Setting the Renormalization Scale in QCD: The Principle of Maximum Conformality

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A key difficulty in making precise perturbative QCD predictions is the uncertainty in determining the renormalization scale $\mu$ of the running coupling $\alpha_s(\mu^2)$. It is common practice to simply guess a physical scale $\mu = Q$ of order of a typical momentum transfer $Q$ in the process, and then vary the scale over a range $Q/2$ and $2Q$. This procedure is clearly problematic since the resulting fixed-order pQCD prediction will depend on the choice of renormalization scheme; it can even predict negative QCD cross sections at next-to-leading-order \cite{1}.

The purpose of the running coupling in any gauge theory is to sum all terms involving the $\beta$ function; in fact, when the renormalization scale is set properly, all non-conformal $\beta \neq 0$ terms in a perturbative expansion arising from renormalization are summed into the running coupling. The remaining terms in the perturbative series are then identical to that of a conformal theory; i.e., the corresponding theory with $\beta = 0$. The resulting scale-fixed predictions using the “principle of maximum conformality” (PMC) are independent of the choice of renormalization scheme – a key requirement of renormalization group invariance. The results avoid renormalon resummation and agree with QED scale-setting in the Abelian limit. The PMC is also the theoretical principle underlying the BLM procedure, commensurate scale relations between observables, and the scale-setting method used in lattice gauge theory. The number of active flavors $n_f$ in the QCD $\beta$ function is also correctly determined. We discuss several methods for determining the PMC scale for QCD processes. We show that a single global PMC scale, valid at leading order, can be derived from basic properties of the perturbative QCD cross section. The elimination of the renormalization scale ambiguity and the scheme dependence using the PMC will not only increase the precision of QCD tests, but it will also increase the sensitivity of collider experiments to new physics beyond the Standard Model.

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I. INTRODUCTION

A key difficulty in making precise perturbative QCD predictions is the uncertainty in determining the renormalization scale $\mu$ of the running coupling $\alpha_s(\mu^2)$. It is common practice to simply guess a physical scale $\mu = Q$ of order of a typical momentum transfer $Q$ in the process, and then vary the scale over a range $Q/2$ and $2Q$. This procedure is clearly problematic since the resulting fixed-order pQCD prediction will depend on the choice of renormalization scheme; it can even predict negative QCD cross sections at next-to-leading-order \cite{1}.

The purpose of the running coupling in any gauge theory is to sum all terms involving the $\beta$ function; in fact, when the renormalization scale $\mu$ is set properly, all non-conformal $\beta \neq 0$ terms in a perturbative expansion arising from renormalization are summed into the running coupling. The remaining terms in the perturbative series are then identical to that of a conformal theory; i.e., the theory with $\beta = 0$. The divergent “renormalon” series of order $\alpha_n^\beta n!$ does not appear in the conformal series. Thus as in quantum electrodynamics, the renormalization scale $\mu$ is determined unambiguously by the “Principle of Maximal Conformality (PMC)”. This is also the principle underlying BLM scale setting \cite{2}.

It should be recalled that there is no ambiguity in setting the renormalization scale in QED. In the standard Gell-Mann–Low scheme for QED, the renormalization scale is simply the virtuality of the virtual photon \cite{3}. For example, in electron-muon elastic scattering, the renormalization scale is the virtuality of the exchanged photon, spacelike momentum transfer squared $\mu^2 = q^2 = t$. Thus

$$\alpha(t) = \frac{\alpha(t_0)}{1 - \Pi(t, t_0)} \quad (1)$$

where

$$\Pi(t, t_0) = \frac{\Pi(t) - \Pi(t_0)}{1 - \Pi(t_0)} \quad (2)$$
sums all vacuum polarization contributions to the dressed photon propagator, both proper and improper. (Here \( \Pi(t) = \Pi(t, 0) \) is the sum of proper vacuum polarization insertions, subtracted at \( t = 0 \).) Formally, one can choose any initial renormalization scale \( \mu_0^2 = t_0 \), since the final result when summed to all orders will be independent of \( t_0 \). This is the invariance principle used to derive renormalization group results such as the Callan-Symanzik equations [4, 5]. However, the formal invariance of physical results under changes in \( t_0 \) does not imply that there is no optimal scale. In fact, as seen in QED, the scale choice \( \mu^2 = q^2 \), the photon virtuality, immediately sums all vacuum polarization contributions to all orders exactly in the conventional Gell-Mann-Low scheme. With any other choice of scale, one will recover the same result, but only after summing an infinite number of vacuum polarization corrections.

Thus, although the initial choice of renormalization scale \( t_0 \) is arbitrary, the final scale \( t \) which sums the vacuum polarization corrections is unique and unambiguous. The resulting perturbative series is identical to the conformal series with zero \( \beta \)-function. In the case of muonic atoms, the modified muon-nucleus Coulomb potential is precisely \(-Z\alpha(-q^{-2})/q^{-2}\); i.e., \( \mu^2 = -q^2 \). Again, the renormalization scale is unique.

One can employ other renormalization schemes in QED, such as the \( \overline{\text{MS}} \) scheme, but the physical result will be the same once one allows for the relative displacement of the scales of each scheme. For example, one can start with the result in the \( \overline{\text{MS}} \) scheme for spacelike argument \( q^2 = -Q^2 \), for the standard one-loop charged lepton pair vacuum polarization contribution to the photon propagator using dimensional regularization:

\[
\log \frac{\mu^2_{\overline{\text{MS}}}}{m^2_f} = 6 \int_0^1 dx \, x(1-x) \log \left( \frac{m^2_f + Q^2 x(1-x)}{m^2_f} \right),
\]

which becomes at large \( Q^2 \)

\[
\log \frac{\mu^2_{\overline{\text{MS}}}}{m^2_f} = \log \frac{Q^2}{m^2_f} - 5/3;
\]

i.e., \( \frac{\mu^2_{\overline{\text{MS}}}}{m^2_f} = \frac{Q^2}{m^2_f} e^{-5/3} \). Thus if \( Q^2 >> 4m^2_f \), we can identify

\[
\alpha_{\overline{\text{MS}}}(-5/3, Q^2) = \alpha_{\text{GM-L}}(Q^2).
\]

The \( e^{-5/3} \) displacement of renormalization scales between the \( \overline{\text{MS}} \) and Gell-Mann–Low schemes is a result of the convention [6] which was chosen to define the minimal dimensional regularization scheme. One can use another definition of the renormalization scheme, but the final physical prediction cannot depend on the convention. This invariance under choice of scheme is a consequence of the transitivity property of the renormalization group [3, 7–9].

