Constructive counterexamples to the additivity of the minimum output Rényi entropy of quantum channels for all \( p > 2 \)

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Abstract
We present a constructive example of the violation of the additivity of minimum output Rényi entropy for each \( p > 2 \). The example is provided by an antisymmetric subspace of a suitable dimension. We discuss the possibility of extension of the result to go beyond \( p > 2 \) and obtain additivity for \( p = 0 \) for a class of entanglement breaking channels.

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1. Introduction
The question whether the minimal output entropy of quantum channels is additive has been open for quite a long time. While the most interesting case is when the entropy is the von Neumann one, the more general Rényi \( p \)-entropies (or equivalently \( p \)-norms) were also studied. Several additivity results were first obtained for the particular classes of channels, including unital qubit channels \([1]\) and entanglement breaking channels \([2]\) for the von Neumann entropy (cf \([3]\) for a more complete list). The first counterexample was obtained for \( p \geq 4.79 \) \([4]\), for the so-called Werner–Holevo channel. Subsequently, Winter \([5]\) proved nonadditivity for all \( p > 2 \) which was pushed by Hayden \([6]\) until all \( p > 1 \) (see also \([7]\)). Finally, Hastings showed nonadditivity for \( p = 1 \) which is the von Neumann entropy case \([8]\) (see \([3, 9]\) in this context). For a concise review of the additivity problem (which does not yet include Hastings’ result), see \([10]\).

The counterexamples to additivity, apart from that of \([4]\), are nonconstructive: they hold for randomly picked channels. The purpose of this paper is to provide explicit counterexamples for all \( p > 2 \). We will have to increase dimension towards infinity, as \( p \) approaches 2, similarly, as in \([5]\). Our channels have the input dimension \( (d^2) \) and the output dimension \(- d\). The channel
2. Constructive counterexamples for \( p > 2 \)

The question of the additivity of the minimum output entropy of a quantum channel is the following. Consider two channels \( \Lambda_1 \) and \( \Lambda_2 \), and fix \( p \) for quantum Rényi entropy \( S_p = \frac{1}{1-p} \log \text{Tr} \rho^p \). Define the minimum output entropy of a channel \( \Lambda \) as

\[
S_p^\min(\Lambda) = \min_{\psi} S_p(\Lambda(|\psi\rangle\langle\psi|)),
\]

where the minimum runs over all pure input states. Then we have additivity if

\[
S_p^\min(\Lambda_1 \otimes \Lambda_2) = S_p^\min(\Lambda_1) + S_p^\min(\Lambda_2),
\]

while additivity is violated when

\[
S_p^\min(\Lambda_1 \otimes \Lambda_2) < S_p^\min(\Lambda_1) + S_p^\min(\Lambda_2)
\]

(3)

(the inequality \( \geq \) always holds).

As explained in [6], using the Stinespring dilation of channel, it is easy to see that the problem of the additivity of output minimal Rényi entropy is equivalent to the following question. We start with the bipartite Hilbert spaces \( \mathcal{H}_{A,B} \), \( \mathcal{H}_{A':B'} \), and consider the subspaces \( \mathcal{H}_{(1)} \subset \mathcal{H}_{A,B} \) and \( \mathcal{H}_{(2)} \subset \mathcal{H}_{A':B'} \), with corresponding projectors \( P^{(1)}_{AB} \) and \( P^{(2)}_{A'B'} \). Then the question is whether for any state \( \psi \) from the subspace \( \mathcal{H}_{(1)} \otimes \mathcal{H}_{(2)} \) with the corresponding projection \( P^{(1)}_{AB} \otimes P^{(2)}_{A'B'} \), we have

\[
S_p(\rho_{BB'}) \geq \min_{\psi_{AB} \in \mathcal{H}_{(1)}} S_p(\rho_B) + \min_{\psi_{A'B'} \in \mathcal{H}_{(2)}} S_p(\rho_B'),
\]

(4)

where \( \rho_{BB'} \) etc are the reduced density matrices of the corresponding pure states. In other words, the question is whether the minimal entanglement of subspace measured by the Rényi entropy of reduction [11] is additive.

Therefore, to construct a counterexample, one should find such projectors \( P^{(1)}_{AB} \) and \( P^{(2)}_{A'B'} \), that

(1) all states from the corresponding subspaces are highly entangled in the cuts \( A : B \) and \( A' : B' \), respectively, and

(2) there exists a state with the support \( P^{(1)}_{AB} \otimes P^{(2)}_{A'B'} \) which has small entanglement in the cut \( AA' : BB' \).

