Periods and elementary real numbers

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Abstract

The periods, introduced by Kontsevich and Zagier, form a class of complex numbers which contains all algebraic numbers and several transcendental quantities. Little has been known about qualitative properties of periods. In this paper, we compare the periods with hierarchy of real numbers induced from computational complexities. In particular we prove that periods can be effectively approximated by elementary rational Cauchy sequences. As an application, we exhibit a computable real number which is not a period.

1 Introduction

In their paper [10], Kontsevich and Zagier introduced the notion of periods:

Definition 1. A period is a complex number whose real and imaginary parts are values of absolutely convergent integral of rational functions with rational coefficients, over domains in $\mathbb{R}^\ell$ given by polynomial inequalities with rational coefficients.

The set of all periods is denoted by $\mathcal{P} \subset \mathbb{C}$. Obviously, $\mathcal{P}$ is a countable set, forms a $\mathbb{Q}$-algebra (because of Fubini’s theorem) and contains all algebraic numbers and several transcendental quantities, like $\pi$ and $\log n$. One of their motivations to introduce this notion is that the structure of $\mathcal{P}$ is directly related to profound theory of motives. See [19] for related problems in transcendental number theory.

Kontsevich and Zagier pose several conjectures and problems on $\mathcal{P}$. However it seems that the qualitative properties of $\mathcal{P}$ have not been well studied.

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so far. For instance, they pose the following “Problem 3 Exhibit at least one number which does not belong to $P$”. We have not had any properties on real numbers which can distinguish non-periods from periods.

The purpose of this paper is to give an answer to this problem by constructing a computable real number which can not be a period.

We approach the problem as follows. Since the real number field $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the Euclidean norm, a positive real number $\alpha \in \mathbb{R}_{>0}$ can be expressed as the limit of a positive rational Cauchy sequence

$$\lim_{n \to \infty} \frac{a(n)}{b(n)} = \alpha,$$

where $a$ and $b$ are functions $\mathbb{N} \to \mathbb{N}$. Therefore a positive real number $\alpha$ is expressed by a pair of functions $a, b : \mathbb{N} \to \mathbb{N}$.

The observation that not all functions $\mathbb{N} \to \mathbb{N}$ are computable “by finite means”, since the set of all functions $\mathbb{N}^\mathbb{N}$ is uncountable, leads us to consider the computability of the functions $a$ and $b$. The idea of computability goes back to the seminal paper [18] by A. Turing. Turing defines computable real numbers as those real numbers with computable decimal expansions. An equivalent definition is that the real numbers which are limits of Cauchy sequences (1) with computable functions $a$ and $b$ (see [13] or §2.1 below). So, refined notions of computability enable us to hierarchize computable real numbers [16, 14, 4, 5].

In this paper, we will focus on a proper sub-class called “elementary functions” $\mathbb{N} \to \mathbb{N}$ introduced in [7, 9] (see §2 below for definitions). The main result (Theorem 18) states that every real period is an elementary real number, that (roughly speaking) is, we can choose $a$ and $b$ from elementary functions. And we will also construct a computable real number which is not elementary (§2.3). The non-elementary real numbers can not be periods by our main result.

Let us briefly describe the idea of the proof. First we show that periods are generated by the volumes $\mathrm{vol}(D)$ of the bounded domains of the form

$$D = \left\{ (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell \mid G_k(x_1, \ldots, x_\ell) > 0, k = 1, \ldots, q \right\},$$

where $G_k \in \mathbb{Z}[x_1, \ldots, x_\ell]$ are polynomials of integer coefficients. To approximate the volume $\mathrm{vol}(D)$, we use the Riemann sum, that is, consider the union of small cubes

$$V_n := \text{Union of cubes contained in } D \text{ with vertices in } \left( \frac{1}{n} \mathbb{Z} \right)^\ell.$$

Then, clearly, $\mathrm{vol}(V_n)$ converges to $\mathrm{vol}(D)$ as $n \to \infty$. However there are two major problems here.
(a) Which small cubes are contained in the domain \( D \)?

(b) In which rate \( \text{vol}(V_n) \) converges to \( \text{vol}(D) \)? (As will be seen in Definition 9, we have to know the rate of convergence elementarily.)

Let \( C \subset \mathbb{R}^\ell \) be a cube. Then the problem (a) above is to ask whether or not the first-order formula

\[
\forall x (x \in C \implies x \in D)
\]

is true. In general, the truth assignment for a first-order formula with quantifiers (\( \forall, \exists \)) is difficult. However, in our situation, Tarski’s quantifier elimination for real closed ordered field tells us that the validity of the above formula can be decided by a quantifier free formula. It is simply a Boolean combination of polynomial inequalities on the coefficients of \( G_k \)'s. This enables us to conclude the rational sequence \( \text{vol}(V_n) \) is elementary.

The other problem (b) is related to count how many small cubes are there near the boundary \( \partial D \)? It is essentially done by bound the Minkowski dimension of the boundary \( \partial D \) by using resolution of singularities of algebraic varieties.

