Lebesgue regularity for differential difference equations with fractional damping

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We provide necessary and sufficient conditions for the existence and uniqueness of solutions belonging to the vector-valued space of sequences $\ell_p(\mathbb{Z}, X)$ for equations that can be modeled in the form

$$\Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + G(u)(n) + f(n), \quad n \in \mathbb{Z}, \quad \alpha, \beta > 0, \quad \lambda \geq 0,$$

where $X$ is a Banach space, $f \in \ell_p(\mathbb{Z}, X)$, $A$ is a closed linear operator with domain $D(A)$ defined on $X$, and $G$ is a nonlinear function. The operator $\Delta^\gamma$ denotes the fractional difference operator of order $\gamma > 0$ in the sense of Grünwald-Letnikov. Our class of models includes the discrete time Klein-Gordon, telegraph, and Basset equations, among other differential difference equations of interest. We prove a simple criterion that shows the existence of solutions assuming that $f$ is small and that $G$ is a nonlinear term.

KEYWORDS
Delay, Differential difference equations, Fractional differences, Lebesgue maximal regularity

1 | INTRODUCTION

This paper is concerned with a wide class of mixed evolution equations that can be considered either as models for partial differential equations that are continuous in space but discrete in time,1 or systems of difference equations.2, Chapter 3;3 An additional feature of these models is that they admit the possibility of two fractional orders in the discrete variable.

Typical models that are included in this article correspond to the discrete time Klein-Gordon equation

$$\Delta^2 u(n, x) = u_{xx}(n, x) - bu(n, x) + G(u)(n, x), \quad n \in \mathbb{Z}, \quad x \in \Omega \subset \mathbb{R}^N, \quad (1)$$

where $\Delta^2 u(n, x) := u(n + 2, x) - 2u(n + 1, x) + u(n, x)$, and the discrete time telegraph equation

$$\tau \Delta^2 u(n, x) + \Delta u(n, x) = \rho u_{xx}(n, x), \quad n \in \mathbb{Z}, \quad \tau > 0, \rho > 0, \quad x \in J \subset \mathbb{R}, \quad (2)$$

as well as fractional versions of them.4,5 The discrete version of the Basset equation6,7

$$\Delta^2 u(n) + \lambda \Delta^{3/2} u(n) + bu(n) = f(n), \quad n \in \mathbb{Z}, \quad \lambda, b > 0 \quad (3)$$

will be also included in our framework. The study of the uniqueness and causality of $p$-summable solutions4 suggests to consider the above equations on $\mathbb{Z}$. More precisely, given a Banach space $X$, we ask the following problem: Is it possible to characterize solely in terms of the data of a given mixed evolution equation, the existence, and uniqueness of solutions that belong to the vector-valued space of sequences $\ell_p(\mathbb{Z}, X)$?
We success solving this open problem for the following abstract model

$$\Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + f(n), \quad n \in \mathbb{Z}, \quad \alpha, \beta > 0, \quad \lambda \geq 0,$$

(4)

where \( f \in \ell_p(\mathbb{Z}, X), A \) is a closed linear operator with domain \( D(A) \) defined on \( X \) and \( \Delta^\gamma \) denotes the fractional difference operator of order \( \gamma > 0 \) as defined recently in Abadias and Lizama.\(^3\) Roughly speaking, it corresponds to a slight variant of the Grünwald-Letnikov derivative. Compare Definition 2.1 below with Ortigueira et al.\(^1\), Section 3.3 formula (27). It is worthwhile to observe that, for instance, the model (1) includes the Basset Equation 3 taking \( X = \mathbb{C}, A = bl, \alpha = 2, \) and \( \beta = 3/2, \) whereas it also includes the linearized Klein-Gordon Equation 1 choosing \( X = L^2(\Omega), A = \partial_{xx} - bl, \alpha = 2, \) and \( \lambda = 0. \)

Modeling with fractional difference equations is a recent and promising area of research that has been developed from different sides of interest. For instance, Atici and Sengul\(^1\) develop some basics results of discrete fractional calculus. These authors introduce and solve Gompertz fractional difference equation for tumor growth models. See also Atici and Eloe\(^}\) for related results in this direction. The methodology used in such discrete fractional calculus was extended in Lizama\(^1\) to the context of abstract models, including in this way the handling of difference differential equations by methods of functional analysis and operator theory. Studies on qualitative properties, as for example, the existence of positive solutions for discrete fractional systems, have been provided by Goodrich.\(^1\) Other interesting contributions are due to Ferreira,\(^1\) Holm,\(^1\) Kovács, Li, and Lubich,\(^1\) Dassios,\(^1\) Wu, Baleanu et al,\(^2\) and Tarasov et al.\(^3\)

Starting with the work of Blunck,\(^3\) the existence and uniqueness of solutions for discrete systems that belong to the Lebesgue space of vector-valued sequences began to be considered by many authors.\(^2\) Some of the studies correspond to a numerical point of view.\(^3\) However, none of them have considered causal solutions, ie, solutions with domain on \( \mathbb{Z} \). On the other hand, some abstract models that are less general than ours have been recently reported in the literature and analyzed from diverse perspectives. For instance, in Abadias and Lizama,\(^9\) it is shown that the semilinear problem

$$\Delta^\alpha u(n) = Au(n + 1) + f(n, u(n)), \quad n \in \mathbb{Z}, \quad 0 < \alpha \leq 1,$$

(5)

admits almost automorphic solutions whenever the operator \( A \) is the generator of a \( C_0 \)-semigroup and \( f \) satisfies Lipschitz conditions of global and local type. We note that in such paper, an application to a model of population of cells is also given.

