1. **Introduction.** Sales of equity-indexed annuities (EIAs) have grown considerably in recent years. An equity-indexed annuity (EIA) is an insurance product issued by an insurance company whose earning rate is closely related to an equity index. EIAs offer a minimum guaranteed interest rate combined with an interest rate linked to a market index. In general, investing directly in the stock market involves high risk, while investing in a fixed deposit usually earns a very low return rate. An equity-indexed annuity is a hybrid between a fixed and variable annuity. Due to the minimum guaranteed return, EIAs have less market risk than variable annuities. EIAs can also earn returns better than traditional fixed annuities when the stock market is advancing. Therefore, they are appealing to moderately conservative investors who usually hesitate to take high risks but want a moderate investment growth.
As an application of EIAs, Gerber and Shiu (1999)\cite{9}, Imai and Boyle (2001)\cite{15}, Gerber and Pafumi (2000)\cite{8} and Gerber and Shiu (2003)\cite{10} propose the concept of dynamic fund protection (DFP). DFP can be seen as an extended put option since the seller provides continuous protection to ensure that the balance in the customer’s account does not fall below the predetermined protection level. If the fund price goes below the protection level at any time during its life, just enough money will be added to the fund so that the price will be upgraded to the protected level. Hence, both individual and institutional investors can use this product against the downside risk of their target portfolio. Since the fund has a more profitable and complex payout structure than the traditional put option for its investors, the value of the fund will be greater than a traditional put option, and be priced more difficultly.

There has been substantial research on DFP. Gerber and Pafumi (2000)\cite{8} obtain the explicit formula for the price of the dynamic guaranteed fund under the Black-Scholes model based on the assumption of constant interest rate. Imai and Boyle (2001)\cite{15} provide an alternative derivation of the Gerber-Pafumi closed form formula for the price of dynamic fund protection within the framework of Black-Scholes model. However, it is well known that the Black-Scholes model can not describe two empirical facts: the asymmetric leptokurtic features and the volatility smile. Therefore, using this not realistic hypothesis of normal returns can lead to mispricing life insurance contracts. Many studies have been conducted to modify the Black-Scholes model in order to incorporate the above-mentioned empirical phenomena. See for example, Fung and Li (2003)\cite{7} investigate the pricing of discretely monitored DFP under a CEV process. Wong and Chan (2007)\cite{22} use an asymptotic method to value DFP when the fund price satisfies a stochastic volatility model.

Jump-diffusion processes are widely used in finance to model asset prices. The jump component can capture the impacts of some major events, such as, major political changes, some abrupt changes in the policy of the company or even a natural disaster, on the asset prices. Some research on the valuation of DFP under a jump-diffusion model has been conducted. See for example, Wong and Lam (2009)\cite{23} investigate the valuation of discrete DFP under Lévy processes and obtain the analytical solution of discrete DFP by using the Fourier transforms technology. The double exponential jump-diffusion process, firstly proposed by Kou (2002)\cite{17}, Kou and Wang (2003, 2004)\cite{18}\cite{19}, is used to model the returns of the stocks. Due to the memoryless property of exponentially distributed random variables, some expressions depending on the first passage times can be analytically obtained. Chang et al. (2012)\cite{2} derive the Laplace transform of DFP when the fund price follows a double exponential jump-diffusion model.

As life insurance contracts are long term products, these instruments should be subject to the changes of economic regimes. Yet, the aforementioned literature focus on the valuations of option-embedded insurance contracts under models without taking into account changes of market regimes. Regime-switching models, introduced by Hamilton (1989)\cite{13} to financial econometrics and economists, have been used by many researchers in different branches in modern financial economics, see Naik (1993)\cite{20}, Guo (2001)\cite{12}, Buffington and Elliott (2002)\cite{1} and Elliott et al. (2005)\cite{6}. One of the main features of these models is that model dynamics are allowed to change over time according to the state of an underlying Markov chain.
Regime switches are often interpreted as structural changes in macro-economic conditions and in different stages of business cycles. By using monthly returns data from the Standard and Poor's 500 and the Toronto Stock Exchange 300 indices, Hardy (2001) [14] finds that the regime-switching lognormal model fits to the monthly returns data much better than other econometric models, such as the independent lognormal model and the ARCH type models.

Motivated by the previous works, in this paper we investigate the valuation of DFP under a regime-switching process. Recently, Siu et al. (2015) [21] investigate the valuation of various equity-linked benefits including DFP under a regime-switching, double exponential jump-diffusion process by using Laplace transform. Jin et al. (2016) [16] study the value of dynamic fund protections under a generalized regime-switching jump-diffusion model. Instead of investigating the Laplace transform of the first passage time in [21], they derive the coupled system of integro-differential equations for the value of DFP. Since it is impossible to find the analytic solutions, they focus on designing a numerical scheme to approximate the value of DFP. In general, the protection level is often set to be a constant or an exponential function with time $t$. However, the protection level is not necessarily chosen to be deterministic. See for example, Dong (2013) [3] proposes a stochastic protection level with the guaranteed rate matching the return of a government bond and derives the fair value of DFP under a Vasicek interest rate environment. Gerber and Shiu (2003) [10] model the protection by a Black-Scholes model. In fact, DFP can be beneficial in that it protects downside risk for investors. If the protection level is modeled by a stochastic process with positive jumps, it can help to provide a higher protection for the investors since the value of the protection level will increase during economic growth.

There is rather abundant research on the valuation of DFP with a deterministic protection level. However, relatively few studies focus on the pricing models using stochastic protection level with jumps. Recently, Xu and Dong (2018) [24] model the protection level by a jump-diffusion process. In this paper, we shall price DFP under a regime-switching jump-diffusion model with a stochastic protection level by using a Laplace transform method, which is similar to [21]. However, comparing with [21], where the authors focus on a regime-switching, double exponential jump-diffusion process and a constant protection level, we consider a more general jump distribution and a stochastic protection level with jumps. Furthermore, we derive the integro-differential equations satisfied by the Laplace transforms of the first passage time. In particular, we present explicit formula for the Laplace transform of the value of the DFP at time 0 by solving integro-differential system when the size of jumps has a regime-switching hyper-exponential distribution, which is more extensive than a regime-switching double exponential distribution in [21]. It is well known that empirically asset return distributions have high peaks and heavy tails. The hyper-exponential distribution is rich enough to approximate many other distributions, including any discrete distribution, the normal distribution, and various heavy-tailed distributions such as Gamma, Weibull and Pareto distributions in the sense of weak convergence. Furthermore, the stochastic protection level with jumps can produce a higher protection level, so that it may be more attracted to the investors.

The main contribution of this paper is that we have proposed a regime-switching jump-diffusion model with a stochastic protection level for pricing DFP, provided a method to value DFP, and characterized explicitly the Laplace transform of the
value of the DFP at time 0 when the size of jumps has a regime-switching hyper-
exponential distribution.

The rest of the paper is organized as follows. Section 2 introduces the models
for the fund price and the protection level. In Section 3, we derive a system of
integro-differential equations for the Laplace transforms of the first passage time.
In particular, closed-form solution for the Laplace transform of the value of the
DFP at time 0 under the regime-switching hyper-exponential jump-diffusion model
is obtained. Section 4 carries out some calculations. Section 5 concludes the paper.
The proofs are presented in the appendix.

2. The model. Consider a continuous-time model with a finite time horizon \( \mathcal{T} = [0, T] \) with \( T < \infty \). Let \( F(t) \) denote the value of the basic fund at time \( t \in [0, T] \). Let \( \{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, Q\} \) be a filtered complete probability space, where \( Q \) is the risk neutral measure such that the discounted value process of \( F(t) \) is a \( Q \)-martingale, and \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) is a filtration satisfying the usual conditions. Throughout the paper, it is assumed that all random variables are well defined on this probability space and \( \mathcal{F}_T \)-measurable.

Let \( X := \{X(t), t \geq 0\} \) be a homogenous continuous-time, finite-state, irreducible
Markov chain with generator \( A = (a_{ij})_{i,j=1,2,\ldots,N} \), where \( a_{ij} \geq 0 \), for \( i \neq j \), \( a_{ii} = -\sum_{j \neq i} a_{ij} \). The states of the Markov chain \( X \) are interpreted as different states of an
macro-economy or different stages of a business cycle. Without loss of generality, we
identify the states of the Markov chain \( X \) with a set of standard unit vectors \( \mathcal{E} = \{e_1, e_2, \ldots, e_N\} \), where \( e_i = (0, \cdot, 0, 1, 0, \cdot, 0)^* \in \mathbb{R}^N \). \(*\) denotes the transpose
of a vector or a matrix. Let \( \langle \cdot, \cdot \rangle \) denote a scalar product in \( \mathbb{R}^N \), that is, for any
\( x, y \in \mathbb{R}^N \), \( \langle x, y \rangle = \sum_{i=1}^N x_i y_i \).