The organization of this paper is as follows. \( \S 2 \) is about elementary functions and elementary real numbers. Section \( \S 2.1 \) begins with the definition of the class \( \mathbb{R}_E \) of real numbers computable by a given class \( E \subset \mathbb{N}^\mathbb{N} \) of functions. Section \( \S 2.2 \) gives the precise definition of elementary functions and elementary real numbers. In \( \S 2.3 \) we algorithmically enumerate all elementary Cauchy sequence. Then by the diagonal argument, we construct a computable real number which is not an elementary real number. In view of the main result in \( \S 3 \) this number can not be a period. In \( \S 3 \) we first state the main result. After stating the main result in \( \S 3.1 \) we will reduce the problem to the bounded cases by employing results from structure theorems of semi-algebraic sets in \( \S 3.2 \). In \( \S 3.3 \) we recall quantifier elimination by Tarski, and by using it, we will construct an elementary rational sequence converging to the volume of bounded semi-algebraic domain. In the rest, \( \S 3.5 \) and \( \S 3.6 \) we prove that the sequence converges elementarily.

2 Elementary real numbers

**Notation.** In this section, \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \) denotes the set of nonnegative integers and \( (\mathbb{N}^n)^\mathbb{N} = \{f : \mathbb{N}^n \rightarrow \mathbb{N}\} \) denotes the set of all functions from \( \mathbb{N}^n \) to \( \mathbb{N} \). We only deal with nonnegative real and rational numbers.
2.1 Computable real numbers

The set $\mathbb{R}$ of real numbers is defined as the completion of the rational number field $\mathbb{Q}$ by the metric $d(x, y) = |x - y|$. In other words, exhibiting a real number is equivalent to exhibit a Cauchy sequence in $\mathbb{Q}$. Hence for a given nonnegative real number $\alpha \in \mathbb{R}$, there exist two functions $a, b : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n \to \infty} \frac{a(n)}{b(n) + 1} = \alpha.$$  

(The term “+1” in the denominator is just for avoiding to be equal to zero.)

In the paper [18], Turing introduced the notion of computable real numbers by restricting the class of functions $a, b : \mathbb{N} \to \mathbb{N}$. Following Turing and subsequent studies [15, 4, 5], we shall set the following definitions.

Definition 2. Let $\mathcal{E} \subset \mathbb{N}^\mathbb{N}$ be a class of functions. A nonnegative real number $\alpha \in \mathbb{R}$ is said to be $\mathcal{E}$-computable if there exist $a(x), b(x), c(x) \in \mathcal{E}$ such that

$$\left| \frac{a(x)}{b(x) + 1} - \alpha \right| < \frac{1}{k}, \text{ for } \forall x \geq c(k).$$

Denote the set of all $\mathcal{E}$-computable real numbers by $\mathbb{R}_\mathcal{E}$.

Example 3. Obviously $\mathbb{R}_\mathcal{E} \subset \mathbb{R}$ depends on the class $\mathcal{E}$.

1. Let $(\text{Const}) \subset \mathbb{N}^\mathbb{N}$ be the set of all constant functions. Then $\mathbb{R}_{(\text{Const})} = \mathbb{Q}$.

2. Let $(\text{Lin}) \subset \mathbb{N}^\mathbb{N}$ be the set of all functions of linear growth, that is,

$$(\text{Lin}) = \{ f \in \mathbb{N} | \exists C > 0, \text{ s.t. } f(n) < C \cdot n \}.$$  

Then $\mathbb{R}_{(\text{Lin})} = \mathbb{R}$. Indeed for given $\alpha \in \mathbb{R}$, define

$$a(n) = \lfloor (n + 1) \cdot \alpha \rfloor$$

$$b(n) = n,$$

which are of linear growth. It is easily shown that

$$\left| \frac{a(n)}{b(n) + 1} - \alpha \right| < \frac{1}{n + 1}.$$  

(3) If $\mathcal{E}$ is the set of all computable or recursive (resp. primitive recursive) functions, then $\mathbb{R}_\mathcal{E}$ is the set of computable (resp. primitive recursive) real numbers. (See [18] and [13, 14] for computable numbers. And see [4] for a recent survey on primitive recursive real numbers.)
2.2 Elementary functions

In order to state the main result, we need the notion of elementary functions (Elem). Here we consider functions having any number of arguments, that is, \( f : \mathbb{N}^n \rightarrow \mathbb{N} \) for \( n = 1, 2, \ldots \).

We begin with the simplest functions and operations on functions.

**Definition 4.** The zero function: \( o(x) = 0 \). The successor function: \( s(x) = x + 1 \). The \( i \)-th projection function: \( P^i_1(x_1, \ldots, x_n) = x_i \). These three functions are called the initial functions.

**Definition 5.** Define the modified subtraction \( m : \mathbb{N}^2 \rightarrow \mathbb{N} \) as follows:

\[
m(x, y) = x - y : = \begin{cases} 
  x - y & \text{if } x \geq y, \\
  0 & \text{if } x < y.
\end{cases}
\]

Let \( f(x_1, \ldots, x_m) \) be a function with \( m \) arguments. Let \( g_i(y_1, \ldots, y_n) \) \((i = 1, \ldots, m)\) be functions of \( n \) arguments. Then the composition

\[
f(g_1(y_1, \ldots, y_n), \ldots, g_m(y_1, \ldots, y_n))
\]

is a function with \( n \) arguments.

Let \( f(t, x_1, x_2, \ldots, x_n) \) be a function with \((n + 1)\) arguments. We define bounded summation by

\[
\sum_{t \leq x} f(t, x_1, \ldots, x_n) = f(0, x_1, \ldots, x_n) + \cdots + f(x, x_1, \ldots, x_n),
\]

and bounded product by

\[
\prod_{t \leq x} f(t, x_1, \ldots, x_n) = f(0, x_1, \ldots, x_n) \times \cdots \times f(x, x_1, \ldots, x_n),
\]

which are functions with \((n + 1)\) arguments.

