Mixing Angle between $^3P_1$ and $^1P_1$ in HQET

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Some claim that there are two independent mixing angles ($\theta = 35.3^\circ, -54.7^\circ$) between $^3P_1$ and $^1P_1$ states of heavy-light mesons in the heavy quark symmetric limit, and others claim there is only one ($\theta = 35.3^\circ$). We clarify the difference between these two and suggest which should be adopted. General arguments on the mixing angle between $^3L_L$ and $^1L_L$ of heavy-light mesons are given in HQET and a general relation is derived in the heavy quark symmetric limit as well as that including the first-order correction in $1/m_Q$.

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§1. Introduction

The $P$ states have a rich structure because, combined with quark spins, they form four $P$ states, i.e., $^3P_0$, $^3P_1$, $^3P_2$ and $^1P_1$, and also because there is an interesting feature of mixing between two $1^+$ states. An explicit study on the mixing between $^3P_1$ and $^1P_1$ states in the context of a heavy-light system is given by Rosner¹) and is restudied a few years later by Godfrey and Kokoski shown in Ref. 2) by taking into account more states, $D$, $D_s$, $B$ and $B_s$ mesons.

Here, we illustrate their idea using notations of Ref. 2) in which, as well as in Ref. 1), they have assumed that the dominant interaction between heavy and light quarks is in nonrelativistic spin-orbit terms. These terms partially contribute to the mass of a heavy-light meson:

$$H_{SO} = \frac{4}{3} \frac{\alpha_s}{r^3} \left( \frac{\vec{S}_q + \vec{S}_Q}{m_q m_Q} \right) \cdot \vec{L} + \frac{1}{4} \left( \frac{4 \alpha_s}{3 r^3} - \frac{b}{r} \right) \left[ \left( \frac{1}{m_q^2} + \frac{1}{m_Q^2} \right) \left( \vec{S}_q + \vec{S}_Q \right) \cdot \vec{L} \right. $$

$$+ \left. \left( \frac{1}{m_q^2} - \frac{1}{m_Q^2} \right) \left( \vec{S}_q - \vec{S}_Q \right) \cdot \vec{L} \right] = H_{SO}^q \vec{S}_q \cdot \vec{L} + H_{SO}^Q \vec{S}_Q \cdot \vec{L}. \quad (1.1)$$

Taking the heavy quark mass limit ($m_Q \to \infty$), we are left only with the term $H_{SO}^q \vec{S}_q \cdot \vec{L}$ proportional to $1/m_q^2$ in Eq. (1.1). Assuming other interaction terms

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including kinetic terms give a constant $M_0$ contribution, then they give the following relation between mass eigenstates and angular momentum eigenfunctions,\(^\text{2)}\)

$$
\begin{pmatrix}
M^{(3P_1)} \\
M^{(1P_1)}
\end{pmatrix} = \begin{pmatrix}
M_0 - \langle H_{SO}^q \rangle & -\sqrt{2} \langle H_{SO}^q \rangle \\
-\sqrt{2} \langle H_{SO}^q \rangle & M_0
\end{pmatrix} \begin{pmatrix}
^3P_1 \\
^1P_1
\end{pmatrix},
$$

(1.2)

which is, as shown later, translated into

$$
\begin{pmatrix}
|j^P = 1^+, j_\ell = \frac{1}{2}\rangle \\
|j^P = 1^+, j_\ell = \frac{3}{2}\rangle
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
^3P_1 \\
^1P_1
\end{pmatrix},
$$

(1.3)

with two mixing angles,

$$\theta = \arctan \left( \frac{1}{\sqrt{2}} \right) = 35.3^\circ \quad \text{or} \quad \theta = \arctan \left( -\sqrt{2} \right) = -54.7^\circ,
$$

(1.4)

where the left-hand side of Eq. (1.3) is the mass eigenstate and is specified in terms of eigenvalues of the total angular momentum $\vec{j}$ of a $Q\bar{q}$ bound state, $\vec{j}_\ell$, which stands for the light quark total angular momentum, $\vec{j}_\ell = \vec{L} + \vec{S}_q$, whose square is conserved in the heavy quark symmetric limit, and the parity $P$. Here, $\vec{S}_q$ is a light quark spin.

