ASYMPTOTICITY OF GRAFTING AND TEICHMÜLLER RAYS I

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ABSTRACT. We show that any grafting ray in Teichmüller space determined by an arational lamination or a multi-curve is (strongly) asymptotic to a Teichmüller geodesic ray. As a consequence the projection of a generic grafting ray to moduli space is dense. We also show that the set of points in Teichmüller space obtained by integer \((2\pi)-\) graftings on any hyperbolic surface projects to a dense set, which implies that complex projective surfaces with any fixed Fuchsian holonomy are dense in moduli space.

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1. Introduction

A complex projective structure on a surface \(S_g\) of genus \(g\) is an atlas of charts to \(\mathbb{CP}^1\) such that the transition maps are in \(PSL_2(\mathbb{C})\), and as this also determines a marked conformal structure, the space \(\mathcal{P}(S_g)\) of such structures forms a bundle over Teichmüller space \(\mathcal{T}_g\). In particular, a hyperbolic structure on a surface can be thought of as a complex projective structure with Fuchsian (or real) holonomy, and the operation of projective grafting on a simple closed curve deforms such a complex projective structure by inserting a projective annulus along a geodesic representative of that curve. By taking limits, this procedure extends to geodesic laminations and gives a geometric parametrization of \(\mathcal{P}(S_g)\) (see for example [KT92], [Tan97], [McM98]). In this paper we shall consider conformal grafting rays which are the conformal locus in \(\mathcal{T}_g\) of such deformations of complex projective structures, and establish a (strong) asymptoticity with Teichmüller geodesics (Theorem 1.1) that is used to show a density result concerning the set of complex projective structures with Fuchsian holonomy (Theorem 1.4).

The conformal grafting ray determined by a pair \((X,\lambda)\) of a hyperbolic surface \(X\) and a measured geodesic lamination \(\lambda\) shall be denoted by \(gr_{\lambda}X\), and is a one-parameter family of conformal structures obtained, roughly speaking, by cutting

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along \(\lambda\) on \(X\) and inserting a euclidean metric whose width increases along the ray (a more precise description is given later). Associated to a pair \((X, \lambda)\) there is also a Teichmüller geodesic ray, and it is known that the conformal structures arising along a grafting ray and a Teichmüller ray share many qualitative features. For example, along such a Teichmüller ray the extremal length of \(\lambda\) decreases monotonically: this is also true along a grafting ray \([DW08]\). The similarity of the geometry of a large-grafted surface and a singular flat surface has been exploited in \([Dum07]\), and one of the aims of this article is to make that similarity precise.

Two rays \(\Theta\) and \(\Psi\) in \(T_g\) are said to be asymptotic if the Teichmüller distance (defined in \(\S2\)) between them goes to zero, after reparametrizing if necessary. More concisely, \(\lim_{t \to \infty} \inf_{Z \in \Theta} d_T(Z, \Psi(t)) = 0\). We shall prove here that:

**Theorem 1.1.** Let \(X \in T_g\) and let \(\lambda \in ML\) be such that \(\lambda\) is arational, or a multicurve. Then there exists a \(Y \in T_g\) such that the grafting ray determined by \((X, \lambda)\) is asymptotic to the Teichmüller ray determined by \((Y, \lambda)\).

Here a measured lamination \(\lambda\) is said to be arational when it is both maximal (complementary regions are all triangular) and irrational (has a single minimal component that is not a closed geodesic). Such laminations are in fact of full measure in \(ML\) - we refer to \(\S2\) for a fuller discussion of the structure theory of geodesic laminations. In a sequel to this paper \([Gup]\) we hope to generalize Theorem 1.1 to the case of a general lamination - the case of laminations with polygonal complements can be already handled (see \(\S3.2\)) but we omit it in this paper for brevity.

The asymptotic behavior of two Teichmüller rays with respect to the Teichmüller metric is well known (see \[Mas80, Iva01, LM10\]). In \[Mas80\] Masur proved that if \(\lambda\) is uniquely ergodic, then for any two initial surfaces \(X, Y \in T_g\) the Teichmüller rays determined by \((X, \lambda)\) and \((Y, \lambda)\) are asymptotic. The comparison of grafting rays and Teichmüller rays has been less explored, however a recent result along these lines (see also \[DK\]) is the following “fellow-traveling” result in \[CDR\]:

**Theorem (Choi-Dumas-Rafi).** For any \(X \in T_g\) and any unit length lamination \(\lambda\), the grafting ray determined by \((X, \lambda)\) and the Teichmüller ray determined by \((X, \lambda)\) are a bounded distance apart, where the bound depends only on the injectivity radius of \(X\).

Theorem 1.1 makes a finer but less uniform comparison involving the stronger notion of asymptoticity defined above. We aim to discuss a more quantitative version of this theorem in our sequel.

A crucial difference between grafting and Teichmüller rays is that the latter is a flow, whereas for grafting it is not:

\[g^{t+\lambda} \circ g^{s\lambda} \neq g^{(t+s)\lambda}\]

Our results show that if one waits till \(t\) is sufficiently large, however, it approximates the Teichmüller geodesic flow.

The proof of Theorem 1.1 is achieved by constructing quasiconformal maps of small dilatation from sufficiently large grafted surfaces along the grafting ray, to singular flat surfaces that lie along a common Teichmüller ray. It involves a comparison of
the **Thurston metric**, a hybrid of a hyperbolic and euclidean metric underlying a complex projective surface on one hand, and a singular flat metric induced by a holomorphic quadratic differential on the other. The case when the lamination \( \lambda \) is arational is handled in §4, and the case when the lamination is a multicurve is dealt with in §5. An outline of the strategy of the proofs of these cases are provided in §3. In a subsequent article we shall describe a more quantitative version of the theorem, and consider the remaining cases when \( \lambda \) is not arational or a multi-curve.

The following are immediate corollaries of Theorem 1.1:

**Corollary 1.2.** Let \( X, Y \) be any two hyperbolic surfaces and let \( \lambda \) be a maximal uniquely-ergodic lamination. Then the grafting rays determined by \((X, \lambda)\) and \((Y, \lambda)\) are asymptotic.

**Corollary 1.3.** For every \( X \in \mathcal{T}_g \) and almost every \( \lambda \in \mathcal{ML} \) in the Thurston measure, the projection of the grafting ray determined by \((X, \lambda)\) is dense in moduli space \( \mathcal{M}_g \).

Let \( \mathcal{S} \) be the set of simple closed geodesic multicurves on \( S_g \) and \( \pi: \mathcal{T}_g \to \mathcal{M}_g \) be the usual projection to moduli space. As a further application, by a careful multicurve approximation of the lamination determining a dense grafting ray (which exists by Corollary 1.3) we show that for any \( X \in \mathcal{T}_g \), the set of integer or \( 2\pi \)-graftings \( \{ \pi(gr_{2\pi, \gamma}X) | \gamma \in \mathcal{S} \} \) is dense. These integer graftings are the conformal “shadow” of projective graftings that preserve the Fuchsian holonomy of the complex projective structure of the initial hyperbolic surface \( X \) (see [Gol87]), and so the following theorem follows:

**Theorem 1.4.** Let \( \mathcal{P}_\rho \) be the set of complex projective structures on a surface \( S_g \) with a fixed holonomy \( \rho \in \text{Rep}(\pi_1(S_g), PSL_2(\mathbb{C})) \). Then for any Fuchsian representation \( \rho \), the projection of \( \mathcal{P}_\rho \) to \( \mathcal{M}_g \) has a dense image.

In [Fal83] Faltings had first conjectured that this projection of the set \( \mathcal{P}_\rho \) is infinite, and this can be thought of as the strongest possible affirmation of that.

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### 2. Preliminaries

**Teichmüller space** \( \mathcal{T}_g \). For a closed orientable genus-\( g \) surface \( S_g \), the Teichmüller space \( \mathcal{T}_g \) is the space of marked conformal (or equivalently, complex) structures on \( S_g \) with the equivalence relation of isotopy. (see [HT92], [Hub96] for a treatment of the subject). Note that although for this article, we assume the surfaces have no punctures, the results still hold for the punctured case with a slight modification of the arguments.

The distance between two points \( X \) and \( Y \) in \( \mathcal{T}_g \) in the **Teichmüller metric** is defined to be

\[
d_{\mathcal{T}}(X, Y) = \frac{1}{2} \inf_f \ln K_f
\]

where \( f : X \to Y \) is a quasiconformal homeomorphism, and \( K_f \) is its quasiconformal dilatation. The infimum is realized by the **Teichmüller map** between the surfaces.
A thorough discussion of the definitions of a quasiconformal map and its dilatation (also referred to as its quasiconformal distortion) is provided in Appendix A. It suffices to point out here that roughly speaking, a quasiconformal map takes infinitesimal circles on the domain to infinitesimal ellipses, and the dilatation is a measure of the maximum eccentricity of the image ellipses. The ‘difference’ or distance between two conformal structures is then measured in terms of the dilatation of the least distorted quasiconformal map between them.

A consequence of the above definition is that if there exists a map \( f : X \to Y \) which is \((1 + O(\epsilon))\)-quasiconformal (i.e, \( K_f = 1 + O(\epsilon) \)), then \( d_T(X, Y) = O(\epsilon) \).

Notation. Here, and throughout this article, \( O(\alpha) \) refers to a quantity bounded above by \( C\alpha \) where \( C > 0 \) is some constant depending only on genus \( g \) (which remains fixed), the exact value of which can be determined a posteriori.

**Definition 2.1.** For the ease of notation, an almost-conformal map shall refer to a map which is \((1 + O(\epsilon))\)-quasiconformal.

**Hyperbolic surfaces and geodesic laminations.** Any conformal structure on a surface of genus \( g \geq 2 \) has a unique hyperbolic structure (a Riemannian metric of constant negative curvature \(-1\)) in its conformal class via uniformization. Thus Teichmüller space is, equivalently, the space of marked hyperbolic structures and this gives rise to a rich interaction between two-dimensional hyperbolic geometry and complex analysis.

A geodesic lamination on a hyperbolic surface is a closed subset of the surface which is a union of disjoint simple geodesics. A maximal lamination is a geodesic lamination such that any component of its complement is an ideal hyperbolic triangle. There is a rich structure theory of geodesic laminations (see [CB88] for example): in particular, any geodesic lamination \( \lambda \) is a disjoint union of sub laminations

\[
\lambda = \lambda_1 \cup \lambda_2 \cup \cdots \lambda_m \cup \gamma_1 \cup \gamma_2 \cup \cdots \gamma_k
\]

where \( \lambda_i \)s are minimal components (with each half-leaf dense in the component) which consist of uncountably many geodesics (a Cantor set cross-section) and the \( \gamma_j \)s are isolated geodesics.

A measured geodesic lamination is equipped with a transverse measure, that is, a measure on arcs transverse to the lamination which is invariant under sliding along the leaves of the lamination. It can be shown that for the support of a measured lamination the isolated leaves in (1) above are weighted simple closed curves (ruling out the possibility of isolated geodesics spiralling onto a closed component). We call a lamination arational if every simple closed curve intersects it, and it can be shown that for a measured lamination this condition is equivalent to being maximal and irrational, which is that it consists of a single minimal component and no isolated leaves. A measured lamination is uniquely-ergodic if such a measure is unique. A maximal, uniquely ergodic lamination is necessarily arational. The set of weighted simple closed curves is dense in \( \mathcal{ML} \), the space of measured geodesic laminations equipped with the weak-* topology. \( \mathcal{ML} \) also has a piecewise-linear structure and
a corresponding Thurston measure.

**Train tracks.** A train-track on a surface is a trivalent graph with a labeling of incoming and outgoing half-edges at every vertex, and an assignment of (non-negative) weights to the edges (or branches) that are compatible, such that at every vertex, the sum of the weights of the incoming edges is equal to the sum of the edges of the outgoing edges. This provides a convenient combinatorial encoding of a lamination (see, for example, [FLP79] or [Thu82]) - in particular, for an assignment of integer weights, one can place a number of strands along each branch equal to the weight, and the compatibility condition ensures that these can be “hooked” together to form a multicurve.

**Quadratic differentials and Teichmüller rays.** Any measured geodesic lamination corresponds to a unique measured foliation of the surface, obtained by ‘filling in’ the complementary components. Conversely, any measured foliation can be ‘tightened’ to a geodesic lamination. The space of measured foliations $\mathcal{MF}$ is homeomorphic to $\mathcal{ML}$ via this correspondence.

A holomorphic quadratic differential $\phi$ is locally of the form $\phi(z)dz^2$ where $\phi(z)$ is a holomorphic function, and has a vertical foliation given by the level sets of $\text{Im}(\int_0^z \sqrt{\phi(z)}dz)$, which has singularities at the zeroes of $\phi(z)$. These singularities are also the cone-points in the singular flat metric given by $|\phi(z)||dz|^2$. For a hyperbolic surface $X$, the map that assigns the vertical measured foliation $F_v(\phi)$ to a quadratic differential $\phi(X)$ defines a homeomorphism between the space of quadratic differentials $Q(X)$ and $\mathcal{MF}$ ([HM79]). Composed with the previous correspondence between measured laminations and foliations, we get a homeomorphism $q_L : Q(X) \to \mathcal{ML}$.

A Teichmüller ray from a point $X$ in $\mathcal{T}_g$ and in a direction determined by a holomorphic quadratic differential $\phi$ (or equivalently the measured geodesic lamination $\lambda = q_L(\phi)$) is a subset $\{X_t\}_{t \geq 0}$ of $\mathcal{T}_g$ where $X_t$ is obtained by starting with the surface $X$ with the horizontal and vertical foliations $F_h(\phi)$ and $F_v(\phi)$ and scaling the horizontal foliation by a factor of $e^t$ and the vertical foliation by a factor of $e^{-t}$. This ray is geodesic in the Teichmüller metric.

**Complex projective structures and grafting.** An excellent exposition of this material can be found in [Dum].

A complex projective structure on a surface $S_g$ is a pair $(\text{dev}, \rho)$ where $\text{dev} : \tilde{S}_g \to \mathbb{C}P^1$ is the developing map of its universal cover and $\rho : \pi_1(S_g) \to PSL_2(\mathbb{C})$ is the holonomy representation that satisfies

$$\text{dev} \circ \gamma = \rho(\gamma) \circ \text{dev}$$

for each $\gamma \in \pi_1(S_g)$.

Equivalently, a complex projective structure is a collection of charts on the surface to $\mathbb{C}P^1$ such that overlapping charts differ by a projective (or Möbius) transformation. Since Möbius transformations are holomorphic, any complex projective structure has an underlying conformal structure. This gives rise to a forgetful map from the space of complex projective structures $\pi : \mathcal{P}(S_g) \to \mathcal{T}_g$. 

A hyperbolic surface arises from a Fuchsian representation \( \rho : \pi_1(S_g) \to PSL_2(\mathbb{R}) \) and thus has a canonical complex projective structure. Grafting, introduced by Thurston (see [KT92], [Tan97], [SW02], [DW08] for subsequent development) can be thought of as a way to deform the Fuchsian complex projective structure.

In the case of grafting along a simple closed geodesic \( \gamma \), the process can be described as follows:

Embed the universal cover \( \tilde{X} \) of the hyperbolic surface \( X \) as the equatorial plane in the ball-model of \( \mathbb{H}^3 \). The hyperbolic Gauss map to the \( \partial \mathbb{H}^3 = \mathbb{C}P^1 \) provides the developing map of the Fuchsian structure. The curve \( \gamma \) lifts to a collection of geodesic lines on this plane. Now we bend along these lifts equivariantly by an angle \( s \) such that one gets a convex pleated plane. Via the Gauss map, this corresponds to inserting crescent-shaped regions of angle \( s \) along the images of the lifts of \( \gamma \) (in Figure 11 in §4.3 this is shown in the upper half-plane model where the imaginary axis is a lift of \( \gamma \)). This defines the image of the developing map of the new projective structure. Of course, for an angle \( s \geq 2\pi \), the image wraps around \( \mathbb{C}P^1 \), and the developing map is not injective.

There is a canonical stratification of this developed image on \( \mathbb{C}P^1 \) ([KP94]), and in particular, one can speak of maximally embedded round disks. One can recover a locally convex pleated plane by taking an envelope of the convex hulls (domes) over these disks. The projective metric on the universal cover is defined by taking the Poincaré metric on the maximal disks.

On the quotient surface, these \( s \)-crescents descend to a annulus inserted at the closed geodesic \( \gamma \). The projective metric descends to the Thurston metric on the surface. In this metric, the inserted annulus is flat (euclidean), and the rest of the surface remains hyperbolic.

This grafted surface is denoted by \( gr_{s\gamma}X \), where \( s \) (the bending angle above) is a nonnegative real parameter, and can be thought of as the ‘weight’ of \( \gamma \).

Grafting for a general measured lamination \( \lambda \) is defined by taking the limit of a sequence of approximations of \( \lambda \) by simple closed curves (the bending angles are determined by the approximating measures). It was an observation of Thurston that the map

\[
(\lambda, X) \mapsto Gr_\lambda X
\]

is a homeomorphism of \( ML \times \mathcal{T}_g \) to \( P_g \), where \( Gr \cdot \) refers to the projective surface obtained by the above operation.

**Grafting rays.** A grafting ray from a point \( X \) in \( \mathcal{T}_g \) in the ‘direction’ determined by a lamination \( \lambda \) is the set of points \( \{X_t\}_{t \geq 0} \) where \( X_t = gr_{t\lambda}(X) \).
Here we are concerned with conformal grafting, where we consider only the conformal structure underlying the complex projective structure. For example, starting with a hyperbolic surface $X$ and grafting along $\lambda$ results in a surface with the Thurston metric (a different conformal structure): if $\lambda$ is a simple closed curve this is the surface with the inserted euclidean annulus as described above, for a general lamination the distance ‘across’ $\lambda$ increases by an amount equal to its transverse measure. It is known that for any fixed lamination $\lambda$, the grafting map $(X \mapsto gr_{\lambda}X)$ is a self-homeomorphism of $T_g$.

3. An overview of the proofs

The proofs of both Theorem 1.1 and Theorem 1.4 involve understanding the geometry of the grafted surfaces $X_t = gr_{t\lambda}X$ along the grafting ray determined by a pair $(X, \lambda)$, for large $t$. By the definition of grafting, as described in §2, these surfaces carry a conformal metric which is euclidean on the grafted region and hyperbolic elsewhere. A convenient way to picture this is to consider a thin “train-track” neighborhood $\mathcal{T} \subset X$ that contains the lamination $\lambda$. As one grafts, the subsurface $\mathcal{T}$ widens in the transverse direction (along the “ties” of the train-track), and conformally approaches a union of wide euclidean rectangles.

