GEOMETRY AND ANALYSIS OF CONTACT INSTANTONS AND
ENTANGLEMENT OF LEGENDRIAN LINKS

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Abstract. The purposes of the present paper are two-fold. Firstly we further develop the interplay between the contact Hamiltonian geometry and the geometric analysis of the new analytical machinery of Hamiltonian-perturbed contact instantons with the Legendrian boundary condition in the study of contact dynamics and topology. We introduce the class of tame contact manifolds \((M, \lambda)\), which includes compact ones but not necessarily compact, and establish uniform a priori \(C^0\)-estimates for the contact instantons. Then we study the problem of estimating the Reeb-untangling energy of one Legendrian submanifold from another, and formulate a particularly designed parameterized moduli space for the study of the problem. We establish the Gromov-Floer-Hofer type convergence results of contact instantons of finite energy and construct its compactification of the moduli space. We do this first by defining the correct energy and then by carrying out bubbling-off analysis and proving uniform a priori energy bounds in terms of the relevant contact Hamiltonian. Secondly, as an application of this geometry and analysis of contact instantons, we prove that the self Reeb-untangling energy of a compact Legendrian submanifold \(R\) in any tame contact manifold \((M, \lambda)\) is greater than that of the period gap \(T_\lambda(M, R)\) of the Reeb chords of \(R\). This is an optimal result in general. In a sequel [Oh22], we also prove Shelukhin’s conjecture specializing to the Legendrianization of contactomorphisms and utilizing its \(Z_2\)-symmetry argument as the fixed point set of anti-contact involution.

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1. Introduction

The general Lagrangian Floer theory in symplectic geometry concerns intersections of Lagrangian submanifolds which largely relies on the study of the moduli spaces of solutions of Hamiltonian-perturbed pseudoholomorphic curves under the Lagrangian boundary condition with finite energy (and of bounded image in addition when the ambient space is noncompact).

In this paper and its sequels, we develop a contact analog to such a theory for the study of contact dynamics which concerns entanglement of Legendrian links...
which relies on the study of the moduli spaces of Hamiltonian-perturbed contact instantons under the Legendrian boundary condition. This nonlinear elliptic boundary value problem was introduced by the present author in [Oh21a] which is based on the analytic study of contact instantons provided in [OW18a, OW18b, Oh21a]. More importantly in ibid, we proved a fundamental vanishing result of the asymptotic charge for finite π-energy contact instantons with Legendrian boundary conditions, which eliminates the phenomenon of the occurrence of spiraling cusp instantons along a Reeb core. This enables us to carry out the Gromov-Floer-Hofer style compactification and the relevant Fredholm theory of the associated moduli space. The present paper and its sequels [Oh22], [Ohb], [OY] are based on this geometric analysis of perturbed contact instantons and the relevant contact geometry and Hamiltonian calculus.

**Remark 1.1.** The equation (with $H = 0$) itself, what we call the contact instanton equation, was first introduced by Hofer [Hof00, p.698] and utilized in [ACH05], [Abb11] in their attempts to attack Weinstein’s conjecture in three dimensions.

Along the way, we also develop the analytic machinery of Hamiltonian-perturbed contact instantons and illustrate its application to a quantitative study of contact dynamics. The heart of the matter lies in the quantitative study of the moduli space of perturbed contact instantons and its interplay with the contact Hamiltonian geometry and calculus. Leaving a full systematic study of perturbed contact instantons and the aforementioned quantitative entanglement study of general Legendrian links in future works, we utilize contact instantons with Legendrian boundary conditions as its probes, and prove an optimal result on the contact displacement problem of a compact Legendrian submanifold as an application of a quantitative investigation of self-entanglement of a compact Legendrian submanifold in the present paper. In a sequel [Oh22], we specialize to the Legendrianization of contactomorphisms of general compact contact manifold and improve Theorem 1.12 by a factor of 2 utilizing a $\mathbb{Z}_2$-symmetry argument.

The identification of correct energy and the relevant compactification of the moduli space developed in Part 3 of the present paper is also the analytic basis for the Floer theoretic construction of Legendrian spectral invariants via contact instanton Floer-type homology given in [OY] for the one-jet bundle, and other applications prepared in [Ohb] and other sequels. Our study provides a flexible analytic tool for general systematic quantitative study of the contact dynamics and topology through the geometric analysis of perturbed contact instantons.

### 1.1. Legendrian isotopy and Reeb chords.

We will always consider cooriented contact manifolds $(M, \xi)$ which admits associated contact forms $\lambda$ satisfying $\ker \lambda = \xi$. We denote by

$$\mathcal{C}(\xi) = \mathcal{C}(M, \xi)$$

(1.1)

the set of contact forms $\lambda$ of $\xi$, i.e., of those satisfying $\ker \lambda = \xi$ and $\lambda \wedge (d\lambda)^n$ is a volume form of $M = M^{2n+1}$.

Contact manifolds equipped with a contact form carry canonical background Reeb dynamics: When $\lambda$ is a contact form of $\xi$, it uniquely defines a contact vector field $R_\lambda$ called the Reeb vector field by the defining condition

$$R_\lambda | d\lambda = 0, R_\lambda | \lambda = 1.$$  

(1.2)
Unlike the Lagrangian submanifolds in symplectic background, the generic characteristic of a Legendrian link is its entanglement structure relative to this background dynamics.

One of the purposes of the present paper is to make the first step towards a systematic quantitative study of this entanglement structure. With this long-term goal of investigation in our minds, we formulate the general problem in this introduction in the scope wider than that of what we actually investigate in the present paper which mainly deals with a two-component link of the type \((\psi(R), R)\) for an arbitrary contactomorphism \(\psi\).

A Legendrian link is a finite disjoint union

\[
R = \bigsqcup_{i=1}^{\ell} R_i
\]

of connected Legendrian submanifolds \(R_i\). We call each \(R_i\) a component of the link \(R\).

**Definition 1.2.** Consider a co-oriented contact manifold \((M, \xi)\) and a Legendrian link \(R\). Let \(\lambda \in \mathcal{C}(\xi)\). A curve \(\gamma : [0, T] \to M\) satisfying

\[
\begin{cases}
\dot{\gamma} = R_\lambda(\gamma(t)), \\
\gamma(0), \gamma(T) \in R
\end{cases}
\]

is called a Reeb chord of \(R\). We call \(\gamma\) a self-chord if its initial and final points land at the same component, and a trans-chord otherwise. We denote by \(\Reeb(M, R; \lambda)\) the set of \(\lambda\)-Reeb chords of \(R\).

Here is the simplest form of nontrivial entanglement.

**Definition 1.3.** Let \((M, \xi)\) be a contact manifold and \(\lambda\) a contact form. We say a Legendrian link \(R\) is dynamically \(\lambda\)-entangled if there exists a Reeb chord of \(R\), i.e., \(\Reeb(M, R; \lambda) \neq \emptyset\).

We say a chord \(\gamma\) primary if \(\gamma((0, T))\) does not intersect \(R\).

**Definition 1.4.** For a given Legendrian link \(R\), we define the \(\lambda\)-chord period gap (or simply a \(\lambda\)-chord gap) \(T(M, R; \lambda)\) of \(R\) is defined by

\[
T(M, R; \lambda) := \inf \left\{ \int_{\gamma} \lambda \mid \lambda \in \Reeb(M, R; \lambda), \gamma \text{ is primary} \right\}.
\]

Obviously if \(R\) is compact, \(T(M, R; \lambda) > 0\).

This definition is a direct generalization to the links of the following standard definition \(T(M, R; \lambda)\) below in contact geometry.

**Definition 1.5.** Let \(\lambda\) be a contact form of contact manifold \((M, \xi)\) and \(R \subset M\) a connected Legendrian submanifold. Denote by \(\Reeb(M, \lambda)\) (resp. \(\Reeb(M, R; \lambda)\)) the set of closed Reeb orbits (resp. the set of self Reeb chords of \(R\)).

1. We define \(\text{Spec}(M, \lambda)\) to be the set

\[
\text{Spec}(M, \lambda) = \left\{ \int_{\gamma} \lambda \mid \lambda \in \Reeb(M, \lambda) \right\}
\]

and call the action spectrum of \((M, \lambda)\).

2. We define the period gap to be the constant given by

\[
T(M, \lambda) := \inf \left\{ \int_{\gamma} \lambda \mid \lambda \in \Reeb(M, \lambda) \right\} > 0.
\]
We define \( \text{Spec}(M, R; \lambda) \) and the associated \( T(M, R; \lambda) \) similarly using the set \( \mathcal{R}_{\text{Reeb}}(M, R; \lambda) \) of Reeb chords of \( R \).

We set \( T(M, \lambda) = \infty \) (resp. \( T(M, R; \lambda) = \infty \)) if there is no closed Reeb orbit (resp. no \((R_0, R_1)\)-Reeb chord). Then we define

\[
T_\lambda(M; R) := \min\{T(M, \lambda), T(M, R; \lambda)\}
\]

and call it the \((\text{chord}) \) period gap of \( R \) in \( M \).

In relation to our solution to the conjecture of Sandon and Shelukhin [San12, She17], we consider the pair \((R_0, R_1)\) of the type \( R_0 = \psi(R) \), \( R_1 = R \) for a contactomorphism \( \psi \) and prove a result in the simplest level of dynamical entanglement, the existence of a Reeb chord of a two-component link \((\psi(R), R)\), by studying the moduli space of contact instantons intertwining the link, and give its application to the aforementioned conjecture.

We denote by \( \mathcal{L}_{\text{eg}}(M, \xi) \) the set of Legendrian submanifold and by \( \mathcal{L}_{\text{eg}}(M, \xi; R) \) its connected component containing \( R \in \mathcal{L}_{\text{eg}}(M, \xi) \), i.e, the set of Legendrian submanifolds Legendrian isotopic to \( R \). We denote by

\[
\mathcal{P}(\mathcal{L}_{\text{eg}}(M, \xi))
\]

the monoid of Legendrian isotopies \([0, 1] \to \mathcal{L}_{\text{eg}}(M, \xi)\). We have natural evaluation maps

\[
ev_0, \ ev_1 : \mathcal{P}(\mathcal{L}_{\text{eg}}(M, \xi)) \to \mathcal{L}_{\text{eg}}(M, \xi)
\]

and denote by

\[
\mathcal{P}(\mathcal{L}_{\text{eg}}(M, \xi), R) = \ev_0^{-1}(R) \subset \mathcal{P}(\mathcal{L}_{\text{eg}}(M, \xi))
\]

and

\[
\mathcal{P}(\mathcal{L}_{\text{eg}}(M, \xi), (R_0, R_1)) = (\ev_0 \times \ev_1)^{-1}(R_0, R_1) \subset \mathcal{P}(\mathcal{L}_{\text{eg}}(M, \xi)).
\]

**Notation 1.6.** Let \( H = H(t, x) \) be a given contact Hamiltonian.

1. We denote by \( \psi_H : t \mapsto \psi_H^t \) the contact Hamiltonian path generated by \( H \).
2. When \( \psi \in \text{Cont}(M, \xi) \), we denote by \( H \mapsto \psi \) when \( \psi = \psi_H^1 \).
3. When \( R_0 = R \) and \( R_1 = \psi(R) \) for a contactomorphism contact isotopic to the identity, the Hamiltonian path \( \psi_H \) defines a natural Legendrian isotopy

\[
\mathcal{R}_{H; R} : t \mapsto \psi_H^t(R).
\]

We denote by

\[
\mathcal{P}(\mathcal{L}_{\text{eg}}(M, \xi), R; H) \subset \mathcal{P}(\mathcal{L}_{\text{eg}}(M, \xi), (\psi(R), R))
\]

the set of Legendrian isotopies which is homotopic to the path \( \mathcal{R}_{H; R} \) relative to the ends.

**1.2. Tame contact manifolds and maximum principle.** Now we specify what kind of contact manifolds we will take as the background geometry in the present paper. Obviously all compact ones will be included which however will not be enough for the proof of Shelukhin’s conjecture given in [Oh22] which inevitably involve noncompact contact manifolds of the type \( Q \times Q \times \mathbb{R} \).

Our introduction of the following class of noncompact contact manifolds is largely motivated to make the relevant contact manifolds are amenable to the maximum principle in the study of contact instantons, especially to include the one-jet bundle
We first introduce the following type of barrier functions.

**Definition 1.7 (Reeb-tame functions)**. Let \((M, \lambda)\) be a contact manifold. A function \(\psi : M \to \mathbb{R}\) is called \(\lambda\)-tame on an open subset \(U \subset M\) if \(\mathcal{L}_{R_\lambda} d\psi = 0\) on \(M \setminus K\) on \(U\).

**Definition 1.8 (Contact J quasi-pseudoconvexity)**. Let \(J\) be a \(\lambda\)-adapted CR almost complex structure. We call a nonnegative function \(\psi : M \to \mathbb{R}\) contact \(J\) quasi-plurisubharmonic on \(U\) if it satisfies

\[
- d(d\psi \circ J) + d\psi \wedge \beta \geq 0 \quad \text{on } \xi, \tag{1.4}
\]

\[
R_\lambda d(d\psi \circ J) = g d\psi \tag{1.5}
\]

for some smooth one-form \(\beta\) and a smooth function \(g\) on \(U\). We call such a pair \((\psi, J)\) a contact quasi-pseudoconvex pair on \(U\).

Here the inequality (1.4) needs some explanation: We recall a CR almost complex structure is an endomorphism \(J(\xi) \subset \xi\) that satisfies

\[
J(R_\lambda) = 0 \quad J^2 = -id|_\xi \oplus 0.
\]

In particular \(J(TM) \subset \xi\). The meaning of (1.4) is that

\[
- d(d\psi \circ J) + d\psi \wedge \beta = h d\lambda \tag{1.6}
\]

on \(\xi\) for some nonnegative function \(h \geq 0\).

**Remark 1.9.** In other words, the two form \(-d(d\psi \circ J)\) (if \(\beta = 0\)) we are considering in (1.4) corresponds to the real almost complex version of the standard Levi-form in several complex variables, when \(M\) is a CR manifold of the hypersurface-type. The condition (1.4) in particular implies pseudoconvexity of the hypersurface at every critical point of \(\psi\). The condition (1.5) is an additional requirement that involves contact geometry. This is responsible for our naming of ‘contact \(J\) quasi-convexity’ and ‘contact quasi-pseudoconvex pairs’. Similar notion of such a pair is also utilized in [Oh21b] in the study of Liouville sectors introduced in [GPS17, GPS18] in symplectic geometry.

The upshot of introducing this kind of barrier functions on contact manifold is the following amenability of the maximum principle to the pair \((\psi, J)\) in the study of contact instantons.

**Theorem 1.10 (Theorem 6.1)**. Let \((M, \xi)\) be a contact manifold and consider the contact triad \((M, \lambda, J)\) associated to it. Let \(\psi\) be a \(\lambda\)-tame contact \(J\) quasi-plurisubharmonic function. Then for any contact instanton \(w : \hat{\Sigma} \to M\) for the triad \((M, \lambda, J)\), the composition \(\psi \circ w\) is a quasi-subharmonic function, i.e., satisfies

\[
\Delta(\psi \circ w) dA + d(\psi \circ w) \wedge \beta \geq 0
\]

for some one-form \(\beta\) on \(\hat{\Sigma}\). In particular, the maximum of \(\psi \circ w\) cannot be achieved on the interior of \(\hat{\Sigma}\). The same holds for the time-dependent contact triad \((M, \lambda_t, J_t)\) and time-dependent \(g_t, \beta_t\).

Motivated by this analytical fact, we introduce the following class of contact manifolds.
Definition 1.11 (Tame contact manifolds). Let \((Q, \xi)\) be a contact manifold, and let \(\lambda\) be a contact form of \(\xi\).

1. We say \(\lambda\) is tame on \(U\) if \((M, \lambda)\) admits a pair \((\psi, J)\) of a \(\lambda\)-adapted CR almost complex structure \(J\) and a \(\lambda\)-tame contact \(J\) quasi-plurisubharmonic exhaustion function \(\psi\) on \(U\).

2. We call an end of \((M, \lambda)\) tame if \(\lambda\) is tame on the end of \(M\).

We say an end of contact manifold \((M, \xi)\) (resp. \((M, \xi)\)) is tame if it admits contact form \(\lambda\) that is tame on the end (resp. at infinity) of \(M\).

The one-jet bundle is tame while the contact product \(Q \times Q \times \mathbb{R}\) is tame for the end with \(\eta > 0\). (See [OY] and [Oh22] for their proofs respectively.)

1.3. Dynamical untangledmnet of Legendrian submanifolds. The first main result of the present paper is the following existence theorem of Reeb chords between \(\psi(R)\) and \(R\) for any compact Legendrian submanifold \(R\) on tame contact manifolds, as an application of the analytic framework of perturbed contact instantons.

Theorem 1.12 (Theorems 10.5 & 10.6). Let \((M, \xi)\) be a contact manifold equipped with a tame contact form \(\lambda\). Let \(\psi \in \text{Cont}_0(M, \xi)\) and consider any compactly supported Hamiltonian \(H = H(t, x)\) with \(H \mapsto \psi\). Assume \(R\) is any compact Legendrian submanifold of \((M, \xi)\). Then the following hold:

1. Provided \(|H| \leq T_{\lambda}(M, R)\), we have
\[
\#\text{Reeb}(\psi(R), R) \neq \emptyset.
\]

2. Provided \(|H| < T_{\lambda}(M, R)\) and \(\psi = \psi^1_H\) is nondegenerate to \((M, R)\), then
\[
\#\text{Reeb}(\psi(R), R) \geq \dim H^*(R; \mathbb{Z}_2).
\]

Remark 1.13. This result is optimal in general: We refer readers to [RS20, Example 1.4 & Lemma 1.6] for an example that shows that Theorem 1.12 is optimal on \(\mathbb{R}^5\) equipped with the standard contact structure. We thank Dylan Cant for pointing out this example to the author.

We would like to readers’ attention that when \(\psi = \text{id}\), we \(\psi(R) \cap R = R\) and so have plenty of constant translated intersection points. To uniformly relate the constant paths to Reeb chords in our study of translated intersection points via contact instantons, the following representation of Reeb chords is useful.

Definition 1.14 (Iso-speed Reeb chords). Let \(R_0, R_1\) be two Legendrian submanifolds, not necessarily disjoint, and consider a curve \(\gamma: [0, 1] \to M\) fixed domain.

We say a pair \((T, \gamma)\) a iso-speed Reeb chord from \(R_0\) to \(R_1\) if it satisfies
\[
\begin{cases}
\dot{\gamma}(t) = TR_\lambda(\gamma(t)) \\
\gamma(0) \in R_0, \quad \gamma(1) \in R_1.
\end{cases}
\]

We say an iso-speed Reeb chord \((T, \gamma)\) is nonnegative if \(T \geq 0\) and negative if \(T < 0\). We denote by \(\mathcal{X}(R_0, R_1)\) the set of nonnegative iso-speed Reeb chords from \(R_0\) to \(R_1\).

Remark 1.15. (1) We highlight that the set of nonnegative iso-speed Reeb chords includes constant curves when \(R_0 \cap R_1 \neq \emptyset\), i.e., allows \(T\) to be 0, while \(T\) in Definition 1.2 must be positive. This set will play the role of generators of the contact instanton Floer complex we utilize in relation
to the proof of Sandon-Shelukhin’s conjecture for the nondegenerate case. (See Theorem 10.6.) By including constant curves enables us to extend the definition of contact instanton Floer homology to the case of Morse-Bott situation e.g., the case with \( R_0 = R_1 = R \) as in the Lagrangian intersection Floer homology theory as in [FOOO09]. The existence result proved in Theorem 1.12 essentially follows from the fact that we have a continuum of solution for this Morse-Bott case.

(2) The \( \gamma \) in the definition of a iso-speed Reeb chord also naturally arises as an asymptotic limit of a finite \( \pi \)-energy contact instanton (Theorem 5.15).

Theorem 1.12 motivates us to introduce the following notion of Reeb-untangling energy of one subset from the Reeb trace of the other: We call the following union

\[
Z_S := \bigcup_{t \in \mathbb{R}} \phi_{R_3}^t(S)
\]

(1.8)

the Reeb trace of a subset \( S \subset M \).

**Definition 1.16.** Let \((M, \xi)\) be a contact manifold, and let \( S_0, S_1 \) of compact subsets \((M, \xi)\).

1. We define

\[
e^{\text{trn}}(S_0, S_1) := \inf_{H} \{ \|H\| \mid \psi_H^1(S_0) \cap Z_{S_1} = \emptyset \}.
\]

(1.9)

We put \( e^{\text{trn}}(S_0, S_1) = \infty \) if \( \psi_H^1(S_0) \cap Z_{S_1} \neq \emptyset \) for all \( H \). We call \( e^{\text{trn}}(S_0, S_1) \) the \( \lambda \)-untangling energy of \( S_0 \) from \( S_1 \) or just of the pair \((S_0, S_1)\).

2. We put

\[
e^{\text{trn}}(S_0, S_1) = \sup_{\lambda \in \mathfrak{e}(\xi)} e^{\text{trn}}_{\lambda}(S_0, S_1).
\]

(1.10)

We call \( e^{\text{trn}}(S_0, S_1) \) the Reeb-untangling energy of \( S_0 \) from \( S_1 \) on \((M, \xi)\).

We mention that the quantity \( e^{\text{trn}}_{\lambda}(S_0, S_1) \) is not symmetric, i.e.,

\[
e^{\text{trn}}_{\lambda}(S_0, S_1) \neq e^{\text{trn}}_{\lambda}(S_1, S_0)
\]

in general.

Theorem 1.12 implies

\[
e^{\text{trn}}_{\lambda}(R, R) \geq T_{\lambda}(M, R) > 0
\]

for all compact Legendrian submanifolds \( R \).

1.4. **Hamiltonian-perturbed contact instantons.** Recall a contact form \( \lambda \) admits a decomposition \( TM = \xi \oplus \mathbb{R}(R_\lambda) \). We denote the associated projection to \( \xi \) by \( \pi : TM \to \xi \) and decompose

\[
v = v^\pi + \lambda(v) R_\lambda, \quad v^\pi := \pi(v).
\]

**Definition 1.17 (Contact triad [OW14]).** Let \((M, \xi)\) be a contact manifold, and \( \lambda \) be a contact form of \( \xi \). An endomorphism \( J : TM \to TM \) is called a \( \lambda \)-adapted CR-almost complex structure if it satisfies

1. \( J(\xi) \subset \xi \), \( JR_\lambda = 0 \) and \( J^2_{\xi} = -id|_{\xi} \),
2. \( g_\xi := d\lambda(\cdot, J\cdot)|_{\xi} \) defines a Hermitian vector bundle \((\xi, J_\xi, g_\xi)\).

We call the triple \((M, \lambda, J)\) a contact triad.
For given such a triad, we first decompose any $TM$-valued one-form $\Xi$ on a Riemann surface $(\Sigma, j)$ into

$$\Xi = \Xi^\pi + \lambda(\Xi) R$$

and then we further decompose

$$\Xi^\pi = \Xi^{\pi(1,0)} + \Xi^{\pi(0,1)}$$

into $J$ linear and $J$ anti-linear parts of $\Xi$.

We now consider $(M, \lambda, J)$ is a contact triad for the contact manifold $(M, \xi)$, and equip with it the contact triad metric

$$g = d\lambda(J \cdot, J \cdot) + \lambda \otimes \lambda.$$ 

In terms of this splitting the contact Hamilton’s equation can be decomposed

$$\dot{x} = X_H(t, x) \iff \begin{cases} (\dot{x} - X_H(t, x))^\pi = 0 \\ \gamma^*(\lambda + H dt) = 0 \end{cases}$$

into the $\xi$-component and the Reeb component of the equation. (See [Oh21a].) We now introduce a Hamiltonian-perturbed contact instanton equation that is introduced in [Oh21a] as the contact counterpart of the celebrated Floer’s Hamiltonian-perturbed Cauchy-Riemann equation in symplectic geometry. For this purpose, we take the following notation throughout the paper.

**Notation 1.18.** In the rest of the paper, $J_0$ always stands for the usual (time-independent) CR-almost complex structure $J_0 = J_0(x)$, $x \in M$, $J$ stands for the domain-dependent one, e.g., $J = \{J_t\}_{t \in [0,1]}$ or $J = \{J_{(s,t)}\}_{(x,t) \in [0,1]^2}$ or even $J = \{J_z\}_{z \in \Sigma}$ for some bordered Riemann surface $\Sigma$. On noncompact contact manifolds, we always assume that outside a compact subset of $M$, there exists a compact subset $K \subset M$ such that pair $(H, J)$ satisfies

$$\text{supp} H \subset K, \quad J_z \equiv J_0$$

for a fixed $\lambda$-adapted CR-almost complex structure $J_0$.

The requirement on $(H, J)$ in (1.12) can be always assumed for the purpose of studying Sandan-Shelukhin type problem of untangling Legendrian submanifolds.

**Definition 1.19** ([Oh21a]). Let $(M, \lambda)$ be a contact manifold equipped with a contact form, and consider the (time-dependent) contact triad

$$(M, \lambda, J), \quad J = \{J_t\}_{t \in [0,1]}.$$ 

Let $H = H(t, x)$ be a time-dependent Hamiltonian. We say $u : \mathbb{R} \times [0,1] \to M$ is a $X_H$-perturbed Legendrian Floer trajectory if it satisfies

$$\begin{cases} (du - X_H \otimes dt)^\pi(0,1) = 0, \\ d(g_H(u)(u^*\lambda + H dt) \circ j) = 0 \\ u(\tau, 0) \in R, \quad u(\tau, 1) \in R \end{cases}$$

where the function $g_H : \mathbb{R} \times [0,1] \to \mathbb{R}$ is defined by

$$g_H(t, x) := g_H^t(\psi_H^{-1}(u(t, x))).$$

Here we refer to Subsection 5.1 for the explanation and the perspective of the transformation $u \mapsto \overline{u}$ where

$$\overline{u}(\tau, t) := (\psi_H^t(\psi_H^{-1})^{-1}(u(t, x)) = \psi_H^t(\psi^{-1}(u(t, x))$$

for a fixed $\lambda$-adapted CR-almost complex structure $J_0$. The requirement on $(H, J)$ in (1.12) can be always assumed for the purpose of studying Sandan-Shelukhin type problem of untangling Legendrian submanifolds.
which we apply above in (1.14). This kind of coordinate change from the *dynamical version* to the *intersection theoretic version* of the Floer homology has been systematically utilized by the present author in the symplectic Floer theory. (See [Oh05b, Oh99, Oh05b] and the book [Oh15b, Section 12.7].) To motivate this transformation, we will provide the general perspective associated to it in Subsection 5.1. In Subsection 5.2, we will also provide the parametric version of the gauge transformation of perturbed contact instanton equations which enters in our proof of the Theorem 1.12.