**Definition 6.** The class (Elem) of elementary functions is the smallest class of functions:

1. containing the initial functions, the addition \( x + y \), the multiplication \( x \cdot y \), the modified subtraction \( x - y \),
2. closed under composition, and
3. closed under bounded summation and product.

**Example 7.** The following are examples of elementary functions.
(1) By definitions, $s(o(x)) = 1$, $s(s(o(x))) = 2$, etc, are elementary. Hence the constant function is elementary. Since $s(x) \cdot s(0) = x$, the identity function is elementary. The power $x^{y+1} = \prod_{k=0}^{y} x$ is also elementary.

(2) The sign function

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is elementary. Indeed, $\text{sgn}(x) = 1 \cdot (1 \cdot x)$.

(3) Recall that a subset $P \subset \mathbb{N}$ is called a predicate. A predicate $P$ is said to be elementary if the characteristic function

$$\chi_P(x) = \begin{cases} 1 & \text{if } x \in P, \\ 0 & \text{if } x \notin P \end{cases}$$

is an elementary function. If $P$ and $Q$ are elementary predicates, then the Boolean connection $P \land Q$, $P \lor Q$ and $\neg P$ are also elementary predicates.

(4) The order predicate

$$f_\geq(x, y) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \leq y \end{cases}$$

is elementary. Indeed $f_\geq(x, y) = \text{sgn}(x - y)$. Other functions $f_\leq, f_\leq$ are similarly elementary.

(5) The quotient $q(x, y) = \left\lfloor \frac{x}{y+1} \right\rfloor$ is elementary. Indeed,

$$q(x, y) = \left( \sum_{i=0}^{x} f_\geq(x, i \cdot (y + 1)) \right) \div 1.$$  

Similarly, the logarithm $l(a, b) = \left\lfloor \log_a b \right\rfloor$ and the square root $\left\lfloor \sqrt{x} \right\rfloor$ are also elementary.

(6) Bounded minimizer

$$(\mu y_1 \leq n)(f(y_1, y_2, \ldots, y_k) = 0)$$

is defined as the least $t \leq n$ such that $f(t, y_2, \ldots, y_k) = 0$ and $n$ if no such $t$. If $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is elementary, then

$$g(n, y_2, \ldots, y_k) = (\mu y_1 \leq n)(f(y_1, y_2, \ldots, y_k) = 0)$$

is also elementary.
The pairing function $J(x, y)$ is defined by

$$J(x, y) = \frac{(x + y)(x + y + 1)}{2} + y.$$  

The inverse pairing functions $L(z), R(z)$ are defined by the following relations

$$J(L(z), R(z)) = z, \ L(J(x, y)) = x, \ R(J(x, y)) = y.$$  

The functions $L, R$ are also elementary.

**Remark 8.** There also exists a computable but non-elementary function, e.g.,

$$f(x) = x^{x^{\ldots^x}} \ (x \ floors),$$

i.e., $f(2) = 2^2 = 4$, $f(3) = 3^{3^3} = 3^{27} = 7625597484987$, $f(4) = 4^{4^{4^4}} = 4^{2^{16}} > 4.34078 \times 10^{154}$. This is a very rapidly growing function, faster than any elementary function with one variable. (See [15] for details.)

Recall that the set of elementary real numbers $\mathbb{R}_{(\text{Elem})}$ is defined as follows.

**Definition 9.** A real number $\alpha \in \mathbb{R}$ is called elementary if there exist $a(x), b(x), c(x) \in \text{(Elem)}$ such that

$$\left| \frac{a(x)}{b(x) + 1} - \alpha \right| < \frac{1}{k}, \ \text{for } \forall x \geq c(k). \quad (3)$$

The following proposition is straightforward.

**Proposition 10.** The set of elementary real numbers $\mathbb{R}_{(\text{Elem})}$ forms a field.

**Definition 11.** A map $g : \mathbb{N} \to \mathbb{Q}$ is said to be elementary if $g$ is expressed as

$$g(x) = \frac{a(x)}{b(x) + 1}$$

for some $a(x), b(x) \in \text{(Elem)}$. An map $g : \mathbb{N} \to \mathbb{Q}$ is said to be fast if it satisfies

$$|g(x) - g(x + 1)| < \frac{1}{7^x + 1},$$

for $\forall x \in \mathbb{N}$.

**Lemma 12.** A real number $\alpha \in \mathbb{R}$ is elementary if and only if there exist an elementary fast map $g : \mathbb{N} \to \mathbb{Q}$ such that

$$\lim_{x \to \infty} g(x) = \alpha. \quad (4)$$
Proof. Suppose we have $a(x), b(x), c(x) \in (\text{Elem})$ satisfying Eq.(3). Set $k = 8^{n+1}$ and $x = c(8^{n+1})$, we have

\[
\left| \frac{a(c(8^{n+1}))}{b(c(8^{n+1})) + 1} - \alpha \right| < \frac{1}{8^{n+1}}.
\]

Since $a(c(8^{n+1})), b(c(8^{n+1}))$ are elementary on $n$, we have $\alpha \in \mathbb{R}_{(\text{Elem})}$. Put $g(x) = a(c(8^{x+1}))/b(c(8^{x+1}) + 1)$. Then $|g(n) - \alpha| < 8^{-n-1}$. Hence $|g(n) - g(n + 1)| < 8^{-n-1} + 8^{-n-2}$, which is less than $7^{-n-1}$. \qed

2.3 A non-elementary real number

In this section, we construct a non-elementary real number, essentially, by the diagonal argument. Together with the main result in the next section, it is an example of real number which is not a period.

