Optical knots and contact geometry II.
From Hopf links to transverse and cosmetic knots

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Abstract

In 1985 Moffatt conjectured that in steady incompressible Euler-type fluids the streamlines could have knots/links of all types. Using methods of contact geometry Etnyre and Ghrist in 2000 developed the existence-type proof of the Moffatt conjecture. The alternative proof, also of existence-type, was proposed by Enciso and Peralta-Salas in 2012. In all three papers the Beltrami equation was used as point of departure. However, only work by Etnyre and Ghrist takes full advantage of contact-geometric nature of the Beltrami equation. In this work we propose the constructive proof of the Moffatt conjecture based on ideas and methods of contact geometry. We discuss in sufficient detail various physical processes generating such knotted structures. By employing the correspondence between ideal hydrodynamics and electrodynamics discussed in part I, the Moffatt conjecture is proved for Maxwellian electrodynamics. The potential relevance of the obtained results for source-free Yang-Mills and gravity fields is also briefly discussed.

PACS numbers: 11.15 Yc; 11.27.+d; 11.30.-j; 42, 45.20 Jj; 47

1. Introduction

The proof (direct or indirect) of Moffatt conjecture (Moffatt 1985) is opening Pandora’s box of all kinds of puzzles. Indeed, from seminal work by Witten (1988) (see also Atiyah (1990)) it is known that the observables for both Abelian and non Abelian source-free gauge fields are knotted Wilson loops. It is believed that only the non Abelian Chern-Simons (C-S) topological field theory is capable of generating nontrivial knots/links. By "nontrivial knots" we mean knots other than unknots, Hopf links and torus-type knots/links. Being topological in nature the C-S functional is not capable of taking into account the boundary conditions. This is true for all known to us path integral treatments of the Abelian C-S field theory responsible for generation of knots in Abelian/Maxwellian electrodynamics (e.g. read part I). In the meantime the boundary conditions do play an important role in the work by Enciso and Peralta-Salas (2012) on solving the Moffatt conjecture. In the work by Etnyre and Ghrist (2000a) the role of boundary conditions is less explicit while in this work we take into account boundary conditions when their use becomes essential. As result, the obtained results are in accord with those by Witten and Atiyah whenever the role of boundary conditions can be ignored and are in agreement with Enciso and Peralta-Salas (2012) whenever their use becomes indispensable. The fact that the Abelian gauge fields are capable of producing
nontrivial knots/links blurs the barriers between the Maxwellian electrodynamics, Yang-Mills fields and gravity. As result, recently, there has been (apparently uncorrelated) visibly large activity in electromagnetism (Arrayas and Trueba 2012), non Abelian gauge fields (Kobayashi and Nitta 2014), and gravity (Thompson et al 2014) producing at least torus-type knots/links by using more or less the same methods. The justification for application of the same methods to apparently different gauge fields is given in (Kholodenko 2011). In this paper a delicate interrelationship between the electromagnetic, Yang-Mills and gravity fields leading to formation of knotted/linked structures is discussed in detail. Although we cited these papers as the most recent and representative, there are many other similar papers describing the same type of knotty structures in these fields and, also in magnetohydrodynamics, in condensed matter physics, etc.

From knot theory and, now proven, geometrization conjecture it follows that complements of knots/links embedded in $S^3$ are spaces of positive, negative and zero curvature. Thus far in physics literature the ability to curve the ambient space was always associated with masses. The famous God’s particle-the Higgs boson- is believed to supply masses to the rest of particles. Now we encounter the situation when the space is being curved by knots/links produced by stable configurations of gauge fields of both Abelian and non Abelian nature. In part I we argued that the electric and magnetic charges can be recreated by the Hopf-like structures of the respective gauge fields. Surely, such charges must also be massive. Accordingly, if such massive formations are created by gauge fields then, apparently, all known masses and charges can be topologically described. In this work we are not going to discuss these issues any further with the exception of the last section.

This work is made of seven sections and seven appendices. Almost book-style manner of presentation in this paper is aimed at making it accessible for readers with various backgrounds: from purely physical to purely mathematical. As in part I, for a quick introduction to ideas and methods of contact geometry/topology our readers may consult either (Kholodenko 2013), aimed mainly at readers with physics background, or (Geiges 2008) aimed at mathematicians.

In section 2 we provide statement of the problem written in the traditional style of boundary value type problem. In the same section we reformulate our problem in the language of contact geometry. In section 3 we introduce the Reeb and the Liouville vector fields and compare them with the Beltrami vector field. Section 4 plays an important role in the whole paper. In it we establish the chain of the following correspondences: Beltrami vector fields $\iff$ Reeb vector fields $\iff$ Hamiltonian vector fields. These correspondences allow us to introduce the nonsingular Morse-Smale (NMS) flows. We connect these flows with the Hamiltonian flows discussed (independently of NMS flows) by Zung and Fomenko (1990). This is done in section 5 which is of major importance. In section 6 we explicitly derive the iterated torus knot structures predicted by Zung and Fomenko in their paper of 1990. These structures are obtained via cascade of bifurcations of Hamiltonian vector fields which we describe in some detail. We also reinterpret the obtained iterated torus knots/links in terms of the transversely simple knots/links known from contact geometry/topology. As a by

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1More information is provided in the longer version [arXiv:1001.0029 v.1](http://arxiv.org/abs/1001.0029).
product, we also introduce the Legendrian and transverse knots and links. Since the Legendrian knots/links were christened by Arnol’d as optical knots/links we use this terminology in the titles of both parts I and II of our work. In section 7 we relate results of Birman and Williams (1983a,1983b) papers with what was obtained already in previous sections in order to obtain knots and links of arbitrary complexity. Along the way we developed new way of designing the Lorenz template. This template was originally introduced in Birman and Williams (1983a) paper in order to facilitate the description of closed orbits occurring in dynamics of Lorenz equations. The simplicity of our derivation of this template enabled us to reobtain the universal template of Ghrist (1997) by methods very different from those by Ghrist. Possible applications of the obtained results to gravity are discussed in section 7 in the context of cosmetic (not cosmic!) knots. Appendices-- from A to E--contain all kinds of support information needed for uninterrupted reading of the main text.

2. Force-free/Beltrami equation from the point of view of contact geometry

From Theorem 4.1. (part I) it follows that force-free/Beltrami vector fields are solutions of the steady Euler flows. At the same time, Corollary 4.2. is telling us that such flows minimize the kinetic energy functional. This is achieved due to the fact that Beltrami/force-free fields have nonzero helicity. The helicity is playing the central role in Ranada’s (1989) and (1992) papers. By studying helicity Ranada discovered his torus-type knots/links. The same type of knots were recently reported by Kedia et al (2013). Based on results of part I, it should be obvious that study of helicity is synonymous with study of knots and links (at least of torus-type). Can the same be achieved by studying the Beltrami/force-free equation? We would like to demonstrate that this is indeed possible. Although the literature on solving the Beltrami equation is large, only quite recently the conclusive results on existence of knots and links in Belrami flows have been published. An example of systematic treatment of Beltrami flows using conventional methods of partial differential equations is given in the pedagogically written monograph by Majda and Bertozzi (2003). Our readers should be aware of many other examples existing in literature. All these efforts culminated in the Annals of Mathematics paper by Enciso and Peralta-Salas (2012). In this paper the authors proved that the equation for (strong) Beltrami fields

\[ \nabla \times \mathbf{v} = \kappa \mathbf{v} \]  

(2.1a)

(that is the Beltrami equation with constant \( \kappa \)) supplemented with the boundary condition

\[ \mathbf{v} \mid_\Sigma = \mathbf{w} \]  

(2.1b)

where \( \Sigma \) is embedded oriented analytic surface in \( \mathbb{R}^3 \) so that the vector \( \mathbf{w} \) is tangent to \( \Sigma \), can have solutions describing knots/links of any type (that is not just torus knots/links). Subsequent studies by the same authors (Enciso and Peralta-Salas 2013) demonstrated that
Σ is actually having a toral shape/topology². These authors were able to prove what Moffatt (1985) proposed/conjectured long before heuristically. The same conclusion was reached earlier, in 2000, by Etnyre and Ghrist (2000a) who were using methods of contact geometry and topology. They also provided a sort of existence-type proof of the Moffatt conjecture. Thus, at the moment of writing of this paper, to author’s knowledge, there are two independent ways (both of existence-type) of proving that the steady Beltami flow fields contain knots/links of arbitrary complexity.

In this paper we discuss yet another proof of Moffatt’s conjecture. It is based on methods of contact geometry and topology. Our results can be considered as some elaboration on the results by Etnyre and Ghrist (2000a). Unlike the existence-type results of previous authors, we were able to find explicitly some of the knots/links being guided (to a some extent) by the seminal works by Birman and Williams (1983a,b), Fomenko (1991) and Ghys (2007). It is appropriate to mention at this point that recently proposed methods of generating knots and links in fluids (Kleckner and Irvine 2013) are compatible with those discussed by Birman and Williams (1983a,b), Ghys (2007) and Enciso and Peralta-Salas (2012).

Following Etnyre and Ghrist (2000b) we begin our derivations by rewriting the Beltrami eq.(2.1a) as

\[ *d\alpha = \kappa \alpha \] (2.2a)

where \( \alpha \) is any contact 1-form and \( * \) is the Hodge operator. Although details of derivation of eq.(2.2a) are given in Chr.5 of (Kholodenko 2013), for uninterrupted reading the basics are outlined in appendix A. Since \( ** = id \), the same equation can be equivalently rewritten as

\[ d\alpha = \kappa * \alpha. \] (2.2b)

Should \( \kappa = 1 \), the above equation would coincide with the standard Hodge relation between 1 and 2 forms. Following Etnyre and Ghrist (2000b) we need the following

**Definition 2.1.** The Beltrami field is called rotational if \( \kappa \neq 0 \).

For this case, we can introduce the volume 3-form \( \mu \) as follows

\[ \mu = \alpha \wedge d\alpha = \kappa \alpha \wedge *\alpha = \kappa dV. \] (2.3a)

The volume form \( \kappa dV \) can be re-normalized so that the factor (function) \( \kappa \) can be eliminated. This is so because the volume form \( dV \) contains the metric factor \( \sqrt{g} \) which can be readjusted. This fact can be formulated as

**Theorem 2.2.** (Chern and Hamilton 1985) Every contact form \( \alpha \) on a 3-manifold has the adapted Riemannian metric \( g \).

The metric is adapted (that is normalized) if eq.(2.2b) can be replaced by

\[ d\alpha = *\alpha \] (2.2.c)

²Recall that any knot is an embedding of \( S^1 \) into \( S^3 \) or \( \mathbb{R}^3 \).
This result can be recognized as the standard statement from the Hodge theory (Wells 2008). For such a case we obtain:

\[ \mu = \alpha \wedge \ast \alpha = dV. \]  

(2.3b)

Consider now the volume integral

\[ I = \int_Y \alpha \wedge \ast \alpha. \]  

(2.4)

On one hand, it can be looked upon as the action functional for the 3d version of the Abelian/Maxwellian gauge field theory as discussed in Sections 2 and 3 of part I, on another, the same functional can be used for description of dynamics of 3+1 Einsteinian gravity (Fischer and Moncrief 2001), (Kholodenko 2008). We believe, that mention of such a connection should be useful for those who specialize in the loop quantum gravity (Gambini and Pullin 2011). In view of Theorem 2.2., eq.(2.2c) can be rephrased now as

**Corollary 2.3.** Every 3-manifold admits a non-singular Beltrami flow for some Riemannian structure on it.

The non singularity of flows is assured by the requirement \( \kappa \neq 0 \). The above statement does not include any mention about the existence of knots/links in the Beltrami flows. Thus, the obtained results are helpful but not constructive. To obtain constructive results we need to introduce the Reeb vector fields associated with contact structures. For our purposes it is sufficient to design the Reeb vector fields only for \( S^3 \).

### 3. Reeb vs Beltami vector fields on \( S^3 \)

Following (Geiges 2008) we begin with the definition of the Liouville vector field \( X \). For this purpose we need to use the symplectic 2-form introduced in (4.15a) of part I defined on \( \mathbb{R}^4 \). Up to a constant it is given by

\[ \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2. \]  

(3.1)

If we use the definition of the Lie derivative \( \mathcal{L}_X \) for the vector field \( X \)

\[ \mathcal{L}_X = d \circ i_X + i_X \circ d, \]  

(3.2)

then, using this definition, we obtain the following

**Definition 3.1.** The vector field \( X \) is called Liouville if it obeys the equation

\[ \mathcal{L}_X \omega = \omega. \]  

(3.3)

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3Equivalently, study of the Beltrami fields on 3-manifolds is equivalent to study of the Hodge theory on 3-manifolds

4With such normalization it coincides with 2-form given in Geiges (2008), page 24.
The contact 1-form $\alpha$ can be defined now as

$$\alpha = i_X \omega. \quad (3.4)$$

This formula connects the symplectic and contact geometries in the most efficient way. To find the Liouville vector field $X$ for $S^3$ we notice that eq.(3.3) may hold for any form and, therefore, such a form could be, say, some function $f$. In this case eq.(3.3) acquires the form (Arnol’d 1989)

$$\sum_i x_i \frac{\partial f}{\partial x_i} = f. \quad (3.5)$$

Using this result, the Liouville vector field $X$ on $S^3$, where $S^3$ is defined by the equation $r^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2$, is given by

$$X = \frac{1}{2}(x_1 \partial_{x_1} + y_1 \partial_{y_1} + x_2 \partial_{x_2} + y_2 \partial_{y_2}). \quad (3.5)$$

To check correctness of this result, by combining eq.s(3.4)-(3.5) and using properly normalized eq.(4.14) of part I, we obtain:

$$L_X \omega = d \circ i_X \omega = d \circ \alpha = d\left[\frac{1}{2} \sum_{i=1}^{2} (x_i dy_i - y_i dx_i)\right] = \omega \quad (3.6)$$

as required. Going back to eq.(3.4) we would like to demonstrate now that the 1-form $\alpha$ can be also obtained differently. This is so because the very same manifold has both the symplectic and the Riemannian structure. In the last case the metric 2-form $g = g_{ij} dx^i \otimes dx^j$ should be defined. Then, for the vector field $\bar{X} = X^i \frac{\partial}{\partial x^i}$ we obtain (using definitions from appendix A): $\bar{X}^b = i_{\bar{X}} g = g(\bar{X}, \cdot) \equiv g_{ij} X^i dx^j$. From the same appendix A we know that if the operator $\flat$ transforms vector fields into 1-forms, then the inverse operator $\sharp$ is transforming 1-forms into vector fields, that is $[X^\flat]^b = g^{ij} X_i \frac{\partial}{\partial x^j} = X^i \frac{\partial}{\partial x^i} = X$. Suppose now that $\alpha = \bar{X}^b = i_{\bar{X}} g$ for some vector field $\bar{X}$ such that $\alpha = \bar{X}^b = i_{\bar{X}} g = i_X \omega$. This can be accomplished as follows. Suppose that $\bar{X}$ is the desired vector field then, we can normalize it as

$$i_{\bar{X}} \alpha = i_{\bar{X}} i_{\bar{X}} g = 1. \quad (3.7)$$

Explicitly, this equation reads $i_{\bar{X}} [g_{ij} \bar{X}^i dx^j] = g_{ij} \bar{X}^i \bar{X}^j = 1$. This result is surely making sense. Furthermore, the above condition can be safely replaced by $g_{ij} \bar{X}^i \bar{X}^j > 0$. This is so, because in the case of $S^3$ the condition given by eq.(3.7) reads: $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$. Therefore, it is clear that this condition can be relaxed to $x_1^2 + y_1^2 + x_2^2 + y_2^2 = r^2 > 0$. The condition, eq.(3.7), is the 1st of two conditions defining the Reeb vector field. The 2nd condition is given by

$$i_{\bar{X}} d\alpha = 0. \quad (3.8)$$

Suppose that indeed $\alpha = i_{\bar{X}} g = i_X \omega$, where $X$ is the Liouville and $\bar{X}$ is the Reeb vector field. Then, we have to require: $i_{\bar{X}} d\alpha = i_{\bar{X}} d \circ i_{\bar{X}} g = i_{\bar{X}} d \circ i_{\bar{X}} \omega = i_{\bar{X}} \omega = 0$. This requirement

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Here we used eq.s(3.6) and (3.8).
allows us to determine the Reeb field. It also can be understood physically. For this purpose we consider the volume 3-form $\mu = \alpha \wedge d\alpha$ and apply to it the Lie derivative, i.e.

$$L_{\tilde{X}} \mu = (L_{\tilde{X}} \alpha) \wedge d\alpha + \alpha \wedge (L_{\tilde{X}} d\alpha)$$

$$= (d \circ i_{\tilde{X}} + i_{\tilde{X}} \circ d)\alpha \wedge d\alpha + \alpha \wedge (d \circ i_{\tilde{X}} + i_{\tilde{X}} \circ d)d\alpha = 0. \quad (3.9)$$

This result is obtained after we used the two Reeb conditions. Clearly, for the Reeb fields the equation

$$L_{\tilde{X}} \alpha = 0 \quad (3.10)$$

is equivalent to the incompressibility condition for fluids, or to the transversality condition $\text{div} A = 0$ for electromagnetic fields. From here we obtain the major Corollary 3.2.

**Corollary 3.2.** From eq.(3.9) it follows that the condition $L_{\tilde{X}} \alpha = 0$ is implying that the Reeb vector field flow preserves the form $\alpha$ and, with it, the contact structure $\xi = \text{ker} \alpha$.

Alternatively, the Reeb vector field $\tilde{X}$ is determined by the condition $L_{\tilde{X}} \alpha = 0$.

Nevertheless, we would like to demonstrate now that the condition $i_{\tilde{X}} \omega = 0$ is also sufficient for determination of the Reeb field. For the tasks we are having in mind, it is sufficient to check this condition for $S^3$ where the results are known (Geiges 2008, Kholodenko 2013). Specifically, it is known that the Reeb vector field for $S^3$ is given by

$$\tilde{X} = 2(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2}) \quad (3.11)$$

By combining eq.s(3.1) and (3.11) we obtain:

$$i_{\tilde{X}} \omega = -2(x_1 dx_1 + y_1 dy_1 + x_2 dx_2 + y_2 dy_2). \quad (3.12)$$

However, in view of the fact that $x_1^2 + y_1^2 + x_2^2 + y_2^2 = r^2$, we obtain as well $2r dr = 2(x_1 dx_1 + y_1 dy_1 + x_2 dx_2 + y_2 dy_2) = 0$ if this result is to be restricted to $S^3$. Thus, at least for the case of $S^3$, we just have obtained $i_{\tilde{X}} \omega = 0$ as required.

Eq.(3.4), when combined with eq.(3.5), yields $\alpha = i_{\tilde{X}} \omega = \frac{1}{2}\{eq.(4.14), \text{part I}\}$ in accord with eq.(3.6). Now we take again $\alpha = \tilde{X}^b = i_{\tilde{X}} g = i_{\tilde{X}} \omega$ and, since $\tilde{X}^b = g_{ij} \tilde{X}^i dx^j$, we obtain

$$\alpha = \tilde{X}^b = g_{ij} \tilde{X}^i dx^j. \quad (3.13)$$

By combining eq.s(3.11) and (3.13) we again recover $\alpha = i_{\tilde{X}} g = \frac{1}{2}\{eq.(4.14), \text{part I}\}$. Therefore, we just demonstrated that indeed, $i_{\tilde{X}} g = i_{\tilde{X}} \omega$ where $\tilde{X}$ is the Reeb and $X$ is the Liouville vector fields. By combining eq.(3.13) with eq.(A.5) and (2.2a) we re obtain now the Beltrami equation

$$* d\tilde{X}^b = \kappa \tilde{X}^b. \quad (3.14)$$

Clearly, it is equivalent to either eq.(2.2a) or (2.2b). Furthermore, in view of the Theorem 2.2., it is permissible to put $\kappa = 1$. 

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Next, suppose that $*\alpha = i_{\hat{X}}\mu$ then, for the r.h.s of this equality we obtain: $i_{\hat{X}}\mu = (i_{\hat{X}}\alpha) \wedge d\alpha - \alpha \wedge i_{\hat{X}}d\alpha = d\alpha$. This result becomes possible in view of the 1st and 2nd Reeb conditions. Thus, we just reobtained eq.(2.2c). The obtained results can be formulated as theorem\textsuperscript{6}. It is of central importance for this work.

**Theorem 3.3.** Any rotational Beltrami field on a Riemannian 3-manifold is Reeb-like and vice versa.

**Corollary 3.4.** Every Reeb-like vector field generates a non-singular steady solution to the Euler equations for a perfect incompressible fluid with respect to some Riemannian structure. Equivalently, every Reeb-like vector field which is solution of the force-free equation generates non-singular solution of the source-free Maxwell equations with respect to some Riemannian structure.

Appendix B provides an illustration of the Corollary 3.4. in terms of conventional terminology used in physics literature.

### 4. Hamiltonian dynamics and Reeb vector fields

Eq.(D.1a) of part I describes the conformation of the single vortex tube. In view of the Beltrami condition, this equation can be equivalently rewritten as

$$[v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}]f = 0. \quad (4.1a)$$

If we add just one (compactification) point to $\mathbb{R}^3$ we can use the stereographic projection allowing us to replace $\mathbb{R}^3$ by $S^3$ and to consider the conformation of the vortex tube in $S^3$. Example 1.9. (page 123) from the book by Arnol’d and Kheshin (1998) is telling us (without proof) that the components of the vector $v$ on $S^3$ are $v = [x_1, -y_1, x_2, -y_2]$. The same source (again without proof) is also telling us that the vector $v$ is the eigenvector of the force-free equation $\text{curl}^{-1}v = \lambda v$ with the eigenvalue $\lambda = 1/2$.

