HIERARCHIES FOR RELATIVELY HYPERBOLIC VIRTUALLY SPECIAL GROUPS

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Abstract. Wise’s Quasiconvex Hierarchy Theorem classifying hyperbolic virtually compact special groups in terms of quasiconvex hierarchies played an essential role in Agol’s proof of the Virtual Haken Conjecture. Answering a question of Wise, we construct a new virtual quasiconvex hierarchy for relatively hyperbolic virtually compact special groups. We use this hierarchy to prove a generalization of Wise’s Malnormal Special Quotient Theorem for relatively hyperbolic virtually compact special groups with arbitrary peripheral subgroups.

1. Introduction

1.1. Background, History and Motivation. One of the main goals of cube complex theory is to use the geometry and combinatorial structure of cube complexes to better understand groups. The study of cubical groups has played an important role in recent developments in the theory of hyperbolic 3-manifold groups, particularly in Agol’s proof of the Virtual Haken Conjecture [1].

Virtually special cube complexes, developed by Wise and his collaborators, are central to the theory of cubical groups. A group is called compact virtually special if it is the fundamental group of a compact virtually special cube complex whose hyperplanes satisfy certain combinatorial conditions. Virtually special cube complexes have desirable separability properties that allow certain immersions to be promoted to embeddings using Scott’s Criterion [26].

A construction in [23] due to Sageev provides a method for constructing a group action on a CAT(0) cube complex using “codimension-1-subgroups;” however, in general, this action may not be proper, cocompact, or have a virtually special quotient. For hyperbolic groups, the situation is much clearer: Bergeron and Wise [5] proved that hyperbolic groups with an ample supply of quasiconvex codimension-1-subgroups have a proper and cocompact action on a CAT(0) cube complex. The key to Agol’s proof of the Virtual Haken Conjecture is that any geometric action of a hyperbolic group on a CAT(0) cube complex has virtually special quotient [1, Theorem 1.1]. In the case of closed 3–manifolds, the ample supply of codimension-1-subgroups comes from immersed surfaces constructed by Kahn and Markovic in [19].
Two key ingredients in Agol’s theorem are Wise’s Quasiconvex Hierarchy Theorem and Malnormal Special Quotient theorem (MSQT). Wise’s Quasiconvex Hierarchy Theorem \cite{Wise12} Theorem 13.3] characterizes the virtually special hyperbolic groups in terms of virtual quasiconvex hierarchies.

**Definition 1.1 ([30 Definitions 11.5])**. Let $QVH$ be the smallest class of hyperbolic groups closed under the following operations.

1. $\{1\} \in QVH$.
2. If $G = A *_B C$ and $A, C \in QVH$ and $B$ is finitely generated and quasi-isometrically embedded in $G$ then $G \in QVH$.
3. If $G = A *_B$, $A \in QVH$ and $B$ is finitely generated and quasi-isometrically embedded in $G$, then $G \in QVH$.
4. If $H \leq G$ with $|G: H| < \infty$ and $H \in QVH$, then $G \in QVH$.

In other words, groups in $QVH$ are hyperbolic groups that can be built from the trivial group by taking finite index subgroups or taking amalgamations and HNN extensions over quasiconvex subgroups.

**Theorem 1.2 ([30 Theorem 13.3], Wise’s Quasiconvex Hierarchy Theorem)**. Let $G$ be a hyperbolic group. Then $G \in QVH$ if and only if $G$ is virtually compact special.

As Wise notes in \cite{Wise12} Section 12], the MSQT is an essential ingredient in the proof of the Quasiconvex Hierarchy Theorem.

**Theorem 1.3** (Wise’s Malnormal Special Quotient Theorem \cite{Wise12} Theorem 12.3]). Let $G$ be a hyperbolic and virtually special group with $G$ hyperbolic relative to a collection of subgroups $\{P_1, \ldots, P_m\}$. Then there exist finite index subgroups $\tilde{P}_1 \leq P_i$ such that if $\tilde{G} = G(N_1, \ldots, N_m)$ is any peripherally finite Dehn filling with $N_i \leq \tilde{P}_i$, then $\tilde{G}$ is hyperbolic and virtually special.

The MSQT together with virtually special amalgamation criteria from \cite{HsuWise11} and \cite{HsuWise12} are used to prove Theorem 1.2.

For relatively hyperbolic groups, much less is known. Wise’s methods from \cite{Wise12} extend to more general situations than hyperbolic groups. In particular, many of the methods for hyperbolic groups extend to finite volume hyperbolic 3-manifolds. Hsu and Wise \cite{HsuWise12} also proved a special combination result for relatively hyperbolic groups albeit with much more restrictive hypotheses.

The main goal of this paper is to prove relatively hyperbolic analogs of important ingredients in the proof of Theorem 1.2. The first result answers \cite{Wise12} Question 16.31] posed by Wise:

**Theorem 1.** Let $(G, \mathcal{P})$ be a relatively hyperbolic pair and let $G$ be a virtually compact special group. Then there exists a finite index subgroup $G_0 \leq G$ and an
induced relatively hyperbolic pair \((G_0, P_0)\) so that \(G_0\) has a quasiconvex, malnormal and fully \(P_0\)-elliptic hierarchy terminating in groups isomorphic to elements of \(P_0\).

Proving that the hierarchy is not only quasiconvex and **malnormal** but also **fully \(P_0\)-elliptic** is a way of ensuring that the hierarchy is compatible with the relatively hyperbolic structure on \(G\) and allows for the use of relatively hyperbolic Dehn filling arguments.

Theorem \(1\) will be used to prove a relatively hyperbolic generalization of the MSQT using relatively hyperbolic Dehn filling techniques similar to those used in \(3\):

**Theorem 2.** Let \((G, P)\) be a relatively hyperbolic pair with \(P = \{P_1, \ldots, P_m\}\). If \(G\) is virtually compact special, then there exist subgroups \(\{P_i \triangleleft \hat{P}_i\}\) where \(\hat{P}_i\) is finite index in \(P_i\) such that if \(\hat{G} = G(N_1, \ldots, N_m)\) is any peripherally finite filling with \(N_i \triangleleft \hat{P}_i\), then \(\hat{G}\) is hyperbolic and virtually special.

While Wise proved a generalized relatively hyperbolic version of the MSQT in \([30, \text{Lemma 16.13}]\) for relatively hyperbolic groups with virtually abelian peripherals, Theorem \(2\) holds for arbitrary peripheral subgroups.

### 1.2. Outline.
Section 2 contains a brief overview of the geometry of relatively hyperbolic groups. Section 3 covers preliminaries about graphs of groups and quasiconvex hierarchies.

Section 4 is devoted to proving a relative fellow traveling result for a CAT(0) spaces with a geometric action by a relatively hyperbolic group, a generalized version of quasigeodesic stability in hyperbolic spaces. The main result is Theorem 4.2. Similar results were proved by Hruska and Kleiner in \([16]\) for CAT(0) spaces with isolated flats, and this result was previously known to experts in the field. However, it was difficult to find an exact formulation of Theorem 4.2 in the literature, so a proof is produced here.

Section 5 contains a combination lemma for certain subspaces of CAT(0) spaces with a geometric action by a relatively hyperbolic group. The main result, Theorem 5.6 shows that subspaces of such a CAT(0) space that are unions of convex cores for peripheral coset orbits and convex subspaces that obey a separation property are quasiconvex. The proof technique is inspired partly by the proof of the combination lemma in \([18]\). These results are use to show in Proposition 5.16 that these subspaces satisfy a strong quasiconvexity property modeled after \([28]\).

Section 6 reviews the properties of special cube complexes. In particular, Section 6.3 will introduce separability and explain how to pass to a finite cover so that each hyperplane’s elevations to the universal cover obey a separation property. Section 6.4 recalls a result of Sageev and Wise \([25]\) used to represent peripheral subgroups of a relatively hyperbolic compact special group \(G\) as immersed complexes in a NPC cube complex \(X\) with \(\pi_1 X = G\).
Section 7 follows the outline of [3, Section 5] and uses Wise’s double dot hierarchy construction to prove Theorem 1. While the general strategy is the same, the hyperbolic geometry used in [3] to prove the edge groups of the hierarchy are $\pi_1$-injective and quasi-isometrically embedded needs to be replaced by relatively hyperbolic geometric results from the preceding sections.

Section 8 uses Theorem 1 along with a relatively hyperbolic Dehn filling argument similar to the one used in a new proof of Wise’s MSQT from [3] to prove Theorem 2, a relatively hyperbolic analog of Wise’s MSQT.

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2. Relatively Hyperbolic Geometry

2.1. The geometry of CAT(0) spaces being acted on by relatively hyperbolic groups. In the situation where a relatively hyperbolic group acts properly and cocompactly on a CAT(0) space, it is reasonable to hope to partially recover the geometric features of a hyperbolic space. There are many equivalent definitions of a relatively hyperbolic group, see [15] for several examples; one definition, originally due to Farb [10], is produced here:

**Definition 2.1** ([15] Definition 3.6). Let $G$ be finitely generated relative to $\mathcal{P}$ with each $P \in \mathcal{P}$ finitely generated. The pair $(G, \mathcal{P})$ is a relatively hyperbolic pair if for some finite relative generating set $S$, the coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{P}, S)$ is hyperbolic and $(G, \mathcal{P}, S)$ has Farb’s bounded coset penetration property (see [10, Section 3.3]).

The elements of $\mathcal{P}$ and their conjugates are called peripheral subgroups and the cosets $\{gP: g \in G, P \in \mathcal{P}\}$ are called peripheral cosets.

Definition 2.1 establishes useful notation to refer to a relatively hyperbolic group pair, but the technical details will be less useful. Instead, most of the arguments involving relatively hyperbolic groups will be made using two key properties: that coarse intersections of peripheral cosets are uniformly bounded and that triangles are relatively thin which will be introduced in Section 2.2.

The following fact is well known:

**Proposition 2.2.** Let $(G, \mathcal{P})$ be a relatively hyperbolic pair and let $k \geq 0$. Let $S$ be a finite generating set for $G$. For all $R \geq 0$, there exists a $M = M(R)$ such if $gP, g'P'$ is a pair of distinct peripheral cosets, then $\text{diam} \mathcal{N}_R(gP) \cap \mathcal{N}_R(g'P') \leq M$ in the word metric on $\Gamma(G, S)$. 

The uniform bounds on coarse intersections of peripheral costs applies nicely to the case where a relatively hyperbolic group acts properly and cocompactly on a geodesic space by isometries:

**Corollary 2.3.** Let \((G, \mathcal{P})\) be a relatively hyperbolic pair with finite generating set \(S\) and let \(G\) act properly and cocompactly on a geodesic space \(X\) by isometries. Let \(x \in X\) be a base point. For all \(R \geq 0\), there exists \(M = M(R,G,\mathcal{P},S)\) such that if \(P, P' \in \mathcal{P}, g, g' \in G\) with \(gP \neq g'P'\), then \(\text{diam}(\mathcal{N}_R(gPx) \cap \mathcal{N}_R(g'P'x)) \leq M\).

Every \(P' \in \mathcal{P}'\) is a finite index subgroup of some \(P \in \mathcal{P}\) and \(H\) is a finite index subgroup of \(G\), so Proposition 2.12 follows from the fact that the inclusion of a Cayley graph for \(H\) into a Cayley graph for \(G\) is a quasi-isometry.

### 2.2. Relatively Thin Triangles

Comparison tripods help compare geodesic triangles in \(X\) with tripods:

**Definition 2.4.** Let \(a, b, c \in X\) and let \(\triangle abc\) be a geodesic triangle. There exists a map \(f : \triangle abc \rightarrow T(a, b, c)\) where \(T(a, b, c)\) is a unique tripod (up to isometry) with center point \(x\) such that \(f\) is isometric on each side of the triangle and the three legs of the tripod are \([f(a), x], [f(b), x]\) and \([f(c), x]\). The tripod \(T(a, b, c)\) is called a **comparison tripod** for \(\triangle abc\). The map \(f\) is the **comparison map**.

A geodesic metric space \(X\) is **hyperbolic** if there exists a \(\delta\) for every geodesic \(\triangle\) in \(X\) so that the preimage of every point in the comparison map has diameter less than \(\delta\).

**Definition 2.5.** Let \(X\) be a geodesic metric space, and let \(\mathcal{B}\) be a collection of subspaces. Let \(\triangle abc\) be a geodesic triangle in \(X\) and let \(\delta > 0\). Let \(T(a, b, c)\) be the comparison tripod, and let \(f : \triangle abc \rightarrow T(a, b, c)\) be the comparison map. If there exists \(F \in \mathcal{B}\) such that for all \(p \in T(a, b, c)\), either

1. \(\text{diam}(f^{-1}(p)) < \delta\) or
2. \(f^{-1}(p) \subseteq N_\delta(F)\),

then \(\triangle abc\) is \(\delta\)-**thin relative to** \(F\).

**Definition 2.6.** Let \(X\) be a geodesic metric space, \(\delta \geq 0\) and let \(\mathcal{B}\) be a collection of subspaces. The space \(X\) has the **\(\delta\)-relatively thin triangle property relative to** \(\mathcal{B}\) if each geodesic triangle \(\triangle\) is \(\delta\)-thin relative to some \(B_\Delta \in \mathcal{B}\).

The space \(X\) may contain triangles that are \(\delta\)-thin. By definition, these triangles are \(\delta\)-thin relative to every element of \(\mathcal{B}\). In the applications, \(X\) will usually be a CAT(0) space with a geometric action by a relatively hyperbolic group \(G\) where the elements of \(\mathcal{B}\) are convex subspaces of \(X\) that lie in uniformly bounded neighborhoods of peripheral coset orbits. If \((G, \mathcal{P})\) is a relatively hyperbolic group pair, a
Let \((G, \mathcal{P})\) be a relatively hyperbolic pair and let \(G\) act properly and cocompactly on a CAT(0) space \(X\) by isometries. Let \(x \in X\) be a base point and set \(\mathcal{F} = \{gPx \mid g \in G, P \in \mathcal{P}\}\).

Then there exists \(\delta > 0\) such that whenever \(\triangle\) is a geodesic triangle in \(X\), \(\triangle\) is \(\delta\)-thin relative to some \(F \in \mathcal{F}\).

Note that when \(X\) has the relatively thin triangle property relative to \(\mathcal{F}\), \(R \geq 0\) and \(\mathcal{F}' = \{N_R(F) : F \in \mathcal{F}\}\), then \(X\) still has the relatively thin triangle property relative to \(\mathcal{F}'\).

The notion of fellow-traveling will be useful for describing behavior of geodesics that issue from the same point. Definitions of fellow-traveling may vary, so the one that will be used is recorded here:

**Definition 2.8.** Let \(\alpha : [a_1, a_2] \to X\) and \(\beta : [b_1, b_2] \to X\) be geodesics, and let \(k \geq 0\). The geodesics \(\alpha\) and \(\beta\) \(k\)-**fellow travel for distance** \(D\) if \(d(\alpha(a_1 + t), \beta(b_1 + t)) \leq k\) for all \(0 \leq t \leq D\). If \(x := \alpha(a_1) = \beta(b_1)\) and \(\alpha\) and \(\beta\) \(k\)-fellow travel for distance \(D\), then \(\alpha\) and \(\beta\) \(k\)-**fellow travel distance** \(D\) from \(x\).
Definition 2.9. Let $X$ be a CAT(0) geodesic metric space with triangles that are $\delta$-thin relative to $B$. Let $\Delta \subseteq X$ with vertices $a, b, c$ with comparison map $f : \Delta abc \to T(a, b, c)$. Let $L_a$ be the closure of the leg of the tripod $T(a, b, c)$ that contains $f(a)$. Let $\text{Thin}_a := \{ f^{-1}(p) : \text{diam}(f^{-1}(p)) < \delta \} \cap f^{-1}(L_a)$. The corner segments of $\Delta$ at $a$ are the subsegments of the sides in $\text{Thin}_a$ and the corner length is the length of a corner segment at $a$.

The fat part of the side $ab \subseteq \Delta$ in $\Delta$ is the closure of $ab \setminus (\text{Thin}_a \cup \text{Thin}_b)$.

The corner segments at $a$ are subsegments of the sides issuing from $a$ that $\delta$-fellow travel. Each of these segments have the same length, which is defined to be the corner length. If $\Delta$ is $\delta$-thin relative to $B_\Delta \in B$, the fat part of each side of $\Delta$ is the maximal subsegment that does not lie in any of the corner segments and hence lies in $N_\delta(B_\Delta)$. Note that the fat part of a side may be empty. Since $X$ is CAT(0), each corner segment or fat part of a side is connected.

Similarly, quasigeodesic triangles in the Cayley graph of a relatively hyperbolic group also satisfy a thinness condition which is used to obtain Proposition 2.7.

Theorem 2.10 ([25] Theorem 4.1, originally due to [8]). Let $(G, \mathcal{P})$ be a relatively hyperbolic pair with Cayley graph $\Gamma$. For all $\lambda \geq 1, \epsilon > 0$ there exists a $\delta > 0$ such that if $\Delta$ is a $(\lambda, \epsilon)$-quasigeodesic triangle in $\Gamma$ with sides $c_0, c_1, c_2$, either:

1. there exists a point $p$ that lies within $\frac{\delta}{2}$ of each side or
2. each side of $c_i$ has a subpath $c_i'$ such that the terminal endpoint of $c_i'$ and the initial point of $c_{i+1}'$ (indices mod 3) are within distance $\delta$ of each other.

Lemma 2.11 is simple but is instrumental for working with relatively thin triangles.

Lemma 2.11. Let $\Delta abc$ be a geodesic triangle that is $\delta$-thin relative to $F$. Let $ab, bc, ac$ denote the sides of $\Delta abc$. If the length of the fat part of $ac$ in $\Delta abc$ is bounded above by $M$, then the length of the fat part of $bc$ and the length of the fat part of $ab$ differ by at most $M + 3\delta$.

The proof involves four applications of the triangle inequality. See Figure [2] for a schematic. When triangles that are thin relative to $B$ where elements of $B$ have uniformly bounded coarse intersections, the bounds on coarse intersections can be used to help bound the fat part of one side of a triangle that is $\delta$-thin relative to $B \in B$. With Lemma 2.11 a bound on the fat part of one side of a relatively thin triangle helps control the lengths of the fat parts of the other two sides. This technique will be used repeatedly, particularly in Section 5.

Relatively hyperbolic groups interact nicely with passing to finite index subgroups:

Proposition 2.12 ([3, Notation 2.9]). Let $G$ be a group and let $\mathcal{P}$ be a finite collection of subgroups of $G$. Let $H \triangleleft G$ be a finite index normal subgroup. For
Figure 2. Applying the triangle inequality four times gives a bound on the difference between the length of \([p_{ab}, p_{ba}]\) and the length of \([p_{bc}, p_{cb}]\) in terms of \(\|[p_{ac}, p_{ca}]\|, \delta\).

Each \(P \in \mathcal{P}\), let \(E_0(P) = \{gPg^{-1} \cap H | g \in G\}\) and let \(E(P)\) be a set of representatives of \(H\)-conjugacy classes in \(E_0(P)\). Let \(\mathcal{P}' = \bigsqcup_{P \in \mathcal{P}} E(P)\).

The pair \((G, \mathcal{P})\) is relatively hyperbolic if and only if \((H, \mathcal{P}')\) is relatively hyperbolic.

Proposition 2.12 follows from [15, Theorem 1.5].

There is also a generalized version of quasiconvexity for relatively hyperbolic groups.

**Definition 2.13** ([15, Definition 6.10]). Let \((G, \mathcal{P})\) be a relatively hyperbolic pair. Let \(H \leq G\). Let \(S\) be any finite set such that \(S \cup \mathcal{P}\) generates \(G\). Suppose there exists \(\kappa(S, d_S)\) such that for any \(\hat{\Gamma}(G, S, \mathcal{P})\)-geodesic \(\gamma\) with endpoints in \(H\), \(\gamma \cap g\) lies in \(N_\kappa(H)\) with respect to \(d_S\). Then \(H\) is relatively quasiconvex in \((G, \mathcal{P})\).

Note that there are other equivalent definitions which are discussed in [15]. The definition is also independent of the choice of finite relative generating set (see [15, Theorem 7.10]).

Relative quasiconvexity will only be needed for the peripheral subgroups:

**Proposition 2.14.** Let \((G, \mathcal{P})\) be a relatively hyperbolic group pair. Then every element of \(\mathcal{P}\) is relatively quasiconvex in \(G\).

**Proof.** In \(\hat{\Gamma}(G, S, \mathcal{P})\) every \(P \in \mathcal{P}\) has diameter 1. \(\square\)

3. Graphs of Groups and Hierarchies

3.1. **Graphs of Groups.** A graph of groups (together with an isomorphism from the fundamental group) is a way of decomposing a group along a finite number of
splittings and HNN extensions. Further decomposing the vertex groups as graphs of groups, decomposing the resulting vertex groups as a graph of groups again and continuing this process a finite number of times yields a kind of “multilevel graph of groups” called a hierarchy which will be defined in Definition 3.5.

Definition 3.1. A graph of groups \((\Gamma, \chi)\) consists of the following data:

1. a finite graph \(\Gamma = (V, E)\) where \(V\) is the vertex set of \(\Gamma\) and \(E\) is the oriented edge set of \(\Gamma\) with an involution \(e \mapsto \overline{e}\) that switches the orientation of each edge,
2. an assignment map \(\chi: V \cup E \rightarrow \text{Grp}\) that assigns a group to each vertex and edge,
3. for all \(e \in E\), \(\chi(e) = \chi(\overline{e})\),
4. attachment homomorphisms \(\psi_e: \chi(e) \rightarrow \chi(t(e))\) where \(t(e)\) is the terminal vertex of the edge \(e\).

\(\Gamma\) is a faithful graph of groups if the attachment homomorphisms \(\psi_e\) are injective.

A graph of spaces is constructed like a graph of groups, except that the assignment map \(\chi\) assigns a topological space instead of a group to each edge and vertex. The attachment homomorphisms are replaced by continuous attachment maps, and a faithful graph of spaces has \(\pi_1\)-injective attachment maps. A graph of spaces realization of a space \(X\) for a graph of spaces \((\Gamma, \chi)\) is a triple \((\Gamma, \chi, f)\) where \(f\) is a homeomorphism from \(X\) to the mapping cylinders of the attachment maps glued along vertex spaces.

Some authors, for example Wise and Serre, take faithfulness to be a part of the definition of a graph of groups. Not requiring faithfulness makes it easier to define graphs of groups in terms of graphs of spaces. For the applications in Section 7, graphs of groups will be constructed first without showing that they are faithful, but these graphs of groups will turn out to be faithful.

If \((\Gamma, \chi)\) is a graph of groups, and \(T\) is a maximal tree in \(\Gamma\), then \(\pi_1(\Gamma, T)\) will denote the fundamental group of the graph of groups \(\Gamma\) with respect to the tree \(T\). See [27] for further details about graphs of groups.

A graph of groups structure is the group theoretic analogue of a graph of spaces realization:

Definition 3.2. Let \(G\) be a group, let \((\Gamma, \chi)\) be a graph of groups where \(T\) is a maximal tree and let \(\phi: G \rightarrow \pi_1(\Gamma, T)\) be an isomorphism. The triple \((\Gamma, \phi, T)\) is a graph of groups structure on \(G\).

The structure \((\Gamma, \phi, T)\) is degenerate if \(\Gamma\) is a single vertex labeled with \(G\) and \(\phi\) is the identity.

While a graph of groups structure determines a splitting of \(G\), the choice of isomorphism and maximal tree affects the precise splitting. In many cases, it suffices
to give a splitting of $G$ up to conjugacy which will be the case in the examples below. When the splitting is given up to conjugacy, the choice of maximal tree also becomes unnecessary.

**Example 3.3.** Figure 3.1 shows a graph of spaces decomposition of a genus 2 surface and a graph of groups splitting of the fundamental group induced by the graph of spaces decomposition.

