Infinite pinning

Patrick Dondl\textsuperscript{1} | Martin Jesenko\textsuperscript{2} | Michael Scheutzow\textsuperscript{3}

\textsuperscript{1}Abteilung für Angewandte Mathematik, Albert-Ludwigs-Universität Freiburg, Freiburg, Germany
\textsuperscript{2}Fakulteta za gradbeništvo in geodezijo, Univerza v Ljubljani, Ljubljana, Slovenia
\textsuperscript{3}Institut für Mathematik, MA 7–5, Fakultät II, Technische Universität Berlin, Berlin, Germany

Correspondence
Martin Jesenko, Univerza v Ljubljani, Fakulteta za gradbeništvo in geodezijo, Jamova 2, 1000 Ljubljana, Slovenia.
Email: martin.jesenko@fgg.uni-lj.si

Abstract
In this work, we address the occurrence of infinite pinning in a random medium. We suppose that an initially flat interface starts to move through the medium due to some constant driving force. The medium is assumed to contain random obstacles. We model their positions by a Poisson point process and their strengths are not bounded. We determine a necessary condition on its distribution so that regardless of the driving force the interface gets pinned.

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1 | INTRODUCTION

In this work, we consider models for the propagation of an interface in a random environment, describing, for example, the behavior of a phase or domain boundary. Generally, we assume that such models include a regularizing term, for example, a line-tension for the interface, a driving force, stemming for example from an external magnetic field in the case of a magnetic domain boundary, as well as a random term describing the heterogeneous body in which the interface is propagating. The main question we are addressing here is under which conditions the interface may become stuck (thus preventing a phase transformation of the body) due to its interaction with the random environment, even in the case of a large external driving force.

A canonical version of our model is given by a semilinear parabolic equation of the form

$$\partial_t u(t, x) = \Delta u(t, x) - f(x, u(t, x)) + F,$$

where, in the nomenclature above:

- the interface is given by the graph of the function $u$;
- the linearized line tension results in a Laplacian;

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• the random environment which locally interacts with the interface is modeled by the function $f$ (evaluated at the graph of $u$); and
• the external driving force is given by the constant $F \geq 0$.

In our setting of random heterogeneous media, $f$ is a given quenched, random field, that is, a function also depending on $\omega \in \Omega$, with $\Omega$ being a probability space, but not on time. For notational simplicity, we suppress this dependence in the introduction. As initial condition, we consider $u(0, \cdot) = 0$. In addition to the continuous model above, we also consider a fully discrete model where the Laplacian is replaced with a finite difference Laplacian.

The effect of pinning in these settings, continuous or discrete, means that, even for $F > 0$, the function $u$ remains (at least locally) bounded for all time, that is, the phase transformation is stopped before the entire body can be transformed. In [6], where the same models as those considered here are studied, it is shown that pinning occurs even in a quenched heterogeneous medium that has vanishing spatial average. More information on the setting and the models, including examples from the physics literature, are also presented there.

The basic idea to prove pinning is to find a stationary viscosity supersolution (for the definition and properties see, for example, [3]), that is, a function $v$ (generally depending on $F$ and the random environment) that satisfies

$$\Delta v(x) - f(x, v(x)) + F \leq 0 \quad \text{and} \quad v(x) \geq 0 \quad \text{for all } x \in \mathbb{R}.$$  

By employing an appropriate comparison principle, this immediately implies that the interface stays below the graph of $v$ for all times since this was the case at $t = 0$. In other words, the comparison principle assures that $u(x, t) \leq v(x)$ for all $t \geq 0$; the function $v$ acts as a barrier for the evolving interface. Our main goal in this work is thus to show the existence of such a non-negative stationary supersolution in the setting of our two models.

In the present work, we consider the question whether infinite pinning can occur, that is, whether such a supersolution $v$ can be found, almost surely, for any arbitrarily large driving force $F$, and for which distributions of obstacles this may occur. In the 1+1-dimensional setting, we show that this can happen even if the expectation of the obstacle strength is finite.