The same principle underlying renormalization scale-setting in QED must also hold in QCD since the \( n_f \) terms in the QCD \( \beta \) function have the same role as the lepton \( N \) vacuum polarization contributions in QED. QCD and QED share the same Yang-Mills Lagrangian. In fact, one can show [10] that QCD analytically continues as a function of \( N_C \) to Abelian theory when \( N_C \to 0 \) at fixed \( \alpha = C_F \alpha_s \) with \( C_F = \frac{N_C^2 - 1}{2 N_C} \). For example, at lowest order \( \beta_0^{QCD} = -\frac{1}{12} \left( \frac{11}{3} N_C - 2 n_f \right) \to -\frac{1}{144} \frac{2}{3} n_f \) at \( N_C = 0 \). Thus the same scale-setting procedure must be applicable to all renormalizable gauge theories.

Thus there is a close correspondence between the QCD renormalization scale and that of the analogous QED process. For example, in the case of \( e^+e^- \to 3g \), the PMC/BLM scale is set by the gluon jet virtuality, just as in the corresponding QED reaction. The specific argument of the running coupling depends on the renormalization scheme because of their intrinsic definitions; however, the actual numerical prediction is scheme-independent.

The basic procedure for PMC/BLM scale setting is to shift the renormalization scale so that all terms involving the \( \beta \) function are absorbed into the running coupling. The remaining series is then identical with a conformal theory scheme. Thus there is a close correspondence between the QCD renormalization scale and that of the analogous QED process. The PMC procedure also agrees with QED in the \( N_C \to 0 \) limit.

The determination of the PMC-scale for exclusive processes is often straightforward. For example, consider the process \( e^+ e^- \to c\bar{c} \to c\bar{c}g^* \to c\bar{c}b\bar{b} \), where all the flavors and momenta of the final-state quarks are identified. The \( n_f \) terms at NLO come from the quark loop in the gluon propagator. Thus the PMC scale for the differential cross section in the \( \overline{\text{MS}} \) scheme is given simply by the \( \overline{\text{MS}} \) scheme displacement of the gluon virtuality:

\[
\mu^2_{\text{PMC}} = e^{-5/3}(p_b + p_\bar{b})^2.
\]

In practice, one can identify the PMC/BLM scale for QCD by varying the initial renormalization scale \( \mu_0^2 \) to identify all of the \( \beta \)-dependent nonconformal contributions. At lowest order \( \beta_0 = \frac{1}{12} (11/3 N_C - 2/3 n_f) \). Thus at NLO one can simply use the dependence on the number of flavors \( n_f \) which arises from the quark loops associated with ultraviolet renormalization as a marker for \( \beta_0 \).

In QCD, the \( n_f \) terms also arise from the renormalization of the three-gluon and four-gluon vertices as well as from gluon wavefunction renormalization.
It is often stated that the argument of the coupling in a renormalization scheme based on dimensional regularization has no physical meaning since the scale $\mu$ was originally introduced as a mass parameter in extended space-time dimensions. However, the QED example above shows that the $\overline{\text{MS}}$ scale is unambiguously related to invariants in physical $3+1$ space. The connection of $\alpha_{\overline{\text{MS}}}$ to the Gell-Mann–Low scheme can be established at all orders. This also provides the analytic extension [34] of the $\alpha_{\overline{\text{MS}}}$ scheme for finite fermion masses as well to timelike arguments where the coupling is complex.

An example which shows how critical is to properly fix the renormalization scale is the three-gluon vertex. The PMC/BLM scale which appears in the three-gluon vertex is a function of the virtuality of the three external gluons $q_1^2$, $q_2^2$, and $q_3^2$. It has been computed in detail in refs. [12]. The results are surprising when the virtualities are very different as in the subprocess $gg \rightarrow g \rightarrow Q\bar{Q}$,

$$\hat{\mu}^2 \propto \frac{q_{\text{min}}^2 q_{\text{med}}^2}{q_{\text{max}}^2}$$  \hspace{1cm} (6)

where $|q_{\text{min}}^2| < |q_{\text{med}}^2| < |q_{\text{max}}^2|$: i.e. $q_{\text{max}}^2$ has the maximal virtuality [13]. The prediction based on simply guessing $\hat{\mu}^2 \simeq q_{\text{max}}^2$ would give misleading results.

The PMC/BLM scale that appears in the three-gluon vertex is the mass scale that controls the number of quark flavors $n_f$ which appears in the triangle graph. This is verified by keeping the quark masses and threshold dynamics in the loop. Thus we accurately determine the number of flavors $n_f$ that appears in the $\beta$ function in the three-gluon coupling. This generalizes for all gluonic processes.

Although these results have been obtained using the pinch-scheme, the final PMC/BLM result is scheme-independent. The pinch scheme is used because it provides a gauge-invariant setting for the analysis. In effect one calculates a scattering amplitude with three on-shell quark currents. One then obtains 14 invariant amplitudes which describe the three-gluon vertex, only one of which is renormalized.

In fact the calculation of the PMC scale for the three-gluon vertex $g_a \rightarrow g_b g_c$ given in Eq.(6) uses the pinch scheme to obtain a gauge invariant result. In effect, one computes the entire gauge invariant on-shell amplitude $q_a + q_b \rightarrow q_b q_c + q_c q_c$ including the triangle loop graph from quark loops with general mass. All 14 invariant amplitudes are computed analytically to one loop, only one of which is renormalized. The PMC scale for the three-gluon vertex as given in Eq.(6) also correctly sets the scale which controls the number of effective flavors which contribute to the $\beta-$ function for the three-gluon vertex. Details are given in refs. [12],[13].

These results show that the usual method of guessing the renormalization scale for processes involving the three-gluon and four-gluon couplings, typically misses this essential physics, assigns $n_f$ incorrectly and mischaracterizes the perturbative prediction. The error which is introduced can be in principle eliminated at infinite order, but only if one can sum the renormalon series.

The explicit result for the PMC/BLM scale is the physical scale controlling the quark threshold in the specific renormalization procedure used, but it is always possible to relate one scheme with another by the transitivity property of the renormalization group. This property is guaranteed by the PMC so there can be a constant displacement between schemes.

The PMC method is a general approach to set the renormalization scale in QCD including purely gluonic processes. It is scheme independent and void of renormalon growth due to the absence of the $\beta-$ function terms in the perturbative expansion. We stress that the $\beta-$function is gauge invariant in any correct renormalization scheme. The resulting conformal series is then gauge invariant. Thus the PMC is a gauge-invariant procedure.

It is sometimes argued that it is advantageous not to fix the renormalization scale at all, since its variation provides a measure of higher-order contributions to the theory predictions. In fact, one obtains sensitivity only to the $\beta-$dependent non-conformal terms by this procedure. In some cases the conformal contributions may be unexpectedly large. For example, the very large electron-loop light-by-light scattering contribution [14] $\simeq 18(\alpha^3/\pi)^3$ to the muon anomalous magnetic moment is unassociated with renormalization or the $\beta$ function. Of course, one can still compute the variation of the prediction around the PMC scale as an indicator of higher order non-conformal terms.

