Novel Symmetry of Non-Einsteinian Gravity in Two Dimensions

Harald Grosse 1, Wolfgang Kummer 2, Peter Prešnajder 3 and Dominik J. Schwarz 2

1) Institut für theoretische Physik
Universität Wien
Boltzmanngasse 5, A-1090 Wien
Austria

2) Institut für theoretische Physik
Technische Universität Wien
Wiedner Hauptstraße 8-10, A-1040 Wien
Austria

3) Matematicko-fyzikálna fakulta
Univerzita Komenského
Mlynská dolina F 2, CS-842 15 Bratislava
Czechoslovakia

Abstract

The integrability of $R^2$-gravity with torsion in two dimensions is traced to an ultralocal dynamical symmetry of constraints and momenta in Hamiltonian phase space. It may be interpreted as a quadratically deformed $iso(2,1)$-algebra with the deformation consisting of the Casimir operators of the undeformed algebra. The locally conserved quantity encountered in the explicit solution is identified as an element of the centre of this algebra. Specific contractions of the algebra are related to specific limits of the explicit solutions of this model.

May 1992

*e-mail: dschwarz@email.tuwien.ac.at
1 Introduction

The beautiful properties of string theory, interpreted as Einstein-gravity in two dimensions with “world-coordinates” as additional fields, are closely related to the Weyl-invariance which arises as an additional “accidental” symmetry in two spacetime dimensions. Recently also a non-Einsteinian model of gravity — without Weyl-symmetry — in two dimensions has attracted interest [1, 2, 3, 4], because its integrability also led to the conjecture of possessing some further symmetry [3]. It belongs to the variety of gravitational theories with higher powers of curvature and with torsion in the action [1, 2]. Restricting the field equations to contain at most second order derivatives of the field variables (zweibein $e^a_{\mu}$ and spin-connection $\omega^a_{\mu b}$), the action invariant under diffeomorphisms and local Lorentz transformations is essentially unique

$$L = -\frac{1}{4M^2} \int d^2x (R^a_{\mu \nu} R^\mu \nu_{ab} + M^2 \beta T^a_{\mu \nu} T^{a\mu \nu} + 4M^2 \lambda)$$

with

$$e = \det(e_{\mu}^a)$$
$$R^a_{\mu \nu b} = (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) \varepsilon^a_{b} =: F_{\mu \nu} \varepsilon^a_{b}$$
$$T^a_{\mu \nu} = \partial_\mu e^a_\nu + \omega_\mu e^a_\nu e^b_\nu - (\mu \leftrightarrow \nu)$$

where

$$\omega^a_{\mu b} =: \omega^a_{\nu b}$$
$$\varepsilon^{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

Greek indices are lowered and raised by the metric $g_{\mu \nu} = e_{\mu}^a e_{\nu}^b \eta_{ab}$ and its inverse $g^{\mu \nu}$, where $(\eta_{ab}) = \text{diag}(+, -)$ is the Lorentz metric operating with the Latin indices. The (topological) Einstein-Hilbert term $eR$, a total divergence, is omitted in (1), $M$ and $\beta$ are free parameters, $\lambda$ represents a cosmological constant. $M, \beta$ and $\lambda$ are chosen to have mass dimension one, zero and two, in order to yield a dimensionless action $L$. For simplicity we shall set $M^2 \equiv 8$ in the following.

The classical solutions of (1) were studied extensively in the conformal gauge $e^a_{\mu} = e^a e^{\mu}_{\varphi} \delta^a_{\mu}$ with $\varphi$ and $\omega^a_{\mu}$ as dynamical variables. Complete integrability was found [1]. The solution consists of two branches, one with constant curvature and vanishing torsion, the other one with nontrivial curvature and torsion. The former is closely related to Liouville theory. Recently all universal coverings of geodesically complete solutions have been classified [2].

The Euclidian version of the model (1) has also been identified with the appropriate geometric formulation of continuous distributions of dislocations and disclinations in a two-dimensional membrane. The three-dimensional version of this equation has been used for the treatment of elastic media in Euclidian $d = 3$ [7].

On the other hand, (1) closely resembles a (noncompact) gauge theory. This suggests also light-cone (LC) components and the use of a LC gauge [3, 4]. Working

