GOWERS' RAMSEY THEOREM FOR GENERALIZED TETRIS OPERATIONS

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Abstract. We prove a generalization of Gowers' theorem for \( \text{FIN}_k \) where, instead of the single tetris operation \( T : \text{FIN}_k \to \text{FIN}_{k-1} \), one considers all maps from \( \text{FIN}_k \) to \( \text{FIN}_j \) for \( 0 \leq j \leq k \) arising from nondecreasing surjections \( f : \{0, 1, \ldots, k+1\} \to \{0, 1, \ldots, j+1\} \). This answers a question of Bartošová and Kwiatkowska. We also prove a common generalization of such a result and the Galvin–Glazer–Hindman theorem on finite products, in the setting of layered partial semigroups introduced by Farah, Hindman, and McLeod.

1. Introduction

Gower’s theorem on \( \text{FIN}_k \) is a generalization of Hindman’s theorem on finite unions where one considers, rather than finite nonempty subsets of \( \omega \), the space \( \text{FIN}_k \) of all finitely supported functions from \( \omega \) to \( \{0, 1, \ldots, k\} \) with maximum value \( k \). Such a space is endowed with a natural operation of pointwise sum, which is defined for pairs of functions with disjoint support. Gowers considered also the tetris operation \( T : \text{FIN}_k \to \text{FIN}_{k-1} \) defined by letting \((Tb)_n = \max\{b(n) - 1, 0\}\) for \( b \in \text{FIN}_k \). Gowers’ theorem can be stated, shortly, by saying that for any finite coloring of \( \text{FIN}_k \) there exists an infinite sequence \((b_n)\) which is a block sequence—in the sense that every element of the support of \( b_n \) precedes every element of the support of \( b_{n+1} \)—with the property that the intersection of \( \text{FIN}_k \) with the smallest subset of \( \text{FIN}_1 \cup \cdots \cup \text{FIN}_k \) that contains the \( b_n \)’s and it is closed under pointwise sum of disjointly supported functions and under the tetris operation, is monochromatic [4]. Gowers then used such a result—or more precisely its symmetrized version where one considers functions from \( \omega \) to \( \{-k, \ldots, k\} \)—to prove an oscillation stability result for the sphere of the Banach space \( c_0 \). Other proof of Gowers’ theorem can be found in [5, 7, 11].

Gowers’ theorem of \( \text{FIN}_k \) as stated above implies through a standard compactness argument its corresponding finitary version. Explicit combinatorial proofs of such a finitary version have been recently given, independently, by Tyros [12] and Ojeda-Aristizabal [9]. Particularly, the argument from [12] yields a primitive recursive bound on the associated Gowers numbers.

A broad generalization of Gowers’ theorem has been proved by Farah, Hindman, and McLeod in [3, Theorem 3.13] in the framework, developed therein, of layered partial semigroups and layered actions. Such a result provides, in particular, a common generalization of Gowers’ theorem and the Hales–Jewett theorem; see [3, Theorem 3.15]. As general as [3, Theorem 3.13] is, it nonetheless does not cover the case where one considers \( \text{FIN}_k \) endowed with the multiple tetris operations described below, since these do not form a layered action in the sense of [3, Definition 3.3].

In [1], Bartošová and Kwiatkowska considered a generalization of Gowers’ theorem, where multiple tetris operations are allowed. Precisely, they defined for \( 1 \leq i \leq k \) the tetris...
operation \( T_i : \text{FIN}_k \rightarrow \text{FIN}_{k-1} \) by

\[
T_i(b) : n \mapsto \begin{cases} 
  b(n) - 1 & \text{if } b(n) \geq i, \text{ and} \\
  b(n) & \text{otherwise.}
\end{cases}
\]

Adapting methods from [12], Bartošová and Kwiatkowska proved in [1] the strengthening of the finitary version of Gowers’ theorem where multiple tetris operations are considered. The authors then provided in [1] applications of such a result to the dynamics of the Lelek fan.

Question 8.3 of [1] asks whether the infinitary version of Gowers’ theorem on \( \text{FIN}_k \) holds when one considers multiple tetris operations. In this paper, we show that this is the case, via an adaptation of Gowers’ original argument using idempotent ultrafilters. In order to precisely state our result, we introduce some terminology, to be used in the rest of the paper.

We denote by \( \omega \) the set of nonnegative integers, and by \( \mathbb{N} \) the set of nonzero elements of \( \omega \). We identify an element \( k \) of \( \omega \) with the set \( \{0, 1, \ldots, k-1\} \) of its predecessors. As mentioned above, \( \text{FIN}_k \) denotes the set of functions from \( \omega \) to \( k + 1 \) with maximum value \( k \) and that vanish for all but finitely many elements of \( \omega \). We also let \( \text{FIN}_{\leq k} \) be the union of \( \text{FIN}_j \) for \( j = 1, 2, \ldots, k \). The support \( \text{Supp}(b) \) of an element \( b \) of \( \text{FIN}_k \) is the set of elements of \( \omega \) where \( b \) does not vanish. For finite nonempty subsets \( F, F' \) of \( \omega \), we write \( F < F' \) if the maximum element of \( F \) is smaller than the minimum element of \( F' \).

