A Subsequence-Histogram Method for Generic Vocabulary Recognition over Deletion Channels

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Abstract—We consider the problem of recognizing a vocabulary—a collection of words (sequences) over a finite alphabet—from a potential subsequence of one of its words. We assume the given subsequence is received through a deletion channel as a result of transmission of a random word from one of the two generic underlying vocabularies. An exact maximum a posterior (MAP) solution for this problem counts the number of ways a given subsequence can be derived from particular subsets of candidate vocabularies, requiring exponential time or space.

We present a polynomial approximation algorithm for this problem. The algorithm makes no prior assumption about the rules and patterns governing the structure of vocabularies. Instead, through off-line processing of vocabularies, it extracts data regarding regularity patterns in the subsequences of each vocabulary. In the recognition phase, the algorithm just uses this data, called subsequence-histogram, to decide in favor of one of the vocabularies. We provide examples to demonstrate the performance of the algorithm and show that it can achieve the same performance as MAP in some situations.

Potential applications include bioinformatics, storage systems, and search engines.

Index terms—Classification, histogram, recognition, search, storage, and subsequence.

I. INTRODUCTION

Consider you have two database of sequences. You observe a subsequence that you know is derived from one of the databases and you would like to determine which one. Alternatively, consider you have two nonlinear codebooks that you do not know their generating rules. Suppose a codeword is chosen from one of the codebooks, punctured with an unknown deletion pattern, and handed to you. Your task is to infer which codebook was chosen. As another example, suppose you have the vocabularies of two languages: Spanish and Italian. You see an abbreviated version of a word in one of them and try to decide which language it belongs to.

We call these applications as generic vocabulary recognition, in which one seeks to identify an underlying vocabulary (collection) based on a potential subsequence of one of its words. What make these problems particularly challenging are: 1) vocabularies are generic and 2) deletion patterns are unknown. Since vocabularies are generic, we do not know a common underlying model or structure to simplify their descriptions. For example, should we know that the words in each vocabulary were generated by an i.i.d. source, then a simple regular histogram method could be used to learn the underlying distribution for each vocabulary and contrast it against that of the received subsequence. Moreover, the existence of an unknown deletion pattern makes the problem way more complex. Should there exist no deletion channel, then a suffix-tree implementation of vocabularies would suffice to deal with the problem [1]. In the existence of deletion, however, one needs exponential space to develop a generalized suffix-tree to enclose subsequences derived from a vocabulary.

Deletion channels, not to be mistaken for erasure channels, have risen in different applications. In a recent survey, Mitzenmacher [2] points out some of these applications as well as the existing open problems, including the capacity of the binary deletion channels. In [3], Diggavi et. al study communication over a finite buffer in the framework of deletion channels, and in [4], they provide upper bounds for the capacity of these channels. As noted by these authors, the main challenge is raised because of the unknown deletion patterns. As a result, a maximum a posterior (MAP) or a maximum likelihood (ML) detector needs to find the transmitted codeword analyzing its subsequences.

Problems involving subsequences are generally hard combinatorial problems whose exact solutions require exponential time or space [5], [6]. This is no exception to decoding over deletion channels [3] and to our case, in which, we formulate the vocabulary recognition problem in an MAP decision framework leading to the same bottleneck as in decoding: count the number of ways that a received subsequence can be obtained from a particular sequence. Unlike the case for decoding, here, we also need to compute the aggregated number of ways that a received subsequence can be derived from certain groups of words.

Since the computation is NP-Hard, we approximate it by the average number of position-wise matches that a received subsequence can find in the multiset of same length subsequences derived from a vocabulary. This approximation led to an algorithm that we call subsequence-histogram method, due to its operational similarities to regular histogram method. In fact, one may view regular histogram method as an special case of subsequence-histogram method for subsequences of unit length. The algorithm requires a space proportional to the cube of the maximum length of words and a time that is upper bounded by the product of the length of the received subsequence and the maximum length of the words. Examples are provided to illustrate the process and performance of the algorithm. In one simple example, we show that the
algorithm can achieve the same performance as the exact MAP. Another example, simulated numerically, shows a case in which the algorithm achieves 10% error rate compared to the regular histogram method that has 50% error rate. A rigorous performance analysis of the algorithm and assessment of its proximity to exact MAP is still work in progress.

