Theoretical and Numerical Aspects of a Third-order Three-point Nonhomogeneous Boundary Value Problem

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ABSTRACT. In this paper we are considering a third-order three-point equation with nonhomogeneous conditions in the boundary. Using Krasnosel’skii’s Theorem and Leray-Schauder Alternative we provide existence results of positive solutions for this problem. Nontrivials examples are given and a numerical method is introduced.

Keywords: numerical solutions, third-order, boundary value problem, Krasnosel’skii’s Theorem.

1 INTRODUCTION

Multi-point boundary value problems there has been attention of several studies mainly focused on the existence of solutions with qualitative and quantitative aspects, we recommend [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15] and the references therein. It is well known that the Krasnosel’skii’s fixed point theorem, Avery-Peterson and Leggett-Williams theorems are massively used in this line.

In this paper, motived by [13], we discuss the existence of a positive solution for the third-order boundary value problem:

\[ u''' + f(t, u, u') = 0, \quad 0 < t < 1, \]
\[ u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda, \]

where \( \eta \in (0, 1), \alpha \in [0, \frac{1}{\eta}) \) are constants and \( \lambda \in (0, \infty) \) is a parameter. Essentially, we combine Leray-Schauder Alternative and Krasnosel’skii’s theorem to show the existence of a positive solution for (1.1)-(1.2) without supposing superlinearity on \( f \). Numerical solutions are poorly

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explored, thus we complement this work presenting a numerical study for (1.1)-(1.2) based on Banach’s Contraction Principle.

2 BACKGROUND MATERIAL

We begin this section by stating the following results.

**Theorem 1.** Let $E$ be a Banach space, $C \subseteq E$ a closed and convex set, $\Omega$ an open set in $C$ and $p \in \Omega$. Then each completely continuous mapping $T : \Omega \to C$ has at least one of the following properties:

(A1) $T$ has a fixed point in $\Omega$.

(A2) There are $u \in \partial \Omega$ and $\lambda \in (0, 1)$ such that $u = \lambda T(u) + (1 - \lambda)p$.

**Theorem 2.** Let $E$ be a Banach space and let $K \subseteq E$ be a cone in $E$. Assume $\Omega_1, \Omega_2$ are bounded open subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that, either

(B1) $\|Tu\| \leq \|u\|, u \in K \cap \partial \Omega_1$, and $\|Tu\| \geq \|u\|, u \in K \cap \partial \Omega_2$, or

(B2) $\|Tu\| \geq \|u\|, u \in K \cap \partial \Omega_1$, and $\|Tu\| \leq \|u\|, u \in K \cap \partial \Omega_2$.

Then $T$ has a fixed point in $K \cap (\overline{\Omega}_1 \setminus \Omega_1)$.

The first theorem is a well-known Leray-Schauder alternative and the second theorem is due to Krasnoselskii, see [1].

Let us set an auxiliary problem that will be useful in our context.

$$u''' + f(t, x, x') = 0, \quad 0 < t < 1, \quad (2.1)$$

$$u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda. \quad (2.2)$$

Related to this problem we have an important lemma.

**Lemma 3.** Let $x \in C^1[0, 1] := \{ x \in C^1[0, 1], t \in [0, 1] \},$ then we have a unique solution for (2.1)-(2.2). Moreover, this solution is expressed by

$$u(t) = \int_0^1 G(t, s)f(s, x(s), x'(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)f(s, x(s), x'(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)}, \quad (2.3)$$

where $G$ is the Green’s function:

$$G(t, s) = \frac{1}{2} \left\{ \begin{array}{ll}
(2t - t^2 - s)s, & s \leq t \\
(1 - s)t^2, & t \leq s
\end{array} \right. \quad (2.4)$$
and

\[ G_1(t, s) = \frac{\partial G(t, s)}{\partial t} = \begin{cases} (1-t)s, & s \leq t \\ (1-s)t, & t \leq s \end{cases}. \] (2.5)

**Proof.** If \( u(t) \) is solution of (2.1), we can suppose that

\[ u(t) = -\frac{1}{2} \int_0^t (t-s)^2 f(s, x, x') ds + At^2 + Bt + C. \]

From condition (2.2), we have \( B = C = 0 \) and

\[
A = \frac{1}{2(1-\alpha \eta)} \int_0^1 (1-s)f(s, x, x') ds - \frac{\alpha}{2(1-\alpha \eta)} \int_0^\eta (\eta-s)f(s, x, x') ds + \frac{\lambda}{(1-\alpha \eta)} \int_0^\eta (\eta-s) ds.
\]

