The chain-level intersection pairing for PL pseudomanifolds revisited

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Abstract

We revisit the construction of the PL intersection pairing for chains on PL manifolds and for intersection chains on PL stratified pseudomanifolds. We first provide a minor correction to the Goresky-MacPherson proof of a version of Poincaré duality on pseudomanifolds that is necessary for the construction of the intersection pairing on pseudomanifolds, and we provide a construction of the intersection product for locally finite intersection chains on non-compact pseudomanifolds that reduces to the pre-existing (corrected) construction on compact pseudomanifolds. Such a version of the intersection pairing is necessary for “sheafifying” the intersection product in intersection homology.

As an application of the techniques developed, we provide a direct proof that the Goresky-MacPherson homology intersection product is Poincaré dual to the cup product pairing on compact oriented PL manifolds, but we leave as an open question, with discussion of the difficulties involved, whether an analogous proof can be formulated to demonstrate such a duality between intersection and cup product pairings for intersection homology of pseudomanifolds.

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Contents

1 Introduction

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2 PL stratified pseudomanifolds

3 Dold duality for manifolds and Goresky-MacPherson duality for pseudomanifolds
   3.1 Dold’s duality .................................................. 7
   3.2 Goresky-MacPherson duality .................................. 11

4 PL chains
   4.1 PL chains .......................................................... 15
   4.2 Useful lemmas ...................................................... 16

5 Chain-level intersection pairings on pseudomanifolds
   5.1 Local umkehr maps and intersection coefficients .................. 18
   5.2 The umkehr map ..................................................... 26
   5.3 The cross product .................................................. 30
   5.4 The chain-level intersection pairing ............................. 32

6 The intersection pairing for intersection homology
   6.1 PL intersection chains ............................................ 32
   6.2 The intersection pairing .......................................... 33

7 Duality of intersection products and cup products .......................... 36

1 Introduction

The notion of taking the intersection product of two simplicial chains that are in general position\(^1\) in a PL (piecewise linear) manifold goes back at least to Lefschetz in \[16\]. As observed in the historical overview presented by McClure in Section 2 of \[18\], Lefschetz’s definition proceeds in the naive way by defining the intersection of \(\xi = \sum a_i \sigma_i\) and \(\eta = \sum b_j \tau_j\) as the bilinear extension of the intersections \(\sigma_i \cap \tau_j\), each of which is a polyhedron that is then subdivided again into a simplicial chain. This process requires the assumption that each such pair \((\sigma_i, \tau_j)\) is in general position\(^2\), and, consequently, the hypotheses do not survive subdivision. In \[17\], Lefschetz provided a second definition of intersection that corrected this defect, but at the expense of making the definition much more complicated. At this point, as McClure notes, “the chain-level intersection pairing became temporarily obsolete when the cup product was discovered,” as the homology intersection pairing on manifolds is Poincaré dual to the cup product.

Interest in intersection products was renewed, however, when Goresky and MacPherson introduced in \[11\] a new version for use on PL stratified pseudomanifolds in order to provide

\(^1\)Recall that two respectively \(i\)- and \(j\)-dimensional PL subspace \(A\) and \(B\) of an \(n\)-dimensional PL manifold \(M\) are in general position if \(\dim(A \cap B) \leq i + j - n\).

\(^2\)In particular, \(\xi\) and \(\eta\) are not necessarily assumed to be taken from a single triangulation of the manifold.
an intersection product for intersection homology. The Goresky-MacPherson product utilizes PL chains rather than simplicial chains, the PL chains living in the direct limit of the usual simplicial chain complexes under subdivision, and they are assumed to be in stratified general position, meaning that general position holds within each manifold stratum of the pseudomanifold. The intersection pairing itself is then defined using the cup product. More specifically, we first identify the chains with certain homology classes by a procedure reviewed below in Section 4. Then we dualizes to cohomology using a pseudomanifold generalization of a version of Poincaré duality that seems to be due to Dold (see [4, Proposition VIII.7.2] and [11, Appendix]). Next, we takes the cup product and finally dualize back to homology and again reinterpret the resulting homology class as a chain. According to McClure, “[this] version of the chain-level intersection pairing is probably equivalent to Lefschetz’s second definition.”

In [18], motivated by applications to the study of Chas-Sullivan operations on the homology of free loop spaces of PL manifolds, McClure extended the intersection pairing on PL manifolds in order to obtain not just a pairing of two chains in general position but a full chain map $\mu : G_\ast \to C_\ast(M)$, where $C_\ast(M)$ is the complex of PL chains on a PL manifold $M$ and the domain $G_\ast \subset C_\ast(M) \otimes C_\ast(M)$ is a subcomplex comprising chains satisfying a suitable notion of general position and such that the inclusion $G_\ast \hookrightarrow C_\ast(M) \otimes C_\ast(M)$ is a quasi-isomorphism, i.e. it induces an isomorphism on homology. The map $\mu$ here is McClure’s intersection product, which is defined via a process kindred to the Goresky-MacPherson product, though now employing a transfer (or umkehr) map. More specifically, $\mu$ is a composition of the chain cross product followed by an umkehr map composed of Dold’s duality isomorphism (again interpreting chains as certain homology classes), pullback by the diagonal, and then duality back to homology. An explicit proof that this process is consistent with the Goresky-MacPherson product on manifolds can be found in [7, Section 4.3]. McClure goes on to show that this pairing gives $C_\ast(M)$ the structure of a partially defined commutative DGA. In [7], the present author extended McClure’s work to the setting of intersection products of intersection chains on PL stratified pseudomanifolds, developing for intersection products of intersection chains a partial restricted algebra structure.

The purpose of this paper is to revisit two aspects of this development. Firstly, it turns out that there is a minor error in the Goresky-MacPherson construction of the intersection product in [11]; more precisely, the issue occurs in [11, Appendix] in the construction of a generalization of Dold’s Poincaré duality to PL pseudomanifolds, which is then used in the definition of the intersection product. We point out the problem and then provide a corrected version of the construction, which provides a duality isomorphism $H_i(K, L) \to$.

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3As reviewed in Section 6.1, intersection homology is defined using a complex of what are called intersection chains, but these are not necessarily the same things as chains that have been “intersected” via the intersection product. Rather, intersection chains are defined based upon how they intersect the singularities of a pseudomanifold. Unfortunately, this can sometimes cause some awkward terminology, such as “the intersection product of intersection chains,” the word “intersection” here playing two different roles.

4Motivation for this definition of an intersection product comes from the set-theoretic observation that the intersection of two subsets $A, B \subset X$ is equal to the intersection of $A \times B \subset X \times X$ with the diagonal $\Delta(X) = \{(x, x) \in X \times X\}$. 

3

4
\( H^{n-i}(X - L, X - K) \) with \( X \) a compact oriented \( n \)-dimensional PL stratified pseudomanifold and \( L \subset K \) compact PL subsets of \( X \) such that \( L \) contains the singular set \( \Sigma \) of \( X \). Our duality construction is slightly less general than that in [11], where \( K \) and \( L \) need only be constructible subspaces, though this restriction is sufficient for all the applications here, in [7], and in [11], itself. Our construction of this duality isomorphism and exploration of some of its needed properties can be found in Section 3 following a detailed treatment of some properties of Dold’s duality isomorphism. Beyond fixing the existing error, we hope that such a detailed treatment will be a useful expository addition to the literature.

Secondly, we consider in detail the intersection pairing of locally finite (Borel-Moore) intersection chains on non-compact PL stratified pseudomanifolds and thus, as a special case, the intersection pairing of ordinary locally finite chains on non-compact PL manifolds. The construction of such a pairing is sketched in [7, Remark 4.7], referring to Spanier’s paper [23] for the manifold versions of the needed duality isomorphisms. However, the necessary modification of the duality results for pseudomanifolds is not discussed in detail. Additionally, invocation of the duality results in Spanier is somewhat unsatisfactory, as the duality results proven there do not utilize the cap product with the fundamental class but rather a dual approach using Thom classes. This makes it difficult to compare Spanier’s construction directly with the work of [11, 18, 7] on compact spaces.

To remedy this lacuna, we here develop from scratch an intersection pairing on non-compact PL stratified pseudomanifolds. Furthermore, it will be clear from the construction that it is compatible with the pre-existing compact space constructions, and we will also demonstrate that it commutes with restrictions to open subsets, which is also a stated property in [7, Section 6] that is never formally verified. This restriction property is crucial in constructing a sheafification of the intersection product, which plays a critical role in comparing the various chain- and sheaf-theoretic intersection homology and intersection cohomology duality pairings. Such sheafifications have already been used in [7] to provide a concrete construction of the sheaf-theoretic intersection pairing map in the derived category and will be applied in [9] to verify that PL intersection products in intersection homology are compatible with other products in intersection (co)homology, including the Goresky-MacPherson sheaf pairing of [12], the cup product of [10], and the wedge product of intersection forms of Brasselet-Hector-Saralegi [3] and Saralegi [21, 22]. While such compatibilities are often stated, and so likely do not come as a surprise, there is a dearth of published detailed proofs even for smooth manifolds, validating the need for a detailed development, of which this paper is a key component.

To construct the intersection product for locally finite chains, rather than proceed using a global version of Poincaré (or Dold/Goresky-MacPherson) duality suited directly to such chains, we instead show that it is possible to proceed by “patching together” local intersection information obtained on compact subsets. This method may suffer a bit in elegance but gains in making it clear the extent to which intersection of chains is completely governed by local information. This makes the compatibility with restrictions to open subsets conceptually obvious, though there are still details to check to provide a complete proof.

Our development of the intersection pairing begins in Section 4 with some review and technical preliminaries. Then, in Section 5, we construct the locally finite chain intersection
pairing as a chain map on relative chains $\mathcal{G}_\infty^\ast(X, \Sigma) \to C_\infty^\ast(X, \Sigma)$, where $\Sigma$ is the singular locus of $X$ and $\mathcal{G}_\infty^\ast(X, \Sigma) \subset C_\infty^\ast(X, \Sigma) \otimes C_\infty^\ast(X, \Sigma)$ consists of chains satisfying the needed stratified general position requirements. Of course if $X = M$ is a manifold (unstratified), then $\Sigma$ is empty and we obtain a pairing $\mathcal{G}_\infty^\ast(M) \to C_\infty^\ast(M)$. The intersection chain version of the pairing is derived as a consequence in Section 6.

In a final section, Section 7 we show that the work of this paper can be applied to give a direct proof that the Goresky-MacPherson intersection pairing of [11] is Poincaré dual to the cup product pairing on PL manifolds. As noted, such a result is certainly not unexpected, but the only thorough proof of which the author is aware is that developed in [9] using some heavy sheaf-theoretic machinery. It is therefore quite reasonable to seek a proof using only classical techniques, and that is provided here. Quite surprisingly, however, the proof given for manifolds does not seem to extend to a proof of the analogous duality between cup and intersection products for intersection (co)homology on pseudomanifolds, and so it seems that the sheaf-theoretic techniques are, for now, necessary. In Section 7 we explain where the argument for manifolds runs into trouble in the pseudomanifold setting and leave open the interesting question of how to demonstrate rigorously the duality of the pairings using non-sheaf techniques.

Sections 2, 4, 5.3, and 6.1 of the paper contain review and foundational development of, respectively, PL stratified pseudomanifolds, PL chains, the PL cross product, and PL intersection chains.

A note on conventions. In [7], following the conventions in [18], the chain complexes involved in the definition of the intersection products incorporate some degree shifts in order for the intersection product itself to be a chain map of degree 0. Additionally, intersection products with more than two input tensor factors were considered, with the goal of studying partial algebra structures. In the present paper, since we will not revisit these algebra structures, we simplify somewhat by restricting to the more traditional two input factors and without shifts, so that our product will be a chain map of degree $-\dim(X)$. The reader should have little difficulty placing the development here back into the context of [7] if desired. We also generalize somewhat from the intersection chains of [7]; there, all perversities (see Section 6.1) were assumed to be traditional perversities, i.e. those perversities satisfying the original definition of Goresky and MacPherson in [11]. Here, we will take advantage of the more general notion of perversity, which often results in some simplifications. In particular, a perversity will be any function from the set of singular strata to $\mathbb{Z}$; additionally, our PL pseudomanifolds will be allowed codimension one strata. Our intersection chains will be those introduced initially for “super-perversities” in [6], though see [8] and, in particular, [5] for the definitive treatment for completely general perversities. In this context, it makes sense to think of the PL intersection chain complexes as subcomplexes not of $C_\ast(X)$ but of $C_\ast(X, \Sigma)$, explaining the aforementioned setting of the intersection pairing in this context.
2 PL stratified pseudomanifolds

All work in this paper is done in the piecewise linear (PL) category of topological spaces; we refer the reader to [20] and [15] for any necessary background concerning PL topology. In this section, we provide a quick review of the essential definitions concerning pseudomanifolds. A review of intersection chains and intersection homology can be found in Section 6.1.

For a compact PL space $K$, we let $c(K)$ denote its open cone; if $vK$ is the usual PL cone [20, page 2] with vertex $v$, then $c(K) = (vK) - K$. Note that $c(\emptyset) = \{v\}$. If $K$ is a filtered PL space, meaning that $K$ is endowed with a family of closed PL subspaces $K = K_n \supseteq K_{n-1} \supseteq \cdots \supseteq K_0 \supseteq K_{-1} = \emptyset$, we define $c(K)$ to be the filtered space with $(c(K))^i = c(K^{i-1})$ for $i \geq 0$ and $(c(K))^{-1} = \emptyset$.

The definition of stratified pseudomanifold is inductive on the dimension:

**Definition 2.1.** A 0-dimensional PL stratified pseudomanifold $X$ is a discrete set of points with the trivial filtration $X = X^0 \supseteq X_{-1} = \emptyset$.

An $n$-dimensional PL stratified pseudomanifold $X$ is a PL space filtered by closed PL subsets

$$X = X^n \supseteq X^{n-1} \supseteq X^{n-2} \supseteq \cdots \supseteq X^0 \supseteq X_{-1} = \emptyset$$

such that

1. $X - X^{n-1}$ is dense in $X$, and
2. for each point $x \in X^i - X^{i-1}$, there exists a neighborhood $U$ of $x$ for which there is a compact $n - i - 1$ dimensional PL stratified pseudomanifold $L$ and a PL homeomorphism $\phi : \mathbb{R}^i \times cL \to U$

that takes $\mathbb{R}^i \times c(L^{j-1})$ onto $X^{i+j} \cap U$. A neighborhood $U$ with this property is called distinguished and $L$ is called a link of $x$.

The $X^i$ are called skeleta. As the number of skeleta is finite, there exist triangulations of $X$ with respect to which each $X^i$ is a subcomplex (apply [15, Theorem 3.6.C] iteratively and see also [15, Lemma 3.7]); we will refer to such triangulations as being compatible with the stratification and assume that all triangulations satisfy this requirement. In general, we say that a triangulation $T$ is compatible with a PL subspace $Y$ if some subcomplex of $T$ triangulates $Y$.

We write $X_i$ for $X^i - X^{i-1}$; this is a PL $i$-manifold that may be empty. We refer to the connected components of the various $X_i$ as strata. If $L$ and $L'$ are links of points in the

---

5 We abuse notation by referring to the “triangulation $T$” without referring separately to the triangulating simplicial complex and the homeomorphism taking the simplicial complex to the space being triangulated. However, this should not cause much confusion.

6 Note: some authors refer to $X_i$ as the stratum and what we call strata as “stratum components.”
same stratum then they are PL homeomorphic \cite{5}. If a stratum $Z$ is a subset of $X_n$ it is called a regular stratum; otherwise it is called a singular stratum. Note that codimension 1 strata are allowed. The union of the singular strata, which is identical to $X^{n-1}$, is often denoted $\Sigma$, or $\Sigma_X$.

3 Dold duality for manifolds and Goresky-MacPherson duality for pseudomanifolds

3.1 Dold’s duality

In this section we revisit Dold’s Poincaré duality isomorphism of \cite{4} Proposition VIII.7.2 in order to obtain some properties we will need and that are not set out explicitly in \cite{4}.

