Another approach to the study of phase noise in electrical oscillators

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(Dated: May 1, 2015)

The mechanism at the base of phase noise generation in electrical oscillators is reexamined from first principles. The well known Lorentzian spectral power distribution is obtained, together with a clearcut expression for the line-width parameter. The mechanism of suppression of the amplitude fluctuations and its effects are discussed and the true role of the figure of merit $Q_0$ of the resonator is restated. The up conversion of low frequency components from a non-white noise source is also considered. A number of simple numerical experiments is presented to illustrate and clarify the mathematical results.

INTRODUCTION

Signal generation is half the world of electronics, the electronic oscillator is ubiquitous, it has a part in all consumer, professional and scientific appliances and most often the quality of the generated signal is crucial to the overall performances.

After the problems of accuracy, stability, adjustability and resolution, one of the main concern in today signal generation is spectral purity, either harmonic content or non-harmonic wide and narrow band noise.

The fact that a sinusoidal signal at the output of an oscillator never is truly monochromatic in the mathematical sense, but has a line-width and noise side bands, has important implications. For example, it reduces the possibility for a radio receiver to discriminate a week signal close to a strong one [1].

The first paper on this subject I am aware of is chapter 15 in Edson’s “Vacuum Tube Oscillators” [2]. In this work Edson assumed that an electronic oscillator could be considered an almost linear system whose nonlinearities gave a negligible contribution to the spectral composition of the signal.

Edson’s model for an oscillator was a sort of enhanced $Q$-multiplier brought to the limit of hugely amplifying and filtering the thermal noise always present in any real circuit. In this model the gain of the positive feedback loop was smaller than unity by an infinitesimal amount, to accommodate for the noise power at the amplifier input. The equivalent $Q$ of the circuit resulted very very high, but not infinite, and its selectivity accounted for the Lorentzian shape of the generated signal spectrum, in agreement with experimental observations.

This model could account for the presence of the narrow noise skirt on sides of the generated signal (the carrier) and for the 20 dB intensity decay per decade of frequency moving away from the carrier.

This work did receive little attention and was little quoted in the literature in the years following. The author himself declared that he “was never fully satisfied with the results from this approach ...” [3], to the point that, a few years after the book, he published a completely new paper on the same argument [4].

Moreover, experimental studies soon demonstrated the presence of a $1/f$ noise component in the frequency region nearest to the carrier, that Edson’s model could not explain, not even including some amount of non linearity. Yet, Edson’s model had the great virtue of being very clear to understand and of directly connecting noise characteristics with electrical circuit parameters.

Other works followed shortly after. In [5] Leeson approached the problem from an heuristic point of view, presenting an equation that described the commonly observed spectrum of feedback oscillators. This paper, though very popular in the following works, did not face the problem of connecting the observed noise spectrum properties with the electrical circuit characteristics. Little later Lax laid the basis for approaching the problem through statistics and interest into the argument was paralleled by an analogous interest for the line width of a laser generated light beam [6].

Recently interest into oscillator noise has greatly increased because of its relevance in the performance of telecommunication networks, both digital and analog, wired and wireless, as documented by the large number of recent contributions to the literature. See, e.g., [7–10] and references therein.

These works, as well as Lax’s one, greatly deepened the theoretical analysis of the stochastic processes behind the phase noise origin. They make large use of concepts and arguments from the world of statistics, at the expense, though, of somewhat missing a clearcut connection with the physical circuit, its parameters and its behaviour.

In [9], e.g., the analogy has been used between the oscillator signal phase fluctuations and the process of molecular diffusion. As suggestive and appropriate as it can be by principle such a connection, its application to the study of phase noise in oscillators can help only from a qualitative point of view. Otherwise, a double effort would be required first to correlate circuit equations with molecular processes and then to bring results from molecular behaviour back into the circuits realm.

A consequence of this state of affairs from a didactic point of view is that the subject is often relegated to books and courses of electronics at the highest spe-
cialization level or is proposed through an heuristic and qualitative only approach, following Leeson’s approach (see, e.g., [1]).

In this paper I will try to approach the problem making profit of the large and good amount of work already available, through a simple but rigorous enough way, making use mostly of those concepts familiar to the world of electronics like Johnson’s noise, modulation, frequency conversion and vectorial (de)composition. Clearly, some mathematical extras to this menu are unavoidable, not to remain at the qualitative and descriptive level only. They are the Central Limit and the Wiener-Khinchin theorems and some properties of gaussian distributions. Luckily enough, currently they are well established and widely known theorems, also at the student level.

The path I shall propose in this paper easily lends itself to the implementation of numerical experiments, very useful from a didactic point of view to illustrate the theoretical concepts and enhance confidence into the results obtained through equations.

THE MODEL OSCILLATOR

It is a common practice in a linear system to group all noise sources into a single one, conveniently placed and with an equivalent effect on the output signal.

In this paper I have chosen not to follow this way for the sake of the simmetry both in the circuit and in the equations that will be written. The cost of this choice will be that most equations will contain the sum of two equivalent terms, one for current, one for voltage, with no further increase in complexity. The advantage will be a clearer splitting of noise into amplitude and phase components and the possibility to introduce into the circuit some non linear behaviour without having to deal with the violation of the primary hypothesis “in a linear system” intrinsic to the common practice. In the last, as expected, it will be clear that the total noise power is the unique meaningful quantity in evaluating phase noise.

In an electronic oscillator, like in every harmonic oscillator, unavoidable sources of noise are the dissipative elements associated with the two energy storing components of the resonator, most often an inductance L and its series resistance R and a capacitance C and its parallel conductance G (fig. 1).

The dissipative elements R and G are responsible both for the introduction of thermal noise into the circuit, a random voltage \( v_n(t) \) and current \( i_n(t) \) associated with resistance R and conductance G, and for the resonator damping.

Actually, in a stationary oscillator values of damping and noise generation are disjoined from the physical values of R and G, because of circuits and devices that provide the positive (regenerative) feedback indispensable to add a negative damp that cancels out the natural (positive) damp of the resonator and allows a persistent oscillation to be sustained.

These circuits and devices introduce their own noise contributions that add to \( i_n \), and/or \( v_n \).