More generally, the pants decomposition of a surface $\Sigma_g$ will give a graph of groups decomposition of $\pi_1(\Sigma_g)$ where each vertex group is a free group of rank 2 and every edge group is an infinite cyclic group.

Graph of groups structures interact naturally with finite index normal subgroups. The following is [3, Proposition 3.18] but is originally due to Bass [4].

**Proposition 3.4.** Suppose $G$ has a graph of groups structure $(\Gamma, \phi, T)$, $H \triangleleft G$ and $H$ is finite index in $G$. Then $H$ has an induced graph of groups structure $(\tilde{\Gamma}, \tilde{\phi}, T')$ so that:

1. Every vertex group of $(\tilde{\Gamma}, T')$ has the form $(K^g \cap H) \triangleleft K^g$ and is finite index in $K^g$ for some vertex group $K$ of $(\Gamma, T)$ and some $g \in G$.
2. Every edge group of $(\tilde{\Gamma}, T')$ has the form $(K^g \cap H) \triangleleft K^g$ and is finite index in $K^g$ for some edge group $K$ of $(\Gamma, T)$ and some $g \in G$.

3.2. Hierarchies. Hierarchies of groups are inductively defined multilevel graphs of groups:
Definition 3.5. A hierarchy of groups of length 0 is a single vertex labeled by a group.

A hierarchy of groups of length $n$ is a graph of groups $(\Gamma_n, \chi_n)$ together with hierarchies of length $n - 1$ on each vertex of $\Gamma_n$.

If $\mathcal{H}$ is a length $n$ hierarchy of groups, the $n$th level of $\mathcal{H}$ is the graph of groups $\Gamma_n$. For $1 \leq k \leq n$, the $(n - k)$th level of $\mathcal{H}$ is the disjoint union of the $(n - k)$th levels of the hierarchies on the vertices of $\Gamma_n$.

The terminal groups are the groups labeling the vertices at level 0.

It will be useful to think of graphs of groups as length 1 hierarchies. Realizing a group as a hierarchy is similar to finding a graph of groups structure for that group:

Definition 3.6. Let $G$ be a group, $\mathcal{H}$ be a hierarchy of length $n$. Let $(\Gamma_n, \chi_n)$ be the level $n$ graph of groups. A hierarchy for $G$ is $\mathcal{H}$ together with a graph of groups structure $(\Gamma_n, \phi, T)$ for $G$. Let $\mathcal{P}$ be a collection of subgroups of $G$. The hierarchy structure terminates in $\mathcal{P}$ if every terminal group of $\mathcal{H}$ is conjugate to $\phi(P)$ for some $P \in \mathcal{P}$.

It will often be convenient to forget the choice of maximal tree and only give a hierarchy structure for a group up to conjugacy. In general, hierarchies will be allowed to contain degenerate splittings, but in order to obtain non-trivial results, it will be necessary to ensure that at least one of the splittings in the hierarchy is non-degenerate.

Wise’s hierarchies in [30] permit only one-edge splittings rather than allowing a graph of groups splitting for each vertex group in the hierarchy. The hierarchies in Definition 3.6 can be converted to hierarchies with one-edge splittings for each vertex group at the expense of increasing the length of the hierarchy. Wise’s hierarchies also terminate in the trivial group while Definition 3.6 allows arbitrary terminal groups. In practice, the goal in Section 7 will be to (virtually) find a hierarchy for a relatively hyperbolic group $(G, \mathcal{P})$ that terminates in groups isomorphic to those in the induced peripheral structure. Section 8 will explore what happens to the hierarchy after quotienting out the peripheral subgroups.

A hierarchy of spaces and a hierarchy realization for a space $X$ can be defined analogously by replacing groups in Definition 3.5 with topological spaces and replacing graph of groups structures by realizations in Definition 3.6.

Malnormality is an important group property which will play a role in Section 8 and is useful for amalgamating virtually special groups to make new virtually special groups (see [18]).

Definition 3.7. Let $G$ be a group and let $H \leq G$. The subgroup $H$ is malnormal in $G$ if for all $g \in G \setminus H$, $g^{-1}Hg \cap H = \{1\}$. Similarly, $H$ is almost malnormal in $G$ if for all $g \in G \setminus H$, $|g^{-1}Hg \cap H| < \infty$. 
Malnormality also extends to collections of subgroups. Let $\mathcal{P}$ be a collection of subgroups of $G$. The collection $\mathcal{P}$ is (almost) malnormal in $G$ if for all $g \in G$ and $P, P' \in \mathcal{P}$ either $g^{-1}Pg \cap P'$ is trivial (finite) or $P = P'$ and $g \in P$.

For example, if $(G, \mathcal{P})$ is a relatively hyperbolic pair and $G$ is finitely generated, then the collection $\mathcal{P}$ is almost malnormal in $G$ by Proposition 2.2.

Definition 3.8. Let $(\Gamma, \chi)$ be a faithful graph of groups and let $(\Gamma, \phi)$ be a graph of groups structure (up to conjugacy) for a group $G$.

1. $\Gamma$ is quasiconvex if every edge attachment map is a quasi-isometric embedding into $\pi_1(\Gamma)$.
2. $\Gamma$ is (almost) malnormal if for every $e \in E$, the image of the attachment homomorphism $\psi_e$ in $\pi_1(\Gamma)$ is (almost) malnormal in $\pi_1(\Gamma)$.

Let $\mathcal{H}$ be a hierarchy for $G$.

1. $\mathcal{H}$ is faithful if every graph of groups at every level of $\mathcal{H}$ is faithful.
2. $\mathcal{H}$ is quasiconvex if every edge group of every graph of groups at every level of $\mathcal{H}$ quasi-isometrically embeds in $G$.
3. $\mathcal{H}$ is (almost) malnormal if every edge group of every graph of groups at every level of $\mathcal{H}$ is (almost) malnormal in $G$.

It may be possible to give a reasonable weaker definition of quasiconvex (or malnormal) hierarchy by only requiring an edge group $G_e$ of a graph of groups $H$ in $\mathcal{H}$ to be quasi-isometrically embedded (malnormal) in each adjacent vertex group, but the stronger definition given here will be needed in Section 8.

Here are some examples to help illustrate the definition of a hierarchy:

Example 3.9. A splitting of the fundamental group of a hyperbolic surface group can be realized along quasiconvex infinite cyclic subgroups by using the pants decomposition. The splitting can be achieved either as a sequence of 1-edge splittings to create a hierarchy or can be achieved a single multi-edge graph of groups splitting.

There are iterated hierarchy splittings that cannot be realized by a single graph of groups splitting:

Example 3.10. Figure 4 shows a length 2 hierarchy for the fundamental group of a genus 2 surface, $\Sigma_2$. Cuts are made along the both the blue and green simple closed curves which intersect, so the iterated splitting of the fundamental group cannot be accomplished by a graph of groups (length 1 hierarchy).

Other notable examples of hierarchies are the Haken Hierarchy for Haken 3-manifolds, see [21] Section 9.4, and the Magnus-Moldvanskii hierarchy for one-relator groups, see [30] Section 18.
A hierarchy for $\pi_1(\Sigma_2)$, the fundamental group of a genus 2 surface $\Sigma_2$, where the iterated splitting of $\pi_1(\Sigma_2)$ cannot be realized by a graph of groups. The first splitting is over the infinite cyclic subgroup of $\pi_1(\Sigma_2)$ corresponding to one of the blue copies of $S^1$. The resulting vertex spaces are punctured tori whose fundamental groups are rank 2 free groups. Cutting along the green arc in each punctured torus makes an annulus. Then the fundamental group of a punctured torus splits as an HNN extension of the fundamental group of an annulus ($\mathbb{Z}$) over the trivial group (corresponding to the green arcs in each annulus which are glued together to make a punctured torus).

Figure 4.

Proposition 3.4 extends to hierarchies by induction on the length of the hierarchy.

Corollary 3.11. Suppose $G$ has a hierarchy $\mathcal{H}$ and $H$ is a finite index normal subgroup of $G$, then $\mathcal{H}$ has an induced hierarchy $\mathcal{H}'$ such that the length of $\mathcal{H}$ is the length of $\mathcal{H}'$ and:

1. every vertex group at level $i$ of the hierarchy $\mathcal{H}'$ is of the form $K^g \cap H$ which is finite index and normal in $K^g$ for some vertex group $K$ of $\mathcal{H}$ at level $i$ and some $g \in G$.

2. every edge group at level $i$ of the hierarchy $\mathcal{H}'$ is of the form $K^g \cap H$ which is finite index and normal in $K^g$ for some edge group $K$ of $\mathcal{H}$ at level $i$ and some $g \in G$. 


Lemma 3.12 follows from Corollary 3.11

**Lemma 3.12.** If $\mathcal{H}$ is a quasiconvex hierarchy for $G$ and $G_0$ is a finite index normal subgroup of $G$, then the induced hierarchy on $\mathcal{H}_0$ on $G_0$ is quasiconvex.

The definition of a quasiconvex hierarchy for a group $G$ only requires that the edge groups are quasi-isometrically embedded in $G$; when a graph of groups $(\Gamma, \phi, T)$ structure for $G$ is quasiconvex, the vertex groups are quasi-isometrically embedded as well.

**Lemma 3.13.** Let $(\Gamma, T)$ be a graph of groups structure for $G$. If the edge groups of $G$ are quasi-isometrically embedded in $G$, then the vertex groups of $\Gamma$ are quasi-isometrically embedded in $G$.

Here is a rough sketch of the proof of Lemma 3.13. A Cayley graph $\Lambda(G, S)$ of $G$ coarsely looks like a “tree of spaces” whose underlying (infinite) graph is the covering tree of $(\Gamma, T)$ where the edge spaces are Cayley graphs of edge groups and the vertex spaces are Cayley graphs of vertex groups. If $\Lambda_v := \Lambda(G_v, S_v)$ is one of the vertex spaces, the coarse tree structure ensures that if a $\Lambda(G, S)$-geodesic shortcut $\gamma$ between two points in $\Lambda_v$ exits $\Lambda_v$ through an edge space $\Lambda_e$, it must return through $\Lambda_e$ and the geodesic in $\Lambda_e$ between the entry and exit points $p_e, p_e'$ in $\Lambda_v \cap \Lambda_e$ is a quasi-geodesic with constants set by the quasi-isometric embedding of $\Lambda_e$ into $\Lambda(G, S)$. If $\gamma$ enters and exists $\Lambda_v$ at points $p_{e_1}, p'_{e_1}, \ldots, p_{e_m}, p'_{e_m}$, then a piecewise geodesic path $\rho$ connecting the initial point of $\gamma$ to $p_{e_1}$, the entry and exit points in order and the endpoint of $\gamma$ to $p'_{e_m}$ lies entirely in $\Gamma_v$ and cannot be much longer than $\gamma$.

**3.3. Fully $\mathcal{P}$-Elliptic Hierarchies.** Given a relatively hyperbolic group pair $(G, \mathcal{P})$ and a hierarchy $\mathcal{H}$ for $G$, the goal in Section 8 will be to strategically find a quotient of $G$ that has a hierarchy induced by $\mathcal{H}$ and inherits a relatively hyperbolic structure from $(G, \mathcal{P})$ that is also compatible with the induced hierarchy structure. Theorem 1.2 can then be used to show the resulting quotient is virtually special. To ensure that this happens, some additional restrictions must be imposed on the interactions between the edge and vertex groups of the hierarchy and the peripheral subgroups of $G$.

**Definition 3.14.** Let $\mathcal{H}$ be a hierarchy for a group $G$ and let $\mathcal{P}$ be a collection of subgroups of $G$. Let $\mathcal{V}$ be the vertex groups of $\mathcal{H}$. For each $H \in \mathcal{V}$, let $\pi_1(\Gamma, \phi_H, T_H)$ be the graph of groups structure for $H$ induced by the hierarchy $\mathcal{H}$. The hierarchy $\mathcal{H}$ is $\mathcal{P}$-elliptic if the following holds: whenever there exists a $g \in G$ such that $P' := gPg^{-1} \subseteq H \in \mathcal{V}$, then there exists an $h \in H$ such that $hP'h^{-1}$ is contained in some vertex group of $\Gamma_H$.

A $\mathcal{P}$-elliptic hierarchy is fully $\mathcal{P}$ elliptic if the following holds: whenever $E$ is an edge group in $\mathcal{H}$, then for all $g \in G$, either $gPg^{-1} \cap E$ is finite or $gPg^{-1} \subseteq E$. 
When \( H \) is a fully \( \mathcal{P} \)-elliptic hierarchy for \( G \) and \( G_0 \) is a finite index normal subgroup of \( G \), the induced hierarchy from Corollary 3.11 for \( H \) is also fully \( \mathcal{P} \)-elliptic in the induced peripheral structure provided by Proposition 2.12.

**Proposition 3.15.** Suppose that \( G_0 \) is finite index normal in \( G \) and let \((G_0, \mathcal{P}_0)\) be the peripheral structure induced on \( G_0 \) by Proposition 2.12. If \( G \) has a fully \( \mathcal{P} \)-elliptic hierarchy, then the induced hierarchy \( H_0 \) of \( G_0 \) is fully \( \mathcal{P}_0 \)-elliptic.

Proposition 3.15 follows immediately from the explicit characterizations of the edge and vertex groups of the induced hierarchies in Corollary 3.11 and from the explicit description of the induced peripheral structure.

4. The Relative Fellow Traveling Property

For this section, fix a relatively hyperbolic pair \((G, \mathcal{P})\), and assume \( G \) is acting geometrically on a CAT(0) space \( \tilde{X} \) with basepoint \( x \in \tilde{X} \).

**Definition 4.1** (Similar to [16, Definition 4.1.4]). Let \( \tilde{X} \) be a CAT(0) space and let \( G \) act geometrically on \( \tilde{X} \) with basepoint \( x \in \tilde{X} \). If for all \( \lambda \geq 1 \) and \( \epsilon \geq 0 \), there exists \( \ell(\lambda, \epsilon) \geq 0 \) such that for all pairs of \( (\lambda, \epsilon) \)-quasigeodesics \( \gamma: [a,b] \to \tilde{X} \) and \( \gamma': [a',b'] \to \tilde{X} \) with the same endpoints, there exist partitions:

\[
a \leq s_0 \leq s_1 \leq \ldots \leq s_m \leq b \quad \text{and} \quad a' \leq t_0 \leq t_1 \leq \ldots t_m \leq b'
\]

such that for all \( i \), \( d(\gamma(s_i), \gamma'(t_i)) \leq \ell \) and:

1. either \( d_{\text{Haus}}(\gamma((s_i, s_{i+1})), \gamma'(t_i, t_{i+1})) \leq \ell \) or
2. \( \gamma((s_i, s_{i+1})), \gamma'(t_i, t_{i+1}) \subseteq N_\ell(g_i P_i x) \) for some peripheral coset \( g_i P_i \),

then \( \tilde{X} \) has the relative fellow traveling property relative to \( \{gPx \mid g \in G, P \in \mathcal{P} \} \).

The main result of the section is Theorem 4.2. In [16, Proposition 4.1.6], Hruska and Kleiner wrote that CAT(0) spaces with isolated flats have the relative fellow traveling property relative to the isolated flats. It had been originally proved by Epstein for truncated hyperbolic spaces associated to finite volume cusped hyperbolic manifolds [9, Theorem 11.3.1]. Theorem 4.2 is a version of relative fellow traveling for CAT(0) spaces with a proper cocompact action by a relatively hyperbolic group. Theorem 4.2 is presumed to be known to experts based on the works of [8], [15] and others, but the exact formulation used here proved difficult to find in the literature. Therefore, a proof is provided here. The proof is quite technical and self-contained, so the reader may wish to understand the statement of Theorem 4.2 and skip to the next section.

**Theorem 4.2.** Let \((G, \mathcal{P})\) be a relatively hyperbolic group acting geometrically on a CAT(0) space \( \tilde{X} \) with basepoint \( x \in \tilde{X} \). If there is an \( R > 0 \) so that \( gPx \) is \( R \)-quasiconvex for all \( P \in \mathcal{P} \), then \( \tilde{X} \) has the relative fellow traveling property relative to \( \{gPx \mid g \in G, P \in \mathcal{P} \} \).
The following definition is based on [8, Lemma 8.10]:

**Definition 4.3.** Let $G$ act geometrically on a CAT(0) space $\tilde{X}$ with base point $x \in \tilde{X}$. Let $\lambda \geq 1$ and $\epsilon \geq 0$ and let $\gamma$ be a $(\lambda, \epsilon)$-quasigeodesic in $\tilde{X}$. Define

$$Sat_{\mu}(\gamma) := \bigcup \{gPx \mid g \in G, P \in \mathcal{P}, \gamma \cap N_{\mu}(gP) \neq \emptyset\}$$

This corollary follows immediately from [8, Lemma 8.10] and the Milnor-Švarc Lemma:

**Corollary 4.4.** Let $G, \tilde{X}$ and $x \in \tilde{X}$ be as in Definition 4.3. Fix $\lambda \geq 1$ and $\epsilon > 0$. There exists $\mu_0(\lambda, \epsilon) \geq 0$ such that for all $\mu \geq \mu_0$, there exists a $D \geq 0$ depending on $\lambda, \mu, \epsilon$ and the action of $G$ on $\tilde{X}$ such that if $\gamma, \gamma'$ are $(\lambda, \epsilon)$-quasigeodesics in $\tilde{X}$ with the same endpoints, then

$$\gamma' \subseteq N_D(\gamma) \cup N_D(Sat_{\mu}(\gamma)).$$

Corollary 4.4 and Theorem 2.10 give some notion of quasigeodesic stability for a CAT(0) space with a geometric action by a relatively hyperbolic group. The next lemma (together with Corollary 4.4) says that when $\gamma$ and $\gamma'$ are $(\lambda, \epsilon)$-quasigeodesics, if $gP \neq g'P'$ are two peripheral cosets and $\gamma'$ enters $gPx$, then $\gamma'$ cannot enter $g'P'x$ without passing within a bounded distance of $\gamma$ where the bound depends only on $\tilde{X}$, the action of $G$ and the quasigeodesic constants $\lambda, \epsilon$.

**Lemma 4.5.** Fix $\lambda \geq 1$ and $\epsilon \geq 0$ and let $\mu$, $D$ be the constants specified by $\lambda, \epsilon$ and Corollary 4.4. Suppose there exists $L' \geq 0$ so that for every peripheral coset $gP, gPx$ is $L'$-quasiconvex in $\tilde{X}$. There exists $D'_0 = D'_0(G, \mathcal{P}, \lambda, \epsilon, \tilde{X}, L')$ so that for all $(\lambda, \epsilon)$-quasigeodesics $\gamma, \gamma'$ with the same endpoints and $D' \geq D'_0$, if $\gamma'(t) \in N_{D' + \mu + \lambda + \epsilon}(g_1P_1x) \cap N_{D' + \mu + \lambda + \epsilon}(g_2P_2x)$ where $g_1P_1$ and $g_2P_2$ are peripheral cosets in the $\mu$-saturation of $\gamma$, then $\gamma'(t) \in N_{D}(\gamma)$.

**Proof.** Since $g_1P_1x$ and $g_2P_2x$ are in the $\mu$ saturation of $\gamma$, there exist points $a \in \gamma \cap N_{\mu}(g_1P_1x)$ and $b \in \gamma \cap N_{\mu}(g_2P_2x)$. Let $\Delta$ be a quasigeodesic triangle with sides $[\gamma'(t), a] \subseteq \mathcal{N}_{L' + D' + \mu + \lambda + \epsilon}(g_1P_1x), [\gamma'(t), b] \subseteq \mathcal{N}_{L' + D' + \mu + \lambda + \epsilon}(g_2P_2x)$ and $(ab) \subseteq \gamma$ where $(ab)$ is a subpath of $\gamma$ between $a$ and $b$.

The quasigeodesic triangle $\Delta$ is $\delta$-thin relative to some $gPx$ by Theorem 2.10 (and the fact that $\tilde{X}$ is CAT(0)) in that:

1. there exists a point $p \in \tilde{X}$ that is $\frac{\delta}{2}$ from all three sides of $\Delta$ or
2. there exist **corner segments** that are $\delta$-fellow traveling subsegments of the geodesic sides of $\Delta$ at $\gamma'(t)$ and **fat segments** that lie in each of the geodesic sides so that each fat segment has one endpoint on a corner segment and one endpoint that is distance $\delta$ from $(ab)$.

The length of the corner segments at $\gamma'(t)$ is at most

$$A := \max \text{diam}(N_{\delta + L' + D' + \mu + \lambda + \epsilon}(g_1P_1x) \cap N_{\delta + L' + D' + \mu + \lambda + \epsilon}(g_2P_2x))$$
and at least one of the fat segments has length at most \( A \) in \( gP \) because at least one of \( g_1P_1 \) and \( g_2P_2 \) does not equal \( gP \). By Theorem 2.10, a path of length \( 2A + \delta \) from \( \gamma'(t) \) to \((ab)\), the third side of the quasigeodesic triangle \( \Delta \), and \((ab)\) lies in a uniformly bounded neighborhood of a single peripheral coset orbit: \( \gamma \). Let \( \lambda, \epsilon \geq 0 \). Choose sufficient \( \mu, D, D' \) with \( D' \geq D \) that make the conclusions of Corollary 4.4 and Lemma 4.5 hold. Let \( L = D' + \mu + \lambda + \epsilon \) and let \( \gamma' \) be continuous \((\lambda, \epsilon)-\)quasigeodesics with the same endpoints.

If \( \gamma'(t_0) \in N_{D+\mu}(gP) \), then the interval \( I = [t_-, t_+] \) where \( t_+ = \inf \{ t \leq t_0 : \gamma'(t) \in N_{D+\mu}(gP) \} \) and \( t_- = \sup \{ t \geq t_0 : \gamma'(t) \in N_{D+\mu}(gP) \} \) has the following properties:

1. \( t_0 \in I \),
2. \( \gamma'(t) \in N_{D+\mu}(gP) \),
3. \( \gamma'(t_-), \gamma'(t_+) \in N_L(\gamma) \).

Proof. By construction \( t_0 \in I \). By continuity, \( \gamma'(I) \subseteq N_{D+\mu}(gP) \).

For some small fixed \( 0 < \rho < 1 \), \( \gamma'(t_+ + \rho) \notin N_{D+\mu}(gP) \). If \( \gamma'(t_+ + \rho) \notin N_{D+\mu}(\gamma') \), then \( \gamma'(t_+) \in N_{D+\mu+\lambda+\epsilon}(\gamma) \). Otherwise by Corollary 4.4, \( \gamma'(t_+ + \rho) \in N_{D+\mu}(g'P') \) for some peripheral coset \( g'P' \neq gP \). Therefore, \( \gamma'(t_+ + \rho) \in N_{D+\mu+\lambda+\epsilon}(gP) \cap N_{D+\mu+\lambda+\epsilon}(g'P') \). By Lemma 4.5, then \( \gamma'(t_+ + \rho) \in N_{D'}(\gamma) \), so \( \gamma'(t_+) \in N_{D'+\mu+\lambda+\epsilon}(\gamma) = N_L(\gamma) \). Similarly, \( \gamma'(t_-) \in N_L(\gamma) \).

The following lemma shows how to partition the domain of \( \gamma' \) for relative fellow traveling:

Lemma 4.7. Let \( L, D, D', \mu \) be as in Lemma 4.6. Let \( \gamma' : [a', b'] \to X \) be a continuous \((\lambda, \epsilon)-\)quasigeodesic, let \( \gamma : [a, b] \to X \) be a continuous \((\lambda, \epsilon)-\)quasigeodesic with the same endpoints and let \( R > 0 \), then there exists \( L_0 = L_0(R, \lambda, \epsilon) \geq 0 \) and a partition \( a' \leq t_0 \leq t_1 \leq \ldots \leq t_m = b' \) such that

1. \( \gamma(t_j) \in N_{L_0}(\gamma) \) for all \( j \),
2. \( |t_{2j} - t_{2j+1}| \geq R \),
3. \( \gamma'(t_{2j+1}) \in N_{D+\mu}(g_1P) \) for some peripheral coset \( g_1P \),
4. \( |t_{2j+1} - t_-| \geq R, \) and \( |t_+ - t_{2j+1}| \geq R, \) then \( \gamma'(t_{2j+1}) \notin N_{D+\mu}(gP) \) for all peripheral cosets \( gP \),
5. \( \gamma'([a, t_0]), \gamma'([t_0, t_{2j+1}]), \gamma'([t_{2j+1}, b]) \in N_{L_0}(\gamma) \).