The following is the duality isomorphism, stated in a slightly modified version:

**Theorem 3.1 (Dold).** If $L \subset K$ are compact subsets of an oriented $n$-manifold $M$, then there is an isomorphism $\tilde{H}^i(K, L) \to H_{n-i}(M - L, M - K)$ induced by the cap product with a fundamental class $\Gamma_K \in H_n(M, M - K)$.

Here $\tilde{H}^i(K, L)$ is the Čech cohomology of the pair $(K, L)$, which is isomorphic to the direct limit $\varinjlim H^*(V, W)$ as $(V, W)$ ranges over open neighborhood pairs of $(K, L)$ and the maps of the direct system are induced by restrictions to subspaces. See \cite{4} Section VIII.6.

More specifically, the Dold duality isomorphism is constructed as follows: Let $(V, W)$ be a pair of open neighborhoods of $(K, L)$ and consider the composite

$$H^i(V, W) \to H^i(V - L, W - L) \xrightarrow{-j_{W,K}^*} H_{n-i}(V - L, V - K) \xrightarrow{\zeta} H_{n-i}(M - L, M - K). \quad (1)$$

Here, the unlabeled maps are induced by inclusions, the rightmost map being an excision isomorphism, $\Gamma_K \in H_n(M, M - K)$ is the fundamental class (see \cite{14} Lemma 3.27), and $j_{W,K}^*$ is the composite

$$H_*(M, M - K) \to H_*(M, (M - K) \cup W) \xrightarrow{\zeta} H_*(V - L, (V - K) \cup (W - L)), \quad (2)$$

the leftward map being an excision isomorphism. Dold’s isomorphism is the direct limit of the composition (1) over all open neighborhoods $(V, W)$ of $(K, L)$.

So that we can later utilize Dold’s isomorphism to obtain chain maps of the appropriate degree, we adopt the following definition:

**Definition 3.2.** Define the Dold duality isomorphism $\mathfrak{D} : \tilde{H}^i(K, L) \to H_{n-i}(M - L, M - K)$ to be $(-1)^{in}$ times the direct limit of the composition (1). The extra sign is necessary to be consistent with duality maps being chain maps of the appropriate degree at the chain level; see \cite{7} Section 1.3] for details.

We demonstrate the naturality of the Dold duality isomorphism with respect to inclusion maps in the variables $K$, $L$, and $M$. 

7
Proposition 3.3. Let \((K, L) \subset (K', L')\) be pairs of compact subsets of an oriented \(n\)-manifold \(M'\) contained in a larger oriented \(n\)-manifold \(M\), i.e. \(M' \subset M\). Then there is a commutative diagram

\[
\begin{array}{ccc}
\tilde{H}^i(K, L) & \overset{\sim}{\longrightarrow} & \tilde{H}^i(K', L') \\
\downarrow & & \downarrow \\
H_{n-i}(M - L, M - K) & \leftarrow & H_{n-i}(M' - L', M' - K'),
\end{array}
\]

in which the horizontal maps are induced by inclusions and the vertical maps are Dold duality isomorphisms.

Proof. Let \((V, W)\) be a pair of open neighborhoods of \((K, L)\), and, similarly, let \((V', W')\) be a pair of open neighborhoods of \((K', L')\). We further suppose that \((V, W) \supset (K, L)\). Now consider the following diagram:

\[
\begin{array}{ccc}
\tilde{H}^i(V, W) & \overset{\sim}{\longrightarrow} & \tilde{H}^i(V', W') \\
\downarrow & & \downarrow \\
H^i(V - L, W - L) & \leftarrow & H^i(V' - L', W' - L') \\
\sim j_*^{W, K} \Gamma_K & \sim j_*^{W', K} \Gamma_K & \sim j_*^{W', K'} \Gamma_K' \\
H_{n-i}(V - L, V - K) & \rightarrow & H_{n-i}(V' - L, V' - K) \\
H_{n-i}(M - L, M - K) & \overset{\sim}{\longrightarrow} & H_{n-i}(M' - L', M' - K').
\end{array}
\]

Here, the unlabeled maps are induced by inclusions, \(\Gamma_K \in H_n(M, M - K)\) and \(\Gamma_{K'} \in H_n(M', M' - K')\) are orientation classes (see [14, Lemma 3.27]), while \(j_*^{W, K}\) is the composite \(2\). The maps \(j_*^{W', K}\) and \(j_*^{W', K'}\) are defined analogously. The direct limit over all such \((V, W) \supset (K, L)\) of the maps down the left side of the diagram is, up to sign \((-1)^m\), the Dold isomorphism \(\tilde{H}^i(K, L) \rightarrow H_{n-i}(M - L, M - K)\), and similarly the direct limit over all such \((V', W') \supset (K', L')\) of the maps down the right side is the Dold isomorphism \(\tilde{H}^i(K', L') \rightarrow H_{n-i}(M' - L', M' - K')\) up to the same sign. That these limits of maps are well-defined is demonstrated in [4].

Let us observe that the diagram commutes. This is immediate for the squares not involving cap products. For the squares involving cap products, the commutativity is due to
the naturality of the cap product [4] §VII.12.6]. For the square on the left, we use naturality with respect to the inclusion map of triples $\langle V - L; V - K, W - L \rangle \rightarrow \langle V' - L; V' - K, W' - L \rangle$, and for the square on the right, this we use naturality with respect to the inclusion map of triples $\langle V' - L'; V' - K', W' - L' \rangle \rightarrow \langle V' - L; V' - K, W' - L \rangle$. To utilize the naturality, we observe that the images of $j^W K(*)$ and $j^W K'(*)$ under the respective maps induced by inclusion $H_n(V - L, (V - K) \cup (W - L)) \rightarrow H_n(V' - L, (V' - K) \cup (W' - L))$ and $H_n(V' - L', (V' - K') \cup (W' - L')) \rightarrow H_n(V' - L, (V' - K) \cup (W' - L))$ are each indeed $j^W K(*)$. This follows from an easy commutative diagram argument, noting that the image of $\Gamma K'$ in $H_n(M, M - K)$ is $\Gamma K$, by [14, Lemma 3.27].

An easy diagram chase now shows that the outer square of the diagram commutes, yielding the commutative square

$$
\begin{array}{c}
H^i(V, W) \\
\downarrow \\
H_{n-i}(M - L, M - K)
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
H_{n-i}(M' - L', M' - K'),
\end{array}
\begin{array}{c}
H^i(V', W') \\
\downarrow \\
\rightarrow \\
H^i(V, W)
\end{array}
$$

and taking the direct limit over $(V, W) \supset (K, L)$ yields a diagram

$$
\begin{array}{c}
\tilde{H}^i(K, L) \\
\downarrow \varnothing \\
\Downarrow \varnothing \\
H_{n-i}(M - L, M - K)
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
H_{n-i}(M' - L', M' - K'),
\end{array}
\begin{array}{c}
\tilde{H}^i(V', W') \\
\downarrow \\
\rightarrow \\
\tilde{H}^i(V, W)
\end{array}
$$

where the lefthand vertical map is Dold’s isomorphism. Lastly, taking the direct limit over $(V', W') \supset (K', L')$ gives the diagram

$$
\begin{array}{c}
\tilde{H}^i(K, L) \\
\downarrow \varnothing \\
\Downarrow \varnothing \\
H_{n-i}(M - L, M - K)
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\tilde{H}^i(K, L')
\end{array}
\begin{array}{c}
\tilde{H}^i(K', L') \\
\downarrow \varnothing \\
\Downarrow \varnothing \\
H_{n-i}(M' - L', M' - K')
\end{array}
$$

We also need to know how Dold’s isomorphism interacts with boundary maps:

**Proposition 3.4.** Let $L \subset K \subset J$ be compact subsets of an oriented $n$-manifold $M$. The
following diagram commutes up to the sign \((-1)^n\):

\[
\begin{array}{ccc}
\hat{H}^i(K, L) & \xrightarrow{d^*} & \hat{H}^{i+1}(J, K) \\
\mathcal{D} & & \mathcal{D} \\
H_{n-i}(M - L, M - K) & \xrightarrow{\partial_*} & H_{n-i-1}(M - K, M - J).
\end{array}
\]

\textit{Proof.} Consider the following diagram, in which \(\mathcal{D}\) is the Dold isomorphism but without our added sign (i.e. \(\mathcal{D}(\alpha) = (-1)^{|\alpha|n}\mathcal{D}(\alpha)\)):

\[
\begin{array}{ccc}
\hat{H}^i(K, L) & \xrightarrow{d^*} & \hat{H}^{i+1}(J, K) \\
\mathcal{D} & & \mathcal{D} \\
H_{n-i}(M - L, M - K) & \xrightarrow{\partial_*} & H_{n-i-1}(M - K, M - J).
\end{array}
\]

The unlabeled maps are induced by inclusions. Each of the squares commute by the arguments in Case 6 in the proof of [4, Proposition VIII.7.2]. The bottom horizontal composition is equal to the boundary map in the exact sequence of the triple \((M - L, M - K, M - J)\), as we see by the natural transformation to the sequence of the triple \((M, M - K, M - J)\):

\[
\begin{array}{ccc}
\hat{H}^i(K, L) & \xrightarrow{d^*} & \hat{H}^{i+1}(J, K) \\
\mathcal{D} & & \mathcal{D} \\
H_{n-i}(M - L, M - K) & \xrightarrow{\partial_*} & H_{n-i-1}(M - K, M - J).
\end{array}
\]

Similarly, the composition across the top is the cohomology coboundary map of the triple \((J, K, L)\) via

\[
\begin{array}{ccc}
\hat{H}^i(J, L) & \xrightarrow{d^*} & \hat{H}^{i+1}(J, K) \\
\mathcal{D} & & \mathcal{D} \\
\hat{H}^i(J) & \xrightarrow{d^*} & \hat{H}^{i+1}(J, K).
\end{array}
\]

So, if \(\alpha \in \hat{H}^i(K, L)\) then

\[
\mathcal{D}(d^*(\alpha)) = (-1)^{(i+1)n}\mathcal{D}(d^*(\alpha))
\]

\[
= (-1)^{(i+1)n}\partial_*(\mathcal{D}(\alpha))
\]

\[
= (-1)^n\partial_*(\mathcal{D}(\alpha)).
\]
3.2 Goresky-MacPherson duality

In the appendix to [11], Goresky and MacPherson provide a generalization to Dold’s duality isomorphism. However, there is an error in their argument. In this section, we provide a slight modification that does not claim the full generality of the Goresky-MacPherson duality but that is sufficient both for the applications here and in [11].

The setting in [11, Appendix] provides a compact oriented $n$-dimensional PL pseudomanifold $X$ with singular locus $\Sigma$ and with $A \subset B$ two constructible subsets, i.e. unions of interiors of simplices in some triangulation of $X$. It is also assumed that $B - A \subset X - \Sigma$, i.e. that $A \cap \Sigma = B \cap \Sigma$. They then construct a duality isomorphism $H^i(B, A) \to H_{n-i}(X - A, X - B)$, induced by the cap product with the fundamental class. However, in constructing their isomorphism\(^7\), the first claimed step of the isomorphism, which occurs in the proof of the lemma on page 162, is “$H^i(B, A) \cong H^i(B \cup \Sigma, A \cup \Sigma)$ by excision of $\Sigma - (A \cap \Sigma)$.” However, this is not always an isomorphism. For example, consider $B = X - \Sigma$ and $A = \emptyset$, in which case $(X - A, X - B) = (X, \Sigma)$; this case is of practical importance to the intersection theory when elements $H_n(X, \Sigma)$ arise as homological representatives of the fundamental class of $X$ in the Goresky-MacPherson version of the intersection pairing. Note that the requirement $B - A = X - \Sigma \subset X - \Sigma$ is satisfied here. But then the claimed excision isomorphism would be $H^i(X - \Sigma) \cong H^i(X, \Sigma)$, which is false, for example, when $i = 0$, $\Sigma$ is neither empty nor all of $X$, and $X$ is connected.

As Proposition 3.6 below, we give a construction of the desired isomorphism, utilizing essentially the same basic ideas of the Goresky-MacPherson construction but avoiding the above difficulty. We simplify matters somewhat by limiting ourselves to the case where, in the Goresky-MacPherson notation, $X - A$ and $X - B$ are subcomplexes of some triangulation of $X$, as this is the only situation required for the construction of intersection products both below and in [11].

Remark 3.5. There is another minor logical problem with the Goresky-MacPherson construction that is cured using the approach of McClure in [18], which is emulated in [7] and below: In Section 2.1 of [11], Goresky and MacPherson define a notion of dimensional transversality and define their intersection product on pairs of chains $C, D$ such that the following pairs are dimensionally transverse: $(C, D)$, $(C, \partial D)$, and $(\partial C, D)$. If we let $C \pitchfork D$ denote the Goresky-MacPherson intersection product\(^8\), it is then claimed that $\partial(C \pitchfork D) = (\partial C) \pitchfork D + (-1)^{n - |C|} C \pitchfork (\partial D)$. The trouble is that in order for, say, $(\partial C) \pitchfork D$ to be defined, one must then also know that the pair $(\partial C, \partial D)$ is dimensionally transverse, which is not a part of the initial assumption. This does not become a serious problem in [11], however, as the primary interest is in the intersection of cycles.

In the following proposition, we do not require that $X$ be a pseudomanifold, but merely that we have a compact PL pair $(X, S)$ such that $X - S$ is an $n$-dimensional oriented manifold.
Proposition 3.6. Let \((X, S)\) be a compact PL space pair such that \(X - S\) is an oriented 
\(n\)-dimensional (topological) manifold, and let \(L \subset K \subset X\) be compact PL subspaces, such 
that \(S \subset L\). Then there is an isomorphism \(D : H^i(X - L, X - K) \to H_{n-i}(K, L)\) composed 
of excisions, isomorphisms induced by inclusions, and Dold's duality isomorphism.

Proof. As \(S, L, \) and \(K\) are PL subspaces of \(X\), there is a triangulation \(T\) of \(X\) with respect 
to which these subspaces arise as subcomplexes [20, Addendum 2.12]. Furthermore, by 
replacing \(T\) with a subdivision, if necessary, we may assume that each is a full subcomplex 
of \(T\), using [20, Lemma 3.3].

Suppose \(Z\) is the underlying space of a full subcomplex of \(T\). Let \(C_Z\) be the the union 
of all simplices in \(T\) that are disjoint from \(Z\). Then, by [19, Lemma 70.1] and its proof, the 
subcomplex corresponding to \(C_Z\) is also a full subcomplex of \(T\), \(Z\) is a deformation retract 
of \(X - C_Z\), and \(C_Z\) is a deformation retract of \(X - Z\). Furthermore, if, as in [19, Section 72] 
(modifying that notation slightly), we let \(St(Z)\) be the union of the interiors of all simplices 
of \(T\) that have a face in \(Z\), then \(X - C_Z = St(Z)\).

Note that, as \(X\) is compact, each \(C_Z\) is also compact. Also, as \(S \subset L \subset K\), both sets \(C_L\) 
and \(C_K\) are subsets of the manifold \(X - S\).

The claimed isomorphism of the lemma is the composite of the following isomorphisms:

\[
H^i(X - L, X - K) \xrightarrow{\cong} H^i(C_L, C_K) \quad \text{homotopy equivalences}
\]

\[
\xrightarrow{D} H_{n-i}((X - S) - C_K, (X - S) - C_L)
\]

\[
= H_{n-i}((X - C_K) - S, (X - C_L) - S) \quad \text{set equality (3)}
\]

\[
= H_{n-i}(St(K) - S, St(L) - S) \quad \text{see above}
\]

\[
\xrightarrow{\cong} H_{n-i}(St(K), St(L)) \quad \text{by excision, see below}
\]

\[
\xleftarrow{\cong} H_{n-i}(K, L) \quad \text{homotopy equivalences.}
\]

For the excision in the fifth line, which excises \(S\), we note that \(S\) is a closed subcomplex 
contained in the interior of \(St(L)\) by construction.