In view of these results and using eq.(3.11) for the Reeb vector field, we replace eq.(4.1a) by

$$\left(\frac{x_1}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2}\right)f = 0 \quad (4.1b)$$

Now, in view of eq.(D.1b) of part I this equation can be equivalently rewritten as

$$\frac{dy_1}{x_1} = \frac{dx_1}{-y_1} = \frac{dy_2}{x_2} = \frac{dx_2}{-y_2} = dt \quad (4.1c)$$

\textsuperscript{6}Our derivation of this result differs from that in (Etnyre and Ghrist 2000b)

\textsuperscript{7}We have relabeled coordinates in Arnol’d Kheshin book so that they match those given in eq.(3.11).
so that we recover the result of Arnol’d and Khesin for \( \mathbf{v} \). In addition, we obtain:

\[
\begin{align*}
\dot{x}_1 &= -y_1; \\
\dot{x}_2 &= -y_2; \\
\dot{y}_1 &= x_1; \\
\dot{y}_2 &= x_2.
\end{align*}
\]

These are the Hamiltonian-type equations describing dynamics of two uncoupled harmonic oscillators. From mechanics it is known that all integrable systems can be reduced by a sequence of canonical transformations to the set of independent harmonic oscillators. The simplicity of the final result is misleading though as can be seen from the encyclopedic book by Fomenko and Bolsinov (2004). It is misleading because the dynamical system described by eq.s (4.2) possesses several integrals of motion. In particular, it has the energy \( h = \frac{1}{2}(p_1^2 + p_2^2 + y_1^2 + y_2^2) \), where \( p_1 = x_1, p_2 = x_2 \), as one of such integrals. The existence of \( h \) indicates that the motion is constrained to \( S^3 \). Thus, the problem emerges of classifying all exactly integrable systems whose dynamics is constrained to \( S^3 \). Surprisingly, there are many dynamical systems fitting such a classification. The full catalog is given in the book by Fomenko and Bolsinov.

Whatever these systems might be, once their description is reduced to the set of eq.s (4.2) supplemented by, say, the constraint of moving on \( S^3 \), their treatment follows the standard protocol. The protocol can be implemented either by the methods of symplectic mechanics (Arnol’d 1989, Jovanović 2011) or by the methods of sub-Riemannian geometry—a discipline which is part of contact geometry (Calin and Der Chang 2009, Kholodenko 2013). The results of, say, symplectic treatment indicate that the trajectories of the dynamical system described by eq.s (4.2) are the linked (Hopf) rings. Furthermore, the same eq.s (4.1b) were obtained by Kamchatnov (1982) whose analysis of these equations demonstrates that, indeed, in accord with the unproven results by Arnol’d and Khesin (1998), the largest eigenvalue \( \lambda \) of the Beltrami equation \( \text{curl}^{-1}\mathbf{v} = \lambda \mathbf{v} \) is \( \frac{1}{2} \). This result follows from eq.(16) of Kamchatnov’s paper where one should replace \( x^2 \) by 1 as required for description of a sphere \( S^3 \) of unit radius. The Example 1.9. in the book by Arnol’d and Khesin exhausts all possibilities available without further use of methods of contact geometry and topology. These methods are needed, nevertheless, if we are interested in obtaining solutions of Hamiltonian eq.s (4.2) more complicated than Hopfian rings.

To begin our discussion of this topic, we would like to use the notion of contactomorphism defined already by eq.(4.9) of part I. Now we are interested in applying it to the standard contact form on \( S^3 \). To do so, we introduce the complex numbers \( z_1 = r_1 \exp\{i\phi_1\} \) and \( z_2 = r_2 \exp\{i\phi_2\} \) so that in terms of these variables the 3-sphere \( S^3 \) is described by the equation \( r_1^2 + r_2^2 = 1 \). The 1-form, eq.(4.14), of part I can be rewritten in terms of just introduced variables as

\[
\alpha = \frac{1}{2}(r_1^2 d\phi_1 + r_2^2 d\phi_2).
\]

By combining eq.s (3.11) with (3.6) we see that the factor \( 1/2 \) is needed if we want to preserve the Reeb condition, eq.(3.7). Accordingly, in terms of just introduced new variables the Reeb vector \( \tilde{X} \) acquires the following form: \( \tilde{X} = 2\left(\frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}\right) \). By design, it satisfies the

\footnotesize
\begin{itemize}
\item[8] Without any uses of contact geometry
\item[9] Here the factor 1/2 is written in accord with eq.(3.6).
\end{itemize}

Reeb condition \( \alpha(\tilde{X}) = 1 \). Consider now yet another Reeb vector \( \tilde{X} = 2(\frac{1}{r_1} \frac{\partial}{\partial \phi_1} + \frac{1}{r_2} \frac{\partial}{\partial \phi_2}) \) and consider a contactomorphism

\[
\frac{1}{2} \varepsilon (r_1^2 d\phi_1 + r_2^2 d\phi_2) = \frac{1}{2} (\tilde{r}_1^2 d\phi_1 + \tilde{r}_2^2 d\phi_2) = \tilde{\alpha}.
\]  

(4.4)

For such defined \( \tilde{\alpha} \) we obtain \( \alpha(\tilde{X}) = 1 \), provided that we can find such \( \varepsilon > 0 \) that \( \varepsilon (r_1 + r_1) = 1 \). But this is always possible!

By analogy with eq.(4.1b) using \( \tilde{X} = 2(\frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}) \) we obtain,

\[
\left( \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} \right) f = 0 \text{ or, equivalently, } \dot{\phi}_1 = 1, \dot{\phi}_2 = 1.
\]

(4.5a)

This solution describes the Hopf link (Jovanović 2011, Kamchatnov 1982).

At the same time, by using \( \tilde{X} \) we obtain:

\[
\left( \frac{1}{r_1} \frac{\partial}{\partial \phi_1} + \frac{1}{r_2} \frac{\partial}{\partial \phi_2} \right) f = 0 \text{ or, equivalently, } \dot{\phi}_1 = \frac{1}{r_1}, \dot{\phi}_2 = \frac{1}{r_2}.
\]

(4.5b)

If both \( r_1 \) and \( r_2 \) are rational numbers, eq.s(4.5b) describe torus knots. Both cases were discussed in detail by Birman and Williams (1983a). Understanding of this paper is substantially facilitated by reading books by Ghrist et al (1997), by Gilmore and Lefranc (2002) supplemented by the review article by Franks and Sullivan (2002).

The above arguments, as plausible as they are, cannot be considered as final. This is so because of the following. According to the Theorem 3.3, the Beltrami fields can be replaced by the Reeb fields and, in view of eq.s(4.1) and (4.2), the Beltrami vector fields are equivalent to the Hamiltonian vector fields. The question arises: can we relate the Beltrami fields to Hamiltonian fields without using specific examples given by eq.s(4.1) and (4.2)? This indeed happens to be the case (Geiges 2008). To facilitate our readers understanding, needed mathematical information about symplectic and contact manifolds is given in appendix C.

Specifically, from this appendix we obtain:

\[- dH = i_{v_H} \omega = \omega(v_H, \cdot). \]

(4.6)

Suppose now that the Reeb vector field \( \tilde{X} \) is just a reparametrization of the Hamiltonian vector field \( v_H \) This makes sense if we believe that examples given in eq.s(4.1) and (4.2) are generic. If this is indeed the case, we obtain: \( i_{v_H} \omega = i_{\tilde{X}} d\alpha = i_{\tilde{X}} \omega = 0 \). Here we used results which follow after eq.(3.8). By looking at eq.(4.6) and by using results of appendix C we conclude that just obtained results are equivalent to the requirement \( dH = 0 \). But, since

\[
dH = \sum_i \left( \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right)
\]

we obtain,

\[
\frac{dp_i}{\partial H} = \frac{dq_i}{\partial H} = dt.
\]

(4.7)

Readers interested in more details are encouraged to read (Kholodenko 2013).
But these are just Hamilton’s equations! Thus our assumption about the (anti)collinearity of the Reeb and Hamiltonian vector fields is correct, provided that both Reeb conditions hold. Since the equation $i_{\tilde{X}} d\alpha = 0$ is the 2nd Reeb condition according to eq.(3.8), we only need to make sure that the 1st Reeb condition $i_{\tilde{X}} \alpha = 1$ also holds. Since according to eq.(3.4) $\alpha = i_X \omega$, where $X$ is the Liouville field, we can write $i_{\tilde{X}} \alpha = i_{\tilde{X}} i_X \omega = \omega(\tilde{X}, X) = 1$. It remains now to check if such a condition always holds. Since we are working with $S^3$, it is sufficient to check this condition for $S^3$. The proof of the general case is given in the book by Geiges (2008), page 25. For $S^3$ the Reeb vector field is given by eq.(3.11) while the Liouville vector field is given by eq.(3.5). Since the symplectic 2-form is given by eq.(3.1), by direct computation we obtain $i_{\tilde{X}} i_X \omega = \omega(\tilde{X}, X) = 1$. Thus, we just obtained the following correspondences of major importance: Beltrami vector fields $\iff$ Reeb vector fields $\iff$ Hamiltonian vector fields.

The obtained correspondence allows us now to utilize all knotty results known for dynamical systems for the present case of Abelian (Maxwellian) gauge fields. We begin our study of this topic in the next section.

5. From Weinstein conjecture to nonsingular Morse-Smale flows

The Weinstein conjecture is just a mathematical restatement of the issue about the existence of closed orbits on constant energy surfaces. These are necessarily manifolds of contact type. Indeed, if $2n$ is the dimension of the symplectic manifold, then the dimension of the constant energy surface embedded in such a manifold is $2n - 1$ which is the odd number. All odd dimensional manifolds are contact manifolds (Geiges 2008, Kholodenko 2013). Following Hofer (1998), we now formulate

**Conjecture 5.1a.** (Weinstein) Let $W$ be symplectic manifold with 2-form $\omega$. Let $H$ be a smooth Hamiltonian $H$ so that $M := H^{-1}(E)$ is compact regular energy surface (for some prescribed energy $E$). If there exist a 1-form $\lambda$ on $M$ such that $\lambda(v_H(x)) \neq 0 \forall x \in M$ and $d\lambda = \omega|_M$, then there exists a periodic orbit on $M$.

According to eq.(3.6) $d\alpha = \omega|_M$ and, surely, we can replace $\lambda$ by $\alpha$. Then, the condition $\lambda(v_H(x)) \neq 0$ is equivalent to the 1st Reeb condition, eq.(3.7), since $v_H$ is equivalent to $\tilde{X}$. The 2nd Reeb condition surely holds too in view of the result $i_{v_H} \omega = i_{\tilde{X}} d\alpha = i_{\tilde{X}} \omega = 0$ obtained in the previous section. Thus, the above conjecture can be restated as (Hutchings 2009)

**Conjecture 5.1b.** (Weinstein) Let $M$ be closed oriented odd-dimensional manifold with a contact form $\alpha$. Then, the associated Reeb vector field has a closed orbit.

Hofer (1993) using theory of pseudoholomorphic curves demonstrated that the Weinstein conjecture is true for $S^3$. Much later the same result was obtained by Taubes, e.g. read (Hutchings 2009) for a review, who used results of Seiberg-Witten and Floer theories. Since
some of Floer’s results were mentioned in part I, our readers might be interested to know how Floer’s ideas were used for proving Weinstein’s conjecture by reading Ginzburg (2005). Thus, we know now that on $S^3$ trajectories of the Reeb vector fields do contain closed orbits. This is surely true in the simplest case of Reeb orbits described by eq.(4.1) and (4.2). The question arises: Is there other Reeb orbits for the Hamiltonian system described by eq.s(4.2)? We had provided some answer in eq.s(4.5). Now the question arises: Is eq.s(4.5) exhaust all possibilities? Surprisingly, the answer is “no”. Hofer, Wysocki and Zehnder (1998) proved the following.

**Theorem 5.2.** (Hofer, Wysocki and Zehnder) Let the standard contact form $\alpha_0$ on $S^3$ be given either by eq.(4.6) of part I or, equivalently, by eq.(4.3) above, then there should be a smooth, positive function $f: S^3 \rightarrow (0, \infty)$, such that if the Reeb vector field associated with the contact form $\alpha = f \alpha_0$ possesses a knotted periodic orbit, then it possesses infinitely many periodic orbits.

Here $\alpha_0$ is the standard contact form, e.g. that given by eq.(4.3). Eq.(4.4) is an example of relation $\alpha = f \alpha_0$. If these orbits are unknotted, then they are all equivalent. Thus, “infinitely many” presupposes nonequivalence of closed orbits which is possible only if they are knotted. Etnyre and Ghrist (2000a) proved that periodic orbits on $S^3$ contain knots/links of all possible types simultaneously. Their proof is of existence-type though. They were not able to find the function $f$ explicitly. In hydrodynamics, this result provides the affirmative answer to the Moffatt conjecture (Moffatt 1985). Subsequently Enciso and Peralta-Salas (2012) reached the same conclusion by designing a different existence-type proof not based on methods of contact geometry.

In this work we shall provide evidence that, in fact, the conclusion that all possible knots and links are obtainable via some kind of contactomorphism is incorrect. This is going to be proved by explicitly designing knots/links occurring in the Hamiltonian-type flows on $S^3$. Our arguments are based on the equivalence established at the end of section 4 between the Beltrami and Hamiltonian vector flows, on one hand, and on the results obtained in paper by Zung and Fomenko (1990), on another. In it, the topological classification of non-degenerate Hamiltonian flows on $S^3$ was developed. Fortunately, there are other methods to obtain all knots and links which, with some adjustments, are consistent with the results of Etnyre and Ghrist (2000a). They are discussed in section 7.

We adopt the results of Zung-Fomenko (Z-F) paper for our case. Specifically, the dynamical system whose equations of motion are given by eq.s(4.2) fits perfectly into Zung-Fomenko general theory. In it, in addition to the Hamiltonian $h = \frac{1}{2}(p_1^2 + p_2^2 + y_1^2 + y_2^2)$ there are several other integrals of motion among which the integral $F = \frac{1}{2}(f_2 - f_1)$, where $f_1 = p_1^2 + y_1^2$ and $f_2 = p_2^2 + y_2^2$, is playing a very special role. In Z-F paper it is called the Bott integral for reasons which will be explained shortly below. Full analysis of this dynamical system was given in the paper by Jovanović (2011). From it, it follows that the solution is made of two interlocked circular trajectories describing the Hopf link. According to Z-F theory other,\footnote{That is in terms of terminology of Conjecture 5.1. in the present case we have $H^{-1}(E) = S^3 = M$.}
more complicated links and torus knots, can be constructed from the Hopf link with help of the following 3 topological operations.

1. A connected sum $\#$ operation
2. A toral winding described as follows. Let $K$ be a link, $K = \{S_1, \ldots, S_k\}$. Select, say, $S_i$ and design a regular tubular neighborhood (that is torus $T^2$) around $S_i$. Draw on $T^2$ a simple closed smooth curve $S_i(T)$. Then the operation $K \rightarrow K \cup S_i(T^2)$ is called toral winding.\(^\text{12}\)
3. A special toral winding is described as follows. Let $S_i \in K$ and let $S_i(T^2)$ be the toral winding around $S_i$ of the type $(2, 2l + 1), l \in \mathbb{Z}$. Then, the operation $K \rightarrow K \cup S_i(T^2) \setminus S_i$ is called a special toral winding.

Zung and Fomenko (1990) proved the following

**Theorem 5.3.** Generalized iterated toral windings are precisely all the possible links of stable periodic trajectories of integrable systems on $S^3$.

**Corollary 5.4.** A generalized iterated torus knot is a knot obtained from trivial knots by toral windings and connected sums. These are the only knots of stable periodic trajectories of integrable systems on $S^3$.

**Remark 5.5.** The above theorem implies that not every link of stable periodic trajectories can be generated by some integrable dynamical system on $S^3$. For instance, there are no dynamically generated knots/links containing figure eight knot. This observation immediately excludes results of Birman and Williams (1983b), of Etnyre and Ghrist (2000a) and of Enciso and Peralta-Salas (2012).

It is very helpful to reinterpret the obtained results in terms of dynamics of the nonsingular Morse-Smale (NMS) flows. Morgan (1978) demonstrated that any iterated torus knot can be obtained as an attracting closed orbit for some NMS flow. He proved the following

**Theorem 5.6.** If $K \subset S^3$ is an attracting closed orbit for a NSM flow on $S^3$, then, $K$ is iterated torus knot.

In appendix D basic results on Morse-Smale flows are provided. Beginning from works by Poincare', it has become clear that the description of flows on manifolds is inseparable from the description of the topology of the underlying manifolds. Morse theory brings this idea to perfection by utilizing the gradient flows. The examples of gradient flows are given in part I, e.g. see eq.s(2.22) and (2.33). The basics on gradient flows can be found, for instance, in the classical book by Hirch and Smale (1974). The basic facts of Morse theory known to physicists, e.g. from (Nash and Sen 1983 and Frankel 1997), are not sufficient for description of the present case. This is so because the standard Morse theory deals with the

\(^{12}\)In knot-theoretic literature this operation is called "cabling operation". It will be discussed in some detail in the next section.

\(^{13}\)Here $(2, 2l + 1)$ is the standard notation for torus knots. In particular, $(2, 3)$ denotes the trefoil knot. Its complement was studied in detail by Ghys (2007). Further details are discussed in section 7.
nondegenerate and well separated critical points. In the present case we need to discuss not the critical points but critical (sub)manifolds. The extension of Morse theory covering the case of critical (sub)manifolds was made by Bott. Further developments are discussed in the review paper by Guest (2001).

In the context of evolution of dynamical systems on manifolds this extension naturally emerges when one is trying to provide answers to the following

**Questions:** How is complete integrability of a Hamiltonian system related to the topology of a phase or configuration space of this system? It is well known that in the action-angle variables the completely integrable system is being decomposed into Arnol’d-Liouville tori. But what is relative arrangement of these tori in phase space? Can such tori be knotted?

These questions can be answered by studying the already familiar dynamical system described by eqs(4.2). It has the Hamiltonian $h$ as the integral of motion. But in addition, it has another (Bott) integral $F$. To move forward, we notice that on $S^3$ grad $h \neq 0$ while grad $F$ can be zero. Since the Bott integral $F$ lives on $h$, the equation grad $F = 0$ is in fact the equation for the critical submanifold. The theory developed by Fomenko (1991) is independent of the specific form of $F$ and requires only its existence. The following theorem is crucial

**Theorem 5.7.** Let $F$ be the Bott integral on some 3-dimensional compact nonsingular isoenegric surface $h$. Then it can have only 3 types of critical submanifolds: $S^1$, $T^2$ and Klein bottles.

**Remark 5.8.** Without loss of generality we can exclude from consideration the Klein bottles by working only with orientable manifolds. In such a case we are left with $S^1$, $T^2$ and these were the only surfaces of Euler characteristic zero discussed in Theorem 4.1. of part I. Thus, using just this observation we can establish the relationship between the Morse-Smale and Beltrami (force-free) flows. In addition, following Wada (1989) we attach index 0 when the orbit is attractor, attach index 1 when the orbit is a saddle and 2 when the orbit is a repeller. Both the repeller, Fig.1a), and the attractor, Fig.1b), are circular orbits: $\alpha = S^1$. They are depicted in Fig.1.

**Fig.1**
Index 2 (repeller) a), and index 0 (attractor) b), $S^1$-type orbits

It does not include the saddle orbit. This orbit happens to be less important as explained in Theorem 5.9. It will be demonstrated below that both $S^1$ and $T^2$ can serve as attractors, repellers or saddles. For $T^2$ the analogous situation is depicted in Fig.2.

**Fig.2**
The”orientable cylinder” whose boundaries are made out of two $T^2$'s.
The repelling b) and the attracting c) tori $T^2$
The "orientable cylinder" whose boundaries is made out of two $T^2$'s. The repelling b) and the attracting c) tori $T^2$.

In the case of $T^2$ In such a case the following theorem by Wada is of importance

**Theorem 5.9.** Every indexed link which consists of all closed orbits of a NMS flow on $S^3$ is obtained from $(0,2)$ Hopf link by applying six operations. Conversely, every indexed link obtained from $(0,2)$ Hopf link by applying these six operations is the set of all the closed orbits of some NMS flow on $S^3$

These six (Wada) operations mentioned in Theorem 5.9. will be discussed in the next section. In the meantime we notice the following. Since by definition the NMS flow does not have fixed points on the underlying manifold $M$, use of the Poincare'-Hopf index theorem leads to $\chi(M) = 0$, where $\chi(M)$ is Euler's characteristic of $M$. It can be demonstrated that Euler characteristic for odd dimensional manifolds without boundary is zero (Prasolov and Sossinsky 1997). The proof for $S^3$ is especially simple and is provided below. Thus, $S^3$ can sustain the NMS flow. The full implications of this fact were investigated by Morgan (1978) (e.g. see Theorem 5.6 above). According to (now proven) geometrization conjecture every 3-manifold can be decomposed into no more than 8 fundamental pieces (Scott 1983, Fomenko and Matveev 1997). Morgan proved that any 3-manifold which does not contain a hyperbolic piece can sustain the NMS flows. This result is in accord with Remark 5.5.

