The Birkhoff theorem
for unitary matrices of prime-power dimension

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Abstract

The unitary Birkhoff theorem states that any unitary matrix with all row sums and all column sums equal unity can be decomposed as a weighted sum of permutation matrices, such that both the sum of the weights and the sum of the squared moduli of the weights are equal to unity. If the dimension $n$ of the unitary matrix equals a power of a prime $p$, i.e. if $n = p^w$, then the Birkhoff decomposition does not need all $n!$ possible permutation matrices, as the epicirculant permutation matrices suffice. This group of permutation matrices is isomorphic to the general affine group $GA(w, p)$ of order only $p^w(p^w - 1)(p^w - p)...(p^w - p^{w-1}) \ll (p^w)!$.

1 Introduction

Let $D(n)$ be the semigroup of $n \times n$ doubly stochastic matrices; let $P(n)$ be the group of $n \times n$ permutation matrices. Birkhoff $[1]$ has demonstrated

Theorem 1 Every $D(n)$ matrix $D$ can be written

$$D = \sum_{\sigma} c_{\sigma} P_{\sigma}$$

with all $P_{\sigma} \in P(n)$ and the weights $c_{\sigma}$ real, satisfying both $0 \leq c_{\sigma} \leq 1$ and $\sum_{\sigma} c_{\sigma} = 1$.

The question arises whether a similar theorem holds for matrices from the unitary group $U(n)$. This question is discussed by De Baerdemacker et al. $[2]$ $[3]$. For this purpose, the subgroup $XU(n)$ of $U(n)$ is introduced $[4]$ $[5]$. It consists of all $U(n)$ matrices with all line sums (i.e. all row sums and all column sums) equal to 1. Whereas $U(n)$ is an $n^2$-dimensional Lie
group, the group XU(n) is only \((n - 1)^2\)-dimensional. A unitary Birkhoff theorem has been proved for XU(n) matrices \([2]\ [3]\). Remarkable is the fact that the case \(n = p\) with \(p\) an arbitrary prime \([3]\) has been treated in a very different way from the case where \(n\) is an arbitrary integer \([2]\). As a result, the decomposition, tailored to prime numbers \([3]\), can be restricted to \(n^2\) terms, whereas the general case \([2]\) leads to a summation over all \(n!\) (or at least over \(n!/2\)) permutation matrices, albeit with a large number of degrees of freedom. In the present paper, we will treat the two cases in a unified way. Moreover, the unified approach will be applied to the case \(n = p^w\), i.e. \(n\) equal to an arbitrary power \(w\) of an arbitrary prime \(p\).

In general, the Birkhoff theorem for unitary matrices is easily proved as follows. Let \(G(n)\) be a finite subgroup of XU(n).

**Lemma 1** If an XU(n) matrix \(X\) can be written

\[
X = \sum_{\sigma} c_{\sigma} G_{\sigma}
\]

with all \(G_{\sigma} \in G(n)\), then the weights \(c_{\sigma}\) satisfy \(\sum_{\sigma} c_{\sigma} = 1\).

The proof is trivial: all line sums of \(G_{\sigma}\) equal unity; therefore, all line sums of the matrix \(c_{\sigma} G_{\sigma}\) equal \(c_{\sigma}\) and thus all line sums of the matrix \(\sum_{\sigma} c_{\sigma} G_{\sigma}\) are equal to \(\sum_{\sigma} c_{\sigma}\). As all line sums of \(X\) are equal to 1, we thus need \(\sum_{\sigma} c_{\sigma} = 1\).

**Lemma 2** If every XU(n) matrix \(X\) can be written

\[
X = \sum_{\sigma} a_{\sigma} G_{\sigma}
\]

with all \(G_{\sigma} \in G(n)\), then there exists a decomposition

\[
X = \sum_{\sigma} b_{\sigma} G_{\sigma},
\]

such that not only \(\sum_{\sigma} b_{\sigma} = 1\), but also \(\sum_{\sigma} |b_{\sigma}|^2 = 1\).

This fact follows from the Klappenecker–Rötteler theorem \([6]\).

2 The group XU(n)

**Remark 1** For sake of convenience, in the present paper, the rows and columns of a matrix are not numbered starting from 1, but instead starting from 0. Thus the upper-left entry of any \(m \times m\) square matrix \(A\) is \(A_{0,0}\) and its lower-right entry is \(A_{m-1,m-1}\).
We recall that the group Xu(n) is an \((n - 1)^2\)-dimensional subgroup of the \(n^2\)-dimensional unitary group U(n). Any member \(X\) of Xu(n) can be written
\[
X = T \begin{pmatrix} 1 \\ U \end{pmatrix} T^{-1},
\]
where \(U\) is a member of U\((n - 1)\) and where the constant unitary matrix \(T\) is \(1/\sqrt{n}\) times a dephased complex Hadamard matrix \([7]\). Thus (1) constitutes a 1-to-1 mapping between \(X\) and \(U\). Because of \(T_{j,0} = T_{0,k} = 1/\sqrt{n}\),
\[
T_{j,0} = T_{0,k} = 1/\sqrt{n},
\]
eqn (1) leads to
\[
X_{k,l} = \frac{1}{n} + \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} T_{k,r} U_{r-1,s-1} (T^{-1})_{s,l}.
\]
With \(T\) being unitary, i.e. with \(T^{-1} = T^\dagger\), this becomes
\[
X_{k,l} = \frac{1}{n} + \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} U_{r-1,s-1} T_{k,r} T_{l,s}.
\]
We thus can write the matrix \(X\) as a sum of \(1 + (n - 1)^2\) matrices:
\[
X = W + \frac{1}{n} \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} U_{r-1,s-1} M_{r,s},
\]
where \(W\) is the \(n \times n\) van der Waerden matrix, i.e. the doubly stochastic matrix with all entries equal to \(1/n\), and where \(M_{r,s}\) is an \(n \times n\) matrix defined by
\[
(M_{r,s})_{k,l} = n T_{k,r} T_{l,s}.
\]
The labels \(r\) and \(s\) of the matrix \(M_{r,s}\) run from 1 to \(n - 1\), in contrast to the indices \(k\) and \(l\) of its entries, which run from 0 to \(n - 1\). We thus have \((n - 1)^2\) such matrices, each having \(n^2\) entries. Each entry of the matrix \(M_{r,s}\) equals the leftmost entry of its row times the uppermost entry of its column. Taking into account (2), one indeed easily checks
\[
(M_{r,s})_{0,l} (M_{r,s})_{k,0} = (M_{r,s})_{k,l}.
\]
Both the first row and the first column of \(M_{r,s}\) equal a line of the Hadamard matrix \(T\) (up to complex conjugation and up to the factor \(\sqrt{n}\)):
\[
(M_{r,s})_{0,l} = \sqrt{n} T_{l,s},
(M_{r,s})_{k,0} = \sqrt{n} T_{k,r}.
\]
Because $T$ is $1/\sqrt{n}$ times a Hadamard matrix, we have $|T_{i,s}| = 1/\sqrt{n}$ and $|T_{k,r}| = 1/\sqrt{n}$, such that $|(M_{r,s})_{0,l}| = 1$ and $|(M_{r,s})_{k,0}| = 1$, and thus, because of (5), we conclude that all entries $(M_{r,s})_{k,l}$ have unit modulus.

