INFALL OF PLANETESIMALS ONTO GROWING GIANT PLANETS: ONSET OF RUNAWAY GAS ACCRETION AND METALLICITY OF THEIR GAS ENVELOPES

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ABSTRACT
We have investigated the planetesimal accretion rate onto giant planets that are growing through gas accretion, using numerical simulations and analytical arguments. We derived the condition for the opening of a gap in the planetesimal disk, which is determined by a competition between the expansion of the planet’s Hill radius due to the planet’s growth and the damping of planetesimal eccentricity due to gas drag. We also derived the semianalytical formula for the planetesimal accretion rate as a function of the ratios of the rates of the Hill radius expansion, the damping, and planetesimal scattering by the planet. The predicted low planetesimal accretion rate due to the opening of the gap in early gas accretion stages quantitatively shows that “phase 2,” which is a long (more than a Myr), slow gas accretion phase before the onset of runaway gas accretion, is not likely to occur. In later stages, rapid Hill radius expansion fills the gap, resulting in significant planetesimal accretion, which is as large as several $M_\oplus$ for Jupiter and Saturn. The efficient onset of runaway gas accretion and the late pollution may reconcile the ubiquity of extrasolar giant planets with the metal-rich envelopes of Jupiter and Saturn inferred from interior structure models. These formulae will give deep insights into the formation of extrasolar gas giants and the diversity in the metallicities of transiting gas giants.

Subject headings: planetary systems: formation — solar system: formation
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1. INTRODUCTION

Models of the interior structure of the Jovian planets in our solar system suggest that Jupiter and Saturn contain a greater quantity of heavy elements in their envelopes than that suggested from assuming the solar metallicity (Saumon & Guillot 2004). This may imply that a significant mass of planetesimals was accreted onto the planets, together with gas accretion from the protoplanetary disk. However, the orbital calculations (e.g., Tanaka & Ida 1997; Zhou & Lin 2007) showed that the coupled effect of the excitation of the planetesimals’ eccentricities due to gravitational scattering by the planet and their damping by aerodynamical and/or dynamical drag tends to open up a gap in the planetesimal disk, resulting in the truncation of planetesimal infall onto the planet’s gas envelope.

Pollack et al. (1996) a priori assumed a very efficient planetesimal accretion during the gas accretion phase. As a result of an increase in the planet’s mass, the width of its feeding zone, which is proportional to the cubic root of the mass, expands. They assumed that planetesimals in the expanded zones are accreted with the fast rate for nearly circular orbits of planetesimals. Their assumption can be consistent with the anticipated metal-rich envelopes of Jupiter and Saturn, but it is inconsistent with the eccentricity excitation and gap formation implied by the above orbital integrations.

Furthermore, the assumption of the very fast planetesimal accretion results in a long “phase 2,” which is a very inefficient gas accretion phase before the onset of runaway gas accretion. As is explained in §2, envelope contraction starts when the core’s mass ($M_\ast$) becomes larger than a critical core mass ($M_{\ast, \text{hydro}}$). For $M_\ast > M_{\ast, \text{hydro}}$, the pressure gradient no longer hydrodynamically supports the envelope gas against the gravity of the increased core (Mizuno 1980; Ikoma et al. 2000). Pollack et al. (1996) showed that heat generation due to the assumed planetesimal accretion associated with gas accretion supports the envelope quasi-hydrodynamically (in other words, it increases the critical core mass; see eq. [1]) after the onset of envelope contraction. The quasi-hydrodynamical state is called “phase 2,” and it may last for a Myr or longer. However, the inefficient gas accretion is inconsistent with the ubiquity of extrasolar giant planets (Ida & Lin 2008).

Recently, the likelihood of phase 2 has been readdressed. Fortier et al. (2007) showed that even if gap formation is neglected, a more realistic planetesimal accretion rate based on oligarchic growth (Kokubo & Ida 1998, 2002; Thommes et al. 2003) significantly suppresses the duration of phase 2. Zhou & Lin (2007) showed that planetesimals around a protoplanet with $M_\ast \sim M_{\ast, \text{hydro}}$ are gravitationally shepherded and cannot be accreted. This suggests further reduction of phase 2. They also showed that the anticipated metal-rich envelopes of Jupiter and Saturn are not inconsistent with this finding, because such shepherding occurs only in early stages. The planet starts accreting planetesimals when its mass becomes comparable to that of gas giants, because the planetesimals trapped in mean motion resonances are released by resonance overlapping due to the increase of the planet’s mass. The planetesimal accretion ceases to halt gas accretion onto such a massive planet. Through numerical simulations with two different simple gas accretion prescriptions, Zhou & Lin estimated that the total accreted mass can be as large as several Earth masses.

The idea by Zhou & Lin (2007) reconciled the efficient formation of gas giants with the anticipated metal-rich envelope. However, since they showed only numerical results with limited prescriptions for gas accretion, it is not clear that the accreted planetesimal mass for a more realistic gas accretion rate is as large as what they obtained. Furthermore, they discussed the suppression of phase 2 only qualitatively. As is shown below, the planetesimal accretion rate depends on both the gas accretion speed and the planet’s mass. Hence, in order to evaluate the amount of planetesimal infall for more realistic gas accretion models and quantitatively discuss the
suppression of phase 2, general formulae for the gas accretion rate as a function of the planet’s mass \( (M) \) and its increase rate \( (\dot{M}) \) are needed.

In the present paper, through orbital integrations, we clarify the physical mechanism with which to determine the planetesimal accretion rate and derive detailed semianalytical formulae for the accretion rate as a function of \( M \) and \( \dot{M} \). We find that the total infall mass of the planetesimals can be as much as several Earth masses even for more realistic gas accretion models, and we quantitatively show that phase 2 is significantly suppressed.

The outline of this paper is as follows. We summarize the processes of gas accretion onto planets in § 2. The method of our calculation and its initial setup is described in § 3. With artificial simple gas accretion models, we clarify the intrinsic physics that determines the planetesimal accretion rate and derive semianalytical formulae for the accretion rate (§§ 4.1–4.3). Applying the formulae to realistic gas accretion models, we discuss the metallicity of the envelopes of Jupiter and Saturn (§ 4.4). We also discuss phase 2 (§ 4.5). The conclusion is presented in § 5.

2. GAS ACCRETION ONTO A CORE

As we mentioned in § 1, when the core mass becomes larger than a critical core mass, the pressure gradient no longer hydrodynamically supports the envelope gas against the core’s gravity, and the hydrostatic atmosphere does not exist. After that, heat generation due to the gas envelope contraction itself supports the envelope against dynamical collapse, and thus the envelope undergoes quasi-static contraction. The contraction allows gas to flow from the disk into the Bondi radius of the planet, such that the contraction rate is almost equal to the gas accretion rate of the planet.

