MODULI OF ELLIPTIC $K3$ SURFACES: MONODROMY AND SHIMADA ROOT LATTICE STRATA

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ABSTRACT. In this paper we investigate two stratifications of the moduli space of elliptically fibred $K3$ surfaces. The first comes from Shimada’s classification of connected components of the moduli of elliptically fibred $K3$ surfaces and is closely related to the root lattices of the fibration. The second is the monodromy stratification defined by Bogomolov, Petrov and Tschinkel. The main result of the paper is a classification of all positive-dimensionalambi-typical strata, that is, strata which are both Shimada root strata and monodromy strata. We also discuss the relationship with moduli spaces of lattice-polarised $K3$ surfaces. The appendix by M. Kirschmer contains computational results about the 1-dimensionalambi-typical strata.

1. Introduction

Elliptically fibred $K3$ surfaces have been studied over a long period from many different angles. We refer the reader in particular to the book [ScSh] where $K3$ surfaces are especially treated in Sections 11 and 12. Due to the work of Miranda [Mi90] it is well known that the moduli space $\mathcal{F}$ of elliptically fibred $K3$ surfaces with a section, also known as Jacobian fibrations, can be described as a GIT quotient of an open subset $V$ in the weighted projective space $\mathbb{P}_{8,12}(9,13)$ by the group $SL(2,\mathbb{C})$. Alternatively, the moduli space $\mathcal{F}$ can also be constructed as the moduli space of lattice polarised $K3$ surfaces, more precisely $U$-quasi-polarised $K3$ surfaces (where $U$ is the hyperbolic plane spanned by the classes of the section and a general fibre).

The moduli space $\mathcal{F}$ itself is a rational variety by a result of Lejarraga [Le]. Geometric properties of elliptically fibred $K3$ surfaces provide this space with many interesting geometric features.

The starting point of our work are two stratifications of the moduli space $\mathcal{F}$. The first of these was introduced by Bogomolov, Petrov and Tschinkel in [BPT]. The strata of this decomposition are defined by the property that they are maximal locally closed irreducible subvarieties with constant monodromy group $\Gamma$ where $\Gamma$ is a fixed subgroup of $SL(2,\mathbb{Z})$ modulo conjugation. Bogomolov, Petrov and Tschinkel prove the remarkable fact that all of these monodromy strata are themselves rational varieties.

The other stratification is due to the work of Shimada [Shi1], [Shi2], [Shi3] in which he classifies all connected components of the moduli of elliptic $K3$ surfaces with fixed combinatorial type. This means the following: Given an elliptically fibred $K3$ surface $f : S \to \mathbb{P}^1$ the components of the singular fibres not meeting the 0-section define a root lattice $R$ which is the direct sum of some $ADE$-lattices. This need not be be saturated in the Néron-Severi group $NS(S)$. Its saturation $L$ corresponds, by lattice theory, to an isotropic subgroup $G$ of the discriminant group of $R$. Shimada’s work provides a complete classification of all connected families of elliptically fibred $K3$ surfaces with given $(R, G)$. This leads to a total of
3932 families and in this way one obtains a second stratification of the moduli space $\mathcal{F}$. We refer to the strata in this stratification as Shimada root strata or, shorter, simply as Shimada strata. A priori the stratifications given by monodromy strata and Shimada strata are not related and none is a refinement of the other. However, both stratifications are refined by a third stratification, namely the one given by a fixed configuration of the singular fibres. We call these the configuration strata. Their properties were investigated by Kloosterman in [Kl].

The starting point of our work is the observation that there are some strata which appear in both the monodromy stratification and Shimada’s stratification. We call these ambidiagonal strata and the main purpose of this paper is to understand these special strata. Our main result (Theorem 4.2) is

**Main Theorem.** There are exactly 50 positive-dimensional ambidiagonal strata. These are listed in Table 11.

We then prove in Theorem 4.3 that the ambidiagonal Shimada strata are, with the exception of two strata, already completely determined by the root lattice $R$ itself. In Theorem 4.4 and Table 12 we further characterise the ambidiagonal strata in terms of local monodromy around the singular fibres and the branching behaviour of the $j$-invariant.

Clearly, there is a connection with moduli spaces of lattice polarised $K3$ surfaces. Indeed, every ambidiagonal stratum gives rise to a priori several moduli spaces of lattice-polarised $K3$ surfaces. It turns out that the total number of possible components of moduli spaces associated to a given ambidiagonal stratum is either 1, 2, or 4, as follows from Corollary 11.4 and Proposition 11.7. For each such component $\mathcal{N}$ associated to an ambidiagonal stratum $\mathcal{M}$ there is a natural finite dominant map $\mathcal{N} \to \mathcal{M}$ and we compute its degree. This is the content of Proposition 12.1, Theorem 12.8 and Corollary 12.9.

We will now discuss the contents of the paper in more detail: In Section 2 we recall basic facts about elliptically fibred $K3$ surfaces and Miranda’s construction of the moduli space $\mathcal{F}$. The monodromy and Shimada stratifications are introduced in Section 3 where we also recall the principal results of Shimada’s theory. In Section 4 we formulate the main results of our paper.

Sections 5 to 10 are dedicated to the proof of the Main Theorem. We start in Section 3 by recalling the configuration strata studied by Kloosterman [Kl]. It is sufficient to look for ambidiagonal configuration strata and this will lead us to the severe restriction that the generic element of an ambidiagonal stratum cannot have singular fibres of type $II, III, IV, II^*$ or $III^*$ (Proposition 5.7). In Section 6 we recall a natural factorisation of the $j$-invariant $j(\mathcal{E}) = j_{\Gamma} \circ j_{\mathcal{E}}$ via the modular curve $X(\Gamma)$ defined by the modular monodromy group $\Gamma \subset \text{PSL}(2,\mathbb{Z})$. The restrictions on fibre configurations then translates into branching properties of $j_{\Gamma}$ and $j_{\mathcal{E}}$ over the points $0, 1, \infty$. In Section 7 we use an Euler number consideration to find a list of possible candidates for the modular monodromy group $\Gamma$. In Proposition 7.1 we prove that the index of the modular group $\Gamma$ in $\text{PSL}(2,\mathbb{Z})$ is bounded by 18. For the rest of the proof we shall distinguish between low index $\leq 6$ and high index $> 6$. The next step in the proof is that we provide Weierstraß data for all possible ambidiagonal strata with modular monodromy group $\Gamma$ of low index in Proposition 8.1. We also obtain some results about the Mordell-Weil groups and the monodromy of the 6 families listed in this proposition. In Sections 9 and 10 we finally complete the proof of the classification in the low and high index case respectively.
Section 11 and 12 are devoted to relating the ambi-typical strata which we have found to moduli spaces of lattice polarised $K3$ surfaces.

The appendix by M. Kirschmer contains explicit calculations concerning the 1-dimensional ambi-typical strata, in particular the genus of the moduli spaces of lattice polarised $K3$ surfaces covering these ambi-typical strata and the degree of the covering map. We find it remarkable that although the ambi-typical 1-dimensional strata all have genus 0, in accordance with the rationality result of Bogomolov, Petrov and Tschinkel, the genus of the moduli space of lattice-polarised $K3$ surfaces can be as high as 13.

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2. Elliptically fibred $K3$ surfaces

A $K3$ surface $S$ is **elliptically fibred** if there exists a surjective morphism $f : S \to \mathbb{P}^1$ whose generic geometric fibre is a genus 1 curve. We say that $f : S \to \mathbb{P}^1$ is a Jacobian fibration if it also has a section $s$. We denote the Néron-Severi group by $\text{NS}(S)$ and its rank, also called the Picard rank by $\rho(S)$. A general fibre of $f$ and the 0-section $s$ define a hyperbolic sublattice $U \subset \text{NS}(S)$. Conversely, the existence of a hyperbolic plane $U$ in $\text{NS}(S)$ guarantees the existence of a Jacobian fibration on $S$. But it should be noted that the class of the fibre and the section need not be contained in $U$. This is only the case if $U$ contains a nef, or equivalently base point free, isotropic class and an irreducible $-2$ curve.

We will typically denote $K3$ surfaces with a Jacobian fibration by $f : \mathcal{E} \to \mathbb{P}^1$ or simply by $\mathcal{E}$, where we think of $\mathcal{E}$ as an elliptic curve over the function field $\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(t)$. The sections of $\mathcal{E}$ form a finitely generated abelian group, the Mordell-Weil group of $\mathcal{E}$ which we denote by $\text{MW}(\mathcal{E})$.

The components of all singular fibres of $\mathcal{E}$ which do not meet the section generate a sublattice $R(\mathcal{E})$ of $\text{NS}(\mathcal{E})$ which is a root lattice, more precisely an orthogonal sum of lattices of $ADE$ type. The **trivial part** of the Néron-Severi group $\text{NS}(\mathcal{E})$ is

$$\text{NS}_{\text{tr}}(\mathcal{E}) = U + R(\mathcal{E})$$

where $U$ is the hyperbolic plane defined by the Jacobian fibration. To simplify notation we write here and in the sequel simply $+$ for a direct orthogonal sum. We will denote the rank of the trivial part by $\rho_{\text{tr}}(\mathcal{E})$. In general $R(\mathcal{E})$ is not saturated in $\text{NS}(\mathcal{E})$ and we will denote its saturation by $L(\mathcal{E})$. It is well known, see [ScSh, Corollary 6.20], that

$$L(\mathcal{E})/R(\mathcal{E}) \cong \text{MW}_{\text{tors}}(\mathcal{E}).$$

We recall that a Jacobian fibration $\mathcal{E}$ is called **extremal** if $\rho_{\text{tr}}(\mathcal{E}) = 20$. Such surfaces correspond to isolated points in $\mathcal{F}$ and will not be considered here, see however [BeMg] for the classification of those with a semi-stable fibration.
Jacobian fibrations can be classified via their Weierstraß models and this is the approach taken by Miranda. For the basic theory of Weierstraß equations we refer the reader to Miranda’s paper [Mi81]. Any Jacobian fibration \( f : \mathcal{E} \to \mathbb{P}^1 \), where \( \mathcal{E} \) is a \( K3 \) surface, is birational to a minimal Weierstraß model

\[
y^2z = 4x^3 - g_2xz^2 - g_3z^3 \quad \text{where} \quad g_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)), g_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12))
\]

and where the following (open) conditions hold:

1. \( \Delta = g_3^3 - 27g_2^2 \neq 0 \).
2. For every point \( q \in \mathbb{P}^1 \) the inequality \( \min\{3\nu_q(g_2), 2\nu_q(g_3)\} < 12 \) holds, where \( \nu_q(g) \) is the vanishing order of a polynomial \( g \) at \( q \).

The latter condition ensures that the Weierstraß equation is minimal in the sense that we cannot write \( g_2 = h^3\overline{g}_2 \) and \( g_3 = h^6\overline{g}_3 \) with \( \overline{g}_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) \) and \( \overline{g}_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)) \).

Conversely, an equation given as above defines a surface \( \mathcal{E}' \) in the projective bundle \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}) \) with at most rational double points. A minimal resolution \( \mathcal{E} \) of \( \mathcal{E}' \) is then a \( K3 \) surface and the projection onto the base of the projective bundle gives a Jacobian fibration \( f : \mathcal{E} \to \mathbb{P}^1 \). Weierstraß equations with coefficients \( g_2, g_3 \) and \( g'_2, g'_3 \) respectively, define isomorphic Jacobian fibrations if and only if there exists a coordinate transformation \( \text{GL}(2, \mathbb{C}) \), which maps the pair \( (g_2, g_3) \) to \( (g'_2, g'_3) \). This allows to describe the moduli of Jacobian fibrations in terms of a GIT quotient.

In order to do this we consider the weighted projective space \( \mathbb{P}_{8,12}(9,13) \) associated to \( H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12)) \). The open conditions described above define an open subset \( V \subset \mathbb{P}_{8,12}(9,13) \) and the group \( \text{SL}(2, \mathbb{C}) \) acts on \( \mathbb{P}_{8,12}(9,13) \) as well as on the open subset \( V \). By [Mi81, Proposition 5.1] all points of \( V \) are stable with respect to the action of \( \text{SL}(2, \mathbb{C}) \). Hence we can form the quotient

\[
\mathcal{F} := V//\text{SL}(2, \mathbb{C})
\]

which is a quasi-projective variety. Its geometric points correspond bijectively to isomorphism classes of Jacobian fibrations \( f : \mathcal{E} \to \mathbb{P}^1 \) where \( \mathcal{E} \) is a \( K3 \) surface.

In Section 11 we will discuss the theory of moduli spaces of lattice (quasi-)polarised \( K3 \) surfaces. Taking the hyperbolic plane \( U \) (which one should think of as being spanned by the class of the section and of a general fibre) one obtains an alternative construction of the moduli space \( \mathcal{F} \) as a quotient of a homogeneous domain of type IV by an arithmetic group. As we shall see, both points of view are relevant for our purposes. An interesting result of Odaka and Oshima [OO, Theorem 7.9] even says that the GIT compactification and the Baily-Borel compactification of this space coincide, i.e. \( \mathcal{F}^{\text{GIT}} \cong \mathcal{F}^{\text{BB}} \).

3. Monodromy and Shimada strata

In this section we will introduce the main protagonists of this paper, namely monodromy strata and Shimada strata.

**Monodromy strata.** Monodromy strata for Jacobian fibrations were first introduced by Bogomolov, Petrov and Tschinkel in [BPTT]. Here we recall some of their results, restricting ourselves to the case in hand, namely Jacobian fibrations of \( K3 \) surfaces. The first step is to consider the open subset \( V' \subset V \) corresponding to non-isotrivial fibrations. It is naturally obtained by replacing condition (1) above by the slightly stronger condition
(1') \( \Delta = g_3^2 - 27g_2^2 \) and \( g_3^2 \) are not proportional (which also excludes the case \( g_2 \equiv 0 \) and the case \( \Delta \equiv 0 \)).

Since the \( j \)-invariant is the quotient of these two expressions, this translates directly into several equivalent, more geometric, characterisations of points \( v \in V' \):

1. The \( j \)-invariant of the Jacobian elliptic fibration \( \mathcal{E}_v \) is non-constant.
2. The Jacobian elliptic fibration \( \mathcal{E}_v \) is not isotrivial.
3. The number of singular fibres of multiplicative type \( I_\nu \) or additive type \( I_\nu^* \), \( \nu > 0 \), is positive.

Clearly \( V' \) is invariant (as a set) under the action of \( \text{SL}(2, \mathbb{C}) \) and we denote the quotient by \( \mathcal{F}' \subset \mathcal{F} \). This is the open subset parameterising all non-isotrivial Jacobian fibrations of \( K3 \) surfaces.

To decompose \( V' \) (and thus \( \mathcal{F}' \)) in a geometrically meaningful way, \([BPT]\) exploit the monodromy group of elliptic fibrations: the complement \( \mathcal{E}' \) of the union of singular fibres of a Jacobian fibration \( \mathcal{E} \) is topologically equivalent to a torus bundle over the base punctured at the critical values of the fibration. The image of the associated monodromy representation in the automorphisms of the first homology of a fibre is orientation preserving. Upon the choice of a basis, it becomes the monodromy group \( \Gamma(\mathcal{E}) \), a subgroup of \( \text{SL}(2, \mathbb{Z}) \), well-defined up to conjugacy, see \([Mi89, VI,3]\). Note that according to this definition we will consider monodromy groups as subgroups of \( \text{SL}(2, \mathbb{Z}) \), and keep in mind that it is the conjugacy class which is the invariant. We will use \( \sim \) to denote conjugate subgroups. The group \( \Gamma(\mathcal{E}) \) determines a group \( \bar{\Gamma}(\mathcal{E}) \subset \text{PSL}(2, \mathbb{Z}) \) which we will call the modular monodromy group.

The initial observation in \([BPT]\) is the following semi-continuity property with respect to the Zariski topology on the quasi-projective variety \( \mathcal{F}' \): the monodromy group can only change if the configuration of singular fibres changes, i.e. if some fibres split or come together. The latter case only occurs on closed subsets of the base of a family and the monodromy group will be a subgroup after the process. Therefore, the property that the monodromy group is a subgroup of a fixed subgroup of \( \text{SL}(2, \mathbb{Z}) \) (up to conjugacy) defines a closed subset.

This can be rephrased as follows. We consider the set \( P \) of all conjugacy classes of subgroups of \( \text{SL}(2, \mathbb{Z}) \) together with the partial ordering induced by inclusion. On this we define the Alexandroff topology for which a set \( U \) is open if and only if \( p \in U \) and \( p \leq q \) implies \( q \in U \). The semicontinuity property observed by Bogomolov, Petrov and Tschinkel then says that the map \( V' \to P \) is continuous with respect to the Alexandroff topology on \( P \) and the Zariski topology on \( V' \). It implies that the fibres \( V'_\Gamma \) of this map are locally closed for a fixed (conjugacy class of a) subgroup \( \Gamma \subset \text{SL}(2, \mathbb{Z}) \). We can decompose each such set into its irreducible components \( V'_\Gamma = \bigcup_i V'_{\Gamma,i} \). Bogomolov, Petrov and Tschinkel showed that the number of possible monodromy groups is finite and in this way obtained the

**Lemma 3.1** (cf. \([BPT]\), Lemma 3.1). The variety \( V' \) is a finite union of locally closed irreducible subvarieties \( V'_{\Gamma,i} \), each preserved under the action of \( \text{SL}(2, \mathbb{C}) \) such that for every \( v \in V'_{\Gamma,i} \) one has \( \Gamma(\mathcal{E}_v) \sim \bar{\Gamma} \).

We denote \( \mathcal{F}'_{\Gamma,i} = V'_{\Gamma,i} \# \text{SL}(2, \mathbb{C}) \) and in this way one obtains a finite decomposition

\[
(3.1) \quad \mathcal{F}' = \bigcup_i \mathcal{F}'_{\Gamma,i}
\]

into irreducible locally closed subvarieties.
Definition 3.2. We refer to [31] as the \textit{monodromy stratification} of the moduli space of $K3$ surfaces with a non-isotrivial Jacobian fibration, and call the subvarieties $\mathcal{F}_{\Gamma,i}$ the \textit{monodromy strata} of $\mathcal{F}'$.

While the objective of [BPT] is the rationality of the monodromy strata $\mathcal{F}_{\Gamma,i}$ in the moduli space, for our arguments it is important to note the following maximality condition
\begin{equation}
 v \in \overline{V_{\Gamma,i}} \setminus V_{\Gamma,i} \implies \Gamma(\mathcal{E}_v) \nsim \Gamma.
\end{equation}

\textbf{Shimada strata.} We have already seen the root lattice $R(\mathcal{E})$ associated to a Jacobian fibration $f : \mathcal{E} \to \mathbb{P}^1$ of a $K3$ surface. In general $R(\mathcal{E})$ is not saturated in $\text{NS}(\mathcal{E})$ and we denoted its saturation by $L(\mathcal{E})$. The overlattice $L(\mathcal{E})$ of $R(\mathcal{E})$ defines, and can be reconstructed, from the finite group $G(\mathcal{E}) = L(\mathcal{E})/R(\mathcal{E}) \subset D(R(\mathcal{E}))$. Here we use standard notation and standard facts form lattice theory: if $L$ is an even lattice we denote by $L^\vee$ the dual lattice which can be defined as $L^\vee = \{ x \in \mathbb{L} \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in L \}$. The \textit{discriminant} of $L$ is the quotient $D(L) = L^\vee/L$. This is a finite group and it is equipped with a quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$. The importance of the discriminant group in lattice theory was fully developed in Nikulin’s seminal paper [Nik]. Overlattices of $L$ correspond to isotropic subgroups $G \subset D(L)$, see [Nik] Proposition 1.4.1.

In his paper [Shi], Shimada investigated the question which pairs $(R, G)$, where $R$ is an $ADE$ root lattice and $G \subset D(R)$ is a finite (isotropic) subgroup of the discriminant group, can occur as $(R(\mathcal{E}), G(\mathcal{E}))$ and how many irreducible families these Jacobian fibrations form. At this point it is important to note that in general a pair $(R, G)$ does not necessarily determine a unique family. There are several reasons for this. Firstly, the abstract group $G$ does not necessarily define a unique overlattice $L$ of $R$, more precisely one must choose a specific isotropic subgroup $G$ in $D(R)$ and it can happen that $G$ can be embedded in several ways as an isotropic subgroup in $D(R)$. Secondly, given $L$ one must also specify a primitive embedding of $M = U \oplus L$ into the $K3$ lattice
\[ L_{K3} = 3U + 2E_8 \]
where $E_8$ is the unique even unimodular negative definite lattice of rank 8 and the lattice $M$ may have several such embeddings (modulo the orthogonal group $O(L_{K3})$). Given such an embedding one can consider the moduli space of \textit{lattice polarised} $K3$ surfaces with lattice polarisation $M$. Note however, that a lattice polarization contains possibly more information than the pair $(R, G)$ as different lattice polarizations can give rise to the same elliptic $K3$ surface with a given configuration of singular fibres. We shall discuss this relationship in detail in Sections 11 and 12. The Shimada strata correspond to finite quotients of (open subsets) of moduli spaces of lattice polarised $K3$ surfaces. We also note that the moduli spaces can have 1 or 2 components and this, thirdly, can also increase the number of strata. If there are 2 such components then they are complex conjugate to each other, i.e. surfaces of one component are complex conjugate to those of the other component, cf. [Fam] p.206.

Shimada’s classification [Shi2] gives the following result.

\textbf{Theorem 3.3} (Shimada). The following holds:
\begin{enumerate}
\item There are 3278 different root lattices which occur as $R(\mathcal{E})$ for a Jacobian fibration $\mathcal{E}$ of a $K3$ surface. Of these 2953 belong to non-extremal fibrations.
\item This decomposes the set of all Jacobian fibrations of $K3$ surfaces into 3932 connected families, of which 3469 belong to non-extremal fibrations.
\end{enumerate}
Shimada’s result defines a stratification of the moduli space $\mathcal{F}'$ into maximal (with respect to inclusion) locally closed irreducible subvariety $\mathcal{M}'_{R,G,j}$ and this defines a finite decomposition
\begin{equation}
\mathcal{F}' = \bigcup_j \mathcal{M}'_{R,G,j}.
\end{equation}

**Definition 3.4.** We call the decomposition (3.3) the **Shimada stratification** of $\mathcal{F}'$ and the subvarieties $\mathcal{M}'_{R,G,j}$ the **Shimada strata** of $\mathcal{F}'$.

For future use we also note the following remark which is simply obtained by counting the conditions imposed by the rank of the Néron-Severi group.

**Remark 3.5.** The dimension of a component $\mathcal{M}'_{R,G,j}$ is given by
\begin{equation}
\dim \mathcal{M}'_{R,G,j} = 18 - \text{rank}(R).
\end{equation}
We notice that this only depends on $R$ and not on $G$ nor the specific component we are considering. Note that all components of $\mathcal{M}'_R$ have the same dimension, which we will also refer to as the dimension of $\mathcal{M}'_R$.

We note that the decomposition in Shimada strata (3.3) is a topological poset stratification in the sense of [YaYo].

### 4. The main classification result

The primary goal of this paper is to compare the two stratifications of the moduli space $\mathcal{F}'$ of non-isotrivial Jacobian fibrations on $K3$ surfaces which we have defined above, namely the monodromy stratification and the Shimada stratification. More precisely, we want to determine all positive dimensional monodromy and Shimada strata whose generic points coincide. In other words, we want to determine all pairs $\mathcal{F}'_{\Gamma,i}$ and $\mathcal{M}'_{R,G,j}$ such there intersection is non-empty and open (and hence dense) in both $\mathcal{F}'_{\Gamma,i}$ and $\mathcal{M}'_{R,G,j}$. This leads us to the definition

**Definition 4.1.** A positive dimensional irreducible closed subset $A \subset \mathcal{F}'$ is called an **ambi-typical stratum** if there is a monodromy stratum $\mathcal{F}'_{\Gamma,i}$ and a Shimada stratum $\mathcal{M}'_{R,G,j}$ such that
\begin{equation}
A = \overline{\mathcal{F}'_{\Gamma,i} = \mathcal{M}'_{R,G,j}}.
\end{equation}
(The monodromy stratum and the Shimada stratum are then uniquely defined.)

Our main result is

**Theorem 4.2.** There are 50 ambi-typical strata in $\mathcal{F}'$.

As we have discussed, a Shimada stratum determines a pair $(R, G)$, but not necessarily the other way round. There are, however, cases where the root lattice defines a unique Shimada stratum. Indeed, this is the case for most of the ambi-typical strata. The data $(R, G)$ for the ambi-typical strata which are referred to in the following theorem are listed in Table 11.

**Theorem 4.3.** In all but 3 cases the root lattice $R$ of an ambi-typical stratum $A$ determines a unique Shimada stratum. The exceptions are:

1. The root lattice $D_4 + 2A_6 + A_1$ in 38/39 determines two Shimada strata, these are conjugate complex to each other.
(2) The root lattice $2A_3 + 8A_1$ in $6$ determines two non-conjugate Shimada strata. Only one of these occurs as an ambi-typical stratum.

By definition a monodromy stratum determines a monodromy group $\Gamma$, but the converse will in general not be true. In fact, as we will see in the rest of the paper, monodromy strata are determined by some fairly involved data. This includes the modular monodromy group $\overline{\Gamma} \subset \text{PSL}(2, \mathbb{Z})$, but also information about the $j$-invariant $j(E): \mathbb{P}^1 \rightarrow \mathbb{P}^1$. We shall see in Section 6 that this can be factored in a unique way as $j(E) = j_{\overline{\Gamma}} \circ j_E$ where $j_{\overline{\Gamma}}$ is a Belyi map.

The additional data which determine a monodromy stratum are given by a space of rational maps $j_E: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and a rational family of divisors on $\mathbb{P}^1$ corresponding to the $\ast$-fibres. However, in the case of ambi-typical strata these data are fully determined by much more accessible ones:

**Theorem 4.4.** The 50 ambi-typical monodromy strata determine the list of invariants given in Table 12, consisting of

1. the local monodromies of $j_{\overline{\Gamma}}$ at $0, 1, \infty$,
2. the branching of $j_E$ for a generic element $E$ at the non-critical pre-images of $0, 1$ and at poles of $j_{\overline{\Gamma}}$,
3. the number of $\ast$-fibres.

Conversely, the data are pairwise distinct and each determines a unique monodromy stratum with the following exception: the strata 38/39 correspond to two distinct maps $j_{\overline{\Gamma}}$ having the same combinatorial type but with monodromy factorizations in distinct conjugation classes.