The map labeled “Dold” is the duality isomorphism defined in the preceding section. 
Note that \(\tilde{H}^i(C_L \cup S, C_K \cup S) \cong H^i(C_L \cup S, C_K \cup S)\) by [4, Proposition CIII.6.12], as both 
spaces are ENRs as subcomplexes of compact simplicial complexes (in fact any finite CW 
complex is an ENR [14, Corollary A.10]).

We now prove some properties of the duality isomorphism \(D\):

Lemma 3.7. The duality isomorphism of Lemma 3.6 is natural in the sense that if \((K, L) \subset \) 
\((K', L')\) are pairs of compact PL subspaces satisfying the criteria of Lemma 3.6, then there 
is a commutative diagram

\[
\begin{array}{ccc}
H^i(X - L, X - K) & \xrightarrow{D} & H^i(X - L', X - K') \\
\downarrow & & \downarrow \\
H_{n-i}(K, L) & \xrightarrow{D} & H_{n-i}(K', L'),
\end{array}
\]

\]
in which the horizontal maps are induced by inclusions and the vertical maps are the isomorphisms of Lemma 3.6.

Proof. With the exception of the Dold duality isomorphism, it is straightforward to verify that all of the maps in the sequence of isomorphisms in Proposition 3.6 commute with appropriate inclusion maps. The commutativity for the Dold isomorphisms is provided by Lemma 3.3.

There are two other observations we will need concerning the duality isomorphism \( \mathbb{D} \). As stated, the lemma and its construction utilize the singular set \( S \) as part of the input data. It will be useful to know that the isomorphism is in fact independent of \( S \), so long as \( S \subset L \) and \( X - S \) is an oriented \( n \)-manifold.

**Lemma 3.8.** The duality isomorphism of Proposition 3.6 does not depend on the choice of compact set \( S \subset L \subset X \) such that \( X - S \) is an oriented \( n \)-manifold.

Proof. Let \( S' \) be an alternative such subspace. We may assume that \( S' \subset S \), for if not then the claim follows by comparing the isomorphism determined by each of \( S \) and \( S' \) to that determined by \( S \cap S' \). So, assuming that \( S' \subset S \), we have the following diagram, continuing to use the notation of the proof of Proposition 3.6

\[
\begin{align*}
H^i(C_L, C_K) & \xrightarrow{=} H^i(C_L, C_K) \\
\mathbb{D} & \quad \mathbb{D} \\
H_{n-i}((X - S) - C_K, (X - S) - C_L) & \longrightarrow H_{n-i}((X - S') - C_K, (X - S') - C_L) \\
\mathbb{D} & \quad \mathbb{D} \\
H_{n-i}((X - C_K) - S, (X - C_L) - S) & \longrightarrow H_{n-i}((X - C_K) - S', (X - C_L) - S') \\
\mathbb{D} & \quad \mathbb{D} \\
H_{n-i}(St(K) - S, St(L) - S) & \longrightarrow H_{n-i}(St(K) - S', St(L) - S') \\
\mathbb{D} & \quad \mathbb{D} \\
H_{n-i}(St(K), St(L)) & \longrightarrow H_{n-i}(St(K), St(L))
\end{align*}
\]

The top square commutes by Proposition 3.3 and the remaining squares clearly commute. Commutativity of the whole diagram, together with the definition of the map in Proposition 3.6 demonstrates the independence of choice of \( S \). \( \square \)
We will also need that the duality isomorphism of Proposition 3.6 is natural with respect to “expansion of the singularity.” In other words, suppose that we enlarge \( X \) to a complex \( X \cup T \) such that \( X \cap T \subset S \). Then, as \( S \subset L \), we have \( H_i(K, L) \cong H_i(K \cup T, L \cup T) \), while \((X - (L \cup T), X - (K \cup T)) = (X - L, X - K)\). We have the following compatibility:

**Lemma 3.9.** Let \((X, S)\) be a compact PL space pair such that \( X - S \) is an oriented \( n \)-dimensional manifold, and let \( L \subset K \subset X \) be compact PL subspaces such that \( S \subset L \). Let \( X \cup T \) be a compact PL space with \( X \cap T \subset S \). Then the following diagram commutes, with each vertical map being the isomorphism of Proposition 3.6:

\[
\begin{array}{ccc}
H^i(X - L, X - K) & \cong & H^i(X - (L \cup T), X - (K \cup T)) \\
\downarrow & & \downarrow \\
H_{n-i}(K, L) & \cong & H_{n-i}(K \cup T, L \cup T).
\end{array}
\]

**Proof.** This time we use the commutative diagram

\[
\begin{array}{cccc}
H^i(C_L, C_K) & \cong & H^i(C_L, C_K) & \\
\downarrow & & \downarrow \\
H_{n-i}((X - S) - C_K, (X - S) - C_L) & \cong & H_{n-i}(((X \cup T) - (S \cup T)) - C_K, ((X \cup T) - (S \cup T)) - C_L) & \\
\downarrow & & \downarrow \\
H_{n-i}((X - C_K) - S, (X - C_L) - S) & \cong & H_{n-i}(((X \cup T) - C_K) - (S \cup T), ((X \cup T) - C_L) - (S \cup T)) & \\
\downarrow & & \downarrow \\
H_{n-i}(St_X(K) - S, St_X(L) - S) & \cong & H_{n-i}(St_{X \cup T}(K \cup T) - (S \cup T), St_{X \cup T}(L \cup T) - (S \cup T)) & \\
\downarrow & & \downarrow \\
H_{n-i}(St_X(K), St_X(L)) & \cong & H_{n-i}(St_{X \cup T}(K \cup T), St_{X \cup T}(L \cup T)) & \\
\downarrow & & \downarrow \\
H_{n-i}(K, L) & \cong & H_{n-i}(K \cup T, L \cup T).
\end{array}
\]

Here the subscript on \( St \) indicates in which space the star is taken.

The next lemma shows how the duality isomorphism \( \mathcal{D} \) interacts with the connecting morphisms:
Lemma 3.10. Let \((X, S)\) be a compact PL space pair such that \(X - S\) is an oriented \(n\)-dimensional manifold, and let \(J \subset L \subset K \subset X\) be compact PL subspaces such that \(S \subset J\). Then the following diagram commutes up to \((-1)^n\):

\[
\begin{array}{ccc}
H^i(X - L, X - K) & \xrightarrow{d^*} & H^{i+1}(X - J, X - L) \\
\downarrow & & \downarrow \\
H_{n-i}(K, L) & \xrightarrow{\partial_*} & H_{n-i-1}(L, J).
\end{array}
\]

Proof. Consider the definition of the duality map of Proposition 3.6 and the evident ladder diagram connecting the maps of the definition for the pair \((K, L)\) with that for the pair \((L, J)\). It is clear from the standard naturality of the connecting maps in the long exact sequences that such a diagram commutes in all squares with the possible exception of the square involving the Dold duality map. But this square commutes up to \((-1)^n\) by Proposition 3.4.

\[
\blacksquare
\]

4 PL chains

In this section, we first review some fundamentals concerning PL chains and then review and develop some useful identifications between groups of PL chains and certain homology groups.

4.1 PL chains

Let \(X\) be a locally compact PL space. Any triangulation \(T\) of \(X\) is locally finite \([19]\, \text{Lemma 2.6}\). Let \(c^T_{i,\infty}(X)\) be the complex of locally finite \(i\)-chains on \(X\) with respect to the triangulation \(T\), i.e. the elements of \(c^T_{i,\infty}(X)\) are formal sums \(\sum \sigma a_{\sigma} \sigma\), where the sum is over all oriented \(i\)-simplices \(\sigma\) of \(X\) in \(T\) and each \(a_{\sigma} \in \mathbb{Z}\). These are called locally finite chains or chains with closed support or Borel-Moore chains. The local-finiteness ensures that the boundary map is well-defined in the usual way (see \([13, 12]\) or \([1\, \text{Section 4.1.3}]\)), and thus \(c^T_{i,\infty}(X)\) is a chain complex with homology groups \(H^T_{i,\infty}(X)\). Relative chain complexes and relative homology are defined in the obvious way. If \(T'\) is a subdivision of \(T\), there is a subdivision map \(c^T_{i,\infty}(X) \to c^{T'}_{i,\infty}(X)\) that (formally) takes each \(i\)-simplex to the sum of the \(i\)-simplices contained in it with compatible orientations. We let \(C^\infty(X)\) be the direct limit of the \(c^T_{i,\infty}(X)\) over all compatible triangulations of \(X\) with maps induced by the subdivision maps. The corresponding homology groups are \(H^\infty_{i}(X)\).

\[\text{More precisely, each simplex comes equipped with two orientations } (\sigma, o) \text{ and } (\sigma, -o), \text{ and we impose in } c^T_{i,\infty}(X) \text{ the relation } -(\sigma, o) = (\sigma, -o). \text{ For the sake of writing chains as sums, we follow standard practice and choose the notation } \sigma \text{ to denote a simplex with one of its orientations, arbitrarily chosen, and then allow the simplex to appear in the sum expression only with this orientation.}\]
Given a chain \( \xi \in C^\infty_i(X) \), its support \(|\xi|\) is defined as follows: write \( \xi \) as an element \( \sum a_\sigma \sigma \in c^T_i(X) \) for some triangulation \( T \) and let \(|\xi|\) be the union of the \( i \)-simplices \( \sigma \) such that \( a_\sigma \neq 0 \). This definition is independent of the choice of \( T \). If \( \xi \in C^\infty_i(X, Y) \), for \( Y \) a PL subspace of \( X \), then we can again write a representative for \( \xi \) as an element \( \sum a_\sigma \sigma \in c^T_i(X) \) for some triangulation \( T \) for which \( Y \) is a subcomplex, then we let \(|\xi|\) be the union of the \( i \)-simplices \( \sigma \) not contained in \( Y \) and such that \( a_\sigma \neq 0 \). In other words, \(|\xi|\) is the support of the minimal chain representing \( \xi \) in \( C^\infty_i(X, Y) \).

If we let \( c^T_i(X) \) be the usual simplicial chain complex in which each chain is a finite sum of simplices with coefficients, then \( c^T_i(X) \subset c^T_* \). Taking subdivisions we obtain the PL chain complex with compact supports, \( C_*^{\infty}(X) \subset C_*^{\infty}(X) \).

4.2 Useful lemmas

By [12, §2.1] and [13, §1], we have the following useful isomorphism. A detailed proof can be found in [18, Lemma 4] (compare also [11, §1.2]) for the corresponding result concerning compactly supported chains, though the proof in the case of locally finite chains is identical:

**Lemma 4.1.** Let \( X \) be a PL space and let \( B \subset A \) be closed PL subspaces of \( X \) such that \( \dim(A) = p \) and \( \dim(B) < p \). Then

1. there is an isomorphism \( \alpha_{A,B} \) from \( H^\infty_p(A, B) \) to the abelian group
   \[
   c^A_{p,B} = \{ \xi \in C^\infty_p(X) \mid |\xi| \subset A, |\partial \xi| \subset B \},
   \]

2. the following diagram commutes:

\[
\begin{array}{ccc}
H^\infty_p(A, B) & \xrightarrow{\alpha_{A,B}} & c^A_{p,B} \\
\partial & \downarrow & \partial \\
H^\infty_{p-1}(B) & \xrightarrow{\alpha_{B,\emptyset}} & c^{B,\emptyset}_{p-1}
\end{array}
\]

**Proof.** As observed in [18], \( H^\infty_p(A, B) \) is the \( p \)th homology of the complex \( C^\infty_* (A, B) \), which is the quotient of the relative cycles by the relative boundaries. But the set \( c^A_{p,B} \) is precisely the set of relative cycles, while the set of relative boundaries is 0 by the dimension hypothesis. The second part of the lemma follows by chasing the definitions.

In particular then, as noted in [13, §1], a chain \( \xi \in C^\infty_p(X) \) is completely described by \(|\xi|, |\partial \xi|\), and the class represented by \( \xi \) in \( H^\infty_p(|\xi|, |\partial \xi|) \).

**Remark 4.2.** Suppose \( \xi \in c^A_{p,B} \) and \( T \) is a triangulation of \( X \) such that \( A \) and \( B \) are subcomplexes and \( \xi \) can be represented as an element of \( c^T_i(X) \). Let \( \sigma \) be an (oriented) \( p \)-simplex of \( A \). Then the coefficient of \( \sigma \) in the representation of \( \xi \) in \( c^T_i(X) \) can be recovered as the image of \( \alpha^{-1}_{A,B}(\xi) \) in \( H^\infty_p(A - int(\sigma)) \cong H_p(\sigma, \partial \sigma) \cong \mathbb{Z} \), the first isomorphism by
excision. The argument in this context for excision is identical to the standard simplicial
proof, e.g. [19, Theorem 9.1]. The claim follows by letting the representation of \( \xi \) in \( c_i^{T,\infty}(X) \) also serve as the chain representative for \( \alpha_{A,B}^1(\xi) \) and then its image in \( H_p^\infty(A, A - \text{int}(\sigma)) \). So, in fact, given an element \( [\xi] \in H_p^\infty(A, B) \) and a triangulation \( T \) compatible with \( A \) and \( B \), we can in this way construct a representative of \( \alpha_{A,B}([\xi]) \) in \( c_p^{T,\infty}(A, B) \subset c_p^{T,\infty}(X) \) by so determining its coefficient at each \( p \)-simplex of \( A \).

In our application below, we will need to consider more general scenarios in which \( A \\) might have arbitrary dimensions but \( A - B \) still has dimension \( p \). For this we have the following lemma. We let \( cl(Y) \) denote the closure of the subspace \( Y \) in its ambient space.

**Lemma 4.3.** Let \( X \) be a PL space and let \( C \subset B \subset A \) be closed PL subspaces of \( X \) such that \( \text{dim}(A - B) = p \) and \( \text{dim}(B - C) < p \), and let \( T \) be a triangulation of \( X \) with respect to which \( A, B, C \) are subcomplexes. Then:

1. There is an isomorphism \( \bar{\alpha}_{A,B} \) from \( H_p^\infty(A, B) \) to the abelian group

   \[
   \bar{\xi}_p^{A,B} = \{ \xi \in C_p^\infty(X) \mid |\xi| \subset cl(A - B), |\partial \xi| \subset B \}.
   \]

   If \( [\xi] \in H_p^\infty(A, B) \) is represented by a simplicial chain \( \xi = \sum a_\sigma \sigma \) with \( p \)-simplices in \( T \), then \( \bar{\alpha}_{A,B}([\xi]) = \sum_{\sigma \subset B} a_\sigma \sigma, \) with the sum being over \( p \)-simplices of \( T \) not contained in \( B \) (though necessarily \( \sigma \subset A \)).

2. If \( [\xi] \in H_p^\infty(A, B) \) with \( \partial \alpha_{A,B}([\xi]) = \sum b_\tau \tau \) and \( \partial_* : H_p(A, B) \to H_{p-1}(B, C) \) is the connecting morphism, then \( \bar{\alpha}_{B,C}([\partial_*([\xi]]) = \sum_{\tau \subset C} b_\tau \tau \).

**Proof.** Let \( \bar{A}_B \) denote the closure of \( A - B \). In the triangulation \( T \), this is just the space of \( p \)-simplices of \( A \) that are not contained in \( B \). We have \( \text{dim}(\bar{A}_B) = p \) and also \( \text{dim}(\bar{A}_B \cap B) < p \), as \( \bar{A}_B \cap B \) consists of those simplices of \( B \) that are faces of the simplices of \( A \) that are not contained in \( B \). Thus, via Lemma 4.1, we have an isomorphism between \( H_p^\infty(\bar{A}_B, \bar{A}_B \cap B) \) and \( \bar{c}_p^{A_B, \bar{A}_B \cap B} \). There is also an excision isomorphism \( H_p^\infty(\bar{A}_B, A_B \cap B) \to H_p^\infty(\bar{A}_B) \). Furthermore, \( \bar{c}_p^{A_B, \bar{A}_B \cap B} = \bar{c}_p^{A_B, \bar{A}_B} \) from the definitions, noting that if \( \xi \) is a chain in \( \bar{A}_B \), then \( \partial \xi \) is in \( B \) if and only if it is in \( B \cap A_B \). We let \( \bar{\alpha}_{A,B} : H_p^\infty(\bar{A}_B) \to \bar{c}_p^{A_B, \bar{A}_B} \) be the composition of these isomorphisms. In addition, if \( [\xi] \in H_p^\infty(A, B) \) is represented by \( \xi = \sum a_\sigma \sigma \), it is also represented by \( \bar{\xi} = \sum_{\sigma \subset B} a_\sigma \sigma \), as the piece of \( \xi \) contained in \( B \) is 0 in \( C_p^\infty(A, B) \). But then \( \bar{\xi} \in \bar{c}_p^{A_B, \bar{A}_B \cap B} \). It follows that \( \bar{\alpha}_{A,B}([\xi]) = \bar{\xi} \) from the proof of Lemma 4.1.

