(In)stability of the charged massive scalar field on the Kerr-Newman spacetime

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(Dated: July 7, 2021)

Abstract

In this work, we investigate the quasibound states of charged massive scalar fields in the Kerr-Newman black hole spacetime by using a new method recently developed, which uses the polynomial conditions of the Heun functions. We calculate the resonant frequencies related to the spectrum of quasibound states and the wave functions, and also we analyze the (in)stability of the system. These results are particularized to the cases of the Schwarzschild and Kerr black holes. Additionally, we compare our analytical results with the numerical ones known in the literature. Finally, we apply this method to the supermassive black hole situated at the center of the M87 galaxy, compute the characteristic times of growth and decay of bosonic particles around the astrophysical object and discuss the results.

PACS numbers: 02.30.Gp, 03.65.Ge, 04.20.Jb, 04.62.+v, 04.70.-s, 04.80.Cc, 47.35.Rs, 47.90.+a

Keywords: quantum gravity, Klein-Gordon equation, confluent Heun function, quasistationary level, eigenfunction
I. INTRODUCING THE KERR-NEWMAN BLACK HOLE SPACETIME

The charged generalization of the Kerr geometry [1] was first found as a solution to the Einstein-Maxwell field equations by Newman et al. [2]. The connection to black holes was later discovered by Boyer and Lindquist [3], as well as the global structure of the Kerr family of gravitational fields by Carter [4]. In this section, we will review the properties of the Kerr-Newman black hole solution, on which we want to investigate the behavior of charged massive scalar fields following our previous work [5].

This paper has two broad aims. First, to obtain general solutions for the charged massive Klein-Gordon equation in the Kerr-Newman black hole spacetime. Second, to show that the resonant frequencies describing quasibound states may be calculated analytically using the Vieira-Bezerra-Kokkoras method [6, 7].

The line element describing a black hole with mass \( M \), charge \( Q \), and angular momentum per unit mass \( a = J/M \) can be written as [8]

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\Delta}{\rho^2}(dt - a \sin^2 \theta \, d\phi)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 \, d\theta^2 - \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a \, dt]^2,
\]

where \( \rho^2 = r^2 + a^2 \cos^2 \theta \), while the function \( \Delta = \Delta(r) \) has the form

\[
\Delta = r^2 - 2Mr + a^2 + Q^2.
\]

Here, \( M \) is the total mass centered at the origin of the system of coordinates. It is obvious that at the limits \( a \to 0, Q \to 0, \) and \( a, Q \to 0 \) the well-known Reissner-Nordström, Kerr, and Schwarzschild black holes are recovered, respectively. The electromagnetic 4-vector of the black hole is given by [9]

\[
A_\mu = \frac{Qr}{\rho^2} (1, 0, 0, -a \sin^2 \theta).
\]

The horizons (null surfaces) of the Kerr-Newman black hole (KNBH) are given by the zeros of \( \Delta(r) = 0 \), that is,

\[
\Delta(r) = (r - r_+)(r - r_-) = 0.
\]

The solutions of this horizon surface equation, for \( M^2 > a^2 + Q^2 \), correspond to two horizons, namely, the interior (Cauchy) and exterior (event) horizons, which are given by \( r_- = M - \sqrt{M^2 - a^2 - Q^2} \) and \( r_+ = M + \sqrt{M^2 - a^2 - Q^2} \), respectively. Note that when \( M^2 = a^2 + Q^2 \), we get an extreme Kerr-Newman black hole, that is, the interior and exterior horizons...
coincide at \( r_e = M \). In this work we will focus on the non-extreme case. The exterior event horizon \( r_+ \) is the outermost marginally trapped surface for the outgoing photons. In fact, that is the last surface from which the light waves could still escape from the black hole. Thus, it is meaningful to study quantum scalar particles that propagate outside the exterior event horizon, whose equation of motion is discussed in the next section.

The outline of this paper is as follows. In Sec. II we solve the Klein-Gordon equation in the background under consideration. In Sec. III we impose the appropriately boundary conditions and then we obtain the Hawking radiation spectrum and the resonant frequencies related to the quasibound states. In Sec. IV we provide the eigenfunctions by using some properties of the confluent Heun functions. In Sec. V we apply our results to the supermassive black hole located at the center of M87. Finally, in Sec. VI the conclusions are given. Here we adopted the natural units where \( G \equiv c \equiv \hbar \equiv 1 \).

II. KLEIN-GORDON EQUATION

The equation of motion which describes charged massive scalar particles propagating in a curved spacetime and in the presence of an electromagnetic field is given by

\[
\frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu) + ie(\partial_\mu A^\mu + \frac{ie}{\sqrt{-g}} A^\mu (\partial_\mu \sqrt{-g}) - e^2 A^\mu A_\mu + \mu^2) \Psi = 0, \quad (5)
\]

where \( \Psi = \Psi(t, r, \theta, \phi) \) is the scalar wave function, \( \mu \) is the mass of the scalar particle, and \( e \) is the charge of the scalar particle, and \( g \equiv \det(g_{\mu\nu}) \).

The equations of motion (5) for the metric (1) lead in a system of two separated ordinary differential equations (ODEs) for the angular and radial parts of the scalar wave function, which due to the spherical symmetry can be written as \( \Psi(t, r, \theta, \phi) = e^{-i\omega t} R(r) S(\theta) e^{im\phi} \), where \( \omega \) is the frequency (energy) of the scalar particle, and \( m \) is now the azimuthal quantum number. The ODEs are given by Eqs. (24) and (25) in Ref. [5], namely,

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS}{d\theta} \right) + \left[ \lambda - \left( \omega \sin \theta - \frac{m}{\sin \theta} \right)^2 - \mu^2 a^2 \cos^2 \theta \right] S = 0 \quad (6)
\]

and

\[
\frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + \left\{ \frac{1}{\Delta} [\omega(r^2 + a^2) - am - eQr]^2 - (\lambda + \mu^2 r^2) \right\} R = 0, \quad (7)
\]

where \( \lambda \) is the separation constant.
Thus, an analytical, general solution for the angular part of the Klein-Gordon equation in the KNBH spacetime is given by

\[ S(x) = (x - 1)^2 \{ C_1 \, \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; x) + C_2 \, x^{-\beta} \, \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta; x) \}, \quad (8) \]

where \( C_1 \) and \( C_2 \) are constants to be determined, and \( x = \cos^2 \theta \). The parameters \( \alpha, \beta, \gamma, \delta, \) and \( \eta \) are given by

\[ \alpha = 0, \]
\[ \beta = -\frac{1}{2}, \]
\[ \gamma = \pm m, \]
\[ \delta = \frac{1}{4} a^2 (\mu^2 - \omega^2), \]
\[ \eta = \frac{1}{4} (1 + m^2 - \lambda + a^2 \omega^2 - 2a \omega m). \]

Since \( \beta \) is not an integer, these two functions form linearly independent solutions of the confluent Heun equation, whose canonical form is given by \[10\]

\[ \frac{d^2 U}{dz^2} + \left( \alpha + \frac{\beta + 1}{z} + \frac{\gamma + 1}{z - 1} \right) \frac{dU}{dz} + \left( \frac{\sigma_0}{z} + \frac{\sigma_1}{z - 1} \right) U = 0, \quad (14) \]

where \( U(z) = \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z) \) are the confluent Heun functions, with the parameters \( \alpha, \beta, \gamma, \delta \) and \( \eta \) related to \( \sigma_0 \) and \( \sigma_1 \) by the following expressions

\[ \sigma_0 = \frac{1}{2} (\alpha - \beta - \gamma + \alpha \beta - \beta \gamma) - \eta, \]
\[ \sigma_1 = \frac{1}{2} (\alpha + \beta + \gamma + \alpha \gamma + \beta \gamma) + \delta + \eta. \]

In fact, these results generalize the ones obtained in our paper \[5\], since the sign of the parameter \( \gamma \) can now be chosen according to some boundary conditions; the positive sign corresponds to the solution obtained in \[5\]. The same procedure can be done for the radial part, as follows.

