Hight-order Noether’s Theorem for Nonsmooth Extremals of Isoperimetric Variational Problems with Time Delay

G. S. F. Frederico\textsuperscript{1,2} and M. J. Lazo\textsuperscript{3}
gastao.frederico@ua.pt

\textsuperscript{1}Department of Mathematics, Federal University of Santa Catarina, Florianopolis, SC, Brazil
\textsuperscript{2}Department of Science and Technology, University of Cape Verde, Praia, Santiago, Cape Verde
\textsuperscript{3}Institute of Mathematics, Statistics and Physics, Federal University of Rio Grande, Rio Grande, RS, Brazil

Abstract

We obtain a nonsmooth higher-order extension of Noether’s symmetry theorem for variational isoperimetric problems with delayed arguments. The result is proved to be valid in the class of Lipschitz functions, as long as the delayed higher-order Euler–Lagrange extremals are restricted to those that satisfy the delayed higher-order DuBois–Reymond necessary optimality condition. The important case of delayed isoperimetric optimal control problems is considered as well.

Keywords: time delays; invariance; symmetries; isoperimetric conservation laws; DuBois–Reymond necessary optimality condition; Noether’s theorem; optimal control.

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1 Introduction

Noether’s theorem, published in 1918 [21] in a seminal work, is a central result of the calculus of variations that explains all physical conservation laws based upon the action principle. It is a very general result, asserting that “to every variational symmetry of the problem there corresponds a conservation law”. Noether’s principle gives powerful insights from the various transformations that make a system invariant. For instance, in mechanics the invariance of a physical system with respect to spatial translation gives conservation of linear momentum; invariance with respect to rotation gives conservation of angular momentum; and invariance with respect to time translation gives conservation of energy [15]. As a consequence, the Noether’s theorem is usually considered the most important mathematical theorem of the twentieth century for Physics. On the other hand, the calculus of variations is now part of a more vast discipline, called optimal control [22], and Noether’s principle still holds in this more general setting [24]. Within the years, this result has been studied by many authors and generalized in different directions: see [1, 5–7, 13, 19, 20, 24] and references therein. In particular, in the recent paper [8,9], Noether’s theorem was formulated for variational problems with delayed arguments. The result is important because problems with delays play a crucial role in the modeling of real-life phenomena in various fields of applications [4][10][12]. In order to prove Noether’s theorem with delays, it was assumed that admissible functions are $C^2$-smooth and that Noether’s conserved quantity holds along all $C^2$-extremals of the Euler–Lagrange equations with time delay [7]. Here we show that to formulate higher-order Noether’s theorem...
with time delays for nonsmooth functions, it is enough to restrict the set of delayed isoperimetric higher-order Euler–Lagrange extremals to those that satisfy the delayed isoperimetric higher-order DuBois–Reymond condition. Moreover, we prove that this result can be generalized to more general isoperimetric optimal control problems.

The text is organized as follows. In Section 2 we give a short review of the results for the fundamental isoperimetric problem of variational calculus with delayed arguments. The main contributions of the paper appear in Sections 3 and 4: we prove an Euler–Lagrange and DuBois–Reymond optimality type conditions for nonsmooth higher-order isoperimetric variational problems with delayed arguments (Theorems 16 and 19, respectively), isoperimetric higher-order Noether symmetry theorem with time delay for Lipschitz functions (Theorem 25) and a delayed Noether’s symmetry theorem (Theorem 33) for isoperimetric optimal control problems. Two examples of application are given in Section 5.

2 Preliminaries

In this section we review necessary results on the calculus of variations with time delay. For more on variational problems with delayed arguments we refer the reader to [2, 3, 12, 14, 16, 17, 23].

We begin by defining the isoperimetric variational problem as in [9].

Problem 1. (The isoperimetric variational problem with time delay) The isoperimetric problem of the calculus of variations consists of minimizing a functional

\[ J^*[q(\cdot)] = \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t), q(t - \tau), \dot{q}(t - \tau)) \, dt \tag{1} \]

subject to the isoperimetric equality constraints

\[ I^*[q(\cdot)] = \int_{t_1}^{t_2} g(t, q(t), \dot{q}(t), q(t - \tau), \dot{q}(t - \tau)) \, dt = l, \quad l \in \mathbb{R}^k, \tag{2} \]

and boundary conditions

\[ q(t) = \delta(t) \text{ for } t \in [t_1 - \tau, t_1] \text{ and } q(t_2) = q_{t_2}. \tag{3} \]

We assume that \( L, g : [t_1, t_2] \times \mathbb{R}^4 \to \mathbb{R} \), are a \( C^2 \)-functions with respect to all their arguments, admissible functions \( q(\cdot) \) are \( C^2 \)-smooth, \( t_1 < t_2 \) are fixed in \( \mathbb{R} \), \( \tau \) is a given positive real number such that \( \tau < t_2 - t_1 \), \( l \) is a specified real constant and \( \delta \) is a given piecewise smooth function on \([t_1 - \tau, t_1]\).

Throughout the text, \( \partial_i L \) denotes the partial derivative of \( L \) with respect to its \( i \)th argument. For convenience of notation, we introduce the operator \( [\cdot]_\tau \) defined by

\[ [q]_\tau(t) = (t, q(t), \dot{q}(t), q(t - \tau), \dot{q}(t - \tau)). \]

The arguments of the calculus of variations assert that, by using the Lagrange multiplier rule, Problem 1 is equivalent to the following augmented problem [11, §12.1]: to minimize

\[ J^*[q(\cdot), \lambda] = \int_{t_1}^{t_2} F[q, \lambda]_\tau(t) \, dt \]

\[ := \int_{t_1}^{t_2} [L[q]_\tau(t) - \lambda \cdot g[q]_\tau(t)] \, dt \tag{4} \]

subject to (3), where \([q, \lambda]_\tau(t) = (t, q(t), \dot{q}(t), q(t - \tau), \dot{q}(t - \tau), \lambda)\).

The augmented Lagrangian

\[ F := L - \lambda \cdot g, \lambda \in \mathbb{R} \tag{5} \]
has an important role in our study.

The notion of extremizer (a local minimizer or a local maximizer) can be found in [11]. Extremizers can be classified as normal or abnormal.

**Definition 2.** An extremizer of Problem [7] that does not satisfy the Euler–Lagrange equations

\[
\begin{aligned}
\frac{d}{dt} & \left\{ \partial_4 F[q, \lambda]_\tau(t) \right\} = \partial_2 F[q, \lambda]_\tau(t) + \partial_5 F[q, \lambda]_\tau(t + \tau), \\
\frac{dx}{dt} & \left\{ \partial_5 g[q]_\tau(t) \right\} = \partial_2 g[q]_\tau(t),
\end{aligned}
\]

is said to be a normal extremizer; otherwise (i.e., if it satisfies (6) for all \( t \in [t_1, t_2] \)), is said to be abnormal.

The following theorem gives a necessary condition for \( q(\cdot) \) to be a solution of Problem [11], under the assumption that \( q(\cdot) \) is a normal extremizer.

**Theorem 3.** If \( q(\cdot) \in C^2([t_1 - \tau, t_2]) \) is a normal extremizer to Problem [11], then it satisfies the following isoperimetric Euler–Lagrange equation with time delay:

\[
\begin{aligned}
\frac{d}{dt} & \left\{ \partial_3 F[q, \lambda]_\tau(t) \right\} + \partial_5 F[q, \lambda]_\tau(t + \tau) \\
= & \partial_2 F[q, \lambda]_\tau(t) + \partial_4 F[q, \lambda]_\tau(t + \tau), \\
\frac{dx}{dt} & \left\{ \partial_5 g[q]_\tau(t) \right\} = \partial_2 g[q]_\tau(t),
\end{aligned}
\]

for all \( t \in [t_1, t_2] \), where \( F \) is the augmented Lagrangian [3] associated with Problem [7].

**Remark 4.** If one extends the set of admissible functions in problem [11]–[3] to the class of Lipschitz continuous functions, then the isoperimetric Euler–Lagrange equations [7] remain valid. This result is obtained from our Corollary [17] by choosing \( m = 1 \).

