A stochastic analysis for a phytoplankton–zooplankton model

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Abstract. A simple phytoplankton-zooplankton nonlinear dynamical model was proposed to study the coexistence of all the species and a Hopf bifurcation was observed. In order to study the effect of environmental robustness on this system, we have stochastically perturbed the system with respect to white noise around its positive interior equilibrium. We have observed that the system remains stochastically stable around the positive equilibrium for same parametric values in the deterministic situation.

1. Introduction

In marine ecosystem, there are a considerable amount of studies on the interaction of nutrient, autotroph (primary producers, like phytoplankton) and herbivore (like, zooplankton) systems. These systems were well studied both mathematically and experimentally [1-8].

But, the effect of environmental robustness on the nutrient–autotroph–herbivore systems has not been considered so far adequately in the above literatures. DW Huang created and presented some nonlinear stochastic dynamical models on typical HAB algae diatom and dianoflagellate densities. He simplified the models through a stochastic averaging method, and the singular boundary theory of diffusion process and the invariant measure theory were applied in analyzing the bifurcation of stability and the Hopf bifurcation of the stochastic system[9-10]. Similar methods were used to study a business cycle model by Wei Li, Wei Xu, Junfeng Zhao[11].

In this paper we are interested in the studying of the dynamics under a certain environmental noise around the system’s equilibrium and finding some way to control the robustness.

2. The original mathematical Model

Here $P(t)$ denotes the concentration of the phytoplankton at time $t$ and $Z(t)$ denote the zooplankton population respectively. Let us consider, $\alpha(>0)$ as the growth rate of the phytoplankton, $\beta(>0)$ and $\epsilon(>0)$ are death rates of phytoplankton and zooplankton, $R_m$ is the maximal consumption rate of $Z$, $G$ is the half saturation constant for consumption of zooplankton on phytoplankton, and $\gamma$ is the proportion of phytoplankton consumed by zooplankton. From the above assumptions the following differential equations can be formed:

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2.1. Boundedness of the solutions

**Theorem 2.1**

If \( 0 < \gamma < 1 \), the solutions of system (1) which initiate in \( \mathbb{R}^2_+ \) are uniformly bounded.

**Proof:** Let \((P(t),Z(t))\) be any solution with positive initial conditions \((P(0),Z(0))\), since

\[
\frac{dP}{dt} = \alpha P - \beta P^2 - \frac{R_m PZ}{G + P} \quad \text{and} \quad \frac{dZ}{dt} = \gamma \frac{R_m PZ}{G + P} - \varepsilon Z
\]

We have \(\lim_{t \to \infty} \sup P(t) = M\), where \(M = \max(P(0), \alpha / \beta)\). Let us define a function

\[ W = P + Z \]  

The time derivative of (3) along the solutions of (1) is

\[
\frac{dW}{dt} = \alpha P - \beta P^2 - \frac{R_m PZ}{G + P} + \gamma \frac{R_m PZ}{G + P} - \varepsilon Z \leq \alpha P - \varepsilon Z = (\alpha + 1)P - P - \varepsilon Z
\]

(Provided \(0 < \gamma < 1\)). For each \(D(>0)\), the following inequality holds

\[
\frac{dW}{dt} + DW \leq (\alpha + 1)P + (D - 1)P + (D - \varepsilon)Z \]

If we take \(D > \max(1, \varepsilon)\), then the equation to:

\[
\frac{dW}{dt} + DW \leq (\alpha + 1)M + D(6)
\]

Thus, we can find a constant \(L(>0)\) (say), such that \(\frac{dW}{dt} + DW \leq L\). Applying the theorem of differential inequality (Birkoff and Rota, [12]), we obtain

\[
0 < W(P, Z) \leq \frac{L}{D}(1 - e^{-Dt}) + W(P(0), Z(0))e^{-Dt}
\]

and for \(t \to \infty\), we have \(0 < W(P, Z) \leq \frac{L}{D}\).

