten’s method to the Bondi energy was achieved.

Similar techniques were used to prove extensions of these results to the case of Einstein-Maxwell (EM) theory \[^8^], supergravity \[^9^] and Kaluza-Klein theory \[^10^]. It was shown \[^11^] that self-gravitating solitons saturated a gravitational version of the Bogomolnyi bound on energy due to inclusion of Yang-Mills and Yang-Mills-Higgs and dilaton interactions. In the case of
the low-energy string theory, the so-called Einstein-Maxwell axion-dilaton gravity (EMAD), the results of positivity of the ADM and Bondi energy were achieved in [12, 13]. The other problem was to extend the positive mass theorem to asymptotically flat manifolds containing black holes [14].

But one of the major problems connected with the Witten’s type of proofs is the existence of solutions to an elliptic system of partial differential equations on a non-compact hypersurface. If the hypersurface is compact we can refer to the known theorems for existence but in the case of non-compactness it should be necessary to devise new mathematical theorems (perhaps quite complicated). In [15] it was shown that positivity of the ADM and Bondi energy in Einstein gravity could be achieved in a slight different way, without proving the existence theorems on non-compact hypersurfaces.

In our paper we generalize and extend the reasoning presented in [15] to the case of EMAD gravity. We shall use two-component spinor notation [16]. Spinor indices will be denoted by capital letters. Our signature of $g_{\mu\nu}$ is $(+−−−)$ and the convention for the curvature tensor is $2\nabla_\alpha \nabla_\beta \eta_\gamma = -R_{\alpha\beta\gamma\delta} \eta^\delta$. The Einstein equations are $G_{\mu\nu} = -T_{\mu\nu}$.

## 2 Positivity of energy in EMAD gravity

The low-energy limit of the heterotic string theory compactified on a six-dimensional torus $T^6$ consists of the pure $N = 4, d = 4$ supergravity coupled to $N = 4$ super Yang-Mills. One can truncate consistently this theory to the simpler pure supergravity one. The truncation is based on introducing always equal numbers of Kaluza-Klein and winding number modes for each cycle. The truncated theory still exhibits $S$ and $T$ dualities. Therefore $N = 4, d = 4$ supergravity provides a simple framework for studying classical solutions which can be considered as solutions of the full effective string theory. The bosonic sector of this theory with a simple vector field is called in literature EMAD gravity. EMAD gravity provides after all a non-trivial generalization of EM gravity, which consists of coupled system containing a metric $g_{\mu\nu}$, $U(1)$ vector fields $A_\mu$, a dilaton $\phi$ and three-index antisymmetric tensor field $H_{\alpha\beta\gamma}$. The action has the form [17]

$$I = \int d^4x \sqrt{-g} \left[-R + 2(\nabla\phi)^2 + \frac{1}{3} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - e^{-2\phi} F_{\alpha\beta} F^{\alpha\beta} \right] + I_{\text{matter}}, \quad (1)$$

where the strength of the gauge fields is described by $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$ and the three index antisymmetric tensor is defined by the relation

$$H_{\alpha\beta\gamma} = \nabla_\alpha B_{\beta\gamma} - A_\alpha F_{\beta\gamma} + \text{cyclic}. \quad (2)$$

In four dimensions $H_{\alpha\beta\gamma}$ is equivalent to the Peccei-Quin pseudo-scalar and implies the following:

$$H_{\alpha\beta\gamma} = \frac{1}{2} F_{\alpha\beta\gamma} e^{4\phi} \nabla^\delta a. \quad (3)$$

A straightforward consequence of the definition (3) is that the action (1) can be written as

$$I = \int d^4x \sqrt{-g} \left[-R + 2(\nabla\phi)^2 + \frac{1}{2} e^{4\phi} (\nabla a)^2 - e^{-2\phi} F_{\alpha\beta} F^{\alpha\beta} - a F_{\mu\nu} \ast F_{\mu\nu} \right] + I_{\text{matter}}, \quad (4)$$

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where \( *F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} F^{\rho\lambda} \).

