Null structure in a system of quadratic derivative nonlinear Schrödinger equations

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Abstract: We consider the initial value problem for a three-component system of quadratic derivative nonlinear Schrödinger equations in two space dimensions with the masses satisfying the resonance relation. We present a structural condition on the nonlinearity under which small data global existence holds. It is also shown that the solution is asymptotically free. Our proof is based on the commuting vector field method combined with smoothing effects.

Key Words: Derivative nonlinear Schrödinger equation; Null condition; Mass resonance.

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1 Introduction

We consider the initial value problem for the system of nonlinear Schrödinger equations

\[
\left(\frac{i}{2m_j} \Delta \right) u_j(t) = F_j(u, \partial_x u), \quad t > 0, \quad x \in \mathbb{R}^d, \quad j = 1, 2, 3
\]

with

\[
u_j(0, x) = \varphi_j(x), \quad x \in \mathbb{R}^d, \quad j = 1, 2, 3,
\]

where \(i = \sqrt{-1}\), each \(m_j\) is a non-zero real constant, \(\partial_t = \partial/\partial t\), and \(\Delta = \sum_{a=1}^d \partial_{x_a}^2\) with \(\partial_{x_a} = \partial/\partial x_a\) for \(x = (x_a)_{a=1,...,d} \in \mathbb{R}^d\). \(\varphi = (\varphi_j)_{j=1,2,3}\) is a prescribed \(\mathbb{C}^3\)-valued function, and

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\[ u = (u_j(t, x))_{j=1,2,3} \] is a \( \mathbb{C}^3 \)-valued unknown function, while \( \partial_x u = (\partial_{x_a} u_j)_{a=1,...,d; j=1,2,3} \) stands for its first order derivatives with respect to \( x \). The nonlinear term \( F = (F_j(u, \partial_x u))_{j=1,2,3} \) is always assumed to be of the form

\[
\begin{cases}
F_1(u, \partial_x u) = \sum_{|\alpha|,|\beta| \leq 1} C_{1,\alpha,\beta} (\partial^\alpha u_2)(\partial^\beta u_3),
F_2(u, \partial_x u) = \sum_{|\alpha|,|\beta| \leq 1} C_{2,\alpha,\beta} (\partial^\alpha u_3)(\partial^\beta u_1),
F_3(u, \partial_x u) = \sum_{|\alpha|,|\beta| \leq 1} C_{3,\alpha,\beta} (\partial^\alpha u_1)(\partial^\beta u_2)
\end{cases}
\]

with some complex constants \( C_{k,\alpha,\beta} \).

The system (1.1) appears in various physical settings (see, e.g., [7], [8]). If the derivatives are not included in \( F \), this system reads

\[
\begin{cases}
i\partial_t + \frac{1}{2m_1} \Delta u_1 = \overline{u_2} u_3,
i\partial_t + \frac{1}{2m_2} \Delta u_2 = u_3 \overline{u_1},
i\partial_t + \frac{1}{2m_3} \Delta u_3 = u_1 u_2,
\end{cases}
\]

\[ t > 0, \ x \in \mathbb{R}^d. \] (1.4)

Note also that the two-component system

\[
\begin{cases}
i\partial_t + \frac{1}{2m_1} \Delta u_1 = \overline{u_1} u_2,
i\partial_t + \frac{1}{2m_2} \Delta u_2 = u_1^2,
\end{cases}
\]

\[ t > 0, \ x \in \mathbb{R}^d \] (1.5)

can be regarded as a degenerate case of (1.4). In the case of \( d = 2 \), Hayashi–Li–Naumkin [16] obtained a small data global existence result for (1.5) under the relation

\[ m_2 = 2m_1. \] (1.6)

The non-existence of the usual scattering state for (1.5) is also proved under (1.6). Higher dimensional case \( (d \geq 3) \) for (1.5) under the relation (1.6) is considered by Hayashi–Li–Ozawa [18] from the viewpoint of small data scattering. Remark that (1.6) is often called the mass resonance relation, which was first discovered in the study of nonlinear Klein-Gordon systems (see [35], [38], [10], [28], [25], etc., and the references cited therein). The above-mentioned results for the two-component system (1.5) can be generalized to the three-component system (1.4) if the mass resonance relation (1.6) is replaced by

\[ m_3 = m_1 + m_2. \] (1.7)

However, it is non-trivial at all whether or not these can be generalized to the case of (1.1) under (1.7), because the presence of the derivatives in the nonlinearity causes a derivative loss in general. On the other hand, the presence of the derivatives in the nonlinearity sometimes
yields extra-decay property. One of the most successful example will be the null condition introduced by Christodoulou [6] and Klainerman [30] in the case of quadratic quasilinear systems of wave equations in three space dimensions. Our aim in this paper is to reveal analogous null structure in the case of quadratic derivative nonlinear Schrödinger systems. Of our particular interest is the case of \( d = 2 \), because the two-dimensional case for the Schrödinger equations corresponds to the three-dimensional case for wave equations from the viewpoint of the decay rate of solutions to the linearized equations.

In what follows, we concentrate our attention on the three-component Schrödinger system \((1.1)\) with \((1.3)\) under the relation \((1.7)\). Remark that \((1.7)\) enables us to use the Leibniz-type rules for the operator \( J_m(t) = x + \frac{t}{m} \partial_x \) (see Lemma 3.2 below), which play a crucial role in our analysis. It should be noted that single quadratic nonlinear Schrödinger equations are distinguished from the present setting because \((1.7)\) is never satisfied when \( m_1 = m_2 = m_3 \neq 0 \). We refer to [2], [9], [14], [20] and the references cited therein for recent results on small data global existence and asymptotic behavior of solutions to single quadratic nonlinear Schrödinger equations in two space dimensions. We hope that our approach might be generalized to more general settings, but we do not pursue it here to avoid several technical complications.

2 Main Results

First of all, we should formulate the null condition for the Schrödinger case in an appropriate way. One way of understanding the null condition for quasi-linear wave equations is the John-Shatah observation: “The requirement that no plane wave solution is genuinely nonlinear leads to the null condition” (see [23] for the detail; see also [1] for the application to elastic waves). Another way is to see the null condition as the condition to guarantee the cancellation of the main part of the nonlinearity, which is sometimes called the Hörmander test ([22]).

Let us take the second way. For the solution \( u^0_j \) to \( \left(i \partial_t + \frac{1}{2m_j} \Delta\right) u^0_j = 0 \) with \( u^0_j(0) = \varphi_j \), it is known that we have

\[
\partial_x^\alpha u^0_j(t, x) \sim \left(\frac{im_j x}{t}\right)^\alpha \left(\frac{m_j}{it}\right)^\frac{\beta}{2} \hat{\varphi}_j \left(\frac{m_j x}{t}\right) e^{\frac{im_j |x|^2}{2t}} =: \left(\frac{im_j x}{t}\right)^\alpha \psi_j(t, x)e^{\frac{im_j |x|^2}{2t}}
\]

as \( t \to \infty \), where \( \hat{\varphi} \) denotes the Fourier transform of \( \varphi \). Then, under the mass resonance relation \((1.7)\), the direct calculation yields

\[
F_j(u^0, \partial_x u^0) \sim p_j \left(\frac{x}{t}\right) \Psi_j(t, x)e^{\frac{im_j |x|^2}{2t}}, \quad j = 1, 2, 3,
\]
where
\[ p_1(\xi) = \sum_{|\alpha|,|\beta| \leq 1} C_{1,\alpha,\beta} (\overline{im_2 \xi})^\alpha (im_3 \xi)^\beta, \]
\[ p_2(\xi) = \sum_{|\alpha|,|\beta| \leq 1} C_{2,\alpha,\beta} (im_3 \xi)^\alpha (\overline{im_1 \xi})^\beta, \]
\[ p_3(\xi) = \sum_{|\alpha|,|\beta| \leq 1} C_{3,\alpha,\beta} (im_1 \xi)^\alpha (im_2 \xi)^\beta, \]
with \( \Psi_1 = \overline{\psi_2 \psi_3}, \Psi_2 = \psi_3 \overline{\psi_1}, \Psi_3 = \psi_1 \psi_2 \). In order that these main parts of \( F_j \) vanish, we must have \( p_1(\xi) = p_2(\xi) = p_3(\xi) = 0 \).

The above observation naturally leads us to the following definition:

**Definition 2.1.** We say that the nonlinear term \( F = (F_j)_{j=1,2,3} \) of the form (1.3) satisfies the null condition if \( p_1(\xi) = p_2(\xi) = p_3(\xi) = 0 \) for all \( \xi \in \mathbb{R}^d \).

**Remark 2.2.** In the case of one-dimensional cubic nonlinear Schrödinger equations, similar structural conditions have been considered in [37], [26], [19], [27], [36], [21], etc. Analogous consideration for quadratic nonlinear Klein-Gordon systems can be found in [10], [28], [25]. However, as far as the authors know, there are no previous papers which concern quadratic derivative nonlinear Schrödinger systems from this viewpoint.

Now we are going to state our results. For a non-negative integer \( s \), we denote by \( H^s(\mathbb{R}^d) \) the standard Sobolev space:
\[ H^s(\mathbb{R}^d) = \{ \phi ; \partial_x^\alpha \phi(x) \in L^2(\mathbb{R}^d) \text{ for all } |\alpha| \leq s \} \]
equipped with the norm
\[ \| \phi \|_{H^s(\mathbb{R}^d)} = \left( \sum_{|\alpha| \leq s} \| \partial_x^\alpha \phi \|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2}. \]
We also introduce the following function space:
\[ \Sigma^s(\mathbb{R}^d) = \{ \phi ; x^\alpha \phi(x) \in H^{s-|\alpha|}(\mathbb{R}^d) \text{ for all } |\alpha| \leq s \} \]
equipped with the norm
\[ \| \phi \|_{\Sigma^s(\mathbb{R}^d)} = \left( \sum_{|\alpha| \leq s} \| x^\alpha \phi \|_{H^{s-|\alpha|}(\mathbb{R}^d)}^2 \right)^{1/2}. \]

The main result of this paper is the following.
Theorem 2.3. Let $\varphi \in \Sigma^s(\mathbb{R}^2)$ with $s \geq 7$, and $F$ be of the form (1.3). Assume that (1.7) is satisfied and the nonlinear term $F$ satisfies the null condition in the sense of Definition 2.1. Then there exists a positive constant $\varepsilon_1$ such that (1.1)–(1.2) admits a unique global solution $u \in C([0, \infty); \Sigma^s(\mathbb{R}^2))$, provided that $\|\varphi\|_{\Sigma^s(\mathbb{R}^2)} \leq \varepsilon_1$. Moreover, the solution $u(t)$ has a free profile, i.e., there exists $\varphi^+ = (\varphi_j^+)_{j=1,2,3} \in \Sigma^{s-1}(\mathbb{R}^2)$ such that

$$\lim_{t \to \infty} \sum_{j=1}^3 \|u_{mj}(-t)u_j(t) - \varphi_j^+\|_{\Sigma^{s-1}(\mathbb{R}^2)} = 0,$$

where $U_m(t) = \exp \left( \frac{it}{2m} \Delta \right)$.

Remark 2.4. As we have mentioned in the introduction, small data global existence for (1.4) has been proved although the nonlinearity in (1.4) does not satisfy the null condition in the sense of Definition 2.1. However, it is also shown in [10] that the solution to (1.4) does not have the free profile, which should be contrasted with Theorem 2.3. If one tries to show the global existence in the case where the null condition is violated, some long-range effects must be taken into account (see [10, 17], etc., for the related works).

If we do not assume the null condition, our approach does not imply the small data global existence in two-dimensional case. However, we can prove the almost global existence.

Theorem 2.5. Let $\varphi \in \Sigma^s(\mathbb{R}^2)$ with $s \geq 7$, and $F$ be of the form (1.3). Assume that (1.7) is satisfied. Then there exist positive constants $\varepsilon_2$ and $\omega$ such that (1.1)–(1.2) admits a unique solution $u \in C([0, T]; \Sigma^s(\mathbb{R}^2))$ with some $T \geq \exp(\omega/\varepsilon)$, provided that $\|\varphi\|_{\Sigma^s(\mathbb{R}^2)} =: \varepsilon \leq \varepsilon_2$.

In the higher dimensional case ($d \geq 3$), we are able to show the following small data global existence result without assuming the null condition:

Theorem 2.6. Let $d \geq 3$, $\varphi \in \Sigma^s(\mathbb{R}^d)$ with $s \geq 2\left\lceil \frac{d}{2} \right\rceil + 5$, and $F$ be of the form (1.3), where $\left\lceil \frac{d}{2} \right\rceil$ is the largest integer not exceeding $\frac{d}{2}$. Assume that (1.7) is satisfied. Then there exists a positive constant $\varepsilon_3$ such that (1.1)–(1.2) admits a unique global solution $u \in C([0, \infty); \Sigma^s(\mathbb{R}^d))$, provided that $\|\varphi\|_{\Sigma^s(\mathbb{R}^d)} \leq \varepsilon_3$. Moreover, there exists $\varphi^+ = (\varphi_j^+)_{j=1,2,3} \in \Sigma^{s-1}(\mathbb{R}^d)$ such that

$$\lim_{t \to \infty} \sum_{j=1}^3 \|u_{mj}(-t)u_j(t) - \varphi_j^+\|_{\Sigma^{s-1}(\mathbb{R}^d)} = 0.$$

Remark 2.7. In the case of $d = 1$, our approach does not imply any global nor almost global existence results because of insufficiency of expected decay in $t$ of the quadratic nonlinear term (see Remarks 5.4 and 6.2 below for the detail). In the recent paper by Ozawa–Sunagawa [32], a small data blow-up result is obtained for a three-component system of quadratic nonlinear Schrödinger equations in one space dimension. More precisely, it is shown in [32] that we can choose $m_j$, $F_j$ and $\varphi_j$ with $\|\varphi_j\|_{\Sigma^s} = \varepsilon$ such that the corresponding solution to (1.1)–(1.2) blows up in finite time no matter how small $\varepsilon > 0$ is. However, the nonlinear term treated in [32] is different from (1.3).
The rest of this paper is organized as follows. The next section is devoted to preliminaries on basic properties of the operator $J_m$. In Section 4, a characterization of the null condition will be given in terms of some special quadratic forms. As a consequence, extra-decay property of the nonlinear term satisfying the null condition under mass resonance will be made clear. In Section 5, we recall the smoothing property of linear Schrödinger equations. After that, the main theorems will be proved in Section 6 by means of a priori estimates. The Appendix is devoted to the proof of technical lemmas.

