Polymer quantization of the CGHS model: I

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Abstract

We present a polymer (loop) quantization of a two-dimensional theory of dilatonic gravity known as the CGHS model. We recast the theory as a parametrized free field theory on a flat two-dimensional spacetime and quantize the resulting phase space using techniques of loop quantization. The resulting (kinematical) Hilbert space admits a unitary representation of the spacetime diffeomorphism group. We obtain the complete spectrum of the theory using a technique known as group averaging and perform quantization of Dirac observables on the resulting Hilbert space. Finally, we argue that the algebra of Dirac observables gets deformed in the quantum theory.

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1. Introduction

In the past two decades, two-dimensional theories of gravity have received quite a bit of attention [1] as toy models to address questions arising in (four-dimensional) quantum gravity. In particular, the CGHS model whose action is inspired from the effective target space action of 2D non-critical string theory constitutes a highly desirable choice, due to its various features such as classical integrability, existence of black hole spacetimes in its solution space and the presence of Hawking radiation and evaporation at a 1-loop level.

Semi-classical analysis of this model has been carried out by a number of authors ([2–5] and references therein). By incorporating a large number of conformal scalar fields, Hawking radiation (arising from trace anomaly) and the back reaction take place at the 1-loop level. However during the final stages of the collapse, the semi-classical approximation breaks down signaling a need to incorporate higher order quantum corrections and non-perturbative effects. It is always believed that a non-perturbative quantum theory is required in order to answer questions regarding final fate of singularity, information loss, etc (see, however, [3]).

In the canonical formulation, the non-perturbative quantization of the CGHS model has been carried out in detail in various papers ([6–9] and references therein). After a rescaling of the metric, the model becomes amenable to Dirac constraint quantization as well as BRST
methods. Although the complete spectrum is known in the BRST approach as well as in the Dirac method (in the so-called Heisenberg picture), so far it has not been possible to ask the questions regarding quantum geometry using this spectrum.

In this paper, we begin the analysis of the rescaled-CGHS (KRV) model using the methods of loop quantum gravity (LQG) \cite{10, 11}, more generally known as polymer quantization \cite{12, 13}. More in detail, we derive a quantum theory of dilaton gravity (starting from the classical CGHS model) which can be used to understand the near-Planckian physics of the CGHS model. The aim of this work is two-fold. First, we would eventually like to understand if the methods of loop quantization shed new light on the structure of quantum geometry close to singularity of the CGHS black holes. Although we do not answer this question in this paper, we set up a framework where this question can be asked. Secondly as the model offers a greater degree of analytic control than its higher-dimensional avatars, we can study in detail various structures which arise in LQG but have so far remained rather formal (physical Hilbert space, Dirac observables, relational dynamics).

We begin by reviewing the classical CGHS model and its canonical formulation in section 2. We recast it as a free parametrized scalar field theory on a fiducial flat spacetime \cite{6}. Parametrized field theories on a fixed background have a very rich mathematical and conceptual structure \cite{14–16}. They are ideal arenas to test the various quantization methods which one hopes to use in the quantization of general relativity. It is partly our aim to show that by combining the ideas from parametrized field theories and LQG, one obtains a potentially interesting quantum theory of dilaton gravity.

In section 3, we kinematically quantize the phase space (i.e. prior to solving the constraints). By choosing an appropriate sub-algebra of the full Poisson algebra and performing the so-called GNS quantization using a positive linear functional (analogous to the Ashtekar–Lewandowski functional used in LQG), we obtain a Hilbert space which carries a unitary representation of the spacetime diffeomorphism group of the theory. We use the group averaging method in section 4 to solve the constraints and obtain the complete spectrum (physical Hilbert space) of the theory.

Parametrized field theories give us a general algorithm to obtain an algebra of Dirac observables (perennials) of the theory. In section 5, we show how to quantize this algebra on the physical Hilbert space and show how the physical Hilbert space is not a representation space for this algebra (in other words the algebra gets deformed in the quantum theory). We finally conclude with a discussion.

This is the first paper in a two-part series. In the second paper, we carry our analysis of polymer-quantized CGHS model further and define the time evolution as well as the physical dilaton operator.

2. Classical theory

In this section, we briefly recall the (rescaled) action of the CGHS model along with the solution of the field equations and the structure of the canonical theory.

The original CGHS action\(^1\) describing a two-dimensional theory of dilatonic gravity is given by

\[
S_{\text{CGHS}} = \frac{1}{4} \int d^2x \sqrt{-g} [e^{-2\phi} (R[g] + 4(\nabla \phi)^2 + 4\lambda^2) - (\nabla f)^2].
\]  

\(^1\) We choose \(c = G = 1\). Thus the only basic dimension in the theory is \(L\) and \([M] = L^{-1}\). In these units \(\hbar\) becomes a dimensionless number.
Here $\phi$ is the dilaton field, $g$ is the spacetime metric (signature $(-, +)$) and $f$ is a conformally coupled scalar field.

Rescaling the metric $g_{\mu\nu} = e^{2\phi} \gamma_{\mu\nu}$ one obtains the KRV action [6]

$$S_{\text{KRV}} = \frac{1}{2} \int d^2X \sqrt{-\gamma} \left[ (\gamma R[\gamma] + 4\lambda^2) - \gamma^{\alpha\beta} \nabla_\alpha f \nabla_\beta f \right],$$

(2)

where $\gamma = e^{2\phi}$.

The field equations obtained by varying $S_{\text{KRV}}$ can be analyzed in the conformal gauge. The solution is as follows. $\gamma_{\alpha\beta}$ is flat. The remaining fields can be described most elegantly in terms of null coordinates $X^\pm = Z \pm T$ on the flat spacetime. The scalar field $f$ is simply the free field propagating on the flat spacetime

$$f(X) = f_+(X^+) + f_-(X^-)$$

(3)

and the dilaton is

$$y(X) = \lambda^2 X^+ X^- - \frac{1}{2} \int X^+ dX^+ \int \overline{X}^+ d\overline{X}^+ \partial_+ f \partial_+ f - \frac{1}{2} \int X^- dX^- \int \overline{X}^- d\overline{X}^- \partial_- f \partial_- f,$$

where $(X^+, \overline{X}^+), (X^-, \overline{X}^-)$ are null coordinates on Minkowski space.

Thus the solution space of the original CGHS model, namely $(g_{\mu\nu}, f)$, is completely determined in terms of the matter field $f$. This space contains black hole spacetimes as well. The easiest way to see this is to look at vacuum solutions. Taking $f(X) = 0$, one can show that the dilaton is given by

$$y(X) = \lambda^2 X^+ X^- - \frac{M}{\lambda}$$

(4)

and the associated physical metric is

$$g_{\mu\nu} = \frac{1}{\lambda^2 X^+ X^- - \frac{M}{\lambda}} \gamma_{\mu\nu}$$

(5)

which corresponds to black holes of mass $M$ in two dimensions ($M = 0$ is the linear dilaton vacuum). The singularity occurs where $y(X) = 0$. One can obtain more generic black hole spacetimes by sending left-moving matter pulses from past null infinity. In all these cases the locus of singularity is defined by $y(X) = 0$.

2.1. Canonical description

The reason for using the rescaled-KRV action rather than the original (and perhaps more interesting) CGHS action is the following. One can perform a canonical transformation on the canonical coordinates of the KRV phase space and obtain a parametrized free field theory on the flat background. This will be our starting point for quantization. The details of this canonical transformation (also known as the Kuchar decomposition [19]) are given in [6]; here, we only summarize the main results.

2 There is an important difference between the CGHS and KRV action at the semi-classical level. In the path integral quantization, Hawking radiation is encoded in a 1-loop term obtained by integrating out the matter field. This term is known as the Polyakov–Liouville term and is zero if one uses the flat metric $\gamma$ (naturally appearing in the KRV action) to define the measure for the matter field. It is, however, non-zero if one uses the physical metric $g$ (which appears in the CGHS action). Whence it is often claimed that the theory defined by the KRV action does not contain Hawking radiation [4].

3 It is interesting to note that even the phase space of the CGHS action can be mapped onto a parametrized scalar field theory on the Kruskal spacetime. However, the canonical transformation is singular in a portion of phase space [20].
The KRV spacetime action can be cast into a canonical form by using an arbitrary foliation
\[ X^\alpha = X^\alpha(x, t) \]
of spacetime by \( (t = \text{const}) \) spacelike hypersurfaces
\[ S_{\text{KRV}} = \int dt \int_{-\infty}^{\infty} dx (\pi_\gamma \dot{\gamma} + \pi_\sigma \dot{\sigma} + \pi_\phi \dot{\phi} - N H - N^1 \tilde{H}_1), \]
(6)
where \((\gamma(x), \sigma(x), f(x))\) are the pullback of the dilaton, spacetime metric \(\gamma_{\mu\nu}\) and the scalar field onto the hypersurface \(\Sigma\) respectively and \(\pi_\gamma, \pi_\sigma, \pi_\phi\) are their conjugate momenta. \((N, N^1)\) are the usual lapse and shift functions and \(H, \tilde{H}_1\) are Hamiltonian and momentum constraints respectively and are constrained to vanish.