By means of the two-component spinor notation, we define a Maxwell spinor by the relation

\[
F_{MN' NN'} = \phi_{MN} \epsilon_{M'N'} + \bar{\phi}_{M'N'} \epsilon_{MN},
\]

where

\[
\phi_{AB} = \frac{1}{2} F_{ABC} C', \quad \bar{\phi}_{AB'} = \frac{1}{2} F_{C'A'B'}.
\]

Equations of motion derived from the variational principle are given by

\[
\nabla_M \phi_{MN} = \nabla_M' \phi_{M'N'},
\]

\[

abla_M \left( z \phi_{MN} \right) - \nabla_M' \left( z' \phi_{M'N'} \right) = J^{N'N}(\text{matter}),
\]

\[
\nabla_{\mu} \phi - \frac{1}{2} e^{2\phi} (\nabla a)^2 - e^{4\phi} \left( \phi_{MN} \phi^{MN} + \bar{\phi}_{M'N'} \bar{\phi}^{M'N'} \right) = 0,
\]

\[
\nabla_{\mu} \nabla^{\mu} a + 4 \nabla_{\mu} \bar{\phi} \nabla^{\mu} a + 2 i e^{-4\phi} \left( \bar{\phi}_{M'N'} \phi^{M'N'} - \phi_{MN} \phi^{MN} \right) = 0,
\]

\[
6 \Lambda \epsilon_{AB} \epsilon_{A'B'} + 2 \Phi_{ABA'B'} = T_{A'A'B'},
\]

where we introduced a complex axial-dilaton \( z = a + i e^{-2\phi} \), while \( \Lambda = \frac{R}{24} \) and \( \Phi_{ABA'B'} \) is the curvature spinor (sometimes called Ricci spinor). The energy momentum tensor \( T_{\mu\nu} = \frac{2M}{\sqrt{-g}g^{\mu\nu}} \) can be written as

\[
T_{MN' NN'}(F, \phi, a) = 8 e^{-2\phi} \phi_{MN} \bar{\phi}_{M'N'} - \epsilon_{MN} \epsilon_{M'N'} \left[ 2(\nabla \phi)^2 + \frac{1}{2} e^{4\phi} (\nabla a)^2 \right]
\]

\[
+ 4 \nabla_{MM'} \phi \nabla_{NN'} \bar{\phi} + e^{4\phi} \nabla_{MM'} a \nabla_{NN'} a,
\]

while \( J^{\mu\nu}(\text{matter}) = \frac{\delta I(\text{matter})}{4g_{4\mu}} \).

To begin with we define the supercovariant derivatives for two-component spinors \( (\alpha^A(i), \beta_{A'}(i)) \) as follows:

\[
\hat{\nabla}_{MM'} \alpha_{K(i)} = \nabla_{MM'} a_{K(i)} + \frac{i}{2} e^{2\phi} \nabla_{MM'} a \alpha_{K(i)} + \sqrt{2} e^{-\phi} \phi_{KM} a_{(ij)} \beta_{M'}^{(j)},
\]

\[
\hat{\nabla}_{MM'} \beta_{A'(i)} = \nabla_{MM'} a_{A'(i)} + \frac{i}{2} e^{2\phi} \nabla_{MM'} a \beta_{A'(i)} - \sqrt{2} e^{-\phi} \phi_{A'M'} a_{(ij)} \alpha_{M}^{(j)}.
\]

The hatted derivative \( \hat{\nabla}_{MM'} \) may be regarded as a supersymmetry transformation about non-trivial gravitational, scalar and \( U(1) \) backgrounds. The two-dimensional antisymmetric matrices \( a_{(ij)} \) where \( i, j = 1, 2 \) and \( a_{(ij)} a^{(jm)} = -\delta_{i}^{m} \) are of the form \( a_{(ij)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). They constitute two-dimensional equivalents of matrices introduced in [18].

As was mentioned the EMAD gravity theory is a truncation of \( N = 4, d = 4 \) supergravity with only one vector field. Having this in mind we shall keep another Latin index in the spinor transformation rules to make transparent linkage to the spinor transformations in the underlying supergravity theory (see [17]-[18] and references therein).

For spinor fields \( (\alpha^A(i), \beta_{A'}(i)) \) we describe the quantities which will be useful for our further considerations and to state our results, namely we define the Nester-Witten two-form [19] by

\[
\Theta(\alpha_{A}, \bar{\alpha}_{A'}, \beta_{A}, \bar{\beta}_{A'}) = -i \left( \bar{\alpha}_{B'(i)} \nabla_{A_{B}(i)} d x^{a} \wedge d x^{b} + \beta_{B'(i)} \nabla_{A_{B}(i)} \bar{\beta}_{B}^{(i)} d x^{a} \wedge d x^{b} \right),
\]