Let \( \mathcal{E}^X := \{\mathcal{E}^X_t | t \in \mathcal{T}\} \) be a filtration generated by the Markov chain \( X \). Elliott et al. (1994)[5] provide the following semi-martingale decomposition for \( X \):

\[
dX(t) = A^* X(t) dt + d\mathcal{M}(t),
\]

where \( \mathcal{M}(t) \) is an \( \mathbb{R}^N \)-valued martingale with respect to the filtration \( \mathcal{E}^X \) under the
measure \( Q \).

Gerber and Shiu (2003)[10] model the basic fund price and the protection level
by a bivariate Winer process. Motivated by them, we shall model the processes
of the basic fund price and the protection level by two regime-switching processes
with jumps. Consider two regime-switching compound Poisson processes \( \mathcal{L}^1 := \{L^1(t) | t \in \mathcal{T}\} \) and \( \mathcal{L}^2 := \{L^2(t) | t \in \mathcal{T}\} \) such that

\[
L^1(t) = \sum_{j=1}^{N_1(t)} Y^1_j - \sum_{j=1}^{N_2(t)} Y^2_j, \quad L^2(t) = \sum_{j=1}^{N_1(t)} Y^3_j,
\]

where \( \mathcal{N}_i = \{N_i(t) | t \in \mathcal{T}\} \) is a regime-switching Poisson process with intensity given
by \( \lambda_i(t) = \langle \lambda_i, X(t) \rangle \) for a constant vector \( \lambda_i = (\lambda_{i1}, \ldots, \lambda_{iN})^* \) with \( \lambda_{ij} > 0 \) for each
\( i = 1, 2, j = 1, 2, \ldots, N \); the non-negative sequences \( \{Y^1_1, Y^1_2, \ldots\} \), \( \{Y^2_1, Y^2_2, \ldots\} \)
and \( \{Y^3_1, Y^3_2, \ldots\} \) are independent and they are independent of \( \mathcal{N}_1, \mathcal{N}_2 \), given
the path of the Markov chain \( X \). Furthermore, given \( X \), the random variables \( Y^i_j, j = 1, 2, \cdots \), are assumed to be independent and identically distributed with the
common conditional distribution \( F^{ij}(y) \). Assume \( F^{ij}(\cdot) = \mathbb{E}^{ij}(\cdot) = \mathbb{E}(\cdot) \)
and the distribution functions \( F^{ij}(\cdot), i = 1, 2, 3, j = 1, \cdots, N \),
satisfy some suitable integrability conditions. That is to say, when \( X(s) = e_j \) for \( s \in (0, t] \), the process \( N_j \) follows a Poisson distribution with parameter \( \lambda_{ij} \), and the corresponding jump amounts have a common distribution \( F^{(j)} \).

Assume the processes of the basic fund price and the protection level are modeled by

\[
F(t) = F_0 e^{Z(t)}, \quad t > 0
\]

with

\[
Z(t) = \int_0^t \left( r(s) - \frac{1}{2} \sigma^2(s) - \lambda_1(s) \xi_1(s) - \lambda_2(s) \xi_2(s) \right) ds + \int_0^t \sigma(s) dW(s) + L^1(t),
\]

and

\[
K(t) = K_0 e^{L^2(t)}.
\]

Here \( F_0 > 0 \) is the primary price of the fund, \( 0 < K_0 < F_0 \) is the initial protection barrier, \( r(t) \) is the risk-free rate, \( \sigma(t) \) is the volatility of the basic fund price at time \( t \), \( \xi_1(t) = \int_0^\infty e^z dF_1^j(z) - 1 < \infty, i = 1, 2, W := \{ W(t) | t \in T \} \) is a standard \( Q \)-Brownian motion. Assume that \( r(t), \sigma(t) \) and \( \xi_i(t) \) are driven by the state of the chain \( X \) as follows:

\[
r(t) = \langle r, X(t) \rangle, \sigma(t) = \langle \sigma, X(t) \rangle, \xi_i(t) = \langle \xi_i, X(t) \rangle
\]

for constant vectors

\[
r = (r_1, r_2, \cdots, r_N)^*, \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)^*, \xi_i = (\xi_{i1}, \xi_{i2}, \cdots, \xi_{iN})^*
\]

with \( \xi_{1j} = \int_0^\infty e^z dF_1^j(z) - 1 < \infty, \xi_{2j} = \int_0^\infty e^{-z} dF_2^j(z) - 1 < \infty \) and \( r_j > 0, \sigma_j > 0 \) for each \( j = 1, 2, \cdots, N \).

We now introduce the DFP. Denote the time-\( t \) account value with dynamic guarantee by \( \hat{F}(t) \). In Gerber and Pafumi (2000)[8], the relation between the two processes \( \hat{F}(t) \) and \( F(t) \) is given by

\[
\hat{F}(t) = F(t) \max \left\{ 1, \max_{0 \leq s \leq t} \frac{K(s)}{F(s)} \right\}.
\]

Hence, the terminal payoff for DFP with a maturity \( T \) is given by

\[
DFP(T) = \hat{F}(T) - F(T) = F(T) \max \left\{ 1, \max_{0 \leq s \leq T} \frac{K_0}{F_0 e^{Z(s) - L^2(s)}} \right\} - F(T).
\]

**Remark 2.1.** Note that, the protection level and the basic fund price process have common upward jumps so that the insurer can provide a higher protection against downside risk for the investors. In fact, DFP can be beneficial in that it protects downside risk for investors. However, in the view of insurers, this implies that they must settle for downside risk that they originally did not face. Therefore, if the protection level is modeled by a stochastic process with downward jumps, then it can help to reduce such an unsatisfying risk for insurers since the value of the protection level will decrease during economic recession. So we can also model the protection level by a regime-switching two-sided jump-diffusion process. But this choice does not bring any essential extension of mathematics.
3. Valuation of DFP in a regime-switching jump-diffusion model. Define the running minimum process:

\[ m_{Z}(t) = \min_{0 \leq s \leq t} (Z(s) - L^2(s)), \]

where \( Z(t) = Z(t) - L^2(t) \). Hence, we can rewrite \( DFP(T) \) as

\[ DFP(T) = F(T) \left( \hat{K} e^{-m_{Z}(T)} - 1 \right)^+, \quad \hat{K} = \frac{K}{F_0} \]

Denote \( DFP_0 \) the value of the dynamic fund protection with maturity \( T \) at time 0. Then, by the fundamental theorem of asset pricing, we have

\[ DFP_0 = E \left[ e^{-\int_0^T r(s)ds} DFP(T) \right] = F_0 E \left[ e^{-\int_0^T r(s)ds + Z(T)} (\hat{K} e^{-m_{Z}(t)} - 1)^+ \right]. \]  \hspace{1cm} (3.1)

To derive the expression for (3.1), we define a new probability \( \hat{Q} \) through Radon-Nikodym derivative

\[
\frac{d\hat{Q}}{dQ} = e^{-\int_0^T r(s)ds + Z(T)} = e^{-\int_0^T (\frac{1}{2} \sigma^2(s) + \lambda_1(s)\xi_1(s) + \lambda_2(s)\xi_2(s))ds + \int_0^T \sigma(s)dW(s) + L^1(T)}.
\]

Then changing measure \( Q \) to \( \hat{Q} \) yields

\[ DFP_0 = F_0 E^{\hat{Q}} \left[ (\hat{K} e^{-m_{Z}(t)} - 1)^+ \right]. \]  \hspace{1cm} (3.2)

In order to value \( DFP \), we shall give the dynamics of \( Z(t) \) under \( \hat{Q} \). From Girsanov theorem, we can obtain the following result.

**Lemma 3.1.** Under the new measure \( \hat{Q} \), the process \( Z(t) \) follows

\[
Z(t) = \int_0^t \mu(s)ds + \int_0^t \sigma(s)d\hat{W}(s) + \sum_{i=1}^{N_1(t)} \hat{Y}^{(1)}_i - \sum_{i=1}^{N_2(t)} \hat{Y}^{(2)}_i - \sum_{i=1}^{N_3(t)} \hat{Y}^{(3)}_i,
\]  \hspace{1cm} (3.3)

where \( \mu(s) = (\mu, X(t)) \), for a vector \( \mu = (\mu_1, \cdots, \mu_N)^\ast \) with \( \mu_i = r_i + \frac{1}{2} \sigma_i^2 - \lambda_{11}\xi_{11} - \lambda_{22}\xi_{22}, i = 1, 2, \cdots, N; \)

(1) \( \hat{W}(t) = W(t) - \int_0^t \sigma(s)ds \) is a \( \hat{Q} \)-standard Brownian motion;

