CLT Variance Associated with Baxendale’s SDE

STEVEN R. FINCH

September 30, 2008

Abstract. Simple analysis of the leftmost eigenvalue of Ince’s equation (a boundary value problem with periodicity) resolves an open issue surrounding a stochastic Lyapunov exponent. Numerical verification is also provided.

Let \( a > b \) and \( \sigma > 0 \). Consider the stochastic differential equation (SDE)
\[
dX_t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} X_t \, dt + \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X_t \circ dW_t, \quad X_t \in \mathbb{R}^2
\]
where \( W_t \) is scalar Brownian motion with unit variance and \( \circ \) denotes the use of Stratonovich calculus. A measure of the stability of such a system is provided by the (almost-sure) Lyapunov exponent
\[
\ell = \lim_{t \to \infty} \frac{1}{t} \ln |X_t|
\]
for \( X_0 \neq 0 \). Baxendale [1, 2] computed a formula for \( \ell \) and, further, proved an associated central limit theorem (CLT)
\[
\lim_{t \to \infty} P \left( \frac{\frac{1}{t} \ln |X_t| - \ell}{s} \leq v \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} \exp \left( -\frac{u^2}{2} \right) \, du.
\]
No formula has been known for the variance \( s^2 \) until now. Our calculation of \( s^2 \) is based on the boundary value problem (BVP)
\[
y''(x) + c \sin(2x)y'(x) + (\lambda - \mu c \cos(2x)) y(x) = 0,
\]
\[
y'(0) = y'(\pi) = 0
\]
due to Ince [3], where \( c = -(a - b)/\sigma^2 \) and \( \mu \approx 0 \). More precisely, if \( \lambda(\mu) \) is the leftmost eigenvalue of the Ince BVP, given \( \mu \), then
\[
\ell = \frac{a + b}{2} - \frac{\sigma^2}{2} \lambda'(0), \quad s^2 = -\frac{\sigma^2}{2} \lambda''(0).
\]
In the next three sections, we discuss how Ince’s equation arises and the details of computing \( \lambda'(0) \) and \( \lambda''(0) \). The final section is devoted to numerical verification of the preceding.

\( \copyright \) 2008 by Steven R. Finch. All rights reserved.
1. Equivalence

Let $\gamma/\sigma^2$ denote the rightmost eigenvalue of the differential operator $[\Pi]$

$$\gamma \frac{d}{dx} - a - b \sin(2x) \frac{d}{dx} + \frac{\mu}{\sigma^2} (a \cos^2 x + b \sin^2 x)$$

(which is obtained, in part, by projecting the solution $X_t$ of Baxendale’s SDE onto the unit circle). By the double angle formula for cosine, the BVP we wish to study is

$$y''(x) - \frac{a - b}{\sigma^2} \sin(2x)y'(x) + \left[ -2\gamma \frac{\mu}{\sigma^2} + \frac{\mu}{\sigma^2} ((a + b) + (a - b) \cos(2x)) \right] y(x) = 0$$

hence $c = -(a - b)/\sigma^2$ follows, as does

$$\lambda = -\frac{2\gamma}{\sigma^2} + \frac{a + b}{\sigma^2} \mu;$$

hence

$$\lambda' = -\frac{2\gamma'}{\sigma^2} + \frac{a + b}{\sigma^2} \Rightarrow \lambda'' = -\frac{2\gamma''}{\sigma^2} \Rightarrow s^2 = -\frac{\sigma^2}{2} \lambda''(0).$$

This argument demonstrates the equivalence of Baxendale’s setting (in terms of $\gamma'(0)$ and $\gamma''(0)$) and our setting (in terms of $\lambda'(0)$ and $\lambda''(0)$).

2. First Derivative

Let $y(x, \mu)$ denote the solution of the Ince BVP with $\lambda = \lambda(\mu)$ that satisfies the initial conditions $y(0) = 1, y'(0) = 0$. Recall that $\lambda(\mu)$ is the leftmost such eigenvalue. Define

$$z(x, \mu) = \frac{\partial}{\partial \mu} y(x, \mu).$$

Differentiate Ince’s equation with respect to $\mu$, yielding

$$z''(x, \mu) + c \sin(2x) z'(x, \mu) + (\lambda'(\mu) - c \cos(2x)) y(x, \mu) + (\lambda(\mu) - \mu c \cos(2x)) z(x, \mu) = 0.$$ Call this $(\ast)$. Set $\mu = 0$, yielding

$$z''(x, 0) + c \sin(2x) z'(x, 0) + \lambda'(0) - c \cos(2x) = 0$$

because $\lambda(0) = 0$ and $y(x, 0) = 1$. Multiply both sides of $(\ast)$ by $\exp(-\frac{\pi}{2} \cos(2x))$, yielding

$$\left[ \exp(-\frac{\pi}{2} \cos(2x)) z'(x, 0) \right]' = (c \cos(2x) - \lambda'(0)) \exp(-\frac{\pi}{2} \cos(2x)).$$
Since \( z'(0, 0) = 0 \),
\[
\exp\left(-\frac{\xi}{2} \cos(2x)\right) z'(x, 0) = \int_{0}^{x} (c \cos(2\theta) - \lambda'(0)) \exp\left(-\frac{\xi}{2} \cos(2\theta)\right) d\theta.
\]

