Spin effects in the quantum many-particles systems
Mariya Iv. Trukhanova[a] and Kuz’menkov L. S. [b]
(Dated: 02.01.2014)

In this paper, we explain a magneto quantum hydrodynamics (MQHD) method for the study of the quantum evolution of a system of spinning particles in an external electromagnetic field. The fundamental equations of microscopic quantum hydrodynamics (the momentum balance equation, the energy evolution equation, the vorticity equation and the magnetic moment density equation) were derived from the many-particle microscopic Schrödinger equation with a Spin-spin and Coulomb modified Hamiltonian. It has been showed that in the absence of external electromagnetic field the system of particles are subject to the new spin-dependent addition (Spin stress) and the magnetic moments are subject to the new Spin Torque. The new spin-dependent Spin Stress and Spin Torque which have been derived taken into account thermal fluctuations of the spins about the macroscopic average. Spin-dependent additions appears even in absence of the external magnetic field. We begin by describing the whistler wave model based on two dimensional electron magnetohydrodynamic equations. The spin effects were taken into account. The spin dynamics and spin torque in quantum plasmas is shown to cause formation of a new type of waves as well as changes in the dispersion characteristics of whistler waves. We also analyzed the dispersion of the collective excitations in the three dimensional samples of the magnetized dielectrics. We show that dynamics of the spin-current leads to formation of new type of stable spin waves. The explicated MQHD approach was used to the nonlinear dynamics of whistler waves. We obtained more general nonlinear dynamical equations for the amplitudes of whistler waves with the quantum corrections due to the Bohm potential and the magnetization energy of electrons due to its spin. The collective Spin stress and Spin torque associated with electron - 1/2 spin effect were taken into account.

PACS numbers: 52.35.We, 67.10.-j
Keywords: Quantum hydrodynamics, spin-spin Interactions, vorticity, spin quantum plasmas

I. INTRODUCTION

The spinning quantum fluid plasma is becoming of increasing current interest. The quantum and magnetization effects in plasma can be represented by two main corrections. The first is a quantum force, the multiparticle quantum Bohm or Madelung potential, proportional to powers of h and produced by density fluctuations. The second is associated with the quantum particle angular momentum spin by the possible inhomogeneity of the external and internal magnetic fields. In the momentum balance equation this force appears through the magnetization energy.

The Hall-magnetohydrodynamic (H-MHD) of electron plasma and (H-MHD) fluctuations in a magnetized plasma have been investigated in and . The high-frequency kinetic-Alfven regime of the plasma and electromagnetic fluctuations have been studied.

Hydrodynamics equations of a spinning fluid for the Pauli equation with the quantum particle angular momentum spin was presented since the pioneering works by Takabayashi and Vigier, . The geometrohydrodynamical formalism and the vector representation of non-relativistic spinning particle lead to appearance of new effects. This effects had been separated as nonlinear terms which arises from the inhomogeneity of spin distribution. The extension of the interpretation to developed approach had been carried in the . In the previous works Kramers have provided the geometrical representation of a spinor. As a result the spin effects in the quantum hydrodynamics of non-relativistic spinning particles have provided the appearance of the nonlinear dynamical terms in the equations of motion. These terms have been interpreted as the "internal potential" and "internal magnetic field".

Recently the method of quantum hydrodynamics (QHD) for the study of the quantum spinning plasmas had been investigated. Fundamental magnetohydrodynamic MQHD equations for charged and neutral particles were derived from the many-particle microscopic Schrodinger equation. Fundamental MQHD equations include the spin force, the multiparticle spin stress and the collective spin torque. The most interesting and defining features of a quantum spinning plasmas can be derived from the vorticity equation.

A quantum mechanics description for systems of interacting particles is based upon the many-particle Schrodinger equation (MPSE) that specifies a wave function in a 3N-dimensional configuration space. As wave processes, processes of information transfer, and other spin transport processes occur in 3D physical space, it becomes necessary to turn to a mathematical method of physically observable values that are determined in a 3D physical space and . The dispersion of waves, existed in the plasma in consequence of dynamic of the...
magnetic moments had been studied in\cite{27}. The instabilities at propagation of the neutron beam through the plasma had been showed. The dispersion of the collective excitations in the three dimensional samples of the magnetized dielectrics had been investigated\cite{28}. The dynamics of the spin-current must lead to formation of new type of the collective excitations in the magnetized dielectrics, which we called spin-current waves.

It had been derived by\cite{29} that the vorticity, constructed from spin field of a quantum spinning plasma, combines with the classical generalized vorticity to yield a new grand generalized vorticity that obeys the standard vortex dynamics. Astrophysics is also a rapidly growing field of research. It is important that the consequences of turbulent plasma movement in the solar photosphere lead to the generation of vorticity, while magnetic vortices are produced by magnetic tension. For example, magnetohydrodynamics (MHD) simulations of magnetoconvection have been used to analyze the generation of small-scale vortex motions in the solar photosphere. Using the vorticity equation, combined with G-band radiative diagnostics, it has been shown that two different types of photospheric vorticity, magnetic and non-magnetic, are generated in the domain\cite{30}. The presence of vortex motions for the astrophysics had been developed in\cite{29} \cite{31}. We propose a method of quantum hydrodynamics that allows one to obtain a description of the collective effects in magnetized quantum plasmas in terms of functions in physical space. The new formalism given in this references is used in this article for studying of vorticity and energy evolution in the magnetized plasma.

In this article for studying of energy evolution and spin vortex effects we generalize and use the method of the many-particle quantum hydrodynamics MQHD approach. We derive the fundamental balance equation, the fundamental energy evolution equation, the magnetic moment evolution equation and new vorticity dynamics equation and the magnetic vortex evolution equation for the magnetized quantum plasmas. The fact that the thermal spin fluctuations were taken into account.

Our paper is organized as follows. In Sect. \[\text{II}\] we present the derivation of the momentum balance equation, the energy evolution equation, we obtain equations of spin density evolution and vorticity dynamics for spin-having charged particles from MPSE in a self-consistent field approximation. An explicit form of the quantum part of the pressure tensor, spin stress tensor and spin part of magnetic flux density tensor are also presented. The spin-thermal effects were taken into account. In Sect. \[\text{III}\] we investigate the whistler wave model based on two dimensional electron magnetohydrodynamic equations. Spin properties of whistler waves are described. New dispersion branch is shown to exist and the contribution of spin torque and magnetization current into the dispersion are estimated. In Sect. \[\text{IV}\] we investigate the equation of spin-current and analyze the spin waves produced by spin-current. We show the existence of stable and not damped spin wave along the external magnetic field. In Sect. \[\text{V}\] the nonlinear equation for vector potential of whistler waves in magnetized spin plasmas was derived.

## II. GENERAL THEORY

In this section we derive the system of magnet quantum hydrodynamics (MQHD) equations for charged and neutral particles from the many-particle microscopic Schrodinger equation

\[i\hbar \frac{\partial \psi_s(R,t)}{\partial t} = (\hat{H}_s)(R,t), \tag{1}\]

where \(R = (\vec{r}_1, ..., \vec{r}_N)\). We consider a system of \(N\) interacting fermions with equal masses \(m_j\), charged and proper magnetic moments in an external electromagnetic field. A state of the system of \(N\) particles is determined by a wave function in the \(3N\)-dimensional configuration space, which is a rank – \(N\) spinor.

\[\psi_s(R,t) = \psi_{s_{1,s_2,....s_N}}(\vec{r}_1, ..., \vec{r}_N, t). \tag{2}\]

The Hamiltonian has the form

\[\hat{H} = \sum_{j=1}^{N} \left( \frac{\hbar^2}{2m_j} + q_j \varphi_{j,ext} - \mu_j \hat{\sigma}^α_j B^α_{j,ext} \right) + \frac{1}{2} \sum_{j\neq k}^{N} q_{jk} G_{jk} - \frac{1}{2} \sum_{j\neq k,k}^{N} \mu_j^2 F^{αβ}_{jk} \hat{σ}^α_j \hat{σ}^β_k, \tag{3}\]

where \(\mu_j = g\mu_B/2\), \(\mu_{jB}\) - is the electron or positron magnetic moment and \(\mu_{jB} = q_j \hbar/2m_j c\) - is the Bohr magneton, \(q_j\) stands for the charge of electrons \(q_e = -e\) or for the charge of positrons \(q_p = e\), and \(\hbar\) - is the Planck constant, \(g \approx 2.0023193\). The covariant derivative operator is

\[\hat{D}^α_j = -i\hbar \hat{\nabla}^α_j - \frac{q_j}{c} A^α_j, \tag{4}\]

where \(\hat{A}_{ext}, \varphi_{j,ext}\) - are the vector and scalar potentials of external electromagnetic field.