1
in LC coordinates
\[ x^\pm = x^0 \pm x^1 \]
simplifies the calculations considerably. Introducing
\[
\begin{align*}
\omega_\pm & = \frac{1}{2}(\omega_0 \pm \omega_1) \\
e^a_\pm & = \frac{1}{2}(e^a_0 \pm e^a_1)
\end{align*}
\]
and grouping the field variables as \((i = 1, 2, 3)\)
\[
\begin{align*}
q_i &= (\omega_-, e_-, e_+^\prime) \\
\bar{q}_i &= (\omega_+, e_+, e_+^\prime)
\end{align*}
\]
(4) can be rewritten as
\[
\mathcal{L} = -\frac{1}{2e}(F_{+-})^2 + \frac{2\beta}{e}T_{++}^+T_{+-}^- - e\lambda,
\]
where
\[
\begin{align*}
e &= q_2\bar{q}_3 - q_3\bar{q}_2 \\
F_{+-} &= \dot{q}_1 - \dot{q}_1^i \\
T_{++}^+ &= \dot{q}_3 - \dot{q}_3^i + \dot{q}_1q_3 - q_1\bar{q}_3 \\
T_{+-}^- &= \dot{q}_2 - \dot{q}_2^i - \dot{q}_1q_2 + q_1\bar{q}_2
\end{align*}
\]
Dot and prime indicate the derivative with respect to \(x^+\) and \(x^-\). The homogeneous LC gauge is characterized by \(\bar{q}_1 = \bar{q}_2 = \bar{q}_3 = 1\) (cf. (4)). It should be emphasized that it can always be reached by appropriate diffeomorphisms and local Lorentz transformations; the proof is straightforward. In this gauge, as shown in [3], the complete integral of the equations of motion can be even expressed in terms of elementary functions of \(x^+\). It was found that the solutions depend on three arbitrary functions of \(x^-\) and one constant, the latter being related to the quantity
\[
Q = \exp\left(-\frac{p_1}{2\beta}\right)\left(\frac{E}{2\beta^2} + \frac{p_1}{\beta} + 2\right)
\]
with
\[
E = \frac{p_1^2}{2} - \frac{p_2p_3}{2\beta} - \lambda
\]
(8)
p_i are the “conjugate momenta” to the “coordinates” \(q_i\) and will be defined in Sec. 2.1. \(p_1\) is proportional to the curvature scalar, \(p_2p_3\) to the torsion scalar, the second term in (4). Thus (7) behaves as a scalar under diffeomorphisms and local Lorentz transformations and therefore holds in any gauge. \(Q\) is constant for any solution, i.e. it is independent of both variables
\[
\dot{Q} = Q' = 0.
\]
The action (1) is nonpolynomial and thus its renormalizibility is by no means evident, the more so, because the usual specific infrared problems of $d = 2$ are relevant also here. However, as shown by two of the present authors [4], at least in a flat background the path-integral quantization can be carried through, and the theory turns out to be renormalizable by fixing one (UV-divergent) constant in front of an infinite number of counterterms.

The purpose of our present work is to investigate the symmetry properties which are responsible for the integrability of (1). Constants of motion like the “local” one (10) must follow from this symmetry.

This symmetry can be found starting from the Hamiltonian formulation. As is well-known, in the latter the algebra of all first class constraints provides an opportunity to identify such a symmetry [5]. We show in Sec. 2.1 that the Hamiltonian can be expressed completely in terms of the secondary constraints. This could have been expected for a theory of gravitation. Actually the constraints resemble the ones in the Ashtekar formulation of general relativity, being polynomials in $p$ and $q$ [9].

In Sec. 2.2 and 2.3 we solve the system of Hamiltonian equations of motion, introducing in a natural manner conserved quantities like (7). Calling the evolution parameter $x^+$ “time”, the equations for the time-evolution of the momenta (Sec. 2.2) and for the coordinates (Sec. 2.3) can be solved. The general solution depends on three arbitrary functions of “space” $x^-$. Exploiting the residual gauge-symmetry after fixing the LC gauge it is possible to introduce a “normalized” solution, were the arbitrary functions are replaced by constants (Sec. 2.4). This shows that the $x^-$-dependence is essentially irrelevant for the dynamics.

After this preparation we relate the different branches of solutions to the dynamical symmetry (Sec. 3) which is the main issue of our present work. This symmetry appears because the algebra of secondary constraints in our present case closes together with the momenta $p$. The algebra is nonlinear and can be considered as a deformed $iso(2, 1)$, with the deformation consisting of the Casimir operators of the undeformed $iso(2, 1)$. The quantity (7) appears as one of the two elements of the centre of this algebra.

## 2 Hamiltonian Formulation

### 2.1 Hamiltonian

Choosing $x^+$ to be the evolution parameter in the Hamiltonian, the canonical momenta corresponding to (10) in LC coordinates become ($p_i = \delta L/\delta \dot{q}_i$)

\[
\begin{align*}
p_1 &= -\frac{1}{e} (\dot{q}_1 - \dot{q}_1') \\
p_2 &= \frac{2\beta}{e} (\dot{q}_3 - \dot{q}_3' + \dot{q}_1 q_3 - q_1 \dot{q}_3)
\end{align*}
\]
\[ p_3 = \frac{2\beta}{e} (\dot{q}_2 - \ddot{q}_2 - \dot{q}_1 q_2 + q_1 \ddot{q}_2) \]

with primary constraints
\[ \bar{p}_i = \frac{\delta L}{\delta \dot{\bar{q}}_i} \sim 0 . \quad (11) \]

We define the Poisson brackets \((x^+ = y^+)\):
\[
\begin{align*}
[q_i(x), p_j(y)] &= \delta(x^- - y^-) \delta_{ij} \\
[\bar{q}_i(x), \bar{p}_j(y)] &= \delta(x^- - y^-) \delta_{ij} \\
[q_i(x), q_j(y)] &= [p_i(x), p_j(y)] = \ldots = 0
\end{align*}
\quad (12)
\]

With the canonical Hamiltonian \(H_C = \int dx^- (\dot{q}_i \bar{p}_i - L) = \int dx^- H_C\) they entail secondary constraints
\[ G_i := \bar{p}_i = -\frac{\partial H_C}{\partial \bar{q}_i} \sim 0 , \quad (13) \]
where
\[
\begin{align*}
G_1 &= \ p'_1 + q_3 p_3 - q_2 p_2 \\
G_2 &= \ p'_2 + q_1 p_2 - q_3 E \\
G_3 &= \ p'_3 - q_1 p_3 + q_2 E
\end{align*}
\quad (14)
\]
with \(E\) already defined in \([8]\). The Hamiltonian turns out to be linear in \(G_i\)
\[ H_C(x^+) = \int dx^- (P' - \bar{q}_i G_i) , \quad (15) \]
where
\[ P := \bar{q}_i p_i . \quad (16) \]

In the course of our study of the algebra of \(G_i\) below (Sec. 3), we shall find that the \(G_i\) are in involution. This means the absence of ternary constraints, all primary and secondary constraints are first class.