Suppose that \( 0 \leq j \leq k \) and \( f : k + 1 \rightarrow j + 1 \) is a nondecreasing surjection. We also denote by \( f \) the generalized tetris operation \( f : \text{FIN}_k \rightarrow \text{FIN}_j \) defined by \( f(b) = f \circ b \). It is clear that the class of generalized tetris operations is precisely the set of mappings that can be obtained as composition of the multiple tetris operations \( T_i \) for \( i = 1, 2, \ldots, k \).

We say that \( (b_n) \) is a block sequence in \( \text{FIN}_k \) if \( b_n \in \text{FIN}_k \) and \( \text{Supp}(b_n) \subset \text{Supp}(b_{n+1}) \) for every \( n \in \omega \). If \( j \in \mathbb{N} \), then we define the tetris subspace \( \text{TS}_j(b_n) \) of \( \text{FIN}_j \) generated by \( (b_n) \) to be the set of elements of \( \text{FIN}_j \) of the form

\[
f_0 \circ b_0 + \cdots + f_n \circ b_n
\]

for some \( n \in \omega \), \( j_0, \ldots, j_n \in j + 1 \) such that \( \max\{j_0, \ldots, j_n\} = j \), and nondecreasing surjections \( f_i : k + 1 \rightarrow j_i + 1 \) for \( i \in n \). A block sequence \( (b'_n) \) in \( \text{FIN}_k \) is a block subsequence of \( (b_n) \) if \( (b'_n) \) is contained in \( \text{TS}_k(b_n) \).

In the following we will use some standard terminology concerning colorings. An \( r \)-coloring (or coloring with \( r \) colors) of a set \( X \) is a function \( c : X \rightarrow r \), and a finite coloring is an \( r \)-coloring for some \( r \in \omega \). A subset \( A \) of \( X \) is monochromatic (for the given coloring \( c \)) if \( c \) is constant on \( A \). Using this terminology, we can state our infinitary Gowers’ theorem for generalized tetris operations as follows.

**Theorem 1.1.** Suppose that \( k \in \mathbb{N} \). For any finite coloring of \( \text{FIN}_{\leq k} \), there exists an infinite block sequence \( (b_n) \) in \( \text{FIN}_k \) such that \( \text{TS}_j(b_n) \) is monochromatic for every \( j = 1, 2, \ldots, k \).

Theorem 3.1 implies via a standard compactness argument its corresponding finitary version. If \( k, n \in \omega \), then we denote by \( \text{FIN}_k(n) \) the set of functions \( f : n \rightarrow k + 1 \) with maximum value \( k \), and by \( \text{FIN}_{\leq k}(n) \) the union of \( \text{FIN}_j(n) \) for \( j = 1, 2, \ldots, k \). The notion of block sequence \( (b_0, \ldots, b_{m-1}) \) and tetris subspace \( \text{TS}_j(b_0, \ldots, b_{m-1}) \) of \( \text{FIN}_k(n) \) generated by \( (b_0, \ldots, b_{m-1}) \) are defined similarly as their infinite counterparts.

**Corollary 1.2.** Given \( k, r, \ell \in \mathbb{N} \) there exists \( n \in \mathbb{N} \) such that for any \( r \)-coloring of \( \text{FIN}_{\leq k}(n) \), there exists a block sequence \( (b_0, b_1, \ldots, b_{\ell-1}) \) in \( \text{FIN}_k(n) \) of length \( \ell \) such that \( \text{TS}_j(b_0, \ldots, b_{\ell-1}) \) is monochromatic for any \( j = 1, 2, \ldots, k \).

We will also prove below a more general statement than Theorem 1.1, where one considers colorings of the space \( \text{FIN}_k^{[m]} \) of block sequences of \( \text{FIN}_k \) of a fixed length \( m \). We will also provide a common generalization of such a result and the Galvin–Glazer–Hindman theorem.

\[1\] We thank Slawomir Solecki for pointing this out.
on finite products, in the setting of layered partial semigroups introduced by Farah, Hindman, and McLeod in [3].