Stevenson has proposed that one should set the renormalization scale at a point where the predicted cross section has minimal variation with respect to $\mu$ – the “principle of minimal sensitivity” (PMS) [15]. However, unlike the PMC, the application of the PMS to jet production gives unphysical results [16] since it sums physics into the running coupling not associated with renormalization. Worse, the PMS prediction depends on the choice of renormalization scheme, and it violates the transitivity property of the renormalization group [17]. Such heuristic scale-setting methods also give incorrect results when applied to Abelian QED.

It should be emphasized that the factorization scale which enters predictions for QCD inclusive reactions is introduced to match nonperturbative and perturbative aspects of the parton distributions in hadrons; it is present even in conformal theory, and thus its determination is a completely separate issue from renormalization scale setting.
II. IDENTIFYING THE RENORMALIZATION SCALE USING THE PRINCIPLE OF MAXIMUM CONFORMALITY

Given the analytic form of the hard process amplitude or cross section as a series in \( \alpha_s(\mu^2_0) \) calculated at an initial scale \( \mu^2_0 \) and at a certain order (NLO, NNLO and so on), one can identify the PMC scale, order by order, in a systematic way:

1. The variation of the cross section with respect to \( \log \mu^2_0 \) can be used to distinguish the conformal terms versus the nonconformal terms proportional to the \( \beta \) function.
2. The identified nonconformal terms have the form \( \beta \times \log p_{ij}/\mu^2_0 \) where \( p_{ij} = p_i \cdot p_j \) are the scalar product invariants \( i \neq j \) which enter the hard subprocess. In practice, these terms can be identified as coefficients of \( n_f \), the number of flavors appearing in the \( \beta \) function; i.e., the flavor dependence arising from quark loops associated with coupling constant renormalization. The \( n_f \) terms in QCD arise from the renormalization if the three-gluon and four-gluon vertices as well as from gluon wavefunction renormalization.
3. The scale is then shifted \( \mu^2_0 \rightarrow \mu^2 \) in order to absorb the non-conformal terms. Thus when the scale is correctly set, the coefficients of \( \alpha_s(\mu^2) \) become independent of the \( \beta \) function and \( \log \mu^2 \).
4. The series is then identical to that of the conformal theory where \( \beta = 0 \) as given by the Banks-Zaks method [18].
5. The PMC scale is fixed for an observable (such as a differential cross section). PMC then can give a single effective global scale for the whole set of skeleton graphs entering the calculations which sums all the non-conformal \( \beta \)-terms associated with renormalization into the running coupling.

Other examples of this procedure will be given in the next sections.

A. The Global PMC Scale

Ideally, as in the BLM method, one should allow for separate scales for each skeleton graph; e.g., for electron-electron scattering, one takes \( \alpha(t) \) and \( \alpha(u) \) for the \( t \)-channel and \( u \)-channel amplitudes, respectively.

Setting separate renormalization scales can be a challenging task for complicated processes in QCD where there are many final-state particles and thus many possible Lorentz scalars \( p^2_{ij} = p_i \cdot p_j \). However, one can obtain a useful first approximation to the full PMC/BLM scale-setting procedure by using a single global scale \( \mu^2 \) which appropriately weights the individual BLM scales.

The global scale can be determined by varying the subprocess amplitude with respect to each invariant, thus determining the coefficients \( f_{ij} \) of \( \log p^2_{ij}/\mu^2_0 \) in the nonconformal terms in the amplitude. The global PMC scale is then

\[
\mu^2 = C \times \Pi_{ij} [p^2_{ij}]^{w_{ij}},
\]

i.e.,

\[
\log \mu^2 = \sum_{i \neq j} w_{ij} \log p^2_{ij} + \log C
\]

where the weight for each invariant is

\[
w_{ij} = \frac{f_{ij}}{\sum_{i \neq j} f_{ij}}.
\]

and \( \sum_{i \neq j} w_{ij} = 1 \). The constant \( C \) is the scheme displacement; e.g., \( C = e^{-5/3} \) for \( \overline{MS} \) for \( \mu^2 \gg 4m^2_f \).

As a specific example of the application of a PMC global scale, consider the electron-electron scattering amplitude in QED. (For simplicity, we will just take the contribution of the convection current to the amplitude, as in scalar QED.) The Lorentz invariant Born amplitude at the initial scale \( t_0 \) is then

\[
M^0(t, u) = 4\pi\alpha(t_0) \left( \frac{s - u}{t} + \frac{s - t}{u} \right).
\]

The running QED coupling \( \alpha(q^2) \) in QED sums all proper and improper vacuum polarization graphs

\[
M(t, u) = 4\pi\alpha(t) \left( \frac{s - u}{t} \right) + 4\pi\alpha(u) \left( \frac{s - t}{u} \right)
\]
where to leading order

$$\alpha(t) = \alpha(t_0)(1 + n\ell \frac{\alpha(t_0)}{3\pi} \log \frac{-t}{t_0}). \quad (12)$$

Aside from power-suppressed contributions involving the lepton masses, the resulting series is identical to the corresponding conformal theory with $\beta = 0$.

In this process we have contributions from both the $t$- and $u$-channel amplitudes which require separate renormalization scales for each skeleton graph. However, at leading order we can weight the amplitudes to obtain a single PMC/BLM scale which still sums the nonconformal $\beta$ terms into the running coupling $\alpha(\mu^2)$ at leading order. For example, using the standard Gell-Mann–Low scheme, we can write

$$M(t, u) = f(t)\alpha(t) + g(u)\alpha(u) = (f(t) + g(u))\alpha(\hat{\mu}^2) \quad (13)$$

where $f(t) = 4\pi(s - u)/t$ and $g(u) = 4\pi(s - t)/u$ are the Born amplitudes for the $t$- and $u$-channels, respectively.

Then in this case we have two basic PMC scales $\alpha(t)$ and $\alpha(u)$ for each skeleton graph in the standard Gell-Mann-Low scheme used in QED. These couplings then sum all of the vacuum polarization corrections to the skeleton graphs to infinite order. The result is then gauge invariant and the logarithm of the global scale is

$$\log \hat{\mu}^2 = \frac{f(t)}{f(t) + g(u)} \log(-t) + \frac{g(u)}{f(t) + g(u)} \log(-u) \quad (14)$$

which duplicates the multi-scale result at NLO.

One can also use the mean value theorem to obtain an effective single scale which analytically reproduces the exact multi-scale result to next to leading order. Since it matches the exact result at NLO, it also retains gauge invariance at this order. Moreover, the PMC single or multi-scale result is independent of the choice of scheme. The single scale result illustrates why it is wrong to guess a single scale like $\mu^2 = p_T^2$ since it fails to agree with this simple example.