Since there is no \(\bar{p}_i\)-dependence in the Hamiltonian, the canonical Heisenberg equations of motion for \(\bar{q}_i\) imply \(\dot{\bar{q}}_i = [\bar{q}_i, H_C] = 0\). If one uses the formalism of extended Hamiltonian \([11]\) \(H_E = H_C + \mu_i \bar{p}_i + \lambda_i G_i\), where all first class constraints are added through Lagrange multipliers \(\mu_i\) and \(\lambda_i\), the analogous equation reads \(\dot{\bar{q}}_i = \mu_i\).

\(\mu_i\) are unphysical fields and may be “gauge fixed”, i.e. we will work at \(\lambda_i = \mu_i = 0\) throughout this paper (the \(\lambda_i\) are “absorbed” by the \(\bar{q}_i\) in \(H_C\)). This leads exactly to the class of general LC gauges \((\bar{q}_i = \bar{q}_i(x^-))\) of the Lagrangian approach. In Refs. \([3, 4]\) we used the homogeneous LC gauge as defined after \((6)\). Thus working in LC gauges, \(H_C\) need not be “extended”. This would be necessary in the conformal gauge \((q_3 = \bar{q}_2 = 0, \ q_2 = \bar{q}_3 = \exp(\varphi))\) where \(\dot{q}_i \neq 0\) in general.

Disregarding the surface term, \(H_C\) can be interpreted as a generator of \(G\)-transformations with parameter \(\bar{q}_i\). On the constraint surface \(H_C \sim 0\).

So far we discussed a LC formulation, because in terms of a corresponding gauge the complete integrability of the equations of motion is rather easy. The case of general axial-type gauges — without specifying in advance the space-time property of the evolution parameter — has been worked out as well \([11]\).
2.2 Evolution of Momenta

The classical equations of motion for the momenta $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ from (15)

\[
\dot{p}_1 = -\bar{q}_3 p_3 + \bar{q}_2 p_2 \\
\dot{p}_2 = -\bar{q}_1 p_2 + \bar{q}_3 E \\
\dot{p}_3 = \bar{q}_1 p_3 - \bar{q}_2 E
\]

for given gauge functions $\bar{q}_i(x^-)$ are closed for the momenta alone and hold together with $G_i = 0$ (14). Moreover they are “ultralocal” in the sense that no space derivatives $\partial_-$ appear. The quantity $P$ (14) is immediately recognized as a constant of motion ($\dot{P} = 0$). In addition, from (18) and (19)

\[
\partial_+ (p_2 p_3) + E \dot{p}_1 = 0
\]

follows which implies $\dot{Q} = 0$ with $Q$ already introduced in (4). From the vanishing of $G_2$ and $G_3$ in (14) an analogous relation

\[
\partial_- (p_2 p_3) + E \dot{p}'_1 = 0
\]

can be obtained, proving that also $Q' = 0$. Thus the result (9) of [3] has been recovered that $Q$ is conserved in space and time.

Eliminating $p_2$ and $p_3$ in (17) by $E$ and $Q$ according to (7) and (8), and using $P$ from (16)

\[
\dot{p}_1^2 = F(p_1)
\]

follows with

\[
F = (P - \bar{q}_1 p_1)^2 - 8\beta \bar{q}_2 \bar{q}_3 \left[ \frac{p_1^2}{2} - \lambda + 2\beta^2 \left( \frac{p_1}{\beta} + 2 - \exp \left( \frac{p_1}{2\beta} \right) Q \right) \right],
\]

which is readily integrated as long as $F > 0$

\[
x^+ = x^+(p_1, x^-) = \pm \int^{p_1} d\bar{p}_1 F^{-\frac{1}{2}}(\bar{p}_1) + \rho(x^-) .
\]

The correct sign has to be fixed with help of Eq. (17). Having inverted (23) for $p_1$, the momenta $p_2$ and $p_3$ are given by the algebraic Eqs. (7) and (8).

In the homogeneous LC gauge $\bar{q}_1 = \bar{q}_2 = \bar{q}_3 - 1 = 0$ of [3] the integrations are trivial and even independent of $Q$

\[
p_3 = P_0 \\
p_1 = -P_0 (x^+ - \rho(x^-))
\]

where $P = P_0 = p_3$ in the homogeneous LC gauge. $p_2$ is determined algebraically by Eqs. (7) and (8). Comparing with Ref. [3], the arbitrary functions $P_0$ and $\rho$ are identified with $f$ and $h$ of that work:

\[
P_0 = 2\beta f \\
\rho = -h/f
\]
The case $P = 0$ must be treated separately. In the homogeneous LC gauge it implies $p_3 = 0$ (i.e. vanishing torsion). As $q_2 = e \neq 0$, $p_1$ is fixed by the third Eq. (14) to be $p_1 = \pm \sqrt{21}$. It exists for $\lambda \geq 0$. Furthermore $G_1 = 0$ requires $p_2 = 0$. This branch was called de Sitter solution in [3].

Another interesting limit for the solutions of (17) – (19) is obtained at $\beta \to \infty$ for $p_2$ and $p_3 \neq 0$. In that case the quantity $E$ (8) becomes independent of $p_2p_3$ and, repeating the steps (20) and (21), the corresponding conserved quantity $Q_\infty$ reads

$$Q_\infty = \frac{p_1^3}{6} - \lambda p_1 + p_2p_3.$$

It may be obtained as well by expanding (7) in powers of $1/\beta$ as

$$Q = 2 - \frac{\lambda}{2\beta^2} - \frac{1}{4\beta^3} Q_\infty + O\left(\frac{1}{\beta^4}\right).$$

The remaining steps are as above. The curvature varies in this case, but the torsion scalar vanishes as $T^2 \propto p_2p_3\beta^{-2}$. We shall call this branch of solutions the “Einstein” branch, because it contains nontrivial curvature with vanishing torsion, reminiscent of Einstein gravity. It should be noted that this limit does not commute with the procedure leading to the equations of motion. In this sense it is a singular one.

