CONVERGENCE RATES OF INERTIAL PRIMAL-DUAL DYNAMICAL METHODS FOR SEPARABLE CONVEX OPTIMIZATION PROBLEMS

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Abstract. In this paper, we propose a second-order continuous primal-dual dynamical system with time-dependent positive damping terms for a separable convex optimization problem with linear equality constraints. By the Lyapunov function approach, we investigate asymptotic properties of the proposed dynamical system as the time $t \to +\infty$. The convergence rates are derived for different choices of the damping coefficients. We also show that the obtained results are robust under external perturbations.

Key words. Separable convex optimization problem, inertial primal-dual dynamical system, Lyapunov analysis, convergence rate

AMS subject classifications. 34D05, 37N40, 46N10, 90C25

1. Introduction.

1.1. Problem statement. Throughout this paper we discuss in the Euclidean spaces with the inner $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let $f : \mathbb{R}^{n_1} \to \mathbb{R}$ and $g : \mathbb{R}^{n_2} \to \mathbb{R}$ be two smooth convex functions. Consider the separable convex optimization problem:

$$
\begin{align*}
\min f(x) + g(y) \\
\text{s.t. } Ax + By = b,
\end{align*}
$$

(1.1)

where $A \in \mathbb{R}^{m \times n_1}$, $B \in \mathbb{R}^{m \times n_2}$ and $b \in \mathbb{R}^m$. This problem plays important roles in diverse applied fields such as, machine learning, signal recovery, structured nonlinear theory and image recovery (see, e.g., [14, 20, 24, 27]).

Denoted by $\Omega$ the KKT point set of the problem (1.1), i.e., $(x^*, y^*, \lambda^*) \in \Omega$ if and only if

$$
\begin{align*}
-A^T \lambda^* &= \nabla f(x^*), \\
-B^T \lambda^* &= \nabla g(y^*), \\
Ax^* + By^* - b &= 0.
\end{align*}
$$

(1.2)

In what follows, we always suppose that $\Omega \neq \emptyset$. It is well-known that $(x^*, y^*)$ solves the problem (1.1) if and only if there exists $\lambda^* \in \mathbb{R}^m$ such that $(x^*, y^*, \lambda^*) \in \Omega$. The augment Lagrangian function $\mathcal{L} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}$, associated with the problem (1.1), is defined by

$$
\mathcal{L}(x, y, \lambda) = f(x) + g(y) + \langle \lambda, Ax + By - b \rangle + \frac{1}{2} \| Ax + By - b \|^2.
$$

(1.3)

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Then, \((x^*, y^*, \lambda^*)\) ∈ Ω if and only if it is a saddle point of \(L\), i.e.,

\[
L(x^*, y^*, \lambda) \leq L(x^*, y^*, \lambda^*) \leq L(x, y, \lambda^*), \quad \forall (x, y, \lambda) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m.
\]

Given a fixed \(t_0 > 0\), in terms of the augment Lagrangian function \(L\), we propose the following inertial primal-dual dynamical system for solving the problem (1.1):

\[
\begin{cases}
\ddot{x}(t) + \gamma(t)\dot{x}(t) = -\nabla_x L(x(t), y(t), \lambda(t) + \delta(t)\dot{\lambda}(t)), \\
\ddot{y}(t) + \gamma(t)\dot{y}(t) = -\nabla_y L(x(t), y(t), \lambda(t) + \delta(t)\dot{\lambda}(t)), \\
\ddot{\lambda}(t) + \gamma(t)\dot{\lambda}(t) = \nabla_\lambda L(x(t) + \delta(t)\dot{x}(t), y(t) + \delta(t)\dot{y}(t), \lambda(t)),
\end{cases}
\]

where \(\gamma, \delta : [t_0, +\infty) \to (0, +\infty)\) are two continuous damping functions. By computations, the inertial primal-dual dynamical system can be rewritten as follows:

\[
\begin{cases}
\ddot{x}(t) + \gamma(t)\dot{x}(t) = -\nabla f(x(t)) - A^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) - A^T(Ax(t) + By(t) - b), \\
\ddot{y}(t) + \gamma(t)\dot{y}(t) = -\nabla g(y(t)) - B^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) - B^T(Ax(t) + By(t) - b), \\
\ddot{\lambda}(t) + \gamma(t)\dot{\lambda}(t) = A(x(t) + \delta(t)\dot{x}(t)) + B(y(t) + \delta(t)\dot{y}(t)) - b.
\end{cases}
\]

In this paper we shall discuss the convergence rate analysis of the proposed inertial primal-dual dynamical method for the problem (1.1) by investigating the asymptotic behavior of the inertial primal-dual dynamical system (1.4) as \(t \to +\infty\).

### 1.2. Historical presentation

In recent years, the second-order dynamical system method is very popular for solving the unconstrained smooth optimization problem

\[
\min \Phi(x),
\]

where \(\Phi(x)\) is a smooth cost function. To solve the problem (1.5), Polyak [33, 34] introduced the heavy ball with friction system

\[
\ddot{x}(t) + \gamma\dot{x}(t) + \nabla \Phi(x(t)) = 0,
\]

where \(\gamma > 0\) is a damping coefficient. Alvarez [1] studied the asymptotic behavior of the heavy ball with friction system (1.6) under the condition that \(\Phi(x)\) is convex. The asymptotic behavior of (1.6) with \(\Phi(x)\) being nonconvex was discussed by Bégout et al. [11]. Haraux and Jendoubi [25] investigated the asymptotic behavior of the following perturbed version of the heavy ball with friction system (1.6):

\[
\ddot{x}(t) + \gamma\dot{x}(t) + \nabla \Phi(x(t)) = \epsilon(t),
\]

where \(\epsilon(t)\) is used as a perturbation. When the positive damping coefficient is dependent upon the time \(t\), (1.6) and (1.7) become, respectively, the following inertial gradient system

\[
(IGS_\gamma) \quad \ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla \Phi(x(t)) = 0,
\]

and its perturbed version

\[
(IGS_{\gamma, \epsilon}) \quad \ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla \Phi(x(t)) = \epsilon(t).
\]

The importance of \((IGS_\gamma)\) and \((IGS_{\gamma, \epsilon})\) has been recognized in the fields of fast optimization methods, control theory, and mechanics. Here, we mention some nice
work concerning fast optimization methods. Su et al. [36] pointed out that \((IGS)_γ\) with \(γ(t) = \frac{1}{t}\) can be viewed as a continuous version of the Nesterov’s accelerated gradient algorithm (see [31, 32]). The convergence rate \(\Phi(x(t)) - \min \Phi = O(\frac{1}{t^2})\) was also obtained in [36] for \((IGS)_γ\) with \(γ(t) = \frac{γ}{t}\) when \(γ \geq 3\). Attouch et al. [4] generalized this result by showing that \(\Phi(x(t)) - \min \Phi = O(\frac{1}{t^2})\) for \((IGS)_{γ,ε}\) with \(γ(t) = \frac{γ}{t}, α ≥ 3, and ε(t) satisfying \(\int_0^{∞} t|ε(t)||ds < +∞\). In the case \(γ(t) = \frac{γ}{t}\) with \(α > 3\), May [30] proved an improved convergence rate \(\Phi(x(t)) - \min \Phi = o(\frac{1}{t^2})\) for \((IGS)_γ\). When \(γ(t) = \frac{1}{t}\) with \(α ≤ 3\), it was shown in [7, 38] that the convergence rate of the values along the trajectory is \(\Phi(x(t)) - \min \Phi = O(\frac{1}{t^∞})\) for \((IGS)_γ\). In the case \(γ(t) = \frac{γ}{t}\) and \(α > 0\), Aujol et al. [8] studied the convergence rate of the values along the trajectory under some additional geometrical conditions on \(Φ(x)\). When \(γ(t) = \frac{1}{t^r}\) with \(r ∈ (0, 1)\), Cabot and Frankel [18] studied the asymptotic behavior of \((IGS)_γ\). Jendoubi and May [26] extended the results of Cabot and Frankel [18] to the perturbed case, and the corresponding convergence results can be found in [9, 29]. The results on asymptotic behaviors of \((IGS)_γ\) and \((IGS)_{γ,ε}\) with a general damping function \(γ(t)\) can be found in [2, 3, 6, 16, 17]. For more results on second-order dynamical system approaches for unstrained optimization problems, we refer the reader to [5, 12, 13, 28, 35].

For the linear equality constrained optimization problem \((1.1)\), popular numerical methods are based on the primal-dual framework (see, e.g., [10, 14, 19, 24]). In recent years, some first-order dynamical system methods based on the primal-dual framework were proposed for solving the problem \((1.1)\) (see, e.g., [21, 22, 23, 40]). However, to the best of our knowledge, second-order dynamical system methods based on the primal-dual framework are less discussed. It is worth mentioning that \(IGS_γ\) and \(IGS_{γ,ε}\) proposed for unstrained optimization problems cannot be directly applied to the primal-dual framework for the problem \((1.1)\). Recently, Zeng et al. [39] proposed the following second-order dynamical system based on the primal-dual framework for solving the problem \((1.1)\) with \(g(x) ≡ 0\) and \(B = 0\):

\[
\begin{align*}
\ddot{x}(t) + \frac{γ}{t^2} \dot{x}(t) &= - ∇f(x(t)) - A^T(λ(t) + βt\dot{λ}(t)) - A^T(Ax(t) - b), \\
\ddot{λ}(t) + \frac{α}{t^2} λ(t) &= A(x(t) + βt\dot{x}(t)) - b
\end{align*}
\]

and proved \(L(x(t), λ^*) - L(x^*, λ^*) = O(1/t^{\min(3, α)})\) and \(\|Ax(t) - b\| = O(1/t^{\min(3, α)})\) with \(α > 0\) and \(β = \frac{3}{2^{min(3, α)}}\).

1.3. Organization. In Section 2, based on new Lyapunov analysis, we obtain the existence and uniqueness of a global solution and discuss the asymptotic properties of the trajectories generated by the dynamic \((1.4)\) when \(γ(t)\) meets certain conditions. The results covers the ones of the Nesterov’s accelerated gradient system in which \(γ(t) = \frac{1}{t}\) with \(α > 0\). In Section 3, we establish the existence and uniqueness of a global solution and investigate the asymptotic properties in the case \(γ(t) = \frac{1}{t^r}\) with \(r ∈ (-1, 1)\). Finally, in Section 4, we complement these results by showing that the results obtained are robust with respect to external perturbations.

2. Asymptotic properties of \((1.4)\) with a general \(γ(t)\). In this section we discuss the asymptotic behavior of \((1.4)\) with a general \(γ(t)\) as the time \(t → +∞\). To do this, we first establish the existence of a global solution of the dynamic \((1.4)\). The following proposition, whose proof follows from the Picard-Lindelof Theorem (see [37, Theorem 2.2]), establishes the existence and uniqueness of a local solution of the dynamic \((1.4)\):
Proposition 2.1. Let \( f \) and \( g \) be two continuously differentiable functions such that \( \nabla f \) and \( \nabla g \) are locally Lipschitz continuous, and let \( \gamma, \delta : [t_0, +\infty) \to (0, +\infty) \) be locally integrable. Then for any \((x_0, y_0, \lambda_0, u_0, v_0, w_0)\), there exists a unique solution \((x(t), y(t), \lambda(t))\) with \( x(t) \in C^2([t_0, T), \mathbb{R}^{n_1}) \), \( y(t) \in C^2([t_0, T), \mathbb{R}^{n_2}) \) and \( \lambda(t) \in C^2([t_0, T), \mathbb{R}^{m}) \) of the dynamic (1.4) satisfying \((x(t_0), y(t_0), \lambda(t_0)) = (x_0, y_0, \lambda_0) \) and \((\dot{x}(t_0), \dot{y}(t_0), \dot{\lambda}(t_0)) = (u_0, v_0, w_0)\) on a maximal interval \([t_0, T) \subseteq [t_0, +\infty)\).

