Stringy scaling of \( n\)-point hard string scattering amplitudes

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Abstract

Motivated by the recent calculation of the \( SL(K + 3, \mathbb{C}) \) symmetry of \( n\)-point Lauricella string scattering amplitudes (SSA) of open bosonic string theory, we calculate ratios of the solvable infinite linear relations of arbitrary \( n\)-point hard SSA (HSSA). We discover a general stringy scaling behavior for all \( n\)-point HSSA to all string loop orders. For the special case of \( n = 4 \), the stringy scaling behavior reduces to the infinite linear relations and constant ratios among HSSA conjectured by Gross\(^8\) and later corrected and calculated by the method of decoupling of zero-norm states\(^{11}\).
Symmetry principle was one of the most important discovery of the 20th century physics. All four fundamental interactions of the universe were based on the symmetry principles. In particular, the principles of the general coordinate symmetry and the gauge symmetry have led to Einstein general relativity theory and Yang-Mills gauge theory, respectively. In string theory, on the contrary, the situation went the other way around. One is given a set of rules through quantum consistency of the extended string to fix the forms of interactions or vertices of the theory. Moreover, in contrast to up to the usual four-point couplings in quantum field theory, e.g. QCD, in string theory, one encounters arbitrary $n$-point couplings which correspond to the infinite number of particles in the spectrum of string. As a result, it is crucial to identify the huge symmetry structure of string theory and uses it to relate these infinite number of couplings of particles with arbitrary higher masses and spins.

One key approach to uncover symmetry of string theory is to explicitly calculate string scattering amplitudes (SSA). Recently, the author of [1] calculated a subset of exact 4-point SSA with three tachyons and one arbitrary string states (Note that SSA of three tachyons and one arbitrary string states with polarizations orthogonal to the scattering plane vanish.)

\[
|r^T_n, r^P_m, r^L_l\rangle = \prod_{n>0} (\alpha^T_{-n}) r^T_n \prod_{m>0} (\alpha^P_{-m}) r^P_m \prod_{l>0} (\alpha^L_{-l}) r^L_l |0, k\rangle
\]  

(1)

where $e^P = \frac{1}{M^2_s} (E_2, k_2, 0) = \frac{k^2}{M^2_s}$ is the momentum polarization, $e^L = \frac{1}{M^2_s} (k_2, E_2, 0)$ is the longitudinal polarization and $e^T = (0, 0, 1)$ is the transverse polarization on the $(2+1)$-dimensional scattering plane, and expressed them in terms of the $D$-type Lauricella functions [1]. In addition to the mass level $M^2_s = 2(N - 1)$ with

\[
N = \sum_{n,m,l>0, \{ r^X_j \neq 0 \}} (m r^T_n + m r^P_m + l r^L_l),
\]

(2)

we define another important index $K$ for the state in Eq.(1)

\[
K = \sum_{n,m,l>0, \{ r^X_j \neq 0 \}} (n + m + l)
\]

(3)

where $X = (T, P, L)$ and we have put $r^T_n = r^P_m = r^L_l = 1$ in Eq.(2) in the definition of $K$. Intuitively, $K$ counts the number of variety of the $\alpha^X_{-j}$ oscillators in Eq.(1). For later use, we also define $k^X_j \equiv e^X \cdot k_j$. 

\[2\]
In addition, it was shown that these Lauricella SSA (LSSA) can be expressed in terms of the basis functions in the infinite dimensional representation of the $SL(K+3, \mathbb{C})$ group \[2, 3\]. This is similar to the well-known spherical harmonics function representation of the $SU(2)$ rotation group in quantum mechanics. Moreover, it was further shown that there existed $K+2$ recurrence relations of the $D$-type Lauricella functions. These recurrence relations can be used to reproduce the Cartan subalgebra and simple root system of the $SL(K+3, \mathbb{C})$ group with rank $K+2$. On the other hand, with the Cartan subalgebra and the simple roots, one can easily deduce the whole Lie algebra of the $SL(K+3, \mathbb{C})$ group. So we have the following correspondences

$$LSSA \iff SL(K+3, \mathbb{C}) \text{ symmetry} \iff \text{Recurrence relations of Lauricella.} \quad (4)$$

As a result, the $SL(K+3, \mathbb{C})$ group with its associated stringy Ward identities (recurrence relations) can be used to solve all the LSSA and express them in terms of one amplitude. See the recent review paper \[5\].

