Driven quantum spin chain in the presence of noise: Anti-Kibble-Zurek behavior

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We study defect generation in a quantum XY-spin chain arising due to the linear drive of the many-body Hamiltonian in the presence of a time-dependent fast Gaussian noise. The main objective of this work is to quantify analytically the effects of noise on the defect density production. In the absence of noise, it is well known that in the slow sweep regime, the defect density follows the Kibble-Zurek (KZ) scaling behavior with respect to the sweep speed. We consider time-dependent fast Gaussian noise in the anisotropy of the spin-coupling term \( \gamma_0 = (J_1 - J_2)/(J_1 + J_2) \) and show via analytical calculations that the defect density exhibits anti-Kibble-Zurek (AKZ) scaling behavior in the slow sweep regime. In the limit of large chain length and long time, we calculate the entropy and magnetization density of the final decohered state and show that their scaling behavior is consistent with AKZ picture in the slow sweep regime. We have also numerically calculated the sub-lattice spin correlators for finite separation by evaluating the Toeplitz determinants and find results consistent with the KZ picture in the absence of noise, while in the presence of noise and slow sweep speeds the correlators exhibit the AKZ behavior. Furthermore, by considering the large \( n \)-separation asymptotes of the Toeplitz determinants, we further quantify the effect of the noise on the spin-spin correlators in the final decohered state. We show that while the correlation length of the sub-lattice correlator scales according to the AKZ picture in the slow sweep regime, we obtain different scaling for the magnetization correlators.

I. INTRODUCTION

A quantum system driven at zero temperature by some system dependent parameter through a quantum critical point (QCP) is subjected to quantum phase transition (QPT) in which the ground state of the system is fundamentally altered with completely different physical properties across the phase transition point. One of the main points of interest is to quantify the generation of excitations or the defect density generation due to the quench through the critical point. Defects are inevitable in a drive through the critical point due to the vanishing energy gap at the critical point where the adiabaticity criterion breaks down and non-adiabatic effects become important. In this regard, the Kibble-Zurek mechanism (KZM) a theory originally proposed to quantify the topological defect production in a cosmological phase transition has been successfully applied in quantifying the defect production in idealized condensed matter systems undergoing QPT \(^1\text{-}^{12}\). The theory predicts that the defect density scales as \( n \propto \tau^{-\beta} \), where \( \tau \) is the quench rate and the universal exponent \( \beta > 0 \) is determined by the critical exponents and the dimension of the system. Recent experimental studies supporting KZM have been reported in well controlled systems involving trapped ions, Bose-Einstein condensates and Rydberg simulator\(^{13\text{-}15}\).

While the study of quantum systems exhibiting KZ behavior remains an area of active interest, scenarios which result in deviations from this universal behavior have also come under increased scrutiny. Recent studies of drive protocols in quantum systems that are coupled to external environment, disorder or are in the presence of noise indicate that the defect density generated exhibit fundamentally different dynamical behavior than the one predicted by the KZ theory\(^{16\text{-}33}\). The focus of our attention has been to understand experimental and numerical studies wherein, unlike the KZ behavior, slower drives beyond a certain optimal quench rate/speed create more defects\(^{34\text{-}40}\). This scaling has been termed as the anti-Kibble-Zurek (AKZ) behavior. In all of them, the AKZ scaling behavior manifested itself in the presence of the noisy control field driving the system through the critical points. In this work, we consider a time dependent quantum XY-spin chain which is driven by a transverse magnetic field with an anisotropy term that contains a fluctuating Gaussian noise term. In the limit of fast noise, we perform exact analytical calculations and derive the universal AKZ scaling behavior of the defect density with respect to the quench rate.

Apart from the study of the defect generation, the consequence of the KZ picture to the entropy, magnetization and the correlation functions have been considered before (for example in the Refs. \([28, 41\text{-}45]\)). We furthermore quantify the effect of noise on the above physical quantities in the final decohered state due to the noisy drive through the QCPs. By considering the large \( n \)-separation asymptotes of the Toeplitz determinants, we show analytically that the correlation lengths of the sub-lattice correlators exhibits the AKZ scaling behavior in the slow sweep regime. The scaling behavior of the magnetization density at the end of the protocol is also consistent with the AKZ scaling behavior in the slow sweep limit. However, the correlation length of the (connected) magnetization correlator in the large \( n \)-separation limit continues to follow the KZ picture.

The organization of the paper is as follows. In Sec. II we discuss the model Hamiltonian and dynamics of the XY-spin chain with transverse magnetic field (varying linearly with time) in the \( z \)-direction in the absence of noise. In Sec. III we consider transverse protocol in the presence of fast Gaussian noise and obtain analytically the AKZ scaling behavior of the defect density in the slow
II. THE MODEL HAMILTONIAN

We consider the quantum XY-spin model driven by the transverse external field \( h(t) \),

\[
H(t) = -\sum_{n=1}^{N} \left[ J_1 \sigma_n^1 \sigma_{n+1}^1 + J_2 \sigma_n^2 \sigma_{n+1}^2 + h(t)\sigma_n^3 \right],
\]

where \( J_1 \) and \( J_2 \) are respectively the spin-spin couplings along the \( x \) and \( y \)-spin directions. Introducing the coefficients \( J = J_1 + J_2 \) and anisotropy, \( \gamma_0 = (J_1 - J_2)/J \), allows us to re-express the Hamiltonian as,

\[
H(t) = -\frac{J}{2} \sum_{n=1}^{N} \left[ (1 + \gamma_0)\sigma_n^1 \sigma_{n+1}^1 + (1 - \gamma_0)\sigma_n^2 \sigma_{n+1}^2 \right] - h(t) \sum_{n=1}^{N} \sigma_n^3.
\]

