KOSZULITY OF COHOMOLOGY
= $K(\pi, 1)$-NESS + QUASI-FORMALITY

LEONID POSITSELSKI

Abstract. This paper is a greatly expanded version of [37, Section 9.11]. A series of definitions and results illustrating the thesis in the title (where quasi-formality means vanishing of a certain kind of Massey multiplications in the cohomology) is presented. In particular, we include a categorical interpretation of the “Koszulity implies $K(\pi, 1)$” claim, discuss the differences between two versions of Massey operations, and apply the derived nonhomogeneous Koszul duality theory in order to deduce the main theorem. In the end we demonstrate a counterexample providing a negative answer to a question of Hopkins and Wickelgren about formality of the cochain DG-algebras of absolute Galois groups, thus showing that quasi-formality cannot be strengthened to formality in the title assertion.

Contents

Introduction 1
1. Koszulity Implies $K(\pi, 1)$-ness 5
2. Koszulity Implies Quasi-Formality 10
3. Noncommutative (Rational) Homotopy Theory 18
4. $K(\pi, 1)$-ness + Quasi-Formality Imply Koszulity 23
5. Self-Consistency of Nonhomogeneous Quadratic Relations 26
6. Koszulity Does Not Imply Formality 33
References 37

Introduction

A quadratic algebra is an associative algebra defined by homogeneous quadratic relations. In other words, a positively graded algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ over a field $k$ is called quadratic if it is generated by its first-degree component $A_1$ with relations in degree 2. A positively graded associative algebra $A$ is called Koszul [41, 4, 32] if one has $\text{Tor}_i^A(k, k) = 0$ for all $i \neq j$, where the first grading $i$ on the Tor spaces is the usual homological grading and the second grading $j$, called the internal grading, is induced by the grading of $A$. In particular, this condition for $i = 1$ means that the algebra $A$ is generated by $A_1$, and the conditions for $i = 1$ and 2 taken together mean that $A$ is quadratic.

Conversely, for any positively graded algebra $A$ with finite-dimensional components $A_n$ the diagonal part $\bigoplus_n \text{Ext}_{A_n}^n(k, k)$ of the algebra $\text{Ext}^*_A(k, k) \simeq \text{Tor}^*_A(k, k)^*$
is a quadratic algebra. When the algebra $A$ is quadratic, the two quadratic algebras $A$ and $\bigoplus_n \text{Ext}^n_A(k,k)$ are called quadratic dual to each other. Without the finite-dimensionality assumption on the grading components, the quadratic duality connects quadratic graded algebras with quadratic graded coalgebras [32, 33].

When one attempts to deform quadratic algebras by considering algebras with nonhomogeneous quadratic relations, one discovers that there are two essentially different ways of doing so. One can either consider relations with terms of the degrees not greater than 2, that is

$$q_2(x) + q_1(x) + q_0 = 0,$$

where $x = (x_\alpha)$, deg $x_\alpha = 1$ denotes the set of generators, and deg $q_n = n$; or relations with terms of the degrees not less than 2, that is

$$q_2(x) + q_3(x) + q_4(x) + q_5(x) + \cdots = 0.$$

In the latter case, it is natural to allow the relations to be infinite power series, that is consider the algebra they define as a quotient algebra of the algebra of formal Taylor power series in noncommuting variables $x_\alpha$, rather than the algebra of noncommutative polynomials. Almost equivalently, this means considering a set of relations of the type (2) as defining a conilpotent coalgebra, while a set of relations of the type (1) defines a filtered algebra. More precisely, of course, one should say that the complete topological algebra defined by the relations (2) is the dual vector space to a discrete conilpotent coalgebra. This is one of the simplest ways to explain the importance of coalgebras in Koszul duality.

The alternative between considering nonhomogeneous quadratic relations of the types (1) and (2) roughly leads to a division of the Koszul duality theory into two streams, the former of them going back to the classical paper [41] and the present author’s work [31], and the latter one originating in the paper [32]. The former theory, invented originally for the purposes of computing the cohomology of associative algebras generally and the Steenrod algebra in particular, eventually found its applications in semi-infinite homological algebra. The latter point of view was being applied to Galois cohomology, the conjectures about absolute Galois groups, and the theory of motives with finite coefficients. As a general rule, the author’s subsequent papers vaguely associated with relations of the type (1) were published on the arXiv in the subject area [math.CT], while the papers having to do with relations of the type (2) were put into the area [math.KT].

This paper is concerned with relations of the type (2). Its subject can be roughly described as cohomological characterization of the coalgebras $C$ defined by the relations (2) with the quadratic principal parts

$$q_2(x) = 0$$

of the relations defining a Koszul graded coalgebra. In fact, according to the main theorem of [32] (see also [23]) a conilpotent coalgebra $C$ is defined by a self-consistent system of relations (2) with Koszul quadratic principal part (3) if and only if its
cohomology algebra

\[ H^*(C) = \text{Ext}^*_C(k, k) \]

is Koszul. Moreover, a certain seemingly weaker set of conditions on the algebra \( H^* = H^*(C) \) is sufficient, and implies Koszulity of algebras of the form \( H^*(C) \). Specifically, the algebra \( H^*(C) \) is Koszul whenever \( H^2(C) \) is multiplicatively generated by \( H^1(C) \), there are no nontrivial relations of degree 3 between elements of degree 1 in \( H^*(C) \), and the quadratic part of the algebra \( H^*(C) \) is Koszul (see Theorem 1.3). When the algebra \( H^*(C) \) is Koszul, it is simply the dual quadratic algebra to the quadratic coalgebra defined by the relations (3).

Given an arbitrary (not necessarily conilpotent) coaugmented coalgebra \( D \) with the maximal conilpotent subcoalgebra \( C = \text{Nilp} \ D \subset D \), the algebra \( H^*(D) = \text{Ext}^*_D(k, k) \) is Koszul if and only if the following two conditions hold [37, Section 9.11]:

(i) the homomorphism of cohomology algebras \( H^*(C) \to H^*(D) \) induced by the embedding of coalgebras \( C \to D \) is an isomorphism;

(ii) a certain family of higher Massey products in the cohomology algebra \( H^*(D) \) vanishes.

In this paper we provide a detailed proof of this result, and discuss at length its constituting components.

In particular, the condition (i) and the implication “Koszulity of \( H^*(D) \) implies (i)” allow numerous analogues and generalizations, including such assertions as

- for any discrete group \( \Gamma \) whose cohomology algebra \( H^*(\Gamma, k) \) with coefficients in a field \( k \) is Koszul, the cohomology algebra \( H^*(\Gamma_k^*, k) \) of the \( k \)-completion of the group \( \Gamma \) is isomorphic to the algebra \( H^*(\Gamma, k) \) [33, Section 5]; or

- any rational homotopy type \( X \) with a Koszul cohomology algebra \( H^*(X, \mathbb{Q}) \) is a rational \( K(\pi, 1) \) space [30].

That is why we call the condition (i) “the \( K(\pi, 1) \) condition”.

More generally, in place of the cochain DG-algebra of a coaugmented coalgebra \( D \), consider an arbitrary nonnegatively cohomologically graded augmented DG-algebra

\[ 0 \to A^0 \to A^1 \to A^2 \to \cdots \]

over a field \( k \) with \( H^0(A^*) \simeq k \). Then the cohomology algebra \( H^*(A^*) \) of the DG-algebra \( A^* \) is Koszul if and only if the following two conditions hold:

(i) the cohomology coalgebra of the bar-construction of the augmented DG-algebra \( A^* \) is concentrated in cohomological degree 0;

(ii) a certain family of higher Massey products in the cohomology algebra \( H^*(A^*) \) vanishes.

Once again, we call the condition (i) “the \( K(\pi, 1) \) condition”.

Furthermore, relation sets of the type (2) are naturally viewed up to variable changes

\[ x_\alpha \to x_\alpha + p_{2,\alpha}(x) + p_{3,\alpha}(x) + p_{4,\alpha}(x) + \cdots, \]

where \( \deg p_{n,\alpha} = n \). Some apparently different systems of relations (2) may be also equivalent, i. e., they may mutually imply each other, or in other words, generate
the same closed ideal in the algebra of noncommutative Taylor power series (see Example 6.5 for an illustration). A natural question is whether or when a system of relations (2) can be homogenized, i.e., transformed into a system of relations equivalent to (3) by a variable change (4). We show that a variable change (4) homogenizing a given system of relations (2) defining a conilpotent coalgebra $C$ with Koszul cohomology algebra $H^*(C)$ exists if and only if the cochain DG-algebra computing $H^*(C)$ is formal, i.e., can be connected with its cohomology algebra by a chain of multiplicative quasi-isomorphisms.

Obviously, formality implies the Massey product vanishing condition (ii), which we accordingly call the quasi-formality condition. Not distinguishing formality from quasi-formality seems to be a common misconception. The above explanations suggest that the cochain DG-algebras of most conilpotent coalgebras $C$ with Koszul cohomology algebras $H^*(C)$ should not be formal but only quasi-formal, as the possibility of homogenizing a system of relations (2) looks unlikely, generally speaking.

Indeed, we provide a simple counterexample of a pro-$l$-group $H$ whose cohomology algebra $H^*(H, \mathbb{Z}/l)$ is Koszul, while the cochain DG-algebra computing it is not formal, as the relations in the group coalgebra $\mathbb{Z}/l(H)$ cannot be homogenized by variable changes. It was conjectured in the papers [32, 37] that the cohomology algebra $H^*(G_F, \mathbb{Z}/l)$ is Koszul for the absolute Galois group $G_F$ of any field $F$ containing a primitive $l$-root of unity; and the question was asked in the paper [16] whether the cochain DG-algebra of the group $G_F$ with coefficients in $\mathbb{Z}/l$ is formal. As the group $H$ in our counterexample is the maximal quotient pro-$l$-group of the absolute Galois group $G_F$ of an appropriate $p$-adic field $F$ containing a primitive $l$-root of unity, our results provide a negative answer to this question of Hopkins and Wickelgren.

Acknowledgements. The mathematical content of this paper was largely worked out around the years 1996–97, when the author was a graduate student at Harvard University. I benefited from helpful conversations with A. Beilinson, J. Bernstein, and D. Kazhdan at that time. Section 9.11, where these ideas were first put in writing, was inserted into the paper [37] shortly before its journal publication under the influence of a question asked by P. Deligne during the author’s talk at the “Christmas mathematical meetings” in the Independent University of Moscow in January 2011. I learned that the question about formality of the cochain DG-algebras of absolute Galois groups was asked in the paper [16] from a conversation with I. Efrat while visiting Ben Gurion University of the Negev in Be’er Sheva in the Summer of 2014. The decision to write this paper was stimulated by the correspondence with P. Schneider in the Summer of 2015. I am grateful to all the mentioned persons and institutions. Finally, I would like to thank J. Stasheff, who read the first version of this paper, made several helpful comments, and suggested a number of relevant references. I also wish to thank the anonymous referee for careful reading of the manuscript and helpful suggestions. The author’s research is supported by the Israel Science Foundation grant # 446/15 at the University of Haifa.
1. Koszulity Implies $K(\pi, 1)$-ness

Postponing the discussion of DG-algebras, DG-coalgebras, and derived nonhomogeneous Koszul duality to Sections 2–4, we devote this section to the formulation of a categorical version of the “Koszulity implies $K(\pi, 1)$” claim. We begin our discussion with recalling some basic definitions and results from [32] and [33, Section 5].

A coassociative counital coalgebra $D$ over a field $k$ is said to be coaugmented if it is endowed with a coalgebra morphism $k \to D$ (called the coaugmentation). The quotient coalgebra (without counit) of a coaugmented coalgebra $D$ by the image of the coaugmentation morphism is denoted by $D_+ = D/k$.

A coaugmented coalgebra $C$ is called conilpotent if for any element $c \in C$ there exists an integer $m \geq 1$ such that $c$ is annihilated by the iterated comultiplication map $C \to C \otimes k D \to C \otimes k D \otimes k D \to \cdots$. (Several references and terminological comments related to this definition can be found in [35, Remark D.6.1].) The maximal conilpotent subcoalgebra $\bigcup_m \ker(D \to D \otimes k D \otimes k D \to \cdots)$ of a coaugmented coalgebra $D$ is denoted by $\text{Nilp}_D \subset D$.

The cohomology algebra of a coaugmented coalgebra $D$ is defined as the Ext algebra $H^*(D) = \text{Ext}^*_D(k, k)$, where the field $k$ is endowed with a left $D$-comodule structure via the coaugmentation map. The cohomology algebra $H^*(D)$ is computed by the reduced cochain DG-algebra of the coalgebra $D$

$$k \longrightarrow D_+ \longrightarrow D_+ \otimes_k D_+ \longrightarrow D_+ \otimes_k D_+ \otimes_k D_+ \longrightarrow \cdots,$$

which is otherwise known as the reduced cobar-complex or the cobar construction of the coaugmented coalgebra $D$ and denoted by $\text{Cob}^*(D)$.

The following result can be found in [33, Corollary 5.3].

**Theorem 1.1.** Let $D$ be a coaugmented coalgebra over a field $k$ and $\text{Nilp}_D \subset D$ be its maximal conilpotent subcoalgebra. Assume that the cohomology algebra $H^*(D) = \text{Ext}^*_D(k, k)$ is Koszul. Then the embedding $\text{Nilp}_D \hookrightarrow D$ induces a cohomology isomorphism $H^*(\text{Nilp}_D) \cong H^*(D)$.

Theorem 1.1 has a version with an augmented algebra $R$ replacing the coaugmented coalgebra $D$ [33, Remark 5.6]. Let $R_+ = \ker(R \to k)$ denote the augmentation ideal, and let $I$ run over all the ideals $I \subset R_+$ in $R$ for which the quotient algebra $R/I$ is finite-dimensional and its augmentation ideal $R_+/I$ is nilpotent. The coalgebra of pronilpotent completion $R^\sim$ of the augmented algebra $R$ is defined as the filtered inductive limit $R^\sim = \lim^\to_{I} (R/I)^*$ of the coalgebras $(R/I)^*$ dual to the finite-dimensional algebras $R/I$. Clearly, the coalgebra $R^\sim$ is conilpotent.

The cohomology algebra $H^*(R) = \text{Ext}^*_R(k, k)$ of an augmented algebra $R$ is computed by its reduced cobar-complex $\text{Cob}^*(R)$

$$k \longrightarrow R_+^* \longrightarrow (R_+ \otimes_k R_+)^* \longrightarrow (R_+ \otimes_k R_+ \otimes_k R_+)^* \longrightarrow \cdots$$

The natural injective morphism of cobar-complexes $\text{Cob}^*(R^\sim) \hookrightarrow \text{Cob}^*(R)$ induces a natural morphism of cohomology algebras $H^*(R^\sim) \hookrightarrow H^*(R)$.

**Theorem 1.2.** Let $R$ be an augmented algebra over a field $k$ and $R^\sim$ be the coalgebra of its pronilpotent completion. Assume that the cohomology algebra

$$...$$
\[ H^*(R) = \text{Ext}^*_R(k, k) \] is Koszul. Then the natural morphism of the cohomology algebras \[ H^*(R^c) \longrightarrow H^*(R) \] is an isomorphism.

The proofs of Theorems 1.1 and 1.2 are based on the following result about the cohomology of conilpotent coalgebras [32, Main Theorem 3.2]. For any positively graded algebra \( H^* \) over a field \( k \), we denote by \( qH^* \) the quadratic part of the algebra \( H^* \), i.e., the universal final object in the category of quadratic algebras over \( k \) endowed with a morphism into \( H^* \). The quadratic algebra \( qH^* \) is uniquely defined by the condition that the morphism of graded algebras \( qH^* \longrightarrow H^* \) is an isomorphism in degree 1 and a monomorphism in degree 2.

**Theorem 1.3.** Let \( C \) be a conilpotent coaugmented coalgebra, i.e., \( \text{Nilp} C = C \). Assume that
- the quadratic part \( qH^*(C) \) of the graded algebra \( H^*(C) \) is Koszul; and
- the morphism of graded algebras \( qH^*(C) \longrightarrow H^*(C) \) is an isomorphism in degree 2 and a monomorphism in degree 3.

Then the graded algebra \( H^*(C) \) is quadratic (and consequently, Koszul).

The proof of Theorem 1.1 can be found in [33, Theorem 5.2 and Corollary 5.3]. The proof of Theorem 1.2 is very similar; let us briefly explain how it works.

**Proof of Theorem 1.2.** One notices that for any augmented algebra \( R \) the morphism of cohomology algebras \( H^*(R^c) \longrightarrow H^*(R) \) is an isomorphism in degree 1 and a monomorphism in degree 2. Indeed, the category of left comodules over \( R^c \) is isomorphic to the full subcategory in the category of left \( R \)-modules consisting of all the ind-nilpotent \( R \)-modules (direct limits of iterated extensions of the trivial \( R \)-module \( k \), the latter being defined in terms of the augmentation of \( R \)). This is a full subcategory closed under subobjects, quotient objects, and extensions in the abelian category of left \( R \)-modules; so the argument of [33, Lemma 5.1] applies.

Now if the algebra \( H^*(R) \) is Koszul, then it follows that the maps \( H^1(R^c) \longrightarrow H^1(R) \) and \( H^2(R^c) \longrightarrow H^2(R) \) are isomorphisms, the composition \( qH^*(R^c) \longrightarrow H^*(R^c) \longrightarrow H^*(R) \) is an isomorphism of graded algebras, and the algebra \( H^*(R^c) \) satisfies the conditions of Theorem 1.3. Hence we conclude that the algebra \( H^*(R^c) \) is quadratic and the morphism \( H^*(R^c) \longrightarrow H^*(R) \) is an isomorphism.

