THE UNBOUNDED KASPAROV PRODUCT BY A DIFFERENTIABLE MODULE

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Abstract. In this paper we investigate the unbounded Kasparov product between a differentiable module and an unbounded cycle of a very general kind that includes all unbounded Kasparov modules and hence also all spectral triples. Our assumptions on the differentiable module are as minimal as possible and we do in particular not require that it satisfies any kind of (smooth) projectivity conditions. The algebras that we work with are furthermore not required to possess a (smooth) approximate identity. The lack of an adequate projectivity condition on our differentiable module entails that the usual class of unbounded Kasparov modules is not flexible enough to accommodate the unbounded Kasparov product and it becomes necessary to twist the commutator condition by an automorphism.

We show that the unbounded Kasparov product makes sense in this twisted setting and that it recovers the usual (bounded) Kasparov product after taking bounded transforms. Since our unbounded cycles are twisted (or modular) we are not able to apply the work of Kucerovsky (nor any of the earlier work on the unbounded Kasparov product) for recognizing unbounded representatives for the bounded Kasparov product. In fact, since we do not impose any twisted Lipschitz regularity conditions on our unbounded cycles, even the passage from an unbounded cycle to a bounded Kasparov module requires a substantial amount of extra care.

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1. Introduction

In a series of papers from the early eighties, Kasparov proved the fundamental results on the $KK$-theory of $C^*$-algebras, [Kas80a, Kas80b, Kas75]. One of the main inventions appearing in these papers is the interior Kasparov product which provides a bilinear and associative pairing 

$$\widehat{\otimes}_B : KK_n(A, B) \times KK_m(B, C) \to KK_{n+m}(A, C)$$

between the $KK$-groups of three (separable) $C^*$-algebras $A$, $B$, and $C$. The interior Kasparov product of two $KK$-classes is computable in many cases, but the main construction remains inexplicit as it relies on Kasparov’s absorption theorem and Kasparov’s technical theorem.

One of the advantages of the $KK$-groups of $C^*$-algebras is the wealth of explicit examples of elements arising from geometric data. Indeed, in the unbounded picture of $KK$-theory the cycles are unbounded Kasparov modules, which are bivariant versions of Connes concept of a spectral triple, and the unbounded Kasparov modules exhaust the $KK$-groups as was proved by Baaj and Julg, [BaJu83].

The problem that we are concerned with in this paper is to construct an unbounded version of the interior Kasparov product. More precisely, starting with two unbounded Kasparov modules, the aim is to find an explicit unbounded Kasparov module that represents the interior Kasparov product. In particular, this construction should bypass the need for invoking both the absorption theorem and the technical theorem. The problem of constructing the unbounded Kasparov product is currently receiving an increasing amount of attention, see [Con96, KALe13, Mes14, MeRe15], as is also witnessed by the quantity of recent applications, see [BMS13, MeGo15, FoRe15, BCR15].

At a deeper level, the unbounded Kasparov product is important because of the loss of geometric information that is inherent in the passage from an unbounded Kasparov module to a class in $KK$-theory. It is thus in our interest to be able to perform a version of the interior Kasparov product while retaining a larger amount of geometric data (as for example the asymptotic behaviour of eigenvalues of differential operators).

In this paper we are focusing on the case where the class in the $KK$-group, $KK(A, B)$, is represented by a $C^*$-correspondence $X$ from $A$ to $B$ and where the action of $A$ from the left factorizes through the $C^*$-algebra of compact operators on $X$. On the other hand, our class in the $KK$-group $KK(B, C)$ will be represented by an unbounded selfadjoint operator $D : \mathcal{D}(D) \to Y$ acting on a $C^*$-correspondence from $B$ to $C$. The unbounded operator $D$ is required to satisfy a couple of extra conditions that will be detailed out in the main text. The first challenge is then to
construct a new unbounded selfadjoint operator
\[ 1 \otimes \nabla D : \mathcal{D}(1 \otimes \nabla D) \to X \hat{\otimes}_B Y \]
that acts on the interior tensor product of the \( C^* \)-correspondences \( X \) and \( Y \). In the main part of the earlier works on the unbounded Kasparov product this step is accomplished by assuming the existence of a (tight normalized) frame \( \{ \zeta_k \} \) for \( X \) (see \cite{FrLa02}) such that the associated orthogonal projection
\[ P := \sum_{n,m=1}^{\infty} \langle \zeta_n, \zeta_m \rangle \delta_{nm} : \ell^2(Y) \to \ell^2(Y) \]
(which acts on the standard module over \( Y \)) has a bounded commutator with the unbounded selfadjoint operator \( D : \mathcal{D}(D) \to Y \) (slightly weaker conditions are applied in \cite{BMS13} and \cite{MeRe15}). The unbounded selfadjoint operator \( 1 \otimes \nabla D : \mathcal{D}(1 \otimes \nabla D) \to X \hat{\otimes}_B Y \) can then be expressed as the infinite sum
\[ 1 \otimes \nabla D := \sum_{n=1}^{\infty} T_{\zeta_n} DT_{\zeta_n}^* \]
where \( T_{\zeta_n} : Y \to X \hat{\otimes}_B Y, \ y \mapsto \zeta_n \otimes_B y, \) is the creation operator associated with the element \( \zeta_n \in X \). It should be noted that the unbounded selfadjoint operator \( 1 \otimes \nabla D \) can be described in an alternative way by using the notion of a densely defined covariant derivative \( \nabla \) on the \( C^* \)-correspondence \( X \). Indeed, the frame \( \{ \zeta_k \} \) gives rise to a Grassmann covariant derivative \( \nabla_{\text{Gr}} \) and the unbounded selfadjoint operator \( 1 \otimes \nabla D \) is then given by the (closure of the) sum \( c(\nabla_{\text{Gr}}) + 1 \otimes D \) where the “\( c \)” refers to an appropriate notion of Clifford multiplication.

One of the main contributions of this paper is that we have been able to entirely remove the above smooth projectivity condition on the \( C^* \)-correspondence \( X \). This radical step is motivated by the detailed investigations of differentiable structures in Hilbert \( C^* \)-modules carried out in \cite{Kaa14, Kaa13}. In particular, we find that the removal of smooth projectivity is necessary for accommodating examples arising from non-complete manifolds.

Instead of smooth projectivity we will simply assume that there exists a sequence of generators \( \{ \xi_k \} \) for \( X \) such that the associated operator
\[ G := \sum_{n,m=1}^{\infty} \langle \xi_n, \xi_m \rangle \delta_{nm} : \ell^2(Y) \to \ell^2(Y) \]
has a bounded commutator with (the diagonal operator induced by) \( D : \mathcal{D}(D) \to Y \). We then obtain a new unbounded selfadjoint operator
\[ D_\Delta := \sum_{n=1}^{\infty} T_{\xi_n} DT_{\xi_n}^* \]
on the interior tensor product \( X \hat{\otimes}_B Y \). We refer to this unbounded selfadjoint operator as the modular lift of \( D : \mathcal{D}(D) \to Y \). The fact that our sequence \( \{ \xi_k \} \) is no longer a frame means that we obtain an extra (non-trivial) bounded adjointable operator
\[ \Delta := \sum_{n=1}^{\infty} T_{\xi_n} T_{\xi_n}^* : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y \]
on the interior tensor product. An investigation of the commutators between the
algebra elements in $A$ and the modular lift now shows that the usual straight com-
mmutator has to be replaced by a twisted commutator where the twist is given by the
(modular) automorphism $\sigma$ obtained from conjugation with the modular operator
$\Delta$. This modular automorphism corresponds to the analytic extensio
n at $-i \in \mathbb{C}$ of the modular group of automorphisms $\sigma_t : T \mapsto \Delta^t T \Delta^{-t}$, $t \in \mathbb{R}$. We remark however that, in spite of the definitions in $[\text{CoMo08}]$, we do not require that the
modular automorphism $\sigma$ is densely defined on the algebra $A$.

Our first main result can now be stated as follows (where we refer to the main
text for the precise definitions):

**Theorem 1.1.** Suppose that $X$ is a differentiable $C^*$-correspondence with left action
factorizing through the compacts and that $(Y, D, \Gamma)$ is an unbounded modular cycle
(with modular operator $\Gamma : Y \to Y$). Then the triple

$$(X \hat{\otimes}_B Y, D_\Delta, \Delta)$$

is an unbounded modular cycle where the new modular operator is defined by $\Delta := \sum_{n=1}^{\infty} T_{\xi_n} \Gamma T_{\xi_n}^*$.

The second central theme of this paper develops around the relationship between the assignment

$$(X, (Y, D, \Gamma)) \mapsto (X \hat{\otimes}_B Y, D_\Delta, \Delta)$$

and the interior Kasparov product $KK_0(A, B) \times KK_m(B, C) \to KK_m(A, C)$. In this respect it is first necessary to understand how to produce a class in $KK$-theory from an unbounded modular cycle. We announce the following theorem:

**Theorem 1.2.** Suppose that $(Y, D, \Gamma)$ is an unbounded modular cycle (between the
$C^*$-algebras $B$ and $C$). Then the pair $(Y, D(1 + D^2)^{-1/2})$ is a bounded Kasparov
module from $B$ to $C$ and we thus have a class $[D] \in KK_m(B, C)$.

Of course, this theorem is a direct analogue of the theorem of Baaj and Julg that shows how to construct a class in $KK$-theory from an unbounded Kasparov module. The proof of this result in the context of unbounded modular cycle is however far more involved. The reason for this extra difficulty can be found in the seemingly innocent change from straight commutators to twisted commutators. Indeed, an examination of the proof appearing in $[\text{BAJu83}]$ shows that the crucial step fails for algebraic reasons when applied to unbounded modular cycles. An alternative approach would be to follow Connes and Moscovici’s method and replace $(1 + D^2)^{-1/2}$ by $(1 + |D|)^{-1}$, see $[\text{CoMo08}]$. This alternative approach does however rely on an extra assumption of twisted Lipschitz regularity and we do not impose this kind of extra regularity conditions on our unbounded modular cycle. Indeed, it is unclear how twisted Lipschitz regularity behaves with respect to the unbounded Kasparov product given in Theorem 1.1. We have therefore found it necessary to develop a novel method of proof that can be applied to non-Lipschitz unbounded modular cycles.

The main new tool appearing in the proof of Theorem 1.2 is the **modular transform**

$G_{D, \Gamma} : \Gamma(\mathcal{D}(D)) \to Y$ which is given by the (absolutely convergent) integral

$$G_{D, \Gamma} : \Gamma(\xi) \mapsto \frac{1}{\pi} \int_0^{\infty} \lambda^{-1/2} \Gamma(1 + \lambda \Gamma^2 + D^2)^{-1} D(\Gamma \xi) d\lambda$$
for all $\xi \in \mathcal{D}(D)$. The modular transform is obtained from the bounded transform by making a non-commutative change of variables corresponding to $\mu := \lambda \Gamma^2$. This change of variables is motivated by the observation that the modular transform (contrary to the bounded transform) has the right commutator properties with elements in the algebra $A$. A substantial part of the proof of Theorem 1.2 is then devoted to a comparison between the bounded transform and the modular transform. Notice that the modular transform does not in general have a bounded extension to $Y$ but that a sufficient condition for this to happen is that the modular operator $\Gamma : Y \to Y$ has a bounded inverse.

With the knowledge of the relationship between unbounded modular cycles and classes in $KK$-theory in place, we can state our second main result:

**Theorem 1.3.** Suppose that $X$ is a differentiable $C^*$-correspondence with left action factorizing through the compacts and suppose that $(Y, D, \Gamma)$ is an unbounded modular cycle. Let $[X] \in KK_0(A, B)$ and $[D] \in KK_m(B, C)$ denote the corresponding classes in $KK$-theory. Let also $[D_\Delta] \in KK_m(A, C)$ denote the $KK$-class of the unbounded modular cycle $(X \hat{\otimes}_B Y, D_\Delta, \Delta)$. Then we have the identity

$$[D_\Delta] = [X] \hat{\otimes}_B [D]$$

in the $KK$-group $KK_m(A, B)$.

The proof of this theorem does again not follow the usual scheme in unbounded $KK$-theory. Indeed, the standard method that is available for recognizing an unbounded representative for the interior Kasparov product is to invoke the machinery invented by Kucerovsky, [KUC97]. However, the results of Kucerovsky does not apply in the context of unbounded modular cycles because of our systematic use of twisted commutators instead of straight commutators. Instead of applying Kucerovsky’s ideas we have found it necessary to rely directly on the notion of an $F_2$-connection as introduced by Connes and Skandalis, [CoSk84].

Let us end this introduction by giving a more tangible corollary to our main theorems. Consider a countable union $U : = \bigcup_{k=1}^\infty I_k$ of bounded open intervals $I_k \subseteq \mathbb{R}$. For each $k \in \mathbb{N}$ we then choose a smooth function $f_k : \mathbb{R} \to \mathbb{R}$ with support equal to the closure $\overline{I_k} \subseteq \mathbb{R}$. After a rescaling we may assume that $\|f_k\| + \|f'_k\| \leq 1/k$ for all $k \in \mathbb{N}$ (where $\| \cdot \|$ denotes the supremum norm). Define the first order differential operator

$$(D_\Delta)_0 := i \sum_{k=1}^\infty f_k^2 \frac{d}{dx} + i \sum_{k=1}^\infty f_k \frac{df_k}{dx} : C_c^\infty(U) \to L^2(U)$$

and let $D_\Delta := \overline{(D_\Delta)_0}$ denote the closure. We then have the following result:

**Corollary 1.1.** The triple $(C_c^\infty(U), L^2(U), D_\Delta)$ is an odd spectral triple and the associated class in the odd $K$-homology group $K^1(C_0(U))$ agrees with the interior Kasparov product of (the $KK$-classes associated with) the $C^*$-correspondence $C_0(U)$ and the (flat) Dirac operator on the real line.

Of course there is a similar kind of corollary where the setting is given by an arbitrary spectral triple $(\mathcal{A}, H, D)$ together with a sequence of elements $\{x_k\}$ in the algebra such that $\|x_k\| + \|[D, x_k]\| \leq 1/k$ for all $k \in \mathbb{N}$. When the algebra $\mathcal{A}$ is non-commutative it is however not true that one obtains a new spectral triple out of this construction. In the general case it becomes necessary to twist all the commutators
appearing by the modular operator $\Delta := \sum_{k=1}^{\infty} x_k x_k^*$ and the framework that we are developing here is therefore fine-tuned for treating this kind of examples.

1.1. Acknowledgement. The union $U := \cup_{k=1}^{\infty} I_k$ appearing in the introduction is referred to as a fractral string when it is bounded and when the open intervals are disjoint. I am grateful to Michel Lapidus for making me aware of this example, \cite{LavF13}.

2. Preliminaries on operator spaces

We begin this paper by fixing our conventions for the analytic properties of the $\ast$-algebras appearing throughout this text. We have found that the conventional setup of Banach spaces is not adequate for capturing the relevant structure on our $\ast$-algebras. Indeed, it will soon become apparent that one needs to fix the analytic behaviour not only of the $\ast$-algebra itself but of all the finite matrices with entries in the $\ast$-algebra. The notion of operator spaces is therefore providing the correct analytic setting and we will now briefly survey the main definitions. For more details we refer the reader to the books by Blecher-Merdy and by Pisier, \cite{BLLM04, Pis03}.

Let $H$ and $G$ be Hilbert spaces, and let $X \subseteq \mathcal{L}(H,G)$ be a subspace (of the bounded operators from $H$ to $G$) which is closed in the operator norm. Then the vector space $M(X) := \lim_{n \to \infty} M_n(X)$ of finite matrices over $X$ has a canonical norm $\| \cdot \|_X$ coming from the identifications $M_n(X) \subseteq M_n(\mathcal{L}(H,G)) \cong \mathcal{L}(H^n,G^n)$. The properties of the pair $(M(X), \| \cdot \|_X)$ are crystallized in the next definition.

Notice that the above construction yields a canonical norm $\| \cdot \|_C : M(\mathbb{C}) \to [0, \infty)$ on the finite matrices over $\mathbb{C}$ since $\mathbb{C} \cong \mathcal{L}(\mathbb{C},\mathbb{C})$. For each $n \in \mathbb{N}$ the norm $\| \cdot \|_C : M_n(\mathbb{C}) \subseteq M(\mathbb{C}) \to [0, \infty)$ coincides with the unique $C^\ast$-algebra norm.

**Definition 2.1.** An operator space is a vector space $X$ over $\mathbb{C}$ with a norm $\| \cdot \|_X$ on the finite matrices $M(X) := \lim_{n \to \infty} M_n(X)$ such that

1. The normed space $X \subseteq M(X)$ is a Banach space.
2. The inequality $\|v \cdot \xi - w \|_X \leq \|v\| \|\xi\|_C \cdot \|\xi\|_C \cdot \|w\|_C$ holds for all $v, w \in M(\mathbb{C})$ and all $\xi \in M(X)$.
3. The equality $\|\xi \oplus \eta\|_X = \max\{\|\xi\|_X, \|\eta\|_X\}$ holds for all $\xi \in M_n(X)$ and $\eta \in M_m(X)$, where $\xi \oplus \eta \in M_{n+m}(X)$ is the direct sum of the matrices.

A morphism of operator spaces is a completely bounded linear map $\alpha : X \to Y$. The term completely bounded means that $\alpha_n : M_n(X) \to M_n(Y)$ is a bounded operator for each $n \in \mathbb{N}$ and that $\sup_n \|\alpha_n\| < \infty$ (where $\| \cdot \|_\infty$ is the operator norm). The supremum is denoted by $\|\alpha\|_{cb} := \sup_n \|\alpha_n\|_\infty$ and is referred to as the completely bounded norm.

By a fundamental theorem of Ruan each operator space $X$ is completely isometric to a closed subspace of $\mathcal{L}(H)$ for some Hilbert space $H$. See \cite[Theorem 3.1]{Rua88}.

We remark that any $C^\ast$-algebra $A$ carries a canonical operator space structure such that $M_n(A)$ becomes a $C^\ast$-algebra for all $n \in \mathbb{N}$.

We will in this text mainly be concerned with dense subspaces of operator spaces. On such a dense subspace $\mathcal{X} \subseteq X$ we will then refer to the norm on the surrounding operator space $X$ as an operator space norm on $\mathcal{X}$.

The next assumption will remain in effect throughout this paper:
Assumption 2.2. Any \( \ast \)-algebra \( \mathcal{A} \) encountered in this text will come equipped with an operator space norm \( \| \cdot \|_1 : \mathcal{A} \to [0, \infty) \) and a \( C^* \)-norm \( \| \cdot \| : A \to [0, \infty) \). We will denote the operator space completion of \( \mathcal{A} \) by \( \mathcal{A}_1 \) and the \( C^* \)-algebra completion by \( A \). It will then be assumed that the inclusion \( \mathcal{A} \to A \) extends to a completely bounded map \( \mathcal{A}_1 \to A \).

In this text we will never assume the existence of a bounded approximate identity in \( \mathcal{A} \) with respect to the norm \( \| \cdot \|_1 : \mathcal{A} \to [0, \infty) \).

2.0.1. Stabilization of operator spaces. Let us consider an operator space \( X \). The following stabilization construction will play a central role in this paper. It does of course not make any sense when \( X \) is merely a Banach space.

Definition 2.3. By the stabilization of \( X \) we will understand the operator space \( K(X) \) obtained as the completion of the vector space of finite matrices \( M(X) \) with respect to the canonical norm
\[
\| \cdot \|_X : M(X) \to [0, \infty)
\]
The matrix norms for \( K(X) \) comes from the matrix norms for \( X \) via the canonical identification (forgetting the subdivisions):
\[
M_n(M_m(X)) \cong M_{n,m}(X) \quad n, m \in \mathbb{N}
\]

3. Unbounded modular cycles

Throughout this section we let \( \mathcal{A} \) be a \( \ast \)-algebra which satisfies the conditions in Assumption 2.2. We let \( A_1 \) denote the operator space completion of \( \mathcal{A} \) and \( A \) denote the \( C^* \)-completion of \( \mathcal{A} \). We let \( B \) be an arbitrary \( C^* \)-algebra.

Let us recall some basic constructions for a Hilbert \( C^* \)-module \( X \) over \( B \), for more details the reader may consult the book by Lance, [LAN95].

The standard module over \( X \) is the Hilbert \( C^* \)-module \( \ell^2(X) \) over \( B \) consisting of all sequences \( \sum_{n=1}^{\infty} x_n \delta_n \) in \( X \) such that the sequence of partial sums \( \{ \sum_{n=1}^{N} (x_n, x_n) \} \) converges in the norm on \( B \). The right module structure is given by \( (\sum_{n=1}^{\infty} x_n \cdot b) \delta_n \) and the inner product is given by
\[
\langle \sum_{n=1}^{\infty} x_n \delta_n, \sum_{n=1}^{\infty} y_n \delta_n \rangle := \sum_{n=1}^{\infty} \langle x_n, y_n \rangle
\]
(where the convergence of the last sum follows from the Cauchy-Schwartz inequality).