As a consequence, the value of R and \( v_n \) and of G and \( i_n \) can be considered uncorrelated, only with the constraint that the spectral density \( V_n(\omega) \) and \( I_n(\omega) \) of \( v_n(t) \) and \( i_n(t) \) cannot be lower than what due respectively to the R and G of the physical circuit, namely

\[
V_n \geq 4kTR \quad I_n \geq \sqrt{4kTG}
\]

where \( k \) is Boltzmann’s constant and T absolute temperature.

So, it is possible to consider the practical case where damping is kept to zero by regeneration, i.e. R and G are nulled by equivalent negative terms, but some noise, both \( v_n \) and \( i_n \), is present in the circuit. After putting some energy \( W_0 \) into L and C there will be a persistent stationary oscillation. For the sake of simplicity, let’s consider the initial conditions \( v(t) = 0 \) V and \( i(t) = 1 \) A at \( t = 0 \) so that, in the absence of noise, oscillation would be

\[
\begin{align*}
v(t) &= v_0 \sin(\omega_0 t) \\
i(t) &= i_0 \cos(\omega_0 t)
\end{align*}
\]

with, as usual, \( \omega_0 = 2\pi f_0 = 1/\sqrt{LC} \) and

\[
\frac{1}{2}Li^2(t) + \frac{1}{2}Cv^2(t) = \frac{1}{2}Li_0^2 = \frac{1}{2}Cv_0^2 = W_0
\]

In the state space described by the \( i, v \) circuit variables in absence of noise the representative point S of the system describes, as well known, a circle with constant angular velocity \( \omega_0 \). In the presence of noise both the trajectory and angular velocity of the point are perturbed, as described in fig. 2.

Noise decomposition

By effect of noise current \( i_n \) and voltage \( v_n \), in a time \( dt \) charge upon capacitor C and current through inductor

\[
\begin{align*}
\frac{1}{2}Li_n^2(t) + \frac{1}{2}Cv_n^2(t) &= \frac{1}{2}Li_0^2 + \frac{1}{2}Cv_0^2 + W_0
\end{align*}
\]
L change by an amount (with effective $G = 0$ and $R = 0$)

$$dv = \frac{i_n}{C} dt \quad \text{and} \quad di = \frac{v_n}{L} dt \quad (1)$$

As $v_n$ and $i_n$ are statistically independent random variables, so are $dv$ and $di$.

Whatever their instantaneous values, $dv$ and $di$ can always be decomposed into a tangential $dT$ and a radial $dP$ part as shown in fig. 2, with:

$$P^2 = \left( \frac{i}{i_0} \right)^2 + \left( \frac{v}{v_0} \right)^2 = \frac{i^2L + v^2C}{2W_0} \quad (2)$$

The effects of $dT$ and $dP$ over the motion of point $S$ are independent each other: $dP$ changes the oscillation amplitude $P$, $dT$ its phase $\varphi$ by the amount $d\varphi = dT/P$.

There is only a small link left between phase and amplitude fluctuations. If the noise generators $i_n$ and $v_n$ are stationary, as usual, their $rms$ amplitudes remain constant. Hence, the same amount of noise from $i_n$ and $v_n$ yields a different effect on $d\varphi = dT/P$, depending upon the instantaneous value of amplitude $P$. But this effect is vanishingly small indeed. In a real system the standard deviation of the $P$ fluctuations is many orders of magnitude smaller than the $P$ average value: nobody would be concerned in a change of one part per million of a noise source relative intensity.

By standard analytical geometry relations, the transformation

$$dP = \left( \frac{i_n}{v_0CP} \cdot \cos \varphi + \frac{v_n}{i_0LP} \cdot \sin \varphi \right) dt \quad (3)$$
$$dT = \left( -\frac{i_n}{v_0CP} \cdot \sin \varphi + \frac{v_n}{i_0LP} \cdot \cos \varphi \right) dt \quad (4)$$

holds. Thereafter, by transforming a couple of values $(v_n, i_n)$ by eqs. (3-4), zeroing alternatively the $dP$ or $dT$ part and transforming back with the inverse of (3-4), two couples of components $(v_{nt}, i_{nt})$ and $(v_{np}, i_{np})$ are obtained that sum up to give the original $(v_n, i_n)$ values, but yield each one only a $dT$ or $dP$ perturbation:

$$v_{nt} = -i_nR_0 \sin \varphi \cos \varphi + v_n \cos^2 \varphi \quad (5)$$
$$i_{nt} = i_n \sin^2 \varphi - \frac{v_n}{R_0} \sin \varphi \cos \varphi \quad (6)$$
$$v_{np} = i_nR_0 \sin \varphi \cos \varphi + v_n \sin^2 \varphi \quad (7)$$
$$i_{np} = i_n \cos^2 \varphi + \frac{v_n}{R_0} \sin \varphi \cos \varphi \quad (8)$$

with $R_0 = \sqrt{L/C}$.

Using these equations it is possible to decouple the $dT$ and $dP$ components of the random motion of the representative point $S$ in the state space and go on studying their effect independently each other.

Clearly, in a completely linear circuit as in fig. 1, such a decomposition is only a mathematical formalism. Actual voltage and current will be given only by the sum of the two components and will be the linear response of the $LRCG$ network under the action of the $v_n, i_n$ generators.

But, when the circuit will be modified in order to suppress one of the two noise components, namely the amplitude fluctuations, and this can be obtained only through the unavoidable introduction of a suitable non linear behaviour, then the remaining tangential component (the phase fluctuations) will describe the true physical current and voltage.

