Polynomial least squares fitting in the Bernstein basis

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Abstract

The problem of polynomial regression in which the usual monomial basis is replaced by the Bernstein basis is considered. The coefficient matrix $A$ of the overdetermined system to be solved in the least squares sense is then a rectangular Bernstein-Vandermonde matrix.

In order to use the method based on the QR decomposition of $A$, the first stage consists of computing the bidiagonal decomposition of the coefficient matrix $A$.

Starting from that bidiagonal decomposition, an algorithm for obtaining the QR decomposition of $A$ is the applied. Finally, a triangular system is solved by using the bidiagonal decomposition of the R-factor of $A$.

Some numerical experiments showing the behavior of this approach are included.

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1 Introduction

Given $\{x_i\}_{1 \leq i \leq l+1}$ pairwise distinct real points and $\{f_i\}_{1 \leq i \leq l+1} \in \mathbb{R}$, let us consider a degree $n$ polynomial

$$P(x) = c_0 + c_1 x + \ldots + c_n x^n$$

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for some $n \leq l$. Such a polynomial is a least squares fit to the data if it minimizes the sum of the squares of the deviations from the data,

$$l+1 \sum_{i=1}^{l+1} |P(x_i) - f_i|^2.$$  

Computing the coefficients $c_j$ of that polynomial $P(x)$ is equivalent to solve, in the least squares sense, the over-determined linear system $Ac = f$, where $A$ is the rectangular $(l + 1) \times (n + 1)$ Vandermonde matrix corresponding to the nodes $\{x_i\}_{1 \leq i \leq l+1}$.

Taking into account that $A$ has full rank $n + 1$, the problem has a unique solution given by the unique solution of the linear system

$$A^TAc = A^Tf,$$

the normal equations.

Since $A$ is usually an ill-conditioned matrix, it was early recognized that solving the normal equations was not an adequate method. Golub [7], following previous ideas by Householder, suggested the use of the QR factorization of $A$, which involves the solution of a linear system with the triangular matrix $R$.

Let us observe that, if $A = QR$ with $Q$ being an orthogonal matrix, then using the condition number in the spectral norm we have

$$\kappa_2(R) = \kappa_2(A),$$

that is, $R$ inherits the ill-conditioning of $A$ while $\kappa_2(A^TA) = \kappa_2(A)^2$.

In addition, as it was already observed by Golub in [8] (see also Section 20.1 of [9]), although the use of the orthogonal transformation avoids some of the ill effects inherent in the use of normal equations, the value $\kappa_2(A)^2$ is still relevant to some extent.

Consequently a good idea is to use, instead of the monomial basis, a polynomial basis which leads to a matrix $A$ with smaller condition number than the Vandermonde matrix.

It is frequently assumed that this happens when bases of orthogonal polynomials, such as the basis of Chebyshev polynomials, are considered. However, this fact is true when special sets of nodes are considered, but not in the case of general nodes. A basis which leads to a matrix $A$ better conditioned than the Vandermonde matrix is the Bernstein basis of polynomials, a widely used basis in Computer Aided Geometric Designed due to the good properties that it possess (see, for instance, [2,10]). We illustrate these facts with Table 1,
where the condition numbers of Vandermonde, Chebyshev-Vandermonde and Bernstein-Vandermonde matrices are presented for the nodes considered in Example 5.1 and in Example 5.2.

| Example | V     | TV    | BV    |
|---------|-------|-------|-------|
| 5.1     | 1.7e+12 | 1.7e+12 | 2.0e+05 |
| 5.2     | 2.5e+14 | 4.0e+14 | 5.3e+08 |

Table 1
Condition numbers of the Vandermonde V, Chebyshev-Vandermonde TV and Bernstein-Vandermonde BV matrices

Let us observe that, without lost of generality, we can consider the nodes \( \{x_i\}_{1 \leq i \leq l+1} \) ordered and belonging to \((0,1)\). So, we will solve the following problem:

Let \( \{x_i\}_{1 \leq i \leq l+1} \in (0,1) \) a set of points such that \( 0 < x_1 < \ldots < x_{l+1} < 1 \). Our aim is to compute a polynomial

\[
P(x) = \sum_{j=0}^{n} c_j b_j^{(n)}(x)
\]

expressed in the *Bernstein basis* of the space \( \Pi_n(x) \) of the polynomials of degree less that or equal to \( n \) on the interval \([0,1]\)

\[
B_n = \{ b_j^{(n)}(x) = \binom{n}{j}(1-x)^{n-j}x^j, \quad j = 0, \ldots, n \},
\]

such that \( P(x) \) minimizes the sum of the squares of the deviations from the data.