where \( \nabla_{A_{B}(i)} = \hat{\nabla}_{MM'} \).
Now, let Σ be a spacelike two-surface spanned by a three-dimensional compact hypersurface \( \mathcal{H} \), with the unit normal \( t^{AA'} \) to \( \mathcal{H} \). Then, using the Stokes’ theorem, after tedious calculations one obtains

\[
\int_{\Sigma} \Theta(\alpha_A, \bar{\alpha}_{A'}, \beta_A, \bar{\beta}_{A'}) = \int_{\mathcal{H}} d\Theta(\alpha_A, \bar{\alpha}_{A'}, \beta_A, \bar{\beta}_{A'})
\]

\[
= \int_{\mathcal{H}} d\mathcal{H} \left[ -\left( \hat{D}_{a}\alpha_{(i)} \hat{D}^{b}\alpha_{A'}^{(i)} + \hat{D}_{b}\bar{\alpha}_{(i)} \hat{D}^{a}\bar{\beta}_{A'}^{(i)} \right) t^{AA'} + \frac{1}{2} T_{ab}(\text{matter}) \xi^a t^b \right.
\]

\[
+ \left( (\delta \lambda_{A(i)}^{(\alpha)}) \check{(\delta \lambda_{A'}^{(\alpha)})} + (\delta \lambda_{A(i)}^{(\beta)}) \check{(\delta \lambda_{A'}^{(\beta)})} \right) t^{AA'}
\]

\[
+ \left( -\sqrt{2} e^{\phi} J_{AA'}(\bar{z}, F) \alpha^{K(i)} \bar{\beta}_{K(i)} + c.c. \right) t^{AA'}
\]

\[- 2 \sqrt{2} J_{AA'}(\phi, F) \bar{\alpha}^{K(i)} \beta_{K(i)} t^{AA'} \right],
\]

where \( \xi^{BB'} = \alpha_{(i)}^{B'}(\alpha_{(i)}^{B'} + \bar{\beta}_{(i)}^{B'}) \) is the future-directed null vector on an asymptotically flat spacetime \( [14] \) while \( \hat{D}_a = \nabla_a - t_a b \check{\nabla}_b \) is the projection of the supercovariant derivative into \( \mathcal{H} \). We choose the initial data for the Weyl equations \( \nabla_{AA'} \alpha^A = 0 \) and \( \nabla_{AA'} \bar{\alpha}^A = 0 \) such that on \( \mathcal{H} \) one has the following:

\[
\hat{D}_{A'A} \alpha^A_{(i)} = 0, \quad \hat{D}_{A'A} \bar{\alpha}^A_{(i)} = 0.
\]

The Witten’s-like equations \([17]\) constitute the first order elliptic equations on \( \alpha_{(i)}^A \) and \( \bar{\alpha}_{(i)}^A \). The energy momentum tensor \( T_{ab}(\text{matter}) \) is defined by the relation

\[
T_{ab}(\text{matter}) = T_{ab}(\text{total}) - T_{ab}(F, \phi, a).
\]

In relation \([16]\) we have also defined the following complex currents:

\[
J_{AA'}(F, \phi) = \nabla_{A'B}(e^{-\phi} \bar{\phi}^B A),
\]

and

\[
J_{AA'}(\bar{z}, F) = \nabla_{A'B}(z \phi^B A).
\]

Moreover, the adequate quantities appearing in equation \([16]\) yield

\[
\delta \lambda_{A'(i)}^{(\alpha)} = \sqrt{2} \nabla_A B^{\phi} \alpha_{(i)}^B + \frac{i}{\sqrt{2}} e^{2\phi} \nabla_A B^{a} \alpha_B a_{(i)} - 2ie^{-\phi} \bar{\phi}_A C^a a_{(i)} \beta_{C(i)},
\]

\[
\delta \lambda_{A(i)}^{(\beta)} = \sqrt{2} \nabla_A B' \phi \beta_{B'(i)} + \frac{i}{\sqrt{2}} e^{2\phi} \nabla_A B' a \beta_{B'(i)} + 2ie^{-\phi} \bar{\phi}_A C a_{(i)} \alpha_{C(i)}.
\]

The motivation for the specific factors that we used in the above definitions was to derive the desired energy bound as well as to have the supergravity transformation laws of the appropriate particles in the associated supergravity model. As was mentioned our model, EMAD gravity, constitutes the bosonic part of \( N = 4, \ d = 4 \) supergravity with only one gauge field. In order to achieve the right-hand side of equation \([16]\) positive one should have the following conditions satisfied:

\[
T_{ab}(\text{matter}) \xi^a t^b \geq \left( \sqrt{2} e^{\phi} J_{AA'}(\bar{z}, F) \alpha^{K(i)} \bar{\beta}_{K(i)} + c.c. \right) t^{AA'} + 2 \sqrt{2} J_{AA'}(\phi, F) \bar{\alpha}^{K(i)} \beta_{K(i)} t^{AA'}.
\]
Condition (23) is stronger than the dominant energy condition normally assumed in general relativity to prove the positivity of energy. It reveals the fact that the local energy density is greater or equal to the densities of the adequate charge densities.