For the second part of the lemma, if \( [\xi] \in H_p^\infty(A, B) \) is represented by the chain \( \xi = \sum a_\sigma \sigma \), then, as above, it is also represented by \( \bar{\xi} = \sum_{\sigma \subset B} a_\sigma \sigma \). So \( \partial_*([\xi]) \) is represented by \( \partial \bar{\xi} \), which is \( \sum b_\tau \tau \) by assumption. The map \( \bar{\alpha}_{B,C} \) thus takes \( \partial_*([\xi]) \) to \( \sum_{\tau \subset C} b_\tau \tau \) by the argument just above. \( \square \)

**Remark 4.4.** As in Remark 4.2, given \( [\xi] \in H_p^\infty(A, B) \) in the setting of Lemma 4.3, as well as a triangulation \( T \) compatible with \( A \) and \( B \) and a \( p \)-simplex \( \sigma \) of \( cl(A - B) \), we can recover the coefficients of \( \sigma \) in the corresponding chain \( \bar{\alpha}_{A,B}([\xi]) \) in \( c_i^{T,\infty}(X) \) by looking at the image of \( [\xi] \) in \( H_p^\infty(A, A - \text{int}(\sigma)) \cong H_p(\sigma, \partial \sigma) \cong \mathbb{Z} \). In this way, given \( [\xi] \in H_p^\infty(A, B) \) and a triangulation \( T \) compatible with \( A \) and \( B \), we can construct the unique chain in \( c_p^{T,\infty}(\bar{A}_B, \bar{A}_B \cap B) \subset c_p^{T,\infty}(X) \) prescribed by the isomorphisms.
5 Chain-level intersection pairings on pseudomanifolds

This section contains our development of the intersection pairing $G^\infty_*(X, \Sigma) \to C^\infty_*(X, \Sigma)$ for locally finite chains on non-compact pseudomanifolds. Following some preparatory remarks, we define intersection coefficients via local umkehr maps in Section 5.1; these are used to define a global umkehr map in Section 5.2. Section 5.3 concerns the chain cross product, and, finally, Section 5.4 contains the definition of the intersection pairing.

The first step toward constructing an intersection pairing map is the construction, generalizing those in [13, 17], of an umkehr map $\Delta_* : C_*^\Delta((X, \Sigma) \times (X, \Sigma)) \to C_{*-n}((X, \Sigma))$. Here $X$ continues to be an oriented $n$-dimensional PL stratified pseudomanifold, not necessarily compact, and $C_*^\Delta((X, \Sigma) \times (X, \Sigma))$ is the subcomplex of chains of $C_*^\infty((X, \Sigma) \times (X, \Sigma))$ that are in general position with respect to the map $\Delta$, where $\Delta : X \to X \times X$ is the diagonal inclusion $\Delta(x) = (x, x)$. Explicitly, this means the following:

**Definition 5.1.** For a set $A \subseteq X \times X$, let $A' = \Delta^{-1}(A)$. Then $C_i^\Delta((X, \Sigma) \times (X, \Sigma)) \subseteq C_i^\infty((X, \Sigma) \times (X, \Sigma))$ is the subcomplex consisting of those chains $\xi$ such that

1. $\dim(|\xi' - \Sigma|) \leq i - n$
2. $\dim(|\partial\xi' - \Sigma|) \leq i - n - 1$.

Observe that this is indeed a chain subcomplex, as if $\xi_1, \xi_2 \in C_i^\Delta((X, \Sigma) \times (X, \Sigma))$ then so is $\xi_1 + \xi_2$, as $|\xi_1 + \xi_2| \subseteq |\xi_1| \cup |\xi_2|$, so $|\xi_1 + \xi_2'| \subseteq |\xi_1'| \cup |\xi_2'|$ and similarly for $\partial(\xi_1 + \xi_2)$. The second condition guarantees that if $\xi \in C_i^\Delta((X, \Sigma) \times (X, \Sigma))$ then so is $\partial\xi$.

**Remark 5.2.** These are weaker conditions than what were required in [7, Definition 4.2], where there were also requirements on the dimensions of intersection with the singular strata in both $X \times X$ and $X$. Our use here of the relative chain complex ultimately makes these additional assumptions unnecessary for the purposes of defining an intersection product on relative chains. However, we will impose some additional constraints below in Section 6.2 when working with intersection chains in order to properly manage the perversities involved.

So let $\xi \in C_i^\Delta((X, \Sigma) \times (X, \Sigma))$. We need to define the chain $\Delta(\xi) \in C_{i-n}((X, \Sigma))$. Due to our focus on relative chains, we only care about simplices that are not contained in the singular locus $\Sigma$, so to define $\Delta(\xi)$ in a given triangulation $T$ of $X$, it suffices to prescribe the coefficient of $\Delta(\xi)$ for each simplex of $T$ not contained in $\Sigma$. More precisely, given a sufficiently refined $T$, we define a simplicial chain $\Delta_t^T(\xi)$. To show that we obtain for $\Delta_t(\xi)$ a well-defined PL chain, we then show that our definition is independent of the choice of triangulation. More precisely, we show that if $T'$ is a subdivision of $T$, then the image of $\Delta_t^T(\xi)$ under the subdivision map is $\Delta_t^{T'}(\xi)$. As any two triangulations of $X$ have a common subdivision (apply [15, Theorem 3.6.C] to the identity map), this suffices to determine a PL chain $\Delta_t(\xi)$.

In fact, we will use triangulations of $X$ of the following form: Suppose $T$ is some triangulation of $X \times X$ compatible with the stratification \footnote{Recall the notation for products of pairs: $(A, B) \times (C, D) = (A \times C, (A \times D) \cup (B \times C))$.} and with respect to which $|\xi|$ and $\Delta(X)$
are subcomplexes (apply \cite{15} Theorem 3.6.C) inductively to obtain such a triangulation). Let $T$ be the induced triangulation on $X$ corresponding to the subcomplex triangulation of $\Delta(X)$. Note that using only such triangulations on $X$ is not a strong restriction, as any arbitrary triangulations $T$ of $X$ and $T'$ of $X \times X$ have subdivisions $T'$ and $T'$ such that $T'$ is a subcomplex of $T'$; this follows from \cite{15} Theorem 3.6.C, as the diagonal inclusion is proper.

Next, let $\sigma$ be an $i-n$ simplex of $T$ with $\sigma \not\subset \Sigma$. As usual, we consider $\sigma$ to be chosen with an orientation, which we suppress from the notation. We will define an intersection coefficient $I_{T,\sigma}(\xi) \in \mathbb{Z}$, which ultimately will not depend on the particular choice of triangulation so long as $\sigma$ is a simplex of the triangulation and the triangulation is obtained as above. Then, for a specific triangulation $T$, we will define $\Delta_T^T(\xi) = \sum_{\sigma \in T, \sigma \not\subset \Sigma} I_{T,\sigma}(\xi) \sigma$, the sum over $i-n$ simplices.

5.1 Local umkehr maps and intersection coefficients

To define the intersection coefficient $I_{T,\sigma}(\xi)$, let $T$, $\xi$, and $\sigma$ all be given as above. Let $Z$ be any finite union of $n$-simplices of $T$ such that the interior of $\sigma$ is contained in the interior of $Z$. Such a $Z$ exists; for example we could choose $Z$ to be the union of all $n$-simplices of $T$ containing $\sigma$ as a face. For a chain $\zeta$ in $X$, let $|\zeta|_Z = |\zeta| \cap Z$, and, similarly, for a chain $\zeta$ in $X \times X$, let $|\zeta|_Z \times Z = |\zeta| \cap (Z \times Z)$. Let $\Sigma_Z = \Sigma \cap Z$. Let $D_Z$ be the union of all $n$-simplices of $T$ not contained in $Z$, and, let $S_Z = Z \cap D_Z$. Let $D_{Z \times Z} = (X \times D_Z) \cup (D_Z \times X)$. Let $J_Z = \Sigma_Z \cup S_Z$, and let $J_{Z \times Z} = (Z \times J_Z) \cup (J_Z \times Z)$. Note that $\Delta^{-1}(J_{Z \times Z}) = J_Z$. Recall that if $A \in X \times X$, then we let $A' = \Delta^{-1}(A) = A \cap \Delta(X)$.

We will show below that $Z - J_Z$ and $Z \times Z - J_{Z \times Z}$ are oriented $n$-dimensional manifolds, with the orientations inherited from $X$ and $X \times X$, respectively. Thus $(Z, J_Z)$ and $(Z \times Z, J_{Z \times Z})$ satisfy the first hypothesis of Proposition 3.6.

**Definition 5.3.** Given the assumptions above, we define the $(Z,T)$-intersection coefficient $I_{Z,T}^T(\xi) \in \mathbb{Z}$ as follows for an $i-n$ simplex $\sigma$, $\sigma \not\subset \Sigma$, of the triangulation $T$: If $\sigma \not\subset \Delta^{-1}(|\xi|) = |\xi'|$, then define $I_{Z,T}^Z(\xi)$ to be 0. Otherwise, let $[\xi] \in H_i^\infty(|\xi|, |\partial \xi|)$ represent $\xi$ under the isomorphism $\alpha_{|\xi|,|\partial \xi|}^{-1}$ of Lemma 4.1 and let $I_{Z,T}^Z(\xi)$ be the image of $[\xi]$ under the following composition:

\[12\text{In this case, the referenced "intersection" is the intersection of } \xi \text{ with the diagonal } \Delta(X).\]
\[ H_i^\infty(|\xi|, |\partial \xi|) \rightarrow H_i^\infty(|\xi| \cup J_{Z \times Z} \cup D_{Z \times Z}, |\partial \xi| \cup J_{Z \times Z} \cup D_{Z \times Z}) \]
\[ \cong H_i(|\xi|_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi|_{Z \times Z} \cup J_{Z \times Z}) \]
\[ \leftarrow H^2_{n-i}(Z \times Z - (|\partial \xi|_{Z \times Z} \cup J_{Z \times Z}), Z \times Z - (|\xi|_{Z \times Z} \cup J_{Z \times Z})) \]
\[ \Delta^* \rightarrow H^2_{n-i}(Z - |\partial \xi|'_Z \cup J_Z, Z - (|\xi|'_Z \cup J_Z)) \]
\[ \nrightarrow \ H_{i-n}(|\xi|'_Z \cup J_Z, |\partial \xi|'_Z \cup J_Z) \]
\[ \rightarrow H_{i-n}((|\xi|'_Z \cup J_Z, (|\xi|'_Z \cup J_Z) - \text{int}(\sigma)) \]
\[ \cong H_{i-n}(\sigma, \partial \sigma) \]
\[ \cong \mathbb{Z}. \]

We will show below that this construction does not depend on the choice of \( Z \) or the triangulation \( T \), given the previous constraints, and so we can define the intersection coefficient \( I_\sigma(\xi) \) to be \( I_{\sigma,T}^Z(\xi) \) for any choice of \( Z \) or of \( T \) containing \( \sigma \).

**Well-definedness.** Let us first observe that \( I_{\sigma,T}^Z(\xi, \eta) \) is well defined:

1. The first map is induced by inclusions.
2. The second map is an excision isomorphism. To see that this is a valid excision, we note that
   \[
   (|\partial \xi| \cup J_{Z \times Z} \cup D_{Z \times Z}) \cap (|\xi|_{Z \times Z} \cup J_{Z \times Z}) = |\partial \xi|_{Z \times Z} \cup J_{Z \times Z},
   \]
as \( D_{Z \times Z} \cap |\xi|_{Z \times Z} \subset D_{Z \times Z} \cap (Z \times Z) = (Z \times S_Z) \cup (S_Z \times Z) \subset J_{Z \times Z} \). So the excision removes \( D_{Z \times Z} - ((Z \times Z) \cap D_{Z \times Z}) \). As \( Z \times Z \) can be triangulated as a finite subcomplex of \( X \times X \), the argument for excision is identical to the standard simplicial homology excision proof, e.g. [19, Theorem 9.1]. Finally, as \( |\xi|_{Z \times Z} \cup J_{Z \times Z} \) and \( |\partial \xi|_{Z \times Z} \cup J_{Z \times Z} \) are compact, in particular they will be finite subcomplexes in any triangulation of \( Z \times Z \), we have \( H_i^\infty(|\xi|_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi|_{Z \times Z} \cup J_{Z \times Z}) = H_i(|\xi|_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi|_{Z \times Z} \cup J_{Z \times Z}) \), which can from here on also be identified with the singular homology group.
3. The third map and fifth maps are the isomorphisms of Proposition [3.6] once we show that \( Z - J_Z \) and \( Z \times Z - J_{Z \times Z} \) are manifolds.
4. The fourth map is the pullback by the diagonal map.
5. The sixth map is induced by inclusion, noting that, by our assumptions, \( \sigma \) is contained in \( |\xi|'_Z \) and the interior of \( \sigma \) is not contained in \( |\partial \xi|'_Z \cup J_Z \). In fact, by assumption of general position, \( \dim(|\xi|'_Z) \leq i - n \) and \( \dim(|\partial \xi|'_Z) \leq i - n - 1 \). So, in particular, if \( \sigma \) is an \( i - n \) simplex contained in \( |\xi|'_Z \), then \( \sigma \) cannot be contained in \( |\partial \xi|'_Z \). It is also outside of \( S_Z \) and \( \Sigma_Z \) by assumption.
6. The seventh map is again an excision.

7. The last isomorphism depends on the orientation of $\sigma$.

To finish showing that the intersection coefficient $I_{\sigma}^{Z,T}(\xi)$ is well defined, we must show that $Z - J_Z$ and $Z \times Z - J_{Z \times Z}$ are manifolds, as claimed. We do this now:

**Lemma 5.4.** $Z - J_Z$ and $Z \times Z - J_{Z \times Z}$ are oriented manifolds with orientations inherited respectively from $X$ and $X \times X$.

*Proof.* We begin with $Z - J_Z = Z - (\Sigma Z \cup S_Z)$. Once we have shown that it is a manifold, the orientability will follow because then $Z - (\Sigma Z \cup S_Z)$ must be a submanifold of $X - \Sigma X$, which is oriented.

Let $x \in Z - (\Sigma Z \cup S_Z)$. We must show that $x$ has an $n$-dimensional Euclidean neighborhood. Clearly this is true at any point that is interior to an $n$-simplex of $Z$. So suppose that $x$ is contained in the interior of a simplex $\tau$ of $Z$ in the triangulation $T$ with respect to which $Z$ is defined and that $\dim(\tau) < n$. Thinking of $x$ as a point of $X$, we have $x \in X - \Sigma X$ and so $x$ has a Euclidean neighborhood, say $U$, in $X$, and we may suppose $U$ is contained in a star neighborhood of $\tau$ in $X$ (in the triangulation $T$). If the star neighborhood of $\tau$ is contained in $X$, then $U$ is a Euclidean neighborhood of $x$ in $Z$. If not, then there is a simplex $\delta$ of $X$ that contains $\tau$ as a face and that is not contained in $Z$. But, as $X$ is a pseudomanifold, every simplex of $X$ is contained in some $n$-simplex, and this implies there is some $n$-simplex $\gamma$, having $\delta$ and $\tau$ as faces and such that $\gamma$ is not contained in $Z$ (if any $n$-simplex having $\delta$ as a face were contained in $Z$, then $\delta$ would be contained in $Z$). So then $\tau$ is contained in $Z$ and in $D_Z$, which was defined as the union of $n$-simplices not contained in $Z$. So $\{x\} \subset \tau \subset S_Z$, a contradiction.