Thus (2.1)-(2.2) has a unique solution. Furthermore \( u(t) = -\frac{1}{2} \int_0^t (t-s)^2 f(s, x, x') ds + \frac{t^2}{2(1-\alpha \eta)} \int_0^1 (1-s)f(s, x, x') ds \)

\[
- \frac{\alpha t^2}{2(1-\alpha \eta)} \int_0^\eta (\eta-s)f(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha \eta)}
\]

\[ = -\frac{1}{2} \int_0^t (t-s)^2 f(s, x, x') ds + t^2 \frac{1}{2} \int_0^1 (1-s)f(s, x, x') ds \]

\[ + \frac{\alpha \eta t^2}{2(1-\alpha \eta)} \int_0^1 (1-s)f(s, x, x') ds - \frac{\alpha t^2}{2(1-\alpha \eta)} \int_0^\eta (\eta-s)f(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha \eta)} \]

\[ = \frac{1}{2} \int_0^t (-t^2 + 2st - s^2) f(s, x, x') ds + \frac{1}{2} \int_0^t (1-s)t^2 f(s, x, x') ds \]

\[ + \frac{1}{2} \int_0^1 (1-s)^2 f(s, x, x') ds + \frac{\alpha t^2}{2(1-\alpha \eta)} \int_0^\eta (1-s) f(s, x, x') ds \]

\[ + \frac{\alpha \eta t^2}{2(1-\alpha \eta)} \int_0^1 (1-s) \eta f(s, x, x') ds - \frac{\alpha t^2}{2(1-\alpha \eta)} \int_0^\eta (\eta-s)f(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha \eta)} \]

\[ = \frac{1}{2} \int_0^t (2t-t^2-s) f(s, x, x') ds + \frac{1}{2} \int_0^1 (1-s)t^2 f(s, x, x') ds \]

\[ + \frac{\alpha t^2}{2(1-\alpha \eta)} \left( \int_0^\eta (1-\eta) f(s, x, x') ds + \int_0^\eta \eta (1-s) f(s, x, x') ds \right) + \frac{\lambda t^2}{2(1-\alpha \eta)} \]

\[ = \int_0^1 G(t, s) f(s, x, x') ds + \frac{\alpha t^2}{2(1-\alpha \eta)} \int_0^1 G_1(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha \eta)}. \]  

\[ \square \]

Defining \( x(t) = u(t) \) in Lemma 3 is easy to see that the solution of (1.1)-(1.2) can be expressed as fixed point of the operator \( T : C^1[0, 1] \rightarrow C^1[0, 1] \) defined by:

\[ Tu(t) = \int_0^t G(t, s) f(s, u, u') ds + \frac{\alpha t^2}{2(1-\alpha \eta)} \int_0^1 G_1(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha \eta)}. \] (2.6)
Remark 4. Related to $G$ and $G_1$ we have useful properties that will be used in the next section.

- For all $(t, s) \in [0, 1] \times [0, 1]:$
  
  $$0 \leq G_1(t, s) \leq (1 - s)s$$

- For all $(t, s) \in [0, 1] \times [0, 1]:$
  
  $$G(t, s) \leq G_1(1, s) = \frac{1}{2}(1 - s)s$$

3 POSITIVE SOLUTIONS

Let $E = \{ u \in C^1[0, 1] : u(0) = 0 \}$, where $C^1[0, 1]$ be the Banach space of continuously differentiable functions in $[0, 1]$ equipped with

$$\| u \|_E = \max \{ \| u \|_\infty, \| u' \|_\infty \}.$$  

Remark 1. If $u \in E$ then $Tu$ satisfies $Tu(0) = 0$. Besides $\|(Tu)'\|_\infty \geq \|Tu\|_E$.

In order to prove the existence we need to consider some basic assumptions.

$(H1)$ There exist positive constants $A, B$ and $\beta$ such that

- $$\max_{(s, v_1, v_2) \in [0, 1] \times [-\beta, \beta] \times [-\beta, \beta]} \{ \| f(s, v_1, v_2) \| \} \leq \beta \frac{(1 - \alpha \eta)6B}{1 + \alpha(1 - \eta)}$$

- $$\lambda \leq A \beta (1 - \alpha \eta)$$

- $$A + B \leq 1.$$  

Lemma 2. Suppose that $(H1)$ holds. Thus the problem (1.1)-(1.2) has a solution $u^* \in E$ with $\|u^*\|_E \leq \beta$.