Phase fluctuations

By definition of angular frequency and using eq. (3-4)

$$d\varphi = \omega_0 \cdot dt + \left( -\frac{i_n}{v_0CP} \cdot \sin \varphi + \frac{v_n}{i_0LP} \cdot \cos \varphi \right) \cdot \frac{dt}{P} \quad (9)$$

This equation expresses how noise voltage and current from the $v_n$ and $i_n$ sources induce fluctuations in the phase $\varphi$ of the oscillator signal. The two quantities $v_0CP$ and $i_0LP$ represent the amplitude of the oscillation (the envelope) and its evolution over time. In any real oscillator of practical interest, the amplitude is controlled by some kind of stabilization mechanism, as described in the next section; hence in eq. (9) amplitude drift can be neglected and $P$ does not move from the initial value $P = 1$. This yields

$$\omega = \frac{d\varphi}{dt} = \omega_0 + \left( -\frac{i_n}{v_0C} \cdot \sin \varphi + \frac{v_n}{i_0L} \cdot \cos \varphi \right) \quad (10)$$

The effect of the phase fluctuations is clearly the introduction of a frequency modulation over $\omega_0$. In the conditions that are common in real (and useful) oscillators

FIG. 2: Decomposition of independent noise contributions $dv = (i_n/C)dt$ and $di = (v_n/L)dt$ into the radial $dP$ and tangential $dT$ parts. Normalization of the $i$, $v$, $di$ and $dv$ variables over $i_0$ and $v_0$ as expressed by eq. (2) is not shown to avoid cluttering the figure and should be implicitly assumed.
this effect is very very small indeed, hence it is safe to write

\[ \omega_n = \omega - \omega_0 = -\frac{i_n}{v_0 C} \cdot \sin(\omega_0 t) + \frac{v_n}{i_0 L} \cdot \cos(\omega_0 t) \] (11)

and

\[ \varphi(t) = \omega_0 t + \varphi_n(t) \quad \varphi_n(t) = \int_0^t \omega_n(t')dt' \] (12)

(but in the numerical experiments the differential equations of the circuit will be integrated without using such an approximation).

If \( i_n \) and \( v_n \) are statistically independent sources of white noise, as it is the case for Johnson’s noise in resistors, they have a uniform spectral density. Hence, the product of their components with the \( \sin() \) and \( \cos() \) oscillations at frequency \( \omega_0 \) has the effect of folding their constant spectral density over itself, yielding again a constant spectral density. Under such conditions, the spectral density as a function of frequency \( f \) of the resulting oscillation around frequency \( f_0 \) is the Lorentzian curve (normalized to unitary power over the full frequency range)

\[ \mathcal{L}(f) = \frac{1}{\pi \sigma} \cdot \frac{1}{(f - f_0)^2 + 1} \] (13)

where \( \sigma \) is the line-width of the generated signal.

The path from eq. (11) to (13) contains all the mathematical difficulties, so widely described in the literature. It is possible to travel this path in three steps, making use of two widely known theorems (Central Limit and Wiener-Khinchin theorem) and an established property of gaussian distributions. All of this will be used here without any demonstration, that is left to appropriate textbooks.

The starting point is the requirement that \( i_n \) and \( v_n \) be white noise sources. It is not necessary they have a gaussian distribution. Usually, \( \omega_n \) will not have a Gaussian distribution, apart some very special conditions, because the product of \( i_n \) and \( v_n \) with the \( \sin() \) and \( \cos() \) functions in eq. (11) would alter their original Gaussian distribution, if any. In any case, \( \varphi_n \) will have a Gaussian distribution, as stated by the Central Limit theorem, because \( \varphi_n \) results from the sum of infinitely many independent random contributions, as seen in eq. (12).

The second step is to obtain the time autocorrelation function \( R_F(\tau) \) for the signal at the output of the oscillator, namely for the \( \sin[\varphi(t)] \) or the \( \cos[\varphi(t)] \) function.

Following [11][12], using the complex notation, this can be written as

\[ R_F(\tau) = \frac{A_0^2}{2} \cdot \text{Re} \left\{ \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{i \varphi(t)} \cdot e^{-i \varphi(t+\tau)} dt \right\} \]

\[ = \frac{A_0^2}{2} \cdot \text{Re} \left\{ e^{-i \omega_0 \tau} \left\{ e^{i[\varphi_n(t) - \varphi_n(t+\tau)]} \right\} \right\} \]

where \( A_0 \) is the oscillation amplitude and the \( \{ \) brackets are a short for the \( \{ \lim f \} \) computation.

Here the properties of the gaussian distribution play their part. Using the gaussian moment theorem, it is possible to write

\[ \left\langle e^{i[\varphi_n(t) - \varphi_n(t+\tau)]} \right\rangle = e^{-\frac{1}{2} \left\{ [\varphi_n(t) - \varphi_n(t+\tau)]^2 \right\}} \] (14)

A short but clear demonstration for this can be found in [12] and references therein.

Evaluation of the exponent in the right hand side of eq. (14) is easier if it is made in a discrete time framework, like in the numerical integration of the circuit differential equations in the numerical experiments. It has been supposed that noise generators are stationary, hence time average can be replaced by variance (indicated by the overline):

\[ \left\langle [\varphi_n(t) - \varphi_n(t+\tau)]^2 \right\rangle = \left[ \varphi_n(0) - \varphi_n(\tau) \right]^2 = \varphi_n^2(\tau) \] (15)

At every \( \Delta t \) time step, \( \varphi_n \) increases by the amount \( \omega_n \Delta t \). Increments are random and uncorrelated, hence sum quadratically. After \( N = \tau/\Delta t \) time steps

\[ \varphi_n^2(\tau) = \omega_n^2 \Delta t^2 \frac{\tau}{\Delta t} = \omega_n^2 \Delta t \tau \] (16)

It is to be recognized that \( 2 \omega_n^2 \Delta t \) is the spectral density of \( \omega_n \) (this will be made clear in the description of the numerical experiments):

\[ \omega_n^2 \Delta t = \Omega_n^2 = \frac{1}{2} \left( \frac{I_n^2}{v_0^2 C^2} + \frac{V_n^2}{i_0^2 L^2} \right) \] (17)

where \( \Omega_n^2 \) has dimensions of \( s^{-1} \).

