SYMPLECTOMORPHISMS OF SURFACES PRESERVING A SMOOTH FUNCTION, I

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Abstract. Let $M$ be a compact orientable surface equipped with a volume form $\omega$, $P$ be either $\mathbb{R}$ or $S^1$, $f : M \to P$ be a $C^\infty$ Morse map, and $H$ be the Hamiltonian vector field of $f$ with respect to $\omega$. Let also $Z_\omega(f) \subset C^\infty(M, \mathbb{R})$ be set of all functions taking constant values along orbits of $H$, and $S_{id}(f, \omega)$ be the identity path component of the group of diffeomorphisms of $M$ mutually preserving $\omega$ and $f$.

We construct a canonical map $\varphi : Z_\omega(f) \to S_{id}(f, \omega)$ being a homeomorphism whenever $f$ has at least one saddle point, and an infinite cyclic covering otherwise. In particular, we obtain that $S_{id}(f, \omega)$ is either contractible or homotopy equivalent to the circle.

Similar results hold in fact for a larger class of maps $M \to P$ whose singularities are equivalent to homogeneous polynomials without multiple factors.

1. Introduction

Let $M$ be a closed oriented surface, $\text{Diff}(M)$ be the group of all $C^\infty$ diffeomorphisms of $M$, and $\text{Diff}_0(M)$ be the identity path component of $\text{Diff}(M)$ consisting of all diffeomorphisms isotopic to the identity.

Let also $\text{Vol}(M, 1)$ be the space of all volume forms on $M$ having volume 1 and $\omega \in \text{Vol}(M, 1)$. Since $\dim M = 2$, $\omega$ is a closed non-degenerate 2-form and so it defines a symplectic structure on $M$. Denote by $\text{Symp}(M, \omega)$ the group of all $\omega$-preserving $C^\infty$ diffeomorphisms, and let $\text{Symp}_0(M, \omega)$ be its identity path component.

Then Moser’s stability theorem [20] implies that for any $C^\infty$ family

$$\{\omega_t\}_{t \in D^n} \subset \text{Vol}(M, 1)$$

of volume forms parameterized by points of a closed $n$-dimensional disk $D^n$, there exists a $C^\infty$ family of diffeomorphisms

$$\{h_t\}_{t \in D^n} \subset \text{Diff}_0(M)$$

such that $\omega_t = h_t^*\omega$ for all $t \in D^n$. In particular, this implies that the map

$$p : \text{Diff}_0(M) \to \text{Vol}(M, 1), \quad p(h) = h^*\omega$$

is a Serre fibration with fiber $\text{Symp}_0(M, \omega)$, see e.g. [19, §3.2], [2], or [21, §7.2].

Since $\text{Vol}(M, 1)$ is convex and therefore contractible, it follows from exact sequence of homotopy groups of the Serre fibration $p$ that $p$ yields isomorphisms of the corresponding homotopy groups $\pi_k\text{Symp}_0(M, \omega) \cong \pi_k\text{Diff}_0(M)$, $k \geq 0$. Hence the inclusion

$$\text{Symp}_0(M, \omega) \subset \text{Diff}_0(M)$$

turns out to be a weak homotopy equivalence. See also [18] for discussions of the inclusion (1.1) for non-compact manifolds.

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Moreover, let $\text{Diff}^+(M)$ be the group of orientation preserving diffeomorphisms. Then we have an inclusion $i : \text{Symp}(M, \omega) \subset \text{Diff}^+(M)$. Indeed, if $h$ preserves $\omega$, then it fixes the corresponding cohomology class $[\omega] \in H^2(M, \mathbb{R}) \cong \mathbb{R}$, and so yields the identity on $H^2(M, \mathbb{R})$. In particular, $h$ preserves orientation of $M$. Hence (1.1) also implies that $i$ yields a monomorphism $i_0 : \pi_0 \text{Symp}(M, \omega) \to \pi_0 \text{Diff}^+(M)$ on the set of isotopy classes.

It is well known that $\pi_0 \text{Diff}^+(M)$ is generated by isotopy classes of Dehn twists, [4], [9], and one easily shows that each Dehn twist can be realized by $\omega$-preserving diffeomorphism. This implies that $i_0$ is also surjective, and so $i$ is a weak homotopy equivalence as well.

On the other hand, let $f : M \to \mathbb{R}$ be a Morse function,

$$\text{Stab}(f) = \{h \in \text{Diff}(M) \mid f \circ h = f\}$$

be the group of $f$-preserving diffeomorphisms, i.e. the stabilizer of $f$ with respect to the right action of $\text{Diff}(M)$ on $C^\infty(M, \mathbb{R})$, and $\text{Stab}_0(f)$ be its identity path component. Let also

$$\mathcal{O}(f) = \{f \circ h \mid h \in \text{Diff}(M)\}$$

be the corresponding orbit of $f$,

$$\text{Stab}(f, \omega) = \text{Stab}(f) \cap \text{Symp}(M, \omega)$$

be the group of diffeomorphisms mutually preserving $f$ and $\omega$, and $\text{Stab}_0(f, \omega)$ be its identity path component.

In a series of papers the author proved that $\text{Stab}_0(f)$ is either contractible or homotopy equivalent to the circle and computed the higher homotopy groups of $\mathcal{O}(f)$, [11], [15]; showed that $\mathcal{O}(f)$ is homotopy equivalent to a finite-dimensional CW-complex, [12]; and recently described precise algebraic structure of the fundamental group $\pi_1 \mathcal{O}(f)$, [17]. E. Kudryavtseva, [7], [8], studied the homotopy type of the space of Morse maps on compact surfaces and using similar ideas as in [11], [15] proved that $\mathcal{O}(f)$ has the homotopy type of a quotient of a torus by a free action of a certain finite group.

The present paper is former in a series subsequent ones devoted to extension of the above results to the case of $\omega$-preserving diffeomorphisms. We will describe here the homotopy type of $\text{Stab}_0(f, \omega)$. In next papers will study the homotopy type of the subgroup of $\text{Stab}(f, \omega)$ trivially acting on the Kronrod-Reeb graph of $f$, see §3.2, and describe the precise algebraic structure of $\pi_0 \text{Stab}(f, \omega)$.

Notice that if $H$ is the Hamiltonian vector field of $f$ and $H : M \times \mathbb{R} \to M$ is the corresponding Hamiltonian flow, then $H_t \in \text{Stab}(f, \omega)$ for all $t \in \mathbb{R}$.

More generally, given a $C^\infty$ function $\alpha : M \to \mathbb{R}$, one can define the map

$$H_\alpha : M \to M,$$,

being in general just a $C^\infty$ map leaving invariant each orbit of $H$, and so preserving $f$. However, $H_\alpha$ is not necessarily a diffeomorphism.