The complement $X \setminus \mathcal{T}$ is unaffected by grafting: in the case when $\lambda$ is arational, it comprises finitely many hyperbolic ideal triangles (the number depends only on genus) and for $\lambda$ non-arational this complement might consist of ideal polygons or subsurfaces with moduli. The former case therefore is simpler and allows for more explicit constructions, and we focus on that first. For the latter case, this paper shall deal with the case when these subsurfaces have boundary components consisting of closed curves, and the general case is deferred to a subsequent paper. In either case, the underlying intuition is that this complementary hyperbolic part becomes negligible compared to the euclidean part of the Thurston metric, for a sufficiently large grafted surface.

3.1. The arational case.

The surface $\hat{X}_t$.

For $\lambda$ arational, one can consider the associated (singular) transverse horocyclic foliation which we denote by $\mathcal{F}$. This is obtained as follows: Since the lamination $\lambda$ is maximal, it lifts to the universal cover of the surface to give a tessellation of the hyperbolic plane $\mathbb{H}^2$ by ideal hyperbolic triangles. Each ideal hyperbolic triangle has a partial foliation by horocyclic arcs belonging to horocycles tangent to each of the three ideal vertices. It can be shown (see [Thu]) that these can be extended across adjacent ideal triangles to form a (partial) $C^1$-foliation of the surface which is transverse to $\lambda$. (For arbitrary laminations one integrates the corresponding Lipschitz vector field.) It is often useful (as in §4.4) to extend this partial foliation to a singular foliation with “3-prong” singularities in the center of each ideal triangle (see Figure 2).

This transverse foliation $\mathcal{F}$ persists under grafting along $\lambda$, with the leaves getting longer along a grafting ray. This is easier to see in the finite grafting case where the grafted euclidean cylinder has a horizontal foliation that appends to $\mathcal{F}$, and the general case is obtained by a limiting argument (see §4.3).
Then there is an associated Riemann surface $\hat{X}_t$ with a singular flat metric obtained by collapsing the ideal triangle components of $X \setminus \lambda$ along the leaves of $\mathcal{F}$, but preserving the transverse measure. The singularities of the flat metric on $\hat{X}_t$ are 3-pronged conical singularities that arise by collapsing the central (unfoliated) region of each ideal triangle in the complement of the lamination.

There are a couple of ways one can think of the collapsed surface: one is to think of an explicit collapsing map (see [CBBSS]) that “blows up” the lamination (that is locally a Cantor set cross interval) by a function that is the Cantor function on each transverse cross section. The other is to think of $\hat{X}_t$ as a singular flat surface obtained by gluing up euclidean rectangles with the same combinatorics of the gluing as dictated by the structure of the lamination (or equivalently the corresponding train-track $\mathcal{T}$) on $X_t$.

The singular flat surfaces $\hat{X}_t$ acquire a horizontal foliation that is measure-equivalent to $\mathcal{F}$, and a vertical foliation that is measure-equivalent to $t\lambda$. Hence as $t$ varies they lie on a common Teichmüller ray. The “collapsing map” itself is far from being quasiconformal (it is not even a homeomorphism), and main idea behind the proof of Theorem 1.1 is to use the additional grafted region to “diffuse out” the collapsing to get a quasiconformal map from $X_t$ to $\hat{X}_t$, which is moreover almost-conformal (see Definition 2.1).

**Outline of the proof (arational case):**

*Step 1. The decomposition of $X_t$.* One first decomposes the grafted surface into rectangles and pentagonal pieces (§4.1) that essentially make up the train track neighborhood $\mathcal{T}$, and a slight thickening of the truncated ideal triangles in its complement, respectively. Lemmas 4.2 and 4.3 are concerned with the dimensions of the resulting pieces.

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**Figure 2.** The partial horocyclic foliation of an ideal hyperbolic triangle.

**Figure 3.** A partial picture of a maximal lamination, with two truncated ideal triangles in the complement shown shaded.
Step 2. Mapping the pieces. Next, one constructs quasiconformal maps that map the pieces in the decomposition to (singular) flat regions, by mapping the leaves of $F$ in a suitable manner. The rectangular pieces of the grafted surface that “carries” the lamination need a finite approximation argument, which is carried out in §4.3, and culminates in Lemma 4.18. The maps for the pentagonal pieces are constructed by first constructing maps of “truncated sectors” (sections §4.5 and 4.4 respectively). Much of these constructions depend on explicit constructions of $C^1$ maps of controlled dilatation that one can build between various hyperbolic or euclidean regions, which are compiled in §4.2. The assumption of $C^1$-regularity is justified by the corresponding regularity of the Thurston metric (see §4.3) and the horocyclic foliation $F$.

Figure 4. In Step 2, the maps of the pieces assemble to give a quasiconformal map of the portion of $X_t$ shown on the left to the singular flat region on $\hat{X}_t$ shown on the right. The (shaded) hyperbolic region is taken to a neighborhood of the central “tripod”.

Step 3. Gluing the maps of the pieces. To assemble the maps of the pieces to a map of the grafted surface $X_t$ to the singular flat surface $\hat{X}_t$, they need to be adjusted on the boundary. This is possible by an additional property of the maps called almost-isometry (Definition 4.10) which allows this adjustment to be made maintaining the almost-conformality (see Lemma 4.14 in §4.2). At this stage one has a quasiconformal map that is almost-conformal for most of the surface (Lemma 4.23).

Step 4. Adjusting to an almost-conformal map. The quasiconformal map from Lemma 4.23 is then adjusted to be almost-conformal everywhere. This relies on the fact that the regions of no control of quasiconformal distortion are contained in portions of the surface surrounded by annuli of large modulus (Lemma 4.24), and a technical lemma on quasiconformal extensions (Lemma 4.25) whose proof we defer to Appendix A.

3.2. The general case. In contrast to the case of an arational (and therefore maximal) lamination, in the general case the complementary subsurface $X \setminus T$ might contain an ideal polygon or non-simply-connected subsurface with a non-trivial parameter space (moduli) of conformal structures, and which is unaffected by the procedure of grafting.
The approach here is to:

**Step A.** Define a decomposition of the large-grafted surface $X_t$ into a subsurface $C$ containing the above complementary subsurface, and a subsurface $B = X_t \setminus C$ where the lamination is maximal. Let $\hat{B}$ be the “collapsed” singular flat surface corresponding to $B$.

**Step B.** Define a singular flat surface $D$ together with an almost-conformal map $G : C \rightarrow D$. This is done by considering an “infinitely grafted surface” corresponding to $C$, prescribing a meromorphic quadratic differential on it, and taking a suitable truncation of the resulting singular flat surface. By careful choice of the principal part at the poles, $D$ “glues up” with $\hat{B}$ to define a singular flat surface $Y_t = \hat{B} \cup D$ that lies along the same Teichmüller ray for different $t$.

**Step C.** There exists an almost conformal map $F$ from $B$ to $\hat{B}$ from the arational case. One now shows that $F$ and $G$ can be suitably interpolated to define a composite almost-conformal map from $X_t$ to $Y_t$.

**Multicurve case.** Following the above strategy, this paper shall consider the case when $\lambda$ is a multicurve in §5. The following is an outline of the argument for the special case when $\lambda$ is a single non-separating simple closed geodesic $\gamma$:

The surface $X_t$ appearing along the grafting ray has a euclidean cylinder of length $t$ inserted at $\gamma$. To find the singular flat surface $\hat{X}_t$ to map the grafted surface to, one first forms the “infinitely grafted” surface $X_\infty$ obtained by gluing semi-infinite cylinders on the two boundary components of $X_t \setminus \gamma$.

By a theorem of Strebel (Proposition 5.7), there is a meromorphic quadratic differential on this surface with two double poles and closed horizontal trajectories, that induces a singular flat metric which makes the surface isometric to two semi-infinite euclidean cylinders glued along the boundary (we call this $Y_\infty$ to distinguish this from $X_\infty$, though they are conformally equivalent).

The singular flat surface $\hat{X}_t$ is then obtained by truncating these infinite cylinders of $Y_\infty$ at some “height” and gluing them along the truncating circles, and this lies along the Teichmüller ray determined by $\lambda$. It only remains to adjust the conformal map between $X_\infty$ to $Y_\infty$ to an *almost-conformal* map between $X_t$ and $\hat{X}_t$ - this is possible because a conformal map “looks” affine at small scales, or (in this case) like an isometry far out a cylindrical end (see Lemma 5.8).

**Minimal, polygonal complement.** In the case when the lamination is minimal with the complementary region $C$ an ideal polygon the “infinitely grafted” surface corresponding to it is obtained by gluing euclidean half-planes along the infinite geodesic boundary components of the hyperbolic completion of $C$. The resulting surface is uniformized by $C$, and the existence of a meromorphic quadratic differential (with a higher order pole and prescribed principal part) on $X_\infty$ is an explicit construction, which allows the above strategy to be carried out. However, details of this are deferred to [Gup] where we hope to handle the most general case.
3.3. **Idea of the proof of Theorem 1.4:** When the lamination $\lambda$ is arational at least one of the weights on its train-track representation is irrational. As described at the beginning of §3, the train track neighborhood $\mathcal{T}$ of $\lambda$ widens along the grafting ray, and a typical rectangular piece (corresponding to a branch of the train track) looks more and more euclidean, with its euclidean width at time $t$ equal to the initial weight times $t$.

A key observation is that since the switch conditions for the train-track reduce to a system of linear equations with integer coefficients, there is always an assignment of *integer* weights that approximate those of the arational lamination, such that for each branch the integer weight is within a bound that depends only on the genus (Lemma 6.16). For sufficiently large $t$, this difference is small in proportion to the entire width, and this allows the construction of an almost-conformal map from a surface along the grafting ray to a surface obtained by grafting along the multi-curve corresponding to the integer system (Lemma 6.20).

This construction together with a choice of $\lambda$ that provides a *dense* grafting ray (Corollary 1.3 of Theorem 1.1) shows that integer graftings are dense in moduli space (Proposition 6.1), and Theorem 1.4 follows as they also preserve the Fuchsian holonomy.

4. **The arational case**

In this section we prove the following proposition, a special case of Theorem 1.1:

**Proposition 4.1.** Let $X \in \mathcal{T}_g$ and let $\lambda$ be an arational (ie. maximal and irrational) geodesic lamination. Then there exists a $Y \in \mathcal{T}_g$ such that the grafting ray determined by $(X, \lambda)$ is asymptotic to the Teichmüller ray determined by $(Y, \lambda)$.

An outline of the proof was provided in §3.1, and we refer to that section for the notation used here.

4.1. **A train-track decomposition of the grafted surface.** We begin by constructing a subsurface $\mathcal{T}_r \subset X$ containing the arational lamination $\lambda$, that is further decomposed into rectangles. This can be thought of as a physical realization of a train track carrying $\lambda$, and the rectangles correspond to the branches of the traintrack.

4.1.1. **The return map.** From §3.1 recall that there is a horocyclic foliation $\mathcal{F}$ with $C^1$ leaves transverse to $\lambda$, obtained by integrating the Lipschitz line field along the horocyclic arcs of the ideal triangles in its complement. Choose an oriented segment $\tau$ from a leaf of $\mathcal{F}$ away from its singularities, such that the endpoints of $\tau$ are on leaves of the lamination $\lambda$ which are isolated on the side away from $\tau$.

In what follows we use the first return map of $\tau$ to itself (following leaves of $\lambda$) to form a collection of rectangles with vertical geodesic sides, and horizontal sides lying on $\tau$. This is similar to the standard method constructing the suspension of interval exchange transformations, and in particular to the decomposition in [Mas80] where it is done for a quadratic differential metric and the associated foliations.
Label the two sides of the transverse arc $\tau$ by $\tau_+$ and $\tau_-$ (one can imagine $\tau$ to be “thickened” a bit). Consider a point $x \in \lambda \cap \tau$. We follow the leaf leaving $\tau$ on the $+$ side, and since the lamination is minimal, this leaf returns to $\tau$ a first time, to either $\tau_+$ or $\tau_-$. We denote that first return point as $T_+(x)$. We can denote a corresponding first return map on the $-$ side as $T_-(x)$.

Let $I^+_1, J^+_1, I^+_2, J^+_2, \ldots, I^+_n, J^+_n$ be a partition of $\tau^+$ into a collection of sub-intervals ordered from left to right, such that for each $1 \leq i \leq n_1$ we have:

1. $\partial I^+_i \cap \lambda \neq \emptyset$.
2. $(I^+_i)^\circ \cap \lambda = \emptyset$.
3. If $T^+(I^+_i)$ denotes the sub-interval of $\tau$ determined by $T^+(\partial I^+_i)$, then $I^+_i$ and $T^+(I^+_i)$ and the leaves of $\lambda$ through $\partial I^+_i$ emanating in the $+$ side bound a simply connected region (a rectangle) on the surface.

Let $I^-_1, J^-_1, I^-_2, J^-_2, \ldots, I^-_{n_2}, J^-_{n_2}$ be a corresponding partition of $\tau^-$ such that $I^-_i$, $T^- (I^-_i)$ and the leaves through $\partial I^-_i$ emanating on the $-$ side bound rectangles, for each $1 \leq i \leq n_2$, and such that if $T^+(I^+_j)$ lies on $\tau^-$ for any $1 \leq j \leq n_1$ then it belongs to the collection $\{I^-_1, I^-_2, \ldots, I^-_{n_2}\}$.

Furthermore, let the above choice of intervals be such that $n_1 + n_2$ is minimal (by amalgamating two adjacent rectangles whenever possible).

This decomposes the surface into a collection of rectangles $R_1, R_2, \ldots, R_n$, and complementary regions which are truncated ideal hyperbolic triangles $T_1, T_2, \ldots T_m$ (where $n$ and $m$ depend only on the genus).

We define

$$\mathcal{T}_c = R_1 \cup R_2 \cup \cdots R_n$$

Along the grafting ray the rectangles get wider, and one gets a decomposition of each grafted surface $X_t$ with the same combinatorics of the gluing.

4.1.2. Labelling sub-arcs. For later use, we denote the sub-arcs of the arc $\tau$ that are the horocyclic edges of $T_1, \ldots T_m$ by $J_1, \ldots J_{3m}$ and the remaining sub-arcs in

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**Figure 5.** The train track decomposition and the labelling of the sub-arcs of $\tau$. 
\[ \tau \setminus \{J_1 \cup \cdots J_{3m}\} \] by \( I'_1, \ldots I'_{3m+1} \). Note that for each \( 1 \leq i \leq n \) we have

\[ I_i = \bigcup_{k \in S_i} I'_k \]

where \( S_i \) is a finite subset of \( \{1, 2, \ldots, 3m+1\} \), and \( S_i \cap S_j = \emptyset \) for \( i \neq j \).

4.1.3. Dimensions. The total width of a rectangle \( R_i \) shall be the supremum of the lengths in the Thurston metric of the segments \( l \cap R_i \), where \( l \) is a leaf of the transverse foliation \( \mathcal{F} \).

The hyperbolic width of a rectangle \( R_i \) is defined to be the supremum of the hyperbolic lengths of the leaves of \( \mathcal{F} \) that intersect \( R_i \).

The height of the rectangle is the hyperbolic length of one of the vertical geodesic sides.

**Lemma 4.2** (Long, thin train-track). If the hyperbolic length of \( \tau \) is sufficiently small, the height of each of the rectangles \( R_1, R_2, \ldots, R_n \) is greater than \( \frac{1}{\epsilon^2} \), and the hyperbolic width is less than \( \epsilon \).

**Proof.** Let the hyperbolic length of the arc \( \tau \) be \( L \). Then the two horizontal sides of a rectangle being intervals on the segment \( \tau \) have length less than \( L \). It follows from hyperbolic geometry that on the (ungrafted) surface \( X \) the two vertical geodesic sides remain within \( L \) of each other, and each piecewise-horocyclic leaf of \( \mathcal{F} \) between them has length at most \( O(L) \). We also have by elementary hyperbolic geometry that if the horocyclic edges of a truncated ideal triangle have length \( O(L) \), then the height at which the triangle is truncated is at least \( \ln \frac{1}{L} \). Clearly \( \ln \frac{1}{L} \to \infty \) as \( L \to 0 \), so we can choose \( L < \epsilon \) small enough so that the statement of the lemma holds.

Henceforth, we shall assume that \( \tau \) was chosen short enough such that the conclusions of the above lemma hold.

**Lemma 4.3** (Total width). There is a \( T > 0 \) such that for all \( t > T \), the total width of each rectangle in the above decomposition is greater than \( 1/\epsilon^2 \), and moreover the modulus of each is greater than \( 1 \).

**Proof.** Each rectangle has a euclidean width equal to the transverse measure \( \mu(t \lambda \cap R) \) which goes to infinity as the grafting time \( t \to \infty \). For large enough time \( t \), the width is greater than \( 1/\epsilon^2 \) and the modulus (the ratio \( \frac{\text{width}}{\text{height}} \)) to be greater than 1.

The decomposition \( \mathcal{D} \). It will be useful later to derive from the above another decomposition of a sufficiently grafted surface into rectangles and pentagons, obtained by what follows.

The construction of the 1-skeleton \( K \) of this derived decomposition is obtained in three steps:
(1) Start with the 1-skeleton $K_0$ of the decomposition into rectangles and truncated ideal triangles as above.

(2) Each truncated ideal triangle $T_j$ has a centroid $p_j$. The geodesic segments from $p_j$ to the midpoints of the horocyclic sides of $T_j$ shall be new edges. Note that this divides the truncated ideal triangle (which is a hexagon) into three pentagons, which are better described as ‘truncated $2\pi/3$-sectors of an ideal hyperbolic triangle’.

(3) Each $T_j$ is adjacent to rectangles from the collection $\{R_i\}$ on its three geodesic sides. By Lemma 4.3 for a sufficiently-grafted surface there is a grafted portion of euclidean width much greater than 4 adjacent to each geodesic side. We replace each geodesic side (which is an edge of $K_0$) with the vertical line which is (euclidean) distance 2 away from it (and has the same height). In other words, subrectangles of euclidean width 2 are trimmed from the adjacent rectangles and appended to the truncated sectors of $T_j$ (obtained in Step 2).

This trimming-and-appending results in the euclidean (and total) widths of the rectangles $\{R_i\}$ decreasing by 4. By abuse of notation, we continue to denote these trimmed rectangles as $\{R_i\}_{1 \leq i \leq n}$, and we denote the pentagons (thickened sectors) as $\{P_j\}_{1 \leq j \leq 3m}$. These form the pieces of this new decomposition, which we shall refer to as the decomposition $\mathcal{D}$.

