WEIGHTED LEBESGUE AND CENTRAL MORREY
ESTIMATES FOR P-ADIC MULTILINEAR HAUSDORFF
OPERATORS AND ITS COMMUTATORS

NGUYEN MINH CHUONG, DAO VAN DUONG, AND KIEU HUU DUNG

Abstract. In this paper, we establish the sharp boundedness of $p$-adic multilinear Hausdorff operators on the product of Lebesgue and central Morrey spaces associated with both power weights and Muckenhoupt weights. Moreover, the boundedness for the commutators of $p$-adic multilinear Hausdorff operators on the such spaces with symbols in central BMO space is also obtained.

1. Introduction

The $p$-adic analysis in the past decades has received a lot of attention due to its important applications in mathematical physics as well as its necessity in sciences and technologies (see e.g. [2, 3, 6, 12, 22, 23, 24, 33, 34, 35, 36] and references therein). It is well known that the theory of functions from $\mathbb{Q}_p$ into $\mathbb{C}$ play an important role in $p$-adic quantum mechanics, the theory of $p$-adic probability in which real-valued random variables have to be considered to solve covariance problems. In recent years, there is an increasing interest in the study of harmonic analysis and wavelet analysis over the $p$-adic fields (see e.g. [1, 6, 10, 20, 21, 24]).

It is crucial that the Hausdorff operator is one of the important operators in harmonic analysis. It is closely related to the summability of the classical Fourier series (see, for instance, [13], [15], [17], and the references therein). Let $\Phi$ be a locally integrable function on $\mathbb{R}^n$. The matrix Hausdorff operator $H_{\Phi,A}$ associated to the kernel function $\Phi$ is then defined in terms of the integral form as follows

\[ H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(A(y)x) dy, \quad x \in \mathbb{R}^n, \quad (1.1) \]

2010 Mathematics Subject Classification. 42B25, 42B99, 26D15.

Key words and phrases. Multilinear Hausdorff operator, commutator, central BMO space, Morrey space, $A_p$ weight, maximal operator, $p$-adic analysis.

This paper is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED).
where $A(y)$ is an $n \times n$ invertible matrix for almost everywhere $y$ in the support of $\Phi$. It is worth pointing out that if the kernel function $\Phi$ is chosen appropriately, then the Hausdorff operator reduces to many classical operators in analysis such as the Hardy operator, the Cesàro operator, the Riemann-Liouville fractional integral operator and the Hardy-Littlewood average operator.

In 2010, Volosivets [37] introduced the matrix Hausdorff operator on the $p$-adic numbers field as follows

$$H_{\varphi, A}(f)(x) = \int_{Q_p^n} \varphi(t) f(A(t)x) dt, \quad x \in Q_p^n,$$

(1.2)

where $\varphi(t)$ is a locally integrable function on $Q_p^n$ and $A(t)$ is an $n \times n$ invertible matrix for almost everywhere $t$ in the support of $\varphi$. It is easy to see that if $\varphi(t) = \psi(t_1)x_{p^n}(t)$ and $A(t) = t_1 I_n$ ($I_n$ is an identity matrix), for $t = (t_1, t_2, ..., t_n)$, where $\psi : Q_p \to \mathbb{C}$ is a measurable function, $H_{\varphi, A}$ then reduces to the $p$-adic weighted Hardy-Littlewood average operator due to Rim and Lee [31]. In recent years, the theory of the Hardy operators, the Hausdorff operators over the $p$-adic numbers field has been significantly developed into different contexts, and they are actually useful for $p$-adic analysis (see e.g. [7], [8], [16], [38]). It is known that the authors in [9] also introduced and studied a general class of multilinear Hausdorff operators on the real field defined by

$$H_{\Phi, \vec{A}}(\vec{f})(x) = \int_{\mathbb{R}^n} \Phi(y) \prod_{i=1}^m f_i(A_i(y)x) dy, \quad x \in \mathbb{R}^n,$$

(1.3)

for $\vec{f} = (f_1, ..., f_m)$ and $\vec{A} = (A_1, ..., A_m)$.

Motivated by above results, in this paper we shall introduce and study a class of $p$-adic multilinear (matrix) Hausdorff operators defined as follows.

**Definition 1.1.** Let $\Phi : Q_p^n \to [0, \infty)$. Let $f_1, f_2, ..., f_m$ be measurable complex-valued functions on $Q_p^n$. The $p$-adic multilinear Hausdorff operator is defined by

$$H_{\Phi, \vec{A}}(\vec{f})(x) = \int_{Q_p^n} \Phi(y) \prod_{i=1}^m f_i(A_i(y)x) dy, \quad x \in Q_p^n,$$

(1.4)

for $\vec{f} = (f_1, ..., f_m)$.

Let $b$ be a measurable function. We denote by $M_b$ the multiplication operator defined by $M_b f(x) = b(x) f(x)$ for any measurable function $f$. If $H$ is a linear operator on some measurable function space, the commutator of Coifman-Rochberg-Weiss type formed by $M_b$ and $H$ is defined by $[M_b, H] f(x) = (M_b H - H M_b) f(x)$. Analogously, let us give the definition...
for the commutators of Coifman-Rochberg-Weiss type of \( p \)-adic multilinear Hausdorff operator.

**Definition 1.2.** Let \( \Phi, \vec{A} \) be as above. The Coifman-Rochberg-Weiss type commutator of \( p \)-adic multilinear Hausdorff operator is defined by

\[
\mathcal{H}^p_{\Phi, \vec{A}, \vec{b}}(f)(x) = \int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|^p} \prod_{i=1}^{m} \left( b_i(x) - b_i(A_i(y)x) \right) \prod_{i=1}^{m} f_i(A_i(y)x) dy,
\]

where \( x \in \mathbb{Q}_p^n, \vec{b} = (b_1, ..., b_m) \) and \( b_i \) are locally integrable functions on \( \mathbb{Q}_p^n \) for all \( i = 1, ..., m \).

The main purpose of this paper is to study the \( p \)-adic multilinear Hausdorff operators and its commutators on the \( p \)-adic numbers field. More precisely, we obtain the necessary and sufficient conditions for the boundedness of \( \mathcal{H}^p_{\Phi, \vec{A}} \) and \( \mathcal{H}^p_{\Phi, \vec{A}, \vec{b}} \) on the product of Lebesgue and central Morrey spaces with weights on \( p \)-adic field. In each case, we estimate the corresponding operator norms. Moreover, the boundedness of \( \mathcal{H}^p_{\Phi, \vec{A}, \vec{b}} \) on the such spaces with symbols in central BMO space is also established. It should be pointed out that all our results are new even in the case of \( p \)-adic linear Hausdorff operators.

Our paper is organized as follows. In Section 2, we present some notations and preliminaries about \( p \)-adic analysis as well as give some definitions of the Lebesgue and central Morrey spaces associated with power weights and Muckenhoupt weights. Our main theorems are given and proved in Section 3 and Section 4.

### 2. Some notations and definitions

For a prime number \( p \), let \( \mathbb{Q}_p \) be the field of \( p \)-adic numbers. This field is the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the non-Archimedean \( p \)-adic norm \(| \cdot |_p\). This norm is defined as follows: if \( x = 0 \), \(|0|_p = 0\); if \( x \neq 0 \) is an arbitrary rational number with the unique representation \( x = p^n \frac{m}{n} \), where \( m, n \) are not divisible by \( p \), \( \alpha = \alpha(x) \in \mathbb{Z} \), then \(|x|_p = p^{-\alpha}\). This norm satisfies the following properties:

(i) \(|x|_p \geq 0\), \( \forall x \in \mathbb{Q}_p \), and \(|x|_p = 0 \iff x = 0\);

(ii) \(|xy|_p = |x|_p |y|_p\), \( \forall x, y \in \mathbb{Q}_p \);

(iii) \(|x + y|_p \leq \max(|x|_p, |y|_p)\), \( \forall x, y \in \mathbb{Q}_p \), and when \(|x|_p \neq |y|_p\), we have \(|x + y|_p = \max(|x|_p, |y|_p)\).

It is also well-known that any non-zero \( p \)-adic number \( x \in \mathbb{Q}_p \) can be uniquely represented in the canonical series

\[
x = p^\alpha(x_0 + x_1p + x_2p^2 + \cdots),
\]

(2.1)
where \(\alpha = \alpha(x) \in \mathbb{Z}\), \(x_k = 0, 1, ..., p - 1, x_0 \neq 0, k = 0, 1, ...\). This series converges in the \(p\)-adic norm since \(|x_k p^k|_p \leq p^{-k}\).

The space \(\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p\) consists of all points \(x = (x_1, ..., x_n)\), where \(x_i \in \mathbb{Q}_p, i = 1, ..., n, n \geq 1\). The \(p\)-adic norm of \(\mathbb{Q}_p^n\) is defined by

\[
|x|_p = \max_{1 \leq j \leq n} |x_j|_p.
\] (2.2)

Let \(A\) be an \(n \times n\) matrix with entries \(a_{ij} \in \mathbb{Q}_p\). For \(x = (x_1, ..., x_n) \in \mathbb{Q}_p^n\), we denote

\[
Ax = \left( \sum_{j=1}^{n} a_{1j}x_j, ..., \sum_{j=1}^{n} a_{nj}x_j \right).
\]

By Lemma 2 in paper [38], the norm of \(A\), regarded as an operator from \(\mathbb{Q}_p^n\) to \(\mathbb{Q}_p^n\), is

\[
\|A\|_p := \max_{1 \leq i \leq n} \max_{1 \leq j \leq n} |a_{ij}|_p.
\]

For simplicity of notation, we write \(k_A = \log_p \|A\|_p\). It is clear to see that \(k_A \in \mathbb{Z}\). It is easy to show that \(|Ax|_p \leq \|A\|_p|x|_p\) for any \(x \in \mathbb{Q}_p^n\). In addition, if \(A\) is invertible, by estimating as Lemma 3.1 in paper [29], then we get

\[
\|A\|^{-n}_p \leq |\det(A^{-1})|_p \leq \|A^{-1}\|_p^n.
\] (2.3)

Let

\[
B_\alpha(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\alpha\}
\]

be a ball of radius \(p^\alpha\) with center at \(a \in \mathbb{Q}_p^n\). Similarly, denote by

\[
S_\alpha(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\alpha\}
\]

the sphere with center at \(a \in \mathbb{Q}_p^n\) and radius \(p^\alpha\). If \(B_\alpha = B_\alpha(0), S_\alpha = S_\alpha(0)\), then for any \(x_0 \in \mathbb{Q}_p^n\) we have \(x_0 + B_\alpha = B_\alpha(x_0)\) and \(x_0 + S_\alpha = S_\alpha(x_0)\).

Since \(\mathbb{Q}_p^n\) is a locally compact commutative group under addition, it follows from the standard theory that there exists a Haar measure \(dx\) on \(\mathbb{Q}_p^n\), which is unique up to positive constant multiple and is translation invariant. This measure is unique by normalizing \(dx\) such that

\[
\int_{B_0} dx = |B_0| = 1,
\]

where \(|B|\) denotes the Haar measure of a measurable subset \(B\) of \(\mathbb{Q}_p^n\). By simple calculation, it is easy to obtain that \(|B_\alpha(a)| = p^{n \alpha}, |S_\alpha(a)| = p^{n \alpha}(1 - p^{-n}) \simeq p^{n \alpha}\), for any \(a \in \mathbb{Q}_p^n\). For \(f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)\), we have

\[
\int_{\mathbb{Q}_p^n} f(x)dx = \lim_{\alpha \to +\infty} \int_{B_\alpha} f(x)dx = \lim_{\alpha \to +\infty} \sum_{-\infty < \gamma \leq \alpha} \int_{S_\gamma} f(x)dx.
\]
In particular, if \( f \in L^1(Q^n_p) \), we can write
\[
\int_{Q^n_p} f(x)\,dx = \sum_{a=\infty}^{+\infty} \int_{S_a} f(x)\,dx,
\]
and
\[
\int f(tx)\,dx = \frac{1}{|t|^n_{Q^n_p}} \int_{Q^n_p} f(x)\,dx,
\]
where \( t \in Q^n_p \setminus \{0\} \). For a more complete introduction to the \( p \)-adic analysis, we refer the readers to [22, 36] and the references therein.

Let \( \omega(x) \) be a weighted function, that is a non-negative locally integrable measurable function on \( Q^n_p \). The weighted Lebesgue space \( L^q_{\omega}(Q^n_p) \) (\( 0 < q < \infty \)) is defined to be the space of all measurable functions \( f \) on \( Q^n_p \) such that
\[
\|f\|_{L^q_{\omega}(Q^n_p)} = \left( \int_{Q^n_p} |f(x)|^q \omega(x)\,dx \right)^{1/q} < \infty.
\]

The space \( L^q_{\omega,\text{loc}}(Q^n_p) \) is defined as the set of all measurable functions \( f \) on \( Q^n_p \) satisfying \( \int_K |f(x)|^q \omega(x)\,dx < \infty \), for any compact subset \( K \) of \( Q^n_p \). The space \( L^q_{\omega,\text{loc}}(Q^n_p \setminus \{0\}) \) is also defined in a similar way as the space \( L^q_{\omega,\text{loc}}(Q^n_p) \).

Through the whole paper, we denote by \( C \) a positive geometric constant that is independent of the main parameters, but can change from line to line. We also write \( a \lesssim b \) to mean that there is a positive constant \( C \), independent of the main parameters, such that \( a \leq Cb \). The symbol \( f \simeq g \) means that \( f \) is equivalent to \( g \) (i.e. \( C^{-1}f \leq g \leq Cf \)). For any real number \( \ell > 1 \), denote by \( \ell' \) conjugate real number of \( \ell \), i.e. \( \frac{1}{\ell} + \frac{1}{\ell'} = 1 \). Denote \( \omega(B)^{\lambda} = (\int_{B} \omega(x)\,dx)^{\lambda} \), for \( \lambda \in \mathbb{R} \). Remark that if \( \omega(x) = |x|^\alpha_p \) for \( \alpha > -n \), then we have
\[
\omega(B_\gamma) = \int_{B_\gamma} |x|^\alpha_p \,dx = \sum_{k \leq \gamma} \int_{S_k} p^{k\alpha} \,dx = \sum_{k \leq \gamma} p^{k(\alpha+n)}(1 - p^{-n}) \simeq p^{\gamma(\alpha+n)}. \tag{2.4}
\]

Next, let us give the definition of weighted \( \lambda \)-central Morrey spaces on \( p \)-adic numbers field as follows.