Thus all points $x \in Z - (\Sigma Z \cup S_Z)$ have $n$-dimensional Euclidean neighborhoods, and $Z - (\Sigma Z \cup S_Z)$ is a subspace of a Hausdorff space, so it is a manifold.

For $Z \times Z - J_{Z \times Z}$, we recall that $J_{Z \times Z} = (Z \times J_Z) \cup (J_Z \times Z)$ so $Z \times Z - J_{Z \times Z} = (Z - J_Z) \times (Z - J_Z)$, and the claim follows. 

*Remark 5.5.* Note that while the triangulation $T$ is utilized in our selection of $\sigma$ and $Z$, once this simplex and subspace have been established, nothing in the chain of maps defining $I_{\sigma}^{Z,T}(\xi)$ depends on the specific triangulation. Thus, we can define $I_{\sigma}^{Z}(\xi) = I_{\sigma}^{Z,T}(\xi)$ for any suitable triangulation. Furthermore, we see that if we replace $T$ with some subdivision $T'$, then $Z$ and the other spaces involved in defining our intersection coefficient are still well-defined as simplicial subspaces of $X$ in this subdivision. Furthermore, if $\tau$ is an $i - n$ simplex of $T'$ with $\tau \subset \sigma$ and with $\tau$ oriented consistently with $\sigma$, then the diagram
shows that $I^Z_\sigma(\xi) = I^{Z,T}_\sigma(\xi)$ agrees with $I^Z_\sigma(\xi) = I^{Z,T}_\sigma(\xi)$.

**Remark 5.6.** By assumption, $\dim(|\xi|_Z) \leq i-n$, and therefore, by Lemma 4.3 (taking $B = C$ there), an element of $H_{i-n}(|\xi|_Z \cup J_Z, |\partial \xi|_Z \cup J_Z)$, as occurs in the middle of our definition of $I^{Z,T}_\sigma(\xi)$, determines a PL chain in $Z$ that consists, in any compatible triangulation, of $i-n$ simplices not contained in $\Sigma_Z$ or $S_Z$ and whose boundary is contained in $|\partial \xi|_Z \cup \Sigma_Z \cup S_Z$. We can think of this chain as the local umkehr image of $\xi$ in $Z$, which we denote $\Delta^Z_\sigma(\xi)$. If we wish to fix a given triangulation, we can refer more specifically to the simplicial chain $\Delta^{Z,T}_\sigma(\xi)$ whose underlying PL chain is $\Delta^Z_\sigma(\xi)$. The further refinement in the definition to $I^{Z,T}_\sigma(\xi)$, itself, is then just an example of the process described in Remarks 4.2 and 4.4 for finding the coefficients of simplices in a simplicial representation of a PL chain. Our construction of $\Delta_\sigma(\xi)$, which is described in the next section, can then be thought of alternatively as patching together these local images to obtain a single global chain in $X$ with boundary in $|\partial \xi| \cup \Sigma$. The independence of choice of $Z$ that we are about to demonstrate implies that this patching is well defined locally, and so determines a unique such global image chain.

**Independence of $Z$.** Next we want to show that $I^Z_\sigma(\xi)$ is independent of the choice of $Z$. It is sufficient to demonstrate this property with respect to an alternative choice, say $Y$, in any triangulation compatible with the stratification and containing $|\xi|$ and $\Delta(X)$ as subcomplexes, and such that $\sigma \subset Y \subset Z$. For if $Z'$ is an arbitrary alternative to $Z$ then we can always find a $Y$ such that $Z \supset Y \subset Z'$ by taking $Y$ to be a star neighborhood of $\sigma$ in a sufficiently fine triangulation. Then the independence of choices between $Z$ and $Y$ and between $Y$ and $Z'$ demonstrate the overall independence of choice of $Z$.

**Proposition 5.7.** The intersection coefficient $I^Z_\sigma(\xi)$ is independent of the choice of $Z$.

**Proof.** The proof will proceed through a series of commutative diagrams, throughout which the top lines will compose to the definition of $I^{Z,T}_\sigma(\xi)$ and the bottom lines will compose to the definition of $I^{Y,T}_\sigma(\xi)$. The middle line will be an intermediary that accepts maps from the top and bottom lines, but the middle and lower lines will be isomorphic throughout, allowing us to reverse the lower vertical maps to obtain a direct comparison between $I^{Z,T}_\sigma(\xi)$ and $I^{Y,T}_\sigma(\xi)$.
The following observations will be useful: As $Y \subset Z$, it follows that $D_Z \subset D_Y$, $S_Z \subset D_Y$, $D_{Z \times Z} \subset D_{Y \times Y}$, if $A$ is any subset of $X$ then $(A_Z)_Y = A_Y$, and if $A$ is any subset of $X \times X$ then $(A_{Z \times Z})_{Y \times Y} = A_{Y \times Y}$. Also, $J_Y \times Y \subset J_{Z \times Z} \cup D_{Y \times Y}$, and, in fact, one can check that $J_Y \times Y \cup D_{Y \times Y} = J_{Z \times Z} \cup D_{Y \times Y}$, using that $\Sigma_Z \subset \Sigma_Y \cup D_Y$. Similarly, $J_Z \cup D_Y = J_Y \cup D_Y$.

First, we have the following:

\[
H^\infty(|\xi| \cup J_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi| \cup J_{Z \times Z} \cup J_{Z \times Z}) \cong H_i(|\xi| \cup J_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi| \cup J_{Z \times Z} \cup J_{Z \times Z})
\]

\[
H^\infty(|\xi|, |\partial \xi|) \cong H^\infty(|\xi| \cup J_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi| \cup J_{Z \times Z} \cup J_{Z \times Z}) \cong H_i(|\xi| \cup J_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi| \cup J_{Z \times Z} \cup J_{Z \times Z})
\]

\[
H^\infty(|\xi| \cup J_{Y \times Y} \cup J_{Y \times Y}, |\partial \xi| \cup J_{Y \times Y} \cup J_{Y \times Y}) \cong H_i(|\xi| \cup J_{Y \times Y} \cup J_{Y \times Y}, |\partial \xi| \cup J_{Y \times Y} \cup J_{Y \times Y})
\]


The commutativity is clear at the space level as all maps are induced by spatial inclusions, using the observations concerning spatial relations given above. Notice also that $|\partial \xi|_{Z \times Z}$ is the union of $|\partial \xi|_{Y \times Y}$ with the simplices of $|\partial \xi|$ in $D_{Y \times Y \cap (Z \times Z)}$. The right horizontal arrow in the middle row excises $(X \times (D_Z - S_Z)) \cup ((D_Z - S_Z) \times X)$, and so is an isomorphism. Thus the bottom right vertical map is also an isomorphism.

Next, we have a diagram

\[
H_i(|\xi| \cup J_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi|_{Z \times Z} \cup J_{Z \times Z}) \cong H^\infty(|\xi| \cup J_{Z \times Z} \cup J_{Z \times Z}, |\partial \xi| \cup J_{Z \times Z} \cup J_{Z \times Z})
\]

\[
H_i(|\xi| \cup J_{Y \times Y} \cup J_{Y \times Y}, |\partial \xi| \cup J_{Y \times Y} \cup J_{Y \times Y}) \cong H^\infty(|\xi| \cup J_{Y \times Y} \cup J_{Y \times Y}, |\partial \xi| \cup J_{Y \times Y} \cup J_{Y \times Y})
\]

\[
The top commutes by Lemma 3.7 using $(Z \times Z, J_{Z \times Z})$ for the pair “$(Z, S)$”. By Lemma 3.8, we can also treat the middle duality isomorphism as utilizing $(Z \times Z, J_{Y \times Y} \cup (D_{Y \times Y \cap (Z \times Z)})$ for the pair “$(Z, S)$.” Then the bottom square commutes by Lemma 3.9, letting the $S$ of that lemma be $J_{Y \times Y}$ and the $T$ of that lemma be $D_{Y \times Y \cap (Z \times Z)}$.

We then have the naturality of the pullback:

\[
H^{\infty}((Z \times (Z \times Z)), \Delta^\ast \Delta^\ast) \cong H^\infty((Z \times (Z \times Z)), \Delta^\ast \Delta^\ast)
\]

\[
H^{\infty}((Z \times (Z \times Z)), \Delta^\ast \Delta^\ast) \cong H^\infty((Z \times (Z \times Z)), \Delta^\ast \Delta^\ast)
\]

This is followed again by a commutative duality map

23
Again, the top commutes by Lemma 3.7 using \((Z, J_Z)\) for the pair \(\"(Z, S)\"\). By Lemma 3.8, we can also treat the middle duality isomorphism as utilizing \((Z, J_Y \cup (D_Y \cap Z))\) for the pair \(\"(Z, S)\"\). Then the bottom square commutes by Lemma 3.9, letting the \(T\) of that lemma be \(D_Y \cap Z\).

Finally, we have the diagram

\[
\begin{array}{ccc}
H_{n-i}(\xi'_{Z} \cup J_Z, \partial \xi'_{Z} \cup J_Z) & \xrightarrow{\cong} & H_{n-i}(\xi'_{Z} \cup J_Z, \partial \xi'_{Z} \cup J_Z) \\
\cong & & \cong \\
H_{n-i}(\xi'_{Y} \cup J_Y \cup (D_Y \cap Z), \partial \xi'_{Y} \cup J_Y \cup (D_Y \cap Z)) & \xrightarrow{\cong} & H_{n-i}(\xi'_{Y} \cup J_Y \cup (D_Y \cap Z), \partial \xi'_{Y} \cup J_Y \cup (D_Y \cap Z)) \\
\end{array}
\]

The lower vertical map in the middle column is an isomorphism from the commutativity of the diagram and from the other excision isomorphisms.

Putting all these pieces together, we see that \(I_{\sigma}^{Z,T}(\xi) = I_{\sigma}^{Y,T}(\xi)\), which is the desired independence result. □

Proposition 5.7 together with Remark 5.5 demonstrates that we are justified in defining \(I_{\sigma}(\xi)\) to be \(I_{\sigma}^{Z,T}(\xi)\) for any appropriate choice of \(Z\) and \(T\), without ambiguity.

Additivity. We would next like to show that if we have \(\xi_1, \xi_2 \in C_{i}^{\Delta,\infty}((X, \Sigma) \times (X, \Sigma))\), then \(I_{\sigma}(\xi_1 + \xi_2) = I_{\sigma}(\xi_1) + I_{\sigma}(\xi_2)\). Below, this will allow us to see that the intersection product is a homomorphism. This point is somewhat neglected in all the previous treatments [11] [18] [7], though it is not completely trivial. The tricky point is that \(I_{\sigma}(\xi_1)\) is defined using a representation of \(\xi_1\) in \(H_{i}^{\infty}(|\xi_1|, |\partial \xi_1|)\), while \(I_{\sigma}(\xi_2)\) is defined using a representation of \(\xi_2\) in \(H_{i}^{\infty}(|\xi_2|, |\partial \xi_2|)\), and \(I_{\sigma}(\xi_1 + \xi_2)\) is defined using a representation of \(\xi_1 + \xi_2\) in \(H_{i}^{\infty}(|\xi_1 + \xi_2|, |\partial (\xi_1 + \xi_2)|)\). Beyond needing to find a way to relate these groups, we also have the issue that \(|\xi_1 + \xi_2| \neq |\xi_1| \cup |\xi_2|\) in general, as there may be cancellation of simplices in the sum. The solution is to observe that the use of \(H_{i}^{\infty}(|\xi|, |\partial \xi|)\) in the first step of the definition of \(I_{\sigma}(\xi)\) is overly restrictive. This is the content of the next lemma.

**Lemma 5.8.** Let \(B \subset A \subset X \times X\) be PL subspaces such that

1. \(\dim(A) = i\) and \(\dim(B) < i\),

\[
H^{2n-i}(Z - |\partial \xi|_{Z} \cup J_{Z}, Z - (|\xi|_{Z} \cup J_{Z})) \xrightarrow{\cong} H_{i-n}(|\xi|_{Z} \cup J_{Z}, |\partial \xi|_{Z} \cup J_{Z})
\]

\[
H^{2n-i}(Z - |\partial \xi|_{Y} \cup J_{Y} \cup (D_{Y} \cap Z), Z - (|\xi|_{Z} \cup J_{Z} \cup (D_{Y} \cap Z))) \xrightarrow{\cong} H_{i-n}(|\xi|_{Z} \cup J_{Y} \cup (D_{Y} \cap Z), |\partial \xi|_{Z} \cup J_{Z} \cup (D_{Y} \cap Z))
\]

\[
H^{2n-i}(Y - |\partial \xi|_{Y} \cup J_{Y}, Y - (|\xi|_{Y} \cup J_{Y})) \xrightarrow{\cong} H_{i-n}(|\xi|_{Y} \cup J_{Y}, |\partial \xi|_{Y} \cup J_{Y}).
\]
2. \(|\xi| \subseteq A\) and \(|\partial \xi| \subseteq B\)

3. \(A\) and \(B\) are in general position with respect to \(\Delta\).

Then for \(\sigma \subseteq A \cap \Delta(X)\) and \(\sigma \nsubseteq \Sigma\), the definition of \(I_\sigma(\xi)\) remains unchanged replacing \([\xi] \in H^\infty_1([\xi],|\partial \xi|)\) with its image in \(H^\infty_1(A,B)\) in the first step of the construction of Definition 5.3 and then replacing \((|\xi|,|\partial \xi|)\) with \((A,B)\) throughout. If \(\sigma \nsubseteq A \cap \Delta(X)\), we set \(I_\sigma(\xi) = 0\), which is consistent with the original definition.

Proof. Definition 5.3 defines \(I_\sigma(\xi)\) in terms of \(I^{Z,T}_{\sigma}(\xi,\eta)\). We show here that \(I^{Z,T}_{\sigma}(\xi,\eta)\) does not change when we replace \((|\xi|,|\partial \xi|)\) by \((A,B)\), but the independence of \(I^{Z,T}_{\sigma}(\xi)\) with respect to choices of \(Z\) and \(T\) will then indicate that the same is true using \((A,B)\).

Choose \(T, \sigma,\) and \(Z\) as in Definition 5.3 though assuming also that \(T\) is compatible with \(A\) and \(B\). We first suppose \(\sigma \subseteq [\xi]\) and \(\sigma \nsubseteq \Sigma X\). Let \(A' = \Delta^{-1}(A) = A \cap \Delta(X)\) and define \(B'\) similarly. Then we have the following diagram
The diagram commutes by the obvious space inclusions and by Lemma 3.7 for the duality maps; we can use \((Z, J_Z)\) and \((Z \times Z, J_{Z \times Z})\) as the pairs \(\langle X, S \rangle\) for the duality isomorphisms. Note that the inclusion map \(H_{i-n}(A'_Z \cup J_Z, B'_Z \cup J_Z) \to H_{i-n}(A'_Z \cup J_Z, (A'_Z \cup J_Z) - \text{int}(\sigma))\) utilizes our usual assumption that \(\sigma\) is not contained in \(J_Z\), while the assumptions on \(B\) assure that \(\dim(B_X') < i-n\). Similarly, the excision isomorphism \(H_{i-n}(\sigma, \partial \sigma) \to H_{i-n}((A'_Z \cup J_Z, (A'_Z \cup J_Z) - \text{int}(\sigma))\) uses the dimension assumptions to assure that \(\sigma \cap ((A'_Z \cup J_Z) - \text{int}(\sigma)) = \partial \sigma\). The diagram shows that, in this case, \(I_{\sigma, T}^Z(\xi)\) can indeed be computing utilizing \((A, B)\) as claimed.

Next, suppose \(\sigma \not\subset A'\). Then \(\sigma \not\subset |\xi'|\), so \(I_{\sigma, T}^Z(\xi) = 0\) by definition, which is consistent with the statement of the lemma here.