The proof of these theorems is quite involved and takes up most of the remaining paper. Here we shall give a rough outline of our strategy:

- In Section 5 we study configurations of singular fibres where we use the work of Kloosterman [Kl, sec.4]. This leads to a semi-continuous invariant which induces a stratification which refines both the stratifications by monodromy and by root lattice. Hence it suffices to look for ambi-typical fibre configuration strata. We find that to be ambi-typical poses severe restrictions on the possible singular fibres.
- In Section 6 we recall the factorisation $j(E) = j_{\overline{\Gamma}} \circ j_E$ of the $j$-invariant. The restrictions on fibre configurations then translates into branching properties of $j_{\overline{\Gamma}}$ and $j_E$ over the points $0, 1, \infty$.
- In Section 7 we use an Euler number restriction to find a list of candidates for the modular monodromy group $\overline{\Gamma}$ and give an upper bound for the dimension of the corresponding strata.
- In Section 8 we parameterise closed subsets of $V'$ which give Weierstraß data for all possible ambi-typical strata with modular monodromy group $\overline{\Gamma}$ of low index, i.e. at most 6.
- In Section 9 we determine the ambi-typical strata among these families of Weierstraß data and their invariants.
- In Section 10 we address the classification of ambi-typical strata with modular monodromy group $\overline{\Gamma}$ of high index, i.e. at least 7. In the first step we determine the possible corresponding root lattices and the topological types of the $j_E$ factor of the $j$-function. In the second step we determine the topology of the maps $j_{\overline{\Gamma}}$ and the corresponding monodromy groups.

\footnote{We will say more on this in Remark 9.6}
This program will finally allow us to complete the proofs of our theorems.

5. Singular fibre configurations and generic invariants

In this section we consider the configuration of singular fibres of elliptic surfaces. The possible singular fibre types have been classified by Kodaira and are listed in Table 1, see [BHPV, Table V.6] with some of their invariants. The relation between singular fibre types and Weierstraß data is given in Table 2. This contains all information necessary to apply the Tate algorithm over the complex numbers, i.e. to determine the fibre types from the vanishing orders of \( \nu_2 = \nu(g_2), \nu_3 = \nu(g_3) \) and \( \nu_\Delta = \nu(\Delta) \) of \( g_2, g_3 \) and \( \Delta \) respectively.

| fibre type | ADE-type | Euler number | local monodromy | local \( j \)-expansion |
|------------|----------|--------------|------------------|-------------------------|
| \( I_0 \)  | –        | 0            | \( \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \) | \( j = s^{3k}, j = 1 + s^{2k} \) or \( j \neq 0, 1 \) |
| \( I_1 \)  | –        | 1            | \( \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \) | pole of order 1         |
| \( I_b \)  | \( A_{b-1} \) | \( b \)    | \( \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \) | pole of order \( b \)   |
| \( I_0^* \) | \( D_4 \) | 6           | \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) | same as in first case   |
| \( I_b^* \) | \( D_{4+b} \) | \( 6+b \) | \( \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} \) | pole of order \( b \)   |
| \( II \)   | –        | 2            | \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) | \( j = s^{3k+1} \)     |
| \( III \)  | \( A_1 \) | 3            | \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) | \( j = 1 + s^{2k+1} \) |
| \( IV \)   | \( A_2 \) | 4            | \( \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \) | \( j = s^{3k+2} \)     |
| \( IV^* \) | \( E_6 \) | 8            | \( \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \) | \( j = s^{3k+1} \)     |
| \( III^* \)| \( E_7 \) | 9            | \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) | \( j = 1 + s^{2k+1} \) |
| \( II^* \)| \( E_8 \) | 10           | \( \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \) | \( j = s^{3k+2} \)     |

Table 1. Fibre types of elliptic surfaces

In the ensuing discussion we will use notation as introduced in Kloosterman’s paper [K]. For a Jacobian fibration \( f : S \to \mathbb{P}^1 \) let \( C(f) \) denote the configuration of its singular fibres. Since we are restricting ourselves to non-isotrivial fibrations, we can assume that \( C(f) \) contains at least one fibre of type \( I_\nu \) or \( I_\nu^* \) with \( \nu > 0 \). For an abstract configuration \( C \) of singular fibres we define

\[
L(C) := \{ [f : S \to \mathbb{P}^1] \in \mathcal{F} \mid C(f) = C \}.
\]

| type | \( I_0 \) | \( I_{k, k > 0} \) | \( I_0^* \) | \( I_{k, k > 0}^* \) | \( II \) | \( III \) | \( IV \) | \( IV^* \) | \( III^* \) | \( II^* \) |
|------|----------|-------------------|----------|-------------------|--------|--------|--------|--------|--------|--------|
| \( j \) | 0 1 gen. | \( \infty \) | 0 1 gen. | \( \infty \) | 0 1 0 0 1 0 |
| \( \nu_2 \) | >0 0 0 0 | 0 >2 2 2 | 2 >0 1 >1 >2 3 | >3 |
| \( \nu_3 \) | >0 0 0 0 | 0 >3 3 3 | 3 1 >1 2 4 >4 5 |
| \( \nu_\Delta \) | 0 0 0 | \( k \) | 6 6 6 | \( k+6 \) | 2 3 4 | 8 9 10 |

Table 2. Tate algorithm (cf. [Mi89, Table IV.3.1])
Remark 5.1. We recall from [KL, Lemma 4.6] that \( L(C) \subset \mathcal{F}' \) is constructible. Fibre configurations are partially ordered by degeneration of several singular fibres into fewer more complicated ones. Degeneration occurs on closed subsets, thus semi-continuity holds and \( \mathcal{F}' \) is topologically stratified by the components of the sets \( L(C) \).

Definition 5.2. We call \( L(C) \) a configuration locus and the components of the sets \( L(C) \) the configuration strata.

As we will need the dimension of the components of \( L(C) \) we recall

Lemma 5.3. [KL, Lemma 4.6, Assumption 4.3] Assume that \( C \) is a configuration of singular fibres arising from a non-isotrivial Jacobian fibration. Then all components of \( L(C) \) have the dimension

\[
\dim L(C) = \#\{\text{singular fibres}\} + \#\{\text{fibres of type II}^*, \text{III}^*, \text{IV}^*, I^*_\nu, \nu \geq 0\} - 6.
\]

By construction, each irreducible component of \( L(C) \) corresponds to Jacobian elliptic fibrations with topologically equivalent complements of singular fibres. In particular all elements in an irreducible component of \( L(C) \) have the same monodromy group and each irreducible component \( \mathcal{F}'_{\Gamma,i} \) is a finite union of irreducible components of certain \( L(C) \). The only one of these, which is open, corresponds thus to the configuration of a generic member of the monodromy stratum. We call this configuration \( C(\mathcal{F}'_{\Gamma,i}) \) and all components of \( L(C(\mathcal{F}'_{\Gamma,i})) \) have the same dimension as \( \mathcal{F}'_{\Gamma,i} \).

By definition all Jacobian fibrations \( \mathcal{E} \in L(C) \) have the same root configuration and hence the same rank \( \rho_{\text{tr}}(\mathcal{E}) \) of the trivial lattice.

From this one easily obtains

Lemma 5.4. [KL, Prop. 4.7] Let \( C \) be a configuration of singular fibres, containing at least one \( I_\nu \) or \( I^*_\nu \)-fibre \((\nu > 0)\) with \( L(C) \neq \emptyset \) and let \( \mathcal{E} \) be any element in \( L(C) \). Then

\[
\dim L(C) = 20 - \rho_{\text{tr}}(\mathcal{E}) - \#\{\text{fibres of type II, III or IV}\}.
\]

Proof. We use the above lemma and apply the fact that the rank of the trivial lattice \( \rho_{\text{tr}}(\mathcal{E}) \) is given by

\[
\rho_{\text{tr}}(\mathcal{E}) = 2 + \sum_{\text{F multiplicative}} (e(F) - 1) + \sum_{\text{F additive}} (e(F) - 2) = 26 - \#\{\text{multiplicative fibres}\} - 2\#\{\text{additive fibres}\}.
\]

Here \( e(F) \) denotes the Euler number of a fibre \( F \). \( \square \)

Any component of a stratum \( L(C) \) defines an embedding \( U + R \hookrightarrow L_{K3} \) (up to isometries in \( O(L_{K3}) \)) and thus an embedding into a root stratum. Hence the stratification given by the configuration strata refines both the monodromy stratification and the root lattice stratification. In particular, every Shimada stratum \( \mathcal{M}'_{R,G,j} \) and every monodromy stratum \( \mathcal{F}'_{\Gamma,i} \) contains a unique configuration stratum which is open and dense in this stratum.

This also allows us to talk about the properties of a generic element of a Shimada stratum or a monodromy stratum. In particular, the following data, in addition to the configuration \( C(\mathcal{E}) \), are invariant for all members \( \mathcal{E} \) of a configuration stratum: the monodromy group \( \Gamma(\mathcal{E}) \) (as we have already pointed out), the root lattice \( R(\mathcal{E}) \), the trivial lattice \( \text{NS}_{\text{tr}}(\mathcal{E}) \), the
saturation $L(\mathcal{E})$ of $R(\mathcal{E})$ in $\text{NS}(\mathcal{E})$ and the torsion of the Mordell-Weil group $\text{MW}_{\text{tors}}(\mathcal{E}) \cong L(\mathcal{E})/R(\mathcal{E})$.

We are now ready to start the classification of ambi-typical strata. For this we first collect necessary conditions which a generic element of such a stratum must fulfill. Indeed, the first restrictions on the possible configurations can be derived easily.

**Proposition 5.5.** Suppose $\Gamma$ is a proper subgroup of $\text{SL}(2,\mathbb{Z})$ of finite index and $\mathcal{E} \in \mathcal{F}_{\Gamma,i}$ a generic element of a monodromy stratum with monodromy group $\Gamma$. Then the following properties are equivalent:

1. The Jacobian fibration $\mathcal{E}$ has no fibres of type $\text{II}, \text{III}$ or $\text{IV}$.
2. The dimensions of the Shimada stratum $\mathcal{M}_{R(\mathcal{E}),G(\mathcal{E}),i}$ which contains $\mathcal{E}$ and of $\mathcal{F}_{\Gamma,i}$ coincide.

In particular, the generic element of an ambi-typical stratum does not have any fibres of type $\text{II}, \text{III}$ or $\text{IV}$.

**Proof.** Since $\mathcal{E}$ is a generic element of $\mathcal{F}_{\Gamma,i}$ it lies in a unique open and dense configuration stratum, namely a component of $L(C(\mathcal{E}))$. Hence

$$\dim \mathcal{F}_{\Gamma,i} = \dim L(C(\mathcal{E})) = 20 - \rho_3(\mathcal{E}) - \# \{\text{fibres of type } \text{II}, \text{III} \text{ or } \text{IV}\}$$

where the last equality follows from Lemma 5.4. Since the Shimada stratum has dimension equal to $20 - \rho_3(\mathcal{E})$ the claim follows immediately.

**Remark 5.6.** The above lemma shows: if the generic element of a monodromy stratum $\mathcal{F}_{\Gamma,i}$ has fibres of type $\text{II}, \text{III}$ or $\text{IV}$, then this monodromy stratum is contained in a Shimada stratum of strictly bigger dimension.

We can find further severe restrictions on ambi-typical strata by studying how the monodromy behaves under certain fibre degenerations.

**Proposition 5.7.** Assume that $\mathcal{E}$ is a generic element of an ambi-typical stratum. Then $\mathcal{E}$ has no singular fibres of type $\text{II}, \text{III}, \text{IV}, \text{II}^* \text{ or } \text{III}^*$. If $-\text{id} \in \Gamma(\mathcal{E})$ then $\mathcal{E}$ also has no singular fibres of type $I_{g_0}^*, \text{IV}^*$.

**Proof.** We have already seen in Proposition 5.3 that fibres of type $\text{II}, \text{III}$ or $\text{IV}$ cannot exist.

Our strategy is the following: suppose that $\mathcal{E}$ has a singular fibre of type $\text{II}^*, \text{III}^*, I_{g_0}^*$ or $\text{IV}^*$, where in the latter two cases we assume that $-\text{id} \in \Gamma(\mathcal{E})$. We will then construct a family containing $\mathcal{E}$ where the monodromy remains constant, but where the rank of the trivial Néron-Severi group drops. This contradicts the assumption that $\mathcal{E}$ is a generic element of a Shimada stratum.

To do this pick a Weierstraß datum $g_2, g_3$ for $\mathcal{E}$. By inspection of the Tate Table 2 the presence of a $^*\text{-fibre}$ implies that $g_2, g_3$ have a common zero at this fibre. More precisely, we can factor $g_2 = \tilde{g}_2 x^2, g_3 = \tilde{g}_3 x^3$ where $x$ is a linear form vanishing at this fibre. We then consider the family of Weierstraß data $\tilde{g}_2(x-t)^2, \tilde{g}_3(x-t)^3$ where $t$ varies. It has the same $j$-invariant as $\mathcal{E}$, hence the projective monodromy group $\Gamma(\mathcal{E}_i)$ is constant, see [BHPV, p.211]. If in addition $-\text{id} \in \Gamma(\mathcal{E})$, then also the monodromy group $\Gamma(\mathcal{E}_i)$ is constant, since it can only get smaller on a closed subset. By Table 11 for fibres of type $\text{II}^*, \text{III}^*$ a power of the local monodromy is $-\text{id}$, so this assumption is fulfilled. For fibres of type $I_{g_0}^*, \text{IV}^*$ this is part of our assumptions. Hence the monodromy group remains constant if we vary $t$, however for $t \neq 0$ the fibre of $^*\text{-type}$ is replaced by an $I_0^*$ fibre and the fibre without $^*$ that
has the same local monodromy up to $-\text{id}$, see Table 1. But then, again by Table 1, this implies that the rank of the trivial Néron-Severi group $\text{NS}_t(E)$ group drops as $t \neq 0$, giving the desired contradiction.

\[\square\]

**Remark 5.8.** A typical example for the situation discussed is when a type $II^*$-fibre splits into a type $IV$- and an $I_0^*$-fibre. Here the rank of the trivial Néron-Severi drops by 2. This gives examples where a Shimada stratum is contained as a proper subset in a bigger monodromy stratum.

### 6. Factorisations of the $j$-invariant

An essential tool for our classification result is the $j$-function associated to a Jacobian fibration $f: \mathcal{E} \to \mathbb{P}^1$. As usual we denote the upper half plane by $\mathbb{H}_1$ and set

\[\mathbb{H}_1 = \mathbb{H}_1 \cup \mathbb{Q} \cup \{\infty\}.\]

The quotient

\[X(1) := \mathbb{H}_1 / \text{PSL}(2, \mathbb{Z}) \cong \mathbb{P}^1\]

is the modular curve of level 1, namely the compactification of the $j$-line. If the Jacobian fibration $f: \mathcal{E} \to \mathbb{P}^1$ has the monodromy group $\Gamma \subset \text{SL}(2, \mathbb{Z})$ then we denote its image in $\text{PSL}(2, \mathbb{Z})$ by $\overline{\Gamma}$. This defines the modular curve

\[X(\overline{\Gamma}) := \mathbb{H}_1 / \overline{\Gamma}\]

which in our case is again isomorphic to $\mathbb{P}^1$. The $j$-function

\[j(\mathcal{E}): \mathbb{P}^1 \to X(1) \cong \mathbb{P}^1\]

has a unique factorisation

\[(6.1) \quad j(\mathcal{E}) = j_{\overline{\Gamma}} \circ j_{\mathcal{E}}\]

up to deck transformations of $j_{\overline{\Gamma}}$, where

\[j_{\overline{\Gamma}}: X(\overline{\Gamma}) \cong \mathbb{P}^1 \to X(1) \cong \mathbb{P}^1\]

is the natural quotient map. We refer the reader also to [BPT, p. 1107]. Another characterization of $\overline{\Gamma}$, using the factorisation of $j(\mathcal{E})$, is provided by

**Lemma 6.1.** The group $\Gamma$ has the following minimality property:

\[(6.2) \quad \text{if } j(\mathcal{E}) \text{ factors through } j_{\overline{\Gamma}}, \text{ then } \Gamma \subset \Gamma' \text{ up to conjugation.}\]

**Proof.** Given a holomorphic surjective map $j: \mathbb{P}^1 \to X(1)$ there exists by [BT, Lemma 2.3] an, up to conjugation unique, subgroup $\overline{\Gamma}_j \subset \text{PSL}(2, \mathbb{Z})$ of finite index such that

\[(6.3) \quad j \text{ factors through } j_{\overline{\Gamma}_j},\]

\[(6.4) \quad \text{if } j \text{ factors through } j_{\overline{\Gamma}}, \text{ then } \overline{\Gamma}_j \subset \overline{\Gamma}' \text{ up to conjugation.}\]

We apply this result to $j(\mathcal{E})$ and thus have to show that $\overline{\Gamma}_{j(\mathcal{E})} = \overline{\Gamma}$. The inclusion $\overline{\Gamma}_{j(\mathcal{E})} \subset \overline{\Gamma}$ (up to conjugacy) follows from (6.4).

For the other direction we use the discussion in [BHPV, p.211]. Every $j$-map gives rise to a representation class $r = r(j)$ with values in $\text{PSL}(2, \mathbb{Z})$ and, in particular, $r(j_{\overline{\Gamma}'})$ has image $\overline{\Gamma}'$. By Kodaira, $r(j(\mathcal{E}))$ is the composition of the monodromy homomorphism with $\text{SL}(2, \mathbb{Z}) \to \text{PSL}(2, \mathbb{Z})$ and hence has image $\overline{\Gamma}$. By property (6.3) the map $j(\mathcal{E})$ factors
Since \( \pi \text{ as the free product } \mathbb{Z} \) correspond with each other via the following maps:

Through \( j \) and hence \( \text{Im } r(j(\mathcal{E})) \subseteq \text{Im } r(j_{\Gamma(\mathcal{E})}) \), which implies that \( \Gamma \subseteq \Gamma(\mathcal{E}) \) (up to conjugation).

\( \square \)

**Remark 6.2.** In the sequel it will be very helpful to investigate subgroups of \( \text{PSL}(2, \mathbb{Z}) \) by means of the following three sets, where \( S_d \) denotes the symmetric group of \( d \) elements:

1. subgroups \( \bar{\Gamma} \) in \( \text{PSL}(2, \mathbb{Z}) \) of index \( d \) up to conjugacy in \( \text{PSL}(2, \mathbb{Z}) \),
2. (connected) branched covers \( j : C \to X(1) \) of degree \( d \) up to equivalence of covers, branched only over \( 0, 1, \infty \) with multiplicities \( 1, 3 \) over 0 and \( 1, 2 \) over 1,
3. homomorphisms \( \mu : \pi_1(\mathbb{C} \setminus \{0, 1\}) \to S_d \) up to conjugacy in \( S_d \), such that simple loops around 0, resp. 1 map to elements of order 1 or 3, resp. 1 or 2, and the image acts transitively.

Since \( \pi_1(\mathbb{C} \setminus \{0, 1\}) \) is freely generated by simple loops around 0 and 1, while \( \text{PSL}(2, \mathbb{Z}) \) as the free product \( \mathbb{Z}/2 \ast \mathbb{Z}/3 \) is generated by \( \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \) and \( \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \), these sets are in bijective correspondence with each other via the following maps:

- **(1)\to(2):** Given \( \bar{\Gamma} \), associate the branched cover \( j_{\bar{\Gamma}} : X(\bar{\Gamma}) \to X(1) \).
- **(1)\to(3):** Given \( \bar{\Gamma}, \) associate left multiplication \( \text{PSL}(2, \mathbb{Z}) \to S(\text{PSL}(2, \mathbb{Z})/\bar{\Gamma}) \cong S_d \) on cosets and compose with \( \pi_1(\mathbb{C} \setminus \{0, 1\}) \to \text{PSL}(2, \mathbb{Z}) \) to get \( \mu \).
- **(2)\to(3):** Given \( j : C \to \mathbb{P}^1, \) restrict to \( \mathbb{C} \setminus \{0, 1\} \), which is a topological cover of degree \( d \) and associate the representation \( \mu : \pi_1(\mathbb{C} \setminus \{0, 1\}) \to S_d \) by permutations of a fibre.
- **(3)\to(1):** Given \( \mu, \) note that by our assumptions this factors through a homomorphism \( \bar{\mu} : \text{PSL}(2, \mathbb{Z}) \to S_d \) and associate the stabiliser subgroup \( \bar{\Gamma} \subset \text{PSL}(2, \mathbb{Z}) \) of 1.
- **(3)\to(2):** Given \( \mu, \) associate the connected topological cover over \( \mathbb{C} \setminus \{0, 1\} \) of degree \( d \) that extends to a branched cover \( j : C \to \mathbb{P}^1 \) by Riemann existence.

Using the remark we obtain a characterisation of the factorisation (6.1) from Lemma 6.1.

**Lemma 6.3.** Given the holomorphic map \( j(\mathcal{E}) : \mathbb{P}^1 \to X(1) \) a factorisation \( j_2 \circ j_1 \) is equivalent to \( j_{\bar{\Gamma}} \circ j_{\mathcal{E}} \) if and only if

\( j_2 \) is branched only over \( 0, 1, \infty \) with multiplicities \( 1, 3 \) over 0 and \( 1, 2 \) over 1,

\( j_1 \) has no proper left factor \( j' \), such that \( j_2 \circ j' \) has the property above.

In the topological analysis of more arbitrary (connected) branched covers of \( \mathbb{P}^1 \) we first fix a base point not in the branch locus. We then use the monodromy homomorphism taking homotopy classes of closed paths inside the complement of the branch locus around this base point to the permutation group of the fibre over the base point. This defines a local monodromy which associates to a branch point the conjugacy class of the monodromy of the positively oriented boundary of a sufficiently small disc centred at the branch point. While the monodromy depends on the choice of a fibre, the conjugacy class does not, and it is thus an invariant, which we may as well give by the cycle type of the permutation or the corresponding partition of the fibre cardinality. The sizes of the parts of that partition are exactly the multiplicities of the pre-images of the branch point.

Accordingly, we can associate to \( j_{\bar{\Gamma}} \) the three conjugacy classes \( C_0, C_1, C_{\infty} \) corresponding to the branch points \( 0, 1, \infty \). In practice we will give the three conjugacy classes as a 3-tuple.
of partitions of degree $\deg j_\Gamma$. The degree of $j_\Gamma$ and the numbers $e_3$ and $e_2$ of fixed points of elements in $C_0$ and $C_1$ respectively, are related by the following congruences

\begin{equation}
\deg j_\Gamma \equiv_2 e_2, \quad \deg j_\Gamma \equiv_3 e_3.
\end{equation}

This follows since the difference of the two numbers in question is the sum of cycle lengths of transpositions and 3-cycles respectively. Note that for the subgroup $\bar{\Gamma}$ of $\text{PSL}(2, \mathbb{Z})$ corresponding to $j_\Gamma$, the numbers $e_2, e_3$ count the non-ramified pre-image of 1 and 0 under $j_\Gamma$, which we call 2-torsion points and 3-torsion points respectively.

For the next lemma, we only require a small part of the branching datum of $j_\mathcal{E}$, namely the partitions corresponding to the local monodromy of $j_\mathcal{E}$ around the 2-torsion and 3-torsion points. In particular, if $f^{-1}(p), p \in \mathbb{P}^1$ is a fibre of type $IV^*$, then the map $j_\mathcal{E}$ is locally unbranched and maps the point $p$ to a 3-torsion point.

**Lemma 6.4.** The following properties hold for the invariants associated to the factors $j_\Gamma$ and $j_\mathcal{E}$ of a generic element $\mathcal{E}$ in an ambi-typical monodromy stratum:

1. partitions associated to $j_\mathcal{E}$ at 2-torsion points have only even size parts,
2. the parts of the partitions associated to $j_\mathcal{E}$ at 3-torsion points all have size divisible by 3 except for a total of $\# IV^*$ parts of size equal to $1 \pmod{3}$,
3. $e_2 < 2$,
4. $e_3 < 2$, if $\# IV^* = 0$,
5. $e_3 < 3$, if $\# IV^* \leq 2$,
6. $\deg j_\mathcal{E} \equiv_3 1$ and $\# IV^* = 2$, if $e_3 = 2$ and $\# IV^* < 3$.

**Proof of 1.** Let $s$ be a local coordinate at a point corresponding to an odd part. Then the $j$-invariant takes value 1 and has odd multiplicity at $s = 0$. Hence the local expansion is $j = 1 + s^{2k+1}$. But then the corresponding fibre of $\mathcal{E}$ would be, according to Table 1, of type $III$, or $III^*$. This we have already excluded in Proposition 5.7.

**Proof of 2)** If $s$ is a local coordinate at a point corresponding to a part of size $\ell$, then the $j$-invariant has value 0 and multiplicity equal to $\ell$. From the local expansion $j = s^\ell$ we can determine the corresponding fibre type again from Table 1. It is $IV, II^*$ in case $\ell \equiv_3 2$, but these are excluded by Proposition 5.7. It is $II$ or $IV^*$ in case $\ell \equiv_3 1$, but the former is excluded again by the same reason and hence the number of $IV^*$ fibres is equal to the number of parts of size $\ell \equiv_3 1$.

**3)** Suppose $e_2 \geq 2$, then there are at least two partitions with only even size parts. By Proposition 6.6 below, $j_\mathcal{E}$ then has a proper factor $j'$ which violates condition (6.6) on the factorisation of the $j$-invariant.

**4)** Suppose $e_3 \geq 2$ and $\# IV^* = 0$ Then there are at least two partitions with parts of size divisible by 3. Again by Proposition 6.6 below, $j_\mathcal{E}$ then has a proper factor $j'$ violating condition (6.6).

**5)** The claim follows since the number of parts of size $\ell \equiv_3 1$ is bounded below by $e_3$ if $\deg j_\mathcal{E}$ is not divisible by 3, otherwise the number of parts of size $\ell \equiv_3 1$ is at least 3 or 4) applies.

**6)** By 4) it is not possible to have $\# IV^* = 0$. In case $\deg j_\mathcal{E} \equiv_3 0$ or 2, the number of parts of size $\ell \equiv_3 1$ is therefore at least 3, respectively 4. So $\deg j_\mathcal{E} \equiv_3 1$ and the number of parts of size $\ell \equiv_3 1$ is 2 and hence $\# IV^* = 2$. 

□
In order to prove the factorisation result used above we now study branched coverings of the Riemann sphere more systematically from a topological point of view. Consider a finite branched covering $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $d$ and let $r$ denote the number of branch points. The fundamental group of the complement with respect to a base point is generated by elements associated to a geometric sequence of $r$ simple loops around these points, i.e. chosen carefully such that they meet only at the base point and their product loop is homotopic to the boundary of a disc containing all base points. Of course the product loop represents the trivial element and is known to suffice as the only relation.

These elements act by permutations on the set $I = \{1, \ldots, d\}$ in bijection to the elements of the fibre over the base point, giving rise to the monodromy elements $\sigma_1, \ldots, \sigma_r \in S(I)$ where $\sigma_i$ is given by monodromy along the simple loop around the $i$-th branch point. Of course the choices of paths form an orbit under the action of the Hurwitz braid group [Hur]. The induced Hurwitz action on $r$-tuples of monodromy elements is generated by transformations on adjacent pairs

$$\left(\sigma_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_r\right) \mapsto \left(\sigma_1, \ldots, \sigma_{i+1}, \sigma_i^{-1} \sigma_{i+1} \sigma_i, \sigma_{i+1}, \ldots, \sigma_r\right).$$

**Lemma 6.5.** Suppose that the covering $g : \mathbb{P}^1 \to \mathbb{P}^1$ has degree $kh$ with $k > 1$, and $I$ has a partition into parts $I_1, \ldots, I_k$ each of cardinality $h$, such that

1. all $\sigma_i, i > 2$ preserve all parts and
2. $\sigma_1, \sigma_2$ permute the parts.

