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The Cauchy method of analytical regularisation in the modelling of plane wave scattering by a flat pre-fractal system of impedance strips

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Abstract
The Cauchy method of analytical regularisation, first in full explanation, is used for the modelling of the plane wave scattering by a flat pre-fractal system of impedance strips. It is based on a modification of the singular integral equation technique and the correct type of inversion formula of the simplest Cauchy singular integral equation. The approach is effective for the plane \(H\)-polarised electromagnetic (or sonic) wave scattering problems. The principle of strict mathematical orderings of the pre-fractal strip grating deals with a five-parts class of perfect fractal sets, which is characterised by a variable positive fractal dimension. Considerable attention is devoted to the asymptotic model of the plane wave scattering by a sparsely filled grating, which has an explicit solution. This result is very important for verification of the mathematical models presented and testing of direct numerical methods. The scattered electromagnetic field in the far zone is successfully analysed with the aid of the asymptotic solution derived for electrically narrow strips.

1 INTRODUCTION

The method of analytical regularisation (MAR) deals with the transformation of a system of the first-kind singular integral equations (IE) into a system of the second-kind IE, inverting the singular part of the original IE system. The merit of such a transformation is in the well-known fact that, with the second-kind IE system with smoother kernels it is easy to build a convergent and efficient numerical algorithm. Various wave-scattering and eigen-value problems numerically solved with this approach have been reviewed in [1]. This approach is still efficient for the analysis of other wave-scattering problems, which have emerged today [2, 3]. What is especially important and useful, the MAR allows us to find analytical solutions in the form of fully grounded asymptotic expansions in terms of certain small parameters. Among the first works, using this opportunity, we can mention [4, 5], where the principal guided waves in the multi-element systems of microstrip lines were analysed. More recently, this asymptotic approach was successfully applied to the plane wave scattering from various gratings of impedance strip (IS) — see, for instance, [6–8]. In these works, the analysis was made on the basis of the first-kind singular IE. Therefore, it is interesting to extend this approach to the second-kind singular IE systems. The idea is based on the assumption that the strips of gratings are electrically narrow and sufficiently distanced between themselves. Besides, we will assume that, in the considered gratings, a finite natural number of flat strips are placed according to a certain stage of one-dimensional (1D) self-similar fractal (SSF) creation process. Such gratings are characterised by the strict mathematical ordering and called pre-fractal ones. This novel type of gratings has some advantages over the classical finite-periodic gratings and has a potential of wide use in microwave devices in the future (details one can see in 2.2).

In the present article, a version of the MAR based on the analytical inversion of the Cauchy operator is explained in full detail and used for the modelling of the plane \(H\)-polarised electromagnetic wave scattering by a flat pre-fractal system of flat IS. Initially, the mathematical model is presented in the
form of an outer boundary-value problem with the third-type boundary condition, specific to this type of the plane wave, for the 2-D Helmholtz equation on a pre-fractal set of segments on a plane. Then, making use of the familiar IE technique, a mathematical model in the form of the first-kind singular IE system is created. Of course, it is simpler and more convenient then the initial 2-D model; however, it needs a certain correction. This correction deals with the transition from the first-kind singular IE system with a logarithmic (weak) singularity to the first-kind singular IE system with a Cauchy type (strong) singularity. As the solution of the latter IE system is not unique, the correct solution has to be singled out with the aid of edge conditions. Using this correct type of unique solution and inverting the simplest singular parts of the original IE system, we arrive at a second-kind of IE system. This is the sense of analytical regularisation of the scattering problem.

To show the practical importance of the MAR version based on the inversion of the Cauchy operator, in the sense of testing and verification of the associated numerical method, we examine the case of sparsely filled grating of electrically narrow flat impedance strips. Here, the corresponding second-kind IE system has a simple, explicit asymptotic solution, which is more correct and convenient than the analysis of the first-kind singular IE system, presented in [8]. In particular, this asymptotic approach creates the possibility to study the dependence of the integral scattering characteristics on the strip impedance and the angle of the plane wave incidence; note that some preliminary results of such an analysis can be found in the conference article [7]. Below, we present full details of the development of the corresponding efficient numerical algorithm and discuss some results of computer simulations, covering the far-field angular scattering patterns of the mentioned grating.