A generalization of the results of Theorems 1.1 and 1.2 to t-structures in triangulated categories [1, n°1.3] was announced in [33, Remark 5.6]. The idea of this generalization can be described as follows.

Recall that for any t-structure \((D^{<0}, D^{\geq 0})\) on a triangulated category \( D \) with the core \( C = D^{<0} \cap D^{\geq 0} \) and for any two objects \( X, Y \in C \) there are natural maps
\[
\theta^n_C(X, Y) : \text{Ext}^n_C(X, Y) \longrightarrow \text{Hom}_D(X, Y[n]), \quad n \geq 0,
\]
from the Ext groups in the abelian category \( C \) to the Hom groups in the triangulated category \( D \). The maps \( \theta^n_C = \theta^n_C(X, Y) \) transform the compositions of Yoneda Ext classes in \( C \) into the compositions of morphisms in \( D \). Furthermore, the maps
\( \theta^n_{C,D} \) are always isomorphisms for \( n \leq 1 \) and monomorphisms for \( n = 2 \) (see [1, Remarque 3.1.17], [3, Section 4.0], or [37, Corollary A.17]).

Starting with a coaugmented coalgebra \( D \), consider the conilpotent coalgebra \( C = \text{Nilp}\ D \) and the abelian category \( C \) of finite-dimensional left \( C \)-comodules. Consider the bounded derived category of left \( D \)-comodules \( \mathbb{D}^b(D-\text{comod}) \), and set \( D \) to be the full subcategory of \( \mathbb{D}^b(D-\text{comod}) \) generated by the abelian subcategory \( C \subset D-\text{comod} \). Then \( C \) is the core of a bounded \( t \)-structure on \( D \).

Analogously, starting with an augmented algebra \( R \), consider the conilpotent coalgebra \( C = \hat{R} \) and the abelian category \( C \) of finite-dimensional left \( C \)-comodules (or, which is the same, finite-dimensional nilpotent \( R \)-modules). Consider the bounded derived category of left \( R \)-modules \( \mathbb{D}^b(R-\text{mod}) \), and set \( D \) to be the full triangulated subcategory of \( \mathbb{D}^b(R-\text{mod}) \) generated by the abelian subcategory \( C \subset R-\text{mod} \). Once again, \( C \) is the core of a bounded \( t \)-structure on \( D \).

In both cases, the assertions of Theorems 1.1 and 1.2 claim that the map

\[
\theta^n_{C,D}(k,k) : \text{Ext}^n_C(k,k) \longrightarrow \text{Hom}_D(k,k[n])
\]

is an isomorphism for all \( n \), provided that the graded algebra \( \text{Hom}_D(k,k[\ast]) \) is Koszul. Here the trivial \( D \)-comodule or \( R \)-module \( k \) is the only irreducible object in \( C \). All objects of the abelian category \( C \) being of finite length, it follows that all the morphisms \( \theta^n_{C,D} \) are isomorphisms for the \( t \)-structures under consideration.

A \( t \)-structure for which all the maps \( \theta^n_{C,D} \) are isomorphisms is called a “\( t \)-structure of derived type” [3, Section 4.0]. This condition is also known as the “\( K(\pi,1) \) condition of Bloch and Kriz” [7] and, in somewhat larger generality, as the “silly filtration condition” [37, Sections 0.2–0.5]. Proving that a given \( t \)-structure is of derived type is sometimes an important and difficult problem (see, e. g., [2]). A standard approach working in some particular cases can be found in [22, Section 12] (see also [39, Section A.2]); the results below in this section provide an alternative way.

Let \( S = \{\alpha\} \) be a set of indices. A big ring (or a “ring with many objects”) \( A \) is a collection of abelian groups \( A^n_{\alpha\beta} \), endowed with the multiplication maps \( A_{\alpha\beta} \times A_{\beta\gamma} \longrightarrow A_{\alpha\gamma} \) and the unit elements \( 1_{\alpha} \in A_{\alpha\alpha} \) satisfying the conventional associativity and unit axioms. A big ring with a set of indices \( S \) is the same thing as a preadditive category with the objects indexed by \( S \) (see [29] or [37, Section A.1]).

The categories of left and right modules over a big ring are defined in the obvious way. A left (resp., right) \( A \)-module is the same thing as a covariant (resp., contravariant) additive functor from the preadditive category corresponding to \( A \) to the category of abelian groups. The tensor product of a right \( A \)-module \( N \) and a left \( A \)-module \( M \) is an abelian group constructed as the cokernel of the difference of the right and left action maps

\[
\bigoplus_{\alpha,\beta} N_{\alpha} \otimes_Z A_{\alpha,\beta} \otimes_Z M_{\beta} \twoheadrightarrow \bigoplus_{\alpha} N_{\alpha} \otimes_Z M_{\alpha}.
\]

The derived functor \( \text{Tor}^A \) of the functor of tensor product \( \otimes_A \) of modules over a big graded ring is defined in the usual way.
A big graded ring (or a “graded ring with many objects”) $A$ is a big ring in which every group $A_{\alpha\beta}$ is graded and the multiplication maps are homogeneous. In other words, it is a collection of abelian groups $A^n_{\alpha\beta}$ endowed with the multiplication maps $A^p_{\alpha\beta} \times A^q_{\beta\gamma} \rightarrow A^{p+q}_{\alpha\gamma}$ and the unit elements $1_{\alpha} \in A^0_{\alpha\alpha}$ satisfying the associativity and unit axioms. We will assume that $A^n_{\alpha\beta} = 0$ for $n < 0$ or $\alpha \neq \beta$ and $n = 0$, and that the rings $A^0_{\alpha\alpha}$ are (classically) semisimple.

The definition of the Koszul property of a nonnegatively graded ring $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ with a semisimple degree-zero component $A_0$ is pretty well known [4], and the definition of a Koszul big graded ring in the above generality is its straightforward extension. Specifically, a big graded ring $A$ is called Koszul if one has $\text{Tor}^n_{A}(A^0_{\alpha\alpha}, A^0_{\beta\beta}) = 0$ for all $\alpha, \beta \in S$ and all $i \neq j$ (where $i$ denotes the homological and $j$ the internal grading on the Tor). One can also define quadratic big graded rings and the quadratic part of a big graded ring, etc. A discussion of the Koszul property of big graded rings in a greater generality with the semisimplicity condition replaced by a flatness condition can be found in [37, Section 7.4], and even without the flatness condition, in the rest of [37, Section 7].

The following theorem is the main result of this section.

**Theorem 1.4.** Let $C$ be the core of a t-structure on a small triangulated category $D$. Suppose that every object of $C$ has finite length and let $I_\alpha$ be the irreducible objects of $C$. Assume that the big graded ring $A$ with the components $A^n_{\alpha\beta} = \text{Hom}_D(I_\alpha, I_\beta[n])$ is Koszul. Then for any two objects $X, Y \in C$ and any $n \geq 0$ the natural map $\theta^n_{C,D}(X,Y): \text{Ext}^n_C(X,Y) \rightarrow \text{Hom}_D(X,Y[n])$ is an isomorphism.

The above theorem is deduced from the following result about the Ext rings between irreducible objects in abelian categories, which is a categorical generalization of Theorem 1.3.

**Theorem 1.5.** Let $C$ be a small abelian category such that every object of $C$ has finite length and let $I_\alpha$ be the irreducible objects of $C$. Consider the big graded rings $B$ with the components $B^n_{\alpha\beta} = \text{Ext}^n_C(I_\alpha, I_\beta)$.

(I) the quadratic part $qB$ of the big graded ring $B$ is Koszul; and

(II) the morphism of big graded rings $qB \rightarrow B$ is an isomorphism in the degree $n = 2$ and a monomorphism in the degree $n = 3$.

Then the big graded ring $B$ is quadratic (and consequently, Koszul).

Actually, the following slightly stronger form of Theorem 1.4, generalizing both Theorems 1.4 and 1.5, can be obtained from Theorem 1.5. It is a categorical generalization of [33, Theorem 5.2].

**Theorem 1.6.** Let $C$ be the core of a t-structure on a small triangulated category $D$. Suppose that every object of $C$ has finite length and let $I_\alpha$ be the irreducible objects of $C$. Consider the big graded rings $A$ and $B$ with the components $A^n_{\alpha\beta} = \text{Ext}^n_C(I_\alpha, I_\beta)$ and $B^n_{\alpha\beta} = \text{Hom}_D(I_\alpha, I_\beta[n])$. Then whenever the big graded ring $B$ satisfies the assumptions (I) and (II) of Theorem 1.5, the natural morphism of big graded rings $A \rightarrow B$ induces an isomorphism $A \simeq qB$. 

Let us clarify the following point, which otherwise might become a source of confusion. It is well-known [1, 3, 37] that for a t-structure with the core \( C \) on a triangulated category \( D \) the natural maps \( \theta^n_{C,D}(X,Y) : \text{Ext}^n_C(X,Y) \to \text{Hom}_D(X,Y[n]) \) are isomorphisms for all \( X, Y \in C, n \geq 0 \) if and only if any element in \( \text{Hom}_D(U,V[1]) \) with \( U, V \in C \). However, this is only true because one considers the degree-one generation condition for \( X \) and \( Y \) running over all the objects of \( C \) and not just the irreducible objects (cf. [37, Proposition B.1]). The Koszulity condition in Theorem 1.4 is much stronger than a degree-one generation condition, but it is applied to a much smaller algebra of homomorphisms between the irreducible objects of \( C \). On the other hand, the big graded ring of Yoneda extensions between all the objects of a given abelian category is always Koszul in an appropriate sense [37, Example 8.3].

**Example 1.7.** Given a discrete group \( \Gamma \) and a field \( k \), set \( C \) to be the coalgebra of (functions on) the proalgebraic completion of \( \Gamma \) over \( k \). Then the category \( C \) of finite-dimensional representations of a group \( \Gamma \) over a field \( k \) is isomorphic to the category of finite-dimensional left \( C \)-comodules. Let \( D \) denote the full triangulated subcategory of the bounded derived category \( \text{D}^b(k[\Gamma]-\text{mod}) \) of arbitrary \( \Gamma \)-modules over \( k \) generated by the subcategory of finite-dimensional modules \( C \subset k[\Gamma]-\text{mod} \). Then \( C \) is the core of a (bounded) t-structure on \( D \), so Theorems 1.4–1.6 are applicable whenever the Koszulity assumption or the assumptions (I–II) are satisfied. This would allow to obtain a comparison between the cohomology of (the proalgebraic group corresponding to) the coalgebra/commutative Hopf algebra \( C \) and the discrete group \( \Gamma \) with finite-dimensional coefficients.

Let us emphasize that we do not assume the abelian category \( C \) or the triangulated category \( D \) to be linear over a field in the above theorems. E. g., the category of finite modules over a profinite group is fine in the role of \( C \), as is the category of modules of finite length over any complete commutative local ring, etc.

**Proof of Theorem 1.6.** This is a particular case of [37, Corollary 8.5]. By a general property of t-structures (see [1, Remarque 3.1.17] or [37, Corollary A.17]), the morphism of big graded rings \( A \to B \) is an isomorphism in degree 1 and a monomorphism in degree 2 (cf. the proof of Theorem 1.2). It follows that if the ring \( B \) satisfies the conditions (I) and (II), then so does the ring \( A \). By Theorem 1.5, one can then conclude that the big graded ring \( A \) is quadratic, and hence \( A \simeq qB \).

**Proof of Theorem 1.4.** According to Theorem 1.6, the map \( \theta^n_{C,D}(X,Y) \) is an isomorphism whenever the objects \( X \) and \( Y \in C \) are irreducible. The general case follows by induction on the lengths.

**Brief sketch of proof of Theorem 1.5.** This is a particular case of [37, Theorem 8.4]. Let us start with a comment on the proof of Theorem 1.3. It is based on two ingredients: the basic theory of quadratic and Koszul graded algebras and coalgebras over a field, and the spectral sequence connecting the cohomology of a conilpotent coalgebra \( C \) with the cohomology of its associated graded coalgebra with respect to
the coaugmentation filtration. In the case of Theorem 1.5, first of all one needs to
develop the basic theory of quadratic and Koszul big graded rings, which is done (in
a greater generality) in [37, Sections 6–7]. Then one has to work out the passage
from the ungraded category $\mathcal{C}$ to its graded version.

One possible approach to the latter task would be to associate a coalgebra-like
algebraic structure with an abelian category consisting of objects of finite length,
then filter that structure and pass to the associated graded one. The required class
of algebraic structures was introduced in [10, § IV.3-4]. An abelian category consisting
of objects of finite length is equivalent to the category of finitely generated discrete
modules over a pseudo-compact topological ring. One would have to embed the
category of finitely generated discrete modules into the category of pseudo-compact
modules in order to do cohomology computations with projective resolutions.

There is a more delicate approach developed in [37, Sections 3–4], which is purely
categorical. One associates with an abelian category $\mathcal{C}$ consisting of objects of finite
length the exact category $\mathcal{F}$ whose objects are the objects of $\mathcal{C}$ endowed with a finite
filtration for which all the successive quotient objects are semisimple. Then one needs
to pass from the filtered exact category $\mathcal{F}$ to the associated graded abelian category $\mathcal{G}$.
The main property connecting the categories $\mathcal{C}$ and $\mathcal{F}$ is the natural isomorphism
\begin{equation}
\lim_{\longrightarrow} \mathcal{F}^{\mathbb{Z}}(X, Y(m)),
\end{equation}
where $u: \mathcal{F} \to \mathcal{C}$ is the functor of forgetting the filtration and $Z \mapsto Z(m)$ denotes
the filtration shift. The main property connecting the categories $\mathcal{F}$ and $\mathcal{G}$ is the long
exact sequence
\begin{equation}
\cdots \to \text{Ext}^{\mathbb{Z}}_F(X, Y(-1)) \to \text{Ext}^{\mathbb{Z}}_F(X, Y) \to \text{Ext}^{\mathbb{Z}+1}_G(\text{gr} X, \text{gr} Y) \to \text{Ext}^{\mathbb{Z}+1}_F(X, Y(-1)) \to \cdots
\end{equation}
for any two objects $X, Y \in \mathcal{F}$, where $\text{gr} : \mathcal{F} \to \mathcal{G}$ is the functor assigning to a filtered
object its associated graded object. The construction of the category $\mathcal{G}$ is a particular
case of a general construction of the reduction of an exact category by a graded center
element developed in the paper [40].

The isomorphism (5) and the long exact sequence (6) taken together are used in
lieu of the spectral sequence connecting the cohomology of a coalgebra $C$ and its
associated graded coalgebra $\text{gr}_N C$ by the coaugmentation filtration $N$ in order to
extend the argument from [32, Main Theorem 3.2] from conilpotent coalgebras to
abelian categories consisting of objects of finite length.

\section{Koszulity Implies Quasi-Formality}

Generally speaking, Massey products are natural partially defined multivalued
polylinear operations in the cohomology algebra of a DG-algebra which are preserved
by quasi-isomorphisms of DG-algebras. There are several different ways to construct
such operations. We start with introducing the construction most relevant for our
purposes, and later explain how it is related to a more elementary construction. The
most relevant reference for us is [25]; see also the earlier paper [43], the heavier [26],
and the later work [12, Section 5].
Let $A^\bullet = (A^*, d: A^i \to A^{i+1})$ be a nonzero DG-algebra over a field $k$; suppose that
it is endowed with an augmentation (DG-algebra morphism) $A^\bullet \to k$ and denote
the augmentation kernel ideal by $A^+_* = \ker(A^* \to k)$. By the definition, the bar
construction $\text{Bar}^\bullet(A^\bullet)$ of an augmented DG-algebra $A^\bullet$ is the tensor coalgebra
$$\text{Bar}(A) = \bigoplus_{n=0}^\infty A^+_[1]^\otimes n$$
of the graded vector space $A^+_[1]$ obtained by shifting by 1 the cohomological grading
of the augmentation ideal $A^+$. The grading on $\text{Bar}(A)$ is induced by the grading of
$A^+_[1]$. Alternatively, one can define the bar construction $\text{Bar}(A)$ as the direct sum
of the tensor powers $A^+_[1]^\otimes n$ of the vector space $A^+$ and endow it with the total grading
equal to the difference $i - n$ of the grading $i$ induced by the grading of $A^+$ and the
grading $n$ by the number of tensor factors.
The differential on $\text{Bar}^\bullet(A^\bullet)$ is the sum of two summands $d + \partial$, the former of them
induced by the differential on $A^+_[1]$ and the latter one by the multiplication in $A^+_[1]$.
One has to work out the plus/minus signs in order to make the total differential on
$\text{Bar}^\bullet(A^\bullet)$ square to zero; this is a standard exercise.
One defines a natural increasing filtration on the complex $\text{Bar}^\bullet(A^\bullet)$ by the rule
$$F_p \text{Bar}^\bullet(A^\bullet) = \bigoplus_{n=0}^p A^+_[1]^\otimes n.$$The associated graded complex $\text{gr}^F \text{Bar}^\bullet(A^\bullet)$ of the complex $\text{Bar}^\bullet(A^\bullet)$ by the filtration
$F$ is naturally identified with the graded vector space $\text{Bar}(A)$ endowed with the
differential $d$ induced by the differential on $A^+_[1]$. The spectral sequence $E^{pq}_r$ of the filtered complex $\text{Bar}^\bullet(A^\bullet)$ has the initial page
$$E^{pq}_1 = (H^*(A^+_[1]^\otimes p))^q,$$where the grading $q$ on the tensor power $H^*(A^+_[1]^\otimes p)$ of the augmentation ideal $H^*(A^+_[1])$
of the cohomology algebra $H^*(A^*)$ of the DG-algebra $A$ is induced by the cohomological grading on $H^*(A^+_[1])$. The differentials are
$$d^{pq}_r: E^{pq}_r \longrightarrow E^{p-r,q-r+1}_r,$$and the limit page is given by the rule
$$E^\infty = \text{gr}^F H^q - p \text{Bar}^\bullet(A^\bullet).$$The cohomology of the complex $\text{Bar}^\bullet(A^\bullet)$ are known as the differential $\text{Tor}$
spaces $[9, 12] H^* \text{Bar}^\bullet(A^\bullet) = \text{Tor}^A_*(k, k)$ (“of the first kind” [17]) over the
DG-algebra $A^\bullet$. This is the derived functor of tensor product of DG-modules over $A^\bullet$ defined on
the “conventional” derived categories of DG-modules (obtained by inverting
the DG-module morphisms inducing isomorphisms of the cohomology modules);
see [21], [14], or [36, Section 1]. The spectral sequence $E^{pq}_r$ is called the algebraic
Eilenberg–Moore spectral sequence associated with a DG-algebra $A^\bullet$ [9, 12, 13].
The differential $d_1$ is induced by the multiplication in the cohomology algebra $H^*(A^*)$, and whole the page $E_1$ is simply the bar-complex of the cohomology algebra $H^*(A^*)$. Hence the page $E_2$ can be computed as

$$E^{pq}_2 = \text{Tor}^{H^*(A^*)}_{pq}(k, k),$$

where the first grading $p$ on the Tor spaces in the right-hand side is the conventional homological grading of the Tor and the second grading $q$ is the “internal” grading induced by the cohomological grading on the algebra $H^*(A^*)$.