The remainder of this article is structured as follows. In Section 2, we study the aforementioned spatially purely discrete variant of the continuous model. The interface is then the graph of a function $u : \mathbb{Z} \to \mathbb{Z}$ and $f : \mathbb{Z}^2 \to \mathbb{R}$ is the function describing the random environment. The notions of the space and time derivative are adapted to the discrete case. This model was studied in, for example, [1], where pinning and depinning results were shown. Here, we focus on results regarding pinning, still in the sense of existence of a stationary, non-negative, discrete supersolution, and give a sufficient condition on $f$ for the existence of such a supersolution for arbitrarily large $F$ in Theorem 2.1.

Section 3 is devoted to the continuous model. In Theorem 3.4, we prove that for a random environment that contains obstacles whose strengths have sufficiently heavy tails, that is, very large values of $f$ occur sufficiently often, again infinite pinning occurs. Finally, in Section 4, our results are briefly summarized and contrasted with work that proves the absence of infinite pinning for less heavy tailed obstacles.
2 | DISCRETE MODEL ON $\mathbb{Z}^2$

In the discrete model on $\mathbb{Z}^2$, the shape of the interface is determined by a function $u_t : \mathbb{Z} \to \mathbb{Z}$, $t \geq 0$, with the initial condition $u_0 \equiv 0^\dagger$. At any time $t$, the function $u$ may jump from its current value $u_t(i)$ only to $u_t(i) \pm 1$ depending on the current jump rate $\lambda$. For $\lambda > 0$, $u$ can only jump to $u_t(i) + 1$ with rate $\lambda$, whereas for $\lambda < 0$, $u$ can only jump to $u_t(i) - 1$ and does so with rate $-\lambda$. The jump rate depends on the local shape of the interface and the obstacle force at the current position. More precisely,

$$\lambda = \Lambda(\Delta_1 u_t(i) - f(i, u_t(i)))$$

where $\Delta_1 u(i) = u(i + 1) + u(i - 1) - 2u(i)$ is the discrete Laplacian, $f(i, j)$ is the obstacle strength at $(i, j) \in \mathbb{Z}^2$ and $\Lambda$ is a strictly increasing and bounded function from $\mathbb{Z}$ to $\mathbb{R}$ which satisfies $\Lambda(0) = 0$. We suppose $f(i, j), i, j \in \mathbb{Z}$, to be independent and identically distributed $\mathbb{N}_0$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The strength 0 simply means the absence of an obstacle. In this article, we only consider the case where $f(i, j) \geq 0$ for all $i, j \in \mathbb{Z}$, that is, the forces are non-negative. For more details and the definition of jump rate, see [6].

We will show the following simple criterion for infinite pinning.

**Theorem 2.1.** Let for all $(i, j) \in \mathbb{Z}^2$ be $f(i, j) \sim X$ for some $\mathbb{N}_0$-valued random variable $X$. If $X$ has unbounded second moment, that is, the expectation $\mathbb{E}(X^2) = \infty$, then infinite pinning occurs.

In [6], see Corollary 2.3, it was shown that if $f(i, j) \sim X$ for all $(i, j)$ for some $\mathbb{Z}$-valued random variable $X$ and if $X$ is such that for any independent random variables $X_0, X_1, \ldots$ having the same distribution as $X$, we have

$$\mathbb{E}(\sup\{X_0, -1 + X_1, -2 + X_2, \ldots\}) > F$$

for some $F \in \mathbb{Z}$, then almost surely there exists a function $v : \mathbb{Z} \to \mathbb{N}_0$ such that $\Delta_1 v(i) \leq f(i, v(i)) - F$. As mentioned in the introduction, this stationary supersolution acts as a barrier for the interface since the jump rate is in every point non-positive. Hence, a sufficient condition for infinite pinning reads

$$\mathbb{E}\left(\sup\{-j + X_j : j \in \mathbb{N}_0\}\right) = \infty.$$

Thus, Theorem 2.1 follows immediately from

**Lemma 2.2.** Let $X$ be a random variable with values in $\mathbb{N}_0$. Then the following statements are equivalent.

- For any independent random variables $X_0, X_1, \ldots$ having the same distribution as $X$, we have $\mathbb{E}(\sup\{-j + X_j : j \in \mathbb{N}_0\}) = \infty$.
- $\mathbb{E}(X^2) = \infty$.