2 | THE PLANE WAVE SCATTERING BY PRE-FRACTAL FIS GRATINGS

A general formulation of the plane wave scattering by a zero-thickness strip grating is well known and widely used [9–12]. Let us recall it briefly for the wave scattering by a pre-fractal flat IS grating.

2.1 | Formulation of the scattering problem

Assume that the plane electromagnetic wave is incident on a finite grating consisting of a zero-thickness flat IS with parallel edges. In Figure 1, one can see the simplest strip grating, which consists of 3 flat IS, placed according to the first stage (or the SSF creator) of a perfect fractal set creation [13].

The corresponding SSF belongs to the 1-D five-part classes (denoted by SSF3); one can see three equal strips and two equal slots. This means that the initiator (a flat straight-line segment) is divided into five specific parts.

In the case of any flat pre-fractal grating of flat IS, the transverse cross-section of the structure is the appropriate number of equal segments, placed according to a certain fixed stage of the 1-D SSF creations.

Since the strips have parallel edges, we assume that they run along the y-axis of the Cartesian coordinates. Then, for the plane electromagnetic wave scattering, the Maxwell equations can be separated into two independent sets of the partial differential equations, which are reduced to the 2-D homogeneous Helmholtz equation [10]. The same equation is obtained in the acoustic plane wave scattering by a strip grating. In both cases, the impedence boundary condition on the segments of the x-axis that correspond to the transverse cross-section of the flat strip grating is the most general condition [11, 12].

In the case of the plane H-polarised electromagnetic wave scattering by a strip grating, the single non-zero component of the magnetic field is defined by the formula $H_x = u(x, z)/\chi$ ($\chi$ is the intrinsic impedance of the host medium). Then the electric field vector is $E = (u_x', 0, -u_y')/ik$, where $i$ is the imaginary unit and $k$ is the wave number.

The impedence boundary condition on the segments of the x-axis corresponding to the strip's cross-sections has the following form: $u_x' + ik(w/\chi)u = 0$, where $w$ is the strip impedence, which is assumed the same for both sides of the strips. In the case of zero value of the strip impedence, the third-type boundary condition transforms into the second-type one, that is to the Neumann boundary condition.

In the case of the acoustic wave scattering by the grating, we have an impedence boundary condition of the form $u_x' + \gamma \cdot u = 0$, where the coefficient $\gamma$ is linearly dependent on the acoustic impedence of the strip [12]. In the case of zero value of the impedence, the mixed boundary condition transforms into the second-type as well. Thus, one can see that the boundary conditions in acoustic and electromagnetic impedences are similar to each other and coincide if we put $\gamma = ikw/\chi$.

Thus, for the wave scattering problems of both types we have the exterior mixed boundary-value problem for the homogeneous Helmholtz equation. These 2-D problems have unique solutions, provided that the radiation condition at infinity (by the Sommerfeld criterion) and the conditions at the end points of the strip (in the Meixner form) are met [9, 12]. The boundary-value problem's formulation is classic and mathematically correct; however the types of considered gratings are not classic in the sense of the Euclidian geometry. Therefore, we have to give the basic information on the fractal and pre-fractal sets, which have numerous applications in the mathematical modelling of various natural objects and processes [14–16].
2.2  |  Types of pre-fractal FIS gratings