### A. Radial equation

Now, we will also provide an analytic general solution for the radial part of the Klein-Gordon equation in the KNBH spacetime. Then, we will use this solution to impose some
boundary conditions in order to obtain the quasibound states, which involves the choice of
signs determining its asymptotic behaviors.

First, we use equation (4), which provides the values for the horizons, in order to write
the radial equation (7) as
\[
\frac{d^2 R}{dr^2} + \left( \frac{1}{r - r_+} + \frac{1}{r - r_-} \right) \frac{dR}{dr} + \left\{ \frac{[\omega(r^2 + a^2) - am - eQr]^2 - (r - r_+)(r - r_-)(\lambda + \mu^2 r^2)}{(r - r_+)^2(r - r_-)^2} \right\} R = 0. \tag{17}
\]

The new form of the radial equation (17) indicates the existence of two finite regular
singularities associated with the two horizons \( r_+ \) and \( r_- \), and a regular singularity at infinity,
which implies that (17) is a confluent Heun-type equation. Let us transform this radial
equation to a more suitable form, by defining a new radial coordinate \( x \) related to \( r_+ \) and
\( r_- \) as
\[
x = \frac{r - r_+}{r_- - r_+}. \tag{18}
\]
Thus, by substituting Eqs. (18) into Eq. (17), we obtain
\[
\frac{d^2 R(x)}{dx^2} + \left( \frac{1}{x} + \frac{1}{x - 1} \right) \frac{dR(x)}{dx} + \left[ E_2 + \sum_{j=0}^{1} \frac{E_j}{(x - x_j)^2} + \sum_{j=0}^{1} \frac{F_j}{x - x_j} \right] R(x) = 0, \tag{19}
\]
where \( j = 0, 1, 2 \) labels the singularities \( x_j = (0, 1, \infty) \), and the coefficients \( E_j \) and \( F_j \) are
given by
\[
E_0 = \frac{[\omega(r_+^2 + a^2) - am - eQr_+]^2}{(r_+ - r_-)^2}, \tag{20}
\]
\[
E_1 = \frac{[\omega(r_+^2 + a^2) - am - eQr_-]^2}{(r_+ - r_-)^2}, \tag{21}
\]
\[
E_2 = -(r_+ - r_-)^2(\mu^2 - \omega^2), \tag{22}
\]
\[
F_0 = \frac{1}{(r_+ - r_-)^2} [2a^2 m^2 + 2aemQ(r_+ + r_-) + 2e^2 Q^2 r_+ r_- + (r_+ - r_-)^2(\lambda + \mu^2 r_+^2) \]
- \( 2\omega(2a^3 m + a^2 eQr_+ + r^2 eQr_- + 2amr_+ r_- - eQr_+^3 + 3eQr_+^2 r_-) \]
+ \( 2\omega^2(a^2 + r_+^2)(a^2 - r_+^2 + 2r_+ r_-) \), \tag{23}
\]
\[
F_1 = (r_+ - r_-)[2e\omega + (r_+ + r_-)(\mu^2 - 2\omega^2)] - F_0. \tag{24}
\]
The form of the radial equation, given by Eq. (19), is almost similar to the confluent
Heun equation, where we just need to reduce the power of the terms \( 1/(x - x_j)^2 \), as well as
eliminate the term $E_2$ from the right side. In order to do this, we perform a $s$-homotopic transformation of the dependent variable $R(x) \mapsto U(x)$, such that

$$R(x) = x^{D_0}(x - 1)^{D_1}e^{D_2x}U(x), \quad (25)$$

where the coefficients $D_j$ are the exponents of the three singular points $x_j$ in Eq. (19). They obey to the following indicial equation

$$G(z) = s(s - 1) + s + E_j = s^2 + E_j = 0, \quad (26)$$

whose roots are given by

$$s_{1,2}^{x=0} = \pm i\frac{\omega(r_+^2 + a^2) - am - eQr_+}{r_+ - r_-} \equiv D_0, \quad (27)$$

$$s_{1,2}^{x=1} = \pm i\frac{\omega(r_-^2 + a^2) - am - eQr_-}{r_+ - r_-} \equiv D_1, \quad (28)$$

$$s_{1,2}^{x=\infty} = \pm (r_+ - r_-)\sqrt{\mu^2 - \omega^2} \equiv D_2. \quad (29)$$

Then, by substituting Eqs. (25)-(29) into Eq. (19), we derive a new equation for the radial function $U(x)$, namely,

$$\frac{d^2U(x)}{dx^2} + \left(2D_2 + \frac{1 + 2D_0}{x} + \frac{1 + 2D_1}{x - 1}\right)\frac{dU(x)}{dx} + \left(\frac{\sigma_0}{x} + \frac{\sigma_1}{x - 1}\right)U(x) = 0, \quad (30)$$

where

$$\sigma_0 = -D_0 - D_1 - 2D_0D_1 + D_2 + 2D_0D_2 + F_0, \quad (31)$$

$$\sigma_1 = D_0 + D_1 + 2D_0D_1 + D_2 + 2D_1D_2 + F_1. \quad (32)$$

Therefore, these substitutions moved the two singularities, $r_+$ and $r_-$, to the points 0 and 1, respectively. Thus, we can conclude that Eq. (30) has the form of a confluent Heun equation (14).

As a conclusion, we can write an analytic general solution for the radial part of the Klein-Gordon equation in the KNBH spacetime can be written as

$$R(x) = x^{D_0}(x - 1)^{D_1}e^{D_2x}\{C_1 \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; x) + C_2 x^{-\beta} \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta; x)\}, \quad (33)$$

where $C_1$ and $C_2$ are constants to be determined. The parameters $\alpha, \beta, \gamma, \delta,$ and $\eta$ are given by

$$\alpha = 2D_2, \quad (34)$$
\[ \beta = 2D_0, \quad (35) \]
\[ \gamma = 2D_1, \quad (36) \]
\[ \delta = F_0 + F_1, \quad (37) \]
\[ \eta = -F_0. \quad (38) \]

Note that the final expressions for these parameters will depend on the signs to be chosen for the coefficients \( D_j \) given by Eqs. (27)-(29); the positive sign corresponds to the solution obtained in [3]. For our purposes in this work, the negative sign is the correct choice for the coefficients \( D_0 \) and \( D_1 \), as well as the positive sign for \( D_2 \), which will be justified in the discussion of the quasibound states. Therefore, from now on we will consider the following expressions for the coefficients \( D_j \):

\[ D_0 = -i\frac{\omega(r_+^2 + a^2) - am - eQr_+}{r_+ - r_-}, \quad (39) \]
\[ D_1 = -i\frac{\omega(r_-^2 + a^2) - am - eQr_-}{r_+ - r_-}, \quad (40) \]
\[ D_2 = (r_+ - r_-) \sqrt{\mu^2 - \omega^2}. \quad (41) \]

In what follows, we will use this analytical solution of the radial equation in the KNBH spacetime, and the properties of the confluent Heun functions to discuss the quasibound states and their corresponding wave functions.

III. QUASIBOUND STATES

In this section we will investigate the quasibound states, which are solutions to the equation of motion corresponding to ingoing waves at the exterior event horizon and tend to zero at infinity. Thus, this means that there exist two boundary conditions associated to the spectrum of quasibound states. To do this, we need to calculate the spectrum of charged massive scalar resonant frequencies on the KNBH spacetime.