The differentials $d^{pq}_r$, $r \geq 2$, in the algebraic Eilenberg–Moore spectral sequence $E^{pq}_r = E^{pq}_r(A^*)$ associated with an augmented DG-algebra $A^*$ are, by the definition, the Massey products in the cohomology algebra $H^*(A^*)$ that we are interested in. An augmented DG-algebra $A^*$ is called quasi-formal if the spectral sequence $E^{pq}_r(A^*)$ degenerates at the page $E_2$, that is all the Massey products $d^{pq}_r$, $r \geq 2$, vanish.

An augmented DG-algebra $A^*$ is called formal (in the class of augmented DG-algebras) if it can be connected with its cohomology algebra $H^*(A^*)$, viewed as an augmented DG-algebra with zero differential and the augmentation induced by that of $A^*$, by a chain of quasi-isomorphisms of augmented DG-algebras.

**Proposition 2.1.** Any formal augmented DG-algebra is quasi-formal.

**Proof.** Notice that any morphism of augmented DG-algebras $A^* \rightarrow B^*$ induces a morphism of spectral sequences $E^{pq}_r(A^*) \rightarrow E^{pq}_r(B^*)$. When the morphism $A^* \rightarrow B^*$ is a quasi-isomorphism, the induced morphism of spectral sequences is an isomorphism on the pages $E_1$, and consequently also on all the higher pages. So the Massey products are preserved by quasi-isomorphisms of augmented DG-algebras.

Therefore, whenever an augmented DG-algebra $A^*$ is connected with an augmented DG-algebra $B^*$ by a chain of quasi-isomorphisms of augmented DG-algebras, an augmented DG-algebra $A^*$ is quasi-formal if and only if an augmented DG-algebra $B^*$ is. Since the Massey products in a DG-algebra with zero differential clearly vanish (as do the higher differentials in the spectral sequence of any bicomplex with one of the two differentials vanishing), the desired assertion follows. \hfill $\square$

An extension of (a part of) the canonical, partially defined, multivalued Massey operations on the cohomology algebra of a DG-algebra $A^*$ to total, single-valued poly-linear maps, which taken together are defined up to certain transformations, is called the $A_\infty$-algebra structure on the cohomology algebra $H^*(A^*)$ of a DG-algebra $A^*$ [43, 19, 20, 11, 24]. The $A_\infty$-algebra structure on $H^*(A^*)$ determines a DG-algebra $A^*$ up to quasi-isomorphism. Thus, while vanishing of the Massey operations in the cohomology only makes a DG-algebra $A^*$ quasi-formal, vanishing of the higher operations in (a certain representative of the $A_\infty$-isomorphism class of) the $A_\infty$-algebra structure on $H^*(A^*)$ would actually mean that the DG-algebra $A^*$ is formal.

Let $m \neq 0$ be an integer. Suppose that the cohomology algebra $H^*(A^*)$ is concentrated in the cohomological gradings $q = mn$, $n = 0, 1, 2, \ldots$, one has $H^0(A^*) = k$, \[12\]
and the algebra $H^*(A^*)$ is Koszul in the grading rescaled by $m$, i.e., one has
$$\Tor_{mp}^{H^*(A^*)}(k, k) = 0 \quad \text{for } mp \neq q.$$ 

Then all the differentials $d^p_q$, $r \geq 2$, vanish for “dimension” (bigrading) reasons, and the DG-algebra $A^*$ is quasi-formal (cf. [5]). One can say that this is an instance of intrinsic quasi-formality, i.e., a situation when any augmented DG-algebra with a given cohomology algebra is quasi-formal. In this paper, we are interested in the case $m = 1$, i.e., the situation when the augmentation ideal $H^*(A^*_m)$ of the cohomology algebra $H^*(A^*)$ is concentrated in the cohomological degrees 1, 2, 3, \ldots

**Corollary 2.2.** Suppose that the cohomology algebra $H^*(A^*)$ of an augmented DG-algebra $A^*$ is positively cohomologically graded and Koszul in its cohomological grading. Then the augmented DG-algebra $A^*$ is quasi-formal.

**Proof.** This is a corollary of the definitions, as explained above. \hfill $\Box$

For lack of a better term, let us call the Massey products discussed above the tensor Massey products. Our next aim is to compare these with a more elementary construction that we call the tuple Massey products.

One reason for our interest in tuple Massey products and this comparison comes from the application to the absolute Galois groups and Galois cohomology. A conjecture of ours claims that the cohomology algebra $H^*(G_F, \mathbb{Z}/l)$ of the absolute Galois group $G_F$ of a field $F$ containing a primitive $l$-root of unity is Koszul [32, 37]. In the paper [38], this conjecture was proven for all the (one-dimensional) local and global fields. On the other hand, there is a series of recent papers [16, 27, 8, 28] discussing and partially proving the conjecture that tuple Massey products of degree-one elements vanish in the cohomology algebra $H^*(G_F, \mathbb{Z}/l)$. The results above in this section and the discussion below show that the Koszulity conjecture implies vanishing of the tensor Massey products in $H^*(G_F, \mathbb{Z}/l)$, but may have no direct implications concerning the problem of vanishing of the tuple Massey products.

Let $A^* = (A^i, d: A^i \to A^{i+1})$ be a DG-algebra over a field $k$; assume for simplicity that $A^i = 0$ for $i < 0$ and $A^0 = k$ (so in particular $d^0: A^0 \to A^1$ is a zero map and the DG-algebra $A^*$ has a natural augmentation $A^* \to k$). Let $B^i \subset Z^i \subset A^i$ denote the subspaces of coboundaries and cocycles in $A^*$, so that $H^i = H^i(A^*) = Z^i/B^i$. The simplest possible construction of a 3-tuple Massey product of degree-one elements in the cohomology algebra $H^*(A^*)$ proceeds as follows.

Let $x, y, z \in H^1(A^*)$ be three elements for which $xy = 0 = yz$ in $H^2(A^*)$. Pick some preimages $\tilde{x}, \tilde{y}, \tilde{z} \in Z^1$ of the elements $x, y, z \in H^1$. Then the products $\tilde{x}\tilde{y}$ and $\tilde{y}\tilde{z}$ are coboundaries in $A^2$; so there exist elements $\zeta$ and $\xi \in A^1$ such that $d\zeta = \tilde{x}\tilde{y}$ and $d\xi = \tilde{y}\tilde{z}$ in $A^2$. Hence one has
$$d(\tilde{x}\xi + \zeta\tilde{z}) = -\tilde{x}d(\xi) + d(\zeta)\tilde{z} = -\tilde{x}\tilde{y}\tilde{z} + \tilde{x}\tilde{y}\tilde{z} = 0,$$

so the element $\tilde{x}\xi + \zeta\tilde{z}$ is a cocycle in $A^2$. By the definition, one sets the 3-tuple Massey product $(x, y, z) \in H^2(A^*)$ to be equal to the cohomology class of the cocycle $\tilde{x}\xi + \zeta\tilde{z} \in Z^2$. 

13
We have made some arbitrary choices along the way, so it is important to find out how does the output depend on these choices. Replacing the cochain $\zeta$ by a different cochain $\zeta'$ with the same differential $d\zeta' = \tilde{x}\tilde{y} \in A^2$ adds the product of two cocycles $(\zeta' - \zeta)\tilde{z} \in Z^1 \cdot \tilde{z} \subset Z^2$ to the cocycle $\tilde{x}\xi + \zeta\tilde{z} \in Z^2$. This means adding an element of the subspace $H^1 \cdot \tilde{z} \subset H^2$ to our 3-tuple Massey product $\langle x, y, z \rangle \in H^2(A^*)$.

Similarly, replacing the cochain $\xi$ by a different cochain $\xi'$ with the same differential $d\xi' = \tilde{y}\tilde{z}$ adds the product of two cocycles $\tilde{x}(\xi' - \xi) \in \tilde{x} \cdot Z^1 \subset Z^2$ to the cocycle $\tilde{x}\xi + \zeta\tilde{z} \in Z^2$, which means adding an element of the subspace $x \cdot H^1 \subset H^2$ to the 3-tuple Massey product $\langle x, y, z \rangle$.

Furthermore, since we have assumed that $A^0 = k$ assumption, one easily checks that the choice of the preimages $\tilde{x}, \tilde{y}, \tilde{z}$ does not introduce any new indeterminacies into the output of our 3-tuple Massey product construction as compared to the ones we already described.

To conclude, the tuple Massey product of three elements $x, y, z \in H^1(A^*)$ with $xy = 0 = yz$ in $H^2(A^*)$ is well-defined as an element of the quotient space $\langle x, y, z \rangle \in H^2(A^*)/(x \cdot H^1 + H^1 \cdot z)$.

Now let us describe the connection with the tensor Massey products. Suppose that we want to extend the above construction to elements of the tensor product space $H^1(A^*)^\otimes 3 = H^1(A^*) \otimes_k H^1(A^*) \otimes_k H^1(A^*)$. With any three vectors $x, y, z \in H^1(A^*)$ one can associate the decomposable tensor $x \otimes y \otimes z \in H^1(A^*)^\otimes 3$; however, not every tensor is decomposable.

Let $m: A^* \otimes_k A^* \to A^*$ denote the multiplication map in the DG-algebra $A^*$. We denote the induced (conventional) multiplication on the cohomology algebra by $m_2: H^1(A^*) \otimes_k H^1(A^*) \to H^1(A^*)$. Let $K^2 \subset H^1(A^*) \otimes_k H^1(A^*)$ denote the kernel of the multiplication map $m_2: H^1(A^*) \otimes_k H^1(A^*) \to H^1(A^*)$. We would like to have our triple Massey product defined on the subspace $K^2 \otimes_k H^1(A^*) \cap H^1(A^*) \otimes_k K^2 \subset H^1(A^*)^\otimes 3$.

The construction proceeds as follows. Given a tensor $\theta \in K^2 \otimes H^1 \cap H^1 \otimes K^2 \subset H^1 \otimes H^1 \otimes H^1$, we lift it to a tensor $\tilde{\theta}$ in $Z^1 \otimes Z^1 \otimes Z^1$ and apply the maps of multiplication of the first two and the last two tensor factors $m^{(12)} = m \otimes \id$ and $m^{(23)} = \id \otimes m$ to obtain a pair of elements $m^{(12)}(\tilde{\theta}) \in B^2 \otimes Z^1$ and $m^{(23)}(\tilde{\theta}) \in Z^1 \otimes B^2$. Then we lift these two elements arbitrarily to elements in $A^1 \otimes Z^1$ and $Z^1 \otimes A^1$, respectively, and finally apply the product map $m$ to each of them and add the results in order to obtain an element in $A^2$. By virtue of a computation similar to the above, this turns out to be an element of $Z^2$. Its image in $H^2(A^*)$, denoted by $m_3(\theta)$, is the triple tensor Massey product of our tensor $\theta$.

What is the subspace in $H^2(A^*)$ up to which the element $m_3(\theta)$ is defined? Let $W_l \subset H^1$ be the minimal vector subspace for which $\theta \in W_l \otimes H^1 \otimes H^1$, and let $W_r$ be the similar minimal subspace for which $\theta \in H^1 \otimes H^1 \otimes W_r$ (hence in fact
\( \theta \in W_l \otimes H^1 \otimes W_r \). If one is careful, one can make the Massey product \( m_3(\theta) \) well-defined up to elements of \( W_l \cdot H^1 + H^1 \cdot W_r \subset H^2(A^*) \). However, generally speaking, for “most” tensors \( \theta \in K^2 \otimes H^1 \cap H^1 \otimes K^2 \) (and certainly for “most” tensors in \( H^1 \otimes H^1 \otimes H^1 \)) one would expect \( W_l = H^1 = W_r \). So the triple Massey product that we have constructed is most simply viewed as a linear map

\[
m_3: K^2 \otimes_k H^1(A^*) \cap H^1(A^*) \otimes_k K^2 \longrightarrow H^2(A^*)/m_2(H^1(A^*) \otimes_k H^1(A^*)),
\]

\[
K^2 = \ker(m_2: H^1(A^*) \otimes H^1(A^*) \rightarrow H^2(A^*)).
\]

Notice that one has

\[
K^2 \otimes_k H^1(A^*) \cap H^1(A^*) \otimes_k K^2 \simeq \Tor^{H^*(A^*)}_{3,3}(k, k) = E_2^{3,3}
\]

and

\[
H^2(A^*)/m_2(H^1(A^*) \otimes_k H^1(A^*)) \simeq \Tor^{H^*(A^*)}_{1,2}(k, k) = E_2^{1,2}
\]

in the Eilenberg–Moore spectral sequence. We have obtained an explicit construction of the differential

\[
d_2^{3,3}: E_2^{3,3} \longrightarrow E_2^{1,2},
\]

which is the simplest example of a tensor Massey product in the sense of our definition.

How is this triple tensor Massey product construction related to the 3-tuple Massey product defined above? On the one hand, a subspace \( K^2 \subset H^1 \otimes H^1 \) may well contain no nonzero decomposable tensors at all, while containing many nontrivial indecomposable tensors. Then there may be also many nontrivial indecomposable tensors in \( K^2 \otimes H^1 \cap H^1 \otimes K^2 \). So the domain of definition of the tensor Massey product may be essentially much wider than that of the tuple Massey product. On the other hand, the latter, more elementary construction may produce its outputs with better precision, i.e., modulo a smaller subspace in \( H^2(A^*) \). Thus the map \( m_3 \) carries both more and less information about the DG-algebra \( A^* \) than the operation \( \langle x, y, z \rangle \) in the cohomology algebra \( H^*(A^*) \).

Similarly, let \( x_1, x_2, x_3, x_4 \in H^1(A^*) \) be four elements for which \( x_1 x_2 = x_2 x_3 = x_3 x_4 = 0 \) in \( H^2(A^*) \). Since we have assumed that \( A^0 = k \), these elements have uniquely defined preimages in \( Z^1 \simeq H^1 \), which we will denote by \( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \). The products \( \tilde{x}_1 \tilde{x}_2, \tilde{x}_2 \tilde{x}_3, \tilde{x}_3 \tilde{x}_4 \) are coboundaries in \( A^2 \), so there exist three elements \( \eta_{12}, \eta_{23}, \eta_{34} \in A^1 \) such that \( d\eta_{rs} = \tilde{x}_r \tilde{x}_s \) for all \( 1 \leq r < s \leq 4, \ s - r = 1 \). The elements

\[
\tilde{x}_1 \eta_{23} + \eta_{12} \tilde{x}_3 \quad \text{and} \quad \tilde{x}_2 \eta_{34} + \eta_{23} \tilde{x}_4 \in Z^2
\]

represent the 3-tuple Massey products \( \langle x_1, x_2, x_3 \rangle \) and \( \langle x_2, x_3, x_4 \rangle \). Suppose that these two cocycles are coboundaries, i.e., there exist two elements \( \zeta_{123} \) and \( \zeta_{234} \in A^1 \) such that \( d\zeta_{rst} = \tilde{x}_r \eta_{st} + \eta_{rs} \tilde{x}_t \) for all \( 1 \leq r < s < t \leq 4, \ t - s = s - r = 1 \). One has

\[
d(\tilde{x}_1 \zeta_{234} + \eta_{12} \eta_{34} + \zeta_{123} \tilde{x}_4)
\]

\[
= -\tilde{x}_1 \tilde{x}_2 \eta_{34} - \tilde{x}_1 \eta_{23} \tilde{x}_4 + \tilde{x}_1 \tilde{x}_2 \eta_{34} - \eta_{12} \tilde{x}_3 \tilde{x}_4 + \tilde{x}_1 \eta_{23} \tilde{x}_4 + \eta_{12} \tilde{x}_3 \tilde{x}_4 = 0,
\]
so the element $\bar{x}_1\zeta_{234} + \eta_{12}\eta_{34} + \zeta_{123}\bar{x}_4$ is a cocycle in $A^2$. By the definition, one sets the 4-tuple Massey product $\langle x_1, x_2, x_3, x_4 \rangle \in H^2(A^*)$ to be equal to the cohomology class of this cocycle.