**Definition 2.1.** Let \( \lambda \in \mathbb{R} \) and \( 1 < q < \infty \). The weighted \( \lambda \)-central Morrey \( p \)-adic spaces \( B^q_{\omega,}\lambda(Q^n_p) \) consists of all Haar measurable functions \( f \in L^q_{\omega,\text{loc}}(Q^n_p) \) satisfying \( \|f\|_{B^q_{\omega,}\lambda(Q^n_p)} < \infty \), where
\[
\|f\|_{B^q_{\omega,}\lambda(Q^n_p)} = \sup_{\gamma \in \mathbb{Z}} \left( \frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q \omega(x)\,dx \right)^{1/q}. \tag{2.5}
\]

Remark that \( B^q_{\omega,}\lambda(Q^n_p) \) is a Banach space and reduces to \( \{0\} \) when \( \lambda < -\frac{1}{q} \).
Let us recall the definition of the weighted central BMO \( p \)-adic space.

**Definition 2.2.** Let \( 1 \leq q < \infty \) and \( \omega \) be a weight function. The weighted central bounded mean oscillation space \( CMO_q^\omega(Q^n_p) \) is defined as the set of all functions \( f \in L^q_{\omega, \text{loc}}(Q^n_p) \) such that

\[
\|f\|_{CMO_q^\omega(Q^n_p)} = \sup_{\gamma \in \mathbb{Z}} \left( \frac{1}{\omega(B_\gamma)} \int_{B_\gamma} |f(x) - f_{B_\gamma}|^q \omega(x) dx \right)^{\frac{1}{q}} < \infty,
\]

where \( f_{B_\gamma} = \frac{1}{|B_\gamma|} \int_{B_\gamma} f(x) dx \).

The theory of \( A_\ell \) weight was first introduced by Benjamin Muckenhoupt on the Euclidean spaces in order to characterise the boundedness of Hardy-Littlewood maximal functions on the weighted \( L^\ell \) spaces (see \([28]\)). For \( A_\ell \) weights on the \( p \)-adic fields, more generally, on the local fields or homogeneous type spaces, one can refer to \([11, 18]\) for more details. Let us now recall the definition of \( A_\ell \) weights.

**Definition 2.3.** Let \( 1 < \ell < \infty \). It is said that a nonnegative locally integrable function \( \omega \in A_\ell(Q^n_p) \) if there exists a constant \( C \) such that for all balls \( B \), we have

\[
\left( \frac{1}{|B|} \int_B \omega(x) dx \right) \left( \frac{1}{|B|} \int_B \omega(x)^{-\ell/(\ell-1)} dx \right)^{\ell-1} \leq C.
\]

It is said that a weight \( \omega \in A_1(Q^n_p) \) if there is a constant \( C \) such that for all balls \( B \), we get

\[
\frac{1}{|B|} \int_B \omega(x) dx \leq C \text{ essinf}_{x \in B} \omega(x).
\]

We denote by \( A_\infty(Q^n_p) = \bigcup_{1 \leq \ell < \infty} A_\ell(Q^n_p) \). Let us give the following standard result related to the Muckenhoupt weights.

**Proposition 2.4.**

(i) \( A_\ell(Q^n_p) \subseteq A_q(Q^n_p) \), for \( 1 \leq \ell < q < \infty \).

(ii) If \( \omega \in A_\ell(Q^n_p) \) for \( 1 < \ell < \infty \), then there is an \( \varepsilon > 0 \) such that \( \ell - \varepsilon > 1 \) and \( \omega \in A_{\ell-\varepsilon}(Q^n_p) \).

A closing relation to \( A_\infty(Q^n_p) \) is the reverse Hölder condition. If there exist \( r > 1 \) and a fixed constant \( C \) such that

\[
\left( \frac{1}{|B|} \int_B \omega(x)^r dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B \omega(x) dx \right),
\]

for all balls \( B \subset Q^n_p \), we then say that \( \omega \) satisfies the reverse Hölder condition of order \( r \) and write \( \omega \in RH_r(Q^n_p) \). According to Theorem 19 and Corollary
21 in [19], \( \omega \in A_\infty(Q^n_p) \) if and only if there exists some \( r > 1 \) such that \( \omega \in RH_r(Q^n_p) \). Moreover, if \( \omega \in RH_r(Q^n_p), r > 1 \), then \( \omega \in RH_{r+\varepsilon}(Q^n_p) \) for some \( \varepsilon > 0 \). We thus write \( r_\omega = \sup\{r > 1 : \omega \in RH_r(Q^n_p)\} \) to denote the critical index of \( \omega \) for the reverse Hölder condition.

An important example of \( A_\ell(Q^n_p) \) weight is the power function \( |x|^\alpha_p \). By the similar arguments as Propositions 1.4.3 and 1.4.4 in [26], we obtain the following properties of power weights.

**Proposition 2.5.** Let \( x \in Q^n_p \). Then, we have

(i) \( |x|^\alpha_p \in A_1(Q^n_p) \) if and only if \( -n < \alpha \leq 0 \);

(ii) \( |x|^\alpha_p \in A_\ell(Q^n_p) \) for \( 1 < \ell < \infty \), if and only if \( -n < \alpha < n(\ell - 1) \).

Let us give the following standard characterization of \( A_\ell \) weights which it is proved in the similar way as the real setting (see [14, 32] for more details).

**Proposition 2.6.** Let \( \omega \in A_\ell(Q^n_p) \cap RH_r(Q^n_p), \ell \geq 1 \) and \( r > 1 \). Then, there exist constants \( C_1, C_2 > 0 \) such that

\[
C_1 \left( \frac{|E|}{|B|} \right)^{\ell} \leq \frac{\omega(E)}{\omega(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r}
\]

for any measurable subset \( E \) of a ball \( B \).

**Proposition 2.7.** If \( \omega \in A_\ell(Q^n_p), 1 \leq \ell < \infty \), then for any \( f \in L^1_{\text{loc}}(Q^n_p) \) and any ball \( B \subset Q^n_p \), we have

\[
\frac{1}{|B|} \int_B |f(x)|dx \leq C \left( \frac{1}{\omega(B)} \int_B |f(x)|^\ell \omega(x)dx \right)^{1/\ell}.
\]

Let us recall the definition of the Hardy-Littlewood maximal operator

\[
\mathcal{M}f(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{p^{n\gamma}} \int_{B_\gamma(x)} |f(y)|dy.
\]

It is useful to remark that the Hardy-Littlewood maximal operator \( M \) is bounded on \( L^p_{\text{loc}}(Q^n_p) \) if and only if \( \omega \in A_\ell(Q^n_p) \) for all \( \ell > 1 \). Finally, we introduce a new maximal operator which is used in the sequel, that is

\[
\mathcal{M}^{\text{mod}}f(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{p^{n\gamma}} \int_{|x|^p \leq p^{n\gamma}} |f(y)|dy.
\]

### 3. The main results about the boundness of \( \mathcal{H}^p_{\Phi, A} \)

Let us now assume that \( q \) and \( q_i \in [1, \infty) \), \( \alpha, \alpha_i \) are real numbers such that \( \alpha_i \in (-n, \infty) \), for \( i = 1, 2, ..., m \) and

\[
\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = \frac{1}{q}.
\]
Proof. Firstly, we will prove for the sufficient condition of the theorem. By
Furthermore, for almost everywhere \( y \in Q^m_p \). Thus, by the property of invertible matrix, it is
easy to show that
\[
\| A_i(y) \|_p \cdot \| A_i^{-1}(y) \|_{p'} \leq p'^{-\sigma}, \text{ for all } i = 1, ..., m, \tag{3.1}
\]
for all \( \sigma \in \mathbb{R} \).

Our first main result is the following.

Theorem 3.1. Let \( \omega(x) = |x|^n_1, ..., \omega_m(x) = |x|^n_m \) and \( \omega(x) = |x|^n \). Then, \( H_{\Phi,A}^p \) is bounded from \( L_{\omega_1}^m(Q^m_p) \times \cdots \times L_{\omega_m}^m(Q^m_p) \) to \( L^2(Q^m_p) \) if and only if
\[
C_1 = \int_{Q^m_p} \Phi(y) \| y \|_p^m \prod_{i=1}^m \| A_i^{-1}(y) \|_{p'} \frac{\alpha_i + n}{\alpha_i} \, dy < \infty.
\]
Furthermore, \( \| H_{\Phi,A}^p \|_{L^2(Q^m_p)} \to L^2(Q^m_p) \approx C_1. \)

Proof. Firstly, we will prove for the sufficient condition of the theorem. By applying the Minkowski inequality and the Hölder inequality, we have
\[
\| H_{\Phi,A}^p (\tilde{f}) \|_{L^2(Q^m_p)} = \left( \int_{Q^m_p} \left| \int_{Q^m_p} \frac{\Phi(y)}{|y|_p^m} \prod_{i=1}^m f_i(A_i(y) x) dy \right|^q \omega(x) dx \right)^{\frac{1}{q}}
\]
\[
\leq \int_{Q^m_p} \frac{\Phi(y)}{|y|_p^m} \prod_{i=1}^m \| f_i(A_i(y) \cdot) \|_{L^2_\omega(Q^m_p)} dy.
\]

By making the change of variables, we get
\[
\| f_i(A_i(y) \cdot) \|_{L^m_\omega(Q^m_p)} = \left( \int_{Q^m_p} |f_i(z)|^\omega |A_i^{-1}(y) z|_{p'}^\omega |\det A_i^{-1}(y)|_{p} dz \right)^{\frac{1}{\omega}}
\]
\[
\leq \max \{ \| A_i^{-1}(y) \|_{p'}, \| A_i(y) \|_{p'}^{-\omega} \} \frac{1}{\omega} |\det A_i^{-1}(y)|_{p} ^\omega \| f_i \|_{L^m_\omega(Q^m_p)}.
\]
Thus,
\[
\|\mathcal{H}_{\Phi,A}^p(\hat{f})\|_{L^p_\nu(Q^n_p)}
\leq \left( \int_{Q^n_p} \frac{\Phi(y)}{|y|_p^\nu} \prod_{i=1}^m \max \{ \|A_i^{-1}(y)\|^\alpha_p, \|A_i(y)\|^{-\alpha}_p \} \frac{1}{\nu} |\det A_i^{-1}(y)| \frac{1}{\nu} dy \right) \prod_{i=1}^m \|f_i\|_{L^p_\nu(Q^n_p)}^{\alpha_i/n_i}.
\]
(4.4)

Note that, by (2.3) and (3.2), we have
\[
\max \{ \|A_i^{-1}(y)\|^\alpha_p, \|A_i(y)\|^{-\alpha}_p \} \frac{1}{\nu} |\det A_i^{-1}(y)| \frac{1}{\nu} \lesssim \|A_i^{-1}(y)\|_p^{\alpha_i/n_i}
\]
This shows that
\[
\|\mathcal{H}_{\Phi,A}^p(\hat{f})\|_{L^p_\nu(Q^n_p)} \lesssim C_{1} \prod_{i=1}^m \|f_i\|_{L^p_\nu(Q^n_p)}^{\alpha_i/n_i}.
\]

Next, to prove the necessary condition of this theorem, for \(i = 1, \ldots, m\) and \(r \in \mathbb{Z}^+\), let us now take
\[
f_{i,r}(x) = \begin{cases} 0, & \text{if } |x|_p \leq p^{-\nu_\delta}^{-1}, \\ |x|_p^{-\frac{n}{n_i} - \frac{\alpha_i}{\nu_i} - p^{-r}}, & \text{otherwise}. \end{cases}
\]

By a simple calculation, we have
\[
\|f_{i,r}\|_{L^p_{\nu_i}(Q^n_p)} = \left( \int_{Q^n_p} |x|_p^{-n-\alpha_i-q_rp^{-r}} \chi_{B_{r^{-\nu_\delta}}^{-1}}(x), |x|_p^{\alpha_i} dx \right)^{\frac{1}{\nu_i}} = \left( \sum_{k \geq -\nu_\delta, S_k} \int_{S_k} p^{n-k-q_rp^{-r}} dx \right)^{\frac{1}{\nu_i}} 
\]
\[
\lesssim \left( \sum_{k \geq -\nu_\delta} p^{n-k-q_rp^{-r}} p^{kn} \right)^{\frac{1}{\nu_i}} = \left( \sum_{k \geq -\nu_\delta} p^{-k-q_rp^{-r}} \right)^{\frac{1}{\nu_i}} = \frac{p^{\nu_\delta p^{-r}}}{(1 - p^{-q_rp^{-r}})^{\frac{1}{\nu_i}}}. 
\]
(3.6)

Next, we define two sets as follows
\[
S_x = \bigcap_{i=1}^m \left\{ y \in Q^n_p : |A_i(y)x|_p \geq p^{-\nu_\delta} \right\},
\]
and
\[
U_r = \left\{ y \in Q^n_p : \|A_i(y)\|_p \geq p^{-r}, \text{ for all } i = 1, \ldots, m \right\}.
\]

From this, we derive
\[
U_r \subset S_x, \text{ for all } x \in Q^n_p \setminus B_{r-1}.
\]
(3.7)

In fact, by letting \(y \in U_r\), we have \(\|A_i(y)\|_p|x|_p \geq 1\), for all \(x \in Q^n_p \setminus B_{r-1}\).