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\dagger Note that, as usual in such discrete settings, the subscript does not indicate a partial derivative. Here $u_t$ is the state of the interface at time $t$.

\ddagger $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. 

Proof. Define

\[ M := \sup\{-j + X_j : j \in \mathbb{N}_0\}. \]

Then, for \( n \in \mathbb{N} \),

\[ \mathbb{P}(M \geq n) \leq \mathbb{P}(X_0 \geq n) + \mathbb{P}(X_1 \geq n + 1) + \mathbb{P}(X_2 \geq n + 2) + \cdots = \sum_{l=0}^{\infty} (l + 1) \mathbb{P}(X = n + l) \]

and therefore

\[ \mathbb{E}(M) = \sum_{n=1}^{\infty} \mathbb{P}(M \geq n) \leq \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \mathbb{P}(X = n + l) = \sum_{k=0}^{\infty} \left( \mathbb{P}(X = k) \sum_{l=0}^{\infty} l \right) < \infty \]

in case \( \mathbb{E}(X^2) < \infty \).

On the other hand, assume \( \mathbb{E}(X^2) = \infty \). If \( \mathbb{E}X = \infty \), then \( \mathbb{E}M \geq \mathbb{E}X_0 = \infty \). Therefore, let us explore the case \( \mathbb{E}X < \infty \).

Let \( \alpha_k := \mathbb{P}(X \geq k) \), \( k \in \mathbb{N} \). Then for each \( n \in \mathbb{N} \)

\[ \mathbb{P}(M \geq n) = 1 - \mathbb{P}(M < n) = 1 - \prod_{k=0}^{\infty} \mathbb{P}(X_k - k < n) = 1 - \prod_{k=0}^{\infty} (1 - \alpha_{n+k}) = 1 - \prod_{k=n}^{\infty} (1 - \alpha_k). \]

Since \( \mathbb{E}X < \infty \), there exists some \( k_0 \in \mathbb{N} \) for which \( \sum_{k=k_0}^{\infty} \alpha_k \leq \frac{1}{2} \). For every \( y \in [0, \frac{1}{2}] \), it holds \( -y \geq \log(1 - y) \geq -2y \). Therefore, for \( n \geq k_0 \),

\[ \mathbb{P}(M \geq n) = 1 - \prod_{k=n}^{\infty} (1 - \alpha_k) \]

\[ \geq \frac{1}{2} \left( 1 - \prod_{k=n}^{\infty} (1 - \alpha_k)^2 \right) \]

\[ = \frac{1}{2} \left( 1 - \exp \left\{ 2 \sum_{k=n}^{\infty} \log (1 - \alpha_k) \right\} \right) \]

\[ \geq \frac{1}{2} \left( 1 - \exp \left\{ -2 \sum_{k=n}^{\infty} \alpha_k \right\} \right) \]

\[ \geq \frac{1}{2} \left( 1 - \exp \left\{ \log \left( 1 - \sum_{k=n}^{\infty} \alpha_k \right) \right\} \right) \]

\[ = \frac{1}{2} \sum_{k=n}^{\infty} \alpha_k, \]
where the first inequality follows since \(1 - u \geq \frac{1}{2}(1 - u^2)\) for \(u \in [0, 1]\). Therefore,

\[
\mathbb{E}M = \sum_{n=1}^{\infty} \mathbb{P}(M \geq n) \geq \sum_{n=k_0}^{\infty} \mathbb{P}(M \geq n) \geq \frac{1}{2} \sum_{n=k_0}^{\infty} \sum_{k=n}^{\infty} \alpha_k = \frac{1}{2} \sum_{k=k_0}^{\infty} (k - k_0 + 1)\alpha_k = \infty
\]

since

\[
\infty = \mathbb{E}(X^2) = \sum_{k=1}^{\infty} (2k - 1)\mathbb{P}(X \geq k) = 2 \sum_{k=1}^{\infty} k\alpha_k - \mathbb{E}X.
\]

\[\Box\]

3 CONTINUOUS MODEL

For the continuous model, we take the setting described in [5] that we now briefly present. We are investigating the behavior of solutions \(u : \mathbb{R}^n \times [0, \infty) \times \Omega \to \mathbb{R}\) of the semi-linear parabolic partial differential equation

\[
\Delta u(t, x, \omega) - f(x, u(t, x, \omega), \omega) + F = \partial_t u(t, x, \omega),
\]

\(u(0, x, \omega) = 0\).