In what follows, we denote several positive constants by the same letter $C$, which may vary from one line to another. Also we will frequently use the following convention on implicit constants: The expression $f = \sum_{\lambda \in \Lambda} g_{\lambda}$ means that there exists a family $\{A_{\lambda}\}_{\lambda \in \Lambda}$ of constants such that $f = \sum_{\lambda \in \Lambda} A_{\lambda} g_{\lambda}$. For $z \in \mathbb{R}^d$ (or $z \in \mathbb{R}$), we write $\langle z \rangle = \sqrt{1 + |z|^2}$.

## 3 The operator $J_m$

For a non-zero real constant $m$, we set $L_m = i\partial_t + \frac{1}{2m} \Delta$ and $J_m(t) = (J_{m,a}(t))_{a=1,\ldots,d}$ with $J_{m,a}(t) = x_a + \frac{im}{m} \partial_{x_a}$. Then we can easily check that

$$[L_m, \partial_{x_a}] = [L_m, J_m(t)] = 0, \quad [J_{m,a}(t), \partial_{x_b}] = -\delta_{ab}, \quad [J_{m,a}(t), J_{m,b}(t)] = 0,$$

where $[\cdot, \cdot]$ denotes the commutator of linear operators, and $\delta_{ab}$ is the Kronecker symbol, i.e.,

$$\delta_{ab} = \begin{cases} 1 & (a = b), \\ 0 & (a \neq b). \end{cases}$$

We write $J_m(t)^\alpha = J_{m,1}(t)^{\alpha_1} \cdots J_{m,d}(t)^{\alpha_d}$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{Z}_+)^d$. Here and the later on as well, $\mathbb{Z}_+$ denotes the set of all non-negative integers. We also set

$$\Gamma_m(t) = (\partial_{x_1}, \ldots, \partial_{x_d}, J_{m,1}(t), \ldots, J_{m,d}(t)),\quad \Gamma_m(t)^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} J_{m,1}(t)^{\alpha_1} \cdots J_{m,d}(t)^{\alpha_d}$$

for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{Z}_+)^d$. For simplicity of exposition, we often write $J_{m,a}$, $J_m$ and $\Gamma_m$ for $J_{m,a}(t)$, $J_m(t)$ and $\Gamma_m(t)$, respectively.

The following identities are quite useful:

$$J_{m,a}(t)f = \frac{it}{m} e^{im\theta} \partial_{x_a}(e^{-im\theta} f) = U_m(t)(x_a U_m(-t)f),$$

where $\theta = \frac{|x|^2}{2t}$ and $U_m(t) = \exp(\frac{im}{2m} \Delta)$. Indeed, we can deduce the following lemmas from (3.2).

**Lemma 3.1.** We have

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C_m}{\langle t \rangle^\frac{d}{2}} \sum_{|\alpha| \leq \lceil \frac{d}{2} \rceil + 1} \|\Gamma_m(t)^\alpha f\|_{L^2(\mathbb{R}^d)}$$

for $t \in \mathbb{R}$, where $C_m > 0$ is independent of $t$. 

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Proof. We have only to consider the case of \( t \geq 1 \), because the opposite case easily follows from the standard Sobolev inequality. By the Gagliardo-Nirenberg-Sobolev inequality (see, e.g., [12]), we have
\[
\|f\|_{L^\infty} \leq C \|f\|_{L^2}^{1-\frac{\varpi}{d}} \left( \sum_{|\alpha|=\sigma} \|\partial^\alpha_x (e^{-im\theta} f)\|_{L^2} \right)^{\frac{\varpi}{d}},
\]
where \( \sigma = \left\lfloor \frac{d}{2} \right\rfloor + 1 \). Also we note that (3.2) yields
\[
J_m^a f = \left( \frac{it}{m} \right)^{|\alpha|} e^{im\theta} \partial^\alpha_x (e^{-im\theta} f)
\]
for any \( \alpha \in (\mathbb{Z}_+)^d \). From them it follows that
\[
\|f\|_{L^\infty} = \|e^{-im\theta} f\|_{L^\infty} \leq C \|e^{-im\theta} f\|_{L^2}^{1-\frac{\varpi}{d}} \left( \sum_{|\alpha|=\sigma} \|\partial^\alpha_x (e^{-im\theta} f)\|_{L^2} \right)^{\frac{\varpi}{d}} \leq C \|f\|_{L^2}^{1-\frac{\varpi}{d}} \left( |m|^\varpi t^{-\varpi} \sum_{|\alpha|\leq\sigma} \|J_m^a f\|_{L^2} \right)^{\frac{\varpi}{d}} \leq C |m|^\frac{\varpi}{d} t^{-\frac{\varpi}{d}} \sum_{|\alpha|\leq\sigma} \|J_m^a f\|_{L^2}.
\]
This completes the proof.

Lemma 3.2. Let \( m_1, m_2, m_3 \) be non-zero real constants satisfying (1.7). Then we have
\[
J_{m_1,a}(t) f = \frac{m_2}{m_1} \overline{(J_{m_2,a}(t) f) g} + \frac{m_3}{m_1} \overline{(J_{m_3,a}(t) g) f},
\]
\[
J_{m_2,a}(t) f g = \frac{m_3}{m_2} (J_{m_3,a}(t) f) g - \frac{m_1}{m_2} f (J_{m_1,a}(t) g),
\]
\[
J_{m_3,a}(t) f g = \frac{m_1}{m_3} (J_{m_1,a}(t) f) g + \frac{m_2}{m_3} f (J_{m_2,a}(t) g)
\]
for \( a \in \{1, \ldots, d\} \) and smooth functions \( f, g \).

Proof. Since \( m_1 = -m_2 + m_3 \), it follows from (3.2) that
\[
J_{m_1,a}(t) f = \frac{it}{m_1} e^{im_2 \theta} e^{im_3 \theta} \partial_{x_a} \left( (e^{-im_2 \theta} f)(e^{-im_3 \theta} g) \right)
\]
\[
= \frac{it}{m_1} \left( e^{im_2 \theta} \partial_{x_a} (e^{-im_2 \theta} f) g + \overline{f} e^{im_3 \theta} \partial_{x_a} (e^{-im_3 \theta} g) \right)
\]
\[
= -\frac{m_2}{m_1} \left( \frac{it}{m_2} e^{im_2 \theta} \partial_{x_a} (e^{-im_2 \theta} f) \right) g + \frac{m_3}{m_1} \overline{f} \left( \frac{it}{m_3} e^{im_3 \theta} \partial_{x_a} (e^{-im_3 \theta} g) \right)
\]
\[
= -\frac{m_2}{m_1} (J_{m_2,a} f) g + \frac{m_3}{m_1} \overline{f} (J_{m_3,a} g).
\]
The other two identities can be shown in the same way. \( \square \)
4 Characterization of the null condition

Throughout this section, we fix non-zero real constants $m_1$, $m_2$, $m_3$ satisfying (1.7). Remember that our null condition depends on the masses. Let us introduce the following quadratic forms:

$$G_{1,a}(f, g) = m_2 f (\partial_{x_a} g) + m_3 (\overline{\partial_{x_a} f}) g,$$

$$G_{2,a}(f, g) = m_3 f (\partial_{x_a} g) + m_1 (\overline{\partial_{x_a} f}) g,$$

$$G_{3,a}(f, g) = m_1 f (\partial_{x_a} g) - m_2 (\overline{\partial_{x_a} f}) g$$

and

$$Q_{a,b}(f, g) = (\partial_{x_a} f)(\partial_{x_b} g) - (\partial_{x_b} f)(\partial_{x_a} g)$$

for $a, b \in \{1, \ldots, d\}$. We call $G_{1,a}$, $G_{2,a}$, $G_{3,a}$ the null gauge forms associated with the mass resonance relation (1.7), while $Q_{a,b}$ is called the strong null forms (cf. [37], [26], [15], [13]).

The objectives of this section are twofold: The first is to give an algebraic characterization of the null condition in terms of the null gauge forms and the strong null forms (Lemma 4.1). The second is to investigate properties of these quadratic forms in connection with the operator $J_m$ (Lemmas 4.3 and 4.4). As a consequence, extra-decay structure of $F$ satisfying the null condition under mass resonance will be made clear (Corollary 4.5).

We start with the following lemma.

**Lemma 4.1.** The nonlinear term $F = (F_j)_{j=1,2,3}$ of the form (1.3) satisfies the null condition if and only if it can be written in the following form:

$$F_1(v, \partial_x v) = \sum_{a=1}^{d'} G_{1,a}(v_2, v_3) + \sum_{a,b=1}^{d} Q_{a,b}(v_2, v_3),$$

$$F_2(v, \partial_x v) = \sum_{a=1}^{d} G_{2,a}(v_3, v_1) + \sum_{a,b=1}^{d} Q_{a,b}(v_3, v_1),$$

$$F_3(v, \partial_x v) = \sum_{a=1}^{d} G_{3,a}(v_1, v_2) + \sum_{a,b=1}^{d} Q_{a,b}(v_1, v_2)$$

for any $C^3$-valued smooth functions $v = (v_j)_{j=1,2,3}$.

**Remark 4.2.** An analogous characterization of the null condition for quadratic nonlinear Klein-Gordon systems can be found in Proposition 5.1 of [25]. See also [34], [1] for related works on nonlinear elastic wave equations.

**Proof.** We will consider only $F_1$ because the others can be shown in the same way. We set $o = (0, \ldots, 0) \in (\mathbb{Z}_+)^d$ and $u(a) = (0, \ldots, 0, 1^a, 0, \ldots, 0) \in (\mathbb{Z}_+)^d$ for $a \in \{1, \ldots, d\}$. By the
The following lemma asserts a gain of extra-decay in $t$

Lemma 4.3. (i) follows immediately from the relation $\partial_t F$. Proof. When $a \neq b$.

These identities imply that $F_1$ is of the form \([4.3]\). In order that this polynomial vanish identically on $\mathbb{R}^d$, we must have

\[
C_{1,o,o} = C_{1,t(a),t(a)} = 0 \quad \text{for } a \in \{1, \ldots, d\},
\]

\[
m_3 C_{1,o,t(a)} - m_2 C_{1,t(a),o} = 0 \quad \text{for } a \in \{1, \ldots, d\},
\]

\[
C_{1,t(a),t(b)} + C_{1,t(b),t(a)} = 0 \quad \text{for } a, b \in \{1, \ldots, d\} \text{ with } a \neq b.
\]

These identities imply that $F_1$ is of the form \([4.3]\). It is easy to see that the converse is also true. \(\square\)

Next we turn our attention to properties of the null gauge forms and the strong null forms.

The following lemma asserts a gain of extra-decay in $t$ with the aid of $J_m$.

**Lemma 4.3.** (i) For $a \in \{1, \ldots, d\}$, we have

\[
G_{1,a}(f,g) = \frac{im_3m_d}{t} \left\{ (J_m \alpha f) g - f (J_m \alpha g) \right\},
\]

\[
G_{2,a}(f,g) = \frac{im_3m_1}{t} \left\{ -(J_m \alpha f) \bar{g} + \bar{f} (J_m \alpha g) \right\},
\]

\[
G_{3,a}(f,g) = \frac{im_1m_2}{t} \left\{ (J_m \alpha f) g - f (J_m \alpha g) \right\}.
\]

(ii) For $a, b \in \{1, \ldots, d\}$ and for $m, \mu \in \mathbb{R} \setminus \{0\}$, we have

\[
Q_{ab}(f,g) = \frac{\mu}{it} \left\{ (\partial_{x_a} f)(J_\mu g) - (\partial_{x_b} f)(J_\mu g) \right\} + \frac{m}{it} \left\{ (J_mJ_\mu f)(\partial_{x_a} g) - (J_mJ_\mu f)(\partial_{x_b} g) \right\} + \frac{m\mu}{it^2} \left\{ (J_mJ_\mu f)(J_\mu g) - (J_mJ_\mu f)(J_\mu g) \right\}.
\]

\[
Q_{ab}(\bar{f},\bar{g}) = \frac{\mu}{it} \left\{ (\partial_{x_a} \bar{f})(J_\mu g) - (\partial_{x_b} \bar{f})(J_\mu g) \right\} - \frac{m}{it} \left\{ (J_mJ_\mu f)(\partial_{x_a} g) - (J_mJ_\mu f)(\partial_{x_b} g) \right\} - \frac{m\mu}{it^2} \left\{ (J_mJ_\mu f)(J_\mu g) - (J_mJ_\mu f)(J_\mu g) \right\}.
\]

Proof. (i) follows immediately from the relation $\partial_{x_a} = \frac{im}{t} x_a - \frac{im}{t} J_m a$. To prove (ii), we just substitute $mx_a = mJ_{m,a} - it\partial_{x_a}$ and $\mu x_b = \mu J_{\mu,b} - it\partial_{x_b}$ into the identities

\[
(mx_a f)(\mu x_b g) - (mx_b f)(\mu x_a g) = 0,
\]

\[
(mx_a f)(\mu x_b g) - (mx_b f)(\mu x_a g) = 0.
\]

This completes the proof. \(\square\)
As for the action of the operator $J_m$ on the null gauge forms and the strong null forms, we have the following:

**Lemma 4.4.** Assume that the mass resonance relation \((1.7)\) is satisfied. Then we have

\[
J_{m_1}^\alpha G_{1,a}(f, g) = \sum_{|\beta|+|\gamma| \leq |\alpha|} G_{1,a}(J_{m_2}^\beta f, J_{m_3}^\gamma g), \quad (4.4)
\]

\[
J_{m_2}^\alpha G_{2,a}(f, g) = \sum_{|\beta|+|\gamma| \leq |\alpha|} G_{2,a}(J_{m_3}^\beta f, J_{m_1}^\gamma g), \quad (4.5)
\]

\[
J_{m_3}^\alpha G_{3,a}(f, g) = \sum_{|\beta|+|\gamma| \leq |\alpha|} G_{3,a}(J_{m_1}^\beta f, J_{m_2}^\gamma g), \quad (4.6)
\]

and

\[
J_{m_1}^\alpha Q_{ab}(\overline{f}, g) = \sum_{|\beta|+|\gamma| \leq |\alpha|} Q_{ab}(J_{m_2}^\beta \overline{f}, J_{m_3}^\gamma g)
\]

\[
+ \sum_{c=1}^d \sum_{|\beta|+|\gamma| \leq |\alpha|-1} G_{1,c}(J_{m_2}^\beta f, J_{m_3}^\gamma g), \quad (4.7)
\]

\[
J_{m_2}^\alpha Q_{ab}(f, \overline{g}) = \sum_{|\beta|+|\gamma| \leq |\alpha|} Q_{ab}(J_{m_3}^\beta f, J_{m_1}^\gamma \overline{g})
\]

\[
+ \sum_{c=1}^d \sum_{|\beta|+|\gamma| \leq |\alpha|-1} G_{2,c}(J_{m_3}^\beta f, J_{m_1}^\gamma \overline{g}), \quad (4.8)
\]

\[
J_{m_3}^\alpha Q_{ab}(f, g) = \sum_{|\beta|+|\gamma| \leq |\alpha|} Q_{ab}(J_{m_1}^\beta f, J_{m_2}^\gamma g)
\]

\[
+ \sum_{c=1}^d \sum_{|\beta|+|\gamma| \leq |\alpha|-1} G_{3,c}(J_{m_1}^\beta f, J_{m_2}^\gamma g) \quad (4.9)
\]

for any $\alpha \in (\mathbb{Z}_+)^d$ and $a, b \in \{1, \ldots, d\}$.

**Proof.** We will show only \((4.6)\) and \((4.9)\), because the others can be shown in the same way.
From Lemma 3.2 and the commutation relation (3.1), it follows that
\[
J_{m_3,b}G_{3,a}(f, g) = m_1 \left\{ \frac{m_1}{m_3} (J_{m_1,b} f)(\partial_x g) + \frac{m_2}{m_3} f(J_{m_2,b} \partial_x g) \right\}
- m_2 \left\{ \frac{m_1}{m_3} (J_{m_1,b} \partial_x f) g + \frac{m_2}{m_3} (\partial_x f) (J_{m_2,b} g) \right\}
= \frac{m_1}{m_3} \left\{ m_1 (J_{m_1,b} f)(\partial_x g) - m_2 (J_{m_1,b} \partial_x f) g \right\}
+ \frac{m_2}{m_3} \left\{ m_1 f(J_{m_2,b} \partial_x g) - m_2 (\partial_x f) (J_{m_2,b} g) \right\}
= \frac{m_1}{m_3} \left\{ G_{3,a}(J_{m_1,b} f, g) + m_2 \delta_{ba} f g \right\}
+ \frac{m_2}{m_3} \left\{ G_{3,a}(f, J_{m_2,b} g) - m_1 f \delta_{ba} g \right\}
= \frac{m_1}{m_3} G_{3,a}(J_{m_1,b} f, g) + \frac{m_2}{m_3} G_{3,a}(f, J_{m_2,b} g).
\]
Similarly we have
\[
J_{m_3,c}Q_{ab}(f, g) = \left\{ \frac{m_1}{m_3} (J_{m_1,c} \partial_x f)(\partial_x g) + \frac{m_2}{m_3} (\partial_x f)(J_{m_2,c} \partial_x g) \right\}
- \left\{ \frac{m_1}{m_3} (J_{m_1,c} \partial_x f)(\partial_x g) - (J_{m_1,c} \partial_x f)(\partial_x g) \right\}
+ \frac{m_2}{m_3} \left\{ (\partial_x f)(J_{m_2,c} \partial_x g) - (\partial_x f)(J_{m_2,c} \partial_x g) \right\}
- \frac{m_1}{m_3} \left\{ Q_{ab}(J_{m_1,c} f, g) - \delta_{ca} f (\partial_x g) + \delta_{cb} f (\partial_x g) \right\}
+ \frac{m_2}{m_3} \left\{ Q_{ab}(f, J_{m_2,c} g) - (\partial_x f) \delta_{cb} g + (\partial_x f) \delta_{ca} g \right\}
= \frac{m_1}{m_3} Q_{ab}(J_{m_1,c} f, g) + \frac{m_2}{m_3} Q_{ab}(f, J_{m_2,c} g) + \frac{\delta_{cb}}{m_3} G_{3,a}(f, g)
- \frac{\delta_{ca}}{m_3} G_{3,b}(f, g).
\]
These equalities imply (4.6) and (4.9) with \(|\alpha| = 1\). By induction on \(\alpha\), we arrive at the desired conclusion. \(\square\)

**Corollary 4.5.** Assume that the masses satisfy (1.7) and that the nonlinear term \(F = (F_j)_{j=1,2,3}\) of the form (1.3) satisfies the null condition. Then we have
\[
\sum_{j=1}^{3} \sum_{|\alpha| \leq s} |\Gamma_{m_j}^\alpha F_j(v, \partial_x v)| \leq \frac{C}{(t)} \left( \sum_{j=1}^{3} \sum_{|\alpha| \leq [\frac{3}{2}] + 1} |\Gamma_{m_j}^\alpha v_j| \right) \left( \sum_{j=1}^{3} \sum_{|\alpha| \leq s + 1} |\Gamma_{m_j}^\alpha v_j| \right)
\]
for any \(s \in \mathbb{Z}_+\) and \(v = (v_j)_{j=1,2,3}\), where the positive constant \(C\) is independent of \(t\).  

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Proof. By Lemma 4.1 Lemma 4.4 and the usual Leibniz rule, we have
\[\Gamma_{m_1}^\alpha F_1(v, \partial_x v)\]
\[= \sum_{|\beta| + |\gamma| \leq |\alpha|} \left\{ \sum_{a=1}^d G_{1,a}(\Gamma_{m_2}^\beta v_2, \Gamma_{m_3}^\gamma v_3) + \sum_{a,b=1}^d Q_{ab}(\Gamma_{m_2}^\beta v_2, \Gamma_{m_3}^\gamma v_3) \right\},\]
\[\Gamma_{m_2}^\alpha F_2(v, \partial_x v)\]
\[= \sum_{|\beta| + |\gamma| \leq |\alpha|} \left\{ \sum_{a=1}^d G_{2,a}(\Gamma_{m_3}^\beta v_3, \Gamma_{m_1}^\gamma v_1) + \sum_{a,b=1}^d Q_{ab}(\Gamma_{m_3}^\beta v_3, \Gamma_{m_1}^\gamma v_1) \right\},\]
\[\Gamma_{m_3}^\alpha F_3(v, \partial_x v)\]
\[= \sum_{|\beta| + |\gamma| \leq |\alpha|} \left\{ \sum_{a=1}^d G_{3,a}(\Gamma_{m_1}^\beta v_1, \Gamma_{m_2}^\gamma v_2) + \sum_{a,b=1}^d Q_{ab}(\Gamma_{m_1}^\beta v_1, \Gamma_{m_2}^\gamma v_2) \right\}\]
for any \(\alpha \in (\mathbb{Z}_+)^{2d}\). To obtain the desired estimate, we have only to apply Lemma 4.3 for each terms on the right-hand sides of the above identities. 

5 Smoothing effect

In this section, we recall smoothing properties of the linear Scrödinger equations. As we have mentioned in the introduction, the system (1.1) does not allow the standard energy estimate because the nonlinearity contains the derivatives of the unknown. smoothing effect is a useful device to overcome this difficulty. Among various versions of smoothing effects, we will mainly follow the approach of [2] with a few modifications to fit our purpose (see also [29], [11], [5], etc., for the history and more information on this subject).

Let \(S(\mathbb{R}^d)\) be the space of rapidly decreasing functions, and the Fourier transform of \(f \in S(\mathbb{R}^d)\) be given by
\[\hat{f}(\xi) = \mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-iy \cdot \xi} f(y) dy.\]

For \(a \in \{1, \ldots, d\}\), we denote by \(\mathcal{H}_a\) the Hilbert transform with respect to \(x_a\), that is,
\[(\mathcal{H}_a f)(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} f(x - \tau 1_a) \frac{d\tau}{\tau}\]
for \(f \in S(\mathbb{R}^d)\), where \(1_a = (\delta_{ab})_{b=1,...,d} \in \mathbb{R}^d\). \(\mathcal{H}_a\) can be uniquely extended to a bounded linear operator on \(L^2(\mathbb{R}^d)\) satisfying \(\mathcal{H}_a f(\xi) = -i \text{sgn}(\xi_a) \hat{f}(\xi)\). With a parameter \(\kappa \in (0, 1]\), we put
\[\Lambda_{\kappa,a}(t, x) = \kappa \arctan \left( \frac{x_a}{(t)} \right)\]
and
\[S_{\pm,a}(t; \kappa) f = (\cosh \Lambda_{\kappa,a}(t, \cdot)) f \mp i(\sinh \Lambda_{\kappa,a}(t, \cdot)) \mathcal{H}_a f\]
for \( f \in L^2(\mathbb{R}^d) \) and \( t \in \mathbb{R} \). We define the operators \( S_\pm(t; \kappa) \) by

\[
S_\pm(t; \kappa) = \prod_{a=1}^{d} S_{\pm,a}(t; \kappa)
\]

for \( t \in \mathbb{R} \). Since \( \| (\tanh \Lambda_{\kappa,a}(t, \cdot)) \mathcal{H}_a \|_{L^2 \to L^2} \leq \tanh(\pi/2) < 1 \), both \( S_\pm(t; \kappa) \) and their inverse operators \( S_\pm(t; \kappa)^{-1} \), which are given by

\[
S_\pm(t; \kappa)^{-1}f = \prod_{a=1}^{d} \left( I \mp i \tanh \Lambda_{\kappa,a}(t, \cdot) \mathcal{H}_a \right)^{-1} \left( \frac{f}{\cosh \Lambda_{\kappa,a}(t, \cdot)} \right),
\]

are bounded operators on \( L^2(\mathbb{R}^d) \) with the estimates

\[
\sup_{t \in \mathbb{R}, \kappa \in (0,1]} \| S_\pm(t; \kappa) \|_{L^2 \to L^2} < \infty, \quad \sup_{t \in \mathbb{R}, \kappa \in (0,1]} \| S_\pm(t; \kappa)^{-1} \|_{L^2 \to L^2} < \infty. \tag{5.1}
\]

Roughly speaking, if we put \( S(t) = S_\pm(t; \kappa) \) when \( m > 0 \) and \( S(t) = S_\pm(t; \kappa) \) when \( m < 0 \), then the operator \( S(t) \) is expected to satisfy

\[
[L_m, S(t)] \simeq -\frac{i\kappa}{|m|(t)} \sum_{a=1}^{d} w_a^2(t, \cdot) |S(t)|_{\partial x_a} + \text{remainder},
\]

where \(|\partial x_a|\) is interpreted as the Fourier multiplier and

\[
w_a(t, x) = \left( \frac{x_a}{\langle t \rangle} \right)^{-\frac{1}{2}} = \left( 1 + \frac{x_a^2}{1 + t^2} \right)^{-\frac{1}{2}}.
\]

This enables us to recover a half-derivative of the solution (Lemma 5.1). By combining with auxiliary estimates (Lemma 5.2), we can get rid of the worst contribution of the nonlinear term (Lemma 5.3).

**Lemma 5.1.** Let \( m \in \mathbb{R} \setminus \{0\}, \kappa \in (0,1], t_0 \in \mathbb{R}, \) and \( T > 0 \). Put \( S(t) = S_+(t; \kappa) \) when \( m > 0 \) and \( S(t) = S_-(t; \kappa) \) when \( m < 0 \). We have

\[
\| S(t) f(t) \|^2_{L^2} + \int_{t_0}^{t} \frac{\kappa}{|m|(<t>)} \sum_{a=1}^{d} \left\| w_a(\tau) S(\tau) |\partial x_a|^{\frac{1}{2}} f(\tau) \right\|^2_{L^2} d\tau
\]

\[
\leq \| S(t_0) f(t_0) \|^2_{L^2} + \int_{t_0}^{t} \left( \frac{C_\kappa}{<\tau>} \| S(\tau) f(\tau) \|^2_{L^2} + 2 |\langle S(\tau) f(\tau), S(\tau) L_0 f(\tau) \rangle_{L^2}| \right) d\tau
\]

for \( t \in [t_0, t_0 + T] \) and \( f \in C([t_0, t_0 + T]; H^2(\mathbb{R}^d)) \cap C^1([t_0, t_0 + T]; L^2(\mathbb{R}^d)) \), where the constant \( C \) is independent of \( \kappa \in (0,1], T > 0 \) and \( t_0 \in \mathbb{R} \).
Lemma 5.2. Let $m \in \mathbb{R} \setminus \{0\}$ and $\kappa \in (0,1]$. Let $S, S'$ be either $S_+(t;\kappa)$ or $S_-(t;\kappa)$. We have
\[
\left| \langle S f, S(g \partial_{x_a} h) \rangle_{L^2} \right| + \left| \langle S f, S(g \overline{\partial_{x_a} h}) \rangle_{L^2} \right| \leq \frac{C}{(t)_{\frac{1}{2}}} \left( \sum_{|\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 3} \| \Gamma_{\alpha} \|_{L^2} \right) \left( \| f \|_{L^2} + \| w_a(t) S|\partial_{x_a} |^\frac{1}{2} f \|_{L^2} \right) \times \left( \| h \|_{L^2} + \| w_a(t) S'|\partial_{x_a} |^\frac{1}{2} h \|_{L^2} \right),
\]
where the constant $C$ is independent of $\kappa \in (0,1]$ and $t \in \mathbb{R}$.

