Structure and Properties of Hughston’s
Stochastic Extension of the Schrödinger Equation

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ABSTRACT

Hughston has recently proposed a stochastic extension of the Schrödinger equation, expressed as a stochastic differential equation on projective Hilbert space. We derive new projective Hilbert space identities, which we use to give a general proof that Hughston’s equation leads to state vector collapse to energy eigenstates, with collapse probabilities given by the quantum mechanical probabilities computed from the initial state. We discuss the relation of Hughston’s equation to earlier work on norm-preserving stochastic equations, and show that Hughston’s equation can be written as a manifestly unitary stochastic evolution equation for the pure state density matrix. We discuss the behavior of systems constructed as direct products of independent subsystems, and briefly address the question of whether an energy-based approach, such as Hughston’s, suffices to give an objective interpretation of the measurement process in quantum mechanics.
I. INTRODUCTION

A substantial body of work [1] has addressed the problem of state vector collapse by proposing that the Schrödinger equation be modified to include a stochastic process, presumably arising from physics at a deeper level, that drives the collapse process. In particular, Gisin [2], Percival [3], and Ghirardi, Pearle, and Rimini [4] have constructed equations that preserve the norm of the state vector, which in the approximation that the usual Schrödinger Hamiltonian dynamics is neglected are shown [4] to lead to state vector collapse with the correct quantum mechanical probabilities. An alternative approach to constructing a stochastic extension of the Schrödinger equation has been pursued by Hughston [5], based on the proposal of a number of authors [6] to rewrite the Schrödinger equation as an equivalent dynamics on projective Hilbert space, i.e., on the space of rays, a formulation in which the imposition of a state vector normalization condition is not needed. Within this framework, Hughston [5] has proposed a simple stochastic extension of the Schrödinger equation, constructed solely from the Hamiltonian function, and has shown that his equation leads to state vector reduction to an energy eigenstate, with energy conservation in the mean throughout the reduction process. In the simplest spin-1/2 case, Hughston exhibits an explicit solution that shows that his equation leads to collapse with the correct quantum mechanical probabilities, but the issue of collapse probabilities in the general case has remained open.

Our purpose in this paper is to further investigate the structure and properties of Hughston’s equation, proceeding from new identities in projective Hilbert space derived in Sec. II. A principal result will be the proof in Sec. III (using the martingale or “gambler’s ruin” argument pioneered by Pearle [7]) that in the generic case, with no approximations,
Hughston’s equation leads to state vector collapse to energy eigenstates with the correct quantum mechanical probabilities. The relation of Hughston’s equation to earlier work on norm-preserving equations is discussed in Sec. IV, and the density matrix form of Hughston’s equation, which gives a manifestly unitary stochastic evolution on pure states, is given in Sec. V. In Sec. VI we examine the stochastic evolution of an initial state that is constructed as the product of independent subsystem states. Finally, in Sec. VII we discuss whether an energy-based approach to stochastic evolution (as opposed to approaches [8] based on spontaneous localization) suffices to give a satisfactory objective description of the evolution of a state during the quantum mechanical measurement process.

II. PROJECTIVE HILBERT SPACE AND SOME IDENTITIES

We begin by explaining the basic elements of projective Hilbert space needed to understand Hughston’s equation, working in an $n + 1$ dimensional Hilbert space. We denote the general state vector in this space by $|z\rangle$, with $z$ a shorthand for the complex projections $z^0, z^1, ..., z^n$ of the state vector on an arbitrary fixed basis. Letting $F$ be an arbitrary Hermitian operator, and using the summation convention that repeated indices are summed over their range, we define

\[
(F) \equiv \frac{\langle z|F|z \rangle}{\langle z|z \rangle} = \frac{z^\alpha F_{\alpha\beta} z^\beta}{z^\gamma z^\gamma}, \tag{1a}
\]

so that $(F)$ is the expectation of the operator $F$ in the state $|z\rangle$, independent of the ray representative and normalization chosen for this state. Note that in this notation $(F^2)$ and $(F)^2$ are not the same; their difference is in fact the variance $[\Delta F]^2$,

\[
[\Delta F]^2 = (F^2) - (F)^2. \tag{1b}
\]
We shall use two other parameterizations for the state $|z\rangle$ in what follows. Since $(F)$ is homogeneous of degree zero in both $z^\alpha$ and $\overline{z}^\alpha$, let us define new complex coordinates $t^j$ by

$$t^j = z^j / z^0, \quad \overline{t}^j = \overline{z}^j / \overline{z}^0, \quad j = 1, \ldots, n,$$

which are well-defined over all states for which $z^0 \neq 0$ [9]. Next, it is convenient to split each of the complex numbers $t^j$ into its real and imaginary part $t^j_R$, $t^j_I$, and to introduce a $2n$ component real vector $x^a$, $a = 1, \ldots, 2n$ defined by $x^1 = t^1_R$, $x^2 = t^1_I$, $x^3 = t^2_R$, $x^4 = t^2_I$, $\ldots$, $x^{2n-1} = t^n_R$, $x^{2n} = t^n_I$. Clearly, specifying the projective coordinates $t^j$ or $x^a$ uniquely determines the unit ray containing the unnormalized state $|z\rangle$, while leaving the normalization and ray representative of the state $|z\rangle$ unspecified.