In the fractal geometry, a fundamental object is the triadic Smith-Cantor set, which is different from the usual sets of the Euclidian geometry [13, 14]. It contains no interval of positive length (its topological dimension = 0), but it is uncountable like the set of real numbers. This means that one-to-one correspondence exists between the points of the triadic Smith-Cantor set and the segment [0, 1]. This basic perfect set has numerous modifications and generalisations in the fractal geometry [15, 16]. In practical applications, for the mathematical modelling of various natural objects and processes only the pre-fractal sets have been used. In particular, a pre-fractal strip grating (PFSG) is a flat strip grating with a fixed number of strips distributed according to the segments of the 1-D SSF creation, first introduced and examined in the conference article [17]. Such a grating is characterised by strict mathematical ordering and has certain advantages over the periodic diffraction gratings (PDG), which have many applications in various branches of science and engineering. These advantages are as follows: (a) PFSG is a grating with a fixed number of strips that reflects the real situation more adequately; (b) PFSG has a higher level of the mathematical ordering (periodic law is the simplest law among a wide variety of mathematical orderings); (c) for some PFSGs, a unification of all frequency ranges is possible, that is, the range of short waves, range of resonances and the range of long waves. They have greater capabilities, especially with respect to many 1-D SSFs with a variable fractal or Hausdorff dimension (HD). Thus, the mentioned advantages of PFSG over the classical PDG can justify the replacement of PDGs by PFSGs in classical microwave devices and provide the improvement of their characteristics. In particular, a comprehensive overview of recent developments in the field of the fractal antenna engineering is provided in [18].

In the present article, the five-part class of the perfect fractal sets will be used. For this class a straight-line segment (the initiator) is divided into five specific parts: three equal sub-segments and two intervals or gaps. These three sub-segments and two gaps (between them) give us the creator for a certain SSF3 construction. The construction process is like the three-part class of SSF2 with a variable fractal dimension, which is defined by the formula $HD = \ln 2/\ln c$, where $c$ is the self-similarity coefficient. One can see the details on the three-part class of the SSF2 construction process in the article [8].

Just like the three segments are active parts of the fractal construction process, the perfect fractal sets are noted by SSF3. Figure 2 shows cross-sections of three types of flat PFSG, where zero-thickness strips are distributed according to segments of three first-creation stages of 1-D SSF3 creation. The simplest type of PFSG consists of three flat IS (Figure 1), which corresponds to the first stage of SSF3 creation or the SSF3 creator (a) level in Figure 2). Two others levels present gratings consisting of nine and twenty seven zero-thickness strips. So, the geometrical interpretation of SSF3 creation is simple and clear. But not simple is the ideal boundary structure referred to as the fractal set of points that belong to every stage of the SSF3’s creation. The fractal set is not convenient to be used for modelling; however, the stages of its creation are quite often used in practice, see, for instance, article [18]. For this reason it is opportune to talk about pre-fractal modelling.

To construct a qualitative mathematical model of the plane wave scattering by a PFSG, it is important to find the output functions of the SSF3’s creation process in as simple a form as possible. For this purpose, the system approach was made to examine creation processes for different types of 1-D SSFs with a variable HD, like in article [19]. Figure 3 shows two first stages of the 1-D SSF3’s creation in analytic form; its HD is defined by the formula $HD = \ln 3/\ln c$, where $c$ is the self-similarity coefficient. In this figure, one can see two sets of output functions: three functions of the first set are in yellow colour and nine are in blue.

The number of the output functions at a certain stage is determined by powers of three, viz. $3^n$, where $n$ represents the number of a creation stage. They are defined by two initial real numbers: $a$ and $b$ ($b > 2a$, see Figure 2 the creator, (a) level); the third number is the self-similarity coefficient $c = 1 + b/a > 3$. So, depending on the real values of these two initial numbers, $a$ and $b$, the variable HD of the SSF3 is changed into the interval $(0, 1)$.