The \( \gamma_0 = 0 \) limit represents the isotropic XY-spin chain, while \( \gamma_0 = \pm 1 \) limits correspond to the quantum Ising case. We consider time dependence in the anisotropy by including Gaussian correlated noise \( \eta(t) \), i.e., \( \gamma_0 \rightarrow \gamma(t) = \gamma_0 + \eta(t) \). The Gaussian noise \( \eta(t) \) is characterized by,

\[
\overline{\eta(t)} = 0, \quad \overline{\eta(t) \eta(t')} = \eta_0^2 e^{-\Gamma |t-t'|},
\]

where \( \eta_0 \) is the noise strength and \( \Gamma \) is the inverse timescale associated with the noise.

In the following, we will summarize the well studied transverse protocol\(^{31,46,47}\). In this protocol, the transverse external field \( h(t) \) is tuned to drive the equilibrium system from \( h \rightarrow -\infty \) at the start of the protocol to \( h \rightarrow \infty \) at the end of the drive protocol. For a linear protocol,

\[
h(t) = vt = Jt/\tau_Q,
\]

where \( v \) is the sweep speed of the drive, \( \tau_Q = J/v \) is the ‘quench time’ and the time \( t \) runs from \( -\infty \) to \( \infty \). The system starts out in the paramagnetic (PM) ground state (GS), \( |\downarrow\downarrow\downarrow \ldots \downarrow\rangle \) with all the spins along the negative \( z \)-axis. As \( t \) is increased the system goes through the QCPs at \( h(t) = \pm J \). The quantum phase transition involves change in the nature of GS from PM to ferromagnetic (FM) at \( h = -J \) and from FM to PM at \( h = J \). The final state that emerges is not the perfect GS of the Hamiltonian, \( |\uparrow\uparrow\ldots \rangle \), but instead is formed out of the quantum superposition of the states of the type \( |... \uparrow\uparrow\uparrow\downarrow\downarrow\uparrow \ldots \rangle \). The reason for such a final state is the inevitable violation of the adiabaticity criteria. This criteria is obtained by comparing two time-scales, the relaxation time scale, which is proportional to the inverse of minimum gap of the system and a time scale for driving the system, \( \tau_Q \). For a perfectly adiabatic dynamics, ‘relaxation time \( \ll \tau_Q \).’ But in the \( N \rightarrow \infty \) limit, the energy gap vanishes at the QCPs \( (h(t) = \pm J) \). Therefore the dynamics becomes non-adiabatic in close proximity to the QCPs, leading to a final state which is formed out of the quantum superposition of the states having kinks/domain walls.
to the spinless free fermion Hamiltonian

\[
H(t) = -J \sum_{n=1}^{N} \left[ (c_{n+1}^d c_n + \gamma_0 c_n c_{n+1} + \text{h.c.}) - h(t)(2c_n^d c_n - 1) \right].
\]

Restricting to the even parity subspace with \(c_{N+1} = -c_1\) and using the Fourier transformation, \(c_n = \frac{e^{-i\pi n}}{\sqrt{N}} \sum_k c_k e^{i kn}\) yields

\[
H(t) = \sum_k \left[ (h(t) + J \cos k) c_k^d c_k + \frac{\Delta_k}{2} c_{-k} c_k \right] + \text{h.c.,}
\]

where \(\Delta_k = 2J \gamma_0 \sin(k)\). The above Hamiltonian can be written as a sum of independent terms, \(H(t) = \sum_{k>0} H_k(t)\), where for each value of \(k\), \(H_k(t)\) acts on the 4-dimensional Hilbert space spanned by the basis vectors: \(|0\rangle, |k\rangle = c_k^d |0\rangle, |-k\rangle = c_k^d |0\rangle\) and \(|k, -k\rangle = c_k^d e^{i k} |0\rangle\).

We note that the Hamiltonian with or without the noise term proportional to \(\eta(t)\) leaves the parity unchanged. Although the 1-particle states \(|k\rangle\) and \(|-k\rangle\) evolve in time with a global phase factor only, the states \(|0\rangle\) and \(|k, -k\rangle\) couple to each other and exhibit LZ dynamics. The projected Hamiltonian in the subspace of \(|0\rangle\) and \(|k, -k\rangle\) has the structure of LZ type Hamiltonian and is given by

\[
\mathcal{H}_k(t) = 2 \begin{bmatrix}
 h(t) + J \cos(k) & \Delta_k/2 \\
 \Delta_k/2 & -h(t) - J \cos(k)
\end{bmatrix},
\]

where note that the off-diagonal term is \(k\)-dependent.

Applying the LZ transition theory, yields the \(k\)-dependent transition/excitation probability (in the large time limit),

\[
p_0^k(z) = e^{-2\pi z \sin^2 k},
\]

where \(z\) is the dimensionless quench parameter given by \(z = J^2 \gamma_0^2 / v = J^2 \gamma_0^2 \tau_Q\) (see Fig. 1).