On the spacelike two-surface Σ we define a spinor basis \((\alpha_A, i^A)\) on Σ in such a manner that \(l^a = o^A\bar{\alpha}^A\) and \(n^a = i^A\bar{\alpha}_A\) are future-directed outgoing and incoming null normals to the considered spacelike two-surface, while \(m^a = o^A\bar{\alpha}^A\) is tangent vector to Σ. The differential operators \(\delta = o^A\bar{\alpha}^A \nabla_{AA'}\) and \(\bar{\delta} = i^A\bar{\alpha}^A \nabla_{AA'}\) and their generalization \(\partial\) and \(\bar{\partial}\) are all intrinsic on Σ [19].

One supposes further, that the number of spinor fields on Σ is restricted so that the space of such fields is isomorphic to a two-dimensional complex vector space \(\mathcal{S}\). Following Dougan and Mason [21] we can define the two-dimensional complex vector space \(\mathcal{S}\) by choosing the propagation equations for the spinor fields \(\alpha_A(i)\) and \(\bar{\beta}_A(i)\) on the spacelike two-surface Σ. Namely, we shall consider spinor fields satisfying equations \(\partial_{A} A = 0\), (or \(\partial_{A} A = 0\)) and \(\partial_{A} \bar{B}_A = 0\), (or \(\partial_{A} \bar{B}_A = 0\)). If \(\alpha_A^A = \beta_A^A\) where \(\underline{A} = 0, 1\) and \(\underline{A}' = 0, 1\) are such two fields, then in the standard way [13, 14, 22] one may define a four-momentum corresponding to the spacelike two-surface Σ by the following expression:

\[
P^{\underline{A}'A}(\Sigma, \alpha_{\underline{A}}, \bar{\alpha}_{\underline{A}'}, \bar{\beta}_{\underline{A}}, \beta_{\underline{A}'}) = \int_{\Sigma} \Theta(\alpha_{\underline{A}}, \bar{\alpha}_{\underline{A}'}, \bar{\beta}_{\underline{A}}, \beta_{\underline{A}'}) ,
\]

where Σ is normally assumed to be homeomorphic to the two-sphere.

When the hypersurface Σ is a cross section of null infinity and when \((\alpha^A_{A(\infty)}, \beta^A_{A(\infty)})\) are required to be asymptotically constant spinors, then \(P^{\underline{A}'A}(\Sigma, \alpha_{\underline{A}}, \bar{\alpha}_{\underline{A}'}, \bar{\beta}_{\underline{A}}, \beta_{\underline{A}'})\) is the Bondi four-momentum. On the other hand, at spacelike infinity one gets the ADM four-momentum.

From now on (for simplicity) we shall drop the underlined indices and denote \(P^{\underline{A}'A} = P\) and \(\alpha^0_{A(i)} = \alpha_{A(i)}\), etc.. The Geroch-Held-Penrose formalism [14, 22] enables us to express the quantity \(P(\Sigma, \alpha_{\underline{A}}, \bar{\alpha}_{\underline{A}'}, \bar{\beta}_{\underline{A}}, \beta_{\underline{A}'})\) in the following form:

\[
P(\Sigma, \alpha_{\underline{A}}, \bar{\alpha}_{\underline{A}'}, \bar{\beta}_{\underline{A}}, \beta_{\underline{A}'}) =
\]

\[
\int_{\Sigma} d\Sigma \left[ -\bar{\alpha}^{(i)} \left( \partial_{0} \alpha_{0(i)} + \rho' \alpha_{0(i)} + \frac{ie^{2\phi}}{2} \partial a \alpha_{0(i)} + \sqrt{2} e^{-\phi} \phi_{(1)} a_{(ij)} \beta^{(j)} \right) + \alpha^{(i)} \left( \bar{\partial}_{0} \alpha_{0(i)} + \rho \alpha_{0(i)} + \frac{ie^{2\phi}}{2} \bar{\partial} a \alpha_{0(i)} + \sqrt{2} e^{-\phi} \phi_{(1)} a_{(ij)} \beta^{(j)} \right) \right]
\]