To analyze the asymptotic behavior of the dynamic (1.4), it is necessary to prove the existence of a global solution. To do so, we introduce the following function \( p : [t_0, +\infty) \to [1, +\infty) \) defined by

\[
p(t) = e^{\int_{t_0}^t \gamma(s)ds}, \quad \forall t \geq t_0,
\]

which will be used for convergence rate analysis. It is easy to verify that

\[
p(t) = p(t)\gamma(t), \quad \forall t \geq t_0.
\]

Fix \((x^*, y^*, \lambda^*) \in \Omega\). Then we have \( \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) \geq 0 \) for all \( t \in [t_0, T)\). Consider the energy function \( \mathcal{E}_{\theta, \eta}^\beta : [t_0, T) \to [0, +\infty) \) defined by

\[
\mathcal{E}_{\theta, \eta}^\beta(t) = \mathcal{E}_0(t) + \mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t),
\]

where

\[
\begin{aligned}
\mathcal{E}_0(t) &= p(t)^{2\beta}(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)), \\
\mathcal{E}_1(t) &= \frac{1}{2}||\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t), (\theta(t) + \beta p(t)^{-1}\dot{p}(t))\dot{\lambda}(t)||^2, \\
\mathcal{E}_2(t) &= \frac{1}{2}||\theta(t)(y(t) - y^*) + p(t)^\beta \dot{y}(t)||^2 + \frac{\eta(t)}{2}||y(t) - y^*||^2, \\
\mathcal{E}_3(t) &= \frac{1}{2}||\theta(t)(\lambda(t) - \lambda^*) + p(t)^\beta \dot{\lambda}(t)||^2 + \frac{\eta(t)}{2}||\lambda(t) - \lambda^*||^2,
\end{aligned}
\]

\( \theta, \eta : [t_0, +\infty) \to [0, +\infty) \) are two suitable functions, and \( \beta \) is a positive constant.

Multiplying the first equation of (1.4) by \( p(t)^\beta \) we get

\[
p(t)^\beta \dot{x}(t) = -p(t)^\beta(\gamma(t)\dot{x}(t) + \nabla f(x(t))) + \mathcal{A}^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) + \mathcal{A}^T(Ax(t) + By(t) - b),
\]

which together with (2.2) yields

\[
\begin{aligned}
\dot{\mathcal{E}}_1(t) &= \langle \dot{\theta}(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t), \dot{\theta}(t)(x(t) - x^*) + (\theta(t) + \beta p(t)^{-1}\dot{p}(t))\dot{\lambda}(t) \rangle \\
&\quad + (\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t), p(t)^\beta \dot{\lambda}(t)) + \frac{\eta(t)}{2}||x(t) - x^*||^2 \\
&\quad + \eta(t)||x(t) - x^*||^2 \\
&\quad = (\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t), \dot{\theta}(t)(x(t) - x^*) + (\theta(t) + (\beta - 1)p(t)^\beta \gamma(t))\dot{\lambda}(t)) \\
&\quad - p(t)^\beta(\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t), \nabla f(x(t))) + \mathcal{A}^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) \\
&\quad - p(t)^\beta(\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t), \mathcal{A}^T(Ax(t) + By(t) - b)) \\
&\quad + \frac{\eta(t)}{2}||x(t) - x^*||^2 + \eta(t)||x(t) - x^*||^2 + \dot{\lambda}(t)||x(t) - x^*||^2 \\
&\quad = (\theta(t)\dot{\theta}(t) + \frac{\eta(t)}{2})||x(t) - x^*||^2 + p(t)^\beta(\theta(t) + (\beta - 1)p(t)^\beta \gamma(t))||\dot{\lambda}(t)||^2 \\
&\quad + (\theta(t)\dot{\theta}(t) + (\beta - 1)p(t)^\beta \gamma(t) + \dot{\theta}(t)p(t)^\beta + \eta(t))(x(t) - x^*, \dot{x}(t)) \\
&\quad - (\theta(t)p(t)^\beta(x(t) - x^*, \nabla f(x(t))) + \mathcal{A}^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) \\
&\quad - \theta(t)p(t)^\beta(Ax(t) - Ax^*, Ax(t) + By(t) - b) \\
&\quad - p(t)^{2\beta}(\dot{x}(t), \nabla f(x(t))) + \mathcal{A}^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) + \mathcal{A}^T(Ax(t) + By(t) - b)).
\end{aligned}
\]
By similar arguments, we have
\[
\dot{E}_2(t) = (\theta(t)\dot{\theta}(t) + \frac{\dot{\eta}(t)}{2})\|y(t) - y^*\|^2 + p(t)\beta(\theta(t) + (\beta - 1)p(t)\gamma(t))\|\dot{y}(t)\|^2 \\
+ (\theta(t)\dot{\theta}(t) + (\beta - 1)p(t)\gamma(t)) + \dot{\theta}(t)p(t)\beta + \eta(t))(y(t) - y^*, \dot{y}(t)) \\
- \theta(t)p(t)\beta(y(t) - y^*, \nabla g(y(t)) + B_T(\lambda(t) + \delta(t)\dot{\lambda}(t)) \\
- \theta(t)p(t)\beta(By(t) - By^*, Ax(t) + By(t) - b) \\
- p(t)^2\beta(y(t), \nabla g(y(t)) + B_T(\lambda(t) + \delta(t)\dot{\lambda}(t)) + B_T(Ax(t) + By(t) - b))
\]
and
\[
\dot{E}_3(t) = (\theta(t)\dot{\theta}(t) + \frac{\dot{\eta}(t)}{2})\|\lambda(t) - \lambda^*\|^2 + p(t)\beta(\theta(t) + (\beta - 1)p(t)\gamma(t))\|\dot{\lambda}(t)\|^2 \\
+ (\theta(t)\dot{\theta}(t) + (\beta - 1)p(t)\gamma(t)) + \dot{\theta}(t)p(t)\beta + \eta(t))(\lambda(t) - \lambda^*, \dot{\lambda}(t)) \\
+ \theta(t)p(t)\beta(\lambda(t) - \lambda^*, A(x(t) + \delta(t)\dot{x}(t)) + By(y(t) + \delta(t)\dot{y}(t)) - b) \\
+ p(t)^2\beta(\dot{\lambda}(t), A(x(t) + \delta(t)\dot{x}(t)) + B(y(t) + \delta(t)\dot{y}(t)) - b).
\]
Adding $\dot{E}_1(t)$, $\dot{E}_2(t)$, $\dot{E}_3(t)$ together, using $Ax^* + By^* = b$ and rearranging the terms, we have
\[
\dot{E}_1(t) + \dot{E}_2(t) + \dot{E}_3(t) = \sum_{i=1}^{5} V_i(t),
\]
where
\[
V_1(t) = \left(\theta(t)\dot{\theta}(t) + \frac{\dot{\eta}(t)}{2}\right)(\|x(t) - x^*\|^2 + \|y(t) - y^*\|^2 + \|\lambda(t) - \lambda^*\|^2),
\]
\[
V_2(t) = (\theta(t)\dot{\theta}(t) + (\beta - 1)p(t)\gamma(t)) + \dot{\theta}(t)p(t)\beta + \eta(t)) \\
\times (\langle x(t) - x^*, \dot{x}(t) \rangle + \langle y(t) - y^*, \dot{y}(t) \rangle + \langle \lambda(t) - \lambda^*, \dot{\lambda}(t) \rangle),
\]
\[
V_3(t) = -\theta(t)p(t)\beta(\langle x(t) - x^*, \nabla f(x(t)) + A_T\lambda^* \rangle + \langle y(t) - y^*, \nabla g(y(t)) + B_T\lambda^* \rangle) \\
+ \theta(t)p(t)\beta(\lambda(t) - \lambda^*, A\dot{x}(t) + By(t)),
\]
\[
V_4(t) = p(t)\beta(\theta(t) + (\beta - 1)p(t)\gamma(t))(\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) \\
- \theta(t)p(t)\beta\|Ax(t) + By(t) - b\|^2,
\]
\[
V_5(t) = (p(t)^2\beta - \theta(t)p(t)\beta)(\|\dot{\lambda}(t), Ax(t) + By(t) - b) \\
- p(t)^2\beta(\langle \dot{x}(t), \nabla f(x(t)) + A_T\lambda(t) + A_T(Ax(t) + By(t) - b) \\
- p(t)^2\beta(\dot{y}(t), \nabla g(y(t)) + B_T\lambda(t) + B_T(Ax(t) + By(t) - b)).
\]
Derivate $\mathcal{E}_0(t)$ to get
\[
\dot{\mathcal{E}}_0(t) = 2\beta p(t)^2\gamma(t)(f(x(t)) - f(x^*) + g(y(t)) - g(y^*) + \langle \lambda^*, Ax(t) + By(t) - b \rangle) \\
+ \beta p(t)^2\gamma(t)(Ax(t) + By(t) - b)^2 + p(t)^2\beta(\|\nabla f(x(t)), \dot{x}(t) \rangle + \|\nabla g(y(t)), \dot{y}(t) \rangle) \\
+ p(t)^2\beta(\langle \lambda^*, Ax(t) + By(t) \rangle + \langle Ax(t) + By(t) - b, A\dot{x}(t) + By(t) \rangle).
\]
Now we are in a position to investigate the existence and uniqueness of a global solution of the dynamic (1.4) with suitable choices of $\gamma(t)$ and $\delta(t)$.

**Theorem 2.2.** Let $f$ and $g$ be two continuously differentiable functions such that $\nabla f$ and $\nabla g$ are locally Lipschitz continuous, $\gamma : [t_0, +\infty) \to (0, +\infty)$ be a nonincreasing and twice continuously differentiable function satisfying
\[
\hat{\gamma}(t) \geq 2\beta^2\gamma(t)^3, \forall t \geq t_0.
\]
for some $\beta \in (0, \frac{1}{3})$, and $\delta(t) = \frac{1}{\beta_0 \gamma(t)}$ with $\beta_0 \in [2\beta, 1 - \beta)$. Let $(x^*, y^*, \lambda^*) \in \Omega$ and $(x(t), y(t), \lambda(t))$ be the unique solution of the dynamic (1.4) defined on a maximal interval $[0, T]$ with $T \leq +\infty$ for some initial value. Then, the following conclusions hold:

(a) There exist positive functions $\theta(t)$ and $\eta(t)$ satisfying

$$\dot{\theta}^\beta_{\theta, \eta}(t) \leq 0, \quad \forall t \in [t_0, T].$$

As a consequence, the function $\mathcal{E}^\beta_{\theta, \eta}(t)$ is nonincreasing on $[t_0, T]$.

