COMPLETE MONOTONICITY PROPERTIES OF A FUNCTION INVOLVING THE POLYGAMMA FUNCTION

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ABSTRACT. In this paper, we study complete monotonicity properties of the function \( f_{a,k}(x) = \psi^{(k)}(x + a) - \psi^{(k)}(x) - \frac{ax^k}{k+1}, \) where \( a \in (0,1) \) and \( k \in \mathbb{N}_0. \) Specifically, we consider the cases for \( k \in \{2n : n \in \mathbb{N}_0\} \) and \( k \in \{2n + 1 : n \in \mathbb{N}_0\}. \) Subsequently, we deduce some inequalities involving the polygamma functions.

1. Introduction and Preliminaries

The classical Gamma function, which is an extension of the factorial notation to noninteger values is usually defined as

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad x > 0,
\]

and satisfying the basic property

\[
\Gamma(x + 1) = x \Gamma(x), \quad x > 0.
\]

Its logarithmic derivative, which is called the Psi or digamma function is defined as (see [1, p. 258-259] and [3, p. 139-140])

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, dt, \quad x > 0,
\]

\[
= -\gamma - \frac{1}{x} + \sum_{k=1}^\infty \frac{x}{k(k+x)}, \quad x > 0,
\]

where \( \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right) = 0.577215664... \) is the Euler-Mascheroni’s constant. Derivatives of the Psi function, which are called polygamma functions are given as [1, p. 260]

\[
\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} \, dt, \quad x > 0,
\]

\[
= (-1)^{n+1} n! \sum_{k=0}^\infty \frac{1}{(k+x)^{n+1}}, \quad x > 0,
\]

satisfying the functional equation [1, p. 260]

\[
\psi^{(n)}(x + 1) = \psi^{(n)}(x) + \frac{(-1)^n n!}{x^{n+1}}, \quad x > 0,
\]

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where $n \in \mathbb{N}_0$ and $\psi^{(0)}(x) \equiv \psi(x)$. Here, and for the rest of this paper, we use the notations: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R} = (-\infty, \infty)$. Also, it is well known in the literature that the integral
\[ \frac{n!}{x^{n+1}} = \int_0^{\infty} t^n e^{-xt} dt, \quad (4) \]
holds for $x > 0$ and $n \in \mathbb{N}_0$. See for instance [1, p. 255].

In [6], Qiu and Vuorinen established among other things that the function
\[ h_1 = \psi (x + 1) - \psi(x) - \frac{1}{2x}, \quad (5) \]
is strictly decreasing and convex on $(0, \infty)$. Motivated by this result, Mortici [2] proved a more generalized and deeper result which states that, the function
\[ f_a = \psi(x + a) - \psi(x) - \frac{a}{x}, \quad a \in (0, 1), \quad (6) \]
is strictly completely monotonic on $(0, \infty)$. In this paper, the objective is to extend Mortici’s results to the polygamma functions. Particularly, we study completeness monotonicity properties of the function $f_{a,k}(x) = \psi^{(k)}(x+a) - \psi^{(k)}(x) - \frac{ak!}{x^{k+1}}$, where $a \in (0, 1)$ and $k \in \mathbb{N}_0$, by considering the cases for $k \in \{2n : n \in \mathbb{N}_0\}$ and $k \in \{2n + 1 : n \in \mathbb{N}_0\}$. Unlike Mortici’s work, the techniques of the present work are simple and do not rely on the Hausdorff-Bernstein-Widder theorem.

2. Main Results

We present our findings in this section by starting with the following lemma.

**Lemma 2.1.** Let a function $q_{\alpha,\beta}(t)$ be defined as
\[ q_{\alpha,\beta}(t) = \begin{cases} \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}, & t \neq 0, \\
\beta - \alpha, & t = 0, \end{cases} \quad (7) \]
where $\alpha, \beta$ are real numbers such that $\alpha \neq \beta$ and $(\alpha, \beta) \notin \{(0,1),(1,0)\}$. Then $q_{\alpha,\beta}(t)$ is increasing on $(0, \infty)$ if and only if $(\beta - \alpha)(1 - \alpha - \beta) \geq 0$ and $(\beta - \alpha)(|\alpha - \beta| - \alpha - \beta) \geq 0$.

**Proof.** See [4, Theorem 1.16] or [5, Proposition 4.1].

**Lemma 2.2.** Let $a \in (0, 1)$. Then the inequality
\[ a < \frac{1 - e^{-at}}{1 - e^{-t}} < 1, \quad (8) \]
holds for $t \in (0, \infty)$.