4.2. A compendium of quasiconformal maps. There are two types of pieces in the decomposition $\mathcal{D}$ of the grafted surface described in the previous section, rectangles $R_i$, and pentagons $P_j$, which can be further decomposed into truncated $2\pi/3$-sectors and width-2 rectangles. We shall eventually construct some quasiconformal maps of these pieces to the euclidean plane which we shall glue to form a quasiconformal map of the grafted surface to the singular flat surface.

We first isolate as lemmas a few quasiconformal maps that will be useful at several steps of the actual construction. Since we would need some control on the quasiconformal dilatation of the final map, we take care to ascertain the distortion at all points of the domain.
We use repeatedly the following facts about quasiconformal maps:

A. For a $C^1$ map $f$ between planar domains the quasiconformal dilatation at a point is $1 + O(\epsilon)$ whenever the partial derivatives satisfy
\[ \|f_x\| \|f_y\| = 1 + O(\epsilon) \]
and
\[ |\langle f_x, f_y \rangle| = O(\epsilon). \]

B. If a map $f$ is a homeomorphism onto its image, and it is quasiconformal on the domain except for a measure zero set (typically a collection of $C^1$ arcs), then $f$ is quasiconformal everywhere on the domain.

C. If a homeomorphism $f$ is $K$-quasiconformal, so is its inverse $f^{-1}$.

**Straightening maps.** We define an arc on the euclidean plane to be $\epsilon$-almost-vertical if it is a portion of a graph $x = g(y)$ where $|g'(y)| < \epsilon$.

**Lemma 4.4.** Let $R$ be a planar region bounded by two parallel horizontal sides at a distance $a$ and two almost-vertical sides of varying width $w(y)$ where $0 \leq y \leq a$. Let $b = \sup_{y\in[0,a]} w(y)$. Then if $\frac{b}{w(y)} = 1 + O(\epsilon)$ for each $0 \leq y \leq a$, there exists a height-preserving almost-conformal map from $R$ to a euclidean rectangle of height $a$ and width $b$.

**Proof.** Let the left edge be given by the graph of a function $g(y)$ for $0 \leq y \leq a$, where $|g'(y)| = O(\epsilon)$. Then the inverse of the required map $f^{-1}$ is:
\[ (x, y) \mapsto \left( \frac{w(y)}{b} x + g(y), y \right) \]
This is easily shown to be almost-conformal by computing the derivatives and using property A above (Note that the sides being almost-vertical implies that the width function satisfies $|w'(y)| = O(\epsilon))$. \qed

**Lemma 4.5.** Let $R$ be a euclidean region bounded by two parallel horizontal sides at a distance $a$ and two equidistant ‘vertical’ sides, which are the graphs $g(y)$ and $g(y) + b$ of some function $g$ over an interval on the $y$-axis. Then there is a height-preserving $K$-quasiconformal map from $R$ to a euclidean rectangle of height $a$ and width $b$ which is also an isometry on each horizontal line. The quasiconformal distortion at height $y$ is $(1 + O(\epsilon))$ if $|\frac{dg(y)}{dy}| = O(\epsilon)$.

**Proof.** It suffices to check this for the inverse map, which is
\[ (x, y) \mapsto (g(y) + x, y) \]
It is easy to see (by computing the derivatives) that the above conditions on quasiconformality hold. \qed
Lemma 4.6. Let $R$ be a euclidean region bounded between a vertical segment of height $a$ on the $y$-axis, and the graph of a $C^1$ function $w$ on that interval. Assume that $w(y) > 1$ for all $0 \leq y \leq a$. Let $b = \sup_{y \in [0, a]} w(y)$. Then there exists a height-preserving quasiconformal map $f$ from $R$ to a rectangle of height $a$ and width $b$, such that the quasiconformal distortion is $1 + O(\epsilon)$ at height $y$ if the quantities $\left| \frac{dw(y)}{dy} \right|$ and $|b - w(y)|$ are $O(\epsilon)$.

Proof. The map $f$ is given by

$$(x, y) \mapsto \left( \frac{b}{w(y)} x, y \right)$$

The rest follows by computing the derivatives and using the estimates

$$\left| \frac{b}{w(y)} - 1 \right| = \left| \frac{b - w(y)}{w(y)} \right| < |b - w(y)| = O(\epsilon)$$

and

$$\left| \frac{bx}{w(y)^2} \frac{dw(y)}{dy} \right| \leq \left| \frac{b^2}{w(y)^2} \frac{dw(y)}{dy} \right| = (1 + O(\epsilon)) \left| \frac{dw(y)}{dy} \right|$$

Maps straightening a horizontal foliation.

Lemma 4.7. Let $R$ be a rectangular region on the hyperbolic plane bounded by two (vertical) geodesic sides and two (horizontal) horocyclic sides such that the hyperbolic width is $O(\epsilon)$. Note that $R$ is foliated by horocyclic segments.

Then the map $f : R \to \mathbb{R}^2$ that

(i) takes the left edge to a vertical segment

(ii) maps each horocyclic leaf to a horizontal line, and

(iii) is distance-preserving along each horocyclic leaf,

is almost-conformal.

Moreover, the right edge of $R$ is mapped to an almost-vertical segment.

Proof. Consider $R$ to be a rectangle in the upper-half plane model of $\mathbb{H}^2$ with the horocyclic sides parallel to the $x$-axis and at a height $y \geq 1$, and the left side lying on the $y$-axis. The map $f$ is

$$(x, y) \mapsto \left( \frac{x}{y}, \ln y \right)$$

Note that the domain has the hyperbolic metric and the hyperbolic width being $O(\epsilon)$ implies that

$$\left| \frac{x}{y} \right| = O(\epsilon)$$

for all points in the domain.

One can compute the derivatives of the above map to get the dilatation:

$$\frac{\|f_x\|}{\|f_y\|} = \frac{1}{y^2} \frac{1}{\sqrt{x^2 + \frac{1}{y^2}}} = \frac{1}{\sqrt{x^2 + 1}} = 1 + O(\epsilon)$$

and

$$|\langle f_x, f_y \rangle| = \frac{x}{y^3} = O(\epsilon)$$
Figure 8. The map to $\mathbb{R}^2$ that straightens the horocyclic foliation of an $\epsilon$-thin hyperbolic region (shown on the left in the upper half plane) is almost-conformal (Lemma 4.7).

This computation also verifies that the image of the right edge is a graph over the $y$-axis of small derivative.

Lemma 4.8 (Inside-out version). Let $R$ be the region as in the previous lemma. Assume that the height of $R$ is greater than 1. Then the map $f : R \to \mathbb{R}^2$ that takes the right edge to a vertical segment, and satisfies (ii) and (iii), is almost-conformal.

Proof. We realize $R$ as a rectangle in the upper half-plane as before. The map to consider now is

$$(x, y) \mapsto (w(y) - \frac{x}{y}, \ln y)$$

where $w(y)$ is the hyperbolic width at height $y$.

Since the two vertical sides are geodesic segments of length greater than 1 and within a distance $O(\epsilon)$, we have $\frac{dw(y)}{dy} = O(\epsilon)$, and the rest of the calculation is similar to the one in the proof of the previous lemma.

Lemma 4.9. Let $R$ be a rectangular region on the hyperbolic plane that is bounded by a (left) side that is a geodesic segment, two (horizontal) sides which are geodesic arcs of equal length and a (right) side that is equidistant from the opposite side. This ‘truncated crescent’ has a horizontal foliation (by geodesic segments).

Then the map $f : R \to \mathbb{R}^2$ that

(i) takes the left edge to an almost-vertical segment
(ii) maps each horizontal leaf to a horizontal line, and
(iii) is distance-preserving along each horizontal leaf,

is almost-conformal.

Proof. Consider $R$ to be a rectangle in the upper-half plane model of $\mathbb{H}^2$ with the left side lying on the $y$-axis and the right side lying on a ray at a fixed angle from the origin. The other two sides are arcs of circles centered at the origin.

It can be checked that the conformal map $z \mapsto \pi/2 + i \ln z$ maps $R$ to a ‘straight’ euclidean rectangle, with sides parallel to the axes, and satisfies (ii) and (iii). By Lemma 4.5 we can map this rectangle to a region with an almost-vertical left (and right) edge maintaining (ii) and (iii), such that the map is almost-conformal. Composing, we get a map of $R$ as required.
Almost-isometries and quasiconformal extensions.

**Definition 4.10.** Let $L$ and $L'$ be two intervals with a metric (eg., two line segments on the plane). Then a homeomorphism $f : L \to L'$ is said to be an $\epsilon$-almost-isometry if

1. $f$ is $C^1$ with dilatation $d$ satisfies $|d - 1| = O(\epsilon)$.
2. for every pair $x \in L, y = f(x) \in L'$, the lengths $l = l(ax)$ and $l' = l(cy)$ satisfy $|l - l'| = O(\epsilon)$

**Remark.** Note that a more concise way of expressing condition (2) is to say that the lengths of any sub-interval of $L$ and its image in $L'$ differ by an additive error of $O(\epsilon)$. Often we shall drop the $\epsilon$ and say ‘almost-isometric’ to mean ‘$\epsilon$-almost-isometry’.

Here are some immediate observations, whose proofs we omit:

**Lemma 4.11.** If $f : L \to L'$ and $g : L' \to L''$ are $\epsilon$-almost-isometries, then so are $f^{-1}$ and $g \circ f$.

**Lemma 4.12.** If $|l(L) - l(L')| = O(\epsilon)$ then the affine map $f : L \to L'$ is an $\epsilon$-almost-isometry.

**Lemma 4.13.** If $L$ is subdivided into subintervals $A_1, A_2, \ldots, A_N$ and $L'$ into subintervals $A'_1, A'_2, \ldots, A'_N$ and the restrictions $f|_{A_i} : A_i \to A'_i$ are $\epsilon$-almost-isometries. Then $f : L \to L'$ is an $N\epsilon$-almost-isometry.

The following lemma is how the condition would be useful for in our constructions.

**Lemma 4.14.** Let $R$ and $R'$ be two planar rectangles of the same height $h > 1$ and moduli $m, m'$ greater than 1. Suppose $f : \partial R \to \partial R'$ is vertex-preserving homeomorphism that maps the left and right edges by an isometry and is a $\epsilon$-almost-isometry on the top and bottom edges. Then $f$ can be extended to an almost-conformal map from $R$ to $R'$.

**Proof.** Let $S$ be a Euclidean rectangle of modulus $m$. Then by mapping the rectangle to a unit disk and applying the Ahlfors-Beurling extension ([AB50]), any vertex-preserving piecewise-affine $C^1$ homeomorphism $f : \partial S \to \partial S$ of dilatation of the order of $1 + C\epsilon$ can be extended to a homeomorphism $f : S \to S$ which is $(1 + K\epsilon)$-quasiconformal, where $K$ depends only on $C$. $K$ depends on the modulus of $R$ (gets worse for $m$ very large or very small), but if the modulus lies in a compact set, say $[1, 2]$, we get a uniform value.
Figure 10. A boundary map that is “almost-isometric” can be extended to an almost conformal map, by subdividing into smaller rectangles (Lemma 4.14).

The strategy is to subdivide $R$ into smaller rectangles of moduli between 1 and 2, and use the above fact.

Let $p$ and $q$ be the top and bottom corners on the left side. Choose a collection of points $p_1, p_2, \ldots, p_n$ on the top side, and $q_1, q_2, \ldots, q_n$ on the bottom such that

(i) $l(p_i q_i) = l(q_i p_i)$ for each $1 \leq i \leq n$.
(ii) The subintervals the points divide the sides into have lengths between $h$ and $2h$.
(This is possible because the modulus $m > 1$.)

Consider the rectangles $R_1, R_2, \ldots, R_n$ obtained by connecting each pair $p_i, q_i$ by a straight line. By (ii) above, the modulus of each $R_i$ is between 1 and 2.

Consider the images $f(p_i)$ and $f(q_i)$ on the top and bottom sides of $\partial R'$. By (i) above, property (2) of the definition of almost-isometry, and the height $h > 1$, we have that the straight line joining $f(p_i)$ and $f(q_i)$ is $\epsilon$-almost-vertical, for each $i$.

We call the resulting collection of almost-rectangles $R'_1, R'_2, \ldots, R'_n$. We extend the boundary map $f$ to map each line from $p_i$ to $q_i$ to the line from $f(p_i)$ and $f(q_i)$ by an affine stretch (of dilatation $1 + O(\epsilon)$).

By Lemma 4.4 and a horizontal affine scaling we have an almost-conformal map $h$ from each $R_i$ to $R'_i$. To correct for this map differing from the map $f$ on $\partial R_i$, we consider the map $f \circ h^{-1} |_{\partial R'_i} : \partial R'_i \to \partial R_i$. This has dilatation $1 + O(\epsilon)$ and can be extended to an almost-conformal map $g$ by the Ahlfors-Beurling extension (the moduli of $R'_i$ also lie in a compact subset slightly larger than $[1, 2]$). The map $g \circ h : R_i \to R'_i$ agrees with $f$ on $\partial R_i$.

These almost-conformal maps of each $R_i$ to $R'_i$ piece together to give an almost-conformal map of $R$ to $R'$. (The property of almost-conformality extends across the intermediate arcs.)

4.3. Map for a rectangular piece. Consider a typical rectangle $R = R_i$ in our decomposition of the grafted surface $X_t$. $R$ is bounded by geodesics on each vertical side and by leaves of the transverse horocyclic foliation on each horizontal side. This horocyclic foliation $F \cap R$ gives a $C^1$ foliation of the rectangle. Geodesic arcs belonging to the lamination $\lambda$ cut across the rectangle transverse to the foliation, and $R \setminus \lambda$ has countably many hyperbolic components, bounded by horizontal horocyclic arcs and vertical geodesic arcs. The goal of the section is to construct a quasiconformally equivalent ‘euclidean’ model for $R$. 


Working in the universal cover. We shall work in the universal cover \( \widetilde{X}_t \) of \( X_t \), where we consider a (fixed) lift \( \tilde{R} \) of \( R \). Moreover we shall assume that the developing map \( \text{dev} : \widetilde{X}_t \to \mathbb{C}P^1 \) of the complex projective structure on \( X_t \) is injective (and a homeomorphism) on \( R \). It is injective whenever the transverse measure across \( R \) is less than \( 2\pi \), so this condition can be ensured by subdividing \( R \) vertically. The map for \( R \) having arbitrary transverse width can then be obtained by piecing these divisions together: properties of quasiconformal extension tell us that if the map for each piece is almost-conformal, so is the concatenated map.

By abuse of notation, we shall identify \( \tilde{R} \) with its homeomorphic image on \( \mathbb{C}P^1 \), and consider it a planar domain (since it is a proper subset of \( \mathbb{C}P^1 \) it lies in an affine chart).

The horizontal foliation \( \mathcal{F}|_R \) lifts to the universal cover and to \( \tilde{R} \) via the developing map. We denote it by \( \tilde{\mathcal{F}} \). The Thurston metric on \( R \subset X_t \) is locally isometric to the projective metric on \( \tilde{R} \) via \( \text{dev} \circ p^{-1} \) where \( p : \widetilde{X}_t \to X_t \) is the universal covering.

A finite approximation. \( \widetilde{X}_t \) is thought of as obtained by grafting the upper half plane identified as the universal cover of \( X_t \), along the lifted measured lamination \( \tilde{\lambda} \). The grafting locus consists of a collection of infinitely many geodesics, but can be approximated by a sequence of finite approximations \( \widetilde{X}_i \), by taking larger and larger finite weighted subsets of this collection. (This can be thought of as approximating the Borel measure induced by \( \lambda \) on \( S^1 \times S^1 \setminus \Delta \) by a sequence of sums of Dirac measures). We can further assume that these finite approximations (i) always include the geodesics \( \gamma_l, \gamma_r \) that form the left and right edge of \( \tilde{R} \) on \( \tilde{X} \), and (ii) the grafting locus is maximal in the sense that the complement consists of ideal triangles.

By (ii), there is a piecewise-horocyclic foliation \( \tilde{\mathcal{F}}_i \) on each finite approximation \( \widetilde{X}_i \).

Notation. In what follows, given a sub-arc \( s \) of a leaf of \( \mathcal{F} \), we shall denote its hyperbolic length as \( l_h(s) \), its total length in the Thurston metric as \( l(s) \), and its euclidean length as \( l_e(s) \), which is defined as the difference \( l(s) - l_h(s) \).

Let \( s \subset \gamma_l \) denote the left edge of \( \tilde{R} \). Then by (i) we can define a rectangle \( \tilde{R}_i \) on \( \tilde{X}_i \) as having \( s \) as the left edge, leaves of the foliation \( \tilde{\mathcal{F}}_i \) as the two horizontal edges, and a segment on \( \gamma_r \) as the right edge. Note that for all \( i \), we can define the leaf \( l^y = \tilde{\mathcal{F}}_i \cap \tilde{R}_i \) at ‘height \( y \)’ to be the leaf intersecting \( s \) at a distance of \( y \) from the lower endpoint of \( s \). We also define \( l^y \) to be the leaf of \( \tilde{\mathcal{F}} \cap \tilde{R} \) at height \( y \).

Lemma 4.15. For the above sequence of finite approximations \( \widetilde{X}_i \) the following are true:

(i) The Thurston (or projective) metric on \( \widetilde{X}_i \) is of class \( C^{1,1} \). They converge pointwise to the Thurston metric on \( \widetilde{X}_i \). Moreover, the \( C^{1,1} \) norms remain bounded as \( i \to \infty \).

(ii) The horocyclic foliations \( \tilde{\mathcal{F}}_i \to \tilde{\mathcal{F}} \) in the sense that for all \( 0 \leq y \leq l(s) \) we have
Figure 11. The map for the finite approximation case. The figure on the left shows a rectangle $\tilde{R}_i$: the hyperbolic part (unshaded) is mapped by the straightening map of Lemma 4.7, and the euclidean part (shown shaded) is then spliced in by the map from Lemma 4.9.

\( l_y \to l_y \) in the Hausdorff metric on compact subsets of \( \mathbb{C} \).

(iii) The rectangles $\tilde{R}_i \to \tilde{R}$ in the Hausdorff metric on compact subsets of \( \mathbb{C} \).

Proof. Part (i) and the regularity of the Thurston metrics is a standard result (see [KP94] or Lemma 2.3.1 in [SW02]). Part (ii) follows from the ‘bending’ description of grafting (in our choice of approximates the corresponding locally convex pleated planes converge in the Gromov-Hausdorff sense by work of Bonahon in [Bon96]), and part (iii) is an immediate consequence. \( \square \)

Map for the finite approximation.

