Extended least action principle for steady flows under a prescribed flux

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Abstract

The extended principle of minimal action is described in the presence of prescribed source and sink points. Under the assumption of zero net flux, it leads to an optimal Monge-Kantorovich transport problem of metric type. We concentrate on action corresponding to a mechanical Lagrangian. The optimal solution turns out to be a measure supported on a graph composed of geodesic arcs connecting pairs of sources and sinks.

1 Introduction

1.1 The extended action principle

Consider the Lagrangian of a mechanical system:

\[ L(p, x) = \frac{1}{2} |p|^2 - V(x) , \]

where \( x, p \in \mathbb{R}^k \) and \( V \) is a smooth bounded potential function.

The minimal action is a function on \( \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R} \), defined by

\[ J(x_0, x_1, t_0, t_1) := \inf_y \int_{t_0}^{t_1} L(\dot{y}(t), y(t))dt , \]

where the infimum is taken on all orbits \( y : [t_0, t_1] \to \mathbb{R}^k \) satisfying the end conditions \( y(t_0) = x_0, y(t_1) = x_1 \).

The extended minimal action principle (EMAP) was introduced by Benamou and Brenier ([BB,BBG]), in the case \( V \equiv 0 \), and by [W] in the general case. Below we review its definition and relation to the differential point of view of optimal mass transportation - see also ch. 8 of [V].

Let \( \mathcal{M} \) be the set of probability Borel measures on \( \mathbb{R}^k \). Let \( \mathcal{E} \) the set of \( \mathbb{R}^k \)-valued Borel measures defined on \( \mathbb{R}^k \). The extended Lagrangian is defined for any \( (\mu, \vec{E}) \in \mathcal{M} \times \mathcal{E} \), provided \( \vec{E} \) is absolutely continuous with respect to \( \mu \) and the Radon-Nykodym derivative \( d\vec{E}/d\mu \in L^2(d\mu) \), as

\[ \mathcal{L}(\vec{E}, \mu) := \frac{1}{2} \left| \frac{d\vec{E}}{d\mu} \right|^2 - V(x) \in L^1(d\mu). \]

The extended action is defined on all orbits \( \mu = \mu_t \) (res. \( \vec{E} = \vec{E}_t \)) of \( \mathcal{M} \) (res. \( \vec{E} \)) valued functions of the real line, which satisfies the continuity equation

\[ \partial_t \mu_t + \nabla_x \cdot \vec{E}_t = 0 \]

(1.2)
in the sense of distributions. For any pair \( \mu_0, \mu_1 \in \mathcal{M} \) and any \( t_0, t_1 \in \mathbb{R} \) we define the EMAP

\[
J(\mu_0, \mu_1, t_0, t_1) := \min_{(\mu, \overrightarrow{E})} \int_{t_0}^{t_1} \int_{\mathbb{R}^k} \mathcal{L}(\overrightarrow{E}_t, \mu_t)(dx)dt
\]

(1.3)

over all pairs \((\mu_t, \overrightarrow{E}_t)\) satisfying (1.2), subjected to the end condition \( \mu_{t_0} = \mu_0, \mu_{t_1} = \mu_1 \).

EMAP is really an extension of the classical minimal action principle, if the end measures \( \mu_0, \mu_1 \) are replaced by the point measures \( \delta(x_0), \delta(x_1) \). In general, a minimizer \((\overrightarrow{E}_t, \mu_t)\) of the EMAP exists and the Radon-Nykodym derivative \( d\overrightarrow{E}_t/d\mu_t \) is a Lipschitz function on \( \text{Supp}(\mu) \) under some general conditions, so that the flow induced by

\[
\dot{x} = \frac{d\overrightarrow{E}_t}{d\mu_t}\bigg|_{x(t)}
\]

(1.4)

exists and is unique for \( \mu \) a.e. point \((x, t)\). Moreover, equation (1.4) is compatible with the Euler-Lagrange equation associated with the Lagrangian \( L \) for each individual orbit \( x = x(t) \) (see [W]).

The motivation of Brenier and Benamou for the introduction of the extended action principle is an attempt to devise an algorithm for solving the Monge optimal mass transportation for quadratic cost: Given two probability measures \( \mu_0, \mu_1 \) on a common space (say \( \mathbb{R}^k \)), find a mapping \( T : \mathbb{R}^k \to \mathbb{R}^k \) which transports \( \mu_0 \) to \( \mu_1 \) so that the cost of transportation

\[
\min_T \int_{\mathbb{R}^k} |T(x) - x|^2 d\mu_0.
\]

(1.5)

Recall that a mapping \( T \) transports \( \mu_0 \) to \( \mu_1 \) \((T_#\mu_0 = \mu_1)\) if and only if \( \mu_0 \left(T^{-1}(A)\right) = \mu_1(A) \) for any (Borel) measurable set \( A \). The existence of a unique minimizer of (1.5) is known if \( \mu_0 \) is absolutely continuous with respect to Lebesgue measure and both \( \mu_0, \mu_1 \) have finite second moment (see [B], and later extension in [GM]). It was the fundamental observation of Benamou and Brenier that (in the case \( V \equiv 0 \)) the flow (1.4) associated with the solution of the EMAP induces the optimal mapping \( T \) of (1.5) under some regularity assumptions. The extension of this result to the case \( V \neq 0 \), introduced in [W], relates the flow (1.4) to the optimal solution of the mass transport with respect to the cost function \( c = c(x, y) \),

\[
\min_T \int_{\mathbb{R}^k} c(x, T(x))d\mu_0
\]

(1.6)

where \( c(x, y) = J(x, y, t_1, t_2) \) given by the classical action (1.1).

1.2 Sources and sinks

In this paper we consider the extended action principle under a set of sources and sinks. Let \( D_i \subset \mathbb{R}^k, i = 1, \ldots, n \) be a disjoint set of compact domains whose boundaries \( S_i = \partial D_i \) are smooth surfaces. On each such surface we assign an integrable function \( \lambda_i : S_i \to \mathbb{R} \) so that

\[
\sum_{i=1}^{N} \int_{S_i} \lambda_i ds = 0.
\]

(1.7)
The extended action principle under the presence of prescribed fluxes \( \lambda_i \) across the surfaces \( S_i \) is defined as (1.3), where the set of pairs \( (\mu, \vec{E}) \) is defined on \( (\mathbb{R}^k - \bigcup_i D_i) \times [t_0, t_1] \) and (1.2) replaced by

\[
\partial_t \mu + \nabla_x \cdot \vec{E} = 0 \quad \text{on} \quad (\mathbb{R}^k - \bigcup_i D_i) \times [t_0, t_1] , \quad \vec{E}_t \cdot \vec{n}_i = \lambda_i \quad \forall (s, t) \in S_i \times [t_0, t_1] ,
\]

where \( \vec{n}_i(s) \) are the outward normal to \( S_i \) at point \( s \).

We shall concentrate on stationary extended minimal action: \( \mu_t \) (and, correspondingly, \( \vec{E}_t \)), are independent of \( t \).