Let $Z(f) = \{\alpha \in C^\infty(M, \mathbb{R}) \mid H(\alpha) = 0\}$ be the algebra of all smooth functions taking constant values along orbits of $H$. Equivalently, $Z(f)$ is the centralizer of $f$ with respect to the Poisson bracket induced by $\omega$, see §2.3. In Lemma 3.2.1 we also identify $Z(f)$ with a certain subset of continuous functions on the Kronrod-Reeb graph of $f$. In particular, $Z(f)$ contains all constant functions.

We will prove in Theorem [3.0.3] that $H_\alpha \in \text{Stab}_0(f, \omega)$ if and only if $\alpha \in Z(f)$. Moreover if $f$ has at least one saddle critical point, then the correspondence $\alpha \mapsto H_\alpha$ is a homeomorphism.
\[ Z(f) \cong \text{Stab}_0(f, \omega) \] with respect to \( C^\infty \) topologies, and so \( \text{Stab}_0(f, \omega) \) is contractible. Otherwise, that correspondence is an infinite cyclic covering map and \( \text{Stab}_0(f, \omega) \) is homotopy equivalent to the circle. It will also follow that the inclusion
\[
\text{Stab}_0(f, \omega) \subset \text{Stab}(f)
\]
is a homotopy equivalence. This statement can be regarded as an analogue of (1.1) for \( f \)-preserving diffeomorphisms.

Again it implies that the inclusion
\[
j : \text{Stab}(f, \omega) \subset \text{Stab}^+(f) \equiv \text{Stab}(f) \cap \text{Diff}^+(M)
\]
yields an injection \( j_0 : \pi_0 \text{Stab}(f, \omega) \to \pi_0 \text{Stab}^+(f) \) on the sets of isotopy classes. However, now \( j_0 \) is not necessarily surjective, see §3.3. The reason is that \( \text{Stab}^+(f) \) has many invariant subsets, e.g. the sets of the form \( M_a = f^{-1}(-\infty, a], a \in \mathbb{R}, \) and so if \( h \in \text{Stab}(f, \omega) \) interchanges connected components of \( M_a, \) then they must have the same \( \omega \)-volume.

In fact, our results hold for a larger class of smooth maps \( f \) from \( M \) into \( \mathbb{R} \) and \( S^1, \) see §2.4. On the other hand, we also provide in §3.1 an example of a function with isolated critical points for which the above correspondence \( \alpha \mapsto H_\alpha \) is not surjective.

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2. Preliminaries

2.1. Shift map. Let \( M \) be a connected \( n \)-dimensional \( C^\infty \) manifold, \( H \) be a \( C^\infty \) vector field tangent to \( \partial M \) and generating a flow \( H : M \times \mathbb{R} \to M. \) For each \( \alpha \in C^\infty(M, \mathbb{R}) \) define the following \( C^\infty \) map \( H_\alpha : M \to M \) by
\[
H_\alpha(x) = H(x, \alpha(x)),
\]
for \( x \in M. \) Evidently, \( H_\alpha \) leaves invariant each orbit of \( H \) and is homotopic to id_\( M \) by the homotopy \( \{H_{t\alpha}\}_{t \in [0, 1]} \). Also notice that if \( \alpha \equiv t \) is a constant function, then \( H_\alpha = H_t \) is a diffeomorphism belonging to the flow \( H \).

For \( \alpha \in C^\infty(M, \mathbb{R}) \) we will denote by \( H(\alpha) \) the Lie derivative of \( \alpha \) along \( H \).

2.1.1. Lemma. [10, Theorem 19] Let \( \alpha \in C^\infty(M, \mathbb{R}), \ y \in M, \) and \( z = H_\alpha(y). \) Then the tangent map \( T_y H_\alpha : T_y M \to T_z M \) is an isomorphism if and only if \( 1 + H(\alpha)(y) \neq 0. \)

2.1.2. Remark. In fact, [10, Lemma 20], if \( \alpha(y) = 0, \) so \( z = H_\alpha(y) = H(y, 0) = y \) is a fixed point of \( H_\alpha, \) then the determinant of \( T_y H_\alpha : T_y M \to T_y M \) does not depend on a particular choice of local coordinates at \( z \) and equals \( 1 + H(\alpha)(y). \) The general case \( \alpha(y) = a \neq 0 \) reduces to \( a = 0 \) by observation that \( H_\alpha = H_{\alpha-a} \circ H_a. \)

To get a global variant of Lemma 2.1.1 notice that the correspondence \( \alpha \mapsto H_\alpha \) can also be regarded as the following mapping
\[
\varphi_H : C^\infty(M, \mathbb{R}) \to C^\infty(M, M), \qquad \varphi_H(\alpha) = H_\alpha.
\]
It will be called the shift map along orbits of \( H, [10], [16]. \) Consider the following subset of \( C^\infty(M, \mathbb{R}) : \)
\[
\Gamma_H = \{ \alpha \in C^\infty(M, \mathbb{R}) \mid 1 + H(\alpha) > 0 \},
\]
and let \( \text{Diff}_0(H) \) be the group of all diffeomorphisms of \( M \) which leave invariant each orbit of \( H \) and isotopic to the identity via an orbit preserving isotopy.
2.1.3. Lemma. [10] Theorem 19] If $M$ is compact, then
\begin{equation}
\varphi(\Gamma_H) \subset \text{Diff}_0(H),
\end{equation}
\begin{equation}
\Gamma_H = \varphi^{-1}(\text{Diff}_0(H)).
\end{equation}
In other words, suppose $\alpha \in C^\infty(M, \mathbb{R})$. Then $\alpha \in \Gamma$ if and only if $H_\alpha \in \text{Diff}_0(H)$.

2.2. Hamiltonian vector field. Let $M$ be a compact orientable surface equipped with a volume form $\omega$ and $P$ be either $\mathbb{R}$ or $S^1$. Since $\dim M = 2$, $\omega$ is a closed 2-form, and therefore it defines a symplectic structure on $M$. Then for each $C^1$ map $f : M \to P$ there exists a unique vector field $H$ on $M$ satisfying
\begin{equation}
df(z)(u) = \omega(u, H(z)),
\end{equation}
for each point $z \in M$ and a tangent vector $u \in T_zM$. This vector field is called the Hamiltonian vector field of $f$ with respect to $\omega$. For the convenience of the reader we recall its construction as it is usually defined for functions $f : M \to \mathbb{R}$ only.

