Optimal convergence rate of the explicit Euler method for convection–diffusion equations II: High dimensional cases

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Abstract
This is the second part of study on the optimal convergence rate of the explicit Euler discretization in time for the convection–diffusion equations [Zhang et al. Appl. Math. Lett. 131 (2022), 108048] which focuses on high-dimensional linear/nonlinear cases under Dirichlet/Neumann boundary conditions. Several new difference schemes are proposed based on the explicit Euler discretization in temporal derivative and centered difference discretization in spatial derivatives. The priori estimate of the improved difference scheme with application to the constant convection coefficients is performed under the maximum norm and the optimal convergence rate four is achieved when the step-ratios along each direction equal to $1/6$. Also we give partial results for the three-dimensional case. The improved difference schemes have essentially improved the CFL condition and the numerical accuracy comparing with the classical difference schemes. Numerical examples involving two-/three-dimensional linear/nonlinear problems under Dirichlet/Neumann boundary conditions such as the Fisher equation, the Chafee–Infante equation and the Burgers’ equation substantiate the good properties claimed for the improved difference scheme.

KEYWORDS
CFL condition, convection–diffusion equation, explicit Euler method, optimal convergence rate, priori estimate
1 | INTRODUCTION

Historically, the explicit Euler method has been one of the oldest and most classical numerical methods, which is compulsory part in the textbooks and monographs, see, for example, [4, 9, 15, 17]. It and other improved versions were frequently used as the time integrator for the numerical solutions of ordinary differential equations or partial differential equations [5, 8, 10, 11, 14, 18].

However, most of the time we ignore a fact that explicit Euler method could generate superconvergence with a specific step-ratio when it is applied to solve the convection–diffusion problems [19]. Subsequently to the result in [19], as the second part of this series of study, we further investigate the optimal convergence rate of the explicit Euler method with application to the convection–diffusion equations in high-dimensional cases.

In this article, we will first consider the numerical procedure for the initial-boundary value problem of the anisotropic convection–diffusion equation with variable convection coefficients as follows

\[
\begin{align*}
    u_t & = \nabla \cdot (A \nabla u) + B \cdot \nabla u + f(x, t), \quad x \in \Omega, \quad t \in (0, T], \quad (1.1a) \\
    u(x, 0) & = \varphi(x), \quad x \in \overline{\Omega}, \quad (1.1b) \\
    u(x, t) & = a(x, t), \quad x \in \Gamma, \quad t \in (0, T], \quad (1.1c)
\end{align*}
\]

where \( x \in \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\), \( A \) is a diagonal matrix with positive diagonal elements and \( B \) is a two- or three-dimensional vector-value function. \( \overline{\Omega} \) is the closure of \( \Omega \) and \( \Gamma \) is the boundary of \( \Omega \). \( f(x, t), \varphi(x) \) and \( a(x, t) \) are given smooth functions satisfying the consistent conditions. Then we will further move our attention to a more general nonlinear convection–diffusion equation

\[
\begin{align*}
    u_t + \nabla \cdot F(u) & = \nabla \cdot (A \nabla u) + r(u), \quad x \in \Omega, \quad t \in (0, T], \quad (1.2a) \\
    u(x, 0) & = \varphi(x), \quad x \in \overline{\Omega}, \quad (1.2b) \\
    u(x, t) & = a(x, t), \quad x \in \Gamma, \quad t \in (0, T], \quad (1.2c)
\end{align*}
\]

where the flux \( F(u) = (f(u), g(u)) \) or \( (f(u), g(u), h(u)) \), which occurs in many applications such as semi-linear or quasi-linear problems: the Fisher equation [13] \((u_t = \Delta u + u(1 - u))\), the Chafee–Infante equation [2] \((u_t = \Delta u + u(1 - u^2))\), the scalar viscous Burgers’ equation [12] \((u_t + uu_x + uu_y = \mu \Delta u, \mu > 0 \) denotes the viscous coefficient), and so on.

Though there have been plenty of discussions for numerical methods to solve these issues, we here focus only on the simplest numerical discretization of the problem (1.1) or (1.2) composing of the standard centered finite difference method for the spatial derivatives and the forward Euler method for the temporal derivative because of its simplicity, time-saving and easy to implement. Scholars generally consider the scheme resulting from such discretization as a low order, conditionally stable and therefore impractical numerical method. However, by applying an improved difference technique to the local truncation errors, we recover the optimal convergence rate four when the specific step-ratios are utilized and the exact solution satisfies a certain of regularity. Our finding makes the improved scheme suitable to simulate a large part of the physical phenomenon with a very high precision. It can also be possible to achieve higher order numerical approximations by increasing the grid points, however, this would lead to increased bandwidth of the discretization matrices and complicates treatments of boundary conditions. These problems do not arise when using a centered finite difference method.

There are several efforts and novelty in the current paper for the improved difference scheme. Specifically:

(I) The explicit Euler method combines with an improved second-order centered difference discretization exports to a superconvergent numerical scheme when the step-ratios \( r_x = r_y = \)}
\( r_z = 1/6 \), where \( r_x \), \( r_y \), and \( r_z \) denote the step-ratios along \( x \)-, \( y \)-, and \( z \)-direction, respectively. Moreover, the improved difference scheme owns higher-order numerical accuracy compared with the standard centered difference scheme even if the step-ratios \((r_x, r_y, r_z) \neq (1/6,1/6,1/6)\).

(II) A priori estimate is demonstrated for the improved difference scheme under the maximum norm in the case of constant convection coefficients, which naturally leads to the CFL condition and optimal convergence for the improved difference scheme. In detail, the stable region for the two-dimensional diffusion problem is doubled compared with the classical difference scheme. For the three-dimensional diffusion problem, the stable region slightly decreases but shows new stable region (triangular pyramid EGIK, see Figure 2 in Section 6).

(III) Most notably, we discover that the improved difference scheme could be extended to solve more general high-dimensional nonlinear convection–diffusion equations such as the Fisher equation, the Chafee–Infante equation and the viscous Burgers’ equation.

(IV) It is worth noting that the improved difference scheme is fully explicit and very convenient to be implemented. Moreover, it is not constrained by the boundary conditions. Numerical examples in a variety of scenarios are carried out for linear/nonlinear problems under Dirichlet and Neumann boundary conditions to confirm the designed convergence rate.

Throughout the whole article, we always assume that the exact solutions to (1.1) or (1.2) satisfy \( u(x, t) \in C^{6,6,4}(\overline{\Omega} \times [0, T]) \) for two-dimensional problems and \( u(x, t) \in C^{6,6,4}(\overline{\Omega} \times [0, T]) \) for three-dimensional problems, respectively.

The rest of the article is arranged as follows. In Section 2, we derive an improved explicit difference scheme for a linear two-dimensional convection–diffusion equation involving several special cases. In Section 3, the priori estimate with constant convection coefficients is discussed and the optimal fourth-order convergence rate is proved. Extending the improved difference scheme to the nonlinear convection–diffusion problem with the Dirichlet boundary conditions and the Neumann boundary conditions are available in Sections 4 and 5, respectively. Our improved technique is also applied to solve a three-dimensional convection–diffusion problem in Section 6. Extensive numerical examples including linear/nonlinear cases under Dirichlet/Neumann boundary conditions are carried out to verify the theoretical results in Section 7 followed by a short concluding remark in Section 8.

## 2 | THE IMPROVED EXPLICIT DIFFERENCE SCHEME

In this section, we focus on the construction of the numerical scheme to solve the two-dimensional convection–diffusion equation (1.1). Here \( x = (x, y) \), \( A = \text{diag}(a, b) \) with \( a \) and \( b \) positive constant diffusion coefficients and \( B = (c(x), d(x)) \) the convection coefficients. The spatial domain is set to be \( \Omega = (L_1, R_1) \times (L_2, R_2) \subset \mathbb{R}^2 \).

We start by introducing some preliminary notations in the context of the finite difference method. First, the domain \( \Omega \times [0, T] \) is subdivided into a number of small elements by passing orthogonal lines through the region. For this purpose, we take three positive integers \( m_1, m_2, n \) and let \( h_x = (R_1 - L_1)/m_1 \), \( h_y = (R_2 - L_2)/m_2 \), \( \tau = T/n \), \( x_i = L_1 + ih_x, 0 \leq i \leq m_1 \), \( y_j = L_2 + jh_y, 0 \leq j \leq m_2 \), \( t_k = kr \), \( 0 \leq k \leq n \). Define the step-ratios \( r_x = a\tau/h_x^2 \), \( r_y = b\tau/h_y^2 \). Denote \( \Omega_{hr} = \{(x_j, t_k) \mid 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq k \leq n \} \) with \( x_j = (x_i, y_j) \), \( \omega = \{(i, j) \mid x_j \in \Omega \} \), \( \gamma = \{(i, j) \mid x_j \notin \Omega \} \) and \( \overline{\omega} = \omega \cup \gamma \). For any grid function \( v = \{v^i_j \mid 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq k \leq n \} \) on \( \Omega_{hr} \), define \( \delta^i v^k_{i,j} = (v^k_{i,j} - v^{k-1}_{i,j})/\tau \), \( \delta^i v^k_{i+1/2,j} = (v^k_{i+1,j} - v^k_{i,j})/h_x \), \( \delta^i v^k_{j} = (v^k_{i+1,j} - 2v^k_{i,j} + v^k_{i-1,j})/h_x^2 \), \( \Delta v^k_{i,j} = (v^k_{i+1,j} - v^k_{i-1,j})/(2h_x) \), \( \|v\|_\infty = \max_{(i,j)\in\omega} |v^i_j| \). Similarly, we could define \( \delta^i v^k_{i,j+1/2} \), \( \delta^i v^k_j \), and \( \Delta v^k_j \).
An improved finite difference scheme for (1.1) reads

\[
\left\{\begin{array}{l}
\delta_{ij}^{k+1} = a + \frac{\tau}{2}(2ac_{ij} + c_{ij}^2)\delta_{ij}^k + b + \frac{\tau}{2}(2bd_{ij} + d_{ij}^2)\delta_{ij}^k \\
+ c_{ij} + \frac{\tau}{2}(ac_{xx} + bc_{yy} + cc_{x} + c_{y}d)\Delta_{x}u_{ij}^k + \frac{\tau}{2}(ad_{xx} + bd_{yy} + cd_{x} + d_{y})\Delta_{y}u_{ij}^k \\
+ \frac{\tau}{2}(2ad_{x} + 2bc_{y} + 2cd_{y})\Delta_{x}\Delta_{y}u_{ij}^k + \frac{\tau}{2}(\alpha + \delta_{ij}^2 \delta_{ij}^k + \tau \Delta_{x}u_{ij}^k + bc_{ij} \delta_{ij}^2 \Delta_{y}u_{ij}^k + p_{ij})^k,
\end{array}\right.\]

(i, j) ∈ ω, 0 ≤ k ≤ n − 1,

\[u_{ij}^0 = \varphi(x_{ij}), \quad (i, j) \in \bar{\omega},\]

\[u_{ij}^k = \alpha(x_{ij}, t_k), \quad (i, j) \in \gamma, 1 \leq k \leq n.\]

The detailed derivation of the difference scheme (2.1a) is postponed to Appendix A. It is easy to see that the local truncation error satisfies

\[
\left\{\begin{array}{l}
O(\tau^2 + h_{x}^4 + h_{y}^4), \quad \text{if } r_x = r_y = 1/6, \\
O(\tau^2 + h_{x}^4 + h_{y}^4), \quad \text{otherwise}.
\end{array}\right.
\]

Several special cases of the difference scheme (2.1) are compared with the corresponding classical difference schemes, see Appendix B.

### 3 THE PRIORI ESTIMATE AND OPTIMAL CONVERGENCE

For brevity, we take the difference scheme (B.3) (see Appendix B) as an example to illustrate the priori estimate. In the meantime, the problem (1.1) becomes

\[
\begin{align*}
&\left\{\begin{array}{l}
u_t = a\nu_{xx} + b\nu_{yy} + cu_x + du_y + f(x, t), & x \in \Omega, \ t \in (0, T], \\
u(x, 0) = \varphi(x), & x \in \bar{\Omega}, \\
u(x, t) = \alpha(x, t), & x \in \Gamma, \ t \in (0, T].
\end{array}\right.
\end{align*}
\]

**Theorem 3.1** (Priori estimate). Let \( \{u_{ij}^k\} \ 0 \leq i \leq m_1, \ 0 \leq j \leq m_2, \ 0 \leq k \leq n \) be the solution of the difference scheme (B.3) with the constraint \( \alpha(x, t) \equiv 0 \). When

\[
\begin{align*}
&\max\{|r_x, r_y\} \leq 1/2, \\
&|c|h_y \leq a \cdot \min\left\{2, \sqrt{1 - 2r_x/(1 - 2r_y)}\right\}, \\
&|d|h_y \leq b \cdot \min\left\{2, \sqrt{1 - 2r_x/(1 - 2r_y)}\right\},
\end{align*}
\]

it holds

\[
\|u^k\|_{\infty} \leq \|\varphi\|_{\infty} + \tau \sum_{i=0}^{k-1}\|r_i\|_{\infty}, \quad 1 \leq k \leq n.
\]

The detailed proof is postponed to Appendix C.