Now \( z'(x, 0) \) has period \( \pi \), hence \( z'(\pi, 0) = 0 \) and therefore
\[
\int_{0}^{\pi} (c \cos(2\theta) - \lambda'(0)) \exp\left(-\frac{\xi}{2} \cos(2\theta)\right) d\theta = 0.
\]

It follows that
\[
\lambda'(0) = \frac{c}{\int_{0}^{\pi} \exp\left(-\frac{\xi}{2} \cos(2\theta)\right) d\theta} \int_{0}^{\pi} (c \cos(2\theta) - \lambda'(0)) \exp\left(-\frac{\xi}{2} \cos(2\theta)\right) d\theta = c \frac{I_1(-\frac{\xi}{2})}{I_0(-\frac{\xi}{2})}
\]
where \( I_0, I_1 \) are modified Bessel functions \([4]\). This reproduces Baxendale’s formula for \( \ell \).

3. Eigenfunction
In the next section, we will need a formula for \( z(x, 0) \). Note that
\[
y(x, \mu) \approx 1 + \mu z(x, 0)
\]
when \( \mu \approx 0 \), thus a byproduct of our work is an approximation of the leftmost eigenfunction of Ince’s equation to first order.

From \([4]\), the following expressions hold:
\[
\exp\left(-\frac{\xi}{2} \cos(2\theta)\right) = I_0(-\frac{\xi}{2}) + 2 \sum_{k=1}^{\infty} I_k(-\frac{\xi}{2}) \cos(2k\theta),
\]
\[
\exp(-\frac{\xi}{2} \cos(2\theta)) \cos(2\theta) = I_1(-\frac{\xi}{2}) + \sum_{k=1}^{\infty} \left( I_{k-1}(-\frac{\xi}{2}) + I_{k+1}(-\frac{\xi}{2}) \right) \cos(2k\theta)
\]

hence the integrand \( (c \cos(2\theta) - \lambda'(0)) \exp\left(-\frac{\xi}{2} \cos(2\theta)\right) \) is equal to
\[
c \sum_{k=1}^{\infty} \left( I_{k-1}(-\frac{\xi}{2}) + I_{k+1}(-\frac{\xi}{2}) \right) \cos(2k\theta)
\]
Integrating once gives
\[
\exp\left(-\frac{c}{2} \cos(2x)\right) z'(x, 0) = \frac{c}{2} \sum_{k=1}^{\infty} \frac{j_k}{k} \sin(2k x).
\]
Integrating twice gives
\[
z(x, 0) = \frac{c}{2} \sum_{k=1}^{\infty} \frac{j_k}{k} \int_0^x \exp\left(\frac{c}{2} \cos(2\theta)\right) \sin(2k \theta) d\theta.
\]
Although we do not explicitly write out the integrals here, they are elementary and can be computed symbolically for arbitrary \(k\).

