Actions of relative Weyl groups I

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General introduction

A theorem of Digne, Lehrer and Michel says that Lusztig restriction of a Gel’fand-Graev character of a finite reductive group $G^F$ is still a Gel’fand-Graev character [DLM2, Theorem 3.7]. However, an ambiguity remains on the character obtained (whenever the center of $G$ is not connected, there are several Gel’fand-Graev characters). The original aim of this series of papers was to drop this ambiguity. For this, we needed to study more deeply the structure of the endomorphism algebra of an induced cuspidal character sheaf: for instance, we wanted to follow the action of a Frobenius endomorphism through this algebra.

This lead us to this first part, in which we develop another approach for computing explicitly this endomorphism algebra. One of the main goal is to construct another isomorphism between this endomorphism algebra and the group algebra of the relative Weyl group involved (a first one was constructed by Lusztig [Lu1, Theorem 9.2]). By comparing both isomorphisms, we get some immediate consequences for finite reductive groups. Note that the results of this part are valid for any cuspidal local system supported by a unipotent class and have a chance to be useful for computing values of characters at unipotent elements.

In the second forthcoming part, we will restrict our attention to the case where the cuspidal local system is supported by the regular unipotent class. We will then be able to compute explicitly the generalized Springer correspondence through this new isomorphism. As an application of these (sometimes fastidious) computations, we will get the desired more precise version of Digne, Lehrer and Michel’s theorem. It must be said that this final result is valid only whenever the cardinality of the finite field is large enough.

Introduction to this first part

Let $G$ be a connected reductive group defined over an algebraically closed field $F$, let $L$ be a Levi subgroup of a parabolic subgroup of $G$, let $C$ be a unipotent class of $L$, and let $v \in C$. We first explain how the action of the finite group $W_G(L, C) = N_G(L, C)/L$ on some varieties introduced by Lusztig [Lu1, §3 and 4] can be extended “by density” to some slightly bigger varieties (see §2). We then generalize this construction to extend the action of $W_G(L, v) = N_G(L, v)/C_L^v(v)$ on other varieties covering the previous ones (see §3.B). We show that, in most cases, the description of some stabilizers is entirely given by a morphism $\varphi^C_{L,v} : W_G(L, C) \to A_L(v) = C_L(v)/C_L^v(v)$ (see §3.C and §3.D). We also provide some reduction arguments to compute explicitly the morphisms $\varphi^C_{L,v}$ (see §4).

From §5 to the end, we assume that $C$ supports a cuspidal local system $\mathcal{E}$ (we denote by $\zeta$ the character of the finite group $A_L(v)$ associated to $\mathcal{E}$). Let $K$ denote the perverse sheaf obtained from the datum $(C, \mathcal{E})$ by parabolic induction [Lu1, 4.1.1], and let $A$ denote its
endomorphism algebra. In this case, $W_G(L,C)$ is equal to $W_G(L)$ (by [Lu1, Theorem 9.2]) and is isomorphic to $W_G^\circ(L,v) = N_G(L) \cap C_G^v(L)/C_G^v(L)$. Lusztig [Lu1, Theorem 9.2] constructed a canonical isomorphism $\Theta : \overline{Q}_\ell W_G(L,v) \to A$. The aim of §5 is to construct an other explicit isomorphism $\Theta' : \overline{Q}_\ell W_G(L,v) \to A$. For this purpose, we use the varieties previously introduced in §3. It turns out that $\Theta$ and $\Theta'$ differs by a linear character $\gamma_{L,v,\zeta}$ (see Corollary 6.5). Moreover, whenever $\zeta$ is linear, and whenever some mild hypothesis are satisfied, we have $\gamma_{L,v,\zeta} = \zeta \circ \varphi_{L,v}^G$.

In §8, we assume further that $F$ is an algebraic closure of a finite field and that $G$ is endowed with a Frobenius endomorphism. We then explain what kind of precisions may be obtained by using the previous results about characteristic functions of character sheaves.

In the future part II, we will assume throughout that $v$ is regular. Under this hypothesis, we will compute explicitly the morphisms $\varphi_{L,v}^G$ in sufficiently many cases to be able to apply successfully this work to Gel’fand-Graev characters. We then follow Digne, Lehrer and Michel’s method: from the knowledge of $\gamma_{L,v,\zeta}$ and from the explicit nature of the isomorphism $\Theta'$, we get the desired precision.

Notation

**Fields, varieties, sheaves.** We fix an algebraically closed field $F$ and we denote by $p$ its characteristic. All algebraic varieties and all algebraic groups will be considered over $F$. We also fix a prime number $\ell$ different from $p$. Let $\overline{Q}_\ell$ denote an algebraic closure of the $\ell$-adic field $Q_\ell$.

If $X$ is an algebraic variety (over $F$), we also denote by $\overline{Q}_\ell$ the constant $\ell$-adic sheaf associated to $\overline{Q}_\ell$ (if necessary, we denote it by $(\overline{Q}_\ell)_X$). By a constructible sheaf (respectively a local system) on $X$ we mean a constructible $\overline{Q}_\ell$-sheaf (respectively a $\overline{Q}_\ell$-local system). Let $\mathcal{D}X$ denote the bounded derived category of constructible sheaves on $X$. If $K \in \mathcal{D}X$ and $i \in \mathbb{Z}$, we denote by $\mathcal{H}^i K$ the $i$-th cohomology sheaf of $K$ and if $x \in X$, then $\mathcal{H}^i_x K$ denotes the stalk at $x$ of the constructible sheaf $\mathcal{H}^i K$. If $K \in \mathcal{D}X$, we denote by $DK$ its Verdier dual. If $L$ is a constructible sheaf on $X$, we identify it with its image in $\mathcal{D}X$, that is the complex concentrated in degree 0 whose 0th term is $L$.

Let $K \in \mathcal{D}X$. We say that $K$ is a **perverse sheaf** if the following two conditions hold:

(a) $\forall i \in \mathbb{Z}$, dim $\supp \mathcal{H}^i K \leq -i$,

(b) $\forall i \in \mathbb{Z}$, dim $\supp \mathcal{H}^i DK \leq -i$.

We denote by $\mathcal{M}X$ the full subcategory of $\mathcal{D}X$ whose objects are perverse sheaves: this is an abelian category [BBD, 2.14, 1.3.6].

Let $Y$ be a locally closed, smooth, irreducible subvariety of $X$ and let $\mathcal{L}$ be a local system on $Y$. We denote by $\mathcal{I}C(Y,\mathcal{L})$ the intersection cohomology complex of Deligne-Goresky-MacPherson of $Y$ with coefficients in $\mathcal{L}$. We often identify $\mathcal{I}C(Y,\mathcal{L})$ with its extension by zero to $X$; $\mathcal{I}C(Y,\mathcal{L})[\dim Y]$ is a perverse sheaf on $X$.

**Algebraic groups.** If $H$ is a linear algebraic group, we will denote by $H^\circ$ the neutral component of $H$, by $H_{\text{uni}}$ the closed subvariety of $H$ consisting of unipotent elements
of $H$, and by $Z(H)$ the center of $H$. If $h \in H$, then $A_H(h)$ denotes the finite group $C_H(h)/C_H^0(h)$, $(h)_H$ denotes the conjugacy class of $h$ in $H$, and $h_s$ (respectively $h_u$) denotes the semisimple (respectively unipotent) part of $h$. If $\mathfrak{h}$ is the Lie algebra of $H$, we denote by $\text{Ad}_h : \mathfrak{h} \rightarrow \mathfrak{h}$ the differential at 1 of the automorphism $H \rightarrow H$, $x \mapsto h_x = hxh^{-1}$.

If $X$ and $Y$ are varieties, and if $X$ (respectively $Y$) is endowed with an action of $H$ on the right (respectively left), then we denote, when it exists, $X \times_H Y$ the quotient of $X \times Y$ by the diagonal left action of $H$ given by $h.(x, y) = (xh^{-1}, hy)$ for any $h \in H$ and $(x, y) \in X \times Y$. If $(x, y) \in X \times Y$, and if $X \times_H Y$ exists, we denote by $x *_H y$ the image of $(x, y)$ in $X \times_H Y$ by the canonical morphism.

Finally, if $X_1, \ldots, X_n$ are subsets or elements of $H$, we denote by $N_H(X_i)$ the intersection of the normalizers $N_H(X_i)$ of $X_i$ in $H$ ($1 \leq i \leq n$).

**Reductive group.** We fix once and for all a connected reductive algebraic group $G$. We fix a Borel subgroup $B$ of $G$ and a maximal torus $T$ of $B$. We denote by $X(T)$ (respectively $Y(T)$) the lattice of rational characters (respectively of one-parameter subgroups) of $T$. Let $W = N_G(T)/T$. Let $\Phi$ denote the root system of $G$ relative to $T$ and let $\Phi^+$ (respectively $\Delta$) denote the set of positive roots (respectively the basis) of $\Phi$ associated to $B$. For each root $\alpha \in \Phi$, we denote by $U_\alpha$ the one-parameter unipotent subgroup of $G$ normalized by $T$ associated to $\alpha$.

We also fix in this paper a parabolic subgroup $P$ of $G$ and a Levi subgroup $L$ of $P$. We denote by $\pi_L : P \rightarrow L$ the canonical projection with kernel $V$, the unipotent radical of $P$. We denote by $\Phi_L$ the root system of $L$ relative to $T$; we have $\Phi_L \subset \Phi$. Finally, $W_L$ denotes the Weyl group of $L$ relative to $T$.

1. Preliminaries

1.A. Centralizers. We start this subsection by recalling two well-known results on centralizers of elements in reductive groups. The first one is due to Lusztig [Lu1, Proposition 1.2], while the second one is due to Spaltenstein [HS, Proposition 3].

**Lemma 1.1 (Lusztig).** (1) Let $l \in L$ and $g \in G$. Then

$$\dim \{xP \mid x^{-1}gx \in (l)_L, V\} \leq \frac{1}{2}(\dim C_G(g) - \dim C_L(l)).$$

(2) If $g \in P$, then $\dim C_P(g) \geq \dim C_L(\pi_L(g))$.

**Lemma 1.2 (Spaltenstein).** If $l \in L$, then $C_V(l)$ is connected.

We will now give several applications of the two previous lemmas. We first need the following technical result:
Lemma 1.3. Let \( l \in L \). Then the following are equivalent:

(a) \( C^{-}_G(l) \subset L \);

(b) \( C^+_G(l) \subset L \);

(c) \( C_V(l) = \{1\} \);

(d) \( \dim C_V(l) = 0 \).

Proof - It is clear that (b) implies (a), and that (a) implies (d). Moreover, by Lemma 1.2, (c) is equivalent to (d). It remains to prove that (c) implies (b).

For this, let \( s \) (respectively \( u \)) denote the semisimple (respectively unipotent) part of \( l \), and assume that \( C^{-}_G(s) \not\subset L \). We want to prove that \( C_V(l) \neq \{1\} \). Without loss of generality, we may, and we will, assume that \( s \in T \) and \( u \in U \cap L \).

Let \( G' = C^{-}_G(s) \), \( U' = C_U(s) \), \( V' = C_V(s) \) and \( B' = C_B(s) \). Then, by [Bor, Corollary 11.12], \( u \in G' \). Moreover, by Lemma 1.2, \( U' \) and \( V' \) are connected. So \( B' = T . U' \) is connected. Let \( \Phi_s \) denote the root system of \( G' \) relative to \( T \), and let \( \Phi^+_s \) be the positive root system associated to the Borel subgroup \( B' \) of \( G' \). Let \( \alpha_s \) denote the highest root of \( \Phi_s \) with respect to \( \Phi^+_s \). Since \( G' \not\subset L \), we have \( \alpha_s \notin \Phi_L \). But \( U_{\alpha_s} \) is central in \( U' \), so \( U_{\alpha_s} \subset C_V(u) = C_V(l) \). Therefore, \( C_V(l) \neq \{1\} \). The proof of Lemma 1.3 is now complete. ■

Now, let \( \mathcal{O} \) be the set of elements \( l \in L \) such that \( C_V(l) = \{1\} \).

Lemma 1.4. The set \( \mathcal{O} \) is a dense open subset of \( L \) and the map

\[
\mathcal{O} \times V \longrightarrow \mathcal{O} V \\
(l, x) \longmapsto xlx^{-1}
\]

is an isomorphism of varieties.

Proof - The group \( V \) acts on \( G \) by conjugation. So, by [Hu, Proposition 1.4], the set

\[
\mathcal{N} = \{ g \in G \mid \dim C_V(g) = 0 \}
\]

is an open subset of \( G \). Therefore, \( \mathcal{N} \cap L \) is an open subset of \( L \). Moreover, \( \mathcal{N} \cap L \) is not empty since any \( G \)-regular element of \( T \) belongs to \( \mathcal{N} \cap L \). But, by Lemma 1.2, \( \mathcal{N} \cap L = \mathcal{O} \). This proves the first assertion of the lemma.

Now, let \( f \) denote the morphism defined in Lemma 1.4. Let \( l \in \mathcal{O} \). Then the map \( f_l : V \to l . V, x \mapsto xlx^{-1} \) is injective by definition of \( \mathcal{O} \), and its image is closed because it is an orbit under a unipotent group [Bor, Proposition 4.10]. By comparing dimensions, we get that \( f_l \) is bijective. As this holds for every \( l \in \mathcal{O} \), \( f \) is bijective.

Moreover, the variety \( \mathcal{O} . V \simeq \mathcal{O} \times V \) is smooth. Hence, to prove that \( f \) is an isomorphism, we only need to prove, by [Bor, Theorems AG.17.3 and AG.18.2], that the differential \( (df)_{(l,1)} \) is surjective for some \( l \in \mathcal{O} \).

Now, let \( l \in T \) be a \( G \)-regular element (so that \( l \in \mathcal{O} \)). The tangent space to \( \mathcal{O} \) at \( t \) may be identified with the Lie algebra \( \mathfrak{l} \) of \( L \) via the translation by \( t \). By writing
$f(l, x) = l.(l^{-1}xl^{-1})$ for every $(l, x) \in \mathcal{O} \times V$, the differential $(df)_{(t,1)}$ may be identified with the map

$$
\begin{align*}
{l \oplus v} & \mapsto l \oplus v \\
{l \oplus x} & \mapsto l \oplus (\text{Ad} t^{-1} - \text{Id}_v)(x).
\end{align*}
$$

Here, $v$ denotes the Lie algebra of $V$. The bijectivity of $(df)_{(t,1)}$ follows immediately from the fact that the eigenvalues of $\text{Ad} t^{-1}$ are equal to $\alpha(t)^{-1}$ for $\alpha \in \Phi^+ - \Phi_L$, so they are different from 1 by the regularity of $t$. ■

Lemma 1.4 implies immediately the following result:

**Corollary 1.5.** Let $S$ be a locally closed subvariety of $\mathcal{O}$. Then the map

$$
S \times V \rightarrow S.V \\
(l, x) \mapsto xlx^{-1}
$$

is an isomorphism of varieties.