Let $z \in M$. Fix local charts $h : U \to M$ and $q : J \to P$ at $z$ and $f(z)$ respectively, where $U$ is an open subset of the upper half-plane $\mathbb{R}^2_+ = \{(x, y) \mid y \geq 0\}$ and $J$ is an open interval in $\mathbb{R}$. Decreasing $U$ one can assume that $f(h(U)) \subset q(J)$. Then the map $\hat{f} = q^{-1} \circ f \circ h : U \to J$ is called a local representation of $f$ at $z$.

Now if in coordinates $(x, y)$ on $U$ we have that $\omega(x, y) = \gamma(x, y)dx \wedge dy$ for some non-zero $C^\infty$ function $\gamma : U \to \mathbb{R} \setminus \{0\}$, then
\begin{equation}
H(x, y) = \frac{1}{\gamma(x, y)}(-\hat{f}'y \frac{\partial}{\partial x} + \hat{f}'x \frac{\partial}{\partial y}).
\end{equation}

A definition of $H$ that does not use local coordinates can be given as follows. Since the restriction of $\omega$ to each tangent space $T_zM$ is a non-degenerate skew-symmetric form, it follows that $\omega$ yields a bundle isomorphism
\[\begin{array}{ccc}
TM & \xrightarrow{\psi} & T^*M \\
\downarrow & & \downarrow \\
M & & \\
\end{array}\]
defined by the formula $\psi(u)(v) = \omega(u, v)$ for all $u, v \in T_zM$ and $x \in M$.

Further notice, that the tangent bundle of $P$ is trivial, so we have the unit section
\[s : P \to TP \equiv P \times \mathbb{R}, \quad s(q) = (q, 1)\].

Now for a $C^1$ map $f : M \to P$ its differential $df : M \to T^*M$ and the Hamiltonian vector field $H : M \to TM$ are unique maps for which the following diagram is commutative:
\[\begin{array}{ccc}
TM & \xrightarrow{\psi} & T^*M \\
& \subset & \downarrow T^*f \\
& & T^*P \equiv P \times \mathbb{R} \\
\downarrow df & & \uparrow s \\
M & \xrightarrow{f} & P \\
& \subset & \\
\end{array}\]
Thus $df = T^*f \circ s \circ f$, and $H = \psi^{-1} \circ df$. It follows that
\begin{equation}
H(z)(f) = \omega(H(z), H(z)) = 0,
\end{equation}
as $\omega$ is skew-symmetric, and so $H$ is tangent to level curves of $f$.

Suppose, in addition, that $f$ takes constant values at boundary components of $M$. Then, due to $(2.5)$, $H$ is tangent to $\partial M$, and therefore it yields a flow $H : M \times \mathbb{R} \to M$. It also
follows from (2.5) that each diffeomorphism $H_t: M \to M$ preserves $f$, in the sense that $f \circ H_t = f$. Moreover, the well known Liouville’s theorem claims that each diffeomorphism $H_t$ also preserves $\omega$. In fact, that theorem is a simple consequence of Cartan’s identity:

\begin{equation}
\mathcal{L}_{H} \omega = d(\iota_{H} \omega) + \iota_{H} d\omega = d(df) + \iota_{H} 0 = 0,
\end{equation}

since $\iota_{H} \omega = \omega(H, \cdot) = df$ by (2.3), and $d\omega = 0$ as dim $\omega = $ dim $M$.

2.3. Poisson multiplication. Let $Q$ be another one-dimensional manifold without boundary, so $Q$ is either $\mathbb{R}$ or $S^1$ as well as $P$. Then $\omega$ yields a Poisson multiplication

\begin{equation}
\{ \cdot, \cdot \}: C^\infty(M,P) \times C^\infty(M,Q) \to C^\infty(M,\mathbb{R})
\end{equation}

defined by one of the following equivalent formulas:

\begin{equation}
\{ f, g \} := \omega(H_f, H_g) = \psi(H_f)(H_g) = H_f(g) = -H_g(f),
\end{equation}

where $H_f$ and $H_g$ are Hamiltonian vector fields of $f \in C^\infty(M,P)$ and $g \in C^\infty(M,Q)$ respectively.

In particular, for each $f \in C^\infty(M,P)$ one can define its annulator with respect to (2.8) by

\begin{equation}
\mathcal{Z}^Q(f) = \{ g \in C^\infty(M,Q) \mid H_f(g) = \{ f, g \} = 0 \}.
\end{equation}

Thus $\mathcal{Z}^Q(f)$ consists of all maps $g \in C^\infty(M,Q)$ taking constant values along orbits of the Hamiltonian vector field $H_f$. It follows from (2.8) that $g \in \mathcal{Z}^Q(f)$ iff $f \in \mathcal{Z}^P(g)$.

When $P = Q = \mathbb{R}$, this multiplication is the usual Poisson bracket, and $\mathcal{Z}^\mathbb{R}(f)$ is the centralizer of $f$, see [19, §3].

2.4. Class $\mathcal{F}(M,P)$. Let $\mathcal{F}(M,P)$ be the subspace of $C^\infty(M,P)$ consisting of maps $f$ satisfying the following two axioms:

Axiom (B) The map $f$ takes a constant value at each connected component of $\partial M$ and has no critical points on $\partial M$.

Axiom (L) For every critical point $z$ of $f$ there is a local presentation $\hat{f}: \mathbb{R}^2 \to \mathbb{R}$ of $f$ near $z$ in which $\hat{f}$ is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

In particular, since the polynomial $\pm x^2 \pm y^2$ (a non-degenerate singularity) is homogeneous and has no multiple factors, we see that $\mathcal{F}(M,P)$ contains an open and everywhere dense subset $\operatorname{Morse}(M,P)$ consisting of maps satisfying Axiom (B) and having non-degenerate critical points only.

Figure 2.1 describes possible singularities satisfying Axiom (L).

2.4.1. Definition. We will say that a vector field $F$ on $M$ is Hamiltonian like for $f \in \mathcal{F}(M,P)$ if

(a) $F(f) = 0$, and, in particular, $F$ is tangent to $\partial M$ and generates a flow on $M$;
(b) $F(z) = 0$ if and only if $z$ is a critical point of $f$;
(c) for each $z$ critical point of $f$ there exists a local representation $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ of $f$ as a homogeneous polynomial without multiple factors such that in these coordinates $F(x, y) = -\hat{f}y \frac{\partial}{\partial x} + \hat{f}x \frac{\partial}{\partial y}$.

One can easily prove that for each $f \in \mathcal{F}(M, P)$ there exists a Hamiltonian like vector field, $\mathbb{[11]} \text{Lemma 5.1].}$

Notice also that every Hamiltonian vector field $H$ of $f$ has properties $[a]$ and $[b]$ of Definition 2.4.1. Moreover, if $H$ is also a Hamiltonian like, then due to (2.4) in the corresponding coordinates satisfying property $[c]$ of Definition 2.4.1 we have that $\omega = dx \wedge dy$.