Since the flux of particles crosses into the exterior horizon surface, the spectrum of quasibound states has complex frequencies, which can be expressed as \( \omega_n = \omega_R + i\omega_I \), where \( \omega_R = \text{Re}[\omega_n] \) and \( \omega_I = \text{Im}[\omega_n] \) are the real and imaginary parts, respectively. Here, \( n \) is the overtone number. The wave solution decays with the time if \( \text{Im}[\omega_n] < 0 \), otherwise, it grows if \( \text{Im}[\omega_n] > 0 \).
The common approach used to derive the characteristic resonance equation is to solve the radial equation in two different asymptotic regions and then, by using a standard matching procedure, evaluate these two solutions in their common overlap region \[11–13\]. Here, we will obtain the spectrum of quasibound states by using the technique recently developed by Vieira and Kokkotas \[7\], which corresponds to impose two boundary conditions to the radial solution and then use the polynomial conditions of the Heun functions as matching procedure; it is an extension of the method developed by Vieira and Bezerra \[6\] used to find general expressions for the resonant frequencies.

The first boundary condition is to demand that the radial solution should describe an ingoing wave at the exterior event horizon. In this limit, which means \(r \to r_+\) (or \(x \to 0\)), the radial solution \(R(r)\) has the following asymptotic behavior \[5\]

\[
\lim_{r \to r_+} R(r) \sim C_1 (r - r_+)^{D_0} + C_2 (r - r_+)^{-D_0},
\]

where all remaining constants have been included in \(C_1\) and \(C_2\). Then, from Eq. \(39\), we get

\[
D_0 = -\frac{i}{2\kappa_+}(\omega - \omega_c),
\]

where \(\omega_c = m\Omega_+ + e\Phi_+\) is a critical value related to the superradiance phenomenon, with \(\Omega_+ = a/(r_+^2 + a^2)\) and \(\Phi_+ = Qr_+/(r_+^2 + a^2)\). The gravitational acceleration on the exterior event horizon \(\kappa_+\) is given by

\[
\kappa_+ \equiv \frac{1}{2(r_+^2 + a^2)} \frac{d\Delta(r)}{dr} \bigg|_{r=r_+} = \frac{r_+ - r_-}{2(r_+^2 + a^2)}.
\]

Thus, Eq. \(42\) can be rewritten as

\[
\lim_{r \to r_+} R(r) \sim C_1 \left(R_{\text{in}}(r > r_+) = (r - r_+)^{-\frac{1}{\kappa_+}(\omega - \omega_c)}\right) + C_2 \left(R_{\text{out}}(r > r_+) = (r - r_+)^{\frac{1}{\kappa_+}(\omega - \omega_c)}\right).
\]
The second boundary condition is such that the radial solution must tend to zero far from the black hole at asymptotic infinity. In this limit, which means \( r \to \infty \) (or \( x \to \infty \)), we will use the asymptotic power-series for the two linearly independent solutions of the confluent Heun equation \([14]\), at \( x = \infty \), to expand (in a sector) the radial solution (33) as

\[
\lim_{r \to \infty} R(r) \sim r^{D_0+D_1} e^{e_{-r+}r} \left\{ C_1 \ r^{-(\frac{\beta+\gamma+2}{2}+\frac{4}{\alpha})} + C_2 \ e^{-\alpha r} \ r^{-(\frac{\beta+\gamma+2}{2}-\frac{4}{\alpha})} \right\}. \quad (49)
\]

However, since \( C_2 = 0 \), we get

\[
\lim_{r \to \infty} R(r) \sim C_1 \ r^{-1} e^{-qr}, \quad (50)
\]

where

\[
p = \frac{eQ\omega + M(\mu^2 - 2\omega^2)}{q}, \quad (51)
\]

\[
q = \sqrt{\mu^2 - \omega^2}. \quad (52)
\]

The sign of the real part of \( q \) determines the behavior of the wave function as \( r \to \infty \). On the other hand, the sign of the real part of \( p \) does not play an important role on determining the asymptotic behavior of the wave function, since the exponential function is dominant in such a limit. In fact, by imposing additionally that \( \text{Re}[p] < 0 \), we can obtain a solution which tends directly to zero (without “oscillations” for small \( r \)). Thus, if \( \text{Re}[q] > 0 \), the solution tends to zero, whereas if \( \text{Re}[q] < 0 \), the solution diverges; by definition, the quasibound state solutions are ingoing at the horizon, and tend to zero at infinity (\( \text{Re}[q] > 0 \)). The final behavior of the scalar wave function will be determined when we know the values of the frequency \( \omega \) (depending on the values of the parameters \( e, Q, M, \mu, \) and \( a \)), which will be obtained in what follows.

\section*{A. Polynomial condition}

Now, we will follow the technique developed by Vieira and Kokkotas \([7]\), in which the quasibound states of scalar fields in a black hole spacetime can be found by imposing the polynomial condition related to the Heun polynomials.

It is known that a confluent Heun polynomial is the solution of the confluent Heun equation which is valid at all its singularities, in the sense of being simultaneously a Frobenius solution at each one of them \([14]\). In fact, this is a kind of standard matching procedure.
Therefore, in order to satisfy the second boundary condition, it is necessary that the radial solution must be written in terms of the confluent Heun polynomials, which means that we need to derive a form of the confluent Heun functions that present a polynomial behavior. This will be done in the next section.

Thus, we will use the fact that the confluent Heun functions become polynomials of degree \( n \) if they satisfy the so-called \( \delta \)-condition \[10\], which is given by

\[
\frac{\delta}{\alpha} + \frac{\beta + \gamma}{2} + 1 = -n,
\]

where \( n = 0, 1, 2, \ldots \) is now the principal quantum number. Such polynomial solutions are denoted by \( H_{p_n}(x) \) and their properties will be discussed in the next section. Then, by imposing the polynomial condition given by Eq. (53), we obtain the following characteristic resonance equation

\[
c_4 \left( \frac{\omega}{\mu} \right)^4 + c_3 \left( \frac{\omega}{\mu} \right)^3 + c_2 \left( \frac{\omega}{\mu} \right)^2 + c_1 \left( \frac{\omega}{\mu} \right) + c_0 = 0,
\]

where

\[
c_0 = 2i(n+1)(eQ + am)\sqrt{1 - a^2 - Q^2 - 2eQam} - a^2[m^2 - (M\mu)^2 + (n+1)^2] - Q^2[e^2 - (M\mu)^2 + (n+1)^2] - (M\mu)^2 + (n+1)^2,
\]
\[
c_1 = 2M\mu\{eQ(a^2 + 1) + (Q^2 - 2)[i(n+1)\sqrt{1 - a^2 - Q^2 - am}]\},
\]
\[
c_2 = -2i(n+1)(eQ + am)\sqrt{1 - a^2 - Q^2 + 2eQam} + a^2[e^2Q^2 + m^2 - 4(M\mu)^2 + (n+1)^2] + Q^2[e^2 - (M\mu)^2] + (n+1)^2(Q^2 - 1),
\]
\[
c_3 = -2M\mu\{eQ(2a^2 + eQ^2) + (Q^2 - 2)[i(n+1)\sqrt{1 - a^2 - Q^2 - am}]\},
\]
\[
c_4 = (M\mu)^2(4a^2 + Q^4).
\]

Note that our results will be given in terms of the gravitational coupling \( M\mu \), which is the (dimensionless) product of the black hole mass \( M \) and the scalar field mass \( \mu \). Furthermore, from here on, we will also set \( M = 1 \), so that \( r \) and \( a \) are measured in units of \( M \), and \( \omega \) and \( \mu \) in units of \( M^{-1} \).

Now, let us do the first validation of this approach. To do this, we compute the massive resonant frequencies in the limit \( M\mu \ll 1 \), which means the lowest order in \( M\mu \). In this limit, the quasibound state spectrum of massive Klein-Gordon fields is the solution of a second order equation in \( \omega/\mu \), and is given by

\[
\frac{\omega}{\mu} = \pm \left( 1 - \frac{M^2\mu^2}{2n^2} \right),
\]

(56)
\[ h \omega_n = \pm \left( 1 - \frac{GM^2 \mu^2}{2\hbar c n^2} \right) \mu c^2, \]  
(57)

where \( \bar{n} = n + 1 \) is the principal quantum number associated to the states. The signs + and − are related to the nonrelativistic particle and antiparticle spectrum, respectively. Therefore, the massive charged scalar field has a hydrogenic spectrum in this limit, which does not depend on the rotation and charge of the black hole.