Under stationarity condition, the extended principle of minimal action takes the form

\[
\min_{(\mu, \vec{E})} \left[ \frac{1}{2} \int_{\mathbb{R}^k - \bigcup_i D_i} \left| \frac{d\vec{E}}{d\mu} \right|^2 d\mu - \int_{\mathbb{R}^k - \bigcup_i D_i} V d\mu \right]
\]

where \( \mu \in \mathcal{M} \) is supported on \( \mathbb{R}^k - \bigcup D_i \) and \( \vec{E} \) is a vector valued measure, absolutely continuous with respect to \( \mu \), which satisfies in addition

\[
\nabla \cdot \vec{E} = 0 \quad \text{on} \quad \mathbb{R}^k - \bigcup D_i ; \quad \vec{E} \cdot \vec{n}_i = \lambda_i \quad \text{on} \ S_i , \quad 1 \leq i \leq n .
\]

**Points sources and sinks:** Letting the domains \( D_j \) shrink to points \( x_j \in \mathbb{R}^k \), we replace the functions \( \lambda_i \) on \( S_i \) by constants \( \lambda_i \in \mathbb{R} \) so that

\[
\sum_{i=1}^{n} \lambda_i = 0.
\]

The admissibility condition (1.10) for the stationary minimal action (1.9) can now be casted into the single condition:

\[
\nabla \cdot \vec{E} = \sum_{i=1}^{n} \lambda_i \delta_{x_i} \quad \text{on} \quad \mathbb{R}^k .
\]

The problem is now reduced to finding the minimizer \( \mu \in \mathcal{M} \) of

\[
\mathcal{J} := \min_{(\mu, \vec{E})} \left[ \frac{1}{2} \int_{\mathbb{R}^k} \left| \frac{d\vec{E}}{d\mu} \right|^2 d\mu - \int_{\mathbb{R}^k} V d\mu \right]
\]

where \( \vec{E} \) is absolutely continuous with respect to \( \mu \) and satisfies (1.11).

### 1.3 Main results

In this paper we concentrate on point sources and sinks. The set of fluxes \( \lambda_1, \ldots, \lambda_n \) is divided into two sets: sources \( (\lambda_i > 0) \) and sinks \( (\lambda_i < 0) \).

Let

\[
I_+ := \{i \mid \lambda_i > 0 \} ; \quad I_- := \{i \mid \lambda_i < 0 \}.
\]

3
The metric Monge transport plant between a pair of probability measures \( \mu_0, \mu_1 \) is the special case of (1.6) where \( c(x, y) \) is a metric, say \( c(x, y) = |x - y| \). It turns out that the solution of the stationary EMAP is related to the metric Monge transport where \( \mu_0, \mu_1 \) are replaced by the point measures

\[
\lambda \equiv -\sum_{i \in I_+} \lambda_i \delta(x_i) \quad -\lambda \equiv -\sum_{j \in I_-} \lambda_j \delta(x_j) \quad \text{for} \quad I_+ \cup I_- = \{1, \ldots, n\}.
\]

where \( |\lambda| := \sum_{i \in I_+} \lambda_i \equiv -\sum_{j \in I_-} \lambda_j \).

In general, however, there are no mappings \( T \) which transport a point measure \( \mu_0 \) to another measure, so there is no sense to define the Monge problem (1.6) for this case. However, the Monge problem can be relaxed to an optimization problem on the set of 2-point probability distributions \( Q = Q(dx, dy) \). This is the celebrated Kantorovich relaxation of the Monge problem [K].

We shall first describe the special case \( V \equiv 0 \), corresponding to the Euclidean metric \( c(x, y) = |x - y| \). In the case of point measures (1.14), the Kantorovich relaxation of the minimal cost (1.6) takes the form of the 1–Wasserstein metric (see, e.g. [R]). It is defined by

\[
W^{(1)}(\lambda) = \sum_{i \in I_+} \sum_{j \in I_-} A_{ij} |x_i - x_j| := \min_{A \in Q_{\lambda}} \sum_{i \in I_+} \sum_{j \in I_-} A_{ij} |x_i - x_j|. \quad \text{(1.15)}
\]

where

\[
Q_{\lambda} := \left\{ A_{ij} \geq 0 ; \quad \sum_{j \in I_-} A_{ij} = \lambda_i \quad \text{if} \quad i \in I_+ ; \quad \sum_{i \in I_+} A_{ij} = \lambda_j \quad \text{if} \quad j \in I_- \right\} \quad \text{for} \quad I_+ \cup I_- = \{1, \ldots, n\}.
\]

The first result states that the minimal action (1.12) is given by

\[
J = \frac{1}{2} \left[ W^{(1)}(\lambda) \right]^2. \quad \text{(1.16)}
\]

Let \( G_0 \) be the bi-graph composed of the set of vertices \( x_i, i = 1, \ldots n \), and edges \( L(i, j) \) are segments connecting \( i \in I_+ \) to \( j \in I_- \).

The second result states that the action minimizer of (1.11, 1.12) is supported in the edges of the bi-graph \( G_0 \) for which \( A_{ij} > 0 \). On each such edge, \( \mu \) is a uniform measure and

\[
\mu(L(i, j)) = |\lambda|^{-1} A_{ij}. \quad \text{(1.17)}
\]

We now consider the case \( V \neq 0 \). We assume that

\[
\lim_{|x| \to \infty} V(x) = 0. \quad \text{(1.18)}
\]

Let

\[
\bar{V} = \sup_{R^n} V. \quad \text{(1.19)}
\]
For any $E > V$ we consider the Riemannian metric associated with the Maupertuis’ action principle (see [Ar]):

$$d\sigma_E = \sqrt{E - V} \, ds,$$

(1.19)

here $ds$ is the Euclidean metric.

The geodesic distance associated with this metric is denoted by $D_E(x, y)$. We recall that a geodesic arc connecting two points $x, y$ coincides with an orbit of the mechanical system

$$\ddot{x} + \nabla_x V(x(t)) = 0 \quad x(0) = x_0, \quad x(T) = x_1,$$

corresponding to the energy level $|\dot{x}|^2/2 + V(x) = E$. Here $T = T_{x_0, x_1}(E)$ is the time of flight from $x_0$ to $x_1$ (which is, of course, a function of $E$ as well).

The bi-graph $G_E$ is defined, analogously to $G_0$, as the collection of vertices $x_i, 1 \leq i \leq n$, and all edges composed of geodesic arcs (with respect to the metric $d\sigma_E$) connecting $x_i, i \in I_+$, to $x_j, j \in I_-$. The 1–Wasserstein metric associated with this distance is given, analogously to $W(1)$, as

$$W_E^{(1)}(\lambda) = \sum_{i \in I_+} \sum_{j \in I_-} A_{i,j}^E D_E(x_i, x_j) := \min_{A \in \mathcal{Q}} \sum_{i \in I_+} \sum_{j \in I_-} A_{i,j} D_E(x_i, x_j).$$

(1.20)

Then the minimal action (1.12) is

$$\mathcal{J} = \max_{E \geq V} \left[ \sqrt{2W_E^{(1)}(\lambda)} - E \right].$$

(1.21)

There exists a minimizer $\mu \in \mathcal{M}$ realizing this action which satisfies the following:

**Case a:** $E_0 > V$ is the maximizer of the RHS of (1.21). Then there exists an action minimizer supported on the bi-graph $G_E$ so that $\mu(L_E(i, j)) > 0$ only if $A_{i,j}^E > 0$. To wit:

$$\mu(L_{E_0}(i, j)) = 2^{-1/2} A_{i,j}^{E_0} T_{i,j}(E_0),$$

(1.22)

where $T_{i,j}(E_0) = T_{x_i, x_j}(E_0)$ as defined below (1.19).