\[
+ \int_{\Sigma} d\Sigma \left[ -\beta^{(i)} \left( \partial_{0} \bar{\beta}_{0(i)} + \rho' \bar{\beta}_{0(i)} - \frac{ie^{2\phi}}{2} \partial a \bar{\beta}_{0(i)} - \sqrt{2} e^{-\phi} \phi_{(1)} a_{(ij)} \bar{\alpha}^{(j)} \right) + \bar{\beta}^{(i)} \left( \bar{\partial}_{0} \bar{\beta}_{0(i)} + \rho \bar{\beta}_{0(i)} - \frac{ie^{2\phi}}{2} \bar{\partial} a \bar{\beta}_{0(i)} - \sqrt{2} e^{-\phi} \phi_{(1)} a_{(ij)} \bar{\alpha}^{(j)} \right) \right],
\]

where \(\phi_{(1)} = \phi_{AB} a^A i^B\) is a complex scalar generated from a symmetric two-spinor describing the Maxwell field, and \(\alpha_{0(i)} = \alpha_{A(i)} a^A\), \(\alpha_{1(i)} = \alpha_{A(i)} i^A\) while \(\beta_{0(i)} = \bar{\beta}_{A(i)} a^A\), \(\beta_{1(i)} = \bar{\beta}_{A(i)} i^A\).

Thus we shall consider the following propagation equations on Σ:

\[
\tilde{\tau}^{\underline{A}'A} \tilde{m}^b \nabla_b \beta^{\underline{A}'(i)} = \partial \beta^{\underline{A}'(i)} + \bar{\rho} \beta^{\underline{A}'(i)} + \frac{ie^{2\phi}}{2} \bar{\partial} a \beta^{\underline{A}'(i)} - \sqrt{2} e^{-\phi} \phi_{(1)} a_{(ij)} \beta^{\underline{A}'(i)} = 0 ,
\]

\[
(26)
\]

5
\[ i^A m^b \hat{\nabla}_b \alpha_{A(i)} = \partial \alpha_{A(i)} + \rho' \alpha_{0(i)} + \frac{ie^{2\phi}}{2} \partial a \alpha_{A(i)} + \sqrt{2} e^{-\phi} \phi_{(1)} a_{(ij)} \beta_{V(j)}^{(i)} = 0, \]  

(27)

and their conjugates. It will be easily verified that by virtue of the equations (26) and (27) and integration by parts. Finally one gets the formula due to the fact that 

\[ P(\Sigma, \alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_{A'}) = \int_{\Sigma} d\Sigma \left( \rho' \bar{\alpha}_{\Psi(i)} \alpha_0^{(i)} + \rho \bar{\alpha}_{V(i)} \alpha_1^{(i)} \right) \]

\[ + \sqrt{2} e^{-\phi} \bar{\phi}_{(1)} a_{(ij)} \alpha_0^{(i)} \beta_1^{(j)} + \sqrt{2} e^{-\phi} \phi_{(1)} \alpha_1^{(i)} a_{(ij)} \beta_1^{(j)} \]

\[ + \int_{\Sigma} d\Sigma \left( \rho' \beta_{\Psi(i)} \bar{\beta}_0^{(i)} + \rho \beta_{V(i)} \bar{\beta}_1^{(i)} \right) \]

\[ - \sqrt{2} e^{-\phi} \bar{\phi}_{(1)} a_{(ij)} \alpha_1^{(i)} \beta_0^{(j)} - \sqrt{2} e^{-\phi} \phi_{(1)} \alpha_0^{(i)} a_{(ij)} \beta_1^{(j)}. \]

(28)