Further, there exist \( s_j \) so that

1. \( d(\gamma(s_j), \gamma'(t_j)) \leq L \),
2. \( |s_j - s_{j+1}| \geq \frac{|(|t_{2j+1} - t_{2j}| - 2L - \epsilon)|}{\lambda} \).
Let \( t_{-1} := a' \). Partition \([a', b']\) inductively as follows:

- let \( t_{2j} = \inf \{ t \in [t_{2j-1}, b']: \gamma'([t, t + R]) \in \mathcal{N}_{D+\mu}(gP_x) \text{ for some } gP_x \} \), halt if no such \( t_{2j} \) exists,
- let \( t_{2j+1} = \sup \{ t \leq b': \gamma'([t, t + R]) \in \mathcal{N}_{D+\mu}(gP_x) \} \).

Let \( m \) be the largest subscript for which \( t_m \) was determined and let \( t_{m+1} := b' \). For each \( t_{2j+1} \), let \( I_{2j} \) be the interval specified by Lemma 4.6 with \( \gamma'(t_{2j+1}) \in \mathcal{N}_{D+\mu}(gP_x) \). Immediately, \( \gamma'(t_{2j+1}) \in \mathcal{N}_L(gP_x) \). Either \( t_{2j} \) is the left hand endpoint of \( I_{2j} \) in which case \( \gamma'(t_{2j}) \in \mathcal{N}_L(gP_x) \) by Lemma 4.6 or \( t_{2j} = t_{2j-1} \), so \( t_{2j} \in \mathcal{N}_L(gP_x) \) by Lemma 4.5. Either way, by construction \( \gamma'([t_{2j}, t_{2j+1}]) \in \mathcal{N}_{D+\mu}(gP_x) \) for some peripheral coset \( gP_j \).

Observe that \([t_{2i+1}, t_{2i+2}]\) cannot contain any length at least \( R \) subintervals whose image under \( \gamma' \) lies in \( \mathcal{N}_{D+\mu}(gP_x) \) for some peripheral coset \( gP \) because otherwise that subinterval would have \( t_{i+1} \) as its left endpoint.

Let \( L_0 := L + R\lambda + \epsilon \). For \( t \in [t_{2i+1}, t_{2i+2}] \), if \( \gamma(t) \notin \mathcal{N}_{D+\mu}(gP) \), then there exists \( t' \) such that \( |t - t'| \leq R \) and \( \gamma'(t') \in \mathcal{N}_L(\gamma) \) by Lemma 4.6 so \( \gamma'(t) \in \mathcal{N}_{L + R\lambda + \epsilon}(\gamma) = \mathcal{N}_{L_0}(\gamma) \) because \( \gamma' \) is a \((\lambda, \epsilon)\)-quasigeodesic.

The final two assertions follow immediately from the fact that \( \gamma'(t_-), \gamma'(t_+) \in \mathcal{N}_L(\gamma) \) and the fact that \( \gamma \) is a \((\lambda, \epsilon)\)-quasigeodesic. \(\square\)

Note that the constant \( L_0 \) can be made arbitrarily larger if desired.

The next two lemmas are devoted to showing that when \( \gamma' \) is \((\lambda, \epsilon)\)-geodesic and \( \gamma \) is geodesic, and \( \gamma' \) does not have a long subpath in any peripheral coset orbit, then they lie within a bounded Hausdorff distance of each other.

**Lemma 4.8.** Fix \( \lambda \geq 1, \epsilon > 0 \). Suppose there exists an \( L' \geq 0 \) so that for any peripheral coset \( gP \), the orbit \( gP_x \) is \( L' \)-quasiconvex in \( \bar{X} \). Let \( \delta \geq 0 \) be sufficiently large so that Theorem 2.10 holds, and let \( D + \mu \) be as before, but possibly enlarged so that \( D + \mu \geq \delta \). Suppose there exists \( R = R(D, \mu, \lambda, \epsilon) \) such that whenever \( a((t_-, t_+)) \in \mathcal{N}_{D+\mu}(gP_x), |t_- - t_+| \leq R \). Let \( a \) be a continuous \((\lambda, \epsilon)\)-quasigeodesic, let \( b \) be a geodesic with \( |b| \leq L \) and let \( c \) be a geodesic such that \( \triangle := \triangle abc \) is a quasigeodesic triangle.

Then there exists \( S \) not depending on the choice of \((\lambda, \epsilon)\)-quasigeodesic triangle \( \triangle \) such that if \( y \in c, d(y, a) \leq S \). In other words, \( c \leq \mathcal{N}_S(a) \).

**Proof.** If \( |c| \leq \delta \), then \( S = \delta \leq D + \mu + R\lambda + \epsilon + 4\delta \) suffices.

Now suppose \( S = D + \mu + R\lambda + \epsilon + 4\delta \) suffices for \( |c| \leq (n\delta) \) for \( n \in \mathbb{N} \).

Suppose \( |c| \leq (n + 1)\delta \). Assume \( c \) is parameterized \( c: [0, \alpha] \rightarrow \bar{X} \) with \( c(0) \) on \( a \), \( \alpha > 0 \) and \( c(\alpha) \) on \( b \). Let \( P = \{ t \in [0, \alpha]: d(c(t), a) = L \} \).

There are four cases:

1. If \( P = \{ \alpha \} \), since \( d(c(0), a) = 0 \), then for all \( y \in c, d(y, a) \leq L \) by continuity.
In the first case, let $c = \{0, q\}$. Then $S = L + R\alpha + \epsilon + 4\delta$ suffices because every point in $c([0, q])$ lies in an $S$ neighborhood of $a$ by the assumption and $c([q, a])$ is in an $L + 2\delta$ neighborhood of $a$ because $|b| \leq L$.

(3) $P \cap [\alpha - 2\delta, \alpha] = \emptyset$. This case requires some additional argument. See below.

(4) There exists $q \in P \cap [\alpha - \delta, \alpha)$ and $P \cap [\alpha - 2\delta, \alpha - \delta] = \emptyset$. For $t \in [q, \alpha]$, $d(c(t), a) \leq \delta + L$ because $|b| \leq L$. If $t \in [0, q)$, consider the triangle formed by $c([0, q])$, the geodesic of length $L$ connecting $c$ to $a$ and the subsegment of $a$ connecting their other two endpoints. This is a triangle satisfying one of the three preceding cases, so $d(c(t), a) \leq L + R\alpha + \epsilon + 4\delta$.

Thus proving $S = L + R\lambda + \epsilon + 4\delta$ suffices in the third case as is all that remains. Choose $0 < t \leq \alpha - 2\delta$ so that $t = \sup\{0 \leq t' \leq \alpha - 2\delta| c(t') \in P\}$. Then for all $y \in c([0, t])$, $d(y, a) \leq S$ because $|t| \leq \alpha - 2\delta \leq (n - 1)\delta$. Let $r \in (t, \alpha)$. The goal is to show $d(c(r), a) \leq L + R\lambda + \epsilon + 4\delta$. By Theorem 2.10 either:

1. there exists a point $z \in \tilde{X}$ such that $z$ is $\frac{\delta}{2}$ from each side of $\Delta$ or
2. each side $a, b, c$ of $\Delta$ has a subpath $a', b', c' \in N_\delta(gP)$ for some peripheral coset $gP$ such that the terminal endpoint of $c'$ closer to $c(0)$ is close to one endpoint of $a'$ and the terminal endpoint of $c'$ close to $c(\alpha)$ is within $2\delta + L$ of the other endpoint of $c'$ (because $|b'| \leq L$).

In the first case, let $s \in (0, \alpha)$ so that $d(c(s), b) \leq \delta$, and $d(c(s), a) \leq \delta$. If $t \leq r \leq s$, then $d(c(s), a) \leq \delta \leq L$, so $d(c(r), a) \leq L$ by continuity because no points of $c((t, \alpha))$ are distance $L$ from $a$. If $s \leq r$, since $\tilde{X}$ is CAT(0), $d(c(r), b) \leq \delta$, so $d(c(r), a) \leq L + \delta$.

In the second case, let $c' = [c(t_-), c(t_+)]$ so that $0 \leq t_- \leq t_+ \leq \alpha$ and $d(c(t_+), b) \leq \delta$. When $r \geq t_+$, let $s = t_+$ and the argument is similar to the previous case where $s \leq r$. When $t \leq r \leq t_-$, then $d(c(t_+), a) \leq \delta$ and the argument is similar to the previous case where $s \geq r$. On the other hand if $t_- \leq r \leq t_+$, by hypothesis the distance between the endpoints of $a' \in N_\delta(gP)$ is at most $\lambda R + \epsilon$ and by Lemma 2.11 $|c'| \leq \lambda R + \epsilon + 3\delta + T$ because $|b'| \leq T$. Therefore $|t_- - t_+| \leq \lambda R + \epsilon + 3\delta + L$, and $d(c(t_-), a) \leq \delta$, so $d(c(t), a) \leq \lambda R + \epsilon + 4\delta + L$. Therefore, by induction on $n$, the statement holds.

Lemma 4.9. Let $\mu, D, D' \geq 0$ be large enough so that the conclusions of Lemma 4.3, Lemma 4.6 and Lemma 4.8 hold for $(\lambda, \epsilon)$-quasigeodesics. Let $gP$ be $L' \geq 0$-quasiconvex in $\tilde{X}$ for all peripheral cosets $gP$. Let $L \geq 0$ satisfy the conclusions of Lemma 4.4 for $(\lambda, \epsilon)$-quasigeodesics, let $\gamma : [0, a] \to \tilde{X}$ be a geodesic and let $\gamma' : [0, b] \to \tilde{X}$ be a $(\lambda, \epsilon)$-quasigeodesic such that $d(\gamma'(0), \gamma(0)), d(\gamma'(b), \gamma(a)) \leq L$.

Suppose there exists $R \geq 0$ such that whenever $I$ is some interval and either $\gamma(I)$ or $\gamma(I) \in N_{D' + T}(gP)$ for some peripheral coset $gP$, $|I| \leq R$. Then there exists $S$ not depending on the choice of $\gamma$ or $\gamma'$ such that $d_{\text{Haus}}(\gamma, \gamma') \leq S$. 

Proof. Since \( \tilde{X} \) is CAT(0), \( \gamma \) and the geodesic connecting the endpoints of \( \gamma' \) are Hausdorff distance \( L \) apart. Therefore, it suffices to assume \( \gamma \) and \( \gamma' \) have the same endpoints.

Let \( p = \gamma(t) \). Let \( D' \) (depending on \( D, \mu \)) be as in the conclusion of Lemma 4.5. By Corollary 4.4, either \( d(p, \gamma) \leq D \) or \( p \in N_{D+\mu}(gP) \) for some peripheral coset \( gP \). In the latter case, by Lemma 4.5 there exists \( t_0 \) such that \( |t - t_0| \leq R + 1 \) and \( d(\gamma(t_0), \gamma) \leq D' \). Therefore, \( d(p, \gamma) \leq D + \lambda(R + 1) + \epsilon + D' \) because \( \gamma' \) is \((\lambda, \epsilon)\)-quasigeodesic.

On the other hand suppose there exists a point \( q \in \gamma \) such that \( d(q, \gamma') \geq L \). Let \( q' \in \gamma' \) with \( d(q, q') \geq L_0 \). Then the geodesic \( \psi := [q, q'] \) splits the quasigeodesic bigon formed by \( \gamma \) and \( \gamma' \) into two quasigeodesic triangles satisfying the hypotheses of Lemma 4.8. Therefore, there exists \( S' \) such that for every \( y \in \gamma \), then \( d(y, \gamma') \leq S' \).

The preceding lemmas are combined to partition the domain of \( \gamma' \) into intervals where \( \gamma' \) is close to \( \gamma \) and where \( \gamma' \) lies in a neighborhood of some peripheral coset orbit. The endpoints of each interval have images close to \( \gamma \) which suggests a way to partition \( \gamma \) using the projections of the endpoints of each interval onto \( \gamma \). However, the projections of these points may not appear in the correct order, so some adjustments will need to be made.

Before proving Theorem 4.2 it is restated here:

**Theorem 4.2 (Restated).** Let \( gPx \) be \( L' \)-quasiconvex in \( \tilde{X} \) for every peripheral coset \( gP \).

For all \( \lambda \geq 1 \) and \( \epsilon \geq 0 \) there exists \( \ell = \ell(\lambda, \epsilon) \geq 0 \) such that for all pairs of \((\lambda, \epsilon)\)-quasigeodesics \( \gamma : [a, b] \rightarrow \tilde{X} \) and \( \gamma' : [a', b'] \rightarrow \tilde{X} \) with the same endpoints, there exist partitions:

\[
a \leq s_0 \leq s_1 \leq \ldots \leq s_m \leq b \quad \text{and} \quad a' \leq t_0 \leq t_1 \leq \ldots t_m \leq b'
\]

such that for all \( i \), \( d(\gamma(s_i), \gamma'(t_i)) \leq \ell \) and:

1. either \( d_{\text{Haus}}(\gamma((s_i, s_{i+1})), \gamma'((t_i, t_{i+1}))) \leq \ell \) or
2. \( \gamma((s_i, s_{i+1})), \gamma'((t_i, t_{i+1})) \subseteq N_{\ell}(g_{P_i}x) \).

**Proof.** It suffices to assume that \( \gamma, \gamma' \) are continuous quasigeodesics because \( \gamma \) and \( \gamma' \) are each within a fixed Hausdorff distance (depending only on \( \lambda, \epsilon \)) of continuous quasigeodesics \( \gamma_0, \gamma_0' \) with the same endpoints whose quasigeodesic constants depend only on \( (\lambda, \epsilon) \) (see [7, Lemma III.H.1.11]).

Let \( L, L_0, D, \mu \) be as in Lemma 4.6 and Lemma 4.7 and let \( D' \) satisfy the conclusions of Lemma 4.5. Let \( M > 0 \) be a constant so that

\[
M \geq \max_{gP, gP'} \text{diam}(N_{D+D'+100L_0+L'+1}(gPx) \cap N_{D+D'+100L_0+L'+1}(gP'x),)
\]

and so that \( M > 2\lambda L_0 + \epsilon \).
First assume \( \gamma \) is geodesic. Let:

\[
a' \leq t'_0 \leq t'_1 \leq t'_2 \leq \ldots \leq t'_m \leq b'
\]

be the partition specified by Lemma 4.7 with \( R = 3\lambda M + \epsilon + 2\lambda L \).

By Lemma 4.7 there exist \( a \leq s_0' \leq s_1', s_2', \ldots, s_m' \leq b \) such that \( d(\gamma'(t'_i), \gamma(s'_i)) \leq L \) and \( d((s'_{2i}, s'_{2i+1})) \geq 3M \).

By Lemma 4.9 whenever \( t_0, t_+ \in (t'_{2i-1}, t'_{2i}) \) and \( |t_- - t_+| \geq 3\lambda M + \epsilon + 2\lambda L \), then \( \gamma'((t_-, t_+)) \notin N_{\lambda M + \epsilon + 2\lambda L}(gP) \) for all peripheral cosets \( gP \).

By Lemma 4.9, there exists \( S \geq 0 \) such that \( d_{\text{Haus}}(\gamma((t'_{2i-1}, t'_{2i})), \gamma(s'_{2i}, s'_{2i+1})) \leq S \). The \( L' \)-quasiconvexity of \( gP \) implies that \( \gamma((s'_{2i}, s'_{2i+1})) \) lies in \( N_{L' + 2L}(gP,x) \).

This nearly completes the proof; however, there is the possibility that \( j > i \) and that \( s'_j < s'_i \).

Then there exist \( i < k_1 < k_2 \) where \( k_2 = k_1 + 1 \) so that \( s'_i \in [s'_{k_1}, s'_{k_2}] \). There are three possibilities:

1. **Case:** \( \gamma([s'_{k_1}, s'_{k_2}]) \) lies in \( N_L(\gamma([t_{k_1}, t_{k_2}])) \). Then \( |t_{k_1} - t_i| \leq \lambda(2L) + \epsilon \), because there exists some \( t_{k_1} \leq t \leq t_{k_2} \) with \( d(\gamma'(t), \gamma(s_i)) \leq L \) so that \( d(\gamma'(t), \gamma'(t)) \leq 2L \) and \( t_i < t \). Therefore, by construction, \( i \) is odd, \( j = i + 1 \), and \( |s_i - s_j| \leq \lambda^2(2L) + \lambda\epsilon + \epsilon \).

2. **Case:** \( \gamma([s'_{k_1}, s'_{k_2}]) \) lies in \( N_{L + \mu}(gP) \) where \( gP \) is a peripheral coset. Either \( \gamma([s'_{k_1-1}, s'_i]) \) or \( \gamma([s'_{i+1}, s'_{k_2+1}]) \) lies in \( N_{L + \mu}(hP) \) where \( hP \) is a peripheral coset and \( hP \neq gP \). If \( gP \neq hP \), then either \( |s'_i - s'_{k_1}| < M \) or \( |s'_i - s'_{k_2}| < M \) because \( M \) bounds the diameter of \( N_{L+\mu}(gP) \cap N_{L+\mu}(hP) \). In either case, there exists \( t \in (t'_{k_1}, t'_{k_2}) \) with \( d(\gamma'(t), \gamma(s_i)) \leq L + M \) so that \( |t'_{k_1} - t'_i| \leq \lambda(2L + M) + \epsilon \). As in the previous case, then \( i \) is odd, \( j = i + 1 \), and \( |s_i - s_j| \leq \lambda^2(2L + M) + \lambda\epsilon + \epsilon \).

3. **Case:** **same as case (2) except that** \( gP = hP \). If \( gP = hP \), then \( \gamma([s'_{k_1}, s'_{k_2}]) \) lies in \( N_{L+\mu}(gP) \), so whenever \( i \leq 2k_3 \leq 2k_1 + 1 \leq k_2 \), then \( gPk_3 \) is \( gP \) because \( \gamma([s'_i, s'_{k_2+1}]) \) must have a length \( 3M \) subsegment in \( N_{L}(gP, k_3) \). Therefore, \( \gamma'([t'_{k_1}, t'_{k_2}]) \in N_{L+\mu}(gP) \).

The partitions \( a' \leq t_0 \leq t'_1 \leq \ldots \leq t'_m \leq b' \) and \( a \leq s'_0, s'_1, \ldots, s'_m \leq b \) can now be reworked. First set \( s'_0 := s'_0 \) and \( t_0 := t'_0 \). Given that \( s_{2i} \) was set equal to some \( s'_{2i} \), set \( s_{2i+1}, s_{2i+2}, t_{2i+1} \), and \( t_{2i+2} \) as follows:

1. If \( g_{j+1}P_{j+1} = g_jP_j \), reset \( s_{2i} := s'_{2i+2} \) and \( t_{2i} := t'_{2i+2} \) and repeat this process to determine \( s_{2i+1} \) and \( s_{2i+2} \).
2. If \( 2j = m - 1 \), then set \( s_{i+1} := s'_{m} \) and \( t_{i+1} := t'_{m} \) and stop.
3. If \( s'_{2j+1} \geq s'_{2j} \), set \( s_{2i+1} := s'_{2j+1} \), set \( s_{2i+2} := s'_{2j+2} \), set \( t_{2i+1} := t'_{2j+1} \) and set \( t_{2i+2} := t'_{2j+2} \).
4. If \( s'_{2j+1} < s'_{2j} \), set \( s_{2i+1} := s'_{2j} \), set \( s_{2i+2} := s'_{2j+2} \), set \( t_{2i+1} := t'_{2j+1} \) and set \( t_{2i+2} := t'_{2j+2} \).
First observe that \(s_{2i+2} \geq s_{2i}\). Indeed, if \(s'_{2j+2} \leq s'_{2j}\), then by the argument in case (3) above, there exist some \(s'_{2k}, s'_{2k+1}\) where \(2k > 2j\) such that \(g_k P_k = g_j P_j\). However, then \(g_j P_j, g_{j+1} P_{j+1}, \ldots, g_k P_k\) are all equal, so step (1) precludes \(s_{2i} > s_{2i+2}\).

By construction and cases (1) and (2) above, \([s_{2i+1}, s_{2i+2}]\) is either a point \(s'_{2j}\) where \(|s'_{2j} - s'_{2j+1}| \leq \lambda^2(2L + M) + \epsilon\) or an interval \([s'_{2j}, s'_{2j+1}]\). In either case, \(d_{\text{Haus}}(\gamma([s_{2i}, s_{2i+1}]), \gamma'([t_{2i}, t_{2i+1}])) \leq L + \lambda^2(2L + M) + \epsilon\). By the argument in Case (3) above and Step (4) in the selection of \(s_{2i+1}\) and \(t_{2i+1}\), both \(\gamma([s_{2i}, s_{2i+1}]) \in N_{2L+L'+S+L+\lambda^2(2L+M)+\epsilon}(gPx)\) and \(\gamma'([t_{2i}, t_{2i+1}]) \in N_{2L+L'+S+L+\lambda^2(2L+M)+\epsilon}(gPx)\) for some (common) peripheral coset \(gP\). Thus setting \(\ell := 2L + L' + S + L + \lambda^2(2L + M) + \epsilon\) completes the case that \(\gamma\) is geodesic.

The case where both \(\gamma\) and \(\gamma'\) are quasigeodesics can be handled by taking \(\rho\) to be the geodesic joining the common endpoints of \(\gamma\) and \(\gamma'\) and using the case where one path is geodesic on the pairs \((\gamma, \rho)\) and \((\gamma', \rho)\). Let \(M' = \max\text{diam}\left(N_{2L+L'+S+L+\lambda^2(2L+M)+\epsilon}(gPx) \cap N_{2L+L'+S+L+\lambda^2(2L+M)+\epsilon}(g'P'x)\right)\) where \(gP \neq g'P'\) are peripheral cosets. After repartitioning \(\gamma, \gamma'\) and \(\rho\) similarly, the constant \(\ell := 2(2L + L' + S + L + \lambda^2(2L + M) + \epsilon + M')\) suffices.

5. A Relatively Hyperbolic Combination Lemma

The construction of hierarchies in Section 7 is quite similar to the hierarchy constructed in [3]. The goal of this section is to prove a combination theorem for the relatively hyperbolic setting that will be used to show the edge groups of the hierarchy are undistorted.

5.1. CAT(0) Relatively hyperbolic pairs.

**Definition 5.1.** Let \(\overline{X}\) be a CAT(0) space, let \(\delta \geq 0\), let \(f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}\) be a function and let \(B\) be a collection of complete convex subspaces. The pair \((\overline{X}, B)\) is a (\(\delta, f\)-)Relatively Hyperbolic pair if

1. every triangle in \(\overline{X}\) is \(\delta\)-thin relative to some \(F \in B\),
2. for all \(r \geq 0\) and \(F_1, F_2 \in B\) with \(F_1 \neq F_2\), \(\text{diam}(N_r(F_1) \cap N_r(F_2)) \leq f(r)\).

The subspaces \(B\) are called **peripheral spaces**.

The first goal is to improve a relatively hyperbolic pair so that geodesics that stay near a peripheral space intersect the peripheral space.

**Definition 5.2.** Let \((\overline{X}, B)\) be a (\(\delta, f\)) relatively hyperbolic pair. Let \(Z\) be a convex subspace of \(\overline{X}\) and let \(K : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}\) be a function. The subspace \(Z\) is **K-attractive** if for all \(R > 0\) whenever \(\gamma\) is a geodesic with endpoints in \(N_R(Z)\) and \(|\gamma| > K(R)\), then \(\gamma \cap Z \neq \emptyset\).