**Remark 3.1.** Based on the analysis in Theorem 3.1, we have the following conclusions.

- The condition (3.2) is just the Courant–Friedrichs–Lewy (CFL) condition. The proof above implies that the difference scheme (B.3) is stable under the CFL condition (3.2).
• When \( c = d = 0 \) in (3.1), the CFL condition (3.2) is simplified to (3.2a). The CFL condition (3.2a) of the improved difference scheme (B.1) for the diffusion problem obviously improves that of (B.2), for which the CFL condition requires \( r_x + r_y \leq 1/2 \). In details, the stability region has doubled for the improved difference scheme (B.1).

Furthermore, we have the convergence result.

**Theorem 3.2.** Let \( \{U^k_{i,j} \mid 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq k \leq n\} \) be the solution of (3.1) and \( \{u^k_{i,j} \mid 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq k \leq n\} \) be the solution of the difference scheme (B.3). Denote \( e^k_{i,j} = U^k_{i,j} - u^k_{i,j} \) (\( 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq k \leq n\)). When (3.2) holds, then there is a constant \( c_1 \) such that

\[
\|e^k\|_\infty \leq \begin{cases} 
   c_1(\tau^2 + h^2_{1,k} + h^2_{2,k}), & \text{if } r_x = r_y = \frac{1}{6}, \ 0 \leq k \leq n, \\
   c_1(\tau^2 + h^2_{3,k} + h^2_{4,k}), & \text{otherwise.}
\end{cases}
\]

The theorem can be proved in combination with Theorem 3.1 and the local truncation error (2.2).

## 4 THE NONLINEAR PROBLEMS

The improved technique can be extended to two-dimensional nonlinear convection–diffusion equations such as semi-linear and quasi-linear parabolic problems. In what follows, we consider the numerical solution of the nonlinear convection–diffusion equation (1.2). The improved difference scheme for (1.2) reads

\[
\delta_t u^{k+1}_{i,j} = \begin{cases} 
   a + \frac{\tau}{2} \left( f'(u^k_{i,j})^2 + 2ar'(u^k_{i,j}) \right) - 2ar''(u^k_{i,j})\Delta_x u^k_{i,j} - arg''(u^k_{i,j})\Delta_y u^k_{i,j} \\
   + \left[ b + \frac{\tau}{2} \left( g'(u^k_{i,j})^2 + 2br'(u^k_{i,j}) \right) - 2brg''(u^k_{i,j})\Delta_x u^k_{i,j} - brf''(u^k_{i,j})\Delta_y u^k_{i,j} \right] \delta^2_x u^k_{i,j} \\
   - \left[ f'(u^k_{i,j}) + \frac{\tau}{2} \left( f''(u^k_{i,j})r(u^k_{i,j}) + 2f'(u^k_{i,j})r'(u^k_{i,j}) \right) \right] \Delta_x u^k_{i,j} \\
   - \left[ g'(u^k_{i,j}) + \frac{\tau}{2} \left( g''(u^k_{i,j})r(u^k_{i,j}) + 2g'(u^k_{i,j})r'(u^k_{i,j}) \right) \right] \Delta_y u^k_{i,j} + r(u^k_{i,j}) \\
   + \tau ab \delta^2_x \delta^2_y u^k_{i,j} - brf'(u^k_{i,j})\Delta_x \delta^2_x u^k_{i,j} - arg'(u^k_{i,j})\Delta_y \delta^2_y u^k_{i,j} \\
   + \frac{\tau}{2} \left[ 2f'(u^k_{i,j})y''(u^k_{i,j}) + ar''(u^k_{i,j}) - af''(u^k_{i,j})\Delta_x u^k_{i,j} - ag''(u^k_{i,j})\Delta_y u^k_{i,j} \right] (\Delta_x u^k_{i,j})^2 \\
   + \frac{\tau}{2} \left[ 2g'(u^k_{i,j})y''(u^k_{i,j}) + br''(u^k_{i,j}) - bg''(u^k_{i,j})\Delta_y u^k_{i,j} - bf''(u^k_{i,j})\Delta_x u^k_{i,j} \right] (\Delta_y u^k_{i,j})^2 \\
   + \tau \left( f'(u^k_{i,j})g'(u^k_{i,j}) - ag''(u^k_{i,j})\Delta_y u^k_{i,j} - bf''(u^k_{i,j})\Delta_x u^k_{i,j} \right) \Delta_x u^k_{i,j} \Delta_y u^k_{i,j} \\
   + \tau \left( f''(u^k_{i,j})g'(u^k_{i,j}) + f'(u^k_{i,j})g''(u^k_{i,j}) \right) \Delta_x u^k_{i,j} \cdot \Delta_y u^k_{i,j} + \frac{\tau}{2} r'(u^k_{i,j})r(u^k_{i,j}), \\
   (i,j) \in \omega, \ 0 \leq k \leq n - 1,
\end{cases}
\]

\[
u^0_{i,j} = \varphi(x_{i,j}), \quad (i,j) \in \omega, \quad (4.1a)
\]

\[
u^k_{i,j} = \alpha(x_{i,j}, t_k), \quad (i,j) \in \gamma, \ 1 \leq k \leq n. \quad (4.1b)
\]

The detailed derivation of the difference scheme (4.1) is moved to Appendix D for brevity.
5 | PROBLEMS UNDER NEUMANN BOUNDARY CONDITIONS

In this section, we extend our idea to the general nonlinear convection–diffusion equation under the Neumann boundary value conditions as

\[
\begin{aligned}
    &u_t + f(u)_x + g(u)_y = au_{xx} + bu_{yy} + r(u), \quad x \in \Omega, \; t \in (0, T], \\
    &u(x, 0) = \varphi(x), \quad x \in \bar{\Omega}, \\
    &u_s(L_1, y, t) = a_1(y, t), \quad u_s(R_1, y, t) = a_2(y, t), \quad y \in [L_2, R_2], \; t \in (0, T], \\
    &u_y(x, L_2, t) = \beta_1(x, t), \quad u_y(x, R_2, t) = \beta_2(x, t), \quad x \in (L_1, R_1), \; t \in (0, T].
\end{aligned}
\]  

(5.1a)

(5.1b)

(5.1c)

(5.1d)

Since (5.1a) is the same to (1.2a), it is necessary to discrete the boundary value conditions (5.1c) and (5.1d) with a fourth-order algorithm. For example, based on the fourth-order backward difference formula [4], we have the discrete schemes for the boundary value conditions (5.1c) and (5.1d) as

\[
\begin{aligned}
    &1 \frac{h_x}{12} \left( -25 u_{0j}^k + 4 u_{1j}^k + 3 u_{2j}^k - \frac{4}{3} u_{3j}^k - \frac{1}{4} u_{4j}^k \right) = a_1(y_j, t_k), \quad 0 \leq j \leq m_2, \; 0 \leq k \leq n, \\
    &\frac{h_x}{1} \left( -\frac{1}{4} u_{m_1-1,j}^k - \frac{4}{3} u_{m_1-2,j}^k - 4 u_{m_1-3,j}^k - \frac{25}{12} u_{m_1,j}^k \right) = a_2(y_j, t_k), \quad 0 \leq j \leq m_2, \; 0 \leq k \leq n, \\
    &\frac{h_y}{1} \left( -25 u_{i0}^k + 4 u_{i1}^k - 3 u_{i2}^k + \frac{4}{3} u_{i3}^k - \frac{1}{4} u_{i4}^k \right) = \beta_1(x_i, t_k), \quad 1 \leq i \leq m_1 - 1, \; 0 \leq k \leq n, \\
    &\frac{h_y}{1} \left( -\frac{1}{4} u_{im_2-1}^k - \frac{4}{3} u_{im_2-2}^k - 4 u_{im_2-3}^k - \frac{25}{12} u_{im_2}^k \right) = \beta_2(x_i, t_k), \quad 1 \leq i \leq m_1 - 1, \; 0 \leq k \leq n.
\end{aligned}
\]  

(5.2a)

(5.2b)

(5.2c)

(5.2d)

In the part of the numerical implement, we compute the numerical solutions at \((k + 1)\)th-level by the following three steps. The sketch map is referred to Figure 1:

(i) Compute the numerical solutions on the solid points by (4.1) at \(k\)th-level;

(ii) Compute the numerical solutions on the hollow points by (5.2a)–(5.2d) and the numerical solutions on solid points;

(iii) Compute the numerical solutions on the star points by (5.2a) and (5.2b) and the numerical solutions on hollow points.

6 | THREE-DIMENSIONAL DIFFUSION PROBLEM

We consider a three-dimensional convection–diffusion problem as

\[
\begin{aligned}
    &u_t = \kappa_1 u_{xx} + \kappa_2 u_{yy} + \kappa_3 u_{zz} + \lambda_1 u_x + \lambda_2 u_y + \lambda_3 u_z + f(x, t), \quad x \in \Omega, \; t \in (0, T], \\
    &u(x, 0) = \varphi(x), \quad x \in \bar{\Omega}, \\
    &u(x, t) = \alpha(x, t), \quad x \in \Gamma, \; t \in (0, T],
\end{aligned}
\]  

(6.1a)

(6.1b)

(6.1c)

where \(x = (x, y, z)\) and \(\kappa_i > 0 \; (i = 1, 2, 3)\) denote constant diffusion coefficients. The spatial domain is set to be \(\Omega = (L_1, R_1) \times (L_2, R_2) \times (L_3, R_3) \subset \mathbb{R}^3\).
To ease the notation, we only state the details for \( \lambda_i = 0 \) \((i = 1, 2, 3)\). In addition to the notation defined in Section 2, we take one more integer \( m_1 \) and let \( h_z = (R_3 - L_3)/m_3, z_l = L_3 + lh_z, 0 \leq l \leq m_3, \) \( r_z = c\tau/h_z^2 \). Denote \( \Omega_{ht} = \{(x_{ijl}, t_k) | 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq l \leq m_3, 0 \leq k \leq n\} \) with \( x_{ijl} = (x_i, y_j, z_l), \omega = \{(i, j, l) | x_{ijl} \in \Omega\}, \gamma = \{(i, j, l) | x_{ijl} \in \Gamma\} \) and \( \omega = \omega \cup \gamma \). For any grid function \( \nu = \{\nu_{ijl} | 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq l \leq m_3, 0 \leq k \leq n\} \) on \( \Omega_{ht} \), define \( \delta_i \nu_{ijl} = (\nu_{ijl}^{k+1/2} - \nu_{ijl}^k)/\tau, \) \( \delta_x^2 \nu_{ijl} = (\nu_{ijl}^{k+1,j} - 2\nu_{ijl}^{k,j} + \nu_{ijl}^{k-1,j})/h_x^2 \). Analogously, we could define \( \delta_y^2 \nu_{ijl}^k \) and \( \delta_z^2 \nu_{ijl}^k \). Denote \( f_{ijl}^k = f(x_{ijl}, t_k) \) and define the grid function \( U = \{U_{ijl}^k | 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq l \leq m_3, 0 \leq k \leq n\} \) with \( U_{ijl}^k = u(x_{ijl}, t_k) \).