**Definition 5** (Isoperimetric extremals with time delay). The solutions \( q(\cdot) \in C^2([t_1 - \tau, t_2]) \) of the Euler–Lagrange equations [7] are called isoperimetric extremals with time delay.

**Theorem 6** (Isoperimetric DuBois–Reymond necessary condition with time delay [9]). If \( q(\cdot) \) is an isoperimetric extremals with time delay such that

\[
\partial_4 F[q]_\tau(t + \tau) \cdot \dot{q}(t) + \partial_5 F[q]_\tau(t + \tau) \cdot \ddot{q}(t) = 0
\]

for all \( t \in [t_1 - \tau, t_2 - \tau] \), then it satisfies the following conditions:

\[
\frac{d}{dt} \left\{ F[q]_\tau(t) - \dot{q}(t) \cdot (\partial_4 F[q]_\tau(t) + \partial_5 F[q]_\tau(t + \tau)) \right\} = \partial_1 F[q]_\tau(t)
\]

for \( t_1 \leq t \leq t_2 - \tau \), and

\[
\frac{d}{dt} \left\{ F[q]_\tau(t) - \dot{q}(t) \cdot \partial_4 F[q]_\tau(t) \right\} = \partial_1 F[q]_\tau(t)
\]

for \( t_2 - \tau \leq t \leq t_2 \), where \( F \) is defined in [5].

**Remark 7.** If we assume that admissible functions in problem [11]–[3] are Lipschitz continuous, then one can show that the DuBois–Reymond necessary conditions with time delay [9] are still valid (cf. Corollary [24] by choosing \( m = 1 \)).

**Definition 8** (Invariance up to a gauge-term). Consider the following \( s \)-parameter group of infinitesimal transformations:

\[
\begin{aligned}
\bar{t} & = t + s \eta(t, q) + o(s), \\
\bar{q}(t) & = q(t) + s \xi(t, q) + o(s),
\end{aligned}
\]

for all \( t \).
where \( \eta \in C^1(\mathbb{R}) \) and \( \xi \in C^1(\mathbb{R}^2) \). We say that functional \((\text{II})\) is invariant under the \( s \)-parameter group of infinitesimal transformations \((\text{II})\) up to the gauge-term \( \Phi \) if

\[
\int_I \Phi[q,\tau](t)dt = \frac{d}{ds} \int_{i(t)} F \left( t + s\eta(t,q(t)), q(t) + s\xi(t,q(t)) + o(s), \frac{\dot{q}(t) + s\dot{\xi}(t,q(t))}{1 + s\dot{\eta}(t,q(t))}, q(t) - \tau + s\xi(t-\tau,q(t-\tau)) + o(s), \frac{\dot{q}(t-\tau) + s\dot{\xi}(t-\tau,q(t-\tau))}{1 + s\dot{\eta}(t-\tau,q(t-\tau))} \right) (1 + s\dot{\eta}(t,q(t)))dt \bigg|_{s=0} \tag{12}
\]

for any subinterval \( I \subseteq [t_1,t_2] \) and for all \( q(\cdot) \in \text{Lip}\left( [t_1-\tau,t_2]\right) \).

**Definition 9** (Isoperimetric constant of motion/isoperimetric conservation law with time delay). We say that a quantity \( C(t, t + \tau, q(t), q(t-\tau), q(t+\tau), \dot{q}(t), \dot{q}(t-\tau), \dot{q}(t+\tau)) \) is an isoperimetric constant of motion with time delay \( \tau \) if

\[
\frac{d}{dt} C(t, t + \tau, q(t), q(t-\tau), q(t+\tau), \dot{q}(t), \dot{q}(t-\tau), \dot{q}(t+\tau)) = 0 \tag{13}
\]

along all the extremals \( q(\cdot) \) (cf. Definition \( \text{IX} \)). The equality \((\text{IX})\) is then an isoperimetric conservation law with time delay.

The next theorem establishes an extension of Noether’s theorem to isoperimetric problems of the calculus of variations with time delay.

**Theorem 10** (Isoperimetric Noether’s symmetry theorem with time delay for Lipschitz functions \((\text{II})\)). If functional \((\text{II})\) is invariant up to \( \Phi \) in the sense of Definition \( \text{IX} \) then the quantity \( C(t, t + \tau, q(t), q(t-\tau), q(t+\tau), \dot{q}(t), \dot{q}(t-\tau), \dot{q}(t+\tau)) \) defined by

\[
- \Phi[q,\tau](t) + (\partial_3 F[q,\tau](t) + \partial_5 F[q,\tau](t+\tau)) \cdot \eta(t,q(t))
+ \left( F[q,\tau] - \dot{q}(t) \cdot (\partial_3 F[q,\tau] + \partial_5 F[q,\tau](t+\tau)) \right) \eta(t,q(t)) \tag{14}
\]

for \( t_1 \leq t \leq t_2 - \tau \) and by

\[
- \Phi[q,\tau](t) + \partial_3 F[q,\tau](t) \cdot \eta(t,q(t)) + \left( F[q,\tau] - \dot{q}(t) \cdot \partial_3 F[q,\tau](t) \right) \eta(t,q(t)) \tag{15}
\]

for \( t_2 - \tau \leq t \leq t_2 \), is a constant of motion with time delay along any \( q(\cdot) \in \text{Lip}\left( [t_1-\tau,t_2]\right) \) satisfying both \((\text{II})\) and \((\text{III})\)–\((\text{VII})\), i.e., along any Lipschitz Euler–Lagrange extremal that is also a Lipschitz DuBois–Reymond extremal that satisfy the condition \((\text{IX})\).

### 3 Nonsmooth higher-order Noether’s theorem for isoperimetric problems of the calculus of variations with time delay

Let \( \mathcal{W}^{k,p} \), \( k \geq 1, 1 \leq p \leq \infty \), denote the class of functions that are absolutely continuous with their derivatives up to order \( k - 1 \), the \( k \)th derivative belonging to \( L^p \). With this notation, the class \( \text{Lip} \) of Lipschitz functions is represented by \( \mathcal{W}^{1,\infty} \). We now extend previous results to isoperimetric problems with higher-order derivatives.

#### 3.1 Higher-order Euler–Lagrange and DuBois–Reymond optimality conditions with time delay

Let \( m \in \mathbb{N} \) and \( q^{(i)}(t) \) denote the \( i \)th derivative of the vector \( q(t) \) defined in \( \mathbb{R}^n \) \(( n \in \mathbb{N}^* \)\), \( i = 0, \ldots, m \), with \( q^{(0)}(t) = q(t) \). For simplicity of notation, we introduce the operator \([\cdot]^m_{\tau}\) by

\[
[q]^m_{\tau}(t) := \left( t, q(t), \dot{q}(t), \ldots, q^{(m)}(t), q(t-\tau), \dot{q}(t-\tau), \ldots, q^{(m)}(t-\tau) \right).
\]

Consider the following higher-order isoperimetric variational problem with time delay:
Problem 11. To minimize
\[ J_m^\tau[q(\cdot)] = \int_{t_1}^{t_2} L[q]_\tau^m(t) dt \] (16)
subject to the isoperimetric equality constraints
\[ I_m^\tau[q(\cdot)] = \int_{t_1}^{t_2} g[q]_\tau^m(t) dt = l, \quad l \in \mathbb{R}^k, \] (17)
boundary conditions \( q^{(i)}(t_2) = q_i^{t_2}, \; i = 1, \ldots, m - 1 \). The functions \( L, g : [t_1, t_2] \times \mathbb{R}^{2n(m+1)} \to \mathbb{R} \) are assumed to be a \( C^{m+1} \) function with respect to all their arguments, admissible functions \( q(\cdot) \) are assumed to be \( \mathcal{W}^m, \infty \), \( t_1 < t_2 \) are fixed in \( \mathbb{R} \), \( \tau \) is a given positive real number such that \( \tau < t_2 - t_1 \), \( q_i^{t_2} \) are given vectors in \( \mathbb{R}^n \), \( i = 1, \ldots, m - 1 \), \( l \) is a specified real constant and \( \delta \) is a given piecewise smooth function on \( [t_1 - \tau, t_1] \).