As discussed in Refs. [6], projective Hilbert space is also a Riemannian space with respect to the Fubini-Study metric $g_{\alpha\beta}$, defined by the line element

$$ds^2 = g_{\alpha\beta} dz^\alpha dz^\beta \equiv 4 \left( 1 - \frac{|\langle z|z + dz\rangle|^2}{\langle z|z\rangle\langle z + dz|z + dz\rangle} \right).$$

Abbreviating $\overline{z}^\gamma z^\gamma \equiv \overline{z} \cdot z$, a simple calculation gives

$$g_{\alpha\beta} = 4(\delta_{\alpha\beta} \overline{z} \cdot z - z^\alpha z^\beta)/(\overline{z} \cdot z)^2 = 4 \frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial z^\beta} \log \overline{z} \cdot z .$$

Because of the homogeneity conditions $\overline{z}^\alpha g_{\alpha\beta} = z^\beta g_{\alpha\beta} = 0$, the metric $g_{\alpha\beta}$ is not invertible, but if we hold the coordinates $\overline{z}^0$, $z^0$ fixed in the variation contained in Eq. (3a) and go over to the projective coordinates $t^j$, we can rewrite the line element of Eq. (3a) as

$$ds^2 = g_{jk} dt^j dt^k ,$$

with the invertible metric [9]

$$g_{jk} = \frac{4[(1 + \overline{t}^j t^k)\delta_{jk} - t^j t^k]}{(1 + \overline{t}^\alpha t^\alpha)^2} .$$
with inverse
\[ g^{jk} = \frac{1}{4} (1 + T^m t^m) (\delta_{jk} + t^l T^k_l) \]  

(4c)

Reexpressing the complex projective coordinates \( t^j \) in terms of the real coordinates \( x^a \), the line element can be rewritten as
\[ ds^2 = g_{ab} dx^a dx^b, \]
\[ g_{ab} = \frac{4 [(1 + x^d x^d) \delta_{ab} - (x^a x^b + \omega_{ac} x^c \omega_{bd} x^d)]}{(1 + x^e x^e)^2}, \]
\[ g^{ab} = \frac{1}{4} (1 + x^e x^e) (\delta_{ab} + x^a x^b + \omega_{ac} x^c \omega_{bd} x^d) \] .

(4d)

Here \( \omega_{ab} \) is a numerical tensor whose only nonvanishing elements are
\[ \omega_{a = 2j - 1 \ b = 2j} = 1 \text{ and } \omega_{a = 2j \ b = 2j - 1} = -1 \text{ for } j = 1, ..., n. \]

As discussed by Hughston, one can define a complex structure \( J^b_a \) over the entire projective Hilbert space for which
\[ J^b_a \delta^{cd} = g_{ab}, \quad J^b_a J^c_b = -\delta^c_a, \]
such that \( \Omega^{ab} = g_{bc} J^c_a \) and \( \Omega^{ab} = g^{ac} J^b_c \) are antisymmetric tensors. At \( x = 0 \), the metric and complex structure take the values
\[ g_{ab} = 4 \delta_{ab}, \quad g^{ab} = \frac{1}{4} \delta_{ab}, \]
\[ J^b_a = \omega_{ab}, \quad \Omega^{ab} = 4 \omega_{ab}, \quad \Omega^{ab} = \frac{1}{4} \omega_{ab}. \]

(5)

Returning to Eq. (1a), we shall now derive some identities that are central to what follows. Differentiating Eq. (1a) with respect to \( z^\alpha \), with respect to \( z^\beta \), and with respect to both \( \bar{z}^\alpha \) and \( z^\beta \), we get
\[ \langle z | z \rangle \frac{\partial (F)}{\partial \bar{z}^\gamma} = F_{\alpha \beta} z^\beta - (F) \bar{z}^\alpha, \]
\[ \langle z | z \rangle \frac{\partial (F)}{\partial z^\beta} = -\bar{z}^\alpha F_{\alpha \beta} - (F) \bar{z}^\beta, \]
\[ \langle z | z \rangle \frac{\partial^2 (F)}{\partial \bar{z}^\alpha \partial z^\beta} = (\langle z | z \rangle \delta_{\alpha \beta} F - (F)) + 2 z^\alpha \bar{z}^\beta (F) - \bar{z}^\gamma F_{\gamma \beta} z^\alpha - \bar{z}^\gamma F_{\alpha \gamma} z^\gamma. \]

(6a)

Writing similar expressions for a second operator expectation \((G)\), contracting in various combinations with the relations of Eq. (6a), and using the homogeneity conditions
\[ \frac{\partial (F)}{\partial \bar{z}^\gamma} = \bar{z}^\alpha \frac{\partial (F)}{\partial z^\beta} = \bar{z}^\beta \frac{\partial^2 (F)}{\partial \bar{z}^\alpha \partial z^\beta} = x^\beta \frac{\partial^2 (F)}{\partial \bar{z}^\alpha \partial z^\beta} = 0 \]

(6b)
to eliminate derivatives with respect to $\overline{z}^0$, $z^0$, we get the following identities,

$$-i(FG - GF) = -i\langle z|z \rangle \left( \frac{\partial(F)}{\partial z^\alpha} \frac{\partial(G)}{\partial \overline{z}^\alpha} - \frac{\partial(G)}{\partial z^\alpha} \frac{\partial(F)}{\partial \overline{z}^\alpha} \right) = 2\Omega^{ab} \nabla_a(F) \nabla_b(G),$$

$$(FG + GF) - 2(F)(G) = \langle z|z \rangle \left( \frac{\partial(F)}{\partial z^\alpha} \frac{\partial(G)}{\partial \overline{z}^\alpha} + \frac{\partial(G)}{\partial z^\alpha} \frac{\partial(F)}{\partial \overline{z}^\alpha} \right) = 2g^{ab} \nabla_a(F) \nabla_b(G),$$

$$(FGF) - (F^2)(G) - (F)(FG + GF) + 2(F)^2(G) = \langle z|z \rangle 2 \left( \frac{\partial(F)}{\partial z^\alpha} \frac{\partial^2G}{\partial z^\alpha \partial \overline{z}^\beta} \frac{\partial(F)}{\partial \overline{z}^\beta} \right) - 2\nabla^a(F) \nabla^b(F) \nabla_a \nabla_b(G),$$

(7a)

with $\nabla_a$ the covariant derivative constructed using the Fubini-Study metric affine connection.

It is not necessary to use the detailed form of this affine connection to verify the right hand equalities in these identities, because since $(G)$ is a Riemannian scalar, $\nabla_a \nabla_b(G) = \nabla_a \partial_b(G)$, and since projective Hilbert space is a homogeneous manifold, it suffices to verify the identities at the single point $x = 0$, where the affine connection vanishes and thus $\nabla_a \nabla_b(G) = \partial_a \partial_b(G)$. Using Eqs. (7a) and the chain rule we also find

$$-\nabla_a[(F^2) - (F)^2] \nabla^a(G) = -\frac{1}{2}(F^2G + GF^2) + (F^2)(G) + (F)(FG + GF) - 2(F)^2(G).$$

(7b)

When combined with the final identity in Eq. (7a) this gives

$$D \equiv \nabla^a(F) \nabla^b(F) \nabla_a \nabla_b(G) - \frac{1}{2} \nabla_a[(F^2) - (F)^2] \nabla^a(G)$$

$$= \frac{1}{4}(2FGF - F^2G - GF^2)$$

$$= -\frac{1}{4}([F, [F, G]])$$

(7c)

with $[,]$ denoting the commutator, from which we see that $D$ vanishes when the operators $F$ and $G$ commute.