### 2.3 Evolution of Coordinates

The Hamiltonian equations for $q_i$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$, read:

$$\dot{q}_1 = \dot{q}_1' - ep_1 \quad \text{(27)}$$
$$\dot{q}_2 = \dot{q}_2' + \dot{q}_1q_2 - \dot{q}_2q_1 + e\frac{1}{2\beta}p_3 \quad \text{(28)}$$
$$\dot{q}_3 = \dot{q}_3' - \dot{q}_1q_3 - \dot{q}_3q_1 + e\frac{1}{2\beta}p_2 \quad \text{(29)}$$

The dependence on space-derivatives is restricted to the gauge-functions $\bar{q}_i$. Thus, for the purpose of dynamics, Eqs. (27) – (29) are as ultralocal as (17) – (19). The integration at general $\bar{q}_i$, with $p_i$ as input from (23), (16) and (7) is simple. We note first that the constraints $G_1 = 0$ and $G_3 = 0$ allow to express $q_3$ and $q_1$ in terms of $q_2$

$$q_1 = \frac{(q_2E + p_3')}{p_3} \quad \text{and} \quad q_3 = \frac{(q_2p_2 - p_1')}{p_3}. \quad \text{(30)}$$

We introduce (30) into (28) and obtain a linear inhomogeneous differential equation for $q_2$. E.g. for $\bar{q}_2 = 0$ its solution is given by

$$q_2(x^+, x^-) = \exp \left[ x^+\bar{q}_1 + \frac{\bar{q}_3p_3(0, x^-)}{2\beta} \left( \frac{\bar{q}_1x^+ - 1}{\bar{q}_1} \right) + \sigma(x^-) \right]. \quad \text{(31)}$$
In the homogeneous LC gauge (31) is further simplified and becomes
\[ e = q_2(x^+, x^-) = \exp \left( \frac{P_0}{2\beta} x^+ + \sigma(x^-) \right). \] (32)

In that case yet another method may be used, because then
\[ V_0 = q_2 \exp \left( \frac{p_1}{2\beta} \right) \] (33)
is readily identified as a constant of motion. It essentially coincides with the arbitrary function \( F \cdot f \) of Ref. [3] (cf. the explicit solution (39) below). Expressing \( q_2 \) by \( p_1 \) according to (33), Eqs. (30) provide the (algebraic) expressions for \( q_1 \) and \( q_3 \).

In the de Sitter case \( P = 0 \) described after Eq. (25) above, the determinant \( e \), Eq. (32), is independent of \( x^+ \)
\[ e = \exp \left( \sigma(x^-) \right) \] (34)
whereas, with \( p_1 = \pm \sqrt{2\lambda} \) and \( p_2 = p_3 = 0 \) from (27)
\[ q_1 = -ep_1 x^+ + \mu(x^-). \] (35)
Finally (28) is trivially integrated
\[ q_3 = \int^{x^+}_{x^-} d\hat{x}^+ q_1(\hat{x}^+) + \nu(x^-). \] (36)

In the Einstein branch \( (\beta \to \infty) \) Eq. (28) yields, working in the homogeneous LC gauge, again (34) as for the de Sitter case. The determination of \( q_1 \) and \( q_3 \), however, again proceeds most suitably using (30) at \( \beta \rightarrow \infty \).

### 2.4 Independent Functions of The Solution

Six first order differential equations in \( x^+ \) minus three constraints yield a dependence on three functions of \( x^- \) in the solution. This counting turns out to be correct for the homogeneous LC gauge, where, in fact, all those relations are independent. E.g. for the general branch of the solution in that gauge they are \( \rho, P_0 \) and \( \sigma \) or \( V_0 \). We show that these functions can be essentially eliminated by appropriate residual diffeomorphisms and Lorentz transformations which do not change the LC gauge \( \omega_+ = \delta_1(x^-), e_+^- = \delta_2(x^-), e_+^+ = \delta_3(x^-) \). A similar argument can be found in [2] for the conformal gauge. In a neighbourhood of a point \( x \), a diffeomorphism \( \tilde{x}^\mu = \tilde{x}^\mu(x) \), together with a Lorentz transformation \( \Lambda^a_b(x) \), induces transformations (going back to the notation \( e^a_\mu, \omega_\mu \) instead of (2))
\[
\tilde{\omega}_\mu(\tilde{x}) = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} (\omega_\nu - \partial_\nu \alpha) \\
\tilde{e}^a_\mu(\tilde{x}) = \Lambda^a_b(x(\tilde{x})) \frac{\partial x^\nu}{\partial \tilde{x}^\mu} e^b_\nu, \quad (37)
\]
where for \(a, b \in \{+, -\}\) \(\Lambda^a_b\) is diagonal with \(\Lambda^+_+ = (\Lambda^-^-)^{-1} = \exp(\alpha(x))\). The residual transformations for \(\tilde{e}^+_+ = e^+_+ = 1, \tilde{e}^-_- = e^-_- = \dot{\omega}_+ = \omega_+ = 0\), with (37) yield the restrictions
\[
\begin{align*}
\alpha &= \alpha(x^-) \\
x^- &= \xi^-(\tilde{x}^-) \\
x^+ &= \exp(-\alpha)\tilde{x}^+ + \xi^+(\tilde{x}^-).
\end{align*}
\] (38)