This problem is equivalent to solve in the least squares sense the overdetermined linear system \( Ac = f \) where now

\[
A = \begin{pmatrix}
\binom{n}{0}(1-x_1)^n & \binom{n}{1}x_1(1-x_1)^{n-1} & \cdots & \binom{n}{n}x_1^n \\
\binom{n}{0}(1-x_2)^n & \binom{n}{1}x_2(1-x_2)^{n-1} & \cdots & \binom{n}{n}x_2^n \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n}{0}(1-x_{l+1})^n & \binom{n}{1}x_{l+1}(1-x_{l+1})^{n-1} & \cdots & \binom{n}{n}x_{l+1}^n
\end{pmatrix}
\]

(1.1)

is the \((l+1) \times (n+1)\) Bernstein-Vandermonde matrix for the Bernstein basis \( B_n \) and the nodes \( \{x_i\}_{1 \leq i \leq l+1} \),

\[
f = (f_1, f_2, \ldots, f_{l+1})^T
\]

(1.2)
is the data vector, and
\[ c = (c_1, c_2, \ldots, c_{n+1})^T \]
is the vector containing the coefficients of the polynomial that we want to compute.

The rest of the paper is organized as follows. Neville elimination and total positivity are considered in Section 2. In Section 3, the bidiagonal factorization of a rectangular Bernstein-Vandermonde matrix is presented. The algorithm for computing the regression polynomial in Bernstein basis is given in Section 4. Finally, Section 5 is devoted to illustrate the accuracy of our algorithm by means of some numerical experiments.

## 2 Basic results on Neville elimination and total positivity

In this section we will briefly recall some basic results on Neville elimination and total positivity which we will apply in Section 3. Our notation follows the notation used in [4] and [5]. Given \( k, n \in \mathbb{N} \) \( (1 \leq k \leq n) \), \( Q_{k,n} \) will denote the set of all increasing sequences of \( k \) positive integers less than or equal to \( n \).

Let \( A \) be an \( l \times n \) real matrix. For \( k \leq l, m \leq n \), and for any \( \alpha \in Q_{k,l} \) and \( \beta \in Q_{m,n} \), we will denote by \( A[\alpha|\beta] \) the submatrix \( k \times m \) of \( A \) containing the rows numbered by \( \alpha \) and the columns numbered by \( \beta \).

The fundamental tool for obtaining the results presented in this paper is the Neville elimination \([4,5]\), a procedure that makes zeros in a matrix adding to a given row an appropriate multiple of the previous one. We will describe the Neville elimination for a matrix \( A = (a_{i,j})_{1 \leq i \leq l; 1 \leq j \leq n} \) where \( l \geq n \). The case in which \( l < n \) is analogous.

Let \( A = (a_{i,j})_{1 \leq i \leq l; 1 \leq j \leq n} \) be a matrix where \( l \geq n \). The Neville elimination of \( A \) consists of \( n - 1 \) steps resulting in a sequence of matrices \( A := A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_n \), where \( A_t = (a_{i,j}^{(t)})_{1 \leq i \leq l; 1 \leq j \leq n} \) has zeros below its main diagonal in the \( t - 1 \) first columns. The matrix \( A_{t+1} \) is obtained from \( A_t \) \( (t = 1, \ldots, n) \) by using the following formula:

\[
a_{i,j}^{(t+1)} := \begin{cases} 
  a_{i,j}^{(t)} & \text{if } i \leq t \\
  a_{i,j}^{(t)} - \frac{a_{i,t}^{(t)}}{a_{i-1,t}^{(t)}} a_{i-1,j}^{(t)} & \text{if } i \geq t + 1 \text{ and } j \geq t + 1 \\
  0 & \text{otherwise.}
\end{cases}
\]