The bounded adjointable operators on \( X \) is the \( C^* \)-algebra \( \mathcal{L}(X) \) consisting of all the bounded operators on \( X \) that admit an adjoint with respect to the inner product on \( X \).

The compact operators on \( X \) is the \( C^* \)-algebra \( \mathcal{K}(X) \) defined as the operator norm closure of the \( \ast \)-subalgebra
\[
\mathcal{F}(X) := \text{span}_\mathbb{C}\{ \theta_{\xi,\eta} \mid \xi, \eta \in X \} \subseteq \mathcal{L}(X)
\]
where \( \theta_{\xi,\eta} : X \to X \) is defined by \( \theta_{\xi,\eta}(\zeta) := \xi \cdot \langle \eta, \zeta \rangle \) for all \( \xi, \eta, \zeta \in X \).

For a bounded adjointable operator \( T : X \to X \) we let \( C^*(T) \subseteq \mathcal{L}(X) \) denote the \( C^* \)-subalgebra generated by \( T \).
We are now ready to introduce the first of the main new concepts of the present paper:

**Definition 3.1.** An odd unbounded modular cycle from $\mathcal{A}$ to $B$ is a triple $(X, D, \Delta)$ where

1. $X$ is a countably generated Hilbert $C^*$-module over $B$ which comes equipped with a $*$-homomorphism $\pi : A \to \mathcal{L}(X)$;
2. $D : \mathcal{D}(D) \to X$ is an unbounded selfadjoint and regular operator on $X$;
3. $\Delta : X \to X$ is a bounded positive and selfadjoint operator with dense image, such that the following holds:
   1. $\pi(a) \cdot (i + D)^{-1} : X \to X$ is a compact operator for all $a \in A$;
   2. $T \Delta(\mathcal{D}(D)) \subseteq \mathcal{D}(D)$ and
      $$DT \Delta - \Delta TD : \mathcal{D}(D) \to X$$
      extends to a bounded adjointable operator $d_\Delta(T) : X \to X$ for all $T \in \pi(\mathcal{A}) + \mathbb{C} \cdot \text{Id}_X$;
   3. The image of $d_\Delta(T)$ is contained in the image of $\Delta^{1/2}$ and the unbounded operator
      $$\Delta^{-1/2}d_\Delta(T)\Delta^{-1/2} : \text{Im}(\Delta^{1/2}) \to X$$
      has a bounded adjointable extension to $X$ for all $T \in \pi(\mathcal{A}) + \mathbb{C} \cdot \text{Id}_X$;
   4. The linear map $\rho_\Delta : \mathcal{A} \to \mathcal{L}(X)$ defined by
      $$a \mapsto \Delta^{-1/2}d_\Delta(\pi(a))\Delta^{-1/2}$$
      is completely bounded;
   5. There exists a countable approximate identity $\{V_n\}_{n=1}^\infty$ for the $C^*$-algebra $C^*(\Delta)$ such that the sequence
      $$\{V_n\pi(a)\}_{n=1}^\infty$$
      converges in operator norm to $\pi(a)$ for all $a \in A$.

We will refer to $\Delta : X \to X$ as the modular operator of our unbounded modular cycle.

An even unbounded modular cycle from $\mathcal{A}$ to $B$ is an odd unbounded modular cycle equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading operator $\gamma : X \to X$ such that

$$\gamma \pi(a) = \pi(a)\gamma \quad \gamma \Delta = \Delta \gamma \quad \text{and} \quad \gamma D = -D\gamma$$

for all $a \in A$.

**Remark 3.2.** The definition of an unbounded Kasparov module (see [BaJu83]) is a special case of the above definition. Indeed, it corresponds to the case where the modular operator $\Delta = \text{Id}_X$.

The concept of a twisted spectral triple (see [CoMo08]) is closely related to the above definition. Indeed, one of the main examples of a twisted spectral triple is obtained by starting from a unital spectral triple $(\mathcal{A}, H, D)$ together with a fixed positive and invertible element $g \in \mathcal{A}$. One then forms the twisted spectral triple $(\mathcal{A}, H, \sqrt{g}Dg)$ where the modular automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ is given by $\sigma(a) := g^2ag^{-2}$ (in this case we have that $\Delta = g^2$). This procedure corresponds to making a conformal change of the underlying metric, see for example [Hu86, Proposition 4.3.1].
Our definition of an unbounded modular cycle is inspired by this construction but there are three important differences:

1. We are considering a bivariant theory, thus the scalars can consist of an arbitrary C*-algebra and not just the complex numbers;
2. The modular operator \( \Delta : X \to X \) can have zero in the spectrum (thus allowing for a treatment of non-compact manifolds);
3. The modular automorphism \( \sigma \) given by conjugation with \( \Delta \) need not map the algebra \( \mathcal{A} \) into itself, in fact it need not even be defined on \( \mathcal{A} \).

For more information about twisted spectral triples we refer to [FaKh11, Mos10, PoWa15].

Let us spend a little extra time commenting on the conditions in Definition 3.1. We let \( \pi : A_1 \to \mathcal{L}(X) \) denote the completely bounded map induced by the inclusion \( A \to A \) and the \( * \)-homomorphism \( \pi : A \to \mathcal{L}(X) \). It then follows by a density argument that the conditions (2) and (3) also hold for all \( T \in \pi(A_1) + \mathcal{C} \cdot \text{Id}_X \). Furthermore, we obtain a completely bounded map \( \rho_\Delta : A_1 \to \mathcal{L}(X) \) which is induced by \( \rho_\Delta : A \to \mathcal{L}(X) \). For condition (5) we notice that the sequence \( \{ V_n \} \) converges strictly to the identity on \( X \) (this holds since \( \text{Im}(\Delta) \) is dense in \( X \)). Furthermore, condition (5) automatically holds for any countable approximate identity for \( C^*(\Delta) \) (once it holds for one of them). In particular we could choose \( V_n = \Delta(\Delta + 1/n)^{-1} \) for all \( n \in \mathbb{N} \). In general we have that condition (3) and (5) are automatic when \( \Delta : X \to X \) is invertible as a bounded operator.

For later use we introduce the following terminology:

**Definition 3.3.** When \( (X, D, \Delta) \) is an unbounded modular cycle (from \( \mathcal{A} \) to \( B \)) we will say that a bounded adjointable operator \( T : X \to X \) is differentiable (with respect to \( (X, D, \Delta) \)) when the following holds:

1. \( T \Delta(\mathcal{D}(D)) \subseteq \mathcal{D}(D) \) and
   \[
   DT\Delta - \Delta TD : \mathcal{D}(D) \to X
   \]
   extends to a bounded adjointable operator \( d_\Delta(T) : X \to X \).
2. The image of \( d_\Delta(T) : X \to X \) is contained in the image of \( \Delta^{1/2} : X \to X \) and the unbounded operator
   \[
   \Delta^{-1/2}d_\Delta(T)\Delta^{-1/2} : \text{Im}(\Delta^{1/2}) \to X
   \]
   has a bounded adjointable extension \( \rho_\Delta(T) : X \to X \).

We remark that the adjoint of a differentiable operator \( T : X \to X \) is automatically differentiable as well and that the identities \( d_\Delta(T)^* = -d_\Delta(T^*) \) and \( \rho_\Delta(T)^* = -\rho_\Delta(T^*) \) are valid.

**3.1. Stabilization of unbounded modular cycles.** Let us fix an unbounded modular cycle \( (X, D, \Delta) \) from the \( * \)-algebra \( \mathcal{A} \) to the C*-algebra \( B \). We let \( \gamma : X \to X \) denote the grading operator in the even case.

The aim of this subsection is to construct a stabilization of \( (X, D, \Delta) \) which is an unbounded modular cycle from the finite matrices over \( \mathcal{A} \) to \( B \). The parity of the stabilization is the same as the parity of \( (X, D, \Delta) \).
To this end, we first notice that the finite matrices over $\mathcal{A}$ comes equipped with a canonical operator space norm and a canonical $C^*$-norm (see Definition 2.3):

$$\| \cdot \|_1, \| \cdot \| : M(\mathcal{A}) \to [0, \infty)$$

The respective completions are the operator space $K(A_1)$ and the $C^*$-algebra $K(A)$. We remark that $K(A)$ is isomorphic to the compact operators on the standard module $\ell^2(A)$ where $A$ is considered as a Hilbert $C^*$-module over itself.

We now consider the standard module $\ell^2(X)$ over $B$ and we equip it with the $*$-homomorphism $K(\pi) : K(A) \to \mathcal{L}(\ell^2(X))$ given by

$$K(\pi) \left( \sum_{n,m=1}^{\infty} a_{nm} \cdot \delta_{nm} \right) \left( \sum_{k=1}^{\infty} x_k \delta_k \right) := \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \pi(a_{nm})(x_m) \right) \delta_n$$

where $a \cdot \delta_{nm} \in K(A)$ denotes the finite matrix with $a \in A$ in position $(n, m)$ and zeroes elsewhere.

Furthermore, on the standard module over $X$, we have the diagonal operators induced by the unbounded selfadjoint and regular operator $D : \mathcal{D}(D) \to X$ and the modular operator $\Delta : X \to X$. The diagonal operator induced by $D : \mathcal{D}(D) \to X$ is given by

$$\text{diag}(D) : \mathcal{D}(\text{diag}(D)) \to \ell^2(X) \quad \sum_{n=1}^{\infty} x_n \delta_n \mapsto \sum_{n=1}^{\infty} D(x_n) \delta_k$$

where the domain $\mathcal{D}(\text{diag}(D)) \subseteq \ell^2(X)$ is defined by

$$\mathcal{D}(\text{diag}(D)) := \left\{ \sum_{n=1}^{\infty} x_n \delta_n \in \ell^2(X) \mid x_n \in \mathcal{D}(D) \text{ and } \sum_{n=1}^{\infty} D(x_n) \delta_n \in \ell^2(X) \right\}$$

The diagonal operator induced by $\Delta : X \to X$ is given by

$$\text{diag}(\Delta) : \ell^2(X) \to \ell^2(X) \quad \sum_{n=1}^{\infty} x_n \delta_n \mapsto \sum_{n=1}^{\infty} \Delta(x_n) \delta_n$$

Likewise (in the even case) we have the diagonal operator $\text{diag}(\gamma) : \ell^2(X) \to \ell^2(X)$ induced by the grading operator $\gamma : X \to X$.

It is well-known (and a good exercise) to check that $\text{diag}(D) : \mathcal{D}(\text{diag}(D)) \to \ell^2(X)$ is again a selfadjoint and regular operator. We also note that $\text{diag}(D)$ has a core given by the algebraic direct sum $\oplus_{n=1}^{\infty} \mathcal{D}(D) \subseteq \ell^2(X)$.

To ease the notation, we write

$$1 \otimes D := \text{diag}(D) \quad 1 \otimes \Delta := \text{diag}(\Delta) \quad \text{and} \quad 1 \otimes \gamma := \text{diag}(\gamma)$$

**Definition 3.4.** By the stabilization of $(X, D, \Delta)$ we will understand the triple $(\ell^2(X), 1 \otimes D, 1 \otimes \Delta)$ with $\mathbb{Z}/2\mathbb{Z}$-grading operator $1 \otimes \gamma$ in the even case.

**Proposition 3.5.** The stabilization $(\ell^2(X), 1 \otimes D, 1 \otimes \Delta)$ is an unbounded modular cycle from $M(\mathcal{A})$ to $B$.

**Proof.** We need to verify the conditions (1)-(5) in Definition 3.1.

(1): This follows by standard compactness arguments.
(2): For any element \( \sum_{n,m=1}^{N} \pi(a_{nm})\delta_{nm} + \lambda \cdot \text{Id}_{\ell^2(X)} \) we easily obtain that (2) holds and that
\[
d_{1\otimes \Delta} \left( \sum_{n,m=1}^{N} \pi(a_{nm})\delta_{nm} + \lambda \cdot \text{Id}_{\ell^2(X)} \right) = \sum_{n,m=1}^{N} d_{\Delta}(\pi(a_{nm}))\delta_{nm} + \lambda \cdot (1 \otimes d_{\Delta}(\text{Id}_{X}))
\]

(3): The assertion in (3) is clear for \( T = \sum_{n,m=1}^{N} \pi(a_{nm})\delta_{nm} + \lambda \cdot \text{Id}_{\ell^2(X)} \). Furthermore, for \( \sum_{n,m=1}^{N} \pi(a_{nm})\delta_{nm} \in M(\mathcal{A}) \) we see that
\[
\rho_{1\otimes \Delta} \left( \sum_{n,m=1}^{N} a_{nm}\delta_{nm} \right) = \sum_{n,m=1}^{N} \rho_{\Delta}(a_{nm})\delta_{nm} = M(\rho_{\Delta}) \left( \sum_{n,m=1}^{N} a_{nm}\delta_{nm} \right)
\]
(where \( M(\rho_{\Delta}) : M(\mathcal{A}) \to \mathcal{L}(\ell^2(X)) \) is the completely bounded map induced by \( \rho_{\Delta : \mathcal{A} \to \mathcal{L}(X)} \)).

(4): This is clear since \( \rho_{1\otimes \Delta} : M(\mathcal{A}) \to \mathcal{L}(\ell^2(X)) \) agrees with the induced map \( M(\rho_{\Delta}) : M(\mathcal{A}) \to \mathcal{L}(\ell^2(X)) \) (see the proof of (3)).

(5): Let \( \{V_n\} \) be a countable approximate unit for \( C^*(\Delta) \) such that \( \pi(a)V_n \to \pi(a) \) in operator norm for all \( a \in A \). The sequence \( \{1 \otimes V_n\}_{n=1}^{\infty} \) is then a countable approximate unit for \( C^*(1 \otimes \Delta) \) such that \( K(\pi)(T)(1 \otimes V_n) \to K(\pi)(T) \) in operator norm for all \( T \in K(A) \).

\[ \Box \]

4. Differentiable Hilbert \( C^* \)-modules

Throughout this section \( \mathcal{A} \) and \( \mathcal{B} \) will be \(*\)-algebras which satisfy the conditions in Assumption 2.2. We let \( A_1 \) and \( B_1 \) denote the operator space completions and we let \( A \) and \( B \) denote the \( C^* \)-completions of \( \mathcal{A} \) and \( \mathcal{B} \), respectively.

The next definition is the second main new concept which we introduce in this paper:

**Definition 4.1.** A Hilbert \( C^* \)-module \( X \) over \( B \) which comes equipped with a \(*\)-homomorphism \( \pi : A \to \mathcal{L}(X) \) is said to be differentiable (from \( \mathcal{A} \) to \( \mathcal{B} \)) when there exists a sequence \( \{\xi_n\}_{n=1}^{\infty} \) in \( X \) such that the following holds:

1. \( \text{span}_c\{\xi_n \cdot b \mid b \in B, n \in \mathbb{N}\} \) is dense in \( X \).
2. \( \langle \xi_n, T\xi_m \rangle \in \mathcal{B} \) for all \( T \in \pi(\mathcal{A}) + \mathbb{C} \cdot \text{Id}_X \) and all \( n, m \in \mathbb{N} \).
3. The sequence of finite matrices
   \[
   \left\{ \sum_{n,m=1}^{N} \langle \xi_n, T\xi_m \rangle\delta_{nm} \right\}_{N=1}^{\infty}
   \]
   is a Cauchy sequence in \( K(B_1) \) for all \( T \in \pi(\mathcal{A}) + \mathbb{C} \cdot \text{Id}_X \).
4. The linear map \( \tau : \mathcal{A} \to K(B_1) \), \( a \mapsto \sum_{n,m=1}^{\infty} \langle \xi_n, \pi(a)\xi_m \rangle\delta_{nm} \) is completely bounded (with respect to the operator space norm on \( \mathcal{A} \)).

We will refer to a sequence \( \{\xi_n\}_{n=1}^{\infty} \) in \( X \) satisfying the above conditions as a differentiable generating sequence.

**Remark 4.2.** The conditions (3) and (4) in Definition 4.1 can be replaced by the following:
(3a): The sequence of finite matrices
\[ \left\{ \sum_{n,m=1}^{N} \langle \xi_n, T \xi_m \rangle \delta_{nm} \right\}_{N=1}^{\infty} \]
is bounded in \( K(B_1) \) for all \( T \in \pi(\mathcal{A}) + C \cdot \text{Id}_X \).

(4a): The linear map \( \mathcal{A} \to M_N(B_1), a \mapsto \sum_{n,m=1}^{\infty} \langle \xi_n, \pi(a) \xi_m \rangle \delta_{nm} \) is completely bounded, where \( M_N(B_1) \) is the operator space of infinite matrices over \( B_1 \), see [BLLM04, Section 1.2.26] for details.

Given a sequence \( \{ \xi_n \} \) that satisfies (1), (2), (3a), and (4a) we obtain a sequence satisfying (1), (2), (3), and (4) by rescaling each \( \xi_n \in X \) by \( \frac{1}{n} \), for example.

4.0.1. Example: Finitely generated Hilbert \( C^* \)-modules. Let us consider a \( * \)-algebra \( \mathcal{B} \) which satisfies the conditions of Assumption 2.2. Let us also consider a dense \( * \)-subalgebra \( \mathcal{A} \) of a \( C^* \)-algebra \( A \). Let now \( X \) be a finitely generated Hilbert \( C^* \)-module over \( B \) with generators \( \xi_1, \ldots, \xi_N \in X \) and let \( \pi : A \to \mathcal{L}(X) \) be a \( * \)-homomorphism. By “finitely generated” we mean that the subspace \( \{ \xi_n \cdot b \mid n \in \{1, \ldots, N\}, b \in B \} \subseteq X \) is dense in the norm-topology on \( X \). Thus, in our context, finitely generated does not imply that \( X \) is finitely generated projective as a right module over \( B \).

Suppose that \( \langle \xi_n, T \xi_m \rangle \in \mathcal{B} \) for all \( T \in \pi(\mathcal{A}) + \mathbb{C} \cdot \text{Id}_X \), \( n, m \in \{1, \ldots, N\} \)

We then have a linear map
\[ \tau : \mathcal{A} \to M_N(\mathcal{B}) \quad \tau(a) := \sum_{n,m=1}^{N} \langle \xi_n, \pi(a) \xi_m \rangle \delta_{nm} \]

Using this linear map we obtain an operator space norm on \( \mathcal{A} \) by defining
\[ \|a\|_1 := \max\{\|a\|, \|\tau(a)\|_1\} \quad \forall a \in M_k(\mathcal{A}), \quad k \in \mathbb{N} \]
where we have suppressed the usual identification \( M_k(M_N(\mathcal{B})) \cong M_{kN}(\mathcal{B}) \) (see Definition 2.3). By construction we then get that \( X \) is a differentiable Hilbert \( C^* \)-module from \( \mathcal{A} \) to \( \mathcal{B} \).

5. The Modular Lift

In this section we will consider two Hilbert \( C^* \)-modules \( X \) and \( Y \) with the same base \( C^* \)-algebra \( A \). We will then fix an unbounded selfadjoint and regular operator \( D : \mathcal{D}(D) \to Y \) on the Hilbert \( C^* \)-module \( Y \) together with a bounded selfadjoint and positive operator \( \Gamma : Y \to Y \) with dense image. Furthermore, we will consider a bounded adjointable operator \( \Phi : X \to Y \) such that the adjoint \( \Phi^* : Y \to X \) has dense image.

The main concern of this section is to “transport” the unbounded selfadjoint and regular operator \( D : \mathcal{D}(D) \to Y \) to an unbounded selfadjoint and regular operator \( D_{\Delta} : \mathcal{D}(D_{\Delta}) \to X \). This transportation will happen via the bounded adjointable operator \( \Phi : X \to Y \).

We will apply the notation:
\[ \Delta := \Phi^* \Gamma \Phi : X \to X \quad \text{and} \quad G := \Phi \Phi^* : Y \to Y \]
We remark that $\Delta : X \to X$ is bounded selfadjoint and positive and that $\text{Im}(\Delta) \subseteq X$ is norm-dense.

The following standing assumptions will be in effect:

**Assumption 5.1.** It is assumed that

1. The bounded adjointable operator $G\Gamma : Y \to Y$ has $\mathcal{D}(D)$ as an invariant subspace.
2. The twisted commutator
   
   $$DG\Gamma - \Gamma GD : \mathcal{D}(D) \to Y$$
   
   has a bounded extension to $Y$. This bounded extension is denoted by $d_{\Gamma}(G) : Y \to Y$.
3. The image of $d_{\Gamma}(G) : Y \to Y$ is contained in the image of $\Gamma^{1/2} : Y \to Y$ and the unbounded operator
   
   $$\Gamma^{-1/2}d_{\Gamma}(G)\Gamma^{-1/2} : \text{Im}(\Gamma^{1/2}) \to Y$$
   
   has a bounded extension $\rho_{\Gamma}(G) : Y \to Y$.