As mentioned above, the original Gowers theorem from [4] was used to prove the following oscillation-stability result for the positive part of the sphere of $c_0$. Recall that $c_0$ denotes the real Banach space of vanishing sequences of real numbers endowed with the supremum norm. Let $\text{PS}(c_0)$ be the positive part of the sphere of $c_0$, which is the set of elements of $c_0$ of norm 1 with nonnegative coordinates. The support $\text{Supp}(f)$ of an element $f$ of $c_0$ is the set $n \in \omega$ such that $f(n) \neq 0$. A normalized positive block basis is a sequence $(f_n)$ of finitely-supported elements of $\text{PS}(c_0)$ such that $\text{Supp}(f_n) < \text{Supp}(f_{n+1})$ for every $n \in \omega$. Gower’s oscillation-stability result asserts that for any Lipschitz map $F : \text{PS}(c_0) \to \mathbb{R}$ and $\varepsilon > 0$ there exists a block basis $(f_n)$ such that the oscillation of $F$ on the positive part of the sphere of the subspace of $c_0$ spanned by $(f_n)$ is at most $\varepsilon$ [4, Theorem 6]. Such a result is proved by considering a suitable discretization of $\text{PS}(c_0)$ that can naturally be identified with $\text{FIN}_k$; see the proof of [4, Theorem 6] and also [11, Corollary 2.26]. Under such an identification, the tetris operation on $\text{FIN}_k$ corresponds to multiplication by positive scalars in $c_0$.

Similarly, one can observe that the multiple tetris operations $T_i$ for $i = 1, 2, \ldots, k$ described above correspond to the following nonlinear operators on $c_0$. Fix $\lambda, t \in [0, 1]$ and consider the operator $S_{t, \lambda}$ on $c_0$ mapping $f$ to the function

$$n \mapsto \begin{cases} \lambda f(n) & \text{if } |f(n)| \geq t, \\ f(n) & \text{otherwise.} \end{cases}$$

Given a normalized positive block basis $(f_n)$, one can consider the smallest subspace of $c_0$ that contains $(f_n)$ and it is invariant under $S_{t, \lambda}$ for every $t, \lambda \in [0, 1]$. Then arguing as in the proof of Gowers’ oscillation-stability theorem one can deduce from Theorem 1.1 the following result.

**Theorem 1.3.** Suppose that $F : \text{PS}(c_0) \to \mathbb{R}$ is a Lipschitz map, and $\varepsilon > 0$. There exists a positive normalized block sequence $(f_n)$ such that the oscillation of $F$ on the positive part of the sphere of the smallest subspace of $c_0$ containing $(f_n)$ and invariant under $S_{t, \lambda}$ for $t, \lambda \in [0, 1]$ is at most $\varepsilon$.

The rest of this paper consists of three sections. In Section 2 we present a proof of Theorem 1.1. In Section 3 we explain how the proof of Theorem 1.1 can be modified to prove its multidimensional generalization. Finally in Section 4 we recall the theory of layered partial semigroups developed in [3], and present in this setting a common generalization of the (multidimensional version of) Theorem 1.1 and the Galvin–Glazer–Hindman theorem on finite products.

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2. Gowers’ theorem for generalized tetris operations

Our proof of Theorem 1.1 uses the tool of idempotent ultrafilters, similarly as Gowers’ original proof from [4]. In the following we will frequently use the notation of ultrafilter quantifiers [11, §1.1], which are defined as follows. If $\mathcal{U}$ is an ultrafilter on $\text{FIN}_k$ and $\varphi(x)$ is a first-order formula, then $(\mathcal{U}b) \varphi(b)$ means that the set of $b \in \text{FIN}_k$ such that $\varphi(b)$ holds belongs to $\mathcal{U}$. A similar notation applies to ultrafilters on an arbitrary set.

Adopting the terminology of [11, §2.4], we say that an ultrafilter $\mathcal{U}$ on $\text{FIN}_k$ is cofinite if $(\forall n \in \omega)(\mathcal{U}b), b(n) = 0$. The set $\gamma \text{FIN}_k$ of cofinite ultrafilters on $\text{FIN}_k$ is endowed with a canonical semigroup operation, defined by setting $A \in \mathcal{U} + \mathcal{V}$ if and only if $((\mathcal{U}b)(\mathcal{V}b'))$, where
Supp \((b) < \text{Supp}(b')\) and \(b + b' \in A\). Furthermore \(\gamma\text{FIN}_k\) is endowed with a canonical compact Hausdorff topology. Such a topology has a clopen basis consisting of sets of the form \(A = \{U \in \gamma\text{FIN}_k : A \subseteq U\}\) for \(A \subseteq \text{FIN}_k\). Endowed with such a topology and semigroup operation, \(\gamma\text{FIN}_k\) is a compact right topological semigroup \([11, \S 2.4]\). This means that the right multiplication map \(U \mapsto U + V\) is continuous for any \(V \in \gamma\text{FIN}_k\). Any generalized tetris operation \(f : \text{FIN}_k \rightarrow \text{FIN}_k\) admits a canonical extension to a continuous homomorphism \(f : \gamma\text{FIN}_k \rightarrow \text{FIN}_k\), defined by letting \(A \in f(U)\) iff \((Ub) \cap f(b) \in A\). In the following we will repeatedly use the well know result, due to Ellis and Namakura, that any compact right topological semigroup contains an idempotent element \([11, \text{Lemma 2.1}]\).

The following lemma can be seen as a refinement of \([4, \text{Lemma 3}]\), and it is the core of the proof of Theorem 1.1.