2.4.2. Lemma. Let $F$ be any Hamiltonian like vector field for $f \in \mathcal{F}(M, P)$, and $H$ be the Hamiltonian vector field for $f$ with respect to $\omega$. Then there exists an everywhere non-zero $C^\infty$ function $\lambda : M \to \mathbb{R} \setminus \{0\}$ such that $H = \lambda F$.

Proof. Denote by $\Sigma_f$ the set of critical point of $f$, being also the set of zeros of $H$ as well as of $F$. Since $F$ and $H$ are parallel and non-zero on $M \setminus \Sigma_f$, it follows that there exists a $C^\infty$ non-zero function $\lambda : M \setminus \Sigma_f \to \mathbb{R}$ such that $H = \lambda F$. It remains to show that $\lambda$ can be defined by non-zero values on $\Sigma_f$ to give a $C^\infty$ function on all of $M$.

Let $z$ be a critical point of $f$. Then by definition of Hamiltonian like vector field there exists a local representation $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ of $f$ such that $z = (0, 0) \in \mathbb{R}^2$, $\hat{f}$ is a homogeneous polynomial without multiple factors, and $F = -\hat{f}y \frac{\partial}{\partial x} + \hat{f}x \frac{\partial}{\partial y}$.

Then $\omega(x, y) = \gamma(x, y)dx \wedge dy$ for some non-zero $C^\infty$ function $\gamma$, and by formula (2.4), we have $H = \frac{1}{\gamma}F$ on $U \setminus z$. Hence $\lambda = 1/\gamma$, and so $\lambda$ smoothly extends to all of $U$ by $\lambda(z) = 1/\gamma(z)$. 

The following statement is a particular case of results of $\mathbb{[13]}$ on parameter rigidity.

2.4.3. Corollary. c.f. $\mathbb{[13]} \text{§4 & Theorem 11.1}]$ For any two Hamiltonian like vector fields $F_1$ and $F_2$ there exists an everywhere non-zero $C^\infty$ function $\mu : M \to \mathbb{R} \setminus \{0\}$ such that $F_1 = \mu F_2$.

Proof. It follows from Lemma 2.4.2 that $H = \lambda_1 F_1 = \lambda_2 F_2$ for some everywhere non-zero $C^\infty$ functions $\lambda_1, \lambda_2 : M \to \mathbb{R} \setminus \{0\}$. Hence $\mu = \lambda_2/\lambda_1$. 

2.5. Topological type of $\text{Stab}_0(f)$. Let $f \in \mathcal{F}(M, P)$, $H$ be a Hamiltonian like vector field for $f$, and $H : M \times \mathbb{R} \to M$ be the corresponding Hamiltonian flow.

2.5.1. Theorem. $\mathbb{[14, 15, 16]}$. Let $\varphi : C^\infty(M, \mathbb{R}) \to C^\infty(M, M)$ be the shift map along orbits of $H$ and 

$\Gamma = \{\alpha \in C^\infty(M, \mathbb{R}) \mid 1 + H(\alpha) > 0\}$, 

see $\mathbb{[2.1]}$.

1. $\varphi(\Gamma) = \text{Stab}_0(f)$ and $\Gamma = \varphi^{-1}(\text{Stab}_0(f))$.

2. Suppose all critical points of $f$ are non-degenerate local extremes, so, in particular, $f \in \text{Morse}(M, P)$. Then the restriction map $\varphi|_\Gamma : \Gamma \to \text{Stab}_0(f)$ is an infinite cyclic covering, and so $\text{Stab}_0(f)$ is homotopy equivalent to the circle. More precisely, in this case there exists $\theta \in \Gamma$ such that

(i) $\theta > 0$ on all of $M$;

(ii) each non-constant orbit $\gamma$ of $F$ is periodic, and $\theta$ takes a constant value on $\gamma$ being an positive integral multiple of the period $\text{Per}(\gamma)$ of $\gamma$.
(iii) there exists a free action of $\mathbb{Z}$ on $\Gamma$ defined by $n \ast \alpha = \alpha + n\theta$, for $n \in \mathbb{Z}$ and $\alpha \in \Gamma$, such that the map $\varphi$ is a composite

$$\varphi : \Gamma \overset{p}{\longrightarrow} \Gamma/\mathbb{Z} \overset{r}{\cong} \text{Stab}_0(f),$$

where $p$ is a projection onto the factor space $\Gamma/\mathbb{Z}$ endowed with the corresponding final topology, and $r$ is a homeomorphism.

(3) Suppose $f$ has a critical point being not a non-degenerate local extreme. Then $\varphi |_\Gamma : \Gamma \to \text{Stab}_0(f)$ is a homeomorphism, and so $\text{Stab}_0(f)$ is contractible.

Proof. In fact, Theorem 2.5.1 is stated and proved in [15] for any Hamiltonian like vector field $F$ of $f$. The advantage of using Hamiltonian like vector fields is that we have precise formulas for $F$ near critical points of $f$.

Let $\lambda : M \to \mathbb{R}$ be everywhere non-zero $C^\infty$ function and $H = \lambda F$. We will deduce from results of [14] that Theorem 2.5.1 also holds for $H$. Due to Lemma 2.4.2 this includes the case when $H$ is Hamiltonian.

Let $F, H : M \times \mathbb{R} \to M$ be the flows of $F$ and $H = \lambda F$ respectively,

$$\varphi_F, \varphi_H : C^\infty(M, \mathbb{R}) \to C^\infty(M, M)$$

be the corresponding shift maps, $\text{image}(\varphi_H), \text{image}(\varphi_F)$ be their images in $C^\infty(M, M)$, and $\Gamma_F, \Gamma_H$ be corresponding the subsets of $C^\infty(M, \mathbb{R})$ defined by (2.1). Define the following $C^\infty$ function

$$\sigma : M \times \mathbb{R} \to \mathbb{R}, \quad \sigma(x, s) = \int_0^s \lambda(H(x, t)) dt.$$ 

Then it is well known and easy to see, e.g. [14], that for each $\alpha \in C^\infty(M, \mathbb{R})$ we have that

$$H(x, \alpha(x)) = F(x, \sigma(x, \alpha(x))).$$

(2.11)

Consider the map

$$\gamma : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), \quad \gamma(\alpha)(x) = \sigma(x, \alpha(x)).$$