We remark that the extension $d_{\Gamma}(G) : Y \to Y$ is automatically adjointable with $d_{\Gamma}(G) = -d_{\Gamma}(G)$. Likewise we have that $\rho_{\Gamma}(G) : Y \to Y$ is adjointable with $\rho_{\Gamma}(G) = -\rho_{\Gamma}(G)$.

The main aim of this section is to show that the composition

$$\Phi^*D\Phi : \mathcal{D}(\Phi^*D\Phi) \to X$$

is essentially selfadjoint and regular, where the domain is given by

$$\mathcal{D}(\Phi^*D\Phi) := \{ x \in X \mid \Phi(x) \in \mathcal{D}(D) \}$$

We immediately remark that $\mathcal{D}(\Phi^*D\Phi) \subseteq X$ is norm-dense. Indeed, this follows since $\Phi^*\Gamma(\mathcal{D}(D)) \subseteq \mathcal{D}(\Phi^*D\Phi)$. Furthermore, it is evident that the unbounded operator $\Phi^*D\Phi : \mathcal{D}(\Phi^*D\Phi) \to X$ is symmetric.

We notice that

$$\Delta(\Phi^*D\Phi \Delta - \Delta^*D\Phi)(\eta) = (\Phi^*d_{\Gamma}(G)\Phi)(\eta)$$

for all $\eta \in \mathcal{D}(\Phi^*D\Phi)$. In particular, this shows that the straight commutator

$$\Phi^*D\Phi \Delta - \Delta^*D\Phi : \mathcal{D}(\Phi^*D\Phi) \to X$$

has a bounded adjointable extension to $X$.

**Definition 5.2.** The modular lift of $D : \mathcal{D}(D) \to Y$ with respect to $\Phi : X \to Y$ is the closure of $\Phi^*D\Phi : \mathcal{D}(\Phi^*D\Phi) \to X$. The modular lift is denoted by $D_{\Delta} : \mathcal{D}(D_{\Delta}) \to X$.

**5.1. Selfadjointness.** In order to show that the modular lift is selfadjoint we need a few preliminary lemmas.

**Lemma 5.3.** Let $\xi \in \mathcal{D}((\Phi^*D\Phi)^*)$. Then $\Delta(\xi) \in \mathcal{D}(\Phi^*D\Phi)$ and

$$(\Phi^*D\Phi)(\Delta\xi) = \Delta(\Phi^*D\Phi)^*(\xi) + \Phi^*d_{\Gamma}(G)\Phi(\xi)$$
Proof. Let \( \eta \in \mathcal{D}(D) \) and compute as follows:
\[
\langle \Phi \Delta(\xi), D(\eta) \rangle = \langle \Phi(\xi), \Gamma G D(\eta) \rangle = \langle \Phi(\xi), D G \Gamma(\eta) \rangle - \langle \Phi(\xi), d_{\Gamma}(G)(\eta) \rangle \\
= \langle (\Phi^* D \Phi)^*(\xi), \Phi^* \Gamma(\eta) \rangle - \langle d_{\Gamma}(G)^*(\Phi(\xi), \eta) \rangle \\
= \langle \Gamma \Phi(\Phi^* D \Phi)^*(\xi), \eta \rangle + \langle d_{\Gamma}(G)(\Phi(\xi), \eta) \rangle
\]
Using the selfadjointness assumption on \( D : \mathcal{D}(D) \to Y \), this implies that \( \Phi \Delta(\xi) \in \mathcal{D}(D) \) and furthermore that
\[
D(\Phi \Delta(\xi)) = \Gamma \Phi(\Phi^* D \Phi)^*(\xi) + d_{\Gamma}(G)(\Phi(\xi))
\]
This clearly implies the result of the lemma. \( \square \)

Lemma 5.4. Let \( \xi \in \mathcal{D}((\Phi^* D \Phi)^*) \) and let \( z \in \mathbb{C} \setminus [0, \infty) \) be given. Then \( (\Delta - z)^{-1}(\xi) \in \mathcal{D}((\Phi^* D \Phi)^*) \) and
\[
(\Phi^* D \Phi)^*(\Delta - z)^{-1}(\xi) = (\Delta - z)^{-1}(\Phi^* D \Phi)^*(\xi) \\
- (\Delta - z)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta - z)^{-1}(\xi)
\]
Proof. We define the two holomorphic functions \( g \) and \( h : \mathbb{C} \setminus [0, \infty) \to X \) by
\[
g(z) := (\Delta - z)^{-1}(\xi) \quad \text{and} \quad h(z) := (\Delta - z)^{-1}(\Phi^* D \Phi)^*(\xi) - (\Delta - z)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta - z)^{-1}(\xi)
\]
for all \( \mathbb{C} \setminus [0, \infty) \).

Let now \( \eta \in \mathcal{D}(\Phi^* D \Phi) \) be fixed. By the uniqueness of holomorphic extensions it is then enough to prove that
\[
\langle g(z), (\Phi^* D \Phi)(\eta) \rangle = \langle h(z), \eta \rangle \tag{5.1}
\]
for all \( z \in \mathbb{C} \setminus [0, \infty) \) with \( |z| > ||\Delta|| \).

Let thus \( z \in \mathbb{C} \setminus [0, \infty) \) with \( |z| > ||\Delta|| \) be given. We then have that
\[
(\Delta - z)^{-1} = -\sum_{n=0}^{\infty} \Delta^n z^{-n+1} : X \to X
\]
where the sum converges absolutely in operator norm. Furthermore, by Lemma 5.3 we obtain that \( -(\sum_{n=0}^{N} \Delta^n z^{-n+1})(\xi) \in \mathcal{D}((\Phi^* D \Phi)^*) \) and that
\[
(\Phi^* D \Phi)^*(\sum_{n=0}^{N} \Delta^n z^{-n+1})(\xi) = -\sum_{n=0}^{N} \Delta^n z^{-n+1}(\Phi^* D \Phi)^*(\xi) \\
- \sum_{n=1}^{N} \sum_{j=0}^{n-1} \Delta^j z^{-j-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta - z)^{-1} \sum_{j=0}^{n-1} \Delta^j z^{-n+j}(\xi)
\]
for all \( N \in \mathbb{N} \). Since the right hand side converges to
\[
(\Delta - z)^{-1}(\Phi^* D \Phi)^*(\xi) - (\Delta - z)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta - z)^{-1}(\xi)
\]
we conclude that \( (\Delta - z)^{-1}(\xi) \in \mathcal{D}((\Phi^* D \Phi)^*) \) and furthermore that
\[
(\Phi^* D \Phi)^*(\Delta - z)^{-1}(\xi) = (\Delta - z)^{-1}(\Phi^* D \Phi)^*(\xi) - (\Delta - z)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta - z)^{-1}(\xi)
\]
This ends the proof of the present lemma. \( \square \)

We are now ready to show that the modular lift \( D_\Delta : \mathcal{D}(D_\Delta) \to X \) is selfadjoint:
Proposition 5.5. The composition
\[ \Phi^* D \Phi : \mathcal{D}(\Phi^* D \Phi) \to X \]
is essentially selfadjoint.

Proof. It is enough to prove that \( \mathcal{D}((\Phi^* D \Phi)^*) \subseteq \mathcal{D}(D_{\Delta}) \). Thus, let \( \xi \in \mathcal{D}((\Phi^* D \Phi)^*) \) be given.

Let us consider the sequence \( \{ \Delta(\Delta + 1/n)^{-1}(\xi) \} \). Since \( \text{Im}(\Delta) \subseteq X \) is norm-dense we obtain that
\[ \Delta(\Delta + 1/n)^{-1}(\xi) \to \xi \]
and furthermore by Lemma 5.3 and Lemma 5.4 that \( \Delta(\Delta + 1/n)^{-1}(\xi) \in \mathcal{D}(\Phi^* D \Phi) \) for all \( n \in \mathbb{N} \).

To show that \( \xi \in \mathcal{D}(D_{\Delta}) \) it therefore suffices to prove that the sequence \( \{ (\Phi^* D \Phi)\Delta(\Delta + 1/n)^{-1}(\xi) \} \) is norm-convergent in \( X \).

For each \( n \in \mathbb{N} \) we use Lemma 5.3 and Lemma 5.4 to compute in the following way:
\[
(\Phi^* D \Phi)\Delta(\Delta + 1/n)^{-1}(\xi) \\
= \Delta(\Phi^* D \Phi)^*(\Delta + 1/n)^{-1}(\xi) + \Phi^* d_{\Gamma}(G)\Phi(\Delta + 1/n)^{-1}(\xi) \\
= \Delta(\Delta + 1/n)^{-1}(\Phi^* D \Phi)^*(\xi) - \Delta(\Delta + 1/n)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta + 1/n)^{-1}(\xi) \\
+ \Phi^* d_{\Gamma}(G)\Phi(\Delta + 1/n)^{-1}(\xi) \\
= \Delta(\Delta + 1/n)^{-1}(\Phi^* D \Phi)^*(\xi) + \frac{1}{n}(\Delta + 1/n)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta + 1/n)^{-1}(\xi)
\]

Since the sequence \( \{ \Delta(\Delta + 1/n)^{-1} \} \) converges strictly to the identity operator on \( X \), the result of the proposition is proved, provided that the sequence
\[ \{ \frac{1}{n}(\Delta + 1/n)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta + 1/n)^{-1}(\xi) \} \]
converges to zero in the norm on \( X \). But this is a consequence of the next lemma. \[ \square \]

Lemma 5.6. The sequence
\[ \{ \frac{1}{n}(\Delta + 1/n)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta + 1/n)^{-1} \} \]
is bounded in operator norm and converges strictly to the zero operator on \( X \).

Proof. We first show that our sequence is bounded in operator norm. To this end, we simply notice that
\[
\frac{1}{n}\| (\Delta + 1/n)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta + 1/n)^{-1} \| \\
\leq \frac{1}{n}\| \Gamma^{1/2}\Phi(\Delta + 1/n)^{-1} \|^2 \cdot \| \rho_{\Gamma}(G) \| \leq \| \rho_{\Gamma}(G) \|
\]
for all \( n \in \mathbb{N} \).

To prove the lemma, we may now limit ourselves to showing that
\[ \frac{1}{n}(\Delta + 1/n)^{-1}\Phi^* d_{\Gamma}(G)\Phi(\Delta + 1/n)^{-1}(\xi) \to 0 \]
for all $\xi$ in a dense subspace of $X$. Since $\text{Im}(\Delta) \subseteq X$ is dense in $X$ we let $\eta \in X$ and remark that
\[
\left\| \frac{1}{n}(\Delta + 1/n)^{-1}\Phi^*d_\Gamma(G)\Phi(\Delta + 1/n)^{-1}(\eta) \right\|
\leq \frac{1}{n} \cdot \left\| (\Delta + 1/n)^{-1}\Phi^*\Gamma^{1/2} \cdot \left\| \Gamma^{1/2}\Phi\Delta + 1/n)^{-1}(\eta) \right\| \cdot \| \rho_\Gamma(G) \|
\leq \frac{1}{\sqrt{n}} \cdot \| \Gamma^{1/2}\Phi \| \cdot \| \eta \| \cdot \| \rho_\Gamma(G) \|
\]
for all $n \in \mathbb{N}$. This computation ends the proof of the present lemma. \hfill \Box

5.2. Regularity. In order to show that the modular lift $D_\Delta : \mathcal{D}(D_\Delta) \to X$ is regular we will use the local-global principle for unbounded regular operators, see [Pie06, KALE12]. We will thus pause for a second and remind the reader how this principle works.

Let $\rho : A \to \mathbb{C}$ be a state on the $C^*$-algebra $A$. We may then define the pairing,
\[
\langle \cdot, \cdot \rangle_\rho : X \times X \to \mathbb{C}, \quad \langle x_0, x_1 \rangle_\rho := \rho(x_0, x_1)
\]
Furthermore, with $N_\rho := \{ x \in X \mid \langle x, x \rangle_\rho = 0 \}$, we obtain that the vector space quotient $X/N_\rho$ has a well-defined norm, $\| [x] \|_\rho := \langle x, x \rangle_\rho$. The completion of $X/N_\rho$ is then a Hilbert space with inner product induced by $\langle \cdot, \cdot \rangle_\rho$. We denote this Hilbert space by $X_\rho$ and let $[\cdot] : X \to X_\rho$ denote the canonical map (quotient followed by inclusion).

The unbounded selfadjoint operator $D_\Delta : \mathcal{D}(D_\Delta) \to X$ yields an induced unbounded symmetric operator
\[
(D_\Delta)_\rho : \mathcal{D}((D_\Delta)_\rho) \to X_\rho, \quad [x] \mapsto [D_\Delta(x)]
\]
where the domain is given by
\[
\mathcal{D}((D_\Delta)_\rho) := \{ [x] \mid x \in \mathcal{D}(D_\Delta) \}
\]
We denote the closure of this unbounded symmetric operator by
\[
D_\Delta \otimes 1 : \mathcal{D}(D_\Delta \otimes 1) \to X_\rho
\]
The local-global principle states that the unbounded selfadjoint operator $D_\Delta$ is regular if and only if $D_\Delta \otimes 1$ is selfadjoint for each state $\rho : A \to \mathbb{C}$, see [KALE12, Theorem 4.2]. We remark that an even stronger result is proved in [Pie06]: It does in fact suffice to prove selfadjointness for every pure state on $A$.

Let us from now on fix a state $\rho : A \to \mathbb{C}$. We are interested in showing that $D_\Delta \otimes 1 : \mathcal{D}(D_\Delta \otimes 1) \to X_\rho$ is selfadjoint. We remark that it already follows by the local-global principle that the unbounded operator $D \otimes 1 : \mathcal{D}(D \otimes 1) \to Y_\rho$ is selfadjoint.

The next lemma is left as an exercise to the reader:

**Lemma 5.7.** The triple $(D \otimes 1, \Gamma \otimes 1, \Phi \otimes 1)$ satisfies the conditions (1), (2), and (3) stated in Assumption 5.1 (where $\Gamma \otimes 1 : Y_\rho \to Y_\rho$ and $\Phi \otimes 1 : X_\rho \to Y_\rho$ are defined using the same recipe as in the unbounded case). Furthermore, we have the identities
\[
d_{(\Gamma \otimes 1)}(G \otimes 1) = d_\Gamma(G) \otimes 1 \quad \text{and} \quad \rho_{(\Gamma \otimes 1)}(G \otimes 1) = \rho_\Gamma(G) \otimes 1
\]
It is a consequence of the above lemma and Proposition 5.5 that the composition
\((\Phi^* \otimes 1)(D \otimes 1)(\Phi \otimes 1) : \mathcal{D}\left((\Phi^* \otimes 1)(D \otimes 1)(\Phi \otimes 1)\right) \to X\_\rho\)
is essentially selfadjoint. We will denote the closure by \((D \otimes 1)_{\Delta \otimes 1}\). We may thus focus our attention on proving the identity
\((D \otimes 1)_{\Delta \otimes 1} := (\Phi^* \otimes 1)(D \otimes 1)(\Phi \otimes 1) = D_{\Delta} \otimes 1\)

We start by proving the easiest of the two inclusions:

**Lemma 5.8.**
\(D_{\Delta} \otimes 1 \subseteq (D \otimes 1)_{\Delta \otimes 1}\)

**Proof.** Let \(\xi \in \mathcal{D}(D_{\Delta} \otimes 1)\). Then there exists a sequence \(\{\xi_n\}\) in \(\mathcal{D}(\Phi^*D\Phi)\) such that
\[
[\xi_n] \to \xi \quad \text{and} \quad [D_{\Delta}(\xi_n)] \to (D \otimes 1)(\xi)
\]
But then we clearly have that \([\xi_n] \in \mathcal{D}\left((\Phi^* \otimes 1)(D \otimes 1)(\Phi \otimes 1)\right)\) and furthermore that
\[
(\Phi^* \otimes 1)(D \otimes 1)(\Phi \otimes 1)[\xi_n] = [D_{\Delta}(\xi_n)]
\]
This proves the lemma. \(\square\)

The proof of the reverse inclusion
\((D \otimes 1)_{\Delta \otimes 1} \subseteq D_{\Delta} \otimes 1\) \hspace{1cm} (5.2)
is more subtle. It will rely on the following lemma:

**Lemma 5.9.** Let \(\xi \in \mathcal{D}\left((D \otimes 1)_{\Delta \otimes 1}\right)\). Then \((D \otimes 1)(\xi) \in \mathcal{D}(D_{\Delta} \otimes 1)\) and furthermore,
\[(D_{\Delta} \otimes 1)(\Delta \otimes 1)(\xi) = (\Delta \otimes 1)(D \otimes 1)_{\Delta \otimes 1}(\xi) + (\Phi^* d_\Gamma(G) \Phi \otimes 1)(\xi)\]

**Proof.** Let first \(\eta \in \mathcal{D}(D \otimes 1)\) be given. Choose a sequence \(\{\eta_n\}\) in \(\mathcal{D}(D)\) such that
\[
[\eta_n] \to \eta \quad \text{and} \quad [D\eta_n] \to (D \otimes 1)(\eta)
\]
Remark that \((\Phi^* \Gamma \otimes 1)[\eta_n] \in \mathcal{D}\left((D_{\Delta})_\rho\right)\) for all \(n \in \mathbb{N}\) and furthermore that
\[(\Phi^* \Gamma \otimes 1)[\eta_n] \to (\Phi^* \Gamma \otimes 1)(\eta)\]
We now compute as follows:
\[(D_{\Delta})_\rho[\Phi^* \Gamma \eta_n] = [\Phi^* D\Gamma \eta_n] = [\Phi^* \Gamma GD\eta_n] + [\Phi^* d_\Gamma(G)\eta_n]\]
This shows that
\[(D_{\Delta})_\rho(\Phi^* \Gamma \otimes 1)[\eta_n] \to (\Phi^* \Gamma G \Gamma \otimes 1)(D \otimes 1)(\eta) + (\Phi^* d_\Gamma(G) \otimes 1)(\eta)\]
We thus have that \((\Phi^* \Gamma \otimes 1)(\eta) \in \mathcal{D}(D_{\Delta} \otimes 1)\) and furthermore that
\[(D_{\Delta} \otimes 1)(\Phi^* \Gamma \otimes 1)(\eta) = (\Phi^* \Gamma G \otimes 1)(D \otimes 1)(\eta) + (\Phi^* d_\Gamma(G) \otimes 1)(\eta)\]

Let now \(\xi \in \mathcal{D}\left((D \otimes 1)(\Phi \otimes 1)\right)\). It then follows from the above considerations \((\Delta \otimes 1)(\xi) \in \mathcal{D}(D_{\Delta} \otimes 1)\) and furthermore that
\[(D_{\Delta} \otimes 1)(\Delta \otimes 1)(\xi) = (D_{\Delta} \otimes 1)(\Phi^* \Gamma \otimes 1)(\Phi \otimes 1)(\xi)\]
\[
= (\Phi^* \Gamma G \otimes 1)(D \otimes 1)(\Phi \otimes 1)(\xi) + (\Phi^* d_\Gamma(G) \Phi \otimes 1)(\xi)
\]
\[
= (\Delta \otimes 1)(D \otimes 1)_{\Delta \otimes 1}(\xi) + (\Phi^* d_\Gamma(G) \Phi \otimes 1)(\xi)
\]
The result of the lemma now follows by using that \((\Phi^* \otimes 1)(D \otimes 1)(\Phi \otimes 1) = (D \otimes 1)_{\Delta \otimes 1}\) by definition. \(\square\)
We are now ready to prove the reverse inclusion which (together with Lemma 5.8) will imply the following:

**Proposition 5.10.** We have the identity of unbounded operators
\[(D \otimes 1)\Delta \otimes 1 = D\Delta \otimes 1\]
on the Hilbert space \(X_\rho\). In particular we obtain that \(D\Delta \otimes 1\) is selfadjoint.

**Proof.** By Lemma 5.8 we only need to show that
\[\mathcal{D}\left((D \otimes 1)\Delta \otimes 1\right) \subseteq \mathcal{D}(D\Delta \otimes 1)\]
Let thus \(\xi \in \mathcal{D}\left((D \otimes 1)\Delta \otimes 1\right)\) be given. For each \(n \in \mathbb{N}\), it is then a consequence of Lemma 5.4 and Lemma 5.9 that
\[(\Delta(\Delta + 1/n^{-1}\otimes 1)(\xi) \in \mathcal{D}(D\Delta \otimes 1)\]
Furthermore, these two lemmas allow us to compute as follows:
\[(D\Delta \otimes 1)((\Delta(\Delta + 1/n^{-1}\otimes 1)\otimes 1)(\Delta + 1/n^{-1}\otimes 1)\otimes 1)\]
\[= (\Delta + 1/n^{-1}\otimes 1)(D\otimes 1)\Delta \otimes 1(\xi)\]
\[= (\Delta(\Delta + 1/n^{-1}\otimes 1)\Delta \otimes 1(\xi)\]
Together with Lemma 5.6 (and Lemma 5.7) this computation shows that
\[(D\Delta \otimes 1)((\Delta + 1/n^{-1}\Delta \otimes 1)\otimes 1)\rightarrow (D \otimes 1)\Delta \otimes 1(\xi)\]
This proves the present proposition. \(\Box\)

The main theorem of this section is now a consequence of the above considerations and Proposition 5.10:

**Theorem 5.1.** Suppose that the conditions in Assumption 5.1 hold. Then the modular lift \(D\Delta : \mathcal{D}(D\Delta) \rightarrow X\) is selfadjoint and regular.