For the sake of simplicity, we suppose that all the obstacles are of the same shape and have the following properties.

- Shape of obstacles is given by the function \(\varphi \in C^\infty_c(\mathbb{R}^n \times \mathbb{R})\) that satisfies

\[
\varphi(x, y) \geq 1 \text{ for } \max\{|x|, |y|\} \leq r_0 \text{ and } \varphi(x, y) = 0 \text{ for } |(x, y)| \geq r_1
\]

for some \(r_0, r_1 > 0\) with \(r_1 > \sqrt{n}r_0\).

- Obstacle positions \(\{(x_i, y_i)\}_{i \in \mathbb{N}}\) are distributed according to an \((n + 1)\)-dimensional Poisson point process on \(\mathbb{R}^n \times [r_1, \infty)\) with intensity \(\lambda > 0\).

- Obstacle strengths \(\{f_i\}_{i \in \mathbb{N}}\) are independent and identically distributed strictly positive random variables \((f_i \sim f_1 \text{ for all } i \in \mathbb{N})\) that are independent of \(\{(x_i, y_i)\}_{i \in \mathbb{N}}\).

Thus, the force of the obstacle field is the random function

\[
f(x, y, \omega) = \sum_i f_i(\omega)\varphi(x - x_i(\omega), y - y_i(\omega)).
\]

Remark 3.1. We note that this specific form is only an example which we focus on for concreteness’ sake and to simplify the exposition. Variants of these obstacle distributions, for example, obstacles centered on lattice sites with random strength, lead to the same results.

Again, our goal is to construct a nonnegative, stationary supersolution for all \(F > 0\). The construction heavily relies on the ideas in [5].

The first step of the construction is to find a stationary supersolution for a related Neumann problem on a ball with an obstacle in its center.
From the requirement of being a supersolution, it is clear that the Laplacian of our constructed function may be positive (if $f$ is sufficiently large there) inside an obstacle and it must be negative (below $-F$) outside, in order to compensate for the driving force $F$.

We thus explicitly construct a function that is radially increasing, has an appropriate Laplacian and becomes flat at the boundary of the ball, see Figure 1.

More precisely, we choose some $0 < r_{\text{in}} < r_{\text{out}}$ and $F_{\text{in}}, F_{\text{out}} > 0$. The parameter $r_{\text{in}}$ determines the cylinder $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : |x| \leq r_{\text{in}}, |y| \leq r_{\text{in}}\}$ where we suppose the obstacle to have the full strength. We are looking for a radially symmetric function $v_{\text{local}}$ that satisfies

$$\Delta v_{\text{local}}(x) \leq \begin{cases} F_{\text{in}}, & |x| < r_{\text{in}}, \\ -F_{\text{out}}, & r_{\text{in}} < |x| < r_{\text{out}}. \end{cases}$$

(2)

It may be non-differentiable on $\partial B_{r_{\text{in}}}$ however, to be a viscosity supersolution, it must fulfill

$$\lim_{x \to x_0} \frac{\partial_r v_{\text{local}}(x)}{x \in B_{r_{\text{in}}}} \geq \lim_{x \to x_0} \frac{\partial_r v_{\text{local}}(x)}{x \notin B_{r_{\text{in}}}} \quad \text{for every } x_0 \in \partial B_{r_{\text{in}}}. $$