The final form of the autocorrelation function is

\[ R_F(\tau) = \frac{A_0^2}{2} \cos(\omega_0 \tau) e^{-\Omega_n^2 \tau/4} \] (18)

The third and last step is the application of Wiener-Khinchin theorem that states that the spectral distribution \( F(\omega) \) of a stationary random sequence is linked to the Fourier transform of its autocorrelation function:

\[ F(\omega) = 4 \int_0^\infty R_F(\tau) \cos(\omega \tau) d\tau \] (19)

Applying the theorem to function (18)

\[ F(\omega) = 2 A_0^2 \int_0^\infty \cos(\omega \tau) \cos(\omega_0 \tau) e^{-\Omega_n^2 \tau/4} \]

\[ = A_0^2 \int_0^\infty e^{-\Omega_n^2 \tau/4} \cos[(\omega_0 - \omega) \tau] + \cos[(\omega_0 + \omega) \tau] \] (20)
The rapidly oscillating term with frequency \( \omega_0 + \omega \) gives a negligible contribution to the integral and can be dropped (the integral goes rapidly to zero when \( \omega \) moves away from \( \omega_0 \)). Developing the integral of the remaining term, the Lorentzian of eq. (13) is obtained, with

\[
\sigma = \frac{\Omega^2}{8\pi} \quad (21)
\]

This result is illustrated in the Numerical Experiments paragraph, in the first numerical experiment, where only the components \((v_{nt}, i_{nt})\) from the full noise sources \((v_n, i_n)\) will be applied to the resonant circuit. The oscillation amplitude will remain constant and the well known Lorentzian noise distribution will appear on sides of \( \omega_0 \).

**Amplitude fluctuations**

As anticipated above, every oscillator of practical interest has some kind of regulation mechanism that controls the oscillation amplitude.

There is a twofold requirement for this. First, the condition for persistent oscillation is that the damping given by \( R \) and \( G \) be canceled out (it does not make a great difference if each one separately, as considered in this work, or together as a whole). This is accomplished, as well known, using a positive feedback, that introduces into the circuit a negative \( R' \) and/or \( G' \) that exactly compensates the \( R \) and \( G \) damping effect.

The word exactly is a term out of the realm of physics or engineering; however accurate a compensation is made, it will not be exact but for a short instant, thereafter a drift of one kind or another will make damping move away from zero and make the oscillation rise or decay.

Only a negative feedback regulation mechanism can continuously compare actual oscillation amplitude with a reference value and adjust the negative damping to contrast any drift.

Indeed, in a mathematical sense damping can be made exactly null. This has been assumed in all the numerical experiments, not inserting into the oscillator circuit either \( R \) or \( G \). But in the second numerical experiment the two noise sources \( i_n \) and \( v_n \) will exchange energy with the \( LC \) resonator and make the oscillation amplitude fluctuate in a random way.

The current generator \( i_n \), for example, will induce a disturbance in the voltage across the capacitor \( C \) whose spectral amplitude will be

\[
V_{cn}(\omega) = I_n(\omega) \cdot \frac{1}{\omega C - \frac{1}{\omega L}} \quad (22)
\]

At the resonance frequency \( \omega_0 = 1/\sqrt{LC} \) expression (22) diverges, and indeed this is what would happen, given an infinitely long time.

But the presence of the regulation mechanism, that cannot be avoided for the reasons explained above, keeps under control and limits the amplitude fluctuations also. And this is the second reason the regulation mechanism is needed.

One of the simplest conceivable regulation mechanisms is a first order control loop described by the equation

\[
\frac{dP}{dt} = K \cdot (P_0 - P) \quad (23)
\]

that has a stationary asymptotic solution \( P = P_0 \) and a transient behaviour given by an exponential relaxation with time constant \( 1/K \).

The dynamical behaviour of the solution of this equation is analogous to the oscillation amplitude decay in a resonant circuit with resonance frequency \( \omega_0 \) and figure of merit \( Q_0 \):

\[
\frac{d}{dt} v_0(t) + \frac{\omega_0}{2Q_0} v_0(t) = 0
\]

with

\[
K = \frac{\omega_0}{2Q_0} \quad (24)
\]

Similar equations hold for \( i_0(t) \).

Hence the behaviour we should expect for the noise induced amplitude fluctuations is the excitation of a resonance around frequency \( \omega_0 \) with a bandpass \( \pm \Delta \omega = K \). This process is akin to the virtual damping in \([9]\) and will be illustrated in the second numerical experiment.

The scenario outlined up to now is completely consistent if it is recognized that considering only the \((v_{nt}, i_{nt})\) couple of components of the \((v_n, i_n)\) noise in the circuit while discarding the \((v_{np}, i_{np})\) components is fully equivalent to suppressing such fluctuations through a very efficient amplitude regulation mechanism, as expressed by eq. (23).

**Upconversion of 1/f noise**

From eqs. (10-11) it is clear that if \( i_n \) and \( v_n \), the coloured noise sources, also the resulting spectrum for \( \omega_n \) will be coloured, hence the main mathematical assumption in derivation of result (15) will no more hold and the spectrum of the generated signal mostly will not be a Lorentzian curve.

It is out of the purpose of this paper to analyze in any detail the implications of eqs. (10-11) with non-white \( v_n \) and \( i_n \) noise sources. For a deeper analysis see, e.g., \([10]\). Only an attempt has been made to illustrate the point, with the third numerical experiment where coloured noise sources have been used for \( v_n \) and \( i_n \).
NUMERICAL EXPERIMENTS

Numerical experiments have been carried out using the circuit of fig. 1 and numerically integrating the circuit differential equations in the presence of the two noise generators $i_n$ and $v_n$. The computational engine of the Spice program [13] is a very well optimized computer application to perform this task. In this work all computations have been made using version 24 of ngspice [14], an actively maintained open source variety of Spice, and the fftw library [15] to compute by Fast Fourier Transform (FFT) frequency spectra from temporal sequences (this library also available as open source).

All computations that will be described can be afforded using a medium-high range PC, but their repetition to improve results quality by averaging noise spectra can be very time consuming. Hence, most of the repetitive work has been performed using an HP ProLiant DL560 Gen8 machine equipped with 64 processors and 128 GB RAM.

The basic machinery used for the first numerical experiment can be seen in the following Spice list:

*** experiment 1 ***

```
11 0 2 159.154943092e-6 ic=1
c1 1 0 159.154943092e-6 ic=0
vn 0 dc 0 trnoise(100m 10u 0 0 0 0 0)
vni 4 0 dc 0 trnoise(100m 10u 0 0 0 0 0)
bwn 5 0 v=v(1)*v(3)-i(bvg)*v(4)
big 1 0 i=-i(bvg)*v(5)
.control
set nobreak
set numdgt=12
tran 10u 10000m 0m 0.1u uic
linearize
print v(1) i(11) > data.out
quit
.endc
```

Values for $L$ and $C$ (11 and c1) have been chosen to give a resonance frequency $f_0$ as close as possible to 1000 Hz to reduce the noise generated by truncation during FFT (in the experiments when phase is not affected) and have been set numerically equal to avoid unneeded computational overheads. This way equal amplitudes on vnn and vni give equal contribution to overall noise power. There should be no loss of generality in this choice.