Then

1. the covering map $g$ is the composition of two factors $g = g_2 \circ g_1$ where
2. the second factor $g_2$ is a cyclic branched cover of degree $k$ branched at the two branch points corresponding to $\sigma_1, \sigma_2$.

Note that the claim and conclusion are insensitive to Hurwitz transformations, except that the indices may be relabelled.

**Proof.** Let us consider the topological cover over the complement of the branch points and an element in $I_1$. Its stabiliser in the fundamental group determines the covering. The setwise stabiliser of $I_1$ determines an intermediate cover $g_2$ which is of degree $k > 1$ over the base. By assumption $\sigma_i, i > 2$ stabilises $I_1$, so $g_2$ is cyclically branched of degree $k$ over the two points corresponding to $\sigma_1, \sigma_2$. \qed

We now prove the factorisation of $j_\mathcal{E}$ into two factors, which we used in the proof of Lemma 6.4 in a more abstract setting.

**Proposition 6.6.** Suppose $P_1, \ldots, P_r$ are the partitions associated to the branch points of a branched covering $g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $hk$ with $k > 1$. If all parts of $P_1$ and $P_2$ have length divisible by $k$ then

1. the covering map $g$ is the composition of two factors $g = g_2 \circ g_1$ and
2. the second factor $g_2$ is a cyclic branched cover of degree $k$ branched at the two branch points corresponding to $P_1, P_2$.

**Proof.** We choose a base point and a geometric sequence of $r$ simple loops around the $r$ points making up the branch locus. Let $\sigma_1, \ldots, \sigma_r$ be the permutations associated to these paths. We recall that the conjugacy class of $\sigma_i$ is determined by $P_i$ and does not depend on the chosen paths. The product of the $\sigma_i$ is the identity. It suffices to show that there is a
decomposition of the fibre $I$ over the base point into $k$ subsets of cardinality $h$, the blocks, such that the hypothesis of Lemma 6.5 is met.

We will rely on the following elementary observation. If $\sigma$ is a permutation in $S_n$ and $\tau$ the transposition of two elements $a, b$, then

either (1): $a, b$ belong to the orbit of the same cycle $c$ of $\sigma$ of length $\ell$, and

(a) there exists a minimal $\ell_1 > 0$ such that $\sigma^{\ell_1}(a) = b$,
(b) $\sigma$ and $\sigma \tau$ have all cycles of $\sigma$ except $c$ in common,
(c) the orbit of $c$ is the union of orbits of two cycles of $\sigma \tau$ which are of lengths $\ell_1$ and $\ell_2 = \ell - \ell_1$.

or (2): $a, b$ belong to orbits of distinct cycles $c_1, c_2$ of $\sigma$ of lengths $\ell_1, \ell_2$, and

(a) $\sigma$ and $\sigma \tau$ have all cycles of $\sigma$ except $c_1, c_2$ in common,
(b) the union of the orbits of $c_1, c_2$ is the orbit of one cycle of $\sigma \tau$ which has length $\ell = \ell_1 + \ell_2$.

Without loss of generality we may assume $\sigma_1, \sigma_2$ to have $h$ cycles of length $k$ each, and all other $\sigma_i, i > 2$ to be transpositions. This case we call the generic case.

In fact, in any other case there are fewer monodromy elements. We can factor each of the first two elements into a permutation with cycles of length $k$ only and a minimal number of transpositions. Any other element can be factored into a minimal number of transpositions. Then the sequence of factors – after a suitable reordering using Hurwitz transformations – is just a sequence of permutations as in the generic case. Thus it suffices to prove the claim in the generic case, since the monodromy elements meet the hypotheses of Lemma 6.5 if the factors do.

Recall that the cycles of $\sigma_i$ correspond bijectively to the points in the $i$-th fibre. The Riemann Hurwitz formula, which relates the Euler number of the domain to the Euler number of the target, the degree and the fibre defects, reads

\[ 2 = 2hk - 2(hk - h) - (r - 2) \quad \Rightarrow \quad r = 2h. \]

Let us write our monodromy elements $\sigma_1, \sigma_2, \tau_3, \ldots, \tau_{2h}$ where the $\tau_i$ are transpositions.

Next we exploit transitivity of the group generated, which needs only the elements $\sigma_2, \tau_i, i > 2$, since the product equals identity. The element $\sigma_2$ has $h$ orbits, so it generates a transitive group only in case $h = 1$. For $h > 1$ there must be a sequence $\tau_{i_1}, \ldots, \tau_{i_{h-1}}$ with $i_1 < \cdots < i_{h-1}$ such that $\rho := \sigma_2 \tau_{i_1} \cdots \tau_{i_{h-1}}$ is an $hk$-cycle. Using Hurwitz transformations we may assume without loss of generality that $i_1 = 3, \ldots, i_{h-1} = h + 1$.

The element $\rho^h$ has order $h$, hence $I$ decomposes into $k$ orbits of length $h$. It remains to show, that this is the decomposition into blocks we need for Lemma 6.5.

Assume to the contrary that $\tau_i$ for some $2 < i \leq h + 1$ does not preserve the blocks. Using Hurwitz transformations we may write

\[ (6.8) \quad \sigma_2 = \rho \tau_{h+1} \cdots \tau_3 = \rho \tau_{i} \tau'_{h} \cdots \tau'_3. \]

The difference in the number of cycles for $\sigma_2$ and $\rho$ is $h-1$, so in each of the $h-1$ compositions with a transposition on the right we are in case (1) where the number of cycles goes up.

Then $\tau_i$ transposes two elements of the cycle of $\rho$. By observation (1.a) the permutation $\rho \tau_i$ has a cycle of length $\ell_i$ which $k$ does not divide, since $\tau_i$ is assumed not to preserve the blocks. We remain in the case (1) for all the following $\tau'$, so also $\sigma_2$ has a cycle of length not divisible by $k$ contrary to the hypothesis of the proposition.
Since blocks are permuted by the element $\rho$ and preserved by the $\tau_i$, $2 < i \leq h + 1$ the element $\sigma_2$ permutes the blocks.

Repeating the discussion for the remaining elements, we assume to the contrary that some $\tau_i$, with $i > h + 1$ does not preserve the blocks. We may then write

$$
\sigma_1^{-1} = \sigma_2 \tau_3 \cdots \tau_{2h} = \rho \tau_{h+2} \cdots \tau_{2h} = \rho \tau_i \tau_{h+3} \cdots \tau_{2h}.
$$

The argument applied above to (6.8) can now be applied here to conclude, that $\sigma_1^{-1}$ and thus $\sigma_1$ has a cycle of length not divisible by $k$ contrary to the hypothesis of the proposition. We conclude, too, that $\sigma_1$ permutes the blocks.

Therefore we are in a position to apply Lemma 6.5 and this concludes the proof. \(\square\)

### 7. Restrictions on possible modular monodromy groups

The results collected so far are sufficient to derive a first characterization of the modular monodromy groups $\bar{\Gamma}$ which can occur for the elliptic surfaces $\mathcal{E}$ we consider.

Since the singular fibres of a generic element are all of the form $I_k$, $I_k^*$ and $IV^*$ we can, using Table 1, write the Euler number formula in the following form:

$$
(7.1) \quad 24 = \sum_{I_k} k + \sum_{I_k^*} 6 + 8\# IV^* = \deg j_\mathcal{F} \cdot \deg j_\mathcal{E} + 6\# I^* + 8\# IV^*.
$$

We will consider all numerically possible combinations of the four integers $\deg j_\mathcal{E} > 0$, $\# I^* \geq 0$, $\# IV^* \geq 0$ and $\deg j_\mathcal{F} > 0$ and we discard the trivial case $\deg j_\mathcal{F} = 1$ corresponding to $\bar{\Gamma} = PSL(2, \mathbb{Z})$ and hence to $\bar{\Gamma} = \text{SL}(2, \mathbb{Z})$. We set up the corresponding table of combinations in two parts, namely low ($\leq 6$) and high ($> 6$) index $[\text{PSL}(2, \mathbb{Z}) : \bar{\Gamma}]$, and we add two rows giving $e_2$ and $e_3$. Since $\# IV^* \leq 2$ in every column, they are determined by

1. $e_2 < 2$ and $e_2 \equiv_2 \deg j_\mathcal{F}$, according to (6.7) and Lemma 6.4(3).
2. $e_3 < 3$ and $e_3 \equiv_3 \deg j_\mathcal{F}$, according to (6.7) and Lemma 6.4(5).

In a last row we mark columns, which we discard from further consideration according to one of the following arguments:

3. $e_3 = 2$ implies $\# IV^* = 2$, according to Lemma 6.4(6).
4. If $\deg j_\mathcal{F} = 1$ then the $j$-invariant of $\mathcal{E}$ is rigid. Thus $\mathcal{E}$ is rigid, except when there are $I_0^*$ fibres, which is obviously excluded for $\# I^* = 0$. We recall here that we are only concerned with positive dimensional strata. But this is also excluded for $\# IV^* > 0$ since the presence of an $I_0^*$-fibre implies that $-id$ is in the monodromy group, which in turn forbids the existence of a $IV^*$ fibre in $\mathcal{E}$ by Proposition 5.7.

| $\deg j_\mathcal{F}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 5 | 5 | 6 | 6 | 6 | 6 |
|---------------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\deg j_\mathcal{E}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 12 | 12 | 24 | 6 | 8 | 1 | 2 | 3 | 4 | 6 | 8 | 1 | 2 | 3 | 4 | 6 |
| $\# I^*$           | 1 | 2 | 3 | 0 | 1 | 2 | 0 | 1 | 0 | 3 | 2 | 1 | 0 | 2 | 0 | 2 | 0 | 0 | 1 | 3 | 2 | 1 | 0 |
| $\# IV^*$          | 2 | 1 | 0 | 2 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| $e_2$              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| $e_3$              | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 0 | 0 | 0 |
| **discard**        | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 3 |

**Table 3.** Combinations of numerical invariants (low index)
Table 4. Combinations of numerical invariants (high index)

|       | deg $j_{\bar{\Gamma}}$ | deg $j_{\bar{\Gamma}}$ | $\#I^*$ | $\#IV^*$ | $e_2$ | $e_3$ | discard |
|-------|-------------------------|-------------------------|---------|----------|-------|-------|---------|
|       | 8                       | 8                       | 8       | 1        | 2     | 0     | 4       |
|       | 8                       | 8                       | 9       | 1        | 2     | 2     | 3       |
|       | 9                       | 10                      | 10      | 2        | 2     | 1     | 3       |
|       | 12                      | 12                      | 12      | 0        | 1     | 2     | 4       |
|       | 16                      | 16                      | 16      | 1        | 0     | 0     | 4       |
|       | 18                      | 18                      | 18      | 0        | 1     | 0     | 4       |
|       | 24                      | 24                      | 24      | 0        | 0     | 0     | 4       |

Note that $e_2 = e_3 = 0$ is equivalent to $\bar{\Gamma}$ being torsion free, and that deg $j_{\bar{\Gamma}}$ is equal to the index of $\bar{\Gamma}$ in $\text{PSL}(2, \mathbb{Z})$. Furthermore, subgroups of $\text{PSL}(2, \mathbb{Z})$ of index at most 6 are congruence subgroups [Wo, Thm.5]. Accordingly, our groups of low index occur in [CuPa, Table 2] with genus 0, the genus of the domain of $j_{\bar{\Gamma}}$, and with $(I = \text{deg } j_{\bar{\Gamma}}, e_2, e_3)$ as in one of the columns in our Table 3. We find the entries $2A^0, 2B^0, 3B^0, 2C^0$ and $4B^0$. By [CuPa, Table 4] they usually go by the standard names $\bar{\Gamma}^{(1)^2}, \bar{\Gamma}^{1(2)}, \bar{\Gamma}^{1(3)}, \bar{\Gamma}^{2(2)}$ and $\bar{\Gamma}^{1(4)}$ – which we recall after the proposition – and we get:

**Proposition 7.1.** Let $\bar{\Gamma}$ be a subgroup of $\text{PSL}(2, \mathbb{Z})$ which appears in one of the columns of Table 3 or Table 4 which have not been discarded. Then it belongs to one of the following, mutually exclusive cases:

1. $\bar{\Gamma}$ has index 18 in $\text{PSL}(2, \mathbb{Z})$ and is torsion free,
2. $\bar{\Gamma}$ has index 12 in $\text{PSL}(2, \mathbb{Z})$ and is torsion free,
3. $\bar{\Gamma}$ has index 9 in $\text{PSL}(2, \mathbb{Z})$ with $e_2 = 1, e_3 = 0$,
4. $\bar{\Gamma}$ is either $\bar{\Gamma}^{(2)}$ or $\bar{\Gamma}^{1(4)}$, has index 6 in $\text{PSL}(2, \mathbb{Z})$ and is torsion free,
5. $\bar{\Gamma}^{1(3)}$ which has index 4 with $e_2 = 0, e_3 = 1$,
6. $\bar{\Gamma}^{1(2)}$ which has index 3 with $e_2 = 1, e_3 = 0$,
7. $\bar{\Gamma}^{(1)^2}$ which has index 2 with $e_2 = 0, e_3 = 2$.

Here we recall the standard notation for certain congruence subgroups of $\text{SL}(2, \mathbb{Z})$ which are of importance for us. The principal congruence subgroup of level $n$ is denoted by $\Gamma(n)$ and defined as

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \mod n \right\}.$$  

Its geometric relevance is the fact that the modular curve $X^0(n) = \mathbb{H}_1/\Gamma(n)$ parameterises elliptic curves with a level $n$ structure, i.e. a symplectic basis with respect to the Weil form, of the group $E[n]$ of $n$-torsion points. Next we recall the group

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1, c \equiv 0 \mod n \right\}$$

whose meaning is that the modular curve $X^1_1(n) = \mathbb{H}_1/\Gamma_1(n)$ parameterises elliptic curves with a fixed point of order $n$. We will also use the group

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \mod n \right\}.$$
Its significance is that the modular curve \( X^0_0(n) = \mathbb{H}_1/\Gamma_0(n) \) is the moduli space of elliptic curves with a distinguished subgroup \( \mathbb{Z}/n\mathbb{Z} \subset E[n] \) of \( n \)-torsion points. By \( \Gamma(n) \), \( \Gamma_1(n) \) and \( \Gamma_0(n) \) we denote the images of these groups in PSL(2, \( \mathbb{Z} \)). Finally we recall that \( \Gamma(1)^2 \) is the subgroup of PSL(2, \( \mathbb{Z} \)) which is generated by all squares.

Remark 7.2. We note here that \( \bar{\Gamma}_0(n) = \bar{\Gamma}_1(n) \) for \( n \leq 4 \). This will be relevant for Proposition 9.3.

The data for the groups \( \bar{\Gamma} \) in Proposition 7.1 can be further exploited to obtain upper bounds for the dimension of any corresponding monodromy stratum. For this purpose we make the following

Definition 7.3. Let \( \Gamma \) be a subgroup of finite index of SL(2, \( \mathbb{Z} \)). We define the maximal dimension of a monodromy stratum associated to \( \Gamma \) by:

\[
m(\Gamma) = \begin{cases} 
\max\{\dim F_{\Gamma,i} | F_{\Gamma,i} \text{ is a monodromy stratum associated to } \Gamma\} \\
-\infty \text{ if there exists no monodromy stratum with monodromy group } \Gamma.
\end{cases}
\]

Together with the explicit bounds in the upcoming Lemma 7.4 this will be used later in the following way:

The closure of a Shimada stratum of dimension \( d \) is ambi-typical
(7.2) if \( d \geq m(\Gamma) \) for its generic monodromy \( \Gamma \).

Lemma 7.4. Let \( \bar{\Gamma} \) be a subgroup of PSL(2, \( \mathbb{Z} \)) which appears in one of the columns of Table 3 or Table 4 which have not been discarded and let \( \Gamma \) be any lift of \( \bar{\Gamma} \) in SL(2, \( \mathbb{Z} \)). The invariants are listed in Table 5.

| \( \deg j_{\bar{\Gamma}} \) | 2 | 3 | 4 | 9 | 6 | 12 | 18 |
| \( e_2 \) | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| \( e_3 \) | 2 | 0 | 1 | 0 | 0 | 0 | 0 |
| \# poles of \( j_{\bar{\Gamma}} \) | 1 | 2 | 2 | 3 | 3 | 4 | 5 |
| \( m(\Gamma) \) | \( \leq 6 \) | \( \leq 10 \) | \( \leq 6 \) | \( \leq 2 \) | \( \leq 6 \) | \( \leq 2 \) | \( \leq 1 \) |

Table 5. Maximal dimension of \( \Gamma \)-strata

Proof. The first row in the table simply lists the possible degrees for \( j_{\bar{\Gamma}} \) which occur in Table 3 or Table 4. The second and third row are also copied from these tables. The number of poles of \( j_{\bar{\Gamma}} \) can be computed via Riemann-Hurwitz and equals \( 2 + \deg j_{\bar{\Gamma}}/6 - 2e_3/3 - e_2/2 \). Thus it remains to prove the upper bound for the dimension of the monodromy strata.

The dimension of a stratum is equal to the open dense irreducible subset of elliptic surfaces sharing the generic configuration of singular fibres. In Lemma 5.3 the dimension augmented by 6 is given as the sum of cardinality of singular fibres and the cardinality of \( * \)-fibres. Thus

(7.3) \( 6 + \dim \leq s_\infty + s_0 + s_1 + s^* + s^* \)

where \( s_\infty, s_0, s_1 \) are the number of singular fibres with \( j = \infty, 0, 1 \) resp. and \( s^* \) is the number of \( * \)-fibres which is also an upper bound for the number of singular fibres with \( j \notin \{0, 1, \infty\} \).
On the other hand the Euler sum gives the bound

\[(7.4) \quad 24 \geq \deg j_{E} \deg j_{F} + 4s_{0} + 3s_{1} + 6s^{*}\]

as follows immediately from Table 1.

Indeed, there are further restrictions on \(s_{\infty}, s_{0}, s_{1}\) due to the map \(j_{F}^{-1} \circ j_{E} \circ j_{F}^{-1}\):

1. \(s_{\infty}\) is the number of poles of the \(j\)-invariant and thus bounded above by \(\deg j_{E} \times \deg j_{F} \times \deg j_{F}^{-1}\).
2. \(s_{0}\) is the number of points with local \(j\)-expansion \(s_{k}, k\) not a multiple of 3, so \(s_{0} = 0\) if \(e_{3} = 0\).
3. \(s_{1}\) is the number of points with local \(j\)-expansion \(1 + s_{k}, k\) odd, so \(s_{1} = 0\) if \(e_{2} = 0\).

We shall now give the proof exemplary for the case of index 9. The other cases can be argued similarly. From (7.4) and using \(s_{0} \geq 0\) we find

\[24 \geq \deg j_{F} \deg j_{E} \deg j_{F}^{-1} + 3s_{1} + 6s^{*}\]

or equivalently

\[8 \geq s_{\infty} + s_{1} + 2s^{*}\]

Since \(e_{3} = 0\) in this case, and hence \(s_{0} = 0\), it then follows immediately from (7.3) that \(m(\Gamma) \leq 2\).

In the case of low index we have ample information on the two factors in the factorisation \(j(\mathcal{E}) = j_{F} \circ j_{E}\). Firstly, since we know the groups \(\overline{\Gamma}\) explicitly we can in fact also write down the maps \(j_{F}^{-1} \circ j_{E} \circ j_{F}^{-1}\) explicitly, as listed in Table 6.

| \(\overline{\Gamma}(2)\) | \(j_{F} = \frac{(z^{2}+3)^{3}}{z^{2}(z-9)^{2}} = 1 + \frac{27(z^{2}-1)^{2}}{z^{2}(z-9)^{2}} \) |
| --- | --- |
| \(\overline{\Gamma}(4)\) | \(j_{F} = \frac{4(z^{2}-4z+1)^{3}}{27(z-4)^{3}} = 1 + \frac{(z-2)^{2}(z^{2}-8z-1)^{2}}{27(z-4)^{3}} \) |
| \(\overline{\Gamma}(1)\) | \(j_{F} = \frac{64(z+1)^{3}}{64z(z-9)^{2}} = 1 + \frac{z(z-9)^{2}}{64(z+1)^{3}} \) |
| \(\overline{\Gamma}(2)\) | \(j_{F} = \frac{4z(z+1)^{2}}{(z+1)^{2}} = 1 - \frac{z^{2}}{(z+1)^{2}} \) |

Table 6. \(j\)-functions

There are various different explicit formulae in the literature, e.g. [FK, MS] since coordinates on the domain can be chosen arbitrarily. Indeed, to check our formulae it suffices to check their degrees and their branching over 0, 1 and \(\infty\) which are determined by the multiplicity sequences of the two numerators and the denominator respectively.

With our choice of \(z\)-coordinate we note that

\[(7.5) \quad \text{row 1 : } z = 0, \pm 3 \text{ are the poles of } j_{F}, \text{ of pole order 2 each}\]
\[(7.6) \quad \text{row 2 : } z = 0, 4, \infty \text{ are the poles of } j_{F}, \text{ of pole order 1, 1, 4 respectively}\]
\[(7.7) \quad \text{row 3 : } e_{2} = 0 \text{ and } z = 0 \text{ is the only 3-torsion point (non-critical point of value 0)}\]
\[(7.8) \quad \text{row 4 : } e_{3} = 0 \text{ and } z = 0 \text{ is the only 2-torsion point (non-critical point of value 1)}\]
\[(7.9) \quad \text{row 5 : } e_{2} = 0 \text{ and } z = 0, \infty \text{ are the 3-torsion points mapping to 0.}\]
Secondly, information from Table 3 also allows us to obtain information about the map \( j_E \). In particular, we can list the possible degrees of this map as well as the branching behaviour over special points, namely the 3-torsion and 2-torsion points, see Lemma 6.4. Using the notation of [BPT, Section 1] we will denote these by \( A \) and \( B \) respectively. So we can derive the number of pre-images of special points and their multiplicities. They are included into Table 7 as the tuple of multiplicities with index the point they map to. The last column of this table gives the most general polynomial expression for \( j_E \) fitting the branching data. Here \( \alpha_i, \beta_i, \gamma_i, \delta_i \) denote coprime homogeneous bivariate polynomials of degree \( i \).

| \( \bar{\Gamma} \) | deg \( j_E \) | special point(s) | branch data | \( j_E \) |
|------------------|-------------|-----------------|-------------|------------------|
| \( \bar{\Gamma}(2), \bar{\Gamma}_1(4) \) | 4 | – | – | \( \alpha_4 : \beta_4 \) |
| | 3 | – | – | \( \alpha_3 : \beta_3 \) |
| | 2 | – | – | \( \alpha_2 : \beta_2 \) |
| | 1 | – | – | \( \alpha_1 : \beta_1 \) |
| \( \bar{\Gamma}_1(3) \) | 6 | \( A = 0 \) | (3, 3)\(_A\) | \( \alpha_3^2 : \beta_6 \) |
| | 4 | \( A = 0 \) | (3, 1)\(_A\) | \( \alpha_1^2 \gamma_1 : \beta_4 \) |
| | 3 | \( A = 0 \) | (3)\(_A\) | \( \alpha_1^3 : \beta_3 \) |
| | 2 | \( A = 0 \) | (1, 1)\(_A\) | \( \gamma_2 : \beta_2 \) |
| \( \bar{\Gamma}_1(2) \) | 8 | \( B = 0 \) | (2, 2, 2)\(_B\) | \( \alpha_2^4 : \beta_8 \) |
| | 6 | \( B = 0 \) | (2, 2)\(_B\) | \( \alpha_2^3 : \beta_6 \) |
| | 4 | \( B = 0 \) | (2, 2)\(_B\) | \( \alpha_2^2 : \beta_4 \) |
| | 2 | \( B = 0 \) | (2)\(_B\) | \( \alpha_2 : \beta_2 \) |
| \( \bar{\Gamma}(1)^2 \) | 4 | \( A_1 = 0, A_2 = \infty \) | (3, 1)\(_A_1\), (3, 1)\(_A_2\) | \( \alpha_1^3 \gamma_1 : \beta_3 \delta_1 \) |

Table 7. \( j_E \)-functions for low index

8. Families of Weierstrass data for low index

In this section we will analyse the Weierstrass data of the Jacobian fibrations with modular monodromy of low index. We will also discuss the Mordell-Weil group in these cases.

We start with the following table of Weierstrass data, whose relevance is that this includes all the Weierstrass data needed to describe the families from Section 7 of low monodromy index.

Note that in this table we still assume the polynomials to have the degree given by their index, but no assumption of coprimality is imposed.

**Proposition 8.1.** Every elliptic surface with data as in Table 7 has Weierstrass datum in one of the families in Table 8. The correspondence between the monodromy group and the Weierstrass data is given by:

\[
\begin{align*}
&i) \quad : \quad \bar{\Gamma}(2) \quad iii) \quad : \quad \bar{\Gamma}_1(3), \deg j_E = 0 \quad v) \quad : \quad \bar{\Gamma}_1(2) \\
&ii) \quad : \quad \bar{\Gamma}_1(4) \quad iv) \quad : \quad \bar{\Gamma}_1(3), \deg j_E = 3 \quad vi) \quad : \quad \bar{\Gamma}(1)^2.
\end{align*}
\]

**Proof.** We have to show that for each row of Table 7 any elliptic surface \( E \) with the data provided by the row can be obtained by some suitable choice of Weierstrass datum from Table 8. Here we shall give the proof for \( E \) with modular monodromy \( \bar{\Gamma}_1(3) \) which is the
This expression can not be obtained as easily. Indeed, we have to re call that we are allowed
only singular fibres of type $I_v$ which is exactly the expression for
$j(E) = j_I \circ j_{E_1} = \frac{\alpha_3^2 \gamma_1 (\alpha_3^2 \gamma_1 + 8 \beta_4)}{64 (\alpha_3^2 \gamma_1 - \beta_4)^3 \beta_4} = 1 + \frac{(\alpha_3^2 \gamma_1 - \beta_4)^2 (\alpha_3^2 \gamma_1 - 8 \beta_4)}{64 (\alpha_3^2 \gamma_1 - \beta_4)^3 \beta_4}.

We look at the general expression of the $j$-function in terms of Weierstraß data
\[ j = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{(g_2/3)^3}{(g_2/3)^3 - g_3^2} = 1 + \frac{g_2^3}{(g_2/3)^3 - g_3^2}. \]

If we plug in the same $\alpha_2$ and $\beta_6$ we get the identical expression for the $j$-invariant as above. Moreover the analysis of common factors of $g_2, g_3$ and their multiplicities according to the Tate algorithm, see Table 2, gives no singular fibres except of type $I_v$ since $\alpha_2$ and $\beta_6$ are coprime by assumption $\deg j_E = 6$. Therefore $E$ and the elliptic surface given by the Weierstraß datum share the functional and the homological invariant and hence are isomorphic.