At this point it is important to note that by choosing appropriate gauge-fixing conditions \((\dot{\rho} = \pi_\theta = 0)\), one obtains a reduced phase space coordinatized by \((f, \pi_f)\) with a true Hamiltonian given by
\[ H = \frac{1}{2} \int dx (\pi_f^2 + (f')^2). \]
(7)
One can then quantize this free field theory on a Fock space and obtain a non-perturbative quantum theory. However as our primary motivation is to gain insights into the structure of LQG, where constraints are solved directly in quantum theory, we do not solve the constraints classically.

A series of non-local canonical transformations maps the above action into that of a parametrized free field theory of flat background [6].
\[ S[X^\pm, \Pi^\pm, f, \pi_f, N, N^1, p, m_R] = \int dt \int_{-\infty}^{\infty} dx \left( \Pi_+ \dot{X}^+ + \Pi_- \dot{X}^- + \pi_f \dot{f} - N \tilde{H} - N^1 \tilde{H}_1 \right) + \int dt p(t) m_R(t), \]
(8)
where \(X^\pm(x)\) are the embedding variables\(^4\) and correspond to the light-cone coordinates on the Minkowski spacetime, \(\Pi_{\pm}\) are conjugate momenta and the Hamiltonian constraint has been rescaled so as to have the same density weight as the momentum constraint\(^5\). The boundary term \(\int p(t) m_R \) arises due to asymptotic conditions (note that there are two boundaries in the problem, left and right infinity but only one boundary term in the action) on the initial data. \(m_R\) is the right mass of spacetime and the conjugate momentum \(p\) is the difference between the parametrized time and the proper time at right infinity when the parametrized time at left infinity is chosen to agree with the proper time\(^6\). This action is the canonical action for a parametrized massless scalar field theory on the flat spacetime.

The two constraints can be combined to form two Virasoro constraints \(\tilde{H}^\pm = \frac{1}{2}(\tilde{H} \pm \tilde{H}_1)\).
These two Virasoro constraints mutually commute with each other. Thus the constraint algebra can be written as a direct sum of two Lie algebras each of which generates \(\text{Diff}(\mathbb{R})\).

We now proceed to the quantum theory.

3. Quantum theory

In this section, we quantize the classical theory using the techniques of polymer quantization. To make this section self-contained we recall the basic idea behind polymer quantization in

\(^4\) Here \(X^\pm(x)\) means a phase-space function evaluated at \(x\).
\(^5\) Here our notation is \(f\) means \(f\) is a density of weight 1.
\(^6\) As the physical metric \(g_{\mu\nu}\) is asymptotically flat, it has an asymptotic stationary Killing field. The proper time is the time measured by a clock along the orbit of this Killing field. The parametrized time is the time defined by the asymptotic value of the lapse function. For more details, see [21].
a manner that is coherent with the recent work on kinematical quantization in LQG ([23, 24] and references therein).

The basic idea behind the polymer quantization can be summarized as follows:

1. Choose an appropriate sub-algebra of the full Poisson algebra (one that does not use any background metric).
2. Define the corresponding quantum algebra. (This is an abstract *-algebra with an associative multiplication and in which the Poisson relations are represented as Lie-relations.) As our quantum algebra does not use any background metric, it turns out that diffeomorphisms act as a group of outer automorphisms on this algebra.
3. Choose a diffeomorphism-invariant positive linear functional to GNS quantize the quantum algebra.

The resulting Hilbert space then carries a unitary and anomaly-free representation of the diffeomorphism group of the theory.

These steps are rather generic to canonical quantization of field theories. The difference between a polymer and Fock quantization lies in the choice of the sub-algebra which we are interested in quantizing and more importantly in the choice of the GNS functional.

We now carry out these steps in detail for our model.

3.1. Embedding sector

The first step toward canonical quantization is a suitable choice of quantum algebra. Let us first describe our choice of quantum algebra for the embedding sector. Recall that \( \Pi^\pm \) are scalar densities of weight +1 (equivalently 1-forms in one dimensional) and \( X^\pm \) are scalars (equivalently densitized vector fields).

Consider a graph \( \gamma \) in the spatial slice \( \Sigma \) as a collection of a finite number of edges and vertices. Define a cylindrical function for both the right-moving (+) and left-moving (−) embedding sectors as

\[
    f^\pm_\gamma = \Pi_{e \in E(\gamma)} \exp \left( i k^\pm_e \int_e \Pi^\pm \right),
\]

where \( k^\pm_e \in \mathbb{R} \).

Define Abelian *-algebras \( \text{Cyl}^\pm = \bigcup_{\gamma \in \Gamma} \text{Cyl}^\pm_\gamma \). Let Vec denote the complexified Lie algebra of vector fields \( X^\pm(x) \) which are maps \( \text{Cyl}^\pm \rightarrow \text{Cyl}^\pm \) (via Poisson brackets) that satisfy the Liebniz rule and annihilate constants.

Consider the Lie-* algebra \( V \) defined by

\[
    \begin{align*}
    &\left[ (f, X(x)), (f', X'(x')) \right] = ([X(x), f'] - [X'(x'), f], 0), \\
    &\text{where } (f, f') \text{ are in } \text{Cyl and } (X, X') \text{ are vector fields.} \\
    &\text{*-operation is just the complex conjugation. (Conjugation of vector fields is defined by } X(x)^* f := (X(x)f)^*.)
    \end{align*}
\]

We now define the quantum algebras for the embedding sector. Our derivation mimics the derivation of quantum algebra for LQG given in [24].

Let us denote the pair \( (f^\pm, X^\pm(x)) \) by a symbol \( a^\pm \). Consider the *-algebra of finite linear combinations of finite sequences of the form \( (a^\pm_1, \ldots, a^\pm_n) \) with an associative product

\[
    (a^\pm_1, \ldots, a^\pm_n) \cdot (a'^\pm_{n+1}, \ldots, a'^\pm_m) = (a^\pm_1, \ldots, a'^\pm_m)
\]

and an involution

\[
    (a^\pm_1, \ldots, a^\pm_n)^* = (a'^\pm_n, \ldots, a'^\pm_1).
\]

We have supressed the ± symbol so as to not clutter the formulae.
Divide this algebra by a two-sided ideal defined by elements of the form
\[(ka^\pm) - k(a^\pm)
(a_1^\pm + a_2^\pm) - (a_1^\pm) - (a_2^\pm)\].
\[(13)\]

The resulting algebras (for both \(\pm\) sectors) are nothing but the free tensor algebras generated by \(a^\pm\). The algebras \(U^\pm_E\) that we will quantize are defined as the free tensor algebras defined above modulo the two-sided ideal generated by elements of the form \(a_1^\pm \otimes a_2^\pm - a_2^\pm \otimes a_1^\pm - [a_1^\pm, a_2^\pm]^8\).

So finally the algebra that we choose for quantization is \(U_E = U^+_E \otimes U^-_E\).

