Einstein-Cartan cosmology and the CMB anisotropies

Davor Palle
ul. Ljudevita Gaja 35, 10000 Zagreb, Croatia
e-mail: davor.palle@gmail.com

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Abstract
We derive linear scalar perturbation equations for Einstein-Cartan field equations of Weyssenhoff fluid, as well as for the corresponding perturbations of Bianchi identity and geodesic equations. The equations are given in both conformal Newtonian and synchronous gauges. They are suitable for numerical implementation when precise evolution of torsion and its perturbation will be extracted from N-body cosmic simulations of the large scale structures in the Universe. A rising number of problems of the concordance cosmological model forces us to include the rotational degrees of freedom realized through torsion in the Einstein-Cartan gravity.

1 Introduction and motivation
Despite the huge success of the concordance ΛCDM model, cosmologists are faced with new challenges of the theory rooted in the surprising observational results known as the Hubble tension, 21cm EDGES anomalous absorption signal or anomalous high-redshift galaxy halo number densities (see [1] and references therein). It seems that some of the above problems could be resolved within the Einstein-Cartan cosmology [1].

Since there are a large number of cosmic observables defined as small perturbations of various physical quantities, we derive in this paper scalar perturbations for the CMB, neutrinos, baryons and CDM within the Einstein-Cartan cosmology.

A detailed description of the framework and all equations can be found in the next chapter and in Appendix A. The last chapter and the Appendix B contain some comments and suggestions for numerical implementation.

2 Linear scalar perturbation equations
We follow closely the definitions of ref. [2] of spatially flat cosmology with metric assignment (-+++) and the standard relation between the proper t and
conformal τ time $d\tau = dt/a(\tau)$, while derivatives are denoted by dots: $\dot{a} \equiv \partial a/\partial \tau$.

All equations will be given in the Fourier k-space with the following definition for any G:

$$G(\vec{x}, \tau) = \int d^3 k e^{i\vec{k} \cdot \vec{x}} G(\vec{k}, \tau).$$

The Einstein-Cartan cosmological model [3, 4, 1] is spatially flat with $\Omega_{tot} = \Omega_m + \Omega_Q + \Omega_\Lambda = 1$, but with very well determinate $\Omega_m = 2$, $\Omega_Q = -1$ and $\Omega_\Lambda = 0$. The gauge invariant perturbed densities in the spacetimes with vorticity or shear have more complex structure [4] than in Friedmann spacetimes. Fortunately, the observations suggest that we can ignore small deviations from isotropy and homogeneity of Friedmann geometry. Thus, we perform perturbations in the conformal Newtonian and synchronous gauges on the Friedmann background [2] with the effective energy-momentum tensor of the Einstein-Cartan (EC) Weyssenhoff fluid model [5]. It is possible to define an effective energy-momentum tensor for any model in the Einstein-Cartan theory of gravity [6].

Appendix A is dedicated to the detailed definitions and discussion of the EC field equations for Weyssenhoff fluid and the corresponding Bianchi identity.

Acknowledging the relations of Appendix A and flat geometry perturbation theory [7, 2] we arrive at the perturbed EC field equations in the conformal Newtonian gauge (equations analogous to eq. (23a)-(23d) of ref.[2]: $\phi$ and $\psi$ are metric perturbations):

$$3\frac{\dot{a}}{a}(\dot{\phi} + \frac{\dot{a}}{a}\dot{\psi}) + k^2 \phi = \frac{a^2}{2} \kappa \delta \rho + a^2 \kappa \delta Q,$$

$$k^2(\phi + \frac{\dot{a}}{a}\dot{\psi}) = \frac{a^2}{2} \kappa (\rho + p) \vec{k} \cdot \vec{v} - a^2 Q^2 \vec{k} \cdot \vec{v},$$

$$\ddot{\phi} + \frac{\dot{a}}{a}(\psi + 2\dot{\phi}) + (2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2})\psi + \frac{k^2}{3}(\phi - \psi) = \frac{1}{2} \kappa a^2 \delta \rho - a^2 Q \delta Q,$$