For two-spinors \( \mu_A, \bar{\mu}_{A'}, \bar{\eta}_A, \eta_{A'} \) on \( H \) satisfying the Witten equations (17) and the condition (23) we have \( P(\Sigma, \mu_A, \bar{\mu}_{A'}, \bar{\eta}_A, \eta_{A'}) \geq 0 \). Having given \( \alpha_{A(i)} \) and \( \beta_{A'(i)} \) and their conjugates satisfying relations (26, 27) one can find such \( \mu_{1(i)} \) and \( \bar{\eta}_{1(i)} \) on \( H \) fulfilling the boundary conditions \( \mu_{1(i)} = \alpha_{1(i)} \) and \( \bar{\eta}_{1(i)} = \bar{\beta}_{1(i)} \) on \( \Sigma \). The existence of such \( \mu_{1(i)} \) and \( \bar{\eta}_{1(i)} \) can be proved \( \{21\} \) by the direct application of the Fredholm alternative (see, e.g., \( \{23\} \)) or by using the known theorems \( \{24\} \) for compact hypersurfaces. Having in mind \( \{28\} \), we shall relate \( P(\Sigma, \alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_{A'}) \) to \( P(\Sigma, \mu_A, \bar{\mu}_{A'}, \bar{\eta}_A, \eta_{A'}) \). First one replaces \( \alpha_{1(i)} \) by \( \mu_{1(i)} \) and \( \bar{\beta}_{1(i)} \) by \( \bar{\eta}_{1(i)} \) in equation (28). Then, we use equations (26) and (27) and integration by parts. Finally one gets the formula

\[ P(\Sigma, \mu_A, \bar{\mu}_{A'}, \bar{\eta}_A, \eta_{A'}) = P(\Sigma, \alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_{A'}) \]

\[ - \int_{\Sigma} d\Sigma \rho' |\alpha_0^{(i)} - \mu_{0(i)}|^2 - \int_{\Sigma} d\Sigma \rho' |\bar{\beta}_{0(i)} - \bar{\eta}_{0(i)}|^2. \]

(29)

We assume that \( \rho' \geq 0 \) on the spacelike two-surface \( \Sigma \) and because of this fact the left-hand side of the above equation is greater or equal to zero on \( \Sigma \), namely

\[ P(\Sigma, \alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_{A'}) \geq 0. \]

(30)

Let us remark that the case of the other possible equations of spinors propagation of the forms

\[ \bar{\sigma}^A m^b \hat{\nabla}_b \beta_{A'i} = \partial \beta_{\Psi'(i)} + \bar{\rho} \beta_{V'(i)} + \frac{ie^{2\phi}}{2} \partial a \beta_{\Psi'(i)} \]

\[ - \sqrt{2} e^{-\phi} \bar{\phi}_{(1')} a_{(ij)} \beta_{V'}^{(i)} = 0, \]

(31)

\[ \sigma^A m^b \hat{\nabla}_b \alpha_{A(i)} = \partial \alpha_{0(i)} + \rho \alpha_{1(i)} + \frac{ie^{2\phi}}{2} \partial a \alpha_{0(i)} + \sqrt{2} e^{-\phi} \phi_{(1)} a_{(ij)} \beta_{V'}^{(i)} = 0, \]

(32)

and their conjugates, follow almost in the same way with a slight modification of the arguments due to the fact that \( \rho = \bar{\rho} \) and the assumption \( \rho \leq 0 \). All these lead us to the same conclusion. Thus, from the above considerations we can deduce the following result:

**Theorem**

Consider a three-dimensional compact spacelike hypersurface \( H \) on which the condition (23) is satisfied. Let \( \Sigma \) be a spacelike two-surface spanned by a three-dimensional hypersurface...
\(\mathcal{H} (\Sigma = \partial\mathcal{H})\). Suppose further, that the propagation equations (20-27) hold on \(\Sigma\) and assume that \(\rho' \geq 0\) (or \(\rho \leq 0\) in the case of equations (31)-(32)). Then, it follows
\[
\int_{\Sigma} \Theta(\alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_{A'}) \geq 0.
\]

Having established the fact that \(P(\Sigma, \alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_{A'}) \geq 0\) we can proceed to the proof of the positivity of energy in EMAD gravity. Thus, let us consider a three-surface extension inwards from infinity (spacelike or null) and foliate it by two-surfaces \(\Sigma_r\) \((r \to \infty)\) at infinity). We notice that one can always find families of two-surfaces approaching spacelike or null infinity with \(\rho' > 0\). Namely, following Chrusciel [25] we shall call a hypersurface asymptotically flat if it contains an asymptotically flat end, i.e., a data set \((\Sigma_{\text{end}}, g_{ij}, K_{ij})\) with gauge and dilaton fields such that \(\Sigma_{\text{end}}\) is diffeomorphic to \(\mathbb{R}^3\) minus a ball and the following asymptotic conditions are fulfilled:

\[
|g_{ij} - \delta_{ij}| + r|\partial_a g_{ij}| + \ldots + r^k|\partial_{a_1 \ldots a_k} g_{ij}| + r|K_{ij}| + \ldots + r^k|\partial_{a_1 \ldots a_k} K_{ij}| \leq O\left(\frac{1}{r}\right),
\]

\[
|F_{a\beta}| + r|\partial_a F_{a\beta}| + \ldots + r^k|\partial_{a_1 \ldots a_k} F_{a\beta}| \leq O\left(\frac{1}{r^2}\right),
\]

\[
\phi = \phi_\infty + O\left(\frac{1}{r}\right).
\]