Green’s functions of the Coulomb and Spin – Spin interaction are

\[G_{jk} = \frac{1}{r_{jk}}, \quad F^{αβ}_{jk} = 4\pi \delta_{α,β} \delta(\vec{r}_{jk}) + \partial^α_j \partial^β_k \frac{1}{r_{jk}}. \tag{5}\]

The first step in the construction of MQHD apparatus is to determine the concentration of particles in the neighborhood of \(\vec{r}\) in a physical space. If we define the concentration of particles as quantum average of the
concentration operator in the coordinate representation
\[ \hat{\rho} = \sum_j \delta(\vec{r} - \vec{r}_j) \] we obtain

\[ \rho(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \psi_s^+(R, t) \psi_s(R, t), \]  \quad (6) \]

Differentiation of \( \rho(\vec{r}, t) \) with respect to time and applying of the Schrödinger equation with Hamiltonian \( \hat{H} \) leads to continuity equation

\[ \frac{\partial \rho(\vec{r}, t)}{\partial t} + \nabla \cdot j(\vec{r}, t) = 0, \]  \quad (7) \]

where the current density takes a form of

\[ j^\alpha(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \frac{1}{2m_j} \times \]

\[ \times (\hat{D}_j^\alpha \psi_s^+(R, t) \psi_s(R, t) + \psi_s^+(R, t) \hat{D}_j^\alpha \psi_s(R, t)), \]

A momentum balance equation can be derived by differentiating current density \( \rho(\vec{r}, t) \) with respect to time

\[ \partial_t j^\alpha(\vec{r}, t) + \frac{1}{m} \partial_j \mathbb{R}^{\alpha\beta}(\vec{r}, t) = \frac{q}{m} \rho(\vec{r}, t) E^\alpha_{\text{ext}}(\vec{r}, t) \]  \quad (9) \]

\[ + \frac{q}{mc} \epsilon^{\alpha\beta\gamma} j_\beta(\vec{r}, t) B^\gamma_{\text{ext}}(\vec{r}, t) - \frac{1}{m} \int d\vec{q} q^2 \partial^\alpha G(\vec{r}, \vec{q}) \rho_2(\vec{r}, \vec{q}, t) \]

\[ + \frac{1}{m} M_\beta(\vec{r}, t) \partial^\alpha B^\beta_{\text{ext}}(\vec{r}, t) + \frac{1}{m} \int d\vec{q} \partial^\alpha F^{\gamma\delta}(\vec{r}, \vec{q}) M^{\gamma\delta}(\vec{r}, \vec{q}, \vec{q}, t), \]

where

\[ \mathbb{R}^{\alpha\beta}(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \frac{1}{4m_j} \times \]  \quad (10) \]

\[ \times (\psi_s^+(R, t) \hat{D}_j^\alpha \hat{D}_j^\beta \psi_s(R, t) + (\hat{D}_j^\alpha \psi_s(R, t))^+ \hat{D}_j^\beta \psi_s(R, t) + \text{h.c.}) \]

represents the momentum current density tensor.

Momentum balance equation \( \mathbb{R}^{\alpha\beta}(\vec{r}, t) \) contains the particle magnetic moment density \( \mathbb{M}^\alpha(\vec{r}, t) \) which is the two-particle probability density for the occurrence of two particles in the neighborhoods of the points \( \vec{r} \) and \( \vec{r}' \) normalized by \( N(N - 1) \), and two-particle tensor of the magnetic moment density

\[ M^{\alpha\beta}(\vec{r}, \vec{r}', t) = \sum_s \int dR \sum_{j \neq k} \delta(\vec{r} - \vec{r}_j) \delta(\vec{r}' - \vec{r}_k) \psi_s^+(R, t) \psi_s(R, t) \]  \quad (13) \]

\[ \times \mu_{\beta\gamma} \mu_{\alpha\delta} \gamma S_{\alpha\beta}^S \psi_s(R, t). \]

Differential of \( M^\alpha \) with respect to time and applying of the Schrödinger equation with Hamiltonian \( \hat{H} \) leads to magnetization equation. The equation representing the non-relativistic evolution of spin \(-1/2\) motion takes a form of

\[ \partial_t M^\alpha(\vec{r}, t) + \partial_\beta \mathbb{M}^{\alpha\beta}(\vec{r}, t) = \frac{2\mu}{\hbar} \epsilon^{\alpha\beta\gamma} M^\beta(\vec{r}, t) B_{\text{ext}}^\gamma(\vec{r}, t) \]  \quad (14) \]

\[ + \frac{2\mu}{\hbar} \epsilon^{\alpha\beta\gamma} \int d\vec{q} F^{\gamma\delta}(\vec{r}, \vec{q}) M^{\beta\delta}(\vec{r}, \vec{q}, \vec{q}, t) \]

where the tensor of the magnetic moment flux density is

\[ \mathbb{M}^{\alpha\beta}(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \frac{\mu_j}{4m_j} (\psi_s^+ \gamma S_{\alpha\beta}^S \psi_s) \]

\[ + (\gamma S_{\alpha\beta}^S \psi_s^+ \psi_s)(R, t). \]

The spinning quantum magnetohydrodynamics should explain the vorticity evolution. The main idea of this paper was to create a hydrodynamics foundation for the vortex dynamic in the context of spinning quantum plasma. We use the MPQHHD approach to receive the equations for the particle vorticity density, obeying the standard vortex dynamics. We determine the vorticity density vector of particles in the neighborhood of \( \vec{r} \) in a physical space as

\[ \Omega^\alpha(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \frac{\epsilon^{\alpha\beta\gamma}}{2m_j} \times \]

\[ \times \gamma S_{\beta\gamma}^S \psi_s^+(R, t) \psi_s(R, t) + \psi_s^+(R, t) \gamma S_{\beta\gamma}^S \psi_s(R, t), \]

where we construct the vorticity density in term of the wave function, we denote the macroscopic vorticity density as \( \Omega = \nabla \times \mathbf{j} \), as will be shown below. The classical generalized vorticity density \( \Omega \) can be defined as the curl
of the current density. But in article[29] the ordinary vorticity of the plasma is proportional to the curl of the flow velocity of the fermions (vorticity have the dimensions of the magnetic field). On the other hand vorticity can be defined as the curl of velocity \( \vec{\omega} = \nabla \times \vec{v} \) similar to[22] and[29] (see below), but contained the information about interactions inside the fluid.

A. Velocity field

The velocity of \( j \)-th particle \( \vec{v}_j \) is determined by equation

\[
\vec{v}_j = \frac{1}{m_j}(\nabla_j S - i\hbar \vec{A}^\perp - \frac{q_j}{m_j c} \vec{A}),
\]

(17)

The quantity \( \vec{v}_s(R, t) \) describe the current of probability connected with the motion of \( j \)-th particle, in general case \( \vec{v}_j(R, t) \) depend on coordinate of all particles of the system \( R \), where \( R \) is the totality of \( 3N \) coordinate of \( N \) particles of the system \( R = (\vec{r}_1, ..., \vec{r}_N) \).

The \( S(R, t) \) value in the formula (17) represents the phase of the wave function and as the electron has spin, the wave function is now be expressed in the form

\[
\psi_s(R, t) = a(R, t)e^{\frac{i}{\hbar}S}\phi_s(R, t),
\]

(18)

where \( \phi_s \), normalized such that \( \phi_s^\dagger \phi_s = 1 \), is the new spinor, defined in the local frame of reference with the origin at the point \( \vec{r} \). The spinor gives the spin part of the wave function.