The vector of the right-hand side of Eq. (1.3) denoted as $|^2S+1L_j\rangle$ is specified in terms of eigenvalues of a light quark angular momentum $\vec{L}$, a sum of intrinsic quark spins $\vec{S} = \vec{S}_q + \vec{S}_Q$, and a total angular momentum $\vec{j}$ of the heavy-light meson.

On the other hand, using the heavy quark symmetry, we have derived the relation\(^\text{*)} equivalent to Eq. (1.3),\(^\text{3),4)\)

$$
\begin{pmatrix}
|j^P = 1^+, j_\ell = \frac{1}{2}\rangle \\
|j^P = 1^+, j_\ell = \frac{3}{2}\rangle
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
^3P_1 \\
^1P_1
\end{pmatrix},
$$

(1.5)

but with only one mixing angle,

$$\theta = \arctan \left( \frac{1}{\sqrt{2}} \right) = 35.3^\circ.
$$

(1.6)

Equations (1.3) and (1.5) are equivalent to each other but Eq. (1.6) is more restrictive than Eq. (1.4). We would like to solve the origin of this discrepancy and give a reasonable interpretation that should be adopted for the heavy-light mesons.

§2. Mass matrix by Rosner or Godfrey and Kokoski

Equation (1.2) is very confusing in the sense that 1) there are no eigenstates on the l.h.s. of the equation, and 2) the eigenvalues on the l.h.s., $M^{(3P_1)}$ and $M^{(1P_1)}$, are written with explicit arguments $^3P_1$ and $^1P_1$. They have assumed that the upper and lower components on the l.h.s. are dominated by $^3P_1$ and $^1P_1$ from the beginning, respectively.

\(^\text{*})\) The sign change of $\sin \theta$ in this equation can be absorbed into state redefinitions so that the form of an orthogonal matrix becomes the same as that of Eq. (1.3).
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To better understand Eq. (1.2), we introduce ket vectors as eigenstates with angular momentum quantum numbers, and an orthogonal matrix, $U$, to diagonalize the mass matrix. We rewrite Eq. (1.2) as an eigenvalue equation in an operator form so that everybody is on the same footing.

\[
\left(M_0 + H_{SO}^q \vec{L} \cdot \vec{S}_q\right) |\psi\rangle = \lambda |\psi\rangle, \quad |\psi\rangle = \alpha |^3P_1\rangle + \beta |^1P_1\rangle, \quad (\alpha^2 + \beta^2 = 1)
\]  

(2.1)

where $|\psi\rangle$ is a wave function expanded in terms of $|^{2S+1}L_j\rangle$, and $\alpha$ and $\beta$ are constant coefficients. The mass Hamiltonian is defined using a $2 \times 2$ matrix of the r.h.s. of Eq. (1.2), whose matrix elements are expectation values of $M_0 + H_{SO}^q \vec{L} \cdot \vec{S}_q$ between $|^3P_1\rangle$ and $|^1P_1\rangle$. Although it might be a rather redundant explanation shown below to solve Eq. (2.1), we believe that it clarifies the reason why two mixing angles appear.

Now, we can reexpress Eq. (2.1) in the following eigenvalue equation in which all the quantities are constant:

\[
MP = \lambda P \quad \text{or} \quad M_D P' = \lambda P',
\]

(2.2)

where

\[
M_D \equiv U M U^T, \quad P' = U P, \quad M = \begin{pmatrix} M_0 - \langle H_{SO}^q \rangle & -\sqrt{2} \langle H_{SO}^q \rangle \\ -\sqrt{2} \langle H_{SO}^q \rangle & M_0 \end{pmatrix},
\]

\[
P = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad P' = U P = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}, \quad U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},
\]

(2.3)

with $\alpha'^2 + \beta'^2 = 1$. When one solves an eigenvalue equation $MP = \lambda P$, we obtain

\[
\lambda = M_0 - 2 \langle H_{SO}^q \rangle \quad \text{or} \quad M_0 + \langle H_{SO}^q \rangle; \quad P' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad P' = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(2.4)

\[
P = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \quad \text{or} \quad P = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix},
\]

respectively, and we have only one mixing angle,

\[
\theta = \arctan \left(1/\sqrt{2}\right) = 35.3^\circ.
\]

(2.5)

Inserting Eq. (2.4) into Eq. (1.2), we have eigenfunctions $|\psi\rangle = |^{2S+1}P_1\rangle'$ as

\[
|^3P_1\rangle' \equiv \frac{1}{\sqrt{3}} \left(\sqrt{2} |^3P_1\rangle + |^1P_1\rangle\right), \quad |^1P_1\rangle' \equiv \frac{1}{\sqrt{3}} \left(-|^3P_1\rangle + \sqrt{2} |^1P_1\rangle\right),
\]