Remark 12. When \( m = 1 \) and \( n = 1 \) the Problem 11 reduces to Problem 7.

In [8] the authors proved the following corollary:

Corollary 13 (Higher-order Euler–Lagrange equations with time delay in differential form). If \( q(\cdot) \in \mathcal{W}^m, \infty ([t_1 - \tau, t_2], \mathbb{R}^n) \) is an extremal of functional \( 16 \), then
\[ \sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \left( \partial_{t+2} L[q]_\tau^m(t) + \partial_{t+m+3} L[q]_\tau^m(t + \tau) \right) = 0 \]
for \( t_1 \leq t \leq t_2 - \tau \)
\[ \sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \partial_{t+2} L[q]_\tau^m(t) = 0 \]
for \( t_2 - \tau \leq t \leq t_2 \).

The previous corollary motivates the following definition.

Definition 14. An admissible function \( q(\cdot) \in \mathcal{W}^m, \infty ([t_1 - \tau, t_2], \mathbb{R}^n) \) is an extremal for problem \( 17 - 3 \) if it satisfies the following Euler–Lagrange equations with time delay:
\[ \sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \left( \partial_{t+2} g[q]_\tau^m(t) + \partial_{t+m+3} g[q]_\tau^m(t + \tau) \right) = 0 \] (18)
for \( t_1 \leq t \leq t_2 - \tau \)
\[ \sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \partial_{t+2} g[q]_\tau^m(t) = 0 \] (19)
for \( t_2 - \tau \leq t \leq t_2 \).

Now we extend notion of normal extremizer (Definition 2) to higher-order normal extremizer for isoperimetric problems of the calculus of variations with time delay.

Definition 15. An extremizer of Problem 11 that does not satisfy (18)–(19) is said to be a higher-order normal extremizer; otherwise (i.e., if it satisfies (18)–(19) for all \( t \in [t_1, t_2] \)), is said to be a higher-order abnormal extremizer.

The next theorem is crucial for our purposes.

Theorem 16 (Isoperimetric higher-order Euler–Lagrange equations with time delay in integral form). If \( q(\cdot) \in \mathcal{W}^m, \infty ([t_1 - \tau, t_2], \mathbb{R}^n) \) is a higher-order normal extremizer of Problem 11, then
Then we can regard \( I \) problem with time delay. Consider the quantity

\[
\kappa h
\]

\[
\partial^{i+2} F[q]^m_{\tau}(s_{m-i})
\]

Proof. Performing the linear change of variables \( t = \sigma + \tau \) in the last term of integral \((22)\), and using the fact that \( h_2(t) = 0 \) if \( t \in [t_1 - \tau, t_1] \), \((22)\) becomes

\[
\hat{I} (0, 0) = 0.
\]

On the other hand, we have

\[
\left. \frac{\partial I}{\partial \kappa_2} \right|_{(0, 0)} = \int_{t_1}^{t_2} \left( \sum_{i=0}^{m} \partial_i + 2g[q]^m_{\tau}(t) \cdot h_2^{(i)}(t) + \sum_{i=0}^{m} \partial_i + 2g[q]^m_{\tau}(t) \cdot h_2^{(i)}(t) \right) dt.
\]

Performing the linear change of variables \( t = \sigma + \tau \) in the last term of integral \((24)\), and using the fact that \( h_2(t) = 0 \) if \( t \in [t_1 - \tau, t_1] \), \((24)\) becomes

\[
\left. \frac{\partial I}{\partial \kappa_2} \right|_{(0, 0)} = \int_{t_1}^{t_2} \left( \sum_{i=0}^{m} \partial_i + 2g[q]^m_{\tau}(t) \cdot h_2^{(i)}(t) \right) dt
\]

\[
+ \int_{t_1}^{t_2 - \tau} \left( \sum_{i=0}^{m} \partial_i + 2g[q]^m_{\tau}(t + \tau) \cdot h_2^{(i)}(t) \right) dt.
\]
By repeated integration by parts one has

\[
\sum_{i=0}^{m} \int_{t_1}^{t_2} \partial_i + 2g[q]_\tau^m(t) \cdot h_2^{(i)}(t) \, dt
\]

\[
= \sum_{i=0}^{m} \left\{ \sum_{j=1}^{m-i} (-1)^{j+1} h_2^{(i+j-1)}(t) \cdot \left( \int_{t_2-\tau}^{t} \int_{t_2-\tau}^{s_2} \cdots \int_{t_2-\tau}^{s_{m-1}} \left( \partial_i + 2g[q]_\tau^m(s_j) \right) ds_j \, ds_{m-i} \right) \right\}_{t_1}^{t_2}
\]

\[
+ (-1)^i \int_{t_1}^{t_2} h_2^{(m)}(t) \cdot \left( \int_{t_2-\tau}^{t} \int_{t_2-\tau}^{s_2} \cdots \int_{t_2-\tau}^{s_{m-1}} \left( \partial_i + 2g[q]_\tau^m(s_{m-i}) \right) ds_m \, ds_{m-i} \right) \, dt \right\}
\]  

\[
(26)
\]

and

\[
\sum_{i=0}^{m} \int_{t_1}^{t_2-\tau} \partial_i + m3g[q]_\tau^m(t+\tau) \cdot h_2^{(i)}(t) \, dt
\]

\[
= \sum_{i=0}^{m} \left\{ \sum_{j=1}^{m-i} (-1)^{j+1} h_2^{(i+j-1)}(t) \cdot \left( \int_{t_2-\tau}^{t} \int_{t_2-\tau}^{s_2} \cdots \int_{t_2-\tau}^{s_{m-1}} \left( \partial_i + m3g[q]_\tau^m(s_j+\tau) \right) ds_j \, ds_{m-i} \right) \right\}_{t_1}^{t_2-\tau}
\]

\[
+ (-1)^i \int_{t_1}^{t_2-\tau} h_2^{(m)}(t) \cdot \left( \int_{t_2-\tau}^{t} \int_{t_2-\tau}^{s_2} \cdots \int_{t_2-\tau}^{s_{m-1}} \left( \partial_i + m3g[q]_\tau^m(s_{m-i}+\tau) \right) ds_m \, ds_{m-i} \right) \, dt \right\}.
\]  

\[
(27)
\]

Because \( h_2^{(i)}(t_2) = 0, i \neq 0, \ldots, m - 1 \), and \( h_2(t) = 0, t \in [t_1 - \tau, t_1] \), the terms without integral sign in the right-hand sides of identities (26) and (27) vanish. Therefore, equation (25) becomes

\[
\left. \frac{\partial I}{\partial \epsilon_2} \right|_{(0,0)} = \int_{t_1}^{t_2-\tau} h_2^{(m)}(t) \cdot \sum_{i=0}^{m} (-1)^i \left( \int_{t_2-\tau}^{t} \int_{t_2-\tau}^{s_2} \cdots \int_{t_2-\tau}^{s_{m-i}} \left( \partial_i + 2g[q]_\tau^m(s_{m-i}) \right) ds_m \, ds_{m-i} \right) \, dt
\]

\[
+ \sum_{i=0}^{m} (-1)^i \left( \int_{t_2-\tau}^{t} \int_{t_2-\tau}^{s_2} \cdots \int_{t_2-\tau}^{s_{m-i}} \left( \partial_i + 2g[q]_\tau^m(s_{m-i}) \right) ds_m \, ds_{m-i} \right) \, dt.
\]  

\[
(28)
\]

For \( i = 0, \ldots, m \) we define functions

\[
\varphi_i(t) = \begin{cases} 
\partial_i + 2g[q]_\tau^m(t) + \partial_i + 3g[q]_\tau^m(t+\tau) & \text{for } t_1 \leq t \leq t_2 - \tau \\
\partial_i + 2g[q]_\tau^m(t) & \text{for } t_2 - \tau \leq t \leq t_2.
\end{cases}
\]