**Proposition 5.3.** Let \((\overline{X}, B')\) be a (\(\delta, f'\)) relatively hyperbolic pair. Let \(B = \{N_{2\delta}(F') : F' \in B'\}\). Let \(f(r) = f'(r + 2\delta)\). Then \((X, B)\) is a (\(\delta, f\)) relatively hyperbolic pair.
Further, if $M = f(5\delta)$, then every $B \in \mathcal{B}$ is $(3M + 6(R + 2\delta) + 9\delta)$-attractive.

The following Lemma will be used to prove Proposition 5.3:

**Proposition 5.4.** Let $F \in \mathcal{B}'$, $R \geq \delta$ and let $\gamma$ be a geodesic. Suppose that for all $F_1, F_2 \in \mathcal{B}'$ with $F_1 \neq F_2$ that $\text{diam}(\mathcal{N}_{2\delta}(F_1) \cap \mathcal{N}_{2\delta}(F_2)) < M$. If $\gamma$ has endpoints in $\mathcal{N}_R(F)$, then $\text{diam}(\gamma \cap \mathcal{N}_{2\delta}(F)) > |\gamma| - (3M + 6R + 9\delta)$.

**Proof.** There is a quadrilateral whose sides are $\gamma$, two geodesics $\sigma_1, \sigma_2$ of length at most $R$ connecting the endpoints of $\gamma$ to points in $F$ and a geodesic $\alpha$ connecting the endpoints of $\sigma_1, \sigma_2$ that are in $F$. By convexity, $\alpha \subseteq F$. Let $\rho$ be a diagonal so that there are two triangles, $\Delta_1, \Delta_2$ so that $\Delta_1$ has sides $\alpha, \rho, \sigma_1$ as a side and $\Delta_2$ has sides $\gamma, \rho, \sigma_2$. Designate vertices $p, q, r, s$ so that $\alpha = [p, q], \sigma_1 = [q, r], \gamma = [r, s], \sigma_1 = [q, s]$.

**Case 1:** $\Delta_1$ is $\delta$-thin relative to some $F' \neq F$.

Since $F' \neq F$ and $\alpha \subseteq F$, the length of the fat part of $\alpha$ in $\Delta_1$ is bounded by $M$.

Let $\rho_1$ be the corner segment of $\rho$ in $\Delta_1$ at $s$. Then $|\rho_1| \leq R$. Let $\rho_2$ be the fat part of $\rho$ in $\Delta_1$. The fat part of $\sigma_1$ in $\Delta_1$ has length at most $R$, so by the triangle

---

**Figure 5.** The quadrilateral constructed in the proof of Proposition 5.3.
inequality, \( |\rho_2| < M + R + 3\delta \). Let \( \rho_3 \) be the corner segment of \( \rho \) in \( \triangle_1 \) at \( q \). By construction, \( \rho_3 \subseteq N_\delta(F) \).

Let \( \gamma_1 \) be the corner segment of \( \gamma \) at \( s \) in \( \triangle_1 \), let \( \gamma_2 \) be the fat part of \( \gamma \) in \( \triangle_2 \) and let \( \gamma_3 \) be the corner segment of \( \gamma \) in \( \triangle_2 \) at \( r \). Observe that \( (\gamma_1 \cap N_\delta(\rho_3)) \subseteq N_{2\delta}(F) \) and

\[
\text{diam}(\gamma_1 \cap N_\delta(\rho_3)) \geq |\gamma_1| - |\rho_1| - |\rho_2| > |\gamma_1| - (M + 2R + 3\delta).
\]

If \( \triangle_2 \) is \( \delta \)-thin relative to \( F \), then \( \gamma_2 \subseteq N_\delta(F) \). If \( \triangle_2 \) is \( \delta \)-thin relative to some other element of \( B' \), the fat part of \( \rho \) in \( \triangle_2 \) has length at most \( |\rho_1| + |\rho_2| + M \leq 2M + 2R + 3\delta \) because \( \rho_3 \subseteq N_\delta(F) \). By the triangle inequality:

\[
|\gamma_2| \leq 2M + 2R + 3\delta + R + 3\delta
\]

because \( |\sigma_2| \leq R \). Finally, \( |\gamma_3| \leq R \).

In summary, less than \( M + 2R + 3\delta \) of \( \gamma_1 \), lies outside of \( N_{2\delta}(F) \), at most \( 2M + 3R + 6\delta \) of \( \gamma_2 \) lies outside of \( N_{2\delta}(F) \), and at most \( R \) of \( \gamma_3 \) lies outside of \( N_{2\delta}(F) \), so:

\[
\text{diam}(\gamma \cap N_{2\delta}(F)) > |\gamma| - (3M + 6R + 9\delta)
\]
as desired.

**Case 2:** \( \triangle_1 \) is \( \delta \)-thin relative to \( F \).

Let \( \rho_1, \rho_2, \rho_3 \) and \( \gamma_1, \gamma_2, \gamma_3 \) be as in the previous case. Here, \( |\rho_1| \leq R, \rho_2 \subseteq N_\delta(F) \) since \( \triangle_1 \) is \( \delta \)-thin relative to \( F \) and \( \rho_1 \subseteq N_\delta(\rho_2) \subseteq N_\delta(F) \). Since \( \gamma_1 \) \( \delta \)-fellow travels a subsegment of \( \rho \) at \( s \), \( \text{diam}(\gamma_1 \cap N_\delta(\rho_2 \cup \rho_3)) \geq |\gamma_1| - R \) because \( |\rho_1| \leq R \). Since \( \rho_2 \cup \rho_3 \subseteq N_\delta(F) \), \( \text{diam}(\gamma_1 \cap N_{2\delta}(F)) \geq |\gamma_1| - R \). If \( \triangle_2 \) is \( \delta \)-thin relative to some \( F'' \neq F \), the fat part of \( \rho \) in \( \triangle_2 \) has length less than \( R + M \) because its intersection with \( \rho_2 \cup \rho_3 \subseteq N_\delta(F) \) has length less than \( M \) and \( |\rho_1| \leq R \). Therefore by the triangle inequality, \( |\gamma_2| < M + 2R + 3\delta \). On the other hand, if \( \triangle_2 \) is \( \delta \)-thin relative to \( F \), then \( \gamma_2 \subseteq N_\delta(F) \) so in both cases, all but a less than \( M + 2R + 3\delta \) subsegment of \( \gamma_2 \) lies in \( N_{2\delta}(F) \).

In summary, \( \text{diam}(\gamma \cap N_{2\delta}(F)) \geq |\gamma| - R \), \( \text{diam}(\gamma_2 \cap N_{2\delta}(F)) \geq |\gamma_2| - M + 2R + 3\delta \) and \( |\gamma_3| \leq R \). Therefore, by the convexity of \( N_{2\delta}(F) \):

\[
|\gamma \cap N_{2\delta}(F)| \geq |\gamma| - (M + 4R + 3\delta)
\]
as desired.

**Proof of Proposition 5.3** Take \( f(r) = f'(r + 2\delta) \). Let \( F_1, F_2 \in B \) with \( F_1 \neq F_2 \). Then

\[
\text{diam}(N_r(F_1) \cap N_r(F_2)) \leq f(r)
\]

Since each \( F \in B \) is a thickening of an element of \( B' \), triangles in \( \tilde{X} \) are \( \delta \)-relatively thin relative to elements of \( B \).

Let \( \gamma \) be a geodesic with endpoints in \( N_R(F) \). Then by convexity, \( \gamma \subseteq N_{R+2\delta}(F') \) for some \( F' \in B' \) where \( F = N_{2\delta}(F') \). By Proposition 5.4 if \( |\gamma| > 3M + 6(R + 2\delta) + 39\delta \), then \( \gamma \cap N_{2\delta}(F') \neq \emptyset \). Noting that \( F' = N_{2\delta}(F) \) completes the proof.
5.2. A combination lemma for relatively hyperbolic pairs. Maintain the following baseline hypotheses for Section 5.2:

**Hypotheses 5.5.** Let $(\tilde{X}, \mathcal{B})$ be a $(\delta, f)$ relatively hyperbolic pair and let $M = f(5\delta)$. Suppose that every $B \in \mathcal{B}$ is $(3M + 6(R + 2\delta) + 9\delta)$-attractive.

**Theorem 5.6.** Assume Hypotheses 5.5. Let $\gamma = b_1a_2b_2a_3b_3\ldots a_nb_n$ be a broken geodesic. Let $\gamma_n$ be the geodesic connecting the endpoints of $\gamma$. Suppose that:

1. For each $1 \leq i \leq n$, there exists some $F_i \in \mathcal{B}$ so that $b_i \subseteq F_i$.
2. If $F_i = F_j$, then $i = j$.
3. There exists $M > 0$ so that for all $F_1, F_2 \in \mathcal{B}$, $\text{diam}(N_{5\delta}(F_1) \cap N_{5\delta}(F_2)) \leq M$.
4. For $1 \leq i \leq n - 1$, $|b_i| \geq 26M + 250\delta$.
5. For $1 \leq i \leq n - 1$, $b_i \subseteq N_\delta(F_i)$ and $b_n \subseteq N_\delta(F_n)$.
6. For $j = i$ and $j = i - 1$, $\text{diam}(a_i \cap N_{3\delta}(F_j)) \leq 3M + 39\delta$.

Then $\gamma_n \subseteq N_{6M + 81\delta}(\gamma \cup \bigcup_{i=1}^n N_{5\delta}(F_i))$. If $b_n \subseteq N_\delta(F_n)$, $\gamma_n$ has a length at least $|b_n| - (16M + 165\delta)$ tail at $p_i$ in $N_{2\delta}(F_n)$. 

![Figure 6. One possible configuration of $\Delta_i^1$ and $\Delta_i^2$ in the proof of Theorem 5.6. Corner segments of triangles at the same point are connected by dotted lines.](image)
Proof. In the case $n = 1$, the proof is straightforward. For each $1 \leq i \leq n$, let $\omega_i$ be the geodesic connecting the endpoints of the broken geodesic $b_1a_2b_2 \ldots b_{i-1}a_i$ and let $\gamma_i$ be the geodesic connecting the endpoints of the broken geodesic $b_1a_2b_2 \ldots b_{i-1}a_i b_i$. For each $1 \leq i \leq n$, let $\Delta_i^1$ be the triangle with sides $\gamma_{i-1}$, $\omega_i$, and $a_i$. For each $1 \leq i \leq n$, let $\Delta_i^2$ be the triangle with sides $\omega_i$, $b_i$, and $\gamma_i$. Label vertices so that $a_i = [p_i, q_i]$ and $b_i = [q_i, p_{i+1}]$. See Figure 5.2 for a visual representation.

The proof of Theorem 5.6 is by induction on $n$, so assume Theorem 5.6 holds for all $1 \leq j < i$.

Let $c_i$ be the corner segment of $\omega_i$ in $\Delta_i^2$ at $q_i$.

**Proposition 5.7.** Under the inductive hypotheses for the proof of Theorem 5.6 there is a point $x \in \gamma_i$ so that $d(x, F_{i-1}) \leq 4\delta$. Further, $|c_i| \leq 8M + 81\delta$. When $\Delta_i^2$ is $\delta$-thin relative to $F_1$, then the length of the fat part of $\omega_i$ in $\Delta_i^2$ is at most $8M + 81\delta$.

Proof. Since $1 \leq i - 1 < n$, $b_{i-1}$ has a length at least $10M + 84\delta$-tail at $p_i \in N_{2\delta}(F_{i-1})$.

**Case:** $\Delta_i^1$ is thin relative to $F \neq F_{i-1}$. The corner segments of $\Delta_i^1$ at $p_i$ have length at most $3M + 39\delta$ because a more than $3M + 39\delta$ tail of $\gamma_{i-1}$ at $p_i$ lies in $N_{3\delta}(F_{i-1})$. Since $\Delta_i^1$ is thin relative to $F \neq F_{i-1}$, the length of the fat part of $\gamma_{i-1}$ in $\Delta_i^1$ is at most $M$. Therefore, there is a point $y \in \gamma_{i-1}$ and a point $y' \in \omega_i$ so that $d(y, p_i) \leq 4M + 39\delta$ and $d(y, y') \leq \delta$ so that $y, y'$ are in corner segments of $\Delta_i^1$ at $p_i$ and further, there exists a subsegment $\sigma$ of the corner segment $[p_1, y'] \subseteq \omega_i$ with endpoint $y'$ so that $|\sigma| > 2M$ and $\sigma \subseteq N_{3\delta}(F_{i-1})$.

The intersection of the corner segment of $\omega_i$ in $\Delta_i^2$ at $q_i$ with $\sigma$ has length at most $M$. The fat part of $\omega_i$ in $\Delta_i^2$ is either contained in $N_\delta(F_{i-1})$ or intersects $\sigma$ in a segment of length at most $M$. Therefore, either there is a point in $\gamma_i$ that is at most $\delta$ from the fat part of $\omega_i$ in $\Delta_i^2$ and the fat part of $\omega_i$ in $\Delta_i^2$ is contained in $N_\delta(F_{i-1})$ or $\sigma$ intersects the corner segment of $\Delta_i^2$ at $p_i$. In the first case, there is a point $x \in \gamma_i$ that lies in $N_{2\delta}(F_{i-1})$ and in the second case, there is a point $x \in \gamma_i$ so that $d(x, \sigma) < \delta$, so $x \in N_{\delta}(F_{i-1})$.

The next tasks are to bound $|c_i|$ from above and to prove that when $\Delta_i^2$ is $\delta$-thin relative to $F_i$, the fat part of $\omega_i$ in $\Delta_i^2$ has length at most $M$. Note that $c_i \subseteq N_{2\delta}(F_i)$.

The intersection of $c_i$ with the corner segment of $\omega_i$ in $\Delta_i^1$ at $q_i$ has length at most $3M + 39\delta$ because $\text{diam}(a_i \cap N_{3\delta}(F_i)) \leq 3M + 39\delta$. If $F \neq F_i$, the intersection of $c_i$ with the fat part of $\Delta_i^1$ is a segment of length at most $M$. Since $|c_i \cap \sigma| \leq M$ and $|\sigma| > 2M$, $|c_i| \leq 5M + 39\delta$. Further, if $\Delta_i^2$ is $\delta$-thin relative to $F_i$, then the fat part of $\omega_i$ in $\Delta_i^2$ intersects $\sigma$ in a segment of length at most $M$, intersects the fat part of $\omega_i$ in $\Delta_i^1$ in a length at most $M$ segment and intersects the corner segment of $\omega_i$ in $\Delta_i^1$ at $q_i$ in a segment of length at most $3M + 39\delta$. Hence the fat part of $\omega_i$ in $\Delta_i^2$ has length at most $5M + 39\delta$ when $\Delta_i^2$ is thin relative to $F_i$.

If $F = F_i$, then the fat parts of $a_i$ and $\gamma_{i-1}$ in $\Delta_i^1$, which are contained in $N_\delta(F_i)$, have length at most $3M + 39\delta$ and $M$ respectively. Therefore, the length of the fat
part of $\omega_i$ in $\Delta_1^i$ is at most $3M + 39\delta + M + 3\delta$. Then $|c_i| \leq 8M + 81\delta$ by a computation similar to the one in the previous case.

When $F = F_i$, the fat part of $\omega_i$ in $\Delta_1^i$ intersects $\sigma$ in a segment of length at most $M$, intersects the fat part of $\omega_i$ in $\Delta_1^i$ in a segment of length at most $4M + 42\delta$ and intersects the corner segment of $\omega_i$ in $\Delta_1^i$ at $\gamma_i$ in a segment of length at most $3M + 39\delta$. Therefore, if $\Delta_1^i$ is thin relative to $F_i$, then the length of the fat part of $\omega_i$ in $\Delta_1^i$ is at most $8M + 81\delta$.

**Case:** $\Delta_1^i$ is thin relative to $F_{i-1}$.

Recall $c_i$ is the corner segment of $\omega_i$ in $\Delta_1^2$. The intersection of $c_i$ with the corner segment of $\omega_i$ in $\Delta_1^i$ again lies in $N_{28}(F_i) \cap N_6(a_i)$ and hence has length at most $3M + 39\delta$. The fat part of $\omega_i$ in $\Delta_1^i$ lies in $N_6(F_{i-1})$. Hence, if the length of the fat part of $\omega_i$ in $\Delta_1^i$ exceeds $M$, then its intersection with $c_i$ has length at most $M$ so $|c_i| \leq 4M + 39\delta$. Hence for the purposes of bounding $|c_i|$ from above, assume the fat part of $\omega_i$ in $\Delta_1^i$ has length at most $M$. The length of the fat part of $a_i$ in $\Delta_1^i$ is at most $3M + 39\delta$. If the length of the fat part of $\omega_i$ in $\Delta_1^i$ is at most $M$, then by the triangle inequality, the length of the fat part of $\gamma_{i-1}$ in $\Delta_1^i$ is at least $4M + 39\delta$. Then there exist $y \in \gamma_{i-1}$ and $y' \in \gamma_i$ so that $y' \in \gamma_i$ and $y'$ are endpoints of the corner segments of $\Delta_1^i$ at $p_1$ and $d(y, p_1) \leq 3M + 39\delta + 4M + 42\delta = 7M + 81\delta$. Therefore there is a tail at $y'$ of the corner segment of $\omega_i$ in $\Delta_1^i$ at $p_1$ called $\sigma$ so that $|\sigma| > 2M$ and $\sigma \subseteq N_{28}(F_{i-1})$ because $\gamma_i$ has a more than $10M + 84\delta$ tail in $N_{28}(F_{i-1})$. Therefore, $c_i$ intersects $[y', p_1]$ in a segment of length at most $M$ because $c_i \subseteq N_{28}(F_i)$. Hence $|c_i| \leq 5M + 39\delta$ because the union of the two corner segments of $\omega_i$ in $\Delta_1^i$ and the fat part of $\omega_i$ in $\Delta_1^i$ is $\omega_i$.

In all cases, $|c_i| \leq 8M + 81\delta$.

If $\Delta_1^i$ is $\delta$-thin relative to $F_i$, the fat part of $\omega_i$ in $\Delta_1^i$ has length at most $4M + 39\delta$ because $c_i \subseteq N_6(a_i)$, and $\sigma$ and the fat part of $\Delta_1^i$ lie in $N_6(F_{i-1})$. In particular, the fat part of $\omega_i$ in $\Delta_1^i$ may only intersect $[p_1, y']$ in $\sigma$ because otherwise its intersection with $\sigma$ has length more than $M$ and lies in $N_{28}(F_{i-1})$. Hence assume $\Delta_1^i$ is thin relative to some $F \in B$ with $F \neq F_{i-1}$.

Let $\omega^1$ be the fat part of $\omega_i$ in $\Delta_1^i$ and let $\omega^2$ be the corner segment of $\omega_i$ in $\Delta_1^i$ at $p_1$. If there exists $r \in \omega^1 \cap \omega^2$, then $d(r, \gamma_i) < \delta$, so there exists an $x \in \gamma_i$ such that $\gamma_i \subseteq N_{28}(F_{i-1})$.

Otherwise, $\omega^1$ intersects $c_i$ in a segment of length at most $M$ because such a corner segment lies in $N_{28}(F_i)$ and intersects the fat part of $\omega_i$ in $\Delta_1^i$ in a segment of length at most $M$ (the fat part of $\omega_i$ in $\Delta_1^i$ lies in $N_6(F)$). Hence $|\omega^1| \leq 2M$. Let $\omega^3$ be the corner segment of $\omega_i$ in $\Delta_1^i$ at $p_1$. Let $z \in \omega_i$ be the point where $\omega^3$ intersects $\omega^3$. By the triangle inequality, the fat part of $\gamma_{i-1}$ in $\Delta_1^i$ has length at most
2M + 3M + 39δ + 3δ = 5M + 42δ because \( \text{diam}(a_i \cap N_{2\delta}(F_i)) \leq 3M + 39\delta \). The corner length of \( \triangle_i \) at \( p_i \) is at most \( 3M + 39\delta \) because any subsegment of \( a_i \) in \( N_{3\delta}(F_i) \) has length at most \( 3M + 39\delta \). Then at least a \( 10M + 84\delta - (3M + 39\delta + 5M + 42\delta) > M \) tail of \( \omega^1 \) at \( z \), which will be called \( \omega' \), lies in \( N_{3\delta}(F_i-1) \) because it \( \delta \)-fellow travels a subsegment of the tail of \( \gamma_{i-1} \) at \( p_{i-1} \) contained in \( N_{2\delta}(F_{i-1}) \). However, like \( \omega^1 \), at most \( 2M \) of \( \omega' \) lies in the union of a corner segment of \( \triangle_i \) at \( q_i \), with the fat part of \( \omega_i \) in \( \triangle_i \). Hence \( \omega' \) intersects \( \omega \). Since \( \omega' \) lies in \( N_{3\delta}(F_1) \) and \( \omega \) is a corner segment of \( \triangle_2 \) at \( p_1 \), there is a point \( x \in \gamma_i \) so that \( x \in N_{4\delta}(F_{i-1}) \).

Lemma 5.8. The segment \( \eta := [p_i, p_{i+1}] \) lies in \( N_{5M+79\delta}(a_i \cup b_i) \).

Proof. Let \( \triangle \) be the triangle whose sides are \( a_i, b_i, \eta \), and suppose \( \triangle \) is \( \delta \)-thin relative to \( F \in \mathcal{B} \). Since \( a_i \) does not contain a subsegment of length more than \( 3M + 39\delta \) in \( N_{2\delta}(F_i) \), \( a_i \) and \( b_i \) \( \delta \)-fellow travel for a length of at most \( 3M + 39\delta \) at \( q_i \). Therefore, the corner length of \( \triangle \) at \( q_i \) is at least \( 3M + 39\delta \). If \( F \neq F_i \), the length of the fat part of \( b_i \) in \( \triangle \) is at most \( M \) because \( \text{diam}(N_{2\delta}(F) \cap N_{2\delta}(F_i)) \leq M \). If \( F = F_i \), then the length of the fat part of \( a_i \) in \( \triangle \) is at most \( 3M + 39\delta \) because \( \text{diam}(N_{2\delta}(F_i) \cap a_i) \leq 3M + 39\delta \). In either case, \( \text{d}(q_i, \eta) \leq 6M + 78\delta + \delta = 6M + 79\delta \). Since \( \bar{X} \) is CAT(0), every point on \( \eta \) lies in \( N_{6M+79\delta}(a_i \cup b_i) \).

Let \( x \in \gamma_i \) be the point closest to \( p_{i+1} \) so that \( \text{d}(x, F_{i-1}) \leq 4\delta \). Let \( \triangle' \) be the triangle whose vertices are \( x, p_i \) and \( p_{i+1} \). Let \( \eta' = [x, p_i] \), \( \eta'' = [x, p_{i+1}] \) so that the three sides of \( \triangle' \) are \( \eta, \eta', \eta'' \).

Proposition 5.9. The segment \( \eta'' \in N_{8M+81\delta}(a_i \cup b_i) \cup N_{5\delta}(F_i-1) \cup N_{\delta}(F_i) \).

Proof. The corner segments of \( \triangle' \) at \( x \) lie in \( N_{3\delta}(F_{i-1}) \) because \( \eta \in N_{4\delta}(F_{i-1}) \) by convexity. Based on Lemma 5.8 it suffices to prove that every point in \( \eta'' \) that is not in a corner segment of \( \triangle' \) at \( x \) lies in \( N_{M+2\delta}(\eta) \cup N_{\delta}(F_i) \).