Here we briefly summarize the prescriptions for this process for later use. The critical core mass depends on the planetesimal accretion rate onto the core \( (\dot{M}_c) \) and the grain opacity \( (\kappa_{\text{gr}}) \) associated with the disk gas. On the basis of a series of numerical models, Ikoma et al. (2000) found that the critical core mass for the breakdown of the hydrostatic atmosphere is

\[
\dot{M}_{\text{c,hydro}} \simeq 10 \left( \frac{\dot{M}_c}{10^{-6} \ M_\odot \ yr^{-1}} \right)^{0.2-0.3} \left( \frac{\kappa_{\text{gr}}}{\kappa_{\text{gr}}} \right)^{0.2-0.3} \ M_\odot, \tag{1}
\]

where \( \kappa_{\text{gr}} \) \((\sim 1 \text{ cm}^2 \text{ g}^{-1})\) is the grain opacity given by Pollack et al. (1985), who assumed dust grains with interstellar abundance and size distributions. Faster accretion and higher opacity (relatively large values of \( \dot{M}_c \) and \( \kappa_{\text{gr}} \)) result in a warmer planetary atmosphere and an enhanced pressure gradient, so \( \dot{M}_{\text{c,hydro}} \) is larger (Stevenson 1982; Ikoma et al. 2000).

Pollack et al. (1996) assumed the most efficient planetesimal accretion induced by the expansion of the feeding zone due to an increase in the planet’s mass, the rate of which is \( \sim 10^{-6} \ M_\odot \ yr^{-1} \), for \( \dot{M}_c \sim 10 \ M_\odot \). When a planet with \( \dot{M}_c \sim 10 \ M_\odot \) becomes isolated as a result of having consumed the planetesimals in its feeding zone, the gas envelope begins to contract, and the induced planetesimal accretion from the expanded region of the feeding zone increases the value of \( \dot{M}_{\text{c,hydro}} \) up to \( \sim \dot{M}_c \) (eq. [1]) and stalls gas accretion. This self-regulated process works for a Myr or longer, until the value of \( \dot{M}_c \) exceeds \( \sim 20 \ M_\odot \). This is called “phase 2.” However, as we will show in § 4.5, the rate of the planetesimal accretion induced by gas accretion is not generally large enough to maintain phase 2. Then the phase in which gas accretion is dominant starts.

For values of \( M_c \sim M_{\text{c,hydro}} \), the heat generated due to planetesimal accretion marginally equilibrates with the core’s gravity. In the quasi-static contraction stage, the heat generated due to the envelope contraction marginally equilibrates with the gravity of the planet with total mass \( M \) (including envelope mass). The Kelvin-Helmholtz contraction timescale is equivalent to the planetary mass increase timescale \( \tau_{\text{g,acc}} = M/\dot{M} \). Replacing \( M_c \) and \( \dot{M}_c \) by \( M \) and \( \dot{M}/\tau_{\text{g,acc}} \) in equation (1), we find that \( \tau_{\text{g,acc}} \) is given by

\[
\tau_{\text{g,acc}} \simeq 10^7 \left( \frac{M}{10 \ M_\odot} \right)^{-2.3} \left( \frac{\kappa_{\text{gr}}}{\kappa_{\text{gr}}} \right)^{0.2} \ M_\odot \ yr. \tag{2}
\]

Detailed numerical simulations of the quasi-static evolution of the gaseous envelope (Ikoma et al. 2000; Ikoma & Genda 2006) show consistent results at the onset of runaway gas accretion in which the envelope and core masses are nearly equal. Although Podolak (2003) suggested a value of \( \kappa_{\text{gr}} \sim 0.01 \kappa_{\text{gr}}^\text{hydro} \) through numerical simulations of coagulation and sedimentation of dust grains in the atmosphere, the total mass and size distributions of dust grains in the atmosphere are highly uncertain. Here we adopt the results by Ikoma & Genda (2006) with \( \kappa_{\text{gr}} = \kappa_{\text{gr}}^\text{hydro} \),

\[
\tau_{\text{g,acc}} = 10^{6.5} \left( \frac{M}{10 \ M_\odot} \right)^{-3.5} \ M_\odot \ yr, \tag{3}
\]

as a fiducial “realistic” gas accretion model.

When equation (3) is extrapolated to large values of \( M \) (\( \gtrsim 100 \ M_\odot \)), it may give an unrealistically fast supply of gas from the disk. Hence, we limit the gas accretion rate as follows. Tanigawa & Watanabe (2002) showed through two-dimensional local hydrodynamic simulations that the mass infall to the circumplanetary subdisk from the protoplanetary disk is limited by

\[
\frac{\dot{M}}{\dot{M}_{\text{in}}} \simeq 6 \times 10^{-4} f_b \left( \frac{a}{5 \ \text{AU}} \right)^{-1.5} \left( \frac{M}{M_\odot} \right)^{0.3} \ M_\odot \ yr^{-1}, \tag{4}
\]

where \( f_b \) is a scaling factor for the disk gas surface density that is defined in equation (7). We use this limit with \( f_b = 0.7 \).

Another limit is Bondi gas accretion, the rate of which is given by \( \dot{M} = \pi \rho_{\text{gas}} \left( \frac{a}{2H} \right) \) (where \( \rho_{\text{gas}} = \Sigma_{\text{gas}}/(2H) \) is the spatial density of the gas disk and \( H \) is the disk scale height, \( r_b = 2GM/c_s^2 \) is the Bondi radius, \( c_s = H/\Omega \) is the sound speed, and \( \Omega = (GM/a^3)^{1/2} \) is the Keplerian angular velocity, where \( M_c \) is the stellar mass. Adopting the temperature distribution in the limit of an optically thin disk (Hayashi 1981), \( T = 2.8 \times 10^2 (r/1 \ \text{AU})^{-1.2} \) K, we find that the Bondi gas accretion limit is

\[
\frac{\dot{M}}{\dot{M}_{\text{Bondi}}} = 0.7 \times 10^{-3} \left( \frac{a}{5 \ \text{AU}} \right)^{-2} \left( \frac{M}{M_\odot} \right) \ M_\odot \ yr^{-1}. \tag{5}
\]

As Figure 1 shows, the timescale \( (\tau_{\text{g,acc}}) \) in equation (4) is generally longer than the Bondi accretion timescale, so an actual lower limit for gas accretion timescale is given by equation (4).

3. CALCULATION SETUP

3.1. Orbital Integration

We numerically calculate the orbital evolution of a swarm of planetesimals in the vicinity of a protoplanet’s orbit embedded in a gaseous disk. The protoplanet grows by accreting gas with a given rate. The planetesimals are treated as massless test particles, and we neglect their interactions. We assume that the protoplanet has a fixed circular orbit.
The planetesimals’ orbits are affected by the gravitational force of the growing protoplanet and the drag force from the disk gas,

\[ f_{\text{gas}} = -\frac{v - v_{\text{gas}}}{\tau_{\text{damp}}}, \]

where \( v \) and \( v_{\text{gas}} \) are the velocities of a planetesimal and the disk gas, respectively. The gas motion is a circular Keplerian motion. In some runs, we adopted a slightly slower rotation speed of the disk gas due to the radial pressure gradient in the disk gas (e.g., Adachi et al. 1976), which induces the planetesimal orbits to migrate inward. However, we found that the inward migration hardly changed the results of the planetesimal accretion rate onto the protoplanet. We here show the results without the inward migration.

We set a damping timescale of the gas drag, \( \tau_{\text{damp}} (= v/e_0) \), as a constant parameter for all the planetesimals throughout a run in order to make the effect of the damping force clear.