We impose

$$|v_{\text{local}}(x)| \leq r_{\text{in}} \quad \text{for } |x| < r_{\text{in}}$$

since the solution must lie within the cylinder modeling an obstacle. Moreover, let

$$\partial_r v_{\text{local}}(x) = 0 \quad \text{for } |x| = r_{\text{out}}$$

and $v_{\text{local}}(x) = \infty$ if $|x| > r_{\text{out}}$. Denote the (blue) part in the inner ball by $v_{\text{in}}$. Let $m \in \mathbb{N}$ be arbitrary. (By choosing an appropriate value for $m$, we will ensure that the supersolution stays in the obstacle.) In contrast to [5], we do not (necessarily) take constant Laplacian but allow for a specific function instead. We choose a radially symmetric function $v_{\text{in}}(x) = \phi(|x|)$ with Laplacian $\Delta v_{\text{in}}(x) = F_{\text{in}} \cdot \left(\frac{|x|}{r_{\text{in}}}\right)^m$ and zero values on $\partial B_{r_{\text{in}}}$, that is,

$$\phi''(r) + \frac{n-1}{r} \phi'(r) = F_{\text{in}} \cdot \left(\frac{r}{r_{\text{in}}}\right)^m \quad \text{with } \phi(r_{\text{in}}) = 0. $$

Then

$$\phi(r) = \frac{F_{\text{in}}}{(m+n)(m+2)r_{\text{in}}^m} \left(r^{m+2} - r_{\text{in}}^{m+2}\right).$$
If we have several translated local supersolutions such that their domains cover $\mathbb{R}^n$, their minimum is a viscosity supersolution of the corresponding equation.

To stay in the cylinder, it must hold that

$$\phi(0) = \frac{-F_{\text{in}}r_{\text{in}}^2}{(m + n)(m + 2)} \geq -r_{\text{in}}.$$

The derivative at the boundary is

$$\phi'(r_{\text{in}}) = \frac{F_{\text{in}}r_{\text{in}}}{m + n}.$$

In the remaining part of the ball, we take the same function $v_{\text{out}}$ as in [5]. Again it is radially symmetric, and therefore we write $v_{\text{out}}(x) = \psi(|x|)$. It should meet $v_{\text{in}}$ at $r_{\text{in}}$, that is, $\psi(r_{\text{in}}) = 0$, and has a zero normal derivative on $\partial B_{r_{\text{out}}}$, therefore, $\psi'(r_{\text{out}}) = 0$. Its Laplacian is simply $-F_{\text{out}}$. Hence,

$$\psi''(r) + \frac{n - 1}{r} \psi'(r) = -F_{\text{out}}.$$

For our construction, the function values of $\psi$ are irrelevant — it is enough to consider its derivative. We obtain

$$\psi'(r) = \frac{F_{\text{out}}}{n} \frac{(r_{\text{out}}^n - r_{\text{in}}^n)}{r_{\text{in}}^{n-1}}.$$

Hence,

$$\psi'(r_{\text{in}}) = \frac{F_{\text{out}}}{nr_{\text{in}}^{n-1}} (r_{\text{out}}^n - r_{\text{in}}^n) = \frac{F_{\text{out}}r_{\text{in}}}{n} \left(\frac{r_{\text{out}}^n}{r_{\text{in}}^n} - 1\right).$$

If we define $v_{\text{local}}$ in such a way, it will be a viscosity supersolution if

$$\phi'(r_{\text{in}}) \geq \psi'(r_{\text{in}}) \quad \text{or} \quad \frac{F_{\text{in}}}{m + n} \geq \frac{F_{\text{out}}}{n} \left(\frac{r_{\text{out}}^n}{r_{\text{in}}^n} - 1\right). \quad (3)$$

If we take the minimum of appropriately translated local supersolutions, as depicted in Figure 2, we obtain a supersolution for a problem with obstacles all having the height-coordinate $y$ equal 0. We call this function the flat supersolution.

