NEWTON POLYGONS FOR $L$-FUNCTIONS OF GENERALIZED KLOOSTERMAN SUMS

CHUNLIN WANG AND LIPING YANG

Abstract. In this paper, we study the Newton polygons for the $L$-functions of $n$-variable generalized Kloosterman sums. Generally, the Newton polygon has a topological lower bound, called the Hodge polygon. In order to determine the Hodge polygon, we explicitly construct a basis of the top dimensional Dwork cohomology. Using Wan’s decomposition theorem and diagonal local theory, we obtain when the Newton polygon coincides with the Hodge polygon. In particular, we concretely get the slope sequence for $L$-function of $F(\overline{\lambda}, x) := \sum_{n=1}^{\infty} x^{a_n, i} + \overline{\lambda} \prod_{i=1}^{n} x^{-1}_i$.

1. Introduction

Let $p > 2$ be a prime, and $\mathbb{F}_q$ be the finite field of $q$ elements with characteristic $p$. For each positive integer $k$, let $\mathbb{F}_q^k$ be the finite extension of $\mathbb{F}_q$ of degree $k$. Let $\mathbb{Q}_p$ be the $p$-adic number field, and $\mathbb{Z}_p$ be the ring of $p$-adic integers. For any Laurent polynomial $f(x) \in \mathbb{F}_q[x_{1}^{\pm}, \ldots, x_{n}^{\pm}]$, the toric exponential sum is defined as

$S_k(f) := \sum_{x \in (\mathbb{F}_q^*)^n} T_{p, k}^{f(x)},$

where $T_{p, k} : \mathbb{F}_q^k \to \mathbb{F}_p$ is the trace of the field extension and $\mathbb{F}_q^*$ denotes the set of non-zero elements in $\mathbb{F}_q$. By a theorem of Dwork–Bombieri–Grothendieck, the following exponential generating $L$-function is a rational function

$L(f, T) := \exp \left( \sum_{k=1}^{\infty} \frac{S_k(f) T^k}{k} \right) = \prod_{i=1}^{d_1} (1 - \alpha_i T) \prod_{j=1}^{d_2} (1 - \beta_j T),$

where $\alpha_i (1 \leq i \leq d_1)$ and $\beta_j (1 \leq j \leq d_2)$ are non-zero algebraic integers. From Deligne’s integrality theorem, one has the following estimate

$|\alpha_i|_p = q^{-r_i}, |\beta_j|_p = q^{-s_j}, r_i \in \mathbb{Q} \cap [0, n], s_j \in \mathbb{Q} \cap [0, n]$

with normalized $p$-adic absolute value $| \cdot |_p$ such that $|q|_p = q^{-1}$. The rational number $r_i$ (resp. $s_j$) is called the slope of $\alpha_i$ (resp. $\beta_j$) with respect to $q$. The $p$-adic Riemann hypothesis for the $L$-function $L(f, T)$ is to determine the slopes of the zeros and poles. This is an extremely hard problem if there is no smoothness condition on $f$.

Write $f(x) = \sum \bar{a}_v x^v \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm]$. The support of $f$, denoted by $\text{Supp}(f)$, is defined as

$\text{Supp}(f) := \{ v : \bar{a}_v \neq 0 \}.$

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Let $\Delta(f)$ be the convex closure of $\text{Supp}(f)$ and the origin. If $\delta$ is a subset of $\Delta(f)$, we define the restriction of $f$ to $\delta$ to be

$$f_{\delta} = \sum_{v \in \delta} a_v x^v.$$  

**Definition 1.1.** The Laurent polynomial $f$ is called nondegenerate if for each closed face $\delta$ of $\Delta(f)$ of arbitrary dimension which does not contain the origin, the $n$ partial derivatives

$$\left\{ \frac{\partial f_{\delta}}{\partial x_1}, \ldots, \frac{\partial f_{\delta}}{\partial x_n} \right\}$$

have no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\mathbb{F}_q$.

Assume that $\Delta(f)$ is of dimension $n$ and $f$ is nondegenerate with respect to $\Delta(f)$. Adolphson and Sperber \cite{1} showed that the $L$-function $L(f,T)$ is a polynomial of degree $n! \text{Vol}(\Delta(f))$, where $\text{Vol}(\Delta(f))$ denotes the volume of $\Delta(f)$. Then

$$L(f,T)^{(-1)^{n-1}} = \prod_{i=1}^{n! \text{Vol}(\Delta(f))} (1 - \alpha_i T) = \sum_{i=0}^{A_i T_i} A_i T_i.$$  

The $q$-adic Newton polygon of $L(f,T)^{(-1)^{n-1}}$, denoted by NP$(f)$, is defined to be the lower convex closure in $\mathbb{R}^2$ of the points

$$(r, \text{ord}_q A_r), \ r = 0, 1, \ldots, n! \text{Vol}(\Delta(f)).$$

Obviously, determining the slope sequence $\{\text{ord}_q(\alpha_1), \ldots, \text{ord}_q(\alpha_{n! \text{Vol}(\Delta(f))})\}$ is equivalent to determining the $q$-adic Newton polygon of $L(f,T)^{(-1)^{n-1}}$.

In general, there is no effectual way to obtain the Newton polygon. A well studied example is the Kloosterman polynomial

$$f^{(n)}(\lambda, x) = x_1 + \cdots + x_n + \overline{\lambda} x_1 \cdots x_n$$

with $\bar{\lambda} \in \mathbb{F}_q^*$. For $n = 1$, Dwork \cite{5} derived that $\{0, 1\}$ is the slope sequence of the $L$-function $L(f^{(1)}(\lambda, x), T)$. Later, Sperber \cite{11, 12} generalized Dwork’s result to the $n$-variable case by showing that the slope sequence of $L(f^{(n)}(\lambda, x), T)^{(-1)^{n-1}}$ is $\{0, 1, \cdots, n\}$. But the methods used by Dwork and Sperber are very complicated.

Adolphson and Sperber \cite{1} proved that the Newton polygon has a topological lower bound called the Hodge polygon, which is easier to calculate. In particular, the polynomial $f$ is called ordinary if the Newton polygon equals to its Hodge polygon. Hence a general way to calculate the Newton polygon is usually to compute the Hodge polygon first, and then to decide when the Newton polygon and the Hodge polygon coincide. Actually, for the diagonal Laurent polynomial, Wan \cite{18} got some criterions to decide when the Newton polygon and the Hodge polygon coincide. For the non-diagonal Laurent polynomial, one main technique is Wan’s decomposition theorem, that is, decomposing $\Delta$ into small pieces which is diagonal and easy to deal with, the related work can be found in \cite{2}, \cite{3}, \cite{15}, \cite{17}, \cite{18}. Further decomposition methods for Newton polygons are developed in \cite{8}.

As a generalization of classical Kloosterman polynomial, the Newton polygon of the following polynomial

$$F(\lambda, x) := \sum_{i=1}^{n} x_i^{s_i} + \overline{\lambda} \prod_{i=1}^{n} x_i^{-d_i}$$
interests many mathematicians. For the case $a_1 = \cdots = a_n = 1$, Wan [15] used decomposition
theorems to prove that $F(\lambda, x)$ is ordinary for all $p$ such that $p - 1$ is divisible by $\text{lcm}(d_1, \ldots, d_n)$, and this result also was proved by Sperber [13] for large $p$, see [12] for the classical case. On the other hand, Bellovin et al. [2] studied the case $d_1 = \cdots = d_n = 1$ and obtained when $F(\lambda, x)$ is ordinary. But the Hodge polygon is hard to calculate. Using the combinatorial definition of Hodge polygon, Bellovin et al. computed the Hodge polygon under certain numeric restriction. Recently, the authors of the present paper studied the slope sequence of the $L$-function $L(F(\lambda, x), T)^{(-1)^{n-1}}$ for $n = 2$ and obtained the explicit slope sequence for the case when $d_1 = d_2 = 1$ and $a_1, a_2$ being coprime [10].

In this paper, we will require that $a_i, d_i$ are positive integers not divisible by $p$ for $i = 1, \ldots, n$. For convenience, we also use $F$ instead of $\bar{F}(\lambda, x)$ sometimes. Our main work is to use the above idea to study the Newton polygon of $L(\bar{F}, T)^{(-1)^{n-1}}$. That is, we compute the Hodge polygon first, then decide when $\bar{F}$ is ordinary, and at last we can derive the Newton polygons under certain conditions.

Instead of the combinatorial definition, we will use the cohomology definition to compute the Hodge polygon. According to Adolphson and Sperber [1], the $L$-function $L(F(\lambda, x), T)^{(-1)^{n-1}}$ can be expressed as the characteristic polynomial of a Frobenius map acting on the top dimensional cohomology space of a certain Koszul complex $\Omega^\bullet(C_0, \nabla(D))$ (constructed as in [2A]). The Hodge polygon is then totally determined by a basis of the top dimensional cohomology space $H^n(\Omega^\bullet(C_0, \nabla(D)))$. This allows us to study the explicit form of the basis.

In order to describe our results, we introduce some notations. Here $\Delta(\bar{F})$ denotes the convex closure of $\text{Supp}(\bar{F})$ and the origin. For $v \in \mathbb{Z}^n$, let $w(v)$ be the least nonnegative rational number such that $v \in w(v)\Delta(\bar{F})$. For every set $A$, let $|A|$ denote the cardinality of $A$. Define set $B$ as follows. If $n = 1$, then let $B := \{v \in \mathbb{Z} : -d_1 < v \leq a_1\}$. If $n \geq 2$, then define $B$ to be the set of $(v_1, \cdots, v_n) \in \mathbb{Z}^n$ such that $-d_i < v_i \leq a_i$ for all $i = 1, \cdots, n$, and

\[
\frac{d_j}{d_i}(v_i - a_i) \leq v_j < \frac{d_j}{d_i}v_i + a_j.
\]

for integers $i, j$ with $1 \leq i < j \leq n$. Let $e^* = \text{lcm}(a_1, d_1)$ for $n = 1$, and

$e^* = \text{lcm}(a_1, \ldots, a_n) \cdot \text{lcm}(d_1, \ldots, d_n)$

for $n \geq 2$.