Finally, suppose \(\sigma \subset A'\) but \(\sigma \not\subset |\xi'|\). Then \(I_{\sigma, T}^Z(\xi) = 0\) by definition, and we must show that the image down the right side of our diagram of \([\xi] \in H_{i-n}^\infty(A, B)\) is 0. But notice that the top part of the diagram does not depend on \(\sigma\), so the image of \([\xi]\) in \(H_{i-n}(A'_Z \cup J_Z, B'_Z \cup J_Z)\) must be in the image of \(H_{i-n}(\sigma' \cup J_{\sigma'}, |\xi'| \cup J_Z)\). Therefore, it can be represented by a chain that does not include \(\sigma\) and so represents 0 in \(H_{i-n}(A'_Z \cup J_Z, (A'_Z \cup J_Z) - \text{int}(\sigma))\), as desired.

\[\text{Proposition 5.9. Suppose } \xi_1, \xi_2 \in C_i^\Delta,\infty((X, \Sigma) \times (X, \Sigma))). \text{ Then } I_\sigma(\xi_1 + \xi_2) = I_\sigma(\xi_1) + I_\sigma(\xi_2).\]

\[\text{Proof. By Lemma 5.8, } I_\sigma(\xi_1 + \xi_2), I_\sigma(\xi_1), \text{ and } I_\sigma(\xi_2) \text{ can all be computed using the formula of Definition 5.3 beginning with representations of the chains } \xi_1, \xi_2, \text{ and } \xi_1 + \xi_2 \text{ as elements } [\xi_1], [\xi_2], [\xi_1 + \xi_2] \in H_i^\infty((\xi_1 \cup \xi_2, |\xi_1| \cup |\xi_2|). \text{ Clearly } [\xi_1] + [\xi_2] = [\xi_1 + \xi_2], \text{ using the isomorphism } \alpha_{[\xi_1 \cup [\xi_2, [\partial \xi_1] \cup [\partial \xi_2]} \text{ of Lemma 4.1. From here, the result follows using that the maps of Definition 5.3 are all homomorphisms.}\]

### 5.2 The umkehr map

We can now officially define the umkehr map \(\Delta _i : C^\Delta \infty_s ((X, \Sigma) \times (X, \Sigma)) \to C^\infty_{i-n}(X, \Sigma)\) as follows:

\[\text{Definition 5.10. Let } X \text{ be an oriented } n \text{-dimensional PL stratified pseudomanifold, not necessarily compact, and let } \xi \in C^\Delta,\infty_i((X, \Sigma) \times (X, \Sigma)). \text{ Define } \Delta_i(\xi) \text{ such that, if } T \text{ is any triangulation of } X \text{ restricting a triangulation of } X \times X \text{ that is compatible with the stratification and for which } |\xi| \text{ and } \Delta(X) \text{ are subcomplexes, then } \Delta_i(\xi) \text{ is represented in } T \text{ as }\]

\[\sum \sigma I_\sigma(\xi)\sigma,\]

where the sum is over all \(i - n\) simplices \(\sigma\) of \(T\) not contained in \(\Sigma_X\).

By Remark 5.5, \(\Delta_i(\xi)\) is well-defined as a PL chain independent of the choice of triangulation used in our defining formula, and it follows from Proposition 5.9 that \(\Delta_i\) is a homomorphism \(C^\Delta,\infty_i((X, \Sigma) \times (X, \Sigma)) \to C^\infty_{i-n}(X, \Sigma)\).

Next we show that \(\Delta_i\) is a chain map of degree \(-n\). Recall (VI.10.5) that this means that \(\Delta_i\) lowers the degree by \(n\) and that \(\partial \Delta_i = (-1)^n \Delta_i \partial\).
Proposition 5.11. Let $X$ be an oriented $n$-dimensional PL stratified pseudomanifold, not necessarily compact, and let $\xi \in C^\Delta_\infty((X, \Sigma) \times (X, \Sigma))$. Then $\Delta_1(\partial \xi) = (-1)^n \partial(\Delta_1(\xi)) \in C_{i-n-1}(X, \Sigma)$.

Proof. It is sufficient to demonstrate that this proposition holds with respect to some fixed triangulation $T$ of the type we have been using. Given such a triangulation, we next observe that all boundary computations are local. In other words, if $\tau$ is an $i-n-1$ simplex of $T$, not contained in $\Sigma$, then the coefficient of $\tau$ in $\partial(\Delta_1(\xi))$ depends only on the coefficients of the simplices of $\Delta_1(\xi)$ that contain $\tau$ as a face. Thus if $Z$ is a neighborhood of $\tau$ containing all the $i-n$ simplices adjacent to $\tau$, we can compute the coefficient of $\tau$ in $\partial(\Delta_1(\xi))$ using only $\Delta^Z,T_1(\xi)$ (see Remark 5.6). Let us first use this observation to show that both $\Delta_1(\partial \xi)$ and $\partial(\Delta_1(\xi))$ are only made up of simplices in $|\partial \xi|$'. This is clear for $\Delta_1(\partial \xi)$ by the construction of $\Delta$. For $\partial(\Delta_1(\xi))$, we know from Remark 5.6 that each $\Delta^Z,T_1(\xi)$ is a chain composed of $i-n$ simplices not contained in $\Sigma_Z$ or $S_Z$ and whose boundary is contained in $|\partial \xi|$' $\cup \Sigma_Z \cup S_Z$. Since we work with relative chains, we are not interested in coefficients of simplices in $\Sigma$, while the spaces $S_Z$ are artifacts of the choice of $Z$. Consequently, as we patch the various $\Delta^Z,T_1(\xi)$ together to form $\Delta_1(\xi)$, we can get a better look at the simplices in $\partial(\Delta_1(\xi))$ that might possibly lie in $S_Z$ by choosing a more appropriate $Z$. Ultimately, by “moving our microscope,” we see that all of $\partial(\Delta_1(\xi))$ must be contained in $|\partial \xi|$' (or $\Sigma$).

Thus, to prove the proposition, it suffices to show for each $\tau \in |\partial \xi|$' with $\tau \not\subseteq \Sigma$ that $\Delta^Z,T_1(\partial \xi)$ and $(-1)^n \partial(\Delta^Z,T_1(\xi))$ agree for a large enough choice of $Z$, say one containing the star neighborhoods of all $i-n$ simplices having $\tau$ as a face. For such a $\tau$, we can now utilize the following diagram:
Except for the squares involving $\mathbb{D}$, this diagram commutes by the standard properties of connecting morphisms. The top square involving $\mathbb{D}$ commutes up to $(-1)^{2n} = 1$ by Lemma 3.10 while the lower square involving $\mathbb{D}$ commutes up to $(-1)^n$ by the same lemma. Starting up the upper left and proceeding right then down in the diagram takes the homology class representing $\xi$ to $\Delta^1(\partial \xi)$ and then to its coefficient for $\tau$. On the other hand, proceeding straight downward takes us to the class representing $\Delta^1_{Z,T}(\xi)$ and then to its boundary. More specifically, applying Lemma 4.3 with $A = |\xi|_Z \cup J_Z$, $B = |\partial \xi|_Z \cup J_Z$, and $C = J_Z$, the $\partial_*$ map here can be thought of as taking $\Delta^1_{Z,T}(\xi)$ to the piece of its boundary consisting of simplices in $|\partial \xi|_Z$ but not in $J_Z$. We know that $\tau$ is such a simplex by assumption. The bottom of the diagram then provides the coefficient of $\tau$ according to Remark 4.4.

It follows that $\tau$ obtains the same coefficient, up to the sign $(-1)^n$, by either route, proving the proposition. \[ \square \]

The next lemma concerns the behavior of $\Delta_1$ under restriction. Recall that if $X$ is a PL space then any open subset $U$ is also PL and that if $X$ is given a triangulation $T$ then $U$
can be given a triangulation $S$ such that every simplex of $S$ is contained in a simplex of $T$ (see [2, Section I.1.3]). This induces a restriction map $c_{*,\infty}^T(X) \to c_{*,\infty}^S(U)$ that takes each $i$-simplex of $T$ to the formal sum of the compatibly oriented $i$-simplices of $S$ contained in it. Such maps then induce a restriction map $r : C_{*,\infty}^S(X) \to C_{*,\infty}(U)$.

**Lemma 5.12.** Let $U \subset X$ be an open subset. Then there is a commutative diagram

$$
\begin{array}{ccc}
C_{*,\infty}^\Delta((X,\Sigma_X) \times (X,\Sigma_X)) & \xrightarrow{\Delta_t} & C_{*-\infty}(X,\Sigma_X) \\
r & & r \\
C_{*,\infty}^\Delta((U,\Sigma_U) \times (U,\Sigma_U)) & \xrightarrow{\Delta_t} & C_{*-\infty}(U,\Sigma_U).
\end{array}
$$

**Proof.** Let $\xi \in C_{*,\infty}^\Delta((X,\Sigma_X) \times (X,\Sigma_X))$, let $T$ be a triangulation $T$ of $X$ satisfying our usual conditions as given above, and let $S$ be a triangulation of $U$ such that every simplex of $S$ is contained in some simplex of $T$. Let $\sigma$ be any $i-n$ simplex of $S$ contained in $|\xi|$ but not contained in $\Sigma_U$. As dim$(|\xi|) \leq i-n$ and as $|\xi|$ is triangulated by $T$, any such $\sigma$ is contained in an $i-n$ simplex $\gamma$ of $|\xi|$ in $T$. By the arguments of Remark 5.5 we see that the coefficient of $\sigma$ in $r\Delta_t(\xi)$ will equal $I_\gamma(\xi)$ (which can be computed as $I_\gamma^T(\xi)$ using $T$ and some appropriate choice of $Z$). We must show that this is equal to the coefficient of $\sigma$ in $\Delta r(\xi)$, which is $I_\gamma(r(\xi))$ (which can be computed as $I_\gamma^S(\xi)$ using $S$ and some appropriate choice of $Y$). As $\sigma$ in $S$ is an arbitrary simplex of $|\xi| \cap U$, this will suffice to prove the lemma.

Next, we take a sufficiently iterated barycentric subdivision $T'$ of $T$ so that

1. there is an $i-n$ simplex $\tau$ of $T'$ contained within $\sigma$ and
2. the star neighborhood of $\tau$ in $T'$ is contained in $U$.

This is possible by the arguments of [19, Section 15]. Then let $S'$ be a subdivision of $S$ such that $\tau$ is also a simplex of $S'$ and such that every simplex of $S'$ is contained in some simplex of $T'$. We can find such an $S'$ by the same procedure described in Footnote 13 form the cell complex on $U$ whose cells are the intersections of the simplices of $S$ and $T'$ (see [20, Example 2.8.5]) and then create the simplicial complex obtained from this cell complex but with the same vertices via the procedure of [20, Proposition 2.9]. As $\tau$ is already a simplex of $T'$ contained in the simplex $\sigma$ of $S$, $\tau$ will be a cell of the cell complex and then remain a simplicial complex on $U$.

---

13 Here’s one way to do this: start with a triangulation $T$ of $X$ and an arbitrary triangulation $S_0$ of $U$ compatible with the PL structure on $U$ inherited from $X$. We can then obtain a cell complex on $U$ whose cells are the intersections of the simplices of $S_0$ with the simplices of $T$ (see [20, Example 2.8.5]). Finally, we can subdivide the cell complex on $U$ to a simplicial complex $S$ triangulating $U$ by using the procedure of [20, Proposition 2.9]; note that the cited proposition works on infinite complexes either by choosing a well-ordering of the vertices or simply by using a partial ordering on the vertices which is locally a total ordering on each cell. Each simplex of $S$ will be contained in a cell of the cell complex and thus in a simplex of $T$.  

29
simplex in the triangulation $S'$. As every simplex of $S'$ is contained in a simplex of $T'$, the star neighborhood of $\tau$ in $S'$ will be a subset of the star neighborhood of $\tau$ in $T'$.

Now, let the star neighborhood of $\tau$ in $T'$ be our space $Z$. As $Z \subset U$, it will also be triangulated as a union of $n$-simplexes in $S'$. We will use this same $Z$ for the computation of the intersection coefficients $I^Z_{T'}(\xi)$ in $X$ and $I^Z_{S'}(r(\xi))$ in $U$. By Remark 5.3 $I^Z_{T'}(\xi) = I_\tau(\xi)$, as $\tau \subset \sigma \subset \gamma$, and $I^Z_{S'}(r(\xi)) = I_\tau(r(\xi))$. So it suffices to show that $I^Z_{T'}(\xi) = I^Z_{S'}(r(\xi))$. This follows from the following commutative diagram and the definitions of these intersection coefficients, noting that the spaces $Z$, $Z \times Z$, $J_Z$, $J_{Z \times Z}$, and $D_{Z \times Z}$ (as a subspace of $X \times X$) do not depend on the particular choice of triangulation used to define $Z$:

$$
\begin{array}{ccc}
H^\infty_i(|\xi|, |\partial \xi|) & \xrightarrow{r} & H^\infty_i(|r(\xi)|, |r(\partial \xi)|) \\
\downarrow \cong & & \downarrow \cong \\
H^\infty_i(|\xi| \cup J_{Z \times Z} \cup D_{Z \times Z}, |\partial \xi| \cup J_{Z \times Z} \cup D_{Z \times Z}) & \xrightarrow{r} & H^\infty_i(U \cap (|\xi| \cup J_{Z \times Z} \cup D_{Z \times Z}), U \cap (|\partial \xi| \cup J_{Z \times Z} \cup D_{Z \times Z})) \\
\cong & & \cong \\
H_i(|\xi| \cup J_{Z \times Z} \cup J_{Z \times Z}) & = & H_i(|\xi| \cup J_{Z \times Z} \cup J_{Z \times Z} \cup J_{Z \times Z}).
\end{array}
$$

Note that the excisions are both still valid as all of the spaces in the bottom row are contained within $U$. \qed

5.3 The cross product

Let $\epsilon : C^\infty_*(X) \otimes C^\infty_*(X) \to C^\infty_*(X \times X)$ denote the chain level PL cross product; see [18, 7]. The cross products in the citations are defined primarily for compact chains, but as the cross product construction is defined locally, there is no difficulty extending it in the obvious way to locally finite chains. This map induces a relative product $\epsilon : C^\infty_*(X, \Sigma) \otimes C^\infty_*(X, \Sigma) \to C^\infty_*(X, \Sigma) \times (X, \Sigma))$.

When an element of $C^\infty_*(X) \otimes C^\infty_*(X)$ can be represented as $\xi \otimes \eta$, we sometimes use the notation $\epsilon(\xi \otimes \eta) = \xi \times \eta$.

Lemma 5.13. Let $U \subset X$ be an open subset. Then there is a commutative diagram

$$
\begin{array}{ccc}
C^\infty_*(X) \otimes C^\infty_*(X) & \xrightarrow{r \otimes r} & C^\infty_*(U) \otimes C^\infty_*(U) \\
\epsilon \downarrow & & \epsilon \downarrow \\
C^\infty_*(X \times X) & \xrightarrow{r} & C^\infty_*(U \times U),
\end{array}
$$

30
where the horizontal maps are induced by restriction. This induces a diagram

\[ C^\infty_*(X, \Sigma_X) \otimes C^\infty_*(X, \Sigma_X) \xrightarrow{\epsilon \otimes \epsilon} C^\infty_*(U, \Sigma_U) \otimes C^\infty_*(U, \Sigma_U) \]

\[ C^\infty_*(\{(X, \Sigma_X) \times (X, \Sigma_X)\}) \xrightarrow{\epsilon} C^\infty_*(\{(U, \Sigma_U) \times (U, \Sigma_U)\}) \]

Proof. The groups \( C^\infty_*(X) \otimes C^\infty_*(X) \) in any fixed degree are generated by elements of the form \( \xi \otimes \eta \) (though not every element can be written in this form). Suppose \( \xi \) is an \( i \)-chain and \( \eta \) is a \( j \)-chain. As observed in [2, Section II.1], a PL chain \( \xi \) is completely determined by \( |\xi|, |\partial \xi| \), and the local sheaf values \([\xi]_x \in (\mathcal{H}_i)_x \cong H_i(|\xi|, |\xi| - \{x\}) \cong H^\infty_i(|\xi|, |\xi| - \{x\})\) determined by the class \([\xi]\) \in \( H_i(|\xi|, |\partial \xi|) \) as \( x \) varies in \( |\xi| - |\partial \xi| \).