For $S^3$ the above results can be proven using elementary arguments. Specifically, let us notice that Euler characteristic of both $S^1$ and $T^2$ is zero. In addition, it is well known that $S^3$ can be made out of two solid tori glued together. In fact, any 3-manifold admits Heegaard splitting (Fomenko and Matveev 1997). This means that it can be made by appropriately gluing together two handlebodies whose surfaces are Riemannian surfaces of genus $g$. In our case $T^2$ is the Riemannian surface of genus one and the handlebody (solid torus) is $V = D^2 \times S^1$. It has $T^2$ as its surface. Consider now the Hopf link. It is made out of two interlocked circles. We can inflate these circles thus making two solid tori out of them. These tori (toric handlebodies) can be glued together. If the gluing $\bar{h}$ is done correctly (Saveliev 2012, Rolfsen 1976), we obtain $S^3$ as result. The following set-theoretic properties of Euler characteristic $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$ and $\chi(M \times N) = \chi(M) \cdot \chi(N)$ can be used now to calculate the Euler characteristic of $S^3$. For the solid torus $V = D^2 \times S^1$ we obtain: $\chi(D^2 \times S^1) = \chi(D^2) \cdot \chi(S^1) = 0$ since $\chi(S^1) = 0$. Next $S^3 = (D^2_1 \times S^1_1) \cup_{\bar{h}} (D^2_2 \times S^1_2)$, where $\bar{h}$ is gluing homeomorphism $\bar{h} : T^2_1 \to T^2_2$. Therefore $\chi(S^3) = \chi(V_1 \cup_{\bar{h}} V_2) = \chi(V_1) + \chi(V_2) = 0$. Furthermore, the solid torus $V$ is the trivial case of the Seifert fibered space (see appendix E). Therefore $S^3$ is also Seifert fibered space (Seifert and Threlfall 1980). According to Theorem 5.7. and Definition D.5 of appendix D the NMS flows are made of finite number of periodic orbits which are either circles or tori. Thus, the NMS flows can take place on Seifert fibered manifolds or on graph manifolds. These are made of appropriately glued together Seifert fibered spaces (Fomenko and Matveev 1997). This conclusion is in accord with that obtained differently by Morgan (1978).
that can be integrated with help of the Bott integrals. Then, the following question arises:

**Question**: Is it always true that \((H) \subset (M)\)?

There is a number of ways to construct 3-manifolds from elementary (prime) blocks (Fomenko and Matveev 1997, Scott 1983). The situation with construction of 3-manifolds resembles to a large extent that in the number theory. From the number theory it is known that every nonnegative integer can be represented as a product of primes (taken in some nonnegative powers). The prime numbers play in number theory the same role as the prime manifolds in topology. The analog of multiplication in number theory is the operation of connected sum in topology.

To answer the above question Fomenko (1991) introduces 5 building blocks. Four out of 5 blocks are associated with orientable manifolds. In this paper we shall discuss only these types of manifolds. They can be described as follows.

1. The solid torus \(D^2 \times S^1\) whose boundary is \(T^2\).
2. The orientable "cylinder" \(T^2 \times D^1\), e.g. see Fig.2a). Its boundary of is made out of two \(T^2\).
3. An "orientable saddle" \(N^2 \times S^1\), e.g. see Fig.3a).

   ![Fig.3](image)
   Orientable saddle a) and its bifurcation b)

   Its boundary is made out of three \(T^2\).
4. A "non-orientable saddle", e.g. see Fig. 4a).

   ![Fig.4](image)
   A non-orientable saddle a) and its bifurcation b)

   Its boundary is made out of two \(T^2\).

Just by looking at these figures, it is clear that: orientable cylinder = orientable saddle + solid torus

**Theorem 5.10.** Let \(H = \text{const}\) be some isoenergy surface of some Hamiltonian system integrable on \(H\) via Bott integral \(F\). Then \(H\) can be made by gluing together a certain number of solid tori \(D^2 \times S^1\) and orientable saddles \(N^2 \times S^1\), that is

\[ H = \alpha(D^2 \times S^1) + \beta(N^2 \times S^1), \]

where \(\alpha\) and \(\beta\) are some nonnegative integers.

Matveev and Fomenko (1988) proved the following theorem (e.g. read Theorem 3 of this reference)

**Theorem 5.11.** A compact orientable 3-manifold with (possibly empty) torus-type boundary belongs to the class \((H)\) if and only if its interior admits a canonical decomposition into
pieces having geometries of the first seven types. In particular, \((H)\) contains no hyperbolic manifolds.

**Remark 5.12.** According to Thurston’s geometrization conjecture (now proven by G. Perelman) every 3-manifold admits a canonical decomposition into 8 basic building blocks/geometries. Out of these, only one is hyperbolic, e.g. read Scott (1983). This result is consistent with earlier made Remark 5.5. since the complement of figure 8 knot is a hyperbolic manifold. This topic will be further discussed in section 7.

**Corollary 5.13.** Theorem 5.11. provides an answer to the question “Is it always true that \((H) \subset (M)\)?”

Theorem 5.11. is of central importance for this paper. We shall return to its content a number of times in what follows. In the meantime, in preparing results for the next section, following Fomenko (1991), we need to describe how Wada’s results are connected with bifurcations of the Arnol’d-Liouville tori. Bifurcations of these tori are caused by changes in the constant \(c\) in the equation \(F = c\) for the Bott integral. These are depicted in Fig.26 of Fomenko’s paper but, again, we do not need all of them and those which we need can be easily described in analogy with 4 basic blocks described above. Our presentation is facilitated by results of Theorem 5.7. Using it, we conclude that in orientable case we have to deal only with \(S^1\) and \(T^2\). Stability or instability of these structures cause us to consider along with them a nearby space foliated, say, by tori. Specifically,

1. A torus \(T^2\) is contracted to the axial circle \(S^1 = \alpha\) and then, it may even vanish, depending upon the value of \(c\). Thus, we get \(T^2 \rightarrow S^1 \rightarrow 0\). Naturally, the process can go in opposite direction as well, e.g. see Fig. 1 a) and b).
2. Two tori \(T^2\) move toward each other (that is they both flow into \(T^2\)) as depicted in Fig.2 b) and c). Let, say, both the outer boundary and the inner boundary be unstable in the sense that there is a \(T^2\) surface somewhere in between the inner and the outer \(T^2\)s. Such a torus is as an attractor since both the inner and the outer \(T^2\)s are being attracted to it. In this case we may have the following process: \(2T^2 \rightarrow T^2 \rightarrow 0\).
   Apparently, the process can go in reverse too. In such a case we are dealing with repeller.
3. Imagine now a pair of pants. Consider a succession of crosssections for such pants.
   On one side, we will have \(D^2\) (the waist). This configuration will continue till it will hit the fork- the place from where the pants begin. The fork crosssection is made of figure 8. After passing that crosssection we are entering the pants. This process is depicted in Fig.3. We begin with the configuration of Fig.2a), then the bifurcation depicted in Fig.3b) is taking place resulting in configuration depicted in fig.3 a).
   Thus, initially we had \(T^2\) and finally we obtained \(2 T^2\). That is now we have: \(T^2 \rightarrow 2T^2\).
4. A non-orientable saddle is obtained if for any crosssection of \(N^2 \times S^1\) we can swap the 1st \(D_1^2 \subset T_1^2\) with \(D_2^2 \subset T_2^2\) as depicted in Fig.4 a) and b). In such a case
we obtain: $T^2 \rightarrow T^2$. Such description of bifurcations is consistent with Theorem 5.7. It can be demonstrated that the bifurcations 2 and 4 can be reduced to 1 and 3. This fact will be used in the next section.

6. Dynamical bifurcations and topological transitions associated with them

6.1. General remarks

In the previous section we mentioned works by Wada, Morgan and Fomenko-Zung related to dynamics of NMS flows. The focus of this paper however is not on these flows as such but rather on descriptions of mechanisms of generation of knotted/linked trajectories. Theorems 5.3. and 5.6. provide us with guidance regarding the types of knots/links which can be generated by the NMS flows. However, the above theorems do not explain how such knots/links are actually generated. Wada’s results, summarized in Theorem 5.9., provide a formal description of the sequence of topological moves producing the iterated knots/links, beginning with the Hopf links. These moves are depicted in Figs 2-7 of the paper by Campos et al (1997). These are, still, just particular kinds of Kirby moves as explained in appendix F. The description of these moves is totally disconnected from the description of dynamical bifurcations depicted in Figs 1-4 above. Fomenko (1991) designed a graphical method helpful for understanding of the sequence of topological transitions. More details on this topic can be found in the monograph by Fomenko and Bolsinov (2004). In spite of this, it is not immediately clear from these sources (at least to the author of this paper) how to connect these results with those by Wada (depicted in the paper by Campos et al). For this reason, in this section we develop our own (different) approach to the description of topological transitions between dynamically generated knots/links. It is based on results of sections 4 and 5.

6.2. Generating cable and iterated torus knots

In view of Theorems 5.3 and 5.6. we need to provide more detailed description of the iterated torus knots/links. For this purpose, following Menasco (2001) we introduce an oriented knot $S^1 \rightarrow K \subset S^3$. Let then $V_K$ be a solid torus neighborhood of $K$. Let $\partial V_K = T^2_K \subset S^3$. As in appendix E, we write

$$J \sim \nu R + \mu L.$$  \hfill (6.1)

This is eq.(E.1a) describing a simple oriented closed curve $J$ going $\nu$ times around the meridian $R$ and $\mu$ times around the longitude $L$ of $T^2_K$. $J$ belongs to the homotopy class $\pi_1(T^2_K)$ if and only if either $\nu = \mu = 0$ or $g.c.d(\mu, \nu) = 1$ (Rolfsen 1976).

**Definition 6.1.** When $K$ is unknot, $K = K_0$, the curve $J$ is called $(\mu, \nu)$ a cable of $K$. Thus, $K(\mu, \nu)$ torus knot is a cable of the unknot. The cabling operation leading to the formation of $K(\mu, \nu)$ torus knot from now on will be denoted as $C(K_0, (\mu, \nu))$.

The above definition allows us to generate the iterated torus knot inductively. Beginning with some unknot $K_0$, we select a sequence of co-prime 2-tuples of integers $(P, Q) =$
\{ (p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n) \}, \text{ where } p_1 < q_1. \text{ This information allows us to construct the oriented } iterated \text{ torus knot}

\[ K(P, Q) = C(C(\cdots C(C(K_0, (p_1, q_1))(p_2, q_2)) \cdots, (p_{n-1}, q_{n-1})), (p_n, q_n)). \] (6.2)

No restrictions on the relative magnitudes of \( p_i \) and \( q_i \) are expected to be imposed for \( n > 1 \).

(Birman and Wrinkle 2000).

These general rules can be illustrated using the trefoil knot \( K(2, 3) \) as an example. It is a cable of the unknot. Since the trefoil is the (simplest) torus knot it can be placed on the surface of the solid torus. This torus has an unknot \( K_0 \) as the core. In Fig.5

Fig.5

One of the ways to depict the trefoil knot

the core \( K_0 \) is depicted as a circle while the longitude of the solid torus around \( K_0 \) is labeled by \( L_0 \). Without loss of generality both circles \( K_0 \) and \( L_0 \) are placed on the same plane \( z = 0 \). The projection of the knot \( K(2, 3) \) into the same \( z \)-plane intersects \( L_0 \) in \( q = 3 \) points. Instead of Fig.5, the same configuration can be interpreted in terms of closed braids. For this purpose, following Murasugi (1996), the representation of torus knot \( K(q, p) \) in terms of closed braids is depicted in Fig.6

Fig.6

Braid-type presentation of the torus knot \( K(p, q) \)

Such a representation is not unique. It is so because \( K(q, p) = K(p, q) \). Furthermore, \( K(-q, p) \) is the mirror image of \( K(q, p) \). If \( \gcd(p, q) = 1 \), then \( K(-q, -p) \) is the same torus knot but with the reverse orientation. Being armed with these results, we need to recall the presentation of the braid group \( B_n \) made out of \( n \) strands. Its generators and relations are

\[ B_n = \left( \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2 \right. \]

\[ \left. \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, i = 1, 2, \ldots, n - 2 \right). \] (6.3)

The connection between the \( K(q, p) \) and its braid analog depicted in Fig.6. can be also established analytically via

\[ K(q, p) \Rightarrow (\sigma_{p-1} \sigma_{p-2} \cdots \sigma_2 \sigma_1)^e, \] (6.4.)

where \( e = \pm 1 \) depending on knot orientation. Here the symbol \( \Rightarrow \) means ”closure of the braid”-an operation converting braids into knots/links (Murasugi 1996). These general results adopted for the trefoil knot are depicted in Fig.7.

Fig.7

Braid-type representation of the torus \( K(2,3) \).

By design, the notations on this figure are meant to facilitate visualization of the iteration process. It is formalized in the following
Definition 6.2. Cabling operation. Let \( K_1 \) be an arbitrary oriented knot in \( S^3 \) and \( N(K_1) \) is its solid torus tubular neighborhood. Let furthermore \( L_{K_1} \) be a longitude for \( K_1 \). It is a simple closed curve on \( \partial N(K_1) \) homologous to \( K_1 \) in \( N(K_1) \) and null-homologous in \( S^3 \setminus K_1 \). Consider now a homeomorphism \( h : N(K_0) \to N(K_1) \) which is also mapping \( L_0 \) into \( L_{K_1} \). By relabeling: \( K_0 \to K_1 \) and \( K_1 \to K_{i+1} \), the cabling operation \( C \) can be formally defined now as \( h : N(K_i) \to N(K_{i+1}) \), with \( L_i \) being mapped into \( L_{i+1} \).

Definition 6.3. A cable space \( C \) is a Seifert fibered manifold obtainable from the solid torus \( S^1 \times D^2 \) by removing \( N(K_1) \subset S^1 \times D^2 \) from \( S^1 \times D^2, D^2 = D^2 \setminus \partial D^2 \). Thus, in accord with results of appendix A, it is a Seifert fibered manifold having no exceptional fibers. Alternatively, following (Jaco and Shalen 1979), page 182, a cable space \( C \) can be defined as follows. Let \( S \) be a Seifert fibered space over a disc \( D^2 \) with one exceptional fiber, then the complement in \( S \) of an open regular neighborhood of a regular fiber is a cable space \( C \).

The validity of the above definition is based on the following

Theorem 6.4. (Hempel 1964) Let \( V_K \subset S^1 \times D^2 \) be a tubular neighborhood of the torus knot \( K \) (including the unknot) in \( S^3 \) and let \( M \) be a simply connected 3-manifold containing a solid torus \( V \). Suppose that there is a homeomorphism \( h : S^3 \setminus \hat{V}_K \) onto \( M \setminus V \), then \( M = S^3 \).

Corollary 6.5. Due to the Heegaard decomposition, \( S^3 \) is always decomposable into two solid tori. Therefore, in view of the above homeomorphism, we can replace \( S^3 \) by the solid torus \( S^1 \times D^2 \) so that \( S^3 \setminus \hat{V}_K \) can be replaced by \( S^1 \times D^2 \setminus \hat{V}_K \). This provides an alternative justification of the operations depicted in Fig.12 (appendix F).

The information we have accumulated allow us now to reobtain one of Zung and Fomenko’s (1990) results. Specifically, we have in mind the special toral windings leading to torus knots of the type \( K(2, 2l+1) \). Now we are in the position enabling us to explain the meaning of this result.

Since \( K(q, p) = K(p, q) \), we obtain: \( K(2, 2l+1) = K(2l+1, 2) \) implying that we are dealing with torus knots made out of just 2 braids twisted \( n = 2l+1 \) times followed by the closure operation making a knot out of them. Adopted for such a case eq.(6.4) now reads: \( K(n, 2) = (\sigma_1) \cdot i+j=1=m. \) Switching of just one crossing in the knot projection, e.g. see Fig.6, makes \( (\sigma_1)^n \) to be replaced by \( \sigma_1^i \sigma_1^{-1} \sigma_1^j \) so that \( i + j - 1 = m \). To find \( m \) it is sufficient to notice that initially (that is before switching) we had \( i + j = n \). Thus, \( i + j = n - 1 \) and, therefore, \( n - 2 = m \). In view of the property \( K(q, p) \simeq K(-q, -p) \) it is sufficient to discuss only the case \( m \geq 0 \) leading to \( n \geq 2 \). Since \( n = 2l+1 \) we obtain \( n = 3 \) when \( l = 1 \) that is we are dealing with the trefoil \( K(2, 3) \). When \( l = 0 \) we are dealing with the unknot and so on. The obtained result is consistent with the Kirby move depicted in Fig.17 (appendix F). Indeed, we can always enclose the unknot into solid torus (Corollary 6.5). If we begin with the Hopf link, which is ±1 framed unknot, and fix our attention at one of the rings which is unknot, then another ring can be looked upon as framed with framing ±1. Such type of framing converts another unknotted ring into the Hopf ring again. This situation is depicted in Fig.8.
The extra ring can always be found in the spirit of Wada’s (1989) paper. Say, we can take the meridian of the solid torus into which the first unknotted ring of the Hopf ring was enclosed as an extra ring. In fact, this is the content of Theorem 2 by Menasco (1984). Thus, the Kirby moves depicted in Fig. 15 generate all special toral windings obtained in Fomenko and Zung paper.

With this result in our hands, we still have to uncover the topological mechanism by which the iterated torus knots are generated in order to recover the rest of Zung-Fomenko results and those obtained by Morgan (e.g. see Theorems 5.3. and 5.6. above). It is important to notice at this stage that torus knots are allowed in the NMS dynamics just because the Kirby-Fen-Rourke moves allow such knots to exist. These moves are not sufficient though for generation of the iterated torus knots/links. Following works by Milnor (1968) and Eisenbud and Neumann (1985) it is possible using methods of algebraic geometry to develop graphical calculus generating all iterated torus knots and links. Incidentally, in current physics literature one can find proposals for generating iterated torus knots/links via methods of algebraic geometry just cited. E.g. read Dennis et al (2010) or Machon and Alexander (2013). While methods of algebraic geometry are very effective for depicting knots/links they are rather formal since they are detached from the topological content/mechanism of dynamical bifurcations generating various iterated torus knots. Thus, we are going to proceed with the topological treatment of dynamical bifurcations. For this purpose, following Jaco (1980), we proceed with a couple of

**Definition 6.6.** In accord with Theorem 6.4, a complement of a torus knot $K(p,q)$ in $S^3$ is the torus knot space.

**Definition 6.7.** An ($n$-fold) composing space is a compact 3-manifolds homeomorphic to $W(n) \times S^1$, where $W(n)$ the disk with n-holes

**Remark 6.8.** Evidently, previously defined cable space is just a special case of the composing space.

The complement of a link in $S^3$ made of a composition of $n$ torus-type knots (including the unknot(s)) is an n-fold composing space. An n-fold composing space is the Seifert fibered space in which there are no exceptional fibers (appendix E) and the base (the orbit space) is a disc with $n$ holes. Such a fibration of composing space is called standard. The following theorem summarizes what had been achieved thus far

**Theorem 6.9.** (Jaco and Shalen 1979), Lemma 6.3.4. A Seifert-fibered 3-manifold $M(K) = S^3 \setminus \hat{N}(K)$ with incompressible boundary $\partial M(K) = N(K) \setminus \hat{N}(K)$ is either a torus knot space, a cable space or a composing space.

Following Jaco (1980), page 32, the incompressibility can be defined as follows. Set $M(K) = S^3 \setminus \hat{N}(K)$, then $\partial M(K) = S^1 \times S^1$ is incompressible in $M(K)$ iff $K$ is not an unknot. A complement of a cabled knot in $S^3$ always contains a cabled space with incompressible boundary (Jaco and Shalen 1989), page 182.

At this point we are having all the ingredients needed for description of the bifurcation cascade creating iterated torus knots/links. The process can be described inductively. We begin with the seed- the cable space depicted in Fig.2a). The first bifurcation is depicted in Fig.3 a),b). It is producing the composing nonsingular Seifert fibered space whose orbit
space is the disc $D^2$ with two holes. The homeomorphism depicted in Fig.12 allows us to twist two strands as many times as needed. Thus, many (but surely not all) iterated torus knots are going to have the same complements in $S^3$. Clearly, the first in line of such type of knots is the trefoil knot $K(2,3)$ depicted in Fig.7a). Evidently, other knots of the type $K(p,q)$ are also permissible. Their complement is still going to be the two-hole composing space depicted in Fig.3.a). The three-hole composing space is generated now as follows. Begin with the composing space depicted in Fig.3.a). Use the bifurcation process depicted in Fig.3.b) and apply it to one of the two holes in Fig.3.a). As result, we obtain the three-hole composing space. Again, we can use the homeomorphisms to entangle the corresponding strands with each other.

The bifurcation sequence leading to creation of all types of iterated torus knots is made of steps just described. It suffers from several deficiencies. The first among them is a regrettable absence of the one-to-one correspondence between the links and their complements, e.g. see Fig.12. The famous theorem by Gordon and Luecke (1989) seemingly guarantees that the degeneracy is removed for the case of knots since it states that knots are being determined by their complements. This happens not always to be the case. Details are given in the next section. The second deficiency becomes evident already at the level of the trefoil knot which is the cable of the unknot, e.g. see Fig.7. In Fig.7a) we see that it is permissible to associate one strand of the two-holed composing space with the unknot while another-with the trefoil knot. At the same time, if we want to use braids, as depicted in Fig.7b), then we obtain 3 braids instead of two strands. Thus, we are coming to the following

Problem: Is there a description of the iterated torus knots in terms of braids as it is done, say, for the torus knots in Fig.6?

A connection between closed braids and knots/links is known for a long time. It is of little use though if we are interested in providing a constructive solution to the problem we had just formulated. Surprisingly, the solution of this problem is very difficult. It was given by Schubert (1953). Recently, Birman and Wrinkle (2000) found an interesting interrelationship between the iterated torus knots, braids and contact geometry. In view of this, we would like to discuss some results of these authors in the next subsection.

6.3. Back to contact geometry/topology. Fascinating interrelationship between the iterated torus and transversely simple knots/links

In sections 5 and 6 the results of contact geometry/topology obtained in sections 2-4 were used but without development. In this subsection we would like to correct this deficiency. For this purpose we need to introduce the notions of the Legendrian and transverse knots. As before, our readers are encouraged to consult books by Geiges (2008) and Kholodenko (2013) for details.