**Lemma 4.16.** If the hyperbolic width of $\tilde{R}_i$ is \( O(\epsilon) \) then there exists a \( (1 + O(\epsilon)) \)-quasiconformal map $\tilde{f}_i$ from $\tilde{R}_i$ to the euclidean plane that satisfies the following:

(i) It takes the lower endpoint of the left vertical geodesic side $s$ to the origin.

(ii) It is an isometry of $s$ onto a segment on the $y$-axis.

(iii) Each horizontal leaf of $\tilde{F} \cap \tilde{R}_i$ is mapped to a horizontal line.

(iv) It preserves the distances along each horizontal leaf.

Proof. The map $\tilde{f}_i$ is uniquely determined and injective by conditions (i)-(iii), and is also \( C^1 \) since the the grafted metrics are \( C^1 \) and the foliation is \( C^1 \), the horizontal leaves being integral curves of a nowhere-zero Lipschitz vector field. In particular it is a quasiconformal map, and it remains to show that the dilatation is \( 1 + O(\epsilon) \), and it is enough to check that almost everywhere.

Recall that the projective metric is the Poincaré metric on the maximal disk at every point. If one starts with a maximal disk in the upper half plane model with metric $dz/Im(z)$, and the $x = 0$ axis is the lift of the grafting curve, then grafting introduces a sector with the (euclidean) metric $dz/|z|$. $\tilde{R}_i$ consists of regions that alternately lie in the hyperbolic part (of width $O(\epsilon)$) and the euclidean part (see the figure) with finitely many separating geodesic arcs.
Collapse the euclidean regions of $\tilde{R}_i$ to get a rectangle $\tilde{R}'$ of hyperbolic width $O(\epsilon)$. By Lemma 4.7 there is a map $f_0$ of $\tilde{R}'$ to $\mathbb{R}^2$ satisfying (i)-(iii) above. Note that the proof of Lemma 4.7 shows that separating geodesic arcs are mapped to almost-vertical arcs on the plane.

Starting with $f_0$ we now inductively splice in each euclidean region to $\tilde{R}'$ and extend the map already constructed, to the larger domain, such that (iii) and (iv) are satisfied. ((i) and (ii) are automatically satisfied for all these extensions.) By the above observation on the interface arcs being almost-vertical, and Lemma 4.9, these extensions are almost-conformal on each region.

Since the interface arcs are of measure zero, the final map $f_n = \tilde{f}_i$ thus constructed is $(1 + O(\epsilon))$-quasiconformal almost everywhere, as required.

Taking a limit. By the above lemma we now have a sequence $\tilde{f}_i : \tilde{R}_i \to \mathbb{R}^2$ of almost-conformal maps.

**Lemma 4.17.** Let $\tilde{f} : \tilde{R} \to \mathbb{R}^2$ be the map defined by (i)-(iv) as in the above lemma. Then

(i) $\tilde{f}_i \to \tilde{f}$ uniformly, and
(ii) $\tilde{f}$ is almost-conformal.

**Proof.** (i) follows from parts (i) and (ii) of Lemma 4.15 part (ii) says the each leaf of $\tilde{R}_i$ converges as a set to the corresponding leaf of $\tilde{R}$, and by part (i) the lengths also converge, and then (i) above follows from the definition of $\tilde{f}$ (distance preserving along the leaves). To get (ii) we employ a trick of considering the sequence of inverse maps $\tilde{g}_i = \tilde{f}_i^{-1}$. For an arbitrary $x \in \tilde{R} \setminus \partial \tilde{R}$ there is, by part (iii) of Lemma 4.15, an open neighborhood $\tilde{U} \subset \mathbb{R}^2$ containing $\tilde{f}(x)$, such that $\tilde{U}$ is contained in $\tilde{f}_i(R_i)$ for sufficiently large $i$. The sequence $\tilde{g}_i|_{\tilde{U}}$ are a uniformly converging sequence of $(1 + O(\epsilon))$-quasiconformal maps from a fixed domain $\tilde{U}$ to $\mathbb{C}$. The limit $\tilde{g} = \tilde{f}^{-1}$ is hence $(1 + O(\epsilon))$-quasiconformal on $\tilde{U}$, and so is $\tilde{f} = \tilde{g}^{-1}$ in a neighborhood of $x$. Note that $\tilde{f}$ is height-preserving: this follows from parts (ii) and (iii) of Lemma 4.16 (which are preserved in the limit).□

The almost-conformal model. By the previous lemma we have obtained an almost-conformal map $\tilde{f}$ from $\tilde{R}$ to $\mathbb{R}^2$. Together with the local isometry $\text{dev} \circ p^{-1}$ this gives an almost-conformal map of $R \subset X_t$ to a planar domain. We conclude the construction of a quasiconformal model for $\tilde{R}$ by noting that this planar domain can be almost-conformally straightened to a rectangle.

**Notation.** Recall that the total width $W$ of the rectangle $R$ is the supremum of the lengths of the leaves of $\mathcal{F} \cap R$, and the euclidean width $W_e$ of $R$ is the supremum of the euclidean lengths of the leaves of $\mathcal{F} \cap R$.

**Lemma 4.18 (Map for a rectangle).** If the total width of $R$ is $W > 1$ and height $h > 1$, there exists a height-preserving almost-conformal map $\tilde{f}$ from $R$ to a euclidean rectangle of width $W_e$ and height $h$. 

Proof. By the previous lemma we have a height-preserving almost-conformal map $f = \tilde{f} \circ \text{dev} \circ p^{-1}$ from $R$ to a planar region $D$. The euclidean width of $W_e(y)$ (total-hyperbolic) of a leaf at height $y$ is a constant independent of height (it only depends on the total transverse measure of $R$). Since the hyperbolic width $W_h(y)$ at height $y$ is $O(\epsilon)$ (Lemma 4.2), we have that $W(y) = W_h(y) + W_e(y)$ is the total width of $D$ (and also of $R$) at height $y$. Since the two vertical sides of $R$ are geodesic segments of length $h > 1$ on the (ungrafted) hyperbolic surface at distance $O(\epsilon)$, it follows from hyperbolic geometry that $|dW_h(y)| = O(\epsilon)$. Hence by an application of Lemma 4.4 we obtain a height-preserving almost-conformal map from $D$ to a euclidean rectangle of width $W_e$, and the required map $\tilde{f}$ from $R$ to $W_e$ is obtained by precomposing this with $f$. □

Corollary 4.19 (Almost-isometric). Let $L$ and $\bar{L}$ be the top and bottom edges of $R$. The map $\bar{f}$ constructed above is an $\epsilon$-almost-isometry on $L$ and $\bar{L}$, and isometric on the other two sides.

Proof. Consider the top edge $L$. The map of Lemma 6.19 is an isometry of $L$ and the map (from Lemma 4.4) used in straightening step in the proof of the previous lemma is affine on horizontal lines, hence the composition (the map $\bar{f}$) is affine on $L$. Since $l(L) = W(h)$ and $l(\bar{f}(L)) = W_e$ differ by $O(\epsilon)$ we can apply Lemma 4.12 to conclude that $\bar{f}$ is an $\epsilon$-almost-isometry on $L$. The proof for the bottom edge $\bar{L}$ is identical.

The property of being “height-preserving” implies that $\bar{f}$ is isometric on the vertical (left and right) sides of $R$. □

4.4. Map for a truncated sector. An ideal hyperbolic triangle $\hat{T}$ can be divided into three $2\pi/3$-sectors by joining the centroid $p$ to the ideal vertices by geodesic rays. These are symmetric by a $2\pi/3$-rotation about $p$. We shall describe a map (to a planar region) of one of these sectors which we call $\hat{S}$. It will be clear from the construction that the maps for two adjacent sectors of $\hat{T}$ agree on the geodesic ray common to both.

We denote by $\gamma$ the geodesic side of $\hat{S}$ which is a boundary component of $\hat{T}$.
An ideal hyperbolic triangle has a partial foliation by horocyclic arcs which restricts to give a partial foliation of the sector $\hat{S}$. When realized in the upper half plane $\mathbb{H}^2$ with $\gamma$ a vertical geodesic and $p$ the point $\frac{i\sqrt{3}}{2}$, the leaves of non-negative height are the horizontal segments starting at $y = 1$. The height of a leaf is the logarithm of its $y$-coordinate of the point where it intersects $\gamma$. This definition will hold for any horizontal foliation of $\hat{S}$.

Let $a$ be the geodesic arc from $p$ which is orthogonal to $\gamma$. Consider a $C^1$ foliation which agrees with the horocyclic foliation for height greater than $D = \ln(1/\epsilon)$ (when the leaves have width less than $\epsilon$), and interpolates in between the leaf at height $D$ and the arc $a$ in such a way that the lengths of the leaves is a $C^1$ function of (non-negative) height that decreases with height. Reflect this foliation across $a$. This extends the foliation to the whole of $\hat{S}$, we call it $\hat{F}$. The above length function is $C^1$ except at height 0.

**Remark:** We can also impose the condition that each leaf of $\hat{F}$ is orthogonal to $\gamma$: this shall ensure that it matches in a $C^1$ way with the partial foliation $F$ on the grafted rectangles on the other side of $\gamma$. However, this shall not be important for what we consider.

We now define a quasiconformal map to the plane that ‘straightens’ the leaves and is height-preserving.

**Lemma 4.20 (Infinite sector).** There is a quasiconformal homeomorphism $f : \hat{S} \rightarrow \mathbb{R}^2$ such that

(i) Each leaf of $\hat{F}$ is mapped isometrically to a horizontal line segment.

(ii) For all $y$, the left endpoint of the image of the leaf at height $y$ is $(0,y)$.

(iii) The quasiconformal distortion of $f$ is $1 + O(\epsilon)$ at all points of $\hat{S}$ at height $|h| > D = \ln(1/\epsilon)$.

(iv) $\gamma$ is mapped to a graph of a function $g$ over the $y$-axis that is $C^1$ except at 0, and is almost-vertical and $|g(y)| = O(\epsilon)$ at points with $|y| > D$. Moreover, $\sup_y g(y) = g(0) < 1$.

**Proof.** We map the part $\hat{S}_+$ above $a$ (of nonnegative height) and extend to $\hat{S}_-$ by reflection. We denote this map of $\hat{S}_+$ by $f_+$. Note that (i) and (ii) uniquely determines $f_+$, and ensures it is injective. The fact that the foliation is $C^1$ and the lengths of the leaves is $C^1$ in (positive) height ensures that $f_+$ is $C^1$, and is hence a homeomorphism to its image. (iii) follows from Lemma 4.8.

The fact that $f_+$ is isometric on the leaves implies that the function $g$ that describes the image of $\gamma$ as a graph is the length-function of the leaves. This is $C^1$, except at 0. Part (iv) again follows from Lemma 4.8 (a calculation similar to Lemma 4.7). In fact, $|g(y)| \rightarrow 0$ exponentially as $|y| \rightarrow \infty$. The last statement of the lemma follows from the fact that the length function was chosen to be decreasing for increasing height (or decreasing height, by reflection) and the length at height 0 is the length of the geodesic arc $a$, which is $\ln(\sqrt{3}) \approx 0.54$. □
The map for the sector $S_H$ which is truncated at height $H$ (greater than $D$) shall be just the restriction of the above map to the truncated part. We summarize this in the following corollary.

**Corollary 4.21** (Truncated sector). There is a quasiconformal homeomorphism $f$ of $S_H$ to the planar region $S'$ that lies between an interval $[-H, H]$ on the y-axis, and a graph of a function $g$ on that interval, such that

(i) $g$ is $C^1$ except at 0, and the graph is almost-vertical and $|g(y)| = O(\epsilon)$ at points with $|y| > D$. Moreover, $\sup_y g(y) = g(0) < 1$.

(ii) $f$ is height-preserving.

(iii) $f$ is isometric on the top and bottom horocyclic sides of $S_H$.

(iv) The quasiconformal distortion of $f$ is $1 + O(\epsilon)$ at all points of $S_H$ at height $|h| > D$.

**Proof.** The map $f$ restricted to the truncated sector $S_H \subset P$ shall be the map from Corollary [1,21]. Note that $f$ satisfies (i), (ii) and (iii) on $S_H$. In fact it is isometric on the top and bottom horocyclic edges, which is stronger than (ii).

Recall also that this map sends $\gamma$ to a graph of a function $g$ that satisfies:

(1) It is $C^1$ except at one point.

(2) It is almost-vertical for height more than $D$.
The map for a pentagonal piece (shown on the left) to a euclidean rectangle. The region $S_H$ to the left of $\gamma$ is mapped by Corollary 4.21 and the region to its right by Lemma 4.6. The map on $\gamma$ is the same as they both preserve height.

\begin{equation}
|g(y)| = O(\epsilon) \text{ for } |y| > D.
\end{equation}

\begin{equation}
\sup_y g(y) = g(0) < 1.
\end{equation}

We can consider this image of $\gamma$ as the graph of the function $w(y) = 2 - g(y)$ on a segment on the vertical line $y = 2$ (by abuse of orientation the ‘positive’ side of the vertical segment is to its left). The function $w(y)$ satisfies the conditions of Lemma 4.6 which we shall use in a moment (in particular (4) above ensures that $w(y) > 1$).

Now by Lemma 4.18 we map the grafted rectangle $S$ to a euclidean rectangle $S'$ of height $2H$ and width 2 by an almost conformal height-preserving map $f_1$. By Lemma 4.6 (or a suitably ‘reflected’ version) there is a height-preserving quasiconformal map $f_2$ of this rectangle to the planar region bounded by a vertical segment on the line $y = 2$ and the graph of $w(y)$ over this segment (which lies to its left).

The map $f$ restricted to $S \subset P$ shall be the composition $f_2 \circ f_1$. By properties (2) and (3) of the image of $\gamma$, the appropriate conditions of Lemma 4.6 are satisfied and $f_2$ is almost-conformal for all points of height more than $D$. Since $f_1$ is almost-conformal everywhere, the composition satisfies (iii). Since both $f_1$ and $f_2$ are height-preserving, (i) is satisfied. Finally, it can be checked that (ii) holds because of Corollary 4.19 and Lemmas 4.11 and 4.12.

Thus the map $f$ is defined on $S_H$ and $S$, and hence on their union $P$ (since it is height-preserving on both they match along $\gamma$). Another application of Lemma 4.12 implies that it satisfies (ii). The usual property of quasiconformal maps (Property B at the beginning of §4.2) implies that it is quasiconformal everywhere, and (iii) is satisfied since it is satisfied on both $S_H$ and $S$.

4.6. Mapping the grafted surface. In this section we shall use the decomposition $D$ of $X_t$ into pentagons $\{P_j\}_{1 \leq j \leq 3m}$ and rectangles $\{R_i\}_{1 \leq i \leq n}$, as described at the end of §4.1.

Recall from §3.1 that $\hat{X_t}$ is the singular flat surface obtained by collapsing the ideal triangle components of $X_t \setminus \lambda$ along the leaves of $\mathcal{F}$ and preserving the transverse measure (one could use a Cantor function, as in [CB88]). Recall that the triangular regions in the complement of the partial foliation $\mathcal{F}$ collapse to the singularities
The pentagons in the decomposition of $X_t$ are mapped by the collapsing map to euclidean rectangles $\{S_j\}$ (of euclidean width 2), which together with the collapsed images $\{R'_i\}$ of the trimmed-rectangles form a rectangular decomposition of $\hat{X}_t$ with the same combinatorics as $\mathcal{D}$.

As before, let $K$ denote the 1-skeleton of the decomposition of $X_t$, and let $\hat{K}$ the corresponding 1-skeleton on $\hat{X}_t$. An edge $E \subset K$ is either horizontal (if it is formed of segments of the horizontal foliation) or vertical, and we denote by $K_H$ the collection of horizontal edges.

When the hyperbolic part is collapsed, a segment of a leaf of $\mathcal{F}$ of total width $W$ collapses to a segment of width $W_e$ (which, recall, is the euclidean width of the segment). In particular, since the hyperbolic width is $O(\epsilon)$ for every horizontal edge $E$, the corresponding edge $\hat{E}$ of $\hat{K}$ differs in length by $O(\epsilon)$.

For each pentagonal piece consider the map $f^P_j : P_j \to S_j$ which is the map $\bar{f}$ obtained from Lemma 4.22. Let $\bar{f}^P_j$ denote the restriction of the map to the horizontal edges of $\partial P_j$. These are $\epsilon$-almost-isometric (part (ii) of Lemma 4.22). Choose a map $\bar{g} : K_H \to \hat{K}_H$ that is equal to $\bar{f}^P_j$ for each horizontal edge of a pentagonal piece, and affine on each remaining horizontal edge. By the above observation on the difference of lengths of the edges, and Lemmas 4.12 and 4.13, $\bar{g}$ satisfies an $M\epsilon$-almost-isometry condition on each horizontal edge of the 1-skeleton for some $M$ that depends on the genus (the number of sub-edges of each horizontal edge is determined by the number of rectangles and pentagons in the decomposition that depends only on the genus). We can henceforth absorb that constant in the $O(\epsilon)$ term in the definition of almost-isometry, and refer to the above map as being $\epsilon$-almost-isometric.

For each rectangular piece in the decomposition, consider the map $f^R_i : R_i \to R'_i$ obtained from Lemma 1.18. Let $\bar{f}^R_i$ denote the restriction of the map to the horizontal edges of $\partial R_i$. Choose a map $\bar{h}_i : \partial R_i \to \partial R'_i$ that is isometric on the vertical edges and agrees with $\bar{g} \circ \bar{f}^R_i^{-1}$ on the horizontal edges. By Corollary 1.19 and Lemma 4.11, this map is $\epsilon$-almost-isometric on the horizontal edges. The modulus of $R'_i$ is greater than 1 by Lemma 4.3, so we can apply Lemma 4.14 and extend $\bar{h}_i$ to an almost-conformal self-map $h_i$ of $R'_i$. The map $h_i \circ \bar{f}^R_i$ now maps $R_i$ to $R'_i$ such that the map agrees with $\bar{g}$ on $K_H$.

The maps $\{h_i \circ \bar{f}^R_i\}$ and $\{f^P_j\}$ agree on each vertical edge $E$ since they are height-preserving. Hence these maps agree on the 1-skeleton $K$ and form a continuous map from $X_t$ to $\hat{X}_t$ that is quasiconformal on each piece $P_i$ and $R_i$, and is hence quasiconformal everywhere.