(2) \( \hat{N}_1(t) \) is a regime-switching Poisson process with intensity given by \( \hat{\lambda}_1(t) = (\hat{\lambda}_1, X(t)) \) for a vector \( \hat{\lambda}_1 = (\hat{\lambda}_{11}, \cdots, \hat{\lambda}_{1N})^\ast, \hat{\lambda}_{1j} = \lambda_{1j}(1 + \xi_{1j}); \) given \( X \), the random variables \( \hat{Y}^{(1)}_j, j = 1, 2, \cdots, \) are assumed to be independent and identically distributed with the common conditional distribution \( \hat{F}^{(1)}_j(y) = (\hat{F}^{(1)}_1(y), X(t)) \) for a vector \( \hat{F}^{(1)}_j \) \( \in \mathbb{R}^N \), where \( d\hat{F}^{(1)}_j(y) = \frac{e^{y}dF^{(1)}_j(y)}{1 + \xi_{1j}}, j = 1, 2, \cdots, N; \)

(3) \( \hat{N}_2(t) \) is a regime-switching Poisson process with intensity given by \( \hat{\lambda}_2(t) = (\hat{\lambda}_2, X(t)) \) for a vector \( \hat{\lambda}_2 = (\hat{\lambda}_{21}, \cdots, \hat{\lambda}_{2N})^\ast, \hat{\lambda}_{2j} = \lambda_{2j}(1 + \xi_{2j}); \) given \( X \), the random variables \( \hat{Y}^{(2)}_j, j = 1, 2, \cdots, \) are assumed to be independent and identically distributed with the common conditional distribution \( \hat{F}^{(2)}_j(y) = (\hat{F}^{(2)}_1(y), X(t)) \) for a vector \( \hat{F}^{(2)}_j \) \( \in \mathbb{R}^N \), where \( d\hat{F}^{(2)}_j(y) = \frac{e^{y}dF^{(2)}_j(y)}{1 + \xi_{2j}}, j = 1, 2, \cdots, N; \)

(4) The distributions of the Markov chain \( X \) and the random variables \( \hat{Y}^{(3)}_j, j = 1, 2, \cdots, \) remain the same.

**Proof.** The result can be directly obtained through Girsanov theorem for the jump-diffusion process. \( \square \)
Lemma 3.2. Let \( \tilde{S}(t) = \sum_{i=1}^{N} \tilde{Y}_i^1 - \sum_{i=1}^{N} \tilde{Y}_i^2 - \sum_{i=1}^{N} \tilde{Y}_i^3 \). Then under the new measure \( \tilde{Q} \), the process \( \tilde{S}(t) \) is a regime-switching compound Poisson process, where \( \tilde{N}(t) \) is a regime-switching Poisson process with intensity given by \( \tilde{\lambda}(t) = \tilde{\lambda}_1(t) + \tilde{\lambda}_2(t) \), and given \( X \), the common conditional distribution of the random variables \( Z_j, j = 1, 2, \cdots \) is given by

\[
\tilde{F}_i(y) = (\tilde{F}_1(\cdot), \cdots, \tilde{F}_N(\cdot))^* \in R^N.
\]

Proof. The proof is presented in the appendix.

From Lemmas 3.1, 3.2, we can conclude that the process \( \tilde{Z}(t) \) is also a regime-switching jump-diffusion process under \( \tilde{Q} \). Although it is difficult to give the distribution of the running minimum process \( m_{\tilde{Z}}(t) \), we can obtain the integro-differential system satisfied by the Laplace transforms of the first passage time under the regime-switching jump-diffusion model.

Let \( u > 0 \) be a given constant and define the first passage time by

\[
\tau_u = \inf\{t \geq 0 : u + \tilde{Z}(t) \leq 0\},
\]

with the convention \( \inf\{\phi\} = \infty \). Then we have

\[
\tilde{Q}(m_{\tilde{Z}}(t) \leq -u) = \tilde{Q}(\tau_u \leq t).
\]

For \( u > 0, \delta > 0 \), define

\[
\Phi_i(u, \delta) = E^{\tilde{Q}}[e^{-\delta \tau_u} 1_{\{\tau_u < \infty\}} | X(0) = e_i, u + \tilde{Z}(0) = u], i = 1, 2, \cdots, N.
\]

It is easy to see

\[
\Phi_i(u, \delta) = 1, u \leq 0. \quad (3.4)
\]

To simplify the notation, we drop \( \delta \) in the parameters. Firstly, we derive the integro-differential equations for \( \Phi_i(u), i = 1, 2, \cdots, N \).

**Theorem 3.1.** Let \( u > 0 \). Then, \( \Phi_i(u) \)'s satisfy the integro-differential system

\[
(\delta - a_{ii} + \tilde{\lambda}_i)\Phi_i(u) - \mu_i\Phi_i'(u) - \frac{\sigma_i^2}{2}\Phi_i''(u) - \tilde{\lambda}_i\tilde{F}_i(-u)
\]

\[
-\tilde{\lambda}_i \int_{-u}^{\infty} \Phi_i(u + x)d\tilde{F}_i(x) = \sum_{j=1, j \neq i}^{N} a_{ij}\Phi_j(u). \quad (3.5)
\]

Proof. The proof is presented in the appendix.

In order to obtain a closed form expression for the Laplace transform \( \Phi_i(u) \), in what follows, we consider a continuous-time Markov chain \( X \) with two states (i.e., \( N = 2 \)). We suppose that state \( e_1 \) (state \( e_2 \)) represents a “bad” (“good”) economic state. Therefore, the intensity matrix can be written as

\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix},
\]
where $a_{12} > 0, a_{21} > 0$. Furthermore, for each $j = 1, 2$, we assume the distributions $F^{1j}$ and $F^{2j}$ follow the hyper-exponential distributions with density functions given by

$$f^{1j}(y) = \sum_{i=1}^{m} p_{ij} \alpha_{ij} e^{-\alpha_{ij} y}, \quad y > 0,$$

and

$$f^{2j}(y) = \sum_{i=1}^{m} q_{ij} \beta_{ij} e^{-\beta_{ij} y}, \quad y > 0,$$

where $0 < p_{ij}, q_{ij} < 1, \sum_{i=1}^{m} p_{ij} = 1, \sum_{i=1}^{m} q_{ij} = 1$ and $1 < \alpha_{i1} < \alpha_{i2}, \beta_{i1} < \beta_{i2}$ for each $i = 1, 2, \cdots, m$. The condition $\alpha_{i1} < \alpha_{i2}, \beta_{i1} < \beta_{i2}$ holds due to the fact that the expectation of the jump size corresponding to the “bad” economic state should be greater than that corresponding to the “good” economic state. Then $\xi_{1j} = \sum_{i=1}^{m} p_{ij} \frac{\alpha_{ij}}{\alpha_{ij} - 1} - 1, \xi_{2j} = \sum_{i=1}^{m} q_{ij} \frac{\beta_{ij}}{\beta_{ij} + 1} - 1, j = 1, 2$. For simplicity, we further assume $\alpha_{i2} < \alpha_{i+1}, \beta_{i2} < \beta_{i+1}, i = 1, 2, \cdots, m - 1$.

From Lemma 3.1, we can obtain that the distributions $F^{1j}$ and $F^{2j}$ have the density functions given by

$$\tilde{f}^{1j}(y) = \sum_{i=1}^{m} \tilde{p}_{ij} \tilde{\alpha}_{ij} e^{-\tilde{\alpha}_{ij} y}, \quad y > 0,$$

and

$$\tilde{f}^{2j}(y) = \sum_{i=1}^{m} \tilde{q}_{ij} \tilde{\beta}_{ij} e^{-\tilde{\beta}_{ij} y}, \quad y > 0,$$

where $\tilde{\alpha}_{ij} = \alpha_{ij} - 1, \tilde{\beta}_{ij} = \beta_{ij} + 1, \tilde{p}_{ij} = \frac{1}{1 + \xi_{ij}} p_{ij} \alpha_{ij}, \tilde{q}_{ij} = \frac{1}{1 + \xi_{ij}} q_{ij} \beta_{ij}$.

For each $j = 1, 2$, we assume the distribution $F^{3j}$ follows an exponential distribution with density function given by

$$f^{3j}(y) = \tilde{\beta}_{0j} e^{-\tilde{\beta}_{0j} y}, \quad y > 0,$$

where $\tilde{\beta}_{m2} < \tilde{\beta}_{01} < \tilde{\beta}_{02}$.

Then the distribution $\tilde{F}$ has the density function

$$\tilde{f}(x) = \begin{cases} \sum_{i=1}^{m} p_{ij} \tilde{\alpha}_{ij} e^{-\tilde{\alpha}_{ij} x}, & \text{if } \sum_{i=1}^{m} \tilde{p}_{ij} \tilde{\alpha}_{ij} x < x, \\ \sum_{i=1}^{m} \tilde{q}_{ij} \tilde{\beta}_{ij} e^{\tilde{\beta}_{ij} x}, & \text{if } \sum_{i=1}^{m} \tilde{q}_{ij} \tilde{\beta}_{ij} x < x, \end{cases}$$

where $p_{ij} = \frac{\lambda_{ij}}{\lambda_{ij} + \lambda_{2j}}, \tilde{p}_{ij} = \frac{\tilde{\lambda}_{ij}}{\tilde{\lambda}_{ij} + \tilde{\lambda}_{0j}}, \tilde{q}_{ij} = \frac{\tilde{\lambda}_{ij}}{\lambda_{ij} + \lambda_{2j}} \tilde{q}_{ij}$.