We begin our discussion with some comments on ”optical knots” which were made in the Introduction section of part I. The term ”optical knots” was invented by Arnol’d (1986) in connection with the following.

Solutions of the eikonal equation $(\partial S/\partial q)^2 = 1$ determine the optical Lagrangian submanifold $p = \partial S/\partial q$ belonging to the hypersurface $p^2 = 1$. Every stable Lagrangian singularity is revealing itself in the projection of the Lagrangian submanifold into the base (q-space),
e.g. read Appendix 12 of Arnol’d (1989) book. If these results are used in \( \mathbb{R}^3 \), then the base is just the whole or part of \( \mathbb{R}^2 \). In such a case we can introduce the notion of a Legendrian knot. It originates from the equation \( p = \partial S/\partial q \) written as \( dS - pq = 0 \) which we had already encountered in part I, section 4. This time to comply with literature on Legendrian knots we are going to relabel the entries in the previous equation as follows

\[
dz - ydx = 0. \tag{6.5a}
\]

The minus sign in front of \( y \) is determined by the orientation of \( \mathbb{R}^3 \). Change in orientation causes change in sign. The standard contact structure \( \xi \) in the oriented 3-space \( \mathbb{R}^3 = (r, \phi, z) \), that is in cylindrical coordinates, is determined by the kernel \( (\xi = \ker \alpha, \text{that is by the condition } \alpha = 0) \) of the 1-form \( \alpha \),

\[
\alpha = r^2 d\phi + dz. \tag{6.5b}
\]

(compare this result against the eq.(4.3).)

**Definition 6.10.** A Legendrian knot \( K_L \) in an oriented contact manifold \( (M, \xi) \) is a circle \( S^1 \) embedded in \( M \) in such a way that it is always tangent to \( \xi \). Let the indeterminate \( \tau \) parametrize \( S^1 \) and choose \( \mathbb{R}^3 \) as \( M \), then the embedding is defined by the map \( f : S^1 \to \mathbb{R}^3 \) specified either by \( \tau \to (x(\tau), y(\tau), z(\tau)) \) or by \( f : \tau \to (r(\tau), \phi(\tau), z(\tau)) \). The coordinates \( r(\tau), \phi(\tau), z(\tau) \) are real-valued periodic functions with period, say, \( 2\pi \). The tangency condition is being enforced by the equation

\[
\frac{dz}{d\tau} = y(\tau) \frac{dx}{d\tau}. \tag{6.6}
\]

Thus, whenever \( \frac{dx}{d\tau} \) vanishes, \( \frac{dz}{d\tau} \) must vanish as well.

**Definition 6.11.** In terms of \( x, y, z \) coordinates it is possible to define either the *front* or the Lagrangian projection of the Legendrian knot. The front projection \( \Pi \) is defined by

\[
\Pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \to (x, z) \tag{6.7}
\]

The image \( \Pi(K_L) \) under the map \( \Pi \) is called the *front projection* of \( K_L \). The condition, eq.(6.6), is causing the front projection not to contain the vertical tangencies to the \( K_L \) projection. Because of this, the front projection of the \( K_L \) is made of a collection of cusp-like pieces (with all cusps arranged in such a way that the cusp axis of symmetry is parallel to \( x \)-axis) joined between each other. The front projection is always having \( 2m \) cusps, \( m \geq 1 \). It is important that these cusps exist only in the \( x-z \) plane, that is not in \( 3\)-space.

*The Lagrangian projection* \( \pi \) of the Legendrian knot \( K_L \) is defined by

\[
\pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \to (x, y). \tag{6.8}
\]

For the sake of space, we shall not discuss details related to the Lagrangian projection. They can be found in Geiges (2008).

---

\[\text{In fact, we can always use the contactomorphic transformation to replace locally } \alpha = \frac{1}{2}(r_1^2 d\phi_1 + r_2^2 d\phi_2) \text{ by } \tilde{\alpha} = \frac{r_1^2}{r_2} d\phi_1 + d\phi_2.\]
Remark 6.12. Arnol’d optical knots are Legendrian knots. In geometrical optics such knots are also known as plane wavefronts. They obey the Hugens principle.

In addition to the Legendrian knots with their two types of projections there are also transverse knots. They are immediately relevant to this paper. It can be shown that the topological knots/links can be converted both to the Legendrian and to the transverse knots so that the transverse knots can be obtained from the Legendrian ones and vice versa (Geiges 2008), page 103. Since the Legendrian knots are also known as optical knots, this fact provides a justification for the titles of both parts I and II of this work.

Going back to the description of transverse knots, we begin with the

Definition 6.13. A transverse knot $K_T$ in contact manifold $(\mathbb{R}^3, \xi)$ is a circle $S^1$ embedded in $\mathbb{R}^3$ in such a way that it is always transverse to $\xi$. The transverse knots are always oriented. The front projection $\Pi(K_T)$ must satisfy two conditions (Etnyre 2004, Geiges 2008) graphically depicted in Fig.9.

Fig.9
Forbidden configurations for the front projection of transverse knots

To formulate the meaning of the notion of transversality and of these conditions analytically, consider a mapping $f: S^1 \rightarrow \mathbb{R}^3: \tau \rightarrow (x(\tau), y(\tau), z(\tau))$ with $x(\tau), y(\tau)$ and $z(\tau)$ being some periodic functions of $\tau$. The transversality condition now reads

$$\frac{dz}{d\tau} - y(\tau)\frac{dx}{d\tau} > 0. \quad (6.9a)$$

This inequality defines the canonical orientation on $K_T$. The front projection $\Pi(K_T)$ is parametrized in terms of the pair $(x(\tau), z(\tau))$. At the vertical tangency pointing down we should have $\frac{dx}{d\tau} = 0$ and $\frac{dz}{d\tau} < 0$. This contradicts eq.(6.9a) thus establishing the condition a) in Fig.9. The condition b) on the same figure is established by using eq.(6.9a) written in the form

$$y(\tau) < \frac{z'}{x'}. \quad (6.9b)$$

This inequality implies that $y$-coordinate is bounded by the slope $dz/dx$ in the $x$-$z$ plane (the positive $y$-axis is pointing into the page). Etnyre (2004) argues that this observation is sufficient for proving that the fragment depicted in Fig.9b) cannot belong to the fragment of the projection of $K_T$.

The importance of transverse knots for this paper is coming from their connection with closed braids studied in previous subsection. To describe this connection mathematically, it is useful to introduce the cylindrical system of coordinates $(r, \phi, z)$ so that any closed braid can be looked upon as a map $f: S^1 \rightarrow \mathbb{R}^3: \tau \rightarrow (r(\tau), \phi(\tau), z(\tau))$ for which $r(\tau) \neq 0$ and $\phi'(\tau) > 0 \forall \tau$ (Overkov and Schevchishin 2003).

Definition 6.14. A link $L$ is transverse if the restriction of $\alpha = r^2d\phi + dz$ to $L$ nowhere vanishes. Any conjugacy class in $B_n$ defines a transverse isotopy class of transversal links/knots. Bennequin (1983) proved that any transverse knot/link is transversely isotopic to a closed braid.
Any knot $K \subset \mathbb{R}^3$ belongs to its topological type $\mathcal{K}$, that is to the equivalence class under isotopy of the pair $(K, \mathbb{R}^3)$. In the case of transverse knots/links one can define the transverse knot type $\mathcal{T}(K)$. It is determined by the requirement $\frac{\dot{z}(\tau)}{\dot{\phi}(\tau)} + r^2(\tau) > 0$ at every stage of the isotopy and at every point which belongs to the knot $K_T$.

In the standard knot theory knots are described with help of topological invariants, e.g. the Alexander or Jones polynomials, etc. Every transverse knot belongs to a given topological type $\mathcal{K}$. This means that the knot/link invariants such as Alexander or Jones polynomials can be applied. In addition, though, the transverse knots have their own invariant $\mathcal{T}(K)$ implying that all invariants for topological knots, both the Legendrian and transverse, should be now supplemented by the additional invariants. For the transverse knots in addition to $\mathcal{T}(K)$ one also has to use the Bennequin (the self-linking) number $\beta(\mathcal{T}(K))$. To define this number, following Bennequin (1983), we begin with a couple of definitions.

**Definition 6.15.**a) The braid index $n = n(K)$ of a closed braid $K$ is the number of strands in the braid

**Definition 6.16.**a) The algebraic length $e(K)$ of the braid $b$

$$b = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_k}^{\varepsilon_k} \in B_n$$

prior to its closure resulting in knot $K$ is defined as

$$e(K) = \varepsilon_1 + \cdots + \varepsilon_k \in \mathbb{Z}.$$ (6.10a)

**Definition 6.17.** The Bennequin number $\beta(\mathcal{T}(K))$ is defined as

$$\beta(\mathcal{T}(K)) = e(K) - n(K).$$ (6.11)

Further analysis (Birman and Wrinkle 2000) of the results obtained by Bennequin ended in the alternative definitions of the braid index and the algebraic length. Specifically these authors came up with the following

**Definition 6.15.**b) The braid index $n = n(K)$ of a closed braid $K$ is the linking number of $K$ with the oriented $z$-axis.\[15\]

**Definition 6.16.**b) The algebraic crossing number $e = e(K)$ of the closed braid is the sum of the signed crossings in the closed braid projection using the sign convention depicted in Fig.13.

A generic (front) projection of $K_T$ onto $r - \phi$ plane is called closed braid projection. Since the braid is oriented, the projection is also oriented in such a way that moving in the positive direction following the braid strand (along the z-axis direction) increases $\phi$ in the projection. To use these definitions effectively, we need to recall the definition of a writhe.

**Definition 6.18.** The writhe $w(K)$ of the knot diagram $D(K)$ (knot projection into plane $\mathbb{R}^2$) of an oriented knot is the sum of the signs of the crossings of $D(K)$ using the sign convention depicted in Fig.13.

\[15\]Recall that we are using the cylindrical system of coordinates for description of braids.
From here it follows that if the z-axis is perpendicular to the plane \( \mathbb{R}^2 \) we obtain \( n(K) = 0 \) and
\[
\beta(T(K)) = w(K).
\] (6.12)
This result is in accord with that listed in Etnyre (2004) and Geiges (2008), page 127, where it was obtained differently.

Now we would like to evaluate \( w(K) \). For this purpose choose the point \( z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in S^3 \) and, at the same time, \( z \in K_T \). It is permissible to think about \( z \) also as a point in \( \mathbb{R}^4 \). Because of this, it is possible to introduce physically appropriate coordinate system in \( \mathbb{R}^4 \) as follows. Using eq.(3.5) we select the components of the Liouville vector field \( \mathbf{X} \), that is \( \mathbf{X} = (x_1, y_1, x_2, y_2) \), as the initial reference direction. Since \( S^3 \) is determined by the equation \( x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1 \), or \( |z_1|^2 + |z_2|^2 = 1 \), we obtain \( z_1 dz_1 + z_2 dz_2 = 0 \). This is an equation for a hyperplane \( \xi (\xi = \ker \alpha) \) is defined by eq.(4.3)).

Recall that, say, in \( \mathbb{R}^3 \) the plane \( \mathcal{P} \) is defined as follows. Let \( X_0 = (x_0, y_0, z_0) \in \mathcal{P} \). Let the normal \( N \) to \( \mathcal{P} \) is given by \( N = (a, b, c) \) where both \( X_0 \) and \( N \) vectors are determined with respect to the common origin. Then, \( \forall X = (x, y, z) \in \mathcal{P} \) the equation for the plane is \( N \cdot (X - X_0) = 0 \). To relate the complex and real cases, we rewrite \( z_1 dz_1 + z_2 dz_2 = 0 \) as \( z \cdot dz = 0 \). This result is not changed if we replace \( dz \) by \( idz \). In such a case, the vector \( \mathbf{X} \) is being replaced by \( \mathbf{\tilde{X}} = (-y_1, x_1, -y_2, x_2) \). Eq.s (3.5) and (3.11) help us to recognize in \( \mathbf{\tilde{X}} \) the Reeb vector field. Evidently, the equation \( z_1 dz_1 + z_2 dz_2 = 0 \) defining the contact structure \( \xi (\xi = \ker \alpha) \) is compatible now with the condition of orthogonality \( \mathbf{X} \cdot \mathbf{\tilde{X}} = 0 \) in \( \mathbb{R}^4 \).

In section 4 we established that the Reeb vector field is proportional to the Hamiltonian vector field and to the Beltrami vector field. Therefore, the knot/link transversality requires us to find a plane \( \mathcal{P} \) such that the Beltrami-Reeb vector field \( \mathbf{\tilde{X}} \) is pointed in the direction orthogonal/transversal to the contact plane \( \mathcal{P} \). To find this plane unambiguously, we have to find a set of mutually orthogonal vectors \( \mathbf{X}, \mathbf{\tilde{X}}, \mathbf{\tilde{X}}, \mathbf{X} \) which span \( \mathbb{R}^4 \). By keeping in mind that \( \mathbb{R}^4 \) is a symplectic manifold into which the contact manifold \( S^3 \) is embedded, we then should adopt these vectors to \( S^3 \). Taking into account that the Liouville field \( \mathbf{X} \) is orthogonal to the surface \( |z_1|^2 + |z_2|^2 = 1 \) we can exclude it from the consideration. Then, the velocity \( \dot{x} \) of any curve \( x(\tau) = (x_1(\tau), y_1(\tau), x_2(\tau), y_2(\tau)) \) in \( S^3 \) admits the following decomposition (Calin and Chang 2009)
\[
\dot{x}(\tau) = a(\tau)\dot{X} + b(\tau)\dot{\mathbf{X}} + c(\tau)\dot{\mathbf{X}}.
\] (6.13)
To understand the true meaning this result, we shall borrow some results from our book, Kholodenko (2013). In it we emphasized that contact geometry and topology is known under different names in different disciplines. In particular, we demonstrated that the basic objects of study in sub-Riemannian and contact geometries coincide. This gives us a permission to re interpret the obtained results in the language of sub-Riemannian geometry.

By means of contactomorphism: \((x, y, z) \rightarrow (x, y, \frac{1}{2}xy - z)\), the standard 1-form of contact geometry \( \alpha = dz + xdy \) after subsequent replacement of \( z \) by \( t/4 \) acquires the following look
\[
\alpha = -\frac{1}{4}dt + \frac{1}{2}(ydx - xdy).
\] (6.14a)

26
This form vanishes on two horizontal vector fields

\[ X_1 = \partial_x + 2y\partial_t \quad \text{and} \quad X_2 = \partial_x - 2y\partial_t. \]  

(6.14b)

Mathematically, the condition of horizontality is expressed as

\[ \alpha(X_i) = 0, \quad i = 1, 2. \]  

(6.14c)

The Reeb vector field \( R \) is obtained in this formalism as a commutator:

\[ [X_1, X_2] = -4\partial_t \equiv R. \]  

(6.14d)

Clearly, this commutator is sufficient for determination of \( R \). It was demonstrated in Kholodenko (2013) that the above commutator is equivalent to the familiar quantization postulate \( [q, p] = i\hbar I \). That is, the commutator, eq. (6.14d), defines the Lie algebra of the Heisenberg group. This group has 3 real parameters so that the Euclidean space \( \mathbb{R}^3 \) can be mapped into the space of Heisenberg group. Because of the commutator, eq. (6.14d), it follows that the motion in the 3rd (vertical) dimension is determined by the motion in the remaining two (horizontal) dimensions so that

\[ \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}. \]

This is the simplest form of the Holographic principle used in high energy physics. It also lies at the foundation of the sub-Riemannian geometry.

Specifically, suppose that we are having a curve \( x(s) = (x(s), y(s), t(s)) \) in \( \mathbb{R}^3 \). Its velocity vector \( \dot{x}(s) \) can be decomposed as

\[
\dot{x}(s) = \dot{x}\partial_x + \dot{y}\partial_y + i\partial_t = \dot{x}(\partial_x + 2y\partial_t) - 2\dot{y}\partial_t + \dot{y}(\partial_x - 2y\partial_t) + 2\dot{x}y\partial_t + i\partial_t = \dot{x}X_1 + \dot{y}X_2 - \frac{1}{4}(i + 2\dot{y}(2x)\partial_t)R
\]

(6.15)

so that the curve \( x(s) \) is horizontal iff \( i + 2\dot{y}(2x)\partial_t = 0 \). This condition is equivalent to the condition \( \xi = \ker\alpha \) introduced before, e.g. see eq. (6.6), in connection with the Legendrian knots/links. Thus, the decomposition given by eq. (6.15) for the contact manifold \( \mathbb{R}^3 \) should be replaced now by the decomposition given by eq. (6.13) for \( S^3 \). In spite of the apparent differences in appearance between these two results, they can be brought into correspondence with each other. For this purpose, following Geiges (2008), pages 76 and 95, we need to construct the neighborhood of a transverse knot/link. Since any knot \( K \) is just an embedding of \( S^1 \) into \( S^3 \) (or \( \mathbb{R}^3 \)), locally we can imagine \( S^1 \) piercing a plane \( \mathcal{P} \) perpendicularly. Such a plane can be spanned, say, by the vectors \( X_1 \) and \( X_2 \) we just had described. Clearly, this makes our knot/link transverse and the neighborhood of \( K_T \) is described by the two conditions

\[ a) \; \gamma_T := (\theta, x = 0, y = 0) \quad \text{and} \quad \theta \in S^1; \quad b) \; d\theta + xdy - ydx = 0. \]  

(6.16)

The contact structure \( \xi = \ker\alpha \) is determined by eq. (6.14a)) in which the "time \( t \)" coordinate is compactified to a circle \( S^1 \). Once this is done we immediately recognize that:
a) The contact structure in the present case is exactly the same as can be obtained from the one-form \( \tilde{\alpha} = \frac{r_1^2}{r_2} d\phi_1 + d\phi_2 \) we have introduced earlier.

b) The locality of the result, eq.(6.16), makes it not sensitive to the specific nature of knot/link. In particular, it remains valid for the Hopf link too.

These observations were proven previously by Bennequin (1983) who derived them differently and formulated them in the form of the

**Theorem 6.19.** (Bennequin 1983, Theorem 10). *Every link transversal to standard contact structure on \( S^3 \)(given by our eq.(4.3)), transversally isotopic to a link \( L \) whose tangent \( TL \) at any point which belongs to \( L \) is arbitrarily close to the tangent to the fibre of the Hopf fibration.*

To take full advantage of this result we have to make several additional steps. For this purpose we have to use the quaternions. Recall, that the quaternion \( q \) can be represented as \( q = z_1 + j \bar{z}_2 = (z_1, \bar{z}_2) \), where \( j \) is another complex number, \( j^2 = -1 \) such that \( ij = -ji \).

Formally, there is also the third complex number \( k \), \( k^2 = -1 \).

In view of the definition of \( q \) and the commutation relation \( ij = -ji \) its use can be by-passed. In view of just defined rules, \( jq = jz_1 + j^2 \bar{z}_2 = -\bar{z}_2 + jz_1 = (-\bar{z}_2, z_1) \). The obtained result we use to encode yet another vector \( \tilde{\mathbf{X}} = (-x_2, y_2, x_1, -y_1) \) in \( \mathbb{R}^4 \). This vector was introduced by Bennequin without derivation. With the vectors \( \mathbf{X}, \tilde{\mathbf{X}} \) and \( \mathbf{X} \) just defined, we notice that they are mutually orthogonal in \( \mathbb{R}^4 \) by design. Instead of 3 tangent vectors \( X_1, X_2 \) and \( R \) which span \( \mathbb{R}^3 \) now we have 4 vectors which span \( \mathbb{R}^4 \). These are \( \mathbf{X}, \tilde{\mathbf{X}}, \mathbf{X} \) and \( \tilde{\mathbf{X}} = (-y_2, -x_2, y_1, x_1) \).

The last vector is written in accord with that used by Hurtado and Rosales (2008). In this paper the remaining three vectors also coincide with ours. When adopted to \( S^3 \), three vectors are tangent to \( S^3 \) and one, that is \( \mathbf{X} \), is normal to \( S^3 \). Thus, it is sufficient to consider only the vectors which span the contact manifold \( S^3 \). In such a case the analog of the commutator, eq.(6.14d), is given by

\[
[\tilde{\mathbf{X}}, \mathbf{X}] = -2\mathbf{X}. \quad (6.17)
\]

In accord with eq.(6.14d)), \( \tilde{\mathbf{X}} \) is the Reeb vector field. The associated contact 1-form was defined in part I, eq.(4.14), as

\[
\alpha = -y_1dx_1 + x_1dy_1 - y_2dx_2 + x_2dy_2. \quad (6.18)
\]

In the present case the horizontality conditions are: \( \alpha(\tilde{\mathbf{X}}) = \alpha(\mathbf{X}) = 0 \). Obtained information is sufficient for explanation of the expansion given in eq.(6.13) (to be compared with eq.(6.15)) and, hence, for utilization of Theorem 6.19. Results of this theorem are in accord with the results obtained by Wada (Theorem 5.9.) stating the the standard Hopf link serves as the seed generating all oriented iterated torus knots. Being armed with these results, now we are in the position to evaluate (to estimate) the writhe in eq.(6.12). Geiges (2008), page 128, proved the following

**Theorem 6.20.** *Every integer can be realized as the self-linking number (that is writhe) of some transverse link*

\[10\] For convenience of our readers we reproduce it again using current numeration.
This theorem is providing only a guidance to what to expect. More useful is the concept of transversal simplicity

**Definition 6.21.** A transverse knot is *transversely simple* if it is characterized (up to transversal isotopy) by its topological knot type \( K \) and by \( \beta(\mathcal{T}(K)) \).