The two generators vnn and vni are white noise voltage sources that yield random sequences of values with a gaussian amplitude distribution and rms value $V_G = 0.1 V$. The cadence of the computed signals has been set to $\Delta t = 10 \mu s$, hence Nyquist’s frequency is $f_N = 1/(2\Delta t) = 50 kHz$ and the resulting noise spectral density is $V_G^2/f_N = 2V_G^2\Delta t = 0.2 \cdot 10^{-6} V_{rms}^2/Hz$ (−67 dBV in a 1 Hz bandwidth).

In a similar way the random variable $\omega_n$ of eq. [14] with variance $\omega_n^2$ has spectral density $\Omega_n^2 = 2\omega_n^2\Delta t$ as given in eq. [17].

Generator bwn is a convenience to obtain an intermediate result and can be used to draw distributions of $\omega_n$, as in fig. 5 below.

Two further controlled generators (bvg and big) transform the independent voltages from vnn and vni into the pair of values $v_{nt}$, $i_{nt}$ given by equations (5-6) and play the role of $v_n$ and $i_n$ in the circuit.

Because the numerical values of $L$ and $C$ have been chosen to be equal, if the amplitude of the oscillation remains constant and equal to 1 (both $V$ and $A$), with the given initial conditions $\sin \varphi$ and $\cos \varphi$ can be written simply as

$$\sin \varphi = i$$
$$\cos \varphi = v$$

namely, $\sin \varphi = i(bvg)$ and $\cos \varphi = v(1)$.

First of all, the computational engine has been tested idle, i.e. performing computations with the amplitude of vnn and vni generators set to zero, for a time length of 1 s.

![Plot of i vs. v without noise (a, left) and with a noise of 0.1 Vrms (b, right). Computational length: 1 s (1000 cycles).](image)

The results are shown in fig. 3a, where values in the data file data.out have been plotted as $i$ vs. $v$, i.e., $i(11)$ vs. $v(1)$. The plot covers 1000 cycles, each cycle containing 100 equally spaced samples and confirms phase coherence and amplitude stability of the computational engine over such a time length (indeed, a more accurate analysis shows a decay in amplitude of 0.04% /s). The noise floor due to the overall arithmetic accuracy is about −145 dBV (or dBA) in a 1 Hz bandwidth.

**Experiment 1: phase fluctuations**

The computations described above have been repeated activating the noise generators vnn and vni at levels of
0.05, 0.1 and 0.2 \( V_{rms} \) over a time length up to 10 s, in order to obtain a frequency resolution of 0.1 Hz. The plot in fig. 3a changes, in that points do not repeat with the periodical cadence of 3.6\(^\circ\), rather scatter continuously all over the circumference. Nevertheless the radius remains unitary, thanks to the behaviour of the bvgs and bg generators that suppress the \( dP \) component from each couple of \( v_n, i_n \) values, leaving only the \(dT \) component (fig. 3b).

![FIG. 4: Spectral amplitude in a 1 Hz bandwidth of the \( v \) signal across the \( C \) capacitor in the presence of the \( v_{nt} \) and \( i_{nt} \) components only of the \( v_n \) and \( i_n \) noise generators. Noise generators intensity: 50, 100 and 200 mV. Results from the numerical experiment (dots) are plotted in comparison with Lorentz’s function (solid lines). The \( \sigma \) parameter computed from eq. (17) and (21) is \( \sigma = 0.0785, 0.314 \) and 1.257 Hz respectively from the lower to the upper curve). No time windowing has been applied with FFT, because the noise contribution due to truncation is 15-20 dB below the curves shown in the plot and becomes visible only for noise generators amplitude of 25 mV or smaller. Each curve obtained averaging results from 15 repetitions of the experiment, with a time length of 10 s.](image)

The spectral amplitude of the generated \( v \) signal across capacitor \( C \) is shown in fig. 4, together with the computed Lorentzian distributions, with the \( \sigma \) values obtained from eq. (17) and (21). The spectra for current \( i \) through inductance \( L \) are identical and are not shown.

\[
\sin \varphi = \frac{i}{v_{n}P}, \quad \cos \varphi = \frac{v}{\sqrt{i_{n}P}}
\]

FIG. 5: Distribution of the \( \varphi_n \Delta t \) values given by eq. (11) (on the left) and Bode plot of the right branch of the spectral amplitudes (on the right) obtained with two generators with uniform distribution (U) and with a single gaussian generator (S). The third plot (G), for comparison, is made from the data of fig. 4. The expected analytical gaussian distribution is plotted as well together with the G curve, on the left, and is practically undistinguishable from the numerical distribution. In all cases the total \( rms \) value of the generators was \( 200 \cdot \sqrt{2} \) mV.

The computations have been repeated again using for \( v_{nv} \) and \( v_{ni} \) two generators with a uniform values distribution instead of a Gaussian one, with \( rms \) amplitude 200 mV (interval \( \pm 346.4 \) mV):

\[
v_{nv} 3 0 \ dc \ 0 \ tr\ random \ (1 \ 10u \ 0 \ 346.4m)
v_{ni} 4 0 \ dc \ 0 \ tr\ random \ (1 \ 10u \ 0 \ 346.4m)
\]

and once more with a single gaussian generator, either \( v_{nv} \) or \( v_{ni} \), with an \( rms \) amplitude \( 200 \cdot \sqrt{2} \) mV.

The distributions of the \( \varphi_n \Delta t \) values given by eq. (11) for the three cases are shown in fig. 5, on the left. Only the special case of the first computation exhibits a Gaussian distribution. Nevertheless, the other two cases also yield the same identical Lorentzian spectral amplitudes, as shown in the Bode plots on the right. In all cases, the Gaussian form of the \( \varphi_n \) distribution is granted by effect of the Central Limit Theorem and can be seen examining the sum of a few consecutive values of \( \varphi_n \Delta t \).