Since we are in the random case, we must first localize sufficiently many obstacles, that is, find almost surely (a.s.) an array of them. Its existence will follow from the result from the percolation theory below.
FIGURE 3  Decomposition. We decompose the upper half-space into cuboids $Q_{a,j}$ of volume $l' h$ with stripes of the width $d$ between them. The centers of obstacles should lie in the smaller cuboids of volume $(l - 2r_1) n h$, denoted by the dotted lines

**Theorem 3.2** [4, Theorem 1]. Suppose that to each $z \in \mathbb{Z}^{n+1}$ a state is assigned that can be open or closed. For every point, the probability that it is open is $p \in (0, 1)$ with different points receiving independent states. If $p > 1 - \frac{1}{(2n+2)^2}$, then there exists a.s. a (random) function $L : \mathbb{Z}^n \to \mathbb{N}$ with the following properties.

- For each $a \in \mathbb{Z}^n$, the site $(a, L(a)) \in \mathbb{Z}^{n+1}$ is open.
- For any $a, b \in \mathbb{Z}^n$ with $\|a - b\|_1 = 1$, we have $|L(a) - L(b)| \leq 1$.

Now, for each $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $j \in \mathbb{N}$, define

$$Q_a := \prod_{i=1}^n \left[ a_i(l + d) - \frac{l}{2}, a_i(l + d) + \frac{l}{2} \right], \quad Q_{a,j} := Q_a \times [(j-1)h + r_1, jh + r_1]$$

for some (still arbitrary) $l > 2r_1$ and $h, d > 0$. We say that a point $(a, j)$ is open if there exists an obstacle with strength at least $M$ (where $M$ will be set later) such that its center $(x, y) \in \mathbb{R}^n \times [r_1, \infty)$ lies in $Q_{a,j}$ and fulfills $|x_i - a_i(l + d)| \leq \frac{l}{2} - r_1$ for every $i \in \{1, \ldots, n\}$. (It lies within the part of $Q_{a,j}$ bounded by the dotted line in Figure 3.)

The probability of the event that the center of an obstacle with strength at least $M$ lies in an cuboid of volume $(l - 2r_1)^n h$ is $1 - \exp[-\lambda(l - 2r_1)^n h \mathbb{P}(f_1 \geq M)]$. Therefore, Theorem 3.2 is applicable if $l, h$ and $M$ are such that

$$1 - \exp[-\lambda(l - 2r_1)^n h \mathbb{P}(f_1 \geq M)] > 1 - \frac{1}{(2n+2)^2},$$

or equivalently if

$$l > 2r_1 + \left( \frac{2 \log(2n + 2)}{\lambda h \mathbb{P}(f_1 \geq M)} \right)^{1/n}. \quad (4)$$
Lifting function. This is a smooth function that has the prescribed value in the boxes and for which we have bounds for the first and second derivative outside the boxes.

The plan to construct a supersolution is to locate sufficiently many obstacles and use the flat supersolution adapted to the height of each such obstacle. This is done by adding a lifting function, see also Figure 4.

**Proposition 3.3** [5, Proposition 2.13]. Let $h, l, d > 0$. Suppose $y : \mathbb{Z}^n \to \mathbb{R}$ has the following property:

$$\forall a, b \in \mathbb{Z}^n : \|a - b\|_1 = 1 \Rightarrow |y(a) - y(b)| < 2h.$$

Then there exists $C_1 = C_1(n) > 0$ and a smooth function $v_{\text{lift}} : \mathbb{R}^n \to \mathbb{R}$ such that:

- $v_{\text{lift}}|_{Q_a} = y(a)$ for every $a \in \mathbb{Z}^n$;
- $\|D^2 v_{\text{lift}}\|_\infty \leq C_1 \frac{h}{d^2}$;
- $\|\nabla v_{\text{lift}}\|_\infty \leq C_1 \frac{h}{d}$.

Now we are in the position to state and prove the main result for the continuous case.

**Theorem 3.4.** If

$$\limsup_{z \to \infty} z^{\frac{1}{2} + \frac{1}{n}} \mathbb{P}(f_1 \geq z) = \infty,$$

then infinite pinning occurs.

**Remark 3.5.** In dimension 1, this condition can be fulfilled by distributions with finite expectation, for example, by a Pareto distribution $P(I)(1, \alpha)$ for some $1 < \alpha < \frac{3}{2}$.

