Supersymmetric AdS$_3$ supergravity backgrounds and holography

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Abstract: We analyse the conditions for AdS$_3 \times \mathcal{M}_7$ backgrounds with pure NS-NS flux to be supersymmetric. We find that a necessary condition is that $\mathcal{M}_7$ is a U(1)-fibration over a balanced manifold. We subsequently classify all $\mathcal{N} = (2,2)$ solutions where $\mathcal{M}_7$ satisfies the stronger condition of being a U(1)-fibration over a Kähler manifold. We compute the BPS spectrum of all the backgrounds in this classification. We assign a natural dual CFT to the backgrounds and confirm that the BPS spectra agree, thus providing evidence in favour of the proposal.
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1 Introduction

AdS$_3$ supergravity backgrounds provide an interesting playground to explore the AdS/CFT correspondence [1]. The case of AdS$_3$ is particularly tractable, since the dual CFT's are often exactly solvable. Moreover, there is an abundance of two-dimensional CFTs — a fact which is reflected in the richness of the corresponding supergravity backgrounds. On the other hand, this makes it hard to classify AdS$_3$-backgrounds.

However, one can make some progress when imposing a sufficient amount of supersymmetry. There are only three $\mathcal{N} = (4,4)$ backgrounds: AdS$_3 \times S^3 \times \mathcal{M}_4$, where $\mathcal{M}_4 = T^4$, K3 or $S^3 \times S^1$. The former two possibilities support the small $\mathcal{N} = (4,4)$ algebra [2], whereas the latter supports the large $\mathcal{N} = (4,4)$ algebra [3–7]. These backgrounds have very simple properties: They can be supported exclusively by NS-NS fields, exclusively by R-R fields or mixed flux. Those possibilities are related by the SL(2, $\mathbb{R}$)-symmetry of type IIB supergravity. NS-NS backgrounds allow for a simple string world-sheet description [8], while R-R backgrounds are believed to have the simplest dual CFT’s [1].

Moving on to less supersymmetry, there is one known $\mathcal{N} = (4,2)$ background [9]. However this background is much more involved, in particular it requires all form-fields to be turned on.

For a smaller amount of supersymmetry, classifications are more difficult. When allowing five-form flux only, the geometry is very restricted. For constant axio-dilaton and $\mathcal{N} = (2,0)$, the internal manifold is a U(1)-fibration over a Kähler manifold [10], which satisfies some additional curvature constraints. This was generalized
in [11] to varying axio-dilaton using an F-theory language. In particular, it was found that for $\mathcal{N} = (4, 0)$-supersymmetry, the most general geometry in this case is $\text{AdS}_3 \times S^3/\mathbb{Z}_M \times \text{CY}_3$. Here $\text{CY}_3$ is an elliptically fibred Calabi-Yau three-fold, where the complex structure of the fiber is given by the axio-dilaton.

One other direction was recently explored in [12]. There, all symmetric space solutions of type IIB supergravity were analysed. Interestingly, it was found that all $\text{AdS}_3$ symmetric space solutions are related via T-duality to one of the aforementioned backgrounds $\text{AdS}_3 \times S^3 \times \mathcal{M}_4$ with $\mathcal{N} = (4, 4)$ supersymmetry. Furthermore, all these backgrounds have $(4, 4)$ or $(4, 0)$ supersymmetry. Thus, both the symmetric space $\mathcal{N} = (4, 0)$ solutions and the $\mathcal{N} = (4, 0)$ solutions with five-form flux only are related to the known $\mathcal{N} = (4, 4)$ solutions by either T-duality or are quotients thereof.

In this note we enlarge the classification result to incorporate $\mathcal{N} = (2, 2)$ supersymmetry. This amount of supersymmetry is particularly attractive, since one still has good control over protected quantities like the BPS spectrum and the elliptic genus. This allows one in particular to determine the dual CFT. While previous structural results in this direction have been obtained [13, 14], they are quite indirect: A background $\text{AdS}_3 \times \mathcal{M}_7$ enjoys $\mathcal{N} = (2, 2)$ supersymmetry when $\mathcal{M}_7$ is a $U(1)$-fibration over a six-dimensional space which is the target of an $\mathcal{N} = (2, 2)$ sigma-model. This result was obtained from string theory. This result is conceptually nice, but provides little intuition on the geometry of $\mathcal{M}_7/U(1)$ or on the dual CFT.

Recently, some $\mathcal{N} = (2, 2)$ backgrounds were discussed [15], mostly from a string perspective. They involved taking specific orbifolds of $\text{AdS}_3 \times S^3 \times T^4$. Most of the orbifold singularities cannot be resolved, which renders the backgrounds non-geometric. However, to all of the string models, a dual CFT can be associated. Comparing BPS spectra and elliptic genera yields very non-trivial evidence for the proposal. To our knowledge, very few $\mathcal{N} = (2, 2)$-backgrounds prior to [15] were known, which demonstrates the scarcity of these backgrounds.

In this paper, we revisit the problem from the point of view of supergravity to understand the scarcity of the $\mathcal{N} = (2, 2)$ backgrounds. Motivated by the string computation of [13–15] and to keep the calculation manageable, we consider the case of pure NS-NS flux and constant dilaton. This subsector of IIB supergravity is also known as heterotic supergravity (with trivial gauge group). Via $\text{SL}(2, \mathbb{R})$-symmetry,

1Note that this is strictly speaking only true for $\mathcal{M}_4 = T^4$, for K3 we cannot perform T-dualities to relate the D3-brane system of [11] to the D1-D5 system considered here.

2In [16], $\mathcal{N} = (2, 2)$ backgrounds were constructed by compactifying $\mathcal{N} = 4$ SYM on a Riemann surface with a suitable twisting. The dual supergravity background flows in the IR to a $\mathcal{N} = (2, 2)$ supergravity background.
this also finds the pure R-R solutions. The full U-duality group can typically also generate solutions with five-form flux turned on.

We investigate \( \mathcal{N} = (2, 2) \) backgrounds in two steps. First, we show that \( \mathcal{N} = (2, 0) \) supersymmetry requires the manifold \( \mathcal{M}_7 \) to be a U(1)-fibration over a balanced manifold \( \mathcal{M}_6 \). \( \mathcal{M}_6 \) fails to be Kähler, as the fundamental \((1,1)\)-form \( J \) of the manifold is not necessarily closed, but \( J \wedge J \) is. This result was already obtained in [19] using heterotic supergravity. There are known backgrounds which happen to be Kähler, namely the small \( \mathcal{N} = (4, 4) \) backgrounds we mentioned above. In contrast, the large \( \mathcal{N} = (4, 4) \) background corresponds to a balanced manifold.

It is interesting that the dual CFTs of the small \( \mathcal{N} = (4, 4) \) cases have been known for such a long time, whereas progress in the large \( \mathcal{N} = (4, 4) \) case was made only very recently [7]. The failure of \( \mathcal{M}_6 \) to be Kähler is just another incarnation of the difficulty. To make progress in a classification, we will consider the case when \( \mathcal{M}_6 \) is a Kähler manifold. In that case, we succeed in completely classifying the possible internal manifolds \( \mathcal{M}_7 \). The main result will be that all are quotients of \( S^3 \times T^4 \) or \( S^3 \times K3 \). This result is even true for \( (2,0) \)-supersymmetry only.

We will subsequently classify all possible quotients of \( S^3 \times T^4 \) and \( S^3 \times K3 \) leading to \( \mathcal{N} = (2, 2) \) supersymmetry. There is a unique such quotient for \( S^3 \times K3 \), whereas there are seven for \( S^3 \times T^4 \). This result has a similar flavour as the \( \mathcal{N} = (4, 0) \) classification results we mentioned above. Also in this case, the backgrounds are all related to \( \mathcal{N} = (4, 4) \) backgrounds by quotients. Note that non-abelian T-dualities typically break supersymmetry of one chirality completely and thus do not yield \( \mathcal{N} = (2, 2) \) backgrounds.

We will then identify the dual CFTs of these backgrounds. The main claim is that \( \text{AdS}_3 \times (S^3 \times \mathcal{M}_4)/G \) is dual to a marginal deformation of the symmetric product orbifold of \( \mathcal{M}_4/G \). Here, \( \mathcal{M}_4 \) is \( T^4 \) or K3 and \( \mathcal{M}_4/G \) is either the Enriques surface or a hyperelliptic surface. To support this claim, we compute the chiral-chiral spectrum and the chiral-antichiral spectrum in string theory and match it to the CFT calculation. The methods and ideas follow our earlier paper [15], in particular one of the solutions we present appeared already in this earlier discussion.

This paper is organized as follows. In Section 2, we discuss the supergravity field equations and the relevant Killing spinor equations. In Section 3, we will continue to analyse the constraints coming from these equations. Starting from Section 4, we discuss the backgrounds which satisfy the aforementioned Kähler condition and classify them completely. In Section 5, we will compute the BPS spectra of these backgrounds, which will be matched to the proposed CFTs in Section 6. Some more technical calculations and conventions are summarized in various appendices.
2 Supergravity equations

We will assume throughout this note that the background geometry is of the form

\[ \text{AdS}_3 \times \mathcal{M}_7, \]  

(2.1)

with NS three-form flux on \text{AdS}_3 and on \mathcal{M}_7. This is no loss of generality, since it was shown in [19] from the point of view of heterotic supergravity that the warp factor is trivial in this case. \mathcal{M}_7 will be assumed to be compact. We will set all other fields to zero (or constant in the case of the dilaton) and we will not consider warp factors. In particular, no R-R fields are turned on. We will derive the possible forms of \mathcal{M}_7 such that the boundary CFT has (2,2) superconformal symmetry.

2.1 Equations of motion

In the following, big latin indices \( M, N, \ldots \) denote ten-dimensional indices, small greek indices \( \mu, \nu, \ldots \) denote \text{AdS}_3 indices, and small latin indices \( a, b, \ldots \) denote \mathcal{M}_7 indices. The relevant part of the type IIB action in string frame is then

\[ S_{\text{NS}} = \int d^{10}x \sqrt{-g}e^{-2\Phi} \left( R - \frac{1}{12} H_{MNP}H^{MNP} + 4 \nabla_M \Phi \nabla^M \Phi \right), \]  

(2.2)

where \( R \) is the scalar curvature of the metric \( g^{MN} \) and \( H \) is the field strength corresponding to the two-form \( B \):

\[ H_{MNP} = 3 \nabla_{[M}B_{NP]} \]  

(2.3)

It is a consistent truncation of the theory to set all R-R fields to zero and omit the Chern-Simons term in the action. This is explained e.g. in [5]. The corresponding equations of motion are

\[ 0 = \frac{1}{4} H^{MPQ} H^N_{\ P Q} - \frac{1}{24} g^{MN} H_{PQR} H^{PQR} - R^{MN} + \frac{1}{2} g^{MN} R - 2 \nabla^{(M} \nabla^{N)} \Phi \]
\[ + 2 g^{MN} \nabla_P \nabla^P \Phi - 2 g^{MN} \nabla_P \Phi \nabla^P \Phi, \]  

(2.4)

\[ 0 = \frac{1}{6} H_{MNP} H^{MNP} - 2 R - 8 \nabla_M \nabla^M \Phi + 8 \nabla_M \Phi \nabla^M \Phi, \]  

(2.5)

\[ 0 = \nabla_P H^{MNP}. \]  

(2.6)

As advertised above, we set the dilaton to a constant, thus eliminating a large portion of the terms. The equations of motion for \( H \) (2.6) together with its definition in (2.3) simply imply that \( H \) should be a harmonic three-form. The metric of \text{AdS}_3 can be easily inserted. Since \text{AdS}_3 is a symmetric space the curvatures satisfy

\[ R_{\mu \nu \rho \sigma} = -\ell^{-2} (g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \rho}), \]
\[ R_{\mu \nu} = -2 \ell^{-2} g_{\mu \nu}. \]  

(2.7)  

(2.8)
Here $\ell$ is the AdS$_3$-radius. For $H$ on AdS$_3$, we can make the ansatz
\[ H_{\mu\nu\rho} = \xi \epsilon_{\mu\nu\rho}, \quad (2.9) \]
with constant $\xi$, since this is the unique harmonic three-form on AdS$_3$. Here, $\epsilon$ denotes the Levi-Civita tensor on AdS$_3$. Inserting the ansatz and solving the equations yields $\xi = 2\ell^{-1}$. The equations of motion imply furthermore that the scalar curvature of AdS$_3 \times M_7$ should vanish and thus we conclude that the scalar curvature of $M_7$ has to equal $6\ell^{-2}$. They require also that
\[ H^2 \equiv \frac{1}{6} H_{abc} H^{abc} = 4\ell^{-2}. \quad (2.10) \]
A harmonic form has always constant norm and $(2.10)$ determines the normalization. Our notations used for forms are summarized in Appendix A. We apply these conventions only to $M_7$-indices. Finally, the Einstein equation remains:
\[ R^{ab} = \frac{1}{4} H^{acd} H_{cd}. \quad (2.11) \]
We will demonstrate below that it is implied by the existence of a single Killing spinor on $M_7$.