Let us briefly describe the tensor version of the quadruple Massey product. Let $K^3 \subset K^2 \otimes H^1 \cap H^1 \otimes K^2 \subset H^1(A^*)^\otimes 3$ denote the kernel of the above map $m_3$. Consider the intersection of two vector subspaces $K^3 \otimes H^1 \cap H^1 \otimes K^3$ inside $H^1(A^*)^\otimes 4$. Then the desired map is

$$m_4: K^3 \otimes H^1(A^*) \cap H^1(A^*) \otimes K^3 \longrightarrow (H^2(A^*)/\text{im } m_2)/\text{im } m_3.$$ 

Its explicit construction is based on the same formulas as the above construction of the 4-tuple Massey product. This is the differential

$$d_3^{4,4}: E_3^{4,4} \longrightarrow E_3^{1,2}$$

in the Eilenberg–Moore spectral sequence.

The $n$-ary tensor Massey product of degree-one elements is a partially defined multivalued linear map

$$m_n: H^1(A^*) \otimes H^1(A^*) \otimes \cdots \otimes H^1(A^*) \longrightarrow H^2(A^*)$$

that can be identified (up to a possible plus/minus sign) with the differential

$$d_{n-1}^{m,n}: E_{n-1}^{m,n} \longrightarrow E_{n-1}^{1,2}.$$ 

As in the case of triple Massey products, the constructions of the $n$-tuple and $n$-ary tensor Massey products agree where the former is defined up to elements of the subspace up to which the latter is defined. However, the domain of definition of the tensor Massey product may be wider than that of the tuple Massey product, while the tuple Massey product may produce its outputs with better precision.

Now let us consider the case when the cohomology algebra $H^*(A^*)$ is generated by $H^1$ (as an associative algebra with the conventional multiplication $m_2$). Then, the map $m_2: H^1(A^*) \otimes H^1(A^*) \longrightarrow H^2(A^*)$ being surjective, the above tensor Massey product maps $m_3, m_4, \ldots$ vanish automatically (as their target spaces are zero). So do the similar Massey products

$$d_{p-1}^{j_1+\cdots+j_p}: H^{j_1}(A^*) \otimes \cdots \otimes H^{j_p}(A^*) \longrightarrow H^{j_1+\cdots+j_p-p+2}(A^*),$$

$j_1, \ldots, j_p \geq 1, p \geq 3$, in the higher cohomology.

Does it mean that all the differentials $d^q_p$ in the Eilenberg–Moore spectral sequence vanish for $r \geq 2$? Not necessarily. The first possibly nontrivial example would be

$$d_2^{4,4}: H^1(A^*)^\otimes 4 \longrightarrow H^1(A^*) \otimes H^2(A^*) \oplus H^2(A^*) \otimes H^1(A^*).$$

This is the map whose source space is actually the kernel $\text{Tor}_{4,4}^{H^*(A^*)}(k, k)$ of the differential $d_2^{4,4}: H^1(A^*)^\otimes 4 \longrightarrow (H^*(A^*)^\otimes 3)^4$, that is, the subspace

$$K^2 \otimes H^1 \otimes H^1 \cap H^1 \otimes K^2 \otimes H^1 \cap H^1 \otimes H^1 \otimes K^2 \subset H^1(A^*)^\otimes 4$$

and whose target space is the middle homology space $\text{Tor}_{2,3}^{H^*(A^*)}(k, k)$ of the sequence

$$H^1 \otimes H^1 \otimes H^1 \longrightarrow H^2 \otimes H^1 \oplus H^1 \otimes H^2 \longrightarrow H^3.$$
formed by the differentials \( d_1^{3,3} \) and \( d_1^{2,3} \). The latter vector space is otherwise known as the space of relations of degree 3 in the graded algebra \( H^*(A^*) \).

What does the map \( d_2^{4,4} \) do? Its source space can be otherwise described as the intersection

\[
(K^2 \otimes H^1 \cap H^1 \otimes K^2) \otimes H^1 \cap H^1 \otimes (K^2 \otimes H^1 \cap H^1 \otimes K^2).
\]

The map \( (m_3 \otimes \id, \id \otimes m_3) \) acts from this subspace to the quotient space of the vector space \( H^2 \otimes H^1 \oplus H^1 \otimes H^2 \) by the image of the map \( (m_2 \otimes \id, \id \otimes m_2) \) coming from the direct sum of two copies of \( H^1 \otimes H^1 \otimes H^1 \). It is claimed that the map \( (m_3 \otimes \id, \id \otimes m_3) \) can be naturally lifted to the quotient space of \( H^2 \otimes H^1 \oplus H^1 \otimes H^2 \) by the image of only one (diagonal) copy of \( H^1 \otimes H^1 \otimes H^1 \), as one can see from the explicit construction of \( m_3 \).

Indeed, let us restrict ourselves to decomposable tensors now (for simplicity of notation). Let \( x_1, x_2, x_3, x_4 \) be four elements in \( H^1(A^*) \) for which \( x_1 x_2 = x_2 x_3 = x_3 x_4 = 0 \) in \( H^2(A^*) \), and let \( \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \) be the liftings of these elements to \( Z^1 \simeq H^1 \). Let \( \eta_{12}, \eta_{23}, \eta_{34} \) be three elements in \( A^1 \) such that \( d \eta_{rs} = \bar{x}_r \bar{x}_s \) in \( B^2 \subset A^2 \) for all \( 1 \leq r < s \leq 4, \ s - r = 1 \). Then the triple Massey products are \( \langle x_1, x_2, x_3 \rangle = (\bar{x}_1 \eta_{23} + \eta_{12} \bar{x}_3 \mod B^2) \) and \( \langle x_2, x_3, x_4 \rangle = (\bar{x}_2 \eta_{34} + \eta_{23} \bar{x}_4 \mod B^2) \) for all \( 1 \leq r < s \leq 4, \ s - r = 1 \). Replacing \( \eta_{rs} \) with \( \eta'_{rs} = \eta_{rs} + \bar{y}_{rs} \) with \( \bar{y}_{rs} \in Z^1 \), one obtains \( \langle x_1, x_2, x_3 \rangle = \bar{x}_1 \eta_{23} + \eta_{12} \bar{x}_3 \mod B^2 = \langle x_1, x_2, x_3 \rangle + x_1 y_{23} + y_{12} x_3 \) and \( \langle x_2, x_3, x_4 \rangle = \bar{x}_2 \eta_{34} + \eta_{23} \bar{x}_4 \mod B^2 = \langle x_2, x_3, x_4 \rangle + x_2 y_{34} + y_{23} x_4 \), where \( y_{rs} \in H^1 \) are the cohomology classes of the elements \( \bar{y}_{rs} \in Z^1 \). Finally, one has

\[
\langle (x_1, x_2, x_3) \otimes x_4, x_1 \otimes (x_2, x_3, x_4) \rangle = ((x_1, x_2, x_3) \otimes x_4, x_1 \otimes (x_2, x_3, x_4))
\]

and

\[
((x_1 y_{23} + y_{12} x_3) \otimes x_4, x_1 \otimes (x_2 y_{34} + y_{23} x_4))
\]

in \( H^2 \otimes H^1 \oplus H^1 \otimes H^2 \), because \( x_3 x_4 = x_1 x_2 = 0 \) in \( H^2(A^*) \) by assumption.

What if the cohomology algebra \( H^*(A^*) \) is not only generated by \( H^1 \), but also defined by quadratic relations? There still can be nontrivial tensor Massey operations (i.e., the differentials \( d_{pq}^r \) with \( r \geq 2 \)), starting from

\[
d_2^{5,5} : H^1(A^*)^\otimes 5 \longrightarrow H^2 \otimes H^1 \oplus H^1 \otimes H^2 \oplus H^1 \otimes H^1 \oplus H^1 \otimes H^2.
\]

This is actually well-defined as a linear map from the source space

\[
\Tor_{5,5}^H(A^*)(k, k) \simeq \bigcap_{i=1}^4 H^1(A^*)^\otimes i \otimes K^2 \otimes H^1(A^*)^\otimes 4-i \subset H^1(A^*)^\otimes 5
\]
to a target space isomorphic to \( \Tor_{3,4}^H(A^*)(k, k) \). The latter Tor space is the first obstruction to Koszulity of a quadratic graded algebra \( H^*(A^*) \).
3. Noncommutative (Rational) Homotopy Theory

From an algebraist’s point of view, \textit{rational homotopy theory} is an equivalence between categories of commutative and Lie DG-(co)algebras satisfying appropriate boundedness conditions and viewed up to quasi-isomorphism. The classical formulation [42] claims an equivalence between the categories of negatively cohomologically graded Lie DG-algebras and augmented cocommutative DG-coalgebras with the augmentation ideals concentrated in the cohomological degrees \( \leq -2 \). The localizations of such two categories of DG-(co)algebras by the classes of (co)multiplicative quasi-isomorphisms are equivalent over any field of characteristic 0; over the field of rational numbers, these are also identified with the localization of the category of connected, simply connected topological spaces by the class of rational equivalences.

Attempting to include a nontrivial fundamental group into the picture, people usually consider nilpotent topological spaces, nilpotent groups, and Malcev completions. Here a discrete group is called nilpotent if its lower central series converges to zero in a finite number of steps. Yet the most natural setup for the nilpotency condition is that of coalgebras rather than algebras, as it allows for infinitary, or “ind”-conilpotency [32, 15]. Thus it appears that the maximal natural generality for “an algebraist’s version of rational homotopy theory” is that of an equivalence between the categories of nonnegatively cohomologically graded conilpotent Lie DG-coalgebras and positively cohomologically graded commutative DG-algebras, considered up to quasi-isomorphism over a field of characteristic 0.

Here the commutative DG-algebra computes the cohomology algebra of the would-be topological space, while the conilpotent Lie DG-coalgebra is, roughly speaking, dual to the derived rational completion of its homotopy groups with their Whitehead bracket (notice the passage to the dual coalgebra in the completion construction of Theorem 1.2 and [33, Remark 5.6]). A Lie DG-coalgebra is called conilpotent if its underlying graded Lie (super)coalgebra is conilpotent; a nonnegatively graded Lie supercoalgebra is conilpotent if it is a union of dual coalgebras to finite-dimensional nilpotent nonpositively graded Lie superalgebras; and a finite-dimensional nonpositively graded Lie superalgebra is nilpotent if its degree-zero component is nilpotent Lie algebra and its action in the components of other degrees is nilpotent. (See [33, Section 8] and [35, Section D.6.1] for a discussion of conilpotent Lie coalgebras and their conilpotent coenveloping coalgebras.)

In this section, we make yet another algebraic generalization/simplification and replace the pair of dual operads \( \text{Com-Lie} \) with that of \( \text{Ass-Ass} \). In other words, we consider a noncommutative version of the above-described theory with commutative DG-algebras replaced by associative ones and conilpotent Lie DG-coalgebras replaced by conilpotent coassociative ones. In this setting, the characteristic 0 restriction becomes unnecessary and one can work over an arbitrary ground field \( k \). Thus our aim is to construct an equivalence between the categories of nonnegatively cohomologically graded conilpotent (coassociative) DG-coalgebras and positively cohomologically graded (associative) DG-algebras.
Let $C^\bullet$ be a nonzero DG-coalgebra over a field $k$; suppose that it is endowed with a coaugmentation (DG-coalgebra morphism) $k \to C^\bullet$ and denote the quotient DG-coalgebra (without counit) by $C^\bullet_+=C^\bullet/k$. By the definition, the cobar construction $\text{Cob}^\bullet(C^\bullet)$ of a coaugmented DG-coalgebra $C^\bullet$ is the free associative algebra
\[ \text{Cob}(C) = \bigoplus_{n=0}^{\infty} C_+[-1]^\otimes n \]
generated by the graded vector space $C_+[-1]$ obtained by shifting by $-1$ the cohomological grading of the coaugmentation cokernel $C_+$. The grading on $\text{Cob}(C)$ is induced by the grading of $C_+[-1]$. Alternatively, one can define the cobar construction $\text{Cob}(C)$ as the direct sum of the tensor powers $C_+^\otimes n$ of the vector space $C_+$ and endow it with the total grading equal to the sum $i+n$ of the grading $i$ induced by the grading of $C_+$ and the grading $n$ by the number of tensor factors.

The differential on $\text{Cob}^\bullet(C^\bullet)$ is the sum of two summands $d+\partial$, the former of them induced by the differential on $C^\bullet_+$ and the latter one by the comultiplication in $C^\bullet_+$. One has to work out the plus/minus signs in order to make the total differential on $\text{Cob}^\bullet(C^\bullet)$ square to zero (cf. the definition of the bar construction in Section 2).

A quasi-isomorphism of augmented DG-algebras $A^\bullet \to B^\bullet$ induces a quasi-isomorphism of their bar constructions $\text{Bar}^\bullet(A^\bullet) \to \text{Bar}^\bullet(B^\bullet)$ (as one can see from the filtration and the Eilenberg–Moore spectral sequence discussed in Section 2). However, the morphism of cobar constructions $\text{Cob}^\bullet(C^\bullet) \to \text{Cob}^\bullet(D^\bullet)$ induced by a quasi-isomorphism of coaugmented (even conilpotent) DG-coalgebras $C^\bullet \to D^\bullet$ may not be a quasi-isomorphism (see Remark 3.6 at the end of this section). The reason is that the filtration of the bar construction by the number of tensor factors is an increasing one, while the similar filtration of the cobar construction is a decreasing one (cf. the discussion of two kinds of differential Cotor functors in [9], [17], and [35, Section 0.2.10]).

As we are interested in the cobar construction as defined above (i.e., the direct sum of the tensor powers of $C_+^\bullet[-1]$) rather than its completion by this filtration (which would mean the direct product of such tensor powers), the related spectral sequence can be viewed as converging to the cohomology of the cobar construction only when it is in some sense locally finite. This includes two separate cases considered below, which roughly correspond to the “conilpotent” and “simply connected” versions of noncommutative homotopy theory as discussed above.

The conilpotent version of the theory, which is of primary interest to us, is based on the following assertion (which does not yet presume conilpotency, but it will be needed further on).

**Proposition 3.1.** Let $C^\bullet = (C^0 \to C^1 \to C^2 \to \cdots)$ and $D^\bullet = (D^0 \to D^1 \to D^2 \to \cdots)$ be two nonnegatively cohomologically graded coaugmented DG-coalgebras. Then any comultiplicative quasi-isomorphism $f: C^\bullet \to D^\bullet$, i.e., a morphism of DG-coalgebras inducing an isomorphism $H^\bullet(C^\bullet) \simeq H^\bullet(D^\bullet)$ of their cohomology coalgebras, induces a quasi-isomorphism of the cobar constructions $\text{Cob}^\bullet(f): \text{Cob}^\bullet(C^\bullet) \to \text{Cob}^\bullet(D^\bullet)$.
Proof. For any coaugmented DG-coalgebra $E^*$, set $G^p \operatorname{Cob}^*(E^*) = \bigoplus_{n \geq p} E_n[-1]^{\otimes n}$. This is a decreasing filtration of the DG-algebra $\operatorname{Cob}^*(E^*)$ compatible with the multiplication and the differential. Clearly, a quasi-isomorphism of coaugmented DG-coalgebras $C^* \to D^*$ induces a quasi-isomorphism of the associated graded algebras $\operatorname{gr}_C \operatorname{Cob}^*(C^*) \to \operatorname{gr}_C \operatorname{Cob}^*(D^*)$. It remains to observe that when the DG-coalgebras $C^*$ and $D^*$ are nonnegatively cohomologically graded, the filtrations $G^\bullet \operatorname{Cob}^*(C^*)$ and $\operatorname{Cob}^*(D^*)$ are finite at every (total) cohomological degree. Indeed, one has $G^{n+1} \operatorname{Cob}^n(C^*) = 0 = G^{n+1} \operatorname{Cob}^n(D^*)$ for every integer $n$.

A coaugmented DG-coalgebra $C^*$ is called conilpotent if its underlying coaugmented (graded) coalgebra is conilpotent (see Section 1 for the definition). The cobar construction $\operatorname{Cob}^*(C^*)$ of a coaugmented DG-coalgebra $C^*$ is naturally an augmented DG-algebra; and the bar construction $\operatorname{Bar}^*(A^*)$ of an augmented DG-algebra $A^*$ is a conilpotent DG-coalgebra.

The two constructions $C^* \to \operatorname{Cob}^*(C^*)$ and $A^* \to \operatorname{Bar}^*(A^*)$, viewed as functors between the categories of augmented DG-algebras and conilpotent DG-coalgebras, are adjoint functors: for any conilpotent DG-coalgebra $C^*$ and augmented DG-algebra $A^*$, there is a bijective correspondence between morphisms of augmented DG-coalgebras $\operatorname{Cob}^*(C^*) \to A^*$ and morphisms of coaugmented DG-coalgebras $C^* \to \operatorname{Bar}^*(A^*)$.

Proposition 3.2. (a) For any augmented DG-algebra $A^*$, the adjunction morphism $\operatorname{Cob}^*(\operatorname{Bar}^*(A^*)) \to A^*$ is a quasi-isomorphism of augmented DG-algebras.

(b) For any conilpotent DG-coalgebra $C^*$, the adjunction morphism $C^* \to \operatorname{Bar}^*(\operatorname{Cob}^*(C^*))$ is a quasi-isomorphism of (conilpotent) DG-coalgebras.

Warning: the assertion of part (b) does not hold without the conilpotency assumption on $C^*$, and in fact, the adjunction morphism does not even exist without this assumption (cf. Remark 4.5 below).

Proof. Part (a): define an increasing filtration on a DG-algebra $A^*$ by the rules $F_0 A^* = k$ and $F_n A^* = A^*$ for $n \geq 1$. This filtration is compatible with the differential and the multiplication on $A^*$, and so it induces a filtration $F$ compatible with the differential and the comultiplication on $\operatorname{Bar}^*(A^*)$, and further, a filtration $F$ compatible with the differential and the multiplication on $\operatorname{Cob}^*(\operatorname{Bar}^*(A^*))$. In fact, the filtration on the (co)bar construction induced by a filtration on a (co)algebra is always compatible with the tensor (co)multiplication, while compatibility of the original filtration with the differential and the (co)multiplication ensures compatibility of the induced filtration with the (co)bar differential. The adjunction morphism $\operatorname{Cob}^*(\operatorname{Bar}^*(A^*)) \to A^*$ is also compatible with the filtrations.