Considering (6.1a) at the point \((x_{ijl}, t_k)\) and similar to that of two dimensions, we have

\[
\begin{align*}
\delta_t U^k_{ijl} &- \kappa_1 \delta_x^2 U^k_{ijl} - \kappa_2 \delta_y^2 U^k_{ijl} - \kappa_3 \delta_z^2 U^k_{ijl} \\
&= f_{ijl}^k + \left( \frac{\tau}{2} u_{nn} - \frac{\kappa_1 h_x^2}{12} u_{xxxx} - \frac{\kappa_2 h_y^2}{12} u_{yyyy} - \frac{\kappa_3 h_z^2}{12} u_{zzzz} \right) (x_{ijl}, t_k) + O(\tau^2 + h_x^4 + h_y^4 + h_z^4) \\
&= p_{ijl}^k + \left( \frac{\kappa_1 h_x^2}{2} \left( r_x - \frac{1}{6} \right) u_{xxxx} + \frac{\kappa_2 h_y^2}{2} \left( r_y - \frac{1}{6} \right) u_{yyyy} + \frac{\kappa_3 h_z^2}{2} \left( r_z - \frac{1}{6} \right) u_{zzzz} \right) (x_{ijl}, t_k) \\
&+ \tau \kappa_1 \delta_x^2 \delta_z^2 U_{ijl}^k + \tau \kappa_2 \delta_y^2 \delta_z^2 U_{ijl}^k + \tau \kappa_3 \delta_x^2 \delta_y^2 U_{ijl}^k + O(\tau^2 + h_x^4 + h_y^4 + h_z^4) + \gamma, \end{align*}
\]

where

\[
p_{ijl}^k = f_{ijl}^k + \frac{\tau}{2} \left[ a(f_{xx})_{ijl}^k + b(f_{yy})_{ijl}^k + c(f_{zz})_{ijl}^k + (f_{ij})_{ijl}^k \right]
\]

with \((f_{xx})_{ijl}^k = f_{xx}(x_{ijl}, t_k), (f_{yy})_{ijl}^k = f_{yy}(x_{ijl}, t_k), (f_{zz})_{ijl}^k = f_{zz}(x_{ijl}, t_k), (f_{ij})_{ijl}^k = f_t(x_{ijl}, t_k)\).

Therefore, an improved difference scheme for (6.1) is constructed as

\[
\begin{align*}
\delta_t U^k_{ijl} &= \kappa_1 \delta_x^2 U_{ijl}^k + \kappa_2 \delta_y^2 U_{ijl}^k + \kappa_3 \delta_z^2 U_{ijl}^k + p_{ijl}^k \\
&+ \tau \kappa_1 \delta_x^2 \delta_z^2 U_{ijl}^k + \tau \kappa_2 \delta_y^2 \delta_z^2 U_{ijl}^k + \tau \kappa_3 \delta_x^2 \delta_y^2 U_{ijl}^k, \quad (i, j, l) \in \omega, \quad 0 \leq k \leq n - 1, \quad (6.2a) \\
u_{ijl}^0 &= \varphi(x_{ijl}), \quad (i, j, l) \in \omega, \quad (6.2b) \\
u_{ijl}^k &= \alpha(x_{ijl}, t_k), \quad (i, j, l) \in \gamma, \quad 0 \leq k \leq n. \quad (6.2c)
\end{align*}
\]
Similar to the proof in two dimensions, we have the convergence result for the improved difference scheme (6.2) under the maximum norm.

**Theorem 6.1.** Let \( \{ U_{ijl}^k \mid 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq l \leq m_3, 0 \leq k \leq n \} \) be the solution of (6.1) with \( \lambda_i = 0 \) (\( i = 1, 2, 3 \)) and \( \{ u_{ijl}^k \mid 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq l \leq m_3, 0 \leq k \leq n \} \) be the solution of the improved difference scheme (6.2). Denote \( e_{ijl}^k = U_{ijl}^k - u_{ijl}^k \), \((0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq l \leq m_3, 0 \leq k \leq n)\). When

\[
\max \{ r_x, r_y, r_z \} \leq 1/4, \tag{6.3}
\]

then there is a constant \( c_2 \) such that

\[
\| e^k \|_\infty \leq \begin{cases} 
  c_2(r^2 + h^3_x + h^3_y + h^3_z), & \text{when } r_x = r_y = r_z = 1/6, \ 0 \leq k \leq n, \\
  c_2(r^2 + h^3_x + h^3_y + h^3_z), & \text{otherwise}.
\end{cases}
\]

**Remark 6.1.** The following results are easily obtained from the proof of the convergence.

- (6.3) is also the CFL condition for the stability of the improved difference scheme (6.2).
- The CFL condition of the classical difference scheme

\[
\begin{aligned}
\delta_t u_{ijl}^{k+1} &= \kappa_1 \delta_x^2 u_{ijl}^k + \kappa_2 \delta_y^2 u_{ijl}^k + \kappa_3 \delta_z^2 u_{ijl}^k + p_{ijl}^k, \tag{6.4a} \\
u_{ijl}^0 &= \varphi(x_{ijl}), \quad (i, j, l) \in \tilde{\omega}, \tag{6.4b} \\
u_{ijl}^k &= \alpha(x_{ijl}, t_k), \quad (i, j, l) \in \gamma, \ 0 \leq k \leq n. \tag{6.4c}
\end{aligned}
\]

requires \( r_x + r_y + r_z \leq 1/2 \), see, for example, [3]. The stability regions between the improved difference scheme (cube) and the classical difference scheme (rectangular triangular pyramid) are shown in Figure 2. Suppose the length of the right edge of the triangular pyramid (OABC) is one half, we easily knows that the stability region decreases from 1/48 of the triangular pyramid (OABC) to 1/64 of the cube (OHJIE-FGK). However, the improved difference scheme shows a new stability region in the triangular pyramid (EGIK).

- For the convection–diffusion problem with constant convection coefficients (6.1) in three dimensions, the CFL condition becomes

\[
\max \{ r_x, r_y, r_z \} \leq 1/4, \quad \begin{cases} 
|\lambda_1|h_x \leq \kappa_1 \cdot \min \left\{ 2, \sqrt{\frac{(1 - 2r_x)(1 - 2r_y)(1 - 2r_z)}{r_x}} \right\}, \\
|\lambda_2|h_y \leq \kappa_2 \cdot \min \left\{ 2, \sqrt{\frac{(1 - 2r_x)(1 - 2r_y)(1 - 2r_z)}{r_y}} \right\}, \\
|\lambda_3|h_z \leq \kappa_3 \cdot \min \left\{ 2, \sqrt{\frac{(1 - 2r_x)(1 - 2r_y)(1 - 2r_z)}{r_z}} \right\}.
\end{cases}
\]

We omit details here for sake of brevity.

7 | NUMERICAL EXAMPLES

In this section, we verify the accuracy and test the CFL condition of the numerical simulation for several problems including linear and nonlinear cases in different scenarios. We first focus on the two-dimensional case. For this purpose, we denote the difference solution \( \{ u_{ij}^k \} \) with the chosen fixed spacial step sizes \((h_x, h_y)\) and temporal step size \( \tau \) by \( \{ u_{ij}^k(h_x, h_y, \tau) \} \). Similarly, \( \{ u_{ij}^k(h_x/2, h_y/2, \tau/4) \} \)
denotes the difference solution with the chosen fixed spatial step sizes \((h_x/2, h_y/2)\) and temporal step size \(\tau/4\). Let \(r_x = r_y := r\). The numerical errors in \(L^\infty\)-norm and global convergence orders are defined as

\[
\text{Ord}_G = \begin{cases} 
\log_2 \left( \frac{E_\infty(h_x, h_y, \tau)}{E_\infty(h_x/2, h_y/2, \tau/4)} \right), & \text{if the exact solution is known;} \\
\log_2 \left( \frac{E_\infty(h_x, h_y, \tau)}{E_\infty(h_x/2, h_y/2, \tau/4)} \right), & \text{where } E_\infty(h_x, h_y, \tau) = \max_{(x_j, t_k) \in \Omega} |u(x_j, t_k) - u_j^k(h_x, h_y, \tau)| \\
-u_{2,2}^k(h_x/2, h_y/2, \tau/4), & \text{if the exact solution is unknown.}
\end{cases}
\]  

(7.1)

In the numerical implementation, we first fix the step-ratio \(r\) and temporal step size \(\tau\), then determine \(h_x\) and \(h_y\) by \(r\) and \(\tau\).

Similarly, the numerical errors and global convergence orders for the three-dimensional case could be defined. The step-ratio is abbreviated as “R” and the grid parameter as “P.”

7.1  |  Linear problems

**Example 7.1.** First, the model problem (1.1) with \(c = d = 0\) is solved by the improved difference scheme (B.1) and classical Euler difference method (B.2), respectively, with the parameters \(L_1 = L_2 = 0, R_1 = R_2 = 1, T = 1\). The initial value condition, the boundary value conditions and the source term \(f(x, t)\) are determined by the exact solution \(u(x, t) = \exp(1/2(x+y) - t)\), see, for example, [15].

Two sets of diffusion coefficients are used. Numerical results are listed in Tables 1 and 2.

**Case I:** \(a = 4, b = 1\); **Case II:** \(a = 1, b = 0.0001\).
TABLE 1 The errors in $L^\infty$-norm versus grid sizes reduction and convergence orders of the improved difference scheme (B.1) for the linear diffusion equation in Example 7.1.

| Case I | Case II |
|--------|---------|
| \(R\) | \(P\) | \(m_1\) | \(m_2\) | \(n\) | \(E_{\infty}(h_x,h_y,\tau)\) | \(\text{Ord}_G\) | \(m_1\) | \(m_2\) | \(n\) | \(E_{\infty}(h_x,h_y,\tau)\) | \(\text{Ord}_G\) |
| \(r = 1/7\) | | 5 | 10 | 700 | 2.2598e-6 | - | 5 | 500 | 175 | 3.7141e-6 | - |
| | | 10 | 20 | 2800 | 5.7458e-7 | 1.9756 | 10 | 1000 | 700 | 8.4007e-7 | 2.1444 |
| | | 20 | 40 | 11,200 | 1.4348e-7 | 2.0016 | 20 | 2000 | 2800 | 2.0296e-7 | 2.0493 |
| | | 40 | 80 | 44,800 | 3.5942e-8 | 1.9971 | 40 | 4000 | 11,200 | 5.0363e-8 | 2.0107 |
| | | 80 | 160 | 179,200 | 8.9853e-9 | 2.0000 | 80 | 8000 | 44,800 | 1.2563e-8 | 2.0032 |
| \(r = 1/6\) | | 5 | 10 | 600 | 2.0937e-8 | - | 5 | 500 | 150 | 7.8615e-7 | - |
| | | 10 | 20 | 2400 | 1.3370e-9 | 3.9690 | 10 | 1000 | 600 | 5.0407e-8 | 3.9631 |
| | | 20 | 40 | 9600 | 8.3554e-11 | 4.0001 | 20 | 2000 | 2400 | 3.1499e-9 | 4.0002 |
| | | 40 | 80 | 38,400 | 5.1935e-12 | 4.0079 | 40 | 4000 | 9600 | 1.9706e-10 | 3.9986 |
| | | 80 | 160 | 153,600 | 1.6276e-13 | 4.9959 | 80 | 8000 | 38,400 | 1.2082e-11 | 4.0277 |
| \(r = 1/5\) | | 5 | 10 | 500 | 3.1255e-6 | - | 5 | 500 | 125 | 3.2597e-6 | - |
| | | 10 | 20 | 2000 | 8.0197e-7 | 1.9624 | 10 | 1000 | 500 | 1.0517e-6 | 1.6320 |
| | | 20 | 40 | 8000 | 2.0072e-7 | 1.9983 | 20 | 2000 | 2000 | 2.7637e-7 | 1.9281 |
| | | 40 | 80 | 32,000 | 5.0310e-8 | 1.9963 | 40 | 4000 | 8000 | 7.0022e-8 | 1.9807 |
| | | 80 | 160 | 128,000 | 1.2579e-8 | 1.9998 | 80 | 8000 | 32,000 | 1.7558e-8 | 1.9957 |
| \(r = 1/2\) | | 5 | 10 | 200 | 3.2085e-5 | - | 5 | 500 | 50 | 3.6891e-5 | - |
| | | 10 | 20 | 800 | 8.0721e-6 | 1.9909 | 10 | 1000 | 200 | 1.0791e-5 | 1.7734 |
| | | 20 | 40 | 3200 | 2.0106e-6 | 2.0054 | 20 | 2000 | 800 | 2.7808e-6 | 1.9563 |
| | | 40 | 80 | 12,800 | 5.0330e-7 | 1.9981 | 40 | 4000 | 3200 | 7.0129e-7 | 1.9874 |
| | | 80 | 160 | 51,200 | 1.2580e-7 | 2.0003 | 80 | 8000 | 12,800 | 1.7565e-7 | 1.9973 |
| \(r = 1/1.99\) | | 5 | 10 | 199 | 3.2334e-5 | - | 5 | 500 | 50 | 9.5017e-5 | - |
| | | 10 | 20 | 796 | 8.1327e-6 | 1.9912 | 10 | 1000 | 199 | 1.0872e-5 | 3.1276 |
| | | 20 | 40 | 3184 | 4.4720e-4 | −5.7810 | 20 | 2000 | 796 | 2.8052e-6 | 1.9544 |
| | | 40 | 80 | 12,736 | 5.5224e+76 | −266.06 | 40 | 4000 | 3184 | 5.0471e+9 | −50.676 |
| | | 80 | 160 | 50,944 | ln | −ln | 80 | 8000 | 12,736 | 1.5555e+9 | −267.38 |

Note: Bold indicates the numerical superiority of the step-ratio \(r = 1/6\).