### 2.2 Killing spinor equations

On top of the equations of motions, Killing spinor equations should be satisfied to guarantee supersymmetry of the background. There are two of these, namely the dilatino and the gravitino Killing spinor equations. The dilatino Killing spinor equation simply reads
\[ H_{MNP} \Gamma^{MNP} \epsilon = 0, \quad (2.12) \]
where $\epsilon$ is the ten-dimensional Majorana-Weyl spinor. In particular, $(2.12)$ is algebraic. We make the following ansatz for $\epsilon$:
\[ \epsilon = \chi \otimes \psi \otimes \eta, \quad (2.13) \]
where $\chi$ is a constant spinor, $\psi$ is a spinor on AdS$_3$ and $\eta$ is a spinor on $M_7$. For details of this decomposition, consult Appendix B. The Weyl-condition (B.5) in ten dimensions imposes
\[ \sigma_3 \chi = \mp \chi, \quad (2.14) \]
i.e. $\chi$ has only one non-zero component. The sign depends on the chirality of the ten-dimensional spinor. The dilatino Killing spinor equation can then be rewritten as
\[ \left( H_{abc} \gamma^{abc} + \frac{12i}{\ell} \right) \eta = 0. \quad (2.15) \]
Let us discuss now the gravitino Killing spinor equation. It reads
\[ \nabla_M \epsilon_\pm \pm \frac{1}{8} H_{MNP} \Gamma^{NP} \epsilon_\pm = 0 . \] (2.16)
Here, one of two ten-dimensional spinors satisfies the equation with a plus sign, the other with a minus sign. Let us first set \( M = \mu \). Then using (2.9), the equation becomes
\[ \nabla_\mu \psi_\pm = \pm \frac{1}{2\ell} \gamma_\mu \psi_\pm , \] (2.17)
where the spin-derivative is the one of \( \text{AdS}_3 \). The solution to this equation has been long known in the literature \[20, 21\] and is reviewed in Appendix C. The important fact is that there are exactly two independent Killing spinors on \( \text{AdS}_3 \) — one for each choice of the sign and the sign corresponds precisely to the chirality of the spinor in the boundary theory.

The remaining gravitino Killing spinor equation on \( \mathcal{M}_7 \) reads
\[ \nabla_a \eta_\pm = \pm \frac{1}{8} H_{abc} \gamma^{bc} \eta_\pm . \] (2.18)
We take the spinor \( \eta_\pm \) to be complex in the following. The ten-dimensional spinor \( \epsilon_\pm \) will be Majorana — this can always be achieved by choosing \( \psi_\pm \) appropriately and taking the real part.

### 2.3 Killing spinor equation implies Einstein equation

We now demonstrate that the Einstein equation is already implied by the existence of a single solution to (2.18) and assuming that \( H \) is harmonic, regardless of the sign choice. The logic of the reasoning follows \[22\] and exploits the integrability condition of the Killing spinor equations and some heavy use of the gamma-algebra. The precise argument is relegated to Appendix D. The main idea is however the following. We compute the expression \( \gamma^b \nabla_a [a \nabla_b \eta_-] \) twice, once using the gravitino Killing spinor equation and once using the fact that the commutator of the spinor derivatives can be expressed in terms of the Riemann tensor. Simplifying the expressions gives the following equation:
\[ R_{ab} \gamma^b \eta_- = \frac{1}{4} H_{acd} H_b^{cd} \gamma^b \eta_- . \] (2.19)
Using the fact that \( \gamma^b \eta_- \) are all independent, (2.11) follows.

### 2.4 Summary of the conditions on \( \mathcal{M}_7 \)

Let us first summarize the conditions on \( \mathcal{M}_7 \) we have to impose for \( \mathcal{N} = (2, 2) \) supersymmetry. First recall that \( \mathcal{N} = (2, 2) \) supersymmetry requires four left and four right-moving real supercharges. Solutions to the Killing spinor equations are Dirac spinors. Using the Majorana representation of the gamma-matrices of Appendix B, the gamma-matrices are purely imaginary. With this it follows that also the complex
conjugate $\eta^*$ is a solution. This can easily seen by taking the complex conjugate of the Killing spinor equations and using that

$$(\gamma^{ab})^* = \gamma^{ab}, \quad (\gamma^{abc})^* = -\gamma^{abc}. \quad (2.20)$$

Thus, every complex solution to the Killing spinor equations actually yields two real solutions. Using the results of Appendix C, the requirement of four left- and right-moving supercharges translates into the following requirements on $\mathcal{M}_7$:

(i) $\mathcal{M}_7$ must admit a harmonic three-form $H$, i.e. $H^3(\mathcal{M}_7; \mathbb{R}) \neq 0$ by Hodge’s theorem. With this three-form, we define two new covariant derivatives:

$$D_a^\pm \equiv \nabla_a \pm \frac{1}{8} H_{abc} \gamma^{bc}. \quad (2.21)$$

(ii) There exists precisely one solution for each sign to the following Killing spinor equations:

$$D_a^\pm \eta^\pm = 0, \quad \left( H_{abc} \gamma^{abc} + \frac{12i}{\ell} \right) \eta^\pm = 0. \quad (2.22)$$

In the following, we shall only work with one such solution and we set $\eta \equiv \eta_-$. This will be strong enough to draw many conclusions.

### 3 Constraining the geometry with spinor bilinears

#### 3.1 Possible forms

We apply the popular strategy of G-structures to further analyse the constraints imposed by the existence of Killing spinors in the spirit of [10, 19, 23–27]. Partially, the obtained results appeared already in [19]. The seven-dimensional Killing spinors are Dirac, so we can form the following bilinears:

$$C \equiv \eta^\dagger \eta, \quad (3.1)$$

$$K_a \equiv \eta^\dagger \gamma_a \eta, \quad (3.2)$$

$$Y_{ab} \equiv i \eta^\dagger \gamma_{ab} \eta, \quad (3.3)$$

$$Z_{abc} \equiv i \eta^\dagger \gamma_{abc} \eta. \quad (3.4)$$

Using the properties of the gamma-matrices under taking the adjoint, one may easily check that these forms are real. We show below that $C$ is constant and we normalize $\eta$ such that $C \equiv 1$.

We may also define some complex forms:

$$\Phi \equiv \eta^T \eta, \quad (3.5)$$

$$\Omega_{abc} \equiv i \eta^T \gamma_{abc} \eta. \quad (3.6)$$
Other forms vanish since we chose the gamma-matrices to be in the Majorana-representation, see Appendix B. Similarly to $C$, $\Phi$ is constant, as we shall see below. We now show that we can redefine the Killing spinor in such a way that we can also choose $\Phi \equiv 0$. For this, write
\[
\eta = \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2) ,
\]
where $\eta_1$ and $\eta_2$ are two Majorana spinors. Now define
\[
\eta_1' = \eta_1 , \quad \eta_2' = \eta_2 - \frac{\eta_2^T \eta_1}{\eta_1^T \eta_1} \eta_1
\]
and then normalize as
\[
\eta_1'' = \frac{\eta_1'}{\sqrt{(\eta_1')^T \eta_1'}}, \quad \eta_2'' = \frac{\eta_2'}{\sqrt{(\eta_2')^T \eta_2'}} .
\]
These are still two Majorana spinors satisfying the Killing spinor equations, since the differential relations imply that all inner products of these two Majorana spinors are constant. Finally set
\[
\eta' \equiv \frac{1}{\sqrt{2}}(\eta_1'' + i\eta_2'')
\]
This Dirac spinor then satisfies
\[
(\eta')^T \eta' = 0 , \quad (\eta')^\dagger \eta' = 1 .
\]
So we may without loss of generality assume that $\Phi \equiv 0$ and $C \equiv 1$.

No higher forms exist, since they are always the Hodge dual of the forms $K$, $Y$, $Z$ and $\Omega$. These forms are by no means independent, they satisfy many relations which we will explore systematically in the following. There are three types of identities between the forms. The first is of an algebraic nature and arises by multiplying two of the forms and using the Fierz identity on the terms of the form $\eta \eta^\dagger$ or $\eta \eta^T$. This will allow us to relate products of forms to other forms. The second kind involves the dilatino Killing spinor equation and is also algebraic. Moreover, it contains also $H$. The third kind of identity involves taking a covariant derivative of the definition of the forms and using the gravitino Killing spinor equations to write the result in terms of the forms and $H$.

### 3.2 Fierz identities

We will in the following list all Fierz identities. They are ordered by the number of free indices or equivalently by the number of gamma-matrices inserted. Some of the Fierz identities with a lower number of free indices are implied by those with a larger number of free indices, but we list all for completeness. Furthermore, we only
list the identities up to three free indices. Beyond this, all identities can be obtained
by Hodge-dualizing those we have presented.

There is one Fierz identity, which deserves special attention. Namely, there is
an identity with three free indices which simply states

\[ Z = K \wedge Y . \]  

(3.12)

Thus, \( Z \) is not an independent form and we have inserted at most places where it
appears \( K \wedge Y \) instead. The other Fierz identities establish relations between the
other forms, but they do not allow one form to be written entirely in terms of the
others. Some conventions regarding the forms are summarized in Appendix A.