An alternative derivation of Eq. (7c) proceeds from the fact, noted by Hughston, that (for $F$ self-adjoint)

$$\xi^a_F \equiv \Omega^{ab} \nabla_b(F)$$

(8a)
is a Killing vector obeying
\[ \nabla_c \xi^a_F + \nabla^a \xi_{cF} = 0 \quad (8b) \]

Using the identity \((F^2) - (F)^2 = \nabla_b(F) \nabla^b(F)\), which is the \(F = G\) case of the middle equality of Eq. (7a), we rewrite \(D\) of Eq. (7c) as

\[ D = \nabla^a(F) \nabla^b(F) \nabla_a \nabla_b(G) - \nabla^b(F) \nabla^a(G) \nabla_a \nabla_b(F) \quad (9a) \]

This can be rewritten, using the identity \(\Omega^{ab} \Omega_{cb} = \delta^a_c\), the antisymmetry of \(\Omega\), the fact that \(\Omega\) commutes with the covariant derivatives, and the Killing vector definition of Eq. (8a), as

\[ D = \Omega^{ac}_\xi e F \Omega_{eF} \nabla_a \nabla_b(G) - \Omega^{bc}_\xi e F \Omega_{eF} \nabla_a \nabla_b(F) \]

\[ = - \xi_{eF} \Omega^{ac}_\xi e F \nabla_a \xi_G + \xi_{eF} \xi_{eG} \Omega^{ae}_\xi \nabla_a \xi_F \quad (9b) \]

We now use the Killing vector identity of Eq. (8b) on the final factor in each term, giving

\[ D = \xi_{eF} \xi_{eF} \nabla^c \xi_{aG} - \xi_{eF} \xi_{eG} \Omega^{ae} \nabla^c \xi_{aF} \quad (9c) \]

Exchanging the labels \(e\) and \(c\) in the first term, and exchanging the labels \(a\) and \(e\) in the second term, we get

\[ D = \xi_{eF} \xi_{eF} \nabla^c \xi_{aG} + \xi_{eF} \xi_{aG} \Omega^{ae} \nabla^c \xi_{eF} \]

\[ = \xi_{eF} \nabla^c [\Omega^{ae}_\xi e F \xi_{aG}] \quad (10a) \]

Substituting the Killing vector definition of Eq. (8a), this becomes

\[ D = \Omega_{cb} \nabla^b(F) \nabla^c [\Omega^{ae}_\xi e_f \nabla^f(F) \Omega_{ag} \nabla^g(G)] \]

\[ = \Omega_{cb} \nabla^b(F) \nabla^c [\Omega_{gF} \nabla^g(G) \nabla^f(F)] \]

\[ = - \frac{1}{4} ([F, [F, G]]) \quad (10b) \]

where to get the final line we have twice used the first identity in Eq. (7a). This completes our geometric derivation of Eq. (7c).
III. HUGHSTON’S EQUATION AND STATE VECTOR

COLLAPSE PROBABILITIES

Let us now turn to Hughston’s stochastic differential equation, which reads

$$dx^a = [2\Omega^{ab}\nabla_b(H) - \frac{1}{4}\sigma^2\nabla^a V]dt + \sigma\nabla^a(H)dW_t$$ ,

(11a)

with $W_t$ a Brownian motion or Wiener process, with $\sigma$ a parameter governing the strength of the stochastic terms, with $H$ the Hamiltonian operator and $(H)$ its expectation, and with $V$ the variance of the Hamiltonian,

$$V = [\Delta H]^2 = (H^2) - (H)^2$$ .

(11b)

When the parameter $\sigma$ is zero, Eq. (11a) is just [6] the transcription of the Schrödinger equation to projective Hilbert space. For the time evolution of a general function $G[x]$, we get by Taylor expanding $G[x + dx]$ and using the Itô stochastic calculus rules [10]

$$[dW_t]^2 = dt , \quad [dt]^2 = dtdW_t = 0$$ ,

(12a)

the corresponding stochastic differential equation

$$dG[x] = \mu dt + \sigma\nabla_a G[x]\nabla^a(H)dW_t$$ ,

(12b)

with the drift term $\mu$ given by

$$\mu = 2\Omega^{ab}\nabla_a G[x]\nabla_b(H) - \frac{1}{4}\sigma^2\nabla^a V\nabla_a G[x] + \frac{1}{2}\sigma^2\nabla^a(H)\nabla^b(H)\nabla_a\nabla_b G[x]$$ .

(12c)

Hughston shows that with the $\sigma^2$ part of the drift term chosen as in Eq. (11a), the drift term $\mu$ in Eq. (12b) vanishes for the special case $G[x] = (H)$, guaranteeing conservation of the expectation of the energy with respect to the stochastic evolution of Eq. (11a). But referring
to Eq. (7c) and the first identity in Eq. (7a), we see that in fact a much stronger result is also true, namely that $\mu$ vanishes [and thus the stochastic process of Eq. (12b) is a martingale] whenever $G[x] = (G)$, with $G$ any operator that commutes with the Hamiltonian $H$.