Proof. Let us consider the Theorem 1 with $p = 0$ and $\Omega = \{ u \in E : \|u\|_E < \beta \}$.

We claim that $T$ is continuous and completely continuous. In fact, the continuity follows immediately from the Lebesgue dominated convergence theorem and noting that

$$|T(u)(t) - T(u_n)(t)| \leq \int_0^1 G(t, s) \left| f(s, u(s), u'(s)) - f(s, u_n(s), u'_n(s)) \right| ds +$$

$$+ \frac{\alpha^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s) \left| f(s, u(s), u'(s)) - f(s, u_n(s), u'_n(s)) \right| ds$$

$$\leq \int_0^1 G_1(1, s) \left| f(s, u(s), u'(s)) - f(s, u_n(s), u'_n(s)) \right| ds.$$
with \( u, u_0 \in E \). To show complete continuity we will use the Arzela-Ascoli's theorem. Let \( \Omega \subseteq E \) be bounded, in other words, there exists \( \Lambda_0 > 0 \) with \( \| u \| \leq \Lambda_0 \) for each \( u \in \Omega \). Now if \( u \in \Omega \) we have

\[
| (Tu)'(t) | = \left| \int_0^1 G_1(t,s)f(s,u,u')ds + \frac{\alpha t}{1-\alpha \eta} \int_0^1 G_1(\eta,s)f(s,u,u') + \frac{\lambda t}{1-\alpha \eta} \right| \\
\leq \int_0^1 |G_1(t,s)f(s,u,u')ds| + \frac{\alpha t}{1-\alpha \eta} \int_0^1 |G_1(\eta,s)f(s,u,u')| + \frac{\lambda t}{1-\alpha \eta} \\
\leq \max_{t \in [0,1]} \frac{1-\alpha \eta + \alpha t}{1-\alpha \eta} \int_0^1 |(1-s)f(s,u,u')ds| + \frac{\lambda t}{1-\alpha \eta} \\
\leq \frac{1+\alpha\left(-\eta+1\right)}{1-\alpha \eta} \int_0^1 |(1-s)f(s,u,u')ds| + \frac{1}{1-\alpha \eta} \\
\leq \frac{1+\alpha\left(-\eta+1\right)}{1-\alpha \eta} \int_0^1 |f(s,v_1,v_2)| + \frac{\lambda}{1-\alpha \eta} \\
\leq \frac{1+\alpha\left(-\eta+1\right)}{1-\alpha \eta} \max |f(s,v_1,v_2)| + \frac{\lambda}{1-\alpha \eta} \\
\leq \frac{1}{1-\alpha \eta} \left[ \frac{1+\alpha(1-\eta)}{6} \max |f(s,v_1,v_2)| + \frac{\lambda}{1-\alpha \eta} \right] \\
\leq \frac{1}{1-\alpha \eta} \left[ \frac{1+\alpha(1-\eta)}{6} \beta(1-\alpha \eta)6B + A \beta(1-\alpha \eta) \right] \\
\leq \beta A + \beta B \leq \beta.
\]

Therefore, \( \| u \| \leq \beta \) and (A2) in Theorem 1 cannot occur. Thus (A1) holds and there is \( u^* \in E \) such that \( \| u^* \| \leq \beta \).

**Theorem 3.** Suppose that (H1) holds and \( f(s,u,v) \geq 0, \forall (s,u,v) \in [0,1] \times [-\beta,\beta] \times [-\beta,\beta] \).
Then (1.1)–(1.2) has at least one positive solution \( u^* \in E \).

**Proof.** We start the proof defining the cone \( K \subseteq E \) by

\[
K = \{ u \in E : u \geq 0, u(0) = 0, u'(0) = 0 \}.
\]

Tend. Mat. Apl. Comput., 20, N. 3 (2019)
From \((H1)\) and the definition of \(G\) and \(G_1\), we have that \(T\) applies \(K\) in \(K\). As seen in the last result, \(T\) is completely continuous.

We shall apply Theorem 2. Thus, we will define \(\Omega_1 = \{ u \in E; \|u\|_E < \beta \}\), \(\Omega_2 = \{ u \in E; \|u\|_E < \alpha \}\) and we will show that the following conditions are true for all \(u \in K\):

\[(a)\] if \(\|u\|_E = \alpha\) then \(\|Tu\|_E \leq \alpha\);

\[(b)\] if \(\|u\|_E = \beta\) then \(\|Tu\|_E \geq \beta\).