Integrating it over the \(k\) modes one obtains the exact result for the total defect density \(n_0\) in the absence of noise,

\[
n_0(z) = \frac{1}{\pi} \int_0^{\pi} dk e^{-2\pi z \sin^2 k} = e^{-\pi z} I_0[\pi z],
\]

where \(I_0(\pi z)\) is the modified Bessel function of the first kind. The limit of large \(z\) (or the small sweep speed regime) reveals that the defect density scales as \(n_0 \sim 1/\sqrt{z}\), which indeed matches with the prediction of the KZ scaling behavior (see Fig. 2).

Interestingly, recent numerical studies have shown that the defect density production exhibits completely different scaling when in addition to the usual linear protocol a small Gaussian noise is present in the control field\(^{36,37}\). In particular, the defect density scales as

\[
n \approx c \tau_Q^{-1/2} + d \eta^2 \tau_Q,
\]

where \(c\) and \(d\) are system-dependent parameters. The first term in the above equation Eq. (10) accounts for the usual KZ scaling behavior. The second term is proportional to the square of noise amplitude and represents increased defect-density production with the increase of \(\tau_Q\) (or is inversely proportional to the sweep speed). This is converse to that of the KZ scaling and is termed as the AKZ scaling behavior.

In the next section we will consider linear protocol with transverse noise and obtain analytical results for the defect density. We will show that for large-\(z\) values AKZ scaling behavior dominates. In the subsequent sections, we will calculate the entropy and correlation functions of a driven quantum XY-spin chain in the presence of noise.

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**FIG. 1.** \(p_0^k\) Vs \(k\) in the absence of noise. In the large \(z\) or the small sweep speed limit, most of the contribution to the defect generation comes from the regions near the critical points \(k = 0, \pm \pi\). It is to be noted that \(z_1 = \log 2/2\pi\) is a special value of \(z\), where \(p_0^k = 1/2\) at \(k = \pm \pi/2\).

**FIG. 2.** \(n_0(z)\) Vs \(z\) in the absence of noise. The approximate result supports KZ mechanism i.e., \(n_0 \sim 1/\sqrt{z}\), and matches with the exact result in the large-\(z\) regime.
III. EFFECT OF NOISE ON THE DEFECT GENERATION

The time dynamics of states spanned by |0⟩ and |k, −k⟩ is governed by the Hamiltonian \( H_k(t) \),

\[
H_k(t) = 2 \begin{bmatrix} h(t) + J \cos(k) & J \gamma_0 \sin(k) \\ J \gamma_0 \sin(k) & -h(t) - J \cos(k) \end{bmatrix} + 2 \eta(t) \begin{bmatrix} 0 & J \sin(k) \\ J \sin(k) & 0 \end{bmatrix},
\]

where the first term in the above Hamiltonian is the usual deterministic part. The second term couples the homogeneous time-dependent Gaussian noise \( \eta(t) \) to each of the \( k \)-mode within the restricted subspace. In this subspace, a general state at any given time \( t \) for a single realization of the noise \( \eta(t) \) can be written as, \(|\Psi_k^n(t)⟩ = u_k^n(t)|0⟩ + v_k^n(t)|k, −k⟩\), where \( u_k^n(t) \) and \( v_k^n(t) \) are the time-dependent amplitudes. The system starts out in the perfect PM state defined by the initial conditions \( u_k^n(−∞) = 1 \) and \( v_k^n(−∞) = 0 \). The time evolution of the general state \(|\Psi_k^n(t)⟩\) for a single noise realization is governed by the stochastic Schrödinger equation,

\[
i \frac{d}{dt} |\Psi_k^n(t)⟩ = H_k^n(t)|\Psi_k^n(t)⟩,
\]

where the solution has to be averaged over all possible realizations (ensemble averaging due to the noise) of the 2-level system corresponding to each \( k \)-mode.

The projected Hamiltonian Eq. (11) is equivalent to the noisy LZ problem with the noise present only in the transverse part of the Hamiltonian. We consider the density matrix, \( \rho_k^n(t) = |\Psi_k^n(t)⟩⟨\Psi_k^n(t)| \), and set up the time evolution equation for the population inversion \( \rho_k^n(τ) \) (difference between the unexcited and the excited density for a given mode \( k \)):

\[
\frac{d}{dτ} \rho_k^n(τ) = -\frac{1}{2} \int_{−∞}^{τ} dτ_1 e^{i \int_{τ_1}^{τ_2} dv_L (v_L)} dτ_2 (v_L, τ_2)/2 \rho_k^n(τ_1)
-\frac{1}{2} \eta(τ)η(τ_1) \rho_k^n(τ_1) + h.c., \]

where \( τ = 2Δ_k(t + \cos k/v_L) \), and \( v_L = u/Δ_k \). Taking the noise average one obtains

\[
\frac{d}{dτ} \rho_k(τ) = -\frac{1}{2} \int_{−∞}^{τ} dτ_1 \cos \left[ \frac{v_L Z}{4} (τ^2 - τ_1^2) \right] \rho_k(τ_1)
-\frac{1}{γ_0} \int_{−∞}^{τ} dτ_1 \cos \left[ \frac{v_L Z}{4} (τ^2 - τ_1^2) \right] η(τ)η(τ_1) \rho_k(τ_1), \]