(2.6)

where we have named eigenstates $|^3P_1\rangle'$ on the l.h.s. of Eq. (2.6) according to which coefficient of eigenstates $|^2S+1P_1\rangle$ on the r.h.s. is larger; for example, on the r.h.s. of the first equation, a coefficient of $|^3P_1\rangle \left(\sqrt{2}/3\right)$ is larger than that of $|^1P_1\rangle \left(\sqrt{1/3}\right)$; thus, we call this $|^3P_1\rangle'$. 

Another way to solve Eq. (2.2) \( M_D P' = \lambda P' \) is to require vanishing off-diagonal elements of \( M_D \), which gives the following two mixing angles \( \theta \) as in Refs. 1) and 2),

\[
\theta_1 = \arctan \left( \frac{1}{\sqrt{2}} \right) = 35.3^\circ \quad \text{or} \quad \theta_2 = \arctan \left( -\sqrt{2} \right) = -54.7^\circ. \tag{2.7}
\]

To check whether they are independent, we may rewrite the eigenvalue equation \( M_D P' = \lambda P' \) for each angle, which is given as follows. In the case of \( \tan \theta = \tan \theta_1 = \frac{1}{\sqrt{2}} \), the diagonalized mass matrix and eigenvectors are given by

\[
M_{D1} = U_1 M U_1^T = \begin{pmatrix}
M_0 - 2\langle H_{SO}^q \rangle & 0 \\
0 & M_0 + \langle H_{SO}^q \rangle
\end{pmatrix},
P_1 = U_1^T P'_1 = \begin{pmatrix}
\frac{1}{\sqrt{3}} (\sqrt{2} & 1) \\
\frac{1}{\sqrt{3}} (-1 & \sqrt{2})
\end{pmatrix}, \tag{2.8}
\]

respectively, with \( U_1 = U(\theta = \theta_1) \) in Eq. (2.3). In the case of \( \tan \theta = \tan \theta_2 = -\sqrt{2} \), those are given by

\[
M_{D2} = U_2 M U_2^T = \begin{pmatrix}
M_0 + \langle H_{SO}^q \rangle & 0 \\
0 & M_0 - 2\langle H_{SO}^q \rangle
\end{pmatrix},
P_2 = U_2^T P'_2 = \begin{pmatrix}
\frac{1}{\sqrt{3}} (-1 & \sqrt{2}) \\
\frac{1}{\sqrt{3}} (\sqrt{2} & 1)
\end{pmatrix}, \tag{2.9}
\]

respectively, with \( U_2 = U(\theta = \theta_2) \) in Eq. (2.3). Multiplying the following matrix \( U_0 \) on \( M_{D2} \) and \( U_2 \) in Eq. (2.9) as

\[
U_0 = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = \begin{pmatrix}
\cos 90^\circ & \sin 90^\circ \\
-\sin 90^\circ & \cos 90^\circ
\end{pmatrix},
M_{D1} = U_0 M_{D2} U_0^T, \quad P_1' = U_0 P_2', \quad U_1 = U_0 U_2, \tag{2.10}
\]

we can reproduce Eq. (2.8). Hence, Eqs. (2.8) and (2.9) are equivalent to each other, which means that the two mixing angles are also equivalent. Actually, \( \theta_2 = \theta_1 - 90^\circ \) as easily seen from Eq. (2.10). This is consistent with the solution given by Eqs. (2.4) \( - (2.6) \) with the mixing angle \( \tan \theta = 1/\sqrt{2} \) when solving the eigenvalue equation \( MP = \lambda P \).

When one tries to identify which eigenstate corresponds to a lower-mass or higher-mass state as in Refs. 1) and 2), it does not matter which angle one adopts. It depends only on the sign of \( \langle H_{SO}^q \rangle \). By looking at Eqs. (2.8) and (2.9), one finds that if \( \langle H_{SO}^q \rangle > 0 \), then the lower-mass state is identified as \( |^3P_1\rangle' \) and the higher-mass state as \( |^1P_1\rangle' \). On the other hand, if \( \langle H_{SO}^q \rangle < 0 \), the lower-mass state is identified as \( |^1P_1\rangle' \) and the higher-mass state as \( |^3P_1\rangle' \) irrespective of mixing angle.