3  |  THE CAUCHY METHOD OF ANALITICAL REGULARISATION

By making use of the IE technique in the standard way, the exterior 2-D mixed boundary-value problem can be transformed into some 1-D problem of a mixed integral differential equation (IDE) solution. Then, the differential and integral parts of the IDE system should be separated carefully. As the result, we obtain an ordinary inhomogeneous differential equation of the second order, in which new unknown functions are presented as integral transformations of old unknown functions. The problem of plane acoustic wave scattering by a flat system of a flat IS has been considered in the article [20]. In particular, the plane acoustic wave scattering problem has been completely solved from the mathematical standpoint for an asymptotic model of a weakly filled grating on the basis of the first-kind singular IE systems only. So, a similar problem of the plane $H$-polarised EM wave scattering by a flat system of IS will be examined with the aid of the Cauchy MAR here.
3.1 The IE technique modification for the $H$-polarisation case

In the case of the plane $H$-polarised electromagnetic wave scattering by a flat system of zero-thickness IS, modification of the IE technique brings the inhomogeneous ordinary DE (ODE) of the second order:

$$\Phi''(x) + k^2(1 - \mu^2) \cdot \Phi(x) = -4k(\mu + q_2) \cdot \exp(i k q x)$$

(1)

Here, $k$ is the wave number and a new measureless impedance parameter $\mu = \omega/\kappa x$; $q_1, q_2$ are components of the plane wave direction vector $\mathbf{q}$ (see Figure 1) and for them the identity $q_1^2 + q_2^2 = 1$ is true. A new function $\Phi(x)$ is represented by the integral transformation of the function, which defines the transverse component of the surface current densities flowing on the strips:

$$\Phi(x) = \int_S \psi(x') \cdot H_0^{(1)}(k|x - x'|) dx'$$

(2)

Here, the domain of integration $S$ is a set of segments—the cross-section of a pre-fractal system of a flat IS, and $H_0^{(1)}(z)$ is the first-kind Hankel function of zero order. In the case of the plane acoustic wave scattering by a single IS we have a similar situation, see, for instance, article [20]. The principal difference is in the coefficients of the inhomogeneous ODE of the second order Equation (1); so, we ought to be careful to fail physical sense.

It will be noted that for zero value of the strips’ impedance (or parameter $\mu = 0$), we obtain a 1-D mathematical model of the plane $H$-polarised wave scattering by a perfectly electrically conducting (PEC) strip grating. It is presented by a similar, easier inhomogeneous ODE; so, we can orient on the IE technique, which was used in the article [21].

First, we find the general solution of the linear homogeneous ODE of the second order with constant coefficients corresponding to the inhomogeneous Equation (1). Then, we seek a particular solution for the inhomogeneous ODE in a special form. As an intermediate result, we obtain the general solution of the inhomogeneous ODE Equation (1),

$$\Phi(x) = A \cdot \sin(\nu \cdot kx) + B \cdot \cos(\nu \cdot kx) + C \cdot \exp(i q_1 kx)$$

(3)

Here, $\nu = \sqrt{1 - \mu^2}$ and, due to expression (2), this relationship is the singular IE (SIE) system with respect to the unknown function $\psi(x)$. It can be considered as an intermediate mathematical model of the plane $H$-polarised EM wave scattering by a flat IS system. One can see that the right-hand part’s function (3) contains two sets of unknown constants: $A, B$; the constant $C$ is known. So, to have a more convenient 1-D mathematical model of the plane wave scattering problem, some correct transformations of the newly obtained SIE system will be used. One can see the details in the article [8].

The differences are in the output functions of the 1D SSF3's creation: $x_n(t) = x_n(0) + a nt$ ($a_2 = a_1/e^\alpha - 1$, $a = ka_2$) and the order of the system ($3^n$ by $3^m$; $n$ is a number of the SSF3 creation stage). As a result, the SIE system takes the following form:

$$\int_{-1}^{1} j_{\epsilon}(t) \cdot \ln |r - t| dt + \sum_{m=1}^{3^n} \prod_{k=1}^{3^m} R_{km}(\tau - t) dt = f_\epsilon(r), |r| < 1, \quad k = 1, \ldots, 3^n.$$  (4)