The organization of this paper is as follow. In Section II, we discuss the setup of the problem. In Section III, we introduce the algorithm and provide examples to demonstrate its operation and performance. Finally, we will discuss the derivation of the algorithm and its error rate analysis in Section IV.

II. SETUP

Let \( n \) denote a discrete time index, and let \( \Sigma \) be a finite alphabet. Assume two finite vocabularies \( \mathcal{V}_{\theta_1}, \mathcal{V}_{\theta_2} \subset \cup_{l=1}^{L} \Sigma^l \) of maximum order \( L \) are given. A vocabulary is picked randomly with a known probability \( P(\theta) \) and a word \( W = (w_1, \ldots, w_n) \) is picked from it uniformly random. The word \( W \) is passed through a deletion channel producing \( S = (s_1, \ldots, s_m) \) at the output. The channel is i.i.d. in which every letter \( w_i \) of the word \( W \) can be deleted with probability \( p \). The problem is to observe \( S \) and infer which underlying vocabulary was more likely picked.

**Example 2.1:** Let \( \Sigma = \{0, 1\} \) and assume
\[
\mathcal{V}_{\theta_1} = \{0101, 1100\}
\]
and
\[
\mathcal{V}_{\theta_2} = \{1010, 0011\}
\]
are two equiprobable underlying vocabularies. If \( S = 01 \) is observed, \( \mathcal{V}_{\theta_2} \) is chosen since the likelihood of \( \mathcal{V}_{\theta_1} \) and \( \mathcal{V}_{\theta_2} \) are \( \frac{1}{4}p^2(1-p)^2 \) and \( \frac{1}{4}p^2(1-p)^2 \), respectively. On the other hand, observing \( S = 10 \), \( \mathcal{V}_{\theta_1} \) is chosen. Intuitively, this is because \( S = 10 \) and \( S = 01 \) are common subsequences in \( \mathcal{V}_{\theta_1} \) and \( \mathcal{V}_{\theta_2} \), respectively.

The MAP formulation of this problem is to maximize
\[
P(\theta|S) \propto \sum_{W \in \mathcal{V}_{\theta}} P(S|W)P(W|\theta)P(\theta), \tag{1}
\]
in which
\[
P(W|\theta) = \frac{1}{N_{\theta}}
\]
for every \( W \in \mathcal{V}_{\theta} \). The challenge in Eq. (1) is to compute or approximate \( P(S|W) \). Should there be no deletion, i.e., \( p = 0 \), then observed sequence \( S \) should exactly match a word \( W \) in either of the dictionaries. In this case, a suffix-tree implementation of the vocabularies would require \( O(|\mathcal{V}_{\theta}|L) \) space and allow to identify an exact match of sequence \( S \) in \( O(mL) \) time. For \( p \neq 0 \), a generalized suffix-tree is not an option as it needs to enclose the multi-set of all possible subsequences of \( \mathcal{V}_{\theta} \), requiring \( O(|\mathcal{V}_{\theta}|2^L) \) space, in the worst case.

III. ALGORITHM

Here, we introduce an approximation algorithm that needs \( O(|\Sigma|L^3) \) space and operates in \( O(mL) \) time. The algorithm has two phases: a learning phase and a recognition phase.