We follow the prescription of the gas surface density distribution given by Ida & Lin (2004),

\[ \Sigma_g = 210 f_0 \left( \frac{a}{5 \text{ AU}} \right)^{-3/2} \text{ g cm}^{-2}, \]

where \( f_0 \) is the scaling parameter and the value of \( f_0 = 0.7 \) corresponds to the gas surface density of the minimum mass solar nebula (MMSN) model, \( \Sigma_g, \text{MMSN} \) (Hayashi 1981). For simplicity, we neglect the gap in the gas disk, which may be opened up by the perturbations from a massive protoplanet (e.g., Lin & Papaloizou 1993), and assume the above unperturbed value of \( \Sigma_g \) everywhere. With this value of \( \Sigma_g \), a given value of \( \tau_{\text{damp}} \) corresponds to an individual planetesimal mass of (Adachi et al. 1976; Tanaka & Ida 1999)

\[ m = 3 \times 10^{17} f_0 \left( \frac{e}{0.1} \right)^3 \left( \frac{\tau_{\text{damp}}}{10^5 \text{ yr}} \right)^3 \frac{\rho_{\text{pl}}}{1 \text{ g cm}^{-3}} \frac{a}{(5 \text{ AU})}^{-39/4} \text{ g}, \]

where \( a, e, \) and \( \rho_{\text{pl}} \) are the semimajor axis, eccentricity, and material density of the planetesimals, respectively. If gravitational drag (e.g., Tanaka & Ward 2004) is considered instead of aerodynamical gas drag, then

\[ m = 4.5 \times 10^6 f_d^{-1} \left( \frac{\tau_{\text{damp}}}{10^5 \text{ yr}} \right)^{-1} \left( \frac{a}{5 \text{ AU}} \right)^2 \text{ g}, \]

although in this case, interactions among the planetesimals could be important.

The orbits of the planetesimals are numerically integrated by using the fourth-order Hermite scheme (Makino & Aarseth 1992) with a hierarchical time step (Makino 1991). The equation of motion of particle \( k \) is given by

\[ \frac{d^2 r_k}{dt^2} = -\frac{GM}{|r_k|^3} \frac{r_k}{|r_k|} - \frac{GM}{|r|^3} + f_{\text{gas}}, \]

where \( M \) and \( r \) are the mass and position of the protoplanet, respectively. The terms on the right-hand side, from left to right, represent the gravity from the central star, the gravitational perturbation from the protoplanet, the indirect term, and the gas drag force, respectively. We set \( M_f = M_c \), where \( M_c \) is the solar mass.

When a planetesimal contacts the surface of the protoplanet, we remove the planetesimal after recording the collision. The planet’s mass is unchanged. The physical radius of the protoplanet is determined by its mass and internal density \( \rho \) as

\[ R = \left( \frac{3M}{4\pi\rho} \right)^{1/3}. \]

We set \( \rho = 1 \text{ g cm}^{-3} \) in all simulations. The dependence of the planetesimal accretion rates on \( \rho \) will be discussed in § 4.4.

Although we neglect the gravitational forces of the planetesimals, the mass of the planetesimals is specified so that we can calculate the amount of mass accreted onto the protoplanet (regarding the “effective” mass for gas drag force, see below). If we assume equal-mass planetesimals, they are initially distributed in the range \( a_{\text{in}} < a < a_{\text{out}} \) so as to satisfy the surface mass density

\[ \Sigma_d = 3.8 f_d \left( \frac{a}{5 \text{ AU}} \right)^{-3/2} \text{ g cm}^{-2}, \]

where \( f_d \) is a scaling factor. As is the case for \( f_0 \), the value of \( f_d = 0.7 \) corresponds to the MMSN model. The inner and outer boundaries are \( a_{\text{in}} = a_p(1 - 5h_f) \) and \( a_{\text{out}} = a_p(1 + 10h_f) \), where \( a_p \) is the semimajor axis of the protoplanet and \( h_f \) is the reduced Hill radius for the final mass of the planet (\( M_f \)). The reduced Hill radius of a protoplanet \( h \) is the Hill radius \( r_H \) scaled by \( a_p \):

\[ h = r_H/a_p = \left( \frac{M}{3M_c} \right)^{1/3}. \]

In all numerical simulations, we adopt the values of \( a = 5 \text{ AU}, M_f = M_1 \) (Jupiter mass), and \( f_d = f_0 \). Accordingly, \( h_f = 6.8 \times 10^{-2}, a_{\text{in}} = 3.3 \text{ AU}, \) and \( a_{\text{out}} = 8.4 \text{ AU.} \) We will derive the dependences on \( a, M_f, \) and \( f_d (= f_0) \) by analytical arguments and discuss the results with different parameter values. The total mass of the planetesimals within the region \( a_{\text{in}} < a < a_{\text{out}} \) is \( \sim 20 f_d M_c \). The number of planetesimals in most runs is \( N = 20,000 \). With the assumption that the planetesimals have equal masses, their individual masses are \( m_{\text{pl}} \approx 1.1 \times 10^{-3} f_d M_c = 6.6 \times 10^2 f_d \text{ g} \). In our simulations, we specify that \( \tau_{\text{damp}} = 10^6 \text{ yr}, 10^5 \text{ yr}, 10^4 \text{ yr}, \) and \( \infty \).
(the gas-free case), independent of the values of $m_{\text{pl}}$. Except for the gas-free case, the above values of $m_{\text{pl}}$ are much larger than the values in equation (8) for the given values of $\tau_{\text{damp}}$, so the planetesimals that we use correspond to “superparticles” representing many smaller planetesimals. Since we neglect the interactions among planetesimals, this superparticle treatment is not consistent. The initial eccentricity and inclination of the planetesimals are taken to be $e_0 = i_0 = 0.001$ for all simulations. Since both $e$ and $i$ are quickly pumped up by perturbations from the protoplanet, the choices of $e_0$ and $i_0$ do not affect the results.

### 3.2. Growth of a Protoplanet

Since we consider the phase after the isolation of the protoplanet, we assume that the growth of the protoplanet is dominated by the accretion of surrounding disk gas rather than the accretion of planetesimals. As we will show later, this assumption is valid because the mass of the accreted gas is much larger than the anticipated mass of the accreted planetesimals.

In § 2, we described the prescription for gas accretion. In the numerical simulations, we use simple artificial gas accretion models in order to clarify the conditions that regulate the planetesimal accretion rate. From the results with the artificial models, we derive semi-analytical formulae for the planetesimal accretion in general forms (§ 3.3). Applying the formulae to the more realistic gas accretion rate in § 2, we calculate the total mass of the planetesimal infall into the envelopes of Jupiter and Saturn in § 4.4.

The simple artificial gas accretion models are expressed by

$$\frac{dM}{dt} = \alpha M^p,$$

where $\alpha$ is the integration constant determined by the boundary condition. We set the condition to be $M = M_0$ for $t = 0$ and $M = M_f$ for $t = t_f$. Following Zhou & Lin (2007), we set the protoplanet at 5 AU with an initial mass of $M_0 = 5.67 M_\odot$ and a final mass of $M_f = M_{\text{J}}$ (Jupiter mass). We adopt a value of $t_f = 10^5$ yr for our numerical simulation, following the nominal cases of Zhou & Lin. The growth with values of $p = 2$ and 0 corresponds to the Bondi and linear models, respectively, in Zhou & Lin (2007). Figure 2 shows the evolution of the mass of a protoplanet due to gas accretion with $p = 2, 1, 0$, and $-2$. Here we assume that $M_0 > M_{\text{c, hydro}}$. The consistency of this assumption is checked in § 4.5.

