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Superhedging and Dynamic Risk Measures under Volatility Uncertainty

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Abstract

We consider dynamic sublinear expectations (i.e., time-consistent coherent risk measures) whose scenario sets consist of singular measures corresponding to a general form of volatility uncertainty. We derive a càdlàg nonlinear martingale which is also the value process of a superhedging problem. The superhedging strategy is obtained from a representation similar to the optional decomposition. Furthermore, we prove an optional sampling theorem for the nonlinear martingale and characterize it as the solution of a second order backward SDE. The uniqueness of dynamic extensions of static sublinear expectations is also studied.

Keywords volatility uncertainty, risk measure, time consistency, nonlinear martingale, superhedging, replication, second order BSDE, G-expectation

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1 Introduction

Coherent risk measures were introduced in [1] as a way to quantify the risk associated with a financial position. Since then, coherent risk measures and sublinear expectations (which are the same up to the sign convention) have been studied by numerous authors; see [13, 24, 25] extensive references. Most of these works consider the case where scenarios are probability measures absolutely continuous with respect to a given reference probability (important early exceptions are [12, 21]). The present paper studies dynamic sublinear expectations and superhedging under volatility uncertainty, which is naturally related to singular measures. The concept of volatility uncertainty was

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introduced in financial mathematics by [2, 11, 19] and has recently received considerable attention due to its relation to $G$-expectations [22, 23] and second order backward stochastic differential equations [6, 29], called 2BSDEs for brevity.

Any (static) sublinear expectation $\mathcal{E}^0_0$, defined on the set of bounded measurable functions on a measurable space $(\Omega, \mathcal{F})$, has a convex-dual representation

$$\mathcal{E}^0_0(X) = \sup_{P \in \mathcal{P}} E^P[X]$$

for a certain set $\mathcal{P}$ of measures which are $\sigma$-additive as soon as $\mathcal{E}^0_0$ satisfies certain continuity properties (cf. [13, Section 4]). The elements of $\mathcal{P}$ can be seen as possible scenarios in the presence of uncertainty and hence (1.1) corresponds to the worst-case expectation. In this paper, we take $\Omega$ to be the canonical space of continuous paths and $\mathcal{P}$ to be a set of martingale laws for the canonical process, corresponding to different scenarios of volatilities. For this case, $\mathcal{P}$ is typically not dominated by a finite measure and (1.1) was studied in [5, 10, 11] by capacity-theoretic methods.

While any set of martingale laws gives rise to a static sublinear expectation via (1.1), we are interested in dynamic sublinear expectations; i.e., conditional versions of (1.1) satisfying a time-consistency property. If $\mathcal{P}$ is dominated by a probability $P_\ast$, a natural extension of (1.1) is given by

$$\mathcal{E}^{0,P_\ast}_t(X) = \operatorname{ess sup}_{P' \in \mathcal{P}(\mathcal{F}^t_\circ, P_\ast)} E^{P'}[X|\mathcal{F}^t_\circ] \quad P_\ast\text{-a.s.},$$

where $\mathcal{P}(\mathcal{F}^t_\circ, P_\ast) = \{P' \in \mathcal{P} : P' = P_\ast \text{ on } \mathcal{F}^t_\circ\}$ and $\mathcal{F}^0 = \{\mathcal{F}^t_\circ\}$ is the filtration generated by the canonical process. Such dynamic expectations are well-studied; in particular, time consistency of $\mathcal{E}^{0,P_\ast}_t$ can be characterized by a stability property of $\mathcal{P}$ (see [7]). In the non-dominated case, we can similarly consider the family of random variables $\{\mathcal{E}^{0,P}_t(X), P \in \mathcal{P}\}$. Since a reference measure is lacking, it is not straightforward to construct a single random variable $\mathcal{E}^0_t(X)$ such that

$$\mathcal{E}^0_t(X) = \mathcal{E}^{0,P}_t(X) = \operatorname{ess sup}_{P' \in \mathcal{P}(\mathcal{F}^t_\circ, P)} E^{P'}[X|\mathcal{F}^t_\circ] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P}. \quad (1.2)$$

This problem of aggregation has been solved in several examples. In particular, the $G$-expectations and random $G$-expectations [20] (recalled in Section 2) correspond to special cases of (1.2). The construction of $G$-expectations is based on a PDE, which directly yields random variables defined for all $\omega \in \Omega$. The random $G$-expectations are defined pathwise using regular conditional probability distributions. A general study of aggregation problems is presented in [28], while [4] will provide a solution for the aggregation (1.2) when $X$ and $\mathcal{P}$ satisfy certain regularity conditions. However, the study of aggregation is not an object of the present paper. In view of the diverse approaches, we shall proceed axiomatically and start with
Given aggregated family \( \{ \mathcal{E}_t(X), t \in [0, T] \} \). Having in mind the example of (random) \( G \)-expectations, this family is assumed to be given in the raw filtration \( \mathbb{F}^0 \) and without any regularity in the time variable.

The main goal of the present paper is to provide basic technology for the study of dynamic sublinear expectations under volatility uncertainty as stochastic processes. Given the family \( \{ \mathcal{E}_t(X), t \in [0, T] \} \), we construct a corresponding càdlàg process \( \mathcal{E}(X) \), called the \( \mathcal{E} \)-martingale associated with \( X \), in a suitably enlarged filtration \( \mathbb{F} \) (Proposition 4.5). We use this process to define the sublinear expectation at stopping times and prove an optional sampling theorem for \( \mathcal{E} \)-martingales (Theorem 4.9). Furthermore, we obtain a decomposition of \( \mathcal{E}(X) \) into an integral of the canonical process and an increasing process (Proposition 4.10), similarly as in the classical optional decomposition [17]. In particular, the \( \mathcal{E} \)-martingale yields the dynamic superhedging price of the financial claim \( X \) and the integrand \( Z^X \) yields the superhedging strategy. We also provide a connection between \( \mathcal{E} \)-martingales and 2BSDEs by characterizing \( (\mathcal{E}(X), Z^X) \) as the minimal solution of such a backward equation (Theorem 4.15). Our last result concerns the uniqueness of time-consistent extensions and gives conditions under which (1.2) is indeed the only possible extension of the static expectation (1.1). In particular, we introduce the notion of local strict monotonicity to deal with the singularity of the measures (Proposition 5.3).

To obtain our results, we rely on methods from stochastic optimal control and the general theory of stochastic processes. Indeed, from the point of view of dynamic programming, \( \mathcal{E}_t(X) \) is the value process of a control problem defined over a set of measures, and time consistency corresponds to Bellman’s principle. Taking the control representation (1.2) as our starting point allows us to consider the measures \( P \in \mathcal{P} \) separately in many arguments and therefore to apply standard arguments of the general theory.

The remainder of this paper is organized as follows. In Section 2 we detail the setting and notation. Section 3 relates time consistency to a pasting property. In Section 4 we construct the \( \mathcal{E} \)-martingale and provide the optional sampling theorem, the decomposition, and the characterization by a 2BSDE. Section 5 studies the uniqueness of time-consistent extensions.

2 Preliminaries

We fix a constant \( T > 0 \) and let \( \Omega = \{ \omega \in C([0, T]; \mathbb{R}^d) : \omega_0 = 0 \} \) be the canonical space of continuous paths equipped with the uniform topology. We denote by \( B \) the canonical process \( B_t(\omega) = \omega_t \), by \( P_0 \) the Wiener measure and by \( \mathbb{F}^0 = \{ \mathcal{F}_t^0 \}_{0 \leq t \leq T}, \mathcal{F}_t^0 = \sigma(B_s, s \leq t) \) the raw filtration generated by \( B \). As in [10, 20, 26, 29] we shall use the so-called strong formulation of volatility uncertainty in this paper; i.e., we consider martingale laws induced by stochastic integrals of \( B \) under \( P_0 \). More precisely, we define \( \mathcal{P}_S \) to be
the set of laws
\[ P^\alpha := P_0 \circ (X^\alpha)^{-1}, \quad \text{where} \quad X^\alpha_t := \int_0^t \alpha_s^{1/2} dB_s, \quad t \in [0, T] \] (2.1)
and \( \alpha \) ranges over all \( \mathbb{F}^0 \)-progressively measurable processes with values in \( \mathbb{S}_{d}^{>0} \) satisfying \( \int_0^T |\alpha_t| \, dt < \infty \) \( P_0 \)-a.s. Here \( \mathbb{S}_{d}^{>0} \subset \mathbb{R}^{d \times d} \) denotes the set of strictly positive definite matrices and the stochastic integral in (2.1) is the Itô integral under \( P_0 \), constructed in \( \mathbb{F}^0 \) (cf. [31, p. 97]). We remark that \( \mathcal{P}_S \) coincides with the set denoted by \( \mathcal{P}_S \) in [28].

The basic object in this paper is a nonempty set \( \mathcal{P} \subseteq \mathcal{P}_S \) which represents the possible scenarios for the volatility. For \( t \in [0, T] \), we define \( L^1_\mathcal{P}(\mathcal{F}_t^\circ) \) to be the space of \( \mathcal{F}_t^\circ \)-measurable random variables \( X \) satisfying
\[ \|X\|_{L^1_\mathcal{P}} := \sup_{P \in \mathcal{P}} \|X\|_{L^1(P)} < \infty, \]
where \( \|X\|_{L^1(P)} := E[|X|] \). More precisely, we take equivalences classes with respect to \( \mathcal{P} \)-quasi-sure equality so that \( L^1_\mathcal{P}(\mathcal{F}_t^\circ) \) becomes a Banach space.

(Two functions are equal \( \mathcal{P} \)-quasi-surely, \( \mathcal{P} \)-q.s. for short, if they are equal up to a \( \mathcal{P} \)-polar set. A set is called \( \mathcal{P} \)-polar if it is a \( \mathcal{P} \)-nullset for all \( \mathcal{P} \in \mathcal{P} \).)

We also fix a nonempty subset \( \mathcal{H} \) of \( L^1_\mathcal{P}(\mathcal{F}_t^\circ) \) whose elements play the role of financial claims. We emphasize that in applications, \( \mathcal{H} \) is typically smaller than \( L^1_\mathcal{P} \).

**Example 2.1.** (i) Given real numbers \( 0 \leq \underline{a} \leq \overline{a} < \infty \), the associated \( G \)-expectation (for dimension \( d = 1 \)) corresponds to the choice
\[ \mathcal{P} = \{ P^\alpha \in \mathcal{P}_S : \underline{a} \leq \alpha \leq \overline{a} \quad P_0 \times dt \text{-a.e.} \}, \] (2.2)
cf. [10, Section 3]. Here the symbol \( G \) refers to the function
\[ G(\gamma) := \frac{1}{2} \sup_{\underline{a} \leq \alpha \leq \overline{a}} a \gamma. \]
If \( X = f(B_T) \) for a sufficiently regular function \( f \), then \( \mathcal{E}_t^{\circ,G}(X) \) is defined via the solution of the nonlinear heat equation 
\[-\partial_t u - G(u_{xx}) = 0 \]
with boundary condition \( u|_{t=T} = f \). In [22], the mapping \( \mathcal{E}_t^{\circ,G} \) is extended to random variables of the form \( X = f(B_{t_1}, \ldots, B_{t_n}) \) by a stepwise evaluation of the PDE and finally to the \( \|\cdot\|_{L^1_\mathcal{P}} \)-completion \( \mathcal{H} \) of the set of all such random variables. For \( X \in \mathcal{H} \), the \( G \)-expectation then satisfies
\[ \mathcal{E}_t^{\circ,G}(X) = \text{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_t^\circ)} E^{P'}[X|\mathcal{F}_t^\circ] \quad P\text{-a.s. for all } P \in \mathcal{P}, \]
which is of the form (1.2). The space \( \mathcal{H} \) coincides with the \( \|\cdot\|_{L^1_\mathcal{P}} \)-completion of \( C_b(\Omega) \), the set of bounded continuous functions on \( \Omega \), and is strictly smaller than \( L^1_\mathcal{P} \) as soon as \( \underline{a} \neq \overline{a} \).
(ii) The random $G$-expectation corresponds to the case where $\xi$, $\pi$ are random processes instead of constants and is directly constructed from a set $\mathcal{P}$ of measures (cf. [20]). In this case the space $\mathcal{H}$ is the $\|\cdot\|_{L^1_P}$-completion of $UC_b(\Omega)$, the set of bounded uniformly continuous functions on $\Omega$. If $\pi$ is finite-valued and uniformly bounded, $\mathcal{H}$ coincides with the space from (i).