Proof. (a): Take

$$\theta(t) = \beta_0 p(t)^{\beta} \gamma(t) \quad \text{and} \quad \eta(t) = -\beta_0 p(t)^{2\beta} ((\beta_0 + 2\beta - 1) \gamma(t)^2 + \dot{\gamma}(t)).$$

Clearly, $\theta(t) > 0$ for all $t \geq t_0$. By assumption, we have

$$\dot{\gamma}(t) \geq 2\beta^2 \gamma(t)^3, \quad \forall t \geq t_0.$$

This together with Lemma A.1 yields

$$\dot{\gamma}(t) \leq -\beta \gamma(t)^2, \quad \forall t \geq t_0.$$

Since $\beta \in (0, \frac{1}{3})$ and $\beta_0 \in [2\beta, 1 - \beta)$, it follows from (2.4) and (2.5) that

$$\eta(t) \geq \beta_0 (1 - \beta - \beta_0) p(t)^{2\beta} \gamma(t)^2 > 0$$

for $t \geq t_0$. By computations, we have

$$\begin{cases}
\dot{\theta}(t) = \beta_0 p(t)^{\beta} (\beta \gamma(t)^2 + \dot{\gamma}(t)), \\
\dot{\eta}(t) = -\beta_0 p(t)^{2\beta} (2\beta (\beta_0 + 2\beta - 1) \gamma(t)^3 + (6\beta + 2\beta_0 - 2) \gamma(t) \dot{\gamma}(t) + \dot{\gamma}(t)).
\end{cases}$$

We shall prove that for any $t \geq t_0$,

$$\theta(t) \dot{\theta}(t) + \frac{\dot{\theta}(t)}{2} \leq 0,$$  

(2.8)

$$p(t)^{2\beta} - \theta(t) p(t)^{\beta} \delta(t) = 0,$$  

(2.9)

$$\theta(t)(\theta(t) + (\beta - 1)p(t)^{\beta} \gamma(t)) + \dot{\theta}(t)p(t)^{\beta} + \eta(t) = 0.$$  

(2.10)

Using (2.4) and (2.7), by simple computations we get (2.9) and (2.10). Next, we shall show (2.8). Again from (2.4) and (2.7) we have

$$\theta(t) \dot{\theta}(t) + \frac{\dot{\theta}(t)}{2} = -\beta_0 \frac{p(t)}{2} (\gamma(t) + (6\beta - 2) \gamma(t) + 2\beta (2\beta - 1) \gamma(t)^3)$$

$$= -\beta_0 \frac{p(t)}{2} (\gamma(t) - 2\beta^2 \gamma(t)^3 + 2(3\beta - 1) \gamma(t) \dot{\gamma}(t) + \beta \gamma(t)^2)).$$

This together with (2.5) and assumption yields (2.8) since $\beta \in (0, \frac{1}{3})$ and $\beta_0 \in [2\beta, 1 - \beta)$. Thus, we have proved (2.8) - (2.10).
From (2.8) and (2.10) we have \( V_1(t) \leq 0 \) and \( V_2(t) = 0 \) for any \( t \in [t_0, T) \). Since \( f \) and \( g \) are convex, it follows from (2.9) that

\[
\dot{E}_{\theta, \eta}^\beta(t) \leq \dot{E}_0(t) + V_3(t) + V_4(t) + V_5(t)
\]

\[
= \beta_0 p(t)^{2\beta_0} \gamma(t) (f(x(t)) - f(x^*) - (x(t) - x^*, \nabla f(x(t)))
\]

\[
+ \beta_0 p(t)^{2\beta_0} \gamma(t) (g(y(t)) - g(y^*) - (y(t) - y^*, \nabla g(y(t)))
\]

\[
- \beta_0 - 2\beta p(t)^{2\beta} \gamma(t) (L(x(t), y(t), \lambda^*) - L(x^*, y^*, \lambda^*))
\]

\[
- \frac{\beta_0}{2} p(t)^{2\beta} \gamma(t) \|Ax(t) + By(t) - b\|^2
\]

\[
- (1 - \beta - \beta_0) p(t)^{2\beta_0} \gamma(t) (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2)
\]

\[
\leq - \beta_0 - 2\beta p(t)^{2\beta} \gamma(t) (L(x(t), y(t), \lambda^*) - L(x^*, y^*, \lambda^*))
\]

\[
- \frac{\beta_0}{2} p(t)^{2\beta} \gamma(t) \|Ax(t) + By(t) - b\|^2
\]

\[
- (1 - \beta - \beta_0) p(t)^{2\beta} \gamma(t) (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2)
\]

\[
\leq 0
\]

for any \( t \in [t_0, T) \). As a consequence, the function \( E_{\theta, \eta}^\beta(t) \) is nonincreasing on \([t_0, T)\).

(b): By (a), \( E_{\theta, \eta}^\beta(t) \) is nonincreasing on \([t_0, T)\). Then,

\[
E_{\theta, \eta}^\beta(t) \leq E_{\theta, \eta}^\beta(t_0), \quad \forall t \in [t_0, T),
\]

This implies that \( E_{\theta, \eta}^\beta(t) \) is bounded on \([t_0, T)\). It follows from (2.3) that

\[
\frac{1}{2} \|\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t)\|^2 + \frac{\eta(t)}{2} \|x(t) - x^*\|^2 \leq E_{\theta, \eta}^\beta(t_0), \quad \forall t \in [t_0, T),
\]

where \( \theta(t) \) and \( \eta(t) \) are defined by (2.4). This implies that

\[
\eta(t) \|x(t) - x^*\|^2 \leq 2E_{\theta, \eta}^\beta(t_0), \quad \forall t \in [t_0, T)
\]

and

\[
\|\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t)\| \leq \sqrt{2E_{\theta, \eta}^\beta(t_0)}, \quad \forall t \in [t_0, T).
\]

Combining (2.12) with (2.6) we get

\[
\beta_0 (1 - \beta - \beta_0) p(t)^{2\beta} \gamma(t) \|x(t) - x^*\|^2 \leq 2E_{\theta, \eta}^\beta(t_0), \quad \forall t \in [t_0, T),
\]

which yields

\[
\sup_{t \in [t_0, T)} p(t)^{\beta} \gamma(t) \|x(t) - x^*\| < +\infty.
\]

It follows from (2.13) and (2.4) that

\[
p(t)^{\beta} \|\dot{x}(t)\| \leq \sqrt{2E_{\theta, \eta}^\beta(t_0) + \beta_0 p(t)^{\beta} \gamma(t) \|x(t) - x^*\|}, \quad \forall t \in [t_0, T).
\]

Since \( p(t) \geq 1 \), we have

\[
\sup_{t \in [t_0, T]} \|\dot{x}(t)\| \leq \sqrt{2E_{\theta, \eta}^\beta(t_0) + \beta_0 \sup_{t \in [t_0, T]} p(t)^{\beta} \gamma(t) \|x(t) - x^*\|} < +\infty.
\]
By similar arguments, we have
\[
\sup_{t \in [t_0, T]} \|\dot{y}(t)\| < +\infty \quad \text{and} \quad \sup_{t \in [t_0, T]} \|\dot{\lambda}(t)\| < +\infty.
\]
Assume on the contrary that \( T < +\infty \). Clearly, the trajectory \((x(t), y(t), \lambda(t))\) is bounded on \([t_0, T]\). By assumption and (1.4), \((\dot{x}(t), \dot{y}(t), \dot{\lambda}(t))\) are bounded on \([t_0, T]\).

It ensues that both \((x(t), y(t), \lambda(t))\) and its derivative \((\dot{x}(t), \dot{y}(t), \dot{\lambda}(t))\) have a limit at \( t = T \), and therefore can be continued, a contradiction. Thus \( T = +\infty \).

\textbf{Remark 2.3}. To establish the existence and uniqueness of a global solution of (1.4), it is assumed in Theorem 2.2 that
\[
\ddot{\gamma}(t) \geq 2\beta^2 \gamma(t)^3, \quad \forall t \geq t_0
\]
for some \( \beta \in (0, \frac{1}{3}) \), and \( \delta(t) = \frac{1}{2\beta \gamma(t)} \) with \( \beta_0 \in [2\beta, 1 - \beta) \). From the proof, it is easy to see that the conclusion (b) of Theorem 2.2 still holds if \( \beta = \frac{1}{3} \) and \( \delta(t) = \frac{3}{2\gamma(t)} \).

Under this condition,
\[
\ddot{\gamma}(t) \geq 2\beta^2 \gamma(t)^3 \geq 2\tilde{\beta}^2 \gamma(t)^3, \quad \forall \tilde{\beta} \in (0, \frac{1}{3}), \quad t \geq t_0
\]
and \( \delta(t) = \frac{3}{2\gamma(t)} = \frac{1}{\beta_0 \gamma(t)} \) with \( \beta_0 = \frac{2}{3} \in [2\tilde{\beta}, 1 - \tilde{\beta}) \) for any \( \tilde{\beta} \in (0, \frac{1}{3}) \). Let us mention that the condition
\[
(2.14) \quad \ddot{\gamma}(t) \geq 2\beta^2 \gamma(t)^3 \quad \text{for some} \quad \beta \in (0, \frac{1}{3})
\]
has been used in [3] for the asymptotic analysis of \((IGS)_\gamma\) associated with the unconstrained optimization problem (1.5). As pointed out in [3], the value \( \beta = \frac{1}{3} \) is crucial and it corresponds to \( \alpha = 3 \) in the case \( \gamma(t) = \frac{t}{t} \). To the best of our knowledge, this is the first time that this condition is applied to the study of primal-dual dynamical systems for constrained optimization problems.

\textbf{Remark 2.4}. The existence and uniqueness of a global solution for \((IGS)_\gamma\) associated with the unconstrained optimization problem (1.1) has been established in [2, Proposition 3.2]). The nonincreasing property of the energy function \( W(t) := \frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) \) on \([t_0, T]\) plays a crucial role in the proof of [2, Proposition 3.2]). As a comparison, in Theorem 2.2 we use the nonincreasing property of the energy function \( E^{\beta}_{\alpha, \eta}(t) \) to prove the existence and uniqueness of a global solution for the dynamic (1.4).

With Theorem 2.2 in hands, we start to discuss the asymptotic behavior of the dynamic (1.4). The following condition on the damp function \( \gamma(t) \) is a common assumption for convergence analysis:
\[
(2.15) \quad \int_{t_0}^{+\infty} \gamma(t)dt = +\infty.
\]
Notice \( p(t) = e^{\int_{t_0}^{t} \gamma(s)ds} \rightarrow +\infty \) as \( t \rightarrow +\infty \) when \( \gamma(t) \) satisfies (2.15).

\textbf{Theorem 2.5}. Let \( \gamma : [t_0, +\infty) \rightarrow (0, +\infty) \) be a nonincreasing and twice continuously differentiable function satisfying (2.14) and (2.15), and \( \delta(t) = \frac{1}{\beta_0 \gamma(t)} \) with \( \beta_0 \in [2\beta, 1 - \beta) \). Suppose that \((x(t), y(t), \lambda(t))\) is a global solution of the dynamic (1.4) and \((x^*, y^*, \lambda^*) \in \Omega \). Then, the following conclusions hold:
(a) $\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(p(t)^{-2\beta}).$
(b) $\|Ax(t) + By(t) - b\| = \mathcal{O}(p(t)^{-\beta}).$
(c) $\int_0^+ p(t)^{2\beta} \gamma(t) \|Ax(t) + By(t) - b\|^2 dt < +\infty.$

Moreover, we have the following results:

Case I : $\beta < \frac{1}{2}$ and $\beta_0 \in (2\beta, 1 - \beta)$. Then

\[ \frac{d}{dt} \int_0^\infty p(t)^{2\beta} \gamma(t)(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty. \]

Case II : $\beta = \frac{1}{2}$ and $\beta_0 = \frac{2}{3}$. Then for any $\tau \in (0, \frac{1}{2})$ we have

\[ \int_0^\infty p(t)^{2\tau} \gamma(t)(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty. \]

Proof. Take $\theta(t)$ and $\eta(t)$ as in (2.4). Consider the energy function $\mathcal{E}_{\theta, \eta}^\beta : [t_0, +\infty) \to [0, +\infty)$ defined by (2.3). From (2.11), we have

\[ (2.16) \quad \mathcal{E}_{\theta, \eta}^\beta(t) \leq \mathcal{E}_{\theta, \eta}^\beta(t_0), \quad \forall t \geq t_0 \]

and

\[ (2.17) \quad \dot{\mathcal{E}}_{\theta, \eta}^\beta(t) + (\beta_0 - 2\beta)p(t)^{2\beta} \gamma(t)(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) + \frac{\beta_0}{2} p(t)^{2\beta} \gamma(t) \|Ax(t) + By(t) - b\|^2 + (1 - \beta - \beta_0)p(t)^{2\beta} \gamma(t)(\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) \leq 0, \quad \forall t \geq t_0. \]

As a consequence of (2.16), $\mathcal{E}_{\theta, \eta}^\beta(\cdot)$ is bounded on $[t_0, +\infty)$. This together with (2.3) implies

\[ (2.18) \quad \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(p(t)^{-2\beta}). \]

Since $f$ and $g$ are convex, it follows from from (1.2) that

\[ (2.19) \quad \begin{aligned} \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) &= f(x(t)) - f(x^*) + g(y(t)) - g(y^*) + \langle \lambda^*, Ax(t) + By(t) - b \rangle \\
&= f(x(t)) - f(x^*) - \langle A^T \lambda^*, x(t) - x^* \rangle \\
&\quad + g(y(t)) - g(y^*) - \langle B^T \lambda^*, y(t) - y^* \rangle + \frac{1}{2} \|Ax(t) + By(t) - b\|^2 \\
&\geq \frac{1}{2} \|Ax(t) + By(t) - b\|^2. \end{aligned} \]

This together with (2.18) yields

\[ \|Ax(t) + By(t) - b\| = \mathcal{O}(p(t)^{-\beta}). \]

Since $\beta \in (0, \frac{1}{2}]$ and $\beta_0 \in [2\beta, 1 - \beta]$, again from (2.17) we have

\[ \int_0^+ p(t)^{2\beta} \gamma(t) \|Ax(t) + By(t) - b\|^2 dt < +\infty. \]
(2.20) \[(\beta_0 - 2\beta) \int_{t_0}^{+\infty} p(t)^{2\beta} \gamma(t) (\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty,\]
and
\[(2.21) \ (1 - \beta - \beta_0) \int_{t_0}^{+\infty} p(t)^{2\beta} \gamma(t) (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) dt < +\infty.\]

Thus we have shown (a) \(-\gamma\).