The passage to the associated graded DG-(co)algebras with respect to the filtration $F$ gets rid of all the information about the multiplication in $A^*$; so the DG-algebra $\operatorname{gr}^F A^*$ is obtained from the DG-algebra $A^*$ by setting the multiplication on $A^*_+$ to be zero, and the DG-algebra $\operatorname{gr}^F \operatorname{Cob}^*(\operatorname{Bar}^*(A^*))$ is the cobar-construction of the bar-construction of the DG-algebra $\operatorname{gr}^F A^*$. From this point one can proceed further and...
rid oneself also of the differential on $A^\bullet$ in addition to the multiplication; but this is unnecessary. It suffices to notice that the component of degree $n$ of the complex $\text{gr}^F \text{Cob}^\bullet(\text{Bar}^\bullet(A^\bullet))$ with respect to the grading by the indices of the filtration $F$ is the total complex of a bicomplex composed of $2^{n-1}$ copies of the complex $A^\otimes m_+$. Proving that such complexes are acyclic for $n \geq 2$ is elementary combinatorics.

Part (b): the argument dual to the one in part (b) is not immediately applicable, as the related filtration $G$ on the DG-coalgebra $C^\bullet$ and the induced filtrations on its cobar and bar constructions would be decreasing ones. Instead, consider the canonical increasing filtration $N_mC^\bullet = \ker(C^m \to C^m_{+1})$ on the conilpotent DG-coalgebra $C^\bullet$ and the induced filtrations on its cobar and bar constructions. This reduces the question to the case of the DG-coalgebra $\text{gr}^N C^\bullet$, which is endowed with an additional positive grading by the indices of the filtration $N$. The induced decreasing filtration $G$ on the DG-coalgebra $\text{Bar}^\bullet(\text{Cob}^\bullet(\text{gr}^N C^\bullet))$ is locally finite in the grading by the indices $m$ of the filtration $N$. This reduces the question to proving that the morphism of DG-coalgebras $\text{gr}^G \text{gr}^N C^\bullet \to \text{Bar}^\bullet(\text{Cob}^\bullet(\text{gr}^G \text{gr}^N C^\bullet))$ is a quasi-isomorphism, which can be done by a combinatorial argument similar to the one in part (a).

We recall that an augmented DG-algebra $A^\bullet$ is called positively cohomologically graded if its augmentation ideal is concentrated in the positive cohomological degrees, that is $A_+^i = 0$ for all $i \leq 0$. Equivalently, a DG-algebra $A^\bullet$ is positively cohomologically graded if $A_0^i = 0$ for all $i < 0$ and $A^0$ is a one-dimensional vector space generated by the unit element of $A$; any such DG-algebra $A^\bullet$ has a unique augmentation (DG-algebra morphism) $A^\bullet \to k$.

**Theorem 3.3.** The functors $A^\bullet \to \text{Bar}^\bullet(A^\bullet)$ and $C^\bullet \to \text{Cob}^\bullet(C^\bullet)$ induce mutually inverse equivalences between the category of positively cohomologically graded DG-algebras $A^\bullet$ with quasi-isomorphisms inverted and the category of nonnegatively cohomologically graded conilpotent DG-coalgebras $C^\bullet$ with quasi-isomorphisms inverted.

**Proof.** Having in mind the results of Propositions 3.1 and 3.2, it suffices to notice that the bar construction takes positively cohomologically graded augmented DG-algebras to nonnegatively cohomologically graded conilpotent DG-coalgebras, while the cobar construction takes nonnegatively cohomologically graded coaugmented DG-coalgebras to positively cohomologically graded augmented DG-algebras.

For comparison, let us now present the simply connected version of the theory, that is the direct noncommutative analogue of “the algebraic part” of [42, Theorem I]. A coaugmented DG-coalgebra $C^\bullet$ is called negatively cohomologically graded if its coaugmentation cokernel $C^\bullet_+$ is concentrated in the negative cohomological degrees, that is $C^i_+ = 0$ for all $i \geq 0$. Clearly, any negatively cohomologically graded DG-coalgebra
is conilpotent. Let us call a negatively cohomologically graded DG-coalgebra simply connected if $C^{-1} = 0$, i.e., $C_i = 0$ for all $i \geq -1$.

**Proposition 3.4.** Any quasi-isomorphism $f : C^* \to D^*$ between two simply connected negatively cohomologically graded DG-coalgebras $C^* = (\cdots \to C^{-3} \to C^{-2} \to 0 \to k)$ and $D^* = (\cdots \to D^{-3} \to D^{-2} \to 0 \to k)$ induces a quasi-isomorphism of the cobar constructions $\text{Cob}^*(f) : \text{Cob}^*(C^*) \to \text{Cob}^*(D^*)$.

**Proof.** The argument is similar to the proof of Proposition 3.1. Once again, one observes that the decreasing filtrations $G$ on the cobar constructions $\text{Cob}^*(C^*)$ and $\text{Cob}^*(D^*)$ are finite at every cohomological degree in our assumptions. Indeed, one has $G^{n+1}\text{Cob}^{-n}(C^*) = 0 = G^{n+1}\text{Cob}^{-n}(D^*)$ for every integer $n$. □

An augmented DG-algebra $A^*$ is called negatively cohomologically graded if one has $A_i = 0$ for all $i \geq 0$.

**Theorem 3.5.** The functors $A^* \mapsto \text{Bar}^*(A^*)$ and $C^* \mapsto \text{Cob}^*(C^*)$ induce mutually inverse equivalences between the category of negatively cohomologically graded augmented DG-algebras $A^*$ with quasi-isomorphisms inverted and the category of simply connected negatively cohomologically graded DG-coalgebras $C^*$ with quasi-isomorphisms inverted.

**Proof.** Here one has to notice that the bar construction takes negatively cohomologically graded augmented DG-algebras to simply connected negatively cohomologically graded DG-coalgebras, while the cobar construction takes simply connected negatively cohomologically graded DG-coalgebras to negatively cohomologically graded augmented DG-algebras. Otherwise the argument is similar to the proof of Theorem 3.3 and based on the result of Proposition 3.4. □

**Remark 3.6.** The assertions of the above theorems can be modified so as to hold for arbitrary augmented DG-algebras and conilpotent DG-coalgebras. One just has to replace the class of quasi-isomorphisms of DG-coalgebras with a finer class of filtered quasi-isomorphisms of conilpotent DG-coalgebras, which are to be inverted in order to obtain a category equivalent to the category of augmented DG-algebras with quasi-isomorphisms inverted (see [15, Section 4] or [36, Section 6.10]). In the form stated above, on the other hand, the assertions of the theorems do not hold already for the negatively cohomologically graded DG-coalgebras that are not simply connected—in fact, this is the class of DG-coalgebras for which the difference between quasi-isomorphisms and filtered quasi-isomorphisms becomes essential. It suffices to consider a morphism between two different augmented $k$-algebras (viewed as DG-algebras concentrated in cohomological degree zero) $A \to B$ inducing an isomorphism of the Tor spaces $\text{Tor}_A^2(k, k) \simeq \text{Tor}_B^2(k, k)$. Then the induced morphism of the bar constructions $\text{Bar}^*(A) \to \text{Bar}^*(B)$ is a quasi-isomorphism of negatively cohomologically graded conilpotent DG-coalgebras that is transformed by the cobar construction into a morphism of DG-algebras $\text{Cob}^*(\text{Bar}^*(A)) \to \text{Cob}^*(\text{Bar}^*(B))$ with two different cohomology algebras $A$ and $B$ [36, Remark 6.10].
4. $K(\pi, 1)$-ness + Quasi-Formality Imply Koszulity

We refer for the definitions of the bar construction $\text{Bar}^\bullet(A^\bullet)$ of an augmented DG-algebra $A^\bullet$ to Section 2 and of the cobar construction $\text{Cob}^\bullet(C^\bullet)$ of a coaugmented DG-coalgebra $C^\bullet$ to Section 3. The definition of the cobar construction $\text{Cob}^\bullet(D)$ of a coalgebra $D$ was given previously in Section 1; it is but the particular case of the construction of Section 3 corresponding to the situation of a DG-coalgebra $C^\bullet = D$ concentrated in the cohomological degree 0. The definition of the conilpotency property of a coalgebra $C$ can be also found in Section 1.

The construction of the (tensor) Massey operations on the cohomology algebra of an augmented DG-algebra $A^\bullet$, understood as the higher differentials in the algebraic Eilenberg–Moore spectral sequence (associated with a natural increasing filtration on the bar-complex $\text{Bar}^\bullet(A^\bullet)$), was introduced and discussed in Section 2. An augmented DG-algebra $A^\bullet$ is called quasi-formal if all these Massey operations vanish.

Finally, we recall that a graded algebra $H^*$ over a field $k$ is called Koszul if it is concentrated in the positive degrees, that is $H^i = 0$ for $i < 0$ and $H^0 = k$, and its bigraded Tor coalgebra (computed by the internally graded DG-coalgebra $\text{Bar}^*(H^*)$) is concentrated in the diagonal grading, i.e., $\text{Tor}^H_{ij}(k, k) = 0$ for $i \neq j$.

Let $A^\bullet$ be an augmented DG-algebra over a field $k$. We will say that a DG-algebra $A^\bullet$ is of the $K(\pi, 1)$ type (or just simply “a $K(\pi, 1)$”) if there exists a conilpotent coalgebra $C$ over $k$ such that the DG-algebra $A^\bullet$ can be connected with the DG-algebra $\text{Cob}^\bullet(C)$ by a chain of quasi-isomorphisms of augmented DG-algebras. Let us emphasize that, in this definition, the coalgebra $C$, if viewed as a DG-coalgebra, must be concentrated in the cohomological degree 0. The conilpotency condition on $C$ is of key importance. Here the (coassociative and counital, but otherwise arbitrary) conilpotent coalgebra $C$ plays the role of the conilpotent coenveloping coalgebra of the conilpotent Lie coalgebra of a $k$-complete fundamental group $\pi$. So one could as well write “$A^\bullet = K(C, 1)$”. (Notice the degree 1 homological shift transforming the homotopy groups of a rational homotopy type endowed with their Whitehead bracket into a graded Lie (super)algebra.)

The connection between Koszulity and the Massey operation vanishing was first pointed out by Priddy [41, Section 8] (cf. [24]), who was working with algebras endowed with an important positive grading (in addition to some other possible gradings whose role was to ensure the locally finite dimension with respect to the multidegree). In our language, the result of [41, Proposition 8.1] can be reformulated as the following theorem, in which one considers a coalgebra $C$ endowed with an “internal”, rather than a “cohomological”, grading. We refer to [32, Section 2] for the background material about positively graded coalgebras (see also the next Section 5).

**Theorem 4.1.** Let $C = k + C_1 + C_2 + C_3 + \cdots$ be a positively (internally) graded coalgebra cogenerated by its first-degree component $C_1$ over a field $k$. Let $A^\bullet = \text{Cob}^\bullet(C)$ be the cobar DG-algebra of the coalgebra $C$ (which is viewed as a DG-coalgebra concentrated entirely in the cohomological degree 0). Then the graded coalgebra $C$ is...
Koszul if and only if the differentials \( d_{p-1, q}^p : E_{p-1, q}^p \to E_{p-1, q-p+2}^p \)

\[(H^*(A^*_+))^q \to H^{q-p+2}(A^*_+)\]

in the Eilenberg–Moore spectral sequence of the DG-algebra \( A^* \) vanish for \( p \geq 3 \).

**Proof.** By [32, Propositions 1 and 2], a positively internally graded coalgebra \( C \) generated by its first-degree component is Koszul if and only if its cohomology algebra \( \text{Ext}^*_C(k, k) = H^1(A^*) \) is generated by \( \text{Ext}^1_C(k, k) = H^1(A^*) \). The latter condition does not depend on the internal grading on the coalgebra \( C \). Furthermore, any coalgebra admitting a positive grading is conilpotent. Hence it remains to apply part (a) of the following proposition. (Notice also that, by [32, Proposition 3], a positively internally graded coalgebra \( C \) generated by its first-degree component \( C_1 \) is Koszul if and only if the algebra \( H^*(A^*) \) is Koszul.) \( \square \)

**Proposition 4.2.** Let \( A^* = \text{Cob}^*(C) \) be the cobar DG-algebra of a conilpotent coalgebra \( C \) (viewed as a DG-coalgebra concentrated in the cohomological degree 0). Then

(a) the cohomology algebra \( H^*(A^*) \) is multiplicatively generated by \( H^1(A^*) \) if and only if the differentials \( d_{p-1}^{n,q} : E_{p-1}^{n,q} \to E_{p-1}^{1,q-p+2} \) vanish for \( p \geq 3 \);

(b) the cohomology algebra \( H^*(A^*) \) is quadratic if and only if the differentials \( d_{p-1}^{n,q} \) as well as the differentials \( d_{p-1}^{p+1,q} : E_{p-1}^{p+1,q} \to E_{p-1}^{q-p+2} \)

\[(H^*(A^*_+))^q \to (H^*(A^*_+))^{q-p+2}\]

vanish for \( p \geq 3 \).

**Proof.** Part (a): if the algebra \( H^*(A^*) \) is multiplicatively generated by \( H^1 \), then \( E_{2,n}^1 = \text{Tor}_1^H(A^*_+)(k, k) = 0 \) for all \( n \geq 2 \); since one also has \( E_{1}^{1,q} = (H^*(A^*_+))^q = 0 \) for all \( p > q \), it follows that the differentials \( d_{p-1}^{n,q} \) vanish for \( p \geq 3 \).

Conversely, by Proposition 3.2(b) the DG-coalgebra \( \text{Bar}^*(A^*) \) is quasi-isomorphic to \( C \). So in the algebraic Eilenberg–Moore spectral sequence \( E_{pq}^r \) from Section 2 we have \( E_{pq}^r = \text{gr}_p^F H^{p-q} \text{Bar}^*(A^*) = 0 \) for \( p \neq q \), and in particular \( E_{1,n}^1 = 0 \) for \( n \geq 2 \). This is the situation which people colloquially describe as “the cohomology \( H^*(A^*) \) is generated by \( H^1(A^*) \) using Massey products”. (One should keep in mind that the DG-algebra \( A^* = \text{Cob}^*(C) \) is positively cohomologically graded by construction.) If all the differentials \( d_{r}^{n,q} \) landing in \( E_{r,n}^1 \) vanish for \( r \geq 2 \), it follows that \( E_{2,n}^1 = 0 \), so \( H^*(A^*) \) is generated by \( H^1 \) using the conventional multiplication.

Part (b): we can assume that the algebra \( H^*(A^*) \) is generated by \( H^1 \). If this algebra is also quadratic, then \( E_{2,n}^2 = \text{Tor}_2^H(A^*_+)(k, k) = 0 \) for all \( n \geq 3 \), so it follows that the differentials \( d_{p-1}^{p+1,q} \) vanish for \( p \geq 3 \). Conversely, as we explained above, \( E_{\infty,n}^2 = 0 \) for \( n \geq 3 \), so if all the differentials landing in \( E_{r,n}^2 \) vanish for \( r \geq 2 \), then we can conclude that \( E_{2,n}^2 = 0 \) for \( n \geq 3 \). \( \square \)

In the nonhomogeneous conilpotent setting we are working in, the implication “\( K(\pi, 1) \)-ness + quasi-formality imply Koszulity” becomes a bit more complicated.
than in Theorem 4.1, as the cohomology algebra $H^*(A^*) = H^* \operatorname{Cob}^*(C)$ being generated by $H^1$ no longer implies it being Koszul (cf. the final paragraphs of Section 2). The following theorem is the main result of this paper.

**Theorem 4.3.** The cohomology algebra $H^* = H^*(A^*)$ of an augmented DG-algebra $A^*$ is Koszul if and only if the augmented DG-algebra $A^*$ is simultaneously quasi-formal and of the $K(\pi, 1)$ type.

**Proof.** It was explained in Section 2 that Koszulity of the cohomology algebra $H^*(A^*)$ implies vanishing of the Massey products. The assertion that $A^*$ is a $K(\pi, 1)$ whenever $H^*(A^*)$ is Koszul, announced in the title of Section 1, was not actually proven there (in our present setting) but rather postponed; so we have to prove it now. We start with the following lemma.

**Lemma 4.4.** Let $A^*$ be an augmented DG-algebra whose cohomology algebra $H^*(A^*)$ is concentrated in the positive cohomological degrees. Then there exists a positively cohomologically graded DG-algebra $P^*$ together with a quasi-isomorphism of augmented DG-algebras $P^* \to A^*$.

**Proof.** The construction of a cofibrant resolution of the DG-algebra $A^*$ in the conventional model structure on the category of augmented DG-algebras (see [14], [18], or [36, Section 9.1]) provides the desired DG-algebra $P^*$. One starts from a free graded algebra with generators corresponding to representative cocycles of a chosen basis in $H^*(A^*_1)$, and then iteratively adds to it new free generators whose differentials kill the cohomology classes annihilated by the morphism into $A^*$.

The observation is that all the cocycles that need to be killed at each step, being linear combinations of products of at least two generators of cohomological degrees $\geq 1$, have cohomological degrees $\geq 2$. So all the new generators that one has to add at this step have cohomological degrees $\geq 1$.