In what follows, as a second validation of this approach, we solve the characteristic resonance equation (54) for the particular cases of the Schwarzschild \((a = Q = 0)\) and Kerr \((Q = 0)\) black holes and compare our analytical results with the numerical ones obtained by Dolan [15]. Next, we will present the quasibound state spectrum in the charged and rotating case, that is, in the KNBH spacetime and then we compare our analytical results with the ones obtained by Furuhashi and Nambu [16]; they used two approaches: numerical and analytic approximation.

**B. Schwarzschild black hole**

In the particular case of \(a = Q = 0\), the metric (1) reduces to the Schwarzschild form. This implies \(a_4 = 0\) in Eq. (54) and means that the characteristic resonance equation has three solutions.

The Wolfram Mathematica© can solve cubic equations exactly. However, the final expressions are quite long, and for this reason no insight is gained by writing them out. Thus, instead of doing this, we will discuss some of their features.

The obtained quasispectrum is complex, as expected, and the eigenvalues will be denoted as \(\omega/\mu\)\(_n^{(j)}\), where \(j = 1, 2, 3\) labels the solutions. As a consequence, the coefficients \(p\) and \(q\), which are given by Eq. (51) and (52), can be denoted as \(p_n^{(j)}\) and \(q_n^{(j)}\).

Some values of the massive scalar resonant frequencies in the Schwarzschild black hole, as well as the corresponding coefficients \(p_n^{(j)}\) and \(q_n^{(j)}\), are shown in Table I as functions of the gravitational coupling \(M\mu\).
TABLE I. Values of the resonant frequencies \((\omega/\mu)^{(j)}_n\) and the real part of the corresponding coefficients \(p^{(j)}_n\) and \(q^{(j)}_n\), in the Schwarzschild black hole spacetime for \(0.01 \leq M\mu \leq 1\). We focus on the fundamental mode \(n = 0\).

| \(M\mu\) | \((\omega/\mu)^{(1)}\) | \(\text{Re}[p^{(1)}_0]\) | \(\text{Re}[q^{(1)}_0]\) | \((\omega/\mu)^{(2)}\) | \(\text{Re}[p^{(2)}_0]\) | \(\text{Re}[q^{(2)}_0]\) | \((\omega/\mu)^{(3)}\) | \(\text{Re}[p^{(3)}_0]\) | \(\text{Re}[q^{(3)}_0]\) |
|---------|------------------|-------------|-------------|------------------|-------------|-------------|------------------|-------------|-------------|
| 0.01    | 0.999950 − 0.000001i | −1.000000 | 0.000000 | −25.00000i | 0.500000 | 0.250200 | −0.999950 − 0.000001i | −1.000000 | 0.000099 |
| 0.10    | 0.995674 − 0.001728i | −0.999654 | 0.000468 | −2.496540i | 0.500601 | 0.268937 | −0.995674 − 0.001728i | −0.999654 | 0.009468 |
| 0.14    | 0.992484 − 0.004207i | −0.998822 | 0.017751 | −1.777300i | 0.502356 | 0.285504 | −0.992484 − 0.004207i | −0.998822 | 0.017751 |
| 0.15    | −1.656650i         | 0.503094 | 0.290260 | −0.991657 − 0.005007i | −0.998498 | 0.020130 | 0.991657 − 0.005007i | −0.998498 | 0.020130 |
| 0.20    | −1.230100i         | 0.507958 | 0.317059 | −0.987637 − 0.009947i | −0.996021 | 0.033529 | 0.987637 − 0.009947i | −0.996021 | 0.033529 |
| 0.30    | −0.787020i         | 0.527788 | 0.381767 | −0.981333 − 0.023156i | −0.986106 | 0.065883 | 0.981333 − 0.023156i | −0.986106 | 0.065883 |
| 0.40    | −0.548101i         | 0.561519 | 0.456143 | −0.977943 − 0.038449i | −0.969241 | 0.103072 | 0.977943 − 0.038449i | −0.969241 | 0.103072 |
| 0.50    | −0.391622i         | 0.608378 | 0.536975 | −0.977047 − 0.054189i | −0.945811 | 0.143487 | 0.977047 − 0.054189i | −0.945811 | 0.143487 |
| 0.60    | −0.277383i         | 0.667140 | 0.622655 | −0.978016 − 0.069641i | −0.916430 | 0.186327 | 0.978016 − 0.069641i | −0.916430 | 0.186327 |
| 0.70    | −0.188125i         | 0.736626 | 0.712279 | −0.980338 − 0.084509i | −0.881687 | 0.231140 | 0.980338 − 0.084509i | −0.881687 | 0.231140 |
| 0.80    | −0.115116i         | 0.815815 | 0.805283 | −0.983635 − 0.098692i | −0.842093 | 0.277642 | 0.983635 − 0.098692i | −0.842093 | 0.277642 |
| 0.90    | −0.053418i         | 0.903848 | 0.901283 | −0.987639 − 0.112180i | −0.798076 | 0.325642 | 0.987639 − 0.112180i | −0.798076 | 0.325642 |
| 1.00    | 0                 | 1.000000 | 1.000000 | −0.992157 − 0.125000i | −0.750000 | 0.375000 | 0.992157 − 0.125000i | −0.750000 | 0.375000 |
From Table I we conclude that all resonant frequencies \((\omega/\mu)^{(j)}_0\) are physically admissible, since \(\text{Re}[q_0^{(j)}] > 0\), which represent the quasibound state energies for massive scalar particles in the Schwarzschild black hole spacetime. In fact, we can further restrict the number of acceptable solutions by requiring that \(\text{Re}[p_0^{(j)}] < 0\) and then we get that \((\omega/\mu)^{(3)}_0\) is the unique physically acceptable solution valid in the entire range \(M\mu \geq 0.01\). However, the solution \((\omega/\mu)^{(1)}_0\) is also physically acceptable for \(0.01 \leq M\mu \lesssim 0.14\), as well as \((\omega/\mu)^{(2)}_0\) for \(M\mu \gtrsim 0.15\). In these cases, the radial solution \((33)\), with \(C_2 = 0\), tends to zero far from the Schwarzschild black hole at asymptotic infinity, as required by the conditions for quasibound states. Among these eigenvalues, at least one, namely, \((\omega/\mu)^{(1)}_0\), may describe an unstable system, since there is a change in the sign of its imaginary part when \(M\mu = 1\). Furthermore, that is an over-damped motion with purely imaginary frequencies for \(0.15 \lesssim M\mu \lesssim 1\). On the other hand, two solutions have the same decay rate \((\text{Im}[(\omega/\mu)^{(1)}_0] = \text{Im}[(\omega/\mu)^{(3)}_0])\) for \(0.01 \leq M\mu \lesssim 0.14\), and \(\text{Im}[(\omega/\mu)^{(2)}_0] = \text{Im}[(\omega/\mu)^{(3)}_0]\) for \(M\mu \gtrsim 0.15\) and opposite oscillation frequency \((\text{Re}[(\omega/\mu)^{(1)}_0] = -\text{Re}[(\omega/\mu)^{(3)}_0])\) for \(0.01 \leq M\mu \lesssim 0.14\), and \(\text{Re}[(\omega/\mu)^{(2)}_0] = -\text{Re}[(\omega/\mu)^{(3)}_0]\) for \(M\mu \gtrsim 0.15\), which may describe the pair production of a particle and its antiparticle from a boson.