**Theorem 1.2.** Let $\hat{B} := \{\bar{z}^e w(u)x^v : v \in B\}$. Then $\hat{B}$ forms a basis of $H^n(\Omega^\bullet(C_0, \nabla(D)))$.

To prove Theorem 1.2, we first show that $\hat{B}$ contains a basis of $H^n(\Omega^\bullet(C_0, \nabla(D)))$ and then show the number of lattice points $|B|$ equals to the rank of $H^n(\Omega^\bullet(C_0, \nabla(D)))$. This is the technical part of this paper. As a direct consequence of Theorem 1.2, we have the following result for the Hodge polygon.

**Corollary 1.3.** Let $k_i := |\{v \in B : w(v) = i/e^*\}|$ for $0 \leq i \leq ne^*$. Then the Hodge polygon $\text{HP}(\Delta(\bar{F}))$ is the convex closure of $(0, 0)$ and

\[
\left(\sum_{i=0}^{m} k_i, \sum_{i=0}^{m} \frac{ik_i}{e^*}\right), \quad m = 0, 1, \ldots, ne^*.
\]

It follows from Corollary 1.3 that the Hodge polygon can be computed concretely. Using Wan’s decomposition theorem and diagonal local theory, we can decide when the Newton polygon of $L(F(\lambda, x), T)^{(-1)^{n-1}}$ coincides with the Hodge polygon. Hence we have
\textbf{Theorem 1.4.} If \( p \equiv 1 \mod e^* \), then \( NP(\bar{F}) \) equals to the Hodge polygon \( HP(\Delta(\bar{F})) \).

In particular, we have

\textbf{Theorem 1.5.} For \( n \geq 1 \), let \( d_i = 1 \) for \( 1 \leq i \leq n \), \( \gcd(a_i, a_j) = 1 \) for any \( 1 \leq i < j \leq n \) and \( p \equiv 1 \mod \prod_{i=1}^{n} a_i \). Then each slope of \( L(\bar{F}(\lambda, x), T)^{(-1)^{n-1}} \) is with multiplicity one and the slope sequence equals to

\[
\left\{ \frac{\sum_{i=1}^{n} u_i}{a_i} : u_i = 0, \cdots, a_i \right\}
\]

as a set.

This paper is organized as follows. In section 2, we review Dwork’s cohomology theory. In section 3, we first prove Theorem 1.2, from which Corollary 1.3 follows. Then by reviewing Wan’s decomposition theorem and diagonal local theory we prove Theorem 1.4 and Theorem 1.5.

\section{Dwork cohomology}

In this section, we give a brief review on \( p \)-adic cohomology theory of Dwork type, for more details, see \cite{1} or \cite{7}. Let \( \bar{F}(\lambda, x) \) be given by (1.1). Recall that \( \Delta(\bar{F}) \) is the convex closure of \( \text{Supp}(\bar{F}) \) and the origin. Note that \( p \not| a_i d_i \) for \( i \in \{1, \cdots, n\} \). Then \( \bar{F} \) is nondegenerate with respect to \( \Delta(\bar{F}) \). Let \( \text{Cone}(\bar{F}) \) be the cone in \( \mathbb{R}^n \) over \( \Delta(\bar{F}) \) and let \( M(\bar{F}) := \text{Cone}(\bar{F}) \cap \mathbb{Z}^n \). Notably \( M(\bar{F}) = \mathbb{Z}^n \). For a point \( v \in \mathbb{R}^n \), the weight \( w(v) \) is defined to be the smallest nonnegative real number \( c \) such that \( v \in c \cdot \Delta(\bar{F}) \). If there is no such \( c \), we then define \( w(v) = \infty \). There is a positive integer \( e \) such that \( w(\mathbb{Z}^n) \subset (1/e)\mathbb{Z}_{\geq 0} \). The weight function \( w(v) \) has the following properties:

(a) \( w(v) = \inf \{ \sum_{u \in \text{Supp}(\bar{F})} a_u : \sum_{u \in \text{Supp}(\bar{F})} a_u u = v, a_u \in \mathbb{Q}, a_u \geq 0 \} \);

(b) \( w(lv) = lw(v) \) for \( l \in \mathbb{Z}_{\geq 0} \);

(c) \( w(v + v') \leq w(v) + w(v') \), with equality holding if and only if \( v \) and \( v' \) are cofacial, i.e., \( v/w(v) \) and \( v'/w(v') \) lie on the same closed face of \( \Delta(\bar{F}) \).

Let \( R := \mathbb{F}_q[\hat{x}_1^\pm, \cdots, \hat{x}_n^\pm] \). One can define an increasing filtration on \( R \) by setting, for \( i \in \mathbb{Z}_{\geq 0} \),

\[
R_{i/e} := \left\{ \sum_{v} b_v x^v : w(v) \leq i/e \text{ for all } v \text{ with } b_v \neq 0 \right\}.
\]

Let \( \bar{R} \) denote the associated graded ring. Multiplication in \( \bar{R} \) obeys the rule

\begin{equation}
\tag{2.1}
x^u x^{u'} = \begin{cases} x^{u+u'}, & \text{if } u \text{ and } u' \text{ are cofacial;} \\ 0, & \text{otherwise.} \end{cases}
\end{equation}

We can construct two complexes as follows. The spaces in both complexes are the same

\begin{equation}
\tag{2.2}
\Omega^i(\bar{R}, \nabla(\bar{F})) := \Omega^i(\bar{R}, \nabla(\bar{D})) := \bigoplus_{1 \leq j_1 < \cdots < j_i \leq n} R_{\bar{D}} \frac{dx_{j_1}}{x_{j_1}} \wedge \cdots \wedge \frac{dx_{j_i}}{x_{j_i}},
\end{equation}

with respective boundary operators given by

\[
\nabla(\bar{F})(\zeta \frac{dx_{j_1}}{x_{j_1}} \wedge \cdots \wedge \frac{dx_{j_i}}{x_{j_i}}) := \left( \sum_{l=1}^{n} x_l \frac{\partial \bar{F}}{\partial x_l} \zeta \frac{dx_{j_1}}{x_{j_1}} \wedge \cdots \wedge \frac{dx_{j_i}}{x_{j_i}} \right)
\]

and

\[
\nabla(\bar{D})(\zeta \frac{dx_{j_1}}{x_{j_1}} \wedge \cdots \wedge \frac{dx_{j_i}}{x_{j_i}}) := \left( \sum_{l=1}^{n} D_l(\zeta) \frac{dx_{j_1}}{x_{j_1}} \wedge \cdots \wedge \frac{dx_{j_i}}{x_{j_i}} \right).
\]
where
\[ \tilde{D}_l := x_l \frac{\partial}{\partial x_l} + x_t \frac{\partial \tilde{F}}{\partial x_t} \]

Since \( \tilde{F}(\lambda, x) \) is nondegenerate with respect to \( \Delta(\tilde{F}) \), we have

**Theorem 2.1** (Theorem 2.2. [7].) The two complexes \( \Omega^*(\tilde{R}, \nabla(\tilde{F})) \) and \( \Omega^*(\tilde{R}, \nabla(\tilde{D})) \) are acyclic except in the top dimension \( n \). In both cases, the top dimensional cohomology \( H^n \) is a finite free \( \mathbb{F}_q \)-algebra of rank \( n! \text{Vol}(\Delta(\tilde{F})) \). For each \( i \in (1/e)\mathbb{Z}_{\geq 0} \), we may choose a monomial basis \( B_i \) consisting of monomials of weight \( i \) for an \( \mathbb{F}_q \)-vector space \( V_i \) such that the \( i \)-th graded piece \( \tilde{R}_i \) of \( \tilde{R} \) may be written as

\[ \tilde{R}_i = V_i \oplus \sum_{l=1}^{n} x_l \frac{\partial \tilde{F}}{\partial x_l} \tilde{R}_{i-1} \]

so that if \( B = \bigcup_{i \in (1/e)\mathbb{Z}_{\geq 0}} B_i \) and \( V = \sum_{i \in (1/e)\mathbb{Z}_{\geq 0}} V_i \) is the \( \mathbb{F}_q \)-vector space with basis \( B \), then

\[ \tilde{R} = V \oplus \sum_{l=1}^{n} x_l \frac{\partial \tilde{F}}{\partial x_l} \tilde{R} \]

and

\[ \tilde{R} = V \oplus \sum_{l=1}^{n} \tilde{D}_l \tilde{R}. \]

Let
\[ H_\Delta(i) := \text{dim}_q V_i. \]

We now give the definition of the Hodge polygon of \( \Delta(\tilde{F}) \).