This paper is organized as follows: In Section 2, we first recall the notions of \( \textit{UMD} \)-spaces, \( \textit{R} \)-boundedness, and the discrete time Fourier transform defined over the space of distributions. These concepts allow us to formulate Blunck’s Fourier multiplier theorem for operator-valued symbols on \( \textit{UMD} \)-spaces.\(^2\) In Section 3, we prove our main result, namely, if

$$\{(1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta \}_{t \in \mathbb{T}} \subset \rho(A), \quad \lambda \geq 0, \quad \alpha, \beta > 0, \quad \mathbb{T} := (-\pi, \pi),$$

where \( \rho(A) \) denotes the resolvent set of \( A \), then the following assertions are equivalent:

(i) For all \( f \in \ell_p(\mathbb{Z}, X) \), the problem \( \Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + f(n), \ n \in \mathbb{Z} \) has a unique solution in \( \ell_p(\mathbb{Z}, [D(A)]) \);

(ii) \( M(t) := (1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta - A)^{-1} \) is an \( \ell_p \)-multiplier from \( X \) to \( [D(A)] \);

(iii) The set \( \{M(t)\}_{t \in \mathbb{T}} \) is \( \textit{R} \)-bounded.

Furthermore, in the context of Hilbert spaces, a simpler criterion is also provided, replacing the condition (iii) above by

$$\sup_{t \in \mathbb{T}} \|M(t)\| < \infty.$$ 

As a consequence, we analyze the nonlinear equation

$$\Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + G(u(n)) + g(n), \quad n \in \mathbb{Z},$$

where \( g \in \ell_p(\mathbb{Z}, X) \) and \( G: \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, X) \) are given. We show that if \( G(0) = G'(0) = 0 \) and \( g \) is small enough, then the nonlinear equation has at least one solution in \( \ell_p(\mathbb{Z}, X) \). Finally, in Section 4, we prove, as an application of our characterization, that for all \( 0 < \alpha, \beta < 2 \) and \( b > 2^\alpha + \lambda 2^\beta \), we can find \( \epsilon > 0 \) such that for all \( \epsilon \in (0, \epsilon^*), \) there exists \( u^* \in \ell_p(\mathbb{Z}, L^2(\mathbb{R})) \) that solves the problem

$$\Delta^\alpha u^*(n, x) + \lambda \Delta^\beta u^*(n, x) = \partial_{xx}(n, x) - bu^*(n, x) + G(u^*)(n, x) + \epsilon f(n, x),$$

for all \( n \in \mathbb{Z}, x \in \Omega \subset \mathbb{R}^N. \)
2 | PRELIMINARIES

In this section, we recall some concepts about fractional derivatives, the discrete time Fourier transform, in short, DTFT, and operator-valued Fourier multipliers theorems defined on UMD spaces. For more details, see Agarwal et al, Denk et al,34,35 and the references therein.

Let \( X \) be a Banach space. \( X \) is said to be a UMD space if for each \( p > 1 \), there exists a constant \( C_p > 0 \) such that for any \( (f_n)_{n \geq 0} \subset L^p(\Omega, \Sigma, \mu; X) \) and any choice of signs \( (\xi_n)_{n \geq 0} \subset (-1, 1) \) and any \( N \in \mathbb{Z}_+ \), we have the following estimate

\[
\left\| f_0 + \sum_{n=1}^{N} \xi_n (f_n - f_{n-1}) \right\|_{L^p(\Omega, \Sigma, \mu; X)} \leq C_p \| f_N \|_{L^p(\Omega, \Sigma, \mu; X)}.
\]

In what follows, we denote by \( B(X, Y) \) the space of bounded linear operators between Banach spaces \( X \) and \( Y \) endowed with the uniform operator topology; when \( X = Y \), we denote it by \( B(X) \).

Let \( X \) and \( Y \) be a Banach spaces. A subset of \( T \subset B(X, Y) \) is called \( R \)-bounded if there is a constant \( c > 0 \) such that

\[
\sum_{\varepsilon_j \in \{-1, 1\}^n} \left\| \sum_{j=1}^{n} \varepsilon_j T_j x_j \right\| \leq c \sum_{\varepsilon_j \in \{-1, 1\}^n} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|
\]

for all \( T_1, \ldots, T_n \in T, x_1, \ldots, x_n \in X \), \( n \in \mathbb{N} \).

Given \( u \in \ell_\mu(\mathbb{Z}; X) \) and \( v \in \ell_1(\mathbb{Z}) \), we define the convolution product

\[
(u * v)(n) := \sum_{j=-\infty}^{n} u(n-j)v(j) = \sum_{j=0}^{\infty} u(j)v(n-j), \quad n \in \mathbb{Z}.
\]

For any \( \alpha \in \mathbb{R} \), we set

\[
k^\alpha(n) = \begin{cases} \frac{n^{\alpha+1} \ldots (\alpha+n-1)}{n!} & \text{if } n \in \mathbb{N}_0, \\ 0 & \text{otherwise,} \end{cases}
\]

Some properties related to the special kernel \( k^\alpha \) can be found in Abadias et al, Section 2 and Lizama.13

**Definition 2.1.** Let \( \alpha > 0 \) be given and \( f : \mathbb{Z} \to X \) a vector-valued sequence. We define the fractional sum of order \( \alpha \) as follows

\[
\Delta^{-\alpha} f(n) := (k^\alpha * f)(n) = \sum_{j=-\infty}^{n} k^\alpha(n-j) f(j), \quad n \in \mathbb{Z},
\]

and the fractional difference of order \( \alpha \) is defined by

\[
\Delta^\alpha f(n) := (k^{-\alpha} * f)(n) = \sum_{j=-\infty}^{n} k^{-\alpha}(n-j) f(j) = \sum_{j=0}^{\infty} k^{-\alpha}(j) f(n-j).
\]