We substituted the wave function in the definition of the basic hydrodynamical quantities. Using that the velocity field \( \vec{v} \) is the velocity of the local center of mass and determined by equation

\[
\vec{j}(\vec{r}, t) = \rho(\vec{r}, t)\vec{v}(\vec{r}, t),
\]

(19)

the vorticity density field[16] and the momentum current density tensor[16] have the new form of

\[
\vec{j}_s(\vec{r}, t) = \frac{1}{m_j}(\nabla_j S - i\hbar \phi^\dagger \nabla_j \phi - \frac{q_j}{m_j c} \vec{A}),
\]

(20)

\[
\vec{\rho}^{\alpha\beta}(\vec{r}, t) = m\rho(\vec{r}, t)\upsilon^{\alpha}(\vec{r}, t)\upsilon^{\beta}(\vec{r}, t) + \phi^{\alpha\beta}(\vec{r}, t) + \Lambda^{\alpha\beta}(\vec{r}, t) + \Upsilon_s^{\alpha\beta}(\vec{r}, t),
\]

(21)

where

\[
\phi^{\alpha\beta}(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) a^2 m_j u_j^\alpha u_j^\beta,
\]

(22)
is the well known kinetic pressure tensor. Value \( u_j^\alpha(\vec{r}, R, t) \) is a quantum equivalent of the thermal speed and \( u_j^\alpha(\vec{r}, R, t) = v_j^\alpha(R, t) - \upsilon(\vec{r}, t) \).

The tensor \( \Lambda^{\alpha\beta} \) is proportional to \( h^2 \), has a purely quantum origin and can therefore be interpreted as an additional quantum pressure[29]

\[
\Lambda^{\alpha\beta}(\vec{r}, t) = -\sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) a^2(R, t) \frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial x_j^\beta \partial x_j^\beta} \ln \rho(\vec{r}, t),
\]

(23)

The quantum tensor (23) is the quantity which can be rewritten in terms of concentration \( \rho \) in the approximation of noninteracting particles, using the definition \( \rho = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) a^2(R, t) \) as

\[
\Lambda^{\alpha\beta}(\vec{r}, t) = -\frac{\hbar^2}{4m}\rho(\vec{r}, t)\partial^\alpha \partial^\beta (\ln \rho)(\vec{r}, t).
\]

(24)

Using simple manipulation with the expression (24) we may substitute it for a large system of noninteracting particles; this tensor is

\[
\Lambda^{\alpha\beta}(\vec{r}, t) = -\frac{\hbar^2}{4m}\rho(\vec{r}, t)\partial^\alpha \partial^\beta \rho(\vec{r}, t) - \frac{1}{\rho(\vec{r}, t)} \{ \partial^\alpha (\rho(\vec{r}, t)) \{ \partial^\beta (\rho(\vec{r}, t)) \}
\]

(25)

It should be explained that the tensor[25] arises as a consequence of the quantum Madelung potential and can be interpreted as an additional quantum pressure.

The tensor \( T_s^{\alpha\beta} \) appears in the theory as a result of representations of rotating electrons as an assembly of bodies continuously distributed in space[13] and for the many-particles systems takes the form

\[
T_s^{\alpha\beta}(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) a^2(R, t) \times
\]

\[
\times \frac{1}{m_j} \nabla^\alpha s_j^\dagger \nabla^\beta s_j^\dagger.
\]

(26)

The thermal spin fluctuations takes the form[32]

\[
\xi_s^\alpha(\vec{r}, R, t) = s_j^\dagger(\vec{r}, R, t) - s^\dagger(\vec{r}, t),
\]

(27)

In the context of quantum hydrodynamics, the force due to a new Spin Stress inside the fluid takes the form[22]

\[
\Upsilon_s^{\alpha\beta}(\vec{r}, t) = -\frac{\hbar^2}{4m\mu^2} M(\vec{r}, t) \partial^\alpha \partial^\beta \left( \frac{M(\vec{r}, t)}{\rho(\vec{r}, t)} \right)
\]

(27)
where the tensor of the thermal spin fluctuations

$$Q_{s}^{\alpha \beta}(r, t) = \sum_{s} \int dR \sum_{j=1}^{N} \delta(r - r_{j}) a^{2}(R, t) \frac{1}{m_{j}} \nabla_{\alpha}^{j} \xi_{j}^{\gamma} \nabla_{\beta}^{j} \xi_{j}^{\gamma}$$

(28)

$$-\frac{\rho}{2m} \nabla^{\alpha} \left\{ \frac{1}{\rho} \sum_{s} \int dR \sum_{j=1}^{N} \delta(r - r_{j}) a^{2}(R, t) \xi_{j}^{\gamma} \xi_{j}^{\gamma} \right\}$$

This new force emerges from the inhomogeneity of spin distribution\(^{[22]}\).

After the presentation of the wave function in the exponential form\(^{[18]}\), using the Madelung decomposition\(^{[17]}\), the tensor of the magnetic moment flux density takes the form

$$\mathcal{Z}_{M}^{\alpha \beta}(r, t) = M^{l} v^{\beta}(r, t) + \gamma_{s}^{\alpha \beta}(r, t),$$

(29)

where

$$\gamma_{s}^{\alpha \beta}(r, t) = -\sum_{s} \int dR \sum_{j=1}^{N} \delta(r - r_{j}) \frac{2 \mu_{j} \varepsilon^{\alpha \mu \nu}}{m_{j} h} \times$$

$$\times a^{2}(R, t) s_{j}^{\alpha} \nabla_{j}^{\beta} s_{j}^{\nu}$$

In the context of quantum hydrodynamics we derive the additional Spin Torque in the form\(^{[22]}\)

$$\gamma_{s}^{\alpha \beta}(r, t) = -\frac{\hbar}{2 m \mu_{j}} \varepsilon^{\alpha \gamma \lambda} M_{s}(r, t) \partial_{\beta} \left( \frac{M^{\lambda}(r, t)}{\rho(r, t)} \right) + \Theta_{s}^{\alpha \beta}(r, t).$$

(31)

The quantum equivalent of the thermal speed contributes to the magnetization current \(d_{T}^{\alpha \beta}\)

$$d_{T}^{\alpha \beta}(r, t) = \sum_{s} \int dR \sum_{j=1}^{N} \delta(r - r_{j}) \frac{2 \mu_{j}}{h} a^{2}(R, t) v_{j}^{\alpha} s_{j}^{\beta}.$$  

(32)

The spin density thermal fluctuations are presented in the form of\(^{[33]}\)

$$\Theta_{s}^{\alpha \beta}(r, t) = -\varepsilon^{\alpha \mu \nu} \sum_{s} \int dR \sum_{j=1}^{N} \delta(r - r_{j}) \frac{2 \mu_{j}}{m_{j} h} \times$$

$$\times a^{2}(R, t) \xi_{j \mu} \nabla_{j}^{\beta} \xi_{j}^{\nu}$$

### B. Energy evolution equation

The energy density taking into account the Coulomb and Spin-Spin interactions is given by\(^{[1,12]}\) and\(^{2}\)

$$\varepsilon(r, t) = \int dR \sum_{j=1}^{N} \delta(r - r_{j}) \frac{1}{4 m_{j}} \{ \psi_{s}^{+} D_{s}^{2} \psi_{s} + (D_{s}^{2} \psi_{s})^{+} \psi_{s} \}$$

(34)

$$+ \int dR \sum_{i \neq k}^{N} \delta(r - r_{j}) \frac{1}{2} \rho_{s} \left\{ q_{j} q_{k} G_{j k} - \mu_{j}^{2} \sigma_{j}^{\alpha} \sigma_{k}^{\beta} F_{j k}^{\alpha \beta} \right\} \psi_{s}(R, t).$$

Differentiation of \(\varepsilon\) with respect to time and application of the Schrödinger equation with Hamiltonian\(^{[3]}\) leads to the energy balance equation

$$\frac{\partial}{\partial t} \varepsilon(r, t) + \nabla \cdot Q(r, t) = q_{j} a(r, t) E_{ext}^{\alpha}(r, t)$$

(35)

$$+ \mathcal{Z}_{M}^{\alpha \beta}(r, t) \partial_{\beta} B_{ext}^{\alpha}(r, t) + A(r, t),$$

where \(A(r, t)\) is the internal force density and \(Q(r, t)\) is the internal energy flux density.