**Definition 2.2.** The Hodge polygon \( HP(\Delta(\tilde{F})) \) of \( \Delta(\tilde{F}) \) is the lower convex polygon in \( \mathbb{R}^2 \) with vertices \((0, 0)\) and

\[ \left( \sum_{k=0}^{m} H_\Delta(k), \sum_{k=0}^{m} \frac{k}{e} H_\Delta(k) \right), \text{ } m = 0, 1, \ldots, n. \]

Let \( \mathbb{Q}_q \) be the unramified extension of \( \mathbb{Q}_p \) of degree \( a \), and \( \mathbb{Z}_q \) be its ring of integers. Let \( \zeta \) be a primitive \( p \)-th root of unity. Then \( \mathbb{Z}_q[\zeta] \) and \( \mathbb{Z}_p[\zeta] \) are rings of integers of \( \mathbb{Q}_q(\zeta) \) and \( \mathbb{Q}_p(\zeta) \), respectively. Let \( \pi \) be an element in an algebraic closure of \( \mathbb{Q}_p \) such that \( \pi^{p-1} = -p \). By Krasner’s lemma, we have \( \mathbb{Q}_p(\pi) = \mathbb{Q}_p(\zeta) \). Adjoining the \( e \)-th root of \( \pi \) in \( \Omega \), say \( \tilde{\pi} \), we obtain totally ramified extensions of \( \mathbb{Q}_q(\zeta) \) and \( \mathbb{Q}_p(\zeta) \), which are denoted by \( K \) and \( K_0 \), respectively. Let \( \mathbb{Z}_q[\tilde{\pi}] \) and \( \mathbb{Z}_p[\tilde{\pi}] \) denote the respective rings of integers of \( K \) and \( K_0 \). Let \( \lambda \) be the Teichmüller representative of \( \lambda \) and \( F(\lambda, x) \) be the Teichmüller lifting of \( F(\lambda, x) \). Define

\[ C_0 := \left\{ \sum_{u \in \mathbb{Z}^n} c_u \tilde{\pi}^{e-u(u)} x^u : c_u \in \mathbb{Z}_q[\tilde{\pi}], c_u \rightarrow 0 \text{ as } u \rightarrow \infty \right\}. \]

Let \( \theta(t) := \exp \pi(t - t^p) \). If we write \( \theta(t) = \sum_{i=0}^{\infty} \lambda_i t^i \), it then follows from \([4]\) that

(2.3) \[ \text{ord}_p \lambda_i \geq \frac{p-1}{p^2} \cdot i \]

for every \( i \geq 0 \). Let
\[ F_0(x) = \theta(\lambda \prod_{i=1}^{n} x_i^{-d_i}) \prod_{i=1}^{n} \theta(x_i^{a_i}) \]
and
\[ F(x) = \prod_{j=0}^{\sigma-1} F_0^{\sigma^j}(x^{p^j}), \]
where \( \sigma \in \text{Gal}(K/K_0) \) is the Frobenius automorphism of \( \text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) \) extended to \( K \) by requiring \( \sigma(\tilde{\pi}) = \tilde{\pi} \) and \( \sigma(\zeta_p) = \zeta_p \).

We may define the Koszul complex \( \Omega^\bullet(C_0, \nabla(D)) \) by letting
\[ \Omega^i(C_0, \nabla(D)) := \bigoplus_{1 \leq j_1 < \ldots < j_i \leq n} C_0 \frac{dx_{j_1}}{x_{j_1}} \wedge \ldots \wedge \frac{dx_{j_i}}{x_{j_i}}, \]
with boundary map
\[ \nabla(D)(\frac{dx_{j_1}}{x_{j_1}} \wedge \ldots \wedge \frac{dx_{j_i}}{x_{j_i}}) = \left( \sum_{l=1}^n D_l(\zeta) \frac{dx_{j_l}}{x_{j_l}} \right) \wedge \frac{dx_{j_1}}{x_{j_1}} \wedge \ldots \wedge \frac{dx_{j_i}}{x_{j_i}}, \]
where
\[ D_l = x_l \frac{\partial}{\partial x_l} + x_l \frac{\pi \partial F(\lambda, x)}{\partial x_l}. \]

Set
\[ \alpha_0 := \sigma^{-1} \circ \psi \circ F_0 \]
and
\[ \alpha := \psi^0 \circ F, \]
where \( \psi \) is defined as
\[ \psi(\sum A_v x^v) = \sum A_{pv} x^v. \]
Then by estimate (2.3) and \( p > 2 \) we conclude that \( \alpha_0 \) is an \( \mathbb{Z}_q[\tilde{\pi}] \)-semilinear morphism acting on \( C_0 \), and \( \alpha \) maps \( C_0 \) to \( C_0 \) linearly over \( \mathbb{Z}_q[\tilde{\pi}] \). It follows from Serre [10] that the operators \( \alpha^i \) and \( \alpha_0^i \) acting on \( C_0 \) have well-defined traces.

By (4, equation(4.35)) the following communication law holds
\[ (2.5) \quad qD_l \circ \alpha = \alpha \circ D_l \]
for \( l = 1, \ldots, n \). Define an endomorphism \( \text{Frob}^i \) on \( \Omega^i(C_0, \nabla(D)) \) by
\[ (2.6) \quad \text{Frob}^i := \bigoplus_{1 \leq j_1 < \ldots < j_i \leq n} q^{n-i} \alpha_0 \frac{dx_{j_1}}{x_{j_1}} \wedge \ldots \wedge \frac{dx_{j_i}}{x_{j_i}}. \]
The commutation rule (2.5) ensures that (2.6) defines a chain map on the Koszul complex \( \Omega^\bullet(C_0, \nabla(D)) \).

Then by the Dwork’s trace formula, we have
\[ S_k(\bar{F}, \bar{\lambda}) = \sum_{i=0}^n (-1)^i \text{Tr}(H^i(\text{Frob})^k | H^i(C_0, \nabla(D))). \]

Using the same argument to Theorem 2.1 as [7], [6], or going back to [1] or [9], we have the following result.

**Proposition 2.3.** The cohomology of \( \Omega^\bullet(C_0, \nabla(D)) \) is acyclic except in top dimension \( n \), and \( H^n(\Omega^\bullet(C_0, \nabla(D))) \) is a free \( \mathbb{Z}_q[\tilde{\pi}] \)-module of rank equal to \( n! \text{Vol}(\Delta(\bar{F})) \). Furthermore,
\[ C_0 = \sum_{v \in B} \mathbb{Z}_q[\tilde{\pi}] \tilde{\pi}^{\nu(v)} x^v \oplus \sum_{l=1}^n D_l C_0, \]
where \( B \) is defined as Theorem 2.1.
It follows that
\[ S_k(\bar{F}, \lambda) = (-1)^n \text{Tr}(H^n(Frob)^k | H^n(C_0, \nabla(D))). \]
Hence
\[ L(\bar{F}(\lambda, x), T)^{(-1)^n-1} = \det(1 - FrobT | H^n(C_0, \nabla(D))). \]

It follows from (2.7) that the Newton polygon of \( L(\bar{F}(\lambda, x), T)^{(-1)^n-1} \) lies over the Newton polygon (using \( ord_q \)) of
\[ \prod_{w \in B} (1 - q^{w(v)} T). \]

In other words,
\[ NP(\bar{F}) \geq HP(\Delta(\bar{F})). \]

In particular, the polynomial \( \bar{F} \) is called ordinary if \( NP(\bar{F}) = HP(\Delta(\bar{F})) \).

3. LOWER BOUND FOR NEWTON POLYGON

3.1. Basis. For \( n = 1 \), we let \( \Delta_0 = \{a_1\} \) and \( \Delta_1 = \{-d_1\} \). For \( n \geq 2 \), we define the following notations. For each \( j = 1, \ldots, n \), by \( A_j \in \mathbb{Z}^n \) we denote the vector such that the \( j \)-th component is \( a_j \) and the other components are 0. Let \( \gamma = (-d_1, \ldots, -d_n) \). Then
\[ supp(\bar{F}) = \{\gamma\} \cup \{A_j\}_{j=1}^n. \]

Let \( \Delta_0 \) denote the convex closure generated by the lattice points \( A_j(1 \leq j \leq n) \). For \( i = 1, \ldots, n \), let \( \Delta_i \) be the convex closure generated by \( A_j(1 \leq j \neq i \leq n) \) and \( \gamma \). We define
\[ C(\Delta_0) := \{u \in \mathbb{Z}^n : u_i \geq 0 \text{ for } i = 1, \ldots, n\}. \]

For \( i = 1, \ldots, n \), we let
\[ C(\Delta_i) := \{u \in \mathbb{Z}^n : u_i \leq 0 \text{ and } u_j - \frac{d_j}{d_i} u_i \geq 0 \text{ for any } 1 \leq j \neq i \leq n\}. \]

From the definition, we can see that if there exists an integer \( i \) such that \( u_i = 0 \) or there is at least one pair of integers \( 1 \leq i \neq j \leq n \) such that \( u_j = \frac{d_j}{d_i} u_i \) and \( u_i < 0 \), then \( u \) is cofacial to all elements in \( Supp(\bar{F}) \) with respect to \( \Delta(\bar{F}) \).

Recall that for \( n = 1 \), \( B = \{v \in \mathbb{Z} : -d_1 < v \leq a_1\} \). For \( n \geq 2 \), the set \( B = \{(v_1, \ldots, v_n) \in \mathbb{Z}^n \} \) such that \( -d_i < v_i \leq a_i \) for all \( i = 1, \ldots, n \), and \((v_i, v_j)\) satisfies that
\[ \frac{d_j}{d_i} (v_i - a_i) \leq v_j < \frac{d_j}{d_i} v_i + a_j \]
for \( 1 \leq i < j \leq n \). And \( \bar{B} = \{x^v : v \in B\} \).