3 Underlying framework

In the present section, we consider an arbitrary doubly transitive group $G(n)$ of $n \times n$ permutation matrices. We denote by $N$ the order of the group. We generalize the ideas and computations in Reference [2], where $G(n)$ is equal to the group $P(n)$ of all $n \times n$ permutation matrices, thus $G(n)$ being isomorphic to the symmetric group $S_n$ and $N$ being equal to $n!$.

In the next three sections, we will apply the Lemmas 1 and 2 to three different choices of $G(n)$:

- In case of arbitrary $n$, we choose the group of all $n \times n$ permutation matrices (i.e. a group isomorphic to the symmetric group $S_n$). See Section 4.

- In case of $n$ equal to some prime $p$, we choose the group of all $n \times n$ supercirculant permutation matrices (i.e. a group isomorphic to a semidirect-product group $C_n : C_{n-1}$). See Section 5.

- In case of $n$ equal to some power $w$ of some prime $p$ (i.e. equal to $p^w$), we choose the group of all $n \times n$ epicirculant permutation matrices (i.e. a group isomorphic to the general affine group $GA(w, p)$). See Section 6.

The meaning of the words ‘supercirculant’ and ‘epicirculant’ will be made clear below. The mentioned groups are doubly transitive, as it is known that the symmetric group $S_n$ is $n$-transitive, the alternating group $A_n$ is $(n-2)$-transitive, and the affine groups are 2-transitive [5], in contrast to e.g. the cyclic group $C_n$, which is only 1-transitive.

In each of the three cases, we will prove below that every $XU(n)$ matrix $X$ can be written as

$$X = \sum_{\sigma} c_{\sigma} G_{\sigma} \quad (7)$$

with all $G_{\sigma}$ member of the appropriate group $G(n)$. Because of Lemmas 1 and 2, we are then allowed to put the case that both $\sum_{\sigma} c_{\sigma} = 1$ and $\sum_{\sigma} |c_{\sigma}|^2 = 1$. For the explicit computation of the weights $c_{\sigma}$, we note that the $G(n)$ matrices form an $n$-dimensional reducible representation of
some abstract group $G$. We assume that $G$ has $\mu$ different irreducible representations. According to Lemma (29.1) of [9], because $G$ is 2-transitive, the $n$-dimensional natural representation is the sum of the 1-dimensional trivial representation and an $(n - 1)$-dimensional irreducible representation, which we will call the standard representation.

We replace eqn (7) by an eqn concerning one of the $\mu$ irreducible representations of $G$:

$$U^{(\nu)} = \sum_\sigma c_\sigma D^{(\nu)}_\sigma,$$

where $\nu$ is the label of the irrep ($0 \leq \nu \leq \mu - 1$), where $D^{(\nu)}_\sigma$ is the $\nu$ th irreducible representation of $G_\sigma$, and $U^{(\nu)}$ is an appropriate $n_\nu \times n_\nu$ unitary matrix, with a special mentioning for $\nu = 0$ and $\nu = 1$ (see further). Here, $n_\nu$ is the dimension of the $\nu$ th representation. We have $\mu$ such matrix equations (8). Each matrix eqn constitutes $n_{\nu}^2$ scalar equations. We thus have a total of $\sum_{\nu=0}^{\mu-1} n_{\nu}^2 = N$ scalar equations with $N$ unknowns $c_\sigma$:

$$\sum_\sigma c_\sigma (D^{(\nu)}(\sigma))_{k,l} = (U^{(\nu)})_{k,l}.$$

Solution of this set of equations is:

$$c_\sigma = \frac{1}{N} \sum_\nu n_\nu \sum_{i=0}^{n_\nu - 1} \sum_{j=0}^{n_\nu - 1} (D^{(\nu)}(\sigma))_{i,j} (U^{(\nu)}(\sigma))_{i,j}$$

$$= \frac{1}{N} \sum_\nu n_\nu \text{Tr} \left( D^{(\nu)}(\sigma)^\dagger U^{(\nu)}(\sigma) \right).$$

We choose for $\nu = 0$ the trivial representation, i.e. the 1-dimensional irreducible representation with all characters equal to 1. We choose for $\nu = 1$ the standard representation, i.e. the $(n - 1)$-dimensional irreducible representation obtained by applying (1) to the permutation matrix $P_\sigma$:

$$P_\sigma = T \begin{pmatrix} 1 \\ D^{(1)}(\sigma) \end{pmatrix} T^{-1}$$

and thus

$$\begin{pmatrix} 1 \\ D^{(1)}(\sigma) \end{pmatrix} = T^{-1} P_\sigma T.$$

In (9), the matrix $U^{(0)}(\sigma)$ equals the $1 \times 1$ unit matrix and the matrix $U^{(1)}(\sigma)$ equals the $(n - 1) \times (n - 1)$ lower-right block of

$$\begin{pmatrix} 1 \\ U \end{pmatrix} = T^{-1} XT.$$
For the remaining matrices $U^{(\nu)}(\sigma)$ with $2 \leq \nu \leq \mu - 1$, we are allowed to choose any unitary matrix of the right dimension $n_\nu$. This usually allows a large number of degrees of freedom. Here, we propose two different strategies to take advantage of this freedom.