where \( \rho_k(τ_1) \) is obtained by performing noise average over \( \rho_k^n(τ_1) \). The fast noise criteria allows us to decouple \( η(τ)η(τ_1) \rho_k(τ_1) \) into a separate product of the noise terms and the density matrix term, \( η(τ)η(τ_1) \rho_k(τ_1) \), \( n \). The solution of the reduced master equation in the \( t \to ∞ \) limit is obtained by following the approach of Ref. \( n \) and is

\[
ρ_k(z) = \frac{1}{2} \left[ 1 + e^{-4πz η_1^2} \sin^2 k (2e^{-2πz η_1^2} - 1) \right],
\]

where \( η_1 = η_0/γ_0 \). It is interesting to note that the excitation probability [Eq. (15)] which is non-zero around the \( k = 0, ±π \) points in the absence of noise, also opens up around \( k = ±π/2 \) regions in the presence of noise and in particular for the large \( z \)-limit or the slow sweep speed scenario, which is the result of dephasing due to the presence of the fast noise which is one of the main reason for more defect generation for slower sweeps (see Fig. 3). Recently, similar results have been reported in Ref. 52 for the noisy drive protocols in a trapped ion experiment. It is worth noting that the fast sweep regime \( z \ll z_1 \) is unaffected by the fast noise.

We next evaluate the defect density by integrating \( n(z) = \int_{0}^{z} dk \rho_k/π \) and obtain

\[
n(z) = \frac{1}{2} + e^{-π(z + γ_1)} I_0[π (z + γ_1)] - \frac{1}{2} e^{-πz} I_0[π z], \]

where \( γ_1 = 2π η_1^2 \). In the limit, \( η_1 \ll 1 \), the defect density
is approximated as

\[ n(z) \approx \frac{1}{\sqrt{2\pi} \sqrt{z}} + \pi \eta_1^2 z, \]  

(17)

which gives the AKZ scaling behavior (see Figs. 4 and 5). From the above expression it can be deduced that the defect density is minimized for an optimal quench rate given by,

\[ z_O \approx 2\sqrt{2\pi} \eta_1^{4/3} \propto \eta_1^{-4/3}. \]  

(18)

We note that the optimal quench time has a universal exponential suppression of the fluctuating coherences.

### IV. DECOHERENCE OF LOCAL OBSERVABLES

In both the noiseless and noisy drive protocols, the XY-spin chain (with N spins) is prepared in a PM state at the initial time \( t_{in} = -T \). This initial state is a pure state i.e., the full system density matrix can be written as \( \rho(t = -T) = |0_N\rangle\langle 0_N| \). Subsequently the system is driven by the transverse magnetic field \( h(t) = t/\pi Q \) through the quantum critical points \( (h = \pm J) \) up to the final time \( t_f = T \). In the noiseless drive scenario the full density matrix of the evolved N-spin chain remains in the pure state due to the unitary time evolution. However, for large system size \( (N \to \infty) \) and in the long time limit i.e., \( T \to \infty \), the coherences of the density matrix develop highly fluctuating phases (dependent on \( k \) and \( T \)) which vanishes when integrated over \( k \) for all local observables. This decohered density matrix corresponds to the nonequilibrium steady state (NESS) which is fundamentally different from the decoherence process due to any external or internal noise. In addition to the internal decoherence the dephasing is further enhanced by the noise in the drive protocol which results in the exponential suppression of the fluctuating coherences.

For \( \eta_k^2 J \ll \Delta_k \) the noise has negligible effect on the system. The crossover region is around \( \eta_k^2 J \sim \Delta_k \) at which the noise begins to play a role in the dynamics of the system. In the limit \( \eta_k^2 J \gg \Delta_k \), the fast noise effects the system the most specifically in the nonadiabatic regions (when the gap \( \Delta_k \to 0 \)) around the quantum critical points \( (k = 0, \pm \pi) \), in addition new critical regions around \( k = \pm \pi/2 \) become important. Overall, nonadiabatic effects are enhanced due to the noise which gives incoherent contributions to the defect density leading to increased defect generation at the end of the protocol.

In the long time limit, the noise averaged off-diagonal terms of the density matrix (coherences), \( \tilde{\rho}_{12}^k \) and \( \tilde{\rho}_{21}^k \) vanish. Therefore the noise averaged decohered density matrix, \( \tilde{\rho}_D = \otimes_{k>0} \tilde{\rho}_{D,k} \) (with \( \tilde{\rho}_{D,k} \) a diagonal \( 2 \times 2 \) matrix in the subspace \( |0\rangle, |k, -k\rangle \)) in the limit \( N \to \infty \) and \( T \to \infty \), with respect to the final decohered state can be written as,

\[ \tilde{\rho}_{D,k} = \begin{pmatrix} p_k & 0 \\ 0 & 1 - p_k \end{pmatrix}. \]  

We will use the above expression of the density matrix to calculate the noise averaged observables in the final decohered state.

#### 2-Point Correlator in the fermionic representation

Similar to Ref. 41, one can define a 2-point correlator in terms of the fermionic operators as,

\[
\langle x - x' \rangle = \langle c_{x}^{\dagger} c_{x'} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk e^{-i k(x-x')} p_k.
\]  

(20)
This correlator is non-zero for \( x - x' = 2n \), where \( n \) is an integer. In the absence of noise the 2-point correlator is

\[
d_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \, e^{-ikn} p_k^0 = e^{-\pi z} I_n(\pi z),
\]

where \( I_n(\pi z) \) is the modified Bessel function of the first kind. The large-\( n \) expansion of \( d_n \) at fixed \( z \) yields

\[
d_n(z) \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \, e^{-\pi z k^2/2} e^{i n k} = \frac{e^{-n^2/2\pi z}}{\sqrt{2\pi^2 z}}.
\]

Thus consistent with the KZ-picture the correlation length of the density correlator is proportional to \( \sqrt{z} \) and the magnitude of the correlator is inversely proportional to \( \sqrt{z} \).