**Lemma 3.1.** Let \( v := (v_1, \ldots, v_n) \in \mathbb{Z}^n \) such that \( -d_i < v_i \leq a_i \) for \( i = 1, \ldots, n \). Then there exists \( u = (u_1, \ldots, u_n) \in B \) and \( c \in \mathbb{F}_q \) such that
\[ x^v \equiv cx^u \mod \sum_{l=1}^{n} \bar{F_l}. \]

**Proof.** Evidently, Lemma 3.1 is true for \( n = 1 \). In what follows, we let \( n \geq 2 \). If \( v \in B \), then Lemma 3.1 is true. Hence we let \( v \notin B \). Then there is at least one pair of integers \((i, j)\) such that (1.2) does not hold. Define the set \( M_v \) as follows:
\[ M_v := \{(i, j) : 1 \leq i < j \leq n \text{ such that } v_j \geq \frac{d_j}{d_i} v_i + a_j \text{ or } v_j < \frac{d_j}{d_i} (v_i - a_i)\}. \]
We claim that for each positive integer $k$, if $|M_v| = k$, then there are $c \in \mathbb{F}_q$ and $v' \in \mathbb{Z}^n$ such that

$$x^v \equiv cx^{v'} \mod \sum_{l=1}^n \bar{F}_l \bar{R},$$

$$-d_i < v'_i \leq a_i$$

for $i = 1, \ldots, n$ and $|M_v| \leq k$. In particular, if $|M_v| = 1$, then $v' \in \mathcal{B}$.

Suppose $M_v = \{(i_1, j_1), \ldots, (i_k, j_k)\}$. Let

$$M_1 = \{(i, j) \in M_v : v_j \geq \frac{d_j}{d_i} v_i + a_j\}$$

and

$$M_2 = \{(i, j) \in M_v : v_j < \frac{d_j}{d_i} (v_i - a_i)\}.$$

Let

$$J = \{j : (i, j) \in M_1 \text{ or } (j, i) \in M_2\}.$$

Let $j_0$ be the largest integer in $J$ such that

$$\frac{v_{j_0} - a_{j_0}}{d_{j_0}} = \max \left\{ \frac{v_j - a_j}{d_j} : j \in J \right\}.$$

Let

$$I = \{i : (i, j_0) \in M_1 \text{ or } (j_0, i) \in M_2\}$$

and $i_0$ be the least integer of $I$ such that

$$\frac{v_{i_0}}{d_{i_0}} = \min \left\{ \frac{v_i}{d_i} : i \in I \right\}.$$

Clearly, one has $(i_0, j_0) \in M_1$ or $(j_0, i_0) \in M_2$. Consider the following two cases.

**Case 1.** $(i_0, j_0) \in M_1$. That is, $v_{j_0} \geq \frac{d_{i_0}}{d_{j_0}} v_{i_0} + a_{j_0}$. Then $v_{i_0} \leq 0$. If $v - A_{j_0}$ is not cofacial to $A_{i_0}$, then

$$\bar{F}_{j_0} \cdot x^{v-A_{j_0}} = a_{j_0} x^v - d_{j_0} \bar{\lambda} x^{v-A_{j_0}-\gamma}$$

and

$$\bar{F}_{i_0} \cdot x^{v-A_{j_0}} = 0 - d_{i_0} \bar{\lambda} x^{v-A_{j_0}-\gamma}.$$

Hence

$$d_{i_0} a_{j_0} x^v \equiv 0 \mod \sum_{l=1}^n \bar{F}_l \bar{R}.$$

If $v - A_{j_0}$ is cofacial to $A_{i_0}$, then

$$\bar{F}_{j_0} \cdot x^{v-A_{j_0}} = a_{j_0} x^v - d_{j_0} \bar{\lambda} x^{v-A_{j_0}-\gamma}$$

and

$$\bar{F}_{i_0} \cdot x^{v-A_{j_0}} = a_{i_0} x^v - A_{j_0} + A_{i_0} - d_{i_0} \bar{\lambda} x^{v-A_{j_0}-\gamma}.$$

Hence

$$d_{i_0} a_{j_0} x^v \equiv d_{j_0} a_{i_0} x^{v-A_{j_0} + A_{i_0}} \mod \sum_{l=1}^n \bar{F}_l \bar{R}.$$

Let $v^{(1)} = v - A_{j_0} + A_{i_0}$. One has that

$$v^{(1)}_{j_0} = v_{j_0} - a_{j_0} \geq \frac{d_{j_0}}{d_{i_0}} (v_{i_0} + a_{i_0} - a_{i_0}) = \frac{d_{j_0}}{d_{i_0}} (v^{(1)}_{i_0} - a_{i_0}).$$
One can check that $-d_{j_0} < v_{(1)}^{(1)} = v_{j_0} - a_{j_0} \leq a_{j_0}$ and $-d_{i_0} < v_{(1)}^{(1)} = v_{i_0} + a_{i_0} \leq a_{i_0}$. Now we prove that if $(i, j_0) \notin M_v$, then $(i, j_0) \notin M_v^{(1)}$. Since $(i, j_0) \notin M_v$, one has that

$$d_{j_0}(v_{i_0} - a_{i_0}) \leq v_{j_0} < d_{j_0}(v_{i} + a_{j_0}).$$

Clearly, one has $v_{j_0} - a_{j_0} \leq d_{j_0}(v_{i_0} - a_{i_0})$. If $v_{j_0} - a_{j_0} < d_{j_0}(v_{i} - a_{i_0})$, then $v_{j_0} < d_{j_0}(v_{i} - a_{i_0})$. If $i > j_0$, then $(i_0, i) \in M_1$. If $i < j_0$, then $(i, i_0) \in M_2$. It follows from the definition of $j_0$ that $\frac{v_{i_0} - a_{i_0}}{d_{j_0}} \geq \frac{v_{i} - a_{i}}{d_{j_0}}$. But $\frac{v_{i_0} - a_{i_0}}{d_{j_0}} < \frac{v_{i} - a_{i}}{d_{j_0}}$, which is a contradiction. Hence

$$v_{j_0} - a_{j_0} \geq \frac{d_{j_0}}{d_{i}} (v_{i} - a_{i}).$$

That is, $(i, j_0) \notin M_v^{(1)}$. On the other hand, if $(j_0, i) \notin M_v$, then

$$d_{j_0}(v_{i} - a_{i}) \leq v_{j_0} < d_{j_0}(v_{i_0} + a_{i_0}).$$

Clearly, $d_{j_0}(v_{j_0} - 2a_{j_0}) \leq v_{j_0}$. If $v_{j_0} - 2a_{j_0} \leq d_{j_0}(v_{i_0} - a_{j_0})$, then $v_{j_0} - a_{j_0} \geq \frac{d_{j_0} v_{j_0}}{d_{j_0} v_{i_0}} v_{j_0}$. Note that $i > j_0 > i_0$, then $(i_0, i) \in M_1$. By the definition, we have $\frac{v_{i_0} - a_{i_0}}{d_{j_0}} < \frac{v_{i} - a_{i}}{d_{j_0}}$, which is a contradiction. Hence

$$v_{j_0} - a_{j_0} \geq \frac{d_{j_0}}{d_{i}} (v_{i} - a_{i}).$$

That is, $(j_0, i) \notin M_v^{(1)}$.

We now show that if $(i, i_0) \notin M_v$, then $(i, i_0) \notin M_v^{(1)}$. Since $(i, i_0) \notin M_v$, one has that

$$d_{i_0}(v_{i} - a_{i}) \leq v_{i_0} < d_{i_0}(v_{i_0} + a_{i}).$$

Clearly, $d_{i_0}(v_{i} - a_{i}) \leq v_{i_0} + a_{i_0}$. If $v_{i_0} + a_{i_0} \geq d_{i_0}(v_{i} - a_{i})$, then

$$v_{j_0} - a_{j_0} \geq \frac{d_{j_0}}{d_{i_0}} v_{i_0} \geq \frac{d_{j_0}}{d_{i}} v_{i}.$$
\[ d_{i_0}a_{j_0}x'' = d_{j_0}a_{i_0}x'' - A_{j_0} + A_{i_0} \mod \sum_{i=1}^{n} \bar{F_i} \bar{R}. \]

Let \( v^{(1)} = v - A_{j_0} + A_{i_0} \). One has that
\[ v^{(1)}_{j_0} - a_{j_0} = v_{i_0} + a_{i_0} - a_{j_0} < \frac{d_{i_0}}{d_{j_0}}(v_{j_0} - a_{j_0}) = \frac{d_{i_0}}{d_{j_0}} v^{(1)}_{j_0}. \]

By the similar argument as Case 1, we can prove that \( |M_{v^{(1)}}| \leq |M_v| \).

For \((i_0, j_0) \in M_1\), if \( v_{j_0} - a_{j_0} < \frac{d_{i_0}}{d_{j_0}}(v_{i_0} + a_{i_0}) + a_{j_0} \), then we let \( v' = v^{(1)} \) and \((i_0, j_0) \notin M_{v'}\). For \((j_0, i_0) \in M_2\), if \( v_{j_0} - a_{j_0} \leq \frac{d_{i_0}}{d_{j_0}}(v_{i_0} + a_{i_0}) + a_{j_0} \), then we let \( v' = v^{(1)} \) and \((j_0, i_0) \notin M_{v'}\). Hence \( |M_{v'}| < |M_v| \).