Noting that \( |r(\xi)| = |\xi| \cap U \) and as \( |\xi \times \eta| = |\xi| \times |\eta| \), we have

\[ |r(\xi \times \eta)| = |\xi \times \eta| \cap (U \times U) \]

\[ = (|\xi| \times |\eta|) \cap (U \times U) \]

\[ = (|\xi| \cap U) \times (|\eta| \cap U) \]

\[ = |r(\xi)| \times |r(\eta)| \]

and so \( r\epsilon(\xi \otimes \eta) \) and \( \epsilon(r(\xi) \otimes r(\eta)) \) have the same support.

Similarly, as \( r \) and \( \epsilon \) are chain maps and \( |\partial(\xi \otimes \eta)| = (|\partial \xi| \times |\eta|) \cup (|\xi| \times |\partial \eta|) \), we have

\[ |\partial r\epsilon(\xi \otimes \eta)| = |r\epsilon((\partial \xi) \otimes \eta \pm \xi \otimes \partial \eta)| \]

\[ = (|\partial \xi| \times |\eta|) \cup (|\xi| \times |\partial \eta|) \cap (U \times U) \]

\[ = (|r(\partial \xi)| \times |r(\eta)|) \cup (|r(\xi)| \times |r(\partial \eta)|) \]

\[ = (|\partial r(\xi)| \times |r(\eta)|) \cup (|r(\xi)| \times |\partial r(\eta)|) \]

\[ = |\partial(r(\xi) \times r(\eta))| \]

\[ = |\partial(\xi \times \eta)|. \]

Do the boundaries of \( r\epsilon(\xi \otimes \eta) \) and \( \epsilon(r(\xi) \otimes r(\eta)) \) have the same support.

It remains to check the local sheaf values, but clearly, via excisions and using standard local chain representatives, we have a commutative diagram at each point \((x, y) \in |r(\xi \times \eta)| - |\partial r(\xi \times \eta)|) :

\[ H_i(|\xi|, |\xi| - \{x\}) \otimes H_j(|\eta|, |\eta| - \{y\}) \xrightarrow{\cong} H_i(|r\xi|, |r\xi| - \{x\}) \otimes H_j(|r\eta|, |r\eta| - \{y\}) \]

\[ H_{i+j}(|\xi| \times |\eta|, |\xi| \times |\eta| - \{(x, y)\}) \xrightarrow{\cong} H_{i+j}(|r\xi| \times |r\eta|, |r\xi| \times |r\eta| - \{(x, y)\}) \]

\[ \epsilon \]

\[ \epsilon \]

\[ \cong \]

\[ \cong \]
5.4 The chain-level intersection pairing

Let \( \mathfrak{G}_\infty^\infty(X) \) be the subcomplex of \( C_\infty^\infty(X, \Sigma) \otimes C_\infty^\infty(X, \Sigma) \) defined by \( \epsilon^{-1}(C_\Delta \infty^\infty((X, \Sigma) \times (X, \Sigma))) \).

**Definition 5.14.** The intersection pairing is defined to be

\[
\mu = \Delta ! \circ \epsilon : \mathfrak{G}_\infty^\infty(X) \rightarrow C_\infty^{\infty-n}(X, \Sigma).
\]

We note that \( \mu \) is a chain map of degree \(-n\) as \( \Delta ! \) is a chain map of degree \(-n\) by Propositions 5.11 and 5.9, while \( \epsilon \) is a degree 0 chain map. This latter claim follows from the arguments in Section 5.2.2 of [5].

The intersection pairing is also compatible with restriction: putting together Lemmas 5.12 and 5.13, we obtain a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{G}_\infty^\infty(X) & \xrightarrow{\mu} & C_\infty^{\infty-n}(X, \Sigma_X) \\
\downarrow & & \downarrow \\
\mathfrak{G}_\infty^\infty(U) & \xrightarrow{\mu} & C_\infty^{\infty-n}(U, \Sigma_U),
\end{array}
\]

where the vertical maps are induced by restriction.

6 The intersection pairing for intersection homology

In this section, we briefly review the definition of intersection chains and then demonstrate how an intersection pairing of intersection chains follows from the work of the previous sections. A thorough introduction to intersection homology can be found in [5].

6.1 PL intersection chains

If \( X \) is a PL stratified pseudomanifold, a perversity on \( X \) is a function \( \bar{p} : \{ \text{singular strata of } X \} \rightarrow \mathbb{Z} \). The perversity \( \bar{p} \) PL intersection chain complex \( I^pC_\infty^i(X) \) can then be defined as the subcomplex of chains of \( C_\infty^i(X, \Sigma) \) consisting of chains \( \xi \) such that if \( \xi \in C_\infty^i(X, \Sigma) \) and \( |\xi| \) denotes the minimal support\(^{14}\) of \( \xi \), i.e. the union in some triangulation (compatible with \( \xi \) and with the stratification of \( X \)) of the \( i \)-simplices of \( \xi \) that are not contained in \( \Sigma \), then the following conditions are satisfied for each singular stratum \( \mathcal{Z} \) of \( X \):

1. \( \dim(|\xi| \cap \mathcal{Z}) \leq i - \operatorname{codim}(\mathcal{Z}) + \bar{p}(\mathcal{Z}) \),
2. \( \dim(|\partial \xi| \cap \mathcal{Z}) \leq i - 1 - \operatorname{codim}(\mathcal{Z}) + \bar{p}(\mathcal{Z}) \).

\(^{14}\)See Section 4.1.
The groups $I^pC_*(X)$ form a chain complex due to the second condition, and the homology groups are the intersection homology groups, denoted $I^pH_*(X)$. Similarly, $I^pC^\infty_*(X)$ and $I^pH^\infty_*(X)$ are defined identically by imposing the dimension conditions on chains in $C^\infty_*(X, \Sigma)$.

The reader familiar with the original definition of intersection chains, e.g. from Goresky-MacPherson [11], might be surprised to see them defined here as a subcomplex of the relative chain complex, but it is shown in [8] that these intersection chain complexes are isomorphic to those of Goresky and MacPherson provided that $\bar{\sigma}$ chain complex, but it is shown in [5] that these intersection chain complexes are isomorphic to those of Goresky and MacPherson provided that $\bar{\sigma}$.

In this section, we demonstrate how the intersection pairing $\mu$ of the prior section restricts to provide an intersection pairing on intersection chains. In addition to limiting ourselves to intersection chains, we will need to impose a stronger notion of general position (cf. Remark 6.2). This stratified general position is formulated in general in [7, Definition 3.1]; we here restrict to the version we will need:

**Definition 6.1.** Let $X$ be a PL stratified pseudomanifold and $\Delta : X \to X \times X$ the diagonal inclusion $\Delta(x) = (x, x)$. We will say that the PL subset $A \subset X \times X$ is in stratified general position (with respect to $\Delta$) if for each stratum $Z \subset X$ we have

$$\dim(A \cap \Delta(Z)) \leq \dim(A \cap (Z \times Z)) - \dim(Z).$$

In other words, noting that $\dim(Z \times Z) = 2\dim(Z)$, we require that, for each stratum $Z$, the PL spaces $A \cap (Z \times Z)$ and $\Delta(Z)$ are in general position in the manifold $Z \times Z$.

If $\xi \in C^\infty_\Delta((X, \Sigma) \times (X, \Sigma))$, then we will say that $\xi$ is in stratified general position if $|\xi|$ is in stratified general position.

**Remark 6.2.** Notice that if $\xi \in C^\infty_\Delta((X, \Sigma) \times (X, \Sigma))$ is in stratified general position and if $R$ is the union of regular strata of $X$, then we have $\dim(\Delta^{-1}(|\xi|) - \Sigma) = \dim(|\xi| \cap \Delta(R)) \leq i - n$. So if $|\xi|$ and $|\partial\xi|$ are in stratified general position, then $\xi \in C^\infty_\Delta((X, \Sigma) \times (X, \Sigma))$.

Next, let $P = (\bar{p}_1, \bar{p}_2)$ be a pair of perversities.

**Definition 6.3.** The domain $G^{\infty,P}_*(X)$ of the intersection product is defined to be the subcomplex of $I^{\bar{p}_1}C^\infty_*(X) \otimes I^{\bar{p}_2}C^\infty_*(X)$ consisting of chains $\xi$ such that $|\epsilon(\xi)|$ and $|\epsilon(\partial\xi)|$ are in stratified general position. We have $G^{\infty,P}_*(X) \subset \Phi^\infty_*(X)$ by Remark 6.2. Let $\mu : G^{\infty,P}_*(X) \to C^\infty_{*-n}(X, \Sigma)$ be the restriction of the intersection product.

Now that we have constructed the intersection product, we can verify its key properties:
**Proposition 6.4.** The inclusion $G^\infty_*P(X) \rightarrow I^\bar{p}_1C^\infty_* (X) \otimes I^\bar{p}_2C^\infty_* (X)$ induces an isomorphism on homology.

**Proof.** As noted in [7, Remark 3.8], this follows directly from the proof of Theorems 3.5 and 3.7 of [7].

**Proposition 6.5.** Let $\bar{p}_1 + \bar{p}_2$ be the perversity that evaluates on the singular stratum $Z$ to $\bar{p}_1(Z) + \bar{p}_2(Z)$. The intersection product $\mu$ restricts to a map $\mu : G^\infty_*P(X) \rightarrow I^{\bar{p}_1 + \bar{p}_2}C^\infty_{s-n}(X)$.

**Proof.** We already know that $\mu : G^\infty_*P(X) \rightarrow C^\infty_{s-n}(X)$ is well defined, so we have only to show that the image lies in $I^{\bar{p}_1 + \bar{p}_2}C^\infty_{s-n}(X)$.

As $G^\infty_*P(X) \subset I^{\bar{p}_1}C^\infty_* (X) \otimes I^{\bar{p}_2}C^\infty_* (X)$, each element $\xi \in G^\infty_*P(X)$ can be written as a sum $\sum a_i \zeta_i \otimes \eta_i$ with $\zeta_i \in I^{\bar{p}_1}C^\infty_* (X)$ and $\eta_i \in I^{\bar{p}_2}C^\infty_* (X)$. Suppose $\zeta_i \in I^{\bar{p}_1}C^\infty_* (X)$ and $\eta_i \in I^{\bar{p}_2}C^\infty_* (X)$ with $j + k = i$, and let $Z$ be a stratum of $X$. Then, by definition, $\dim(\zeta_i \cap Z) \leq j - \codim_X(Z) + \bar{p}_1(Z)$ and similarly for $\eta_i$. Thus

$$\dim(|\epsilon(\zeta_i \otimes \eta_i)| \cap (Z \times Z)) = \dim(|\epsilon(\zeta_i)| \times |\epsilon(\eta_i)| \cap (Z \times Z))$$

$$\leq \dim(|\epsilon(\zeta_i)| \cap (Z \times Z)) \times \dim(|\epsilon(\eta_i)| \cap Z)$$

$$= \sum j - \codim_X(Z) + \bar{p}_1(Z) + k - \codim_X(Z) + \bar{p}_2(Z)$$

$$= i - 2\codim_X(Z) + \bar{p}_1(Z) + \bar{p}_2(Z).$$

It follows that $\dim(|\epsilon(\xi)| \cap (Z \times Z)) \leq i - 2\codim_X(Z) + \bar{p}_1(Z) + \bar{p}_2(Z)$. Therefore, as $|\epsilon(\xi)|$ is in stratified general position by assumption,

$$\dim(|\epsilon(\xi)| \cap \Delta(Z)) \leq i - 2\codim_X(Z) + \bar{p}_1(Z) + \bar{p}_2(Z) - \dim(Z)$$

$$= i - 2\codim_X(Z) + \bar{p}_1(Z) + \bar{p}_2(Z) - (n - \codim_X(Z))$$

$$= i - n - \codim_X(Z) + \bar{p}_1(Z) + \bar{p}_2(Z).$$

Therefore, as $\mu(\xi)$ is supported in $\Delta^{-1}(|\epsilon(\xi)|) = |\epsilon(\xi)| \cap \Delta(X)$, we see that $\mu(\xi)$ satisfies the conditions to be $\bar{p}_1 + \bar{p}_2$ allowable. An equivalent computation shows that $\partial \mu(\xi) = \mu(\partial \xi)$ is also $\bar{p}_1 + \bar{p}_2$ allowable, so $\mu(\xi) \in I^{\bar{p}_1 + \bar{p}_2}C^\infty_{s-n}(X)$.

**Remark 6.6.** Notice that the full power in Proposition 6.5 of the stratified general position assumption is in making sure that $\mu(\xi)$ lies in the intersection chain complex with the smallest possible perversity, $\bar{p}_1 + \bar{p}_2$. Consequently, $\mu(\xi)$ also lies in $I^{\bar{r}}C^\infty_{s-n}(X)$ for all perversities $\bar{r}$ with $\bar{p}_1 + \bar{p}_2 \leq \bar{r}$. In fact, it follows directly from the definitions that $I^\bar{r}C^\infty_* (X) \subset I^qC^\infty_* (X)$ if $\bar{r} \leq q$, meaning that $\bar{r}(\bar{Z}) \leq \bar{q}(\bar{Z})$ for all singular strata $Z$. If our goal is simply to have $\mu(\xi) \in I^{\bar{r}}C^\infty_{s-n}(X)$ for some $\bar{r} \geq \bar{p}_1 + \bar{p}_2$, we could relax the stratified general position requirement accordingly on the singular strata. In fact, this is the case in the original definition of the intersection product for intersection chains in [11 Section 2.1], where, in addition to general position on the regular strata, it is only required for the intersection of a pair of chains to satisfy the dimension requirements recalled in Section 6.1 with respect to $\bar{r}$ (see [11] for full details).
Proposition 6.7. There is a commutative diagram

\[ G^\infty_\ast P(X) \xrightarrow{\mu} I^{p_1+p_2}C^\infty_\ast n(U) \]

where the vertical maps are induced by restriction.

Proof. This follows from Lemmas 5.12 and 5.13 and the observation that the restriction maps \( r : C^\infty_\ast (X, \Sigma_X) \to C^\infty_\ast (U, \Sigma_U) \) take the subcomplex \( I^pC^\infty_\ast (X) \) to \( I^pC^\infty_\ast (U) \) for any perversity \( p \).

For the next proposition, we need the notion of stratified general position for two chains.

Definition 6.8. We say that two chains \( \xi, \eta \) in \( X \) are in stratified general position if \( \varepsilon(\xi \otimes \eta) \) is in stratified general position, i.e. if \( |\xi| \times |\eta| \) is in stratified general position with respect to \( \Delta \); see [7] Proof of Proposition 4.9. As \( (|\xi| \times |\eta|) \cap \Delta(Z) = |\xi| \cap |\eta| \cap Z \) and \( \dim(B \times C) = \dim(B) + \dim(C) \) in general (if neither \( B \) nor \( C \) is empty), this is equivalent to requiring that \( |\xi| \cap Z \) and \( |\eta| \cap Z \) are in general position in \( Z \), i.e. \( \dim(|\xi| \cap |\eta| \cap Z) \leq \dim(|\xi| \cap Z) + \dim(|\eta| \cap Z) - \dim(Z) \).

Proposition 6.9. If \( X \) is compact and \( \bar{p}_1, \bar{p}_2 \) are traditional perversities (see [7], [11]), then \( \mu \) is consistent with the intersection pairing we defined in [7] (replacing the Goresky-MacPherson duality map with the one constructed here). Therefore, if \( \xi \in I^{p_1}C_j(X) \) and \( \eta \in I^{p_2}C_j(X) \) are such that the pairs \( (\xi, \eta), (\xi, \partial \eta) \), and \( (\partial \xi, \eta) \) are in stratified general position, then \( \mu(\xi \otimes \eta) \) is equal to the Goresky-MacPherson intersection product of \( \xi \) and \( \eta \), up to sign.