### 3.1 First strategy

For each matrix $U^{(\nu)}(\sigma)$ with $2 \leq \nu \leq \mu - 1$, we choose the $n_\nu \times n_\nu$ unit matrix. Then (9) becomes

$$c_\sigma = \frac{1}{N} \left[ n_0 \text{Tr} \left( D^{(0)}(\sigma) \right) + n_1 \text{Tr} \left( D^{(1)}(\sigma)U \right) + \sum_{\nu=2}^{\mu-1} n_\nu \text{Tr} \left( D^{(\nu)}(\sigma) \right) \right].$$

We take advantage of Shur’s orthogonality relation:

$$\sum_\nu n_\nu \text{Tr} \left( D^{(\nu)}(\sigma) \right) = \sum_\nu n_\nu \text{Tr} \left( D^{(\nu)}(\sigma) \right) D^{(\nu)}(\epsilon) = \delta_\sigma N,$$

where $\epsilon$ is the trivial identity permutation and where $\delta_\epsilon = 1$ while $\delta_\sigma = 0$ if $\sigma \neq \epsilon$. Because moreover $D^{(1)}(\sigma) = D^{(1)}(\sigma^{-1})$ and $n_1 = n - 1$, we obtain the explicit expression for the weight:

$$c_\sigma = \delta_\sigma + \frac{n - 1}{N} \text{Tr} \left( D^{(1)}(\sigma^{-1})U \right) - \frac{n - 1}{N} \chi^{(1)}(\sigma^{-1}).$$

The number $\chi^{(\nu)}(G)$ denotes the character of the element $G$ of the group $G$ according to the $\nu$th representation. It is equal to $\text{Tr}(D^{(\nu)}(G))$. In particular, we have $\text{Tr}(D^{(1)}(G)) = \text{Tr}(G) - 1.$

### 3.2 Second strategy

The second strategy is only applicable if the group $G$ has an anti-standard irreducible representation, non-equivalent to the standard representation. The anti-standard representation, which we will assign the label $\nu = 2$ (if it exists), has the same characters as the standard representation (with label $\nu = 1$), except for a factor $-1$ if the corresponding permutation is an odd permutation. A necessary condition for the second strategy is

$$N \geq 2 + 2(n - 1)^2.$$  

As in the first strategy, we again choose the $1 \times 1$ unit matrix for $U^{(0)}(\sigma)$ and the $(n - 1) \times (n - 1)$ matrix $U$ for $U^{(1)}(\sigma)$. However, in this second
strategy, we also choose the matrix $U$ for each matrix $U^{(2)}(\sigma)$. For each matrix $U^{(\nu)}(\sigma)$ with $3 \leq \nu \leq \mu - 1$, we choose the $n_{\nu} \times n_{\nu}$ unit matrix. Then (9) becomes

$$c_\sigma = \frac{1}{N} \left[ n_0 \text{Tr} \left( D^{(0)}(\sigma) \right) + n_1 \text{Tr} \left( D^{(1)}(\sigma) \right) + n_2 \text{Tr} \left( D^{(2)}(\sigma) \right) + \sum_{\nu=3}^{\mu-1} n_\nu \text{Tr} \left( D^{(\nu)}(\sigma) \right) \right].$$

(13)

Again taking advantage of Shur’s orthogonality relation and $n_1 = n_2 = n - 1$, we obtain

$$c_\sigma = \delta_{\sigma} + \frac{2(n-1)}{N} \text{Tr} \left( D^{(1)}(\sigma^{-1})U \right) - \frac{2(n-1)}{N} \chi^{(1)}(\sigma^{-1}) \quad \text{if } \sigma \text{ even}$$

$$= 0 \quad \text{if } \sigma \text{ odd}. \quad (14)$$

In the second strategy, the group $G \cap A_n$ thus takes over the role of $G$ and $N/2$ takes over the role of $N$.

4 The case of arbitrary dimension $n$

**Lemma 3** Every $XU(n)$ matrix $X$ can be written

$$X = \sum_\sigma c_\sigma P_\sigma$$

with all $P_\sigma \in P(n)$.

The proof is provided by [3], by means of induction on $n$. Combining Lemmas 1, 2, and 3 leads to the unitary Birkhoff theorem:

**Theorem 2** Every $XU(n)$ matrix $X$ can be written

$$X = \sum_\sigma c_\sigma P_\sigma$$

with all $P_\sigma \in P(n)$, such that both $\sum_\sigma c_\sigma = 1$ and $\sum_\sigma |c_\sigma|^2 = 1$.

4.1 First strategy

We can apply result (11) with $N = n!$. The only possible values of $\chi^{(1)}$ are $\text{Tr}(P_\sigma) - 1$ and thus $-1, 0, 1, 2, \ldots, n - 1$, with exception of $n - 2$. 7
4.2 Second strategy

The character tables of the groups $S_2$ and $S_3$ show no anti-standard representation. For $n > 3$, the group $S_n$ has an anti-standard representation. In this case, we can apply result (14) with $N = n!$. The restriction $n > 3$ is not surprising, as (12) with $N = n!$ is fulfilled neither if $n = 2$ nor if $n = 3$.

5 The case of prime dimension $n = p$

We call an $n \times n$ matrix $A$ supercirculant iff each row $k$ equals row $k - 1$ shifted $x$ positions to the right. Thus $A_{k,l} = A_{k-1,l-x}$, where addition and subtraction are modulo $n$. We equivalently may write

$$A_{0,a} = A_{k,a+kx} .$$

We call $x$ the pitch of the matrix. If $x = 1$, then the supercirculant matrix is called circulant; if $x = n - 1$, then the supercirculant matrix is called anticirculant.