3 Time Consistency and Pasting

In this section, we consider time consistency as a property of the set $\mathcal{P} \subseteq \mathcal{P}_S$ and obtain some auxiliary results for later use. The set $\mathcal{H} \subseteq L^1_P$ is fixed throughout. Moreover, we let $T(\mathcal{F}^\circ)$ be the set of all $\mathcal{F}^\circ$-stopping times taking finitely many values; this choice is motivated by the applications in the subsequent section. However, the results of this section hold true also if $T(\mathcal{F}^\circ)$ is replaced by an arbitrary set of $\mathcal{F}^\circ$-stopping times containing $\equiv 0$; in particular, the set of all stopping times and the set of all deterministic times. Given $A \subseteq \mathcal{F}^\circ_T$ and $P \in \mathcal{P}$, we use the standard notation $P(A, P) = \{P' \in \mathcal{P} : P' = P \text{ on } A\}$. At the level of measures, time consistency can then be defined as follows.

**Definition 3.1.** The set $\mathcal{P}$ is $\mathcal{F}^\circ$-time-consistent on $\mathcal{H}$ if

$$\text{ess sup}^P E^P\left[ \text{ess sup}^{P'} E^{P'}[X | \mathcal{F}^\circ_{\tau} \upharpoonright \mathcal{F}^\circ_{\sigma}^\circ] \right] = \text{ess sup}^P E^P[X | \mathcal{F}^\circ_{\sigma}^\circ] \quad P\text{-a.s.}$$

for all $P \in \mathcal{P}$, $X \in \mathcal{H}$ and $\sigma \leq \tau$ in $T(\mathcal{F}^\circ)$.

We shall relate the previous property to the following notion of stability, also called m-stability, fork-convexity, stability under concatenation, etc.

**Definition 3.2.** The set $\mathcal{P}$ is stable under $\mathcal{F}^\circ$-pasting if for all $P \in \mathcal{P}$, $\tau \in T(\mathcal{F}^\circ)$, $\Lambda \in \mathcal{F}^\circ_T$ and $P_1, P_2 \in \mathcal{P}(\mathcal{F}^\circ_T, P)$, the measure $\bar{P}$ defined by

$$\bar{P}(A) := E^P[P_1(A | \mathcal{F}_T^\circ)1_{\Lambda} + P_2(A | \mathcal{F}_T^\circ)1_{\Lambda^c}], \quad A \in \mathcal{F}_T^\circ$$

(3.2)

is again an element of $\mathcal{P}$.

As $\mathcal{F}^\circ$ is the only filtration considered in this section, we shall sometimes omit the qualifier “$\mathcal{F}^\circ$”.

**Lemma 3.3.** The set $\mathcal{P}_S$ is stable under pasting.

**Proof.** Let $P, P_1, P_2, \tau, \Lambda, \bar{P}$ be as in Definition 3.2. Using the notation (2.1), let $\alpha, \alpha^i$ be such that $P^\alpha = P$ and $P^{\alpha^i} = P_i$ for $i = 1, 2$. Setting

$$\bar{\alpha}_u(\omega) := 1_{[0, \tau(X^\alpha)])}^T(u) \alpha_u(\omega) + 1_{[\tau(X^\alpha), T]}^T(u) \left[ \alpha_{\omega}^1(\omega)1_\Lambda(X^\alpha(\omega)) + \alpha_{\omega}^2(\omega)1_{\Lambda^c}(X^\alpha(\omega)) \right],$$

we have $\bar{P} = P^{\bar{\alpha}} \in \mathcal{P}_S$ by the arguments in [26, Appendix].
The previous proof also shows that the set appearing in (2.2) is stable under pasting. The following result is classical.

**Lemma 3.4.** Let \( \tau \in T(\mathbb{F}) \), \( X \in L^1_P \) and \( P \in \mathcal{P} \). If \( \mathcal{P} \) is stable under pasting, then there exists a sequence \( P_n \in \mathcal{P}(\mathcal{F}_\tau^\circ, P) \) such that

\[
\operatorname{ess \ sup} P \ E^{P'} [X|\mathcal{F}_\tau^\circ] = \lim_{n \to \infty} E^{P_n} [X|\mathcal{F}_\tau^\circ] \quad P\text{-a.s.},
\]

where the limit is increasing \( P\text{-a.s.} \).

**Proof.** It suffices to show that the family \( \{ E^{P'}[X|\mathcal{F}_\tau^\circ] : P' \in \mathcal{P}(\mathcal{F}_\tau^\circ, P) \} \) is \( P\text{-a.s.} \) upward filtering (cf. [13, Theorem A.32]). Given \( P_1, P_2 \in \mathcal{P}(\mathcal{F}_\tau^\circ, P) \), we set

\[
\Lambda := \{ E^{P_1}[X|\mathcal{F}_\tau^\circ] > E^{P_2}[X|\mathcal{F}_\tau^\circ] \} \in \mathcal{F}_\tau^\circ
\]

and define \( \tilde{P}(A) := E^P[P_1(A|\mathcal{F}_\tau^\circ)1_A + P_2(A|\mathcal{F}_\tau^\circ)1_{A^c}] \). Then \( \tilde{P} = P \) on \( \mathcal{F}_\tau^\circ \) and \( \tilde{P} \in \mathcal{P} \) by the stability. Moreover, \( E^\tilde{P}[X|\mathcal{F}_\tau^\circ] = E^{P_1}[X|\mathcal{F}_\tau^\circ] \lor E^{P_2}[X|\mathcal{F}_\tau^\circ] \) \( P\text{-a.s.} \), showing that the family is upward filtering. \( \square \)

To relate time consistency to stability under pasting, we introduce the following closedness property.

**Definition 3.5.** We say that \( \mathcal{P} \) is maximally chosen for \( \mathcal{H} \) if \( \mathcal{P} \) contains all \( P \in \mathcal{P}_S \) satisfying \( E^P[X] \leq \sup_{P' \in \mathcal{P}} E^{P'}[X] \) for all \( X \in \mathcal{H} \).

If \( \mathcal{P} \) is dominated by a reference probability \( P_\ast \), then \( \mathcal{P} \) can be identified with a subset of \( L^1(P_\ast) \) by the Radon-Nikodym theorem. If furthermore \( \mathcal{H} = L^\infty(P_\ast) \), the Hahn-Banach theorem implies that \( \mathcal{P} \) is maximally chosen if and only if \( \mathcal{P} \) is convex and closed for weak topology of \( L^1(P_\ast) \). Along these lines, the following result can be seen as a generalization of [7, Theorem 12]; in fact, we merely replace functional-analytic arguments by algebraic ones.

**Proposition 3.6.** With respect to the filtration \( \mathbb{F}^\circ \), we have:

(i) If \( \mathcal{P} \) is stable under pasting, then \( \mathcal{P} \) is time-consistent on \( L^1_P \).

(ii) If \( \mathcal{P} \) is time-consistent on \( \mathcal{H} \) and maximally chosen for \( \mathcal{H} \), then \( \mathcal{P} \) is stable under pasting.

**Proof.** (i) This implication is standard; we provide the argument for later reference. The inequality “\( \geq \)” in (3.1) follows by considering \( P'' := P' \) on the left hand side. To see the converse inequality, fix an arbitrary \( P \in \mathcal{P} \) and choose a sequence \( P_n \in \mathcal{P}(\mathcal{F}_\tau^\circ, P) \subseteq \mathcal{P}(\mathcal{F}_\tau^\circ, P) \) as in Lemma 3.4. Then monotone convergence yields

\[
E^P \left[ \operatorname{ess \ sup} P \ E^{P'} [X|\mathcal{F}_\tau^\circ] \bigg| \mathcal{F}_\sigma^\circ \right] = \lim_{n \to \infty} E^{P_n} [X|\mathcal{F}_\sigma^\circ] \\
\leq \operatorname{ess \ sup} P \ E^{P'} [X|\mathcal{F}_\sigma^\circ] \quad P\text{-a.s.}
\]
(ii) Let $\mathcal{P}$ be time-consistent and let $P, P_1, P_2, \tau, \Lambda, \tilde{P}$ be as in Definition 3.2. For any $X \in \mathcal{H}$, we have

$$E_{\tilde{P}}[X] = E^P \left[ E^{P_1}[X|\mathcal{F}_\tau^\circ]1_\Lambda + E^{P_2}[X|\mathcal{F}_\tau^\circ]1_{\Lambda^c} \right]$$

$$\leq E^P \left[ \text{ess sup}_{P'' \in \mathcal{P}(\mathcal{F}_\tau^\circ, P)} E^{P''}[X|\mathcal{F}_\tau^\circ] \right]$$

$$\leq \sup_{P' \in \mathcal{P}} E^{P'} \left[ \text{ess sup}_{P'' \in \mathcal{P}(\mathcal{F}_\tau^\circ, P')} E^{P''}[X|\mathcal{F}_\tau^\circ] \right]$$

$$= \sup_{P' \in \mathcal{P}} E^{P'}[X],$$

where the last equality uses (3.1) with $\sigma \equiv 0$. Since $\mathcal{P}$ is maximally chosen and $\tilde{P} \in \mathcal{P}_S$ by Lemma 3.3, we conclude that $\tilde{P} \in \mathcal{P}$. □

4 $\mathcal{E}$-Martingales

As discussed in the introduction, our starting point in this section is a given family $\{\mathcal{E}_t^\circ(X), t \in [0, T]\}$ of random variables which will serve as a raw version of the $\mathcal{E}$-martingale to be constructed. We recall that the sets $\mathcal{P} \subseteq \mathcal{P}_S$ and $\mathcal{H} \subseteq L_1^P$ are fixed.

**Assumption 4.1.** Throughout Section 4, we assume that

(i) for all $X \in \mathcal{H}$ and $t \in [0, T]$, there exists an $\mathcal{F}_t^\circ$-measurable random variable $\mathcal{E}_t^\circ(X)$ such that

$$\mathcal{E}_t^\circ(X) = \text{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_t^\circ, P)} E^{P'}[X|\mathcal{F}_t^\circ] \quad P\text{-a.s. for all } P \in \mathcal{P}. \quad (4.1)$$

(ii) the set $\mathcal{P}$ is stable under $\mathbb{F}^\circ$-pasting.

The first assumption was discussed in the introduction; cf. (1.2). With the motivating Example 2.1 in mind, we ask for (4.1) to hold at deterministic times rather than at stopping times. The second assumption is clearly motivated by Proposition 3.6(ii), and Proposition 3.6(i) shows that $\mathcal{P}$ is time-consistent in the sense of Definition 3.1. (We could assume the latter property directly, but stability under pasting is more suitable for applications.) In particular, we have

$$\mathcal{E}_s^\circ(X) = \text{ess sup}_{P'' \in \mathcal{P}(\mathcal{F}_s^\circ, P)} E^{P''}[\mathcal{E}_t^\circ(X)|\mathcal{F}_s^\circ] \quad P\text{-a.s. for all } P \in \mathcal{P}, \quad (4.2)$$

$0 \leq s \leq t \leq T$ and $X \in \mathcal{H}$. If we assume that $\mathcal{E}_s^\circ(X)$ is again an element of the domain $\mathcal{H}$, this amounts to $\{\mathcal{E}_s^\circ\}$ being time-consistent (at deterministic times) in the sense that the semigroup property $\mathcal{E}_s^\circ \circ \mathcal{E}_t^\circ = \mathcal{E}_s^\circ$ is satisfied. However, $\mathcal{E}_s^\circ(X)$ need not be in $\mathcal{H}$ in general; e.g., for certain random
$G$-expectations. Inspired by the theory of viscosity solutions, we introduce the following extended notion of time consistency, which is clearly implied by (4.2).

**Definition 4.2.** A family $(E_t)_{0 \leq t \leq T}$ of mappings $E_t: \mathcal{H} \to L^1_P(\mathcal{F}_t^\circ)$ is called $\mathcal{F}^\circ$-time-consistent at deterministic times if for all $0 \leq s \leq t \leq T$ and $X \in \mathcal{H}$,

$$E_s(X) \leq (\geq) E_s(\varphi) \quad \text{for all } \varphi \in L^1_P(\mathcal{F}_t^\circ) \cap \mathcal{H} \text{ such that } E_t(X) \leq (\geq) \varphi.$$  

One can give a similar definition for stopping times taking countably many values. (Note that $E_t(X)$ is not necessarily well defined for a general stopping time $\tau$.)