No free index:

\[ K^2 = 1 \, , \quad Y^2 = 3 \, , \quad |\Omega|^2 = 8 \, , \quad \Omega^2 = 0 \, , \quad \Omega_{abc}K^a Y^{bc} = 0 . \]  

(3.13)

One free index:

\[ \iota_K Y = 0 \, , \]  

(3.14)

\[ \star (\overline{\Omega} \wedge \Omega) = 8i K \, , \]  

(3.15)

\[ \iota_Y \Omega = 0 \, . \]  

(3.16)

Two free indices:

\[ Y_{a} Y_{b}^{c} = g_{ab} - K_{a} K_{b} \, , \]  

(3.17)

\[ \overline{\Omega}_{a}^{cd} \Omega_{bcd} = -4i Y_{ab} - 4 K_{a} K_{b} + 4 g_{ab} \, , \]  

(3.18)

\[ \Omega_{a}^{cd} \Omega_{bcd} = 0 \, , \]  

(3.19)

\[ \star (Y \wedge \Omega) = 0 \, , \]  

(3.20)

\[ \iota_K \Omega = 0 \, . \]  

(3.21)

Three free indices:

\[ \epsilon_{abcdefg} \overline{\Omega}_{a}^{deh} \Omega_{b}^{fg} = -32 (K \wedge Y)_{abc} \, , \]  

(3.22)

\[ \star (Y \wedge Y) = -2K \wedge Y \, , \]  

(3.23)

\[ \epsilon_{abcdefg} \Omega_{c}^{deh} \Omega_{fg}^{h} = 0 \, , \]  

(3.24)

\[ \star (K \wedge \Omega) = -i \Omega \, , \]  

(3.25)

\[ Y_{[a} d \Omega_{bc]d} = -i \Omega_{abc} \, . \]  

(3.26)

3.3 Dilatino Killing spinor equations

To fully exploit the dilatino Killing spinor equation (2.15), we note that we have the
series of identities

\[ H_{abc} \eta^\dagger \gamma^{d_1 \cdots d_k} \gamma_{abc} \eta = -\frac{12i}{\ell} \eta^\dagger \gamma^{d_1 \cdots d_k} \eta \, , \]  

(3.27)

\[ H_{abc} \eta^T \gamma^{d_1 \cdots d_k} \gamma_{abc} \eta = -\frac{12i}{\ell} \eta^T \gamma^{d_1 \cdots d_k} \eta \]  

(3.28)
for every $k$. One may then use the gamma-algebra to write the left hand side as a linear combination of known forms. We will refer to the first line as real equations (involving $K$, $Y$ and $H$) and the second one (involving $\Omega$ and $H$) as complex equations. Real equations give always two equations, since their real and imaginary parts have to vanish separately. This is because the involved fields are all real.

We will again order the equations by the number of free indices they have. Recall that $Z = K \wedge Y$ in all of the following equations.

**Real equations**

One free index:

\[
\iota_Y H = 2\ell^{-1}K ,
\]

\[
\star(H \wedge Z) = 0 .
\] (3.29)

Two free indices:

\[
Y = \frac{1}{2}\ell (\iota_K H - \star(H \wedge Y)) ,
\]

\[
0 = H_{[a \, cd}Z_{b]cd} .
\] (3.30)

Three free indices:

\[
Z_{abc} = \frac{1}{2}\ell \left( H_{abc} - \frac{3}{4}\epsilon_{[ab|defgh]}H_{c]}^{de}K^{f}Y^{gh} \right) ,
\]

\[
0 = 3H_{[ab \, \, d}Y_{c]d} - \star(H \wedge K)_{abc} .
\] (3.31)

**Complex equations**

No free index:

\[
\iota_\Omega H = 0 .
\] (3.32)

One free index:

\[
\star(H \wedge \Omega) = 0 .
\] (3.33)

Two free indices:

\[
0 = H_{[a \, cd}\Omega_{b]cd} .
\] (3.34)

Three free indices:

\[
\Omega_{abc} = -\frac{1}{8}\ell \epsilon_{[ab|defgh]}H_{c]}^{de}\Omega^{fgh} .
\] (3.35)

**3.4 Gravitino Killing spinor equations**

Finally, we make use of the gravitino Killing spinor equation (2.18)

\[
\nabla_a \eta = \frac{1}{8}H_{abc}\gamma^{bc}\eta
\] (3.36)
to derive differential relations among the different forms. The differentials read:

\[
\nabla_a C = 0 , \\
\nabla_a \Phi = 0 , \\
\n\nabla a K^b = \frac{1}{2} (\iota_K H)_{ab} , \\
\n\nabla_a Y_{bc} = - H_{a[b} Y_{c]d} , \\
\n\nabla a \Omega_{bcd} = \frac{3}{2} H_{a[b} \Omega_{cd]e} .
\]

The first relations (3.40) and (3.41) obviously imply that $C$ and $\Phi$ are constant. As argued above, we can redefine $\eta$ in such a fashion that $C \equiv 1$ and $\Phi \equiv 0$.

The next relation (3.42) implies that $K$ is a Killing vector

\[
\nabla_{(a} K_{b)} = 0 .
\]

The fact that $K$ is a Killing vector means that there is a continuous symmetry associated to the existence of the Killing spinor. Looking ahead, this has to be the U(1)-symmetry of the $\mathcal{N} = 2$ superconformal symmetry, since there is no other candidate.\(^3\)

For the antisymmetric part of the differential of $K^4$ we obtain:

\[
dK = \iota_K H ,
\]

so upon taking another exterior derivative and using Cartan’s relation as well as $H$ being closed, we obtain:

\[
\mathcal{L}_K H = 0 .
\]

Thus, $K$ preserves not only the metric, but all supergravity fields. Thus it is an isometry in the sense of generalized geometry. This construction is essentially the one of the Killing superalgebra [28].

3.5 Writing $\mathcal{M}_7$ as a U(1)-fibration

The existence of the Killing vector $K$ allows us to choose local coordinates such that

\[
K = \frac{\partial}{\partial \psi} .
\]

The metric then has the form

\[
ds_7^2 = e^{2\phi} (d\psi + B)^2 + ds_6^2 .
\]

\(^3\)Note that for a real spinor, this Killing vector would vanish, matching the fact that $\mathcal{N} = 1$ supersymmetry has no R-symmetry.

\(^4\)Here and in the following, we identify vectors with one-forms.
Here, $ds^2_6$ is the metric on a six-dimensional manifold, which we call $\mathcal{M}_6$. $B$ and $\phi$ are a one-form and a function on $\mathcal{M}_6$, respectively. We have $K = B + d\psi$. Calculating the norm of $K$ with this metric gives $e^{2\phi}$. We saw however in (3.13) that the norm of $K$ is one. Hence we conclude that $\phi \equiv 0$ and the metric is of the form

$$ds^2_7 = (d\psi + B)^2 + ds^2_6.$$  (3.50)

In other words, $\mathcal{M}_7$ is a U(1)-fibration over $\mathcal{M}_6$. Many fields will turn out to actually live on $\mathcal{M}_6$. Latin indices $i, j, \ldots$ will refer to coordinates on $\mathcal{M}_6$.

### 3.6 Inner products and Lie derivatives

We have $\iota_K \omega = 0$ and $\iota_K \Omega = 0$ by the Fierz identities (3.14) and (3.21). Moreover, it follows straightforwardly from (3.43) and (3.34) that

$$dY = \iota_K \ast H,$$  (3.51)

which implies by Cartan’s relation that $\mathcal{L}_K Y = 0$. Thus, $Y$ is a field living on $\mathcal{M}_6$.

It is true that $\iota_K \Omega = 0$, but $\mathcal{L}_K \Omega \neq 0$ in general. Thus, also $\Omega$ lives on $\mathcal{M}_6$, but it still depends on the coordinate $\psi$. However, one can derive from (3.44) and (3.38) that

$$d\Omega = -2i \ell^{-1} (K \wedge \Omega).$$  (3.52)

Hence, we can define $\omega \equiv e^{2i \ell^{-1} \psi} \Omega$. This form then satisfies

$$d\omega = 2i \ell^{-1} d\psi \wedge \omega - 2i \ell^{-1} K \wedge \omega = -2i \ell^{-1} B \wedge \omega.$$  (3.53)

So it follows $\mathcal{L}_K \omega = 0$, since $\iota_K B = 0$. Thus $\omega$ is also a field living purely on $\mathcal{M}_6$. Note that the multiplication by a phase does not change any of the above algebraic identities. So in the following, it will be more convenient to work with $\omega$ instead of $\Omega$.

### 3.7 Almost complex structure and holomorphic 3-form

We define now

$$J_{ab} \equiv -Y_{ab},$$  (3.54)

then the Fierz identity (3.17) directly implies that $J_{ij}$ defines an almost complex structure on $\mathcal{M}_6$. Furthermore, we have the following set of Fierz identities, which follow from repeatedly applying (3.26):

$$J_d^d [a \omega_{bc}]d = i \omega_{abc},$$  (3.55)

$$J_d^d [a J^b_b \omega_{cd}e] = -\omega_{abc},$$  (3.56)

$$J_d^d J^b_a J^c_f \omega_{def} = -i \omega_{abc}.$$  (3.57)
which implies that
\[ \omega_{\text{def}} (\bar{\delta}_a^d + \lambda J_a^d) (\bar{\delta}_b^e + \lambda J_b^e) (\bar{\delta}_b^e + \lambda J_b^e) = (1 + i \lambda)^3 \omega_{\text{def}} , \] (3.58)

and hence \( \Omega \) defines a holomorphic \((3,0)\)-form on \( \mathcal{M}_6 \). \( J \) is compatible with the metric induced by it, which is equivalent [29] to
\[ J \wedge \omega = 0 . \] (3.59)

This is precisely the Fierz identity (3.20). \( \omega \) then defines an SU(3)-structure on \( \mathcal{M}_6 \). As shown e.g. in [29], the fact that
\[ d\omega = -2i \ell^{-1} B \wedge \omega \] (3.60)

implies that the almost complex structure is integrable and hence complex. Note also that \( \omega \) is normalized in the standard fashion [29]:
\[ \omega \wedge \bar{\omega} = -\frac{4i}{3} J \wedge J \wedge J . \] (3.61)

### 3.8 Torsion classes

Let us recall the definition of torsion classes [30–32]:
\[ dJ = -\frac{3}{2} \Im(\bar{W}_1 \omega) + W_4 \wedge J + W_3 , \] (3.62)
\[ d\omega = W_1 J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \omega . \] (3.63)

Clearly \( W_1 = 0, W_2 = 0 \) and \( W_4 = 0 \). For the other forms we have
\[ W_3 = \iota_K \ast H , \quad W_5 = 2i \ell^{-1} B . \] (3.64)

As required, (3.14) and (3.29) imply that \( W_3 \) satisfies the primitivity condition
\[ W_3 \wedge J = 0 . \] (3.65)

It follows from (3.14), (3.29) and (3.51) that
\[ d(J \wedge J) = 0 \] (3.66)

These properties are precisely the definition of balanced manifolds [17, 18]. The situation is hence very reminiscent of type IIA or heterotic string compactifications. A non-vanishing \( H \)-field led also in this case to a non-Kähler geometry [18]. It should be stressed that this analysis actually holds already for \( \mathcal{N} = (2,0) \) supersymmetry, since we have only used one chirality of Killing spinors.
3.9 The Kähler condition

Let us briefly comment on the fact that $M_6$ need not be Kähler as it was in [10]. In fact, one can easily construct an example where $M_6$ is not Kähler, namely $M_7 \cong S^3 \times S^3 \times S^1$. Then using e.g. the Serre spectral sequence one can show that $M_6$ has the cohomology of $S^3 \times S^3$ or $S^3 \times S^2 \times S^1$. However, the Betti numbers reveal that these manifolds cannot be Kähler. The background $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ has large $N = (4,4)$ supersymmetry [3–7], which contains in particular also an $N = (2,2)$ superconformal algebra. This shows that in general the manifold $M_6$ will not be Kähler.

However, for small $N = (4,4)$ supersymmetry, the relevant manifolds are $M_6 = S^2 \times T^4$ and $M_6 = S^2 \times K3$, which are both Kähler. Those examples are much better understood. Also from the perspective pursued here, these examples are much simpler, since $K \wedge H = 0$ for dimensional reasons. So again from (3.64), it follows that they are Kähler.