Next we prove (d),(e) and (f) in the case \(\beta < \frac{1}{\beta} \) and \(\beta_0 \in (2\beta, 1 - \beta)\). Clearly, \(\beta_0 - 2\beta > 0\) and \(1 - \beta - \beta_0 > 0\). So (d) and (e) follow directly from (2.20) and (2.21), respectively.

As shown in the proof of (b) of Theorem 2.2, we have
\[\sup_{t \geq t_0} p(t)^\beta \gamma(t) \|x(t) - x^*\| < +\infty\]
and
\[p(t)^\beta \|\dot{x}(t)\| \leq \sqrt{2\mathcal{L}_0(t_0) + \beta_0 p(t)^\beta \gamma(t) \|x(t) - x^*\|}, \quad \forall t \geq t_0.\]

Then
\[\sup_{t \geq t_0} p(t)^\beta \|\dot{x}(t)\| \leq \sqrt{2\mathcal{L}_0(t_0) + \beta_0 \sup_{t \geq t_0} p(t)^\beta \gamma(t) \|x(t) - x^*\| < +\infty},\]
this implies
\[\|\dot{x}(t)\| = O(p(t)^{-\beta}).\]

By similar arguments, we have
\[\|\dot{y}(t)\| = O(p(t)^{-\beta}) \quad \text{and} \quad \|\dot{\lambda}(t)\| = O(p(t)^{-\beta}).\]

This proves (f).

In the case \(\beta = \frac{1}{\beta} \) and \(\beta_0 = \frac{2}{\beta} \). For any \(\tau \in (0, \frac{1}{\beta})\), we have \(\dot{\gamma}(t) \geq 2\tau^2 \gamma(t)^3 \) and \(\beta_0 \in (2\tau, 1 - \tau)\). So (d'), (e') and (f') follow directly from (d),(e) and (f), respectively.

Remark 2.6. It is assumed in Theorem 2.5 that \(\beta_0 \in [2\beta, 1 - \beta] \). In fact, for any \(\beta_0 \in (0, 1)\), we can prove convergence rates as in Theorem 2.5 by substituting \(\bar{\beta}\) for \(\beta\), where \(\bar{\beta} = \min\{\beta, \frac{2}{\beta}, 1 - \beta_0\}\). It is easy to verify that
\[\dot{\gamma}(t) \geq 2\bar{\beta}^2 \gamma(t)^3, \quad \bar{\beta} \in (0, \frac{1}{\beta}], \quad \text{and} \quad \beta_0 \in [2\bar{\beta}, 1 - \bar{\beta}].\]

Remark 2.7. Theorem 2.5 can be viewed as analogs of the results in [3, Theorem 2.1, Proposition 3, Proposition 4], where the convergence rate analysis of (IGS,\(\gamma\)) associated with the unconstrained optimization problem (1.5) were derived. In [3, Theorem 2.1], they assumed that \(x(t)\) is bounded on \([t_0, +\infty)\) to get \(\|\dot{x}(t)\| = O(p(t)^{-\beta})\).

Theorem 2.5 shows that the boundedness assumption is redundant both in the IGS,\(\gamma\) and in our primal-dual dynamical system.

In the rest of this section, we apply the results of Theorem 2.5 to two special damping functions: \(\gamma(t) = \frac{\alpha}{r} \) with \(\alpha > 0\) and \(\gamma(t) = \frac{1}{(\ln t)^r} \) with \(r \in [0, 1]\).

Case \(\gamma(t) = \frac{\alpha}{r} \) with \(\alpha > 0\). In this case, \(\dot{\gamma}(t) = \frac{\alpha}{r^2} \) and
\[\dot{\gamma}(t) \geq 2\beta^2 \gamma(t)^3 \iff \alpha \beta^2 \leq 1.\]
Assumption on \(\gamma(t)\) in Theorem 2.5 is satisfied if we take
\[0 < \beta \leq \frac{1}{\alpha} \min\left\{\frac{1}{3}, \frac{1}{\alpha}\right\}.\]
COROLLARY 2.8. Suppose that \( \gamma(t) = \frac{\alpha}{t} \) with \( \alpha > 0 \) and \( \delta(t) = \frac{1}{t^3} \) with \( \beta_0 > 0 \). Let \((x^*, y^*, \lambda^*) \in \Omega\) and \((x(t), y(t), \lambda(t))\) be a global solution of the dynamic (1.4). Then we have the following results:

i) If \( \alpha \leq 3 \) and \( \beta_0 = \frac{2}{3} \), then
\[
\begin{align*}
(\text{a}) & \quad \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(t^{-\frac{2\alpha}{3}}). \\
(\text{b}) & \quad \|Ax(t) + By(t) - b\| = \mathcal{O}(t^{-\frac{2}{3}}). \\
(\text{c}) & \quad \int_{t_0}^{t} \frac{2}{3\alpha - 1} \|Ax(t) + By(t) - b\|^2 dt < +\infty. \\
(\text{d}) & \quad \int_{t_0}^{t} t^m(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty, \quad \forall m \in (-1, \frac{2}{3\alpha - 1}). \\
(\text{e}) & \quad \int_{t_0}^{t} t^m(\|\hat{x}(t)\|^2 + \|\hat{y}(t)\|^2 + \|\hat{\lambda}(t)\|^2) dt < +\infty, \quad \forall m \in (-1, \frac{2}{3\alpha - 1}). \\
(\text{f}) & \quad \|\hat{x}(t)\| + \|\hat{y}(t)\| + \|\hat{\lambda}(t)\| = \mathcal{O}(t^{-m}), \quad \forall m \in (0, \frac{2}{3\alpha}).
\end{align*}
\]
Moreover if \( \alpha < 3 \), then
\begin{align*}
(\text{g}) & \quad \|\hat{x}(t)\| + \|\hat{y}(t)\| + \|\hat{\lambda}(t)\| = \mathcal{O}(t^{-\frac{2}{3\alpha}}).
\end{align*}

ii) If \( \alpha > 3 \) and \( \beta_0 \in \left(\frac{2}{3}, 1 - \frac{1}{\alpha}\right)\), then
\begin{align*}
(\text{a'}) & \quad \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(t^{-2}). \\
(\text{b'}) & \quad \|Ax(t) + By(t) - b\| = \mathcal{O}(t^{-1}). \\
(\text{c'}) & \quad \int_{t_0}^{t} \|Ax(t) + By(t) - b\|^2 dt < +\infty. \\
(\text{d'}) & \quad \int_{t_0}^{t} t(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty. \\
(\text{e'}) & \quad \int_{t_0}^{t} t(\|\hat{x}(t)\|^2 + \|\hat{y}(t)\|^2 + \|\hat{\lambda}(t)\|^2) dt < +\infty. \\
(\text{f'}) & \quad \|\hat{x}(t)\| + \|\hat{y}(t)\| + \|\hat{\lambda}(t)\| = \mathcal{O}(t^{-1}).
\end{align*}

Proof. Since \( \gamma(t) = \frac{\alpha}{t} \), by computation we have
\begin{equation}
(2.22) \quad p(t) = e^{\int_{t_0}^{t} \gamma(s) ds} = \left(\frac{t}{t_0}\right)^{\alpha}.
\end{equation}

Take \( \beta = \min \left\{\frac{4}{3}, \frac{1}{\alpha}\right\} \). It is easy to verify that all the assumptions in Theorem 2.5 are satisfied. So \( (\alpha) - (f') \) and \( (a') - (f') \) follow directly from Theorem 2.5.

Now we prove (g). Notice that \( \alpha < 3 \), \( \beta = \frac{1}{3} \), and \( \beta_0 = \frac{2}{3} \). Consider the functions \( \theta(t) \) and \( \eta(t) \) defined by (2.4). By computations we get
\begin{equation}
(2.23) \quad \theta(t) = \frac{2\alpha}{3t_0^{\frac{2}{3\alpha - 1}}} \quad \text{and} \quad \eta(t) = \frac{2\alpha}{3t_0^{\frac{2}{3\alpha - 1}}} \left(1 - \frac{\alpha}{3}\right) t^{\frac{2}{3\alpha - 2}}.
\end{equation}
Then,
\[ \mathcal{E}^\beta_{\theta, \eta}(t) \leq \mathcal{E}^\beta_{\theta, \eta}(t_0), \quad \forall t \geq t_0, \]
where \( \mathcal{E}^\beta_{\theta, \eta}(t) \) is the energy function defined by (2.3). As a consequence, we have
\begin{align*}
\frac{1}{2}(\|\theta(t)(x(t) - x^*) + p(t)\| + \eta(t)\|x(t) - x^*\|) + \|\hat{\lambda}(t)\|^2 \leq \mathcal{E}^\beta_{\theta, \eta}(t_0), \quad \forall t \geq t_0.
\end{align*}
This implies that for any \( t \geq t_0 \)
\begin{equation}
(2.24) \quad \eta(t)\|x(t) - x^*\|^2 \leq 2\mathcal{E}^\beta_{\theta, \eta}(t_0)
\end{equation}
and
\begin{equation}
(2.25) \quad p(t)\|\hat{x}(t)\| - \theta(t)\|x(t) - x^*\| \leq \|\theta(t)(x(t) - x^*) + p(t)\| \leq \sqrt{2\mathcal{E}^\beta_{\theta, \eta}(t_0)}.
\end{equation}
It follows from (2.22)-(2.25) that for any \( t \geq t_0 \),
\begin{equation*}
\frac{t_0^{\frac{2}{3\alpha - 1}}\|x(t) - x^*\|}{\sqrt{\alpha(3 - \alpha)}} \leq \frac{3t_0^{\frac{2}{3\alpha - 1}}}{\sqrt{\alpha(3 - \alpha)}} \sqrt{\mathcal{E}^\beta_{\theta, \eta}(t_0)}
\end{equation*}
and

\[ \left( \frac{t}{t_0} \right)^{\alpha\beta} \left\| \dot{x}(t) \right\| \leq \sqrt{2c_{\alpha,\beta}^3(t_0)} + \frac{2\alpha}{3t_0^2} t^{2-1} \left\| x(t) - x^* \right\| \leq \left( \sqrt{2} + \frac{\alpha}{3-\alpha} \right) \sqrt{c_{\alpha,\beta}^3(t_0)}. \]

This means

\[ \left\| \dot{x}(t) \right\| = \mathcal{O}(t^{-\frac{\alpha}{2}}). \]

By similar arguments, we get

\[ \left\| \dot{y}(t) \right\| = \mathcal{O}(t^{-\frac{\beta}{2}}) \quad \text{and} \quad \left\| \dot{\lambda}(t) \right\| = \mathcal{O}(t^{-\frac{\alpha}{2}}). \]

This proves (g).

\[ \Box \]

Remark 2.9. Corollary 2.8 improves [39, Theorem 3.1 and Theorem 3.2] where convergence rates of a second-order dynamical system based on the primal-dual framework for the problem (1.1) with \( g(x) \equiv 0 \) and \( B = 0 \) were established.