From Theorem 3.1, we can directly obtain the following result.

**Corollary 3.1.** Let $u > 0$. Then, $\Phi_1(u)$’s satisfy the integro-differential system

$$\begin{align*}
(\delta - a_{11} + \tilde{\lambda}_1) \Phi_1(u) - \mu_1 \Phi_1'(u) - \frac{\sigma_2^2}{2} \Phi_2''(u) - \tilde{\lambda}_1 \int_{-\infty}^{-u} \tilde{f}(x) dx \\
- \tilde{\lambda}_1 \int_{-u}^{\infty} \Phi_1(u + x) \tilde{f}(x) dx = -a_{11} \Phi_2(u).
\end{align*}$$

(3.7)
and

\[
(\delta - a_{22} + \tilde{\lambda}_2)\Phi_2(u) - \mu_2 \Phi_2''(u) - \frac{\sigma_2^2}{2} \Phi_2''(u) - \tilde{\lambda}_2 \int_{-\infty}^{\infty} \Phi_2(u + x) f^2(x) dx
\]

\[
- \tilde{\lambda}_2 \int_{-\infty}^{\infty} \Phi_2(u + x) f^2(x) dx = -a_{22} \Phi_1(u). \tag{3.8}
\]

In order to solve (3.7)-(3.8), we let I and D denote the identity operator and the differential operator, respectively. Define the differential operator polynomial operators

\[
h_i(D) = \frac{1}{2} \sigma_i^2 D^2 + \mu_i D - (\delta - a_{ii} + \tilde{\lambda}_i) I.
\]

Inserting (3.6) into (3.7) and (3.8) gives

\[
h_1(D) \Phi_1(u) = a_{11} \Phi_2(u) - \tilde{\lambda}_1 \left( \sum_{i=0}^{m} \eta_{1i} e^{-\tilde{\beta}_{1i} u} + \int_{0}^{u} \Phi_1(s) \sum_{i=0}^{m} \eta_{1i} \tilde{\beta}_{1i} e^{\tilde{\beta}_{1i} (s-u)} ds \right)
\]

\[
+ \int_{u}^{\infty} \Phi_1(s) \sum_{i=1}^{m} \eta_{1i} \tilde{\alpha}_{1i} e^{-\tilde{\alpha}_{1i} (s-u)} ds, \tag{3.9}
\]

and

\[
h_2(D) \Phi_2(u) = a_{22} \Phi_1(u) - \tilde{\lambda}_2 \left( \sum_{i=0}^{m} \eta_{2i} e^{-\tilde{\beta}_{2i} u} + \int_{0}^{u} \Phi_2(s) \sum_{i=0}^{m} \eta_{2i} \tilde{\beta}_{2i} e^{\tilde{\beta}_{2i} (s-u)} ds \right)
\]

\[
+ \int_{u}^{\infty} \Phi_2(s) \sum_{i=1}^{m} \eta_{2i} \tilde{\alpha}_{2i} e^{-\tilde{\alpha}_{2i} (s-u)} ds. \tag{3.10}
\]

Similar to Gerber and Shiu (2005) and Dong et al. (2011), applying the differential operator polynomials \( \prod_{i=1}^{m} (D + \tilde{\beta}_{1i} I)(D - \tilde{\alpha}_{1i} I) \) and \( \prod_{i=1}^{m} (D + \tilde{\beta}_{01} I)(D - \tilde{\alpha}_{2i} I)(D + \tilde{\beta}_{02} I) \) to the both sides of (3.9) and (3.10), respectively,

\[
\prod_{i=1}^{m} [(D + \tilde{\beta}_{1i} I)(D - \tilde{\alpha}_{1i} I)](D + \tilde{\beta}_{01} I) h_1(D) \Phi_1(u)
\]

\[
= a_{11} \prod_{i=1}^{m} [(D + \tilde{\beta}_{1i} I)(D - \tilde{\alpha}_{1i} I)](D + \tilde{\beta}_{01} I) \Phi_2(u)
\]

\[
+ \tilde{\lambda}_1 \sum_{i=1}^{m} \eta_{1i} \tilde{\alpha}_{1i} \left( (D + \tilde{\beta}_{01} I)(D + \tilde{\beta}_{1i} I) \prod_{j=1, j \neq i}^{m} [(D + \tilde{\beta}_{j1} I)(D - \tilde{\alpha}_{j1} I)] \right) \Phi_1(u)
\]

\[
- \tilde{\lambda}_1 \sum_{i=1}^{m} \eta_{1i} \tilde{\beta}_{1i} \left( (D - \tilde{\alpha}_{1i} I)(D + \tilde{\beta}_{01} I) \prod_{j=1, j \neq i}^{m} [(D + \tilde{\beta}_{j1} I)(D - \tilde{\alpha}_{j1} I)] \right) \Phi_1(u)
\]

\[
- \tilde{\lambda}_1 \eta_{01} \tilde{\beta}_{01} \prod_{j=1}^{m} [(D + \tilde{\beta}_{j1} I)(D - \tilde{\alpha}_{j1} I)] \Phi_1(u), \tag{3.11}
\]
and
\[
\prod_{i=1}^{m}[(D + \tilde{\beta}_{i,2}I)(D - \hat{\alpha}_{i,2}I)\Phi_2(u) + a_{22} \prod_{i=1}^{m}[(D + \tilde{\beta}_{0,2}I)(D - \hat{\alpha}_{0,2}I)\Phi_1(u)
\]
\[
+ \dot{\lambda}_2 \sum_{i=1}^{m} \tilde{\beta}_{i,2}\tilde{\alpha}_{i,2} \prod_{j=1,j\neq i}^{m} ((D + \tilde{\beta}_{j,2}I)(D - \hat{\alpha}_{j,2}I)\Phi_2(u)
\]
\[
- \dot{\lambda}_2 \sum_{i=1}^{m} \tilde{\alpha}_{i,2}\tilde{\beta}_{i,2} \prod_{j=1,j\neq i}^{m} ((D - \hat{\alpha}_{i,2}I)(D + \tilde{\beta}_{i,2}I)\Phi_2(u)
\]
\[
- \tilde{\lambda}_2 \tilde{\alpha}_{0,2}\tilde{\beta}_{0,2} \prod_{j=1}^{m}((D + \tilde{\beta}_{j,2}I)(D - \hat{\alpha}_{j,2}I)) \Phi_2(u).
\] (3.12)

Define operators
\[
\hat{h}_j(D) = \prod_{i=1}^{m}[(D + \tilde{\beta}_{i,j}I)(D - \hat{\alpha}_{i,j}I)\Phi_2(D)]
\]
\[
- \dot{\lambda}_j \left( \sum_{i=1}^{m} \tilde{\alpha}_{i,j}(D + \tilde{\beta}_{i,j}I)(D - \hat{\alpha}_{i,j}I) \prod_{l=1, l\neq i}^{m} \Phi_2(D - \hat{\alpha}_{l,j}I)\right)
\]
\[
- \sum_{i=1}^{m} \tilde{\beta}_{i,j}(D + \tilde{\beta}_{i,j}I)(D - \hat{\alpha}_{i,j}I) \prod_{l=1, l\neq i}^{m} \Phi_2(D - \hat{\alpha}_{l,j}I)
\]
\[
- \tilde{\alpha}_{0,j}\tilde{\beta}_{0,j} \prod_{l=1}^{m} \Phi_2(D - \hat{\alpha}_{l,j}I)\right), j = 1, 2.
\]

Similarly, we define
\[
\tilde{h}_j(x) = \prod_{i=1}^{m}(x + \tilde{\beta}_{i,j})(x - \hat{\alpha}_{i,j})(x - \tilde{\alpha}_{i,j})h_j(x)
\]
\[
- \dot{\lambda}_j \left( \sum_{i=1}^{m} \tilde{\alpha}_{i,j}(x + \tilde{\beta}_{i,j})(x + \tilde{\alpha}_{i,j}) \prod_{l=1, l\neq i}^{m} \Phi_2(x - \hat{\alpha}_{l,j})\right)
\]
\[
- \sum_{i=1}^{m} \tilde{\beta}_{i,j}(x + \tilde{\beta}_{i,j})(x - \hat{\alpha}_{i,j}) \prod_{l=1, l\neq i}^{m} \Phi_2(x - \hat{\alpha}_{l,j})
\]
\[
- \tilde{\alpha}_{0,j}\tilde{\beta}_{0,j} \prod_{l=1}^{m} \Phi_2(x - \hat{\alpha}_{l,j})\right), j = 1, 2,
\]

where \( h_i(x) = \frac{1}{2} \sigma_i^2 x^2 + \mu_i x - (\delta - \alpha_{ii} + \lambda_i). \)