If $p$ denotes a prime, then the $p \times p$ supercirculant permutation matrices are denoted $S_{a,x}$, where $x$ is the pitch and $a$ (called the shift) is the column with the unit entry in the upper row (i.e. row 0). The unit entries of such $p \times p$ permutation matrix thus are located at the $p$ positions $(0, a)$, $(1, a+x)$, $(2, a+2x)$, ..., and $(p-1, a+(p-1)x)$, where sums are to be taken modulo $p$. Because $x$ and $p$ are co-prime, the consecutive columns with a 1, i.e. the columns $a$, $a+x$, $a+2x$, ..., and $a+(p-1)x$, are all different.

If $n$ equals some prime $p$, then we choose for the $p \times p$ Hadamard matrix $T$ of Section 2 the $p \times p$ discrete Fourier transform $F$, with entries

$$F_{k,l} = \frac{1}{\sqrt{p}} \omega^{kl} ,$$

where $\omega$ is equal to the $p$th root of unity. Thus (4) becomes

$$(M_{r,s})_{k,l} = \omega^{kr-ls} .$$

From [3], we know that $M$ can be written as a weighted sum of $p$ supercirculant permutation matrices:

$$M_{r,s} = \sum_{a=0}^{p-1} (M_{r,s})_{0,a} S_{a,x(r,s)} ,$$

(15)
where the pitch $x$ of the matrix $S_{a,x}$ is a function of $r$ and $s$. Indeed, the condition

$$(M_{r,s})_{k,a+kx} = (M_{r,s})_{0,a}$$

yields

$$kr - (a + kx)s = -as$$

and thus $r - xs = 0$. Thus $x$ has to satisfy the eqn

$$sx = r \mod p,$$

This eqn has one solution:

$$x = rs^{-1} \mod p,$$

where $s^{-1}$ is the inverse of $s$ modulo $p$. As $p$ is prime, each non-zero integer has exactly one inverse. With $(M_{r,s})_{0,a} = \omega^{-as}$, we finally obtain

$$M_{r,s} = \sum_{a=0}^{p-1} \omega^{-as} S_{a,rs^{-1}}.$$

The supercirculant $p \times p$ permutation matrices form a group $S(p)$, subgroup of $P(p)$ (proof in Appendix A), isomorphic to the semidirect product of the cyclic group of order $p$ and the multiplicative group of integers modulo $p$. The group thus is isomorphic to the semidirect product of two cyclic groups:

$$C_p : C_{p-1},$$

a non-Abelian group of order $p(p - 1)$.

Lemma 4 If $n$ is prime, then every $XU(n)$ matrix $X$ can be written

$$X = \sum_{\sigma} c_{\sigma} S_{\sigma}$$

with all $S_{\sigma} \in S(n)$.  

The proof is as follows. If $n$ is a prime $p$, then all matrices $M_{r,s}$ are supercirculant with a pitch $x = rs^{-1}$ modulo $p$. Also the van der Waerden matrix $W$ is supercirculant, as it is circulant:

$$W = \sum_{a=0}^{n-1} \frac{1}{n} S_{a,1}.$$

Hence, according to [3], $X$ is a weighted sum of supercirculant permutation matrices.

Combining Lemmas 1, 2, and 4 leads to
**Theorem 3** If $n$ is prime, then every $XU(n)$ matrix $X$ can be written

$$X = \sum_{\sigma} c_{\sigma} S_{\sigma}$$

with all $S_{\sigma} \in S(n)$, such that both $\sum_{\sigma} c_{\sigma} = 1$ and $\sum_{\sigma} |c_{\sigma}|^2 = 1$.

5.1 First strategy

We can apply result (11) with $N = p(p-1)$. The only possible values of $\chi^{(1)}$ are $-1, 0,$ and $p−1$, as demonstrated in Appendix B. Thus we find a unitary Birkhoff decomposition with only $p(p−1)$ terms. For a prime exceeding 3, this number is substantially smaller than the number $p!/2$ of Subsection 4.2. The resulting unitary Birkhoff theorem is also slightly stronger than the theorem in [3], where the Birkhoff decomposition consists of $p^2$ terms.

5.2 Second strategy

The group $S(2)$, isomorphic to the cyclic group $C_2$, has only two irreducible representations: the trivial one and the standard one. Also the group $S(n)$ with $n$ equal to an odd prime $p$, has no inequivalent anti-standard representation. Indeed, because all odd supercircular permutations have non-unit pitch (see Appendix C) and thus have unit trace (see Appendix B) and hence have zero character $\chi^{(1)}$, all characters of the anti-standard representation equal the corresponding characters of the standard representation. Therefore, the standard and anti-standard representations are equivalent. We conclude that we cannot apply the second strategy of Subsection 3.2. The absence of any inequivalent anti-standard representation is no surprise, as $N = n(n−1)$ does not satisfy (12).

6 The case of prime-power dimension $n = p^w$

For $n = p^w$ with arbitrary positive $w$, we can choose for $T$ of Section 2 the Kronecker product of $w$ small (i.e. $p \times p$) Fourier matrices $F$:

$$T = F \otimes F \otimes \ldots \otimes F = F^{\otimes w}.$$ 

The $n \times n$ matrix $T$ has following entries:

$$T_{a,b} = \frac{1}{\sqrt{n}} \omega^{f(a,b)},$$

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where $f(x, y)$ is the sum of the ditwise product of the $p$-ary numbers $x$ and $y$:

$$f(x, y) = \sum_j x_j y_j \mod p.$$ 

As a consequence, we have

$$(M_{r,s})_{k,l} = \omega^{f(k,r)-f(s,l)} . \quad (16)$$

Among the $n^2$ entries of this matrix, $n^2/p$ are equal to 1, $n^2/p$ are equal to $\omega$, ..., and $n^2/p$ are equal to $\omega^{p-1}$.

**Remark 2** For sake of convenience, below, the rows and the columns of a matrix will sometimes be pointed at, not by a number, but instead by a vector. This will allow matrix computations for the row and column numbers. For this purpose, any number $z = z_0 + z_1 p + z_2 p^2 + ... + z_{w-1} p^{w-1}$ has an associated boldfaced $w \times 1$ vector $\mathbf{z} = (z_0, z_1, z_2, ..., z_{w-1})^T$, consisting of the $w$ dits of the number $z$.