The internal energy flux density is given by

$$Q(r, t) = \int dR \sum_{j}^{N} \delta(r - r_{j}) \frac{1}{8 m_{j}} \{ \psi_{s}^{+} D_{j} D_{s}^{2} \psi_{s}$$

(36)

$$+ (D_{j} D_{s}^{2} \psi_{s})^{+} \psi_{s} + D_{j}^{+} \psi_{s}^{+} D_{j}^{2} \psi_{s} + (D_{j}^{2} \psi_{s})^{+} D_{j} \psi_{s} \} \}$$

$$+ \int dR \sum_{j}^{N} \delta(r - r_{j}) \frac{1}{4 m_{j}} \{ \psi_{s}^{+} (R, t)(q_{j} q_{k} G_{j k}$$

$$- \mu_{j}^{2} \sigma_{j}^{\alpha} \sigma_{k}^{\beta} F_{j k}^{\alpha \beta} \} D_{j} \psi_{s}(R, t) + k.c.$$\}

The internal force density \(A(r, t) = A_{s-s}(r, t) + A_{s,r}(r, t) + A_{s,d}(r, t)\) in\(^{[35]}\) consists of the Coulomb force density \(A_{cl}(r, t)\) and Spin-Spin force density \(A_{s-s}(r, t)\)
\[ + \frac{1}{m_k} \{ (\sigma_j^a \sigma_k^b \bar{D}_k^a \psi)^+ \psi + \psi^+(\sigma_j^a \sigma_k^b \bar{D}_k^a \psi) \}(R, t) \}\]

and

\[ A_{cl}(r, t) = - \int dR \sum_{i \neq k} \delta(r - r_j) \frac{1}{4}(q_j q_k \nabla \alpha G_{jk}) \times \]

\[ \times \left\{ \frac{1}{m_j} (\hat{D}_j^a \psi)(R, t) \psi(R, t) + \psi^+(R, t) \hat{D}_j^a \psi(R, t) \right\} + \frac{1}{m_k} \{ (\hat{D}_k^a \psi)^+ \psi(R, t) + \psi^+(R, t) \hat{D}_k^a \psi(R, t) \}\]

\[ A_s(r, t) = \sum_s \int dR \sum_{i \neq k} \delta(r - r_j) \frac{\mu_{jk}}{2} (F^{\alpha \beta}_{jk}) e^{\mu \nu} \times \]

\[ \times \left\{ \frac{2 \mu_j}{\hbar} B^\mu_{ext} \psi^+_s(R, t) \hat{\sigma}^\alpha \hat{\sigma}^\beta \psi_s(R, t) + \frac{2 \mu_k}{\hbar} B^\mu_{ext} \psi^+_s(R, t) \hat{\sigma}^\alpha \hat{\sigma}^\beta \psi_s(R, t) \right\} - \sum_s \int dR \sum_{i \neq k} \delta(r - r_j) \frac{\mu_{jk}}{2} (\nabla_j F^{\alpha \beta}_{jk}) e^{\mu \nu} \times \]

\[ \times \left\{ \frac{2 \mu_j}{\hbar} F^{\mu \nu}_{jk} \psi^+_s(R, t) (\sigma_\mu^\alpha \sigma_\nu^\beta + \sigma_\nu^\alpha \sigma_\mu^\beta) \psi_s(R, t) + \frac{2 \mu_k}{\hbar} F^{\mu \nu}_{jk} \psi^+_s(R, t) (\sigma_\mu^\alpha \sigma_\nu^\beta + \sigma_\nu^\alpha \sigma_\mu^\beta) \psi_s(R, t) \right\}. \]

Using the fact that the velocity field is the velocity of the local center of mass and is determined by \[ \text{(19)} \] and using the definitions \[ \text{(27)} \] and \[ \text{(31)} \] the energy evolution equation reads

\[ \rho \frac{\partial}{\partial t} + v \nabla \psi + \bar{\nabla} \bar{q}(r, t) + \frac{\hbar^2}{4m \rho(r, t)} \{ \partial_\alpha \rho(r, t) \} \{ \partial_\beta \rho(r, t) \} \partial_\beta v^\alpha(r, t) \]

\[ - \frac{\hbar^2}{4m} \partial_\alpha (\rho(r, t) \partial_\alpha v^\beta(r, t)) + \rho \frac{\hbar^2}{4m} \partial_\alpha \partial_\beta \rho(r, t) \partial_\beta v^\alpha(r, t) - \frac{\hbar^2}{4m} \partial_\alpha (M^\gamma(r, t) \rho(r, t)) \partial_\beta v^\alpha(r, t) \]

\[ + Q_s^{\alpha \beta}(r, t) \nabla_\beta v^\alpha(r, t) + p^{\alpha \beta}(r, t) \partial_\beta v_\alpha(r, t) = d^{\alpha \beta}(r, t) \partial_\beta B^\mu_{ext}(r, t) + \nabla_\beta B^\mu_{ext}(r, t) \Theta_s^{\alpha \beta}(r, t) \]

\[ - \frac{\hbar}{2m} \epsilon^{\alpha \mu \nu} M_\rho(r, t) \partial_\beta \left( \frac{M^\nu(r, t)}{\rho(r, t)} \right) \nabla_\beta B^\alpha_{ext}(r, t) + \eta(r, t). \]

Let us discuss the physical significance of the terms on the left-hand side of the energy equation \[ \text{(39)} \]. The third, fourth and fifth terms on the left-hand side in Eq. \[ \text{(39)} \] describe a quantum force produced by density fluctuations, which has its origin in the so-called Bohm-Madelung potential. The eighth term represents the well-known pressure tensor influence. The sixth term on the left-hand side of \[ \text{(39)} \] characterizes the energy density generation by the Spin Stress and the third term on the right-hand side describes the Spin torque influence. The seventh term on the left-hand side, the first and second terms on the right-hand side describe the thermal effects. The forth term on the right-hand side in Eq. \[ \text{(39)} \] is the total force field density.

The specific "heat" energy density of the spinning fermions in an external electromagnetic field takes the form

\[ \rho \epsilon(r, t) = \int dR \sum_j \delta(r - r_j) a^2(R, t) \left( \frac{m_j u_j^2}{2} - \frac{\hbar^2}{2m_j} \left( \frac{\nabla_j a}{a} \right)^2 \right) + \frac{1}{2m_j} (\nabla_\alpha s^\alpha)^2 \]

\[ \text{(40)} \]
The first term on the right-hand side of the expression (40) describes the quantum equivalent of the thermal speed contribution, the second term characterizes the quantum Madelung potential contribution and the third term presents the internal spin potential influence. The forth and fifth terms are the potential energy density of the particle interaction, namely the Coulomb interaction of charges and spin-spin interactions.

The density of the heat kinetic energy flow takes the form

\[
q^\alpha(r,t) = \int dR \sum_j \delta(r - r_j) a^2(R,t) \{ u_j^\alpha \left( \frac{m_j u_j^2}{2} - \frac{\hbar^2}{2m_j} \Delta_j a + \frac{1}{2m_j} |\nabla a s_j^\alpha|^2 \right) \}
\]

Note that to simplify the problem we consider that the thermal spin-interactions are neglected and microscopic spin \( s_j^\alpha = s^\alpha \) is equal to macroscopic average \( s^\alpha \). Taken in the approximation of self-consistent field, from (9), (14) we have the set of MQHD equation for the electrons and ions (p=e, i):

| continuity equation |
| \( \partial_t \rho_p + \vec{\nabla} (\rho_p \vec{v}_p) = 0 \) |

| momentum balance equation |
| \[
m_p \partial_t \vec{v}_p + \frac{e}{m_p c} \vec{E}_{ext} + \vec{B}_{ext} \times \vec{v}_p = -\vec{\nabla} P_p + \frac{\mu_p}{2m_p} \frac{\hbar^2}{m_p} \vec{\nabla} B_{ext} + \frac{e}{m_p} \left( \frac{M_p}{\rho_p} \right) \partial_\gamma \vec{B}_{ext} \]
\]

| magnetic moment density equation |
| \[
(\partial_t + v_\beta^\gamma \partial_\gamma) \vec{M}_p = \frac{2\mu_p}{\hbar} \vec{M}_p \times \vec{B}_{ext} + \frac{\hbar}{2m_p \mu_p} \partial_k \left( \vec{M}_p \times \partial^k \left( \frac{M_p}{\rho_p} \right) \right) + \frac{2\mu_p}{\hbar} \epsilon^{\alpha\beta\gamma} M^\beta_p \int d\vec{r}' F^{\gamma\delta}(\vec{r}, \vec{r}') M^\delta_p(\vec{r}', t)
\]