Evidently, $\gamma$ is continuous with respect to $C^\infty$ topologies. Moreover, (2.11) means that

$$H_\alpha = F_\gamma(\alpha)$$

for all $\alpha \in C^\infty(M, \mathbb{R})$. Hence $\varphi_H = \varphi_F \circ \gamma$, image$(\varphi_H) \subset \text{image}(\varphi_F)$, and we get the following commutative diagram:

$$\begin{array}{ccc}
\Gamma_H & \overset{\varphi_H}{\longrightarrow} & \text{image}(\varphi_H) \\
\gamma \downarrow & & \downarrow \\
\Gamma_F & \overset{\varphi_F}{\longrightarrow} & \text{image}(\varphi_F)
\end{array} \overset{\sim}{\longrightarrow} \begin{array}{ccc}
\Gamma_H & \overset{\varphi_H}{\longrightarrow} & \text{image}(\varphi_H) \\
\gamma \downarrow & & \downarrow \\
\Gamma_F & \overset{\varphi_F}{\longrightarrow} & \text{image}(\varphi_F)
\end{array} \overset{\sim}{\longrightarrow} C^\infty(M, M)
$$

Since $\lambda \neq 0$ everywhere, one can interchange $F = \frac{1}{\lambda} H$ and $H$. Hence by the same arguments as above we get that $\text{image}(\varphi_H) = \text{image}(\varphi_F)$ and $\gamma$ is a homeomorphism. Also notice that the orbit structures of $F$ and $H$ coincide. Hence $\text{Diff}_0(F) = \text{Diff}_0(H)$, and so

$$\begin{align*}
\Gamma_H & \overset{(2.2)}{\longrightarrow} \varphi_H^{-1}(\text{image}(\varphi_H) \cap \text{Diff}_0(H)) = \varphi_H^{-1}(\text{image}(\varphi_F) \cap \text{Diff}_0(F)) \\
& = \gamma^{-1} \circ \varphi_F^{-1}(\text{image}(\varphi_F) \cap \text{Diff}_0(F)) \overset{(2.2)}{\longrightarrow} \gamma^{-1}(\Gamma_F).
\end{align*}$$
Thus $\gamma$ yields a homeomorphism of $\Gamma_H$ onto $\Gamma_F$. Since Theorem 2.5.1 holds for $F$, we get the following commutative diagram

$$
\begin{array}{ccc}
\Gamma_H & \xrightarrow{\gamma} & \Gamma_F \\
\downarrow{\varphi_H} & & \downarrow{\varphi_F} \\
\text{Stab}_0(f) & & \text{Stab}_0(f)
\end{array}
$$

implying that $\varphi_H|_{\Gamma_H}$ has the same topological properties as $\varphi_F|_{\Gamma_F}$, and so Theorem 2.5.1 holds for $H$ as well. □

2.5.2. Remark. Let us discuss the case (2) of Theorem 2.5.1 which is realized precisely for the following four types of Morse maps, see [11, Theorem 1.9]:

(A) $M = S^2$ is a 2-sphere and $f : S^2 \to P$ has exactly two critical points: non-degenerate local minimum and maximum;

(B) $M = D^2$ is a 2-disk and $f : D^2 \to P$ has exactly one critical point being a non-degenerate local extreme;

(C) $M = S^1 \times [0, 1]$ is a cylinder and $f : S^1 \times [0, 1] \to P$ has no critical points;

(D) $M = T^2$ is a 2-torus, $P = S^1$ is a circle, and $f : T^2 \to P$ has no critical points.

Due to (i) and (ii) each regular point $x \in M$ of $f$ is periodic of some period $\text{Per}(x)$, and there exists $k_x \in \mathbb{N}$ depending on $x$ such that $\theta(x) = k_x \text{Per}(x)$. Hence

$$
H_{\theta}(x) = H(x, \theta(x)) = H(x, k_x \text{Per}(\gamma)) = x,
$$

and so $H_{\theta} = \text{id}_M$. Moreover, if $\alpha \in \Gamma$, then

$$
H_{\alpha+n\theta}(x) = H(x, \alpha(x) + n\theta(x)) = H(H(x, n\theta(x)), \alpha(x)) = H(x, \alpha(x)) = H_{\alpha}(x).
$$

This implies correctness of the $\mathbb{Z}$-action from (iii) of Theorem 2.5.1 and existence of decomposition (2.10) with continuous $p$ and $r$. The principal difficulty was to prove that $r$ is a homeomorphism.

The aim of the present paper is to deduce from Theorem 2.5.1 a description of the homotopy type of $\text{Stab}_0(f, \omega)$, see Theorem 3.0.3 below.

3. Main result

Let $M$ be a compact orientable surface equipped with a volume form $\omega$, $f \in \mathcal{F}(M, P)$, $H$ be the Hamiltonian vector field of $f$ with respect to $\omega$, $H : M \times \mathbb{R} \to M$ be the corresponding Hamiltonian flow, and

$$
\varphi : C^\infty(M, \mathbb{R}) \to C^\infty(M, M), \quad \varphi(\alpha)(x) = H(x, \alpha(x))
$$

be the shift map along orbits of $H$. Let also

$$
\mathcal{Z} = \mathcal{Z}_\omega^H(f) = \{\alpha \in C^\infty(M, \mathbb{R}) \mid H(\alpha) = 0\}
$$

be the space of functions taking constant values along orbits of $H$, see (2.9). Then $\mathcal{Z}$ is a linear subspace of $C^\infty(M, \mathbb{R})$ and is contained in $\Gamma$, see (2.1). In particular, $\mathcal{Z}$ is contractible as well as $\Gamma$.

3.0.3. Theorem. The following statements hold true.

(1) $\varphi(\mathcal{Z}) = \text{Stab}_0(f, \omega) = \text{Stab}_0(f) \cap \text{Stab}(f, \omega)$ and $\mathcal{Z} = \varphi^{-1}(\text{Stab}_0(f, \omega))$. 
(2) If all critical points of \( f \) are non-degenerate local extremes, then the restriction \( \varphi|_Z : Z \to \text{Stab}_0(f, \omega) \) is an infinite cyclic covering, and \( \text{Stab}_0(f, \omega) \) is homotopy equivalent to the circle.

(3) Otherwise, \( \varphi|_Z : Z \to \text{Stab}_0(f, \omega) \) is a homeomorphism, and so \( \text{Stab}_0(f, \omega) \) is contractible.

(4) The inclusion \( \text{Stab}_0(f, \omega) \subset \text{Stab}(f) \) is a homotopy equivalence.

(5) The inclusion map \( j : \text{Stab}(f, \omega) \subset \text{Stab}(f) \) induces an injection \( j_0 : \pi_0\text{Stab}(f, \omega) \to \pi_0\text{Stab}(f) \).

Proof. First we need the following lemma.