(2.1)

In this process the element
\[
p_{i,j} := a_{i,j}^{(j)} \quad 1 \leq j \leq n, \quad j \leq i \leq l
\]
is called pivot \((i, j)\) of the Neville elimination of \(A\). The process would break
down if any of the pivots \(p_{i,j}\) (\(1 \leq j \leq n, j \leq i \leq l\)) is zero. In that case
we can move the corresponding rows to the bottom and proceed with the new
matrix, as described in [4]. The Neville elimination can be done without row
exchanges if all the pivots are nonzero, as it will happen in our situation. The
pivots \(p_{i,i}\) are called diagonal pivots. If all the pivots \(p_{i,j}\) are nonzero, then
\(p_{i,1} = a_{i,1} \forall i\) and, by Lemma 2.6 of [4]

\[
p_{i,j} = \frac{\det A[i-j+1, \ldots, i]1, \ldots, j]}{\det A[i-j+1, \ldots, i-1|1, \ldots, j-1]} \quad 1 < j \leq n, j \leq i \leq l. \quad (2.2)
\]

The element

\[
m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}} \quad 1 \leq j \leq n, j < i \leq l \quad (2.3)
\]

is called multiplier of the Neville elimination of \(A\). The matrix \(U := A_n\) is
upper triangular and has the diagonal pivots in its main diagonal.

The complete Neville elimination of a matrix \(A\) consists on performing the
Neville elimination of \(A\) for obtaining \(U\) and then continue with the Neville
elimination of \(U^T\). The pivot (respectively, multiplier) \((i, j)\) of the complete
Neville elimination of \(A\) is the pivot (respectively, multiplier) \((j, i)\) of the
Neville elimination of \(U^T\), if \(j \geq i\). When no row exchanges are needed in the
Neville elimination of \(A\) and \(U^T\), we say that the complete Neville elimination
of \(A\) can be done without row and column exchanges, and in this case the
multipliers of the complete Neville elimination of \(A\) are the multipliers of the
Neville elimination of \(A\) if \(i \geq j\) and the multipliers of the Neville elimination
of \(A^T\) if \(j \geq i\).

A matrix is called totally positive (respectively, strictly totally positive) if all
its minors are nonnegative (respectively, positive). The Neville elimination
characterizes the strictly totally positive matrices as follows [4]:

**Theorem 2.1.** A matrix is strictly totally positive if and only if its complete
Neville elimination can be performed without row and column exchanges,
the multipliers of the Neville elimination of \(A\) and \(A^T\) are positive, and the
diagonal pivots of the Neville elimination of \(A\) are positive.

It is well known [2] that the Bernstein-Vandermonde matrix is a strictly totally
positive matrix when the nodes satisfy \(0 < x_1 < x_2 < \ldots < x_{l+1} < 1\), but this
result will also be shown to be a consequence of our Theorem 3.2.
In this section we consider the bidiagonal factorization of the Bernstein-Vandermonde matrix $A$ of (1.1).

Let us observe that when $l = n$ this matrix $A$ is the coefficient matrix of the linear system associated with a Lagrange interpolation problem in the Bernstein basis $B_n$ whose interpolation nodes are $\{x_i : i = 1, \ldots, n + 1\}$. A fast and accurate algorithm for solving this linear system, and therefore the corresponding Lagrange interpolation problem in the Bernstein basis can be found in [13]. A good introduction to the interpolation theory can be seen in [3].

The following two results will be the key to construct our algorithm.