Still in case $\Gamma(1)(3)$ but with $\deg j_E = 4$ we obtain from Table 6 and Table 7 that
\[ j(E) = j_I \circ j_{E_1} = \frac{\alpha_3^2 \gamma_1 (\alpha_3^2 \gamma_1 + 8 \beta_4)}{64 (\alpha_3^2 \gamma_1 - \beta_4)^3 \beta_4} = 1 + \frac{(\alpha_3^2 \gamma_1 - \beta_4)^2 (\alpha_3^2 \gamma_1 - 8 \beta_4)}{64 (\alpha_3^2 \gamma_1 - \beta_4)^3 \beta_4}.

This expression can not be obtained as easily. Indeed, we have to recall that we are allowed to plug in polynomials into the families which are not necessarily coprime. Doing this with $\beta_6 = \gamma_1^2 \beta_4$ and $\alpha_2 = \alpha_1 \gamma_1$ in family $i i i$ we get
\[ j = \frac{\alpha_3^2 \gamma_1 (\alpha_3^2 \gamma_1 + 8 \beta_4 \gamma_1^2)}{64 (\alpha_3^2 \gamma_1 - \beta_4 \gamma_1^2)^3 \beta_4 \gamma_1} = 1 + \frac{(\alpha_3^2 \gamma_1 - \beta_4 \gamma_1^2)^2 (\alpha_3^2 \gamma_1 - 8 \gamma_1 \beta_4^2)}{64 (\alpha_3^2 \gamma_1 - \beta_4 \gamma_1^2)^3 \beta_4 \gamma_1^2},\]
which is exactly the expression for $j(E)$ expanded by $\gamma_1^8$. The Weierstraß datum is thus
\[ g_2 = 3 \alpha_1 \gamma_1 (\alpha_1^2 \gamma_1 + 8 \beta_4), \quad g_3 = \gamma_1^4 (\alpha_5^6 \gamma_1^2 - 20 \alpha_5^3 \gamma_1 \beta_4 - 8 \beta_4^2). \]

The analysis with the Tate algorithm shows the existence of one $IV^*$ fibre and otherwise only singular fibres of type $I_v$ since $\alpha_1 \gamma_1$ and $\beta_4$ are coprime and we may conclude again that $E$ is isomorphic to a surface given by Weierstraß datum from family $i i i$).

| $j$ | $\Gamma(2)$ | $g_2$ | $g_3$ | $\Delta = g_2^2 - 27g_3^2$ |
|-----|--------------|--------|--------|-------------------------|
| i)  | $\Gamma(2)$ | $\alpha_1^2 + 3 \beta_4^2$ | $\beta_4 (\alpha_1^2 - \beta_4^2)$ | $\alpha_4^2 (\alpha_1^2 - 9 \beta_4^2)^2$ |
| ii) | $\Gamma(1)(4)$ | $12 \alpha_1^2 - 4 \alpha_4 \beta_4 + \beta_4^2$ | $4 (\alpha_4 - 2 \beta_4)(2 \alpha_4^2 - 8 \alpha_4 \beta_4 - \beta_4^2)$ | $2^4 \cdot 3^6 \alpha_4 (\alpha_4 - 4 \beta_4) \beta_4^4$ |
| iii) | $\Gamma(1)(3)$ | $3 \alpha_2 (\alpha_1^2 + 8 \beta_4)$ | $\alpha_6^2 - 20 \alpha_2^3 \beta_6 - 8 \beta_6^2$ | $2^6 \cdot 3^3 (\alpha_2^2 - \beta_6)^3 \beta_6$ |
| iv) | $\Gamma(1)(3)$ | $3 \alpha_1 \gamma_1^2 (\alpha_1^2 + 8 \beta_3)$ | $\gamma_2^2 (\alpha_1^2 - 20 \alpha_2 \beta_3 - \beta_3^2)$ | $2^6 \cdot 3^3 (\alpha_1^2 - \beta_3)^3 \beta_6 \gamma_6^2$ |
| v)  | $\Gamma(1)(2)$ | $3 \alpha_1^2 + 9 \beta_8$ | $\alpha_4 (\alpha_1^2 - 9 \beta_8)$ | $3^6 (\alpha_1^2 - \beta_8)^2 \beta_8$ |
| vi) | $\Gamma(1)(1)^2$ | $-12 \alpha_1 \beta_1 \gamma_1^2 \delta_1^2$ | $4 \gamma_1^3 \delta_1^3 (\alpha_1^2 \gamma_1 - \beta_1^2 \delta_1)$ | $-2^3 \cdot 3^3 (\alpha_1^2 \gamma_1 + \beta_1^2 \delta_1)^2 \gamma_1^2 \delta_1^2$ |

Table 8. Weierstraß families
The case with \( \Gamma_1(3) \) and \( \text{deg} \ j_E = 2 \) is very similar but with \( \beta_6 = \gamma_2^2 \beta_2 \) and \( \alpha_2 = \gamma_2 \) sharing even a factor of degree two. The Weierstraß datum
\[
g_2 = 3\gamma_2^3(\gamma_2 + 8\beta_2), \quad g_3 = \gamma_2^6(\gamma_2^2 - 20\gamma_2\beta_2 - 8\beta_2^2)
\]
defines then an elliptic surface sharing the functional and homological invariant with \( E \) again.

This leaves us with \( \Gamma_1(3) \) and \( \text{deg} \ j_E = 3 \). From Table 6 and Table 7 we get
\[
j(E) = j \circ j_E = \frac{\alpha_1^3(\alpha_1^3 + 8\beta_3)^3}{64(\alpha_1^3 - \beta_3)^3\beta_3} = 1 + \frac{(\alpha_1^6 - 20\alpha_1^3\beta_3 - 8\beta_2^2)^2}{64(\alpha_1^3 - \beta_3)^3\beta_3}.
\]
This time we choose to plug the coprime \( \alpha_1 \) and \( \beta_3 \) from an expression for \( j(E) \) together with a \( \gamma_2 \) still to be determined into the family \( iv \) and obtain the \( j \)-function of this Weierstraß datum to be
\[
j = \frac{\alpha_1^3\gamma_2^6(\alpha_1^3 + 8\beta_3)^3}{64(\alpha_1^3 - \beta_3)^3\gamma_2^6\beta_3} = 1 + \frac{\gamma_2^6(\alpha_1^6 - 20\alpha_1^3\beta_3 - 8\beta_2^2)^2}{64(\alpha_1^3 - \beta_3)^3\gamma_2^6\beta_3}
\]
which is exactly the expression for \( j(E) \) expanded by \( \gamma_2^6 \). Thus we get the \( j \)-invariant \( j(E) \) with the Weierstraß datum
\[
g_2 = 3\alpha_1\gamma_2^2(\alpha_1^3 + 8\beta_3), \quad g_3 = \gamma_2^3(\alpha_1^6 - 20\alpha_1^3\beta_3 - 8\beta_2^2).
\]
Finally we choose \( \gamma_2 \) to vanish at the two \( I_0^* \) fibres of \( E \). Then \( \gamma_2 \) is coprime to \( \beta_3(\alpha_1^3 - \beta_3) \) since the \( j \)-invariant of an \( I_0^* \) fibre is finite.

The analysis of this datum with the Tate algorithm shows the existence of \( I_0^* \) fibres precisely at the zeros of \( \gamma_2 \). Again we may conclude, since functional and homological invariant are shown to coincide.

We next determine for the generic members of each family whether \(-\text{id} \in \Gamma \) or not. The following lemma will be used as a tool to determine the Mordell-Weil torsion from the monodromy group.

In Section 7 we already introduced the principal congruence subgroup \( \Gamma(n) \). As we said there, the modular curve \( X^0(n) = \mathbb{H}_1/\Gamma(n) \) is the classifying space of elliptic curves with a level-\( n \) structure. This carries a universal family if \( n \geq 3 \). If \( n = 2 \), we no longer have a universal family of elliptic curves, but a universal Kummer family still exists. We denote by \( X(n) \) the compactification of \( X^0(n) \) which is obtained by adding the cusps, i.e.
\[
X(n) = \overline{\mathbb{H}_1}/\Gamma(n).
\]
The universal family over \( X^0(n) \) can be extended to \( X(n) \), the extension is known as Shioda modular surface. This has \( n^2 \) sections which restrict to the \( n \)-torsion points on the smooth fibres of the universal family.

We had also introduced the group \( \Gamma_1(n) \) and the curve \( X^0_1(n) = \mathbb{H}_1/\Gamma_1(n) \). This is the classifying space of elliptic curves with a fixed \( n \)-torsion point. As above we can compactify the curve \( X^0_1(n) \) by adding the cusps and obtain a curve \( X_1(n) \).

If \( m \) divides \( n \) we consider the group
\[
\Gamma_m(n) := \Gamma(m) \cap \Gamma_1(n).
\]
Obviously, \( X^0_m(n) := \mathbb{H}_1/\Gamma_m(n) \) parameterises elliptic curves with a level-\( m \) structure and additionally an \( n \)-torsion point. Again by adding the cusps we obtain the compactification \( X_m(n) \). The universal family over \( X^0_m(n) \) extends to \( X_m(n) \) and in addition to the sections

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Lemma 8.2. Let \( \mathcal{E} \) be a Jacobian fibration with monodromy group \( \Gamma \) and assume that \( m \) and \( n \) are positive integers with \( m \mid n \). Then the following are equivalent:

1. \( \Gamma \) is, up to conjugation, contained in \( \Gamma_m(n) \).
2. The Mordell-Weil group \( \text{MW}(\mathcal{E}) \) contains a subgroup \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \).

Proof. We shall first prove the implication form (1) to (2). For this let \( U \) be the subset of the base \( \mathbb{P}^1 \) of \( \mathcal{E} \) which is given by removing the points \( j(\mathcal{E})^{-1}\{0, 1, \infty\} \) and the base points of the singular fibres. We want to construct sections which form a subgroup \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \) of \( \text{MW}(\mathcal{E}) \). It is enough to do this over \( U \) as the sections can then be extended to \( \mathbb{P}^1 \) (since the base has dimension 1). We choose a base point \( x_0 \in U \). For each point \( x \in U \) we can choose a small disc \( U(x) \) such that \( \mathcal{E}|_{U(x)} \cong \mathbb{C} \times U(x)/(\mathbb{Z} + \mathbb{Z}\tau_x) \) where \( \tau_x : U(x) \to \mathbb{H}_1 \) is a local lift of the \( j \)-function \( j(\mathcal{E}) \) and \( \mathbb{Z} + \mathbb{Z}\tau_x \) acts fibrewise on \( \mathbb{C} \times U(x) \) by translation on \( \mathbb{C} \times \{t\} \) with the lattice \( \mathbb{Z} + \mathbb{Z}\tau_x(t) \).

In particular, the fibre \( \mathcal{E}_{x_0} \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_{x_0}(x_0)) \) where we have chosen a fixed lift \( \tau_{x_0} \). Let \( a, b \in \mathbb{Z} \) and \( z_0 = (a + b\tau_{x_0}(x_0))/\ell \) be an \( \ell \)-torsion point in \( \mathcal{E}_{x_0} \) (where \( \ell \) will become either \( m \) or \( n \). Using the above local uniformisation of \( \mathcal{E} \) we can extend this to a local \( \ell \)-torsion section of \( \mathcal{E}|_{U(x_0)} \). We want to extend such a section to \( U \). Given any point \( y \in U \) we can choose a path \( s : [0, 1] \to U \) connecting \( x_0 \) with \( y \). We can cover this path with finitely many open sets \( U(x_i), i = 0, \ldots, N \) with \( x_N = y \) and choose local lifts \( \tau_{x_i} \) with \( \tau_{x_i}|_{U_i \cap U_{i+1}} = \tau_{x_{i+1}}|_{U_i \cap U_{i+1}} \). Then we can move the point \( z_0 \) along the path \( s \) to an \( \ell \)-torsion point \( z_0(y) \in \mathcal{E}_y \). Clearly, this will a priori depend on the chosen path \( s \). The point \( z_0(y) \) will be independent of this choice if all elements in the monodromy group \( \Gamma(\mathcal{E}) \) fix the point \( (a + b)/\ell \in \mathbb{Z}/\ell \) (where we have chose a fixed representation \( \Gamma(\mathcal{E}) \to \text{SL}(2, \mathbb{Z}) \) and hence consider \( \Gamma(\mathcal{E}) \) as a subgroup of \( \text{SL}(2, \mathbb{Z}) \)). Given this observation, the claim now follows immediately from the definition of the group \( \Gamma_m(n) \). The converse implication (2) to (1) follows by the same argument.

As a consequence of the above discussion we can now obtain some first results on the torsion Mordell-Weil group.

Lemma 8.3. Let \( \mathcal{E} \) be a member of the family iii). Then the following holds:

1. \( -\text{id} \notin \Gamma(\mathcal{E}) \),
2. \( \mathbb{Z}/3\mathbb{Z} \subset \text{MW}(\mathcal{E}) \).

Moreover, for a generic element \( \mathcal{E} \) the equality \( \text{MW}(\mathcal{E}) = \mathbb{Z}/3\mathbb{Z} \) holds.

Proof. Consider the following Weierstraß datum of a rational elliptic surface \( \mathcal{E} \) in homogeneous coordinates \( z_1, z_0 \)

\[
\begin{align*}
g_2 &= 3z_1^2(z_1 + 8z_0), \\
g_3 &= z_1^4(z_1^2 - 20z_1z_0 - 8z_0^2).
\end{align*}
\]

A generic member \( \mathcal{E} \) of family iii) is given by coprime \( \beta_0 \) and \( \alpha_2 \). To compute the datum of the pull-back of \( \mathcal{E} \) along \( j \) we plug in \( \alpha_2^3 \) and \( \beta_0 \) into \( z_1 \) and \( z_0 \) resp. resulting in

\[
\begin{align*}
g'_2 &= 3\alpha_2^9(\alpha_2^3 + 8\beta_0), \\
g'_3 &= \alpha_2^{12}(\alpha_2^6 - 20\alpha_2^3\beta_0 - 8\beta_0^2).
\end{align*}
\]

The proper Weierstraß datum of the normalised and possibly blown down pull-back is then

\[
\begin{align*}
g_2 &= 3\alpha_2(\alpha_2^3 + 8\beta_0), \\
g_3 &= \alpha_2^{6} - 20\alpha_2^3\beta_0 - 8\beta_0^2.
\end{align*}
\]
which is exactly that of $E$. By the Tate algorithm we find that the fibre configuration on the rational elliptic surface $\bar{E}$ is $IV^*, I_1, I_3$.

Due to the singular fibres the trivial lattice is $E_6 + A_2$, which implies that the surface is no. 69 in the list of Oguiso-Shioda [OS] and thus has Mordell-Weil torsion $\mathbb{Z}/3$. In particular $-id$ does not belong to the monodromy.

Since torsion sections are pulled back to torsion sections, the torsion Mordell-Weil group of $E$ also contains $\mathbb{Z}/3$. Moreover, the monodromy group of $E$ is generated by the monodromy elements of the rational elliptic surface $\bar{E}$ associated to loops liftable along $j_E$. So the monodromy of $E$ is contained in that of the rational elliptic surface $\bar{E}$ and hence does not contain $-id$.

Conversely, the generic $E$ does not have torsion Mordell-Weil properly containing $\mathbb{Z}/3$. Otherwise $j(E)$ factors through $j_F$ for some $\Gamma$ properly contained in $\Gamma_1(3)$. This contradicts $j_F \circ j_E$ being the canonical factorisation.

In the above arguments we have used that $E$ is a generic element of the family. It thus remains to prove that $-id$ is never contained in the monodromy group and $\mathbb{Z}/3$ is always contained in the torsion Mordell-Weil. But this both follows by semicontinuity, since the monodromy group can only get smaller under specialization and the torsion Mordell-Weil can only get bigger.

\begin{lemma}
Let $E$ be a member of family ii). Then the following holds:

1. If $\alpha_4$ or $\beta_4$ is a perfect square, then $-id \notin \Gamma(E)$.
2. In the two cases the generic element has $\text{MW}(E) = \mathbb{Z}/2\mathbb{Z}$ and $\text{MW}(E) = \mathbb{Z}/4\mathbb{Z}$ respectively.
\end{lemma}

\begin{proof}
In both cases the elliptic fibration is a pull-back along $j_E$ of degree four of a rational elliptic fibration whose fibre configuration is one of the following

$I^*_a + 2 I_1, \quad I^*_1 + I_4 + I_1$

and with two simple ramification points over the $*$-fibre. Again, it suffices to investigate the rational elliptic fibrations. The monodromy is freely generated by two elements, so $-id$ does not belong to the monodromy, since it is torsion.

The surfaces occur as nos. 64, 72 in the list of Oguiso-Shioda [OS] and have Mordell-Weil torsion $\mathbb{Z}/2$ and $\mathbb{Z}/4$ respectively. So via the pull-back these groups are contained in the Mordell-Weil torsion. Equality holds for all $j_E$ such that $j_F \circ j_E$ is the canonical factorisation, hence generically.

\end{proof}

\begin{lemma}
If $E$ is a generic member of either family i) or ii), then $-id \in \Gamma(E)$.
\end{lemma}

\begin{proof}
Here we use the semicontinuity argument for the monodromy in the other direction. Since the group can only get smaller under specialisation, the generic monodromy for the family contains $-id$, if it does so for some member.

If $\alpha_4 = \alpha_3 \gamma_1$ and $\beta_4 = \beta_3 \gamma_1$ with $\alpha_3, \beta_3, \gamma_1$ pairwise coprime, then the Tate algorithm shows that for either family we get an $I^*_0$ fibre corresponding to $\gamma_1$. Hence $-id$ is in the monodromy of such a special member.

\end{proof}

9. **Classification for $j_F$ of low degree**

We are now ready to classify the ambi-typical strata with $\bar{\Gamma}$ of index at most 6. At the same time we determine the corresponding root lattices and Mordell-Weil torsion.
Proposition 9.1. There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}_1(2)$. Its monodromy group is $\Gamma_1(2)$. A generic element of this stratum has the following root lattice and Mordell-Weil torsion:

$$8A_1, \mathbb{Z}/2\mathbb{Z}.$$  

Proof. Since $-\text{id}$ is the only 2-torsion element in $\text{SL}(2,\mathbb{Z})$, there is no splitting map to $\text{SL}(2,\mathbb{Z})$ from any subgroup of $\text{PSL}(2,\mathbb{Z})$ containing 2-torsion such as $\Gamma_1(2)$. In particular, every $\bar{\Gamma}_1(2)$ monodromy stratum is actually a $\Gamma_1(2)$ monodromy stratum.

With our explicit knowledge of $j_{\bar{\Gamma}}$ and $j_E$ for the family $v$) we can use the Tate algorithm and find that the generic fibre configuration is $8I_1 + 8I_2$. By Lemma 5.3 we find that the dimension of the configuration locus is $\text{dim } L(8I_1 + 8I_2) = 10$. The corresponding generic root lattice is $8A_1$. The Shimada stratum with this root lattice and saturation given by the generic element of family $v$) also has dimension 10. This stratum and family $v$) therefore determine the same irreducible closed subset of $\mathcal{F}'$. It is ambi-typical by the argument of (7.2), since $\Gamma_1(2)$ belongs to the second column in Table 5 and thus $m(\Gamma_1(2)) \leq 10$ by Lemma 7.4.

Any other Shimada stratum with $\bar{\Gamma}_1(2)$ monodromy corresponds by Proposition 8.1 to a stratum in family $v$) and must be of dimension less than 10. But its monodromy group must still be $\Gamma_1(2)$, so it belongs to the monodromy stratum above, see (3.2), and cannot be a monodromy stratum of its own.

By Lemma 8.2 the Mordell-Weil torsion always contains $\mathbb{Z}/2\mathbb{Z}$ and for a generic element this is equal to $\mathbb{Z}/2\mathbb{Z}$. The proof is analogous to that in the proof of Lemma 8.3. □

Proposition 9.2. There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}(2)$ and monodromy group $\Gamma(2)$. A generic element of this stratum has the following root lattice and Mordell-Weil torsion:

$$12A_1, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$  

Proof. $\Gamma(2)$ is the pre-image of $\bar{\Gamma}(2)$ in $\text{SL}(2,\mathbb{Z})$ and contains $-\text{id}$, thus by Lemma 8.3 it is the monodromy group of the generic surface in family $i$). We can now argue very much as in the proof of the previous proposition. The generic fibre configuration is $12I_2$ which gives a generic root lattice $12A_1$, so both the corresponding strata have dimension 6.

On the other hand $\Gamma(2)$ belongs to Column 5 in Table 5. Hence $m(\Gamma(2)) \leq 6$ by Lemma 7.4 and the irreducible closed subset of $\mathcal{F}'$ corresponding to family $i$ is ambi-typical by (7.2).

Any other Shimada stratum with $\Gamma(2)$ monodromy corresponds to a stratum in family $i$) by Proposition 8.1 and must be of dimension less than 6. Due to (3.2) it belongs to the monodromy stratum above, and cannot be a monodromy stratum of its own.

The claim about the Mordell-Weil torsion again follows as in the previous proof. □

Proposition 9.3. There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}_1(4)$ and monodromy group $\Gamma_0(4)$. A generic element of this stratum has the following root lattice and Mordell-Weil torsion:

$$4A_3, \mathbb{Z}/2\mathbb{Z}.$$  

Proof. $\Gamma_0(4)$ is the pre-image in $\text{SL}(2,\mathbb{Z})$ of $\bar{\Gamma}_1(4) = \bar{\Gamma}_0(4)$ (see Remark 7.2, thus contains $-\text{id}$ and by Lemma 8.3 is the monodromy group of the generic surface in family $ii$). The argument then is as above, only applied to the family $ii$). □
To complete the classification of monodromy strata for the two torsion free subgroups of PSL$(2, \mathbb{Z})$, we have to exploit the fact, that for fibrations with monodromy $\Gamma$ not containing $-\text{id}$ the quotient map $\text{SL}(2, \mathbb{Z}) \rightarrow \text{PSL}(2, \mathbb{Z})$ defines an isomorphism $\Gamma \cong \bar{\Gamma}$.

**Proposition 9.4.** There is a unique ambi-typical stratum with modular monodromy $\bar{\Gamma}(2)$ and monodromy group $\Gamma$ such that $\Gamma \rightarrow \bar{\Gamma}(2)$ is an isomorphism. The generic root lattice and Mordell-Weil torsion are

$$2A_3 + 8A_1, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$ 

**Remark 9.5.** The group $\Gamma$ in the proposition can be described as

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| \begin{array}{c} b, c \equiv 0 \mod 2, \quad a, d \equiv 1 \mod 4 \end{array} \right\}. $$

**Proof.** By Proposition 8.1 the surfaces of any $\Gamma$ monodromy stratum are contained in family $i$) where $j_\Gamma$ by (7.5) of Table 6 has three poles at 0 and $\pm 3$. The splitting isomorphism gives a map from $\pi_1^{orb}(\mathbb{H}_1/\bar{\Gamma}) \cong \bar{\Gamma}$ to $\Gamma \subset \text{SL}(2, \mathbb{Z})$ and $\mathbb{H}_1/\bar{\Gamma}$ is identified with the complement in $\mathbb{P}^1$ of these poles. The positive loop at each pole is mapped by ($^1_0 -^2_0$) or ($^1_0 -^2_0$) corresponding to 2 and 8 in $\mathbb{Z}/12\mathbb{Z} = \text{SL}(2, \mathbb{Z})_{ab}$ respectively. Now the sum of the three elements in $\text{SL}(2, \mathbb{Z})_{ab}$ must be zero, since the sum in homology of the three loops is zero. Under the assumption of a splitting isomorphism we therefore can assume that the conjugacy class associated to the loop around 0 contains $(^1_0 -^2_0)$ possibly after applying a deck transformation in $\bar{\Gamma}(1)/\bar{\Gamma}(2)$ of $j_\bar{\Gamma}(2)$ acting by the full permutation group on $\{0, \pm 3\}$.

We now consider the map $j_\mathcal{E} = (\alpha_4 : \beta_4)$ and try to understand the possible fibres over pre-images of 0. Such a pre-image corresponds to a linear factor $\gamma_1$ of $\alpha_4$ which may occur with multiplicity $1 \leq k \leq 4$. It may occur in $\beta_4$ with multiplicity $l = 0$ or $l = 1$, as long as $k > l$. Indeed, in case $k \leq l$ the image would not be 0, in case $k > l \geq 1$ the Weierstraß coefficients $g_2, g_3$ would share a common factor of $\gamma_1^2$ which is not allowed.

In all possible combinations we can compute the local fibre type at the zero of $\gamma_1$ from the local monodromy. Recall that we made an assumption on the matrix associated to the loop around 0 in the codomain of $j_\mathcal{E}$. Since the multiplicity of $j_\mathcal{E}$ is $k - l$ at the zero of $\gamma_1$ in the domain of $j_\mathcal{E}$, the matrix associated to the loop around this zero must be conjugate to $(^1_0 -^2_0)^{k-l}$. These monodromy matrices determine the local fibre type by Table 11 in the way recorded in the last row of Table 9.

The other row gives the local fibre types determined by the Tate algorithm: The pole order of $j(\mathcal{E})$ is the product of the multiplicities of $j_\mathcal{E}$ at the zero of $\gamma_1$ and of $j_\bar{\Gamma}$ at 0, hence $2(k-l)$. The vanishing orders of $g_2, g_3$ at the zero of $\gamma_1$ are determined by row $i)$ in Table 8 so for $l = 0$ they do not vanish simultaneously, and $\nu_2 = 2, \nu_3 = 3$ for $l = 1$.

| $(k, l)$ | 1, 0 | 2, 0 | 3, 0 | 4, 0 | 2, 1 | 3, 1 | 4, 1 |
|---------|-----|-----|-----|-----|-----|-----|-----|
| Tate    | $I_2$ | $I_4$ | $I_6$ | $I_8$ | $I_2^*$ | $I_4^*$ | $I_6^*$ |
| cover   | $I_2^*$ | $I_4^*$ | $I_6^*$ | $I_8^*$ | $I_2^*$ | $I_4^*$ | $I_6^*$ |

**Table 9.** Comparisons of fibre type calculations
Since the fibre types in both rows of Table 9 must coincide, our assumption on the splitting implies that \( k \) is even. Hence we may conclude, that \( \alpha_4 = \gamma_2^2 \). This condition generically corresponds to a map \( j_\mathcal{E} \) branched outside the other poles of \( j_{\mathcal{F}} \). Therefore \( j_\mathcal{E} \) induces a surjection on orbifold fundamental groups and thus the monodromy group is \( \Gamma \).

Generically there are \( I_4 \) fibres at the zeroes of \( j_\mathcal{E} \) and an \( I_2 \) fibre at each of the four unramified pre-images of each of the other two poles \( \pm 3 \) of \( j_{\mathcal{F}} \). This yields generic fibre type \( 2I_4 + 8I_2 \) and generic root lattice \( 2A_3 + 8A_1 \) with corresponding strata of dimension 4. So we found the unique ambi-typical \( \Gamma \) stratum of the claim, since every stratum of root lattice type with constant \( \Gamma \) monodromy was shown to belong to the closed set in family \( i \) defined by \( \alpha_4 = \gamma_2^2 \) and we can use conclusion (3.2) again.