One important application of Eq. (4) in the hard string scattering limit was to reproduce infinite linear relations with constant coefficients among all 4-point hard SSA ($HSSA$) and solve the ratios among them. This high energy symmetry of string theory \[6, 7\] was first conjectured by Gross \[8\] and later corrected and proved \[9–12\] by the method of decoupling of zero norm states (ZNS) \[13–15\]. ZNS was also shown to carry $w_\infty$ symmetry charges of 2D string ($c = 1$, 2D quantum gravity) \[16\]. See the review papers \[17, 18\]. More importantly, since the decoupling of ZNS or stringy Ward identities persist to all string loop orders \[19\], it is conceivable that the infinite linear relations obtained in the hard scattering limit for arbitrary mass $M^2 = 2(N - 1)$ levels for 4-point SSA are also valid for all string loop amplitudes. On the other hand, one notes that these linear relations are not shared by amplitudes of quantum field theories which depend on polarizations of the scattering particles \[20\].

More recently, it was demonstrated \[21, 22\] that by using the string on-shell recursion relation of SSA \[23–25\], one can show that all $n$-point SSA can be expressed in terms of the Lauricella functions. This result extends the previous exact $SL(K+3, \mathbb{C})$ symmetry of the 4-point LSSA of three tachyons and one arbitrary string states in Eq. (4) to the whole tree-level open bosonic string theory. Motivated by this remarkable result \[21, 22\], one is tempted to believe that for the higher point ($n \geq 5$) LSSA, there exist hard scattering
kinematics regimes for which the infinite linear relations persist and the ratios can be solved accordingly.

In general, the explicit calculation of higher point \( (n \geq 5) \) SSA is very lengthy \[21, 22\]. In this letter, we will generalize the method of decoupling of ZNS and calculate the solvable infinite linear relations and ratios of arbitrary \( n \)-point HSSA. The case of \( n = 4 \) corresponds to the previous Gross conjecture. In addition, we will use saddle point method to re-calculate these infinite linear relations and ratios of a class of \( n \)-point HSSA to justify our results.

For each \( n \) with \( n \geq 5 \), we discover new ratios among \( n \)-point HSSA. Moreover, we discover the reduction of both the number of kinematics variables dependence on the ratios and the number of independent HSSA. These stringy scaling behaviors reminiscent of Bjorken scaling \[26\] and the Callan-Gross relation \[27\] in deep inelastic scattering of electron and proton in the quark-parton model of QCD are stringy phenomena with energy much higher than the Planck energy \( (\gg M^2_{\text{Planck}} \sim 1/\alpha') \), while the QCD scaling behavior is considered to be a low energy \( (\ll M^2_{\text{Planck}}) \) string theory.

Nevertheless, there is another view to study hard scattering of QCD in string theory, namely, string realization of QCD through gauge/string duality or string on warped space-time geometry \[30–32\]. In this picture, it was shown that the hard string scattering is power-law and thus QCD-like. However, since the scattering energy is of order of the string scale, it involves physics beyond the supergravity approximation, and one encounters the difficulty of solving string theory on curved spacetime. We believe that the stringy scaling behavior discovered in this paper on flat spacetime background and its possible extension to warped spacetime geometry may find application on these issues, in particular, the realization of QCD scaling behavior in string theory.

The stringy scaling behavior also reminds us of renormalization group (RG) approach to the critical phenomena of phase transition \[28\] such as ferromagnet, percolation, chaos, polymers and wind fetch. However, although both theories are fixed-point theories \( (D = 26 \) for the string case), the stringy scaling or reduction of number of kinematics variables shows up only in the hard scattering limit. See more discussion following Eq.(32).

On the other hand, there exists mechanism of (Spontaneously) broken symmetry in the Landau-Wilson theory of phase transition where the low-temperature phase has lost some symmetry. For the string theory side, there are two different concepts of broken symmetry. The first one was provided by the no-ghost theorem which says that the string spectrum is
ghost-free provided that the intercept \( a = 1 \) and \( D = 26 \) or \( a \leq 1 \) and \( D \leq 25 \). For the case of \( D = 26 \) comparing to the latter case, there are enhanced \( SL(K + 3, \mathbb{C}) \) symmetries due to the existence of the type II zero-norm states. The second one was discovered more recently that the exact \( SL(K + 3, \mathbb{C}) \) symmetry of \( LSSA \) at \( D = 26 \) was broken down to the \( SL(5, \mathbb{C}) \) symmetry in the Regge limit and to the \( SL(4, \mathbb{C}) \) symmetry in the nonrelativistic limit.