Since the corner segments of \( \triangle' \) at \( p_{i+1} \) (which are contained in \( \eta \) and \( \eta'' \)) \( \delta \)-fellow travel, it suffices to show that the fat part of \( \eta'' \) in \( \triangle' \) either lies in \( N_{\delta}(F_i) \), has length 0 or lies in \( N_{M+2\delta}(\eta) \). The triangle \( \triangle' \) is thin relative to some \( F \in \mathcal{B} \). If \( F = F_{i-1} \), then the length of the fat part of \( \eta'' \) in \( \triangle' \) is 0 by the choice of \( x \). If \( F = F_i \), then the fat part of \( \eta'' \) in \( \triangle' \) lies in \( N_{\delta}(F_i) \). If \( F \neq F_i, F_{i-1} \), then the fat part of \( \eta'' \) in \( \triangle' \) has length at most \( M \) because \( \text{diam}(N_{4\delta}(F_{i-1}) \cap N_{\delta}(F_i)) \leq M \). Therefore, the endpoints of the fat part of \( \eta'' \) are distance \( M + 2\delta \) and \( \delta \) from \( \eta \), so the fat part of \( \eta'' \) lies in \( N_{M+2\delta}(\eta) \).

Proposition 5.10. Let \( A_i = b_1a_2b_2 \ldots a_ib_i \cup \bigcup_{j=1}^{i} N_{\delta}(F_j) \). For all \( x \in \gamma_i \), \( d(z, A_i) \leq 8M + 81\delta \).

Proof. Let \( x \) be the point in \( \gamma_i \) contained in \( N_{4\delta}(F_{i-1}) \) that is closest to \( p_{i+1} \).

Suppose first that \( z \in [p_i, x] \). Let \( b'_{i-1} = [q_{i-1}, x] \in N_{4\delta}(F_i) \). Then by applying the inductive hypothesis to the broken geodesic \( b_1a_2b_2 \ldots a_{i-1}b'_{i-1} \), \( d(z, A_{i-1}) \leq 8M + 81\delta \).
If $z \notin [p_1, x]$, then $z \in \eta''$, so the result follows by Proposition 5.9. \hfill \Box

**Proposition 5.11.** If $b_i \in N_\delta(F_i)$, then the geodesic $\gamma_i$ has a $|b_i| - (16M + 165\delta)$ tail at $p_{i+1}$ that is contained in $N_{2\delta}(F_i)$.

**Proof.** There are two cases:

**Case 1:** $\Delta_i^2$ is $\delta$-thin relative to some $F \neq F_i$.

The corner length of $\Delta_i^2$ at $q_i$ is at most $8M + 81\delta$ by Proposition 5.7. The length of the fat part of $b_i$ in $\Delta_i^2$ is at most $M$ because $b_i \in N_\delta(F)$. Therefore, the corner length of $\Delta_i^2$ at $p_{i+1}$ is at least $|b_i| - (7M + 42\delta)$. Thus the corner segment of $\gamma_i$ at $p_{i+1}$ has length at least $|b_i| - (7M + 42\delta)$ and lies in $N_\delta(b_i) \subseteq N_{2\delta}(F_i)$.

**Case 2:** $\Delta_i^2$ is $\delta$-thin relative to $F_i$.

The corner length of $\Delta_i^2$ at $q_i$ is at most $8M + 81\delta$. Let $s$ be the length of the fat part of $b_i$ in $\Delta_i^2$. Then the corner length of $\Delta_i^2$ at $p_{i+1}$ is at least $|b_i| - s - (8M + 81\delta)$. By Proposition 5.7, the length of the fat part of $\omega_i$ in $\Delta_i^2$ is at most $8M + 81\delta$. By Lemma 2.11, the fat part of $\gamma_i$ in $\Delta_i^2$ has length at least $s - (8M + 81\delta + 3\delta)$. The corner segment of $\gamma_i$ at $p_{i+1}$ in $\Delta_i^2$ and the fat part of $\gamma_i$ in $\Delta_i^2$ both lie in $N_{2\delta}(F_i)$ and their combined length is at least $s - (8M + 84\delta) + |b_i| - s - (8M + 81\delta) = |b_i| - (16M + 165\delta)$.

Proposition 5.11 and Proposition 5.10 complete the inductive proof of Theorem 5.6. \hfill \Box

**Definition 5.12.** Let $\mathcal{A}$ be a collection of subsets of a geodesic metric and let $K \geq 0$. Suppose that for all $A_1, A_2 \in \mathcal{A}$ with $A_1 \neq A_2$, $d(A_1, A_2) \geq K$, then the collection $\mathcal{A}$ is $K$-separated.

The paths in Theorem 5.6 are of a special type to facilitate the inductive proof. Proposition 5.14 uses Theorem 5.6 to prove a quasiconvexity result for certain subspaces of $\bar{X}$ with some additional assumptions:

**Hypotheses 5.13.** Assume Hypotheses 5.5 and assume the following:

1. Let $Z := \max(8f(\max(R, 40M + 405\delta)), 8(6R + 259M + 2967\delta))$.
2. Let $\mathcal{A}$ be a $Z$-separated collection of convex subspaces of $\bar{X}$.
3. Let $B_0 \subseteq B$.
4. Let $S$ be any connected component of $(\bigcup_{A \in \mathcal{A}} A) \cup (\bigcup_{B \in B_0} B)$.
5. If $A \in \mathcal{A}$ and $B \in B$ with $A \cap B \neq \emptyset$, then $B \in B_0$.

**Proposition 5.14.** Under Hypotheses 5.13, $S$ is $40M + 405\delta$-quasiconvex in $\bar{X}$. Further, any geodesic in $S$ is not mapped to a loop in $\bar{X}$.

**Proof.** Let $\gamma$ be the image in $\bar{X}$ of a geodesic in $S_0$ and let $\gamma'$ be the $\bar{X}$-geodesic between its endpoints.

Then $\gamma$ can be written as a piecewise geodesic of one of the following piecewise geodesic forms:
(1) $b_1a_2b_2 \ldots a_nb_n$ and $|b_1| \geq 26M + 250\delta$
(2) $b_1a_2b_2 \ldots a_nb_n$ and $|b_n| \geq 26M + 250\delta$
(3) $a_1b_1a_2b_2 \ldots b_{n-1}a_n$
(4) $b_1a_2b_2 \ldots a_n$
(5) $a_1b_1a_2b_2 \ldots a_nb_n$
(6) $b_1a_2b_2 \ldots a_nb_n$, where one of $|b_1|$, $|b_n|$ is less than $26M + 250\delta$.

where for each $1 \leq i \leq n$, $a_i \in A_i \in \mathcal{A}$, for all $1 \leq i \leq n$, $b_i \in B_i \in \mathcal{B}$, and for $2 \leq i \leq n-1$, $|b_i| \geq 18M + 204\delta$ because $\mathcal{A}$ is a $\mathcal{Z}$-separated collection. Assume also that $n$ is minimal and that $\gamma$ is subdivided so that the sum of the $b_i$.

Note that if $i \neq j$, then $B_i \neq B_j$ because otherwise the subsegment $b_i \ldots b_j$ of $\gamma$ could be replaced by a single geodesic segment contradicting either the maximality of the lengths of the $b_i$ or contradicting the fact that $\gamma$ is geodesic in $S$. By the maximality of the lengths of the $b_i$, $\text{diam}(a_i \cap \mathcal{N}_{3\delta}(B_i)) \leq 3M + 39\delta$ and if $i \geq 2$, $\text{diam}(a_i \cap \mathcal{N}_{3\delta}(B_{i-1})) \leq 3M + 39\delta$ because otherwise the $a_i$ intersect either $B_i$ or $B_{i-1}$ so that $b_i$ or $b_{i-1}$ respectively could be made longer by convexity.

**Case:** $\gamma = b_1a_2b_2 \ldots a_nb_n$ and $|b_1| \geq 26M + 250\delta$.

By Theorem [5.6] $\gamma' \in \mathcal{N}_{5M+81\delta}(S)$ and $|\gamma'| \geq 10M + 48\delta$.

**Case:** $\gamma = b_1a_2b_2 \ldots a_nb_n$ and $|b_n| \geq 26M + 250\delta$.

Rewrite $\gamma$ as a piecewise geodesic $b_1'a_2'b_2 \ldots a_n'b_n$ where $a_i' = a_{n-i+1}$ and $b_i' = b_{n-i+1}$. By the previous case, it follows that $\gamma' \in \mathcal{N}_{5M+81\delta}(S)$ and $|\gamma'| \geq 10M + 48\delta$.

**Case:** $\gamma = a_1b_1a_2b_2 \ldots b_{n-1}a_n$.

Let $\rho = b_1a_2b_2 \ldots b_{n-1}$ and let $\rho'$ be the geodesic between its endpoints. By the preceding cases, $\rho' \in \mathcal{N}_{5M+81\delta}(S)$. Let $p_1, p_n, q_0, q_n$ be endpoints so that $a_1 = [q_0, p_1]$, $a_n = [p_n, q_n]$, $\rho' = [p_1, p_{n+1}]$, and $\gamma = [q_0, q_n]$. By Theorem [5.6], $\rho'$ has a length $10M + 48\delta$ tail at $p_1$ contained in $\mathcal{N}_{2\delta}(B_1)$, and $\rho'$ has a length $10M + 48\delta$ tail at $p_n$ contained in $\mathcal{N}_{2\delta}(B_{n-1})$. Let $\rho''$ be the diagonal $[q_0, p_n]$. Let $\triangle$ be the triangle with sides $a_1, \rho', \rho''$, and suppose that $\triangle$ is $\delta$-thin relative to $F \in \mathcal{B}$. The corner length of $\triangle$ at $p_1$ is at most $3M + 39\delta$ because $\rho'$ has a $10M + 48\delta$ tail at $p_1$ contained in $\mathcal{N}_{2\delta}(B_1)$ and $\text{diam}(a_1 \cap \mathcal{N}_{3\delta}(B_{n-1})) \leq 3M + 39\delta$.

If $F = B_1$, then the length of the fat part of $a_1$ in $\triangle$ is at most $3M + 39\delta$. On the other hand, if $F \neq B_1$, then the length of the fat part of $\rho'$ in $\triangle$ is at most $M$ because the corner length of $\triangle$ at $p_1$ is at most $3M + 39\delta$ and a $10M + 48\delta$ tail of $\rho'$ at $p_1$ lies in $\mathcal{N}_{2\delta}(B_1)$. Therefore, $d(p_1, \rho'') \leq 6M + 7\delta$, so $\rho'' \in \mathcal{N}_{5M+73\delta}(a_1 \cup \gamma') \subseteq \mathcal{N}_{14M+164\delta}(S)$ because $\tilde{X}$ is CAT(0).

Observe that $|\rho'| \geq 19M + 96\delta$ because $\text{diam}(\mathcal{N}_{2\delta}(B_{n-1}) \cap \mathcal{N}_{2\delta}(B_1)) \leq M$. By the preceding, if $\triangle$ is $\delta$-thin relative to $B_n$, then the length of the fat part of $\rho'$ in $\triangle$ is at most $M$. Therefore, the corner length of $\triangle$ at $p_n$ is at least $16M + 57\delta$. If $\triangle$ is $\delta$-thin relative to $B_1$, then the corner length of $\triangle$ at $p_n$ is at least $9M + 48\delta$ because at most $M$ of the fat part of $\rho'$ can intersect the $10M + 48\delta$ tail of $\rho'$ at $p_n$ that is
contained in $\mathcal{N}_{2\delta}(B_{n-1})$. In either case, $\rho''$ has a $9M + 48\delta$ tail at $p_n$ contained in $\mathcal{N}_{3\delta}(B_{n-1})$.

By Proposition 5.3, $\text{diam}(a_n \cap \mathcal{N}_{\delta}(B_{n-1})) \leq 3M + 45\delta$. By an argument similar to the one showing that $d(\rho, \rho'') \leq 6M + 79\delta$, $d(p_n, \gamma') \leq 6M + 79\delta$. Consequently, $\gamma' \subseteq \mathcal{N}_{6M+79\delta}(\rho'' \cup a_n) \subseteq \mathcal{N}_{20M+243\delta}(S)$. Further, the corner length of the triangle formed by $\rho''$, $\gamma'$ and $a_n$ at $p_n$ is at most $3M + 45\delta$ while the length of $\rho''$ is at least $9M + 48\delta$, so $|\gamma'| > 0$.

**Case:** $\gamma = b_1a_2b_2 \ldots b_{n-1}a_n$.

Let $\rho = b_1a_2b_2 \ldots b_{n-1}$ and let $\rho'$ be the geodesic between its endpoints. By the first case, $\rho' \subseteq \mathcal{N}_{8M+8\delta}(S)$. Let $p_1, p_2, p_n, q_n$ be endpoints so that $b_1 = [p_1, p_2]$, $a_n = [p_n, q_n]$, $\gamma' = [p_1, q_n]$ and $\rho' = [p_1, p_n]$. By an argument similar to the one in the previous case, $d(p_1, \gamma') \leq 6M + 79\delta$. Since $\bar{X}$ is CAT(0), $\gamma' \subseteq \mathcal{N}_{6M+79\delta}(a_n \cup \rho')$. Therefore, $\gamma' \subseteq \mathcal{N}_{14M+165\delta}(S)$.

Here, $\rho'$ has a $10M + 48\delta$-tail in $\mathcal{N}_{2\delta}(B_{n-1})$ while $\text{diam}(a_n \cap \mathcal{N}_{\delta}(B_{n-1})) \leq 3M + 39\delta$, bounding the corner length at $p_n$ of the triangle formed by $\rho'$, $a_n$ and $\gamma'$. Therefore, $|\gamma'| > 0$ because $|\rho'| > 10M + 48\delta$.

**Case:** $\gamma = a_1b_1a_2b_2 \ldots a_nb_n$.

By reindexing the pieces of $\gamma$, the proof is the same as in the previous case.

**Case:** $b_1a_2b_2 \ldots a_nb_n$, where one of $|b_1|$, $|b_n|$ is less than $26M + 250\delta$.

Up to a reindexing, assume $|b_1| \leq 26M + 250\delta$. Let $\rho = a_2b_2 \ldots a_nb_n$. Then by a previous case, $\rho \subseteq \mathcal{N}_{14M+165\delta}(S)$. Since $\bar{X}$ is CAT(0), $d_{\text{Haus}}(\gamma', \rho) \leq 26M + 250\delta$. Therefore, $\gamma' \subseteq \mathcal{N}_{40M+405\delta}(S)$.

If $A_2 = A_n$ then by minimality, $\gamma = \gamma'$ (and $\gamma$ does not have the specified form), so $A_1 \neq A_n$. Since $d(A_2, A_n) > 36M + 408\delta$ and $|b_1|, |b_n| \leq 26M + 250\delta$, the image of $\gamma$ is not a loop in $\bar{X}$.

**Proposition 5.15.** Assume Hypotheses 5.13. Let $F \in \mathcal{B}$ and $R \geq 0$ and let $\gamma$ be a geodesic contained in $\mathcal{N}_R(F)$ with endpoints in $S$. If $|\gamma| > Z$, then $F \in \mathcal{B}_0$.

**Proof.** There is an $S$-geodesic that can be expressed as a piecewise geodesic in $\bar{X}$ connecting the endpoints of $\gamma$ of the form:

$$a_1b_1a_2b_2 \ldots a_nb_na_{n+1}$$

where each $b_i \in B_i \in \mathcal{B}_0$, for $2 \leq i \leq n-1$, $|b_i| \geq Z$. By Proposition 5.14, $\gamma \in \cup_{i=1}^n(\mathcal{N}_{40M+405\delta}(a_i \cup B_i))$.

By the pigeonhole principle, there exists a subsegment $\gamma_0$ so that the length of $\gamma_0$ is as follows:

$$|\gamma_0| \geq \begin{cases} \frac{1}{4}Z & n \leq 2 \\ \frac{n-2}{2n+2}Z \geq \frac{1}{8}Z & n > 2 \end{cases}$$

and $\gamma_0 \subseteq \mathcal{N}_{40M+405\delta}(a_i)$ for some $i$ or $\gamma_0 \subseteq \mathcal{N}_{40M+405\delta}(B_i)$. If $\gamma_0 \subseteq \mathcal{N}_{10M+405\delta}(B_i)$, then $F = B_i$ because $\frac{1}{8}Z > f(\max(R, 40M + 405\delta))$. 

\[\square\]
Otherwise $a_i$ has a length $\frac{1}{5}Z - 2(40M + 405\delta)$ subsegment in $N_{R+40M+405\delta}(F)$. By Proposition 5.3 if $\frac{1}{5}Z - 2(40M + 405\delta) > 3M + 6(R + 32M + 371\delta) + 3\delta$, then $a_i$ intersects $F$. Therefore $F \in B_0$. \hfill \Box

**Proposition 5.16.** Assume Hypotheses 5.13 assume that $\bar{X}$ admits a geometric action by a relatively hyperbolic group and assume that for each peripheral coset orbit $gP_x$ relative to some fixed basepoint $x$, there exists $B \in \mathcal{B}$ so that $gP_x \subseteq B$.

Let $\lambda \geq 1$ and $\epsilon \geq 0$. Then there exists $L = L(\lambda, \epsilon)$ so that for every $(\lambda, \epsilon)$-quasigeodesic $\gamma$ with endpoints in $S$, $\gamma \subseteq N_L(S)$.

**Proof.** By Theorem 4.2 let $\ell \geq 0$ be a constant so that every $(\lambda, \epsilon)$-quasigeodesic $\ell$-relatively fellow travels relative to $B$ in $\bar{X}$. Let $\rho$ be the geodesic connecting the endpoints of $\gamma$. By Theorem 1.2 there are subpaths $\rho_1, \rho_2, \ldots, \rho_k$ of $\rho$ and $(\lambda, \epsilon)$-quasigeodesics $\gamma_1, \ldots, \gamma_k$ with $\gamma_i \subseteq \gamma$ so that:

1. $\bigcup_{i=1}^k \rho_i = \rho$ and $\bigcup_{i=1}^k \gamma_i = \gamma$, and
2. either $d_{Haus}(\rho_i, \gamma_i) \leq \ell$ or $\rho_i, \gamma_i \subseteq N_\ell(B_i)$ for some $B_i \in \mathcal{B}$.

If $d_{Haus}(\rho_i, \gamma_i) \leq \ell$, then by Proposition 5.14 $\gamma_i \subseteq N_{\ell+40M+405\delta}(S)$. Otherwise, $\rho_i, \gamma_i \subseteq N_\ell(B_i)$ for some $B_i \in \mathcal{B}$. If $B_i \in B_0$, then $\gamma_i \subseteq N_\ell(B_i) \subseteq N_\ell(S)$. On the other hand, if $B_i \notin B_0$, then the length of $\rho_i$ is at most $Z$, so for any $x \in \gamma_i$, $d(x, \rho_i)$ is at most $\ell + \lambda(2\ell + Z) + \epsilon$, so $\gamma_i \subseteq N_{\ell+\lambda(2\ell+Z)+\epsilon+40M+405\delta}(S)$. \hfill \Box

6. **THE GEOMETRY OF SPECIAL CUBE COMPLEXES**

6.1. **Non-Positively Curved Cube complexes.** A **cube complex** is a union of Euclidean cubes $[0, 1]^n$ of possibly varying dimensions glued isometrically along faces. A **non-positively curved** (NPC) cube complex is a cube complex such that the link of every vertex is a flag simplicial complex. See [29] Section 2.1 for details.

In each cube $[0, 1]^n$, fixing one coordinate at $\frac{1}{2}$ makes a **codimension-1** midcube. A **hyperplane** $H$ is a connected union of midcubes glued isometrically along faces so that the intersection of $H$ with any cube is either a codimension-1 midcube or empty. See Figure 7 for an example of an NPC cube complex and the link of a vertex.

6.2. **Special cube complexes and separability.** A **special cube complex** is a type of NPC cube complex developed by Wise and others whose hyperplanes are embedded, are 2-sided and avoid two other pathologies, see [29] Definition 4.2]. The important properties of special cube complexes that will be used in the following are the embeddedness and 2-sidedness of the hyperplanes and the fact that hyperplane subgroups of special cube complexes are separable (see Proposition 6.3).

A group is **special** if it is the fundamental group of a special cube complex. By work of Haglund and Wise [13], special groups embed into right angled Artin groups and are hence residually finite. Recall that if $G$ is a group and $H$ is a subgroup, $H$...
is separable in $G$ if it is the intersection of the finite index subgroups containing $H$.

Passing to finite index subgroups is compatible with separability:

**Lemma 6.1.** Let $G$ be a group, let $G_0$ be a finite index subgroup of $G$ and let $H \leq G$. Then $H$ is separable in $G$ if and only if $H \cap G_0$ is separable in $G_0$.

**Theorem 6.2** (Scott’s Criterion, [26]). Let $X$ be a connected complex, $G = \pi_1 X$ and $H \leq G$. Let $p : X^H \rightarrow X$ be the cover corresponding to $H$. The subgroup $H$ is separable in $G$ if and only if for every compact subcomplex $Y \subseteq X^H$, there exists an intermediate finite cover $X^H \rightarrow \hat{X} \rightarrow X$ such that $Y \rightarrow \hat{X}$.

Every finitely generated subgroup of a free group is separable. Likewise, special groups have an ample supply of separable subgroups. For example, the hyperplane subgroups of a special cube complex are separable:

**Proposition 6.3.** Let $X$ be a virtually special compact and non-positively curved cube complex. Let $W$ be a hyperplane of $X$. Then $\pi_1(W)$ is separable in $\pi_1(X)$.

Proposition 6.3 follows from Haglund and Wise’s canonical completion and retraction (see [29, Construction 4.12] or [13, Corollary 6.7]).

6.3. **Elevations and $R$-embeddings.** This subsection builds up the technical tools and terminology used to obtain finite covers whose hyperplanes elevate to sufficiently separated images in the universal cover.

The first step is to formalize the notion of an elevation:
**Definition 6.4.** Let $W$ be a connected topological space and let $\phi : W \to Z$ be a continuous map. Let $p : \hat{Z} \to Z$ be a covering map. An **elevation of $W$ to $\hat{Z}$** is a minimal covering $\hat{p} : \hat{W} \to W$ such that the map $\hat{\phi} := \phi \circ \hat{p}$ lifts to a map $\hat{W} \to \hat{Z}$.

Often, the map $\hat{W} \to \hat{Z}$ will be implied and an elevation of $\phi$ will instead refer to the image of some elevation.

Elevations may not be unique: two elevations of the same map are **distinct** if they have different images.

When $\phi : W \to Z$ is an inclusion map, then the distinct elevations of $\phi$ are precisely the components of $p^{-1}(W)$.

**Definition 6.5.** Let $X$ be a metric space, $R \geq 0$ and let $Y \subseteq X$ be connected. Let $p : X^Y \to X$ be the covering space associated to $\pi_1(Y)$ so that the inclusion $Y \to X$ lifts canonically to $X^Y$. The subspace $Y$ is **$R$-embedded in $X$** if $p$ is injective on $N_R(Y) \subseteq X^Y$.

The following lemma is straightforward but will be important:

**Lemma 6.6.** Let $p : \hat{X} \to X$ be a finite regular cover. If $A$ is $R$-embedded in $X$, then each component of $p^{-1}(A)$ is $R$-embedded in $\hat{X}$.

The main application of hyperplane separability is to show that every compact virtually special cube complex has a finite cover where every hyperplane is $R$-embedded.

**Proposition 6.7.** Let $X$ be a compact virtually special non-positively curved cube complex. Given $R \geq 0$, then there exists a finite regular (compact) special cover $C$ such every hyperplane $V \subseteq C$ is $R$-embedded in $C$.

If $\hat{W}_1, \hat{W}_2$ are distinct elevations of a hyperplane $V$ of $C$ to the universal cover $\hat{X}$, then $d_\hat{X}(\hat{W}_1, \hat{W}_2) \geq 2R$.

**Proof.** For each hyperplane $W$ of $X$, $\pi_1(W)$ is separable by Proposition 6.3. By theorem 6.2, there exists a finite covering $\hat{p} : \hat{X} \to X$ such that there is an embedding $i_W : N_R(W) \to \hat{X}$.