We will give a sketch of the proof of these lemmas in the Appendix.

Let $m_j \in \mathbb{R} \setminus \{0\}$ for $j \in \{1,2,3\}$, $t_0 \in \mathbb{R}$ and $T > 0$. We consider the system
\[
L_m v_j = \sum_{k=1}^{3} \sum_{a=1}^{d} (\Phi_{j,a} \partial_{x_a} v_k + \Psi_{j,a} \partial_{x_a} w_k) + \Theta_j, \quad (t, x) \in (t_0, t_0 + T) \times \mathbb{R}^d \quad (5.2)
\]
for $j = 1,2,3$ with given functions $\Phi_{j,a}, \Psi_{j,a}$ and $\Theta_j$. For $s \in \mathbb{Z}^+$ and an interval $I$, we denote by $\mathcal{B}^s(I \times \mathbb{R}^d)$ the space of functions of $C^s$-class on $I \times \mathbb{R}^d$ with bounded derivatives of order up to $s$.

Lemma 5.3. Let $d \geq 2$, and let $\lambda_{jk}, \mu_{jk} \in \mathbb{R} \setminus \{0\}$ be given. Suppose that
\[
\Phi_{j,a}, \Psi_{j,a} \in \mathcal{B}^1([t_0, t_0 + T] \times \mathbb{R}^d) \cap C \left( [t_0, t_0 + T]; \Sigma \left[ \frac{d}{2} \right] + 3(\mathbb{R}^d) \right),
\]
and $\Theta = (\Theta_j)_{j=1,2,3} \in L^1([t_0, t_0 + T]; L^2(\mathbb{R}^d))$. Let $v \in C([t_0, t_0 + T]; L^2(\mathbb{R}^d))$ satisfy (5.2). There is a positive constant $\delta$ such that if
\[
e_{t_0,T} := \sup_{t \in [t_0, t_0 + T]} \sum_{j,k=1}^{3} \sum_{a=1}^{d} \sum_{|\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 3} \left( \| \Gamma_{\lambda_{jk}}(t)^\alpha \Phi_{j,a}(t) \|_{L^2} + \| \Gamma_{\mu_{jk}}(t)^\alpha \Psi_{j,a}(t) \|_{L^2} \right)
\]
\leq \delta,
\]
then we have
\[
\| v(t) \|_{L^2}^2 \leq C \| v(t_0) \|_{L^2}^2 + C \int_{t_0}^{t} \left( \frac{e_{t_0,T}}{\tau} \| v(\tau) \|_{L^2}^2 + \| v(\tau) \|_{L^2} \| \Theta(\tau) \|_{L^2} \right) d\tau
\]
for $t \in [t_0, t_0 + T]$. Here the positive constants $\delta$ and $C$ depend only on $|m_j|, |\lambda_{jk}|$ and $|\mu_{jk}|$. In particular, they are independent of $t_0$ and $T$. 

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Proof. Here we prove the estimate only for the case where
\[ \Theta \in C([t_0, t_0 + T]; L^2(\mathbb{R}^d)) \]
and
\[ v \in C([t_0, t_0 + T]; H^2(\mathbb{R}^d)) \cap C^1([t_0, t_0 + T]; L^2(\mathbb{R}^d)). \]
The general case follows from the standard argument of mollifiers (in time and space variables), where we use the assumption that the coefficients belong to \( B^1([t_0, t_0 + T] \times \mathbb{R}^d) \) (see [31] for instance).

We may assume \( \epsilon_{t_0, T} > 0 \), because the standard energy estimate gives the desired result if \( \epsilon_{t_0, T} = 0 \). For each \( j \in \{1, 2, 3\} \), we put \( S_j(t) = S_+(t; \kappa) \) if \( m_j > 0 \), and \( S_j(t) = S_-(t; \kappa) \) if \( m_j < 0 \), where the parameter \( \kappa \in (0, 1] \) will be fixed later. By Lemma 5.2, we get
\[
2 \sum_{j=1}^{3} \int_{t_0}^{t} \left| \langle S_j(\tau)v_j(\tau), S_j(\tau)L_{m_j}v_j(\tau) \rangle_{L^2} \right|^2 d\tau
\]
\[
\leq C_* \sum_{j=1}^{3} \sum_{a=1}^{d} \frac{\epsilon_{t_0, T}}{|m_j|}\left( \|w_a S_j(\tau)|\partial_{x_a}|^2 v_j(\tau)\|_{L^2}^2 + \|v_j(\tau)\|_{L^2}^2 \right) + 2 \sum_{j=1}^{3} \|S_j(\tau)v_j(\tau)\|_{L^2} \|S_j(\tau)\Theta_j(\tau)\|_{L^2}
\]
with some positive constant \( C_* \) depending only on \(|m_j|, |\lambda_{jk}|\) and \(|\mu_{jk}|\) (in particular, \( C_* \) is independent of \( \kappa \)). Note that we have \( \|v_k(t)\|_{L^2} \leq C\|S_k(t)v_k(t)\|_{L^2} \) by (5.1). We put
\[ \kappa := C_* \epsilon_{t_0, T}, \quad \delta = \frac{1}{C_*}. \]
If \( 0 < \epsilon_{t_0, T} \leq \delta \), we have \( 0 < \kappa \leq 1 \) and it follows from Lemma 5.1 that
\[
\sum_{j=1}^{3} \|S_j(t)v_j(t)\|_{L^2}^2
\]
\[
\leq - \sum_{j=1}^{3} \int_{t_0}^{t} \frac{\kappa}{|m_j|}\sum_{a=1}^{d} \left| w_a S_j(\tau)|\partial_{x_a}|^2 v_j(\tau)\right|^2_{L^2} d\tau + \sum_{j=1}^{3} \|S_j(t_0)v_j(t_0)\|_{L^2}^2
\]
\[
+ \sum_{j=1}^{3} \int_{t_0}^{t} \left( \frac{C\kappa}{|\langle \tau \rangle|}\|S_j(\tau)v_j(\tau)\|_{L^2}^2 + 2 \langle S_j(\tau)v_j(\tau), S_j(\tau)L_{m_j}v_j(\tau) \rangle_{L^2} \right) d\tau
\]
\[
\leq \sum_{j=1}^{3} \int_{t_0}^{t} \frac{C_* \epsilon_{t_0, T} - \kappa}{|m_j|}\sum_{a=1}^{d} \left| w_a S_j(\tau)|\partial_{x_a}|^2 v_j(\tau)\right|^2_{L^2} d\tau + \sum_{j=1}^{3} \|S_j(t_0)v_j(t_0)\|_{L^2}^2
\]
\[
+ \sum_{j=1}^{3} \int_{t_0}^{t} \left( \frac{C\epsilon_{t_0, T}}{|\langle \tau \rangle|}\left( \|S_j(\tau)v_j(\tau)\|_{L^2}^2 + \|v_j(\tau)\|_{L^2}^2 \right) \right) + 2\|S_j(\tau)v_j(\tau)\|_{L^2} \|S_j(\tau)\Theta_j(\tau)\|_{L^2} d\tau
\]
\[
\leq \sum_{j=1}^{3} \|S_j(t_0)v_j(t_0)\|_{L^2}^2 + C \int_{t_0}^{t} \left( \frac{\epsilon_{t_0, T}}{|\langle \tau \rangle|}\|v(\tau)\|_{L^2}^2 + \|v(\tau)\|_{L^2} \|\Theta(\tau)\|_{L^2} \right) d\tau \tag{5.3}
\]
for $t \in [t_0, t_0 + T]$, since $d \geq 2$. Recalling (5.1) again, we obtain the desired result.

**Remark 5.4.** When $d = 1$, we need to assume that $(\sup_{t \in [t_0, t_0 + T]} (t))^{1/2} \epsilon_{t_0, T}$ is sufficiently small in order that the above proof works, because $(\tau)^{-1/2} \geq (\tau)^{-1}$.

### 6 Proof of the theorems

Throughout this section, we always suppose that the mass resonance relation (1.7), as well as (1.3), is satisfied. For $v : \mathbb{R}^d \ni x \mapsto (v_j(x))_{j=1,2,3} \in \mathbb{C}^3$ and $s \in \mathbb{Z}_+$, we set

$$|v(x)|_{\Gamma(t), s} = \left( \sum_{j=1}^{3} \sum_{|\alpha| \leq s} |\Gamma_m(t)^{\alpha} v_j(x)|^2 \right)^{1/2}$$

and

$$\|v\|_{\Gamma(t), s} = \|v(\cdot)|_{\Gamma(t), s}\|_{L^2(\mathbb{R}^d)}.$$

Note that $\|v\|_{\Gamma(t), s}$ and $\sum_{j=1}^{3} \|U_{m_j}(-t)v_j\|_{L^2}$ are equivalent as a norm, where $U_m(t) = \exp \left( \frac{t}{2m} \Delta \right)$ for $m \in \mathbb{R} \setminus \{0\}$. Indeed, (3.2) implies

$$\partial^\alpha J_{m_j}(t)^{\beta} v_j = U_{m_j}(t) \partial^\alpha \left( x^\beta U_{m_j}(-t)v_j \right), \quad \alpha, \beta \in (\mathbb{Z}_+)^d.$$ 

Moreover, for any bounded interval $I(\subset \mathbb{R})$ and $s \in \mathbb{Z}_+$, there is a positive constant $C_{I,s}$ such that

$$\frac{1}{C_{I,s}} \|v\|_{\Gamma(t), s} \leq \|v\|_{\Sigma^s} \leq C_{I,s} \|v\|_{\Gamma(t), s}, \quad t \in I.$$

In fact, given $m \in \mathbb{R} \setminus \{0\}$, there are polynomials $p^{\alpha,m}_{\beta,\gamma}$ and $q^{\alpha,m}_{\beta,\gamma}$ of $t \in \mathbb{R}$ such that we have

$$\Gamma_m(t)^{\alpha} f(x) = \sum_{|\beta| + |\gamma| \leq |\alpha|} p^{\alpha,m}_{\beta,\gamma}(t) \partial^\beta_x (x^\gamma f(x)), \quad \alpha \in (\mathbb{Z}_+)^{2d},$$

and

$$\partial^\gamma_x (x^\gamma f(x)) = \sum_{|\alpha| \leq |\beta| + |\gamma|} q^{\alpha,m}_{\beta,\gamma}(t) \Gamma_m(t)^{\alpha} f(x), \quad \beta, \gamma \in (\mathbb{Z}_+)^d$$

for any sufficiently smooth function $f$. In the sequel, we often write $|u(t, x)|_{\Gamma, s}$ and $\|u(t)\|_{\Gamma, s}$ for $|u(t, x)|_{\Gamma(t), s}$ and $\|u(t, \cdot)\|_{\Gamma(t), s}$, respectively. We also write $u^{(\alpha)} = (u_j^{(\alpha)})_{j=1,2,3}$, $F^{(\alpha)} = (F_j^{(\alpha)})_{j=1,2,3}$ with $u_j^{(\alpha)} = \Gamma_m^\alpha u_j$, $F_j^{(\alpha)} = \Gamma_m^\alpha (F_j(u, \partial_x u))$ for the simplicity of the exposition.

First we state the local existence result for (1.1) (considered for $t > t_0$) with the initial condition

$$u_j(t_0, x) = \psi_j(x), \quad x \in \mathbb{R}^d, \quad j = 1, 2, 3,$$

with some $t_0 \in \mathbb{R}$, instead of (1.2). For the convenience of the readers, we will give an outline of the proof in the Appendix.
Lemma 6.1. Let \( d \geq 2, t_0 \in \mathbb{R} \), and \( B \geq 0 \). For any \( s \geq 2 \left[ \frac{d}{2} \right] + 4 \), there is a positive constant \( \delta_s \), which is independent of \( t_0 \) and \( B \), such that for any \( \psi = (\psi_j)_{j=1,2,3} \in \Sigma^s(\mathbb{R}^d) \) with

\[
\| \psi \|_{\Gamma(t_0), \left[ \frac{d}{2} \right] + 4} \leq \varepsilon < \delta_s,
\]

and \( \| \psi \|_{\Gamma(t_0), 2\left[ \frac{d}{2} \right] + 4} \leq B \), the initial value problem (1.1)-(6.1) possesses a unique solution \( u \in C([t_0, t_0 + T^*_s]; \Sigma^s(\mathbb{R}^d)) \) with

\[
\sup_{t \in [t_0, t_0 + T^*_s]} \| u(t) \|_{\Gamma, \left[ \frac{d}{2} \right] + 4} < \delta_s,
\]

where \( T^*_s = T^*_s(\varepsilon, B) \) is a positive constant which can be determined only by \( s, \varepsilon \) and \( B \), and is independent of \( t_0 \).

Remark 6.2. In the case of \( d = 1 \), this claim is true except that \( T^*_s \) may depend also on \( t_0 \) (see Remark 5.4 as well as the proof of Lemma A.6 in the Appendix). It is unclear whether this exception is just a technical one or not, but this problem is out of the purpose of the present work.

By virtue of this lemma, it suffices to obtain an \textit{a priori} estimate for \( \| u(t) \|_{\Gamma, 2\left[ \frac{d}{2} \right] + 4} \) with a sufficiently small upper bound in order to show the global or almost global existence for (1.1)-(1.2). Let \( u \) be a solution to (1.1)-(1.2) satisfying \( u \in C([0, T^*]; \Sigma^s(\mathbb{R}^d)) \) with some \( T^* > 0 \) and some positive integer \( s(\geq 2 \left[ \frac{d}{2} \right] + 4) \).

We define

\[
E_k(T) = \sup_{t \in [0, T]} \| u(t) \|_{\Gamma, k}
\]

for \( T \in [0, T^*] \) and \( k \in \mathbb{Z}_+ \). Then we have the following:

Lemma 6.3. Let \( d \geq 1, s \geq 2 \left[ \frac{d}{2} \right] + 4 \) and \( T \in [0, T^*] \).