II. STRINGY SCALING BY DECOUPLING OF ZNS

The use of method of decoupling of ZNS (or stringy Ward identities) to calculate infinite linear relations for the 4-point \( HSSA \) was first adapted in the "physical state basis" with the Virasoro constraints imposed. It was then extended to the general mass levels \( M^2 = 2(N - 1) \) in the "oscillator state basis" without imposing the constraints. We will generalize the method to the higher point \( (n \geq 5) \) \( HSSA \) cases as will describe in this section.

In the old covariant first quantized open bosonic string spectrum, there are two types of physical ZNS. While type I states have zero-norm at any spacetime dimension, type II states have zero-norm only at \( D = 26 \). The starting point is to apply the \( n \)-point \( l \)-loop stringy on-shell Ward identities in the hard scattering limit.

\[
\langle V_1 \chi V_3 \cdots V_n \rangle_{l-loop} = 0 \tag{5}
\]

are leading order in energy. In Eq.5, \( V_j \) can be any string vertex and the second vertex \( \chi \) is the vertex of a ZNS. In the hard scattering limit of scattering processes on the scattering plane, the space part of momenta \( k_j \ (j = 3, 4, \cdots, n) \) form a closed 1-chain with \((n - 2)\) sides due to momentum conservation and it can be shown that at each fixed mass level \( M^2 = 2(N - 1) \) only states of the following form are leading order in energy.

\[
|N, 2m, q\rangle = \left( \alpha_{-1}^{T} \right)^{N-2m-2q} \left( \alpha_{-1}^{L} \right)^{2m} \left( \alpha_{-2}^{L} \right)^{q} |0; k\rangle \tag{6}
\]

are leading order in energy. In Eq.6 we have defined \( e^P = \frac{1}{M_2} (E_2, k_2, 0) = \frac{k_2}{M_2} \) the momentum polarization, \( e^L = \frac{1}{M_2} (k_2, E_2, 0) \) the longitudinal polarization and the transverse polarization \( e^T = (0, 0, \omega) \) where

\[
\omega_i = \cos \theta_i \prod_{\sigma=1}^{i-1} \sin \theta_{\sigma} \text{ with } i = 1, \cdots, r, \theta_r = 0 \tag{7}
\]
are the solid angles in the transverse space spanned by 24 transverse directions \( e^{T_i} \). Note that \( \alpha^T_1 = \alpha_1 \cdot e^T \) etc. Most importantly, to the leading hard scattering limit in energy, one can identify \( e^P \simeq e^L \). This is different from the Regge regime where \( e^P \not\equiv e^L \). It is a straightforward calculation to show that, at each fixed mass level \( N \), the decoupling of ZNS leads to ratios among \( n \)-point HSSA.

\[
\frac{T^{(N,m,q)}}{T^{(N,0,0)}} = \left(\frac{2m!}{m!}\right)^2 \left(\frac{-1}{2M}\right)^{2m+q} \tag{8}
\]

where \( T^{(N,m,q)} \) is the \( n \)-point HSSA of any string vertex \( V_j \) with \( j = 1, 3, \ldots n \), and \( V_2 \) is the high energy state in Eq.(6); while \( T^{(N,0,0)} \) is the \( n \)-point HSSA of any string vertex \( V_j \) with \( j = 1, 3, \ldots n \), and \( V_2 \) is the leading Regge trajectory string state at mass level \( N \). Note that we have omitted the tensor indice of \( V_j \) with \( j = 1, 3, 4 \) and keep only those of \( V_2 \) in \( T^{(N,2m,q)} \).

### A. Examples

For \( n = 4 \), Eq.(8) implies the ratios are independent of 1 scattering angle \( \varphi \) which reproduces Gross conjecture [9]. For \( n = 5 \), Eq.(8) implies the ratios are independent of 3 kinematics variables (2 angles and 1 fixed ratio of two infinite energies) or, for simplicity, 3 scattering ”angles”. For \( n = 6 \), there are 5 scattering ”angles”. See the \( r = 1 \) special case of Eq.(21) at the end of this section. To illustrate the infinite linear relations, historically, the first example of Eq.(8) was calculated for \( n = 4 \) and \( M^2 = 4, 6 \) in the ”physical state basis” with the Virasoro constraints imposed. At mass level \( M^2 = 4 \), there are two physical positive-norm states, one symmetric spin-3 state and one anti-symmetric spin-2 state. The decoupling of ZNS leads to 3 linear relations among 4 leading order 4-point HSSA:

\[
T_{5 \to 3}^{LT} + T_{3}^{LT} = 0, \tag{9}
\]

\[
10T_{5 \to 3}^{LT} + T_{3}^{TTT} + 18T_{3}^{LT} = 0, \tag{10}
\]

\[
T_{5 \to 3}^{LT} + T_{3}^{TTT} + 9T_{3}^{LT} = 0, \tag{11}
\]

which can be solved to get the ratios [9, 10, 33]

\[
T_{TTT} : T_{LLT} : T_{(LT)} : T_{[LT]} = 8 : 1 : -1 : -1. \tag{12}
\]
Eq. (12) calculated in the ”physical state basis” is consistent with the result of Eq.(8) calculated in the ”oscillator state basis” in Eq.(6). It was also demonstrated that the next to leading order amplitudes are in general not proportional to each other [9, 10].

In general scattering processes out of the scattering plane, the general high energy states at each fixed mass level \( M^2 = 2(N - 1) \) can be written as

\[
|\{p_i\}, 2m, 2q \rangle = (\alpha_{-1}^{T_1})^{N+p_1} (\alpha_{-1}^{T_2})^{p_2} \cdots (\alpha_{-1}^{T_r})^{p_r} \left( \alpha_{-2}^L \right)^{2m} \left( \alpha_{-2}^L \right)^q |0; k \rangle
\]

where \( \sum_{i=1}^r p_i = -2(m + q) \) with \( r \leq 24 \). With \( (\alpha_{-1}^{T_i}) = (\alpha_{-1}^{T}) \omega_i \), we easily obtain

\[
(\alpha_{-1}^{T_1})^{N+p_1} (\alpha_{-1}^{T_2})^{p_2} \cdots (\alpha_{-1}^{T_r})^{p_r} \left( \alpha_{-2}^L \right)^{2m} \left( \alpha_{-2}^L \right)^q |0; k \rangle = \left( \omega^N \prod_{i=1}^r \omega_i^{p_i} \right) \left( \alpha_{-1}^{T} \right)^{N-2m-2q} \left( \alpha_{-2}^L \right)^{2m} \left( \alpha_{-2}^L \right)^q |0; k \rangle,
\]

which leads to the ratios of \( n \)-point \( HSSA \)

\[
\frac{T(\{p_i\}, 2m, 2q)}{T(\{0\}, 0, 0)} = (2m)! \left( \frac{-1}{2M} \right)^{2m+q} \prod_{i=1}^r \omega_i^{p_i}
\]

where \( T(\{0\}, 0, 0) \) is the \( HSSA \) of leading Regge trajectory state at mass level \( M^2 = 2(N - 1) \). These ratios are valid to all string loop orders.

We are now ready to discuss the stringy scaling behavior of these \( n \)-point \( HSSA \). First, we would like to count the number of independent kinematics variables of the \( n \)-point \( HSSA \). For the simple case with \( n = 4 \) and \( r = 1 \) in Eq.(13) or Eq.(6), one has two variables, \( s \) and \( t \), or energy \( E \) and the scattering angle \( \varphi \), and Eq.(8) means that all \( HSSA \) at each fixed mass level are proportional to each other and the ratios are independent of the scattering angle \( \varphi \). For the general \( n \)-point \( HSSA \) with \( r \leq 24 \) in Eq.(13), we have \( k_j \) vector with \( j = 1, \cdots, n \) and \( k_j \in \mathbb{R}^{d-1,1} \). We can count the number of independent kinematics variables to be \( n (d - 1) - \frac{d(d+1)}{2} \).