Thus, by applying the condition (3.1), one has
\[
|A_i(y)x|_p \geq \|A_i^{-1}(y)\|^{-1}_p |x|_p = \frac{\|A_i(y)\|_p|x|_p}{\|A_i^{-1}(y)\|_p \|A_i(y)\|_p} \geq p^{-\nu_\delta},
\]
which confirms the relation (3.7). Now, by taking \( x \in \mathbb{Q}_p^n \setminus B_{r-1} \) and using the relation (3.7), we get

\[
\mathcal{H}_{\Phi,A}^p(f)(x) \geq \int_{S_x} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m |A_i(y)x|_p^{-\frac{n}{q_i}} \prod_{i=1}^m |A_i(y)x|_p^{-\frac{\alpha_i}{q_i}} dy \geq \int_{U_r} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m |A_i(y)x|_p^{-\frac{n}{q_i}} \prod_{i=1}^m |A_i(y)x|_p^{-\frac{\alpha_i}{q_i}} dy.
\]

From this, by (3.3), one has

\[
\mathcal{H}_{\Phi,A}^p(f)(x) \geq\left( \int_{U_r} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m |A_i^{-1}(y)||_p^{\frac{n+\alpha_i}{q_i}} dy \right) \cdot |x|_p^{\frac{(n+\alpha)}{q}} - mp^-r \cdot \chi_{\mathbb{Q}_p^n \setminus B_{r-1}}(x).
\]

where

\[
\mathcal{A}_r = \int_{U_r} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m |A_i^{-1}(y)||_p^{\frac{n+\alpha_i}{q_i}} \prod_{i=1}^m |A_i^{-1}(y)||_p^{p^{-r}} dy
\]

and \( g(x) = |x|_p^{\frac{(n+\alpha)}{q}} - mp^-r \cdot \chi_{\mathbb{Q}_p^n \setminus B_{r-1}}(x) \). By estimating as (3.6) above, we also have

\[
\|g\|_{L^2_q(\mathbb{Q}_p^n)} \simeq \frac{p^{-mp^-r}}{(1-p^{-qmp^-r})^\frac{1}{q}}.
\]

As a consequence above, by (3.6), we find that

\[
\|\mathcal{H}_{\Phi,A}^p(f)\|_{L^2_q(\mathbb{Q}_p^n)} \geq \mathcal{A}_r \cdot T_r \cdot \prod_{i=1}^m \|f_{i,r}\|_{L^q_{m_i}(\mathbb{Q}_p^n)},
\]

where

\[
T_r = \frac{\prod_{i=1}^m (1-p^{-q_{mp^-r}})^{\frac{1}{q_i}}}{(1-p^{-q_{mp^-r}})^\frac{1}{q} \prod_{i=1}^m p^{m_i \cdot p^{-r}}}.
\]

Note that from \( \frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q} \), it is clear to obtain that

\[
\lim_{r \to +\infty} T_r = a > 0.
\]

Therefore, because \( \mathcal{H}_{\Phi,A}^p \) is bounded from \( L^{q_i}_{m_i}(\mathbb{Q}_p^n) \times \cdots \times L^{q_m}_{m_m}(\mathbb{Q}_p^n) \) to \( L^q(\mathbb{Q}_p^n) \), there exists \( M > 0 \) such that \( \mathcal{A}_r \leq M \), for \( r \) sufficiently big. On the other hand, by letting \( r \) sufficiently large, \( y \in U_r \) and by (3.1), we get

\[
p^{-mp^-r} \cdot \prod_{i=1}^m |A_i^{-1}(y)||_p^{p^{-r}} \lesssim p^{p^-r} \lesssim 1.
\]
MULTILINEAR HAUSDORFF OPERATORS AND COMMUTATORS

From this, by the dominated convergence theorem of Lebesgue, we obtain
\[
\int_{\mathbb{Q}_p^n} \Phi(y) \prod_{i=1}^m |\text{det} A_i^{-1}(y)| |A_i(y)|^{\frac{s}{p}} dy < \infty,
\]
which finishes the proof of the theorem.

\[\square\]

**Theorem 3.2.** Let \(1 \leq q^* \leq \infty\) and \(\omega \in A_\zeta\) with the finite critical index \(r_\omega\) for the reverse Hölder and \(\omega(B_\gamma) \approx 1\), for all \(\gamma \in \mathbb{Z}\). Assume that \(q > q^*\zeta r_\omega/(r_\omega - 1)\), \(\delta \in (1, r_\omega)\) and the following condition holds:

\[
C_2 = \int_{\mathbb{Q}_p^n} \Phi(y) \prod_{i=1}^m |\text{det} A_i^{-1}(y)| |A_i(y)|^{\frac{s}{p}} \times
\]
\[
\times \left( \chi_{\{\|A_i(y)\|_p \leq 1\}}(y)\|A_i(y)\|^{\frac{n-\zeta}{p}}_p + \chi_{\{\|A_i(y)\|_p > 1\}}(y)\|A_i(y)\|^{\frac{n}{p}}_p \right) dy < \infty.
\]

Then, we have \(H_{\Phi,A}^p\) is bounded from \(L_{\omega}^q(\mathbb{Q}_p^n) \times \cdots \times L_{\omega}^{q_m}(\mathbb{Q}_p^n)\) to \(L_{\omega}^q(\mathbb{Q}_p^n)\).

**Proof.** For any \(R \in \mathbb{Z}\), by the Minkowski inequality, we have

\[
\|H_{\Phi,A}^p(f)\|_{L_{\omega}^q(B_R)} \leq \int_{\mathbb{Q}_p^n} \Phi(y) \left( \int_{B_R} \prod_{i=1}^m |f_i(A_i(y))x|^{q^*} \omega(x) dx \right)^\frac{1}{q^*} dy.
\]

From the inequality \(q > q^*\zeta r_\omega/(r_\omega - 1)\), there exists \(r \in (1, r_\omega)\) such that \(q = \zeta q^* r^*\). Then, by the Hölder inequality and the reverse Hölder condition, we have

\[
\left( \int_{B_R} \prod_{i=1}^m |f_i(A_i(y))x|^{q^*} \omega(x) dx \right)^\frac{1}{q^*} \leq \left( \int_{B_R} \prod_{i=1}^m |f_i(A_i(y))x|^{q^*} \omega(x) dx \right)^\frac{1}{q^*} \left( \int_{B_R} \omega(x) dx \right)^\frac{1}{q^*} \lesssim \left( \int_{B_R} \prod_{i=1}^m |f_i(A_i(y))x|^{q^*} \omega(B_R) \right)^\frac{1}{q^*} |B_R|^{-\frac{\zeta}{q^*}}.
\]

Next, by making the Hölder inequality and the change of variables formula, and applying Proposition 2.7, we have

\[
\left( \int_{B_R} \prod_{i=1}^m |f_i(A_i(y))x|^{q^*} \omega(x) dx \right)^\frac{1}{q^*} \leq \prod_{i=1}^m \left( \int_{B_R} |f_i(A_i(y))x|^{q^*} dx \right)^\frac{1}{q^*_i} \lesssim \prod_{i=1}^m \left( \int_{B_R} |\text{det} A_i^{-1}(y)|^{q^*_i} dy \right)^\frac{1}{q^*_i} \left( \int_{B_R+k_{A_i}} |f_i| L_{\omega}^{q^*_i}(B_R+k_{A_i}) \right)^\frac{1}{q^*_i} \lesssim \prod_{i=1}^m \left( |\text{det} A_i^{-1}(y)|^{q^*_i} \omega(B_R+k_{A_i}) \right)^\frac{1}{q^*_i} |f_i| L_{\omega}^{q^*_i}(B_R+k_{A_i}),
\]

where
Hence, by letting $|B_{R+kA_i}| \leq 1$, we obtain that

$$\|H_{\Phi,A}^p(\vec{f})\|_{L^p_q(B_R)} \lesssim \omega(B_R) \omega(B_{R+kA_i})^{-\frac{1}{q_i}} \|A_i(y)\|_{L^p_q(B_{R+kA_i})}dy$$

Next, for $i = 1, \ldots, m$, by using Proposition 2.6, we have

$$\left(\frac{|Q_n|}{\omega(B_{R+kA_i})}\right)^{\frac{1}{q_i}} \leq \begin{cases} \frac{|B_R|}{|B_{R+kA_i}|} - p \frac{(R-R+kA_i)n}{q_i} & \text{if } \|A_i(y)\|_{p} \leq 1, \\ \frac{|B_R|}{|B_{R+kA_i}|} - p \frac{(R-R+kA_i)\delta}{q_i} & \text{otherwise.} \end{cases}$$

Hence, by letting $R \to +\infty$ and applying the monotone convergence theorem of Lebesgue, we obtain that

$$\|H_{\Phi,A}^p(\vec{f})\|_{L^p_q(\mathbb{Q})^\circ} \lesssim C_2 \prod_{i=1}^m \|f_i\|_{L^p_q(\mathbb{Q})},$$

which completes the proof of the theorem.

Theorem 3.3. Let $\omega, \omega_i$ be as Theorem 3.1 and $\lambda_i \in (\frac{1}{q_i}, 0)$ for all $i = 1, \ldots, m$. Assume that

$$(\alpha + n)\lambda = (\alpha_1 + n)\lambda_1 + \cdots + (\alpha_m + n)\lambda_m. \quad (3.10)$$
Then, $\mathcal{H}^p_{\Phi, \mathcal{A}}$ is bounded from $\dot{B}^{q_1\lambda_1}_{\mathcal{A}_1}(Q^n_p) \times \cdots \times \dot{B}^{q_m\lambda_m}_{\mathcal{A}_m}(Q^n_p)$ to $\dot{B}^{q\lambda}_{\mathcal{A}}(Q^n_p)$ if and only if

$$C_3 = \int_{Q^n_p} \Phi(y) \frac{m}{|y|^p} \prod_{i=1}^m \|A_i^{-1}(y)\|_{L^p}^{(\alpha_i + n)\lambda_i} dy < \infty.$$  

Furthermore, $\|\mathcal{H}^p_{\Phi, \mathcal{A}}\|_{L^2(Q^n_p)} \simeq C_3$.

**Proof.** We start with the proof for the sufficient condition of the theorem. For $\gamma \in \mathbb{Z}$, by estimating as $(3.1)$ and $(3.5)$ above, we have

$$\|\mathcal{H}^p_{\Phi, \mathcal{A}}(f)\|_{L^2(B_r)} \leq \int_{Q^n_p} \frac{\Phi(y)}{|y|^p} \prod_{i=1}^m \max\{\|A_i^{-1}(y)\|_{L^p}^{\alpha_i}, \|A_i(y)\|_{L^p}^{-\alpha_i}\} \frac{1}{|y|^p} \prod_{i=1}^m \|f_i\|_{L^2(B_r(y))} dy.$$  

This implies that

$$\frac{1}{\omega(B_r)}^{\frac{1}{q} + \lambda} \|\mathcal{H}^p_{\Phi, \mathcal{A}}(f)\|_{L^2(B_r)} \lesssim \int_{Q^n_p} \frac{\Phi(y)}{|y|^p} \left( \prod_{i=1}^m \|A_i^{-1}(y)\|_{L^p}^{(\alpha_i + n)\lambda_i} \right) B_i(y) \times$$  

$$\times \left( \prod_{i=1}^m \frac{1}{\omega_i(B_{r + k_i})^{\frac{1}{q_i} + \lambda_i}} \|f_i\|_{L^p(B_{r + k_i})} \right) dy, \quad (3.11)$$

where $B_i(y) = \prod_{i=1}^m \frac{\omega_i(B_{r + k_i})^{\frac{1}{q_i} + \lambda_i}}{\omega(B_r)^{\frac{1}{q} + \lambda}}$.

On the other hand, by hypothesis $(3.10)$, we immediately get

$$\sum_{i=1}^m (\alpha_i + n) \left( \frac{1}{q_i} + \lambda_i \right) = (\alpha + n) \left( \frac{1}{q} + \lambda \right).$$

Consequently, by the estimation $(2.4)$ and $(3.1)$, we have

$$B_i(y) \lesssim \frac{\sum_{i=1}^m (\gamma + k_i)(\alpha_i + n)(\frac{1}{q_i} + \lambda_i)}{p^{(\gamma + n)\frac{1}{q} + \lambda}} = \frac{\sum_{i=1}^m k_i (\alpha_i + n) (\frac{1}{q_i} + \lambda_i) \sum_{i=1}^m \lambda_i (\alpha_i + n) (\frac{1}{q_i} + \lambda_i)}{p^{(\gamma + n)\frac{1}{q} + \lambda}}$$

$$= \prod_{i=1}^m \|A_i(y)\|_{L^p}^{(\alpha_i + n)\lambda_i} \lesssim \prod_{i=1}^m \|A_i^{-1}(y)\|_{L^p}^{-\alpha_i (\alpha_i + n)\frac{1}{q_i} + \lambda_i}.$$
Hence, by (3.11), one has
\[ \| H_{\Phi, \mathbf{A}}^{p, \lambda}(\tilde{f}) \|_{B_{q, \lambda}^{\gamma} \omega((Q_n^p)_{p})} \lesssim C_3 \prod_{i=1}^{m} \| f_i \|_{B_{\omega_i}^{q_i, \lambda_i}((Q_n^p)_{p})}. \]