Thus we can assume our DG-algebra $A^*$ to be positively cohomologically graded; then its bar construction $\operatorname{Bar}^*(A^*)$ is nonnegatively cohomologically graded. Now if the cohomology algebra $H^*(A^*)$ is Koszul, then it follows from the Eilenberg–Moore spectral sequence that the cohomology coalgebra $H^* \operatorname{Bar}^*(A^*)$ of the DG-coalgebra $\operatorname{Bar}^*(A^*)$ is concentrated in cohomological degree 0.

Hence the embedding $C \to \operatorname{Bar}^*(A^*)$ of the subcoalgebra $C = \ker(d^0 : \operatorname{Bar}^0(A^*) \to \operatorname{Bar}^1(A^*))$ of the DG-coalgebra $\operatorname{Bar}^*(A^*)$ is a quasi-isomorphism. By Proposition 3.1, the induced morphism of the cobar constructions $\operatorname{Cob}^*(C) \to \operatorname{Cob}^*(\operatorname{Bar}^*(A^*))$ is a quasi-isomorphism, too. By Proposition 3.2(a), so is the adjunction morphism $\operatorname{Cob}^*(\operatorname{Bar}^*(A^*)) \to A^*$. Finally, the coalgebra $C$ is conilpotent, since its ambient DG-coalgebra $\operatorname{Bar}^*(A^*)$ is. We have shown that the DG-algebra $A^*$ is a $K(\pi, 1)$.

Now suppose, as the title of this section suggests, that the augmented DG-algebra $A^*$ is a $K(\pi, 1)$ and the Massey products in its cohomology algebra $H^*(A^*)$ vanish. Then the augmented DG-algebra $A^*$ is connected by a chain of quasi-isomorphisms with the DG-algebra $\operatorname{Cob}^*(C)$ for a certain conilpotent coalgebra $C$; we can simply assume that $A^* = \operatorname{Cob}^*(C)$. In particular, the cohomology algebra $H^*(A^*)$ is concentrated in the positive cohomological degrees.

25
Applying Proposition 3.2(b), we can conclude that the DG-coalgebra $\text{Bar}^\bullet(A^\bullet)$ is quasi-isomorphic to $C$, so its cohomology coalgebra $H^\bullet \text{Bar}^\bullet(A^\bullet)$ is concentrated in cohomological degree 0. On the other hand, the Massey product vanishing means that one has $E^{pq}_2 = E^{pq}_\infty = 0$ for $p \neq q$, it follows that $E^{pq}_2 = \text{Tor}^{H^\bullet(A^\bullet)}(k, k) = 0$. We have proven that the cohomology algebra $H^\bullet(A^\bullet)$ is Koszul. □

Remark 4.5. Applying the cobar and bar constructions to a nonconilpotent coaugmented coalgebra $D$ produces a conilpotent DG-coalgebra $\text{Bar}^\bullet(\text{Cob}^\bullet(D))$ with the zero-degree cohomology coalgebra $H_0 \text{Bar}^\bullet(\text{Cob}^\bullet(D))$ isomorphic to the maximal conilpotent subcoalgebra $C = \text{Nilp} D$ of the coaugmented coalgebra $D$ (see Section 1 for the definitions and notation here and below). The DG-coalgebra $\text{Bar}^\bullet(\text{Cob}^\bullet(D))$ can be called the DG-coalgebra of derived conilpotent completion of a coaugmented coalgebra $D$. The DG-algebra $\text{Cob}^\bullet(D)$ is a $K(\pi, 1)$ (i. e., the cohomology coalgebra of the DG-coalgebra $\text{Bar}^\bullet(\text{Cob}^\bullet(D))$ is concentrated in cohomological degree 0) if and only if the embedding $C \longrightarrow D$ induces a cohomology isomorphism $\text{Ext}_C^\bullet(k, k) \simeq \text{Ext}_D^\bullet(k, k)$.

Similarly, applying the cobar and bar constructions to an augmented algebra $R$ produces a conilpotent DG-coalgebra $\text{Bar}^\bullet(\text{Cob}^\bullet(R))$ with the zero cohomology algebra $H^0 \text{Bar}^\bullet(\text{Cob}^\bullet(R))$ isomorphic to the coalgebra of pronilpotent completion $C = R^\wedge$ of the augmented DG-algebra $R$. The DG-coalgebra $\text{Bar}^\bullet(\text{Cob}^\bullet(R))$ can be called the DG-coalgebra of derived pronilpotent completion of an augmented algebra $R$. The DG-algebra $\text{Cob}^\bullet(R)$ is a $K(\pi, 1)$ (i. e., the cohomology coalgebra of the DG-coalgebra $\text{Bar}^\bullet(\text{Cob}^\bullet(R))$ is concentrated in cohomological degree 0) if and only if the natural map of the cohomology algebras $\text{Ext}_C^\bullet(k, k) \longrightarrow \text{Ext}_R^\bullet(k, k)$ is an isomorphism. These are noncommutative analogues of the procedure of rational completion of the space $K(\Gamma, 1)$ with a discrete group $\Gamma$ in rational homotopy theory.

These observations show, in particular, how to deduce the assertions of Theorems 1.1 and 1.2 from the “Koszulity implies $K(\pi, 1)$-ness” claim in Theorem 4.3.

5. Self-Consistency of Nonhomogeneous Quadratic Relations

The aim of this section is to explain the thesis, formulated in the introduction, about the connection between self-consistency of systems of nonhomogeneous quadratic relations (2) with Koszul principal parts (3) and Koszulity of the cohomology algebra $H^\bullet(C)$ of the coalgebra $C$ defined by such relations.

Self-consistency of algebraic relations is a fundamental concept in algebra that is too general to allow a precise general definition. On the other hand, examples of non-self-consistent systems of relations are easily demonstrated. E. g., consider the following system of nonhomogeneous quadratic relations of the type (1) in two variables $x$ and $y$:

\[
\begin{align*}
xy - y + 1 &= 0, \\
yx - y &= 0.
\end{align*}
\]
Looking on the relations (7), one might expect them to define an ungraded associative algebra of the “size” of the graded algebra with the relations \( xy = yx = 0 \). However, proceeding to derive consequences of (7), one obtains

\[(xy)x = (y - 1)x = y - x\]

and

\[x(yx) = xy = y - 1\]

hence \( x - 1 = 0 \). Substituting \( x = 1 \) into the first relation, one comes to \( 1 = 0 \). So the whole algebra defined by (7) vanishes.

Self-consistency of nonhomogeneous quadratic relations of the type (1) is studied in the paper [31] and the book [34, Chapter 5]. The main result, called “the Poincaré–Birkhoff–Witt theorem for nonhomogeneous quadratic algebras”, claims that it suffices to perform computations with expressions of degree \( \leq 3 \) when checking self-consistency of relations of the type (1) with Koszul quadratic principal parts (3).

Consequences of non-self-consistency of relations of the type (2) are a bit less dramatic. A typical example would be the single relation

\[(x^2 - y^3 = 0)\]

for two (noncommutative) variables \( x \) and \( y \), implying

\[(x^2)x = y^3x \quad \text{and} \quad x(x^2) = xy^3,\]

hence

\[(xy^3 - y^3x = 0)\]

No consequences comparable to (9) can be derived from the quadratic principal part \( x^2 = 0 \) (3) of the relation (8).

The system of relations

\[(x^2 - y^3 = 0, \quad xy - yx = 0)\]

is self-consistent, on the other hand (as we will see below).

To sum up this informal discussion, one can say that a system of nonhomogeneous relations (1) or (2) is called self-consistent if it defines an ungraded (co)algebra of the same “size” as the graded (co)algebra defined by the homogeneous relations (3). When the nonhomogeneous relations are not self-consistent, the ungraded object they define is “smaller” than the graded object defined by the homogeneous principal parts of the relations. So self-consistency of relations, considered in this context, appears as a species or a variation of the notion of a flat deformation.

In order to formally define self-consistency of the relations (2) in the sense we are interested in, let us start with the following setup of complete algebras before moving to coalgebras. Let \( U \) be a (possibly) infinite-dimensional vector space over a field \( k \), and let \( V = U^* \) be the dual vector space endowed with its natural locally linearly
compact (profinite-dimensional) topology. Consider the cofree conilpotent (tensor) coalgebra
\[ F = \bigoplus_{n=0}^{\infty} U^{\otimes n} \]
cogenerated by \( U \) and the dual topological algebra
\[ F^* = \prod_{n=0}^{\infty} (U^{\otimes n})^* = k \cap V \cap V \hat{\otimes} V \cap V \hat{\otimes} V \cap \cdots \]
where, by the definition, the completed tensor product of the dual vector spaces to discrete vector spaces \( U' \) and \( U'' \) is the dual vector space to the tensor product, \( U'^* \otimes U''^* = (U' \otimes U'')^* \). The conilpotent coalgebra \( F \) is endowed with its natural increasing coaugmentation filtration
\[ N_m F = \bigoplus_{n=0}^{m} U^{\otimes m}, \]
and the topological algebra \( F^* \) is endowed with the dual decreasing augmentation (topological adic) filtration
\[ N^m F^* = \prod_{n=m}^{\infty} (U^{\otimes n})^* = (U^{\otimes m})^* \cap (U^{\otimes m+1})^* \cap \cdots \]

Let \( R \subset N^2 F^* \) be a closed vector subspace and \( J \subset N^2 F^* \) be the closed two-sided ideal in the algebra \( F^* \) generated by the subspace \( R \). The quotient algebra \( A = F^*/J \) is endowed with the quotient topology and the induced filtration \( N^m A = N^m F^*/(N^m F^* \cap J) \). The associated graded algebra \( \text{gr}_N A \) can be defined as the infinite product
\[ \text{gr}_N A = \prod_{m=0}^{\infty} \text{gr}_N^m A = \prod_{m=0}^{\infty} N^m A/N^{m+1} A. \]
The algebra \( \text{gr}_N A \) is the quotient algebra of the algebra \( \text{gr}_N F^* \simeq F^* \) by the closed ideal \( \text{gr}_N J = \prod_{m=0}^{\infty} N^m J/N^{m+1} J \), where \( N^m J = N^m F^* \cap J \).

When the space of generating relations \( R \subset N^2 F^* \) is homogeneous, that is \( R = \prod_{n=0}^{\infty} R \cap (U^{\otimes n})^* \), the algebra \( A \simeq \text{gr}_N A \) is the product of its grading components, \( A = \prod_{n=0}^{\infty} A^n \). One has \( A^0 = k \), \( A^1 = U^* = V \), and \( A^n = (U^{\otimes n})^*/R \cap (U^{\otimes n})^* \). When one can choose the subspace \( R \) so that \( R \subset V \hat{\otimes} V \subset N^2 F^* \), the algebra \( A \) is said to be defined by (homogeneous) quadratic relations.

One says that the topological algebra \( A \) is defined by nonhomogeneous quadratic relations if the subspace \( R \subset N^2 F^* \) can be chosen in such a way that the projection map \( R \rightarrow N^2 F^*/N^3 F^* = V \hat{\otimes} V \) is injective. The image \( \overline{R} \subset V \hat{\otimes} V \) of this map is the space of principal quadratic parts (3) of the nonhomogeneous quadratic relations (2) from \( R \). The quotient space \( (V \hat{\otimes} V)/\overline{R} \) is the component \( \text{gr}_N^2 A = N^2 A/N^3 A \) of the associated graded algebra \( \text{gr}_N A \), while the lower grading components are, as above, \( \text{gr}_N^1 A = V \) and \( \text{gr}_N^0 A = k \).

Denote by \( A \) the quotient algebra \( F^*/J \) of the algebra \( F^* \) by the closed two-sided ideal \( J \) generated by the subspace \( R \subset V \hat{\otimes} V \subset N^2 F^* \). Then the identification of the spaces of generators \( A \supset V \simeq \text{gr}_N^1 A \subset \text{gr}_N A \) extends uniquely to a surjective homomorphism of topological graded algebras
\[ A \longrightarrow \text{gr}_N A. \]
The map (11) is always an isomorphism in the degrees $n \leq 3$. In particular, one has $\overline{A}^0 = k = \text{gr}_N^0 A$, $\overline{A}^1 = V = \text{gr}_N^1 A$, $\overline{A}^2 = (V \otimes V)/\overline{R} = \text{gr}_N^2 A$, and
\[
\overline{A}^3 = (V \otimes V \otimes V)/(V \otimes \overline{R} + \overline{R} \otimes V) = \text{gr}_N^3 A.
\]
The example of the relation (8) illustrates how the map (11) can fail to be an isomorphism in degree 4. One says that the system of nonhomogeneous quadratic relations $R \subset N^2F^*$ is self-consistent if the algebra $\text{gr}_N A$ is defined by quadratic relations, or equivalently, if the map (11) is an isomorphism in all the degrees.

To see that the system of relations (10) is self-consistent, one identifies the algebra $A$ defined by (10) with the subalgebra in the algebra of formal power series $k[[z]]$ topologically spanned by the monomials $z^i$, $i = 0$ or $i \geq 2$, where $x = z^3$ and $y = z^2$.

The decreasing filtration $N$ on the algebra $A$ is described by the rules that $N^0 A = A$, $N^1 A$ is spanned by $z^i$, $i \geq 2$, and $N^m A = (N^1 A)^m$ is spanned by $z^i$, $i \geq 2m$.

One can check that the associated graded algebra $\text{gr}_N A$ is indeed isomorphic to the quotient algebra of the algebra of noncommutative formal power series $k\{\{x,y\}\}$ by the closed ideal generated by the quadratic principal parts (3) of the relations (10)
\[
(12) \quad \begin{cases}
    x^2 = 0, \\
x y - y x = 0,
\end{cases}
\]
or, which is the same, the quotient algebra of the algebra of power series $k[[x,y]]$ by the ideal generated by $x^2$.

For a more technical discussion of nonhomogeneous relations of the type (2), the language of conilpotent coalgebras is preferable. Let $C$ be a conilpotent coalgebra over a field $k$, in the sense of the definition in Section 1. The algebra $A$ above is just the topological algebra dual to $C$, that is $A = C^*$.

Let $N_m C = \ker(C \to (C/k)^{\otimes m+1})$ denote the canonical increasing coaugmentation filtration on a conilpotent coalgebra $C$ (cf. the proof of Proposition 3.2(b)). The filtration $N$ is compatible with the comultiplication on $C$, so the associated graded vector space $\text{gr}^N C = \bigoplus_m N_m C/N_{m-1} C$ is endowed with a natural structure of positively graded coalgebra.

We refer to [33, Section 2.1] for the definition of the cohomology of comodules over a coaugmented coalgebra $C$. In particular, for a left $C$-comodule $M$, the space $H^0(C,M)$ is the kernel of the coaction map $M \to C_+ \otimes_k M$, where $C_+ = C/k$. The cofree left $C$-comodule cogenerated by a $k$-vector space $U$ is the $C$-comodule $C \otimes_k U$.

**Lemma 5.1.** Let $M$ be a left $C$-comodule. Then

(a) morphisms of left $C$-comodules $M \to C \otimes_k U$ correspond bijectively to morphisms of $k$-vector spaces $M \to U$;

(b) a morphism of left $C$-comodules $f : M \to C \otimes_k U$ is injective if and only if the morphism $H^0(C,f) : H^0(C,M) \to H^0(C,C \otimes_k U) = U$ is injective.

**Proof.** Part (a) does not depend on the conilpotency assumption on $C$. To a morphism of left $C$-comodules $f : M \to C \otimes_k U$, one assigns the composition $M \to C \otimes_k U \to U$ of the morphism $f$ with the morphism $C \otimes_k U \to U$ induced by the
counit map $C \rightarrow k$. To a morphism of $k$-vector spaces $g: M \rightarrow U$, one assigns the composition $M \rightarrow C \otimes_k M \rightarrow C \otimes_k U$ of the coaction map $M \rightarrow C \otimes_k M$ with the map $C \otimes_k g: C \otimes_k M \rightarrow C \otimes_k U$.

In part (b), one has $H^0(C, M) \subset M$ and $H^0(C, C \otimes_k U) \subset C \otimes_k U$, so injectivity of the map $f$ clearly implies injectivity of $H^0(C, f)$. Conversely, let $L$ denote the kernel of the morphism $f$; so $L$ is also a left $C$-comodule. If the map $H^0(C, f)$ is injective, then $H^0(C, L) = 0$. We have to show that this implies $L = 0$.

Indeed, let $N_m L = \ker(L \rightarrow C/N_m C \otimes_k L)$ denote the filtration on $L$ induced by the coaugmentation filtration $N$ on $C$. One has $L = \bigcup_m N_m L$. The subspace $H^0(C, L) = N_0 L \subset L$ is the maximal subcomodule of $L$ with the trivial coaction of $C$. The subspaces $N_m L \subset L$ are also subcomodules of $L$, and the coaction of $C$ in the quotient comodules $N_m L/N_{m-1} L$ is trivial. Hence $N_0 L = 0$ implies by induction $N_m L = 0$ for all $m \geq 0$ and $L = 0$.  

Lemma 5.1 purports to explain why the vector space $U = H^0(C, M)$ is called the space of cogenerators of a left $C$-comodule $M$. Choosing an arbitrary $k$-linear map $M \rightarrow U$ equal to the identity isomorphism in restriction to $H^0(C, M) \subset M$ and applying Lemma 5.1(a), one obtains a left $C$-comodule morphism $M \rightarrow C \otimes_k U$, which is injective according to Lemma 5.1(b). This is the minimal possible way to embed $M$ into a cofree left $C$-comodule, in the sense made precise by Lemma 5.1.

As above, let $F = \bigoplus_{n=0}^{\infty} U^\otimes n$ be the tensor coalgebra of a $k$-vector space $U$.

**Lemma 5.2.** Let $C$ be a conilpotent coalgebra over $k$. Then

(a) morphisms of (coaugmented) coalgebras $C \rightarrow F$ correspond bijectively to morphisms of $k$-vector spaces $C_+ \rightarrow U$;

(b) a morphism of coalgebras $f: C \rightarrow F$ is injective if and only if the morphism $H^1(f): H^1(C) \rightarrow H^1(F) = U$ is injective.