**1.5. Definition of horizontal and vertical energies.** As in the study of the moduli space of Floer trajectories, especially the kind of parameterized autonomous Floer trajectories appearing in the study of displacement energy of Lagrangian submanifolds presented in [Oh97a], one needs to develop the whole analytic package for the study of the moduli space of solutions of (1.13) such as the a priori coercive estimates, Fredholm theory and compactification of the moduli spaces. The first has been established in [OW18a, OW18b] for the closed string case, and then in [Oh21a] for the open string case of Legendrian boundary conditions. The Fredholm theory and the first step of compactification for the closed string case is carried out in [Oh23] *under the asymptotic charge vanishing hypothesis*. The present author then proved in [Oh21a] that this asymptotic charge always vanishes for the open string case of Legendrian boundary condition. In these works, precise tensorial calculations utilizing the canonical connection of the contact triad introduced in [OW14] play an important role.

In Part 3, we establish all these ingredients for this moduli space up to the level of what are needed to complete the proof of Theorem 1.12. We leave a full account of the Fredholm theory and the compactification of general contact instanton moduli spaces in a sequel.

To make the study of Gromov-Floer-Hofer type compactness result, it is crucial to identify a correct choice of energy in the framework of contact instantons. Such an identification of the Hofer-type energy is given in [Oh23] for the closed string case of contact instantons. (See also [OS] for such a study in the context of lcs-instantons on locally conformal symplectic manifolds.) We adapt this study to the open-string context of Legendrian boundary conditions for the perturbed equation, and prove an a priori energy bound and develop the relevant bubbling argument and $C^1$-estimates in Part 3.

It turns out that the correct choice of the horizontal part of the energy, which we call the $\pi$-energy, is the following.

**Definition 1.20** (The $\pi$-energy of perturbed contact instanton). Let $(J,H)$ be as in Definition 1.19. Let $u: \mathbb{R} \times [0,1] \to M$ be any smooth map. We define

\[
E^\pi_{J,H}(u) := \frac{1}{2} \int e^{g_H(u)}(|du - X_H(u) \otimes dt|)^2 \]

(1.15)

call it the *off-shell* $\pi$-energy, where $g_H(u)$ is the function given in (1.14).

We apply the gauge transformation $\Phi_{H}^{-1}$ to define a map $\pi$ by

\[
\pi(\tau,t) = (\psi_H(\psi_H^{-1})^{-1}(u(\tau,t)) = \psi_H^{-1}(\psi_H^{-1}(u(\tau,t))) \]

(1.16)
and
\[ J' = \{ J'_t \}, \quad J'_t = (\psi^*_H(\psi^1_H)^{-1})^* J, \quad (1.17) \]
\[ \lambda'_t = (\psi^*_H(\phi^1_H)^{-1})^* \lambda. \quad (1.18) \]

Then for the given contact triad \((M, \lambda, J)\), the triple
\[ (M, \{ \lambda'_t \}, \{ J'_t \}) \]
forms a \(t\)-dependent family of contact triads. We denote the associated contact triad metric by
\[ g'_t = (\psi^*_H(\phi^1_H)^{-1})^* g. \]

The upshot of this coordinate change is that if \(u\) satisfies (1.13) with respect to \(J_t\), then \(\overline{u}\) is a nonautonomous contact instanton that satisfies
\[ \overline{\partial}_{J', \overline{u}} = 0, \]
with the boundary condition
\[ \overline{u}(\tau, 0) \in \psi_H^1(R_0), \quad \overline{u}(\tau, 1) \in R_1, \]
where we have
\[ (\overline{\partial}_{J', \overline{u}})(\tau, t) = (d\overline{u})^{(0,1)}(\tau, t). \]

The following identity justifies the presence of weight function \(e^{g_H(u)}\) in the definition of energy (1.15) with \(g_H(u)\) defined in (1.14). (See also Remark 7.3 for its naturality and further necessity of the presence of the weight factor.)

**Proposition 1.21** (Proposition 7.2 & 7.4). Let \(u : \mathbb{R} \times [0, 1] \to M\) be any smooth map and \(\overline{u}\) be as above. Then
\[ E_{J,H}^\pi(u) = E_{J',\overline{u}}^\pi = \int \overline{u}^* \lambda. \quad (1.19) \]

Another crucial component of Hofer-type energy, the vertical part of energy, is more nontrivial to describe. We refer readers to Section 12, Part 3 for the details of its construction (See [Oh23] also for the same definition considered in the closed string context.) We will show in Part 3 that these two energies will govern the convergence behavior of perturbed contact instantons, similarly as in the case of pseudoholomorphic curves in the symplectization [Hof93, BEH⁺03].

1.6. **Cut-off Hamiltonian perturbed contact instanton equation.** For the purpose of relating the existence question of Reeb chords between the pair \((\psi(R), R)\) to the oscillation norm of the Hamiltonian \(H \mapsto \psi\), we will adapt the scheme laid out in [Oh97a] to the context of perturbed contact instantons with Legendrian boundary conditions in contact manifolds and prove Theorem 1.12: The scheme of [Oh97a] was used for the study of displacement energy of Lagrangian submanifolds on symplectic manifolds.

We consider the perturbed contact instanton with moving Legendrian boundary condition, and set-up the deformation-cobordism framework of parameterized moduli space of perturbed contact instantons by adapting a similar parameterized Floer moduli spaces used in [Oh97a].

**Notation 1.22** (The family \((H^s, J^s)\)). Consider the two parameter family of CR-almost complex structures and Hamiltonian functions:
\[ J = \{ J(s,t) \}, \quad H = \{ H(s,t) \} \quad \text{for} \ (s, t) \in [0, 1]^2. \]
We denote by $H^s$ the Hamiltonian given by $H^s(t, x) := H(s, t, x)$ and $J^s$ given by $J^s(t, x) = J(s, t, x)$.

We will be particularly interested in the case for which the domain dependent Hamiltonian arises as

$$H_K(\tau, t, x) := H(\chi_K(\tau), t, x)$$

where $\chi_K : \mathbb{R} \times [0, 1] \to [0, 1]$ is the family of cut-off functions used in [Oh97a]: For each $K \in \mathbb{R}_+ = [0, \infty)$, we define a family of cut-off functions $\chi_K : \mathbb{R} \to [0, 1]$ so that for $K \geq 1$, they satisfy

$$\chi_K = \begin{cases} 0 & \text{for } |\tau| \geq K + 1 \\ 1 & \text{for } |\tau| \leq K. \end{cases}$$

We also require

$$\chi'_K \geq 0 \text{ on } [-K - 1, -K]$$
$$\chi'_K \leq 0 \text{ on } [K, K + 1].$$

(1.22)

For $0 \leq K \leq 1$, define $\chi_K = K \cdot \chi_1$. Note that $\chi_0 \equiv 0$.

Especially, we will concern the two-parameter family $H$ of the form $H \in [0, 1]^2$. We consider the 2-parameter family of contactomorphisms $\psi_{s,t} : \psi_{s,t} H \in \psi_{s,t}$. Obviously we have the $t$-developing Hamiltonian $Dev_\lambda(t \mapsto \psi_{s,t}) = H^s$. We then consider the elongated two parameter family

$$H_K(\tau, t, x) = \chi_K H(t, x)$$

and write the $\tau$-developing Hamiltonian

$$G_K(\tau, t, x) = Dev_\lambda(\tau \mapsto \psi^K_{\tau,t})$$

where $\psi^K_{\tau,t} = \psi^{\chi_K(\tau), t}$.

More generally, we will need to consider the following general form of perturbed contact instanton equations given by

$$\begin{cases} (du - P_K(u)) \tau^{(0,1)} = 0, \\ d(e g_K(u)(u^* \lambda_K) \circ j) = 0, \\ u(\partial \Theta_{K+1}) \in R, \end{cases}$$

(1.23)

where $P_K(u)$ is a $u^*TM$-valued one form on the domain $\Theta_K$ and $g_K(u)$ is some function on $\Theta_{K+1}$. (We refer readers to Section 8 for the precise expression of this equation.)

Then, to each $K \in \mathbb{R}_+ = [0, \infty)$, we associate a parameterized moduli space denoted by

$$\mathcal{M}^{para}_{[0, K_0]}(M, R; J, H) = \bigcup_{K \in [0, K_0]} \{K\} \times \mathcal{M}_K(M, R; J, H)$$

for a sufficiently large $K_0 > 0$. (See Section 9 for the precise definition thereof.) Then we prove the following fundamental a priori energy bounds

**Theorem 1.23** (Propositions 8.7 & 8.8). Assume $(M, \lambda)$ be a tame contact manifold. Let $(K, u) \in \mathcal{M}^{para}(M, \lambda; J, H)$ be any element and $\pi_K$ be its gauge transform. Then we have

1. [The horizontal energy bound]

$$E_{\pi_K}^H(\pi_K) \leq \|H\|.$$
(2) \textit{[The vertical energy bound]}

\[ E_{K}^{\perp}(\pi_{K}) \leq \|H\| \]

We would like to emphasize that the proofs of these a priori bounds much rely on the particular form of Hamiltonian-perturbed contact instanton equation (1.13) under the Legendrian boundary condition and utilize the calculations based on the contact Hamiltonian calculus. (See Proposition 11.3 and Proposition 13.1 respectively.) These calculi are systematically developed and organized in [Oh21a, Section 2] with coherent signs, notations and conventions, and further developed in Subsection 2 of the present paper.

Then the following is a sample Gromov-Floer-Hofer type convergence result that we use in the proof of Theorem 1.12. Here we denote by

\[ M(M; \alpha_{j}) \quad \text{(resp. } M(M, R; \beta_{k}) \text{)} \]

the moduli spaces of contact instantons on the plane \( \mathbb{C} \) with \( \alpha_{j} \) as its asymptotic Reeb orbits (resp. that of contact instantons on the half plane \( \mathbb{H} \) with boundary in \( R \) and with \( \beta_{k} \) as its asymptotic Reeb chord).

**Theorem 1.24** (Theorem 15.1). Consider the moduli space \( M^{\text{para}}(M, R; J, H) \). Then one of the following alternatives holds:

1. There exists some \( C > 0 \) such that

\[ |d\overline{u}|_{C^{0}; R \times [0,1]} \leq C \quad (1.24) \]

where \( C \) depends only on \( (M, R; J, H) \) and \( \lambda \).

2. There exists a sequence \( u_{\alpha} \in M_{K_{\alpha}}(M, R; J, H) \) with \( K_{\alpha} \to K_{\infty} \leq K_{0} \) and a finite set \( \{\gamma_{z}^{+}\} \) of closed Reeb orbits of \( (M, \lambda) \) such that \( u_{\alpha} \) weakly converges to the union

\[ \overline{u}_{\infty} = \overline{u}_{0} + \sum_{j=1}^{\infty} v_{j} + \sum_{k} w_{k} \]

in the Gromov-Floer-Hofer sense, where

\[ \overline{u}_{0} \in M_{K_{\infty}}(M, R; J, H), \]

\[ v_{j} \in M(M, J_{z_{j}}^{\prime}; \alpha_{j}); \quad \alpha_{j} \in \mathcal{R}(M, \lambda_{z_{j}}^{\prime}), \]

and

\[ w_{k} \in M(M, \psi_{z_{j}}(R)), J_{z_{j}}^{\prime}; \beta_{k}); \quad \beta_{k} \in \mathcal{R}(M, R; \lambda). \]

Here the domain point \( z_{j} \in \partial \hat{\Theta}_{K_{\infty}+1} \) is the point at which the corresponding bubble is attached.

The proof of this theorem will occupy Part 3.

1.7. **Discussion.** One natural avenue to pursue in the future is to amplify our quantitative study given in the present paper to define the contact instanton spectral invariants of general Legendrian links and investigate their entanglement structure. We also hope to further apply this machinery to problems of contact topology and thermodynamics. (See [BLMN15], [MNSS90] and [LO], for example, in the midst of many articles on contact geometric formulation of thermodynamics in physics literature. Entov and Polterovich applied ideas from contact dynamics to a problem of non-equilibrium thermodynamics in a recent article [EP21].) The first step towards this goal together with some applications is given in [OY] where Legendrian spectral invariants, the analog of Floer theoretic construction of Lagrangian
spectral invariants given in [Oh97b], are constructed. Further study in general and other applications will be given in [Ohb].

In the present paper, we consider the special family $J'$ given in (1.17) so that the perturbed equation (1.13) for the family $J$ can be converted to the (unperturbed) contact instanton equations for $J'$ by the gauge transformation $\Phi_H$ and utilized the analysis established in [Oh21a, OY23] for $H = 0$. In [Oha, OY23], we establish the coercive elliptic estimates for the perturbed equation (1.13) itself for the general family $J'$ not necessarily of the type given in (1.17) so that one cannot convert to the (unperturbed) contact instanton equations by the gauge transformation $\Phi_H$. Such an analysis will be important for the general study and applications of the perturbed equation in contact topology such as construction of contact analogs of the spectral invariants [Oh97b, Oh05a, Oh05b], similarly as Floer’s Hamiltonian-perturbed Cauchy-Riemann equation does in symplectic topology.

The context of of the present paper in Part 3 together with the gluing theory of contact instantons developed in [Ohc] and basic elliptic regularity theory for the equation (1.13) developed in [Oha], are the analytic foundation for the applications of the perturbed contact instantons (with Legendrian boundary condition) in the sequels [Oh22], [Ohb]. It is also the analytic basis for the Floer theoretic construction of Legendrian spectral invariants via contact instanton Floer-type homology given in [OY] for the one-jet bundle.

Throughout the paper, neither symplectization nor pseudoholomorphic curves in symplectic geometry is involved. We refer readers to the survey article [OK] for the detailed relationship between the analysis of (unperturbed) contact instantons and that of pseudoholomorphic curves on symplectization.

We would like to emphasize that with the analytical foundations of (perturbed) contact instantons in our disposal, the remaining study of contact Hamiltonian dynamics utilizing perturbed contact instantons e.g., construction of relevant contact spectral invariants on contact manifolds is largely geometric-topological and dynamical. This enables us to carry out such a study in an optimal way because the perturbed contact instanton equation (5.1) interacts with contact Hamiltonian calculus in a straightforward canonical fashion as illustrated by those in the present paper and in [Oh22]: Many calculations given in the present paper and in its sequels such as [Oh22] are of the nature of contact Hamiltonian calculus, not those arising from homogeneous Hamiltonian dynamics. By its purely contact nature of contact instanton equation, it is supposed to interact best with the contact Hamiltonian dynamics and calculus. In this regard many calculations provided in the present paper are purely of contact nature, and appear for the first time that have no their precedent in the literature. (See the proof of Proposition 8.7, for example.) We refer readers to our survey paper [OK] (or the latest arXiv version of the present paper) for the detailed exposition on the relationship between the analysis of contact instantons and that of pseudoholomorphic curves on symplectization.

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both ends of \( \{ \eta > 0 \} \) and \( \{ \eta < 0 \} \) in [Oh22]. This prompted us to have modified our definition of tameness by weakening the defining conditions to the present ones given in Definition 1.11.

**Convention and Notations:**

- (Contact Hamiltonian) We define the contact Hamiltonian of a contact vector field \( X \) to be \( -\lambda(X) =: H \).
- For given time-dependent function \( H = H(t, x) \), we denote by \( X_H \) the associated contact Hamiltonian vector field whose associated Hamiltonian \( -\lambda(X_t) \) is given by \( H = H(t, x) \), and its flow by \( \psi^H_t \).
- (Developing map) \( \text{Dev}(t \mapsto \psi_t) \): denotes the time-dependent contact Hamiltonian generating the contact Hamiltonian path \( t \mapsto \psi_t \).
- (Reeb vector field) We denote by \( R_\lambda \) the Reeb vector field associated to \( \lambda \) and its flow by \( \phi^R_t \).
- (Contact instanton homology) We denote by \( CI^*_\lambda(\mathcal{R}_0, \mathcal{R}_1) \) the \( \lambda \)-contact instanton complex and \( HI^*_\lambda(\mathcal{R}_0, \mathcal{R}_1) \) its cohomology, when defined. (See Subsection 10.2 for the definition.)
- \( \dot{\Sigma}, \mathcal{J} \): a punctured Riemann surface (with boundary) and \( (\Sigma, \mathcal{J}) \) the associated compact Riemann surface.
- We always regard the tangent map \( du \) as a \( u^*TM \)-valued one-form and write \( du = d\pi u + u^*\lambda \otimes R_\lambda \) with respect to the decomposition \( TM = \xi \oplus \mathbb{R}(R_\lambda) \).
- \( g_H(u) \): the function defined by \( g_H(u)(\tau, t) := g_{\psi^H_1(\psi^{-1}_t(u(\tau, t)))} \).
- \( T(M, \lambda) \): the infimum of the action of closed \( \lambda \)-orbits.
- \( T(M, R; \lambda) \): the infimum of the action of self \( \lambda \)-chords.
- \( T_\lambda(M, R) = \min\{T(M, \lambda), T(M, R; \lambda)\} \).

## 2. Summary of contact Hamiltonian calculus

In this section, we summarize basic contact Hamiltonian calculus we are going to use. We follow the exposition of [Oh21a, Section 2] and its sign conventions, and amplify the calculus further which will be needed for the purpose of the present paper and its sequels. We will always assume that \( (M, \xi) \) is cooriented.

**Definition 2.1.** For given coorientation preserving contact diffeomorphism \( \psi \) of \( (M, \xi) \) we call the function \( g \) appearing in \( \psi^*\lambda = e^g\lambda \) the **conformal exponent** for \( \psi \) and denote it by \( g = g_\psi \).

**Definition 2.2.** Let \( \lambda \) be a contact form of \( (M, \xi) \). For each contact vector field \( X \), the associated function \( H \) is given by

\[
H = -\lambda(X)
\]

is called the **\( \lambda \)-contact Hamiltonian** of \( X \). We also call \( X \) the **\( \lambda \)-contact Hamiltonian vector field** associated to \( H \).

When \( (M, \xi) \) is cooriented, the line bundle \( L = \mathbb{R}_M \) is trivial and the associated Jacobi bracket is also called the **Lagrange bracket** in some literature. (See [AG01].) When a contact form \( \lambda \) is given the map \( X \mapsto -\lambda(X) \) induces a Lie algebra isomorphism, which does not necessarily satisfy the Leibnitz rule. Partly due to different sign conventions literature-wise, we fix the sign convention of the bracket following that of [Oh21a].
Proposition 2.3 (Compare with Proposition 5.6 [LOTV18], Proposition 9 [dLLV19]). Let \((M, \xi)\) be cooriented and \(\lambda\) be an associated contact form. Define a bilinear map 
\[
\{ \cdot, \cdot \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M).
\]
by
\[
\{ H, G \} := -\lambda([X_H, X_G]).
\]
Then it satisfies Jacobi identity and the assignment \(X \mapsto -\lambda(X)\) defines an Lie algebra (anti-)isomorphism \(\mathfrak{x}(M, \xi)\) to \(C^\infty(M)\).

Under our sign convention, the \(\lambda\)-Hamiltonian \(H\) of the Reeb vector field \(R_\lambda\) as a contact vector field becomes the constant function \(H = -1\).

Next let \(\psi_t\) be a contact isotopy of \((M, \xi = \ker \lambda)\) with \(\psi_t^* \lambda = e^{\varphi_t} \lambda\) and let \(X_t\) be the time-dependent vector field generating the isotopy. Let \(H : [0, 1] \times M \to \mathbb{R}\) be the associated time-dependent contact Hamiltonian \(H_t = -\lambda(X_t)\).

We denote by \(\text{Cont}_0(M, \xi)\) the identity component of the group \(\text{Cont}(M, \xi)\) of (orientation-preserving) contactomorphisms. We denote by
\[
\mathcal{P}(\text{Cont}(M, \xi))
\]
the groupoid of (Moore) paths of contactomorphisms \(\ell : [0, T] \to \text{Cont}(M, \xi)\). We call an element thereof a contact Hamiltonian path. We have three obvious maps, the assignment \(\ell \to T\) of the domain length of \(\ell\), and the source and the target map
\[
s, t : \mathcal{P}(\text{Cont}(M, \xi)) \to \text{Cont}(M, \xi); \quad s(\ell) = \ell(0), \ t(\ell) = \ell(T)
\]
as usual. (When we consider an element of Moore path \(\ell\), one often writes it as a pair \((T, \ell)\).)

When \((M, \xi)\) is equipped with a contact form \(\lambda\) and we consider a path on \([0, 1]\), i.e., with \(T = 1\), we can associated another map the \(\lambda\)-developing map
\[
\text{Dev}_\lambda : \mathcal{P}(\text{Cont}(M, \xi)) \to C^\infty([0, T] \times M, \mathbb{R})
\]
by assigning its \(\lambda\)-contact Hamiltonian functions
\[
\text{Dev}_\lambda(\ell)(t, x) := -\lambda \left( \frac{\partial \ell}{\partial t}(t, \ell^{-1}_t(x)) \right),
\]
In the present paper, we will always assume \(T = 1\) unless otherwise said. Unravelling this definition, we have \(\text{Dev}_\lambda(\ell)(t, x) = H(t, x)\) where \(X_t\) is the vector field generating the path \(\ell\), i.e.,
\[
X_t(x) = \frac{\partial \ell}{\partial t}(t, \ell^{-1}_t(x)), \quad H = -\lambda(X_t).
\]
We denote by \(X \mapsto \psi\) if \(\psi = \psi^1_H\).

The following formulae for the contact Hamiltonians can be derived by a straightforward calculation. (See [MS15] for its first appearance.)

Lemma 2.4. Let \(\Psi = \{ \psi_t \} \in \mathcal{P}(\text{Cont}(M, \xi))\) be a contact isotopy satisfying \(\psi_t^* \lambda = e^{\varphi_t} \lambda\) with \(g_{\Psi} := g(t, x)\) and generated by the vector field \(X_t\) with its contact Hamiltonian \(H(t, x) = H_t(x)\), i.e., with \(\text{Dev}_\lambda(\Psi) = H\).

(1) Then the (timewise) inverse isotopy \(\Psi^{-1} := \{ \psi_t^{-1} \}\) is generated by the (time-dependent) contact Hamiltonian
\[
\text{Dev}_\lambda(\Psi^{-1}) = -e^{-g_{\Psi} \circ \Psi} \text{Dev}_\lambda(\Psi) \tag{2.5}
\]
where the function \(g_{\Psi} \circ \Psi\) is given by \((g_{\Psi} \circ \Psi)(t, x) := g_{\Psi}(t, \psi_t(x)).\)
(2) If $\Psi' = \{\psi'_t\}$ is another contact isotopy with conformal exponent $g_{\Psi'} = \{g'_t\}$, then the timewise product $\Psi' \Psi$ is generated by the Hamiltonian

$$\text{Dev}_{\lambda}(\Psi' \Psi) = \text{Dev}_{\lambda}(\Psi') + e^{-g_{\Psi'} \circ (\Psi')^{-1}} \text{Dev}_{\lambda}(\Psi) \circ (\Psi')^{-1}. \quad (2.6)$$

In particular, if $\Psi$ is a strict contactomorphism like the Reeb flow $\phi_t^{R_{\lambda}}$, then we have

$$\text{Dev}_{\lambda}(\Psi' \Psi) = \text{Dev}_{\lambda}(\Psi') + \text{Dev}_{\lambda}(\Psi) \circ (\Psi')^{-1}$$

which is an immediate generalization of the symplectic case.

Now we prove the following contact analog of well-known Banyaga’s formula from [Ban78] in symplectic geometry. For the derivation of this contact counterparts, we need to employ somewhat different argument to derive these formulae, especially because the bracket $\{G, H\}$, called the Lagrange bracket [AG01], is defined differently from the symplectic Poisson bracket and carries different properties, e.g., that the bracket does not satisfy the Leibnitz rule any more. (See [LOTV18, Proposition 5.6], [dLLV19, Proposition 9].)

**Proposition 2.5.** Let $\psi_{s,t}$ be a two-parameter family of contactomorphisms and $H$ and $G$ the $t$-Hamiltonian and the $s$-Hamiltonian respectively. Then we have

$$\frac{\partial H}{\partial s} - \frac{\partial G}{\partial t} + \{G, H\} = 0, \quad (2.7)$$

$$\frac{\partial}{\partial t}(G(s, t, \psi^t_H(x))) = \frac{\partial H}{\partial s}(s, t, \psi^t_H(x)). \quad (2.8)$$

**Proof.** We first consider the map $\Phi : (-\delta, \delta) \times [0, T] \to M$ defined by

$$\Phi(s, t) = (s, t, \psi_{s,t}(x)).$$

Then we have

$$d\Phi(\partial_s) = \partial_s \oplus X_G(\psi_{s,t}(x))$$

$$d\Phi(\partial_t) = \partial_t \oplus X_H(\psi_{s,t}(x))$$

on $(-\delta, \delta) \times [0, T] \oplus TM$. Therefore the vector fields

$$\partial_s \oplus X_G, \quad \partial_t \oplus X_H$$

on $(-\delta, \delta) \times [0, T] \times M$ are $\Phi$-related to $\partial_s$ and $\partial_t$ we have

$$[\partial_s \oplus X_G, \partial_t \oplus X_H] = 0.$$  

By expanding the left hand side, we obtain

$$\frac{\partial X_H}{\partial s} - \frac{\partial X_G}{\partial t} + [X_G, X_H] = 0. \quad (2.9)$$

By evaluating the one-form $\lambda$ against the equation and recalling the definition of Lagrange bracket $\{G, H\}$ (see [Oh21a, Proposition 2.19] for the definition thereof with our sign convention)

$$\{G, H\} = -\lambda([X_G, X_H]),$$

the above is equivalent to

$$-\frac{\partial H}{\partial s} + \frac{\partial G}{\partial t} - \{G, H\} = 0.$$

This finishes the proof of (2.7).
For the proof of (2.8), we rewrite (2.9) as
\[ 0 = \frac{\partial X_H}{\partial s}(s, t, \psi^t_H(x)) - \frac{\partial X_G}{\partial t}(s, t, \psi^t_H(x)) \]
\[ - L_{X_H} X_G(s, t, \psi^t_H(x)) = \frac{\partial X_H}{\partial s}(s, t, \psi^t_H(x)) - \frac{\partial}{\partial t} \left( X_G(s, t, \psi^t_H(x)) \right). \]
for all \((s, t, x)\). We then apply the one-form \(\lambda\) and get
\[ - \frac{\partial H}{\partial s}(s, t, \psi^t_H(x)) + \frac{\partial}{\partial t} (G(s, t, \psi^t_H(x))) = 0 \]
which finishes the proof of (2.8). \(\square\)

**Remark 2.6.** We would like to mention that in this contact case the equality (2.7) holds in the nose while in the symplectic case, we only have
\[ - \frac{\partial H}{\partial s}(s, t, \psi^t_H(x)) + \frac{\partial}{\partial t} (G(s, t, \psi^t_H(x))) = c(s, t) \]
for some function \(c = c(s, t) \) depending only on the parameters \((s, t)\).