| energy evolution equation |
| \[
\rho \left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \epsilon(r,t) + \vec{\nabla} \cdot \vec{q}(r,t) + \frac{\hbar^2}{4m_p \rho(r,t)} \{ \partial_\alpha \rho(r,t) \} \{ \partial_\beta \rho(r,t) \} \partial_\beta \epsilon^\alpha(r,t) - \frac{\hbar^2}{4m} \partial_\alpha (\rho(r,t) \partial_\beta v_\beta(r,t)) \]
\]
- \frac{\hbar^2}{4m} \partial_{\alpha} \partial_{\beta} \rho(\mathbf{r}, t) \partial_{\beta} v^{\alpha}(\mathbf{r}, t) - \frac{\hbar^2}{4m \mu_0^2} \mathbf{M}_e(\mathbf{r}, t) \partial_{\alpha} \partial_{\beta} \left( \frac{\mathbf{M}^\alpha(\mathbf{r}, t)}{\rho(\mathbf{r}, t)} \right) \partial_{\beta} v^{\alpha}(\mathbf{r}, t)

+p^{\alpha\beta}(\mathbf{r}, t) \partial_{\beta} v_{\alpha}(\mathbf{r}, t) = -\frac{\hbar}{2m \mu_0} \varepsilon^{\alpha\mu\nu} \mathbf{M}_\mu(\mathbf{r}, t) \partial_{\beta} \left( \frac{\mathbf{M}_\nu(\mathbf{r}, t)}{\rho(\mathbf{r}, t)} \right) \nabla_{\beta} B_{ext}^{\alpha}(\mathbf{r}, t) + \kappa(\mathbf{r}, t).

\]

Let's discuss the physical significance of terms on the right side of the system of MQHD equations obtained above \([42]-[45]\). The first and second terms in Eq. \([43]\) describe the well-known interaction with the external electromagnetic field, where the first term represents the effect of the external electric field on the charge density and the second term is the Lorentz force field. The fourth term is a quantum force produced by density fluctuations, which has its origin in the so-called \textit{Madelung potential}. The fifth term appears in the equation of motion \([43]\) through the magnetization energy and depends on the spin or magnetic moment density of particles. The sixth term represents a force produced by spin fluctuations. This \textit{spin stress} appears even in the absence of the electromagnetic fields and arises from the inhomogeneity of the spin distribution. Other terms in \([43]\) describe a force field that represents interactions between particles, namely the \textit{Coulomb} interaction of charges and \textit{Spin – Spin} interactions.

The second term in the equation of magnetic moment density motion \([44]\) represents the additional \textit{spin torque} effect on the magnetic moment density evolution and tends to align spins parallel. It's important that the second term has a similar form respectively to the contribution of exchange interaction in ferromagnetic media for isotropic cubic ferromagnetic.

Using the definition \([16]\) and the Madelung decomposition \([17]\), the hydrodynamics classical vorticity dynamical equation \([46]\),

\[
\partial_t \tilde{\omega}_p = \nabla \times (\tilde{v}_p \times \tilde{\omega}_p) - \nabla \left( \frac{1}{\rho_p} \right) \times \nabla \varphi_p
\]

\[
+ \frac{1}{m_p} \nabla \left( \frac{\mathbf{M}_p}{\rho_p} \right) \times \nabla \mathbf{B}_{ext}^k + \frac{1}{c_m p} \nabla \times \left( \frac{1}{\rho_p} \right) \nabla \times \left( \mathbf{J}_{pe}^k \times \mathbf{B} \right)
\]

\[
+ \frac{\hbar^2}{4m_p} \nabla \left( \frac{\mathbf{M}_\nu^\alpha}{\rho_p} \right) \times \nabla \left\{ \frac{1}{\rho_p} \nabla_k (\rho_p \nabla_k \{ \frac{\mathbf{M}_\nu}{\rho_p} \}) \right\}
\]

\[
+ \frac{1}{m_p} \nabla \left( \frac{\mathbf{M}_\nu^\alpha}{\rho_p} \right) \times \nabla \int d^3 \mathbf{r}^\delta F^\gamma(\mathbf{r}, \mathbf{r}^\gamma) \mathbf{M}_\nu^\delta(\mathbf{r}^\gamma, t).
\]

The vorticity evolution equation \([46]\) shows the different physical factors associated with the generation of vorticity. The second term on the right side of \([46]\) is proportional to the gas pressure and is responsible for the hydrodynamic baroclinic vorticity generation of the classical vortex field. The third term represents the magnetic baroclinic vorticity and is associated with the anisotropic magnetic pressure effect. The fourth term contains information about the vorticity generated by the magnetic tension. The fifth term is associated with the magnetic vorticity generation, even in the absence of the magnetic field. The seventh term characterizes the effect of \textit{Spin – Spin} interactions in the vorticity evolution. Equation \([46]\) contains the normal electrons or ions current density \(J_{ep} = q_e n_p \nu_p \), and the magnetic moment density \(\mathbf{M}_p = \rho_p \mu_p \). The vorticity evolution equation \([46]\) is a generalization of classical vorticity equation which had been presented in works \([32]-[35]\) and \([36]\). At first, Eq. \([46]\) combines the erstwhile generalized classical vorticity, but in contrast to \([33]\) and \([36]\) contains the information about interactions inside the quantum vortical fluid and have been derived using the MQHD method.

Note, that for a 3D system of particles the momentum balance equation \([43]\), the magnetic density equation \([44]\) and the vorticity evolution equation \([46]\) may be written down in terms of magnetic intensity of the field that is created by charges \(q_p\) and spins \(\tilde{s}_p\) of the particle system

\[
m_p (\partial_t + \nu_p \partial_{\beta}) \tilde{v}_p = q_p \tilde{E} + \frac{q_p}{c} \tilde{v}_p \times \tilde{B} - \frac{\nabla \varphi_p}{\rho_p} \times \tilde{B} + \frac{\hbar^2}{2m_p} \nabla \left( \frac{\Delta \sqrt{\rho_p}}{\sqrt{\rho_p}} \right)
\]

\[
+ \frac{2\mu_p}{\hbar} s_p \tilde{B}_{eff}^k \times \tilde{B}_{eff}^k,
\]

\[
(\partial_t + \nu_p \partial_{\beta}) \tilde{s}_p = \frac{2\mu_p}{\hbar} \tilde{s}_p \times \tilde{B}_{eff}^k,
\]

and

\[
\partial_t \tilde{\omega}_p = \nabla \times (\tilde{v}_p \times \tilde{\omega}_p) - \nabla \left( \frac{1}{\rho_p} \right) \times \nabla \varphi_p + \frac{2\mu_p}{\hbar} \tilde{s}_p \times \tilde{B}_{eff}^k \times \tilde{B}_{eff}^k
\]

where \(\tilde{\omega}_p = \omega_p + \frac{q_p}{m_p} \tilde{B} \) is the generalized vorticity and the effective magnetic field \(\tilde{B}_{eff} = \tilde{B} + \tilde{B}_{in}\) includes the total magnetic field and internal magnetic field \(\tilde{B}_{in}\).
\[ \vec{B}_{\text{int}} = \frac{c}{q_p \rho_p} \nabla k (\rho_p \nabla^k \vec{s}_p) \quad (50) \]

The total magnetic field \( \vec{B} \) consists of the field generated by the charge and the field generated by the spins. Ampère's law including the magnetization spin current \( j_m = 2\mu / h \vec{\nabla} \times (\rho \vec{s}) \) takes the form of

\[ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \sum_p j_p + \frac{8\pi \mu}{h} \vec{\nabla} \rho \times \vec{s}_p + \frac{8\pi \mu}{h} \sum_p \rho_p \vec{\nabla} \times \vec{s}_p, \quad (51) \]