**Proof.** In part (a) already, it is important that the coalgebra $C$ is conilpotent. It is not difficult to see that any morphism of conilpotent coalgebras preserves the coaugmentations. To a morphism of coalgebras $f: C \rightarrow F$, one assigns the composition $C_+ \rightarrow F_+ \rightarrow U$ of the morphism $f$ with the projection $F_+ = \bigoplus_{n=1}^{\infty} U^\otimes n \rightarrow U$.

To a morphism of $k$-vector spaces $g: C_+ \rightarrow U$, one assigns the morphism of coalgebras $C \rightarrow F$ with the components $C_+ \rightarrow U^\otimes n$ constructed as the compositions $C_+ \rightarrow C_+^\otimes n \rightarrow U^\otimes n$ of the iterated comultiplication map $C_+ \rightarrow C_+^\otimes n$ with the map $g^\otimes n: C_+^\otimes n \rightarrow U^\otimes n$. The assumption of conilpotency of the coalgebra $C$ guarantees that the map with such components lands inside $\bigoplus_{n=1}^{\infty} U^\otimes n \subset \prod_{n=1}^{\infty} U^\otimes n$.

In part (b), one has $H^1(C) \subset C_+$ and $H^1(F) \subset F_+$, so injectivity of the map $f$ implies injectivity of $H^1(f)$. Conversely, let $c \in C$ be an element belonging to $N_n C$ but not to $N_{n-1} C$, where $n \geq 1$. Then the image of $c$ in $C_+^\otimes n$ is a nonzero element of the subspace $(N_1 C_+)^\otimes n \subset C_+^\otimes n$, where $N_m C_+$ denotes the filtration on the quotient coalgebra without counit $C_+ = C/k$ induced by the filtration $N$ on the coalgebra $C$.

In order to show that $f(c) \neq 0$ in $F$, it suffices to check that the map 

$$(N_1 f_+)^\otimes n: (N_1 C_+)^\otimes n \rightarrow (N_1 F_+)^\otimes n$$

is injective. It remains to recall that $N_1 C_+ = H^1(C)$ and $N_1 F_+ = H^1(F)$.  

Lemma 5.2(a) purports to explain why the tensor coalgebra $F$ is called the cofree conilpotent coalgebra cogenerated by a vector space $U$. Furthermore, Lemma 5.2 explains why the vector space $U = H^1(C)$ is called the space of cogenerators of a conilpotent coalgebra $C$. Choosing an arbitrary $k$-linear map $C_+ \to U = H^1(C) \subset C_+$ and applying Lemma 5.2(a), one obtains a coalgebra morphism $C \to F$ from $C$ into the tensor coalgebra $F = \bigoplus_{n=0}^{\infty} U^{\otimes n}$, which is injective by Lemma 5.2(b). This is the minimal possible way to embed $C$ into a tensor (cofree conilpotent) coalgebra.

Let $f: C \to D$ be an injective morphism of conilpotent coalgebras. The quotient space $D/C$ has a natural structure of bicomodule over the coalgebra $D$, that is a left comodule over the coalgebra $D \otimes_k D^\text{op}$, where $D^\text{op}$ denotes the opposite coalgebra to $D$. In other words, the vector space $D/C$ is endowed with natural left and right coactions $D/C \to D \otimes_k D/C$ and $D/C \to D/C \otimes_k D$ of the coalgebra $D$, which commute with each other, so they can be united in a bicoaction map $D/C \to D \otimes_k D/C \otimes_k D$.

Indeed, $D$ is naturally a bicomodule over itself, and the coalgebra morphism $f$ endows $C$ with a structure of bicomodule over $D$, so $D/C$ is the cokernel of a bicomodule morphism $f: C \to D$.

We will call the vector space of cogenerators of the bicomodule $D/C$ over $D$ $R = R(C, D) = H^0(D \otimes_k D^\text{op}, D/C)$ the space of defining corelations of the subcoalgebra $C$ in $D$. This is the dual point of view to constructing the defining relations of a quotient algebra $B$ as the generators of the kernel ideal of the morphism $B \to A$. The next theorem is the coalgebra version of [34, proof of Proposition 5.2 of Chapter 1].

**Theorem 5.3.** For any injective morphism of conilpotent coalgebras $f: C \to D$, there is a natural exact sequence

$$0 \to H^1(C) \to H^1(D) \to R(C, D) \to H^2(C) \to H^2(D).$$

**Proof.** Consider the short exact sequence of complexes

$$0 \to C_+ \to D_+ \to D/C \to 0$$

$$0 \to C^{\otimes 2}_+ \to D^{\otimes 2}_+ \to D^{\otimes 2}_+/C^{\otimes 2}_+ \to 0$$

$$0 \to C^{\otimes 3}_+ \to D^{\otimes 3}_+ \to D^{\otimes 3}_+/C^{\otimes 3}_+ \to 0,$$

two of which are just the initial fragments of the cobar-complexes of $C$ and $D$. The related long exact sequence of cohomology is the desired one, with the only difference that the kernel of the map $D/C \to D^{\otimes 2}_+/C^{\otimes 2}_+$ stands in place of the vector space $R$. To see that these are the same, one first notices that $R$ is the maximal subbicomodule...
of $D/C$ where both the left and the right coactions of $D$ are trivial. This can be alternatively constructed as the kernel of the map $D/C \rightarrow D_{+} \otimes_{k} D/C \oplus D/C \otimes_{k} D_{+}$ whose components are the left and the right coaction maps. Finally, the natural map $D_{+}^{\otimes 2}/C_{+}^{\otimes 2} \rightarrow D_{+} \otimes_{k} D/C \oplus D/C \otimes_{k} D_{+}$ is injective.

Assume that the map between the spaces of cogenerators $H^{1}(C) \rightarrow H^{1}(D)$ of the coalgebras $C$ and $D$ is an isomorphism. Then the space of corelations $R(C, D)$ is identified with the kernel of the map $H^{2}(C) \rightarrow H^{2}(D)$. In particular, when $D = F$ is a cofree conilpotent coalgebra, we have $H^{2}(F) = 0$ and $R(C, F) = H^{2}(C)$. This explains why the vector space $H^{2}(C)$ is called the space of defining corelations of a conilpotent coalgebra $C$ (in a cofree conilpotent coalgebra).

The following rule allows to construct a natural filtration $N$ on the space of corelations $R(C, D)$. The components of the coaugmentation filtration $N_{m}C \subset C$ are subcoalgebras in $C$. Set $N_{m}R(C, D) = R(N_{m}C, N_{m}D)$. By assumptions, one has $N_{0}C = N_{0}D$ and $N_{1}C = N_{1}D$, hence $N_{m}R(C, D) = 0$ for $m \leq 1$. One has $N_{m}C = C \cap N_{m}D$, so the induced morphism $N_{m}D/N_{m}C \rightarrow D/C$ is injective, and it follows that the natural morphism $N_{m}R(C, D) \rightarrow R(C, D)$ is injective, too. One has $D/C = \bigcup_{m} N_{m}D/N_{m}C$, hence $R(C, D) = \bigcup_{m} N_{m}R(C, D)$.

We set $N_{m}H^{2}(C) = N_{m}R(C, F)$. This construction of a filtration does not depend on the choice of an embedding of the coalgebra $C$ into a cofree conilpotent coalgebra $F$ with $H^{1}(F) = H^{1}(C)$, because all such embeddings only differ by an automorphism of the coalgebra $F$. A subcoalgebra $C \subset D$ is said to be defined by nonhomogeneous quadratic corelations in $D$ if $R(C, D) = N_{2}R(C, D)$. A coalgebra $C$ is defined by nonhomogeneous quadratic corelations if it is defined by nonhomogeneous quadratic corelations as a subcoalgebra in $F$.

**Proposition 5.4.** For any conilpotent coalgebra $C$, the subspace $N_{2}H^{2}(C) \subset H^{2}(C)$ coincides with the image of the multiplication map $H^{1}(C) \otimes_{k} H^{1}(C) \rightarrow H^{2}(C)$.

**Proof.** The vector space $N_{2}H^{2}(C) = N_{2}R(C, F)$ is computed as the cokernel of the differential $N_{2}C_{+} \rightarrow L$, where $L$ is the subspace of all elements in $(N_{2}C_{+})^{\otimes 2}$ whose images in $(N_{2}F_{+})^{\otimes 2}$ are coboundaries, i. e., come from elements of $N_{2}F_{+}$. The image of the differential $N_{2}F_{+} \rightarrow (N_{2}F_{+})^{\otimes 2}$ is equal to $(N_{1}F_{+})^{\otimes 2} \subset (N_{2}F_{+})^{\otimes 2}$. Hence the subspace $L \subset (N_{2}C_{+})^{\otimes 2}$ coincides with $(N_{1}C_{+})^{\otimes 2}$. It remains to recall that $H^{1}(C) = N_{1}C_{+}$.

**Corollary 5.5.** A conilpotent coalgebra $C$ is defined by nonhomogeneous quadratic corelations if and only if the multiplication map $H^{1}(C) \otimes_{k} H^{1}(C) \rightarrow H^{2}(C)$ is surjective.

Let $D = \bigoplus_{n=0}^{\infty} D_{n}$, $D_{0} = k$ be a positively graded coalgebra. Then the cohomology spaces $H^{i}(D)$ are endowed with the internal grading $H^{i}(D) = \bigoplus_{j=1}^{\infty} H^{i,j}(D)$ induced by the grading of $D$. Assume that $D$ is cogenerated by $D_{1}$, i. e., the iterated comultiplication maps $D_{n} \rightarrow D_{n}^{\otimes n}$ are injective. Equivalently, this means that $H^{1}(D) = H^{1,1}(D)$. Then the above-defined filtration $N$ on $H^{2}(D)$ is associated with the internal grading, that is $N_{m}H^{2}(D) = \bigoplus_{j=2}^{m} H^{2,j}(D)$. In particular, one has
$N_2H^2(D) = H^2(D)$ if and only if $H^{2,j}(D) = 0$ for $j > 2$, that is, if and only if the graded coalgebra $D$ is quadratic.

A conilpotent coalgebra $C$ is said to be defined by self-consistent nonhomogeneous quadratic corelations if the graded coalgebra $\text{gr}^N C$ is quadratic. According to the main theorem of [32] (see [33, Theorem 4.2] for the relevant formulation), any conilpotent coalgebra $C$ with a Koszul cohomology algebra $H^*(C)$ is defined by self-consistent nonhomogeneous quadratic corelations with Koszul quadratic principal part (3). Moreover, the seemingly weaker conditions on the algebra $H^*(C)$ formulated in the above Theorem 1.3 are sufficient in lieu of the Koszulity condition.

The proof of this result in [32] is based on the spectral sequence converging from $H^*(\text{gr}^N C)$ to $H^*(C)$. The above filtration $N$ on $H^2(C)$ is a part of the filtration $N$ on $H^*(C)$ induced by the filtration $N$ on the cobar-complex $\text{Cob}^\bullet(C)$ induced by the natural filtration $N$ on a conilpotent coalgebra $C$.

The condition of surjectivity of the map $qH^*(C) \rightarrow H^*(C)$ in degree 2 in Theorem 1.3 means that the coalgebra $C$ is defined by nonhomogeneous quadratic corelations, as we have explained. The condition of injectivity of this map in degree 3 means, basically, that syzygies of degree 3 between the homogeneous quadratic parts (3) of nonhomogeneous quadratic corelations in $C$ do not lead to non-self-consistencies (as it happens in the example with the relation (8)). In this sense, Theorem 1.3 can be viewed as an analogue for relations of the type (2) of the Poincaré–Birkhoff–Witt theorem for relations of the type (1) [31, 34] (cf. [23]).

Conversely, if a conilpotent coalgebra $C$ is defined by self-consistent nonhomogeneous quadratic corelations with Koszul quadratic principal parts, that is the graded coalgebra $\text{gr}^N C$ is Koszul, then the algebra $H^*(C)$ is isomorphic to $H^*(\text{gr}^N C)$, as it easily follows from the same spectral sequence, and consequently Koszul.

6. Koszulity Does Not Imply Formality

Examples of quasi-formal DG-algebras that are not formal are well known in the conventional (commutative) rational homotopy theory [13, Examples 8.13]. In this section we, working in the noncommutative homotopy theory of Section 3, over a field of prime characteristic, present a series of counterexamples of quasi-formal, nonformal DG-algebras with Koszul cohomology algebras. We also present a family of commutative DG-algebras with the similar properties defined over an arbitrary field (of zero or prime characteristic).

Recall that a DG-algebra $A^*$ is called formal if it can be connected by a chain of quasi-isomorphisms of DG-algebras with its cohomology algebra $H^*(A^*)$, viewed as a DG-algebra with zero differential (cf. Section 2). The following lemma shows that there is no ambiguity in this definition as applied to DG-algebras with the cohomology algebras concentrated in the positive cohomological degrees. We refer to Section 3 for a short discussion of positively cohomologically graded DG-algebras.

Lemma 6.1. Let $A^*$ and $B^*$ be two augmented DG-algebras with the cohomology algebras concentrated in the positive cohomological degrees, connected by a chain of
quasi-isomorphisms of DG-algebras over a field $k$. Then there exists a positively cohomologically graded DG-algebra $P^\bullet$ together with two quasi-isomorphisms of augmented DG-algebras $P^\bullet \to A^\bullet$ and $P^\bullet \to B^\bullet$. In particular, the chain of quasi-isomorphisms between the DG-algebras $A^\bullet$ and $B^\bullet$ can be made to consist of augmented quasi-isomorphisms of augmented DG-algebras.

**Proof.** It suffices to choose a positively cohomologically graded cofibrant model of either DG-algebra $A^\bullet$ or $B^\bullet$ in the role of $P^\bullet$ (see Lemma 4.4).

Recall that any DG-algebra $A^\bullet$ with a Koszul cohomology algebra $H^*(A^\bullet)$ is “a $K(\pi,1)$”, i.e., admits a quasi-isomorphism $\text{Cob}^\bullet(C) \to A^\bullet$ from the cobar construction of a conilpotent coalgebra $C$ (see Theorem 4.3 and its proof). The conilpotent coalgebra $C$ can be recovered as the degree-zero cohomology coalgebra of the bar construction of the DG-algebra $A^\bullet$, i.e., $C = H^0 \text{Bar}^\bullet(A^\bullet)$.

As above, let $N_mC = \ker(C \to (C/k)^{\otimes m+1})$ be the canonical increasing filtration on a conilpotent coalgebra $C$ and let $\text{gr}^N C = \bigoplus_m N_mC/N_{m-1}C$ be the associated graded coalgebra (see Section 5). The following theorem characterizes those DG-algebras with Koszul cohomology algebras that are not only quasi-formal but actually formal.

**Theorem 6.2.** Let $A^\bullet$ be an augmented DG-algebra with a Koszul cohomology algebra $H^*(A^\bullet)$. Then the DG-algebra $A^\bullet$ is formal if and only if the conilpotent coalgebra $C = H^0 \text{Bar}^\bullet(A^\bullet)$ is isomorphic to its associated graded coalgebra $\text{gr}^N C$ with respect to the canonical increasing filtration $N$.

**Proof.** By (the proof of) Theorem 4.3, the DG-coalgebra $\text{Bar}^\bullet(A^\bullet)$ is quasi-isomorphic to its degree-zero cohomology coalgebra $C$. The coalgebra $C$ is conilpotent, and its cohomology algebra $\text{Ext}^*_C(k,k) = H^* \text{Cob}^\bullet(C)$, being isomorphic to the algebra $H^*(A^\bullet)$, is Koszul. By [33, Theorem 4.2], it follows that the graded coalgebra $\text{gr}^N C$ is Koszul and quadratic dual to $H^*(A^\bullet)$. By the definition of a Koszul graded coalgebra, there is a natural quasi-isomorphism $\text{Cob}^\bullet(\text{gr}^N C) \to H^*(A^\bullet)$.

Hence, whenever the coalgebras $C$ and $\text{gr}^N C$ are isomorphic, the DG-algebras $A^\bullet$ and $H^*(A^\bullet)$ are connected by a pair of quasi-isomorphisms $\text{Cob}^\bullet(C) \to A^\bullet$ and $\text{Cob}^\bullet(C) \to H^*(A^\bullet)$. Conversely, suppose that there is a chain of quasi-isomorphisms of DG-algebras connecting $A^\bullet$ with $H^*(A^\bullet)$. By Lemma 6.1, this can be assumed to be a chain of quasi-isomorphisms of augmented DG-algebras. Applying the bar construction, we obtain a chain of comultiplicative quasi-isomorphisms connecting the DG-coalgebras $\text{Bar}^\bullet(A^\bullet)$ and $\text{Bar}^\bullet(H^*(A^\bullet))$. It follows that the degree-zero cohomology coalgebras $H^0 \text{Bar}^\bullet(A^\bullet) = C$ and $H^0 \text{Bar}^\bullet(H^*(A^\bullet)) = \text{gr}^N C$ of these two DG-coalgebras are isomorphic.

The following series of examples [37, Section 9.11] provides a negative answer to a question of Hopkins and Wickelgren [16, Question 1.4].

**Example 6.3.** Let $l$ be a prime number and $G$ be a profinite group; denote by $G^{(l)}$ the maximal quotient pro-$l$-group of $G$. Let $k$ be a field of characteristic $l$; then the $k$-vector space $D = k(G)$ of locally constant $k$-valued functions on $G$ is
endowed with a natural structure of coalgebra over $k$ with respect to the convolution comultiplication. We will call this coalgebra the group coalgebra of a profinite group $G$ over a field $k$. The maximal conilpotent subcoalgebra $C = \text{Nilp} D \subset D$ is naturally identified with the group coalgebra $k(G^{(l)})$ of the pro-$l$-group $G^{(l)}$. The cohomology map $H^*(G^{(l)}, k) \rightarrow H^*(G, k)$ is known to be an isomorphism, at least, whenever either the cohomology algebra $H^*(G, k)$ is Koszul [33, Corollary 5.5], or $G = G_F$ is the absolute Galois group of a field $F$ containing a primitive $l$-root of unity [44].