Conversely, suppose that \( H_{\Phi, \mathbf{A}}^{p, \lambda} \) is bounded from \( B_{\omega_1}^{q_1, \lambda_1}(\mathbb{Q}_p^n) \times \cdots \times B_{\omega_m}^{q_m, \lambda_m}(\mathbb{Q}_p^n) \) to \( B_{\omega}^{q, \lambda}(\mathbb{Q}_p^n) \). For \( i = 1, \ldots, m \), let us choose the functions as follows
\[ f_i(x) = |x|^{(\alpha_i + n)\lambda_i}. \]
Then, by (2.4), it is not difficult to show that
\[ \| f_i \|_{B_{\omega_i}^{q_i, \lambda_i}((Q_n^p)_{p})} = \sup_{\gamma \in \mathbb{Z}} \frac{1}{B_{\gamma} \omega_i((B_\gamma)_{n}^{1 + \lambda_i})} \left( \int_{B_\gamma} |x|^{(\alpha_i + n)\lambda_i + \alpha_i} \, dx \right)^{1/q_i} \]
\[ \simeq \sup_{\gamma \in \mathbb{Z}} \frac{p^{\gamma((\alpha_i + n)\lambda_i + \alpha_i + n)\frac{1}{q_i}}}{p^{\gamma(\alpha_i + n)(\frac{1}{q_i} + \lambda_i)}} = 1, \]
and similarly, we also have
\[ \| | \cdot |^{(\alpha_i + n)\lambda} \|_{B_{\omega}^{q, \lambda}(\mathbb{Q}_p^n)} \simeq 1. \] (3.12)
Next, by choosing \( f_i \)'s and using (3.3) and (3.10), we have
\[ H_{\Phi, \mathbf{A}}^{p, \lambda}(\tilde{f})(x) = \int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_{p}^\lambda} \prod_{i=1}^{m} |A_i(y)x|^{(\alpha_i + n)\lambda_i} \, dy \]
\[ \gtrsim \int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_{p}^\lambda} \prod_{i=1}^{m} \| A_i^{-1}(y) \|_{p^{-1}(\alpha_i + n)\lambda_i} \| x |^{(\alpha_i + n)\lambda_i} \, dy = C_3 \| x |^{(\alpha + n)\lambda}. \]
Thus, by (3.12), it follows that
\[ \| H_{\Phi, \mathbf{A}}^{p, \lambda}(\tilde{f}) \|_{B_{q, \lambda}^{\gamma} \omega((Q_n^p)_{p})} \lesssim C_3 \| | \cdot |^{(\alpha_i + n)\lambda} \|_{B_{\omega}^{q, \lambda}(\mathbb{Q}_p^n)} \gtrsim C_3 \prod_{i=1}^{m} \| f_i \|_{B_{\omega_i}^{q_i, \lambda_i}((Q_n^p)_{p})}. \]
This gives that \( C_3 < \infty \). Hence, the theorem is completely proved. \( \square \)

**Theorem 3.4.** Let \( 1 \leq q^*, \zeta < \infty, \lambda_i \in (-\frac{1}{q'}, 0) \), for all \( i = 1, \ldots, m \) and \( \omega \in A_\zeta \) with the finite critical index \( r_\omega \) for the reverse H"older. Assume that \( q > q^* \zeta r_\omega/(r_\omega - 1) \), \( \delta \in (1, r_\omega) \) and the following two conditions are true:
\[ \lambda = \lambda_1 + \cdots + \lambda_m. \] (3.13)
\[ C_4 = \int_{Q^p} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^m \left| \det A_i^{-1}(y) \right| \alpha_n \| A_i(y) \| \times \]
\[ \times \left( \chi(\| A_i(y) \|_p \leq 1) \| A_i(y) \|^{n\zeta_i} + \chi(\| A_i(y) \|_p > 1) \| A_i(y) \|^{n\lambda_i} \right) dy < \infty. \]

Then, \( \mathcal{H}_{\Phi, \tilde{A}}^p \) is bounded from \( B^{q_{1}, \lambda_1}_\omega (Q^p) \times \cdots \times B^{q_{m}, \lambda_m}_\omega (Q^p) \) to \( B^{q', \lambda}_\omega (Q^p) \).

Proof. For \( \gamma \in \mathbb{Z} \), by estimating as (3.8) above and using the relation (3.13), we obtain that
\[
\frac{1}{\omega(B_\gamma)} \left\| \mathcal{H}_{\Phi, \tilde{A}}^p (\tilde{f}) \right\|_{L^s_p (B_\gamma)}
\leq \int_{Q^p} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^m \left| \det A_i^{-1}(y) \right| \alpha_n \| A_i(y) \| \left( \frac{\omega(B_{\gamma+kA_i})}{\omega(B_\gamma)} \right)^{\lambda_i} \frac{1}{\omega(B_{\gamma+kA_i})} \left\| f_i \right\|_{L^p_\gamma (B_{\gamma+kA_i})} dy
\leq \left( \int_{Q^p} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^m \left| \det A_i^{-1}(y) \right| \alpha_n \| A_i(y) \| \left( \frac{\omega(B_{\gamma+kA_i})}{\omega(B_\gamma)} \right)^{\lambda_i} \| f_i \|_{L^p_\gamma (B_{\gamma+kA_i})} \right)^m dy \prod_{i=1}^m \| f_i \|_{B^{q_i, \lambda_i}_\omega (Q^p)}.
\]

In addition, for \( i = 1, \ldots, m \), by making Proposition 2.6 again and \( \lambda_i < 0 \), we infer
\[
\left( \frac{\omega(B_{\gamma+kA_i})}{\omega(B_\gamma)} \right)^{\lambda_i} \leq \begin{cases} \left( \frac{|B_{\gamma+kA_i}|}{|B_\gamma|} \right)^{\zeta_i} \lesssim p^{(\gamma+kA_i-\gamma)n\zeta_i} = \| A_i(y) \|^{n\zeta_i}, & \text{if } \| A_i(y) \|_p \leq 1, \\ \left( \frac{|B_{\gamma+kA_i}|}{|B_\gamma|} \right)^{\lambda_i} \lesssim p^{(\gamma+kA_i-\gamma)n\lambda_i} = \| A_i(y) \|^{n\lambda_i}, & \text{otherwise.} \end{cases}
\]

Thus, we have \( \left\| \mathcal{H}_{\Phi, \tilde{A}}^p (\tilde{f}) \right\|_{B^{q', \lambda}_\omega (Q^p)} \lesssim C_4 \prod_{i=1}^m \| f_i \|_{B^{q_i, \lambda_i}_\omega (Q^p)} \), which gives that the proof of this theorem is ended. \( \square \)

4. THE MAIN RESULTS ABOUT THE BOUNDEDNESS OF \( \mathcal{H}_{\Phi, \tilde{A}}^p \)

Before stating our next results, we introduce some notations which will be used throughout this section. Let \( q, q_i \in [1, \infty) \), and let \( \alpha, \alpha_i, r_i \) be real numbers such that \( r_i \in (1, \infty), \alpha_i \in (-n, \frac{nr_i}{r_i}) \), \( i = 1, 2, \ldots, m \). Denote
\[
\left( \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} \right) + \left( \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m} \right) = \frac{1}{q}.
\]
\[
\left(\frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2} + \cdots + \frac{\alpha_m}{q_m}\right) + \left(\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2} + \cdots + \frac{\alpha_m}{r_m}\right) = \frac{\alpha}{q}
\]

**Lemma 4.1.** Let \(\omega(x) = |x|_p^\alpha\), \(\omega_i(x) = |x|_p^{\alpha_i}\) and \(b_i \in \text{CMO}_{\omega_i}(\mathbb{Q}_p^n)\), for all \(i = 1, \ldots, m\). Then, for any \(\gamma \in \mathbb{Z}\), we have

\[
\|\mathcal{H}_{\Phi, A_i, b_i}(\vec{f})\|_{L^q(B_\gamma)} \lesssim \sum_{i=1}^m \frac{\gamma(n+\alpha_i)}{\gamma} \int_{\mathbb{Q}_p^n} \Phi(y) \sum_{i=1}^m |\psi_i(y) \cdot \mu_i(y)\|_{L^q_i(B_{\gamma+k,A_i})} dy,
\]

where

\[
\psi_i(y) = 1 + \left(\max\{|A_i^{-1}(y)|_p^{\alpha}, |A_i(y)|_p^{-\alpha}\}\right) |\det A_i^{-1}(y)|_p^{\frac{\gamma}{q}} |A_i(y)|_p^{\frac{\gamma}{q}} + |\log p| |A_i(y)| |A_i(y)|_p^{\frac{1}{\gamma}} \mu_i(y) = \left(\max\{|A_i^{-1}(y)|_p^{\alpha}, |A_i(y)|_p^{-\alpha}\}\right) |\det A_i^{-1}(y)|_p^{\frac{1}{q}} \text{ and } B_{r,\omega} = \prod_{i=1}^m \|b_i\|_{\text{CMO}_{\omega_i}(\mathbb{Q}_p^n)}.
\]

**Proof.** By the Minkowski inequality and the Hölder inequality, for any \(\gamma \in \mathbb{Z}\), we get

\[
\|\mathcal{H}_{\Phi, A_i, b_i}(\vec{f})\|_{L^q(B_\gamma)} \lesssim \int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m |b_i(\cdot) - b_i(A_i(y)\cdot)|_{L^q_i(B_{\gamma})} \|f_i(A_i(y)\cdot)\|_{L^q_i(B_{\gamma})} dy.
\]

(4.1)

To prove this lemma, we need to show that the following inequality holds

\[
\|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^q_i(B_{\gamma})} \lesssim p^{\frac{\gamma(n+\alpha_i)}{q}} \psi_i(y) \|b_i\|_{\text{CMO}_{\omega_i}(\mathbb{Q}_p^n)} \text{ for all } i = 1, \ldots, m.
\]

(4.2)

We put \(I_{1,i} = \|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^q_i(B_{\gamma})}, I_{2,i} = \|b_i(A_i(y)\cdot) - b_i(A_i(y)B_i\cdot)\|_{L^q_i(B_{\gamma})}\) and \(I_{3,i} = \|b_i(B_i\cdot) - b_i(A_i(y)B_i\cdot)\|_{L^q_i(B_{\gamma})}\). It is obvious that

\[
\|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^q_i(B_{\gamma})} \leq I_{1,i} + I_{2,i} + I_{3,i} \text{ for all } i = 1, \ldots, m.
\]

(4.3)

By the definition of the space \(\text{CMO}_{\omega_i}(\mathbb{Q}_p^n)\) and the estimation (2.3), we have

\[
I_{1,i} \leq \omega_i(B_{\gamma})^{\frac{1}{r_i}} \|b_i\|_{\text{CMO}_{\omega_i}(\mathbb{Q}_p^n)} \lesssim p^{\frac{\gamma(n+\alpha_i)}{q}} \|b_i\|_{\text{CMO}_{\omega_i}(\mathbb{Q}_p^n)}.
\]

(4.4)
To estimate $I_{2,i}$, we deduce that

$$I_{2,i} = \left( \int_{B_{\gamma}} \left| b_i(A_i(y)x) - b_i,A_i(y)B_{\gamma} \right|^\alpha \omega_i(x)dx \right)^{\frac{1}{\alpha}}$$

$$\leq \omega_i(B_{\gamma})^{\frac{1}{\alpha}} \left| b_i,A_i(y)B_{\gamma} - b_i,B_{\gamma+k}A_i \right| + \left( \int_{B_{\gamma}} \left| b_i(A_i(y)x) - b_i,B_{\gamma+k}A_i \right|^\alpha \omega_i(x)dx \right)^{\frac{1}{\alpha}},$$

where $k_{A_i}(y) = \log_p ||A_i(y)||_p$. Note that, by the formula for change of variables, we get

$$|A_i(y)B_{\gamma}| = \int_{A_i(y)B_{\gamma}} dx = \int_{B_{\gamma}} |\det A_i(y)|_p dz \simeq |\det A_i(y)|_p p^{-n}. \tag{4.6}$$

Thus, by using the Hölder inequality and (2.4), it is clear to see that

$$|b_i,A_i(y)B_{\gamma} - b_i,B_{\gamma+k}A_i| \leq \frac{1}{|A_i(y)B_{\gamma}|} \int_{A_i(y)B_{\gamma}} |b_i(x) - b_i,B_{\gamma+k}A_i| dx$$

$$\leq \frac{1}{|A_i(y)B_{\gamma}|} \left( \int_{B_{\gamma+k}A_i} |b_i(x) - b_i,B_{\gamma+k}A_i|^\alpha \omega_i(x)dx \right)^{\frac{1}{\alpha}} \left( \int_{B_{\gamma+k}A_i} \omega_i^{\frac{\alpha}{\alpha'}} \frac{1}{\alpha'} dx \right)^{\frac{1}{\alpha'}}$$

$$\leq \frac{\omega_i(B_{\gamma+k}A_i)^\frac{\alpha}{\alpha'} \left( \int_{B_{\gamma+k}A_i} \frac{1}{x^{\frac{n+\alpha}{\alpha'}}} dx \right)^{\frac{1}{\alpha}}}{|A_i(y)B_{\gamma}|} \|b_i\|_{L^\alpha(Q^p)}$$

$$\leq \frac{p^{(\gamma+kA_i)(\frac{n+\alpha}{\alpha'})} ||b_i\|_{L^\alpha(Q^p)}}{|\det A_i(y)|_p p^{\gamma n}} \|b_i\|_{L^\alpha(Q^p)} = \frac{||A_i(y)||_p^\alpha}{|\det A_i(y)|_p} \|b_i\|_{L^\alpha(Q^p)}. \tag{4.7}$$

By using the formula for change of variables again, one has

$$\left( \int_{B_{\gamma}} \left| b_i(A_i(y)x) - b_i,B_{\gamma+k}A_i \right|^\alpha \omega_i(x)dx \right)^{\frac{1}{\alpha}}$$

$$= \left( \int_{A_i(y)B_{\gamma}} |b_i(z) - b_i,B_{\gamma+k}A_i|^\alpha |A_i^{-1}(y)z|_p^{\alpha} |\det A_i^{-1}(y)|_p dz \right)^{\frac{1}{\alpha}}$$

$$\leq \left( \max \{ ||A_i^{-1}(y)||_p^{\alpha}, ||A_i(y)||_p^{-\alpha} \} |\det A_i^{-1}(y)|_p \omega_i(B_{\gamma+k}A_i) \right)^{\frac{1}{\alpha}} \times$$

$$\times \left( \frac{1}{\omega_i(B_{\gamma+k}A_i)} \int_{B_{\gamma+k}A_i} |b_i(z) - b_i,B_{\gamma+k}A_i|^\alpha \omega_i(z)dz \right)^{\frac{1}{\alpha}}. \tag{4.8}$$
This leads to
\[
\left( \int_{B_\gamma} \left| b_i(A_i(y)x) - b_{i,A_i(y)} \right|^\gamma_1 \omega_i(x) dx \right)^\frac{1}{\gamma_1} \leq p^{\frac{(n-\alpha_i)}{r_i}} \left( \max \left\{ \|A_i^{-1}(y)\|_p^{\alpha_i}, \|A_i(y)^{-\alpha_i}\|_p \right\} \|\det A_i^{-1}(y)y\|_p \right)^\frac{1}{r_i} \|A_i(y)y\|_p^{\frac{(n-\alpha_i)}{r_i}} \|b_i\|_{CMO^{n}_i(Q^n_p)}.
\]