The group generated by the two Virasoro constraints which is a direct product of two copies of \(\text{Diff}(\mathbb{R})\) has a natural representation as a group of outer automorphisms on \(U_E\). Abusing the standard nomenclature we refer to this group as Virasoro group
\[a^\pm(f^\pm) = f^\pm_{\phi^{-1}(\gamma)}\]
\[a^\pm(X^\pm(x)) = X^\pm(\phi^{-1}(x)).\]
\[(14, 15)\]

The representation of \(U_E\) should be such that the outer automorphisms of \(U_E\) are represented via unitary operators as inner automorphisms, i.e.
\[U_\gamma(\phi)\pi(a^\pm)U_\gamma(\phi)^{-1} = \pi(\alpha\phi(a^\pm))\forall a \in U_E.\]
\[(16)\]

By GNS quantization of the C* sub-algebra generated by \(\text{Cyl}^\pm\) [23], one obtains a diffeomorphism-invariant irreducible representation of \(U_E\). The representation is given by \(\mathcal{H}_E^\pm = L^2(\pi^\pm, d\mu_0)\) where \(\pi^\pm\) is the spectrum of C* sub-algebra generated by \(\text{Cyl}^\pm\) and \(d\mu_0\) is a regular Borel probability measure given by
\[\int f^\pm d\mu_0 = \omega^+_0(f^\pm),\]
\[(17)\]

where \(\omega^+_0\) is a positive linear functional which is motivated by the A–L positive linear functional of LQG
\[\omega^+_0(f^\pm) = \delta_{\gamma,0}.\]
\[(18)\]

On \(\mathcal{H}_E^\pm\) cylindrical functions act as multiplication operators and one can show that the embedding variables act as derivations:
\[\hat{X}^\pm(x) f^\pm_{\gamma}(\Pi^\pm) = (-\hat{\gamma})ik_x f^\pm_{\gamma}(\Pi^\pm) \quad \text{if} \quad x \in e
= (-\hat{\gamma})i(k_x + k_x') f^\pm_{\gamma}(\Pi^\pm) \quad \text{if} \quad x \in e \cap e'
= 0 \quad \text{otherwise.}\]
\[(19)\]

The Virasoro group acts unitarily on \(\mathcal{H}_E^\pm\) as
\[\hat{U}^\pm(\phi^\pm) f^\pm_{\gamma}(\pi^\pm) = f^\pm_{\phi^{-1}(\gamma)}(\pi^\pm)
\hat{U}^\pm(\phi^\pm) f^\pm_{\gamma} \hat{U}^\pm(\phi^\pm) = f^\pm_{\phi^{-1}(\gamma)}
\hat{U}^\pm(\phi^\pm) \hat{X}^\pm(x) \hat{U}^\pm(\phi^\pm) = \hat{X}^\pm(\phi^\pm(x)).\]
\[(20)\]

The complete embedding Hilbert space is, of course, given by \(\mathcal{H}_E = \mathcal{H}_E^+ \otimes \mathcal{H}_E^-\).

\[\mathcal{H}_E^\pm\] is nothing but the universal enveloping algebra of the Lie algebra \(V^\pm\).
3.2. Matter sector

Now we consider the kinematical quantization of the matter sector. The quantization given here is unitarily inequivalent to the Bohr quantization of the scalar field, but it is the same quantization that is used by Thiemann to quantize the Bosonic string. For more details we refer the reader to [25].

Once again the choice of quantum algebra will be motivated by the fact that we want the Virasoro group to act as a group of outer automorphisms on this algebra. Following observations helps us make such a choice.

Consider the canonical transformation $(\pi_f, f) \rightarrow (Y^\pm = \pi_f \pm f')$. $Y^\pm$ satisfy the Poisson bracket relations:

$$
\{Y^\pm(x), Y^\pm(x')\} = \mp \delta(x', x) - \delta(x, x')
$$

$$
\{Y^\pm(x), Y^\mp(x')\} = 0.
$$

(21)

In terms of these variables the Virasoro constraints are given by

$$
H^+(x) = \Pi_+ X^+ + \frac{1}{2}(\pi_f + f')^2 \quad H^-(x) = \Pi_- X^- - \frac{1}{2}(\pi_f - f')^2.
$$

(22)

Whence one can see that under the Lie derivative along the Hamiltonian vector field of the constraints

$$
L_{H^+(N_4)} Y^\pm(x) = (N_4 Y^\pm)'(x) \quad L_{H^-(N_4)} Y^\mp(x) = 0.
$$

(23)

Thus it is clear that the two generators of the Virasoro algebra $H^\pm$ act as generators of spatial diffeomorphisms on $Y^\pm$. These considerations motivate the following.

Once again let $\Gamma$ be the set of all graphs $\gamma$ embedded in $\Sigma$ consisting of a finite number of edges and vertices. We start by defining the momentum network (similar to the spin network in LQG) as a pair $(\gamma, l(\gamma) := (l_e, \ldots, l_{e_\gamma}))$ where $l_e$ are real numbers. A momentum network operator for both the right- and left-moving sectors is defined as

$$
W^\pm(s) := \exp \left( i \sum_{e \in E(\gamma)} l_e^\pm \int_e Y^\pm \right)
$$

(24)

The Weyl relations obeyed by $W^\pm(s)$ can be easily derived from (21) (using BHC formula),

$$
W^\pm(s_1) W^\pm(s_2) = e^{\pi/4 [\alpha(s_1, s_2)]} W^\pm(s_1 + s_2)
$$

$$
W^\pm(s)^* = W^\pm(-s),
$$

(25)

where

$$
\alpha(s_1, s_2) = \sum_{e_1 \in E(\gamma)} \sum_{e_2 \in E(\gamma)} l^{e_1} l^{e_2} \alpha(s_1, e_1, e_2)
$$

(26)

with $\alpha(e_1, e_2) = [\kappa_x]_{e_1 - e_2}$. Here $\kappa_x$ is the characteristic function of $e$. $\kappa_x(x) = 1$ for $x \in \text{Int}(e)$, $\kappa_x(x) = \frac{1}{2}$ for $x \in \text{boundary}(e)$ and 0 otherwise. In (25) notation $(s_1 + s_2)$ means we decompose all edges $e_1$ and $e_2$ in their maximal mutually non-overlapping segments and assign $l^{e_1}$ to $e_1 \cap e_2$, $l^{e_1}$ to $e_1 - \gamma(s_2)$ and $l^{e_2}$ to $e_2 - \gamma(s_1)$ respectively.

Now we define the algebra that we will be interested in quantizing. Consider an associative algebra generated by formal finite linear combinations of formal sequences of the form $(W_{s_1}^\pm, \ldots, W_{s_n}^\pm)$ with the associative multiplication given by

$$
\left( W_{s_1}^\pm, \ldots, W_{s_n}^\pm \right) \cdot \left( W_{s_{n+1}}^\pm, \ldots, W_{s_{2n}}^\pm \right) := \left( W_{s_1}^\pm, \ldots, W_{s_{2n}}^\pm \right).
$$

(27)

We give this algebra tensor product structure by moding out two-sided ideals generated by elements of the form

$$
(\alpha W^\pm(s)) - \alpha(W^\pm(s)) \alpha \in \mathbb{C}
$$

$$
(W^\pm(s_1) + W^\pm(s_2)) - (W^\pm(s_1)) + (W^\pm(s_2)).
$$

(28)
We refer to this tensor algebra as \( \text{Cyl}_M^{\pm} \). The *-algebra that we will quantize is \( \text{Cyl}_M^{\pm} \) modulo the two-sided ideal implied by (25). We denote this algebra as \( \mathcal{U}_M^\pm \). Finally, the full algebra for both sectors is given by \( \mathcal{U}_M = \mathcal{U}_M^+ \otimes \mathcal{U}_M^- \).

As emphasized earlier, the reason for choosing this particular algebra for quantization is its covariance properties under the action of the Virasoro group.

For all momentum-network functions \( W^\pm(s) \),

\[
\alpha^\pm_\phi(W^\pm(s)) = W^\pm_{\phi(s)},
\]

where \( \phi(s) := (\phi^{-1}(y), \hat{t}(y)) \).

Now just like for the embedding sector we perform a GNS quantization of \( \mathcal{U}_M \) using a Virasoro-invariant positive linear functional,

\[
\omega^\pm(W^\pm(s)) = \delta_{s,0},
\]

where \( 0 \) in \( \delta_{s,0} \) stands for a graph with zero edges and an empty label set.

This functional is clearly motivated by the Ashtekar–Lewandowski functional used in LQG. It can be easily shown to be Virasoro invariant. The resulting Hilbert space for both \( \pm \) sectors is given by the Cauchy completion of \( \mathcal{U}_M^\pm \) and the representation is

\[
\hat{W}^\pm(s_1)W^\pm(s_2)(Y^\pm) = W^\pm(s_1)(Y^\pm)W^\pm(s_2)(Y^\pm) = e^{\pm i\ell_{s_1,s_2}} W^\pm(s_1 + s_2)(Y^\pm).
\]

As \( \omega^\pm \) is Virasoro invariant, it implies that the Virasoro group acts unitarily and anomaly freely on \( \mathcal{H}_M^\pm \).

\[
\hat{U}^\pm(\phi^\pm)W^\pm(s)(Y^\pm) = W^\pm((\phi^\pm)s)(Y^\pm)
\]

\[
\hat{U}^\pm(\phi^\pm)\hat{W}^\pm(s)\hat{U}^{\pm,-1}(\phi^\pm) = \hat{W}^\pm(\phi^\pm s).
\]

Finally, the matter Hilbert space is \( \mathcal{H}_M = \mathcal{H}_M^+ \otimes \mathcal{H}_M^- \).