Let us now make two applications of this fact. First, taking $G[x] = V = (H^2) - (H)^2$, we see that the contribution from $(H^2)$ to $\mu$ vanishes, so the drift term comes entirely from $-(H)^2$. Substituted this into $\mu$ gives $-2(H)$ times the drift term produced by $(H)$, which is again zero, plus an extra term

$$-\sigma^2 \nabla^a(H) \nabla^b(H) \nabla_a(H) \nabla_b(H) = -\sigma^2 V^2 \;,$$  \hspace{1cm} \text{(13a)}

where we have used the relation $V = \nabla_a(H) \nabla^a(H)$ which follows from the $F = G = H$ case of the middle identity of Eq. (7a). Thus the variance $V$ of the Hamiltonian satisfies the stochastic differential equation, derived by Hughston by a more complicated method,

$$dV = -\sigma^2 V^2 dt + \sigma \nabla_a V \nabla^a(H) dW_t \;.$$ \hspace{1cm} \text{(13b)}

This implies that the expectation $E[V]$ with respect to the stochastic process obeys

$$E[V(t)] = E[V(0)] - \sigma^2 \int_0^t ds E[V(s)^2] \;,$$  \hspace{1cm} \text{(13c)}

which using the inequality $0 \leq E[\{V - E[V]\}^2] = E[V^2] - E[V]^2$ gives the inequality

$$E[V(t)] \leq E[V(0)] - \sigma^2 \int_0^t ds E[V(s)]^2 \;.$$  \hspace{1cm} \text{(13d)}

Since $V$ is necessarily positive, Eq. (13d) implies that $E[V(\infty)] = 0$, and again using positivity of $V$ this implies that $V(s)$ vanishes as $s \to \infty$, apart from a set of outcomes of probability measure zero. Thus, as concluded by Hughston, the stochastic term in his equation drives the system, as $t \to \infty$, to an energy eigenstate.
As our second application of the vanishing of the drift term \( \mu \) for expectations of operators that commute with \( H \), let us consider the projectors \( \Pi_e \equiv |e\rangle\langle e| \) on a complete set of energy eigenstates \( |e\rangle \). By definition, these projectors all commute with \( H \), and so the drift term \( \mu \) vanishes in the stochastic differential equation for \( G[x] = (\Pi_e) \), and consequently the expectations \( E[(\Pi_e)] \) are time independent; additionally, by completeness of the states \( |e\rangle \), we have \( \sum_e (\Pi_e) = 1 \). But these are just the conditions for Pearle’s [7] gambler’s ruin argument to apply. At time zero, \( E[(\Pi_e)] = (\Pi_e) \equiv p_e \) is the absolute value squared of the quantum mechanical amplitude to find the initial state in energy eigenstate \( |e\rangle \). At \( t = \infty \), the system always evolves to an energy eigenstate, with the eigenstate \( |f\rangle \) occurring with some probability \( P_f \). The expectation \( E[(\Pi_e)] \), evaluated at infinite time, is then

\[
E[(\Pi_e)] = 1 \times P_e + \sum_{f \neq e} 0 \times P_f = P_e ;
\]

hence \( p_e = P_e \) for each \( e \) and the state collapses into energy eigenstates at \( t = \infty \) with probabilities given by the usual quantum mechanical rule applied to the initial wave function.

This conclusion clearly generalizes to the stochastic equation

\[
dx^a = [2\Omega^{ab} \nabla_b(H) - \frac{1}{4} \sigma^2 \sum_j \nabla^a V_j]dt + \sigma \sum_j \nabla^a(H_j) dW^j_t ,
\]

(15a)

with the \( H_j \) a set of mutually commuting self-adjoint operators that commute with \( H \), with \( V_j = (H_j^2) - (H_j)^2 \), and with the \( dW_i^j \) independent Wiener processes obeying \( dW_i^j dW_i^k = \delta^{jk}dt \). Following the same method used in obtaining Eq. (13b), and defining \( C_{kj} = \nabla_a(H_k) \nabla^a(H_j) = (H_k H_j) - (H_k)(H_j) \), one finds

\[
dV_k = -\sigma^2 \sum_j C_{kj}^2 dt + \sigma \nabla_a V_k \sum_j \nabla^a(H_j) dW_i^j ,
\]

(15b)
and therefore

\[ E[V_k(t)] = E[V_k(0)] - \sigma^2 \int_0^t ds \sum_j E[C_{kj}(s)^2] \quad . \]  

(15c)

Since \( E[C_{kj}^2] \geq E[C_{kj}]^2 \), we have as before

\[ E[V_k(t)] \leq E[V_k(0)] - \sigma^2 \int_0^t ds \sum_j E[C_{kj}(s)]^2 \quad , \]  

(15d)

which implies that each \( E[C_{kj}(s)] \) approaches zero as \( s \to \infty \). Hence for each \( k, j \) we have at large times

\[ E[(H_k H_j)] - E[(H_k)(H_j)] \to 0 \quad , \]  

(15e)

and so there is an effective quantum decorrelation of commuting observables. Moreover, for \( k = j \) Eq. (15e) implies that at large times \( E[V_k] \to 0 \), which since \( V_k \) is nonnegative implies that \( V_k \) approaches zero apart from a set of outcomes of probability measure zero, and so the state evolves to a simultaneous eigenstate of all the commuting observables entering the process of Eq. (15a).

IV. RELATION OF HUGHSTON’S EQUATION TO OTHER 
STOCHASTIC NORM-PRESERVING EQUATIONS

Let us now specialize Eqs. (12b) and (12c) to the case in which \( G[x] \) is simply the expectation \( (G) \) of an operator \( G \). Then by substituting Eqs. (7c) and the second equality in Eq. (7a), we find

\[ d(G) = \mu dt + \kappa dW_t \quad , \]  

(16a)

with

\[ \kappa = \frac{1}{2} \sigma \left[ \{G, H\} - 2(G)(H) \right] = \frac{1}{2} \sigma \{G, H - (H)\} \quad , \]  

(16b)
and with

$$
\mu = (-i[G, H]) - \frac{1}{8}\sigma^2([H, [H, G]])
$$

(16c)

where we have used \{ , \} to denote the anticommutator.