**Case b:** If $E_0 = V$ is the maximizer of (1.21) then the following holds: Let $\mu_0$ be the measure supported on $G_V$ subject to (1.22). Then there exists $\beta \in [0, 1]$ so that

$$2^{-1/2} \sum_{i \in I_+} \sum_{j \in I_-} A_{i,j}^{E_0} T_{i,j}(V) = 1 - \beta \leq 1.$$

Let $x_0$ be a maximizer of $V$, that is, $\nabla V = V(x_0)$. Then

$$\mu = \mu_0 + \beta \delta_{x_0}$$

is an action minimizer.

In both cases (a) and (b), the following claim is valid:

**Time/Flux duality:** The expectation of the inverse flow time, $E_\mu(T^{-1})$, is proportional to the total in(out) flux $|\lambda| := \sum_{i \in I_+} \lambda_i = -\sum_{j \in I_-} \lambda_j$:

$$E_\mu(T^{-1}) := \sum_{i \in I_+} \sum_{j \in I_-} \mu(L_{E_0}(i, j)) T_{i,j}^{-1}(E_0) = \frac{|\lambda|}{\sqrt{2}}.$$

(1.23)
1.4 Outline

In section 2 we derive the weak formulation of the stationary EMAP, which leads to a dual problem:

\[ \inf_{\mu \in \mathcal{M}} \sup_{\phi \in C^1} \left[ -\int_{\mathbb{R}^k - \bigcup D_i} \left( V + \frac{1}{2} |\nabla \phi|^2 \right) \mu(dx) + \sum_i \oint_{S_i} \phi(s) \lambda_i(s) ds \right] . \]

In section 3 we concentrate in the case of point sources and sinks, where \( D_i \) shrink to points \( x_i \). It contains some definitions and a preliminary lemma on the Wasserstein metric and its dual representation.

Section 4 is the most technical part of this paper. It contains a sequence of auxiliary lemmas, which are needed to the proof of the main result, as described in section 1.3. The proof itself is given at the end of this section. It is given for the general case \( V \neq 0 \), since the case \( V = 0 \) follows easily from the general one. Finally, the short section 5 summarizes the results of this paper.

2 Weak formulation

2.1 Notations

Given \( n \) disjoint compact sets \( D_i \subset \mathbb{R}^k \) and \( S_i \equiv \partial D_i \) smooth surfaces, set

\[ \Omega := \mathbb{R}^k - \bigcup_i D_i . \]

Let \( \mathcal{M} \) stands for the set of probability Borel measures on \( \Omega \). Let

\[ \mathcal{M} := \left\{ \nu ; \nu(dx dp) \text{ Borel Probability measure on } \Omega \times \mathbb{R}^k , \int_{\Omega \times \mathbb{R}^k} |p|^2 \nu(dx dp) < \infty . \right\} \]

Let \( \mathcal{P} : \Omega \times \mathbb{R}^k \rightarrow \Omega \) be the projection \( \mathcal{P}(x, p) = x \). For \( \nu \in \mathcal{M} \), \( \mathcal{P} \# \nu \) is the push-forward to the marginal measure \( \mu = \mu(dx) \in \mathcal{M} \):

\[ \int_{\Omega \times \mathbb{R}^k} \psi(x) \nu(dx dp) = \int_{\Omega} \psi(x) \mathcal{P} \# \nu(dx) ; \forall \psi \in C(\Omega \times \mathbb{R}^k) . \]

Let also

\[ \mathcal{E}_\nu(dx) := \int_{\mathbb{R}^k} p \nu(dp dx) . \]

For \( \mu \in \mathcal{M} \) let

\[ \mathcal{M}_\mu := \{ \nu \in \mathcal{M} ; \mathcal{P} \# \nu = \mu \} . \]

\[ \Lambda := \left\{ \nu \in \mathcal{M} ; \int_{\Omega} \mathcal{E}_\nu(dx) \cdot \nabla \phi(x) - \sum_i \oint_{S_i} \phi(s) \lambda_i(s) ds \mu(dx) = 0 \forall \phi \in C^1(\mathbb{R}^k) \right\} . \]

\[ \Lambda_\mu := \Lambda \cap \mathcal{M}_\mu . \]
The Lagrangian is now defined as a function on \( \mathcal{M} \) via:

\[
L(\nu) = \frac{1}{2} \int_{\Omega \times \mathbb{R}^k} |p|^2 \nu(dxpdp) - \int_{\Omega} V(x) \nu #(\nu)(dx) .
\] (2.1)

### 2.2 Weak form of the minimal action

We now describe the weak form of the stationary action principle:

\[ \min_{\nu \in \Lambda} L(\nu) . \] (P)

Equivalently, if

\[ \mathcal{L}(\mu) := \inf_{\nu \in \Lambda_\mu} L(\nu) \]

then (P) is equivalent to

\[ \inf_{\mu \in \mathcal{M}} \mathcal{L}(\mu) . \] (P*)

### 2.3 Dual representation

Let now define \( \mathcal{L} : \mathcal{M} \times C^\infty(\Omega) \rightarrow \mathbb{R} \cup \{ \infty \} \) by

\[
\mathcal{L}(\nu, \phi) := L(\nu) - \int_{\Omega} \widetilde{E}_\nu(dx) \cdot \nabla \phi(x) + \sum_i \int_{S_i} \phi(s) \lambda_i(s) dpds .
\]

Next, we use an appropriate version of the minmax principle to obtain the dual formulation:

**Proposition 2.1.** For any \( \mu \in \overline{\mathcal{M}} \),

\[
\mathcal{L}(\mu) = \sup_{\phi \in C^1(\Omega)} \inf_{\nu \in \mathcal{M}_\mu} \mathcal{L}(\nu, \phi) .
\]

**Proof.** First, note that

\[
\mathcal{L}(\mu) = \inf_{\nu \in \mathcal{M}_\mu} \sup_{\phi \in C^1(\Omega)} \mathcal{L}(\nu, \phi) .
\]

Indeed, by definition, \( \sup_{\phi \in C^1(\Omega)} \mathcal{L}(\nu, \phi) = \infty \) if \( \nu \not\in \Lambda \), while \( \mathcal{L}(\nu, \phi) = L(\nu) \) otherwise. Next, note that \( \mathcal{L} \) is an affine function on each of the domains \( \mathcal{M}_\mu \) and \( C^1(\overline{\Omega}) \), separately. As such, it is a convex functional on \( \mathcal{M}_\mu \) and concave on \( C^1(\overline{\Omega}) \). In addition, \( \mathcal{M}_\mu \) is a compact set with respect to the weak topology (in which \( \mathcal{L} \) is continuous). The Minmax theorem, then, can be applied (see, e.g. [Ro]), and the claim follows. \( \square \)

Next we evaluate

\[
\inf_{\nu \in \mathcal{M}_\mu} \mathcal{L}(\nu, \phi) = \inf_{\nu \in \mathcal{M}_\mu} \int_{\Omega \times \mathbb{R}^k} \left[ \frac{1}{2} |p|^2 - p \cdot \nabla \phi - V \right] \nu(dxpdp) + \sum_i \int_{S_i} \phi(s) \lambda_i(s) dpds .
\]

\[
= \inf_{\nu \in \mathcal{M}_\mu} \int_{\Omega \times \mathbb{R}^k} \left[ \frac{1}{2} |p - \nabla \phi|^2 - \frac{1}{2} |\nabla \phi|^2 - V \right] \nu(dxpdp) + \sum_i \int_{S_i} \phi(s) \lambda_i(s) dpds .
\]
\[
- \int_{\mathbb{R}^k} \left[ \frac{1}{2} |\nabla \phi|^2 + V \right] \mu(dx) + \sum_i \oint_{S_i} \phi(s) \lambda_i(s) dp ds .
\]

(2.2)

where the infimum is obtained at \( \nu(dx dp) = \mu(dx) \delta(p - \nabla \phi) \).