Using (2.8), we derive the following formula for the \(s\)-Hamiltonian \(\text{Dev}(s \mapsto \phi^t_{sH})\) of the two-parameter family of contactomorphisms
\[ \mu(s) := \phi^t_{sH} \] (2.10)
for any \(t\)-dependent Hamiltonian \(H = H(t, x)\). Unlike the \(t\)-Hamiltonian which is manifest from the expression \(\phi^t_{sH}\), the \(s\)-Hamiltonian is not manifest therefrom.

The following formula will play a fundamental role in our derivation of the optimal inequality that enters in our proof of the Sandon-Shelukhin conjecture. (See the proof of the \(\pi\)-energy identity stated in Proposition 11.3.)

**Lemma 2.7.** Let \(L\) be the two-parameter Hamiltonian given by \(L(s, t, x) = \text{Dev}(s \mapsto \phi^t_{sH})(s, x)\). Then
\[ L(s, t, \psi^t_H(x)) = \int_0^t H(r, \psi^r_{sH}(x)) \, dr. \] (2.11)
In particular we have \(L(s, 0, x) \equiv 0\) and
\[ L(s, 1, \psi^1_{sH}(x)) = \int_0^1 H(r, \psi^r_{sH}(x)) \, dr. \] (2.12)

**Proof.** We also denote the \(t\)-Hamiltonian of the family \(\mu = \mu(s, t)\) by \(F = F(s, t, x)\) such that
\[ F = F(s, t, x) := \text{Dev}(t \mapsto \mu(s, t))(t, x) \]
which is nothing but
\[ F(s, t, x) := sH(t, x). \]
Using (2.8), we compute the \(\text{Dev}(s \mapsto \mu(s, t))\) by integrating the following
\[ \frac{\partial}{\partial t} (L \circ \mu)(s, t, x) = \left( \frac{\partial F}{\partial s} \right)(s, t, \mu(s, t)(x)) \]
\[ = H(t, \mu(s, t)(x)) = H(t, \psi^t_{sH}(x)) \] (2.13)
over \(t\). Then we obtain
\[ L(s, t, \mu(s, t)(x)) = \int_0^t \frac{\partial F}{\partial s} (s, r, \mu(s, r)(x)) \, dr = \int_0^t H(r, \psi^r_{sH}(r, x)) \, dr \]
which proves (2.11). By substituting \( t = 0 \) and \( t = 1 \) respectively, we have finished the proof. \( \square \)

Part 1. Tame contact manifolds, maximum principle and contact instantons

3. Co-Legendrian submanifolds

The notion of pre-Lagrangian submanifolds was introduced in [EHS95] which we recall now. Let \( SM = M \times \mathbb{R} \) be the (conical) symplectization of \( M \) which is the \( \mathbb{R}_+ \)-subbundle of \( T^*M \) which is formed by contact forms compatible with the given co-orientation of \((M, \xi)\). We rephrase the definition given in [EHS95] as follows.

**Definition 3.1.** Let \( SM = M \times \mathbb{R} \) be the symplectization of \( M \). A submanifold \( K \subset M \) is called pre-Lagrangian if there is a Lagrangian section \( \hat{k} : K \rightarrow SM \).

By setting \( \hat{K} = \text{Image} \hat{k} \) and \( \tilde{k} = (\pi|_\hat{K})^{-1} \), it can be easily seen that this definition is equivalent to the definition given in [EHS95] the condition of which reads that there is a Lagrangian lift \( \hat{K} \subset SM \) such that the restriction \( \pi|_{\hat{K}} : \hat{K} \rightarrow K \) is a diffeomorphism.

We now introduce a more intrinsic notion of co-Legendrian submanifolds which does not involve symplectization and includes that of pre-Lagrangian submanifolds as a special case.

3.1. **Definition of co-Legendrian submanifolds.** We start with recalling the definition of coisotropic submanifolds in contact manifolds \((M, \xi)\). (We refer to [LOTV18] for the definition thereof in the general, not necessarily coorientable, case.) Majority of the discussion given in this subsection will not be directly relevant to the main purpose of the present paper except for the purpose of providing some general perspective for a future purpose with the Reeb trace of Legendrian submanifold. (We also find that the notion itself is interesting and so worthwhile to describe its geometry in some detail for a future purpose.) The Reeb traces of Legendrian submanifolds are the prototypes of co-Legendrian submanifolds and will enter in the intersection theoretic translation of Sandon’s translated points.

Recall by definition of \( \lambda \) that \((M, d\lambda)\) is a presymplectic manifold.

**Definition 3.2.** Let \((M, \xi)\) be a contact manifold equipped with a contact form \( \lambda \). A \((\lambda)\)-coisotropic submanifold \( C \subset M \) (with respect to \( \lambda \)) is one that satisfies

\[
(TC)^{d\lambda} \subset TC.
\]

It is easy to show that this definition does not depend on the choice of contact forms \( \lambda \) and depends only on the contact structure \( \xi \).

**Lemma 3.3.** Let \( C \) be a coisotropic submanifold of \((M^{2n+1}, \lambda)\), and let \( \lambda \) be any contact form of \( \xi \).

1. \( R_\lambda \in TC \).
2. We have a natural exact sequence vector bundles

\[
0 \rightarrow \mathbb{R}\langle R_\lambda \rangle \rightarrow (TC)^{d\lambda} \rightarrow TC \rightarrow TC/TC^{(d\lambda)} \rightarrow 0
\]

where \( TC/TC^{(d\lambda)} \rightarrow C \) is a symplectic vector bundle.
Proof. Since $R_\lambda \in \ker d\lambda$, we have
\[ \mathbb{R}(R_\lambda) \subset (TC)^{d\lambda}. \]
Since $(TC)^{d\lambda} \subset TC$ by definition of coisotropic submanifolds, we derive that $R_\lambda$ is tangent to $C$. On the other hand, since $C$ is coisotropic, $TC/(TC)^{d\lambda}$ is symplectic.
This finishes the proof. □

Definition 3.4. A coisotropic submanifold $C \subset (M,\lambda)$ is called co-Legendrian if $C$ is minimally coisotropic, i.e., if $TC = (TC)^{d\lambda}$.

Proposition 3.5. Let $Z$ be a co-Legendrian submanifold of $M$. Then we have $\dim Z = n + 1$, and
\[ TZ = (TZ)^{d\lambda} = (\xi \cap TZ) \oplus \mathbb{R}(R_\lambda). \]
In particular, $\xi \cap TZ \subset TM|_Z$ is a Legendrian subbundle.

Proof. By Lemma 3.3, we know $\mathbb{R}(R_\lambda) \subset (TZ)^{d\lambda} = TZ$.
The latter also implies $TZ/\mathbb{R}(R_\lambda) = (TZ/\mathbb{R}(R_\lambda))^{d\lambda}$ with respect to the induced fiberwise symplectic bilinear form $[d\lambda]$ on the quotient bundle $TM/\mathbb{R}(R_\lambda) \cong \xi$ of rank $2n$. In other words, $TZ/\mathbb{R}(R_\lambda)$ is a Lagrangian subbundle of the quotient and hence rank $TZ/\mathbb{R}(R_\lambda) = n$.
Combining the above, we have finished the proof. □

Remark 3.6. It is shown in [OP05], [Zam08] (for the symplectic case) and in [LOTV18] (for the contact case) that the local deformation problem of general coisotropic submanifolds is obstructed. In particular the set of general coisotropic submanifolds is not a smooth (Frechet) manifold [Zam08]. However for the corresponding deformation problem of the co-Legendrian case, this obstruction vanishes and so the set of co-Legendrian submanifolds forms a smooth manifold: The quadratic term in the defining equation of coisotropic subspace given in [OP05, Proposition 2.2] vanishes for the co-Legendrian case!

Proposition 3.7. Any pre-Lagrangian submanifold is co-Legendrian for any contact form $\lambda$ of a given contact structure $(M,\xi)$ on $M$. 

We need some preparation to start with the proof.
Let $\lambda$ be a contact form compatible to the co-orientation of $(M,\xi)$. Then we have canonical splitting
\[ T_\alpha(SM) = \tilde{\xi} \oplus \mathbb{R}(\frac{\partial}{\partial r})|_\alpha \oplus \mathbb{R}(\tilde{R}_\lambda(\alpha)) \] (3.1)
where $\tilde{R}_\lambda$ is the projectable vector field on $SM = M \times \mathbb{R}$ whose $\pi$-projection to $M$ is the Reeb vector field $R_\lambda$. Similarly $\tilde{\xi}$ is the lift of contact distribution $\xi$ of $M$ to $SM$. Both $\mathbb{R}(\frac{\partial}{\partial r})|_\alpha \oplus \mathbb{R}(\tilde{R}_\lambda(\alpha))$ and $\tilde{\xi}$ are symplectic subspaces of $T_\alpha(SM)$ with respect to the symplectic form $\omega = d(\pi^*\lambda)$.

Let $K \subset M$ be any submanifold and let $\hat{k} : K \rightarrow SM$ be a section, i.e, a smooth map with $\pi \circ \hat{k} = id|_K$. We write the tangent vectors of $K$ at $y$ as
\[ v = v' + aR_\lambda, \quad u = u' + bR_\lambda \in T_yK \]
for some $a, b \in \mathbb{R}$. At $\alpha \in \hat{k}^{-1}(y)$, we have canonical lifts $\hat{v}, \hat{u}$ thereof so that
\[
dr(\hat{v}) = 0 = dr(\hat{u}), \quad \hat{v}', \hat{u}' \in \xi_{\alpha}
\]
by the splitting.

**Lemma 3.8.** Let $\alpha = r^*\lambda_y \in \hat{k}^{-1}(y)$. Then for any $v, w \in T_yK$, we have
\[
\omega(\hat{v}, \hat{u}) = rd\lambda(v, u).
\]

**Proof.** We compute
\[
\omega(\hat{v}, \hat{u}) = d(r^*\lambda)(\hat{v}, \hat{u})
= (dr \wedge \pi^*\lambda + rd^*\lambda)(\hat{v}', \hat{u}') + a\hat{R}_\lambda, \hat{u}' + b\hat{R}_\lambda)
= r^*d\lambda(\hat{v}', \hat{u}') + a\hat{R}_\lambda, \hat{u}' + b\hat{R}_\lambda) = rd\lambda(v, u).
\]

\[
\square
\]

We are now ready to give a proof of the proposition.

**Proof of Proposition 3.7.** Let $K$ be a pre-Lagrangian submanifold. Pick a Lagrangian section $\hat{k} : K \to SM$ and let $\hat{K}$ be its image. Then since $T_a\hat{K}$ is Lagrangian, so is $(T_a\hat{K})^w$ since $(T_a\hat{K})^w = (T_a\hat{K})$. Therefore
\[
\omega(\hat{v}, \hat{u}) = 0
\]
for all $\hat{v}, \hat{u}$ with $v, u \in (T_yK)^d\lambda$, since the latter implies $\hat{v}, \hat{u} \in T_a\hat{K} = (T_a\hat{K})^w$.

Therefore by (3.2) $d\lambda(v, u) = 0$ for all $v, u \in (T_yK)^d\lambda$ and hence $K$ is isotropic with respect to $d\lambda$. Since dim $K = \dim \hat{K} = n$, $K$ must be co-Legendrian. This finishes the proof.

The following proposition shows that none of closed (i.e., compact without boundary) co-Legendrian submanifold are pre-Lagrangian.

**Proposition 3.9.** No closed co-Legendrian submanifold is pre-Lagrangian.

**Proof.** Denote by $SM \to M$ the symplectization of $M$. Let $i : K \to M$ be a closed co-Legendrian submanifold. We need to show that no section $\hat{k} : K \to i^*SM$ can be Lagrangian.

Suppose to the contrary that there is a section $\hat{k} : K \to SM$ over $i$ whose image is a Lagrangian submanifold in $SM$. Note that any section $\hat{k}$ can be written as
\[
\hat{k}(y) = (y, e^{g(y)}) \in K \times \mathbb{R}_+\]
for some function $g : K \to \mathbb{R}$. Since $\hat{k}$ is assumed to be Lagrangian,
\[
d(e^{g(\hat{k})}i^*\lambda) = 0
\]
on $K$, which is equivalent to
\[
0 = dg \wedge (\pi \circ \hat{k})^*i^*\lambda + d(\pi \circ \hat{k})^*i^*\lambda = dg \wedge i^*\lambda + i^*d\lambda
\]
on $K$. Since $K$ is assumed to be compact without boundary, $g$ must have at least one critical point $y_0 \in K$ where we have $dg(y_0) = 0$. Therefore $d\lambda|_{r_{y_0}K} = 0$ at $y_0$. In particular, we have
\[
T_{y_0}K \subset \xi_{y_0}
\]
and so $T_{y_0}K$ is isotropic in $\xi_{y_0}$. But this is a contradiction to the fact that dim $K = n + 1$. Therefore there cannot be any Lagrangian section $\hat{k}$ of $SM \to M$. This finishes the proof.

\[
\square
\]
3.2. Examples of co-Legendrian submanifolds.

**Example 3.10 (Conormal jets).** The most natural examples of co-Legendrian submanifolds are the conormal one-jets \( \tilde{\nu}^* N \subset J^1 B \) for any submanifolds \( N \subset B \) where we define \( \tilde{\nu}^* N \) by

\[
\tilde{\nu}^* N := \pi^{-1}(\nu^* N) = \{ (q, p, z) | (q, p) \in \nu^* N \}.
\] (3.3)

In fact, any co-Legendrian submanifold of general contact manifold is locally of this form.

We now show that the set of co-Legendrian submanifolds is strictly bigger than that of pre-Lagrangian submanifolds.

**Example 3.11.** We have only to provide an example of compact co-Legendrian submanifolds without boundary which then will not be pre-Lagrangian by Proposition 3.9. Consider the contact manifold \( S^1 \times T^* S^1 \) equipped with the contact one form \( \lambda = dt - pdq \) where \( (q, p) \) is the coordinates of \( T^* S^1 \sim = S^1 \times \mathbb{R} \) and \( t \) is the standard coordinate \( S^1 \sim = \mathbb{R}/\mathbb{Z} \). Then it is easy to see that \( S^1 \times 0_{T^* S^1} \sim = S^1 \times S^1 \) is a co-Legendrian submanifold but it is not pre-Lagrangian by Proposition 3.9.

The next examples are the ones of our main interest in the present paper.

**Example 3.12 (Reeb trace of Legendrian submanifolds).** Let \( R \) be any Legendrian submanifold of \( (M, \lambda) \) and consider its Reeb trace

\[
Z_R := \bigcup_{t \in \mathbb{R}} \phi^t_{R_0}(R).
\] (3.4)

This is an immersed co-Legendrian submanifold in general. *Under the hypothetical situation* that there is no self Reeb chord of \( R \) i.e., that Arnold’s Reeb chord conjecture fails for \( R \), this immersion becomes a one-one immersion.

**Remark 3.13 (Control manifolds).** The class (3.4) of submanifolds has been considered in the geometric formulation of thermodynamics and the information theory in the physics literature with the name of *control manifold*. (See [MNSS90], [BLM15], [BLMN15] for example.) (Strictly speaking, the control manifolds in these references are considered only in the one-jet bundle and also equipped with a (pseudo-)Riemannian metric in addition.) Co-Legendrian submanifolds are defined on general contact manifold, not just in the one-jet bundle case, without being equipped with a metric.

We now introduce the intersection theoretic version of translated fixed points.

**Definition 3.14 (Translated intersection points).** Let \( (M, \xi) \) be a contact manifold. Let \( (R_0, R_1) \) a pair of Legendrian submanifolds.

1. Let \( \lambda \in \mathcal{C}(\xi) \). We call a pair \( (x, \eta) \in R_0 \times \mathbb{R}_+ \) a \( \lambda \)-translated intersection point of \( R_0 \) and \( R_1 \) if there is a Reeb chord \( \gamma \) satisfying

\[
\gamma(0) \in R_0, \quad \gamma(\eta) = x \in R_1.
\]

i.e., if \( x \in R_0 \cap (\phi_{R_0}^\eta)^{-1}(R_1) \subset R_0 \cap Z_R \).

2. We say the pair \( (R_0, R_1) \) are dynamically \( \lambda \)-entangled if there is a Reeb chord of \( \lambda \) from \( R_0 \) to \( R_1 \), and just dynamically Reeb-entangled if there is \( \lambda \in \mathcal{C}(\xi) \) for which \( \Reeb(M, R; \lambda) \neq \emptyset \).
By definition, for each given translated intersection point \((x, \eta)\) with \(x \in \psi(R)\), there is a Reeb chord \(\gamma_x : [0, \eta] \rightarrow M\) defined by
\[
\gamma_x(t) := \phi^{t\eta}_{R_1}(x)
\]
which satisfies
\[
\gamma_x(0) = x \in \psi(R), \quad \gamma_x(\eta) \in R
\]  
(3.5)
and vice versa.

We summarize the relationship between the set of translated intersection points between \(R_0\) and \(R_1\) and the intersection set \(\psi(R_0) \cap Z_{R_1}\) in terms of the following general correspondence lemma.

**Lemma 3.15.** Let \((M, \lambda)\) be a contact manifold, and let \(\psi : M \rightarrow M\) be a contactomorphism. Consider any compact subset \(S_0, S_1\) of \(M\).

\[
\psi(R) \cap Z_{S_1} = \emptyset \iff \psi(S_0) \cap S_1 = \emptyset \quad \& \quad \text{Reeb}(\psi(S_0), S_1) = \emptyset. \tag{3.6}
\]

We remark that for Legendrian pair \((R_0, R_1)\), the condition \(\psi(R_0) \cap R_1 = \emptyset\) holds for a generic contactomorphism \(\psi\) by dimensional reason. This translation applied to a pair of compact Legendrian submanifolds \((R_0, R_1)\) will be what enables us to study the existence question of translated intersection points through the study of the moduli space of contact instantons with Legendrian boundary conditions which is the main analytical framework that we employ in the present paper.

### 4. Tame contact manifolds

In this paper, our analysis of (perturbed) contact instantons will be performed on a general class of contact manifolds which may not be necessarily compact. As in other geometric analysis problems such as that of pseudoholomorphic curves, compactness of the ambient manifolds is not needed as long as relevant \(C^0\) confinement results can be achieved.

We first introduces a class of barrier functions which will control the \(C^0\) bounds of contact instantons on noncompact contact manifolds.

**Definition 4.1 (Reeb-tame function).** Let \((M, \xi)\) be contact manifold equipped with contact form \(\lambda\). A function \(\psi : M \rightarrow \mathbb{R}\) is called \(\lambda\)-tame (at infinity) if
\[
\mathcal{L}_{R_1} d\psi = 0 \text{ on } M \setminus K
\]
for some smooth one-form \(\beta\) and a smooth function \(g\) on \(U\).

**Definition 4.2 (Contact \(J\) quasi-pseudoconvexity).** Let \(J\) be a \(\lambda\)-adapted CR almost complex structure. We call a function \(\psi : M \rightarrow \mathbb{R}\) contact \(J\) quasi-plurisubharmonic on \(U\)
\[
-d(d\psi \circ J) + d\eta \wedge \beta \geq 0 \quad \text{on } \xi, \tag{4.1}
\]
\[
R_1 d(d\psi \circ J) = g \, d\psi \tag{4.2}
\]
for some smooth one-form \(\beta\) and a smooth function \(g\) on \(U\). We call such a pair \((\psi, J)\) a contact quasi-pseudoconvex pair on \(U\).

Using this, we introduce the following class of contact manifolds for which the \(C^0\) estimates for the (perturbed) contact instantons will be available.

**Definition 4.3 (Tame contact manifolds).** Let \((Q, \xi)\) be a contact manifold, and let \(\lambda\) be a contact form of \(\xi\).
(1) We say $\lambda$ is tame on $U$ if $(M, \lambda)$ admits a pair $(\psi, J)$ of a $\lambda$-adapted CR almost complex structure $J$ and a $\lambda$-tame contact $J$ quasi-plurisubharmonic exhaustion function $\psi$ on $U$.

(2) We call an end of $(M, \lambda)$ tame if $\lambda$ is tame on the end of $M$.

We say an end of contact manifold $(M, \xi)$ (resp. $(M, \xi)$) is tame if it admits contact form $\lambda$ that is tame on the end (resp. at infinity) of $M$.

**Example 4.4.**  (1) Obviously any contact form $\lambda$ on any compact contact manifold is tame.

(2) We will show in [Oh22] that the contact form

$$A = -\frac{\eta}{2\pi} \psi_1^* \lambda + \frac{\eta}{2\pi} \psi_2^* \lambda$$

on $Q \times Q \times \mathbb{R}$ is tame on $\{\eta > 0\}$ for any compact contact manifold $(Q, \lambda)$ by proving that the coordinate function $\eta$ is $\tilde{J}$ quasi-plurisubharmonic with respect to some natural class of $A$-adapted CR almost complex structures $\tilde{J}$.

(3) Any one-jet bundle $J^1 B$ with the standard contact form $dz - pdq$ for a compact manifold $M$ is tame. In fact, the coordinate function $z$ is a contact $J$-convex Reeb tame exhaustion function. (See [OY] for its proof and usage in the study of Legendrian spectral invariants.)

## 5. Hamiltonian-Perturbed Contact Instantons

Consider a time-dependent function $H = H(t, x) : \mathbb{R} \times M \to \mathbb{R}$. Denote by $X_H$ the associated contact vector field. Then we have $\lambda(X_H) = -H$ by definition.

**Definition 5.1.** Let $(M, \lambda)$ be a contact manifold equipped with a contact form, let $(R_0, R_1)$ be a pair of Legendrian submanifolds in $(M, \xi)$. We say a map $u : \mathbb{R} \times [0, 1] \to M$ is a $X_H$-perturbed contact instanton trajectory with Legendrian boundary condition if it satisfies

$$\begin{cases}
(du - X_H \otimes dt)^{\pi(0,1)} = 0, \\
d(e^{g_H(u)}(u^* \lambda + H \gamma) \circ j) = 0
\end{cases}$$

(5.1)

together with the boundary condition

$$u(\tau, 0) \in R_0, \quad u(\tau, 1) \in R_1.$$  

(5.2)

**Remark 5.2.** Note that it is easy to see that any asymptotic limit $\gamma_\infty$ of a solution $u$ with finite $\pi$-energy $E^\pi_{\lambda, H}(u)$ satisfies

$$(\dot{\gamma}(t) - X_H(t, \gamma(t)))^\tau = 0.$$ 

We refer to [Oh21a, Proposition 3.4] for the general description of such a curve which is consistent with the asymptotic convergence result of the equation (5.1).

We will also need to consider the full domain-dependent parameterized Hamiltonians of the type

$$H = H_K(\tau, t, x)$$

and consider the 2-parameter family of contactomorphisms $\Psi_{s,t} := \psi_{s}^{\tau}$. Then we have non-trivial $t$-developing Hamiltonian $\text{Dev}_\lambda(t \mapsto \Psi_{(s,t)}) = H^s$. We also need to consider the $\tau$-developing Hamiltonian

$$G_K(\tau, t, x) = \text{Dev}_\lambda(\tau \mapsto \Psi_{\tau,t}^K)$$
Remark 5.3. In symplectic geometry, thanks to the equality $(\phi_H^t)^*\omega = \omega$, a (symplectic) Hamiltonian diffeomorphism $\phi_H^t$ induces a filtration preserving msp between two Floer complexes $CF(\phi_H^t(R_0), R_1)$ and $CF(R_0, (\phi_H^t)^{-1}(R_1))$. Therefore one can freely go back and forth between the two without changing the relevant quantitative invariants. (See [Oh99].) However in the current contact context, the transformation changes the overall filtration structure between the two. We would like to emphasize that we will consistently use $\Phi_H$ and its inverse and never use the transformation $\Phi_H'$ in the present paper. This is because the conformal exponent $e^{g\omega}$ would appear if $\Phi_H'$ were used. This appearance of conformal exponent is also responsible for the phenomenon that the Reeb-entangling energy $e^{trn(S_0, S_1)}$ is not symmetric. To keep the importance of this coordinate change in readers’ minds, we name and call $\Phi_H$ and its inverse gauge transformation.

where $\Psi_{\chi_K}^t = \Psi_{\chi_K}(\tau,t)$. Then we consider the 2-parameter perturbed contact instanton equation given by

$$\begin{cases}
(du - X_{H_K}(u))dt + X_{G_K}(u) ds = 0, \\
d(e^{gK}(u)(u^*\lambda + u^*H_Kdt - u^*G_K d\tau) \circ j) = 0,
\end{cases} \quad (5.3)$$

where $g_K(u)$ is the function defined by

$$g_K(u)(\tau,t) := g_{\psi_{H_K}(\bar{\psi}_{H_K})^{-1}}(u(\tau,t)) \quad (5.4)$$

for $0 \leq K \leq K_0$.

5.1. Gauge transformation: autonomous version. We first consider two different representations of the trajectory of the ODE $\dot{x} = X_H(x)$, one in terms of the initial point and the other in terms of the final point. The first one is given by

$$z_H^p(t) := \psi_{H}^t(q), \quad q \in M \quad (5.5)$$

and the other given by

$$z_H^p(t) = \psi_{H}^t((\psi_{H}^1)^{-1}(p)). \quad (5.6)$$

For a given pair $(R_0, R_1)$ of Legendrian submanifolds in $M$ we denote by $\Omega(R_0, R_1)$ the set of smooth paths from $R_0$ to $R_1$.