**Proposition 3.1.** (See [13]) Let $A$ be the square Bernstein-Vandermonde matrix of order $n + 1$ for the Bernstein basis $B_n$ and the nodes $x_1, x_2, \ldots, x_{n+1}$. We have:

$$\det A = \begin{pmatrix} n \\ 0 \end{pmatrix} \begin{pmatrix} n \\ 1 \end{pmatrix} \cdots \begin{pmatrix} n \\ n \end{pmatrix} \prod_{1 \leq i < j \leq n+1} (x_j - x_i).$$

**Theorem 3.2.** Let $A = (a_{i,j})_{1 \leq i \leq l+1; 1 \leq j \leq n+1}$ be a Bernstein-Vandermonde matrix for the Bernstein basis $B_n$ whose nodes satisfy $0 < x_1 < x_2 < \ldots < x_l < x_{l+1} < 1$. Then $A$ admits a factorization in the form

$$A = F_l F_{l-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n \quad (3.1)$$

where $G_j$ are $(n + 1) \times (n + 1)$ upper triangular bidiagonal matrices ($j = 1, \ldots, n$), $F_i$ are $(l + 1) \times (l + 1)$ lower triangular bidiagonal matrices ($i = 1, \ldots, l$), and $D$ is a $(l + 1) \times (n + 1)$ diagonal matrix.

**Proof.** The matrix $A$ is strictly totally positive (see [2]) and therefore, by Theorem 2.1, the complete Neville elimination of $A$ can be performed without row and column exchanges providing the following factorization of $A$ (see [6]):

$$A = F_l F_{l-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n,$$
where $F_i \ (1 \leq i \leq l)$ are $(l + 1) \times (l + 1)$ bidiagonal matrices of the form

$$F_i = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 & m_{i+1,1} & 1 \\ 0 & 1 & \cdots & 0 & 1 & m_{i+2,2} & 1 & \cdots & m_{l,l-i} & 1 \end{pmatrix}, \quad (3.2)$$

$G^T_i \ (1 \leq i \leq n)$ are $(n + 1) \times (n + 1)$ bidiagonal matrices of the form

$$G^T_i = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 & \tilde{m}_{i+1,1} & 1 \\ 0 & 1 & \cdots & 0 & 1 & \tilde{m}_{i+2,2} & 1 & \cdots & \tilde{m}_{n,n-i} & 1 \end{pmatrix}, \quad (3.3)$$

and $D$ is the $(l + 1) \times (n + 1)$ diagonal matrix whose $i$th $(1 \leq i \leq n + 1)$ diagonal entry is the diagonal pivot $p_{i,i} = a^{(i)}_{i,i}$ of the Neville elimination of $A$:

$$D = (d_{i,j})_{1 \leq i \leq l+1; 1 \leq j \leq n+1} = \text{diag}\{p_{1,1}, p_{2,2}, \ldots, p_{n+1,n+1}\}. \quad (3.4)$$

Taking into account that the minors of $A$ with $j$ initial consecutive columns and $j$ consecutive rows starting with row $i$ are

$$\det A[i, \ldots, i+j-1; 1, \ldots, j] = \binom{n}{0} \binom{n}{1} \cdots \binom{n}{j-1} (1-x_i)^{n-j+1}(1-x_{i+1})^{n-j+1} \cdots (1-x_{i+j-1})^{n-j+1} \prod_{i \leq k < h \leq i+j-1} (x_h - x_k),$$

a result that follows from the properties of the determinants and Proposition 3.1, and that $m_{i,j}$ are the multipliers of the Neville elimination of $A$, we obtain
that
\[ m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}} = \frac{(1 - x_i)^{n-j+1}(1 - x_{i-j}) \prod_{k=1}^{j-1}(x_i - x_{i-k})}{(1 - x_{i-1})^{n-j+2} \prod_{k=2}^{j}(x_{i-1} - x_{i-k})}, \] (3.5)
where \( j = 1, \ldots, n + 1 \) and \( i = j + 1, \ldots, l + 1 \).

As for the minors of \( A^T \) with \( j \) initial consecutive columns and \( j \) consecutive rows starting with row \( i \), they are:

\[
\det A^T[i, \ldots, i+j-1|1, \ldots, j] = \binom{n}{i-1} \binom{n}{i} \cdots \binom{n}{i+j-2} x_i^{i-1} x_i^{i-1} \cdots x_i^{i-1}
\]
\[
(1 - x_1)^{n-i-j+2} (1 - x_2)^{n-i-j+2} \cdots (1 - x_j)^{n-i-j+2} \prod_{1 \leq k < h \leq j} (x_h - x_k).
\]