### 3.3. Quantizing the asymptotic degrees of freedom

The final component of the kinematical Hilbert space is the Schrödinger representation of the boundary data \((m, p)\). As the Virasoro group has trivial action on the corresponding Heisenberg algebra, we do not need to loop quantize these asymptotic degrees of freedom. Thus we choose for the boundary Hilbert space \( L^2(\mathbb{R}, dm) \)

\[
\hat{m}\Psi(m) = m\Psi(m)
\]

\[
\hat{p}\Psi(m) = -i\frac{\partial}{\partial m}\Psi(m).
\]

This finishes the construction of the kinematical Hilbert space of the quantum theory. We rewrite the final Hilbert space as a tensor product of three Hilbert spaces,

\[
\mathcal{H}_{\text{kin}} = \mathcal{H}_M \otimes \mathcal{H}_E \otimes \mathcal{H}_m = (\mathcal{H}_M^+ \otimes \mathcal{H}_M^-) \otimes (\mathcal{H}_E^+ \otimes \mathcal{H}_E^-) \otimes \mathcal{H}_m = (\mathcal{H}_M^+ \otimes \mathcal{H}_E^+) \otimes (\mathcal{H}_M^- \otimes \mathcal{H}_E^-) \otimes \mathcal{H}_m.
\]

one for the left-moving (−) sector (\( \mathcal{H}_M^- \)), one for the right-moving (+) sector (\( \mathcal{H}_M^+ \)) and one for the asymptotic sector. The Virasoro group acts unitarily on the Hilbert space as two mutually commuting copies of spatial diffeomorphisms.

\( \text{Note that this Hilbert space is not of the form } L^2(T^\mathbb{R}, dm). \) It is (the completion of) algebra itself considered as a vector space with the inner product defined by the GNS state.

\( \text{More precisely, we need to perform quantization on a half-line in order to restrict ourselves to } m \geq 0 \text{ configurations.} \)
We define the basis for $\mathcal{H}$ as follows.
Consider a graph $\gamma$ with a set of pairs of real numbers $(k_1^\pm, l_1^\pm), \ldots, (k_N^\pm, l_N^\pm)$ where in outermost edges $(k_1^\pm, l_1^\pm)$ and $(k_N^\pm, l_N^\pm)$ either $k_i$ or $l_i$ can be zero but not both. In the interior edges $(e_2, \ldots, e_{N-1})$ we even allow both the charges $(k, l)$ to be zero.

We call the pair $(\gamma, ((k_1^\pm, l_1^\pm), \ldots, (k_N^\pm, l_N^\pm)))$ charge network (in analogy with spin networks in LQG) and denote it by $s$. The state associated with $s$ is given by

$$f_s^\pm = \sum_{\text{edges} \in E(\gamma)} ik_i^\pm f_i^\pm e^{\sum_{\text{edges} \in E(\gamma)} l_i^\pm f_i^\pm y^\pm}. \tag{35}$$

4. Physical Hilbert space

The motivation behind choosing a particular quantum algebra and a peculiar GNS functional has been the unitary and anomaly-free representation of the Virasoro group on the Hilbert space. This is analogous to the representation of spatial diffeomorphisms in LQG whence we use the same method that is used there to solve the diffeomorphism constraint to solve the Virasoro constraints. It is more commonly known as group averaging [26]. The idea is to construct a rigging map $\eta$ from a dense subspace $\Phi_\text{kin}$ of $\mathcal{H}_\text{kin}$ to its algebraic dual $\Phi^*_\text{kin}$ such that the image of the map is solutions to the Virasoro constraints in the following sense:

$$\Psi(\hat{U}(\phi^\pm) f_s^\pm) = \Psi(f_s^\pm) \forall \phi \in \text{Diff}(\mathbb{R}). \tag{36}$$

The rigging map is defined as follows. Given a charge network $s$, define $[s] = \phi \cdot s |\phi \in \text{Diff}(\mathbb{R})$.

Then the rigging map (which is tied to the charge-network basis) is given by

$$\eta(f_s^+ \otimes f_s^-) = \sum_{\phi \in \text{Diff}(\mathbb{R})} (f_s^+ \hat{U}^+(\phi)) \otimes \sum_{\phi \in \text{Diff}(\mathbb{R})} (f_s^- \hat{U}^-(\phi)), \tag{37}$$

where the sum is over all the diffeomorphisms $\phi$ which take a charge network $s = (\gamma, (k_1^\pm, l_1^\pm))$ to a different charge network $s' = (\phi^{-1}(\gamma), (k_1^\pm, l_1^\pm))$. In the higher dimensions (i.e. when the dimension of the spatial slice is greater than 1) this map does not work as the orbits (set of diffeomorphisms) over which we are averaging can be infinitely different even for two states based on the same graph. This in turn poses a problem in defining the inner product on the vector space of group-averaged states [10, 27]11. However, in one dimension the above map yields solutions to the Virasoro constraints. As can be clearly seen from the definition of the rigging map, the solution space is a tensor product of two vector spaces. The inner product on both of them can be defined as

$$[\eta^\pm(f_s^\pm) | \eta^\pm(f_s'^\pm)] = [\eta^\pm(f_s^\pm)](f_s'^\pm). \tag{38}$$

The physical Hilbert space is thus characterized by a diffeomorphism-equivalence class of charge networks which in one dimensions can be classified by the following data:

1. number of edges $|E(\gamma)| = N$,
2. the ordered set $((k_1^\pm, l_1^\pm), \ldots, (k_N^\pm, l_N^\pm))$.

Thus we can write a ket in $\mathcal{H}_\text{phy}^\pm$ as

$$|\Psi\rangle_\pm = |N, (k_1^\pm, l_1^\pm), \ldots, (k_N^\pm, l_N^\pm)\rangle. \tag{39}$$

Finally as the Virasoro group acts trivially on $\mathcal{H}_m$ it remains unchanged under group averaging, whence complete $\mathcal{H}_\text{phy} = \mathcal{H}_\text{phy}^+ \otimes \mathcal{H}_\text{phy}^- \otimes \mathcal{H}_m$.

11 In fact, the rigging map in LQG is not even derived in full generality to the best knowledge of the author, and is only defined for the so-called strongly diffeo-invariant observables.
At this point we would like to comment on the anomaly freeness of our representation. In the Fock-space quantization of the model \[6, 8\], one obtains a Virasoro anomaly in the constraint algebra due to the Schwinger term in the commutator of the energy momentum tensor for the matter field. In \[6\], the anomaly is removed by modifying the embedding sector of the theory whereas when one uses BRST methods to quantize the model, the anomaly is removed by adding background charges (enhancing the central charge) and ghost fields (which define so-called bc-CFT). Our choice of Poisson sub-algebra coupled with an unusual choice of GNS functional results in a discontinuous (but anomaly-free) representation of the Virasoro group. Here it is important to note that even in the Fock space one can normally order the constraints with respect to the so-called squeezed vacuum state \[8\] such that the central charge is zero. However these states have peculiar properties like the action of finite gauge transformations is ill-defined on them\[12\].

5. A complete set of Dirac observables

By group averaging the Virasoro constraints we have obtained a physical Hilbert space \(H^\text{phy} = H^\text{phy}_+ \otimes H^\text{phy}_-\). Now we encounter (what one always encounters at some stage in the canonical quantization of diffeo-invariant theories) the problem of time. There is a priori no dynamics on the physical Hilbert space. In order to ask the dynamical questions about, for example, singularity resulting from the collapse of scalar field in quantum theory, some notion of dynamics should be defined on \(H^\text{phy}\). We do this by employing ideas due to \[14, 17\], which goes back to the old idea of evolving constants of motion by \[28\].

However first we show how to define a complete set of Dirac observables (Perennials) for our model and how to represent them as well-defined operators on \(H^\text{phy}\). For the classical theory these perennials have been known for a long time \[6, 29\] and are analogous to the DDF observables of bosonic string theory \[30\].

The basic idea behind constructing Dirac observables in the parametrized field theory is fairly simple. (This is a general algorithm for defining Dirac observables in parametrized field theories and is also known as Kuchar decomposition \[19\].) Given the phase space of the theory coordinatized by \((X^\pm, \Pi^\pm, f, \pi_f)\), one can perform a canonical transformation to the so-called Heisenberg chart \((X^\pm, \Pi^\pm, f, \pi_f)\) \[6\] where \(\Pi^\pm\) are the two Virasoro constraints and \((f, \pi_f)\) are the scalar field data on an initial (fixed) slice. \((X^\pm, \Pi^\pm)\) and \((f, \pi_f)\) form a mutually commuting canonically conjugate pair whence it is clear that \((f, \pi_f)\) are Dirac observables.