First we recall a simpler description of elementary functions, due to Mazzanti.

Proposition 13. (Mazzanti [12]) All elementary functions can be generated from the following four functions by composition:

- The successor, $x \mapsto S(x) = x + 1$.
- The modified subtraction, $(x, y) \mapsto x - \cdot y$.
- The quotient, $(x, y) \mapsto \lfloor \frac{x}{y+1} \rfloor$.
- The exponential function, $(x, y) \mapsto x^y$.

Next we enumerate all elementary functions \{f : \mathbb{N} \to \mathbb{N} \mid \text{elementary}\} of one variable by using the pairing functions $J, L, R$ in Example 7 (7). For each $e \in \mathbb{N}$ we attach an elementary function $f_e : \mathbb{N} \to \mathbb{N}$ as follows.

(0) If $L(e) = 0$, then $f_e(x) = x$. (That is, $f_{J(0,k)}(x) = x$).

(1) If $e = J(1, k)$, then $f_e(x) = S(f_k(x)) = f_k(x) + 1$.

(2) If $e = J(2, k)$, then $f_e(x) = f_{L(k)} \cdot f_{R(k)}$.

(3) If $e = J(3, k)$, then $f_e(x) = \lfloor \frac{f_{L(k)}}{f_{R(k)}+1} \rfloor$.

(4) If $e = J(4, k)$, then $f_e(x) = (f_{L(k)})^{f_{R(k)}}$.

(5) If $e = J(c, k)$ with $c \geq 5$, then $f_e(x) = 0$.  

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**Example 14.** Here are some examples. \( f_0(x) = x \) by (0). Since 1 = \( J(1, 0) \), \( f_1 = S \circ f_0(x) = x + 1 \) by (1). Since 2 = \( J(0, 1) \), again \( f_2 = x \). Since 3 = \( J(2, 0) = J(2, J(0, 0)) \), \( f_3 = f_0 \circ f_0 = 0 \). Since 4 = \( J(1, 1) \), \( f_4 = S \circ f_1 = f_1 + 1 = x + 2 \). Since 169 = \( J(1, 16) = J(1, J(4, 1)) = J(1, J(4, J(1, 0))) \), \( f_{169} = f_{16} + 1 = (x + 1)^2 + 1 \), etc.

Now we can enumerate all elementary maps
\[ g : \mathbb{N} \rightarrow \mathbb{Q}_{\geq 0}, \]
by the following way:
\[ g_e(n) := \frac{f_{L(e)}(n)}{f_{R(e)}(n) + 1}. \] (5)

Obviously the sequence \( \{g_e(x)\}_{x \in \mathbb{N}} \) is not Cauchy in general. We enforce being fast on these sequences. For an elementary sequence \( g : \mathbb{N} \rightarrow \mathbb{Q} \), define
\[ g(n) = \begin{cases} g(n) & \text{if } (\forall i < n)(|g(i) - g(i + 1)| < 7^{-i-1}), \\ g(n_0) & \text{otherwise, where } n_0 := (\mu i < n)(|g(i) - g(i + 1)| \geq 7^{-i-1}). \end{cases} \]

The map \( \overline{g} : \mathbb{N} \rightarrow \mathbb{Q} \) is a fast elementary map by definition, and \( g \) is fast if and only if \( g = \overline{g} \).

**Definition 15.** For \( e \in \mathbb{N} \), define the \( e \)-th elementary real number by
\[ \beta_e := \lim_{n \to \infty} \overline{g_e}(n). \]

From Lemma 12 every elementary real number is the limit of a fast sequence, we have
\[ \{\beta_0, \beta_1, \ldots, \beta_e, \ldots\} = \mathbb{R}_{\text{Elem}}. \]

**Example 16.** First several terms are \( \beta_0 = 0, \beta_1 = 1, \beta_2 = \beta_3 = 0, \beta_4 = 1/2 \) etc. Let us compute \( \beta_{40} \). Since 40 = \( J(4, 4) \), \( L(40) = R(40) = 4 \). Thus \( g_{40} = f_{40}/(f_{40} + 1) \). Recall Example 14 that we already have \( f_4(x) = x + 2 \). Hence \( g_{40}(x) = \frac{x + 2}{x + 3} \). This is not fast, the enforced one is
\[ \overline{g}_{40}(x) = \begin{cases} 2/3 & \text{if } x = 0, \\ 3/4 & \text{if } x > 0. \end{cases} \]

At the end we obtain \( \beta_{40} = \frac{3}{4} \).

Now we construct a non-elementary computable real number \( \alpha \in \mathbb{R} \) as the limit of sequence
\[ \alpha_n = \frac{2 \varepsilon_1}{3^1} + \frac{2 \varepsilon_2}{3^2} + \frac{2 \varepsilon_3}{3^3} + \cdots + \frac{2 \varepsilon_n}{3^n}, \] (6)
defined as follows. Put $\alpha_0 = 0$ and define $\varepsilon_n (n \geq 1)$ inductively as

$$
\varepsilon_{n+1} = \begin{cases} 
0 & \text{if } \overline{g_n}(n) > \alpha_n + \frac{1}{2 \cdot 3^n} \\
1 & \text{if } \overline{g_n}(n) \leq \alpha_n + \frac{1}{2 \cdot 3^n} 
\end{cases}
$$

(7)

**Proposition 17.** Set $\alpha = \lim_{n \to \infty} \alpha_n$, then $\alpha \notin \mathbb{R}^{(\text{Elem})}$.

*Proof.* We shall prove $\alpha \neq \beta_e$ for any $e \in \mathbb{N}$. By the definition of $\alpha_n$,

$$
\alpha \leq \alpha_n + 2(3^{-n-1} + 3^{-n-2} + \cdots) = \alpha_n + 3^{-n}.
$$