By definition, in the case of a link \( \beta(\mathcal{T}(K)) \) is the sum of \( \beta(\mathcal{T}(K)) \) for each component of the link. Thus, the \( m \)-component unlink is transversely simple since the unknot is transversely simple. To go beyond these obvious results requires introduction of the concept of *exchange reducibility*. Leaving details aside (e.g. read Birman and Wrinkle (2000)), it is still helpful to notice the following.

Closed \( n \)-braid isotopy classes are in one-to-one correspondence with the conjugacy classes of the braid group \( B_n \). This result survives under the transverse isotopy. Such isotopy preserves both \( n(K) \) and \( e(K) \). In addition to the isotopy moves, there are positive/negative (±) *destabilization moves*. The destabilization move reduces the braid index from \( n \) to \( n - 1 \) by removing a trivial loop. If such a loop contains a positive crossing, the move is called positive (+). The move reduces \( e(K) \) by one and thus preserves \( \beta(\mathcal{T}(K)) \). The negative destabilization (−) increases \( \beta(\mathcal{T}(K)) \) by 2. Finally, there is an *exchange* move. It is similar to the 2nd Reidemeister move. It changes the conjugacy class and, therefore, it cannot be replaced by the braid isotopy. Nevertheless, an exchange move preserves both \( n \) and \( e \) and, therefore, also \( \beta(\mathcal{T}(K)) \). Clearly, these moves allow us to untangle the braid of \( n \) strands and to obtain the unlink whose projection is made of \( m \) unknots.

**Definition 6.22.** A knot of type \( K \) is *exchange reducible* if a closed \( n \)-braid representing this knot can be changed by the finite sequence of braid isotopies, exchange moves and ± destabilizations to the minimal \( m \)-component unlink \( m = n_{\text{min}}(K) \). In such a case \( \max \beta(\mathcal{T}(K)) \) is determined by \( n_{\text{min}}(K) \). For the iterated torus knots the exact value of \( \max \beta(\mathcal{T}(K)) \) is known\(^{17}\). Thus, it can be used for estimation of the self-linking number, that is of writhe.

Based on this result, it is possible to prove the following

**Theorem 6.23.** If \( \mathcal{T}(K) \) is a transverse knot type such that \( K \) is exchange reducible, then \( TK \) is transversely simple.

Proof of this theorem allows to prove the theorem of central importance

**Theorem 6.24.** The oriented iterated torus knots are exchange reducible. Thus they are transversely simple.

**Question:** Are all knot types \( K \), when converted to transverse knot types \( \mathcal{T}(K) \), transversely simple?

**Answer:** No!

It happens, (Etnyre 2004) that transversely simple knots comprise a relatively small subset of knots/links. It is being made of:

a) the unknot;

b) the torus and iterated torus knots;

\(^{17}\)E.g. read Birman and Wrinkle (2000), Corollary 3.
c) the figure eight knot.

From the results we obtained thus far we know that the figure eight knot cannot be dynamically generated (Theorem 5.11). Thus, the transversal simplicity is not equivalent to the previously obtained correspondence for flows: Beltrami $\rightarrow$ Reeb $\rightarrow$ Hamiltonian.

**Question:** Is this fact preventing generation of hyperbolic optical knots of which the figure eight knot is the simplest representative?

**Answer:** No!
The explanation is provided in the next section.

7. **From Lorenz equations to cosmetic knots/links**

7.1. Birman-Williams (1983a) treatment of the Lorenz equations

Using methods of topological and symbolic dynamics Birman and Williams (1983a) (BWa)) explained the fact that flows generated by eq.s(4.5b) are special cases of those which follow from the description of periodic orbits of Lorenz equations. These equations were discovered by Lorenz in 1963 who obtained them as finite-dimensional reduction of the Navier-Stokes equation. Lorenz equations are made of a coupled system of three ordinary differential equations.

\[
\begin{align*}
\dot{x} &= -10x + 10y, \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= -8/3z + xy.
\end{align*}
\]

Here $r$ is real parameter, the Rayleigh number. It is typically taken to be about 24. The equations are not of the Hamiltonian-type since they contain dissipative term. Accordingly, the results of previous sections cannot be immediately applied. Nevertheless, the results of BWa) paper are worth discussing because of the following key properties of these equations. They can be summarized as follows.

1. There are infinitely many non equivalent Lorenz knots/links generated by closed/periodic orbits of eq.s(7.1).
2. Every Lorenz knot is fibered.\(^{18}\)
3. Every algebraic knot is a Lorenz knot.
   From the previous section it follows that all iterated torus knots/links are of Lorenz-type since they are algebraic.
4. There are Lorenz knots which are not iterated torus knots;\(^{19}\) there are iterated torus knots which are Lorenz but not algebraic.\(^{20}\)
5. Every Lorenz link is a closed positive braid,\(^{21}\) however there are closed

---

\(^{18}\)This concept will be explained below
\(^{19}\)This means that such knots/links cannot be generated via mechanism of previous section. For example, these are some hyperbolic knots/links described on page 72 of BW a).
\(^{20}\)This is claim which is proven in Theorem 6.5. of BW a). It is compatible with results of previous sections.
\(^{21}\)That is all $\varepsilon_i > 0$ in eq.(6.10a).
positive braids which are not Lorenz.

The last statement is obvious since all knots/links can be obtained as closures of the associated with them braids and, surely, not all knots are torus/iterated torus knots/links.

6. Every non-trivial Lorenz link which has 2 or more components is non splittable.
7. Non-trivial Lorenz links are non-amphicheiral. The link \( L \) is amphichiral if there is an orientation-reversing homeomorphism \( H: (S^3, L) \to (S^3, L) \) reversing orientation of \( L \). The Lorenz links are non-amphicheiral. They are oriented links.

From the Definition 6.13 we know that all transverse knots/links are oriented. Thus all Lorenz knots/links are subsets of transverse knots/links.

8. Non-trivial Lorenz links have positive signature. This technical concept is explained in Murasugi (1996).

7.2. Birman-Williams (1983b) paper, the universal template and the Moffatt conjecture

BWa) study of periodic orbits was greatly facilitated by the template construction invented in their work. See also Ghrist et al (1997), Gilmore and Lefranc (2002) and Ghrist (1998).

**Definition 7.1.** A template \( T \) is a compact branched two-manifold with boundary built from finite number of branch line charts.

The key justification for introduction of templates can be formulated as follows. Upon embedding of \( T \) into \( S^3 \) the periodic orbits of (Lorenz) semiflow tend to form knots/links. In BWa) we find a

**Conjecture 7.2.** There does not exist an embedded \( T \) supporting all (tame) knots/links as periodic orbits, that is there are no universal template.

**Definition 7.3.** An universal template is an embedded template \( T \subset S^3 \) among whose close orbits can be found knots of every type.

In 1995 guided by Birman and Williams (1983b) (BWb)) Ghrist designed an universal template thus disproving the Birman-Williams conjecture. The full proof was published in (Ghrist 1997). Etnyre and Ghrist (2000a) took the full advantage of this fact and came up with the existence-type proof of the Moffatt conjecture (Moffatt 1985). Recall, that this conjecture is claiming that in steady Euler flows there could be knots of any type. From the results presented thus far in parts I and II it should be clear that the same conjecture is applicable to optical knots. More recently, Enciso and Peralta-Salas (2012) also produced the
existence-type proof of the Moffatt conjecture for Eulerian steady flows\footnote{See also Enciso and Peralta-Salas (2013).}. These existence-type results are in contradiction with the Beltrami→Reeb→Hamiltonian flows correspondence \footnote{Incidentally, this correspondence was also established by Etnyre and Ghrist (2000b) by different methods} and, accordingly, with the Theorem 5.3. and its numerous corollaries.

**Question:** Is there way out of the existing controversy?

**Answer:** Yes, there is. It is based on discussion of specific experimentally realizable set ups.

The first example of such a set up was discussed in great detail already in the B-Wb) paper in which the following experiment was considered. Suppose we are given a knotted piece of wire with steady current flowing through the wire. The task lies in describing the magnetic field in the complement of such knot.

7.3. Physics and mathematics of experimental design

In BWb) paper the authors discussed knotted magnetic field configurations surrounding a piece of wire coiled in the shape of figure 8 knot. From knot theory it is known that the Seifert surface of figure 8 knot is punctured torus. Remarkably, but the punctured torus is also Seifert surface for the trefoil knot. The trefoil knot is a torus knot and, therefore, it can be generated mechanically. Thus the discussion of similarities and differences between these knots will help us to resolve the controversy described in the previous subsection. In fact, discussions related to the trefoil knot will also be helpful for description of the Lorenz knots.

The experimental set up not necessarily should involve compliance with eq.s(2.1a,b). For instance, if we use wires which are ordinary conductors, then we immediately lose both eq.s(2.1a,b) and all machinery we discussed thus far is going to be lost. This fact is also is consistent with Theorem 5.3. The correspondence between physics of incompressible ideal Euler fluids and physics of superconductors was noticed by Fröhlich (1966) but, apparently, was left non appreciated till publication of our book (Kholodenko 2013). The first 3 chapters of our book provide needed background for recognition of the fact that the correct set up realizing the requirements given by eq.s(2.1a,b) is possible only if we are using superconducting wires. In part I and at the beginning of part II we discussed the Hopf links made of two linked unknotted rings. If one of the rings is made out of superconducting wire, then the magnetic field in this wire will be collinear with the direction of the superconducting current. The superconducting current, in turn, will create yet another magnetic field and, since the magnetic field does not have sources or sinks, we end up with the Hopf link. In the present case, we have instead of the superconducting circle (that is an unknot) a superconducting coil $K$ in the form of a trefoil or figure 8 knot. The magnetic field collinear with the current in the circuit obeys eq.s(2.1a,b) as required while the surrounding magnetic field will live in $S^3 \setminus K$ space and should not be complacent with eq.s(2.1a,b). Thus, we are left with the problem of describing all knotted/linked structures in the space $S^3 \setminus K$. We begin this description in the next subsection.
7.4. Some important facts about fibered knots

Both the trefoil and figure eight knots are the simplest examples of fibered knots. We discussed both of them in detail in Kholodenko (2001) in the context of dynamics of 2+1 gravity. This circumstance greatly simplifies our task now. From previous sections we know that the trefoil knot can be generated with help of methods of Hamiltonian and contact dynamics while the figure eight knot cannot. Nevertheless, both of them are having the same Seifert surface \( S - \) the punctured/holed torus. An orientation-preserving surface homeomorphism \( h : S \rightarrow S \) of the holed torus should respect the presence of a hole. This is so because the circumference of this hole is our knot (trefoil or figure eight).

The mapping torus fiber bundle \( T_h \) can be constructed as follows. Begin with the products \( S \times 0 \) (the initial state) and \( S \times 1 \) (the final state) so that for each point \( x \in S \) we have \((x, 0)\) and \((h(x), 1)\) respectively. After this, the interval \( I = (0, 1) \) can be made closed (to form a circle \( S^1 \)). This is achieved by identifying 0 with 1 causing the identification \( h : (x, 0) = (h(x), 1) \).

The fiber bundle \( T_h \) is constructed now as a quotient

\[
T_h = \frac{S \times I}{h}. \tag{7.2}
\]

It is a 3-manifold which fibers over the circle \( S^1 \). Such constructed 3-manifold in \( S^3 \) is complimentary to the (fibered) knot, in our case the figure eight or trefoil.

It happens that every closed oriented 3-manifold \( M \) admits an open book decomposition. This means the following. An oriented link \( L \subset M \), called binding can be associated with locally trivial bundle \( p: M \setminus L \rightarrow S^1 \) whose fibers are open (Seifert) surfaces \( F_S \), called pages. Technically speaking, \( L \) is also required to have a tubular neighborhood \( L \times D^2 \) so that the restricted map \( p: L \times (D^2 \setminus \{0\}) \rightarrow S^1 \) is of the form \((x, y) \rightarrow y/|y| \). The closure of each page \( F_S \) is then a connected compact orientable surface with boundary \( L \). Any oriented link \( L \subset M \) that serves as a binding of an open book decomposition of \( M \) is called fibered link/knot. Importance of the open book decomposition for contact geometry and topology is summarized in the excellent review article by Etnyre (2005).

There is a deep relationship between the Alexander polynomial \( \Delta_K(t) \) for the fibered knot \( K \) and the associated with it 3-manifold \( M \). If \( V \) be the Seifert matrix of linking coefficients for \( K \), then \( \Delta_K(t) = \det(V^T - tv) \). If the knot \( K \) is fibered, the polynomial \( \Delta_K(t) \) possess additional symmetries. Specifically, in such a case we obtain

\[
\Delta_K(t) = \sum_{i=0}^{2g} a_i t^i \tag{7.3}
\]

and it is known that \( \Delta_K(0) = a_0 = a_{2g} = \pm 1 \). Here \( g \) is the genus of the associated Seifert surface. Thus, fibered knots can be recognized by analyzing their Alexander polynomial. It is monic for fibered knots. Furthermore, using just described properties of \( \Delta_K(t) \), we obtain as well \( \Delta_K(0) = \det(V^T) = \det(V) = \pm 1 \). Because of this, we obtain:

\[
\Delta_K(t) = \det(V^{-1}V^T - tE) \equiv \det(M - tE). \tag{7.4}
\]
Here $E$ is the unit matrix while $M = V^{-1}V^T$ is the monodromy matrix responsible for the surface homeomorphisms $h : S \to S$. In the case of figure 8 knot BWb) use

$$M_a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

while below, eq.(7.20), it is shown that there is yet another matrix

$$M_b = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

participating in these homeomorphisms. Both matrices produce $\Delta_{K_8}(t) = t^2 - 3t + 1$. The matrix $M$ for the trefoil knot $T$ is different. It is either

$$M_a = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{or} \quad M_b = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

producing $\Delta_{K_T}(t) = t^2 - t + 1$. The 3-manifolds $T_b$ associated with the figure 8 and trefoil knots are different because of the difference in the respective monodromy matrices. The 3-manifold for the figure 8 knot is hyperbolic while for the trefoil it is Seifert-fibered.

Since locally, the complement of the figure 8 knot is behaving as the space of constant negative curvature the figure 8 knot is hyperbolic knot. In hyperbolic space the nearby dynamical trajectories diverge. Nevertheless, Ghrist (1997) using BWb) results was able to prove the following

**Theorem 7.4.** (Ghrist) Any fibration of the complement of the figure-eight knot in $S^3$ over $S^1$ induces a flow on $S^3$ containing every tame knot and link as closed orbits.

The proof of this result involves explicit design of the universal template which was used subsequently by Etnyre and Ghrist (2000a) in proving the Moffatt conjecture. These knots and links are not generated as Hamiltonian flows though. Nevertheless, in view of the above theorem the complement in $S^3$ of the superconducting current carrying wire coiled in the shape of figure 8 knot is expected to contain magnetic lines of any (tame) knot and link.

In developing their proof Etnyre and Ghrist (2000a) needed to combine the results of (Ghrist 1997) with those coming from contact geometry. The specific contactomorphisms described in section 4 are sufficient only for generation of torus and the iterated torus knots. The experimental set up discussed in previous subsection provides an opportunity to enlarge the collection of optical knots. In the following subsections we provide some details of how this can be achieved.

7.5. Lorenz knot/links living in the complement of the trefoil knot

We begin by noticing that the trefoil knot can be generated dynamically, on one hand, and can be created by a superconducting piece of wire coiled in the form of the trefoil knot,
on another hand. In both cases the complements of the trefoil in $S^3$ are Seifert-fibered 3-manifolds (Jaco 1980). The well known theorem by Gordon and Luecke (1989) is telling us that knots are determined by their complements. This restriction is not extendable to links though as discussed in appendix F. Following Ghys (2007), and Ghys and Leys (2011), we shall focus now at the complement of the trefoil knot. Since this is not a review paper, our exposition is not going to be based exclusively on results of these authors.

We begin by discussing in some detail the fiber bundle construction for the trefoil knot whose Seifert surface $S$ is the punctured torus. The homeomorphisms of $S$ are generated by the sequence of Dehn twists. The torus can be visualized in a variety of ways. For example, as some cell of the square-type lattice whose opposite sides are being identified. The punctured torus would require us in addition to get rid of the vertices and of neighborhoods of these vertices. The sequence of Dehn twists acting on the punctured torus relates different square-type lattices to each other. Should the puncture be absent, the elementary cells of these different lattices upon identification of opposite sides would be converted to different tori. Closed geodesics on these tori would correspond to different torus knots. Presence of the puncture complicates matters considerably, e.g. read Minsky (1999).

Let $G = \pi_1(S)$ be a fundamental group of surface and $P \subseteq G$ be the set of peripheral elements, that is those elements of $G$ which correspond to loops freely homotopic to the boundary components. For the punctured torus $G$ is just a free group of two generators $a$ and $b$. That is $G = \langle a, b \rangle$. The subset $P$ is determined by the commutator $aba^{-1}b^{-1} \equiv [a, b]$. If generators $a$ and $b$ are represented by the matrices, then it can be shown that the presence of the puncture is reflected in the fact that $\text{tr}[a, b] = -2$. The question arises: is it possible to find $a$ and $b$ explicitly based on this information? Notice, because of the puncture Euler characteristic $\chi$ of the torus becomes -1. This means that the covering space is $\mathbb{H}^2$ that is it is the Poincare' upper half plane or, which is equivalent, the Poincare disc $\mathcal{D}$. The group of isometries of $\mathcal{D}$ is $\text{PSL}(2, \mathbb{R})$. Its closest relative is $\text{SL}(2, \mathbb{R})$. It is possible to do all calculations using $\text{SL}(2, \mathbb{R})$ and only in the end to use the projective representation. For matrices of $\text{SL}(2, \mathbb{R})$ the following identities are known

$$2 + \text{tr}[a, b] = (\text{tra})^2 + (\text{trb})^2 + (\text{trab})^2 - [\text{tra}] \cdot [\text{trb}] \cdot [\text{trc}],$$

$$[\text{tra}] \cdot [\text{trb}] = \text{trab} + \text{trab}^{-1}. \quad (7.7)$$

Since we know already that $\text{tr}[a, b] = -2$ the above identities can be rewritten as

$$x^2 + y^2 + z^2 = xyz; \quad xy = z + w. \quad (7.8)$$

Clearly, $x = \text{tra}, y = \text{trb}, z = \text{trab}$ and $w = \text{trab}^{-1}$. It happens that already the first identity is sufficient for restoration of matrices $a$ and $b$. These are given by

$$a = \frac{1}{z} \begin{pmatrix} xz - y & x \\ x & y \end{pmatrix}, \quad b = \frac{1}{z} \begin{pmatrix} yz - x & -y \\ -y & x \end{pmatrix}. \quad (7.9)$$

For the integer values of $x$, $y$ and $z$ the first of identities in eq.(7.8) is known as equation for the Markov triples. It was discovered in the number theory by Markov.\textsuperscript{24} By introducing

\textsuperscript{24}E.g. read all needed references in Kholodenko (2001).
the following redefinitions

\[ x = 3m_1, \quad y = 3m_2, \quad z = 3m_3 \]  

(7.10) 

the first identity in eq.(7.8) acquires standard form used in number theory

\[ m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3. \]  

(7.11) 

The simplest solution of this equation is \( m_1^2 = m_2^2 = m_3^2 = 1 \). To generate additional solutions it is convenient to introduce the notion of trace maps. In the present case we have to investigate the map \( F \) defined as

\[
F : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} 3yz - x \\ y \\ z \end{pmatrix}
\]  

(7.12) 

which possess an "integral of motion" \( I(x, y, z) = x^2 + y^2 + z^2 - 3xyz \). By definition, it remains unchanged under the action of \( F \). From the theory of Teichmüller spaces (Imayoshi and Taniguchi 1992) it is known that the length \( l(\gamma) \) of closed geodesics associated with \( \gamma \in G \) is given by

\[ tr^2 \gamma = 4 \cosh^2(\frac{1}{2}l(\gamma)). \]  

(7.13) 

In the present case, we have

\[ x^2 = 4 \cosh^2(\frac{1}{2}l(\gamma)) \]  

(7.14) 

with analogous results for \( y \) and \( z \). In view of eq.(7.10) for the Markov triple (1,1,1) we obtain:

\[ l(\gamma) = 2 \cosh^{-1}(\frac{3}{2}) = 2 \ln(\frac{1}{2}(3 + \sqrt{5})) \equiv 2 \ln \lambda. \]  

(7.15) 

Here \( \lambda \) is the eigenvalue of the monodromy matrix to be determined momentarily. For this purpose we notice that the larger root of the equation

\[ \Delta_K(t) = t^2 - 3t + 1 = 0 \]  

(7.16) 

for the Alexander polynomial of figure 8 knot is equal to \( \lambda \). Taking into account that eq.(7.4) can be equivalently presented as

\[ \Delta_K(t) = t^2 - (trM) t + \det M \]  

(7.17a) 

and, in view of the fact that \( \Delta_K(0) = a_0 = a_{2g} = \pm 1 \), eq.(7.17 a) can be rewritten as

\[ \Delta_K(t) = t^2 - (trM) t \pm 1. \]  

(7.17b) 

From the knot theory (Rolfsen 1976) it is known that for any knot \( \Delta_K(1) = \pm 1 \). Therefore, we are left with the condition

\[ \Delta_K(t) = 1 - (trM) t \pm 1 = \pm 1. \]  

(7.18)
This equation leaves us with two options: \( trM = 3 \) or 1. In view of eq. (7.17), the first option leads us to the Alexander polynomial for the figure eight knot while the 2nd- to the Alexander polynomial for the trefoil knot. From here we recover the monodromy matrices \( M \) for the figure 8 and trefoil knots. These are given by

\[
M_8 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.
\] (7.19)

Obtained results are in accord with BWa) and BWb) where they were given without derivation. In addition, notice that the obtained result for \( M_8 \) linked with the motion along the geodesics is not applicable for \( M_T \). From here several conclusions can be drawn:

a) closed geodesics in hyperbolic space are related to hyperbolic knots;

b) not all (even hyperbolic) knots can be associated with closed geodesics (Kholodenko 2001, Miller 2001). In this work we are not going to discuss these, exceptional, cases.