Let us now compare this with the evolution of \( G \) implied by the stochastic state vector evolution

$$
d|z\rangle = [\alpha dt + \beta dW_t]|z\rangle
$$

(17a)

with

$$
\alpha = -iH - \frac{1}{8}\sigma^2[A - (A)]^2 \quad ,
$$

$$
\beta = \frac{1}{2}\sigma[A - (A)] \quad ,
$$

(17b)

where \( A \) is a general self-adjoint operator and \((A)\) is defined, as in Eq. (1a), by \((A) = \langle z|A[z]/\langle z|z\rangle\). For the evolution of \( \langle z|G|z\rangle \), we find by the Itô rules,

$$
d\langle z|G|z\rangle = \langle z|[\alpha^\dagger G + G\alpha + \beta^\dagger G\beta]dt + [\beta^\dagger G + G\beta]dW_t|z\rangle
$$

(18a)

$$
= \langle z| - i[G, H]dt - \frac{1}{8}\sigma^2[[G, [A - (A)]^2} - 2[A - (A)]G[A - (A)]]dt
$$

$$
+ \frac{1}{2}\sigma\{G, [A - (A)]\}dW_t|z\rangle
$$

$$
= \langle z| - i[G, H]dt - \frac{1}{8}\sigma^2[A, [A, G]]dt + \frac{1}{2}\sigma\{G, [A - (A)]\}dW_t|z\rangle
$$

When \( G = 1 \), the right hand side of Eq. (18a) vanishes, since the commutator terms vanish trivially and \( \langle z|A - (A)|z\rangle = \langle z|A|z\rangle - \langle z|z\rangle(A) = 0 \). Therefore the state vector evolution of Eqs. (17a, b) is norm preserving, and so it is consistent to choose the normalization \( \langle z|z\rangle = 1 \) in conjunction with this evolution. For general \( G \) we then have \( d\langle z|G|z\rangle = d(G) \), and so Eq. (18a) gives an expression for \( d(G) \), which we see is identical to Eqs. (16a) - (16c) when the operator \( A \) is taken as the Hamiltonian \( H \). In particular, when \( A = G = H \) we learn from Eq. (18a) that \( d(H) = \sigma V dW_t \), in agreement with Eq. (12b), because

$$
\langle z|\{H, H - (H)\}|z\rangle = 2[(H^2) - (H)^2] = 2V
$$

(18b)
and so the convergence argument of Eqs. (13a)-(13d) follows directly from Eq. (18a). Apart from minor changes in notation, the norm preserving evolution of Eqs. (17a) and (17b) is the one given by Gisin [2], Percival [3], and Ghirardi, Pearle, and Rimini [4], and so we see that this evolution is equivalent [11] to the state vector evolution in projective Hilbert space given by Hughston’s equation.

The evolution of Eq. (17a) can be generalized [as was done for the Hughston equation in Eq. (15)] to read

\[ d|z\rangle = \left[ \alpha dt + \sum_j \beta_j dW^j \right]|z\rangle, \]

\[ \alpha = -iH - \frac{1}{8}\sigma^2 \sum_j (A_j - (A_j))^2, \]

\[ \beta_j = \frac{1}{2}\sigma [A_j - (A_j)], \]

(19)

with the \( A_j \) any set of mutually commuting operators. When the \( A_j \) do not all commute with the Hamiltonian \( H \), it is necessary to make the approximation of neglecting the Hamiltonian evolution (the \(-iH\) term in \( \alpha \)) in proving that Eq. (19) implies state vector reduction to the mutual eigenstates of the \( A_j \) with probabilities given by the usual quantum mechanical rule. Such a proof, very similar to the one given for Hughston’s equation in Sec. III above, has been given by Ghirardi, Pearle, and Rimini [4]. In order to carry through the proof with no approximations, it is necessary to assume that the \( A_j \) are operators in the mutually commuting set \( H_j \) that all commute with \( H \), as was done in Sec. III.

V. DENSITY MATRIX EVOLUTION

Let us now define the pure state density matrix \( \rho \) by

\[ \rho = \frac{|z\rangle\langle z|}{\langle z|z\rangle}, \]

(20a)
in terms of which \((G)\) is given by

\[
(G) = \text{Tr} \rho G .
\] (20b)

Since \(G\) is a fixed operator, Eq. (18a) for \(d(G)\) can be rewritten as

\[
\text{Tr} G d\rho = \text{Tr} \rho \left[ -i[G, H] dt - \frac{1}{8} \sigma^2 [A, [A, G]] dt + \frac{1}{2} \sigma \{G, [A - (A)]\} dW_t \right]
\]

(20c)

where in the final line we have cyclically permuted terms under the trace. Since Eq. (20c) holds for arbitrary self-adjoint operators \(G\), it implies that the density matrix obeys the stochastic differential equation (each term of which is self-adjoint)

\[
d\rho = -i[H, \rho] dt - \frac{1}{8} \sigma^2 [A, [A, \rho]] dt + \frac{1}{2} \sigma \{\rho, [A - (A)]\} dW_t .
\] (21a)

This equation can be written in an alternative form by observing that since \(\rho\) is a pure state density matrix obeying \(\rho^2 = \rho\), we have \(\rho(A) = \rho \text{Tr} \rho A = \rho A \rho\). These facts imply that

\[
\{\rho, [A - (A)]\} = \rho A + A \rho - 2\rho(A)
\]

(21b)

\[
= \rho^2 A + A \rho^2 - 2\rho A \rho = [\rho, [\rho, A]] ,
\]

and so we can rewrite Eq. (21a) as

\[
d\rho = -i[H, \rho] dt - \frac{1}{8} \sigma^2 [A, [A, \rho]] dt + \frac{1}{2} \sigma [\rho, [\rho, A]] dW_t .
\] (21c)

Equations (21a) and (21c) have the following properties for general \(A:\)

(i) Since \(\text{Tr} d\rho = 0\), the condition \(\text{Tr} \rho = 1\) is preserved by the time evolution.

(ii) After some algebra using the Itô rules, one finds that \(\rho^2 = \rho\) implies that

\[
\{\rho, d\rho\} + [d\rho]^2 = d\rho ,
\] (21d)
which can be rewritten as \([\rho + d\rho]^2 = \rho + d\rho\). Hence the evolution of Eqs. (21a, c) is consistent with the pure state condition. This is required by the fact that Eqs. (21a, c) may be derived as consequences of Eqs. (17a, b), which are a norm preserving pure state evolution. The condition of Eq. (21d) determines the coefficient of the \(\sigma^2 dt\) drift term in terms of the coefficient of the \(\sigma \cdot dW_t\) stochastic term, and so the ratio of these two coefficients in Eq. (21c) cannot be treated as an additional adjustable parameter.