**Remark 4.3.** If Assumption 4.1 is weakened by requiring $\mathcal{P}$ to be stable only under $\mathcal{F}^\circ$-pastings at deterministic times (i.e., Definition 3.1 holds with $T(\mathcal{F}^\circ)$ replaced by the set of deterministic times), then all results in this section remain true with the same proofs, except for Theorem 4.9, Lemma 4.14 and the last statement in Theorem 4.15.

### 4.1 Construction of the $\mathcal{E}$-Martingale

Our first task is to turn the collection $\{\mathcal{E}^\circ_t(X), t \in [0, T]\}$ of random variables into a reasonable stochastic process. As usual, this requires an extension of the filtration. We denote by

$$\mathcal{F}^+ = \{(\mathcal{F}^+)_t\}_{0 \leq t \leq T}, \quad (\mathcal{F}^+)_t := \mathcal{F}_t^\circ$$

the minimal right continuous filtration containing $\mathcal{F}^\circ$; i.e., $(\mathcal{F}^+)_t := \bigcap_{s > t} \mathcal{F}^\circ_s$ for $0 \leq t < T$ and $(\mathcal{F}^+)_T := \mathcal{F}_T^\circ$. We augment $\mathcal{F}^+$ by the collection $\mathcal{N}^P$ of $(\mathcal{P}, \mathcal{F}^\circ)$-polar sets to obtain the filtration

$$\mathbb{F} = \{(\mathcal{F}_t)\}_{0 \leq t \leq T}, \quad (\mathcal{F}_t) := (\mathcal{F}^+)_t \cup \mathcal{N}^P.$$  

Then $\mathbb{F}$ is right continuous and a natural analogue of the “usual augmentation” that is standard in the case where a reference probability is given. More precisely, if $\mathcal{P}$ is dominated by some probability measure, then one can find a minimal dominating measure $P_*$ (such that every $\mathcal{P}$-polar set is a $P_*$-nullset) and then $\mathbb{F}$ coincides with the $P_*$-augmentation of $\mathcal{F}^+$. We remark that $\mathbb{F}$ is in general strictly smaller than the universal augmentation $\bigcap_{P \in \mathcal{P}} \mathcal{F}^\circ_P$, which seems to be too large for our purposes. Here $\mathcal{F}^\circ_P$ denotes the $P$-augmentation of $\mathcal{F}^\circ$.

Since $\mathbb{F}$ and $\mathbb{F}^+$ differ only by $\mathcal{P}$-polar sets, they can be identified for most purposes; note in particular that $\mathcal{F}_T = (\mathcal{F}^+)_T = \mathcal{F}_T^\circ$ $\mathcal{P}$-a.s. We also recall the following result (e.g., [15, 28]), which shows that $\mathbb{F}$ and $\mathbb{F}^\circ$ differ only by $P$-nullsets for each $P \in \mathcal{P}$. 

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Lemma 4.4. Let $P \in \mathcal{P}$. Then $\mathbb{F}_P$ is right continuous and in particular contains $\mathbb{F}$. Moreover, $(P, B)$ has the predictable representation property; i.e., for any right continuous $(\mathbb{F}_P^P, P)$-local martingale $M$ there exists an $\mathbb{F}_P^P$-predictable process $Z$ such that $M = M_0 + (P)[Z dB]$, $P$-a.s.

Proof. We sketch the argument for the convenience of the reader. We define a predictable process $\hat{a}_t = d(B)_t/\alpha$ taking values in $\mathbb{S}^0_\alpha P \times dt$-a.e. and consider $W_t := (P)[\hat{a}_u]^{-1/2} dB_u$. Let $\mathbb{F}^W$ be the raw filtration generated by $W$. Since $W$ is a $P$-Brownian motion by Lévy’s characterization, the $P$-augmentation $\mathbb{F}_W^P$ is right continuous and $W$ has the representation property. Moreover, as $P \in \mathcal{P}_S$, [28, Lemma 8.1] yields that $\mathbb{F}_W^P = \mathbb{F}_P$. Thus $\mathbb{F}_P$ is also right continuous and $B$ has the representation property since any integral of $W$ is also an integral of $B$.

We deduce from Lemma 4.4 that for $P \in \mathcal{P}$, any (local) $(\mathbb{F}, P)$-martingale is a (local) $(\mathbb{F}, P)$-supermartingale. In particular, this applies to the canonical process $B$. Note that Lemma 4.4 does not imply that $\mathbb{F}$ and $\mathbb{F}_P$ coincide up to $P$-polar sets. E.g., consider the set

$$A := \left\{ \limsup_{t \to 0} t^{-1}(B)_t = \liminf_{t \to 0} t^{-1}(B)_t = 1 \right\} \in \mathcal{F}^C_0. \quad (4.3)$$

Then the lemma asserts that $P(A) \in \{0, 1\}$ for all $P \in \mathcal{P}$, but not that this number is the same for all $P$. Indeed, $P^\alpha(A) = 1$ for $\alpha = 1$ but $P^\alpha(A) = 0$ for $\alpha = 2$.

We can now state the existence and uniqueness of the stochastic process derived from $\{\mathcal{E}^\alpha_t(X), t \in [0, T]\}$. For brevity, we shall say that $Y$ is an $(\mathbb{F}, \mathcal{P})$-supermartingale if $Y$ is an $(\mathbb{F}, P)$-supermartingale for all $P \in \mathcal{P}$; analogous notation will be used in similar situations.

Proposition 4.5. Let $X \in \mathcal{H}$. There exists an $\mathbb{F}$-optional process $(Y_t)_{0 \leq t \leq T}$ such that all paths of $Y$ are càdlàg and

(i) $Y$ is the minimal $(\mathbb{F}, \mathcal{P})$-supermartingale with $Y_T = X$; i.e., if $S$ is a càdlàg $(\mathbb{F}, \mathcal{P})$-supermartingale with $Y_T = X$, then $S \geq Y$ up to a $\mathcal{P}$-polar set.

(ii) $Y_t = \mathcal{E}^\alpha_{t+}(X) := \lim_{r \downarrow t} \mathcal{E}^\alpha_r(X) \mathcal{P}$-q.s. for all $0 \leq t < T$, and $Y_T = X$.

(iii) $Y$ has the representation

$$Y_t = \operatorname{ess sup}^P E^{P'}[X|\mathcal{F}_t] \quad \mathcal{P}$-a.s. \quad \text{for all } P \in \mathcal{P}. \quad (4.4)$$

Any of the properties (i),(ii),(iii) characterizes $Y$ uniquely up to $\mathcal{P}$-polar sets. The process $Y$ is denoted by $\mathcal{E}(X)$ and called the (càdlàg) $\mathcal{E}$-martingale associated with $X$. 

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Proof. We choose and fix representatives for the classes \( \mathcal{E}_r^0(X) \) \( \in L_P^1(\mathcal{F}_t^r) \) and define the \( \mathbb{R} \cup \{ \pm \infty \} \)-valued process \( Y \) by

\[
Y_t(\omega) := \limsup_{r \to t} \mathcal{E}_r^0(X)(\omega) \quad \text{for } 0 \leq t < T \quad \text{and} \quad Y_T(\omega) := X(\omega)
\]

for all \( \omega \in \Omega \). Since each \( \mathcal{E}_r^0(X) \) is \( \mathcal{F}_r^0 \)-measurable, \( Y \) is adapted to \( \mathbb{F}^+ \) and in particular to \( \mathbb{F} \). Let \( N \) be the set of \( \omega \in \Omega \) for which there exists \( t \in [0, T) \) such that \( \lim_{r \to t} \mathcal{E}_r^0(X)(\omega) \) does not exist as a finite real number. For any \( P \in \mathcal{P} \), (4.2) implies the \( (\mathbb{F}, P) \)-supermartingale property

\[
\mathcal{E}_r^0(X) \geq E^P[\mathcal{E}_t^0(X)|\mathcal{F}_r^0] \quad \text{P-a.s.}, \quad 0 \leq s \leq t \leq T.
\]

Thus the standard modification argument for supermartingales (see [9, Theorem VI.2]) yields that \( P(N) = 0 \). As this holds for all \( P \in \mathcal{P} \), the set \( N \) is \( \mathcal{P} \)-polar and thus \( N \in \mathcal{F}_0 \). We redefine \( Y := 0 \) on \( N \). Then all paths of \( Y \) are finite-valued and càdlàg. Moreover, the resulting process is \( \mathbb{F} \)-adapted and therefore \( \mathbb{F} \)-optional by the càdlàg property. Of course, redefining \( Y \) on \( N \) does not affect the \( P \)-almost sure properties of \( Y \). In particular, [9, Theorem VI.2] shows that \( Y \) is an \( (\mathbb{F}, P) \)-supermartingale.

Let \( P' \in \mathcal{P}(\mathcal{F}_t, P) \). Using the above observation with \( P' \) instead of \( P \), we also have that \( Y \) is an \( (\mathbb{F}, P') \)-supermartingale. As \( X = Y_T \), this yields that \( E^{P'}[X|\mathcal{F}_t] = E^{P'}[Y_T|\mathcal{F}_t] \leq Y_t \) \( P' \)-a.s., and also \( P \)-a.s. because \( P' = P \) on \( \mathcal{F}_t \). Since \( P' \in \mathcal{P}(\mathcal{F}_t, P) \) was arbitrary, we conclude that

\[
Y_t \geq \operatorname{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_t, P)} E^{P'}[X|\mathcal{F}_t] \quad \text{P-a.s.} \tag{4.5}
\]

To see the converse inequality, consider a strictly decreasing sequence \( t_n \downarrow t \) of rationals. Then \( \mathcal{E}_{t_n}^0(X) \to Y_t \) \( P \)-a.s. by the definition of \( Y_t \), but as \( E^P[\mathcal{E}_{t_n}^0(X)] \leq E^P[X] < \infty \), the backward supermartingale convergence theorem [9, Theorem V.30] shows that this convergence holds also in \( L_1^1(P) \) and hence

\[
Y_t = \lim_{n \to \infty} E^P[\mathcal{E}_{t_n}^0(X)|\mathcal{F}_t] \quad \text{in } L_1^1(P) \quad \text{and } P \text{-a.s.} \tag{4.6}
\]

For fixed \( n \), let \( P^n_k \in \mathcal{P}(\mathcal{F}_{t_n}, P) \) be a sequence as in Lemma 3.4. Then monotone convergence yields

\[
E^P[\mathcal{E}_{t_n}^0(X)|\mathcal{F}_t] = E^P \left[ \operatorname{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_{t_n}, P)} E^{P'}[X|\mathcal{F}_{t_n}^0]|\mathcal{F}_t \right]
\]

\[
= \lim_{k \to \infty} E^{P^n_k}[X|\mathcal{F}_t]
\]

\[
\leq \operatorname{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_t, P)} E^{P'}[X|\mathcal{F}_t] \quad \text{P-a.s.},
\]

since \( P^n_k \in \mathcal{P}(\mathcal{F}_{t_n}, P) \subseteq \mathcal{P}(\mathcal{F}_{t_n}^0, P) = \mathcal{P}(\mathcal{F}_t, P) \) for all \( k \) and \( n \). In view of (4.6), the inequality converse to (4.5) follows and (iii) is proved.
To see the minimality property in (i), let $S$ be an $(\mathbb{F}, \mathcal{P})$-supermartingale with $S_T = X$. Exactly as in (4.5), we deduce that

$$S_t \geq \text{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_t, P)} E^{P'}[X | \mathcal{F}_t] \quad P\text{-a.s. for all } P \in \mathcal{P}.$$ 

By (iii) the right hand side is $P$-a.s. equal to $Y_t$. Hence $S_t \geq Y_t \ P\text{-q.s. for all } t$ and $S \geq Y \ P\text{-q.s. when } S$ is càdlàg.

Finally, if $Y$ and $Y'$ are processes satisfying (i) or (ii) or (iii), then they are $P$-modifications of each other for all $P \in \mathcal{P}$ and thus coincide up to a $P$-polar set as soon as they are càdlàg.