Moreover, we need to take into account the wave equations

\[ \Box \vec{A} = \frac{4\pi}{c} \sum_p q_p \rho_p \vec{v}_p - 4\pi \sum_p \frac{2\mu_p}{h} \rho_p \vec{\nabla} \times \vec{s}_p, \quad (52) \]

in the Lorentz gauge

\[ \frac{1}{c} \frac{\partial \phi}{\partial t} + \vec{\nabla} \vec{A} = 0 \quad (53) \]

We must note that the spin stress term have notably interesting nature, exist even in absence of magnetic field, have only quantum foundation and arising out from the spin part of the wave function. Equation (49) was rewritten by separating the magnetic and non-magnetic terms\[22\]

### III. THE ELECTROMAGNETIC WAVES

Two-dimensional turbulence has been studied in a magnetized plasma involving incompressible electrons and immobile ion\[23\]. We consider that the electrons carry currents, while the immobile ions provide a neutralizing background to a quasi-neutral spinning plasma. Using the fact that the electron fluid velocity is associated not only with the rotational magnetic field but also with the magnetization spin current \( j_m = 2\rho_0 \mu_e \vec{\nabla} \times \vec{s} / h \), which is determined by the spin vector \( \vec{s} \), we have, from Ampere's law

\[ \vec{v}_e = - \frac{c}{4\pi \rho_0 e} \vec{\nabla} \times \vec{B} - \frac{g}{2m_e \vec{\nabla} \times \vec{s}} \quad (54) \]

where \( \mu_e = -g\mu_B / 2, \mu_B = e^2 / 2m_e, m_e \) is the electron mass, and \( \rho_0 \) is the electron density. The 3D equation (49) closed by (48) becomes 2D by regarding the variation in the \( z \)-direction as ignorable or \( \partial / \partial z = 0 \) and using the separation of the total magnetic field into two scalar variables\[22\]

\[ \vec{B} = \vec{\zeta} \times \vec{\nabla} \psi + b\vec{\zeta} \]

In this case the linear approximation of the 2D equation (49) can be read

\[ \omega \left( 1 + d_e^2 k^2 \right) + \frac{d_e^2 \mu \tanh(\alpha)}{- \omega^2 - \omega_g^2} \left( k^2 \omega_g - k^2 \omega_c \right) b = 0 \quad (55) \]

and

\[ \omega \left( 1 + d_e^2 k^2 \right) + \frac{d_e^2 \mu \tanh(\alpha)}{- \omega^2 - \omega_g^2} \left( k^2 \omega_c - k^2 \omega_g \right) \psi = 0 \quad (56) \]

where \( d_e = c / \omega_{pe} \) - the electron skin depth, \( \omega_{pe} = 4\pi e^2 \rho_0 / m_e \) - is the electron plasma frequency, \( \omega_c \) - is the electron cyclotron frequency. The function \( \tanh(\alpha) \) is the Brillouin function due to the magnetization of a spin distribution in thermodynamic equilibrium with \( \alpha = \mu_B B_0 / k_B T_e \). The spin-precession frequency taken into account the spin torque influence is

\[ \omega_g = \frac{g \omega_e}{2} + \frac{k^2 h}{2m_e} \tanh(\alpha), \]

\( \omega_e = g^2 h / 8m_e d_e^2 \) - is a frequency which involves the spin correction due to plasma magnetization current, and \( h \) - is the Planck constant.

### A. Parallel propagation of the wave

We will assume propagation of the waves along an external magnetic field. A linearized set of equations in this case gives us the dispersion equation

\[ (1 + d_e^2 k^2) \omega_k^3 - (\omega_e - \omega_i) d_e^2 k^2 \omega_k^2 \]

\[ + ((\omega_i \omega_g - \omega_g^2 - \omega_i \omega_c) d_e^2 k^2 - \omega_g^2 \omega_k) = 0, \quad (57) \]

The effect of the frequency that involves the spin correction due to the plasma magnetization current is small and the cubic expression may be expanded to yield formulae in the form...
corresponds to a regime of normal magnetic field in which the external field strength approaches $B_0 \sim 10^4 \text{G}$. The second term involving $\omega_\mu$ appears even in absence of the external magnetic field. The low of dispersion (59) represents the new wave as a result of spin torque effect.

![Figure 1](image1.png)

**FIG. 1.** Figure shows the electromagnetic mode (58), the external magnetic field $B_0 = 2 \times 10^4 \text{G}$, the electron equilibrium concentration $\rho_0 = 5 \cdot 10^{23} \text{ cm}^{-3}$, $m_e = 9.1093 \times 10^{-28} \text{g}$, the electron temperature $T_e \sim 10 \text{K}$. The blue branch characterizes the classical whistler wave and the red branch represents the magnetization current and spin torque contribution.

![Figure 2](image2.png)

**FIG. 2.** Figure shows the electromagnetic mode (58), the external magnetic field $B_0 = 2 \times 10^4 \text{G}$, the electron equilibrium concentration $\rho_0 = 5 \cdot 10^{23} \text{ cm}^{-3}$, $m_e = 9.1093 \times 10^{-28} \text{g}$, the electron temperature $T_e \sim 10 \text{K}$. The blue branch characterizes the classical whistler wave and the red branch represents the magnetization current and spin torque contribution, the green mode shows the magnetization current effect only.

\[
\omega_1 = \frac{\omega_c d^2 k^2}{1 + d^2 k^2} \left(1 - \frac{\omega_\mu \tanh(\alpha)}{\omega_g (\omega_g - \omega_\mu) d^2 k^2}\right), \quad (58)
\]

\[
\omega_2 = \omega_g - \frac{\hbar}{2m_c} \frac{\omega_\mu \tanh(\alpha) d^2 k^2}{\omega_g (\omega_g - \omega_\mu) d^2 k^2}, \quad (59)
\]

We see that the second term in Eq. (58) is proportional to the frequency $\omega_\mu$. The magnetization current and spin torque contribution might be important when the electron equilibrium concentration $\sim \rho_{0e} \geq 10^{25} \text{cm}^{-3}$. This corresponds to a regime of normal magnetic field in which the external field strength approaches $B_0 \sim 10^4 \text{G}$. The second term involving $\omega_\mu$ appears even in absence of the external magnetic field. The low of dispersion (59) represents the new wave as a result of spin torque effect.

![Figure 3](image3.png)

**FIG. 3.** Figure shows the electromagnetic mode (59), the external magnetic field $B_0 = 2 \times 10^4 \text{G}$, the electron equilibrium concentration $\rho_0 = 5 \cdot 10^{23} \text{ cm}^{-3}$, $m_e = 9.1093 \times 10^{-28} \text{g}$, the electron temperature $T_e \sim 10 \text{K}$. The blue branch characterizes the classical whistler wave and the red branch represents the spin torque contribution.

**IV. SPIN WAVE EXCITATION BY SPIN CURRENT**

The most interesting results in the direction of spin-current investigation were obtained in [35][36]. General form of the force field appearing in the spin-current equation and introducing of the self-consistent field approximation emerges in the following explicit form (for electrons and ions $p = e, p = i$)

\[
\frac{\partial \Sigma_\alpha^\beta(r, t)}{\partial t} + \frac{\partial \Sigma_\alpha^\beta(r, t)}{\partial x^\gamma} = \frac{q_p}{m_p} M_\alpha^p(r, t) E^\beta(r, t) \quad (60)
\]

\[+ \frac{q_p}{m_p \epsilon} \delta_{\beta\gamma} \Sigma_\alpha^\gamma (r, t) B^\eta(r, t) + \frac{\mu_p^2}{m_p} \rho_p(r, t) \nabla^\beta B(r, t) \]

\[- \frac{2 \mu_p}{\hbar} \delta_{\alpha\gamma} B^\gamma(r, t) \Sigma_\alpha^\eta M(r, t),
\]

Equation for the magnetization evolution has the form

\[
\partial_t M^\alpha(r, t) + \partial_\beta \Sigma_\beta^\alpha = \frac{2 \mu}{\hbar} \epsilon^{\alpha\beta\gamma} M^\beta(r, t) B^\gamma(r, t) \quad (61)
\]

One of the primary goals of this article is to derive dispersion characteristics of spin waves in spinning systems of neutral and charged particles that take account
of the spin-current dynamics. In this section we consider the systems of particles in an external uniform magnetic field. We can investigate the ferromagnetic order and small perturbations of physical variables from the stationary state. We can mark in this section and below, all physical quantities are presented in the form of sum of equilibrium part and small perturbations $f = f_0 + \delta f$. The external magnetic field as $B_0 = B_0(0, \cos \theta, \sin \theta)$ with respect of the propagation direction determined by the wavenumber $k = ky$, where the spin of the electrons is $s_0 = s_0(0, \cos \theta, \sin \theta)$, and the equilibrium spin-current components are $\Im \alpha = \Im s_0 \sin \theta$, $\Im \beta = \Im s_0 \cos \theta$. The equilibrium spin state $s_0 = -\hbar \tanh(\alpha)/2$, antiparallel to the background magnetic field, where the magnetization of a spin distribution in thermodynamic equilibrium with $\tanh(\alpha) = \tanh(\mu_B B_0/k_B T_\infty)$.