For a given Hamiltonian $H \mapsto \psi$, the above representations (5.5) and (5.6) provide two different one-to-one correspondences

$$\Phi_H : \ell(t) \mapsto \psi_{H}^t(\psi_{H}^1)^{-1}(\ell(t))$$

and

$$\Phi_H' : \tilde{\ell}(t) \mapsto \psi_{H}^t(\tilde{\ell}(t))$$

$\Phi_H$ defines a bijective map

$$\Phi_H : \Omega(\psi_{H}^1(R_0), R_1) \to \Omega(R_0, R_1) \quad (5.7)$$

and $\Phi_H'$ defines

$$\Phi_H' : \Omega(R_0, (\psi_{H}^1)^{-1}(R_1)) \to \Omega(R_0, R_1). \quad (5.8)$$

Obviously the composition

$$\Phi_H^{-1} \circ \Phi_H' : \Omega(R_0, (\psi_{H}^1)^{-1}(R_1)) \to \Omega(\psi_{H}^1(R_0), R_1)$$

is induced by the diffeomorphism $\phi_H^t : M \to M$ in that

$$(\Phi_H^{-1} \circ \Phi_H')(\ell)(t) = (\psi_H^1 \circ \tilde{\ell})(t).$$

Remark 5.3. In symplectic geometry, thanks to the equality $(\phi_H^t)^*\omega = \omega$, a (symplectic) Hamiltonian diffeomorphism $\phi_H^t$ induces a filtration preserving msp between two Floer complexes $CF(\phi_H^t(R_0), R_1)$ and $CF(R_0, (\phi_H^t)^{-1}(R_1))$. Therefore one can freely go back and forth between the two without changing the relevant quantitative invariants. (See [Oh99].) However in the current contact context, the transformation changes the overall filtration structure between the two. We would like to emphasize that we will consistently use $\Phi_H$ and its inverse and never use the transformation $\Phi_H'$ in the present paper. This is because the conformal exponent $e^{g\omega}$ would appear if $\Phi_H'$ were used. This appearance of conformal exponent is also responsible for the phenomenon that the Reeb-entangling energy $e^{trn(S_0, S_1)}$ is not symmetric. To keep the importance of this coordinate change in readers’ minds, we name and call $\Phi_H$ and its inverse gauge transformation.
Now let \((M, \lambda, J)\) be a contact triad for the contact manifold \((M, \xi)\), and equip with it the \textit{contact triad metric}

\[
g = d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda.
\]

We also consider the time-dependent contact triads and \(H = H(t, x)\) be a time-dependent Hamiltonian, and consider

\[
\begin{align*}
\left\{ (du - X_H \otimes dt)^{\pi(0,1)} = 0, & \quad d(\epsilon g_H(u)(u^*\lambda + H dt) \circ j) = 0 \\
u(\tau, 0) \in R, & \quad u(\tau, 1) \in R
\end{align*}
\]

(5.9)

where the function \(g_H : \mathbb{R} \times [0, 1] \to \mathbb{R}\) is defined by

\[
g_H(t, x) := g_{\psi^1_H(\psi^1_H)^{-1}(u(t, x))}
\]

(5.10)

already introduced in (1.14). Now we take the following coordinate change (1.16)

\[
u(\tau, t) := \Phi^{-1}_H(u(t, x)) = (\psi^1_H(\psi^1_H)^{-1})^{-1}(u(\tau, t))
\]

(5.11)

and consider the following particular time-dependent family of \(J\)'s.

\textbf{Choice 5.4 (CR almost complex structures of \(J\)).} Let \(J_0 \in J(\lambda)\). For given contact Hamiltonian \(H = H(t, x)\), we fix a time-dependent CR almost complex structures given by

\[
J = \{J_t\}_{0 \leq t \leq 1}, \quad J_t := (\psi^1_H(\psi^1_H)^{-1})J_0.
\]

of \(\lambda\)-admissible almost complex structures.

Now we apply the discussion given in the previous subsection to \(w = \overline{\eta}\) which is defined by (1.16), i.e.,

\[
\overline{\eta}(\tau, t) = (\psi^1_H(\psi^1_H)^{-1})^{-1}(u(\tau, t)).
\]

\textbf{Choice 5.5 (CR almost complex structures).} Let \(J_0 \in \mathcal{J}(\lambda)\). For given contact Hamiltonian \(H = H(t, x)\), we fix a time-dependent CR almost complex structures given by

\[
J = \{J_t\}_{0 \leq t \leq 1}, \quad J_t := (\psi^1_H(\psi^1_H)^{-1})J_0.
\]

of \(\lambda\)-admissible almost complex structures.

The (translated) intersection theoretic version of the contact instanton complex for the pair \((R_0, R_1)\) is generated by the set of Reeb chords

\[
\mathcal{N}eb(R_0, R_1)
\]

between them and its boundary map is constructed by the moduli space of unperturbed contact instanton equation (5.19).

On the other hand, the dynamical version of the complex is generated by some set of solutions of Hamilton’s equation \(\dot{x} = X_H(t, x)\) and its boundary map is constructed by the moduli space of (5.9). (See Lemma 5.6 and Proposition 5.16 below for the details of this correspondence.) These two frameworks are related by the bijective map \(\Phi_H\) via the correspondence

\[
\ell(t) = \psi^1_H((\psi^1_H)^{-1}(\ell(t))), \quad u(s, t) = \psi^1_H((\psi^1_H)^{-1}(\overline{\eta}(s, t))).
\]

(5.12)

In particular we have

\[
\overline{\eta} = \Phi^{-1}_H(u)
\]

when we regard \(u\) as a path on \(\Omega(R_0, R_1)\) and \(\overline{\eta}\) as one on \(\Omega(\psi^1_H(R_0), R_1)\).
For the given pair \((R_0, R_1)\) of compact Legendrian submanifolds and a Hamiltonian \(H\), we also consider the family (5.11),

\[ J = \{ J_t \}_{0 \leq t \leq 1}, \quad J_t := (\psi_H^t (\psi_H^1)^{-1})_* J_0 \]

of \(\lambda\)-admissible almost complex structures.

A straightforward calculation also gives rise to the following.

**Lemma 5.6.** Let \(J_0 \in \mathfrak{J}(\lambda)\) and \(J_t\) defined as in (5.11). We equip \((\Sigma, j)\) a Kähler metric \(h\). Let \(g_H(u)\) be the function defined in (1.14). Suppose \(u\) satisfies

\[
\begin{cases}
(du - X_H \otimes dt)_{J_t}^{\pi(0,1)} = 0, \\
u(t, 0) \in R_0, \quad u(t, 1) \in R_1
\end{cases}
\]

with respect to \(J_t\). Then \(\overline{\pi}\) satisfies

\[
\begin{cases}
\overline{\partial}_{\overline{J}_t} \overline{\pi} = 0, \\
d(\overline{\pi} \circ \lambda) = 0 \\
\overline{\pi}(\tau, 0) \in \psi_H^1(R_0), \quad \overline{\pi}(\tau, 1) \in R_1
\end{cases}
\]

for \(J_0\).

### 5.2. Gauge transformation: non-cutonomous version

In relation to the parametric study of moduli spaces considered in the present paper, we also need to consider the parametric gauge transformation too. We make the following specific choice of two-parameter Hamiltonians associated to each time-dependent Hamiltonian \(H = H_K(\tau, t, x)\) with slight abuse of notations. We then consider the 2-parameter family of contactomorphisms \(\Psi_{s,t} : = \psi^s_H \). We need to consider both \(t\)-developing Hamiltonian \(\text{Dev}_\lambda(t \mapsto \Psi_{s,t}) = H^x\). We consider the elongated two parameter family

\[ H_K(\tau, t, x) = \chi_K(\tau) H(t, x) \]

and the \(\tau\)-developing Hamiltonian

\[ G_K(\tau, t, x) = \text{Dev}_\lambda(\tau \mapsto \Psi_{\tau, t}^s) \]

where \(\Psi_{\tau, t}^s = \Psi_{\chi_K(\tau), t}\). Now we take the following coordinate change (1.16)

\[ \overline{\pi}(\tau, t) := \Phi_{H_K}^{-1}(u)(t, x) = (\psi_H^1(\psi_H^1)^{-1})(u(\tau, t)) = \psi_H^1(\psi_H^1)^{-1}(u(\tau, t)) \]

and consider the following particular time-dependent family of \(J_t\)'s.

**Choice 5.7** (CR almost complex structures of \(J\)). Let \(J' = J'_{(s,t)}\) be a given smooth family in \(\mathfrak{J}(\lambda)\). For given contact Hamiltonian \(H_K = H_K(s, t, x)\), we fix a time-dependent CR almost complex structures given by

\[ J = \{ J_{(s,t)} \}_{(s,t) \in [0,1]^2}, \quad J_{(s,t)} := (\psi_{H_K}^s (\psi_{H_K}^1)^{-1})_* J'_{(s,t)} \]

of \(\lambda\)-admissible almost complex structures.

By construction, \(J'\) satisfies

\[ J'_{(s,t)} \equiv J_0 \in \mathfrak{J}(\xi) \]

near \(s = 0, 1\). For the purpose of the present paper, especially for the purpose of establishing the \(C^0\)-estimates by the maximum principle, we will need to consider only the case

\[ J' \equiv J_0 \]
but we discuss this general case since it makes no difference for the study of gauge transformation and the energy estimates.

Through the parametric gauge transformation, we have the following parametric analog to Lemma 5.8

**Lemma 5.8.** Let \( J' \in \mathcal{J}(\lambda) \) and \( J \) defined as in (5.11). We equip \((\Sigma, j)\) a Kähler metric \( h \). Let \( g_H(u) \) be the function defined in (5.4). Suppose \( u \) satisfies
\[
\begin{align*}
     (du - X_H \otimes dt + X_G \otimes dr)^{(0,1)} &= 0, \\
     d(e^{g_H(u)}(u^* \lambda + H dt) \circ j) &= 0 \\
     u(\tau, 0) &\in R_0, \quad u(\tau, 1) &\in R_1
\end{align*}
\]
with respect to \( J_1 \). Then \( \overline{\tau} \) satisfies
\[
\begin{align*}
     \partial_t \overline{\tau} &= 0, \\
     d(\overline{\tau}^* \lambda \circ j) &= 0 \\
     \overline{\tau}(\tau, 0) &\in \psi_{H,K}^{-1}(R_0), \quad \overline{\tau}(\tau, 1) \in R_1
\end{align*}
\]
for \( J' \).

**Remark 5.9.** We will consider a variation the domain-dependent family \( J' = J'(\tau, t) \) such that
\[ J'(\tau, t) \equiv J_0 \]
for \(|\tau|\) sufficiently large. See Choice 9.6. Then we will also consider the 2-parameter perturbed contact instanton equation given by (5.3).

5.3. **Coercive elliptic estimates and subsequence convergence.** In this section, we fix a contact triad
\[ (M, \lambda, J) \]
where \( J \in \mathcal{J}(\lambda) \) is a \( \lambda \)-adapted CR-almost complex structure. All relevant estimate is in terms of the associated triad metric \( g = g_{\lambda, J} \) given by
\[ g = d\lambda(\cdot, J \cdot) + \lambda \otimes \lambda. \]

Then we consider the equation of (unperturbed) contact instantons
\[
\begin{align*}
     \overline{\tau}^* w &= 0, \\
     d(w^* \lambda \circ j) &= 0, \\
     w(\tau, 0) &\in R_0, \quad w(\tau, 1) \in R_1,
\end{align*}
\]
for a general Legendrian pair \((R_0, R_1)\) on general contact manifold \((M, \lambda)\). We will apply the result to the case of \((\psi_H^1(R), R)\) with \( w = \overline{\tau} \).

We collect some basic results on the analysis of the equation established in [OW18a] or in [Oh21a]. We start with the following local \( W^{2,2} \)-estimates.

**Theorem 5.10** (Theorem 1.3 [Oh21a]). Let \( w : \mathbb{R} \times [0, 1] \to M \) satisfy (5.19). Then for any relatively compact domains \( D_1 \) and \( D_2 \) in \( \Sigma \) such that \( \overline{D_1} \subset D_2 \) with \( w(\partial D_2) \subset R_0 \) or \( w(\partial D_2) \subset R_1 \), we have
\[ \|dw\|_{W^{2,2}_1(D_1)} \leq C_1\|dw\|_{L^2(D_2)} + C_2\|dw\|_{L^4(D_2)} \]
where \( C_1, C_2 \) are some constants which depend only on \( D_1, D_2 \) and \((M, \lambda, J)\) and \( C_3 \) is a constant which also depends on \( R_0 \) or \( R_1 \).

Once this \( W^{2,2} \)-estimate is established, we then proceed the following higher regularity estimates.
Theorem 5.11 (Theorem 1.4 [OY23]). Let $w$ be a contact instanton satisfying (5.19). Then for any pair of domains $D_1 \subset D_2 \subset \check{\Sigma}$ such that $D_1 \subset D_2$ with $w(\partial D_2) \subset R_0$ or $w(\partial D_2) \subset R_1$, we have

$$\|dw\|_{C^{k,\alpha}(D_1)} \leq C_\delta(\|dw\|_{\dot{W}^{1,2}(D_2)})$$

for some function $C_\delta = C_\delta(r) > 0$ for $r > 0$ and continuous at $r = 0$ with $C(0) = 0$ which depends on $J, \lambda, D_1, D_2, R_0$ or $R_1$, and $(k, \alpha)$ but independent of $w$.

Let $w : \mathbb{R} \times [0, 1] \to M$ be any smooth map. As in [OW18a], we define the total $\pi$-harmonic energy $E^\pi(w)$ by

$$E^\pi(w) = E^\pi_{(\lambda, J; \check{\Sigma}, h)}(w) = \frac{1}{2} \int |w|^2$$

where the norm is taken in terms of the given metric $h$ on $\check{\Sigma}$ and the triad metric on $M$.

Next we study the asymptotic behavior of contact instantons $w$ satisfying the following hypotheses.

Hypothesis 5.12. Assume $w : \mathbb{R} \times [0, 1] \to M$ satisfies the contact instanton equation (5.19) and

1. $E^\pi_{(\lambda, J; \check{\Sigma}, h)}(w) < \infty$ (finite $\pi$-energy);
2. $\|dw\|_{C^0(\check{\Sigma})} < \infty$.

For any $w$ satisfying Hypothesis 5.12, we associate two natural asymptotic invariants at $\tau \pm \infty$.

Definition 5.13. The asymptotic action is defined to be

$$T := \lim_{\tau \to \infty} \int_{\{\tau\} \times [0, 1]} (w|_{\{0\} \times [0, 1]})^* \lambda$$

and the asymptotic charge is by

$$Q := \lim_{\tau \to \infty} \int_{\{\tau\} \times [0, 1]} (w|_{\{0\} \times [0, 1]})^* \lambda \circ j.$$

provided they exist. (Here we only look at the positive end. The case of negative end is similar.)

The above finite $\pi$-energy and $C^0$ bound hypotheses imply

$$\int_{[0, \infty) \times [0, 1]} |d^\pi w|^2 \, d\tau \, dt < \infty, \quad \|dw\|_{C^0([0, \infty) \times [0, 1])} < \infty.$$  

Remark 5.14. In general there is no reason why these limits exist and even if the limits exist, they may also depend on the choice of subsequences under Hypothesis 5.12. In the closed string case, [OW18a] shows that the asymptotic charge $Q$ may not vanish which is the key obstacle to the compactification and the Fredholm theory of contact instantons for $Q \neq 0$.

As in [OW18a], [Oh21a], we call $T$ the asymptotic contact action and $Q$ the asymptotic contact charge of the contact instanton $w$ at the given puncture.
Theorem 5.15 (Vanishing asymptotic charge; Theorem 6.7 [Oh21a]). Let \( w : [0, \infty) \times [0, 1] \to M \) satisfy the contact instanton equations (5.19) and Hypothesis 5.12. Then for any sequence \( s_k \to \infty \), there exists a subsequence, still denoted by \( s_k \), and a massless instanton \( w_\infty(\tau, t) \) (i.e., \( E^*(w_\infty) = 0 \)) on the cylinder \( \mathbb{R} \times [0, 1] \) that satisfies the following:

1. On any given compact subset \( K \subset [0, \infty) \), we have
   \[
   \lim_{k \to \infty} w(s_k + \tau, t) = w_\infty(\tau, t)
   \]
   in the \( C^l(K \times [0, 1], M) \) sense for any \( l \).
2. \( w_\infty \) has \( Q = 0 \) and the formula \( w_\infty(\tau, t) = \gamma(T t) \), where \( \gamma \) is some Reeb chord joining \( R_0 \) and \( R_1 \) with action \( T \).

We mention that the asymptotic action \( T \) could be either positive, negative or zero, i.e., the pair \((T, \gamma)\) is a iso-speed Reeb chord in the sense of Definition 1.14.

5.4. Obstruction to existence of translated intersections. We apply the above discussion to the case \( R_0 = R_1 = R \) and then the subsequence convergence result, Theorem 5.15, to the contact instantons \( w = \varphi \). In the rest of the present paper, we will investigate the simplest dynamical entanglement question for the two-component link \((\psi(R), R)\), which concerns existence of Reeb chords from \( \psi(R) \) to \( R \).

The following proposition illustrates how the analytical study of the above considered perturbed Cauchy-Riemann equation gives rise to an existence result of Reeb chords between \( R \) and \( R \). It is the converse of Lemma 5.6 which holds for the case \( R_0 = R_1 \).

Proposition 5.16. Let \( u : [0, \infty) \times [0, 1] \to M \) be a smooth map satisfying the boundary condition
\[
u(\tau, 0), \quad u(\tau, 1) \in R.
\]
Denote by \( \varphi \) the map defined in (1.16). Suppose that \( \varphi \) is a contact instanton of finite \( \pi \)-energy with uniform \( C^1 \)-bound satisfying the boundary condition
\[
\varphi(\tau, 0) \in \psi_H(R), \quad \varphi(\tau, 1) \in R.
\]
Then the followings hold:
1. There exist a pair of translated points \( (\eta_\pm, x_\pm) \) with
   \[
x_\pm \in \psi_H(R), \quad \phi_{R_\pm}^\eta(x_\pm) \in R
   \]
   such that the asymptotic Reeb chords of \( \varphi \) have the form
   \[
t \mapsto \phi_{R_\pm}^\eta(x_\pm).
   \]
2. We have a contact Hamiltonian trajectory \( \gamma_\pm \) from \( R \) to \( \psi_H^1(R) \) given by
   \[
   \gamma_\pm(t) = z_\pm^H(t) = \psi_H^1((\psi_H^1)^{-1}(x_\pm)).
   \]
(See (5.24).)

Proof. Statement (1) is the consequence of [Oh21a, Theorem 1.6]. (See Theorem 5.15 below for the precise statement.)

Statement (2) is just a translation of Statement (1) by the definition of translated points \( (\eta_\pm, x_\pm) \) associated the asymptotic Reeb chords at \( \pm \infty \) respectively. \( \square \)
The following diagram describes the relationship between translated intersection points and Reeb chords between $R$ and $\psi(R)$ for a Hamiltonian $H \mapsto \psi$:

![Diagram](5.25)

Figure 1. Translated intersection

Combining Lemma 5.6 and Proposition 5.16, we have the following which shows that nonexistence of intersection $\psi^1_H(R) \cap Z_R = \emptyset$ is an obstruction to the existence of finite energy solution to (5.13). This is the contact analog to a similar obstruction appearing in [Oh97a, Lemma 2.2].

The following rephrased form of the last statement will play an important role as an obstruction to compactness of the moduli space of solutions of a suitably cut-off version of Floer trajectory equation.

**Corollary 5.17** (Obstruction to existence). Suppose $\psi^1_H(R) \cap Z_R = \emptyset$. Then the equation (5.13) has no solution of finite $\pi$-energy.

6. Maximum principle and $C^0$ estimates on tame contact manifolds

The following theorem is the reason why we introduce the class of tame contact manifolds.

**Theorem 6.1.** Let $(M, \lambda)$ be a contact manifold and consider a $\lambda$ quasi-pseudocconvex pair $(\psi, J)$. Let $\tilde{R} = \{R_i\}$ be any Legendrian link consisting of compact Legendrian submanifolds $R_i$. Suppose that there exists a compact subset $K \subset M$ containing $\{R_i\}$ and $\{\gamma_i\}$. Then for any contact instanton $w : \Sigma \to M$ satisfying

\[
\begin{cases}
\overline{D} w = 0, & d(w^* \lambda \circ j) = 0, \\
w(\gamma_iz_{i+1}) \subset R_i, & i = 0, \ldots, k \\
w(\infty_i) = \gamma_i, & i = 0, \ldots, k
\end{cases}
\]

we have $\text{Image } w \subset K$.

**Proof.** By definition of tame contact manifolds, there exists a compact subset $K' \subset M$ such that on $M \setminus K'$ $\psi$ satisfies

\[
\mathcal{L}_{R_\lambda} d\psi = 0,
\]

and

\[
-d(d\psi \circ J) + d\eta \wedge \beta = h \, d\lambda \quad \text{on } \xi,
\]

for some smooth one-form $\beta$ and smooth functions $g$ and $h \geq 0$ on $M \setminus K$. By enlarging $K$, we may assume that $K$ itself satisfies the above property.

Let $w$ be a solution to (6.1). By the standing hypothesis, it will be enough to prove that the maximum $\psi \circ w$ cannot be achieved at a point in $M \setminus K$.

Suppose to the contrary that a maximum is achieved at $z_0$ with $w(z_0) \in M \setminus K$. We decompose

\[
dw = d^s w + w^* \lambda \otimes R_\lambda
\]
which we often just write $dw = d^w w + w^* \lambda R_\lambda$ as a $T^* M$-valued one-form on $\hat{\Sigma}$ in the calculation below and henceforth.

We compute

$$\Delta (\psi \circ w) \, dA = -d(d(\psi \circ w) \circ j) = -d(d(\psi \circ dw) \circ j) = -d(d(w^* \lambda \otimes R_\lambda(w)) \circ j)$$

Using the $\lambda$-tameness of $\psi$ and the closedness of $w^* \lambda \circ j$, we compute the second term

$$d(d(w^* \lambda \otimes R_\lambda(w)) \circ j) = d((w^*(d(\psi(R_\lambda))))(w^* \lambda \circ j)) = w^*(d(\psi(R_\lambda))) \wedge w^* \lambda \circ j.$$ 

This vanishes by (6.2) since

$$d(d(\psi(R_\lambda))) = \mathcal{L}_{R_\lambda} d\psi.$$

For the first term of (6.5), using the equation $Jd^w w = d^w w \psi$ and the properties $J(R_\lambda) = 0, \quad \text{Image } J = \xi,$

of CR almost complex structure $J$, we compute

$$-d(d(\psi(d^w w \circ j))) = -w^*(d(\psi \circ J))(v, jv) = -d(\psi \circ J)(dw(v), dw(jv)).$$

Again we decompose $dw = d^w w + w^* \lambda R_\lambda$ and evaluate

$$d(\psi \circ J)(dw(v), dw(jv)) = d(\psi \circ J)(d^w w(v) + w^* \lambda(v) R_\lambda(w(z_0)), d^w w(jv) + w^* \lambda(jv) R_\lambda(w(z_0)))$$

Now we first evaluate the last three terms. Obviously the fourth term vanishes.

**Lemma 6.2.** The sum of the second and the third vanishes.

**Proof.** We evaluate the first term

$$d(\psi \circ J)(d^w w(v), w^* \lambda(jv) R_\lambda(w(z_0)))$$

by the condition (6.4) for the second equality. Similar computation leads to

$$d(\psi \circ J)(w^* \lambda(v) R_\lambda(w(z_0)) = w^* \lambda(v) g \circ w(z_0) d(\psi \circ w)(v) - w^* \lambda(v) R_\lambda,$$

By summing them over, we get

$$-w^* \lambda(jv) g \circ w(z_0) d(\psi \circ w)(v) + w^* \lambda(v) g \circ w(z_0) d(\psi \circ w)(jv)$$

which vanish at $z_0$ since $z_0$ is a critical point of $\psi \circ w$. □
Finally, using the definition (6.3) of quasi-plurisubharmonicity of \( \psi \), we rewrite the first term above into

\[
-d(\psi \circ J)(dw(v), dw(jv)) = (h\lambda - d\psi \wedge \beta)(dw(v), dw(jv))
\]

where the second equality holds at the critical point \( z_0 \) of \( \psi \circ w \). In turn, we rewrite

\[
h\lambda(dw(v), dw(jv)) = h\lambda(d^e w(v), d^e w(jv)) = h\lambda(d^e w(v), d^e w(jv)) \geq 0
\]

for all \( v \in T_{z_0} \dot{\Sigma} \), where the last positivity follows by (6.3). This proves

\[
\Delta(\psi \circ w) \geq 0.
\]

Once we have this differential inequality, the classical strong maximum principle applies to get the inequality

\[
\sup_{\dot{\Sigma}} |\psi \circ w| = \max \left\{ \sup_{\partial \dot{\Sigma}} |\psi \circ w|, \sup_{k=0}^k |\psi \circ \gamma_i| \right\}
\]

(See [GT70, Theorem 3.1] for example). Since \( w(\partial \dot{\Sigma}) \subset R \) and \( \gamma_i \) are given, this proves

\[
\sup_{\dot{\Sigma}} |\psi \circ w| \leq \max \left\{ \sup_{\partial \dot{\Sigma}} |\psi||_{R, \sup_{k=0}^k |\psi \circ \gamma_i|} \right\}.
\]

By setting

\[
C = \max \left\{ \sup_{\partial \dot{\Sigma}} |\psi||_{R, \sup_{k=0}^k |\psi \circ \gamma_i|} \right\}, \quad K = \psi^{-1}([-C, C])
\]

we have now finished the proof. \( \square \)

## 7. The \( \pi \)-energy identity

Let \( H = H(t, x) \) be a given Hamiltonian. It turns out that the correct definition of the \( \pi \)-energy for the perturbed contact instanton is the following.