**Proof.** Let $K > 0$ be arbitrary. According to our assumption, there exists $M \geq 2K$ such that

$$M^\frac{1}{2} + \frac{1}{n} \mathbb{P}(f_1 \geq M) \geq K.$$

From now on, we will use the notation for the data $r_0, r_1, \lambda$, etc. as introduced in the beginning of the section. By choosing

$$l := 2r_1 + C_0 \sqrt{\frac{M^\frac{1}{2} + \frac{1}{n}}{hK}} \quad \text{with} \quad C_0 := \sqrt{\frac{3 \log(2n + 2)}{\lambda}}$$

(5)
(with $h > 0$ still arbitrary), the condition \eqref{eq:4} is fulfilled. Hence, by Theorem 3.2 there exist an array of percolating open points $(a, ja)_{a \in \mathbb{Z}^n}$. Let us denote the centers of corresponding obstacles by $(x_a, y_a)$.

Now we choose a local supersolution. We take $m \geq \max\{n, 2\}$ such that

$$\frac{Mr_0}{2} \geq (m + n)(m + 2) \geq \frac{Mr_0}{4}$$

(if necessary we take a larger $M$ at the start so that such $m$ exists), and set the radii

$$r_{\text{in}} := r_0, \quad r_{\text{out}} := \sqrt{n \left( l + \frac{d}{2} - r_1 \right)} = \sqrt{n \left( C_0 n^{\frac{1}{n} + \frac{1}{\pi}} \right)^{\frac{1}{hK}} + \frac{d}{2} + r_1}$$

and the forces

$$F_{\text{in}} := \frac{(m + n)(m + 2)}{r_0}, \quad F_{\text{out}} := 2C_1 \frac{h}{d^2}$$

with $C_1$ as Proposition 3.3 and $h, d > 0$ still free. We denote the corresponding solution by $v_{\text{local}}$. In order to be a local supersolution, it must fulfill inequality \eqref{eq:3}, which reads for our choice of $F_{\text{in}}$ and $F_{\text{out}}$

$$m + 2 \geq 2C_1 \frac{r_0}{n} \frac{h}{d^2} \left( -1 + \frac{r_{\text{out}}}{r_0} \right).$$

Since for $m \geq \max\{n, 2\}$ it holds $M \leq \frac{16}{r_0} m^2$, and assuming $d \geq 2r_1$, we may estimate

$$2C_1 \frac{r_{\text{out}}}{n r_{\text{out}}^{n-1}} \leq 2C_1 \frac{n^{n/2} 2^{n-1}}{n r_0^{n-1}} \left( C_0^{\frac{1}{n}} M^{\frac{1}{n} + \frac{1}{\pi}} + \left( \frac{d}{2} + r_1 \right)^n \right) \leq C_2 \left( \frac{m^{1 + \frac{n}{n}}}{{hK}} + d^n \right)$$

with $C_2 = C_2(n, \lambda, r_0)$. Inequality \eqref{eq:7} will surely hold if the following is fulfilled:

$$m \geq C_2 \frac{h}{d^2} \left( \frac{m^{1 + \frac{n}{n}}}{{hK}} + d^n \right) = C_2 \left( \frac{m^{1 + \frac{n}{n}}}{{d^2 K}} + hd^{n-2} \right).$$

Let us simply fulfill this condition by setting both summands at $\frac{m^{1}}{2}$. Hence, we define

$$d := \sqrt{\frac{2C_2}{{K}}} m^{\frac{1}{n}}$$

(where, if necessary, we take appropriate larger $M$ and $m$ so that $d \geq 2r_1$) and

$$h := K^{\frac{n}{2} - 1} (2C_2)^{-\frac{n}{2}} m^{\frac{2}{n}}.$$

Now all the scales are set, and $v_{\text{local}}$ is a local (viscosity) supersolution.
We chose \( r_{\text{out}} \) sufficiently large so that the domain of the flat supersolution

\[
v_{\text{flat}}(x) := \min_{a \in \mathbb{Z}^n} v_{\text{local}}(x - x_a)
\]

is \( \mathbb{R}^n \). We take \( v_{\text{lift}} \) as in Proposition 3.3 with \( y(a) := y_a \) for each \( a \in \mathbb{Z}^n \).