The assertion about the Mordell-Weil torsion follows from Lemma 8.2 since the modular monodromy is \( \bar{\Gamma}(2) \) and only contained in \( \bar{\Gamma}_n^m(\mathbb{Z}/2\mathbb{Z}) \) if \( m, n \) divide 2.

**Remark 9.6.** Note that the generic root lattice and the Mordell-Weil torsion do not necessarily determine a unique Shimada stratum. In fact, Entry 95 in Table 3 of Shimada [Shi2] shows that there are two families with root lattice \( 2A_3 + 8A_1 \) and Mordell-Weil torsion \( \mathbb{Z}/4\mathbb{Z} \), which can be distinguished algebraically. That – in the terminology of Shimada – means, that the saturations of the root lattices are non-isomorphic for the two families.

Shimada also gives a recipe how to analyse such a situation in Section 6.1 and his Example 6.4 corresponding to number 91 in his table is similar to our case in many aspects. By lattice theory the two possible saturations in case 95 correspond to two isotropy subgroups of the discriminant group of \( 2A_3 + 8A_1 \) up to isometry. Shimada computes these and provides his result on his home page [Shi3]. The subgroup belonging to our family consists of the trivial element and the isotropic elements in \( \text{discr}(2A_3 + 8A_1) = (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^8 \) given by

\[
(0, 0, 1, 1, 1, 1, 1, 1, 1, 1), \quad (2, 2, 0, 0, 0, 0, 1, 1, 1, 1), \quad (2, 2, 1, 1, 1, 1, 0, 0, 0, 0),
\]

where a basis for the discriminant group is chosen in the obvious way. To prove that we are in that case requires a detailed analysis of the torsion sections and their intersections with fibre components. We shall not give the details here.

**Proposition 9.7.** There are two ambi-typical strata with modular monodromy \( \bar{\Gamma}_1(4) \) and monodromy group \( \Gamma \) such that \( \Gamma \to \bar{\Gamma}_1(4) \) is an isomorphism. Their generic root lattice and Mordell-Weil torsion are respectively

1. \( 4A_3 + 2A_1, \mathbb{Z}/4\mathbb{Z} \),
2. \( 2A_7, \mathbb{Z}/2\mathbb{Z} \).

The two corresponding subgroups in \( \text{SL}(2, \mathbb{Z}) \) are not conjugate.

**Remark 9.8.** The group is \( \Gamma_1(4) \) in case (1) and in case (2) can be given by

\[
\Gamma = \left\{ M \mid M \equiv \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}^k \mod 4, \quad k = 0, 1, 2, 3 \right\}.
\]

**Proof.** We argue as in the last proof, but with family \( \text{ii}) \) and the associated map \( j_{\mathcal{F}} \) which by (7.6) has poles at 0, 4 and \( \infty \). The analysis of the splitting isomorphism again provides information about the monodromy at these poles. However, this time only the poles at 0 and 4 are equivalent under deck transformation, so we have to consider two cases instead of one.
(1) the conjugacy class associated to the pole at 0 contains \((-1, -1)\),
(2) the conjugacy class associated to the pole at \(\infty\) contains \((-1, -4)\).

In the first case, we apply the Tate algorithm to \(j_\infty = (\alpha_4 : \beta_4)\) and compare local monodromies at its zeroes. We infer \(\alpha_j\) which is the composition of a proper closed subset determined by the specialization to \(\alpha_2^j\), then compute the generic fibre configuration \(2I_2 + 4I_1 + 4I_4\), the generic root lattice \(4A_3 + 2A_1\) and the dimension 4 of the corresponding strata. In the second case we look at the poles instead and get \(2I_8 + 8I_1, 2A_7\) and again dimension 4.

We can argue as before that these yield the ambi-typical strata of the claim and that there are no more.

Lemma 8.4 provides the Mordell-Weil torsion groups, showing in particular that the two subgroups are not conjugate.

**Proposition 9.9.** There is a unique ambi-typical stratum with modular monodromy \(\bar{\Gamma}_1(3)\). Its monodromy group is \(\Gamma_1(3)\). Its generic root lattice and Mordell-Weil torsion are \(6A_2, \mathbb{Z}/3\).

**Proof.** Any ambi-typical stratum with \(\bar{\Gamma}_1(3)\) modular monodromy belongs to family iii) or iv).

If it belongs to family iii) then by Lemma 8.3 the Mordell-Weil torsion contains \(\mathbb{Z}/3\) and \(-\text{id} \not\in \Gamma\). By the former property and Lemma 8.2 family iii) has monodromy contained in \(\Gamma_1(3)\). Since \(-\text{id} \not\in \Gamma_1(3)\), no proper subgroup surjects onto \(\bar{\Gamma}_1(3)\), thus \(\Gamma_1(3)\) is the monodromy group for the generic surface in family iii).

We follow the proofs of Propositions 9.1 and 9.2 and get generic fibre configuration \(6I_3 + 6I_1\), generic root lattice \(6A_2\), corresponding strata dimension 6 and \(m(\Gamma_1(3)) \leq 6\) by Lemma 7.4. Using (7.2) and (3.2) we conclude that family iii) constitutes a unique ambi-typical stratum, with the invariants given in the claim.

To address possible strata in family iv) we look at the additional family with Weierstraß datum
\[ g_2 = 3\alpha_1\alpha_2(\alpha_1^3\alpha_2 + 8\beta_5), \quad g_3 = \alpha_2(\alpha_1^6\alpha_2^2 - 20\alpha_1^3\alpha_2\beta_5 - 8\beta_5^2), \]
that has associated \(j\)-invariant
\[ j = \frac{(g_2/3)^3}{(g_2/3)^3 - g_3^2} = \frac{\alpha_1^3\alpha_2(\alpha_1^3\alpha_2 + 8\beta_5)^3}{64(\alpha_1^4\alpha_2 - \beta_5)^3\beta_5^2}, \]
which is the composition of \(j_\Gamma\) for \(\bar{\Gamma}_1(3)\) with \(j_\infty = \alpha_3^3\alpha_2/\beta_5\). It contains the family iv) as a proper closed subset determined by the specialization to \(\alpha_2 = \gamma_2, \beta_5 = \gamma_2\beta_3\). A zero of \(\gamma_2\) generically corresponds to a fibre of type \(I_0^6\) so both have generic monodromy \(\Gamma_0(3)\), the pre-image in \(\text{SL}(2, \mathbb{Z})\) of \(\bar{\Gamma}_1(3) = \bar{\Gamma}_0(3)\).

Repeating the argument above using (3.2) in particular, we deduce that no stratum of monodromy \(\Gamma_0(3)\) is ambi-typical, except possibly that corresponding to the generic fibre configuration \(2II + 5I_3 + 5I_1\) in the additional family. However, by Proposition 5.3 the fibre \(II\) may not occur in generic surfaces of an ambi-typical stratum.

We are left to look for an ambi-typical stratum in family iv) with \(\Gamma \to \bar{\Gamma}_1(3)\) an isomorphism. The splitting isomorphism gives a map from \(\pi_1^{arb}(\mathbb{H}/\Gamma) \cong \bar{\Gamma}\) to \(\Gamma \subset \text{SL}(2, \mathbb{Z})\). The positive loops around the 3-torsion point and the poles of \(j_\Gamma\) of order 1 and 3 are mapped to 2, 1 and 3 in \(\mathbb{Z}/6 = \text{PSL}(2, \mathbb{Z})_{ab}\), so the splitting isomorphism associates to these loops the elements 2 or 8, 1 or 7, and 3 or 9 respectively in \(\mathbb{Z}/12 = \text{SL}(2, \mathbb{Z})_{ab}\).
In fact, we can be more precise. On the one hand 2 can not be associated to a 3-torsion point, since then the corresponding monodromy is conjugate to \((-\frac{1}{0} \frac{1}{1})\) contradicting \(-\text{id} \not\in \Gamma\). On the other hand 1 and 3 can not be associated to the two poles. Otherwise the corresponding elements of \(\Gamma\) are conjugate to powers of \((-\frac{1}{1} \frac{1}{1})\) and so are all monodromies at poles of \(j(\mathcal{E})\). Then the \(I^*\) fibres – present according to Table 3 – are \(I_0^*\) fibres contradicting again \(-\text{id} \not\in \Gamma\).

We use again the fact that the sum of the three elements in \(\text{SL}(2, \mathbb{Z})_{ab}\) must be zero, since the sum in homology of the three loops is zero. Thus the elements must be 8, 7 and 9, and \(\mathcal{E}\) is a pullback (up to normalization and blow down) of an elliptic surface with singular fibre configuration \(I_1^* + I_3^* + IV^*\) along \(j_\mathcal{E}\).

By Table 7 the map \(j_\mathcal{E}\) has degree 3 and ramification datum (3) at the 3-torsion point. By Table 3 the surface \(\mathcal{E}\) has two \(I^*\) fibres, so the ramification datum of \(j_\mathcal{E}\) at the two poles must be \((2, 1)\). But then \(j_\mathcal{E}\) is unramified outside these points and therefore violates the maximality condition \((6, 6)\).

Proposition 9.10. There is no ambi-typical stratum with \(\bar{\Gamma} = \bar{\Gamma}(1)^2\).

Proof. We first assume that there is such a stratum with \(\Gamma \rightarrow \bar{\Gamma}(1)^2\) an isomorphism. By Lemma 8.1 we know that there is a corresponding generic surface \(\mathcal{E}\) in the family \(vi)\).

The splitting isomorphism gives a map from \(\pi_1^{orb}(\mathbb{H}/\Gamma) \cong \bar{\Gamma}\) to \(\Gamma \subset \text{SL}(2, \mathbb{Z})\). The positive loops around both 3-torsion points and the pole of \(j_\Gamma\) are mapped to \(2 \in \mathbb{Z}/6 = \text{PSL}(2, \mathbb{Z})_{ab}\), so the splitting isomorphism associates to each loop either 2 or 8 in \(\mathbb{Z}/12 = \text{SL}(2, \mathbb{Z})_{ab}\). However, 2 can not be associated to a 3-torsion point, since then the corresponding monodromy is conjugate to \((-\frac{1}{1} \frac{1}{1})\) contradicting \(-\text{id} \not\in \Gamma\).

Again we use the fact that the sum of the three elements in \(\text{SL}(2, \mathbb{Z})_{ab}\) must be zero. Thus all elements must be 8 and \(\mathcal{E}\) is a pullback (up to normalization and blow down) of an elliptic surface with singular fibre configuration \(I_2^* + 2IV^*\) along \(j_\mathcal{E}\).

By Table 7 the map \(j_\mathcal{E}\) has degree 4 and ramification datum (3, 1) at the two 3-torsion points. By Table 3 the surface \(\mathcal{E}\) has no \(I^*\) fibre, so the ramification datum of \(j_\mathcal{E}\) at the pole must be \((2, 2)\). But then \(j_\mathcal{E}\) is unramified outside these points and therefore violates the maximality condition \((6, 6)\).

To show that there is neither a stratum with \(-\text{id} \in \Gamma\) we look at the new family with Weierstraß datum given by

\[
g_2 = -12\alpha_1\alpha_3\beta_1\beta_3, \quad g_3 = 4\alpha_3\beta_3(\alpha_1^3\alpha_3 - \beta_1^3\beta_3)
\]

that has associated \(j\)-invariant

\[
j = \frac{(g_2/3)^3}{(g_2/3)^3 - g_3} = \frac{4\alpha_3\beta_3(\alpha_1\beta_1)^3}{(\alpha_1^3\alpha_3 + \beta_1^3\beta_3)^2}
\]

which is the composition of \(j_\Gamma\) for \(\bar{\Gamma}(1)^2\) with \(j_\mathcal{E} = (\alpha_1^3\alpha_3 : \beta_1^3\beta_3)\). It contains the family \(vi)\) as a proper closed subset determined by the specialization to \(\alpha_3 = \gamma_1^2\delta_1, \beta_3 = \gamma_1\delta_1^2\), so both have the same generic monodromy \(\Gamma\).

Repeating the argument above, using in particular \((3, 2)\), we deduce that no stratum of that monodromy is ambi-typical, except possibly that corresponding to the generic fibre configuration \(6I + 6I_2\) of the new family. However, again by Proposition 5.5, the fibre \(II\) may not occur in generic surfaces of an ambi-typical stratum. \(\square\)
10. Classification in the high index cases

In this section we will complete the classification of ambi-typical strata of high index. More precisely, we will classify the subgroups $\bar{\Gamma}$ of PSL($2, \mathbb{Z}$) of index at least 9 for which an ambi-typical stratum exists and give a description of the possible monodromy groups $\Gamma$ in SL($2, \mathbb{Z}$) which cover $\bar{\Gamma}$.

10.1. Uniqueness of strata. We shall proceed as follows. First, given a group $\bar{\Gamma}$, we shall determine the possible groups $\Gamma$ and prove the uniqueness of the $\Gamma$ ambi-typical strata. Second, we will determine the root lattices, $j_{\bar{\Gamma}}$ branch data and the number of $*$-fibres of their generic members in terms of the ramification data of $j_{\bar{\Gamma}}$. In fact, we use the notation as in [BPT] for the three cases that $j_{\bar{\Gamma}}$ can belong to:

(2)

B

2 :

$j_{\bar{\Gamma}}$ is of degree 2 with branching in a 2-torsion point and a non-torsion point,

2

2 :

$j_{\bar{\Gamma}}$ is of degree 2 with branching in two non-torsion points,

1 :

$j_{\bar{\Gamma}}$ is of degree 1 with no branching.

The actual list of ramification data of $j_{\bar{\Gamma}}$, and thus the list of possible groups $\bar{\Gamma}$ is postponed to the next subsection, as is the computation of the corresponding Mordell-Weil torsion.

Proposition 10.1. Suppose $\bar{\Gamma}$ has index 9 in PSL($2, \mathbb{Z}$). Any such subgroup defines at most one ambi-typical stratum and in this case the monodromy group $\Gamma$ is the pre-image of $\bar{\Gamma}$ in SL($2, \mathbb{Z}$).

The generic invariants, root lattice, $j_{\bar{\Gamma}}$ ramification and number of $*$-fibres are

$$D_4 + 2A_{i_1} + \cdots + 2A_{i_k}, \quad (2)_B, 2, \quad 1,$$

with $k$ the number of poles of $j_{\bar{\Gamma}}$ of order at least two, $i_1 + 1, \ldots, i_k + 1$ their orders and root lattice rank $4 + 2i_1 + \cdots + 2i_k = 16$.

Proof. By Column 4 of Table 4 the map $j_{\bar{\Gamma}}$ is a double cover and we have $e_2 = 1$ and $e_3 = 0$. So $\bar{\Gamma}$ contains 2-torsion and has no splitting map to SL($2, \mathbb{Z}$), since $-\text{id}$ is the only 2-torsion element in SL($2, \mathbb{Z}$). In particular, every $\Gamma$ monodromy stratum is actually a $\bar{\Gamma}$ monodromy stratum with $\bar{\Gamma}$ the pre-image.

The map $j_{\bar{\Gamma}}$ is branched over the unique 2-torsion point according to Lemma 6.4(1). The family of such coverings is 1-dimensional and irreducible.

Moreover, it follows from Proposition 5.7 that the $I^*$-fibre in the generic member of the stratum must be $I^*_0$. Therefore we get a 2-dimensional irreducible family of elliptic surfaces with monodromy $\Gamma$, where the second parameter is the position of the $I^*_0$-fibre. If we have an ambi-typical stratum the rank of the root lattice must be 16.

The generic fibre type consists of the $I^*_0$-fibre and the $I^*_\nu$-fibres corresponding to pole orders of the generic $j$-map $j_{\bar{\Gamma}} \circ j_{\bar{\Gamma}}$, which coincides with the lengths of the ramification partition at infinity. Since $j_{\bar{\Gamma}} \circ j_{\bar{\Gamma}}$ is the canonical factorisation, the map $j_{\bar{\Gamma}}$ is a double cover not branched over $j_{\bar{\Gamma}}^{-1}(\infty)$, and the ramification partition is twice that of $j_{\bar{\Gamma}}$.

We can now use the argument in (7.2) to conclude that $\Gamma$ defines a monodromy stratum. This is the case since our family has varying moduli and is of maximal dimension by Column 4 of Table 4. □

Proposition 10.2. Suppose $\bar{\Gamma}$ is of index 12 in PSL($2, \mathbb{Z}$). Let $k$ be the number of poles of $j_{\bar{\Gamma}}$ of order at least two and $i_1 + 1, \ldots, i_k + 1$ their orders. If $\bar{\Gamma}$ gives rise to an ambi-typical stratum then it is torsion free and there are two possibilities:
(1) there is a unique ambi-typical \( \Gamma \) stratum with \( \Gamma \to \Gamma \) an isomorphism. The generic invariants, root lattice of rank 16, \( j_\ell \) ramification and number of \( * \)-fibres are

\[
2A_i + \cdots + 2A_k, \quad 2^2, \quad 0,
\]

(2) there is a unique ambi-typical \( \Gamma \) stratum with \( \Gamma \subset \text{SL}(2, \mathbb{Z}) \) the pre-image of \( \bar{\Gamma} \). The generic invariants, root lattice of rank 16, \( j_\ell \) ramification and number of \( * \)-fibres are

\[
2D_4 + A_i + \cdots + A_k, \quad 1, \quad 2.
\]

Proof. By the hypothesis on \( \bar{\Gamma} \) we are either in either the cases of Column 6 or Column 7 of Table 4 and we already know that \( \bar{\Gamma} \) is torsion free by Proposition 7.1(2).

In case of Column 6 of Table 4 the map \( j_\ell \) is an isomorphism. Moreover the two \( I^* \)-fibres in the general member of the stratum must be of type \( I_0^* \) according to Proposition 5.7, since otherwise the corresponding surface were rigid. Therefore we get a 2-dimensional irreducible family of elliptic surfaces with monodromy \( \Gamma \) containing \( -\text{id} \), where the parameters are the positions of the \( I^* \)-fibres, given by a reduced divisor of degree two on \( \mathbb{P}^1 \).

The generic fibre type consists of the \( I_0^* \)-fibres and the \( I_\nu \)-fibres corresponding to the poles of the generic \( j \)-map \( j_\bar{\Gamma} \circ j_\ell \). Since \( j_\ell \) is an isomorphism, the pole orders are those of \( j_\Gamma \).

In case of Column 7 of Table 4 the map \( j_\ell \) is a double cover. It is branched generically at two points, since Lemma 6.4 imposes no additional conditions. The family of such coverings is 2-dimensional and irreducible. The generic fibre type consists of the \( I_\nu \)-fibres corresponding to the pole orders of the generic \( j \)-map \( j_\bar{\Gamma} \circ j_\ell \), which coincide with the lengths of the ramification partition at infinity. Since \( j_\bar{\Gamma} \circ j_\ell \) is the canonical factorisation, the map \( j_\ell \) is a double cover not branched over \( j_\bar{\Gamma}^{-1}(\infty) \), and the ramification partition is twice that of \( j_\Gamma \).

By Column 6 of Table 4 our family is not contained in a monodromy stratum of larger dimension.

Since the dimension of both families is two, the rank of the root lattice must be 16. \( \square \)

**Proposition 10.3.** Suppose \( \bar{\Gamma} \) is of index 18 in \( \text{PSL}(2, \mathbb{Z}) \). If \( \bar{\Gamma} \) is associated to an ambi-typical stratum, then it is torsion free and the monodromy group \( \Gamma \) is the pre-image of \( \bar{\Gamma} \) in \( \text{SL}(2, \mathbb{Z}) \).

The generic invariants, root lattice of rank 17, \( j_\ell \) ramification and number of \( * \)-fibres are

\[
D_4 + A_i + \cdots + A_k, \quad 1, \quad 1,
\]

with \( k \) the number of poles of \( j_\bar{\Gamma} \) of order at least two, and \( i_1 + 1, \ldots, i_k + 1 \) their orders.

Proof. By Proposition 7.1(1) we know that \( \bar{\Gamma} \) is torsion free. We also note that the \( I^* \)-fibre in the general member of the stratum must be of type \( I_0^* \), since \( j_\ell \) is of degree 1 by Column 9 of Table 4 thus positive dimensionality of the stratum must be due to moduli of a movable \( I_0^* \)-fibre. Accordingly \( -\text{id} \in \Gamma \) and \( \Gamma \) is the pre-image in \( \text{SL}(2, \mathbb{Z}) \) of \( \bar{\Gamma} \).

Therefore we get a 1-dimensional irreducible family of elliptic surfaces with monodromy \( \Gamma \), where the parameter is the position of the \( I_0^* \)-fibre, consequently the rank of the root lattice must be 17.

The generic fibre type consists of the \( I_0^* \)-fibre and the \( I_\nu \)-fibres corresponding to the poles of the generic \( j \)-map \( j_\bar{\Gamma} \circ j_\ell \). Since \( j_\ell \) is an isomorphism, the pole orders are those of \( j_\Gamma \).

The property to be a monodromy stratum follows as before by (7.2) and Table 5. \( \square \)
10.2. **Classification of relevant subgroups in the high index cases.** In the previous subsection we have related the monodromy strata uniquely, resp. in a two to one way, with subgroups of $\text{PSL}(2, \mathbb{Z})$ in the high index case. So we have to classify these subgroups next.

To this end we note that subgroups $\bar{\Gamma}$ are in bijective correspondence with maps $j_{\bar{\Gamma}}$ and such maps are in turn determined by triples of permutations $\mu_0, \mu_1, \mu_\infty$ whose product is the identity and which generate a group acting transitively on the set of 9, 12 and 18 elements respectively. Since Condition (6.5) and the value of $e_2, e_3$ impose restrictions on $\mu_0$ and $\mu_1$, we deduce:

1. In the index 9 case, $\mu_0$ has only 3-cycles and $\mu_1$ only 2-cycles except for one fixed point.
2. In the torsion free cases, where the index is 12 or 18, $\mu_0$ has only 3-cycles and $\mu_1$ only 2-cycles.

Moreover, concerning the fibres of type $I_\nu, \nu > 0$:

3. The number of parts of $\mu_\infty$ is the number of poles and the pole orders are the lengths of the parts and determine the corresponding fibre.

**Proposition 10.4.** In the index 9 subcase there are 4 relevant subgroups $\bar{\Gamma}$ of $\text{PSL}(2, \mathbb{Z})$. These are in bijection with the following 4 partitions of $\mu_\infty$ of the corresponding map $j_{\bar{\Gamma}}$:

\[(7, 1, 1), (6, 2, 1), (5, 3, 1), (4, 3, 2).\]

This translates into the following root lattices:

\[D_4 + 2A_6, D_4 + 2A_5 + 2A_1, D_4 + 2A_4 + 2A_2, D_4 + 2A_3 + 2A_2 + 2A_1.\]

with corresponding Mordell-Weil torsion

- trivial, $\mathbb{Z}/2$,
- trivial, $\mathbb{Z}/2$.

**Proof.** Since each relevant subgroup gives an irreducible stratum their number is bounded above by the number of Shimada components which have root lattice as described in Proposition [10.1 above. The list of Shimada [Shi1] published in the arXiv version shows that there are four such root lattices, entries 2171, 2179, 2190 and 2198, which uniquely determine the Mordell-Weil torsion as claimed. They do not appear in the list of root lattices corresponding to multiple components, see Shimada [Shi2, Table II, p.38], therefore each of these contributes one component.

To see that these Shimada strata are indeed ambi-typical it suffices to exhibit the four groups, which we do in terms of four tuples of permutations

\[
\begin{align*}
(10.1) & \quad (123)(456)(789), \quad (14)(27)(56)(89), \quad (1643297)(5)(8) \\
(10.2) & \quad (123)(456)(789), \quad (14)(26)(57)(89), \quad (16)(259743)(8) \\
(10.3) & \quad (123)(456)(789), \quad (14)(25)(67)(89), \quad (16975)(243)(8) \\
(10.4) & \quad (123)(456)(789), \quad (14)(27)(59)(68), \quad (167)(2943)(58)
\end{align*}
\]

where we use the correspondence from (3) to (1) in Remark [6.2].

**Remark 10.5.** One can give an alternative proof which avoids the use of Shimada’s list by performing an exhaustive search for all homomorphisms as in (3) of Remark [6.2]. This is straightforward though cumbersome and we omit the details.
In the index 12 subcase, we get, by Proposition 10.2 two monodromy strata for each relevant subgroup of $\text{PSL}(2, \mathbb{Z})$. As it turns out Beauville [Be] has classified exactly these torsion free subgroups in his study of semi-stable rational elliptic fibrations. Note that Beauville uses the notation $\Gamma_0^0(n)$ for our groups $\Gamma_1(n)$.

**Proposition 10.6.** In the index 12 subcase there are 6 relevant subgroups of $\text{PSL}(2, \mathbb{Z})$ with $j_{\Gamma}$ in bijection with the partition corresponding to $\mu_\infty$ being one of 

$$ (9, 1, 1, 1), (8, 2, 1, 1), (6, 3, 2, 1), (5, 5, 1, 1), (4, 4, 2, 2), (3, 3, 3, 3). $$

This claim is proved in [Be, p.658f], which is used also in the proof of the following proposition addressing case (1) of Proposition 10.2.

**Proposition 10.7.** If $j_E$ is generic of degree 2, so $E$ is the pull-back along $j_E$ of a rational modular elliptic surface without *-fibre, then the monodromy group $\Gamma$ is one of the following

$$ \Gamma_0(9) \cap \Gamma_1(3), \Gamma_0(8) \cap \Gamma_1(4), \Gamma_1(6), \Gamma_1(5), \Gamma_1(4) \cap \Gamma(2), \Gamma(3) $$

with corresponding Mordell-Weil torsion

$$ \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6, \mathbb{Z}/5, \mathbb{Z}/4 \times \mathbb{Z}/2, \mathbb{Z}/3 \times \mathbb{Z}/3. $$

**Proof.** The list for the monodromy groups of the rational modular elliptic surfaces is taken from [Be]. Indeed, they do not change under generic pull-back, since $(j_E)_*$ induces a surjection on fundamental groups of the complements of singular values. The claim of the Mordell-Weil torsion is then obtained using Lemma 8.2. It can also be verified by a check of the corresponding lines 2242, 2262, 2322, 2345, 2368, 2373 in [Shi1]. \hfill $\square$

The other families of surfaces with torsion free modular monodromy of index 12 are obtained from the Beauville elliptic surfaces, which are rational and rigid for deformations preserving the monodromy group, by replacing two smooth fibres by fibres of type $I^*$. This *generic quadratic twisting* corresponds to the transformation of the Weierstraß datum $(g_2', g_3')$ of a rational Beauville elliptic surface into Weierstraß data $(\gamma_2^2 g_2', \gamma_3^3 g_3')$, where the zeroes of $\gamma_2$ avoid the singular fibres. Geometrically this means the following. We first take the double cover branched at the smooth fibres over the zeroes of $\gamma_2$. This double cover is acted on by the deck-transformation and the involution which is $-\text{id}$ on each fibre. The quotient by the product of these two involutions gives the twisted surface after resolution of its singularities. The choice of $\gamma_2$, or equivalently the position of the $I^*$ fibres, gives the two moduli of these families.