As proposed in [5], it is convenient to take \( \mathcal{H}_{AB} \) isomorphic to \( \mathcal{H}_{A'B'} \), and \( P^{(1)}_{AB} = P^{(2)}_{A'B'} \), where the bar denotes the complex conjugation in the standard product basis. The trial joint state is a maximally entangled state in the \( AA' : BB' \) cut, with local density matrices proportional to the projections.

There is a lemma by Hayden [6], which provides bound for the entanglement of such a state in the \( AA' : BB' \) cut:

**Lemma 1.** Consider a projector \( P = \sum_{i=1}^k |\psi_i\rangle\langle\psi_i| \) on the Hilbert space \( \mathcal{H}_{A} \otimes \mathcal{H}_{B} \), \( k \leq d_A d_B \), with \( d_A, d_B \) being the dimensions of \( \mathcal{H}_{A}, \mathcal{H}_{B} \). Denote

\[
\psi^*(P) = \frac{1}{\sqrt{k}} \sum_{i=1}^k |\psi_i\rangle_{AB} |\psi_i^*\rangle_{A'B'},
\]

(5)
where $A'$ and $B'$ are isomorphic to $A$ and $B$, $^*$ is complex conjugation in the standard product basis of $\mathcal{H}_A \otimes \mathcal{H}_B$. Then the square of the maximal Schmidt coefficient $a_{\text{max}}$ in the cut $AA': BB'$ satisfies

$$\lambda_{\text{max}} \equiv a_{\text{max}}^2 \geq \frac{\dim P}{d_A d_B}. \quad (6)$$

We shall now take $P_{A B} = P_{A' B'} = P_a$ where $P_a$ is a projector onto the antisymmetric subspace. Since $P_{a} = P_{a}^*$, we can apply the above lemma to get

$$\lambda_{\text{max}} \geq \frac{1}{2} \frac{d - 1}{d}. \quad (7)$$

From this we obtain that

$$S_p(\rho_{BB'}) = \frac{1}{1 - p} \log \sum_i \lambda_i^p \leq \frac{p}{1 - p} \log(\lambda_{\text{max}}) \leq \frac{p}{1 - p} \left( \log \frac{d - 1}{d} - 1 \right). \quad (8)$$

To bound $S_p(\rho_B)$ we note that an arbitrary vector from antisymmetric subspace is more entangled than a two-qubit maximally entangled state, in the sense of majorization [12]. Namely, we have

**Proposition 1.** Any vector from antisymmetric subspace satisfies $a_{\text{max}}^2 \leq 1/2$, where $a_{\text{max}}$ is the maximal Schmidt coefficient of the vector.

**Proof.** We shall use the following simple fact (see e.g. [13]): for any Hilbert space $\mathcal{H}$, its subspace $\mathcal{H}'$ and arbitrary vector $\phi \in \mathcal{H}$, we have

$$\sup_{\psi \in \mathcal{H}'} |\langle \psi | \phi \rangle|^2 = \langle \phi | P_{\mathcal{H}'} | \phi \rangle, \quad (9)$$

where $P_{\mathcal{H}'}$ is the projector onto the subspace $\mathcal{H}'$. Thus, it is enough to show that

$$\sup_{\psi \otimes \phi} |\langle \psi \otimes \phi | P_{a s} | \psi \otimes \phi \rangle| \leq 1/2. \quad (9)$$

Writing $P_{as} = \frac{I - V}{2}$ where $V$ is the swap operator and using $\text{Tr}(AB^\Gamma) = \text{Tr}(A^\Gamma B)$ where $\Gamma$ denotes the partial transpose, we obtain the required inequality. The proposition implies that

$$S_p(\rho_B) \geq 1, \quad (10)$$

for any $p$.