Notice that the same results can be obtained differently. There is a good reason for doing so as we would like to demonstrate now. For instance, use of solution \( x = y = z = 3 \) in eq.s (7.9) results in the following matrices

\[
a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}
\] (7.20)

yielding the trace of commutator \([a, b]\) being equal to \(-2\) as required. To understand the "physical" meaning of these results, we introduce two new matrices

\[
L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\] (7.21)

These are easily recognizable as Logitudinal and Meridional elementary Dehn twists (Rolfsen 1976), page 24. Remarkably, now we are having the following chain of identities

\[
a = RL = M_8.
\] (7.22)

Also,

\[
b = L^{-1}aR^{-1},
\] (7.23)

where

\[
L^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad R^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.
\]

Because of this, it is convenient to introduce yet another matrix

\[
\hat{I} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \hat{I}^2 = \mp \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mp I
\] (7.24)

giving us a chance to represent \( L^{-1} \) and \( R^{-1} \) as

\[
L^{-1} = \hat{I}R\hat{I} \quad \text{and} \quad R^{-1} = \hat{I}L\hat{I}
\] (7.25a)

37
and, in addition, to obtain
\[ L\hat{I}L = R, \quad R\hat{I}R = L, \quad R\hat{I}L = \hat{I} \quad \text{and} \quad L\hat{I}R = \hat{I}. \] (7.25b)

Consider now the "word" of the type
\[ W_1 = L^{\alpha_1} R^{\beta_1} \cdots L^{\alpha_r} R^{\beta_r} \] (7.26)
in which \( \alpha_i \) and \( \beta_i \) are some integers. In addition, we could consider words with insertions of \( \hat{I} \). The analysis done in Kholodenko (2001) spares us from the necessity of doing so thanks to eq.s(7.25). In the end, only words of the type \( W_1 \) and \( \hat{I}W_1 \) should be considered.

**Remark 7.5.** The obtained results are useful to compare against those in the paper by Ghys (2007). In it, Ghys takes \( U = -\hat{I} \) and \( V = \pm R^{-1} \) as two letters used in creation of the word \( W \) of the type
\[ W = UV^\varepsilon_1 UV^\varepsilon_2 \cdots UV^\varepsilon_n \] (7.27)
with each \( \varepsilon_i = \pm 1 \). It is clear, however, that our \( W_1 \) is equivalent to \( W \) (e.g. see Birman (2009), page 35). To relate \( W \) (or \( W_1 \)) to Lorenz knots/links is a simple matter at this point. This is done by realizing that encoding of dynamical semiflows on Lorenz template \( T \) is isomorphic with the sequence of letters \( LR \) in eq.(7.26). Details are given in Ghys (2007) and Birman (2009).

Instead of copying results of these authors, it is of interest to arrive at these results differently. This will help us to shorten our description of semiflows connected with figure 8 knot by adding few details to the paper by Miller (2001) in which results of BWb reobtained and simplified.

To describe the detour, we do not need to do more than we did already. We only need to present results in the appropriate order. The key observations are.

1. Words of the type \( W_1 \) are represented as matrices
\[ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \gamma\beta = 1. \]

2. Matrix entries are determined by the Markov triples.

3. Traces of these matrices are related to the lengths \( l(\gamma) \) of closed geodesics via eq.(7.14).

4. At the same time, the remarkable relation \( l(\gamma) = 2\ln \lambda \) connects these lengths with the largest eigenvalue of the monodromy matrix \( M \) entering the definition of the Alexander polynomial which itself is a topological invariant.

4. The representation of the fundamental group of the complement of the trefoil knot in \( S^3 \) is given either as
\[ < x, y \mid x^3 = y^2 > \] (7.28a)
or as
\[ < a, b \mid aba = bab >= < a, b, c \mid ca = bc, ab = c > . \] (7.28b)

In view of eq.(6.3) the 2nd form of representation is easily recognizable as \( B_3 \). The centralizer \( Z \) of \( B_3 \) is the combination \( (aba)^4 \). However, already the combination \( \Delta_3 = (aba)^2 \) is of
importance since, on one hand, it can be proven (Kassel and Turaev 2008) that $\sigma_1 \Delta_3 = \Delta_3 \sigma_3$ while, on another, this operation physically means the following. While keeping the top of the braid fixed, the bottom is turned by an angle $\pi$. Accordingly, for $Z$ we have to perform a twist by $2\pi$. If we consider a quotient $B_3/\mathbb{Z}_3$ it will bring us back to the braids representation associated with transverse knots/links discussed in section 6. In such a case Kassel and Turaev (2008) demonstrate that the presentation for the quotient can be written as

$$<a, b \mid a^3 = b^2 = 1> \quad (7.29)$$

This result can be realized in terms of matrices we had obtained already. Specifically, we can identify $b$ with $\hat{I}$ and $a$ with $\hat{I}R$. By direct computation we obtain: $\hat{I}^2 = (\hat{I}R)^3$. The presentation, eq.(7.29), is graphically depicted in Fig. 10a).

Fig.10
From random walks on the graph for the trefoil knot to dynamics on the Lorenz template $T$

Following our previous work (Kholodenko 1999) we also can consider some kind of a random walk on the figure eight (e.g. see Fig.10b). This process also can be looked upon as a diagram of states for finite state automaton. From here is the connection with the symbolic dynamics (e.g. subshifts) (Ghrist, Holmes and Sullivan 1997). Alternatively, the same process can be considered as depicting a motion on the (Lorenz) template (Fig.10c)). The thick dark horizontal line on Lorenz template is made out of two parts: left (L) and right (R). The representative closed trajectory passing through such a template is encoded by the sequence of L and R’s (Fig.10d) left). It is in one-to-one correspondence with the (Lorenz) knot (Fig.10d) right). The word $W$ in eq.(7.27) keeps track record of wandering of the walker (of the mover) on the figure eight or, which is equivalent, on the Lorenz template.

7.6. Geodesic knots in the figure eight knot complement

By analogy with eq.s(7.28) we begin with presentation of the fundamental group for the figure eight knot complement. The standard form can be found on page 58 of Rolfsen (1976)

$$\pi(S^3 - K_8) =< a, b, c, d \mid r_1, r_2, r_3, r_4 > \quad (7.30)$$

Here $r_1 = bcb^{-1} = a, r_2 = ada^{-1} = b, r_3 = d^{-1}bd = c, r_4 = c^{-1}ac = d$. This presentation is not convenient however for the tasks we are having in mind. To avoid ambiguities caused by some mistakes we found in literature, we derive needed presentations in appendix G. Our first task is to find a presentation which looks analogous to that given by eq.(7.28b). This is given by eq.(G.7) which we rewrite here as

$$< a, b, w \mid wa = bw, w = b^{-1}aba^{-1} > . \quad (7.31)$$

By achieving the desired correspondence with eq.(7.28b) we are interested in obtaining the analog of eq.(7.29) as well. This is achieved in several steps. They would be unnecessary,
should all presentations done in Miller (2001) be correct. Unfortunately, they are not. For this reason her results are reconsidered in appendix D. The analog of presentation eq.(7.29) now is given by generators and relations of the dihedral group $D_2$

$$D_2 =< t, s | t^2 = s^2 = 1, ts = st >$$ (7.32)

As in the case of a trefoil, where the Lorenz template was recovered, the above presentation allows us to reobtain the universal template $T$ of Ghrist. The successive steps are depicted in Fig.11. Their description is analogous to those depicted in Fig.10.

Fig.11
From random walks on the graph for the figure eight knot to dynamics on the universal template $T$ of Ghrist

To deal with the fundamental group of the figure eight knot, Bowditch (1998) extended theory of Markov triples to hyperbolic space $H^3$. To do so requires equations for Markov triples, eq.s(7.8), to be solved in the complex domain. In $H^3$ the isometry group is $PSL(2, \mathbb{C})$ as compared with the isometry group of $H^2$ which is $PSL(2, \mathbb{R})$. As before, we can do all calculations using $SL(2, \mathbb{C})$ and only at the end of calculations switch to $PSL(2, \mathbb{C})$. Because of this, it is useful to know (Elstrodt et al 1998) that $SL(2, \mathbb{C})$ can be generated by just two elements

$$SL(2, \mathbb{C}) = \{ V = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \hat{I} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \alpha \in \mathbb{C} \}. $$ (7.33)

It is also helpful to compare these elements with earlier derived matrices $\hat{I}$ and $R$. Clearly, $V \rightarrow R$ when $\alpha \rightarrow 1$.

In the case of figure eight knot it can be checked directly (Kholodenko 2001) that matrices $a$ and $b$ in eq.(7.31) can be presented as follows:

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix},$$ (7.34)

where $\omega$ is one of the solutions of the equation $\omega^2 + \omega + 1 = 0$. Typically, one chooses $\omega = \frac{1}{2}(-1 + 3i)$. This result has number-theoretic significance explained in the appendix B of Kholodenko (2001). More deeply, theory of arithmetic hyperbolic 3-manifolds is explained in the book by Maclachlan and Reid (2003). Some physical (cosmological) implications of this arithmeticity are discussed in Kholodenko (2001). Interested readers are encouraged to read these references.

By noticing that any matrix

$$SL(2, \mathbb{C}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha \delta - \gamma \beta = 1$$ (7.35)

can be represented in terms of $V$ and $\hat{I}$ as demonstrated in (Elstrodt et al 1998), it is clear that now it is possible to represent all words in the form analogous to that given in eq.(7.26) or
(7.27) (if the dihedral presentation, eq.(7.32), is being used). Furthermore, the fundamental relation, eq.(7.13) for closed geodesics can be used for $\mathbf{H}^3$ also so that now we are dealing with the geodesic hyperbolic knots. The difference between the complements of the trefoil and that for figure eight knots lies in the fact that now the length $l(\gamma)$ of geodesics becomes a complex number. It can be demonstrated (Bowditch 1998), page 717, that $\text{Re}(l(\gamma)) > 0$.

**Remark 7.6.** Although the dihedral presentation is given in Miller (2001) incorrectly, it is possible to prove (Elstrodt et al 1998) that the finite subgroups of $\text{PSL}(2, \mathbb{C})$ contain all group of orientation-preserving isometries of Euclidean regular solids and their subgroups. Evidently, the presence of actual symmetries depends upon the values of complex parameters $\alpha, \beta, \gamma, \delta$. The choice of $D_2$ leads to the universal template of Ghrist.

**Remark 7.7.** Instead of the real hyperbolic space $\mathbf{H}^3$ in which the complement of the figure eight knot lives, it is possible to use the complex hyperbolic space $\mathbf{H}^3_{\mathbb{C}}$ advocated by Goldman (1999). It is possible to consider a complement of the figure eight in $\mathbf{H}^3_{\mathbb{C}}$ as well (Falbel 2008). It happens that isometry group of the boundary of $\mathbf{H}^3_{\mathbb{C}}$ is the Heisenberg group, e.g. see eq.s(6.14). Thus, in dealing with problems related to the complement of figure eight knot the contact geometry and topology again can be successfully used.

### 7.7. Cosmetic knots and gravity

Using results of part I the famous Gordon and Luecke (1989) theorem stating that in $S^3$ knots are determined by their complements can be restated now in physical terms. For this we need to recall some facts from general relativity. The original formulation by Einstein makes heavy emphasis on study of the Schwarzschild solution of Einstein equations. These are field equations without sources in which the mass enters as adjustable parameter (Stephani 1990). Using these equations the curvature tensor can be obtained. If masses are associated with knots, then knot complements producing curved space around knots can be used for parametrizing the masses. In which case the Gordon and Luecke theorem is telling us that the mass spectrum is discrete. In studying knotted geodesics in the complement of the trefoil and figure eight knot we followed Einsteinian methodology without recognizing Einsteinian influence. Indeed, we completely ignored the fact that each nontrivial knot in the complement of the trefoil/figure eight knot creates its own complement. Thus, the space around, say, the trefoil knot is not just a complement of the trefoil but the space made out of complements of all knots existing in the trefoil knot complement! This fact is ignored in calculations presented above. Therefore, these "complementary" knots are, in fact, closed geodesics on the punctured torus. Very much like in Einsteinian theory of gravity the motion of point-like objects whose mass is ignored is taking place along the geodesics around the massive body whose field of gravity is described by the Schwarzschild solution. To account for extended sizes of particles in both general relativity and Yang-Mills field theory is always a great challenge, e.g. read page 97 of (Kholodenko 2013). In the case of knotted geodesics the effect of extended size (that is of finite thickness of a geodesic) is also a difficult problem which is not solved systematically as can be seen from (Miller 2001).

To be specific, we would like to talk about the complement of the figure eight knot $K_8$. We begin with $S^3 \setminus K_8$. This is a hyperbolic manifold. Next, we drill in it a closed curve $c$ (a
tube of finite thickness, a tubular neighborhood, e.g. see Fig.3a)) which is freely homotopic to some geodesic $\gamma$. Such closed curve forms a link with $K_8$. When we remove the content of knotted/linked tube from $S^3 \setminus K_8$ we obtain a manifold which is complementary to the link $K_8 \cup c$ and, if we are interested in computation of the hyperbolic volume of such hyperbolic link, the result is going to depend very sensitively on the tube thickness. Thus, indeed, we end up with the situation encountered in gravity. Furthermore, additional complications can arise because of the following.

The Gordon and Luecke (1989) theorem was proven for $S^3$ only. In the present case, say, if we are dealing with $M = S^3 \setminus K_8$ the knots $K_i$ living in $S^3 \setminus K_8$ also will have their complements $M - K_i$ and then we arrive at the following

**Question:** Suppose in some 3-manifold $M$ (other than $S^3$) there are two knots $K_1$ and $K_2$ so that the associated complements are $M \setminus K_1$ and $M \setminus K_2$. By analogy with Fig.12 suppose that there is a homeomorphism $h$: $h(M \setminus K_1) = (M \setminus K_2)$. Will such a homeomorphism imply that $K_1 = K_2$?

If the answer to this question is negative, this then would imply that different knots would have the same complement in $M$. If we are interested in attaching some physics to these statements then, we should only look for situations analogous to those in $S^3$. This means that all physical processes should be subject to selection rules (just like in quantum mechanics) making such degenerate cases physically forbidden. Evidently, all these considerations about the selection rules presuppose that the degeneracy just described can be realized in Nature. And indeed, it can! This leads us to the concept of **cosmetic knots**.

**Definition 7.8.** Let $M$ be some 3-manifold (other than $S^3$ and $S^2 \times S^1$) and $K$ be some non-trivial knot living in $M$. If performing Dehn surgery on $K$ results in the same manifold $M$, then, the pair $(M, K)$ is called **cosmetic pair**.

**Remark 7.9.** To perform Dehn surgery physically (most likely) is not possible. However, one can imagine a situation when there are two different knots $K_1 \neq K_2$ such that their complements $M - K_1$ and $M - K_2$ are homeomorphic while knots themselves are not. If such situation can be realized in practice then, if we want to keep the knot-mass correspondence alive, one should impose selection rules which will forbid transitions into such cosmetic states.

Following Mathieu (1992), consider few details relevant for this subsection. To begin, we have to notice similarities between the operations of cabling as depicted in Fig.7 and Dehn surgery. After reading the description for Fig.7 in the main text. and the Definition F.1., it is appropriate to consider some knot $K$ in $S^3$ so that its complement $S^3 - K = M$ is some 3-manifold with boundary. Let $k$ and $k'$ be cores of the respective surgeries for these knots in $M$. There should be a homeomorphism from $S^3 - K$ to $M - k$ sending the curve $J$ (for $K$) onto meridian $m$ of $k$. At the same time, there should be a homeomorphism from $S^3 - K$ to $M - k'$ sending another curve $\tilde{J}$ (for $K$) onto meridian $m'$ of $k'$. Should $k$ and $k'$ be equivalent, then by using composition of mappings it would be possible to find a homeomorphism in $M = S^3 - K$ -from $J$ to $\tilde{J}$ -without preserving the meridian of $K$. Notice that, by design, $J \neq \pm \tilde{J}$ making such homeomorphism is impossible. Therefore, $k$ and $k'$ are not equivalent while the complements $M - k$ and $M - k'$ are since they are homeomorphic.

**Remark 7.10.** Let $K$ be a trefoil knot. Then in its complement there will be cosmetic
knots. In fact, all Lorenz knots are cosmetic. The same is true for all knots living in the complement of the figure eight knot. Evidently, such cosmetic knots cannot be associated with the physical masses.

**Appendices**

Appendix A  The Beltrami equation and contact geometry

Consider 3-dimensional Euclidean space. Let $\mathbf{A}$ be some vector in this space so that $\mathbf{A} = \sum_{i=1}^{3} A_i \mathbf{e}_i$. Introduce next the $b$-operator via

$$ (\mathbf{A})^b = \sum_{i=1}^{3} A_i dx_i. \quad (A.1) $$

Use of $b$ allows us now to write

$$ d(\mathbf{A})^b = (\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2})dx_1 \wedge dx_2 + (\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1})dx_3 \wedge dx_1 + (\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3})dx_2 \wedge dx_3. \quad (A.2) $$

Next, we obtain,

$$ *d(\mathbf{A})^b = (\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2})dx_3 + (\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1})dx_2 + (\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3})dx_1. \quad (A.3) $$

Finally, we get

$$ [*d(\mathbf{A})^b]^b = (\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2})\mathbf{e}_3 + (\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1})\mathbf{e}_2 + (\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3})\mathbf{e}_1 = \text{curl}\mathbf{A}. \quad (A.4) $$

Using these results we obtain as well

$$ *d(\mathbf{A})^b = (\nabla \times \mathbf{A})^b. \quad (A.5) $$

If $(\mathbf{A})^b = \alpha$, where $\alpha$ is contact 1-form, then the Beltrami equation reads as

$$ *d\alpha = \kappa \alpha. \quad (A.6) $$

Appendix B  Derivation of the force-free equation from the source-free
Maxwell’s equations

In Appendix D of part I we mentioned about works by Chu and Ohkawa (1982) and Brownstein (1987) in which the force-free equation is derived from the source-free Maxwell’s equations. In this appendix we use some results from the paper by Uehara, Kawai and Shimoda (1989) in which the same force-free equation was obtained in the most transparent
way. In the system of units in which $\varepsilon_0 = \mu_0 = 1$ the source-free Maxwell’s equations acquire the following form

$$\frac{\partial}{\partial t} \mathbf{E} = \nabla \times \mathbf{B} \quad \text{and} \quad -\frac{\partial}{\partial t} \mathbf{B} = \nabla \times \mathbf{E},$$

(B.1)

provided that $\text{div}\mathbf{B} = \text{div}\mathbf{E} = 0$. Let $\mathbf{E} = \mathbf{v}(r, t) \cos f(r, t)$ and $\mathbf{B} = \mathbf{v}(r, t) \sin f(r, t)$. In view of the fact that $\nabla \times (a \mathbf{v}) = a \nabla \times \mathbf{v} - \mathbf{v} \times \nabla a$ use of $\mathbf{E}$ and $\mathbf{B}$ in Maxwell’s equations leads to the following results

$$-\dot{f} \mathbf{v} \sin f + \dot{\mathbf{v}} \cos f = \sin f(\nabla \times \mathbf{v}) - (\mathbf{v} \cdot \nabla f) \cos f$$

(B.2a)

and

$$\dot{f} \mathbf{v} \cos f + \dot{\mathbf{v}} \sin f = -\cos f(\nabla \times \mathbf{v}) - (\mathbf{v} \cdot \nabla f) \sin f$$

(B.2b)

where, as usual, the dot over $f$ denotes time differentiation. Analysis shows that eq.(B.2a) and (B.2b) equivalent to the following set of equations

$$\dot{f} \mathbf{v} = -\nabla \times \mathbf{v}$$

(B.3a)

and

$$\dot{\mathbf{v}} = -\mathbf{v} \times \nabla f$$

(B.3b)

while the incompressibility equations $\text{div}\mathbf{B} = \text{div}\mathbf{E} = 0$ are converted into the requirements

$$\text{div}\mathbf{v} = 0$$

(B.4a)

and

$$\mathbf{v} \cdot \nabla f = 0.$$  

(B.4b)

If now we identify $\mathbf{v}$ with the fluid velocity, then both eq.s(B.4a) and (B.4b) can be converted into one equation $\text{div}(f \mathbf{v}) = 0$ which we had encountered already in section 4 of part I. E.g. read the text below the eq.(4.3). This observation allows us to reinterpret eq.s(B.3a) and (B.3b). Specifically, if we let $f = -\kappa(r)$ in eq.(B.3a), then the force-free equation

$$\nabla \times \mathbf{v} = \kappa \mathbf{v}$$

(B.5)

is obtained. By applying the curl operator to both sides of this equation and taking into account eq.(B.4a) we obtain

$$-\nabla^2 \mathbf{v} = \kappa^2 \mathbf{v} + (\nabla \kappa) \times \mathbf{v}. \quad \text{(B.5)}$$

Now eq.(B.3b) can be rewritten as $\dot{\mathbf{v}} = t(\mathbf{v} \times \nabla \kappa)$. This result can be formally used in eq.(B.5). The Hemholtz equation

$$\nabla^2 \mathbf{v} + \kappa^2 \mathbf{v} = 0 \quad \text{(B.6)}$$

is obtained if $(\nabla \kappa) \times \mathbf{v} = 0$. Since in addition $\kappa$ must also satisfy eq.(B.4b), we conclude that $\kappa = \text{const}$ is the only admissible solution. As discussed in (Kholodenko 2013), the Hemholtz eq.(B.6) is related to the force-free eq.(B.5) as the Klein-Gordon equation is related to the Dirac equation. This means that every solution of eq.(B.5) is also a solution of eq.(B.6) but not another way around.
Using just obtained results, the energy density, eq.(2.11) of part I, manifestly time-independent. In the non-Abelian case of Y-M fields this would correspond to the case of monopoles in accord with statements made in part I.