Let $\hat{p} : \hat{X} \to X$, $p^W : \hat{X}^W \to X$ and $p : \hat{X} \to X^W$ be canonical covering maps so that $\hat{p} = p^W \circ p$. Let $\hat{W} \to \hat{W}_1, \hat{W} \to \hat{W}_2$ be distinct elevations of $W$ to $\hat{X}$, and let $\hat{W}_1 \in \hat{W}_1$ and $\hat{W}_2 \in \hat{W}_2$.

Suppose there exists a path $\gamma \subseteq \hat{X}$ with $|\gamma| \leq 2R$ between $\hat{W}_1$ and $\hat{W}_2$. Let $\hat{x} \in \gamma$ such that $d(\hat{x}, \hat{W}_1) < R$ and $d(\hat{x}, \hat{W}_2) < R$.

There exists $g \in \pi_1(X) \setminus \pi_1(W)$ such that $g \cdot \hat{W}_1 \in \hat{W}_2$, and $g \notin \pi_1(W)$ because otherwise $g \cdot \hat{W}_1 \in \hat{W}_1 \cap \hat{W}_2$ in which case, $\hat{W}_1 \in \hat{W}_2$ but $\hat{W}_1 \notin \hat{W}_2$. Now $d(\hat{x}, \hat{W}_2) \leq R$. Since $g \notin \pi_1(W)$, $p(x) \neq p(g \cdot x)$. By definition of an elevation, $p(W_2)$ is contained in the image of the inclusion of $W$ into $X^W$. Also $p(x), p(g \cdot x)$ lie in an $R$-neighborhood of the image of $W$ in $X^W$. However,

$$p^W \circ p(x) = \hat{p}(x) = \hat{p}(g \cdot x) = p^W \circ p(g \cdot x)$$
contradicting the fact that $W$ is $R$-embedded in $X^W$.

Suppose $X$ has $n$ hyperplanes. By passing to a finite cover if necessary, assume $X^W$ is regular. The number of hyperplane orbits under deck transformations of $X^W$ is at most $n$, and every hyperplane in the orbit of an elevation of $W$ to $X^W$ is $R$-embedded. Therefore, performing this procedure at most $n$ times, will produce a finite cover $C \to X$ where every hyperplane is $R$-embedded. □

Proposition 6.7 will be used later in Section 7 to make the elevations of a hyperplane a $2R$-separated family in the sense of Definition 5.12.

6.4. Convex Cores. Specialness also plays a role in building a geometric representation of the peripheral structure. In the hyperbolic case, Wise and others (see [12], [24], see also [13, Proposition 7.2] proved that quasiconvex subgroups of virtually special groups have “convex cores” in the CAT(0) universal cover. This fact and canonical completion and retraction can be used to show that hyperbolic special groups are QCERF or quasiconvex extended residually finite [13, Theorem 1.3] meaning that if $G$ is hyperbolic and special, then every quasiconvex subgroup of $G$ is separable.

A similar result exists in the relatively hyperbolic case. One might imagine that replacing the quasiconvex subgroup $H$ by a relatively quasiconvex subgroup might yield a generalization; however, some care is required. Consider the following example:

Example 6.8. Take the standard action of $\mathbb{Z}^2 = \langle (1,0), (0,1) \rangle$ on $\mathbb{R}^2$ by translation. The diagonal $D := \{(r,r) : r \in \mathbb{R}\}$ is a subspace stabilized by $L := \langle (1,1) \rangle \leq \mathbb{Z}^2$. The subgroup $L$ is $(2,0)$-quasi-isometrically embedded in the given presentation of $\mathbb{Z}^2$, but the convex hull of $D$ is all of $\mathbb{R}^2$.

Full relatively quasiconvex subgroups eliminate these pathologies:

Definition 6.9 ([25, Section 4]). Let $(G,P)$ be a relatively hyperbolic group pair and let $H$ be a relatively quasiconvex subgroup of $G$. The subgroup $H$ is a full relatively quasiconvex subgroup of $G$ if for each $g \in G$ and $P \in P$, either $gPg^{-1} \cap H$ is finite or $gPg^{-1} \cap H$ is finite index in $gPg^{-1}$.

Theorem 6.10 ([25, Theorem 1.1]). Let $X$ be a compact non-positively curved cube complex with $G = \pi_1(X)$ hyperbolic relative to subgroups $P_1, \ldots, P_n$. Let $\tilde{X}$ be the CAT(0) universal cover of $X$. If $H$ is a full relatively quasiconvex subgroup of $G$, then for any compact $U \subseteq \tilde{X}$, then there exists an $H$-cocompact convex subcomplex $\tilde{Y} \subseteq \tilde{X}$ with $U \subseteq \tilde{Y}$.

By Proposition 2.14 if $(G,P)$ is a relatively hyperbolic group pair, the elements of $P$ and their conjugates are relatively quasiconvex. By Proposition 2.2 the elements of $P$ and their conjugates are full relatively quasiconvex. Therefore:
**Lemma 6.11.** Let \( X \) be a non-positively curved cube complex with \( \text{CAT}(0) \) universal cover \( \tilde{X} \) and \( G := \pi_1(X) \). Let \((G, \mathcal{P})\) be a relatively hyperbolic pair. Let \( x \in \tilde{X} \) be a base point in the universal cover. For each \( P \in \mathcal{P} \), there exists a \( Z'(P, x) \) such that \( Z'(P, x) \) is a \( P \)-cocompact convex subcomplex of \( \tilde{X} \) containing \( x \).

It follows immediately that there exists a \( Q \geq 0 \) such that the cubical convex hull of \( P x \) is contained in \( \mathcal{N}_Q(P x) \).

### 7. A Malnormal Quasiconvex Fully \( \mathcal{P} \)-Elliptic Hierarchy

For the following section, let \( X \) be a non-positively curved cube complex with \( \text{CAT}(0) \) universal cover \( \tilde{X} \) and \( G = \pi_1(X) \) hyperbolic relative to subgroups \( \mathcal{P} := \{P_1, \ldots, P_n\} \). Fix a base point \( x \in \tilde{X} \). For each \( P \in \mathcal{P} \), there is a convex subcomplex \( Z(P, x) \) that is a \( P \)-cocompact subcomplex of \( \tilde{X} \) containing \( P x \).

By Proposition 2.7, there is a \( \delta \geq 0 \) so that triangles in \( \tilde{X} \) are \( \delta \)-thin relative to \( B_0 := \{g P x : g \in G, P \in \mathcal{P} \} \). Since \((G, \mathcal{P})\) is a relatively hyperbolic pair, there exists \( s_0 : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \text{diam}(N_t(g_1 P_1 x) \cap N_t(g_2 P_2 x)) \leq s_0(t) \). Let \( Z(P, x) \) be the cubical convex hull of \( \mathcal{N}_{2\delta}(Z'(P, x)) \). Theorem 6.10 implies \( Z(P, x) \) is also \( P \)-cocompact. By convexity and Proposition 2.2 there exists \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) so that

\[
\text{diam}(N_t(g_1 Z(P_1, x) \cap N_t(g_2 Z(P_2, x)))) \leq f(t)
\]

for all \( g_1, g_2 \in G, P_1, P_2 \in \mathcal{P} \) with \( g_1 Z(P_1, x) \neq g_2 Z(P_2, x) \). Then by Proposition 5.3 \( Z(P, x) \) is \((3M + 6(R + 2\delta) + 9\delta)\)--attractive in the sense of Definition 5.2.

#### 7.1. Superconvexity, Peripheral Complexes and Augmented Complexes.

Bi-infinite geodesics contained in a bounded neighborhood of \( Z(P, x) \) actually lie in \( Z(P, x) \).

**Definition 7.1.** Let \( X \) be a non-positively curved cube complex and let \( \phi : Z \to X \) be a local isometry. The map \( \phi \) is superconvex if for any elevation \( \bar{\phi} : \bar{Z} \to \bar{X} \) of \( Z \) to the universal cover \( \tilde{X} \) of \( X \) and any bi-infinite geodesic \( \gamma \) in \( \tilde{X} \) such that \( d_{\text{Haus}}(\gamma, \bar{Z}) \) is bounded, then \( \gamma \subseteq \bar{Z} \).

If the immersion \( \phi : Z \to X \) is superconvex, then \( Z \) is said to be superconvex in \( X \) (with respect to \( \phi \)).

**Proposition 7.2.** The immersed quotient \( \bar{Z}(P, x) := P \backslash Z(P, x) \) of \( Z(P, x) \) in \( X \) is superconvex.

**Proof.** Suppose \( \gamma \) is a bi-infinite geodesic contained in \( \mathcal{N}_R(Z(P, x)) \) and \( p \in \gamma \). There exist \( s_1, s_2 \in \gamma \) so that \( p \in [s_1, s_2] \) and \( d(s_1, p) > 3M + 6R + 9\delta \). Hence there exist points \( t_1, t_2 \) so that \( t_1 \in [s_1, p] \) and \( t_2 \in [p, s_2] \) so that \( t_1, t_2 \in Z(P, x) \). Therefore by convexity \( p \in Z(P, x) \). Hence \( \gamma \subseteq Z(P, x) \).

The immersed complexes \( \bar{Z}(P, x) \) constructed in Proposition 7.2 are called peripheral complexes. There is a convenient way to upgrade the immersion to an embedding:
Definition 7.3. Let \( X \) be a non-positively curved cube complex with \( \text{CAT}(0) \) universal cover \( \tilde{X} \) and \( G := \pi_1(X) \). Let \((G, P)\) be a relatively hyperbolic pair. Let \( \mathcal{Z} := \bigsqcup_{P \in \mathcal{P}} \tilde{Z}(P, x) \). The augmented cube complex for the pair \((X, \mathcal{Z})\) is the complex:

\[
C(X, \mathcal{Z}) = X \cup \left( \bigsqcup_{P \in \mathcal{P}} \tilde{Z}(P, x) \times [0, 1] \right)/\langle \tilde{Z}(P, x) \times \{1\} \rangle \sim \phi_{P, x}(Z(P, x)),
\]

consisting of the mapping cylinders of the \( \phi_{P, x} \).

The hyperplanes \( \tilde{Z}(P, x) \times \frac{1}{2} \) are called peripheral hyperplanes while the remaining hyperplanes of \( C(X, \mathcal{Z}) \) are non-peripheral.

Note that the non-peripheral hyperplanes of \( C(X, \mathcal{Z}) \) are in one-to-one correspondence with the hyperplanes of \( X \).

Since \( \pi_1G \cong \pi_1(C(X, \mathcal{Z})) \), a hierarchy for \( \pi_1(C(X, \mathcal{Z})) \) determines a hierarchy of \( \pi_1G \).

Lemma 7.4. If \( X \) is special, then \( C(X, \mathcal{Z}) \) is special.

Technically, the definition of \( C(X, \mathcal{Z}) \) depends on the base point, but since the following results are given up to conjugacy, there is no need to keep track of base points.

For the remainder of this section, let \( M = f(5\delta) \).

Proposition 7.5. Let \( C(X, \mathcal{Z}) \) be the augmented cube complex for the pair \((X, \mathcal{Z})\) described in Definition 7.3. Let \( \tilde{C} \) be the universal cover of \( C(X, \mathcal{Z}) \). Let \( \mathcal{B} \) be the collection of (images of) elevations of \( \phi_{P, x} \) to \( \tilde{C} \).

The following hold:

1. Each \( B \in \mathcal{B} \) is convex,
2. \((\tilde{C}, \mathcal{B})\) is a \((\delta, f)\) relatively hyperbolic pair (recall Definition 5.1),
3. Every \( B \in \mathcal{B} \) is \((3M + 6(R + 2\delta) + 9\delta)\)-attractive (recall Definition 5.2).

Proof. The universal cover \( \tilde{X} \) of \( X \) embeds as a convex subset of \( \tilde{C} \) so that each \( B \in \mathcal{B} \) intersects \( \tilde{X} \) in a translate of some \( Z(P, x) \). Since \( B \) intersects \( \tilde{X} \) in a closed convex subspace, \( B \) is convex in \( \tilde{C} \).

Since triangles in \( \tilde{X} \) are \( \delta \)-thin relative to translates of \( Z(P, x) \), triangles in \( \tilde{C} \) are \( \delta = \delta \) thin relative to \( \mathcal{B} \). For every \( B_1, B_2 \) in \( \mathcal{B} \) with \( B_1 \neq B_2 \), \( \mathcal{N}_\delta(B_1) \cap \mathcal{N}_\delta(B_2) \) coincides with the intersection of \( g_1Z(P_1, x) \) and \( g_2Z(P_2, x) \) for some \( g_1, g_2 \in G \) and \( P_1, P_2 \in \mathcal{P} \), so by the properties of \( f \), \((\tilde{X}, \mathcal{B})\) is a \((\delta, f)\) relatively hyperbolic pair.

Attractiveness follows from the attractiveness of the \( Z(P, x) \). \( \square \)

7.2. The Double Dot Hierarchy. The construction of a hierarchy will use a finite cover called the double dot cover whose construction is originally due to Wise [30] Construction 9.1. This treatment of the double dot cover is similar to the one in [3] Section 5].
Definition 7.6 ([30, Construction 9.1]). Let $X$ be a cube complex, let $W \subseteq X$ be a hyperplane of $X$. Let $\gamma$ be a based loop and let $[\gamma] \in \pi_1 X$. Then $[\gamma]$ has a well defined (mod 2) intersection number with $W$.

Let $\mathcal{W}$ be the set of embedded, 2-sided, non-separating hyperplanes of $X$. Then there exist maps $i_W : \pi_1 X \to \mathbb{Z}/2\mathbb{Z}$, a map:

$$\Phi : \pi_1 X \to \bigoplus_{W \in \mathcal{W}} \mathbb{Z}/2\mathbb{Z} \quad \Phi = \bigoplus_{W \in \mathcal{W}} i_W$$

The **double dot cover** of $X$ is the cover corresponding to the subgroup $\ker \Phi \subseteq \pi_1 X$.

The double dot cover of a cube complex is usually a high degree cover. Therefore, constructing examples can be quite difficult. Fortunately, the double dot cover of a rose with 2 petals is easy to construct:

Example 7.7. See Figure 8 for the double dot cover of the figure 8 loop.

An important feature of the double dot cover is that the cover is taken over non-separating hyperplanes. This serves two purposes: first, making sure that double dot cover is not trivial and second, making sure that the double dot hierarchy constructed later has non-trivial splittings.

Fortunately, there is a way to obtain a complex where every hyperplane is non-separating:

Theorem 7.8 ([6, Proposition 2.12]). Let $X$ be a compact special NPC cube complex, then $X$ is homotopy equivalent to a compact special NPC cube complex whose hyperplanes are all non-separating.

Let $X$ be a special cube complex with finitely many hyperplanes $\mathcal{W} := \{W_1, \ldots, W_n\}$ where every hyperplane is non-separating and let $p : \tilde{X} \to X$ be the double dot cover of $X$. The hyperplanes of $\tilde{X}$ are elevations of hyperplanes of $X$, and they divide $\tilde{X}$ in a natural way. Let $x \in \tilde{X} \setminus \bigcup p^{-1}(\mathcal{W})$. Any two paths $\gamma_1, \gamma_2$ between $x$ and a lift of $p(x)$ to $\tilde{X}$ represent the same element of $\ker \phi$ precisely when the number of
times $\gamma_1$ and $\gamma_2$ cross elevations of $W$ agree (mod 2) for every $W \in \mathcal{W}$. Therefore, $\tilde{X} \setminus \bigcup p^{-1}(\mathcal{W})$ has components labeled by an element of $\bigoplus_{W \in \mathcal{W}} \mathbb{Z}/2\mathbb{Z}$.

These labels will help organize the vertex spaces of a hierarchy. Given a compact special non-positively curved cube complex $X$ whose hyperplanes are all non-separating, a local isometric immersion $\Phi: \mathcal{Z} \to X$ and an ordering on the hyperplanes of $C_\Phi$, the augmented cube complex induced by $\Phi$ (recall Definition [7,3]), the double dot hierarchy will produce a hierarchy of spaces for $C_\Phi$. When these inputs satisfy certain criteria discussed in Section [7,3] the double dot hierarchy gives rise to a quasiconvex, and fully $P$-elliptic hierarchy of groups for $\pi_1(\tilde{C}_\Phi)$ which is isomorphic to a finite index subgroup of $\pi_1X$. Passing to a particular finite cover will produce an induced hierarchy that is also malnormal. The next several paragraphs outline the construction of the double dot hierarchy as it is presented in [3, Section 5].

Let $X$ be a compact special NPC cube complex whose hyperplanes are all non-separating. Let $\mathcal{Z} = \bigcup_{i=1}^n \mathcal{Z}_i$ and let $\Phi: \mathcal{Z} \to X$ be a local isometric immersion of NPC cube complexes. Let $C := C_\Phi$ be the augmented cube complex. Recall that $\Phi$ can now be canonically regarded as an embedding $\Phi: \mathcal{Z} \to C$. Let $\mathcal{W}$ be the set of non-peripheral hyperplanes of $C$ and let $W_1, \ldots, W_n$ be an ordering of the elements of $\mathcal{W}$. Let $p: \tilde{C} \to C$ be the double dot cover and let $\tilde{Z} := p^{-1}(\mathcal{Z})$.

As above, choose a basepoint $x \in \tilde{C}$ with $p(x) \notin \bigcup \mathcal{W}$ so that each component of $C \setminus p^{-1}(\bigcup \mathcal{W})$ is labeled by a vector $\hat{t} \in \bigoplus_{i=1}^n \mathbb{Z}/2\mathbb{Z}$. For each $1 \leq i \leq n$, let $W_i$ be the first $i$ hyperplanes and let $M_i = \bigoplus_{j=1}^i \mathbb{Z}/2\mathbb{Z}$, and as before, the complementary components of $\bigcup W_i$ are labeled by elements of $M_i$. For each $\hat{t} \in M_i$, let $K_\hat{t}$ be the closure of the union of components labeled by $\hat{t}$.

For each $\hat{t} \in M_i$, a $\hat{t}$-vertex space at level $n-i+1$ consists of components of $K_\hat{t} \cup \tilde{Z}$ that intersect $K_{\hat{t}}$. In the construction of the double dot hierarchy, the set of components of $\hat{t}$-vertex spaces at level $n-i+1$ specifies all of the vertex spaces at each level, but the actual graph of spaces structure at each level must be described.

If $A$ is the closure of a component of $e^{-(i)}(W_i) \setminus \bigcup_{j<i} p^{-1}(W_j)$, then $A$ is called a partly-cut-up elevation of $W_i$. The double dot hierarchy is constructed by cutting along an elevation of a hyperplane $W_i$ to $\tilde{C}$ and any elements of $\tilde{Z}$ that intersect $W_i$, but the elevation of the hyperplane $W_i$ may have already been cut by one of the other hyperplane elevations of $W_j$ with $j < i$.

By construction, any two $\hat{t}$-vertex spaces at level $n-i+1$ are either disjoint or intersect in a union of components of $\tilde{Z}$ and disjoint partly-cut-up elevations of $W_i$.

Now it is time to construct the graph of spaces structures at each level. Let $V$ be a vertex space at level $n-i+1$ so that $V$ is the $\hat{t}$-vertex space for some $\hat{t} \in M_i$. Consider the canonical projection $\pi: M_{i+1} \to M_i$, let $\hat{t}^+$ and $\hat{t}^-$ be the preimages of $\hat{t}$ under $\pi$. Let $\hat{V}^+ := \{V_1^+, \ldots, V_p^+\}$ and $\hat{V}^- := \{V_1^-, \ldots, V_m^-\}$ be the components labeled by $\hat{t}^+$ and $\hat{t}^-$ respectively. Then $V = \bigcup \hat{V}^+ \cup \hat{V}^-$. By construction, elements
of $\hat{V}^+$ are pairwise disjoint and similarly, elements of $\hat{V}^-$ are pairwise disjoint. The elements of $\hat{V}^+$ and $\hat{V}^-$ are the vertex spaces in the graph of spaces for $V$ and this graph of spaces will have a bipartite underlying graph $\Gamma$. For convenience, the edges representing multiple components can be repeated so that each edge space is connected and the edges of $\Gamma$ are in one-to-one correspondence with components of $\bigcup(\hat{V}^+) \cap (\bigcup \hat{V}^-)$. The attaching maps are the inclusion maps of edge spaces into vertex spaces while the realization is provided by a homotopy equivalence collapsing the mapping cylinders of the edge spaces onto the images of the edge spaces.

Let $\hat{t} \in M_n$. Then the components of the $\hat{t}$-vertex spaces are the vertex spaces of level 1 of the hierarchy, so the terminal spaces of the hierarchy are precisely these spaces.

**Definition 7.9.** The hierarchy $\mathcal{H}$ constructed in the preceding paragraphs with vertex spaces is called the **double dot hierarchy for the pair $(X, Z)$**.

The double dot hierarchy actually depends on an ordering on the hyperplanes, but the applications that follow only need an existence of a hierarchy given some local isometric immersion $Z \to X$, so this complication will be henceforth ignored.

For general NPC cube complexes, the double dot hierarchy may fail to be faithful and even if it is, may fail to be quasiconvex or malnormal. Also, the terminal spaces may not be useful. However, when hyperplanes are embedded, nonseparating and two-sided, the terminal spaces are easy to understand:

**Lemma 7.10** ([3, Lemma 5.2]). Let $\Phi : Z \to X$ be a local isometric immersion of NPC special compact cube complexes. Let $C := C_\Phi$ be the augmented cube complex and let $p : \tilde{C} \to X$ be the double dot cover. Let $\tilde{Z} = p^{-1}(Z)$. Suppose that every hyperplane of $X$ is nonseparating. If $Y$ is a terminal space of the double dot hierarchy for $(X, Z)$, then $Y$ has a graph of spaces structure $(\Gamma, \chi)$ such that:

1. $\Gamma$ is bipartite with vertex set $V = V^+ \cup V^-,$
2. if $v \in V^+$, $\chi(v)$ is contractible,
3. if $v \in V^-$, $\chi(v)$ is a component of $\tilde{Z}$ and
4. every edge space is contractible.

**Corollary 7.11.** Under the same assumptions as Lemma 7.10, the fundamental group of a terminal space of the double dot hierarchy is a free product of the form $(\ast_{i=1}^n G_i) \ast F$ where $F$ is a finitely generated free group and each $G_i := \pi_1(Z_i)$ where $Z_i$ is a component of $\tilde{Z}$.

### 7.3. A fully $P$-elliptic malnormal quasiconvex hierarchy.

When $Z$ is a union of the complexes constructed in Proposition 7.2 and $X$ is compact special, strategically passing to finite covers and building the double dot hierarchy will produce a faithful, quasiconvex and fully $P$-elliptic hierarchy. Let $C := C_\Phi$ be the augmented complex for the pair $(X, Z)$ and let $\tilde{C}$ be its universal cover. Each edge space of
the double dot hierarchy consists of unions of components of \( \tilde{Z} \) and partly-cut-up hyperplane elevations of a single hyperplane of \( C \).

For the following, let \( X_0 \) be a NPC compact special cube complex. By Theorem \([7,8]\) there exists a homotopy equivalent compact special cube complex \( X \) whose hyperplanes are all non-separating. Let \( \tilde{X} \) be the universal cover of \( X \). Let \( G := \pi_1X \cong \pi_1X_0 \) and suppose that \((G,P)\) is a relatively hyperbolic group pair. Let \( x \in \tilde{X} \) be a base point in \( \tilde{X} \) not in any hyperplane of \( \tilde{X} \).

By Proposition \([7,2]\) for each \( P \in \mathcal{P} \) there exists a complex \( Z_P \) and superconvex local isometric immersions \( \phi_P : Z_P \to X \) such that \( \pi_1Z_P \cong P \) and the image of \( \pi_1Z_P \) in \( G \) is conjugate to \( P \) in \( G \). Let \( \Phi : \bigcup_{P \in \mathcal{P}} Z_P \to \tilde{X} \) so that \( \Phi \vert_{Z_i} = \phi_i \). The map \( \Phi \) is still a superconvex local isometric immersion.