Then one can write equation (28) as follows:

\[
\left. \frac{\partial I}{\partial \epsilon_2} \right|_{(0,0)} = \int_{t_1}^{t_2} h_2^{(m)}(t) \cdot \sum_{i=0}^{m} (-1)^i \left( \int_{t_2-\tau}^{t} \int_{t_2-\tau}^{s_2} \cdots \int_{t_2-\tau}^{s_{m-i}} (\varphi_i(s_{m-i})) ds_m \, ds_{m-i} \right) \, dt.
\]
Since \( q(\cdot) \in W^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n) \) is a higher-order normal extremizer of Problem 11 by the fundamental lemma of the calculus of variations (see, e.g., [27]), there exists a function \( h_2 \) such that

\[
\frac{\partial \tilde{J}}{\partial \epsilon_2} \bigg|_{(0,0)} \neq 0 .
\]  

(29)

Using (23) and (29), the implicit function theorem asserts that there exists a function \( h_2(\cdot) \), defined in a neighborhood of zero, such that \( \hat{J} \equiv \tilde{J}(\epsilon_1, \epsilon_2) = J_m[q(\cdot)] \). By hypothesis, \( J \) has minimum (or maximum) at \((0,0)\) subject to the constraint \( \hat{J}(0,0) = 0 \), and we have proved that \( \nabla \hat{J}(0,0) \neq 0 \). We can appeal to the Lagrange multiplier rule (see, e.g., [27, p. 77]) to assert the existence of a number \( \lambda \) such that \( \nabla (\hat{J}(0,0) - \lambda \cdot \hat{I}(0,0)) = 0 \).

Repeating the calculations as before,

\[
\frac{\partial \tilde{J}}{\partial \epsilon_1} \bigg|_{(0,0)} = \int_{t_1}^{t_2} h_1^{(m)}(t) \cdot \left[ \sum_{i=0}^{m} (-1)^i \left( \int_{t_2 - \tau}^{t} \int_{s_{i+1} - \tau}^{s_i} \ldots \int_{s_{m+1} - \tau}^{s_1} \left( \phi_i(s_{m-i}) \right) ds_{m-i} \right) ds_1 \right] dt .
\]

and

\[
\frac{\partial \tilde{I}}{\partial \epsilon_1} \bigg|_{(0,0)} = \int_{t_1}^{t_2} h_1^{(m)}(t) \cdot \left[ \sum_{i=0}^{m} (-1)^i \left( \int_{t_2 - \tau}^{t} \int_{s_{i+1} - \tau}^{s_i} \ldots \int_{s_{m+1} - \tau}^{s_1} \left( \phi_i(s_{m-i}) \right) ds_{m-i} \right) ds_1 \right] dt ,
\]

where for \( i = 0, \ldots, m \)

\[
\phi_i(t) = \begin{cases} 
\partial_{i+2}L[q]_\tau^m(t) + \partial_{i+m+3}L[q]_\tau^m(t + \tau) & \text{for } t_1 \leq t \leq t_2 - \tau \\
\partial_{i+2}L[q]_\tau^m(t) & \text{for } t_2 - \tau \leq t \leq t_2 .
\end{cases}
\]

Therefore, one has

\[
\int_{t_1}^{t_2} h_1^{(m)}(t) \cdot \left[ \sum_{i=0}^{m} (-1)^i \left( \int_{t_2 - \tau}^{t} \int_{s_{i+1} - \tau}^{s_i} \ldots \int_{s_{m+1} - \tau}^{s_1} \left( \phi_i(s_{m-i}) \right) ds_{m-i} \right) ds_1 \right] dt = 0 .
\]  

(30)

Applying the higher-order DuBois–Reymond lemma in (30) (see, e.g., [15, 26]), one arrives to (20) and (21).

**Corollary 17** (Isoperimetric higher-order Euler–Lagrange equations with time delay in differential form). If \( q(\cdot) \in W^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n) \) is a higher-order normal extremizer of Problem 11, then

\[
\sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \left( \partial_{i+2}F[q]_\tau^m(t) + \partial_{i+m+3}F[q]_\tau^m(t + \tau) \right) = 0
\]  

for \( t_1 \leq t \leq t_2 - \tau \) and

\[
\sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \partial_{i+2}F[q]_\tau^m(t) = 0
\]  

for \( t_2 - \tau \leq t \leq t_2 \).
Proof. We obtain (31) and (32) applying the derivative of order \( m \) to (20) and (21), respectively.

Remark 18. If \( m = 1 \) and \( n = 1 \), then the higher-order Euler–Lagrange equations (31)–(32) reduce to (27).

Associated to a given function \( q(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n) \), it is convenient to introduce the following quantities (cf. [25]):

\[
\psi_1^j = \sum_{i=0}^{m-j} (-1)^i \frac{d^i}{dt^i} \left( \partial_{t+j+2} F[q]\tau^m(t) + \partial_{t+j+m+3} F[q]\tau^m (t + \tau) \right)
\]

for \( t_1 \leq t \leq t_2 - \tau \), and

\[
\psi_2^j = \sum_{i=0}^{m-j} (-1)^i \frac{d^i}{dt^i} \partial_{t+j+2} F[q]\tau^m(t)
\]

for \( t_2 - \tau \leq t \leq t_2 \), where \( j = 0, \ldots, m \). These operators are useful for our purposes because of the following properties:

\[
\frac{d}{dt} \psi_1^j = \partial_{j+1} F[q]\tau^m(t) + \partial_{j+m+3} F[q]\tau^m(t + \tau) - \psi_1^{j-1}
\]

for \( t_1 \leq t \leq t_2 - \tau \), and

\[
\frac{d}{dt} \psi_2^j = \partial_{j+1} F[q]\tau^m(t) - \psi_2^{j-1}
\]

for \( t_2 - \tau \leq t \leq t_2 \), where \( j = 1, \ldots, m \). We are now in conditions to prove a higher-order DuBois–Reymond optimality condition for isoperimetric problems with time delay.

Theorem 19 (Isoperimetric higher-order delayed DuBois–Reymond condition). If \( q(\cdot) \in \mathbb{W}^{m,\infty}([t_1 - \tau, t_2], \mathbb{R}^n) \) is a higher-order normal extremizer of Problem 11 such that

\[
\sum_{j=0}^{m} \partial_{j+m+3} F[q]\tau^m(t + \tau) \cdot q^{(j+1)}(t) = 0
\]

for all \( t \in [t_1 - \tau, t_2 - \tau] \), then it satisfies the following conditions:

\[
\frac{d}{dt} \left( F[q]\tau^m(t) - \sum_{j=1}^{m} \psi_1^j \cdot q^{(j)}(t) \right) = \partial_1 F[q]\tau^m(t)
\]

for \( t_1 \leq t \leq t_2 - \tau \) and

\[
\frac{d}{dt} \left( F[q]\tau^m(t) - \sum_{j=1}^{m} \psi_2^j \cdot q^{(j)}(t) \right) = \partial_1 F[q]\tau^m(t)
\]

for \( t_2 - \tau \leq t \leq t_2 \), where \( F \) is the augmented Lagrangian 5 associated with Problem 11, \( \psi_1^j \) given by (33) and \( \psi_2^j \) by (34).

Proof. We prove the theorem in the interval \( t_1 \leq t \leq t_2 - \tau \). The proof is similar for \( t_2 - \tau \leq t \leq t_2 \). We derive equation (37) as follows:
Let an arbitrary \( x \in [t_1, t_2 - \tau] \). Note that

\[
\int_{t_1}^{x} \frac{d}{dt} \left( F[q]^m \right) (t) - \sum_{j=1}^{m} \psi_j^1 \cdot q^{(j)}(t) \right) dt = \int_{t_1}^{x} \frac{d}{dt} \left( L[q]^m(t) - \lambda \cdot g[q]^m(t) - \sum_{j=1}^{m} \psi_j^1 \cdot q^{(j)}(t) \right) dt
\]

\[
= \int_{t_1}^{x} \left( \partial_1 (L[q]^m(t) - \lambda \cdot g[q]^m(t)) + \sum_{j=0}^{m} \partial_j+2 (L[q]^m(t) - \lambda \cdot g[q]^m(t)) \cdot q^{(j+1)}(t) \right.
\]

\[
- \sum_{j=1}^{m} \left( \psi_j^1 \cdot q^{(j)}(t) + \psi_j^1 \cdot q^{(j+1)}(t) \right) dt
\]

\[
+ \int_{t_1}^{x} \sum_{j=0}^{m} \partial_j+m+3 (L[q]^m(t) - \lambda \cdot g[q]^m(t)) \cdot q^{(j+1)}(t - \tau) dt. \quad (39)
\]