The restriction on perversities is necessary simply because the intersection pairing in [7] is defined only for traditional perversities.\(^{15}\)

Proof. Given that \( X \) is compact, the entire construction of \( \Delta_i \) in Sections 5.1 and 5.2 can be carried out with a single neighborhood space \( Z \), namely \( X \) itself. In this case, \( D_Z = D_{Z \times Z} = S_Z = \emptyset \), so \( J_Z = \Sigma_Z \) and \( J_{Z \times Z} = (X \times \Sigma_X) \cup (\Sigma_X \times X) = \Sigma_{X \times X} \). So then if \( \xi \in C^\Delta_\ast((X, \Sigma) \times (X, \Sigma)) \), by following Definition 5.3 in this case, the chain \( \Delta_i(\xi) \) corresponds under the isomorphism of Lemma 4.3 to the image of \([\xi]\) under the composition

\[ H_i(|\xi|, |\partial \xi|) \to H_i(|\xi| \cup \Sigma_{X \times X}, |\partial \xi| \cup \Sigma_{X \times X}) \]

\[ \overset{\Delta^*}{\to} H^{2n-i}(X \times X - (|\partial \xi| \cup \Sigma_{X \times X}), X \times X - (|\xi| \cup \Sigma_{X \times X})) \]

\[ \overset{\Delta^*}{\to} H^{2n-i}(X - |\partial \xi'| \cup \Sigma_X, X - (|\xi'| \cup \Sigma_X)) \]

\[ \overset{\Delta^*}{\to} H_{i-n}(|\xi'| \cup \Sigma_X, |\partial \xi'| \cup \Sigma_X). \]

\(^{15}\)Note, however, that without the use of relative chain complexes, which are utilized here but not in [7], there would be some difficulties for large perversities about how the intersection pairing is defined concerning simplices that are contained in \( \Sigma \).
But, up to indexing shifts and the new corrected version of the map $\mathbb{D}$, this is precisely the map defined in Section 4.2 of [7]. The cross product part of the intersection pairing remains identical.

The comparison with the Goresky-MacPherson intersection product is then outlined\footnote{That argument notes $(\alpha \times \beta) \sim (\xi \times \eta) = (-1)^{|\beta|(|\xi|)}(\alpha \times \beta) \sim (\xi \times \eta)$, as provided by [4, §VII.12.17], but does not trace through the full compatibility between cross products and the Dold and Goresky-MacPherson duality maps. However, that is a routine, though tedious, exercise in applying naturality of cross products, as well as the formula just quoted, to the defining diagrams of the duality isomorphisms.} in the proof of Proposition 4.9 of [7].

\section{Duality of intersection products and cup products}

One nice corollary of Lemma \ref{lem:duality_map}, which demonstrated the naturality of the duality map $\mathbb{D}$, is the following result about PL manifolds. For a subset $A \subset M$, let $i_A : H_i(A) \to H_i(M)$ and $i^A : H^i(M, M - A) \to H^i(M)$ be induced by inclusion.

\begin{corollary}
Let $M$ be a compact oriented PL $n$-manifold, and let $\xi$ be a PL $i$-cycle in $M$. The following diagram commutes:

\begin{equation}
\begin{array}{ccc}
H^{n-i}(M, M - |\xi|) & \xrightarrow{i|\xi|} & H^{n-i}(M) \\
\mathbb{D} \downarrow & & \downarrow \mathbb{D} \\
H_i(|\xi|) & \xrightarrow{i|\xi|} & H_i(M).
\end{array}
\end{equation}

In other words, $i|\xi|\mathbb{D} = \mathbb{D}i|\xi|$.
\end{corollary}

In this case, the pair playing the role of “$(K', L')$” in Lemma \ref{lem:duality_map} is simply $(M, \emptyset)$. Furthermore, tracing through the definitions, the vertical map on the right is just the (signed) cap product with the fundamental class. This corollary then demonstrates a compatibility between the dualities we’ve been using to explore chains and classical Poincaré duality: If $\xi$ is an $i$-cycle in $M$, it can be represented by a class $[\xi] \in H_i(|\xi|)$ by Lemma \ref{lem:chain_intersect} and this maps by inclusion to the class represented by $\xi$ in $H_i(M)$. The Poincaré dual in $H^{n-i}(M)$ of this class in $H_i(M)$ is then the image of $\mathbb{D}^{-1}([\xi]) \in H^{n-i}(M, M - |\xi|)$ under the inclusion-induced $H^{n-i}(M, M - |\xi|) \to H^{n-i}(M)$.

Beyond being a pleasant verification of consistency, this observation can be used to see that the Goresky-MacPherson intersection product on homology really is Poincaré dual to the usual cup product pairing on manifolds. In fact, suppose $\xi \in C_i(M)$ and $\eta \in C_j(M)$ are two cycles in general position in the compact oriented PL $n$-manifold $M$. In this case, “stratified” general position reduces to the single general position requirement that $\dim(|\xi| \cap |\eta|) \leq i + j - n$. If $[\xi] \in H_i(|\xi|)$ and $[\eta] \in H_j(|\eta|)$ are the homology classes corresponding to the chains by Lemma \ref{lem:chain_intersect} then the Goresky-MacPherson intersection pairing of \cite[Section 2.1]{36}
(though replacing the Goresky-MacPherson duality map with our map \(D\)) reduces in this case to the chain corresponding to the image of \([\xi] \otimes [\eta]\) under the composition of maps

\[
H_i(|\xi|) \otimes H_j(|\eta|) \xrightarrow{(D \otimes D)^{-1}} H^{n-i}(M, M - |\xi|) \otimes H^{n-j}(M, M - |\eta|)
\]

\[
\xrightarrow{\cong} H^{2n-i-j}(M, M - (|\xi| \cap |\eta|))
\]

\[
\xrightarrow{D} H_{i+j-n}(|\xi| \cap |\eta|).
\]

We denote this intersection product by \(\xi \triangleleft \eta\). It is shown in [11, Theorem 1] that this intersection pairing leads to a well-defined pairing on homology

\[
H_i(|\xi|) \otimes H_j(|\eta|) \rightarrow H_{i+j-n}(M),
\]

using that any two PL cycles are homologous to a pair of cycles in general position and that the homology class of the image depends only on the homology classes of the inputs.

We show that the homology intersection pairing is dual to the cup product pairing. By the Goresky-MacPherson result just cited, it is sufficient to verify this utilizing two representative cycles in general position.

**Theorem 7.2.** Let \(M\) be a compact oriented PL \(n\)-manifold, and let \(\xi \in C_i(M)\) and \(\eta \in C_j(M)\) be PL cycles in general position and represented by classes \([\xi] \in H_i(|\xi|)\) and \([\eta] \in H_j(|\eta|)\). Then we have a commutative diagram

\[
\begin{array}{ccc}
H_i(|\xi|) \otimes H_j(|\eta|) & \xrightarrow{i_{|\xi|} \otimes i_{|\eta|}} & H_i(M) \otimes H_j(M) \\
\downarrow \Phi & & \downarrow (D \otimes D)^{-1} \\
H_{i+j-n}(|\xi| \cap |\eta|) & \xrightarrow{i_{|\xi| \cap |\eta|}} & H_{i+j-n}(M)
\end{array}
\]

**Corollary 7.3.** Let \(M\) be a compact oriented PL manifold of dimension \(n\). The following

\footnote{In fact, more generally, such a pairing is defined between intersection homology groups on PL stratified pseudomanifolds.}
Diagram commutes:

\[
H^{n-i}(M) \otimes H^{n-j}(M) \xrightarrow{\sim} H^{2n-i-j}(M)
\]

\[
\begin{array}{c}
\mathcal{D} \otimes \mathcal{D} \\
\downarrow \\
H_i(M) \otimes H_j(M) \xrightarrow{\cap} H_{i+j-n}(M).
\end{array}
\]

Proof of Theorem 7.2. Applying the definition of \(\cap\) and Corollary 7.1 we have

\[
i_{[\xi] \cap [\eta]}[\xi \cap \eta] = [i_{[\xi] \cap [\eta]} \circ \mathcal{D} \circ \mathcal{D}^{-1}][([\xi] \otimes [\eta])]
\]

\[
= \mathcal{D} \circ i_{[\xi] \cap [\eta]} \circ \mathcal{D}^{-1}([\xi] \otimes [\eta])
\]

\[
= (-1)^n[\mathcal{D} \circ i_{[\xi] \cap [\eta]} \circ (\mathcal{D}^{-1} \otimes \mathcal{D}^{-1})][([\xi] \otimes [\eta])]
\]

\[
= (-1)^{n+|\xi|}[\mathcal{D} \circ i_{[\xi] \cap [\eta]} \circ \mathcal{D}^{-1}([\xi]) \otimes \mathcal{D}^{-1}([\eta])]
\]

\[
= (-1)^{n+|\xi|}[\mathcal{D} \circ (i_{[\xi]} \otimes i_{[\eta]}) \circ \mathcal{D}^{-1}([\xi] \otimes [\eta])]
\]

\[
= \mathcal{D} \circ \mathcal{D}^{-1}([\xi] \otimes [\eta]).
\]

In the fifth line, we use naturality of the cap product coming from the maps \((M; \emptyset, \emptyset) \to (M; M - |\xi|, M - |\eta|)\). In particular, \(i_{[\xi] \cap [\eta]}\) restricts on pairs to \(i_{[\xi]}\) and \(i_{[\eta]}\). To explain the signs, we also recall that if \(f, g\) are chain maps then the Koszul convention has \(f \otimes g\) acting on an element \(x \otimes y\) by \((f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)\), where \(|g|\) is the degree of \(g\) as a chain map and \(|x|\) is the degree of \(x\) as a chain element. It also follows that

\[
\alpha \otimes \beta = (\mathcal{D}^{-1}(\mathcal{D}(\alpha)) \otimes (\mathcal{D}^{-1} \mathcal{D}(\beta))
\]

\[
= (-1)^{|\alpha|+|\beta|}(\mathcal{D}^{-1} \otimes \mathcal{D}^{-1})(\mathcal{D}(\alpha) \otimes (\mathcal{D}(\beta))
\]

\[
= (-1)^{|\alpha|-|\beta|}(\mathcal{D}^{-1} \otimes \mathcal{D}^{-1})(\mathcal{D}(\alpha) \otimes (\mathcal{D}(\beta))
\]

\[
= (-1)^{|\alpha|-|\beta|}(\mathcal{D}^{-1} \otimes \mathcal{D}^{-1})(\mathcal{D} \otimes \mathcal{D})(\alpha \otimes \beta)
\]

\[
= (-1)^{|\alpha|-|\beta|}(\mathcal{D}^{-1} \otimes \mathcal{D}^{-1})(\mathcal{D} \otimes \mathcal{D})(\alpha \otimes \beta),
\]

so \((\mathcal{D} \otimes \mathcal{D})^{-1} = (-1)^{|\alpha|-|\beta|}(\mathcal{D}^{-1} \otimes \mathcal{D}^{-1})\).

The next reasonable question to ask is whether versions of Corollary 7.1 and Theorem 7.2 hold for intersection homology on PL stratified pseudomanifolds, utilizing the intersection homology Poincaré duality isomorphism and intersection homology cup and cap products of [10]. In particular, if \(X\) is a compact oriented PL stratified pseudomanifold and if \(\xi \in I^pC_i(X)\) is a cycle, then we have a version of \(i_{[\xi]}\) that runs \(H_i(|\xi|) \to I^pH_i(X)\) and takes the homology class representing \(\xi\) in \(H_i(|\xi|)\) to the intersection homology class represented by \(\xi\).
in $I^p H_i(X)$. One could then first seek a diagram

$$
\begin{array}{ccc}
H^{n-i}(X, X - |\xi|) & \xrightarrow{i|\xi|} & I^p H^{n-i}(X) \\
\downarrow & & \downarrow \\
H_i(|\xi|) & \xrightarrow{i|\xi|} & I^p H_i(X),
\end{array}
$$

analogous to the diagram of Corollary 7.1 but with the map $\mathbb{D}$ on the right now being the duality isomorphism of [10]; as for the manifold case above, this isomorphism is simply the signed (intersection homology) cap product with the fundamental class. Then one could attempt to proceed on through the argument of Theorem 7.2 using these $i$ maps.

But there are problems. First of all, the map $\mathbb{D}$ on the left here is not well defined because the hypotheses for Proposition 3.6 are not met, as the subspace in the pair $(|\xi|, \emptyset)$ does not contain $\Sigma_X$. It fact, it is easy to construct examples where $H_i(|\xi|)$ and $H^{n-i}(X, X - |\xi|)$ are not even abstractly isomorphic. For example, let $X = S^n \vee S^n$ with some orientation on the spheres, and let $\xi$ be a fundamental class. Then $|\xi| = X$ and $H_n(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, but $H^0(X, X - X) = H^0(X) \cong \mathbb{Z}$. Of course this does not immediately rule out the commutativity of some diagram of the form (6), but it does mean that we cannot define an intersection product by the composition (4).

Indeed, this is not how Goresky and MacPherson define their intersection pairing for intersection chains. Instead, simplifying the construction of [11, Section 2.1] to cycles and using our duality map, their intersection product is determined by the composition

$$
H_i(|\xi|) \otimes H_j(|\eta|) \rightarrow H_i(|\xi| \cup \Sigma, \Sigma) \otimes H_j(|\eta| \cup \Sigma, \Sigma)
$$

$$(\mathbb{D} \otimes \mathbb{D})^{-1} \rightarrow H^{n-i}(X - \Sigma, X - (|\xi| \cup \Sigma)) \otimes H^{n-j}(X - \Sigma, X - (|\eta| \cap \Sigma))
$$

$$(\mathbb{D} \otimes \mathbb{D})^{-1} \rightarrow H^{2n-i-j}(X - \Sigma, X - (|\xi| \cap |\eta|) \cup \Sigma))
$$

$$
|\xi| \otimes |\eta| \otimes \Sigma
$$

by excision.

With the assumptions on the perversities in [11], this last group is isomorphic to $H_{i+j-n}(|\xi| \cap |\eta|)$, though we also know from our work here (see Propositions 6.5 and 6.9) that an element of $H_{i+j-n}((|\xi| \cap |\eta|) \cup \Sigma, \Sigma)$ itself determines an intersection chain if $\xi$ and $\eta$ are intersection chains in stratified general position. Thus it seems that our replacement for the diagram of
Corollary 7.1 might instead be something like

\[ H^{n-i}(X - \Sigma, X - (|\xi| \cup \Sigma)) \to I_\beta H^{n-i}(X) \]

As both \( \mathbb{D} \) maps are now isomorphisms again, we can still talk sensibly about commutativity of this diagram. And, in fact, if \( p(Z) \leq \text{codim}(Z) - 2 \) for all singular strata \( Z \), as is the case for all perversities in [11], then the map \( H_i(|\xi|) \to H_i(|\xi| \cup \Sigma, \Sigma) \) is also an isomorphism: It is equal to the composition

\[ H_i(|\xi|) \to H_i(|\xi|, |\xi| \cap \Sigma) \to H_i(|\xi| \cup \Sigma, \Sigma), \]

the second map being an excision isomorphism and the first being an isomorphism from the long exact sequence of the pair, as the perversity condition ensures \( H_i(|\xi| \cap \Sigma) = H_{i-1}(|\xi| \cap \Sigma) = 0 \).

But now there is a different problem: what is the natural definition of the map \( i_{|\xi|} : H^{n-i}(X - \Sigma, X - (|\xi| \cup \Sigma)) \to I_\beta H^{n-i}(X) \)? There is no obvious chain map \( I_\beta C_*(X) \to C_*(X - \Sigma, X - (|\xi| \cup \Sigma)) \) that would induce this cohomology map. We could conceivably define \( i_{|\xi|} \) via diagram (7) in the case where all vertical maps are isomorphisms, but that’s not good enough for a version of Theorem 7.2 as we have no reason to believe that a collection of \( i \) maps defined this way is natural with respect to the cup product.

So we are stuck.

Now, it is shown in [9] using sheaf-theoretic techniques that there is a duality relation between the Goresky-MacPherson intersection product and the cup product pairing of [10], but we do not currently know how to prove this result without sheaf theory but instead via a method in the spirit of the proof of Theorem 7.2 for manifolds. We therefore can conclude only with a conjecture that such a proof exists by some alternate route.

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