(i) We have

\[
E_{s-1}(T) \leq C_1 \| \varphi \|_{\Sigma^{s-1}} + C_2 E_{s-1}(T) \int_0^T \frac{\| u(t) \|_{\Gamma, s}}{(1 + t)^{\frac{d}{2}}} \, dt,
\]

where the positive constants \( C_1, C_2 \) are independent of \( T \).

(ii) If the null condition is satisfied, then we have

\[
E_{s-1}(T) \leq C_1 \| \varphi \|_{\Sigma^{s-1}} + C_3 E_{s-1}(T) \int_0^T \frac{\| u(t) \|_{\Gamma, s}}{(1 + t)^{1 + \frac{d}{2}}} \, dt,
\]

where the positive constant \( C_3 \) is independent of \( T \).

Proof. By Lemmas 3.3 and 3.2, we have

\[
\sum_{|\alpha| \leq s-1} \| F^{(\alpha)}(t) \|_{L^2} \leq C \| u(t) \|_{\Gamma, \left[ \frac{d}{2} \right] + 4} \| u(t) \|_{\Gamma, s} \leq \frac{C}{\langle t \rangle^{\frac{d}{2}}} \| u(t) \|_{\Gamma, s-1} \| u(t) \|_{\Gamma, s} \leq C E_{s-1}(T) \frac{\| u(t) \|_{\Gamma, s}}{\langle t \rangle^{\frac{d}{2}}}
\]
for $t \in [0, T]$. Here we have used the relation $\left\lfloor \frac{s+1}{2} \right\rfloor + \left\lfloor \frac{d}{2} \right\rfloor + 1 \leq s - 1$ for $s \geq 2 \left\lfloor \frac{d}{2} \right\rfloor + 4$. Also it follows from the commutation relation (3.1) that $L_{m_j}u_j^{(α)} = F_j^{(α)}$. Therefore the standard energy inequality leads to

$$\sum_{|α| \leq s-1} \|u^{(α)}(t)\|_{L^2} \leq \sum_{|α| \leq s-1} \|u^{(α)}(0)\|_{L^2} + \int_0^t \sum_{|α| \leq s-1} \|F^{(α)}(τ)\|_{L^2} dτ$$

$$\leq C\|φ\|_{S^{s-1}} + CE_{s-1}(T) \int_0^T \frac{\|u(τ)\|_{r,s}}{\langle τ \rangle^{1+\frac{d}{2}}} dτ$$

for $t \leq T$. This implies (6.3). If the null condition is satisfied, we can use Corollary 4.5 instead of Lemma 5.2 to obtain

$$\sum_{|α| \leq s-1} \|F^{(α)}(t)\|_{L^2} \leq \frac{C}{|t|} \|u(t)|_{Γ,\left\lfloor \frac{s+1}{2} \right\rfloor}\|_{L^∞} \|u(t)\|_{r,s}$$

$$\leq \frac{C}{|t|^{1+\frac{d}{2}}} \|u(t)\|_{r,s} \|u(t)\|_{r,s}$$

$$\leq CE_{s-1}(T) \frac{\|u(t)\|_{r,s}}{\langle t \rangle^{1+\frac{d}{2}}}$$ (6.5)

for $t \in [0, T]$, which, together with the energy inequality, yields (6.4).

**Lemma 6.4.** Let $d \geq 2$, $s \geq 2 \left\lfloor \frac{d}{2} \right\rfloor + 5$ and $T \in [0, T^*]$. We put $s_0 := \left\lfloor \frac{d}{2} \right\rfloor + 4$. There is a positive constant $\tilde{δ}$ such that if $E_{s_0}(T) \leq \tilde{δ}$, we have

$$\|u(t)\|_{r,s} \leq C_4\|φ\|_{S^s}(1 + t)^{C_5E_{s-1}(T)}$$ (6.6)

for $t \in [0, T]$, where the positive constants $C_4$, $C_5$ and $\tilde{δ}$ are independent of $T$.

**Proof.** For $α \in (\mathbb{Z}_+)^d$, we write $α = (α', α'')$ with $α', α'' \in (\mathbb{Z}_+)^d$. Let $|α| \leq s$. By using Lemma 3.2, we can split $F_j^{(α)}$ into the following form:

$$F_1^{(α)} = \sum_{a=1}^d \left( g_{1,a}^α \partial_{x_a} u_3^{(α)} + h_{1,a}^α \partial_{x_a} u_2^{(α)} \right) + ν_1^α,$$

$$F_2^{(α)} = \sum_{a=1}^d \left( g_{2,a}^α \partial_{x_a} u_1^{(α)} + h_{2,a}^α \partial_{x_a} u_2^{(α)} \right) + ν_2^α,$$

$$F_3^{(α)} = \sum_{a=1}^d \left( g_{3,a}^α \partial_{x_a} u_2^{(α)} + h_{3,a}^α \partial_{x_a} u_1^{(α)} \right) + ν_3^α.$$
where

\[
\begin{align*}
g_{1,a}^\alpha &= \left(\frac{m_3}{m_1}\right)^{\alpha''} \sum_{|\alpha'| \leq 1} C_{1,\alpha',a}(\partial^3 u_2), \\
h_{1,a}^\alpha &= \left(\frac{-m_3}{m_1}\right)^{\alpha''} \sum_{|\beta| \leq 1} C_{1,\beta,a}(\partial^3 u_3), \\
g_{2,a}^\alpha &= \left(\frac{-m_1}{m_2}\right)^{\alpha''} \sum_{|\beta| \leq 1} C_{2,\beta,a}(\partial^3 u_3), \\
h_{2,a}^\alpha &= \left(\frac{m_3}{m_2}\right)^{\alpha''} \sum_{|\beta| \leq 1} C_{2,\beta,a}(\partial^3 u_1), \\
g_{3,a}^\alpha &= \left(\frac{m_2}{m_3}\right)^{\alpha''} \sum_{|\beta| \leq 1} C_{3,\beta,a}(\partial^3 u_1), \\
h_{3,a}^\alpha &= \left(\frac{m_1}{m_3}\right)^{\alpha''} \sum_{|\beta| \leq 1} C_{3,\beta,a}(\partial^3 u_2),
\end{align*}
\]

and \(r_j^\alpha\) satisfies

\[|r_j^\alpha| \leq C|u|\Gamma,|\frac{d}{2}|+4|u|\Gamma,6.\]

From [10] we get \(u \in C^1([0, T]; \Sigma^{d-2}(\mathbb{R}^d))\), which, together with the Sobolev embedding theorem, implies \(g_{j,a}^\alpha, h_{j,a}^\alpha \in B^1([0, T] \times \mathbb{R}^d)\), since we have \(\left[\frac{d}{2}\right] + 2 \leq s - 2\). We set \((\lambda_1, \mu_1) = (-m_2, m_3), (\lambda_2, \mu_2) = (m_3, -m_1)\) and \((\lambda_3, \mu_3) = (m_1, m_2)\). Then we get

\[
\sum_{|a| \leq s} \sum_{j=1}^3 \sum_{a=1}^d \sum_{|\gamma| \leq \left|\frac{d}{2}\right|+3} \left(\|\Gamma_{\lambda_j,\mu_j}^\alpha g_{j,a}(t)\|_{L^2} + \|\Gamma_{\lambda_j,\mu_j}^\alpha h_{j,a}(t)\|_{L^2}\right) \leq C\|u(t)\|_{\Gamma,|\frac{d}{2}|+4}
\]

\[
\leq CE_{s,0}(T) \tag{6.7}
\]

for \(0 \leq t \leq T\). We also have

\[
\|r_j^\alpha(t)\|_{L^2} \leq \frac{C}{\langle t \rangle^{\frac{d}{2}}} \|u(t)\|_{\Gamma,|\frac{d}{2}|+|\frac{d}{2}|+2}\leq CE_{s-1}(T) \frac{\|u(t)\|_{\Gamma,6}}{\langle t \rangle^{\frac{d}{2}}}
\]

for \(0 \leq t \leq T\). Here we have used the relation \(\left[\frac{d}{2}\right] + \left[\frac{d}{2}\right] + 2 \leq s - 1\) for \(s \geq 2 \left[\frac{d}{2}\right] + 5\). In view of [6.7], we can apply Lemma 5.3 if \(E_{s,0}(T)\) is sufficiently small, and we obtain

\[
\|u(t)\|_{\Gamma,6}^2 \leq C\|\varphi\|_{\Sigma^2}^2 + C \int_0^t \left(\frac{E_{s,0}(T)}{\langle \tau \rangle}\|u(\tau)\|_{\Gamma,6}^2 + \sum_{j=1}^3 \|r_j^\alpha(\tau)\|_{L^2}\|u(\tau)\|_{\Gamma,6}\right) d\tau
\]

\[
\leq C\|\varphi\|_{\Sigma^2}^2 + CE_{s-1}(T) \int_0^t \frac{\|u(\tau)\|_{\Gamma,6}^2}{\langle \tau \rangle} d\tau
\]

for \(t \in [0, T]\). By the Gronwall lemma, we obtain the desired estimate. \(\square\)
Now we are in a position to complete the proof of the main theorems.

**Proof of Theorem** 2.3 First we show the global existence for (1.1)–(1.2). Let \( d = 2 \) and \( s \geq 7 = (2 \left\lceil \frac{d}{2} \right\rceil + 5) \big|_{d=2} \). We put \( A = 4C_1 \) and \( \varepsilon_0 = \min \left\{ \frac{\delta}{A}, \frac{1}{2C_5A}, \frac{1}{8C_3C_4} \right\} \), where the constants \( \delta \) and \( C_j \) with \( 1 \leq j \leq 5 \) come from the previous lemmas. We also define

\[
T^{**} = \sup \{ T \in [0, T^*) : E_{s-1}(T) \leq A\varepsilon \}
\]

for \( \varepsilon = \| \varphi \|_{\Sigma^s} \leq \varepsilon_0 \) (note that we have \( T^{**} > 0 \) because \( E_{s-1}(0) \leq C_1\varepsilon \leq A\varepsilon/4 \)). Then it follows from (6.6) and (6.4) that \( T^{**} = T^* \). Indeed, if \( T^{**} < T^* \), we have \( E_{s-1}(T^{**}) \leq A\varepsilon \) and thus \( E_{s_0}(T^{**}) \leq A\varepsilon_0 \leq \tilde{\delta} \). Hence (6.6) implies \( \| u(t) \|_{s, \Gamma^s} \leq C_4 \varepsilon (1 + t)^\frac{3}{2} \), and (6.4) yields

\[
E_{s-1}(T^{**}) \leq \frac{A}{4} \varepsilon + C_3 A\varepsilon \int_0^\infty \frac{C_4 \varepsilon}{(1 + t)^\frac{3}{2}} \, dt \leq \frac{A}{2\varepsilon}.
\]

By the continuity of \([0, T^*) \ni T \mapsto E_{s-1}(T)\), we can choose \( \tilde{T} > T^{**} \) such that \( E_{s-1}(\tilde{T}) \leq A\varepsilon \), which contradicts the definition of \( T^{**} \), and the desired identity is shown. Consequently, we see that

\[
\sup_{t \in [0, T^*)} \| u(t) \|_{s, \Gamma^s} \leq E_{s-1}(T^*) \leq A\varepsilon.
\]

This *a priori* bound and Lemma 6.1 imply the global existence. Moreover, the global solution \( u(t) \) satisfies

\[
\| u(t) \|_{r, s-1} \leq C\varepsilon, \quad \| u(t) \|_{r, s} \leq C\varepsilon (t)^\frac{C\varepsilon}{2}
\]  

(6.8)

for all \( t \geq 0 \), where the second inequality follows from the first one and (6.6).

Next we turn our attention to the asymptotic behavior. We write \( F_j(t) = F_j(u(t), \partial_x u(t)) \) for simplicity. We observe that Corollary 4.5 and (6.8) yield

\[
\sum_{j=1}^3 \| U_{m_j}(-t)F_j(t) \|_{\Sigma^{s-1}} \leq C \| F(t) \|_{r, s-1} \leq \| C\varepsilon_2 \frac{\varepsilon^2}{\langle t \rangle^{2-}\varepsilon} \| \in L^1(0, \infty)
\]

if \( \varepsilon \) is small enough (cf. (5.5)). This allows us to define

\[
\varphi_j^+ := \varphi_j - i \int_0^\infty U_{m_j}(-\tau)F_j(\tau) \, d\tau \in \Sigma^{s-1}(\mathbb{R}^2).
\]

Then it follows from the Duhamel formula that

\[
u_j(t) = U_{m_j}(t)\varphi_j + i \int_0^t U_{m_j}(t-\tau)F_j(\tau) \, d\tau
\]

\[
u_j(t) = U_{m_j}(t) \left( \varphi_j^+ + i \int_t^\infty U_{m_j}(-\tau)F_j(\tau) \, d\tau \right).
\]
Therefore we have
\[ \|U_m(t)u_j(t) - \varphi_j\|_{\Sigma_s} \leq \int_0^\infty \|U_m(\tau)F_j(\tau)\|_{\Sigma_s}d\tau \]
\[ \leq \int_0^\infty \frac{C\varepsilon^2}{(\tau)^2-C\varepsilon}d\tau \]
\[ \leq \frac{C\varepsilon^2}{(t)^1-C\varepsilon}. \]

This completes the proof. \[\square\]

**Proof of Theorem 2.6.** We put \( A = 4C_1 \) and choose \( \omega > 0 \) such that
\[ C_2C_4e^{C_5A\omega} \omega \leq 1/4. \]
Then, it follows from (6.6) and (6.3) with \( d = 2 \) that the inequalities \( E_{s-1}(T) \leq A\varepsilon \) and \( \log(1 + T) \leq \omega/\varepsilon \) imply
\[ E_{s-1}(T) \leq C_1\varepsilon + C_2A\varepsilon \int_0^T \frac{C_4\varepsilon e^{C_5A\varepsilon \log(1 + T)}}{(1 + t)} dt \]
\[ \leq \frac{4}{4} \varepsilon + C_2C_4e^{C_5A\varepsilon} A\varepsilon^2 \log(1 + T) \]
\[ \leq \left( \frac{1}{4} + C_2C_4e^{C_5A\varepsilon} \omega \right) A\varepsilon \leq \frac{A}{2}\varepsilon, \]
provided that \( \varepsilon \) is small enough. This means that we can keep \( E_{s-1}(T^*) \) dominated by \( A\varepsilon \) as long as \( \log(1 + T^*) \leq \omega/\varepsilon \). This *a priori* bound and Lemma 6.1 imply the desired almost global existence. \[\square\]

**Proof of Theorem 2.7.** We skip it since the essential idea is exactly the same as that of the preceding ones. We only point out that \( \langle t \rangle^{-\frac{3}{2}+C\varepsilon} \in L^1(0, \infty) \) if \( d \geq 3 \) and \( \varepsilon \) is small enough. \[\square\]

### A Proof of Lemmas 5.1, 5.2 and 6.1

This Appendix is devoted to the proof of Lemmas 5.1, 5.2 and 6.1 which are essentially not new but less trivial.