$$k^2(\phi - \psi) = 12\pi G N a^2(\rho + p) \sigma. \quad (1)$$

The effective energy-momentum tensor appears as:

$$T_{\mu\nu}^{eff} = (p - \kappa S^2 - \Lambda)g_{\mu\nu} + U_\mu U_\nu(p + \rho - 2\kappa S^2) - 2(-g^{\alpha\beta} + U^\alpha U^\beta) \nabla_\alpha [U(\mu S_\nu)_{\beta}],$$

only $Q_{12} = -Q_{21} \neq 0$, $Q^2 = \frac{1}{2} Q_{\mu\nu} Q^{\mu\nu}$, $\kappa = 8\pi G N$, $Q = \kappa S$. \quad (2)

We apply the assumption of isotropy (Friedmann geometry) deriving the above equations neglecting the terms proportional to torsion such as $k_2 v_1 - k_1 v_2$ or $k_1 k_2 (v_1 - v_2)$. Perturbations of the EC field equations in the synchronous gauge contain the same torsion terms as in the conformal Newtonian gauge.

Perturbation of the EC Bianchi identities described in Appendix A leads us to the following equations for CDM and baryon density contrasts and velocity.
gradients in the conformal Newtonian gauge:

\[
\dot{\delta}_c = -\Theta_c + 3\dot{\phi},
\]

\[
\dot{\Theta}_c = \left[1 + \left(-7 + \frac{1}{3}Q^2\right)\right]^{-1}\left\{-\frac{a}{a}\dot{\Theta}_c + k^2\psi\right\} - \frac{1}{\kappa\rho_c}\left[2k^2Q\delta Q + k^2(5 - \frac{1}{3})Q^2\psi - 8Q\dot{Q}\Theta_c - 12Q^2\frac{\dot{a}}{a}\Theta_c\right],
\]

\[
\dot{\delta}_b = -\Theta_b + 3\dot{\phi},
\]

\[
\dot{\Theta}_b = \left[1 + \left(-7 + \frac{1}{3}Q^2\right)\right]^{-1}\left\{-\frac{a}{a}\dot{\Theta}_b + k^2\psi + c^2k^2\delta_b\right\} - \frac{1}{\kappa\rho_b}\left[2k^2Q\delta Q + k^2(5 - \frac{1}{3})Q^2\psi - 8Q\dot{Q}\Theta_b - 12Q^2\frac{\dot{a}}{a}\Theta_b\right] + \frac{4\rho_\gamma}{3\rho_b}a_n\sigma_T(\Theta_\gamma - \Theta_b),
\]

\[
\Theta \equiv \imath\vec{k} \cdot \vec{v}.
\]

Density contrasts in the synchronous gauge do not contain torsion terms, just like in the Newtonian gauge. \(\dot{\Theta}_c\) (synch) vanishes, while \(\dot{\Theta}_b\) (synch) has the same form as \(\dot{\Theta}_b\) (conf), but without terms proportional to \(\psi\). We discard terms that should vanish owing to the isotropy and put \(\imath k_3 v_3 = \frac{1}{3}\imath\vec{k} \cdot \vec{v}\) using the same argument.

The Boltzmann equations for the phase-space distributions require the resolution of the perturbed geodesic equations in the EC cosmology:

\[
P_0 \frac{D^\mu}{d\tau} + \tilde{\Gamma}_{(\nu\kappa)}^\mu P^\nu P^\kappa = 0,
\]

\[
\tilde{\Gamma}_{(\nu\kappa)}^\mu = \left\{\begin{array}{l}
\mu \\
\nu
\end{array}\right\} + Q_{\nu\kappa}^\mu + Q_{\kappa\nu}^\mu, (\mu\nu) = \frac{1}{2}(\mu\nu + \nu\mu),
\]

\[
\tilde{\Gamma}_{\nu\kappa}^\mu = \left\{\begin{array}{l}
\mu \\
\nu
\end{array}\right\} + Q_{\nu\kappa}^\mu + Q_{\kappa\nu}^\mu + Q_{\nu\kappa}^\mu,
\]

\[
torsion tensor = Q_{\nu\kappa}^\mu = \frac{1}{2}(\tilde{\Gamma}_{\nu\kappa}^\mu - \tilde{\Gamma}_{\kappa\nu}^\mu).
\]