4. Second Derivative

Define
\[
w(x, \mu) = \frac{\partial^2}{\partial^2 \mu} y(x, \mu) = \frac{\partial}{\partial \mu} z(x, \mu).
\]
Differentiate equation (*) with respect to \(\mu\), yielding
\[
w''(x, \mu) + c \sin(2x) w'(x, \mu) + \lambda''(\mu) y(x, \mu) + 2(\lambda'(\mu) - c \cos(2x)) z(x, \mu)
+ (\lambda(\mu) - \mu c \cos(2x)) w(x, \mu) = 0.
\]
Set \(\mu = 0\), yielding
\[
w''(x, 0) + c \sin(2x) w'(x, 0) + \lambda''(0) + 2(\lambda'(0) - c \cos(2x)) z(x, 0) = 0
\]
because \(\lambda(0) = 0\) and \(y(x, 0) = 1\). We again multiply both sides by \(\exp\left(-\frac{c}{2} \cos(2x)\right)\):
\[
\left[\exp\left(-\frac{c}{2} \cos(2x)\right) w'(x, 0)\right]' = \left[2(\cos(2x) - \Lambda'(0)) z(x, 0) - \lambda''(0)\right] \exp\left(-\frac{c}{2} \cos(2x)\right).
\]
Since \(w'(0, 0) = 0\),
\[
\exp\left(-\frac{c}{2} \cos(2x)\right) w'(x, 0) = \int_0^x \left[2(\cos(2\theta) - \Lambda'(0)) z(\theta, 0) - \lambda''(0)\right] \exp\left(-\frac{c}{2} \cos(2\theta)\right) d\theta.
\]
Now \(w'(x, 0)\) has period \(\pi\), hence \(w'(\pi, 0) = 0\) and therefore
\[
\int_0^\pi \left[2(\cos(2\theta) - \Lambda'(0)) z(\theta, 0) - \lambda''(0)\right] \exp\left(-\frac{c}{2} \cos(2\theta)\right) d\theta = 0.
\]
It follows that
\[
\lambda''(0) = \frac{2}{\pi I_0\left(-\frac{c}{2}\right)} \int_0^\pi (\cos(2\theta) - \Lambda'(0)) z(\theta, 0) \exp\left(-\frac{c}{2} \cos(2\theta)\right) d\theta
\]
where \(z(\theta, 0)\) is defined via an infinite series in the preceding section.
5. Numerical Verification

One way to compute $\lambda(\mu)$ is to construct the infinite tridiagonal matrix

$$M = \begin{pmatrix}
  r_0 & \sqrt{2q_0} & 0 & 0 & 0 \\
  \sqrt{2q_0} & r_1 & q_{-2} & 0 & 0 \\
  0 & q_1 & r_2 & q_{-3} & 0 \\
  0 & 0 & q_2 & r_3 & q_{-4} \\
  0 & 0 & q_3 & r_4 & \ddots
\end{pmatrix}$$

where $r_n = 4n^2$ and $q_n = (-n + \mu/2)c$. The leftmost eigenvalue of $M$ is $\lambda(\mu)$.

Another way to compute $\lambda(\mu)$ is to solve the continued fraction equation

$$-\frac{\lambda}{2} = \frac{p_0}{4 \cdot 1^2 - \lambda} - \frac{p_1}{4 \cdot 2^2 - \lambda} - \frac{p_2}{4 \cdot 3^2 - \lambda} - \frac{p_3}{4 \cdot 4^2 - \lambda} - \frac{p_4}{4 \cdot 5^2 - \lambda} - \cdots$$

where $p_n = (-n + \mu/2)(n + 1 + \mu/2)c^2$.

High precision estimates of the derivatives of $\lambda(\mu)$ at zero are found via

$$\lambda'(0) \approx \frac{\lambda(\mu)}{\mu}, \quad \lambda''(0) \approx \frac{\lambda(\mu)}{\mu} - \frac{c}{\mu} \frac{I_1(-\frac{\mu}{2})}{I_0(-\frac{\mu}{2})}$$

for $\mu \approx 0$. For example, when $a = 1$, $b = -2$ and $\sigma = 10$, we obtain

$$\ell = -0.4887503163943852244580286..., \quad s^2 = 0.0112485762885419873084837...$$

as the CLT parameter values. As another example,

$$\ell = 0.3941998582469360577816389..., \quad s^2 = 0.3841476218435126147382099...$$

when $\sigma = 1$ (and $a$, $b$ remain unchanged). The same results were obtained using exact formulation in sections 2 and 4, confirming our work.

6. Acknowledgements

Peter Baxendale suggested that I work on Ince’s equation, Hans Volkmer sketched out the derivation of $\lambda'(0)$ and $\lambda''(0)$, and Joe Keane found the generating function in [4] for $\exp(r \cos(\theta))$. My grateful thanks go to all three!
References

[1] P. H. Baxendale, Moment stability and large deviations for linear stochastic differential equations, *Probabilistic Methods in Mathematical Physics*, Proc. 1985 Katata/Kyoto conf., ed. K. Itô and N. Ikeda, Academic Press, 1987, pp. 31–54; MR0933817 (89c:60068).

[2] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, 1992, pp. 545–548; MR1214374 (94b:60069).

[3] E. L. Ince, A linear differential equation with periodic coefficients, *Proc. London Math. Soc.* 23 (1925) 56–74.

[4] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1972, pp. 375–376, eqns. (9.6.10) and (9.6.34); MR1225604 (94b:00012).

[5] H. Volkmer, Coexistence of periodic solutions of Ince’s equation, *Analysis* 23 (2003) 97–105; MR1983977 (2004c:34074).

Steven R. Finch
Department of Statistics
Harvard University
Science Center
1 Oxford Street
Cambridge, MA 02138
Steven.Finch@inria.fr