Let us now define, for any \( \mu \in \overline{\Omega} \) and \( \phi \in C^1(\overline{\Omega}) \):

\[
J_\mu(\phi) := - \int_{\Omega} \left[ V + \frac{1}{2} |\nabla \phi|^2 \right] \mu(dx) + \sum_i \oint_{S_i} \phi(s) \lambda_i(s) ds .
\]

By Proposition 2.1 we get

Corollary 2.1. \( \overline{\text{L}}(\mu) = \sup_{\phi \in C^1(\overline{\Omega})} J_\mu(\phi) \)

Proposition 2.2. If \( \overline{\Omega} \) is compact, then there exists \( \mu \in \overline{\mathcal{M}} \) which solves problem \( P^* \).

Proof. If \( \overline{\Omega} \) is compact, so is the set \( \overline{\mathcal{M}} \) with respect to the weak topology, as the set of Probability Borel measures on a compact set. In addition, \( \overline{\text{L}} \) is lower semi continuous, since it is a supremum of the affine functionals \( J_{(\cdot)}(\phi) \) by Corollary 2.1. Hence, a minimizing sequence of \( \overline{\text{L}} \) in \( \overline{\mathcal{M}} \) contains a subsequence which converges to a minimum of \( \overline{\text{L}} \). \qed

3 Point sources and sinks

Assume now that the surfaces \( S_i \) degenerate to points \( x_i \in \mathbb{R}^k \). The fluxes functions \( \lambda_i \) defined on \( S_i \) degenerate, then, to constants \( \lambda_i \in \mathbb{R} \). The total flux condition takes the form

\[
\sum_{i=1}^n \lambda_i = 0 .
\]

(3.1)

In this case, the functional \( J_\mu \) is written as

\[
J_\mu(\phi) := - \int_{\mathbb{R}^k} \left[ V + \frac{1}{2} |\nabla \phi|^2 \right] \mu(dx) + \sum_i \lambda_i \phi(x_i) .
\]

Recall that \( D_E \) be the distance metric induced by the Riemannian metric \( d\sigma_E \) (1.19), that is, for \( x, x' \in \mathbb{R}^k \):

\[
D_E(x, x') := \inf_q \int_0^S \sqrt{E - V(q(s))} ds
\]

(3.2)

where the infimum above is taken over all orbits \( q(s) \) in Euclidian arc-length parameterization connecting \( x \) to \( x' \). A minimizer orbit \( q \) of (3.2) is called an \( E \)- geodesic for the metric \( D_E(x, x') \).

Given \( E > \overline{V} \) (1.18), \( i, j \in (1, \ldots, n) \), let \( q_{i,j}(s) \) be the \( E \)-geodesic of \( D_E(x_i, x_j) \), parameterized by the Euclidian arc-length. So, \( q_{i,j}(0) = x_i, q_{i,j}(S_{i,j}) = x_j \) where \( S_{i,j} \) is the Euclidian length of the \( D_E(x_i, x_j) \) geodesic curve. Let

\[
T_{i,j}(E) = \int_0^{S_{i,j}} (E - V(q_{i,j}(s)))^{-1/2} ds .
\]

(3.3)

Note that \( T_{i,j}(E) \) is the time interval of existence of the orbit which connects \( x_i \) to \( x_j \) at energy \( E \). By differentiation of \( D_E(x_i, x_j) \) with respect to \( E \) we obtain
Lemma 3.1. If $E > \bar{V}$ then 
\[
\frac{d}{dE} D_E(x_i, x_j) = \frac{1}{2} T_{i,j}(E) < \infty .
\]

For each $E > \bar{V}$ define
\[
\rho_{i,j}(s) := \frac{1}{T_{i,j}(E) \sqrt{E - V(q_{i,j}(s))}} ; 0 \leq s \leq S_{i,j} .
\]  \tag{3.4}

Then $\rho_{i,j}$ is a probability density on the interval $[0, S_{i,j}]$. Then
\[
\mu_{i,j} := q_{i,j} \# (\rho_{i,j}(s) ds)
\]  \tag{3.5}

is a push forward of this probability density to a probability measure $\mu_{i,j}$ supported on the $E$–geodesic arc connecting $x_i$ and $x_j$ in $\mathbb{R}^k$. It is defined, for any test function $\phi \in C(\mathbb{R}^k)$, via
\[
\int_{\mathbb{R}^k} \phi(x) d\mu_{i,j} = \int_0^{S_{i,j}} \phi(q_{i,j}(s)) \rho_{i,j}(s) ds .
\]

We now recall the definition of the 1–Wasserstein metric (1.20). By duality formulation of the Wasserstein metric we also obtain (see \cite{R}):
\[
W_1^E (\lambda) := \max_{\phi \in \mathbb{R}^n} \left\{ \sum_{i,j=1}^{n} \lambda_i \phi_i ; \frac{|\phi_j - \phi_k|}{D_E(x_j, x_k)} \leq 1 , \forall j, k \in \{1, \ldots n\} \right\} . \tag{3.6}
\]

Below we collect some useful results:

Lemma 3.2. $A \in Q^{-\lambda} (1.16)$ is an optimal solution of (1.20) iff there exists a minimizer $\overrightarrow{\phi}$ of (3.6) so that $A_{i,j} = 0$ for any pair $i, j$ satisfying $|\phi_i - \phi_j| < D_E(x_i, x_j)$.

Proof. For any $\overrightarrow{\phi}$ satisfying $\max_{i \neq j} |\phi_i - \phi_j| / D_E(x_i, x_j) \leq 1$ and $A \in Q^{-\lambda}$ we obtain 
\[
\sum_{i \in I_+} \sum_{j \in I_+} A_{i,j} D_E(x_i, x_j) \geq \sum_{i \in I_+} \sum_{j \in I_+} A_{i,j} (\phi_i - \phi_j)
\]
\[
= \sum_{i \in I_+} \left( \sum_{j \in I_+} A_{i,j} \phi_i \right) - \sum_{j \in I_-} \left( \sum_{i \in I_+} A_{i,j} \phi_j \right) = \sum_{i \in I_+} \lambda_i \phi_i - \sum_{j \in I_-} \lambda_j \phi_j = \sum_{i=1}^{n} \lambda_i \phi_i ,
\]

where we used $A_{i,j} \equiv 0$ if either $i, j \in I_+$ or $i, j \in I_-$. Since equality holds for the optimal $A$ and $\overrightarrow{\phi}$, it follows that $A_{i,j} > 0$ implies $\phi_i - \phi_j = D_E(x_i, x_j)$.