In a flat spacetime \(\rho' = \frac{1}{r}\) for spheres around \(r = 0\). Having in mind the results presented in [24] one can deduce that in asymptotically flat case near infinity we have \(\rho' = \frac{1}{r} + O\left(\frac{1}{r^2}\right)\). The same situation takes place for cross sections of a null hypersurface approaching a given cross section of future null infinity [19].

On \(\Sigma_r\) we assume that \(\rho' \geq 0\) for \(r\) large enough. Suppose further that we have given values of asymptotically constant spinors \((\alpha_{A(\infty)}, \bar{\beta}_{A'(\infty)})\) at infinity. Let us extend the components of \(\alpha_{1(i)}\) and \(\bar{\beta}_{1(i)}\) to all hypersurfaces \(\Sigma_r\) in an arbitrary differentiable way. Next define \(\alpha_{0(i)}\) and \(\bar{\beta}_{0(i)}\) such that

\[
\alpha_{0(i)} = \frac{1}{\rho'} \left( -\partial_\alpha_{1(i)} - \frac{ie^{2\phi}}{2} \partial_a \alpha_{1(i)} - \sqrt{2} e^{-\phi} \phi_{(1)} a_{1(i)} \beta_{1(j)}^{(j)} \right),
\]

and

\[
\bar{\beta}_{0(i)} = \frac{1}{\rho'} \left( -\partial_{\bar{\beta}_{1(i)}} + \frac{ie^{2\phi}}{2} \partial_a \bar{\beta}_{1(i)} + \sqrt{2} e^{-\phi} \phi_{(1)} a_{1(i)} \bar{\alpha}_{1(j)}^{(j)} \right).
\]

Hence, from the previous theorem we have that \(\int_{\Sigma_r} \Theta(\alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_{A'}) \geq 0\) for all \(r > R\). In an asymptotically flat spacetime we have introduced \(\Sigma_r\) a family of two-surfaces approaching spacelike infinity or a cross section of future null infinity as \(r \to \infty\). For asymptotically constant spinors \((\alpha_{A(\infty)}, \beta_{A'(\infty)})\) at spacelike or null infinity we can define

\[
P_l k^l = \lim_{r \to \infty} \int_{\Sigma_r} \Theta(\alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_{A'}) \geq 0,
\]

which is respectively the ADM or Bondi momentum, while \(k^a = \lim_{r \to \infty} \left( \alpha_{A(i)} \bar{\alpha}_{A'(i)} + \bar{\beta}_{A(i)} \beta_{A'(i)} \right)\) is a future pointing asymptotic null translation. Using a linear combinations of null vectors, the quantity \(P_l\) can be determined completely.
Then, in order to prove the positivity of energy it is necessary to get the relation that $P_i u^i \geq 0$, for all future pointed asymptotic translations $u^i$. On the other hand, since a translation $u^i$ can be expressed as a linear combination of two future pointed null vectors, it will be sufficient to find that $P_i k_i \geq 0$, for all null $k^i$.

One can notice that the above limit exist because it is energy. In the above proof spinor fields defining the energy need to be defined near infinity and obey the propagation equations (26 -27).

Positivity of it is proved by having in mind the previous theorem and the Witten’s propagation equation for auxiliary spinor fields $\mu_{A(i)}$ and $\eta_{A(i)}$ on a compact set.

We can therefore assert the following estimate:

**Theorem**

Suppose that the condition (23) is satisfied in an asymptotically flat spacetime. Then, $P_i k_i \geq 0$, where $P_i$ is the ADM or Bondi momentum.