By using \( \bar{h}_i(D) \), Eqs. (3.11) and (3.12) become
\[
\hat{h}_1(D)\Phi_1(u) = a_{11} \prod_{i=1}^{m}[(D + \tilde{\beta}_{i,1}I)(D - \hat{\alpha}_{i,1}I)(D + \tilde{\beta}_{0,1}I)\Phi_2(u), \quad (3.13)
\]
Then it follows from (3.13) and (3.14) that
\[
\hat{h}_2(D)\Phi_2(u) = a_{22} \prod_{i=1}^{m} (D + \tilde{\beta}_i)(D - \tilde{\alpha}_i)(D + \tilde{\beta}_0)\Phi_1(u).
\] (3.14)

Then it follows from (3.13) and (3.14) that
\[
\hat{h}_2(D)\hat{h}_1(D)\Phi_1(u) = a_{11} a_{22} \prod_{j=1}^{m} (D + \tilde{\beta}_j)(D - \tilde{\alpha}_j)(D + \tilde{\beta}_0)\Phi_1(u),
\] (3.15)
and
\[
\hat{h}_1(D)\hat{h}_2(D)\Phi_2(u) = a_{11} a_{22} \prod_{j=1}^{m} (D + \tilde{\beta}_j)(D - \tilde{\alpha}_j)(D + \tilde{\beta}_0)\Phi_2(u).
\] (3.16)

The characteristic equation of (3.15) and (3.16) is
\[
\hat{h}_1(x)\hat{h}_2(x) = a_{11} a_{22} \prod_{j=1}^{m} (x + \tilde{\beta}_j)(x - \tilde{\alpha}_j)(x + \tilde{\beta}_0)\Phi_1(u) = 0.
\] (3.17)

Define
\[
\hat{h}_j(x) = \frac{\sigma_j^2}{2} x^2 + \mu_j x + \lambda_j \sum_{i=1}^{m} \frac{\beta_{ij} \alpha_{ij}}{\alpha_{ij} - x} + \sum_{i=0}^{m} \frac{\beta_{ij} \tilde{\beta}_0}{\beta_{ij} + x} - (\delta - a_{jj} + \tilde{\lambda}_j).
\]

Then (3.17) can be written as
\[
\hat{h}_1(x)\hat{h}_2(x) = a_{11} a_{22}.
\] (3.18)

In order to derive the solution for \(\Phi_j(u)\), it remains to investigate the roots of the equation (3.18). Dong et al. (2011) have proved that the equation
\[
\hat{h}_j(x) = 0
\]
has \(2m + 3\) roots satisfying
\[
-\infty < y_{m+2j} < -\tilde{\beta}_0 < y_{m+1j} < -\tilde{\beta}_m < y_{m+1j} < \cdots < -\tilde{\beta}_1 < y_{1j} < 0 < x_{1j} < \tilde{\alpha}_0 < \cdots < x_{m+1j} < \tilde{\alpha}_m < x_{m+1j} < \infty.
\]

For simplicity, we only consider the case when all \(x_{ij}\)'s are distinct and all \(y_{ij}\)'s are also distinct since the analysis of the other case is more tedious.

**Lemma 3.3.** For \(\delta > 0\), the equation (3.18) has \(2m + 2\) distinct positive real roots, and \(2m + 4\) distinct negative real roots.

**Proof.** The proof is presented in the appendix. \(\square\)

Denoted by \(\rho_1 > \rho_2 > \cdots > \rho_{2m+4}\) the \(2m + 4\) distinct negative real roots of the equation (3.18). Note that \(\lim_{u \to +\infty} \Phi_1(u) = 0\). Then the Laplace transforms \(\Phi_1(u)\) and \(\Phi_2(u)\) have the forms
\[
\Phi_1(u) = \sum_{j=1}^{2m+4} c_j e^{\rho_j u}, \quad \Phi_2(u) = \sum_{j=1}^{2m+4} d_j e^{\rho_j u},
\]
where \(c_j\)’s, \(d_j\)’s are arbitrary constants.
We first determine the coefficients \( c_j \)'s. Substituting \( \Phi_1(u) = \sum_{j=1}^{2m+4} c_j e^{\rho_j u} \) into (3.5) yields

\[
a_{11} \Phi_2(u) = \sum_{j=1}^{2m+4} c_j e^{\rho_j u} \hat{h}_1(\rho_j) + \lambda_1 \sum_{i=0}^{2m+2} q_{1i} e^{-\tilde{\beta}_{11} u} (1 - \sum_{j=1}^{2m+4} \frac{c_j \hat{\beta}_{11}}{\beta_{11} + \rho_j}) (3.19)
\]

Inserting Eq. (3.19) into Eq. (3.10) and equating the coefficients of \( e^{-\tilde{\beta}_{ij} u} \), \( i = 0, 1, \ldots, m \) yield

\[
\begin{align*}
\left\{ \begin{array}{l}
a_{11} - \sum_{j=1}^{2m+4} \frac{c_j \hat{\beta}_{11} \hat{h}_1(\rho_j)}{\beta_{11} + \rho_j} - \sum_{i=0}^{m} \lambda_i \frac{\hat{\beta}_{ij} \hat{q}_{ij}}{\beta_{ij} - \beta_{11}} (1 - \sum_{j=1}^{2m+4} \frac{c_j \hat{\beta}_{11}}{\beta_{11} + \rho_j}) = 0 \\
(1 - \sum_{j=1}^{2m+4} \frac{c_j \hat{\beta}_{11}}{\beta_{11} + \rho_j}) \hat{h}_2(-\tilde{\beta}_{11}) = 0
\end{array} \right.
\end{align*}
\]

Note that \( \hat{h}_2(-\tilde{\beta}_{11}) \neq 0 \). Therefore, we obtain that

\[
\sum_{j=1}^{2m+4} \frac{c_j \hat{\beta}_{11}}{\beta_{11} + \rho_j} = 1, i = 0, 1, \ldots, m,
\]

(3.20)

and

\[
\sum_{j=1}^{2m+4} \frac{c_j \hat{\beta}_{11} \hat{h}_1(\rho_j)}{\beta_{11} + \rho_j} = a_{11}, i = 0, 1, \ldots, m.
\]

(3.21)

Furthermore, letting \( u = 0 \) in Eq. (3.19) and using Eq. (3.20), with the boundary condition \( \Phi_2(0) = 1 \) yield that

\[
\sum_{j=1}^{2m+4} c_j = 1,
\]

(3.22)

and

\[
\sum_{j=1}^{2m+4} c_j \hat{h}_1(\rho_j) = a_{11}.
\]

(3.23)

Let \( H \) denote the \((2m+4) \times (2m+4)\) coefficient matrix of the linear system (3.20)-(3.23). We have

\[
H = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\hat{h}_1(\rho_1) & \hat{h}_1(\rho_2) & \cdots & \hat{h}_1(\rho_{2m+4}) \\
\beta_{01} & \beta_{01} & \cdots & \beta_{01} \\
\beta_{01} + \rho_1 & \beta_{01} + \rho_2 & \cdots & \beta_{01} + \rho_{2m+4} \\
\cdots & \cdots & \cdots & \cdots \\
\beta_{11} & \beta_{11} & \cdots & \beta_{11} \\
\beta_{11} + \rho_1 & \beta_{11} + \rho_2 & \cdots & \beta_{11} + \rho_{2m+4} \\
\cdots & \cdots & \cdots & \cdots \\
\beta_{m1} & \beta_{m1} & \cdots & \beta_{m1} \\
\beta_{m1} + \rho_1 & \beta_{m1} + \rho_2 & \cdots & \beta_{m1} + \rho_{2m+4} \\
\cdots & \cdots & \cdots & \cdots \\
\beta_{02} & \beta_{02} & \cdots & \beta_{02} \\
\beta_{02} + \rho_1 & \beta_{02} + \rho_2 & \cdots & \beta_{02} + \rho_{2m+4} \\
\cdots & \cdots & \cdots & \cdots \\
\beta_{m2} & \beta_{m2} & \cdots & \beta_{m2} \\
\beta_{m2} + \rho_1 & \beta_{m2} + \rho_2 & \cdots & \beta_{m2} + \rho_{2m+4}
\end{pmatrix}
\]
Denote the determinant of the matrix $H$ by $\det H$. Let $H_j$ denote the matrix obtained from $H$ by replacing its $j$th column by column vector 

$$(1, a_{11}, 1, \cdots, 1, a_{11}, \cdots, a_{11})^*.$$ 

Parallel to the first passage time distribution under the regime-switching Brownian motion framework in Guo (2001)[12] and under the regime-switching double-exponential jump-diffusion model in Siu et al. (2015)[21], the uniqueness and existence of the solution to the linear system (3.20)-(3.23) can be established by noting that the linear system (3.20)-(3.23) is the Vandermonde system of linear equations, which is uniquely solvable if and only if Eq. (3.18) has $2m + 4$ distinct negative real roots. Therefore, $H$ is non-singular and we have

$$c_j = (\det H)^{-1}(\det H_j), j = 1, 2, \cdots, 2m + 4. \quad (3.24)$$

Similarly, $d_j$’s satisfy

$$\left\{ \begin{array}{ll}
\sum_{j=1}^{2m+4} d_j &= 1, \\
\sum_{j=1}^{2m+4} d_j \hat{h}_2(\rho_j) &= a_{22}, \\
\sum_{j=1}^{2m+4} d_j \hat{\beta}_{12} &= 1, i = 0, 1, \cdots, m, \\
\sum_{j=1}^{2m+4} d_j \hat{\beta}_{11} \hat{h}_2(\rho_j) &= a_{22}, i = 0, 1, \cdots, m.
\end{array} \right. \quad (3.25)$$