### A. Perpendicular propagation $\theta = \pi/2$

The spin wave is obtained as the wave which propagates perpendicular to the background magnetic field. In this case, it appears for the magnetic susceptibility $M^\alpha = \chi^{\alpha\beta} B_\beta$

$$\chi^{\alpha\beta} = \frac{2\mu_0^2 \rho_0}{h} \left( \begin{array}{ccc} \frac{\omega_1^2 - \omega_9^2}{2} + \frac{k^2 m}{2m (\omega_1^2 - \omega_9^2 - \omega_2^2)} & i \frac{k^2 m}{2m (\omega_1^2 - \omega_9^2 - \omega_2^2)} & 0 \\ i \frac{k^2 m}{2m (\omega_1^2 - \omega_9^2 - \omega_2^2)} & \frac{\omega_2^2 - \omega_9^2}{2} + \frac{k^2 m}{2m (\omega_1^2 - \omega_9^2 - \omega_2^2)} & 0 \\ 0 & 0 & \frac{\omega_1^2}{2m} \end{array} \right)$$

### V. WHISTLERON GAS IN SPIN QUANTUM PLASMAS

Let us consider the propagation of EM whistlers along a constant magnetic field $B_0 = B_0 \hat{e}_z$ in a quantum electron-ion plasma where any equilibrium drift velocity is zero. Lets involve small perturbations of physical variables from the stationary state

$$\tilde{A} \to \epsilon \tilde{A} \quad \phi \to \epsilon \phi \quad \tilde{v}_e \to \epsilon \tilde{v}$$

$$n_e \to n_0 + \epsilon n, \quad \tilde{s} \to \tilde{S}_0 + \epsilon \tilde{S},$$

where $\epsilon$ is a formal small parameter, and will be set equal to one at the end of the calculations. The stationary states have the form of $n_0 = \text{const}$, $\tilde{v}_0 = 0$, and the stationary field amplitudes $\tilde{A}_0 = 0$, $\phi_0 = 0$ are equal to zero. Also, $\tilde{S}_0$ is the stationary spin angular momentum with its absolute value $|\tilde{S}_0| = \hbar/2$. Letting the unperturbed spin field lie along the x-axis $\tilde{S}_0 = S_0 \hat{e}_x$.

Following the standard procedure we represent the quantities $\rho$, $\tilde{A}$, $\phi$, $\tilde{s}$ and $\tilde{v}$ as the perturbation expansions

$$\Psi = \hat{\Psi}(\tilde{x}, \tilde{X}, t) = \sum_{j=0}^{\infty} \epsilon^j \hat{\Psi}_j \quad \Psi = (n, \tilde{A}, \phi, \tilde{v}, \tilde{S})$$

where, all hydrodynamic and field variables depend on the spatial variables, where $\tilde{X} = \epsilon \tilde{x}$ - is a slow spatial variable. Using Eqs. (52), (53), (42), (17) and (48), we find for all orders the perturbation equations in the form of...

$$\omega_1^2 = \omega_9^2, \quad \omega_2^2 = \omega_9^2 + \omega s \omega g \tanh(\alpha) - \frac{4\pi \mu_0^2 \rho_0}{m} k^2.$$
\[ \partial_t \rho_j + \partial_\beta (\rho_j v_j^\beta) = \delta_j, \quad (67) \]

\[ \partial_t v_j^\alpha - \frac{v_j^2}{\rho_0} \nabla^\alpha \rho_j + \frac{\hbar^2}{4m_e^2 \rho_0} \nabla^\alpha \Delta \rho_j - \frac{e}{m_e} F_j^\alpha + \frac{1}{m_e^2} \pi e^2 \rho_0 \nabla^\alpha \delta_j - \omega_c v_j^\alpha x + W_j^\alpha, \quad (68) \]

\[ \square \frac{\partial^2 \tilde{A}_j}{\partial t^2} + \omega_c \square \frac{\partial \tilde{A}_j}{\partial t} \times \tilde{e}_x - \frac{\omega_c^2}{c^2} \frac{\partial^2 \tilde{A}_j}{\partial t^2} + \omega_p^2 \nabla \cdot (\nabla \tilde{A}_j) - v_j^2 \nabla \cdot (\nabla \tilde{A}_j) \]

\[ + \eta_0 \nabla \times \frac{\partial^2 \tilde{S}_j}{\partial t^2} + \omega_0 \eta_0 \epsilon_{\alpha \beta \gamma} (\nabla \times \frac{\partial S_j}{\partial t})_\beta + \frac{\hbar^2}{4m_e^2} \Delta \nabla \cdot (\nabla \tilde{A}_j) + \frac{\hbar^2}{4m_e} \eta_0 \Delta \nabla \cdot (\nabla, \nabla \times S_j) \]

\[ - v_j^2 \eta_0 \nabla \cdot (\nabla, \nabla \times S_j) - \frac{4\pi e \rho_0}{m_e^2 c} S_0 \nabla \Delta \frac{\partial S_j}{\partial t} = 4\pi e \rho_0 \frac{\partial W_j^x}{\partial t} + \frac{\partial^2 \xi_j^x}{\partial t^2} + \omega_c \frac{\partial \xi_j^x}{\partial t} \times \tilde{e}_x \]

\[ + \omega_p^2 \nabla \omega_j - v_j^2 \nabla \cdot (\nabla \tilde{S}_j) - \frac{4\pi e}{c} v_j^2 \nabla \delta_j + \frac{\hbar^2}{4m_e^2} \Delta \nabla \cdot (\nabla \tilde{S}_j) + \frac{\hbar^2}{4m_e^2} \frac{4\pi e}{c} \Delta \nabla \delta_j, \]

\[ \partial_t S_j^\alpha = \frac{2\mu}{\hbar} \epsilon^{\alpha \beta \gamma} (S_j^\beta B^\gamma_j + S_j^\beta B_0^\gamma) \]

\[ + \frac{1}{m_e} \epsilon^{\alpha \beta \gamma} S_j^\beta \Delta S_j^\gamma + \gamma_j, \quad (69) \]

\[ \square \tilde{A}_j = \frac{4\pi}{c} \epsilon \rho_0 \tilde{v}_j - \frac{8\pi \mu}{\hbar} \rho_0 \nabla \times \tilde{S}_j + \xi_j, \quad (70) \]

where, we involve the electron cyclotron frequency \( \omega_c = eB_0/m_e c \), \( v_j^2 = k_BT_e/m_e \) - is the thermal speed.

We find a single equation for the amplitudes of whistler waves \( \tilde{A}_j \).

\[ \tilde{A}_0 = \sum_k \tilde{A}_k e^{i(kz - \omega_k t)}, \quad (73) \]

where, \( \omega_k \) is an infinite set of constant complex amplitudes, moreover the amplitudes depend on the slow spatial variables \( \tilde{X} = \epsilon \tilde{x} \), but do not depend on the fast spatial variable \( \tilde{x} \) and time \( t \). Using Eq. \((72)\) and \((48)\) the following linear dispersion relation \((71)\) is obtained for the whistler modes along an external magnetic field \( \tilde{B}_0 = B_0 \epsilon \tilde{x} \).

\[ n^2_k = 1 - \frac{\omega_k^2}{\omega(\omega \pm \omega_c)} - \frac{g^2 \omega_p^2 |S_0| \tanh(\alpha) k^2}{4\omega^2 m_e (\omega \pm \tilde{\omega}_g)}, \quad (74) \]
where, \( n_R = \frac{ck}{\omega} \) is the refractive index, \( \omega_p^2 = 4\pi n_0 e^2/m_e \) and \( \omega_c = eB_0/m_ec \). \( \omega_\theta = g/2\omega_c \) are respectively the electron plasma, cyclotron and the spin-precession frequency, \( g \) is the electron g-factor and \( |S_0| \) - is an unperturbed spin state.