Using kinematical constraints such as the total momentum conservation $s + t + u = 0$ the weighted scale dependence can be confined into the $\log(t/u)$ term inside the running coupling. The global scale $\hat{\mu}^2$ is maximal at $\theta_{CM} = \pi/2$ ($\hat{\mu}^2 = \sqrt{t/u} = -t = -u$) and vanishes at the boundaries $(0, \pi)$ where $\tan^2(\theta_{CM}/2) = t/u$. The effective renormalization scale for electron-electron scattering in Eq. 14 is weighted by the respective scattering amplitudes. The $t$-channel amplitude strongly dominates at $\Theta_{CM} = 0$, and the renormalization scale is thus $t$. Similarly, the $u$-channel amplitude strongly dominates at $\Theta_{CM} = \pi$, and the effective renormalization scale in that domain is $u$. Thus in both limits the effective renormalization scale $\hat{\mu}$ vanishes.

The results are shown in Fig. 1.

![Graph](image-url)  

FIG. 1: The PMC/BLM scale as function of the CM angle $\theta_{CM}: e e \rightarrow e e$ scalar QED
III. A PMC EXAMPLE FOR QCD: APPLICATION TO JET CROSS SECTIONS IN ELECTRON-POSITRON ANNIHILATION

As an example of the application of the PMC to QCD, we will show how the renormalization scale can be determined for the cross sections for $e^+e^-$ annihilation into two and three jets in \( \overline{\text{MS}} \) scheme.

The two-jet cross section has only infrared divergences:

\[
\sigma^{(2)} = \sigma_0 \left( \frac{4\pi \mu^2}{q^2} \right)^{\lambda/2} \left( 1 - \lambda/2 \right) \frac{\Gamma(1 - \lambda/2)}{\Gamma(2 - \lambda)} \tag{15}
\]

where \( \sigma_0 = \frac{4\pi^2}{3\alpha^2} N_C \sum_{i=1}^{N_f} e_i^2 \).

Here \( \lambda \equiv 4 - n \) is the number of extra space-time dimensions used to regulate infrared and ultraviolet divergent integrals. Eventually all of the infrared divergences and the factors involving \( \lambda \) will cancel out. In dimensional regularization the scale \( \mu \) is introduced as a mass scale to restore the correct dimension of the coupling. The gauge coupling \( g_R \) is related to the renormalized coupling constant \( \alpha_R \) by

\[
\frac{g_R^2}{(4\pi)^{1-\lambda/2}} = \frac{\alpha_s(\mu^2)}{4\pi} (\mu^2)^{\lambda/2} e^{\gamma_E \lambda/2} \tag{16}
\]

and here \( \gamma_E \) is the Euler constant.

As discussed in the introduction, the mass scale of schemes defined by dimensional regularization attains its physical meaning when it is applied to QED. The renormalized gauge coupling is also related to the bare coupling by:

\[
g_R = \sqrt{Z_3 Z_2 / Z_1 g_0}, \tag{17}
\]

where \( Z_1 \) is the renormalization constant for the quark-antiquark-gluon vertex, \( Z_2 \) for the quark field and \( Z_3 \) for the gluon field. The renormalization constants are:

\[
Z_1 = 1 - \frac{g_0^2}{16\pi^2} \left( N_c + C_F \right) \left( 2 \frac{\lambda_{UV}}{\lambda_{IR}} - \frac{2}{\lambda_{IR}} \right) \tag{18}
\]

\[
Z_2 = 1 - \frac{g_0^2}{16\pi^2} C_F \left( 2 \frac{\lambda_{UV}}{\lambda_{IR}} - \frac{2}{\lambda_{IR}} \right) \tag{19}
\]

\[
Z_3 = 1 + \frac{g_0^2}{16\pi^2} \left( 5 \frac{N_c}{3} - \frac{2}{3} N_f \right) \left( 2 \frac{\lambda_{UV}}{\lambda_{IR}} - \frac{2}{\lambda_{IR}} \right) \tag{20}
\]

where \( \lambda_{UV}, \lambda_{IR} \) are related respectively to the \( UV \)–ultraviolet and \( IR \)–infrared poles. In the \( MS \) only the pole associated with UV renormalization is subtracted out, and this leads us to a redefinition of the gauge coupling:

\[
\frac{1}{g_R} \delta g_0 = \frac{g_R^2}{16\pi^2} \left( 2 \frac{N_f}{3} - \frac{11}{3} N_c \right) \frac{1}{\lambda_{UV}} \tag{21}
\]

A suitable renormalization scheme is the \( \overline{\text{MS}} \) which differs from \( MS \) by a constant term and the respective counterterm can be inserted in the Born cross section by shifting the coupling constant:

\[
\alpha_s^0 = \alpha_s^\overline{\text{MS}} \left\{ 1 - \left( \frac{11}{6} N_c - \frac{2}{3} T_R \right) \frac{\alpha_s^\overline{\text{MS}}}{2\pi} \left( \frac{1}{\epsilon} + (\ln 4\pi - \gamma_E) \right) \right\} = \alpha_s^\overline{\text{MS}} \left\{ 1 - \beta_0 \alpha_s^\overline{\text{MS}} \left( \frac{1}{\epsilon} \right) \right\} \tag{22}
\]

where:

\[
\frac{1}{\epsilon} = \frac{1}{\epsilon} + (\ln 4\pi - \gamma_E), \tag{23}
\]

\[
\beta_0 = \frac{1}{2\pi} \left( \frac{11}{6} N_c - \frac{2}{3} T_R \right) \tag{24}
\]

with \( T_R = N_f/2, \epsilon = \lambda_{UV}/2 \).
The Born cross section for $e^+e^- \rightarrow q(p_1)\bar{q}(p_2)g(p_3)$ for massless quarks and gluons is

$$\left. \frac{d\sigma^{(3)}(\mu^2)}{dx_1dx_2} \right|_{\text{Born}} = \sigma^{(2)} \left( \frac{4\pi\mu^2}{q^2} \right)^{\lambda/2} \frac{1}{\Gamma(1-\lambda/2)} F_\lambda(x_1,x_2) \frac{\alpha_s^{MS}(\mu^2)}{2\pi} C_F B^{V-\lambda/2S}(x_1,x_2) \quad (25)$$

Here

$$F_\lambda(x_1,x_2) = [(x_1 + x_2 - 1)(1 - x_1)(1 - x_2)]^{-\lambda/2} \quad (26)$$

and

$$B^{V-\lambda/2S}(x_1,x_2) = B^V(x_1,x_2) - \frac{\lambda}{2} B^S(x_1,x_2) \quad (27)$$

$$B^V(x_1,x_2) = \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \quad (28)$$

$$B^S(x_1,x_2) = \frac{x_3^2}{(1 - x_1)(1 - x_2)} \quad (29)$$

where $x_i = \frac{2p_i^2}{q^2}$ in the $e^+e^-$ CM. In terms of invariants: $y_{ij} = s_{ij}/q^2 = (p_i + p_j)^2/q^2$. Then $x_1 = 1 - y_{23}$, $x_2 = 1 - y_{13}$, $x_3 = 1 - y_{12}$, $x_1 + x_2 + x_3 = 2$.