Let $G$ denote the $(2m + 4) \times (2m + 4)$ coefficient matrix of the above linear system. We have

$$G = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\hat{h}_2(\rho_1) & \hat{h}_2(\rho_2) & \cdots & \hat{h}_2(\rho_{2m+4}) \\
\beta_{02} & \beta_{02} & \cdots & \beta_{02} \\
\beta_{02} \hat{h}_2(\rho_1) & \beta_{02} \hat{h}_2(\rho_2) & \cdots & \beta_{02} \hat{h}_2(\rho_{2m+4}) \\
\beta_{12} & \beta_{12} & \cdots & \beta_{12} \\
\beta_{12} \hat{h}_2(\rho_1) & \beta_{12} \hat{h}_2(\rho_2) & \cdots & \beta_{12} \hat{h}_2(\rho_{2m+4}) \\
\beta_{m2} & \beta_{m2} & \cdots & \beta_{m2} \\
\beta_{m2} \hat{h}_2(\rho_1) & \beta_{m2} \hat{h}_2(\rho_2) & \cdots & \beta_{m2} \hat{h}_2(\rho_{2m+4}) \\
\beta_{11} h_2(\rho_1) & \beta_{11} h_2(\rho_2) & \cdots & \beta_{11} h_2(\rho_{2m+4}) \\
\beta_{11} \hat{h}_2(\rho_1) & \beta_{11} \hat{h}_2(\rho_2) & \cdots & \beta_{11} \hat{h}_2(\rho_{2m+4}) \\
\beta_{m1} h_2(\rho_1) & \beta_{m1} h_2(\rho_2) & \cdots & \beta_{m1} h_2(\rho_{2m+4}) \\
\beta_{m1} \hat{h}_2(\rho_1) & \beta_{m1} \hat{h}_2(\rho_2) & \cdots & \beta_{m1} \hat{h}_2(\rho_{2m+4})
\end{pmatrix}$$

Denote the determinant of the matrix $G$ by $\det G$. Let $G_j$ denote the matrix obtained from $G$ by replacing its $j$th column by column vector 

$$(1, a_{22}, 1, \cdots, 1, a_{22}, \cdots, a_{22})^*.$$ 

By using the same arguments as in the proof of the non-singularity of the matrix $H$, we can prove the matrix $G$ is invertible. Then we have

$$d_j = (\det G)^{-1}(\det G_j), j = 1, 2, \cdots, 2m + 4. \quad (3.26)$$
Let $\pi_1 = Q(X(0) = e_1)$ and $\pi_2 = Q(X(0) = e_2)$ with $\pi_1 + \pi_2 = 1$. Note that, the distribution of the Markov chain $X$ under $Q$ remains the same. We have

$$E^Q[e^{-\delta \tau_\infty}] = \pi_1 \Phi_1(u) + \pi_2 \Phi_2(u) = \sum_{j=1}^{2m+4} (\pi_1 c_j + \pi_2 d_j) e^{\rho_j u}. \tag{3.27}$$

Now we are ready to derive the Laplace transform of the value of DFP at time 0. Firstly, we present a lemma.

**Lemma 3.4.** For any $t > 0$, $\lim_{y \to -\infty} e^{-y \hat{Q}(\tau \infty(t) \leq y)} = 0$.

**Proof.** The proof is presented in the appendix. \hfill \square

**Theorem 3.2.** For any $\delta > 0$, the Laplace transform of $DFP_0$ is given by

$$\int_0^\infty e^{-\delta t} DFP_0 dt = -\frac{K_0}{\delta} \sum_{i=1}^{2m+4} \frac{\pi_1 c_i + \pi_2 d_i}{\rho_i + 1} \left(\frac{F_0}{K_0}\right)^{(\rho_i + 1)}, \tag{3.28}$$

where $c_i, d_i$ are defined in (3.24) and (3.26), respectively.

**Proof.** The proof is presented in the appendix. \hfill \square

4. **Numerical results.** In this section we present some numerical results for the value of DFP at time 0 by inverting (3.28) via the Gaver-Stehfest algorithm, which is used in Kou and Wang (2003)[18] and Dong et al. (2011)[4]. For the details of the implementation of the Gaver-Stehfest algorithm, we refer to Section 5 in [18] or Section 4 in [4].

For all the computations, the values of certain parameters are held fixed except otherwise indicated: we take $T = 5$, $F_0 = 100$, $K_0 = 70$, $a_{11} = a_{22} = -0.1$, $r_1 = 0.02$, $r_2 = 0.05$, $\sigma_1 = 0.6$, $\sigma_2 = 0.3$, $\lambda_{11} = \lambda_{21} = 2$, $\lambda_{12} = \lambda_{22} = 1$, $m = 1$, $\alpha_{11} = \beta_{11} = 25$, $\alpha_{12} = \beta_{12} = 50$, $\alpha_{21} = 30$, $\beta_{22} = 60$. For the distribution of $X(0)$, we consider two choices: $\pi_1 = 1$, $\pi_2 = 0$ and $\pi_1 = 0$, $\pi_2 = 1$, which correspond to the cases that the chain starts from $e_1$ ("good" economic state) and $e_2$ ("bad" economic state), respectively.

Figures 1-6 present the effects of maturity $T$ and model parameters on the values of $DFP_0$ for $\pi_1 = 1$ and $\pi_1 = 0$. From them we can see the value of $DFP_0$ is much larger for $\pi_1 = 1$. That is to say, the value of $DFP_0$ is much higher when we start at the state $e_1$ ("bad" economic state) at time $t = 0$.

![Figure 1. $DFP_0$ versus $T$](image1)

![Figure 2. $DFP_0$ versus $F_0$](image2)
Figure 1 displays the relationship between $DFP_0$ and $T$ for the stochastic protection level and the constant protection level. From it we can observe that $DFP_0$ increases with $T$. This is because a longer protection period can lead to a larger protection cost. We can also see that the value of $DFP_0$ for the stochastic protection level with upward jumps is much larger than that of the constant protection level. This is because the insurer provides a higher protection against the downside risk for the investors.

Figure 2 plots $DFP_0$ as a function of $F_0$. It can be seen that $DFP_0$ is a decreasing function of $F_0$. The reason is that a higher value of $F_0$ leads to a decreasing probability of the asset price hitting the protection level.

Figure 3 represents the relationship between $K_0$ and $DFP_0$. It can be seen that $DFP_0$ increases with $K_0$, since a higher value of $K_0$ leads to a higher protection level.

Figure 4 presents the impact of transition intensity $a_{12}$ on $DFP_0$ with $a_{21} = a_{12}$. From it we can conclude that a higher $a_{12}$ with $a_{12} = a_{21}$ results in a higher value of $DFP_0$ if we start at the “good” economy at time $t = 0$. This is because higher $a_{21}$ leads to an increasing probability of switching to the “bad” economy. On the other hand, if we start at the “bad” economy at time $t = 0$, the value of $DFP_0$ decreases with $a_{12}$. This is due to an increasing probability of switching to the “good” economy.

Figure 5. $DFP_0$ versus $\lambda_{11}$

Figure 6. $DFP_0$ versus $\sigma_1$
Figure 5 plots the impact of jump intensity \( \lambda_{11} \) on the values of \( DPF_0 \) with \( \lambda_{11} = 2 \lambda_{12} \). It can be seen that \( DPF_0 \) increases with jump intensity, which implies that a higher jump risk can lead to a larger protection cost.

Figure 6 graphs \( DPF_0 \) as a function of the basic fund’s volatility \( \sigma_1 \) with \( \sigma_1 = 2 \sigma_2 \). From it we can conclude that \( DPF_0 \) is an increasing function of the volatility, in line with stylized features and financial intuition: the option price increases as the volatility increases.

To sum up, numerical results indicate that changes of market regimes and the stochastic protection level have material effects on \( DPF_0 \).

5. Conclusions. In this paper, we investigate the valuation of DFP under a regime-switching jump-diffusion model. Furthermore, we consider a stochastic protection level with positive jumps, which provide a higher protection, so that it may be more appealing to the investors. Since the Laplace transform of the price of DFP is associated with the Laplace transform of the first passage time, we derive integro-differential system for the Laplace transforms of the first passage time. Explicit solutions for the system are obtained when the jumps follow a regime-switching hyper-exponential distribution. Based on the results, we give numerical calculations for the value of DFP by inverting Laplace transforms via Gaver-Stehfest algorithm. Numerical results illustrate the regime-switching effects have a significant effect on the value of DFP. Therefore, we should incorporate changes of market regimes into models for pricing long term insurance products.

