A brief abstract and introduction to the paper are as follows:

**Abstract.** We show that for a class of operator algebras satisfying a natural condition the $C^*$-envelope of the universal free product of operator algebras $A_i$ is given by the free product of the $C^*$-envelopes of the $A_i$. We apply this theorem to, in special cases, the $C^*$-envelope of the semicrossed products for multivariable dynamics in terms of the single variable semicrossed products of Peters.

An important invariant for non-selfadjoint operator algebras is the $C^*$-envelope. This is a minimal $C^*$-algebra containing the operator algebra in a completely isometric manner. The utility of such a $C^*$-algebra was laid out in [1] and [2] and its existence was proved by Hamana in [10] using injective envelopes.

Unfortunately as with most universal objects identifying the requisite $C^*$-algebra is often difficult, and is often carried out on a case by case basis. There have been several important classes of operator algebras which have received intensive study: the semigroupoid algebras of [13] as special cases of the tensor algebras over $C^*$-correspondences of [15], and the semicrossed products of [18]. Both algebras try to encode some sort of dynamics on an underlying $C^*$-algebra. In both of these cases however the dynamics are constrained significantly by either avoiding interactions between morphisms as in the first algebras, or by constraining the dynamics to a single variable in the case of the second algebra.

Recently a new attempt at multivariable dynamics has been initiated in [8]. There, two possible universal objects related to a multivariable system of dynamics are defined and studied. In particular they let $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ be a tuple of continuous self maps of a locally compact Hausdorff space. They then study universal operator algebras which encode these dynamics. To do this they look at the universal operator algebra generated by $C_0(X)$ and contractions $S_i$ encoding the dynamics of $\tau_i$ via a covariance relation $S_i f(x) = f(\tau_i(x))S_i$ for all $f \in C_0(X)$.

There are two universal operator algebras they study the first they call the semicrossed product, and the second the tensor algebra. The only difference being an additional constraint on the tensor algebra, that the contractions $S_i$ are a family of row contractions. This additional constraint allows a cleaner analysis and more concrete theorems. In particular, building on the groundbreaking work in [12] and [15], the $C^*$-envelopes of the tensor algebras are identified in Theorem 5.1 of [8].

The semicrossed products however are less tractable since they lack this constraint. While some results can be proved in these examples they are often less satisfying. In particular a good understanding of the $C^*$-envelopes is lacking. In this paper we begin to address this issue by recognizing the semicrossed products.
for multivariable dynamics as universal free products of the semicrossed products of \[18\]. We then apply a result for $C^*$-envelopes of free products in the context of certain semicrossed products.

In the first section we remind the reader of the universal objects we study and their universal properties. First are the universal free products which were constructed intrinsically in \[7\]. We then focus on the $C^*$-envelope a now standard object in the non-selfadjoint operator algebra literature. We refer the reader to the books \[17\] and \[6\] for very readable accounts on both of these objects.

In the second section we prove some easy facts about the unique extension property for completely positive maps following \[3\] and \[9\]. We then prove in the third section Theorem \[1\] that allows us to relate the $C^*$-envelope and universal free products. In the final two sections we bring these results together and apply them to certain semicrossed products for multivariable dynamics to calculate the $C^*$-envelope of the semicrossed products.

1. **Free products and $C^*$-envelopes of operator algebras**

We begin by reminding the reader of the universal properties for free products and $C^*$-envelopes.

We assume the following, \(\{A_i\}\) is a collection of operator algebras, sharing a common $C^*$-subalgebra which we will call $D$. We will denote by \(\ast_D A_i\) the universal operator algebra free product with amalgamation over $D$. In particular, we mean the universal operator algebra satisfying the following universal property.

**Universal Property for $\ast_D A_i$:**

1. there exists completely isometric isomorphisms \(\iota_i : A_i \rightarrow \ast_D A_i\)
2. given \(\pi_i : A_i \rightarrow S\) completely contractive homomorphisms into the operator algebra $S$, such that for all $i$ and $j$ we have \(\pi_i(x) = \pi_j(x)\) for all $x \in D$
3. there is a unique completely contractive homomorphism \(\ast \pi_i : \ast_D A_i \rightarrow S\) such that \((\ast \pi_i) \circ \iota_i = \pi_i\).

Notice that when the $A_i$ are $C^*$-algebras then $\ast_D A_i$ is a $C^*$-algebra.

Given an operator algebra $A$ there is a unique $C^*$-algebra, denoted $C^*_e(A)$ satisfying the following universal property.