In Case I, we see clearly that when the step-ratio \(r = 1/6\), the improved difference scheme (B.1) obtains the fourth-order accuracy approximatively. Otherwise, it is only second-order accurate globally. When the step-ratio increases to \(r = 1/2\) gradually, the improved difference scheme (B.1) still works. However, the classical difference scheme (B.2) fails even if the step-ratio is \(r = 1/3.99\) because of the restriction of the CFL condition. These numerical observations are surprisingly consistent with the theoretical results. Moreover, the numerical results in Table 1 are more accurate than those in Table 2 even using the same step-ratio, which display the advantage of the improved Euler difference scheme (B.1). In Case II, similar results to Case I are observed from the third column in Tables 1 and 2 even though the diffusion coefficient varies greatly from direction to direction.

Example 7.2. Then we consider the problem (1.1) by the improved difference scheme (B.2) and the classical difference scheme (B.3). The parameters are taken as \(L_1 = L_2 = 0\), \(R_1 = R_2 = 1\) and \(T = 1\). The initial value condition, the boundary value conditions and
the source term are determined by the exact solution \( u(x,t) = \exp(x+y-t) \). We take coefficients \( a, \ b, \ c, \ d \) as

**Case I:** \( a = 4, \ b = 1, \ c = -10, \ d = 20; \quad \textbf{Case II:} \ a = 1, \ b = 0.01, \ c = -1, \ d = 2.\)

Numerical results are listed in Tables 3 and 4. When the step-ratio is 1/6, the convergence rate is approximately fourth-order for the above two sets of parameters, which confirm the theoretical findings. For other step-ratios, the convergence rate is only two globally in both cases. Moreover, the improved difference scheme (B.2) is more accurate than the classical difference scheme (B.3) whatever the step sizes are, which displays the superiority of the improved difference scheme (B.2).

**Example 7.3.** We test the convergence rate and solution behavior to the problem (1.1) with the variable coefficients \( c(x) = \sin(x+y) \) and \( d(x) = \cos(x+y) \) by the difference scheme (2.1). The initial condition is taken as \( \varphi(x) = \exp(-x^2-y^2) \) and boundary value conditions are homogeneous. The exact solution is unknown. The parameters are taken as \( L_1 = L_2 = -5, \ R_1 = R_2 = 5, \ T = 1 \) and the diffusion coefficients are respectively taken as

### Table 2

| \( P \) | \( \text{Case I} \) | \( \text{Case II} \) |
|------|----------------|-----------------|
| \( R \) | \( m_1 \) | \( m_2 \) | \( n \) | \( E_{\infty}(h_x,h_y,\tau) \) | \( \text{Ord}_{G} \) | \( m_1 \) | \( m_2 \) | \( n \) | \( E_{\infty}(h_x,h_y,\tau) \) | \( \text{Ord}_{G} \) |
| \( r = 1/7 \) | 5 | 10 | 700 | 3.0144e-6 | - | 5 | 500 | 175 | 2.7831e-4 | - |
| | 10 | 20 | 2800 | 7.7458e-7 | 1.9604 | 10 | 1000 | 700 | 7.1417e-5 | 1.9623 |
| | 20 | 40 | 11,200 | 1.9395e-7 | 1.9977 | 20 | 2000 | 2800 | 1.7854e-5 | 2.0000 |
| | 40 | 80 | 44,800 | 4.8617e-8 | 1.9961 | 40 | 4000 | 11,200 | 4.4692e-6 | 1.9981 |
| | 80 | 160 | 179,200 | 1.2156e-8 | 1.9998 | 80 | 8000 | 44,800 | 1.1173e-6 | 2.0000 |
| \( r = 1/6 \) | 5 | 10 | 600 | 9.2716e-7 | - | 5 | 500 | 150 | 3.2822e-4 | - |
| | 10 | 20 | 2400 | 2.3637e-7 | 1.9718 | 10 | 1000 | 600 | 8.4248e-5 | 1.9620 |
| | 20 | 40 | 9600 | 5.9067e-8 | 2.0006 | 20 | 2000 | 2400 | 2.1063e-5 | 1.9999 |
| | 40 | 80 | 38,400 | 1.4799e-8 | 1.9969 | 40 | 4000 | 9600 | 5.2727e-6 | 1.9981 |
| | 80 | 160 | 153,600 | 3.6999e-9 | 1.9999 | 80 | 8000 | 38,400 | 1.3182e-6 | 2.0000 |
| \( r = 1/5 \) | 5 | 10 | 500 | 1.9943e-6 | - | 5 | 500 | 125 | 3.9804e-4 | - |
| | 10 | 20 | 2000 | 5.1709e-7 | 1.9474 | 10 | 1000 | 500 | 1.0221e-4 | 1.9614 |
| | 20 | 40 | 8000 | 1.2977e-7 | 1.9945 | 20 | 2000 | 2000 | 2.5556e-5 | 1.9998 |
| | 40 | 80 | 32,000 | 3.2546e-8 | 1.9953 | 40 | 4000 | 8000 | 6.3975e-6 | 1.9981 |
| | 80 | 160 | 128,000 | 8.1388e-9 | 1.9998 | 80 | 8000 | 32,000 | 1.5994e-6 | 2.0000 |
| \( r = 1/4 \) | 5 | 10 | 400 | 6.3753e-6 | - | 5 | 500 | 100 | 5.0264e-4 | - |
| | 10 | 20 | 1600 | 1.6472e-6 | 1.9525 | 10 | 1000 | 400 | 1.2914e-4 | 1.9606 |
| | 20 | 40 | 6400 | 4.1301e-7 | 1.9958 | 20 | 2000 | 1600 | 3.2294e-5 | 1.9996 |
| | 40 | 80 | 25,600 | 1.0356e-7 | 1.9957 | 40 | 4000 | 6400 | 8.0846e-6 | 1.9980 |
| | 80 | 160 | 102,400 | 2.5897e-8 | 1.9997 | 80 | 8000 | 25,600 | 2.0212e-6 | 2.0000 |
| \( r = 1/3.99 \) | 5 | 10 | 399 | 6.4302e-6 | - | 5 | 500 | 100 | 4.3540e-4 | - |
| | 10 | 20 | 1596 | 1.6614e-6 | 1.9525 | 10 | 1000 | 399 | 1.2948e-4 | 1.7496 |
| | 20 | 40 | 6384 | 4.1656e-7 | 1.9958 | 20 | 2000 | 1596 | 3.2379e-5 | 1.9996 |
| | 40 | 80 | 25,536 | 2.1130e+18 | -82.069 | 40 | 4000 | 6384 | 1.1756e-5 | 1.4617 |
| | 80 | 160 | 102,144 | 1.0174e+18 | -550.39 | 80 | 8000 | 25,536 | 3.4068e+34 | -131.09
TABLE 3  The errors in $L^{\infty}$-norm versus grid sizes reduction and convergence orders of the improved difference scheme (B.3) for the linear convection–diffusion equation in Example 7.2.

| R  | Case I | Case II |
|----|--------|---------|
|   | $m_1$ | $m_2$ | $n$  | $E_{\infty}(h_x, h_y, \tau)$ | $\text{Ord}_G$ | $m_1$ | $m_2$ | $n$  | $E_{\infty}(h_x, h_y, \tau)$ | $\text{Ord}_G$ |
| $r = 1/7$ | 5  | 10  | 700 | 9.8608e$-5$ | - | 5  | 50  | 175 | 7.2462e$-5$ | - |
|   | 10  | 20  | 2800 | 1.8204e$-5$ | 2.4374 | 10  | 100 | 700 | 1.7282e$-5$ | 2.0680 |
|   | 20  | 40  | 11,200 | 4.1716e$-6$ | 2.1256 | 20  | 200 | 2800 | 4.2680e$-6$ | 2.0176 |
|   | 40  | 80  | 44,800 | 1.0174e$-6$ | 2.0358 | 40  | 400 | 11,200 | 1.0653e$-6$ | 2.0023 |
|   | 80  | 160 | 179,200 | 2.5287e$-7$ | 2.0084 | 80  | 800 | 44,800 | 2.6613e$-7$ | 2.0011 |
| $r = 1/6$ | 5  | 10  | 600 | 4.0615e$-5$ | - | 5  | 50  | 150 | 5.1442e$-6$ | - |
|   | 10  | 20  | 2400 | 2.5579e$-6$ | 3.9890 | 10  | 100 | 600 | 3.2166e$-7$ | 3.9994 |
|   | 20  | 40  | 9600 | 1.6117e$-7$ | 3.9883 | 20  | 200 | 2400 | 2.0103e$-8$ | 4.0000 |
|   | 40  | 80  | 38,400 | 1.0079e$-8$ | 3.9993 | 40  | 400 | 9600 | 1.2583e$-9$ | 3.9978 |
|   | 80  | 160 | 153,600 | 6.3068e$-10$ | 3.9982 | 80  | 800 | 38,400 | 7.8984e$-11$ | 3.9938 |
| $r = 1/5$ | 5  | 10  | 500 | 4.0159e$-5$ | - | 5  | 50  | 125 | 3.2297e$+2$ | - |
|   | 10  | 20  | 2000 | 1.9326e$-5$ | 1.0552 | 10  | 100 | 500 | 2.3429e$-5$ | 2371.7 |
|   | 20  | 40  | 8000 | 5.4520e$-6$ | 1.8257 | 20  | 200 | 2000 | 5.9274e$-6$ | 1.9828 |
|   | 40  | 80  | 32,000 | 1.4000e$-6$ | 1.9613 | 40  | 400 | 8000 | 1.4885e$-6$ | 1.9936 |
|   | 80  | 160 | 128,000 | 3.5249e$-7$ | 1.9898 | 80  | 800 | 32,000 | 3.7239e$-7$ | 1.9989 |

Note: Bold indicates the numerical superiority of the step-ratio $r = 1/6$.

TABLE 4  The errors in $L^{\infty}$-norm versus grid sizes reduction and convergence orders of the classical Euler difference scheme (B.4) for the linear convection–diffusion equation in Example 7.2.

| R  | Case I | Case II |
|----|--------|---------|
|   | $m_1$ | $m_2$ | $n$  | $E_{\infty}(h_x, h_y, \tau)$ | $\text{Ord}_G$ | $m_1$ | $m_2$ | $n$  | $E_{\infty}(h_x, h_y, \tau)$ | $\text{Ord}_G$ |
| $r = 1/7$ | 5  | 10  | 700 | 4.6670e$-4$ | - | 5  | 50  | 175 | 8.9757e$-4$ | - |
|   | 10  | 20  | 2800 | 1.1650e$-4$ | 2.0021 | 10  | 100 | 700 | 2.2506e$-4$ | 1.9957 |
|   | 20  | 40  | 11,200 | 2.9304e$-5$ | 1.9912 | 20  | 200 | 2800 | 5.6308e$-5$ | 1.9989 |
|   | 40  | 80  | 44,800 | 7.3252e$-6$ | 2.0001 | 40  | 400 | 11,200 | 1.4101e$-5$ | 1.9975 |
|   | 80  | 160 | 179,200 | 1.8322e$-6$ | 1.9993 | 80  | 800 | 44,800 | 3.5254e$-6$ | 1.9999 |
| $r = 1/6$ | 5  | 10  | 600 | 4.6947e$-4$ | - | 5  | 50  | 150 | 9.6777e$-4$ | - |
|   | 10  | 20  | 2400 | 1.1270e$-4$ | 2.0021 | 10  | 100 | 600 | 2.4273e$-4$ | 1.9953 |
|   | 20  | 40  | 9600 | 2.9479e$-5$ | 1.9912 | 20  | 200 | 2400 | 6.0733e$-5$ | 1.9988 |
|   | 40  | 80  | 38,400 | 7.3691e$-6$ | 2.0001 | 40  | 400 | 9600 | 1.5209e$-5$ | 1.9975 |
|   | 80  | 160 | 153,600 | 1.8431e$-6$ | 1.9993 | 80  | 800 | 38,400 | 3.8026e$-6$ | 1.9999 |
| $r = 1/5$ | 5  | 10  | 500 | 4.7336e$-4$ | - | 5  | 50  | 125 | 1.0660e$-3$ | - |
|   | 10  | 20  | 2000 | 1.1817e$-4$ | 2.0020 | 10  | 100 | 500 | 2.6747e$-4$ | 1.9947 |
|   | 20  | 40  | 8000 | 2.9724e$-5$ | 1.9912 | 20  | 200 | 2000 | 6.6929e$-5$ | 1.9987 |
|   | 40  | 80  | 32,000 | 7.4305e$-6$ | 2.0001 | 40  | 400 | 8000 | 1.6761e$-5$ | 1.9975 |
|   | 80  | 160 | 128,000 | 1.8585e$-6$ | 1.9993 | 80  | 800 | 32,000 | 4.1906e$-6$ | 1.9999 |
TABLE 5  The errors in $L^\infty$-norm versus grid sizes reduction and convergence orders of the improved difference scheme (2.1) for the linear convection–diffusion equation with the variable coefficients in Example 7.3.