Hence, all the solutions of (1) that initiate in \(\{R^2_+ / 0\}\) are confined in the region \(\Omega = \{(P, Z) \in \mathbb{R}^2_+ : W = \frac{L}{D} + \varepsilon\}\), for any \(\varepsilon > 0\) and for \(t\) large enough.

Next, we will try to investigate the local asymptotic stability of different steady states of the system.

2.2. Qualitative analysis of the model

The model is simplified by scaling the variables as \(x = \frac{P}{\alpha}, \ y = \frac{Z\alpha^2}{\beta}\) and considering the dimensionless time \(\tau = tr\). The dimensionless equations which are equal to model (1) topologically are given by

\[
\begin{align*}
\frac{dx}{d\tau} &= x(1 - x) - \frac{xy}{a + x} \\
\frac{dy}{d\tau} &= b \frac{xy}{a + x} - cy
\end{align*}
\]

where \(a = G\beta / \alpha > 0\), \(b = \gamma R_m / \alpha > 0\), \(c = \varepsilon / \alpha > 0\). For convenience, time \(\tau\) is replaced by \(t\) as the dimensionless time. Then the number of the parameters of the preceding model can be reduced from 6 to 3.
We can find out the equilibrium states of the system (7):

\[ E_1 = (0,0) \]
\[ E_2 = (1,0) \]
\[ E_3 = (ac/(b-c),(ab^2 - abc - a^2 bc)/(b-c)^2) \]

the existence condition of positive interior equilibrium \( E_i \) is: \( b > c(a+1) \)

The stability of the equilibrium state is determined by the nature of eigen values of the Jaccobi matrix at these equilibriums.

\[
J_{accobi} = \begin{bmatrix}
1 - 2x - \frac{y}{a+x} + \frac{xy}{(a+x)^2} & -x \\
\frac{by}{a+x} - \frac{bxy}{(a+x)^2} & \frac{bx}{a+x} - c
\end{bmatrix}
\]

(8)

Let the \( J_i (i=1,2,3) \) denote the Jaccobi matrixes of the \( E_i (i=1,2,3) \) we gain

\[
J_1 = \begin{bmatrix}
1 & c \\
0 & -c
\end{bmatrix}, \quad J_2 = \begin{bmatrix}
-1 & -1 \\
0 & \frac{a+1}{b+c}
\end{bmatrix}, \quad J_3 = \begin{bmatrix}
\frac{(a+1)c}{b} & \frac{2ac}{b-c-ac} & c \\
\frac{b}{b-c-ac} & \frac{c}{b}
\end{bmatrix}
\]

Obviously \( E_1 = (0,0) \) is a saddle. Considering the existence condition of positive interior equilibrium \( E_3 \): \( b > c(a+1) \), One eigenvalue of \( J_2 \) is \( b/(a+1) - c > 0 \). So \( E_2 = (1,0) \) is a saddle too. The eigenvalues of \( J_3 \) are:

\[
\lambda_{i,2} = \frac{-A_1 \pm \sqrt{A_1^2 - 4A_2}}{2}, \text{ where } A_1 = \frac{2ac}{b-c} - \frac{(a+1)c}{b} \text{ and } A_2 = \frac{c}{b} (b-c-ac) > 0 , \text{ then, for } \\
\lambda_{i,2} < 0 , \text{the equilibrium } E_3 \text{ is asymptotically stable.}
\]

When \( A_1 > 0 \) and \( A_1^2 - 4A_2 < 0 \), that is to say the real parts of the eigenvalues \( \lambda_{i,2} \) is negative. And the image parts of \( \lambda_{i,2} \) implies that the trajectory will spiral to the equilibrium \( E_3 \) in the phase space.