**Experiment 2: amplitude fluctuations and stabilization**

In order to apply amplitude only fluctuations to the circuit, without affecting the phase, generators bvgs and bg are to be modified according to eq. (18). In this case \( P \) cannot be considered any more a constant, so that

\[
\sin \varphi = \frac{i}{v_{nt}P}, \quad \cos \varphi = \frac{v}{\sqrt{i_{nt}P}}
\]

Thanks to the numerical equivalence of \( L \) and \( C \) the work can be done substituting the buns, bvgs and big generators in the Spice list of the first numerical experiment with
proceeds at the constant angular speed of 3 (left). From the figure it is clear that now the phase, unless a phase lock mechanism to some time

and using this quantity as an error signal to actuate the radial only correction

This can be split into the $dv$ and $di$ components:

$$dv = v_0 \cos \varphi \cdot dP = v(t)dP$$

$$di = i_0 \sin \varphi \cdot dP = i(t)dP$$

Combining these expressions with equation (1) and (25), the correction current and voltage $i_c$ and $v_c$ result:

$$dv = \frac{i_c}{C}dt = v(t)dP \quad \rightarrow \quad i_c = KCv(t)\mathcal{E}(t) \quad (26)$$

$$di = \frac{v_c}{L}dt = i(t)dP \quad \rightarrow \quad v_c = KLi(t)\mathcal{E}(t) \quad (27)$$

These corrections can be applied to the circuit adding their values to generators $\text{bvg}$ and $\text{big}$, e.g. through generator $\text{bpp}$, modifying its definition into

$$\text{bpp} 21 0 \ v=(v(4)*v(1)+v(3)*i(\text{bvg}))/v(20)+ + (\sqrt(v(20))-1)*K$$

and adding the command “.param K=...” to set the appropriate value for the $K$ constant.

Computations have been performed with an amplitude of $10 \ mV$ for the $\text{vnv}$ and $\text{vni}$ noise generators and values of $K$ from $10^{-3}$ to 1. Results are shown in fig. 7, together with the expected resonance curves. The value of $Q_0$ for each value of $K$ has been computed according to eq. (28), taking into account that the $C$ and $L$ factors in eqs. (24) and (26) actually are contained in the computational value of $K$.

$$Q_0 = \frac{\omega_0}{2K/C} \quad (28)$$

FIG. 7: Spectral amplitudes in a 1 Hz bandwidth of the $v$ signal in the presence of the $v_{np}$ and $i_{np}$ components only of the $v_n$ and $i_n$ noise generators. The noisy curves have been obtained with values of the $K$ constant in the amplitude regulation mechanism of 1, 0.1, 0.01 and 0.001, from the lower to the upper curve (each one is the average of 25 repetitions of the numerical experiment, with a time length of 1 s). The smooth lines over the noisy ones are the resonance curves computed with the $Q_0$ values given by eq. (28).

In any case, an amplitude regulation mechanism is necessary for the reasons already explained. Even the purely computational approach of Spice applied to an ideal circuit is not completely free from drift, as quoted in the introduction to the Numerical Experiments.

The amplitude control expressed by eq. (24) has been implemented computing the instantaneous deviation of energy in the circuit with respect to the reference value $W_0$

$$\mathcal{E}(t) = \sqrt{\frac{1}{2}v^2(t) \cdot C + \frac{1}{2}i^2(t) \cdot L} - 1$$

and using this quantity as an error signal to actuate the radial only correction

$$dP = K\mathcal{E}(t)dt \quad (25)$$

The frequency spectra clearly show the presence of a really monochromatic oscillation of amplitude $0 \ dBV$, as set by the initial conditions, and the response of a
resonator with different values of $Q_0$, excited by the noise signals of $v_n$ and $i_n$.

As already pointed out by most authors, an efficient amplitude regulation mechanism can reduce the contribution of the amplitude fluctuations to the overall noise to a level negligible in front of the phase fluctuations contribution.

However, it is not correct to conclude that half the overall noise injected by $v_n$ and $i_n$ into the circuit can be suppressed by the regulation mechanism. This happens only in this very particular and abstract example in which the regulation mechanism uses two synchronized generators for voltage and current, that act only over the amplitude, avoiding any interference with the phase.

While it is not impossible by principle to implement practically such a machinery, it is not that easy. A possible way, as suggested in [9] is to make the regulating machinery, it is not that easy. A possible way, as suggested in [9], is to make the regulating network, usually behaving as a voltage or a current generator, to interact with the resonator only when the oscillation phase is at an appropriate point and the correction would require only a voltage or a current injection.

Most often it happens that the regulating injected signal contains a $dT$ component over the $dP$ one and this makes the noise contribution suppressed as an amplitude fluctuation to appear again as a phase fluctuation.

Numerical experiments have been performed to verify this point (results not shown). Applying both the $dT$ and $dP$ noise components to the resonator with the amplitude control active (with $K = 10$), noise spectra identical to fig. 4 are obtained. Thereafter, suppressing the regulating action of one of the bvg or big generators, correction is no more only radial and phase noise spectrum shows the resulting tangential increment. This is the common behaviour of real oscillators, where the splitting of the regulation signal into a $dT$ and a $dP$ components is not usually feasible.

Experiment 3: upconversion of low frequency and $1/f$ noise

Experiments in this section have been performed in the conditions of both experiment 1 and 2, i.e. with no amplitude control, applying only the $dT$ noise component, and with the amplitude control active (with $K = 10$), applying both the $dT$ and $dP$ components to the resonator. This last condition has been implemented using the following definitions for the b.. generators:

$$\begin{align*}
\text{bpq} & \quad 2 \quad 0 \quad v = v(1) \cdot v(1) + i(bvg) \cdot i(bvg) \\
\text{bpp} & \quad 21 \quad 0 \quad v = (\text{sqrt}(v(20)) - 1) \cdot K \\
\text{bvg} & \quad 21 \quad 0 \quad v = i(bvg) \cdot v(21) + v(3) \\
\text{big} & \quad 1 \quad 0 \quad i = v(1) \cdot v(21) + v(4)
\end{align*}$$

Identical results have been obtained with both implementations of the experiment (apart a different level of the noise floor due to the arithmetics), because in both cases the $dP$ perturbation component was effectively removed.