In fact, the demonstration of \((a)\) is similar to the proof of the Lemma 2. To prove \((b)\) is necessary to verify that there is \(\gamma > 0\) with \(\|Tu\|_E \geq \|u\|_E\), \(\forall u \in K \cap \partial \Omega_3\), where \(\Omega_3 = \{ u \in E; \|u\|_E < \gamma \}\).

Let us assume that the inequality is false, that is, for every \(\gamma\) such that \(\beta > \gamma > 0\) there exists \(u \in E\) with \(\|u\|_E = \gamma\) and \(\|Tu\|_E < \gamma\). Thus for all \(n \in \{1,2,\cdots\}\) with \(\frac{1}{n} < \alpha\), we can find \(u_n \in K\) such that

\[\|u_n\|_E = \frac{1}{n} \quad \text{and} \quad \|Tu_n\|_E < \frac{1}{n}.\]

Then \(\|u_n\|_E \rightarrow 0\) and \(\|Tu_n\|_E \rightarrow 0\), when \(n \rightarrow \infty\). Being \(T\) continuous, we have \(\|T0\|_E = 0\). On the other hand, using \((H1)\) and the definition of \(G\) and \(G_1\) we have

\[\|T0\|_\infty \geq \max_{t \in [0,1]} \left\{ \frac{\lambda t^2}{2(1-\alpha n)} \right\},\]

\[\geq \frac{\lambda}{2(1-\alpha n)} > 0\]

which is a contradiction. Therefore we have the result. \(\Box\)

**Remark 4.** Note that the most important step in the proof of Theorem 3 is to impose conditions to conclude that 0 is not fixed point of \(T\).

**Example 3.1.** Let us consider (1.1)-(1.2) with

\[f(t,u,v) = \frac{1}{4}t + u^2 + v^2\]

\[\eta = \frac{1}{10}, \quad \alpha = \frac{1}{3}, \quad \lambda = \frac{1}{4}\]

Choosing the constants

\[\beta = 10, \quad A = 0.54, \quad B = 0.45,\]

we can easily verify that in these conditions the hypotheses \((H1)\) are satisfied.
Example 3.2. Let us define

\[ f(t, u, v) = \frac{1}{4}t + \sin(u) + \frac{1}{4}\cos(v) \]

\[ \eta = \frac{1}{9}, \; \alpha = \frac{1}{6}, \; \lambda = \frac{14}{10} \]

As before, choosing the constants

\[ \beta = 2, \; A = 0.75, \; B = 0.2, \]

we can verify that (H1) is satisfied.

4 NUMERICAL SOLUTIONS

In this section we show the existence and uniqueness for (1.1)-(1.2) using Banach Fixed Point Theorem. This approach is classical but very important to define numerical methods for our problem. Let us consider the iterative sequence

\[ u^{k+1} = T(u^k) \]

and the basic assumptions

(H2) \[ |f(s, u, u') - f(s, v, v')| \leq A \max \{|u(s) - v(s),|u'(s) - v'(s)|\} \]

(H3) \[ -\frac{r^2 + r}{2} + \frac{\alpha \eta (-\eta + 1)}{2(1-\alpha \eta)} \leq \frac{1}{A}. \]

Theorem 1. Suppose that (H1), (H2) and (H3) are satisfied. Then (1.1)- (1.2) has a unique solution \( u \) with \( ||u||_E \leq \beta \). Moreover, \( u^{k+1} = T(u^k) \rightarrow u \).

Proof. Let us consider \( u, v \in \Omega \) with \( ||u||_E \leq \beta \) and \( ||v||_E \leq \beta \). Then

\[ ||Tu - Tv||_E = ||(Tu - Tv)'||_\infty \]

\[ = \left| \int_0^1 G_1(t,s)[f(s, u, u') - f(s, v, v')]ds + \frac{\alpha t}{1-\alpha \eta} \int_0^1 G_1(t,s)[f(s, u, u') - f(s, v, v')]ds \right| \]

\[ \leq A \max_s \{|u(s) - v(s),|u'(s) - v'(s)|\} \left( \int_0^1 G_1(t,s)ds + \frac{\alpha t}{1-\alpha \eta} \int_0^1 G_1(\eta, s)ds \right) \]

\[ \leq A \max_s \{|u(s) - v(s),|u'(s) - v'(s)|\} \left( -\frac{r^2 + r}{2} + \frac{\alpha \eta (-\eta + 1)}{2(1-\alpha \eta)} \right) \]

Using (H3) we obtain

\[ \leq A \max_s \{|u(s) - v(s),|u'(s) - v'(s)|\} \frac{1}{A} \]

\[ \leq \max_s \{|u(s) - v(s),|u'(s) - v'(s)|\} = ||u - v||_E \]

Motivated by the last result we can define Algorithm 1.

