K-theory of torus manifolds

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Abstract

The torus manifolds have been defined and studied by Masuda and Panov ([7]) who in particular also describe its cohomology ring structure. In this note we shall describe the topological $K$-ring of a class of torus manifolds (those for which the orbit space under the action of the compact torus is a homology polytope whose nerve is shellable) in terms of generators and relations. Since these torus manifolds include the class of quasi-toric manifolds this is a generalisation of our earlier results ([11]).

1 Introduction

We shall first briefly recall the notations and basic definitions from [7].

Let $M$ be a $2n$-dimensional closed connected orientable smooth manifold with an effective smooth action of an $n$-dimensional torus $T = (S^1)^n$ such that $MT \neq \emptyset$. Since $\dim(M) = 2\dim(T)$ and $M$ is compact, the fixed point set $M^T$ is a finite set of isolated points.

A closed connected codimension-two submanifold of $M$ is called characteristic if it is the connected component of the set fixed pointwise by a certain circle subgroup of $T$ and contains at least one $T$-fixed point. Since $M$ is compact there are only finitely many characteristic submanifolds. We denote them by $M_i (i = 1, \ldots, m)$. Note that each $M_i$ is orientable. We say that $M$ is omnioriented if an orientation is fixed for $M$ and for every characteristic submanifold $M_i$. Further, $M$ is called a torus manifold when it is omnioriented.

Let $Q := M/T$ denote the orbit space of $M$ and $\pi : M \to Q$ the quotient projection. We define the facets of $Q$ to be the orbit spaces of the characteristic submanifolds: $Q_i := \pi(M_i), i = 1, \ldots, m$. Every facet is a closed connected subset in $Q$ of codimension 1. We refer to a non-empty intersection of $k$-facets as a codimension-$k$ preface, $k = 1, \ldots, n$. Hence a preface is the orbit space of some non-empty intersection $M_{i_1} \cap \cdots \cap M_{i_k}$ of characteristic submanifolds. We refer to the connected components of prefaces as faces. We also regard $Q$ itself as a codimension-zero face; other faces are called proper faces. A space $X$ is acyclic if $H_i(X) = 0$ for all $i$. We say that $Q$ is face-acyclic if all of its faces (including $Q$ itself) are acyclic. We call $Q$ a homology polytope if all its prefaces are acyclic (in particular, connected). Note that $Q = M/T$ is a homology polytope if and only if it is face-acyclic and all non-empty multiple intersections of characteristic submanifolds $M_i$ are connected.

We say that a torus manifold $M$ is locally standard if every point in $M$ has an invariant neighbourhood $U$ weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ (invariant under the standard $T$-action on $\mathbb{C}^n$). The latter means that there is an automorphism $\psi : T \to T$ and a diffeomorphism $f : U \to W$ such that $f(ty) = \psi(t)f(y)$ for all $t \in T, y \in U$.

Any point in the orbit space $Q$ of a locally standard torus manifold $M$ has a neighbourhood diffeomorphic to an open subset in the positive cone

$$\mathbb{R}^n_+ = \{(y_1, \ldots, y_n) \in \mathbb{R} : y_i \geq 0, i = 1, \ldots, n\}.$$

Moreover, this local diffeomorphism preserves the face structures in \(Q\) and \(\mathbb{R}^m\) (that is, a point from a codimension-k face of \(Q\) is mapped to a point with at least \(k\) zero coordinates). By the definition, this identifies \(Q\) as a manifold with corners. In particular \(Q\) is a manifold with boundary \(\partial Q = \cup_i Q_i\). Let \(K\) denote the nerve of the covering of \(\partial Q\) by the facets. Thus \(K\) is an \((n-1)\)-dimensional simplicial complex on \(m\)-vertices. The \((k-1)\)-dimensional simplices of \(K\) are in one-one correspondence with the codimension-\(k\) prefaces of \(Q\).

We assume that a torus manifold \(M\) is locally standard. Then the orbit space \(Q\) is a manifold with corners. The facets of \(Q\) are the quotient images \(Q_i\) of characteristic submanifolds \(M(i)\) (\(i\) = 1, ..., \(m\)). Let \(\Lambda : \{1, \ldots, m\} \to H_2(BT) = Hom(S^1, T) \cong \mathbb{Z}^n\) be a map sending \(i\) to \(a_i\), where the circle subgroup determined by \(a_i\), that is, \(a_i(S^1)\), is the one which fixes \(M_i\) (see Prop. 2.5 and §3.2 of \([11]\)). Further, the characteristic map \(\Lambda\) satisfies the following non-singular condition:

If \(Q_{i_1} \cap \cdots \cap Q_{i_k}\) is non-empty, then \(\Lambda(i_1), \ldots, \Lambda(i_k)\) span a \(k\)-dimensional unimodular subspace (i.e. extend to a \(Z\)-basis) of \(Hom(S^1, T) \cong \mathbb{Z}^n\).