Recall the definition of the subgradient of a function $h$ at $x \in \mathbb{R}^k$:
\[
\partial_x h = \left\{ p \in \mathbb{R}^k ; h(x') - h(x) \geq p \cdot (x' - x) \forall x' \in \mathbb{R}^k \right\} .
\]

The following result can be found in, e.g., \cite{HL}:
Lemma 3.3.

i) If $h$ is a convex function defined on $\mathbb{R}^k$, then $\partial_x h \neq \emptyset$ for all $x \in \mathbb{R}^k$.

ii) If $h$ is convex, then $p \in \partial_x h$ if and only if $x \in \partial_p h^*$ where $h^*$ is the Legendre transform of $h$.

iii) If $h$ is convex, then for any $x, p \in \mathbb{R}^k$, the inequality $h(x) + h^*(p) \geq x \cdot p$ holds with equality if and only if $p \in \partial_x h$ if and only if $x \in \partial_p h^*$.

4 Proof of the main result

We shall prove the main result of section 1.3 for the general case $V \not\equiv 0$ satisfying (1.17).

The special case $V \equiv 0$ follows easily.

Let $\phi := (\phi_1, \ldots, \phi_n) \in \mathbb{R}^n$ and

$$B_{\phi} := \left\{ \phi \in C^1(\mathbb{R}^k) : \phi(x_i) = \phi_i \right\} .$$

For $\mu \in \overline{M}$ set

$$H(\phi ; \mu) := \inf_{\phi \in B_{\phi}} \left[ \frac{1}{2} \int_{\mathbb{R}^k} |\nabla \phi|^2 d\mu + \int_{\mathbb{R}^k} Vd\mu \right] ,$$

and

$$\overline{H}(\phi) := \sup_{\mu \in \overline{M}} H(\phi ; \mu) .$$

Remark: Note that $H(\phi ; \mu)$ is not necessarily a convex function on $\mathbb{R}^k$ for each $\mu \in \overline{M}$. However, $\overline{H}$ is convex, as we shall see later on in Corollary 4.4.

Lemma 4.1. For any $\phi \in \mathbb{R}^n$,

$$\overline{H}(\phi) = \min_{E \geq V} \left\{ E, \max_{i \neq j} \frac{|\phi_i - \phi_j|}{D_E(x_i, x_j)} \leq \sqrt{2} \right\} . \tag{4.1}$$

Proof. Let $E > V$. Assume there exists $\phi \in C^1(\mathbb{R}^k)$ so that

$$|\nabla \phi| \leq \sqrt{2} \sqrt{E - V} \text{ on } \mathbb{R}^k \text{ and } \phi(x_i) = \phi_i , \ 1 \leq i \leq n . \tag{4.2}$$

Now, if such a function satisfies (4.2), then for any $i \neq j$ and any geodesic arc $y(t), 0 \leq t \leq 1$, connecting $x_i$ to $x_j$ we obtain:

$$|\phi(x_i) - \phi(x_j)| \leq \int_0^1 |\nabla_{x(t)} \phi| \cdot |\dot{x}(t)| dt \leq \sqrt{2} D_E(x_i, x_j) .$$

In addition, for any $\mu \in M$

$$H(\phi ; \mu) \leq \frac{1}{2} \int_{\mathbb{R}^k} |\nabla \phi|^2 d\mu + \int_{\mathbb{R}^k} Vd\mu \leq E .$$
We now show the existence of \( \phi \) satisfying (4.4), provided

\[
\max_{i \neq j} \frac{|\phi_i - \phi_j|}{D_E(x_i, x_j)} < \sqrt{2}
\]

(4.3)

Given \( \varepsilon > 0 \), let \( D_E^\varepsilon(x, x_i) := \max \{ D_E(x, x_i), \varepsilon \} \). Set

\[
\phi^\varepsilon(x) = \min_{1 \leq i \leq n} \left\{ \sqrt{2}D_E^\varepsilon(x, x_i) + \phi_i - \varepsilon \right\}.
\]

Note that \( D_E^\varepsilon(x, x_i) \) is a Lipschitz function on \( \mathbb{R}^k \) (recall \( E > V \)) and satisfies \(|\nabla D_E(\cdot, x_i)| \leq \sqrt{E - V}\) for almost any \( x \in \mathbb{R}^k \). In addition, the condition (4.3) implies that \( \phi^\varepsilon(x) = \phi_j \) if \( |x - x_j| < \varepsilon \) for any \( j \in \{1, \ldots, n\} \), if \( \varepsilon \) is sufficiently small. Now, let us take the smoothing kernel

\[
\eta \in C^\infty(\mathbb{R}^k) ; \quad \eta \geq 0, \quad \int_{\mathbb{R}^k} \eta = 1, \quad \eta(x) = 0 \text{ if } |x| > 1.
\]

(4.4)

Let \( \eta_\delta(x) = \delta^{-k} \eta(x/\delta) \). Let

\[
\phi(x) := \eta_\delta \ast \phi^\varepsilon.
\]

Evidently, the inequality \(|\nabla \phi| \leq \sqrt{2} \sqrt{E - V}\) holds now everywhere for \( \phi \). If \( \delta < \varepsilon \) then, by the last condition in (4.3), also the condition \( \phi^\varepsilon(x_i) = \phi_i \) are preserved for \( \phi \). So

\[
\Pi(\phi) \leq \inf_{E > V} \left\{ E, \max_{i \neq j} \frac{|\phi_i - \phi_j|}{D_E(x_i, x_j)} \leq \sqrt{2} \right\}
\]

(4.5)

is verified.

To prove the opposite inequality we construct a probability measure as follows. Let \( \overline{E} \) defined by

\[
\sqrt{2}D_{\overline{E}}(x_m, x_k) = |\phi_k - \phi_m| ; \quad \sqrt{2}D_{\overline{E}}(x_i, x_j) \geq |\phi_i - \phi_j| \forall i \neq j.
\]

(4.6)

Assume \( \overline{E} > V \). Let \( \mu = \mu_{k,m} \) as defined in (3.3). We now calculate the minimizer \( \phi \in B_{\overline{E}} \) of \( \int_{\mathbb{R}^k} |\nabla \phi|^2 d\mu_{k,m} \). Note that for any \( \phi \in C^1(\mathbb{R}^k) \) we may define \( \eta \in C^1 [0, S_{k,m}] \) via \( \eta(s) = \phi(q_{k,m}(s)) \), where \( q_{k,m} \) is the parameterization of the geodesic arc connecting \( x_m \) to \( x_k \) (see paragraph preceding 3.3). Then

\[
\int_{\mathbb{R}^k} |\nabla \phi|^2 d\mu_{k,m} \geq \int_0^{S_{k,m}} |\dot{\eta}(s)|^2 \rho_{k,m}(s) ds
\]

(4.7)

where \( \rho_{k,m} \) as defined in (3.4). Now, the minimizer on the RHS of (4.7) subject to the condition \( \eta(0) = \phi_m \), \( \eta(S_{k,m}) = \phi_k \) is attained for

\[
\eta(s) = \phi_m + \frac{\phi_k - \phi_m}{\int_0^s \rho_{k,m}^{-1} \int_0^t \rho_{k,m}(u) \, du} \int_0^s \rho_{k,m}(t) \, dt.
\]

By (4.7) it follows that

\[
\frac{1}{2} \int_{\mathbb{R}^k} |\nabla \phi|^2 d\mu_{k,m} \geq \frac{1}{2} \int_0^{S_{k,m}} \frac{|\phi_k - \phi_m|^2}{\int_0^s \rho_{k,m}^{-1}}.
\]

(4.8)
must holds for any \( \phi \in B_{\overrightarrow{\sigma}} \).