6. Compactness of resolvents

We will in this section remain in the general setting presented in Section 5 and the conditions in Assumption 5.1 will therefore be in effect. In particular, it follows by Theorem 5.1 that the modular lift \(D\Delta := \Phi^*D\Phi : \mathcal{D}(D\Delta) \rightarrow X\) is a selfadjoint and regular unbounded operator. We recall that \(\Delta := \Phi^*\Gamma\Phi\).

Our principal interest is now to study the compactness properties of the resolvent \((i + D\Delta)^{-1} : X \rightarrow X\) of the modular lift.

**Lemma 6.1.** We have the identity
\[
\Delta^2(i + D\Delta)^{-1} = \Phi^*\Gamma(i + D)^{-1} \cdot (i(iG - 1)\Gamma + d\Gamma(G))\Phi(i + D\Delta)^{-1} + \Gamma\Phi
\]
Proof. Let $\xi \in \mathcal{D}(D\Phi)$. Since the unbounded operator $(i + \Phi^* D\Phi) : \mathcal{D}(D\Phi) \to X$ has dense image (by Theorem 5.1) it is enough to verify that

$$\Delta^2(\xi) = \Phi^* \Gamma(i + D)^{-1} \left( (i(G - 1)\Gamma + d_\Gamma(G))\Phi + \Gamma\Phi(i + \Phi^* D\Phi) \right)(\xi)$$

But this follows from the computation

$$\Phi^* \Gamma(i + D)^{-1} \left( (i(G - 1)\Gamma + d_\Gamma(G))\Phi + \Gamma\Phi(i + \Phi^* D\Phi) \right)(\xi)
= \Phi^* \Gamma(i + D)^{-1} (i\Gamma + D\Gamma)\Phi(\xi)
= \Phi^* \Gamma G \Gamma \Phi(\xi) = \Delta^2(\xi) \quad \square$$

**Proposition 6.2.** Suppose that $\Phi^* \Gamma(i + D)^{-1} \in \mathcal{K}(Y, X)$. Then $\Delta(i + D_\Delta)^{-1} \in \mathcal{K}(X)$.

**Proof.** It is an immediate consequence of Lemma 6.1 that

$$\Delta^2(i + D_\Delta)^{-1} \in \mathcal{K}(X)$$

The result of the lemma therefore follows by noting that the sequence $\{\Delta^2(\Delta + 1/n)^{-1}\}$ converges to $\Delta : X \to X$ in operator norm. \( \square \)

For later use, we shall also be interested in the relationship between the resolvents of the squares $D_\Delta^2 : \mathcal{D}(D_\Delta^2) \to X$ and $D^2 : \mathcal{D}(D^2) \to Y$. In order to study these two resolvents we will need the following extra assumption:

**Assumption 6.3.** It is assumed that $\Gamma(\mathcal{D}(D)) \subseteq \mathcal{D}(D)$ and that the straight commutator

$$D\Gamma - \Gamma D : \mathcal{D}(D) \to Y$$

has a bounded adjointable extension $d(\Gamma) : Y \to Y$.

We start with a preliminary lemma:

**Lemma 6.4.** We have the identity

$$(D^2(1 + D^2)^{-1}\Phi \Delta^2 - (1 + D^2)^{-1}\Gamma^2\Phi D_\Delta^2)(\xi)
= \left( D(1 + D^2)^{-1}d_\Gamma(G)G\Phi + D(1 + D^2)^{-1}\Gamma Gd_\Gamma(G)\Phi + (1 + D^2)^{-1}d_\Gamma(\Gamma G)\Phi D_\Delta \right)(\xi)$$

for all $\xi \in \mathcal{D}(D_\Delta^2)$.

**Proof.** Consider first an element $\eta \in \mathcal{D}(D\Phi)$. We then have that

$$- (1 + D^2)^{-1}\Gamma^2\Phi D_\Delta(\eta) = -(1 + D^2)^{-1}\Gamma^2 G D\Phi(\eta)
= -D(1 + D^2)^{-1}\Gamma G\Phi(\eta) + (1 + D^2)^{-1}d_\Gamma(\Gamma G)\Phi(\eta)$$

Hence, since $\mathcal{D}(D\Phi) \subseteq X$ is a core for the modular lift $D_\Delta : \mathcal{D}(D_\Delta) \to X$ we obtain that

$$-(1 + D^2)^{-1}\Gamma^2\Phi D_\Delta^2(\xi) = -D(1 + D^2)^{-1}\Gamma G\Phi D_\Delta(\xi)
+ (1 + D^2)^{-1}d_\Gamma(\Gamma G)\Phi D_\Delta(\xi)$$
for all $\xi \in \mathcal{D}(D_\Delta^2)$. Thus to prove the lemma we only need to show that

$$
\left( (D^2(1 + D^2)^{-1} \Phi \Delta^2 - D(1 + D^2)^{-1} \Gamma G \Phi D_{\Delta}) \right)(\xi)
= \left( D(1 + D^2)^{-1} d_T(G) \Gamma \Phi + D(1 + D^2)^{-1} \Gamma G d_{\Gamma}(G) \Phi \right)(\xi)
$$

(6.1)

We will prove the stronger statement that this identity holds for all $\xi \in \mathcal{D}(D_{\Phi})$. To obtain this, we may focus on the case where $\xi \in \mathcal{D}(D_{\Phi})$. A straightforward computation then implies that

$$
- D(1 + D^2)^{-1} \Gamma G G \Phi D(\xi)
= -D^2(1 + D^2)^{-1} \Gamma \Phi D(\xi) + D(1 + D^2)^{-1} \Gamma G d_{\Gamma}(G) \Phi(\xi)
+ D(1 + D^2)^{-1} d_T(G) \Gamma \Phi(\xi)
$$

Since $\Gamma G \Phi D = \Phi \Delta^2$ this proves the identity in (6.1) and hence the lemma. \hfill \square

Let us apply the notation

$$
T_{\lambda} := (1 + \lambda^2/r + D^2)^{-1} : Y \to Y \quad \text{and} \quad S_{\lambda} := (1 + \lambda \Delta^2/r + D_\Delta^2)^{-1} : X \to X
$$

for all $\lambda \geq 0$ where $r \in (\|\Delta\|^2 + \|\Gamma\|^2, \infty)$ is a fixed constant.

The next result will play an important role in our later investigations of the relationship between the unbounded Kasparov product and the interior Kasparov product:

**Proposition 6.5.** We have the identity

$$
\Phi \Delta^2 S_{\lambda} - T_{\lambda} \Gamma^2 \Phi = T_{\lambda} \left( \Phi \Delta^2 - \Gamma^2 \Phi + d_T(\Gamma G) \Phi D_{\Delta} \right) S_{\lambda}
+ (DT_{\lambda})^*(d_T(G) \Phi D(\Gamma) + \Gamma G d_{\Gamma}(G) \Phi) S_{\lambda}
$$

for all $\lambda \geq 0$.

**Proof.** Let $\xi \in \mathcal{D}(D_\Delta^2)$. To prove the lemma, it clearly suffices to check that

$$
\left( \Phi \Delta^2 - T_{\lambda} \Gamma^2 \Phi (1 + \lambda \Delta^2/r + D_\Delta^2) \right)(\xi)
= \left( T_{\lambda}(\Phi \Delta^2 - \Gamma^2 \Phi + d_T(\Gamma G) \Phi D_{\Delta}) \right)(\xi)
+ (DT_{\lambda})^*(d_T(G) \Phi D(\Gamma) + \Gamma G d_{\Gamma}(G) \Phi) \Phi(\xi)
$$

However, by Lemma 6.4 we have that

$$
T_{\lambda}(\Phi \Delta^2 - \Gamma^2 \Phi + d_T(\Gamma G) \Phi D_{\Delta})(\xi) + (DT_{\lambda})^*(d_T(G) \Phi D(\Gamma) + \Gamma G d_{\Gamma}(G) \Phi)(\xi)

= T_{\lambda}(\Phi \Delta^2 - \Gamma^2 \Phi)(\xi) + (D^2 T_{\lambda})^* \Phi \Delta^2(\xi) - T_{\lambda} \Gamma^2 \Phi D_\Delta^2(\xi)

= \left( (1 + D^2) T_{\lambda} \right)^* \Phi \Delta^2(\xi) - T_{\lambda} \Gamma^2 \Phi (1 + D_\Delta^2)(\xi)
$$

The result of the present lemma then follows since

$$
(1 + D^2) T_{\lambda} \Phi \Delta^2 = \Phi \Delta^2 - T_{\lambda} \Gamma^2 \Phi \lambda \Delta^2/r
$$

\hfill \square
7. The unbounded Kasparov product

Throughout this section we let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras which satisfy the conditions in Assumption 2.2. As usual we denote the $C^*$-completions by $A$ and $B$ and the operator space completions by $A_1$ and $B_1$. Furthermore, we will fix a third $C^*$-algebra $C$. On top of this data, we shall consider:

1. An unbounded modular cycle $(Y, D, \Gamma)$ from $B$ to $C$ (with grading operator $\gamma : Y \to Y$ in the even case).
2. A differentiable Hilbert $C^*$-module $X$ from $\mathcal{A}$ to $B$ with a fixed differentiable generating sequence $\{\xi_n\}_{n=1}^\infty$.

We let $\pi_A : A \to \mathcal{L}(X)$ and $\pi_B : B \to \mathcal{L}(Y)$ denote the $*$-homomorphisms associated with the above data. It will then be assumed that

$\pi_A(a) \in \mathcal{K}(X)$ for all $a \in A$.

It is the principal goal of this section to apply the above data to construct a new (and explicit) unbounded modular cycle from $\mathcal{A}$ to $C$:

$X \otimes_\mathcal{A}(Y, D, \Gamma) := (X \otimes_B Y, D_\Delta, \Delta)$

We shall refer to this new unbounded modular cycle as the unbounded Kasparov product of the differentiable Hilbert $C^*$-module $X$ and the unbounded modular cycle $(Y, D, \Gamma)$.

Let us return to the interior tensor product $X \hat{\otimes}_B Y$ of Hilbert $C^*$-modules. We recall that this is the Hilbert $C^*$-module over $C$ defined as the completion of the algebraic tensor product of modules $X \otimes B Y$ with respect to the norm coming from the $C$-valued (pre) inner product

$\langle \cdot, \cdot \rangle : X \otimes B Y \times X \otimes B Y \to C \quad \langle x_0 \otimes_B y_0, x_1 \otimes_B y_1 \rangle := \langle y_0, \pi_B(\langle x_0, x_1 \rangle)(y_1) \rangle$

The interior tensor product comes equipped with a $*$-homomorphism

$(\pi_A \otimes 1) : A \to \mathcal{L}(X \hat{\otimes}_B Y) \quad (\pi_A \otimes 1)(a) := \pi_A(a) \otimes \text{Id}_Y$

For each $N \in \mathbb{N}$ we may then define the bounded adjointable operator

$\Phi_N : X \hat{\otimes}_B Y \to \ell^2(Y) \quad z \mapsto \sum_{n=1}^N T_{\xi_n}^*(z)\delta_n$
Lemma 7.1. The sequence of bounded adjointable operators
\[ \{\Phi_N\}_{N=1}^{\infty} \quad \Phi_N : X \hat{\otimes}_B Y \to \ell^2(Y) \]
converges in operator norm to a bounded adjointable operator \( \Phi : X \hat{\otimes}_B Y \to \ell^2(Y) \). Moreover, we have that \( \Phi^* : \ell^2(Y) \to X \hat{\otimes}_B Y \) has dense image.

Proof. Let us prove that the sequence \( \{\Phi_N\} \) is Cauchy in operator norm. To this end, we let \( M > N \) be given and notice that
\[ \|\Phi_M - \Phi_N\|^2 = \|\Phi_M^* - \Phi_N^*\|^2 = \|\Phi_M \Phi_M^* + \Phi_N \Phi_N^* - \Phi_M \Phi_N^* - \Phi_M^* \Phi_N\| \]
Furthermore, it may be verified that
\[ \Phi_M \Phi_M^* + \Phi_N \Phi_N^* - \Phi_M \Phi_N^* = \sum_{n=N+1}^{M} \sum_{m=N+1}^{M} \pi_B(\langle \xi_n, \xi_m \rangle) \delta_{nm} \]
Since the sequence
\[ \left\{ \sum_{n,m=1}^{K} \langle \xi_n, \xi_m \rangle \delta_{nm} \right\}_{K=1}^{\infty} \]
is a Cauchy sequence in \( K(B_1) \) (by our assumption on the differentiable generating sequence \( \{\xi_n\} \)) and since the canonical map \( B_1 \to B \) is completely bounded, this shows that \( \{\Phi_N\} \) is a Cauchy sequence as well.

To see that the image of \( \Phi^* : \ell^2(Y) \to X \hat{\otimes}_B Y \) is dense it suffices (since \( \{\xi_n\} \) generates \( X \)) to check that \( \xi_n b \otimes_B y \in \text{Im}(\Phi^*) \) for all \( n \in \mathbb{N}, b \in B \) and \( y \in Y \). But this is clear since
\[ \Phi^*(\pi_B(b)(y) \cdot \delta_n) = \xi_n \otimes_B \pi_B(b)(y) = \xi_n b \otimes_B y \]
for all \( n \in \mathbb{N}, b \in B \) and \( y \in Y \). This ends the proof of the lemma. \( \square \)

Let us recall from Subsection 3.1 that the notation
\[ 1 \otimes D : \mathcal{D}(1 \otimes D) \to \ell^2(Y) \quad \text{and} \quad 1 \otimes \Gamma : \ell^2(Y) \to \ell^2(Y) \]
refers to the diagonal operators induced by \( D : \mathcal{D}(D) \to Y \) and \( \Gamma : Y \to Y \).

The next lemma explains how \( \Phi : X \hat{\otimes}_B Y \to \ell^2(Y) \) creates a link between the \( * \)-algebra \( \mathcal{A} \) and the unbounded modular cycle \( (Y, D, \Gamma) \).

Lemma 7.2. Let \( T \in \pi_A(\mathcal{A}) + \mathbb{C} \cdot \text{Id}_X \) be given. Then the following holds:

1. The bounded adjointable operator \( \Phi(T \otimes 1) \Phi^*(1 \otimes \Gamma) : \ell^2(Y) \to \ell^2(Y) \) preserves the domain of \( 1 \otimes D : \mathcal{D}(1 \otimes D) \to \ell^2(Y) \).
2. The twisted commutator
\[ (1 \otimes D) \Phi(T \otimes 1) \Phi^*(1 \otimes \Gamma) - (1 \otimes \Gamma) \Phi(T \otimes 1) \Phi^*(1 \otimes D) : \mathcal{D}(1 \otimes D) \to \ell^2(Y) \]
extends to a bounded adjointable operator
\[ d_{1 \otimes \Gamma} \Phi(T \otimes 1) \Phi^* : \ell^2(Y) \to \ell^2(Y) \]
3. The image of \( d_{1 \otimes \Gamma} \Phi(T \otimes 1) \Phi^* : \ell^2(Y) \to \ell^2(Y) \) is contained in the image of \( (1 \otimes \Gamma)^{1/2} : \ell^2(Y) \to \ell^2(Y) \) and the unbounded operator
\[ (1 \otimes \Gamma)^{-1/2} d_{1 \otimes \Gamma} \Phi(T \otimes 1) \Phi^*(1 \otimes \Gamma)^{-1/2} : \text{Im}((1 \otimes \Gamma)^{1/2}) \to \ell^2(Y) \]
extends to a bounded adjointable operator.
Furthermore, the linear map \( \mathcal{A} \to \mathcal{L}(\ell^2(Y)) \) defined by
\[
a \mapsto (1 \otimes \Gamma)^{-1/2}d_{1\otimes \Gamma}(\Phi(\pi_A(a) \otimes 1)\Phi^*)(1 \otimes \Gamma)^{-1/2}
\]
is completely bounded (with respect to the operator space norm on \( \mathcal{A} \)).

Proof. Let \( \tau : \mathcal{A} \to K(B_1) \) denote the completely bounded map defined by
\[
\tau(a) := \sum_{n,m=1}^{\infty} \langle \xi_n, \pi_A(a)(\xi_m) \rangle \delta_{nm} \quad a \in \mathcal{A}
\]
(where the complete boundedness is understood with respect to the operator space norm on \( \mathcal{A} \)). Let also \( g \in K(B_1) \) be given by \( g := \sum_{n,m=1}^{\infty} \langle \xi_n, \xi_m \rangle \delta_{nm} \). Finally, we let \( K(\pi_B) : K(B) \to \mathcal{L}(\ell^2(Y)) \) denote the \( * \)-homomorphism defined by
\[
K(\pi_B)(\sum_{n,m=1}^{\infty} b_{nm} \delta_{nm})(\sum_{k=1}^{\infty} y_k \delta_k) := \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \pi_B(b_{nm})(y_m) \right) \delta_n
\]
For each \( T = \pi_A(a) + \lambda \cdot \text{Id}_X \in \pi_A(\mathcal{A}) + \mathbb{C} \cdot \text{Id}_X \) we then have the identity
\[
\Phi(T \otimes 1)\Phi^* = K(\pi_B)\left( \tau(a) + \lambda \cdot g \right) : \ell^2(Y) \to \ell^2(Y)
\]
(where we are suppressing the canonical map \( K(B_1) \to K(B) \)). The result of the lemma is now a consequence of Proposition 3.5 (and the remarks following Definition 3.1).

It follows by Lemma 7.2 (with \( T = \text{Id}_X \)) that the triple \((\Phi, (1 \otimes \Gamma), (1 \otimes D))\) satisfies the conditions applied in Section 5. In particular, we may form the modular lift
\[
(1 \otimes D)_\Delta : \mathcal{D}((1 \otimes D)_\Delta) \to X \hat{\otimes}_B Y
\]
We define the bounded adjointable operator
\[
\Delta := \Phi^*(1 \otimes \Gamma)\Phi : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y
\]

**Theorem 7.1.** Suppose that the conditions outlined in the beginning of this section are satisfied. Then the triple \((X \hat{\otimes}_B Y,(1 \otimes D)_\Delta, \Delta)\) is an unbounded modular cycle from \( \mathcal{A} \) to \( C \). The parity of \((X \hat{\otimes}_B Y,(1 \otimes D)_\Delta, \Delta)\) is the same as the parity of \((Y,D,\Gamma)\) and the grading operator is given by \( 1 \otimes \gamma : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y \) in the even case.

Proof. We will verify each of the points in Definition 3.1 separately.

The fact that \( X \hat{\otimes}_B Y \) is a countably generated Hilbert C*-module follows since both \( X \) and \( Y \) are countably generated by assumption.

The modular lift \((1 \otimes D)_\Delta : \mathcal{D}((1 \otimes D)_\Delta) \to X \hat{\otimes}_B Y\) is selfadjoint and regular by Theorem 5.1.

The bounded operator \( \Delta := \Phi^*(1 \otimes \Gamma)\Phi : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y \) is clearly positive and selfadjoint and it has dense image since \((1 \otimes \Gamma) : \ell^2(Y) \to \ell^2(Y)\) and \( \Phi^* : \ell^2(Y) \to X \hat{\otimes}_B Y \) have dense images.

It is finally clear that the grading operator \( 1 \otimes \gamma : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y \) satisfies the constraints
\[
(1 \otimes \gamma)(\pi_A(a) \otimes 1) = (\pi_A(a) \otimes 1)(1 \otimes \gamma) \quad (1 \otimes \gamma)(1 \otimes D)_\Delta = -(1 \otimes D)_\Delta(1 \otimes \gamma)
\]
for all \( a \in A \) in the even case.

We will now focus on the conditions (1)-(5) in Definition 3.1.
(5): Let \( a \in A \) be given. We need to show that \( 1/n(\Delta + 1/n)^{-1}(\pi_A(a) \otimes 1) \to 0 \) in the operator norm on \( \mathcal{L}(X \hat{\otimes}_B Y) \). To this end, we remark that there exists a positive and selfadjoint compact operator \( K : X \to X \) with dense image such that \( \Phi^* \Phi = K \otimes 1 : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y \). (In fact, we may choose \( K := \sum_{n=1}^{\infty} \theta_{e_n,e_n} \).