In the hard scattering limit, for the kinematics parameter space \( \mathcal{M} \) defined by

\[
\omega_j \text{ (kinematics parameters with } E \to \infty) = \text{fixed constant } \quad (j = 2, \cdots, r),
\]

we can count the dimension of \( \mathcal{M} \) to be

\[
\dim \mathcal{M} = n (d - 1) - \frac{d(d+1)}{2} - 1 - (r - 1) = \frac{(r + 1) (2n - r - 6)}{2}.
\]
where \( r = d - 2 \) is the number of transverse directions \( e^{T_i} \). In sum, the ratios among \( n \)-point HSSA with \( r \leq 24 \) calculated in Eq.(15) are constants and independent of the scattering "angles" in \( \mathcal{M} \). For \( n = 6 \) and \( r = 3 \), as an example, \( \mathcal{M} \) is defined by

\[
\theta_j \ (8 \ \text{kinematics parameters}) = \text{fixed constant}, \quad j = 1, 2, \quad (18)
\]

and we have \( \dim \mathcal{M} = 6 \). For this case, the ratios

\[
\frac{T((p_1,p_2,p_3),m,q)}{T(\{0,0,0\},0,0)} = \left( \frac{2m!}{m!} \right) \left( \frac{1}{2M} \right)^{2m+q} (\cos \theta_1)^{p_1} (\sin \theta_1 \cos \theta_2)^{p_2} (\sin \theta_1 \sin \theta_2)^{p_3} \quad (19)
\]

are independent of kinematics parameters in the space \( \mathcal{M} \). For example, for say \( \theta_1 = \frac{\pi}{4} \) and \( \theta_2 = \frac{\pi}{6} \), we get the ratios among 6-point HSSA

\[
\frac{T((p_1,p_2,p_3),m,q)}{T(\{0,0,0\},0,0)} = \left( \frac{1}{M} \right)^{2m+q} (2m - 1)!! \left( \frac{1}{2} \right)^{p_2+p_3} \left( \sqrt{3} \right)^{p_3}. \quad (20)
\]

We conclude that, in the hard scattering limit, the number of scattering "angles" dependence on ratios of \( n \)-point HSSA with \( r \leq 24 \) reduces by \( \dim \mathcal{M} \). For a given \( (n, r) \), we list some numbers of \( \dim \mathcal{M} \) calculated in Eq.(17) as following

\[
\begin{array}{cccc}
\text{dim} \mathcal{M} & r = 1 & r = 2 & r = 3 & r = 4 \\
n = 4 & 1 & 1 & 1 & 1 \\
n = 5 & 3 & 3 & 3 & 3 \\
n = 6 & 5 & 6 & 6 & 6 \\
n = 7 & 7 & 9 & 10 & 10 \\
\end{array} \quad (21)
\]

We see from Eq.(15) that form of the ratios calculated for a given \( r \) are valid for arbitrary \( n \)-point HSSA and to all string loop orders. Note that for a given \( n \), \( \max(r) = n - 3 \), we have \( \dim \mathcal{M} (r = n - 4) = \dim \mathcal{M} (r = n - 3) = \frac{(n-3)(n-2)}{2} \).

The reduction of both the number of scattering "angles" dependence on ratios and independent SSA in the hard scattering limit suggests a very interesting phenomenon of stringy scaling behavior.

### III. STRING AMPLITUDES IN THE HARD LIMIT

To do a consistent check of the ratios calculated in Eq.(15) by the decoupling of ZNS, in this section, we will use saddle point method to calculate the \( n \)-point HSSA. Since the
ratios in Eq. (15) are independent of the choices of \( V_j \) with \( j = 1, 3, \cdots n \), we choose them to be tachyons and \( V_2 \) is chosen to be Eq. (13). On the other hand, since the ratios are independent of the loop order, we choose to calculate \( l = 0 \) loop. In the hard scattering limit, \( p = E \to \infty \), we define the 26-dimensional momenta in the CM frame to be

\[
\begin{align*}
k_1 &= (E, -E, 0^r), \\
k_2 &= (E, +E, 0^r), \\
&\vdots \\
k_j &= (-q_j, -q_j\Omega_1^j, -q_j\Omega_2^j, \cdots, -q_j\Omega_{r-1}^j, -q_j\Omega_{r+1}^j)
\end{align*}
\]

where \( j = 3, 4, \cdots, n \), and

\[
\Omega_i^j = \cos \varphi^j_i \prod_{s=1}^{i-1} \sin \varphi^j_s \text{ with } \varphi^j_{j-1} = 0, \varphi^j_{i>r} = 0 \text{ and } r \leq \min\{n - 3, 24\}
\]