**Learning Phase:** In the learning phase, it computes two sets of matrices. The first set of matrices are what we call as **positional histograms**. That is for every \( n \leq L \), we compute \( |\Sigma| \times n \) matrices \( \Phi_n \) whose elements are
\[
\Phi_n(\sigma, j) = \sum_{W \in \mathcal{V}_{\theta}} 1(w_j = \sigma)
\]
for \( \sigma \in \Sigma \) and \( 1 \leq j \leq n \). Here,
\[
\mathcal{V}_{\theta}(n) = \{W \in \mathcal{V}_{\theta} : |W| = n\}
\]
denotes the subset of \( \mathcal{V}_{\theta} \) containing all words of length \( n \). Eq. (2) denotes the number of \( n \)-length words whose \( j \)-th position (from left) is \( \sigma \). Through an off-line process, one can adaptively compute \( \Phi_n(\sigma, j) \) for every \( \sigma \) and \( j \) and store it in a \( |\Sigma| \times n \) matrix \( \Phi_n \). The total space required for this purpose is \( O(|\Sigma|L^2) \). This process is summarized in Table I.

**Example 3.1 (Positional histograms):** Consider the two given vocabularies in Example 2.1. Then, for \( n = 4 \),
\[
\Phi_n,\theta_1 = \begin{bmatrix}
1 & 0 & 2 & 1 \\
1 & 2 & 0 & 1 \\
\end{bmatrix}, \quad \Phi_n,\theta_2 = \begin{bmatrix}
1 & 2 & 0 & 1 \\
1 & 0 & 2 & 1 \\
\end{bmatrix}
\]
For \( n = 1, 2, 3 \), the positional histograms are zero.

The second set of matrices are what we call as **subsequence histograms**. For every \( n \leq L \), and \( m \leq n \), we compute \( |\Sigma| \times m \) matrices \( \Psi_n,\sigma \) whose elements are
\[
\Psi_n,\sigma(m, i) = \sum_{j=1}^{n} \Phi_n(\sigma, j)\alpha_{n,m}(j, i)
\]
where
\[
\alpha_{n,m}(j, i) = \binom{j-1}{i-1} \binom{n-j}{m-i}
\]

| TABLE I |
|-----------------------------|
| **ILLUSTRATION OF THE LEARNING PHASE OF ALGORITHM.** |
|-----------------------------|

| I. COMPUTING \( \Phi_n \) |
|-----------------------------|
| 1. Initialization: |
| - For \( n \leq L \), \( \sigma \in \Sigma \), and \( j \leq n \), |
| \( \Phi_n(\sigma, j) = 0 \). |
| 2. Recursion: |
| - For \( W \in \mathcal{V}_{\theta} \), let \( n = |W| \). For any \( j \leq n \), |
| \( \Phi_n(w_j, j) = \Phi_n(w_j, j) + 1 \). |
| 3. Termination: For \( n \leq L \), |
| output \( |\Sigma| \times n \) matrix \( \Phi_n \). |

| II. COMPUTING \( \Psi_n,\sigma \) |
|-----------------------------|
| 1. Recursion: |
| - For \( n \leq L \), \( m \leq n \), \( i \leq m \), and \( \sigma \in \Sigma \), |
| \( \Psi_n,\sigma(m, i) = \sum_{j=1}^{n} \Phi_n(\sigma, j)\alpha_{n,m}(j, i) \). |
| 3. Termination: For \( n \leq L \) and \( m \leq n \), |
| output \( |\Sigma| \times m \) matrix \( \Psi_n,\sigma \). |
for $\sigma \in \Sigma$ and $1 \leq i < m$ denote the number of ways that an $m$-length subsequence can be derived from $V_\theta(n)$ such that its $i$-th element is $\sigma$. Each matrix $\Psi_{n,m}$ may be viewed as positional histogram of the multiset of all $m$-subsequences of $V_\theta(n)$. These matrices can be computed off-line and stored in $O(|\Sigma|^L \cdot 3)$ space. Table II illustrates this process.