| R    | Case I | Case II |
|------|--------|---------|
|      | $m_1$  | $m_2$  | $n$  | $E_\infty(h_x, h_y, \tau)$ | $\text{Ord}_h$ | $E_\infty(h_x, h_y, \tau)$ | $\text{Ord}_h$ |
| $r = 1/7$ | 10 | 20 | 28 | 3.5063e-04 | - | 10 | 100 | 7 | 4.7662e-03 | - |
| | 20 | 40 | 112 | 9.5061e-05 | 1.8830 | 20 | 200 | 28 | 1.0877e-03 | 2.1315 |
| | 40 | 80 | 448 | 2.4605e-05 | 1.9499 | 40 | 400 | 112 | 3.3119e-04 | 1.7156 |
| | 80 | 160 | 1792 | 6.1907e-06 | 1.9908 | 80 | 800 | 448 | 8.6900e-05 | 1.9303 |
| | 160 | 320 | 7168 | 1.5513e-06 | 1.9966 | 160 | 1600 | 1792 | 2.2006e-05 | 1.9815 |
| $r = 1/6$ | 10 | 20 | 24 | 1.3708e-04 | - | 10 | 100 | 6 | 1.0013e-02 | - |
| | 20 | 40 | 96 | 8.6521e-06 | 3.9858 | 20 | 200 | 24 | 5.7511e-04 | 4.1218 |
| | 40 | 80 | 384 | 5.4266e-07 | 3.9949 | 40 | 400 | 96 | 3.5329e-05 | 4.0249 |
| | 80 | 160 | 1536 | 3.3796e-08 | 4.0051 | 80 | 800 | 384 | 2.2085e-06 | 3.9997 |
| | 160 | 320 | 6144 | 2.1103e-09 | 4.0013 | 160 | 1600 | 1536 | 1.3778e-07 | 4.0027 |
| $r = 1/5$ | 10 | 20 | 20 | 7.2761e-04 | - | 10 | 100 | 5 | 2.1408e-02 | - |
| | 20 | 40 | 80 | 1.4741e-04 | 2.3033 | 20 | 200 | 20 | 2.7078e-03 | 2.9830 |
| | 40 | 80 | 320 | 3.5263e-05 | 2.0636 | 40 | 400 | 80 | 5.4266e-04 | 2.3190 |
| | 80 | 160 | 1280 | 8.7176e-06 | 2.0161 | 80 | 800 | 320 | 1.2652e-04 | 2.1007 |
| | 160 | 320 | 5120 | 2.1751e-06 | 2.0028 | 160 | 1600 | 1280 | 3.1116e-05 | 2.0236 |

Note: Bold indicates the numerical superiority of the step-ratio $r = 1/6$.

Case I: $a = 4, b = 1$; Case II: $a = 1, b = 0.01$;  
Case III: $a = 100, b = 100$; Case IV: $a = 0.01, b = 0.01$.

The numerical results are shown in Tables 5 and 6. Since the exact solution is unknown, we use the second method in (7.1) to test the convergence rate. As we see from Tables 5 and 6, all the results are in agreement with our theoretical findings. It is worth mentioning that the smaller diffusion coefficients $a$ and $b$ are, the more dense grids required (see Case IV in Table 6).

7.2 Nonlinear problems

We will test Examples 7.4 and 7.5 under Dirichlet boundary conditions by the difference scheme (4.1) and under Neumann boundary conditions by the difference scheme (4.1a), (4.1b), (5.2a)–(5.2d), respectively.

Example 7.4. We first consider semi-linear diffusion reaction equations as

$$u_t = \Delta u + r(u), \quad x \in (0, 1)^2, \quad t \in (0, T],$$

where the initial and boundary value conditions are determined by the exact solution.

Case I: Fisher equation [13]: $r(u) = u(1 - u)$. The exact solution is

$$u(x, t) = \left[1 + \exp \left(-\frac{5}{6}t + \frac{\sqrt{3}}{6}x + \frac{\sqrt{3}}{6}y\right)\right]^{-2};$$
The errors in $L^\infty$-norm versus grid sizes reduction and convergence orders of the improved difference scheme (2.1) for the linear convection–diffusion equation with the variable coefficients in Example 7.3.

| Case III | Case IV |
|----------|---------|
| $r = 1/7$ | $r = 1/6$ |
| $r = 1/5$ | $r = 1/5$ |

Note: Bold indicates the numerical superiority of the step-ratio $r = 1/6$.

**Case II:** Chafee–Infante equation [2]: $r(u) = u(1 - u^2)$. The exact solution is

$$u(x, t) = \frac{1}{2} \tanh \left( \frac{1}{4} x + \frac{1}{4} y + \frac{3}{4} t \right) + \frac{1}{2}.$$  

The numerical results for these two problems with $\tau = h^2$ and $T = 1$ are listed in Table 7 for the Dirichlet boundary conditions and in Table 8 for the Neumann boundary conditions. We clearly see that when the step-ratio $r = 1/6$, the numerical accuracy is fourth-order. Meanwhile, the numerical results are much better than that calculated by using other step-ratios. For example, when $m_1 = m_2 = 160$, the maximum numerical error $10^{-14}$ is achieved with the optimal step-ratio $r = 1/6$, which is about 1/1000 of that obtained by the same spatial step sizes and smaller temporal step size. A similar phenomenon is observed under the Neumann boundary conditions. The numerical solution behavior for both problems and corresponding error surfaces are displayed in Figure 3 with the optimal step-ratio $r = 1/6$.

**Example 7.5.** Next, we solve the two-dimensional scalar quasi-linear Burgers’ equation [12] as

$$u_t + (u^2/2)_x + (u^2/2)_y = \mu \Delta u, \quad x \in \Omega, \quad t \in (0, 1],$$

where $\mu$ is the viscous coefficient. The initial and boundary value conditions are taken from the exact solution

$$u(x, t) = \frac{2\mu \pi \sin(\pi(x + y)) \exp(-2\mu \pi^2 t)}{2 + \cos(\pi(x + y)) \exp(-2\mu \pi^2 t)}.$$  

The following two sets of viscous coefficients are considered.
### TABLE 7

The errors in \(L^\infty\)-norm versus grid sizes reduction and convergence orders of the improved difference scheme (4.1) for the Fisher equation and Chafee–Infante equation under the Dirichlet boundary conditions in Example 7.4.

| \(r\) | Case I | Case II |
|-------|--------|--------|
| \(m_1\) | \(m_2\) | \(n\) | \(E_{\infty}(h_x, h_y, \tau)\) | \(\text{Ord}_{G}\) | \(m_1\) | \(m_2\) | \(n\) | \(E_{\infty}(h_x, h_y, \tau)\) | \(\text{Ord}_{G}\) |
| \(r = 1/7\) | 10 | 10 | 700 | 3.2948e-9 | - | 10 | 10 | 700 | 2.9377e-8 | - |
|       | 20 | 20 | 2800 | 1.2599e-9 | 1.3869 | 20 | 20 | 2800 | 7.9078e-9 | 1.8934 |
|       | 40 | 40 | 11,200 | 3.4607e-10 | 1.8642 | 40 | 40 | 11,200 | 2.0121e-9 | 1.9745 |
|       | 80 | 80 | 44,800 | 8.8370e-11 | 1.9694 | 80 | 80 | 44,800 | 5.0524e-10 | 1.9937 |
|       | 160 | 160 | 179,200 | 2.2219e-11 | 1.9918 | 160 | 160 | 179,200 | 1.2645e-10 | 1.9984 |
| \(r = 1/6\) | 10 | 10 | 600 | 4.2836e-9 | - | 10 | 10 | 600 | 4.4406e-9 | - |
|       | 20 | 20 | 2400 | 2.6769e-10 | \textbf{4.0002} | 20 | 20 | 2400 | 2.7710e-10 | \textbf{3.9890} |
|       | 40 | 40 | 9600 | 1.6730e-11 | \textbf{4.0000} | 40 | 40 | 9600 | 1.7312e-11 | \textbf{3.9883} |
|       | 80 | 80 | 38,400 | 1.0456e-12 | \textbf{4.0001} | 80 | 80 | 38,400 | 1.0825e-12 | \textbf{3.9993} |
|       | 160 | 160 | 153,600 | 6.5337e-14 | \textbf{4.0003} | 160 | 160 | 153,600 | 6.7946e-14 | \textbf{3.9982} |
| \(r = 1/5\) | 10 | 10 | 500 | 1.3295e-8 | - | 10 | 10 | 500 | 5.1364e-8 | - |
|       | 20 | 20 | 2000 | 2.3143e-9 | 2.5223 | 20 | 20 | 2000 | 1.1709e-8 | 2.1331 |
|       | 40 | 40 | 8000 | 5.1745e-10 | 2.1611 | 40 | 40 | 8000 | 2.8569e-9 | 2.0351 |
|       | 80 | 80 | 32,000 | 1.2585e-10 | 2.0398 | 80 | 80 | 32,000 | 7.0982e-10 | 2.0089 |
|       | 160 | 160 | 128,000 | 3.1237e-11 | 2.0103 | 160 | 160 | 128,000 | 1.7719e-10 | 2.0022 |

**Note:** Bold indicates the numerical superiority of the step-ratio \(r = 1/6\).

### TABLE 8

The errors in \(L^\infty\)-norm versus grid sizes reduction and convergence orders of the improved difference scheme (4.1) for the Fisher equation and Chafee–Infante equation with the Neumann boundary conditions in Example 7.4.

| \(r\) | Case I | Case II |
|-------|--------|--------|
| \(m_1\) | \(m_2\) | \(n\) | \(E_{\infty}(h_x, h_y, \tau)\) | \(\text{Ord}_{G}\) | \(m_1\) | \(m_2\) | \(n\) | \(E_{\infty}(h_x, h_y, \tau)\) | \(\text{Ord}_{G}\) |
| \(r = 1/7\) | 10 | 10 | 700 | 3.7818e-7 | - | 10 | 10 | 700 | 8.5997e-7 | - |
|       | 20 | 20 | 2800 | 9.9509e-8 | 1.9262 | 20 | 20 | 2800 | 2.0788e-7 | 2.0486 |
|       | 40 | 40 | 11,200 | 2.5240e-8 | 1.9791 | 40 | 40 | 11,200 | 5.1631e-8 | 2.0094 |
|       | 80 | 80 | 44,800 | 6.3342e-9 | 1.9945 | 80 | 80 | 44,800 | 1.2890e-8 | 2.0020 |
|       | 160 | 160 | 179,200 | 1.5852e-9 | 1.9985 | 160 | 160 | 179,200 | 3.2215e-9 | 2.0005 |
| \(r = 1/6\) | 10 | 10 | 600 | 7.7947e-8 | - | 10 | 10 | 600 | 6.8543e-8 | - |
|       | 20 | 20 | 2400 | 5.0469e-9 | \textbf{3.9490} | 20 | 20 | 2400 | 4.6223e-9 | \textbf{3.8903} |
|       | 40 | 40 | 9600 | 3.2070e-10 | \textbf{3.9761} | 40 | 40 | 9600 | 2.9933e-10 | \textbf{3.9488} |
|       | 80 | 80 | 38,400 | 2.0186e-11 | \textbf{3.9898} | 80 | 80 | 38,400 | 1.9060e-11 | \textbf{3.9731} |
|       | 160 | 160 | 153,600 | 1.2221e-12 | \textbf{4.0459} | 160 | 160 | 153,600 | 1.2601e-12 | \textbf{3.9190} |
| \(r = 1/5\) | 10 | 10 | 500 | 6.3087e-7 | - | 10 | 10 | 500 | 1.1272e-6 | - |
|       | 20 | 20 | 2000 | 1.4610e-7 | 2.1104 | 20 | 20 | 2000 | 2.8694e-7 | 1.9739 |
|       | 40 | 40 | 8000 | 3.5778e-8 | 2.0299 | 40 | 40 | 8000 | 7.2064e-8 | 1.9934 |
|       | 80 | 80 | 32,000 | 8.8960e-9 | 2.0078 | 80 | 80 | 32,000 | 1.8034e-8 | 1.9986 |
|       | 160 | 160 | 128,000 | 2.2209e-9 | 2.0020 | 160 | 160 | 128,000 | 4.5095e-9 | 1.9997 |

**Note:** Bold indicates the numerical superiority of the step-ratio \(r = 1/6\).
FIGURE 3 Numerical surfaces and error surfaces at the terminal time \( t = 2 \). (a, b) The Fisher equation, the domain are taken as \( \Omega = [-20, 20] \times [-20, 20] \) and mesh parameters \( h_x = h_y = 1/8, \tau = 1/384; \) (c, d) The Chafee–Infante equation, the domain are taken as \( \Omega = [-10, 10] \times [-10, 10] \) and mesh parameters \( h_x = h_y = 1/16, \tau = 1/1536. \) (a, c) Numerical surface. (b, d) Error surface.