The renormalized one-loop corrected cross section for $e^+e^- \rightarrow q(p_1)\bar{q}(p_2)g(p_3)$ is given by Eq. (2.11) of Fabricius et al. [21] For our purposes it is sufficient to quote only the term proportional to $\beta_0$ in the $\overline{MS}$-scheme:

$$\left. \frac{d\sigma^{(3)}}{dx_1dx_2} \right|_{\text{one-loop}} = \left. \frac{d\sigma^{(3)}(\mu^2)}{dx_1dx_2} \right|_{\text{Born}} \left[ 1 + \alpha_s(\mu^2) \Gamma(1-\lambda/2) \frac{4\pi\mu^2}{q^2} \frac{\lambda/2}{\Gamma(1-\lambda)} \beta_0 \left( \log \frac{\mu^2}{q^2} \right) + \cdots \right] \quad (30)$$

where the coupling is defined as in Eq. 22: $\alpha_{MS}(\log 4\pi - \gamma_e \mu^2) \equiv \alpha_{MS}(\mu^2)$. The remaining contributions are independent of $n_f$ and $\beta_0$.

We can eliminate the non-conformal log-term proportional to $\beta_0$ by shifting the renormalization scale $\alpha_{MS}(\mu^2)$ in the Born cross section Eq. 25

$$\alpha_s(\mu^2) \simeq \alpha_s(\mu^2_0) \left( 1 - \alpha_s(q^2) \beta_0 \log \frac{\mu^2}{q^2} \right) ;$$

however, it is first convenient to shift the scale to $\mu^2 \rightarrow (\mu^2_0)$. Then

$$\left. \frac{d\sigma^{(3)}}{dx_1dx_2} \right|_{\text{one-loop}} = \left. \frac{d\sigma^{(3)}(\mu^2_0)}{dx_1dx_2} \right|_{\text{Born}} \left[ 1 + \alpha_s(\mu^2_0) \Gamma(1-\lambda/2) \frac{4\pi\mu_0^2}{q^2} \frac{\lambda/2}{\Gamma(1-\lambda)} \beta_0 \left( \log \frac{\mu^2}{q^2} \right) + \cdots \right] \quad (31)$$

Naively one could simply fix the scale to $\sqrt{q^2}$, but the 3-jet cross section will still be affected by IR divergences; in order to apply the PMC/BLM prescription we will first need to include the 4-jet contributions.

IV. NUMERICAL SCALE FIXING

The complete differential 3-jet cross section has been calculated by Fabricius et al. [21], and we quote here the results for the $\beta_0$-dependent terms:

$$\frac{d^2\sigma^{(3)}(\epsilon,\delta)}{dx_1dx_2} = \sigma_0 \frac{\alpha_s(q^2)}{2\pi} C_F \times$$

$$\left\{ B^V(x_1,x_2) \left[ 1 - \alpha_s(q^2) \beta_0 \left( \log \frac{1 - \cos \delta}{2} + \log \frac{\tilde{x}_3}{3} \right) - B^S(x_1,x_2) \alpha_s(q^2) \frac{\beta_0}{2} \right] + \mathcal{O}(\delta^4) \right\} + \cdots \quad (32)$$

$$\left\{ \frac{d^2\sigma^{(3)}(\epsilon,\delta)}{dx_1dx_2} = \sigma_0 \frac{\alpha_s(q^2)}{2\pi} C_F \times$$

$$\left\{ B^V(x_1,x_2) \left[ 1 - \alpha_s(q^2) \beta_0 \left( \log \frac{1 - \cos \delta}{2} + \log \frac{\tilde{x}_3}{3} \right) - B^S(x_1,x_2) \alpha_s(q^2) \frac{\beta_0}{2} \right] + \mathcal{O}(\delta^4) \right\} + \cdots \quad (33)$$
where \( \tilde{x}_3 = (2 - x_1 - x_2) \) and

\[
d\sigma^{(3)}(\epsilon, \delta) = d\sigma^{(3)} + d\sigma^{(4)}(\epsilon, \delta)
\]

is the sum of the 3- and the 4-jets contributions. The cancellation of the IR-poles is guaranteed by the KLN theorem [19, 20].

The variables \((\epsilon, \delta)\) are small quantities introduced in the virtual amplitude in order to define the soft and collinear 4-jet contributions to the 3-jet cross section. In particular these quantities refer respectively to the fraction of the total energy and to the cone opening angle which define the phase volume for a 3-jet event (for more details, see Ref. [21]).

In order to extract the PMC/BLM scale we first work in the \( \overline{\text{MS}} \)-scheme, fixing an arbitrary renormalization scale: \( \mu^2 = \mu_0^2 \). It turns out that \( \beta_0 \) term of the 3-jet differential IR safe cross section has the form:

\[
\frac{d^2\sigma^{(3)}(\epsilon, \delta)}{dx_1dx_2} = \frac{\alpha_s(\mu_0^2)}{2\pi} C_F \times \\
\left\{ B^V(x_1, x_2) \left[ 1 - \alpha_s(\mu_0^2) \beta_0 \left( \log \left( \frac{1 - \cos \delta}{2} \right) + 2 \log (2 - x_1 - x_2) - \frac{13}{3} + \log \frac{q^2}{\mu_0^2} \right) \right] \\
- B^S(x_1, x_2)\alpha_s(\mu_0^2)\frac{\beta_0}{2} \right\} + \mathcal{O}(\delta^2) + \cdots .
\]

In principle we can extract information on the terms in this formula performing a detailed analysis of the dependence of the \( \beta_0 \)-coefficient on the invariants. Performing a blindfold study we can single out the \( \beta_0 \)-coefficient by means of the \( \beta_0 \)-derivative of the whole cross section or either by the \( n_f \)-derivative since:

\[
\frac{df}{d\beta_0} = \frac{df}{dn_f} \times \frac{d\beta_0}{dn_f}^{-1}
\]