Since \( \pi_A(a) \in \mathcal{K}(X) \) we thus have that \( \Phi^* \Phi(1/m + \Phi^* \Phi)^{-1}(\pi_A(a) \otimes 1) \to \pi_A(a) \otimes 1 \) where the convergence takes place in operator norm. It therefore suffices to check that \( 1/n(\Delta + 1/n)^{-1}\Phi^* \Phi \to 0 \) in operator norm. To prove this, we notice that \( \Phi \Phi^*: \ell^2(Y) \to \ell^2(Y) \) lies in the image of the \(*\)-homomorphism \( K(\pi_B) : K(B) \to \mathcal{L}(\ell^2(Y)) \). By Proposition 3.5 it thus follows that \( (1 \otimes \Gamma(\Gamma + 1/m)^{-1})\Phi \to \Phi \) in operator norm. We may therefore restrict our attention to showing that \( 1/n(\Delta + 1/n)^{-1}\Phi^* (1 \otimes \Gamma) \to 0 \) in operator norm. But this is clear since \( \Delta = \Phi^* (1 \otimes \Gamma) \Phi \) by definition.

(1): Let again \( a \in A \) be given. To verify that \( (\pi_A(a) \otimes 1) \cdot (i + (1 \otimes D)\Delta)^{-1} : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y \) is a compact operator it suffices (by (5) and Proposition 6.2) to check that \( \Phi^*(1 \otimes \Gamma)(i + (1 \otimes D))^{-1} : \ell^2(Y) \to \ell^2(Y) \) is compact. But this is clear by Proposition 3.5 since \( \Phi \Phi^*: \ell^2(Y) \to \ell^2(Y) \) lies in the image of \( K(\pi_B) : K(B) \to \mathcal{L}(\ell^2(Y)) \).

(2): Consider an element \( T \in (\pi_A(1) \otimes \mathcal{A}) + \mathbb{C} \cdot \text{Id}_{X \hat{\otimes}_B Y}, \) thus \( T = \pi_A(a) \otimes 1 + \lambda \cdot \text{Id}_{X \hat{\otimes}_B Y} \) for some \( a \in \mathcal{A} \) and some \( \lambda \in \mathbb{C} \).

Let \( z \in \mathcal{D}(1 \otimes D) \Phi \) be given (thus \( \Phi(z) \in \mathcal{D}(1 \otimes D) \)). Then \( T \Delta(z) = \Phi \Phi^*(1 \otimes \Gamma)(\Phi z) \in \mathcal{D}(1 \otimes D) \) (by Lemma 7.2) and therefore \( T \Delta(z) \in \mathcal{D}(1 \otimes D) \Phi \). Furthermore, we have that

\[
\Phi^*(1 \otimes D) \Phi T \Delta(z) = \Phi^*(1 \otimes \Gamma) \Phi T \Phi^*(1 \otimes D) \Phi(z) + \Phi^* d_{1 \otimes \Gamma}(\Phi T \Phi^*) \Phi(z)
\]

But this implies that

\[
(1 \otimes D) \Delta T \Delta - \Delta T(1 \otimes D) \Delta)(z) = \Phi^* d_{1 \otimes \Gamma}(\Phi T \Phi^*) \Phi(z)
\]  

(7.1)

Since \( \mathcal{D}(1 \otimes D) \Phi \subseteq X \hat{\otimes}_B Y \) is a core for the modular lift \( (1 \otimes D)_\Delta : \mathcal{D}(1 \otimes D)_\Delta \to X \hat{\otimes}_B Y \) this proves the relevant statement about twisted commutators (again by Lemma 7.2).

(3): Let \( T = \pi_A(a) \otimes 1 + \lambda \cdot \text{Id}_{X \hat{\otimes}_B Y} \). It follows by (2) that \( d_\Delta(T) = \Phi^* d_{1 \otimes \Gamma}(\Phi T \Phi^*) \Phi. \) Thus to prove the third condition it is enough (by Lemma 7.2) to show that \( \text{Im}(\Phi^*(1 \otimes \Gamma)^{1/2}) \subseteq \text{Im}(\Delta^{-1/2}) \) and that \( \Delta^{-1/2} \Phi^*(1 \otimes \Gamma)^{1/2} : \ell^2(Y) \to X \hat{\otimes}_B Y \) is a bounded adjointable operator. To prove this, we first remark that

\[
\Delta^{-1/2} \Phi^*(1 \otimes \Gamma)^{1/2} = ((1 \otimes \Gamma)^{1/2} \Phi \Delta^{-1/2})^*
\]

Thus, it suffices to check that \( (1 \otimes \Gamma)^{1/2} \Phi \Delta^{-1/2} : \mathcal{D}(\Delta^{-1/2}) \to \ell^2(Y) \) is the restriction of a bounded operator \( \Omega : X \hat{\otimes}_B Y \to \ell^2(Y) \). But this is clear since

\[
\langle (1 \otimes \Gamma)^{1/2} \Phi \Delta^{-1/2} \xi, (1 \otimes \Gamma)^{1/2} \Phi \Delta^{-1/2} \xi \rangle = \langle \Delta^{-1/2} \Delta \Delta^{-1/2} \xi, \xi \rangle = \langle \xi, \xi \rangle
\]

for all \( \xi \in \mathcal{D}(\Delta^{-1/2}). \)

(4): Recall first that the linear map \( \tau : \mathcal{A} \to K(B_1), a \mapsto \sum_{n,m=1}^{\infty} \langle \xi_n, \pi_A(a) \xi_m \rangle \delta_{nn} \) is completely bounded. Furthermore, it follows by the above considerations that

\[
d_\Delta(\pi_A(a) \otimes 1) = \Phi^*(1 \otimes \Gamma)^{1/2} \rho_{1 \otimes \Gamma}(\tau(a)) (1 \otimes \Gamma)^{1/2} \Phi : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y
\]
for all $a \in \mathcal{A}$. In particular we have that
\[
\Delta^{-1/2}d_\Delta(\pi_A(a) \otimes 1)\Delta^{-1/2} = \Omega^* \rho_1 \otimes \Gamma(\tau(a))\Omega
\]
where $\Omega = (1 \otimes \Gamma)^{1/2} \Phi \Delta^{-1/2} : X \hat{\otimes}_B Y \to \ell^2(Y)$ is a bounded adjointable operator (see the proof of (3)). Since $\rho_1 \otimes \Gamma : K(B_1) \to \mathcal{L}(\ell^2(Y))$ is completely bounded by Proposition 3.5 we have proved condition (4).

\[\Box\]

8. The modular transform

Throughout this section we will consider the following data:

(1) An unbounded selfadjoint and regular operator $D : \mathcal{D}(D) \to Y$ acting on a fixed Hilbert $C^*$-module $Y$.

(2) A positive selfadjoint bounded operator $\Delta : Y \to Y$ such that $\text{Im}(\Delta) \subseteq Y$ is dense.

We will then make the following standing assumption:

**Assumption 8.1.** It is assumed that

(1) The domain $\mathcal{D}(D) \subseteq Y$ is an invariant submodule for $\Delta : Y \to Y$ and the commutator

\[
D\Delta - \Delta D : \mathcal{D}(D) \to Y
\]

is the restriction of a bounded adjointable operator $d(\Delta) : Y \to Y$.

(2) The image of $d(\Delta) : Y \to Y$ is contained in the image of $\Delta^{1/2} : Y \to Y$ and the unbounded operator

\[
\Delta^{-1/2}d(\Delta)\Delta^{-1/2} : \text{Im}(\Delta^{1/2}) \to Y
\]

is the restriction of a bounded adjointable operator $\rho(\Delta) : Y \to Y$.

Let us choose

\[r \in (\|\Delta\|^2, \infty)\]

For each $\lambda \geq 0$ we then apply the notation:

\[
S_\lambda := (\lambda \Delta^2 / r + 1 + D^2)^{-1} \quad \text{and} \quad R_\lambda := (\lambda + 1 + D^2)^{-1}
\]

We are then interested in studying the modular transform of the pair $(D, \Delta)$. This is the unbounded operator defined by

\[
G_{D,\Delta} : \Delta(\mathcal{D}(D)) \to Y \quad G_{D,\Delta}(\Delta \xi) := \frac{1}{\pi} \int_0^{\infty} (\lambda r)^{-1/2} S_\lambda \Delta D(\Delta \xi) d\lambda
\]

for all $\xi \in \mathcal{D}(D)$. In particular we are interested in comparing the modular transform with the bounded transform of $D : \mathcal{D}(D) \to Y$. We recall here that the bounded transform of $D$ can be defined by $F_D := D(1 + D^2)^{-1/2} : Y \to Y$ and it follows that the bounded transform is a bounded extension of the unbounded operator

\[
F_D|_{\mathcal{D}(D)} : \mathcal{D}(D) \to Y \quad \eta \mapsto \frac{1}{\pi} \int_0^{\infty} \lambda^{-1/2} R_\lambda D(\eta) d\lambda
\]

The modular transform will play a key role in our later proof of one of the main theorems in this paper, namely that the bounded transform of an unbounded modular cycle yields a bounded Kasparov module (Theorem 9.1) and hence a class in $KK$-theory.
We notice that the modular transform has been obtained from the bounded transform by making a non-commutative change of variables in the integral over the half-line. Indeed, the idea is just to replace the scalar-valued variable $\lambda \geq 0$ by the operator-valued variable $\lambda \Delta^2/r$. In the case where $D$ and $\Delta$ actually commute it can therefore be proved that the modular transform is just a restriction of the bounded transform to \( \Delta(D(D)) \subseteq Y \). However, in the case of real interest, thus when $d(\Delta) \neq 0$, there is a substantial error-term appearing and a great deal of this section is devoted to controlling the size of this error-term. There are easier proofs of the main results of this section when the modular operator $\Delta : Y \to Y$ is assumed to be invertible (as a bounded operator). One of the important points of the whole theory that we are developing here does however lie in the fact that $\Delta : Y \to Y$ is allowed to have zero in the spectrum. This condition should therefore not be relaxed.

8.1. Preliminary algebraic identities. Let us apply the notation $K := 1 - \Delta^2/r$ and $X_\lambda := \lambda \cdot R_\lambda K$ for all $\lambda \geq 0$.

We start our work on understanding the modular transform

$$G_{D,\Delta} : \Delta(D(D)) \to Y, \quad G_{D,\Delta}(\Delta \xi) = \frac{1}{\pi} \int_0^\infty (\lambda r)^{-1/2} \Delta S_\lambda D(\Delta \xi) \, d\lambda$$

by rewriting the (modular) resolvent $S_\lambda = (\lambda \Delta^2/r + 1 + D^2)^{-1}$ in a way that is more amenable to a computation of the integral appearing in the expression for the modular transform. More precisely, we will first expand the resolvent $S_\lambda : Y \to Y$ as a power-series involving the (standard) resolvent $R_\lambda : Y \to Y$ and the bounded adjointable operator $K : Y \to Y$. We will then reorganize this power-series by moving all the $K$-terms to the left and all the $R_\lambda$-terms to the right (and hence picking up an error-term). This will be accomplished in the present subsection.

**Lemma 8.2.** For each $\lambda \geq 0$ we have the identities

$$S_\lambda = \sum_{n=0}^{\infty} X_\lambda^n \cdot R_\lambda = (1 - X_\lambda)^{-1} R_\lambda$$

where the sum converges absolutely.

**Proof.** Let $\lambda \geq 0$ be given. By the resolvent identity we have that

$$R_\lambda - S_\lambda = (\lambda + 1 + D^2)^{-1} - (\lambda \Delta^2/r + 1 + D^2)^{-1} = -X_\lambda \cdot S_\lambda$$

Since $\|\Delta^2\| < r$ we have that $\|X_\lambda\| \leq \lambda(1 + \lambda)^{-1} < 1$. We may thus conclude that $1 \to X_\lambda : Y \to Y$ is invertible with $(1 - X_\lambda)^{-1} = \sum_{n=0}^{\infty} X_\lambda^n$ where the sum converges absolutely. From the above we deduce that

$$S_\lambda = (1 - X_\lambda)^{-1} R_\lambda = \sum_{n=0}^{\infty} X_\lambda^n \cdot R_\lambda$$

This proves the lemma. \qed
We will from now on apply the notation
\[ I(T) := [D^2, T] : \mathcal{D}(D^2) \to Y \]
whenever \( T : Y \to Y \) is a bounded adjointable operator such that \( T(\mathcal{D}(D^2)) \subseteq \mathcal{D}(D^2) \).

**Lemma 8.3.** Let \( \lambda \geq 0 \), \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \). Then we have that
\[ X^n_\lambda \cdot \Delta^k = (X^n_\lambda)^{n-1} K \Delta^k R_\lambda - I((X^n_\lambda \Delta^k) R_\lambda) \]

**Proof.** The proof runs by induction on \( n \in \mathbb{N} \) using the identity
\[ R_\lambda K \Delta^k = K \Delta^k R_\lambda - I(R_\lambda K \Delta^k) R_\lambda \]
Notice that it is convenient to do the cases \( k = 0 \) and \( k \in \mathbb{N} \) separately, starting with \( k = 0 \).

**Lemma 8.4.** Let \( \lambda \geq 0 \), \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) be given. Then we have that
\[ X^n_\lambda \cdot \Delta^k = \Delta^k \lambda^n R^n_\lambda - \sum_{j=0}^{n-1} K^n_\lambda \cdot I((X^n_\lambda \Delta^k) R^{j+1}_\lambda) \]

**Proof.** The proof runs by induction using the identity in Lemma 8.3.

For each \( m \in \mathbb{N} \) and each \( \lambda \geq 0 \) we define the bounded adjointable operator
\[ L_\lambda(m) := I((1 - X^n_\lambda) S_\lambda K \Delta^3) R_\lambda : Y \to Y \]

**Lemma 8.5.** Let \( \lambda \geq 0 \) and \( N \in \mathbb{N} \) be given. We then have that
\[ \sum_{n=0}^{N} X^n_\lambda \cdot \Delta^3 \cdot R_\lambda = \sum_{n=0}^{N} \Delta^3 \lambda^n R^{n+1}_\lambda - \sum_{n=0}^{N-1} K^n \cdot L_\lambda(N - n) \cdot R^{n+1}_\lambda \lambda^n \]

**Proof.** By an application of Lemma 8.4 (and a reordering of terms) we obtain that
\[
\sum_{n=0}^{N} X^n_\lambda \cdot \Delta^3 \cdot R_\lambda = \sum_{n=0}^{N} \Delta^3 \lambda^n R^{n+1}_\lambda - \sum_{n=0}^{N-1} \sum_{j=0}^{n-1} K^n \cdot I((X^n_\lambda \Delta^3) R^{j+2}_\lambda) \lambda^n \\
= \sum_{n=0}^{N} \Delta^3 \lambda^n R^{n+1}_\lambda - \sum_{j=0}^{N-1} \sum_{m=1}^{N-j} K^n \cdot I((X^m_\lambda \Delta^3) R^{j+2}_\lambda) \lambda^n \\
\]
The result of the lemma now follows by noting that
\[ \sum_{m=1}^{N-j} X^m_\lambda = \sum_{m=0}^{N-j-1} X^m_\lambda R_\lambda K \lambda = (1 - X^{N-j}_\lambda) S_\lambda K \lambda \]

For each \( \lambda \geq 0 \) we define the bounded adjointable operator
\[ L_\lambda := I(S_\lambda K \lambda \Delta^3) R_\lambda : Y \to Y \]

**Lemma 8.6.** Let \( \lambda \geq 0 \) be given. Then the sequence \( \{L_\lambda(m)\}_{m=1}^{\infty} \) converges to \( L_\lambda : Y \to Y \) in operator norm.
Proof. Using the Leibniz rule we see that it suffices to verify that the sequence \( \{ I(X^m)S_\lambda \}_{m=1}^\infty \) converges to zero in operator norm. However, using the Leibniz rule one more time, we obtain that

\[
I(X^m)S_\lambda = - \sum_{j=0}^{m-1} X^j I(R_\lambda \Delta^2) X^{m-1-j} S_\lambda \cdot \lambda/r
\]

The result of the lemma now follows easily by noting that \( \|X_\lambda\| \leq \lambda(1 + \lambda)^{-1} < 1 \). Indeed, we may then find a constant \( C > 0 \) such that

\[
\|I(X^m)S_\lambda\| \leq C \cdot m \cdot (\lambda(1 + \lambda)^{-1})^{m-1}
\]

for all \( m \in \mathbb{N} \).

We are now ready to prove the main result of this subsection. It provides an expansion of \( S_\lambda \Delta^3 : Y \to Y \) where the first power-series appearing can be directly related (after integration over the half-line) to the bounded adjointable operator \((1 + D^2)^{-1/2} : Y \to Y \). The exponent 3 that appears here (and earlier in this section) is not special, we will only need that it is large enough for certain estimates to carry through later on.

Proposition 8.7. Let \( \lambda \geq 0 \) be given. Then we have the identity

\[
S_\lambda \Delta^3 = \sum_{n=0}^{\infty} \Delta^3 K^n R_\lambda^{r+1} \lambda^n - \sum_{n=0}^{\infty} K^n L_\lambda R_\lambda^{r+1} \lambda^n - (1 - X_\lambda)^{-1} I(R_\lambda \Delta^3) R_\lambda
\]

where each of the sums converges absolutely in operator norm.

Proof. It is clear that the sums converge absolutely in operator norm. Indeed, this follows since \( \|K\| \leq 1 \) and since \( \|R_\lambda \cdot \lambda\| \leq \lambda(\lambda + 1)^{-1} < 1 \).

To continue, we notice that

\[
S_\lambda \Delta^3 = (1 - X_\lambda)^{-1} \Delta^3 R_\lambda + (1 - X_\lambda)^{-1} [R_\lambda, \Delta^3]
\]

\[
= (1 - X_\lambda)^{-1} \Delta^3 R_\lambda - (1 - X_\lambda)^{-1} I(R_\lambda \Delta^3) R_\lambda
\]

Now, by an application of Lemma 8.5, we see that we may restrict our attention to proving that the sequence

\[
\{ \sum_{j=0}^{N-1} K^j \cdot L_\lambda (N - j) \cdot R_\lambda^{j+1} \lambda^j \}_{N=1}^\infty
\]

converges in operator norm to \( \sum_{j=0}^{\infty} K^j L_\lambda R_\lambda^{j+1} \lambda^j \). To this end, we define

\[
C_0 := \sup_{n \in \mathbb{N}} \|L_\lambda(n)\| \quad \text{and} \quad C_1 := \sum_{j=0}^{\infty} \|R_\lambda^{j+1} \lambda^j\|
\]

Both of these constants are of course finite. Let now \( \epsilon > 0 \) be given. By Lemma 8.6 we may then choose \( N_0, M_0 \in \mathbb{N} \) such that

\[
\|L_\lambda - L_\lambda(n)\| < \frac{\epsilon}{3(C_1 + 1)} \quad \forall n \geq N_0 \quad \text{and} \quad \sum_{j=M_0}^{\infty} \|R_\lambda^{j+1} \lambda^j\| < \frac{\epsilon}{3(C_0 + 1)}
\]
It is then straightforward to verify that
\[
\left\| \sum_{j=0}^{\infty} K_j L R_j^{j+1} \lambda_j - \sum_{j=0}^{N-1} K_j L (N - j) R_j^{j+1} \lambda_j \right\| < \varepsilon
\]
for all \( N \geq N_0 + M_0 \). This proves the present proposition. \( \square \)

8.2. Integral formulae for the square root. The aim of this subsection is to compute the integral over the half line of the continuous map
\[
f : (0, \infty) \to \mathcal{L}(X) \quad f : \lambda \mapsto (\lambda r)^{-1/2} \sum_{n=0}^{\infty} \Delta^3 K^n R_{\lambda}^{n+1} \lambda^n
\]
which appears (up to a factor of \((\lambda r)^{-1/2}\)) in the expression for \( \Delta^3 S \Delta^3 : Y \to Y \) obtained in Proposition 8.7. The main result of this subsection is then the explicit formula
\[
\frac{1}{\pi} \int_0^\infty f(\lambda) \, d\lambda = \Delta^5 (1 + D^2)^{-1/2}
\]
which is proved in Proposition 8.13.

We start by recalling a general result on integral formulae for powers of resolvents:

**Lemma 8.8.** Let \( \Lambda : \mathcal{D}(\Lambda) \to Y \) be an unbounded selfadjoint regular operator and let \( p, q > 0 \). Then we have the identity
\[
B(p, q) \cdot (1 + \Lambda^2)^{-q} = \int_0^\infty \lambda^{p-1} (1 + \lambda + \Lambda^2)^{-p+q} \, d\lambda
\]
where the integral converges absolutely and where
\[
B(p, q) = \frac{\Gamma(p) \cdot \Gamma(q)}{\Gamma(p+q)} = \int_0^\infty \mu^{p-1} (1 + \mu)^{-p+q} \, d\mu
\]
is the beta function.