**Case** \( \gamma(t) = \frac{3}{t(\ln t)^r} \) with \( r \in [0, 1] \). In this case,

\[ \tilde{\gamma}(t) = \frac{2(\ln t)^2 + 3\ln t + r(r + 1)}{t^3(\ln t)^{r+2}}. \]

It is easy to verify that (2.14) holds for all \( \beta \in (0, \frac{1}{2}] \) and \( t_0 \geq e \). As a consequence of Theorem 2.5, we have

**Corollary 2.10.** Suppose that \( \gamma(t) = \frac{3}{t(\ln t)^r} \) with \( r \in [0, 1] \), \( \delta(t) = \frac{3(\ln t)^2}{2} \) and \( t_0 \geq e \). Let \((x(t), y(t), \lambda(t))\) be a global solution of the dynamic (1.4) and \((x^*, y^*, \lambda^*) \in \Omega\). Then we have the following results:

(a) \( \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(p(t)^{-\frac{1}{2}}). \)

(b) \( \left\| Ax(t) + By(t) - b \right\| = \mathcal{O}(p(t)^{-\frac{1}{2}}). \)

(c) \( \int_{t_0}^{t} \frac{p(t)}{t(\ln t)^r} \left\| Ax(t) + By(t) - b \right\|^2 dt < +\infty. \)

(d) \( \int_{t_0}^{t} \frac{p(t)}{t(\ln t)^r} \left( \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) \right) dt < +\infty, \quad \forall m \in (0, \frac{1}{2}) \).

(e) \( \int_{t_0}^{t} \frac{p(t)}{t(\ln t)^r} \left( \left\| \dot{x}(t) \right\|^2 + \left\| \dot{y}(t) \right\|^2 + \left\| \dot{\lambda}(t) \right\|^2 \right) dt < +\infty, \quad \forall m \in (0, \frac{1}{2}). \)

(f) \( \left\| \dot{x}(t) \right\| + \left\| \dot{y}(t) \right\| + \left\| \dot{\lambda}(t) \right\| = \mathcal{O}(p(t)^{-m}), \quad \forall m \in (0, \frac{1}{2}). \)

Here \( p(t) = e^{\int_{t_0}^{t} \frac{1}{\ln s} ds}. \)

Remark 2.11. Corollary 2.10 can be viewed as analogs of the results in [3, Subsection 4.2], where the convergence rate analysis of \((IGS_{\gamma})\) associated with the unstrained optimization problem (1.5) has been discussed when \( \gamma(t) = \frac{1}{t(\ln t)^r} \) with \( r \in [0, 1] \).

**3. Asymptotic properties for** \( \gamma(t) = \frac{\alpha}{t^r}. \) In this section, we investigate the asymptotic behavior of the dynamic (1.4) with \( \gamma(t) = \frac{\alpha}{t^r}, \ r \in (-1, 1) \) and \( \alpha > 0 \). Let us mention that the results of this section cannot be obtained as applications of the results in Section 2. In the case \( \gamma(t) = \frac{\alpha}{t^r} \) with \( r \in (-1, 1) \) and \( \alpha > 0 \), the condition (2.14) required in Section 2 is not satisfied. Indeed, in this case, we have \( \tilde{\gamma}(t) = \frac{\alpha r(1+r)}{t^{2r+2}}. \) If there exists \( \beta \in (0, \frac{1}{3}] \) satisfying (2.14), i.e., \( \tilde{\gamma}(t) \geq 2\beta^2 \gamma(t)^3. \) Then we have

\[ 0 < 2\beta^2 \alpha^2 \leq \frac{r(r+1)}{t^{2(1-r)}} \to 0, \quad \text{as} \ t \to +\infty, \]

a contradiction.
Next, we first prove the existence and uniqueness of a global solution of the
dynamic (1.4) with \( \gamma(t) = \frac{1}{t^p} \).

Throughout this section, we always suppose that \( t_0 \geq 1 \) and \((x^*, y^*, \lambda^*) \in \Omega\). Then we have \( \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) \geq 0 \) for \( t \geq t_0 \). We consider the energy function \( E^p_{\theta, \eta} : [t_0, +\infty) \to [0, +\infty) \) defined by
\[
E^p_{\theta, \eta}(t) = E_0(t) + E_1(t) + E_2(t) + E_3(t),
\]
where
\[
\begin{align*}
E_0(t) &= t^{2p}(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)), \\
E_1(t) &= \frac{1}{2}\|\theta(t)(x(t) - x^*) + t^p\dot{x}(t)\|^2 + \frac{\eta(t)}{2}\|x(t) - x^*\|^2, \\
E_2(t) &= \frac{1}{2}\|\theta(t)(y(t) - y^*) + t^p\dot{y}(t)\|^2 + \frac{\eta(t)}{2}\|y(t) - y^*\|^2, \\
E_3(t) &= \frac{1}{2}\|\theta(t)(\lambda(t) - \lambda^*) + t^p\dot{\lambda}(t)\|^2 + \frac{\eta(t)}{2}\|\lambda(t) - \lambda^*\|^2,
\end{align*}
\]
\( \theta, \eta : [t_0, +\infty) \to [0, +\infty) \) are two suitable functions, and \( \rho > 0 \). Multiplying the first equation of (1.4) with \( \gamma(t) = \frac{1}{t^p} \) by \( t^p \), we have
\[
t^p\dot{x}(t) = -\alpha t^{p-\gamma} \dot{x}(t) - t^p(\nabla f(x(t)) + A^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) + A^T(Ax(t) + By(t) - b)).
\]
This yields
\[
\begin{align*}
\dot{E}_1(t) &= \theta(t(x(t) - x^*) + t^p\dot{x}(t), \dot{\theta}(t(x(t) - x^*) + \theta(t)\dot{x}(t) + \rho t^{p-1}\dot{x}(t) + t^p\ddot{x}(t)) \\
&+ \frac{\eta(t)}{2}\|x(t) - x^*\|^2 + \eta(t|x(t) - x^*, \dot{x}(t)|
\end{align*}
\]
\[
= \langle \theta(t)(x(t) - x^*) + t^p\dot{x}(t), \dot{\theta}(t(x(t) - x^*) + \theta(t) + \rho t^{p-1} - \alpha t^{p-\gamma})\dot{x}(t) + t^p(\nabla f(x(t)) + A^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) + A^T(Ax(t) + By(t) - b))
\]
\[
+ \frac{\eta(t)}{2}\|x(t) - x^*\|^2 + \eta(t|x(t) - x^*, \dot{x}(t)|
\end{align*}
\]
\[
= \langle \theta(t)(x(t) - x^*) + t^p\dot{x}(t), \dot{\theta}(t(x(t) - x^*) + \theta(t) + \rho t^{p-1} - \alpha t^{p-\gamma})\dot{x}(t) + t^p(\nabla f(x(t)) + A^T(\lambda(t) + \delta(t)\dot{\lambda}(t)) + A^T(Ax(t) + By(t) - b))
\]

Similarly, we have
\[
\begin{align*}
\dot{E}_2(t) &= \frac{\eta(t)}{2}\|y(t) - y^*\|^2 + t^p(\theta(t) + \rho t^{p-1} - \alpha t^{p-\gamma})\|\dot{y}(t)\|^2
\end{align*}
\]
and
\[
\begin{align*}
\dot{E}_3(t) &= \frac{\eta(t)}{2}\|\lambda(t) - \lambda^*\|^2 + t^p(\theta(t) + \rho t^{p-1} - \alpha t^{p-\gamma})\|\dot{\lambda}(t)\|^2
\end{align*}
\]
Adding $\dot{E}_1(t)$, $\dot{E}_2(t)$, $\dot{E}_3(t)$ together, using $Ax^* + By^* = b$ and rearranging the terms, we get

$$\dot{E}_1(t) + \dot{E}_2(t) + \dot{E}_3(t) = \sum_{i=1}^{5} V_i(t),$$

where

$$V_1(t) = \left(\theta(t)\dot{\vartheta}(t) + \frac{\eta(t)}{2}\right)(\|x(t) - x^*\|^2 + \|y(t) - y^*\|^2 + \|\lambda(t) - \lambda^*\|^2),$$

$$V_2(t) = (\theta(t)\theta(t) + pt^{\rho - 1} - \alpha t^{\rho - r}) + \eta(t) + t^\rho \dot{\vartheta}(t)$$

$$\times ((x(t) - x^*, \dot{x}(t)) + (y(t) - y^*, \dot{y}(t)) + (\lambda(t) - \lambda^*, \dot{\lambda}(t))),$$

$$V_3(t) = -\theta(t)t^\rho((x(t) - x^*, \nabla g(x(t)) + A^T\lambda^*) + (y(t) - y^*, \nabla g(y(t)) + B^T\lambda^*))$$

$$+ \theta(t)\delta(t)\rho\|\lambda(t) - \lambda^*, Az(t) + By(t)\|,$$

$$V_4(t) = -\theta(t)t^\rho\|Ax(t) + By(t) - b\|^2$$

$$+ t^\rho(\theta(t) + pt^{\rho - 1} - \alpha t^{\rho - r})(\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2),$$

$$V_5(t) = (t^{2\rho} - \theta(t)\delta(t)t^\rho)(\dot{\lambda}(t), Ax(t) + By(t) - b)$$

$$- t^{2\rho}(\dot{x}(t), \nabla f(x(t)) + A^T\lambda(t) + A^T(Ax(t) + By(t) - b))$$

$$- t^{2\rho}(\dot{y}(t), \nabla g(y(t)) + B^T\lambda(t) + B^T(Ax(t) + By(t) - b)).$$

Derivative of $E_0(t)$ to get

$$\dot{E}_0(t) = 2\rho t^{2\rho - 1}(f(x(t)) - f(x^*)) + g(y(t)) - g(y^*) + (\lambda^*, Ax(t) + By(t) - b)$$

$$+ \rho t^{2\rho - 1}\|Ax(t) + By(t) - b\|^2$$

$$+ t^{2\rho}(\nabla f(x(t)), \dot{x}(t)) + (\nabla g(y(t)), \dot{y}(t)) + (\lambda^*, Ax(t) + By(t))$$

$$+ t^{2\rho}(Ax(t) + By(t) - b, Az(t) + By(t)).$$

The existence and uniqueness of a local solution of the dynamic (1.4) can be derived from Proposition 2.1 when $\gamma(y) = \frac{\rho t^\rho}{\alpha}$ with $r \in (-1, 1)$ and $\delta(t)$ is locally integrable. In the following, we will further investigate the existence and uniqueness of its global solution.

**Theorem 3.1.** Let $f$ and $g$ be two continuously differentiable functions such that $\nabla f$ and $\nabla g$ are locally Lipschitz continuous. Suppose that $\gamma(t) = \frac{\alpha}{\rho}$ with $r \in (-1, 1)$ and $\delta(t)$ is locally integrable.

(a) $r_0 > \frac{1}{2\rho}$ and $\alpha > \max\{0, (4r_0 + r + 1)t_0^{\rho - 1}\}$ when $r \in (-1, 0]$;

(b) $r_0 > r$ and $\alpha > \max\{0, (4r_0 + 2r)t_0^{\rho - 1}\}$ when $r \in (0, 1)$.

Then for any initial value $(x_0, y_0, \lambda_0, u_0, v_0, w_0)$, there exists a unique solution $(x(t), y(t), \lambda(t))$ with $x(t) \in C^2([t_0, +\infty), \mathbb{R}^n)$, $y(t) \in C^2([t_0, +\infty), \mathbb{R}^n)$ and $\lambda(t) \in C^2([t_0, +\infty), \mathbb{R}^n)$ of the dynamic (1.4) satisfying $(x(t_0), y(t_0), \lambda(t_0)) = (x_0, y_0, \lambda_0)$ and $(\dot{x}(t_0), \dot{y}(t_0), \dot{\lambda}(t_0)) = (u_0, v_0, w_0)$.