Each map from the collection $\{h_i \circ \bar{f}^R_i\}$ is almost-conformal (since $h_i$ is almost-conformal from the above construction, and $\bar{f}^R_i$ is almost-conformal by Lemma 4.18). Each map $f^R_i$ is almost-conformal away from a set of diameter $O(\ln(1/\epsilon))$, by part
Figure 15. A typical pair $(D_j, K_j)$ on the grafted surface. The map in Lemma 4.23 is almost-conformal outside $K_j$ (shaded darker). The annular region $D_j \setminus K_j$ (shaded lighter) has large modulus (Lemma 4.24).

(iii) of Lemma 1.22

We summarize this discussion in the following lemma.

**Lemma 4.23** (Map of grafted surface). There exists a $T > 0$ such that for all $t > T$ there is a quasiconformal homeomorphism $f : X_t \to \hat{X}_t$ such that the quasiconformal distortion is $1 + O(\epsilon)$ away from finitely many simply-connected subsets $K_1, K_2, \ldots, K_m \subset X_t$ of diameter $O(\ln(1/\epsilon))$ in the Thurston metric. Moreover, each subset $K_j$ is contained in a simply connected set $D_j$ of diameter $\frac{1}{\epsilon^2}$.

Each $2\pi/3$-angled vertex of a pentagonal piece in the decomposition $\mathcal{D}$ is a centre of an ideal triangle complement of the lamination $\lambda$ (in fact each such center is common to three pentagons). The subsets $K_1, K_2, \ldots, K_m \subset X_t$ in the lemma are the $O(\ln(1/\epsilon))$-neighborhoods of the finitely many centres of the ideal triangle complements of $\lambda$, by part (iii) of Lemma 1.22. The last statement follows from Lemmas 4.2 and 4.3: the three rectangles adjacent to a truncated ideal triangle $T_j$ are of height and width at least $\frac{1}{\epsilon^2}$ when the grafting time $t > T$ is sufficiently large, hence one can embed a disk $D_j$ of diameter $\frac{1}{\epsilon^2}$ centered at the center of $T_j$, containing $K_j$.

**Lemma 4.24.** For each $1 \leq j \leq m$, the annular region $D_j \setminus K_j$ where $D_j$ and $K_j$ are the simply-connected subsets in the above lemma, has modulus greater than $2\pi \ln(1/\epsilon)$.

**Proof.** Consider the annular region that is the image of $f(D_j \setminus K_j)$ on the singular flat surface $\hat{X}_t$. From the construction of $f$ (in particular its property of being height-preserving and almost-isometric along $F$) it follows that one can embed a flat annulus $A_j$ of large modulus in $f(D_j \setminus K_j)$. In fact, since the diameter of $f(D_j)$ is greater than $\frac{1}{\epsilon^2}$ and the diameter of $f(K_j)$ is at most $O(\ln \frac{1}{\epsilon})$, the modulus of $A_j$ satisfies

$$\text{mod}(A_j) = \frac{1}{3} \cdot \frac{2\pi}{2\pi} \ln \frac{R_{out}}{R_{inn}} \geq A \ln \frac{1/\epsilon^2}{\ln \frac{1}{\epsilon}} + B > 4\pi \ln \frac{1}{\epsilon}$$

for $\epsilon$ sufficiently small ($A$ and $B$ are some positive constants).

Since $f$ is almost-conformal on $D_j \setminus K_j$, we have an embedded annulus $f^{-1}(A_j)$ of large modulus in $D_j \setminus K_j$ (in the Thurston metric). In particular, the modulus

$$\text{mod}(D_j \setminus K_j) > \frac{4\pi}{1 + \epsilon} \ln \frac{1}{\epsilon} > 2\pi \ln \frac{1}{\epsilon}$$

$\square$
4.7. Modifying the map to almost-conformality. In this section we modify the map \( f : X_t \to X_t \) from Lemma 4.23 such that it is almost-conformal. To do this we shall redefine the map in the subsets \( D_1, D_2, \ldots, D_m \). The crucial fact that allows this modification is that the lack of almost-conformality for \( f \) is contained in the subsets \( K_j, 1 \leq j \leq m \) which have diameter much smaller than the diameter of \( D_j \).

4.7.1. A quasiconformal extension lemma. Let \( \mathbb{D} \) be the unit disk in \( \mathbb{C} \) and let \( B_r \) be a closed ball of radius \( r \) about the origin.

The following result is probably well-known to experts, however it does not seem to be readily available in the literature on the subject. A proof is provided in Appendix A.

**Lemma 4.25.** For any \( \epsilon > 0 \) sufficiently small and any \( 0 \leq r \leq \epsilon \) if \( f : \mathbb{D} \to \mathbb{D} \) satisfies

1. \( f \) is a quasiconformal map,
2. The quasiconformal distortion is \((1 + C\epsilon)\) on \( \mathbb{D} \setminus B_r \),

then the map \( f \) extends to a \((1 + C'\epsilon)\)-quasisymmetric map of the boundary, where \( C' \) is a constant depending only on \( C \).

An immediate consequence of the Ahlfors-Beurling extension is

**Corollary 4.26.** Let \( \epsilon > 0 \) be sufficiently small, \( r \leq \epsilon \) and let \( f : \mathbb{D} \to \mathbb{D} \) satisfy (1) and (2) as in the previous lemma. Then there exists an almost-conformal map \( g : \mathbb{D} \to \mathbb{D} \) such that \( f|_{\partial \mathbb{D}} = g|_{\partial \mathbb{D}} \).

We shall use the above for modifying a map between Riemann surfaces:

**Corollary 4.27.** Let \( \Sigma, \Sigma' \) be homeomorphic Riemann surfaces. Let \( f : \Sigma \to \Sigma' \) be a quasiconformal map and \( K \subset D \subset \Sigma \) be concentric embedded disks such that

1. The modulus of the annulus \( D \setminus K \) is at least \( 2\pi \ln \frac{1}{\epsilon} \).
2. \( f \) is almost-conformal on \( D \setminus K \).

Then there is a quasiconformal map \( g : \Sigma \to \Sigma' \) such that \( f|_{\Sigma \setminus D} = g|_{\Sigma \setminus D} \) and \( g \) is almost-conformal on \( D \).

**Proof.** Let \( \phi : D \to \mathbb{D} \) and \( \psi : f(D) \to \mathbb{D} \) be uniformizing maps to the unit disk, normalized such that the centers are taken to \( 0 \in \mathbb{D} \). Then by (i) above \( \phi(K) \) and \( \psi(f(K)) \) have diameter \( O(\epsilon) \) (this is an application of equation (28) in Lemma A.15) and contain \( 0 \in \mathbb{D} \). The map \( g = \psi \circ f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D} \) satisfy the requirements of Corollary 4.26 and hence can be replaced by an almost-conformal map \( h : \mathbb{D} \to \mathbb{D} \) that has the same map as \( g \) on \( \partial \mathbb{D} \). We replace \( f : D \to f(D) \) by the almost-conformal map \( \psi^{-1} \circ h \circ \phi : D \to f(D) \). This restricts to the same map as \( f \) on \( \partial D \). Together with the map \( f \) on \( \Sigma \setminus D \) it defines a continuous map of \( \Sigma \) to \( \Sigma' \). This map is quasiconformal on \( D \) and \( \Sigma \setminus D \), and is hence quasiconformal, since \( \partial D \) is a measure zero set. \( \square \)

4.8. Proof of Proposition 4.1

**Proof of Proposition 4.1.** We start with the map from Lemma 4.23 and modify it by applying Corollary 4.27 (taking \( \Sigma = X_t, \Sigma' = \tilde{X}_t, D = D_j, K = K_j \)), for each \( 1 \leq j \leq m \) in succession. This is possible since each annulus \( D_j \setminus K_j \) has modulus at least \( 2\pi \ln \frac{1}{\epsilon} \) by Lemma 4.24 (so (i) of Cor. 4.27 holds). The final map \( f : X_t \to \tilde{X}_t \)
is almost-conformal on $D_1 \cup D_2 \cup \cdots \cup D_m$ and agrees with the original $f$, and is hence almost-conformal on the complement $X_t \setminus D_1 \cup D_2 \cup \cdots \cup D_m$. The property of almost-conformality extends across the measure zero set consisting of the union of the $\partial D_j$s.

This completes the construction of an almost-conformal map $f : X_t \to \hat{X}_t$, for all $t$ sufficiently large.

The singular flat surface $\hat{X}_0$ has a horizontal foliation, and a vertical foliation which is measure equivalent to $\lambda$. Now $\hat{X}_t$ can be obtained from $\hat{X}_0$ by scaling the horizontal foliation on $\hat{X}_0$ by a factor of $t$, and keeping the vertical foliation the same, which is conformally equivalent to scaling the horizontal foliation by a factor of $\sqrt{t}$ and the vertical foliation by a factor of $1/\sqrt{t}$. The surface $\hat{X}_t$ thus lies on the Teichmüller ray from $\hat{X}_0$ determined by $\lambda$, at a distance of $\frac{1}{2} \ln t$.

Since our choice of $\epsilon > 0$ throughout was arbitrary, this shows the grafting ray based at $X = X_0$ determined by the lamination $\lambda$ is asymptotic to the Teichmüller ray based at $Y = \hat{X}_0$. $\square$

Remark. When $\lambda$ is also uniquely-ergodic, then for any other choice of basepoint $Y$, the Teichmüller ray $Y_t$ determined by $\lambda$ is asymptotic to the above Teichmüller ray, by the result of Masur ([Mas80]), and hence by the triangle inequality, the grafting ray based at $X$ is asymptotic to the Teichmüller ray $Y_t$.

The proofs of Corollaries 1.2 and 1.3 only require Theorem 1.1 in the arational case, hence we provide them here:

Proof of Corollary 1.2. Pick any Teichmüller ray determined by such an arational $\lambda$. By Proposition 4.1, the two grafting rays are both asymptotic to it, and hence to each other. $\square$

Proof of Corollary 1.3. It is known that for a generic choice of $\lambda$ (with respect to the Thurston measure on $\mathcal{ML}$) and any choice of basepoint the corresponding Teichmüller ray is dense in moduli space (this follows from the ergodicity of the Teichmüller geodesic flow, proved in [Mas82] - see [Mas] for explicit examples of such rays). Arational laminations form a full measure set in $\mathcal{ML}$. By Proposition 4.1, a grafting ray determined by a generic $\lambda$ is then asymptotic to a dense Teichmüller ray, and is hence itself dense. $\square$

5. The multi-curve case

In this section we prove Theorem 1.1 in the case when $\lambda$ is a multi-curve (Proposition 5.9) following the outline in §3.2.

5.1. A quasiconformal toolkit. As in §4.1, we first collect some constructions and extensions of quasiconformal maps that shall be useful later. Most are probably well-known to experts, however in our setting we need care to maintain almost-conformality of the maps, and this aspect seems to be absent in the literature. A glossary of known results is included in Appendix A.1.
Throughout, $D$ shall denote the unit closed disk on the complex plane, and $B_r$ shall denote the closed disk of radius $r$ centered at 0. Note that any quasiconformal map defined on the interior of a Jordan domain extends to a homeomorphism of the boundary.

**Interpolating maps.** We start with the following observation about the Ahlfors-Beurling extension that was used in a construction in [AJKS10]:

**Lemma 5.1** (Interpolating with identity). Let $h : \partial D \to \partial D$ be a $(1 + O(\epsilon))$-quasisymmetric map. Then there exists an $0 < s < 1$ and a homeomorphism $H : D \to D$ such that

1. $H$ is almost-conformal.
2. $H|_{\partial D} = h|_{\partial D}$.
3. $H$ restricts to the identity map on $B_s$.

Moreover, $s$ depends only on the quasi-symmetry constant of $h$.

**Proof.** As in §2.4 of [AJKS10], lift $h$ to a homeomorphism $\tilde{h} : \mathbb{R} \to \mathbb{R}$ that satisfies $\tilde{h}(x + 1) = \tilde{h}(x) + 1$, and consider the Ahlfors-Beurling extension of $\tilde{h}$ to the upper half plane $\mathbb{H}$:

$$F(x + iy) = \frac{1}{2} \int_{0}^{1} \tilde{h}(x + ty) + \tilde{h}(x - ty) dt + i \int_{0}^{1} \tilde{h}(x + ty) - \tilde{h}(x - ty) dt$$

It follows from the periodicity that

$$F(x + i) = x + i + c_0$$

where $c_0 = \frac{1}{2} \int_{0}^{1} \tilde{h}(t) dt - 1/2 \in [-1/2, 1/2]$.

We note that since we have used a locally conformal change of coordinates $w \mapsto e^{2\pi i w}$ between $\mathbb{H}$ and $D$, we have that $\tilde{h}$ is also $(1 + O(\epsilon))$-quasisymmetric, and $F$ is almost-conformal.

For $H > 1$ we can define a map $F_1 : \mathbb{H} \to \mathbb{H}$ which restricts to $F$ on $\mathbb{R} \times [0, 1]$, and the identity map for $y \geq H$ and interpolates linearly on the strip $\mathbb{R} \times [1, H]$:

$$F_1(x + iy) = x + iy + c_0 \left( \frac{H - y}{H - 1} \right)$$

For $H$ sufficiently large (greater than $1/(K' - 1)$ where the quasi-symmetry constant is $K'$), $F_1$ is almost-conformal everywhere as can be checked by computing derivatives on the interpolating strip. Since $F_1(z + k) = F_1(z) + k$ for all $z \in \mathbb{H}$ and $k \in \mathbb{Z}$, it descends to an almost-conformal map $H : \mathbb{D} \to \mathbb{D}$ that restricts to $h$ on $\partial \mathbb{D}$ and the identity map on $B_s$ for $s = e^{-2\pi H}$. \qed

The following corollary of the above lemma interpolates a quasiconformal map with the identity map on the outer boundary:

**Lemma 5.2.** Let $f : \mathbb{D} \to \mathbb{D}$ be an almost-conformal map such that $f(0) = 0$. Then there exists an $0 < r_0 < 1$ and a map $F : \mathbb{D} \to \mathbb{D}$ such that

1. $F$ is almost-conformal.
2. $F|_{B_{r_0}} = f|_{B_{r_0}}$.
3. $F|_{\partial \mathbb{D}}$ is the identity.
Proof. Since \( f \) is almost-conformal, so is \( f^{-1} \), and the latter extends to the boundary and restricts to a homeomorphism \( h : \partial \mathbb{D} \to \partial \mathbb{D} \) that is \((1 + O(\epsilon))\)-quasisymmetric. By Lemma 5.1 there exists an almost-conformal extension \( H : \mathbb{D} \to \mathbb{D} \) of \( h \) that restricts to the identity on \( B_{r_0} \) for sufficiently small \( r_0 \). The composition \( H \circ f \) then is the required map \( F \) that restricts to \( f \) on \( B_{r_0} \) and is identity on \( \partial \mathbb{D} \).

We shall generalize the previous lemma to obtain an interpolation of an almost-conformal map with a given conformal map at the outer boundary. We first show that we can fix such a conformal map to be the identity near 0, without too much quasiconformal distortion.

Lemma 5.3. Let \( g : \mathbb{D} \to g(\mathbb{D}) \subset \mathbb{C} \) be a conformal map such that \( g(0) = 0 \) and \( g'(0) = 1 \). Then there exists an \( 0 < s < 1 \) and a map \( G : \mathbb{D} \to g(\mathbb{D}) \) such that

1. \( G \) is almost-conformal.
2. \( G \) restricts to the identity map on \( B_s \).
3. \( G|_{\partial \mathbb{D}} = g|_{\partial \mathbb{D}} \).

Proof. Since \( g(0) = 0 \) and \( g \) is conformal, there exists an an expansion

\[
g(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \psi(z)
\]

where \( a_i \in \mathbb{C} \) for \( i \geq 2 \).

We shall have \( s = \epsilon \). Let \( \phi_\epsilon : [0, 1] \to [0, 1] \) be a smooth bump function such that

1. \( \phi_\epsilon(t) = 0 \) for \( 0 \leq t \leq \epsilon \).
2. \( \phi_\epsilon(t) = 1 \) for \( 2\epsilon \leq t \leq 1 \).
3. \( |\phi_\epsilon'(t)| = O(1/\epsilon) \) for all \( t \in [\epsilon, 2\epsilon] \).

and define

\[
\psi_\epsilon(z) = \phi_\epsilon(|z|)\psi(z)
\]

for all \( z \in \mathbb{D} \).

Define the map \( G : \mathbb{D} \to \mathbb{C} \) as

\[
G(z) = z + \psi_\epsilon(z)
\]

for \( z \in \mathbb{D} \).

From the Koebe distortion theorem (see for example Theorem 1.3 of \textit{Pom}) we have

\[
\frac{1}{(1 + |z|)^2} \leq \left| \frac{g(z)}{z} \right| \leq \frac{1}{(1 - |z|)^2}
\]

and that implies

\[
\left| \frac{g(z)}{z} - 1 \right| \leq \frac{4|z|}{(1 - |z|^2)^2} \quad \Rightarrow \quad |\psi(z)| \leq \frac{4|z|^2}{(1 - |z|^2)^2}
\]

For \( \epsilon \leq |z| \leq 2\epsilon \) we therefore have

\[
|\psi(z)| \leq C\epsilon^2
\]

for some universal constant \( C \) (we know \( \epsilon \) is sufficiently small).

From this and (3) above it is easy to check that

\[
|\partial z G| = |\partial \psi_\epsilon| = O(\epsilon)
\]

and

\[
|\partial z G| = |1 + \partial \psi_\epsilon| = 1 + O(\epsilon)
\]
In Lemma 5.4 we interpolate from the map $g$ on $\partial B_r$ to the almost-conformal map $f$ between the shaded regions. In the proof we introduce an intermediate circle $\partial B_s$ and apply Lemma 5.2 to its “inside” and Lemma 5.3 to remaining annulus.

for each $z \in \mathbb{D}$.

$G$ is hence a diffeomorphism of quasiconformal dilatation of $1 + O(\epsilon)$ that restricts to the identity map on $B_s$ and to $g$ on $\partial \mathbb{D}$ as required. $\square$

Remark. By conjugating by the dilation $z \mapsto (1/r)z$ the above result holds (for some $0 < s < r$) if the conformal map $g$ is defined only on $B_r \subset \mathbb{D}$.