**Universal Property for $C^*_e(A)$:**

1. there is a completely isometric isomorphism $\iota_A : A \rightarrow C^*_e(A)$
2. the set $\iota_A(A)$ generates $C^*_e(A)$ as a $C^*$-algebra and
3. given a completely isometrically isomorphism $\pi : A \rightarrow C$ where $C$ is a $C^*$-algebra generated by $\pi(A)$, there is a unique onto $*$-homomorphism $\tilde{\pi} : C \rightarrow C^*_e(A)$ such that $\tilde{\pi} \circ \pi = \iota_A$.

2. **The unique extension property and the main theorem**

We refer the reader to \[3\] for a discussion of boundary representations, Silov ideals, and the unique extension property of unital completely positive maps. Here we remind the reader of some definitions and then prove a simple proposition that is helpful in the context of universal free products.

If $A$ is a unital operator algebra and $C$ is a $C^*$-algebra such that there is a completely isometric representation $\iota : A \rightarrow C$, with $C^*(\iota(A)) = C$ we say that $C$...
is generated by $A$. In this case we will usually drop reference to the map $\iota$ and just denote the $C^*$-algebra by $C^*(A)$. Given such a $C^*$-algebra and using the universal property for $C^*_e(A)$ there is a $*$-representation $\sigma : C^*(A) \to C^*_e(A)$. The kernel of this representation is called the Silov ideal.

Now we say a unital $*$-representation $\pi : C^*(A) \to B(\mathcal{H})$ is a unique extension if given $\tau : C^*(A) \to B(\mathcal{H})$ a unital completely positive map such that $\tau|_A = \pi|_A$, then $\tau = \pi$. It is shown in [3] that the Silov ideal will be contained in the kernel of any representation which is a unique extension. In the terminology of Arveson we are defining $\pi$ to be a unique extension if the map $\pi|_A$ has the unique extension property.

We will need a result relating the $C^*$-envelope, and the unique extension property for unital completely positive maps.

**Proposition 1.** If $A$ is a unital operator algebra and $\pi : C^*(A) \to B(\mathcal{H})$ is a faithful representation such that $\pi$ is a unique extension, then $C^*(A) \cong C^*_e(A)$.

**Proof.** Since $\pi$ is a unique extension we know [3] Proposition 2.2 that $\pi$ is a maximal unital completely positive map. Hence, from the proof of [3] Corollary 3.3 we see that the Silov ideal for $A$ is contained in ker $\pi$. But ker $\pi$ is trivial since $\pi$ is faithful and hence $C^*(A) \cong C^*_e(A)$.

3. THE MAIN THEOREM

In this section we present our main theorem relating the $C^*$-envelope of a universal free product to the free product of the $C^*$-envelopes. First let us say that $A$ has the unique extension property if every faithful $*$-representation of $C^*_e(A)$ is a unique extension.

**Theorem 1.** Let $A_i$ be a collection of unital operator algebras, with common unital $C^*$-subalgebra $D$. If $A_i$ has the unique extension property and $C^*_e(A_i) \cap C^*(A_j) = D$ for all $i \neq j$ when viewed as subalgebras of $C^*_e(A_i)$ then $C^*_e(*_DA_i)$ is $*$-isomorphic to $*_DC^*_e(A_i)$.

**Proof.** We let $\pi : *_DC^*_e(A_i) \to B(\mathcal{H})$ be a faithful $*$-representation of $*_DC^*_e(A_i)$. We wish to show that $\pi$ is a unique extension. We then apply Proposition 1 to get the result.

To do this let $\tau : *_DC^*_e(A_i) \to B(\mathcal{H})$ be a unital completely positive map such that $\tau|_{*_DA_i} = \pi|_{*_DA_i}$. We need to show that $\tau = \pi$. First notice that $\pi_i := \pi|_{C^*_e(A_i)}$ is a faithful representation of $C^*_e(A_i)$ and hence $\pi_i$ is a unique extension relative to $A_i$. Next we see that $\tau|_{A_i} = \pi_i$, and hence $\tau_i := \tau|_{C^*_e(A_i)} = \pi_i$. Now $\tau$ is a unital completely positive map, hence when we apply Theorem 3.18 of [17] we see that $\tau$ is a $*$-representation on the $C^*$-algebra generated by $\{C^*_e(A_i)\}$. It follows that $\tau$ is a $*$-representation and hence $\tau = \pi$.