(iii) Since \(d\rho = d\rho^\dagger\), the self-adjointness of \(\rho\) is preserved by the time evolution.

(iv) Time reversal invariance is violated by the stochastic terms, since when \(dt\) and \(i\) are reversed in sign, the term \(-i[H, \rho]dt\) is invariant, but the term \(-\frac{1}{8}\sigma^2[A, [A, \rho]]dt\) reverses sign.

(v) When we take the stochastic expectation of Eq. (21c), the \(dW_t\) term drops out, and we get

\[
\frac{dE[\rho]}{dt} = -i[H, E[\rho]] - \frac{1}{8}\sigma^2[A, [A, E[\rho]]] ,
\]

(22a)

which as pointed out by Percival [3] and Ghirardi, Pearle, and Rimini [4] is a quantum dynamical semigroup evolution of the completely positive Lindblad [12] form. The stochastic expectation \(E[\rho]\) is what is usually termed the density matrix; it starts off at \(t = 0\) as a pure state density matrix but then evolves, through the stochastic process, into a mixed state density matrix.

(vi) The conditions for Eq. (22a) to admit stationary solutions \(E[\rho]_S\) with \(dE[\rho]_S/dt = 0\) are very stringent, since when the left hand side of Eq. (22a) is zero, multiplying by
$E[\rho]_S$ and taking the trace gives

$$0 = -i \text{Tr} E[\rho]_S[H, E[\rho]_S] - \frac{1}{8} \sigma^2 \text{Tr} E[\rho]_S[A, [A, E[\rho]_S]] \ .$$

(22b)

Using cyclic permutation under the trace, the first term on the right hand side vanishes, while the second term becomes

$$\frac{1}{8} \sigma^2 \text{Tr}[A, E[\rho]_S]^2 \ ,$$

(22c)

which can only vanish when $[A, E[\rho]_S] = 0$. Substituting this equation back into Eq. (22a) then further implies that $[H, E[\rho]_S] = 0$. When $A$ and $H$ commute, these conditions can be satisfied with $E[\rho]_S$ a general function of $H$ [see the further discussion of this case in (xiii) below], but when $A$ and $H$ do not commute, one can have situations in which either $E[\rho]_S$ must be a multiple of the unit operator, which trivially commutes with both $H$ and $A$, or else there are no stationary solutions and $E[\rho]$ diverges at large times. The latter case is found in spontaneous localization models, as discussed for example in Section III.B.3 of Ref. [4].

(vii) The evolution implied by Eq. (22a) leads [13] to a monotonic increase of the von Neumann entropy, a result that can be demonstrated directly from Eq. (22a) as follows. Letting

$$S = -\text{Tr} E[\rho] \log E[\rho]$$

(22d)

be the von Neumann (or information) entropy, we find by substituting Eq. (22a) and using cyclical permutation of factors under the trace, that

$$\frac{dS}{dt} = -\text{Tr} \frac{dE[\rho]}{dt} [1 + \log E[\rho]]$$

$$= \frac{1}{8} \sigma^2 \text{Tr}[\log E[\rho], A][A, E[\rho]] \ .$$

(22e)
Substituting complete sets of eigenstates $|n\rangle$, $|m\rangle$ of the nonnegative density matrix $E[\rho]$, this becomes

$$\frac{dS}{dt} = \frac{1}{8} \sigma^2 \sum_n \sum_m \log E[\rho]_n - \log E[\rho]_m A_{nm} A_{mn} [E[\rho]_n - E[\rho]_m]$$

$$= \frac{1}{8} \sigma^2 \sum_n \sum_m |A_{nm}|^2 [\log E[\rho]_n - \log E[\rho]_m] [E[\rho]_n - E[\rho]_m] \geq 0 . \quad (22f)$$

(viii) Since Eqs. (21a) and (21c) are nonlinear in $\rho$, the Schrödinger dynamics described by them cannot be represented as an equivalent Heisenberg or dual dynamics on the operator $G$. On the other hand, Eq. (22a) is linear in $\rho$, and so as noted by Lindblad, the Schrödinger dynamics for $E[\rho]$ can be represented as a dual Heisenberg dynamics for $E[G]$, given by

$$\frac{dE[G]}{dt} = i[H,E[G]] + \frac{1}{8} \sigma^2 [A, [A,E[G]]] . \quad (22g)$$

(ix) The evolution of Eq. (21c) can be written (after some algebra, and again using $\rho^2 = \rho$) in the manifestly unitary form

$$\rho + d\rho = U\rho U^\dagger , \quad U = e^{dK} , \quad (23a)$$

with the infinitesimal anti-self-adjoint generator $dK$ given by

$$dK = \left[ -iH - \frac{1}{8} \sigma^2 [A^2 - 2A\rho A, \rho] \right] dt - \frac{1}{2} \sigma [\rho, A] dW_t . \quad (23b)$$

Equations (21c) and (23b) thus give the stochastic unitary extension of the Lindblad evolution of Eq. (22a) [14].

Specializing to the Hughston case $A = H$, Eq. (21c) (which uses the pure state condition $\rho^2 = \rho$) becomes

$$d\rho = -i[H,\rho]dt - \frac{1}{8} \sigma^2 [H, [H, \rho]] dt + \frac{1}{2} \sigma [\rho, [\rho, H]] dW_t , \quad (24a)$$
while Eq. (22a) becomes

$$\frac{dE[\rho]}{dt} = -i[H, E[\rho]] - \frac{1}{8}\sigma^2[H, [H, E[\rho]]], \quad (24b)$$

and the following further properties are evident:

(x) When $\rho = \Pi_e$, the projector on an energy eigenstate, then since all commutators in Eq. (24a) vanish we have $d\rho = 0$.

(xi) For $G$ commuting with $H$, $E[d(G)] = \text{Tr} G dE[\rho] = 0$, since by cyclic permutation inside the trace each term arising from substituting Eq. (24b) into the expectation of Eq. (20b) can be rearranged to have a factor $[G, H]$.