3.0.4. Lemma. Let \( \alpha \in C^\infty(M, \mathbb{R}) \). Then the action of \( H_\alpha \) on \( \omega \) is given by

\[
H_\alpha^*\omega = (1 + H(\alpha)) \cdot \omega.
\]

(3.1) Proof. Since the set of critical points is finite and so nowhere dense, it suffices to check this relation at regular points of \( f \) only.

So let \( p \) be a regular point of \( f \). Then \( H(p) \neq 0 \), whence there are local coordinates \( (x, y) \) at \( p \) in which \( p = (0, 0) \), \( H(x, y) = \frac{\partial}{\partial x} \), and \( H(x, y, t) = (x + t, y) \) for sufficiently small \( x, y, t \).

In particular, \( H(\alpha) = \frac{\partial \alpha}{\partial x} \). We also have that \( \omega(x, y) = \gamma(x, y) dx \wedge dy \) for some \( C^\infty \) function \( \gamma \).

Notice that one may also assume that \( \alpha(p) = 0 \). Indeed, let \( b = \alpha(p) \). Then \( H_\alpha = H_{\alpha-b} \circ H_b \).

Since \( H_b \) preserves \( \omega \), see (2.6), it follows that

\[
H_\alpha^*\omega = H_{\alpha-b}^* \circ H_b^* \omega = H_{\alpha-b}^*\omega.
\]

Thus suppose \( \alpha(p) = 0 \), whence \( H_\alpha(p) = p \). Then

\[
H_\alpha^*\omega(x, y) = \gamma \circ H_\alpha(x, y) d(x + \alpha) \wedge dy = \gamma(x + \alpha, y) (1 + \alpha') dx \wedge dy = \gamma(x + \alpha, y) (1 + H(\alpha)) dx \wedge dy.
\]

In particular, at \( p \) we have that

\[
H_\alpha^*\omega(p) = (1 + H(\alpha(p))) \cdot \omega(p),
\]

which proves (3.1). \( \square \)

Now we can complete Theorem 3.0.3.

Let us check that

(3.2) \( \varphi(Z) \subset \text{Stab}_0(f, \omega) \).

Let \( \alpha \in Z \). As \( H_\alpha \) leaves invariant each orbit of \( H \), and therefore it preserves \( f \), we have that \( H_\alpha \in \text{Stab}(f) \).

Moreover, by formula (3.1), \( H_\alpha^*\omega = \omega \), so \( H_\alpha \in \text{Stab}(f, \omega) \).

Now notice that \( t\alpha \in Z \) for all \( t \in \mathbb{R} \), and so \( H_{t\alpha} \in \text{Stab}(f, \omega) \) as well. Thus the homotopy \( H_{t\alpha} : M \to M, t \in [0, 1], \) is in fact an isotopy in \( \text{Stab}(f, \omega) \) between \( \text{id}_M = H_0 \) and \( H_\alpha \).

Hence \( H_\alpha \in \text{Stab}_0(f, \omega) \).

Further we claim that

(3.3) \( Z \supset \varphi^{-1}(\text{Stab}_0(f) \cap \text{Stab}(f, \omega)) \).
Indeed, let $h \in \text{Stab}_0(f) \cap \text{Stab}(f, \omega)$. Then by (1) of Theorem 2.5.1 $h = H_\alpha$ for some $\alpha \in \Gamma$. As $\omega$ is everywhere non-zero on $M$, it follows from formula (3.1) that $H(\alpha) = 0$ on all of $M$, that is $\alpha \in \mathcal{Z}$.

Hence

$$\varphi(\mathcal{Z}) \subset \text{Stab}_0(f, \omega) \subset \text{Stab}_0(f) \cap \text{Stab}(f, \omega) \subset \varphi(\mathcal{Z}),$$

$$\mathcal{Z} \subset \varphi^{-1}(\text{Stab}_0(f, \omega)) \subset \varphi^{-1}(\text{Stab}_0(f) \cap \text{Stab}(f, \omega)) \subset \mathcal{Z}.$$  

This proves (1).

(2) Suppose $\varphi : \Gamma \to \text{Stab}_0(f)$ is an infinite cyclic covering map, and let $\theta \in \Gamma$ be the function from (2) of Theorem 2.5.1.

Then due to property (iii) in Theorem 2.5.1, $\theta$ takes constant values along orbits of $H$, and therefore $\theta \in \mathcal{Z}$. Since, in addition, $\mathcal{Z}$ is a group, it follows that $\mathcal{Z}$ is invariant with respect to the $\mathcal{Z}$-action on $\Gamma$, i.e. $\alpha + n\theta \in \mathcal{Z}$ for all $\alpha \in \mathcal{Z}$. Therefore $\mathcal{Z} = \varphi^{-1}(\text{Stab}_0(f, \omega))$. Hence $\varphi|_\mathcal{Z} : \mathcal{Z} \to \text{Stab}_0(f, \omega)$ is a $\mathcal{Z}$-covering as well as $\varphi|_\Gamma$. As $\mathcal{Z}$ is contractible, we obtain that the quotient $\text{Stab}_0(f, \omega)$ is homotopy equivalent to the circle.

Consider the following path $\tau : [0, 1] \to \mathcal{Z} \subset \Gamma$, $\tau(t) = t\theta$. Then $\varphi \circ \tau$ is a loop in $\text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$, since

$$\varphi \circ \tau(1)(x) = F(x, \theta(x)) = x = F(x, 0) = \varphi \circ \tau(0)(x).$$

This loop is a generator of $\pi_1\text{Stab}_0(f, \omega) \cong \mathcal{Z}$ as well as a generator of $\pi_1\text{Stab}_0(f) \cong \mathcal{Z}$. Hence the inclusion $j : \text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$ yields an isomorphism of fundamental groups. Since these spaces homotopy equivalent to the circle, we obtain that $j$ is a homotopy equivalence.

(3) If $\varphi : \Gamma \to \text{Stab}_0(f)$ is a homeomorphism, then due to (1) it yields a homeomorphism of $\mathcal{Z}$ onto $\text{Stab}_0(f, \omega)$. In particular, both $\text{Stab}_0(f, \omega)$ and $\text{Stab}_0(f)$ are contractible, and so the inclusion $\text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$ is a homotopy equivalence.

(5) Injectivity of $j_0$ follows from the relation $\text{Stab}_0(f, \omega) = \text{Stab}_0(f) \cap \text{Stab}(f, \omega)$. Theorem 3.0.3 is completed.