In sequence we are presenting some examples in order to establish the effectiveness of Algorithm 1. In tables, \( \varepsilon^k_u \) denotes \( ||u^k - u^*||_\infty \) where \( u^* \) is the exact solution, \( \varepsilon^k \) denotes \( ||u^{k+1} - u^k||_\infty \) and \( \varepsilon^k = \frac{||u^{k+1} - u^k||_\infty}{||u^{k+1}||_\infty} \). Still, “It” denotes “iteration”.

Tend. Mat. Apl. Comput., 20, N. 3 (2019)
Algorithm 1 Fixed-Point

1: Define an uniformly distributed mesh \( \{x_j\} \) in \([0, 1]\)
2: Define an initial approximation \( u_j^0 = u^0(x_j) \)
3: for \( k = 0, 1, 2, \ldots \), do
4: Compute \( u_j^k \) using finite differences
5: Compute \( u_j^{k+1} \) using
   \[ u^{k+1} = T(u^k) \text{ and Trapezoidal Rule} \]
6: Test the convergence
7: end for

Example 4.1. In this example, we consider

\[ f(x, u, u') = -u' \]
\[ \eta = \frac{\pi}{4}, \alpha = \frac{1}{10}, \lambda = 0.770760306689242 \]

The analytical solution is \( u^*(x) = 1 - \cos(x) \). The Table 1 contains results of application in Example 4.1.

We can make additional tests. From Theorem 3 we have a solution for Examples 3.1 and 3.2 but in both case, we do not know which they are. Let us apply Algorithm 1 in these problems. For this purpose, we can consider the condition

\[ \frac{\|u^{k+1} - u^k\|}{\|u^{k+1}\|_\infty} < 10^{-4} \]

as stopping criterion for the algorithm. The results for these examples are presented in Table 2 and 3, respectively. The illustrations of these results are given in Figure 1 and 2.
Table 1: Algorithm 1 considering Example 4.1.

| It | $\varepsilon^k_{iu}$ | $\varepsilon^k$ | $\bar{e}^k$ |
|----|-----------------------|-----------------|-------------|
| 1  | 0.104585227251908    | 0.355112466879952 | 1.000000000000000 |
| 2  | 0.072538564385106    | 0.03204662866802  | 0.082773878760819 |
| 3  | 0.069264937033799    | 0.003273627351307 | 0.008384612437847 |
| 4  | 0.068925441261629    | 0.000339495772170 | 0.000868781674432 |
| 5  | 0.068890166416009    | 0.00035274845620  | 0.00090261428243  |

Table 2: Algorithm 1 considering Example 3.1.

| It | $\varepsilon^k_{iu}$ | $\varepsilon^k$ | $\bar{e}^k$ |
|----|-----------------------|-----------------|-------------|
| 1  | -                     | 0.168278823890335 | 1.000000000000000 |
| 2  | -                     | 0.007563461402919 | 0.043012756518181 |
| 3  | -                     | 0.000744660869474 | 0.004216964422657 |
| 4  | -                     | 0.000077049209989 | 0.000436134198292 |

Table 3: Algorithm 1 considering Example 3.2.

| It | $\varepsilon^k_{iu}$ | $\varepsilon^k$ | $\bar{e}^k$ |
|----|-----------------------|-----------------|-------------|
| 1  | -                     | 0.740971458506793 | 1.000000000000000 |
| 2  | -                     | 0.010141509530254 | 0.013876702158219 |
| 3  | -                     | 0.000276799190473 | 0.000378602975785 |
| 4  | -                     | 0.000007307338952 | 0.000009995000056 |

Figure 1: Numerical solution obtained from Example 1 using Algorithm 1.

Figure 2: Numerical solution obtained from Example 2 using Algorithm 1.
RESUMO. Neste artigo, consideramos uma equação com três pontos de fronteira de terceira ordem com condições de contorno não homogêneas. Com uso do Teorema de Krasnoselskii e da Alternativa de Leray-Schauder, apresentamos resultados de existência para soluções positivas. Exemplos não triviais são fornecidos e um método numérico é introduzido.

Palavras-chave: soluções numéricas, terceira-ordem, problema de valor de contorno, Teorema de Krasnoselskii.

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