**Definition 7.1** (The \( \pi \)-energy of perturbed contact instanton). Let \( u : \mathbb{R} \times [0, 1] \to M \) be any smooth map. We define

\[
E_{J,H}^\pi(u) := \frac{1}{2} \int e^{g_H(u)}(du - X_H(u) \otimes dt)^\pi|_J^2
\]

call it the \textit{off-shell} \( \pi \)-energy.

Now we apply the gauge transformation \( \Phi_H^{-1} \) to \( u \) and define \( \pi := \Phi_H^{-1}(u) \) which has the expression

\[
\pi(\tau, t) = (\psi_H^1(\psi_H^1)^{-1})^{-1}(u(\tau, t)) = \psi_H^1(\psi_H^1)^{-1}(u(\tau, t)). \tag{7.1}
\]

The following identity justifies the presence of weight \( e^{g_H(u)} \) in this definition. (Also see Remark 7.3 for further justification.)

**Proposition 7.2.** Let \( u : \mathbb{R} \times [0, 1] \to M \) be any smooth map and \( \pi \) be as above. Then

\[
E_{J,H}^\pi(u) = E_J^\pi(\pi). \tag{7.2}
\]
Proof. We first note that \( u \) satisfies
\[
0 = \frac{\partial}{\partial t} \pi(\partial_r) = \left( \frac{\partial \pi}{\partial \tau} \right)^\pi + J' \left( \frac{\partial \pi}{\partial t} \right)^\pi.
\]
We compute \(|d^r \pi(\partial_r)|_J\) and \(|d^r \pi(\partial_t)|_J\) separately. By definition, we have
\[
u(t, u) = \psi_H^{-1}(\pi(t, u)).
\]
Then
\[
du(\partial_r) = \frac{\partial u}{\partial \tau} d(\psi_H^{-1})(\frac{\partial \pi}{\partial \tau} + X_H(u(\tau, t)))
\]
and hence we have
\[
\frac{\partial \pi}{\partial \tau} = (d(\psi_H^{-1}))^{-1} \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right).
\]
And more easily, we compute
\[
\frac{\partial \pi}{\partial t} = (d(\psi_H^{-1}))^{-1} \left( \frac{\partial u}{\partial t} \right).
\]
We recall that \( J(\xi) \subset \xi \) for any \( \lambda \)-adapted CR-almost complex structure \( J \), and
\[
d(\psi_H^{-1})(\xi) = \xi
\]
for the conformal exponent of the contactomorphism \( \psi \) in the following calculation.

Then we have
\[
\left| \left( \frac{\partial \pi}{\partial t} \right)^\pi \right|_{J'}^2 = \left| d\psi_t \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right) \right|_{J'}^2
\]
\[
= d\lambda \left( d\psi_t \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right)^\pi, J' \left( \frac{\partial u}{\partial t} - X_H(u(\tau, t)) \right)^\pi \right)
\]
\[
= \psi_t^* d\lambda \left( \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right)^\pi, J_t \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right)^\pi \right)
\]
\[
= d(e^{\gamma t} \lambda) \left( \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right)^\pi, J_t \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right)^\pi \right)
\]
\[
= e^{\gamma t} \left( \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right)^\pi, J_t \left( \frac{\partial u}{\partial \tau} - X_H(u(\tau, t)) \right)^\pi \right) \left| J_t \right|
\]
Here we used the vanishing
\[
\lambda(\cdot)^\pi = 0
\]
for the penultimate equality.

More easily, we also derive
\[
\left| \left( \frac{\partial \pi}{\partial \tau} \right)^\pi \right|_{J'}^2 = e^{\gamma t} \left| \left( \frac{\partial u}{\partial \tau} \right)^\pi \right|_{J_t}.
\]
By adding the two, we have finished the proof.
Remark 7.3. (1) Unless there were aforementioned exponential weight-factor, it would be possible to achieve the kind of a priori estimates neither for the \(\pi\)-energy nor for the \(\lambda\)-energy we are going to define in Part 3. In this regard, the presence of the exponential weight factor in the definition of \(\pi\)-energy is essential.

(2) In fact, the most natural explanation of the appearance of this weighting factor can be given in terms of the contact mapping tori construction. See [OS, Section 2.1]. We will elaborate this point of view when we consider contact fibration elsewhere.

The following proposition is one of the key energy estimates for the solutions \(u\) of (5.1) in terms of the geometry of Legendrian boundary conditions and its asymptotic chords.

Proposition 7.4. Let \(\psi_t\) be as in (7.3). Let \(u\) be any finite energy solution of (5.1) with the limits
\[
\gamma_{\pm}(t) := \lim_{\tau \to \pm \infty} u(\tau, t)
\]
and let \(\overline{u}\) be as above. Consider the paths given by
\[
\gamma_{\pm}(t) = \psi_t(\gamma_{\pm}(t)).
\]
Then \(\gamma_{\pm}\) are Reeb chords from \(\psi_1^H(R_0)\) to \(R_1\) and satisfy
\[
E_{\gamma_{\pm}}^\pi J_H(u) = \int_0^1 (\gamma_+)^* \lambda - \int_0^1 (\gamma_-)^* \lambda. \tag{7.4}
\]

Proof. The first statement immediately follows from definition of the gauge transformation \(\Phi_H\) and by the subsequence convergence theorem, Theorem 5.15.

For the energy identity, it is enough to compute \(E_{\gamma_{\pm}}^\pi(J)\) by Proposition 7.2. Similarly as in the proof of Proposition 7.2, we compute, this time using the equation
\[
Jd\overline{u}(\partial_t) = d\overline{u}(\partial_t),
\]
\[
\int_{-\infty}^{\infty} \int_0^1 |d\overline{u}(\partial_{\tau})|^2 d\tau d\tau
= \int_{-\infty}^{\infty} \int_0^1 d\lambda(d\overline{u}(\partial_{\tau}), Jd\overline{u}(\partial_{\tau})) d\tau d\tau
= \int_{-\infty}^{\infty} \int_0^1 d\lambda(d\overline{u}(\partial_{\tau}), d\overline{u}(\partial_{\tau})) d\tau d\tau = \int (\overline{\pi})^* d\lambda
= \left( \int_0^1 (\gamma_+)^* \lambda - \int_0^1 (\gamma_-)^* \lambda \right)
+ \int_{-\infty}^{\infty} \lambda \left( \frac{\partial \overline{\pi}}{\partial \tau}(\tau, 0) \right) d\tau - \int_{-\infty}^{\infty} \lambda \left( \frac{\partial \overline{\pi}}{\partial \tau}(\tau, 1) \right) d\tau
= \int_0^1 (\gamma_+)^* \lambda - \int_0^1 (\gamma_-)^* \lambda.
\]

Here the last equality follows from the Legendrian boundary condition
\[
\overline{\pi}(\tau, 0) \in \psi_1^H(R), \quad \overline{\pi}(\tau, 1) \in R
\]
for \(i = 0, 1\). This finishes the proof. \qed
Part 2. Study of Reeb-untangling energy: Proof of Theorem 1.12

The discussion given in the previous section applies more generally when the pair \((J, H)\) depends on the domain parameter.

In this part, we set-up the deformation-cobordism framework of parameterized moduli space of perturbed contact instantons by adapting a similar parameterized Floer moduli spaces appearing in the study of the displacement energy of compact Lagrangian submanifolds in [Oh97a].

8. Cut-off Hamiltonian-perturbed contact instanton equation

We will consider the two-parameter family of CR-almost complex structures and Hamiltonian functions:

\[ J = \{ J(s,t) \}, \quad H = \{ H^s_t \} \text{ for } (s,t) \in [0,1]^2. \]

We write \( H_t^s(x) := H(s,t,x) \) in general for a given two-parameter family of functions \( H = H(s,t,x) \). Note that \([0,1]^2\) is a compact set and so \( J, H \) are compact families. We always assume the \((s,t)\)-family \( J \) or \( H \) are constant near \( s = 0, 1 \). Then we take the \( K \)-family of cut-off functions \( \chi_K \) introduced in (1.21).

8.1. Setting up the parameterized moduli space. Knowing that \( H_K \equiv 0 \) when \( |\tau| \) is sufficiently large and having Proposition 8.10 in our disposal which we will prove later, we consider a one-parameter family of domains defined as follows.

Consider the following capped semi-infinite cylinders

\[ \Theta_- = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \cup \{ z \in \mathbb{C} \mid \text{Re}z \geq 0, |\text{Im}z| \leq 1 \} \]
\[ \Theta_+ = \{ z \in \mathbb{C} \mid \text{Re}z \leq 0, |\text{Im}z| \leq 1 \} \cup \{ z \in \mathbb{C} \mid |z| \leq 1 \}. \]

We will fix the \( K_0 \) once and for all and consider \( K \) with \( 0 \leq K \leq K_0 \). (See Proposition 8.10 below for its required condition to satisfy.) For such given \( K_0 \), we define the spaces

\[ \Theta_{-, K_0+1} := \{ z \in \Theta_- \mid \text{Re}z \leq K_0 + 1 \}, \]
\[ \Theta_{+, K_0+1} := \{ z \in \Theta_- \mid \text{Re}z \geq -K_0 - 1 \}. \]

We glue the three spaces

\[ \Theta_{-, K_0+1}, \quad [-K_0 + 1, K_0 + 1] \times [0, 1], \quad \Theta_{+, K_0+1} \]

subdomains of \( \mathbb{R} \times [0,1] \), by making the identification

\[ (0,t) \in \Theta_{-, K_0+1} \iff (-K_0 - 1, t) \in [-K_0 - 1, K_0 + 1] \times [0,1], \]
\[ (0,t) \in \Theta_{+, K_0+1} \iff (K_0 + 1, t) \in [-K_0 - 1, K_0 + 1] \times [0,1], \]

respectively. We denote the resulting domain as

\[ \Theta_{K_0+1} := \Theta_- \#_{K_0+1}(\mathbb{R} \times [0,1]) \#_{K_0+1} \Theta_+ \subset \mathbb{C} \]

and equip it with the natural complex structure induced from \( \mathbb{C} \). (See [FOOO09, Figure 8.1.2] for the visualization of this domain.) We can also decompose \( \Theta_{K_0+1} \) into the union

\[ \Theta_{K_0+1} := D^- \cup [-2K_0 - 1, 2K_0 + 1] \cup D^+ \]

(8.2)
where we denote
\[ D^\pm = D^\pm_{K_0} := \{ z \in \mathbb{C} \mid |z| \leq 1, \pm \text{Im}(z) \leq 0 \} \pm (2K_0 + 1) \] respectively.

**Remark 8.1** (Domain \( \Theta_{K_0+1} \)). We highlight the obvious fact that the domain \( \Theta_{K_0+1} \) is a compact disk-like domain, where any closed one-form is exact. In particular any contact instanton \( w \) on \( \Theta_{K_0+1} \) can be lifted to a pseudoholomorphic curve \( \tilde{u} = (w, f) \) for a uniquely defined function \( f : \Theta_{K_0+1} \to \mathbb{R} \) satisfying
\[ df = w^* \lambda \circ j \]
where \( w^* \lambda \circ j \) is closed by the defining equation of contact instantons.

We make the following specific choice of two-parameter Hamiltonians associated to each time-dependent Hamiltonian \( H = H(t, x) \) with slight abuse of notations.

**Choice 8.2.** Take the family \( H = H(s, t, x) \) given by
\[ H^s(t, x) = sH(t, x). \] (8.4)

We consider the 2-parameter family of contactomorphisms \( \Psi_{s,t} := \psi_{t,s}^j \). Obviously we have the \( t \)-developing Hamiltonian \( \text{Dev}_t(\tau \mapsto \Psi_{(s,t)}) = H^s \). We then consider the elongated two parameter family
\[ H_K(\tau, t, x) = \chi_K(\tau)H(t, x) \]
and write the \( \tau \)-developing Hamiltonian
\[ G_K(\tau, t, x) = \text{Dev}_\tau(\tau \mapsto \Psi_{\tau, t}^K) \]
where \( \Psi_{\tau, t}^K = \Psi_{\chi_K(\tau), t} \).

Then we consider the 2-parameter perturbed contact instanton equation for a map \( u : \Theta_{K_0+1} \to M \) given by
\[ \begin{cases} \left( du - X_{H_K}(u) \, dt + X_{G_K}(u) \, ds \right)^{\pi(0,1)} = 0, \\ d( e^{g_K(u)} (u^* \lambda + u^* H_K dt - u^* G_K d\tau) \circ j) = 0, \\ u(\tau, 0) \in R, u(\tau, 1) \in R. \end{cases} \] (8.5)

where \( g_K(u) \) is the function on \( \Theta_{K_0+1} \) defined by (5.4) for \( 0 \leq K \leq K_0 \). We note that if \( |\tau| \geq K + 1 \), the equation becomes
\[ \overline{\partial} u = 0, \quad d(u^* \lambda \circ j) = 0. \] (8.6)

It will be useful to introduce the full version of vector-valued one-form on \( \Theta_{K_0+1} \) that is familiar in the study of Hamiltonian fibrations in symplectic topology. (See [Sei97], [Ent01], [Oh15b, Section 20.2].) In this spirit, we regard
\[ P_{H_K}(u) := X_{H_K}(u) \, dt - X_{G_K} \, ds \] (8.7)
as a \( u^* TM \) valued one-form on \( \Theta_{K_0+1} \) for which we express using the basis of one-forms \( \{ ds, dt \} \) coming from the inclusion map \( \Theta_{K_0+1} \subset \mathbb{R} \times [0, 1] \). This being said, we define the relevant \( \pi \)-energy as follows.

**Definition 8.3.** Let \( (J, H) \) be a Floer data with \( J = J(t, x), H = H(t, x) \). Fix \( K_0 > 0 \). For \( 0 \leq K \leq K_0 \), we define the (perturbed) off-shell \( \pi \)-energy \( E^\pi_{J, H} \) by
\[ E^\pi_{J, H}(u) = \int_{\Theta_{K_0+1}} e^{g_K(u)} (du - P_K(u))^2 |J_K| dA \] (8.8)
for any smooth map \( u : \Theta_{K_0+1} \to M \).
We also define the corresponding vertical energy $E_{\lambda, J, K, H}$ in Section 12, which is also called the $\lambda$-energy. Then we define the total energy by

$$E_{J, K, H}(u) = E^\pi_{J, K, H}(u) + E^\lambda_{J, K, H}(u).$$

Then we introduce the following moduli space of finite energy solutions.

**Definition 8.4.** Let $K_0 \in \mathbb{R} \cup \{\infty\}$ be given. For $0 \leq K \leq K_0$, we define

$$M_K(M, R; J, H) = \{u : \mathbb{R} \times [0, 1] \to M \mid u \text{ satisfies (8.5) and } E_{J, K, H}(u) < \infty\}$$

and

$$M^\text{para}_{[0, K_0]}(M, R; J, H) = \bigcup_{K \in [0, K_0]} \{K\} \times M_K(M, R; J, H).$$

The Fredholm theory developed in [Oh23] provides a natural smooth structure with

$$M_K(M, R; J, H), \quad M^\text{para}_{[0, K_0]}(M, R; J, H).$$

[Oh23] deals with the closed string case which can be easily adapted to the current case with boundary, whose details will be given elsewhere.

**Remark 8.5.**

1. We would like to attract readers’ attention to the difference in the setting-up of the domain-varying parameterized moduli space: In [Oh97a], we fix the domain to be the total strip $\mathbb{R} \times [0, 1]$ while here we compactify the domain by using the $K$-dependent family of capped-strips and vary $K \to \infty$. The reason for this is because for the pseudoholomorphic curves for the Lagrangian boundary condition the removal singularity theorem automatically compactifies the domain of any finite energy solution defined on $\mathbb{R} \times [0, 1]$, while such a removable singularity theorem does not apply to the contact instanton equation. Using the fact that the contact instanton equation is still coordinate free, we consider the family of compact domains $\Theta_{K_0+1}$ and repeat the same kind of proof given in the proof of [Oh97a, Lemma 2.2]. Indeed we can also apply the same domain-changing family to the latter case and rewrite the proof of [Oh97a, Lemma 2.2] as in the way how the current proof goes.

2. Because of the aforementioned difference working on the compact domains $\Theta_{K_0+1}$ instead of the full strip $\mathbb{R} \times [0, 1]$, strictly speaking, we need to smooth out the seam $|\tau| = K + 1$ on a small neighborhood thereof, say on

$$\{(\tau, t) \in \Theta_{K_0+1} \mid \tau \in [K + 1 - \delta, K + 1 + \delta]\}$$

for some given $\delta > 0$. (Note that $\partial \Theta_{K_0+1}$ is $C^1$ but not smooth.) We replace $\chi_K$ by $\chi_{K+\delta}$ so that the equation (8.5) still becomes (8.6) for $0 \leq K \leq K_0$. This being said, we will ignore non-smoothness of the boundary of $\Theta_{K_0+1}$ at $|\tau| = K + 1$ and directly work with it instead of the smoothed one for the simplicity of exposition.

### 8.2. Obstruction to existence of finite energy solution.

We recall the following correspondence

$$\psi(R) \cap Z_R = \emptyset \iff \psi(R) \cap R = \emptyset \quad \& \quad \mathcal{Reb}(\psi(R), R) = \emptyset$$

from Lemma 3.15 applied to $(S_1, S_2) = (R, R)$. We will show that the condition $\psi(R) \cap Z_R = \emptyset$ will play the role of an obstruction to the existence of finite energy solutions to the contact instanton equation (5.9). In this subsection, we assume
the priori energy bound which we will establish in Section 11 for the λ-energy $E_{\lambda, u}^J(u)$ and Section 13.

**Remark 8.6.** This obstruction is the analog to the obstruction employed by in [Che98], [Oh97a] for the study of displacement energy of compact Lagrangian submanifolds. Similarly as in the present paper, this kind of non-intersection played the role of an obstruction to compactness of certain parameterized moduli space of Hamiltonian-perturbed Floer trajectories in symplectic geometry. (See [Oh97a] for the details.)

To state the aforementioned a priori energy bounds, we also consider the family $J' = J'(s,t)$ defined by

$$J'_{(s,t)} = (\psi^t_H(\psi^s_H))^{-1} J_{(s,t)} = (\psi^t_H(\psi^s_H))^{-1} J_{(s,t)}$$  \hspace{1cm} (8.12)

for a given $J \in \{J_{(s,t)}\}$. We also consider the gauge transformation of $u$

$$\pi_K(\tau,t) := \psi^t_{x_K}(\tau) H(\psi^s_{x_K}(\tau))^{-1} u(\tau,t).$$  \hspace{1cm} (8.13)

We will prove the following propositions in Section 11 and Sections 12, 13 of Part 3 respectively. We recall the definition of oscillation

$$osc(H_t) = \max H_t - \min H_t.$$

**Proposition 8.7.** Let $u$ be any finite energy solution of (8.5). Then we have

$$E_{\pi}^{J'}(\pi_K) \leq \int_0^1 osc(H_t) dt =: \|H\|$$  \hspace{1cm} (8.14)

**Proposition 8.8.** Let $u$ be any finite energy solution of (8.5). Then we have

$$E_{\perp}^{J'}(\pi_K) \leq \|H\|.$$  \hspace{1cm} (8.15)

Let

$$E_{J'}(\pi_K) = E_{\pi}^{J'}(u) + E_{\perp}^{J'}(\pi_K)$$

be the total energy. Then using these energy bounds, we will prove the following Gromov-type weak convergence theorem, the details of which we postpone till Section 13 of Part 3. We just mention here that the above explicit upper bound of $E_{\pi}(J_K, u)$ plays a fundamental role in our quantitative study while that of $E_{\perp}(J_K, u)$, other than the existence of uniform finite upper bound, does not play any role.

**Theorem 8.9.** Suppose $K_\alpha \to K_\infty \leq K_0 \in \mathbb{R}$ and let $\pi_\alpha$ be solutions of (5.3) for $K = K_\alpha$ with uniform energy bound

$$E_{J_{K_\alpha}}(\pi_\alpha) < C < \infty$$

for $C$ independent of $\alpha$. Let $\pi$ be as above.

Then, there exist a subsequence again enumerated by $\nu_\alpha$ and a cusp-trajectory $(\pi, \nu, w)$ such that

1. $u$ is a solution of (5.17) for $H_K$ with $K = K_\infty$.
2. $\nu = \{\nu_i\}_{i=1}^k$ where
   - each $\nu_i$ is a contact instanton on the plane $\mathbb{C}$ with end converging to a closed Reeb orbit, and
   - each $w_j$ is a contact instanton on the half plane $\mathbb{H}$ with its boundary lying on $R$ with end converging to a Reeb chord of the pair $(\psi(R), R)$.
(3) We have
\[ \lim_{\alpha \to \infty} E^7_{J, K_{\alpha}}(\pi_{\alpha, K_{\alpha}}) = E^7_{J, K_{\infty}}(\pi_{K_{\infty}}) + \sum_{i} v^*_i d\lambda_{z_i} + \sum_{j} w^*_j d\lambda_{z_j}. \]

(4) And \( \pi_{\alpha, K_{\alpha}} \) weakly converges to \((\pi, v, w)\) in the sense of Gromov-Floer-Hofer and converges in compact \(C^\infty\) topology away from the nodes. Furthermore, if \( v = w = \emptyset \), then \( u_{\alpha} \to u \) smoothly on \( \Theta_{K_{\alpha}+1} \).

With the above energy bound and convergence result in our disposal, we prove the following obstruction result. The following proposition is the analog to [Oh97a, Lemma 2.2]: It shows that non-intersection \( \psi^j_H(R) \cap Z_R = \emptyset \) is an obstruction to the asymptotic existence of finite energy solutions for the autonomous equation (5.13).

**Proposition 8.10.** Let \( H_t \) be the Hamiltonian such that \( \psi^j_H(R) \cap Z_R = \emptyset \) is empty and \( H = sH_t \) and \( J \) as before. Suppose \( \|H\| < T_\lambda(M, R) \). Then there exists \( K_0 > 0 \) sufficiently large such that \( M_{K}(M, R; J, H) \) is empty for all \( K \geq K_0 \).

**Proof.** We give the proof by contradiction by closely following the scheme used in [Oh97a].

Suppose to the contrary that there exists a sequence \( K_{\alpha} \to \infty \) and a solution \( u_{\alpha} \in M_{K_{\alpha}}(M, R; J, H) \). By the a priori energy bound from Propositions 8.7, 8.8 for \( M_{K_{\alpha}}(M, R; J, H) \), there exists \((u, v, w)\) such that \( u_{\alpha} \to (u, v, w) \) in the sense of Theorem 8.9. On the other hand, by the hypothesis \( \|H\| < T_\lambda(M, R) \), we derive \( v = \emptyset = w \).

In particular, we have produced the \( C^1 \)-limit \( u \) of \( u_{\alpha} \) which satisfies the equation (5.9), i.e.,
\[
\begin{cases}
(du - X_H(u))^{\tau(0,1)} = 0, & d(e^{\partial_H}(u^* \lambda + u^* H\,dt) \circ j) = 0, \\
u(\tau, 0), & u(\tau, 1) \in R
\end{cases}
\]
with \( \pi \)-energy bound. In particular we also have \( \|du\|_{C^0} < \infty \). Furthermore we also have
\[ E^7_{J, H}(u) \leq \limsup_{\alpha \to \infty} E^7_{J, K_{\alpha}, u_{\alpha}}(u_{\alpha}) \leq \|H\| < \infty. \]
Then by Theorem 5.15 applied to \( w = \pi \), we derive \( \Reb(\psi(R), R) \neq \emptyset \) for \( \psi = \psi^j_H \) which in turn implies
\[ \psi(R) \cap Z_R \neq \emptyset. \]
This contradicts to the standing hypothesis and hence finishes the proof. \( \square \)

9. Deformation-cobordism analysis of parameterized moduli space

With the \( \pi \)-energy bound (8.14) and the \( \lambda \)-energy bound (8.15), we are now ready to make a deformation-cobordism analysis of \( M_{[0, K_0+1]}(M; \lambda; R, H) \). The logical scheme of this analysis is similar to that of [Oh97a].

9.1. The case \( K = 0: J = J_0 \) and \( H = 0 \). In this case, the equation (5.9) becomes
\[
\begin{cases}
\mathcal{J}^\pi u = 0, & d(u^* \lambda \circ j) = 0 \\
u(\tau, 0), & u(\tau, 1) \in R
\end{cases}
\]
with \( E_{J,\pi}(u) = E^\pi_{J_0}(u) + E^\lambda_{J_0}(u) < \infty. \)
The following is the open string version of [Abb11, Proposition 1.4], [OW18a, Proposition 3.4].

**Proposition 9.1.** Assume \( w : (\Sigma, \partial \Sigma) \to (M, R) \) is a smooth contact instanton from a compact connected Riemann surface \((\Sigma, j)\) genus zero with one boundary component. Then \( w \) is a constant map.

**Proof.** For contact Cauchy–Riemann maps, we have

\[
|d^{\pi} w|^2 dA = d(2 w^* \lambda).
\]

By Stokes’ formula and the Legendrian boundary condition, we derive

\[
\frac{1}{2} \int_{\Sigma} |d^\pi w|^2 = \int_{\Sigma} dw^* \lambda = \int_{\partial \Sigma} w^* \lambda = 0
\]

when \( \Sigma \) is compact with \( w(\partial \Sigma) \subset R \). This implies \( |d^\pi w|^2 = 0 \) which in turn implies \( dw^* \lambda = 0 \) by the above equality. Combining the defining equation \( d(w^* \lambda \circ j) = \) for contact instantons, this vanishing implies that \( w^* \lambda \) (so is \( *w^* \lambda \)) is a harmonic one-form on the compact Riemann surface \( \Sigma \) satisfying

\[
w^* \lambda |_{\partial \Sigma} = 0.
\]

If the genus of \( \Sigma \) is zero, i.e., when \( \Sigma = D^2 \). By the reflection argument, we prove \( w^* \lambda \equiv 0 \) on \( \Sigma \). This together with \( d^\pi w = 0 \) implies \( w \) must be a constant map valued at a point of \( R \). This finishes the proof.

**Remark 9.2.** Similarly as in [Abb11, Proposition 1.4], [OW18a, Proposition 3.4], we can prove that for the case with \( g(\Sigma) \geq 1 \), \( w \) is either a constant or the locus of its image is either a closed Reeb chord or a finite union of Reeb chords of \( R \) contained in a single leaf. Since this is not used in the present paper, we do not elaborate its proof referring the similar result for the closed string case to ibid.