**Notation** - If $S$ is a locally closed subvariety of $L$, we denote by $S_{\text{reg}}$ (or $S_{\text{reg},G}$ if there is some ambiguity) the open subset $S \cap \mathcal{O}$ of $S$. It might be empty. □

1.B. **Steinberg map.** Let $\nabla : G \rightarrow T/W$ be the Steinberg map. Let us recall its definition. If $g \in G$, then $\nabla(g)$ is the intersection of $T$ with the conjugacy class of the semisimple part of $g$. Then $\nabla$ is a morphism of varieties [Ste, §6]. To compute the Steinberg map, we need to determine semisimple parts of elements of $G$. In our situation, the following well-known lemma will be useful [Lu1, 5.1]:

**Lemma 1.6.** If $g \in P$, then the semisimple part of $g$ is $V$-conjugate to the semisimple part of $\pi_L(g)$. In particular, $\nabla(g) = \nabla(\pi_L(g))$.

**Proof** - Let $s$ be the semisimple part of $g$. Then the semisimple part of $\pi_L(g)$ is $\pi_L(s)$. But, $s$ belongs to some Levi subgroup $L_0$ of $P$. Let $x \in V$ be such that $L = xL_0$. Then $x sx^{-1} \in L$ and $x sx^{-1}$ is the semisimple part of $xgx^{-1}$. Therefore, the semisimple part of $\pi_L(xgx^{-1}) = \pi_L(g)$ is $\pi_L(x sx^{-1}) = x sx^{-1}$. ■

**Lemma 1.7.** $\nabla(Z(L)^\circ)$ is a closed subset of $T/W$ and $\nabla(Z(L)^\circ_{\text{reg}})$ is an open subset of $\nabla(Z(L)^\circ)$. 

**Proof** - The restriction of $\nabla$ to $T$ is a finite quotient morphism. In particular, it is open and closed. Since it is closed, $\nabla(Z(L)^\circ)$ is a closed subset of $T/W$. Since it is open, $\nabla(T_{\text{reg}})$ is an open subset of $T/W$. But $\nabla(Z(L)^\circ_{\text{reg}}) = \nabla(T_{\text{reg}}) \cap \nabla(Z(L)^\circ)$. So the Lemma 1.7 is proved. ■
Let $\nabla_L : L \to T/W_L$ denote the Steinberg map for the group $L$. By Lemma 1.3, we have

$$\mathcal{O} = \nabla_L^{-1}(T_{\text{reg}}/W_L).$$

1.C. **A family of morphisms.** If $S$ is a locally closed subvariety of $L$ stable under conjugation by $L$, then $S.V$ is a locally closed subvariety of $P$ stable under conjugation by $P$. We can therefore consider the quotients $G \times_L S$ and $G \times_P S.V$. With these varieties are associated the maps $G \times_L S \to G$, $g *_L l \mapsto glg^{-1}$ and $G \times_P S.V \to G$, $g *_P x \mapsto gxg^{-1}$: they are well-defined morphisms of varieties.

**Remark 1.9 -** If $S$ is contained in $O$, then the map $G \times_L S \to G \times_P S.V$, $g *_L l \mapsto g *_P l$ is an isomorphism of varieties (by Corollary 1.5). □

The next result is well-known:

**Lemma 1.10.** The map $G \times_P P \to G$, $g *_P x \mapsto gxg^{-1}$ is a projective surjective morphism of varieties. In particular, if $F$ is a closed subvariety of $P$ stable under conjugation by $P$, then the map $G *_P F \to G$, $g *_P x \mapsto gxg^{-1}$ is a projective morphism.

**Proof -** Let $\tilde{X} = \{(x, gP) \in G \times G/P \mid g^{-1}xg \in P\}$. Then $\tilde{X}$ is a closed subvariety of $G \times G/P$. Moreover, the variety $G/P$ is projective. Therefore, the projection $\pi : \tilde{X} \to G$, $(x, gP) \mapsto x$ is a projective morphism. Since every element of $G$ is conjugate to an element of $B$, $\pi$ is surjective.

But the maps $G \times_P P \to \tilde{X}$, $g *_P x \mapsto (gxg^{-1}, gP)$ and $\tilde{X} \to G \times_P P$, $(x, gP) \mapsto g *_P g^{-1}xg$ are morphisms of varieties which are inverse of each other. Moreover, through these isomorphisms, the map constructed in Lemma 1.10 may be identified with $\pi$. The proof is now complete. ■

The next result might be known but the author have never seen such a statement.

**Lemma 1.11.** The morphisms of varieties

$$G \times_L \mathcal{O} \quad \longrightarrow \quad G$$

$$g *_L l \quad \longmapsto \quad glg^{-1}$$

and

$$G \times_P \mathcal{O}.V \quad \longrightarrow \quad G$$

$$g *_P x \quad \longmapsto \quad gxg^{-1}$$

are étale.

**Proof -** By Remark 1.9, it is sufficient to prove that the morphism

$$f : G \times_P \mathcal{O}.V \quad \longrightarrow \quad G$$

$$g *_P x \quad \longmapsto \quad gxg^{-1}$$

is étale.
Since $G \times \mathfrak{p} \mathcal{O} \cdot V$ and $G$ are smooth varieties, $f$ is étale if and only if the differential of $f$ at any point of $G \times \mathfrak{p} \mathcal{O} \cdot V$ is an isomorphism [Ha, Proposition III.10.4]. By $G$-equivariance of the morphism $f$ ($G$ acts on $G \times \mathfrak{p} \mathcal{O} \cdot V$ by left translation on the first factor, and acts on $G$ by conjugation), it is sufficient to prove that $(df)_{1 \times \mathfrak{p} x}$ is an isomorphism for every $x \in \mathcal{O} \cdot V$.

For this, let $P^-$ denote the parabolic subgroup of $G$ opposed to $P$ (with respect to $L$), and let $V^-$ denote its unipotent radical. Then $V^- \times \mathcal{O} \cdot V$ is an open neighborhood of $1 \times P^\mathfrak{p} x$ in $G \times \mathfrak{p} \mathcal{O} \cdot V$. Therefore, it is sufficient to prove that the differential of $f^-$ at $(1, x)$ is an isomorphism for every $x \in \mathcal{O} \cdot V$.

Let $\mathfrak{g}, \mathfrak{v}^-, I$ and $\mathfrak{p}$ denote the Lie algebra of $G$, $V^-$, $L$ and $P$ respectively. Since $\mathcal{O}$ is open in $L$, we may identify the tangent space to $\mathcal{O} \cdot V$ at $x$ with $\mathfrak{p}$ (using left translation by $x$). Similarly, we identify the tangent space to $G$ at $x$ with $\mathfrak{g}$ using left translation. Using these identifications, the differential of $f^-$ at $(1, x)$ may be identified with the map

$$
\delta : \mathfrak{v}^- \oplus \mathfrak{p} \longrightarrow \mathfrak{g},
A \oplus B \longmapsto (\text{ad} x)^{-1}(A) - A + B.
$$

For dimension reasons, we only need to prove that $\delta$ is injective.

For this, let $\lambda \in Y(T)$ be such that $L = C_G(\text{Im} \lambda)$ and $P = \{ g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \}.$

For the definition of $\lim_{t \to 0} \lambda(t),$ see [DLM], Page 184. We then define, for each $i \in \mathbb{Z},$

$$
\mathfrak{g}(i) = \{ X \in \mathfrak{g} \mid (\text{ad} \lambda(t))(X) = t^i X \}.
$$

Then

$$
\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i),
$$

$$
\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i),
$$

and

$$
\mathfrak{v}^- = \bigoplus_{i < 0} \mathfrak{g}(i)
$$

(see [DLM], 5.14]). For each $X \in \mathfrak{g}$, we denote by $X_i$ its projection on $\mathfrak{g}(i)$.

Now, let $l = \pi_L(x)$. Then it is clear that we have, for any $i_0 \in \mathbb{Z}$ and any $X \in \mathfrak{g}(i_0)$,

$$
(\text{ad} x)^{-1}(X) \in (\text{ad} l)^{-1}(X) + (\bigoplus_{i \geq i_0} \mathfrak{g}(i)).
$$

(1)

Now, let $A \oplus B \in \text{Ker} \delta$, and assume that $A \neq 0$. Then there exists $i_0 < 0$ minimal among all $i < 0$ such that $A_i \neq 0$. Then, by equality (1), the projection of $\delta(A \oplus B)$ on $\mathfrak{g}(i_0)$ is equal to $(\text{ad} l)^{-1}(A_{i_0}) - A_{i_0}$. But, $\delta(A \oplus B) = 0$, so $(\text{ad} l)^{-1}(A_{i_0}) - A_{i_0} = 0$. Therefore, $C_{\mathfrak{g}}(l) \not\subseteq I$. So $C_{\mathfrak{g}}(s) \not\subseteq I$, where $s$ denotes the semisimple part of $l$. However, $l$ lies in $\mathcal{O}$, so its semisimple part $s$ lies also in $\mathcal{O}$ by Lemma I.3. But, by [Bor], Proposition 9.1 (1)], $C_{\mathfrak{g}}(s)$ is the Lie algebra of $C_G(s)$ which is contained in $L$ by Lemma I.3. We get a contradiction.

So, this discussion shows that $A = 0$. But $0 = \delta(A, B) = (\text{ad} x)^{-1}(A) - A + B$, so $B = 0$. This completes the proof of Lemma I.11. ■
1.D. **Isolated class.** An element $g \in G$ is said \((G-)\text{-isolated}\) if the centralizer of its semisimple part is not contained in a Levi subgroup of a proper parabolic subgroup of $G$.

Let $L_{\text{iso}}$ denote the subset of $L$ consisting of $L$-isolated elements, and let $T_{\text{iso}} = T \cap L_{\text{iso}}$. Then $T_{\text{iso}}$ is a closed subset of $T$. Therefore $\nabla_L(T_{\text{iso}})$ is a closed subset of $T/W_L$. Moreover, we have $L_{\text{iso}} = \nabla_L^{-1}(\nabla(T_{\text{iso}}))$, so $L_{\text{iso}}$ is a closed subset of $L$.

As a consequence of Lemma 1.10, the image of the morphism $G \times_P L_{\text{iso}}V \to G$, $g \ast_P x \mapsto gxg^{-1}$ is a closed subvariety of $G$ : we denote it by $X_{G,L}$. On the other hand, by Lemma 1.11, the image of the morphism $G \times_L \mathcal{O} \to G$, $g \ast_L x \mapsto gxg^{-1}$ is an open subset of $G$, which will be denoted by $\mathcal{O}_{G,L}$. Finally, we denote by $Y_{G,L}$ the intersection of $X_{G,L}$ and $\mathcal{O}_{G,L}$.

Note that $W_G(L) = N_G(L)/L$ acts (on the right) on the variety $G \times_L L_{\text{iso},\text{reg}}$ in the following way. If $w \in W_G(L)$ and if $g, l \in G \times_L L_{\text{iso},\text{reg}}$, then

$$(g \ast_L l).w = gw \ast_L \hat{w}^{-1}w,$$

where $\hat{w} \in N_G(L)$ is any representative of $w$.

**Proposition 1.12.** The map $G \times L_{\text{iso},\text{reg}} \to Y_{G,L}$. $g \ast_L l \mapsto glg^{-1}$ is a Galois \'{e}tale covering with group $W_G(L)$.

**Proof.** Set $\pi : G \times L_{\text{iso},\text{reg}} \to Y_{G,L}$, $g \ast_L l \mapsto glg^{-1}$, and let $\gamma : G \times L \mathcal{O} \to G$, $g \ast_L l \mapsto glg^{-1}$. By Lemma 1.11, $\gamma$ is an \'{e}tale morphism. If we prove that the square

\[
\begin{array}{ccc}
G \times L_{\text{iso},\text{reg}} & \longrightarrow & G \times L \mathcal{O} \\
\pi \downarrow & & \gamma \downarrow \\
Y_{G,L} & \longrightarrow & G
\end{array}
\]

is cartesian, then, by base change, we get that $\pi$ is an \'{e}tale morphism.

Since $\gamma$ is smooth, the fibred product (scheme) of $G \times L \mathcal{O}$ and $Y_{G,L}$ over $G$ is reduced (because $Y_{G,L}$ is), so we only need to prove that $G \times L_{\text{iso},\text{reg}} = \gamma^{-1}(Y_{G,L})$.

So, let $g \ast_L l \in G \times L \mathcal{O}$ be such that $glg^{-1} \in Y_{G,L}$. Then there exist $h \in G$, $m \in L_{\text{iso}}$ and $v \in V$ such that $hmvh^{-1} = glg^{-1}$. Let $s$ (respectively $t$, respectively $t'$) denote the semisimple part of $l$, $m$ and $mv$. By Lemma 1.4, there exists $x \in V$ such that $t' = xt$. Therefore $hx = t'$.

Since $t$ is $L$-isolated, we have $Z(C^o_L(t)) = Z(L)$. Therefore, $Z(C^o_G(t)) \subset Z(L)$. On the other hand, since $s \in \mathcal{O}$, we have $C^o_G(s) \subset L$, so $Z(L) \subset Z(C^o_G(s))$. This proves that $\phi Z(L) \subset h_x Z(L)$. For dimension reasons, we have $\phi Z(L) = h_x Z(L)$, so $Z(L) = Z(C^o_G(s))$. Hence, $l$ is isolated. So, we have proved that $\pi$ is \'{e}tale.

Now $W_G(L)$ acts freely on $G \times L_{\text{iso},\text{reg}}$. So the quotient morphism $G \times L_{\text{iso},\text{reg}} \to G \times N_G(L) L_{\text{iso},\text{reg}}$ is a Galois \'{e}tale covering with group $W_G(L)$. Moreover, $\pi$ clearly factorizes through this quotient morphism. We get an \'{e}tale morphism $\pi_0 : G \times N_G(L) L_{\text{iso},\text{reg}} \to Y_{G,L}$. Proving Proposition 1.12 is now equivalent to prove that $\pi_0$ is an isomorphism of varieties. Since it is \'{e}tale, we only need to prove that it is bijective.
First $\pi_0$ is clearly surjective. We just need to prove that it is injective. Let $(g, l)$ and $(g', l')$ in $G \times L_{iso, reg}$ be such that $glg^{-1} = g'l'g'^{-1}$. Let $s$ and $s'$ be the semisimple parts of $l$ and $l'$ respectively. Then $Z(C^0_G(s)) = Z(L)^0$ since $l$ is $L$-isolated. Since $l \in O$, we also have $C^0_G(s) = C^0_L(s)$, so $Z(C^0_G(s)) = Z(L)^0$. Similarly, $Z(C^0_G(s')) = Z(L)^0$. So $g^{-1}g' Z(L)^0 = Z(L)^0$. Therefore, $g^{-1}g' \in N_G(L)$. This completes the proof. ■

2. Action of the relative Weyl group

2.A. The set-up. From now on, and until the end of this paper, we denote by $\Sigma$ the inverse image of an $L/Z(L)^0$-isolated class of $L/Z(L)^0$ under the canonical projection $L \to L/Z(L)^0$. We also fix an element $v \in \Sigma$.