Lemma 5.4 (Interpolating between $f$ and $g$). Let $f : \mathbb{D} \to \mathbb{D}$ be an almost-conformal map such that $f(0) = 0$, and let $g : B_r \to g(B_r) \subset \mathbb{D}$, for some $0 < r < 1$, be a conformal map such that $g(0) = 0$ and $g'(0) = 1$. Then there exists an $0 < r' < r < 1$ and a map $F : B_r \to g(B_r)$ such that

1. $F$ is almost-conformal.
2. $F|_{B_s} = f|_{B_s}$.
3. $F|_{\partial B_s} = g|_{\partial B_r}$.

Proof. By Lemma 5.3 (see also the above remark) there exists an $0 < s < r < 1$ and an almost-conformal map $G : B_r \to g(B_r)$ that restricts to $g$ on $\partial B_r$ and is identity on $B_s$. Now we can apply Lemma 5.2 to the rescaled map $f_s = (1/s)f|_{B_s} : \mathbb{D} \to \mathbb{D}$ to get a map $F_1 : \mathbb{D} \to \mathbb{D}$ that restricts to $f_s$ on some $B_{s'}$ for $0 < s' < 1$ and the identity map on the boundary $\partial \mathbb{D}$. Rescaling back, we get a map $F_1 : B_s \to B_r$ that restricts to the identity map on $\partial B_s$ and to $f$ on $B_{r'}$ where $r' = s's < s$. Since $F_1$ and $G$ are both identity on $\partial B_s$, $F_1$ together with the restriction of $G$ on $B_r \backslash B_s$ defines the required interpolation $F : B_r \to g(B_r)$. $\square$

Boundary correspondence for annuli. The following is a reformulation of Lemma 4.25 proved in Appendix A:

Proposition 5.5. Let $A$ be an annulus on the plane, with $\partial B_R$ as one of the boundary components, and let $F : A \to A$ be a $(1 + C\epsilon)$-quasiconformal map. Then there exists a universal constant $M > 0$ such that if $\text{mod}(A) > M$ then $F$ extends to a $(1 + C'\epsilon)$-quasisymmetric map of the boundary component $\partial B_R$ to itself, where $C'$ is a constant depending only on $C$.

Proof. Since dilation ($z \mapsto \lambda z$) and inversion ($z \mapsto 1/z$) are conformal maps we can assume without loss of generality that $R = 1$ (so $B_R = \mathbb{D}$) and $\partial B_R = \partial \mathbb{D}$ is the outer boundary of the annulus $A$. Let $E$ denote the bounded component of the
complement of \( A \), and \( d \) denote its diameter: it is known (see §A.3 of the Appendix) that

\[
\text{mod}(A) < \frac{1}{2\pi} \log \frac{16}{d}
\]

This implies that if \( \text{mod}(A) \) is sufficiently large, then \( d < \epsilon \). The proof of Proposition A.1 mentioned above now shows that the boundary extension of \( F \) to \( \partial \mathbb{D} \) is \((1+C\epsilon)\)-quasisymmetric.

There is also a converse to this, which we prove using Lemma 5.1.

**Lemma 5.6.** Let \( A \) be an annulus on the plane bounded by circles \( \partial B_r \) and \( \partial \mathbb{D} \) where \( 0 < r < 1 \). Let \( f_1 : \partial \mathbb{D} \to \partial \mathbb{D} \) and \( f_2 : \partial B_r \to \partial B_r \) be \((1+C\epsilon)\)-quasisymmetric maps. Then if \( r \) is sufficiently small there is an almost-conformal map \( F : A \to A \) such that \( F|_{\partial B_r} = f_2 \) and \( F|_{\partial \mathbb{D}} = f_1 \).

**Proof.** By the Ahlfors-Beurling extension and Lemma 5.1, we can extend \( f_1 \) to an almost-conformal map \( F_1 : \mathbb{D} \to \mathbb{D} \) that is the identity map on \( B_s \), for some \( 0 < s < 1 \). If \( r < s \) then by inverting across the circle \( \partial B_r \) and scaling by a factor of \( r \), we can apply the previous lemma again to construct a map \( F_2 : B_r \to B_r \) such that \( F_2 \) restricts to \( f_2 \) on \( \partial B_r \) and is the identity map on \( \partial \mathbb{D} \). By the Ahlfors-Beurling extension again, there exists an extension \( F_2^2 : B_r \to B_r \) that is almost-conformal and restricts to \( f_2 \) on \( \partial B_r \). The maps \( F_2^1 \mid_{\mathbb{D} \setminus B_r} \) and \( F_2^2 \) then define a map \( F_2 : \mathbb{D} \to \mathbb{D} \) that restricts to \( f_2 \) on \( B_r \) and is identity on \( \partial \mathbb{D} \). The required map \( F : A \to A \) is then the restriction of the composition \( F_2 \circ F_1 \) to the annular region \( A \). \( \square \)

### 5.2. Proof of the multi-curve case.

Let \( \lambda \) be a multicurve, namely a collection of (weighted) disjoint simple closed geodesics \( \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \) on the closed hyperbolic surface \( X \), with weights \( c_i > 0 \) and lengths \( l_i \), where \( 1 \leq i \leq n \).

Let \( X^\infty \) denote the infinitely grafted surface, obtained by cutting \( X \) along \( \lambda \) and gluing a semi-infinite euclidean cylinder at each of the resulting \( 2n \) boundary components. This resulting Riemann surface can be thought of as the conformal limit of the grafting ray \( X_t = \text{graft}_t X \). (Recall that a time-\( t \) grafting along a simple closed curve \( \gamma \) inserts a euclidean cylinder of width \( t \) at \( \gamma \).

We shall use the following result due to Strebel ([Str84]):

**Proposition 5.7.** Let \( \Sigma \) be a Riemann surface of genus \( g \), and \( x_1, x_2, \ldots, x_m \) be marked points on \( Z \) such that \( 2g - 2 + m > 0 \). Then for any collection of real numbers \( p_1, p_2, \ldots, p_m \) there exists a quadratic differential \( \phi \) on \( \Sigma \) such that

(i) \( \phi \) is holomorphic on \( \Sigma \setminus \{x_1, \ldots, x_m\} \).

(ii) \( \phi \) has a double pole at \( x_i \) with residue \(-\left(\frac{p_i}{2\pi}\right)^2\), for each \( 1 \leq i \leq m \).

(iii) All horizontal trajectories of \( \phi \) are closed, and they foliate \( m \) annular domains which are disks with punctures at \( x_1, \ldots, x_m \).

Applying the above proposition with \( \Sigma = X^\infty \), we can obtain such a *Jenkins-Strebel differential* \( \phi \) on a Riemann surface conformally equivalent (as marked conformal structures in \( T_{g, 2n} \)) to \( X^\infty \), with \( n \) pairs of marked points, each pair having residue \(-\left(\frac{p_i}{2\pi}\right)^2\). We denote this surface equipped with the quadratic differential metric as \( Y^\infty \); this then is a singular flat surface comprising \( n \) pairs of infinite euclidean cylinders, each pair having circumference \( l_i \). Let \( g \) be the conformal map from \( X^\infty \) to
Figure 17. The surface $X^\infty$ on the left is the “infinitely” grafted surface. The conformally equivalent singular flat surface $Y^\infty$ has a quadratic differential metric with a pair of double poles, and metrically it is equivalent to two semi-infinite euclidean cylinders glued along the boundary.

Let $Y$ denote the surface obtained from $Y^\infty$ by truncating the infinite cylinders and gluing up the pairs (of matching lengths) so that they give euclidean cylinders of circumference $l_i$ and height $c_i$, in the homotopy class of $\gamma_i$, for each $1 \leq i \leq m$. This will be the basepoint of the Teichmüller ray $Y_t$ that we shall show is asymptotic to the grafting ray $X_t$. Recall that the surface $Y_t$ is obtained from $Y$ (which is also $Y_0$ in our notation) by stretching along the horizontal foliation, and in particular the euclidean cylinders on $Y_t$ have height $c_i e^{2t}$.

Notation. For a semi-infinite cylinder (homeomorphic to $S^1 \times [0, \infty)$) denoted by $C$, we denote by $C_{\geq h}$ (resp. $C_{\leq h}$) the infinite subcylinder of all points of $C$ of height greater (resp. lesser) than $h$.

Lemma 5.8. Let $C, C'$ be two semi-infinite Euclidean cylinders of the same circumference, and let $f : C \to C'$ be a conformal map which is a homeomorphism onto its image. Then for any $H_0 > 0$ there exists an $H_1 > H_0$ and a map $F : C \to f(C) \subset C'$ such that

1. $F$ is almost-conformal.
2. $F$ is isometric on $C_{\geq H_1}$.
3. $F$ restricts to $f$ on $C_{\leq H_0}$.

Proof. The cylinders $C$ and $C'$ are conformally equivalent to the punctured unit disk $\mathbb{D}^*$ by a conformal map $\phi$ that takes $\infty$ to $0$, and maps the round circle at height $h$ (for each $0 \leq h < \infty$) to a circle of radius $r(h) = e^{-2\pi h}$ centered at the origin. Thus the conformal map $f$ conjugates to a conformal map $g = \phi \circ f \circ \phi^{-1}$ from the punctured disk to itself, which can be extended to a conformal map of $\mathbb{D}$ into itself, such that $g(0) = 0$ and $g'(0) = c$.

We can now apply Lemma 5.3 to the rescaled conformal map $(1/c)g$ to obtain an almost-conformal map $\tilde{G}$ that restricts to a dilatation $z \mapsto cz$ on $B_s$ for some $0 < s < 1$ and agrees with $g$ on $\partial \mathbb{D}$. Since a dilatation conjugates back to an isometric translation of the semi-infinite cylinders, the map $F = \phi^{-1} \circ G \circ \phi$ satisfies (1) to (3).

Proposition 5.9. For any $\epsilon > 0$ there exists a $T > 0$ such that for every $t > T$ there is a $(1 + O(\epsilon))$-quasiconformal map from the grafted surface $X_t$ to a singular flat surface $Y_s$ for some $s$. 

□
Lemma 5.8 allows one to adjust $g$ such that it isometric on the circle at height $H$. Discarding the shaded regions and gluing up along the truncating circles gives a grafted surface $X_t$ on the left and the surface along the Teichmüller ray on the right.

**Proof.** We start with the conformal map $g : X^\infty \to Y^\infty$. Consider its restriction to one of the semi-infinite euclidean cylinders $C$, and let $C'$ be the corresponding infinite cylinder on $Y^\infty$ such that $g(C) \cap C' \neq \emptyset$. By properness, for $H_0$ sufficiently large, the image under $g$ of the $C \geq H_0$ will be strictly contained in $C'$. Applying Lemma 5.8 we have some $H_1 > H_0$ and a map $F_C$ on the truncated cylinder $C \leq H_1$ that agrees with $g$ on $C \leq H_1$ and is isometric on the circle at height $H_1$.

Repeating this for each infinite cylinder on $X^\infty$, we obtain truncations of each and almost-conformal maps such that together we have an almost-conformal map $F$ to $Y^\infty$, with each of its cylinders also truncated at a circle at some height. The map $F$ is isometric on the boundary circles, and in particular its restrictions to truncated paired cylinders agree on the truncating round circles. On gluing these maps we obtain an almost-conformal map from the truncated $X^\infty$ glued along the boundaries of the paired cylinders, to the truncated $Y^\infty$ glued along the boundaries of its paired cylinders. By definitions of the surfaces, the latter is $Y_s$ for some $s$, and the former is $X_T$ for some $T$. By choosing to truncate at higher heights (greater than $H_1$ as in $C$ above) one can obtain a almost-conformal maps from $X_t$ for any $t > T$, to a surface along the Teichmüller ray starting at $Y$. □

Recall that the surfaces $X_t$ form the grafting ray determined by the multicurve $\lambda$, and the surfaces $Y_s$ lie along the Teichmüller ray determined by $\lambda$ and with basepoint $Y = Y_0$. Thus together with Proposition 4.1 this concludes the proof of Theorem 1.1.

### 6. Proof of Theorem 1.4

In this section we prove the following theorem stated in §1:

**Theorem 1.4.** Let $\mathcal{P}_\rho$ be the set of complex projective structures on a surface $S_g$ with a fixed holonomy $\rho \in \text{Rep}(\pi_1(S_g), PSL_2(\mathbb{C}))$. Then for any Fuchsian representation $\rho$, the projection of $\mathcal{P}_\rho$ to $\mathcal{M}_g$ has a dense image.

Recall that $\pi : T_g \to \mathcal{M}_g$ denotes the projection to moduli space, and $\mathcal{P}_\rho \subset \mathcal{P}(S_g)$ denotes the set of complex projective structures that have holonomy $\rho$. Let $p : \mathcal{P}(S_g) \to T_g$ be the “forgetful” projection.

The following is a consequence of the work of Goldman (see Theorem C and §2.14 in [Gol87]):
Theorem. \([\text{Gol87}]\) Let \(X\) be a hyperbolic surface, and let \(\rho : \pi_1(S_g) \to PSL_2(\mathbb{R})\) be the corresponding Fuchsian representation. Then for any other element \(Y \in \mathcal{P}_\rho\) one has that \(p(Y) = gr_{2\pi\gamma}(X)\) for some multi-curve \(\gamma\) on \(X\).

Remark. As is standard, we refer to grafting a hyperbolic surface along a multi-curve with weight \(2\pi\) as integer-grafting or \(2\pi\)-grafting.

It immediately follows that Theorem 1.4 is equivalent to the following proposition:

**Proposition 6.1.** For any \(X \in T_g\), the set of integer graftings \(\{p(gr_{2\pi\gamma}X) | \gamma \in S\}\) is dense in \(\mathcal{M}_g\).

Fix any \(X \in T_g\) and pick an arbitrary \(Y \in \mathcal{M}_g\) and a sufficiently small \(\epsilon > 0\). To establish Proposition 6.1 it is enough to show that for this choice we have:

**Proposition 6.2.** There exists a \(\gamma \in S\) such that \(d_T(p(gr_{2\pi t_i\lambda}(X)), Y) = O(\epsilon)\).

By Corollary 1.3 we have a \(\lambda \in ML\) such that

1. \(\lambda\) is arational, and
2. The projection of the grafting ray determined by \((X, \lambda)\) to moduli space is dense.

In particular, we have a sequence of times \(t_i \to \infty\) such that

\[(8) \quad d_T(p(gr_{2\pi t_i\lambda}(X)), Y) < \epsilon\]

The argument for the proof of Proposition 6.2 carried out in §6.1-5 consists of choosing an appropriate approximation of one of the \(2\pi t_i\lambda\) by a multicurve and showing that the corresponding grafted surface is close to the surface obtained by grafting along this multicurve.

### 6.1. Almost-conformal constructions.

In §4.2 we developed the notion of “almost-isometries” (see Definition 4.10) and some related constructions of almost-conformal maps. In this section, we weaken the definition by introducing a further bounded additive factor \(C\) (see Definitions 6.3 and 6.7). This factor shall be small relative to the other dimensions, however, and will still permit the construction of almost-conformal maps (as in Lemma 6.8).

**Definition 6.3.** A map \(f\) between two \(C^1\)-arcs on a conformal surface is an \((\epsilon, C)\)-almost-isometry if \(f\) is continuously differentiable with dilatation \(d\) such that \(|d - 1| = O(\epsilon)\) and such that the lengths of any subinterval and its image differ by an additive error of at most \(C\).

The following are analogues of Lemmas 4.11, 4.12 and 4.13 and we omit their (easy) proofs:

**Lemma 6.4.** Let \(I_1, I_2\) be arcs of lengths \(l_1\) and \(l_2\) such that \(|l_1 - l_2| < C\) and \(C/l_1 = O(\epsilon)\). Then the affine (stretch) map \(f : I_1 \to I_2\) is \((\epsilon, C)\)-almost-isometric.

**Lemma 6.5.** Let \(f, g : I \to I\) be maps of an arc \(I\) that are \((\epsilon, C)\)-almost-isometric and \((\epsilon, C')\)-almost-isometric respectively. Then \(f^{-1}\) is \((\epsilon, C)\)-almost-isometric, and \(f \circ g\) is \((\epsilon, C + C')\)-almost-isometric.

**Lemma 6.6.** Let \(I = I_1 \cup I_2 \cup \cdots I_N\) be a partition of the arc \(I\) into sub-arcs with disjoint interior. Then any continuously differentiable map \(f : I \to I\) with \((\epsilon, C)\)-almost-isometric restrictions to \(I_1, \ldots I_N\) is \((\epsilon, NC)\)-almost-isometric on \(I\).
Conversely, the restriction of an $(\epsilon, C)$-almost-isometry to a sub-arc is also an $(\epsilon, C)$-almost-isometry to its image.

**Definition 6.7.** A map $f$ between two rectangles is $(\epsilon, C)$-good if it is isometric on the vertical sides and $(\epsilon, C)$-almost-isometric on the horizontal sides.

**Remark.** A “rectangle” in the above definition refers to four arcs in any metric space intersecting at right angles, with a pair of opposite sides of equal length being identified as *vertical* and the other pair also of identical length called *horizontal*.

6.1.1. *Almost-conformal extension.* The following lemma is a slight generalization of Lemma 4.14:

**Lemma 6.8.** Let $R_1$ and $R_2$ be two euclidean rectangles with vertical sides of length $h$ and horizontal sides of lengths $l_1$ and $l_2$, respectively, such that $l_1, l_2 > h$ and $|l_1 - l_2| < C$, where $C/h = O(\epsilon)$. Then any $(\epsilon, C)$-good map $f : \partial R_1 \to \partial R_2$ has an almost-conformal extension $F : R_1 \to R_2$.

**Proof.** The proof follows by rescaling by a factor of $1/h$ and applying Lemma 4.14 to the resulting map between the resulting pair of rectangles. \(\square\)

6.1.2. *Finitely grafted rectangle.* Let $R$ be a rectangular region in the hyperbolic plane bounded by two “vertical” geodesic sides of length $l$ and two “horizontal” horocyclic sides of length $w$. Assume henceforth that $l > 1/\epsilon$ and $w < \epsilon$.

Let $a_1, a_2, \ldots, a_k$ be a finite collection of geodesic arcs with endpoints on the horizontal sides, with corresponding weights $w_1, w_2, \ldots, w_k$. Then one can obtain a *finitely grafted* rectangle $R'$ by inserting euclidean rectangles in the shape of truncated “crescents” (see figure) of widths $w_1, w_2, \ldots, w_k$ at the arcs $a_1, a_2, \ldots, a_k$ respectively.

**Lemma 6.9.** There is an almost-conformal map $f$ from $R'$ to a euclidean rectangle of vertical height $l$ and horizontal width $w_1 + w_2 + \cdots + w_k$. Moreover, $f$ is $(\epsilon, \epsilon)$-good on the boundary.