TABLE 9 The errors in \( L^\infty \)-norm versus grid sizes reduction and convergence orders of the improved difference scheme (4.1) for the viscous Burgers’ equation under the Dirichlet boundary conditions in Example 7.5.

| R     | Case I | Case II |
|-------|--------|---------|
|       | \( m_1 \) | \( m_2 \) | \( n \) | \( E_{\infty}(h_x, h_y, \tau) \) | \( \text{Ord}_{G} \) | \( m_1 \) | \( m_2 \) | \( n \) | \( E_{\infty}(h_x, h_y, \tau) \) | \( \text{Ord}_{G} \) |
| \( r = 1/7 \) | 10 | 10 | 700 | 8.3065e−12 | - | 10 | 10 | 7 | 3.0122e−5 | - |
|       | 20 | 20 | 2800 | 2.0419e−12 | 2.0243 | 20 | 20 | 28 | 1.0200e−5 | 1.5622 |
|       | 40 | 40 | 11,200 | 5.0834e−13 | 2.0061 | 40 | 40 | 112 | 2.7123e−6 | 1.9110 |
|       | 80 | 80 | 44,800 | 1.2710e−13 | 1.9998 | 80 | 80 | 448 | 6.8987e−7 | 1.9751 |
|       | 160 | 160 | 179,200 | 3.1782e−14 | 1.9997 | 160 | 160 | 1792 | 1.7308e−7 | 1.9949 |
| \( r = 1/6 \) | 10 | 10 | 600 | 1.2581e−13 | - | 10 | 10 | 6 | 2.0126e−5 | - |
|       | 20 | 20 | 2400 | 7.8107e−15 | 4.0096 | 20 | 20 | 24 | 1.2675e−6 | 3.9890 |
|       | 40 | 40 | 9600 | 4.8736e−16 | 4.0024 | 40 | 40 | 96 | 7.8155e−8 | 4.0195 |
|       | 80 | 80 | 38,400 | 3.0483e−17 | 3.9989 | 80 | 80 | 384 | 4.8675e−9 | 4.0051 |
|       | 160 | 160 | 153,600 | 1.9059e−18 | 3.9994 | 160 | 160 | 1536 | 3.0437e−10 | 3.9993 |
| \( r = 1/5 \) | 10 | 10 | 500 | 1.1283e−11 | - | 10 | 10 | 5 | 9.2303e−5 | - |
|       | 20 | 20 | 2000 | 2.8372e−12 | 1.9916 | 20 | 20 | 20 | 1.7206e−5 | 2.4235 |
|       | 40 | 40 | 8000 | 7.1034e−13 | 1.9979 | 40 | 40 | 80 | 3.9883e−6 | 2.1091 |
|       | 80 | 80 | 32,000 | 1.7785e−13 | 1.9978 | 80 | 80 | 320 | 9.7742e−7 | 2.0287 |
|       | 160 | 160 | 128,000 | 4.4489e−14 | 1.9992 | 160 | 160 | 1280 | 2.4304e−7 | 2.0078 |

Note: Bold indicates the numerical superiority of the step-ratio \( r = 1/6 \).

**Case I**: \( \mu = 1; \) **Case II**: \( \mu = 0.01; \)

The numerical convergence results on the domain \( \Omega = [0, 1] \times [0, 1] \) are listed in Table 9 for the Dirichlet boundary conditions and in Table 10 for the Neumann boundary conditions. They confirm that the improved difference scheme is still fourth-order accuracy when the step-ratio \( r = 1/6 \) and second-order accuracy for other step-ratios, which fully demonstrate the good performance in accuracy of the improved difference scheme (4.1). Comparing the results in Tables 9 and 10, we see that the numerical errors for the Neumann boundary conditions are larger than those for the Dirichlet boundary conditions. This is mainly caused by the discretization on the boundary conditions. The numerical simulation for the Burgers’ equation with viscous coefficients \( \mu = 1 \) (\( t = 0.1 \)), \( \mu = 0.01 \) (\( t = 0.5 \)) is demonstrated in Figure 4, respectively on the domain \( \Omega = [-3, 3] \times [-3, 3] \). We see that the numerical error surfaces keeps very low whether the viscous coefficient \( \mu \) is large or small.
TABLE 10 The errors in $L^\infty$-norm versus grid sizes reduction and convergence orders of the improved difference scheme (4.1) for the viscous Burgers’ equation under the Neumann boundary in Example 7.5.

| P | Case I | Cas II | | Case II |
|---|--------|--------|---|--------|
| $r = 1/7$ | | | | |
| $m_1$ | $m_2$ | $n$ | $E_\infty(h_x, h_y, \tau)$ | $\text{Ord}_G$ | $m_1$ | $m_2$ | $n$ | $E_\infty(h_x, h_y, \tau)$ | $\text{Ord}_G$ |
| 10 | 10 | 700 | 1.3885e - 7 | - | 40 | 40 | 112 | 2.7129e - 6 | - |
| 20 | 20 | 2800 | 1.2689e - 7 | 0.1300 | 80 | 80 | 448 | 7.2243e - 7 | 1.9089 |
| 40 | 40 | 11,200 | 3.9364e - 8 | 1.6886 | 160 | 160 | 1792 | 1.9385e - 7 | 1.8979 |
| 80 | 80 | 44,800 | 1.0255e - 8 | 1.9406 | 320 | 320 | 7168 | 4.9489e - 8 | 1.9698 |
| 160 | 160 | 179,200 | 2.5871e - 9 | 1.9869 | 640 | 640 | 28,672 | 1.2443e - 8 | 1.9918 |
| $r = 1/6$ | | | | |
| $m_1$ | $m_2$ | $n$ | $E_\infty(h_x, h_y, \tau)$ | $\text{Ord}_G$ | $m_1$ | $m_2$ | $n$ | $E_\infty(h_x, h_y, \tau)$ | $\text{Ord}_G$ |
| 10 | 10 | 600 | 7.1556e - 7 | - | 40 | 40 | 96 | 1.5489e - 6 | - |
| 20 | 20 | 2400 | 4.0472e - 8 | 4.1441 | 80 | 80 | 384 | 9.9905e - 8 | 3.9546 |
| 40 | 40 | 9600 | 2.2378e - 9 | 4.1768 | 160 | 160 | 1536 | 3.9461e - 10 | 3.9955 |
| 80 | 80 | 38,400 | 1.2872e - 10 | 4.1198 | 320 | 320 | 5120 | 2.4688e - 11 | 3.9985 |
| 160 | 160 | 153,600 | 7.6769e - 12 | 4.0676 | 640 | 640 | 24,576 | 2.0423e - 8 | 2.0062 |
| $r = 1/5$ | | | | |
| $m_1$ | $m_2$ | $n$ | $E_\infty(h_x, h_y, \tau)$ | $\text{Ord}_G$ | $m_1$ | $m_2$ | $n$ | $E_\infty(h_x, h_y, \tau)$ | $\text{Ord}_G$ |
| 10 | 10 | 500 | 1.6401e - 6 | - | 40 | 40 | 80 | 5.9410e - 6 | - |
| 20 | 20 | 2000 | 2.7321e - 7 | 4.1198 | 80 | 80 | 320 | 1.2168e - 6 | 2.2877 |
| 40 | 40 | 8000 | 6.0382e - 8 | 2.1778 | 160 | 160 | 1280 | 2.8568e - 7 | 2.0906 |
| 80 | 80 | 32,000 | 1.4660e - 8 | 2.0423 | 320 | 320 | 5120 | 7.0216e - 8 | 2.0245 |
| 160 | 160 | 128,000 | 3.6400e - 9 | 2.0098 | 640 | 640 | 20,480 | 1.7479e - 8 | 2.0062 |

Note: Bold indicates the numerical superiority of the step-ratio $r = 1/6$.

Example 7.6. Consider the solution profile to the nonlinear problem (see e.g., [1, 6, 7, 16])

$$u_t + f(u)_x + g(u)_y = \varepsilon \Delta u, \quad x \in \Omega, \; t > 0.$$

**Case I.** The initial data is given by

$$u_0(x) = \begin{cases} 
1, & \text{for } (x - 0.25)^2 + (y - 2.25)^2 < 0.5, \\
0, & \text{otherwise},
\end{cases}$$

and the fluxes given by

$$f(u) = (u - 0.25)^3,$$

$$g(u) = u + u^2.$$
Case II. The initial data is given by

\[ u_0(x) = \begin{cases} 
1, & \text{for } x^2 + y^2 < 0.5, \\
0, & \text{otherwise,} 
\end{cases} \]

and the fluxes given by

\[ \begin{align*}
    g(u) &= \frac{u^2}{u^2 + (1-u)^2}, \\
    f(u) &= g(u)(1 - 5(1 - u)^2). 
\end{align*} \]

In Case I, the boundary conditions are set to be zeros to keep the consistency and the diffusion coefficient is taken as \( \epsilon = 0.1 \). The temporal step size is \( \tau = 0.001 \) and the simulation domain is on \( \Omega = [-6, 2] \times [0, 8] \), which could contain a complete evolution surface. The numerical surfaces and corresponding contours are displayed in Figure 5 for different terminal time with the optimal step-ratio \( r = 1/6 \). We see clearly that the numerical solutions are diffusive and move from the bottom right to the upper left.

In Case II, we describe a problem motivated from two-phase flow in porous media with a gravitation pull in the \( x \)-direction. The flux functions \( f(u) \) and \( g(u) \) are “S-shape” with \( f(0) = g(0) = 0 \) and \( f(1) = g(1) = 1 \). Boundary value conditions are again put equal to zero. The diffusion coefficient is \( \epsilon = 0.01 \) and the nonlinearity is very strong. We calculate the problem on the domain \( \Omega = [-3, 3] \times [-3, 3] \) with a temporal step-size \( \tau = 0.00005 \). The numerical results are displayed in Figure 6. In order to demonstrate the numerical surfaces easily, the surfaces are drawn every ten lines to reduce image storage. We observe that the peak decreases gradually from the center to the upper left corner, which is consistent with the result in [6].

### 7.3 Three-dimensional case

**Example 7.7.** Finally, we consider a three-dimensional diffusion problem (6.1) with the parameters \( L_1 = L_2 = L_3 = 0, R_1 = R_2 = R_3 = 1 \) and \( T = 1 \). The initial and boundary
FIGURE 6 Numerical evolution surfaces (a–d) and contours (e–h). Here the parameters are taken as (a, e) $t=0$, $m_1 = m_2 = 3464, n = 10,000$; (b, f) $t = 0.5$, $m_1 = m_2 = 3464, n = 10,000$; (c, g) $t = 1.5$, $m_1 = m_2 = 3464, n = 50,000$; (d, h) $t = 3$, $m_1 = m_2 = 3464, n = 60,000$.