Postponing derivation of the full index formula for the linearization operator \( D \Upsilon(w) \) of the equation (9.1) till elsewhere, we just prove the following proposition. (See [Oh23, Section10-11] for the linearization and the full index formula for the closed string case.)

**Proposition 9.3.** Let \( w_p : (D^2, \partial D^2) \to (M, R) \) be the constant map valued at \( p \in R \subset M \) regarded as a constant solution to (9.1). Consider the map

\[
\Upsilon : w \mapsto (\overline{\partial}^\pi w, d(w^* \lambda \circ j))
\]

and its linearization operator

\[
D \Upsilon(w) : \Omega^0(w^* TM, (\partial w)^* TR) \to \Omega^{(0,1)}(w^* \xi) \oplus \Omega^2(\Sigma, \mathbb{R}).
\]

Then we have

\[
\ker D \Upsilon(w_p) = n (= \dim R), \quad \operatorname{Coker} D \Upsilon(w_p) = 0.
\]

In particular, we have

\[
\text{Index } D \Upsilon(w) = n
\]

for any element \( w \in \mathcal{M}_0(M, R; J_0, H_K) \) homotopic to a constant map relative to \( R \) for all \( K \).
Proof. We denote by

\[ D \Upsilon(w) : \Omega^{(0,1)}_{k,p}(w^*TM, (\partial w)^*TR) \to \Omega^{0,1}_{k-1,p}(w^*\xi) \oplus \Omega^2_{k-2,p}(\Sigma, \mathbb{R}) \]

the \( W^{(k,p)} \)-completion of \( \Omega(w^*TM, (\partial w)^*TR) \), the set of vector fields over the map \( w \) and similarly for other completions. When \( w_p \) is the constant map valued at \( p \in \mathbb{R} \), it is easy to see that we have

\[ D \Upsilon(w_p)(\eta) = (\overline{\partial} \eta^\tau, -\ast \Delta f), \quad \eta = \eta^\tau + f R \lambda \]

where

- \( \eta^\tau \) is a map \( D^2 \to (\xi_p, J_p) \) satisfying the totally real boundary condition \( \eta^\tau_{\xi}(\partial D^2) \subset T_p \mathbb{R} \),
- \( f \) is a real-valued function on \( D^2 \) that satisfies the Dirichlet boundary condition, which follows from the Legendrian boundary condition \( R \) along \( \partial D^2 \).

(We refer to [Oh23, Theorem 10.1] for the precise formula for the linearization operator \( \Upsilon(w) \).)

It is well-known that the Laplacian with Dirichlet boundary condition on the disc has zero kernel and the corresponding index is zero.

When we identity \( (\xi_p, J_p) \) with \( (\mathbb{C}^n, i) \), \( \overline{\partial} \) is nothing but the standard Cauchy-Riemann operator. Then the first statement of the proposition is a well-known result whose proof can be found in [Oh95]. The second statement follows from the homotopy invariance of the Fredholm index.

An immediate corollary of the above two propositions (with \( g = 0 \)) is the following description of the moduli space \( \mathcal{M}_0(M, R; J_0, H) \cong \mathcal{M}(M, R; J_0, 0) \).

Corollary 9.4. The evaluation map \( \text{ev}_{(0,0)} : \mathcal{M}_0(M, R; J_0, H) \to R \) is a diffeomorphism. In particular its \( \mathbb{Z}_2 \)-degree is nonzero.

9.2. The case \( K \to K_0 \). Let \( K_0 > 0 \) be the constant that satisfies Proposition 8.10 so that

\[ \mathcal{M}_{K_0}(M, R; J, H) = \emptyset. \quad (9.2) \]

The following is the basic structure theorem of \( \mathcal{M}_K(M, R; J, H) \) given in (8.9) whose proof is a variation of the generic transversality theorem and so is omitted. (See [Oh23] for a proof of similar transversality result proven for the closed string case.)

Theorem 9.5. (1) For each fixed \( K > 0 \), there exists a generic choice of \( (J, H) \) such that \( \mathcal{M}_K(M, R; J, H) \) becomes a smooth manifold of dim \( n \) if non-empty. In particular, \( \dim \mathcal{M}_K(M, R; J, H) = n \) if non-empty.

(2) For the case \( K = 0 \), all solutions are constant and Fredholm regular and hence \( \mathcal{M}_K(M, R; J, H) \cong R \). Furthermore the evaluation map \( \text{ev} : \mathcal{M}_0(M, R; J, H) \to R : u \mapsto u(0, 0) \)

is a diffeomorphism.

(3) The parameterized moduli space \( \mathcal{M}^{\text{para}}_{[0,K_0]}(M, R; J, H) \to [0, K_0] \) is a smooth manifold of dimension \( n + 1 \) with boundary

\[ \{0\} \times \mathcal{M}_0(M, R; J, H) \bigsqcup \mathcal{M}_{K_0}(M, R; J, H) \]

and the evaluation map

\[ \text{Ev} : \mathcal{M}^{\text{para}}_{[0,K_0]}(M, R; J, H) \times \mathbb{R} \to L \times \mathbb{R}_+ \times \mathbb{R} : ((K, u), \tau) \mapsto (K, u(\tau), \tau) \]

is smooth.
\[ \mathcal{M}_K(M, R; J, H) = \emptyset \text{ for all } K \geq K_0. \]

### 9.3. Corbodism-deformation of triads.

The following upper bound for the bubble energy is a key ingredient in the proof of Theorem 1.12 in which consideration of domain dependent family of contact triads

\[ \{ (M, \lambda_z, J_z) \}_{z \in \mathbb{R} \times [0,1]}, \quad z = (x_K(\tau), t) \]

is a crucial ingredient.

**Choice 9.6.** We consider the following two parameter families of \( J \) and \( \lambda \):

\[ J'_{(s,t)} = (\psi_H^t J_H^s - 1)^* J, \quad (9.3) \]
\[ \lambda'_{(s,t)} = (\psi_H^t J_H^s - 1)^* \lambda, \quad (9.4) \]

See Remark 9.9. Once this set-up is carefully introduced, the proof of the upper bound is an easy consequence of Propositions 8.10 and 8.7.

**Proposition 9.7.** Let \( (K_\alpha, u_\alpha) \) be a bubbling-off sequence with

\[ u_\alpha \in \mathcal{M}_{K_\alpha}(J, H) \]

with

\[ u_\alpha \to (u, v, w) \]

in the sense of Theorem 8.9. Then any bubble must have positive asymptotic action less than \( \|H\| \).

**Proof.** By the way how the bubble is constructed, there exists a subsequence, still denoted by \( u_\alpha \), we have

\[ \limsup \alpha E^\pi_{(\lambda_K, J_K)}(u_\alpha, K_\alpha) = E^\pi_{(\lambda_\infty, J_\infty)}(\pi_\infty, K_\infty) + \sum_i E^\pi_{(\lambda_{z_i}, J_{z_i})}(v_i) + \sum_j E^\pi_{(\lambda_j, J_{z_j})}(w_j) \]

where each bubble \( v_i \) (resp. \( w_j \)) is a contact instanton for the triad

\[ (M, \lambda_{z_i}, J_{z_i}), \quad z_i = (\tau_i, t_i), \]

and the norm is taken with respect to the associated triad metrics:

\[ g_{z_i} = d\lambda_{z_i}(\cdot, J_{z_i}) + \lambda_{z_i} \otimes \lambda_{z_i}. \]

We first consider the disc bubbles \( w_j \). We also assume that the bubble point is contained in \( \{ t = 0 \} \). The other cases with \( \{ t = 1 \} \) is easier and can be treated in the same way.

But by the definition of the triad metric, we have

\[ \frac{1}{2} |d^w w_j|^2_{(\lambda_{z_j}, J_{z_j})} dA = w_j^* d\lambda_{z_j} \geq 0 \]

since \( d^w w_j \) is \( (J_{z_j}, J_{z_j}) \)-complex linear and \( J_{z_j} \) is \( d\lambda_{z_j} \)-adapted. Therefore it follows from Propositions 8.7 that

\[ E^\pi_{J_{z_j}}(w_j) \leq \limsup \alpha E^\pi_{(\lambda_K, J_K)}(\pi_\alpha, K_\alpha) \leq \|H\|, \]

we have derived

\[ \int_{\mathbb{H}} dw_j^* \lambda_{z_j} \leq \|H\|. \]
Now the proof will be complete once we prove the following key lemma. We recall the definition of the action spectrum \( \text{Spec}(M, \lambda) \) (resp. \( \text{Spec}(M, R; \lambda) \)) and the period gap \( T(M, \lambda) \) (resp. \( T(M, R; \lambda) \)) from Definition 1.5.

**Lemma 9.8.** Let \( \gamma_j \) be the asymptotic \( \lambda z_j \)-Reeb chord of \( w_j \). Then the value

\[
\int_{H} dw^*_j \lambda z_j
\]

is contained in \( \text{Spec}(M, R; \lambda) \).

**Proof.** By finiteness of the energy \( E^\pi_{\lambda z_j} (w_j) < \infty \) and the Legendrian boundary condition, we derive

\[
\int_{H} dw^*_j \lambda z_j = \int_{\partial H} \gamma^* \lambda z_j
\]

with

\[
\gamma(t) = \lim_{\tau \to \infty} w(e^{\pi (t+\tau)})
\]

where \((\tau, t) \in [0, \infty) \times [0, 1] \subset \Sigma \setminus \{z_j\}\) is the strip-like coordinate around the bubble point \( z_j = (\tau_j, 0) \in \Sigma := \mathbb{R} \times [0, 1] \).

Recall that \( \gamma \) satisfies the boundary condition

\[
\gamma(0), \gamma(1) \in \psi^1_{H^\pi} (R)
\]  

(9.5)

for \( s_j = \chi_{K_j}(\tau_j) \), since it is an asymptotic limit of a disc bubble \( w_j \) at \( t = 0 \) where \( w_j \) satisfies

\[
w_j(\{t = 0\}) \subset \psi^1_{H^\pi} (R), \quad s_j = \chi_{K_j}(\tau_j).
\]

On the other hand, we compute

\[
\gamma^* \lambda z_j = \gamma^* (\psi^0_{H^\pi} (\psi^1_{H^\pi})^{-1})^* \lambda = \gamma^* ((\psi^1_{H^\pi})^{-1})^* \lambda = ((\psi^1_{H^\pi})^{-1} \circ \gamma)^* \lambda.
\]

If we set

\[
\gamma(y) := (\psi^1_{H^\pi})^{-1} (\gamma(y)),
\]

(9.5) implies

\[
\gamma(0), \gamma(1) \in R.
\]

Furthermore \( \gamma \) is a \( \lambda \)-Reeb chord, since \( \gamma \) is a \( \lambda(\tau_j) \)-Reeb chord and

\[
\lambda(\tau_j) = (\psi^1_{H^\pi})^* \lambda.
\]

Therefore we have proved

\[
\int_{\partial H} \gamma^* \lambda z_j = \int_{\partial H} \gamma^* \lambda \in T(M, R; \lambda).
\]

This finishes the proof. \( \square \)

Easier proof also applies to the sphere bubble \( v_i \), which now finishes the proof of the proposition. \( \square \)

**Remark 9.9.** For the above action bound to hold for the bubbles, it is essential to vary the contact triads depending on the domain parameters by simultaneously varying the pairs

\((\lambda, J)\).

For example, if one fixed \( \lambda \) while \( J \) varies, one would not be able to prove this bound of the asymptotic period for the given contact form \( \lambda \) in terms of the oscillation norm \( \|H\| \): The only thing one could get is the kind of statement that there is a
$s_1 \in [0, 1]$ and a bubble whose $(\psi^*_H)^\ast \lambda$-period is less than or equal to $\|H\|$. Since general contactomorphism does not preserve $\lambda$ and hence $(\psi^*_H)^\ast \lambda \neq \lambda$, the two periods are generally different. This is a fundamental difference from the symplectic case where the symplectic form can be fixed while the almost complex structure varies since symplectomorphism preserves the symplectic form.

10. Existence of Reeb chords and translated intersections

We start with the following definition.

**Definition 10.1.** We say the pair 
$$(\lambda, (R_0, R_1))$$
is nondegenerate if all closed $\lambda$-Reeb orbits of $M$ and $\lambda$-Reeb chords from $R_0$ to $R_1$ are nondegenerate.

Note that the latter nondegeneracy is equivalent to the transversality of the intersection
$$R_0 \pitchfork Z_{R_1}.$$  

The following is a standard lemma in contact geometry, which is an easy consequence of nondegeneracy.

**Lemma 10.2.** Let $(M, \xi)$ be a closed contact manifold. Then we have the following:

1. $\text{Spec}(M, \lambda)$ (resp. $\text{Spec}(M, \lambda; R_0, R_1)$) is either empty or countable nowhere dense in $\mathbb{R}_+$. 
2. $T(M, \lambda) > 0$ (resp. $T_\lambda(R_0, R_1)$).
3. When the pair $(\lambda, (R_0, R_1))$ is nondegenerate, then each of the sets
$$\text{Spec}^K(M, \lambda) = \text{Spec}(M, \lambda) \cap (0, K]$$
and
$$\text{Spec}^K(M, \lambda; R_0, R_1) = \text{Spec}(M, \lambda; R_0, R_1) \cap (0, K]$$
are finite for each $K > 0$.

**Definition 10.3** (Relative period gap). For given Legendrian submanifold $R \subset M$, we define the constant $T_\lambda(M, R) > 0$ by 
$$T_\lambda(M, R) = \min\{T(M, \lambda), T(M, R; \lambda)\}$$
and call it the *period gap* of the pair $(M, R)$.

We now introduce the notion of *Reeb-untangling energy* of one Legendrian submanifold from the Reeb trace of another Legendrian submanifold.

**Definition 10.4** (Reeb-untangling energy). Let $(M, \xi)$ be a contact manifold, and let $R_0, R_1$ of compact Legendrian submanifolds $(M, \xi)$.

1. We define
$$e^{\text{trn}}_\lambda(R_0, R_1) := \inf\{\|H\| \mid \psi^*_H(R_0) \cap Z_{R_1} = \emptyset\}. \quad (10.1)$$
We put $e^{\text{trn}}_\lambda(R_0, R_1) = \infty$ if $\psi^*_H(R_0) \cap Z_{R_1} \neq \emptyset$ for all $H$. We call $e^{\text{trn}}_\lambda(R_0, R_1)$ the $\lambda$-untangling energy between them.
2. We put
$$e^{\text{trn}}(R_0, R_1) = \inf_{\lambda \in C(\xi)} e^{\text{trn}}_\lambda(R_0, R_1). \quad (10.2)$$
We call $e^{\text{trn}}(R_0, R_1)$ the Reeb-untangling energy between $(R_0, R_1)$ on $(M, \xi)$. 
10.1. Without nondegeneracy assumption. The following proof is the contact analog to [Oh97a, Theorem]. (See also [Che98].)

**Theorem 10.5.** Let \( R \subset (M, \lambda) \) be a compact Legendrian submanifold. Then we have
\[
e^{\text{trn}}_\lambda(R, R) \geq T_\lambda(M, R)
\]

**Proof.** If \( R \) is not displaceable from \( Z_R \), by definition, we have \( e^{\text{trn}}_\lambda(R, R) = \infty \). Then there is nothing to prove.

Now suppose to the contrary that for any Hamiltonian \( H \) with \( \|H\| < T_\lambda(M, R) \) (10.3)
we have \( \psi^1_\lambda(R) \cap R \neq \emptyset \).

Let \( H \) be any such Hamiltonian. We consider the parameterized moduli space
\[
\mathcal{M}^{\text{para}}_{[0, K_0]}(M, R; J, H) = \bigcup_{K \in [0, K_0]} \{K\} \times \mathcal{M}(M, R; J, H)
\]
which is fibered over \([0, K_0]\), and consider the evaluation map
\[
Ev: \mathcal{M}^{\text{para}}_{[0, K_0]}(M, R; J, H) \to L \times [0, K_0]; \quad u \mapsto (u(0, 0), K).
\]
The transversality theorem (see [Oh23, Section 12]) implies that for generic choice of \( J \), \( \mathcal{M}^{\text{para}}_{[0, K_0]}(M, R; J, H) \) is a smooth manifold of dimension \((\dim R + 1)\) with boundary
\[
\mathcal{M}_0(M, R; J, H) \coprod \mathcal{M}_{K_0}(M, R; J, H).
\]
We also know that
\[
ev_0: \mathcal{M}_0(M, R; J, H) \to R \text{ is a diffeomorphism.}
\]
In particular, its \( \mathbb{Z}_2 \)-degree of \( ev_0 \) is 1. On the other hand we have
\[
\mathcal{M}_{K_0}(M, R; J, H) = \emptyset
\]
by Proposition 8.10, hence the \( \mathbb{Z}_2 \)-degree of the map \( ev_{K_0} \) is zero. But the \( \mathbb{Z}_2 \)-degree is invariant under a compact cobordism. Therefore \( \mathcal{M}^{\text{para}}_{[0, K_0]}(M, R; J, H) \) cannot be compact. By Theorem 8.9, a bubble must develop, i.e., there exists subsequences of \( K_\alpha, u_\alpha \) again denoted by the same such that
\[
K_\alpha \to K_\infty \in [0, K_0],
\]
and \( u_\alpha \in \mathcal{M}_{K_\alpha}(M, R; J, H) \) converging to some cusp-curve \((u, v, w)\) in the sense of Theorem 8.9 with either \( v \neq \emptyset, w \neq \emptyset \). Recall Corollary 9.7 implies that the \( \pi \)-energy of the bubble is always less than \( \|H\| \), and hence
\[
T_\lambda(M, R) \leq \|H\|.
\]
Now taking the infimum of \( \|H\| \) over all \( H \) with \( Z_R \cap \psi^1_\lambda(R) = \emptyset \), we obtain
\[
0 < T_\lambda(M, R) \leq \inf_{H} \{\|H\| \mid Z_R \cap \psi^1_\lambda(R) = \emptyset\}.
\]
But the quantity in the right hand side of this inequality is nothing but twice of the Reeb-untangling energy \( e^{\text{trn}}_\lambda(R, R) \) of \( R \).

On the other hand, by Lemma 9.8, there is no bubble whose energy is less than \( T_\lambda(M, R) \). It implies that the moduli space
\[
\mathcal{M}^{\text{para}}_{[0, K]}(M, R; J, H)
\]
is compact for every $K \geq 0$ which in turn implies
\[ M_K(M, R; J, H) \neq \emptyset. \]
Choose $K_\alpha \to \infty$ as $\alpha \to \infty$ and $u_\alpha \in M K_\alpha(M, R; J, H)$. By the energy estimate given in Proposition 8.7,
\[ E^\pi(u_\alpha) \leq \|H\| \]
for all $\alpha \to \infty$. And we also have the uniform bound of the vertical energy from Proposition 8.8.

Now we consider the translated curve $\bar{u}_\alpha$ defined by
\[ \bar{u}_\alpha(\tau, t) := u_\alpha(\tau + K_\alpha, t) \]
on the semi-strip $[0, \infty) \times [0, 1]$. We observe
\[ A(\bar{u}_\alpha(0, \cdot)) = 0 \]
and hence by the action identity
\[ A(\bar{u}_\alpha(K_\alpha, \cdot)) = A(\bar{u}_\alpha(K_\alpha, \cdot) - A(\bar{u}_\alpha(0, \cdot)) \]
\[ = \int_0^{K_\alpha} \int_0^1 |d\bar{u}_\alpha|^2 dt d\tau \leq E^\pi(u_\alpha) \leq \|H\|. \quad (10.6) \]
Therefore $\bar{u}_\alpha$ on $[0, \infty) \times [0, 1]$ carries a subsequence still denoted by $\bar{u}_\alpha$ such that $\bar{u}_\alpha \to w_\infty$ on any compact subset of $[0, \infty) \times [0, 1]$ for a map $w_\infty : [0, \infty \times [0, 1] \to M$ satisfying
\[ \mathcal{J}_{J, H}^\pi = 0, \quad w_\infty(\tau, 0) \in \Gamma_\psi, \quad w_\infty(\tau, 1) \in \Gamma_{id} \]
with finite energy. This shows that it carries a Reeb chord $\gamma = u(\cdot, \infty)$ with $A(\gamma) < \|H\|$. In particular, we have derived $T(M, \lambda) < \|H\|$, a contradiction to the standing hypothesis (10.3). This finishes the proof. \hfill \Box

10.2. With nondegeneracy assumption: the lower bound of $\text{Fix}^\text{fr}_\lambda(\psi)$. In this subsection, we assume that $(\lambda, (\psi(R), R))$ is nondegenerate in the sense of Definition 10.1.

We will prove the following lower bound adapting the argument used by Chekanov [Che98] in the context of Lagrangian Floer theory. (See [FOOO13] for the relevant energy estimates and the proof of lower bound.) The algebraic arguments leading to the lower bound is based on a purely algebraic homological machinery equally applies to the current context, as long as bubbling does not occur which is ensured by the inequality
\[ \|H\| < T_\lambda(M, R). \quad (10.7) \]
Therefore we will be brief in the details of the proof leaving full details to [Ohc]. (See the relevant algebraic argument to [Che98] or [FOOO13] also.)

**Theorem 10.6.** Suppose $\psi$ is nondegenerate and let $H \mapsto \psi$ with $\|H\| < T_\lambda(M, R)$. Then
\[ \#(\text{Fix}^\text{fr}_\lambda(\psi)) \geq \dim H_\psi(R; \mathbb{Z}_2). \]

**Proof.** Under the nondegeneracy assumption, we consider the $\mathbb{Z}_2$-vector space
\[ CI^*_\lambda(\psi(R), R) := \mathbb{Z}_2(X(\psi(R), R)) \]
where $X(\psi(R), R)$ is the set of nonnegative iso-speed Reeb chords introduced in Definition 1.14. (Here $CI^*_\lambda$ stands for contact instanton for $\lambda$ as well as the letter $C$ also stands for complex at the same time. It follows by definition that when $T \neq 0$
and $\psi (R) \cap R = \emptyset$, each element $(T; \gamma) \in X(R_0, R_1)$ gives rise to a Reeb chord from $R_0$ to $R_1$ of period $T$ given by $\gamma_T := \gamma (\cdot) / T$.)

We define a $\mathbb{Z}_2$-linear map

$$\delta_{\psi (R); R, H} : CI^*_\lambda (\psi (R), R) \to CI^*_\lambda (\psi (R), R)$$

by its matrix element

$$\langle \delta_{\psi (R), R; H} (\gamma^-), \gamma^+ \rangle := \#Z_2 (M(\gamma^-, \gamma^+)).$$

Here $M(\gamma^-, \gamma^+)$ is the moduli space

$$M(\gamma^-, \gamma^+) = M(M, \lambda; R; \gamma^-, \gamma^+)$$

of contact instanton Floer trajectories $u$ satisfying $u (\pm \infty) = \gamma \pm$.

Similarly we define the maps

$$\delta_{(R, R; 0)} : CI^*_\lambda (R, R) \to CI^*_\lambda (R, R).$$

This is the Morse-Bott case in that we have decomposition

$$X(R, R) = X_0 (R, R) \sqcup X_{>0} (R, R)$$

by definition of $X(R, R)$ in Definition 1.14 where $X_0 (R, R) \cong R$ is the set of constant paths and $X_{>0} (R, R)$ is ones with $T > 0$. (In practice, we take

$$CI^*_\lambda (R, R) := C^* (R) \oplus Z X_{>0} (R, R)$$

with any model $C^* (R)$ of cochain complex of $R$. We will take the Morse homology complex $CM^* (f, f)$ for a $C^2$-small function $f$ on $R$. We refer readers to [Ohc] for full explanation on this process. This being said, we will just work with the Morse-Bott case $(R, R)$ in the following exposition.)

We denote by $\overline{H}$ the inverse Hamiltonian of $H$ given by the formula (2.5) and consider the maps

$$\Psi_H = \overline{H} : CI^*_\lambda (R, R) \to CI^*_\lambda (\psi (R), R)$$

and

$$\Psi_{\overline{H}} = (\Psi^{-1}_H) : CI^*_\lambda (\psi (R), R) \to CI^*_\lambda (R, R)$$

where $\Phi_H$ is the gauge transformation defined in Subsection 5.1. Recall $\psi^{-1}_H = \psi^{-1}$. Then we consider the composition

$$\Psi_{\overline{H}} \circ \Psi_H : CI^*_\lambda (R, R) \to CI^*_\lambda (R, R).$$

By considering the one-parameter family of Hamiltonians $H_K$ defined in (1.20), we define a family of homomorphisms

$$\Psi_{H_K} : CI^*_\lambda (R, R) \to CI^*_\lambda (R, R)$$

with $0 \leq K \leq K_0$ which defines a chain homotopy map

$$\delta : CI^*_\lambda (R, R) \to CI^*_{\lambda - 1} (R, R)$$

between $\Psi_{\overline{H}} \circ \Psi_H$ and $id$ on $CI^*_\lambda (R, R)$, i.e., it satisfies

$$\Psi_{\overline{H}} \circ \Psi_H - id = \delta \delta + \delta \delta$$

on $CI^*_\lambda (R, R)$ provided the parameterized moduli space $M_{\text{para}}^{\text{para}} (M, R; J, H)$ that we considered in the previous section does not bubble-off. A standard algebraic argument [Flo89], [Che98], [FOOO13] then shows that the latter homotopy identity follows as long as no bubbling occurs: The inequality (10.7) ensures no bubbling by Proposition 8.7 and Theorem 8.9. (The relevant gluing result leading to such an
algebraic consequence is obtained in [Ohc] for the current case of contact instantons.

Therefore we have shown that

\[ HI^*_λ(ψ(R), R) \cong HI^*_λ(R, R). \]

It remains to show that \( HI^*_λ(R, R) \cong H^*(R) \). This can be shown by a couple of ways either by the Morse-Bott argument (or the PSS-type argument) or by the direct comparison argument with the Morse complex of the generating function as done in [FO97, Mil00] in a Darboux-Weinstein neighborhood after localizing the cohomology as in [Oh96]. (We refer readers to [Ohc] for the reduction to the case of one-jet bundle and to [San12], [OY] for the details of computation for the case of one-jet bundle in the current contact context.) This finishes the proof. □

Part 3. Energy bounds, bubbling analysis and weak convergence

In this part, we assume that \((M, λ)\) is a tame contact manifold. Then we consider

\[ JK(τ, t) = (ψ^1_H(τ))(ψ^1_H(τ))^{-1} J'_K(τ, t) \]

which is associated to the family given in (8.12), and the two-parameter family of Hamiltonians \( H_K = H^{λK} \) given in (1.20). By the boundary-flatness assumed in (5.16), \( J'_K = J_0 \) for all \( τ \) with \(|τ| \) sufficiently large.