One can ask whether $\mathcal{E}(X)$ is a $\mathcal{P}$-modification of $\{\mathcal{E}_t^0(X), t \in [0, T]\}$; i.e., whether

$$\mathcal{E}_t(X) = \mathcal{E}_t^0(X) \quad P\text{-q.s. for all } 0 \leq t \leq T.$$ 

It is easy to see that $\mathcal{E}(X)$ is a $\mathcal{P}$-modification as soon as there exists some càdlàg $\mathcal{P}$-modification of the family $\{\mathcal{E}_t^0(X), t \in [0, T]\}$, and this is the case if and only if $t \mapsto E^{P}[\mathcal{E}_t^0(X)]$ is right continuous for all $P \in \mathcal{P}$. We also remark that the argument given for (4.5) yields

$$\mathcal{E}_t(X) \leq \mathcal{E}_t^0(X) \quad P\text{-q.s. for all } 0 \leq t \leq T \quad (4.7)$$

and so the question is only whether the converse inequality holds true as well. The answer is positive in several important cases (e.g., for $G$-expectations when $X$ is sufficiently regular [30, Theorem 5.3]). However, the following (admittedly degenerate) example shows that the answer is negative in general; this reflects the fact that the set $\mathcal{P}(\mathcal{F}_t, P)$ in the representation (4.4) is smaller than the set $\mathcal{P}(\mathcal{F}_t^0, P)$ in (4.1).

**Example 4.6.** We shall consider a $G$-expectation defined on a set of irregular random variables. Let $\underline{a} = 1$, $\bar{a} = 2$ and let $\mathcal{P}$ be as in (2.2). We take $\mathcal{H} = L^1_\mathbb{P}(\mathcal{F}_0^0)$ and define

$$\mathcal{E}_t^0(X) := \begin{cases} \sup_{P \in \mathcal{P}} E^P[X], & t = 0, \\ X, & 0 < t \leq T \end{cases}$$

for $X \in \mathcal{H}$. Then $\{\mathcal{E}_t^0\}$ trivially satisfies (4.1) since $X$ is $\mathcal{F}_t^0$-measurable for all $t > 0$. As noted after Lemma 3.3, the second part of Assumption 4.1 is also satisfied. Moreover, the càdlàg $\mathcal{E}$-martingale is given by

$$\mathcal{E}_t(X) = X, \quad t \in [0, T].$$

Consider $X := 1_A$, where $A$ is defined as in (4.3). Then $\mathcal{E}_0^0(X) = 1$ and $\mathcal{E}_0(X) = 1_A$ are not equal $P^2$-a.s. (i.e., the measure $P^\alpha$ for $\alpha \equiv 2$). In fact, there is no càdlàg $\mathcal{P}$-modification since $\{\mathcal{E}_t^0(X)\}$ coincides $P^2$-a.s. with the deterministic function $t \mapsto 1_{\{0\}}(t).$
We remark that the phenomenon appearing in the previous example is due to the presence of singular measures rather than the fact that \( \mathcal{P} \) is not dominated. E.g., [20, Example 6.1] is based on only two measures and yields another case where \( \mathcal{E}(X) \) is not a \( \mathcal{P} \)-modification of \( \{\mathcal{E}_t(X), t \in [0, T]\} \), for a large class of random variables \( X \).

### 4.2 Stopping Times

The direct construction of \( G \)-expectations at stopping times is an unsolved problem. Indeed, stopping times are typically fairly irregular functions and it is unclear how to deal with this in the existing constructions (see also [18]). On the other hand, we can easily evaluate the càdlàg process \( \mathcal{E}(X) \) at a stopping time \( \tau \) and therefore define the corresponding sublinear expectation at \( \tau \). In particular, this leads to a definition of \( G \)-expectations at general stopping times. We show in this section that the resulting random variable \( \mathcal{E}(X) \) indeed has the expected properties and that the time consistency extends to arbitrary \( \mathcal{F} \)-stopping times; in other words, we prove an optional sampling theorem for \( \mathcal{E} \)-martingales. Besides the obvious theoretical interest, the study of \( \mathcal{E}(X) \) at stopping times will allow us to verify integrability conditions of the type “class (D)”; cf. Lemma 4.14 below. We start by explaining the relations between the stopping times of the different filtrations.

**Lemma 4.7.** (i) Let \( P \in \mathcal{P} \) and let \( \tau \) be an \( \mathcal{F} \)-stopping time taking countably many values. Then there exists an \( \mathcal{F}^0 \)-stopping time \( \tau^0 \) (depending on \( P \)) such that \( \tau = \tau^0 \) \( P \)-a.s. Moreover, for any such \( \tau^0 \), the \( \sigma \)-fields \( \mathcal{F}_\tau \) and \( \mathcal{F}_\tau^0 \) differ only by \( P \)-nullsets.

(ii) Let \( \tau \) be an \( \mathcal{F} \)-stopping time. Then there exists an \( \mathcal{F}^+ \)-stopping time \( \tau^+ \) such that \( \tau = \tau^+ \) \( P \)-q.s. Moreover, for any such \( \tau^+ \), the \( \sigma \)-fields \( \mathcal{F}_\tau \) and \( \mathcal{F}_\tau^+ \) differ only by \( P \)-polar sets.

**Proof.** (i) Note that \( \tau \) is of the form \( \tau = \sum t_i \mathbf{1}_{A_i} \) for \( A_i = \{\tau = t_i\} \in \mathcal{F}_{t_i} \).

Since \( \mathcal{F} \subseteq \mathcal{F}^0 \) by Lemma 4.4, we can find \( A^0 \in \mathcal{F}_{t_i} \) such that \( A_i = A^0 \) \( P \)-a.s. and take \( \tau^0 := \sum t_i \mathbf{1}_{A^0} \). This ends the proof of the first assertion.

Let \( A \in \mathcal{F}_\tau \). By first part, there exists an \( \mathcal{F} \)-stopping time \( (\tau_A)^0 \) such that \( (\tau_A)^0 = \tau_A := \tau \mathbf{1}_A + T \mathbf{1}_{A^c} \) \( P \)-a.s. Moreover, we choose \( A' \in \mathcal{F}_{t_i} \) such that \( A = A' \) \( P \)-a.s. Then

\[
A^0 := (A' \cap \{(\tau_A)^0 = T\}) \cup \{(\tau_A)^0 = \tau^0 < T\}
\]

satisfies \( A^0 \in \mathcal{F}_{\tau^0} \) and \( A = A^0 \) \( P \)-a.s. A similar but simpler argument shows that for given \( A \in \mathcal{F}_{\tau^0} \) we can find \( A' \in \mathcal{F}_\tau \) such that \( A = A' \) \( P \)-a.s.

(ii) If \( \tau \) is an \( \mathcal{F} \)- (resp. \( \mathcal{F}^+ \))-stopping time, we can find \( \tau^n \) taking countably many values such that \( \tau^n \) decreases to \( \tau \) and since \( \mathbb{E} (\mathcal{F}^+) \) is right continuous, \( \mathcal{F}_{\tau^n} \rightarrow (\mathcal{F}_{\tau^+}) \) decreases to \( \mathcal{F}_\tau (\mathcal{F}_{t_i}^+) \). As a result, we may assume without loss of generality that \( \tau \) takes countably many values.
Let $\tau = \sum_{i=1}^\infty t_i \Lambda_i$, where $\Lambda_i \in \mathcal{F}_i$. The definition of $\mathcal{F}$ shows that there exist $\Lambda_i^+ \in \mathcal{F}_i^+$ such that $\Lambda_i = \Lambda_i^+ \mathcal{P}$-q.s. and the first part follows. The proof of the second part is as in (i); we now have quasi-sure instead of almost-sure relations.

If $\sigma$ is a stopping time taking finitely many values $(t_i)_{1 \leq i \leq N}$, we can define $\mathcal{E}_\sigma(X) := \sum_{i=1}^N \mathcal{E}_i^\sigma(X) 1_{\{\sigma=t_i\}}$. We have the following generalization of (4.1).

**Lemma 4.8.** Let $\sigma$ be an $\mathcal{F}_\sigma$-stopping time taking finitely many values. Then

$$
\mathcal{E}_\sigma^\sigma(X) = \text{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_\sigma^\sigma)} E^{P'} [X|\mathcal{F}_\sigma^\sigma] \quad \text{P-a.s. for all } P \in \mathcal{P}.
$$

**Proof.** Let $P \in \mathcal{P}$ and $Y_i^\sigma := \mathcal{E}_i^\sigma(X)$. Moreover, let $(t_i)_{1 \leq i \leq N}$ be the values of $\sigma$ and $\Lambda_i := \{\tau = t_i\} \in \mathcal{F}_i^\sigma$.

(i) We first prove the inequality “$\geq$”. Given $P' \in \mathcal{P}$, it follows from (4.2) that $\{Y_i^\sigma\}_{1 \leq i \leq N}$ is a $P'$-supermartingale in $(\mathcal{F}_i^\sigma)_{1 \leq i \leq N}$ and so the optional sampling theorem implies $Y_i^\sigma \geq E^{P'}[X|\mathcal{F}_i^\sigma] \quad P'$-a.s. In particular, this also holds $P$-a.s. for all $P' \in \mathcal{P}(\mathcal{F}_\sigma, P)$, hence the claim follows.

(ii) We now show the inequality “$\leq$”. Note that $\sigma = \sum_{i=1}^N t_i 1_{\Lambda_i}$ and that $(\Lambda_i)_{1 \leq i \leq N}$ form an $\mathcal{F}_\sigma^\sigma$-measurable partition of $\Omega$. It suffices to show that

$$
Y_i^\sigma 1_{\Lambda_i} \leq \text{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_\sigma^\sigma)} E^{P'}[X|\mathcal{F}_\sigma^\sigma] 1_{\Lambda_i} \quad \text{P-a.s. for } 1 \leq i \leq N.
$$

In the sequel, we fix $i$ and show that for each $P' \in \mathcal{P}(\mathcal{F}_i^\sigma, P)$ there exists $\bar{P} \in \mathcal{P}(\mathcal{F}_i^\sigma, P)$ such that

$$
\bar{P}(A \cap \Lambda_i) = P'(A \cap \Lambda_i) \quad \text{for all } A \in \mathcal{F}_i^\sigma.
$$

(4.8)

In view of (4.1) and $E^{P'}[X|\mathcal{F}_\sigma^\sigma] 1_{\Lambda_i} = E^{P'}[X|\mathcal{F}_i^\sigma] 1_{\Lambda_i} \quad P'$-a.s., it will then follow that

$$
Y_i^\sigma 1_{\Lambda_i} = \text{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_i^\sigma, P)} E^{P'}[X 1_{\Lambda_i}|\mathcal{F}_i^\sigma] \leq \text{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_i^\sigma, P)} E^{\bar{P}}[X 1_{\Lambda_i}|\mathcal{F}_i^\sigma] \quad \text{P-a.s.}
$$

as claimed. Indeed, given $P' \in \mathcal{P}(\mathcal{F}_i^\sigma, P)$, we define

$$
\bar{P}(A) := P'(A \cap \Lambda_i) + P(A \setminus \Lambda_i), \quad A \in \mathcal{F}_i^\sigma.
$$

(4.9)

then (4.8) is obviously satisfied. If $\Lambda \in \mathcal{F}_\sigma^\sigma$, then $\Lambda \cap \Lambda_i = \Lambda \cap \{\tau = t_i\} \in \mathcal{F}_i^\sigma$ and $P' \in \mathcal{P}(\mathcal{F}_i^\sigma, P)$ yields $P'(\Lambda \cap \Lambda_i) = P(\Lambda \cap \Lambda_i)$. Hence $\bar{P} = P$ on $\mathcal{F}_\sigma^\sigma$. Moreover, we observe that (4.9) can be stated as

$$
\bar{P}(A) = E^{P'}[P'(A|\mathcal{F}_\sigma^\sigma)] 1_{\Lambda_i} + P(A|\mathcal{F}_i^\sigma) 1_{\Lambda_i}, \quad A \in \mathcal{F}_i^\sigma,
$$

which is a special case of the pasting (3.2) applied with $P_2 := P$. Hence $\bar{P} \in \mathcal{P}$ by Assumption 4.1 and we have $\bar{P} \in \mathcal{P}(\mathcal{F}_\sigma^\sigma, P)$ as desired. □
We can now prove the optional sampling theorem for $\mathcal{E}$-martingales; in particular, this establishes the $\mathbb{F}$-time-consistency of $\{\mathcal{E}_t\}$.