Given \( \alpha \in \mathbb{R} \setminus \{-1, -2, \ldots\} \), we have \( k^\alpha(n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha) \Gamma(n+1)} \), where \( \Gamma \) is the Euler function. see Ortigueira et al, formula (27), with \( h = 1 \). The numbers \( k^\alpha \) are known as Cesàro numbers of order \( \alpha \). They were introduced by Zygmund37, p. 77 and rediscovered in several instances. The above definition of fractional difference operator of order \( \alpha \) (for \( n \in \mathbb{Z} \)) was first introduced by Abadias and Lizama,9 after previous work of Lizama,13 as follows:

\[
W^\alpha f(n) := (-1)^m \Delta^m W^{-(m-\alpha)} f(n), \quad n \in \mathbb{Z},
\]

where \( m := [\alpha] + 1 \). In the above cited references, it is also called Weil difference operator of order \( \alpha \) and is denoted by \( W \) instead of \( \Delta \). Their equivalence with (2.1) was recently proved.
In what follows, we detail the definition and properties of the DTFT in the vector-valued Lebesgue space of sequences \( \ell_p(\mathbb{Z}, X) \). We denote by \( S(\mathbb{Z}; X) \) the space of all vector-valued sequences \( f : \mathbb{Z} \to X \) such that for each \( k \in \mathbb{N}_0 \), there exists \( C_k \) with \( p_k(f) := \sup_{n \in \mathbb{Z}} |n|^k \| f(n) \| < C_k \).

We also denote by \( C_{per}^n(\mathbb{R}; X), n \in \mathbb{N}_0 \), the space of all \( 2\pi \)-periodic \( X \)-valued and \( n \)-times continuously differentiable functions defined in \( \mathbb{R} \).

Let \( \mathbb{T} := (-\pi, \pi) \) and \( \mathbb{T}_0 := (-\pi, \pi) \setminus \{0\} \). We introduce the space of test functions as \( C_{per}^\infty(\mathbb{T}; X) := \bigcap_{p \in \mathbb{N}_0} C_{per}^p(\mathbb{T}; X) \) endowed with the topology induced by the countable family of seminorms:

\[
q_k(\varphi) = \max_{k \in \mathbb{N}_0} \sup_{t \in [-\pi, \pi]} \| \varphi^{(k)}(t) \|.
\]

If \( X = \mathbb{C} \), we simply denote \( C_{per}^\infty(\mathbb{T}; X) = C_{per}^\infty(\mathbb{T}) \) and \( S(\mathbb{Z}; X) = S(\mathbb{Z}) \). We also consider the following spaces of vector-valued distributions

\[
S'(\mathbb{Z}; X) := \{ T : S(\mathbb{Z}) \to X : T \text{ is linear and continuous} \}
\]

and

\[
D'(\mathbb{T}; X) := \{ T : C_{per}^\infty(\mathbb{T}) \to X : T \text{ is linear and continuous} \}.
\]

Observe that we can identify \( \ell_p(\mathbb{Z}, X) \) with a subspace of \( S'(\mathbb{Z}; X) \) via the mapping

\[
T_f(\psi) := \langle T_f, \psi \rangle := \sum_{n \in \mathbb{Z}} f(n)\psi(n), \quad \forall \psi \in S(\mathbb{Z}),
\]

and we have \( T_f \in S'(\mathbb{Z}, X) \). The space \( C_{per}^\infty(\mathbb{T}; X) \) can be also identified with a subspace of \( D'(\mathbb{T}; X) \) via the linear map

\[
L_S(\varphi) := \langle L_S, \varphi \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)S(t)dt, \quad \forall \varphi \in C_{per}^\infty(\mathbb{T}).
\]

and we get \( L_S \in D'(\mathbb{T}; X) \).

The discrete time Fourier transform \( F : S(\mathbb{Z}; X) \to C_{per}^\infty(\mathbb{T}; X) \) is given by

\[
F \varphi(t) \equiv \hat{\varphi}(t) := \sum_{j=-\infty}^{\infty} e^{-ijt} \varphi(j), \quad t \in (-\pi, \pi].
\]

It is an isomorphism whose inverse is defined by

\[
F^{-1} \varphi(n) \equiv \tilde{\varphi}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)e^{int}dt, \quad n \in \mathbb{Z},
\]

where \( \varphi \in C_{per}^\infty(\mathbb{T}; X) \). This isomorphism allows to define the discrete time Fourier transform (DTFT) between the spaces of distributions \( S'(\mathbb{Z}; X) \) and \( D'(\mathbb{T}; X) \) as follows

\[
\langle FT, \psi \rangle \equiv F(T)(\psi) := \hat{T}(\psi) \equiv \langle T, \hat{\psi} \rangle, \quad T \in S'(\mathbb{Z}; X), \quad \psi \in C_{per}^\infty(\mathbb{T}),
\]

whose inverse \( F^{-1} : D'(\mathbb{T}; X) \to S'(\mathbb{Z}; X) \) is given by

\[
\langle F^{-1}L, \psi \rangle \equiv F^{-1}(L)(\psi) := \hat{L}(\psi) \equiv \langle L, \hat{\psi} \rangle, \quad L \in D'(\mathbb{T}; X), \quad \psi \in S(\mathbb{Z}).
\]

In particular, we get

\[
\langle FT_f, \varphi \rangle = \langle T_f, \hat{\varphi} \rangle = \sum_{n \in \mathbb{Z}} f(n)\hat{\varphi}(n), \quad \forall \varphi \in C_{per}^\infty(\mathbb{T}), \quad f \in \ell_p(\mathbb{Z}, X),
\]
The convolution of a distribution \( T \in S'(\mathbb{Z}, X) \) with a function \( a \in \ell_1(\mathbb{Z}) \) is defined by

\[
\langle T \ast a, \varphi \rangle := \langle T, a \circ \varphi \rangle, \quad \varphi \in S(\mathbb{Z}),
\]

where

\[
(a \circ \varphi)(n) := \sum_{j=0}^{\infty} a(j) \varphi(j + n).
\]

From Lizama\textsuperscript{13}, the following generation formula holds

\[
\sum_{j=0}^{\infty} k^j(j)z^j = \frac{1}{(1 - z)^{1 - \beta}}, \beta \in \mathbb{R}, |z| < 1,
\]

see also Zygmund\textsuperscript{37}, p.42 formulae (1) and (8). In particular, for all \( \alpha \in \mathbb{R}_+ \), we have that the radial limit exists and

\[
\hat{k}^{-\alpha}(t) = \sum_{j=0}^{\infty} k^{-\alpha}(j)e^{-jt} = \frac{1}{(1 - e^{-it})^{-\alpha}} = (1 - e^{-it})^{-\alpha}, \quad t \in \mathbb{T}_0.
\] (10)

Observe that \( k^{-\alpha} \in \ell_1(\mathbb{Z}) \) (see also Zygmund\textsuperscript{37}, p.42 formula (2)). We also recall the following lemma stated in Lizama and Murillo-Arcila\textsuperscript{38} which will be used in the proof of our main result.