Since \( v_{\text{local}} \) satisfies (2) and \( \|\Delta v_{\text{lift}}\|_{\infty} \leq C_1 \frac{h}{d^2} \), the function \( v := v_{\text{flat}} + v_{\text{lift}} \) satisfies

\[
0 \geq \Delta v(x, \omega) - f(x, v(x, \omega), \omega) + F
\]

for any \( 0 < F \leq \min\{F_{\text{out}} - C_1 \frac{h}{d^2}, M - F_{\text{in}}\} \). By (6),

\[
M - F_{\text{in}} \geq \frac{M}{2} \geq K \quad \text{and} \quad F_{\text{out}} - C_1 \frac{h}{d^2} = C_1 \frac{h}{d^2} = \frac{C_1}{(2C_2)^{\frac{n}{2}} + 1} K^\frac{n}{2}.
\]

To conclude, for a given \( F \), we choose \( K \geq F \) such that also

\[
F \leq \frac{C_1}{(2C_2)^{\frac{n}{2}} + 1} K^\frac{n}{2}
\]

and make the construction above. Hence, for any \( F \) pinning takes place. \( \square \)

**Remark 3.6.** A sufficient condition for Theorem 3.4 to hold is that \( \mathbb{E}(f_1^b) = \infty \) for some \( 0 < b < \frac{1}{2} + \frac{1}{n} \). This is a consequence of the following lemma.

**Lemma 3.7.** Let \( a > b > 0 \) and suppose \( X \geq 0 \) is a real-valued random variable with \( \mathbb{E}(X^b) = \infty \). Then \( \limsup_{z \to \infty} \mathbb{P}(X \geq z)z^a = \infty \).

**Proof.** We may rewrite the assumption as

\[
\infty = \mathbb{E}(X^b) = \int_0^\infty \mathbb{P}(X^b \geq z) \, dz = \int_0^\infty \mathbb{P}(X \geq z^{1/b}) \, dz.
\]

If the claim were wrong, there would exist a \( C < \infty \) such that \( \mathbb{P}(X \geq z)z^a \leq C \) for all \( z \geq 0 \) which would lead to contradiction as

\[
\mathbb{E}(X^b) \leq 1 + \int_1^\infty \mathbb{P}(X \geq z^{1/b}) \, dz \leq 1 + C \int_1^\infty z^{-a/b} \, dz < \infty.
\]

The two conditions are actually very close.

**Lemma 3.8.** Let \( X \geq 0 \) be a real-valued random variable. If for some \( b > 0 \) it holds that \( \limsup_{z \to \infty} z^b \mathbb{P}(X \geq z) = \infty \), then \( \mathbb{E}(X^b) = \infty \).
Proof. For every $M > 0$, there exists a $y > 0$ such that $\mathbb{P}(X \geq y) \geq M y^{-b}$. Therefore,

$$
\mathbb{E}(X^b) = \int_0^\infty \mathbb{P}(X^b \geq z) \, dz = \int_0^\infty \mathbb{P}(X \geq z^{1/b}) \, dz \geq \int_0^{y^b} \mathbb{P}(X \geq z^{1/b}) \, dz \geq y^b \mathbb{P}(X \geq y) \geq y^b M y^{-b} = M.
$$

Since $M$ was arbitrary, the claim follows. \qed

4 | CONCLUSION

We have shown that in our discrete setting in one space dimension, infinite pinning arises if the random, independent obstacles’ strengths have infinite second moment. This should be compared to the continuous case, where infinite pinning in one space dimension occurs if for some $p < \frac{3}{2}$, the $p$th moment of the obstacles’ strengths is infinite. This difference mainly seems to arise due to the fact that not the full obstacle strength can be used in continuum models, as otherwise the pinned interface would not wholly lie inside an obstacle. The question of infinite pinning in discrete models in more than one space dimension remains open. The main question is, however, whether the second moment condition is indeed also necessary for infinite pinning, as the absence of infinite pinning has, to this date, only been shown for models with bounded exponential moment \([1, 2, 7, 8]\).

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