### 4. RESULTS OF ORBITAL CALCULATION

#### 4.1. Overall Evolution

Figure 3 shows the snapshots of the distributions of planetesimals on the $b$-$e/h$ plane, where $h$ is defined in equation (13). The scaled orbital separation $b$ is defined by

$$b = \frac{a - a_p}{\hbar a_p},$$

where $a_p$ is the semimajor axis of the protoplanet. The protoplanet is fixed at the origin of this plane (i.e., $b = 0$ and $e = 0$). The growth rate of the protoplanet, $M$, is $\propto M^2$. To avoid busy plots, we show only 1000 planetesimals in this figure. We integrate the evolution of the planetesimals for $3 \times 10^5$ yr. Since $t_f = 10^5$ yr, we set $M = 0$ for $t = 1-3 \times 10^5$ yr, which corresponds to the termination of gas accretion due to the formation of a gap in the gas disk, although we neglect the effect of the gas density depletion on the drag force. The damping timescale is $\tau_{\text{damp}} = 10^4$ yr.

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**Figure 2.**— Evolution of the protoplanet mass according to the simple power-law gas accretion models ($M \propto M^p$). The initial and final masses are $M_0 = 5.67 M_\odot$ and $M_f = M_{\text{J}}$, where $M_{\text{J}}(=10^{-3} M_\odot)$ is a Jupiter mass. The growth timescale is $t_f = 10^5$ yr. After $t > t_f$, we set $M = M_f = \text{const}$. [See the electronic edition of the Journal for a color version of this figure.]

**Figure 3.**— The evolution of the protoplanet mass according to the simple power-law gas accretion models ($M \propto M^p$). The initial and final masses are $M_0 = 5.67 M_\odot$ and $M_f = M_{\text{J}}$, where $M_{\text{J}}(=10^{-3} M_\odot)$ is a Jupiter mass. The growth timescale is $t_f = 10^5$ yr. After $t > t_f$, we set $M = M_f = \text{const}$. [See the electronic edition of the Journal for a color version of this figure.]

In the figure, we also show the Jacobi energy $E_J$ (Nakazawa & Ida 1988).

$$E_J = \frac{1}{2} \left[ (e/h)^2 + (i/h)^2 \right] - \frac{3}{8} b^2 + \frac{9}{2} + O(h).$$

In the figure, we also include higher order terms of $h$ in $E_J$. In the circular restricted three-body problem, $E_J$ is conserved before and after scattering by the protoplanet (on average, both $(e^2 + i^2)$ and $b^2$ increase). Since only planetesimals with $E_J \geq 0$ can enter the Hill sphere of the protoplanet (e.g., Hayashi et al. 1977), we regard the region in which $E_J > 0$ as the feeding zone of the protoplanet. When $e/h$ and $i/h \leq 1$, the width of the feeding zone is $b \simeq 2\sqrt{3}$.

In the top panel of Figure 3 ($t = 1000$ yr), the planetesimals in the vicinity of the protoplanet are scattered and their values of $e$ and $b$ increase along a constant-$E_J$ curve. The planetesimal eccentricities $e$ are damped in the panel in which $t = 3 \times 10^4$ yr because $\tau_{\text{damp}} = 10^4$ yr. Since the gas drag damps the value of $e$ but keeps the value of $b$ almost constant, all the planetesimals except for those trapped in horseshoe orbits go out of the feeding zone. As the protoplanet grows, its feeding zone expands. Since $b \propto M^{1/3}$, the values of $b$ of the planetesimals decrease with time, but scattering opposes this process. Since $M \propto M^2$, the expansion accelerates with time. Eventually, the expansion overwhelms the opening of the gap due to a coupling effect of scattering and gas drag damping, such that planetesimals go into the feeding zone in the panel in which $t = 1 \times 10^5$ yr. After $t = 1 \times 10^5$ yr, the increase of $M$ has stopped, so a gap in the planetesimal disk is again produced (bottom). Planetesimals that have sufficiently large values of $b$ are captured in proper mean motion resonances. Although we neglect the effects of inward migration caused by a slightly slower than Keplerian rotation of the gas due to the pressure gradient, the damping of $e$ results in a small decrease in $b$ due to the conservation of angular momentum. Such inward migration causes resonance trapping. The evolution of the gap width is consistent with the result by Zhou & Lin (2007).
The numbers of planetesimals are 998 (10^3 yr), 992 (3 b = 878 (3 b of the feeding zone (i.e., where the Jacobi energy E is this factor), we conclude that the planetesimal accretion rate 1000 in the range 3 A U /C20 /

The planetesimal accretion rate is very low until efficient planetesimal accretion restarts in a later stage. Zhou & Lin (2007) showed through orbital simulations of planetesimals that planetesimal accretion occurs only in a late stage of gas accretion. They suggested that in this late stage, the planet’s mass becomes large and the mean motion resonances overlap and release the planetesimals captured in the resonances. Thus, they concluded that the mass of the protoplanet controls the accretion rate of planets.

4.2. Dependence of Planetesimal Accretion Rate on Planet’s Mass

The evolution of the planetesimal accretion rate for \( \tau_{\text{damp}} = 10^6, 10^5, \) and \( 10^4 \) yr and the gas-free case is plotted as a function of \( M \) in Figure 4. The four lines in each panel represent various gas accretion models (\( p = 2, 1, 0, \) and \( -2 \)). The initial mass of the protoplanet, \( M_0, \) is set to 5.67 \( M_{\odot} \). Starting with 20,000 planetesimals (i.e., a planetesimal mass of \( \sim 1.1 \times 10^{-3} f_d M_0 \)), we ran the calculations for \( 10^5 \) yr (the growth timescale \( f_d = 10^5 \) yr).

To see the dependence on \( M \) more clearly, we also plot the scaled planetesimal accretion rate \( M/R^2 \), where \( R \) is the physical radius of the protoplanet, in Figure 4. Through the numerical simulations, we found that the planetesimals are likely to experience accretion in two dimensions rather than three. The two-dimensional accretion rate is

\[
\frac{dM}{dt} \sim 2 R \Sigma_d \frac{v_{\text{esc}}}{v_{\text{rel}}} \frac{v_{\text{rel}}}{3} \sqrt{\frac{32 \pi G \rho}{3 \Sigma_d R^2}},
\]

where \( \Sigma_d, \frac{v_{\text{esc}}}{v_{\text{rel}}} = (2GM/R)^{1/2}, \) and \( v_{\text{rel}} \) are the surface density of the planetesimals, the escape velocity from the protoplanet’s surface, and the relative velocity between the protoplanet and the planetesimals, respectively. The scaled accretion rate \( (M/R^2) \) is determined by the effective value of \( \Sigma_d \) in the feeding zone for a fixed internal density of the protoplanet \( (\rho) \). In our simulations, the total mass of the planetesimals is not significantly decreased, so the effective value of \( \Sigma_d \) is determined from scattering by the planet, gas drag, and Hill radius expansion due to the planet’s growth.

Figure 4 shows the planetesimal accretion rates for \( f_d = 0.7 \). The planetesimal accretion rates as a function of \( M \) depend on the parameter \( p \). For \( p = -2 \) and 0, the scaled planetesimal accretion rate decreases with \( M \), which suggests that a gap in the planetesimal disk is formed when \( M \) becomes large. On the other hand, for \( p = 2 \), the protoplanet may grow so fast in the late stage that the feeding zone expansion overwhelms the gap formation, as is shown in Figure 3. The dependence on \( p \) implies that the accretion rate is not a function solely of \( M \), but depends on \( M \) as well, because \( p \) determines the \( M-M^2 \) relation.