In order to compare our analytical results with the numerical ones obtained by Dolan [15], we show the behavior of the massive scalar resonant frequencies \((\omega/\mu)^{(j)}_n\) in Fig. 1 as a function of the gravitational coupling \(M\mu\).
FIG. 1. Massive scalar resonant frequencies in the Schwarzschild black hole spacetime. The right plots show the oscillation frequency $\text{Re}[\left(\frac{\omega}{\mu}\right)^{(j)}_n]$, while the left plots show the decay (or growth) rate $\text{Im}[\left(\frac{\omega}{\mu}\right)^{(j)}_n]$. Thus, in Fig. 1 we can see that the resonant frequencies $\left(\frac{\omega}{\mu}\right)^{(3)}_n$ have the same behavior as the ones presented in Fig. 3 of Dolan’s paper [15]; here, $n$ “plays the role” of $l$ in comparison with Dolan’s paper. Therefore, it indicates that the Vieira-Bezerra-Kokkotas method is a (analytical) generalization of the (numerical) approaches known in the literature (see [15] and references therein). Furthermore, this method gives the complete set of resonant frequencies; in this case, we get three frequency (energy) eigenvalues, which describes the
full behavior of massive scalar particles in the background under consideration. In addition, the graph in the upper right corner of Fig. 1 may describe the instability of such a system.

C. Kerr black hole

In the particular case when $Q = 0$, the metric (1) reduces to the Kerr form. This implies that the characteristic resonance equation (54) has four solutions, which can also be found by using the Wolfram Mathematica®.

The obtained quasispectrum is also complex, and the eigenvalues are denoted by $(\omega/\mu)_{n,m}^{(j)}$, where $j = 1, 2, 3, 4$ labels the solutions. As a consequence, the coefficients $p$ and $q$ given by Eq. (51) and (52), will be denoted as $p_{n,m}^{(j)}$ and $q_{n,m}^{(j)}$.

Some values of the massive scalar resonant frequencies in the Kerr black hole, as well as the corresponding coefficients $p_{n,m}^{(j)}$ and $q_{n,m}^{(j)}$, are shown in Table II as functions of the gravitational coupling $M\mu$. 

TABLE II. Values of the resonant frequencies $(\omega/\mu)_{n,m}^{(j)}$, and the real part of the corresponding coefficients $p_{n,m}^{(j)}$ and $q_{n,m}^{(j)}$, in the Kerr black hole spacetime for $a = 0.1$ and $0.01 \le M \mu \le 1$. We focus on the fundamental mode $n = m = 0$.

| $M \mu$ | $(\omega/\mu)_{0,0}^{(1)}$ | Re$[p_{0,0}^{(1)}]$ | Re$[q_{0,0}^{(1)}]$ | $(\omega/\mu)_{0,0}^{(2)}$ | Re$[p_{0,0}^{(2)}]$ | Re$[q_{0,0}^{(2)}]$ | $(\omega/\mu)_{0,0}^{(3)}$ | Re$[p_{0,0}^{(3)}]$ | Re$[q_{0,0}^{(3)}]$ | $(\omega/\mu)_{0,0}^{(4)}$ | Re$[p_{0,0}^{(4)}]$ | Re$[q_{0,0}^{(4)}]$ |
|---------|-----------------|----------------|----------------|-----------------|----------------|----------------|-----------------|----------------|----------------|-----------------|----------------|----------------|
| 0.01    | $-0.9999 - 2 \cdot 10^{-6}i$ | 1.0000 | 9.10^{-5} | $-24.937i$ | 0.4987 | 0.2483 | $-9924.9i$ | 198.49 | 98.751 | 0.9999 - 2.10^{-6}i | 1.0000 | 9.10^{-5} |
| 0.10    | $-0.9956 - 0.0017i$ | 0.9996 | 0.0094 | $-2.4902i$ | 0.4994 | 0.2670 | $-992.49i$ | 198.49 | 98.751 | 0.9956 - 0.0017i | 0.9996 | 0.0094 |
| 0.14    | $-0.9924 - 0.0042i$ | 0.9988 | 0.0176 | $-1.7727i$ | 0.5011 | 0.2835 | $-708.92i$ | 198.49 | 98.752 | 0.9924 - 0.0042i | 0.9988 | 0.0176 |
| 0.15    | $-661.66i$ | 198.49 | 98.752 | $-1.6524i$ | 0.5017 | 0.2882 | $-991.6 - 0.0050i$ | 0.9984 | 0.0200 | 0.9916 - 0.0050i | 0.9984 | 0.0200 |
| 0.20    | $-496.24i$ | 198.49 | 98.752 | $-1.2269i$ | 0.5067 | 0.3149 | $-987.6 - 0.0099i$ | 0.9959 | 0.0333 | 0.9876 - 0.0099i | 0.9959 | 0.0333 |
| 0.30    | $-330.83i$ | 198.49 | 98.752 | $-0.7848i$ | 0.5267 | 0.3794 | $-9814 - 0.0232i$ | 0.9860 | 0.0654 | 0.9814 - 0.0232i | 0.9860 | 0.0654 |
| 0.40    | $-248.12i$ | 198.49 | 98.752 | $-0.5464i$ | 0.5606 | 0.4535 | $-9781 - 0.0384i$ | 0.9690 | 0.1023 | 0.9781 - 0.0384i | 0.9690 | 0.1023 |
| 0.50    | $-198.49i$ | 198.49 | 98.753 | $-0.3903i$ | 0.6077 | 0.5340 | $-9774 - 0.0542i$ | 0.9455 | 0.1423 | 0.9774 - 0.0542i | 0.9455 | 0.1423 |
| 0.60    | $-165.41i$ | 198.49 | 98.753 | $-0.2763i$ | 0.6666 | 0.6193 | $-9785 - 0.0696i$ | 0.9160 | 0.1847 | 0.9785 - 0.0696i | 0.9160 | 0.1847 |
| 0.70    | $-141.78i$ | 198.49 | 98.754 | $-0.1873i$ | 0.7363 | 0.7086 | $-9810 - 0.0844i$ | 0.8812 | 0.2290 | 0.9810 - 0.0844i | 0.8812 | 0.2290 |
| 0.80    | $-124.06i$ | 198.49 | 98.755 | $-0.1146i$ | 0.8156 | 0.8012 | $-9845 - 0.0985i$ | 0.8415 | 0.2749 | 0.9845 - 0.0985i | 0.8415 | 0.2749 |
| 0.90    | $-110.27i$ | 198.49 | 98.756 | $-0.0531i$ | 0.9038 | 0.8967 | $-9887 - 0.1119i$ | 0.7974 | 0.3222 | 0.9887 - 0.1119i | 0.7974 | 0.3222 |
| 1.00    | $-99.249i$ | 198.49 | 98.756 | 0 | 1.0000 | 0.9949 | $-9934 - 0.1246i$ | 0.7493 | 0.3709 | 0.9934 - 0.1246i | 0.7493 | 0.3709 |
From Table II we conclude that all resonant frequencies \((\omega/\mu)_{0,0}^{(j)}\) are physically admissible, since \(\text{Re}[q_{0,0}^{(j)}] > 0\), which represent the quasibound state energies for massive scalar particles in the Kerr black hole spacetime. In fact, we can further restrict the number of acceptable solutions by requiring that \(\text{Re}[p_{0,0}^{(j)}] < 0\) and then we get that \((\omega/\mu)_{0,0}^{(4)}\) is the unique physically acceptable solution valid in the entire range \(M\mu \geq 0.01\). However, the solution \((\omega/\mu)_{0,0}^{(1)}\) is also physically acceptable for \(0.01 \lesssim M\mu \leq 0.14\), as well as \((\omega/\mu)_{0,0}^{(3)}\) for \(M\mu \gtrsim 0.15\). Among these eigenvalues, at least one \(((\omega/\mu)_{0,0}^{(2)})\) may describe an unstable system, since there is a change in the sign of its imaginary part when \(M\mu = 1\). Furthermore, that is an over-damped motion with purely imaginary frequencies for \(0.01 \leq M\mu \lesssim 1\). On the other hand, two solutions have the same decay rate \((\text{Im}[(\omega/\mu)_{n,m}^{(1)}] = \text{Im}[(\omega/\mu)_{n,m}^{(4)}])\) for \(0.01 \leq M\mu \lesssim 1.4\), and \((\text{Im}[(\omega/\mu)_{n,m}^{(3)}] = \text{Im}[(\omega/\mu)_{n,m}^{(4)}])\) for \(M\mu \gtrsim 0.15\) and opposite oscillation frequency \((\text{Re}[(\omega/\mu)_{n,m}^{(1)}] = -\text{Re}[(\omega/\mu)_{n,m}^{(4)}])\) for \(0.01 \leq M\mu \lesssim 1.4\), and \((\text{Re}[(\omega/\mu)_{n,m}^{(3)}] = -\text{Re}[(\omega/\mu)_{n,m}^{(4)}])\) for \(M\mu \gtrsim 0.15\), which may describe the pair production of a particle and its antiparticle from a boson.