This expression also follows from the properties of the determinants and Proposition 3.1. Since the entries \( \tilde{m}_{i,j} \) are the multipliers of the Neville elimination of \( A^T \), using the previous expression for the minors of \( A^T \) with initial consecutive columns and consecutive rows, it is obtained that

\[
\tilde{m}_{i,j} = \frac{(n - i + 2) \cdot x_j}{(i-1)(1-x_j)}, \quad j = 1, \ldots, n; \quad i = j + 1, \ldots, n + 1.
\] (3.6)

Finally, the \( i \)th diagonal element of \( D \)

\[
p_{i,i} = \frac{n}{i-1} (1 - x_i)^{n-i+1} \prod_{k<i}(x_i - x_k)
\]
\[
\prod_{k=1}^{i-1}(1 - x_k), \quad i = 1, \ldots, n + 1
\] (3.7)
is obtained by using the expression for the minors of \( A \) with initial consecutive columns and initial consecutive rows. \( \Box \)

Moreover, by using the same arguments of [14] it can be seen that this factorization is unique among factorizations of this type, that is to say, factorizations in which the matrices involved have the properties shown by formulae (3.2), (3.3) and (3.4).

**Remark 3.3.** The formulae obtained in the proof of Theorem 3.3 for the minors of \( A \) with \( j \) initial consecutive columns and \( j \) consecutive rows, and for the minors of \( A^T \) with \( j \) initial consecutive columns and \( j \) consecutive rows show that they are not zero and so, the complete Neville elimination of \( A \) can be performed without row and column exchanges. Looking at equations (3.5)-(3.7) is easily seen that \( m_{i,j}, \tilde{m}_{i,j} \) and \( p_{i,i} \) are positive. Therefore, taking into account Theorem 2.1, this confirms that the matrix \( A \) is strictly totally positive.

**Remark 3.4.** In the square case, the matrices \( F_i \) (\( i = 1, \ldots, l \)) and the matrices \( G_j \) (\( j = 1, \ldots, n \)) are not the same bidiagonal matrices that appear in the bidiagonal factorization of \( A^{-1} \) presented in [13], nor their inverses. The
multipliers of the Neville elimination of \( A \) and \( A^T \) give us the bidiagonal factorization of \( A \) and \( A^{-1} \), but obtaining the bidiagonal factorization of \( A \) from the bidiagonal factorization of \( A^{-1} \) (or vice versa) is not straightforward \([6]\). The structure of the bidiagonal matrices that appear in both factorizations is not preserved by the inversion, that is, in general, \( F_i^{-1} \) \( (i = 1, \ldots, l) \) and \( G_j^{-1} \) \( (j = 1, \ldots, n) \) are not bidiagonal matrices. See \([6]\) for a more detailed explanation.

4 The algorithm

In this section we present an accurate and efficient algorithm for solving the problem of polynomial regression in Bernstein basis we have presented in Section 1. As we introduced there, our algorithm is based on the solution of the least squares problem \( \min_c \| Ac - f \| \), where \( A \), \( f \) and \( c \) are given by (1.1), (1.2) and (1.3), respectively. Taking into account that \( A \) is a strictly totally positive matrix, it is full rank, and the method based on the QR decomposition is the most adequate for solving this least squares problem \([1]\).

The following result (see Section 1.3.1 in \([1]\)) will be essential in the construction of our algorithm.

**Theorem 4.1.** Let \( Ac = f \) a linear system where \( A \in \mathbb{R}^{(l+1) \times (n+1)} \), \( l \geq n \), \( c \in \mathbb{R}^{n+1} \) and \( f \in \mathbb{R}^{l+1} \). Assume that \( \text{rank}(A) = n + 1 \), and let the QR decomposition of \( A \) given by

\[
A = Q \begin{pmatrix} R \\ 0 \end{pmatrix},
\]

where \( Q \in \mathbb{R}^{(l+1) \times (l+1)} \) is an orthogonal matrix and \( R \in \mathbb{R}^{(n+1) \times (n+1)} \) is an upper triangular matrix with nonnegative diagonal entries.