Then we can factorize out the Born amplitude Eq.25:

\[
\frac{d\sigma^{(3)}(\mu_0^2)}{dx_1dx_2} \bigg|_{\text{Born}}^{-1} \cdot \frac{d}{d\beta_0} \frac{d^2\sigma^{(3)}(\epsilon, \delta; \mu_0^2)}{dx_1dx_2} = \\
\left\{ -\alpha_s(\mu_0^2) \left( \log \left( \frac{1 - \cos \delta}{2} \right) + 2 \log (2 - x_1 - x_2) - \frac{13}{3} + \log \frac{q^2}{\mu_0^2} \right) \\
+ B^S(x_1, x_2) \frac{\beta_0}{2} B^V(x_1, x_2) \right\} + \mathcal{O}(\delta^2) + \cdots .
\]

and at the first order approximation the PMC/BLM scale can be fixed numerically imposing:

\[
\left. \left[ \frac{d\sigma^{(3)}(\mu_0^2)}{dx_1dx_2} \right]^{-1} \cdot \left( \frac{d}{dn_f} \frac{d^2\sigma^{(3)}(\epsilon, \delta; \mu_0^2)}{dx_1dx_2} \right) \right|_{n_f=0} = 0
\]

In the numerical procedure at NLO the analytic form of the cross section is not needed; one must only keep track of the appearance of number of flavors \( n_f \) arising from loop diagrams involving renormalization. This procedure, which has been shown at NLO here, can also be iterated to higher orders in \( \alpha_s \), by keeping track of the \( n_f \)-terms entering the \( \beta \)-function, leading us to an improvement of the accuracy of the PMC/BLM scale \( \mu_{\text{PMC}}^2 \).

Following this procedure we can include all the non-conformal \( \beta \) terms into the running coupling constant for every physical process, setting the renormalization scale at the PMC/BLM scale without necessarily knowing the PMC/BLM analytic form. Thus we end up with a cross section which is formally equal to the corresponding conformal expansion with \( \beta = 0 \). In this particular case the PMC/BLM scale has the form:

\[
\mu_{\text{PMC}}^2 \simeq q^2 (2 - x_1 - x_2)^2 \frac{\delta^2}{4} e^{-\frac{\pi^2}{12} \frac{\beta(x_1, x_2)}{\mu_0^2}}.
\]

In this case the coefficient depends on the parton energies \( x_1, x_2 \), on the angle parameter \( \delta \), and on the scale ratio \( q^2/\mu_0^2 \) (all these quantities can be written in the form of Lorentz invariants). The different contributions to the coefficient can be also identified, term by term, by considering the most differential cross section (i.e. for the 3-jet case the triple differential cross section), by performing the derivative (or logarithmic derivative) with respect to the corresponding invariant, and then isolating the constant term. This procedure will be discussed in detail in the next section.
V. THE PMC/BLM SCALE AS A FUNCTION OF THE JET MASS RESOLUTION PARAMETER

As shown by Kramer and Lampe [16], one can define a QCD jet by defining a resolution parameter \( y \cdot s \) as its maximal virtuality. The jet then consists of particles with total invariant mass squared smaller than \( y \cdot s \). Using this definition, we will perform the integration of the entire three-jet differential cross section, including real, \( d\sigma^{(s)} \), and virtual, \( d\sigma^{(3)} \), contributions in order to have a IR safe quantity. This gives a \( y \)-dependent integrated formula with \( \beta_0 \) dependent terms which can be absorbed into the argument of the running coupling, according to the PMC/BLM prescription.

The entire differential three-jet cross section [22]:

\[
\frac{1}{\sigma_0} \frac{d\sigma^{(s)} + d\sigma^{(3)}}{dy} = \int_y^{-1-2y} dz \int_z^{1-y-z} dx \frac{T[1-x-z,x,z]\alpha_s(Q^2)(1-\beta_0 \alpha_s(Q^2)(\log[x]+\log[z]-\frac{5}{3})))}{y}
\]

\[
= \alpha_s(Q^2)(T(y) - \beta_0 \alpha_s(Q^2)(C(y) + ..)) \tag{39}
\]

\[
\equiv T(y)\alpha_s(Q^2)(1-\beta_0 \alpha_s(Q^2)2 \log[\frac{\mu_{BLM}}{\sqrt{s}}]) = T(y)\alpha_s(\mu_{BLM}^2) \tag{40}
\]

where : \( \sigma_0 = \sigma_0 C_F Q^2/2\pi, s = Q^2, x = y_{13}, z = y_{23}, \)

\[
T[x_1, x_2, x_3] = \frac{2x_1^2 + x_2^2 + x_3^2 + 2x_1(x_2 + x_3)}{x_2 x_3} \tag{41}
\]

and \( T(y), C(y) \) result from the partial integration of the LO- and NLO- terms of the 3-jet cross section (for more details see Ref. [22][16]).

Then in the 3-jet case, the BLM-PMC scale as function of the jet-virtuality \( y \), has the analytic form:

\[
\hat{\mu}^2 = \mu_{PMC/BLM}^2 = s \times e^{-\frac{\gamma_E}{\beta_0} + \frac{\alpha_s(Q^2)}{\beta_0}} \tag{42}
\]

A plot of the PMC/BLM scale against \( y \), the virtuality resolution of the jet, in \( e^+e^- \rightarrow q\bar{q}g \) is shown in Fig. 2. The result agrees with the BLM scale calculated by Kramer and Lampe in the \( \overline{MS} \) scheme. The PMC/BLM prediction is scheme-independent; the specific value of the renormalization scale is rescaled according to the choice of scheme so that all results are commensurate. The PMC/BLM scale also accurately determines \( n_f \), the effective number of flavors in the \( \beta \)-function. As is clear from the QED analog, the renormalization scale reflects the virtuality of the gluon jet; it thus must vanish when the resolution \( y s \) vanishes. As noted by Kramer and Lampe [16], the renormalization scales determined by the \textit{ad hoc} PMS and FAC (Fastest Apparent Convergence) [23] procedures have the wrong physical behavior at \( y s \rightarrow 0 \), since they become infinite \( \mu^2 \rightarrow \infty \) as the jet resolution and gluon virtuality vanish.

VI. PMC/BLM SCALE FIXING IN THE 3 JET CASE: THE COMPLETE DIFFERENTIAL CROSS SECTION

In the case of the complete differential cross section; i.e., the most differential cross section for a given process without any constrained variables, the PMC/BLM scales depend on the number of flavors \( n_f \) and on the independent invariants entering the process. In the case of the three jets, we notice that the cross section depends on the color and flavor parameters \( n_f, N_C, C_F \) and on the kinematical invariants \( s_{12}, s_{13}, s_{23} \) where the label 3 refers to the gluon momentum, and the indices 1, 2 refer to the quark and anti-quark momenta. On the other hand, the nonconformal terms entering the running coupling depend only on the number of flavors \( n_f \) and on a reduced number of kinematical invariants. These terms can be identified by first varying the number of flavors \( n_f \) and then the invariant \( s_{ij} \), whereas the constant term can be extracted by simply subtraction at the final step. Starting with the triple differential cross section for three jets, which is given by the sum of the singular part of 4-jet differential cross section \( d\sigma^{(s)} \) and the real 3-jet cross section \( d\sigma^{(3)} \) ( for more details see Ref.[22]):

\[
\frac{d\sigma^{(s)} + d\sigma^{(3)}}{dz dy dx} = \hat{\sigma}_0 \frac{\alpha_s(Q^2)}{2\pi} \delta(1-x-y-z) \left\{ T[z,y] \left[ 1 + \frac{\alpha_s(Q^2)}{2\pi} C_F(...) \right] \right. + \left. \frac{\alpha_s(Q^2)}{2\pi} N_C(...) \right\}
\]
FIG. 2: The PMC/BLM scale, \( \mu_{PMC} \) (plane line) as a function of the jet resolution parameter \( y \), for \( e^+e^- \rightarrow q\bar{q}g \).