3.10 Trivial fibration

We would like to comment on another curious fact, namely we show that a trivial $U(1)$-fibration is not possible. Indeed, if the fibration is trivial, $K$ is covariantly constant, yielding a cohomology class representing the trivial circle. We then have $dK = 0$ and hence $\iota_K H = 0$ by (3.42). Then it follows from (3.30) that $H \wedge Y = 0$, whence (3.31) gives $Y = 0$. This is obviously a contradiction to (3.17). In fact, when looking at the list of [12], one sees that all trivial fibration solutions have to carry non-trivial five-form flux.

4 The Kähler case

We will now consider the case where $M_6$ is Kähler, which amounts to the condition $K \wedge H = 0$, since then $J$ is closed. We will refer to these backgrounds in the following as ‘Kähler backgrounds’.

4.1 Further conditions imposed by $H$

The condition allows us also immediately to express $H$ in terms of the other fields. For this, define

$$
\Psi \equiv H - K \wedge \iota_K H ,
$$

then $\Psi$ satisfies $K \wedge \Psi = 0$ and $\iota_K \Psi = 0$. But this implies

$$
\Psi = \iota_K (K \wedge \Psi) = 0 .
$$

Hence

$$
H = K \wedge \iota_K H = K \wedge dK = K \wedge dB ,
$$

...
where we used (3.42).

Furthermore, the form $dB$ may be identified with the Ricci-form $\rho$ of the Kähler manifold $[10, 31]$. Thus, $B$ is fixed by $\rho$, so in particular the $U(1)$-fibration is uniquely fixed up to the addition of a parallel vectorfield.\(^5\) This again demonstrates that the fibration cannot be trivial, since $dB$ is non-vanishing. Thus, we have

$$H = K \wedge \rho \, .$$  \hspace{1cm} (4.5)

The Ricci-form is also of type $(1, 1)$. Finally, we also need to require $H$ to be closed, since this does not follow from the conditions we have imposed so far. Since $dK = \rho$, we need to require

$$\rho \wedge \rho = 0 \, .$$  \hspace{1cm} (4.6)

Finally, we remark that the norm of the Ricci form of $\mathcal{M}_6$ is constant. Indeed, we have by (2.10) and (4.5):

$$H^2 = \rho^2 = 4\ell^{-2} \, .$$  \hspace{1cm} (4.7)

4.2 Reducing to a local product of Kähler-Einstein spaces

Let us diagonalize $\rho$ when viewed as an endomorphism acting on the tangent space. Since $\rho$ is a $(1, 1)$-form, it has the standard form

$$\rho = \sum_{i=1}^{3} \lambda_i dz_i \wedge d\overline{z}_i \, ,$$  \hspace{1cm} (4.8)

for some choice of coordinates $z_1$, $z_2$ and $z_3$. The condition $\rho \wedge \rho = 0$ implies then that only one of the three eigenvalues $\lambda_1$, $\lambda_2$ and $\lambda_3$ can be non-zero, say $\lambda \equiv \lambda_1$. Moreover, since the norm of $\rho$ has to be constant by (4.7), we conclude that $\lambda$ is constant. Thus, the Ricci-tensor has only constant non-negative eigenvalues.

Now we can use the theorem proved in [33] to conclude that $\mathcal{M}_6$ is locally the product of two Kähler-Einstein manifolds.\(^6\) One manifold is of dimension 2 with positive curvature, the other of dimension 4 with vanishing curvature, i.e. a Calabi-Yau manifold. In other words, we have demonstrated that $\mathcal{M}_6$ is of the form

$$\mathcal{M}_6 \cong (S^2 \times CY_2)/G \, ,$$  \hspace{1cm} (4.9)

\(^5\)The argument goes as follows. Since $dB = \rho$, there is locally a remaining gauge freedom of $B \rightarrow B + d\Lambda$ for some function $\Lambda$. Switching our view to $\mathcal{M}_7$, $K$ is fixed up to $K \rightarrow K + d\Lambda$. However, $K$ is required to be Killing, so the function $\Lambda$ must also satisfy

$$\nabla_{(a} \nabla_{b)} \Lambda = \nabla_a \nabla_b \Lambda = 0 \, .$$  \hspace{1cm} (4.4)

Thus, $d\Lambda$ is a parallel vectorfield. This is quite restrictive, and we easily see that there are no more conditions.

\(^6\)This is actually the only place where we are using the assumption that $\mathcal{M}_7$ is compact.
where $G$ is some group of isometries preserving all the relevant structures. Furthermore, for $\mathcal{M}_6$ to be smooth, $G$ has to act freely. Here we used that $S^2$ is the only two-dimensional Kähler-Einstein manifold with positive Kähler-Einstein constant. Finally, we can change our viewpoint again to $\mathcal{M}_7$. Namely, since the $U(1)$-fibration is parametrized by $\rho$, it is actually a fibration over the two-sphere $S^2$, up to the aforementioned ambiguity of adding a parallel vectorfield. To eliminate this possibility, we make a case-by-case analysis. There are two possible choices for $CY_2$:

1. $CY_2 = T^4$. Choosing the coordinates in the appropriate fashion, we can assume that we have a $U(1)$-fibration over $S^2 \times S^1$. However, through the canonical isomorphism $H^2(S^2 \times S^1; \mathbb{Z}) \cong H^2(S^2; \mathbb{Z})$ and using the fact that the second cohomology group classifies $U(1)$-bundles, we see that all $U(1)$-fibrations are actually only over $S^2$. Thus, in this case the freedom of adding a parallel vectorfield is trivial.

2. $CY_2 = K3$. Since $K3$ has no parallel vectorfields, this question does not arise.

As we have demonstrated above, the fibration cannot be trivial, thus it must be the Hopf-fibration over $S^2$. This can also be seen explicitly, since we have now uniquely fixed $K$.

Thus, we finally conclude that $\mathcal{M}_7$ is a finite quotient of $S^3 \times T^4$ or $S^3 \times K3$. It is very well-known that $\mathcal{M}_7 \cong S^3 \times CY_2$ leads to $N = (4, 4)$ supersymmetry, so the group action has to be non-trivial. This also finally demonstrates that the requirements we imposed were sufficient, since $S^3 \times CY_2$ satisfies the supergravity equations.

4.3 The case of $\mathcal{M}_7 \cong S^3 \times CY_2$

To continue, it is advantageous to have a good understanding of $\mathcal{M}_7 \cong S^3 \times T^4$, so we review here the background following [15]. We have $H \propto \text{vol}_{S^3}$ in this case. Thus the above gravitino Killing spinor reduces to the standard one on $S^3$, while on $T^4$ we are searching for parallel Killing spinors. $H_{abc} \gamma^{abc}$ commutes with all gamma-matrices on $S^3$, but anticommutes with all on $T^4$. Thus, the dilatino spinor equation imposes a definite chirality on $T^4$.

It is a mathematical fact that Killing spinors with non-vanishing Killing constant are in one-to-one correspondence with parallel Killing spinors on the Riemannian cone [34]. The chirality of the spinor on the cone translates into the sign of the Killing constant. For the case of $S^3$, its Riemannian cone is $\mathbb{R}^4$, so the problem simply reduces to finding parallel Killing spinors on $\mathbb{R}^4 \times T^4$. In addition, they have to satisfy the dilatino Killing spinor equation.

Now standard counting tells us that $\mathbb{R}^4 \times T^4$ possesses $2^4 = 16$ parallel spinors, 8 of them satisfy the chirality constraint on $T^4$. Further, 4 have negative chirality.
on \( \mathbb{R}^4 \), while 4 have positive chirality. Thus, we see that this leads to \( \mathcal{N} = (4, 4) \) supersymmetry.

The case of \( \mathcal{M}_7 \cong S^3 \times K3 \) works in essentially the same way. Here the Killing spinors already have a definite chirality on K3, since the holonomy group is SU(2). Thus, the dilatino equation is superfluous, and the same argument as before yields 4 negative chirality and 4 positive chirality spinors on \( \mathbb{R}^4 \), so we get again \( \mathcal{N} = (4, 4) \) supersymmetry.

4.4 All Kähler possibilities

We will now systematically explore all possibilities of quotients of \( S^3 \times CY_2 \) which preserve \( \mathcal{N} = (2, 2) \) supersymmetry. For this, let \( G \subset \text{Isom}(S^3 \times CY_2) \) some group of isometries by which we want to quotient. Obviously,

\[
\text{Isom}(S^3 \times CY_2) \cong O(4) \times \text{Isom}(CY_2) ,
\]

so we may look on the action on \( S^3 \) and \( CY_2 \) separately. To keep things simple, we only consider actions which are orientable on both \( S^3 \) and on \( CY_2 \), since otherwise supersymmetry turns out to be completely broken. The spin double cover of SO(4) is SU(2) \( \times \) SU(2), where the two factors correspond to the two different chiralities. Clearly the group action has to be non-trivial in both factors, since otherwise we would not obtain \( \mathcal{N} = (2, 2) \) supersymmetry.

Let us first consider a cyclic subgroup of the group of isometries. We can choose the coordinates in such a way that the \( S^3 \)-part (or rather its lift to the spin-bundle) lies in the standard Cartan-subalgebra \( U(1) \times U(1) \subset SU(2) \times SU(2) \). Now we claim that the \( S^3 \)-action actually lies in the diagonal or anti-diagonal combination of \( U(1) \times U(1) \), or it lies entirely in one of the \( U(1) \)'s. If this would not be the case, the group element had four different eigenvalues on the spinor representation \((2,1) \oplus (1,2)\). There is an additional phase which can be produced by the action of the group element on \( CY_2 \), but it is the same for all four states in the representation. Thus, at least three states get projected out, and we remain at most with \((2,0)\)-supersymmetry. The case where the isometries lie completely in one of the \( U(1) \)'s can be discarded, since it destroys one chirality of spinors completely and leaves the other untouched. Thus, it is associated with \((4,0)\)-supersymmetry, as discussed in [35]. Without loss of generality, we may assume that the cyclic subgroup lies in the diagonal \( U(1) \subset U(1) \times U(1) \). This however means in the SO(4)-language that the action on \( S^3 \) is given by a rotation. In particular it has a fixed point.

Each group element has to act fix-point free on \( S^3 \) or on \( CY_2 \) for the quotient space to be smooth. We have however seen above that each group element has to act with fixed points on \( S^3 \). Consequently the action on \( CY_2 \) must be free. This in turn implies that the quotient space \( CY_2/G \) is a Calabi-Yau manifold in the weak sense. This means that it is only required to have a vanishing first Chern-class in real
cohomology, but not in integer cohomology. As a consequence, these manifolds are actually not spin manifolds — only the complete $\mathcal{M}_7$ will be a spin manifold. This is extremely restrictive and these quotients are all classified by mathematicians. A standard reference is [36]. There are two classes, belonging to $\mathbb{T}^4$ and $\text{K3}$, respectively.