The data \((Q, \Lambda)\) determines the torus manifold \(M\) if the orbit quotient \(Q\) of \(M\) satisfies \(H^2(Q) = 0\) (see Lemma 3.6 of \([7]\) and Prop. 1.8 of \([11]\)).

Let \(Q\) be a homology polytope (or even a simple convex polytope) with \(m\) facets \(Q_1, \ldots, Q_m\). Let \(k\) be a commutative ground ring with unit. Then the Stanley-Reisner face ring of its nerve \(K\) can be identified with the ring

\[k[Q] = k[v_{Q_1}, \ldots, v_{Q_m}] / (v_{Q_{i_1} \cap \cdots \cap Q_{i_k}} \quad \text{if} \quad Q_{i_1} \cap \cdots \cap Q_{i_k} = \emptyset)\]

called the face ring of \(Q\) (see §4 of \([7]\)).

# 2 Main Theorem

This section is devoted to proving our main result Theorem 2.3.

**Proposition 2.1.** There exists a complex line bundle \(L_j\) on \(M\) admitting a section \(s_j\) whose zero locus is the characteristic submanifold \(M_j\) for \(1 \leq j \leq m\).

**Proof:** Let \(\nu_j\) denote the normal bundle of \(M_j\) in \(M\) and let \(p : \nu_j \to M_j\) be the canonical projection and let \(E(\nu_j)\) denote the total space of \(\nu_j\). The rank 2 real vector bundle \(\nu_j\) on \(M_j\) admits a Riemannian metric (since \(M_j\) is compact) and in fact its structure group can be reduced to \(O(2)\). Fixing orientations on \(M\) and \(M_j\) determines a canonical orientation for every \(\nu_j\) for \(1 \leq j \leq m\). Therefore the normal bundle \(\nu_j\) admits reduction of structure group to \(SO(2)\). We can identify \(SO(2)\) with \(S^1\) so that the principal \(SO(2)\) bundle associated to \(\nu_j\) is in fact an \(S^1\) bundle and the complex line bundle associated to it by the standard action of \(S^1\) on \(\mathbb{C}\) has \(\nu_j\) as its underlying real vector bundle.

By the tubular neighbourhood theorem, \(E(\nu_j)\) is diffeomorphic to a tubular neighbourhood \(D_j\) of \(M_j\) and the diffeomorphism maps the image of the zero section of \(\nu_j\) onto \(M_j\). Further, the total space of the principal \(S^1\) bundle can be identified with \(\partial(D_j)\), the boundary of the tubular neighbourhood. Let \(p^*(\nu_j)\) be the pull back of \(\nu_j\) to \(D_j\). Since \(\nu_j\) is associated to the principal \(S^1\) bundle, its pull back to \(\partial(D_j)\) (its total space) is trivial and since \(\partial(D_j)\) is a deformation retract of \(D_j - M_j\), by the homotopy property of vector bundles it follows that the restriction of \(p^*(\nu_j)\) to \(D_j - M_j\) is trivial.

The vector bundle \(p^*(\nu_j)\) is endowed with a section \(\sigma_j\) namely the diagonal section whose zero locus \(Z(\sigma_j) = M_j\).

Let \(\epsilon\) be the trivial complex line bundle on \(M - int(D_j)\). Note that \(p^*(\nu_j)\) and \(\epsilon\) agree on a neighbourhood of \(D_j \cap (M - int(D_j)) = \partial(D_j)\). Thus we can construct a complex line bundle \(L_j\) on the whole of \(M\) which
agrees with $p^*(\nu_j)$ on $D_j$ and with $\epsilon$ on $M - \text{int}(D_j)$ (see [6]). Further, the section $\sigma_j$ of $\nu_j$ extends to give a section $s_j$ for $L_j$ whose zero locus $Z(s_j) = M_j$. □

We now recall the notion of a shellable simplicial complex (see Def 2.1, page 79 of [3]).