For comparison, the behavior \( \hat{\mu} \simeq \sqrt{y} \) is also shown (dashed line).

\[
- \alpha_s(Q^2) \beta_0 \left( \log[x \ast y] - \frac{5}{3} \right) + \frac{\alpha_s(Q^2)}{2\pi} F[z, y, x] \]

(43)

with \( \tilde{\sigma}_0 = \sigma_0 C_F s \). For simplicity sake we are using the notation \((z, x, y)\) for respectively the final state gluon-, quark-, antiquark-energy. In order to extract the first order terms related to the \( \beta_0 \) function we can start performing an \textit{ab initio} analysis of the cross section. We can first single out the \( \beta_0 \) coefficient by means of the \( \beta_0 \)-derivative, or either by the number of flavors \( n_f \)-derivative, using Eq. 36 and then we can factorize out the Born amplitude:

\[
\frac{d\sigma^{(3)}(Q^2)}{dz \: dy \: dx} \bigg|_{\text{Born}}^{-1} \frac{1}{\alpha_s(Q^2) d\beta_0} \left( d\sigma^{(s)} + d\sigma^{(3)} \right) = \left[ \log[x \ast y] - \frac{5}{3} \right] + O(\alpha_s),
\]

(44)

\[
\frac{d\sigma^{(3)}(Q^2)}{dz \: dy \: dx} \bigg|_{\text{Born}} = \tilde{\sigma}_0 \frac{\alpha_s(Q^2)}{2\pi} T[z, x, y] \delta(1 - x - y - z).
\]

Finally, we can extract the weight for each invariant by taking the logarithmic derivative:

\[
\omega_i = \frac{d}{d \log(x_i)} \left( \frac{d\sigma^{(3)}(Q^2)}{dz \: dy \: dx} \bigg|_{\text{Born}}^{-1} \frac{1}{\alpha_s(Q^2) d\beta_0} \left( d\sigma^{(s)} + d\sigma^{(3)} \right) \right)
\]

(45)

where \( x_i = (x, y, z) \). The constant term can be identified by subtracting out all the logarithm terms from the \( \beta_0 \) coefficient. Then at first order approximation in the coupling constant, the \( \mu_{PMC} \)-scale for the 3-jet differential cross section has the analytic form:

\[
\mu_{PMC}^2 \simeq Q^2 \times C \times \prod_i x_i^{\omega_i} = Q^2 \: x \: y \: e^{-\frac{5}{3}}.
\]

(46)

A. Commensurate Scale Relations

Relations between observables must be independent of the choice of scale and renormalization scheme. Such relations, called “Commensurate Scale Relations” (CSR) [24–26] are thus fundamental tests of theory, devoid of theoretical conventions. One can compute each observable in any convenient renormalization scheme, such as the \( \overline{\text{MS}} \) scheme using dimensional regularization. However, the relation between the observables cannot depend on this choice - this is the transitivity property of the renormalization group [3, 7–9]. For example, the PMC relates the effective charge \( \alpha_s(Q^2) \), determined by measurements of the Bjorken sum rule, to the effective charge \( \alpha_R(s) \), measured in the
total $e^+e^-$ annihilation cross section: $[1 - \alpha_s(Q^2)/\pi] \times [1 + \alpha_R(s^*)/\pi] = 1$. The ratio of PMC scales $\sqrt{s}/Q \simeq 0.52$ is set by physics; it guarantees that each observable goes through each quark flavor threshold simultaneously as $Q^2$ and $s$ are raised. Because all $\beta \neq 0$ nonconformal terms are absorbed into the running couplings using PMC, one recovers the conformal prediction [25]; in this case, it is the Crewther relation [27-31]. Thus by applying the PMC, the conformal commensurate scale relations between observables, such as the Crewther relation, become valid for non-conformal QCD at leading twist.

VII. CONCLUSIONS

As we have shown, the principle of maximal conformality (PMC) provides a consistent method for setting the optimal renormalization scale in pQCD. The PMC scale is determined by identifying the $\beta$ terms in the next-to-leading contributions and making the appropriate shift in order to include the $\beta$-terms into the running coupling. This can be done most simply by identifying the $n_f$ terms which come from quark loops of skeleton graphs. This includes the $n_f$ terms which renormalize the three and four gluon couplings. This procedure has been used to identify the correct PMC scale for the three-gluon vertex [12] [13]. The resulting series is identical to that of the corresponding conformal theory with $\beta = 0$ as given, for example, by the Banks-Zaks method [18].

The global PMC renormalization scale is particularly useful for very complex processes; one only requires the dependence of the calculated subprocess amplitudes on the initial renormalization scale $\mu_0$ and $n_f$, the number of quark flavors appearing from quark loops associated with renormalization. The single global PMC scale, valid at leading order, can thus be derived from basic properties of the perturbative QCD cross section.

We have discussed specific methods for efficiently determining the PMC renormalization scale analytically or numerically for QCD hard subprocesses. The analytic form of the PMC renormalization scale can be determined by varying the subprocess amplitude with respect to each invariant, thus determining the coefficients $f_{1ij} \log(p_{1ij}^2/\mu_0^2)$ in the nonconformal terms in the amplitude. This result can be used to fix the renormalization scales for each contributing skeleton graph. However, we have shown that a single PMC global scale can then determined at NLO by appropriate weighting. Alternatively the numerical value of the PMC scale can be determined without specific information on the analytic form from the $n_f$-derivative of the cross section. The two methods give rise to the same results at NLO.

The factorization scale, in contrast, is the scale entering the structure and fragmentation functions. Unlike the renormalization scale, a factorization scale ambiguity occurs even in a conformal theory. The factorization scale should be chosen to match the nonperturbative bound state dynamics with perturbative DGLAP evolution. This could be done explicitly using nonperturbative models such as AdS/QCD and light-front holography where the light-front wavefunctions of the hadrons are known.

Note that one applies the PMC method to renormalizable hard subprocesses (including the associated radiation diagrams required for IR finiteness) which enter the pQCD leading-twist factorization procedure. The initial and final quark and gluon lines are taken to be on-shell so that the calculation of the hard subprocess amplitude is gauge invariant. Thus the application of the PMC to hard subprocesses does not involve the factorization scale, and thus no double or single logarithms which involve the factorization scale enter.