A similar calculation for the 2-point correlator in the presence of noise, \( p_k \) yields

\[
d_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \, e^{-i k n} p_k
\]

\[
= \frac{1}{2} \int_{z,0} - \frac{1}{2} e^{-\pi z I_n(\pi \bar{z})} + e^{-\pi (z+\bar{z}) I_n(\pi(z+\bar{z}))}. (23)
\]

In the limit of \( \eta_1 \ll 1 \) and \( \pi \bar{z} < 1 \) and for large-\( n \) expansion \( (n > \sqrt{4 \pi \eta_1^2 \ln 1/\eta_1}) \) the above expression is approximated as,

\[
d_n(z) \approx \frac{1}{\sqrt{2\pi^2 (z + \bar{z})}} e^{-n^2/(2\pi (z + \bar{z}))}.
\]

Thus the correlation length in the presence of noise \( l_n = \sqrt{\pi (z + \bar{z})} \) remains proportional to \( \sqrt{z} \) and increases with the strength of the noise.

**V. ENTROPY DENSITY IN THE FINAL DECOHERED STATE**

The decohered state has a finite entropy density which is a clear indication that the final state is a mixed state. To quantify the amount of information lost in the decoherence process (at the end of the drive protocol) we calculate the Von-Neumann entropy \( S = -N \text{tr} \rho_D \ln \rho_D \) in terms of the excitation probability as follow,

\[
S = -\frac{N}{2\pi} \int_{-\pi}^{\pi} dk \ln p_k + (1 - p_k) \ln(1 - p_k),
\]

where \( p_k \) is given in Eq [15]. The above integration is performed by expanding both the log terms in terms of \( e^{-4\pi z \eta_1^2 \sin^2 k(2e^{-2\pi z \sin^2 k} - 1)} \) and integrating each of the terms individually. The final result of the entropy density \( (S/N) \) can be expressed in the following series form,

\[
S/N = \ln 2 - \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \frac{2^{2m-r}(1)^r}{2m(2m-1)} \frac{1}{2m} e^{-\pi z Y_{m,0}^r} I_0(\pi z Y_{m,0}^r),
\]

where \( Y_{m,0}^r = (2m - r + 4m \eta_1^2) \). In Fig. 6 we plot Eq. (26) for different noise strengths. One can observe from the figure that the finite entropy density depends on the sweep speed and also that for each noise strength their exists an optimal quench rate either side of which the entropy density increases. In particular for \( z > z_1 \) (where \( z_1 \) is the optimal quench rate) the entropy increases which is the signature of AKZ behavior of defect production. For \( \eta_1 \neq 0 \), the entropy density asymptotically approaches \( \ln 2 \) for large-\( z \) value i.e., a fully mixed state is formed or in other words, the system approaches an asymptotic infinite temperature steady state. However, for the fast sweep speeds \( (z < z_1) \), the driven system is not affected by the noise. In the intermediate region the system has some finite non-zero entropy density which signifies a partially mixed state.

**VI. MAGNETIZATION AND SPIN-SPIN CORRELATIONS**

The expectation value of the spin-spin correlators with respect to the decohered state is conveniently obtained in terms of the pair products of Majorana fermion operators. Consider the Majorana fermion operators\textsuperscript{49,41,53,54}

\[
A_x = c_x^\dagger + c_x, \quad B_x = c_x^\dagger - c_x.
\]

The pair product of spins \( \sigma_\alpha^x \sigma_\alpha^{x+n} \) and that of the Jordan-Wigner string variable \( \tau_x \tau_{x+n} \) (with \( \tau_x = \Pi_{x<x'}(-\sigma_x^z) \))
can be expressed as follows\textsuperscript{41},
\begin{align}
\sigma_x^1 \sigma_x^1 &= B_x A_{x+1} B_{x+1} \ldots A_{x+n-1} B_{x+n-1} A_{x+n}, \quad (28) \\
\sigma_x^2 \sigma_x^2 &= A_x A_{x+1} B_{x+1} \ldots A_{x+n-1} B_{x+n-1} B_{x+n}, \quad (29) \\
\tau_x \tau_{x+1} &= A_x B_x A_{x+1} B_{x+1} \ldots A_{x+n} B_{x+n}. \quad (30)
\end{align}

Due to the sub-lattice structure of the decohered matrix i.e. decoupling of the decohered matrix, $\rho_D = \rho_E \otimes \rho_O$, with $\rho_E$ and $\rho_O$ acting correspondingly on the even and odd sub-lattice only, the expectation values of the full lattice and the sublattice, respectively. The sublattice correlators are obtained using the Wick’s theorem which requires one to evaluate correlators of the form $\langle B_x A_{x'} \rangle$, $\langle A_x A_{x'} \rangle$, $\langle A_x B_{x'} \rangle$ and $\langle B_x B_{x'} \rangle$. Out of these only the correlators of the type $\langle B_x A_{x'} \rangle$, $\langle A_x B_{x'} \rangle$ are needed to calculate the sublattice correlators which can be expressed in the form of the Toeplitz determinants.