There remain many open questions for the valuation of DFP under regime-switching models. For example, the results obtained under a regime-switching hyper-exponential jump-diffusion model are useful for the valuation of DFP, and can also provide a useful benchmark for more complicated regime-switching models. Based on the derived results, we may investigate the pricing of DFP under a general regime-switching jump-diffusion model by resorting to some numerical procedures. Note that the pricing measure in this paper is fixed. We may investigate how to choose a risk-neutral measure since there are many risk-neutral probability measures. We leave these and other questions for future research.

6. Appendix. Proof of Lemma 3.2. For \( y \in R \), let

\[
G(y) = (G_1(y), \cdots, G^N(y))^*,
\]

where

\[
G^i(y) = \tilde{\lambda_{1i}} \int_0^\infty e^{yz} d\tilde{F}^{1i}(z) \int_0^\infty e^{-yz} dF^{3i}(z) + \tilde{\lambda_{2i}} \int_0^\infty e^{-yz} d\tilde{F}^{2i}(z) - 1, \quad (A.1)
\]

with \( \int_0^\infty e^{yz} d\tilde{F}^{1i}(z) < \infty, \int_0^\infty e^{-yz} dF^{3i}(z) < \infty, \int_0^\infty e^{-yz} dF^{2i}(z) < \infty, \) for \( i = 1, \cdots, N \).

Since the path of the Markov chain \( (X_s)_{s \leq t} \) is known to us, we denote the jump times in the interval \([0, t]\) of the Markov chain \( X_s \) by \( 0 = \bar{T}_0 < \bar{T}_1 < \cdots < \bar{T}_k = t \).
Then we have that
\[
E^Q \left[ e^{\sum_{i=1}^{N(t)} \tilde{Y}_i - \sum_{i=1}^{N(t)} Y_i^2 - \sum_{i=1}^{N(t)} Y_i^3} \right] = \prod_{l=1}^{k} \exp\left\{ \lambda(l) (T_l - T_{l-1}) \int_{0}^{\infty} e^{\gamma x} d\hat{F}_l(x) \right\} \int_{0}^{\infty} e^{-\gamma x} d\hat{F}_l(x) \right\}
\]
\[
\times \exp\left\{ \int_{0}^{T_l - T_{l-1}} e^{-\gamma x} d\hat{F}_l(x) \right\}
\]
\[
= \exp\left\{ \int_{0}^{T_l - T_{l-1}} (G(y), X(s)) ds \right\}
\] (A.2)

From [1], we obtain
\[
E^Q \left[ e^{\frac{\lambda(t)}{2}} \right] = \exp\left\{ (\text{diag}(G(y)) + A) t \right\} X_0,
\] (A.3)
where \( \text{diag}(\theta) \) denotes a diagonal matrix with the diagonal entries given by the vector \( \theta \) and \( 1 = (1, \ldots, 1)^t \). Similarly, we can also prove that
\[
E^Q \left[ e^{\sum_{i=1}^{N(t)} Z_i} \right] = \exp\left\{ (\text{diag}(G(y)) + A) t \right\} X_0,
\] (A.4)
which concludes the proof. \( \square \)

**Proof of Theorem 3.1.** Considering a small time interval \([0, h]\), with \( h > 0 \), there are four possible events regarding to the occurrence of the jump and the change of the environment: (i) no jump and no change of environment occur in \([0, h]\), (ii) a jump occurs in \([0, h]\) (it can either cause hitting the barrier or not), (iii) the environment changes in \([0, h]\), and (iv) two or more events occur in \([0, h]\).

Conditioning on the event occurs in the interval \([0, h]\), we have
\[
\Phi_i(u) = (1 - \lambda_i h + a_i h) e^{-\delta h} E^Q[\Phi_i(u + \mu_i h + \sigma_i W(h))]
\]
\[
+ e^{-\delta h} \lambda_i h E^Q \left[ \int_{-\infty}^{\infty} \Phi_i(u + \mu_i h + \sigma_i W(h) + x) d\hat{F}_i(x) \right]
\]
\[
+ \int_{-\infty}^{\infty} d\hat{F}_i(x)
\]
\[
+ \sum_{j=1, j \neq i}^{N} a_{ij} h e^{-\delta h} E^Q[\Phi_j(u + \mu_i h + \sigma_i W(h))] + o(h),
\] (A.5)

By Taylor’s expansion, we have
\[
E^Q[\Phi_i(u + \mu_i h + \sigma_i W(h))] = \Phi_i(u) + \mu_i \Phi_i'(u) h + \frac{\sigma_i^2}{2} \Phi_i''(u) h + o(h),
\]
and
\[
E^Q[\Phi_j(u + \mu_i h + \sigma_i W(h))] = \Phi_j(u) + \mu_i \Phi_j'(u) h + \frac{\sigma_i^2}{2} \Phi_j''(u) h + o(h).
\]

Plugging the above formulas into (A.5), dividing both sides by \( h \) and letting \( h \to 0 \) we obtain (3.5). \( \square \)
Proof of Lemma 3.3. Let
\[ h(x) = \hat{h}_1(x)\hat{h}_2(x) - a_{11}a_{22}. \]
It is easy to conclude that \( h(0) = (\delta - a_{11} + \hat{\lambda}_1)(\delta - a_{22} + \hat{\lambda}_2) - a_{11}a_{22} > 0, \)
\[ h(x_{ij}) = h(y_{ij}) = -a_{11}a_{22} < 0, h(-\infty) = +\infty, h(+\infty) = +\infty, \]
\[ h(\hat{\alpha}_{ij}+ - \hat{\beta}_{ij}+)h(\hat{\beta}_{ij}+) = -\infty. \]
In what follows, we focus only on arguing that for \( \delta > 0, \) the function \( h(x) = 0 \) has \( 2m + 2 \) distinct positive roots, since the proof for \( h(x) = 0 \) having \( 2m + 4 \) distinct negative roots is similar.

Let \( k_{i1} \) count the number of all of \( x_{ij} \)'s in the interval \( (\hat{\alpha}_{i1}, \hat{\alpha}_{i2}) \) for each \( i = 1, \ldots, m \) and let \( k_{i2} \) count the number of all of \( x_{ij} \)'s in the interval \( (\hat{\alpha}_{i2}, \hat{\alpha}_{i+1}) \) for each \( i = 0, \ldots, m \) with \( \hat{\alpha}_{02} = 0 \) and \( \hat{\alpha}_{m+1} = \infty \). Then it should hold that \( \sum_{i=1}^{m} (k_{i1} + k_{i2}) + k_{02} = 2m + 2 \). Furthermore, \( 0 \leq k_{i1} \leq 2 \) for each \( i = 1, \ldots, m, \)
\[ 0 \leq k_{i2} \leq 2 \]
for each \( i = 1, \ldots, m - 1 \) and \( 1 \leq k_{02}, k_{m+2} \leq 2. \)
For \( i = 1, \ldots, m, \) if \( k_{i1} = 0, \) then we have
\[ h(\hat{\alpha}_{i1}+) = h(\hat{\alpha}_{i2}-) = -\infty; \]
If \( k_{i1} = 1, \) then
\[ h(\hat{\alpha}_{i1}+) = \infty, h(\hat{\alpha}_{i2}-) = -\infty. \]
Recall that \( h(x_{ij}) = h(y_{ij}) = -a_{11}a_{22} < 0. \) Thus, there exists at least one root at the interval \( (\hat{\alpha}_{i1}, \hat{\alpha}_{i2}); \)
If \( k_{i1} = 2, \) then
\[ h(\hat{\alpha}_{i1}+) = h(\hat{\alpha}_{i2}-) = +\infty, \]
which implies there exists at least two roots at the interval \( (\hat{\alpha}_{i1}, \hat{\alpha}_{i2}). \)

Therefore, there exists at least \( \sum_{i=1}^{m} k_{i1} \) roots in the intervals \( (\hat{\alpha}_{i1}, \hat{\alpha}_{i2}), i = 1, \ldots, m \)
(See Figure 7). Similarly, we can prove that there there exists at least \( \sum_{i=0}^{m} k_{i2} \) roots in the intervals \( (\hat{\alpha}_{i2}, \hat{\alpha}_{i+1}), i = 0, 1, \ldots, m, \) which yields that there exists at least \( 2m + 2 \) roots in the real right half-line. By using the same arguments, we can also prove that there exists at least \( 2m + 4 \) roots in the real left half-line. Then the result follows from the fact that \( h(x) \) is a polynomial of degree \( 4m + 6. \)