4.3. Dependence of Planetesimal Accretion Rate on Gap Formation Parameters

Here we show that competition among the processes of feeding zone expansion, scattering, and eccentricity damping regulates the flux of planetesimals across the boundaries of the feeding zone \((E_3 = 0)\); that is, the planetesimal accretion rate. We consider change rates of \( h^2 \) and \( (e/h)^2 \) of the planetesimals (we neglect the contribution from \( i \), because \( i \) is usually correlated to \( e \) and \( i < e \)).

The evolution of planetesimals in the \( h^2 - (e/h)^2 \) space due to gravitational scattering by the protoplanet, damping of eccentricity by gas drag, and expansion of the Hill radius by the increase in the mass of the protoplanet is expressed by the change rates \( v_{\text{esc}}, v_{\text{damp}}, \) and \( v_{\text{H}} \) on the plane. Since \( b \propto h^{-1} \propto M^{-1/3} \),

\[
v_{\text{H}} \equiv \frac{dh^2}{dt} \text{(growth)} = \left\{ \frac{b^2}{M} \right\} \frac{dh^2}{dt} = -2 \frac{b^2}{3} \frac{M}{M} = - \frac{8}{\tau_{\text{damp}}^{-1}},
\]

FIG. 3.—Orbital evolution of a swarm of planetesimals on the \( b-(e/h) \) plane. We adopt \( p = 2, \tau_{\text{damp}} = 10^4 \) yr, and \( t_f = 10^5 \) yr. The planet is fixed at \( e/h = b = 0 \) \( (a_p = 5 \text{ AU}) \). The horizontal axis \( b \) expresses \( (a - a_p)/h \), where \( a \) is the semi-major axis of the planetesimals. The solid and dotted lines represent the boundaries of the feeding zone (i.e., where the Jacobi energy \( E_3 = 0 \)) and those at \( t = 0 \), respectively. The time evolution of the latter is caused by increases in \( h \). The selected number of planetesimals is 1000 in the range 3.3 \( \text{ AU} \leq a \leq 8.4 \text{ AU} \) at \( t = 0 \). The numbers of planetesimals are 998 (10^5 yr), 992 (3 \times 10^4 yr), 904 (10^4 yr), and 878 (3 \times 10^2 yr). [See the electronic edition of the Journal for a color version of this figure.]
where $\tau_{\text{damp}} \equiv M/M_{\text{acc}}$ is the timescale of the planet’s mass increase. In the last equation, we used $b \simeq 2\sqrt{3}$, which is the location of the feeding zone for $e/h \leq 1$, for simplicity. With $\tau_{\text{damp}} = e/\dot{e}$,

$$v_{\text{damp}} \equiv \frac{1}{2} \frac{d(e/h)^2}{dt} \quad \text{(damping)} = \frac{(e/h)^2}{\tau_{\text{damp}}}.$$

The factor of $1/2$ in the definition is added in order to get the simpler form of the final expression and a better fit with the numerical results. Scattering increases $b^2$ and $(e/h)^2$ on average, along a constant-$E_j$ curve ($E_j \sim 0$). The corresponding change rate is

$$v_{\text{scat}} \equiv \frac{db}{dt} \quad \text{(scattering)} = \frac{4}{3} \frac{d(e/h)^2}{dt} \quad \text{(scattering)}.$$

If long-range gravitational interaction with $(e/h) \leq 1$ is assumed, linear calculation (Goldreich & Tremaine 1982; Hasegawa & Nakazawa 1990) has showed that the value of $b$ of a planetesimal increases by $\delta b \simeq 30b^{-5}$ during each encounter. Numerical calculation has showed that for values of $b \sim 3-4$, $\delta b$ is overestimated by a factor of $\sim 10$ (Ida 1990). Since the scattering occurs at every synodical time $T_{\text{syn}} \simeq 2\pi a_p/[(3/2)b\eta K\Omega_K]$, these effects can suppress the growth of the protoplanets, before they attain their isolation masses (Tanaka & Ida 1997, 1999).

$$v_{\text{scat}} = 2b \frac{db}{dt} \quad \text{(scattering)} \simeq 2b \frac{0.16b}{T_{\text{syn}}} \simeq \frac{6}{b^4} \frac{(3/2)b}{T_{\text{K}}} \simeq 0.22 \frac{h}{T_{\text{K}}},$$

where $T_{\text{K}} = 2\pi/\Omega_K$ is the Keplerian period and $b \simeq 2\sqrt{3}$ is again used.

Since the feeding zone is defined by $E_j$ and the scattering does not change its value, the condition for which a gap would open would be $v_{\text{damp}} \gtrsim v_{\text{H}}$. If inward migration of planetesimals due to gas drag or type I migration of the protoplanet is considered but the growth of the protoplanet is neglected, the gap formation condition is similarly derived, with $v_{\text{H}}$ being replaced by $db^2/dt$ due to gas drag (Tanaka & Ida 1997) or type I migration (Tanaka & Ida 1999). These effects can suppress the growth of the protoplanets before they attain their isolation masses (Tanaka & Ida 1997, 1999).
When \( v_{\text{damp}} \leq v_H \), the gap is not created and the planetesimals are engulfed by the expanding feeding zone. The engulfment rate is determined by \( v_H / v_{\text{scat}} \), because \( v_H \) and \( v_{\text{scat}} \) have opposite directions to each other in the \( b^2 \) components. Thus, it is expected that for \( v_{\text{damp}} < v_H \), the accretion rate would be regulated by

\[
\dot{M} = \frac{v_{\text{damp}}}{v_H} \left( \frac{M}{M_\oplus} \right)^{1/2} \left( \frac{\tau_{\text{g, acc}}}{10^4 \text{ yr}} \right)^{-1/3} \left( \frac{a_p}{5 \text{ AU}} \right)^{1/4},
\]

while for \( v_{\text{damp}} > v_H \), the accretion rate would be regulated by

\[
\eta \equiv \frac{v_H}{v_{\text{damp}}} \simeq \frac{8}{(e/h)^2} \frac{\tau_{\text{damp}}}{\tau_{\text{g, acc}}},
\]

\[
\simeq 0.8 \left( \frac{\tau_{\text{damp}}}{10^4 \text{ yr}} \right)^{1/2} \left( \frac{\tau_{\text{g, acc}}}{10^4 \text{ yr}} \right)^{-1} \left( \frac{M}{M_\oplus} \right)^{-1/6} \left( \frac{a_p}{5 \text{ AU}} \right)^{3/4},
\]

where we used equation (A3) in the Appendix to make the substitution for \((e/h)^2\). Since \( \tau_{\text{g, acc}} = M/M \) and \( \tau_{\text{damp}} \) and \( h \) are functions of \( M \), the planetesimal accretion rate would depend on both \( M \) and \( M \). We show that the numerical results agree with the above argument and derive formulae for the planetesimal accretion rate as a function of \( \xi \) and \( \eta \).

Figure 5 shows the evolution of the scaled planetesimal accretion rate as a function of \( \xi = v_H/v_{\text{scat}} \) for \( p = 2, 1, 0, \) and \(-2\). Results are shown for (a) the gas-free case and the cases of (b) \( \tau_{\text{damp}} = 10^6 \text{ yr} \), (c) \( 10^5 \text{ yr} \), and (d) \( 10^4 \text{ yr} \). The fitting formula, eq. (24), is indicated by the thick solid lines. [See the electronic edition of the Journal for a color version of this figure.]