Following [Lu1, §§3 and 4], we consider the varieties

\[ \hat{Y} = G \times \Sigma_{reg}, \]
\[ \tilde{Y} = G \times L \Sigma_{reg}, \]
\[ \hat{X} = G \times \Sigma V \]
\[ \tilde{X} = G \times P \Sigma V. \]

In these definitions, the group $L$ (respectively $P$) acts on $G$ by right translations, and acts on $\Sigma_{reg}$ (respectively $\Sigma V$) by conjugation. We also set

\[ Y = \bigcup_{g \in G} g \Sigma_{reg} g^{-1} \]
and
\[ X = \bigcup_{g \in G} g \Sigma V g^{-1}. \]

By Lemma 1.6, we have

(2.1) $X \subset \nabla^{-1}(\nabla(Z(L)^0))$
and

(2.2) $Y \subset \nabla^{-1}(\nabla(Z(L)^0))$.

Moreover, $X$ is the image of $G \times_P \Sigma V$ under the morphism $G \times_P P \to G$, $g \times_P l \mapsto gxg^{-1}$ studied in Subsection 1.C. So, by Lemma 1.10, $X$ is a closed irreducible subvariety of $G$. We set

\[ \pi : G \times_P \Sigma V \longrightarrow X \]
\[ g \times_P x \mapsto gxg^{-1}. \]

It is a projective morphism of varieties.

On the other hand, $Y^+ = \bigcup_{g \in G} g(\Sigma)_{reg} g^{-1}$ is in fact the intersection of $X$ with $\nabla^{-1}(\nabla(Z(L)^0))$, so it is an open subset of $X$ (by Lemma 1.7). But, by Proposition 1.12, the map $G \times_L (\Sigma)_{reg} \to Y^+$, $g \times_L l \mapsto glg^{-1}$ is a Galois étale covering with group $W_G(L, \Sigma) = N_G(L, \Sigma)/L$. Since $G \times_L \Sigma_{reg}$ is open in $G \times_L (\Sigma)_{reg}$, its image $Y$ under
this étale morphism is open in $Y^+$. This proves that $Y$ is open in $X$. Moreover, since the map

$$
\pi : \ G \times_L \Sigma_{\text{reg}} \longrightarrow \ Y \\
g \star_L l \longmapsto \ g l g^{-1}
$$

is a Galois étale covering with group $W_G(L, \Sigma)$, we get that $Y$ is smooth (indeed, $G \times L \Sigma_{\text{reg}}$ is smooth).

Recall that $G \times L \Sigma_{\text{reg}} \rightarrow \ G \times_P \Sigma_{\text{reg}} V$, $g \star_L l \mapsto g \star_P l$ is an isomorphism (see Corollary L5). Moreover, it is clear that $\bar{\pi}^{-1}(Y) = G \times \Sigma_{\text{reg}} V$. We summarize all these facts in the following proposition.

**Proposition 2.3 (Lusztig [Lu1, §3.1, 3.2 and Lemma 4.3]).** With the above notation, we have:

1. $X$ is a closed irreducible subvariety of $G$ and $Y$ is open in $X$.
2. The natural map $\hat{Y} \rightarrow \hat{X}$, $g \star_L x \mapsto g \star_P x$ is an open immersion and the square

$$
\begin{array}{ccc}
\hat{Y} & \xrightarrow{\pi} & Y \\
\downarrow \alpha & & \downarrow \pi \\
\hat{X} & \xrightarrow{\bar{\pi}} & X
\end{array}
$$

is cartesian.

3. $\bar{\pi}$ (hence $\pi$) is a projective morphism.
4. $\pi$ is an étale Galois covering with group $W_G(L, \Sigma)$. In particular, $Y$ is smooth.

Note that $G$ acts on $\hat{Y}$, $\hat{Y}$, $\hat{X}$ and $\hat{X}$ by left translation on the first factor, and that it acts on $Y$ and $X$ by conjugation. Also, the group $Z(G) \cap Z(L)^o$ acts on the varieties $\hat{Y}$, $\hat{Y}$, $\hat{X}$ and $\hat{X}$ by left translation on the second factor, and it acts on $Y$ and $X$ by left translation. These actions of $G$ and $Z(G) \cap Z(L)^o$ commute.

We have the following commutative diagram

$$
\begin{array}{cccc}
\Sigma_{\text{reg}} & \xrightarrow{\alpha} & \hat{Y} & \xrightarrow{\beta} & \tilde{Y} & \xrightarrow{\pi} & Y \\
\downarrow \bar{\alpha} & & \downarrow \bar{\beta} & & \downarrow \bar{\pi} & & \\
\Sigma & \xrightarrow{\bar{\alpha}} & \bar{X} & \xrightarrow{\bar{\beta}} & \tilde{X} & \xrightarrow{\bar{\pi}} & X
\end{array}
$$

(2.4)

where $\alpha$ and $\bar{\alpha}$ are the canonical projections, and $\beta$ and $\bar{\beta}$ are the canonical quotient morphisms. Moreover, the vertical maps are the natural ones. Note that the morphisms $\beta$, $\bar{\beta}$, $\pi$ and $\bar{\pi}$ are $G \times (Z(G) \cap Z(L)^o)$-equivariant, and that the vertical maps $\hat{Y} \rightarrow \hat{X}$, $\tilde{Y} \rightarrow \tilde{X}$ and $Y \rightarrow X$ are also $G \times (Z(G) \cap Z(L)^o)$-equivariant.
2.B. Extension of the action of $W_G(L, \Sigma)$. We must notice that the action of the group $W_G(L, \Sigma)$ is defined only on $\tilde{Y}$. However, we will see in this subsection that it is possible to extend it to an open subset $\tilde{X}_{\min}$ of $\tilde{X}$ which, in general, contains strictly $\tilde{Y}$.

We first need some preliminaries to construct this extension. If $l \in \Sigma$, then
\[ \dim C_L(l) \geq \dim C_L(v) \]
and equality holds if and only if $l \in \Sigma$. Consequently, if $g \in X$, then
\[ \dim C_G(g) \geq \dim C_L(v) \]
(cf. Lemma 1.1 (2)). Moreover, if the equality holds, then, by Lemma 1.1 and the previous remark, $\tilde{\pi}^{-1}(g)$ is contained in
\[ \tilde{X}_0 = G \times_p \Sigma.V, \]
which is a smooth open subset of $\tilde{X}$.

Let us define now
\[ X_{\min} = \{ g \in X \mid \dim C_G(g) = \dim C_L(v) \} \]
and
\[ \tilde{X}_{\min} = \tilde{\pi}^{-1}(X_{\min}). \]

By [Hu, Proposition 1.4], $X_{\min}$ is an open subset of $X$, and, by the previous discussion, $\tilde{X}_{\min} \subset \tilde{X}_0$. Also, $Y \subset X_{\min}$, so $\tilde{Y} \subset \tilde{X}_{\min}$. Now, let $\pi_{\min} : \tilde{X}_{\min} \to X_{\min}$ denote the restriction of $\tilde{\pi}$: it is a projective morphism.

Moreover, if $g \in X_{\min}$, then, by Lemma 1.1 (1), $\pi_{\min}^{-1}(g)$ is a finite set. The morphism $\pi_{\min}$ being projective and quasi-finite, it is finite [Ha, Exercise III.11.2]. We gather these facts in the next proposition.

**Proposition 2.5.** With the above notation, we have:

1. $X_{\min}$ is a $G \times (Z(G) \cap Z(L)^o)$-stable open subset of $X$ containing $Y$.
2. $\tilde{X}_{\min}$ is a $G \times (Z(G) \cap Z(L)^o)$-stable smooth open subset of $\tilde{X}$ containing $\tilde{Y}$.
3. The morphism $\pi_{\min} : \tilde{X}_{\min} \to X_{\min}$ is finite.

The Proposition 2.5 has the following immediate consequence:

**Theorem 2.6.** (a) The variety $\tilde{X}_{\min}$ is the normalization of $X_{\min}$ in the variety $\tilde{Y}$. Therefore, there exists a unique action of the finite group $W_G(L, \Sigma)$ on the variety $\tilde{X}_{\min}$ extending its action on $\tilde{Y}$.

(b) This action is $G \times (Z(G) \cap Z(L)^o)$-equivariant, and the morphism $\pi_{\min}$ factorizes through the quotient $\tilde{X}_{\min}/W_G(L, \Sigma)$.

(c) If the variety $X_{\min}$ is normal, then $\pi_{\min}$ induces an isomorphism of varieties $\tilde{X}_{\min}/W_G(L, \Sigma) \simeq X_{\min}$.

**Notation** - (1) Let $(\Sigma.V)_{\min}$ denote the open subset $\Sigma.V \cap X_{\min}$ of $\Sigma.V$. Then $\tilde{X}_{\min} = G \times_p (\Sigma.V)_{\min}$.

(2) If there is some ambiguity, we will denote by $?^G_L$ the object $?$ defined above (for instance, $\tilde{Y}^G_L$, $\tilde{X}^G_{\min,L}$, $X^G_L$, $\pi^G_{\min,L}$...).
2.C. Unipotent classes. From now on, and until the end of this paper, \( \Sigma \) is the inverse image of a unipotent class of \( L/Z(L)^0 \). Note that a unipotent class is isolated. Let \( C \) denote the unique unipotent class contained in \( \Sigma \). From now on, the element \( v \) introduced in the previous section is chosen to denote the unique unipotent class contained in \( \Sigma \). We need only to determine which elements of \( \Sigma \) are contained in \( C \). We denote by \( \Sigma \) the unipotent radical of \( \Sigma \) and that \( \Sigma = Z(L)^0 \cdot C \simeq Z(L)^0 \times C \) and that \( \Sigma_{reg} = Z(L)^0_{reg} \cdot C \).

**Notation** - (1) We denote by \( C^G \) the induced unipotent class of \( C \) from \( L \) to \( G \), that is the unique unipotent class \( C_0 \) of \( G \) such that \( C_0 \cap C.V \) is dense in \( C.V \).

(2) If \( z \in Z(L)^0 \), the group \( C^G_p(z) = L.C_V(z) \) is connected (cf. Lemma [12]). So it is a parabolic subgroup of \( C^G_p(z) \) by [DM, Proposition 1.11 (ii)], with unipotent radical \( C_V(z) \) and Levi factor \( L \). We denote by \( u_z \) an element of \( C^G_p(z) \cap vC_V(z) \). We set \( \tilde{u}_z = 1 * P \cdot z u_z \in X_{min} \). (3) For simplicity, the unipotent element \( u_1 \) will be denoted by \( u \), and \( \tilde{u} \) stands for \( \tilde{u}_1 \).

**Remark 2.7** - Let us investigate here what are the elements of \( X_{min} \). Since \( X_{min} \subset X_0 \), we need only to determine which elements of \( \Sigma.V \) belong to \( X_{min} \). Let \( g \in \Sigma.V \). Let \( z \) (respectively \( u' \)) be the semisimple (respectively unipotent) part of \( g \). By Lemma [12], we may assume that \( z \) belongs to \( Z(L)^0 \). Now, let \( G' = C^g_z(z) \), \( P' = C^g_p(z) \), and \( V' = C_V(z) \). Then \( G' \) is a reductive subgroup of \( G \) containing \( L \), \( P' \) is a parabolic subgroup of \( G' \), and \( V' \) is its unipotent radical. Then, by [Bor, Corollary 11.12], we have \( u' \in G' \). On the other hand, \( C^g_G(g) = C^g_{G'}(u') \). Now, by Lemma [12] (2) and by [Spa, Proposition II.3.2 (b) and (e)], \( g \in X_{min} \) if and only if \( u' \in C^G \).

Hence, we have proved that

\[
\nabla^{-1}(\nabla(z)) \cap X_{min} = (zu_z)_G
\]

for every \( z \in Z(L)^0 \), and that

\[
X_{min} = \bigcup_{z \in Z(L)^0} (zu_z)_G. \quad \square
\]

If \( z \in Z(L)^0 \), we denote by \( H_G(L, \Sigma, z) \) the stabilizer of \( \tilde{u}_z \) in \( W_G(L, \Sigma) \). We first investigate what is the group \( H_G(L, \Sigma, 1) \). Recall that the group \( C^g_G(u) \) is contained in \( P \) [Spa, Proposition II.3.2 (e)], so that \( C_G(u)/C_P(u) \) is a finite set.

**Lemma 2.9.** We have \( \pi_{min}^{-1}(u) = \{ g * P \cdot u \mid g \in C_G(u) \} \). In particular,

\[
|\pi_{min}^{-1}(u)| = |C_G(u)/C_P(u)| = |A_G(u)/A_P(u)|.
\]

**Proof** - It is clear that \( \{ g * P \cdot u \mid g \in C_G(u) \} \) is contained in \( \pi_{min}^{-1}(u) \). Conversely, let \( g * P \cdot x \in \pi_{min}^{-1}(u) \). By replacing \( (g, x) \) by \( (gl^{-1}, lx) \) for a suitable choice of \( l \in L \), we may assume that \( \pi_L(x) = v \). Since \( gxg^{-1} = u \), this means that \( x \in vV \cap C^G \). By [Spa, Proposition II.3.2 (d)], there exists \( y \in P \) such that \( yxy^{-1} = u \). Therefore, \( gy^{-1} \in C_G(u) \) and \( g * P \cdot x = gy^{-1} * P \cdot u \). \( \square \)
Corollary 2.10. If $C_G(u) \subset P$, then $\pi_{\min}^{-1}(u) = \{\tilde{u}\}$. In particular, $W_G(L, \Sigma)$ stabilizes $\tilde{u}$, that is $H_G(L, \Sigma, 1) = W_G(L, \Sigma)$.

Now, let us consider the general case. The second projection $\tilde{X} \simeq G \times Z(L)^o \times \overline{U} \times V \to Z(L)^o$ factorizes through the quotient morphism $\tilde{X} \to \tilde{X}$. We denote by $z : \tilde{X} \to Z(L)^o$ the morphism obtained after factorization. The group $W_G(L)$ acts on $Z(L)^o$ by conjugation, and it is easy to check that the restriction $z_{reg} : \tilde{Y} \to Z(L)^o_{reg}$ of $z$ to $\tilde{Y}$ is $W_G(L, \Sigma)$-equivariant. Hence, the morphism $z_{\min} : \tilde{X}_{\min} \to Z(L)^o$ obtained by restriction from $z$ is $W_G(L, \Sigma)$-equivariant.