If \( v_{j_0} - a_{j_0} \geq \frac{d_{i_0}}{d_{j_0}}(v_{i_0} + a_{i_0}) + a_{j_0} \) for \((i_0, j_0) \in M_1\), or \( \frac{d_{i_0}}{d_{j_0}}(v_{j_0} - 2a_{j_0}) > v_{i_0} + a_{i_0} \) for \((j_0, i_0) \in M_2\), then we repeat the above process for \( v^{(1)} = v - A_{j_0} + A_{i_0} \). Suppose
\[ M_{v^{(1)}} = \{(i_1^{(1)}, j_1^{(1)}), \ldots, (i_{k-1}^{(1)}, j_{k-1}^{(1)})\} \]
with \( k^{(1)} \leq k \). Let
\[ M_1^{(1)} = \{(i, j) \in M_{v^{(1)}} : v^{(1)}_j \geq \frac{d_j}{d_i} v^{(1)}_i + a_j \} \]
and
\[ M_2^{(1)} = \{(i, j) \in M_{v^{(1)}} : v^{(1)}_j < \frac{d_j}{d_i} v^{(1)}_i + a_j \}. \]

Let
\[ J^{(1)} = \{j : (i, j) \in M_1^{(1)} \text{ or } (j, i) \in M_2^{(1)}\}. \]

Let \( j_0^{(1)} \) be the largest element in \( J^{(1)} \) such that
\[ \frac{v^{(1)}_{j_0} - a_{j_0}}{d_{j_0}} = \max \left\{ \frac{v^{(1)}_j - a_j}{d_j} : j \in J^{(1)} \right\}. \]

Let
\[ I^{(1)} = \{i : (i, j_0^{(1)}) \in M_1^{(1)} \text{ or } (j_0^{(1)}, i) \in M_2^{(1)}\} \]
and \( i_0^{(1)} \) be the least element of \( I^{(1)} \) such that
\[ \frac{v^{(1)}_{i_0}}{d_{i_0}} = \min \left\{ \frac{v^{(1)}_i}{d_i} : i \in I^{(1)} \right\}. \]

Clearly, one has \((i_0^{(1)}, j_0^{(1)}) \in M_1^{(1)} \) or \((j_0^{(1)}, i_0^{(1)}) \in M_2^{(1)}\). Note that \(-d_i < v_i \leq a_i \) for \( i = 1, \ldots, n \). We observe that
\[ \frac{v^{(1)}_{j_0} - a_{j_0}}{d_{j_0}} \leq \frac{v_{j_0} - a_{j_0}}{d_{j_0}} \]
and
\[ \frac{v^{(1)}_{i_0}}{d_{i_0}} \geq \frac{v_{i_0}}{d_{i_0}}. \]

Hence after finite steps one can obtain \( v' \in \mathbb{Z}^n \) such that \( x'' \equiv cx' \mod \sum_{i=1}^{n} \bar{F_i} \bar{R}, \)
\(-d_i < v'_i \leq a_i \) for \( i = 1, \ldots, n \) and \( |M_{v'}| < |M_v| \).
particularly, if \(|M_v| = 1\), then there are \(c \in \mathbb{F}_q\) and \(v' \in \mathbb{Z}^n\) such that \(x^v \equiv cx^{v'} \mod \sum_{l=1}^n \tilde{F}_l \bar{R}\), \(-d_i < v'_i \leq a_i\) for \(i = 1, \ldots, n\) and \(|M_{v'}| = 0\). That is, \(v' \in B\). This finishes the proof of the claim.

Since \(v \notin B\), there is at least one pair of integers \(i, j\) with \(1 \leq i < j \leq n\) satisfying \(v_j < \frac{d_j}{d_i}(v_i - a_i)\) or \(v_j \geq \frac{d_j}{d_i}v_i + a_j\). We prove Lemma 3.1 by induction on \(|M_v|\). For \(|M_v| = 1\), it follows from the claim that Lemma 3.1 is true when \(|M_v| = 1\).

Let \(m\) be a positive integer with \(m \geq 2\). Suppose that Lemma 3.1 holds for \(|M_v| \leq m-1\). We consider \(|M_v| = m\). It follows from the claim that there are \(v' \in \mathbb{F}_q\) and \(u' \in \mathbb{Z}^n\) such that

\[ x^v \equiv c'x^{v'} \mod \sum_{l=1}^n \tilde{F}_l \bar{R}, \]

\(-d_i < v'_i \leq a_i\) for \(i = 1, \ldots, n\) and \(|M_{v'}| \leq m - 1\). By the hypothesis, there are \(u = (u_1, \ldots, u_n) \in B\) and \(c \in \mathbb{F}_q\) such that

\[ x^v \equiv c'x^{v'} \equiv cc'x^u \mod \sum_{l=1}^n \tilde{F}_l \bar{R}. \]

Hence Lemma 3.1 holds for \(|M_v| = m\). This finishes the proof of Lemma 3.1. \(\square\)

**Lemma 3.2.** Let \(v \in \mathbb{Z}^n\). Then \(x^v\) is a linear combination of elements in \(\tilde{B}\) over \(\mathbb{F}_q\) modulo \(\sum_{l=1}^n \tilde{F}_l \bar{R}\).

**Proof.** First, we prove Lemma 3.2 is true for \(n = 1\). If \(v > a_1\), then \(v - a_1\) is not cofacial to \(-d_1\). Hence \(\tilde{F}_1(x^{v-a_1}) = a_1x^v\) and

\[ a_1x^v \equiv 0 \mod \tilde{F}_1 \bar{R}. \]

If \(v < -d_1\), then \(v + d_1\) is not cofacial to \(a_1\). Thus \(\tilde{F}_1(x^{v+d_1}) = -d_1x^v\). That is,

\[ d_1x^v \equiv 0 \mod \tilde{F}_1 \bar{R}. \]

If \(v = -d_1\), then \(v + d_1\) is cofacial to \(a_1\) and \(d_1\). It follows that \(\tilde{F}_1(x^{v+d_1}) = a_1x^{a_1} - d_1x^v\) and

\[ a_1x^{a_1} \equiv d_1x^v \mod \tilde{F}_1 \bar{R}. \]

Hence Lemma 3.2 is true when \(n = 1\).

In the following, we let \(n \geq 2\) and divide the proof into two cases.

**Case 1.** \(v \in C(\Delta_0)\). Let

\[ S_v := \{k : v_k = 0\}. \]

If \(v_i \leq a_i\) for all \(i\), it then follows from Lemma 3.1 that Lemma 3.2 is true. Hence in what follows, we assume that there is at least one integer \(j\) such that \(v_j > a_j\). If \(|S_v| = 0\), then \(v_k > 0\) for all \(i\). Let \(j\) be the integer such that \(v_j > a_j\). One has that \(v - A_j\) is not cofacial to \(\gamma\). Then \(F_j \cdot x^{v-A_j} = x^v\). Hence \(x^v \equiv 0 \mod \sum_{l=1}^n \tilde{F}_l \bar{R}\). Thus Lemma 3.2 is true if \(|S_v| = 0\).

If \(|S_v| = 1\), then there is one integer \(k\) such that \(v_k = 0\). Let \(j\) be the integer such that \(v_j > a_j\). Then \(v - A_j\) is cofacial to all elements in \(\text{supp}(\tilde{F})\). Hence

\[ \tilde{F}_j \cdot x^{v-A_j} = a_jx^v - d_j\bar{\lambda}_j x^{v-A_j} \gamma \]

and

\[ \tilde{F}_k \cdot x^{v-A_j} = a_kx^{v-A_j} + A_k - d_k\bar{\lambda}_j x^{v-A_j} \gamma. \]
Hence
\[ a_jd_kx^v \equiv a_kd_jx^{v-A_j+A_k} \mod \sum_{l=1}^{n} F_l \bar{R}. \]

We can see that \( v' = v - A_j + A_k \in C(\Delta_0) \) and \( |S_{v'}| = 0 \). Hence we conclude that Lemma 3.2 is true for \( |S_v| = 1 \). Then by induction on \( |S_v| \), we can prove Lemma 3.2 is true in this case.

Case 2. \( v \in C(\Delta_i) \) for \( i \in \{1, \ldots, n\} \). Then \( v \leq 0 \) and \( v \geq \frac{d_i}{d_j}v_i \) for \( 1 \leq j \neq i \leq n \). If \( v_i = 0 \), then \( v \geq 0 \) for \( j = 1, \ldots, n \). Hence \( v \in C(\Delta_0) \). Lemma 3.2 has been proved to be true when \( v_i = 0 \). In what follows, we let \( v_i < 0 \).

Assume that \( v_j > \frac{d_i}{d_j}v_i \) for \( 1 \leq j \neq i \leq n \). If there is an integer \( k \) such that \( k \neq i \) and \( v_k > a_k \), then \( v - A_k \in C(\Delta_i) \) and \( v - A_k \) is not cofacial to \( A_i \). Then
\[ F_k \cdot x^{v-A_k} = a_kx^v - d_k\lambda x^{v-A_k}, \]
and
\[ F_k \cdot x^{v-A_k} = -d_k\lambda(v - A_k - \gamma). \]

Hence
\[ x^v \equiv 0 \mod \sum_{l=1}^{n} F_l \bar{R}. \]

In what follows, we suppose \( v_j \leq a_j \) for all \( j = 1, \ldots, n \). If \( -d_i < v_i < 0 \), then \( v \geq \frac{d_j}{d_i}v_i \) for all \( j \neq i \). It follows from Lemma 3.2 that Lemma 3.2 is true for \( -d_i < v_i < 0 \). If \( v_i < -d_i \), then \( v_i + d_i < 0 \). It is easy to check that \( v_j + d_j > \frac{d_j}{d_i}(v_i + d_i) \). Hence \( (v_1 + d_1, \ldots, v_n + d_n) \in C(\Delta_i) \) and is not cofacial to \( A_i \). Then
\[ F_i \cdot x^{v+\gamma} = -d_i\lambda x^v. \]

Hence \( x^v \equiv 0 \mod \sum_{l=1}^{n} F_l \bar{R} \). Thus Lemma 3.2 is true for \( v_i < -d_i \). If \( v_i = -d_i \), then \( v_i + d_i = 0 \) and
\[ v_j + d_j > \frac{d_j}{d_i}(v_i + d_i) = 0 \]
for all \( j = 1, \ldots, n \). Hence \( v + \gamma \) is cofacial to all elements in \( \text{supp}(F) \). Then
\[ F_i \cdot x^{v+\gamma} = a_i x^{v+\gamma+A_i} - d_i\lambda x^v. \]

Evidently, \( v + \gamma + A_i \in C(\Delta_0) \) and
\[ a_i x^{v+\gamma+A_i} \equiv d_i\lambda x^v \mod \sum_{l=1}^{n} F_l \bar{R}. \]

Hence by Case 1 one has that Lemma 3.2 is true for \( v_i = -d_i \).