Here, regular kernel functions $R_{km}(\mu)$ have the same representations as in the article [8]. The right-hand part functions are defined by the following expressions:

$$f_\epsilon(r) = c_{1\epsilon} \cdot \sin(\nu a_\epsilon r) + c_{2\epsilon} \cdot \cos(\nu a_\epsilon r) + c_{0\epsilon} \cdot \exp(i q_1 a_\epsilon r)$$

(5)

Here, the known constants are determined by the formulas $c_{0\epsilon} = -2 \pi i \cdot \exp(i q_1 x_n(0)/k(q_2 + \nu)$, where the constants $c_{1\epsilon}, c_{2\epsilon}$ are unknown. The geometrical measureless parameters, $x_n(0), k = 1, \ldots, 3^n$, determine the segments’ positions in the $n$-stage of the SSF3 creation (see Figures 2, 3). For instance, there are three segments for the first stage and three corresponding geometrical measureless values as follows: $x_1(0) = -kb$, $x_2(0) = 0$, $x_3(0) = kb$.

The SIE system Equation (4), presented above, can be regarded as an intermediate mathematical model of the plane $H$-polarised EM wave scattering by a flat pre-fractal system of a flat IS. It reflects the main points of pre-fractal nature, but is not convenient for use with a direct numerical method because of the presence of unknown constants in the right-hand parts. Thus, we ought to transform the SIE system Equation (4) further to obtain an adequate mathematical model, which will be convenient for practical applications. To improve the mathematical model, the correct transition to a system of SIE with Cauchy-type singularity will be used, like in the article [21].

Unfortunately, this article contains mathematical models in the form of the first-kind SIE system only. Then, the Cauchy
3.2 | The second-kind IE system

Due to the inconvenience of the intermediate mathematical model, presented by Equation (4), for practical applications, it is necessary to perfect it further. For this purpose, we put in the mentioned SIE system $\tau = 0$ (fixation for the restoration of the original equations’ system) and obtain some restoring conditions for finding unknown constants:

$$c_{2k} = \int_{-1}^{1} j_k(t) \ln|t| dt + \sum_{m=1}^{3^n} \int_{-1}^{1} j_m(t) R_m(t) dt - c_{0k}. \quad \kappa = 1, ..., 3^n. \quad (6)$$

The differentiation of the same equations’ systems with respect to the outer variable $\tau$ brings us to the new SIE system with the Cauchy-type singularity:

$$\int_{-1}^{1} j_k(t)(\tau - t)^{-1} dt + \sum_{m=1}^{3^n} \int_{-1}^{1} j_m(t) R_{km}(\tau - t) dt = f_k(\tau). \quad (7)$$

This is the second intermediate mathematical model of the plane $H$-polarised EM wave scattering with the restoring conditions Equation (6) for finding the unknown constants $c_{2k}$. The system of Equation (7) has a lot of solutions; so, the unknown constants $c_{1k}$ must be used to separate a unique solution typical for this case. Let us rewrite this SIE system in the simplest Cauchy-type SIE form:

$$\int_{-1}^{1} j_k(t)(\tau - t)^{-1} dt = F_k(\tau). \quad (8)$$

Here, the right-hand part’s functions are defined by the following formulas:

$$F_k(\tau) = f_k'(\tau) - \sum_{m=1}^{3^n} \int_{-1}^{1} j_m(t) R_{km}(\tau - t) dt \quad (9)$$

If the right-hand functions $F_k(\tau)$ are given and have certain properties, the simplest Cauchy-type Equation (8) has solutions of three types [22, 23]. If we want to separate the unique solution that is accurate in the case of the plane $H$-polarised EM wave scattering (according to the conditions at the strip edges), we have to use the following conditions:

$$\int_{-1}^{1} F_k(\tau)(1 - \tau)^{-0.5} \cdot d\tau = 0 \quad \kappa = 1, ..., 3^n \quad (10)$$