So we have

$$
\alpha \in [\alpha_n, \alpha_n + 3^{-n}],
$$

(8)

for all $n \in \mathbb{N}$. Since $|\overline{g_e}(n) - \overline{g_e}(n+1)| < 7^{-n-1}$,

$$
|\overline{g_e}(n) - \beta_e| < 7^{-n-1}(1 + 7^{-1} + 7^{-2} + \cdots) = \frac{1}{7^n \cdot 6}.
$$

Thus we have

$$
\beta_e \in \left( \overline{g_e}(n) - \frac{1}{6 \cdot 7^n}, \overline{g_e}(n) + \frac{1}{6 \cdot 7^n} \right),
$$

(9)

If $\overline{g_e}(e) \leq \alpha_e + 2^{-1}3^{-e}$, then $\alpha_{e+1} = \alpha_e + 2 \cdot 3^{-e-1}$. Hence

$$
\alpha \in \left[ \alpha_e + \frac{2}{3^{e+1}}, \alpha_e + \frac{3}{3^{e+1}} \right].
$$

$$
\beta_e < \overline{g_e}(e) + \frac{1}{6 \cdot 7^e}
\leq \alpha_e + \frac{1}{2 \cdot 3^e} + \frac{1}{6 \cdot 7^e}
\leq \alpha_e + \frac{1}{2 \cdot 3^e} + \frac{1}{6 \cdot 3^e}
= \alpha_e + \frac{2}{3^{e+1}}
= \alpha_{e+1} \leq \alpha.
$$

In particular, $\alpha \neq \beta_e$. If $\overline{g_e}(e) > \alpha_e + 2^{-1}3^{-e}$ we can prove $\beta_e > \alpha$ similarly. In conclusion we have $\alpha \notin \mathbb{R}^{(\text{Elem})}$. \qed
The first 80 terms of the sequence $\varepsilon_n$ are the following.

| $n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\varepsilon_n$ | 1  | 0  | 1  | 1  | 1  | 1  | 1  | 0  | 1  | 0  | 1  | 1  | 0  | 1  | 1  | 1  |
| $n$ | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
| $\varepsilon_n$ | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 0  | 1  | 1  | 0  | 1  | 0  | 1  | 1  |
| $n$ | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 |
| $\varepsilon_n$ | 1  | 1  | 0  | 1  | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 0  |
| $n$ | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 |
| $\varepsilon_n$ | 1  | 1  | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 0  | 1  | 1  | 0  | 1  | 0  |
| $n$ | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| $\varepsilon_n$ | 0  | 1  | 0  | 1  | 0  | 1  | 1  | 0  | 1  | 1  | 1  | 0  | 1  | 1  | 1  | 1  |

The real number $\alpha/2 = \sum_{i=1}^{\infty} 3^{-i} \cdot \varepsilon_i$ is not elementary. The first 30 digits are the following.

$$\frac{\alpha}{2} = 0.38883221773824641256243009581\ldots \quad (10)$$

## 3 Periods are elementary

### 3.1 Main result

Now we can state the main result.

**Theorem 18.** Real periods are elementary real numbers, i.e.,

$$\mathcal{P} \subseteq \mathbb{R}_{\text{Elem}}.$$  

So the real number $\alpha$ constructed above (10) is not a period.

To prove this theorem, we need to show that a given absolutely convergent integration is an elementary real number. First we will reduce the problem to the cases of volumes of bounded semi-algebraic domains. Namely, in §3.2 we will prove that $\mathcal{P}$ is generated by volumes $\text{vol}(D)$ of bounded semi-algebraic open domains $D \subset \mathbb{R}^\ell$. The proof is based on Hironaka’s rectilinearization theorem on semi-algebraic sets. Another result, uniformization theorem of semi-algebraic sets, is also mentioned for later purposes.

Next step is to construct an elementary sequence $\text{vol}(V_n)$ converging to the volume $\text{vol}(D)$ of a semi-algebraic domain $D$. In §3.3 and §3.4, this is done by using Riemann sum, that is, approximating the domain by small cubes. The fact that the sequence is elementary is proved by using Tarski’s quantifier elimination theorem.
Finally, in \[3.5\] and \[3.6\] we will prove the convergence $\text{vol}(V_n) \to \text{vol}(D)$ is elementary. The main task is to count small cubes within a $\varepsilon$-neighborhood of the boundary $\partial D$. It is closely related to estimate the Minkowski dimension and the Minkowski content of $\partial D$. It is done with the help of uniformization theorem for semi-algebraic sets. This completes the proof that $\text{vol}(D)$ is an elementary real number.

### 3.2 Uniformization and rectilinearization

In this section, we recall uniformization and rectilinearization theorem on subanalytic sets by Hironaka. Our main references are [8] and [2]. First let us recall the notions of semi-algebraic set and basic open semi-algebraic set. (See [3] for details.)

**Definition 19.** A *semi-algebraic subset* of $\mathbb{R}^\ell$ is a finite union of subsets of the form

$$\{ x \in \mathbb{R}^\ell \mid F_1(x) = \cdots = F_p(x) = 0, \ G_1(x) > 0, \ldots, G_q(x) > 0 \},$$

where $F_j, G_k \in \mathbb{R}[x_1, \ldots, x_\ell]$.