Assume that \( v_j = \frac{d_j}{d_i}v_i \) for some \( 1 \leq j \neq i \leq n \). If \( -d_i \leq v_i < 0 \), then \( v_k \geq -d_k \) for all \( 1 \leq k \neq i \leq n \). Then
\[ F_i \cdot x^{v+\gamma} = a_i x^{v+\gamma+A_i} - d_i\lambda x^v. \]

Clearly, \( v + \gamma + A_i \in C(\Delta_0) \). It follows from Case 1 that Lemma 3.2 is true.

Suppose \( v_i < -d_i \). Let \( J = \{j_1, \ldots, j_m\} \) such that \( v_j = \frac{d_j}{d_i}v_i \) for \( l = 1, \ldots, m \). Then
\[ v_{j_l} < -d_{j_l} \] for \( l = 1, \ldots, m \). Clearly, \( v_{j_l} + d_{j_l} = \frac{d_{j_l}}{d_i}(v_i + d_i) \) for \( l = 1, \ldots, m \). Then using \( F_{j_l} \) to act on \( x^{v+\gamma} \), we have
\[ F_{j_l} \cdot x^{v+\gamma} = a_{j_l}x^{v+\gamma+A_{j_l}} - d_{j_l}\lambda x^v. \]
Clearly, \( v_j + d_j \alpha + a_j > \frac{d_j}{d_i}(v_i + d_i) \). If \( v_i + d_i \geq -d_i \), then we have done. If \( v_i + d_i < -d_i \), then we continue use \( F_{j_0} \) to act on \( x^{v_i+2\gamma_i}A_{j_0} \). The process will stop after finite steps since \( J \) is a finite set. After at most \( m \) steps, we get

\[
v_j + md_{j_i} + a_j > \frac{d_j}{d_i}(v_i + md_i)
\]

for \( l = 1, \cdots, m \). Then by the above discussion that Lemma 3.2 holds for \( x^{v_i+m\gamma_i+\sum_{j=1}^m A_{j_0}} \). So does \( x^\gamma \). Hence Lemma 3.2 is true in this case.

This finishes the proof of Lemma 3.2 \( \square \)

To prove Theorem 1.2 is true, it remains to show that the number of lattices in \( \mathcal{B} \) is equal to \( n!Vd\Delta(F) \). For \( n \geq 2 \), we denote \( v = (v_1, \cdots, v_n) \), and define sets

\[
\mathcal{A} := \{ v \in \mathbb{Z}^n : (v_i, v_j) \text{ satisfies (\ref{eq:1.2}) for } 1 \leq i < j \leq n-1, -d_i < v_i \leq a_i \text{ for } 1 \leq i \leq n \}
\]

and

\[
\mathcal{A}_0 := \{ v \in \mathcal{A} : v \notin \mathcal{B} \}.
\]

That is, if \( v \in \mathcal{A}_0 \), then there is an integer \( 1 \leq i \leq n-1 \) such that \( v_n \geq \frac{d_n}{d_i}v_i + a_n \) or \( v_n < \frac{d_n}{d_i}(v_i - a_i) \). Let

\[
\mathcal{T} := \{ v \in \mathcal{A} : \exists 1 \leq i \leq n-1 \text{ such that } v_n \geq \frac{d_n}{d_i}v_i + a_n \}
\]

and

\[
\mathcal{S} := \{ v \in \mathcal{A} : \exists 1 \leq i \leq n-1 \text{ such that } \frac{d_n}{d_i}(v_i - a_i) > v_n \}.
\]

Hence \( \mathcal{A}_0 = \mathcal{T} \cup \mathcal{S} \). Suppose \( v \in \mathcal{S} \) and \( i \) is the integer such that \( v_n < \frac{d_n}{d_i}(v_i - a_i) \).

For any integer \( j \) such that \( 1 \leq i < j \leq n-1 \), one has \( \frac{d_j}{d_i}(v_i - a_i) \leq v_j < \frac{d_j}{d_i}v_i + a_j \), then by \( v_n < \frac{d_n}{d_j}(v_j - a_j) \), we conclude that \( v_n < \frac{d_n}{d_j}v_j + a_n \). Similarly, we can show for \( j < i \leq n-1 \), one has \( v_n < \frac{d_n}{d_j}v_j + a_n \). Hence \( v \notin \mathcal{T} \) and \( \mathcal{T} \cap \mathcal{S} = \emptyset \). Let

\[
\mathcal{M} = \{ v \in \mathcal{A} : a_n - d_n < v_n \leq a_n, \exists 1 \leq i \leq n-1 \text{ such that } -d_i < v_i \leq 0 \}
\]

and

\[
\mathcal{T} := \{ v \in \mathcal{A} : a_n - d_n < v_n \leq a_n, \exists 1 \leq i \leq n-1 \text{ such that } -d_i < v_i \leq 0 \}
\]

and \( v_n < \frac{d_n}{d_j}v_j + a_n \) for all \( j = 1, \cdots, n-1 \).

Hence \( \mathcal{T} \subset \mathcal{M} \). If \( v \in \mathcal{T} \), then there is an integer \( i \) such that \( v_n \geq \frac{d_n}{d_i}v_i + a_n \). One can check that \(-d_i < v_i \leq 0\) and \( a_n - d_n < v_n \leq a_n \). Then \( \mathcal{T} \subset \mathcal{M} \) and \( \mathcal{M} = \mathcal{T} \cup \mathcal{T} \).

The number of lattice points in \( \mathcal{B} \) can be obtained by counting points in \( \mathcal{A} \) and \( \mathcal{A}_0 \). Specifically, \( |\mathcal{B}| = |\mathcal{A}| - |\mathcal{A}_0| \). In the following, we first show that \( |\mathcal{A}_0| = |\mathcal{M}| \) by building a bijection between \( \mathcal{T} \) and \( \mathcal{S} \), then get \( |\mathcal{M}| \).

For \( v \in \mathcal{T} \), let \( \mathcal{I}_v := \{ i : -d_i < v_i \leq 0 \} \). Let \( i_0 \) be the least element of \( \mathcal{I}_v \) such that

\[
\frac{v_{i_0}}{d_{i_0}} = \min \left\{ \frac{v_i}{d_i} : i \in \mathcal{I}_v \right\}.
\]

Define \( \psi : \mathcal{T} \rightarrow \mathcal{S} \) by

\[
\psi(v) := v + A_{i_0} - A_n.
\]
For \( u \in S \), let \( J_u := \{ j : \frac{d_n}{d_j} (u_j - a_j) > u_n \} \). Let \( j_0 \) be the largest element of \( J_u \) such that
\[
\frac{u_{j_0} - a_{j_0}}{d_{j_0}} = \max \left\{ \frac{u_j - a_j}{d_j} : j \in J_u \right\}.
\]
Define \( \varphi : S \rightarrow T \) by
\[
\varphi(u) = u - A_{j_0} + A_n.
\]
Then we have

**Lemma 3.3.** \( \psi(T) \subseteq S \) and \( \varphi(S) \subseteq T \).

**Proof.** Let \( v \in T \). Let \( u = (u_1, \ldots, u_n) = \psi(v) \). Clearly,
\[
u_n = v_n - a_n < \frac{d_n}{d_{i_0}} (v_{i_0} + a_{i_0} - a_{i_0}) = \frac{d_n}{d_{i_0}} (u_{i_0} - a_{i_0}),
\]
so to show \( \psi(T) \subseteq S \), it remains to show that \( u \in A \). Obviously, \( -d_n < u_n \leq a_n \) and \( -d_{i_0} < u_{i_0} \leq a_{i_0} \). For \( i < n \), if \( i_0 \neq i \), then \( -d_i < u_i = v_i \leq a_i \). We also have \( (1.2) \) is true for \( i_0 \neq i, j < n \).

For \( 1 \leq i < i_0 \), since \( v \in T \), one has
\[
\frac{d_{i_0}}{d_i} (v_i - a_i) < v_{i_0} - a_{i_0},
\]
and immediately, \( \frac{d_{i_0}}{d_i} (v_i - a_i) \leq v_{i_0} + a_{i_0} \). If \( i \in J_v \), then \( \frac{v_{i_0}}{d_{i_0}} < \frac{v_i}{d_i} \) by the definition of \( i_0 \). So \( v_{i_0} + a_{i_0} < \frac{d_{i_0}}{d_i} v_i + a_{i_0} \). If \( i \notin I_v \), then from \( v_i > 0 \geq v_{i_0} \) one also has \( v_{i_0} + a_{i_0} < \frac{d_{i_0}}{d_i} v_i + a_{i_0} \). Thus we obtain
\[
\frac{d_{i_0}}{d_i} (v_i - a_i) < u_{i_0} < \frac{d_i}{d_{i_0}} u_i + a_{i_0}
\]
for \( 1 \leq i < i_0 \). Similarly, we can prove
\[
\frac{d_i}{d_{i_0}} (u_{i_0} - a_{i_0}) \leq u_i < \frac{d_i}{d_{i_0}} u_{i_0} + a_i
\]
for \( i_0 < i \leq n - 1 \). Hence \( u \in A \) and \( \psi(T) \subseteq S \).