**Lemma 2.2.** Let \( u, v \in \ell_p(\mathbb{Z}; X) \) and \( a \in \ell_1(\mathbb{Z}) \). The following assertions are equivalent:

(i) \( (a \ast u)(n) = v(n) \) for all \( n \in \mathbb{Z} \).

(ii) \( < u, \phi > = < v, (\phi \cdot \hat{\alpha}) > \) for all \( \phi \in C^{\infty}((-\pi, \pi), \mathbb{R}) \), where

\[
(\phi \cdot \hat{\alpha})(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\alpha}(-t)\phi(t)e^{int} dt, \quad n \in \mathbb{Z}.
\]

We can now introduce the following notion of \( \ell_p \)-multiplier.

**Definition 2.3.** Let \( X, Y \) be Banach spaces, \( 1 < p < \infty \). A function \( M \in C_{\text{per}}(\mathbb{T}, B(X, Y)) \) is an \( \ell_p \)-multiplier (from \( X \) to \( Y \)) if there exists a bounded operator \( T : \ell_p(\mathbb{Z}; X) \to \ell_p(\mathbb{Z}; Y) \) such that

\[
\sum_{n \in \mathbb{Z}} (Tf)(n) \hat{\varphi}(n) = \sum_{n \in \mathbb{Z}} (\varphi \cdot M \hat{\varphi})(n) f(n)
\] (11)

for all \( f \in \ell_p(\mathbb{Z}; X) \) and all \( \varphi \in C^{\infty}_{\text{per}}(\mathbb{T}) \). Here,

\[
(\varphi \cdot M \hat{\varphi})(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} \varphi(t)M(-t) dt, \quad n \in \mathbb{Z}.
\]

We finally recall the following Fourier multiplier theorem for operator-valued symbols due to Blunck, see previous studies\textsuperscript{29,34} for more details. This theorem will be crucial for showing our main characterization. Blunck’s theorem and its converse establish an equivalence between \( R \)-bounded sets and \( l_p \)-multipliers.

**Theorem 2.4.** (Blunck\textsuperscript{28}, Theorem 1.3)

Let \( p \in (1, \infty) \) and let \( X, Y \) be UMD spaces. Let \( M \in C^{\infty}_{\text{per}}(\mathbb{T}, B(X; Y)) \) such that the sets

\[
\{ M(t), (1 - e^{t})(1 + e^{t})M'(t) : \ t \in \mathbb{T}_0 \}
\]

are both \( R \)-bounded. Then \( M \) is an \( \ell_p \)-multiplier (from \( X \) to \( Y \)) for \( 1 < p < \infty \).

The converse of Blunck’s theorem holds without any restriction on the Banach spaces \( X, Y \) in the following sense.
Theorem 2.5. (Blunck29, Proposition 1.4)
Let \( p \in (1, \infty) \) and let \( X, Y \) be Banach spaces. Let \( M : T \to B(X, Y) \) be an operator valued function. Suppose that there is a bounded operator \( T_M : L_p(\mathbb{Z}; X) \to L_p(\mathbb{Z}; Y) \) such that (11) holds. Then the set \( \{ M(t) : t \in T \} \) is R-bounded.

3 | A CHARACTERIZATION OF MAXIMAL REGULARITY

In this section, we first provide a characterization on the existence and uniqueness of solutions in \( \ell_p(\mathbb{Z}; [D(A)]) \) for the general model

\[
\Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + f(n),
\]

where \( \alpha, \beta > 0, \lambda \geq 0, A \) is a closed linear operator defined on a Banach space \( X \) and \( f : \mathbb{Z} \to X \) is a vector-valued sequence. Recall that the above model is an abbreviated form to write a partial differential equation, which is continuous in space but discrete in time. For example, the equation

\[
u(n + 2, x) - 2u(n + 1, x) + u(n, x) + \lambda[\nu(n + 1, x) - u(n, x)] = \partial_x \nu + f(n, x),
\]

where \( n \in \mathbb{Z}, x \in \Omega \subset \mathbb{R}^N \), fits in the abstract setting of the model (12) with \( \alpha = 2, \beta = 1, \) and \( A = \partial_x \).

We introduce the following definition, also called \( \ell_p \)-well-posedness in the literature.

Definition 3.1. Let \( 1 < p < \infty \). We say that (12) has maximal \( \ell_p \)-regularity if for each \( f \in \ell_p(\mathbb{Z}; X) \), there exists a unique \( u \in \ell_p(\mathbb{Z}; [D(A)]) \) that satisfies (12).

We are ready to prove our main result in this paper.

Theorem 3.2. Let \( A \) be a closed linear operator defined on an UMD space \( X \). Set \( \alpha, \beta > 0, \) and \( \lambda \geq 0 \). Suppose that

\[
\{(1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta \}_{t \in \mathbb{T}} \subset \rho(A)
\]

and define \( M(t) := ((1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta - A)^{-1} \). Then the following assertions are equivalent

(i) (12) has maximal \( \ell_p \)-regularity.
(ii) \( M(t) \) is an \( \ell_p \)-multiplier from \( X \) to \([D(A)]\).
(iii) The set \( \{M(t)\}_{t \in \mathbb{T}} \) is R-bounded.