We verify that the torsion terms cancel out in the perturbed geodesic equations to linear order in both gauges. As a consequence, the Boltzmann equations for photons, massless and massive neutrinos retain their forms as in the Einstein cosmology (see ref. [2]).

Now we have a complete set of coupled equations for the CDM, baryons, photons and neutrinos in the EC cosmology.

3 Conclusion and comments

Inspecting the form of the perturbation equations in the EC cosmology, one can notice that we need the knowledge not only of the torsion, but also of its
time derivative and its perturbation (in the Zeldovich model is $\delta Q = Q(\Omega_\text{c} + \Omega_\text{m} \delta_\text{c} + \Omega_\text{b} \delta_\text{b})$). We can achieve this objective only with the extensive N-body numerical simulations within the EC cosmology starting at large redshifts with a primordial vorticity of the Universe that causes the nonvanishing angular momentum of the Universe which is a nonrelativistic limit of torsion. In Appendix B we suggest how to improve the numerical codes for the CMB anisotropy calculations.

Recent analysis of a parity violation in polarization data of Planck [8] refers to the right-handed characteristic. We show in ref. [9] that the Universe must have a preference to right-handedness of its vorticity as a consequence of a left-handed weak interactions and the resulting abundant right-handed helicity light Majorana neutrinos.

The appearance of the primordial cosmic magnetic field is a inevitable consequence of the existence of the primordial vorticity. All these new phenomena have to be studied both theoretically and observationally.

**Appendix A**

In this appendix we adopt metric assignment (+ - - -), as well as all definitions and conventions as in ref. [5] with intention that a reader can verify some corrections.

The Riemann-Cartan connection can be expressed as:

$$\tilde{\Gamma}^\alpha_{\beta \mu} = \Gamma^\alpha_{\beta \mu} + Q^\alpha_{\beta \mu} + Q^\alpha_{\beta \mu} + Q^\alpha_{\mu \beta}.$$  

The symmetric part of the EC field equations is:

$$\tilde{R}(\mu \nu) - \frac{1}{2} \tilde{R}g_{\mu \nu} = \kappa T(\mu \nu).$$

The contracted Bianchi identity has the following form (note the wrong sign in eqs. (2.15),(4.13),(4.14) and (4.16) of ref. [5] in front of the Mathisson-Papapetrou force):

$$(\tilde{\nabla}_\nu - 2Q_\nu)T^\nu_\mu + 2Q^\alpha_{\mu \beta}T^\beta_\alpha - S^\alpha_{\alpha \beta} \tilde{R}^\alpha_{\mu \nu} = 0.$$  

In the equation preceding eq.(5.1) of ref. [5] the last term is missing:

$$\tilde{R}(\mu \nu) - \frac{1}{2} \tilde{R}g_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + 2\kappa \nabla_\alpha [u(\mu S^\alpha_\nu)]$$

$$+ \kappa^2 S^2(2u_\mu u_\nu - g_{\mu \nu}) - \kappa g_{\mu \nu} g^{\lambda \psi} \nabla_\alpha [u(\lambda S^\alpha_\psi)].$$

However, the effective energy-momentum tensor in eq.(5.2) is correct. The relation (2.13) of ref. [5] is not generally fulfilled:

$$(\tilde{\nabla}_\alpha - 2Q_\alpha)S^\alpha_{\mu \nu} = T_{[\mu \nu]}.$$  

This is not an obstacle since we have, in general, a free choice to define the energy-momentum tensor [6]. Anyhow, instead of the tensor in eq. (2.19) of ref. [5], we can choose the following one:

$$T_{\mu \alpha} = (u_\mu + z_\mu)P_{\alpha}, \; u^\mu z_\mu = 0, \; z^\mu z_\mu = -1, \; u^\mu P_{\mu} = \rho.$$  

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Inserting the above tensor into the eq.(4), we get the algebraic equations for vector $z^\mu$. However, our choice is the "minimal" tensor eq.(5.2) in ref. [5].