**Theorem 4.9.** Let $0 \leq \sigma \leq \tau \leq T$ be $\mathbb{F}$-stopping times, $X \in \mathcal{H}$, and let $\mathcal{E}(X)$ be the càdlàg $\mathcal{E}$-martingale associated with $X$. Then
\[
\mathcal{E}_\sigma(X) = \operatorname{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_{\sigma}, P)} E^{P'}[\mathcal{E}_\tau(X) | \mathcal{F}_\sigma] \quad P\text{-a.s. for all } P \in \mathcal{P} \tag{4.10}
\]
and in particular
\[
\mathcal{E}_\sigma(X) = \operatorname{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_{\sigma}, P)} E^{P'}[X | \mathcal{F}_\sigma] \quad P\text{-a.s. for all } P \in \mathcal{P} \tag{4.11}
\]
Moreover, there exists for each $P \in \mathcal{P}$ a sequence $P_n \in \mathcal{P}(\mathcal{F}_\sigma, P)$ such that
\[
\mathcal{E}_\sigma(X) = \lim_{n \to \infty} E^{P_n}[X | \mathcal{F}_\sigma] \quad P\text{-a.s.} \tag{4.12}
\]
with an increasing limit.

**Proof.** Fix $P \in \mathcal{P}$ and let $Y := \mathcal{E}(X)$.

(i) We first show the inequality “$\geq$” in (4.11). By Proposition 4.5(i), $Y$ is an $(\mathbb{F}, P')$-supermartingale for all $P' \in \mathcal{P}(\mathcal{F}_\sigma, P)$. Hence the (usual) optional sampling theorem implies the claim.

(ii) We now show the inequality “$\leq$” in (4.11). In view of Lemma 4.7(ii) we may assume that $\sigma$ is an $\mathbb{F}^+$-stopping time. Then $(\sigma + 1/n) \land T$ is an $\mathbb{F}^n$-stopping time for each $n \geq 1$. Let $D_n = \{k2^{-n} : k = 0, 1, \ldots \} \cup \{T\}$ and define
\[
\sigma^n(\omega) := \inf\{t \in D_n : t \geq \sigma(\omega) + 1/n\} \land T.
\]
Each $\sigma^n$ is an $\mathbb{F}^n$-stopping time taking finitely many values and $\sigma^n(\omega)$ decreases to $\sigma(\omega)$ for all $\omega \in \Omega$. Hence it follows from Proposition 4.5(ii) that $\mathcal{E}_{\sigma^n}(X) \to Y_\sigma$ $P$-a.s. Again, backward supermartingale arguments (cf. [9, Theorem V.30]) show that this convergence holds also in $L^1(P)$ and
\[
Y_\sigma = \lim_{n \to \infty} E^P[\mathcal{E}_{\sigma^n}(X) | \mathcal{F}_\sigma] \quad P\text{-a.s.} \tag{4.13}
\]
By Lemma 4.8 and Lemma 3.4, there exists for each $n$ a sequence $(P_k^n)_{k \geq 1}$ in $\mathcal{P}(\mathcal{F}_{\sigma^n}, P)$ such that
\[
\mathcal{E}_{\sigma^n}(X) = \operatorname{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_{\sigma^n}, P)} E^{P'}[X | \mathcal{F}_{\sigma^n}] = \lim_{k \to \infty} E^{P_k^n}[X | \mathcal{F}_{\sigma^n}] \quad P\text{-a.s.},
\]
where the limit is increasing. Moreover, using that
\[
\mathcal{F}_{\sigma^n+1}^+ = \{A \in \mathcal{F}_T^+ : A \cap \{\sigma^{n+1} < t\} \in \mathcal{F}_t^+ \text{ for } 0 \leq t \leq T\},
\]
the fact that $\sigma^n > \sigma^{n+1}$ on $\{\sigma^n < T\}$ is seen to imply that $\mathcal{F}^+_{\sigma^{n+1}} \subseteq \mathcal{F}^0_{\sigma^n}$. Together with $\sigma \leq \sigma^{n+1}$ and Lemma 4.7(ii) we conclude that

$$\mathcal{F}_{\sigma} \subseteq \mathcal{F}_{\sigma^{n+1}} \overset{\text{P.a.s.}}{=} \mathcal{F}^+_{\sigma^{n+1}} \subseteq \mathcal{F}^0_{\sigma^n} \quad \text{and hence} \quad \mathcal{P}(\mathcal{F}_{\sigma}, P) \supseteq \mathcal{P}(\mathcal{F}^0_{\sigma^n}, P) \tag{4.14}$$

for all $n$. Now monotone convergence yields

$$E^{P} [\mathcal{E}_{\sigma^n} (X) | \mathcal{F}_{\sigma}] = \lim_{k \to \infty} E^{P^k} [X | \mathcal{F}_{\sigma}] \leq \text{ess sup}_{P' \in \mathcal{P}(\mathcal{F}_{\sigma}, P)} E^{P'} [X | \mathcal{F}_{\sigma}] \quad \text{P-a.s.}$$

In view of (4.13), this ends the proof of (4.11).

(iii) We now prove (4.12); note that the assertion is nontrivial since $\mathcal{P}$ is stable under $\mathbb{F}$-pasting and not under $\mathbb{F}$-pasting. We have just constructed $P^k_n \in \mathcal{P}(\mathcal{F}^0_{\sigma^n}, P)$ such that

$$Y_n = \lim_{n \to \infty} \lim_{k \to \infty} E^{P^k_n} [X | \mathcal{F}_{\sigma}] \quad \text{P-a.s.}$$

Fix $n$. Since $\sigma^n$ is an $\mathbb{F}$-stopping time taking finitely many values and since $\mathcal{F}_{\sigma} \subseteq \mathcal{F}^0_{\sigma^n}$, P-a.s. by (4.14), it follows from the stability under $\mathbb{F}$-pasting that the set $\{E^{P'} [X | \mathcal{F}_{\sigma}] : P' \in \mathcal{P}(\mathcal{F}^0_{\sigma^n}, P)\}$ is P-a.s. upward filtering, exactly as in the proof of Lemma 3.4. In view of $\mathcal{P}(\mathcal{F}^0_{\sigma^n}, P) \subseteq \mathcal{P}(\mathcal{F}^0_{\sigma^{n+1}}, P)$, it follows that for each $N \geq 1$ there exists $P^{(N)} \in \mathcal{P}(\mathcal{F}^0_{\sigma^N}, P)$ such that

$$E^{P^{(N)}} [X | \mathcal{F}_{\sigma}] = \max_{1 \leq n \leq N} \max_{1 \leq k \leq n} E^{P^k_n} [X | \mathcal{F}_{\sigma}] \quad \text{P-a.s.}$$

Since $\mathcal{P}(\mathcal{F}^0_{\sigma^N}, P) \subseteq \mathcal{P}(\mathcal{F}_{\sigma}, P)$ by (4.14), this yields the claim.

(iv) To prove (4.10), we first express $\mathcal{E}_{\sigma}(X)$ and $\mathcal{E}_{\tau}(X)$ as essential suprema by using (4.11) both for $\sigma$ and for $\tau$. The inequality “$\leq$” is then immediate. The converse inequality follows by a monotone convergence argument exactly as in the proof of Proposition 3.6(i), except that the increasing sequence is now obtained from (4.12) instead of Lemma 3.4.

4.3 Decomposition and 2BSDE for $\mathcal{E}$-Martingales

The next result contains the semimartingale decomposition of $\mathcal{E}(X)$ under each $P \in \mathcal{P}$ and can be seen as an analogue of the optional decomposition [17] used in mathematical finance. In the context of $G$-expectations, such a result has also been referred to as “$G$-martingale representation theorem”; see [14, 27, 30, 32]. Those results are ultimately based on the PDE description of the $G$-expectation and are more precise than ours; in particular, they provide a single increasing process $K$ rather than a family $(K^P)_P \in \mathcal{P}$. On the other hand, we obtain an $L^1$-theory whereas those results require more integrability for $X$. 

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Proposition 4.10. Let $X \in \mathcal{H}$. There exist

(i) an $\mathbb{F}$-predictable process $Z^X$ with $\int_0^T |Z^X_s|^2 \, d\langle B \rangle_s < \infty \, \mathbb{P}$-q.s.,

(ii) a family $(K^P)_{P \in \mathcal{P}}$ of $\mathbb{F}^P$-predictable processes such that all paths of $K^P$ are càdlàg nondecreasing and $E^P[|K^P_T|] < \infty$,

such that

$$
\mathcal{E}_t(X) = \mathcal{E}_0(X) + \int_0^t Z^X_s \, dB_s - K^P_t \quad \text{for all } 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \quad (4.15)
$$

for all $P \in \mathcal{P}$. The process $Z^X$ is unique up to $\{ds \times P, P \in \mathcal{P}\}$-polar sets and $K^P$ is unique up to $\mathbb{P}$-evanescence.

Proof. We shall use arguments similar to the proof of [26, Theorem 4.5].

Let $P \in \mathcal{P}$. It follows from Proposition 4.5(i) that $Y := \mathcal{E}(X)$ is an $(\mathbb{F}^P, P)$-supermartingale. We apply the Doob-Meyer decomposition in the filtered space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies the usual conditions of right continuity and completeness. Thus we obtain an $(\mathbb{F}^P, P)$-local martingale $M^P$ and an $\mathbb{F}^P$-predictable increasing integrable process $K^P$, càdlàg and satisfying $M^P = K^P = 0$, such that

$$
Y = Y_0 + M^P - K^P.
$$

By Lemma 4.4, $(P, B)$ has the predictable representation property in $\mathbb{F}^P$. Hence there exists an $\mathbb{F}^P$-predictable process $Z^P$ such that

$$
Y = Y_0 + \int Z^P \, dB - K^P.
$$

The next step is to replace $Z^P$ by a process $Z^X$ independent of $P$. Recalling that $B$ is a continuous local martingale under each $P$, we have

$$
\int Z^P \, d\langle B \rangle^P = \langle Y, B \rangle^P = BY - \int B \, dY - \int Y_- \, dB \quad \mathbb{P}\text{-a.s.} \quad (4.16)
$$

(Here and below, the statements should be read componentwise.) The last two integrals are Itô integrals under $P$, but they can also be defined pathwise since the integrands are left limits of càdlàg processes which are bounded path-by-path. This is a classical construction from [3, Theorem 7.14]; see also [16] for the same result in modern notation. To make explicit that the resulting process is $\mathbb{F}$-adapted, we recall the procedure for the example $\int Y_- \, dB$. One first defines for each $n \geq 1$ the sequence of $\mathbb{F}$-stopping times $\tau^n_0 := 0$ and $\tau^n_{i+1} := \inf\{t \geq \tau^n_i : |Y_t - Y_{\tau^n_i}| \geq 2^{-n}\}$. Then one defines $I^n$ by

$$
I^n_t := Y_{\tau^n_t} (B_t - B_{\tau^n_t}) + \sum_{i=0}^{k-1} Y_{\tau^n_i} (B_{\tau^n_{i+1}} - B_{\tau^n_i}) \quad \text{for } \tau^n_k < t \leq \tau^n_{k+1}, \quad k \geq 0;
$$

Equation (4.15) then takes the form

$$
\mathcal{E}_t(X) = \mathcal{E}_0(X) + \int_0^t Z^X_s \, dB_s - K^P_t \quad \text{for all } 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \quad (4.15)
$$

for all $P \in \mathcal{P}$. The process $Z^X$ is unique up to $\{ds \times P, P \in \mathcal{P}\}$-polar sets and $K^P$ is unique up to $\mathbb{P}$-evanescence.
clearly $I^n$ is again $\mathcal{F}$-adapted and all its paths are càdlàg. Finally, we define

$$I_t := \limsup_{n \to \infty} I^n_t, \quad 0 \leq t \leq T.$$

Then $I$ is again $\mathcal{F}$-adapted and it is a consequence of the Burkholder-Davis-Gundy inequalities that

$$\sup_{0 \leq t \leq T} \left| I^n_t - (P)_{t \downarrow} \int^t_0 Y_- dB \right| \to 0 \quad \text{P-a.s.}$$

for each $P$. Thus, outside a $\mathcal{P}$-polar set, the limsup in the definition of $I$ exists as a limit uniformly in $t$ and $I$ has càdlàg paths. Since $\mathcal{P}$-polar sets are contained in $\mathcal{F}_0$, we may redefine $I := 0$ on the exceptional set. Now $I$ is càdlàg $\mathcal{F}$-adapted and coincides with the Itô integral $(P)_{t \downarrow} Y_- dB$ up to $P$-evanescence, for all $P \in \mathcal{P}$.