Proof. We first show (iii) \( \Rightarrow \) (ii). Let \( \{M(t) : t \in \mathbb{T}\} \) be R-bounded. We will show that the set \( \{(1 - e^{it})(1 + e^{it})M(t) : t \in \mathbb{T}\} \) is also R-bounded. Defining for each \( t \in \mathbb{T}, f_\alpha(t) := (1 - e^{-it})^\alpha \) and \( f_\beta(t) := (1 - e^{-it})^\beta \), it can be shown that

\[
M'(t) = -M(t)(f'_\alpha(t) + \lambda f'_\beta(t)).
\]

Since \( f'_\alpha(t) = i\alpha f_\alpha(t)\frac{1}{1 - e^{it}}, \) it follows that \( (1 - e^{it})(1 + e^{it})M'(t) = i(\alpha f_\alpha(t) + \beta f_\beta(t))(1 + e^{it})M(t)^2. \) From Agarwal et al., Proposition 2.2.5 we deduce that the set \( \{(1 - e^{it})(1 + e^{it})M(t) : t \in \mathbb{T}\} \) is R-bounded and the claim is proved. Consequently, by Theorem 2.4, we obtain (i). The implication (ii) \( \Rightarrow \) (iii) follows immediately from Theorem 2.5. Let us now show that (i) \( \Rightarrow \) (ii). Let \( f \in \ell_p(\mathbb{Z}, X) \) be given. Then there exists a unique \( u_f \in \ell_p(\mathbb{Z}, [D(A)]) \) solution of (12). We define \( T_{u,f} : \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, [D(A)]) \) the linear operator given by \( T_{u,f}(f) = u_f. \) By the Closed Graph Theorem, we get that \( T_{u,f} \) is bounded.

Let \( \varphi \in C_{per}^\infty(\mathbb{R}), f \in \ell_p(\mathbb{Z}; X) \) and \( u = T_{u,f}f. \) Since \( k^{-\alpha} \in \ell_1(\mathbb{Z}), \) we obtain the following identity,

\[
(k^{-\alpha} \ell S)(n) = \sum_{j=0}^{\infty} k^{-\alpha}(j) \ell(j + n) = \sum_{j=0}^{\infty} k^{-\alpha}(j) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it(j+n)} S(t)dt
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\pi} \left( \sum_{j=0}^{\infty} e^{it(j)} \right) S(t)dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\pi} (-t) S(t)dt = (k^{-\alpha} \ell S)(n),
\]

(13)
valid for any $S \in C^\infty_{per}(\T,B(X,Y))$. Therefore, using the hypothesis and the observation that we have $M \in C^\infty_{per}(\T,B(X,[D(A)])$, we get

$$
\langle T_{a,f},\tilde{\varphi} \rangle = \langle u,\tilde{\varphi} \rangle = \sum_{n \in \Z} \tilde{\varphi}(n)u(n) = \sum_{n \in \Z} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int}\varphi(t)u(n)dt
$$

$$
= \sum_{n \in \Z} \frac{1}{2\pi} \int_{-\pi}^{\pi} [(1-e^{-it})^\alpha e^{int}\varphi(t)((1-e^{it})^\alpha + \lambda(1-e^{-it})^\beta - A)^{-1}u(n)dt
$$

$$
+ \lambda \sum_{n \in \Z} \frac{1}{2\pi} \int_{-\pi}^{\pi} (1-e^{it})^\beta e^{int}\varphi(t)((1-e^{it})^\alpha + \lambda(1-e^{-it})^\beta - A)^{-1}u(n)dt
$$

$$
- \sum_{n \in \Z} \frac{1}{2\pi} \int_{-\pi}^{\pi} ((1-e^{it})^\alpha + \lambda(1-e^{-it})^\beta - A)^{-1}Au(n)e^{int}\varphi(t)dt
$$

$$
= \sum_{n \in \Z} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int}\tilde{k}^{-\alpha}(-t)\varphi(t)M(-t)u(n)dt
$$

$$
+ \lambda \sum_{n \in \Z} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int}\tilde{k}^{-\beta}(-t)\varphi(t)M(-t)u(n)dt
$$

$$
- \sum_{n \in \Z} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int}\varphi(t)M(-t)Au(n)dt
$$

$$
= \langle u,((k^{-\alpha} + \lambda k^{-\beta})\varphi \cdot M(t)) \rangle - \langle Au,(\varphi \cdot M(t)) \rangle
$$

$$
= \langle u,(k^{-\alpha} + \lambda k^{-\beta})\sigma(\varphi \cdot M(t)) \rangle - \langle Au,(\varphi \cdot M(t)) \rangle.
$$

where in the last equality, we have used (13) with $S = \varphi \cdot M(t)$. Therefore

$$
\langle u,\tilde{\varphi} \rangle = \langle (k^{-\alpha} + \lambda k^{-\beta}) \ast u,(\varphi \cdot M(t)) \rangle - \langle Au,(\varphi \cdot M(t)) \rangle
$$

$$
= \langle \Delta^\alpha u + \lambda \Delta^\beta u - Au,(\varphi \cdot M(t)) \rangle.
$$

We conclude that $\langle T_{a,f},\tilde{\varphi} \rangle = \langle f,(\varphi \cdot M(t)) \rangle$ and then $M(t)$ is an $\ell_p$-multiplier.