#### A.1 Proof of Lemma 5.1

The following lemma, which will be used repeatedly, is a special case of Lemma 2.1 in [2] and we refer the readers to it for the proof.
**Lemma A.1.** We have
\[
\left\| \left[ \partial_{x_a} \frac{i}{2}, g \right] f \right\|_{L^2} + \left\| \left[ \partial_{x_a} \frac{i}{2} \mathcal{H}_a, g \right] f \right\|_{L^2} \leq C \| g \|_{L^\infty} \| \partial_{x_a} g \|_{L^\infty} \| f \|_{L^2}
\]
for \( f \in L^2(\mathbb{R}^d) \), \( g \in W^{1, \infty}(\mathbb{R}^d) \) and \( a \in \{1, \ldots, d\} \).

We put \( L_{m, \nu} = L_m - i \nu \Delta \) for \( m \in \mathbb{R} \ \setminus \{0\} \) and \( \nu \geq 0 \). For the later purpose, we show a slightly generalized version of Lemma 5.1.

**Lemma A.2.** Let \( m \in \mathbb{R} \ \setminus \{0\} \), \( \kappa \in (0, 1) \), \( t_0 \in \mathbb{R} \), \( T > 0 \), and \( \nu \in [0, 1] \). Put \( S(t) = S_+ (t; \kappa) \) when \( m > 0 \) and \( S(t) = S_- (t; \kappa) \) when \( m < 0 \). We have
\[
\| S(t) f(t) \|_{L^2}^2 + \int_{t_0}^t \frac{\kappa}{\| m \| (\tau)} \sum_{a=1}^d \left\| w_a(\tau) S(\tau) | \partial_{x_a} f(\tau) | \right\|_{L^2}^2 \ d\tau 
\leq \| S(t_0) f(t_0) \|_{L^2}^2 + \int_{t_0}^t \left( \frac{C \kappa}{\| m \| (\tau)} \| S(\tau) f(\tau) \|_{L^2}^2 + 2 \| (S(\tau) f(\tau), S(\tau) L_{m, \nu} f(\tau))_{L^2} \| \right) \ d\tau
\]
for \( t \in [t_0, t_0 + T] \) and \( f \in C([t_0, t_0 + T]; H^{2}(\mathbb{R}^d)) \cap C^1([t_0, t_0 + T]; L^2(\mathbb{R}^d)) \), where the constant \( C \) is independent of \( \kappa \in (0, 1) \), \( \nu \in [0, 1] \), \( T > 0 \) and \( t_0 \in \mathbb{R} \).

**Proof of Lemma A.2.** As in the usual energy method, we first compute
\[
\frac{1}{2} \frac{d}{dt} \| Sf \|_{L^2}^2 = \Im \langle L_{m, \nu} Sf, Sf \rangle_{L^2} - \nu \| \nabla Sf \|_{L^2}^2
\leq \Im \langle SL_{m, \nu} f, Sf \rangle_{L^2} + \Im \langle [L_{m, \nu}, S] f, Sf \rangle_{L^2}.
\]
Also the straightforward calculation yields
\[
[L_{m, \nu}, S] = - \left( \frac{i}{|m|} + 2 \text{sgn}(m) \nu \right) \sum_{a=1}^d (\partial_{x_a} \Lambda_{\kappa, a}) S | \partial_{x_a} | + \sum_{a=1}^d R_a,
\]
where
\[
R_a = \text{sgn}(m) (\partial_{x_a} \Lambda_{\kappa, a}) S \mathcal{H}_a + \left( \frac{1}{2m} - i \nu \right) \left\{ (\partial_{x_a} \Lambda_{\kappa, a})^2 S - i \text{sgn}(m) (\partial_{x_a}^2 \Lambda_{\kappa, a}) S \mathcal{H}_a \right\},
\]
Remark that \( | \partial_{x_a} | = \mathcal{H}_a \partial_{x_a} = \partial_{x_a} \mathcal{H}_a \) and \( \mathcal{H}_a^2 = -1 \). Since
\[
| \partial_{x} \Lambda_{\kappa, a} | + | \partial_{x_a} \Lambda_{\kappa, a} |^2 + | \partial_{x_a}^2 \Lambda_{\kappa, a} | \leq \frac{C \kappa}{\| m \| (t)} + \frac{C \kappa^2}{\| m \| (t)^2} \leq \frac{C \kappa}{\| m \| (t)}
\]
and \( | - i \nu + \frac{1}{2m} | \leq 1 + \frac{1}{2|m|} \), we have
\[
\| R_a f \|_{L^2} \leq \frac{C \kappa}{\| m \| (t)} \| f \|_{L^2} \leq \frac{C \kappa}{\| m \| (t)} \| Sf \|_{L^2}.
\]
Finally, integrating this inequality, we obtain the desired result.

We also note that
\[ \partial_{x_a} \Lambda_{\kappa,a} = \frac{\kappa}{\langle t \rangle} w_a^2. \]

Summing up, we obtain
\[
\frac{d}{dt} \| Sf \|_{L^2}^2 + \frac{2\kappa}{|m| \langle t \rangle} \sum_{a=1}^d \text{Re} \langle w_a S | \partial_{x_a} | f, w_a S f \rangle_{L^2} \leq 2 \langle SL_{m,\nu} f, Sf \rangle_{L^2} \bigg| + \frac{C\kappa}{\langle t \rangle} \| Sf \|_{L^2}^2 + \frac{4\kappa}{\langle t \rangle} \sum_{a=1}^d \text{Im} \langle w_a S | \partial_{x_a} | f, w_a S f \rangle_{L^2} \bigg|.
\]

Since we have
\[
w_a S | \partial_{x_a} | f
= \partial_{x_a} (w_a S \mathcal{H}_a f) + [w_a S, \partial_{x_a}] \mathcal{H}_a f
= -|\partial_{x_a}|^{\frac{1}{2}} \left( \left[ |\partial_{x_a}|^{\frac{1}{2}} \mathcal{H}_a, w_a S \right] \mathcal{H}_a f - w_a S | \partial_{x_a} |^{\frac{1}{2}} f \right) + [w_a S, \partial_{x_a}] \mathcal{H}_a f,
\]
we get
\[
\| w_a S | \partial_{x_a} |^{\frac{1}{2}} f \|_{L^2}^2 - \left\langle w_a S | \partial_{x_a} | f, w_a S f \right\rangle_{L^2} = \left\langle w_a S | \partial_{x_a} |^{\frac{1}{2}} f, \left[ w_a S, |\partial_{x_a}|^{\frac{1}{2}} \right] f \right\rangle_{L^2} + \left\langle \left[ |\partial_{x_a}|^{\frac{1}{2}} \mathcal{H}_a, w_a S \right] \mathcal{H}_a f, w_a S | \partial_{x_a} |^{\frac{1}{2}} f \right\rangle_{L^2} + \left\langle \left[ |\partial_{x_a}|^{\frac{1}{2}} \mathcal{H}_a, w_a S \right] \mathcal{H}_a f, \left[ |\partial_{x_a}|^{\frac{1}{2}} \mathcal{H}_a, w_a S \right] f \right\rangle_{L^2} - \left\langle [w_a S, \partial_{x_a}] \mathcal{H}_a f, w_a S f \right\rangle_{L^2}.
\]

Using Lemma [A.1] with \( g = w_a \cosh \Lambda_{\kappa,a}, \ w_a \sinh \Lambda_{\kappa,a}, \) etc., we can show that all the commutators above are bounded operators on \( L^2. \) Hence we obtain
\[
\| w_a S | \partial_{x_a} |^{\frac{1}{2}} f \|_{L^2}^2 - \text{Re} \left\langle w_a S | \partial_{x_a} | f, w_a S f \right\rangle_{L^2} \leq C \| w_a S | \partial_{x_a} |^{\frac{1}{2}} f \|_{L^2} \| f \|_{L^2} + C \| f \|_{L^2}^2 \leq \frac{1}{4} \| w_a S | \partial_{x_a} |^{\frac{1}{2}} f \|_{L^2}^2 + C \| Sf \|_{L^2}^2,
\]
and
\[
|\text{Im} \langle w_a S | \partial_{x_a} | f, w_a S f \rangle_{L^2}| \leq C \| w_a S | \partial_{x_a} |^{\frac{1}{2}} f \|_{L^2} \| f \|_{L^2} + C \| f \|_{L^2}^2 \leq \frac{1}{8} \| w_a S | \partial_{x_a} |^{\frac{1}{2}} f \|_{L^2}^2 + C \| Sf \|_{L^2}^2.
\]

Therefore we get
\[
\frac{d}{dt} \| Sf \|_{L^2}^2 + \frac{\kappa}{|m| \langle t \rangle} \sum_{a=1}^d \| w_a S | \partial_{x_a} |^{\frac{1}{2}} f \|_{L^2}^2 \leq 2 \| SL_{m,\nu} f \|_{L^2}^2 + \frac{C\kappa}{\langle t \rangle} \| Sf \|_{L^2}^2.
\]

Finally, integrating this inequality, we obtain the desired result. \( \square \)
A.2 Proof of Lemma 5.2

We follow the similar line as the proof of Lemma 2.3 in [2]. Denoting by $S^*$ the adjoint operator of $S$, we have

$$
\langle Sf, S(g\partial_x h) \rangle_{L^2} = \langle S^*Sf, (\partial_x w^{-1}_a)w_a h \rangle_{L^2} - \langle |\partial_x|^\frac{1}{2} \left( w^{-1}_a S^*Sf \right), H_a |\partial_x|^\frac{1}{2} (w_a h) \rangle_{L^2}
$$

$$
= I_1 + I_2,
$$

where we have used the identity

$$
\partial_x h = (\partial_x w^{-1}_a)w_a h + w^{-1}_a \partial_x (w_a h)
$$

$$
= (\partial_x w^{-1}_a)w_a h - w^{-1}_a |\partial_x|^\frac{1}{2} H_a |\partial_x|^\frac{1}{2} (w_a h).
$$

It is easy to see $|I_1| \leq C \|g\|_{L^\infty} \|f\|_{L^2} \|h\|_{L^2}$. Since $H_a(S')^{-1}$ is a bounded operator on $L^2$ and $H_a = (H_a(S')^{-1})S'$, we get

$$
\left\| H_a |\partial_x|^\frac{1}{2} (w_a h) \right\|_{L^2} \leq C \left\| S' |\partial_x|^\frac{1}{2} (w_a h) \right\|_{L^2}
$$

$$
\leq C \left( \left\| w_a S' |\partial_x|^\frac{1}{2} h \right\|_{L^2} + \left\| S' |\partial_x|^\frac{1}{2} w_a h \right\|_{L^2} \right).
$$

Using Lemma A.1, one sees that the second term on the right-hand side can be bounded by $C \|h\|_{L^2}$. Writing

$$
|\partial_x|^\frac{1}{2} \left( w^{-1}_a S^*Sf \right) = \left[ |\partial_x|^\frac{1}{2}, w^{-1}_a g \right] S^*Sf + w^{-1}_a g \left[ |\partial_x|^\frac{1}{2}, S^* \right] Sf
$$

$$
+ w^{-2}_a g \left[ w_a, S^* \right] |\partial_x|^\frac{1}{2} Sf + w^{-2}_a g S^* \left( \left[ |\partial_x|^\frac{1}{2}, w_a \right] Sf \right)
$$

$$
+ w^{-2}_a g S^* \left( w_a \left[ |\partial_x|^\frac{1}{2}, S^* \right] f \right) + w^{-2}_a g S^* \left( w_a S |\partial_x|^\frac{1}{2} f \right),
$$

and using Lemma A.1 to estimate the commutators, we obtain

$$
\left\| |\partial_x|^\frac{1}{2} \left( w^{-1}_a S^*Sf \right) \right\|_{L^2} \leq C \left( \left\| \frac{1}{2} \right\|_{L^\infty} + \left\| \frac{1}{2} \right\|_{L^\infty} \right) \left( \left\| f \right\|_{L^2} + \left\| w_a S |\partial_x|^\frac{1}{2} f \right\|_{L^2} \right).
$$

To sum up, we obtain

$$
\left\| \langle Sf, S(g\partial_x h) \rangle \right\|_{L^2} \leq C \left( \left\| \frac{x_a}{(t)} \right\|_{L^\infty}^{2} + \left\| \frac{x_a}{(t)} \right\|_{L^\infty} \left\| \partial_x g \right\|_{L^\infty} \right)
$$

$$
\times \left( \left\| \frac{1}{2} f \right\|_{L^2} + \left\| f \right\|_{L^2} \right) \left( \left\| \frac{1}{2} h \right\|_{L^2} + \left\| h \right\|_{L^2} \right).
$$
Similarly, we have
\[
\left| \langle S_f, S(g\partial_x h) \rangle \right|_{L^2} \leq C \left( \left\| \frac{\langle x_a \rangle}{(t)} \right\|_{L^\infty}^2 + \left\| \frac{\langle x_a \rangle}{(t)} \right\|_{L^\infty} \partial_{x_a} g \right)_{L^\infty} \times \left( \left\| w_a S|\partial_{x_a}|^\frac{m}{2} f \right\|_{L^2} + \| f \|_{L^2} \right) \left( \left\| w_a S|\partial_{x_a}|^\frac{m}{2} h \right\|_{L^2} + \| h \|_{L^2} \right).
\]
On the other hand, it follows from the relation \( \frac{\langle x_a \rangle}{(t)} = \frac{1}{(t)} J_{m,a} - \frac{u}{m(t)} \partial_{x_a} \) and Lemma 6.4, we obtain

\[
\text{Lemma A.3. Suppose that } u \text{ and } \tilde{u} \text{ be solutions to } (1.1) \text{ satisfying}
\]
\[
\begin{align*}
\left\| \frac{\langle x_a \rangle}{(t)} g \right\|_{L^\infty} + \left\| \frac{\langle x_a \rangle}{(t)} \partial_{x_a} g \right\|_{L^\infty} & \leq C \sum_{|\alpha| \leq 2} \| \Gamma_m^\alpha g \|_{L^\infty} \\
& \leq C \frac{1}{(t)^{\frac{s}{2}}} \sum_{|\alpha| \leq \frac{d}{2}+3} \| \Gamma_m^\alpha g \|_{L^2}.
\end{align*}
\]
By piecing them together, we arrive at the desired conclusion. 