Next, by (4.6) and (3.4):

\[
\frac{|\phi_m - \phi_k|^2}{2\rho^2_{k,m}} + V = \overline{E} .
\] (4.9)

Multiply (4.9) by \( \rho_{k,m} \), integrate from 0 to \( S_{k,m} \) and use (4.8) to obtain

\[
\int_{\mathbb{R}} \left[ \frac{|
abla \phi|^2}{2} + V \right] d\mu_{k,m} \geq E .
\]

for any \( \phi \in B_{\overrightarrow{\sigma}} \). This implies the reverse inequality of (4.5), provided \( E > V \).

Finally, we observe that the choice \( \mu = \delta_{x_0} \) where \( V(x_0) = V \) guarantees:

\[
\overline{H}(\overrightarrow{\phi}) \geq H(\overrightarrow{\phi}; \mu) = V .
\]

Let now \( \overline{H}^* \) be the Legendre transform of \( \overline{H} \):

\[
\overline{H}^*(\overrightarrow{\lambda}) := \sup_{\overrightarrow{\phi} \in \mathbb{R}^n} \left\{ -\overline{H}(\overrightarrow{\phi}) + \sum_{i=1}^n \lambda_i \phi_i \right\} \in \mathbb{R} \cup \{ \infty \} .
\]

Similarly

\[
H^*(\overrightarrow{\lambda}; \mu) := \sup_{\overrightarrow{\phi} \in \mathbb{R}^n} \left\{ -H(\overrightarrow{\phi}; \mu) + \sum_{i=1}^n \lambda_i \phi_i \right\} \in \mathbb{R} \cup \{ \infty \} .
\]

Note that both \( \overline{H}^* \) and \( H^*(\cdot; \mu) \) are convex by definition. From Lemma 4.1 we also obtain

**Corollary 4.1.** \( \overline{H}^*(\overrightarrow{\lambda}) \in \mathbb{R} \) for each \( \overrightarrow{\lambda} \in \mathbb{R}^k \) provided \( \sum_{i=1}^n \lambda_i = 0 \).

**Proof.** We first note that \( D_E(x_i, x_j) \approx \sqrt{E} \) as \( E \to \infty \). From Lemma 4.1 it follows that \( \overline{H} \) is at least quadratic with respect to \( |\phi_i - \phi_j| \) for any \( i \neq j \). So, if we fix, say, \( \phi_1 = 0 \), then

\[
\lim_{|\overrightarrow{\phi}| \to \infty, \phi_1 = 0} \frac{\overline{H}(\overrightarrow{\phi})}{|\overrightarrow{\phi}|} = \infty .
\]

In addition, \( \overline{H} \) is invariant under the shift \( \overrightarrow{\phi} \to \overrightarrow{\phi} + t \overrightarrow{1} \) where \( \overrightarrow{1} = (1, \ldots, 1) \in \mathbb{R}^k \) and \( t \in \mathbb{R} \). This implies that \( \sum_{i=1}^n \lambda_i \phi_i - \overline{H}(\overrightarrow{\phi}) \) is bounded from above on \( \mathbb{R}^k \), provided \( \sum_{i=1}^n \lambda_i = 0 \). \( \square \)

From Proposition 2.1 and (2.2) we observe that

\[
\overline{L}(\mu) = H^*(\overrightarrow{\lambda}; \mu) .
\] (4.10)

By definition, \( H^*(\overrightarrow{\lambda}; \mu) \geq \overline{H}^*(\overrightarrow{\lambda}) \) for any \( \mu \in \overline{M} \). So, we obtain:
Corollary 4.2. The minimal action is not smaller than $\overline{H}^*(\overrightarrow{\lambda})$.

We now state

**Lemma 4.2.** Let $\overrightarrow{\phi} \in \partial_{\overrightarrow{\lambda}} H^*(\cdot; \mu)$. Assume $H(\overrightarrow{\phi}; \mu) = \overline{H}(\overrightarrow{\phi})$ and $H(\cdot; \mu)$ is a convex function in the first variable. Then $\mu$ is an action minimizer (that is, a minimizer of (4.10)).

**Proof.** Consider the chain of inequalities:

\[
H^*(\overrightarrow{\lambda}; \mu) + \overline{H}(\overrightarrow{\phi}) \geq H^*(\overrightarrow{\lambda}; \mu) + H(\overrightarrow{\phi}; \mu) \geq \overrightarrow{\phi} \cdot \overrightarrow{\lambda},
\]

which holds for any $\mu \in \overline{\mathcal{M}}$, $\overrightarrow{\phi} \in \mathbb{R}^n$ and $\overrightarrow{\lambda}$ in the domain of $\overline{H}^*$. The left inequality follows from the definition of $\overline{H}$ as a maximizer of $H$ over $\mathcal{M}$, while the right inequality follows by the convexity of $H(\cdot; \mu)$ and Lemma 3.3-iii. The conditions of the Lemma and Lemma 3.3-iii again imply that both inequalities are, in fact, equalities, so

\[
H^*(\overrightarrow{\lambda}; \mu) + \overline{H}(\overrightarrow{\phi}) = \overrightarrow{\phi} \cdot \overrightarrow{\lambda}.
\]

Now, by definition

\[
H^*(\overrightarrow{\lambda}; \mu) \geq \overline{P}^*(\overrightarrow{\lambda}) \tag{4.11}
\]

so

\[
\overline{P}^*(\overrightarrow{\lambda}) + \overline{H}(\overrightarrow{\phi}) \leq \overrightarrow{\phi} \cdot \overrightarrow{\lambda}. \tag{4.12}
\]

Now, $\overline{P}^{**} \leq \overline{P}$ by definition, so

\[
\overline{P}^*(\overrightarrow{\lambda}) + \overline{P}^{**}(\overrightarrow{\phi}) \leq \overrightarrow{\phi} \cdot \overrightarrow{\lambda} \tag{4.13}
\]

holds as well. Now, Lemma 3.3-iii implies that the reverse inequality must hold in (4.13). Hence there must be an equality in (4.13), which induces the equalities in (4.12) and (4.11) as well. It follows that $\mu$ is an action minimizer by (4.10) and Corollary 4.2. \hfill \Box

**Corollary 4.3.** $\overline{P}^*(\overrightarrow{\lambda}) = \sup_{E \geq \overline{V}} \left\{ \sqrt{2} W^{(1)}_E(\overrightarrow{\lambda}) - E \right\}$.