**Proof.** By Proposition 2.1, there exists a unique solution $(x(t), y(t), \lambda(t))$ with $x(t) \in C^2([t_0, T), \mathbb{R}^n)$, $y(t) \in C^2([t_0, T), \mathbb{R}^n)$ and $\lambda(t) \in C^2([t_0, T), \mathbb{R}^n)$ of the dynamic (1.4) defined on a maximal interval $[t_0, T)$ with $T \leq +\infty$ satisfying the initial condition: $(x(t_0), y(t_0), \lambda(t_0)) = (x_0, y_0, \lambda_0)$ and $(\dot{x}(t_0), \dot{y}(t_0), \dot{\lambda}(t_0)) = (u_0, v_0, w_0)$. We shall show $T = +\infty$ in both cases.

Case (a). In this case, in (3.2), we take $\rho = \frac{\alpha}{\rho}$,

$$\theta(t) = 2r_0t^{\frac{\rho - 1}{\rho}} \quad \text{and} \quad \eta(t) = 2r_0(\alpha - (2r_0 + r)t^{\rho - 1}).$$
Clearly, \( \theta(t) > 0 \) for all \( t \geq t_0 \). We claim that

\[
\alpha - (2r_0 + r)t^r - 1 \geq \frac{\alpha}{2}, \quad \forall t \geq t_0,
\]

which yields

\[
\eta(t) \geq r_0 \alpha > 0, \quad \forall t \geq t_0.
\]

Indeed, we have \( \alpha - (2r_0 + r)t^r - 1 \geq \frac{\alpha}{2} \) for all \( t \geq t_0 \) when \( 2r_0 + r < 0 \). When \( 2r_0 + r \geq 0 \), we have \( \alpha - (2r_0 + r)t^r - 1 \geq \alpha - (2r_0 + r)t_0^r - 1 \geq \alpha \) for all \( t \geq t_0 \) since \( \alpha > (4r_0 + r + 1)t_0^r - 1 > (4r_0 + 2r_0)t_0^r - 1 \).

Next we shall prove that the energy function \( E_{\theta, \eta}^\rho(t) \) defined by (3.1) is nonincreasing on \([t_0, T]\). By computations, we have

\[
\dot{\theta}(t) = r_0(r - 1)t^{\frac{r - 2}{2}}, \quad \text{and} \quad \dot{\eta}(t) = -(4r_0^2 + 2r_0r)(r - 1)t^r - 2.
\]

This together with (3.3) and \( r \in (-1, 0] \) yields

\[
\theta(t)\dot{\theta}(t) + \frac{\dot{\eta}(t)}{2} = -rr_0(r - 1)t^r - 2 \leq 0
\]

and

\[
\theta(t)(\theta(t) + \rho t^\rho - 1 - \alpha t^\rho - r) + \eta(t) + t^\rho \dot{\theta}(t) = 0.
\]

Since \( \alpha > (4r_0 + r + 1)t_0^r - 1 \), we have

\[
\theta(t) + \rho t^\rho - 1 - \alpha t^\rho - r = (2r_0 + \frac{r + 1}{2}) \frac{t}{r_0} - \alpha \frac{t}{r_0}
\]

\[
= t^{\frac{r - 1}{2}} \left( \frac{1}{2}((4r_0 + r + 1)t^{r - 1} - \alpha) - \frac{\alpha}{2} \right)
\]

\[
\leq t^{\frac{r - 3}{2}} \left( \frac{1}{2}((4r_0 + r + 1)t_0^{r - 1} - \alpha) - \frac{\alpha}{2} \right)
\]

\[
< - \frac{\alpha}{2} t^{\frac{r - 3}{2}}.
\]

From (3.3) we get

\[
t^{2\rho} - \theta(t)\delta(t)\rho^\rho = t^{\rho + 1} - 2r_0 t^{\frac{r - 1}{2}} \times \frac{t}{2r_0} \times t^{\frac{r - 3}{2}} = 0.
\]

By (3.5) and (3.6), \( V_1(t) \leq 0 \) and \( V_2(t) = 0 \) for all \( t \geq t_0 \). Since \( f \) and \( g \) are convex, by using (3.7) and (3.8) and similar arguments as in (2.11), we have

\[
\dot{E}_{\theta, \eta}^\rho(t) \leq \dot{E}_0(t) + \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t)
\]

\[
\leq -r_0 \delta(t) \| Ax(t) + By(t) - b \|^2 - \frac{\alpha}{2} t \left( \| \dot{x}(t) \|^2 + \| \dot{y}(t) \|^2 + \| \dot{z}(t) \|^2 \right)
\]

\[
- (2r_0 - r - 1)t^r (\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*))
\]

\[
\leq 0
\]

for \( t \in [t_0, T] \). As a consequence, the function \( E_{\theta, \eta}^\rho(\cdot) \) is nonincreasing on \([t_0, T]\), and so

\[
E_{\theta, \eta}^\rho(t) \leq E_{\theta, \eta}^\rho(t_0), \quad \forall t \in [t_0, T].
\]
From (3.1) we have
\[
\frac{1}{2}||\theta(t)(x(t) - x^*) + t^{r+1}x(t)||^2 + \frac{\eta(t)}{2} ||x(t) - x^*||^2 \leq \mathcal{E}_{\theta,\eta}(t_0), \quad \forall t \in [t_0, T).
\]
This implies
\[
\sqrt{\eta(t)} ||x(t) - x^*|| \leq \sqrt{2\mathcal{E}_{\theta,\eta}(t_0)}, \quad \forall t \in [t_0, T)
\]
and
\[
||\theta(t)(x(t) - x^*) + t^{r+1}x(t)|| \leq \sqrt{2\mathcal{E}_{\theta,\eta}(t_0)}, \quad \forall t \in [t_0, T).
\]
Combining (3.10) with (3.4) we get
\[
||x(t) - x^*|| \leq \frac{2}{r_0 \alpha} \sqrt{\mathcal{E}_{\theta,\eta}(t_0)} \quad \forall t \in [t_0, T).
\]
Since \( \dot{\theta}(t) = r_0(r - 1)t^{r-3} \leq 0, \) \( \dot{\theta}(t) \) is nonincreasing. It follows from (3.11) and (3.12) that for any \( t \in [t_0, T) \)
\[
t^{r+1}||\dot{x}(t)|| \leq \sqrt{2\mathcal{E}_{\theta,\eta}(t_0)} + \theta(t)||x(t) - x^*|| \leq \sqrt{2\mathcal{E}_{\theta,\eta}(t_0)} + \theta(t)\sqrt{\frac{2}{r_0 \alpha} \mathcal{E}_{\theta,\eta}(t_0)}.
\]
This together with \( r \in (-1, 0) \) and \( t_0 \geq 1 \) yields
\[
\sup_{t \in (t_0, T]} ||\dot{x}(t)|| < +\infty.
\]
By similar arguments, we have \( \sup_{t \in [t_0, T]} ||\dot{y}(t)|| < +\infty \) and \( \sup_{t \in [t_0, T]} ||\dot{\lambda}(t)|| < +\infty. \)
Now assume on the contrary \( T < +\infty. \) Clearly, the trajectory \((x(t), y(t), \lambda(t))\) is bounded on \([t_0, T).\) By (1.4) and assumption, \((\dot{x}(t), \dot{y}(t), \dot{\lambda}(t))\) is bounded on \([t_0, T).\) It ensues that the solution \((x(t), y(t), \lambda(t))\) together with its derivative \((\dot{x}(t), \dot{y}(t), \dot{\lambda}(t))\) have a limit at \( t = T \) and therefore can be continued, a contradiction. Thus \( T = +\infty. \)
Case (b). In this case, in (3.2), we take \( \rho = r, \)
\[
\theta(t) = 2r_0 t^{r-1} \quad \text{and} \quad \eta(t) = 2r_0 t^{r-1}(1 - 2r - 2r_0) t^{r-1} + \alpha.
\]
Clearly, \( \theta(t) > 0 \) for all \( t \geq t_0. \) Now we show that
\[
\eta(t) \geq r_0 \alpha t^{r-1} > 0, \quad \forall t \geq t_0.
\]
When \( 2r_0 + 2r \leq 1, \) we have
\[
\eta(t) = 2r_0 t^{r-1}(1 - 2r - 2r_0) t^{r-1} + \alpha \geq 2r_0 \alpha t^{r-1} \geq r_0 \alpha t^{r-1} > 0.
\]
When \( 2r_0 + 2r > 1, \) we have \( \alpha - (2r_0 + 2r - 1) t^{r-1} \geq \alpha - (2r_0 + 2r - 1) t_0^{r-1} \geq \frac{\alpha}{2} \) since \( \alpha > (4r_0 + 4r - 2)t_0^{r-1} > (4r_0 + 4r - 2)t_0^{r-1}. \) Thus (3.14) holds.
By computations, we have
\[
\dot{\theta}(t) = 2r_0(r - 1)t^{r-2}
\]
and
\[
\dot{\eta}(t) = 2r_0(r - 1)t^{r-2}((2 - 4r - 4r_0)t^{r-1} + \alpha).
\]
This together with (3.13) yields

\begin{equation}
(3.15) \quad \theta(t)(\theta(t) + \rho \theta^{\rho-1} - \alpha \theta^{\rho-1}) + \eta(t) + t^\prime \dot{\theta}(t) = 0
\end{equation}

and

\begin{equation}
(3.16) \quad \theta(t) \dot{\theta}(t) + \frac{\dot{\eta}(t)}{2} = r_0(r - 1)t^{r-2}(2 - 4r)t^{r-1} + \alpha.
\end{equation}

We claim that

\begin{equation}
(3.17) \quad \theta(t) \dot{\theta}(t) + \frac{\dot{\eta}(t)}{2} < 0.
\end{equation}

Indeed, since \( \alpha > \max\{0, (4r_0 + 2r)t_0^{r-1}\} \), \( r_0 > r \) and \( r \in (0, 1) \), we have \( \alpha > (4r - 2)t_0^{r-1} \). In the case \( r \in (0, \frac{1}{2}) \), we get \( (2 - 4r)t^{r-1} + \alpha > 0 \) while in the case \( r \in [\frac{1}{2}, 1) \), we get \( (2 - 4r)t^{r-1} + \alpha > \alpha - (4r - 2)t_0^{r-1} > 0 \). So (3.17) follows from (3.16). Since \( r_0 > r \) and \( \alpha > (4r_0 + 2r)t_0^{r-1} \) with \( t_0 > 1 \), \( r \in (0, 1) \), we have

\begin{equation}
(3.18) \quad \theta(t) + rt^{r-1} - \alpha = (2r_0 + r)t^{r-1} - \alpha \leq (2r_0 + r)t_0^{r-1} - \alpha < -\frac{\alpha}{2}, \forall t \geq t_0.
\end{equation}

By computations, we have

\begin{equation}
(3.19) \quad t^2 \rho - \theta(t)\delta(t) = t^{2r} - 2r_0t^{r-1} \times \frac{t}{2r_0} \times t^r = 0.
\end{equation}

By (3.15)-(3.19) and similar arguments as in (a), we get

\begin{equation}
(3.20) \quad \dot{E}_{\theta, \eta}(t) + r_0 t^{2r-1}\|Ax(t) + By(t) - b\|^2 + \frac{\alpha}{2} t^{r}(\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2)
\end{equation}

\[ +2(r_0 - r)t^{2r-1}(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) \leq 0, \]

for \( t \in [t_0, T) \). This implies that the function \( E_{\theta, \eta}^p(\cdot) \) is nonincreasing on \([t_0, T)\), and

\[ E_{\theta, \eta}^p(t) \leq E_{\theta, \eta}^p(t_0), \quad \forall t \in [t_0, T). \]

By similar arguments as in (a), we have

\begin{equation}
(3.21) \quad \sqrt{\eta(t)}\|x(t) - x^*\| \leq \sqrt{2E_{\theta, \eta}^p(t_0)}, \quad \forall t \in [t_0, T)
\end{equation}

and

\begin{equation}
(3.22) \quad \|\theta(t)(x(t) - x^*) + t' \dot{x}(t)\| \leq \sqrt{2E_{\theta, \eta}^p(t_0)}, \quad \forall t \in [t_0, T).
\end{equation}

Combining (3.21) with (3.14) we obtain

\[ t^{\frac{\rho-1}{2}} \|x(t) - x^*\| \leq \sqrt{\frac{2}{r_0\alpha}} \sqrt{E_{\theta, \eta}^p(t_0)}, \quad \forall t \in [t_0, T). \]

It follows from (3.22) and (3.13) that for any \( t \in [t_0, T) \)

\begin{equation}
(3.23) \quad t' \|\dot{x}(t)\| \leq \sqrt{2E_{\theta, \eta}^p(t_0)} + 2r_0t^{r-1}\|x(t) - x^*\| \leq \sqrt{2E_{\theta, \eta}^p(t_0)} + \sqrt{\frac{8r_0}{\alpha}} \sqrt{t^{\frac{r-1}{2}}} \sqrt{E_{\theta, \eta}^p(t_0)}.
\end{equation}

Since \( r \in (0, 1) \) and \( t_0 \geq 1 \), we have \( t^{\frac{r-1}{2}} \leq 1 \) and \( t' \geq 1 \) for \( t \geq t_0 \). The rest of the proof is same as the one in case (a). \( \square \)
Next, we discuss the asymptotic behavior of the dynamic (1.4) with \( \gamma(t) = \frac{\beta}{t} \) and \( r \in (-1, 1) \). We first consider the case \( r \in (-1, 0) \).