**Example 3.2 (Subsequence histograms):** For the vocabulary $V_{\theta_1}$ in Example 2.1, the subsequence histograms are

$$
\Psi_{4,1} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \Psi_{4,2} = \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix}, \quad \Psi_{4,3} = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 4 & 3 \end{bmatrix}, \quad \Psi_{4,4} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}.
$$

For vocabulary $V_{\theta_2}$, on the other hand, they are

$$
\Psi_{4,1} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \Psi_{4,2} = \begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix}, \quad \Psi_{4,3} = \begin{bmatrix} 5 & 4 & 3 \\ 3 & 4 & 5 \end{bmatrix}, \quad \Psi_{4,4} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix}.
$$

Note that $\Psi_{4,1}$ is exactly the same data of the regular histogram. Thus, we may view regular histogram as an special case of subsequence-histogram for subsequences of unit length. Moreover, since $\Psi_{4,1}$ is the same for both vocabularies, the regular histogram method would be no better than tossing a fair coin. In contrast, $\Psi_{4,2}, \Psi_{4,3}, \Psi_{4,4}$ reflect the differences between $V_{\theta_1}$ and $V_{\theta_2}$, a phenomenon that empowers the proposed algorithm enabling it to extract some regularity patterns of a vocabulary.

**Recognition Phase:** Observing $S = (s_1, \ldots, s_m)$, we compute the subsequence similarity score of order $n$

$$
\Psi_{n,m}(S) = \frac{1}{m} \sum_{i=1}^{m} \Psi_{n,m}(s_i, i).
$$

This score is an upper bound that is used as an approximation for the number of ways that $S$ can be derived from $V_\theta(n)$. Computing this score for all $n \in \{m, \ldots, L\}$, we will obtain the total similarity score

$$
\tilde{J}_\theta(S) = \frac{P(\theta)}{N_\theta} \sum_{n=m}^{L} \Psi_{n,m}(S)p^n
$$

and choose the vocabulary with maximum score.

**Example 3.3 (Achieving MAP Performance):** Consider the same setup as in Example 2.1. For an observed sequence $S$, the algorithm computes the total similarity score (6) and makes the following decisions.

- $S \in \{0, 1\}$ → draw.
- $S \in \{00, 1\}$ → draw; $S = 01 \rightarrow V_{\theta_1}$; $S = 10 \rightarrow V_{\theta_2}$.
- $S \in \{100, 110\} \rightarrow V_{\theta_1}$; $S \in \{001, 011\} \rightarrow V_{\theta_2}$; $S \in \{010, 101\}$ → draw.
- $S \in \{0101, 10001\} \rightarrow V_{\theta_1}$; $S \in \{1010, 0011\} \rightarrow V_{\theta_2}$.

Should we have used exact MAP, the decision results were exactly the same as the algorithm’s. Thus, this example shows that the algorithm can achieve the same performance as MAP. We, however, have no sufficient conditions for it, yet. One can also verify that the regular histogram method for the preceding example would be no better than tossing a coin.

**Example 3.4 (All combinations):** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. Let $V_{\theta_1} = \Sigma_1^{L_1}$ and $V_{\theta_2} = \Sigma_2^{L_2}$. Assume that the vocabularies are equiprobable and let $S \in (\Sigma_1 \cap \Sigma_2)^m$ be the observed sequence. Then, decision criterion would be to choose the vocabulary with maximum

$$
\tilde{J}_{\theta_k}(S) = \frac{(L_k)^m}{|\Sigma_k|} p^{L_k}.
$$

In other words, we the algorithm decides in favor of a vocabulary based on

$$
\frac{(L_1)^m}{(L_2)^m} p^{L_1-L_2} \geq \frac{\theta_1}{\theta_2} \frac{|\Sigma_1|^m}{|\Sigma_2|^m}.
$$

Should have we used exact MAP, then recognition results would have been based on

$$
\frac{(L_1)^m}{(L_2)^m} p^{L_1-L_2} \geq \frac{\theta_1}{\theta_2} \frac{|\Sigma_1|^m}{|\Sigma_2|^m}.
$$

These two criteria only differ in the right hand sides. Depending on the values of parameters, the two methods may or may not have the same conclusion. In the confusional case where $|\Sigma_1| > |\Sigma_2|$ and $L_1 > L_2$, both methods decide in favor of $V_{\theta_2}$, which has a smaller size alphabet. In this case, regular histogram method has the same results as the vocabularies have no specific pattern.