The solution of the least squares problem \( \min_c \| Ac - f \|_2 \) is obtained from

\[
\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = Q^T f, \quad Rc = d_1, \quad r = Q \begin{pmatrix} 0 \\ d_2 \end{pmatrix},
\]

where \( d_1 \in \mathbb{R}^{n+1} \), \( d_2 \in \mathbb{R}^{l-n} \) and \( r = f - Ac \). In particular \( \| r \|_2 = \| d_2 \|_2 \).

An accurate and efficient algorithm for computing the QR decomposition of a strictly totally positive matrix \( A \) is presented in \([12]\). This algorithm is called \textsc{TNQR} and can be obtained from \([11]\). Given the bidiagonal factorization of \( A \),
TNQR computes the matrix $Q$ and the bidiagonal factorization of the matrix $R$. Let us point out here that if $A$ is strictly totally positive, then $R$ is strictly totally positive. TNQR is based on Givens rotations, has a computational cost of $O(l^2n)$ arithmetic operations if the matrix $Q$ is required, and its high relative accuracy comes from the avoidance of subtractive cancellation.

A fast and accurate algorithm for computing the bidiagonal factorization of the rectangular Bernstein-Vandermonde matrix that appears in our problem of polynomial regression in the Bernstein basis can be developed by using the expressions (3.5), (3.6) and (3.7) for the computation of the multipliers $m_{i,j}$ and $\tilde{m}_{i,j}$, and the diagonal pivots $p_{i,i}$ of its Neville elimination. The algorithm is an extension to the rectangular case of the one presented in [13] for the square Bernstein-Vandermonde matrices. Given the nodes $\{x_i\}_{1\leq i\leq l+1} \in (0,1)$ and the degree $n$ of the Bernstein basis, it returns a matrix $M \in \mathbb{R}^{(l+1)\times(n+1)}$ such that

\[
M_{i,i} = p_{i,i} \quad i = 1, \ldots, n+1,
\]
\[
M_{i,j} = m_{i,j} \quad j = 1, \ldots, n+1; \quad i = j+1, \ldots, l+1,
\]
\[
M_{i,j} = \tilde{m}_{j,i} \quad i = 1, \ldots, n; \quad j = i+1, \ldots, n+1.
\]

The algorithm, that we call it TNBDBV, has a computational cost of $O(ln)$ arithmetic operations, and high relative accuracy because it only involves arithmetic operations that avoid subtractive cancellation (see [13] for the details). The implementation in MATLAB of the algorithm in the square case can be taken from [11].

In this way, the algorithm for solving the least squares problem $\min_c \| Ac - f \|$ corresponding to our polynomial regression problem will be:

**INPUT:** The nodes $\{x_i\}_{1\leq i\leq l+1} \in (0,1)$, the data vector $f$ and the degree $n$ of the Bernstein basis.

**OUTPUT:** A vector $c = (c_j)_{1\leq j\leq n+1}$ containing the coefficients of the polynomial $P(x)$ in the Bernstein basis $\mathcal{B}_n$ and the minimum residual $r$.

- **Step 1:** Computation of the bidiagonal factorization of $A$ by means of TNBDBV.
- **Step 2:** Given the matrix $M$ obtained in Step 1, computation of the QR decomposition of $A$ by using TNQR.
- **Step 3:** Computation of

\[
d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = Q^T f.
\]

- **Step 4:** Solution of the upper triangular system $Rc = d_1$. 

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- Step 5: Computation of

\[ r = Q \begin{pmatrix} 0 \\ d_2 \end{pmatrix}. \]

Step 3 and Step 5 are carried out by using the standard matrix multiplication command of MATLAB. As for Step 4, it is done by means of the algorithm \texttt{TNSolve} of P. Koep [11]. Given the bidiagonal factorization of a totally positive matrix \( A \), \texttt{TNSolve} solves a linear system whose coefficient matrix is \( A \) by using backward substitution.

Let us observe that \( A \) is not constructed, although we are also computing the residual \( r = f - Ac \).

