Spontaneous scalarization within Scalar-Tensor theory using new conformal function in quark stars

M I Fauzi, H S Ramadhan and A Sulaksono*
Department of Physics, Faculty of Mathematics and Natural Science, University of Indonesia, Indonesia
*Corresponding author: anto.sulaksono@sci.ui.ac.id

Abstract. General relativity had been succeeding to describe gravitational phenomenon in weak-field gravity but some deviation had been observed in compact objects such as neutron stars and quark stars. This phenomenon of spontaneous scalarization which shown for the first time by Damour and Espasito-Farese appears in one of the modified gravity theories i.e., scalar tensor theory of gravity. We will study such novel phenomenon in quark stars. Furthermore, different to that of previous study, here we introduce new conformal function, $A(\phi) = 1 - \ln[1 - 0.5\beta(\phi - \phi_0)^2]$ We found that spontaneous scalarization using such conformal function appears for the first time when $\beta \approx -4.5$. Furthermore, the scalarization effect depends on the equation of state (EoS) of the quark stars. The corresponding effect appears rather different when scalar Coulomb EoS or vector Coulomb EoS are used.

1. Introduction
Around a century ago, Einstein proposes a theory which describes gravitational phenomenon. His revolutionary idea assumes gravitational field as the curvature of space-time caused by existence of massive object. The theory is called General Relativity (GR). This concept was really different with Newtonian gravity which the interpretation is the force between two objects with particular distance. Although Einstein’s theory is very successful, there are difficulties to describe some cosmological phenomena especially in strong gravitational field regime. This makes people to search alternative theory to modified GR. One of the candidates is Scalar Tensor Theory (STT).

The STT assumes gravitational fields occur not only by tensor, as in GR, but also with scalar fields [1]. This theory is constrained by solar system experiment or weak-gravitational field’s object $\alpha_0^2 < 10^{-3}$ [2], where $\alpha$ is coupling constant. The experiment is based on light deflection and time-delay experiment. The simplest and most popular STT’s theory is Brans-Dicke theory [3]. The theory has different phenomenon with GR as shown by Damour and Espasito-Farese [4]. The phenomenon is known as “spontaneous scalarization”[5,6], an analogue with spontaneous magnetization. In this following study, we focus on the impact of choosing the coupling function $A(\phi)$. Then, the impact of Equation of State (EoS) and mass-radius relation of Quark Star (QS) also will be studied. Here, QS is assumed to have spherical symmetry, static, and isotropic matter inside it.

2. Method
2.1. Formalism
The action in scalar tensor theory could be formulated as [4,5]
\[ S = \int d^4x \sqrt{-g} \left( \mathcal{L} + \frac{\omega}{\sqrt{2}} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right) + S_M, \]

where \( S_M \) is the matter part of the action. Here the Ricci scalar coupled directly with scalar field. The matter part contains non-gravitational field variable. The action of that form is in Jordan frame and reduced to general relativity when \( \omega \to \infty \).

We will perform “conformal transformation” to the action (see Ref. [7] and the references therein). This will make the action looks similar with general relativity [8,9]. The transformation will create new metric which it couples with scalar field

\[ \mathcal{g}_{\mu\nu} \rightarrow \tilde{\mathcal{g}}_{\mu\nu} = \Omega^2(\Phi) \mathcal{g}_{\mu\nu}, \]

Here, metric \( \mathcal{g}_{\mu\nu} \) is non-physical but could be used in STT representation. Following Damour and Espasito-Farese[4], these transformations are used

\[ \Omega(\rho) = \frac{1}{\sqrt{3+2\omega(\rho)}}, \]

\[ \tilde{\rho} = \frac{d\Omega(\rho)}{d\rho}. \]

The action could be written in new frame called Einstein frame

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - 2 \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right] + S_M, \]

where \( \Phi \) is the new scalar field. As could be seen now, the Ricci scalar does not coupled with the scalar field but the scalar field combined with the matter part of the action.

Using variation principle, we will obtain field equation of STT in Einstein frame

\[ \tilde{R}_{\mu\nu} = 2 \partial_\mu \Phi \partial_\nu \Phi + \frac{8\pi G}{c^4} \left( \tilde{T}_{\mu\nu} - \frac{1}{2} \tilde{T} \tilde{g}_{\mu\nu} \right), \]

\[ \Box g \phi = -\frac{4\pi G}{c^4} \alpha(\rho) \tilde{T}, \]

with \( \tilde{T}^{\mu\nu} = 2c \tilde{g}^{-\frac{3}{2}} \partial S_M / \partial \tilde{g}_{\mu\nu} \) indicates the stress-energy tensor in Einstein frame.

In this study, we will introduce new coupling function as[6]

\[ A(\rho) = 1 - ln[1 - 0.5\beta(\rho - \rho_0) \beta(\rho - \rho_0)], \]

\[ \alpha(\rho) = (1 - 0.5\beta(\rho - \rho_0)^2)(1 - ln[1 - 0.5\beta(\rho - \rho_0)^2]), \]

with \( \rho_0 = 2.4 \times 10^{-5} \). This value is taken from the \( \rho_0^{\max} \) value of Damour and Espasito-Farese[4]. STT theory will reduce to general relativity when \( \beta = 0 \). This will makes \( \alpha(\rho) = 1 \) and \( \alpha(\rho) = 0 \) which indicates there is no coupling between matter and scalar field.