**Proof.** Notice that a change of variables \( \lambda = \mu \cdot t \) implies that
\[
\int_0^\infty \lambda^{p-1} (1 + \lambda + \Lambda^2)^{-p+q} \, d\lambda = t^{-q} \cdot \int_0^\infty \mu^{p-1} (1 + \mu)^{-p+q} \, d\mu
\]
for all \( t > 0 \). The result now follows by an application of the continuous functional calculus for unbounded selfadjoint regular operators, see [Wor91, WoNa92]. \( \square \)

Let us fix two elements \( \xi, \eta \in Y \) together with a state \( \rho : B \to \mathbb{C} \) on the base \( C^* \)-algebra. We will often apply the notation
\[
\langle y_0, y_1 \rangle_{\rho} := \rho(\langle y_0, y_1 \rangle) \quad y_0, y_1 \in Y
\]
for the localized inner product.

The next lemma reduces the computation of the integral \( \frac{1}{\pi} \int_0^\infty f(\lambda) \, d\lambda \) to a (delicate) matter of interchanging an infinite sum and an integral.

**Lemma 8.9.** The sequence of partial sums
\[
\left\{ \frac{1}{\pi} \sum_{n=0}^{N} \int_0^\infty (\lambda r)^{-1/2} \langle \Delta^2 K^n R_{\lambda}^{n+1} \lambda^n \xi, \eta \rangle_{\rho} \, d\lambda \right\}_{N=1}^{\infty}
\]
converges to \( \langle \Delta (1 + D^2)^{-1/2} \xi, \eta \rangle_{\rho} \).
Proof. By Lemma 8.8 we have that
\[
\frac{1}{\pi \sqrt{r}} \sum_{n=0}^{N} \int_{0}^{\infty} \lambda^{-1/2} \cdot \Delta^2 K^n \lambda^n R_{\lambda}^{n+1} d\lambda
\]
\[
= \frac{1}{\pi \sqrt{r}} \sum_{n=0}^{N} \Delta^2 K^n \cdot \int_{0}^{\infty} \lambda^{n-1/2} \cdot R_{\lambda}^{n+1} d\lambda
\]
\[
= \frac{1}{\pi \sqrt{r}} \sum_{n=0}^{N} \Delta^2 K^n \cdot B(n + 1/2, 1/2) \cdot (1 + D^2)^{-1/2}
\]
for all \(N \in \mathbb{N}\). Thus, it suffices to check that the sequence of complex numbers
\[
\left\{ \frac{1}{\pi \sqrt{r}} \sum_{n=0}^{N} \langle \Delta^2 K^n \xi, \eta \rangle \rho \cdot B(n + 1/2, 1/2) \right\}_{N=1}^{\infty}
\]
converges to \(\langle \Delta \xi, \eta \rangle \rho \in \mathbb{C}\).

Since \(\Delta^2 K^n \cdot B(n + 1/2, 1/2) \geq 0\) for all \(n \in \mathbb{N}_0\) we may assume (without loss of generality) that \(\xi = \eta\).

Let now \(\mu > 0\) be fixed and notice that
\[
\sum_{n=0}^{\infty} \Delta^2 K^n \cdot (1 + \mu)^{-n-1} = \Delta^2 (1 + \mu)^{-1} \cdot \sum_{n=0}^{\infty} (K(1 + \mu)^{-1})^n
\]
\[
= \Delta^2 (1 + \mu)^{-1} (1 - K(1 + \mu)^{-1})^{-1}
\]
\[
= \Delta^2 (1 - K + \mu)^{-1} = \Delta^2 (\Delta^2 / r + \mu)^{-1} = r(1 + \mu \Delta^{-2} r)^{-1}
\]
Next, by a change of variables \((\lambda = \mu \Delta^{-2} r)\), we obtain that
\[
\sqrt{r} \int_{0}^{\infty} \mu^{-1/2} \langle (1 + \mu \Delta^{-2} r)^{-1} \xi, \xi \rangle \rho d\mu = \frac{1}{\pi} \int_{0}^{\infty} \lambda^{-1/2}(1 + \lambda)^{-1} \langle \Delta \xi, \xi \rangle \rho d\lambda = \langle \Delta \xi, \xi \rangle \rho
\]
Therefore, by the Lebesgue monotone convergence theorem, we may conclude that
\[
\frac{1}{\pi \sqrt{r}} \cdot \lim_{N \to \infty} \left( \sum_{n=0}^{N} \langle \Delta^2 K^n \xi, \xi \rangle \rho \cdot B(n + 1/2, 1/2) \right)
\]
\[
= \frac{1}{\pi \sqrt{r}} \cdot \lim_{N \to \infty} \left( \int_{0}^{\infty} \mu^{-1/2} \sum_{n=0}^{N} \langle \Delta^2 K^n (1 + \mu)^{-n-1} \xi, \xi \rangle \rho d\mu \right)
\]
\[
= \sqrt{r} \int_{0}^{\infty} \mu^{-1/2} (1 + \mu \Delta^{-2} r)^{-1} \langle \xi, \xi \rangle \rho d\mu
\]
\[
= \langle \Delta \xi, \xi \rangle \rho
\]
This proves the lemma. \(\square\)

In order to compute the integral of \(f : (0, \infty) \to \mathcal{L}(X)\) (and to show that this function is integrable) we now want to apply the Lebesgue dominated convergence theorem. Or in other words we need to find a positive integrable function \(g : (0, \infty) \to [0, \infty)\) such that
\[
(\lambda r)^{-1/2} \left\| \sum_{n=0}^{N} \Delta^6 K^n R_{\lambda}^{n+1} \lambda^n \right\| \leq g(\lambda) \quad \text{for all } \lambda > 0, \ N \in \mathbb{N}
\]
This turns out to be a subtle problem and the solution will rely on the algebraic identities of Subsection 8.1 and the detailed estimates that we carry out in the appendix to this paper. On top of these estimates we will need the following two lemmas:

**Lemma 8.10.** Let $p \in [0, 2]$ be given. Then we have that
\[
\sum_{n=0}^{\infty} (1 + D^2)^p R^{2n+2}_\lambda \lambda^{2n} = (1 + D^2)^{p-1}(2\lambda + 1 + D^2)^{-1}
\]
where the sum converges absolutely in operator norm for all $\lambda \geq 0$.

**Proof.** It is clear the the sum converges absolutely for all $\lambda \geq 0$. To prove the relevant identity we let $\lambda \geq 0$ be given and compute as follows:
\[
\sum_{n=0}^{\infty} (1 + D^2)^p R^{2n+2}_\lambda \lambda^{2n} = (1 + D^2)^p R^2_\lambda (1 - R^2_\lambda \lambda^2)^{-1}
\]
\[
= (1 + D^2)^p (1 + D^2)^{-1}(2\lambda + 1 + D^2)^{-1}
\]
\[
\Box
\]

**Lemma 8.11.** The sequence of partial sums
\[
\left\{ \sum_{n=0}^{N} \langle (\Delta^2/r)^2 K^{2n} \eta, \eta \rangle \right\}_{N=0}^{\infty}
\]
is bounded in operator norm.

**Proof.** This follows from the identities
\[
\sum_{n=0}^{N} (\Delta^2/r) K^{2n} (2 - \Delta^2/r) = \sum_{n=0}^{N} (\Delta^2/r)(2 - \Delta^2/r)(1 - (\Delta^2/r)(2 - \Delta^2/r))^n
\]
\[
= 1 - (1 - \Delta^2/r)^{2(N+1)}
\]
by noting that $2 - \Delta^2/r : Y \rightarrow Y$ is invertible and that $\|1 - \Delta^2/r\| \leq 1$. \boxed{}

**Lemma 8.12.** There exists a positive integrable function $g : (0, \infty) \rightarrow [0, \infty)$ such that
\[
(\lambda r)^{-1/2}\| \sum_{n=0}^{N} \Delta^6 K^n R^{n+1}_\lambda \lambda^n \| \leq g(\lambda)
\]
for all $\lambda \in (0, \infty)$ and all $N \in \mathbb{N}$.

**Proof.** By an application of Lemma 8.5 we obtain that
\[
\sum_{n=0}^{N} \Delta^6 K^n R^{n+1}_\lambda \lambda^n = \sum_{n=0}^{N} \Delta^3 X^n R_\lambda \Delta^3 + \sum_{n=0}^{N} \Delta^3 X^n I(R_\lambda \Delta^3) R_\lambda
\]
\[
+ \sum_{n=0}^{N-1} \Delta^3 K^n L_\lambda (N-n) R^{n+1}_\lambda \lambda^n
\]
for all $\lambda \geq 0$ and all $N \in \mathbb{N}$. We estimate the operator norm of each of these terms separately.
For the first term in (8.1) we apply Lemma 11.3 to obtain that
\[ \left\| \sum_{n=0}^{N} \Delta^3 X_n^{\star} R_{\lambda} \Delta^3 \right\| \leq \left\| \Delta^3 S_{\lambda} \Delta^4 \right\| \leq 2r^3(1 + \lambda)^{-1} \]
for all \( \lambda \geq 0 \) and all \( N \in \mathbb{N} \).

For the second term in (8.1) we apply Lemma 11.3 and Lemma 11.1 to find a constant \( C_1 > 0 \) such that
\[ \left\| \Delta^3 \sum_{n=0}^{N} X_n^{\star} I(R_{\lambda} \Delta^3) R_{\lambda} \right\| \leq \left\| \Delta^3 (1 - X_{\lambda}^{N+1})(DS_{\lambda})^{\star} \cdot d(\Delta^3) R_{\lambda} \right\| + \left\| \Delta^3 (1 - X_{\lambda}^{N+1}) S_{\lambda} d(\Delta^3) D R_{\lambda} \right\| \]
\[ \leq C_1 \cdot (1 + \lambda)^{-3/4} \]
for all \( \lambda \geq 0 \) and all \( N \in \mathbb{N} \) (recall that \( d(\Delta^3) = \Delta^1/2 \rho(\Delta^3) \Delta^{1/2} \)).

For the third term in (8.1) we apply the Cauchy-Schwartz inequality to obtain that
\[ \left\| \sum_{n=0}^{N-1} \Delta^2 K^n L_{\lambda}(N - n) R_{\lambda}^{n+1} \lambda^n \right\| \leq \left\| \sum_{n=0}^{N-1} \Delta^2 K^n L_{\lambda}(N - n)(1 + D^2)^{-1/2} L_{\lambda}(N - n)^{\star} K^n \Delta^3 \right\|^{1/2} \]
\[ \cdot \left\| \sum_{n=0}^{N-1} (1 + D^2) R_{\lambda}^{2n+2} \lambda^{2n} \right\|^{1/2} \]
for all \( N \in \mathbb{N} \) and all \( \lambda \geq 0 \). This provides an adequate norm-estimate of the final term in (8.1) and the lemma is therefore proved. \[ \square \]

The main result of this subsection now follows by Lemma 8.9, Lemma 8.12, and the Lebesgue dominated convergence theorem:

**Proposition 8.13.** The continuous function
\[ f : (0, \infty) \to \mathcal{L}(X) \quad f(\lambda) := (\lambda r)^{-1/2} \sum_{n=0}^{\infty} \Delta^6 K^n R_{\lambda}^{n+1} \lambda^n \]
is absolutely integrable (with respect to Lebesgue measure on \([0, \infty)\) and the operator norm). Furthermore, the integral is given explicitly by
\[ \frac{1}{\pi} \int_{0}^{\infty} f(\lambda) \, d\lambda = \Delta^5(1 + D^2)^{-1/2} \]
8.3. Comparison with the bounded transform. We are now ready to prove the main theorem of this section. The interpretation of this result is that the bounded transform \( F_D \) has the same summability properties as the modular transform

\[
G_{D,\Delta} : \Delta(\xi) \mapsto \frac{1}{\pi} \int_0^\infty (\lambda r)^{-1/2} \Delta S_{\lambda} D \Delta(\xi) \, d\lambda \quad \xi \in \mathcal{D}(D)
\]

after multiplication from the left with a sufficiently large power of the modular operator. It is also appropriate to remark that the exponent \( p \in [0,1/2) \) appearing in the theorem below is the “best” exponent possible (a part from possibly the limit case \( p = 1/2 \)). Indeed, if we were interested in carrying out a more detailed analysis of summability properties in relation to the unbounded Kasparov product we would be able to show that, in the situation we consider, there is only an infinitesimal loss of summability. For the present study it does however largely suffice to limit ourselves to the question of compactness of resolvents and we will therefore (for the moment) not go into a deeper study of the decay properties of eigenvalues.

**Theorem 8.1.** Let \( p \in [0,1/2) \) be given. Then the difference of unbounded operators

\[
\Delta^5 F_D \cdot (1 + D^2)^p
\]

\[
- \frac{1}{\pi} \int_0^\infty (\lambda r)^{-1/2} \Delta^3(1 - X_{\lambda})^{-1} \Delta^3 R_{\lambda} \cdot D(1 + D^2)^p : \mathcal{D}(|D|^{2p+1}) \to Y
\]

has a bounded extension to \( Y \).

*Proof.* By an application of Proposition 8.7 and Proposition 8.13 we may focus our attention on proving that the unbounded operator

\[
\frac{1}{\pi} \int_0^\infty (\lambda r)^{-1/2} \sum_{n=0}^\infty \Delta^3 K^n L_{\lambda} R_{\lambda}^{n+1} \lambda^n \, d\lambda \cdot D(1 + D^2)^p : \mathcal{D}(|D|^{2p+1}) \to Y
\]

(8.3)

has a bounded extension to \( Y \).

To this end, we apply the Cauchy-Schwartz inequality to obtain that

\[
\left\| \sum_{n=0}^\infty \Delta^3 K^n L_{\lambda} D(1 + D^2)^p R_{\lambda}^{n+1} \lambda^n \right\|
\]

\[
\leq \sup_{N \in \mathbb{N}} \left\| \sum_{n=0}^N \Delta^3 K^n L_{\lambda} R_{\lambda}^{n+1} \lambda^n \right\|^{1/2} \left\| \sum_{n=0}^\infty D^2(1 + D^2)^p R_{\lambda}^{n+2} \lambda^n \right\|^{1/2}
\]

for all \( \lambda \geq 0 \). Next, by an application of Proposition 11.8 and Lemma 8.11 we may find a constant \( C_1 > 0 \) such that

\[
\sup_{N \in \mathbb{N}} \left\| \sum_{n=0}^N \Delta^3 K^n L_{\lambda} R_{\lambda}^{n+1} \lambda^n \right\|^{1/2} \leq C_1 \cdot (1 + \lambda)^{-1/2}
\]

Furthermore, by Lemma 8.10 we have that

\[
\left\| \sum_{n=0}^\infty D^2(1 + D^2)^p R_{\lambda}^{n+2} \lambda^n \right\|^{1/2} \leq \left\| (1 + D^2)^2p (2\lambda + 1 + D^2)^{-1} \right\|^{1/2} \leq (2\lambda + 1)^{p-1/2}
\]

These estimates imply that the integral

\[
\frac{1}{\pi} \int_0^\infty (\lambda r)^{-1/2} \sum_{n=0}^\infty \Delta^3 K^n L_{\lambda} \cdot D(1 + D^2)^p R_{\lambda}^{n+1} \lambda^n \, d\lambda
\]
converges absolutely in operator norm and the theorem is therefore proved. □

9. The Kasparov module of an unbounded modular cycle

Throughout this section we let $\mathcal{A}$ be a $*$-algebra which satisfies the conditions of Assumption 2.2. We will then consider a fixed unbounded modular cycle $(X, D, \Delta)$ from $\mathcal{A}$ to an arbitrary $C^*$-algebra $B$. As usual we will assume that $(X, D, \Delta)$ is either of even or odd parity and in the even case we will denote the $\mathbb{Z}/2\mathbb{Z}$-grading operator by $\gamma : X \to X$. We will apply the notation

$$F_D := D(1 + D^2)^{-1/2}$$

for the bounded transform of the unbounded selfadjoint and regular operator $D : \mathcal{D}(D) \to X$.

The aim of this section is to show that the pair $(X, F_D)$ is a bounded Kasparov module from $A$ to $B$ and hence that our unbounded modular cycle gives rise to a class in the $KK$-group, $KK_p(A, B)$ (where $p = 0, 1$ according to the parity of $(X, D, \Delta)$).

We will thus prove (see Theorem 9.1) that the following holds for all $a \in A$:

1. $\pi(a)(F_D^2 - \text{Id}_X) \in \mathcal{K}(X)$;
2. $\pi(a)(F_D - F_D^*) \in \mathcal{K}(X)$;
3. $[F_D, \pi(a)] \in \mathcal{K}(X)$;
4. $F_D\gamma = -\gamma F_D$ and $\pi(a)\gamma = \gamma \pi(a)$ in the even case.

For more information about $KK$-theory we refer the reader to the book by Blackadar, [BLA98].

The main difficulty is to show that the commutator condition (3) and it is to this end that we have introduced and studied the modular transform in Section 8. To explain why this was necessary we first recall the notation

$$S_\lambda := (\lambda \Delta^2/r + 1 + D^2)^{-1} : X \to X$$

where $r \in ([\|\Delta^2\|, \infty)$ is a fixed constant. The next lemma then presents the main algebraic reason for working with the modular resolvent $S_\lambda$ instead of the ordinary resolvent $R_\lambda = (\lambda + 1 + D^2)^{-1}$. Indeed, when the computation below is carried out with $R_\lambda$ in the place of $S_\lambda$ then the commutator $[\Delta^2, T]$ has to be replaced by the commutator $[(1 + \lambda)\Delta^2, T]$ and there is then no gain in the decay properties when the variable $\lambda$ tends to infinity. This makes the usual proof ([BAJu83]) of condition (3) from the above list fail utterly.

**Lemma 9.1.** Let $T : X \to X$ be differentiable with respect to $(X, D, \Delta)$ (as in Definition 3.3). We then have the identity

$$S_\lambda \Delta^2 T - T \Delta^2 S_\lambda = S_\lambda [\Delta^2, T] S_\lambda - (DS_\lambda)^* d_\Delta(T \Delta) S_\lambda - S_\lambda d_\Delta(\Delta T) DS_\lambda$$

for all $\lambda \geq 0$.

**Proof.** Let first $\xi \in \mathcal{D}(D^2)$ and notice that

$$S_\lambda \Delta^2 T D^2(\xi) - (D^2 S_\lambda)^* T \Delta^2(\xi)$$

$$= (DS_\lambda)^* T \Delta^2(\xi) - S_\lambda d_\Delta(\Delta T) D(\xi) - (D^2 S_\lambda)^* T \Delta^2(\xi)$$

$$= (DS_\lambda)^* T \Delta^2(\xi) - (DS_\lambda)^* d_\Delta(T \Delta)(\xi) - S_\lambda d_\Delta(\Delta T) D(\xi) - (D^2 S_\lambda)^* T \Delta^2(\xi)$$

$$= -(DS_\lambda)^* d_\Delta(T \Delta)(\xi) - S_\lambda d_\Delta(\Delta T) D(\xi)$$
The result of the lemma then follows since
\[ S_\lambda \Delta^2 T - T \Delta^2 S_\lambda = S_\lambda \Delta^2 (D^2 + 1 + \lambda \Delta^2/r) S_\lambda \]
\[ - S_\lambda (1 + \lambda \Delta^2/r) T \Delta^2 S_\lambda - (D^2 S_\lambda)^* T \Delta^2 S_\lambda \]
\[ = S_\lambda [\Delta^2, T] S_\lambda + S_\lambda \Delta^2 T D^2 S_\lambda - (D^2 S_\lambda)^* T \Delta^2 S_\lambda \]
\[ \square \]

In the next two lemmas we show that we may replace the bounded transform \( F_D \) (up to a compact perturbation) by the modular transform \( G_{D,\Delta} \) (in a slight disguise).

**Lemma 9.2.** Let \( T \in \mathcal{L}(X) \) and suppose that \((1 + D^2)^{-1} T \in \mathcal{K}(X) \) and that \((T \Delta)(\mathcal{D}(D)) \subseteq \mathcal{D}(D)\). Then the unbounded operator
\[ \Delta^5 F_D \cdot T - \frac{1}{\pi} D \cdot \int_0^\infty (\lambda r)^{-1/2} \cdot \Delta S_\lambda \Delta^5 d\lambda \cdot T : \Delta(\mathcal{D}(D)) \to X \]
has a compact extension to \( X \).

**Proof.** It follows by Theorem 8.1 that the difference
\[ \Delta^5 F_D T - \frac{1}{\pi} \int_0^\infty (\lambda r)^{-1/2} \cdot \Delta^3 (1 - X_\lambda)^{-1} \Delta^3 R_\lambda d\lambda \cdot DT : \Delta(\mathcal{D}(D)) \to X \]
has a compact extension to \( X \).

Furthermore, we notice that the difference
\[ \Delta^3 (1 - X_\lambda)^{-1} \Delta^3 D R_\lambda T - D \Delta S_\lambda \Delta^5 T : X \to X \]
is a compact operator for all \( \lambda \geq 0 \) (in fact each of the two terms is compact).