The expectation value of the magnetization operator, $m_z$, evaluated with respect to the final decohered state and in terms of the Majorana operators is given by,
\begin{equation}
m_z = \langle \sigma_x^3 \rangle = \langle A_x B_x \rangle, \quad (33)
\end{equation}
and the expectation value of the magnetization correlator is given by,
\begin{equation}
\langle \sigma_x^3 \sigma_{x+2n}^3 \rangle = \langle A_x B_x A_{x'} B_{x'} \rangle. \quad (34)
\end{equation}

We first evaluate the magnetization and magnetization correlators following which we discuss $\sigma_x^2$ spin-spin correlators.

**A. Magnetization Density and Magnetization Correlation**

The average magnetization density $\langle \sigma_x^3 \rangle$ at the end of the quench protocol is given by,
\begin{equation}
m^z = \langle A_x B_x \rangle = 1 - 2n(z), \quad (35)
\end{equation}
where $n(z)$ is the noise dependent defect density [Eq. (16)] and in terms of which,
\begin{equation}
m^z = e^{-z} I_0[\pi z] - 2 e^{-\pi(z+z)} I_0[\pi(z+z)]. \quad (36)
\end{equation}

Thus the large-$z$ limit of the magnetization density given by,
\begin{equation}
m_z - 1 \approx - \frac{1}{\sqrt{2} \pi \sqrt{z}} - \pi z n^2_0, \quad (37)
\end{equation}
is consistent with the AKZ picture or in other words, it decreases after the optimal quench time, Eq (18), when the defect production starts to increase due to the noise (see Fig. 7).

Consider next the magnetization correlator in the $z$-direction, $\langle \sigma_x^3 \sigma_{x+2n}^3 \rangle$, given by,
\begin{equation}
\langle \sigma_x^3 \sigma_{x+2n}^3 \rangle = \langle \sigma_x^3 \rangle^2 - \left( 2 \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikn} \rho_k \right)^2. \quad (38)
\end{equation}
The connected correlator $C_n(z)$ is obtained by subtracting the position independent part, $\langle \sigma_x^3 \rangle^2$ from $\langle \sigma_x^3 \sigma_{x+2n}^3 \rangle$ and is given by $C_n(z) = -4d_n^2(z)$, where $d_n(z)$ is given by the equation Eq. (23). Therefore, the magnetization correlation for large-$n$ retains the KZ scaling relation with the correlation length given by
\begin{equation}
l_{noisy} = \sqrt{\pi (z + \bar{z})^2}, \quad (39)
\end{equation}
where we note that the magnetization correlation length increases as compared to the noiseless scenario. It is interesting to note that the presence of noise in the anisotropy decreases the magnetization density due to increased defect production, the correlation length of the magnetization correlator, however, increases with the strength of the noise. The amplitude of the magnetization correlator nevertheless decreases with the noise.

**B. Spin correlators: $\langle \sigma_x^1 \sigma_{x+1}^2 \rangle$**

As shown in Eq. (31) the spin correlators can be expressed as product of sublattice correlators. One can represent the sublattice correlators at $n$-separation in terms of the determinants of the Toeplitz matrices:
\begin{align}
\langle \langle \sigma_x^1 \sigma_x^{1+1} \rangle \rangle &= D_n[g^{+1,z}], \quad (40) \\
\langle \langle \sigma_x^2 \sigma_x^{2+1} \rangle \rangle &= D_n[g^{-1,z}], \quad (41)
\end{align}
where \( g^{m,z} \) are the generating functions defined as,

\[
g^{m,z}(\xi) = (-\xi)^m (1 - 2 p_k),
\]

where \( \xi = e^{2ik} \) and \( D_n[g^{m,z}] \) are the corresponding Toeplitz matrix determinants for different sub-lattice spin correlators. Given the generating function \( g^{m,z} \) the determinant \( D_n[g^{m,z}] \) are defined as \(^{28,41}\)

\[
D_n[g^{m,z}] = \frac{\prod_{k=0}^{n-1} f^{(m)}_k}{\prod_{k=0}^{n-1} f^{(m)}_{-k}}
\]

where the elements of the determinant \( f^{(m)}_l \) is the \( l^{th} \) cumulant of the generating function \( g^{m,z}(\xi) = \sum_l f^{(m)}_l \xi^l \) and are obtained by performing the following contour integration,

\[
f^{(m)}_l = \oint_C \frac{d\xi}{2\pi i} \xi^{-l} g^{m,z}(\xi),
\]

where \( C \) is a unit circle contour with \( |\xi| = 1 \). The above integral in terms of the \( k \)-variable acquires the form,

\[
f^{(m)}_l = \int_{-\pi/2}^{\pi/2} \frac{dk}{\pi} e^{-ik} g^{m,z}(e^{ik}).
\]