**Theorem 3.2.** Suppose that \( \gamma(t) = \frac{\beta}{t} \) with \( r \in (-1, 0) \) and \( \alpha > 0 \), and \( \sigma(t) = \frac{t}{r^2} \) with \( r_0 > \frac{r+1}{1-r} \). Suppose that \( (x(t), y(t), \lambda(t)) \) is a global solution of the dynamic (1.4) and \((x^*, y^*, \lambda^*) \in \Omega \). Then, the following conclusions hold:

(a) \( \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(t^{-(r+1)}) \).

(b) \( \|Ax(t) + By(t) - b\| = \mathcal{O}(t^{-\frac{r+1}{2}}) \).

(c) \( \int_{t_1}^{+\infty} t^r \|Ax(t) + By(t) - b\|^2 dt < +\infty \).

(d) \( \int_{t_1}^{+\infty} t^r (\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty \).

(e) \( \int_{t_1}^{+\infty} t (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) dt < +\infty \).

(f) \( \|\dot{x}(t)\| + \|\dot{y}(t)\| + \|\dot{\lambda}(t)\| = \mathcal{O}(t^{-\frac{r+1}{2}}) \).

**Proof.** Take \( \theta(t) \) and \( \eta(t) \) as in (3.3). Consider the energy function \( E^\rho_{\theta, \eta}(t) \) defined by (3.1) with \( \rho = \frac{r^2}{1-r} \). Since \( \alpha > 0 \) and \( r \in (-1, 0) \), there exists \( t_1 \geq t_0 \) such that \( \alpha > (4r_0 + r + 1)t_1^{-1} \). From (3.9) we get

\[
E^\rho_{\theta, \eta}(t) = t^{r^2}(Ax(t) + By(t) - b) + \frac{\alpha}{2}t (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) + (2r_0 - r - 1)t^r (\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) \leq 0
\]

for any \( t \geq t_1 \), which implies

\[
E^\rho_{\theta, \eta}(t) \leq E^\rho_{\theta, \eta}(t_1), \quad \forall t \geq t_1.
\]

This together with (3.1) implies (a), i.e.,

\[
\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(t^{-(r+1)}).
\]

By the same arguments as in (2.19), we have (b):

\[
\|Ax(t) + By(t) - b\| = \mathcal{O}(t^{-\frac{r+1}{2}}).
\]

Since \( r_0 > \frac{r+1}{1-r} > 0 \) and \( \alpha > 0 \), integrating the inequality (3.24) on \([t_1, +\infty)\), we have

\[
\int_{t_1}^{+\infty} t^r \|Ax(t) + By(t) - b\|^2 dt < +\infty,
\]

\[
\int_{t_1}^{+\infty} t (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) dt < +\infty,
\]

\[
\int_{t_1}^{+\infty} t^r (\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty.
\]

Since \( \alpha > (4r_0 + r + 1)t_1^{-1} \), from (3.4) we have

\[
\eta(t) \geq r_0 \alpha, \quad \forall t \geq t_1.
\]

This yields (c)-(e). As shown in the proof of (a) of Theorem 3.1, we have

\[
\|x(t) - x^*\| \leq \sqrt{\frac{2}{\eta(t)}} \sqrt{E^\rho_{\theta, \eta}(t_1)} \leq \sqrt{\frac{2}{r_0 \alpha}} \sqrt{E^\rho_{\theta, \eta}(t_1)}, \quad \forall t \geq t_1,
\]
and 
\[ t^{\frac{2r-1}{2}}\|\dot{x}(t)\| \leq \sqrt{2E^0_{\theta,\eta}(t_1) + \theta(t_1)}\|x(t) - x^*\|, \quad \forall t \geq t_1. \]

This implies 
\[ \|\dot{x}(t)\| = \mathcal{O}(t^{-\frac{1}{2r-1}}). \]

By similar arguments, we have 
\[ \|\dot{y}(t)\| = \mathcal{O}(t^{-\frac{1}{2r-1}}) \quad \text{and} \quad \|\dot{\lambda}(t)\| = \mathcal{O}(t^{-\frac{1}{2r-1}}). \]

So we have (f): 
\[ \|\dot{x}(t)\| + \|\dot{y}(t)\| + \|\dot{\lambda}(t)\| = \mathcal{O}(t^{-\frac{1}{2r-1}}). \]

Now we investigate the asymptotic behavior of the dynamic (1.4) with \( \gamma(t) = \frac{\omega}{r} \) and \( r \in (0, 1) \).

**Theorem 3.3.** Suppose that \( \gamma(t) = \frac{\omega}{r} \) with \( r \in (0, 1) \) and \( \alpha > 0 \), and \( \sigma(t) = \frac{\omega}{2r} \) with \( r_0 > r \). Suppose that \( (x(t), y(t), \lambda(t)) \) is a global solution of the dynamic (1.4) and \( (x^*, y^*, \lambda^*) \in \Omega \). Then, the following conclusions hold:

(a) \( \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(t^{-2r}) \).

(b) \( \|Ax(t) + By(t) - b\| = \mathcal{O}(t^{-r}) \).

(c) \( \int_{t_1}^{t_0} t^{2r-1}\|Ax(t) + By(t) - b\|^2 dt < +\infty \).

(d) \( \int_{t_1}^{t_0} t^r (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) dt < +\infty \).

(e) \( \int_{t_1}^{t_0} t^{2r-1}(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty \).

(f) \( \|\dot{x}(t)\| + \|\dot{y}(t)\| + \|\dot{\lambda}(t)\| = \mathcal{O}(t^{-r}) \).

**Proof.** Take \( \theta(t) \) and \( \eta(t) \) as in (3.13). Consider the energy function \( E^0_{\theta,\eta} \) defined by (3.1) with \( \rho = r \). Since \( \alpha > 0 \) and \( r \in (0, 1) \), there exists \( t_1 \geq t_0 \) such that \( \alpha > (4r_0 + 2r)t_1^{-1} \). Then, it follows from (3.20) that 
\[ \dot{E}^0_{\theta,\eta}(t) + r_0 t^{2r-1}\|Ax(t) + By(t) - b\|^2 + \frac{\alpha}{2} t^r (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) \]

\[ + 2(r_0 - r) t^{2r-1}(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) \leq 0 \]

for any \( t \geq t_1 \). This implies 
\[ E^0_{\theta,\eta}(t) \leq E^0_{\theta,\eta}(t_1), \quad \forall t \geq t_1. \]

By same arguments as in the proof of Theorem 3.2, we can prove (a) and (b). Since \( r_0 > r > 0 \), and \( \alpha > 0 \), integrating the inequality (3.26) on \([t_1, +\infty)\), we have 
\[ \int_{t_1}^{+\infty} t^{2r-1}\|Ax(t) + By(t) - b\|^2 dt < +\infty, \]
\[ \int_{t_1}^{+\infty} t^r (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) dt < +\infty, \]
\[ \int_{t_1}^{+\infty} t^{2r-1}(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty, \]

which implies (c)-(e). Since \( \alpha > (4r_0 + 2r)t_1^{-1} \), from (3.14) we have 
\[ \eta(t) \geq r_0 \alpha t^{-1}, \quad \forall t \geq t_1. \]
Then, as shown in the proof of (b) of Theorem 3.1, we have
\[ t \to \frac{2}{r_0 \alpha} \sqrt{E^o_{\theta, \eta}(t_1)} \leq \forall t \geq t_1, \]
and
\[ t' \| \dot{x}(t) \| \leq \sqrt{2E^o_{\theta, \eta}(t_1) + 2r_0 t'^{-1} \| x(t) - x^* \|}, \quad t \geq t_1. \]
This yields
\[ t' \| \dot{x}(t) \| \leq \frac{2}{r_0 \alpha} \sqrt{E^o_{\theta, \eta}(t_1)} \leq \sqrt{2E^o_{\theta, \eta}(t_1) + \frac{2}{r_0 \alpha} \sqrt{E^o_{\theta, \eta}(t_1)}} \]
since \( t \to 1 \leq 1 \) for all \( t \geq t_1 \) when \( r \in (0, 1) \). This yields
\[ \| \dot{x}(t) \| = O(t^{-r}). \]
By similar arguments, we have
\[ \| \ddot{y}(t) \| = O(t^{-r}) \quad \text{and} \quad \| \dot{\lambda}(t) \| = O(t^{-r}). \]
This proves (f).

Remark 3.4. In the case \( \gamma(t) = \frac{\alpha}{t} \) with \( \alpha > 0 \), asymptotic behaviors of \((IGS)_\gamma\) and \((IGS)_{\gamma, \epsilon}\) have been discussed in [2] and [9] respectively.

4. The perturbed case. In this section, we analyze the asymptotic behavior of the following inertial primal-dual dynamical system with external perturbations:
\[
(4.1) \quad \begin{cases}
\dot{x}(t) + \gamma(t) \dot{x}(t) = -\nabla f(x(t)) - A^T (\lambda(t) + \sigma(t) \dot{\lambda}(t)) - A^T (Ax(t) + By(t) - b) + \epsilon(t), \\
\dot{y}(t) + \gamma(t) \dot{y}(t) = -\nabla g(y(t)) - B^T (\lambda(t) + \sigma(t) \dot{\lambda}(t)) - B^T (Ax(t) + By(t) - b) + \epsilon(t), \\
\dot{\lambda}(t) + \gamma(t) \dot{\lambda}(t) = A(x(t) + \sigma(t) \dot{x}(t)) + B(y(t) + \sigma(t) \dot{y}(t)) - b.
\end{cases}
\]

When \( \epsilon(t) \) decays rapidly enough to zeros as \( t \to +\infty \), we will show that asymptotical properties established in the pervious sections are preserved.