A map from a semi-algebraic subset $X \subset \mathbb{R}^p$ to a semi-algebraic subset $Y \subset \mathbb{R}^q$ is called semi-algebraic if its graph is a semi-algebraic subset of $\mathbb{R}^{p+q}$.

**Definition 20.** A *basic open semi-algebraic subset* of $\mathbb{R}^\ell$ is a set of the form

$$\{ x \in \mathbb{R}^\ell \mid G_1(x) > 0, \ldots, G_q(x) > 0 \},$$

where $G_k \in \mathbb{R}[x_1, \ldots, x_\ell]$.

**Proposition 21.** [2, Thm. 5.1.] Let $X$ be a closed analytic subset of a real analytic manifold $M$. Then there is a real analytic manifold $N$ (of the same dimension as $X$) and a proper real analytic map $\varphi : N \to M$ such that $\varphi(N) = X$.

**Proposition 22.** [8, (2.4)] Let $X$ be a real-analytic space countable at infinity. Let $A$ be a globally defined semi-analytic set in $X$, i.e., there exists a finite system of real analytic functions $g_{ij}$ and $f_{ij}$ on $X$ such that

$$A = \bigcup_i \{ x \in X \mid g_{ij}(x) = 0, f_{ij}(x) > 0, \forall j \}.$$

Then there exists a real-analytic map $\pi : \hat{X} \to X$ such that

1. $\hat{X}$ is smooth and $\pi$ is proper surjective,
(2) for every point $\xi \in \hat{X}$, there exists a local coordinate system $(z_1, \ldots, z_n)$ of $\hat{X}$ centered at $\xi$ for which we have: within some neighborhood of $\xi$ in $\hat{X}$, $\pi^{-1}(X)$ is a union of quadrants with respect to $(z_1, \ldots, z_n)$, where a quadrant means a set defined by a system of relations $z_1 \sigma_1 0, z_2 \sigma_2 0, \ldots, z_n \sigma_n 0$ with $\sigma_i$ is either “=”, “>” or “<”.

We note that the map $\pi$ above can be taken to be a composition of a finite sequence of blowing-ups with smooth centers.

The following apparently more general description of $\mathcal{P}$ is equivalent to Definition 1 [1, Thm. 2.5, Prop. 4.2]:

**Proposition 23.** The ring $\mathcal{P}$ is exactly the ring generated by the numbers of the form $\int_{\Delta} \omega$, where $X$ is a smooth algebraic variety of dimension $\ell$ defined over $\mathbb{Q}$, $E \subset X$ is a divisor with normal crossings, $\omega \in \Omega^\ell(X)$ is a top degree algebraic differential form on $X$, and $\Delta \subset X$ is a $\ell$-dimensional compact real semi-algebraic set with $\partial \Delta \subset E$.

In view of Proposition 22, we may assume that the semi-algebraic cycle $\Delta$ in Proposition 23 is smooth and locally (analytically) a union of quadrants.

Now we come to prove that real periods are elementary. We first reduce the problem to the volumes of bounded semi-algebraic sets.

**Lemma 24.** Periods $\mathcal{P}$ is generated by

$$\big\{ \text{vol}(D) \mid D \subset \mathbb{R}^k \text{ is bounded basic open semi-algebraic set} \big\}.$$ (13)

**Proof.** We will prove:

(i) $\mathcal{P}$ is generated by

$$\big\{ \text{vol}(D) \mid D \subset \mathbb{R}^k \text{ is bounded open semi-algebraic set} \big\},$$

and

(ii) The volumes of open semi-algebraic subsets of $\mathbb{R}^\ell$ are generated by those of basic ones.

The second one (ii) is easy. Indeed, a semi-algebraic subset of the form $\{ D \}^\ell$ with $p > 0$ has measure zero. As far as we are interested in volumes, we can ignore the measure zero sets. We may consider an open semi-algebraic subset as a disjoint union of basic ones modulo measure zero sets.

Now we prove (i). We use the description in Proposition 23 with $\Delta$ smooth and locally (analytically) isomorphic to a union of quadrants. Fix a semi-algebraic triangulation $\Delta = \bigcup_{\alpha} \Delta_\alpha$ and also fix base points $p_\alpha \in \Delta_\alpha$.
in each simplex. By taking the triangulation small enough, we may assume
that the orthogonal projections
\[ \pi_\alpha : \Delta_\alpha \longrightarrow T_{p_\alpha} \Delta_\alpha \] (14)
induce the isomorphism \[ \pi_\alpha : \Delta_\alpha \overset{\cong}{\longrightarrow} \pi_\alpha(\Delta_\alpha) \subset T_{p_\alpha} \Delta_\alpha. \] Then the image \[ K_\alpha := \pi_\alpha(\Delta_\alpha) \] is also a semi-algebraic set. Denote the inverse of the projection by \[ \psi_\alpha : K_\alpha \overset{\cong}{\longrightarrow} \Delta_\alpha, \] which is a semi-algebraic \( C^\infty \)-map. Fix a coordinate \((z_1, \ldots, z_\ell)\) of the affine space \( T_{p_\alpha} \Delta_\alpha \). Then the pull-back of \( \omega \) by \( \psi_\alpha \) is of the form
\[ (\psi_\alpha)^* \omega = H_\alpha(z) dz_1 \wedge \cdots \wedge dz_\ell. \] (15)
Since composition and differentiations of semi-algebraic functions are also
semi-algebraic, \( H(z) \) is a semi-algebraic \( C^\infty \)-function. So the integration
\[ \int_{\Delta_\alpha} \omega = \int_{K_\alpha} H(z) dz_1 \cdots dz_\ell \]
is equal to the volume \( \text{vol}(D_\alpha) \) of the bounded semi-algebraic domain
\[ D_\alpha = \{(x,t) \in \mathbb{R}^\ell \times \mathbb{R} \mid x \in K_\alpha, \ 0 \leq t \leq H_\alpha(x)\}. \] (16)
Thus we have (i).