There is a way to determine which state \( |^3P_1\rangle' \) or \( |^1P_1\rangle' \) corresponds to which heavy quark symmetric state \( |j^P = (1/2)^+ \) or \( (3/2)^+\) \). This is done by expanding heavy quark symmetric states \( |j, j_\ell, j_z \rangle \) in terms of states \( |j, S, j_z \rangle \) with \( \vec{S} = \vec{S}_q + \vec{S}_Q \),

\[
\end{pmatrix}. 
\]
i.e., by calculating the $6 - j$ symbols, which is given in Appendix of Ref. 9) as

\[
\begin{pmatrix}
| j = L, j_\ell = L - 1/2, m \rangle \\
| j = L, j_\ell = L + 1/2, m \rangle
\end{pmatrix} = \frac{1}{\sqrt{2L + 1}} \begin{pmatrix}
| \sqrt{L + 1} & \sqrt{L} \\
-\sqrt{L} & \sqrt{L + 1}
\end{pmatrix} \begin{pmatrix}
| j = L, S = 0, m \rangle \\
| j = L, S = 1, m \rangle
\end{pmatrix}.
\]

(2.11)

By substituting $L = 1$, we obtain Eq. (1.3). Therefore, even discussions given in Refs. 1) and 2) are sufficient to uniquely determine the relation between heavy quark symmetric states $|j^P, j_\ell^P \rangle$ and nonrelativistic states $|^{2S+1}L_j \rangle$ in the heavy quark symmetric limit, which is given by Eq. (1.3) with only one mixing angle (Eq. (1.6)).

§3. Mixing between $^3L_L$ and $^1L_L$ in HQET

In the relativistic potential model that we studied more than ten years ago, we have derived the relativistic equation for a $Q\bar{q}$ bound state in the heavy quark symmetric limit ($m_Q \to \infty$) treating a light quark as relativistic and a heavy quark as static.\(^4\) In that equation, the angular component is completely solved and is given by the eigenfunction $y^j_{jm}$. Because a heavy quark is treated as static in the heavy quark limit, a bound state wave function can be separated into (heavy-quark) energy positive and negative components, and the lowest nontrivial order wave function is naturally given by a positive energy component, which has $2 \times 4$ spinor components. To classify the states in terms of a nonrelativistic $^{2S+1}L_j$ state, only the upper $2 \times 2$ component of the wave function is necessary. The relation between $y^j_{jm}$ and angular momentum eigenfunctions is uniquely determined to be

\[
\begin{pmatrix}
| j = (j+1)^{-}, s \rangle \\
| j = j, s \rangle
\end{pmatrix} = U \begin{pmatrix}
| j = j^+, \bar{\sigma} \cdot \bar{Y}^{(M)}_j \rangle \\
| j = j^-, \bar{\sigma} \cdot \bar{Y}^{(E)}_j \rangle
\end{pmatrix},
\]

\[
U = \frac{1}{\sqrt{2j + 1}} \begin{pmatrix}
\sqrt{j + 1} & \sqrt{j} \\
-\sqrt{j} & \sqrt{j + 1}
\end{pmatrix}.
\]

That is, this is the definition of the eigenfunction $y^j_{jm}$. When $j = 1$, we have the following relation between the eigenstates (l.h.s.) respecting the heavy quark symmetry and the nonrelativistic states (r.h.s.) described in terms of $^3P_1$ and $^1P_1$.

\[
\begin{pmatrix}
| j = 1^{-}, s \rangle \\
| j = 1, s \rangle
\end{pmatrix} = U \begin{pmatrix}
| 1^+, \bar{\sigma} \cdot \bar{Y}^{(M)}_1 \rangle \\
| 1^-, \bar{\sigma} \cdot \bar{Y}^{(E)}_1 \rangle
\end{pmatrix},
\]

\[
U = \frac{1}{\sqrt{3}} \begin{pmatrix}
\sqrt{2} & 1 & 1 \\
-1 & 1 & \sqrt{2}
\end{pmatrix}.
\]

\[
(3.1)
\]

\[
(3.2)
\]