TABLE 11 The errors in $L^\infty$-norm versus grid sizes reduction and convergence orders of the improved difference scheme (6.2) for the three-dimensional linear diffusion equation in Example 7.7.

| R          | $m_1$ | $m_2$ | $m_3$ | $n$ | $E_{\infty}(h_x,h_y,h_z,r)$ | $\text{Ord}_G$ |
|------------|-------|-------|-------|-----|----------------------------|----------------|
| Case I     |       |       |       |     |                           |                |
| $r = 1/7$  | 5     | 5     | 5     | 175 | 4.4376e-6                 | -              |
|            | 10    | 10    | 10    | 700 | 1.0501e-6                 | 2.0793         |
|            | 20    | 20    | 20    | 2800| 2.6476e-7                 | 1.9878         |
|            | 40    | 40    | 40    | 11,200 | 6.5958e-8       | 2.0051         |
| Case II    |       |       |       |     |                           |                |
| $r = 1/6$  | 5     | 5     | 5     | 150 | 4.4890e-7                 | -              |
|            | 10    | 10    | 10    | 600 | 2.7992e-8                 | 4.0033         |
|            | 20    | 20    | 20    | 2400| 1.7891e-9                 | 3.9677         |
|            | 40    | 40    | 40    | 9600| 1.1179e-10                | 4.0003         |
| $r = 1/4$  | 5     | 5     | 5     | 100 | 1.4041e-5                 | -              |
|            | 10    | 10    | 10    | 400 | 3.5797e-6                 | 1.9171         |
|            | 20    | 20    | 20    | 1600| 9.2056e-7                 | 1.9593         |
|            | 40    | 40    | 40    | 6400| 2.3047e-7                 | 1.9979         |
| $r = 1/3.99$ | 5    | 5     | 5     | 100 | 7.6526e-5                 | -              |
|            | 10    | 10    | 10    | 399 | 3.6070e-6                 | 4.4071         |
|            | 20    | 20    | 20    | 1596| 9.2751e-7                 | 1.9594         |
|            | 40    | 40    | 40    | 6384| 6.5576e-7                 | 0.5002         |
| $r = 1/3.99$ | 5    | 5     | 5     | 100 | 7.6526e-5                 | -              |
|            | 10    | 10    | 10    | 399 | 3.6070e-6                 | 4.4071         |
|            | 20    | 20    | 20    | 1596| 9.2751e-7                 | 1.9594         |
|            | 40    | 40    | 40    | 6384| 6.5576e-7                 | 0.5002         |

Note: Bold indicates the numerical superiority of the step-ratio $r = 1/6$. 

...
value conditions and the source term \( f(x,t) \) are determined by the exact solution \( u(x,t) = \exp(1/2(x+y+z) - t) \).

Two sets of coefficients including

**Case I**: (isotropic) \( a = b = c = 1; \)

**Case II**: (anisotropic) \( a = 1, \ b = 0.01, \ c = 0.04; \)

are utilized to test the convergence rate and the CFL condition for the improved difference scheme and the classical difference scheme, respectively. Numerical results are listed in Tables 11 and 12.

We clearly see that the improved difference scheme is fourth-order convergent when the step-ratio \( r = 1/6 \) and second-order convergent in other cases. When the step-ratio arrives at the critical value \( r = 1/4 \) of the CFL condition, the numerical results still have second-order convergence. Once the step-ratio \( r > 1/4 \), the numerical results will irreversibly deviate from the exact solution and the numerical error tends to blow up. On the other hand, for the classical difference scheme, we have tested two sets of data with \( r = 1/6 \) and not stable when \( r = 1/5.9 \). In short, all the data in Tables 11 and 12 are consistent with Theorem 6.1 and the restrictive condition (6.3).

## 8 CONCLUDING REMARKS

In closing, we propose an improved technique to construct high-order explicit numerical schemes for convection–diffusion problems in high dimension based on the forward Euler discretization. We obtain the superconvergence with the step-ratio \( r = 1/6 \) for the improved difference scheme and display much better numerical behavior, which serve to the theoretical results. Another advantage of the present difference scheme is that it is an fully explicit numerical method without any matrix by vector operation and pretty convenient in practical implementation.
TABLE 13  The CFL conditions for the diffusion problems in different dimensions with different numerical schemes.

| D  | Improved difference scheme | Classical difference scheme [9] |
|----|-----------------------------|---------------------------------|
| 1D | $rs \leq \frac{1}{2}$ [19]  | $rs \leq \frac{1}{2}$           |
| 2D | $\max\{rx, ry\} \leq \frac{1}{2}$ | $rs + r_s \leq \frac{1}{2}$     |
| 3D | $\max\{rx, ry, rz\} \leq \frac{1}{2}$ | $rs + r_s + r_s \leq \frac{1}{2}$ |

Note: “S” denotes the difference scheme and “D” the dimension.

TABLE 14  The convergence rates for the diffusion problems in different dimensions with different numerical schemes.

| D  | Improved difference scheme | Classical difference scheme [9] |
|----|-----------------------------|---------------------------------|
| 1D | $r \neq \frac{1}{6}$       | $r = \frac{1}{6}$               |
|    | 2, [19]                     | 2                               |
| 2D | $r \neq \frac{1}{6}$       | $r = \frac{1}{6}$               |
|    | 2                           | 2                               |
| 3D | $r \neq \frac{1}{6}$       | $r = \frac{1}{6}$               |
|    | 2                           | 2                               |

Moreover, the improved difference schemes have essentially improved the CFL condition and convergence rate of the classical difference scheme, see, for example, [15] or [9]. The detailed theoretical results with respect to the CFL conditions and the convergence rate of the improved difference scheme and classical difference scheme in different settings are listed in Tables 13 and 14.

In terms of theoretical analysis, we only discuss a constant convection case in current paper. As for the general nonlinear convection–diffusion equations, it remains a challenge for the convergence and stability analysis because of the complex discretization of the nonlinear terms.

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CONFLICT OF INTEREST STATEMENT

The authors have no relevant financial or non-financial interests to disclose.

DATA AVAILABILITY STATEMENT

All data or codes generated or used during the study are available from the corresponding author by request.

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REFERENCES

[1] R. Abedian and M. Dehghan, A RBF-WENO finite difference scheme for non-linear degenerate parabolic equations, J. Sci. Comput. 93 (2022), 60. https://doi.org/10.1007/s10915-022-02022-3.
The derivation of the difference scheme (2.1).

First, using (1.1a), we have

\[ u_{tt} = a(u_t)_{xx} + b(u_t)_{yy} + c(u_t)_x + d(u_t)_y + f_t(x, t) \]

\[ = a \left( a u_{xx} + b u_{yy} + c(x)u_x + d(x)u_y + f(x, t) \right)_{xx} \]
\[ + b \left( a u_{xx} + b u_{yy} + c(x)u_x + d(x)u_y + f(x, t) \right)_{yy} \]
\[ + c \left( a u_{xx} + b u_{yy} + c(x)u_x + d(x)u_y + f(x, t) \right)_x \]
\[ + d \left( a u_{xx} + b u_{yy} + c(x)u_x + d(x)u_y + f(x, t) \right)_y + f_t(x, t) \]

\[ = a^2 u_{xxxx} + 2ab u_{xxyy} + b^2 u_{yyyy} \]
\[ + 2ac(x)u_{xxxx} + 2ad(x)u_{xxyy} + 2bc(x)u_{xyxy} + 2bd(x)u_{xyyy} \]
\[ + \left( 2ac_x(x) + c^2(x) \right) u_{xx} + \left( 2bd_y(x) + bd_{yy}(x) \right) u_{yy} \]
where \((A1)\) is used in the second equality and with 

\[
U_k \quad \text{with}
\]

\[
\text{difference quotient for the spatial derivatives in (A2), which results in}
\]

\[
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\]

\[
U = \{ U^k_{ij} | 0 \leq i \leq m_1, 0 \leq j \leq m_2, 0 \leq k \leq n \}
\]

with \(U^k_{ij} = u(x_{ij}, t_k)\). Considering (1.1a) at the node point \((x_{ij}, t_k)\) and using the Taylor formula, we have

\[
\begin{align*}
&u_t(x_{ij}, t_k) = au_{xx}(x_{ij}, t_k) + bu_{yy}(x_{ij}, t_k) + c_{ij}u_t(x_{ij}, t_k) + d_{ij}u_y(x_{ij}, t_k) + f_{ij}^k, \\
&(i, j) \in \omega, 0 \leq k \leq n, \quad (A2)
\end{align*}
\]

where 

\[
c_{ij} = c(x_{ij}), \quad d_{ij} = d(x_{ij}), \quad f_{ij}^k = f(x_{ij}, t_k).
\]

The forward difference quotient is utilized to discrete the temporal derivative and the centered difference quotient for the spatial derivatives in (A2), which results in

\[
\begin{align*}
\delta_t U^k_{ij} & = a\delta_x^2 U^k_{ij} - b\delta_y^2 U^k_{ij} - c_{ij}\Delta_x U^k_{ij} - d_{ij}\Delta_y U^k_{ij} \\
& = f_{ij}^k + t \left[ \frac{u_t(x_{ij}, t_k) - ah^2}{12} u_{xxx}(x_{ij}, t_k) - bh^2 u_{yyy}(x_{ij}, t_k) \right] \\
& \quad - \frac{h^2}{6} c_{ij} u_{xxx}(x_{ij}, t_k) - \frac{h^2}{6} d_{ij} u_{yyy}(x_{ij}, t_k) + O(t^2 + h_x^4 + h_y^4) \\
& = p_{ij}^k + \frac{ah^2}{2} \left( r_x - \frac{1}{6} \right) u_{xxx}(x_{ij}, t_k) + \frac{bh^2}{2} \left( r_y - \frac{1}{6} \right) u_{yyy}(x_{ij}, t_k) \\
& \quad + c_{ij} h^2 \left( r_x - \frac{1}{6} \right) u_{xxx}(x_{ij}, t_k) + d_{ij} h^2 \left( r_y - \frac{1}{6} \right) u_{yyy}(x_{ij}, t_k) \\
& \quad + \frac{\tau}{2} \left[ \left( 2ac_x + c^2 \right) \delta_x^2 U^k_{ij} + \left( 2ad_x + 2bc_y + 2cd_y \right) \Delta_x \Delta_y U^k_{ij} + \left( 2bd_x + d^2 \right) \delta_y^2 U^k_{ij} \right] \\
& \quad + \left( ac_{xx} + bc_{yy} + cc_x + dc_y \right) \Delta_x U^k_{ij} + \left( ad_{xx} + bd_{yy} + cd_x + d d_y \right) \Delta_y U^k_{ij} \\
& \quad + \tau ab\delta_x^2 \delta_y^2 U^k_{ij} + \tau ad_{ij}\delta_y^2 \Delta_x U^k_{ij} + \tau bc_{ij}\delta_x^2 \Delta_y U^k_{ij} + O(t^2 + \tau h_x^2 + \tau h_y^2 + h_x^4 + h_y^4) \quad (A3)
\end{align*}
\]

where (A1) is used in the second equality and

\[
p_{ij}^k = f_{ij}^k + \frac{\tau}{2} \left[ a(f_{xx})_{ij} + b(f_{yy})_{ij} + (cf_y)_{ij} + (df_x)_{ij} + (f)_{ij} \right]
\]

with \((f_{xx})_{ij} = f_x(x_{ij}, t_k), (f_{yy})_{ij} = f_y(x_{ij}, t_k), (f_{xx})_{ij} = f_x(x_{ij}, t_k), (f_{yy})_{ij} = f_y(x_{ij}, t_k), (f)_{ij}^k = f(x_{ij}, t_k)\).

Omitting the small terms in (A3), we have the difference scheme (2.1).

**APPENDIX B**

Special cases of the difference scheme (2.1).

1. \(c = d = 0\): the difference scheme (2.1) reduces to

\[
\begin{align*}
\delta_t U^k_{ij} & = a\delta_x^2 U^k_{ij} + b\delta_y^2 U^k_{ij} + \tau ab\delta_x^2 \delta_y^2 U^k_{ij} + q_{ij}^k, \quad (i, j) \in \omega, 0 \leq k \leq n - 1, \quad (B.1a) \\
u_{ij}^0 & = \varphi(x_{ij}), \quad (i, j) \in \partial \omega, \quad (B.1b) \\
u_{ij}^k & = \alpha(x_{ij}, t_k), \quad (i, j) \in \gamma, 1 \leq k \leq n, \quad (B.1c)
\end{align*}
\]
where
\[ q^k_{ij} = f^k_{ij} + \frac{\tau}{2} \left[ a(f_{xx})^k_{ij} + b(f_{yy})^k_{ij} + (f_i)^k_{ij} \right]. \]

Comparing with the classical Euler difference scheme
\[
\begin{align*}
\delta_i u^k_{ij} &= \frac{\tau}{2} \left( \frac{a}{2} \right) \delta^2_{ij} u_{ij}^k + \frac{\tau}{2} \left( \frac{b}{2} \right) \delta^2_{ij} u_{ij}^k + c \Delta_x u_{ij}^k + d \Delta_y u_{ij}^k + \tau cd \Delta_x \Delta_y u_{ij}^k + r^k_{ij}, \quad (i, j) \in \omega, \ 0 \leq k \leq n - 1, \\
\delta_i u^0_{ij} &= \varphi(x_{ij}), \quad (i, j) \in \tilde{\omega}, \\
\delta_i u^k_{ij} &= \alpha(x_{ij}, t_k), \quad (i, j) \in \gamma, \ 1 \leq k \leq n, \\
\end{align*}
\]
the local truncation error for the difference scheme (B.1) is only two globally whatever the step-ratios are.