### 3.3 Quantifier elimination

Let \( \mathcal{L}_{OR} \) be the language
\[ \mathcal{L}_{OR} = (+, -, \cdot, 0, 1, \langle, \rangle) \]
of ordered rings. We consider the theory \( T \) of real number field \( \mathbb{R} \) with the
language \( \mathcal{L}_{OR} \). Recall that a quantifier free formula \( \psi(x_1, \ldots, x_n) \) is a Boolean
combination of inequalities \( p(x_1, \ldots, x_n) > 0 \), where \( p \in \mathbb{Z}[x_1, \ldots, x_n] \). The
following is due to Tarski [17], see also [6].

**Theorem 25.** (Tarski) On the real number field, every \( \mathcal{L}_{OR} \)-formula \( \varphi(x_1, \ldots, x_n) \)
is equivalent to a quantifier free formula \( \varphi^*(x_1, \ldots, x_n) \), i.e.,
\[ T \models \forall x_1 \forall x_2 \cdots \forall x_n (\varphi(x_1, \ldots, x_n) \Rightarrow \varphi^*(x_1, \ldots, x_n)). \]

Let
\[ D = \{ x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell \mid G_k(x) > 0, \ k = 1, \ldots, q\}, \] (17)
be a domain in \( \mathbb{R}^\ell \), where \( G_k(x) \in \mathbb{Z}[x_1, \ldots, x_\ell] \) and set
\[ G_k(x) = \sum_j a_{k,j} x^j, \]
where \( J = (j_1, \ldots, j_\ell) \) is multi-index and denoting \( x^J = x_1^{j_1} \cdots x_\ell^{j_\ell} \).

Let us consider the next predicates with variables \( s_i, t_i, a_{kJ} \):

\[
R(s_i, t_i, a_{kJ} : 1 \leq i \leq \ell, 1 \leq k \leq q, J):
\]

\[
\forall x_1 \ldots \forall x_\ell ((s_i \leq x_i \leq t_i), i = 1, \ldots, \ell \Rightarrow (G_k(x) > 0), k = 1, \ldots, q) \quad (18)
\]

The above formula means that the box \( \prod_{i=1}^\ell [s_i, t_i] \) is contained in the domain \( D \),

\[
\prod_{i=1}^\ell [s_i, t_i] = [s_1, t_1] \times \cdots \times [s_\ell, t_\ell] \subset D. \quad (19)
\]

From Theorem 25, we have a quantifier free formula \( R^*(s_i, t_i, a_{kJ}) \) which satisfies for \( \forall s_i, t_i, a_{kJ}, \)

\[
R^*(s_i, t_i, a_{kJ}) \iff [s_1, t_1] \times \cdots \times [s_\ell, t_\ell] \subset D. \quad (20)
\]

### 3.4 Riemann sum

Let \( D \subset \mathbb{R}^\ell \) be a basic open semi-algebraic subset as in (12). Now we assume that \( D \) is bounded and contained in a large cube \( [0, r]^\ell, r > 0 \).

Then the volume \( \text{vol}(D) \) is approximated by the inner Riemann sum.

For positive an integer \( n > 0 \) and \( k_1, \ldots, k_\ell \in \mathbb{N} \), define a small cube \( C_n(k_1, \ldots, k_\ell) \) of size \( r/n \) by

\[
C_n(k_1, \ldots, k_\ell) = \left[ \frac{k_1r}{n}, \frac{(k_1+1)r}{n} \right] \times \cdots \times \left[ \frac{k_\ell r}{n}, \frac{(k_\ell+1)r}{n} \right].
\]

Trivially these cubes subdivides the large cube \( [0, r] = \bigcup_{0 \leq k_i < n} C_n(k_1, \ldots, k_\ell) \).

Let us denotes by \( V_n \) the union

\[
V_n = \bigcup_{C_n(k) \subset D} C_n(k_1, \ldots, k_\ell)
\]

of small cubes which are contained in \( D \). We will prove that \( \text{vol}(V_n) \rightarrow \text{vol}(D) \) (as \( n \rightarrow \infty \)) determines an elementary real number.

**Lemma 26.** The function

\[
\mathbb{N} \rightarrow \mathbb{Q} \quad (n \mapsto \text{vol}(V_n))
\]

is elementary.
Proof. To compute Riemann sum \( \text{vol}(V_n) \), we have to know for which \((k_1, \ldots, k_\ell)\) the small cube \( C_n(k_1, \ldots, k_\ell) \) is contained in \( D \). From Theorem 25 in the previous section, this is decided by a quantifier free formula \( R^*(s_i, t_i, a_{k,j}) \).