We proceed similarly with the integral $(P)_{t \downarrow} B dB$ and obtain a definition for the right hand side of (4.16) which is $\mathcal{F}$-adapted, continuous and independent of $P$. Thus we have defined $\langle Y, B \rangle$ simultaneously for all $P \in \mathcal{P}$, and we do the same for $\langle B \rangle$. Let $\dot{a} = d\langle B \rangle/dt$ be the (left) derivative in time of $\langle B \rangle$, then $\dot{a}$ is $\mathbb{F}^0$-predictable and $\mathbb{S}^2_{\mathbb{F}}$-valued $P \times dt$-a.e. for all $P \in \mathcal{P}$ by the definition of $\mathcal{P}_S$. Finally, $Z^X := \dot{a}^{-1} d\langle Y, B \rangle/dt$ is an $\mathcal{F}$-predictable process such that

$$Y = Y_0 + \int^t_0 Z^X dB - K^P \quad \text{P-a.s. for all } P \in \mathcal{P}.$$

We note that the integral is taken under $P$; it is not clear whether it can be defined for all $P \in \mathcal{P}$ simultaneously.

The previous proof shows that a decomposition of the type (4.15) exists for all càdlàg $(\mathcal{F}, \mathcal{P})$-supermartingales, and not just for $\mathcal{E}$-martingales. As a special case of Proposition 4.10, we obtain a representation for symmetric $\mathcal{E}$-martingales. The following can be seen as a generalization of the corresponding results for $G$-expectations given in [27, 30, 32].

**Corollary 4.11.** Let $X \in \mathcal{H}$ be such that $-X \in \mathcal{H}$. The following are equivalent:

(i) $\mathcal{E}(X)$ is a symmetric $\mathcal{E}$-martingale; i.e., $\mathcal{E}(-X) = -\mathcal{E}(X) \ P$-q.s.

(ii) There exists an $\mathbb{F}$-predictable process $Z^X$ with $\int^T_0 |Z^X|^2 dB_s < \infty \ \mathcal{P}$-q.s. such that

$$\mathcal{E}_t(X) = \mathcal{E}_0(X) + \int^t_0 Z^X_s dB_s \quad \text{for all } 0 \leq t \leq T, \quad \mathcal{P}$-q.s.$$

where the integral can be defined universally for all $P$ and $\int Z^X dB$ is an $(\mathcal{F}, \mathcal{P})$-martingale for all $P \in \mathcal{P}$.

In particular, any symmetric $\mathcal{E}$-martingale has continuous trajectories $\mathcal{P}$-q.s.
Proof. The implication (ii)⇒(i) is clear from Proposition 4.5(iii). Conversely, given (i), Proposition 4.5(i) yields that both $\mathcal{E}(X)$ and $-\mathcal{E}(X)$ are $\mathcal{P}$-supermartingales, hence $\mathcal{E}(X)$ is a (true) $\mathcal{P}$-martingale. It follows that the increasing processes $K^P$ have to satisfy $K^P \equiv 0$ and (4.15) becomes $\mathcal{E}(X) = \mathcal{E}_0(X) + (P)\int Z^X dB$. In particular, the stochastic integral can be defined universally by setting $\int Z^X dB := \mathcal{E}(X) - \mathcal{E}_0(X)$.

\[ \blacksquare \]

Remark 4.12. (a) Without the martingale condition in Corollary 4.11(ii), the implication (ii)⇒(i) would fail even for $\mathcal{P} = \{P_0\}$, in which case Corollary 4.11 is simply the Brownian martingale representation theorem.

(b) Even if it is symmetric, $\mathcal{E}(X)$ need not be a $\mathcal{P}$-modification of the family $\{\mathcal{E}_t^\circ(X), t \in [0, T]\}$; in fact, the $\mathcal{E}$-martingale in Example 4.6 is symmetric. However, the situation changes if the symmetry assumption is imposed directly on $\{\mathcal{E}_t^\circ(X)\}$. We call $\{\mathcal{E}_t^\circ(X)\}$ symmetric if $\mathcal{E}_t^\circ(-X) = -\mathcal{E}_t^\circ(X) \mathcal{P}$-a.s. for all $t \in [0, T]$.

- If $\{\mathcal{E}_t^\circ(X)\}$ symmetric, then $\mathcal{E}(X)$ is a symmetric $\mathcal{E}$-martingale and a $\mathcal{P}$-modification of $\{\mathcal{E}_t^\circ(X)\}$.

Indeed, the assumption implies that $\{\mathcal{E}_t^\circ(X)\}$ is an $(\mathbb{F}^0, \mathbb{P})$-martingale for each $P \in \mathcal{P}$ and so the process $\mathcal{E}(X)$ of right limits (cf. Proposition 4.5(ii)) is the usual càdlàg $\mathcal{P}$-modification of $\{\mathcal{E}_t^\circ(X)\}$, for all $P$.

Next, we represent the pair $(\mathcal{E}(X), Z^X)$ from Proposition 4.10 as the solution of a 2BSDE. The following definition is essentially from [29].

Definition 4.13. Let $X \in L^2_\mathbb{P}$ and consider a pair $(Y, Z)$ of processes with values in $\mathbb{R} \times \mathbb{R}^d$ such that $Y$ is càdlàg $\mathcal{F}$-adapted while $Z$ is $\mathcal{F}$-predictable and $\int_0^T |Z_s|^2 dB_s < \infty \mathcal{P}$-a.s. Then $(Y, Z)$ is called a solution of the 2BSDE (4.17) if there exists a family $(K^P)_{P \in \mathcal{P}}$ of $\mathbb{F}^0$-adapted increasing processes satisfying $E^P ||K^P_T|| < \infty$ such that

\[ Y_t = X - \int_t^T Z_s dB_s + K^P_T - K^P_t, \quad 0 \leq t \leq T, \quad \mathcal{P}\text{-a.s.} \quad \text{for all } P \in \mathcal{P} \tag{4.17} \]

and such that the following minimality condition holds for all $0 \leq t \leq T$:

\[ \operatorname{ess} \inf_{P \in \mathcal{P}(\mathcal{F}_t, P)} E^P \left[ K^P_T - K^P_t \mid \mathcal{F}_t \right] = 0 \quad \mathcal{P}\text{-a.s.} \quad \text{for all } P \in \mathcal{P}. \tag{4.18} \]

We remark that (4.18) is essentially the $\mathcal{E}$-martingale condition (4.4): if the processes $K^P$ can be aggregated into a single process $K$ and $K_T \in \mathcal{H}$, then $-K = \mathcal{E}(-K_T)$.

A second notion is needed to state the main result. A càdlàg process $Y$ is said to be of class $(D, \mathcal{P})$ if the family $\{Y^\sigma\}_\sigma$ is uniformly integrable under $P$ for all $P \in \mathcal{P}$, where $\sigma$ runs through all $\mathbb{F}$-stopping times. As an
example, we have seen in Corollary 4.11 that all symmetric $\mathcal{E}$-martingales are of class $(D, \mathcal{P})$. (Of course, it is important here that we work with a finite time horizon $T$.) For $p \in [1, \infty)$, we define $\|X\|_{L^p_{\mathcal{P}}} := \sup_{P \in \mathcal{P}} E|X|^p_\mathcal{P}$ as well as $\mathcal{H}^p := \{X \in \mathcal{H} : \|X\|^p_{\mathcal{P}} \in \mathcal{P}\}.

**Lemma 4.14.** If $X \in \mathcal{H}^p$ for some $p \in (1, \infty)$, then $\mathcal{E}(X)$ is of class $(D, \mathcal{P})$.

**Proof.** Let $P \in \mathcal{P}$. If $\sigma$ is an $\mathbb{F}$-stopping time, Jensen’s inequality and (4.11) yield that

$$|\mathcal{E}_\sigma(X)|^p \leq \text{ess sup}_{P \in \mathcal{P}(\mathcal{F}_\tau, P)}^\mathcal{P} E_P^\mathcal{P}(|X|^p_{\mathcal{F}_\sigma}) = \mathcal{E}_\sigma(|X|^p) \quad P\text{-a.s.}$$

In particular, $\|\mathcal{E}_\sigma(X)\|^p_{L^p_{\mathcal{P}}(P)} \leq E_P^\mathcal{P}[\mathcal{E}_\sigma(|X|^p)]$ and thus Lemma 4.4 yields

$$\|\mathcal{E}_\sigma(X)\|^p_{L^p_{\mathcal{P}}(P)} \leq E_P^\mathcal{P}[\mathcal{E}_\sigma(|X|^p)|\mathcal{F}_0] \leq \text{ess sup}_{P \in \mathcal{P}(\mathcal{F}_0, P)}^\mathcal{P} E_P^\mathcal{P}[\mathcal{E}_\sigma(|X|^p)|\mathcal{F}_0] \quad P\text{-a.s.}$$

The right hand side $P$-a.s. equals $\mathcal{E}_0(|X|^p)$ by (4.10), so we conclude with (4.7) that

$$\|\mathcal{E}_\sigma(X)\|^p_{L^p_{\mathcal{P}}(P)} \leq \mathcal{E}_0(|X|^p) \leq \sup_{P \in \mathcal{P}} E_P^\mathcal{P}[|X|^p] = \|X\|^p_{L^p_{\mathcal{P}}} < \infty \quad P\text{-a.s.}$$

Therefore, the family $\{\mathcal{E}_\sigma(X)\}_\sigma$ is bounded in $L^p_{\mathcal{P}}(P)$ and in particular uniformly integrable under $P$. This holds for all $P \in \mathcal{P}$.

We can now state the main result of this section.

**Theorem 4.15.** Let $X \in \mathcal{H}$.

(i) The pair $(\mathcal{E}(X), Z^X)$ is the minimal solution of the 2BSDE (4.17); i.e., if $(Y, Z)$ is another solution, then $\mathcal{E}(X) \leq Y$ $P$-a.s.

(ii) If $(Y, Z)$ is a solution of (4.17) such that $Y$ is of class $(D, \mathcal{P})$, then $(Y, Z) = (\mathcal{E}(X), Z^X)$.

In particular, if $X \in \mathcal{H}^p$ for some $p > 1$, then $(\mathcal{E}(X), Z^X)$ is the unique solution of (4.17) in the class $(D, \mathcal{P})$.

**Proof.** (i) Let $P \in \mathcal{P}$. To show that $(\mathcal{E}(X), Z^X)$ is a solution, we only have to show that $K^P$ from the decomposition (4.15) satisfies the minimality condition (4.18). We denote this decomposition by $\mathcal{E}(X) = \mathcal{E}_0(X) + M^P - K^P$.

It follows from Proposition 4.5(i) that $\mathcal{E}(X)$ is an $(\mathbb{F}^P, P)$-supermartingale. As $K^P \geq 0$, we deduce that

$$\mathcal{E}_0(X) + M^P \geq \mathcal{E}(X) \geq E_P^P[X|\mathbb{F}^P] \quad P\text{-a.s.},$$

where $E_P^P[X|\mathbb{F}^P]$ denotes the càdlàg $(\mathbb{F}^P, P)$-martingale with terminal value $X$. Hence $M^P$ is a local $P$-martingale bounded from below by a $P$-martingale.
and thus $M^P$ is an $(\mathbb{F}, P)$-supermartingale by a standard argument using Fatou’s lemma. This holds for all $P \in \mathcal{P}$. Therefore, (4.4) yields

$$0 = \mathcal{E}_t(X) - \operatorname{ess \sup}_{P \in \mathcal{P}} E^P [X | \mathcal{F}_t]$$

where

$$= \operatorname{ess \inf}_{P \in \mathcal{P}} E^P [\mathcal{E}_t(X) - \mathcal{E}_t(X) | \mathcal{F}_t]$$

$$= \operatorname{ess \inf}_{P \in \mathcal{P}} E^P [M_t^P - M_t^P + K_t^P - K_t^P | \mathcal{F}_t]$$

$$\geq \operatorname{ess \inf}_{P \in \mathcal{P}} E^P [K_t^P - K_t^P | \mathcal{F}_t] \quad P\text{-a.s. for all } P \in \mathcal{P}.$$}

Since $K_t^P$ is nondecreasing, the last expression is also nonnegative and (4.18) follows. Thus $(\mathcal{E}(X), Z_X)$ is a solution.