**Lemma 2.1.** There exists a sequence \((U_k)\) of cofinite ultrafilters \(U_k\) on \(\text{FIN}_k\) such that for any \(0 < j \leq k\) and for any nondecreasing surjection \(f : k + 1 \rightarrow j + 1\), \(U_k + U_{k+1} = U_{j+1} + U_k = U_k\) and \(f(U_k) = U_j\).

**Proof.** We define by recursion on \(k\) sequences \((p_j^{(k)})\) of idempotent ultrafilters \(p_j^{(k)} \in \gamma\text{FIN}_j\) such that, for every \(k, i, j \in \mathbb{N}\) and nondecreasing surjection \(f : j + 1 \rightarrow i + 1\),

1. \(f(p_j^{(k)}) = p_j^{(k)}\),
2. \(p_j^{(k+1)} = p_j^{(k)}\) for \(j \leq k\), and
3. \(p_j^{(k)} + p_{j-1}^{(k)} = p_j^{(k)}\) for \(2 \leq j \leq k\).

Granted the construction of the sequences \((p_j^{(k)})\), we can set \(U_k := p_1^{(k)} + p_2^{(k)} + \cdots + p_k^{(k)}\). Observe that \((3)\) implies that \(p_k^{(k)} + p_j^{(k)} = p_j^{(k)}\) for \(j \leq k\). Hence \(U_k\) is idempotent, and \(U_k + U_{k+1} = U_{k+1} + U_k = U_{k+1}\) for every \(k \in \mathbb{N}\) by \((2)\). Furthermore it follows from \((1)\) and \((3)\) that \(f(U_k) = U_j\) for any nondecreasing surjection \(f : k + 1 \rightarrow j + 1\). Indeed suppose that \(f : k + 1 \rightarrow j + 1\) is a nondecreasing surjection. Then \(f|_{i+1} : i + 1 \rightarrow f(i) + 1\) is a nondecreasing surjection for every \(i \in k + 1\). Hence we can conclude by \((1),(3)\), and the fact that the \(p_i^{(k)}\)'s are idempotent that

\[
f(U_k) = f(p_1^{(k)}) + \cdots + f(p_k^{(k)}) = f(p_1^{(1)}) + f(p_2^{(1)}) + \cdots + f(p_k^{(1)}) = p_1^{(j)} + \cdots + p_j^{(j)} = U_j.
\]

We now show how to construct the sequences \((p_j^{(k)})\). In the following we will convene that \(p_0^{(k)}\) is the function on \(\omega\) constantly equal to 0, which can be seen as the unique element of \(\text{FIN}_0\). We let \(\Pi\) be the product of \(\gamma\text{FIN}_j\) for \(j \in \mathbb{N}\). Observe that \(\Pi\) has a natural compact right topological semigroup structure, where the topology is the product topology and the operation is the entrywise sum.

For \(k = 1\), consider the compact semigroup \(\Sigma_1 \subseteq \Pi\) of sequences \((q_j)\) satisfying \(f(q_j) = q_i\) for any \(i, j \in \mathbb{N}\) and nondecreasing surjection \(f : j + 1 \rightarrow i + 1\). We observe that \(\Sigma_1\) is nonempty. Define for \(j \in \mathbb{N}\) the set \(M_j := \{b \in \text{FIN}_j : \forall n \in \omega, b(n) \in \{0, j\}\}\). Fix for any \(j \geq 2\) a nondecreasing surjection \(f_j : j + 1 \rightarrow j\). Observe that \(f_j\) maps \(M_j\) bijectively onto \(M_{j-1}\). Furthermore, for any \(0 < i < j\) and nondecreasing surjection \(f : j + 1 \rightarrow i + 1\), one has that \(f|_{M_j} = (f_{j+1} \circ f_i \circ \cdots \circ f_j)|_{M_j}\). We denote by \((f_j|_{M_j})^{-1} : M_{j-1} \rightarrow M_j\) the inverse map of \(f_j\). Let \(p_1\) be any element of \(\gamma\text{FIN}_1\). One can define a sequence \((q_j)\) that belongs to \(\Sigma_1\) and such that \(M_j \subseteq q_j\) by recursion on \(j \in \mathbb{N}\), by letting \(q_{j+1} = (f_{j+1}|_{M_{j+1}})^{-1}(q_j)\). This concludes the proof that \(\Sigma_1\) is nonempty. We can then let \((p_j^{(1)})\) be an idempotent element of \(\Sigma_1\). This concludes the construction for \(k = 1\).