We also show the behavior of the massive scalar resonant frequencies \((\omega/\mu)_{n,m}^{(j)}\) in Fig. 2 as a function of the gravitational coupling \(M\mu\).
FIG. 2. Massive scalar resonant frequencies in the Kerr black hole spacetime for $m = 0$ and $a = 0.99$. The left plots show the oscillation frequency $\text{Re}[(\omega/\mu)_{n,0}^{(j)}]$, while the right plots show the decay (or growth) rate $\text{Im}[(\omega/\mu)_{n,0}^{(j)}]$.

In Fig. 2 we see that the resonant frequencies $(\omega/\mu)_{n,m}^{(4)}$ have a behavior similar to the results presented in Fig. 4 of Dolan’s paper [15]: here, $n$ “plays the role” of $m$ in comparison with Dolan’s paper. Anyway, it also indicates that the Vieira-Bezerra-Kokkotas method
generalizes the approaches known in the literature (see [15] and references therein). In the present case, the spectrum has four frequency (energy) eigenvalues, which describes the full behavior of massive scalar particles in the background under consideration.

As in the Dolan’s paper, “zooming in” on the lower plot of Fig. 2 reveals the instability of the Kerr back hole, since the imaginary part of the resonant frequencies is actually positive at couplings \(M \mu \sim 1\), and high rotating \(a\) as well. Therefore, it is meaningful to study the (in)stability of such a system for some interesting \(m\) states, which is shown in Table IV and Fig. 4.

D. Kerr-Newman black hole

Now, we present the quasibound state spectrum in the charged and rotating case, that is, in the Kerr-Newman black hole. In this case, the characteristic resonance equation (54) has four solutions, which are denoted by \((\omega/\mu)_{n,m}^{(j)}\), where \(j = 1, 2, 3, 4\) labels the solutions.

Some values of the massive scalar resonant frequencies in the Kerr-Newman black hole, as well as the corresponding coefficients \(p_{n,m}^{(j)}\) and \(q_{n,m}^{(j)}\), are shown in Table III as functions of the gravitational coupling \(M \mu\).
TABLE III. Values of the resonant frequencies \( (\omega/\mu)_{n,m}^{(j)} \), and the real part of the corresponding coefficients \( p_{n,m}^{(j)} \) and \( q_{n,m}^{(j)} \), in the Kerr-Newman black hole spacetime for \( a = 0.1, e = 0.01, Q = 0.1 \) and\( 0.01 \leq M\mu \leq 1 \). We focus on the fundamental mode \( n = m = 0 \).

| \( M\mu \) | \( (\omega/\mu)_{0,0}^{(1)} \) | \( \text{Re}[p_{0,0}^{(1)}] \) | \( \text{Re}[q_{0,0}^{(1)}] \) | \( (\omega/\mu)_{0,0}^{(2)} \) | \( \text{Re}[p_{0,0}^{(2)}] \) | \( \text{Re}[q_{0,0}^{(2)}] \) | \( (\omega/\mu)_{0,0}^{(3)} \) | \( \text{Re}[p_{0,0}^{(3)}] \) | \( \text{Re}[q_{0,0}^{(3)}] \) | \( (\omega/\mu)_{0,0}^{(4)} \) | \( \text{Re}[p_{0,0}^{(4)}] \) | \( \text{Re}[q_{0,0}^{(4)}] \) |
|------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 0.01       | -0.9999 – 2.10^{-6}i | -1.0000              | 0.0001              | 0.9995 – 9800.5i     | 196.01               | 97.020               | 0.0501 – 24.936i     | 0.4987               | 0.2470               | 0.9999 – 1.10^{-6}i | -1.0000              | 8.10^{-5}            |
| 0.10       | -0.9955 – 0.0017i    | -0.9996              | 0.0094              | 0.9999 – 980.05i     | 196.01               | 97.020               | 0.0048 – 2.4901i     | 0.4994               | 0.2656               | 0.9957 – 0.0016i    | -0.9996              | 0.0092               |
| 0.14       | -0.9924 – 0.0042i    | -0.9987              | 0.0176              | 0.9971 – 700.03i     | 196.01               | 97.020               | 0.0033 – 1.7727i     | 0.5011               | 0.2820               | 0.9925 – 0.0041i    | -0.9988              | 0.0174               |
| 0.15       | -0.9915 – 0.0051i    | -0.9984              | 0.0200              | 0.9917 – 0.0049i     | -0.9985              | 0.0197               | 0.0031 – 1.6523i     | 0.5017               | 0.2867               | 0.9966 – 0.0066i    | 653.361              | 196.01               | 97.020               |
| 0.20       | -0.9875 – 0.0100i    | -0.9959              | 0.0332              | 0.9877 – 0.0098i     | -0.9960              | 0.0330               | 0.0023 – 1.2268i     | 0.5067               | 0.3133               | 0.0049 – 490.02i   | 196.01               | 97.020               |
| 0.30       | -0.9813 – 0.0233i    | -0.9859              | 0.0652              | 0.9815 – 0.0230i     | -0.9860              | 0.0649               | 0.0014 – 0.7848i     | 0.5267               | 0.3775               | 0.0033 – 326.68i   | 196.01               | 97.020               |
| 0.40       | -0.9781 – 0.0386i    | -0.9689              | 0.1019              | 0.9782 – 0.0383i     | -0.9691              | 0.1015               | 0.0011 – 0.5464i     | 0.5606               | 0.4512               | 0.0024 – 245.01i   | 196.01               | 97.020               |
| 0.50       | -0.9773 – 0.0543i    | -0.9453              | 0.1418              | 0.9774 – 0.0540i     | -0.9457              | 0.1414               | 0.0008 – 0.3903i     | 0.6076               | 0.5313               | 0.0019 – 196.01i   | 196.01               | 97.021               |
| 0.60       | -0.9785 – 0.0698i    | -0.9158              | 0.1840              | 0.9786 – 0.0694i     | -0.9162              | 0.1835               | 0.0007 – 0.2763i     | 0.6666               | 0.6162               | 0.0016 – 163.34i   | 196.01               | 97.021               |
| 0.70       | -0.9810 – 0.0846i    | -0.8809              | 0.2280              | 0.9810 – 0.0842i     | -0.8814              | 0.2276               | 0.0006 – 0.1873i     | 0.7363               | 0.7050               | 0.0014 – 140.00i   | 196.01               | 97.022               |
| 0.80       | -0.9845 – 0.0987i    | -0.8412              | 0.2737              | 0.9845 – 0.0983i     | -0.8418              | 0.2732               | 0.0005 – 0.1146i     | 0.8156               | 0.7971               | 0.0012 – 122.50i   | 196.01               | 97.023               |
| 0.90       | -0.9887 – 0.1121i    | -0.7971              | 0.3208              | 0.9887 – 0.1117i     | -0.7978              | 0.3203               | 0.0005 – 0.0531i     | 0.9038               | 0.8922               | 0.0011 – 108.89i   | 196.01               | 97.024               |
| 1.00       | -0.9934 – 0.1248i    | -0.7490              | 0.3692              | 0.9934 – 0.1244i     | -0.7497              | 0.3687               | 0.0005               | 1.0000               | 0.9899               | 0.0010 – 98.005i   | 196.01               | 97.025               |
From Table[III] we conclude that all resonant frequencies \((\omega/\mu)_{0,0}^{(j)}\) are physically admissible, since \(\text{Re}[q_{0,0}^{(j)}] > 0\), which represent the quasibound state energies for massive scalar particles in the Kerr-Newman black hole spacetime. We can further restrict the number of acceptable solutions by requiring that \(\text{Re}[p_{0,0}^{(j)}] < 0\) and then we get that \((\omega/\mu)_{0,0}^{(1)}\) is the unique physically acceptable solution valid in the entire range \(M\mu \geq 0.01\). However, the solution \((\omega/\mu)_{0,0}^{(4)}\) is also physically acceptable for \(0.01 \lesssim M\mu \leq 0.14\), as well as \((\omega/\mu)_{0,0}^{(2)}\) for \(M\mu \gtrsim 0.15\). Among these eigenvalues, at least one \((\omega/\mu)_{0,0}^{(3)}\) may describe an unstable system, since there is a change in the sign of its imaginary part when \(M\mu = 1\). In this scenario, there is not over-damped motion and all solutions are different from each other.