Thus, we obtain the violation of additivity if

$$S_p(\rho_{BB'}) < 2S_p(\rho_B), \quad (11)$$

i.e.

$$\frac{p}{p - 1} \log \left( \frac{d}{d - 1} + 1 \right) < 2. \quad (12)$$

Since for any $p < 2$, we have $\frac{p}{p - 1} < 2$, then for any $p < 2$, if we can choose $d$ large enough, this inequality is satisfied. \hfill \square

### 3. Towards generalizations

The simplicity of the provided counterexample is due to the fact that the lower bound for entropy for a single subspace (e.g. $P_{AB}$) is given by majorization (cf [12]), a condition...
which is easy to check, while in the Hastings proof of the non-additivity of minimal von Neumann entropy, the lower bound was the hardest part. Here we shall discuss possible ways to strengthen the result, e.g. by finding some other subspaces with a similar majorization property. We shall first provide a lemma which puts some limitations on the method. Let us introduce the following notation. For a given subspace \( \mathcal{H}_{(1)} \), we define

\[
\lambda_{\max}^{(1)} = \max_{\psi_A \in \mathcal{H}_{(1)}} a_{\max}^2(\psi_{A:B}),
\]

where \( a_{\max}(\psi) \) is the largest Schmidt coefficient of the state \( \psi \). Let also

\[
\lambda_{\max}^{(1:2)} = \max_{\psi_{AA'} \in \mathcal{H}_{(1)} \otimes \mathcal{H}_{(2)}} a_{\max}^2(\psi_{AA':BB'}).
\]

If we take such subspaces, that corresponding projectors are related via complex conjugate, we have by Hayden’s lemma\(^1\) that

\[
\lambda_{\max}^{(1:2)} \geq \frac{d}{d_A d_B},
\]

where \( d = \dim \mathcal{H}_{(1)} \). Now, we will also prove that

**Lemma 2.** With the above notation, we have

\[
\lambda_{\max}^{(1)} \geq \frac{d}{d_A d_B}.
\]

**Proof.** According to formula (9), we have to provide bound for

\[
\sup_{\psi \otimes \phi} \langle \psi \otimes \phi | P | \psi \otimes \phi \rangle.
\]

Obviously, we have

\[
\sup_{\psi \otimes \phi} \langle \psi \otimes \phi | P | \psi \otimes \phi \rangle \geq \langle \langle \psi \otimes \phi | P | \psi \otimes \phi \rangle \rangle,
\]

where \( \langle \cdot \rangle \) denotes the average over product states (obtained by applying random unitaries to a standard one). Computing the average, we obtain

\[
\sup_{\psi \otimes \phi} \langle \psi \otimes \phi | P | \psi \otimes \phi \rangle \geq \frac{\dim P}{d_A d_B}.
\]

This ends the proof. \( \square \)

Using the above lemma it is easy to see that if we want to base on the estimates made in the proof, no subspace that has minimal entanglement equal to 1 (or, even more generally \( \log k \) for integer \( k \)) can lead to violation for \( p \leq 2 \). Indeed, suppose that \( \lambda_{\max}^{(1)} = 1/k \). Then \( S_{\min}^{(1)} = \log \lambda_{\max}^{(1)} \), and by (16) we have

\[
2 \log S_{\min}^{(1)} \leq 2 \log \frac{d}{d_A d_B}.
\]

Now, if we apply the estimate used in (8)

\[
\frac{1}{1-p} \sum_i \lambda_i^p \leq \frac{p}{1-p} \lambda_{\max},
\]

we obtain

\[
\frac{p}{1-p} \log \lambda_{\max}^{(1:2)} \leq 2 \log \frac{d}{d_A d_B}.
\]
If we now use lemma 1 to estimate $\lambda_{\text{max}}^{(1:2)}$, inequality (4) becomes $p/(1 - p) \geq 2$ which can be violated only for $p > 2$.

There is some gap here due to two non-tight estimates: the one of lemma 1 and that of (21). In the case of antisymmetric subspace, this does not enlarge the region of violation, and it seems that we have additivity for $p \leq 2$ in the case of antisymmetric subspace. In general, however, it turns out that if we keep the estimate of lemma 1, there is still a chance for the violation of additivity even for the von Neumann entropy. For example, we get the following sufficient condition for the violation of additivity for the von Neumann entropy.

Additivity is violated if there exist $D$ and $d$ such that in the space $C^D \otimes C^D$ there exists a subspace $\mathcal{H}'$ of dimension $d$, such that

$$2\left(1 - \frac{d}{D^2}\right)\log D + h\left(\frac{d}{D^2}\right) < 2,$$

and any vector in $\mathcal{H}'$ has the entropy of entanglement no less than 1.