**Appendix B**

A line-of-sight method [10] reduces significantly the time to solve the coupled system of equations with photon anisotropies. Let us write the multipole expansion of the temperature anisotropy:

$$F_\gamma(\tau, k, \mu) = \sum_{l=0}^{\infty} F_{\gamma,l}(\tau, k)(-i)^l(2l + 1)P_l(\mu) .$$

It fulfills the following differential equation:

$$\frac{dF_\gamma}{d\tau} + (ik\mu + \frac{d\kappa}{d\tau})F_\gamma = K_F(\phi, \psi, F_{\gamma,0}, \Theta_b, F_{\gamma,2}, G_{\gamma,0}, G_{\gamma,2}),$$

$K_F$ is well known function, $G_\gamma$ is polarization anisotropy.

The coupled equations are solved up to some $l_\gamma = \mathcal{O}(10)$ and then the rest of multipoles are evaluated by the line-of-sight integrals [10] up to some $l_{max} = \mathcal{O}(1000)$.

To avoid the problems with k-sampling and precision, one can instead separate $F_\gamma$ into the known $\tilde{F}_\gamma(l_\gamma)$ and unknown part $\Delta F_\gamma(l_\gamma)$:

$$F_\gamma(\tau, k, \mu) = \tilde{F}_\gamma(l_\gamma, \tau, k, \mu) + \Delta F_\gamma(l_\gamma, \tau, k, \mu),$$

$$\tilde{F}_\gamma(l_\gamma, \tau, k, \mu) = \sum_{l=0}^{l_\gamma} F_{\gamma,l}(\tau, k)(-i)^l(2l + 1)P_l(\mu),$$

$$\Delta F_\gamma(l_\gamma, \tau, k, \mu) = \sum_{l=l_\gamma+1}^{\infty} F_{\gamma,l}(\tau, k)(-i)^l(2l + 1)P_l(\mu) .$$

The unknown part satisfies the differential equation that has an explicit solution in the form of integrals:

$$\frac{d\Delta F_\gamma}{d\tau} + (ik\mu + \frac{d\kappa}{d\tau})\Delta F_\gamma = K_F - \frac{d\tilde{F}_\gamma}{d\tau} - (ik\mu + \frac{d\kappa}{d\tau})\tilde{F}_\gamma .$$

Namely, for known $P(\tau)$ and $R(\tau)$, the equation:

$$\frac{dy(\tau)}{d\tau} + P(\tau)y(\tau) = R(\tau), \text{ has a solution :}$$

$$y(\tau) = y(\tau_1)\exp[-\int_{\tau_1}^{\tau} P(u)du] + \exp[-\int_{\tau_1}^{\tau} P(u)du] \int_{\tau_1}^{\tau} dt R(t)\exp[\int_{\tau_1}^{t} P(v)dv] .$$

It follows then:

$$F_{\gamma,l}(\tau, k, \mu) = \frac{1}{2}(-i)^{-l} \int_{-1}^{+1} d\mu \Delta F_\gamma(l_\gamma, \tau, k, \mu) P_l(\mu), \text{ for any } l > l_\gamma .$$
Denoting the initial power spectrum with $P_{\text{init}}(k)$, the anisotropy spectrum is obtained:

$$C_l(\tau) = N \int d^3k P_{\text{init}}(k) |F_{\gamma,l}(\tau, k)|^2.$$ 

Thus, instead of performing the double derivatives, as in ref. [10], one has to evaluate double integrals.

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