When \( b = (a+1)c/(1-a) \), we have \( A_1 = 0 \), considering \( A_2 > 0 \), the eigenvalues \( \lambda_{i,2} \) of \( J_3 \) have only image parts, then \( \lambda_{i,2} = \pm i \sqrt{c(b-c-ac)/b} \), the trajectory will spiral to a limit cycle in the phase space. That implies the system have a Hopf bifurcation at equilibrium \( E_3 \).

2.3. The numerical simulation
To study the stability of equilibrium \( E_1 \) in phase space we do numerical experiment that follow for the model. The initial condition is set to \((x(0),y(0)) = (0.5,1)\). The parameter estimates were taken arbitrarily as \( a = 0.5 \), \( c = 0.5 \), take \( b = 1.3 \) to confirm that the \( A_1^2 - 4A_2 < 0 \),blue solid line denotes the \( x(t) \) and the red solid line denotes the \( y(t) \). Then the numerical results were shown as fig1.
Fig1 numerical results of this system depicting the $E_i$ is asymptotically stable with parameter $(a=0.5, b=1.3, c=0.5)$.

When $b$ was chosen to be equal to $\frac{(a+1)c}{1-a}$, the system (7) would have a Hopf bifurcation around $E_3$. We take $b_{Hopf} = 1.5$, and the numerical results were shown in fig2.

Fig2 numerical results of this system depicting the $E_i$ is a limit cycle with parameter $(a=0.5, b=1.5, c=0.5)$.

3. The stochastic model

In model (7), we assume that stochastic perturbation of the variables around their values at $E_i$ are of white noise type, which are proportional to the distances of $x, y$ from values $x_*, y_*$. So, system (7) results in

$$\begin{align*}
\frac{dx}{d\tau} &= x(1-x) - \frac{xy}{a+x} + \sigma_1(x-x_*) d\xi^1_t \\
\frac{dy}{d\tau} &= b - \frac{xy}{a+x} - cy + \sigma_2(y-y_*) d\xi^2_t
\end{align*}$$

(8)

Where $\sigma_i (i = 1, 2)$ are real constants and can be defined as the intensities of the stochasticity, $\xi^i_t = \xi^i(t)(i = 1, 2)$ are independent from each other standard Wiener process. We wonder whether the dynamical behavior of model (7) is robust with respect to such a kind of stochasticity by investigating the asymptotic stochastic stability behavior of equilibrium $E_i$ for (8) and comparing the results with those obtained for (7).
Introducing the new variables: \( u_1 = x - x^* , u_2 = y - y^* \) The linearised version of the system \([8]\) is:

\[
dU(t) = F(U(t))dt + g(U(t))d\xi(t)
\] (10)

With \( U(t) = (u_1(t), u_2(t))^T , F(U(t)) = \begin{bmatrix} a_{11}u_1 & a_{12}u_2 \\ a_{21}u_1 & a_{22}u_2 \end{bmatrix} , g(U(t)) = \begin{bmatrix} \sigma_1u_1 \\ 0 \end{bmatrix} \)

Where \( a_{11} = \frac{a + c}{b} , a_{12} = -c , a_{21} = b - c - ac , a_{22} = 0 \)

With reference to the book by Afanasev et al.\([13]\), the following theorem holds.

**Theorem 3.1** Suppose there exists a function \( V(U, t) \in C_2(\Omega) \) satisfying the inequalities

\[
K_1 |U|^\alpha \leq V(U, t) \leq K_2 |U|^\alpha \tag{7}
\]

\[
LV(U, t) \leq -K_1 |U|^\alpha , \quad K_i > 0 , i = 1, 2, 3 , \quad \alpha > 0 \quad \tag{8}
\]

Then the trivial solution of system \( dU(t) = F(U(t))dt + g(U(t))d\xi(t) \) is exponentially \( \alpha \)-stable for all time \( t \geq 0 \).

\[
LV(U, t) = \frac{\partial V(U, t)}{\partial t} + F^T(U) \frac{\partial V(U, t)}{\partial U} + \frac{1}{2} \text{Tr}[g^T(U) \frac{\partial^2 V(U, t)}{\partial U^2} g(U)] \quad \tag{9}
\]

Where \( \frac{\partial V(U, t)}{\partial U} = \left( \frac{\partial V}{\partial u_1} , \frac{\partial V}{\partial u_2} \right)^T , \frac{\partial^2 V(U, t)}{\partial U^2} = \left( \frac{\partial^2 V}{\partial u_i \partial u_j} \right)_{i,j=1,2} \) We can prove the following.