**Fig. 5.—** Evolution of the planetesimal accretion rate as a function of \( \xi = v_H/v_{\text{scat}} \) for \( p = 2, 1, 0, \) and \(-2\). Results are shown for (a) the gas-free case and the cases of (b) \( \tau_{\text{damp}} = 10^6 \text{ yr} \), (c) \( 10^5 \text{ yr} \), and (d) \( 10^4 \text{ yr} \). The fitting formula, eq. (24), is indicated by the thick solid lines. [See the electronic edition of the Journal for a color version of this figure.]

Vol. 684
accretion rate is determined by $\xi$ for $\eta > 1$ (nongap cases). The planetesimal accretion rate in this case is given by

$$\frac{dM_{\text{solid}}}{dt} = 10^3 \left( \frac{R}{R_{\oplus}} \right)^2 f_d \left( \frac{v_{\text{H}}}{v_{\text{damp}}} \right)^\alpha M_{\oplus} \text{ yr}^{-1},$$  \ \ (24)$$

using the values of $\alpha \approx 0.8$ and $\beta \approx -6$ obtained from our numerical results by a least-squares fitting. The fitting line, equation (24), is expressed by thick solid lines in Figure 5. For $\eta < 1$, the accretion rate declines, which corresponds to the opening of a gap in the planetesimal disk.

Figure 6 shows the evolution of the scaled planetesimal accretion rate as a function of $\eta = v_{\text{H}}/v_{\text{damp}}$. In the range of $\eta < 1$, the scaled planetesimal accretion rate is independent of planetary gas accretion models with different $M$-$M$ relations and different values of $v_{\text{damp}}$. This confirms that the accretion rate is determined by $\eta$ for $\eta < 1$. From our numerical results, the planetesimal accretion rate in this case is given by

$$\frac{dM_{\text{solid}}}{dt} = 10^3 \left( \frac{R}{R_{\oplus}} \right)^2 f_d \left( \frac{v_{\text{H}}}{v_{\text{damp}}} \right)^\alpha M_{\oplus} \text{ yr}^{-1},$$  \ \ (25)$$

with $\alpha' \approx 1.4$ and $\beta' \approx -6$.

When the planet’s mass has grown to $M$, the total mass of the planetesimals that undergo infall in the envelope $[M_{\text{solid}}(M)]$ is obtained by integrating

$$\frac{dM_{\text{solid}}}{dM} = \frac{dM_{\text{solid}}}{dt} \frac{\tau_{g,\text{acc}}}{M}$$  \ \ (26)$$

from 0 to $M$. In Figure 7, the value of $M_{\text{solid}}(M)$ evaluated by the above semianalytical formulae is compared with that obtained from orbital calculations for the individual gas accretion models in the cases of $\tau_{\text{damp}} = 10^6$, $10^5$, and $10^4$ yr. The semianalytical formulae reproduce the results of the orbital calculations well, except for the early stages, in which $M_{\text{solid}}$ is so small that the statistical fluctuation is large. The formulae also reproduce the numerical results shown in Figure 9b of Zhou & Lin (2007).

4.4. Application to Jupiter and Saturn

In the preceding subsection, we investigated planetesimal accretion onto growing protoplanets with artificial gas accretion models and obtained semiempirical formulae of the planetesimal accretion rate. Applying these formulae to the more realistic gas accretion models in § 2, we discuss the metallicity of the envelopes of Jupiter and Saturn.

Integrating equation (26) with equation (3) up to $M_f$, we estimate the total mass of the accreted planetesimals in the cases of Jupiter ($M_f = 318 M_{\oplus}$; $a = 5.2$ AU) and Saturn ($M_f = 95 M_{\oplus}$; $a = 9.55$ AU). The evolution of $M_{\text{solid}}$ is plotted in Figure 8. The three curves show the results with $\tau_{\text{damp}} = 10^4$, $10^5$, and $10^6$ yr. It is likely that gas giants were inflated during the gas accretion phase. For a fixed value of $M$, $dM/dt \propto f_d \sqrt{\rho R^2} \propto f_d \sqrt{R}$ (eq. [17]). In the figure, we plot the accreted planetesimal mass $M_{\text{solid}}^* = (R/2R_1)^{1/2} (f_d/2) M_{\text{solid}}$. All the results show similar qualitative features of the evolution of $M_{\text{solid}}$. Planetary accretion is inhibited by the formation of the gap in the early stages, but rapid planetary growth due to gas accretion in later stages allows planetesimal accretion to occur. With a value of $\tau_{\text{damp}} = 10^6$ yr, $M_{\text{solid}} \approx 6(R/2R_1)^{1/2} f_d/2 M_{\oplus}$ for both Jupiter and Saturn. For shorter values of $\tau_{\text{damp}}$, $M_{\text{solid}}$ is smaller because the gap can more easily form. We also performed calculations that started from different core masses. The resultant value of $M_{\text{solid}}$ hardly changed, because $dM_{\text{solid}}/dM$ is negligibly small when $M$ is small and a gap has been opened. The predicted value of $M_{\text{solid}}$ can be as large as that inferred from the internal structure model of Saumon & Guillot (2004), if the planets are inflated and/or a relatively large value of $f_d$ is considered.
For the same values of $M, \rho, \text{and } f_d$, the value of $M_{\text{solid}}$ is larger for larger values of $a_p$. Although the mass of Saturn is $1/3$ that of Jupiter, our model predicts that the mass of the planetesimals falling into the Saturnian envelope is comparable to that for the Jovian envelope. More detailed internal structure models will test our prediction.

4.5. Phase 2

In this subsection, phase 2 with a long duration (more than a Myr) is simply referred to as “phase 2.” So far, we have assumed that gas accretion immediately starts when $M_c$ exceeds $M_c^{\text{hydro}}$, without undergoing phase 2. In the previous subsection, we predicted the planetesimal accretion rate as a function of planetary mass on the basis of the realistic gas accretion model. With this accretion rate, we show that phase 2 is not likely to occur.

In the nominal model (J1) in Pollack et al. (1996), $f_d \approx 2.5$, $a_p = 5.2$ AU, and $M_c \approx 10 M_\oplus$. They then found that a value of $\dot{M}_c \approx 10^{-6} M_\oplus \text{yr}^{-1}$ is maintained during phase 2 with their very efficient planetesimal accretion model. As is shown in equation (1), this value of $\dot{M}_c$ can marginally support a gas envelope around a $10 M_\oplus$ core.

---

**Fig. 7.**—Comparison between the numerical simulations and the semianalytical results. The left and right panels show the results for gas accretion with $p = 2$ and $p = 0$, respectively. The thin and thick curves represent the numerical and semianalytical results, respectively. [See the electronic edition of the Journal for a color version of this figure.]