**Proposition 10.8.** If $E$ is a rational modular elliptic surface without *-fibre twisted at two smooth fibres, then the monodromy group $\Gamma$ is one of the following

$$ \Gamma_0(9), \Gamma_0(8), \Gamma_0(6), \Gamma_1(5)\{\pm \text{id}\}, \Gamma_0(4) \cap \Gamma(2), \Gamma(3)\{\pm \text{id}\} $$

with corresponding Mordell-Weil torsion

$$ \text{trivial}, \mathbb{Z}/2, \mathbb{Z}/2, \text{trivial}, \mathbb{Z}/2 \times \mathbb{Z}/2, \text{trivial}. $$

**Proof.** The monodromy groups are the groups $\Gamma\{\pm \text{id}\}$ generated by the groups from [Be] and $-\text{id}$ due to the presence of $I^*$ fibres. The claim of the Mordell-Weil torsion is then obtained with Lemma 8.2. It is also immediate by a check of the corresponding lines 2148 – 2153 in [Shi1]. \hfill $\square$
Proposition 10.9. In the index 18 subcase there are 26 relevant subgroups of $\text{PSL}(2, \mathbb{Z})$ corresponding to root lattices and Mordell-Weil torsion in the list of Shimada given in lines 2762 – 2786.

The lattice appearing in line 2776 gives rise to two different ambi-typical strata.

Remark 10.10. We remark without proof that the two components corresponding to line 2776 are complex conjugate, as are the maps $j_\Gamma$ for the corresponding modular monodromy groups. More precisely, this means that the corresponding maps $\mu : \pi_1(\mathbb{C} \setminus \{0, 1\}) \to S_{18}$ modulo conjugation in $S_{18}$ differ only by the automorphism of $\pi_1(\mathbb{C} \setminus \{0, 1\})$ induced by complex conjugation.

Proof. Since each relevant subgroup gives an irreducible stratum, their number is bounded above by the number of Shimada strata which have root lattice of rank 17 of the form $D_4 + A_i + \cdots + A_k$ as in Proposition 10.3 above. The list of Shimada [Shi1] shows that there are 25 such root lattices and the component count of Shimada [Shi2] shows that each of these contributes one component except for that of line 2776 where two components exist.

To see that all these components are related to torsion free subgroups we provide the corresponding list of 26 triples of monodromy permutations together with a proof, that the two triples corresponding to line 2776 are not equal under conjugation. We give representatives for all orbits, with $\mu_1 = (12)(34)(56)(78)(910)(1112)(1314)(1516)(1718)$ and $\mu_0, \mu_\infty$ in the following Table 10.

| Table 10. Monodromy data of $j_\Gamma$ |
|----------------------------------------|
| 1 (123)(567)(91011)(131415)(4817)(121618), (1)(5)(9)(13)(2317161415121011188674) | 114111 |
| 2 (123)(567)(91011)(4817)(121517)(141816), (1)(5)(9)(13)(231716121011171418674) | 132111 |
| 3 (123)(567)(91011)(41318)(81514)(121716), (1)(5)(9)(13)(23171612101118667144) | 123111 |
| 4 (123)(458)(697)(101114)(121513)(161718), (1)(17)(57)(1113)(238914151816121064) | 122211 |
| 5 (123)(567)(489)(101113)(121715)(141618), (1)(5)(1617)(111514)(23913812108674) | 113211 |
| 6 (123)(567)(91011)(4814)(131517)(121618), (1)(5)(9)(1011181512)(231417161318674) | 105111 |
| 7 (123)(567)(4911)(81310)(121517)(141816), (1)(5)(1518)(9131612)(23111714867109) | 104211 |
| 8 (123)(567)(4911)(81315)(101617)(121814), (1)(5)(1718)(131816)(23111814867104) | 103311 |
| 9 (123)(457)(698)(101113)(121715)(141618), (1)(58)(1617)(111514)(23791318121064) | 103221 |
| 10 (123)(567)(4911)(81318)(101517)(161718), (1)(5)(18)(9171412)(23713161064) | 94321 |
| 11 (123)(567)(4911)(81315)(101617)(121814), (1)(5)(18)(9171412)(23713161064) | 94321 |
| 12 (123)(497)(698)(101113)(121715)(141618), (1)(5)(9)(6715171838)(231812101116144) | 96111 |
| 13 (123)(567)(4911)(81318)(101517)(161718), (1)(5)(18)(9171412)(23713161064) | 94321 |
| 14 (135)(274)(6810)(9113)(151417)(121816), (14)(1518)(376)(111614)(251013171298) | 83232 |
| 15 (123)(567)(4911)(81318)(101517)(161718), (1)(5)(18)(9171412)(23713161064) | 77211 |
| 16 (123)(567)(4911)(81315)(101417)(121816), (1)(5)(9)(6715181518)(23111713161410) | 95211 |
| 17 (123)(497)(6911)(81315)(101712)(141816), (1)(912)(1316)(5117148)(23715181064) | 85221 |
| 18 (123)(497)(6911)(81318)(101517)(161718), (1)(5)(18)(9171412)(23713161064) | 84321 |
| 19 (123)(497)(6911)(81315)(101417)(121816), (1)(5)(18)(9171412)(23713161064) | 83232 |
| 20 (123)(567)(4911)(81318)(101517)(161718), (1)(5)(18)(9171412)(23713161064) | 83232 |
| 21 (135)(274)(6911)(81315)(101417)(121814), (1)(5)(9)(6715181518)(23111713161410) | 95211 |
| 22 (123)(497)(6911)(81318)(101517)(161718), (1)(5)(18)(9171412)(23713161064) | 77211 |
| 23 (123)(567)(4911)(81315)(101417)(121816), (1)(5)(9)(6715181518)(23111713161410) | 77211 |
| 24 (123)(497)(6911)(81318)(101517)(161718), (1)(5)(18)(9171412)(23713161064) | 77211 |
| 25 (123)(567)(4911)(81315)(101417)(121816), (1)(5)(9)(6715181518)(23111713161410) | 77211 |

Table 10. Monodromy data of $j_\Gamma$
Supposing the factorizations in line 15 and 16 are conjugate, there is a permutation $\sigma$ which commutes with $\mu_1$ and which conjugates $\mu_0, \mu_\infty$ of line 15 to those of line 16, which for this proof we denote by $\rho_0, \rho_\infty$. From this we deduce the conditions
\[\sigma(\mu^*_n(n)) = \rho^*_n(\sigma(n)) \quad \text{for all } n, k \text{ and for } * = 0, 1, \infty.\]

In particular we get for $1 = \mu_\infty(1)$
\[\sigma(1) = \sigma(\mu_\infty(1)) \implies \sigma(1) = \rho_\infty(\sigma(1)).\]

Thus $\sigma(1)$ is a fixed point of $\rho_\infty$ and thus either 1 or 5. Then for $2 = \mu_1(1)$ we get
\[\sigma(2) = \rho_1(\sigma(1)) \in \{\rho_1(1), \rho_1(5)\} = \{2, 6\}.
\]

On the other hand the 2-cycle $(16 17)$ is unique in both $\mu_\infty, \rho_\infty$ so
\[(\sigma(16), \sigma(17)) = (16, 17) \implies \sigma(17) \in \{16, 17\}.
\]

Consequently we get for $18 = \mu_1(17)$ that
\[\sigma(18) = \rho_1(\sigma(17)) \in \{\rho_1(16), \rho_1(17)\} = \{15, 18\}.
\]

But in this way we arrive at a contradiction since
\[\sigma(\mu^*_\infty(2)) = \sigma(18) \in \{15, 18\}\]

while
\[\rho^*_\infty(\sigma(2)) \in \{\rho^*_\infty(2), \rho^*_\infty(6)\} = \{14, 11\}.
\]

This finally concludes the proof of Theorems 4.2, 4.3 and 4.4.

10.3. Conclusion. We obtain a list of 48 root lattices with uniquely associated torsion Mordell-Weil groups. The latter appear in Table 11 using the notation of Shimada [Shi1] with $[n]$ for $\mathbb{Z}/n$ and $[n, m]$ for $\mathbb{Z}/n \times \mathbb{Z}/m$. In addition the Table 11 gives the dimensions of the 50 ambi-typical strata, the index of $\bar{\Gamma}$ in $\text{PSL}(2, \mathbb{Z})$, the cardinality of the kernel of $\Gamma \to \bar{\Gamma}$ and the corresponding number in the list of Shimada. Although it is not needed in this paper we also state for completeness which of the monodromy groups are congruence subgroups. This turns out to be the case if and only if the index is not divisible by 9. For index $\leq 6$ this follows from [Wo, Theorem 5]. For index 9 this follows from [CuPa, Table 2], alternatively one can give an independent argument using the amplitudes of the cusps. Finally, for indices 12 and 18 the claim can be deduced from Sebbar’s classification [Sé, Table 1].

In Table 12 we list the branch behaviour of the maps $j_\Gamma$ and $j_\mathcal{E}$.  

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| # | root lattice         | MW | dim | ind | |ker| | [Shi] | # | Remarks                           |
|---|---------------------|----|-----|-----|-----|-----|------|------|---|---------------------------------|
| 0 | 8A₁                | [1] | 18  | 1   | 2   | 99  | 1     | Γ = SL(2, ℤ)                  |
| 1 | 6A₂                | [2] | 10  | 3   | 2   | 559 | 1     | Γ = Γ₁(2)                     |
| 2 | 4A₃                | [3] | 6   | 4   | 1   | 547 | 2     | Γ = Γ₃(3)                     |
| 3 | 12A₁               |     | 6   | 6   | 2   | 505 | 2     | Γ = Γ₈(4)                     |
| 4 | 2A₇                | [2] | 4   | 6   | 1   | 1134| 1     | Γ = Γ₆(4), − id / Γ           |
| 5 | 2A₃ + 8A₁          | [2,2] | 4   | 6   | 1   | 1223| 1     | Γ = Γ₂(2), − id / Γ          |
| 6 | 4A₃ + 2A₁          | [4] | 4   | 6   | 1   | 1215| 2     | Γ = Γ₄(4)                     |
| 7 | D₄ + 2A₆            |     | 2   | 9   | 2   | 2171| 1     | non-congruence               |
| 8 | D₄ + 2A₅ + 2A₁     |     | 2   | 9   | 2   | 2179| 1     | "                           |
| 9 | D₄ + 2A₃ + 2A₂     |     | 2   | 9   | 2   | 2190| 1     | "                           |
| 10| D₄ + 2A₅ + 2A₂ + 2A₁|     | 2   | 9   | 2   | 2198| 1     | "                           |
| 11| 2D₄ + A₈            |     | 2   | 12  | 2   | 2148| 1     | twisted Beauville            |
| 12| 2D₄ + A₇ + A₁       |     | 2   | 12  | 2   | 2149| 1     | "                           |
| 13| 2D₄ + A₃ + A₂ + A₄  |     | 2   | 12  | 2   | 2150| 1     | "                           |
| 14| 2D₄ + 2A₃ + 2A₁     |     | 2   | 12  | 2   | 2151| 1     | "                           |
| 15| 2D₄ + 4A₂           |     | 2   | 12  | 2   | 2152| 1     | "                           |
| 16| 2D₄ + 4A₃ + 2A₁     |     | 2   | 12  | 2   | 2153| 1     | "                           |
| 17| 8A₂                | [3,3] | 2   | 12  | 1   | 2242| 1     | "                           |
| 18| 2A₈                |     | 2   | 12  | 1   | 2242| 2:1 Beauville             |
| 19| 2A₇ + 2A₁          | [4] | 2   | 12  | 1   | 2262| 2:1 Beauville             |
| 20| 2A₅ + 2A₂ + 2A₁    | [6] | 2   | 12  | 1   | 2322| 2:1 Beauville             |
| 21| 4A₄                | [5] | 2   | 12  | 1   | 2345| 2:1 Beauville             |
| 22| 4A₃ + 4A₁          | [4,2]| 2   | 12  | 1   | 2368| 2:1 Beauville             |
| 23| 8A₂                |     | 2   | 12  | 1   | 2373| 2:1 Beauville             |
| 24| D₄ + A₁₃           |     | 1   | 18  | 2   | 2762| 1     | non-congruence               |
| 25| D₄ + A₁₂ + A₁      |     | 1   | 18  | 2   | 2763| 1     | "                           |
| 26| D₄ + A₁₁ + A₂      |     | 1   | 18  | 2   | 2764| 1     | "                           |
| 27| D₄ + A₁₁ + 2A₁     |     | 1   | 18  | 2   | 2765| 1     | "                           |
| 28| D₄ + A₁₀ + A₂ + A₁  |     | 1   | 18  | 2   | 2766| 1     | "                           |
| 29| D₄ + A₉ + A₄       |     | 1   | 18  | 2   | 2767| 1     | "                           |
| 30| D₄ + A₉ + A₃ + A₁  |     | 1   | 18  | 2   | 2768| 1     | "                           |
| 31| D₄ + A₉ + 2A₂      |     | 1   | 18  | 2   | 2769| 1     | "                           |
| 32| D₄ + A₈ + A₂ + 2A₁ |     | 2   | 18  | 2   | 2770| 1     | "                           |
| 33| D₄ + A₈ + A₅       |     | 1   | 18  | 2   | 2771| 1     | "                           |
| 34| D₄ + A₈ + A₄ + A₁  |     | 1   | 18  | 2   | 2772| 1     | "                           |
| 35| D₄ + A₇ + A₄ + 2A₁ |     | 2   | 18  | 2   | 2773| 1     | "                           |
| 36| D₄ + A₇ + A₃ + A₂ + A₁|     | 2   | 18  | 2   | 2774| 1     | "                           |
| 37| D₄ + A₇ + 2A₂ + 2A₁|     | 2   | 18  | 2   | 2775| 1     | "                           |
| 38| D₄ + 2A₆ + A₁      |     | 1   | 18  | 2   | 2776| 1     | "                           |
| 39| D₄ + 2A₆ + A₁      |     | 1   | 18  | 2   | 2776| 1     | "                           |
| 40| D₄ + A₆ + A₅ + A₂  |     | 1   | 18  | 2   | 2777| 1     | "                           |
| 41| D₄ + A₆ + A₃ + A₂ + A₁|     | 1   | 18  | 2   | 2778| 1     | "                           |
| 42| D₄ + A₆ + A₂ + 2A₁ |     | 1   | 18  | 2   | 2779| 1     | "                           |
| 43| D₄ + 2A₅ + A₁      |     | 2   | 18  | 2   | 2780| 1     | "                           |
| 44| D₄ + 2A₃ + 3A₁     | [2,2]| 1   | 18  | 2   | 2781| 1     | "                           |
| 45| D₄ + A₃ + 2A₁      |     | 1   | 18  | 2   | 2782| 1     | "                           |
| 46| D₄ + A₂ + A₁ + A₃ + A₁|     | 2   | 18  | 2   | 2783| 1     | "                           |
| 47| D₄ + A₅ + 2A₃ + 2A₁|     | [2,2]| 2   | 18  | 2784| 1     | "                           |
| 48| D₄ + 2A₄ + 2A₂ + A₁|     | 1   | 18  | 2   | 2785| 1     | "                           |
| 49| D₄ + 3A₂ + 2A₁     |     | 2   | 18  | 2   | 2786| 1     | "                           |

Table 11. Data of ambi-typical strata
| #  | $\mathcal{I}^{\ast}$ | $\Gamma$ | $j_e$ | $I_0^*$ fibres |
|----|-----------------|----------|------|----------------|
| 0  | 1               | $\text{PSL}(2,\mathbb{Z})$ | $(3^2)_A$, $(2^2)^2$ | - |
| 1  | (3), (2, 1), (2, 1) | $\Gamma_1(2)$ | $(2, 2, 2, 2)_B$, $2^{10}$ | - |
| 2  | (3, 1), (2, 2), (3, 1) | $\Gamma_1(3)$ | $(3, 3)_A$, $2^6$ | - |
| 3  | (3, 3), (2, 2, 2), (4, 1, 1) | $\Gamma_1(4)$ | $2^{10}$ | - |
| 4  | (3, 3), (2, 2, 2), (2, 2, 2) | $\Gamma(2)$ | $2^6$ | - |
| 5  | (3, 3), (2, 2, 2), (4, 1, 1) | $\Gamma_1(4)$ | $(2, 2)_{1_{00}}$, $2^4$ | - |
| 6  | (3, 3), (2, 2, 2), (2, 2, 2) | $\Gamma(2)$ | $(2, 2)_{3_{00}}$, $2^2$ | - |
| 7  | (3, 3), (2, 2, 2), (4, 1, 1) | $\Gamma_1(4)$ | $(2, 2)_{1_{000}}$, $2^0$ | - |
| 8  | (3, 3, 3), (2, 2, 2, 2, 1), (7, 1, 1) | (2) $B$, 2 | 1 |
| 9  | (3, 3, 3), (2, 2, 2, 2, 1), (6, 1, 2) | (2) $B$, 2 | 1 |
| 10 | (3, 3, 3), (2, 2, 2, 2, 1), (5, 1, 1) | (2) $B$, 2 | 1 |
| 11 | (3, 3, 3), (2, 2, 2, 2, 1), (4, 3, 2) | (2) $B$, 2 | 1 |
| 12 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (9, 1, 1, 1) | 1 | 2 |
| 13 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (8, 2, 1, 1) | 1 | 2 |
| 14 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (6, 3, 2, 1) | 1 | 2 |
| 15 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (5, 5, 1, 1) | 1 | 2 |
| 16 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (4, 4, 2, 2) | 1 | 2 |
| 17 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (3, 3, 3, 3) | 1 | 2 |
| 18 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (9, 1, 1, 1) | $2^{10}$ | - |
| 19 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (8, 2, 1, 1) | $2^6$ | - |
| 20 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (6, 3, 2, 1) | $2^6$ | - |
| 21 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (5, 5, 1, 1) | $2^6$ | - |
| 22 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (4, 4, 2, 2) | $2^6$ | - |
| 23 | (3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (3, 3, 3, 3) | $2^6$ | - |
| 24 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 2), (14, 1, 1, 1) | 1 | 1 |
| 25 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2, 2), (13, 2, 1, 1) | 1 | 1 |
| 26 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (12, 3, 1, 1) | 1 | 1 |
| 27 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (12, 2, 1, 1) | 1 | 1 |
| 28 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (11, 3, 2, 1) | 1 | 1 |
| 29 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (10, 5, 1, 1) | 1 | 1 |
| 30 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (10, 4, 2, 1) | 1 | 1 |
| 31 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (10, 3, 3, 1) | 1 | 1 |
| 32 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (10, 3, 2, 1) | 1 | 1 |
| 33 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (9, 6, 1, 1) | 1 | 1 |
| 34 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (9, 5, 2, 1) | 1 | 1 |
| 35 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (8, 5, 2, 1) | 1 | 1 |
| 36 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (8, 4, 3, 2) | 1 | 1 |
| 37 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (8, 3, 3, 2) | 1 | 1 |
| 38 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (7, 7, 2, 1) | 1 | 1 |
| 39 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (7, 7, 2, 1) | 1 | 1 |
| 40 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (7, 6, 3, 1) | 1 | 1 |
| 41 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (7, 5, 3, 2) | 1 | 1 |
| 42 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (7, 4, 3, 1) | 1 | 1 |
| 43 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (6, 6, 4, 1) | 1 | 1 |
| 44 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (6, 6, 2, 2) | 1 | 1 |
| 45 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (6, 5, 5, 1) | 1 | 1 |
| 46 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (6, 5, 4, 2) | 1 | 1 |
| 47 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (6, 4, 4, 2) | 1 | 1 |
| 48 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (5, 5, 3, 2) | 1 | 1 |
| 49 | (3, 3, 3, 3, 3), (2, 2, 2, 2, 2, 2), (4, 4, 4, 3) | 1 | 1 |

**Table 12:** $j$-factorisation data of ambi-typical strata
11. Moduli spaces of lattice polarised $K3$ surfaces and Shimada strata

In this section we want to start our discussion about the precise relationship between moduli spaces of lattice polarised $K3$ surfaces and ambi-typical strata. For this we first recall some basic facts about lattice polarised $K3$ surfaces and their moduli spaces. Our aim is to understand how many moduli spaces of lattice polarised $K3$ surfaces dominate a given ambi-typical stratum and to compute the degree of the finite map from a component of such a moduli space to the ambi-typical stratum it dominates.

In Section 2 we encountered the following situation: we started with an elliptically fibred $K3$ surface $f : \mathcal{E} \to \mathbb{P}^1$ and root lattice $R(\mathcal{E})$ whose saturation in the $K3$ lattice was denoted by $L(\mathcal{E})$. Then $M(\mathcal{E}) = U + L(\mathcal{E})$ is a hyperbolic lattice contained in the Néron-Severi group of $\mathcal{E}$. This gives rise to a lattice polarization.

To explain this in more detail, let $M$ be an even lattice of signature $(1,t)$ which admits a primitive embedding $\iota : M \to L_{K3}$ into the $K3$ lattice $L_{K3}$. By $T = T(\iota) = \iota(M)_{K3}^\perp \subset L_{K3}$ we denote the orthogonal complement of the image of $\iota$. The lattice $T$ has signature $(2,19-t)$. We shall for the rest of this section assume that the rank of $T$ is at least $3$ (which is the case in our situation). Then $T$ defines a type IV homogeneous domain

$$\Omega_T = \{ [x] \in \mathbb{P}(T \otimes \mathbb{C}) \mid (x,x) = 0, (x,\bar{x}) > 0 \} = D_T \cup D'_T$$

of dimension $19-t$ consisting of two connected components, namely $D_T$ and $D'_T$.

We also consider the real cone

$$C_M = \{ x \in M_\mathbb{R} \mid (x,x) > 0 \} = C^+_M \cup C^-_M$$

which again consists of two connected components, of which we choose one, say $C^+_M$. Removing from $C^+_M$ all hyperplanes orthogonal to roots $\Delta_M = \{ d \in M \mid d^2 = -2 \}$ subdivides $C^+_M$ into different connected components, the so called Weyl chambers, of which we choose one and call it $C^\text{pol}_M$. An $(M,\iota)$-polarised $K3$ surface is a pair $(S,\tilde{\iota})$ where $\tilde{\iota} : M \to \text{NS}(S) \subset H^2(S,\mathbb{Z})$ is a primitive embedding which is isomorphic to $\iota$ with respect to a suitable marking $\varphi : H^2(S,\mathbb{Z}) \to L_{K3}$ and such that $\tilde{\iota}(C^\text{pol}_M)$ contains an ample class. In this case we call $\varphi : H^2(S,\mathbb{Z}) \to L_{K3}$ an $M$-polarised marking. We call two $M$-polarised $K3$ surfaces $(S_1,\tilde{\iota}_1)$ and $(S_2,\tilde{\iota}_2)$ isomorphic if there is an isomorphims $f : S_1 \to S_2$ with $f^* \circ \tilde{\iota}_2 = \tilde{\iota}_1$.

In order to describe the moduli space of $M$-polarised $K3$ surfaces we consider the group

$$O(L_{K3},M,\iota) := \{ g \in O(L_{K3}) \mid g|_{\iota(M)} = id_{\iota(M)} \}$$

and recall the definition of the stable orthogonal group

$$\tilde{O}(T) := \{ g \in O(T) \mid g|_{D(T)} = id_{D(T)} \}$$

consisting of all orthogonal transformations of $T$ which act trivially on the discriminant $D(T)$. It is well known, see [Nik, Corollary 1.5.2], that there is an isomorphism

$$O(L_{K3},M,\iota) \cong \tilde{O}(T).$$

The group $O(T)$, and hence also $\tilde{O}(T)$, acts properly discontinuously on $\Omega_T$ as well as on the open subset

$$\Omega_T^{\text{pol}} = \Omega_T \setminus \bigcup_{d \in T, d^2 = -2} (H_d \cap \Omega_T)$$
where $H_d = \langle d \rangle^\perp \subset \mathbb{P}(T \otimes \mathbb{C})$ is the hyperplane orthogonal to $d$.

**Proposition 11.1.** The quotient

$$N_{M,t}^a = \Omega_T^{pol} / \tilde{O}(T)$$

is the moduli space of $(M, \iota)$-polarised $K3$ surfaces.

**Proof.** See [Do, Section 3] or [BHPV, p. 360] \hfill \Box

It is sometimes also useful to consider a weakening of $(M, \iota)$-polarizations. Recall that a line bundle $L$ is said to be a quasi-polarization if it is nef and big. We say that $(S, \tilde{\iota})$ is an $(M, \iota)$-quasi-polarised $K3$ surface if $\tilde{\iota}(C_M^{pol})$ contains a big and nef class. By [Do, Section 3] the quotient

$$N_{M,t} = \Omega_T / \tilde{O}(T).$$

is in 1 : 1 correspondence with the set of isomorphism classes of $M$-quasi-polarised $K3$ surfaces. It can be viewed as the moduli space of $M$-polarized $K3$ surfaces with ADE singularities. This contains $N_{M,t}^a$ as an open subset.

At this point some remarks are in order. In the literature it is often tacitly assumed that the lattice $M$ has a unique primitive embedding into the $K3$ lattice. This assumption then justifies to talk about the moduli space of $M$-polarised $K3$ surfaces. For us it will be important to also allow the possibility that $M$ possesses different primitive embeddings into the $K3$ lattice (the number of such embeddings modulo $O(L_{K3})$ is always finite). We will then consider the union

$$N_M = \cup_\iota N_{M,t}$$

where $\iota$ runs over all classes of different primitive embeddings of $M$ into $L_{K3}$. Moreover, Dolgachev has formulated arithmetic conditions which lead to the notion of $m$-admissible lattices. This means in particular that the transcendental lattice $T$ splits off a summand $U(m)$, i.e. a multiple of a hyperbolic plane. This is relevant for mirror symmetry and the discussion of the Yukawa-coupling, but it plays no role for our purposes since the construction of the moduli space of lattice (quasi-)polarised $K3$ surfaces does not require this condition. In fact, a number of the lattices which we consider, in particular when $M$ has rank 19, do not fulfill this condition, see the Appendix. In these cases the lattice $T$ does not split off a summand $U$ over $\mathbb{Q}$ resulting in compact moduli spaces of $M$-polarised $K3$ surfaces. Finally, we recall that the case $M = U$ gives us another construction for the moduli space $\mathcal{F}$ of elliptically fibred $K3$ surfaces with a section or Jacobian fibrations.

Since $O(T)$ acts properly discontinuously on $\Omega_T$ the quotient $N_{M,t}$ has at most finite quotient singularities. By a well know result of Baily-Borel this is a quasi-projective variety. We also note that for small rank of the transcendental lattice $T$ there is a relation with Siegel spaces: if $t = 18$ then $D_T \cong \mathbb{H}_1$ is the upper half plane, if $t = 17$, then $D_T \cong \mathbb{H}_1 \times \mathbb{H}_1$, and if $t = 16$, then $D_T \cong \mathbb{H}_2$, the Siegel upper half plane of genus 2. We will discuss this in more detail in the cases of $t = 18$ in Appendix.