Experiment 3a

First, the oscillator behaviour has been studied under the application of a 10 mV low frequency sinusoidal signal with frequency $f_1 = 0.25, 0.5, 1, 2, 4, 8$ Hz in place of the vnv and vni noise generators. With $f_1 = 1$ Hz the generators were

$$\begin{align*}
\text{vnv} & \quad 3 \quad 0 \quad dc \quad 0 \quad \text{sin}(0 \quad 10 \quad 1 \quad 0 \quad 0) \\
\text{vni} & \quad 3 \quad 0 \quad dc \quad 0 \quad \text{sin}(0 \quad 10 \quad 1 \quad 0 \quad 0)
\end{align*}$$

and the results are shown in fig. 8.

![Power spectral density of the oscillator](graph.png)

**FIG. 8:** Power spectral density of the oscillator variable in the presence of a sinusoidal perturbation at frequency 1 Hz and amplitude 10 mV for $v_n$ and $i_n$. The computation has been extended over a time length of 20 s. A Blackmann-Harris window has been applied with FFT in order to reduce truncation effects. This spectrum has been obtained applying only the $dT$ perturbation component. The spectrum obtained the other way, i.e. with the amplitude control active, shows identical peaks and a noise floor 20 dB higher.

Experiment 3b

Thereafter, two coloured noise sources have been used for the $v_n$ and $i_n$ generators, obtained summing the usual white gaussian noise with a $1/f^n$ noise component:

$$\begin{align*}
\text{vnvg} & \quad 3 \quad 30 \quad dc \quad 0 \quad \text{trnoise} \quad (0.1 \quad 10u \quad 0 \quad 0 \quad 0 \quad 0) \\
\text{vnvf} & \quad 30 \quad 0 \quad dc \quad 0 \quad \text{trnoise} \quad (0.1 \quad 10u \quad 1.9 \quad 1m \quad 0 \quad 0) \\
\text{vnig} & \quad 40 \quad 0 \quad dc \quad 0 \quad \text{trnoise} \quad (0.1 \quad 10u \quad 0 \quad 0 \quad 0 \quad 0) \\
\text{vnif} & \quad 40 \quad 0 \quad dc \quad 0 \quad \text{trnoise} \quad (0.1 \quad 10u \quad 1.9 \quad 1m \quad 0 \quad 0)
\end{align*}$$

A value close to 2 has been used for the $n$ exponent in the $1/f^n$ distribution to improve the separation among
the excess noise region and the oscillation frequency $f_0$, while keeping an high enough excess noise amplitude.

Results from these computations are shown in fig. 9, together with the Lorentzian curve associated with the white gaussian-only noise source.

An analysis of results in fig. 8 and 9 will be given in the Discussion.

**DISCUSSION**

Results shown in fig. 4 and 7 are well known. Fig. 7 shows that the fluctuations in the oscillation amplitude can be deeply reduced, to the level of being negligible, by a suitable circuit regulation mechanism. When the regulation is tight enough ($K \geq 1$) only a flat noise floor is left, whose level decreases by $20\, \text{dB}$ for each tenfold increase in $K$.

The phase component of the noise induced signal fluctuations, instead, in the case of white noise sources, impresses to the generated signal spectral distribution the shape of a Lorentzian curve (fig. 4).

The most relevant and new aspect of this part of the work is the neat and direct mathematical path that starts from the splitting of the noise into the $dP$ and $dT$ components and leads to equations (15), (21) and (24). These express both the functional form of the noise distribution and the values of the constants it depends on, directly expressed in terms of the unique two relevant circuit characteristics: the energy $W_0$ of the oscillation in the resonator and the rate of energy exchange ($\text{power}$) of the noise source(s) $I_n^2 C + V_n^2 L$.

Remaining with the simple case considered in the first numerical experiment, in which

$$
\frac{I_n}{v_0 C} = \frac{V_n}{i_0 L}
$$

eq. (17) becomes, e.g.,

$$
\Omega_n^2 = \frac{P_n^2}{2W_0}
$$

Substituting this last expression into eq. (24)

$$
\sigma = \frac{1}{16\pi} \cdot \frac{P_n^2}{W_0}
$$

(30)

If the noise sources were only due to the losses in the resonator by effect of the parallel conductance $G$ (or serial resistance $R$, or both), one should have $P_n^2 = 4KTG$ and, by common formulas, $Q_0 = \omega_0 C/G$.

Combining these expressions into eq. (30):

$$
\sigma = \frac{1}{2\pi} \cdot \frac{KT \omega_0}{Q_0}
$$

(31)

This result is identical to analogous results reported in the literature, e.g., eq. (25) in [8] and eq. (70) in [4]; but in the form of eq. (30) it is more pregnant and far-reaching. In a more general case, it is possible through eq. (17) to sum up into the $i_n$ and $v_n$ generators all noise contributions that exist in a circuit, then compute $\Omega_n$ and the resulting line-width $\sigma$. From the details of the amplitude regulation mechanism, through eq. (31), it is possible to compute how much of the amplitude noise component is converted into phase noise, to be added to the native phase noise component.

It is worth noting that the figure of merit $Q_0$ of the resonator does not appear in eq. (30), while it appears when passing to eq. (31). It has always been believed, and verified experimentally, that an high value for $Q_0$ is beneficial to reduce phase noise. Equations (30) and (31) tell this is true, but only when the losses in the resonator are the main contribution as noise sources; otherwise, in passing from eq. (30) to (31) $I_n^2$ cannot be replaced by $4KTG = 4KT \omega_0 C/Q_0$, but should become

$$
I_n^2 = I_{n0}^2 + 4KT \omega_0 C
$$

(32)

where $I_{n0}^2$ stands for all other sources of noise current different from the parallel $G$ of the resonator. This equation expresses the same concept as the $Q_{\text{loaded}}$ in [8]. Also in [8] it had been observed that an increase in $Q_0$ by any means, that also increase the noise, is vain.

As already observed, what is truly relevant for the spectral purity is the noise $\text{power}$ to signal energy ratio as expressed by eq. (30).