As a consequence, we get

\begin{equation}
\text{Stab}_{W_G(L, \Sigma)}(\tilde{g}) \subset \text{Stab}_{W_G(L, \Sigma)}(z(\tilde{g}))
\end{equation}

for every $\tilde{g} \in \tilde{X}_{\min}$. Also, note that $z(\tilde{g})$ is conjugate in $G$ to the semisimple part of $g = \pi_{\min}(\tilde{g})$ (cf. Proposition 1.6). In fact, one can easily get a better result :

Proposition 2.12. Let $z \in Z(L)^o$. Then :

(1) $H_G(L, \Sigma, z) = H_{C_G^o(z)}(L, \Sigma, 1)$.

(2) If $C_{C_G^o(z)}(u_z) \subset P$, then $H_G(L, \Sigma, z) = W_{C_G^o(z)}(L, \Sigma)$.

Proof - For the proof of (1), the reader may refer to the proof of Proposition 3.7 of this paper: indeed, the situations are quite analogous, and the arguments involved are exactly similar. However, since the situation in Proposition 3.7 is a little more complicated, we have decided to give a detailed proof only in this case. (2) follows from (1) and from Corollary 2.10.

2.D. An example. Assume in this subsection, and only in this subsection, that $L = T$. Then $C = 1$, $\Sigma = T$, $X = G \times_B B$, $X = G$ and $\pi : \tilde{X} \to G$ is the well-known Grothendieck map. Also, $W_G(L, \Sigma) = W$ in this case. Moreover, $X_{\min}$ is the open subset of $G$ consisting of regular elements. As an open subset of $G$, it is smooth. So the action of $W$ on $X$ extends to $X_{\min}$ and $X_{\min}/W = X_{\min}$.

Now, let $\tilde{g} \in \tilde{X}_{\min}$, $g = \pi_{\min}(\tilde{g})$, and $t = z(\tilde{g}) \in T$. We denote by $W^o(t)$ the Weyl group of $C_G^o(t)$ relative to $T$. The fiber $\pi_{\min}^{-1}(g)$ may be identified with the set of Borel subgroups of $G$ containing $g$. Since $X_{\min}/W = X_{\min}$, $W$ acts transitively on $\pi_{\min}^{-1}(g)$. But, $u_t$ is a regular unipotent element of $C_G^o(t)$. Therefore, $C_{C_G^o(t)}(u_t) \subset B$. So, by Proposition 2.12, we have

\[ \text{Stab}_{W}(\tilde{g}) = W^o(t). \]

As a consequence, we get the well-known result

\begin{equation}
|\{xB \in G/B \mid g \in \mathfrak{t}B\}| = |W|/|W^o(t)|.
\end{equation}

Remark - It is not the easiest way to prove 2.13.

Example 2.14 - Assume in this example that $G = GL_2(F)$, that

$L = T = \{\text{diag}(a, b) \mid a, b \in F^\times\}$,
and that $\Sigma = T$. Let us denote by $P^1$ the projective line. Then

$$\tilde{X} \simeq \{(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [x, y]) \in G \times P^1 \mid [ax + by, cx + dy] = [x, y]\},$$

$$X = G,$$

and $\pi : \tilde{X} \to X$ is identified with the first projection. Moreover, $X_{\text{min}}$ is the open subset of $G$ consisting of non-central elements. We shall give a precise formula for describing the action of $W$ on $X_{\text{min}}$ in this little example.

Let $w$ denote the unique non-trivial element of $W$. It has order 2. We define the right action of $w$ on $(g, [x, y]) \in \tilde{X}_{\text{min}}$ by

$$(g, [x, y]).w = \begin{cases} (g, [bx, (d-a)x - by]) & \text{if } (bx, (d-a)x - by) \neq (0, 0) \\
(g, [(a-d)y - cx, cy]) & \text{if } ((a-d)y - cx, cy) \neq (0, 0), \end{cases}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. One can check that, if $\tilde{X}_1$ (respectively $\tilde{X}_2$) is the open subset of $\tilde{X}$ defined by the first condition (respectively the second condition), then $\tilde{X}_1 \cup \tilde{X}_2 = \tilde{X}_{\text{min}}$, and that the formulas given above coincide on $\tilde{X}_1 \cap \tilde{X}_2$. So we have defined a morphism of varieties. The fact that it is an automorphism of order 2 is obvious, and the reader can check that it extends the action of $W$ on $\tilde{Y}$.

One can also check, as it is expected by 2.13, that $W$ acts trivially on the elements $(g, [x, y]) \in \tilde{X}_{\text{min}}$ such that $g$ is not semisimple. □

3. A morphism $W_G(L, \Sigma) \to A_L(v)$

The restriction of an $L$-equivariant local system on $C$ through the morphism $L/C^o_L(v) \to C$, $lC^o_L(v) \mapsto lvl^{-1}$ is constant. That is why this morphism is interesting when one is working with character sheaves (which are equivariant intersection cohomology complexes). This morphism can be followed along the diagram 2.13, and it gives rise to new varieties on which the group $W_G(L, v) = N_G(L, v)/C^o_L(v)$ acts (note that $W_G(L, v)/A_L(v) \simeq W_G(L, \Sigma)$). Following the method of the previous section, these actions can be extended to some variety $\tilde{X}'_{\text{min}}$ lying over $\tilde{X}_{\text{min}}$. We show in this section that some stabilizers under this action can be described in terms of a morphism of groups $W_G(L, \Sigma) \to A_L(v)$ (under some little hypothesis). In 3.4, elementary properties of this morphism will be investigated.

3.A. Notation. Let $\Sigma' = L/C^o_L(v) \times Z(L)^o$, and let $\Sigma'_{\text{reg}} = L/C^o_L(v) \times Z(L)^o_{\text{reg}}$. We denote by $f : \Sigma' \to \Sigma$, $(lC^o_L(v), z) \mapsto lvl^{-1} = zlvl^{-1}$. Then $f$ is a finite surjective $L$-equivariant morphism (here, $L$ acts on $\Sigma'$ by left translation on the first factor). We denote by $f_{\text{reg}} : \Sigma'_{\text{reg}} \to \Sigma_{\text{reg}}$ the restriction of $f$.

Now, let

$$\tilde{Y}' = G \times \Sigma'_{\text{reg}},$$

and

$$\tilde{Y}' = G \times L \Sigma'_{\text{reg}} = G/C^o_L(v) \times Z(L)^o_{\text{reg}}.$$
We then get a commutative diagram

\[
\begin{array}{ccc}
\Sigma'_{\text{reg}} & \xrightarrow{\alpha'} & \hat{Y}' \\
\downarrow{f_{\text{reg}}} & \swarrow{\beta'} & \downarrow{\hat{f}} \\
\Sigma_{\text{reg}} & \xrightarrow{\alpha} & \tilde{Y}'
\end{array}
\]

(3.1)

\[
\begin{array}{ccc}
\Sigma'_{\text{reg}} & \xrightarrow{\alpha'} & \hat{Y}' \\
\downarrow{f_{\text{reg}}} & \swarrow{\beta'} & \downarrow{\hat{f}} \\
\Sigma_{\text{reg}} & \xrightarrow{\alpha} & \tilde{Y}'
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma'_{\text{reg}} & \xrightarrow{\alpha'} & \hat{Y}' \\
\downarrow{f_{\text{reg}}} & \swarrow{\beta'} & \downarrow{\hat{f}} \\
\Sigma_{\text{reg}} & \xrightarrow{\alpha} & \tilde{Y}'
\end{array}
\]

where the vertical maps are induced by \( f_{\text{reg}} \), \( \alpha' \) is the projection on the first factor, \( \beta' \) is the quotient morphism, and \( \pi' = \pi \circ \tilde{f} \). The group \( G \) acts on \( \hat{Y}' \) and \( \tilde{Y}' \) by left translation on the first factor, and acts on \( Y \) by conjugation. The group \( Z(G) \cap Z(L)^{\circ} \) acts on \( \Sigma'_{\text{reg}} \) by translation on the second factor: it induces an action on \( \hat{Y}' \) and \( \tilde{Y}' \). The morphisms \( \hat{f} \), \( \tilde{f} \), \( \beta' \) and \( \pi' \) are \( G \times (Z(G) \cap Z(L)^{\circ}) \)-equivariant. Moreover, all the squares of diagram 3.1 are cartesian.

Now, we define

\[
W_G(L, v) = N_G(L, v) / C^o_L(v)
\]

(note that \( N_G(L, v)^{\circ} = C^o_L(v) \)). The group \( C_L(v) \) is a normal subgroup of \( N_G(L, v) \) so \( A_L(v) \) is a normal subgroup of \( W_G(L, v) \). Note that

\[
W_G(L, v) / A_L(v) \simeq W_G(L, \Sigma).
\]

The group \( N_G(L, v) \) acts freely on the right on the variety \( \hat{Y}' \) in the following way: if \( w \in N_G(L, v) \) and if \( (g, lC^o_L(v), z) \in \hat{Y}' \), then

\[
(g, lC^o_L(v), z).w = (gw, w^{-1}lwC^o_L(v), w^{-1}zw).
\]

This induces a free right \( G \times (Z(G) \cap Z(L)^{\circ}) \)-equivariant action of \( W_G(L, v) \) on \( \hat{Y}' \). Moreover, the fibers of the morphism \( \pi' \) are \( W_G(L, v) \)-orbits.

**Remark 3.2** - If \( a \in A_L(v) \) and \( g * L(lC^o_L(v), z) \in \tilde{Y}' \), then

\[
(g * L(lC^o_L(v), z)).a = g * L(laC^o_L(c), z).
\]

\( \square \)

### 3.B. Normalization.

Let \( \tilde{X}' \) be the normalization of the variety \( \tilde{X} \) in \( \hat{Y}' \). We denote by \( \tilde{f} : \tilde{X}' \to \tilde{X} \) the corresponding morphism of varieties. Let \( \tilde{X}'_0 \) (respectively \( \tilde{X}'_{\text{min}} \)) denote the inverse image, in \( \tilde{X}' \), of the variety \( \tilde{X}_0 \) (respectively \( \tilde{X}_{\text{min}} \)). We denote by \( \tilde{f}_0 : \tilde{X}'_0 \to \tilde{X}_0 \) (respectively \( \tilde{f}_{\text{min}} : \tilde{X}'_{\text{min}} \to \tilde{X}_{\text{min}} \)) the restriction of \( \tilde{f} \) to \( \tilde{X}'_0 \) (respectively \( \tilde{X}'_{\text{min}} \)). Then \( \tilde{X}'_0 \) (respectively \( \tilde{X}'_{\text{min}} \)) is the normalization of \( \tilde{X}_0 \) (respectively \( \tilde{X}_{\text{min}} \)) in \( \hat{Y}' \).
We can summarize the notation in the following commutative diagram

\[
\begin{array}{cccccc}
\tilde{Y}' & \longrightarrow & \tilde{X}'_{\min} & \longrightarrow & \tilde{X}'_0 & \longrightarrow & \tilde{X}' \\
\tilde{f} & \downarrow & \pi' & \downarrow & \pi'_{\min} & \downarrow & \tilde{f}_0 \\
\tilde{Y} & \longrightarrow & \tilde{X}_{\min} & \longrightarrow & \tilde{X}_0 & \longrightarrow & \tilde{X} \\
\downarrow & \pi & \downarrow & \pi_{\min} & \downarrow & & \\
Y & \longrightarrow & X_{\min} & \longrightarrow & X_0 & \longrightarrow & X \\
\end{array}
\]

In this diagram, all the horizontal maps are open immersions, and all the squares are cartesian. Since \(\tilde{X}_{\min}\) is the normalization of \(X_{\min}\) in \(\tilde{Y}\), we get:

**Theorem 3.3.** We have:

1. The variety \(\tilde{X}'_{\min}\) is the normalization of \(X_{\min}\) in \(\tilde{Y}'\). Therefore, the action of \(W_G(L,v)\) on \(Y'\) extends uniquely to an action of \(W_G(L,v)\) on \(\tilde{X}'_{\min}\).

2. \(\tilde{X}'\) inherits from \(Y'\) an action of \(G \times (Z(G) \cap Z(L)^o)\), and this action commutes with the one of \(W_G(L,v)\) on \(\tilde{X}_{\min}\).

**Remark 3.4** - We do not know how to determine in general the variety \(\tilde{X}'\). However, it is possible to give an explicit description of \(\tilde{X}'_0\). This can be done as follows. The parabolic subgroup \(P\) acts on \(\Sigma' \times V\) by the following action: if \(l, l_0 \in L, x, x_0 \in V\), and \(z_0 \in Z(L)^o\), then

\[
I^\Sigma (l_0 C_L^o (v), z_0, x_0) = (l_0 C_L^o (v), z_0, l_0^{-1} x_0^{-1} x_0).
\]

The reader can check that this defines an action of \(P\). Moreover, the morphism \(f \times \text{Id}_V : \Sigma' \times V \to \Sigma V, (l_0 C_L^o (v), z_0, x_0) \mapsto l_0 z_0 v l^{-1}_0 x\) induced by \(f\) is \(P\)-equivariant.

By Corollary 1.3, \(G \times_L \Sigma^\text{reg} \simeq G \times_P (\Sigma^\text{reg} \times V)\) is an open subset of \(G \times_P (\Sigma' \times V)\) isomorphic to \(\tilde{Y}'\). Moreover, the morphism

\[
G \times_P (\Sigma' \times V) \longrightarrow \tilde{X} = G \times_P \Sigma V
\]

induced by \(f\) is finite, as it can be checked by restriction to an open subset of the form \(g V^- P \times_P (\Sigma' \times V)\). Here, \(V^-\) is the unipotent radical of the opposite parabolic subgroup \(P^-\) of \(P\) with respect to \(L\). Finally, by the same argument, \(G \times_P (\Sigma' \times V)\) is smooth. Hence

\[(3.5) \quad \tilde{X}'_0 = G \times_P (\Sigma' \times V).\]

Since \(\tilde{X}'\) is the normalization of \(\tilde{X}\) in \(\tilde{Y}'\), it inherits an action of the group \(A_L(v)\). It is very easy to describe this action on \(\tilde{X}'_0\) by using 3.3. It acts on \(\tilde{X}'_0\) by right translation on the factor \(L/c_L^o (v)\) of \(\Sigma'\). This is a free action and the fibers of \(f_0\) are \(A_L(v)\)-orbits. \(\square\)
We will denote by \((\Sigma' \times V)_{\text{min}}\) the inverse image, under \(f \times \text{Id}_V\), of the open subset \((\Sigma V)_{\text{min}}\) of \(\Sigma V\). Then \(\tilde{X}'_{\text{min}} = G \times_P (\Sigma' \times V)_{\text{min}}\). The action of \(W_G(L, v)\) is quite mysterious, but the action of its subgroup \(A_L(v)\) is understandable. It is obtained by restriction from its action on \(\tilde{X}'_0\) which is described at the end of Remark 3.4.