**Proof.** We give a sketch of the argument, and refer to §4.2 and §4.3 for details and similar constructions. We always work in the upper-half-plane model of the hyperbolic plane.

First, we can map the (ungrafted) rectangle $R$ to the euclidean plane by a map that “straightens” the horocyclic foliation across $R$. Since $w < \epsilon$, this straightening map is almost-conformal (Lemma 4.7). It also follows from some elementary hyperbolic geometry that the geodesic arcs are $\epsilon$-almost-vertical, and so are their images under...
Figure 20. Grafting along a geodesic arc intersecting almost, but not at right angles gives a finitely grafted rectangle $R'$ with the top edge having “corners” (the grafted euclidean region is shown shaded). This can be smoothed to a $C^1$-arc together with an almost-conformal map from $R'$ to the resulting rectangle $R''$ (Lemma 6.11).

Next, the truncated “crescents” are spliced in: their straightening maps to the plane are in fact conformal with rectangular images (Lemma 4.9) and hence can be adjusted by almost-conformal maps (Lemma 4.4) to fit with the almost-vertical image arcs above.

This gives a composite map that is almost-conformal with image a rectangle of height $l$ and width $w + w_1 + w_2 + \cdots + w_k$. Since $w < \epsilon$ and $w_1 + w_2 + \cdots + w_k > 1$, one can finally compose by an almost-conformal horizontal affine stretch to a rectangle of width $w_1 + w_2 + \cdots + w_k$ as required.

The statement about the almost-isometry of the sides follows from Lemma 6.4 since prior to the final affine dilatation the map is isometric on the horizontal sides, and the final affine stretch is to a rectangle of width differing by $w < \epsilon$.

6.1.3. Smoothing the horizontal sides. In the finitely grafted rectangle $R'$ above, the horizontal sides may not be differentiable arcs since the geodesic arcs $a_1, \ldots, a_k$ may not intersect the horizontal sides of $R$ at right angles. However, since the rectangle $R$ prior to grafting is thin ($w < \epsilon$) and long ($l > 1/\epsilon$) some elementary hyperbolic geometry implies that the geodesic arcs intersect the horizontal sides at an angle that differs from $\pi/2$ by a quantity bounded by $O(\epsilon)$.

The horizontal sides can then be “smoothed” to be $C^1$ by adjusting the horizontal sides of each euclidean “truncated crescent”: each such horizontal segment is replaced by a $C^1$ arc whose derivatives are small, and have specified values at the endpoints that make the entire arc $C^1$. Denote the resulting new “smoothed” rectangle by $R''$.

Lemma 6.10. Let $S = [0, w] \times [0, l]$ be a euclidean rectangle and let $S'$ be the region enclosed by the left, right and bottom sides of $S$ and the arc given by $y = g(x)$ for $0 \leq x \leq w$ where $g(0) = g(w) = l$. Then if $|g'(x)| < \epsilon$ and $1 - g(x)/l = O(\epsilon)$ for all $0 \leq x \leq w$, there exists an almost-conformal map from $S$ to $S'$ which is $(\epsilon, \epsilon)$-good on the boundary.
Proof. Similar to the proofs of Lemmas 4.4-6, the required map is given by a straightening map:

\[(x, y) \mapsto (x, \frac{g(x)y}{l})\]

and the fact that it is almost-conformal everywhere and almost-isometric on the boundary follows by computing the derivatives. □

By a repeated application of the above lemma on each of the grafted strips, we have:

**Lemma 6.11.** There exists an almost-conformal map from \(R'\) to \(R''\) which is \((\epsilon, \epsilon)\)-good on the boundary.

Moreover, by precomposing with the almost-conformal map \(f\) of Lemma 6.9 we have:

**Corollary 6.12.** There exists an almost-conformal map from \(R''\) to a euclidean rectangle of vertical height \(l\) and horizontal width \(w_1 + w_2 + \cdots + w_k\) which is \((\epsilon, \epsilon)\)-good on the boundary.

### 6.2. Thickening the train-track.

Recall from §4.1 that we have the subsurface \(T_\epsilon \subset X\) containing the arational lamination \(\lambda\) which is its “\(\epsilon\)"-train-track neighborhood. From that section, we have a decomposition of the surface into a collection of rectangles \(R_1, R_2, \ldots, R_n\), and complementary regions which are truncated ideal hyperbolic triangles \(T_1, T_2, \ldots, T_m\).

We now describe a “thickening” to ensure that \(\lambda\) is contained properly in \(T_\epsilon\):

For each \(T_1, T_2, \ldots, T_m\), choose thin strips adjacent to the geodesic sides and bounded by another geodesic segment “parallel" to the sides and append them to the rectangles adjacent to the sides. We continue to denote the collection of this slightly thickened rectangles by \(R_1, R_2, \ldots, R_n\), and their union by \(T_\epsilon\).

### 6.3. Approximating \(\lambda\) by multicurves.

Let \(w_1, w_2, \ldots, w_n\) be the weights of the train track \(T_\epsilon\), that is, \(w_i\) denotes the total transverse measure of the rectangle \(R_i\). By our assumption of the maximality of \(\lambda\), these weights are all positive reals.

By the above construction of the train track it follows that the sub-arcs \(I'_1, \ldots, I'_{3m+1}\) (see [3] in §4.1) also have a positive transverse measures \(w'_1, \ldots, w'_{3m+1}\). This follows from the minimality of \(\lambda\): the only way such a sub-arc will carry no measure is if \(\lambda\) intersected it only at the endpoints, but the leaf of \(\lambda\) passing through an endpoint of one of these sub-arcs is isolated on the complementary side (it is part of the boundary of one of \(T_1, \ldots, T_m\)) and cannot be isolated inside the sub-arc also.

For each \(1 \leq i \leq n\) the weights

\[(9) \quad w_i = \sum_{k \in S_i} w'_k\]

where \(S_i\) is a finite subset of \(\{1, 2, \ldots, 3m + 1\}\) as in equation [3].

**Definition 6.13.** A \(3m + 1\)-tuple of (non-negative) real numbers \((c_1, c_2, \ldots, c_{3m+1})\) is an admissible weighting of \(T_\epsilon\) if the corresponding weights on the train track given by [9] satisfy the switch conditions for the train track.

We begin with the following observations in elementary linear algebra:
Lemma 6.14. Let $S$ be a homogeneous system of linear equations in $N$ variables, with all coefficients in the set $\{0,1,-1\}$. Then there exists a constant $C > 0$ depending only on $N$ such that for any $N$-tuple $(x_1,x_2,\ldots,x_N)$ of real numbers that satisfies $S$ there is an integer $N$-tuple $(k_1,k_2,\ldots,k_N)$ with $|x_i - k_i| < C$ for each $1 \leq i \leq N$, which is also a solution.

Proof. Since the coefficients of the linear system $S$ are integers, by Gauss-Jordan elimination there is a basis $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_M\}$ of the vector space $V$ of solutions such that each $\vec{v}_i$ is a vector with rational entries. Let $L$ be the integer that is the least common multiple of all the denominators of the rational entries, such that $\vec{w}_i = L\vec{v}_i$ is an integer vector for each $1 \leq i \leq M$. Then this set of $M$ linearly-independent integer vectors $W$ spans a lattice in $V$. Let $C$ be the diameter of the torus $T^M$ that is a quotient of $V$ by the action of $W$, or equivalently, the radius of a fundamental domain in $V$. Clearly then, any solution in $V$ is less than $C$ away from an integer vector. The constant $C$ at this point depends on the linear system $S$, but notice that since there are $N$ variables, and each coefficient is from the finite set $\{0,1,-1\}$, there are only finitely many possible choices of $S$ (depending only on $N$), and hence we can choose $C$ to be the maximum value as we vary over all of them. \qed

Corollary 6.15. Let $S$ be a homogeneous system as above, and let $\vec{x} = (x_1,x_2,\ldots,x_N)$ be a solution where each entry is a positive real number. Then there exists a $T_0 > 0$ such that for any $t > T_0$, there is an integer solution $(k_1,k_2,\ldots,k_N)$ with each entry positive, such that $|tx_i - k_i| < C$ for each $1 \leq i \leq N$.

Proof. Note that since $S$ is homogenous the vector $t\vec{x}$ is also a solution. Each entry of this vector will be greater than $C$ when $t > T_0 = \frac{C}{\min_{1\leq i \leq N} x_i}$. Let $(k_1,k_2,\ldots,k_N)$ be the integer solution close to $t\vec{x}$ that the previous lemma guarantees. Since for each $i$, we have $|tx_i - k_i| < C$ and $tx_i > C \implies k_i > 0$ we see that each entry of this integer solution is positive. \qed

We now apply this to our setting:

Lemma 6.16. There exists a $C > 0$ and $T_0 > 0$ such that for any $t > T_0$ there is a tuple $\vec{k} = (k_1,k_2,\ldots,k_{3m+1})$ of positive integers such that

1. $\vec{k}$ is an admissible weighting of $T_\epsilon$.
2. $|tu'_j - k_j| < C$ for each $1 \leq j \leq 3m + 1$.

Moreover, $C$ is a constant that is independent of $\epsilon$.

Proof. The admissible weights on the train track $T_\epsilon$ satisfy a linear system $S$ in $3m + 1$ variables corresponding to the switch conditions and equations (9), which have coefficients in the set $\{0,1,-1\}$. Hence one can apply Corollary 6.15 with $\vec{x}$ being the positive solution $(w_1,w_2,\ldots,w_{3m+1})$ corresponding to the transverse measures of the lamination $\lambda$, and this yields (1) and (2). Also by the lemma, $C$ depends only on $m$, which in turn depends only on the topology of the surface (see §4.1), and hence is independent of $\epsilon$. \qed

Definition 6.17. For $t > T_0$, let $\gamma_t$ denote the geodesic multicurve corresponding to the admissible integer weighting $\vec{k}$ on the train-track $T_\epsilon$ satisfying (2) of Lemma 6.16.
Lemma 6.18. There exists a $T_1 > T_0$ such that for any $t > T_1$ the multicurve $\gamma_t$ is contained in $\mathcal{T}_\epsilon \subset X$.

Proof. Notice that the induced weights $\bar{k}_1, \ldots, \bar{k}_n$ on the branches $R_1, \ldots, R_n$ of the train track $\mathcal{T}_\epsilon$ (obtained from equation (9)) satisfy:

$$|tw_i - \bar{k}_i| < (3m + 1)C$$

for each $1 \leq i \leq n$.

This implies that $[\gamma_t] \to [\lambda]$ in $\mathcal{PM\mathcal{L}}$ as $t \to \infty$, and hence the corresponding geodesic representatives on the surface converge to $\lambda$ in the Hausdorff topology. (The maximality of $\lambda$ is used here too, since in general it is only true that the supports $[\gamma_t] \to [\lambda'] \supset [\lambda]$) Since $\lambda$ is a proper subset of the closed set $\mathcal{T}_\epsilon$ (this uses the “thickening” defined in §6.2), so is $\gamma_t$ for large enough $t$. □

6.4. Model rectangles. Recall that the train-track decomposition of $X$ (described in §2.1) persists as we graft along $\lambda$, with the total width of the rectangles $R_1, R_2, \ldots, R_n$ increasing along the $\lambda$-grafting ray. We denote the grafted rectangles on $gr_{2\pi\tau\lambda}X$ by $R^t_1, R^t_2, \ldots, R^t_n$.

Here, the total width $w_i(t)$ is the maximum width of the rectangle $R^t_i$ in the Thurston metric on $gr_{2\pi t\tau\lambda}X$, and we have

$$|w_i(t) - 2\pi tw_i| < \epsilon$$

since the initial hyperbolic widths of the rectangles on $X$ is less than $\epsilon$ by Lemma 4.2.

We shall use the construction of an almost-conformal euclidean model rectangle for a rectangular piece $R^t$ from the collection $\{R^t_1, R^t_2, \ldots, R^t_n\}$ proved in §4.3. We restate the results of that section as follows:

Lemma 6.19 (Lemma 4.18 and Corollary 4.19). For any $t > 1$, there is an almost-conformal map from $R^t$ to a euclidean rectangle of width $2\pi tw_i$ which is $(\epsilon, \epsilon)$-good on the boundary.

We recapitulate the proof briefly: one first approximates the uncountable collection of geodesic arcs $R^t \cap \lambda$ by a sequence of finite, weighted collections of arcs. For each such finite approximation, one can show that if $t$ is large in proportion to the hyperbolic width the map to the complex plane that straightens the transverse foliation is an almost-conformal map, and then one takes a limit.

6.5. Grafted surfaces are close. Let $T_1 > 0$ be as in Lemma 6.18. The goal of this section is to prove:

Lemma 6.20. There exists a $T_2 > T_1$ such that for any $t > T_2$, we have that $d_{\mathcal{T}}(gr_{2\pi t\lambda}X, gr_{2\pi \gamma_t}X) = O(\epsilon)$.

Here’s a brief summary of the proof prior to the details:

On the initial (ungrafted) surface $X$ one has a train-track $\mathcal{T}_\epsilon$ containing the lamination $\lambda$ which is decomposed into rectangles (corresponding to the branches) by a choice of transverse arc $\tau$. The multi-curve approximation $\gamma_t$ is also contained in $\mathcal{T}_\epsilon$. 
On the grafted surface $gr_{2\pi t\lambda}X$ the rectangles in the train-track decomposition widen to have more (euclidean) width. Similarly on $gr_{2\pi\gamma_t}X$ the rectangles in the initial decomposition are wider, though not with $C^1$-boundary as the arcs of $\gamma_t$ might intersect $\tau$ at an angle slightly off $\pi/2$. $\tau$ is then replaced by its “smoothed” arc which gives the correct rectangle decomposition on $gr_{2\pi\gamma_t}X$.

Each rectangle on $gr_{2\pi t\lambda}X$ is then mapped almost-conformally to the corresponding one on $gr_{2\pi\gamma_t}X$ via their euclidean “models”, by first mapping the boundary and then using the almost-conformal extension lemma. The complement of the train-tracks are isometric as grafting leaves them unaffected, and these put together give the required almost-conformal map between the two surfaces.

**Proof of Lemma 6.20.** Since $t > T_1$ we have that $\gamma_t \subset T_\epsilon$ by Lemma 6.18. We let $R_1', R_2', \ldots, R_n'$ be the rectangles obtained by grafting $R_1, R_2, \ldots, R_n$ along $2\pi\gamma_t$. Note that $\gamma_t \cap R_i$ are a finite collection of geodesic arcs, and grafting along them gives finitely grafted rectangles as in §6.1.2.

Recall that all horizontal sides of the rectangles $R_1, R_2, \ldots, R_n$ (on the surface $X$) lie on the arc $\tau$ which is a segment of a leaf of the horocyclic foliation. We chose the arc $\tau$ to be sufficiently small (see the comment following Lemma 4.2) and we can assume that its hyperbolic length $w$ is less than $\epsilon$.

After grafting along $2\pi t\lambda$, this arc is converted to an arc on $gr_{2\pi t\lambda}X$ of length $w + 2\pi tw_1 + 2\pi tw_2 + \cdots + 2\pi tw_n$ which we denote by $\tau_\lambda$ (recall that as one grafts, the horocyclic foliation extends to a foliation on the grafted surface). After grafting along $2\pi\gamma_t$, the same arc $\tau$ is converted to an arc $\tau'$ on $gr_{2\pi\gamma_t}X$ of length $w + 2\pi k_1 + 2\pi k_2 + \cdots + 2\pi k_n$ which we can smooth to a $C^1$ arc (see §6.1.3) on the grafted surface that we denote by $\tau_\gamma$. This simultaneously “smooths” the finitely grafted rectangles $R_1', R_2', \ldots, R_n'$ to a new collection $R_1'', R_2'', \ldots, R_n''$ which we use for the rest of the construction.

Recall also that there are the sub-arcs $J_1, J_2, \ldots, J_{3m}$ of $\tau$ that are the horocyclic edges of the complementary regions $T_1, \ldots, T_n$. These remain isometrically embedded in the arcs $\tau_\lambda$ and $\tau'$, and also in $\tau_\gamma$ (smoothing of $\tau'$ affects only the segments lying in the grafted part).

**Claim.** For sufficiently large $t$ there is an $(\epsilon, 50mC)$-almost-isometry $h$ from $\tau_\gamma$ to $\tau_\lambda$ that restricts to an isometry between the sub-arcs corresponding to $J_1, J_2, \ldots, J_{3m}$.

**Proof of claim.** Recall that the sub-arcs inbetween the $J_1, \ldots, J_{3m}$ are the sub-arcs $I_1', \ldots, I_{3m+1}'$ that had weights $w_1', \ldots, w_{3m+1}'$ on the surface $X$. On $\tau_\gamma$ and $\tau_\lambda$, the lengths of these become $2\pi k_i$ and $2\pi t w_i'$ respectively, which differ by at most $2\pi C$ by (2) of Lemma 6.16. For large $t$, we have that by Lemma 6.4 the affine maps between these sub-arcs are $(\epsilon, 2\pi C)$-almost-isometries. By Lemma 6.6 the concatenated map of these together with isometries between the sub-arcs corresponding to $J_1, J_2, \ldots, J_{3m}$ give an $(\epsilon, 7m \cdot 2\pi C)$-almost-isometry from the entire arc $\tau_\gamma$ to $\tau_\lambda$. □
By Lemma 4.2, for $t$ sufficiently large the height of the finitely grafted rectangle $R_{t}^{i}$ is sufficiently large and the width is sufficiently small so that one can apply Corollary 6.12 and obtain, for each $1 \leq i \leq n$, an almost-conformal map $g_{i}$ from $R_{t}^{i}$ to a euclidean rectangle of width $2\pi k_{i}$ which is $(\epsilon, \epsilon)$-good on the boundary.

Recall from §6.4 that $R_{t}^{i}$ denotes the rectangle $R_{t}$ on $X$ after grafting along $2\pi\lambda$. By Lemma 6.19 for $t$ sufficiently large there exists, for each $1 \leq i \leq n$, an almost-conformal map $g_{i}$ from $R_{t}^{i}$ to a euclidean rectangle of width $2\pi w_{i}$ which is $(\epsilon, \epsilon)$-good on the boundary.

From the above claim, an $(\epsilon, 50mC)$-good map $h_{i}$ from the rectangle $\partial S_{t}$ on $gr_{2\pi\gamma_{i}X}$ to the rectangle $\partial R_{t}^{i}$ on $gr_{2\pi\lambda}X$ is the map that is isometric on the vertical sides and restricts to the map $h$ on the horizontal sides (which lie on the arc $\tau_{\gamma}$).