Therefore, by (4.3) and (4.7), we have
\[
I_{2,i} \lesssim \left( \frac{\|A_i(y)y\|_p^n}{|\det A_i(y)|_p} + \max \left\{ \|A_i^{-1}(y)\|_p^{\alpha_i}, \|A_i(y)^{-\alpha_i}\|_p \right\} \|\det A_i^{-1}(y)y\|_p \right)^\frac{1}{r_i} \|A_i(y)y\|_p^{\frac{(n-\alpha_i)}{r_i}} \times p^{\frac{(n-\alpha_i)}{r_i}} \|b_i\|_{CMO^{n}_i(Q^n_p)}. \tag{4.9}
\]

Next, we consider the term $I_{3,i}$. We have
\[
I_{3,i} \leq \omega_i(B_\gamma)^\frac{1}{\gamma_1} |b_i,B_\gamma - b_{i,A_i(y)}B_\gamma|. \tag{4.10}
\]

Fix $y \in Q^n_p$. We set
\[
S_{k,A_i} = \begin{cases} \{j \in \mathbb{Z} : 1 \leq j \leq k_{A_i} \}, & \text{if } k_{A_i} \geq 1, \\ \{j \in \mathbb{Z} : k_{A_i} + 1 \leq j \leq 0 \}, & \text{otherwise}. \end{cases}
\]

As mentioned above, we have
\[
|b_i,B_\gamma - b_{i,A_i(y)}B_\gamma| \leq \sum_{j \in S_{k,A_i}} |b_{i,B_{\gamma+j-1}} - b_{i,B_{\gamma+j}}| + |b_{i,B_{\gamma+k_{A_i}}} - b_{i,A_i(y)}B_\gamma|. \tag{4.11}
\]

Combining the Hölder inequality, the definition of the space $CMO^{n}_i(Q^n_p)$ and (2.4), one has
\[
|b_{i,B_{\gamma+j-1}} - b_{i,B_{\gamma+j}}| \leq \frac{1}{|B_{\gamma+j-1}|} \int_{B_{\gamma+j}} |b_i(z) - b_{i,B_{\gamma+j}}| dz \lesssim \frac{1}{|B_{\gamma+j}|} \int_{B_{\gamma+j}} |b_i(z) - b_{i,B_{\gamma+j}}| dz \leq \frac{\omega_i(B_{\gamma+j})^{\frac{1}{\gamma_1}}}{|B_{\gamma+j}|} \left( \int_{B_{\gamma+j}} \omega_i^{\frac{\gamma_1}{r_1}}(z) \frac{1}{\omega_i(B_{\gamma+j})} \int_{B_{\gamma+j}} |b_i(z) - b_{i,B_{\gamma+j}}|^{\gamma_1} \omega_i(x) dx \right)^\frac{1}{\gamma_1} \leq \frac{b_p^{(\gamma+j)(\frac{\alpha_i}{r_i} + \frac{n}{r_i})}}{p^{(\gamma+j)(\frac{\alpha_i}{r_i} + \frac{n}{r_i})}} \|b_i\|_{CMO^{n}_i(Q^n_p)} = \|b_i\|_{CMO^{n}_i(Q^n_p)}.
\]

Thus,
\[
|b_i,B_\gamma - b_{i,A_i(y)}B_\gamma| \lesssim |k_{A_i}| |b_i|_{CMO^{n}_i(Q^n_p)} + |b_{i,B_{\gamma+k_{A_i}}} - b_{i,A_i(y)}B_\gamma|. \tag{4.12}
\]
In addition, by the Hölder inequality again and (4.10), we get
\[ |b_{i,B_{\gamma+kA_i}} - b_{i,A(y)B_y}| \leq \frac{1}{|A_i(y)B_y|} \int_{A_i(y)B_y} |b_i(x) - b_{i,B_{\gamma+kA_i}}| \, dx \]
\[ \leq \frac{\omega_i(B_{\gamma+kA_i})^{\frac{1}{r_i}}}{|A_i(y)B_y|} \left( \int_{B_{\gamma+kA_i}} \omega_i^{\frac{1}{r_i}} \, dx \right)^{\frac{1}{r_i}} \left( \frac{1}{\omega_i(B_{\gamma+kA_i})} \int_{B_{\gamma+kA_i}} |b_i(x) - b_{i,B_{\gamma+kA_i}}|^{r_i} \omega_i(x) \, dx \right)^{\frac{1}{r_i}} \]
\[ \leq 2^p \eta \left( \frac{|\det A_i(y)|}{p \cdot 2^m} \right)^{\frac{\eta}{r_i}} (\gamma + kA_i) \left( \frac{|\det A_i(y)|}{p \cdot 2^m} \right)^{\frac{\eta}{r_i}} \|b_i\|_{\text{CMO}^\gamma_i(Q^p_n)} = \frac{\|A_i(y)\|_p}{|\det A_i(y)|} \|b_i\|_{\text{CMO}^\gamma_i(Q^p_n)}. \]
Consequently, by (4.10)-(4.11), it follows that
\[ I_{\delta,3,i} \lesssim \frac{1}{p} \left( |\log_p|A_i(y)||_p + \frac{|\det A_i(y)||_p}{p} \right) \|b_i\|_{\text{CMO}^\gamma_i(Q^p_n)}. \]
This together with (4.3), (4.4) and (4.9) follow us to have the proof of the inequality (4.2). Finally, by estimating as (4.8), we immediately have
\[ \|f_i(A_i(y))\|_{L^p_{\omega}(B_{\gamma})} \leq \left( \max \{ \|A_i^{-1}(y)\|_{p,\alpha_i}^\alpha, \|A_i(y)\|_{p,\alpha_i}^{1-\alpha_i} \} |\det A_i^{-1}(y)| \right)^{1/p} \|f_i\|_{L^p_{\omega}(B_{\gamma+kA_i})}. \]
In view of (4.11) and (4.2), the proof of this lemma is ended. \[ \square \]

**Lemma 4.2.** Let \( 1 \leq q^*, r_1^*, \ldots, r_m^*, q_1^*, \ldots, q_m^*, \zeta < \infty, \omega \in A_\zeta \) with the finite critical index \( r_\omega \) for the reverse Hölder condition, \( \delta \in (1, r_\omega), \lambda_i \in \left( \frac{1}{q_i^*}, 0 \right), \zeta \leq r_\omega \) and \( b_i \in \text{CMO}^\gamma_i(Q^p_n) \) for all \( i = 1, \ldots, m \). Assume that the following condition holds:
\[ \frac{1}{q^*} > \left( \frac{1}{r_1^*} + \cdots + \frac{1}{r_m^*} + \frac{1}{q_1^*} + \cdots + \frac{1}{q_m^*} \right) \zeta \frac{r_\omega}{r_\omega - 1}. \] (4.13)
Then, we have
\[ \| \mathcal{H}^p_{\Phi,A,\delta}(\vec{f})\|_{L^p_{\omega}(B_{\gamma})} \lesssim \omega(B_{\gamma})^{\frac{1}{\delta}} B_{\mathfrak{p},\omega}^{\frac{\delta}{\omega}} \left( \int_{Q^p_n} \frac{\Phi(y)}{|y|^\delta} \prod_{i=1}^m \psi_i^*(y) \mu_i^*(y) \times \frac{1}{\omega(B_{\gamma+kA_i})^{\frac{1}{q_i^*}}} \|f_i\|_{L^p_{\omega}(B_{\gamma+kA_i})} \, dy \right), \text{ for all } \gamma \in \mathbb{Z}. \]
Here
\[ \psi_i^*(y) = 1 + \frac{2}{|\det A_i(y)|_p} + |\det A_i^{-1}(y)|_p^{\frac{1}{r_i}} \|A_i(y)\|_{p,\alpha_i}^{\frac{2\alpha_i}{r_i}} + |\log_p|A_i(y)||_p, \]
\[ \mu_i^*(y) = |\det A_i^{-1}(y)|_p^{\frac{1}{r_i}} \|A_i(y)\|_{p,\alpha_i}^{\frac{2\alpha_i}{r_i}} \text{ and } B_{\mathfrak{p},\omega} = \prod_{i=1}^m \|b_i\|_{\text{CMO}^\gamma_i(Q^p_n)}. \]
Proof. By virtue of the inequality (4.13), there exist \( r_1, ..., r_m, q_1, ..., q_m \) such that
\[
\frac{1}{q_i} > \frac{\zeta}{q_i^*} \frac{r_{\omega}}{r_i - 1} \quad \text{and} \quad 1 > \frac{\zeta}{r_i^*} \frac{r_{\omega}}{r_i^* - 1} \quad \text{for all} \quad i = 1, ..., m,
\]
and
\[
\frac{1}{q_1} + \cdots + \frac{1}{q_m} + \frac{1}{r_1} + \cdots + \frac{1}{r_m} = \frac{1}{q^*}.
\]
As mentioned above, for any \( \gamma \in \mathbb{Z} \), by the same argument (4.1), we also get
\[
\| H_{\Phi, \vec{A}, \vec{b}}(\vec{f}) \|_{L^q(B_\gamma)} \lesssim \int_{Q^n_p} \frac{\Phi(y)}{|y|^m} \prod_{i=1}^m \| b_i(\cdot) - b_i(A_i(y) \cdot) \|_{L^q_i(B_\gamma)} \| f_i(A_i(y) \cdot) \|_{L^{q^*}_i(B_\gamma)} dy.
\]
(4.14)

In particular, we need to show the following result
\[
\| b_i(\cdot) - b_i(A_i(y) \cdot) \|_{L^q_i(B_\gamma)} \lesssim \omega(B_\gamma)^{\frac{1}{r_i^*}} \psi_i^*(y) \cdot b_i \|_{CMO^*_{L^q_i}(Q^n_p)},
\]
(4.15)
for all \( i = 1, ..., m \). Indeed, we see that
\[
\| b_i(\cdot) - b_i(A_i(y) \cdot) \|_{L^q_i(B_\gamma)} \leq \| b_i(\cdot) - b_i(B_\gamma) \|_{L^q_i(B_\gamma)} + \| b_i(A_i(y) \cdot) - b_i(A_i(y) B_\gamma) \|_{L^q_i(B_\gamma)}
\]
\[
+ \| b_i(B_\gamma) - b_i(A_i(y) B_\gamma) \|_{L^q_i(B_\gamma)} := J_{1,i} + J_{2,i} + J_{3,i}.
\]
(4.16)

By virtue of the inequality \( r_1 < r_i^* \), it is easy to show that
\[
J_{1,i} \leq \omega(B_\gamma)^{\frac{1}{r_i^*}} \| b_i \|_{CMO^*_{L^q_i}(Q^n_p)},
\]
(4.17)

Next, by estimating as (4.15) above, we get
\[
J_{2,i} \leq \omega(B_\gamma)^{\frac{1}{r_i^*}} \| b_i(A_i(y) B_\gamma) - b_i(B_\gamma + A_i) \|_{CMO^*_{L^q_i}(Q^n_p)} + \left( \int_{B_\gamma} | b_i(A_i(y) x) - b_i(B_\gamma + A_i) |^{r_i^*} \omega(x) dx \right)^{\frac{1}{r_i^*}}.
\]
(4.18)
By having the inequality \( \zeta \leq r_i^* \) and applying Proposition 2.7 and (4.6), we infer

\[
|b_{i,B_{\gamma+k_{A_i}}} - b_{i,A_i(y)B_\gamma}| \leq \frac{1}{|A_i(y)B_\gamma|} \int_{B_{\gamma+k_{A_i}}} |b_i(x) - b_{i,B_{\gamma+k_{A_i}}}| \, dx
\]

\[
\leq \frac{|B_{\gamma+k_{A_i}}|}{|A_i(y)B_\gamma|} \frac{1}{|B_{\gamma+k_{A_i}}|} \int_{B_{\gamma+k_{A_i}}} |b_i(x) - b_{i,B_{\gamma+k_{A_i}}}| \, dx
\]

\[
\leq \frac{|B_{\gamma+k_{A_i}}|}{|A_i(y)B_\gamma|} \frac{1}{\omega(B_{\gamma+k_{A_i}})} \int_{B_{\gamma+k_{A_i}}} |b_i(x) - b_{i,B_{\gamma+k_{A_i}}}|^{\zeta} \omega(x) \, dx \]

\[
\leq \frac{p_i^{(\gamma+k_{A_i}) + \eta}}{|\det A_i(y)| p - p_i} \|b_i\|_{CMO(\gamma, p_i)} \leq \frac{\|A_i(y)\|_p |A_i(y)| |b_i|_{CMO(\gamma, p_i)}}{|\det A_i(y)| p - p_i}
\]

By \( \frac{1}{r_i^*} > r_i - 1 \), there exists \( \beta_i \in (1, r_i) \) satisfying \( \beta_i^* = r_i \beta_i^* \). Thus, by combining the Hölder inequality and the reverse Hölder condition again, we have

\[
\left( \int_{B_{\gamma}} |b_i(A_i(y)x) - b_{i,B_{\gamma+k_{A_i}}}|^{r_i} \omega(x) \, dx \right)^{\frac{1}{r_i}}
\]

\[
\leq \left( \int_{B_{\gamma}} |b_i(A_i(y)x) - b_{i,B_{\gamma+k_{A_i}}}|^{\frac{r_i}{\beta_i}} \, dx \right)^{\frac{\beta_i}{r_i}} \left( \int_{B_{\gamma}} \omega(x)^{\beta_i} \, dx \right)^{\frac{1}{\beta_i}}
\]

\[
\lesssim |B_{\gamma}|^{\frac{\beta_i}{r_i}} \omega(B_{\gamma})^{\frac{1}{r_i}} \left( \int_{B_{\gamma}} |b_i(A_i(y)x) - b_{i,B_{\gamma+k_{A_i}}}|^{\frac{r_i}{\beta_i}} \, dx \right)^{\frac{\beta_i}{r_i}}
\]