3.C. Stabilizers. If \(z \in Z(L)\), let \(\bar{u}_z' = 1 \ast_P (C^0_L(v), z, v^{-1}u_z) \in \tilde{X}'_{\text{min}}\). Recall that \(u_z\) is an element of \(vC_V(z) \cap C^c_G(z)\). Note that \(f'_{\text{min}}(\bar{u}_z') = \bar{u}_z\), so that \(\pi'_{\text{min}}(\bar{u}_z') = u_z\). For simplification, we denote by \(\bar{u}'\) the element \(\bar{u}_z'\). The stabilizer of the element \(\bar{u}_z'\) in \(W_G(L, v)\) is denoted by \(H_G(L, v, z)\). The aim of this subsection is to get some informations about these stabilizers.

The first result comes from the fact that \(A_L(v)\) acts freely on \(\tilde{X}'_{\text{min}}\):

\[(3.6) \quad H_G(L, v, z) \cap A_L(v) = \{1\}.
\]

The second one is analogous to Proposition 2.12: it may be viewed as a kind of Jordan decomposition.

**Proposition 3.7.** If \(z \in Z(L)\), then \(H_G(L, v, z) = H_{C^c_G(z)}(L, v, 1)\).

**Proof -** Let \(z'\) denote the composite morphism of varieties \(\tilde{X}' \xrightarrow{f} \tilde{X} \xrightarrow{z} Z(L)\), and let \(z'_{\text{min}} : \tilde{X}'_{\text{min}} \to Z(L)\) denote the restriction of \(z'\). Then \(z'_{\text{min}}\) is a \(W_G(L, v)\)-equivariant morphism (as it can be checked by restriction to \(\tilde{Y}'\)). So, the group \(H_G(L, v, z)\) is contained in \(W_z = W_{C^c_G(z)}(L, v)\).

Let \(A_z = \{t \in Z(L) \mid C^0_G(t) \subset C^c_G(z)\}\); it is an open subset of \(Z(L)\) containing \(z\) and \(Z(L)_{\log}\). Now let \(\Sigma_z = A_z \cdot C\) and let \(\Sigma'_z = L/C^0_L(v) \times A_z\). Then

\[
\tilde{X}'_z = G \times_P (\Sigma'_z \times V)_{\text{min},L}
\]
is an open subset of \(\tilde{X}_{\text{min}}\) containing \(\bar{u}_z\), and it is stable under the action of \(W_z\), since \(A_z\) is and since \(\tilde{X}'_z = z'_{\text{min}^{-1}}(A_z)\). Now, let

\[
\tilde{X}'(z) = C_G(z) \times_{C^0_P(z)} (\Sigma'_z \times C_V(z))_{\text{min},L}^{C^c_G(z)}.
\]

The natural morphism \(\tilde{X}'(z) \to \tilde{X}'_z\) is injective and \(W_z\)-equivariant. This proves that the stabilizer \(H_G(L, v, z)\) is equal to the stabilizer of \(1 \ast_{C^0_P(z)} (C^0_L(v), z, v^{-1}u_z) \in \tilde{X}'(z)\) in \(W_z\). But this stabilizer must stabilize the connected component of \(1 \ast_{C^0_P(z)} (C^0_L(v), z, v^{-1}u_z)\), which is equal to \(C^0_G(z) \times_{C^0_P(z)} (\Sigma'_z \times V)_{\text{min},L}^{C^c_G(z)}\) (because \(C_P(z)\) is connected). Hence, it is contained in \(W_z^0 = W_{C^c_G(z)}(L, v)\), so it is equal to \(H_{C^c_G}(L, v, z)\) because this last variety is an open subset of \((\tilde{X}'_{\text{min}})_{L}^{C^c_G(z)}\).

Now, the action of \(W_z^0\) on \((\tilde{X}'_{\text{min}})_{L}^{C^c_G(z)}\) commutes with the translation by \(z\). Hence \(H_{C^c_G}(L, v, z) = H_{C^c_G}(L, v, 1)\).

**Remark -** The reader may be surprised by the fact that \(C^c_G(z)\) is not necessarily connected. However, he can check directly that the previous constructions \((\tilde{Y}, \tilde{X}_{\text{min}}, W_G(L, v)...)\) remains valid whenever \(G\) is not connected, provided that the parabolic subgroup \(P\) of \(G\) is connected. □
Proposition 3.8. If \( C_\mathbb{G}(u) \subset \mathbb{P} \), then:

1. \( \pi_{\min}^t(u) \) is the \( A_L(v) \)-orbit of \( \tilde{u}' \). In particular, \( |\pi_{\min}^t(u)| = |A_L(v)| \).
2. \( W_\mathbb{G}(L, v) = A_L(v) \times H_\mathbb{G}(L, v, 1) \).

Proof - (1) follows immediately from Corollary 2.10: indeed, \( A_L(v) \) acts freely and transitively on \( \pi_{\min}^t(u) \), so \( W_\mathbb{G}(L, v) \) acts transitively on \( \pi_{\min}^t(u) \). (2) follows from this remark and from 3.6.

3.D. Further investigations. The group \( C_\mathbb{G}^\circ(v) \cap L = C_{C_\mathbb{G}^\circ(v)}(\mathbb{L})^\circ \) is connected, because it is the centralizer of a torus in a connected group [Bor, Corollary 11.12]. Therefore, we have the well-known equality

\[ C_\mathbb{G}^\circ(v) \cap L = C_\mathbb{L}^\circ(v). \]

So the natural morphism \( C_L(v) \hookrightarrow C_\mathbb{G}(v) \) induces an injective morphism

\[ A_L(v) \hookrightarrow A_\mathbb{G}(v). \]

Let \( W_\mathbb{G}^\circ(L, v) = N_\mathbb{G}(L, v) \cap C_\mathbb{G}(v)/C_\mathbb{L}(v) \). Since \( C_\mathbb{G}(v) \cap L = C_\mathbb{L}(v) \), we have \( W_\mathbb{G}^\circ(L, v) \cap A_L(v) = 1 \). Moreover, \( W_\mathbb{G}^\circ(L, v) \) and \( A_L(v) \) are normal subgroups of \( W_\mathbb{G}(L, v) \), therefore \( W_\mathbb{G}^\circ(L, v) \times A_L(v) \) is naturally a subgroup of \( W_\mathbb{G}(L, v) \). This discussion has the following immediate consequence:

Lemma 3.11. If \( A_L(v) = A_\mathbb{G}(v) \), then \( W_\mathbb{G}(L, v) = W_\mathbb{G}^\circ(L, v) \times A_L(v) \).

Corollary 3.12. Assume that \( C_\mathbb{G}(u) \subset \mathbb{P} \), and that \( A_L(v) = A_\mathbb{G}(v) \). Then there exists a unique morphism of groups \( \varphi_{L,v}^G : W_\mathbb{G}^\circ(L, v) \rightarrow A_L(v) \) such that

\[ H_\mathbb{G}(L, v, 1) = \{ (w, a) \in W_\mathbb{G}^\circ(L, v) \times A_L(v) \mid a = \varphi_{L,v}^G(w) \}. \]

Proof - This follows from Proposition 3.8 (2) and from Lemma 3.11.

Corollary 3.13. Assume that \( C_\mathbb{G}(u) \subset \mathbb{P} \), that \( A_L(v) = A_\mathbb{G}(v) \), and that \( |A_L(v)| \) is odd. Then \( H_\mathbb{G}(L, v, 1) = W_\mathbb{G}^\circ(L, v) \).

The morphism \( \varphi_{L,v}^G \) is the central object of this paper. In Part II, we will compute it explicitly whenever \( v \) is a regular unipotent element under some restriction on \( L \).
3.E. **Separability.** Let $C^{\text{ét}}$ denote the separable closure of $C$ in $L/C_L^o(v)$ (under the morphism $L/C_L^o(v) \to C$, $l \mapsto lv^{-1}$). Note that $C^{\text{ét}}$ is smooth. The variety $C^{\text{ét}}$ inherits from $L/C_L^o(v)$ the action of $L$ by left translation, and the action of $A_L(v)$ by right translation. Then we have a sequence of $L \times A_L(v)$-equivariant morphisms

$$L/C_L^o(v) \longrightarrow C^{\text{ét}} \longrightarrow C.$$ 

The first morphism is bijective and purely inseparable, the second one is a Galois étale covering with group $A_L(v)$. We then define $\Sigma^{\text{ét}} = Z(L)^o \times C^{\text{ét}}$, $\Sigma_{\text{reg}}^{\text{ét}} = Z(L)_{\text{reg}}^o \times C^{\text{ét}}$, $\hat{Y}^{\text{ét}} = G \times \Sigma^{\text{ét}}_{\text{reg}}$, and $\check{Y}^{\text{ét}} = G \times_L \Sigma^{\text{ét}}_{\text{reg}}$. We have a commutative diagram with cartesian squares

$$\begin{array}{ccc}
\Sigma'_{\text{reg}} & \xrightarrow{\alpha'} & \check{Y}' \\
\downarrow f^{\text{ins}}_{\text{reg}} & & \downarrow \check{f}^{\text{ins}} \\
\Sigma^{\text{ét}}_{\text{reg}} & \xrightarrow{\alpha^{\text{ét}}} & \hat{Y}^{\text{ét}} \\
\downarrow f^{\text{ét}}_{\text{reg}} & & \downarrow \hat{f}^{\text{ét}} \\
\Sigma_{\text{reg}} & \xrightarrow{\alpha} & Y \\
\downarrow \beta & & \downarrow \pi \\
Y' & \xrightarrow{\beta'} & \check{Y}' \\
\end{array}$$

Here the maps $\check{\alpha}^{\text{ét}}$ and $\check{\beta}^{\text{ins}}$ are induced by the maps $\check{\alpha}$ or $\check{\beta}'$. Moreover, all the morphisms $\alpha^{\text{ét}}$ are Galois étale coverings, and all the morphisms $\beta^{\text{ins}}$ are bijective purely inseparable morphisms.

By the same argument as in Remark 3.4, the group $P$ acts on the variety $\Sigma^{\text{ét}} \times V$ and the quotient $\check{X}^{\text{ét}}_0 = G \times_P (\Sigma^{\text{ét}} \times V)$ exists: it is the separable closure of $\check{X}_0$ in $\check{X}'_0$. If we denote by $(\Sigma^{\text{ét}} \times V)_{\text{min}}$ the inverse of $(\Sigma, V)_{\text{min}}$ under the morphism $f^{\text{ét}} \times \text{Id}_V$, then $\check{X}^{\text{ét}}_{\text{min}} = G \times_P (\Sigma^{\text{ét}} \times V)_{\text{min}}$ is the normalization of $X_{\text{min}}$ in $\check{Y}^{\text{ét}}$. So it inherits an action of $W_G(L, v)$ and the bijective purely inseparable morphism $\check{f}^{\text{min}}_{\text{ét}} : \check{X}^{\text{ét}}_{\text{min}} \to \check{X}^{\text{ét}}_{\text{min}}$ induced by $\check{f}^{\text{min}}_{\text{fin}}$ is $W_G(L, v)$-equivariant. Moreover, the morphism $\check{f}^{\text{ins}}_{\text{min}} : X_{\text{min}}^{\text{ét}} \to X_{\text{min}}$ induced by $\check{f}^{\text{min}}_{\text{fin}}$ is a Galois étale covering with group $A_L(v)$. We summarize the notation in the next diagram.
Remark 3.15 - If \( z \in \mathbb{Z}(L)^{\circ} \), we denote by \( \tilde{u}_{z}^{\text{ét}} \) the image of \( \tilde{u}_{z}' \in \tilde{X}'_{\text{min}} \) in \( \tilde{X}_{\text{min}}^{\text{ét}} \) under the morphism \( \tilde{f}_{\text{min}}^{\text{ins}} \). Since \( \tilde{f}_{\text{min}}^{\text{ins}} \) is bijective and \( W_{G}(L, v) \)-equivariant, the stabilizer of \( \tilde{u}_{z}^{\text{ét}} \) in \( W_{G}(L, v) \) is equal to \( H_{G}(L, v, z) \). □

Example 3.16 - It may happen that the variety \( C^{\text{ét}} \) is different from \( L/C_{L}^{\circ}(v) \), so that the construction above is not irrelevant. Of course, it only occurs in positive characteristic. The smallest example is given by the group \( L = G = \text{SL}_{2}(\mathbb{F}) \), whenever \( p = 2 \) and

\[
v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Nevertheless, this is quite an unusual phenomenon. Indeed, if \( G = \text{SL}_{n}(\mathbb{F}) \) and if \( p \) does not divide \( n \), then the morphism \( L/C_{L}^{\circ}(v) \to C \) is always étale. Also, if \( G \) is a quasisimple group of type different from \( A \) and if \( p \) is good for \( G \), then again the morphism \( L/C_{L}^{\circ}(v) \to C \) is always étale. □

4. Elementary properties of \( \varphi_{L,v}^{G} \)

As it will be shown in §5, the knowledge of the morphism \( \varphi_{L,v}^{G} \) will be of fundamental importance in the description of the endomorphism algebra of an induced cuspidal character sheaf (cf. Corollary 5.10). That is the reason why we devote a section to gather the properties of this morphism. These properties may help to reduce the computations to small groups.
4.A. **Product of groups.** Assume in this subsection, and only in this subsection, that 
\( G = G_1 \times G_2 \). Let \( L = L_1 \times L_2, v = (v_1, v_2), P = P_1 \times P_2 \). Then, for every \( z = (z_1, z_2) \in Z(L)^o \), we have

\[
H_G(L, v, z) = H_{G_1}(L_1, v_1, z_1) \times H_{G_2}(L_2, v_2, z_2).
\]

Moreover, if \( A_L(v) = A_G(v) \) and if \( C_G(u) \subseteq P \), then

\[
\varphi^G_{L,v} = \varphi^G_{L_1,v_1} \times \varphi^G_{L_2,v_2}.
\]

4.B. **Changing the group.** Let \( G_1 \) be a connected reductive algebraic group such that there exists a morphism of algebraic groups \( \sigma : G_1 \to G \) such that

1. The kernel of \( \sigma \) is central in \( G_1 \),
2. The image of \( \sigma \) contains the derived group of \( G \).

We put \( L_1 = \sigma^{-1}(L) \) and \( P_1 = \sigma^{-1}(P) \), so that \( L_1 \) is a Levi subgroup of the parabolic subgroup \( P_1 \) of \( G_1 \). Let \( v_1 \) denote the unique unipotent element of \( G_1 \) such that \( \sigma(v_1) = v \) and let \( C_1 \) be the class of \( v_1 \) in \( G_1 \). Note that \( \sigma(C_1) = C \).

**Lemma 4.3.** (a) \( \sigma(C_{L_1}(v_1))Z(G)^o = C_L(v) \).

(b) The morphism \( A_{L_1}(v_1) \to A_L(v) \) induced by \( \sigma \) is surjective.