**A.3 Proof of Lemma 6.1**

Before we proceed to the proof of Lemma 6.1, we give several lemmas. Concerning the uniqueness, using Lemma 5.3 and going similar lines to the proof of Lemma 6.4, we obtain the following (observe that \( \left[ \frac{s}{2} \right] + \left[ \frac{d}{2} \right] + 2 = s \) if \( s = 2 \left[ \frac{d}{2} \right] + 3 \)).

**Lemma A.3.** Suppose that \( d \geq 2 \). Let \( t_0 \in \mathbb{R} \) and \( T > 0 \). There is a positive constant \( \delta_0 \), which is independent of \( t_0 \) and \( T \), such that if \( u \) and \( \tilde{u} \) be solutions to (1.1) - (6.1) satisfying
\[
u \in C([t_0, t_0 + T]; \Sigma^{2[\frac{d}{2}]+3}(\mathbb{R}^d)) \cap L^\infty((t_0, t_0 + T); \Sigma^{2[\frac{d}{2}]+4}(\mathbb{R}^d)),
\]
and if
\[
\sup_{t_0 \leq t \leq t_0 + T} \| u(t, \cdot) \|_{\Gamma(t), \left[ \frac{d}{2} \right]+4} \leq \delta_0,
\]
then we have \( u(t, x) = \tilde{u}(t, x) \) for all \( (t, x) \in [t_0, t_0 + T] \times \mathbb{R}^d \).

In what follows, for each \( j \in \{1, 2, 3\} \), we put \( S_j = S_+(t; 1) \) if \( m_j > 0 \), and \( S_j = S_-(t; 1) \) if \( m_j < 0 \). Using Lemma A.2 instead of Lemma 5.1, we can easily modify the proof of Lemma 5.3 to obtain the following (compare with (A.1)).

**Lemma A.4.** Let \( d \geq 2 \) and \( \nu \in (0, 1] \). Suppose that \( \Phi_{jk,a}, \Psi_{jk,a} \) and \( \Theta = (\Theta_j)_{j=1,2,3} \) be as in Lemma 5.3. Let \( \epsilon_{t_0,T} \) be defined as in Lemma 5.5 with given constants \( \lambda_{jk}, \mu_{jk} \in \mathbb{R} \setminus \{0\} \). Suppose that \( v = (v_j)_{j=1,2,3} \in C([t_0, t_0 + T]; L^2(\mathbb{R}^d)) \) satisfies
\[
L_{m_j,\nu} v_j = \sum_{k=1}^{3} \sum_{a=1}^{d} \left( \Phi_{jk,a} \partial_{x_a} v_k + \Psi_{jk,a} \partial_{x_a} v_k \right) + \Theta_j.
\]

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Then there is a positive constant $\delta$ such that $e_{t_0,T} \leq \delta$ implies

$$
3 \sum_{j=1}^3 \|S_j(t)v_j(t)\|_{L^2}^2 \leq 3 \sum_{j=1}^3 \|S_j(t_0)v_j(t_0)\|_{L^2}^2 + C \int_{t_0}^t \left( \frac{\|v(\tau)\|_{L^2}^2}{\langle \tau \rangle} + \|v(\tau)\|_{L^2} \|\Theta(\tau)\|_{L^2} \right) d\tau \quad (A.1)
$$

and

$$
\|v(t)\|_{L^2}^2 \leq C \|v(t_0)\|_{L^2}^2 + C \int_{t_0}^t \left( \frac{\|v(\tau)\|_{L^2}^2}{\langle \tau \rangle} + \|v(\tau)\|_{L^2} \|\Theta(\tau)\|_{L^2} \right) d\tau \quad (A.2)
$$

for $t \in [t_0, t_0 + T]$. Here the positive constants $\delta$ and $C$ depend only on $|m_j|$, $|\lambda_{jk}|$ and $|\mu_{jk}|$. In particular, they are independent of $\nu \in (0, 1)$, $t_0 \in \mathbb{R}$, and $T$.

$C_w(I; X)$ denotes the space of $X$-valued weakly continuous functions on an interval $I$, where $X$ is a Hilbert space. As in the argument of Chapter 5 in [33], we can show the following lemma which we will use to obtain the strong continuity of the solution.

**Lemma A.5.** If $f = (f_j)_{j=1,2,3} \in C_w([t_0, t_0 + T]; L^2(\mathbb{R}^d))$ and

$$
\lim_{t \to t_0} \sup_{t \to t_0+0} \sum_{j=1}^3 \|S_j(t)f_j(t)\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{j=1}^3 \|S_j(t_0)f_j(t_0)\|_{L^2(\mathbb{R}^d)}^2,
$$

then we have

$$
\lim_{t \to t_0+0} \|f(t) - f(t_0)\|_{L^2(\mathbb{R}^d)} = 0.
$$

**Proof of Lemma A.5.** We follow the approach of [3], [4], [5] (see also the appendix of [26]). The proof is divided into three steps:

**Step 1.** First we consider the auxiliary problem

$$
\left\{
\begin{array}{ll}
L_{m_j,\nu} u_j^\nu = F_j(u^\nu, \partial_x u^\nu), & t > t_0, \quad x \in \mathbb{R}^d, \quad j = 1, 2, 3, \\
u_j^\nu(t_0, x) = \psi_j(x), & x \in \mathbb{R}^d, \quad j = 1, 2, 3
\end{array}
\right. \quad (A.3)
$$

with $\nu \in (0, 1]$. Due to the stronger smoothing property of $e^{\nu \Delta}$, we can easily solve it in some interval $[t_0, t_0 + T_\nu]$ by the standard contraction mapping principle. More precisely, if $\psi = (\psi_j)_{j=1,2,3} \in \Sigma^s(\mathbb{R}^d)$ with some $s \geq 2 \left[ \frac{d}{2} \right] + 4$, we have the solution

$$
u_j^\nu = (u_j^\nu)_{j=1,2,3} \in C([t_0, t_0 + T_\nu]; \Sigma^s(\mathbb{R}^d))
$$

with $\nabla \Gamma_{m_j}^{\alpha,\nu} u_j^\nu \in L^2((t_0, t_0 + T_\nu) \times \mathbb{R}^d)$ for $j \in \{1, 2, 3\}$ and $\alpha \in (\mathbb{Z}_+)^{2d}$ satisfying $|\alpha| \leq s$, where $T_\nu$ is a positive number depending only on $\nu$ and $\|\psi\|_{\Gamma(t_0, 2\left[\frac{d}{2}\right]+4}$. Observe that we have $\left[ \frac{k+1}{2} \right] + \left[ \frac{d}{2} \right] + 2 \leq k$ for $k \geq 2 \left[ \frac{d}{2} \right] + 4$, which enables us to estimate $\sup_{t \in [t_0, t_0+T]} \|u(t)\|_{\Gamma^{k}}$ for $k \geq 2 \left[ \frac{d}{2} \right] + 5$ inductively if we have the estimate for $\sup_{t \in [t_0, t_0+T]} \|u(t)\|_{\Gamma^{\left[\frac{d}{2}\right]+4}}$. 

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Step 2. Next we show that we can solve (A.3) up to some time which is independent of $\nu$. For a solution $u^\nu$ to (A.3) in $[t_0, t_0 + T]$ and $k \in \mathbb{Z}_+$, we put

$$E_{\nu,k}(T) = \sup_{t \in [t_0, t_0 + T]} \|u^\nu(t)\|_{\Gamma,k}.$$ 

For simplicity of exposition, we write $S_j \Gamma_{m_j}^\alpha u_j(t)$ and $(S_j \Gamma_{m_j}^\alpha)(t_0) \psi_j$ for $S_j(t) \Gamma_{m_j}^\alpha u_j(t)$ and $S_j(t_0) \Gamma_{m_j}^\alpha(t_0) \psi_j$, respectively. Let $\delta_0$ be the constant coming from Lemma A.3. The next lemma is the goal of this step.

**Lemma A.6.** (i) Let $d \geq 2$, $t_0 \in \mathbb{R}$, and $B > 0$. For any $s \geq 2 \left[ \frac{d}{2} \right] + 4$, there is a positive constant $\delta_0(\leq \delta_0)$, which is independent of $t_0$ and $B$, such that for any $\nu \in (0, 1]$ and any $\psi \in \Sigma^s(\mathbb{R}^d)$ with

$$\|\psi\|_{\Gamma(t_0),\left[ \frac{d}{2} \right] + 4} \leq \varepsilon < \delta_s$$

and $\|\psi\|_{\Gamma(t_0),\left[ \frac{d}{2} \right] + 4} \leq B$, the initial value problem (A.3) admits a unique solution $u^\nu \in C([t_0, t_0 + T_\ast]; \Sigma^s(\mathbb{R}^d))$ with $\nabla \Gamma_{m_j}^\alpha u_j^\nu \in L^2((t_0, t_0 + T_\ast) \times \mathbb{R}^d)$ for any $|\alpha| \leq s$ and $j = 1, 2, 3$, where $T_\ast = \sup_{t \in [t_0, \infty)} \|\psi\|_{\Gamma(t_0),\left[ \frac{d}{2} \right] + 4}$. Moreover, there are positive constants $C_s$ and $D_s$, which are independent of $\nu$, such that

$$E_{\nu,\left[ \frac{d}{2} \right] + 4}(T_\ast) \leq \frac{\varepsilon + \delta_s}{2} (< \delta_s),$$

$$E_{\nu,s}(T_\ast) \leq C_s,$$  \hspace{1cm} (A.4)  \hspace{1cm} (A.5)

$$\sum_{j=1}^3 \left\| S_j \Gamma_{m_j}^\alpha u_j^\nu(t) \right\|_{L^2}^2 \leq \sum_{j=1}^3 \left\| (S_j \Gamma_{m_j}^\alpha)(t_0) \psi_j \right\|_{L^2}^2 + D_s(t - t_0), \quad |\alpha| \leq s$$  \hspace{1cm} (A.6)

for $t \in [t_0, t_0 + T_\ast]$.

(ii) If we replace $m_j$ by $-m_j$, and $F_j(u, \partial_x u)$ by $-F_j(u, \partial_x u)$ for $j = 1, 2, 3$ in (A.3), then the assertion of (i) remains true with the same constants.

**Proof.** We put $s_0 = \left[ \frac{d}{2} \right] + 4$ and $s_1 = 2 \left[ \frac{d}{2} \right] + 4$. Let $u^\nu$ be the solution to (A.3) for some $T > 0$. We write $u_j^{\nu,|\alpha|} = \Gamma_{m_j}^\alpha u_j^\nu$ for $|\alpha| \leq s$, and $u_j^{\nu,|\alpha|} = (u_j^{\nu,|\alpha|})_{j=1,2,3}$. Then we have

$$L_{m_j, \nu} u_j^{\nu,|\alpha|} = \Gamma_{m_j}^\alpha \left( F_j(u^\nu, \partial_x u^\nu) \right) - i\nu[\Delta, \Gamma_{m_j}^\alpha] u_j^{\nu,|\alpha|}.$$ 

We modify the proof of Lemma 6.4. Let $g_j^{\nu,|\alpha|}$ and $h_j^{\nu,|\alpha|}$ be given by replacing $u_k$ with $u_k^\nu$ in the definition of $g_j^{\nu,|\alpha|}$ and $h_j^{\nu,|\alpha|}$, respectively. Let $\lambda_j$ and $\mu_j$ be defined as in the proof of Lemma 6.4. Then, similarly to (6.7), we get

$$\sum_{|\alpha| \leq s} \sum_{j=1}^3 \sum_{\lambda=1}^d \sum_{|\gamma| \leq \left[ \frac{d}{2} \right] + 3} \left( \| \Gamma_{\lambda_j} g_j^{\nu,|\alpha|} \|_{L^2} + \| \Gamma_{\mu_j} h_j^{\nu,|\alpha|} \|_{L^2} \right) \leq C_s E_{s_0}(T).$$
with some positive constant $C^*_s$. Let $\delta$ be the constant from Lemma \(A.\delta\) If $C^*_s E_{s_0}(T) \leq \delta$, then (A.2) in Lemma \(A.4\) leads to

$$
\|u''(t)\|^2_{\Gamma,k} \leq A_s \left( \|\psi\|^2_{\Gamma(t_0),k} + \left(1 + E_{\nu,\left[\frac{d}{2}\right] + \left[\frac{d}{2}\right]}(T)\right) \int_{t_0}^t \|u''(\tau)\|^2_{\Gamma,k} d\tau \right) \quad (A.7)
$$

for $0 \leq k \leq s$ with some positive constant $A_s \geq 1$. We set $F_j'' = F_j(u'', \partial_x u'')$. Similarly to the proof of Lemma \(A.\delta\), the standard energy inequality implies

$$
\|u''(t)\|^2_{\Gamma,s_0} \leq \|\psi(t_0)\|^2_{\Gamma(t_0),s_0} + 2 \sum_{j=1}^3 \sum_{|s| \leq s_0} \int_{t_0}^t \left\langle \left( \Gamma^\alpha_{m_j} F_j''(\tau), \Gamma^\alpha_{m_j} u''_j(\tau) \right) \right\rangle_{L^2} d\tau
$$

$$
\leq \|\psi(t_0)\|^2_{\Gamma(t_0),s_0} + C' \int_{t_0}^t \|u''(\tau)\|^2_{\Gamma,s_1} d\tau + C'' \int_{t_0}^t \|u''(\tau)\|^2_{\Gamma,s_0} d\tau \quad (A.8)
$$

with a positive constant $C'$, where we have used $s_0 + 1 \leq s_1$ and $\left[\frac{d}{2}\right] + \left[\frac{d}{2}\right] + 2 \leq s_1$ for $d \geq 2$.