To prove the lemma, it therefore suffices to find a constant \( C > 0 \) such that
\[ \| \Delta^3 (1 - X_\lambda)^{-1} \Delta^3 D R_\lambda - D \Delta S_\lambda \Delta^5 \| \leq C \cdot (1 + \lambda)^{-3/4} \]
for all \( \lambda \geq 0 \). This amounts to providing operator norm estimates of the three bounded adjointable operators
\[ -d(\Delta^3 (1 - X_\lambda)^{-1} \Delta^3) R_\lambda \quad , \quad D \Delta^3 (1 - X_\lambda)^{-1} [\Delta^3, R_\lambda] \quad \text{and} \quad D \Delta [\Delta^2, S_\lambda] \Delta^3 : X \to X \]
This can be carried out by an application of the results in the Appendix (Subsection 11.1). The details are left to the careful reader. \( \square \)

**Lemma 9.3.** Let \( T \in \mathcal{L}(X) \) and suppose that \((1 + D^2)^{-1} T \in \mathcal{K}(X) \) and that \((T \Delta)(\mathcal{D}(D)) \subseteq \mathcal{D}(D)\). Then the unbounded operator
\[ F_D \Delta^5 T - \frac{1}{\pi} D \cdot \int_0^\infty (\lambda r)^{-1/2} \Delta^4 S_\lambda \Delta^2 d\lambda \cdot T : \Delta(\mathcal{D}(D)) \to X \]
has a compact extension to \( X \).

**Proof.** We start by noting that
\[ [F_D, \Delta^5] T \in \mathcal{K}(X) \]
Indeed, this follows by using the integral formula
\[ F_D = D \cdot \frac{1}{\pi} \int_0^\infty \lambda^{-1/2}(\lambda + 1 + D^2)^{-1} d\lambda \]
and the fact that $[D, \Delta] : \mathcal{D}(D) \to X$ has a bounded extension to $X$.

Now, by Lemma 9.2 we obtain that the difference of unbounded operators
\[
\Delta^5 F_D T - \frac{1}{\pi} D \cdot \int_0^\infty (\lambda r)^{-1/2} \Delta S_\lambda \Delta^5 d\lambda \cdot T : \Delta(\mathcal{D}(D)) \to X
\]
has a compact extension to $X$.

We then remark that the difference
\[
D \Delta S_\lambda \Delta^5 T - D \Delta^4 S_\lambda \Delta^2 T : X \to X
\]
is a compact operator (again we do in fact have that each of the two terms is compact).

To prove the lemma, it therefore suffices to find a constant $C > 0$ such that
\[
\|D\Delta[S_\lambda, \Delta^3]\Delta^2\| \leq C \cdot (1 + \lambda)^{-3/4}
\]
But this follows again by the techniques developed in the Appendix (Subsection 11.1) and the details are therefore not provided here.

\[\square\]

**Proposition 9.4.** Let $T_0, T_1 \in \mathcal{L}(X)$ and suppose that the following holds:

1. $T_0$ is differentiable with respect to $(X, D, \Delta)$ and $(1 + D^2)^{-1} T_0 \in \mathcal{K}(X)$.
2. $(1 + D^2)^{-1} T_1 \in \mathcal{K}(X)$ and $(T_1 \Delta)(\mathcal{D}(D)) \subseteq \mathcal{D}(D)$.

Then the bounded adjointable operator $[F_D, \Delta^5 T_0 \Delta^5]T_1 : X \to X$ is compact.

**Proof.** By Lemma 9.2 and Lemma 9.3 it suffices to show that the difference
\[
\frac{1}{\pi} D \cdot \int_0^\infty (\lambda r)^{-1/2} \Delta^4 S_\lambda \Delta^2 d\lambda \cdot T_0 \Delta^5 T_1
\]
\[
- \frac{1}{\pi} \Delta^5 T_0 D \cdot \int_0^\infty (\lambda r)^{-1/2} \Delta S_\lambda \Delta^5 d\lambda \cdot T_1 : \Delta(\mathcal{D}(D)) \to X
\]
To this end, we notice that
\[
K_\lambda := D \Delta^4 S_\lambda \Delta^2 T_0 \Delta^5 T_1 - \Delta^5 T_0 D \Delta S_\lambda \Delta^5 T_1 : X \to X
\]
is compact for all $\lambda \geq 0$. In order to prove the proposition, it therefore suffices to find a constant $C > 0$ such that
\[
\|K_\lambda\| \leq C \cdot (1 + \lambda)^{-3/4}
\]
for all $\lambda \geq 0$. To show that this is indeed possible, we notice that
\[
D \Delta^4 S_\lambda \Delta^2 T_0 \Delta^5 - \Delta^5 T_0 D \Delta S_\lambda \Delta^5
\]
\[
= D \Delta^4 (S_\lambda \Delta^2 T_0 - T_0 \Delta^2 S_\lambda) \Delta^5 + (D \Delta^4 T_0 \Delta - \Delta^5 T_0 D) \Delta S_\lambda \Delta^5
\]
The relevant estimate then follows by Lemma 9.1 and the results in Subsection 11.1. \[\square\]

**Theorem 9.1.** Let $(X, D, \Delta)$ be an unbounded modular cycle from $\mathcal{A}$ to the $C^*$-algebra $B$ (with grading operator $\gamma : X \to X$ in the even case). Then the bounded transform $(X, D(1 + D^2)^{-1/2})$ is a bounded Kasparov module from the $C^*$-algebra $A$ to the $C^*$-algebra $B$ of the same parity as $(X, D, \Delta)$ and with grading operator $\gamma : X \to X$ in the even case.
Proof. The only non-trivial issue is the compactness of the commutator $[F_D, \pi(a)] : X \to X$ for all $a \in A$. However, it already follows by Proposition 9.4 that $[F_D, \Delta \pi(a) \Delta^\delta \pi(b) : X \to X$ is compact for all $a,b \in \mathcal{A}$. Using the density of $\mathcal{A}$ in $\mathcal{A}$ and the fact that $\Delta(1/n + \Delta)^{-1} \pi(a) \to \pi(a)$ in operator norm for all $a \in \mathcal{A}$ we obtain that $[F_D, \pi(a)] \pi(b) \in \mathcal{K}(X)$ for all $a,b \in A$. It then follows that $[F_D, \pi(a)] \in \mathcal{K}(X)$ for all $a \in A$ by a standard trick in $KK$-theory. \hfill \Box

Remark 9.5. There is a much easier proof of Theorem 9.1 in the case where the unbounded modular cycle is Lipschitz regular thus when the twisted commutator $[D] \pi(a) \Delta - \Delta \pi(a) [D] : \mathcal{Q}(D) \to X$ has a bounded extension for all $a \in \mathcal{A}$. Indeed, it is then possible to follow [CoMo08, Proposition 3.2] more or less to the letter. It is however highly unclear whether the condition of Lipschitz regularity is compatible with the unbounded Kasparov product construction given in Section 7. In fact, to our knowledge, this problem is not even decided in the case of the passage from $D$ to $a D g$ (see Remark 3.2 and [CoMo08, Section 2.2]). We have therefore in this text chosen to avoid the extra Lipschitz regularity condition altogether.

10. Relation to the bounded Kasparov product

Throughout this section we let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras which satisfy the conditions in Assumption 2.2.

We will consider an unbounded modular cycle $(Y, D, \Gamma)$ from $\mathcal{B}$ to an auxiliary $C^*$-algebra $C$. The parity of $(Y, D, \Gamma)$ is denoted by $p \in \{0,1\}$. Furthermore, we let $X$ be a differentiable Hilbert $C^*$-module from $\mathcal{A}$ to $\mathcal{B}$ with differentiable generating sequence $\{\xi_n\}_{n=1}^\infty$. We will finally suppose that the $*$-homomorphism $\pi_A : A \to \mathcal{L}(X)$ factorizes through the compact operators $\mathcal{K}(X) \subseteq \mathcal{L}(X)$.

As a consequence of Theorem 7.1 we then obtain that the triple

$$(X \hat{\otimes}_B Y, (1 \otimes D)\Delta, \Delta)$$

is an unbounded modular cycle from $\mathcal{A}$ to $C$ of the same parity as $(Y, D, \Gamma)$. Thus, by an application of Theorem 9.1 we obtain a bounded Kasparov module

$$\left(X \hat{\otimes}_B Y, (1 \otimes D)\Delta \cdot \left(1 + (1 \otimes D)^2\right)^{-1/2}\right)$$

from $A$ to $C$ and hence a class $[F_\Delta]$ in the $KK$-group $KK_p(A, C)$.

On the other hand, since $\pi_A(a) \in \mathcal{K}(X)$ for all $a \in A$, our differentiable Hilbert $C^*$-module $X$ defines an even bounded Kasparov module $(X, 0)$ from $A$ to $B$, and hence a class $[X]$ in the even $KK$-group $KK_0(A, B)$. The grading operator is here just the identity operator on $X$. On top of this, we know from Theorem 9.1 that our original unbounded modular cycle $(Y, D, \Gamma)$ yields a bounded Kasparov module

$$(Y, D(1 + D^2)^{-1/2})$$

from $B$ to $C$ and therefore we also have a class $[F]$ in the $KK$-group $KK_p(B, C)$.

Under the condition that $A$ is separable and $B$ is $\sigma$-unital, we prove in this final section that the identity

$$[F_\Delta] = [X] \hat{\otimes}_B [F] \quad (10.1)$$

holds inside the $KK$-group $KK_p(A, C)$, where

$$\hat{\otimes}_B : KK_0(A, B) \times KK_p(B, C) \to KK_p(A, C)$$

denotes the interior Kasparov product in $KK$-theory.
To ease the notation, we define
\[ F_\Delta := (1 \otimes D) \cdot (1 + (1 \otimes D)^2)^{-1/2} \in \mathcal{L}(X \hat{\otimes} B Y) \quad \text{and} \quad F := D(1 + D^2)^{-1/2} \in \mathcal{L}(Y) \]

For the rest of this paper we will assume that the \( C^* \)-algebra \( A \) is separable and that the \( C^* \)-algebra \( B \) has a countable approximate identity (thus that \( B \) is \( \sigma \)-unital).

**Remark 10.1.** We would like to emphasize that even though the interior Kasparov product in \( KK \)-theory is only constructed under the assumption that \( A \) is separable and \( B \) is \( \sigma \)-unital we do not rely on these assumptions for the construction of the unbounded Kasparov product. The bounded Kasparov module \((X \hat{\otimes} B Y, F_\Delta)\) therefore exists regardless of these assumptions on the \( C^* \)-algebras \( A \) and \( B \).

Due to a result of Connes and Skandalis we may focus on proving that \( F_\Delta \) is an \( F \)-connection, [CoSk84, Theorem A.3]. Or in other words, if we can show that
\[ FT^*_\xi - T^*_\xi F_\Delta \in \mathcal{K}(X \hat{\otimes} B Y, \ell^2(Y)) \quad (10.2) \]
for all \( \xi \in X \) we may conclude that the identity in (10.1) holds. We recall here that \( T^*_\xi : X \hat{\otimes} B Y \to Y \quad T^*_\xi : x \otimes_B y \mapsto \pi_B(\langle \xi, x \rangle)(y) \) for all \( x \in X, y \in Y \).

**Remark 10.2.** In the work of Kucerovsky, [Kuc97, Theorem 13], conditions are given for recognizing unbounded representatives for the interior Kasparov product. These conditions can not be applied in our setting since our unbounded cycles are not unbounded Kasparov modules in the sense of [BAJu83]. Indeed, the main difference is that we are considering a twisted commutator condition (see Definition 3.1) instead of the straight commutator condition applied in [BAJu83].

We start by replacing the connection condition in (10.2) by something more manageable. Let us recall that \( \Phi : X \hat{\otimes} B Y \to \ell^2(Y) \) is defined by \( \Phi : x \otimes_B y \mapsto \sum_{n=1}^\infty \pi_B(\langle \xi_n, x \rangle)(y)\delta_n \) for all \( x \in X, y \in Y \). Furthermore, we have that \( \Delta := \Phi^*(1 \otimes \Gamma) \Phi : X \hat{\otimes} B Y \to X \hat{\otimes} B Y \).

**Lemma 10.3.**
\[ (i + (1 \otimes D))^{-1} \Delta \in \mathcal{K}(X \hat{\otimes} B Y) \]

**Proof.** This follows by Proposition 6.2 since \( \Phi \Phi^* \in \text{Im}(K(\pi_B) : K(B) \to \mathcal{L}(\ell^2(Y))) \), see also Proposition 3.5. \( \square \)

**Lemma 10.4.** Suppose that there exists a \( k \in \mathbb{N} \) such that
\[ (1 \otimes FT^k)\Phi \Delta^k - (1 \otimes \Gamma^k)\Phi \Delta^k F_\Delta \in \mathcal{K}(X \hat{\otimes} B Y, \ell^2(Y)) \]
then we have that
\[ FT^*_\xi - T^*_\xi F_\Delta \in \mathcal{K}(X \hat{\otimes} B Y, Y) \]
for all \( \xi \in X \).

**Proof.** We first show that
\[ (1 \otimes FT^k)\Phi - (1 \otimes \Gamma^k)\Phi F_\Delta \in \mathcal{K}(X \hat{\otimes} B Y, \ell^2(Y)) \quad (10.3) \]
To this end, we notice that
\[
(1 \otimes FT^k) \Phi \Delta^k(\Delta^k + 1/n)^{-1} - (1 \otimes \Gamma^k) \Phi \Delta^k(\Delta^k + 1/n)^{-1} F_\Delta
\]
\[
= (1 \otimes FT^k) \Phi \Delta^k(\Delta^k + 1/n)^{-1} - (1 \otimes \Gamma^k) \Phi \Delta^k F_\Delta(\Delta^k + 1/n)^{-1}
\]
\[
- (1 \otimes \Gamma^k) \Phi \Delta^k(\Delta^k + 1/n)^{-1} [F_\Delta, \Delta^k](\Delta^k + 1/n)^{-1}
\]}
\[
\in \mathcal{K}(X \hat{\otimes}_B Y, \ell^2(Y))
\]
for all \( n \in \mathbb{N} \). Indeed, this is a consequence of the assumptions of the present lemma and the fact that \( \Delta[F_\Delta, \Delta] : X \hat{\otimes}_B Y \to X \hat{\otimes}_B Y \) is compact (this last assertion follows by Lemma 10.3 and Proposition 9.4). The inclusion in (10.3) then follows by noting that the sequence \( \{ (1 \otimes \Gamma)^{1/2} \Phi \Delta^k(\Delta^k + 1/n)^{-1} \}_{n=1}^\infty \) converges to \( (1 \otimes \Gamma)^{1/2} \Phi : X \hat{\otimes}_B Y \to \ell^2(Y) \) in operator norm.

Our next step is to show that
\[
\Phi F_\Delta - (1 \otimes F) \Phi \in \mathcal{K}(X \hat{\otimes}_B Y, \ell^2(Y)) \tag{10.4}
\]
In this respect, we remark that
\[
(1 \otimes FT^k(\Gamma^k + 1/n)^{-1}) \Phi - (1 \otimes \Gamma^k(\Gamma^k + 1/n)^{-1}) \Phi F_\Delta
\]
\[
= (1 \otimes (\Gamma^k + 1/n)^{-1} FT^k) \Phi - (1 \otimes \Gamma^k(\Gamma^k + 1/n)^{-1}) \Phi F_\Delta
\]
\[
- (1 \otimes (\Gamma^k + 1/n)^{-1} [F, \Gamma^k]\Gamma^k(\Gamma^k + 1/n)^{-1}) \Phi \in \mathcal{K}(X \hat{\otimes}_B Y, \ell^2(Y))
\]
for all \( n \in \mathbb{N} \). Indeed, this is a consequence of the inclusion in (10.3) and the fact that \( (1 \otimes [F, \Gamma^k]\Gamma^k(\Gamma^k + 1/n)^{-1}) \Phi \in \mathcal{K}(X \hat{\otimes}_B Y, \ell^2(Y)) \) (as above this last assertion follows by Proposition 9.4). The inclusion in (10.4) now follows since the sequence \( \{ (1 \otimes \Gamma^k(\Gamma^k + 1/n)^{-1}) \Phi \}_{n=1}^\infty \) converges to \( \Phi : X \hat{\otimes}_B Y \to \ell^2(Y) \) in the operator norm.

By the definition of \( \Phi : X \hat{\otimes}_B Y \to \ell^2(Y) \) we see from (10.4) that
\[
T^*_{\xi_n} F_\Delta - FT^*_{\xi_n} \in \mathcal{K}(X \hat{\otimes}_B Y, Y) \tag{10.5}
\]
for all \( n \in \mathbb{N} \). Let now \( b \in B \) and \( n \in \mathbb{N} \) be given. We then have that
\[
T^*_{\xi_n,b} F_\Delta - FT^*_{\xi_n,b} = \pi_B(b^*) T^*_{\xi_n} F_\Delta - F \pi_B(b^*) T^*_{\xi_n}
\]
\[
= \pi_B(b^*) \left( T^*_{\xi_n} F_\Delta - FT^*_{\xi_n} \right) - [F, \pi_B(b^*)] T^*_{\xi_n}
\]
Thus, since \( (Y, F) \) is a bounded Kasparov module we deduce from (10.5) that
\[
T^*_{\xi_n,b} F_\Delta - FT^*_{\xi_n,b} \in \mathcal{K}(X \hat{\otimes}_B Y, Y) \tag{10.6}
\]
Since the sequence \( \{ \xi_n \}_{n=1}^\infty \) generates \( X \) as a Hilbert \( C^* \)-module over \( B \) we conclude from (10.6) that
\[
T^*_{\xi} F_\Delta - FT^*_{\xi} \in \mathcal{K}(X \hat{\otimes}_B Y, Y)
\]
for all \( \xi \in X \). This proves the lemma. \( \square \)

Let us apply the notation
\[
S_\lambda := (\lambda \Delta^2/r + 1 + (1 \otimes D)^2)_{\Delta}^{-1} \quad \text{and} \quad T_\lambda := (\lambda(1 \otimes \Gamma)^2/r + 1 + (1 \otimes D)^2)_{\Delta}^{-1}
\]
where \( r \in (\|\Delta\|^2 + \|\Gamma\|^2, \infty) \) is a fixed constant.

The next lemma relates these two modular resolvents to one another. We will in the following often put
\[
D := 1 \otimes D : \mathcal{D}(1 \otimes D) \to \ell^2(Y) \quad \text{and} \quad \Gamma := 1 \otimes \Gamma : \ell^2(Y) \to \ell^2(Y)
\]
Lemma 10.5. The difference
\[ \Gamma^5 \Phi \Delta^2 S_\lambda \Delta^4 \Delta \lambda - \Gamma^5 DT_\lambda \Gamma^2 \Phi \Delta^5 : \mathcal{D}(D\Phi) \to \ell^2(Y) \] (10.7)
extends to a compact operator \( K_\lambda : X \hat{\otimes}_B Y \to \ell^2(Y) \) and there exists a constant \( C > 0 \) such that
\[ \| K_\lambda \| \leq C \cdot (1 + \lambda)^{-3/4} \] (10.8)
for all \( \lambda \geq 0 \).

Proof. It is not hard to see that the difference in (10.7) has a compact extension \( K_\lambda : X \hat{\otimes}_B Y \to \ell^2(Y) \) for all \( \lambda \geq 0 \) (in fact we have that this holds for each of the two terms). We may thus focus our attention on providing the operator norm-estimate in (10.8).

Our first step in this direction is to notice that it is enough to consider the difference
\[ \Gamma^5 \Phi \Delta^2 S_\lambda \Delta^4 - \Gamma^5 DT_\lambda \Gamma^2 \Phi \Delta^5 : \mathcal{D}(D\Phi) \to \ell^2(Y) \]
of unbounded operators. This follows since we may dominate the operator norm (uniformly in \( \lambda \geq 0 \)) of each of the bounded adjointable operators
\[ \Gamma^5 \Phi \Delta^2 S_\lambda \Delta^4 - \Gamma^5 DT_\lambda \Gamma^2 \Phi \Delta^5 : X \hat{\otimes}_B Y \to \ell^2(Y) \]
by \( C_0 \cdot (1 + \lambda)^{-3/4} \) for some constant \( C_0 > 0 \). To see that this is indeed the case it suffices to apply the elementary estimates in the Appendix (Subsection 11.1).