(c) The morphism \( W_{G_1}(L_1, \Sigma_1) \to W_G(L, \Sigma) \) induced by \( \sigma \) is an isomorphism.

(d) The morphism \( W_{G_1}^o(L_1, v_1) \to W_G^o(L, v) \) induced by \( \sigma \) is surjective.

(e) The morphism \( W_{G_1}^o(L_1, v_1) \to W_G^o(L, v) \) induced by \( \sigma \) is an isomorphism.

**Proof.** First, note that (b) is an immediate consequence of (a). Let \( l \in C_L(v) \). Then there exists \( l_1 \) in \( L_1 \) and \( z \in Z(G)^o \) such that \( \sigma(l_1) = lz \). So there exists \( x \in \text{Ker} \sigma \) such that \( l_1 v_1 l_1^{-1} = xv_1 \). But \( l_1 v_1 l_1^{-1} \) is unipotent and \( \text{Ker} \sigma \) consists of central semisimple elements of \( L_1 \), so \( x = 1 \), that is \( l_1 \) centralizes \( v_1 \). Hence (a) follows.

(c), (d) and (e) follow by similar arguments (note that \( \sigma(C_{G_1}(v_1))Z(G)^o = C_G^o(v) \)).

We will denote by a subscript \( ? \) the object associated to the datum \( (L_1, v_1) \) and defined in the same way as the object \( ? \) in \( G \) (for instance, \( \Sigma_1 = Z(L_1)^o . C_1, \tilde{X}_1, \tilde{Y}_1'... \)).

The reader may check that \( \sigma((\Sigma_1, V_1)_{\text{min}}) \subseteq (\Sigma, V)_{\text{min}} \), so that \( \sigma \) induces a morphism \( \tilde{X}_{1,\text{min}} \to \tilde{X}_{\text{min}} \). This morphism is \( W_{G_1}(L_1, \Sigma_1) \)-equivariant as it can be checked by restriction to \( Y_1 \) (here, \( W_{G_1}(L_1, v_1) \) and \( W_G(L, \Sigma) \) are identified via the morphism \( \sigma \) by Lemma 4.3(c)). Similarly, \( \sigma \) induces a \( W_{G_1}(L_1, v_1) \)-equivariant morphism \( \tilde{X}_{1,\text{min}} \to \tilde{X}_{\text{min}} \) (here, \( W_{G_1}(L_1, v_1) \) acts on \( \tilde{X}_{\text{min}} \) via the surjective morphism \( W_{G_1}(L_1, v_1) \to W_G(L, v) \)).

**Proposition 4.4.** If \( z_1 \in Z(L_1)^o \), then :

(a) \( H_{G_1}(L_1, \Sigma_1, z_1) = H_G(L, \Sigma, \sigma(z_1)) \).

(b) \( \sigma \) induces an isomorphism \( H_{G_1}(L_1, v_1, z_1) \simeq H_G(L, v, \sigma(z_1)) \).

Moreover,
(c) If $C_G(u_1) \subset P_1$ and if $A_L(v_1) = A_G(v_1)$, then $C_G(u) \subset P$ and $A_L(v) = A_G(v)$. In this case, the diagram

\[
\begin{array}{ccc}
W^\alpha_{G_1}(L_1,v_1) & \xrightarrow{\sigma} & W^\alpha_G(L,v) \\
\varphi_{L_1,v_1} & & \varphi^G_L \\
A_{L_1}(v_1) & \xrightarrow{\sigma} & A_L(v)
\end{array}
\]

is commutative.

**Proof** - It is clear that $H_{G_1}(L_1, \Sigma_1, z_1) \subset H_G(L, \Sigma, \sigma(z_1))$. To prove the reverse inclusion, we use Proposition 2.12 to reduce the problem to the case where $z_1 = 1$. Then if an element $w \in W_G(L, \Sigma)$ stabilizes $1 * P u$, this proves that there exists $z_1 \in \text{Ker} \sigma$ such that $(1 * P_1 u_1).w = 1 * P_1 z_1 u_1$. But, by 2.11, $z_1 = 1$. This proves (a).

To prove (b), we must notice that $\sigma(H_{G_1}(L_1, v_1, z_1)) \subset H_G(L, v, \sigma(z_1))$, so $\sigma$ induces a morphism $H_{G_1}(L_1, v_1, z_1) \rightarrow H_G(L, v, \sigma(z_1))$. This morphism is injective by 3.6. It is surjective by the same argument as the one used in (a). The proof of (b) is complete.

The first assertion of (c) follows from Lemma 4.3, (a), while the second follows from (b). ■

4.C. **Parabolic restriction.** In this subsection, we show that the above constructions are compatible with “restriction” to parabolic subgroups of $G$. We need some notation. Let $P'$ denote a parabolic subgroup of $G$ containing $P$ and let $L'$ denote the unique Levi subgroup of $P'$ containing $L$. Let $V'$ denote the unipotent radical of $P'$, and let $v' = \pi_{L'}(u)$. Then $v' \in v(V \cap L')$. We start by some elementary properties.

**Proposition 4.5.** (1) $v' \in C_{L'}$.

(2) If $A_L(v) = A_G(v)$, then $A_{L'}(v) = A_{L'}(v)$.

(3) If $C_G(u) \subset P$, then $C_{L'}(v') \subset P \cap L'$.

**Proof** - By Lemma 1.1, we have $\dim C_P(u) \geq \dim C_{P \cap L'}(v') \geq \dim C_L(v)$. But

$$\dim C_P(u) = \dim C_L(v)$$

because $u \in C^G$. So, $\dim C_{P \cap L'}(v') = \dim C_L(v)$. This proves that $\dim (v')_{P \cap L'} = \dim C + \dim V \cap L'$. So $(v')_{P \cap L'}$ is dense in $C.(V \cap L')$, that is $v' \in C_{L'}$. Hence (1) is proved.

(2) follows from 3.9 applied to the Levi subgroup $L'$. Let us now prove (3). Let $m \in C'_L(v')$. We only need to prove that $m \in P$. But $m_u \in v'.V' \cap C^G \subset vV \cap C^G$. So, by [Spa, Proposition 2.3.2 (d)], there exists $x \in P$ such that $m_u = x u$. So $x^{-1}m \in C_G(u)$. But $C_G(u) \subset P$ by hypothesis, so $m \in P$. ■

**Proposition 4.6.** If $C_G(u) \subset P$, then $H_G(L, v, 1) \cap W_L(L, v) = H_{L'}(L, v, 1)$. 
Proof - By Proposition 3.8 (2), the subgroups \( H_G(L, v, 1) \cap W_{L'}(L, v) \) and \( H_{L'}(L, v, 1) \) have the same index in \( W_{L'}(L, v) \) (this index is equal to \( |A_L(v)| \)). Consequently, it is sufficient to prove that

\[
H_G(L, v, 1) \cap W_{L'}(L, v) \subset H_{L'}(L, v, 1).
\]

Let

\[
\tilde{F}' = P' \times P (\Sigma' \times V)^{G}_{min,L}.
\]

It is an irreducible closed subvariety of \( \tilde{X}' \), and it is stable under the action of \( W_{L'}(L, v) \) (indeed, the open subset \( \tilde{O}' = P' \times_L \Sigma'_{reg} \) is obviously \( W_{L'}(L, v) \)-stable).

By Lemma [1.1] (2), the projection \( \pi_{L'}: P' \to L' \) sends an element of \( (\Sigma, V)^{G}_{min,L} \) to an element of \( (\Sigma, (V \cap L'))^{L'}_{min,L} \). So, it induces a map \( \gamma: \tilde{F}' \to (\tilde{X}'_{min})^{L'}_L \). Moreover, the diagram

\[
\begin{array}{ccc}
\tilde{O}' & \to & \tilde{F}' \\
\downarrow & & \downarrow \gamma \\
(\tilde{Y'})^{L'}_L & \to & (\tilde{X}'_{min})^{L'}_L 
\end{array}
\]

is commutative. The first vertical map is \( W_{L'}(L, v) \)-equivariant, so, by density, the second vertical map is also \( W_{L'}(L, v) \) equivariant. This proves (\#).  

Corollary 4.7. If \( C_G(u) \subset P \) and if \( A_L(v) = A_G(v) \), then

\[
\varphi^{L'}_L = \text{Res}_{W_{L'}(L, v)}^{W_G(L, v)} \varphi^G_{L,v}.
\]

Proof - By Proposition 4.5, (2) and (3), \( \varphi^{L'}_L \) is well-defined. So Corollary 4.7 follows from Proposition 4.6.  

Remark 4.8 - If \( G' \) is a connected reductive subgroup of \( G \) containing \( L \), then it may happen that

\[
\varphi^{G'}_L \neq \text{Res}_{W_{G'}(L, v)}^{W_G(L, v)} \varphi^G_L.
\]

An example is provided in Part II of this paper.
Hypothesis and notation: From now on, and until the end of this first part, we assume that \( C \) supports an irreducible cuspidal \([Lu1, \text{Definition 2.4}] \) local system \( E \). To this local system is associated an irreducible character \( \zeta \) of \( A_L(v) \), via the Galois étale covering \( C^\text{et} \rightarrow C \). Let \( F = \mathbb{Q}_\ell \boxtimes E \) (\( F \) is a local system on \( \Sigma \)) and let \( F_{\text{reg}} \) denote the restriction of \( F \) to \( \Sigma_{\text{reg}} \). Let \( K \) be the perverse sheaf on \( G \) obtained from the triple \((L, v, \zeta)\) by parabolic induction \([Lu2, 4.1.1]\) and let \( A \) denote its endomorphism algebra.

In \([Lu1, \text{Theorem 9.2}]\), Lusztig constructed an isomorphism \( \Theta : \mathbb{Q}_\ell W_G(L) \rightarrow A \). This isomorphism is very convenient for computing the generalized Springer correspondence. On the other hand, Lusztig’s construction is canonical but not explicit. The principal aim of this paper is to construct an explicit isomorphism \( \Theta' : \mathbb{Q}_\ell W_G(L) \rightarrow A \) by another method. It turns out that this isomorphism differs from Lusztig’s one by a linear character \( \gamma \) of \( W_G(L) \). The knowledge of \( \gamma \) would allow us to combine the advantages of both isomorphisms \( \Theta \) and \( \Theta' \). However, we are not able to determine it explicitly in general, although we can relate it with the morphism \( \varphi_{L,v}^G \) defined in the previous section. Note that \( \gamma = 1 \) whenever \( L = T \).

In this section we recall some well-known facts about cuspidal local systems, parabolic induction and endomorphism algebra. Most of these results may be found in \([Lu1]\) or \([Lu2]\). However, Theorem 3.4, which is of fundamental step for constructing the isomorphism \( \Theta' \) defined above, was proved in full generality in \([Bon2]\). The isomorphism \( \Theta' \) will be constructed in \( \S 6 \). We will also prove in \( \S 6 \) the existence of \( \gamma \) and its relation with \( \varphi_{L,v}^G \).

We will provide in \( \S 7 \) some properties of \( \gamma \) which allows us to reduce its computation in the case where \( G \) is semisimple, simply-connected, quasi-simple, and \( P \) is a maximal parabolic subgroup of \( G \). In Part II, we will use the knowledge of the morphism \( \varphi_{L,v}^G \) for \( v \) regular to determine explicitly the linear character \( \gamma \). However, we will need to assume that \( p \) is good for \( L \). As an application, we will get some precision on Digne, Lehrer and Michel’s theorem of Lusztig restriction of Gel’fand-Graev characters \([DLM2, \text{Theorem 3.7}]\).

5.A. Parabolic induction. For the convenience of the reader, we reproduce here the diagram 2.4.

\[
\begin{array}{cccccc}
\Sigma_{\text{reg}} & \xrightarrow{\alpha} & \hat{Y} & \xrightarrow{\beta} & \hat{\hat{Y}} & \xrightarrow{\pi} & Y \\
\Sigma & \xrightarrow{\alpha} & \hat{X} & \xrightarrow{\beta} & \hat{\hat{X}} & \xrightarrow{\pi} & X
\end{array}
\]

We define \( \hat{F}_{\text{reg}} = \alpha^* F_{\text{reg}} \) : it is a local system on \( \hat{Y} \). Moreover, since \( E \) is \( L \)-equivariant, there exists a local system \( \hat{F}_{\text{reg}} \) on \( \hat{Y} \) such that \( \beta^* \hat{F}_{\text{reg}} \simeq \hat{F}_{\text{reg}} \). By \([Lu1, 3.2]\), the morphism \( \pi : \hat{Y} \rightarrow Y \) is a Galois covering with Galois group \( W_G(L) = N_G(L)/L \), so \( \pi_* \hat{F}_{\text{reg}} = \pi_1 \hat{F}_{\text{reg}} \).
(because $\pi$ is finite hence proper) is a local system on $Y$. We denote by $K$ the following perverse sheaf on $G$:

\[(5.1) \quad K = IC(\overline{Y}, K)[\dim Y].\]

where $\pi_\ast \tilde{\mathcal{F}}_{\text{reg}} = K$. Recall that $\overline{Y} = X$, so that $\dim Y = \dim X$.

We shall give, following [Lu1, §4], an alternative description of the perverse sheaf $K$. Let $A$ be the following perverse sheaf on $L$:

\[A = IC(\Sigma, \mathcal{F})[\dim \Sigma].\]

Note that $\Sigma = Z(L)^0C$ so $A = \overline{\mathcal{C}}[\dim Z(L)^0] \boxtimes IC(C, \mathcal{E})[\dim C]$. Since $A$ is $L$-equivariant, there exists a perverse sheaf $\tilde{K}$ on $\tilde{X}$ such that

\[\alpha^* A[\dim G + \dim V] = \beta^* \tilde{K}[\dim P].\]

The perverse sheaf $\tilde{K}$ is in fact equal to $IC(\tilde{X}, \tilde{\mathcal{F}}_{\text{reg}})[\dim \tilde{X}]$. By [Lu1, Proposition 4.5], we have

\[(5.2) \quad K = R\pi_\ast \tilde{K}.\]

5.B. Recollection. The fact that $C$ admits a cuspidal local system has a lot of consequences. We gather some of them in the next theorem.

**Theorem 5.3 (Lusztig).** (a) $v$ is a distinguished unipotent element of $L$ (that is, $v$ is not contained in a Levi subgroup of a proper parabolic subgroup of $L$).

(b) $N_G(L)$ stabilizes $C$ and $\mathcal{E}$.

**Proof** - cf. [Lu1, Proposition 2.8] for (a) and [Lu1, Theorem 9.2] for (b) and (c). ■

The next theorem has been proved by Lusztig provided that $p$ is large enough by using the classification of cuspidal pairs [Lu4]. In [Bon2, Corollary to Proposition 1.1], the author gave a proof in the general case without using the classification.