The quotient $\Omega_T / \tilde{O}(T)$ can have one or two components. If it has two components then these are complex conjugate to each other. An element $g \in O(T)$ can either fix the two components $D_T$ and $D_T'$ or interchange them, depending on its spinor norm. We recall that the real spinor norm is a homomorphism

$$\text{sn}_R : O(T) \rightarrow \mathbb{R}^*/(\mathbb{R}^*)^2 = \{ \pm 1 \}.$$
For a precise definition we refer the reader to [GHS1, Section 1]. Our normalization of
the spinor norm is such that in the case of signature \((2, n)\), in which we here are, the
transformation \(g\) fixes the two components of \(\Omega_T\) if and only if \(s_{nR}(g) = 1\) and it interchanges
them if and only if \(s_{nR}(g) = -1\). We define the groups

\[
O^+(T) = \{ g \in O(T) \mid s_{nR}(g) = 1 \}
\]

and

\[
\tilde{O}^+(T) = O^+(T) \cap \tilde{O}(T).
\]

The quotient \(\Omega_T/\tilde{O}(T)\) then has two components if any only if \(\tilde{O}(T) = \tilde{O}^+(T)\).

We first want to discuss the number of moduli spaces of \(M\)-polarised \(K3\) surfaces and the
number of connected components. Clearly, there is only one such moduli space if there is
a unique primitive embedding of \(\iota:\ M \to L_{K3}\) (up to \(O(L_{K3})\)). In general, however, such
an embedding need not be unique. Assume that there is at least one primitive embedding
\(\iota:\ M \to L_{K3}\) and let \(T_\iota = \iota(M)_{L_{K3}}^\perp\). Then \(T_\iota\) may depend on \(\iota\), but its genus does not. It is
determined by

\[
\text{sign}(T_\iota) = (2, 18 - \text{rank}(M)) \quad \text{and} \quad (D(T_\iota), q_{T_\iota}) \cong (D(M), -q_M).
\]

We call this the genus orthogonal to \(M\) and denote it by \(G_T\).

**Proposition 11.2.** Let \(T \in \mathcal{G}_T\) and let \(\tilde{O}(T)\) be the image of \(O(T)\) in \(O(D(T))\). Then the
index \([O(D(T)) : \tilde{O}(T)]\) depends only on the genus \(G_T\) of \(T\) and not on \(T\) itself.

**Proof.** This follows from the theory of Miranda and Morrison, in particular [MM, Chapter
VIII, Proposition 6.1. (2)]. Here we recall that we are always in the situation that \(T\)
is indefinite since we assume in this section that it has rank at least 3. \(\square\)

It follows from the theory of Miranda-Morrison [MM, Theorem VIII.7.2], see also [Shi2,
p. 514], that there is an exact sequence, which is completely determined by \(M\), of the form

\[
0 \to \text{coker} \left( O(T) \to O(D(T)) \right) \to \mathcal{M}_T \to \mathcal{G}_T \to 0
\]

where \(\mathcal{M}_T\) is a finite group which is in \(1:1\) correspondence with the primitive embeddings
\(\iota:\ M \to L_{K3}\) and thus with the moduli spaces of lattice polarised \(K3\) surfaces with lattice polarisation \(M\).

The following result is far less obvious. We will not need it for the lattices we are concerned
with, as in our cases the genus always consists of one element only, but state it to complete
the picture. The proof of this was communicated to us by Simon Brandhorst, here we only
give a sketch.

**Proposition 11.3.** The index \([\tilde{O}(T) : \tilde{O}^+(T)]\) depends only on the genus \(G_T\) of \(T\) and not
on \(T\) itself.

**Proof.** (Sketch) This is a consequence of the strong approximation theorem. It can be de-
duced from [MM, Proposition VIII.6.1 (2)] with extra bookkeeping of the real spinor norm
using the fact that (in the terminology of [MM]) the objects \(\Gamma_S, \Sigma(T)\) and \(\Sigma^\chi(T)\) all depend
only on the genus \(G_T\) and not on the lattice \(T\) itself. \(\square\)

As a corollary of the theory of Miranda and Morrison and in particular Sequence (11.1)
together with Proposition 11.3 we thus obtain
Corollary 11.4. Let $M$ be a hyperbolic lattice which admits a primitive embedding into the $K3$-lattice $L_{K3}$ and let $G_T$ be the genus of the orthogonal complement of one, and hence any such embedding of $M$ into $L_{K3}$.

1. The number of moduli spaces of $M$-polarised $K3$ surfaces is given by
   $$|\mathcal{M}_T| = [O(D(T)) : \overline{O}(T)] \cdot |G_T|.$$

2. The number of connected components of these moduli spaces is given by
   $$|\mathcal{M}_T|^c = [O(D(T)) : \overline{O}(T)] \cdot |G_T| \cdot \frac{2}{[\overline{O}(T) : \overline{O}^+(T)]}.$$

We shall now discuss which values these numbers can have in our cases. The relevant data for the ambivalent strata are collected in Table 11. The lattice data consist first of a root lattice $R$ and an isotropic subgroup $G \subset D(R)$ in the discriminant group of $R$. This determines an overlattice $L$ of $R$ and the hyperbolic lattice $M = U + L$.

We shall first prove that in our situation the genus $G_T$ always consists of a single element.

Proposition 11.5. Let $M$ be a hyperbolic lattice associated to one of the families listed in Table 11. Then the genus given by the signature $(2, 20 - \text{rank}(M))$ and discriminant group $(D(M), -q_M)$ consists of one element only.

Proof. The claim for the entries (1),(2) and (3) of Table 11 follows immediately from the numerical conditions of Nikulin’s theorem [Nik, Theorem 1.14.2].

For the other lattices we shall use [CS, Chapter 15]. Since we have indefinite lattices, it suffices, according to [CS, Section 15.9.7], to show that there are no non-tractable primes (for a definition of non-tractable primes see [CS, Chapter 15.9.6]). Let $d$ be the discriminant of $M$ and let $n = 22 - \text{rank}(M)$ be the rank of the orthogonal complement. According to [CS, Theorem 15.20] a necessary condition for an odd prime $p$ to be non-tractable is

$$d \text{ divisible by } p^{\frac{n}{2}}.$$  

Note that $n = 8$ in the case of entry (4), i.e. Shimada’s case 565, and $n = 4$ or $n = 3$ in all other cases. One can now check by hand that this condition is never fulfilled for the lattices coming from Table 11. In most cases this follows already from the discriminants of the root lattices $R$ in the second column of this table. However, in some cases one has to be more careful. An example is entry (23) which is Shimada’s family 2373. Here $n = 4$ and we must not have divisors of $d$ of the form $p^6$. Now the lattice $8A_2$ has discriminant $3^8$. However, the lattice $L$ is an overlattice of $8A_2(-1) + U$ with torsion group $[3, 3]$. But this means that the order of the discriminant group of $M$ is $3^8 / 3^4 = 3^4$ and hence 3 is not a non-tractable prime. The other cases can be treated in the same way.

Thus the only possible non-tractable prime is $p = 2$. By [CS, Theorem 15.20] this implies that

$$4^{\frac{n}{2}}d \text{ is divisible by } 8^{\frac{n}{2}}.$$  

Again, this can be checked by hand. For example in case 2784 one has $4^{\frac{n}{2}}d(D_4 + A_5 + 2A_3 + 2A_1 + U) = 2^{11} \cdot 3$. However, taking the torsion into account we obtain that $4^{\frac{n}{2}}d = 2^7 \cdot 3$ which is not divisible by $8^3 = 2^9$. The other cases are similar.  

$\square$
Corollary 11.6. Let $M$ be a hyperbolic lattice associated to one of the families listed in Table [7]. Then the orthogonal complement of a primitive embedding $\iota : M \to L_{K3}$ depends only on $M$ and not on the chosen embedding $\iota$.

This corollary allows us to speak of the orthogonal complement $T$ of the lattice $M$ in the $K3$ lattice $L_{K3}$, even if the primitive embedding $\iota : M \to L_{K3}$ is not uniquely defined (which can occur).

Proposition 11.7. Let $M$ be a hyperbolic lattice associated to one of the families listed in Table [7] and let $T$ be the unique element in the genus orthogonal to $M$. Then the map $O(T) \to O(D(T))$ is always surjective with the exception of the following five root lattices $R$:

(17) $2D_4 + 4A_2$,
(21) $4A_4$,
(37) $D_4 + A_7 + 2A_2 + 2A_1$,
(48) $D_4 + 2A_4 + 2A_2 + A_1$,
(49) $D_4 + 3A_3 + 2A_2$.

In these cases the index $[O(D(T)) : O(T)] = 2$, in particular, there are exactly two non-isomorphic primitive embeddings of $M$ into $L_{K3}$.

Proof. For the cases (1), (2) and (3) from Table [11] the surjectivity of $O(T) \to O(D(T))$ can be seen directly by Nikulin’s criterion [Nik, Theorem 1.16.10]. In general this follows from computations of Shimada, which are available from his website, see [Shi4]. For the rank 17 cases this follows independently from Kirschmer’s computations, see Table [14] Column 5 in the Appendix. 

It now follows that the number of connected components of $M$-polarised $K3$ surfaces, where $M$ is a lattice corresponding to an ambi-typical stratum, is either 1, 2 or 4. It is at least 2 in the cases listed in Proposition [11.7]. To determine the exact number one has to compute the index $[O(T) : \tilde{O}(T)]$. Indeed, all three possible cases occur as the following example shows.

Example 11.8. Assume that the root lattice $R$ has rank 17. Then the following holds: out of the 25 rank 17 lattices we have $|\mathcal{M}_T|^c = 1$ in 18 cases. In the four cases $D_4 + A_7 + A_4 + 2A_1$, $D_4 + 2A_6 + A_1$, $D_4 + A_6 + A_3 + 2A_2$ and $D_4 + 2A_5 + 3A_1$ we have $|\mathcal{M}_T| = 1$ and $[\tilde{O}(T) : \tilde{O}^+(T)] = 1$ and thus $|\mathcal{M}_T|^c = 2$. Finally, in the three cases $D_4 + A_7 + 2A_2 + 2A_1$, $D_4 + 2A_4 + 2A_2 + A_1$ or $D_4 + 3A_3 + 2A_2$ we have $|\mathcal{M}_T| = 2$ and $[\tilde{O}(T) : \tilde{O}^+(T)] = 1$ and hence $|\mathcal{M}_T|^c = 4$. This follows from Kirschmer’s computations, see Table [14] in the Appendix. Note that we have 2 components if either $[\tilde{O}(T) : \tilde{O}^+(T)] = 1$ or $[O(D(T)) : \tilde{O}(T)] = 2$ and 4 components if both of these indices are 1 and 2 respectively. In all other cases we have 1 component.

Remark 11.9. It is interesting to compare this with the Shimada’s list of non-connected moduli of elliptic $K3$ surfaces, see [Shi2, Corollary 1.5 and Table II]. We first observe that the lattices which appear in Proposition [11.7] do not appear in Shimada’s list. The reason is that different moduli spaces of lattice-polarised $K3$ surfaces can lead to the same Shimada stratum. This is due to symmetries of the root lattice $R$ and we shall discuss this in the next section. On the other hand, there are three root lattices which appear in both Table [11] of our paper and Table 3 in [Shi2, p.555-557]. These are $D_4 + 2A_6 + A_1$, which are our cases (38/39), as well as $2A_3 + 8A_1$ and $4A_3 + 2A_1$ which are our cases (6) and (7) respectively. In
the case of $D_4 + 2A_6 + A_1$ we have two complex conjugate components. The lattice $4A_3 + 2A_1$ appears in [Shi2, Table 3] in connection with the Mordell-Weil torsion $\mathbb{Z}/2\mathbb{Z}$. In this case one has two components. However, in our situation we have Mordell-Weil torsion $\mathbb{Z}/4\mathbb{Z}$ and this leads to one component only. Finally, in the case of $2A_3 + 8A_1$ the two components in Shimada’s list come from inequivalent isotropic subgroups $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ which correspond to different intersection behaviour of the torsion sections. Only one of these cases leads to an ambi-typical stratum, see the discussion in Remark 9.6.

12. The moduli maps

The aim of this section is to investigate the precise relationship between moduli spaces of lattice-polarised $K3$ surfaces and ambi-typical strata. In particular we show that there is a finite map from certain moduli spaces of lattice polarised $K3$ surfaces to ambi-typical strata and compute the degree of this map.

For this we consider an ambi-typical stratum

$$\mathcal{A} = \mathcal{F}_{\Gamma,i}^\prime = \mathcal{M}_{R,G,i}^\prime.$$  

As before we denote by $L$ the lattice defined by the isotropic group $G \subset D(R)$ and set $M = U + L$. Our first goal is to associate to such a given ambi-typical stratum certain moduli spaces of lattice-polarised $K3$ surfaces. First of all, given any surface $S$ in such a stratum, we identify the sublattice spanned by the section and the fibre with a fixed summand $U$ of the $K3$ lattice $L_{K3}$. More precisely, we first choose the standard basis $e, f$ of $U$ with $e^2 = f^2 = 0$ and $e.f = 1$ and identify the fibre class with $f$ and the section $s$ with $e - f$. This can be done once and for all simultaneously for all surfaces in this stratum. Now choose a generic surface $S$ in $\mathcal{M}_{R,G,i}$ and a marking $\varphi : H^2(S, \mathbb{Z}) \to L_{K3}$ where the sublattice spanned by the fibre and the section are identified with a given summand $U$ as described above. The components of the singular fibres not meeting the section define a sublattice isomorphic to $R$ whose saturation in the $K3$ lattice is isomorphic to $L$. We note that the isomorphism with $R$ is not canonical, but depends on an identification of these components with a set of simple roots of $R$. There are finitely many ways of choosing such an identification. Adding the sublattice spanned by the fibre and the section one obtains an embedding

$$\iota : M \to \text{NS}(S) \subset H^2(S, \mathbb{Z}) \cong L_{K3},$$

and thus an element $(S, \iota)$ in a connected component $\mathcal{N}_{M,\iota}$ of the moduli space $\mathcal{N}_{M,\iota}$ of lattice polarised $K3$ surfaces. We shall refer to such a component, as well as to the moduli space $\mathcal{N}_{M,\iota}$ itself, as associated to $\mathcal{A}$. Note that we do not claim that $\mathcal{N}_{M,\iota}$ or $\mathcal{N}_{M,\iota}$ are uniquely determined by the ambi-typical stratum since there is no canonical way of identifying the fibre components with a root system. Nor do we claim that we obtain a morphism from (an open part of) the stratum $\mathcal{A}$ to such a moduli space $\mathcal{N}_{M,\iota}$. Indeed, we always have (at least) the ambiguity given by the symmetries $S_R$ of the Dynkin diagram associated to $R$. Changing the isomorphism by such an element can have two effects. One is that it defines different points in the same moduli space of $M$-lattice polarised $K3$ surfaces. The other is that it leads to different moduli spaces of lattice polarised $K3$ surfaces. Note however, that two such moduli spaces of $M$-polarised $K3$ surfaces define the same Shimada stratum as the combinatorial type of the singular fibres is the same. As we shall see, both cases occur. We shall now discuss this in detail. Before doing this we establish the following convention. Given such a moduli space $\mathcal{N}_{M,\iota}$ of lattice polarised $K3$ surfaces and a $K3$ surface $S$ which appears in this
moduli space, we always have a distinguished copy \( U \). In each component \( N_{M,\ell} \) we consider the non-empty open part of the moduli space where this copy of \( U \) contains a nef isotropic class and an irreducible \((-2)\)-curve. This determines a Jacobian fibration \( S \to \mathbb{P}^1 \) whose fibre and section are contained in \( U \). Unless stated otherwise we will always work with this elliptic fibration. By \( N_{M,\ell}^0 \) we further denote the open subset where the configuration of the singular fibres is generic.

**Proposition 12.1.** The following holds:

1. Given an ambi-typical stratum \( A \) there are only finitely many components of moduli spaces of lattice-polarised \( K3 \) surfaces associated to \( A \).
2. Let \( N_{M,\ell} \) be a component of a moduli space \( N_{M,\iota} \) of lattice polarised \( K3 \) surfaces associated to \( A \) and let \( N_{M,\ell}^0 \) be the open subset where the configuration of the singular fibres is generic. Then there is a natural finite to one dominant morphism \( N_{M,\ell}^0 \to A \).

**Proof.** Claim (1) follows from Nikulin’s theory since there are only finitely many inequivalent primitive embeddings \( \iota : L \to L_{K3} \).

To prove (2) we recall that we have for every element in \( N_{M,\iota} \) a well defined Jacobian fibration \( S \to \mathbb{P}^1 \) which can be written in Weierstraß form. The fact that \( F \) is a coarse moduli space of \( K3 \) surfaces with a Jacobian fibration defines a morphism \( N_{M,\ell} \to F \). On the open subset \( N_{M,\ell}^0 \) the monodromy is constant with monodromy group \( \Gamma \) and hence we obtain a morphism \( N_{M,\ell}^0 \to A \). This map has finite fibres since there are only finitely many ways of identifying the components of the singular fibres not intersecting the 0-section with a basis of the root lattice \( R \). The map is dominant by the definition of an ambi-typical stratum. \( \square \)

We now want to understand these maps better. The next two statements will be useful for this.

**Lemma 12.2.** Let \( (R,G) \) be a pair consisting of a root lattice and an isotropic subgroup \( G \subset D(R) \) with associated overlattice \( L \), arising from one of the families listed in Table 11. Then the roots of \( R \) and \( L \) coincide.

**Proof.** A proof can be found in [Shi2, Proposition 3.2]. Indeed, this is a simple geometric argument. We can use the fact that \( M = U + L \) is isomorphic to the Néron-Severi group \( \text{NS}(S) \) of some (sufficiently general) \( K3 \) surface \( S \) and that \( R \) is the subgroup of \( \text{NS}(S) \) generated by all fibre components which do not meet the 0-section. Then the claim is geometrically clear: assume that \( r \) is a root of \( L \), which is not a root of \( R \). Then \( \pm r \) is effective and meets neither the 0-section nor a general fibre, since it is orthogonal to the summand \( U \) which contains the classes of the 0-section and a general fibre. Hence \( \pm r \) defines a union of rational curves on the associated elliptic \( K3 \) surfaces consisting of components of singular fibres, which do not intersect the 0-section. But these roots are already contained in \( R \). \( \square \)

**Remark 12.3.** Here we use the specific geometric situation. In general such a statement is false. Indeed, let \( R = 4A_1 \) and let \( H \) be the subgroup generated by the diagonal \((1,1,1,1)\) of \( D(R) = 4\mathbb{Z}/2\mathbb{Z} \). Then the overlattice is \( D_4 \) and one obtains a new root, namely \( r = \frac{1}{2} \sum_{i=1}^{4} r_i \).

Using the above lemma we can now prove
Proposition 12.4. Let \((R, G)\) be a pair of a root lattice and an isotropic subgroup arising from one of the families listed in Table 11 with associated overlattice \(L\). Then

\[ O(L) \cong \{ g \in O(R) \mid \overline{g}(G) = G \} \]

Proof. Clearly, the elements of \(O(R)\), which leave \(G\) invariant (as a subgroup), define isometries of \(L\), by construction of this lattice. Conversely, let \(g \in O(L)\). Then \(g\) maps roots of \(L\) to roots of \(L\). By the above Lemma 12.2 these are exactly the roots of \(R\) and hence \(R\) is mapped to itself. Hence \(g\) is an isometry of the lattice inclusion \(R \subset L\) and thus \(\overline{g}\) maps the isotropic subgroup \(G\) to itself. \(\square\)

Let \(R\) be a root lattice. We recall that the Weyl group \(W_R \subset O(R)\) is the group generated by the reflections with respect to the roots \(r \in R\). We denote by \(S_R\) the subgroup of \(O(R)\) which is induced by symmetries of the Dynkin diagram. We also recall from [Hum, Theorem 12.2] that the isometry group of \(R\) is the semi-direct product of the Weyl group \(W_R\) and the group \(S_R\):

\[ O(R) = W_R \rtimes S_R. \]

Elements in the Weyl group \(W_R\) act trivially on the discriminant \(D(R)\) and hence, in our situation, lift to isometries of \(L\). In particular, we can consider \(W_R \subset O(L)\). We further denote the subgroup of \(S_R\) which leaves \(G\) invariant (as a subgroup) by \(S^G_R\). It now follows from Proposition 12.4 that \(O(L)\) is generated by the Weyl group \(W_R\) together with the subgroup \(S^G_R\) of diagram isometries which fix the subgroup \(G\), i.e.

\[ (12.2) \quad O(L) = W_R \rtimes S^G_R. \]

We can extend all elements in \(O(L)\) to isometries of \(M = U + L\) by taking the identity on the first factor \(U\). By doing this we can consider \(S^G_R\) as a subgroup of \(O(M)\) and to simplify the notation we shall denote the image of \(S^G_R\) in \(O(M)\) by \(S_M\). This notation is unambiguous in our situation as there is no lattice \(R\) in Table 11 which comes with more than one subgroup \(G \subset D(R)\).

The lattice \(M\) has signature \((1, \text{rank}(M) - 1) = (1, \text{rank}(R) + 1)\). As before we fix a connected component \(C^+(M)\) of the positive cone in \(M_R\). We have already mentioned that the hyperplanes orthogonal to the roots \(r \in M\) subdivide \(C^+(M)\) into connected components, the Weyl chambers of \(M\). It is well known, see [Huy, Proposition 8.2.6] that the Weyl group \(W_M\) of \(M\) acts simply transitively on the Weyl chambers in \(C^+(M)\). By Lemma 12.2 we have

\[ W_R = W_L \subset W_M \subset \tilde{O}^+(M). \]

(12.3)

In particular, \(W_R\) also acts faithfully on the set of Weyl chambers of \(M\).

For our applications it will be essential that the group \(S_M\) maps Weyl chambers to themselves:

Proposition 12.5. Let \((R, G)\) be a pair of a root lattice and an isotropic subgroup arising from one of the families listed in Table 11. The elements of \(S_M\), i.e. the symmetries of the Dynkin diagram of \(R\) fixing \(G\) as a group, map all Weyl chambers of \(C^+(M)\) to themselves.

Proof. Here we make again use of the special situation, namely the fact that there exists an \(M\)-polarised \(K3\) surface \(S\) with \(\text{NS}(S) \cong M\) and such that the components of the singular fibres which do not intersect the 0-section generate the root lattice \(R\). In fact these define a set of simple roots which gives rise to the Dynkin diagram associated to \(R\) and \(S_R\) acts on
these. More precisely, we can choose a primitive embedding \( \iota : M \rightarrow L_{K3} \) and a marking \( \varphi : H^2(S, \mathbb{Z}) \rightarrow L_{K3} \) such that \( \varphi(\text{NS}(S)) = \iota(M) \). Let \( \Delta^+ \) be the set of positive roots (i.e. effective \((-2)\)-classes in \( \text{NS}(S) \)). By [Huy, Section 8.2.3] it is enough to show that every element of \( S_M \) maps positive roots to positive roots. Let \( g \in S_M \) and let \( f \) be the class of a fibre. Then \( g(f) = f \). If \( s \) is a positive root with \( (f, s) \neq 0 \), then \( (f, s) > 0 \). Since \( (f, g(s)) = (g(f), g(s)) = (f, s) > 0 \) it follows that \( g(s) \) is again positive. By definition of the group \( S_M \) we have \( g(s_0) = s_0 \) where \( s_0 \) is the class of the 0-section. Then the same argument also shows that \( g(s) \) is positive for every positive root \( s \) with \( (s, s_0) \neq 0 \). It remains to consider the positive roots which are orthogonal to \( f \) and \( s_0 \), but these are exactly the positive roots of \( R \), which are given by non-negative combinations of components of singular fibres which do not intersect the 0-section. Since \( g \) permutes these, the claim follows.

We shall now discuss the action of the group of symmetries of the Dynkin diagram on the moduli spaces of \( M \)-lattice polarised \( K3 \) surfaces in more detail. Let \( (S, \tilde{i}) \) be a general element in \( N_{M,\iota} \), more precisely an element in \( N_{M,\iota}^0 \), where \( N_{M,\iota} \) is a connected component of \( N_{M,\iota} \) and \( N_{M,\iota}^0 \) denotes the open set where the fibre configuration is constant. Then \( \tilde{i} : M \rightarrow \text{NS}(S) \subset H^2(S, \mathbb{Z}) \) is a primitive embedding and there exists a marking \( \varphi : H^2(S, \mathbb{Z}) \rightarrow L_{K3} \) such that \( \tilde{i} = \varphi^{-1} \circ \iota \) and such that \( \tilde{i}(C^\text{pol}_M) \) contains an ample class on \( S \) where \( C^\text{pol}_M \) is the fixed Weyl chamber in \( C^+(M) \) which we have chosen once and for all. Every element \( g_M \in S_M \subset \text{O}(M) \) has, by Proposition 12.5, the property that it fixes the Weyl chambers in \( C^+(M) \). Hence \( \tilde{i} \circ g_M : M \rightarrow \text{NS}(S) \) defines again an \( M \)-polarization on \( S \). Now two cases can occur. The first is that \( \tilde{i} \) and \( \tilde{i} \circ g_M \) define isomorphic embeddings of \( M \) into the \( K3 \) lattice \( L_{K3} \). In this case \( (S, \tilde{i}) \) and \( (S, \tilde{i} \circ g_M) \) define elements in the same moduli space \( N_{M,\iota} \) and \( g_M \) induces a map from \( N_{M,\iota} \) to itself identifying \( (S, \tilde{i}) \) and \( (S, \tilde{i} \circ g_M) \). Note that the moduli space \( N_{M,\iota} \) can have one or two components. If it has two components, then \( g_M : N_{M,\iota} \rightarrow N_{M,\iota} \) can either fix the components or interchange them. By the discussion in Remark 11.8 all of these possibilities discussed actually occur in our situation. In the second case \( \iota \) and \( \tilde{i} \circ g_M \) define different embeddings and \( g \) induces an isomorphism of moduli spaces \( N_{M,\iota} \rightarrow N_{M,\iota} \). The lattices where more than one embedding exists are listed in Proposition 11.7. In these cases we have two different embeddings, but there exist symmetries of the Dynkin diagram which defines isomorphisms \( N_{M,\iota} \cong N_{M,\iota} \) (by Remark 11.9). Again, note that \( N_{M,\iota} \) can have one or two components, but the latter does not occur among our cases.

We can also formulate the above discussion in more group theoretic terms. The choice of a primitive embedding \( \iota : M \rightarrow L_{K3} \) with orthogonal complement \( T \) (up to isomorphism of embeddings) is equivalent to the choice of an isomorphism \( \alpha_i : (D(M), q_M) \cong (D(T), -q_T) \) (modulo \( \overline{O}(T) \)). Recall the definitions of the groups \( S_R^0 \) and \( S_M \) from \{12.2\} and the subsequent paragraph. An element \( g_M \in S_M \) defines an isometry \( \overline{g}_M \in \text{O}(D(M)) \) and, via \( \alpha_i \), an isometry \( \alpha_i(\overline{g}_M) \in \text{O}(D(T)) \). The morphism which maps \( (S, \tilde{i}) \) to \( (S, \tilde{i} \circ g_M) \) maps \( N_{M,\iota} \) to itself if and only if \( \alpha_i(\overline{g}_M) \in \overline{O}(T) \).