**Example 3.5 (i.i.d. sources):** Consider two i.i.d. sources over alphabet $\Sigma = \{a, c, g, t\}$. Let

$$
P_{\theta_1} = \left[ \begin{array}{c} 1 - \frac{i}{4} \\ \frac{1}{50} \end{array} \right],
\quad P_{\theta_2} = \left[ \begin{array}{c} 1 - \frac{i}{4} \\ \frac{1}{50} \end{array} \right].
$$

denote the density function of two i.i.d. sources on $\Sigma$, in which parameter $i \in \{0, 1, 2, 3, 4\}$ is a deviation parameter. For each vocabulary and for each $i$, we use the given densities to generate 128,000 random words of size 20 to 40. A Monte Carlo error analysis, with 2000 trials, was then conducted. Table III summarizes the obtained error rate of the algorithm for four different values of the probability of deletion $p$. As $i$ increases, the KL distance between the two distributions...
Table III
Error results for i.i.d. sources. Each vocabulary contains 128000 words, generated as described in Example 3.5.

| p   | 0.1 | 0.2 | 0.3 | 0.4 |
|-----|-----|-----|-----|-----|
| i   |     |     |     |     |
| 0   | 0.50| 0.50| 0.50| 0.49|
| 1   | 0.37| 0.38| 0.39| 0.40|
| 2   | 0.25| 0.27| 0.27| 0.29|
| 3   | 0.16| 0.18| 0.19| 0.21|
| 4   | 0.09| 0.10| 0.12| 0.14|

increases and error rate decreases. In this example, regular histogram method has comparable error performance, as words are generated completely i.i.d.

**Example 3.6 (Capturing inherent structures):** Using the same alphabet of the previous example, two vocabularies are generated where the second one is the exact horizontal mirror of the first one. The number of words is 128,000. For the first vocabulary, the words were created in five different cases. In the first case, for each word, a random length of size 20 to 40 is chosen and each letter of the word is chosen uniformly randomly from Σ. In the second to fifth cases, we simply add a prefix ‘a’, ‘ac’, ‘acg’, ‘acgt’, and ‘acgta’, to all words, respectively. In each case, we then truncate from the end of the words as many letters as the length of the prefix added. Table [V] summarizes the results for different cases and for four different probability of deletions. Since the vocabularies are the exact mirror of each other, regular histogram method resulted to 50% error rate in all cases. Note the big reduction in error rate across all deletion probability by adding just one letter of prefix. For example, for deletion probability of \( p = 0.4 \), the error rate is reduced by half. This shows that the algorithm is successful in capturing existing patterns in the vocabularies and sensitive to words permutation.

**IV. Analysis of the Algorithm**

Maximum a posterior (MAP) method is equivalent to maximizing Eq. (1) in which the main challenge is to compute \( P(S|W) \). Here, we discuss the derivation of an approximate algorithm for it.

Assume a sequence \( S \) of size \( m = |S| \) is observed and let \( W \) be a word of length \( n = |W| \). Sequence \( S \) is a subsequence of \( W \), should there exists a warping function \( ^4 \)

\[
\phi_m = (i_1, \ldots, i_m), \quad i_1 < i_2 < \ldots < i_m \leq n
\]

such that \( S = \phi_m(W) = (w_{i_1}, \ldots, w_{i_m}) \). The function \( \phi_m \) may be viewed as an ordered \( m \)-subset in \([n]\). Thus, \( S \) is a subsequence of \( W \), if the number of such maps, i.e.,

\[
\psi(S, W) = \{ \phi_m \subset [n] : S = \phi_m(W) \}
\]

is non-zero. Using this expression, we can describe the probability of observing \( S \) conditioned on \( W \) as

\[
P(S|W) = \psi(S, W)p^{n-m}(1-p)^m.
\]

Consequently, plugging (13) in Eq. (1), we obtain a discriminant function

\[
J_0(S) = \frac{P(\theta)}{N_0} \sum_{n=m}^{L} p^n \sum_{W \in \mathcal{V}_{\theta}(n)} \psi(S, W).
\]

(14)

Exact computation of \( \psi(S, W) \) requires dynamic programming that has \( O(mn) \) time complexity. Thus, the computation of (13) is \( O(mL|\Sigma|^L) \) when \( |\mathcal{V}_{\theta}| = O(|\Sigma|^L) \).