In order to compare our analytical results with the numerical ones obtained by Furuhashi and Nambu[16], we show the behavior of the massive scalar resonant frequencies \((\omega/\mu)_{n,m}^{(j)}\) in Fig. 3, as a function of the gravitational coupling \(M\mu\).
FIG. 3. Massive scalar resonant frequencies in the Kerr-Newman black hole spacetime for $n = 0$, $m = 1$, $a = 0.98$ and $Q = 0.01$. The left plots show the oscillation frequency $\text{Re}[(\omega/\mu)_{0,1}^{(j)}]$, while the right plots show the decay (or growth) rate $\text{Im}[(\omega/\mu)_{0,1}^{(j)}]$.

In Fig. 3 we see that the resonant frequencies $(\omega/\mu)_{0,1}^{(2)}$ have the same behavior as the ones presented in Fig. 6 of Furuhashi and Nambu’s paper [16]. Again, it indicates that the Vieira-Bezerra-Kokkotas method generalizes the approaches known in the literature (see [16]).
and references therein).

The (in)stability of the Kerr-Newman black hole is presented in Table IV and Fig. 9 for the $m = -1, 0, +1$ states, as functions of the gravitational coupling $M\mu$. 
TABLE IV. Maximum instability growth rates, $\tau^{-1} = M \text{Im}[\omega]$, of the $m = \pm 1$ states in the Kerr and Kerr-Newman (with $e = 1$ and $Q = 0.01$) black holes for $n = 0$. 

| $a$  | $m = -1$ | $m = +1$ | $m = -1$ | $m = +1$ |
|------|----------|----------|----------|----------|
|      | $M\mu$   | $\tau^{-1}$ | $M\mu$   | $\tau^{-1}$ | $M\mu$   | $\tau^{-1}$ | $M\mu$   | $\tau^{-1}$ |
| 0.500| 0.166885 | 0.004246 | 0.166885 | 0.004246 | 0.160316 | 0.004500 | 0.173477 | 0.004003 |
| 0.600| 0.201837 | 0.007167 | 0.201837 | 0.007167 | 0.195359 | 0.007517 | 0.208343 | 0.006830 |
| 0.700| 0.239344 | 0.011192 | 0.239344 | 0.011192 | 0.233014 | 0.011647 | 0.245710 | 0.010753 |
| 0.800| 0.282288 | 0.016656 | 0.282288 | 0.016656 | 0.276186 | 0.017221 | 0.288437 | 0.016109 |
| 0.900| 0.338554 | 0.024248 | 0.338554 | 0.024248 | 0.332809 | 0.024919 | 0.344368 | 0.023597 |
| 0.950| 0.380566 | 0.029213 | 0.380566 | 0.029213 | 0.375095 | 0.029917 | 0.386142 | 0.028530 |
| 0.990| 0.442628 | 0.032213 | 0.442628 | 0.032213 | 0.437558 | 0.032835 | 0.447966 | 0.031589 |
| 0.999| 0.481321 | 0.025162 | 0.481321 | 0.025162 | 0.476740 | 0.025359 | 0.486849 | 0.024539 |
FIG. 4. Instability of the \( m = -1, 0, +1 \) states for \( n = 0 \). The growth rate of the quasibound states is shown as a function of the gravitational coupling \( M\mu \), for a range of black hole rotations \( a \). The left plots show the instability of a Kerr black hole, while the right plots are related to the Kerr-Newman black hole for \( e = 1 \) and \( Q = 0 \). The solid and dashed lines are the resonant frequencies \( (\omega/\mu)_{n,m}^{(2)} \) and \( (\omega/\mu)_{n,m}^{(3)} \), respectively.

From Table IV and Fig. 4 we can conclude that the maximum instability growth rates of the Kerr black hole do not depend on the sign of the azimuthal quantum number \( m \), that is, they are the same for \( m = -1 \) and \( m = +1 \). On the other hand, the maximum instability growth rates of the Kerr-Newman black hole are sensitive to the sign of \( m \), as can be seen by the fact that their peaks are highest for \( m = -1 \).
IV. WAVE FUNCTIONS

In this section we will derive the eigenfunctions related to charged massive scalar particles propagating in the Kerr-Newman background. This is possible if one uses some properties of the confluent Heun functions and then obtains their polynomial expressions.

The polynomial solutions of the confluent Heun equation (14) will be denoted by $HC_{p_n}(x)$ and can be written as

$$HC_{p_n}(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where the coefficients $c_n$ are given by

$$P_n c_n = T_n c_{n-1} + X_n c_{n-2},$$

with the initial conditions $c_{-1} = 0$ and $c_0 = 1$. The expressions for $P_n$, $T_n$ and $X_n$ are given by

$$P_n = 1 + \frac{\beta}{n},$$
$$T_n = 1 + \frac{-\alpha + \beta + \gamma - 1}{n} + \frac{\eta - (-\alpha + \beta + \gamma)/2 - \alpha\beta/2 + \beta\gamma/2}{n^2},$$
$$X_n = \frac{\alpha}{n^2} \left( \frac{\delta}{\alpha} + \frac{\beta + \gamma}{2} + n - 1 \right).$$

These equations are consistent if and only if the parameter $\sigma_0$ (see Eq. (14)) is chosen properly, which means that is calculated via a polynomial equation of degree $n + 1$, namely, $c_{n+1} = 0$. We will use for these eigenvalues the notation $\sigma_0(n;N)$, where $N$ runs from 0 to $n$. In this case, the corresponding confluent Heun polynomials will be denoted as $HC_{p_n;N}(x)$.

Let us obtain the explicit form of the first two confluent Heun polynomials as follows.

For $n = 0$, we have

$$HC_{p_0;N}(x) = c_0 = 1,$$

The eigenvalues $\sigma_0(0;N)$ must obey

$$c_1 = 0,$$

which implies

$$c_1 = \frac{T_1}{P_1},$$

and then we have that

$$\sigma_0(0;0) = 0.$$
Thus, the confluent Heun polynomial for the fundamental mode is given by

\[ \text{HCP}_{0,N}(x) = 1. \]  

(65)

Now, for \( n = 1 \), we have

\[ \text{HCP}_{1,N}(x) = c_0 + c_1 x = 1 - \frac{\sigma_{0(1;N)}}{1 + \beta} x. \]  

(66)

The eigenvalues \( \sigma_{0(1;N)} \) must obey

\[ c_2 = 0, \]  

(67)

where

\[ P_2 c_2 = T_2 c_1 + X_2 c_0, \]  

(68)

which implies that

\[ \sigma_{0(1;N)} = \frac{-(\alpha - 2 - \beta - \gamma) \pm \sqrt{\Delta}}{2}, \]  

(69)

with \( \Delta = (\alpha - 2 - \beta - \gamma)^2 + 4\alpha(1 + \beta) \). Here, the signs \(-, +\) stand for \( N = 0, 1 \), respectively.