We now show that \( \varphi(S) \subseteq T \). Let \( u' \in S \). Denote by \( u' := \varphi(u') \) and write \( u' := (u'_1, \ldots, u'_n) \). Clearly,
\[
u'_n = u'_n + a_n < \frac{d_n}{d_{j_0}} (u'_{j_0} - a_{j_0}) + a_n = \frac{d_n}{d_{j_0}} u'_{j_0} + a_n.
\]
Note that \( -d_n < u'_n < 0 \). Hence \( -d_{j_0} < u'_{j_0} \leq 0 \). For \( 1 \leq i < j_0 \), one has
\[
\frac{d_{j_0}}{d_i} (u'_i - a_i) \leq u'_{j_0} < \frac{d_i}{d_{j_0}} u'_i + a_{j_0}.
\]
It then follows from (3.1) that
\[
u'_n = u'_n + a_n < \frac{d_n}{d_i} u'_i + a_n = \frac{d_n}{d_i} v'_i + a_n
\]
for \( 1 \leq i < j_0 \). For \( j_0 < i \leq n - 1 \), one has that
\[
\frac{d_i}{d_{j_0}} (u'_{j_0} - a_{j_0}) \leq u'_i < \frac{d_i}{d_{j_0}} u'_{j_0} + a_i.
\]
From (3.1), one has
\[
u'_n = u'_n + a_n < \frac{d_n}{d_i} u'_i + a_n = \frac{d_n}{d_i} v'_i + a_n
\]
for $j_0 < i < n - 1$. Hence to show $\varphi(S) \subseteq T$, it remains to show that $v' \in \mathcal{A}$. Obviously, $-d_i < v'_i \leq a_i$ for $i = 1, ..., n$. If $j_0 \neq i, j < n$, then $v'_i = u'_i, v'_j = u'_j$. Hence (1.2) is true for $v'_i, v'_j$.

For $1 \leq i < j_0$, one has

$$d_{j_0}(u'_i - a_i) \leq u'_j < \frac{d_{j_0}}{d_i} u'_i + a_{j_0}.$$ 

Immediately, $u'_{j_0} - a_{j_0} < \frac{d_{j_0}}{d_i} u'_i + a_{j_0}$. If $i \in J_{u'}$, then $\frac{u'_{j_0} - a_{j_0}}{d_{j_0}} \geq \frac{u'_i - a_i}{d_i}$ by the definition of $j_0$. So $u'_{j_0} - a_{j_0} \geq \frac{d_{j_0}}{d_i} (u'_i - a_i)$. If $i \notin J_{u'}$, it then follows from $u'_{j_0} \geq \frac{d_{j_0}}{d_i} (u'_i - a_i)$ that $u'_{j_0} - a_{j_0} \geq \frac{d_{j_0}}{d_i} (u'_i - a_i)$. Thus

$$\frac{d_{j_0}}{d_i} (u'_i - a_i) \leq u'_{j_0} - a_{j_0} < \frac{d_{j_0}}{d_i} u'_i + a_{j_0}$$

for $1 \leq i < j_0$. By some similar arguments we can prove

$$\frac{d_i}{d_{j_0}} (u'_{j_0} - 2a_{j_0}) \leq u'_i < \frac{d_i}{d_{j_0}} (u'_{j_0} - a_{j_0}) + a_i$$

for $j_0 < i < n - 1$. That is, $v' \in \mathcal{A}$. Hence $\varphi(S) \subseteq T$.

This finishes the proof of Lemma 3.4. □

Lemma 3.4. $\varphi \circ \psi = Id|_T$ and $\psi \circ \varphi = Id|_S$.

Proof. First, we show $\varphi \circ \psi = Id|_T$. Let $v \in T$ and $i_0 \in \mathcal{I}_v$ be the least integer such that $\frac{u_{i_0}}{d_{i_0}} = \min \{ \frac{u_i}{d_i} : i \in \mathcal{I}_v \}$. Then $\psi(v) = u := v + A_{i_0} - A_n \in S$. For $i < i_0$, one has

$$\frac{d_{i_0}}{d_i} (v_i - a_i) \leq u_{i_0} < \frac{d_{i_0}}{d_i} (v_i + a_i).$$

Hence

$$v_{i_0} + a_{i_0} - a_{i_0} \geq \frac{d_{i_0}}{d_i} (v_i - a_i)$$

for $i < i_0$. For $i > i_0$, one has

$$\frac{d_i}{d_{i_0}} (v_{i_0} - a_{i_0}) \leq v_i < \frac{d_i}{d_{i_0}} (v_i + a_i).$$

Hence

$$v_{i_0} + a_{i_0} - a_{i_0} > \frac{d_{i_0}}{d_i} (v_i - a_i)$$

for $i > i_0$. We conclude that $i_0$ is the largest element of $\mathcal{J}_u$ such that

$$\frac{u_{i_0} - a_{i_0}}{d_{i_0}} = \max \{ \frac{u_j - a_j}{d_j} : j \in \mathcal{J}_u \}.$$ 

It follows that

$$\varphi \circ \psi(v) = \varphi(v + A_{i_0} - A_n) = v + A_{i_0} - A_n - A_{i_0} + A_n = v.$$ 

Thus $\varphi \circ \psi = Id|_T$.

By the same argument, we can show that if $u \in S$, then $\psi \circ \varphi(u) = u$. That is, $\psi \circ \varphi = Id|_S$. This finishes the proof of Lemma 3.4. □

It follows that $|\mathcal{A}_0| = |\mathcal{M}|$, from which we have the following.

Lemma 3.5. For $n \geq 1$, $|\mathcal{B}| = n!Vol\Delta(F)$.  


obtained by computing the weights of elements in set $B$. Let $u \in \mathcal{B}$.
Then by Proposition 2.3 one has that Theorem 1.2 is true.

In what follows, we compute $|\mathcal{B}_n|$ by induction. Let $n = 1$. It is evidently that $|\mathcal{B}_1| = a_1 + d_1$. Let $n = 2$. Then we count that

$$|\mathcal{B}_2| = a_1a_2 + a_1d_2 + a_2d_1.$$ 

Consider $n \geq 3$. Assume that

$$|\mathcal{B}_{n-1}| = \prod_{i=1}^{n-1} a_i + \sum_{j=1}^{n-1} d_j \prod_{i=1, i \neq j}^{n-1} a_i.$$ 

The number of lattices in $\mathcal{B}_{n-1}$ satisfying $v_i > 0$ for all $i = 1, \ldots, n-1$ is $\prod_{i=1}^{n-1} a_i$. Hence the number of lattices in $\mathcal{B}_{n-1}$ such that $-d_i < v_i \leq 0$ for some $1 \leq i \leq n-1$ is $\sum_{j=1}^{n-1} d_j \prod_{i=1, i \neq j}^{n-1} a_i$. Hence

$$|\mathcal{M}| = d_n \sum_{j=1}^{n-1} d_j \prod_{i=1, i \neq j}^{n-1} a_i.$$ 

Clearly,

$$|\mathcal{A}| = (\prod_{i=1}^{n-1} a_i + \sum_{j=1}^{n-1} d_j \prod_{i=1, i \neq j}^{n-1} a_i)(a_n + d_n).$$ 

Note that $\mathcal{B}_n = \mathcal{A} - \mathcal{A}_0$. It then follows from $|\mathcal{M}| = |\mathcal{A}|$ that

$$|\mathcal{B}_n| = \left(\prod_{i=1}^{n-1} a_i + \sum_{j=1}^{n-1} d_j \prod_{i=1, i \neq j}^{n-1} a_i\right)(a_n + d_n) - |\mathcal{M}| = \prod_{i=1}^{n} a_i + \sum_{j=1}^{n} d_j \prod_{i=1, i \neq j}^{n} a_i.$$ 

Clearly, $|\mathcal{B}| = |\mathcal{B}_1| = a_1 + d_1 = \Delta(\bar{F})$ for $n = 1$. For $n \geq 2$, we compute

$$n!Vol\Delta(\bar{F}) = \prod_{i=1}^{n} a_i + \sum_{j=1}^{n} d_j \prod_{i=1, i \neq j}^{n} a_i = |\mathcal{B}_n| = |\mathcal{B}|.$$ 

This finishes the proof of Lemma 3.5.

Now we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. It follows from Lemma 3.2 and Lemma 3.5 that $\bar{B}$ is a basis of

$$H^n(\Omega^*(\bar{R}, \nabla(\bar{F}))) = \bar{R}/ \sum_{i=1}^{n} x_i \partial \bar{F}/ \partial x_i \bar{R}.$$ 

Then by Proposition 2.3 one has that Theorem 1.2 is true.