(II) \( c = d = \text{constant} \): the difference scheme (2.1) simplifies into
\[
\begin{align*}
\delta_i u^k_{ij} &= \left( a + \frac{\tau}{2} c^2 \right) \delta^2_{ij} u_{ij}^k + \left( b + \frac{\tau}{2} d^2 \right) \delta^2_{ij} u_{ij}^k + c \Delta_x u_{ij}^k + d \Delta_y u_{ij}^k + \tau cd \Delta_x \Delta_y u_{ij}^k + r^k_{ij}, \quad (i, j) \in \omega, \ 0 \leq k \leq n - 1, \\
\delta_i u^0_{ij} &= \varphi(x_{ij}), \quad (i, j) \in \tilde{\omega}, \\
\delta_i u^k_{ij} &= \alpha(x_{ij}, t_k), \quad (i, j) \in \gamma, \ 1 \leq k \leq n, \\
\end{align*}
\]
where
\[ r^k_{ij} = f^k_{ij} + \frac{\tau}{2} \left[ a(f_{xx})^k_{ij} + b(f_{yy})^k_{ij} + c(f_i)^k_{ij} + d(f_j)^k_{ij} + (f_i)^k_{ij} \right]. \]

The classical Euler difference scheme for the problem (1.1) is
\[
\begin{align*}
\delta_i u^k_{ij} &= \frac{\tau}{2} \left( \frac{a}{2} \right) \delta^2_{ij} u_{ij}^k - \frac{\tau}{2} \left( \frac{b}{2} \right) \delta^2_{ij} u_{ij}^k - c \Delta_x u_{ij}^k - d \Delta_y u_{ij}^k = f^k_{ij}, \quad (i, j) \in \omega, \ 0 \leq k \leq n - 1, \\
\delta_i u^0_{ij} &= \varphi(x_{ij}), \quad (i, j) \in \tilde{\omega}, \\
\delta_i u^k_{ij} &= \alpha(x_{ij}, t_k), \quad (i, j) \in \gamma, \ 0 \leq k \leq n. \\
\end{align*}
\]

Comparing (B.3) with (B.4), similar theoretical results can be obtained for the improved difference scheme (B.3). (2.1), (B.1), and (B.3) are called the **improved difference schemes**.

**APPENDIX C**

**The proof of Theorem 3.1.**

**Proof.** The difference scheme (B.3a) can be rewritten as
\[
\begin{align*}
\delta_i u^k_{ij} &= S_1 u^k_{i+1,j} + S_2 u^k_{i-1,j} + S_3 u^k_{i,j+1} + S_4 u^k_{i,j-1} + S_5 u^k_{ij} \\
&\quad + S_6 u^k_{i+1,j+1} + S_7 u^k_{i-1,j+1} + S_8 u^k_{i+1,j-1} + S_9 u^k_{i-1,j-1} + r^k_{ij}, \\
\end{align*}
\]

\[ (i, j) \in \omega, \ 0 \leq k \leq n - 1, \]

where
\[ S_1 = (1 - 2r_y) \left( 1 + \frac{c}{2a} h_x \right) r_x + \frac{c^2}{2a^2 r_x^2} h_x^2, \quad S_2 = (1 - 2r_y) \left( 1 - \frac{c}{2a} h_x \right) r_x + \frac{c^2}{2a^2 r_x^2} h_x^2. \]
\[ S_3 = (1 - 2r_x) \left( 1 + \frac{d}{2b}h_y \right) r_y + \frac{d^2}{2b^2}r_y^2, \quad S_4 = (1 - 2r_x) \left( 1 - \frac{d}{2b}h_y \right) r_y + \frac{d^2}{2b^2}r_y^2, \]

\[ S_5 = (1 - 2r_x)(1 - 2r_y) - \frac{c^2r_x^2}{a^2}h_x^2 - \frac{d^2r_y^2}{b^2}h_y^2, \]

\[ S_6 = \left( 1 + \frac{c}{2a}h_x \right) \left( 1 + \frac{d}{2b}h_y \right) r_x r_y, \quad S_7 = \left( 1 + \frac{c}{2a}h_x \right) \left( 1 - \frac{d}{2b}h_y \right) r_x r_y, \]

\[ S_8 = \left( 1 - \frac{c}{2a}h_x \right) \left( 1 + \frac{d}{2b}h_y \right) r_x r_y, \quad S_9 = \left( 1 - \frac{c}{2a}h_x \right) \left( 1 - \frac{d}{2b}h_y \right) r_x r_y. \]

When (3.2) holds, all the coefficients \( S_i \) \((i = 1, 2, \ldots, 9)\) on the right-hand side of (C.1) are nonnegative. Meanwhile \( \sum_{i=1}^{9} S_i = 1 \). Therefore it holds

\[ |u_{ij}^{k+1}| \leq \|u^k\|_{\infty} + \tau \|r^k\|_{\infty}, \quad (i, j) \in \Omega, \quad 0 \leq k \leq n - 1. \]

Whence

\[ \|u^{k+1}\|_{\infty} \leq \|u^k\|_{\infty} + \tau \|r^k\|_{\infty}, \quad 0 \leq k \leq n - 1. \]

By recursion, we have

\[ \|u^k\|_{\infty} \leq \|u^0\|_{\infty} + \tau \sum_{l=0}^{k-1} \|r^l\|_{\infty} = \|\varphi\|_{\infty} + \tau \sum_{l=0}^{k-1} \|r^l\|_{\infty}, \quad 1 \leq k \leq n. \]

\[ \boxed{} \]

**APPENDIX D**

The derivation of the difference scheme 4.1.

With the help of (1.2a) we have

\[ u_{it} = -f(u)_{it} - g(u)_{it} + au_{xx} + bu_{yy} + r(u)_t \]

\[ = - \left( f'(u)u_t \right)_x - \left( g'(u)u_t \right)_y + a(u)_{xx} + b(u)_{yy} + r'(u)u_t \]

\[ = a^2u_{xxx} + b^2u_{yyy} + 2abu_{xxyy} - 2af'(u)u_{xxx} \]

\[ - 2bg'(u)u_{xyy} - 2bf'(u)u_{xxy} - 2ag'(u)u_{xy} \]

\[ - 4af''''(u)u_{t}u_{xx} - 4bg''''(u)u_{t}u_{xy} - 2ag''''(u)u_{t}u_{xx} \]

\[ - 2bf''''(u)u_{t}u_{xy} - 2ag''''(u)u_{t}u_{xy} \]

\[ + (f'(u)^2 + 2ar'(u))u_{xx} + (g'(u)^2 + 2br'(u))u_{xy} + 2f'(u)g'(u)u_{xy} \]

\[ - af''''(u)u_{t}^2 - bg''''(u)u_{t}^2 - ag''''(u)u_{t}u_{xx} - 2bf''''(u)u_{t}u_{xx} \]

\[ + (2f'(u)f''''(u) + ar''''(u))u_{t} + (2g'(u)g''''(u) + br''''(u))u_{t} \]

\[ + 2(f''''(u)g'(u) + f'(u)g''''(u))u_{xy} - (f''''(u)r(u) + 2f'(u)r'(u))u_{xy} \]

\[ - (g''''(u)r(u) + 2g'(u)r'(u))u_{xy} + r'(u)r(u). \]

Considering (1.2a) at the point \((x_0, t_k)\) and combining the forward difference quotient for the temporal derivative with the centered difference discretization for the spatial derivatives, it follows that

\[ \delta_t U_{ij}^{k+\frac{1}{2}} = -f'(U_{ij}^k)\Delta_x U_{ij}^k - g'(U_{ij}^k)\Delta_x U_{ij}^k + a\delta_x^2 U_{ij}^k + b\delta_y^2 U_{ij}^k + r(U_{ij}^k) \]

\[ + \left\{ \frac{\tau}{2}u_{it} + \frac{h_x^2}{6}f'(u)u_{xxx} + \frac{h_y^2}{6}g'(u)u_{yyy} - \frac{ah_x^2}{12}u_{xxx} - \frac{bh_y^2}{12}u_{yyy} \right\} (x_{ij}, t_k) \]

\[ + O(\tau^2 + h_x^4 + h_y^4) \]
where the second equality has utilized (D.1). Further applying the centered difference discretization to the remaining partial derivatives including first-order, second-order, and mixed derivatives in space, and then rearranging the corresponding result, we have

\[
\delta_t U^{k+\frac{1}{2}}_{ij} = 
\left[ a + \frac{\tau}{2} \left( f'(U^k_{ij})^2 + 2ar'(U^k_{ij}) \right) \right] \delta_x^2 U^k_{ij} + \left[ b + \frac{\tau}{2} \left( g'(U^k_{ij})^2 + 2br'(U^k_{ij}) \right) \right] \delta_y^2 U^k_{ij}
\]

\[- \left( f'(U^k_{ij}) + \frac{\tau}{2} \left( f''(U^k_{ij})r(U^k_{ij}) + 2f'(U^k_{ij})r'(U^k_{ij}) \right) \right) \Delta_x U^k_{ij}
\]

\[- \left( g'(U^k_{ij}) + \frac{\tau}{2} \left( g''(U^k_{ij})r(U^k_{ij}) + 2g'(U^k_{ij})r'(U^k_{ij}) \right) \right) \Delta_y U^k_{ij} + r(U^k_{ij})
\]

\[+ \left\{ \frac{h^2}{6} \left( \frac{a}{2} u_{xxxx} - u_{xx} \right) + \frac{h^2}{6} \left( \frac{b}{2} u_{yyyy} - u_{yy} \right) \right\} (x_{ij}, t_k)
\]

\[+ \tau \left[ ab\delta_x^2 U^k_{ij} - bf'(U^k_{ij})\Delta_x \delta_x^2 U^k_{ij} - ag'(U^k_{ij})\delta_x^2 U^k_{ij}
\]

\[- 2af''(U^k_{ij})\Delta_x U^k_{ij}\delta_x^2 U^k_{ij} - 2bg''(U^k_{ij})\Delta_y U^k_{ij}\delta_y^2 U^k_{ij} - ag''(U^k_{ij})\Delta_y U^k_{ij}\delta_y^2 U^k_{ij}
\]

\[- bf''(U^k_{ij})\Delta_x U^k_{ij}\delta_x^2 U^k_{ij} - ag''(U^k_{ij})\Delta_y U^k_{ij}\delta_y^2 U^k_{ij}
\]

\[+ f'(U^k_{ij})g'(U^k_{ij})\Delta_x U^k_{ij}\Delta_x U^k_{ij} - \frac{a}{2} f''(U^k_{ij})\Delta_x U^k_{ij}^3 - \frac{b}{2} g''(U^k_{ij})\Delta_y U^k_{ij}^3
\]

\[- \frac{a}{2} g'''(U^k_{ij})\Delta_x U^k_{ij}(\Delta_x U^k_{ij})^2 - \frac{b}{2} f'''(U^k_{ij})\Delta_y U^k_{ij}(\Delta_y U^k_{ij})^2
\]

\[+ \frac{1}{2} \left( 2f'(U^k_{ij})f''(U^k_{ij}) + ar''(U^k_{ij}) \right) \Delta_x U^k_{ij}^2 + \frac{1}{2} \left( 2g'(U^k_{ij})g''(U^k_{ij}) + br''(U^k_{ij}) \right) \Delta_y U^k_{ij}^2
\]

\[+ \left( f''(U^k_{ij})g'(U^k_{ij}) + f'(U^k_{ij})g''(U^k_{ij}) \right) \Delta_x U^k_{ij}\Delta_y U^k_{ij}
\]

\[+ \frac{1}{2} r'(U^k_{ij})r(U^k_{ij}) \right] + O(\tau^2 + \tau h^2 + \tau h^2 + \tau h^2 + h^4) + h^4).
\]

Omitting the small terms in the above equation, it deduces to the difference scheme (4.1).