**A. Approximation**

We have

\[
\sum_{W \in \mathcal{V}_{\theta}(n)} \psi(S, W) = \sum_{m} \sum_{W \in \mathcal{V}_{\theta}(n)} \mathbb{I}(\phi_m(W) = S)
\]

in which \( \mathbb{I}() \) is equality indicator function. Approximating \( \mathbb{I}() \) with the following upper bound

\[
\mathbb{I}(\phi_m(W) = S) \leq \frac{1}{m} \sum_{i=1}^{m} \mathbb{I}(\phi_{m,i}(W) = s_i)
\]

and substituting in Eq. (15), we obtain

\[
\frac{1}{m} \sum_{m} \sum_{W \in \mathcal{V}_{\theta}(n)} \mathbb{I}(\phi_m(W) = S) \leq \frac{1}{m} \sum_{i=1}^{m} \sum_{W \in \mathcal{V}_{\theta}(n)} \mathbb{I}(\phi_{m,i}(W) = s_i).
\]

(16)

The right most summation returns the number of all \( n \)-length words whose \( i \)-th position under a warping \( \phi_m \) matches \( s_i \). A number

\[
\alpha_{n,m}(j, i) = \binom{j-1}{i-1} \binom{n-j}{m-i}
\]

(17)

of these warping functions map the \( j \)-th position of a word \( W \) to the \( i \)-th position of the observed sequence \( S \). This is the number of placement of \( n \) distinguishable objects orderly into \( m \) bins that place object \( j \) into bin \( i \) satisfying the identity

\[
\binom{n}{m} = \sum_{j=1}^{m} \alpha_{n,m}(j, i)
\]

(18)

for every \( i = 1, \ldots, m \). For any symbol \( \sigma \in \Sigma \), Eq. (2), i.e.,

\[
\Phi_n(\sigma, j) = \sum_{W \in \mathcal{V}_{\theta}(n)} \mathbb{1}(w_j = \sigma)
\]

denotes the number of words whose \( j \)-th position (from left) is \( \sigma \). Eq. [3] defined as

\[
\Psi_{n,m}(\sigma, i) = \sum_{j=1}^{n} \Phi_n(\sigma, j) \alpha_{n,m}(j, i)
\]

denotes the number of \( m \)-length subsequences, in the subsequence multiset derived from \( \mathcal{V}_{\theta}(n) \), whose \( i \)-th element is \( \sigma \).

Using these notations, we can simplify the expression of right hand side of (16) as

\[
\frac{1}{m} \sum_{i=1}^{m} \sum_{\phi_m \in \mathcal{V}_{\theta}(n)} \sum_{i=1}^{m} \mathbb{1}(\phi_{m,i}(W) = s_i) = \frac{1}{m} \sum_{i=1}^{m} \Psi_{n,m}(s_i, i),
\]

(19)

A monotonically increasing sequence of indices.
which is the similarity score of $S$ and $V_0(n)$ measured as the average number of $m$-length subsequences that are derived from $V_0(n)$ and that match $S$ in different positions. This is Eq. (5) in the recognition phase that results to the total similarity score $6$.

### B. Error analysis

Probability of error for this algorithm does not have a closed form solution. In an special case, in which vocabularies are equiprobable and have the same number of words, $N$, all of the same length, $n$, useful insights can be obtained. In such case, observing $S$ of length $m$, the conditional probability of error for an exact MAP solution is

$$P_{\text{MAP}}(\text{error}) = \sum_{m=0}^{n} \frac{1}{2N^{n-m}} (1-p)^m \mu(m) \quad (19)$$

where

$$\mu(m) \triangleq \sum_{|S|=m} \min_{W \in V_1} \sum_{i} \psi(S,W) \quad (20)$$

is the cardinality of the intersection of multisets of subsequences of length $m = |S|$ derived from vocabularies. As $\mu(m)$ increases, $P_{\text{MAP}}(\text{error})$ increases reaching a maximum of $\frac{1}{2}$ when the two vocabularies become identical.