**Fig. 8.**—Evolution of the cumulative mass of the accreted planetesimals as a function of $M$ in the case with $M_c = 10 M_\oplus$. The left and right panels show the results for $a = 5.2$ AU and $a = 9.55$ AU, which correspond to the distances for Jupiter and Saturn, respectively. Here the values $R = 2R_1$ and $f_d = 2$ are assumed, where $R_1$ is the physical radius for mass $M$ and $\rho = 1 \text{ g cm}^{-2}$. For other values of $R$ and $f_d$, the accreted mass is multiplied by $(R/2R_1)^{1/2}(f_d/2)$ [See the electronic edition of the Journal for a color version of this figure.]
First, we derive the condition for the opening of the gap with a realistic value of \( \tau_{g, \text{acc}} \) given by equation (3). Substituting equation (3) into equation (23), we find that

\[
\eta \simeq 0.8 \times 10^{-6} \left( \frac{\tau_{\text{damp}}}{10^4 \ \text{yr}} \right)^{1/2} \left( \frac{M}{M_\odot} \right)^{3.3} \left( \frac{a_p}{5 \ \text{AU}} \right)^{3/4}.
\] \tag{27}

With \( a_p = 5.2 \ \text{AU}, M \sim M_c \sim 10 \ M_\odot, \) and \( \tau_{\text{damp}} = 10^6 \ \text{yr}, \) we obtain \( \eta \simeq 2 \times 10^{-2} \ll 1. \) Then the gap should be opened up. Our formula for \( \eta < 1 \) gives

\[
M_{\text{solid}} \simeq 2.2 \times 10^{-6} f_d \left( \frac{\rho}{1 \ \text{g cm}^{-3}} \right)^{-1/6} \left( \frac{\tau_{\text{damp}}}{10^4 \ \text{yr}} \right)^{7/10} \times \left( \frac{\tau_{g, \text{acc}}}{10^4 \ \text{yr}} \right)^{-7/5} \left( \frac{M}{M_\odot} \right)^{13/30} \left( \frac{a_p}{5 \ \text{AU}} \right)^{21/20} M_\odot \ \text{yr}^{-1}.
\] \tag{28}

Substituting equation (3) into this equation yields

\[
\dot{M}_{\text{solid}} \simeq 0.9 \times 10^{-14} f_d \left( \frac{\rho}{1 \ \text{g cm}^{-3}} \right)^{-1/6} \left( \frac{\tau_{\text{damp}}}{10^4 \ \text{yr}} \right)^{7/10} \times \left( \frac{M}{M_\odot} \right)^{16/3} \left( \frac{a_p}{5 \ \text{AU}} \right)^{21/20} M_\odot \ \text{yr}^{-1}.
\] \tag{29}

For the values of \( \tau_{\text{damp}} = 10^6 \ \text{yr}, f_d \simeq 2.5, a_p = 5.2 \ \text{AU}, \) and \( M \simeq 10 \ M_\odot, M_c \simeq 1.1 \times 10^{-7} M_\odot \ \text{yr}^{-1}, \) which is 1 order smaller than the planetesimal accretion rate that Pollack et al. (1996) assumed.

We examine the possibility of the existence of phase 2 for other values of \( f_d \) and \( a_p. \) Phase 2 to occur, \( M_c \) must be maintained to be as large as \( M \) for \( M_c \sim M_c \), by hypothesis. The core mass can be approximately identified by the core isolation mass beyond the ice line (Kokubo & Ida 1998, 2002; Ida & Lin 2004),

\[
M_c, \text{iso} \simeq 4.6 f_d^{3/2} \left( \frac{a_p}{5 \ \text{AU}} \right)^{3/4} M_\odot.
\] \tag{30}

From equation (1), with the exponent derived from equation (3), the accretion rate required by the occurrence of phase 2 is

\[
\dot{M}_{\text{solid},2} \simeq 10^{-6} \left( \frac{M_c}{10 M_\odot} \right)^{4.5} M_\odot \ \text{yr}^{-1}
\]

\[
\sim 3 \times 10^{-8} f_d \left( \frac{a_p}{5 \ \text{AU}} \right)^{3.4} M_\odot \ \text{yr}^{-1}.
\] \tag{31}

Substituting \( M_c, \text{iso} \) into \( M \) in equation (27) yields

\[
\eta \simeq 1.3 \times 10^{-4} f_d^{5/2} \left( \frac{\tau_{\text{damp}}}{10^4 \ \text{yr}} \right)^{1/2} \left( \frac{a_p}{5 \ \text{AU}} \right)^{13/4}.
\] \tag{32}

Thus, \( \eta < 1 \) is equivalent to

\[
f_d < 3.8 \left( \frac{\tau_{\text{damp}}}{10^6 \ \text{yr}} \right)^{-1/10} \left( \frac{a_p}{5 \ \text{AU}} \right)^{-13/20}.
\] \tag{33}

For this range of values of \( f_d \) and for \( a_p > 3 \ \text{AU} \) (the ice line), equation (28) with \( M \) replaced by \( M_c, \text{iso} \) is always smaller than the value of \( M_{\text{solid},2} \) given by equation (31) (see Fig. 9). For \( \eta > 1, \) on the other hand,

\[
M_{\text{solid}} \simeq 1.5 \times 10^{-5} f_d \left( \frac{\rho}{1 \ \text{g cm}^{-3}} \right)^{-1/6} \left( \frac{\tau_{g, \text{acc}}}{10^4 \ \text{yr}} \right)^{-4/5} \times \left( \frac{M}{M_\odot} \right)^{2/5} \left( \frac{a_p}{5 \ \text{AU}} \right)^{6/5} M_\odot \ \text{yr}^{-1}.
\] \tag{34}

In the range of \( f_d \) and \( a_p \) that satisfy \( \eta > 1, \) equation (34) can reach this value of \( M_{\text{solid},2} \) only at \( a_p > 15 \ \text{AU} \) and \( f_d \sim 1, \) in which case gas giant formation is unlikely (Ida & Lin 2004). Thus, the predicted value of \( M_c \) never reaches the values that would be required for the onset of phase 2. We conclude that phase 2 is not likely to occur for the formation of giant planets. This conclusion is consistent with the ubiquity of extrasolar gas giant planets.

5. CONCLUSION

We have investigated the planetesimal accretion rate onto growing giant planets through numerical simulations and analytical arguments. The planet mass \( M \) increases with the assumed gas accretion rate onto the planet, and the orbits of the planetesimals in the vicinity of the planet’s orbit are integrated with the effect of gas drag, but without self-gravity of the planetesimals.

We first performed simulations with several different artificial gas accretion rates in order to clarify the intrinsic physics that determines the planetesimal accretion rate. A gap in the planetesimal disk is opened by a coupling effect of gravitational scattering by the planet and gas drag damping. Here the gap formation means that most planetesimals are out of the feeding zone of the planet. The scattering increases both \( e \) and \( b, \) keeping the Jacobi energy constant, where \( e \) is the orbital eccentricity and \( b \) is the difference in the semimajor axis between the planet and the planetesimals. Changes in \( e \) and \( bh \) are of the same order, where \( h \) is the reduced Hill radius, defined by \( (M/3M_\odot)^{1/3}. \) Since the gas drag predominantly damps \( e \) after the scattering, the gap is formed. On the other hand, the width of the feeding zone is proportional to \( h. \) Thus, the planet’s growth inhibits the formation of the gap and competes with the scattering/damping process.