### 5 Numerical experiments and final remarks

Two numerical experiments illustrating the good properties of our algorithm are reported in this section. We solve the least squares problem \( \min_c \| Ac - f \| \) corresponding to the computation of the regression polynomial in exact arithmetic by means of the command \texttt{leastsqrs} of \textit{Maple 10} and we denote this solution by \( c_e \). We also compute the minimum residual \( r_e \) in exact arithmetic by using \textit{Maple 10}. We use \( c_e \) and \( r_e \) for comparing the accuracy of the results obtained in MATLAB by means of:

1. The algorithm presented in Section 4.
2. The command \( A\backslash f \) of MATLAB.

The relative errors obtained when using the approaches (1) and (2) for computing the coefficients of the regression polynomial in the experiments described in this section \( (ec_1 \) and \( ec_2 \), respectively) are included in the first and in the third column of Table 2. The relative errors corresponding to the computation of the minimum residual by using the approaches (1) and (2) \( (er_1 \) and \( er_2 \), respectively) are presented in the second and in the fourth column of Table 2.

We compute the relative error of a solution \( c \) of the least squares problem \( \min_c \| Ac - f \| \) by means of the formula

\[ ec = \frac{\| c - c_e \|_2}{\| c_e \|_2}. \]

The relative error of a minimum residual \( r \) is computed by means of

\[ er = \frac{\| r - r_e \|_2}{\| r_e \|_2}. \]
Example 5.1. Let $B_{15}$ the Bernstein basis of the space of polynomials with degree less than or equal to 15 in $[0, 1]$. We will compute the polynomial

$$P(x) = \sum_{j=0}^{15} c_j b_j^{(n)}(x)$$

that minimizes

$$\sum_{i=1}^{21} |P(x_i) - f_i|^2,$$

where

$$\{x_i\}_{1 \leq i \leq 21} = \left\{ \frac{i}{22} \right\}_{1 \leq i \leq 21},$$

and

$$f = (3, 4, 0, -2, 5, 0, 1, 9, -3, 7, -1, 0, 2, 2, -4, -2, 3, 8, -6, 4, 1)^T.$$  

Let us observe that, the condition number of the Bernstein-Vandermonde matrix $A$ of the least squares problem corresponding to the regression polynomial we are interested in computing is $\kappa_2(A) = 2.0e + 05$.

The following example shows how the algorithm we have presented in this paper keeps the accuracy when the condition number of the Bernstein-Vandermonde matrix involved in the regression problem increases, while the accuracy of the general approach (2) which does not exploit the structure of this matrix goes down.

Example 5.2. We consider a regression problem such that the Bernstein basis $B_{15}$ and the data vector $f$ are the same as in Example 5.1. The points $\{x_i\}_{1 \leq i \leq 21}$ are now:

$$\left\{ \frac{1}{22}, \frac{1}{20}, \frac{1}{18}, \frac{1}{16}, \frac{1}{14}, \frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{34}, \frac{1}{32}, \frac{1}{30}, \frac{1}{28}, \frac{1}{26}, \frac{1}{24}, \frac{1}{22}, \frac{1}{20}, \frac{1}{18}, \frac{1}{16}, \frac{1}{14}, \frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2} \right\}.$$  

The condition number of the Bernstein-Vandermonde matrix $A$ involved in this experiment is $\kappa_2(A) = 5.3e + 08$.

| Example | $ec_1$ | $er_1$ | $ec_2$ | $er_2$ |
|---------|--------|--------|--------|--------|
| 5.1     | 1.4e-15| 1.3e-15| 9.0e-12| 5.2e-12|
| 5.2     | 2.0e-15| 2.3e-15| 1.9e-09| 1.4e-08|

Table 2
Relative errors in Example 5.1 and Example 5.2

Remark 5.3. The accuracy of our algorithm is obtained by exploiting the structure of the Bernstein-Vandermonde matrix. Every step of our algorithm,
except the ones in which the standard matrix multiplication command of MATLAB is used, are developed with high relative accuracy because only arithmetic operations that avoid subtractive cancellation are involved [12,13].

**Remark 5.4.** Our algorithm has the same computational cost ($O(l^2 n)$ arithmetic operations) as the conventional algorithms that solve the least squares problem by means of the QR decomposition ignoring the structure of the matrix, when $Q$ is explicitly required (see Section 2.4.1 of [1]).

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