These conditions give us the second set of unique solution conditions for finding the unknown constants $c_{1k}$. In such a case, the simplest Cauchy-type Equations (8, 9) has a bounded solution for both values of $t = \pm 1$ and can be presented in the following form [23]:

$$j_k(t) = \sqrt{1 - t^2} \int_{-1}^{1} F_k(\tau)(1 - \tau)^{-1}(1 - \tau^2)^{-0.5} . d\tau / \pi. \quad (11)$$

This solution (according to the conditions at the strip edges) is valid in the case of the plane $H$-polarised EM wave scattering by a flat pre-fractal system of a flat IS. Expression (11) can be useful for obtaining the second-kind IE system of the Fredholm type as well. To make it, we substitute Expression (9) into the Equation (11). Then, on changing the order of integration for the newly obtained iterated integral, it is not difficult to obtain a regular IE of the second-kind.

$$j_k(x) + \sum_{m=1}^{3^n} \int_{-1}^{1} j_m(t) R_{km}(x, t) dt = b_k(x) \quad (12)$$

Here, the kernel functions are defined by improper integrals (understood in the principal value sense) of the type,

$$R_{km}(x, t) = \frac{1 - x^2}{\pi^2} \int_{-1}^{1} \frac{\partial R_m(\tau - t)}{(x - \tau) \sqrt{1 - \tau^2}} \quad (13)$$

The right-hand part functions are defined by similar improper integrals.

$$b_k(x) = \frac{1 - x^2}{\pi^2} \int_{-1}^{1} \frac{df_k(\tau)}{(x - \tau) \sqrt{1 - \tau^2}} \quad (14)$$

The second-kind system of integral Equation (12), together with two sets of conditions (the restoring Expression (6) and the unique solution Expression (10)), is an adequate full-wave mathematical model of the plane $H$-polarised EM wave scattering by a flat pre-fractal system of an IS. Having the second-kind IE system with smoother kernels, it is easy to create a correct and effective calculation algorithm. But now, to be sure of the full-wave mathematical model, let us study a weakly filled grating model. Thus, for testing the manipulations we made and verifying the numerical methods of computer simulation in future, it is useful to find an accurate, but simple enough, solution for the scattering problem. The newly obtained adequate mathematical model will be sufficient and convenient for working up an asymptotic model. Using this mathematical model, it is easy to find a solution for the scattering problem in an explicit form.

4 | THE PLANE WAVE SCATTERING BY A SPARSELY FILLED GRATING

This asymptotical approach was first used to examine the principal guided waves in the multi-element systems of micro strip lines, in which a narrow PEC strip grating was placed on a screened dielectric layer [4, 5]. It was successfully realised to examine the plane wave scattering by non-classical, weakly filled IS gratings as well [6–8]. The idea is based on the...
assumptions that the grating strips are electrically narrow and sufficiently spaced. It is very convenient to realise this idea by making use of the regular IE of the second-kind system, presented by Equation (12).

4.1 | The pre-fractal sparsely filled strip grating

For the asymptotic mathematical model of sparsely filled strips’ grating, two basic assumptions have to be as follows: \( \alpha_n = a/c^n - 1 < 1 \) (grating strips are electrically narrow) and \( \beta_n = \beta/c^n - 1 = O(1) \) (strips are sufficiently spaced). In the case of a pre-fractal strip grating, these two basic assumptions are true if the fractal dimension of the corresponding SSF is sufficiently small. The variable HD of the SSF3 changes to the interval \((0, 1)\), depending on a real value of the ratio: \( b/a \) (see Figure 3); so, at first the inequality: \( b > a \) has to be fulfilled. Second, due to the formula \( \alpha_n = a/c^n - 1 \) and the self-similarity coefficient \( c = 1 + b/a > 3 \), one of the inequalities, \( a = ka < < 1 \) or \( n > 1 \), has to be true.