If \( g_M \) induces a morphism from \( N_{M,\iota} \) to itself, then we can describe this map explicitly. In this case \( \alpha_i(\overline{g}_M) \in \overline{O}(T) \) and we can lift this to an element \( g_T \in \text{O}(T) \) such that the pair \((g_M, g_T)\) extends to an isometry of \( L_{K3} \). The lift \( g_T \) is uniquely determined up to \( \overline{O}(T) \). The action of \( g_T \) on \( N_{M,\iota} = \Omega_T / \overline{O}(T) \) then induces the map on \( N_{M,\iota} \) in question. Now \( N_{N,\iota} \) has two components if and only if \( \overline{O}^+(T) = \overline{O}(T) \) and in this case \( g_T \) interchanges the two
components if and only if $g_T$ has real spinor norm $-1$, i.e. if and only if $g_T \notin \tilde{O}^+(T)$ (which in this case is independent of the chosen lift).

Let $\pi_M : O(M) \to O(D(M))$ and $\pi_T : O(T) \to O(D(T))$ be the canonical projections. We define
\begin{equation}
\underline{S}_M = \pi_M(S_M)
\end{equation}
which we will, via $\alpha_i : D(M) \to D(T)$, also consider as a subgroup $\underline{S}_M \subset O(D(T))$. As a subgroup this depends on the embedding $\iota$ as it is defined via the isomorphism $\alpha_i$. This becomes important when we define
\begin{equation}
\underline{S}_{M,i} = \underline{S}_M \cap \underline{O}(T), \underline{S}_{M,i}^+ = \underline{S}_M \cap \underline{O}^+(T)
\end{equation}
and their pre-images
\begin{equation}
\Gamma_{M,i} = \pi_T^{-1}(\underline{S}_{M,i}) \subset O(T), \Gamma_{M,i}^+ = \pi_T^{-1}(\underline{S}_{M,i}^+) \subset O^+(T).
\end{equation}
The elements in $\Gamma_{M,i}$ are those isometries of $T$ which can be extended to the overlattice $L_{K3}$ of $T \oplus \iota(M)$ (where we do not ask that these isometries act trivially on $\iota(M)$). The group $\Gamma_{M,i}$ acts on the period domain $\Omega_T$ and induces an action on the moduli space $N_{N,i}$. An element in $\Gamma_{M,i}$ fixes the components of $N_{N,i}$ if and only if it is in $\Gamma_{M,i}^+$.

By the above discussion the group $\underline{S}_M$ operates on $O(D(T))/\underline{O}(T)$ via $h \mapsto h \circ \alpha_i(\mathbf{7})$. It will be important for us to know whether this action is transitive. The following is essentially a reformulation of Remark 11.9.

**Proposition 12.6.** If $M$ is a hyperbolic lattice arising from one of the families listed in Table II, then $\underline{S}_M$ acts transitively on $O(D(T))/\underline{O}(T)$ unless we are in the case where $R = 2A_3 + 8A_1$.

**Proof.** As we have said before (see Remark 11.9) comparing the lists in [Sh2, Corollary 1.5] and [Sh2, Table 3] with our Table II we find three lattices, namely $4A_3 + 2A_1$, $2A_3 + 8A_1$ and $D_4 + 2A_6 + A_1$. The first lattice is irrelevant for us as it appears with Mordell-Weil torsion $\mathbb{Z}/2\mathbb{Z}$ in Shimada’s lists, whereas we have torsion $\mathbb{Z}/4\mathbb{Z}$. The reason that $D_4 + 2A_6 + A_1$ appears in Shimada’s lists is that there are two connected component, but they belong to the same primitive lattice embedding. Finally, for $2A_3 + 8A_1$ there exist two combinatorially different components of the moduli space coming from different lattice embeddings (but only one of them appears in our classification). \hfill \Box

Our previous discussion can now be summarised as follows. Let $\mathcal{A}$ be an ambi-typical stratum, resp. let $\mathcal{A} \cup \mathcal{A}$ be the union of the two complex-conjugated components 38/39. Then there are finitely many components of moduli spaces $N_{N,i}$ of lattice-polarised $K3$ surfaces which are associated to $\mathcal{A}$ or $\mathcal{A}$. The group $\underline{S}_M$ acts transitively on the set of all moduli spaces $N_{M,i}$ whose components are associated to $\mathcal{A}$ or $\mathcal{A} \cup \mathcal{A}$ respectively. The groups $\Gamma_{M,i}$ act on the moduli spaces $N_{M,i}$ (and their elements may interchange connected components of these moduli spaces).

**Remark 12.7.** By inspection one sees that for all our lattices $-1 \in \underline{S}_{M,i}$ and hence also $-1 \in \Gamma_{M,i}$ and hence $\Gamma_{M,i}/(\pm \tilde{O}(T)) \cong \underline{S}_{M,i}/(\pm 1)$.

**Theorem 12.8.** Let $\mathcal{A}$ be an ambi-typical stratum and let $N_{M,i}$ be a moduli space of lattice polarised $K3$ surfaces associated to $\mathcal{A}$. Then the following holds:
(1) If $\mathcal{A}$ is one of the ambi-typical strata different from 38/39, then the dominant map $\mathcal{N}_{M,\ell}^0 \rightarrow \mathcal{A}$ is given by the action of the finite group $\Gamma_{M,\ell}/(\pm \tilde{O}(T))$ which acts faithfully.

(2) Let $\mathcal{A} \cup \overline{\mathcal{A}}$ be the union of the two complex conjugated strata 38/39. Then $\mathcal{N}_{M,\ell}$ has two connected components $\mathcal{N}_{M,\ell}$ and $\overline{\mathcal{N}}_{M,\ell}$ and the dominant map $\mathcal{N}_{M,\ell}^0 \cup \overline{\mathcal{N}}_{M,\ell} \rightarrow \mathcal{A} \cup \overline{\mathcal{A}}$ is given by the group $\Gamma_{M,\ell}/(\pm \tilde{O}(T))$ which acts faithfully.

**Proof.** We start with a surface $S \in \mathcal{A}$ (or $S \in \overline{\mathcal{A}}$). As we have explained before, identifying the fibre components not meeting the 0-section of the singular fibres with simple roots of $R$ and the lattice spanned by the fibre and section with a copy of $U$ we obtain a lattice polarization $i : M \rightarrow NS(S) \subset H^2(S, \mathbb{Z})$. Let $\varphi : H^2(S, \mathbb{Z}) \rightarrow L_{K3}$ be a marking and set $i = \varphi \circ i$. We have to determine when two lattice-polarised $K3$ surfaces in $\mathcal{N}_{M,\ell}$ are mapped to the same point in $\mathcal{A}$ or $\mathcal{A} \cup \overline{\mathcal{A}}$ respectively. Then there is an isomorphism $f : S_2 \rightarrow S_1$ which respects the elliptic fibration. This induces a map $f^* : NS(S_1) \rightarrow NS(S_2)$. Let $\varphi_1 : H^2(S_1; \mathbb{Z}) \rightarrow L_{K3}$ be markings with $i = i_1 \circ \varphi_1$ for $i = 1, 2$. Then $\varphi_2 \circ f^* \circ \varphi_1$ defines an isometry of $M = U + L$. This is the identity on $U$ as the 0-section and the general fibre are mapped to the 0-section and the general fibre respectively. By restriction this then defines an isometry of $L$. Recall from ([12.2]) that $O(L) = W_R \rtimes S_R^{\oplus}$ and from ([12.3]) that $W_R = W_L \subset W_M \subset \tilde{O}^+(L)$. Since the isometry $\varphi_2 \circ f^* \circ \varphi_1|_M$ maps $C_M^{\text{pol}}$ to itself and since the Weyl group acts faithfully on the Weyl chambers it follows that $\varphi_2 \circ f^* \circ \varphi_1|_M$ defines an element in $\Gamma_{M,\ell}$. Conversely, if two lattice polarised $K3$ surfaces in $\mathcal{N}_{N,\ell}^0$ are conjugate under $\Gamma_{M,\ell}$ the underlying elliptic $K3$ surfaces are isomorphic and hence define the same point in $\mathcal{A}$ or $\mathcal{A} \cup \overline{\mathcal{A}}$ respectively. Finally, since $\pm \text{id}_T$ are the only elements in $O(T)$ which act trivially on $\Omega_M$ it follows that the group $\Gamma_{M,\ell}/(\pm \tilde{O}(T))$ acts faithfully on $\mathcal{N}_{M,\ell}$.

**Corollary 12.9.** The following holds:

(1) If $\mathcal{A}$ is an ambi-typical stratum different from 38/39, then the degree of the dominant map $\mathcal{N}_{M,\ell}^0 \rightarrow \mathcal{A}$ is given by

$$|\overline{\mathcal{S}}_M/(\pm 1)|/|\mathcal{M}_T|^c = |\overline{\mathcal{S}}_{M,\ell}/(\pm 1)|.$$  

(2) If $\mathcal{A}$ (or $\overline{\mathcal{A}}$ respectively) is one of the strata 38/39, then the degree of the dominant map $\mathcal{N}_{M,\ell}^0 \cup \overline{\mathcal{N}}_{M,\ell} \rightarrow \mathcal{A} \cup \overline{\mathcal{A}}$ is given by

$$2|\overline{\mathcal{S}}_M/(\pm 1)|/|\mathcal{M}_T|^c = |\overline{\mathcal{S}}_{M,\ell}/(\pm 1)| = 24.$$  

**Proof.** The first equality follows from the fact that $\overline{\mathcal{S}}_M$ acts transitively on the connected components of $M$-polarised $K3$ surfaces with the exception of the cases 38/39 where we have two orbits. The second equality follows from the definition of the group $\overline{\mathcal{S}}_{M,\ell}$ and the constructin of the covering map. 

In the case of 1-dimensional strata one can compute all the data of the covering map from the moduli space of lattice polarised $K3$ surfaces to the ambi-typical stratum. In Table [12] we give these data for all strata of dimension 1 with more than one connected component. We list the case number, the root lattice, the number of connected components,
| Number | root lattice | $|\mathcal{M}_T|^c$ | $|\overline{S}_M/(\pm 1)|$ | $d = |\overline{S}_{M,+}/(\pm 1)|$ | $g$ | $g_{\text{BPT}}$ |
|--------|-------------|----------------|-----------------|-----------------|----|-------------|
| 35     | $D_4 + A_7 + A_4 + 2A_1$ | 2       | 8               | 4               | 0  | 0           |
| 37     | $D_4 + A_7 + 2A_2 + 2A_1$ | 4       | 32              | 8               | 0  | 0           |
| 38/39  | $D_4 + 2A_6 + A_1$         | 2       | 24              | 24              | 1  | 0           |
| 42     | $D_4 + A_6 + A_3 + 2A_2$   | 2       | 96              | 48              | 13 | 0           |
| 44     | $D_4 + 2A_5 + 3A_1$        | 2       | 24              | 12              | 0  | 0           |
| 48     | $D_4 + 2A_4 + 2A_2 + A_1$  | 4       | 192             | 48              | 1  | 0           |
| 49     | $D_4 + 3A_3 + 2A_2$        | 4       | 384             | 96              | 5  | 0           |

Table 13. Covering data for all 1-dimensional ambi-typical strata with more than one component.

the order of the group $\overline{S}_M$ coming from the symmetries of the Dynkin diagram, the degree of the covering map, the genus of the modular curve parameterising the lattice polarised $K3$ surfaces and finally the genus $g_{\text{BPT}}$ of the ambi-typical stratum. The later is always 0 in accordance with [BPT, Theorem] which says that all monodromy strata are rational. We find it remarkable that in contrast the genus of the components of the associated moduli spaces of lattice-polarised $K3$ surfaces can be as high as 13. Note that the group $\overline{S}_M$ is a subgroup of the symmetry group of the Dynkin diagram of the root lattice. If the isotropic group $G$ is trivial, then the two groups coincide. Otherwise the condition that $G$ must be fixed can impose nontrivial extra conditions. This is the case for numbers 37, 48, 49 where $\overline{S}_{M,\iota}$ is a proper subgroup of index 2 of $\overline{S}_M$. Furthermore, in these cases $\overline{S}_{M,\iota}^+$ is also an index 2 subgroup of $\overline{S}_{M,\iota}$. Altogether $\overline{S}_{M,\iota}^+$ can have index 1, 2 or 4 in $\overline{S}_M$. The relevant computations, in particular of the genera $g$, were performed by Markus Kirschmer and are presented in the appendix.

Appendix. Numerical calculations

Markus Kirschmer

In this appendix we summarise some explicit computations concerning the 25 rank 17 lattices in Table 11. All computations were done in MAGMA [Magma] and the complete code is available from [www.math.rwth-aachen.de/~Markus.Kirschmer/magma/K3.html](http://www.math.rwth-aachen.de/~Markus.Kirschmer/magma/K3.html).

Let $L$ be one of the rank 17 lattices in Table 11 and set $M := U \oplus L$. Further let $\iota: M \hookrightarrow L_{K3}$ be a primitive embedding.

**Constructing $T$.** We first need to construct an integral lattice $T$ isometric to $\iota(M)^\perp$. This can be done without constructing an embedding $\iota$ as follows:

Let $(V, q)$ be the ambient quadratic space of $T$. The fact that $M \oplus T$ and $L_{K3}$ lie in isometric quadratic spaces shows that $(V, q)$ has signature $(2, 1)$ and determinant $\det(M)$. It also uniquely determines the Hasse-Witt-invariants of $(V, q)$. Using [Kir Alg. 3.4.3] we can construct a rational quadratic space isometric to $(V, q)$. Let $X$ be a maximal even lattice in $(V, q)$, i.e. $q(X) \subseteq 2\mathbb{Z}$ and no lattice properly containing $X$ has that property, cf. [Kir Alg. 3.5.5]. By [OM Theorem 91:2], the genus of $X$ is unique, thus we may assume that $T \subseteq X$.

For any prime $p$ dividing $\#D(L)$, let $S_p$ be the $p$-Sylow subgroup of $D(L)$. Then

$$\{Y \subseteq X \mid D(Y) \cong S_p \text{ and } \#q_Y^{-1}(\{a\}) = \#q_L^{-1}(\{-a\}) \text{ for all } a \in \mathbb{Q}/2\mathbb{Z}\}$$
consists of a single genus. Let \( Y^{(p)} \) be any representative. By Proposition 11.5, the lattice
\[
T := \bigcap_p Y^{(p)}
\]
is isometric to \( \iota(M)^\perp \).

**Computing \( O(T) \) and its subgroups.** A finite generating set of \( O(T) \) can be constructed using a variation of Voronoi’s algorithm by M.H. Mertens \[ Me \]. A slight modification of Mertens’ algorithms also yields a finite presentation of \( O(T) \) using Bass-Serre theory \[ BCNS \]. This modification was provided to us by S. Schönnenbeck.

The group \( \hat{O}(T) \) is the kernel of the homomorphism \( \pi_T: O(T) \to O(D(T), q_T) \). Since the group \( O(D(T), q_T) \) is finite, we can construct a finite generating set of \( O(T) \) using the standard orbit stabiliser algorithm. Similarly, the spinor norm map \( \text{sn}_R: O(T) \to \{±1\} \) yields finite generating sets for \( O^+(T) \) and \( \hat{O}^+(T) \). Note that spinor norms can be computed using Zassenhaus’ trick \[ Za \]. But one has to keep in mind that our normalization of the spinor norm on \( (T, q) \) corresponds to Zassenhaus’ spinor norm on the lattice \( (T, -q) \).

Next we want to compute generators for the group \( \Gamma_{M,\iota} \), cf. equation (12.0). The group \( D(M) \) is finite and so is its automorphism group \( \text{Aut}(D(M)) \). Thus the subgroup
\[
O(D(M), q_M) = \{ f \in \text{Aut}(D(M)) \mid q_M(f(\mathbf{x})) = q_M(\mathbf{x}) \text{ for all } \mathbf{x} \in D(M) \}
\]
can be enumerated by brute force. Similarly, we enumerate the subgroup \( \overline{S}_M \subseteq O(D(M), q_M) \) induced by the automorphism group of the Dynkin diagram of the root lattice \( R \). If \( \overline{S}_M = O(D(M), q_M) \), then \( \Gamma_{M,\iota} = O(T) \). Suppose now \( \overline{S}_M \not\subseteq O(D(M), q_M) \). In these cases, it just happens that \( \pi_T: O(T) \to O(D(T), q_T) \) is onto. We start by computing any isomorphy \( \alpha: (D(M), q_M) \to (D(T), -q_T) \) using a backtrack approach. This gives us an isomorphism \( O(D(T)) \cong O(D(M)) \). The fact that \( O(T) \to O(D(T), q_T) \) is onto shows that there exists some \( f \in O(T) \) such that \( \pi_T(f) \circ \alpha = \alpha_\iota \). In particular, the pre-image of \( \overline{S}_{M,\iota} \) under \( O(T) \to O(D(T)) \cong O(D(M)) \) must be conjugate to \( \Gamma_{M,\iota} \) and we find generators for this group using the orbit stabiliser algorithm.

**Fuchsian groups.** Let \( SO^+(T) = \{ \varphi \in O^+(T) \mid \det(\varphi) = 1 \} \). We denote by \( SO^+(2,1) \) the connected component of the special orthogonal group of a real quadratic space of signature \( (2,1) \). The sporadic isomorphism between \( SO^+(2,1) \) and \( \text{PSL}(2,\mathbb{R}) \) implies that the type IV homogeneous domain associated to \( T \) is isomorphic to the upper half plane \( \mathbb{H}_1 \). Moreover, it induces an injection \( \text{SO}^+(T) \hookrightarrow \text{PSL}(2,\mathbb{R}) \) and thus an action of \( \text{SO}^+(T) \) on the upper half plane \( \mathbb{H}_1 \). This action is properly discontinuous. Hence any finite index subgroup \( G \) of \( \text{SO}^+(T) \) is a Fuchsian group, cf. \[ Ka \] Theorem 2.2.6].

Suppose \( G \) is a finite index subgroup of \( \text{SO}^+(T) \) and let \( g := g(\mathbb{H}_1/G) \) be the genus of the (compactified) curve \( \mathbb{H}_1/G \). By \[ Ka \] Section 4.3, the group \( G \) admits a presentation
\[
(A.1) \quad G \cong \left\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_d, y_1, \ldots, y_l \mid x_1^{m_1} = \ldots = x_d^{m_d} = 1 \text{ and } x_1 \cdots x_d y_1 \cdots y_l [a_1, b_1] \cdots [a_g, b_g] = 1 \right\rangle
\]
with \( t = 0 \) if and only if \( \mathbb{H}_1/G \) is compact.

Since the index of \( G \) in \( O(T) \) is finite, we can obtain a finite presentation of \( G \) from the presentation of \( O(T) \) using the Reidemeister-Schreier method \[ MKS \] Section 2.3]. Even though this presentation might not be in the form of eq. (A.1), it is good enough to determine the genus of \( \mathbb{H}_1/G \):
Lemma A.1. Let $L$ be one of the rank 17 lattices of Table 11, and let $\iota: L \oplus U \to L_{K3}$ be a primitive embedding. Further, let $T := \iota(L \oplus U)^\bot$ and let $G$ be a finite index subgroup of $SO^+(T)$.

1. The space $\mathbb{H}_1/G$ is compact.
2. The torsion free part of $G/G'$ has rank $2g(\mathbb{H}_1/G)$.

Proof. It suffices to prove the first statement for $G = SO^+(T)$. An explicit computation shows that the abelian group $SO^+(T)/SO^+(T)' \cong (\mathbb{Z}/2\mathbb{Z})^r$ is an elementary abelian 2-group. Suppose $\mathbb{H}_1/\mathbb{O}^+(T)$ is not compact, i.e. the parameter $t$ in eq. (A.1) is non-zero. The isomorphism type of $SO^+(T)/SO^+(T)'$ implies that $g(\mathbb{H}_1/\mathbb{O}^+(T)) = 0$ and $SO^+(T) \cong \mathbb{Z}/2\mathbb{Z} \ast \ldots \ast \mathbb{Z}/2\mathbb{Z}$ is a free product of $r$ copies of $\mathbb{Z}/2\mathbb{Z}$. An explicit computation shows that for all lattices $T$, the groups $SO^+(T)$ and $\mathbb{Z}/2\mathbb{Z} \ast \ldots \ast \mathbb{Z}/2\mathbb{Z}$ have different numbers of subgroups of small index. This proves the first assertion.

The second assertion follows immediately from the fact that the parameter $t$ in eq. (A.1) is zero as $\mathbb{H}_1/G$ is compact. □

Suppose now $G$ is a subgroup of $O^+(T)$. We denote by $\mathbb{H}_1/G$ the space $(\mathbb{H}_1/\pm G \cap \mathbb{O}^+(T))$, where $\pm G$ is the subgroup of $O(T)$ generated by $G$ and $-I$. Results. For each rank 17 lattice in Table 11 we constructed a lattice $T$ as well as generating sets for $O(T)$, $O^+(T)$, $\mathbb{O}(T)$ and $\mathbb{O}^+(T)$. It turns out that in all cases $[O(T) : O^+(T)] = 2$ and $g(\mathbb{H}_1/O^+(T)) = 0$. Table 14 lists the indices $I_1 := [O^+(T) : \mathbb{O}^+(T)]$, $I_2 := [\mathbb{O}(T) : \mathbb{O}^+(T)]$ and $I_3 := [O(D(T)) : O(T)]$, the order of $\mathbb{M}_{M,\iota}/(\pm 1)$ and the genus of the moduli space of lattice polarised $K3$ surfaces. We note that in all but three cases $\mathbb{M}_{M,\iota}/(\pm 1) = \mathbb{M}_{M,\iota}/(\pm 1)$. The only cases where this is not the case are 37, 48 and 49 where $\mathbb{M}_{M,\iota}/(\pm 1)$ has index 2 in $\mathbb{M}_{M,\iota}/(\pm 1)$. Furthermore, in these cases, as well as in 35, 42 and 44 the group $\mathbb{M}_{M,\iota}/(\pm 1)$ has index 2 in $\mathbb{M}_{M,\iota}/(\pm 1)$.

There are only four lattices $T$ for which $\Gamma_{M,\iota} \neq O(T)$. In these cases, the genus of $\hat{C}_T := \mathbb{H}_1/\Gamma_{M,\iota}^+$, which is birational to the ambitypical stratum, is again 0, once more in concordance with [BPT] Theorem. Table 15 lists the indices $[O^+(T) : \Gamma_{M,\iota}^+]$ and $[\Gamma_{M,\iota}^+ : \Gamma_{M,\iota}^+]$ in these cases.
| #  | root lattice | $I_1$ | $I_2$ | $I_3$ | $[S_M/(\pm 1)]$ | $g(\mathbb{H}_1/\tilde{O}^+(T))$ |
|----|-------------|-----|-----|-----|----------------|-----------------|
| 24 | $D_4 + A_{13}$ | 12  | 2   | 1   | 6              | 1               |
| 25 | $D_4 + A_{12} + A_1$ | 12  | 2   | 1   | 6              | 1               |
| 26 | $D_4 + A_{11} + A_2$ | 16  | 2   | 1   | 4              | 0               |
| 27 | $D_4 + A_{11} + 2A_1$ | 8   | 2   | 1   | 4              | 0               |
| 28 | $D_4 + A_{10} + A_2 + A_1$ | 24  | 2   | 1   | 12             | 1               |
| 29 | $D_4 + A_9 + A_4$ | 72  | 2   | 1   | 12             | 4               |
| 30 | $D_4 + A_9 + A_3 + A_1$ | 8   | 2   | 1   | 4              | 1               |
| 31 | $D_4 + A_9 + 2A_2$ | 96  | 2   | 1   | 48             | 7               |
| 32 | $D_4 + A_9 + A_2 + 2A_1$ | 8   | 2   | 1   | 4              | 0               |
| 33 | $D_4 + A_8 + A_5$ | 72  | 2   | 1   | 12             | 4               |
| 34 | $D_4 + A_8 + A_4 + A_1$ | 24  | 2   | 1   | 12             | 1               |
| 35 | $D_4 + A_7 + A_4 + 2A_1$ | 8   | 1   | 1   | 8              | 0               |
| 36 | $D_4 + A_7 + A_3 + A_2 + A_1$ | 16  | 2   | 1   | 8              | 1               |
| 37 | $D_4 + A_7 + 2A_2 + 2A_1$ | 32  | 1   | 2   | 32             | 0               |
| 38/39 | $D_4 + 2A_6 + A_1$ | 48  | 1   | 1   | 24             | 1               |
| 40 | $D_4 + A_6 + A_5 + A_2$ | 48  | 2   | 1   | 24             | 7               |
| 41 | $D_4 + A_6 + A_4 + A_2 + A_1$ | 48  | 2   | 1   | 24             | 0               |
| 42 | $D_4 + A_6 + A_3 + 2A_2$ | 96  | 1   | 1   | 96             | 13              |
| 43 | $D_4 + 2A_5 + A_3$ | 32  | 2   | 1   | 16             | 3               |
| 44 | $D_4 + 2A_5 + 3A_1$ | 24  | 1   | 1   | 24             | 0               |
| 45 | $D_4 + A_5 + 2A_2$ | 96  | 2   | 1   | 48             | 7               |
| 46 | $D_4 + A_5 + A_4 + A_3 + A_1$ | 16  | 2   | 1   | 8              | 1               |
| 47 | $D_4 + A_5 + 2A_3 + 2A_1$ | 16  | 2   | 1   | 8              | 0               |
| 48 | $D_4 + 2A_4 + 2A_2 + A_1$ | 192 | 1   | 2   | 192            | 1               |
| 49 | $D_4 + 3A_3 + 2A_2$ | 384 | 1   | 2   | 384            | 5               |

| # | root lattice | $[O^+(T) : \tilde{O}^+(T)]$ | $\Gamma_{M,\ell} : \Gamma_{M,\ell}^+$ |
|----|-------------|--------------------------|----------------------------------|
| 26 | $D_4 + A_{11} + A_2$ | 2                        | 2                                |
| 29 | $D_4 + A_9 + A_4$ | 3                        | 2                                |
| 33 | $D_4 + A_8 + A_5$ | 3                        | 2                                |
| 38/39 | $D_4 + 2A_6 + A_1$ | 1                        | 1                                |

Table 14. Indices $I_1 = [O^+(T) : \tilde{O}^+(T)]$, $I_2 = [\tilde{O}(T) : \tilde{O}^+(T)]$, $I_3 = [O(D(T)) : O(T)]$, order of $S_{M,\ell}/(\pm 1)$ and genus of the moduli space of lattice polarised $K3$ surfaces.

Table 15. Indices of $\Gamma_{M,\ell}^+$ in $O^+(T)$ and $\Gamma_{M,\ell}$.

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