\(^{a)}\) Some papers adopt an erroneous relation between heavy quark symmetric eigenstates and $|^3P_1 \rangle$ and $|^1P_1 \rangle$ as\(^{1)}\)

\[
\begin{pmatrix}
| j^P = 1^+, j_\ell^P = 1^+, s \rangle \\
| j^P = 1^+, j_\ell^P = 3^+, s \rangle
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
| ^3P_1 \rangle \\
| ^1P_1 \rangle
\end{pmatrix},
\]

with two mixing angles,

\[
\tan \theta = \frac{1}{\sqrt{2}} \text{ or } -\sqrt{2}.
\]

Compare these with Eqs. (1.3) and (1.6).
Here, $Y(E), (M), (L)$ are spinor representations of an intrinsic spin $s = 1$ particle with a total angular momentum $j$, i.e., photon’s wave function with a total angular momentum $j$. Here, $Y(E), (M), (L)$ have parities, $(-)^{j+1}$, $(-)^j$ and $(-)^{j+1}$, respectively, and $Y_j^m$ has parity, $(-)^j$, i.e., the same as $Y(M)$. That is, $Y(M)$ is a spinor representation of $^3P_1$ and so is $1_{2\times2} \times Y_j^m$ that of $^1P_1$, while wave functions on the l.h.s., $y_{1/2}^{1/2}$ and $y_{1/2}^{1/2}$ correspond to $j_\ell = 3/2$ and $1/2$ with $j^P = 1^+$, respectively. Here, we have used the relation between $j_\ell$ and $k,$\(^5\)

\[ j_\ell = |k| - \frac{1}{2}, \tag{3.3} \]

Equation (3.2) is our result which is equivalent to Eq. (1.5). The mixing angle is given by $\theta = \arctan(1/\sqrt{2}) = 35.3^\circ$ that is not a “magic number” as called in Refs. 6) and 7), which is derived from the relation between eigenstates with a $k$ quantum number and $2S+1L_j$ states.

Using the first equation of Eq. (3.1), we can write down a general relation between heavy quark symmetric states and nonrelativistic states $^3L_L$ and $^1L_L$ as

\[
\begin{pmatrix}
|y_L^{\ell_m}\rangle \\
|y_L^{(L+1)}\rangle 
\end{pmatrix}
= \frac{1}{\sqrt{2L+1}} \begin{pmatrix}
\sqrt{L+1} & -\sqrt{L} \\
\sqrt{L} & \sqrt{L+1}
\end{pmatrix}
\begin{pmatrix}
|^3L_L\rangle \\
|^1L_L\rangle 
\end{pmatrix}, \quad P = (-1)^{L+1}, \tag{3.4}
\]

which gives Eq. (1.5) when $j = L = 1$. Here, we have used $P = (-1)^{|k|+1} k/|k|$ with $k = L.$\(^5\)

In our model,$^4$ a spin doublet $(0^+, 1^+)$ degenerates and so does another spin doublet $(1^+, 2^+)$ in the heavy quark symmetric limit, which correspond to $j^P = (1/2)^+$ and $(3/2)^+$ multiplets, respectively. Our most recent numerical calculations\(^8\) show that $M((1/2)^+) < M((3/2)^+)$ in the cases of $c\bar{q}$ and $b\bar{q}$, which is equivalent to $M\left(|^3P_1\rangle^\ell\right) < M\left(|^1P_1\rangle^\ell\right)$. These values of $M$ are degenerate eigenvalues of a first-order differential equation and cannot be predicted beforehand by just looking at the equation.

There appear several quantum numbers to distinguish heavy-light mesons, as summarized in Table I. Here, $j$ stands for total angular momentum, $P$ its parity, $k$ a quantum number whose relation with other quantum numbers is given by, for example, Eq. (3.3),\(^5,10\) $j^P_\ell$ the total angular momentum of a light quark with parity $P$, and $2S+1L_j$ a nonrelativistic quantum number describing a total intrinsic spin $S$, an internal angular momentum $L$, and a total angular momentum $j$.