Let \( \lambda, \epsilon \) be the augmented cube complex of the pair \((X,Z)\).

**Lemma 7.12.** Let \( C' \) be a finite regular cover of \( C_1 \). Then:

1. There exists a finite cover \( X' \) of \( X \) with \( G' := \pi_1X' \) and a superconvex local isometric immersion \( \Phi' : Z' \to X' \) such that \((G',\mathcal{P}')\) is the induced relatively hyperbolic group pair (see Proposition \([2,12]\)) and \( C' \) is the augmented cube complex of the pair \((X',\mathcal{P}')\). The components of \( Z' \) have fundamental group isomorphic to elements of \( \mathcal{P}' \) and for each component \( Z \) of \( Z' \), the image of \( \pi_1Z \) is conjugate to an element of \( \mathcal{P}' \) in \( G' \).
2. Every nonperipheral hyperplane of \( C_1 \) is nonseparating.

Let \( Z'_1, \ldots, Z'_n \) be the components of \( Z' \).

Recall that \( B \) is the collection of elevations of the mapping cylinders of the \( Z_i \in Z' \) and that \( \tilde{C}, B \) is a \((\delta, f)\) relatively hyperbolic pair.

The next statement will not be used until later, but is needed now to help strategically fix constants. Fix a finite generating set and let \( \Gamma \) be some Cayley graph of \( G \). Set \( \lambda \geq 1, \epsilon \geq 0 \) so that the orbit map takes geodesics in \( \Gamma \) to \((\lambda, \epsilon)\)-quasigeodesics in \( \tilde{C} \) and set \( \ell \) so that every \((\lambda, \epsilon)\)-quasigeodesic in \( \tilde{C} \) has the \( \ell \)-fellow traveling property relative to \( B \) (recall Theorem \([4,2]\)).

**Proposition 7.13.** Let \( L \geq 0, \lambda \geq 1, \) and \( \epsilon \geq 0 \). There exist \( \alpha \geq 1, \beta \geq 0 \) (depending only on \( G, \delta, M, L \) ) so that the following holds. Let \( E \to C_\Phi \) be a \( \pi_1 \)-injective immersion so that every \((\lambda, \epsilon)\)-quasigeodesic with endpoints in an elevation \( \tilde{E} \) of \( E \) to \( \tilde{C} \) lies in \( \mathcal{N}_L(\tilde{E}) \). Then \( \tilde{E} \to \tilde{C} \) is an \((\alpha, \beta)\)-quasiisometric embedding.

**Proof.** Let \( \Gamma \) be the Cayley graph of \( G \) with respect to some finite generating set. Choose \( \lambda, \epsilon \) so that the orbit map \( \Gamma \to \tilde{C} \) is a \((\lambda, \epsilon)\)-quasi-isometry. Then \( \pi_1(E) \) is quasi-convex in \( \Gamma \) with constants depending on \( \lambda, \epsilon, L \). Therefore, by \([7]\) Lemma III.1.3.5, the map \( \pi_1(E) \to \Gamma \) is a quasi-isometric embedding. By passing through the orbit map \( \Gamma \to \tilde{C}, \tilde{E} \) is quasi-isometrically embedded in \( \tilde{C} \) where the constants again depend only on \( L, \lambda, \epsilon \).
The quasi-isometric correspondence between $\Gamma$ and $\tilde{C}$ implies that $\pi_1(E)$ is quasiconvex in $\Gamma$ with constants depending only on $L$ and the quasi-isometry. \hfill \Box

Fix $\lambda, \epsilon$ as in the proof of Proposition 7.13. Fix $L$ so that any subspace of $\tilde{C}$ that satisfies Proposition 5.10 (with respect to $B, M, \delta$) has the property that any geodesic lies in the $L-1$ neighborhood of that subspace. Fix $\alpha, \beta$ as in Proposition 7.13 (with respect to $L$). Finally, set $\ell$ so that both $(\alpha, \beta)$ and $(\lambda, \epsilon)$-quasigeodesics $\ell$-fellow travel relative to $B$ (recall Proposition 4.2).

Let $R_0 > \max(8f(\max(\ell, 40M + 40\delta)), 8(6\ell + 259M + 296\delta))$. Using Proposition 6.7 let $C_2$ be a finite regular cover of $C_1$ such that every non-peripheral hyperplane of $C_2$ is $R_0$-embedded and nonseparating. Let $C_2$ be the augmented cube complex of a pair $(X_2, Z'')$ where $X_2$ is a finite cover of $X$. Note that $\tilde{X}$ naturally embeds in $\tilde{C}$, the universal cover of $C_2$ so that $\tilde{C}$ has triangles that are $\delta$-thin relative to $B$. Let $G_2 = \pi_1(C_2)$ and let $(G_2, \mathcal{P}'')$ be the induced peripheral structure. Let $(\tilde{G}_2, \tilde{\mathcal{P}}'')$ be the induced peripheral structure $\tilde{G}_2 := \pi_1\tilde{C}_2$.

The next few statements will show that the double dot hierarchy on $c: \tilde{C}_2 \to C_2$, the double dot cover of $C_2$, is faithful, quasiconvex and fully $\tilde{\mathcal{P}}''$-elliptic hierarchy for $\pi_1\tilde{C}_2$. Passing to a finite regular cover will later yield a hierarchy which is also malnormal.

Recall, $\tilde{C}_2$ is an augmented cube complex with respect to a pair $(\tilde{X}_2, \tilde{Z}_2)$ where $\tilde{Z}_2$ consists of components of $c^{-1}(Z'')$. Let $B$ be the collection of elevations of the mapping cylinders of $\tilde{Z}_2$ (with respect to the augmentation) and let $\tilde{Z}$ be the collection of elevations of elements of $\tilde{Z}_2$ to $\tilde{C}$. Let $E$ be an edge space of the double dot hierarchy on $\tilde{C}_2$. Let $W$ be a partly-cut-up elevation of a non-peripheral hyperplane to $\tilde{C}_2$ so that $E$ is a union of $W$ and elements of $\tilde{Z}_2$.

Let $\tilde{E}$ be an elevation of $E$ to $\tilde{C}$. There exist $A_E$ and $B_E$ so that $A_E$ is a collection of convex partially cut up hyperplane elevations of $W$ to $\tilde{C}$ and $B_E \subseteq \tilde{Z}$ so that $\tilde{E}$ is a union of the elements of $A_E$ and $B_E$. Let $B'_E \subseteq B$ be the collection of elevations of mapping cylinders of components of $\tilde{Z}_2$ into $\tilde{C}_2$ that intersect elements of $B_E$ non-trivially. Let $\tilde{E}'$ be the image of $(\sqcup A_E) \cup (\sqcup B'_E)$ in $\tilde{C}$.

**Proposition 7.14.** Let $E$ be an edge space of the double dot hierarchy on $\tilde{C}_2$. Then the map $E \to \tilde{C}_2$ is $\pi_1$ injective.

**Proof.** Suppose not toward a contradiction. Then there exists a loop $\gamma$ in $E$ such that $\gamma$ is essential in $E$ but has trivial image in $\pi_1(\tilde{C}_2)$.

Since $\gamma$ is $\pi_1$ trivial in $\pi_1(\tilde{C}_2)$, $\gamma$ elevates to a loop $\tilde{\gamma} \subseteq \tilde{E}$ in $\tilde{C}$. Since $\tilde{E}$ is homotopy equivalent to $\tilde{E}'$, there is a loop $\tilde{\gamma}'$ that is the image of a geodesic in $\tilde{E}'$. Recall that the elements of $A_E$ are convex and for all distinct pairs of $A_1, A_2 \in A_E$, $d(A_1, A_2) \geq 2R_0$ by Proposition 6.7. Since $\tilde{E}$ is the image of $(\sqcup A_E) \cup (\sqcup B'_E)$ in $\tilde{C}$, $\tilde{\gamma}'$ cannot be a loop by Proposition 5.14. \hfill \Box
The next step is to prove that the double dot hierarchy on $\tilde{C}_2$ is quasiconvex. Recall $\alpha, \beta$ as in Proposition 7.13.

**Proposition 7.15.** If $E$ is an edge space of the double dot hierarchy on $\tilde{C}_2$ and $\tilde{E}$ is the universal cover of $E$, then any elevation $\tilde{E} \rightarrow \tilde{C}$ of $E$ to $\tilde{C}$ is an $(\alpha, \beta)$-quasiisometric embedding.

**Proof.** Since every hyperplane in $\tilde{C}_2$ is $R_0$-embedded, for any pair $A_1, A_2 \in A_E$, $d(A_1, A_2) \geq 2R_0$. Also $\tilde{C}$ is a $(\delta, f)$ relatively hyperbolic pair relative to $B$ with $M \geq f(5\delta)$, and each $A \in A_E$ and $B \in B'_E$ is convex in $\tilde{C}$. Let $\gamma$ be a $(\lambda, \epsilon)$-quasigeodesic (in $\tilde{C}$) with endpoints in $\tilde{E}$. Then $\gamma$ is a $(\lambda, \epsilon)$-quasigeodesic with end endpoints in $\tilde{E}'$, so by Proposition 5.16, there exists $L \geq 1$ so that $\gamma \in N_{L-1}(\tilde{E}') \subseteq N_{L}(\tilde{E})$. Therefore by Proposition 7.13, the map $\tilde{E} \rightarrow \tilde{C}$ is an $(\alpha, \beta)$-quasiisometric embedding. □

Proposition 7.14 and Proposition 7.15 together yield the following:

**Corollary 7.16.** The double dot hierarchy induced on $\pi_1 \tilde{C}_2$ is faithful and quasiconvex.

The next step is to prove that the double dot hierarchy on $\tilde{C}_2$ is fully $\tilde{P}''$-elliptic. Definition 7.17 introduces geometric terminology for the situation where a subgroup of a relatively hyperbolic group pair $(G, P)$ contains an element $g$ conjugate into a peripheral subgroup $P$ such that no positive power of $g$ lies in $E \cap P$.

**Definition 7.17.** Let $(\tilde{X}, B)$ be an $(\delta, M)$-relatively hyperbolic pair and let $\tilde{X} \rightarrow X$ be a covering. Let $B$ be a locally convex subspace of $X$. Let $E \subseteq X$. The subspace $E$ has an accidental $B$ loop if there exists a homotopically essential loop, $\gamma$, which is both freely homotopic to a geodesic loop in $B$ and has no positive power homotopic in $E$ to a geodesic loop in $B$.

The next few statements will show that the edge spaces of the double dot hierarchy for $\tilde{C}_2$ have no accidental $\tilde{Z}''$-loops. This will imply the hierarchy is fully $\tilde{P}''$-elliptic. The first step is to show that elevations of partly-cut-up hyperplanes do not have accidental $\tilde{Z}''$-loops.

**Lemma 7.18 (Lemma 5.15).** Let $(X, Z)$ be a superconvex pair where each component of $Z$ is embedded and let $C$ be the corresponding augmented cube complex. For $n \geq 1$, let $\{W_1, \ldots, W_n\}$ be a collection of embedded, 2-sided, nonseparating hyperplanes of $C$. Let $Q$ be a component of $W_n \setminus \cup_{i<n} W_i$. Then $Q$ has no accidental $Z$-loops.

**Proposition 7.19.** Let $E$ be an edge space of the double dot hierarchy for $\tilde{C}_2$. Then $E$ has no accidental $\tilde{Z}''$-loops.
Proof. Recall that $E$ is a union of a partly-cut-up hyperplane elevation $Q$ and components of $\tilde{Z}''$ that intersect $Q$. By Lemma 7.18, $Q$ has no accidental $\tilde{Z}''$-loops.

Suppose there exists a $C_2$-essential loop $\gamma$ in $E$ such that $\gamma$ is freely homotopic in $\tilde{C}_2$ into $\tilde{Z}''$. Then a representative of the homotopy class of $\gamma$ lifts to a bi-infinite $\tilde{E}$-geodesic $\hat{\gamma}$ where $\tilde{E}$ is an elevation of $E$ to $\tilde{C}$, and a representative of the homotopy class of $\gamma$ lifts to a bi-infinite $\tilde{C}$-geodesic $\rho \in \tilde{Z}$, an elevation of a component of $\tilde{Z}''$ and there exists $R \geq 0$ so that $\hat{\gamma} \in N_R(\rho)$.

Since $\gamma$ is a $\tilde{E}$-geodesic, $\hat{\gamma}$ is a $(\alpha,\beta)$-quasigeodesic in $\tilde{C}$ by Proposition 7.15.

Let $\gamma_0$ be a subsegment of $\hat{\gamma}$ with $|\gamma_0| = |\gamma|$ (e.g. take $\hat{\gamma}$ to be the subsegment between two consecutive lifts of a point of $\hat{\gamma}$). If $\gamma_0 \subseteq \tilde{Z}'$ where $\tilde{Z}'$ is an elevation of a component of $\tilde{Z}''$, then $\hat{\gamma} \subseteq \tilde{Z}'$ and $\hat{\gamma}$ is geodesic in $\tilde{C}$. Then $\tilde{Z} = \tilde{Z}'$, because $\text{diam}(N_R(\tilde{Z}) \cap N_R(\tilde{Z})) = \infty$ in which case $\gamma$ was not an accidental $\tilde{Z}$ loop.

On the other hand, if $\gamma_0 \subseteq \tilde{Q}$ where $\tilde{Q}$ is some elevation of $Q$ to $\tilde{C}$, then $Q$ has an accidental $\tilde{Z}$-loop, contradicting the fact that there are no such accidental $\tilde{Z}$ loops.

Therefore, there exist subsegments $\hat{\gamma}$ of $\gamma$ with $|\gamma_0| = |\gamma|$ (e.g. take $\gamma_0$ to be the subsegment between two consecutive lifts of a point of $\gamma$) such that $\gamma_0 \subseteq \tilde{Z}'$ where $\tilde{Z}'$ is an elevation of a component of $\tilde{Z}''$, then $\hat{\gamma} \subseteq \tilde{Z}'$ and $\hat{\gamma}$ is geodesic in $\tilde{C}$. Then $\tilde{Z} = \tilde{Z}'$ because $\text{diam}(N_R(\tilde{Z}) \cap N_R(\tilde{Z})) = \infty$ in which case $\gamma$ was not an accidental $\tilde{Z}$ loop.

By construction there is a unique $\tilde{B} \in \mathcal{B}$ so that $\tilde{Z} \subseteq \tilde{B}$. Let $\tau_m$ be the $\tilde{C}$-geodesic connecting the endpoints of $\gamma_m$. Since $\tau_m \subseteq N_R(B)$, all but a connected subsegment of length $3M + 6R + 9\delta$ of $\tau_m$ lies in $B$.

Since $(\alpha,\beta)$ quasigeodesics have the $\ell$-relative fellow traveling property, each $b_{m,i}$ lies in $N_{\ell + M}(\tilde{Z}_{m,i})$. For each $2 \leq i \leq k_m$, $\tau_m$ has a length $2R_0 - 2(\ell + M)$ segment in $N_{M+\ell}(\tilde{Z}_{m,i})$. Since $2R_0 - 2(\ell + M) \geq 3f(M + \ell)$ and the intersection of any of these segments has length at most $f(M + \ell)$, for $m >> 0$, there are subsegments $\tau_{m,1}$ and $\tau_{m,2}$ so that $\tau_{m,i} \subseteq N_{M+\ell}(\tilde{Z}_{m,i}) \cap B$ (since $(m - 2)f(\ell + M) > 3M + 6R + 9\delta$ for $m >> 0$). Therefore, $\tilde{Z}_{m,1}, \tilde{Z}_{m,2} \subseteq \tilde{B}$ because the diameter of $B \cap N_L(\tilde{Z}_{m,i})$ is bounded above by $f(\ell + M)$ unless $B$ contains $\tilde{Z}_{m,i}$. This is impossible because $\tilde{Z}_{m,1} \neq \tilde{Z}_{m,2}$, but each $B \in \mathcal{B}$ contains a unique $\tilde{Z}_{m,i}$. Therefore, $\gamma$ cannot be an accidental $\tilde{Z}''$-loop. □

Corollary 7.20. The double dot hierarchy on $\tilde{C}_2$ is fully $\tilde{P}''$-elliptic.

Faithfulness, quasiconvexity and full $\mathcal{P}$-ellipticity are preserved by taking the induced hierarchy of a finite regular cover of $\tilde{C}_2$. The final step is to show that there exists a finite cover of $\tilde{C}_2$ whose induced hierarchy is also a malnormal hierarchy.

The following lemma is straightforward:

Lemma 7.21. Suppose $H \leq G$ and $G_0$ is a finite index subgroup of $G$ and let $H_0 = H \cap G_0$. If $H$ is malnormal in $G$, then $H_0$ is malnormal in $H$. 

Proposition 7.22. Let $G$ be the fundamental group of a relatively hyperbolic special compact NPC cube complex, and let $H \leq G$ be full relatively quasiconvex. Then $H$ is separable in $G$.

The idea is to follow the proof of Theorem 7.3 of \cite{13} except to use Theorem 6.10 to produce a cocompact convex core for $H$. With a convex core for $H$, there is a compact cube complex $A$, special cube complex $X$ with $\pi_1(X) = G$ and local isometry $f : A \to X$.

Proposition 7.23. (Hruska-Wise \cite{17, Theorem 9.3}) If $G$ is relatively hyperbolic and $H \leq G$ is relatively quasiconvex and separable, then there exists a finite index subgroup $K \leq G$ containing $H$ such that for every $g \in K$ either $gHg^{-1} \cap H$ is finite or $gHg^{-1} \cap H$ is parabolic in $K$.

Therefore, if $H$ is also full relatively quasiconvex, then $H$ is almost malnormal in $K$.

The following is based on \cite{3} (Corollary 3.29) and follows immediately from the two preceding statements and the fact that when $G$ is virtually special, $G$ is linear and hence virtually torsion free.

**Corollary 7.24.** If $G$ is hyperbolic relative to $\mathcal{P}$ and special, and $H \leq G$ is full relatively quasiconvex, then $H$ is virtually malnormal.

**Theorem 7.25.** Let $G$ be special, virtually torsion-free and let $(G, \mathcal{P})$ be a relatively hyperbolic pair. Let $\mathcal{H}$ be a fully $\mathcal{P}$-elliptic quasiconvex hierarchy for $G$, then there exists a finite index subgroup $G_0 \leq G$ with induced fully $\mathcal{P}$-elliptic quasiconvex hierarchy $\mathcal{H}_0$ of $G_0$ which is malnormal and fully $\mathcal{P}$-elliptic.

The proof here is nearly the same as in Theorem 3.30 of \cite{3}.

**Proof.** Because $\mathcal{H}$ is fully $\mathcal{P}$-elliptic, the edge subgroups are full. By \cite{3} Lemma 3.24, if $H \leq G$ is almost malnormal in $G$ and $G_0$ is a finite index normal subgroup of $G$, then $H \cap G_0$ is malnormal in $G_0$. Since there are finitely many edge groups, by Corollary 7.24 there exists some $G_0$ such that for every edge group $E$ of $\mathcal{H}$, $E \cap G_0$ is malnormal in $G_0$. Since $G_0$ is normal, conjugation by $g \in G$ is an automorphism of $G_0$, so in particular, these edge groups $E \cap G_0$ are malnormal in $G$. \hfill $\square$

At last, it is time to prove Theorem 1.

**Theorem 1.** Let $(G, \mathcal{P})$ be a relatively hyperbolic pair and let $G$ be a virtually compact special group. Then there exists a finite index subgroup $G_0 \leq G$ and an induced relatively hyperbolic pair $(G_0, \mathcal{P}_0)$ so that $G_0$ has a quasiconvex, malnormal and fully $\mathcal{P}_0$-elliptic hierarchy terminating in groups isomorphic to elements of $\mathcal{P}_0$.

**Proof of Theorem** \cite{1} Let $X$ be a NPC virtually compact special cube complex with $G = \pi_1(X)$. 

First, pass to a finite index regular cover of $X$, $X_1$ that is special. By applying a homotopy equivalence, $X_1$ is homotopy equivalent to a cube complex where every hyperplane gives a nontrivial splitting of $\pi_1 X_1$ (see [3, Lemma 5.17]).

By Corollary 7.16 there exists a special cube complex $X'_1$ homotopy equivalent to $X_1$ with a finite regular cover $X_2$ such that $G_2 := \pi_1 X_2$ with induced peripheral structure $(G_2, \mathcal{P}_2)$ has a faithful, quasiconvex, fully $\mathcal{P}_2$-elliptic hierarchy terminating in $\mathcal{P}_2 * F_k$ where $F_k$ is a free group.

By Theorem 7.25 there exists a finite regular cover $X_0$ with $G_0 := \pi_1 X_0$ and induced peripheral structure $(G_0, \mathcal{P}_0)$ such that the induced hierarchy on $G_0$ is malnormal as well and terminates in free products of free groups and elements of $\mathcal{P}_0$. The hierarchy can then be continued to a malnormal, quasiconvex, fully-$\mathcal{P}_0$-elliptic one that terminates in $\mathcal{P}_0$. □

8. A Relatively Hyperbolic Version of the Malnormal Special Quotient Theorem

Recall Wise’s Malnormal Special Quotient Theorem (see Theorem 1.3 above or [30, Theorem 12.3]) mentioned in the introduction. The purpose of this section is to apply Theorem 1 to obtain a relatively hyperbolic version of Wise’s MSQT using techniques from [3, Sections 6-9].

Wise’s Quasiconvex Hierarchy Theorem [30, Theorem 13.3] has the following useful consequence:

**Corollary 8.1.** Let $G$ be a hyperbolic group with a quasiconvex hierarchy terminating in finite groups. Then $G$ is virtually special.

The technique for proving a relatively hyperbolic analog of Theorem 1.3 will be to start with the hierarchy provided by Theorem 1 and strategically take quotients using group theoretic Dehn fillings (see Definition 8.2). These quotients can be constructed to be hyperbolic, and with some care, the hierarchy structure can be passed down to the quotient so that Corollary 8.1 can be used. In [3], the authors avoided using Corollary 8.1 because their account aimed to give a new proof of auxiliary results used to prove Corollary 8.1. Consequently, they needed to ensure that the hierarchy structure on the quotient is also a malnormal hierarchy. Here, by using Corollary 8.1 it will only be necessary to produce a quasiconvex hierarchy for such a quotient.

8.1. Group Theoretic Dehn Filling. For this section, let $(G, \mathcal{P})$ be a relatively hyperbolic pair and let $\mathcal{P} = \{P_1, \ldots, P_m\}$ unless stated otherwise. When $M$ is a finite volume hyperbolic 3–manifold with torus cusps, a Dehn filling of $M$ is a gluing of solid tori $T_i \cong D \times S^1$ by a diffeomorphism to the boundary components. The result of the gluing depends only on the isotopy class of the curve $\gamma_i \subseteq \partial M$ that each copy of $\partial D \times \{p\} \subseteq T_i$ is glued to (see e.g. [21, Section 10.1]). In this situation
π₁M is hyperbolic relative to a collection of copies of \( \mathbb{Z}^2 \), one for each boundary component of \( M \).

The next definition is a group theoretic analog of Dehn filling

**Definition 8.2.** Let \( \{ N_i \triangleleft P_i : 1 \leq i \leq m \} \). Then there exists a **group theoretic Dehn filling** of \( G \) with **filling map** \( \pi \) defined by the quotient:

\[
\pi : G \to G(N_1, \ldots, N_m) := G/(\bigcup N_i).
\]

The subgroups \( N_i \) are called **filling kernels**.

A filling is called **peripherally finite** if each filling kernel \( N_i \) is finite index in \( P_i \).