About the third experiment, the application of a small low frequency perturbation to the circuit, as made in

![FIG. 9: Power spectral density of the $v$ (and $i$) oscillator variable in the presence of coloured noise sources in the role of the $v_n$ and $i_n$ generators. The plot inside the inset reports the spectral density of the noise generators (averages from 34 computations 10 s long).](image-url)
Experiment 3a, reveals the deeply non-linear behaviour induced by any efficient amplitude regulation mechanism. Either the radial component \(dP\) of the perturbation be removed \textit{a priori}, as in Experiment 1, or be suppressed by the amplitude regulation, the identical result is the appearance of the same side-band components around the frequency of oscillation.

The very interesting aspect to be observed is that this is not a simple effect of amplitude modulation (AM): the plot in fig. 9 clearly shows that lateral components are at frequency \(1000 \pm 2\) Hz, \(\pm 4\) Hz and \(\pm 6\) Hz, hence at an offset frequency twice the 1 Hz frequency of the perturbation signal (and higher order harmonics). True AM sidebands can be obtained, instead, adding the modulating signal to the expression \(\text{sqr}(20)-1\) in the \textbf{ppp} generator definition (results not shown).

A close analysis of the data in fig. 9 clearly shows that the central peak is no more exactly at 1000 Hz, as it systematically was in all previous computations; rather, it is slightly moved to the left by a small amount that can be appreciated also in the plot. As a fact, the \textbf{data.out} file obtained from this experiment does contain a signal with average frequency of 999.95 Hz instead of 1000.00. This offset in the oscillator frequency depends upon the amplitude of the perturbing signal but does not depend upon its frequency. All frequencies from 0.25 to 8 Hz gave exactly the same offset; varying the amplitude in the 1 – 10 mV range a dependence of the offset with the square of the amplitude was observed.

Reducing the value of the \(K\) constant while performing the third experiment in the condition of Experiment 2, the sidebands and the frequency offset progressively reduce and vanish with \(K \to 0\).

This makes sense, because with \(K \to 0\) every non-linear behaviour is removed from the circuit and only a linear resonator is left, excited by a voltage/current at an offset frequency twice the 1 Hz frequency of the perturbation signal (and higher order harmonics). True AM sidebands can be obtained, instead, adding the modulating signal to the expression \(\text{sqr}(20)-1\) in the \textbf{ppp} generator definition (results not shown).

The frequency doubling cannot be an artifact due to the particular amplitude regulation algorithm adopted; it appears identical with no regulation mechanism at all, only by effect of noise splitting and \(dP\) component suppression.

These findings are to be kept in mind when observing the behaviour of an oscillator in the presence of a \textit{coloured} noise source, as it is the case for \(1/f\) noise components. Results in fig. 9 (the noisy line) show the behaviour in such a condition, compared to the white noise behaviour of fig. 4. It is not a simple process of upconversion and appearance of an upper and lower band on side of \(f_0\); rather, it is a deformation and a shift to the left of the distribution and its peak due to a process of frequency pulling as described in \[11\] [12].

In view of these results, it is doubtful that this process can be considered the source (or the unique source) of \(1/f\) components observed in oscillators on side of the carrier. Other effects should be considered; for example, the direct modulation of the generated signal in passing through a component with equilibrium value affected by \(1/f\) fluctuations \[17\]. But this is argument for a new investigation and is out of the purposes of this paper.

**ACKNOWLEDGMENTS**

The Florence section of INFN (Istituto Nazionale di Fisica Nucleare) and Prof. Maurizio Bini are gratefully acknowledged for granting the access to the 64-core computing machine used in most of the described computations.

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[1] D.M. Pozar, Microwave Engineering, IV ed., J. Wiley & Sons, 2012.
[2] W.A. Edson, “Vacuum tube oscillators”, John Wiley & Sons, Inc., N.Y., 1953.
[3] Current Contents Eng. Tech. Appl. Sci., N. 26, 27 Jun. 1983.
[4] W.A. Edson, “Noise in oscillators”, Proc. IRE, vol. 48-8, pp. 1454-1466, August 1960.
[5] D.B. Leeson, “A simple model of feedback oscillator noise spectrum”, Proc. IEEE, Vol. 54 No. 2, Feb. 1966, pp. 329-330.
[6] Melvin Lax, “Classical Noise v. noise in self-sustained oscillators” Phys. Rev., Vol. 160 No. 2, pp. 290-307, 10 Aug. 1967.
[7] Ali Hajimiri and Thomas H. Lee, “A general theory of phase noise in electrical oscillators”, IEEE J. Solid-State Circuits, Vol. 33 No. 2, pp. 179-194, Feb. 1998.
[8] Thomas H. Lee and Ali Hajimiri, “Oscillator phase noise: a tutorial”, IEEE J. Solid-State Circuits, Vol. 35 No. 3, pp. 326-336, March 2000.
[9] Donhee Ham and Ali Hajimiri, “Virtual damping and Einstein relation in oscillators”, IEEE J. Solid-State Circuits, Vol. 38 No. 3, pp. 407-418, March 2003.
[10] Ahmad Mirzaei and Asad A. Abidi, “The spectrum of a noisy free-running oscillator explained by random frequency pulling”, IEEE Trans. Circuits Syst. I, Vol. 57 No. 3, pp. 642-653, March 2010.
[11] G. Di Domenico, S. Schilt and P. Thomann, “Simple approach to the relation between laser frequency noise and laser line shape”, Applied Optics, Vol. 49, Iss. 25, pp. 4801-4807, 2010.
[12] D.S. Elliott, Rajarshi Roy, and S.J. Smith, “Extracavity laser band-shape and bandwidth modification”, Phys. Rev. A 26, pp. 12-18 (1982).
[13] *Spice* homepage at the Berkeley Wireless Research Center: [http://bwrcs.eecs.berkeley.edu/Classes/IcBook/SPICE](http://bwrcs.eecs.berkeley.edu/Classes/IcBook/SPICE)
[14] *ngspice* homepage: [http://ngspice.sourceforge.net/](http://ngspice.sourceforge.net/)
[15] homepage of the *fftw* library: [http://www.fftw.org/](http://www.fftw.org/)
[16] Behzad Razavi, “A study of injection locking and pulling in oscillators”, IEEE J. Solid-State Circuits, Vol. 39, N. 9, pp. 1415-1424, September 2004.
[17] Richard F. Voss and John Clarke, “1/f noise from systems in thermal equilibrium”, Phys. Rev. Lett., Vol. 36 N. 1, pp. 42-45, 1976.