The usual heuristic method of guessing the renormalization scale and varying it over a range of a factor of two gives scheme-dependent results, leaves the non-convergent perturbative series and gives the wrong result when applied to QED processes. In fact, varying the renormalization scale around such a guess only exposes nonconformal contributions involving the $\beta$ function; it gives no information on the conformal contributions. The PMS method [15] has similar faults – it violates the transitivity property of the renormalization group, depends on the choice of scheme, is wrong for QED, and as shown by Kramer and Lampe [16], leads to unphysical results. In contrast, the PMC method, which has no such disadvantages, and satisfies all principles of renormalization theory, gives the optimal prediction for pQCD at each finite order.

The PMC is the theoretical principle underlying the BLM procedure and commensurate scale relations between observables - the rigorous scale-fixed scheme-independent relations in QCD between observables, such as the Generalized Crewther relation; it is also the scale-setting method used for precision determinations of $\alpha_s$ in lattice gauge theory [32]. In addition, it has been recently shown that for certain observables in 2-jet production the results of the MOM-BLM method are very similar to those of MSYM theory [33][36]? ].

In the case of the BLM method, one deals with separate renormalization scales for each skeleton diagram, as is done in QED. The PMC method provides a single effective renormalization scale which reproduces the BLM scales at NLO, even for rather complex processes that are in our list of important projects, such as $W+$Jets, $e^+e^-$ annihilation, $tt$ production, and for general observables; e.g. differential cross sections, asymmetries.

If one considers a process with high multiplicity, then one confronts separate BLM scale for each of the multiple skeleton diagrams; thus the number of BLM scales will appear as the jet multiplicity increases. The PMC method replaces these multiple scales with an effective single scale at NLO.
We have discussed in this paper an illustration of the PMC procedure for 3-jet production in $e^+e^-$ annihilation where the $n_f$ terms arise from the inclusive 4 jet cross section after IR cancellation: these terms are included in the PMC scale with the effect of lowering its value.

The PMC method provides the correct renormalization scale from first principles without ambiguity or renormalization scheme dependence. The residual errors from the resulting conformal series provide an accurate assessment of higher order errors. The PMC/BLM uncertainty is zero at the order computed. The PMC is equivalent to the standard method used to eliminate the renormalization scale ambiguity in precision tests of QED.

The PMC method gives results which are renormalization scheme independent at each finite order. The PMC also determines the correct number of flavors $n_f$: this is particularly important when one uses a renormalization scheme which is analytic in the quark masses such as the analytic extension of the $\hat M S$ scheme [34]; one can then include the correct flavor threshold dependences and transitions as one evolves the QCD coupling. The correct displacement between the argument of the schemes is also automatically determined.

We stress that PMC does not capture all higher-order effects. One still has higher order corrections in the conformal series. These can never be discovered by varying the renormalization scale, since this variation only exposes terms proportional to the $\beta-$function. It is incorrect to require the scale choice to remove all higher order terms. For example, in QED, the muon anomalous moment receives a large contribution at order $\alpha^3$ from the electron-loop light-by-light insertion. This is due to the physics of the higher-order processes — not the running QED coupling. It is thus incorrect to vary the renormalization scale to minimize the effect of higher order corrections, since the variation of $\mu_R$ cannot expose large terms in the conformal series. Thus the PMC correctly and unambiguously exposes higher order terms which are intrinsic to physical effects, unrelated to the QCD running coupling.

We emphasize that the PMC method for setting the renormalization scale gives predictions for observables which are independent of the choice of renormalization scheme — a key requirement for a valid prediction for a physical quantity. The argument of the running coupling in a given scheme which appears in the resulting conformal series has the correct displacement so that the result is scheme-independent. The number of active flavors $n_f$ in the QCD $\beta$ function is also correctly determined, and the renormalization agrees with QED scale-setting in the $N_C \to 0$ Abelian limit. Furthermore, the resulting conformal series avoids the need for renormalon resummation.

A consistent application of the BLM/PMC procedure to B-decays, including $B \to X_s + \gamma$, has been developed including resummation to all orders in the strong coupling constant. A review and extension of this procedure is given by Melnikov and Mitov [35].

The PMC procedure has recently been extended to the four-loop level, [38] demonstrating that it provides a consistent, systematic and scheme-independent procedure for setting the renormalization scales up to NNLO. The explicit application for determining the renormalization scale of $R_{e^+e^-}(Q)$ up to four loops has also been presented [38].

The PMC is the principle underlying the BLM scale-setting procedure, a method which has been applied to many pQCD predictions. For example, the PMC/BLM procedure for setting the renormalization scale is the standard method for determining the intercept of the BFKL pomeron [36, 37].

A systematic and scheme-independent procedure for setting the PMC/BLM scales up to NNLO has also been demonstrated, including an explicit application for determining the scale for $R_{e^+e^-}(Q)$ up to four loops [38]. The PMC procedure has recently been applied to the $t\bar{t}$ hadroproduction cross section [39, 40]: and the $t\bar{t}$ asymmetry [41] major tests of the Standard Model at colliders [39, 40]. The PMC prediction for the total cross-section $\sigma_{t\bar{t}}$ agrees well with the present Tevatron and LHC data. The initial scale-independence of the PMC prediction is found to be satisfied to high accuracy at the NNLO level: the total cross-section remains almost unchanged even when taking very disparate initial scales. After PMC scale setting, the pQCD predictions are within $1\sigma$ of the CDF [42] and D0 measurements [43] since the relevant renormalization scale is less than conventional estimate; the large discrepancy of the top quark forward-backward asymmetry between the Standard Model prediction and the data is thus greatly reduced.

It should also be noted that the Principle of Maximum Conformality satisfies all of the consequences of renormalization group invariance - reflectivity, symmetry, and transitivity [44]. Using the PMC, all non-conformal in the perturbative expansion series are summed into the running coupling, and one obtains a unique, scale-fixed, scheme-independent prediction at any finite order. The PMC scales and the resulting finite-order PMC predictions are both to high accuracy independent of the choice of initial renormalization scale, consistent with RG invariance. Moreover, after PMC scale-setting, the residual initial scale-dependence at fixed order due to unknown higher-order $\{\beta_i\}$-terms can be substantially suppressed. The PMC thus eliminates a serious systematic scale error in pQCD predictions, greatly improving the precision of tests of the Standard Model and the sensitivity to new physics at collider and other experiments. Further discussion is given in ref. [44].

Clearly, the elimination of the renormalization scheme ambiguity using the PMC will greatly increase the precision of QCD tests and increase the sensitivity of measurements at the LHC and Tevatron to new physics beyond the Standard Model.
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