Suppose that the sequences \((p_j^{(\ell)})\) have been defined for \(\ell = 1, 2, \ldots, k - 1\) satisfying \((1),(2),(3)\) above. We explain how to define \((p_j^{(k)})\). Consider the compact semigroup \(\Sigma_k \subseteq \Pi\) of sequences \((q_j)\) such that, for any \(i, j \in \mathbb{N}\) and nondecreasing surjection \(f : j + 1 \rightarrow i + 1\),
for \( j \in k \), and \( q_j + p_i^{(k-1)} = q_j \) for \( j \in \mathbb{N} \) and \( i \leq \min \{ k-1, j \} \). We need to show that \( \Sigma_k \) is nonempty. Set
\[
q_j := p_j^{(k-1)} + p_{j-1}^{(k-1)} + \cdots + p_1^{(k-1)}
\]
for every \( j \in \mathbb{N} \). Observe that for \( j \leq k-1 \) one has that \( q_j = p_j^{(k-1)} \) in view of (3). If \( j \geq k \) then
\[
q_j = p_j^{(k-1)} + p_{j-1}^{(k-1)} + \cdots + p_{k-1}^{(k-1)},
\]
again by (3). Since \( p_i^{(k-1)} \) is idempotent for any \( i \in \mathbb{N} \), it follows from (1) that \( f(q_j) = q_j \) for any nondecreasing surjection \( f : j+1 \to i+1 \). Furthermore for \( j \in \mathbb{N} \) and \( i \leq \min \{ k-1, j \} \) one has that \( q_j + p_i^{(k-1)} = q_j \) in view of (2). This shows that the sequence \( (q_j) \) belongs to \( \Sigma_k \). One can then let \( (p_j^{(k)}) \) be any idempotent element of \( \Sigma_k \). This concludes the recursive construction. \( \square \)

Theorem 1.1 can now be deduced from Lemma 2.1 through a standard argument. We present a sketch of the proof, for convenience of the reader.

Proof of Theorem 1.1. Suppose that \( \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k \) are the cofinite ultrafilters constructed in Lemma 2.1. Fix a finite coloring \( c \) of \( \mathcal{F} \) for \( 1 \leq j \leq k \), and let \( A_j \) be an element of \( \mathcal{U}_j \) such that \( c \) is constant on \( A_j \) for \( j = 1, 2, \ldots, k \). We define, by recursion on \( n \in \omega \), \( b_n \in \mathcal{F} \) with \( \text{Supp} \ (b_i) < \text{Supp} \ (b_j) \) for \( i < j \), such that the following conditions hold: for any \( j_0, \ldots, j_{n+2} \in k+1 \) and nondecreasing surjections \( f_i : k+1 \to j_i+1 \) for \( i \in n+3 \),

1. \( f_0 \circ b_0 + \cdots + f_n \circ b_n \) belongs to \( A_{\max \{j_0, \ldots, j_n\}} \),
2. \( (\mathcal{U}_k y), f_0 \circ b_0 + \cdots + f_n \circ b_n + f_{n+1} \circ y \) belongs to \( A_{\max \{j_0, \ldots, j_{n+1}\}} \), and
3. \( (\mathcal{U}_k y) (\mathcal{U}_k z), f_0 \circ b_0 + \cdots + f_n \circ b_n + f_{n+1} \circ y + f_{n+2} \circ z \) belongs to \( A_{\max \{j_0, \ldots, j_{n+2}\}} \).

Suppose that such a sequence has been defined up to \( n \). From (2) and (3), we can conclude that there exists \( b_{n+1} \in \mathcal{F} \) such that \( \text{Supp} \ (b_{n+1}) > \text{Supp} \ (b_n) \) satisfying (1) and (2). Then (3) follows from (2), the properties of ultrafilter quantifiers, and the facts that, for \( 1 \leq j \leq k \), \( \mathcal{U}_j + \mathcal{U}_k = \mathcal{U}_k + \mathcal{U}_j = \mathcal{U}_j \) and \( f(\mathcal{U}_k) = \mathcal{U}_j \) for any nondecreasing surjection \( f : k+1 \to j+1 \). This concludes the recursive construction. In view of (1), the sequence \( (b_n) \) obtained through this construction has the property that \( \mathcal{F} (b_n) \) is contained in \( A_j \), and hence \( c \) is constant on \( \mathcal{F} (b_n) \) for \( j = 1, 2, \ldots, k \). \( \square \)

3. A multidimensional generalization

Gowers’ theorem on \( \mathcal{F} \) can be seen as a generalization of Hindman’s theorem for sets of finite unions [6]. Such a theorem asserts that for any finite coloring of \( \mathcal{F} \), there exists a block sequence \( (b_n) \) in \( \mathcal{F} \) such that \( \mathcal{F} (b_n) \) is monochromatic. Observe that one can identify \( \mathcal{F} \) with the set of nonempty finite subsets of \( \omega \). Then \( \mathcal{F} (b_n) \) is just the collection of all finite unions of the elements of the given sequence. Hindman’s theorem on finite unions is the particular instance of Gowers’ theorem for \( k = 1 \).