$\text{K3}$ has a unique such quotient, called the Enriques surface. We will denote the Enriques surface by $\text{ES}$. The quotient group is a $\mathbb{Z}_2$. The Enriques surface has Hodge-diamond

\[
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & 10 & 0 & \\
0 & 0 & & \\
& & & 1
\end{array}
\]

(4.11)

$\mathbb{T}^4$ has a family with seven members of such quotients, which go by the name of (irregular) hyperelliptic surface. They are also called bi-elliptic surfaces, since they admit an elliptic fibration over an elliptic curve. Thus, they are best viewed as a finite quotient of a product of an elliptic curve $E = \mathbb{C}/\Gamma$ with an elliptic curve $C \cong S^1 \times S^1$. We will denote them generically by $\text{HS}$ and write e.g. $\text{HS}_{b_2}$ to indicate a specific one. They have all the Hodge-diamond

\[
\begin{array}{cccc}
1 & & & \\
1 & 1 & & \\
0 & 2 & 0 & \\
1 & 1 & & \\
& & & 1
\end{array}
\]

(4.12)

For the convenience of the reader, we have listed the different possibilities for the group actions in Table 1. Further properties of surfaces are presented in Appendix E. As a last step, we have to determine the corresponding actions of the quotient groups on $S^3$. For the Enriques surface, this is immediate, since we argued before that every group element acting on $S^3$ has to lie in the diagonal or anti-diagonal $U(1)$. As there is only one non-trivial group element, it can either act trivially or by a rotation by $\pi$ around one axis. The former is not possible, since then $\mathcal{M}_7 \cong S^3 \times \text{ES}$, which is not a spin-manifold. Thus, the action on $S^3$ is uniquely determined. The eigenvalues of the $U(1) \times U(1)$ on the spin representation $(2, 1) \oplus (1, 2)$ are given by $(i, -i)$ for both the left and right chirality. Thus, precisely half of the left and right chirality Killing spinors survive the projection, since the eigenvalues on the Enriques surface are either $i$ or $-i$, depending on the choice of the lift of the group action to the spinor-bundle.

We can similarly argue for the hyperelliptic surfaces a 1) and b 1), where the group actions are given by a rotation by $\pi$ and $2\pi/3$, respectively. For the hyperelliptic surfaces c 1) and d 1), the same argument reveals again that the group actions

\[\text{Not to be confused with hyperelliptic Riemann surface.}\]
| Type | Γ           | $G$        | Action of $G$ on $E$ |
|------|-------------|------------|---------------------|
| a 1) | arbitrary   | $\mathbb{Z}_2$ | $\lambda \mapsto -\lambda$ |
| a 2) | arbitrary   | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\lambda \mapsto -\lambda$  

$\lambda \mapsto \lambda + \mu$ with $-\mu = \mu$ |
| b 1) | $\mathbb{Z} \oplus \mathbb{Z}_\omega$ | $\mathbb{Z}_3$ | $\lambda \mapsto \omega \lambda$ |
| b 2) | $\mathbb{Z} \oplus \mathbb{Z}_\omega$ | $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ | $\lambda \mapsto \omega \lambda$  

$\lambda \mapsto \lambda + \mu$ with $\omega \mu = \mu$ |
| c 1) | $\mathbb{Z} \oplus \mathbb{Z}_i$ | $\mathbb{Z}_4$ | $\lambda \mapsto i \lambda$ |
| c 2) | $\mathbb{Z} \oplus \mathbb{Z}_i$ | $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ | $\lambda \mapsto i \lambda$  

$\lambda \mapsto \lambda + \mu$ with $i \mu = \mu$ |
| d)   | $\mathbb{Z} \oplus \mathbb{Z}_\omega$ | $\mathbb{Z}_6$ | $\lambda \mapsto -\omega \lambda$ |

**Figure 1.** The classification of bi-elliptic surfaces. $\omega$ is a cubic unit root. This table is taken from [36].

are given by rotations. However the angle is no longer uniquely fixed. Looking again at the eigenvalues of the group action on the Killing spinors, we see that the angles must be $\pi/2$ and $\pi/3$, respectively, since otherwise no supersymmetry would survive.

Finally, we come to the surfaces of type 2). We see that the second generator acts trivially on the Killing spinors, hence to preserve supersymmetry, it also has to act trivially on $S^3$. Thus, the action on $S^3$ for the surfaces of type 2) are precisely the same as those for their type 1) counterparts. This completes the classification of backgrounds coming from Kähler geometries. We have seen that the action on $S^3$ is in all cases uniquely fixed.

Moreover, the example of the hyperelliptic surface which was mentioned in [15] fits into this classification. It is given by the hyperelliptic surface of type a 1).

### 4.5 Induced action on $S^3$

In this subsection, we will see that the action on $S^3$ is actually very natural. For this, we remember that the background $\text{AdS}_3 \times S^3 \times \text{CY}_2$ supports small $\mathcal{N} = (4, 4)$ superconformal symmetry. Thus, in particular, it has a spacetime $\text{SU}(2) \times \text{SU}(2)$-symmetry, which is simply given by rotations of $S^3$.

We now remember the AdS/CFT-correspondence for $\text{AdS}_3 \times S^3 \times \text{CY}_2$. It states that supergravity on this background lies on the same moduli space as the infinite symmetric product CFT

$$\text{Sym}^\infty(\text{CY}_2).$$

All CY$_2$ manifolds are actually hyperkähler manifolds and hence also support $\mathcal{N} = (4, 4)$ superconformal symmetry. If we now act by some isometries on CY$_2$, we consequently get an induced action on the $\text{SU}(2) \cong S^3$-current algebra the theory supports. By the AdS/CFT correspondence, we expect this action to be precisely the
one we determined in the previous section by brute-force. Since the quotients of CY$_2$ we are considering are still Kähler and Ricci-flat, they still support an $\mathcal{N} = (2,2)$ superconformal field theory. Thus, we conclude that the action on the SU(2)-current has to leave invariant an U(1) $\subset$ SU(2). So the remaining group of automorphisms is only a U(1), in other words, the group acts by rotations on S$^3$.

5 The BPS spectra of the Kähler backgrounds

We have established that all Kähler $\mathcal{N} = (2,2)$ backgrounds are of the form

$$\text{AdS}_3 \times (S^3 \times \text{CY}_2)/G,$$

where a complete list of the possibilities was provided in the last Section. It is the next logical step to compute the supergravity and BPS spectra of these backgrounds.

Since the backgrounds inherit many properties from their $\mathcal{N} = (4,4)$ cousins, we can use the techniques of [37]. For this, we use the fact that the states are still secretly sitting in $\mathcal{N} = (4,4)$ multiplets, but some states of the multiplets are projected out. We have collected some relevant background for this in Appendix F. We have already applied a similar technique in [15]. We will denote by $(m,n)^\alpha$ a modified SU(2) $\times$ SU(2)-multiplet, where $\alpha$ is a unit root of the order of the cyclic group action on S$^3$. Furthermore, we denote by $(m,n)^g_{\alpha}$ a short modified $\mathcal{N} = (4,4)$ multiplet. The refinement of the $\mathcal{N} = (4,4)$ multiplets with insertions of $\alpha$ helps us to keep track of the transformation properties under G.

5.1 The Enriques surface

Let us first begin with the K3 case and the associated Enriques surface. In this case $\alpha$ is a second root of unity, since the group is $\mathbb{Z}_2$. In the following we let $\alpha$ be a formal variable satisfying $\alpha^2 = 1$. The action of $\mathbb{Z}_2$ on the Hodge-diamond of K3 is

$$
\begin{array}{ccc}
1 & & \\
0 & 0 & \\
\alpha & 10(1 + \alpha) & \alpha \\
0 & 0 & \\
1 & & \\
\end{array}
$$

(5.2)

The invariant part is the constant part in $\alpha$, which gives the Hodge-diamond of the Enriques surface (4.11). Following [37], we first compactify to six dimensions and perform subsequently the Kaluza-Klein reduction on S$^3$. During this procedure, we keep track of the eigenvalues of the projection and in the end we only keep invariant states. Furthermore, it will suffice to determine the bosonic field content, since the fermionic fields will be fixed by $\mathcal{N} = (2,2)$ supersymmetry. Compactifying to six dimensions, we obtain the bosonic field content indicated in Table 2. It is not
| type               | number                          |
|-------------------|---------------------------------|
| scalar            | $27 + 20\alpha + \text{dim}(\mathcal{M}_{\text{ES}})$ |
| vector            | 0                               |
| self-dual 2-form  | $3 + 2\alpha$                  |
| anti self-dual 2-form | $11 + 10\alpha$             |
| metric            | 1                               |

**Figure 2.** Six-dimensional fields after the compactification on the Enriques surface. We included also the number of odd fields under the projection, they can still contribute to the three-dimensional field content. To fix whether the six-dimensional two-forms are self- or anti self-dual, we used the signatures of the Enriques surface and K3, see Appendix E.

necessary to determine the dimension of the moduli space of string compactifications $\mathcal{M}_{\text{ES}}$ from first principles — it will also be fixed by $\mathcal{N} = (2, 2)$ supersymmetry.

In the next step, we perform the KK-reduction on the sphere $S^3$. The quotient has the effect of replacing the standard multiplets $(\mathbf{m}, \mathbf{n})$ by the twisted multiplets $(\mathbf{m}, \mathbf{n})^\alpha$, for more details on those consult Appendix F. In this case, $\alpha$ has order two and hence we have to decide whether we replace the multiplet $(\mathbf{m}, \mathbf{n})$ by $(\mathbf{m}, \mathbf{n})^\alpha$ or by $\alpha(\mathbf{m}, \mathbf{n})^\alpha$. The answer is simple: Even spin particles are clearly invariant under the group action on $S^3$, whereas odd spin particles are not. Thus vectors will be multiplied by an additional $\alpha$ in the end. Hence, following [37], the three-dimensional bosonic field content is

$$\bigoplus_{\mathbf{m}} (\mathbf{m}, \mathbf{m} \pm 4)^\alpha \oplus (12 + 16\alpha)(\mathbf{m}, \mathbf{m} \pm 2)^\alpha \oplus (44 + 32\alpha + \text{dim}(\mathcal{M}_{\text{ES}}))(\mathbf{m}, \mathbf{m})^\alpha \, .$$

This can be uniquely fitted into modified $\mathcal{N} = (4, 4)$ multiplets as described in Appendix F with the result

$$\bigoplus_{\mathbf{m}} \alpha(\mathbf{m}, \mathbf{m} \pm 2)_S^\alpha \oplus (12 + 10\alpha)(\mathbf{m}, \mathbf{m})_S^\alpha \, .$$

It is clear that there will be some exceptional cases for small values of $\mathbf{m}$, which we have not treated here. This fixes also uniquely the dimension of the moduli space of the compactification:

$$\text{dim}(\mathcal{M}_{\text{ES}}) = 26 + 32\alpha \, .$$

In particular, the chiral-chiral BPS spectrum reads

$$\bigoplus_{\mathbf{m}=0}^{\infty} 12(\mathbf{m}, \mathbf{m}) \, .$$

---

8This can also be seen in a less hand-wavy manner. The representations we wrote down are SO(4)-representations. A rotation by 180 degrees can be represented by the element \(\text{diag}(-1, -1, 1, 1)\) in SO(4). This is in the Cartan-torus and the sign picked up under this rotation is then \((-1)^{\frac{1}{2}(n-m)}\). Hence we conclude again that vectors receive an additional $\alpha$, whereas the other fields are invariant.
| type               | number                                        |
|--------------------|-----------------------------------------------|
| scalar             | $4 \alpha^{-1} + 7 + 4\alpha + \dim(\mathcal{M}_{\text{HS}})$ |
| vector             | $4(\alpha^{-1} + 2 + \alpha)$                |
| self-dual 2-form   | $3 + 2\alpha$                                |
| anti self-dual 2-form | $2\alpha^{-1} + 3$                         |
| metric             | $1$                                           |

**Figure 3.** Six-dimensional fields after the compactification on the hyperelliptic surface.

We can also extract the chiral-antichiral ring:

$$\bigoplus_{m=0\text{ even}} \infty (m, -m \pm 2) \oplus 10(m, -m) \oplus \bigoplus_{m=0\text{ odd}} \infty 12(m, -m) .$$  \hspace{1cm} (5.7)

There is clearly a quite non-trivial structure in these invariants.