Expression for the error probability of sequence-histogram algorithm replaces $\mu(m)$ with

$$\lambda(m) = \sum_{|S|=m} \sum_{W \in V_1} \psi(S,W) 1(\bar{\Psi}_{\theta_1}(S) < \bar{\Psi}_{\theta_2}(S))$$

$$+ \sum_{W \in V_2} \psi(S,W) 1(\bar{\Psi}_{\theta_2}(S) < \bar{\Psi}_{\theta_1}(S)). \quad (21)$$

Using the inequality $1(x < 1) \leq \frac{1}{\sqrt{x}}$, we will have

$$\lambda(m) \leq \sum_{|S|=m} \sqrt{\bar{\Psi}_{\theta_1}(S)\bar{\Psi}_{\theta_2}(S)}$$

$$\left( \sum_{W \in V_1} \frac{\psi(S,W)}{\bar{\Psi}_{\theta_1}(S)} + \sum_{W \in V_2} \frac{\psi(S,W)}{\bar{\Psi}_{\theta_2}(S)} \right)^{\frac{1}{2}}. \quad (22)$$

This implies the following upper bound on the probability of error of the algorithm:

$$P(\text{error}) \leq \sum_{m=0}^{n} \frac{1}{N^{n-m}} (1-p)^m$$

$$\sum_{|S|=m} \sqrt{\bar{\Psi}_{\theta_1}(S)\bar{\Psi}_{\theta_2}(S)} \quad (23)$$

Thus, we conclude that subsequences with equivalently large $\bar{\Psi}_{\theta_1}(S)$ and $\bar{\Psi}_{\theta_2}(S)$ have bigger impact on error.

### V. Conclusion

The aim of this paper is to demonstrate an approximation algorithm for the problem of generic vocabulary recognition over deletion channels. Without any prior assumption on the structure of vocabularies, the sequence-histogram algorithm seeks to extract regularity patterns of a vocabulary through an off-line analysis. The algorithm uses this data to choose the more likely underlying vocabulary for a received subsequence in polynomial time and space.

A regular histogram method, may be viewed as an special case of this algorithm. Unlike a regular histogram, however, this algorithm is successful in extracting the structure of a vocabulary to dramatically boost its performance. In some situations, the algorithm can achieve the same performance as exact MAP. However, sufficient conditions to characterize such situations are not known, yet.

Some immediate future directions are: 1) analyzing the performance of the algorithm and its proximity to MAP, 2) exploring applications in bioinformatics, sequence segmentation, storage systems, and search engines, 3) extending work to multiple observations, represented by subsequences of different words within one vocabulary, and 4) generalizing model to include substitution and insertion errors.

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### REFERENCES

[1] D. Gusfield, *Algorithms on Strings, Trees and Sequences: Computer Science and Computational Biology*. Cambridge University Press, 1997.

[2] M. Mitzenmacher, “A survey of results for deletion channels and related synchronization channels,” *Lecture Notes in Computer Science*, vol. 5124, pp. 1–3, 2008, algorithm Theory-SWAT 2008.

[3] S. Diggavi and M. Grossglauser, “On information transmission over a finite buffer channel,” *IEEE Trans. Inform. Theory*, vol. 52, no. 3, pp. 1226–1237, Mar. 2006.

[4] S. Diggavi, M. Mitzenmacher, and H. D. Pfister, “Capacity upper bounds for the deletion channel,” *IEEE Intern. Symp. Inform. Theory*, pp. 1716–1720, Jun. 2007.

[5] D. Maier, “The complexity of some problems on subsequences and supersequences,” *Journal of the Association for Computing Machinery*, vol. 25, no. 2, pp. 322–336, Apr. 1978.

[6] V. I. Levenshtein, “Efficient reconstruction of sequences,” *IEEE Trans. Inform. Theory*, vol. 47, no. 1, pp. 2–23, Jan. 2001.