Now we take up a question of the stronger inequality the ADM or Bondi mass should satisfy. In order to do so one considers the situation when $\alpha_{A(i)}$ and $\bar{\beta}_{A(i)}$ approach the constant spinors at infinity. Using the exact form of the two-surface bivector $dS_{ab} = l_{[a} n_{b]} dS$, one obtains

$$
\int_{\Sigma} \Theta(\alpha_A^A, \alpha_A^{A'}, \bar{\beta}_A^A, \bar{\beta}_A^{A'}) = \hat{P}^m(\alpha_A^A, \alpha_A^{A'}, \bar{\beta}_A^A, \bar{\beta}_A^{A'})k_m + \frac{1}{\sqrt{2}} Q_{(F-\phi)}\left( \bar{\alpha}_i^{A'(i)} \beta_{A(i)(\infty)} \right) + \frac{1}{\sqrt{2}} P_{(F-\phi)}\left( \bar{\alpha}_{(\infty)}^{A'(i)} \beta_{A(i)(\infty)} \right),
$$

(39)

where the dilaton-electric charge is defined as [13]

$$
Q_{(F-\phi)} = 2 \int_{S_\infty} dS e^{-\phi_{\infty}} \Re \phi_{(1)},
$$

(40)

and consequently the magnetic-dilaton charge is expressed as follows:

$$
P_{(F-\phi)} = 2 \int_{S_\infty} dS e^{-\phi_{\infty}} \Im \phi_{(1)},
$$

(41)

while $\hat{P}_i k_i$ is denoted by the expression

$$
\hat{P}_i k_i = \lim_{r \to \infty} \int_{\Sigma_r} \Theta(\alpha_A^A, \alpha_A^{A'}, \bar{\beta}_A^A, \bar{\beta}_A^{A'}).
$$

(42)

In equation (42) we have used the following definition:

$$
\Theta(\alpha_A, \bar{\alpha}_A, \bar{\beta}_A, \beta_A) = -i \left( \bar{\alpha}_{B(i)} \nabla_a \alpha_{B(i)} \, dx^a \wedge dx^b + \beta_{B(i)} \nabla_a \beta_{B(i)} \, dx^a \wedge dx^b \right).
$$

(43)

Next following [21] let us define quantity $m(\Sigma)$ of the form

$$
m^2(\Sigma) = \hat{P}^A A'(\Sigma, \alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_A) \hat{P}^{B B'}(\Sigma, \alpha_A, \bar{\alpha}_{A'}, \bar{\beta}_A, \beta_A) \epsilon_{AB} \epsilon_{A'B'},
$$

(44)

where $\epsilon_{AB} = (\alpha_A^A \alpha_{B}^B + \bar{\beta}_A^A \bar{\beta}_B^B) \epsilon_{AB}$. From the propagation equations for spinors, i.e., $\delta \alpha_A^A = \delta \bar{\beta}_A^A = 0$ or $\delta \bar{\alpha}_A^{A'} = \delta \bar{\beta}_A^{A'} = 0$ one has that $\delta \epsilon_{AB} = 0$ or $\delta \epsilon_{A'B'} = 0$. So that $\epsilon_{AB}$ is anti-holomorphic ($\epsilon_{A'B'}$ is holomorphic) and global function on the sphere and by means of the
Liouville’s theorem it is constant. We shall call \( m(\Sigma) \) the mass. If \( \Sigma \) is a cross section of null infinity we shall have the Bondi mass, while at spatial infinity one gets the ADM mass. Moreover one should demand that the two-dimensional complex vector space will be the space of asymptotically constant spinors.

Thus, from equation (39) we deduce that the mass \( m(\Sigma) \) must be positive for all spinors \( \bar{\alpha}^{(i)}_{\infty} \) and \( \beta^{(i)}_{\infty} \), and the following inequality binding mass with dilaton-electric and dilaton-magnetic charges should be fulfilled:

\[
m^2(\Sigma) \geq Q^2_{(F-\phi)} + P^2_{(F-\phi)}.
\] (45)

3 Conclusions

In our work we considered the positivity of the ADM and Bondi energy in EMAD gravity. Extending the reasoning presented in [15, 21] we gave the proof of positivity of energy without proving the existence of asymptotically constant spinors satisfying differential equations on a non-compact hypersurface. The spinor fields defining the energy \( (\alpha_{A(i)}, \beta_{A'(i)}) \) need only to be defined near infinity (near spacelike infinity to get the ADM energy or near null infinity to achieve the Bondi energy). They also should fulfill propagation equations on a spacelike two-surface \( \Sigma \) and the condition (23) for \( T_{ab}(\text{matter}) \) and fields in the theory under consideration. We have also establish the stronger inequality binding the ADM and Bondi mass with the dilaton-electric and dilaton-magnetic charges. However, our new proof of the positive energy theorem does not simplify the proof for the flat spacetime. We hope to return to this problem elsewhere.

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