**Remark 1.** If the operator algebras $A_i$ have contractive approximate identities, then we can as in [14] adjoin a unit to $A_i$ in a unique way such that $A_i$ imbeds completely isometrically into the unitization $(A_i)^+$. The free product $*_D A_i$ will then be the subalgebra of $*_D (A_i^+)$ generated by the $A_i$. In a similar manner we have that $C^*_e(*_D (A_i))$ will be the $C^*$-subalgebra of $*_D C^*_e(A_i^+)$ generated by the $A_i$. This $C^*$-algebra, however is completely isometrically isomorphic to $*_D A_i$ and hence the result still carries through for algebras with contractive approximate identities.
This theorem can be seen as an analogue of a similar result for the maximal $C^*$-dilation of the free products, see [4]. The proof is more complicated however, since the $C^*$-envelope has the opposite universal property that one would want.

Of course to apply this theorem we need to know when an operator algebra has the unique extension property. In the paper [5] a seemingly stronger property was shown to hold for the hierarchy of classes of operator algebras:

\[
\{\text{Operator algebras with factorization}\} 
\cap 
\{\text{Logmodular algebras}\} 
\cap 
\{\text{Logrigged algebras}\}.
\]

These classes include: certain nest algebras [19] and the finite maximal subdiagonal algebras of a von Neumann algebra [11].

We now show that the unique extension property for an operator algebra is in fact not weaker than the property used in [5].

**Proposition 2.** Let $A$ be a unital operator algebra then the following are equivalent:

(a) $A$ has the unique extension property.

(b) Every $\ast$-representation of $C_0^\ast(A)$ is a unique extension.

**Proof.** That $b$ implies $a$ is trivial. To see the other direction let $\pi : C_0^\ast(A) \to B(\mathcal{H})$ be a completely contractive representation. Letting $\tau$ be a faithful representation $\tau : C_0^\ast(A) \to B(\mathcal{K})$ then notice that $\tau \circ \pi : C_0^\ast(A) \to B(\mathcal{K} \circ \mathcal{H})$ is faithful and hence is a unique extension. It now follows that $\pi$ is a unique extension. \qed

Another class of algebras with the unique extension property are the Dirichlet algebras. Recall that a Dirichlet algebra is an operator algebra $A$ such that $A + A^*$ is dense in $C_0^\ast(A)$, see [5]. In this case the unique extension property is obvious since for any unital completely positive map: $\pi : A \to B(\mathcal{H})$ we have that $\pi|_A$ uniquely defines $\pi|_A$, and hence it uniquely defines the extension $\tilde{\pi} : C_0^\ast(A) \to B(\mathcal{H})$.

4. **Semicrossed products for multivariable dynamics**

The semicrossed product algebras for multivariable dynamics were defined in [8] as a generalization of the semicrossed products of [15]. Given a locally compact Hausdorff space $X$ and $\tau_i$ a collection of continuous maps from $X$ to $X$ there is a universal nonselfadjoint operator algebra generated by $C_0(X)$ and contractions $S_i$ such that $f(x)S_i = S_i f(\tau_i(x))$ for all $i$. We denote this universal algebra by $C_0(X) \times_\ast \mathbb{F}_n^+$, where $\mathbb{F}_n^+$ represents the free semigroup generated by $n$ copies of $\mathbb{Z}^+$ amalgamated over the identity.

In the case of a single continuous map this is the semicrossed product defined by Peters. We first show that these algebras can be written as universal free products.

**Theorem 2.** Let $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ denote a collection of continuous self maps of $X$, a locally compact Hausdorff space. Then $C_0(X) \times_\ast \mathbb{F}_n^+$ is completely isometrically isomorphic to $*_{C_0(X)}(C_0(X) \times_\tau \mathbb{Z}^+)$. 

**Proof.** This is an application of universal properties. Notice that the algebra $C_0(X) \times_\tau \mathbb{Z}^+$ is generated by $C_0(X)$ and a contraction $S_i$ satisfying $f(x)S_i = S_i f(\tau_i(x))$ for each $i$. It follows that $*_{C_0(X)}(C_0(X) \times_\tau \mathbb{Z}^+)$ is generated by $C_0(X)$ and contractions $S_i$ satisfying the covariance conditions. By universality there exists...
a completely contractive homomorphism \( \pi : C_0(X) \times_\tau \mathbb{F}_n^+ \to *_{C_0(X)}(C_0(X) \times_\tau \mathbb{Z}^+) \) which is onto.

Similarly since \( C_0(\mathcal{X}) \times_\tau \mathbb{F}_n^+ \) is generated by \( C_0(\mathcal{X}) \) and contractive operators \( S_i \) satisfying the covariance conditions, there is for each \( i \) a completely contractive homomorphism \( \pi_i : C_0(\mathcal{X}) \times_\tau \mathbb{Z}^+ \to C_0(\mathcal{X}) \times_\tau \mathbb{F}_n^+ \). Using the universal property of free products it follows that there is a completely contractive representation \( *\pi_i : *_{C_0(X)}(C_0(\mathcal{X}) \times_\tau \mathbb{Z}^+) \to C_0(\mathcal{X}) \times_\tau \mathbb{F}_n^+ \). Which is onto a generating set for \( C_0(X) \times_\tau \mathbb{F}_n^+ \).