**Proof.** Note that the function $h(t) = \frac{1 - e^{-at}}{1 - e^{-t}}$ which is obtained from Lemma 2.1 by letting $\alpha = 0$ and $\beta = a \in (0, 1)$ is increasing on $(0, \infty)$. Also,
\[ \lim_{t \to 0^+} h(t) = a \quad \text{and} \quad \lim_{t \to \infty} h(t) = 1. \]

Then for $t \in (0, \infty)$, we have
\[ a = \lim_{t \to 0^+} h(t) = h(0) < h(t) < h(\infty) = \lim_{t \to \infty} h(t) = 1, \]
which gives (8).

**Theorem 2.3.** Let \( a \in (0, 1) \) and \( k \in \{2n : n \in \mathbb{N}_0\} \). Then the function

\[
  f_{a,k}(x) = \psi^{(k)}(x + a) - \psi^{(k)}(x) - \frac{ak!}{x^{k+1}},
\]

is strictly completely monotonic on \((0, \infty)\).

**Proof.** Recall that a function \( f : (0, \infty) \to \mathbb{R} \) is said to be completely monotonic on \((0, \infty)\) if \( f \) has derivatives of all order and \( (-1)^{n}f^{(n)}(x) \geq 0 \) for all \( x \in (0, \infty) \) and \( n \in \mathbb{N}_0 \). Let \( a \in (0, 1) \) and \( k \in \{2n : n \in \mathbb{N}_0\} \). Then, by repeated differentiation and by using (2) and (4), we obtain

\[
  f_{a,k}^{(n)}(x) = \psi^{(k+n)}(x + a) - \psi^{(k+n)}(x) - \frac{(1)^{n}a(k + n)!}{x^{k+n+1}}
  
  = (-1)^{k+n+1} \int_{0}^{\infty} \frac{t^{k+n}e^{-(x+a)t}}{1-e^{-t}} dt - (-1)^{k+n+1} \int_{0}^{\infty} \frac{t^{k+n}e^{-xt}}{1-e^{-t}} dt
  
  - (-1)^{n}a \int_{0}^{\infty} t^{k+n}e^{-xt} dt.
\]

Then

\[
  (-1)^{n}f_{a,k}^{(n)}(x) = - \int_{0}^{\infty} \frac{t^{k+n}e^{-xt}e^{-at}}{1-e^{-t}} dt + \int_{0}^{\infty} \frac{t^{k+n}e^{-xt}}{1-e^{-t}} dt - a \int_{0}^{\infty} t^{k+n}e^{-xt} dt
  
  > 0,
\]

as a result of Lemma 2.2. Alternatively, we could proceed as follows.

\[
  (-1)^{n}f_{a,k}^{(n)}(x) = \int_{0}^{\infty} \left[ \frac{1-e^{-at}}{1-e^{-t}} - a \right] t^{k+n}e^{-xt} dt
  
  = a \int_{0}^{\infty} \left[ \frac{1-e^{-at}}{at} - \frac{1-e^{-t}}{t} \right] \frac{t^{k+n+1}e^{-xt}}{1-e^{-t}} dt
  
  > 0.
\]

Notice that, since the function \( \frac{1-e^{-t}}{t} \) is strictly decreasing on \((0, \infty)\), then for \( a \in (0, 1) \), we have \( \frac{1-e^{-at}}{at} > \frac{1-e^{-t}}{t} \).

**Remark 2.4.** Since every completely monotonic function is convex and decreasing, it follows that \( f_{a,k}(x) \) is strictly convex and strictly decreasing on \((0, \infty)\).

**Corollary 2.5.** The inequality

\[
  \frac{ak!}{x^{k+1}} < \psi^{(k)}(x + a) - \psi^{(k)}(x) < \psi^{(k)}(a) - \psi^{(k)}(1) + k! \left( \frac{a}{x^{k+1}} + \frac{1}{a^{k+1} - a} \right),
\]

holds for \( a \in (0, 1) \), \( k \in \{2n : n \in \mathbb{N}_0\} \) and \( x \in (1, \infty) \).
**Proof.** Since \( f_{a,k}(x) \) is decreasing, then for \( x \in [1, \infty) \) and by (3), we obtain

\[
0 = \lim_{x \to \infty} f_{a,k}(x) < f_{a,k}(x) < f_{a,k}(1) = \psi^{(k)}(a + 1) - \psi^{(k)}(1) - ak!
\]

\[
= \psi^{(k)}(a) - \psi^{(k)}(1) + \frac{k!}{a^{k+1}} - ak!,
\]

which completes the proof.