The use of asymptotic expressions for the kernel and right-hand part functions of the regular system of the second-kind integral Equation (12) leads to the explicit solution in the main approximation:

\[
j_e(x) = 2iae^\kappa r (\mu - q_2) \exp(iq_1 x_0(0)) \sqrt{1 - x^2} + O(a^3).
\] (15)

By putting \( \mu = 0 \), we obtain the corresponding explicit solution for the case of the plane \( H \)-polarised EM wave scattering by a flat pre-fractal PEC strip grating. See, for instance, the simplest case of a two-strip system in the article [21]. It is very important to be sure of the explicit solution, presented by Expression (15). In the case of the plane acoustic wave scattering by a flat pre-fractal narrow IS grating, we have a similar explicit solution, but it is in a rather different way with the aid of the first-kind SIE system [20].

The solution in a closed form is itself practically useful and is important for the validation of any direct numerical method, based on a full-wave mathematical model — projection methods, Nyström method and so on [24].

Having the solution in the closed form, it is easy to examine the scattered EM field around the pre-fractal IS grating and in the far-field zone, just like in the article [8]. In particular, the scattered EM field in the far zone is determined in the (polar coordinate system \((r, \theta)\)) by the approximate formula as follows:

\[
H_y \times = n'(r, \theta) \approx -\sqrt{i} \cdot A(\theta) \cdot e^{ikr}/\sqrt{kr}, \quad r \to \infty.
\] (16)

where the dependence on polar angle is presented by the following approximate expression \((\alpha_n < < 1)\):

\[
A(\theta) \approx \sqrt{i} \pi a^2 \cdot (\mu + \sin \theta)(\mu - q_2) \sum_{m=1}^{n} \exp(i\omega_m(0)) \left((q_1 - \cos \theta)/2\right).
\] (17)

One can see two multipliers, representing the explicit dependence on the measureless impedance parameter \( \mu \) (or strip impedance \( w \)). The multipliers are linear with respect to the strip impedance parameter \( \mu \). So, it is easy to examine the strips’ impedance influence on the scattered EM field in the far zone (see, for instance, article [8]).

The sum in the approximate formula (17) can be simplified according to the central symmetry of the SSF3’s creation (see Figure 2). For example, in the case of the second stage of the SSF3’s construction (see Figure 3), the sum will be of the following form: \( 1 + 2\sum_{m=1}^{n} \cos(x_m(0))(q_1 - \cos \theta) \). The geometrical measureless parameters \( x_m(0) \) are defined by the following expressions: \( kb/c, kb(1-1/c)/c, kb, kb(1 + 1/c)/c, \) here, \( a \) and \( b \) define output functions of the SSF3’s creator (marked in yellow colour in Figure 3). Let us perform some computer simulations making use of the approximate Expression (17), which is simple and convenient to make effective examination of the basic integral scattering characteristic in the far-field zone — the direction patterns of the pre-fractal IS grating.

4.2 | Computer simulations

An effective calculation algorithm has been developed with the aid of the approximate Expression (17). The fourth figure presents the dependences of \( |A(\theta)|/\pi a^2 \) on the polar angle calculated for the simplest pre-fractal system of three IS (Figure 1), corresponding to the SSF3’s creator (Figure 2). The geometrical measureless parameters in the first case are as follows: \( a = ka = \pi/27 \) and \( b = kb = 242\pi/27 \); so, the self-similarity coefficient \( c = 1 + \beta/\alpha = 243 \) and \( HD = ln 3/243 = 1/5 \) (Figure 4a). The cross-sectional size of the grating is equal to \( 4\lambda \); here, \( \lambda \) is the plane EM wave length. For the second case these geometrical measureless parameters are: \( a = \pi/27 \) and \( b = 80\pi/21 \); so \( c = 81 \) and \( HD = 1/4 \) (Figure 2b). The cross-sectional size is equal to \( 3\lambda \). For both cases, the solid lines correspond to non-zero values of strip impedance: \( \mu = 0.5 + 0.5i \) (marked by black colour) and \( \mu = 0.25 + 0.25i \) (marked in blue colour); while the red points correspond to the value \( w = 0 \) (grating of three PEC strips). The angle of the plane \( H \)-polarised wave incidence \( \theta = \pi/6 \).