are the solid angles in the \((j - 2)\)-dimensional spherical space with \( \sum_{i=1}^{j-2} (\Omega_i^j)^2 = 1 \). In Eq. (22), \( 0^r \) denotes the \( r \)-dimensional null vector. The condition \( \varphi^j_{j-1} = 0 \) in Eq. (23) was chosen to fix the frame by using the rotational symmetry. Note that the space part of \( k_j \) (\( j = 3, 4, \cdots, n \)) form a closed 1-chain with \((n - 2)\) sides in the 25-dimensional space due to momentum conservation. In the hard scattering limit \( E \to \infty \), the independent kinematics variables counted in the last section can be chosen to be some \( \varphi^i_j \) and some fixed ratios of infinite \( q_j \) defined in Eq. (22). We consider the \( n \)-point HSSA in 26D spacetime with \( n - 1 \) tachyons and 1 state defined in Eq. (13). With the change of variables \( z_i = \frac{x_i}{x_{i+1}} \) or \( x_i = z_i \cdots z_{n-2} \), the \( n \)-point HSSA can be calculated to be

\[
\mathcal{T}(\{p_i\}, m, q) = \int_0^1 dx_{n-2} \cdots \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 e^{-Kf} = \left( \prod_{i=3}^{n-2} \int_0^1 dz_i \; z_i^{2-N} \right) \int_0^1 dz_2 e^{-Kf}
\]

where

\[
f = -\sum_{i<j} \frac{k_i \cdot k_j}{K} \ln(x_j - x_i)
\]

\[
= -\sum_{i<j} \frac{k_i \cdot k_j}{K} [\ln(z_j \cdots z_{n-2}) + \ln(1 - z_i \cdots z_{j-1})], \quad K = -k_1 \cdot k_2,
\]

\[
u = (k^T_1)^{N+p_1} (k^T_2)^{p_2} \cdots (k^T_r)^{p_r} (k^L)^{2m} (k'^L)^q \cdot (k'^L = \frac{\partial k'^L}{\partial x_2})
\]
In Eq. (27), we have defined
\[ k = - \sum_{i \neq 2, n}^n k_i (z_i \cdots z_{n-2} - z_2 z_3 \cdots z_{n-2}) = \sum_{i \neq 2, n}^n k_i x_i - x_2 \] (28)
and \( k^X \) is the momentum \( k \) projected on the \( X \) polarization. We also define \( k_\perp = |k_\perp| \sum_{i=1}^r e^{T_i} \omega_i = |k_\perp| e^{\tilde{T}} \). We will show that the new defined vector \( k \) is the key quantity to connect the stringy scaling variables \( \omega_i \) or \( \theta_i \) introduced in Eq. (7) and the kinematic variables \( \Omega_i^j \) or \( \varphi_i^j \) introduced in the CM frame defined in Eq. (23).

To do the integration in Eq. (24), we define the saddle point \( \tilde{z}_i = (\tilde{z}_2, \cdots, \tilde{z}_{n-2}) \) to be the solution of
\[ \frac{\partial f}{\partial z_2} = 0, \cdots, \frac{\partial f}{\partial z_{n-2}} = 0. \] (29)
Note that Eq. (29) implies
\[ \tilde{k}^L = \tilde{k} \cdot k_2 \] 
\[ = M \frac{k_{12}}{M} \frac{\partial f}{\partial x_2} \bigg|_{z_i = \tilde{z}_i} \] 
\[ = M \frac{k_{12}}{ \partial x_2 \partial z_j} \bigg|_{z_i = \tilde{z}_i} = 0, \] \[ |\tilde{k}| = |k_\perp|. \] (30)
To proceed, let us introduce the following key identity
\[ \tilde{k}^2 + 2M\tilde{k}^{LL} = 0, \] (31)
which can be interpreted as a mathematical identity of the geometric parameters of a closed 1-chain with \( n - 2 \) sides. For \( n = 4 \), one can easily solve the saddle point \( \tilde{z}_2 = \sec^2 \varphi \) to prove Eq. (31) analytically. For the cases of \( n = 5 \) and 6, we are able to prove the identity numerically. Indeed, we will see that the identity is a result of the calculation of decoupling of ZNS in Eq. (15). Eq. (31) is crucial to show the stringy scaling behavior of \( HSSA \). On the other hand, Eq. (31) can be used to express \( \omega_i \) (or \( \theta_i \)) defined in Eq. (7) in terms of independent kinematics variables defined in Eq. (22) if one can analytically solve the saddle point \( \tilde{z}_i = (\tilde{z}_2, \cdots, \tilde{z}_{n-2}) \). Unfortunately, it turns out to be nontrivial except for the case of \( n = 4 \). For \( n = 6 \) and \( r = 3 \), see Eq. (18) as an example, we can only formally express \( \theta_i \) in terms of kinematics variables defined in Eq. (22) as
\[ \theta_1 = \arctan \left( \frac{\sqrt{\left( \tilde{k}_1 T_2 \right)^2 + \left( \tilde{k}_2 T_3 \right)^2}}{\tilde{k}_1 \tilde{T}_1} \right), \theta_2 = \arctan \left( \frac{\sqrt{\left( \tilde{k}_1 T_3 \right)^2 + \left( \tilde{k}_2 T_1 \right)^2}}{\tilde{k}_1 \tilde{T}_2} \right). \] (32)
It is important to note from Eq. (19) that the ratios of 6-point HSSA depends only on 2 variables $\theta_1$ and $\theta_2$ instead of 8. This stringy scaling behavior is reminiscent of the model-independent relations or *scaling relations* among critical exponents through RG analysis of Widom’s hypothesis in statistical mechanics where a set of 6 exponents $\alpha$, $\beta$, $\gamma$, $\delta$, $\nu$ and $\eta$ are related by 4 scaling relations, and the number of independent exponents reduces from 6 to 2.