We derived the condition for the gap formation by comparing the eccentricity damping rate \( (\tau_{\text{damp}}) \) and the rate of expansion of the feeding zone due to the planet’s growth \( (v_{\text{H}}) \). When \( v_{\text{H}}/v_{\text{damp}} > 1, \) the gap is not formed. Then the planetesimal accretion rate
(dM\text{solid}/dt) is scaled by the ratio of the scattering rate, \(v_{\text{scat}}\), to \(v_H\). The numerical results are fitted as

\[
\frac{dM_{\text{solid}}}{dt} = 10^{-6} \left( \frac{R}{R_\odot} \right)^2 f_d \left( \frac{v_H}{v_{\text{scat}}} \right)^{0.8} M_\odot \text{ yr}^{-1},
\]

where \(R\) is the physical radius of the planet and \(f_d\) is a scaling factor for the surface density of the planetesimals (eq. [12]). When the gap is formed (\(v_H/v_{\text{damp}} < 1\)), the accretion rate is significantly depleted. We found that the accretion rate is scaled by \(v_H/v_{\text{damp}}\) as

\[
\frac{dM_{\text{solid}}}{dt} = 10^{-6} \left( \frac{R}{R_\odot} \right)^2 f_d \left( \frac{v_H}{v_{\text{damp}}} \right)^{1.4} M_\odot \text{ yr}^{-1}.
\]

Applying these formulae to the more realistic gas accretion models described in § 2, we found the following:

1. In early stages, when \(M \sim O(10) M_\odot\), a gap is opened in the planetesimal disk. The planetesimal accretion rate is smaller than that required for phase 2 to be maintained. This ensures the efficient formation of gas giants, which may be consistent with the ubiquity of extrasolar giant planets.

2. In later stages \([M \gtrsim O(100) M_\odot]\), the expansion of the feeding zone overwhelms the gap-opening process, so the gap is filled. Then the planetesimal accretion becomes efficient.

3. The mass of the infalling planetesimals into the envelopes of Jupiter and Saturn in the late stages can be as large as several \(M_\odot\), which may be consistent with interior models for these planets.

In this "realistic" model, we assumed that planetesimals are infinitely supplied. However, if the accreted mass is significant, the planetesimals distributed in the regions inside isolated strong mean motion resonances can be consumed. In that case, the release of planetesimals from the resonance capture by resonance overlapping due to the increase in the mass of the planet may also become a significant factor (Zhou & Lin 2007).

Guillot et al. (2006) pointed out the correlation that the mass of the solid components of extrasolar transiting gas giants increases with the metallicity of their host stars, which is proportional to \(f_d\). This trend is consistent with our formulae, because \(dM_{\text{solid}}/dt \propto f_d\). As this example shows, the analysis here will give deep insights into the formation of extrasolar gas giants and their diversity.

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APPENDIX

EQUILIBRIUM ECCENTRICITY

The magnitude of \((e/h)^2\) in § 4.3 is determined by the balance between damping due to gas drag and excitation due to the planet’s perturbations. Since in the nongap case, planetesimals are engulfed by the feeding zone mainly in the parameter range of \((e/h) \lesssim 1\), we use equation (21) to define the parameter \(\xi\). However, the opening of the gap is caused by the damping of the relatively high orbital eccentricities, so we use the formula for the excitation of planetesimal eccentricity due to the protoplanet’s perturbations for \((e/h) \gtrsim 1\) in evaluating \((e/h)^2\). Then the scattering timescale is given approximately by Chandrasekhar’s two-body scattering formula (e.g., Stewart & Ida 2000; Ohtsuki et al. 2002),

\[
\tau_{e, \text{scat}} \approx \frac{1}{n_p \pi \left[ GM/(e v_K) \right]^2 e v_K \ln \Lambda},
\]

where \(\ln \Lambda \sim 3\) and \(n_p\) is the spatial density of the protoplanet, which is given by the inverse of the volume of the planetesimal disk in the feeding zone, \(1/(2\pi a_p)(4\sqrt{3} h_{\text{eq}})(e v_K/\Omega_K)\). Then

\[
\tau_{e, \text{scat}} \approx \frac{8\sqrt{3} \pi (e/h)^4}{27 \pi} h^{-1} \frac{T_K}{2 \pi} \approx 1 \times 10^4 (e/h)^4 \left( \frac{M}{M_\odot} \right)^{-1/3} \left( \frac{a_p}{5 \text{ AU}} \right)^{3/2} \text{ yr}.
\]

From \(\tau_{e, \text{scat}} = \tau_{\text{damp}}\), we obtain

\[
(e/h)^2 \approx 10 \left( \frac{\tau_{\text{damp}}}{10^4 \text{ yr}} \right)^{1/2} \left( \frac{M}{M_\odot} \right)^{1/6} \left( \frac{a_p}{5 \text{ AU}} \right)^{-3/4}.
\]

REFERENCES

Adachi, I., Hayashi, C., & Nakazawa, K. 1976, Prog. Theor. Phys., 56, 1756
Fortier, P., Benvenuto, O. G., & Brunini, A. 2007, A&A, 473, 311
Goldreich, P., & Tremaine, S. 1982, ARA&A, 20, 249
Guillot, T., Santos, N. C., Pont, F., Iro, N., Melo, C., & Ribas, I. 2006, A&A, 453, L21
Hasegawa, M., & Nakazawa, K. 1990, A&A, 227, 619
Hayashi, C. 1981, Prog. Theor. Phys. Suppl., 70, 35
Hayashi, C., Nakazawa, K., & Adachi, I. 1977, PASJ, 29, 163
Ida, S. 1990, Icarus, 88, 129
Ida, S., & Lin, D. N. C. 2004, ApJ, 604, 388
———. 2008, ApJ, 673, 487
Ikoma, M., & Genda, H. 2006, ApJ, 650, 1150
Ikoma, M., Nakazawa, K., & Emori, H. 2000, ApJ, 537, 1013
Kokubo, E., & Iida, S. 1998, Icarus, 131, 171
———, 2002, ApJ, 581, 666
Lin, D. N. C., & Papaloizou, J. C. B. 1993, in Protostars and Planets III, ed. E. H. Levy & J. I. Lunine (Tucson: Univ. Arizona Press), 749
Makino, J. 1991, PASJ, 43, 859
Makino, J., & Aarseth, S. J. 1992, PASJ, 44, 141
Mizuno, H. 1980, Prog. Theor. Phys., 64, 544
Nakazawa, K., & Ida, S. 1988, Prog. Theor. Phys. Suppl., 96, 167
Ohtsuki, K., Stewart, G. R., & Ida, S. 2002, Icarus, 155, 436
Pollack, M. 2003, Icarus, 165, 428
Pollack, J. B., Hubickyj, O., Bodenheimer, P., Lissauer, J. J., Podolak, M., & Greenzweig, Y. 1996, Icarus, 124, 62
Pollack, J. B., McKay, C. P., & Christofferson, B. M. 1985, Icarus, 64, 471
Saumon, D., & Guillot, T. 2004, ApJ, 609, 1170
Stevenson, D. J. 1982, Planet. Space Sci., 30, 755
Stewart, G. R., & Ida, S. 2000, Icarus, 143, 28
Tanaka, H., & Ida, S. 1997, Icarus, 125, 302
———. 1999, Icarus, 139, 350
Tanaka, H., & Ward, W. R. 2004, ApJ, 602, 388
Tanigawa, T., & Watanabe, S. 2002, ApJ, 580, 506
Thomas, E. W., Duncan, M. J., & Levison, H. F. 2003, Icarus, 161, 431
Zhou, J.-L., & Lin, D. N. C. 2007, ApJ, 666, 447