**Theorem 4.1.** Let \( \gamma : [t_0, +\infty) \to (0, +\infty) \) be a nonincreasing and twice continuously differentiable function satisfying (2.14) and (2.15), \( \delta(t) = \frac{1}{\beta_0} \) with \( \beta_0 \in [2\beta, 1 - \beta] \) and let \( \epsilon : [t_0, +\infty) \to \mathbb{R} \) be a locally integrable function such that
\[
\int_{t_0}^{+\infty} p(t) \beta \| \epsilon(t) \| dt < +\infty,
\]
where \( p(t) = e^{\int_{t_0}^{t} \gamma(s) ds} \) is defined in (2.1). Suppose that \((x(t), y(t), \lambda(t))\) is a global solution of the dynamic (4.1) and \((x^*, y^*, \lambda^*) \in \Omega\). Then, the following conclusions hold:
(a) \( \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = O(p(t)^{-2\beta}). \)
(b) \( \| Ax(t) + By(t) - b \| = O(p(t)^{-\beta}). \)
(c) \( \int_{t_0}^{+\infty} p(t)^{2\beta} \gamma(t) \| Ax(t) + By(t) - b \|^2 dt < +\infty. \)
Moreover, we have the following results:
\textbf{Case I :} \( \beta < \frac{1}{\alpha} \) and \( \beta_0 \in (2\beta, 1 - \beta) \). Then
(d) \( \int_{t_0}^{+\infty} p(t)^{2\beta} \gamma(t)(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*)) dt < +\infty. \)
Case II : $\beta = \frac{1}{2}$ and $\beta_0 = \frac{3}{2}$. Then for any $\tau \in (0, \frac{1}{\tau})$ we have

$$\int_{t_0}^{t_\infty} p(t)^{2\beta} \gamma(t)(\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) dt < +\infty.$$  

(f) $\|\dot{x}(t)\| + \|\dot{y}(t)\| + \|\dot{\lambda}(t)\| = O(p(t)^{-\beta}).$

Proof. Define the function $E_{\theta, \eta}^{\beta, \epsilon}(t) = E_{\theta, \eta}(t) - \int_{t_0}^{t} \langle \dot{\theta}(s)(x(s) - x^* + p(s)^\beta \dot{x}(s), p(s)^\beta \epsilon(s)) ds$, 

where $E_{\theta, \eta}(t)$ is defined by (2.3), $\theta(t)$ and $\eta(t)$ are taken as in (2.4), i.e.,

$$\theta(t) = \beta_0 p(t)^{2\beta}(\gamma(t) - \gamma(t)) \quad \text{and} \quad \eta(t) = -\beta_0 p(t)^{2\beta}(\beta_0 + 2\beta - 1)\gamma(t) \quad \gamma(t).$$

By similar arguments as in (2.11), we have

$$\dot{E}_{\theta, \eta}^{\beta, \epsilon}(t) + (\beta_0 - 2\beta)p(t)^{2\beta} \gamma(t)(\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*))$$

$$+ \frac{\beta_0}{2} p(t)^{2\beta} \gamma(t) \|A x(t) + B y(t) - b\|^2$$

$$+ (1 - \beta - \beta_0)p(t)^{2\beta} \gamma(t)(\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2) \leq 0$$

for $t \geq t_0$. Then

$$E_{\theta, \eta}^{\beta, \epsilon}(t) \leq E_{\theta, \eta}^{\beta, \epsilon}(t_0), \quad \forall t \geq t_0,$$

which gives, by definition of $E_{\theta, \eta}^{\beta, \epsilon}(\cdot)$

$$\frac{1}{2} \|\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t)\|^2 + \frac{1}{2} \|\theta(t)(y(t) - y^*) + p(t)^\beta \dot{y}(t)\|^2$$

$$\leq \int_{t_0}^{t} \langle \dot{\theta}(s)(x(s) - x^*) + p(s)^\beta \dot{x}(s), \theta(s)(y(s) - y^*) + p(s)^\beta \dot{y}(s), p(s)^\beta \epsilon(s) \rangle ds$$

$$+ E_{\theta, \eta}^{\beta, \epsilon}(t_0)$$

for any $t \geq t_0$. Applying triangle inequality and Cauchy-Schwarz inequality, we get

$$\frac{1}{2} \left( \|\theta(t)(x(t) - x^*) + p(t)^\beta \dot{x}(t)\| + \|\theta(t)(y(t) - y^*) + p(t)^\beta \dot{y}(t)\| \right)^2$$

$$\leq 2 \int_{t_0}^{t} \left( \|\theta(s)(x(s) - x^*) + p(s)^\beta \dot{x}(s)\| + \|\theta(s)(y(s) - y^*) + p(s)^\beta \dot{y}(s)\| \right) ||p(s)^\beta \epsilon(s)|| ds$$

$$+ 2 |E_{\theta, \eta}^{\beta, \epsilon}(t_0)|.$$
for any \( t \geq t_0 \). This together with (4.2) implies
\[
\sup_{t \geq t_0} (|\theta(t)(x(t) - x^*) + p(t)\beta \dot{x}(t)| + |\theta(t)(y(t) - y^*) + p(t)\beta \dot{y}(t)|) < +\infty.
\]

From (4.3), we have
\[
\mathcal{E}_{\theta, \eta}^{\beta, \epsilon}(t) + \int_{t_0}^{t} (\theta(s)(x(s) - x^*) + p(s)\beta \dot{x}(s), p(s)\beta \epsilon(s))ds
\]
\[
+ \int_{t_0}^{t} (\theta(s)(y(s) - y^*) + p(s)\beta \dot{y}(s), p(s)\beta \epsilon(s))ds = \mathcal{E}_{\theta, \eta}^{\beta}(t) \geq 0, \forall t \geq t_0.
\]
This implies
\[
\inf_{t \geq t_0} \mathcal{E}_{\theta, \eta}^{\beta, \epsilon}(t) \geq -\sup_{t \geq t_0} (|\theta(t)(x(t) - x^*) + p(t)\beta \dot{x}(t)| + |\theta(t)(y(t) - y^*) + p(t)\beta \dot{y}(t)|)
\]
\[
\times \int_{t_0}^{+\infty} p(s)\beta \|\epsilon(s)\|ds > -\infty.
\]
This together with (4.5) implies that \( \mathcal{E}_{\theta, \eta}^{\beta, \epsilon}(t) \) is bounded on \([t_0, +\infty)\). The rest of the proof is similar as the one of Theorem 2.5, and so we omit it.

Remark 4.2. The condition (4.2) assumed in Theorem 4.1 is mild and it has been used in [3] for asymptotic analysis of IGS.\( \gamma, \epsilon \). Especially, in the case \( \gamma(t) = \frac{\delta}{t} \) with \( \alpha > 0 \), the condition (4.2) becomes
\[
\int_{t_0}^{+\infty} t^p \|\epsilon(t)\|dt < +\infty \quad \text{with} \quad p = \min\{1, \frac{\alpha}{\delta}\},
\]
which has been used in [3] and [7].

Remark 4.3. Suppose that \( \nabla f \) and \( \nabla g \) are locally Lipschitz continuous, \( \gamma(t), \delta(t), \) and \( \epsilon(t) \) are locally integrable, By Proposition 2.1, there exists a unique local solution \((x(t), y(t), \lambda(t))\) of the dynamic (4.1) defined on a maximal interval \([t_0, T] \) with \( T \leq +\infty \). Additionally, suppose that \( \gamma(t), \delta(t), \epsilon(t) \) satisfy the assumptions of Theorem 4.1. By similar argument as in the proof of Theorem 2.2, we can prove \( T = +\infty \). So the existence and uniqueness of a global solution of the dynamic (4.1) is established.

Similarly, we can extend the convergence rate results established in Section 3 to the dynamic (4.1) with \( \gamma(t) = \frac{\delta}{t} \).

**Theorem 4.4.** Let \( t_0 \geq 1 \), \( \gamma(t) = \frac{\delta}{t^r} \) with \( r \in (-1, 0] \) and \( \alpha > 0 \), \( \sigma(t) = \frac{\alpha}{2r_0} \) with \( r_0 > \frac{r+1}{2} \). Let \( \epsilon : [t_0, +\infty) \to \mathbb{R} \) be a locally integrable function satisfying
\[
\int_{t_0}^{+\infty} t^{\frac{r+1}{2}} \|\epsilon(t)\|dt < +\infty.
\]
Suppose that \((x^*, y^*, \lambda^*) \in \Omega \) and that \((x(t), y(t), \lambda(t))\) is a global solution of the dynamic (4.1). Then, the following conclusions hold:
(a) \( \mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = O(t^{-(r+1)}) \).
(b) \( \|Ax(t) + By(t) - b\| = O(t^{-\frac{r}{2}}) \).
(c) \( \int_{t_0}^{+\infty} t^r \|Ax(t) + By(t) - b\|^2dt < +\infty \).
(d) \( \int_{t_0}^{+\infty} t^{r+1} (\|\dot{x}(t)\|^2 + \|\dot{y}(t)\|^2 + \|\dot{\lambda}(t)\|^2)dt < +\infty \).
(e) \( \int_{t_0}^{+\infty} t^{r+1} (\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*))dt < +\infty \).
(f) \( \|\dot{x}(t)\| + \|\dot{y}(t)\| + \|\dot{\lambda}(t)\| = O(t^{-\frac{r}{2}}) \).
Theorem 4.5. Let $t_0 \geq 1$, $\gamma(t) = \frac{r}{t^r}$ with $r \in [0,1)$ and $\alpha > 0$, $\sigma(t) = \frac{1}{2e^a}$ with $r_0 > r$. Let $\epsilon : [t_0, +\infty) \to \mathbb{R}$ be a locally integrable function satisfying

$$\int_{t_0}^{+\infty} t^r |\epsilon(t)||dt < +\infty.$$  

Suppose that $(x^*, y^*, \lambda^*) \in \Omega$ and that $(x(t), y(t), \lambda(t))$ is a global solution of the dynamic (4.1). Then, the following conclusions hold:

(a) $\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*) = \mathcal{O}(t^{-2r})$.
(b) $\|Ax(t) + By(t) - b\| = \mathcal{O}(t^{-r})$.
(c) $\int_{t_0}^{+\infty} t^{2r-1} |\dot{x}(t)||dt < +\infty$.
(d) $\int_{t_0}^{+\infty} t^{2r-1} (|\dot{x}(t)|^2 + |\dot{y}(t)|^2 + |\dot{\lambda}(t)|^2)|dt < +\infty$.
(e) $\int_{t_0}^{+\infty} t^{2r-1} (\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda^*))|dt < +\infty$.
(f) $\|\dot{x}(t)|| + |\dot{y}(t)|| + |\dot{\lambda}(t)|| = \mathcal{O}(t^{-r})$.

Remark 4.6. When $\gamma(t) = \frac{\alpha}{t^r}$ with $r \in (0,1)$ and $\alpha > 0$, the assumptions

$$\int_{t_0}^{+\infty} t^r |\epsilon(t)||dt < +\infty \quad \text{and} \quad \int_{t_0}^{+\infty} t^{2r-1} |\epsilon(t)||dt < +\infty$$

have been used in [9] for convergence rate analysis of $IGS_{\gamma, \epsilon}$. For more results on asymptotic analysis of dynamical systems with perturbations associated with unstrained optimization problems, we refer the reader to [26, 25, 35].

5. Conclusion. In this paper, we have proposed an inertial primal-dual dynamical system for a separable convex optimization problem with linear equality constraints. By using the Lyapunov analysis approach, we investigate the convergence rates of the trajectories generated by the dynamical system under different choices of the damping functions. We have also shown that convergence rate results established are preserved when small perturbations are added to the inertial primal-dual dynamical system. The results obtained improves the results of Zeng et al. [39], where convergence rates of a second-order dynamical system based on the primal-dual framework for the problem (1.1) with $g(x) \equiv 0$ and $B = 0$ were established. Our main results can be also viewed as analogs of the ones in [3], where the convergence rate analysis of $IGS_{\gamma, \epsilon}$ associated with the unconstrained optimization problem (1.5) were derived.

Appendix A. Some auxiliary results. The following lemmas have been used in the analysis of the convergence properties of the dynamical systems.

Lemma A.1. [3, Theorem 2.1] Let $t_0 \geq 0$, $\gamma : [t_0, +\infty) \to (0, +\infty)$ be a nonincreasing and twice continuously differentiable function satisfying $\gamma(t) \geq 2\beta^2 \gamma(t)^3$ for some $\beta > 0$. Then $\dot{\gamma}(t) \leq -\beta \gamma(t)^2$.

Lemma A.2. [15, Lemma A.5] Let $\omega : [t_0, T] \to [0, +\infty)$ be integrable, and $C \geq 0$. Suppose $\mu : [t_0, T] \to R$ is continuous and

$$\frac{1}{2} \mu(t)^2 \leq \frac{1}{2} C^2 + \int_{t_0}^{t} \omega(s) \mu(s)ds$$

for all $t \in [t_0, T]$. Then $|\mu(t)| \leq C + \int_{t_0}^{t} \omega(s)ds$ for all $t \in [t_0, T]$.

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