**Remark 2.6.** If \( a = \frac{1}{2} \) and \( k = 0 \) in Corollary 2.5, then we obtain

\[
\frac{1}{2x} < \psi \left( x + \frac{1}{2} \right) - \psi(x) < \frac{1}{2x} + \frac{3}{2} - 2 \ln 2, \quad x \in (1, \infty).
\]

**Remark 2.7.** If \( a = \frac{1}{2} \) and \( k = 2 \) in Corollary 2.5, then we obtain

\[
\frac{1}{x^3} < \psi'' \left( x + \frac{1}{2} \right) - \psi''(x) < \frac{1}{x^3} + 15 - 12\zeta(3), \quad x \in (1, \infty),
\]

where \( \zeta(x) \) is the Riemann zeta function.

**Theorem 2.8.** For \( a \in (0, 1) \) and \( k \in \{2n + 1 : n \in \mathbb{N}_0\} \), let

\[
h_{a,k}(x) = \psi^{(k)}(x + a) - \psi^{(k)}(x) - \frac{ak!}{x^{k+1}}.
\]

Then \( -h_{a,k}(x) \) is strictly completely monotonic on \((0, \infty)\).

**Proof.** Similarly, for \( a \in (0, 1) \) and \( k \in \{2n + 1 : n \in \mathbb{N}_0\} \), we have

\[
-h_{a,k}^{(n)}(x) = \left( -1 \right)^n a(k + n)! \frac{1}{x^{k+n+1}} + \psi^{(k+n)}(x) - \psi^{(k+n)}(x + a)
\]

\[
= (-1)^n a \int_0^\infty \frac{t^{k+n} e^{-xt}}{1 - e^{-t}} dt + (-1)^{k+n+1} \int_0^\infty \frac{t^{k+n} e^{-xt}}{1 - e^{-t}} dt
\]

\[
- (-1)^{k+n+1} \int_0^\infty \frac{t^{k+n} e^{-(x+a)t}}{1 - e^{-t}} dt.
\]

Then,

\[
(-1)^n (-h_{a,k})^{(n)}(x) = a \int_0^\infty \frac{t^{k+n} e^{-xt}}{1 - e^{-t}} dt + \int_0^\infty \frac{t^{k+n} e^{-xt}}{1 - e^{-t}} dt - \int_0^\infty \frac{t^{k+n} e^{-xt} e^{-at}}{1 - e^{-t}} dt
\]

\[
= \int_0^\infty \left[ a + \frac{1 - e^{-at}}{1 - e^{-t}} \right] t^{k+n} e^{-xt} dt
\]

\[
> 0,
\]

which completes the proof.

**Remark 2.9.** Since \( -h_{a,k}(x) \) is strictly completely monotonic on \((0, \infty)\), it follows that \( h_{a,k}(x) \) is strictly concave and strictly increasing on \((0, \infty)\).

**Corollary 2.10.** The inequality

\[
\psi^{(k)}(a) - \psi^{(k)}(1) + k! \left( \frac{1}{a^{k+1}} - \frac{1}{a^{k+1}} - a \right) < \psi^{(k)}(x + a) - \psi^{(k)}(x) < \frac{ak!}{x^{k+1}},
\]

holds for \( a \in (0, 1) \), \( k \in \{2n + 1 : n \in \mathbb{N}_0\} \) and \( x \in (1, \infty) \).
Proof. Since $h_{a,k}(x)$ is increasing, then for $x \in (1, \infty)$, and by using (3), we obtain
\[ \psi^{(k)}(a) - \psi^{(k)}(1) - \frac{k!}{a^{k+1}} - ak! = h_{a,k}(1) < h_{a,k}(x) < \lim_{x \to \infty} h_{a,k}(x) = 0, \]
which yields (14).

**Remark 2.11.** Particularly, if $a = \frac{1}{2}$ and $k = 1$ in Corollary 2.10, then we obtain
\[ \frac{1}{2x^2} + \frac{\pi^2}{3} - \frac{9}{2} < \psi'(x + \frac{1}{2}) - \psi'(x) < \frac{1}{2x^2}, \quad x \in (1, \infty). \quad (15) \]

**Remark 2.12.** Likewise, if $a = \frac{1}{2}$ and $k = 3$ in Corollary 2.10, then we obtain
\[ \frac{3}{x^4} + \frac{14\pi^4}{15} - 99 < \psi'''(x + \frac{1}{2}) - \psi'''(x) < \frac{3}{x^4}, \quad x \in (1, \infty). \quad (16) \]

**CONFLICTS OF INTEREST**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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