\section*{§4. Breaking of heavy quark symmetry}

Let us briefly discuss what the mixing angle tells us when the heavy quark symmetry is broken. A general mixing angle between $^3L_L$ and $^1L_L$ in HQET is given by $\tan \theta = \sqrt{L/(L+1)}$ as readily seen from Eq. (3.1), and when one takes
into account the breaking of the heavy quark symmetry, it is given by

$$\tan (\theta_1 + \delta \theta) = \sqrt{\frac{L}{L+1}} + \frac{(2L+1)}{L+1} \delta \theta, \quad \tan \theta_1 = \sqrt{\frac{L}{L+1}},$$  

(4.1)

where $\delta \theta = O(1/m_Q)$. Because $\tan \theta_1 = \sqrt{L/(L+1)}$ is the result of the heavy quark symmetry, $\delta \theta$ gives mixing between heavy quark symmetric states with different $j_\ell$ as

$$\begin{pmatrix}
|L^P, j_\ell = L - \frac{1}{2}\rangle' \\
|L^P, j_\ell = L + \frac{1}{2}\rangle'
\end{pmatrix} = \begin{pmatrix} 1 & -\delta \theta \\
\delta \theta & 1 \end{pmatrix} \begin{pmatrix}
|L^P, j_\ell = L - \frac{1}{2}\rangle \\
|L^P, j_\ell = L + \frac{1}{2}\rangle
\end{pmatrix},$$  

(4.2)

where $P = (-1)^{L+1}$ and $k = j = L$ is assumed. See Ref. 12) for discussions on this kind of mixing.

§5. Summary

We conclude from the previous sections’ results that the heavy quark symmetry can uniquely determine the relation between heavy quark symmetric eigenstates and states with $^{2S+1}P_1$ with the mixing angle $\theta = 35.3^\circ$ between $^3P_1$ and $^1P_1$ as shown by Eq. (1.5) as

$$\begin{pmatrix}
|1^+, j_\ell = \frac{1}{2}\rangle \\
|1^+, j_\ell = \frac{3}{2}\rangle
\end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix}
|^3P_1\rangle \\
|^1P_1\rangle
\end{pmatrix} \quad \text{with} \quad \tan \theta = \frac{1}{\sqrt{2}}.$$

In the heavy quark symmetric limit, our relativistic potential model\(^8\) predicts that the lower-mass state is $|^3P_1\rangle'$ and the higher-mass state is $|^1P_1\rangle'$, while the model with the Breit-Fermi-type nonrelativistic potential model\(^1,2\) predicts either $|^3P_1\rangle'$ or $|^1P_1\rangle'$ as the lower mass state depending on the sign of $\langle H^{q}_{SO} \rangle$.

Finally, let us clarify the reason why the mass matrix given by Rosner\(^1\) or Godfrey and Kokoski\(^2\) gives the same eigenstates $|^3P_1\rangle'$ and $|^1P_1\rangle'$ as our model.
Interactions including only the spin-orbit terms can be diagonalized by $y^{k}_{jm}$ because these are eigenfunctions of the operator $\vec{L} \cdot \vec{\sigma}$ as
\[
\vec{L} \cdot \vec{\sigma} y^{k}_{jm} = -(k + 1) y^{k}_{jm}.
\] (5.1)
Because
\[
|y^{\pm 2}_{jm}\rangle \equiv |^1 P_1\rangle, \quad |y^{1}_{jm}\rangle \equiv |^3 P_1\rangle,
\]
the operator $\vec{L} \cdot \vec{\sigma}$ has the following eigenvalues,
\[
\vec{L} \cdot \vec{\sigma} |^3 P_1\rangle = -2 |^3 P_1\rangle, \quad \vec{L} \cdot \vec{\sigma} |^1 P_1\rangle = |^1 P_1\rangle.
\]
Hence, we have
\[
\vec{j} \ell^2 = \left(\vec{L} + \vec{S}_q\right)^2 = \frac{3}{4}, \quad \frac{15}{4}, \quad \text{or} \quad j \ell = \frac{1}{2}, \quad \frac{3}{2},
\] (5.2)
respectively. Here, $L = 1$ and $\vec{S}_q = \vec{\sigma}$/2. This simply is the reason why they have obtained the same eigenstates $|^3 P_1\rangle'$ and $|^1 P_1\rangle'$ as our model. The functions $y^{k}_{jm}$ specified by $k$, $j$ and $m$ quantum numbers are equivalent to the heavy quark eigenstates specified by $j^P$ and $j\ell$ as
\[
|y^{k}_{jm}\rangle = |j^P, j\ell\rangle,
\] (5.3)
with
\[
j = |k| \quad \text{or} \quad |k| - 1, \quad j\ell = |k| - \frac{1}{2}, \quad P = \frac{k}{|k|}(-1)^{|k|+1},
\]
where $k \neq 0$ and we have omitted a quantum number $m$ on the r.h.s. of Eq. (5.3).

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