Choosing the initial slice as \(X^\pm_0(x) = x\) we can expand these observables in terms of an orthonormal set of mode functions \(e^{ikx}\),

\[f(x) = \int_{-\infty}^{\infty} \frac{dk}{|k|} e^{ikx} a_k + \text{c.c.}\]

It is clear that \((a_k, a_k^*)\) are also Dirac observables. It is also clear in the Heisenberg chart that they form a complete set (describe true degrees of freedom of the theory). Now by expressing \(a_k\) in terms of the original (Schrödinger) canonical chart we will obtain Dirac observables that will be promoted to operators on \(H^\text{phy}\).

In order to write \(a_k = a_k[X^\pm, \Pi^\pm, f, \pi_f]\) one has to appeal to the spacetime picture of a parametrized field theory propagating on flat background. We just summarize the main results and refer the reader to \[29\] for details.

The scalar field \(f(X)\) in spacetime satisfies

\[\Box f(X) = 0\]

\[12\] Contrast this with our representation where infinitesimal gauge transformations are ill-defined.
The solution can be expanded as
\[ f(X) = \int \frac{dk}{|k|} e^{ik \cdot X} a_k + \text{c.c.} \]

\( a_k \)'s can be projected out of the solutions \( f(X) \)'s on any hypersurface using the Klein–Gordon inner product
\[ a_k = i \int \Sigma \sqrt{g} [u_k n^a \nabla_a f(x) - f(x) n^a \nabla_a u_k^*] \]
\[ = \int \left[ u_k^* \pi f - \sqrt{g} f \left( -ikX^+ - ikX^- \right) \right] \]
\[ = \int [\pi f + ikX^+ f - ikX^- f] \]
where we have used \( n^+ = \sqrt{X^+}, n^- = -\sqrt{X^-} \) and \( \sqrt{g} = \sqrt{X^+X^-} \).

Using \( k^\pm = \frac{1}{2}(k \pm |k|) \) we can show that
\[ a_k = \int e^{-ikX^- Y^-} \quad k > 0 \]
\[ a_k = \int e^{-ikX^+ Y^+} \quad k < 0 \]
\[ a_0 = \int \pi f. \]

\( a_k^* \) are defined by complex conjugating \( a_k \).

By explicit calculations one can check that these functions are Dirac observables. Their Poisson algebra is given by
\[ \{a_k, a_l\} = 0. \]
\[ \{a_k^*, a_l^*\} = 0. \]
\[ \{a_k, a_l^*\} = |k| \delta(k, l). \]  

5.1. Quantization of \( a_k \)

In this section, we show how to promote \((a_k, a_k^*)\) to densely defined operators on \( \mathcal{H}_{\text{phy}} \). This prescription can at best be viewed as an ad hoc way of trying to promote regulated expressions from \( \mathcal{H}_{\text{kin}} \) to \( \mathcal{H}_{\text{phy}} \). We hope that a better scheme for doing this emerges in future or that the one given here is more justified.

Given a (strong) Dirac observable (one that strongly commutes with the Virasoro constraints), the ideal way to promote it to an operator on \( \mathcal{H}_{\text{phy}} \) is as follows. One first defines an operator on \( \mathcal{H}_{\text{kin}} \) and if this operator is \( G \)-equivariant (where \( G \) is the direct product of two copies of diffeomorphisms acting on \( \mathbb{R} \)), then one can define an operator on \( \mathcal{H}_{\text{phy}} \) simply by dual action. We will show how this procedure fails here [25]. (This is analogous to a generic problem in LQG of defining connection-dependent operators on \( \mathcal{H}_{\text{diff}} \)). For \( k > 0 \),
\[ a_k = \int e^{ikX^- Y^-}. \]  

\( a_k^* \) are defined by complex conjugating \( a_k \).

By explicit calculations one can check that these functions are Dirac observables. Their Poisson algebra is given by
\[ \{a_k, a_l\} = 0. \]
\[ \{a_k^*, a_l^*\} = 0. \]
\[ \{a_k, a_l^*\} = |k| \delta(k, l). \]
In order to represent $a_k$ on $\mathcal{H}_{\text{kin}}$ we have to triangulate our spatial slice $\Sigma$ by 1-simplices (closed intervals). Let $T$ be a triangulation of $\sigma$. Given a state $f_s$ for the left-moving sector, we choose a triangulation $T(\gamma(s))$ adapted to $\gamma(s)$, i.e. the triangulation is such that all the vertices of $\gamma(s)$ are vertices of $T(\gamma(s))$. Classically, we know that

$$\frac{h_{\Delta_m}(Y^-) - h_{\Delta_{m+1}}(Y^-)}{|\Delta_m|} = Y^- \left( \frac{v_m + v_{m+1}}{2} \right) + O(|\Delta_m|^2),$$

(45)

where $\Delta_m \in T(\gamma(s))$ (it is a closed interval in, say, Cartesian coordinate system), and $(v_m, v_{m+1})$ are the beginning and terminating vertices of $\Delta_m$ respectively.

Now we can write $a_k$ as the limit of a Riemann sum,

$$a_k = \lim_{T \to \Sigma} a_{k,T(\gamma(s))},$$

(46)

where

$$a_{k,T(\gamma(s))} = \sum_{\Delta_m \in T(\gamma(s))} e^{ik\tilde{X}^{-}(v_m)} [h_{\Delta_m} Y^- - h_{\Delta_{m+1}}(Y^-)].$$

(47)

$a_{k,T(\gamma(s))}$ can be represented on $\mathcal{H}_{\text{kin}}$ as follows:

$$\hat{a}_{k,T(\gamma(s))} f_s^- = \sum_{\Delta_m \in T(\gamma(s))} e^{ik\tilde{X}^{-}(v_m)} [h_{\Delta_m} - h_{\Delta_{m+1}}] f_s^-.$$

(48)

A similar expression holds for $k < 0$ with $(X^-, Y^-)$ replaced by $(X^+, Y^+)$ and the resulting operator acting on $f_s^+$. Also one can (densely) define $\hat{a}_{k,T(\gamma(s))}$ using the inner product on $\mathcal{H}_{\text{kin}}$:

$$\hat{a}_{k,T(\gamma(s))} f_s^+ := \sum_{\Delta_m \in T(\gamma(s))} e^{-ik\tilde{X}^{-}(v_m)} [h_{\Delta_m} - h_{\Delta_{m+1}}] f_s^-.$$

(49)

At finite triangulation (i.e. when the number of simplices in $T$ is finite) $\hat{a}_{k,T(\gamma(s))}$ is not Virasoro equivariant,

$$\hat{U}(\phi^-) a_{k,T(\gamma(s))} \hat{U}^{-1}(\phi^-) = \hat{U}(\phi^-) \sum_{\Delta_m \in T(\gamma(s))} e^{ik\tilde{X}^{-}(v_m)} [h_{\Delta_m} - h_{\Delta_{m+1}}] \hat{U}^{-1}(\phi^-).$$

$$= \sum_{\Delta_m} \hat{U}(\phi^-) e^{ik\tilde{X}^{-}(v_m)} \hat{U}^{-1}(\phi^-) \hat{U}(\phi^-) [h_{\Delta_m} - h_{\Delta_{m+1}}] \hat{U}^{-1}(\phi^-)$$

$$= \sum_{\Delta_m} e^{ik\tilde{X}^{-}(\phi^{-1}(v_m))} [h_{\Delta_m} - h_{\Delta_{m+1}}]$$

$$= \sum_{\Delta_m \in T(\gamma(s))} e^{ik\tilde{X}^{-}(\gamma(s))} [h_{\Delta_m} - h_{\Delta_{m+1}}]$$

$$= a_{k,\phi(T(\gamma(s)))}.$$  

(50)

Thus, $\hat{a}_{k,T(\gamma(s))}$ cannot be promoted to an operator on $\mathcal{H}_{\text{phy}}$ simply by dual action. This problem was also encountered by Thiemann in [25]. As argued by him, if we try to remove the triangulation by taking the continuum limit then we either get zero (in weak operator topology) or infinity (in strong operator topology).

There are two ways to get around this problem. The first way is due to Thiemann.

There the idea was to use the graph (underlying a state) itself as a triangulation and define a strongly Virasoro-invariant operator on $\mathcal{H}_{\text{kin}}$. Here we propose a different way. Essentially we use a sort of gauge fixing in the space of (diff) equivalence class of charge networks (defined as the triple $(\gamma, \tilde{T}(\gamma), \tilde{k}(\gamma))$) to define an operator corresponding to $a_k$ on $\mathcal{H}_{\text{phy}}$. As will be argued later, Thiemann’s proposal can be considered as a special case of ours.