**Theorem 5.4.** The injective morphism $C_L(v) \to C_G(v)$ induces an isomorphism of finite groups $A_L(v) \to A_G(v)$.

By Theorem 5.4 and Lemma 3.11, we have

\[(5.5) \quad W_G(L, v) = A_L(v) \times W_G^0(L, v),\]

and, by Theorem 5.3 (b), we have

\[(5.6) \quad W_G^0(L, v) \simeq W_G(L) = N_G(L)/L.\]
5.C. Lusztig’s description of $\mathcal{A}$. For each $w$ in $W^0_G(L,v)$, we choose a representative $\check{w}$ of $w$ in $N_G(L) \cap C_G^w(v)$. By Theorem 5.3 (b), the local systems $\mathcal{F}$ and $(\text{int } \check{w})^* \mathcal{F}$ are isomorphic. Let $\theta_w$ denote an isomorphism of $L$-equivariant local systems $\mathcal{F} \to (\text{int } \check{w})^* \mathcal{F}$. Then $\theta_w$ induces an isomorphism $\check{\theta}_w : \check{\mathcal{F}}_{\text{reg}} \to \gamma_w^* \check{\mathcal{F}}_{\text{reg}}$ where $\gamma_w : \check{Y} \to \check{Y}$, $(g,xL) \mapsto (g,x\check{w}^{-1}L)$ (cf. [Lu1, proof of Proposition 3.5]). But $\pi_* \gamma_w^* = \pi_*$. Hence $\pi_* \theta_w$ is an automorphism of $\mathcal{K}$. By applying the functor $\text{IC}(X, \cdot)[\dim Y]$, $\pi_* \theta_w$ induces an automorphism $\Theta_w$ of $K$. The automorphism $\Theta_w$, as well as $\theta_w$, is defined up to multiplication by an element of $\mathbb{Q}_l^\times$. By [Lu1], Proposition 3.5, $(\Theta_w)_{w \in W^0_G(L,v)}$ is a basis of the endomorphism algebra $\mathcal{A}$ of $K$; moreover, by [Lu1, Remark 3.6], there exists a family of scalar $(a_{w,w'})_{w,w' \in W^0_G(L,v)}$ of elements of $\mathbb{Q}_l^\times$ such that $\Theta_w \Theta_{w'} = a_{w,w'} \Theta_{ww'}$ for all $w$ and $w'$ in $W^0_G(L,v)$. Lusztig proved that it is possible to choose in a canonical way the family $(\theta_w)_{w \in W^0_G(L,v)}$ such that $\Theta_w \Theta_{w'} = \Theta_{ww'}$ for all $w$ and $w'$ in $W^0_G(L,v)$. The next theorem [Lu1, Theorem 9.2] explains his construction.

**Theorem 5.7 (Lusztig).** There exists a unique family of isomorphisms of local systems $\theta_w : \mathcal{F} \to (\text{int } \check{w})^* \mathcal{F}$, $w \in W^0_G(L,v)$ such that the following condition holds: for each $w' \in W^0_G(L,v)$, $\Theta_w$ acts trivially on $H_{\dim Y} K$, where $u$ is any element of $C^G$.

In the previous theorem, the uniqueness of the family $(\theta_w)_{w \in W^0_G(L,v)}$ follows from the fact that $H_{\dim Y} K \neq 0$ for each $u \in C^G$. As a consequence, one gets that the linear mapping

$$\Theta : \bigoplus_{w \in W^0_G(L,v)} \mathbb{Q}_l^\times \chi_{\mathcal{F}} \longrightarrow \mathcal{A} = \text{End}_{MG}(K)$$

is an isomorphism of algebras. If $\chi$ is an irreducible character of $W^0_G(L,v)$, we denote by $K_{\chi}$ an irreducible component of $K$ associated to $\chi$ via the isomorphism $\Theta$.

**Corollary 5.8 (Lusztig).** For each $u \in C^G$, we have:

(a) $H_{\dim Y} K_1 = H_{\dim Y} K$.

(b) $H_{\dim Y} K_\chi = 0$ for every non-trivial irreducible character $\chi$ of $W^0_G(L,v)$.

5.D. Restriction to the open subset $X_{\min}$. The restriction $\tilde{K}_0$ of $\tilde{K}[-\dim X]$ to $\tilde{X}_0$ is a local system [Lu1, 4.4], that is, a complex concentrated in degree 0, whose 0th term is a local system. In fact, $\tilde{K}_0$ is the local system on $\tilde{X}_0$ associated to the Galois étale covering $\tilde{X}_0^{et} \to \tilde{X}_0$ and to the character $\zeta$ of $A_L(v)$. Therefore, the restriction $K_{\min}$ of $\tilde{K}[-\dim X]$ to $\tilde{X}_{\min}$ is a local system. More precisely, it is the local system associated to the Galois étale covering $\tilde{X}_{\min}^{et} \to \tilde{X}_{\min}$ and to the character $\zeta$. Let $K_{\min}$ denote the restriction of $K[-\dim X]$ to $X_{\min}$. We have the following result.

**Proposition 5.9.** We have $K_{\min} = \pi_{\min,*} \tilde{K}_{\min}$. So, $K_{\min}$ is a constructible sheaf, that is a complex concentrated in degree 0.
**Proof** - Since $\pi_{\text{min}}$ is finite, the functor $\pi_{\text{min}*}$ is exact. The proposition follows from this remark and the Proper Base Change Theorem. ■

6. Another isomorphism between $\mathcal{A}$ and $\mathcal{Q}_c W_G^{\circ}(L, v)$

The aim of this section is to construct an explicit isomorphism $\Theta'$ between the endomorphism algebra $\mathcal{A}$ and the group algebra $\mathcal{Q}_c W_G^{\circ}(L, v)$. Our strategy is the following. First, note that the endomorphism algebra $\mathcal{A}$ of $\mathcal{K}$ is canonically isomorphic to the endomorphism algebra of the local system $\mathcal{K}$ on $\mathcal{Y}$. To this local system is associated a representation of the fundamental group $\pi_1(Y, y)$ of $Y$ (here, $y$ is any point of $Y$). This representation and its endomorphism algebra are easy to describe (cf. 6.1).

6.A. Representations of the fundamental group. Let $V_\zeta$ denote an irreducible left $\mathcal{Q}_c A_L(v)$-module affording the character $\zeta$. We may, and we will, assume that

$$\mathcal{F} = (f_{\text{et}}^*)_* \mathcal{Q}_c \otimes \mathcal{Q}_c A_L(v) V_\zeta,$$

$$\mathcal{F}_{\text{reg}} = (f_{\text{et}}^*)_* \mathcal{Q}_c \otimes \mathcal{Q}_c A_L(v) V_\zeta,$$

$$\hat{\mathcal{F}}_{\text{reg}} = (f_{\text{et}}^*)_* \mathcal{Q}_c \otimes \mathcal{Q}_c A_L(v) V_\zeta,$$

and

$$\hat{\mathcal{F}}_{\text{reg}} = (f_{\text{et}}^*)_* \mathcal{Q}_c \otimes \mathcal{Q}_c A_L(v) V_\zeta.$$

Here, $V_\zeta$ is identified with the constant sheaf with values in $V_\zeta$. From the third equality, we deduce that

$$\mathcal{K} = \pi_* \hat{\mathcal{F}}_{\text{reg}} = \pi_* (f_{\text{et}}^*)^* \mathcal{Q}_c \otimes \mathcal{Q}_c W_G^{\circ}(L, v) \text{Ind}_{A_L(v)}^{W_G^{\circ}(L, v)} V_\zeta.$$

Therefore, the endomorphism algebra of $\mathcal{K}$ is canonically isomorphic to the endomorphism algebra of the $\mathcal{Q}_c W_G^{\circ}(L, v)$-module $\text{Ind}_{A_L(v)}^{W_G^{\circ}(L, v)} V_\zeta$. But, by 5.3, this endomorphism algebra is canonically isomorphic to $\mathcal{Q}_c W_G^{\circ}(L, v)$. Since the functor $IC(Y, \cdot)[\dim Y]$ is fully faithful, it induces an isomorphism

$$\Theta': \mathcal{Q}_c W_G^{\circ}(L, v) \longrightarrow \mathcal{A}.$$

This isomorphism may be constructed in another way. The action of an element $\dot{w} \in N_G(L) \cap C_G^0(v)$ on $\Sigma_{\text{reg}}, \hat{\Sigma}_{\text{et}}, \hat{Y}_{\text{et}}$, and $\hat{Y}_{\text{et}}$ commutes with the action of $A_L(v)$. Therefore, there exists an isomorphism $\theta_{\dot{w}}': \mathcal{F} \to (\text{int } \dot{w})^* \mathcal{F}$ (respectively $\tau_{\dot{w}}': \hat{\mathcal{F}}_{\text{reg}} \to (\text{int } \dot{w})^* \hat{\mathcal{F}}_{\text{reg}}$) which induces the identity on the stalks at $zv$ (respectively $1*_{L} zv$) for every $z \in \mathcal{Z}(L)^0$ (respectively for every $z \in \mathcal{Z}(L)_{\text{reg}}^0$). Then

$$\beta^* \tau_{\dot{w}}' = \alpha^* \theta_{\dot{w}}'|_{\Sigma_{\text{reg}}}.$$

Now, let $\Theta'_{\dot{w}} = IC(Y, \pi_* \tau_{\dot{w}}')[\dim Y] : K \xrightarrow{\sim} K$. By 5.2, there exists an element $\gamma_{\dot{w}} \in \mathcal{Q}_c^\times$ such that

$$\Theta'(\dot{w}) = \gamma_{\dot{w}} \Theta'_{\dot{w}}.$$

By looking at the action on the stalk at $zv \in \mathcal{Y}$, one can immediately get the following result.
Proposition 6.4. With the above notation, we have $\Theta_w' = \Theta'(w)$ for every $w \in W^\circ_G(L, v)$.

Corollary 6.5. There exists a linear character $\gamma^G_{L,v,\zeta}$ of the Weyl group $W^\circ_G(L, v)$ such that

$$\Theta'(w) = \gamma^G_{L,v,\zeta}(w)\Theta(w)$$

for every $w \in W^\circ_G(L, v)$.

\textbf{Proof -} This follows from Theorem 5.7, from 6.3, and from Proposition 6.4. ■

If $\chi$ is an irreducible character of $W^\circ_G(L, v)$, we denote by $K'_{\chi}$ the irreducible component of $K$ associated to $\chi$ via the isomorphism $\Theta'$. By Corollary 6.5, we have

$$K'_{\chi} = K^G_{L,v,\zeta,\chi}.$$  \hfill (6.6)

6.B. Links between $\gamma^G_{L,v,\zeta}$ and $\varphi^G_{L,v}$. The following proposition is an immediate consequence of 6.6 and Corollary 5.8:

Proposition 6.7. The linear character $\gamma^G_{L,v,\zeta}$ of $W^\circ_G(L, v)$ is the unique irreducible character $\gamma$ of $W^\circ_G(L, v)$ satisfying $\dim Y K'_{\chi} \neq 0$ for some (or any) $u \in C^G$.

If $\chi$ is an irreducible character of $W^\circ_G(L, v)$, we denote by $K_{\min, \chi}$ (respectively $K'_{\min, \chi}$) the irreducible component of $K_{\min}$ associated to $\chi$ via the isomorphism $\Theta$ (respectively $\Theta'$). Now, let $V_{\chi}$ be an irreducible $W^\circ_G(L, v)$-module affording $\chi$ as character. Then, since $W_G(L, v)$ acts on $\tilde{X}^\text{ét}_{\min}$, it also acts on the constructible sheaf $(\pi^\text{ét}_{\min})_* \mathcal{Q}_{\ell}$ and we have by construction

$$K'_{\min, \chi} = (\pi^\text{ét}_{\min})_*^{\text{opp}} \otimes W_G(L, v) (V_{\chi} \otimes V_{\zeta}).$$

(6.8)

Since $u \in X_{\min}$, the stalk of $\mathcal{H}_u^{-\dim Y} K'_{\chi}$ at $u$ may easily be deduced from 6.8: we have

$$\dim_{\mathcal{O}_u} \mathcal{H}_u^{-\dim Y} K'_{\chi} = \dim(K'_{\min, \chi})_u = \langle \text{Res}_{H_G(L,v,1)}^W(\chi \otimes \zeta), 1_{H_G(L,v,1)} \rangle$$

We deduce immediately from this and from Proposition 6.7 the following result.

Proposition 6.9. The linear character $\gamma^G_{L,v,\zeta}$ is the unique irreducible character $\gamma$ of $W^\circ_G(L, v)$ such that $\langle \text{Res}_{H_G(L,v,1)}^W(\gamma \otimes \zeta), 1_{H_G(L,v,1)} \rangle \neq 0$.

Corollary 6.10. If $C_G(u) \subset P$, then

$$\gamma^G_{L,v,\zeta} = \frac{1}{\zeta(1)} \zeta \circ \varphi^G_{L,v}.$$
Corollary 6.11. If \( G(u) \subset P \), and if \( |A_L(v)| \) is odd, then \( \gamma_{L,v,\zeta}^G = 1 \).

Corollary 6.12. \( \gamma_{T,1,\mathfrak{P}_d}^G = 1 \).

Proof - Indeed, if \( P = B \) then \( u \) is a regular unipotent element and \( v = 1 \). Hence \( C_G(u) \subset B \) so Corollary 6.10 applies. But \( A_L(v) = 1 \), so \( \gamma_{T,1,\mathfrak{P}_d}^G = 1 \). ■

Example 6.13 - Assume in this example that \( v \) is a regular unipotent element of \( L \). In this case, \( u \) is a regular unipotent element of \( G \), so \( C_G(u) \subset P \). Moreover, \( A_L(v) \) is abelian [Spr]. So \( \zeta \) is a linear character and Corollary 6.10 applies. We get

\[
\gamma_{L,v,\zeta}^G = \zeta \circ \varphi_{L,v}^G.
\]

This case will be studied in full details in Part II. □

7. Elementary properties of the character \( \gamma_{L,v,\zeta}^G \)

7.A. Product of groups. We assume in this subsection that \( G = G_1 \times G_2 \) where \( G_1 \) and \( G_2 \) are reductive groups. Let \( L = L_1 \times L_2 \), \( v = (v_1, v_2) \), \( C = C_1 \times C_2 \) and \( \zeta = \zeta_1 \otimes \zeta_2 \).

Then it is clear that

\[
W_\sigma^G(L,v) = W_\sigma^{G_1}(L_1,v_1) \times W_\sigma^{G_2}(L_2,v_2)
\]

and that

\[
\gamma_{L,v,\zeta}^G = \gamma_{L_1,v_1,\zeta_1}^{G_1} \otimes \gamma_{L_2,v_2,\zeta_2}^{G_2}.
\]

7.B. Changing the group. We use here the notation introduced in §4.3. In particular, we still denote by a subscript \( ?_1 \) for the object in \( G_1 \) corresponding to \( ? \) in \( G \) (e.g. \( L_1, \mathcal{F}_1, \mathcal{F}_{\text{reg},1}, \hat{Y}_1, K_1, A_1, \Theta_1 \ldots \)). We have \( \sigma^{-1}(Y) = Y_1 \). Note that the groups \( W_\sigma^{G_1}(L_1,v_1) \) and \( W_\sigma^{G_2}(L,v) \) are isomorphic via \( \sigma \) by Lemma 4.3 (e).