We will assume the quark star has spherical symmetry property with metric

\[ ds^2 = -e^{2\tilde{\psi}(r)} dt^2 + \left( 1 - \frac{2M(r)}{c^2} \right)^{-1} dr^2 + r^2 d\Omega^2. \]

Thus, the matter inside the star follows ideal fluid characteristic. Its stress-energy tensor reads

\[ \mathcal{T}^{\mu\nu}_{\text{eff}} = (\tilde{\epsilon} + \tilde{\rho}) u^\mu u^\nu + \tilde{\rho} g^{\mu\nu}, \]

following equations are obtained by deriving from field equation and spherical symmetry metric

\[ \frac{dM}{dr} = 4\pi r^2 A^4 \tilde{\epsilon} + \frac{r}{2} (r - 2M) \psi^2, \]

\[ \frac{d\tilde{\psi}}{dr} = \psi, \]

\[ \frac{d\tilde{\rho}}{dr} = -\frac{2}{3} (\tilde{\epsilon} + \tilde{\rho}) \left[ \frac{4\pi r^2 A^4 \tilde{\rho}}{r-2M} + \frac{r}{2} \tilde{\psi}^2 + \frac{M}{r(r-2M)} + \alpha \psi \right], \]

\[ \frac{d\psi}{dr} = 4\pi r^4 A^4 \left[ \alpha(\tilde{\epsilon} - 3\tilde{\rho}) + \frac{2M(r)}{(r-2M)^r} \right]. \]

We will get Tolmann-Oppenheimer-Volkoff equation, as shown in equation (2.15), for STT in Einstein frame. This equation describes the structure of the star.

To solve those equations numerically, we use shooting method and fourth order Runge-Kutta [5]. The methods are commonly used to solve differential equations with two boundary conditions. First, we arrange the equation into first order differential equation. Second, we integrate from the center of the star with following boundary conditions.
\[
M(\Delta r) = 0, \\
\varphi(\Delta r) = \varphi_c, \\
\bar{p}(\Delta r) = 0, \\
\psi(\Delta r) = \frac{4\pi}{3} \Delta r A^4(\varphi_c) a(\varphi_c) (\tilde{e}_c - 3\tilde{p}_c), \\
\bar{M}(\Delta r) = 0.
\]

Those boundary conditions are defined at the centre of the star except for scalar field which defines at infinity, \(\varphi_0^{\text{max}} = 2.4 \times 10^{-3}\). The scalar field value at the center initially is an arbitrary guess with the same order of \(\varphi_0^{\text{max}}\). The exact value will be obtained when the boundary value is found. In every program’s step, the integration moves with distance \(\Delta r\). The process will iterate until the pressure at the surface is zero, \(p(R) = 0\). Some quantities were calculated when iteration stop.

\[
R \equiv r_s, \\
\alpha_s \equiv \frac{2\varphi_c}{\varphi'_c}, \\
Q_1 \equiv (1 + \alpha_s^2)^{1/2}, \\
Q_2 \equiv \left(1 - \frac{2M_s}{r}\right)^{1/2}, \\
\hat{p}_s = \frac{2}{Q_1} tanh^{-1}\left(\frac{Q_1}{1 + 2[Rr_s]}\right), \\
v'_s = R\psi'_s + \frac{2M_s}{R(R-2M_s)}, \\
\varphi_3 = \varphi_s - \frac{1}{2} \alpha_A \tilde{p}_s, \\
m_A = \frac{1}{3} v'_s R Q_2 \tilde{e}^2 \tilde{p}_s, \\
\bar{m}_A = \bar{M}_s,
\]

where \(m_A\) is ADM mass from object A, and \(\omega_A\) is the scalar charge. Symbol \(\bar{m}_A\) denotes mass \(m_A\) in Jordan frame.

3. Result and Discussion

![Figure 1](image_url). Relation \(m_b [M_\odot]\) with \(R [km]\) scalar Coulomb EoS.
The equations used in QS are Scalar Coulomb EoS and Vector Coulomb EoS. Mass-radius relations for both EoS are shown in Figure 1 and Figure 2. In Figure 1, by using minimum scalar Coulomb (red line) and maximum scalar Coulomb (black line) EoSs, QS experienced a phenomenon called “spontaneous scalarization” when $\beta = -4.5$. Scalarization effect for both EoS look similar and baryonic mass of minimum scalar Coulomb EoS is larger than maximum scalar Coulomb EoS. Scalarization makes the mass and the radius of QS are larger than those of GR case.

![Figure 1](image1.png)

**Figure 1.** Relation $m_b [M_\odot]$ with $R [km]$ by using scalar Coulomb EoS

In Figure 2, by using minimum vector Coulomb and maximum vector Coulomb EoSs, QS also experienced spontaneous scalarization when $\beta = -4.5$. Baryonic mass for minimum vector Coulomb EoS (red line) is larger than maximum vector Coulomb EoS (black line) even though the scalarization effect seems similar for both EoS. From the graph, it can be seen that spontaneous scalarization makes baryonic mass larger than GR case.

![Figure 2](image2.png)

**Figure 2.** Relation $m_b [M_\odot]$ with $R [km]$ by using vector Coulomb EoS

4. Conclusion
From the previous result, spontaneous scalarization phenomenon as deviation from GR, appears in our new coupling function by using scalar Coulomb and vector Coulomb EoSs when $\beta \approx -4.5$. Scalarization appears both in minimum and maximum scalar Coulomb EoSs. This also happen in minimum and maximum vector Coulomb ones. The scalarization effect for both EoS looks similar and makes baryonic mass larger than in GR case. This result indicates scalarization depends on EoSs of the quark star.

Acknowledgement
We would like to thank University of Indonesia for supporting this research through PITTA grant No. 2230/UN2.R3.1/HKP.05.00/2018 and No. 2267/UN2.R3.1/HKP.05.00/2018.

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