13 The idea of defining regulated operators on $\mathcal{H}_{\text{phy}}$ in this way was suggested to us by Madhavan Varadarajan.
Given an orbit of diffeomorphism equivalence class of charge networks, \([s] = [\phi \cdot s] | \phi \in \text{Diff } \Sigma \) we fix once and for all a network \(s_0 = (\gamma_0, \mathcal{I}(\gamma), \hat{k}(\gamma)) \) and a triangulation \(T(\gamma_0(s_0)) \) adapted to it. Now for any \(s \) in the orbit, such that \( s = \phi \cdot s_0 = (\phi^{-1}(\gamma_0), \mathcal{I}(\gamma), \hat{k}(\gamma)) \) we choose the corresponding triangulation \(T(\gamma(s))\) such that \( \hat{a}_{\gamma(s)} = \hat{U}(\phi)\hat{a}_{T(\gamma_0(s_0))}\hat{U}^{-1}(\phi) \). Now let \( \Psi \in \mathcal{H}_{\text{phy}} \). One can show that this family of operators is cylindrically consistent and defines an operator on \( \mathcal{H}_{\text{kin}} \). The resulting operator on \( \mathcal{H}_{\text{phy}} \) defined by the dual action turns out to be densely defined,

\[
(\hat{a}_k \Psi)[f_s] = \Psi[\hat{a}_{k,T(\gamma)} f_{s_0}]
\]

where, for the sake of simplicity, we have suppressed ± labels indicating left(right) moving sectors. Here as defined earlier \( \gamma_0(s_0) \) is the graph which is fixed in the orbit of \( \gamma(s) \), and \( T(\gamma_0(s_0)) \) is a fixed triangulation adapted to it. This proposal (of defining \( \hat{a}_k \) on \( \mathcal{H}_{\text{phy}} \)) is, as we emphasized earlier, rather ad hoc as it involves an arbitrary choice \( s_0 \) and triangulation \( T(\gamma_0(s_0)) \). It nonetheless results in a ‘regulated’ and densely defined operator on \( \mathcal{H}_{\text{phy}} \).

We will now argue that Thiemann’s proposal of defining a Virasoro-invariant operator directly on \( \mathcal{H}_{\text{kin}} \) (\([25], \text{p } 28\)) can be subsumed by the prescription given above. (Note that in \([25] \), spatial topology is compact \((S^3)\), whence we have to modify the proposal given there accordingly, as in our case the spatial manifold is \( \mathbb{R}^3 \).) Let us first note how Thiemann’s prescription applies to our perennials.

1. Choose the graph underlying a state itself as a triangulation (by adding fiducial edges if necessary).
2. Then the operator (in our case \( \hat{a}_{k,\gamma(s)} \)) acting on a basis state \( f_s \) results in a linear combination of states \( f_{s'} \) such that \( \gamma(s') \subset \gamma(s) \). Thus,

\[
\hat{U}(\phi)\hat{a}_{k,\gamma(s)} f_s = \sum b_I \hat{U}(\phi) f_{s_I} = \sum b_I f_{\phi(s_I)}.
\]

Using (47) one can write this more explicitly as

\[
\hat{U}(\phi) \left( \sum_{e \in \mathcal{E}(\gamma)} e^{ik \cdot \vec{e} - (k \cdot e)} \left[ h_e(Y^-) - h_{e^{-1}}(Y^-) \right] \right) f_s
\]

\[
= \hat{U}(\phi) \left( \sum_{e \in \mathcal{E}(\gamma')} e^{ik \cdot \vec{e} - (k \cdot e)} e^{\Delta h_{\phi(e)}} \left[ f_{s'} - f_{s''} \right] \right)
\]

\[
= \sum_{e \in \mathcal{E}(\gamma')} e^{ik \cdot \vec{e} - (k \cdot e)} e^{\Delta h_{\phi(e)}} \left[ f_{s'} - f_{s''} \right],
\]

where \( s' = (\gamma, ((k_1, l_1), \ldots, (k_e, l_e + 1), \ldots, (k_N, l_N))) \) and \( s'' = (\gamma, ((k_1, l_1), \ldots, (k_e, l_e - 1), \ldots, (k_N, l_N))) \) and in the last line we have made use of the fact that \( (k_{\phi^{-1}(e)}, l_{\phi^{-1}(e)}) = (k_e, l_e) \).

However it is easy to convince oneself that the last line in (53) equals

\[
\hat{a}_{k,\gamma(\phi(s))} \phi f_s = \hat{a}_{k,T} \hat{U}(\phi) f_s.
\]

This shows Virasoro invariance of \( \hat{a}_{k,\gamma(s)} \).

The above proof crucially relies on the fact that the triangulation used to regulate the operator is the same as the graph (underlying the state on which the operator acts) itself. We now show how to achieve this by adding fiducial edges to the graph. (This is where Thiemann’s prescription has to be slightly modified, as in \([25] \) spatial topology is that of \( S^3 \).)
Figure 1.

Given any basis state $f_s$ we can always write it as a state $\tilde{f}_s \in \mathcal{H}_{\text{kin}}$ such that $\tilde{\gamma}(\tilde{s}) = e_L \cup \gamma(s) \cup e_R$, where $e_L$ and $e_R$ are the edges from $-\infty$ to the initial vertex of $\gamma(s)$ and from the final vertex of $\gamma(s)$ to $\infty$ respectively (see the figure below). In fact we can define a new basis for $\mathcal{H}_{\text{kin}}$ as follows. Any element of the basis $f_{\tilde{s}}$ is defined to be based on a graph which is of the form $\tilde{\gamma}(\tilde{s}) = e_L \cup \gamma \cup e_R$ where $\gamma$ is a subgraph of $\tilde{\gamma}(\tilde{s})$ such that $e_L$ and $e_R$ are as defined above. The charge pairs $(k_{e_L}, l_{e_L})$ are allowed to be $(0, 0)$, but $(k_{e_N}, l_{e_N})$ are not allowed to be $(0, 0)$. (Here $e_1$ and $e_N$ are the initial and final edges of $\gamma$ respectively.)

Thus the graphs on which the new basis is defined become the triangulation of $\Sigma$ and Thiemann’s prescription follows.

How does our definition of $\hat{a}'_k$ subsume Thiemann’s definition as a special case? The answer is as follows. Once we choose an $s_0$ in the orbit of $s$ choose $T(\gamma_0(s_0)) = e_L \cup \gamma_0 \cup e_R$ (as shown in figure 1).

The resulting operator $\hat{a}_k^{\dagger}T(\gamma_0(s_0))$ is Virasoro invariant on $\mathcal{H}_{\text{kin}}$ whence $\hat{a}'_k$ is the dual of a linear operator $\hat{a}^{\dagger}_k$ obtained on $\mathcal{H}_{\text{kin}}$ via cylindrical consistency.

We now show that the adjoint of $\hat{a}'_k$ defined on $\mathcal{H}_{\text{phy}}$ using the inner product is consistent with the definition

$$ (\hat{a}'_k^* \Psi)[f_s] = \Psi[\hat{a}_k T(\gamma_0(s_0)), f_{s_0}]. $$

Let $\Psi_1, \Psi_2 \in \mathcal{H}_{\text{phy}}$, then

$$ \langle \Psi_1, \hat{a}'^* \Psi_2 \rangle = \langle \Psi_1 \hat{\varepsilon}'_1, \Psi_2 \rangle $$

$$ = \langle \Psi_2, \hat{a}' \Psi_1 \rangle^* $$

$$ = [\hat{a}' \Psi_1]_{f_{s_0}}^* $$

$$ = \left[ \Psi_1(\hat{a}^{\dagger}_{T(\gamma_1)}), \right]^* $$

$$ = \sum_\phi \langle \hat{U}(\phi) f_{s_2}, \hat{a}^{\dagger}_{T(\gamma_2)} f_{s_1} \rangle^* $$

$$ = \sum_\phi \langle \hat{U}(\phi) f_{s_2}, \hat{a}^{\dagger}_{T(\gamma_1)}, \hat{U}(\phi)^{-1} f_{\phi^{-1}(s_1)} \rangle $$

$$ = \sum_\phi \langle f_{s_2}, \hat{U}(\phi) \hat{a}^{\dagger}_{T(\gamma_1)}, f_{\phi^{-1}(s_1)} \rangle $$

$$ = \sum_\phi \langle \hat{U}(\phi)^{-1} f_{s_2}, \hat{a}_{T(\gamma_1)} f_{s_1} \rangle $$

$$ = \Psi_2(\hat{a}_{T(\gamma_1)} f_{s_1}), $$

where $\eta(f_{s_0}) = \Psi_I$ for $I = 1, 2$ and we have suppressed the $\pm$ indices for clarity.