To prove the minimality, let $(Y, Z)$ be another solution of (4.17). It follows from (4.17) that $Y$ is a local $(\mathbb{F}, P)$-supermartingale for all $P \in \mathcal{P}$. As above, the integrability of $X$ implies that $Y_0 + \mathbb{E}^P [Z dB]$ is bounded below by a $P$-martingale. Noting also that $Y_0$ is $P$-a.s. equal to a constant by Lemma 4.4, we deduce that $\mathbb{E}^P [Z dB]$ and $Y$ are $(\mathbb{F}, P)$-supermartingales. Since $Y$ is càdlàg and $Y_T = X$, the minimality property in Proposition 4.5(i) shows that $Y \geq \mathcal{E}(X) \quad P\text{-a.s.}$

(ii) If in addition $Y$ is of class $(\mathbb{D}, \mathcal{P})$, then $\mathbb{E}^P [Z dB]$ is a true $P$-martingale by the Doob-Meyer theorem and we have

$$0 = \operatorname{ess \inf}_{P \in \mathcal{P}} E^P [K_t^P - K_t^P | \mathcal{F}_t]$$

$$= Y_t - \operatorname{ess \sup}_{P \in \mathcal{P}} E^P [X | \mathcal{F}_t]$$

$$= Y_t - \mathcal{E}_t(X) \quad P\text{-a.s. for all } P \in \mathcal{P}.$$}

The last statement in the theorem follows from Lemma 4.14.

4.4 Application to Superhedging and Replication

We now turn to the interpretation of the previous results for the superhedging problem. Let $H$ be an $\mathbb{R}^d$-valued $\mathbb{F}$-predictable process satisfying $\int_0^T |H_s|^2 \, dB_s < \infty \quad P\text{-a.s.}$ Then $H$ is called an admissible trading strategy if $\mathbb{E}^P [H dB]$ is a $P$-supermartingale for all $P \in \mathcal{P}$. (We do not insist that the integral be defined without reference to $P$: this should not be the trader’s concern!) As usual in continuous-time finance, this definition excludes “doubling strategies”. We have seen in the proof of Theorem 4.15 that $Z_X$ is admissible for $X \in \mathcal{H}$. The minimality property in Proposition 4.5(i) and the existence of the decomposition (4.15) yield the following conclusion: $\mathcal{E}_0(X)$ is the minimal $\mathcal{F}_0$-measurable initial capital which allows to superhedge $X$; i.e., $\mathcal{E}_0(X)$ is the $P$-q.s. minimal $\mathcal{F}_0$-measurable random variable.
such that there exists an admissible strategy $H$ satisfying
\[ \xi_0 + \int_0^T H_s dB_s \geq X \quad P\text{-a.s. for all } P \in \mathcal{P}. \]

Moreover, the “overshoot” $K^P$ for the strategy $Z^X$ satisfies the minimality condition (4.18).

As seen in Example 4.6, the $\mathcal{F}_0$-superhedging price $\mathcal{E}_0(X)$ need not be a constant, and therefore it is debatable whether it is a good choice for a conservative price, in particular if the raw filtration $\mathbb{F}^0$ is seen as the initial information structure for the model. Indeed, the following illustration shows that knowledge of $\mathcal{F}_0$ can be quite significant. Consider a collection $(a_i)$ of positive constants and $\mathcal{P} = \{P^\alpha : \alpha \equiv a_i \text{ for some } i\}$. Such a set $\mathcal{P}$ can indeed satisfy the assumptions of this section; cf. [20, Example 6.1]. In this model, knowledge of $\mathcal{F}_0$ completely removes the volatility uncertainty since $\mathcal{F}_0$ contains the sets
\[ A_i := \left\{ \limsup_{t \to 0} t^{-1} (B)_t = \liminf_{t \to 0} t^{-1} (B)_t = a_i \right\} \in \mathcal{F}_0^0, \]
which form a $\mathcal{P}$-q.s. partition of $\Omega$. Hence, one may want to use the more conservative choice
\[ x = \mathcal{E}_0^*(X) = \sup_{P \in \mathcal{P}} E^P[X] = \inf\{y \in \mathbb{R} : y \geq \mathcal{E}_0(X)\} \]
as the price. This value can be embedded into the $\mathcal{E}$-martingale as follows. Let $\mathcal{F}_{0-}$ be the smallest $\sigma$-field containing the $\mathcal{P}$-polar sets, then $\mathcal{F}_{0-}$ is trivial $\mathcal{P}$-q.s. If we adjoin $\mathcal{F}_{0-}$ as a new initial state to the filtration $\mathbb{F}$, we can extend $\mathcal{E}(X)$ by setting
\[ \mathcal{E}_{0-}(X) := \sup_{P \in \mathcal{P}} E^P[X], \quad X \in \mathcal{H}. \]
The resulting process $\{\mathcal{E}_t(X)\}_{t \in [-0, T]}$ satisfies the properties from Proposition 4.5 in the extended filtration and in particular the constant $x = \mathcal{E}_{0-}(X)$ is the $\mathcal{F}_{0-}$-superhedging price of $X$. (Of course, all this becomes superfluous in the case where $\mathcal{E}(X)$ is a $\mathcal{P}$-modification of $\{\mathcal{E}^0_t(X)\}$.)

In the remainder of the section, we discuss replicable claims and adopt the previously mentioned conservative choice.

**Definition 4.16.** A random variable $X \in \mathcal{H}$ is called *replicable* if there exist a constant $x \in \mathbb{R}$ and an $\mathbb{F}$-predictable process $H$ with $\int_0^T |H_s|^2 d\langle B \rangle_s < \infty$ $\mathcal{P}$-q.s. such that
\[ X = x + \int_0^T H_t dB_t \quad P\text{-a.s. for all } P \in \mathcal{P} \quad (4.19) \]
and such that $(P)\int H dB$ is an $(\mathbb{F}, P)$-martingale for all $P \in \mathcal{P}$. 21
The martingale assumption is needed to avoid strategies which “throw away” money. Moreover, as in Corollary 4.11, the stochastic integral can necessarily be defined without reference to $P$, by setting $\int H dB := \mathcal{E}(X) - x$. The following result is an analogue of the standard characterization of replicable claims in incomplete markets (e.g., [8, p. 182]).

**Proposition 4.17.** Let $X \in \mathcal{H}$ be such that $-X \in \mathcal{H}$. The following are equivalent:

(i) $\mathcal{E}(X)$ is a symmetric $\mathcal{E}$-martingale and $\mathcal{E}_0(X)$ is constant $\mathcal{P}$-q.s.

(ii) $X$ is replicable.

(iii) There exists $x \in \mathbb{R}$ such that $E^P[X] = x$ for all $P \in \mathcal{P}$.

**Proof.** The equivalence (i)$\Rightarrow$(ii) is immediate from Corollary 4.11 and the implication (ii)$\Rightarrow$(iii) follows by taking expectations in (4.19). Hence we prove (iii)$\Rightarrow$(ii). By (4.7) we have $\mathcal{E}_0(-X) \leq \sup_{P \in \mathcal{P}} E^P[-X] = -x$ and similarly $\mathcal{E}(X) \leq x$. Thus, given $P \in \mathcal{P}$, the decompositions (4.15) of $\mathcal{E}(-X)$ and $\mathcal{E}(X)$ show that

$$-X \leq -x + (P) \int_0^T Z^{-X} dB \quad \text{and} \quad X \leq x + (P) \int_0^T Z^{X} dB \quad \text{P.-a.s.} \quad (4.20)$$

Adding the inequalities yields $0 \leq (P) \int_0^T (Z^{-X} + Z^{X}) dB$ P.-a.s. As we know from the proof of Theorem 4.15 that the integrals of $Z^X$ and $Z^{-X}$ are supermartingales, it follows that $(P) \int_0^T Z^{-X} dB = -(P) \int_0^T Z^{X} dB$ P.-a.s. Now (4.20) yields that $X = x + (P) \int_0^T Z^X dB$. In view of (iii), this integral is a supermartingale with constant expectation, hence a martingale. \qed

## 5 Uniqueness of Time-Consistent Extensions

In the introduction, we have claimed that $\{\mathcal{E}_t^s(X)\}$ as in (1.2) is the natural dynamic extension of the static sublinear expectation $X \mapsto \sup_{P \in \mathcal{P}} E^P[X]$. In this section, we add some substance to this claim by showing that the extension is unique under suitable assumptions.

The setup is as follows. We fix a nonempty set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F}_T)$; it is not important whether $\mathcal{P}$ consists of martingale laws. On the other hand, we impose additional structure on the set of random variables. In this section, we consider a chain of vector spaces $(\mathcal{H}_t)_{0 \leq t \leq T}$ satisfying

$$\mathbb{R} = \mathcal{H}_0 \subseteq \mathcal{H}_s \subseteq \mathcal{H}_t \subseteq \mathcal{H}_T =: \mathcal{H} \subseteq L_p^0, \quad 0 \leq s \leq t \leq T.$$  

We assume that $X, Y \in \mathcal{H}_t$ implies $X \wedge Y, X \vee Y \in \mathcal{H}_t$, and $XY \in \mathcal{H}_t$ if in addition $Y$ is bounded. As before, $\mathcal{H}$ should be seen as the set of financial claims. The elements of $\mathcal{H}_t$ will serve as “test functions”; the main example
to have in mind is $H_t = \mathcal{H} \cap L^p_{\mathbb{P}}(\mathcal{F}_t^\omega)$. We consider a family $(E_t)_{0 \leq t \leq T}$ of mappings

$$E_t : \mathcal{H} \to L^1_p(\mathcal{F}_t^\omega)$$

and think of $(E_t)$ as a dynamic extension of $E_0$. Our aim is to find conditions under which $E_0$ already determines the whole family $(E_t)$, or more precisely, determines $E_t(X)$ up to a $\mathcal{P}$-polar set for all $X \in \mathcal{H}$ and $0 \leq t \leq T$.

**Definition 5.1.** The family $(E_t)_{0 \leq t \leq T}$ is called $(\mathcal{H}_t)$-positively homogeneous if for all $t \in [0, T]$ and $X \in \mathcal{H}$,

$$E_t(X \varphi) = E_t(X) \varphi \quad \mathbb{P}\text{-q.s.} \quad \text{for all bounded nonnegative } \varphi \in \mathcal{H}_t.$$

Note that this property excludes trivial extensions of $E_0$. Indeed, given $E_0$, we can always define the (time-consistent) extension

$$E_t(X) := \begin{cases} E_0(X), & 0 \leq t < T, \\ X, & t = T, \end{cases}$$

but this family $(E_t)$ is not $(\mathcal{H}_t)$-positively homogeneous for nondegenerate choices of $(\mathcal{H}_t)$.

To motivate the next definition, we first recall that in the classical setup under a reference measure $P_*$, strict monotonicity of $E_0$ is the crucial condition for uniqueness of extensions; i.e., $X \geq Y$ $P_*$-a.s. and $P_*\{X > Y\} > 0$ should imply that $E_0(X) > E_0(Y)$. In our setup with singular measures, the corresponding condition is too strong. E.g., for $E_0(\cdot) = \sup_{P \in \mathcal{P}} E^P[\cdot]$, it is completely reasonable to have random variables $X \geq Y$ satisfying $E_0(X) = E_0(Y)$ and $P_1\{X > Y\} > 0$ for some $P_1 \in \mathcal{P}$, since the suprema can be attained at some $P_2 \in \mathcal{P}$ whose support is disjoint from $\{X > Y\}$. In the following definition, we allow for an additional localization by a test function.

**Definition 5.2.** We say that $E_0$ is $(\mathcal{H}_t)$-locally strictly monotone if for every $t \in [0, T]$ and any $X, Y \in \mathcal{H}_t$ satisfying $X \geq Y$ $\mathbb{P}$-q.s. and $P(X > Y) > 0$ for some $P \in \mathcal{P}$, there exists $f \in \mathcal{H}_t$ such that $0 \leq f \leq 1$ and

$$E_0(Xf) > E_0(Yf).$$

Here the delicate point is the regularity required for $f$. Indeed, one is tempted to try $f := 1_{\{X > Y + \delta\}}$ (for some constant $\delta > 0$), but in applications the definition of $\mathcal{H}_t$ will typically exclude this choice and require a more refined construction. We defer this task to Proposition 5.5 and first show how local strict monotonicity yields uniqueness.