**Theorem 3.2** Suppose that \( a_{12} \times a_{21} < 0 , a_{11} < 0 \) and \( \sigma_1^2 < -2a_{11} , \sigma_2^2 = 0 \) exists. Then the solution of (8) is asymptotically mean square stable.

**Proof:** we define the function \( V(U, t) \in C_2(\Omega) = \frac{1}{2} (\omega_1u_1^2 + \omega_2u_2^2) \), then

\[
F^T(U) \frac{\partial V(U, t)}{\partial U} = a_{11}\omega_1u_1^2 + (a_{12}\omega_1 + a_{22}\omega_2)u_1u_2 + a_{22}\omega_2u_2^2
\]

\[
\frac{1}{2} \text{Tr}[g^T(U) \frac{\partial^2 V(U, t)}{\partial U^2} g(U)] = \frac{1}{2} (\omega_1\sigma_1^2u_1^2 + \omega_2\sigma_2^2u_2^2)
\]

\[
LV(U, t) = -\omega_1(-a_{11} - \frac{1}{2}\sigma_1^2)u_1^2 + (a_{12}\omega_1 + a_{22}\omega_2)u_1u_2 - \omega_2(-a_{22} - \frac{1}{2}\sigma_2^2)u_2^2
\]

We choose \( \omega_1 = -\frac{a_{21}}{a_{12}} \omega_2 > 0 \), then \( LV(U, t) = -\omega_1(-a_{11} - \frac{1}{2}\sigma_1^2)u_1^2 - \omega_2(-a_{22} - \frac{1}{2}\sigma_2^2)u_2^2 \)

\( LV(U, t) \) can be expressed as \( LV(U) = -U^TQU \), where

\[
Q = \begin{bmatrix} (-a_{11} - \frac{1}{2}\sigma_1^2)\omega_1 & 0 \\ 0 & (-a_{22} - \frac{1}{2}\sigma_2^2)\omega_2 \end{bmatrix}
\]
The two eigenvalues $\lambda_1, \lambda_2$ of the matrix $Q$ will be positive. If we choose $\lambda_{m} = \min(\lambda_1, \lambda_2)$, so from (14) we can get, $LV(U, t) \leq -\lambda_{m}|U|^2$.

According to Theorem 3.1 the proof is completed.

4. Conclusion

This paper attempts to establish the effect of environmental robustness on a autotroph–herbivore system and also tries to find out a suitable control measure to maintain the stability of the system. To establish these we have considered the system in two ways: deterministic situation and stochastic perturbations around the positive interior equilibrium of the system. Our theoretical results show that for certain values of the system parameters, the system posses asymptotic stability around the positive interior equilibrium which depicts the coexistence of all the species. On the other hand, with some parameter combinations this system has a periodical solution which implies the biomass of the species oscillating ecologically.

Further to study the effect of environmental robustness, we have stochastically perturbed the system with respect to white noise around its positive interior equilibrium. We have observed that the system remains stochastically stable around the positive equilibrium for same parametric values. This can be well implemented in the marine ecosystem, which is in a phytoplankton–zooplankton system under the stress of artificial eutrophication, to control the algal blooms which has adverse effects in marine food web as well as human. The main objective of this paper is to study the effect of environmental stochasticity on the autotroph–herbivore system and to obtain a suitable threshold of the intensity of the stochasticity to control the system under the influence of environmental stress. This has been successfully established through our analytical study.

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