A simplicial complex $\Delta$ is said to be pure if each of its facets (or maximal face) has the same dimension. We say that a pure simplicial complex $\Delta$ is shellable if its facets can be ordered $F_1, \ldots, F_s$ such that the following condition holds: Let $\Delta_j$ be the subcomplex generated by $F_1, \ldots, F_j$, i.e:

$$\Delta_j = 2F_1 \cup \cdots \cup 2F_j$$

where $2F = \{G : G \subseteq F\}$. Then we require that for all $1 \leq i \leq s$ the set of faces of $\Delta_i$ which do not belong to $\Delta_{i-1}$ has a unique minimal element (with respect to inclusion). (When $i = 1$, we have $\Delta_0 = \emptyset$ and $\Delta_1 = \Delta_0$ has the unique minimal element $\emptyset$.) The linear order $F_1, \ldots, F_s$ is called a shelling order or a shelling of $\Delta$. Given a shelling $F_1, \ldots, F_s$ of $\Delta$, we define the restriction $r(F_i)$ of $F_i$ to be the unique minimal element of $\Delta_i - \Delta_{i-1}$.

Henceforth we assume that $M$ is a torus manifold with orbit space a homology polytope $Q$ whose nerve $K$ is a shellable simplicial complex.

Let $d$ be the number of vertices of $Q$ so that the simplicial complex $K$ has $d$ facets (or maximal dimensional faces). Let $F_1, \ldots, F_d$ be a shelling of $K$ (see Def 2.1, page 79 of [3]) and let $r(F_i)$ denote the restriction of $F_i$. Thus (by immediate consequence of the definition of a shelling) we have a disjoint union:

$$K = [r(F_1), F_1] \sqcup \cdots \sqcup [r(F_d), F_d].$$

Let $S_1, \ldots, S_d$ denote the vertices of the polytope which correspond respectively to $F_1, \ldots, F_d$. Further, let $T_i$ be the face of $Q$ corresponding to the face $r(F_i)$ of $K$ for $1 \leq i \leq d$. Thus it follows that every face of $Q$ belongs to $[S_i, T_i]$ for a unique $1 \leq i \leq d$ (where $[S_i, T_i]$ stands for the faces of $Q$ which contain the vertex $S_i$ and lie on the face $T_i$). We isolate this property as follows: For every face $Q_I = Q_{j_1} \cap \cdots \cap Q_{j_k}$ of $Q$ where $I = \{j_1, \ldots, j_k\}$ there is a unique $1 \leq i \leq d$ such that:

$$Q_I \in [S_i, T_i]$$

(\#)

Let $\tau_i := \dim(T_i)$. Further, if $\hat{T}_i$ denotes the subset of $T_i$ obtained by deleting all faces of $T_i$ not containing $S_i$. Since $Q$ is a manifold with corners, $\hat{T}_i$ is identified with $\mathbb{R}_{\geq 0}^{\tau_i}$. Then we note that $\pi^{-1}(\hat{T}_i)$ identified with $\mathbb{C}^{\tau_i}$ for $1 \leq i \leq d$ give a cellular decomposition of $M$ where $\pi : M \to Q$ is the quotient projection (see Construction 5.15 on page 66 of [3] and §3 of [3]). Hence we can summarise as follows:

**Lemma 2.2.** Let $M$ be a torus manifold with orbit space $Q$ a homology polytope. Let $K$ be the nerve of $Q$. If $K$ is shellable then the shelling gives a perfect cellular decomposition of $M$ with cells only in even dimensions.

**Theorem 2.3.** Let $Q_1, \ldots, Q_m$ denote the facets of $Q$. Let $a_i$ denote the element $\Lambda(i)$ in $H_2(BT)$, where $\Lambda$ is the characteristic homomorphism. Consider the polynomial algebra $\mathbb{Z}[v_{Q_1}, \ldots, v_{Q_m}]$. We denote by $I$ the ideal generated by the following two types of elements:

$$v_{Q_{j_1}} \cdots v_{Q_{j_k}}, \ 1 \leq j_p \leq m,$$

where $\cap_{i=1}^k Q_{j_i} = \emptyset$ in $Q$, and the elements

$$\prod_{j, (t,a_j) > 0} (1 - v_{Q_j})^{(t,a_j)} - \prod_{j, (t,a_j) < 0} (1 - v_{Q_j})^{-(t,a_j)}$$

(\#)

for $t \in H^2(BT)$. Let $R = \mathbb{Z}[v_{Q_1}, \ldots, v_{Q_m}] / I$ and let $K^*(M)$ denote the topological $K$-ring of $X$. Then the map $\psi : R \to K^*(M)$ sending $v_{Q_j}$ to $[L_j] - 1$ is a ring isomorphism.
We now state the following lemma which is used for proving Theorem 2.3.

Lemma 2.4. The monomials \( v_{T_i} \), \( 1 \leq i \leq d \) span \( R \) as a \( \mathbb{Z} \)-module.