According to Proposition 2.7, we have

\[
\left( \int_{B_{\gamma}} |b_i(A_i(y)x) - b_{i,B_{\gamma+k_{A_i}}}|^{\frac{r_i}{\beta_i}} \, dx \right)^{\frac{\beta_i}{r_i}}
\]

\[
\leq |\det A_i^{-1}(y)|^{\frac{\beta_i}{p_i}} \left( \int_{B_{\gamma+k_{A_i}}} |b_i(z) - b_{i,B_{\gamma+k_{A_i}}}|^{\frac{r_i}{\beta_i}} \, dz \right)^{\frac{\beta_i}{r_i}}
\]

\[
\leq |\det A_i^{-1}(y)|^{\frac{\beta_i}{p_i}} \frac{|B_{\gamma+k_{A_i}}|^{\frac{r_i}{\beta_i}}}{\omega(B_{\gamma+k_{A_i}})^{\frac{1}{\beta_i}}} \left( \int_{B_{\gamma+k_{A_i}}} |b_i(z) - b_{i,B_{\gamma+k_{A_i}}}|^{\frac{r_i}{\beta_i}} \omega(z) \, dz \right)^{\frac{\beta_i}{r_i}}.
\]
In view of (2.4), one has $\frac{|B_{r} + \text{A}|}{|B_{r}|} \simeq \|A_{r}(x)\|_{p}^{n}$. From this, we give
\[
\left( \int_{B_{r}} |b_{i}(A_{r}(y)x) - b_{i,B_{r}+\text{A}, r}|^{r} \omega(x)dx \right)^{\frac{1}{r}} \leq \omega(B_{r})^{\frac{1}{r}} \|\text{det}A_{r}^{-1}(y)\|_{p}^{\frac{2}{r}} \|A_{r}(y)\|_{p}^{\frac{n}{r}} \times \left( \frac{1}{\omega(B_{r} + \text{A}, r)} \int_{B_{r} + \text{A}, r} |b_{i}(z) - b_{i,B_{r}+\text{A}, r}|^{r} \omega(z)dz \right)^{\frac{1}{r}},
\]
(4.20)

As a consequence, by (4.18) and (4.19), we infer
\[
J_{2,i} \lesssim \omega(B_{r})^{\frac{1}{r}} \left( \|A_{r}(y)\|_{p}^{n} \frac{n}{\text{det}A_{r}(y)} + \|\text{det}A_{r}^{-1}(y)\|_{p}^{\frac{2}{r}} \|A_{r}(y)\|_{p}^{\frac{n}{r}} \right) \|b_{i}\|_{CMO_{r}^{n}(Q_{p}^{n})}.
\]
(4.21)

Now, we will estimate $J_{3,i}$. By a same argument as (4.10), (4.11) and (4.19), we also have
\[
J_{3,i} \leq \omega(B_{r})^{\frac{1}{r}} \left( \sum_{j \in S_{k,A_{r}}} |b_{i,B_{r}+j-1} - b_{i,B_{r}+j}| + |b_{i,B_{r}+kA_{r}} - b_{i,A_{r}(y)B_{r}}| \right) \leq \omega(B_{r})^{\frac{1}{r}} \left( \sum_{j \in S_{k,A_{r}}} \frac{1}{|B_{r}+j|} \int_{B_{r}+j} |b_{i}(z) - b_{i,B_{r}+j}|dz + 1 \right) \leq \omega(B_{r})^{\frac{1}{r}} \left( \sum_{j \in S_{k,A_{r}}} \|b_{i}\|_{CMO_{r}^{n}(Q_{p}^{n})} + \frac{\|A_{r}(y)\|_{p}^{n}}{\text{det}A_{r}(y)} \|b_{i}\|_{CMO_{r}^{n}(Q_{p}^{n})} \right) \leq \omega(B_{r})^{\frac{1}{r}} \left( |\log_{p}|A_{r}(y)|_{p}| + \frac{\|A_{r}(y)\|_{p}^{n}}{\text{det}A_{r}(y)} \|b_{i}\|_{CMO_{r}^{n}(Q_{p}^{n})} \right).
\]
This together with (4.17) and (4.21) yields that the inequality (4.15) is finished. In other words, by estimating as (4.20) above, we also get
\[
\|f_{i}(A_{r}(y)\cdot)\|_{L_{Q_{p}}^{n}(B_{r})} \lesssim \omega(B_{r})^{\frac{1}{r}} |\text{det}A_{r}^{-1}(y)\|_{p}^{\frac{2}{r}} \|A_{r}(y)\|_{p}^{\frac{n}{r}} \omega(B_{r}+kA_{r})^{\frac{1}{r}} \|f_{i}\|_{L_{Q_{p}}^{n}(B_{r+kA_{r}})}^{\frac{1}{r}},
\]
and
\[
= \omega(B_{r})^{\frac{1}{r}} \mu_{r}^{*}(y). \omega(B_{r}+kA_{r})^{\frac{1}{r}} \|f_{i}\|_{L_{Q_{p}}^{n}(B_{r+kA_{r}})}^{\frac{1}{r}}.
\]
Hence, by (4.14) and (4.15), we conclude that the proof of this lemma is finished. □
Theorem 4.3. Let the assumptions of Lemma 4.1 hold and
\[ C_5 = \int_{Q_p^m} \frac{\Phi(y)}{|y|^p} \prod_{i=1}^m \psi_i(y) \cdot \mu_i(y) dy < \infty. \]

Then, for any \( \gamma \in B_\gamma \), we have \( \mathcal{H}_{\Phi, A, b}^{p, \gamma} \) is bounded from \( L^{q_1}_{\omega_1}(Q_p^n) \times \cdots \times L^{q_m}_{\omega_m}(Q_p^n) \) to \( L^{q_\gamma}_{\omega}(B_\gamma) \).

Proof. In view of Lemma 4.2, for any \( \gamma \in \mathbb{Z} \), by using Lemma 4.1, we infer
\[ \| \mathcal{H}_{\Phi, A, b}^{p, \gamma}(\vec{f}) \|_{L^{q_\gamma}_{\omega}(B_\gamma)} \leq \sum_{p=1}^m \frac{\gamma(n+1)}{q_i} B_{\gamma, \omega} \int_{Q_p^m} \frac{\Phi(y)}{|y|^p} \prod_{i=1}^m \psi_i(y) \cdot \mu_i(y) \| f_i \|_{L^{q_i}_{\omega_i}(Q_p^n)} dy. \]

Thus, we have \( \| \mathcal{H}_{\Phi, A, b}^{p, \gamma}(\vec{f}) \|_{L^{q_\gamma}_{\omega}(B_\gamma)} \leq C_5 \cdot B_{\gamma, \omega} \cdot \prod_{i=1}^m \| f_i \|_{L^{q_i}_{\omega_i}(Q_p^n)}, \) which shows that the proof of this theorem is completed.

\[ \square \]

Theorem 4.4. Let the assumptions of Lemma 4.2 hold. Suppose \( \omega(B_\gamma) \lesssim 1 \), for all \( \gamma \in \mathbb{Z} \), and
\[ C_6 = \int_{Q_p^m} \frac{\Phi(y)}{|y|^p} \prod_{i=1}^m \psi_i^*(y) \cdot \mu_i^*(y) \left( \chi_{\{||A_i(y)||\leq 1\}}(y) \| A_i(y) \|_{\rho_i} \right)^{-\frac{n_i}{q_i}} + \]
\[ \quad + \chi_{\{||A_i(y)||> 1\}}(y) \| A_i(y) \|_{\rho_i} \left( \chi_{\{||A_i(y)||\leq 1\}}(y) \| A_i(y) \|_{\rho_i} \right)^{-\frac{n_i}{q_i}} \) \] \( dy < \infty. \)

Then, we have \( \mathcal{H}_{\Phi, A, b}^{p, \gamma} \) is bounded from \( L^{q_1}_{\omega_1}(Q_p^n) \times \cdots \times L^{q_m}_{\omega_m}(Q_p^n) \) to \( L^{q_\gamma}_{\omega}(Q_p^n) \).

Proof. In view of Lemma 4.2, for any \( R \in \mathbb{Z} \), it is clear to see that
\[ \| \mathcal{H}_{\Phi, A, b}^{p, \gamma}(\vec{f}) \|_{L^{q_\gamma}_{\omega}(B_R)} \leq \omega(B_R)^{\frac{1}{q^*}} B_{\gamma, \omega}. \left( \int_{Q_p^m} \frac{\Phi(y)}{|y|^p} \prod_{i=1}^m \psi_i^*(y) \cdot \mu_i^*(y) \left( \frac{1}{\omega(B_R)} \right)^{\frac{n_i}{q_i}} \| f_i \|_{L^{q_i}_{\omega_i}(B_R)} \right)^{\frac{1}{q^*}} dy. \]

Next, by having inequality \( \frac{1}{q^*} > \frac{1}{q_1} + \cdots + \frac{1}{q_m} \) and assuming \( \omega(B_R) \lesssim 1 \) for any \( R \in \mathbb{Z} \), we have \( \omega(B_R)^{\frac{1}{q^*}} \leq \prod_{i=1}^m \omega(B_R)^{\frac{1}{q_i}}. \) Thus,
\[ \| \mathcal{H}_{\Phi, A, b}^{p, \gamma}(\vec{f}) \|_{L^{q_\gamma}_{\omega}(B_R)} \leq B_{\gamma, \omega}. \left( \int_{Q_p^m} \frac{\Phi(y)}{|y|^p} \prod_{i=1}^m \psi_i^*(y) \cdot \mu_i^*(y) \left( \frac{\omega(B_R)}{\omega(B_R+kA_i)} \right)^{\frac{1}{q_i}} \right)^{\frac{1}{q^*}} dy \cdot \prod_{i=1}^m \| f_i \|_{L^{q_i}_{\omega_i}(Q_p^n)} \]
\[ \leq C_6 \cdot B_{\gamma, \omega} \cdot \prod_{i=1}^m \| f_i \|_{L^{q_i}_{\omega_i}(Q_p^n)}. \]
Consequence, by letting $R \to +\infty$ and applying the monotone convergence theorem of Lebesgue, we also have

$$\|\mathcal{H}^p_{\Phi, \tilde{A}, \tilde{b}}(\tilde{f})\|_{L^p_\omega(Q^n_p)} \lesssim C_0 B_{r^\omega} \prod_{i=1}^m \|f_i\|_{L^{p^*_i}_\omega(Q^n_p)}.$$ 

Therefore, the theorem is completely proved. \hfill \Box

**Theorem 4.5.** Let $1 < \zeta < \infty$, $1 \leq q, q_i, r^*_i < \infty$, $-n < \alpha_i < n(\zeta - 1)$, $\omega(x) = |x|^{\alpha_i}, \omega_i(x) = |x|^{\alpha_i}$, for all $i = 1, \ldots, m$, such that

$$\frac{\alpha_1}{q_1} + \cdots + \frac{\alpha_m}{q_m} = \frac{\zeta \alpha_i}{q^*}, \quad 1 \frac{\alpha_1}{q_1} + \cdots + \frac{1}{q_m} = \frac{\zeta}{q^*}, \quad (4.22)$$

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} + \frac{1}{r^*_1} + \cdots + \frac{1}{r^*_m} = 1. \quad (4.23)$$

If $b_i \in CMO^{\omega_i}(Q^n_p)$, for all $i = 1, \ldots, m$, and

$$C_7 = \int_{Q^n_p} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \Gamma_i(y) \|A_i(y)\|_p^{-\frac{\zeta + n}{q_i n}} \, dy < \infty,$$

where

$$\Gamma_i(y) = \left(1 + |\log_p |A_i(y)||_p| + \frac{2\|A_i(y)||_p^n}{|\det A_i(y)|_p} + \frac{\frac{n}{n_i}}{\|A_i(y)\|_p} \right) \times \|\det A_i^{-1}(y)|_p^{\frac{1}{n_i}} ||A_i(y)||_p^{\frac{n_i}{n_i}}, \quad (4.24)$$

then we have

$$\|\mathcal{M}^{mod}(\mathcal{H}^p_{\Phi, \tilde{A}, \tilde{b}})(\tilde{f})\|_{L^{p^*_i}_\omega(Q^n_p)} \lesssim C_7 \left(\prod_{i=1}^m \|b_i\|_{CMO^{\omega_i}(Q^n_p)}\right) \prod_{i=1}^m \|f_i\|_{L^{p^*_i}_\omega(Q^n_p)}.$$

**Proof.** For the sake of simplicity, we denote $B_{r^\omega} = \prod_{i=1}^m \|b_i\|_{CMO^{\omega_i}(Q^n_p)}$. Now, let $x \in Q^n_p$ and fix a ball $B_\gamma$ such that $x \in B_\gamma$. In view of (4.23), by using the H"older inequality, we have

$$\frac{1}{|B_\gamma|} \int_{B_\gamma} |\mathcal{H}^p_{\Phi, \tilde{A}, \tilde{b}}(\tilde{f})(z)| \, dz$$

$$\leq \frac{1}{|B_\gamma|} \int_{Q^n_p} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \|f_i(A_i(y)\cdot)||_{L^p(B_\gamma)} \prod_{i=1}^m \|b_i(\cdot) - b_i(A_i(y)\cdot)||_{L^{p^*_i}(B_\gamma)} \, dy.$$
Thus, we infer that
$$
\left\| b_i(\cdot) - b_i(A_i(y)\cdot) \right\|_{L^p(\mathbb{R}^n)} \lesssim \left| B_\gamma \right|^\frac{1}{p'} \left( 1 + \log_p \| A_i(y) \|_p \right) + \frac{2\| A_i(y) \|^n_p}{|\det A_i(y)|} + \| A_i(y) \|^\frac{1}{p'} |\det A_i^{-1}(y)|^{\frac{1}{p'}} \right\| b_i \right\|_{CMO^p(Q^n_p)}.
$$