3.0.5. Remark. Though the inclusion $\text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$ is a homotopy equivalence, it seems to be a highly non-trivial task to find precise formulas for a strong deformation retraction of $\text{Stab}_0(f, \omega)$ onto $\text{Stab}_0(f)$. For the case (3) of Theorem 3.0.3 this is equivalent to a construction of a strong deformation retraction of $\Gamma$ onto $\mathcal{Z}$. In fact, it suffices to find a retraction $r : \Gamma \to \mathcal{Z}$, so to associate to each $\alpha \in \Gamma$ a function $r(\alpha)$ taking constant values along orbits of $H$ so that each $\beta \in \mathcal{Z}$ remains unchanged. Then a strong deformation $r_t : \Gamma \to \mathcal{Z}$, $t \in [0, 1]$, of $\Gamma$ onto $\mathcal{Z}$ can be given by $r_t(\alpha) = (1 - t)\alpha + tr(\alpha)$.

3.1. Counterexample for maps $g \not\in \mathcal{F}(M, P)$. Let $D^2 = \{|z| \leq 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\omega = dx \wedge dy$ be the standard symplectic form. Consider the following two functions $f, g : D^2 \to [0, 1]$ defined by

$$f(x, y) = x^2 + y^2 = |z|^2, \quad g(x, y) = (x^2 + y^2)^2 = |z|^4.$$  

Then the foliations by level sets of $f$ and $g$ coincide, whence

$$\mathcal{Z}_\omega^R(f) = \mathcal{Z}_\omega^R(g), \quad \text{Stab}_0(f, \omega) = \text{Stab}_0(g, \omega), \quad \text{Stab}_0(f) = \text{Stab}_0(g).$$

However, $f \in \mathcal{F}(D^2, \mathbb{R})$, while $g$ does not belong to $\mathcal{F}(D^2, \mathbb{R})$ since it is a polynomial with multiple factors.
Notice that the Hamiltonian vector fields $F$ and $G$ of $f$ and $g$ are given by

$$F(x, y) = -2y \frac{∂}{∂x} + 2x \frac{∂}{∂y}, \quad G(x, y) = 2(x^2 + y^2)F(x, y).$$

In particular, the Hamiltonian flow $F : D^2 \times \mathbb{R} \to D^2$ of $f$ is given by $F(z, t) = e^{2it}z$, and so the tangent map $T_0F : T_0D^2 \to T_0D^2$ is not the identity for $t \neq \pi n, n \in \mathbb{Z}$.

On the other hand, the linear part of $G$ at 0 vanishes, whence for the Hamiltonian flow $G : D^2 \times \mathbb{R} \to D^2$ of $g$ the corresponding tangent map $T_0G : T_0D^2 \to T_0D^2$ is always the identity. Hence for any $C^\infty$ function $\alpha$ the tangent map at 0 of $G_\alpha$ is the identity as well. Therefore for $t \neq \pi n, n \in \mathbb{Z}$, then $F_t \neq G_\alpha$ for any $\alpha \in C^\infty(M, \mathbb{R})$.

By Theorem 3.0.3 the shift map $\varphi_\gamma : \mathcal{Z}_\omega^R(f) \to \text{Stab}_0(f, \omega)$ of $F$ is an infinite cyclic covering, while the shift map

$$\varphi_\gamma : \mathcal{Z}_\omega^R(g) \equiv \mathcal{Z}_\omega^R(f) \longrightarrow \text{Stab}_0(f, \omega) \equiv \text{Stab}_0(g, \omega)$$

of $G$ turns out to be not surjective, since its image does not contain $F_t$ for $t \neq \pi n, n \in \mathbb{Z}$.

Thus we see that the centralizer of $g$ does not “detect” all the diffeomorphisms from $\text{Stab}_0(g)$, while the centralizer of $f$ does so. This shows that the assumption $f \in \mathcal{F}(M, P)$ in Theorem 3.0.3 is essential.

3.2. Kronrod-Reeb graph of $f$. Now we will give an interpretation of $\mathcal{Z}$ in terms of functions on the Kronrod-Reeb graph of $f$.

Let $f \in \mathcal{F}(M, P)$. Consider the partition $\Delta$ of $M$ into connected components of level-sets of $f$. Let $K := M/\Delta$ be the corresponding quotient space and $p : M \to K$ be the factor map. Then we have a natural decomposition

$$f = \hat{f} \circ p : M \overset{p}{\longrightarrow} K \overset{\hat{f}}{\longrightarrow} P.$$  

Endow $K$ with the final topology, so a subset $U \subset K$ is open if and only if $p^{-1}(U)$ is open in $M$. Then it is well known that $K$ has a natural structure of a one-dimensional CW-complex. It is called a Lyapunov or Kronrod-Reeb graph of $f$, [1], [22], [6], [5], [3].

We will briefly recall the correspondence between elements of $\Delta$ (i.e. points of $K$) and orbits of $H$. Let $\gamma \in \Delta$. If $\gamma$ contains at least one critical point of $f$, then it follows from Axiom (L) that $\gamma$ is a connected 1-dimensional CW-complex such that each of its vertices has even (possibly zero) degree, and $p(\gamma)$ is a vertex of $K$. In this case the vertices of $\gamma$ are critical points of $f$ being also zeros of $H$, while edges of $\gamma$ are non-closed orbits of $H$.

If $\gamma$ has no critical point of $f$, then $\gamma$ is a closed orbit of $H$.

3.2.1. Lemma. Each $\alpha \in \mathcal{Z}_\omega^Q(f)$ yields a unique continuous function $\hat{\alpha} : K \to Q$ such that $\alpha = \hat{\alpha} \circ p$. Moreover, the correspondence $\alpha \mapsto \hat{\alpha}$ is a continuous injective map $\eta : \mathcal{Z}_\omega^Q(f) \to C(K, Q)$ with respect to $C^0$ topology on $C(K, Q)$.

Proof. Let $\alpha \in \mathcal{Z}_\omega^Q(f)$, so $\alpha$ takes constant values along orbits of $H$. First we should show that $\alpha$ takes constant value at each element of $\Delta$.

Consider any element $\gamma \in \Delta$. If $\gamma$ contains no critical point of $f$, then $\gamma$ is a closed orbit of $H$, and so $\alpha$ takes a constant value at $\gamma$, see Figure 3.1.

Otherwise, $\gamma$ is a connected 1-dimensional CW-complex whose vertices and edges are orbits of $H$. Then $\alpha$ takes constant values along edges of $\gamma$, and it follows from continuity of $\alpha$ and connectedness of $\gamma$ that $\alpha$ is constant on all of $\gamma$.

Thus $\alpha$ yields a unique function $\hat{\alpha} : K \to Q$ such that $\alpha = \hat{\alpha} \circ p$. 
Since $K$ has final topology with respect to $p$ and $\alpha$ is continuous, it easily follows and is well known that $\hat{\alpha}$ is continuous. Continuity of the correspondence $\alpha \mapsto \hat{\alpha}$ is left for the reader. \hfill $\square$

Lemma 3.2.1 together with Theorem 3.0.3 implies that for $f \in \mathcal{F}(M, P)$ the elements of $\text{Stab}_0(f, \omega)$ are parametrized by continuous functions on the Kronrod-Reeb graph $K$ of $f$, see Figure 3.1.