Remark: It follows from Theorem 1.2 that the Hodge polygon $HP(\Delta(\bar{F}))$ can be obtained by computing the weights of elements in set $\mathcal{B}$. The following is how to compute the Hodge polygon. Let $u \in \mathcal{B}$. For $n = 1$, then $w(u) = \frac{u_1}{a_1}$ if $0 \leq u \leq a_1$, and $w(u) = \frac{u}{a_1}$ if $-d_1 < u < 0$. For $n \geq 2$, if $u \in C(\Delta_0)$, then let

$$w(u) = \sum_{i=1}^{n} \frac{u_i}{a_i}.$$
If \( u \in C(\Delta_k) \), then we let
\[
w(u) = \frac{u_k}{1 + \frac{d_0}{a_1} + \cdots + \frac{d_{n-2}}{a_n}} + \sum_{i=1}^{n} \frac{u_i}{a_i} - \frac{u_k}{a_k}.
\]
Hence \( e = \text{lcm}(a_1, d_1) \) for \( n = 1 \), and \( e = \text{lcm}(a_1, \ldots, a_n) \cdot \text{lcm}(d_1, \ldots, d_n) \) for \( n \geq 2 \). Then \( e = e^* \). Denote \( H_\Delta(k) = |\{ u \in \mathcal{B} : w(u) = \frac{k}{e} \}| \). The Hodge polygon is the lower convex polygon in \( \mathbb{R}^2 \) with vertices \((0, 0)\) and
\[
\left( \sum_{k=0}^{m} H_\Delta(k), \sum_{k=0}^{m} \frac{k}{e} H_\Delta(k) \right), \ m = 0, 1, \ldots, ne.
\]
Thus Corollary 1.3 follows.

3.2. Newton polygon.

**Theorem 3.6.** (Facial decomposition theorem, [14]) Let \( f \) be a nondegenerate Laurent polynomial with \( n \) variables over \( \mathbb{F}_q \). Assume \( \Delta = \Delta(f) \) is \( n \)-dimensional and \( \Delta_1, \ldots, \Delta_h \) are all the codimension \( 1 \) faces of \( \Delta \) which do not contain the origin. Let \( f^{\Delta_i} \) denote the restriction of \( f \) to \( \Delta_i \). Then \( f \) is ordinary if and only if \( f^{\Delta_i} \) is ordinary for \( 1 \leq i \leq h \).

In what follows, we introduce some criteria to determine the nondegenerate and ordinary property.

A Laurent polynomial \( f \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm] \) is called diagonal if \( f \) has exactly \( n \) nonconstant terms and \( \Delta(f) \) is \( n \)-dimensional. Let \( f(x) = \sum_{j=1}^{n} a_j x^j \), with \( a_j \in \mathbb{F}_q^* \). The square matrix of \( \Delta \) is defined to be
\[
M(\Delta) = (V_1, \ldots, V_n),
\]
where each \( V_j \) is written as a column vector. If \( f \) is diagonal, then \( \det M(\Delta) \neq 0 \).

Suppose \( f \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm] \) is diagonal with \( \Delta(f) \). It is well-known that \( f \) is nondegenerate if and only if \( \gcd(p, \det M(\Delta)) = 1 \).

Let \( S(\Delta) \) be the solution set of the following linear system
\[
M(\Delta) \cdot (r_1, r_2, \ldots, r_n)^t \equiv 0 \pmod{1}, \ r_i \in \mathbb{Q} \cap [0, 1),
\]
where \( (r_1, r_2, \ldots, r_n)^t \) means transposition of \( (r_1, r_2, \ldots, r_n) \). Then \( S(\Delta) \) is an abelian group and its order is given by \( |\det M(\Delta)| \). By the fundamental structure of finite abelian group, \( S(\Delta) \) can be decomposed into a direct product of invariant factors
\[
S(\Delta) = \bigoplus_{i=1}^{n} \mathbb{Z}/s_i(\Delta)\mathbb{Z},
\]
where \( s_i(\Delta)|s_{i+1}(\Delta) \) for \( i = 1, 2, \ldots, n - 1 \). Then Wan proved the following ordinary criterion.

**Proposition 3.7.** [15] Suppose \( f \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm] \) is a nondegenerate diagonal Laurent polynomial with \( \Delta(f) \). Let \( s_n(\Delta) \) be the largest invariant factor of \( S(\Delta) \). If \( p \equiv 1 \mod s_n(\Delta) \), then \( f \) is ordinary.

Now we can use Wan’s results to prove Theorem 1.4.

**Proof of Theorem 1.4.** First we consider \( n = 1 \). Then \( F^{\Delta_0} = x_1^{a_1} \) and \( F^{\Delta_1} = x_1^{-d_1} \). Note that \( p \nmid a_1d_1 \). Hence \( F^{\Delta_0} \) and \( F^{\Delta_1} \) are nondegenerate. Clearly, \( s_1(\Delta_0) = a_1 \) and \( s_1(\Delta_1) = d_1 \). From Theorem 3.6 and Proposition 3.7, one has that if \( p \equiv 1 \mod \text{lcm}(a_1, d_1) \), then \( F \) is ordinary.
In what follows, we let $n \geq 2$. It is easy to see that $\Delta_0, \Delta_1, \ldots, \Delta_n$ are all codimension 1 faces of $\Delta(\hat{F})$ which do not contain the origin. Then let $\hat{F}^\Delta = \sum_{i=1, i \neq j}^n x_i^{a_i} + \sum_{i=1}^\lambda \prod_{c=1}^l x_i^{b_i}$ for $j = 1, \ldots, n$ and $\hat{F}^{\Delta_0} = \sum_{i=1}^n x_i^{a_i}$. We compute $|\det M(\Delta_j)| = d_j \prod_{i=1, i \neq j}^n a_i$ and $|\det M(\Delta_0)| = \prod_{i=1}^n a_i$. Note that $p \mid \prod_{i=1}^n a_i d_i$. Hence $\hat{F}^{\Delta_0}, \ldots, \hat{F}^{\Delta_n}$ are nondegenerate.

We compute $s_n(\Delta_0) = \text{lcm}(a_1, \ldots, a_n)$ and $\text{lcm}(a_1, \ldots, a_j, \ldots, a_n) d_j$ is divisible by $s_n(\Delta_j)$ for $j = 1, \ldots, n$, where $\hat{a}_j$ means that $a_j$ is omitted. It follows from Proposition 3.7 that if

$$p \equiv 1 \mod \text{lcm}(a_1, \ldots, a_n),$$

for $j = 1, \ldots, n$, and $p \equiv 1 \mod \text{lcm}(a_1, \ldots, a_n)$, then

$$NP(\hat{F}^{\Delta_j}) = HP(\Delta_j)$$

for $j = 0, 1, \ldots, n$. It then follows from Theorem 3.6 that if

$$p \equiv 1 \mod \text{lcm}(a_1, \ldots, a_n) \cdot \text{lcm}(d_1, \ldots, d_n),$$

then

$$NP(\hat{F}) = HP(\Delta(\hat{F})).$$

This finishes the proof of Theorem 1.4. \hfill \□

Hence under the condition of Theorem 1.4, the Newton polygon of the $L$-function $L(\hat{F}(\lambda, x), T)^{(-1)^{n-1}}$ can be computed by the Hodge polygon given by Corollary 1.5. Finally, we prove Theorem 1.6.

Proof of Theorem 1.6. It follows from Theorem 1.4 that $NP(\hat{F}) = HP(\Delta(\hat{F}))$. Note that $d_i = 1$ for all $i \in \{1, \ldots, n\}$. Let $u \in B$. For $n = 1$, then $0 \leq u \leq a_1$ and $w(u) = \frac{u}{a_1}$. Hence the slope sequence of $L(\hat{F}(\lambda, x), T)$ is the set

$$\left\{ \frac{u}{a_1} : u = 0, \ldots, a_1 \right\}.$$  

For $n \geq 2$, Theorem 1.2 tells us that the basis of $H^n(C_0, \nabla(D))$ is

$$B = \left\{ \prod_{i=1}^n x_i^{u_i} \right\}_{0 \leq u_i \leq a_i} - \left\{ \prod_{i=1}^n x_i^{v_i} \right\}_{(v_i, v_j) = (0, a_j) \text{ for } 0 \leq i < j \leq n}.$$  

Then $w(u) = \sum_{i=1}^n \frac{u_i}{a_i}$. Suppose there exists $v \in B$ such that $v \neq u$ and $w(v) = w(u)$, that is,

$$\sum_{i=1}^n \frac{u_i}{a_i} = \sum_{i=1}^n \frac{v_i}{a_i}.$$  

Note that $\gcd(a_i, a_j) = 1$ for any $1 \leq i \neq j \leq n$. Then we have $a_i \mid (u_i - v_i)$ for all $i = 1, \ldots, n$. Note that $0 \leq u_i, v_i \leq a_i$ for $i = 1, \ldots, n$. It follows that if $v_i \neq 0, a_i$, then $u_i = v_i$. Hence $v_i = 0, a_i$ for some $i = 1, \ldots, n$. Suppose that $i_1, \ldots, i_k$ are the integers such that $v_{i_l} = 0, a_{i_l}$ for $l = 1, \ldots, k$. Let $v' = (v_{i_1}, \ldots, v_{i_k})$. By the fact that $(v_i, v_j) \neq (0, a_j)$ for $1 \leq i < j \leq n$, one has that $v'$ should be

$$(0, \ldots, 0), (a_{i_1}, 0, \ldots, 0), \ldots, (a_{i_k}, \ldots, a_{i_k}),$$

which have different weights. Hence if $w(u) = w(v)$, then $u = v$. Thus each element in $B$ has different weight. It follows that the slope sequence of $L(\hat{F}(\lambda, x), T)^{(-1)^{n-1}}$ equals to

$$\left\{ \sum_{i=1}^n \frac{u_i}{a_i} : u_i \in \{0, \ldots, a_i\} \right\}$$

as set. This finishes the proof of Theorem 1.6. \hfill \□
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