By $x \in B_\gamma$, we imply that $\| A_i(y) \|^{-1} x \in B_{\gamma+k_A_i}$. Thus, by definition of the Hardy-Littlewood maximal operator, one has

$$
\| f_i(A_i(y)\cdot) \|_{L^q(Q^n_p)} = |\det A_i^{-1}(y)|^{\frac{1}{p'}} \left( \int_{A_i(y)B_\gamma} |f_i(t)|^{q_i} dt \right)^{\frac{1}{q_i}} \leq |\det A_i^{-1}(y)|^{\frac{1}{p'}} \left( \int_{B_{\gamma+k_A_i}} B_{\gamma} |f_i(t)|^{q_i} dt \right)^{\frac{1}{q_i}} \lesssim |B_{\gamma} \| \det A_i^{-1}(y)|^{\frac{1}{p'}} \| A_i(y) \|^{\frac{1}{p}} \left( \mathcal{M}(\| f_i \|^{q_i})(\| A_i(y) \|^{-1} x) \right)^{\frac{1}{q_i}}.
$$

As mentioned above, we give

$$
\frac{1}{|B_{\gamma}|} \int_{B_\gamma} |\mathcal{H}_{\Phi,A_i}^p(\vec{f})|^q(z) dz \lesssim B_{\gamma^p} \int_{Q^n_p} \frac{\Phi(y)}{|y|^{n_p}} \prod_{i=1}^m \Gamma_i(y) \left( \mathcal{M}(\| f_i \|^{q_i})(\| A_i(y) \|^{-1} x) \right)^{\frac{1}{q_i}} dy.
$$

Thus, we infer that

$$
\mathcal{M}^{mod}(\mathcal{H}_{\Phi,A_i}^p(\vec{f}))(x) \lesssim B_{\gamma^p} \int_{Q^n_p} \frac{\Phi(y)}{|y|^{n_p}} \prod_{i=1}^m \Gamma_i(y) \left( \mathcal{M}(\| f_i \|^{q_i})(\| A_i(y) \|^{-1} x) \right)^{\frac{1}{q_i}} dy.
$$

Thus, by using the assumption (4.22) and the Minkowski inequality and the Hölder inequality, we obtain

$$
\left\| \mathcal{M}^{mod}(\mathcal{H}_{\Phi,A_i}^p(\vec{f})) \right\|_{L^p(Q^n_p)} \leq B_{\gamma^p} \int_{Q^n_p} \frac{\Phi(y)}{|y|^{n_p}} \prod_{i=1}^m \Gamma_i(y) \left( \int_{Q^n_p} \prod_{i=1}^m \left( \mathcal{M}(\| f_i \|^{q_i})(\| A_i(y) \|^{-1} x) \right)^{\frac{1}{q_i}} \omega_i(x) dx \right)^{\frac{1}{p'}} dy
$$

$$
= B_{\gamma^p} \int_{Q^n_p} \frac{\Phi(y)}{|y|^{n_p}} \prod_{i=1}^m \Gamma_i(y) \left( \int_{Q^n_p} \prod_{i=1}^m \left( \mathcal{M}(\| f_i \|^{q_i})(\| A_i(y) \|^{-1} x) \right)^{\frac{1}{q_i}} \omega_i(x) dx \right)^{\frac{1}{p'}} dy
$$

$$
\leq B_{\gamma^p} \int_{Q^n_p} \frac{\Phi(y)}{|y|^{n_p}} \prod_{i=1}^m \Gamma_i(y) \prod_{i=1}^m \left( \int_{Q^n_p} \mathcal{M}(\| f_i \|^{q_i})(\| A_i(y) \|^{-1} x) \omega_i(x) dx \right)^{\frac{1}{q_i}} dy. \quad (4.25)
$$

For $i = 1, \ldots, m$, by Proposition [2.5] we have $\omega_i \in A_\xi$. From this, by virtue of the boundedness of the Hardy-Littlewood maximal operator on the Lebesgue
spaces with the Muckenhoupt weights, we have that
\[
\left( \int \mathcal{M}(\|f_i\|^q)^\frac{1}{q'}(\|A_i(y)\|_{L^p}^{-1}\cdot x)\omega_i dx \right) \leq \left( \int \mathcal{M}(\|f_i\|^q)^\frac{1}{q'}(\|A_i(y)\|_{L^p}^{-1}\cdot y)\omega_i dy \right) \leq \left( \int \mathcal{M}(\|f_i\|^q)^\frac{1}{q'}(\|A_i(y)\|_{L^p}^{-1}\cdot y)\omega_i dy \right)
\]
\[
= \|A_i(y)\|_p^{-\frac{\alpha_i+\mu_i}{q'}} \left( \int \mathcal{M}(\|f_i\|^q)^\frac{1}{q'}(\|A_i(y)\|_{L^p}^{-1}\cdot y)\omega_i dy \right) \leq \|A_i(y)\|_p^{-\frac{\alpha_i+\mu_i}{q'}} \|f_i\|_{L_{\omega_i}^{q'}(Q_p^m)}.
\]
This together with (4.25) yields that the proof of this theorem is completed. \( \square \)

In what follows, we set
\[
C_S = \int_{Q_p^m} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \|A_i^{-1}(y)\|_{p,-(\alpha_i+\mu_i)\lambda_i} \log \|A_i(y)\|_p^0 dy.
\]

**Theorem 4.6.** Suppose the hypothesis in Lemma 4.1 holds. Let \( \lambda_i \in \left( \frac{1}{q'}, 0 \right) \) for all \( i = 1, \ldots, m \), and conditions (3.7) and (3.10) be true. Assume that
\[
supp(\Phi) \subset \bigcap_{i=1}^m \{ y \in Q_p^n : \|A_i(y)\|_p < 1 \}. \quad (4.26)
\]

(i) If \( C_S < \infty \), then \( \mathcal{H}_{\Phi, \vec{A}, \vec{b}}^p \) is bounded from \( B_{\omega_1}^{q_1, \lambda_1}(Q_p^n) \times \cdots \times B_{\omega_m}^{q_m, \lambda_m}(Q_p^n) \) to \( B_{\omega}^{q, \lambda}(Q_p^n) \).

(ii) If \( \mathcal{H}_{\Phi, \vec{A}, \vec{b}}^p \) is bounded from \( B_{\omega_1}^{q_1, \lambda_1}(Q_p^n) \times \cdots \times B_{\omega_m}^{q_m, \lambda_m}(Q_p^n) \) to \( B_{\omega}^{q, \lambda}(Q_p^n) \) for all \( \vec{b} = (b_1, \ldots, b_m) \in CMO_{\omega_1}^{q_1}(Q_p^n) \times \cdots \times CMO_{\omega_m}^{q_m}(Q_p^n) \), then \( C_S < \infty \).

**Proof.** Firstly, we prove the part (i) of the theorem. For any \( R \in \mathbb{Z} \), by Lemma 4.1 we get
\[
\begin{align*}
\frac{1}{\omega(B_R)^{\frac{1}{q'}+\lambda}} \| \mathcal{H}_{\Phi, \vec{A}, \vec{b}}^p(f) \|_{L_{\omega}^q(B_R)} & \lesssim \sum_{i=1}^m \int_{Q_p^m} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i(y) \cdot \mu_i(y) \times \\
& \times \left( \int_{\omega_1(B_{R+k}^{\frac{1}{q'}+\lambda})} \frac{\prod_{i=1}^m \omega_i(B_{R+k}^{\frac{1}{q'}+\lambda})^{\frac{1}{q'}+\lambda} dy}{\omega(B_R)^{\frac{1}{q'}+\lambda}} \right) \\
& \leq \mathcal{B}_{\vec{r}, \vec{\alpha}} \left( \int_{Q_p^m} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i(y) \cdot \mu_i(y) \times \left( \int_{\omega_1(B_{R+k}^{\frac{1}{q'}+\lambda})} \frac{\prod_{i=1}^m \omega_i(B_{R+k}^{\frac{1}{q'}+\lambda})^{\frac{1}{q'}+\lambda} dy}{\omega(B_R)^{\frac{1}{q'}+\lambda}} \right) \prod_{i=1}^m \|f_i\|_{B_{\omega_i}^{q_i, \lambda_i}(Q_p^n)}. \right)
\end{align*}
\]
Now, by (2.4) and (3.10), we calculate
\[
\left\| \sum_{i=1}^{m} \frac{R_{n+i,\alpha_i}}{r_i} \int_{B_{R+k,L}} \omega_i(B_{R+k,L}) \int_{\mathbb{R}^n} |f_i|^\nu \right\|_{B_{p,\infty}^{\psi + \lambda}} \geq \left(1 + \left| \omega(B_{R}) \right|^{\psi + \lambda} \right) \left( \int_{B_{R+k,L}} \omega_i(B_{R+k,L}) \int_{\mathbb{R}^n} |f_i|^\nu \right) \left( \int_{B_{R+k,L}} \omega_i(B_{R+k,L}) \int_{\mathbb{R}^n} |f_i|^\nu \right)
\]
\[
= \sum_{i=1}^{m} \frac{R_{n+i,\alpha_i}}{r_i} \int_{B_{R+k,L}} \omega_i(B_{R+k,L}) \int_{\mathbb{R}^n} |f_i|^\nu \right\|_{B_{p,\infty}^{\psi + \lambda}} \geq \left(1 + \left| \omega(B_{R}) \right|^{\psi + \lambda} \right) \left( \int_{B_{R+k,L}} \omega_i(B_{R+k,L}) \int_{\mathbb{R}^n} |f_i|^\nu \right) \left( \int_{B_{R+k,L}} \omega_i(B_{R+k,L}) \int_{\mathbb{R}^n} |f_i|^\nu \right)
\]
Hence, one has
\[
\left\| \mathcal{H}_{\Phi,A,b}^{p}(f) \right\|_{B_{p,\infty}^{\psi + \lambda}(Q^n_p)} \leq B_{p,\infty} \left( \int_{Q^n_p} \psi(y) \mu_i(y) \| A_i(y) \|_{p}^{(\alpha_i+n)(\frac{1}{n}+\lambda_i)} dy \right) \times \prod_{i=1}^{m} \| f_i \|_{B_{p,\infty}^{\psi + \lambda}(Q^n_p)}.
\]
Note that, by the hypothesis (4.26), we see that
\[
\| \log_p \| A_i(y) \|_p \| \geq 1, \text{ for all } y \in \text{supp}(\Phi).
\]
As mentioned above, by (2.3) and (3.11), we make
\[
\psi_i(y) \mu_i(y) \| A_i(y) \|_{p}^{(\alpha_i+n)(\frac{1}{n}+\lambda_i)} \leq \left(1 + \| A_i^{-1}(y) \|_{p}^{(\alpha_i+n)} \| A_i^{-1}(y) \|_{p}^{-\frac{1}{n}} + |\log_p \| A_i(y) \|_p | + 2p^\lambda \right) \times \| A_i^{-1}(y) \|_{p}^{(\alpha_i+n)} \| A_i^{-1}(y) \|_{p}^{-(\alpha_i+n)(\frac{1}{n}+\lambda_i)} \leq \left| \log_p \| A_i(y) \|_p \right| \| A_i(y) \|_{p}^{-(\alpha_i+n)\lambda_i}.
\]
As an application, by (4.27), we obtain that
\[
\left\| \mathcal{H}_{\Phi,A,b}^{p}(f) \right\|_{B_{p,\infty}^{\psi + \lambda}(Q^n_p)} \leq C_8 B_{p,\infty} \prod_{i=1}^{m} \| f_i \|_{B_{p,\infty}^{\psi + \lambda}(Q^n_p)}.
\]
To give the proof for the part (ii) of the theorem, for \( i = 1, \ldots, m \), let us choose \( b_i(x) = \log_p |x|^\alpha \) for all \( x \in \mathbb{Q}^n_p \setminus \{0\} \), and \( f_i(x) = |x|^{\gamma} \) for all \( x \in \mathbb{Q}^n_p \). Now, we need to prove that
\[
\left\| b_i \right\|_{CMO^\gamma_{p}(\mathbb{Q}^n_p)} < \infty, \text{ for all } i = 1, \ldots, m.
\]
In fact, for any \( R \in \mathbb{Z} \), we see that
\[
b_{i,R} = \frac{1}{|B_R|} \int_{B_R} \log_p |x|^\alpha dx = \frac{1}{|B_R|} \int_{B_R} \log_p |x|^\alpha dx = \frac{1}{|B_R|} \int_{B_R} \gamma dx = \frac{1}{|B_R|} \int_{B_R} \gamma p^\gamma dx = \frac{1}{|B_R|} \int_{B_R} \gamma p^\gamma (1 - p^{-\gamma}) dx
\]
\[
= \frac{1}{|B_R|} \int_{B_R} \gamma p^\gamma (1 - p^{-\gamma}) dx = \frac{1}{|B_R|} \int_{B_R} \gamma p^\gamma (1 - p^{-\gamma}) dx = \frac{1}{|B_R|} \int_{B_R} \gamma p^\gamma (1 - p^{-\gamma}) dx
\]
Thus, we get
\[
\frac{1}{\omega_1(B_R)} \int_{B_R} |b_i(x) - b_{i,B_R}|^{r_i} \omega_i dx = p^{-R(\alpha_i + n)} \sum_{\gamma \leq R} \int_{\mathbb{R}^n} \left| \gamma - (R - \frac{1}{p^n - 1}) \right|^{r_i} p^{\gamma \alpha_i} dx
\]
\[
= p^{-R(\alpha_i + n)} \sum_{\gamma \leq R} \left| \gamma - (R - \frac{1}{p^n - 1}) \right|^{r_i} p^{(\gamma \alpha_i + n)(1 - p^{-n})}
\]
\[
\lesssim p^{-R(\alpha_i + n)} \sum_{\gamma \leq R} \left| \gamma - (R - \frac{1}{p^n - 1}) \right|^{r_i} p^{\gamma \alpha_i} = p^{-R(\alpha_i + n)} \sum_{\ell \leq 0} \left| \ell + \frac{1}{p^n - 1} \right|^{r_i} p^{(R + \ell)(\alpha_i + n)}
\]
\[
\lesssim \sum_{\ell \leq 0} \left( |\ell|^{r_i} + \frac{1}{(p^n - 1)^{r_i}} \right) p^{\ell(\alpha_i + n)} < \infty,
\]
uniformly for $R \in \mathbb{Z}$. As an application, it immediately follows that the inequality (4.28) holds. By choosing $b_i$ and $f_i$, we have
\[
\mathcal{H}_{\Phi, \lambda, \mathbf{B}}^p (\tilde{f})(x) = \int_{\mathbb{Q}_p^m} \Phi(y) |y|_p^{-n} \prod_{i=1}^m |A_i(y)|_p^{(\alpha_i + n)\lambda_i} \left( \log_p \frac{|x|_p}{|A_i(y)|_p} \right) dy.
\]
By the hypothesis (4.26), we have $\|A_i(y)\|_p < 1$, for all $y \in \text{supp}(\Phi)$. Thus,
\[
|A_i(y)|_p \leq \|A_i(y)\|_p |x|_p < |x|_p.
\]
This gives that
\[
0 < |\log_p \|A_i(y)\|_p| = \log_p \frac{1}{|A_i(y)|_p} \leq \log_p \frac{|x|_p}{|A_i(y)|_p}.
\]
Consequently, by (3.11) and (4.1), we lead to
\[
\mathcal{H}_{\Phi, \lambda, \mathbf{B}}^p (\tilde{f})(x) \gtrsim \left( \int_{\mathbb{Q}_p^m} \Phi(y) |y|_p^{-n} \prod_{i=1}^m |A_i(y)|_p^{-(\alpha_i + n)\lambda_i} \left| \log_p \|A_i(y)\|_p \right| dy \right) |x|_p^{(\alpha_i + n)\lambda}
\]
\[
= C_8 \cdot |x|_p^{(\alpha_i + n)\lambda}.
\]
From this, by (3.12) above, we infer
\[
\|\mathcal{H}_{\Phi, \lambda, \mathbf{B}}^p (\tilde{f})\|_{B_{\omega^n, \lambda}^q (\mathbb{Q}_p^n)} \gtrsim C_8 \| \cdot \|_{B_{\omega^n, \lambda}^q (\mathbb{Q}_p^n)} \gtrsim C_8 \prod_{i=1}^m \| f_i \|_{B_{\omega_i^n, \lambda_i}^q (\mathbb{Q}_p^n)}.
\]
Therefore, since $\mathcal{H}_{\Phi, \lambda, \mathbf{B}}^p$ is bounded from $\dot{B}_{\omega_1}^{q_1, \lambda_1} (\mathbb{Q}_p^n) \times \cdots \times \dot{B}_{\omega_m}^{q_m, \lambda_m} (\mathbb{Q}_p^n)$ to $\dot{B}_\omega^{q, \lambda} (\mathbb{Q}_p^n)$, it implies that $C_8 < \infty$. This leads to that the theorem is completely proved.