It only remains to prove that (ii) implies (i). We first claim that $N(t) := (1-e^{-it})^\alpha((1-e^{it})^\alpha + \lambda(1-e^{-it})^\beta - A)^{-1}$ and $S(t) = (1-e^{it})^\beta((1-e^{it})^\alpha + \lambda(1-e^{-it})^\beta - A)^{-1}$ are $\ell_p$-multipliers. Indeed, since $N(t) = f_\alpha(t)M(t)$ and $S(t) = f_\beta(t)M(t)$ where $f_\alpha(t) = (1-e^{it})^\alpha$ and $f_\beta(t) = (1-e^{-it})^\beta$, the $R$-boundedness of $N(t)$ and $S(t)$ follows. On the other hand, the identities

$$
(1-e^{it})(1+e^{it})N'(t) = -i\alpha N(t)(1+e^{it})
$$

$$
+ i\alpha N(t)^2(1+e^{it}) + i\beta N(t)S(t)(1+e^{it})
$$

$$
(1-e^{-it})(1+e^{-it})S'(t) = -i\beta S(t)(1+e^{-it})
$$

$$
+ i\beta S(t)N(t)(1+e^{-it}) + i\beta S(t)^2(1+e^{-it})
$$

show that the sets $\{(1-e^{it})(1+e^{it})N'(t) : t \in \T\}$ and $\{(1-e^{-it})(1+e^{-it})S'(t) : t \in \T\}$ are $R$-bounded and then the claim holds by Theorem 2.4. Let $f \in \ell_p(\Z;X)$ be given. By hypothesis, there exists $u \in \ell_p(\Z;[D(A)])$ such that

$$
\sum_{n \in \Z} u(n)\tilde{\varphi}(n) = \sum_{n \in \Z} (\varphi \cdot M(t))(n)f(n)
$$

(15)

for all $\varphi \in C^\infty_{per}(\T)$. On the other hand, there exist $v, w \in \ell_p(\Z;[D(A)])$ such that

$$
\sum_{n \in \Z} v(n)\tilde{\psi}(n) = \sum_{n \in \Z} (\psi \cdot N(t))(n)f(n)
$$

(16)
for all \( \psi, \eta \in C^\infty_{per}(\mathbb{T}) \). Since \( N(t) = \hat{k}^{-\alpha}(t)M(t) \), we have

\[
\sum_{n \in \mathbb{Z}} w(n)\tilde{\eta}(n) = \sum_{n \in \mathbb{Z}} (\eta \cdot S_{-\lambda})(n)f(n)
\]

(17)

Choosing \( \varphi(t) = \psi(t)k^{-\alpha}(-t) \) in (15), we obtain \( \langle v, \psi \rangle = \langle u, (\varphi \cdot k^{-\alpha}) \rangle \), and hence, by Lemma 2.2, we get that \( k^{-\alpha} \ast u \in \ell_p(\mathbb{Z}; X) \) and

\[
\Delta^\alpha u(n) = k^{-\alpha} \ast u(n) = v(n), \quad n \in \mathbb{Z}.
\]

(18)

Analogously, since \( S(t) = \hat{k}^{-\beta}(t)M(t) \), we can choose \( \varphi(t) = \eta(t)k^{-\beta}(-t) \) in (15) and then, by Lemma 2.2, we get that \( k^{-\beta} \ast u \in \ell_p(\mathbb{Z}; X) \) and

\[
\Delta^\beta u(n) = k^{-\beta} \ast u(n) = v(n), \quad n \in \mathbb{Z}.
\]

(19)

Now, from the identity \( N(t) + \lambda S(t) = AM(t) + I \), we obtain after multiplication by \( e^{int}\varphi(t) \) and then integration on the interval \((-\pi, \pi)\), the identity \( (\varphi \cdot N_\lambda)(n) + \lambda(\varphi \cdot S_{-\lambda})(n) = A(\varphi \cdot M_{-\lambda})(n) + \hat{\varphi}(n)I \), for all \( \varphi \in C^\infty_{per}(\mathbb{T}) \). Then we get

\[
\langle f, (\varphi \cdot N_{-\lambda}) \rangle + \langle f, \lambda(\varphi \cdot S_{-\lambda}) \rangle = \langle f, A(\varphi \cdot M_{-\lambda}) \rangle + \langle f, \hat{\varphi} \rangle.
\]

Replacing (15), (16), and (17) in the above identity, we obtain

\[
\sum_{n \in \mathbb{Z}} v(n)\hat{\varphi}(n) + \lambda \sum_{n \in \mathbb{Z}} w(n)\hat{\varphi}(n) = \sum_{n \in \mathbb{Z}} Au(n)\hat{\varphi}(n) + \sum_{n \in \mathbb{Z}} \hat{\varphi}(n)f(n),
\]

for all \( \varphi \in C^\infty_{per}(\mathbb{T}) \). Considering (18), (19), and replacing \( \varphi_k(t) := e^{-ikt}, \; k \in \mathbb{Z} \) in the above identity, we conclude that \( u \) satisfies the Equation (12).

The following statement follows from the closed graph theorem and Theorem 3.2.

**Corollary 3.3.** In the context of Theorem 3.2, if condition (iii) is valid, we have \( u, \Delta^\alpha u, \Delta^\beta u \in \ell_p(\mathbb{Z}, X) \). Moreover, there exists a constant \( C > 0 \) such that

\[
\|\Delta^\alpha u\|_p + \lambda\|\Delta^\beta u\|_p + \|Au\|_p \leq C\|f\|_p.
\]

(20)

As a consequence of Theorem 3.2, we easily have a corresponding one in the case of Hilbert spaces, where \( R \)-boundedness is equivalent to norm boundedness.\(^{35, \text{Chapter 3, Remark 3.2}}\)

**Corollary 3.4.** Let \( H \) be a Hilbert space and \( \alpha, \beta > 0, \lambda \geq 0 \). Suppose that \( \{(1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta\}_{t \in \mathbb{T}} \subseteq \rho(A) \). The following assertions are equivalent:

(i) For all \( f \in \ell_p(\mathbb{Z}, H) \), there exists a unique \( u \in \ell_p(\mathbb{Z}, H) \) such that \( u(n) \in D(A) \) for all \( n \in \mathbb{Z} \) and \( u \) satisfies (12);

(ii) \( \sup_{n \in \mathbb{T}} \|((1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta - A)^{-1}\| < \infty \).