Observe that, by hypothesis (36), the last integral of (39) is null and using (35), the equation (39) becomes

\[
\int_{t_1}^{x} \frac{d}{dt} \left( L[q]^m(t) - \lambda \cdot g[q]^m(t) - \sum_{j=1}^{m} \psi_j^1 \cdot q^{(j)}(t) \right) dt
\]

\[
= \int_{t_1}^{x} \left[ \partial_1 (L[q]^m(t) - \lambda \cdot g[q]^m(t)) + \sum_{j=0}^{m} \partial_j+2 (L[q]^m(t) - \lambda \cdot g[q]^m(t)) \cdot q^{(j+1)}(t) \right.
\]

\[
- \sum_{j=1}^{m} \left( \partial_j+1 L[q]^m(t) + \partial_j+m+2 L[q]^m(t + \tau - \psi_j^{-1}) \cdot q^{(j)}(t) + \psi_j^1 \cdot q^{(j+1)}(t) \right) dt. \quad (40)
\]

We now simplify the second term on the right-hand side of (40):

\[
\sum_{j=1}^{m} \left( \partial_j+1 L[q]^m(t) + \partial_j+m+2 L[q]^m(t + \tau - \psi_j^{-1}) \cdot q^{(j)}(t) + \psi_j^1 \cdot q^{(j+1)}(t) \right)
\]

\[
= \sum_{j=0}^{m-1} \left( \partial_j+2 L[q]^m(t) + \partial_j+m+3 L[q]^m(t + \tau - \psi_j^{-1}) \cdot q^{(j+1)}(t) + \psi_j^{m+1} \cdot q^{(j+2)}(t) \right)
\]

\[
= \sum_{j=0}^{m-1} \left[ \partial_j+2 L[q]^m(t) + \partial_j+m+3 L[q]^m(t + \tau) \cdot q^{(j+1)}(t) \right) - \psi_j^0 \cdot \dot{q}(t) + \psi_j^m \cdot q^{(m+1)}(t). \quad (41)
\]

Substituting (41) into (40) and using the higher-order Euler–Lagrange equations with time delay (41), and since, by definition,

\[
\psi_j^m = \partial_j+m+2 L[q]^m(t) + \partial_j+m+3 L[q]^m(t + \tau),
\]

\[
\psi_j^0 = \sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \left( \partial_j+2 L[q]^m(t) + \partial_j+m+3 L[q]^m(t + \tau) \right) = 0
\]

and by hypothesis (36)

\[
\sum_{j=0}^{m-1} \partial_j+m+3 L[q]^m(t + \tau) \cdot q^{(j+1)}(t) = -\partial_j+2 L[q]^m(t + \tau) \cdot q^{(m+1)}(t),
\]
we conclude that

\[
\int_{t_1}^{x} \frac{d}{dt} \left( L[q]_\tau^m(t) - \sum_{j=1}^{\infty} \psi_1^j \cdot q^{(j)}(t) \right) dt = \int_{t_1}^{x} \left[ \partial_1 L[q]_\tau^m(t) + (\partial_{m+2} L[q]_\tau^m(t) + \partial_{2m+3} L[q]_\tau^m(t + \tau)) \cdot q^{(m+1)} \right. \\
+ \psi_1^0 \cdot \dot{q}(t) - \psi_1^m \cdot q^{(m+1)}(t) \bigg] dt = \int_{t_1}^{x} \partial_1 L[q]_\tau^m(t) dt.
\]

We finally obtain (37) by the arbitrariness \( x \in [t_1, t_2 - \tau] \).

In the particular case when \( m = 1 \) and \( n = 1 \), we obtain from Theorem 19 an extension of Theorem 6 to the class of Lipschitz functions.

**Corollary 20** (Nonsmooth isoperimetric DuBois–Reymond conditions). If \( q(\cdot) \in \text{Lip}([t_1 - \tau, t_2]; \mathbb{R}) \) is a normal isoperimetric extremals with time delay, then the DuBois–Reymond conditions with time delay (9) and (10) hold true.

**Proof.** For \( m = 1 \), the hypothesis (36) is reduced to (8), condition (37) to

\[
\frac{d}{dt} \left( F[q]_\tau(t) - \psi_1^1 \cdot \dot{q}(t) \right) = \partial_1 F[q]_\tau(t)
\]

for \( t_1 \leq t \leq t_2 - \tau \), and (38) to

\[
\frac{d}{dt} \left( F[q]_\tau(t) - \psi_2^1 \cdot \dot{q}(t) \right) = \partial_1 F[q]_\tau(t)
\]

for \( t_2 - \tau \leq t \leq t_2 \). Keeping in mind (33) and (34), we obtain

\[
\psi_1^1 = \partial_3 F[q]_\tau(t) + \partial_5 F[q]_\tau(t + \tau)
\]

and

\[
\psi_2^1 = \partial_3 F[q]_\tau(t).
\]

One finds the intended equalities (9) and (10) by substituting the quantities (14) and (15) into (42) and (43), respectively.

### 3.2 Isoperimetric higher-order Noether’s symmetry theorem with time delay

Now, we generalize the Noether-type theorem (Theorem 10) to the more general case of delayed isoperimetric variational problems with higher-order derivatives.

We know (see Section 3.1) that by using the Lagrange multiplier rule, Problem 11 is equivalent to the following augmented problem: to minimize

\[
J_m^\tau[q(\cdot), \lambda] = \int_{t_1}^{t_2} F[q, \lambda]_\tau^m(t) dt
\]

subject to boundary conditions (3) and \( q^{(i)}(t_2) = q_i(t_2), \ i = 1, \ldots, m - 1 \).

The notion of variational invariance for Problem 11 is defined with the help of the equivalent augmented Lagrangian (46).
Remark 22. Expressions \(\Phi\) and \(\dot{q}^{(i)}\) in equation (47), \(i = 1, \ldots, m\), are interpreted as

\[
\dot{\Phi} = \frac{d}{dt} \Phi, \quad \dot{q} = \frac{dq}{dt}, \quad \dot{q}^{(i)} = \frac{d^{i}q}{dt^{i}}, \quad i = 2, \ldots, m.
\]

The next lemma gives a necessary condition of invariance for functional (40).

Lemma 23 ( Necessary condition of invariance for (40). ) If functional (40) is invariant up to the gauge-term \(\Phi\) under the \(s\)-parameter group of infinitesimal transformations (11), then

\[
\int_{t_{1}}^{t_{2} - \tau} \left[ -\frac{d}{dt} \Phi + \partial_{t} F[q]^{m}_{\tau} (t) \eta(t, q) + F[q]^{m}_{\tau} (t) \dot{\eta}(t, q) \right. \\
\left. + \sum_{i=0}^{m} \left( \partial_{i+2} F[q]^{m}_{\tau} (t) + \partial_{i+m+3} F[q]^{m}_{\tau} (t + \tau) \right) \cdot \dot{\rho}^{(i)}(t) \right] dt = 0
\]

for \(t_{1} \leq t \leq t_{2} - \tau\) and

\[
\int_{t_{2} - \tau}^{t_{2}} \left[ -\frac{d}{dt} \Phi + \partial_{t} F[q]^{m}_{\tau} (t) \eta(t, q) + F[q]^{m}_{\tau} (t) \dot{\eta}(t, q) + \sum_{i=0}^{m} \partial_{i+2} F[q]^{m}_{\tau} (t) \cdot \dot{\rho}^{(i)}(t) \right] dt = 0
\]

for \(t_{2} - \tau \leq t \leq t_{2}\), where

\[
\begin{aligned}
\rho^{0}(t) &= \xi(t, q) , \\
\rho^{(i)}(t) &= \frac{d^{i-1}}{dt^{i-1}} \left[ \left( \dot{q}^{(i-1)}(t) - q^{(i-1)}(t) \dot{\eta}(t, q) \right) \right], \quad i = 1, \ldots, m.
\end{aligned}
\]