**Proof.** By definition of $\overline{P}^*$ and Lemma 4.1

\[
\overline{P}^*(\overrightarrow{\lambda}) = -\min_{\overrightarrow{\phi}} \left[ \overline{P}(\overrightarrow{\phi}) - \overrightarrow{\lambda} \cdot \overrightarrow{\phi} \right] = -\min_{\overrightarrow{\phi}} \inf_{E \geq \overline{V}} \left[ E - \overrightarrow{\lambda} \cdot \overrightarrow{\phi} ; \max_{i \neq j} \frac{|\phi_i - \phi_j|}{D_E(x_i, x_j)} \leq \sqrt{2} \right]
\]

\[
= \sup_{E \geq \overline{V}} \left[ \sqrt{2} W^{(1)}_E(\overrightarrow{\lambda}) - E \right], \tag{4.14}
\]

where we used the duality relation given by 3.30. \hfill \Box

**Lemma 4.3.** Suppose $E_0 > \overline{V}$ is the minimizer of (4.14). Then there exists $A^{E_0} \in Q_{\overrightarrow{\lambda}}$ which minimize the Wasserstein cost $W^{(1)}_{E_0}$ i.e. $\sum_{i \in I_+} \sum_{j \in I_-} A^{E_0}_{i,j} D_{E_0}(x_i, x_j) = W^{(1)}_{E_0}(\overrightarrow{\lambda})$, and

\[
\sum_{i \in I_+} \sum_{j \in I_-} A^{E_0}_{i,j} T_{i,j}(E_0) = \sqrt{2}
\]
is satisfied. If \( E_0 = \nabla \) then for any such \( A^\nabla \in \mathcal{Q}_\lambda \), the inequality
\[
\sum_{i \in I_+} \sum_{j \in I_-} A^\nabla_{i,j} T_{i,j}(\nabla) \leq \sqrt{2}
\]
holds.

**Proof.** Let \( E_n \searrow E_0 \). For each \( n \), set \( A^{E_n} \in \mathcal{Q}_\lambda \) be a minimizer of the Monge-Kantorovich problem associated with \( W^{(1)}_{E_n}(\lambda) \). Note that such a minimizer may not be unique. We choose a subsequence so that the limit
\[
A^{E_0^+} := \lim_{n \to \infty} A^{E_n}
\]
exists. Evidently, \( A^{E_0^+} \) is a minimizer of the Monge-Kantorovich problem associated with \( W^{(1)}_{E_0}(\lambda) \) (again, possibly not the only one).

Next, since \( E_0 \) is a maximizer of (4.14),
\[
\sqrt{2} \left( W^{(1)}_{E_n}(\lambda) - W^{(1)}_{E_0}(\lambda) \right) \leq E_n - E_0,
\]
so
\[
\sum_{i \in I_+} \sum_{j \in I_-} A^{E_n}_{i,j} (D_{E_n}(x_i, x_j) - D_{E_0}(x_i, x_j)) \leq \sum_{i \in I_+} \sum_{j \in I_-} A^{E_n}_{i,j} D_{E_n}(x_i, x_j) - \sum_{i \in I_+} \sum_{j \in I_-} A^{E_0^+}_{i,j} D_{E_0}(x_i, x_j)
\]
\[
= W^{(1)}_{E_n}(\lambda) - W^{(1)}_{E_0}(\lambda) \leq \frac{E_n - E_0}{\sqrt{2}}.
\]
Take the limit \( n \to \infty \) and use (4.15) and Lemma 3.1 to obtain from (4.16)
\[
\sum_{i \in I_+} \sum_{j \in I_-} A^{E_0^+}_{i,j} T_{i,j}(E_0) \leq \sqrt{2}.
\]
This completes the proof for the case \( E_0 = \nabla \).

Now, let \( E_n \nearrow E_0 \), and let us consider the subsequence along which the limit
\[
A^{E_0^-} := \lim_{n \to \infty} A^{E_n}
\]
exists. By following the preceding argument we obtain
\[
\sum_{i \in I_+} \sum_{j \in I_-} A^{E_0^-}_{i,j} T_{i,j}(E_0) \geq \sqrt{2}.
\]
Finally, if the maximizer \( A^{E_0} \) of \( W^{(1)}(\lambda) \) is unique, then \( A^{E_0^-}_{i,j} = A^{E_0^+}_{i,j} := A^{E_0} \) and the proof follows from (4.17, 4.19). Otherwise, since both \( A^{E_0^\pm}_{i,j} \) are minimizers, so is the convex combination thereof. Now, we utilize (4.17, 4.19) to choose \( \alpha \in [0, 1] \) for which \( A^{E_0} := \alpha A^{E_0^-}_{i,j} + (1 - \alpha) A^{E_0^+}_{i,j} \) satisfies the desired equality
\[
W^{(1)}_{E_0}(\lambda) \equiv \sum_{i \in I_+} \sum_{j \in I_-} A^{E_0}_{i,j} T_{i,j}(E_0) = \sqrt{2}.
\]
Given $\vec{\phi} \in \mathbb{R}^n$ so that $\overline{H}(\vec{\phi}) = E$. Let

$$I_E := \left\{ (i, j) \in I_+ \times I_- \ : \ |\phi_i - \phi_j| = \sqrt{2}D_E(x_i, x_j) \right\}.$$  \hspace{1cm} (4.20)

Let $C_E$ be the convex hall of the set of measures $\mu_{i,j}$ as defined in (3.5) where $(i, j) \in I_E$, that is

$$\mu \in C_E \text{ iff } \mu = \sum_{i \in I_+} \sum_{j \in I_-} \alpha_{i,j} \mu_{i,j} \quad ; \quad \alpha_{i,j} \geq 0 , \quad \sum_{i \in I_+} \sum_{j \in I_-} \alpha_{i,j} = 1.$$  \hspace{1cm} (4.21)

**Lemma 4.4.** For any $\vec{\phi} \in \mathbb{R}^n$ satisfying $\overline{H}(\vec{\phi}) = E > \overline{V}$ holds for any $\mu \in C_E$.

**Proof.** The proof of Lemma 4.4 is, basically, identical to the proof of the second part of Lemma 4.1. \hfill $\Box$

**Lemma 4.5.** Let $E > \overline{V}$ and $\mu \in C_E$ for some $\vec{\phi} \in \mathbb{R}^n$ satisfying $\overline{H}(\vec{\phi}) = H(\vec{\phi}; \mu) = E$. Then for any $\vec{\zeta} \in \mathbb{R}^k$,

$$H(\vec{\zeta}; \mu) = \frac{1}{2} \sum_{i \in I_+} \sum_{j \in I_-} \alpha_{i,j} \frac{(\zeta_i - \zeta_j)^2}{D_E(x_i, x_j)T_{i,j}(E)} + \int Vd\mu.$$  \hspace{1cm} (4.22)

In particular, $H(\cdot, \mu)$ is convex for any $\mu$ of the form (4.21).

**Proof.** Given $\vec{\zeta} \in \mathbb{R}^n$, let $\zeta \in C^1(\mathbb{R}^k)$ be an optimal solution corresponding to $\mu$. We push it backward to a function on the graph composed of $\bigcup_{i,j}[0, S_{i,j}]$ via

$$\eta_{i,j}(s) := \zeta(q_{i,j}(s)),$$

so

$$H(\vec{\zeta}; \mu) \leq \frac{1}{2} \sum_{i,j} \alpha_{i,j} \int_0^{S_{i,j}} \rho_{i,j}(s)\dot{\eta}_{i,j}^2 ds + \int Vd\mu.$$  \hspace{1cm} (4.22)

The equality in (4.22) is achieved if we minimize the RHS on each branch separately, subjected to the prescribed end conditions $\eta_{i,j}(0) = \zeta_i$, $\eta_{i,j}(S_{i,j}) = \zeta_j$. Hence $\rho_{i,j}\dot{\eta}_{i,j}$ is constant on each branch. Taking the end conditions and the definition of $\rho_{i,j}$ (3.3) we obtain that

$$\rho_{i,j}(s)\dot{\eta}_{i,j}(s) = \left( \int_0^{S_{i,j}} \frac{ds}{\rho_{i,j}} \right)^{-1} (\zeta_j - \zeta_i) = \frac{\zeta_j - \zeta_i}{T_{i,j}D_E(x_i, x_j)}. \hspace{1cm} (4.23)$$