### 5.2 The hyperelliptic surface

We now repeat the analysis for the hyperelliptic surface. The $\mathbb{Z}_n$-action\(^9\) on the Hodge-diamond of $\mathbb{T}^4$ is now

$$\begin{array}{cccccc}
1 & & & & 1 + \alpha^{-1} & 1 + \alpha \\
\alpha & 2 + \alpha + \alpha^{-1} & \alpha^{-1} & \alpha^{-1} & 1 + \alpha^{-1} & 1 \\
1 + \alpha & & & & & 1
\end{array} .$$  \hspace{1cm} (5.8)

The six-dimensional field content is then determined as before and is collected in Table 3. Now we can perform the KK-reduction as before. Fixing the sign is a bit trickier as before, since $\alpha$ does not necessarily square to one. However, the argument of footnote 8 still works and the prefactor of the representation $(m, n)$ is $\alpha^\frac{n}{2}(n-m)$.

$$\bigoplus_m \alpha^{\pm 2}(m, m \pm 4)^\alpha \oplus \alpha^\pm (6\alpha^{-1} + 16 + 6\alpha) (m, m \pm 2)^\alpha$$

$$\oplus (14\alpha^{-1} + 32 + 14\alpha + \dim(\mathcal{M}_{\text{HS}}))(m, m)^\alpha .$$  \hspace{1cm} (5.9)

Again, we can fit this uniquely into modified multiplets with the result

$$\bigoplus_m \alpha^{\pm 1}(m, m \pm 2)^{\alpha}\oplus 2(1 + \alpha^{\pm 1})(m, m \pm 1)^{\alpha}\oplus (\alpha^{-1} + 4 + \alpha)(m, m)^{\alpha} .$$  \hspace{1cm} (5.10)

Again, there are some exceptional cases at low spin. Furthermore, this tells us

$$\dim(\mathcal{M}_{\text{HS}}) = \alpha^{-2} + 2\alpha^{-1} + 4 + 2\alpha + \alpha^2 .$$  \hspace{1cm} (5.11)

\(^9\)We have not included the second $\mathbb{Z}_m$ which appears in the type 2) surface, since it acts trivial.
From the supergravity spectrum we can now straightforwardly extract the chiral-chiral primary spectrum:

\[
\bigoplus_m 2(m, m \pm 1) \oplus 4(m, m). \tag{5.12}
\]

The chiral-antichiral primary spectrum is very interesting in this case. It can in particular distinguish different hyperelliptic surfaces. It is in general given by the constant part of

\[
\bigoplus_m \alpha^{-m}(m, -m - 2) \oplus \alpha^{2-m}(m, -m + 2) \oplus 2\alpha^{-m}(1 + \alpha)(m, -m - 1) \\
\oplus 2(1 + \alpha)\alpha^{1-m}(m, -m + 1) \oplus \alpha^{-m}(1 + 4\alpha + \alpha^2)(m, -m). \tag{5.13}
\]

It has hence a periodicity in \(m\) of period equal to the order of the quotient group.

6 Dual CFTs for the Kähler backgrounds

There are almost canonical candidates for dual CFTs to the Kähler backgrounds. First note that the Enriques surface and the hyperelliptic surfaces are the only geometric backgrounds besides \(T^4\) and K3 which support an \(\mathcal{N} = (2, 2)\) superconformal algebra at \(c = 6\). It is thus very natural that the dual CFTs should in analogy to the case of \(T^4\) and K3 correspond to the symmetric orbifold of the respective seed theories. This should also work, since we have argued in Section 4.5 that we have identified the same group actions on both sides of the small \(\mathcal{N} = (4, 4)\) dualities.

We hence propose that the supergravity backgrounds we analysed above lie on the same moduli space as the symmetric orbifolds

\[
\text{Sym}^\infty(\text{ES}), \quad \text{Sym}^\infty(\text{HS}) \tag{6.1}
\]

of the Enriques surface and the corresponding hyperelliptic surface, respectively.\(^{10}\) The same proposal was made in [15] for the first of the hyperelliptic surfaces, so this is the natural generalization of the idea presented there.

To support the claim, we will show in this section that the chiral-chiral and chiral-antichiral primary spectrum we calculate from these CFTs agree with the ones we computed in the previous section.

6.1 The DMVV-formula

Denote by \(Z(z|\tau)\) the partition function of the seed theory ES or HS with the insertion of \((-1)^F\):

\[
Z(z|\tau) = \text{tr} \left( (-1)^F y^J \bar{y}^{J_\bar{J}} q^{L_\alpha} \bar{q}^{\bar{L}_\bar{\alpha}} \right). \tag{6.2}
\]

\(^{10}\)We expect that this correspondence continues to hold for a finite number of copies, where the CFT should be dual to a string theory on the respective background. This is in the spirit of what was found in [15] from the point of view of string theory.
Here, we included a chemical potential for the $U(1)$-charges. As usual,
\[ q = e^{2\pi i \tau}, \quad y = e^{2\pi i z}. \] (6.3)

No holomorphicity on $\tau$ or $z$ is assumed. We add a subscript `NSNS` or `RR` to indicate whether the trace is taken in the NS-NS sector or in the R-R sector. We add a superscript $N$ to refer to the symmetric orbifold theory with $N$ copies. As one can see from the definition of the partition function, we suppressed ground state energies. We write
\[ Z_{RR}(z|\tau) = \sum_{m,\ell} c(m, \bar{m}, \ell, \bar{\ell}) q^m \bar{q}^\bar{m} y^\ell \bar{y}^\bar{\ell}. \] (6.4)

In [38] and [39], a formula was given for the partition function of the symmetric orbifold:
\[ \sum_{N=0}^{\infty} p^N Z_{RR}^N(z|\tau) = \prod_{n=1}^{\infty} \prod_{\ell,m,\bar{\ell},\bar{m}} \frac{1}{(1 - p^n q^m \bar{q}^\bar{m} y^\ell \bar{y}^\bar{\ell})^c(nm,n\ell,\bar{\ell},\bar{m})}. \] (6.5)

It is convenient to let this formula flow to the NS-NS sector:
\[ \sum_{N=0}^{\infty} p^N Z_{NSNS}^N(z|\tau) = \prod_{n=1}^{\infty} \prod_{\ell,m,\bar{\ell},\bar{m}} \frac{1}{(1 - p^n q^m \bar{q}^\bar{m} y^\ell \bar{y}^\bar{\ell})^c(nm,n\ell,\bar{\ell},\bar{m})}. \] (6.6)

We note that the right hand side of (6.6) contains exactly one factor of the form $(1 - p)^{-1}$. Following the argument of [40], we can extract $Z_{NSNS}^\infty(z|\tau)$ as follows. The right hand side of (6.6) is of the form
\[ \frac{1}{1 - p} \sum_{i=0}^{\infty} x_i p^i = \sum_{i=0}^{\infty} \sum_{j=0}^{i} x_j p^i. \] (6.7)

Hence, we can extract $Z_{NSNS}^\infty(z|\tau) = \sum_{j=0}^{\infty} x_j$ by omitting the factor of $(1 - p)^{-1}$ and setting $p = 1$. We will indicate the fact that this factor is omitted by a prime in the product. Thus, we have
\[ Z_{NSNS}^\infty(z|\tau) = \prod_{n=1}^{\infty} \prod_{\ell,m,\bar{\ell},\bar{m}} \frac{1}{(1 - q^m \bar{q}^\bar{m} y^\ell \bar{y}^\bar{\ell})^c(nm,n\ell,\bar{\ell},\bar{m})}. \] (6.8)

This is actually not the expression with which we should compare the supergravity answer. The reason is that this partition function also counts multi-particle states, whereas we only dealt with single-particle states in supergravity. The transition between the two partition functions is simple, they are related by
\[ Z_{multi}(z|\tau) = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} Z_{single}(kz|k\tau) \right). \] (6.9)
It is then easy to see that the single-particle version of (6.8) is

\[ Z_{\text{NSNS, single}}^\infty(z|\tau) = \sum_{n=1}^{\infty} \sum_{\ell, \bar{\ell}, \bar{\ell}} c(n(m - \ell), n(\bar{m} - \bar{\ell}), \ell - \frac{1}{2}, \bar{\ell} - \frac{1}{2}) q^m \bar{q}^\bar{m} y^\ell \bar{y}^{\bar{\ell}}. \]  

(6.10)

Here, we omitted the prime on the sum, since it simply corresponds to the vacuum in this partition function.

### 6.2 The chiral-(anti)chiral spectrum

We now extract chiral-chiral primary states of (6.10). Clearly, only terms with \( m = \ell \) and \( \bar{m} = \bar{\ell} \) contribute in the sum. Then the sum localizes onto the Ramond ground states of the seed theory. These in turn correspond via spectral flow to chiral-chiral primary states in the seed theory. We use the same trick as in supergravity to determine the chiral-chiral and the chiral-antichiral primary spectrum in one go. For this, we consider the modified supergravity spectrum of K3 and \( \mathbb{T}^4 \) with the insertions of \( \alpha \)'s.

**Enriques surface.** Using the Hodge-diamond (5.2), we see that

\[ c(0,0,\pm \frac{1}{2}, \pm \frac{1}{2}) = 1, \quad c(0,0,\pm \frac{1}{2}, \mp \frac{1}{2}) = \alpha, \quad c(0,0,0,0) = 10(1 + \alpha), \]  

(6.11)

and all other ground state coefficients vanish. Thus, the modified K3 supergravity spectrum reads after translating to the supergravity conventions:

\[ (1,1)^2_3 \oplus (11 + 10\alpha) (2,2)^2_3 \oplus \alpha (1,3)^2_3 \oplus \bigoplus_{m \geq 3} \alpha (m, m \pm 2) \]  

\( \oplus (12 + 10\alpha) (m, m \pm 10\alpha) (m, m) \]  

(6.12)

which is in perfect agreement with (5.4), up to the aforementioned exceptions at low spin. As a corollary also the chiral-chiral and chiral-antichiral primary spectrum will agree.

**Hyperelliptic surface.** The Hodge-diamond (5.8) tells us this time that

\[ c(0,0,\pm \frac{1}{2}, \pm \frac{1}{2}) = 1, \quad c(0,0,\pm \frac{1}{2}, \mp \frac{1}{2}) = \alpha^{\pm 1}, \quad c(0,0,0,0) = 1 + \alpha^{\mp 1}, \]

\[ c(0,0,0,\pm \frac{1}{2}) = \alpha^{\pm 1} + 1, \quad c(0,0,0,0) = \alpha^{-1} + 2 + \alpha. \]  

(6.13)

This translates into the following supergravity spectrum from the symmetric orbifold:

\[ (1,1)^2_3 \oplus (1 + \alpha) (1,2)^2_3 \oplus (\alpha^{-1} + 1) (2,1)^2_3 \oplus (\alpha^{-1} + 3 + \alpha) (2,2)^2_3 \oplus (1 + \alpha) (2,3)^2_3 \]

\[ \bigoplus_{m \geq 3} \alpha^{\pm} (m, m \pm 2) \]  

\( \oplus (1 + \alpha^{\pm}) (m, m \pm 1) \)  

\( \oplus (\alpha^{-1} + 4 + \alpha) (m, m)\]  

(6.14)

which is again in agreement with (5.10), up to low-lying exceptions. Consequently, also the chiral-chiral primary and chiral-antichiral primary spectrum will agree.