Proof. Without loss of generality, we take \(I = [t_{1}, t_{2}]\). Then, (47) is equivalent to

\[
\int_{t_{1}}^{t_{2}} \left[ -\frac{d}{dt} \Phi + \partial_{t} \left( L[q]^{m}_{\tau}(t) - \lambda \cdot g[q]^{m}_{\tau}(t) \right) \eta(t, q) \right. \\
\left. + \sum_{i=0}^{m} \partial_{i+2} \left( L[q]^{m}_{\tau}(t) - \lambda \cdot g[q]^{m}_{\tau}(t) \right) \cdot \frac{\partial}{\partial s} \left( \frac{d^{i}q}{dt^{i}} \right) \bigg|_{s=0} \right. \\
\left. + \sum_{i=0}^{m} \partial_{i+m+3} \left( L[q]^{m}_{\tau}(t) - \lambda \cdot g[q]^{m}_{\tau}(t) \right) \cdot \frac{\partial}{\partial s} \left( \frac{d^{i}q(t - \tau)}{dt^{i}(t - \tau)} \right) \bigg|_{s=0} \right. \\
\left. + \left( L[q]^{m}_{\tau}(t) - \lambda \cdot g[q]^{m}_{\tau}(t) \right) \dot{\eta} \right] = 0.
\]

Using the fact that (48) implies

\[
\frac{\partial}{\partial s} \left( \frac{d^{i}q}{dt^{i}} \right) \bigg|_{s=0} = \frac{d^{i}}{dt^{i}} \left[ \frac{\partial}{\partial s} \left( \frac{d^{i-1}q}{dt^{i-1}} \right) \bigg|_{s=0} \right] - q^{(i)}(t) \dot{\eta}(t, q), \quad i = 2, \ldots, m,
\]
then equation (52) becomes
\[
\int_{t_1}^{t_2} \left[ -\dot{\Phi} [q]_\tau^m (t) + \partial_t (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \eta(t, q) + (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \dot{\eta}(t, q) \\
+ \sum_{i=0}^{m} \partial_{i+2} (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \cdot \rho^i(t) \\
+ \sum_{i=0}^{m} \partial_{i+m+3} (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \cdot \rho^i(t - \tau) \right] dt = 0. \tag{53}
\]
Performing the linear change of variables \( t = \sigma + \tau \) in the last integral of (53), and keeping in mind that \( \xi = \eta = 0 \) on \([t_1 - \tau, t_1]\), equation (53) becomes
\[
\int_{t_1}^{t_2 - \tau} \left[ -\dot{\Phi} [q]_\tau^m (t) + \partial_t (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \eta(t, q) + (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \dot{\eta}(t, q) \\
+ \sum_{i=0}^{m} \partial_{i+2} (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \cdot \rho^i(t) \right] dt \\
+ \int_{t_2 - \tau}^{t_2} \left[ -\dot{\Phi} [q]_\tau^m (t) + \partial_t (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \eta(t, q) + (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \dot{\eta}(t, q) \\
+ \sum_{i=0}^{m} \partial_{i+2} (L[q]_\tau^m (t) - \lambda \cdot g[q]_\tau^m (t)) \cdot \rho^i(t) \right] dt = 0. \tag{54}
\]
Equations (49) and (50) follow from the fact that (54) holds for an arbitrary \( I \subseteq [t_1, t_2] \).  

**Definition 24 (Isoperimetric Higher-order constant of motion/isoperimetric conservation law with time delay).** A quantity
\[
C\{q\}_\tau^m (t) := C\left(t, t + \tau, q(t), q(t), \ldots, q^{(m)}(t), q(t - \tau), \dot{q}(t - \tau), \ldots, q^{(m)}(t - \tau),
q(t + \tau), \dot{q}(t + \tau), \ldots, q^{(m)}(t + \tau)\)
\]
is a higher-order constant of motion with time delay \( \tau \) if
\[
\frac{d}{dt} C\{q\}_\tau^m (t) = 0, \tag{55}
\]
\( t \in [t_1, t_2] \), along any \( q(\cdot) \in W^{m, \infty} ([t_1 - \tau, t_2], \mathbb{R}^n) \) satisfying both Theorem 16 and Theorem 19. The equality (55) is then said to be a higher-order conservation law with time delay.

**Theorem 25 (Isoperimetric higher-order Noether’s symmetry theorem with time delay).** If functional (46) is invariant up to the gauge-term \( \Phi \) in the sense of Definition 21 such that satisfy condition (30), then the quantity \( C\{q\}_\tau^m (t) \) defined by
\[
\sum_{j=1}^{m} \psi_1^j \cdot \rho^{j-1}(t) + \left( F[q]_\tau^m (t) - \sum_{j=1}^{m} \psi_1^j \cdot q^{(j)}(t) \right) \eta(t, q) - \Phi [q]_\tau^m (t) \tag{56}
\]
for \( t_1 \leq t \leq t_2 - \tau \) and by
\[
\sum_{j=1}^{m} \psi_2^j \cdot \rho^{j-1}(t) + \left( F[q]_\tau^m (t) - \sum_{j=1}^{m} \psi_2^j \cdot q^{(j)}(t) \right) \eta(t, q) - \Phi [q]_\tau^m (t) \tag{57}
\]
for \( t_2 - \tau \leq t \leq t_2 \), is a higher-order constant of motion with time delay (cf. Definition 24) satisfying the hypothesis (30), where \( \psi_1^j \) and \( \psi_2^j \) are given by (33) and (34), respectively.
Proof. We prove the theorem in the interval \( t_1 \leq t \leq t_2 - \tau \). The proof is similar in the interval \( t_2 - \tau \leq t \leq t_2 \). Equation (56) follows by direct calculations:

\[
0 = \int_{t_1}^{t_2 - \tau} \frac{d}{dt} \left[ \psi_1^1 \cdot \rho^0 + \sum_{j=2}^{m} \psi_j^1 \cdot \rho^{j-1}(t) \right] \, dt \\
+ \left( L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t) - \sum_{j=1}^{m} \psi_1^1 \cdot q^{(j)}(t) \right) \eta(t, q) - \Phi[q]_r^m(t) \right] \, dt \\
= \int_{t_1}^{t_2 - \tau} \left[ -\Phi[q]_r^m(t) + \rho^0(t) \cdot \frac{d}{dt} \psi_1^1 + \psi_1^1 \cdot \frac{d}{dt} \rho^0(t) + \sum_{j=2}^{m} \left( \rho^{j-1}(t) \cdot \frac{d}{dt} \psi_j^1 + \psi_j^1 \cdot \frac{d}{dt} \rho^{j-1}(t) \right) \right] \\
+ \eta(t, q) \frac{d}{dt} \left( L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t) - \sum_{j=1}^{m} \psi_j^1 \cdot q^{(j)}(t) \right) \\
+ \left( L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t) - \sum_{j=1}^{m} \psi_j^1 \cdot q^{(j)}(t) \right) \dot{\eta}(t, q) \right] \, dt. \\
(58)
\]