In view of Eq. (27) and Eq. (30), all up to $(2^m)$-order differentiations of $u$ function in Eq. (27) at the saddle point vanish except

$$\frac{\partial^{2m} u}{\partial z_2^{2m}} \bigg|_{z_i = \tilde{z}_i} = \frac{(2m)!}{(-2M)^{2m+q}} \left| \bar{k} \right|^N \prod_{i=1}^{r} \omega_i^{p_i}$$

where we have used the *identity* Eq. (31). With the above inputs, we can calculate the $n$-point HSSA

$$\mathcal{T}({\{p_i}\}}_{2m,2q}) \simeq \left( \prod_{i=3}^{n-2} \int_0^1 dz_i \tilde{z}_i^{i-2-N} \right) \int_0^1 dz_2 \left( \frac{\partial^{2m} \tilde{u}}{\partial z_2^{2m}} (z_2 - \tilde{z}_2)^{2m} \right) e^{-Kf}$$

$$\simeq \frac{1}{(2m)!} \frac{\partial^{2m} \tilde{u}}{\partial z_2^{2m}} \left( \prod_{i=3}^{n-2} \tilde{z}_i^{i-2-N} \right) \int_0^\infty dz_2 (z_2 - \tilde{z}_2)^{2m} e^{-Kf(z_2)}$$

$$= 2 \sqrt{\pi} e^{-Kf} \left| \bar{k} \right|^N \left( \prod_{i=3}^{n-2} \tilde{z}_i^{i-2-N} \right) \omega_1^N \frac{(2m)!}{m!} \left( \frac{-1}{2M} \right)^{2m+q} \prod_{i=1}^{r} \omega_i^{p_i}$$

where $f(z_2) = f(z_2, \tilde{z}_3, \cdots, \tilde{z}_{n-2})$. Eq. (34) gives the ratios of $n$-point HSSA in Eq. (15). If we confine the $n$-point HSSA on a scattering plane, i.e. $r = 1$ and $\omega_1 = 1$, the ratios in Eq. (15) reduces to the ratios of $n$-point HSSA in Eq. (8). For the special $n = 4$ case, one reproduces the ratios obtained in [11, 12] for the Gross conjecture on 4-point HSSA. Thus Gross conjecture is just a special 4-point case of ratios of $n$-point HSSA with $r = 1$ in Eq. (8).

To illustrate the calculation in Eq. (34), the ratios of $n$-point HSSA for the case of $r = 2$
can be calculated to be

\[
\frac{\mathcal{T}^{(p_1, p_2, m, q)}}{\mathcal{T}^{(N, 0, 0, 0)}} = \frac{(2m)!}{m!} \left( -\frac{1}{2M} \right)^{2m+q} \frac{(2K \tilde{f}_{22})^{m+q}}{\left( \sum_{i\neq 2, n} \frac{k_{i1}^{T_1}}{x_i - \bar{x}_2} \right)^{2m+2q+p_2}} \frac{\left( \sum_{i\neq 2, n} \frac{k_{i2}^{T_2}}{x_i - \bar{x}_2} \right)^{-p_2}}{\left( \sum_{i\neq 2, n} \frac{k_{i1}^{T_1}}{x_i - \bar{x}_2} \right)^{2m+2q}}
\]