Then we study the equation (5.3) which we recall here:

\[
\begin{aligned}
(du - X_{H^{λK}}(u))^{π(0,1)} &= 0, \\
d(e^{g_K(u)}(u^*λ + u^*H_K dt) \circ j) &= 0,
\end{aligned}
\]

(10.8)

where \( g_K \) is the function on \( Θ_{K_0+1} \) given in (5.4) for \( 0 ≤ K ≤ K_0 \).

We will develop the necessary analytic package which provides the definition of relevant off-shell energy and the bubbling argument and construct the compactification of the moduli space

\[ \mathcal{M}(M, R; J, H), \quad \mathcal{M}^{para}(M, R; J, H) \]

of solutions of (10.8), and other relevant moduli space of (perturbed) contact instantons.

For the purpose of compactification of the moduli space of contact instantons, identifying a suitable notion of the \( λ \)-energy that controls the \( C^1 \)-estimate is the key element in the compactness study of moduli space of contact instantons. The framework we will use is the one from [Oh23] where the closed string case is studied. In the closed string case, the asymptotic charge \( Q \) may not be zero which is the key obstacle to the compactification and the Fredholm theory. Since in our current case, the asymptotic charge vanishes by Theorem 5.15, which is proved in [Oh21a], this bubbling-off analysis applies to general Legendrian boundary condition.

11. A PRIORI UNIFORM π-ENERGY Bound

In this section and henceforth, we simplify the notation \( E_{J_K, H} \) to \( E \) since the pair \( (J_K, H) \) will not be changed. And we will also highlight the domain almost complex structure dependence thereof by considering the pair \( (j, w) \) instead of \( w \).
Definition 11.1. For a smooth map $\dot{\Sigma} \to M$, we define the $\pi$-energy of $w$ by

$$E_{\pi}(j, w) = \frac{1}{2} \int_{\dot{\Sigma}} |d_{\pi} w|^2.$$  \hfill (11.1)

The following a priori $\pi$-energy identity is a key ingredient in relation to the lower bound of the Reeb-untangling energy. The proof will be carried out by the calculation similar to the one which we extract from the calculation made in [Oh97a] in the context of Lagrangian Floer theory. However the current calculation is significantly much more complex than and varies from that of [Oh97a] because of the presence of the new conformal exponent factor $e^{g_K(u)}$ which makes the calculation much more involved that of [Oh97a].

Remark 11.2. We would like to emphasize that the calculation leading to the proof of the energy identity is the heart of the matter that relates the $\pi$-energy and the oscillation norm similarly as in [Oh97a]. This is one of the key ingredients in the quantitative symplectic and contact topology.

Proposition 11.3. Let $u$ be any finite energy solution of (5.3). Then we have

$$E_{(J_K, H_K)}(u) = \int_{-2K-1}^{2K+1} -\chi_K'(\tau) \left( \int_0^1 H(r, \phi_{\chi_K(\tau)}^r H(u(\tau, 0))) \, dr \right) d\tau \quad \text{and} \quad \int_{2K}^{2K+1} -\chi_K'(\tau) \left( \int_0^1 H(r, \phi_{\chi_K(\tau)}^r H(u(\tau, 0))) \, dr \right) d\tau. \hfill (11.2)$$

Proof. By the energy identity given in Proposition 7.2, we have

$$E_{\pi}(J_K, H_K)(u) = E_{\pi}(\Pi) = \int_{\Theta_{K+1}} \Pi^* d\lambda = \int_{\partial \Theta_{K+1}} \Pi^* \lambda.$$  

We note

$$\Theta_{K+1} \setminus [-2K-1, 2K+1] \times [0, 1] = D^- \cup D^+$$

where $D^\pm$ are two semi-discs given by

$$D^\pm_{K_0} = \{ z \in \mathbb{C} \mid |z| \leq 1, \pm \Re z > 0 \} \pm (2K_0 + 1, 0) \hfill (11.3)$$

with their boundaries given by

$$\partial D^\pm_{K_0} = \{ z \in \mathbb{C} \mid |z| = 1, \pm \Re z > 0 \} \pm (2K_0 + 1, 0).$$

We decompose the integral into

$$\int_{\partial \Theta_{K+1}} \Pi^* \lambda = \int_{[-2K-1, 2K+1]} \Pi^* \lambda |z=0 - \int_{[-2K-1, 2K+1]} \Pi^* \lambda |z=1 \quad \text{and} \quad \int_{\partial D^+_{K_0}} \Pi^* \lambda - \int_{\partial D^-_{K_0}} \Pi^* \lambda. \hfill (11.4)$$

We now examine the four summands of the right hand side separately.

The last two terms vanish by the Legendrian boundary condition on $u$, since $H_K \equiv 0$ and so $\Pi = u$ on the relevant integration domains. It remains to estimate the first two terms.

We recall the definition

$$\Psi_K(\tau, t) := \psi_{H_K(\tau)}^1 (\psi_{H_K(\tau)}^t)^{-1} (u(\tau, t))$$
from (8.13) for the \((\tau, t)\)-family of Hamiltonians
\[
\text{Dev}_\lambda \left( \tau \mapsto \psi^t_{H_K(\tau)} \right) (\tau, x) = \chi'_K(\tau)H^{x_K(\tau)}(t, x)
\]
where \(H^s(t, x) = \text{Dev} \left( s \mapsto \psi^s_{H_K} \right) (s, x) \) is given \(H^s(t, x) = sH(t, x)\) by definition of \(\text{Dev}_\lambda\).

We denote by \(\nu^K_\tau\) the \(\tau\)-path associated to the two-parameter family
\[
\tau \mapsto \nu^K_{(\tau, t)} := \psi^1_{\chi_K(\tau)H} \circ (\psi^t_{\chi_K(\tau)H})^{-1}
\]
and let \(X_\tau\) be the vector field generating the contact Hamiltonian path. Then by definition we have
\[
\text{Dev} \left( \tau \mapsto \nu^K_{(\tau, t)} \right) = -\lambda(X_\tau).
\]

We denote
\[
G_K(\tau, t, x) = \text{Dev} \left( \tau \mapsto \nu^K_{(\tau, t)} \right)(\tau, x).
\]

Then we compute
\[
\frac{\partial \pi^K}{\partial \tau}(\tau, 0) = \psi^1_{\chi_K(\tau)H},
\]
and so
\[
\pi^K(\tau, 0) = \psi^1_{\chi_K(\tau)H}(u(\tau, 0))
\]
from (8.13). We compute
\[
\frac{\partial \pi^K}{\partial \tau}(\tau, 0) = du^1_{\chi_K(\tau)H} \left( \frac{\partial u}{\partial \tau}(\tau, 0) \right) + \lambda(\chi'_K(\tau)X_{G_K(\chi_K(\tau), 0, \pi^K(\tau, 0))})
\]
keeping the moving boundary condition we required at \(t = 0\) in mind. Therefore
\[
\left. \frac{\partial \pi^K}{\partial \tau} \right|_{\tau = 0} = \lambda \left( \frac{\partial u}{\partial \tau}(\tau, 0) \right)
\]
\[
= \lambda \left( \left. \frac{\partial u}{\partial \tau} \right|_{\tau = 0} \right) + \lambda(\chi'_K(\tau)X_{G_K(\chi_K(\tau), 0, \pi^K(\tau, 0))})
\]
\[
= -\chi'_K(\tau)G^K_K(\tau, 0, \pi^K(\tau, 0))
\]
The first term in the second line of the equation vanishes by the Legendrian boundary condition imposed on \(u\) at \(t = 0\) since \(\nu^K_\tau(\tau, 0) = \psi^1_{\chi_K(\tau)H}\) are contact diffeomorphisms which preserve contact distribution.

We note
\[
\nu^K_\tau(\tau, 0) = \mu(\chi_K(\tau), 1)
\]
where \( \mu(s,t) = \phi^t_{sH} \) that already appeared in Lemma 2.7. Therefore we can compute

\[
G_K(\tau, 0, \bar{\pi}_K(\tau, 0))
\]

in terms of the \( s \)-Hamiltonian

\[
L(s, t, x) := \text{Dev}(s \mapsto \psi^t_{sH})(s, x)
\]

which was already computed in Lemma 2.7

\[
L(s, 1, \psi^{1}_{sH}(x)) = \int_{0}^{1} H\left(r, \psi^{r}_{sH}(s, x)\right) dr.
\]

Using (11.9), we have the equality

\[
G_K(\tau, 0, u_K(\tau, 0)) = \chi'_K(\tau) L(\chi_K(\tau), 1, \bar{\pi}_K(\tau, 0))
\]

\[
= \chi'_K(\tau) L(\chi_K(\tau), 1, \psi^{1}_{\chi_K(\tau)} H(u(\tau, 0)))
\]

\[
= \chi'_K(\tau) \int_{0}^{1} H\left(r, \psi^{r}_{\chi_K(\tau)} H(u(\tau, 0))\right) dr
\]

from Lemma 2.7.

Since \( \chi'_K \) is supported in \([-2K - 1, -2K] \cup [2K, 2K + 1]\), the second integral of (11.4) is reduced to

\[
\left( \int_{-2K}^{2K+1} \sum_{i=0}^{2K} \right) (\bar{\pi}_K)^* \lambda|_{t=0} d\tau
\]

\[
= \int_{-2K}^{-2K-1} -\chi'_K(\tau) \left( \int_{0}^{1} H\left(r, \psi^{r}_{\chi_K(\tau)} H(u(\tau, 0))\right) dr \right) dr
\]

\[
+ \int_{2K}^{2K+1} -\chi'_K(\tau) \left( \int_{0}^{1} H\left(r, \psi^{r}_{\chi_K(\tau)} H(u(\tau, 0))\right) dr \right) dr
\]

Combining the above calculations, we have finished the proof of (11.2). \( \square \)

Once we have (11.2), we immediately obtain

\[
E_{\pi, H}(u) \leq \int_{0}^{1} (\max H_t - \min H_t) dt = \|H\|
\]

since

\[
\chi'_K(\tau) \begin{cases} 
\geq 0 & \text{on } [-2K - 1, -2K], \\
\leq 0 & \text{on } [2K, 2K + 1].
\end{cases}
\]

This then finishes the proof of the inequality (11.2).

12. Definition of off-shell vertical energy of contact instantons

Let \( w : \hat{\Sigma} \to M \) be a smooth map for a punctured Riemann surface \( (\hat{\Sigma}, j) \).

Hofer’s definition of the so called \( \lambda \)-energy \( E^\lambda(j, w) \), which we denote by \( E^\lambda(j, w) \) instead, strongly relies on the presence of the additional symplectization factor and the associated radial coordinates \( s \), and the associated pseudoholomorphic curve equation requires the form \( w^* \lambda \circ j = d(w^* s) \) which in particular implies that the form \( w^* \lambda \circ j \) is always a (globally) exact one-form.

Defining the vertical part of the energy is not as straightforward as in the symplectization case introduced by Hofer [Hof93], because the form \( w^* \lambda \circ j \) is only
required to be closed but not exact in general and we do not have the presence of
the background radial function $s$ for the case of contact instanton equation.

Luckily, the Riemann surfaces that are relevant to the purposes of the present
paper are of the following three types:

**Situation 12.1** (Charge vanishing). (1) First, we mention that the starting
Riemann surface will be an open Riemann surface of genus zero with a
finite number of boundary punctures, and mostly

$$\hat{\Sigma} \cong \mathbb{R} \times [0, 1]$$

together with an contact instanton with Legendrian pair boundary condition
$(R_0, R_1)$, or

$$\hat{\Sigma} \cong D^2$$

with moving Legendrian boundary condition.

(2) $\mathbb{C}$ which will appear in the bubbling analysis at an interior point of $\hat{\Sigma}$,

(3) $\mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im } z \geq 0 \}$ which will appear in the bubbling analysis at a
boundary point of $\hat{\Sigma}$.

An upshot is that the asymptotic charges vanish in all these three cases. Therefore in the
present paper, it will be enough to consider the punctures (both interior
or boundary) where the asymptotic charges vanish, which we will assume from now
on.

This being said, we follow the procedure exercised in [Oh23] for the closed string
case. We introduce the following class of test functions following the modification
made in [BEH+03] of Hofer’s original definition [Hof93].

**Definition 12.2.** We define

$$\mathcal{C} = \left\{ \varphi : \mathbb{R} \to \mathbb{R}_{\geq 0} \mid \sup \text{supp } \varphi \text{ is compact, } \int_{\mathbb{R}} \varphi = 1 \right\} \quad (12.1)$$

Then on the given cylindrical neighborhood $D_\delta(p) \setminus \{p\}$, we can write

$$w^* \lambda \circ j = df$$

for some function $f : [0, \infty) \times S^1 \to \mathbb{R}$.

**Definition 12.3** (Contact instanton potential). We call the above function $f$ the
contact instanton potential of the contact instanton charge form $w^* \lambda \circ j$.

We denote by $\psi$ the function determined by

$$\psi' = \varphi, \quad \psi(-\infty) = 0, \quad \psi(\infty) = 1. \quad (12.2)$$

**Definition 12.4.** Let $w$ satisfy $d(w^* \lambda \circ j) = 0$. Then we define

$$E_c(j, w; p) = \sup_{\varphi \in \mathcal{C}} \int_{D_\delta(p) \setminus \{p\}} df \circ j \wedge d(\psi(f))$$

$$= \sup_{\varphi \in \mathcal{C}} \int_{D_\delta(p) \setminus \{p\}} (-w^* \lambda) \wedge d(\psi(f)).$$

We note that

$$df \circ j \wedge d(\psi(f)) = \psi'(f) df \circ j \wedge df = \varphi(f) df \circ j \wedge df \geq 0$$

since

$$df \circ j \wedge df = |df|^2 d\tau \wedge dt.$$
Therefore we can rewrite $E_C(j, w; p)$ into
\[ E_C(j, w; p) = \sup_{\varphi \in \mathcal{C}} \int_{D_3(p) \setminus \{p\}} \varphi(f) df \odot j \wedge df. \]

We remark that when $w$ is given, the function $f$ on $D_3(p) \setminus \{p\}$ is uniquely determined modulo the shift by a constant. However the following proposition shows that the definition of $E_C(j, w)$ does not depend on the constant shift in the choice of $f$.

**Proposition 12.5.** For a given smooth map $w$ satisfying $d(\nu^* \lambda \circ j) = 0$, we have $E_C(f(w)) = E_C(g(w))$ for any pair $(f, g)$ with $df = \nu^* \lambda \circ j = dg$ on $D_3(p) \setminus \{p\}$.

**Proof.** Certainly $df$ or $df \odot j$ are independent of the addition by constant $c$. On the other hand, we have $\varphi(g) = \varphi(f + c)$ and the function $a \mapsto \varphi(a + c)$ still lie in $\mathcal{C}$. Therefore after taking the supremum over $\mathcal{C}$, we have derived
\[ E_{C, f}(j, w; p) = E_{C, g}(j, w; p). \]
This finishes the proof. \[ \square \]

This proposition enables us to introduce the following

**Definition 12.6 (Vertical energy).**

1. We denote the common value of $E_{C, f}(j, w; p)$ by $E^\lambda_p(w)$, and call the $\lambda$-energy at $p$.
2. We define the vertical energy, denoted by $E^\perp(j, w)$, to be the sum
\[ E^\perp(j, w) = \sum_{l=1}^{k} E^\lambda_p(w). \]

Now we define the final form of the off-shell energy.

**Definition 12.7 (Total energy).** Let $w : \hat{\Sigma} \to Q$ be any smooth map. We define the total energy to be the sum
\[ E(j, w) = E^\pi(j, w) + E^\perp(j, w). \] \hspace{1cm} (12.3)

In the rest of the paper, we suppress $j$ from the arguments of the energy $E(j, w)$ and just write $E(w) = E^\pi(w) + E^\perp(w)$.

13. A PRIORI UNIFORM VERTICAL ENERGY BOUND

In this section, we establish the $C^1$-estimates and weak convergence result for the parameterized moduli space
\[ M^{\text{para}}(M, R; J, H) \]
defined in (10.4) for $K_0 = \infty$.

For this purpose, we need to establish the fundamental a priori energy bound for the energy
\[ E_{J^p}(\nu_K) = E_{J^p}^\pi(\nu_K) + E_{J^p}^\perp(\nu_K) \]
where $\nu_K(\tau, t) = (\psi^H_{\tau} \circ \psi^H_t)^{-1}(u(\tau, t))$. 
We have already shown $E_{J,H}^\pi (u) = E_{J,H}^\pi (\overline{\pi})$ in Proposition 7.2, and already proved the $\pi$-energy bound

$$E_{J,H}^\pi (\overline{\pi}_K) \leq \|H\|$$

in Proposition 8.7.

In this section, we complete the study of energy bounds by proving the bound for the $\lambda$-energy as well.

**Proposition 13.1.** Let $u$ be any finite energy solution of (5.3). Then we have

$$E_{+,J}^\pi (\overline{\pi}_K) \leq \|H\|. \quad (13.1)$$

**Proof.** For the simplicity and to highlight that $\overline{\pi}_K$ is an unperturbed contact instanton, we denote

$$\overline{\pi}_K =: w$$

unless the original notation $\overline{\pi}_K$ needs to be used for clarity.

By the defining equation $d(w^* \lambda \circ j) = 0$ of contact instantons, there exists a global function $f: \Theta_{K_0 + 1} \to \mathbb{R}$ such that

$$w^* \lambda \circ j = df$$

since $\Theta_{K_0 + 1}$ is simply connected. By definition of $E_{+,J}^\pi$, we need to get a uniform bound for the integral

$$\int_{\Theta_{K_0 + 1}} (-w^* \lambda) \wedge d(\psi(f)) \geq 0.$$  

(See Definition 12.4.) By integration by parts, we rewrite

$$\int_{\Theta_{K_0 + 1}} (-w^* \lambda) \wedge d(\psi(f)) = \int_{\Theta_{K_0 + 1}} d(\psi(f)w^* \lambda) - \psi(f)dw^* \lambda.$$  

Recall that $dw^* \lambda = \frac{1}{2}|d^\pi w|^2$ for any $w$ satisfying $\overline{\nabla} w = 0$. Then similarly as we proved Proposition 8.7, we derive

$$0 \leq \int_{\Theta_{K_0 + 1}} (-w^* \lambda) \wedge d(\psi(f)) \leq \int_{\Theta_{K_0 + 1}} d(\psi(f)w^* \lambda)$$

$$= \int_{\partial \mathcal{D}_{K_0}^2} \psi(f(\infty,t))(\overline{\nabla}_+)^* \lambda - \int_{\partial \mathcal{D}_{K_0}^2} \psi(f(\overline{\nabla}_-)^* \lambda)$$

$$+ \int_{-2K_0 - 1}^{2K_0 + 1} \psi(f(\tau,0)) \lambda \left( \frac{\partial \pi_K}{\partial \tau} (\tau,0) \right) d\tau$$

$$- \int_{-2K_0 - 1}^{2K_0 + 1} \psi(f(\tau,1)) \lambda \left( \frac{\partial \pi_K}{\partial \tau} (\tau,1) \right) d\tau$$

$$= \int_{-2K_0 - 1}^{2K_0 + 1} \psi(f(\tau,0)) \lambda \left( \frac{\partial \pi_K}{\partial \tau} (\tau,0) \right) d\tau$$

$$- \int_{-2K_0 - 1}^{2K_0 + 1} \psi(f(\tau,1)) \lambda \left( \frac{\partial \pi_K}{\partial \tau} (\tau,1) \right) d\tau$$

$$\leq \int_0^1 \min \psi(f(\cdot,t)) H_t \, dt + \int_0^1 \max \psi(f(\cdot,t)) H_t \, dt$$

$$\leq \int_0^1 (\max H_t - \min H_t) \, dt = \|H\|. $$
Here for the penultimate inequality, we employ the following:

- We use the same calculations as the ones performed in the proof of Proposition 11.3, especially in the course of evaluation of the integral

\[
\int_{[-2K-1,2K+1]} \pi^* \lambda|_{t=0}
\]

in (11.4), we apply (11.8) and (11.10).

- Moreover, we also have used the inequality

\[
\chi_K \begin{cases} 
\geq 0 & \text{for } \tau \in [-2K_0 - 1, -2K_0] \\
\leq 0 & \text{for } \tau \in [2K_0, 2K_0 + 1].
\end{cases}
\]

Then for the last equality, we use the fact \(0 \leq \psi(f) \leq 1\). Combining all the above discussion, we have finished the proof of

\[
\int_{\Theta_{K_0+1}} (-w^* \lambda) \wedge d(\psi(f)) \leq \|H\|
\]

for any \(\psi \in \mathcal{C}\) and hence the proof of the proposition by definition of \(E^\perp\). \(\square\)

14. Bubbling analysis for the contact instantons

As in Hofer’s bubbling-off analysis in pseudo-holomorphic curves on symplectization [Hof93], it turns out that the study of contact instantons on the plane for the closed string case, and on the half plane in addition for the open string case, plays a crucial role in the bubbling-off analysis of contact instantons. Overall bubbling arguments is by now standard which we apply to the new case of conformally invariant elliptic boundary value problem, \textit{the contact instanton equation with Legendrian boundary condition}. We refer readers to [Oh15a, Section 8.4] for the full details of this bubbling argument, especially for the process of disc bubblings for pseudoholomorphic curves with Lagrangian boundary condition in symplectic geometry.

However there are two marked differences between the current blowing-up argument and that of pseudoholomorphic curves:

\begin{remark}
(1) We need to replace the standard harmonic energy by the \(\pi\)-harmonic energy for the blowing-up argument in the proofs of the \(\epsilon\)-regularity or of the period-gap theorem. Of course, we also need the uniform bound for the \(\lambda\)-energy which itself, however, does not enter in this proof other than the presence of uniform bound.

(2) Because of the presence of the equation

\[
d(w^* \lambda \circ j) = 0
\]

in the defining equation of the contact instanton map which is \textit{of the second order, not of the first order}, the local \(C^{k,\alpha}\) a priori estimate for \(k = 2\) given in [Oh21a] is the minimum regularity needed to apply the bubbling argument to establish that the limit map of a subsequence obtained via application of Ascoli-Arzela theorem still satisfies the equation

\[
\bar{\partial}^* w = 0, \quad d(w^* \lambda \circ j) = 0.
\]

The upshot is that the \(C^{1,\alpha}\) bound is not enough to carry out the bubbling process and produce a limit for the contact instanton map, unlike the case of pseudoholomorphic curves.
We recall the following useful lemma from [HV92] whose proof we refer thereto.

**Lemma 14.2.** Let \((X,d)\) be a complete metric space, \(f : X \to \mathbb{R}\) be a nonnegative continuous function, \(x \in X\) and \(\delta > 0\). Then there exists \(y \in X\) and a positive number \(\epsilon \leq \delta\) such that

\[
d(x, y) < 2\delta, \quad \max_{B(y, \epsilon)} f \leq 2f(y), \quad \epsilon f(y) \geq \delta f(x).
\]

We start with the case of \(C\).

14.1. **Contact instantons on the plane.** We start with a proposition which is an analog to [Hof93, Theorem 31]. We refer to [Oh23] or [OS] for its proof. We also refer the proof of the corresponding statement of Proposition 14.8 for the case of contact instantons on the half place, which is harder to prove than the case of the plane.

**Proposition 14.3.** Let \(w : C \to Q\) be a contact instanton. Regard \(\infty\) as a puncture of \(C = \mathbb{C}P^1 \setminus \{\infty\}\). Suppose \(\|dw\|_{C^0} < \infty\) and

\[
E^\pi(w) = 0, \quad E^\perp_\infty(w) < \infty. \tag{14.1}
\]

Then \(w\) is a constant map.

Using the above proposition, we prove the following fundamental result.

**Theorem 14.4.** Let \(w : C \to Q\) be a contact instanton. Suppose

\[
E(w) = E^\pi(w) + E^\perp_\infty(w) < \infty. \tag{14.2}
\]

Then \(\|dw\|_{C^0} < \infty\).

**Proof.** Suppose to the contrary that \(\|dw\|_{C^0} = \infty\) and let \(z_\alpha\) be a blowing-up sequence. We denote \(R_\alpha = \|dw(z_\alpha)\| \to \infty\). Then by applying Lemma 14.2, we can choose another such sequence \(z'_\alpha\) and \(\epsilon_\alpha \to 0\) such that

\[
|dw(z'_\alpha)| \to \infty, \quad \max_{z \in D_{R_\alpha}(z'_\alpha)} |dw(z)| \leq 2R_\alpha, \quad \epsilon_\alpha R_\alpha \to 0. \tag{14.3}
\]

We consider the re-scaling maps \(v_\alpha : D^2_{R_\alpha}(0) \to Q\) defined by

\[
v_\alpha(z) = w \left( z'_\alpha + \frac{z}{R_\alpha} \right).
\]

Then we have

\[
\|dv_\alpha\|_{C^{0,\epsilon_\alpha} R_\alpha} \leq 2, \quad |dv_\alpha(0)| = 1.
\]

Applying Ascoli-Arzela theorem, there exists a continuous map \(v_\infty : C \to Q\) such that \(v_\alpha \to v_\infty\) uniformly on compact subsets. Then by the a priori \(C^{k,\alpha}\)-estimates, [Oh21a, Section 5], the convergence is in compact \(C^\infty\) topology and \(v_\infty\) is smooth. Furthermore \(v_\infty\) satisfies \(\overline{\partial} v_\infty = 0 = d(v_\infty^* \lambda \circ j) = 0\), \(E^\perp(v_\infty) \leq E(w) < \infty\) and

\[
\|dv_\infty\|_{C^{0,\epsilon} \mathcal{C}} \leq 2, \quad |dv_\infty(0)| = 1. \tag{14.4}
\]

On the other hand, by the finite \(\pi\)-energy hypothesis and density identity

\[
\frac{1}{2} |d^\pi w|^2 dA = d(w^* \lambda),
\]
we derive

\[
0 = \lim_{\alpha \to \infty} \int_{D_{z_{\alpha}}(z_{\alpha}')} d(\nu^* \lambda) = \lim_{\alpha \to \infty} \int_{D_{z_{\alpha}}(z_{\alpha}')} d(v_{\alpha}^* \lambda) = \lim_{\alpha \to \infty} \int_{D_{z_{\alpha}}(z_{\alpha}')} |d^\pi v_{\alpha}|^2 = \int_C |d^\pi v_{\infty}|^2 < \infty.
\]

Therefore we derive

\[E^\pi(v_{\infty}) = 0.\]

Then Proposition 14.3 implies \(w_{\infty}\) is a constant map which contradicts to \(|dv_{\infty}(0)| = 1\) in (14.4). This finishes the proof. \(\square\)

An immediate corollary of this theorem and Proposition 14.3 is the following

**Corollary 14.5.** For any non-constant contact instanton \(w : \mathbb{C} \to Q\) with the energy bound \(E(w) < \infty\), we obtain

\[E^\pi(w) = \int \gamma_{w}^* \lambda > 0\]

for \(\gamma_w = \lim_{R \to \infty} w(Re^{2\pi it})\). In particular \(E^\pi(w) \geq T(M, \lambda) > 0\).