In the homogeneous LC gauge the solution \([30], [32]\) may be written in the notation of \([3]\) as (cf. (5), \(\Lambda = \frac{\Lambda}{2\beta}\) and \(C_0 = 8\beta Q\))
\[
\begin{align*}
e_- &= Ff e^{\hat{R}} \\
\omega_- &= 2\beta F e^{\hat{R}} (i\hat{R} - 1) + \beta QF + \frac{f'}{f} \\
e^+_+ &= \frac{\beta F}{f} e^R ((\hat{R} - 1)^2 + 1 - \Lambda) - \frac{\beta QF}{f} + \frac{\hat{R}'}{f}
\end{align*}
\] (39)
with
\[
\hat{R} = f x^+ + h.
\] (40)

The arbitrary functions of \(x^-\) in \([32]\) and \([40]\) are \(F > 0, f > 0\) and \(h\). Let us define a “normal” solution with \(F = f = 1, h = 0 (\hat{R}_0 = x^+)\)
\[
\begin{align*}
e_- &= \hat{e}^{\hat{R}_0} \\
\omega_- &= 2\beta e^{\hat{R}_0} (\hat{R}_0 - 1) + \beta Q \\
e^+_+ &= \beta e^{\hat{R}_0} ((\hat{R}_0 - 1)^2 + 1 - \Lambda) - \beta Q.
\end{align*}
\] (41)

It is straightforward to show that the transformation of the dynamical variables
\[
\begin{align*}
\tilde{e}^-_- &= \exp(-\alpha) \frac{\partial x^-_-}{\partial \tilde{x}^-} e^-_- \\
\tilde{e}^+_+ &= \exp(\alpha) \left( \frac{\partial x^+_+}{\partial \tilde{x}^-} + \frac{\partial x^-_-}{\partial \tilde{x}^-} e^+_+ \right) \\
\tilde{\omega}_- &= \frac{\partial x^-_-}{\partial \tilde{x}^-} (\omega_- - \partial_\alpha)
\end{align*}
\] (42)
with \((38)\) and \((41)\) reproduces the general expression \((39)\) with \(\tilde{F}, \tilde{f}^\prime\) and \(\tilde{h}\) given in terms of \(\alpha\) and \(\xi^\pm\) and their first derivatives. From the solution of the equation of motion and of the constraints leading to \((39)\) \([3]\), \(\tilde{F}, \tilde{f}^\prime\) and \(\tilde{h}\) must allow at least one differentiation. The class of functions produced by \(\alpha\) and \(\xi^\pm\) is certainly larger. Therefore (once differentiable) functions \(\tilde{F}, \tilde{f}^\prime\) and \(\tilde{h}\) can be always produced.

For the de Sitter solution, again in the notation of \([3]\),
\[
e^-_- = q(x^-)
\]
\[ \omega_+ = \frac{R_0}{4} qx^+ + l \]
\[ e_+ = \frac{R_0}{8} qx^2 + lx^+ + s \]

with \( R_0 = \pm 8\beta\sqrt{\Lambda} \). The “normal” solution may be chosen as

\[ e^- = 1 \]
\[ \omega_+ = \frac{R_0}{4} x^+ \]
\[ e_+ = \frac{R_0}{8} x^{+2} \]  

Again applying (42) with (44) produces (43) in terms of variables \( \tilde{x}^+, \tilde{x}^- \) with

\[ \tilde{q} = \exp(-\tilde{\alpha})\xi^{-'} \]
\[ \tilde{l} = -\tilde{\alpha}' + \frac{R_0}{4} \xi^+ \xi^{-'} \]
\[ \tilde{s} = \exp(\tilde{\alpha}) \left( \xi^{+'} + \frac{R_0}{8} \xi^2 \xi^{-'} \right) \]  

Our argument, of course, implies that the inverse transformation can be performed as well in local patches, i.e. that \( x^- \) can be eliminated from the general solution in regions bounded by coordinate singularities.

The present argument works for the homogeneous LC gauge and with the explicit solutions. It is equally true in more general gauges (cf. the proof for the conformal gauge). We checked it in the case where \( \bar{q}_1 \neq 0, \bar{q}_3 \neq 0 \) but \( \bar{q}_2 = 0 \). (38) is replaced by

\[ \alpha(x^+, x^-) = -\ln \left[ 1 + \exp(-\bar{q}_1 x^+) \left( \exp(-\alpha) - 1 \right) \right] \]
\[ x^+ = \frac{1}{\bar{q}_1} \ln \left[ \exp \left( \bar{q}_1 \tilde{x}^+ + \bar{q}_1 B(x^-) - \alpha \right) + 1 - \exp(-\alpha) \right] \]
\[ x^- = \xi^{-}(\tilde{x}^-) \]

where \( \alpha = \alpha(0, x^-) \). In the limit \( \bar{q}_1 \to 0 \) we obtain (38) by identifying \( B = \xi^+ \exp(\alpha) \). It is then straightforward but tedious to verify that there exists locally a transformation of the form of Eq. (37) which maps two solutions of the equations of motions to the same \( Q \), differing by the initial conditions, into each other.