Consider now briefly the case when \( \mathbf{v} \) is time-dependent. Then eq.(B.5) can be equivalently rewritten as

\[
\dot{\mathbf{v}} = -t (\nabla^2 \mathbf{v} + \kappa^2 \mathbf{v}).
\]  
(B.7)

Let now \( \mathbf{v}(r,t) = \mathbf{V}(r)T(t) \). Using this result in eq. (B.7) we obtain

\[
-\frac{1}{t^2(T)} \ddot{T}(t) = \frac{1}{\mathbf{V}} (\nabla^2 \mathbf{V} + \kappa^2 \mathbf{V}).
\]  
(B.8)

If \( \nabla^2 \mathbf{V} + \kappa^2 \mathbf{V} = -E \mathbf{V} \), then the Helmholtz eq.(B.6) should be replaced by the same type of equation in which we have to make a substitution \( \kappa^2 \to \kappa^2 + E = K^2 \) and then to treat \( K^2 \) as an eigenvalue of thus modified eq.(B.6) with the same boundary conditions, e.g. see eq.(2.1b). The situation in the present case is completely analogous that for the Navier-Stokes equation, e.g. read (Majda and Bertozzi 2002), Chr.2, or (Constantin and Majda 1988). Therefore, the solution \( \mathbf{v}(r,t) = \mathbf{V}(r)T(t) \) of the Navier-Stokes equation is represented as the Fourier expansion over the eigenfunctions of the force-free (or Beltrami) eq.(B.5) \( \mathbf{V}(r) \to \mathbf{V}_m(r) \). Using eq.(B.8) \( T(t) \) is straightforwardly obtainable. By applying the appropriate reparametrization \( T(t) \) can be brought to the form (e.g. see eq.(3.8) of part I and comments next to it) suggested by Donaldson (2002). Thus, even in the case of time-dependent solutions the obtained results can be made consistent with those suggested by Floer and Donaldson discussed in part I. Because of this, it is sufficient to treat only the time-independent case. In electrodynamics this corresponds to use of the constant electric and magnetic transversal fields.

Appendix C  Some facts from symplectic and contact geometry

Consider a symplectic 2-form \( \omega = \sum_{j=1}^{m} dp_j \wedge dq_j \). Introduce a differentiable function \( f^{(v)} \) associated with the vector field \( \mathbf{v} \) via skew-gradient

\[
\mathbf{v} = (sgrad) f^{(v)} = \sum_i \left[ \frac{\partial f^{(v)}}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f^{(v)}}{\partial q_i} \frac{\partial}{\partial p_i} \right]
\]  
(C.1a)

and such that \(-df^{(v)} = i_{\mathbf{v}} \omega\). This result can be checked by direct computation. By definition, a vector field \( \mathbf{v} \) is symplectic if \( \mathcal{L}_v \omega = 0 \). Since \( \mathcal{L}_v = d \circ i_v + i_v \circ d \) we obtain: \( \mathcal{L}_v \omega = -ddf^{(v)} = 0 \) as required. Introduce now another vector field \( \mathbf{w} \) via

\[
\mathbf{w} = (sgrad) f^{(w)} = \sum_i \left[ \frac{\partial f^{(w)}}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f^{(w)}}{\partial q_i} \frac{\partial}{\partial p_i} \right]
\]  
(C.1b)

and calculate \( i_{\mathbf{w}} \circ i_{\mathbf{v}} \omega \). We obtain:

\[
i_{\mathbf{w}} \circ i_{\mathbf{v}} \omega = i_{\mathbf{w}} (-df^{(v)}) = i_{\mathbf{w}} \sum_i \left( -\frac{\partial f^{(v)}}{\partial p_i} dp_i - \frac{\partial f^{(v)}}{\partial q_i} dq_i \right) = \sum_i \left[ \frac{\partial f^{(v)}}{\partial p_i} \frac{\partial f^{(w)}}{\partial q_i} - \frac{\partial f^{(v)}}{\partial q_i} \frac{\partial f^{(w)}}{\partial p_i} \right]
\equiv \{ f^{(v)}, f^{(w)} \} = \omega(\mathbf{v}, \mathbf{w}) = \mathbf{v} f^{(w)} = -\mathbf{w} f^{(v)}.
\]  
(C.2)
Dynamical systems (flows) select some point \( \tilde{x} \). We distinguish between topological conjugacy and structural stability. For example, in (Anosov et al 1997, Arnol’d 1988) we shall not make a mistake. The topological conjugacy associated with structural stability. Omitting some fine details will be having another manifold \( N \) so that there is a map (a homomorphism) \( h: M \to N \). Dynamical systems (flows) \( g^t \) on \( M \) and \( \tilde{g}^t \) on \( N \) are topologically conjugate if \( h \circ g^t = \tilde{g}^t \circ h \). With the topological conjugacy associated with structural stability. Omitting some fine details which can be found, for example, in (Anosov et al 1997, Arnol’d 1988) we shall not make distinctions between topological conjugacy and structural stability.

The analysis of the flow described by \( \dot{x} = v(x) \) begins with linearization. For this purpose select some point \( \tilde{x} \) along the trajectory. Typically such a point is the equilibrium point. Let \( x = \tilde{x} + x - \tilde{x} = \tilde{x} + \delta x \). Using this result in \( \dot{x} = v(x) \) we obtain

\[
\delta \dot{x} = \sum_j \frac{\partial v_j}{\partial x_j} |_{x=\tilde{x}} \delta x_j.
\]

The equilibrium point \( \tilde{x} \) is called hyperbolic if all the eigenvalues of the matrix \( A_{ij} = \frac{\partial v_i}{\partial x_j} |_{x=\tilde{x}} \) are real (that is not complex). It is called nondegenerate if all eigenvalues are nonzero. Since eq.(D.1) is in \( TM \) space it is clear that replacing \( \mathbb{R}^n \) by \( M \) and vice versa introduces nothing new into local study of the system of equations \( \dot{x} = v(x) \). The question of central importance is the following. How can solution of eq.(D.1) help us in solving equation \( \dot{x} = v(x) \)? The answer is given by the following (Guckenheimer and Holmes 1983)

**Theorem D.1. (Grobman -Hartman)** If the point \( \tilde{x} \) is hyperbolic, then there is a homeomorphism defined on some neighborhood \( U \) of \( \tilde{x} \) in \( \mathbb{R}^n \) locally taking orbits of the nonlinear flow \( g^t \) to those of linear flow defined by \( \exp(tA_{ij}) \).
The above theorem is being strengthened by the following

**Theorem D.2.** Suppose that equation \( \dot{x} = v(x) \) has a hyperbolic fixed point \( \tilde{x} \). Then there exist local stable and unstable manifolds \( W^s(\tilde{x}), W^u(\tilde{x}) \) of the same dimensions \( d_s = d_u \) as those of eigenspaces \( E^s(\lambda_l \) negative) and \( E^u(\lambda_l \) positive), \( \lambda_l \) are eigenvalues of the matrix \( A_{ij} \). \( W^s(\tilde{x}) \) and \( W^u(\tilde{x}) \) are tangent to \( E^s \) and \( E^u \) respectively.

**Remark D.3.** Eigenspaces \( E^s \) and \( E^u \) are made of respective eigenvector spaces. As in quantum mechanics, it is possible to make eigenvectors mutually orthogonal. If this is possible, the associated stable and unstable manifolds are transversal to each other. These manifolds are considered to be transversal to each other even when the associated eigenvectors are not orthogonal (since they can be made orthogonal). The existence and uniqueness of the solution of \( \dot{x} = v(x) \) ensures that two stable/unstable manifolds of distinct fixed points, say \( \tilde{x}_1 \) and \( \tilde{x}_2 \), cannot intersect. For the same reason \( W^s(\tilde{x}) \) and \( W^u(\tilde{x}) \) cannot self-intersect. However, intersections of stable and unstable manifolds of distinct fixed points or even of the same fixed point are allowed. To define the Morse-Smale flows we need to introduce the following.

**Definition D.4.** A point \( p \) is called nonwandering for the flow \( g^t \) if for any neighborhood \( U \) of \( p \) there exists arbitrary large \( t \) such that \( g^t(U) \cap U \neq \emptyset \).

From here it follows, in particular, that fixed points and periodic orbits are non wandering.

**Definition D.5.** A Morse-Smale dynamical system is the one for which the following holds.
1. The number of fixed points and periodic orbits is finite and each is hyperbolic.
2. All stable and unstable manifolds intersect transversely.
3. The non wandering set consists of fixed points and periodic orbits alone.

**Corollary D.6.** The Morse-Smale systems are structurally stable. Furthermore, compact manifolds can possess only a finite number of periodic orbits and fixed points.

**Definition D.7.** The nonsingular Morse-Smale flow (NMS) does not contain fixed points.

**Corollary D.8.** The Poincaré-Hopf index theorem permits NMS flow only on manifolds whose Euler characteristic is zero. Therefore, such flows cannot be realized, say, on \( S^2 \) but can be realized on \( T^2 \). On \( S^3 \) such flows also can exist. Details are discussed in the main text.

**Appendix E** Some basic facts about Seifert fibered spaces

A Seifert fibered manifold \( M \) is a fiber bundle \( (M, N, \pi, S^1) \) with \( M \) being the total space, \( N \) being the base, called the orbit manifold/orbit space. Typically, it is an orbifold, that is the manifold with corners/cones). \( \pi \) is the projection, \( \pi : M \to N \), and \( S^1 \) is the fiber. Let \( \mathcal{F} \) be one of such fibers. A Seifert fibered manifold is characterized by the property that each \( \mathcal{F} \) has a fibered neighborhood which can be mapped into a fibered solid torus. It is made
out of a fibered cylinder $D^2 \times I$ in which the fibers are $x \times I$, $x \in D^2$. By rotating $D^2 \times I$ through the angle $2\pi (\nu/\mu)$ while keeping $D^2 \times 0$ fixed and identifying $D^2 \times 0$ with $D^2 \times I$ a fibered solid torus is obtained. Upon such an identification the fibers of $D^2 \times I$ are being decomposed into classes in such a way that each class contains exactly $\mu$ lines which match together to give one fiber of the solid torus, e.g. see Fig. 4a), except that which is belonging to the class containing the axis of $D^2 \times I$. This fiber remains unchanged after identification. It belongs to the class called singular (exceptional, middle) (Seifert and Threlfall 1980). When $\mu = 1$ the fiber bundle (still to be constructed) is trivial. The number $\mu$ is called multiplicity of the singular fiber. There always exist a fiber-preserving homeomorphism into fibered solid torus such that $F$ can be mapped into a singular fiber. Reciprocally, from this property it follows that each fibered neighborhood of $F$ should have a boundary. Naturally, it should be $T^2$. This means that there should be a homeomorphism placing $F$ onto $T^2$ in the form of some closed curve $J$. Such a curve is being defined by its meridian $\mathcal{M}$ and longitude $\mathcal{L}$. The longitude $\mathcal{L}$ of the solid torus is a simple closed curve on $T^2$ which intersects $\mathcal{M}$ in exactly one point. Thus formally, we can write

$$J \sim \nu \mathcal{M} + \mu \mathcal{L}.$$  

(E.1a)

The numbers $\nu$ and $\mu$ are the same for all fibered neighborhoods of $F$. Furthermore, from the way the solid torus was constructed, it easily follows that the number $\nu$ is defined not uniquely: $\nu_1 = \nu_2 \pmod{\mu}$. Therefore fibrations $2\pi (\nu/\mu)$ and $2\pi (\nu + n\mu/\mu)$ are indistinguishable. In view of previous results, we shall call the nonsingular fiber $F$ for $\mu > 1$ as regular. The following theorem is of central importance

**Theorem E.1.** A closed (compact) Seifert fibered space contains at most finitely many singular fibers.

From here it follows that all singular fibers are isolated. It is possible to design Seifert manifolds without singular fibers though. In view of their importance, we briefly sketch their design. Let $N$ be an orbit space. Suppose that it is a compact orientable surface with non-empty boundary (made of $\partial D^2_i$, $i = 1, ..., n$). Then $M = N \times S^1$ is a Seifert fibered manifold (a trivial fiber bundle) with fibers $x \times I$, $x \in N$. An orientable saddle, Fig. 3a), defined in section 5 is an example of such manifold.

Using thus designed (nonsingular) Seifert manifold, it is possible now to design a manifold with singular fibers. For this purpose we select $n$ pairs of coprime integers $(\alpha_i, \beta_i)$, then we select a base $N$ which we had already described. It is a compact orientable surface from which $n$ open discs $D^2_i = D^2_i \setminus \partial D^2_i$ are removed so that the obtained surface $\tilde{N} = N - (D^2_1 \cup \cdots \cup D^2_n)$. By design, such Seifert manifold $\tilde{M} = \tilde{N} \times S^1$ is Seifert manifold without singular fibers. Because of Theorem E.1, it is appropriate to associate the locations of singular fibers with the locations of the discs $D^2_i$. By doing so we are going to construct a Seifert manifold with singular fibers. For this purpose we need to notice that with the boundary $\partial D^2_i$ of each disc $i$ the map $\pi^{-1}(\partial D^2_i) \to T^2_i$ can be associated. Choose on $T^2_i$ the meridian $\mathcal{M}_i = \partial D^2_i \times \{\ast\}$ and the longitude $\mathcal{L}_i = \{\ast\}_i \times S^1$ consistent with the orientation of $\tilde{M}$. At the boundary
\[ \partial D_i^2 \times S^1 = T_i^2 \]

we can construct a curve \( J_i \) of the same type as given by eq.(E.1a)

\[ J_i \sim \alpha_i \mathcal{M}_i + \beta_i \mathcal{L}_i \]  

(E.1b)

(we use the pair \((\alpha_i, \beta_i)\) for its description). Take now trivially fibered solid torus \( D^2 \times S^1 \) and glue it to \( T_i^2 \) in such a way that its meridian \( \mathcal{M}_i \) is glued to \( J_i \in T_i^2 \). For each \( i \) the image under gluing of the curve \( \{0\} \times S^1 \in D^2 \times S^1 \) is going to become the \( i \)-th singular fiber. This gluing homeomorphism is the last step in designing the Seifert fibered manifold with \( n \) singular fibers. A particular Seifert fibered manifold can be now described via the following specification:

\[ M = M(N; (\alpha_1, \beta_1), ..., (\alpha_n, \beta_n)) \]

In section 5 an example of such a manifold is given by the non-orientable saddle, e.g. see Fig.4a). It is the Seifert fibered manifold of the type \( M = M(D^2; (2, 1)) \) with one singular fiber. Incidentally, if the orbit manifold is \( S^2 \), then the obtained Seifert manifold is already familiar Hopf fibration. It is a Seifert fibered manifold without singular fibers. Other Seifert-fibered manifolds without singular fibers having \( S^2 \) as their orbit space are \( S^2 \times S^1 \) and the Lens spaces of the type \( L(n, 1) \). For \( n = 1 \) we get back the Hopf fibration, for \( n = 2 \) we get the projective space \( \mathbb{P}^3 \), while the \( n = 0 \) case is identified with \( S^2 \times S^1 \). Finally, a complement of any torus knot in \( S^3 \) is Seifert fibered space too (Jaco 1980), page 87.

Appendix F Some facts about Kirby moves/calculus

In the famous theorem by Gordon and Luecke (1989) it is proven that knots are determined by their complements. That is to say, a non-trivial Dehn surgery on a non-trivial knot in \( S^3 \) does not yield back \( S^3 \). This theorem does not cover the case of links as Fig.12 illustrates.

Fig.12

The generic case (Rolfsen1976), page 49, of generating topologically different links having the same complement in \( S^3 \)

In section 7.7 we argued that the theorem by Gordon and Luecke (1989) could be also violated for the case of some knots. In Fig.12 the initial state is a). It is made out of two unknots (e.g. think also about the Hopf link). The final state is d). To reach d) from a) we have to perform a reversible homeomorphism- from b) to c). The trick is achieved by enclosing the original link located in \( S^3 \) into a solid torus, just like it is done in the case of Fig.3a), and then performing the Dehn twist (or the sequence of Dehn twists). This is further illustrated in going from c) to f).

This generic example can be extended in several directions. For instance, one of these directions was developed by Robion Kirby (1978). He invented moves, now known as Kirby moves/calculus, broadly generalizing situation depicted in Fig.12. To explain what he did we have to introduce several definitions. We begin with

**Definition F.1.** *(Dehn surgery)* Let \( K \) be some knot in some 3-manifold \( M \). Let \( N(K) \) be a tubular neighborhood of \( K \) (that is trivially fibered (that is without a twist) solid torus). Remove now this solid torus \( N(K) \) from \( M \). Then, what is left is a 3-manifold with boundary
\[ M = M \setminus \text{int}N(K) \] where \( \text{int} \) means "interior" and the solid torus \( N(K) = D^2 \times S^1 \). \( \tilde{M} \) is the manifold whose boundary is \( T^2 \) that is \( \partial \tilde{M} = T^2 \). Apparently, \( M = \tilde{M} \cup (D^2 \times S^1) \). To complicate matters we can glue the solid torus back into \( \tilde{M} \) via some homeomorphism \( h : \partial D^2 \times S^1 \to T^2 \). In such a case, instead of \( M = \tilde{M} \cup (D^2 \times S^1) \), we obtain \( \tilde{M} = M \cup h(D^2 \times S^1) \). Thus, a 3-manifold \( \tilde{M} \) is obtained from \( M \) via Dehn surgery along \( K \).

The manifold \( \tilde{M} \) depends upon the specification of \( h \). It can be demonstrated that it is sufficient to use the following type of (gluing) homeomorphism. Choose \( J \sim \alpha \mathfrak{M} + \beta \mathcal{L} \) at the boundary \( T^2 \) of \( \tilde{M} \). Let \( \mathfrak{M} \) be the meridian \( \partial D^2 \times \{*\} \) of the solid torus \( N(K) \). Then the \( h \) is defined by the map: \( J = h(\partial D^2 \times \{*\}) \). Since the pairs \( (\alpha, \beta) \) and \( (-\alpha, -\beta) \) define the same curve \( J \), it is clear that the ratio \( r = \alpha/\beta \) determines \( J \). In particular, since \( 1/0 = \infty \) determines the meridian, the result of any \( 1/0 \) surgery leaves \( M \) unchanged. Incidentally, the surgery determined by \( r = 0 \) along the trivial knot switchers meridian with parallel (the torus switch). This type of surgery leads to the 3-manifold \( S^2 \times S^1 \). The types of surgeries determined by the rational \( r \) are called rational. The surgery is integral if \( \beta = \pm 1 \). As demonstrated by Lickorish and Wallace (Saveliev 2012) any closed orientable 3-manifold can be obtained by an integral surgery along a link \( \mathcal{L} \subset S^3 \).

**Remark F.2.** Operation of Dehn surgery is technically closely related to the cabling operation, e.g. see Fig.7.

**Definition F.3.** The procedure of assigning of \( r_i \) for each component \( \mathcal{L}_i \) of \( \mathcal{L} \) is called framing.

It is very helpful to think about the integral surgery/framing in terms of linking numbers. The linking number for oriented knots/links can be easily defined via knot/link projection. Using the convention as depicted in Fig.13

\[
\text{Fig.13}
\]

The sign convention used for calculation of a) linking and b) sel-linking (writhe) numbers for oriented links(for a)) and knots(for b)) respectively.

The linking number \( \text{lk}(L_1, L_2) \) for oriented curves(links) \( L_1 \) and \( L_2 \) is defined as

\[
\text{lk}(L_1, L_2) = \sum \varepsilon_i, \quad \varepsilon = \pm 1.
\]

(E.1)

Evidently, \( \text{lk}(L_1, L_2) = \text{lk}(L_2, L_1) \) and \( \text{lk}(-L_1, L_2) = -\text{lk}(L_2, L_1) \). Here \(-L_1 \) means link \( L_1 \) with orientation reversed.

Notice now that in the case of integral surgery the meridian \( \mathfrak{M} \) is being mapped into the curve \( J = \alpha \mathfrak{M} + \mathcal{L} \). From here it follows that \( J \) is making exactly one revolution in the direction parallel to \( \mathcal{L} \). Locally this situation can be illustrated with help of a ribbon depicted in Fig.14a).

\[
\text{Fig.14}
\]

From here we obtain the following

\[25\]It is essentially the same as was used above for description of Seifert fibered manifolds with singular fibers
**Definition F.4.** The integral framing of a knot $K$ corresponds to the choice $\text{lk}(K, \Sigma) = \alpha = n$, where $n$ is an integer. An example of integral framing is depicted in Fig.14b). In appendix E it was stated that both Hopf links and complements of any torus knots in $S^3$ are Seifert fibered spaces. This does not mean that these spaces must be the same. Following Lickorish and Wallace (Prasolov and Sossinsky 1997), we can construct these Seifert fibered 3-manifolds using some sequence of integral surgeries. This brings us to

**The main question:** (Kirby) How to determine when two differently framed links produce the same 3-manifold?

The answer to this question is given in terms of two (Kirby) moves. They are designed as equivalence relations between links with different framings which produce the same 3-manifolds. It is possible to inject some physics into these equivalence moves using results of part I. Specifically, following conventions, the value of integral framing is depicted next to the respective projection of knot/link. For instance, if we supply the framing $\pm 1$ to the unknot, it is becoming the Hopf link. It was identified in part I with the magnetic or electric charge. Accordingly, the 1st Kirby move depicted in Fig.15 represents the charge conservation (e.g. see Fig.8).