Now we have the following refinement of the asymptotic convergence result from [Hof93] and [OW14]. It is a refinement of [OW14, Theorem 6.3] in that the derivative bound \(\|dw\|_{C^0} < \infty\) imposed therein is replaced by the more natural energy bound \(E(w) < \infty\).

Combining Theorem 5.15 and Theorem 14.4, we immediately derive

**Corollary 14.6.** Let \(w\) be a non-constant contact instanton on \(\mathbb{C}\) with \(E(w) < \infty\).

(14.5)

Then there exists a sequence \(R_j \to \infty\) and a Reeb orbit \(\gamma\) such that \(z_{R_j} \to \gamma(T(\cdot))\) with \(T \neq 0\) and

\[T = E^\pi(w), \quad Q = \int z w^* \lambda \circ j = 0.\]

**Proof.** If \(T = 0\), the above theorem shows that there exists a sequence \(\tau_i \to \infty\) such that \(w(\tau_i, \cdot)\) converges to a constant in \(C^\infty\) topology and so

\[\int_{\{\tau = \tau_i\}} w^* \lambda \to 0\]

as \(i \to \infty\). By Stokes’ formula, we derive

\[\int_{D_{z_{\tau_i}}(0)} w^* d\lambda = \int_{\tau = \tau_i} w^* \lambda \to 0.\]

On the other hand, we have

\[E^\pi(w) = \lim_{i \to \infty} \frac{1}{2} \int_{D_{z_{\tau_i}}(0)} |d^\pi w|^2 = \lim_{i \to \infty} \int_{D_{z_{\tau_i}}} w^* d\lambda = 0.\]

This contradicts to Corollary 14.5, which finishes the proof. \(\square\)

The following is the analog to [Hof93, Proposition 30].
Corollary 14.7. Let \( w \) be a contact instanton on \( \mathbb{R} \times S^1 \) with \( E(w) < \infty \). Then \( \|dw\|_{C^0} < \infty \).

Proof. As in Hofer’s proof of [Hof93, Proposition 30], we apply the same kind of bubbling-off argument as that of Theorem 14.4 and derive the same conclusion. Since the arguments are essentially the same, we omit the details by referring readers to the proof of [Hof93, Proposition 30]. \( \square \)

Now we consider the case \( w : H \to M \) with \( w(\partial H) \subset R \) (resp. \( w(\partial H) \subset Z \)) with Legendrian boundary condition (resp. with co-Legendrian boundary condition).

14.2. Contact instantons on \( H \) with Legendrian boundary condition. Let \( R \) be a compact Legendrian submanifold on general contact manifold \((M, \lambda)\).

Proposition 14.8. Let \( w : (H, \partial H) \to (M, R) \) be a contact instanton

\[
\begin{cases}
\mathbb{P}^\pi w = 0, \\ d(w^* \lambda \circ j) = 0
\end{cases}
\tag{14.6}
\]

Regard \( \infty \) as a puncture of \( D^2 \setminus \{1\} \cong \mathbb{H} \). Suppose \( \|dw\|_{C^0} < \infty \) and

\[
E^\pi(w) = 0, \quad E^\perp_{\infty}(w) < \infty.
\tag{14.7}
\]

Then \( w \) is a constant map valued in \( R \).

Proof. By the same argument used in the beginning of the proof of Proposition 9.1 using the vanishing \( E^\pi(w) = 0 \), we derive

\[ dw = w^* \lambda \otimes R_\lambda(w) \]

with \( w^* \lambda \) a bounded harmonic one-form. Since \( \mathbb{H} \) is connected, the image of \( w \) must be contained in a single leaf of Reeb foliation.

Then there is a smooth function \( b = b(z) \) such that

\[ w(z) = \gamma(b(z)) \]

for a Reeb trajectory \( \gamma = \gamma(t) \) as before but this time, \( w \) satisfies the boundary condition

\[ w(\partial \Theta) = \gamma(b(\partial \Theta)) \subset R. \]

In particular, we have \((w|_{\partial \mathbb{H}})^* \lambda = 0\). We compute

\[ w^* \lambda = b^* \gamma^* \lambda = b^* (dt) = db \]

which is bounded on \( \mathbb{H} \). We also have

\[ db|_{\partial \mathbb{H}} \equiv 0 \]

and so \( b|_{\partial \mathbb{H}} \) is a constant function. Let \( b|_{\partial \mathbb{H}} \equiv b_0 \). Since we also have \( d(w^* \lambda \circ j) = 0, \)

\[ d(db \circ j) = 0 \]

i.e., \( b : \mathbb{H} \to \mathbb{R} \) is a harmonic function hence \( b \) is the imaginary part of a holomorphic function, say \( f \), with \( f(z) = a(z) + ib(z) \) whose gradient is bounded as before. Furthermore \( b \) satisfies

\[ b|_{\partial \mathbb{H}} \equiv b_0. \]

this time. By applying the reflection principle, we obtain a holomorphic function

\[ \tilde{f} : \mathbb{C} \to \mathbb{C} \]
which has bounded gradient on $\mathbb{C}$. Therefore $\tilde{f}(z) = \alpha z + \beta$ for some constants $\alpha, \beta \in \mathbb{C}$ as before. Then $b(z) \equiv b_0$ on $\partial \mathbb{H}$ implies $\text{Im} \tilde{f} \mid_{\partial \mathbb{H}} \equiv b_0$. By the unique continuation applied to holomorphic functions, we conclude $\tilde{f}$ must be constant on $\mathbb{C}$. This in turn implies $b(z) \equiv b_0$ on $\mathbb{H}$ in particular. Thanks to the Legendrian boundary condition $w(\partial \mathbb{H}) \subset R$, $w$ must be a constant map valued in $R$. This finishes the proof. \hfill $\square$

Again using the above proposition, we prove the following fundamental $C^1$-bound.

**Theorem 14.9.** Let $w : (\mathbb{H}, \partial \mathbb{H}) \to (M, R)$ be a solution of (14.6) with

$$E(w) = E^p(w) + E^\perp(w) < \infty. \quad (14.8)$$

Then $\|dw\|_{C^0} < \infty$.

**Proof.** Suppose to the contrary that $\|dw\|_{C^0} = \infty$ and let $z_\alpha$ be a blowing-up sequence. We denote $R_\alpha = |dw(z_\alpha)| \to \infty$. Again by applying Lemma 14.2, we can choose another such sequence $z'_\alpha$ and $\epsilon_\alpha \to 0$ such that

$$|dw(y_\alpha)| \to \infty, \quad \max_{z \in B_{\epsilon_\alpha}(y_\alpha)} |dw(z)| \leq 2R_\alpha, \quad \epsilon_\alpha R_\alpha \to 0. \quad (14.9)$$

We consider the re-scaling maps $v_\alpha : D^2_{\epsilon_\alpha R_\alpha}(0) \to M$ defined by

$$v_\alpha(z) = w \left( y_\alpha + \frac{z}{R_\alpha} \right).$$

Then we have

$$|dv_\alpha|_{C^{0,\alpha R_\alpha}} \leq 2, \quad |dv_\alpha(0)| = 1$$

as before. Up until now, the proof is the same as that of Theorem 14.4.

Due to the presence of the boundary $\partial \mathbb{H}$, we consider two cases separately:

1. The case $y_\alpha \to y_\infty \in \text{Int} \mathbb{H}$,
2. The case $y_\alpha \to y_\infty \in \partial \mathbb{H}$.

We denote

$$d_\alpha := \text{dist}(y_\alpha, \partial \mathbb{H}).$$

The case (1) can be treated in the same way as in Theorem 14.9 to produce a non-constant contact instanton defined on $\mathbb{C}$.

Therefore we will focus on the case (2) from now on. In this case, the map $v_\alpha$ is defined at least for those $z$’s satisfying

$$\text{Im} \left( y_\alpha + \frac{z}{R_\alpha} \right) \geq 0.$$

In particular, $v_\alpha(z)$ is defined at least on the domain

$$\Theta_\alpha := \{ z \in \mathbb{C} \mid |z| \leq \epsilon_\alpha R_\alpha, \text{Im } y_\alpha \geq \min\{ \epsilon_\alpha R_\alpha, R_\alpha(d_\alpha - \text{Im } y_\alpha) \} \}. $$

At this stage, after applying Ascoli-Arzela theorem, there are two cases to consider:

- there exists a continuous map $v_\infty : \mathbb{C} \to M$ or,
- there exists a continuous map $v_\infty : (\mathbb{H}, \partial \mathbb{H}) \to (M, R)$.
such that $v_\alpha \to v_\infty$ uniformly on compact subsets.

Since the first case can be studied as before to produce a sphere-bubble, we now focus on the latter case. By the a priori $C^{k,\alpha}$-estimates, Theorem 5.11, the convergence is in compact $C^\infty$ topology and $v_\infty$ is smooth. Furthermore $v_\infty$ satisfies

$$\mathcal{F} v_\infty = 0 = d(v_\infty^* \lambda \circ j) = 0, \quad E^\pi(v_\infty) \leq E(w) < \infty$$

and

$$\|dv_\infty\|_{C^0;C} \leq 2, \quad |dv_\infty(0)| = 1$$

and $v_\infty$ also satisfies the boundary condition $v_\infty(\partial \mathbb{H}) \subset R$.

On the other hand, Combining the finite $\pi$-energy hypothesis, Legendrian boundary condition, and the density identity

$$\frac{1}{2} |d^\pi w|^2 dA = d(w^* \lambda),$$

we again derive

$$0 = \lim_{\alpha \to \infty} \int_{D_{\alpha}(y_\alpha)} d(w^* \lambda) = \lim_{\alpha \to \infty} \int_{D_{\alpha}(y_\alpha)} d(v_\alpha^* \lambda)$$

and hence $E^\pi(v_\infty) = 0$. Then Proposition 14.8 implies $w_\infty$ is a constant map which contradicts to $|dw_\infty(0)| = 1$. This finishes the proof. □

An immediate corollary of this theorem and Proposition 14.8 is the following

**Corollary 14.10.** For any non-constant contact instanton $w : (\mathbb{H}, \partial \mathbb{H}) \to (M, R)$ with the energy bound $E(w) < \infty$, we obtain

$$E^\pi(w) = \int z^* \lambda > 0$$

for $z = \lim_{R \to \infty} w(R e^{\pi it})$. In particular $E^\pi(w) \geq T_\lambda(M, R) > 0$.

Combining Theorem 5.15 and Theorem 14.9, we immediately derive

**Corollary 14.11.** Let $w : (\mathbb{H}, \partial \mathbb{H}) \to (M, R)$ be a non-constant contact instanton with

$$E(w) < \infty.$$

(14.10)

Then there exists a sequence $R_j \to \infty$ and a Reeb orbit $\gamma$ such that $z_{R_j} \to \gamma(T(\cdot))$ with $T \neq 0$ and

$$T = E^\pi(w), \quad Q = \int z w^* \lambda \circ j = 0.$$

**Proof.** By considering the coordinates $(\tau, t) \in [0, \infty) \times [0, 1]$ with $e^{\tau + it} \in \mathbb{H}$ which is a strip-like coordinate of $\mathbb{H} \equiv D^2 \setminus \{1\}$ near $\infty$ of $\mathbb{H}$, the charge vanishing theorem 5.15 implies $Q = 0$.

We then prove $T \neq 0$ by contradiction. If $T = 0$, the above theorem shows that there exists a sequence $\tau_i \to \infty$ such that $w(\tau_i, \cdot)$ converges to a constant in $C^\infty$ topology and so

$$\int_{\{\tau = \tau_i\}} w^* \lambda \to 0$$
as \( i \to \infty \). By Stokes’ formula and the Legendrian boundary condition, we derive
\[
\int_{D_{\varepsilon_i}(0)} w^* d\lambda = \int_{\partial D_{\varepsilon_i}(0)} w^* \lambda \to 0.
\]
On the other hand, this implies
\[
E^\pi (w) = \lim_{i \to \infty} \int_{D_{\varepsilon_i}(0)} |d^\pi w|^2 = \lim_{i \to \infty} \int_{D_{\varepsilon_i}(0)} w^* d\lambda = 0.
\]
This contradicts to Corollary 14.5, which finishes the proof. \( \Box \)

We also have the following.

**Corollary 14.12.** Let \( w \) be a contact instanton on \( \mathbb{R} \times [0,1] \) with \( E(w) < \infty \). Then \( \| dw \|_{C^0} < \infty \).

**Proof.** We apply the same kind of bubbling-off argument as that of Theorem 14.9 and derive the same conclusion. \( \Box \)

**Remark 14.13.** A priori we cannot rule out the possibility \( \text{Spec}(M, \lambda) = \emptyset \) or \( \text{Spec}(M, R; \lambda) = \emptyset \). Nonemptiness of this set is precisely the content of Weinstein’s conjecture or of the Arnold chord conjecture: The conjecture has been proved by Taubes [Tau07] (resp. by Hutchings-Taubes [HT13] respectively in the three dimensional cases after other scattered results obtained earlier.

15. \( C^1 \)-estimates and weak convergence of contact instantons

Combining all the results established in the previous sections of Parts 2 and 3, we prove the following alternatives. This will be also the key step towards a full compactification of the moduli space of contact instantons similarly as in the case of pseudoholomorphic curves.

**Theorem 15.1.** Consider the parameterized moduli space
\[
M^{\text{para}}(M, R; J, H) = \bigcup_{K \in [0,\infty)} \{ K \} \times \mathcal{M}_K(M, R; J, H).
\]
As before we consider the gauge-transformed map \( \overline{\pi} : \Theta_{K+1} \to M \) given by \( \overline{\pi} = \Phi^{-1}_H(u) \). Then one of the following alternative holds:

1. There exists some \( C > 0 \) such that
\[
\|d\overline{\pi}\|_{C^0, \Theta_{K+1}} \leq C \tag{15.1}
\]
for all \( u \in \mathcal{M}_K(M, R; J, H) \) for all \( K \geq 0 \).

2. There exists a sequence \( u_\alpha \in \mathcal{M}_K(M, R; J, H) \) such that \( \|d\overline{\pi}_\alpha\| \to \infty \) that gives rise to a non-constant finite \( \pi \)-energy contact instanton of the following two types:

\[
v : \mathbb{C} \to M, \quad v : (\mathbb{H}, \partial \mathbb{H}) \to (M, R)
\]

as a bubble.

**Proof.** We denote \( \overline{w} = \overline{\pi} \) for the simplicity of notation.

Then we have established
\[
E^\pi (w), \ E^\perp (w) \leq ||H|| \tag{15.2}
\]
in Proposition 8.7 and Proposition 8.8, respectively.
This being said, suppose that the case (1) fails to hold so that there exist a sequence of pairs \((K_\alpha, x_\alpha)\) with \(K_\alpha \rightarrow K_0\) and \(x_\alpha \in \Theta_{K_\alpha}\) such that
\[
|dw_\alpha(x_\alpha)| \rightarrow \infty, \quad x_\alpha = (r_\alpha, t_\alpha).
\]
We divide our proof into the two cases, after choosing a subsequence if necessary, one with \(K_\alpha \rightarrow K_0\) for some \(K_0 > 0\) and the other with \(K_\alpha \rightarrow \infty\). We start with the first case \(K_\alpha \rightarrow K_0\) for some \(K_0 > 0\). In this case, we have \(|r_\alpha| \leq K_0 + 1\) by definition of the domain \(\Theta_{K_0+1}\). We denote
\[
d_\alpha := \text{dist}(x_\alpha, \partial \Theta_{K_0+1}).
\] (15.3)
Similarly as in the proof of Theorem 14.9, the first case will gives rise to a non-constant contact instanton on \(\mathbb{C}\).

Again we will focus on the second case where \(d_\alpha \rightarrow 0\). In this case, regarding \(\Theta_{K_0+1}\) as a subset of \(\mathbb{C}\), we can find a pair of nested discs in \(\mathbb{C}\)
\[
D' \subset D \subset \mathbb{C}
\]
centered at \(x_\infty\) with \(\overline{\mathcal{D}} \subset \hat{D}\) and a sequence \(\{w_\alpha\}\) such that the map \(w_\alpha\) is defined on
\[
\Theta_{K_0+1} \cap D
\]
and
\[
\overline{\mathcal{D}} w_\alpha = 0, \quad d(w_\alpha \circ j) = 0
\]
thereon. It also satisfies
\[
E^\infty_{\lambda, j, D}(w_\alpha) \leq \|H\|, \quad E^\infty_{j, D}(w_\alpha) \leq \|H\|, \quad \|dw_\alpha\|_{C^0, D' \cap \Theta_{K_0+1}} \rightarrow \infty
\] (15.4)
as \(\alpha \rightarrow \infty\). Obviously we have \(d(x_\alpha - x_\infty) = |x_\alpha - x_\infty| \rightarrow 0\).

We choose sufficiently small constants \(\delta_\alpha \rightarrow 0\) so that
\[
d_\alpha|dw_\alpha(x_\alpha)| \rightarrow \infty.
\]
We adjust the sequence \(x_\alpha\) to \(y_\alpha\) by applying Lemma 14.2, so that \(d(y_\alpha, x_\infty) \rightarrow 0\) and
\[
\max_{x \in B_{y_\alpha}(\epsilon_\alpha)} |dw_\alpha| \leq 2|dw_\alpha(y_\alpha)|, \quad \delta_\alpha|dw_\alpha(y_\alpha)| \rightarrow \infty.
\] (15.5)
We denote \(R_\alpha = |dw_\alpha(y_\alpha)|\) and consider the re-scaled map
\[
\tilde{w}_\alpha(z) = w_\alpha(y_\alpha + \frac{z}{R_\alpha}).
\]
Then the domain of \(w_\alpha\) at least includes \(z \in \Theta_{K_0+1} \subset \mathbb{C}\) such that
\[
y_\alpha + \frac{z}{R_\alpha} \in D^2(\delta) \cap \Theta_{K_0+1}.
\]
In particular, \(\tilde{w}_\alpha(z)\) is defined at least on the domain
\[
\Theta_\alpha := \{z \in \mathbb{C} \mid |z| \leq \epsilon_\alpha R_\alpha, \text{Im} y_\alpha \geq \min\{\epsilon_\alpha R_\alpha, R_\alpha (d_\alpha - d(y_\alpha, x_\infty))\}
\]
The case where
\[
\epsilon_\alpha R_\alpha \leq R_\alpha (d_\alpha - d(y_\alpha, x_\infty))
\]
(with \(\text{Im} y_\alpha \rightarrow 0\)) corresponds to a ‘sphere bubble’ which is easier to handle than the rest. Therefore we will focus on the remaining cases henceforth. We divide our discussion on the remaining cases into three after choosing a subsequence if necessary:

1. The case \(\text{dist}(y_\alpha, \{t = 1\}) \rightarrow 0\) and so \(\text{dist}(y_\alpha, \{t = 1\}) < \text{dist}(y_\alpha, \{t = 0\})\)
(2) The case $\text{dist}(y_{\alpha}, \{t = 0\}) \to 0$ and so $\text{dist}(y_{\alpha}, \{t = 1\}) > \text{dist}(y_{\alpha}, \{t = 0\})$.

(3) The case $\text{dist}(y_{\alpha}, \partial D_{R_{\alpha}+1}^{I_0}) \to 0$.

The arguments needed to study these cases have little difference and so we focus on the Case (1).

Note that $\delta_{\alpha} R_{\alpha} \to \infty$ by (15.5) for any given $R > 0$, $\tilde{w}_{\alpha}(z)$ is defined eventually on

$$\Theta_{R+1}(0) := \{z \in B_{R+1}(0) \mid \Im z \geq I_0\}$$

for some $\{-\infty\} \cup I_0 \in \mathbb{R}$. Furthermore, we may assume,

$$\Theta_{R+1}(0) \subset \left\{z \in \mathbb{C} \mid \eta_{\alpha} z + y_{\alpha} \in \overline{\mathcal{D}} \right\}$$

Therefore, the maps

$$\tilde{w}_{\alpha} : \Theta_{R+1}(0) \subset \mathbb{C} \to M$$

satisfy the following properties:

(i) $E^\pi(\tilde{w}_{\alpha}) \leq \|H\|$, $\overline{\mathcal{T}} \tilde{w}_{\alpha} = 0$, $E^\perp(\tilde{w}_{\alpha}) \leq \|H\|$, (from the scale invariance)

(ii) $|d\tilde{w}_{\alpha}(0)| = 1$ by definition of $\tilde{w}_{\alpha}$ and $R_{\alpha}$.

(iii) $\|d\tilde{w}_{\alpha}\|_{C^0, B_{1}(z)} \leq 2$ for all $x \in B_{R}(0) \subset D^2(\epsilon_{\alpha} R_{\alpha})$,

(iv) $\overline{\mathcal{T}} \tilde{w}_{\alpha} = 0$ and $d(\tilde{w}_{\alpha}^* \lambda \circ j) = 0$,

(v) $\tilde{w}_{\alpha}(\partial \Theta_{R+1}) \subset R$.

For each fixed $R$, we take the limit of $\tilde{w}_{\alpha}|_{B_{R}}$, which we denote by $w_{R}$. Applying (iii) and then the local $C^{2,\alpha}$ estimates, Theorem 5.11, we obtain

$$\|d\tilde{w}_{\alpha}\|_{2,\alpha; B_{\mathcal{D}}(x)} \leq C$$

for some $C = C(R)$. Therefore we have a subsequence that converges in $C^2$ in each $B_{\mathcal{D}}(x), x \in \overline{\mathcal{D}}$. Then we derive that the convergence is in $C^2$-topology on $B_{\mathcal{D}}(x)$ for all $x \in \overline{\mathcal{D}}$ and in turn on $\Theta_{R}(0)$.

Therefore the limit $w_{R} : B_{R}(0) \to M$ of $\tilde{w}_{\alpha}|_{B_{R}(0)}$ satisfies

(1) $\overline{\mathcal{T}} w_{R} = 0$, $d(w_{R}^* \lambda \circ j) = 0$ and

$$E^\pi(w_{R}), E^\perp(\tilde{w}_{\alpha}) \leq \|H\|,$$

(2) $E^\pi(w_{R}) \leq \limsup_{\alpha} E^\pi_{(\lambda,j;B_{R}(0))}(\tilde{w}_{\alpha}) \leq \|H\|,$

(3) Since $\tilde{w}_{\alpha} \to w_{R}$ converges in $C^2$, we have

$$\|dw_{R}\|_{p, B_{1}(0)}^2 = \lim_{\alpha \to \infty} \|d\tilde{w}_{\alpha}\|_{p, B_{1}(0)}^2 \geq \frac{1}{2}.$$ By letting $R \to \infty$ and taking a diagonal subsequence argument, we have derived nonconstant contact instanton map $w_{\infty} : (\mathbb{H}, \partial \mathbb{H}) \to (M, Z)$.

On the other hand, the bound $E^\pi(w_{R}) \leq \|H\|$ for all $R$ and again by Fatou’s lemma implies

$$E^\pi(w_{\infty}) \leq \|H\|.$$ (By definition of $T_{\lambda}$, we must have $E^\pi(w_{\infty}) \geq T_{\lambda}$.) Now we examine the effect of the equation

$$d(w_{\infty}^* \lambda \circ j) = 0$$
on $E^\pi(w_{\infty})$.  

Using the identity $d(w^*_\infty \lambda) = \frac{1}{2} |d^\pi w_\infty|^2$ and Fatou’s lemma, we have

$$E^\pi (w_\infty) = \int_{\mathbb{H}} d(w^*_\infty \lambda) = \lim_{R \to \infty} \int_{\mathbb{H}_R} d(w^*_\infty \lambda)$$

$$= \lim_{r \to \infty} \int_{\{|z| \leq r, y=0\} \cup \mathbb{H} \cap \{|z|=r\}} w^*_\infty \lambda.$$

On the other hand, since $\text{Image} \ w_\infty |_{\partial \mathbb{H}} \subset R$, a Legendrian submanifold,

$$(w_\infty |_{\partial \mathbb{H}})^* \lambda = 0.$$

Therefore we have shown that

$$E^\pi (w_\infty) = \lim_{r \to \infty} \int_{\{|z|=r, y \geq 0\}} w^* \lambda = \int \alpha^* \lambda > 0$$

where $\alpha$ is the asymptotic self Reeb chord of $R$ given by

$$\alpha(t) := \lim_{r \to \infty} w(e^{\pi(r+it)}, \quad x + iy = e^{\pi(r+it)} \in \mathbb{H}$$

Hence we have produced a bubble map $w_\infty : (\mathbb{H}, \partial \mathbb{H}) \to (M, R)$ with its asymptotic chord given by $\alpha$.

An examination of the above proof in fact shows that the whole argument can be repeated equally applies to the case of $K_\alpha \to \infty$ and $x_\alpha \to \infty$, except that $J_{K_\alpha}$ is replaced by $J_\infty = J_0$. Hence we have established the two alternatives of the statement of the theorem.

By applying the standard procedure based on Theorem 15.1, we have finally finished the proof of Theorem 1.24.

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