**Proposition 5.3.** Let $E_0$ be $(\mathcal{H}_t)$-locally strictly monotone. Then there exists at most one extension of $E_0$ to a family $(E_t)_{0 \leq t \leq T}$ which is $(\mathcal{H}_t)$-positively homogeneous and satisfies $E_t(\mathcal{H}) \subseteq \mathcal{H}_t$ and $E_0 \circ E_t = E_0$ on $\mathcal{H}$. 

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Proof. Let \((E_t)\) and \((\tilde{E}_t)\) be two such extensions and suppose for contradiction that \(E_t(X) \neq \tilde{E}_t(X)\) for some \(X \in \mathcal{H}\); i.e., there exists \(P \in \mathcal{P}\) such that either \(P\{E_t(X) > \tilde{E}_t(X)\} > 0\) or \(P\{E_t(X) < \tilde{E}_t(X)\} > 0\). Without loss of generality, we focus on the first case. Define

\[
\varphi := (\lfloor E_t(X) - \tilde{E}_t(X) \rfloor \vee 0) \wedge 1.
\]

Then \(\varphi \in \mathcal{H}_t\), since \(\mathcal{H}_t\) is a lattice containing the constant functions; moreover, \(0 \leq \varphi \leq 1\) and \(\{\varphi = 0\} = \{E_t(X) \leq \tilde{E}_t(X)\}\). Setting \(X' := X\varphi\) and using the positive homogeneity, we arrive at

\[
E_t(X') \geq \tilde{E}_t(X') \quad \text{and} \quad P\{E_t(X') > \tilde{E}_t(X')\} > 0.
\]

By local strict monotonicity there exists \(f \in \mathcal{H}_t\) such that \(0 \leq f \leq 1\) and \(E_0(E_t(X')f) > E_0(\tilde{E}_t(X')f)\). Now \(E_0 = E_0 \circ E_t\) yields that

\[
E_0(X'f) = E_0(E_t(X')f) > E_0(\tilde{E}_t(X')f) = \tilde{E}_0(X'f),
\]

which contradicts \(E_0 = \tilde{E}_0\).

We can extend the previous result by applying it on dense subspaces. This relaxes the assumption that \(E_t(\mathcal{H}) \subseteq \mathcal{H}_t\) and simplifies the verification of local strictly monotonicity since one can choose convenient spaces of test functions. Consider a chain of spaces \((\mathcal{H}_t)_{0 \leq t \leq T}\) satisfying the same assumptions as \((\mathcal{H}_t)_{0 \leq t \leq T}\) and such that \(\mathcal{H}_T\) is a \(\|\cdot\|_{L^p_T}\)-dense subspace of \(\mathcal{H}\). We say that \((E_t)_{0 \leq t \leq T}\) is \(L^p_T\)-continuous if

\[
E_t : (\mathcal{H}, \|\cdot\|_{L^p_T}) \to (L^p_T(\mathcal{F}_t^0), \|\cdot\|_{L^p_T})
\]

is continuous for every \(t\). We remark that the motivating example \((E_t^\circ)\) from Assumption 4.1 satisfies this property (it is even Lipschitz continuous).

**Corollary 5.4.** Let \(E_0\) be \((\mathcal{H}_t)\)-locally strictly monotone. Then there exists at most one extension of \(E_0\) to an \(L^p_T\)-continuous family \((E_t)_{0 \leq t \leq T}\) on \(\mathcal{H}\) which is \((\mathcal{H}_t)\)-positively homogeneous and satisfies \(E_t(\mathcal{H}_T) \subseteq \mathcal{H}_t\) and \(E_0 \circ E_t = E_0\) on \(\mathcal{H}_T\).

**Proof.** Proposition 5.3 shows that \(E_t(X)\) is uniquely determined for \(X \in \mathcal{H}_T\). Since \(\mathcal{H}_T \subseteq \mathcal{H}\) is dense and \(E_t\) is continuous, \(E_t\) is also determined on \(\mathcal{H}\). \(\square\)

In our last result, we show that \(E_0(\cdot) = \sup_{P \in \mathcal{P}} E_P[\cdot]\) is \((\mathcal{H}_t)\)-locally strictly monotone in certain cases. The idea here is that we already have an extension \((E_t)\) (as in Assumption 4.1), whose uniqueness we try to establish. We denote by \(C_b(\Omega)\) the set of bounded continuous functions on \(\Omega\) and by \(C_b(\Omega_t)\) the \(\mathcal{F}_t\)-measurable functions in \(C_b(\Omega)\), or equivalently the bounded functions which are continuous with respect to \(\|\cdot\|_{\ell_t} := \sup_{0 \leq s \leq t} |\omega_s|\). Similarly, \(UC_b(\Omega)\) and \(UC_b(\Omega_t)\) denote the sets of bounded uniformly continuous...
functions. We also define $L^1_{\mathbb{P}}$ to be the closure of $UC_b(\Omega)$ in $L^1_{\mathbb{P}}$. Finally, $L^\infty_{\mathbb{P}}$ denotes the $\mathcal{P}$-q.s. bounded elements of $L^1_{\mathbb{P}}$ and $L^1_{\mathbb{P}}(\mathcal{F}_t^\infty)$, $L^\infty_{\mathbb{P}}(\mathcal{F}_t^\infty)$ are obtained similarly from $UC_b(\Omega_t)$.

**Proposition 5.5.** Let $\mathbb{E}_0(\cdot) = \sup_{P\in\mathcal{P}} E^P[\cdot]$. Then $\mathbb{E}_0$ is $(\mathcal{H}_t)$-locally strictly monotone for each of the cases

(i) $\mathcal{H}_t = C_b(\Omega_t)$,
(ii) $\mathcal{H}_t = UC_b(\Omega_t)$,
(iii) $\mathcal{H}_t = L^\infty_{\mathbb{P}}(\mathcal{F}_t^\infty)$, if in addition $\mathcal{P}$ is tight.

Before giving the proof, we indicate some examples covered by this result; see also Example 2.1. In all these examples, the domain of $(\mathbb{E}_t)$ is $\mathcal{H} = L^1_{\mathbb{P}}$.

(This statement implicitly uses the fact that $L^1_{\mathbb{P}}$ coincides with the $L^1_{\mathbb{P}}$-closure of $C_b(\Omega)$ when $\mathcal{P}$ is tight.)

(a) Let $(\mathbb{E}_t)$ be the $G$-expectation as introduced in [22, 23]. In this case, $\mathcal{P}$ is tight (see also [10]) and Corollary 5.4 applies: if $\mathcal{H}_t$ is any of the spaces in (i)-(iii), the invariance property $\mathbb{E}_t(\mathcal{H}_T) \subseteq \mathcal{H}_t$ is satisfied and of course $\mathcal{H}_T$ is dense in $\mathcal{H}$.

(b) Using the construction given in [20], the $G$-expectation can be extended to the case when there is no finite upper bound for the volatility. This corresponds to a possibly infinite (but still deterministic) function $G$ and then $\mathcal{P}$ need not be tight. Here Corollary 5.4 applies with $\mathcal{H}_t = UC_b(\Omega_t)$ since $\mathbb{E}_t(\mathcal{H}_T) \subseteq \mathcal{H}_t$ is satisfied by the remark stated after [20, Corollary 3.6].

(c) For the random $G$-expectations, we have to assume that $\mathcal{P}$ is tight; in particular, this covers the case when there is a uniform upper bound for the volatility. Then Corollary 5.4 can be applied with $\mathcal{H}_t = L^\infty_{\mathbb{P}}(\mathcal{F}_t^\infty)$ since $\mathbb{E}_t(\mathcal{H}_T) \subseteq \mathcal{H}_t$ holds by [20, Proposition 5.5].

**Proof of Proposition 5.5.** Fix $t \in [0, T]$. All topological notions in this proof are expressed with respect to $d(\omega, \omega') := ||\omega - \omega'||_t$. Let $X, Y \in \mathcal{H}_t$ be such that $X \geq Y$ $\mathcal{P}$-q.s. and $P_*(X > Y) > 0$ for some $P_* \in \mathcal{P}$. By translating and multiplying with positive constants, we may assume that $1 \geq X \geq Y \geq 0$. For the sake of clarity, we first prove the case (i) separately:

(i) Choose $\delta > 0$ small enough so that $P_*(X \geq Y + 2\delta) > 0$ and let

$$A_1 := \{X \geq Y + 2\delta\}, \quad A_2 := \{X \leq Y + \delta\}.$$  

Then $A_1$ and $A_2$ are disjoint closed sets and

$$f(\omega) := \frac{d(\omega, A_2)}{d(\omega, A_1) + d(\omega, A_2)}$$  

is a continuous function satisfying $0 \leq f \leq 1$ as well as $f = 0$ on $A_2$ and $f = 1$ on $A_1$. It remains to check that

$$\mathbb{E}_0(Xf) > \mathbb{E}_0(Yf), \quad \text{i.e.,} \quad \sup_{P \in \mathcal{P}} E^P[Xf] > \sup_{P \in \mathcal{P}} E^P[Yf].$$

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If $E_0(Yf) = 0$, the observation that $E_0(Xf) \geq E_{P*}[Xf] \geq 2\delta P_*(A_1) > 0$ already yields the proof.

Hence, we may assume that $E_0(Yf) > 0$. For $\varepsilon > 0$, let $P_\varepsilon \in \mathcal{P}$ be such that $E_{P_\varepsilon}[Yf] \geq E_0(Yf) - \varepsilon$. Since $X \geq Y + \delta$ on $\{f > 0\}$ and since $0 \leq Y \leq 1$, we have $Xf \geq (Y + \delta)f \geq (Y + \delta Y)f$ and therefore

$$E_0(Xf) \geq \limsup_{\varepsilon \to 0} E_{P_\varepsilon}[(Y + \delta Y)f]$$

$$= \limsup_{\varepsilon \to 0} (1 + \delta)E_{P_\varepsilon}[Yf]$$

$$= (1 + \delta)E_0(Yf).$$

As $\delta > 0$ and $E_0(Yf) > 0$, this ends the proof of (i).

(ii) The proof for this case is the same; we merely have to check that the function $f$ defined in (5.1) is uniformly continuous. Indeed, $Z := X - Y$ is uniformly continuous since $X$ and $Y$ are. Thus there exists $\varepsilon > 0$ such that $|Z(\omega) - Z(\omega')| < \delta$ whenever $d(\omega, \omega') \leq \varepsilon$. We observe that $d(A_1, A_2) \geq \varepsilon$ and hence that the denominator in (5.1) is bounded away from zero. One then checks by direct calculation that $f$ is Lipschitz continuous.

(iii) We first recall that $L^P_\infty(\mathcal{F})$ coincides with the set of bounded $\mathcal{P}$-quasi uniformly continuous functions (cf. [20, Proposition 5.2]). That is, a bounded $\mathcal{F}$-measurable function $h$ is in $L^P_\infty(\mathcal{F})$ if and only if for all $\varepsilon > 0$ there exists a closed set $\Lambda \subset \Omega$ such that $P(\Lambda) > 1 - \varepsilon$ for all $P \in \mathcal{P}$ and such that the restriction $h|_\Lambda$ is uniformly continuous.

For $\delta > 0$ small enough, we can thus find a closed set $\Lambda \subset \Omega$ such that $X$ and $Y$ are uniformly continuous on $\Lambda$ and $P_*(\{X \geq Y + 2\delta\} \cap \Lambda) > 0$. We then define the disjoint closed sets

$$A_1 := \{X \geq Y + 2\delta\} \cap \Lambda, \quad A_2 := \{X \leq Y + \delta\} \cap \Lambda.$$ 

Let $g$ be the function defined in (5.1), then $g$ is uniformly continuous on $\Lambda$ by the argument given in (ii). Moreover, $1_\Lambda$ is upper semicontinuous since $\Lambda$ is closed, and as $\mathcal{P}$ is tight, this implies that $1_\Lambda$ is $\mathcal{P}$-quasi uniformly continuous (see, e.g., the proof of [20, Proposition 5.2]). Therefore, the function $f := g1_\Lambda$ is in $L^P_\infty(\mathcal{F})$, and now the rest of the proof is as in (i). □

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