Using the Euler–Lagrange equation (31), the DuBois–Reymond condition (37), and relations (35) and (31) into (58), we obtain:

\[
\int_{t_1}^{t_2 - \tau} \left[ -\Phi[q]_r^m(t) + (\partial_2 (L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t)) \\
+ \partial_{m+3} (L[q]_r^m(t + \tau) - \lambda \cdot g[q]_r^m(t + \tau)) \cdot \xi(t, q) + \psi_1^1 \cdot (\rho^1(t) + \dot{q}(t) \dot{\tau}(t, q)) \\
+ \sum_{j=2}^{m} \left( \partial_{j+1} (L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t)) + \partial_{j+m+2} (L[q]_r^m(t + \tau) - \lambda \cdot g[q]_r^m(t + \tau)) \cdot (\psi_1^1 \cdot \rho^{j-1}(t) \\
+ \psi_j^1 \cdot \rho^{j-1}(t) + \dot{q}(t) \dot{\tau}(t, q)) \right) \right] \eta(t, q) \\
+ \partial_1 (L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t)) \eta(t, q) + \left( L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t) - \sum_{j=1}^{m} \psi_j^1 \cdot q^{(j)}(t) \right) \dot{\eta}(t, q) \right] \, dt \\
= \int_{t_1}^{t_2 - \tau} \left[ \partial_1 (L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t)) \eta(t, q) + (L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t)) \dot{\eta}(t, q) \\
+ (\partial_2 (L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t)) + \partial_{m+3} (L[q]_r^m(t + \tau) - \lambda \cdot g[q]_r^m(t + \tau))) \cdot \xi(t, q) \\
+ \psi_1^1 \cdot (\rho^1(t) + \dot{q}(t) \dot{\tau}(t, q)) - \psi_1^1 \cdot \rho^1(t) - \psi_1^1 \cdot \dot{q}(t) \eta(t, q) + \psi_1^1 \cdot \rho^0(t) + \sum_{j=2}^{m} \left( \partial_{j+1} (L[q]_r^m(t) - \lambda \cdot g[q]_r^m(t)) + \partial_{j+m+2} (L[q]_r^m(t + \tau) - \lambda \cdot g[q]_r^m(t + \tau)) \cdot \rho^{j-1}(t) \\
- \Phi[q]_r^m(t) \right] \, dt = 0. \quad (59)
\]

Simplification of (59) leads to the necessary condition of invariance (19). \( \Box \)
4 Noether’s theorem for isoperimetric problems of the optimal control with time delay

Using Theorem 10 we obtain here a Noether’s theorem for the isoperimetric optimal control problems with time delay introduced in 9: to minimize

\[ I^*[q(\cdot),u(\cdot)] = \int_{t_1}^{t_2} L(t,q(t),u(t),q(t-\tau),u(t-\tau)) \, dt \]  

(60)

subject to the delayed control system

\[ \dot{q}(t) = \varphi(t,q(t),u(t),q(t-\tau),u(t-\tau)), \]  

(61)

isoperimetric equality constraints

\[ \int_{t_1}^{t_2} g(t,q(t),u(t),q(t-\tau),u(t-\tau)) \, dt = l, \quad l \in \mathbb{R}^k \]  

(62)

and initial condition

\[ q(t) = \delta(t), \quad t \in [t_1 - \tau,t_1], \]  

(63)

where \( q(\cdot) \in C^1([t_1 - \tau,t_2],\mathbb{R}^n) \), \( u(\cdot) \in C^0([t_1 - \tau,t_2],\mathbb{R}^m) \), the Lagrangian \( L : [t_1,t_2] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) and the velocity vector \( \varphi : [t_1,t_2] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) are assumed to be \( C^1 \)-functions with respect to all their arguments, \( t_1 < t_2 \) are fixed in \( \mathbb{R} \), and \( \tau \) is a given positive real number such that \( \tau < t_2 - t_1 \). As before, we assume that \( \delta \) is a given piecewise smooth function.

Remark 26. In the particular case when \( \varphi(t,q,u,q_v,u_v) = u \), problem (60)–(63) is reduced to the Problem 4 The problems of the calculus of variations with higher-order derivatives are also easily written in the optimal control form (60)–(63). For example, the delayed isoperimetric problem of the calculus of variations with derivatives of second order,

\[ I[q(\cdot)] = \int_{t_1}^{t_2} F(t,q(t),\dot{q}(t),q(t-\tau),\dot{q}(t-\tau),\ddot{q}(t-\tau)) \, dt \]  

(64)

is equivalent to problem

\[ I[q^0(\cdot),q^1(\cdot),u(\cdot)] = \int_{t_1}^{t_2} F(t,q^0(t),q^1(t),u(t),q^0(t-\tau),q^1(t-\tau),u(t-\tau)) \, dt \]  

(65)

where \( F = L - \lambda \cdot g \) and \( g = g(t,q(t),\dot{q}(t),q(t-\tau),\dot{q}(t-\tau),\ddot{q}(t-\tau)) \).

Notation. We introduce the operators \([\cdot,\cdot]_\tau\) and \([\cdot,\cdot,\cdot]_\tau\) defined by

\[ [q,u]_\tau(t) = (t,q(t),u(t),q(t-\tau),u(t-\tau)), \]

where \( q(\cdot) \in C^1([t_1 - \tau,t_2],\mathbb{R}^n) \) and \( u(\cdot) \in C^0([t_1 - \tau,t_2],\mathbb{R}^m) \); and

\[ [q,u,p,\lambda]_\tau(t) = (t,q(t),u(t),q(t-\tau),u(t-\tau),p(t),\lambda), \]

where \( q(\cdot) \in C^1([t_1 - \tau,t_2],\mathbb{R}^n) \), \( p(\cdot) \in C^1([t_1,t_2],\mathbb{R}^n) \), \( u(\cdot) \in C^0([t_1 - \tau,t_2],\mathbb{R}^m) \) and \( \lambda \in \mathbb{R}^k \).
Definition 27. The delayed differential control system (61) is called an isoperimetric control system with time delay.

Definition 28. (Isoperimetric process with time delay) An admissible pair \((q(\cdot), u(\cdot))\) that satisfies the isoperimetric control system (61) and the isoperimetric constraints (2) is said to be a isoperimetric process with time delay.

Theorem 29. (Isoperimetric Pontryagin maximum principle [9]) If \((q(\cdot), u(\cdot))\) is a minimizer of (60)–(63), then there exists a covector function \(p(\cdot) \in C^1([t_1, t_2], \mathbb{R}^n)\) such that for all \(t \in [t_1 - \tau, t_2]\) the following conditions hold:

- the isoperimetric Hamiltonian systems with time delay
  \[
  \begin{align*}
  \dot{q}(t) &= \partial_2 H[q, u, p, \lambda](t) \\
  \dot{p}(t) &= -\partial_1 H[q, u, p, \lambda](t) - \partial_4 H[q, u, p, \lambda](t + \tau)
  \end{align*}
  \]
  for \(t_1 \leq t \leq t_2 - \tau\), and

- the isoperimetric stationary conditions with time delay
  \[
  \begin{align*}
  \partial_4 H[q, u, p, \lambda](t) &= 0 \\
  \partial_3 H[q, u, p, \lambda](t) &= 0
  \end{align*}
  \]
  for \(t_2 - \tau \leq t \leq t_2\):

where the isoperimetric Hamiltonian \(H\) is defined by

\[
H[q, u, p, \lambda](t) = L[q, u]_{\tau}(t) - \lambda \cdot q[g, u]_{\tau}(t) + p(t) \cdot \varphi[q, u]_{\tau}(t).
\]

Definition 30. A triplet \((q(\cdot), u(\cdot), p(\cdot))\) satisfying the conditions of Theorem 29 is called an isoperimetric Pontryagin extremal with time delay.

We define the notion of invariance for Problem (60)–(63) in terms of the Hamiltonian, by introducing the augmented functional as in [3]:

\[
\mathcal{J}[q(\cdot), u(\cdot), p(\cdot)] = \int_{t_1}^{t_2} [H(t, q(t), u(t), q(t - \tau), u(t - \tau), p(t)) - p(t) \cdot \dot{q}(t)] dt
\]
subject to (63), where \(H\) is given by (69). The notion of invariance for (60)–(61) is defined using the invariance of (70).

Definition 31 (cf. Definition 3). Consider the following s-parameter group of infinitesimal transformations:

\[
\begin{align*}
\dot{t} &= t + s \eta(t, q, u) + o(s), \\
\dot{q}(t) &= q(t) + s \xi(t, q, u) + o(s), \\
\dot{u}(t) &= u(t) + s g(t, q, u) + o(s), \\
\dot{p}(t) &= p(t) + s \varphi(t, q, u) + o(s),
\end{align*}
\]

where \(\eta \in C^1(\mathbb{R}^{1+n+m}, \mathbb{R}), \xi, \varsigma \in C^1(\mathbb{R}^{1+n+m}, \mathbb{R}^n), g \in C^0(\mathbb{R}^{1+n+m}, \mathbb{R}^n)\) are given functions. The functional (70) is said to be invariant under (71) if

\[
\frac{d}{ds} \bigg|_{s=0} \int_{t_1}^{t_2} \left[H \left(t + s \eta, q + s \xi, u + s g, q(t - \tau) + s \xi_{\tau}, u(t - \tau) + s g_{\tau}\right) - (p + s \varphi)(1 + s \eta)\right] dt = 0
\]
for any subinterval $I \subseteq [t_1, t_2]$, where $q_\tau(t) = q(t - \tau, q(t - \tau), u(t - \tau))$ and $\xi_\tau(t) = \xi(t - \tau, q(t - \tau), u(t - \tau))$.