Now, we can consider the nonlinear perturbed version of (12) given by

\[
\Delta^\alpha u(n) + \lambda\Delta^\beta u(n) = Au(n) + G(u(n)) + \rho f(n),
\]

(21)

where \( \rho > 0, f \in \ell_p(\mathbb{Z}, X) \) and \( G : \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, X) \). We can show a result concerning the existence of \( \ell_p(\mathbb{Z}, X) \)-solutions of (21) in terms of the symbol of the equation and the regularity of \( G \).
**Theorem 3.5.** Let $X$ be a UMD space, $1 < p < \infty$, $\alpha, \beta > 0$, and $\lambda \geq 0$. Suppose that \((1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta\)_{t \in \mathbb{R}} \subset \rho(A)$. If the following conditions hold

(i) the set \{(1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta - A\}_{t \in \mathbb{R}} is R-bounded;
(ii) $G$ is continuously Fréchet differentiable at $u = 0$ and $G(0) = G'(0) = 0$.

Then there exists $\rho^*$ such the Equation 21 has a solution $u = u_\rho \in \ell_p(\mathbb{Z}, X)$ for each $\rho \in [0, \rho^*)$.

**Proof.** We note that $|||u||| := \|\Delta^u \| + \lambda \|\Delta^f u\| + \|Au\| + \|u\|$ defines a norm in $\ell_p(\mathbb{Z}, [D(A)])$, and hence, \((\ell_p(\mathbb{Z}, [D(A)]), |||\cdot|||)\) becomes a Banach space. Let $L : \ell_p(\mathbb{Z}, [D(A)]) \rightarrow \ell_p(\mathbb{Z}, [D(A)])$ be defined as $(Lu)(n) := \Delta^u(n) + \lambda \Delta^f u(n) - Au(n)$. By Corollary 3.4, and since hypothesis (i) holds, we have that the inequality (20) holds, and then we obtain $|||u||| \leq C \|Lu\|$ for some constant $C > 0$. Also, by definition of $L$, we have $\|Lu\| \leq |||u|||$. Then $L$ defines an isomorphism. Given $\rho \in [0, 1)$, we define:

$$H[u, \rho] = -Lu + G(u) + \rho f.$$  

By hypothesis (ii), we have $H[0, 0] = 0$ and $H$ is continuously differentiable at $(0, 0)$. In addition, $H^1_{(0,0)} = -L$, which is invertible. Therefore, using the implicit function theorem, we deduce the existence of $\rho^*$ such that for all $\rho \in [0, \rho^*)$, there exists $u_\rho \in \ell_p(\mathbb{Z}, X)$ such that $H[u_\rho, \rho] = 0$. This proves the theorem. 

**4 | EXAMPLES**

We verify the conditions provided in Theorem 3.5 to show the existence and uniqueness of $\ell_p(\mathbb{Z}, X)$ solutions for the following equation

$$\Delta^u(n, x) + \lambda \Delta^f u(n, x) + bu(n, x) - \frac{\partial^2}{\partial x^2} u(n, x) = \cos(u(n, x)) \frac{e^{-x^2}}{n^2} + \epsilon f(n, x),$$

where $\lambda \geq 0$ is fixed, $b$ is a real number, $\epsilon > 0$ and $f \in \ell_p(\mathbb{Z}, X)$ is an external force whose size is controlled by $\epsilon$. Note that the linear part of the Equation 22 corresponds to the discrete time Telegraph equation when $\alpha = 2$, $\beta = 1$, $b = 0$, $\lambda = \frac{\epsilon}{\pi}$, and $A = \frac{\epsilon}{\pi} \partial_{xx}$. Also, it coincides with the discrete time Klein-Gordon equation for $\alpha = 2$, $\lambda = 0$, and $A = \partial_{xx} - bI$.

Equation 22 can be modeled as (21) for $Au = u'' - bu$ defined on $L^2(\mathbb{R})$ and $G(u)(n, x) = \cos(u(n, x)) \frac{e^{-x^2}}{n^2}$, for instance. It is well known that the operator $Bu = u''$ with domain $D(B) = H_0^2(\mathbb{R})$ generates a contraction $C_0$-semigroup on $L^2(\mathbb{R})$; therefore, the following estimate for their resolvent operator holds

$$\|(\mu - B)^{-1}\| \leq \frac{1}{\Re(\mu)} \text{, for all } \Re(\mu) > 0.$$  

We observe that, for all $0 < \alpha, \beta \leq 2$ we have

$$\Re((1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta) = (2 - 2 \cos(t))^\alpha \cos \left( \alpha \arctan \left( \frac{\sin(t)}{1 - \cos(t)} \right) \right)$$

$$+ \lambda (2 - 2 \cos(t))^\beta \cos \left( \beta \arctan \left( \frac{\sin(t)}{1 - \cos(t)} \right) \right)$$

$$> (2 - 2 \cos(t))^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + \lambda (2 - 2 \cos(t))^\beta \cos \left( \frac{\beta \pi}{2} \right)$$

$$> -(2^\alpha + \lambda 2^\beta).$$

As a consequence, for all $b \geq 2^\alpha + \lambda 2^\beta$, we get that $(1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta \in \rho(A)$, and (23) shows that

$$\sup_{t \in \mathbb{R}} \|\{(1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta - A\}_{t \in \mathbb{R}} \leq \frac{1}{b - (2^\alpha + \lambda 2^\beta)} < \infty.$$
Furthermore, $G$ is a Fréchet differentiable function at $u = 0$ and clearly satisfies that $G'(0) = 0$. Then, by Theorem 3.5, we conclude that whenever $b > 2^\alpha + 2^\beta$, there exists a number $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$, there exists a solution $u^\epsilon \in \ell_p(\mathbb{Z}, X)$ of the perturbed fractional damping difference Equation 22.

**Remark 4.1.** In particular, this example shows that the discrete Klein-Gordon Equation 1 admits nontrivial square-summable solutions defined on $\mathbb{Z}$, for small and square-summable external forcing terms whenever $b > 4$. In the case of the generalized discrete Basset equation

$$\Delta^2 u(n) + \lambda \Delta^\beta u(n) + bu(n) = f(n), \quad n \in \mathbb{Z}, \quad \lambda, b > 0, \quad \beta > 0,$$

we obtain that for any $f \in \ell_p(\mathbb{Z})$, there exist $p$-summable solutions whenever $b > 4$ and $\lambda < \frac{b-4}{2^\beta}$.