We put

$$
\delta_s := \frac{\delta}{C^*_s}.
$$

Suppose that $\|\psi\|_{\Gamma(t_0),s_0} \leq \varepsilon < \delta_s$ and $\|\psi\|_{\Gamma(t_0),s_1} \leq B$. We set

$$
M_0 := \frac{\varepsilon + \delta_s}{2}, \ M_1 := 2 \left(1 + \sqrt{A_s}\right) B,
$$

and

$$
T^* = \sup \left\{ \tau \in [0, T]; \sup_{t_0 \leq t \leq t_0 + \tau} \|u''(t)\|_{\Gamma(t),s_0} \leq M_0, \sup_{t_0 \leq t \leq t_0 + \tau} \|u''(t)\|_{\Gamma(t),s_1} \leq M_1 \right\}.
$$

Since $\|u''(t_0)\|_{\Gamma,s_0} \leq \varepsilon < M_0$ and $\|u''(t_0)\|_{\Gamma,s_1} \leq B < M_1$, we see that $T^* > 0$ by the continuity of $\|u''(t)\|_{\Gamma,s}$ for $s = s_0, s_1$ with respect to $t$. Observing that $C^*_s M_0 < C^*_s \delta_s \leq \delta$ and $\left[\frac{d}{2}\right] + \left[\frac{d}{2}\right] + 2 = s_1$, it follows from (A.7) with $k = s_1$ that

$$
\|u''(t_0 + T^*)\|^2_{\Gamma,s_1} \leq A_s \left( B^2 + \int_{t_0}^{t_0 + T^*} (1 + M_1) M_1^2 d\tau \right)
$$

$$
\leq \frac{M_1^2}{4} + A_s (1 + M_1) M_1^2 T^*.
$$

\(A.\delta\) yields

$$
\|u''(t_0 + T^*)\|^2_{\Gamma,s_0} \leq \varepsilon^2 + C' \int_{t_0}^{t_0 + T^*} M_0 M_1^2 d\tau \leq \varepsilon^2 + C' M_0 M_1^2 T^*.
$$

Now we put

$$
T^*_s := \min \left\{ \frac{1}{2A_s(1 + M_1)}, \frac{M_0^2 - \varepsilon^2}{2C' M_0 M_1^2} \right\},
$$

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and suppose that $T \leq T_s^*$. Then we have $T^* = T$, because (A.9) and (A.10) imply
\[
\|u''(t_0 + T^*)\|^2_{\Gamma,s_1} \leq \frac{3}{4} M_1^2 < M_1^2, \quad \|u''(t_0 + T^*)\|^2_{\Gamma,s_0} \leq \frac{\varepsilon^2 + M_0^2}{2} < M_0^2.
\]

We have proved that, if $\|\psi\|_{\Gamma(t_0),s_0} \leq \varepsilon < \delta_s$, $\|\psi\|_{\Gamma(t_0),s_1} \leq B$, and $T \leq T_s^*$, then we have
\[
\sup_{t_0 \leq t \leq t_0 + T} \|u''(t)\|_{\Gamma,s_0} \leq M_0(\varepsilon < \delta_s)
\] (A.11) and
\[
\sup_{t_0 \leq t \leq t_0 + T} \|u''(t)\|_{\Gamma,s_1} \leq M_1.
\] (A.12)

Therefore we see that there is a unique solution $u''$ to (A.3) on $[t_0, t_0 + T_s^*] \times \mathbb{R}^d$.

(A.11) implies (A.4). We are going to prove that, for $s_1 \leq k \leq s$, there is a positive constant $C_{s,k}$ such that
\[
\sup_{t_0 \leq t \leq t_0 + T_s^*} \|u''(t)\|_{\Gamma,k} \leq C_{s,k}.
\] (A.13)

If $k = s_1$, (A.13) follows from (A.12). Suppose that (A.13) is true for some $k$ with $s_1 \leq k \leq s - 1$. Noting that we have $\frac{k + 1}{2} + \left(\frac{k}{2}\right)^2 + 2 \leq k$ for $k \geq s_1$, (A.4) and (A.7) yield
\[
\|u''(t)\|^2_{\Gamma,k+1} \leq A_s \left(\|\psi\|^2_{\Gamma(t_0),k+1} + (1 + C_{s,k}) \int_{t_0}^{t} \|u''(\tau)\|^2_{\Gamma,k+1} d\tau\right).
\]

Now the Gronwall Lemma implies
\[
\sup_{t_0 \leq t \leq t_0 + T_s^*} \|u''(t)\|_{\Gamma,k+1} \leq \left(A_s \|\psi\|^2_{\Gamma(t_0),k+1} e^{A_s(1+C_{s,k})T_s^*}\right)^{\frac{1}{2}} =: C_{s,k+1},
\]
which shows (A.13) with $k$ replaced by $k + 1$. (A.13) with $k = s$ shows (A.3).

Finally, in view of (A.4) and (A.5), going similar lines to the proof of (A.7) but using (A.1) instead of (A.2), we obtain (A.6).

Investigating the proof above, we can easily check the assertion (ii). This completes the proof of Lemma A.6. \qed

**Step 3.** We write $I = (t_0, t_0 + T_s^*)$ and $\bar{T} = [t_0, t_0 + T_s^*]$ for simplicity of exposition.

(A.5) implies the boundedness of $u''$ in $L^\infty(I; \Sigma^s)$, and the weak* compactness of $L^\infty(I; \Sigma^s)$ implies that there is a sequence $\{\nu_k\} \subset (0, 1]$ with $\lim_{k \to \infty} \nu_k = 0$ such that $u^{\nu_k}$ converges to some function $u$ weakly* in $L^\infty(I; \Sigma^s)$ as $k \to \infty$. By (A.4), we get
\[
\sup_{t \in I} \|u(t)\|_{\Gamma,s_0} \leq \liminf_{k \to \infty} \sup_{t \in I} \|u^{\nu_k}(t)\|_{\Gamma,s_0} \leq \frac{\varepsilon + \delta_s}{2} (< \delta_s),
\] (A.14)
which is nothing but (A.2).

(A.5) and (A.3) show that $\{u^{\nu_k}\}$ is bounded in $C^{0,1}(\bar{T}; H^{s-2})$, which, together with the boundedness of $\{u^{\nu_k}\}$ in $L^\infty(I; H^s)$, implies that $\{u^{\nu_k}\}$ is bounded in $C^{0, \frac{1}{2}}(\bar{T}; H^{s-1})$. In view of the Rellich-Kondrachov theorem, we obtain by the abstract version of the Ascoli-Arzelà
theorem that \( \{u^\alpha\} \) converges to \( u \) strongly in \( C(\overline{\mathcal{T}}; H^{s-2}_{\text{loc}}) \). This convergence is strong enough to show the convergence of \( F(u^\alpha, \partial_x u^\alpha) \) to \( F(u, \partial_x u) \) in the distribution sense, and thus \( u \) is the local solution to (1.1)–(6.1).

From (1.1), we see that \( \partial_t u \in L^\infty(I; \Sigma^{s-2}) \), which shows \( u \in C(\overline{\mathcal{T}}; \Sigma^{s-2}) \). Moreover, since \( (t, x) \mapsto F_j(u(t, x), \partial_x u(t, x)) \in L^\infty(I; \Sigma^{s-1}) \subset L^1(I; \Sigma^{s-1}) \), the well-posedness in \( \Sigma^{s-1} \) of the linear Schrödinger equations implies \( u \in C(\overline{\mathcal{T}}; \Sigma^{s-1}) \). Since \( u \in L^\infty(I; \Sigma^s) \), by the weak compactness of the Hilbert space \( \Sigma^s(\mathbb{R}^d) \), we see that \( u \in C_w(\overline{\mathcal{T}}; \Sigma^s) \).

What is left to show is the strong continuity of \( u \) as a \( \Sigma^s \)-valued function. Let \( \tau > 0 \) and \( |\alpha| \leq s \). By (A.6), we obtain

\[
\sup_{t \in [t_0, t_0 + \tau]} \sum_{j=1}^{3} \left\| S_j \Gamma_{m_j} u^\alpha_j(t) \right\|_{L^2}^2 \leq \liminf_{k \to \infty} \sup_{t \in [t_0, t_0 + \tau]} \sum_{j=1}^{3} \left\| S_j \Gamma_{m_j} u^\alpha_k(t) \right\|_{L^2}^2 \leq \sum_{j=1}^{3} \left\| S_j \Gamma_{m_j} u_j(t_0) \right\|_{L^2}^2 + D_s \tau.
\]

Hence we get

\[
\limsup_{t \to t_0 + 0} \sum_{j=1}^{3} \left\| S_j \Gamma_{m_j} u_j(t) \right\|_{L^2}^2 \leq \sum_{j=1}^{3} \left\| S_j \Gamma_{m_j} u_j(t_0) \right\|_{L^2}^2,
\]

and Lemma (A.5) yields

\[
\lim_{t \to t_0 + 0} \|u(t) - u(t_0)\|_{\Sigma^s} = 0. \tag{A.15}
\]

We fix \( t_1 \in (t_0, t_0 + T_s^*) \). We consider the problem

\[
\begin{align*}
L_{m_j} v_j &= F_j(v, \partial_x v), \\
v_j(t_1) &= u_j(t_1).
\end{align*}
\]

(A.14) implies that \( \|v(t_1)\|_{H^{s-4}(\mathbb{R}^d)} < \delta_s \). Because of what we have proved so far, (A.16) admits a solution

\[
v \in C([t_1, t_1 + T]; \Sigma^{s-1} \Sigma^s(\mathbb{R}^d)) \cap C_w([t_1, t_1 + T]; \Sigma^s(\mathbb{R}^d))
\]

for some \( T > 0 \), with

\[
\lim_{t \to t_1 + 0} \|v(t) - v(t_1)\|_{\Sigma^s} = 0.
\]

We may assume \( t_1 + T \leq t_0 + T_s^* \). In view of (A.14), we can apply Lemma (A.3) to conclude that \( v(t, x) = u(t, x) \) for \( (t, x) \in [t_1, t_1 + T] \times \mathbb{R}^d \). Hence, recalling (A.15), we find

\[
\lim_{t \to t_1 + 0} \|u(t) - u(t_1)\|_{\Sigma^s} = 0, \quad t_1 \in [t_0, t_0 + T_s^*]. \tag{A.17}
\]

We fix \( t_1 \in (t_0, t_0 + T_s^* \), and consider the problem

\[
\begin{align*}
L_{-m_j} w_j &= -F_j(w, \partial_x w), \\
w_j(-t_1) &= u_j(t_1).
\end{align*}
\]

(A.18)
Note that if we put \( \tilde{u}_j(t, x) = u(-t, x) \) for \(-t_0 - T^*_s \leq t \leq -t_0\), then \( \tilde{u} = (\tilde{u}_j)_{j=1,2,3} \) is a solution to (A.18). Moreover, we have

\[
\Gamma_{-m_j}(-t_1)^\alpha \tilde{u}_j(t, x) = \Gamma_{m_j}(t_1)^\alpha u_j(t_1, x), \quad t \in [-t_0 - T^*_s, -t_0].
\]

In particular, we have \( \Gamma_{-m_j}(-t_1)^\alpha w_j(-t_1, x) = \Gamma_{m_j}(t_1)^\alpha u_j(t_1, x) \). Now, thanks to (A.14) again, (A.18) admits a solution \( w \in C([-t_1, -t_1 + T']; \Sigma^{s-1}(\mathbb{R}^d)) \cap C_w([-t_1, -t_1 + T']; \Sigma^s(\mathbb{R}^d)) \) for some \( T' > 0 \), with

\[
\lim_{t \to -t_1 + 0} \|w(t) - w(-t_1)\|_{\Sigma^s} = 0.
\]

We may assume \(-t_1 + T' \leq -t_0\). As we have observed, (A.14) yields

\[
\sup_{t \in [-t_0 - T^*_s, -t_0]} \left( \sum_{j=1}^3 \sum_{|\alpha| \leq s} \left\| \Gamma_{-m_j}(t)^\alpha \tilde{u}_j(t) \right\|^2_{L^2} \right)^{1/2} < \delta_s \leq \delta_0,
\]

and Lemma [A.3] leads to \( u(t, x) = \tilde{u}(t, x) = u(-t, x) \) for \((t, x) \in [-t_1, -t_1 + T'] \times \mathbb{R}^d\). Hence we get

\[
\lim_{t \to t_1 - 0} \|u(t) - u(t_1)\|_{\Sigma^s} = 0, \quad t_1 \in (t_0, t_0 + T^*_s]. \tag{A.19}
\]

By (A.17) and (A.19), we obtain \( u \in C([\tilde{T}; \Sigma^s]) \) as desired.

Because of (6.2), uniqueness of the solution follows from Lemma [A.3].

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