Now, we perturb $\zeta_i \to \zeta_i + \varepsilon$. Let $\hat{\zeta}_{i,j}$ be defined on the $(i, j)$ branch $[0, S_{i,j}]$ where $\hat{\zeta}(0) = 1$, $\hat{\zeta}(S_{i,j}) = 0$. Let $e_i$ be the canonical $i$– unit vector in $\mathbb{R}^n$, then, evaluating $H(\vec{\zeta} + \varepsilon e_i; \mu)$ via
the function $\eta_{i,j} + \varepsilon \hat{\zeta}_{i,j}$ on each branch $[0, S_{i,j}]$ where $\alpha_{i,j} > 0$ (fixed $i$) and integration by parts yields

$$H(\vec{\zeta} + \varepsilon e_i; \mu) - H(\vec{\zeta}; \mu) = \varepsilon \sum_j \alpha_{i,j} \rho_{i,j}(0) \eta_{i,j}(0) + O(\varepsilon^2),$$

From (4.23) it follows that

$$\frac{\partial H(\vec{\zeta}; \mu)}{\partial \zeta_i} = \left\{ \begin{array}{ll}
\sum_{j \in I^-} \alpha_{i,j} \frac{\zeta_i - \zeta_j}{D_E(x_i, x_j) T_{i,j}} & i \in I^+ \\
\sum_{j \in I^+} \alpha_{i,j} \frac{\zeta_i - \zeta_j}{D_E(x_i, x_j) T_{i,j}} & i \in I^-
\end{array} \right.$$

and the proof follows by integration.

**Corollary 4.4.** The function $\overline{H}$ is convex on $\mathbb{R}^k$.

**Proof.** By definition, $\overline{H}(\vec{\phi})$ is the maximum of the set $H(\vec{\phi}; \mu)$ where $\mu$ run on $\overline{M}$. From Lemma 4.5 it follows that this maximum is obtained at a convex function. Hence, $\overline{H}$ is the maximum of a family of convex functions, so it is convex.

**Proof of the Main Result:**

We prove the main result in full generality ($V \neq 0$).

We obtain from (4.10) and Corollary 4.3 that (1.21) is a lower bound for the minimal action. By Corollary 4.4, $\vec{\lambda}$ is in the domain of $\overline{H}$ if $\sum_1^n \lambda_i = 0$, so there exists $\vec{\phi} \in \partial_\vec{\lambda} \overline{H}$. Let $\overline{H}(\vec{\phi}) = E$ and assume that $E > V$. By Lemma 4.4

$$\max_{i \neq j} \frac{|\phi_i - \phi_j|}{D_E(x_i, x_j)} \leq \sqrt{2}. \tag{4.24}$$

Since $\overline{H}(\vec{\lambda}) + E = \overline{H}(\vec{\phi}) + \overline{H}(\vec{\lambda}) = \sum_1^n \lambda_i \phi_i$ we obtain the equality in (4.25) below from Corollary 4.3

$$W_E^{(1)}(\vec{\lambda}) = 2^{-1/2} \sum_1^n \lambda_i \phi_i. \tag{4.25}$$

Let

$$\mu = 2^{-1/2} \sum_{i \in I^+} \sum_{j \in I^-} A^E_{i,j} T_{i,j}(E) \mu_{i,j} \tag{4.26}$$

where $\mu_{i,j}$ as defined in (3.4, 3.5), corresponding to the same energy $E$. By Lemma 4.3, $\mu$ is a convex combination of $\mu_{i,j}$, hence a probability measure.

Now let $A \in Q_{\vec{\lambda}}$. By (4.24, 4.25)

$$\sum_{i \in I^+} \sum_{j \in I^-} A_{i,j} D_E(x_i, x_j) \geq 2^{-1/2} \sum_{i \in I^+} \sum_{j \in I^-} A_{i,j} |\phi_i - \phi_j|$$

$$\geq 2^{-1/2} \sum_{i \in I^+} \sum_{j \in I^-} A_{i,j} (\phi_i - \phi_j) = 2^{-1/2} \sum_1^n \lambda_i \phi_i = W_E^{(1)}(\vec{\lambda}).$$
Now substitute the minimizer $A^E \in \mathcal{Q}_\lambda$ for $A$ above. Then the chain of inequalities turn into equalities. In particular we obtain that, for any $i \in I_+$, $j \in I_-$, either \( A^E_{i,j} = 0 \) or \( \phi_i - \phi_j = \sqrt{2} D_E(x_i, x_j) \). As a result we can apply Lemma 4.4 to obtain
\[
H(\vec{\phi}; \mu) = E. \tag{4.27}
\]
In addition, we substitute the equalities \( \phi_i - \phi_j = \sqrt{2} D_E(x_i, x_j) \) and \( \alpha_{i,j} = 2^{-1/2} A^E_{i,j} T_{i,j}(E) \) in Lemma 4.5 to obtain
\[
\nabla_{\vec{\phi}} H(\cdot, \mu) = \vec{\lambda},
\]
which implies
\[
\vec{\phi} \in \partial_{\vec{\lambda}} H^*(\cdot; \mu) \tag{4.28}
\]
By (4.27,4.28) and Lemma 4.2 we obtain that $\mu$ is an action minimizer. In particular, it follows that (1.21) is the minimal action.

Finally, if $E = \nabla$, then let $\beta = 2^{-1/2} \sum_{i \in I_+} \sum_{j \in I_-} A^E_{i,j} T_{i,j}(E)$. By lemma 4.3 $\beta \leq 1$. define
\[
\mu = 2^{-1/2} \sum_{i \in I_+} \sum_{j \in I_-} A^E_{i,j} T_{i,j}(E) \mu_{i,j} + (1 - \beta) \delta_{x_0} \tag{4.29}
\]
where $x_0$ is a maximizer of $V$, so $V(x_0) = V$. Since $E = \nabla$ the equality (4.27) holds for $\mu$ given by (4.29). In addition, (4.28) is also verified for this $\mu$ by Lemma 4.5. Hence, $\mu$ is an action minimizer via Lemma 4.2 as well.

5 Conclusion

We considered the extended minimal action principle for stationary actions in the presence of point sources and sinks. The main conclusion of this paper is that this minimal, stationary action is obtained as a minimization of a metric Monge-Kantorovich for the (non-normalized) pair of discrete measures (1.14). This stands in contrast to the non-metric Monge-Kantorovich transport (1.5,1.6) which is related to the extended minimal action in the non-stationary case.

Another interesting conclusion is the relation (1.23) between the expectation of the inverse-time of the minimal orbits to the given flux. This equality follows from substitution of (1.22) and using the fact $A^E \in \mathcal{Q}_\lambda$ in the middle term of (1.23). We stress that this relation is preserved in the case $E = \nabla$, since the point measure $\beta \delta_{(x_0)}$ corresponds to orbits of infinite time length-these are the orbits which converges to the maximum of $V$, but never get there, since the metric $\sqrt{E - V} ds$ is degenerate at the point $x_0$ where $V(x_0) = V$.

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