Expression (74) can be rewritten as

\[
n^2_R (1 + \frac{\omega \mu \tanh(\alpha)}{(\omega \pm \frac{1}{2} \omega_c \pm \frac{\hbar k^2 \tanh(\alpha)}{2m_e})}) = 1 - \frac{\omega_p^2}{\omega (\omega \pm \omega_c)}, \tag{75}
\]

where

\[
\omega_\mu = \frac{g \omega_s^2 \hbar}{8c^2m_e} \tag{76}
\]

- is a frequency which involves the spin correction due to plasma magnetization current. The two first terms in the right-hand side of Eq. (74) appear due to the electron current and represent the well-known classical dispersion relation. The third term appears due to the influence of the magnetization current. The third term leads to a modification of the dispersion relation for transverse plasma oscillations and represents the spin-magnetization contribution as a result of the electron spin perturbation. The spin-modified cyclotron frequency shows the internal spin torque \( \omega_\theta = g\omega_s/2 + k^2\hbar \tanh(\alpha)/2m_e \). The expression (74) generalized the results of article[39] and includes the contribution of the spin torque into the wave process dynamics.

VI. CONCLUSIONS

In this paper we analyzed vorticity and energy evolution caused by the magnetic moment density dynamics in systems of charged 1/2—spin particles. MQHD equations are a consequence of MPSE in which particles interaction is directly taken into account. In our work we consider the Coulomb and Spin—Spin interactions. The system of MQHD equations we have constructed comprises equations of continuity (12), of the momentum balance (13), equation for the energy evolution (45), equation for the magnetization evolution (44), and of the vorticity density dynamics (46). In our studies of wave processes we have used a self-consistent field approximation of the MQHD equations.

The equations we are interested in, determining the system dynamics, are the hydrodynamic equations for the spinning plasma. This equations (43 and 44) have an additional new spin-dependent Spin Stress and Spin Torque which have been derived taken into account thermal fluctuations of the spins about the macroscopic average. Spin-dependent additions appears even in absence of the external magnetic field. The new spin-dependent Spin Stress and Spin Torque represent the exchange interactions inside the plasma.

The main objective of this paper was to construct an appropriate a new generalized vorticity equation (40) for spin quantum plasmas that contains the magnetic, non-magnetic terms and the spin dependent forces being non potential. The turbulent processes in plasmas had been investigated using the vorticity equation[25,26,29] (40). We had derived the vortex dynamic formulation of spinning non-relativistic quantum plasma, using the method of magneto quantum hydrodynamics (MQHD). We had generalized the classical vorticity equation for a spinning quantum fluid plasma and derived the vorticity equation (40) in which particles interactions (Coulomb and Spin—Spin) is directly taken into account. Important that the quantum Madelung potential do not contribute to the vorticity evolution.

We also derived the energy density evolution equation (39) for spin quantum plasmas. It was showed that in the absence of external fields the dynamics of energy density is subject to the quantum Bohm potential and Spin Stress influence. The thermal effects have been taken into account.

Using MQHD equations we analyzed elementary excitations in various physical systems in a linear approximation. At first, we have analyzed elementary excitations in the spin quantum plasma in a linear approximation. We focused on the electron whistlers. Furthermore, the effect of spin torques on the dispersion characteristics of already known wave mode was studied (58). The effect of magnetization current on the dispersion characteristics of whistlers is discussed. Dispersion branch of a novel type that occurs due to new spin torques was discovered (59).

We used the renormalization group method to derive dynamical equations for the vector potential of whistler waves (72). It was showed how contribution of spin quantum effects influence on the amplitudes of eigenmodes. We found the quantum and spin corrections and derived more general system of dynamical equations for the magnetized plasma with two main quantum corrections. One is a quantum force produced by density fluctuations, or Bohm potential and another one which is considered in the equation of motion through the magnetization energy. The Spin Stress and Spin Torque influence was predicted using the whistler waves dispersion relation (74).

Also, we investigate the equation of spin-current and analyze the spin waves produced by spin-current. We show the existence of stable and not damped spin wave along the external magnetic field (64).

1. L. S. Kuzmenkov, S. G. Maksimov, and V. V. Fedoseev, Theor. Math. Phys. 126, 110 (2001).
2. L. S. Kuzmenkov, S. G. Maksimov, and V. V. Fedoseev, Theor. Math. Phys. 126, 212 (2001).
3. G. Brodin and M. Marklund, Phys. Rev. Lett. 98, 025001 (2007).
4. G. Brodin and M. Marklund, New J. Phys. 9, 277 (2007).
5. M. Dvornik et al., J. Phys. A: Math. Gen. 36, 5921 (2003).
6. P. K. Shukla, Phys. Lett. A 369, 312 (2007).
7. K. Shukla, Nature Phys. 5, 92 (2009).
8. E. Madelung, Z. Phys. 40, 322 (1927).
9. L. S. Kuzmenkov and S. G. Maksimov, Theor. Math. Phys. 118, 287 (1999).
10. Jens Zamanian, Gert Brodin, Mattias Marklund, New J. Phys. 11, 073017 (2009).
11. G. Brodin, M. Marklund, Phys. Rev. E, 76, 055403 (2007).
12. T. Takabayasi, Prog. Theor. Phys. 14, 283 (1955).
13. T. Takabayasi and J. P. Vigier, Prog. Theor. Phys. 18, 573 (1957).
14. T. Takabayasi, Prog. Theor. Phys. 70, 1 (1983).
15. T. Takabayasi, Prog. Theor. Phys. Suppl. 4, 2 (1957).
16. P. R. Holland, Phys. Lett. A. 91, 275 (1982).
17. P. R. Holland, Found. Phys. 16, 701 (1987).
18. P. R. Holland, Found. Phys. 22, 1287 (1992).
19. P. R. Holland and J. P. Vigier, Phys. Rev. Lett. 67, 402 (1991).
20. P. R. Holland and P. N. Kyprianidis, Ann. Inst. Henri Poincar 49, 325 (1988).
21. H. A. Kramers, Quantentheorie des Electrons und des Strahlung (Leipzig, 1938) 259.
22. M. I. Trukhanova, Prog. Theor. Exp. Phys. 2013, 11101 (2013).
23. L. S. Kuz’menkov, S. G. Maksimov, and V. V. Fedoseev, Russian Phys. Jour. 43, 718 (2000).
24. P. A. Andreev and L. S. Kuzmenkov, Russ. Phys. J. 50, 1251 (2007).
25. M. I. Trukhanova, Int. J. Mod. Phys. B. 26, 1250004 (2012).
26. P. A. Andreev and L. S. Kuz’menkov, Int. J. Mod. Phys. B. 1250186 (2012).
27. P. A. Andreev and L. S. Kuz’menkov, arXiv:1210.1090.
28. S. M. Mahajan and F. A. Asenjo, Phys. Rev. Lett. 107, 195003 (2011).
29. S. Shelyag, P. Keys, M. Mathioudakis, and F. P. Keenan, arXiv:1010.5604.
30. T. Emonet and F. Moreno-Insertis, Astrophys. J. 492, 804 (1998).
31. A. Guha, M. Rahmani, and G. A. Lawrence, Phys. Rev. E 87, 013020 (2013).
32. R. F. Stein and A. Nordlund, Astrophys. J. 499, 914 (1998).
33. Pavel A. Andreev, Felipe A. Asenjo, Swadesh M. Mahajan, [arXiv:1301.5780].
34. D. Shaikh and P. K. Shukla, arXiv:0903.0906.
35. Pavel A. Andreev, arXiv:1210.1090.
36. P. A. Andreev, L. S. Kuzmenkov, Int. J. Mod. Phys. B 26, 1250186 (2012).
37. Salvatore De Martino, Mariarosaria Falanga, Stephan I. Tzenov, Phys. Plasmas 12, 072308 (2005).
38. A. P. Misra, G. Brodin, M. Marklund, P. K. Shukla, arXiv:1006.4878.