In the case of PEC strips, one can see the symmetry of the graphs with respect to polar axes and the asymmetry of the far-field scattering patterns in the case of non-PEC strip grating.

The fifth figure shows a comparison of the direction patterns of the pre-fractal grating with the same number of strips (corresponding to the SSF3’s creator (Figure 2)); the initial geometrical measureless parameters \( a = \pi/27, b = 8\pi/27, c = 9 \). The cross-sectional size of the grating is equal to \( \lambda/3 \) (\( \beta + \alpha = \pi/3 \)).
The angles of the plane $H$-polarised electromagnetic wave incidence upon the grating are oblique angles and take the following values: $\theta_0 = \pi/3$ (Figure 5a) and $\theta_0 = \pi/18$ (Figure 5b). The solid lines correspond to non-zero values of strip impedance: $\mu = 0.5 + 0.5i$ (marked in black colour) and $\mu = 0.25 + 0.25i$ (marked in blue colour); while the red points correspond to the value $\mu = 0$ (grating of PEC strips). One can see that the direction patterns of the grating essentially dependent on the angles of the plane EM wave incidence on the grating and the values of the strip impedance. Comparing line graphs between point graph gives us the possibility to make the conclusion that direction patterns of impedance strip gratings are essentially greater than PEC strip gratings when the angle of the plane wave incidence on the grating is small (near-sliding incidence).

The sixth figure presents a comparison of the direction patterns of the pre-fractal gratings corresponding to the first and second stages of the SSF3's creation process (Figure 2). The cross-sectional size of both the gratings is the same and is equal to $\lambda$. The black lines correspond to the first stage (three equal IS of normalised strip width $\alpha = \pi/27$ and normalised distance between them $\beta = 26\pi/27$; so, the self-similarity coefficient $c = 27$ and $HD = 3/27 = 1/3$) while the blue lines correspond to the second stage (nine equal IS of normalised strip width $\alpha/c$). The last grating of nine IS consists of three sub-gratings of three IS (Figure 2).

The angles of the plane $H$-polarised EM wave incidence are as follows: $\theta_0 = \pi/6$ Figure 6a), $\pi/9$ Figure 6b and $\pi/90$ (near-sliding incidence) Figure 6c). The pattern shapes are the same for the three strip pre-fractal grating and the nine strip pre-fractal grating. So, in this case, one can see the fractal nature of pre-fractal gratings.

5 | CONCLUSIONS

The Cauchy method of analytical regularisation, for the first time in full explanation, has been successfully developed for the modelling of the plane wave scattering by a flat pre-fractal system of impedance strips. The principle of strict mathematical ordering for the pre-fractal strip grating deals with the five-part class of perfect fractal sets. This class is characterised by a variable positive fractal dimension, which is changed into the interval (0, 1). The Cauchy method is a powerful one for solutions in two cases: the plane $H$-polarised electromagnetic problem and the plane sonic wave scattering problem. It is based on a modification of the singular integral equation technique and the correct type of inversion formula for the simplest Cauchy integral equation.

In addition, the plane $H$-polarised electromagnetic wave scattering problem has been solved analytically in the closed form for the case of sparsely filled grating of impedance strips. It is very important to be sure of the mathematical model presented and for testing direct numerical methods. By making use of the asymptotic solution, the scattered electromagnetic field in the far zone was considered and analysed. Calculation algorithms have been developed for direction patterns of the sparsely filled grating and computer simulations have been performed.

A similar approach can be successfully developed for mathematical modelling of plane wave scattering by a finite multi-level coplanar system of flat impedance strips [25]. Such modelling will be promising for the creation of 2-D domains with given, effective electric and magnetic properties by embedding into a homogeneous medium a lot of small impedance segments, in the sense of [26].

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