We call a matrix $A$ epicirculant if row $k$ equals row 0, ‘shifted to the right’ according to

$$A_{0,a} = A_{k,a+xk} ,$$

where $a$ is the $w \times 1$ vector associated with the column number $a$ and where $\mathbf{x}$ is a $w \times w$ matrix called the pitch matrix, consisting of $w^2$ entries, all $\in \{0, 1, ..., p - 1\}$. A matrix of the form (16) is automatically epicirculant. It is a weighted sum of epicirculant permutation matrices $E$: we have

$$M_{r,s} = \sum_{a=0}^{p-1} (M_{r,s})_{0,a} E_{a,\mathbf{x}(r,s)} . \quad (17)$$

Here, $\mathbf{x}$ is an appropriate $w \times w$ pitch matrix, depending on $r$ and $s$. Proof is in Appendix D. We note that vector $a$ and matrix $\mathbf{x}$ constitute a pair, fully specifying an affine transformation [10].

If $n$ is a prime power, say $n = p^w$, then the epicirculant $p^w \times p^w$ permutation matrices form a group $E(n)$, subgroup of $P(n)$ (proof in Appendix E), isomorphic to the general affine group $GA(w, p)$, a semidirect product of the direct product of cyclic groups of order $p$ and the general linear group $GL(w, p)$:

$$GA(w, p) = C_p^w \rtimes GL(w, p)$$

of order

$$p^w (p^w - 1)(p^w - p)(p^w - p^2) ... (p^w - p^{w-1}) . \quad (18)$$
We note that \( \text{GA}(w, p) \) is a maximal subgroup of the symmetric group \( S_{p^w} \) (O’Nan–Scott theorem) \([11]\).

Each of the \( w \) subgroups \( C_p \) consists of \( p \) matrices, each a Kronecker product with a total of \( w \) factors:

\[
I \otimes I \otimes ... \otimes I \otimes M \otimes I \otimes ... \otimes I,
\]

where \( I \) denotes the \( p \times p \) unit matrix and \( M \) a \( p \times p \) circulant permutation matrix \( S_{a,1} \).

**Lemma 5** If \( n \) is a prime power, then every \( XU(n) \) matrix \( X \) can be written

\[
X = \sum_{\sigma} c_\sigma E_\sigma
\]

with all \( E_\sigma \in E(n) \).

The proof is as follows. If \( n \) is a prime power \( p^w \), then all matrices \( M_{r,s} \) are epicirculant with an invertible pitch matrix \( x \). Also the van der Waerden matrix \( W \) is epicirculant, as it is circulant:

\[
W = \frac{1}{n} \sum_{a=1}^{n-1} E_{a,1},
\]

where the pitch matrix \( 1 \) denotes the \( w \times w \) unit matrix. Hence, according to \((3)\), \( X \) is a weighted sum of epicirculant permutation matrices.

Combining Lemmas 1, 2, and 5 leads to

**Theorem 4** If \( n \) is a prime power, then every \( XU(n) \) matrix \( X \) can be written

\[
X = \sum_{\sigma} c_\sigma E_\sigma
\]

with all \( E_\sigma \in E(n) \), such that both \( \sum_{\sigma} c_\sigma = 1 \) and \( \sum_{\sigma} |c_\sigma|^2 = 1 \).

### 6.1 First strategy

We can apply result \((11)\) with \( N \) given by \((18)\). The only possible values of \( \chi^{(1)} \) are \(-1, 0, p - 1, p^2 - 1, p^3 - 1, \ldots, \) and \( p^w - 1 \), as demonstrated in Appendix F.
6.2 Second strategy

For \( w > 1 \) and \( p > 2 \), the general affine groups have, besides the standard representation, also an inequivalent anti-standard representation. For a proof, it suffices to point to a single example of an odd epicirculant permutation matrix with trace different from unity. We choose the \( p^w \times p^w \) matrix

\[ E = I \otimes I \otimes \ldots \otimes I \otimes M, \]

i.e. the Kronecker product of \( w - 1 \) matrices \( I \) (i.e. the \( p \times p \) unit matrix) and the \( p \times p \) supercirculant matrix \( M = S_{0,q} \). The \( w \times w \) pitch matrix associated with \( E \) is the diagonal matrix \( \text{diag}(q, 1, 1, \ldots, 1) \).

On the one hand, we have the following property of the Kronecker product of two square matrices:

\[
\text{Det}(A \otimes B) = [\text{Det}(A)]^{\text{dim}(B)} [\text{Det}(B)]^{\text{dim}(A)}. \tag{19}
\]

Therefore, we have \( \text{Det}(E) = \text{Det}(M)^{(p^w-1)} \). We choose the number \( q \) such that \( \text{Det}(M) = -1 \) and thus \( \text{Det}(E) = -1 \). This is always possible. Suffice it to choose \( q \) equal to \( g(p) \), where \( g \) is a generator of the modulo \( p \) multiplication group \[12\]. Unfortunately, there is no algorithm known for finding such generator except brute force \[13\]. Nevertheless, we can prove that \( \text{Det}(S_{0,g(p)}) = -1 \), without a priori knowing the value of \( g(p) \): see Appendix C.

On the other hand, we have \( \text{Tr}(E) = p^w - 1 \text{Tr}(M) = p^w - 1 = p^w - 1 \). Because \( w > 1 \), we have \( \text{Tr}(E) > 1 \) and thus \( \chi^{(1)} > 0 \). We thus conclude that we can apply result \[14\] with \( N \) according to \[18\].