The integral is evaluated by taking the integral representation of the modified Bessel function of the first kind,

\[
I_\nu(z) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} d\theta z^{\nu} e^{-i\theta} \sin \theta,
\]

where \( \nu \) is an integer and Re(\( z \)) > 0. Using the above integral identity yields

\[
f^{(m)}_l = \frac{e^{-\pi z}}{2\pi i} I_{l-m}(\pi z) - \frac{e^{-\pi(z+\bar{z})}}{2\pi i} I_{l-m}(\pi(z+\bar{z})),
\]

with \( m = 0, \pm 1 \). In the large-\( z \) limit (and with \( z\eta^2 \neq 0 \)), the expression reduces to

\[
f^{(m)}_l = (-1)^m \frac{1}{\pi} \left[ \frac{e^{-(l-m)^2/2\pi z}}{\sqrt{2\pi z}} + \sqrt{2} \frac{e^{-(l-m)^2/2\pi(z+\bar{z})}}{\sqrt{z+\bar{z}}} \right].
\]

In the Figs. 8 and 9 we plot the numerically calculated determinants for the sub-lattice correlators \( \langle \sigma_x^{1,2} \sigma_{x+r}^{1,2} \rangle \). Figure 8 corresponds to the noiseless case where we observe the KZ scaling behavior for the correlation length, i.e., larger correlation length for smaller sweep speeds. In Fig. 9 we consider fast noise and use the elements of the Toeplitz matrix given by Eq. (48) to evaluate the determinants. Here we notice the signature of the AKZ behavior, i.e., decrease in the correlation length with the increasing strength of the noise for slower sweeps. Similar behavior is found for the \( \langle \tau_z \tau_{z+n} \rangle \) correlator.

![FIG. 8. Sub-lattice spin correlators \( \langle \sigma_x^{1,2} \sigma_{x+r}^{1,2} \rangle \) have been plotted with respect to the separation-\( r \) in the noiseless scenario. The correlators in the figure are normalized from their respective value at separation \( r = 2 \). From the figure one can observe that the larger \( z \)-values correspond to the large correlation length which is consistent with the KZ behavior. The correlation length \( l_\sigma \) becomes small (short ranged spin correlations) in the fast sweep regime. From the inset one can notice the KZ behavior at relatively larger separation.](image)

![FIG. 9. Normalized \( \langle \sigma_x^{1,2} \sigma_{x+r}^{1,2} \rangle \) Vs. \( r \) for different noise strengths (\( \eta_1 = 0.01 \) and \( \eta_1 = 0.05 \)) at \( z = 200 \) (slow sweep regime). The correlation length \( l_\sigma \) decreases with the increased noise strength, which is the signature of AKZ scaling behavior.](image)

### C. Spin Correlations at Large Separation

In this section, we investigate the behavior of spin correlators at large separation in the presence of the fast noise. In the asymptotic limit, following Szeg"{o}'s limit theorem, the Toeplitz determinant acquires the form \(^{28,41,53-36}\)

\[
D_n[g^{m,z}(\xi)] \approx e^{\exp \left[ \frac{n}{\pi} \int_0^\pi d\theta \ln g^{m,z}(\xi^{2\theta}) \right]},
\]

where \( g^{m,z}(\xi) \) is the generating function given in Eq. (43).
The zeroes of the generating function plays an important role in the analyticity of the asymptotic behavior of the spin correlators and it can be easily verified that it has the same set of zeroes as the noiseless generating function. Therefore, the effect of zeroes of the generating function on the analyticity of the spin correlators with respect to \( z \) will remain the same as for the noiseless drive case. The difference is that the generating function with noisy drive is multiplied by an extra noise dependent exponential factor, \( e^{-2\pi z \eta(z)} \). We are mainly interested in the role of this extra term on the correlation lengths of the sublattice and the full lattice spin correlators. In the following discussion we will use the method developed by Cherng and Levitov (41) to show that this extra term is responsible for the AKZ scaling behavior of the correlation lengths.

The generating function for the noisy drive can be written as

\[
g_{\text{in}}(\xi) = -(-\xi)^m \lambda_0^{-1} (1 - \lambda_0 \xi)(1 - \lambda_0^{-1}) e^{H(\xi)},
\]

where

\[
H(\xi) = h(\xi) - 2\pi z \eta_1^2 (1 - x),
\]

and

\[
h(\xi) = \ln \left[ \frac{1 - e^{-\pi z(1-2 z_1/z-x)}}{2(1-2 z_1/z-x)} \right].
\]

The zeroes closest to the unit circle are denoted by \( \lambda_0 \) and \( \lambda_0^{-1} \), where \( \lambda_0 = 1/\exp \left[ \cosh^{-1} \left( 1 - 2 z_1/z \right) \right] \). Both \( h(\xi) \) and \( H(\xi) \) can be expanded as a function of \( x = (\xi + \xi^{-1})/2 \) as follows,

\[
h(x) = \sum_{n \geq 0} h_n x^n, \quad H(x) = \sum_{n \geq 0} H_n x^n,
\]

where we notice that \( H_0 = h_0 - 2\pi z \eta_1^2 \) and \( H_1 = h_1 + 2\pi z \eta_1^2 \), while for \( n \neq 0, 1 \), all other coefficients satisfy \( H_n = h_n \). The correlation length of the sub-lattice spin correlators are given by the following expression

\[
l^{-1}_{\sigma/r} = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} H(x) = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} h(x) + \pi^2 z,
\]

where the full lattice correlator is given by \( l^{-1} = l^{-1}_{\sigma} + l^{-1}_{\tau} \). The result of the above integration yields

\[
l^{-1}_{\sigma/r} \approx \sum_{m=0}^{\infty} \frac{\Gamma(m+1/2)^2 \text{Re} \left[ L_{0m+3/2}(2) \right]}{\pi^{1/2} \Gamma(m+1)(2\pi z)^{m+1/2} + \pi^2 z},
\]