Let us mention that we have also applied the techniques developed in [15, 40] to the present case. We have found that the supergravity elliptic genera of the backgrounds agree with the elliptic genera of the symmetric product orbifold theory. While for the hyperelliptic surfaces, the elliptic genus is vanishing, it equals half of the K3 elliptic genus (E.3) in the case of the Enriques surface.
7 Conclusions

In this paper, we have discussed the conditions imposed by supersymmetry on AdS$_3$ backgrounds with pure NS-NS flux. We found that supersymmetry implies that the internal manifold is a U(1)-fibration over a balanced manifold. Moreover, we found also the explicit expressions for the non-vanishing torsion classes. Strengthening the condition to Kähler, we were able to give a complete classification of these backgrounds. Moreover, it was relatively easy to identify their dual CFTs.

Several directions for future work seem promising. First, it would be interesting to understand also non-Kähler backgrounds on the same level as the Kähler backgrounds — maybe also there a classification could be possible. This would greatly enhance our understanding of AdS$_3$ backgrounds. Furthermore, one can consider warped products of AdS$_3$ with $\mathcal{M}_7$, add a non-trivial dilaton profile, or turn on RR-fields. For the latter case, [10, 41] gives some classification results. Each of these complications adds new interesting ingredients, but the dual CFT will be substantially harder to identify. However, we feel that $\mathcal{N} = (2,2)$ supersymmetry is particularly suited for exploring the landscape of AdS$_3$/CFT$_2$ dualities.

Another exciting direction is black hole counting, particularly for the Enriques surface. The background can be viewed as a near horizon limit of a black hole sitting at a boundary of a five-dimensional space-time. While not a black hole in flat space, one can still perform microscopic state counting. Since the black hole is sitting on the boundary of space-time, the surface of its horizon is precisely half of its original value. This is reflected on the CFT side by the fact that the elliptic genus is half of the K3 value (E.3). It would certainly be interesting to explore this in more detail.

Furthermore, one should embed the background into string theory. In particular, it would be interesting to find a suitable D-brane construction, which may provide some insight on how to construct other $\mathcal{N} = (2,2)$ backgrounds.

The symmetric orbifold of the hyperelliptic surface supports a higher spin symmetry. However, it is unknown whether the same holds true for the symmetric orbifold of the Enriques surface at least at special points in the moduli space. For K3, this is possible thanks to free field constructions [42]. In a similar vein, it would be interesting to see whether the corresponding higher spin symmetry can also be realized as (possibly an orbifold of) a coset [43].

Finally, there is still moonshine to be found in the Enriques surface elliptic genus [44]. It seems that the Mathieu group $\text{M}_{12}$ acts on the BPS states of this compactification. Hence our construction provides another geometric example of moonshine.
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A Notations and conventions

We use a mostly plus metric for $\text{AdS}_3 \times \mathcal{M}_7$. Hence $\mathcal{M}_7$ has a standard Riemannian metric. We define a generalized inner product between forms. Assuming $p > q$, it reads:

$$ (\iota_{\alpha \beta})_{a_{q+1} \cdots a_p} \equiv \frac{1}{(p-q)!} \alpha^{a_1 \cdots a_q} \beta_{a_1 \cdots a_p} .$$

(A.1)

For a $p$-form $\alpha$, the Hodge-norm is defined by

$$ \alpha^2 \equiv \iota_{\alpha} \alpha = \frac{1}{p!} \alpha_{a_1 \cdots a_p} \alpha^{a_1 \cdots a_p} .$$

(A.2)

We have moreover

$$ \alpha \wedge \star \alpha = \alpha^2 \text{vol} ,$$

(A.3)

where vol is the canonical volume form. For a complex form, we have the natural analog

$$ |\alpha|^2 \equiv \iota_{\alpha} \bar{\alpha} .$$

(A.4)

The Hodge dual in $n$ dimensions and Euclidean signature of a $k$-form is defined to be

$$ (\star \alpha)_{a_1 \cdots a_{n-k}} = \frac{1}{k!} \epsilon_{b_1 \cdots b_k} b_{k+1} \cdots b_n \alpha^{b_1 \cdots b_k} .$$

(A.5)

B Gamma-matrices for $\text{AdS}_3 \times \mathcal{M}_7$

B.1 Piecing together the gamma-algebra

The here presented facts are standard and can be found e.g. in [45]. We denote $\text{AdS}_3$-indices by $\mu, \nu, \ldots$ and $\mathcal{M}_7$-indices by $a, b, \ldots$. We can choose the following representation of the ten-dimensional gamma-matrices in terms of the three- and seven-dimensional ones, which we will denote by small $\gamma$’s:

$$ \Gamma_{\mu} = \sigma_1 \otimes \gamma_{\mu} \otimes 1_{8 \times 8} ,$$

(B.1)

$$ \Gamma_{a} = \sigma_2 \otimes 1_{2 \times 2} \otimes \gamma_{a} .$$

(B.2)
One may easily check that this prescription realizes the ten-dimensional Dirac algebra. Furthermore, the dimension of the matrices is $2 \times 2 \times 2^3 = 32$, as expected. Since there is no chirality operator in odd dimensions, the three and seven-dimensional gamma-matrices satisfy

$$\gamma_0 \gamma_1 \gamma_2 = -\mathds{1}_{2 \times 2}, \quad \gamma_3 \gamma_4 \cdots \gamma_9 = -i \mathds{1}_{8 \times 8}. \quad (B.3)$$

In fact, we choose the following representation of the three-dimensional gamma-algebra:

$$\gamma_0 = i \sigma_1, \quad \gamma_1 = \sigma_2, \quad \gamma_2 = \sigma_3, \quad (B.4)$$

where $\sigma_i$ denote the Pauli-matrices. The seven-dimensional gamma-algebra is constructed in the next subsection.

The ten-dimensional chirality operator is defined by $\Gamma_0 \cdots \Gamma_9$ and equals

$$\Gamma = -\sigma_3 \otimes \mathds{1}_{2 \times 2} \otimes \mathds{1}_{8 \times 8}. \quad (B.5)$$

### B.2 The seven-dimensional gamma-algebra

The seven-dimensional gamma-matrices are denoted by $\gamma_a$. They fulfill the Euclidean Dirac algebra

$$\{\gamma_a, \gamma_b\} = 2 \delta_{ab}. \quad (B.6)$$

The $\gamma$-matrices are self-adjoint:

$$\gamma_a^\dagger = \gamma_a. \quad (B.7)$$

Furthermore, they satisfy

$$\gamma_{a_1 \cdots a_k} = \frac{i}{(7-k)!} \epsilon_{a_1 \cdots a_7} \gamma^{a_{k+1} \cdots a_7}. \quad (B.8)$$

We can choose the gamma-matrices to be in a Majorana-representation. Then they are purely imaginary and hence antisymmetric. Such a representation is explicitly given by

$$\gamma_1 = i \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
\[
\gamma_3 = i \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
est
\end{pmatrix},
\gamma_4 = i \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
est
\end{pmatrix},
\gamma_5 = i \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 
est
\end{pmatrix},
\gamma_6 = i \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 
est
\end{pmatrix},
\gamma_7 = i \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 
est
\end{pmatrix}.
\]

\((B.9)\)

\(C\) Killing spinors on \(\text{AdS}_3\)

In this section, we construct explicitly the Killing spinors on \(\text{AdS}_3\), following [21]. Importantly, we determine their chiralities in the boundary theory. We choose the following coordinates for \(\text{AdS}_3\) in Minkowskian signature:

\[ds^2 = \frac{1}{z^2}(-dt^2 + dx^2 + dz^2).\]  \((C.1)\)

For simplicity, we have chosen the radius of \(\text{AdS}_3\) to be of unit size. The boundary of \(\text{AdS}_3\) corresponds to \(z \to 0\) and is parametrized by the standard flat coordinates \(t\) and \(x\). We use the conventions for the gamma-matrices as described in Appendix B. These have to be complemented by the following dreibein:

\[e^a_\mu = \frac{1}{z} \delta^a_\mu.\]  \((C.2)\)

The spin connection is given by

\[\omega_0 = \frac{2i}{z} \sigma_2, \quad \omega_1 = \frac{2}{z} \sigma_1, \quad \omega_2 = 0.\]  \((C.3)\)

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Solving the Killing spinor equation
\[ \left( \partial_\mu - \frac{i}{4} \omega_\mu \right) \psi = \pm \frac{1}{2} \psi \] (C.4)
gives the following solutions:
\[ \psi_+(z, t, x) = \left( \frac{c_1 z^{1/2}}{(c_2 - i c_1 (t - x)) z^{-1/2}} \right), \] (C.5)
\[ \psi_-(z, t, x) = \left( \frac{(c_2 + i c_1 (t + x)) z^{-1/2}}{c_1 z^{1/2}} \right), \] (C.6)
where \( c_1 \) and \( c_2 \) are free parameters. The boundary theory lives at \( z \to 0 \) and has the gamma-subalgebra generated by \( \gamma_0 \) and \( \gamma_1 \). The corresponding chirality operator is given by
\[ \gamma_0 \gamma_1 = -\sigma_3. \] (C.7)
Hence as expected, the solutions have a definite chirality in the boundary theory: The first solution has positive chirality, whereas the second one has negative chirality. Thus, we make the important conclusion that the sign in the Killing spinor equation corresponds precisely to the chirality in the boundary theory. Furthermore, each solution has two free parameters.

D Integrability conditions

In this appendix, we use the integrability of the Killing spinor equations to derive the Einstein equation (2.11). For definiteness, we choose \( \eta \equiv \eta_- \) in the derivation. The \( \eta_+ \) calculation follows from this one by replacing \( H_{abc} \to -H_{abc} \). Using the gravitino Killing-spinor equation (2.18), we have
\[ \frac{1}{4} R_{abcd} \gamma^{cd} \eta = \nabla_{[a} \nabla_{b]} \eta = \frac{1}{8} \nabla_{[a} \left( H_{b]cd} \gamma^{cd} \eta \right). \] (D.1)
Now we multiply by \( \gamma^b \) from the left. Then the left hand side of (D.1) becomes
\[ \frac{1}{4} R_{abcd} \gamma^{cd} \eta = \frac{1}{4} R_{abcd} (2g^{bc} \gamma^d + \gamma^{bcd}) \eta = \frac{1}{2} R_{ab} \gamma^d \eta = -\frac{1}{2} R_{ad} \gamma^d \eta. \] (D.2)
Here, we used the first Bianchi identity \( R_{a[bcd]} = 0 \). The right hand side of (D.1) multiplied from the left by \( \gamma^b \) can also be evaluated further:
\[ \frac{1}{8} \nabla_{[a} \left( H_{b]cd} \gamma^{cd} \eta \right) = \frac{1}{8} \nabla_{[a} \left( H_{b]cd} (2g^{bc} \gamma^d + \gamma^{bcd}) \eta \right) \]
\[ = \frac{1}{16} \nabla_{a} \left( H_{bcd} \gamma^{bcd} \eta \right) - \frac{1}{8} \nabla_{b} \left( H_{a} \gamma^d \eta \right) - \frac{1}{16} \nabla_{b} \left( H_{acd} \gamma^{bcd} \eta \right) \]
\[ = -\frac{3i}{4\ell} \nabla_{a} \eta - \frac{1}{8} H_{a} \gamma^d \nabla_{b} \eta - \frac{1}{16} \nabla_{b} \left( H_{acd} \gamma^{bcd} \eta \right). \] (D.5)
We used the dilatino Killing spinor equation in the first term and the fact that $H$ is coclosed in the second. For the third term, note that we may write

$$
\nabla_b \left( H_{acd} \gamma^{bcd} \eta \right) = \frac{1}{3} \left[ \nabla_b \left( H_{acd} \gamma^{bcd} \eta \right) + \nabla_c \left( H_{dba} \gamma^{bcd} \eta \right) - \nabla_d \left( H_{bac} \gamma^{bcd} \eta \right) \right]
$$