The ultralocal structure of the canonical equations of motion agrees with the fact that the fields at each point \( x^- \) in space evolve independently (in the homogeneous LC gauge). Thus for all times by having fixed one transformation of space the latter may be expressed in a trivial manner. Having eliminated \( x^- \) altogether from the solution of (17) – (19) and (27) – (29), with vanishing constraints (14) and constant
\( \bar{q}_i \), we may conclude that, in order to understand the dynamics of this system, it is enough to consider a phase space \( p_i(x^+), q_i(x^+) \), i.e. to regard (17) – (19) and (27) – (29) at \( \bar{q}_i = \text{const} \) with (14) at \( p_i' = 0 \) as the dynamics of a three dimensional point particle with time \( x^+ \). It may be related to a Hamiltonian \( H^{eff} = -\bar{q}_i G_i \big|_{p_i'=0} \), with Poisson brackets of a point particle. Strictly speaking, also here \( \bar{q}_i \) have to be interpreted as Lagrange-multipliers to be fixed by a gauge condition.

This simplification of the solutions allows too a concise discussion of the different possibilities of nontrivial evolution in \( x^+ \). The expressions for the curvature scalar and the squared torsion with (14) read (41)

\[
\hat{R} = x^+ \\
\hat{T}^2 = Q \exp \left( -x^+ \right) - \left( x^+ - 1 \right)^2 + \Lambda - 1
\]

Starting, e.g. from \( x^+ = 0 \) where

\[
\hat{R} \big|_{x^+ = 0} = 0 \\
\hat{T}^2 \big|_{x^+ = 0} = Q + \Lambda - 2 ,
\]

the curvature always changes linearly with a singularity at \( x^+ \to \infty \). Dynamically different branches of the general solution are simply classified by the value of \( Q \) and of \( \Lambda - 1 \) which determine the behaviour of \( \hat{T}^2 \). We have at \( Q = 0 \) the possibilities for the evolution of \( \hat{T}^2 \)

\[
\Lambda < 1 \Rightarrow \hat{T}^2 < 0 \\
\Lambda = 1 \Rightarrow \text{one zero of } \hat{T}^2 \\
\Lambda > 1 \Rightarrow \text{two zeros of } \hat{T}^2 ,
\]

whereas for \( Q < 0 \)

\[
\Lambda \leq 1 \Rightarrow \hat{T}^2 < 0 \\
\Lambda > 1 \Rightarrow \text{two zeros of } \hat{T}^2 .
\]

If \( Q > 0 \) the number of zeros of \( \hat{T}^2 \) depends on the number of intersections of an exponential with a parabola (\( \leq 3 \) zeros).

Thus, having \( Q \) fixed by one initial value condition, e.g. at the origin of the coordinate system, for a given set of parameters in (14) the dynamics of the system is fully determined.

### 3 Symmetry Generated by Secondary Constraints

#### 3.1 Algebra of Constraints

The Poisson brackets (12) of the secondary constraints \( G_i \) (14) yield

\[
\begin{align*}
[G_1, G_2] &= -G_2^2 \delta \\
[G_1, G_3] &= G_3^2 \\
[G_2, G_3] &= -\left[ p_1 G_1 - \frac{1}{2q} \left( p_3 G_2 + p_2 G_3 \right) \right] \delta ,
\end{align*}
\]

(50)
where $\delta(x^- - y^-)$ has been abbreviated by $\delta$. (50) implies the absence of ternary constraints, used already above. The generator of local $G$-transformations

$$G_\xi = \int dx^- \xi(x^-) G_i(x^+, x^-)$$

induces for any $\xi(x^-)$ variations of a variable $F(p, q)$

$$\delta F = [F, G_\xi] .$$

If we disregard surface effects, assuming appropriate boundary conditions (or periodicity) so that the total divergence $P'$ in (15) may be dropped, the Hamiltonian coincides with (51) for the identification $\xi_i \rightarrow \bar{q}_i$ (the gauge functions).

(52) may be evaluated for any $F$ in terms of $[G_i, p_j]$

$$[G_1, p_1] = 0$$
$$[G_1, p_2] = [p_1, G_2] = -p_2 \delta$$
$$[G_1, p_3] = [p_1, G_3] = p_3 \delta$$
$$[G_2, p_3] = [p_2, G_3] = -E \delta$$

and of $[q_i, G_j]$

$$[q_1, G_1] = -\delta'$$
$$[q_1, G_2] = -q_3 p_1 \delta$$
$$[q_1, G_3] = q_2 p_1 \delta$$
$$[q_2, G_1] = -q_2 \delta$$
$$[q_2, G_2] = -\delta' + \left(\frac{2}{3}q_3 p_1 + q_1\right) \delta$$
$$[q_2, G_3] = -\frac{2}{3} q_2 p_3 \delta$$
$$[q_3, G_1] = q_3 \delta$$
$$[q_3, G_2] = \frac{2}{3} q_3 p_2 \delta$$
$$[q_3, G_3] = -\delta' - \left(\frac{2}{3}q_2 p_2 + q_1\right) \delta$$

where $\delta' = \partial_- \delta(x^- - y^-)$. The $G$-algebra (50) is linear in $G$, with $p$-dependent structure functions on the r.h.s. It is in weak involution, but not closed and resembles the algebra of secondary constraints in general relativity using the Ashtekar formulation with polynomial $G_i$. In contrast to that case, however, a minimal closed algebra involving (50) is obtained taking the “$G$-p-algebra” (50) and (53). The corresponding minimal set of elements for which this algebra is closed is

$$A = A_0(p) + A_i(p) G_i ,$$

where $A_0$ and $A_i$ are polynomials (or analytic functions or formal power series) in $p_i$. A minimal requirement for the $A_i(p)$’s is that they should be elements of a commutative ring containing a unit element. The algebra of elements (55), endowed with the brackets (54) and (53) of the G-p-algebra is ultralocal and shall be denoted as $A$.  