In another direction, Hindman’s theorem on finite unions was generalized, independently, by Milliken and Taylor [8,10]; see also [2]. Fix \( m \in \mathbb{N} \) and consider the set \( \mathcal{F}^{[m]} \) of block sequences in \( \mathcal{F} \) of length \( m \). The Milliken-Taylor theorem on finite unions asserts that, for any finite coloring of \( \mathcal{F}^{[m]} \), there exists an infinite block sequence \( (b_n) \) in \( \mathcal{F} \) such that the set \( \mathcal{F} (b_n)^{[m]} \) of \( m \)-tuples of the form
\[
(b_{n_0} + \cdots + b_{n_{\ell_0}-1}, b_{n_0} + \cdots + b_{n_{\ell_1}-1}, \ldots, b_{n_{\ell_{m-1}}-1} + \cdots + b_{n_{\ell_{m-1}}}),
\]
for \( 0 < \ell_0 < \ell_2 < \cdots < \ell_m \) and \( 0 < n_1 < n_2 < \cdots < n_{\ell_{m-1}} \), is monochromatic.

The multidimensional analog of Gowers’ theorem for a single tetris operation is proved in [11, Corollary 5.26]. The corresponding finite version is considered in [12]. In a similar spirit, one can consider a multidimensional generalization of Theorem 3.1. Let \( \mathcal{F}^{[m]} \) be
the space of block sequences in FIN$_k$ of length $m$, and FIN$_{\leq k}^m$ be the union of FIN$_j^m$ for
$j = 1, 2, \ldots, k$. If $(b_n)$ is a block sequence in FIN$_k$ and $1 \leq j \leq k$, then we define the tetris
subspace $TS_j(b_n)^m$ of FIN$_j^m$ generated by $(b_n)$ to be the set of elements of FIN$_j^m$ of the
form $(a_0, \ldots, a_{m-1})$, where $a_d$ for $d \in m$ is equal to
\[ f_{n_d} \circ b_{n_d} + \cdots + f_{n_{d+1}-1} \circ b_{n_{d+1}-1} \]
for some $n_0 = 0 < n_1 < n_2 < \cdots < n_m$, $0 \leq j_i \leq k$ and nondecreasing surjections
$f_i : k + 1 \to j_i + 1$ for $i \in n_m$ such that $\max \{j_{n_d}, \ldots, j_{n_{d+1}-1}\} = j$. We can then state the
multidimensional generalization of Theorem 1.1 as follows:

**Theorem 3.1.** Suppose that $m, k \in \mathbb{N}$. For any finite coloring of FIN$_{\leq k}^m$, there exists
an infinite block sequence $(b_n)$ in FIN$_k$ such that $TS_j(b_n)^m$ is monochromatic for every
$j = 1, 2, \ldots, k$.

In order to prove the Milliken-Taylor theorem, one can consider an idempotent cofinite
ultrafilter $U_1$ on FIN$_1$, and then the Fubini power $V_1 := U_1^{\otimes m}$. This is defined as the cofinite
ultrafilter on FIN$_1^m$ such that $A \in V_1$ if and only if $(U_1 b_1) \cdots (U_1 b_m)$, $(b_1, \ldots, b_m) \in A$;
see [11, §1.2]. Then any element of $V_1$ witnesses that the Milliken-Taylor theorem holds.
A similar approach works for Theorem 3.1. Indeed, consider the cofinite ultrafilter $U_k$ on
FIN$_k$ given by Lemma 2.1 and its Fubini power $V_k := U_k^{\otimes m}$ on FIN$_k^m$. Then any element of
$V_k$ witness that Theorem 3.1 holds. The proof of such a fact is analogous to the proof of
Theorem 1.1, and only notationally heavier. The details are left to the interested reader.

As usual, it follows by compactness from Theorem 3.1 the corresponding finite version,
which recovers Corollary 2.3 of [1]

**Corollary 3.2.** Suppose that $m, k, \ell, r \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that for any $r$-coloring
of FIN$_j(n)^m$, there exists a block sequence $(b_0, \ldots, b_{\ell-1})$ in FIN$_k$ of length $\ell$ such that
$TS_j(b_0, \ldots, b_{\ell-1})^m$ is monochromatic for $j = 1, 2, \ldots, k$.

4. A generalization for layered partial semigroups

Recall that a partial semigroup [3, Definition 1.2] is a set $S$ endowed with a partially
defined binary operation $(x, y) \mapsto xy$ satisfying $(xy) z = x (yz)$. This equation should be
interpreted as asserting that the left hand side is defined if and only if the right hand side
is defined, and in such a case the equality holds. Suppose that $x$ is an element of a partial
semigroup $S$. Following [3, Definition 2.1], we let $\varphi_S(x)$ be the set of elements $y$ of $S$ such
that $xy$ is defined. More generally, for a subset $A$ of $S$, we let $\varphi_S(A)$ be the set of elements
$y$ of $S$ such that $xy$ is defined for every $x \in A$. As in [3, Definition 2.1], we say that a partial
semigroup $S$ is adequate—also called directed in [11, §2.2]—if $\varphi_S(A)$ is nonempty for every
finite subset $A$ of $S$.