Now, we consider $A_i(y) = s_i(y)I_n$, for $i = 1, ..., m$. By the similar arguments, we then obtain the following useful result.
Corollary 4.7. Suppose the hypothesis in Lemma 4.1 and (3.11) hold. Denote
\[ C_9 := \int_{Q_p^n} \left( \prod_{i=1}^{m} \psi_i^*(y) \cdot \mu_i^*(y) \times \left( \lambda \left( \| A_i(y) \|_p \leq 1 \right) \| A_i(y) \|^{\alpha_i} + \chi \left( \| A_i(y) \|_p > 1 \right) \| A_i(y) \|^{\alpha_i} \right) \right) dy < \infty, \]
for all \( (i) \) and (ii) we have \( H_{\Phi, A, b}^p \) is bounded from \( B_{\omega}^{q_1, \lambda_1} (Q_p^n) \times \cdots \times B_{\omega}^{q_m, \lambda_m} (Q_p^n) \) to \( B_{\omega}^{q, \lambda} (Q_p^n) \), for any \( \omega \in \mathcal{B} (Q_p^n) \). Let \( \| \omega \|_p \leq 1 \), \( \| \omega \|_p > 1 \), \( C \) and condition (3.13) in Theorem 3.4 hold. Then, if
\[ C_{10} := \int_{Q_p^n} \left( \prod_{i=1}^{m} \psi_i^*(y) \cdot \mu_i^*(y) \times \left( \lambda \left( \| A_i(y) \|_p \leq 1 \right) \| A_i(y) \|^{\alpha_i} + \chi \left( \| A_i(y) \|_p > 1 \right) \| A_i(y) \|^{\alpha_i} \right) \right) dy < \infty, \]
we have \( H_{\Phi, A, b}^p \) is bounded from \( B_{\omega}^{q_1, \lambda_1} (Q_p^n) \times \cdots \times B_{\omega}^{q_m, \lambda_m} (Q_p^n) \) to \( B_{\omega}^{q, \lambda} (Q_p^n) \).

Proof. For any \( R \in \mathbb{Z} \), by Lemma 4.2 and (3.13), we infer
\[ \frac{1}{\omega(B_R)^{\frac{1}{\alpha_i} + \lambda_i}} \| H_{\Phi, A, b}^p (f) \|_{L^q_{\omega}(B_R)} \leq B_{R, \omega} \left( \sum_{i=1}^{m} \psi_i^*(y) \cdot \mu_i^*(y) \right) \frac{1}{\omega(B_{R+kA_i})^{\frac{1}{\alpha_i} + \lambda_i}} \| f_i \|_{L^q_{\omega}(B_{R+kA_i})} \left( \frac{\omega(B_{R+kA_i})}{\omega(B_R)} \right)^{\lambda_i} dy \]
\[ \leq B_{R, \omega} \left( \sum_{i=1}^{m} \psi_i^*(y) \cdot \mu_i^*(y) \right) \left( \frac{\omega(B_{R+kA_i})}{\omega(B_R)} \right)^{\lambda_i} dy \sum_{i=1}^{m} \| f_i \|_{L^q_{\omega}(B_{R+kA_i})}. \]

From this, by (3.14), we have \( \| H_{\Phi, A, b}^p (f) \|_{L^q_{\omega}(Q_p^n)} \leq C_{10} \cdot B_{R, \omega} \sum_{i=1}^{m} \| f_i \|_{L^q_{\omega}(Q_p^n)} \),
which implies that the proof of this theorem is finished.

References
1. S. Albeverio, A. Yu. Khrennikov, V. M. Shelkovich, Harmonic analysis in the \( p \)-adic Lizorkin spaces: fractional operators, pseudo-differential equations, \( p \)-wavelets, Tauberian theorems, J. Fourier Anal. Appl. 12(4) (2006), 393-425.
2. A. V. Avetisov, A. H. Bikulov, S. V. Kozyrev, V. A. Osipov, \( p \)-adic models of ultrametric diffusion constrained by hierarchical energy landscapes, J. Phys. A: Math. Gen. 35 (2002), 177-189.
3. A. V. Avetisov, A. H. Bikulov, V. A. Osipov,\textit{ p-adic description of characteristic relaxation in complex systems}, J. Phys. A: Math. Gen. 36 (2003), 4239-4246.
4. J. Chen, D. Fan and J. Li, \textit{Hausdorff operators on function spaces}, Chinese Annals of Mathematics, Series B. 33(2012), 537-556.
5. N. M. Chuong, \textit{Pseudodifferential operators and wavelets over real and p-adic fields}, Springer-Basel, 2018.
6. N. M. Chuong, Yu. V. Egorov, A. Yu. Khrennikov, Y. Meyer, D. Mumford, \textit{Harmonic, wavelet and p-adic analysis}, World Scientific, (2007).
7. N. M. Chuong, D. V. Duong, \textit{Weighted Hardy-Littlewood operators and commutators on p-adic functional spaces}, $ p $-Adic numbers, Ultrametric Anal. Appl., 5(1)(2013), 65-82.
8. N. M. Chuong, D. V. Duong, \textit{The p-adic weighted Hardy-Cesàro operators on weighted Morrey-Herz space}, $ p $-Adic numbers, Ultrametric Anal. Appl., 8(3)(2016), 204-216.
9. N. M. Chuong, D. V. Duong and K. H. Dung, \textit{Multilinear Hausdorff operators on some function spaces with variable exponent}, 2017, arxiv.org/abs/1709.08185.
10. N. M. Chuong, N. V. Co, \textit{The Cauchy problem for a class of pseudo-differential equations over p-adic field}, J. Math. Anal. Appl. 340(2008), 1, 629-643.
11. N. M. Chuong and H. D. Hung, \textit{Maximal functions and weighted norm inequalities on Local Fields}, Appl. Comput. Harmon. Anal. 29 (2010), 272-286.
12. B. Dragovich, A. Yu. Khrennikov, S. V. Kozyrev, I. V. Volovich, \textit{On p-adic mathematical physics, $ p $-Adic numbers, Ultrametric Anal. Appl.,1(1)(2009), 1-17.}
13. M. Dyachenko, E. Nursultanov, S. Tikhonov, \textit{Hardy-Littlewood and Pitt’s inequalities for Hausdorff operators, Bull. Sci. Math. 147(2018), 40-57.}
14. L. Grafakos, \textit{Modern Fourier analysis}, Second Edition, Springer, (2008).
15. F. Hausdorff, \textit{Summation methoden und Momentfolgen}, I, Math. Z. 9 (1921), 74-109.
16. H. D. Hung, \textit{The p-adic weighted Hardy-Cesàro operator and an application to discrete Hardy inequalities}, J. Math. Anal. Appl. 409(2014), 868-879.
17. W. A. Hurwitz, L. L. Silverman, \textit{The consistency and equivalence of certain definitions of summabilities}, Trans. Amer. Math. Soc. 18 (1917), 1-20.
18. T. Hytönen, C. Pérez and E. Rela, \textit{Sharp reverse Hölder property for $ A_\infty $ weights on spaces of homogeneous type}, J. Funct. Anal. 263 (2012), 3883-3899.
19. S. Indratno, D. Maldonado and S. Silwal, \textit{A visual formalism for weights satisfying reverse inequalities}, Expo. Math. 33 (2015), 1-29.
20. S. Haran, \textit{Riesz potentials and explicit sums in arithmetic}, Invent. Math. 101(1990), 697-703.
21. S. Haran, \textit{Analytic potential theory over the p-adics}, Ann. Inst. Fourier (Grenoble) 43(4) (1993), 905-944.
22. A. Yu. Khrennikov, \textit{p-adic valued distributions in mathematical physics}, Kluwer Academic Publishers, Dordrecht-Boston-London, (1994).
23. A. N. Kochubei, \textit{Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields}, Marcel Dekker, New York, 2001.
24. S. V. Kozyrev, \textit{Methods and applications of ultrametric and p-adic analysis: From wavelet theory to biophysics}, Proc. Steklov Inst. Math., 274(2011), 1-84.
25. Z. W. Fu, Z. G. Liu, S. Z. Lu, \textit{Commutators of weighted Hardy operators}, Proc. Amer. Math. Soc. 137(2009), 3319-3328.
26. S. Lu, Y. Ding and D. Yan, \textit{Singular integrals and related topics}, World Scientific Publishing Company, Singapore, 2007.
27. C. Morrey, \textit{On the solutions of quasi-linear elliptic partial differential equations}, Trans. Amer. Math. Soc. 43(1938), 126-166.
28. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207-226.
29. J. Ruan, D. Fan and Q. Wu, *Weighted Herz space estimates for Hausdorff operators on the Heisenberg group*, Banach J. Math. Anal. 11 (2017), 513-535.
30. J. Ruan, D. Fan and Q. Wu, *Weighted Morrey estimates for Hausdorff operator and its commutator on the Heisenberg group*, 2017, arXiv:1712.10328
31. K. S. Rim and J. Lee, *Estimates of weighted Hardy-Littlewood averages on the p-adic vector space*, J. Math. Anal. Appl. 324(2)(2006), 1470-1477.
32. E. M. Stein, *Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, (1993).
33. V. S. Varadarajan, *Path integrals for a class of p-adic Schrödinger equations*, Lett. Math. Phys. 39(1997), 97-106.
34. V. S. Vladimirov, *Tables of Integrals of Complex-Valued Functions of p-Adic Arguments*, Proc. Steklov Inst. Math. 284(2), 2014, 1-59.
35. V. S. Vladimirov, I. V. Volovich, *p-adic quantum mechanics*, Comm. Math. Phys. 123(1989), 659-676.
36. V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-adic analysis and mathematical physics*, World Scientific, (1994).
37. S. S. Volosivets, *Multidimensional Hausdorff operator on p-adic field*, p-Adic numbers, Ultrametric Anal. Appl. 2 (2010), 252-259.
38. S. S. Volosivets, *Hausdorff operators on p-adic linear spaces and their properties in Hardy, BMO, and Hölder spaces*, Mathematical Notes, 3(2013), 382-391.
39. J. Xiao, *L^p and BMO bounds of weighted Hardy-Littlewood averages*, J. Math. Anal. Appl. 262(2001), 660-666.

**Institute of mathematics, Vietnamese Academy of Science and Technology, Hanoi, Vietnam.**

*E-mail address: nmchuong@math.ac.vn*

**School of Mathematics, MienTrung University of Civil Engineering, Phu Yen, Vietnam**

*E-mail address: daovanduong@muce.edu.vn*

**School of Mathematics, University of Transport and Communications, Hanoi, Vietnam**

*E-mail address: khdung@utc2.edu.vn*