Let $l \neq p$ be two prime numbers and $F$ be a finite extension of the field of $p$-adic numbers $\mathbb{Q}_p$, or the field of formal Laurent power series $\mathbb{F}_p((z))$ with coefficients in the prime field $\mathbb{F}_p$. Assume that the field $F$ contains a primitive $l$-root of unity if $l$ is odd, or a square root of $-1$ if $l = 2$. In other words, the cardinality $q$ of the residue field $f = \mathcal{O}_F/\mathfrak{m}_F$ of the field $F$ should be such that $q - 1$ is divisible by $l$ if $l$ is odd and by $4$ if $l = 2$. Then the maximal quotient pro-$l$-group $G_F^{(l)}$ of the absolute Galois group $G_F$ is isomorphic to the semidirect product of two copies of the group of $l$-adic integers $\mathbb{Z}_l$ with one of them acting in the other one by the multiplication with $q$.

So, in the exponential notation, the group $H = G_F^{(l)}$ is generated by two symbols $s$ and $t$ with the relation $sts^{-1} = t^4$, or, redenoting $s = 1 + x$ and $t = 1 + y$ and recalling that we are working over a field of characteristic $l$,

$$
(1 + x)(1 + y)(1 + x)^{-1}(1 + y)^{-1} = (1 + y^l)^{\frac{1}{l-1}} \quad \text{for } l \text{ odd, or}
$$

$$
(1 + x)(1 + y)(1 + x)^{-1}(1 + y)^{-1} = (1 + y^4)^{\frac{1}{4-1}} \quad \text{for } l = 2.
$$

This is a single nonhomogeneous quadratic relation of the type (2) defining the conilpotent group coalgebra $C = k(H)$. The quadratic principal part (3) of this relation is simply $xy - yx = 0$; this in fact defines the associated graded coalgebra $\text{gr}^N C$, which turns out to be the symmetric coalgebra in two variables.

Alternatively, one can easily compute the cohomology algebra $H^*(H, k) \simeq H^*(G_F, k)$ to be the exterior algebra in two generators of degree 1; then the graded coalgebra $\text{gr}^N C$ is recovered as the quadratic dual. Either way, the coalgebra $\text{gr}^N C$ is cocommutative and the coalgebra $C$ is not (as the group $H$ is not commutative), so $C$ cannot be isomorphic to $\text{gr}^N C$. Applying Theorem 6.2, we conclude that the cochain DG-algebra $\text{Cob}^*(C)$ of the pro-$l$-group $H$ is not formal. The cochain DG-algebra $\text{Cob}^*(D) = \text{Cob}^*(k(G_F))$ of the absolute Galois group $G_F$, being quasi-isomorphic to the DG-algebra $\text{Cob}^*(C)$ via the natural quasi-isomorphism $\text{Cob}^*(C) \rightarrow \text{Cob}^*(D)$ induced by the embedding of coalgebras $C \rightarrow D$, is consequently not formal, either.

To sum up this example in a more abstract fashion, for any noncommutative pro-$l$-group $G$ such that $H^*(G, \mathbb{Z}/l)$ is the exterior algebra over $H^1(G, \mathbb{Z}/l)$ and any field $k$ of characteristic $l$, the cochain DG-algebra $\text{Cob}^*(k(G))$ is not formal.

We are not aware of any example of a field $F$ containing all the $l$-power roots of unity (that is, all the roots of unity of the powers $l^n$, $n \geq 1$) whose cochain DG-algebra $\text{Cob}^*(\mathbb{Z}/l(G_F))$ over the coefficient field $\mathbb{Z}/l$ is not formal. In particular, it would be interesting to know if there is a field $F$ containing an algebraically closed
The following family of examples of nonformal commutative DG-algebras over an arbitrary field is obtained by a modification of Example 6.3.

**Example 6.4.** Consider a single nonhomogeneous quadratic Lie relation of the type (2) for $m + 2$ variables $x, y, z_1, \ldots, z_m$

$$[x, y] + q_3(z_1, \ldots, z_m) + q_4(z_1, \ldots, z_m) + q_5(z_1, \ldots, z_m) + \cdots = 0,$$

where $q_n$ are homogeneous Lie expressions of degree $n$ in the variables $z_1, \ldots, z_m$ over a field $k$. The relation (14) can be viewed as defining a pronilpotent Lie algebra $L$, or its dual conilpotent Lie coalgebra, or its conilpotent coenveloping coalgebra $C$, or its dual topological associative algebra, which is simply the quotient algebra of the algebra of noncommutative formal Taylor power series in the $m + 2$ variables $x, y, z_1, \ldots, z_m$ by the closed ideal generated by the single power series (14).

The homogeneous part (3) of the relation (14) has the form $[x, y] = 0$, and the relation (14) is self-consistent, i.e., the associated graded coalgebra $\text{gr}^N C$ is indeed the conilpotent coalgebra cogenerated by the $m + 2$ variables with the single relation $xy - yx = 0$ (and not a smaller coalgebra). One can check this, e.g., by a trivial application of the Diamond Lemma [6] for noncommutative power series (the single relation (14) starts with $xy$, so there are no ambiguities to resolve). The graded coalgebra $\text{gr}^N C$ is Koszul, and its quadratic/Koszul dual algebra $H^*(C) \simeq H^*(\text{gr}^N C)$ is the connected direct sum of the exterior algebra in two generators of degree 1 and $m$ copies of the exterior algebra in one generator of degree 1.

Now setting $A^* = (\wedge(L^*), \delta)$ to be the Chevalley–Eilenberg complex of the cofinite-dimensional Lie algebra $L$ (i.e., the inductive limit of the Chevalley–Eilenberg cohomological complexes of the finite-dimensional quotient Lie algebras of $L$ by its open ideals), one obtains a commutative DG-algebra endowed with a natural quasi-isomorphism $\text{Cob}^*(C) \rightarrow A^*$ from the cobar construction of the coalgebra $C$. The cohomology algebra $H^*(A^*)$ is the connected direct sum of the exterior algebra in two generators and $m$ copies of the exterior algebra in one generator of degree 1, while the bar construction $\text{Bar}^*(A^*)$ is quasi-isomorphic to $C$. So the commutative DG-algebra $A^*$ cannot be connected with its cohomology algebra $H^*(A^*)$ by a chain of quasi-isomorphisms, even in the class of noncommutative DG-algebras, unless the coalgebra $C$ is isomorphic to $\text{gr}^N C$. The latter is easily seen to be impossible, e.g., when the degree-three Lie form $q_3(z_1, \ldots, z_m)$ does not vanish.

Indeed, consider a variable change (4) of the form $x \rightarrow x + \sum_{n \geq 2} p_n,x(x, y, z)$, $y \rightarrow y + \sum_{n \geq 2} p_n,y(x, y, z)$, and $z_i \rightarrow z_i + \sum_{n \geq 2} p_n,i(x, y, z)$, where $z$ denotes the collection of variables $(z_1, \ldots, z_m)$ and $\deg p_n,x = \deg p_n,y = \deg p_n,i = n$. The relation (14) gets transformed by this variable change into the relation

$$[x + \sum_n p_n,x(x, y, z), y + \sum_n p_n,y(x, y, z)] + q_3(z_1 + \sum_n p_n,1(x, y, z), \ldots, z_m + \sum_n p_n,m(x, y, z)) + \cdots = 0.$$
The relation (15) is never equivalent to \([x, y] = 0\), as reducing both of these modulo all the expressions of degree \(\geq 4\) and all the expressions divisible by \(x\) or \(y\), the former takes the form \(q_3(z_1, \ldots, z_m) = 0\), while the latter vanishes entirely. So they cannot generate the same closed ideal in the ring of noncommutative formal power series.

**Example 6.5.** The following naïve attempt to construct a commutative version of Example 6.3 illustrates one of the intricacies of relations sets (2). Consider, instead of the \((m + 2)\)-variable Lie relation in Example 6.4, a two-variable Lie relation

\[
x + q_3(x, y) + q_4(x, y) + q_5(x, y) + \cdots = 0,
\]

where \(q_n\) are homogeneous Lie expressions of degree \(n\) in the variables \(x\) and \(y\). We claim that the relation (16) is always equivalent to the relation \([x, y] = 0\) in the world of (Lie or associative) formal power series in \(x\) and \(y\), so the conilpotent (coenveloping) coalgebra \(C\) defined by (16) is in fact cocommutative and isomorphic to \(gr^N C\).

Indeed, the innermost bracket in any Lie monomial in \(x\) and \(y\) is always \(\pm [x, y]\). Substituting the expression for \([x, y]\) obtained from (16) in place of the innermost bracket in every term of degree \(\geq 3\) in (16), one deduces from (16) a new Lie relation in \(x\) and \(y\) with every term of degree \(n \geq 3\) replaced by an (infinite) linear combination of terms of degrees higher than \(n\). Continuing in this fashion and passing to the limit in the formal power series topology, one concludes that the relation (16) implies \([x, y] = 0\) (which, in turn, implies (16)). E.g., the relation \([x, y] = [x, [x, [x, y]]]\) would imply \([x, y] = [x, [x, [x, y]]]\), which would lead to \([x, y] = [x, [x, [x, [x, y]]]]\), etc., and passing to the limit one would finally come to \([x, y] = 0\).

In fact, the exterior algebra in two variables of degree 1, which is the cohomology algebra \(H^*(C)\) of the coalgebra \(C\) defined by (16), is a free (super)commutative graded algebra, so it is intrinsically formal as a commutative graded algebra (i.e., any commutative DG-algebra with such cohomology algebra is formal). Example 6.3 shows that a noncommutative DG-algebra with such cohomology algebra does not have to be formal, while Example 6.4 provides a (super)commutative Koszul graded algebra that is not intrinsically formal in the commutative world already.

**References**

[1] A. A. Beilinson, J. Bernstein, P. Deligne. Faisceaux pervers. *Astérisque* 100, p. 5–171, 1982.
[2] A. A. Beilinson. On the derived category of perverse sheaves. *Lecture Notes in Math.* 1289, p. 27–41, 1987.
[3] A. A. Beilinson, V. A. Ginzburg, V. V. Schechtman. Koszul duality. *Journ. of Geometry and Physics* 5, #3, p. 317–350, 1988.
[4] A. Beilinson, V. Ginzburg, W. Soergel. Koszul duality patterns in representation theory. *Journ. of the American Math. Society* 9, #2, p. 473–527, 1996.
[5] A. Berghlund. Koszul spaces. *Trans. of the Amer. Math. Soc.* 366, #9, p. 4551–4569, 2014. arXiv:1107.0685 [math.AT]
[6] G. Bergman. The diamond lemma for ring theory. *Advances in Math.* 29, #2, 178–218, 1978.
[7] S. Bloch, I. Kriz. Mixed Tate motives. *Annals of Math.* 140, #3, p. 557–605, 1994.
[8] I. Efrat, E. Matzri. Triple Massey products and absolute Galois groups. Electronic preprint arXiv:1412.7265 [math.NT], to appear in *Journ. of the European Math. Society.*
[9] S. Eilenberg, J. C. Moore. Homology and fibrations. I. Coalgebras, cotensor product and its derived functors. *Commentarii Math. Helvetici* **40**, #1, p. 199–236, 1966.

[10] P. Gabriel. Des catégories abéliennes. *Bulletin de la Soc. Math. de France* **90**, #3, p. 323–448, 1962.

[11] V. K. A. M. Gugenheim. On a perturbation theory for the homology of the loop-space. *Journ. of Pure and Appl. Algebra* **25**, #2, p. 197–205, 1982.

[12] V. K. A. M. Gugenheim, J. P. May. On the theory and applications of differential torsion products. *Memoirs of the American Math. Society* #142, 1974. x+94 pp.

[13] S. Halperin, J. Stasheff. Obstructions to homotopy equivalences. *Advances in Math.* **32**, #3, p. 233–279, 1979.

[14] V. Hinich. Homological algebra of homotopy algebras. *Communications in Algebra* **25**, #10, p. 3291–3323, 1997. [arXiv:q-alg/9702015v1]. Erratum, [arXiv:math.QA/0309453v3].

[15] V. Hinich. DG coalgebras as formal stacks. *Journ. of Pure and Appl. Algebra* **162**, #2-3, p. 209–250, 2001. [arXiv:math.AG/9812034]

[16] M. J. Hopkins, K. G. Wickelgren. Splitting varieties for triple Massey products. *Journ. of Pure and Appl. Algebra* **219**, #5, p. 1304–1319, 2015. [arXiv:1210.4964 [math.AT]]

[17] D. Husemoller, J. C. Moore, J. Stasheff. Differential homological algebra and homogeneous spaces. *Journ. of Pure and Appl. Algebra* **5**, #2, p. 113–185, 1974.

[18] J. F. Jardine. A closed model structure for differential graded algebras. *Cyclic Cohomology and Noncommutative Geometry*, Fields Institute Communications, 17, American Mathematical Society, Providence, RI, 1997, p. 55–58.

[19] A. Kadeishvili. On the homology theory of fibre spaces. *Russian Math. Surveys* **35**, #3, p. 231–238, 1980.

[20] T. V. Kadeishvili. The algebraic structure in the homology of an $A_\infty$-algebra. (Russian.) *Soobshch. Akad. Nauk Gruzin. SSR* **108**, #2, p. 249–252, 1982.

[21] B. Keller. Deriving DG-categories. *Ann. Sci. de l'École Norm. Sup. (4)* **27**, #1, p. 63–102, 1994.

[22] B. Keller. Derived categories and their uses. In: M. Hazewinkel, Ed., *Handbook of Algebra*, vol. 1, 1996, p. 671–701.

[23] P. Lee. The pure virtual braid group is quadratic. *Selecta Math. (New Series)* **19**, #2, p. 461–508, 2013. [arXiv:1110.2356 [math.QA]]

[24] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, J. J. Zhang. $A$-infinity structure on Ext-algebras. *Journ. of Pure and Appl. Algebra* **213**, #11, p. 2017–2037, 2009. [arXiv:math.KT/0606144]

[25] J. P. May. The cohomology of augmented algebras and generalized Massey products for DGA-algebras. *Trans. of the Amer. Math. Soc.* **122**, #2, p. 334–340, 1966.

[26] J. P. May. Matrix Massey products. *Journ. of Algebra* **12**, #4, p. 533–568, 1969.

[27] J. Minač, N. D. Tân. Triple Massey products and Galois theory. *Journ. of the European Math. Society* **19**, #1, p. 255–284, 2017. [arXiv:1307.6624 [math.NT]]

[28] J. Minač, N. D. Tân. Triple Massey products vanish over all fields. *Journ. of the London Math. Society* **94**, #3, p. 909–932, 2016. [arXiv:1412.7611 [math.NT]]

[29] B. Mitchell. Rings with several objects. *Selecta Math. (New Series)* **19**, #2, p. 1–161, 1972.

[30] S. Papadima, S. Yuzvinsky. On rational $K[\pi,1]$ spaces and Koszul algebras. *Journ. of Pure and Appl. Algebra* **144**, #2, p. 157–167, 1999.

[31] L. Positselski. Nonhomogeneous quadratic duality and curvature. *Functional Analysis and its Appl* **27**, #3, p. 197–204, 1993. [arXiv:1411.1982 [math.RA]]

[32] L. Positselski, A. Vishik. Koszul duality and Galois cohomology. *Math. Research Letters* **2**, #6, p. 771–781, 1995. [arXiv:alg-geom/9507010]

[33] L. Positselski. Koszul property and Bogomolov’s conjecture. *Internat. Math. Research Notices* **2005**, #31, p. 1901–1936. [arXiv:1405.0965 [math.KT]]
[34] A. Polishchuk, L. Positselski. Quadratic algebras. University Lecture Series, 37. American Math. Society, Providence, RI, 2005.

[35] L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. arXiv:0708.3398 [math.CT]

[36] L. Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. Memoirs of the American Math. Society 212, #996, 2011. vi+133 pp. arXiv:0905.2621 [math.CT]

[37] L. Positselski. Mixed Artin–Tate motives with finite coefficients. Moscow Math. Journal 11, #2, p. 317–402, 2011. arXiv:1006.4343 [math.KT]

[38] L. Positselski. Galois cohomology of a number field is Koszul. Journ. of Number Theory 145, p. 126–152, 2014. arXiv:1008.0095 [math.KT]

[39] L. Positselski. Contraherent cosheaves. Electronic preprint arXiv:1209.2995 [math.CT].

[40] L. Positselski. Categorical Bockstein sequences. Electronic preprint arXiv:1404.5011 [math.KT].

[41] S. Priddy. Koszul resolutions. Trans. of the Amer. Math. Soc. 152, #1, p. 39–60, 1970.

[42] D. Quillen. Rational homotopy theory. Annals of Math. (2) 90, #2, p. 205–295, 1969.

[43] J. D. Stasheff. Homotopy associativity of H-spaces. II. Trans. of the Amer. Math. Soc. 108, #2, p. 293–312, 1963.

[44] V. Voevodsky. On motivic cohomology with $\mathbb{Z}/l$-coefficients. Annals of Math. 174, #1, p. 401–438, 2011. arXiv:0805.4430 [math.AG]

Department of Mathematics, Faculty of Natural Sciences, University of Haifa, Mount Carmel, Haifa 31905, Israel; and Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 117312; and Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127051, Russia

E-mail address: posic@mccme.ru