When $S$ is a directed partial semigroup, the set $\gamma S$ of ultrafilters $U$ over $S$ with the
property that $(\forall x \in S) (U y) xy$ is defined, is a closed nonempty subset of the space $\beta S$ of
ultrafilters over $S$. One can define a compact right topological semigroup operation on $\gamma S$
by setting $A \in U V$ if and only if $(U y) (V z) yz \in A$ [11, Corollary 2.7]. More generally, it
is shown in [3, Theorem 2.2] that the operation on $S$ extends to a continuous map from
$\beta S \times \gamma S$ to $\beta S$.

Suppose that $S, T$ are partial semigroups, and $\sigma : S \to T$ is a function. We say that $\sigma$
is a partial semigroup homomorphism if for any $x, y \in S$, $\sigma(x) \sigma(y)$ is defined whenever
$xy$ is defined, and in such a case $\sigma(xy) = \sigma(x) \sigma(y)$ [3, Definition 2.8]. We say that $\sigma : S \to T$
is an adequate partial semigroup homomorphism if it is a partial semigroup homomorphism
with the property that for any finite subset $A$ of $S$ there exists a finite subset $B$ of $T$ such
that $\varphi_T(B)$ is contained in the image under $\sigma$ of $\varphi_S(A)$. 
If \( T \) is a partial semigroup and \( S \subset T \), then \( S \) is an **adequate partial subsemigroup** if the inclusion map \( S \to T \) is an adequate partial semigroup homomorphism [3, Definition 2.10]. We say that a subset \( S \) of a partial semigroup \( T \) is an **adequate ideal** if it is an adequate partial subsemigroup, and for any \( x \in S \) and \( y \in T \) one has that \( xy \) and \( yx \) belong to \( S \) whenever they are defined [3, Definition 2.15]. Lemma 2.14 and Lemma 2.16 of [3] show that, if \( S \subset T \) is an adequate partial subsemigroup, then \( \gamma S \) can be canonically identified with a subsemigroup of \( \gamma T \). If furthermore \( S \) is an adequate ideal of \( T \), then \( \gamma S \) is an ideal of \( \gamma T \). We now recall the definition of layered partial semigroup from [3, §3]. An element \( e \) of a partial semigroup is an **identity element** if \( ex \) and \( xe \) are defined and equal to \( x \) for any \( x \in S \).

**Definition 4.1.** A **layered partial semigroup** with \( k \) layers is a partial semigroup \( S \) endowed with a partition \( \{ S_0, \ldots, S_k \} \) such that \( S_0 = \{ e \} \) for some identity element \( e \) for \( S \), and for every \( n = 1, 2, \ldots, k \), letting \( S_{\leq n} = S_0 \cup \cdots \cup S_k \), one has that \( S_{\leq n} \) is an adequate partial semigroup, \( S_n \) is an adequate partial subsemigroup of \( S \), and an adequate ideal of \( S_{\leq n} \).

In the following we will assume that \( S \) is a layered partial semigroup with \( k \) layers as witnessed by the partition \( \{ S_0, \ldots, S_k \} \), and set \( S_{\leq n} = S_0 \cup \cdots \cup S_n \). Observe that it follows from the definition of layered partial semigroup that \( \gamma S_n \) is an ideal of \( \gamma S_{\leq n} \), and a subsemigroup of \( \gamma S \) for \( n = 1, 2, \ldots, k \).

**Definition 4.2.** Suppose that \( \mathcal{A} = (\mathcal{F}_1, M_1, \mathcal{F}_2, M_2, \ldots, \mathcal{F}_k, M_k) \) is a tuple such that for every \( n = 1, 2, \ldots, k \), \( \mathcal{F}_n \) is a nonempty finite collection of partial semigroup homomorphisms from \( S_{\leq n} \) to \( S_{\leq n-1} \), and \( M_n \) is an adequate subsemigroup of \( S_n \) for \( n = 1, 2, \ldots, k \). We say that \( \mathcal{A} \) is a **tetris action** on \( S \) if and only if it satisfies for any \( n = 2, 3, \ldots, k \), and \( \sigma \in \mathcal{F}_n \) the following conditions:

1. the image of \( M_n \) under \( \sigma \) is an adequate partial subsemigroup of \( M_{n-1} \);
2. the image of \( S_n \) under \( \sigma \) is an adequate partial subsemigroup of \( S_{n-1} \);
3. the restriction of \( \sigma \) to \( S_{\leq n-1} \) either belongs to \( \mathcal{F}_{n-1} \), or it is the identity map of \( S_{\leq n-1} \), and
4. for any \( \sigma_1, \sigma_2 \in \mathcal{F}_n \) one has that \( \sigma_1|M_n = \sigma_2|M_n \).