It follows by keeping track of \( S_i \) and \( C_0(X) \) under the maps \( \pi \) and \( *\pi_i \) that the two algebras are completely isometrically isomorphic.

Unfortunately it is not immediate that given a semicrossed product \( C_0(X) \times_\alpha \mathbb{Z}^+ \) the algebra has the unique extension property. We will show in the next section that this fact is indeed true.

We focus first on the simple case where \( \alpha \) is surjective. It is well known under these circumstances \( C_0(X) \times_\alpha \mathbb{Z}^+ \) can be imbedded completely isometrically isomorphically into a crossed product algebra \( C_0(Y) \times_{\alpha^\prime} \mathbb{Z} \). In particular the isometry \( S \) lifts to a unitary and hence every element in \( C_0(X) \times_\alpha \mathbb{Z}^+ \) can be written as the norm limit of finite polynomials given by linear combinations of elements of the form \( U^n f_{n,m}(U^m)^* \) where \( U \) is a unitary.

**Proposition 3.** If \( \alpha \) is surjective then the operator algebra \( C_0(X) \times_\alpha \mathbb{Z}^+ \) is Dirichlet.

**Proof.** Looking at \( x = U^n f_{n,m}(U^m)^* \), since \( U \) is a unitary we have two cases. If \( n \geq m \) we have \( x = f_{n,m} \circ \alpha^n U^m - f_{n,m} \in C_0(X) \times_\alpha \mathbb{Z}^+ \) or if \( n < m \) we get \( x = (U^n)^m f_{n,m} \circ \alpha^m \in (C_0(X) \times_\alpha \mathbb{Z}^+)^* \). The result now follows.

5. **Semicrossed products with the unique extension property**

For the case that \( X \) is metrizable but the \( \alpha_i \) are not necessarily onto we will need a result of [13] and a characterization of when this result applies to semicrossed products. To do this we remind the reader of the following definition.

Given Hilbert module \( \mathcal{K}, \mathcal{M}, \) and \( \mathcal{Q} \) over an operator algebra \( \mathcal{A} \) we say \( \mathcal{K} \) is **orthogonally injective** if every contractive short exact sequence of the form

\[
0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{Q} \to 0
\]

has a contractive splitting. We say \( \mathcal{Q} \) is **orthogonally projective** if any contractive short exact sequence as above, has a contractive splitting. Muhly and Solel showed in [16] that a representation \( \pi : \mathcal{A} \to B(\mathcal{H}) \) has the unique extension property if and only if \( \mathcal{H} \) is both orthogonally injective and orthogonally projective.

For the semicrossed products of [8] there is a relatively simple characterization of when a representation \( \pi : \mathcal{A} \to B(\mathcal{H}) \) is orthogonally projective and orthogonally injective. Given a representation \( \pi : C_0(X) \times_\alpha \mathbb{Z}^+ \to B(\mathcal{H}) \) there is an induced representation \( \mathbf{\pi} \) on the algebra of Borel sets on \( X \), denoted \( B(X) \). The image of \( \chi_\alpha(X) \) is a spectral projection denoted \( E(\alpha(X)) \). The representation is said to be full if \( \pi(S)\pi(S)^* = E(\alpha(X)) \). Now we find in Proposition 6.6 of [8] that if \( \pi : C_0(X) \times_\alpha \mathbb{F}_n^+ \) is a completely contractive representation then \( H \) is orthogonally projective and orthogonally injective if and only if \( \pi \) is a full isometric representation. To apply Theorem [1] we will need to verify that a faithful representation of \( C_0^*(X) \times_\alpha \mathbb{Z}^+ \) is a full isometric representation for \( C_0(X) \times_\alpha \mathbb{Z}^+ \).
Theorem 3. Let $X$ be metrizable and let $\pi : C^+_c(C_0(X) \times_\alpha \mathbb{Z}^+) \to B(\mathcal{H})$ be a faithful $*$-representation. Then $\pi|_{C_0(X) \times_\alpha \mathbb{Z}^+}$ is a full isometric representation and hence $C_0(X) \times_\alpha \mathbb{Z}^+$ has the unique extension property.