Fig.15
The first Kirby move

This move can be interpreted as follows: It is permissible to add or delete an unknot with framing $\pm 1$ which does not intersect the other components of $L_i$ to a given link $\mathcal{L}$. The 2nd Kirby move is depicted in Fig.16.

Fig.16
The second Kirby move

Physically, it can be interpreted in terms of interaction between charges. Mathematically this move can be interpreted as follows. Let $L_1$ and $L_2$ be two link components framed by the integers $n_1$ and $n_2$ respectively and $L_2'$ a longitude defining the framing of $L_2$ that is $\text{lk}(L_2, L_2') = n_2$. Replace now the pair $L_1 \cup L_2$ by another pair $L_1' \cup L_2$ in which $L_1' = L_1 \#_b L_2'$ and $b$ is 2-sided band connecting $L_1$ with $L_2'$ and disjoint from another link components. While doing so, the rest of the link $\mathcal{L}$ remains unchanged. The framings of all components, except $L_1$, are preserved while the framing of $L_1$ is being changed into that for $L_1'$ and is given by $n_1 + n_2 + 2\text{lk}(L_1, L_2)$. The computation of $\text{lk}(L_1, L_2)$ proceeds in the standard way as described above (e.g. see Fig.13), provided that both $L_1$ and $L_2$ are oriented links.

**Remark F.5.** Using the 1st and the 2nd Kirby moves it is possible now to extend/improve results depicted in Fig.12. These improvements are summarized and depicted in Fig.17.

Fig.17
Blow up and blow down Kirby-Fenn-Rourke moves

The Fenn-Rourke moves are equivalent to blow ups/downs and are equivalent to the combined first and second Kirby-type moves (Prasolov and Sossinsky 1997).
Remark F.6. By comparing Kirby moves just described against those suggested by Wada (1989) (e.g. see Fig.s 2-7 of paper by Campos et al (1997)) it is clear that the moves suggested by Wada are just specific adaptations of Kirby (or Fenn-Rourke) moves.

Appendix G Calculations of the various presentations for the fundamental group of the figure eight knot

a) We begin with the set of relations

\[ r_1 : bcb^{-1} = a, \rightarrow c = b^{-1}ab \] (G.1)

\[ r_2 : ada^{-1} = b, \rightarrow d = a^{-1}ba \] (G.2)

\[ r_3 : d^{-1}bd = c, \rightarrow (a^{-1}ba)^{-1} b (a^{-1}ba) = c = b^{-1}ab \] (G.3)

\[ r_4 : e^{-1}ac = d, \rightarrow (b^{-1}ab)^{-1} a(b^{-1}ab) = d = a^{-1}ba \] (G.4)

Noticing now that \((a^{-1}ba)^{-1} = a^{-1}b^{-1}a\) and \((b^{-1}ab)^{-1} = b^{-1}a^{-1}b\) we can rewrite eq.(G.3) as

\[ a^{-1}b^{-1}aba^{-1}ba = b^{-1}ab \rightarrow a = ba^{-1}b^{-1}aba^{-1}bab^{-1} \] (G.5)

and eq.(G.4) as

\[ b^{-1}a^{-1}bab^{-1}ab = a^{-1}ba \rightarrow a = ba^{-1}b^{-1}aba^{-1}bab^{-1} \] (G.6)

Notice that the r.h.s. of eq.s(G.5) and (G.6) coincide. This circumstance allows us to get rid of generators \(c\) and \(d\) in view of eq.s(G.3) and (G.4). This means that we now can rewrite the presentation for \(\pi_1(S^3 - K_8)\) as follows

\[ < a, b, w | wa = bw, w = b^{-1}aba^{-1} > \] (G.7)

b) In Miller (2001) and in her PhD thesis Miller (2005) the following presentation of the first fundamental group of the figure eight knot complement was given

\[ \pi_1(S^3 - K_8) = < x, y, z | zx^{-1}yz^{-1}x = 1, xy^{-1}z^{-1}y = 1 > \] (G.8)

without derivation. To avoid ambiguities, we checked this result and found an error. The correct result is

\[ \pi_1(S^3 - K_8) = < x, y, z | zx^{-1}yz^{-1}x = 1, xy^{-1}z^{-1}y = 1 > \] (G.9)

Proof:
From eq.(G.8b) we obtain

\[ x^{-1}yxx^{-1}x = z^{-1} \] (G.9a)

and

\[ y^{-1}z^{-1}y = x^{-1} \] (G.9b)
By comparing eq.(G.9b) with eq.(G.3) we make the following identifications

\[ x^{-1} = c, y = d, z^{-1} = b \]  

(G.10)

Now we can rewrite all relations in eq.s(G1-G.4) accordingly. Thus, we get:

\[ r_1 : z^{-1}x^{-1}z = a, \]

\[ r_2 : ay^{-1} = z^{-1}, \]

\[ r_3 : y^{-1}z^{-1}y = x^{-1}, \]

\[ r_4 : xax^{-1} = y. \]  

(G.11)

Using eq.s(G.11), we obtain: \( a = x^{-1}yx \). Using this result in \( r_2 \) we obtain: \( x^{-1}yxyx^{-1}y^{-1}x = z^{-1} \). Using \( r_3 \) in this relation, we obtain: \( x^{-1}yxy(y^{-1}z^{-1}y)y^{-1}x = z^{-1} \) or, \( x^{-1}yxyx^{-1}x = z^{-1} \). But this result coincides with eq.(G.9a)! Next, going to \( r_1 \) we obtain \( z^{-1}x^{-1}z = x^{-1}yx \) or \( x^{-1}yxyx^{-1}xz = 1 \) in view of eq.(G.9a).

QED

c) Use of symmetry enables us to embed the standard presentation eq(7.28) for the trefoil into much more manageable representation, eq.(7.29). The same philosophy was used by Miller (2001,2005). Unfortunately, its actual implementation is plagued by mistakes. Specifically, Miller tried to embed her presentation, eq.(G.8), into the dihedral group \( D_2 \). For any dihedral group \( D_n \), \( n = 1, 2, ... \) the presentation is well known (Grosman and Magnus 1964) and is given by

\[ D_n = \langle s, t \mid t^n = 1, s^2 = 1, (ts)^2 = 1 \rangle. \]  

(G.12)

Miller choose \( n = 4 \) for which she wrote the following presentation

\[ D_4 = \langle s, t \mid t^4 = 1, s^2 = 1, s^{-1}ts = t^{-1} \rangle \]  

(G.13)

Not only is the last relation incorrect, but, more importantly, the graphical representation for this group given as Fig.14 of Miller (2001) is also incorrect. The depicted graph corresponds to the graph for \( D_2 \! \). Since this graph matches well with BWb) graph for the for the figure eight knot (e.g. see Fig.1.2 of BW b)), we would like to demonstrate that, indeed \( D_2 \) is the correct group. Thus, we need to find if we can embed \( \pi_1(S^3 - K_8) \) into \( D_2 \). For \( D_2 \) we have the following presentation

\[ D_2 = \langle s, t \mid t^2 = 1, s^2 = 1, tst = s^{-1} \rangle \]  

(G.14)

Before proceeding with calculations, it is important to notice that eq.(G.14) can be rewritten differently in view of the fact that for \( D_2 \) we have \( ts = st \). Indeed,

\[ 1 = (ts)^2 = tsts = tsst = t^2 \] since \( s^2 = 1 \).  

(G.15)

Clearly, the relation \( ts = st \) is compatible with rewriting of eq.(G.15) as \( ts = 1 \) and \( st = 1 \). In view of these results, using eq.(G.9a), we obtain:

\[ xz^{-1}x = y^{-1}xz^{-1}. \]  

(G.16)
By looking at eq.(G.14) the following identification can be made: \( x = t, z^{-1} = s \) and \( y^{-1}xz^{-1} = s^{-1} \). The last result is equivalent to \( y = ts^2 \). Using eq.(G.16) rewritten in just defined notations we obtain: \( tst = ts^2 \) or \( st = s^2 \). If \( s^2 = 1 \), then \( st = 1 \).

At the same time, using eq.(G.9b) we obtain as well: \((ts^2)^{-1}s(ts^2) = t^{-1}\). Since \( s^2 = 1 \), we can convert this result into \( tst = t^{-1} \) and, if \( t^2 = 1 \), then we obtain: \( ts = 1 \). But we already obtained \( st = 1 \). Therefore, we get finally: \( ts = st \).

QED

References

Anosov D, Aranson S, Arnol’d V, Bronstein I, Grines V and Il’yashenko Y 1997
Ordinary Differential Equations and Smooth Dynamical Systems (Berlin: Springer-Verlag)
Arrayas M and Trueba J 2012 Torus-knotted electromagnetic fields [arXiv:1106.1122] [hep-th]
Arnol’d V 1989 Mathematical Methods of Classical Mechanics (Berlin: Springer-Verlag)
Arnol’d V and Khesin B 1998 Topological Methods in Hydrodynamics (Berlin: Springer-Verlag)
Arnol’d V 1988 Geometrical Methods in the Theory of Ordinary Differential Equations (Berlin: Springer-Verlag)
Arnol’d V 1986 First steps in symplectic topology, Russian Mathematical Surveys 41 1-21
Atiyah M 1990 The Geometry and Physics of Knots (Cambridge, UK: Cambridge University Press)
Benequin D 1983 Entrelacements et equations de Pfaff Asterisque 197-108 87-161
Birman J and Williams R 1983a Knotted periodic orbits in dynamical systems I: Lorenz equations Topology 22 47-82
Birman J and Williams R 1983b Knotted periodic orbits in dynamical system II: Knot holders for fibered knots Contemporary Math. 20 1-60
Birman J and Wrinkle N 2000 On transversally simple knots J.Diff.Geom. 55 325-354
Birman J 2009 Lorenz knots and links (transparencies of the talk given on Feb13, 2009)
Bowditch B 1998 Markov triples and quasifuchsian groups Proc.London Math.Soc.77 697-736
Brownstein K 1987 Transformation properties of the equation \( \nabla \times V = kV \)
Phys. Rev. A 35 4856-4858

Calin O and Chang Der-Ch 2009 Sub-Riemannian Geometry
(Cambridge UK, Cambridge University Press)

Campos B, Martinez-Alfaro J and Vindel P 1997 Links and bifurcations in nonsingular Morse-Smale systems, J. of Bifurcation and Chaos 7 1717-1736

Chern S and Hamilton R 1985 On Riemannian metric adapted to three-dimensional contact manifolds LNM 1111 279-305

Constantin P and Majda A 1988 The Beltrami spectrum for incompressible flows Comm.Math.Phys. 115 435-456

Dennis M, King R, Jack B, O’Holleran K and Padgett M 2010 Isolated optical vortex knots Nature Physics 6 118-121

Donaldson S 2002 Floer Homology Groups in Yang-Mills Theory (Cambridge UK: Cambridge University Press)

Eisenbud D and Neumann W 1985 Three-dimensional Link Theory and Invariants of Plane Curve Singularities (Princeton NJ, Princeton university Press)

Elstrodt J, Grunewald F and Mennicke J 1998 Groups Acting on Hyperbolic Space (Berlin: Springer-Verlag)

Enciso A and Peralta-Salas D 2012 Knots and links in steady solutions of the Euler equation Ann.Math. 175 345-367

Enciso A and Peralta-Salas D 2013 Knots and links in fluid mechanics Procedia IUTAM 7 13-20

Etnyre J 2004 Legendrian and transversal knots arxiv:math/0306256 v2

Etnyre J 2005 Lectures on open book decompositions and contact structures arXiv: math/0409402

Etnyre J and Ghrist R 1999 Gradient flows with in plane fields Comm.Math.Helv. 74 507-529

Etnyre J and Ghrist R 2000a Contact topology and hydrodynamics III: Knotted orbits Transactions AMS 352 5781-5794

Etnyre J and Ghrist R 2000b Contact topology and hydrodynamics I: Beltrami fields and Seifert conjecture Nonlinerarity 13 441-458

Falbel E 2008 A spherical CR structure on the complement of the figure eight knot with discrete holonomy J.Diff.Geometry 79 69-110

Fischer A and Moncrief V 2001 The reduced Einstein equations and conformal volume collaps of 3-manifolds Class.Quantum Grav. 18 4493-4515

Frankel T 1997 The Geometry of Physics
Franks J and Sullivan M 2002 Flows with knotted closed orbits in
*Handbook of Geometric Topology*, pp 471-497
(Amsterdam, North-Holland)

Fomenko A 1991 Topological classification of all integrable Hamiltonian
differential equations of general type with two degrees
of freedom in *The Geometry of Hamiltonian Systems*,
T. Ratiu Editor
(Berlin:Springer-Verlag) pp.131-340

Fomenko A and Bolsinov A 2004 *Integrable Hamiltonian Systems:*
*Geometry, Topology and Classification*
(Boca Ratton, Florida: CRC Press LLC)

Fomenko A and Matveev S 1997 *Algorithmic and Computer Methods*
*for Three-Manifolds*
(Boston: Kluver Academic Publishers)

Fröhlich H 1966 Macroscopic wave functions in superconductors
*Proc.Phys.Soc.London* 87 330-332

Gambini R and Pullin J 2011 *A First Course in Loop Quantum Gravity*
(Oxford UK: Oxford U.Press)

Geiges H 2008 *An Introduction to Contact Topology*
(Cambridge UK: Cambridge University Press)

Ghrist R, Holmes P and Sullivan M 1997 Knots and links in three dimensional flows
*Lecture Notes in Mathematics* 1654
(Berlin: Springer-Verlag)

Ghrist R 1997 Branched two-manifolds supporting all links
*Topology* 36 423-448

Ghrist R 1998 Chaotic knots and wild dynamics
*Chaos, Solitons and Fractals* 9 583-598

Ghys E 2007 Knots and Dynamics
*Proceedings of the International Congress of*
*Mathematicians Madrid Spain 2006* pp 247-277
(Zurich: European Mathematical Society)

Ghys E and Leys J 2011 Lorenz and modular flows: A visual introduction.
*Monthly Essays on Mathematical Topics*,
AMS Feature Column

Gilmore R and Lefranc M 2002 *The Topology of Chaos*
(New York: Wiley-Interscience Inc.)

Ginzburg V 2005 The Weinstein conjecture and theorems of nearby
and almost existence, in
*The Breadth of Symplectic and Poisson Geometry*
pp139-172 (Boston: Birkhäuser)

Goldman W 1999 *Complex Hyperbolic Geometry*
(Oxford: Clarendon Press)

Gordon C and Luecke J 1989 Knots are determined by their complements. 
*J.Amer.Math.Soc.** 2 371-415

Grosman I and Magnus W 1964 Groups and Their Presentations. (New York: The random House)

Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Berlin: Springer-Verlag)

Guest M 2001 Morse theory in the 1990s, in *Invitation to Geometry and Topology* (Oxford: Oxford U.Press) pp 207.

Hempel J 1964 A simply connected 3-manifold is $S^3$ if it is the sum of a solid torus and the complement of a torus knot *Proceedings of AMS** 15 154-158

Hirch M and Smale S 1974 *Differential Equations, Dynamical Systems and Linear Algebra* (New York: Academic Press)

Hofer H 1993 Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. *Inv.Math.** 114 515-563

Hofer H, Wysocki K and Zehnder E 1998 The dynamics of three-dimensional strictly convex energy surfaces *Ann.Math.** 148 197-289

Hofer H 1998 Dynamics, topology and holomorphic curves *Documenta Mathematica, Extra Volume* ICM 1-27

Hurtado A and Rosales C 2008 Area-stationary surfaces inside the sub-Riemannian three-sphere *Math.Ann.** 340 675-708

Hutchings M 2009 Taubes's proof of the Weinstein conjecture in dimension three *AMS Bulletin** 47 73-125

Imayoshi Y and Taniguchi M 1992 *An introduction to Teichmüller Spaces* (Berlin: Springer-Verlag)

Jaco W and Shalen P 1979 *Seifert Fibered Spaces in 3-Manifolds* *AMS Memoirs** 21 Number 220 (Providence,RI: AMS Publishers)

Jaco W 1980 *Lectures on Three-Manifold Topology* (Providence RI: AMS Publishers)

Jovanović B 2011 What are completely integrable Hamiltonian systems *The Teaching of Mathematics** 13 1-14

Kamchatov A 1982 Topological solitons in magnetohydrodynamics *Sov.Phys. JETP** 55 69-73

Kassel C and Turaev V 2008 *Braid Groups* (Berlin: Springer-Verlag)

Kedia H, Bialynicki-Birula I, Peralta-Salas D and Irvine W 2013 Tying knots in light fields *Phys.Rev.Lett** 111 150404
Kholodenko A 2013 *Applications of Contact Geometry and Topology in Physics*  
(Singapore:World Scientific)

Kholodenko A 2011 Gravity assisted solution of the mass gap problem for pure Yang-Mills fields *International Journal of Geometric Methods in Modern Physics* **8** 1355-1418

Kholodenko A 2008 Towards physically motivated proofs of the Poincare’ and geometrization conjectures *J.Geom.Phys.* **58** 259-290

Kholodenko A 2001 Statistical mechanics of 2+1 gravity from Riemann zeta function and Alexander polynomial: exact results *J.Geom.Phys.* **38** 81-139

Kholodenko A 1999 Random walks on figure eight: from polymers through chaos to gravity and beyond arXiv: cond. mat/9905221

Kirby R 1978 A calculus for framed links in $S^3$  
*Inv.Math.* **45** 36-56

Kleckenber D and Irvine W 2013 Creation and dynamics of knotted vortices in fluid  
*Nature Physics* **9** 253-258

Kobayashi M and Nitta M 2014 Torus knots as Hopfions  
*Phys.Lett.B* **728** 314-318

Machon T and Alexander G 2013 Nematics, knots and non-orientable surfaces  
*PNAS* **110** (35) 14174-14179

Majda A and Bertozzi A 2003 *Vorticity and Incompressible Flow*  
(Cambridge UK: Cambridge University Press)

Maclachlan C and Reid A 2003 *The Arithmetic Hyperbolic 3-Manifolds*  
(Berlin:Springer-Verlag)

Mathieu Y 1992 Closed 3-manifolds unchanged by Dehn surgery  
*J.of Knot Theory and Its Ramifications* **1** 279-296

Matveev S and Fomenko A 1988 Constant energy surfaces of Hamiltonian systems, enumeration of three-dimensional manifolds in increasing order of complexity, and computation of volumes of closed hyperbolic manifolds *Russian Math.Surveys* **43** 3-24

Menasco W 1984 Closed incompressible surfaces in alternating knot and link complements *Topology* **23** 37-44

Menasco W 2001 On iterated torus knots and transversal knots  
*Geometry and Topology* **5** 651-682

Miller S 2001 Geodesic knots in the figure eight knot complement  
*Experimental Mathematics* **10** 419-436

Miller S 2005 Geodesic Knots in Hyperbolic 3-Manifolds  
PhD Thesis, Department of Mathematics and Statistics, U.of Melbourne, Australia

Milnor J 1968 *Singular Points of Complex Hypersurfaces*  
(Princeton NJ, Princeton University Press)

Minsky Y 1999 The classification of punctured-torus groups
Moffatt H 1985 Magnetostatic equilibria and analogous Euler flows of arbitrary complex topology. I Fundamentals. 
*J.Fluid Mech.* 159 359-378

Morgan J 1978 Non-singular Morse-Smale flows on 3-dimensional manifolds. 
*Topology* 18 41-53

Murasugi K 1996 *Knot Theory and Its Applications* 
(Boston: Birkhäuser)

Nash C and Sen S 1983 *Topology and Geometry for Physicists* 
(New York: Academic Press Inc.)

Overkov S and Shevchisin V 2003 Markov theorem for transverse links. 
*J.Knot Theory and Ramifications* 12 905-913

Prasolov V and Sossinsky A 1997 *Knots, Links, Braids and 3-Manifolds* 
(Providence ,RI, AMS Publishers)

Ranada A 1989 A Topological theory of the electromagnetic field. 
*Lett.Math.Phys.* 18 97-106

Ranada A 1992 Topological electromagnetism. 
*J.Phys.A* 25 1621-1641

Rolfsen D 1976 *Knots and Links* 
(Houston Tx: Publish or Perish, Inc.)

Saveliev N 2012 *Lectures on the Topology of 3-Manifolds* 
(Berlin: Walter de Gruyter GmbH &Co)

Schubert H 1953 Knoten and Vollringe. 
*Acta Math.* 90 131-226

Scott P 1983 The geometries of 3-manifolds. 
*Bull London Math. Soc.* 15 401-487

Seifert H and Threlfall W 1980 *A Textbook on Topology* 
(New York: Academic Press)

Stephani H 1990 *General Relativity* 
(Cambridge, UK: Cambridge University Press)

Thompson A, Swearngin J, Wickes A, Dalhuisen A and Bouwmeester D 2014 
Linked and knotted gravitational radiation, 
[arXiv:1402.3806] [gr-qc]

Uehara K, Kawai T and Shimoda K 1989 Non-transverse electromagnetic waves with parallel electric and magnetic fields. 
*J.Phys.Soc.Japan* 58 3570-3575

Wada M 1989 Closed orbits of non-singular Morse-Smale flows on $S^3$. 
*J.Math.Soc.Japan* 41 405-413

Wells R 2008 *Differential Analysis on Complex Manifolds* 
(Berlin: Springer-Verlag)

Witten E 1989 Quantum field theory and Jones polynomial. 
*Comm.Math.Phys.* 121 351-399
Zung N and Fomenko A 1990 Topological classification of integrable non-degenerate Hamiltonians on a constant energy three-dimensional sphere
Uspekhi Math.Nauk 45 91-111