From now on we assume that \( (\mathcal{F}_1, M_1, \mathcal{F}_2, M_2, \ldots, \mathcal{F}_k, M_k) \) is a tetris action on \( S \) as in Definition 4.2. It follows from [3, Lemma 2.4] that for any \( n = 2, 3, \ldots, k \), any element \( \sigma \) of \( \mathcal{F}_n \) admits a continuous extension \( \sigma: \beta S_{\leq n} \to \beta S_{\leq n-1} \) such that:

- if \( p \in \beta S_n \), \( q \in \gamma S_{\leq n-1} \), and \( \sigma(q) \in \gamma S_{\leq n-1} \), then \( \sigma(pq) = \sigma(p)\sigma(q) \);
- if \( p \in \beta S_{\leq n-1} \), \( q \in \gamma S_n \), and \( \sigma(q) \in \gamma S_n \), then \( \sigma(pq) = \sigma(p)\sigma(q) \);
- \( \sigma \) maps \( \gamma S_n \) to \( \gamma S_{n-1} \) and \( \gamma M_n \) to \( \gamma M_{n-1} \).

In particular, \( \sigma \) induces continuous semigroup homomorphism \( \sigma: \gamma S_n \to \gamma S_{n-1} \) mapping the subsemigroup \( \gamma M_n \) to \( \gamma M_{n-1} \). The same proof as Lemma 2.1 shows the following:

**Lemma 4.3.** There exist idempotent elements \( \mathcal{U}_n \in \gamma S_n \) for \( n = 1, 2, \ldots, k \) such that \( \sigma(\mathcal{U}_n) = \mathcal{U}_{n-1} \) and \( \mathcal{U}_n \mathcal{U}_{n-1} = \mathcal{U}_{n-2} \mathcal{U}_n = \mathcal{U}_n \) for every \( n = 2, \ldots, k \) and \( \sigma \in \mathcal{F}_n \).

Given a tetris action, one can define as in [3, Definition 3.9] the collection \( \mathcal{G}_n \) of maps from \( S_k \) to \( S_n \) of the form \( \sigma_{n+1} \circ \sigma_{n+2} \circ \cdots \circ \sigma_k \), where \( \sigma_j \in \mathcal{F}_j \) for \( j = n+1, \ldots, k \). We also let \( \mathcal{G} \) be the union of \( \mathcal{G}_n \) for \( n = 1, 2, \ldots, k \).

**Definition 4.4.** A **block sequence** in \( S_k \) is a sequence \( (b_n) \) such that \( f_0(b_0) \cdots f_n(b_n) \) is defined for any \( n \in \omega \) and \( f_0, \ldots, f_n \in \mathcal{G} \).

The notion of block sequence in \( S_n \) for some \( n \leq k \) is defined similarly. We let \( S_{[m]}^{[n]} \) be the set of block sequences in \( S_n \) of length \( m \), and \( S_{\leq n}^{[m]} \) be the union of \( S_j^{[m]} \) for \( j = 1, 2, \ldots, n \). If \( (b_n) \) is a block sequence in \( S_k \), then we define the tetris subspace \( TS_j(b_n) \subset S_{\leq n}^{[m]} \) of the \( j \)-th layer generated by \( (b_n) \) to be the set of elements of \( S_j^{[m]} \) of the form \( (a_0, \ldots, a_{m-1}) \) where
for some $0 = n_0 < n_1 < \cdots < n_m \in \omega$, $j_i \in k + 1$, and $f_i \in G_{j_i}$ for $i \in n_m$ one has that for every $d \in m$, $\max\{j_{n_d}, \ldots, j_{n_{d+1}-1}\} = j$ and $a_d = f_{n_d}(b_{n_d}) \cdots f_{n_{d+1}-1}(b_{n_{d+1}-1})$. Then using Lemma 4.3 one can prove as in Theorem 3.1 the following result, which is a common generalization of the Galvin-Glazer theorem and Theorem 3.1.

**Theorem 4.5.** Suppose that $S$ is a layered partial semigroup endowed with a tetris action as above. Fix $m \in \mathbb{N}$ and a finite coloring of $S^{[m]}$. Then there exists an infinite block sequence $(b_n)$ in $S_k$ such that $TS_j(b_n)^{[m]}$ is monochromatic for every $j = 1, 2, \ldots, k$.

It is clear that Theorem 4.5 has the Galvin-Glazer theorem [11, Theorem 2.20] as a particular case. Set now $S_j := \text{FIN}_j$ for $j = 0, 1, \ldots, k$, and $S := S_0 \cup \cdots \cup S_k$. Define a partial semigroup operation on $S$ by $(b, b') \mapsto b + b'$ whenever $\text{Supp}(b) < \text{Supp}(b')$, where $b + b'$ is the pointwise sum. Then $S = S_0 \cup \cdots \cup S_k$ is a layered partial semigroup in the sense of Definition 4.1. Denote by $F_n$ for $n = 1, 2, \ldots, k$ the collection of multiple tetris operations $T_1, \ldots, T_n : \text{FIN}_n \to \text{FIN}_{n-1}$ defined in the introduction. Let also $M_n \subset S_n$ be the set of $b \in S_n$ such that $b(i) \in \{0, n\}$ for every $i \in \omega$. It is then easy to see that $(F_n, M_n)^{[k]}$ is a tetris action on $S$ in the sense of Definition 4.2. Furthermore the conclusions of Theorem 4.5 in the particular case of such a tetris action yields Theorem 3.1.

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