Proof: The proof of this lemma is exactly as of Prop. 2.1 of [11] (also see Lemma 2.2 of [10]). Thanks to the key observation (*), the arguments work analogously in this setting too (the setting of a torus manifold with quotient a homology polytope whose nerve is shellable). We omit the details. \( \square \)

Proof of Theorem 2.3. By Lemma 2.1 there exists a complex line bundle \( L_j \) on \( M \) with section \( s_j \) whose zero locus \( Z(s_j) = M_j \). Thus its first chern class, \( c_1(L_j) = [M_j] \) for \( 1 \leq j \leq m \), where \([M_j]\) denotes the fundamental class of \( M_j \) in \( H^2(M;\mathbb{Z}) \). Further, \( M \) has a cellular decomposition with cells only in even dimensions (see Lemma 2.2). Now, by Corollary 6.8 of [7], \( H^*(M;\mathbb{Z}) \) is generated by \( c_1(L_1), \ldots, c_1(L_m) \in H^2(X;\mathbb{Z}) \). Hence by Theorem 4.1 of [10], it follows that \( K^*(M) = K^0(M) \) is generated by \([L_1], \ldots, [L_m]\) in \( K^0(M) \).

Let \( L_t := \prod_{j=1}^m L_j^{(t,a_j)} \). Since \( c_1(L_t) = \sum_{j=1}^m (t,a_j)[M_j] = 0 \) it follows that \( L_t \) is a trivial line bundle for every \( t \in H^2(BT) \).

Let \( Q_j \cap \cdots \cap Q_jk = \emptyset \) in \( Q \). Then we have \( M_j \cap \cdots \cap M_jk = \emptyset \) in \( M \). Therefore the vector bundle \( V = L_j1 \oplus \cdots \oplus L_jk \) admits a section \( s = (s_j, \ldots, s_jk) \) which is nowhere vanishing. Hence applying the \( \gamma^k \) operation in \( K(M) \) we obtain \( \gamma^k(\oplus_{p=1}^k L_jp - k) = \gamma^k(\oplus_{p=1}^k (L_jp - 1)) = \prod_{p=1}^k [L_jp] - 1 \). Since \( V \) has geometric dimension at most \( k - 1 \) we have: \( \prod_{p=1}^k ([L_jp] - 1) = 0 \).

By the above arguments it follows that the map \( \psi : R \to K^*(X) \) which sends \( v_{Q_j} \) to \([L_j] - 1 \) is well defined and surjective.

Since \( M \) has a cell decomposition with cells only in even dimensions it follows that \( K^*(M) = K^0(M) \) is free abelian of rank \( d \) which is the number of even dimensional cells (see [1]). Further, by Lemma 2.4 we know that \( R \) is generated by \( d \) elements \( v_{T_1}, \ldots, v_{T_d} \). Hence it follows that the map \( \psi \) is a ring isomorphism. \( \square \)

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References

[1] M.F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Proc. Symp. Pure Math., pp. 7-38, Vol III (1961) AMS Providence, RI.
[2] A. Bronsted, An introduction to convex polytope, (1983), Springer-Verlag, NY.
[3] V.M. Buchstaber and T.E. Panov, Torus actions and their applications in topology and combinatorics, Univ. Lect. Series-24,(2002), AMS, Providence, RI.
[4] M. W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. Jour. 62,(1991), 417-451.
[5] W. Fulton, Introduction to toric varieties, Ann Math Studies 131,(1993), Princeton Univ. Press, Princeton, NJ.
[6] M. Karoubi, K-Theory, Grundlehren der Mathematischen Wissenschaften 226, Springer-Verlag, Berlin, 1978.
[7] M. Masuda and T. Panov, On the cohomology of torus manifolds, arXiv:math.AT/0306100 v1 2003.
[8] J.W. Milnor, J.D. Shasheff, Characteristic classes, Ann. Math. Studies 76,(1974), Princeton Univ. Press, Princeton, NJ.
[9] R.P. Stanley, Combinatorics and commutative algebra, Progress in Mathematics 41, Birkhauser, Boston.

[10] P. Sankaran and V. Uma, Cohomology of toric bundles, Comment. Math. Helv., 78,(2003), 540-554. Errata, 79,(2004), 840-841.

[11] P. Sankaran and V. Uma, K-theory of quasi-toric manifolds, Osaka Journal of Mathematics, to appear.

[12] G. Ziegler, Lectures on Polytopes, Graduate Texts in Mathematics, 152. Springer-Verlag, New York, 1995.

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