(xii) For $V = [\Delta H]^2 = \text{Tr} \rho H^2 - [\text{Tr} \rho H]^2$, use of Eq. (24a) and the Itô calculus imply that $E[dV/dt] = -E[R^2]$, with

$$R = \frac{1}{2}\sigma \text{Tr}[\rho, [\rho, H]] H = \sigma V. \quad (24c)$$

(xiii) Items (x) through (xii) are the ingredients used in Sec. III to prove state vector collapse to energy eigenstates $|e\rangle$ with the correct quantum mechanical probabilities $p_e$. Hence at large times, as noted by Hughston, $E[\rho] \to \sum_e p_e \Pi_e$, which explicitly exhibits the role of $E[\rho]$ as the density matrix that evolves, under the stochastic process, from a pure to a mixed state form. The fact that $E[\rho]$ for Hughston’s equation approaches a stationary limit at large times is in accord with the general stationarity discussion given in (vi) above.
VI. BEHAVIOR OF SYSTEMS CONSTRUCTED FROM INDEPENDENT SUBSYSTEMS

Let us next examine the structure of Hughston’s equation for a Hilbert space constructed as the direct product of independent subsystem Hilbert spaces, so that initially at time \( t = 0 \) the state vector is

\[
|z\rangle = \prod_\ell |z_\ell\rangle .
\]

(25a)

We assume the Hamiltonian

\[
H = \sum_\ell H_\ell ,
\]

(25b)

with \( H_\ell \) acting as the unit operator on the states \( |z_k\rangle , \ k \neq \ell \). Then a simple calculation shows that the expectation of the Hamiltonian \( (H) \) and its variance \( V \) are both additive over the subsystem Hilbert spaces,

\[
(H) = \sum_\ell (H_\ell)_\ell ,
\]

(25c)

\[
V = \sum_\ell V_\ell = \sum_\ell [(H^2_\ell)_\ell - (H_\ell)_\ell^2] ,
\]

with \( (F_\ell)_\ell \) the expectation of the operator \( F_\ell \) formed according to Eq. (1a) with respect to the subsystem wave function \( |z_\ell\rangle \). In addition, the Fubini-Study line element is also additive over the subsystem Hilbert spaces, since

\[
1 - ds^2 / 4 = \frac{|\langle z| z + dz\rangle|^2}{\langle z| z + dz\rangle \langle z + dz| z + dz\rangle} = \prod_\ell \frac{|\langle z_\ell| z_\ell + dz_\ell\rangle|^2}{\langle z_\ell| z_\ell + dz_\ell\rangle \langle z_\ell + dz_\ell| z_\ell + dz_\ell\rangle}
\]

(26)

\[
= \prod_\ell [1 - ds^2_\ell / 4] = 1 - \sum_\ell ds^2_\ell / 4 + O(ds^4) .
\]

[An alternative way to see this is to use the identity \( \log z \cdot z = \log \prod_\ell z_\ell \cdot z_\ell = \sum_\ell \log z_\ell \cdot z_\ell \) in Eq. (3b), along with a change of variable from \( z \) to the \( z_\ell \)'s.] As a result of Eq. (26), the metric \( g^{ab} \) and complex structure \( \Omega^{ab} \) block diagonalize over the independent subsystem subspaces.
Equations (25a)-(25c) then imply that Hughston’s stochastic extension of the Schrödinger equation given in Eq. (11a) separates into similar equations for the subsystems, that do not refer to one another’s $x^a$ coordinates, but are correlated only through the common Wiener process $dW_t$ that appears in all of them. These correlations result in the entanglement of the states $|z_\ell\rangle$, so that the product form of Eq. (25a) is not maintained for times $t > 0$, but subsystems $|z_\ell\rangle$ already in energy eigenstates remain unentangled for all time, since the coefficient of $dW_t$ vanishes in their stochastic evolution equations.

These same conclusions follow from the density matrix form of Hughston’s equation given in Eq. (24a), in which the entanglements arising from the action of the same Wiener process on all subsystems are already evident, because the density matrix depends quadratically on the normalized state vector. Considering for simplicity the case of two independent subsystems, substituting the $t = 0$ form

$$\rho = \rho_1\rho_2$$

into Eq. (24a), with $H = H_1 + H_2$, we get

$$d\rho = d\rho_1\rho_2 + \rho_1 d\rho_2 = \frac{1}{4}\sigma^2[H_1, \rho_1][H_2, \rho_2]dt \quad , (27a)$$

with $d\rho_1$ the evolution predicted by Eq. (24a) within subsystem 1,

$$d\rho_1 = -i[H_1, \rho_1]dt - \frac{1}{8}\sigma^2[H_1, [H_1, \rho]]dt + \frac{1}{2}\sigma[\rho_1, [\rho_1, H_1]]dW_t \quad , (27c)$$

and similarly for $d\rho_2$. The entangling term proportional to $[H_1, \rho_1][H_2, \rho_2]dt$ comes from the $[dW_t]^2$ contribution from the state vector evolution equation to the density matrix equation; it is in general nonzero, but vanishes when either $[H_1, \rho_1] = 0$ or $[H_2, \rho_2] = 0$, that is, when either of the two subsystems is in an energy eigenstate. When more than two subsystems are
present, the entangling term coupling $\rho_L$ to $\rho_\ell$, $\ell \neq L$ is more complicated in structure, but still has a factor $[H_L, \rho_L]$ and so vanishes when the subsystem $L$ is in an energy eigenstate. Thus the endpoints of the stochastic evolution under Hughston’s equation, which are the energy eigenstates, can persist indefinitely as unentangled independent subsystems in a larger system.

This conclusion does not extend to the more general evolution of Eq. (21c), in which the stochastic process is driven by an operator $A$ differing from the Hamiltonian, with $A$ taken to be additive over subsystems. The reason is that there is now a competition between the stochastic terms, which are constructed from double commutators with an innermost commutator $[A, \rho]$, and the Schrödinger evolution term, which involves the commutator $[H, \rho]$; the stochastic terms tend to drive the system to $A$ eigenstates, while the Schrödinger term coherently mixes $A$ eigenstates, leading to evolution away from $A$ eigenstates. Thus, a subsystem cannot remain indefinitely in an $A$ eigenstate, and as a result does not persist indefinitely as an unentangled independent subsystem in a larger system. [These statements are in accord with the conclusions reached in the stationarity discussion of (vi) in Sec. V.]