Consider the composition $g_{i|\partial} \circ h_{i} \circ f_{i}|\partial^{-1}$ where $f_{i}|\partial$ and $g_{i|\partial}$ are the restrictions of $f_{i}$ and $g_{i}$ to the boundary of the rectangles where they are defined. By Lemma 6.5 this is an $(\epsilon, 50mC + 2\epsilon)$-good map between two euclidean rectangles and hence by Lemma 6.8 (the height of these rectangles is sufficiently large when $\epsilon$ is sufficiently small) it extends to an almost-conformal map $H_{i}$ between them. The composition $g_{i}^{-1} \circ H_{i} \circ f_{i} : S_{t} \rightarrow R_{t}^{i}$ is an almost conformal map that restricts to $h : \tau_{\gamma} \rightarrow \tau_{\lambda}$ on the horizontal sides and is isometric on the vertical sides.

The collection of maps $\{H_{1}, H_{2}, \ldots, H_{n}\}$ give an almost-conformal map from $S_{1} \cup \cdots \cup S_{n} \subset gr_{2\pi\gamma_{i}X}$ to $R_{1}^{i} \cup \cdots \cup R_{n}^{i} \subset gr_{2\pi\lambda}X$ that is isometric on the geodesic sides. Since grafting does not affect the surface $X$ in the complement of $T_{\epsilon}$, there is an isometry between $gr_{2\pi\gamma_{i}X} \setminus S_{1} \cup \cdots \cup S_{n}$ and $gr_{2\pi\lambda}X \setminus R_{1}^{i} \cup \cdots \cup R_{n}^{i}$. Together with the above collection of maps this isometry defines the almost-conformal map between $gr_{2\pi\gamma_{i}X}$ and $gr_{2\pi\lambda}X$.

6.6. Completing the proof.

Proof of Proposition 6.2. We choose a $t_{1} > T_{2}$ (where $T_{2}$ is as in Lemma 6.20) that satisfies equation (8). By the triangle inequality,

$$d_{\mathcal{C}}(\pi(gr_{2\pi\gamma_{i}X}), Y) \leq d_{\mathcal{C}}(\pi(gr_{2\pi\lambda}X)), Y) + d_{\mathcal{C}}(gr_{2\pi\lambda}X, gr_{2\pi\gamma_{i}X})$$

The first term on the right is less than $\epsilon$ by equation (8) and the second term is $O(\epsilon)$ by Lemma 6.20.

From the discussion at the beginning of §6, this proves Proposition 6.1 and hence Theorem 1.4.

Appendix A. Proof of Lemma 4.25

The purpose of this section is to provide a proof of the following:

Proposition A.1 (Lemma 4.25). For any $\epsilon > 0$ sufficiently small and any $0 \leq r \leq \epsilon$

(1) $f$ is a quasiconformal map,

(2) The quasiconformal distortion is $(1 + C\epsilon)$ on $\mathbb{D} \setminus B_{r}$,

then the map $f$ extends to a $(1 + C'\epsilon)$-quasisymmetric map of the boundary, where $C'$ is a constant depending only on $C$. 

For $r = 0$ the above result is an easy consequence of the work of Ahlfors and Beurling that we recall as Lemma A.10 below.

We begin by recalling the definitions and relevant known results in §A.1, and prove a lemma about moduli of quadrilaterals in §A.2, from which the proof of the theorem follows.

A.1. Background. A starting point for the rich theory of quasiconformal mappings can be Ahlfors’ lectures [Ahl06].

Let $\Gamma$ be a family of (rectifiable) curves in $D$, and let $\rho$ be a non-negative measurable function on the disk $D$ satisfying

\begin{equation}
    l_\gamma(\rho) = \int_\gamma \rho \geq 1
\end{equation}

for all $\gamma \in \Gamma$ and

\begin{equation}
    A(\rho) = \iint_D \rho^2 dzd\bar{z} \neq 0, \infty
\end{equation}

Note that $\rho$ can be thought of as the conformal factor for a metric conformally equivalent to the standard metric on the unit disk.

Definition A.2. The extremal length of $\Gamma$ is denoted by $\lambda(\Gamma)$ is defined as

\begin{equation}
    \lambda(\Gamma) = \sup_{\rho} A(\rho)^{-1}
\end{equation}

where $\rho$ varies over all non-negative measurable functions satisfying (12) and (13). This is a conformal invariant.

Definition A.3. A quadrilateral $Q$ is the unit disk $D$ together with two disjoint arcs $A$ and $B$ on its boundary. There is a conformal map from $Q$ to a rectangle in $\mathbb{R}^2$ of height 1 and length $m$ that takes the two arcs to vertical sides. The positive real number $m$ is the modulus of the quadrilateral, denoted $\text{mod}(Q)$.

One of the basic results asserts:

Lemma A.4. (Grötzsch) If $\Gamma$ is the collection of all rectifiable curves in $D$ joining the boundary arcs $A$ and $B$, then $\lambda(\Gamma) = \text{mod}(Q)$.

If $S \subset D$ is a closed subset containing the boundary $\partial D$, then we can restrict $\Gamma$ in the lemma above to a collection $\Gamma'$ of curves that are contained in $S$. We denote the corresponding extremal length $\lambda_S(Q) = \lambda(\Gamma')$. We shall also call this the extremal length restricted to $S$. Note that by the above lemma $\lambda_D(Q) = \text{mod}(Q)$.

From the definition of extremal length, it is easy to check:

Lemma A.5. If $S \subset D$ then $\lambda_S(Q) \geq \lambda_D(Q)$.

Definition A.6. Let $\Omega, \Omega'$ be two domains in $\mathbb{C}$. Then a homeomorphism $f : \Omega \to \Omega'$ is said to be $K$-quasiconformal if

(1) $f$ has locally integrable distributional derivatives.

(2) The ratio

\begin{equation}
    \frac{|f_z| - |f_{\bar{z}}|}{|f_z| + |f_{\bar{z}}|} \leq K
\end{equation}

almost everywhere in $\Omega$.

The quasiconformal distortion at a point in $\Omega$ is defined to be the value of the left hand side of equation (15).

**Definition A.7.** A homeomorphism $g : \mathbb{R} \to \mathbb{R}$ is $M$-quasisymmetric if for every $x, t \in \mathbb{R}$, we have

$$\frac{1}{M} \leq \frac{f(x + t) - f(x)}{f(x) - f(x - t)} \leq M$$

**Definition A.8.** A homeomorphism $g : \partial \mathbb{D} \to \partial \mathbb{D}$ is $M$-quasisymmetric if $h \circ g \circ h^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ is $M$-quasisymmetric when restricted to $\mathbb{R}$, where $h$ is a conformal map from the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{H}^2$.

The following lemma can culled from the discussion in section 4 of [AB56].

For $A, B$ two intervals in $\mathbb{R}$, let $\lambda(A, B)$ denote the extremal length of the set of rectifiable paths in $\mathbb{H}^2$ from $A$ to $B$. (To use the above definition of extremal length we first map $\mathbb{H}^2$ conformally to $\mathbb{D}$.)

**Lemma A.9.** If $g : \mathbb{R} \to \mathbb{R}$ be a homeomorphism such that

$$\frac{1}{m} \leq \frac{\lambda(g(A), g(B))}{\lambda(A, B)} \leq m$$

for all disjoint intervals $A, B$.

Then $g$ is $M$-quasisymmetric, where $M = e^{A(m-1)}$ where $A \approx 0.228$ is a universal constant.

The following are the fundamental results of Ahlfors and Beurling ([AB56]) (for the version stated here see [Bis02]). Briefly, quasisymmetric maps of the boundary circle extend to quasiconformal maps of the unit disk, and vice versa, with the distortion constants ($K$ and $M$ as above) being close to 1 if one of them is.

**Lemma A.10.** For any $K > 1$ there is an $M > 1$ such that if $f : \mathbb{D} \to \mathbb{D}$ is a $K$-quasiconformal map then it extends to an $M$-quasisymmetric homeomorphism of the boundary. Moreover, there is a $K_0 > 1$ and $C_0 < \infty$ such that if $K = 1 + \epsilon < K_0$ then we can take $M \leq 1 + C_0 \epsilon$.

**Lemma A.11.** Any $M$-quasisymmetric homeomorphism of $\partial \mathbb{D}$ can be extended to a $K$-quasiconformal map of $\mathbb{D}$. Moreover, there is a $M_1 > 1$ and $C_1 < \infty$ such that if $M = 1 + \epsilon < M_1$ then we can take $K \leq 1 + C_1 \epsilon$.

Finally, we note the following standard consequence of quasiconformality (eg, see Chapter II of [Ahl06]):

**Lemma A.12.** Let $\Omega, \Omega' \subset \mathbb{D}$ be domains. If $f : \Omega \to \Omega'$ is a $K$-quasiconformal map, then for any collection $\Gamma$ of rectifiable curves in $\Omega$, we have

$$\frac{1}{K} \leq \frac{\lambda_{\Omega'}(f(\Gamma))}{\lambda_{\Omega}(\Gamma)} \leq K$$

A.2. **An extremal length lemma.** Let $A, B \subset \mathbb{D}$ be two disjoint boundary arcs, and $\Gamma$ the collection of rectifiable paths from $A$ to $B$ and let $E = B_r$ be the ball of radius $r < 1$.

In this section we show (Lemma A.14) that the extremal length of $\Gamma$ changes by a multiplicative factor of $1 + O(r)$ when $E$ is excised, that is, when we restrict to the
family of curves joining $A$ and $B$ and avoiding $E$. The constant in the $O(r)$ term is independent of the arcs $A$, $B$.

The following consequence of the Ko"ebe distortion theorem is used in its proof.

**Lemma A.13.** Let $E = B_r \subset \mathbb{D}$ and let $\phi : \mathbb{D} \to \mathbb{C}$ be a conformal embedding such that $\phi(0) = 0$. Then

$$diam(\phi(E)) < C r dist(\phi(E), \partial \phi(\mathbb{D}))$$

for all $r$ sufficiently small, for some universal constant $C$.

**Proof.** Let $\delta = dist(0, \partial \phi(\mathbb{D}))$.

By a consequence of the Ko"ebe distortion theorem (see Corollary 1.4 of [Pom]) we have

$$|\phi'(0)| \leq 4\delta$$

and by the other direction of the distortion theorem (Theorem 1.3 of [Pom]) we have

$$|\phi(z)| \leq |\phi'(0)| \frac{|z|}{(1 - |z|)^2}$$

which using (18) gives

$$|\phi(z)| < 4\delta r/(1 - r)^2 < 8r\delta$$

for any $z \in E$, since then $|z| \leq r$ and $r$ is sufficiently small. Hence

$$diam(\phi(E)) < 16r\delta$$

Now for $\omega \in E$, let $d_w = dist(\phi(\omega), \partial \phi(\mathbb{D})) = |\phi(\omega) - \phi(s)|$ for some $\phi(s) \in \partial \phi(\mathbb{D})$. We have

$$\delta \leq |\phi(s)| \leq d_w + |\phi(\omega)| \leq d_w + 8r\delta$$

where the last inequality is by (20) and the second the triangle inequality. Rearranging, and taking an infimum over $\omega \in E$ on the left hand side, we obtain

$$\delta(1 - 8r) \leq dist(\phi(E), \partial \phi(\mathbb{D}))$$

which implies, for $r$ sufficiently small,

$$\delta \leq (1 + C'r)dist(\phi(E), \partial \phi(\mathbb{D}))$$

for some constant $C'$. The proof is complete on combining (21) and (23).

**Lemma A.14.** Let $E = B_r \subset \mathbb{D}$ and $Q$ the quadrilateral defined by the two boundary arcs $A$, $B$ as above. Then

$$1 \leq \frac{\lambda_{\mathbb{D}}\setminus E(Q)}{\lambda_{\mathbb{D}}(Q)} \leq 1 + C'r$$

where $C'$ is a constant independent of $Q$.

**Proof.** The first inequality of (24) follows from Lemma A.5 so we need only to prove the other inequality.

Let $\phi$ be a conformal map from $\mathbb{D}$ to a rectangle $R$ of vertical height 1 and horizontal length $m = mod(Q)$ such that the arcs $A$ and $B$ are taken to the left and right vertical sides respectively. We further require that $\phi(0) = 0$. Such a map can be defined using elliptic integrals (see for example Chapter III of [Ahl06]).
It is well-known (see Definition A.3 and Lemma A.4) that this conformal domain realizes the extremal length of $Q$: the conformal metric $\rho \equiv 1$ on the rectangle pulled back via $\phi$ realizes the supremum in Definition A.2.

Let $\Gamma$ be the set of all rectifiable paths in $R$ between the vertical sides, and let $\Gamma'$ be the subcollection of $\Gamma$ of paths disjoint from $\phi(E)$.

We shall adapt the Grötzsch argument to show that $\rho$ is close to being extremal for the collection $\Gamma'$.

Let $S$ be a strip $S = [0, m] \times J$ of vertical ($y$-) height $|J| = \text{diam}(\phi(E))$ and horizontal ($x$-) range $m$ that contains $\phi(E)$. By Lemma A.13 we know that

\begin{equation}
\text{diam}(\phi(E)) < Cr
\end{equation}

for $r$ small, since $\phi(E) \subset R$ implies $\text{dist}(\phi(E), \partial \phi(D)) \leq \min\{1, m\} \leq 1$.

Let $\rho'$ be a conformal factor for $R$ that satisfies

\begin{equation}
l_\gamma(\rho') \geq 1
\end{equation}

for all $\gamma \in \Gamma'$.

In particular,

\begin{equation}
\int_0^m \rho'(x,y)dx \geq 1
\end{equation}

for any $y$ in $[0, 1] \setminus J$.

By integrating (26) over $y$ ranging over $[0, 1] \setminus J$, we get

\[
1 - Cr \leq \int_{[0,1] \setminus J} 1 dy \leq \int_{[0,1] \setminus J} \int_0^m \rho'(x,y)dx dy = \int_{R \setminus S} \rho'(x,y) dx dy
\]

Squaring, and using the Cauchy-Schwarz inequality for the right hand term, we get

\[
(1 - Cr)^2 \leq \int_{R \setminus S} (\rho')^2 dx dy \int_{R \setminus S} 1^2 dx dy \leq (\int_{R} (\rho')^2 dx dy)m
\]

So

\begin{equation}
(\int_{R} (\rho')^2 dx dy)^{-1} \leq m/(1 - Cr)^2 \leq m(1 + O(r))
\end{equation}

for $r$ sufficiently small.

Taking a supremum over $\rho'$ as in Definition A.2 we get

\[
\lambda_{D\setminus E}(Q) = \lambda_{R\setminus \phi(E)}(Q) = \lambda(\Gamma') \leq m(1 + O(r))
\]

where the first equality holds since extremal length is a conformal invariant. Since $m = \lambda_D(Q)$, this is the right hand equality of (24) and the proof is complete. \qed
A.3. **Proof of proposition.** Henceforth let \( f : \mathbb{D} \to \mathbb{D} \) be the quasiconformal map with quasiconformal distortion \( 1 + C \epsilon \) off a small ball \( B_r \), as in Proposition A.1.

We need the following "quasiconformal" version of Lemma A.13 for maps of the unit disk:

**Lemma A.15.** \( \text{diam}(f(B_r)) = O(r^{1-\epsilon}). \)

**Proof.** Let \( d = \text{diam}(f(B_r)) \). For convenience we shall assume \( C = 1 \).

It is well-known (eg. see III.A of [Ahl06]) that for an annular domain on the plane that contains 0, 1 in the bounded component of its complement, and the interval \([c, \infty)\) in the unbounded component where \( c > 1 \), we have

\[
\lambda(P) < \frac{1}{2\pi} \ln 16c
\]

where \( P \) is the set of rectifiable curves connecting the inner boundary component to the outer boundary component.

Since \( f \) is a homeomorphism, if \( A = \mathbb{D} \setminus B_r \) and \( \Gamma \) the set of paths between the boundary components of \( A \) then \( f(A) \) is topologically an annulus, and by rotation and scaling we see that we see that it satisfies the above condition with \( c = 1/d \). So we have

\[
\lambda(f(\Gamma)) < \frac{1}{2\pi} \ln \frac{16}{d}
\]

By Lemma A.12 and the fact that \( f \) is \((1 + \epsilon)\)-quasiconformal on \( \mathbb{D} \setminus B_r \) we know that

\[
(1 - \epsilon)\lambda(\Gamma) \leq \frac{1}{1 + \epsilon} \lambda(\Gamma) \leq \lambda(f(\Gamma))
\]

but since

\[
\lambda(\Gamma) = \frac{1}{2\pi} \ln \frac{1}{r}
\]

we obtain from (29) and (30) that

\[
d \leq 16r^{1-\epsilon}
\]

\( \square \)

**Corollary A.16.** If \( r \leq \epsilon \) then \( \text{diam}(f(B_r)) = O(\epsilon) \).

**Proof.** The maximum of \( x^{-x} \) is \( e^{1/e} \approx 1.44 \). So \( r^{1-\epsilon} \leq e^{1-\epsilon} < 1.45\epsilon \). \( \square \)

**Proof of Proposition 1.1.** By Lemma A.9 it is enough to show that for any pair of disjoint arcs \( A \) and \( B \) on \( \partial D \), we have

\[
1 - O(\epsilon) \leq \frac{\lambda_B(f(Q))}{\lambda_B(Q)} \leq 1 + O(\epsilon)
\]

where \( Q \) is the corresponding quadrilateral, and the constant in \( O(\epsilon) \) is independent of \( Q \).

This is because in the notation of Lemma A.9 if \( m = 1 + O(\epsilon) \) then \( M = e^{A(m-1)} = 1 + O(\epsilon) \).
Let $r \leq \epsilon$. By Lemma A.14 we have
\[ 1 \leq \frac{\lambda_{\mathbb{D}\setminus B_r}(Q)}{\lambda_{\mathbb{D}}(Q)} \leq 1 + O(\epsilon) \]
and by Corollary A.16 and Lemma A.14 we have
\[ 1 \leq \frac{\lambda_{\mathbb{D}\setminus f(B_r)}(f(Q))}{\lambda_{B}(f(Q))} \leq 1 + O(\epsilon) \]
Now by Lemma A.12 since $f$ is almost-conformal on $\mathbb{D}\setminus B_r$, we have
\[ 1 - O(\epsilon) \leq \frac{\lambda_{\mathbb{D}\setminus f(B_r)}(f(Q))}{\lambda_{\mathbb{D}\setminus B_r}(Q)} \leq 1 + O(\epsilon) \]
The required (31) follows easily from the above three inequalities.

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