Our next step is to define the unbounded operator
\[
M_\lambda := \Gamma^3 T_\lambda \left( \Phi \Delta^2 - \Gamma^4 + \Gamma^2 d_\Gamma(G) \Phi D_\Delta \right) S_\lambda D_\Delta \\
+ \Gamma^3 \left( (DT_\lambda)^\ast (d_\Gamma(G) \Gamma T_\lambda \Phi + \Gamma Gd_\Gamma(G) \Phi) S_\lambda D_\Delta \right) : \mathcal{D}(D_\Delta) \to \ell^2(Y)
\]
(where we recall the notation \( G := \Phi \Phi^* : \ell^2(Y) \to \ell^2(Y) \)). It then follows by the estimates in the Appendix (Subsection 11.1) that there exists a constant \( C_1 > 0 \) such that
\[ \| M_\lambda(\xi) \| \leq C_1 (1 + \lambda)^{-3/4} \cdot \| \xi \| \] (10.9)
for all \( \lambda \geq 0 \) and all \( \xi \in \mathcal{D}(D_\Delta) \subseteq X \hat{\otimes}_B Y \). Furthermore, by Proposition 6.5 we have that
\[ \Gamma^3 (\Phi \Delta^2 S_\lambda - T_\lambda \Gamma^2 \Phi) D_\Delta = M_\lambda \]
for all \( \lambda \geq 0 \). In order to provide the relevant estimate on \( K_\lambda : X \hat{\otimes}_B Y \to \ell^2(Y) \) it therefore suffices to analyze the difference
\[ \Gamma^5 T_\lambda \Gamma^2 \Phi D_\Delta \Delta^4 - \Gamma^5 DT_\lambda \Gamma \Phi \Delta^5 : \mathcal{D}(D\Phi) \to \ell^2(Y) \]
of unbounded operators.

However, we have that
\[ T_\lambda \Gamma^2 \Phi D_\Delta (\xi) - DT_\lambda \Gamma \Phi \Delta (\xi) = -d(T_\lambda \Gamma) \Phi \Delta (\xi) - T_\lambda \Gamma d_\Gamma(G) \Phi (\xi) \]
for all \( \xi \in \mathcal{D}(D\Phi) \) and the result of the lemma therefore follows by one more application of the operator norm estimates in the Appendix (Subsection 11.1). \( \square \)
Lemma 10.6. The unbounded operator
\[ \int_0^\infty (\lambda r)^{-1/2} \cdot \Gamma^5 \Phi \Delta^2 S_\lambda \Delta^4 d\lambda \cdot D_\Delta - D \cdot \int_0^\infty (\lambda r)^{-1/2} \cdot \Gamma^4 T_\lambda \Gamma^2 d\lambda \cdot \Phi \Delta^5 \]
\[ : \mathcal{D}(D\Phi) \to \ell^2(Y) \]
is the restriction of an operator in \( \mathcal{K}(X \hat{\otimes} B Y, \ell^2(Y)) \).

Proof. This follows in a straightforward way by an application of Lemma 10.5. \( \square \)

We are now ready to prove our final main theorem:

Theorem 10.1. The bounded adjointable operator \( F_\Delta : X \hat{\otimes} B Y \to X \hat{\otimes} B Y \) is an \( F \)-connection. In particular, we have the identity
\[ [F_\Delta] = [X] \hat{\otimes} [F] \]
inside the \( KK \)-group \( KK_p(A, C) \).

Proof. By Lemma 10.4, we only need to show that
\[ (1 \otimes F T^5) \Phi \Delta^5 - (1 \otimes \Gamma^5) \Phi \Delta^5 F_\Delta \in \mathcal{K}(X \hat{\otimes} B Y, \ell^2(Y)) \]
However, by Lemma 9.3 we see that it suffices to check that the difference
\[ (1 \otimes D) \cdot \int_0^\infty (\lambda r)^{-1/2} (1 \otimes \Gamma^4) T_\lambda (1 \otimes \Gamma^2) d\lambda \cdot \Phi \Delta^5 \]
\[ - \int_0^\infty (\lambda r)^{-1/2} (1 \otimes \Gamma^5) \Phi \Delta^2 S_\lambda \Delta^4 d\lambda \cdot (1 \otimes D) \Delta \]
\[ : \mathcal{D}((1 \otimes D)\Phi) \to \ell^2(Y) \]
is the restriction of an element in \( \mathcal{K}(X \hat{\otimes} B Y, \ell^2(Y)) \). But this is a consequence of Lemma 10.6. \( \square \)

11. Appendix: Norm estimates of error terms

In this appendix we have collected various operator norm estimates needed in the treatment of the modular transform (Section 8) and for the comparison result between the unbounded Kasparov product and the bounded Kasparov product (Section 10).

The general setting will be exactly as in Section 8 and the conditions in Assumption 8.1 will in particular be in effect. We recall the notation for a few bounded adjointable operators acting on the Hilbert \( C^* \)-module \( Y \):
\[ S_\lambda := (\lambda \Delta^2 / r + 1 + D^2)^{-1}, \quad R_\lambda := (\lambda + 1 + D^2)^{-1} \quad \text{and} \quad K := 1 - \Delta^2 / r, \quad X_\lambda := \lambda \cdot R_\lambda K \]
where \( r \in (\|\Delta\|^2, \infty) \) is a fixed constant and \( \lambda \geq 0 \) is variable.

11.1. Preliminary operator norm estimates. We start with a string of elementary operator norm estimates that will be needed throughout this appendix (and in many places in the main text as well).

Lemma 11.1. The unbounded operator \( S_\lambda^{1/2} D : \mathcal{D}(D) \to Y \) has a bounded adjointable extension \( \Omega_\lambda : Y \to Y \) and we have the operator norm estimate
\[ \|\Omega_\lambda\| \leq 1 \quad \text{for all } \lambda \geq 0 \]
Proof. Let $\lambda \geq 0$ be given. Consider the unbounded operator $E_\lambda : \text{Im}(S^{1/2}_\lambda) \to Y$ defined by $E_\lambda : S^{1/2}_\lambda \xi \mapsto DS_\lambda \xi$. It is then clear that $S^{1/2}_\lambda D \subseteq E_\lambda^*$. Furthermore, for each $\xi \in Y$ we have that
\[
\langle E_\lambda S^{1/2}_\lambda \xi, E_\lambda S^{1/2}_\lambda \xi \rangle = \langle S_\lambda (\lambda \Delta^2/r + 1 + D^2)S_\lambda \xi, \xi \rangle = \langle S^{1/2}_\lambda \xi, S^{1/2}_\lambda \xi \rangle
\]
It therefore follows that $E_\lambda : \text{Im}(S^{1/2}_\lambda) \to Y$ has a bounded extension to $Y$, $E_\lambda : Y \to Y$ and furthermore that $\|E_\lambda\| \leq 1$. But this implies that $E_\lambda^*$ is everywhere defined and that $E_\lambda^* = (E_\lambda)^*$. We may then conclude that $S^{1/2}_\lambda D = E_\lambda^*$ and that $\|S^{1/2}_\lambda D\| \leq 1$. This proves the lemma. □

**Lemma 11.2.** Let $\lambda \geq 0$ be given. We then have the identities
\[
D(\Delta S_\lambda - S_\lambda \Delta) = \Omega_\lambda^* \Omega_\lambda d(\Delta)S_\lambda + DS_\lambda d(\Delta)DS_\lambda
\]
and
\[
\Delta S_\lambda - S_\lambda \Delta = S^{1/2}_\lambda \Omega_\lambda d(\Delta)S_\lambda + S_\lambda d(\Delta)DS_\lambda
\]
Proof. We will only prove the first of these two identities. The second identity can be proved by a similar but easier argument.

Using that $\mathcal{D}(D^2) \subseteq Y$ is a core for $D : \mathcal{D}(D) \to Y$ it follows that
\[
\Omega_\lambda^* \Omega_\lambda D = D - DS_\lambda (\lambda \Delta^2/r + 1)
\]
on the common domain $\mathcal{D}(D) \subseteq Y$. The desired identity then follows by a direct computation. □

**Lemma 11.3.** Let $\lambda \geq 0$ be given. We then have the operator norm estimate
\[
\|\Delta S^{1/2}_\lambda\| \leq \frac{\sqrt{2r}}{\sqrt{1 + \lambda}}
\]
Proof. This follows by noting that
\[
0 \leq S^{1/2}_\lambda (\lambda + 1)(\Delta^2/r)S^{1/2}_\lambda \leq 2
\]
□

**Lemma 11.4.** There exists a constant $C > 0$ such that
\[
\|\Delta^2 S_\lambda \Delta^{1/2}\| \leq C \cdot (1 + \lambda)^{-1}
\]
for all $\lambda \geq 0$.

Proof. Using Lemma 11.2 we obtain that
\[
\Delta^2 S_\lambda \Delta^{1/2} = \Delta S_\lambda \Delta^{3/2} + \Delta \cdot S^{1/2}_\lambda \Omega_\lambda d(\Delta)S_\lambda \cdot \Delta^{1/2} + \Delta \cdot S_\lambda d(\Delta)DS_\lambda \cdot \Delta^{1/2}
\]
The desired estimate now follows by Lemma 11.3, Lemma 11.1, and the standing Assumption 8.1:
\[
d(\Delta) = \Delta^{1/2} \rho(\Delta) \Delta^{1/2}
\]
□

**Lemma 11.5.** There exists a constant $C > 0$ such that
\[
\|\Delta DS_\lambda \Delta^{1/2}\| \leq C \cdot (1 + \lambda)^{-1/2}
\]
for all $\lambda \geq 0$. 
Theorem 11.6. Let \( m \geq 2 \) be given. There exists a constant \( C > 0 \) such that
\[
\| DS_\lambda \Delta^m(i + D)^{-1} \| \leq C \cdot (1 + \lambda)^{-1/8}
\]
for all \( \lambda \geq 0 \).

Proof. Let \( \lambda \geq 0 \). We compute as follows:
\[
DS_\lambda(\Delta^m/r)(i + D)^{-1} = D\Delta^{m-2}(i + D)^{-1} - DS_\lambda\Delta^{m-2}(i + D)^{-1} - \Omega^*_\lambda \Omega_\lambda D\Delta^{m-2}(i + D)^{-1}
\]
Since \( D\Delta^{m-2}(i + D)^{-1} : Y \to Y \) is a bounded adjointable operator by Assumption 8.1, we obtain the relevant estimate by an application of Lemma 11.1. \( \square \)

Lemma 11.7. Let \( m \geq 3 \) be given. There exists a constant \( C > 0 \) such that
\[
\| S_{\lambda}^{3/2} \Delta^m(i + D)^{-1} \| \leq C \cdot (1 + \lambda)^{-1/8} \quad \text{and} \quad \| S_{\lambda}^{1/2}(1 - X^*_\lambda)^{-1} \Delta^m(i + D)^{-1} \| \leq C \cdot (1 + \lambda)^{-1/8}
\]
for all \( \lambda \geq 0 \).

Proof. To prove the first of the two estimates we apply Lemma 11.2 to obtain that
\[
S_{\lambda}^{3/2} \Delta^m(i + D)^{-1} = S_{\lambda}^{1/2} DS_\lambda \Delta^{m-1}(i + D)^{-1} - S_\lambda \Omega_\lambda d(\Delta) S_\lambda \Delta^{m-1}(i + D)^{-1} - S_{\lambda}^{3/2} d(\Delta) DS_\lambda \Delta^{m-1}(i + D)^{-1}
\]
After a consultation of Lemma 11.3 and Lemma 11.6 (together with Assumption 8.1) we then see that it suffices to find a constant \( C_1 > 0 \) such that
\[
\| S_{\lambda}^{1/2} \Omega_\lambda \Delta^{1/2} \| \leq C_1 \cdot (1 + \lambda)^{-1/8}
\]
But this follows by noting that \( \| S_{\lambda}^{1/2} \Omega_\lambda \Delta^{1/2} \|^2 = \| S_\lambda D\Delta DS_\lambda \| \) (see the proof of Lemma 11.5).

In order to prove the second of the two estimates, we remark that
\[
(1 - X^*_\lambda)^{-1} = \sum_{n=0}^{\infty} (X^*_\lambda)^n = 1 + \lambda KR_\lambda \sum_{n=0}^{\infty} (X^*_\lambda)^n = 1 + \lambda KS_\lambda
\]
The result then follows from the first estimate, which we already proved above. \( \square \)
11.2. **Norm-estimates of limit error terms.** Let us recall (from Subsection 8.1) that

\[ L_\lambda = I(S_\lambda K \lambda \Delta^3) R_\lambda : Y \to Y \]

for all \( \lambda \geq 0 \) (where \( I(\cdot) = [D^2, \cdot] \)).

**Proposition 11.8.** There exists a constant \( C > 0 \) such that

\[ \| \Delta^2 \cdot L_\lambda \| \leq C \cdot (1 + \lambda)^{-1/2} \]  \hspace{1cm} (11.1)

for all \( \lambda \geq 0 \).

**Proof.** For each \( \lambda \geq 0 \) we rewrite \( L_\lambda : Y \to Y \) in the following way:

\[
\begin{align*}
L_\lambda &= Dd(S_\lambda K \lambda \Delta^3) R_\lambda + d(S_\lambda K \lambda \Delta^3) DR_\lambda \\
&= -DS_\lambda d(\Delta^2/r) \lambda S_\lambda K \lambda \Delta^3 R_\lambda - DS_\lambda d(\Delta^5/r) \lambda R_\lambda \\
&\quad - S_\lambda d(\Delta^2/r) \lambda S_\lambda K \lambda \Delta^3 DR_\lambda - S_\lambda d(\Delta^5/r) \lambda DR_\lambda
\end{align*}
\]  \hspace{1cm} (11.2)

It is then not hard to see that the desired estimate follows by the results in Subsection 11.1. \( \square \)

11.3. **Operator norm estimates of truncated error terms.** Let us recall (again from Subsection 8.1) that

\[ L_\lambda(m) = I((1 - X_\lambda^m) S_\lambda K \lambda \Delta^3) R_\lambda : Y \to Y \]

for all \( \lambda \geq 0 \) and all \( m \in \mathbb{N} \).

**Proposition 11.9.** There exists a constant \( C > 0 \) such that

\[ \| \Delta^2 L_\lambda(m)(i + D)^{-1} \| \leq C \cdot (1 + \lambda)^{-1/8} \]

for all \( \lambda \geq 0 \) and all \( m \in \mathbb{N} \).

**Proof.** We first notice that

\[ L_\lambda(m) = (1 - X_\lambda^m) L_\lambda - I(X_\lambda^m) S_\lambda K \lambda \Delta^3 R_\lambda \]  \hspace{1cm} (11.3)

for all \( \lambda \geq 0 \) and all \( m \in \mathbb{N} \). We now estimate each of these two terms separately.

We begin with the easiest one: \( (1 - X_\lambda^m) L_\lambda : Y \to Y \). Using the identity in (11.2) we obtain that

\[
\begin{align*}
L_\lambda \cdot (i + D)^{-1}
&= -DS_\lambda d(\Delta^2/r) \lambda^2 S_\lambda K \lambda \Delta^3 R_\lambda(i + D)^{-1} - DS_\lambda d(\Delta^5/r) \lambda R_\lambda(i + D)^{-1} \\
&\quad - S_\lambda d(\Delta^2/r) \lambda^2 S_\lambda K \lambda \Delta^3 DR_\lambda(i + D)^{-1} - S_\lambda d(\Delta^5/r) \lambda DR_\lambda(i + D)^{-1}
\end{align*}
\]

It then follows by the results in Subsection 11.1 that there exists a constant \( C_0 > 0 \) such that

\[ \| L_\lambda \cdot (i + D)^{-1} \| \leq C_0 \cdot (1 + \lambda)^{-1/4} \]

for all \( \lambda \geq 0 \). Since \( \| 1 - X_\lambda^m \| \leq 2 \) for all \( \lambda \geq 0 \) and all \( m \in \mathbb{N} \) we obtain the relevant estimate for the first term in (11.3).

To take care of the second term in (11.3) we let \( l \geq 3 \) be given. It then suffices to estimate the norm of the operator

\[ \Delta^2 I(X_\lambda^m) S_\lambda \Delta^l(i + D)^{-1} : Y \to Y \]
uniformly in \( m \in \mathbb{N} \) and \( \lambda \geq 0 \). In order to achieve this goal we notice that

\[
I(X^m_\lambda)S_\lambda = \sum_{j=0}^{m-1} X^j_\lambda(Dd(X_\lambda) + d(X_\lambda)D)X^{m-1-j}_\lambda S_\lambda \\
= -\lambda \cdot \sum_{j=0}^{m-1} X^j_\lambda(DR_\lambda d(\Delta^2/r) + R_\lambda d(\Delta^2/r)D)X^{m-1-j}_\lambda S_\lambda
\]

We now define

\[
A_\lambda(m) := \sum_{j=0}^{m-1} \Delta^2 X^j_\lambda \cdot DR_\lambda d(\Delta^2) \cdot X^{m-1-j}_\lambda \Delta^l(i + D)^{-1} \quad \text{and}
\]

\[
B_\lambda(m) := \sum_{j=0}^{m-1} \Delta^2 X^j_\lambda \cdot R_\lambda d(\Delta^2)D \cdot X^{m-1-j}_\lambda \Delta^l(i + D)^{-1}
\]

and it follows that \( \Delta^2 I(X^m_\lambda)S_\lambda \Delta^l(i + D)^{-1} = -(\lambda/r) \cdot A_\lambda(m) - (\lambda/r) \cdot B_\lambda(m) : Y \to Y \) for all \( \lambda \geq 0 \) and all \( m \in \mathbb{N} \).

Our next step is to estimate the operator norm of each of the terms \( A_\lambda(m) : Y \to Y \) and \( B_\lambda(m) : Y \to Y \) uniformly in \( \lambda \geq 0 \) and \( m \in \mathbb{N} \). We start with \( A_\lambda(m) \). Using the Cauchy-Schwartz inequality we obtain that

\[
\|A_\lambda(m)\|^2 \leq \|\Delta^2 \sum_{j=0}^{m-1} X^j_\lambda DR_\lambda DR_\lambda(X^*_\lambda)^j \Delta^2\| \\
\cdot \|((-i + D)^{-1} \Delta^l S_\lambda \cdot \sum_{j=0}^{m-1} (X^*_\lambda)^j d(\Delta^2) d(\Delta^2) X^j_\lambda \cdot S_\lambda \Delta^l(i + D)^{-1}\| \\
\leq \|\Delta^2 S_\lambda \Delta^2\| \cdot \|d(\Delta^2)\|^2 \cdot \|((-i + D)^{-1} \Delta^l(1 - X_\lambda)^{-1} \\
\cdot \sum_{j=0}^{m-1} X^j_\lambda R_\lambda^2(1 - X^*_\lambda)^j \cdot (1 - X^*_\lambda)^{-1} \Delta^l(i + D)^{-1}\| \\
\leq \|\Delta^2 S_\lambda \Delta^2\| \cdot \|d(\Delta^2)\|^2 \cdot (1 + \lambda)^{-1} \\
\cdot \|((-i + D)^{-1} \Delta^l(1 - X_\lambda)^{-1} S_\lambda(1 - X^*_\lambda)^{-1} \Delta^l(i + D)^{-1}\|
\]

It then follows by Lemma 11.3 and Lemma 11.7 that there exists a constant \( C_1 > 0 \) such that \( \|A_\lambda(m)\| \leq C_1 \cdot (1 + \lambda)^{-1-1/8} \) for all \( m \in \mathbb{N} \) and all \( \lambda \geq 0 \).
We continue with $B_\lambda(m)$. Another application of the Cauchy-Schwartz inequality yields that
\[
\|B_\lambda(m)\|^2 \leq \left\| \Delta^2 \sum_{j=0}^{m-1} X_\lambda^j R_\lambda d(\Delta^2) d(\Delta^2) R_\lambda (X_\lambda^*)^j \Delta^2 \right\|
\]
\[
\cdot \left\| (-i + D)^{-1} \Delta^l S_\lambda \cdot \sum_{j=0}^{m-1} (X_\lambda^*)^j D^2 X_\lambda^j \cdot S_\lambda \Delta^l (i + D)^{-1} \right\|
\]
\[
\leq (1 + \lambda)^{-1} \cdot \|d(\Delta^2)\|^2 \cdot \|\Delta^2 S_\lambda \Delta^2\| \cdot \left\| (-i + D)^{-1} \Delta^l (1 - X_\lambda)^{-1} \right\|
\]
\[
\cdot \left\| \sum_{j=0}^{m-1} X_\lambda^j R_\lambda D^2 R_\lambda (X_\lambda^*)^j \cdot (1 - X_\lambda^*)^{-1} \Delta^l (i + D)^{-1} \right\|
\]
\[
\leq (1 + \lambda)^{-1} \cdot \|d(\Delta^2)\|^2 \cdot \|\Delta^2 S_\lambda \Delta^2\| \cdot \left\| (-i + D)^{-1} \Delta^l (1 - X_\lambda)^{-1} S_\lambda (1 - X_\lambda^*)^{-1} \Delta^l (i + D)^{-1} \right\|
\]
As a consequence of Lemma 11.3 and Lemma 11.7 we may then find a constant $C_2 > 0$ such that $\|B_\lambda(m)\| \leq C_2(1 + \lambda)^{-1-1/8}$ for all $m \in \mathbb{N}$ and all $\lambda \geq 0$. Combining these estimates we find that
\[
\|\Delta^2 I(X_\lambda^m) S_\lambda \Delta^l (i + D)^{-1}\| \leq (C_2/r + C_3/r) \cdot (1 + \lambda)^{-1/8}
\]
for all $m \in \mathbb{N}$ and all $\lambda \geq 0$. This ends the proof of the proposition. \hfill \square

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