14 The introduction of a new basis is only to show how Thiemann’s prescription is consistent with ours and will not be used in the rest of the paper anywhere.
5.2. Commutation relations

Next we study the commutator algebra generated by the Dirac observables \((a_k, a^*_l)\) in the quantum theory. Contrary to the classical Poisson algebra which closes, we show that in the quantum theory even \((a_k, a_l)\) do not in general commute with each other. It is plausible that this will have serious implications on the causal structure of the quantum theory and the issue is far from being resolved. Recall that the physical content of the parametrized free field theory (at least classically) is the same as that of the ordinary free field theory on flat spacetime. Whence we could have started with the reduced phase space coordinatized by \((a_k, a^*_l)\), and its representation on Fock space will result in a quantum theory in which fields separated by the spacelike interval will commute. Also the two-point functions will decay exponentially outside the light cone. However, if the commutator algebra of \((\hat{a}_k, \hat{a}_l^*)\) gets deformed in the quantum theory then it is not clear in what sense the causal structure defined by the background spacetime is preserved. In fact, as we are not aware of a state (or a class of states) in \(\mathcal{H}^{\text{phy}}\) which correspond to the Fock vacuum, it is not even known how to define two-point functions (using which we can study causal relations).

5.3. The commutator

When \(k < 0\) and \(l > 0\) it is clear that \([\hat{a}_k, \hat{a}_l]\) will be trivially zero as \(\hat{a}_k\) acts on right-moving sector \((\mathcal{H}^{\text{phy}}_k)\) and \(\hat{a}_l\) acts on left-moving sector \((\mathcal{H}^{\text{phy}}_l)\) whence they commute.

Let us consider the case when \(k, l < 0\). The remaining case \((k, l > 0)\) can be handled similarly.

As \(a_k = \int Y^*(x) e^{i\hat{X}^*(x)}\), we only look at the right-moving (+) sector of \(\mathcal{H}^{\text{phy}}\)

\[
(\hat{a}_k, \hat{a}_l |\Psi^+\rangle f^+_s = ((\hat{a}_k \hat{a}_l - \hat{a}_l \hat{a}_k) \Psi^+) f^+_s = (\hat{a}_k \hat{a}_l \Psi^+) f^+_s - (\hat{a}_l \hat{a}_k \Psi^+) f^+_s = (\hat{a}_l \Psi^+) (\hat{a}^*_k T_{(y_0(s_0))} f^+_s) - (\hat{a}_k \Psi^+) (\hat{a}^*_l T_{(y_0(s_0))} f^+_s),
\]

where \(f^+_s\) is an arbitrary state in the kinematical Hilbert space of the right-moving sector \(\mathcal{H}^{\text{kin}}_k\) and as before \(s_0\) is a fixed charge network in the orbit of \(s\). Let us look at both the terms separately.

5.4. Term 1 - \((\hat{a}_l \Psi^+) (\hat{a}^*_k T_{(y_0(s_0))} f^+_s)\)

Now we employ a specific choice of triangulation \(T(y_0(s_0))\). This choice is motivated by the requirement of simplicity. We will argue shortly that the result (at least qualitatively) does not depend on this particular choice.

So let us choose \(T(y_0(s_0)) = y_0 \cup e_L \cup e_R\), where \(e_L\) and \(e_R\) are as shown in figure 1.

(Remark: with this choice of the triangulation the continuum limit is approached only when \(|E(y_0)|\) tends to \(\infty\).) So,

\[
\hat{a}^*_k T_{(y_0(s_0))} = \frac{1}{2i} \sum_{e_I \in (y_0(s_0) \cup L_R)} e^{-i\hat{X}^*(v_I)} [h_{e_I} - h_{e_I}^{-1}],
\]

where \(v_I = b(e_I)\).

Similarly we choose \(T(y_0(s_0)) = y_0 \cup e_L \cup e_R\), which implies

\[
\Psi (\hat{a}^*_l T_{(y_0(s_0))} \hat{a}^*_k T_{(y_0(s_0))} f^+_s) = -\frac{1}{4} \Psi \left( \sum_{e_I} e^{-i\hat{X}^*(v_I)} [h_{e_I} - h_{e_I}^{-1}] \sum_{e_I} e^{-i\hat{X}^*(v_I)} [h_{e_I} - h_{e_I}^{-1}] \right).
\]

(59)
The second term \((\hat{a}_I, \Psi^*)(\hat{a}_I^\dagger T_{(j)(m)}) f^+_0\) can be evaluated similarly and we get

\[
([\hat{a}_I^\dagger, \hat{a}_I])\Psi(f^+_0) = \Psi \left( \sum_{e_I, e_J} e^{-i\hat{X}^*(e_I)} e^{-i\hat{X}^*(e_J)} \left[ (h_{e_I} - h_{e_J}), (h_{e_I} - h_{e_J}) \right] \right). 
\] (60)

In the above double sum only those edges \((e_I, e_J)\) contribute for which \(e_I \cap e_J \neq 0\). Consider the following two pairs \((I = M, J = M + 1)\) and \((I + M + 1, J = M)\) for some fixed \(M\).

The contribution to the commutator coming from the above pairs is

\[
-\frac{1}{4} \Psi \left( e^{-i\hat{X}^*(e_M)} e^{-i\hat{X}^*(e_M)} \left[ (h_{e_M}, h_{e_M} - h_{e_M}) \right] + e^{-i\hat{X}^*(e_M)} e^{-i\hat{X}^*(e_M)} \left[ (h_{e_M} - h_{e_M}), (h_{e_M} - h_{e_M}) \right] f^+_0 \right)
\]

\[
= -\frac{1}{4} \Psi \left( e^{-i\hat{X}^*(e_M)} e^{-i\hat{X}^*(e_M)} \left[ (h_{e_M}, h_{e_M} - h_{e_M}) \right] f^+_0 \right)\]

\[
\times (e^{-i\hat{X}^*(e_M)} e^{-i\hat{X}^*(e_M)} e^{-i\hat{X}^*(e_M)} f^+_0). 
\] (61)

Now using (25) one can show that

\[
[h_{e_I}, h_{e_J}] = \frac{1}{2i} \sin(hs(e_I, e_J)) h_{e_I e_J}. 
\] (62)

It is now straightforward to evaluate the commutators in (61), whence given a pair of successive edges which lie within the graph \((e_I, e_{I+1}) (I = 1, \ldots, N - 1)\) their contribution to \(([\hat{a}_I^\dagger, \hat{a}_I])\Psi(f^+_0)\) is

\[
-\frac{1}{4} \Psi \left[ \frac{1}{2i} \sin \left( \frac{h}{2} \right) \sum_{l=1}^{N-1} \left( h_{e_I e_{I+1}} + h_{e_{I+1} e_{I+1}} + h_{e_{I+1} e_{I+1}} + h_{e_{I+1} e_{I+1}} \right) \right]
\]

\[
\times \left[ e^{-i\hat{X}^*(e_I)} e^{-i\hat{X}^*(e_I)} - e^{-i\hat{X}^*(e_I)} e^{-i\hat{X}^*(e_I)} \right] f^+_0
\]

\[
= -\frac{1}{4} \Psi \left[ \left[ \frac{1}{2i} \sin \left( \frac{h}{2} \right) \sum_{l=1}^{N-1} \left( h_{e_I e_{I+1}} + h_{e_{I+1} e_{I+1}} + h_{e_{I+1} e_{I+1}} + h_{e_{I+1} e_{I+1}} \right) \right] f^+_0 \right]
\]

\[
\times \left[ e^{-i\hat{X}^*(e_I)} e^{-i\hat{X}^*(e_I)} - e^{-i\hat{X}^*(e_I)} e^{-i\hat{X}^*(e_I)} \right] f^+_0. 
\] (63)

where we have used \(\hat{X}^*(e_I) f^+_0 = \frac{1}{2i} (k_{e_I}^+ + k_{e_I}^+) f^+_0\) and defined \(k_{e_0} = 0\).