(35)

where \( \tilde{f}_{22} \) is defined in the following

\[
f_2 \equiv \frac{\partial f}{\partial \bar{z}_2}, \quad f_{22} \equiv \frac{\partial^2 f}{\partial \bar{z}_2^2}, \quad \tilde{f} = f(\bar{z}_2, \cdots, \bar{z}_{n-2}), \quad \tilde{f}_{22} = \frac{\partial^2 f}{\partial \bar{z}_2^2} \bigg|_{(\bar{z}_2, \cdots, \bar{z}_{n-2})}.
\]

(36)

On the other hand, the decoupling of ZNS of Eq.(15) gives

\[
\frac{\mathcal{T}^{(p_1, p_2, m, q)}}{\mathcal{T}^{(N, 0, 0, 0)}} = \frac{(2m)!}{m!} \left( -\frac{1}{2M} \right)^{2m+q} \omega_1^{p_1} \omega_2^{p_2} = \frac{(2m)!}{m!} \left( -\frac{1}{2M} \right)^{2m+q} \frac{(\tan \theta_1)^{p_2}}{(\cos \theta_1)^{2m+2q}}.
\]

(37)

The saddle point calculation in Eq.(35) and the ZNS calculation in Eq.(37) can be identified for any \( p_2, m \) and \( q \) if

\[
\left( \sum_{i\neq 2, n} \frac{k_{i1}^{T_1}}{x_i - \bar{x}_2} \right) = \sqrt{2K \tilde{f}_{22}} \cos \theta_1, \quad \left( \sum_{i\neq 2, n} \frac{k_{i2}^{T_2}}{x_i - \bar{x}_2} \right) = \sqrt{2K \tilde{f}_{22}} \sin \theta_1,
\]

(38)

which implies the identity

\[
\left( \sum_{i\neq 2, n} \frac{k_{i1}^{T_1}}{x_i - \bar{x}_2} \right)^2 + \left( \sum_{i\neq 2, n} \frac{k_{i2}^{T_2}}{x_i - \bar{x}_2} \right)^2 = 2K \tilde{f}_{22}.
\]

(39)

The identity derived in Eq.(39) is the \( r = 2 \) special case of the identity in Eq.(31).

IV. CONCLUSION

In this paper, we extend the previous calculation of ratios among \( HSSA \) for \( n = 4 \) with \( r = 1 \) in Eq.(21), which were conjectured by Gross [8] and calculated in [9 –12], to obtain new ratios for general \( n \)-point \( HSSA \) with \( r \leq 24 \). Moreover, we discover the reduction of both the number of kinematics variables dependence on the ratios and the number of independent SSA in the hard string scattering limit. These stringy scaling behaviors are
reminiscent of deep inelastic scattering of electron and proton where the two structure
functions \( W_1(Q^2, \nu) \) and \( W_2(Q^2, \nu) \) scale, and become not functions of kinematics variables
\( Q^2 \) and \( \nu \) independently but only of their ratio \( Q^2 / \nu \). That is, the structure functions scale as

\[
MW_1(Q^2, \nu) \rightarrow F_1(x), \quad \nu W_2(Q^2, \nu) \rightarrow F_2(x)
\] (40)

where \( x \) is the Bjorken variable and \( M \) is the proton mass. Moreover, due to the spin-\( \frac{1}{2} \)
assumption of quark, Callan and Gross derived the relation [27]

\[
2xF_1(x) = F_2(x).
\] (41)

These scaling behaviors in the hard scattering limit of quark-parton model in QCD seems
to persist in some way in the \( HSSA \) of string theory although the latter are stringy phe-
nomena with energy much higher than the Planck energy, while the QCD scaling behavior
is considered to be a low energy (\( \ll M_{\text{Planck}} \)) string theory. Nevertheless, there seems to
be interesting similarities between the two scaling behaviors. For given \( n \)-point \( HSSA \) with
\( r \leq 24 \), the number of independent kinematics variables on the ratios of \( HSSA \) reduced by
dim\( M \) as was calculated in Eq.(21). Moreover, the number of independent \( n \)-point \( HSSA \)
for a given \( r \) also reduces for each fixed mass level as can be seen in Eq.(15).

We believe that, comparing to hard QCD scaling, hard string theory in general has not
been well studied yet in the literature [35]. More new phenomena of stringy scaling proposed
in this letter remain to be uncovered.

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