(D.6)

$$
= \frac{4}{3} \nabla_b \left( H_{acd} \gamma^{bcd} \eta \right) + \frac{1}{3} \nabla_a \left( H_{bcd} \gamma^{bcd} \eta \right)
$$

(D.7)

$$
= \frac{4}{3} H_{[acd]^b \gamma^{bcd} \nabla_b \eta} - \frac{4i}{\ell} \nabla_a \eta
$$

(D.8)

where we again used the dilatino Killing spinor equation and that $H$ is closed. Continuing with this the calculation at (D.5), we obtain

$$
(D.5) = \frac{i}{2\ell} \nabla_a \eta - \frac{1}{8} H_{a b c d} \gamma^{a b c d} \nabla \eta - \frac{1}{12} H_{[acd] b c d} \gamma^{b c d} \nabla b \eta
$$

(D.9)

$$
= -\frac{i}{16\ell} H_{a b c d} \gamma^{a b c d} \eta - \frac{1}{64} H_{a b d} H_{b c f} \gamma^{a b d} \gamma^{c e f} \eta - \frac{1}{96} H_{[acd] b e f} \gamma^{b c d} \gamma^{e f} \eta
$$

(D.10)

$$
= \frac{1}{192} H_{acd} H_{b e f} \gamma^{a c d} \gamma^{b e f} \eta - \frac{1}{64} H_{a b d} H_{b c f} \gamma^{a b d} \gamma^{c e f} \eta - \frac{1}{96} H_{[acd] b e f} \gamma^{b c d} \gamma^{e f} \eta
$$

(D.11)

$$
= \frac{1}{384} H_{acd} H_{b e f} \left( 2 \gamma^{a c d} \gamma^{b e f} - 6 g^{b e c} \gamma^{a d} \gamma^{c f} - 3 \gamma^{b c d} \gamma^{e f} \gamma^{c e f} \eta + \gamma^{b e f} \gamma^{a c d} \eta \right)
$$

(D.12)

$$
= \frac{1}{384} H_{acd} H_{b e f} \left( -48 g^{b c e} \gamma^{d f} \gamma^{a f} \right) \eta
$$

(D.13)

$$
= \frac{1}{8} H_{acd} H_{b e f} \gamma^{a d} \gamma^{d f} \eta
$$

(D.14)

$$
= \frac{1}{8} H_{acd} H_{b c d} \gamma^{a b} \eta
$$

(D.15)

In (D.11), we used the dilatino Killing spinor equation backwards. In the last steps, only the algebraic properties of the gamma-matrices were used. Putting together the simplified left and right hand side of (D.2), we finally conclude

$$
R_{ab} \gamma^{b} \eta = \frac{1}{4} H_{acd} H_{b c d} \gamma^{a b} \eta
$$

(D.16)

but since $\gamma^{b} \eta$ are all linearly independent,\(^{11}\) this also implies

$$
R_{ab} = \frac{1}{4} H_{acd} H_{b c d}
$$

(D.17)

E Some properties of complex surfaces

In this appendix, we collect some interesting and useful properties of the complex surfaces we use in the main text, namely the four-torus $T^4$, K3, the Enriques surface ES and the hyperelliptic surfaces HS.

\(^{11}\)The spinor space is eight-dimensional, but there are only seven possible values for $b$.  

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All of these surfaces are projective and therefore Kähler. They are furthermore distinguished among other complex surfaces, since their first Chern class vanishes in real cohomology, i.e. it is a torsion element in integer cohomology. This suffices for Yau’s Theorem [46] to hold and consequently these surfaces support a Ricci-flat metric. It is furthermore possible to use them as target spaces for $\mathcal{N} = (2, 2)$ sigma-models, since there is no axial anomaly.

We should note that there are other complex surfaces with vanishing first Chern class in real cohomology. These are the primary and secondary Kodaira surfaces. However, these are not algebraic and hence not Kähler, so they are unsuitable for our purposes.

E.1 $\mathbb{T}^4$

$\mathbb{T}^4$ is certainly the most explicit of the four surfaces. In particular the Ricci-flat metric is the canonical metric inherited from $\mathbb{C}^2$, when thinking of $\mathbb{T}^4$ as a quotient thereof. The Hodge-diamond reads

\[
\begin{array}{c}
1 \\
2 & 2 \\
1 & 4 & 1 \\
2 & 2 \\
1
\end{array}
\]

(E.1)

and the cohomology ring is the exterior algebra over four generators – two of degree $(1, 0)$ and two of the degree $(0, 1)$. To determine the action of group actions on the cohomology, it hence suffices to determine the action on these four generators. The Euler-characteristic, the signature and all other genera of the surface vanish. The canonical bundle is trivial. $\mathbb{T}^4$ is a spin manifold.

E.2 K3

K3 is the unique simply-connected Calabi-Yau surface. It can be realized as various orbifolds of $\mathbb{T}^4$. However away from these orbifold points, the Ricci-flat metric is not explicitly known, but exists thanks to Yau’s Theorem. The Hodge-diamond reads

\[
\begin{array}{c}
1 \\
0 & 0 \\
1 & 20 & 1 \\
0 & 0 \\
1
\end{array}
\]

(E.2)

The Euler-characteristic is 24, while the signature is $-16$ — the intersection lattice is the unimodular lattice $\Pi_{3,19}$. The holonomy group equals $\text{SU}(2)$. The canonical
bundle is again trivial. K3 is also a spin manifold. The elliptic genus of string theory is non-vanishing and equals
\[
Z_{K3}(z|\tau) = 8 \left( \frac{\theta_2(z|\tau)^2}{\theta_2(\tau)^2} + \frac{\theta_3(z|\tau)^2}{\theta_3(\tau)^2} + \frac{\theta_4(z|\tau)^2}{\theta_4(\tau)^2} \right). \tag{E.3}
\]

### E.3 HS

Hyperelliptic surfaces are finite quotients of tori — we gave an overview of the different possibilities in Table 1. The Hodge-diamond reads for all possibilities

\[
\begin{array}{ccc}
1 & & \\
1 & 2 & 0 \\
0 & & \\
1 & & \\
1 & & \\
\end{array}
\tag{E.4}
\]

Hyperelliptic surfaces are elliptic fibrations over elliptic curves. For this reason, they are also called bi-elliptic surfaces. The Euler-characteristic vanishes in all cases, as does the signature. The holonomy group is \(\mathbb{Z}_n\), of which the generator is a rotation by an angle of \(\frac{2\pi}{n}\). Here, \(n = 2, 3, 4\) and 6 for type a, b, c and d, respectively. The canonical bundle is a torsion bundle, i.e. it is not trivial but its \(n\)-th power is. Finally, hyperelliptic surfaces are not spin manifolds.

### E.4 ES

Enriques surfaces can be realized as \(\mathbb{Z}_2\)-quotients of K3 surfaces. They have Hodge-diamond

\[
\begin{array}{ccc}
1 & & \\
0 & 10 & 0 \\
0 & & \\
1 & & \\
\end{array}
\tag{E.5}
\]

The Euler-characteristic is 12, the signature is \(-8\). The intersection lattice is the unimodular lattice \(\Pi_{1,9}\). The canonical bundle is a torsion bundle of order two. The holonomy group is a semidirect product \(\text{SU}(2) \rtimes \mathbb{Z}_2\). Finally, Enriques surfaces are not spin manifolds. The string theory elliptic genus is half of the K3 elliptic genus.

### F Modifying \(\mathcal{N} = 4\) multiplets

To determine the BPS and supergravity spectrum in the main text, we used the underlying \(\mathcal{N} = 4\) multiplet structure of the compactification. In this appendix, we provide some details of the modifications of the \(\mathcal{N} = 4\) multiplet structure we used.
We first pick an $\mathcal{N} = 2$ subalgebra inside the $\mathcal{N} = 4$ algebra of which the corresponding supercharges will be denoted by $G^+_r$ and $G^-_r$. The remaining two supercharges will be denoted by $\tilde{G}^+_r$ and $\tilde{G}^-_r$. They are not invariant under the quotient we are performing — they have eigenvalues $\alpha$ and $\alpha^{-1}$, respectively. Here, $\alpha$ denotes a unit root of the same order as the group by which we are performing the quotient. Similarly, the Cartan-element of the $\mathfrak{su}(2)$-current algebra is invariant under the quotient, while the two raising and lowering operators $J^+_m$ pick up the eigenvalues $\alpha^{\pm}$.

Let us denote by $\chi_\ell(y)$ an $\mathfrak{su}(2)$ character of spin $\ell$. We further denote by $\chi^\alpha_\ell(y)$ an $\mathfrak{su}(2)$-character twisted by $\alpha$:

$$\chi^\alpha_\ell(y) = \sum_{j=-\ell}^\ell \alpha^{\ell-j} y^j. \quad (F.1)$$

The corresponding multiplet will be denoted by $(m)^\alpha$ in the main text, where $m = 2\ell + 1$. When combining left- with right-movers, we write $(m,n)^\alpha$ for the twisted multiplet. One has to pay attention that one has to use $\alpha$ for the left-movers and $\alpha^{-1}$ for the right-movers, i.e. we have

$$(m,n)^\alpha \cong (m)^\alpha \otimes (n)^{\alpha^{-1}}. \quad (F.2)$$

It is simple to write down a short $\mathcal{N} = 4$ character of the global $\mathfrak{su}(1,1|2)$-algebra twisted by $\alpha$:

$$\chi^{\mathcal{N}=4,\alpha}_\ell(q,y) = \frac{q^\ell}{1-q} \left( \chi^\alpha_\ell(y) - q^{\frac{1}{2}} (1 + \alpha) \chi^\alpha_{\ell-\frac{1}{2}}(y) + q\alpha \chi^\alpha_{\ell-1}(y) \right). \quad (F.3)$$

Here, we inserted a $(-1)^F$ in the definition of the character. An $\mathcal{N} = 2$ character is by definition invariant under the projection and reads

$$\chi^{\mathcal{N}=2}_{\ell,h}(q,y) = \frac{q^h}{1-q} \begin{cases} y^j + q^{\frac{1}{2}} y^{j-\frac{1}{2}} & j = h, \\ y^j + q^{\frac{1}{2}} y^{j+\frac{1}{2}} & j = -h, \\ y^j + q^{\frac{1}{2}} (y^{j-\frac{1}{2}} + y^{j+\frac{1}{2}}) + qy^j & -h < j < h, \end{cases} \quad (F.4)$$

where the three cases correspond to chiral primary, anti-chiral primary and long representations, respectively. It is then simple to check that we have the following decomposition:

$$\chi^{\mathcal{N}=4,\alpha}_\ell(q,y) = \sum_{j=-\ell}^\ell \alpha^{-j} \chi^{\mathcal{N}=2}_{\ell,\ell}(q,y), \quad (F.5)$$

where the bottom and top components of the sum are short.
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