11
In terms of the new basis
\[
X_0 = G_1, \\
X_\mp = \frac{1}{2\beta|1/2}G_2, \\
1 + \nu Z_0 = -\frac{1}{2\beta}p_1, \\
\nu Z_\mp = -\frac{1}{2\beta(2\beta)^{1/2}}p_3.
\]
(56)

the defining relations of the G-p-algebra are particularly transparent if the quadratic form of \(iso(2,1)\)
\[
(Y, X) = Y_0X_0 - \text{sgn} \beta \left( Y_+X_- + Y_-X_+ \right)
\]
is introduced \((a, b = 0, +, -)\):
\[
\begin{align*}
[X_0, X_\pm] &= \pm X_\pm \delta \\
[X_+, X_-] &= -\text{sgn} \beta \left( X_0 + \nu \langle Z, X \rangle \right) \delta \\
[X_\pm, Z_a] &= 0 \\
[X_0, Z_\pm] &= [Z_0, X_\pm] = \pm Z_\pm \delta \\
[X_-, Z_+] &= [Z_-, X_+] = \text{sgn} \beta \left( Z_0 + \frac{\nu}{2} \langle Z, Z \rangle - \frac{\Lambda - 1}{2\nu} \right) \delta \\
[Z_a, Z_b] &= 0
\end{align*}
\]
(58)

In Eqs. (58) we recognize an \(iso(2,1)\) Lie-Poisson algebra, deformed by \(\nu\)-dependent terms. The deformation is quadratic and is identified with the Casimir operators of an undeformed \(iso(2,1)\) plus a central term depending on the cosmological constant.

The de Sitter solution corresponds \((\lambda \geq 0)\) to the orbit \(p_1 = \pm \sqrt{2\lambda}, p_2 = p_3 = 0\). On this orbit the algebra (50) is itself closed: for \(\lambda > 0\) we recognize the algebra \(sl(2, \mathbb{R})\) and for \(\lambda = 0\) the two dimensional Poincaré algebra \(iso(1,1)\) is recovered. This is as expected, because the de Sitter solution is related to the Liouville equation \(\square\) and \(sl(2, \mathbb{R})\) plays a central rôle in Liouville theory \([12]\).

The Einstein solution \((\beta \rightarrow \infty)\) possesses an algebra which can be read off from (50) and (53), simply dropping the quadratic terms with \(p_2p_3, G_2p_3\) and \(p_2G_3\) on the respective r.h.s. The only remaining quadratic terms are \(p_1G_1\) and \(p_1^2\), i.e. involve mutually commuting elements. Nonlinear generalizations of Lie algebras involving precisely such quadratic expressions of the elements of the Cartan subalgebra have been considered already in the literature \([13]\). Here an example for this idea emerges in a natural manner as the dynamical symmetry of a specific model.

### 3.2 Centre of \(A\)

We call the functions (55) which commute with all generators \(F_a = (G_i, p_j)\) “elements of the centre” of \(A\). In order to determine their number, it is convenient to consider the completely gauge-fixed situation of Sec. 2.4 where the \(x^{-}\)-dependence has been eliminated. It is then sufficient to consider the algebra (50) and (53) also with respect
to this six-dimensional phase space only. Obviously the elements of the centre are solutions of
\[ M_{\alpha\beta} \frac{\partial A}{\partial F_\beta} = 0 , \]  
(59)
where
\[ M_{\alpha\beta} = [F_\alpha, F_\beta] . \]  
(60)

The antisymmetric matrix \( M_{\alpha\beta} \) consists of the blocks \([G, G], [G, p], [p, G] \) and \([p, p]\) with (50) and (53), without delta-functions on the r.h.s. It is easy to see that \( M_{\alpha\beta} \) has rank four, i.e. that there are at most two independent elements \( A \) solving (59) in a nontrivial manner.† The two solutions of (59) are \( Q \) and \( K \) with \( Q \) given by (7), and
\[ K = \frac{1}{4\beta^2} \exp \left( -\frac{p_1}{2\beta} \right) [EG_1 + p_3G_2 + p_2G_3] \]  
(61)
as can be verified easily. Whereas \( Q \) has appeared already in the solution of the classical problem, (61) is new, but it vanishes identically on the constraint surface \( G_i = 0 \). From the explicit expression (17), it is evident that both \( Q \) and \( K \) commute with \( H_C \). In terms of the rescaled basis (56), \( Q \) and \( K \) read
\[ Q = \nu^2 \exp(1 + \nu Z_0) \left( \langle Z, Z \rangle - \frac{\Lambda - 1}{\nu^2} \right) \]  
(62)
\[ K = \nu \exp(1 + \nu Z_0) \langle Z, X \rangle + \frac{1}{2} Q X_0 . \]  
(63)

The general proof for \( x^- \)-dependent \( M_{\alpha\beta} \) is straightforward too. Every \( F(X_a, Z_a) \in U(A) \) which commutes with \( X_0 \) may be written as
\[ F = F(X_0, Z_0, Q, K, X_+X_-, X_+Z_- - X_-Z_+) . \]

Evaluating \([X_\pm, F] = 0 \) and \([Z_a, F] = 0 \) one shows easily that \( F = F(Q, K) \).

The algebra (58) for \( \nu \to 0 \) and \( \Lambda = 1 \) reduces to the linear algebra iso(2,1) and the nonpolynomial expressions (52) and (53) become its Casimir operators. For \( \Lambda \neq 1 \) during this limit at one point a redefinition of \( p_1 \) in Eq. (56) with a factor \( \sqrt{\Lambda} \) is necessary in order to arrive at the same result.