The above reasoning is not valid for \( p = 2 \), because, in that case, \( \text{Det}(M) = -1 \) does not imply \( \text{Det}(E) = -1 \). For the case \( p = 2 \), we will prove that all \( 2^w \times 2^w \) epicirculant matrices are even permutations. For this purpose, it is sufficient to demonstrate that all group generators are even. From reversible computing \[14\] \[15\] \[16\], it is known that the group \( \text{GA}(w; 2) \) is generated by following matrices:

\[
A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \otimes I \otimes I \otimes \ldots \otimes I
\]

\[
B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes I \otimes I \otimes \ldots \otimes I
\]
\[
C = I \otimes I \otimes \ldots \otimes I \otimes \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \otimes I \otimes I \otimes \ldots \otimes I ,
\]

with a total of \( w - 1 \) (for \( A \)) or \( w - 2 \) (for \( B \) and \( C \)) factors \( I \). In the context of computing, these matrices represent \( \text{NOT} \) gates, respectively controlled \( \text{NOT} \) gates. Applying (19), we have:

\[
\det(A) = \left[ \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]^{(p^w - 1)} = (-1)^{2^{w-1}} = 1
\]

\[
\det(B) = \left[ \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right]^{(p^w - 2)} = (-1)^{2^{w-2}} = 1
\]

\[
\det(C) = \left[ \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right]^{(p^w - 2)} = (-1)^{2^{w-2}} = 1 ,
\]

except if \( w = 2 \). Thus, for \( w > 2 \), all members of \( \text{GA}(w,2) \) represent even permutations and the second strategy (Subsection 3.2) is not applicable.

This leaves us with the case \( p = 2 \) and \( w = 2 \). The epicirculant matrices form a group \( \text{E}(4) \) isomorphic to the symmetric group \( \text{S}_4 \). As stated in Section 4.2, the second strategy is applicable. The results on the applicability of the second strategy are summarized in Table I.

Table 1: Applicability of the second strategy for the Birkhoff decomposition of an \( \text{XU}(n) \) matrix with \( n = p^w \).

| \( p \) = 2 | \( p \geq 3 \) |
|------------|------------|
| \( w = 1 \) | no | no |
| \( w = 2 \) | yes | yes |
| \( w \geq 3 \) | no | yes |

7 Conclusion

According to [2], every unit-linesum \( n \times n \) unitary matrix can be decomposed as a weighted sum of the \( n \times n \) permutation matrices, such that both the sum
of the weights and the sum of the squared moduli of the weights equal unity. Such Birkhoff sum contains $n!$ terms. In the present paper, we demonstrate the following:

- If $n \geq 4$, then $n!/2$ terms suffice.
- If $n = p^w$ with $p$ an arbitrary prime and $w$ an arbitrary integer, then
  \[ p^w(p^w - p^{w-1})(p^w - p^{w-2}) \ldots (p^w - p)(p^w - 1) \]
  suffice.
- If $n = p^w$ with $p$ an arbitrary odd prime and $w$ an integer $\geq 2$, then
  \[ p^w(p^w - p^{w-1})(p^w - p^{w-2}) \ldots (p^w - p)(p^w - 1)/2 \]
  suffice.

For numerical examples, see Table 2.

Table 2: Number of Birkhoff terms in the decomposition of an arbitrary $n \times n$ unit-linesum unitary matrix.

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     | 1   | 2   | 6   | 12  | 20  | 360 | 42  | 1,344 | 216 | 1,814,400 | 110 |

| $n$ | 12  | 13  | 14  | 15  | 16  | 17  |
|-----|-----|-----|-----|-----|-----|-----|
|     | 239,500,800 | 156 | 43,589,145,600 | 653,837,184,000 | 322,560 | 272 |

The case of $n$ equal to the product of two different primes is left for further investigation.

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A The group of supercirculant permutation matrices

The supercirculant $n \times n$ permutation matrices form a group. Indeed, the product of two such matrices (say $S_{a,x}$ and $S_{b,y}$) yields a third such matrix. In order to prove this fact, we compute the matrix entry at position $(u, v)$:

$$
(S_{a,x} S_{b,y})_{u,v} = \sum_f (S_{a,x})_{u,f} (S_{b,y})_{f,v} = \sum_f \delta_{f,a+ux} \delta_{v,b+fy} = \delta_{v,b+(a+ux)y} = (S_{b+ay,xy})_{u,v}
$$

and hence

$$
S_{a,x} S_{b,y} = S_{b+ay,xy}. \quad (20)
$$

If $n$ is a prime $p$, each non-zero number $x$ has an inverse number $x^{-1}$. Applying (20), we find

$$
S_{a,x} S_{-ax^{-1},x^{-1}} = S_{0,1}.
$$

The right-hand side being the $p \times p$ unit matrix, the result proves that each supercirculant matrix has an inverse matrix that also is supercirculant:

$$
(S_{a,x})^{-1} = S_{-ax^{-1},x^{-1}}.
$$

We conclude by considering two applications of eqn (20):

- choosing $x = y = 1$ leads to

  $$
  S_{a,1} S_{b,1} = S_{a+b,1}
  $$

  illustrating that the $p$ matrices $S_{a,1}$ are isomorphic to the addition modulo $p$;

- choosing $a = b = 0$ leads to

  $$
  S_{0,x} S_{0,y} = S_{0,xy}
  $$

  illustrating that the $p-1$ matrices $S_{0,x}$ are isomorphic to the multiplication modulo $p$.

Each supercirculant matrix can be decomposed as the product of a zero-shift matrix and a unit-pitch matrix:

$$
S_{a,x} = S_{0,x} S_{a,1} = S_{ax^{-1},1} S_{0,x}.
$$
The trace of a supercirculant permutation matrix

We compute the trace of the supercirculant permutation matrix $S_{a,x}$:

$$\text{Tr}(S_{a,x}) = \sum_u (S_{a,x})_{u,u} = \sum_u \delta_{u,a+ux}.$$  

If the eqn 

$$u(1-x) = a$$

is fulfilled, then the corresponding number $u$ points to a unit entry in position $(u,u)$ of the matrix $S_{a,x}$. We notice:

- If $x \neq 1$, then $u = a(1-x)^{-1}$ is the one and only solution;
- if $x = 1$ and $a \neq 0$, then the eqn has no solution $u$;
- if $x = 1$ and $a = 0$, then $u$ may have any value from \{0, 1, 2, ..., $p-1$\}.

Thus we conclude:

- $\text{Tr}(S_{a,x}) = 1$, if $x \neq 1$,
- $\text{Tr}(S_{a,1}) = 0$, if $a \neq 0$, and
- $\text{Tr}(S_{0,1}) = p$.