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**REFERENCES**

1. Bateman H. Some simple differential difference equations and the related functions. *Bull Amer Math Soc*. 1943;49:494-512.
2. Agarwal RP. *Difference Equations and Inequalities. Theory, Methods, and Applications.* Second edition. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 228. New York: Marcel Dekker, Inc.; 2000.
3. Elaydi S. *An Introduction to Difference Equations.* 3rd ed., Undergraduate Texts in Mathematics. New York: Springer-Verlag; 2005.
4. Atanackovic TM, Pilipovic S, Zorica D. A diffusion wave equation with two fractional derivatives of different order. *Phys, J A: Math Theory*. 2007;40:5319-5333.
5. Golmankhaneh AK, Golmankhaneh A, Baleanu D. On nonlinear fractional Klein-Gordon equation. *Signal Process*. 2011;91:446-451.
6. Mainardi F. *Fractional Calculus, Some Basic Problems in Continuum and Statistical Mechanics.* International Centre for Mechanical Sciences Palazzo del Torso, Piazza Garibaldi, Udine, Italy: CISM lectures; 1997.
7. Baleanu D, Garra R, Petras I. A fractional variational approach to the fractional Basset-type equation. *Rep Math Phys*. 2013;72(1):57-64.
8. Beyer H, Kempfle S. Definition of physically consistent damping laws with fractional derivatives. *Z Angew Math Mech*. 1995;75(8):623-635.
9. Abadias L, Lizama C. Almost automorphic mild solutions to fractional partial difference-differential equations. *Appl Anal*. 2016;95(6):1347-1369.
10. Ortigueira MD, Coito FJV, Trujillo JJ. Discrete-time differential systems. *Signal Process*. 2015;107:198-217.
11. Atici FM, Eloe PW. Initial value problems in discrete fractional calculus. *Proc Amer Math Soc*. 2009;137(3):981-989.
12. Lizama C. The Poisson distribution, abstract fractional difference equations, and stability. *Proc Amer Math Soc*. 2017;145(9):3809-3827.
13. Dahal R, Goodrich CS. A monotonicity result for discrete fractional difference operators. *Arch Math (Basel)*. 2014;102(3):293-299.
14. Goodrich CS. Existence of a positive solution to a system of discrete fractional boundary value problems. *Appl Math Comput*. 2011;217:4740-4753.
15. Goodrich CS. A convexity result for fractional differences. *Appl Math Lett*. 2014;35:58-62.
16. Ferreira R. Existence and uniqueness of solution to some discrete fractional boundary value problems of order less than one. *J Differ Equ Appl*. 2013;19:712-718.
17. Holm MT. The Laplace transform in discrete fractional calculus. *Comp Math Appl*. 2011;62(3):1591-1601.
18. Kovács B, Li B, Lubich C. A-Stable time discretizations preserve maximal parabolic regularity. *SIAM J Numer Anal*. 2016;54(6):36003624.
19. Dassios IK. Stability and robustness of singular systems of fractional nabla difference equations. *Circuits Syst Signal Process*. 2016;36(1):49-64.
20. Dassios IK. Geometric relation between two different types of initial conditions of singular systems of fractional nabla difference equations. *Math Meth Appl Sci*. 2017;40(17):6085-6095.
21. Dassios IK, Baleanu DI. Duality of singular linear systems of fractional nabla difference equations. *Appl Math Modell*. 2015;39(14):4180-4195.
22. Wu GC, Baleanu D. Discrete chaos in fractional delayed logistic maps. *Nonlinear Dyn*. 2016;80(4):1697-1703.
23. Wu GC, Baleanu D, Xie H-P. Riesz Riemann-Liouville difference on discrete domains. *Chaos*. 2016;26(8):084308, 5.
24. Wu GC, Baleanu D, Deng ZG, Zeng SD. Lattice fractional diffusion equation in terms of a Riesz-Caputo difference. *Phys A*. 2015;438:335-339.
25. Tarasov VE. Fractional-order difference equations for physical lattices and some applications. *J Math Phys*. 2015;56(10):1-19.
26. Tarasov VE. Fractional Liouville equation on lattice phase-space. *Phys A*. 2015;421:330-342.
27. Tarasova VV, Tarasov VE. Logistic map with memory from economic model. *Chaos Solitons Fractals*. 2017;95:84-91.
28. Blunck S. Maximal regularity of discrete and continuous time evolution equations. *Studia Math*. 2001;146(2):157-176.
30. Jin B, Li B, Zhou Z. Discrete maximal regularity of time-stepping schemes for fractional evolution equations. arXiv:1606.07587v1 [math.NA].
31. Kemmochi T. Discrete maximal regularity for abstract Cauchy problems. Studia Math. 2016;234(3):241-263.
32. Lizama C. $\ell_p$-maximal regularity for fractional difference equations on UMD spaces. Math Nach. 2015;288(17/18):2079-2092.
33. Lizama C, Murillo-Arcila M. $\ell_p$-maximal regularity for a class of fractional difference equations on UMD spaces: the case $1 < \alpha < 2$. Banach. J Math Anal. 2017;11(1):188-206.
34. Agarwal RP, Cuevas C, Lizama C. Regularity of Difference Equations on Banach Spaces. Cham: Springer-Verlag; 2014.
35. Denk R, Hieber M, Prüss J. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem Amer Math Soc. 2003;166(788).
36. Abadías L, Lizama C, Miana PJ, Velasco MP. Cesàro sums and algebra homomorphisms of bounded operators. Israel J Math. 2016;216(1):471-505.
37. Zygmund A. Trigonometric Series. 2nd ed., Vol. I, II. New York (NY): Cambridge University Press; 1959.
38. Lizama C, Murillo-Arcila M. Maximal regularity in $l_p$ spaces for discrete time fractional shifted equations. J Differ Equ. 2017;263(6):3175-3196.

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