Thus, the confluent Heun polynomials for the first excited mode are given by

\[ \text{HCP}_{1;0}(x) = 1 - \frac{-(\alpha - 2 - \beta - \gamma) - \sqrt{\Delta}}{2(1 + \beta)} x, \]  

(70)

\[ \text{HCP}_{1;1}(x) = 1 - \frac{-(\alpha - 2 - \beta - \gamma) + \sqrt{\Delta}}{2(1 + \beta)} x. \]  

(71)

Then, the radial eigenfunctions, for charged massive scalar particles propagating in a Kerr-Newman black hole spacetime, can be written as

\[ R_{n,m,N}^{(j)}(x) = C_{n,m,N}^{(j)} x^D_0 (x - 1)^D_1 e^{D_2 x} \text{HCP}_{n,N}^{(j)}(x), \]  

(72)

where \( j = 1, 2, 3, 4 \) labels the resonant frequencies solutions, and \( C_{n,m,N}^{(j)} \) is a constant to be determined by using some additional boundary condition, as for example, that the wave function should be appropriately normalized in the range between the exterior event horizon and the infinity. This expression is also valid to massive scalar particles in a Kerr spacetime.

On the other hand, in the case of a Schwarzschild black hole spacetime, the radial eigenfunctions can be written as

\[ R_{n,N}^{(j)}(x) = C_{n,N}^{(j)} x^D_0 (x - 1)^D_1 e^{D_2 x} \text{HCP}_{n,N}^{(j)}(x), \]  

(73)

where \( j = 1, 2, 3 \) labels the resonant frequencies solutions in the background under consideration.
In order to obtain an “unique” wave solution for the quasibound states, in the sense of only one value of $N$ is physically acceptable, we have to analyze the behavior of the radial eigenfunctions. Therefore, by using Eqs. (72) and (73), we can plot the first three squared wave functions, which are presented in Figs. 5-7.
FIG. 5. The first three squared eigenfunctions in the Schwarzschild black hole spacetime for $M\mu = 0.5$. The units are in multiples of $C_{n;N}^{(j)}$.

From Fig. 5 we conclude that the radial eigenfunctions for $(n; N) = (1; 1)$ (the dotted lines) are not physically acceptable for the quasibound states in the Schwarzschild black hole, since they do not tend to zero far from the black hole at asymptotic infinity (for the resonance solutions $j = 2, 3$). Therefore, in general, the radial eigenfunctions $R_{n;N}^{(j)}(r)$, with $j = 1, 2, 3$ for $(n; N) = (n; 0)$, describe quasibound states in the Schwarzschild black hole spacetime, since they present the desired behavior, that is, the radial solution tends to zero at infinity and diverges at the exterior event horizon; it (mathematically reaches a maximum value and then) crosses into the black hole.
FIG. 6. The first three squared eigenfunctions in the Kerr black hole spacetime for \( M \mu = 0.5, m = 0 \) and \( a = 0.1 \). The units are in multiples of \( C_{n,m:n}^{(j)} \).

From Fig. 6, we conclude that the radial eigenfunctions for \((n;N) = (1;1)\) (the dotted lines) are not physically acceptable for the quasibound states in the Kerr black hole, since they do not tend to zero far from the black hole at asymptotic infinity (for the resonance solutions \( j = 3, 4 \)). Therefore, in general, the radial eigenfunctions \( R_{n,m:0}^{(j)}(r) \), with \( j = 1, 2, 3, 4 \) for \((n;N) = (n;0)\), describe quasibound states in the Kerr black hole spacetime, since they present the desired behavior.
FIG. 7. The first three squared eigenfunctions in the Kerr-Newman black hole spacetime for $M\mu = 0.5$, $m = 0$, $a = 0.1$, $e = 0.01$ and $Q = 0.1$. The units are in multiples of $C_{n,m,N}^{(j)}$.

From Fig. 7 we conclude that the radial eigenfunctions for $(n; N) = (1; 1)$ (the dotted lines) are not physically acceptable for the quasibound states in the Kerr-Newman black hole, since they do not tend to zero far from the black hole at asymptotic infinity (for the resonance solutions $j = 4$). Note that the radial eigenfunctions $j = 1, 2, 3$ for $(n; N) = (1; 1)$, as well as $j = 3$ for $(n; N) = (1; 0)$, increase near the exterior event horizon and then they tend to zero at asymptotic infinity. In particular, the radial eigenfunctions for $j = 4$ is not physically acceptable for $N = 0, 1$. Therefore, in general, the radial eigenfunctions $R_{n,m;0}^{(j)}(r)$, with $j = 1, 2, 3$ for $(n; N) = (0; 0)$, describe quasibound states in the Kerr-Newman black hole spacetime, since they present the desired behavior.

V. THE SUPERMASSIVE BLACK HOLE AT THE CENTER OF M87

Lastly, we will apply our previous results for an actual astrophysical object, by computing the decay time of a ultra-light bosonic particle around the supermassive Kerr black hole situated at the center of M87 galaxy, whose shadow was recently observed [17]. We will consider the relativistic approximation $\omega/\mu \gg 1$. In this case, according to Eq. (54), we have

$$\omega_I \approx -\frac{c^3(n+1)\sqrt{1-(a^*)^2}}{GM(a^*)^2},$$

(74)
which does not depend on the particle mass, $\mu$, nor on the angular quantum number, $m$. It is worth calling attention to the fact that in Eq. (74), we have reintroduced the fundamental constants and taken into account the fact that $M\mu \ll 1$, since $M \approx 5 \times 10^9$ solar masses, $\mu \approx 10^{-21}$ eV/$c^2$, and the dimensionless parameter $a^* = a/MG \approx 0.9^{18}$.

Therefore, the decay characteristic time for these ultra light bosons is given by

$$\tau \approx \frac{1.3 \times 10^{39}}{(n + 1)} \text{ seconds.}$$

(75)

The huge magnitude of this time represents a stability for the system, even for large principal quantum numbers.

VI. FINAL REMARKS

In this paper we revisited the analytical solutions for the charged massive Klein-Gordon equation in a Kerr-Newman black hole spacetime. In addition, we get general solutions for this equation of motion, which are given in terms of the confluent Heun functions and can be chosen according to the boundary conditions to be imposed.

The study of the asymptotic behaviors of the radial solution led to the quasibound state phenomena. In the vicinity of the exterior event horizon, the radial solution diverges by reaching a maximum value, which indicates that the scalar particles cross into the black hole. On the other hand, far from the black hole, at the asymptotic infinity, the radial solution tends to zero, which means that the probability of finding some particles there is null.

We obtained the quasispectrum of resonant frequencies for charged massive scalar particles propagating in the Kerr-Newman black hole spacetime. Furthermore, we analyzed the particular cases of the Kerr and Schwarzschild backgrounds. This become possible by imposing two boundary conditions according to the Vieira-Bezerra-Kokkotas method [6, 7] developed to study the quasibound states. We compared our results with the ones known in the literature, and then we conclude that this new method generalizes the numerical approaches used by some authors [15, 16].

We have found a set of frequencies that are solutions of a characteristic resonant equation, which was derived from the polynomial condition of the confluent Heun functions. The physically acceptable solutions are obtained by analyzing the (asymptotic) behavior of these
resonant frequencies, and the radial eigenfunctions as well.

Finally, we have applied the model to an actual black hole, the supermassive one situated at the center of the M87 galaxy, computing the characteristic time of decay of ultra-light bosonic particles around the black hole, in the relativistic approximation. We verified thus that the magnitude of this time reflects the stability of the bosonic particles around the compact object, which could constitute, for instance, the cold dark matter situated in this region (see [19, 20] and references therein).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ACKNOWLEDGMENTS

H.S.V. is funded by the Alexander von Humboldt-Stiftung/Foundation (Grant No. 1209836). This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. V.B.B. is partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) through the Research Project No. 307211/2020-7. C.R.M. is partially supported from CNPq and Fundação Cearense de Apoio ao Desenvolvimento Científico e Tecnológico (FUNCAP) under the grant PRONEM PNE-0112-00085.01.00/16.

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