For a classical filling, if every \( T_i \) is filled by gluing along the curves \( \gamma_i \) that are sufficiently long, Thurston’s Dehn filling theorem says that the resulting manifold is hyperbolic. The group theoretic analog of a sufficiently long classical Dehn filling is a group theoretic Dehn filling where the filling kernels avoid a finite set of elements:

**Definition 8.3.** A statement \( \Psi \) holds for all sufficiently long fillings if there exists a finite \( B \subseteq G \setminus 1 \) such that whenever \( B \cap N_i = \emptyset \) for all \( 1 \leq i \leq m \), the filling \( G(N_1, \ldots, N_m) \) has \( \Psi \).

Osin showed that sufficiently long Dehn fillings of relatively hyperbolic groups are relatively hyperbolic, have kernels which intersect each peripheral subgroup \( P_i \) precisely in \( N_i \) and can be manipulated so that any finite set of elements are not killed by the filling map.

**Theorem 8.4** ([22, Theorem 1.1]). Let \( F \subseteq G \) be any finite subset of \( G \). Then for all sufficiently long Dehn fillings:

1. \( \ker(\phi|_{P_i}) = N_i \) for \( i = 1, 2, \ldots, m \),
2. the pair \( (G(N_1, \ldots, N_m), \{ \phi(P_1), \ldots, \phi(P_m) \}) \) is a relatively hyperbolic pair, and
3. \( \phi|_{F} \) is injective.

The edge subgroups of the hierarchy from Theorem 1 will need to be full relatively quasiconvex subgroups of \( G \). The quasiconvexity of the hierarchy will ensure that these subgroups are relatively quasiconvex.

**Theorem 8.5** ([15, Theorem 1.5]). Let \( H \leq G \) be a quasi-isometrically embedded subgroup. Then \( H \) is relatively quasiconvex in \( G \).

**Theorem 8.6** ([15, Theorem 1.2]). Let \( H \leq G \) be relatively quasiconvex. Then there exists a relatively hyperbolic structure \( (H, \mathcal{D}) \) where \( \mathcal{D} \) is finite and every element of \( \mathcal{D} \) is conjugate into an element of \( \mathcal{P} \).
For the following, it is convenient to introduce equivalent formulations of relative hyperbolicity and relative quasiconvexity. Let $\Gamma(G, S)$ be a Cayley graph for $G$ and let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a finite collection of subgroups of $G$. For each $i$, let $T_i$ be a left transversal for $P_i$.

**Definition 8.7** (see [11, Section 3]). For each pair $(i, t)$ with $1 \leq i \leq m$ and $t \in T_i$, let $\Gamma(i, t)$ be the full subgraph of $\Gamma(G, S)$ containing $tP_i$. Define the combinatorial horoball $\mathcal{H}(i, t)$ as follows: the zero skeleton of $\mathcal{H}(i, t)$ is the set $\Gamma(i, t) \times \{1\}$ or $\mathbb{N})$. Add a single edge between the unordered pair $\{(g_1, y), (g_2, y)\}$ whenever $0 < d_S(g_1, g_2) \leq 2^y$, and for each $g \in tP_i$ and $y \geq 0$, add an edge connecting the unordered pair $\{(g, y), (g, y + 1)\}$.

For $n \in \mathbb{Z}^+$, a point $(g, n)$ is $n$-deep in $\mathcal{H}(i, t)$.

The cusped graph $(X, \mathcal{P}, S)$ is the space formed by taking $\Gamma(G, S) \sqcup \bigcup_{i, t} \Gamma(i, t)$ and gluing each $(tP_i, 0) \subseteq \mathcal{H}(i, t)$ to $tP_i \subseteq \Gamma(G, S)$.

**Theorem 8.8** ([11, Theorem 3.25]). The pair $(G, \mathcal{P})$ is relatively hyperbolic if and only if $(X, \mathcal{P}, S)$ is a hyperbolic graph.

By [2] Lemma 3.1, if $H \leq G$ and $(H, \mathcal{D})$ is a relatively hyperbolic pair so that every $D \in \mathcal{D}$ is conjugate into some $P \in \mathcal{P}$, then there exists a generating set $T$ such that the inclusion $\phi : H \to G$ extends to an $H$-equivariant Lipschitz map $\hat{\phi} : X(H, \mathcal{D}, T)^{(0)} \to X(G, \mathcal{P}, S)^{(0)}$.

**Definition 8.9.** Fix a relatively hyperbolic pair $(G, \mathcal{P})$. Let $H \leq G$ and let $(H, \mathcal{D})$ be a relatively hyperbolic pair so that each $D \in \mathcal{D}$ is $G$-conjugate into some $P \in \mathcal{P}$. The subgroup $H$ is horoball relatively quasiconvex if the inclusion map $\phi : H \to G$ induces a map $\hat{\phi}$ whose image is quasiconvex.

**Theorem 8.10** ([20, Theorem A.10]). A subgroup $H \leq G$ is relatively quasiconvex if and only if $H$ is horoball relatively quasiconvex.

**Proposition 8.11.** Let $(G, \mathcal{P})$ be a relatively hyperbolic pair. Let $(H, \mathcal{D})$ be a relatively hyperbolic pair such that there exists a relatively hyperbolic pair $(H, \mathcal{D})$ where the embedding $H \to G$ induces a map $\hat{\phi} : X(H, \mathcal{D}, T)^{(0)} \to X(G, \mathcal{P}, S)^{(0)}$ with quasiconvex image. Then there are finitely many $H$-conjugacy classes of infinite intersections of $H$ with a conjugate of some $P \in \mathcal{P}$.

**Proof.** Let $H \cap gPg^{-1}$ be infinite for some $P \in \mathcal{P}$ and $g \in G$. Then $H \cap gPg^{-1}$ stabilizes an infinite diameter subset of $gP$ in $\Gamma(G, S)$. Let $d_P$ denote distance in $X(G, \mathcal{P}, S)$. Then there exists $g_n \in H \cap gPg^{-1}$ and $x \in H \cap gP$ such that $d_{\mathcal{P}}(x, g_n x), d_T(x, g_n x) \to \infty$ and there is a geodesic path $\gamma_n$ connecting $x$ to $y_n := g_n x$ so that $x, y_n$ are the only points in $\gamma_n$ that are 0-deep and $\gamma_n$ has a point that is at least $n$-deep in the combinatorial horoball glued to $gP$. By quasiconvexity, for $n >> 0$, then there exists a peripheral coset $h_n D_n$ for some $h_n \in H, D_n \in \mathcal{D}$ such that $x, y_n \in h_n D_n$. There
exist only finitely many $D$ cosets containing $x$, so $H \cap gPg^{-1}$ has infinite intersection with $hDh^{-1}$ for some $h \in H$ and $D \in D$. Therefore, $hDh^{-1}$ is conjugate into $P$ by $g$ and $hDh^{-1}$ and $gPg^{-1}$ fix the same parabolic point in $\partial X(G, \mathcal{P}, S)$. By the relative quasiconvexity of $H$, there are only finitely many $H$-orbits of such points, so there exist only finitely many $H$-conjugacy classes of conjugates of elements of $\mathcal{P}$ that have infinite intersection with $H$. □

**Corollary 8.12.** The collection $D$ can be modified so that:

1. Every element of $D$ is infinite.
2. Every infinite intersection of $H$ with a conjugate of some $P \in \mathcal{P}$ is conjugate in $H$ to some element of $D$.

**Proof.** For the first statement, simply remove all finite elements of $D$. Since $D$ was finite, removing finite sized elements of $D$ will affect distances between points of $G$ in the coned-off Cayley graph by at most a fixed constant.

The second statement follows immediately from Proposition 8.11 □

When a filling of $G$ interacts nicely with a subgroup $H$, it is possible to induce a filling on the subgroup $H$.

**Definition 8.13** ([20, Definition B.1]). Let $H \leq G$. A filling $G \to G(N_1, \ldots, N_m)$ is an $H$-filling if whenever $gP_i g^{-1} \cap H$ is infinite for some $P_i \in \mathcal{P}$, then $gN_i g^{-1} \subseteq H$.

**Definition 8.14.** If $H \leq G$ is a relatively quasiconvex subgroup and $(H, D)$ is the relatively hyperbolic structure from Theorem 8.6 and Corollary 8.12. Let $\pi : G \to G(N_1, \ldots, N_m)$ be an $H$-filling. Let $D_j \in D$. Then there exists some $P_i \in \mathcal{P}$ and $g \in G$ with $g^{-1}D_j g \subseteq P_i$. Let $K_j := gN_i g^{-1}$. Since $\pi$ is an $H$-filling, $K_j \triangleleft D_j$, so the groups $K_j$ determine a filling:

$$\pi_H : H \to H(K_1, \ldots, K_N)$$

called the induced filling of $H$ with respect to $G(N_1, \ldots, N_m)$.

Since $N_i$ is normal in $P_i$, then groups $K_j$ (and hence the filling) do not depend on the choice of $g \in G$.

The following theorem appears as stated in [3] as Theorem 7.11 and collects results about induced Dehn fillings from [2]:

**Theorem 8.15.** Let $H \leq G$ be a full relatively quasiconvex subgroup and let $F \subseteq G$ be a finite subset. For all sufficiently long $H$-fillings, $\phi : G \to G(N_1, \ldots, N_m)$ of $G$:

1. $\phi(H)$ is a full relatively quasiconvex subgroup of $G(N_1, \ldots, N_m)$,
2. $\phi(H)$ is isomorphic to the induced filling in that if $\phi_H : H \to H(K_1, \ldots, K_m)$ is the induced filling map, then $\ker \phi_H = \ker \phi \cap H$, and
3. $\phi(F) \cap \phi(H) = \phi(F \cap H)$.
8.2. The filled hierarchy. Let $\mathcal{H}$ be a quasiconvex fully $\mathcal{P}$-elliptic hierarchy. By Lemma 3.13 and Theorem 8.5 and the full $\mathcal{P}$-ellipticity of the hierarchy, the edge and vertex groups of the hierarchy are full relatively quasiconvex. Let $\pi : G \to \mathcal{G}$ be a filling and let $(\mathcal{G}, \mathcal{P})$ be the relatively hyperbolic structure induced on the filling by Theorem 8.4. The goal of this subsection is to build an induced hierarchy $\mathcal{H}$ (which may not be faithful) for $G$ based on $\mathcal{H}$ where the vertex and edge groups of $\mathcal{H}$ are induced fillings of vertex and edge groups of $\mathcal{H}$. The hierarchy $\mathcal{H}$ will be called a filled hierarchy for $(G, \mathcal{P})$.

The filled hierarchy is built by starting at the top level and building the hierarchy inductively downward.

At the top level, let $\mathcal{H}$ have the degenerate graph of groups decomposition for $G$ consisting of a single vertex labeled $G$. Let $n$ be the length of $\mathcal{H}$. Suppose the filled hierarchy has been filled down to the $(n - i)$th level and let $A$ be a vertex group at level $n - i$ so that $A$ is the induced filling of a vertex group $A$ at level $n - i$ of $\mathcal{H}$.

Let $\chi$ be the assignment map for the graph of groups $(\Gamma, \chi, T)$. If $x$ is a vertex or edge of $\Gamma$, let $A_x := \chi(x)$, the corresponding vertex or edge group. Let $A := A_x$ where $A_x$ is the induced filling $\pi_x : A_x \to A_x$. The problem is that the pair $(\Gamma, \chi)$ still needs attachment homomorphisms to be a graph of groups.

Let $\phi_e : A_e \to A_v$ be an attachment homomorphism of an edge group $A_e$ to a vertex group $A_v$. Two details need to be checked: first there need to be attachment maps $\overline{\phi}_e : A_e \to A_v$ such that $\overline{\phi}_e \circ \pi_e = \pi_v \phi_e$. Then there will need to be an isomorphism $\overline{\alpha} : \pi_1(\Gamma, \chi, T) \to \mathcal{A}$ so that $(\Gamma, \overline{\alpha}, T)$ is a graph of groups structure for $\mathcal{A}$ where $\overline{\alpha} \circ \pi_\Gamma = \pi_A \circ \alpha$.

Completing the square:

\[
\begin{array}{ccc}
A_e & \xrightarrow{\pi_e} & \mathcal{A}_e \\
\downarrow{\phi_e} & & \downarrow{\mathcal{A}_v} \\
A_v & \xrightarrow{\pi_v} & \mathcal{A}_v
\end{array}
\]

with a map $\overline{\phi}_e : A_e \to \mathcal{A}_v$ is straightforward because $\pi_e$ is surjective and $\ker \pi_e \subseteq \ker \pi_v \circ \phi_e$. Even when $\mathcal{H}$ is a faithful hierarchy, the map $\phi_e$ may fail to be injective.

Constructing the desired isomorphism $\overline{\alpha} : \pi_1(\Gamma, \chi, T) \to \mathcal{A}$ amounts to completing the square:

\[
\begin{array}{ccc}
\pi_1(\Gamma, \chi, T) & \xrightarrow{\pi_\Gamma} & \pi_1(\Gamma, \chi, T) \\
\downarrow{\alpha} & & \downarrow{\pi_A} \\
A & \xrightarrow{\pi_A} & \mathcal{A}
\end{array}
\]

Lemma 8.16. There exists an isomorphism $\overline{\alpha} : \pi_1(\Gamma, \chi, T) \to \mathcal{A}$ that completes the diagram.
Proof. It suffices to show that $\ker \pi_A = \ker(\pi_A \circ \alpha)$.

The first step is to show that $\ker \pi_\Gamma \leq \ker(\pi_A \circ \alpha)$. Let $k \in \ker \pi_\Gamma$. Then $k$ can be written as:

$$k = \prod_i k_i^{g_i}$$

where each $g_i \in \pi_1(\Gamma, \chi, T)$ and $k_i \in \ker \pi_v$ where $v_i$ is a vertex of $\Gamma$.

It then suffices to show that $\ker \pi_v \leq \ker(\pi_A \circ \alpha)$ for each vertex $v$, so assume $k \in \ker \pi_v$.

The vertex group $A_v$ is full relatively quasiconvex in $(G, \mathcal{P})$ as noted above, and by Corollary 8.12 there is an induced peripheral structure $D_v := \{D_1, \ldots, D_l\}$ on $A_v$ such that each (infinite) $D_i \in (P_{j_i})^{g_i}$, and $D_i = (P_{j_i})^{g_i}$ by fullness. The element $k$ can be written as:

$$k = \prod_\beta n_{i_\beta}$$

where $n_{j_\beta}$ lies in a filling kernel $N_{j_\beta} \triangleleft D_{j_\beta} \leq (P_{j_\beta})^{g_{j_\beta}}$. By fullness, $(P_{j_\beta})^{g_{j_\beta}}$ is conjugate in $A$ to an element of the peripheral structure on $A$ induced by $(G, \mathcal{P})$, so $n_{j_\beta}$ is conjugate to an element of some filling kernel of the induced filling $\pi_A$. Therefore $k \in \ker(\pi_A \circ \alpha)$.

On the other hand, if $k \in \ker(\pi_A \circ \alpha)$, then $k = \prod_{i=1}^l \alpha^{-1}(k_i^{g_i})$ where each $k_i \in K_{j_i} \triangleleft D_{j_i}$ and $K_{j_i}$ is a filling kernel for the induced filling $\pi_A$. By full $\mathcal{P}$-ellipticity, $\alpha^{-1}(D_{j_i})$ is conjugate into some vertex group $A_v$ of $(\Gamma, \chi)$, so $\alpha^{-1}(k_i)$ is conjugate into $\ker \pi_v$ for some $v \in V$. Therefore, $\alpha^{-1}(k_i) \in \ker \pi_\Gamma$, and $\ker(\pi_A \circ \alpha) \subseteq \ker \pi_\Gamma$. $\square$

For the following, let $(G, \mathcal{P})$ be a relatively hyperbolic pair and let $\mathcal{H}$ be a quasiconvex fully $\mathcal{P}$–elliptic hierarchy for $G$. The next lemma ties together some definitions:

**Lemma 8.17.** If $A \trianglelefteq G$ is an edge or vertex group of $\mathcal{H}$, then $A$ is a full relatively quasiconvex subgroup of $(G, \mathcal{P})$ and every filling is an $A$-filling.

**Proof.** That $A$ is full relatively quasiconvex follows immediately from the definition of full $\mathcal{P}$-ellipticity and Theorem 8.5.

Whenever $gP_ig^{-1} \cap A$ is infinite, then $gP_ig^{-1} \subseteq A$, so if $N_i \triangleleft P_i$, then $gN_ig^{-1} \triangleleft A$. $\square$

**Lemma 8.18.** Let $A$ be an edge or vertex group of $\mathcal{H}$. Then for all sufficiently long fillings:

$$\pi : (G, \mathcal{P}) \to (\overline{G}, \overline{\mathcal{P}})$$

the following hold:

1. The subgroup $\overline{A} := \phi(A)$ is full relatively quasiconvex in $(\overline{G}, \overline{\mathcal{P}})$.
2. If $\overline{G}$ is hyperbolic, then $\overline{A}$ is quasiconvex in $\overline{G}$.
3. The subgroup $\overline{A}$ is isomorphic to the induced filling of $A$. 


Proof. There are only finitely many edge and vertex groups, so the first and third statements follow from Theorem 8.15.

If \( \overline{A} \) is full relatively quasiconvex in \( (\overline{G}, \overline{P}) \), then \( \overline{A} \) is undistorted in \( \overline{G} \) by [15, Theorem 10.5] and by [7, Corollary III.Γ.3.6], \( \overline{A} \) is quasiconvex in \( \overline{G} \). □

The third point also makes the filled hierarchy \( \overline{H} \) faithful:

Corollary 8.19. For all sufficiently long fillings \( \pi : (G, P) \rightarrow (\overline{G}, \overline{P}) \), the filled hierarchy \( \overline{H} \) on \( \overline{G} \) is faithful.

Proof. Let \( \phi_e : A_e \rightarrow A_v \) be an attachment homomorphism mapping an edge group \( A_e \) to a vertex group \( A_v \). Since \( \pi|_{A_e} = \ker \pi \cap \ker \pi \cap A_v \) and \( \ker \pi|_{A_v} = \ker \pi \cap \ker A_e \). Let \( g_e \in A_v \) and let \( \overline{\phi}_e : \overline{A}_e \rightarrow \overline{A}_v \) be the induced edge homomorphism. Then \( \overline{\phi}_e \pi(g_e) = \pi \phi_e(g_e) \). If \( \pi \phi_e(g_e) = 1 \), then \( \phi_e(g_e) \in \ker \pi \), so \( g_e \in \ker \pi \). Therefore, \( \pi(g_e) = 1 \). Therefore, \( \overline{\phi}_e \) is injective. □

The preceding results combine to produce a quasiconvex hierarchy:

Theorem 8.20 (see [3, Theorem 2.12]). Let \( (G, P) \) be a relatively hyperbolic pair and let \( H \) be a quasiconvex fully \( P \)-elliptic hierarchy terminating in \( P \). For all sufficiently long peripherally finite fillings \( \pi : (G, P) \rightarrow (\overline{G}, \overline{P}) \) so that every \( P \in \overline{P} \) is hyperbolic, the group \( \overline{G} \) is hyperbolic and has a quasiconvex hierarchy terminating in \( \overline{P} \).

Proof. By Corollary 8.19, the quotient \( \overline{G} \) has a faithful hierarchy \( \overline{H} \) where the underlying graphs and every vertex or edge group of \( \overline{H} \) is the image of a vertex or edge group (respectively) of \( \overline{H} \) under \( \pi \).

By Lemma 8.18 (2), every edge and vertex group of \( \overline{H} \) is quasiconvex in \( \overline{G} \) and is hence also quasi-isometrically embedded in \( \overline{G} \), so the hierarchy \( \overline{H} \) is quasiconvex.

By construction, the terminal groups are fillings of the terminal groups of \( H \), so the terminal groups of \( \overline{H} \) are in \( \overline{P} \). □

Theorem 8.20 works for a group with a quasiconvex hierarchy, but Theorem 1 only gives a hierarchy for a finite index subgroup. When the filling kernels are chosen carefully, a filling of a finite index subgroup \( G' \trianglelefteq G \) can be promoted to a filling of \( G \).

Definition 8.21. Let \( (G, P) \) be a relatively hyperbolic pair and let \( G' \trianglelefteq G \) be a finite index normal subgroup with induced peripheral structure \( (G', P') \). Let \( \{N'_j \trianglelefteq P'_j \mid P'_j \in P'_1\} \) be a collection of filling kernels. The collection \( \{N'_j\} \) is **equivariantly chosen** if

1. whenever \( gP'_jg'^{-1} \) and \( hP'_kh'^{-1} \) both lie in \( P_i \), then \( gN'_jg'^{-1} = gN'_kh'^{-1} \) and
2. every such \( gN'_jg'^{-1} \) is normal in \( P_i \).
An equivariant filling of \((G', P')\) is a filling with equivariantly chosen filling kernels.

An equivariant filling of \((G', P')\) will induce a nice equivariant filling of \((G, P)\):

**Proposition 8.22.** An equivariant filling \((G', P') \rightarrow (\overline{G'}, \overline{P})\) determines a filling \((G, P) \rightarrow (\overline{G}, \overline{P})\) so that \(\overline{G'}\) is finite index normal in \(\overline{G}\) and \((\overline{G'}, \overline{P'})\) is the peripheral structure induced by \((G, P)\).

For the reader’s convenience, here is a restatement of Theorem 2.

**Theorem 2.** Let \((G, P)\) be a relatively hyperbolic pair with \(P = \{P_1, \ldots, P_m\}\). If \(G\) is virtually compact special, then there exist subgroups \(\{P_i N_i\} / \bigcup_{i} P_i\) where \(P_i\) is finite index in \(P_i\) such that if \(G = G(N_1, \ldots, N_m)\) is any peripherally finite filling with \(N_i < P_i\), then \(G\) is hyperbolic and virtually special.

**Proof.** By Theorem 1, there exists \((G', P')\) such that \(G' / \bigcup_{i} P_i\) is finite index and \((G', P')\) has a quasiconvex, malnormal fully \(P'\)-elliptic hierarchy terminating in \(P\). Since \(G\) is virtually special and hence residually finite, there exist arbitrarily long peripherally finite fillings of \((G', P')\) that are sufficiently long for Theorem 8.20 to hold.

Let \(G(K_1, \ldots, K_M)\) be such a peripherally finite filling which is also sufficiently long so that Theorem 8.4 holds. Now pass to subgroups of the filling kernels to obtain an equivariant filling: let:

\[ K'_i = \bigcap \{K^j_i \mid g \in G, \#(K^g_j \cap P_i) = \infty \}. \]

The new filling kernels \(K'_i \leq K_i\), so the new filling \(G'(K'_1, \ldots, K'_M)\) is still sufficiently long and remains peripherally finite. By Proposition 8.22, the filling \(G'(K'_1, \ldots, K'_M)\) determines a filling of \(G\).

Consider a filling \(G(N_1, \ldots, N_m)\) so that for each \(i\):

1. \(N_i < P_i\)
2. \(N_i \leq P_i\) and
3. \(P_i / N_i\) is virtually special and hyperbolic.

with an induced equivariant filling:

\[ G' \rightarrow G'(N'_1, \ldots, N'_M) \]

so that \(N'_i \leq K'_i\) and \(N'_i \leq P'_i\) for each \(j\). Such a filling is sufficiently long so that Theorem 8.20 holds.

Therefore \(G'(N'_1, \ldots, N'_M)\) has a quasiconvex hierarchy terminating in \(\overline{P'} = \{P'_j / N'_j\}\). By Theorem 8.4, the pair \((G', \overline{P'})\) is relatively hyperbolic, so \(G'\) is hyperbolic because the elements of \(P'\) are finite.

Then \(G'(N'_1, \ldots, N'_M)\) is a hyperbolic group with an malnormal quasiconvex hierarchy that terminates in finite groups (which are hence hyperbolic and virtually special). So by Corollary 8.1 (see [30] Theorem 13.3), \(G'(N'_1, \ldots, N'_M)\) is
virtually special. By Proposition 8.22, $G'(N'_1, \ldots, N'_M)$ is finite index normal in $G(N_1, \ldots, N_M)$, so the filling $G(N_1 \ldots N_m)$ is also virtually special. □

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