In order to violate additivity it would then be enough to get such subspace for which $d/D^2 \to 1$ with entanglement bounded from below by a constant. We have checked numerically the subspaces such as the $(d-1)^2$-dimensional space constructed by Parthasarathy [14] which does not contain any product vector, as well as those provided in [15]—the largest subspaces, which contain only vectors with the Schmidt rank no less than the given value $k$. Unfortunately, the subspaces do not exhibit required properties.

Finally, one may ask the question of multiple additivity, when one takes more than two subspaces (e.g. the Werner–Holevo channel was shown to be multiple-additive for $1 \leq p \leq 2$ in [16]). We have checked numerically the multiple additivity of the antisymmetric subspaces for $p = \infty$ and obtained evidence that the optimal entangled input is bipartite, i.e. even for more copies it does not pay to use multipartite entangled states. For even number of copies, the optimal suggested state is just a product of bipartite maximally entangled states, while for odd number of copies one takes the product of such pairs and a single-partite state of entanglement equal to 1. As an example of the inefficiency of multipartite states, consider $n = d(d-1)/2$ copies of the antisymmetric subspace on the $d \otimes d$ system. One could think that a totally antisymmetric state is a good candidate. However, we checked the cases $d = 3$ and $d = 4$, and obtained that reduced density matrices are proportional to projections, and any $p$-entropy is equal to 4 and 10, respectively, which is larger than the sum of single-copy minimal entropy, being 3 and 6, respectively.

4. On Rényi entropy for $p = 0$ for some entanglement breaking channels

In [2], it was shown that minimum output entropy is additive for $p = 1$ if one of the channels is an entanglement breaking channel. This was extended to all $p \geq 1$ in [17]. Further, King has announced a proof for $0 < p < 1$ [18]. The question whether it is true for $p = 0$ is still open. Here we consider $p = 0$ and a special class of entanglement breaking channels, namely those for which the Choi–Jamiołkowski state is proportional to projection on a subspace spanned by the so-called unextendible product basis [19] (a set of product orthogonal vectors that cannot be extended to such a larger set). Recall that the Choi–Jamiołkowski state of a channel is given by

$$(I \otimes \Lambda)|\Phi_+\rangle|\Phi_+\rangle,$$  

where $|\Phi_+\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle |i\rangle$. To prove additivity, one can e.g. show that all output states of two copies of a channel have maximal rank. We shall apply the following observation of [20] where the first counterexample for $p = 0$ was provided: if the orthogonal complement of the
Choi–Jamiołkowski state for a channel $\Lambda$ has no product vector, then the output of the channel $\Lambda(|\phi\rangle \langle \phi|)$ has the maximal rank $d$ for every input state $|\phi\rangle$.

We shall now take the tensor product of two such Choi–Jamiołkowski states and show that its orthogonal complement does not have a product vector.

To this end, it is enough to show that the product of two unextendible product bases is again an unextendible product basis. However, we are not able to show it for a general unextendible product basis.

According to lemma 1 of [19], we have to show that there is no partition of basis states into two disjoint sets such that the local rank $r_i$ of the $i$th subset is less than the dimension $d_i$ of the $i$th party’s Hilbert space. We consider the bipartite case and assume that the dimensions of Alice and Bob’s Hilbert spaces are equal, i.e. $d_i = d$. Let us take $2(d - 1) + 1$ product states such that any set containing $d$ of them has the following property: it has full rank for both Alice and Bob Hilbert spaces (it is clear that this is minimal set with this property). These states form an unextendible product base, because if we divide them into two disjoint sets, then one set has to contain at least $d$ states.

We want to prove that this construction is closed under the tensor product. We have the following basis states: $|\psi_i\rangle \otimes |\phi_i\rangle$ for $AB$ and $|\psi'_i\rangle \otimes |\phi'_i\rangle$ for $A'B'$. Taking the tensor product we obtain $|\psi_i\rangle |\psi'_j\rangle \otimes |\phi_i\rangle |\phi'_j\rangle$, and for convenience we denote each state by $|ij\rangle$. We can divide these states into $2(d - 1) + 1$ partitions. The $i$th partition contains the states $|ij\rangle$ for each $j$, i.e. it has $2(d - 1) + 1$ elements. Let us take the states of the first partition and divide them into two sets. It is clear that one of these sets contains at least $d$ states. Next, we take the second partition, the third and so on. We see that at the end one set has to contain $d$ partitions which contain at least $d$ elements each. This results from the fact that there are $2(d - 1) + 1$ partitions. Hence, this set has full rank both for Alice and Bob’s Hilbert space. This completes the proof.

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