where the first term is due to the noiseless case, Ref. (41), which in the limit of slow sweep speed results in the correlation length or the domain size proportional to \( \sqrt{z} \) thus satisfying the KZ scenario. On the other hand, for the fast noise and large \( z \) scenario the scaling of the correlation length is consistent with the AKZ picture suggesting that the domain size reduces, this result quantifies how the fast noise randomizes the driven system spatially at the end of the protocol. The sub-lattice correlation lengths have been plotted in Fig. 10 with respect to \( z \) for different noise strengths, both the \( l^{-1}_{\sigma} \) and \( l^{-1}_{\tau} \) show the anti-Kibble-Zurek scaling behavior i.e., the sub-lattice correlation lengths \( l^{-1}_{\sigma,\tau} \) decreases after the optimal quench rate which depends on the noise strength. Furthermore, the \( l^{-1}_{\tau} \) shows non-analytical behavior at \( z = z_1 \) and beyond \( z > z_1 \) both have same values. Therefore, it is understood that the increased noise strength of the fast noise further decreases \( l \), making the spin correlators relatively short ranged in the presence of the fast noise. The \( l^{-1} = l^{-1}_{\sigma} + l^{-1}_{\tau} \) also shows the non-analytical behavior at \( z = z_1 \) as a result of the non-analytical behavior of \( l^{-1}_{\tau} \), this is the indication of the crossover of different behaviors of the spin correlators i.e. non-oscillatory monotonically decreasing behavior for \( z < z_1 \) regime and exponentially suppressed oscillatory behavior of spin correlators for \( z > z_1 \) regime with respect to the spatial separation.

FIG. 10. The sub-lattice inverse correlation lengths have been plotted as a function of \( z \) for the different noise strengths, both the \( l^{-1}_{\sigma} \) and \( l^{-1}_{\tau} \) show the anti-Kibble-Zurek scaling behavior i.e., the sub-lattice correlation lengths \( l^{-1}_{\sigma,\tau} \) decreases after the optimal quench rate which depends on the noise strength. Furthermore, the \( l^{-1}_{\tau} \) shows non-analytical behavior at \( z = z_1 \) and beyond \( z > z_1 \) both have same values. Therefore, it is understood that the increased noise strength of the fast noise further decreases \( l \), making the spin correlators relatively short ranged in the presence of the fast noise. The \( l^{-1} = l^{-1}_{\sigma} + l^{-1}_{\tau} \) also shows the non-analytical behavior at \( z = z_1 \) as a result of the non-analytical behavior of \( l^{-1}_{\tau} \), this is the indication of the crossover of different behaviors of the spin correlators i.e. non-oscillatory monotonically decreasing behavior for \( z < z_1 \) regime and exponentially suppressed oscillatory behavior of spin correlators for \( z > z_1 \) regime with respect to the spatial separation.

For small-\( z \) values the fast noise has negligible effect.
VII. CONCLUDING REMARKS

In this work we quantify analytically the effects of the fast Gaussian noise in the driven quantum XY-spin chain. We have considered transverse protocol in which the external transverse magnetic field drives the system linearly with respect to time, starting from the paramagnetic phase to a region with the ferromagnetic phase and finally back to the paramagnetic phase in the presence of the time dependent Gaussian noise in the anisotropy term. In this protocol, the system passes through two quantum critical points at \( h = \mp J \) where the energy gap vanishes resulting in non-adiabatic effects. We map the problem to the noisy LZ problem and in terms of the density matrix formalism we obtain a reduced master equation for the population inversion. The solution of the equation in the \( T \to \infty \) limit has been utilized to obtain the final excitation probability.

The implications for the correlators due to the non-equilibrium dynamics of the noisy transverse drive protocol are as follows: first, the fast fluctuating coherences vanishes in the large time limit (even without the noise) due to the internal decoherence arising from the large system size (\( N \to \infty \)) and this allows the course graining in momentum space and transforms the pure state into an entropic state with finite non-zero entropy. The time dependent Gaussian fast noise further exponentially suppresses the highly fluctuating coherences and in particular affects the system most when the system passes through the quantum critical points. Finally, the noise can heat up the population to the asymptotic infinite temperature state for the slower sweeps or large-\( n \) case, maximizing the entropy density to \( \log 2 \) and at the same time minimizing the average magnetization density to zero. The effects of the noise are minimized when driven at an optimal sweep rate which turns out to scale universally with the strength of the fast noise.

We have analyzed the spin-spin correlation functions (in the presence of noise) at the end of the protocol for large separation using the Toeplitz determinant asymptotes at large-\( n \). For slow sweep speeds the effect of the fast Gaussian noise on the correlation lengths of the spin correlators at large-separation reveals behavior consistent with the anti-Kibble-Zurek picture. The sub-lattice correlation lengths for \( \sigma^z \)-spin correlators decreases with the strength of the noise according to the anti-Kibble-Zurek scaling behavior. We have also analysed the effect of the fast noise on the magnetization \( \sigma^z \)-spin correlator. For large \( n \)-separation we find that the correlation length of the magnetization correlator increases with the strength of noise when \( \eta_1 \ll 1 \) and \( \pi^2 < 1 \).

VIII. ACKNOWLEDGEMENT

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