The determinant of a supercirculant permutation matrix

As mentioned in Appendix A, each supercirculant matrix can be decomposed as follows:

$$S_{a,x} = S_{0,x} S_{a,1}.$$  

Hence:

$$\text{Det}(S_{a,x}) = \text{Det}(S_{0,x}) \text{Det}(S_{a,1}).$$

We have $S_{a,1} = (S_{1,1})^a$ and therefore $\text{Det}(S_{a,1}) = (\text{Det}(S_{1,1}))^a$. If $p$ is odd, then $\text{Det}(S_{1,1}) = 1$, such that $\text{Det}(S_{a,1}) = 1$. In other words: for odd primes, all of the $p$ circulant permutation matrices have unit determinant. The situation is different for the $p-1$ matrices $S_{0,x}$. Half of them have unit determinant and half of them have determinant equal to $-1$. In order
to prove this fact, the key observation is the fact that the cyclic group is Abelian; so there exists a similarity transformation that diagonalizes all matrices $S_{0,x}$. We now prove that the following matrix $F$ serves our purpose:

$$F_{u,v} = \begin{cases} 
1 & \text{if } u = v = 0 \\
0 & \text{if } u = 0 \text{ and } v \neq 0 \\
0 & \text{if } u \neq 0 \text{ and } v = 0 \\
\frac{\omega^{v\varphi(u)}}{\sqrt{p-1}} & \text{if } u \neq 0 \text{ and } v \neq 0,
\end{cases}$$

where $\omega = \exp\left(\frac{2\pi i}{p-1}\right)$ is the $(p-1)$ th root of unity, and the function $\varphi(a)$ gives the ‘position’ of the number $a$ in the cyclic group $\mathbb{Z}_{p-1}$ (multiplicative group modulo $p$), as a power of the (a priori unknown) generator $g$, i.e.

$$a = g^{\varphi(a)}.$$ 

From this definition, the following interesting properties of $\varphi$ can be deduced:

$$\begin{align*}
\varphi(1) &= 0 \\
\varphi(g) &= 1 \\
\varphi(ab) &= \varphi(a) + \varphi(b).
\end{align*}$$

These properties are key in the following derivation. We compute the similarity transformation given by $F^\dagger S_{0,x} F$. Because both $F$ and $S_{0,x}$ are block diagonal with a single 1 in the upper-left corner, we only need to compute the lower-right part:

$$(F^\dagger S_{0,x} F)_{u,v} = \sum_{k=1}^{p-1} \sum_{l=1}^{p-1} F_{k,u} (S_{0,x})_{k,l} F_{l,v}$$

$$= \frac{1}{p-1} \sum_{k=1}^{p-1} \sum_{l=1}^{p-1} \omega^{-u\varphi(k)} \delta_{l,xk} \omega^{v\varphi(l)}$$

$$= \frac{1}{p-1} \sum_{k=1}^{p-1} \omega^{-u\varphi(k)+v\varphi(xk)}$$

$$= \frac{1}{p-1} \sum_{k=1}^{p-1} \omega^{-u\varphi(k)+v\varphi(x)+v\varphi(k)}$$

$$= \omega^{u\varphi(x)} \delta_{u,v}.$$

This result leads to two conclusions:
• By choosing \( x = 1 \), we find that \((F^†F)_{u,v} = \delta_{u,v}\) and thus that \( F \) is unitary.

• By choosing \( x \) arbitrary, we find that the matrix \( S_{0,x} \) has the eigenvalues \( \omega^v\varphi(x) \) plus an additional 1 from the upper-left matrix block.

The determinant is just the product of all eigenvalues:

\[
\text{Det}(S_{0,x}) = \prod_{u=1}^{p-1} \omega^v\varphi(x) = \omega^\varphi(x) \sum_{v=1}^{p-1} v = \omega^\varphi(x) \frac{p(p-1)}{2} = e^{\frac{2\pi i}{p-1} \varphi(x)} \frac{p(p-1)}{2} = e^{\pi i \varphi(x)} p.
\]

Now, if \( p \) is an odd prime, then \( e^{\pi ip} = -1 \), such that \( \text{Det}(S_{0,x}) = (-1)^{\varphi(x)} \), which proves that the sign of the determinant of \( S_{0,x} \) alternates in the chain of successive elements of \( \mathbb{C}_{p-1} \). More in particular, the position of \( x = g \) always is \( \varphi(g) = 1 \), so we have \( \text{Det}(S_{0,g}) = -1 \).

We note that the above results for both \( S_{a,1} \) and \( S_{0,x} \) are only valid for odd primes \( p \). If \( p \) is even, i.e. if \( p = 2 \), then there exist only two supercirculant matrices \( S_{0,1} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \), with determinant equal to 1, and \( S_{1,1} = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \), with determinant equal to \(-1\).

## D The pitch matrix

In (17), the epicirculant matrix \( E_{a,x} \) needs a unit entry in position \((k,a+xk)\) if

\[
(M_{r,s})_{k,a+xk} = (M_{r,s})_{0,a}
\]

implying

\[
f(k,r) - f(s,a + \sum_u \sum_v x_{u,v}kvp^u) = f(0,r) - f(s,a)
\]

or

\[
\sum_j k_j r_j - \sum_j s_j \left( a_j + \sum_u \sum_v x_{u,v}kvp^u \right) _j = - \sum j s_j a_j
\]

and thus

\[
\sum_j s_j \sum_v x_{j,v}k_v = \sum_j k_j r_j
\]

or

\[
\sum_v k_v \sum_j x_{j,v} s_j = \sum_v k_v r_v
\]
and thus
\[ \sum_v k_v \left( \sum_j x_{j,v} s_j - r_v \right) = 0. \]

We fulfil this condition by the set of \( w \) non-coupled eqns
\[ \sum_j s_j x_{j,v} = r_v. \quad (21) \]

For each eqn, we expect \( p^{w-1} \) solutions (as we can choose \( w - 1 \) out of the \( w \) dits \( x_{j,v} \) arbitrarily from \( \{0, 1, \ldots, p-1\} \)). However, many solutions have to be rejected. Indeed, each column of the matrix \( E_{a,x} \) in (17) should contain one and only one unit entry. For this purpose, it is necessary and sufficient that the matrix \( x \) is invertible. Proof is as follows. We require that for any two different row numbers \( k' \neq k \) the unit entry of the permutation matrix is in another column:

\[ a + xk' \neq a + xk \]

and thus \( x(k' - k) \neq 0 \). This requires that for any non-zero number \( K \) we have

\[ xK \neq 0. \]

This, in turn, requires that the rows of \( x \) are linearly independent and thus that the matrix \( x \) is invertible.