Lemma 7.3. The local system \( \sigma^*\mathcal{E}_1 \) on \( C_1 \) is cuspidal. It is associated to the irreducible character \( \zeta_1 \) of \( A_{L_1}(v_1) \) obtained from \( \zeta \) by composing with the surjective morphism \( A_{L_1}(v_1) \to A_L(v) \) (cf. Lemma 4.3 (b)).

Proof - This is immediate from the alternative definition of a cuspidal local system given in terms of permutation representations [Lu1, introduction]. ■
Proposition 7.4. \( (a) \) The restriction of \( \sigma^*(K) \) to \( Y_1 \) is isomorphic to \( K_1 \).
\( (b) \) \( \sigma \) induces an isomorphism \( \hat{\sigma} : A_1 \simeq A \).
\( (c) \) The diagrams

\[
\begin{array}{ccc}
\overline{Q}_\ell W_{G_1}^\sigma(L_1, v_1) & \xrightarrow{\Theta_1} & A_1 \\
\sigma \downarrow & & \hat{\sigma} \downarrow \\
\overline{Q}_\ell W_G(L, v) & \xrightarrow{\Theta} & A
\end{array}
\]

\( \text{and} \)

\[
\begin{array}{ccc}
\overline{Q}_\ell W_{G_1}^\sigma(L_1, v_1) & \xrightarrow{\Theta'_1} & A_1 \\
\sigma \downarrow & & \hat{\sigma} \downarrow \\
\overline{Q}_\ell W_G(L, v) & \xrightarrow{\Theta'} & A
\end{array}
\]

are commutative.

Proof - \( (a) \) follows from the Proper Base Change Theorem \([\text{[M]}\text{, Chapter VI, Corollary 2.3]} \) applied to the Cartesian square

\[
\begin{array}{ccc}
\tilde{Y}_1 & \xrightarrow{\pi_1} & Y_1 \\
\tilde{\sigma} \downarrow & & \sigma \downarrow \\
\tilde{Y} & \xrightarrow{\pi} & Y.
\end{array}
\]

\( (b) \) follows from Lusztig’s description of \( A \). The commutativity of the first diagram in \( (c) \) follows from the fact that \( \sigma(C_{G_1}^{G_1}) = C^G \) and from Theorem \([\text{[L]}\text{, Theorem 5.7]} \) while the commutativity of the second one follows from Proposition \([\text{[L]}\text{, Proposition 6.4]} \).

Corollary 7.5. \( \gamma_{L,v,\zeta}^G \circ \sigma = \gamma_{L_1,v_1,\zeta_1}^{G_1} \).

7.C. Parabolic restriction. Let \( Q \) be a parabolic subgroup of \( G \) containing \( P \) and let \( M \) be the Levi subgroup of \( Q \) containing \( L \). It follows from \([\text{[Lu]}\text{, Theorem 8.3 (b)}] \) that:

Proposition 7.6. \( \gamma_{L,v,\zeta}^M = \text{Res}_{W_G^\sigma(L,v)}^{W_M^\sigma(L,v)} \gamma_{L,v,\zeta}^G \).
Remark 7.7 - If $G'$ is a connected reductive subgroup of $G$ which contains $L$, then
it may happen that $\gamma_{L,v,\zeta}^{G'} \neq \text{Res}_{W_0^G(L,v)}^{W_0^G(L,v)} \gamma_{L,v,\zeta}^G$. An example is provided by the group $G = \text{Sp}_4(\mathbb{F})$, as it will be shown in Part II of this paper. □

Remark - Whenever $C_G(u) \subset P$, then Corollary 6.10 shows that $[\gamma]$, Proposition 7.4, and Proposition 7.6 may be deduced from 4.2, Proposition 4.4, and Proposition 4.6 respectively. □

8. Introducing Frobenius

8.A. Hypothesis and notation. In this section, and only in this section, we assume that $\mathbb{F}$ is an algebraic closure of a finite field. In particular, $p > 0$. We fix a power $q$ of $p$ and we denote by $\mathbb{F}_q$ the subfield of $\mathbb{F}$ with $q$ elements. We assume also that $G$ is defined over $\mathbb{F}_q$ and we denote by $F : G \to G$ the corresponding Frobenius endomorphism. If $g \in G^F$, we denote by $[g]$ (or $[g]_G^F$ if necessary) the $G^F$-conjugacy class of $g$.

We keep the notation introduced in §4 (L, C, v, E, K, Θ, $\gamma_{L,v,\zeta}^G$...). We assume that $L$ is $F$-stable. Then, by Theorem 5.3(e), there exists $n \in N_G(L)$ such that $F(P) = ^gP$. Now, by Lang theorem, we can pick an element $g \in G$ such that $g^{-1}F(g) = n^{-1}$. Then $^gL$ and $^gP$ are $F$-stable. Since we are interested in the family of all $F$-stable subgroups of $G$ which are conjugate to $L$ under $G$, we may, and we will assume that $L$ and $P$ are both $F$-stable. Without loss of generality, we may also assume that $B$ and $T$ are $F$-stable.

We also assume that $v$ and $E$ are $F$-stable. Let $w \in W_0^G(L,v)$. We choose an element $g_w \in G$ such that $g_w^{-1}F(g_w) = \bar{w}^{-1}$ (recall that $\bar{w}$ is a representative of $w$ in $N_G(L) \cap C_G^e(v)$). We then put :

$$L_w = g_w L,$$
$$u_w = g_w u,$$
$$C_w = g_w C,$$
$$E_w = (\text{ad } g_w^{-1})^* E, \quad \text{and} \quad F_w = (\text{ad } g_w^{-1})^* F.$$

Then $L_w$ is an $F$-stable Levi subgroup of a parabolic subgroup of $G$, $v_w \in L_w^F$ is conjugate to $v$ in $G^F$ (because $g_w^{-1}F(g_w) \in C_{G^F}(v)$), $C_w$ is the conjugacy class of $v_w$ in $L_w$, $E_w$ is an $F$-stable cuspidal local system on $C_w$ and $F_w = \overline{Q}_l \otimes E_w$ (as a local system on $\Sigma_w = Z(L_w)^\circ \times C_w$).

8.B. A conjugacy result. In [Bon2, Proposition 2.1], the author proved the following result :

Proposition 8.1. Let $M$ and $M'$ be two $F$-stable Levi subgroups of (non necessarily $F$-stable) parabolic subgroups of $G$ which are geometrically conjugate and let $u$ be a unipotent element of $M^F$. Assume the following conditions holds :

(a) $u$ is a distinguished element of $M$,
(b) $N_G(M)$ stabilizes the class $[u]_M$, and
(c) $A_M(u) = A_G(u)$.

Then $[u]_{G^F} \cap M'$ is a single $M^F$-conjugacy class.
Corollary 8.2. If \( w \in W_G(L, v) \), then \([v]_G \cap L_w^F = [v_w]_L^G\).

**Proof** - This follows from Theorem 5.3 (a) and (c), from Theorem 5.4 and from the previous Proposition 8.1. \(\blacksquare\)

8.C. **Characteristic functions.** We choose once and for all an isomorphism of local systems \(\varphi : F^*E \to E\) and we denote by \(X_{\varphi}^E\) the class function on \(L_F\) defined by

\[
X_{\varphi}^E(l) = \begin{cases} 
\text{Tr}(\varphi_l, E_l) & \text{if } l \in C_F, \\
0 & \text{otherwise},
\end{cases}
\]

for any \(l \in L_F\). Using the isomorphism \(\Theta : \overline{T}W_G^0(L, v) \to A\), Lusztig defined an isomorphism of local systems \(\varphi_w : F^*E \to E_w\). We recall his construction \[Lu3, 9.3\]. Let \(\theta_w : E \to (\text{ad } \varphi)^*E\) be the isomorphism of local systems defined in Theorem 5.7. Then \(\theta_w\) induces an isomorphism of local systems

\[
F^* \circ (\text{ad } g_w^{-1})^* \theta_w : F^*E_w \longrightarrow (\text{ad } g_w^{-1})^* \circ F^*E.
\]

Moreover, \(\varphi\) induces an isomorphism

\[
(\text{ad } g_w^{-1})^* \varphi : (\text{ad } g_w^{-1})^* \circ F^*E \longrightarrow E_w.
\]

By composition of the two previous isomorphisms, we get an isomorphism

\[
\varphi_w : F^*E_w \longrightarrow E_w.
\]

Once the isomorphism \(\varphi\) is chosen, the isomorphism \(\varphi_w\) depends only on the construction of the isomorphism \(\theta_w\). By Corollary 8.3, the knowledge of the isomorphism \(\theta_w\) is equivalent to the knowledge of the linear character \(\gamma_{G, L, v, \zeta}^G\).

If \(\chi\) is an \(F\)-stable irreducible character of \(W_G^0(L, v)\), then we denote by \(\bar{\chi}\) the preferred extension of \(\chi\) to \(W_G^0(L, v) \rtimes < F >\) (the preferred extension has been defined by Lusztig \[Lu2\]). The choice of \(\varphi\) and \(\bar{\chi}\) induces a well-defined isomorphism \(\varphi_\chi : F^*K^\sim \longrightarrow K^\sim\). Then we set, for every \(g \in G_F\),

\[
X_{K, \chi, \varphi}(g) = \begin{cases} \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(H_g^i(\varphi_\chi), H_g^iK) & \text{if } g \text{ is unipotent,} \\
0 & \text{otherwise} \end{cases}
\]

The importance of the knowledge of the characteristic functions \(\chi_{E_w, \varphi_w}\) (therefore, of the linear character \(\gamma_{G, L, v, \zeta}^G\)) is given by the following theorem.

**Theorem 8.3 (Lusztig).** If \(p\) is almost good for \(G\) and if \(q\) is large enough, we have

\[
X_{K, \chi, \varphi} = \frac{1}{|W_G^0(L, v)|} \sum_{w \in W_G^0(L, v)} \bar{\chi}(wF) R^G_{L_w}(X_{E_w, \varphi_w}).
\]
Remark 8.4 - The expression “q is large enough” in Theorem 8.3 means that there exists a constant $q_0$ depending only on the root datum of $G$ such that Theorem 8.3 holds if $q \geq q_0$. We will say that the pair $(G, F)$ is friendly if the formula given by Theorem 8.3 holds in every connected reductive subgroup of $G$ having the same rank.

Theorem 8.3 says that, if $p$ is almost good for $G$ and if $q$ is large enough, then $(G, F)$ is friendly. □

Remark 8.5 - If $(G, F)$ is friendly, then [Bon1, Proof of Theorem 6.1.1] Mackey formula for Lusztig functors holds in $G$. Consequently, the Lusztig functor $R^G_{L_w \subset P_w}$ is known [Bon1, Proposition 7.1.1] to be independent of the choice of the parabolic subgroup $P_w$ of $G$ which admits $L_w$ as a Levi subgroup. That is why we denoted it simply by $R^G_{L_w}$. □

We conclude this section by determining explicitly the characteristic functions $X_{E^w, \varphi^w}$ using the linear character $\gamma^G_{L_w, v, \zeta}$. It follows from Lang’s theorem that the set of rational conjugacy classes contained in $C^F_w$ is in one-to-one correspondence with $H^1(F, A_L(v_w)) \cong H^1(\hat{w}F, A_L(v)) = H^1(F, A_L(v))$ (the last equality follows from the fact that $W^F_G(L, v)$ acts trivially on $A_L(v)$). Let $a \in H^1(F, A_L(v))$. We denote by $\hat{a}$ a representative of $a$ in $A_L(v)$ and by $v_{w,a}$ a representative of the rational conjugacy class contained in $C^F_w$ parameterized by $a$. If $w = 1$, we denote by $v_a$ the element $v_{w,a}$. We have $v_a \in L^F$. It must be noticed that $[v_{w,a}]_{E^w} = [v_a]_{G^F} \cap L^F$ (cf. Corollary 8.2).

By following step by step the construction of the isomorphisms $\varphi_w$, we obtain that the link between the class functions $X_{E^w, \varphi^w}$ and $X_{E_w, \varphi^w}$ is given in terms of the linear character $\gamma^G_{L_v, v, \zeta}$. More precisely, we get:

**Proposition 8.6.** Let $w \in W^G_G(L, v)$ and let $a \in H^1(F, A_L(v))$. Then
\[
X_{E^w, \varphi^w}(v_{w,a}) = X_{E^w, \varphi^w}(v_a)\gamma^G_{L_w, v, \zeta}(w).
\]

Assume now until the end of this section that $\zeta$ is a linear character. In this case, we have
\[
X_{E^w, \varphi^w}(v_{w,a}) = X_{E^w, \varphi^w}(v_w)\zeta(\hat{a}).
\]
(8.7)

Note that $\zeta(\hat{a})$ does not depend on the choice of $\hat{a}$ because $\zeta$ is $F$-stable. Hence, we deduce from Proposition 8.6 the following result:

**Corollary 8.8.** Assume that $\zeta$ is a linear character, and let $w \in W^G_G(L, v)$ and $a \in H^1(F, A_L(v))$. Then
\[
X_{E^w, \varphi^w}(v_{w,a}) = X_{E^w, \varphi^w}(v_w)\gamma^G_{L_w, v, \zeta}(w)\zeta(\hat{a}).
\]

On the other hand, if $l \notin C^F_w$, then
\[
X_{E^w, \varphi^w}(l) = 0.
\]
Remark 8.9 - In the theory of character sheaves applied to finite reductive groups, the characteristic functions $X_{\mathcal{E},w,\varphi}$ play a crucial role, as it is shown in Theorem 8.3. The Corollary 8.8 shows the importance of the determination of the linear character $\gamma_{L,v,\zeta}^G$. We will show in Part II how the knowledge of the linear character $\gamma_{L,v,\zeta}^G$ whenever $v$ is regular and $p$ is good for $L$ leads to an improvement of Digne, Lehrer and Michel’s theorem on Lusztig restriction of Gel’fand-Graev characters [DLM2, Theorem 3.7]. □

Remark 8.10 - The characteristic function $X_{\mathcal{E},w,\varphi}$ depends on the choice of the isomorphism $\varphi$. Since $\mathcal{E}$ is an irreducible local system, two isomorphisms between $F^*\mathcal{E}$ and $\mathcal{E}$ differ only by a scalar. Hence the two characteristic functions they define differ also by the same constant. This shows that the formula in Corollary 8.8 cannot be improved, because the factor $X_{\mathcal{E},\varphi}(v)$ depends on the choice of the isomorphism $\varphi$. We can give it, by multiplying $\varphi$ by a scalar, any value we want. □

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