Proof of Lemma 3.4. The proof is similar to that of Lemma 2.6 in [16]. Fix a \( \theta > 1, \) then
\[ E^Q[e^{-\theta Z(t)}] = E^Q[E^Q[e^{-\theta Z(t)}|\mathcal{F}_t]] \]
\[ = E^Q[e^{\int_0^t (-\theta \mu(s) + \frac{1}{2} \theta^2 \sigma_i^2(s) + \hat{\lambda}(\eta(s)-1))ds}], \]
where \( \eta(s) = \langle \eta, X(s) \rangle \) for a vector \( \eta = (\eta^1, \ldots, \eta^N)^* \) with \( \eta^i = \int_{-\infty}^{\infty} e^{-\theta y} d\tilde{F}^i(y) < \infty. \)
Let \( J_i \) denote the occupation time of the Markov chain \( X \) in state \( e_i \) over \([0, t], \) and \( M = \max_i | -\theta \mu_i + \frac{1}{2} \theta^2 \sigma_i^2 + \hat{\lambda}(\eta^i - 1)|. \)
Then
\[ E^Q[e^{-\theta Z(t)}] = E^Q[e^{\sum_{i=1}^N (-\theta \mu_i + \frac{1}{2} \theta^2 \sigma_i^2 + \hat{\lambda}(\eta^i - 1))J_i}] \leq e^{M t}. \]
Figure 7. For example, if \( k_{i1} = 1, i = 1, \cdots, m, \ k_{i2} = 1, i = 0, 1, \cdots, m - 1, k_{m2} = 2 \), then we have \( h(\hat{\alpha}_{ij}^{-}) = -\infty, h(\hat{\alpha}_{ij}^{+}) = +\infty \). Therefore, there exists at least one root at each of the \( 2m \) intervals, \((0, \hat{\alpha}_{11}), (\hat{\alpha}_{11}, \hat{\alpha}_{12}), (\hat{\alpha}_{12}, \hat{\alpha}_{21}), \cdots, (\hat{\alpha}_{m1}, \hat{\alpha}_{m2})\) and there exists at least two roots at the interval \((\hat{\alpha}_{m2}, +\infty)\).

Note that the process \( \{e^{-\theta t}Q(m_{\bar{Z}}(t)) : t \geq 0\} \) is a martingale. Thus

\[
e^{-y}Q(m_{\bar{Z}}(t) \leq y) = e^{(\theta - 1)y}e^{-\theta y}Q(m_{\bar{Z}}(t) \leq y) \\
\leq e^{(\theta - 1)y}E^Q[Q(m_{\bar{Z}}(t \wedge T_y))]
\]

\[
e^{(\theta - 1)y}E^Q[Q(m_{\bar{Z}}(t \wedge T_y))]
\]

\[
e^{(\theta - 1)y}e^{\lambda t}.
\]

By letting \( y \to -\infty \), we obtain the result.

**Proof of Theorem 3.2.** Let \( b = \ln(K_0/F_0) \). Then from (3.2), we have

\[
DFP_0 = F_0 \left( \hat{K} \int_{-\infty}^{b} e^{-y}d\tilde{Q}(m_{\bar{Z}}(T) \leq y) - \tilde{Q}(m_{\bar{Z}}(T) \leq b) \right)
\]

\[
= K_0 \left( e^{-y}Q(m_{\bar{Z}}(T) \leq y) \right)_{-\infty}^{b} + \int_{-\infty}^{b} e^{-y}Q(m_{\bar{Z}}(T) \leq y)dy \\
- F_0 \tilde{Q}(m_{\bar{Z}}(T) \leq b).
\]

It follows from Lemma 3.3 that

\[
DFP_0 = K_0 \int_{-\infty}^{b} e^{-y}Q(m_{\bar{Z}}(T) \leq y)dy.
\]
Then the Laplace transform of $DFP_0$ is
\[
\int_0^\infty e^{-\delta T} DFP_0 dT = K_0 \int_0^\infty \int_{-\infty}^b e^{-\delta T} e^{-y} \hat{Q}(m_{\mathcal{P}}(T) \leq y) dy dT
\]
\[
= K_0 \int_{-\infty}^b e^{-y} \int_0^\infty e^{-\delta T} \hat{Q}(\tau_y \leq T) dT dy
\]
\[
= K_0 \int_{-\infty}^b \frac{E[\hat{Q}[e^{-\delta \tau_y}]]}{\delta} e^{-y} dy
\]
\[
= K_0 \int_{-\infty}^b \frac{1}{\delta} \sum_{i=1}^{2n+4} (\pi_1 c_i + \pi_2 d_i) e^{-(\rho_i+1)y} dy. \quad (A.6)
\]
Note that for all $j = 1, \ldots, 2n+4$, $\beta_{ij} = \beta_{ij} + 1 > 1$. Since $\hat{h}_1(0) = -(\delta - a_{11} + \lambda_1) < 0$ and $\hat{h}_1(-\beta_{11}) = +\infty$, there exists a sufficient large $\delta$, such that $y_{11} < -1$. Furthermore, as $h(0) = (\delta + \lambda_1)(\delta + \lambda_2) - a_{21}(\delta + \lambda_2) - a_{22}(\delta + \lambda_1) > 0$, and $h(y_{11}) < 0$, there exists a sufficient large $\delta$, such that $\rho_1 < -1$. Thus, the result can be obtained immediately from (A.6).

REFERENCES

[1] J. Buffington and R. J. Elliott, American options with regime switching, *International Journal of Theoretical and Applied Finance*, 5 (2002), 497–514.
[2] C. C. Chang, Y. H. Lian and M. H. Tsay, Pricing dynamic guaranteed funds under a double exponential jump diffusion model, *Academia Economica Papers*, 40 (2012), 269–306.
[3] Y. H. Dong, Pricing dynamic guaranteed funds with stochastic barrier under Vasicek interest rate model, *Chinese Journal of Applied Probability and Statistics*, 29 (2013), 237–245.
[4] Y. H. Dong, G. J. Wang and R. Wu, Pricing the zero-coupon bond and its fair premium under a structural credit risk model with jumps, *Journal of Applied Probability*, 48 (2011), 404–419.
[5] R. J. Elliott, L. Aggoun and J. B. Moore, *Hidden Markov Models: Estimation and Control*, Springer-Verlag, Berlin-Heidelberg-New York, 1994.
[6] R. J. Elliott, C. Leunglung and T. K. Siu, Option pricing and Esscher transform under regime switching, *Annals of Finance*, 1 (2005), 423–432.
[7] H. K. Fung and L. K. Li, Pricing discrete dynamic fund protections, *North American Actuarial Journal*, 7 (2003), 23–31.
[8] H. U. Gerber and G. Pafumi, Pricing dynamic investment fund protection, *North American Actuarial Journal*, 4 (2000), 28–37.
[9] H. U. Gerber and E. S. Shiu, From ruin theory to pricing reset guarantees and perpetual put options, *Insurance: Mathematics and Economics*, 24 (1999), 3–14.
[10] H. U. Gerber and E. S. Shiu, Pricing perpetual fund protection with withdrawal option, *North American Actuarial Journal*, 7 (2003), 60–77.
[11] H. U. Gerber and E. S. W. Shiu, The time value of ruin in a Sparre Andersen model, *North American Actuarial Journal*, 9 (2005), 49–84.
[12] X. Guo, *Information and option pricings*, *Quantitative Finance*, 1 (2001), 38–44.
[13] J. D. Hamilton, Rational-expectations econometric analysis of changes in regime: An investigation of the terms tructure of interest rates, *Journal of Economic Dynamics and Control*, 12 (1998), 385–423.
[14] M. R. Hardy, A regime-switching model of long-term stock returns, *North American Actuarial Journal*, 5 (2001), 41–53.
[15] J. Imai and P. P. Boyle, Dynamic fund protection, *North American Actuarial Journal*, 5 (2001), 31–47.
[16] Z. Jin, L. Y. Qian, W. Wang and R. M. Wang, Pricing dynamic fund protections with regime switching, *Journal of Computational and Applied Mathematics*, 297 (2016), 13–25.
[17] S. G. Kou, A jump-diffusion model for option pricing, *Management Science*, 48 (2002), 1086–1101.
[18] S. G. Kou and H. Wang, First passage times of a jump diffusion process, *Advance in Applied Probability*, 35 (2003), 504–531.
[19] S. G. Kou and H. Wang, Option pricing under a double exponential jump diffusion model. *Management Science*, 50 (2004), 1178–1192.
[20] V. Naik, Option valuation and hedging strategies with jumps in the volatility of asset returns, *Journal of Finance*, 48 (1993), 1969–1984.
[21] C. C. Siu, S. C. P. Yam and H. Yang, Valuing equity-linked death benefits in a regime-switching framework, *ASTIN Bulletin*, 45 (2015), 355–395.
[22] H. Y. Wong and C. M. Chan, Lookback options and dynamic fund protection under multiscale stochastic volatility, *Insurance: Mathematics and Economics*, 40 (2007), 357–385.
[23] H. Y. Wong and K. W. Lam, Valuation of discrete dynamic fund protection under Lévy processes, *North American Actuarial Journal*, 13 (2009), 202–216.
[24] C. Xu and Y. H. Dong, Pricing dynamic fund protections under a stochastic boundary, *Journal of Suzhou University of Science and Technology*, 2 (2018), 21–25.

Received February 2018; revised March 2019.

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