More precisely, due to Theorem 3.0.3 for each $h \in \text{Stab}_0(f, \omega)$ there exists $\alpha \in \mathcal{Z}_Q(f, \omega)$ such that $h = H_{\alpha}$. This function takes constant values on connected components of level-sets of $f$, and therefore induces a continuous function $\hat{\alpha} : K \to \mathbb{R}$. Then the value of $\hat{\alpha}$ at some point $v \in K$ equals to the common time shift induced by $h$ on all the orbits of $H$ constituting $p^{-1}(v)$.

3.3. **Non-surjectivity of the map** $j_0 : \pi_0 \text{Stab}(f, \omega) \to \pi_0 \text{Stab}^+(f)$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $g(x, y) = ((x + 1)^2 + y^2)((x - 1)^2 + y^2)$. It has three critical points: one saddle $p_0 = (0, 0)$ and two local minimums $p_1 = (-1, 0)$ and $p_2 = (1, 0)$. Let $D = g^{-1}[0, 2]$, and $f = g|_D : D \to \mathbb{R}$ be the restriction of $g$ to $D$. Then $D$ is a 2-disk and $f$ belongs to the class $\mathcal{F}(D, \mathbb{R})$.

Consider the following subset $A = f^{-1}[0, 0.5] \subset D$, see Figure 3.2. It consists of two connected components $A_1$ and $A_2$ containing the points $p_1$ and $p_2$ respectively. Notice that $h(A) = A$ for each $h \in \text{Stab}(f)$, whence $h$ either preserves both $A_1$ and $A_2$ or interchanges them. Also notice that if $h, k \in \text{Stab}(f)$ and $h(A_1) = A_2$, while $k(A_1) = A_1$, then $h$ and $k$ belong to distinct path components of $\text{Stab}(f)$.

Let $h : D \to D$ be a diffeomorphism defined by $D(x, y) = (-x, -y)$. Evidently, $h$ belongs to $\text{Stab}^+(f)$ and interchanges $A_1$ and $A_2$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.1.png}
\caption{Figure 3.1.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.2.png}
\caption{Figure 3.2.}
\end{figure}
3.3.1. **Lemma.** Let $\omega$ be any volume form on $D$ such that $\text{Vol}_\omega(A_1) \neq \text{Vol}_\omega(A_2)$. Then the isotopy class $[h] \in \pi_0\text{Stab}^+(f)$ of $h$ does not contain any $k \in \text{Stab}(f, \omega)$. Hence for such an $\omega$ the map $j_0 : \pi_0\text{Stab}(f, \omega) \to \pi_0\text{Stab}^+(f)$ is not surjective.

**Proof.** Each $k \in \text{Stab}(f, \omega)$ preserves $\omega$-volume. Since $\text{Vol}_\omega(A_1) \neq \text{Vol}_\omega(A_2)$, it follows that $k(A_i) = A_i$ for $i = 1, 2$. But $h(A_1) = A_2$, whence $h$ and $k$ are not isotopic in $\text{Stab}(f)$. □

**References**

[1] G. M. Adelson-Welsky and A. S. Kronrode. Sur les lignes de niveau des fonctions continues possédant des dérivées partielles. *C. R. (Doklady) Acad. Sci. URSS (N.S.),* 49:235–237, 1945.

[2] Augustin Banyaga. Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique. *Comment. Math. Helv.,* 53(2):174–227, 1978.

[3] A. V. Bolsinov and A. T. Fomenko. *Vvedenie v topologiyu integraliruyemykh gamiltonovykh sistem (Introduction to the topology of integrable Hamiltonian systems).* “Nauka”, Moscow, 1997.

[4] M. Dehn. Die Gruppe der Abbildungsklassen. *Acta Mathematica,* 69:135–206, 1938.

[5] John Franks. Nonsingular Smale flows on $S^3$. *Topology,* 24(3):265–282, 1985.

[6] A. S. Kronrod. On functions of two variables. *Uspehi Matem. Nauk (N.S.)*, 5(1(35)):24–134, 1950.

[7] E. A. Kudryavtseva. The topology of spaces of Morse functions on surfaces. *Math. Notes,* 92(1-2):219–236, 2012. Translation of Mat. Zametki 92 (2012), no. 2, 241–261.

[8] E. A. Kudryavtseva. On the homotopy type of spaces of Morse functions on surfaces. *Mat. Sb.,* 204(1):79–118, 2013.

[9] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. *Proc. Cambridge Philos. Soc.,* 60:769–778, 1964.

[10] Sergiy Maksymenko. Smooth shifts along trajectories of flows. *Topology Appl.,* 130(2):183–204, 2003.

[11] Sergiy Maksymenko. Homotopy types of stabilizers and orbits of Morse functions on surfaces. *Ann. Global Anal. Geom.,* 29(3):214–285, 2006.

[12] Sergiy Maksymenko. Homotopy dimension of orbits of Morse functions on surfaces. *Travaux Mathématiques,* 18:39–44, 2008.

[13] Sergiy Maksymenko. $\infty$-jets of diffeomorphisms preserving orbits of vector fields. *Cent. Eur. J. Math.,* 7(2):272–298, 2009.

[14] Sergiy Maksymenko. Reparametrization of vector fields and their shift maps. *Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos.,* 6(2):489–498, arXiv:math/0907.0354, 2009.

[15] Sergiy Maksymenko. Functions with isolated singularities on surfaces. *Geometry and topology of functions on manifolds. Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos.,* 7(4):7–66, 2010.

[16] Sergiy Maksymenko. Local inverses of shift maps along orbits of flows. *Osaka Journal of Mathematics,* 48(2):415–455, 2011.

[17] Sergiy Maksymenko. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. page arXiv:math/1311.3347v3, 2016.

[18] Dusa McDuff. Remarks on the homotopy type of groups of symplectic diffeomorphisms. *Proc. Amer. Math. Soc.,* 94(2):348–352, 1985.

[19] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.

[20] Jürgen Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.,* 120:286–294, 1965.

[21] Leonid Polterovich. *The geometry of the group of symplectic diffeomorphisms.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001.

[22] Georges Reeb. *Sur certaines propriétés topologiques des variétés feuilletées.* Actualités Sci. Ind., no. 1183. Hermann & Cie., Paris, 1952. Publ. Inst. Math. Univ. Strasbourg 11, pp. 5–89, 155–156.