**Definition 32.** A quantity $C(t, q(t), q(t - \tau), u(t), u(t - \tau), p(t))$, constant for $t \in [t_1, t_2]$ along any isoperimetric Pontryagin extremal with delay $(q(\cdot), u(\cdot), p(\cdot))$ of problem (60)-(63), is said to be an isoperimetric constant of motion with delay for (60)-(63).

Theorem 33 gives a Noether-type theorem for isoperimetric optimal control problems with time delay.

**Theorem 33 (Isoperimetric Noether symmetry theorem with time delay in Hamiltonian form).** If we have invariance in the sense of Definition 32, then

\[
C(t, q(t), q(t - \tau), u(t), u(t - \tau), p(t)) = -p(t) \cdot \xi(t, q(t), u(t)) + H(t, q(t), u(t), q(t - \tau), u(t - \tau), p(t)) \eta(t, q(t), u(t))
\]  

(72)

is an isoperimetric constant of motion with delay (cf. Definition 32) for (60)-(63).

**Proof.** The constant of motion with delay (72) is obtained by applying Theorem 10 to problem (60).

**Remark 34.** The constant of motion with time delay (72) has the same expression in the two intervals $t_1 \leq t \leq t_2 - \tau$ and $t_2 - \tau \leq t \leq t_2$.

**Remark 35.** For the isoperimetric problem of the calculus of variations (11)-(13), the Hamiltonian (69) takes the form $H = L + p \cdot u$, with $u = \dot{q}$ and $p(t) = -\partial_t L[q]_*(t) - \partial_q L[q]_*(t + \tau)$. In this case the isoperimetric constant of motion with delay (72) reduces to (14) in the interval $t_1 \leq t \leq t_2 - \tau$ and to (15) in the interval $t_2 - \tau \leq t \leq t_2$ with $\Phi \equiv 0$.

**Corollary 36.** (Isoperimetric Noether's theorem for problems of the calculus of variations with second-order derivatives) For the second-order problem of the calculus of variations (14), the isoperimetric constant of motion with delay (72) is equivalent to

\[
F[q]^2(t)\tau + \left(\partial_3 F[q]^2(t) + \partial_q F[q]^2(t + \tau) - \frac{d}{dt} \left(\partial_4 F[q]^2(t) + \partial_q F[q]^2(t + \tau)\right)\right) \cdot (\xi_0 - \dot{q} \tau)
\]  

(73)

for $t_1 \leq t \leq t_2 - \tau$, and

\[
F[q]^2(t)\tau + \left(\partial_3 F[q]^2(t) - \frac{d}{dt} \partial_4 F[q]^2(t)\right) \cdot (\xi_0 - \dot{q} \tau) + \partial_4 F[q]^2(t) \cdot (\xi_1 - \dot{q} \tau)
\]  

(74)

for $t_2 - \tau \leq t \leq t_2$

**Proof.** For simplicity, we only prove the corollary in the interval $t_2 - \tau \leq t \leq t_2$. For the problem of the calculus of variations with second-order derivatives, one has

\[
H(t, q^0(t), q^1(t), u(t), q^0(t - \tau), q^1(t - \tau), u(t - \tau), p^0, p^1)
\]  

\[
= F(t, q^0, q^1, u, q^0(t - \tau), q^1(t - \tau), u(t - \tau)) + p^0 q^1 + p^1 u,
\]

and

\[
\begin{align*}
q^0(t) &= q(t) \\
q^1(t) &= \dot{q}(t) \\
u(t) &= \ddot{q}(t) \\
q^0(t - \tau) &= q(t - \tau) \\
q^1(t - \tau) &= \dot{q}(t - \tau) \\
u(t - \tau) &= \ddot{q}(t - \tau)
\end{align*}
\]
Using these equalities, it follows from the isoperimetric Pontryagin Maximum Principle (Theorem 20) that
\[
\begin{align*}
\frac{\partial H}{\partial u} = 0 & \iff p^1 = \frac{\partial F}{\partial q}, \\
p^0 & = \frac{\partial H}{\partial \dot{q}^0} = \frac{\partial F}{\partial \dot{q}}, \\
p^1 & = \frac{\partial H}{\partial \dot{q}^1} \iff p^0 = \frac{\partial F}{\partial q} - \frac{d}{dt} \frac{\partial F}{\partial \dot{q}}.
\end{align*}
\]
In this case, the constant of motion (72) takes the form
\[
C = H\tau - p^0 \cdot \xi_0 - p^1 \cdot \xi_1,
\]
and substituting \(H, p^0\) and \(p^1\) by its expressions, the intended result is obtained. \(\square\)

**Remark 37.** We can easily verify that for \(m = 2\) the quantities (73) and (74) coincide with (56) and (57), respectively, in the case \(\Phi \equiv 0\).

### 5 Illustrative examples

In this section we consider two examples where the isoperimetric problems do not depend explicitly on the independent variable \(t\) (autonomous case).

**Example 38.** Consider the second-order isoperimetric problem of the calculus of variations with time delay
\[
J^2_I[q(\cdot)] = \int_0^2 (\ddot{q}(t) + \dot{q}(t-1))^2 \, dt \to \min,
\]
subject to isoperimetric equality constraints
\[
I^1[q(\cdot)] = \int_0^2 (\ddot{q} + \dot{q}(t-1))^2 \, dt = l
\]
in the class of functions \(q(\cdot) \in \text{Lip}([-1, 2]; \mathbb{R})\). For this example, the augmented Lagrangian \(F\) is given as
\[
F = (\ddot{q}(t) + \dot{q}(t-\tau))^2 - \lambda (\ddot{q} + \dot{q}(t-1))^2.
\]
From Corollary 17 with \(m = 2\), one obtains that any solution to problem (75)-(79) must satisfy
\[
2q^{(iv)}(t) + q^{(iv)}(t-1) + q^{(iv)}(t+1) + 2\lambda (\ddot{q}(t) + \dot{q}(t-1) + \ddot{q}(t+1)) = 0, \quad 0 \leq t \leq 1,
\]
subject to isoperimetric Noether’s constant of motion with time delay (78)-(79) coincides with the DuBois–Reymond condition (81)-(82) with \(m = 2\):
where \( c_1 \) and \( c_2 \) are constants.

One can easily check that function \( q(\cdot) \in \text{Lip}([-1, 2]; \mathbb{R}^n) \) defined by

\[
q(t) = \begin{cases} 
-t^4 & \text{for } -1 < t \leq 0 \\
 t^4 & \text{for } 0 < t \leq 1 \\
 t^4 + 2 & \text{for } 1 < t \leq 2 
\end{cases} 
\tag{82}
\]

is an isoperimetric Euler–Lagrange extremal, i.e., satisfies (78)–(79) and is also a isoperimetric DuBois–Reymond extremal, i.e., satisfies (80)–(81). Corollary 36 asserts the validity of Noether’s constant of motion, which is here verified: (73)–(74) holds along (82) with \( \eta \equiv 1 \), and \( \xi \equiv 0 \).

**Example 39.** Let us consider an isoperimetric autonomous optimal control problem with time delay, i.e., the situation when \( L, \varphi \) and \( g \) in (60)–(62) do not depend explicitly on \( t \). In this case one has invariance, in the sense of Definition 37, for \( \eta \equiv 1 \) and \( \xi = \varrho = \varsigma \equiv 0 \). It follows from Theorem 33 that

\[
H(q(t), u(t), q(t-\tau), u(t-\tau), p(t)) = \text{constant} 
\tag{83}
\]

along any isoperimetric Pontryagin extremal with delay \((q(\cdot), u(\cdot), p(\cdot))\) of the problem. In the language of mechanics (83) is called conservation of energy.

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