We now prove that, for any pair \( (r, s) \), the set (21) has at least one acceptable solution, i.e. a solution such that the matrix \( x \) is invertible. Indeed:

- Because both \( r \) and \( s \) are non-zero, at least one dit \( r_u \) is non-zero and at least one dit \( s_j \) is non-zero. Let \( r_\alpha \) be the least-significant non-zero dit of \( r \); let \( s_\beta \) be the least-significant non-zero dit of \( s \).
- We choose all dits \( x_{j,v} = 0 \), except the dits \( x_{v,v}, x_{\beta,v}, \) and \( x_{\alpha,\beta} \). Thus eqns (21) become
  \[ s_v x_{v,v} + s_\beta x_{\beta,v} = r_v \mod p \text{ if } v \neq \beta \]
  \[ s_\alpha x_{\alpha,\beta} + s_\beta x_{\beta,\beta} = r_\beta \mod p. \quad (22) \]
- For \( v \neq \alpha \) and \( v \neq \beta \), we choose \( x_{v,v} = 1 \). Further we choose \( x_{\alpha,\alpha} = 0 \) and \( x_{\alpha,\beta} = 1 \). Thus eqns (22) become
  \[ s_\beta x_{\beta,v} = r_v - s_v \mod p \text{ if } v \neq \alpha \text{ and } v \neq \beta \]
  \[ s_\beta x_{\beta,\alpha} = r_\alpha \mod p \]
  \[ s_\beta x_{\beta,\beta} = r_\beta - s_\alpha \mod p \quad (23) \]
  which lead to a single solution set \( x_{\beta,v} \).
The resulting pitch matrix \( \mathbf{x} \) consists of a non-zero diagonal, one non-zero row, and one extra unit entry. E.g. for \( w = 7, \alpha = 2, \) and \( \beta = 4, \) we have:

\[
\begin{pmatrix}
1 & 1 & 0 & 1 \\
& & & \\
& & & 1 \\
x_{4,0} & x_{4,1} & x_{4,2} & x_{4,4} & x_{4,5} & x_{4,6} \\
& & & & & & 1
\end{pmatrix}.
\]

We note that here \( \text{Det}(\mathbf{x}) = x_{4,2}. \) In general, we have

\[
\text{Det}(\mathbf{x}) = \pm x_{\beta, \alpha} = \pm r_{\alpha} s_{\beta}^{-1}.
\]

Because \( \text{Det}(\mathbf{x}) \neq 0, \) we have that \( \mathbf{x} \) is invertible.

### E The group of epicirculant permutation matrices

The epicirculant permutation matrices form a group. An arbitrary entry (at location \((k, l)\)) of such matrix \( E_{a,x} \) is \( \delta_{l, a+xk}. \) The product of two such matrices yields a third such matrix. Indeed:

\[
(E_{a,x} E_{b,y})_{u,v} = \sum_{f} (E_{a,x})_{u,f} (E_{b,y})_{f,v}
\]

\[
= \sum_{f} \delta_{f, a+xu} \delta_{v, b+yzf}
\]

\[
= \delta_{v, b+ya+yxu}
\]

\[
= (E_{b+ya,yx})_{u,v}
\]

and hence

\[
E_{a,x} E_{b,y} = E_{b+ya,yx}.
\]

Straightforward application of this result leads to

\[
E_{a,x} E_{-x^{-1}a,x^{-1}} = E_{0,1}.
\]

The right-hand side being the \( p^w \times p^w \) unit matrix, the result proves that each epicirculant matrix has an inverse matrix that also is epicirculant:

\[
(E_{a,x})^{-1} = E_{-x^{-1}a,x^{-1}}.
\]
Each epicirculant matrix can be decomposed as the product of a matrix with zero shift vector $a$ and a matrix with unit pitch matrix $x$:

$$E_{a,x} = E_{0,x} E_{a,1} = E_{x^{-1}a,1} E_{0,x}.$$ 

F The trace of an epicirculant permutation matrix

We compute the trace of the epicirculant permutation matrix $E_{a,x}$:

$$\text{Tr}(E_{a,x}) = \sum_u (E_{a,x})_{u,u} = \sum_u \delta_{u,a+ux}.$$ 

If the eqn

$$(1 - x)u = a$$

is fulfilled, then the corresponding number $u$ points to a unit entry in position $(u, u)$ of the matrix $E_{a,x}$. Here, $\mathbf{1}$ denotes the $w \times w$ unit matrix. We notice:

- If $(1 - x)$ is invertible, then $u = (1 - x)^{-1}a$ is the one and only solution;
- if $(1 - x) = \mathbf{0}$ and $a \neq \mathbf{0}$, then the eqn has no solutions $u$;
- if $(1 - x) = \mathbf{0}$ and $a = \mathbf{0}$, then $u$ may have any value from $\{0, 1, 2, ..., p^w - 1\}$;
- if $(1 - x)$ is neither invertible nor zero, then $(1 - x)$ has rank $\lambda$ with $1 \leq \lambda \leq w - 1$ and $u$ can have as many values as there are solutions of the eqn $(1 - x)u = 0$, i.e. as the size of the kernel of $(1 - x)$, i.e. $p^{w-\lambda}$.

Thus we conclude:

- $\text{Tr}(E_{a,1}) = 0$, if $a \neq \mathbf{0}$,
- $\text{Tr}(E_{0,1}) = p^w$, and
- $\text{Tr}(E_{a,x}) = p^{w-\lambda}$, if $(1 - x)$ has rank $\lambda \neq 0$. 

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