Proof. That $\pi$ is isometric is trivial. We now apply Theorem 6.5 of \cite{8} to see that $\pi|_{C_0(X) \times_\alpha \mathbb{Z}^+}$ has a dilation $\tilde{\pi} : C_0(X) \times_\alpha \mathbb{Z}^+ \to B(K)$ such that $\tilde{\pi}$ is a full isometric representation. In particular if we denote by $E(\alpha(X))$ the projection in $B(K)$ corresponding to the Borel function $\chi_{\alpha(X)} \in B(X)$, then we know that $\tilde{\pi}(S)\tilde{\pi}(S^*) = E(\alpha(X))$. Now since $\tilde{\pi}$ is isometric we know that there is a contractive $*$-representation $\theta : C^+(\tilde{\pi}(C_0(X) \times_\alpha \mathbb{Z}^+)) \to \pi(C^+_c(C_0(X) \times_\alpha \mathbb{Z}^+))$. This representation will induce a $*$-representation $\overline{\theta}$ of $\mathcal{R}(B(X))$ onto $\mathcal{P}(B(X))$. It will follow by uniqueness of spectral measures that $\mathcal{P}(E(\alpha(X))) = E(\alpha(X))$. Now $\overline{\theta}(E(\alpha(X))) = \theta(\tilde{\pi}(S)\tilde{\pi}(S)^*)$ and hence $E(\alpha(X)) = \pi(S)\pi(S)^*$. It follows that $\pi$ is a full isometric representation.

We now have the simple corollary that applies to all the semicrossed products for multivariable dynamics where $X$ is metrizable.

Corollary 1. Let $X$ be a metrizable topological space and $\alpha_i$ a collection of continuous self maps. Then $C^+_c(C(X) \times_\alpha \mathbb{R}^+)^+ \cong \ast_{C(X)} C^+_c(C(X) \times_\alpha \mathbb{Z}^+)$.

References

[1] W. Arveson, Subalgebras of $C^*$-algebras, Acta Math. 123 (1969) 141-224.
[2] W. Arveson, Subalgebras of $C^*$-algebras II, Acta Math. 128 (1972) 271-308.
[3] W. Arveson, Notes on the unique extension property, Unpublished notes (2003), available from http://www.math.berkeley.edu/~arveson
[4] D. Blecher, Modules over operator algebras, and the maximal $C^*$-dilation, J. Funct. Anal. 169 (1999) 251-288.
[5] D. Blecher and L. Labuschagne, Logmodularity and isometries of operator algebras, Trans. Amer. Math. Soc. 355 (2002) 1621-1646.
[6] D. Blecher and C. LeMerdy, Operator algebras and their modules, Oxford Science Publications, Oxford, 2004.
[7] D. Blecher and V. Paulsen, Explicit construction of universal operator algebras and applications to polynomial factorizations, Proc. Amer. Math Soc. 112 (1991), 839-850.
[8] K. Davidson and E. Katsoulis, Operator algebras for multivariable dynamics, preprint, 2007.
[9] M. Dritschel and S. McCullough, Boundary representations for families of representations of operator algebras and spaces, J. Operator Theory 52 (2005), 159-167.
[10] M. Hamana, Injective envelopes of operator systems, Publ. R. I. M. S. Kyoto Univ. 15 (1979) 773-785.
[11] G. Ji, and K. -S. Saito, Factorization in subdiagonal algebras, J. Funct. Anal. 159 (1998) 191-202.
[12] E. Katsoulis and D. Kribs, Tensor algebras of $C^*$-correspondences and their $C^*$-envelopes J. Funct. Anal. 234 (2006) 226-233.
[13] D. Kribs, and S. Power, Free semigroupoid algebras, J. Ramanujan Math. Soc. 19 (2004) 75-114.
[14] R. Meyer, Adjoining a unit to an operator algebra, J. Operator Theory 46 (2001), 281-288.
[15] P. Muhly and B. Solel, Tensor algebras over $C^*$-correspondences: representations, dilations, and $C^*$-envelopes, J. Funct. Anal. 158 (1998) 389-457.
[16] P. Muhly and B. Solel, An algebraic characterization of boundary representations, in Nonselfadjoint operator algebras, operator theory, and related topics, Oper. Theory Adv. Appl. 104 189-196, Birkhauser, Basel, 1998.
[17] V. Paulsen, Completely bounded maps and operator algebras Cambridge University Press, Cambridge, 2002.
[18] J. Peters, Semicrossed products of $C^*$-algebras, J. Funct. Anal. 59 (1984) 498-534.
[19] D. Pitts, Factorization problems for nests: factorization methods and characterizations of the universal factorization property, *J. Funct. Anal.* **79** (1988) 57-90.

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