In the Einstein limit \( \beta \to \infty \), the matrix \( M_{\alpha\beta} \) simplifies considerably, because terms with \( p_2 \) and \( p_3 \) are dropped. Again its rank is four and the two elements are \( Q_\infty \) of (26) and
\[ K_\infty = E_\infty G_1 + p_3G_2 + p_2G_3 , \]  
(64)
where \( E_\infty = \frac{\Lambda^2}{2} - \lambda \).

†The theory of elements \( A \) of a Poisson algebra \( A \) has been developed a long time ago by Engels and Lie under the title “function groups”, the elements of the centre were called “singular elements” in this context. A short but comprehensive introduction into this work and further references are provided by [14].
4 Conclusions

We have shown that the reason for the complete integrability of $R^2$-gravity with torsion in $d = 2$ is a new dynamical symmetry of Hamiltonian phase space. This symmetry can be written as a (nonlinear) closed Lie-Poisson algebra with the nonlinear deformation proportional to Casimir operators of undeformed $iso(2, 1)$ and with a central extension involving the cosmological constant. In the de Sitter limit the deformation vanishes. For another special limit with vanishing torsion but with non-trivial curvature the deformation reduces to quadratic terms of the Cartan subalgebra of $iso(2, 1)$.

Among the (two) elements of the centre of the new algebra we recovered the locally conserved quantity, encountered previously in the course of the explicit solutions, whereas the second element vanishes identically on the constraint surface. Fixing the homogeneous LC gauge completely allows the reduction to a certain mechanical system of a point particle in three dimensions, depending on time alone.

The novel symmetry is a Lie-Poisson algebra and fulfils Jacobi’s identities. Replacing Poisson-brackets by commutators at fixed $x^+$ in (12)

$$[\cdot, \cdot]_{\text{Poisson}} \rightarrow -i[\cdot, \cdot]_{\text{commutator}}$$

and using Heisenberg equations of motion $\dot{p}_i = i[H_c, p_i]$ etc. should lead to a quantized version of the present model. The notorious problem of locality on a light-like plane in the presence of massless excitations precludes such a step without further analysis which is outside the scope of our present work [11, 15]. Disregarding this complication, we still have to deal with operator ordering ambiguities in the Hamiltonian. It turns out that the only potentially dangerous terms are of the type $p_i q_i$, to be simply defined as $\frac{1}{2} \{p_i, q_i\}$ in order to obtain a hermitian $H_C$. With correspondingly ordered operators in (14) the G-p-algebra (Sec. 3) can be viewed alternatively as a nonlinear commutator-algebra.

As a consequence of this Lie-Poisson algebra $A$, $R^2$-gravity with torsion in $d = 2$ possesses a dynamical symmetry, which leads to a set of solutions which, in contrast to Einstein gravity in $d=2$, are not just topological. On the other hand, the presence of such a symmetry improves the changes of renormalizability and perhaps finiteness, if such a symmetry can be maintained when further fields are added.

It should be emphasized again that this symmetry is visible at the Hamiltonian level. As far as the action is concerned, it has been shown that no further continuous symmetry beside the original ones (diffeomorphisms plus local Lorentz transformations) are present in this model [11].

Apart from possible applications in physics, the nonlinear Lie-Poisson algebra, encountered in the context of this model may also be suggestive for attempts deforming Lie algebras other than $iso(2, 1)$.

Among the open problems the prominent ones are quantization and the treatment of finite boundaries which is closely related to nontrivial topologies of the solution.
This question is especially important because in d=2 Green functions at large distances are not damped naturally.

**Acknowledgement:** The authors appreciate discussions with Th. Strobl.

**References**

[1] M.O. Katanaev and I.V. Volovich, Phys. Lett. B 175, 413 (1986); Ann. Phys. (N.Y.) 197, 1 (1990); M.O. Katanaev, J. Math. Phys. 31, 882 (1990)

[2] M.O. Katanaev, J. Math. Phys. 32, 2483 (1991); M.O. Katanaev, *All universal coverings of two-dimensional gravity with torsion*, preprint

[3] W. Kummer and D.J. Schwarz, Phys. Rev. D 45 (1992), to be published

[4] W. Kummer and D.J. Schwarz, *Renormalization of R²-gravity with dynamical torsion in d=2*, preprint TUW-91-09

[5] F.W. Hehl, P. von der Heyde, G.D. Kerlick and J. Nester, Rev. Mod. Phys. 48, 393 (1976); K.S. Stelle, Phys. Rev. D 16, 953 (1977); D.E. Neville, Phys. Rev. D 18, 3535 (1978); Phys. Rev. D 21, 867 (1980); E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. D 21, 3269 (1980); R. Kuhfuss and J. Nitsch; Gen. Rel. Grav. 18, 1207 (1986)

[6] J. Gegenberg, P.F. Kelly, R.B. Mann and D. Vincent, Phys. Rev. D 37, 3463 (1988); H.J. Schmidt, J. Math. Phys. 32, 1562 (1991)

[7] M.O. Katanaev and I.V. Volovich, Ann. Phys. (N.Y.) 216, 1 (1992)

[8] M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. B 332, 169 (1990)

[9] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986); Phys. Rev. D 36, 1587 (1987)

[10] M. Henneaux, Phys. Rep. 126, 1 (1985)

[11] Th. Strobl, private communication

[12] E. D’Hooker, Mod. Phys. Lett. A 6, 745 (1991)

[13] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Commun. Math. Phys. 124, 87 (1989); Phys. Lett. B 255, 549 (1991)

[14] L.P. Eisenhart, *Continuous groups of transformations*, Dover Pub., Inc. (1961) New York

[15] F. Haider, *QED in 1+1 Dimensionen in der Lichtkegelformulierung*, Vorbereitungspraktikum 1991, TU Wien, unpublished