By definition, it is a Boolean combination of the predicates of the form

\[
p(s_i, t_i, a_{k,j}) > 0,
\]

with \( p \in \mathbb{Z}[s_i, t_i, a_{k,j}] \). The truth value of the statement \( C_n(k) \subset D \) is decided by checking the truth values of Boolean combination of predicates of the form

\[
p \left( \frac{k_ir}{n}, \frac{(k_i+1)r}{n}, a_{k,j} \right) > 0.
\]

Thus the relation \( C_n(k_1, \ldots, k_\ell) \subset D \) can be decided elementarily, that is, there exists an elementary function

\[
\varphi : \mathbb{N}^{\ell+1} \to \mathbb{N}, \quad ((n, k_1, \ldots, k_\ell) \mapsto \varphi(n, k_1, \ldots, k_\ell))
\]

such that

\[
\varphi(n, k_1, \ldots, k_\ell) = \begin{cases} 1 & \text{if } C_n(k_1, \ldots, k_\ell) \subset D, \\ 0 & \text{otherwise.} \end{cases}
\]

Thus the volume \( \text{vol}(V_n) \) of the union of small cubes is expressed as

\[
\text{vol}(V_n) = \left( \frac{r}{n} \right)^\ell \sum_{0 \leq k_i \leq n} \varphi(n, k_1, \ldots, k_\ell),
\]

which is an elementary function on \( n \).

Next we have to estimate the rate of convergence

\[
\lim_{n \to \infty} \text{vol}(V_n) = \text{vol}(D).
\]

3.5 Minkowski content

In this subsection we recall notations on Minkowski dimension and Minkowski contents from [11].

First let \( B = \{ x \in \mathbb{R}^N \mid |x| < 1 \} \). Then

\[
\Upsilon_N := \text{vol}(B) = 2 \pi^{N/2} \frac{N}{\Gamma(N/2)}.
\]

Here \( \mathcal{L}^N \) denotes the \( N \)-dimensional Lebesgue measure.
Definition 27. Suppose $A \subset \mathbb{R}^N$ and $0 \leq K \leq N$. The $K$-dimensional upper Minkowski content of $A$, denoted by $\mathcal{M}^*(A)$, is defined by

$$\mathcal{M}^*(A) = \limsup_{\varepsilon \downarrow 0} \frac{L^N\{x \mid \text{dist}(x, A) < \varepsilon\}}{\Upsilon_{N-K}\varepsilon^{N-K}}$$

Proposition 28. ([11 Prop 3.5.5]) Let $f : \mathbb{R}^K \rightarrow \mathbb{R}^N$ be a $C^1$-map. $A \subset \mathbb{R}^K$ is compact with

$$A \subset \{x \mid |D(f)| \leq \rho\},$$

then

$$\mathcal{M}^*(f(A)) \leq \rho^K L^K(A).$$

3.6 Proof, completion

Now we return to the proof of Theorem 18, $P \subset \mathbb{R}(\text{Elem})$. In view of Lemma 24, it is enough to show that the sequence $\text{vol}(V_n) \rightarrow \text{vol}(D)$ constructed in §3.4 converges effectively. The following lemma concludes $\text{vol}(D) \in \mathbb{R}(\text{Elem})$.

Lemma 29. There exists a constant $L = L(D)$ depending only on $D$, such that if $k$ and $n$ satisfy

$$4rL\sqrt{k} < n,$$

then $|\text{vol}(D) - \text{vol}(V_n)| < 1/k$.

Proof. Set $P(x) = \prod_{k=1}^n G_k(x)$. Then $\partial D \subset \{P = 0\}$. By the Uniformization theorem (Proposition 21), $X = \{P = 0\}$ is an image $\pi(X)$ of a proper analytic map $\pi : X \rightarrow \mathbb{R}^\ell$. Since $\partial D \subset \mathbb{R}^\ell$ is compact, from Proposition 28 the $(\ell - 1)$-dimensional Minkowski content $\mathcal{M}^{*(\ell-1)}(\partial D)$ of the boundary $\partial D$ is finite. There is a constant $L > 0$ and $\varepsilon_0 > 0$ such that

$$\frac{\mathcal{L}^\ell(\{y \in \mathbb{R}^\ell \mid \text{dist}(y, \partial D) < \varepsilon\})}{2\varepsilon} < L,$$

for $0 < \varepsilon < \varepsilon_0$. Equivalently we have

$$\frac{\mathcal{L}^\ell(\{y \in \mathbb{R}^\ell \mid \text{dist}(y, \partial D) < \varepsilon\})}{2\varepsilon} < 2L.$$

Choose $n$ large enough and $\varepsilon$ as

$$\frac{r\sqrt{\ell}}{n} = \frac{\varepsilon}{2},$$

note that the LHS is exactly the diagonal length of the small cube $C_n(k_1, \ldots, k_\ell)$.

Let us consider the subset of $D$ which is $\varepsilon$-away from the boundary (or removing $\varepsilon$-neighborhood of the boundary)

$$D_{>\varepsilon} = \{x \in D \mid \text{dist}(x, \partial D) > \varepsilon\}.$$
It is easily seen that, under (26),

\[ D_{>\varepsilon} \subset V_n \subset D. \quad (28) \]

Instead of \( \text{vol}(D - V_n) \), we will estimate \( \text{vol}(D - D_{>\varepsilon}) \).

Hence if we choose \( n \) as in Eq. (26),

\[
| \text{vol}(D) - \text{vol}(V_n) | < | \text{vol}(D) - \text{vol}(D_{>\varepsilon}) |
\]

\[
= \mathcal{L}^\ell(\{ y \in D \mid \text{dist}(y, \partial D) < \varepsilon \})
\]

\[
< \mathcal{L}^\ell(\{ y \in \mathbb{R}^\ell \mid \text{dist}(y, \partial D) < \varepsilon \})
\]

\[
< 2\varepsilon L = \frac{4r\sqrt{7}L}{n}.
\]

Thus if (25) is satisfied, we have \( | \text{vol}(D) - \text{vol}(V_n) | < 1/k \).

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