Finally there are contributions from the pair \((e_L, e_I)\) and \((e_N, e_R)\),

\[
-\frac{1}{4} \Psi \left( e^{-i\hat{X}^*(e_L)} e^{-i\hat{X}^*(e_L)} \sin \left( \frac{h}{2} \right) \left[ h_{e_L e_L} + h_{e_L e_L} + h_{e_L e_L} + h_{e_L e_L} \right] f^+_0 \right)
\]

\[
-\frac{1}{4} \Psi \left( e^{-i\hat{X}^*(e_R)} e^{-i\hat{X}^*(e_R)} \sin \left( \frac{h}{2} \right) \left[ h_{e_R e_R} + h_{e_R e_R} + h_{e_R e_R} + h_{e_R e_R} \right] f^+_0 \right)
\]

\[
\times \left[ e^{-i\hat{X}^*(e_L)} e^{-i\hat{X}^*(e_L)} - e^{-i\hat{X}^*(e_L)} e^{-i\hat{X}^*(e_L)} \right] f^+_0. 
\]

Let \(e_L = e_0\) and \(e_R = e_{N+1}, k_{e_0} = k_{e_{N+1}} = 0\) we finally get

\[
([\hat{a}_e, \hat{a}_I])\Psi(f^+_0) = \frac{1}{4} \Psi \left( \frac{1}{2i} \sin \left( \frac{h}{2} \right) \sum_{l=0}^{N} \left[ h_{e_I e_{I+1}} + h_{e_{I+1} e_{I+1}} + h_{e_{I+1} e_{I+1}} + h_{e_{I+1} e_{I+1}} \right] \right)
\]

\[
\times \left[ e^{-i\hat{X}^*(e_L)} e^{-i\hat{X}^*(e_L)} - e^{-i\hat{X}^*(e_L)} e^{-i\hat{X}^*(e_L)} \right] f^+_0. 
\] (64)
Thus it is clear that in general the commutator $[\hat{a}_k, \hat{a}_l]$ ($k, l < 0$) does not vanish on $\mathcal{H}_{\text{phy}}$. The commutator for $[\hat{a}_k, \hat{a}_l]$ with ($k, l > 0$) is exactly similar with all operators acting on the left-moving sector.

Now we give a heuristic proposal showing the existence of (a class of) states on which the commutator $[\hat{a}_k, \hat{a}_l]$ vanishes. Ideally one would like to do a semi-classical analysis of the expectation value of the commutators to see if the non-zero contributions are sub-leading. This is an open question that we have not addressed in the present paper. In what follows we argue for the existence of states (possibly in an infinite tensor product (ITP extension [31]) of $\mathcal{H}_{\text{phy}}$) on which the commutator vanishes.

Note that given a $\Psi \in \mathcal{H}_{\text{phy}}$ $[\hat{a}_k, \hat{a}_l] |\Psi \rangle$ is non-zero iff the ‘embedding component’ of $\Psi$ is the group-averaged distribution obtained from the ‘embedding component’ of $\hat{a}_k$. In other words if $\Psi = \{2N + 1, 2N + 2, \{l_{-N}, l_1, \ldots, [N, l_{2N}]\}$ where the matter charges $(l_1, \ldots, l_{2N})$ are arbitrary but non-zero then $[\hat{a}_k, \hat{a}_l] |\Psi \rangle$ is non-zero iff $[E(\gamma)] = 2N + 1$; the embedding charges on the edges of $\gamma$ constitute the set $(-N, \ldots, N)$ and the matter charges form a set $(l_1, \ldots, l_I \pm 1, l_{I+1} \pm 1, \ldots, l_{2N})$ for some $I$.

\[
[\hat{a}_k, \hat{a}_l] \Psi \rangle = -\frac{1}{4} \Psi \sin \left(\frac{1}{2} \hbar \tilde{l}\right) \left(\sum_{l=0}^{N} [h_{e_i} e_{e_i} + h_{e_{i-1}} e_{e_{i+1}} + h_{e_{i+1}} e_{e_{i-1}} + h_{e_i}] \right) \times \sum_{n=-N}^{N} \left[ e^{-\frac{1}{2} \hbar (l+k) n} e^{-\frac{1}{2} \hbar (l-k) n} - e^{-\frac{1}{2} \hbar (l+k) n} e^{-\frac{1}{2} \hbar (l-k) n} \right] f_n^+ \right),
\]

(65)

Now as $N \to \infty$ and each $e_l$ shrinks to its vertex $v_l$, and if we assume that to leading order in $\frac{1}{N}$, $h_{e_i} \to 1$ then one gets

\[
[\hat{a}_k, \hat{a}_l] \Psi \rangle = -\frac{1}{4} \Psi \left(\sum_{n \in \mathbb{Z}} \left[ e^{-\frac{1}{2} \hbar (l+k) n} e^{-\frac{1}{2} \hbar (l-k) n} - e^{-\frac{1}{2} \hbar (l+k) n} e^{-\frac{1}{2} \hbar (l-k) n} \right] f_n^+ \right)
= -\frac{1}{4} \Psi \left( \sin \left(\frac{1}{2} \hbar (l-k) \right) f_n^+ \right),
\]

(66)

which equals 0 for $l, k < 0$.

A couple of comments are in order:

1. We have not displayed semi-classicality in the sense that we have not shown that the non-vanishing terms in $[\hat{a}_k, \hat{a}_l]$ are sub-leading corrections on a class of states in $\mathcal{H}_{\text{phy}}$.

2. The above result does not depend on our choice of triangulation $T(\gamma_0(\mathcal{S}_0)) = \gamma_0 \cup e_L \cup e_R$.

Consider any triangulation, $T$, which is adapted to $\gamma_0$ in the sense that the vertex set of $\gamma_0$ is a subset of the vertex-set of $T$. Then it can be shown that only those edges which intersect the vertices of the graph contribute. Contributions from all other edges cancel out pairwise.

The calculation of $[\hat{a}_k, \hat{a}_l]$ proceeds similarly,

\[
[\hat{a}_k, \hat{a}_l] \Psi \rangle = \frac{1}{4} \Psi \sin \left(\frac{1}{2} \hbar \tilde{l}\right) \left(\sum_{l=0}^{N} [h_{e_i} e_{e_i} + h_{e_{i-1}} e_{e_{i+1}} + h_{e_{i+1}} e_{e_{i-1}} + h_{e_i}] \right) \times \left( e^{\frac{1}{2} \hbar (k \pm k_i) n} e^{-\frac{1}{2} \hbar (k \pm k_i) n} - e^{-\frac{1}{2} \hbar (k \pm k_i) n} e^{\frac{1}{2} \hbar (k \pm k_i) n} \right) f_n^+ \right),
\]

(67)
which as $N \to \infty$ and each $e_I$ shrinks to its vertex $v_I$ gives

\[
(\hat{a}_k \hat{a}_l^* \hat{a}_m^* \hat{a}_n^* \Psi)(f_s^*) = \frac{1}{\sqrt{\hbar}} \sin \left( \frac{\hbar}{2} \right) \Psi \left( \delta(l - k) \sin \left( \frac{1}{2} \hbar (l + k) \right) f_m^* \right), \tag{68}
\]

which is a specific quantum deformation of the classical Poisson bracket.

Our heuristic calculations show that it is plausible that on a specific class of states with countably infinite edges the commutator algebra generated by $(a^k, a^k \ast$ and 1) closes and is a specific deformation of the Poisson algebra. Such states cannot lie in $\mathcal{H}_{phy}$ but in the infinite tensor product extension thereof [31].

This suggests that in [25], semi-classical states have been defined by using graphs with a large but finite number of edges. However, based on the heuristic calculations displayed above, we believe that when the spatial slice is non-compact, the ideal home for semi-classical states is the ITP extension of $\mathcal{H}_{phy}$.

6. Discussion

The primary aim of this work has been to obtain a quantum theory of dilaton gravity by combining the ideas of the parametrized field theory and polymer (loop) quantization. We started with a parametrized field theory which is canonically equivalent to the KRV action. By choosing appropriate quantum algebras for the embedding and matter sectors, we obtained a Hilbert space which carries a unitary (and anomaly-free) representation of the spacetime diffeomorphism group. Using the so-called group averaging method, we were able to get rid of the quantum gauge degrees of freedom and obtain the physical spectrum of the theory in a rather straightforward manner. The parametrized field theory framework gave us a complete set of Dirac observables which we could promote to well-defined operators on $\mathcal{H}_{phy}$. This required rather ad hoc choices of triangulations and the final operators are dependent on the choice of triangulation. This ad hoc-ness permeates all the consequent constructions and calculations. We hope that eventually a better (regularization-independent) scheme emerges to quantize such observables or at least the regularization-dependent scheme introduced in this paper can find more physical justification.

Unlike the Fock space which by definition is an irreducible representation of the Poisson algebra of mode oscillators $(a_k, a^\ast_k)$, $\mathcal{H}_{phy}$ carries a representation of a deformed algebra. It is a faithful deformation of the classical algebra in the sense that all the corrections are $O(\hbar)$. It is an interesting open question to hunt for the full quantum algebra and try to find a physical interpretation of its elements which do not have a well-defined classical limit (the commutator $[\hat{a}_k, \hat{a}_l]$ defines one such element).

We would once again like to emphasize that the most worrisome feature of our work is the regularization dependence of observables. It is imperative that one finds a physical interpretation for this dependence or give a more sophisticated (regularization-independent) quantization scheme.

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