VII. DOES AN ENERGY-BASED EQUATION SUFFICE?

In the preceding sections we have seen how Hughston’s equation fits into the general framework of stochastic modifications of the Schrödinger equation that have been studied in the past. Its distinguishing feature is that the general operator $A$ of Eqs. (17a, b) and (21a, c) driving the stochastic terms is chosen, in Hughston’s case, to be the Hamiltonian $H$. This choice confers the advantage that the proof of reduction of the state vector to $A$ eigenstates (i.e., in Hughston’s case, to energy eigenstates) with the correct quantum
mechanical probabilities becomes exact, since it is not necessary to neglect the Hamiltonian evolution term. Moreover, since for Hughston’s equation the stochastic expectation of the Hamiltonian operator $E[H]$ is conserved in time, and since convergence to $H$ eigenstates preserves the quantum mechanical predictions, any statistical test of energy conservation performed on the endpoint of the stochastic process will agree with the quantum mechanical prediction. To justify these advantages, we must now address the issue of whether an energy-based stochastic equation is sufficient to give an objective interpretation of state vector reduction [15].

First, we must deal with the objection that in most measurements, the quantum attribute being measured is not an energy; for example, in a Stern-Gerlach experiment, it is typically the $z$ component of a spin. However, to perform a measurement, it is always necessary to couple the quantum attribute being measured to the apparatus through an interaction energy term $H_I$, in such a way that the macroscopic state of the apparatus is ultimately determined by the quantum attribute being measured. Thus, in the first instance, what is being measured is an energy, even though after amplification to macroscopic scale this can be converted to other forms of indication, such as pointer displacements. So from the point of view of the variety of quantum attributes that can be measured, Hughston’s equation appears to be as viable as localizing approaches [8] in which $A$ is chosen as an operator that produces spatial localization.

We must next deal with the issue of whether an energy-based approach can prevent the occurrence of macroscopic quantum superpositions. For example, take a macroscopic object and displace it a macroscopic distance; the two states have the same energy, and so in Hughston’s approach such superpositions would appear to be allowed, whereas in localizing
approaches they are strongly forbidden. However, this objection neglects the interactions of the macroscopic object with its environment, of the same type that are important in studies of decoherence. When such effects are taken into account, macroscopic displacement of a macroscopic object results in an energy shift $\Delta E$, reflecting the altered environment, which is sufficient, from the point of view of Hughston’s equation, to lead to rapid state vector reduction to one displaced alternative or the other. To study this quantitatively, let us consider the following two environmental effects: (i) thermal energy fluctuations, and (ii) the surface adsorption of surrounding molecules. Hughston proposes, as have other authors [16], that the parameter governing the stochastic terms is of order $\sigma \sim M_{\text{Planck}}^{-1/2}$ in microscopic units with $\hbar = c = 1$, which he shows leads to state vector reduction in a time $t_R$ given by

$$t_R \sim \left( \frac{2.8\text{MeV}}{\Delta E} \right)^2 \text{sec}.$$  \hspace{1cm} (28)

Hence to get a reduction time of order, say, $10^{-6}$ seconds, one needs a $\Delta E \sim 3\text{GeV} \sim 3$ nucleon masses.

Considering first the effect of thermal fluctuations, let us consider a macroscopic object with $N \sim 10^{23}$ nucleon masses, so that $\Delta E \sim N^{\frac{1}{4}}kT \sim 8\text{GeV}$ at room temperature (300° Kelvin) and $\Delta E \sim .08\text{Gev}$ at the 3° temperature of the cosmic microwave background. For such an object, thermal energy driven state vector reduction will occur in $10^{-7}$ seconds at room temperature and in $10^{-3}$ seconds at the temperature of the microwave background. Examining next the effect of adsorbed molecules, consider an object with a surface area of 1 cm$^2$ at room temperature in an extreme vacuum of $10^{-14}$Torr (less [17] than the nighttime pressure at the surface of the moon.) Then the flux of molecules bombarding its surface is [17] $4 \times 10^6$ per second, so assuming a high probability for the molecules to stick, a $\Delta E$ of 3GeV is attained in of order $10^{-6}$ seconds, permitting a $10^{-6}$ second state vector reduction.
time driven by the change in energy produced by surface adsorption. One can scale to other sizes of macroscopic object from these examples, but they suffice to show that in the normal range of laboratory operating conditions for measuring apparatus, environmental interactions produce a large enough spread of energy values to give rapid state vector reduction through an energy driven stochastic equation.

From a formal point of view, it is instructive to cast the above discussion of environmental effects in terms of the analysis of the measurement process given by Zurek [18], starting from Eq. (24b) for the evolution of the stochastic expectation of the density matrix. Zurek assumes that the total Hamiltonian $H$ describes the system $S$ being measured, the apparatus $A$ doing the measuring, and the environment $E$. Thus, he writes the Hamiltonian as a sum of 6 terms,

$$H = H_S + H_A + H_E + H_{SA} + H_{AE} + H_{SE}, \quad (29)$$

with the first three terms giving the Hamiltonians of the system, apparatus, and environment in isolation from one another, and with the second three terms giving the corresponding interaction Hamiltonians. Zurek assumes that the interaction $H_{SE}$ between system and environment can be neglected, and that the interaction $H_{SA}$ between system and apparatus acts only briefly while entanglement of the system and apparatus states is established, but is unimportant during the subsequent evolution of the density matrix that results in the actual measurement. He also makes the simplifying assumption that the states which actually distinguish between quantities being measured have equal eigenvalues of the non-interaction part of the Hamiltonian $H_S + H_A + H_E$, which implies that for the submatrix of $E[\rho]$ spanned by these states, the commutator $[H_S + H_A + H_E, E[\rho]]$ is zero, and so these commutator terms
in Eq. (24b) can be neglected. With these simplifications, Eq. (24b) becomes

$$\frac{dE[\rho]}{dt} = -i[H_{AE}, E[\rho]] - \frac{1}{8}\sigma^2[H_{AE}, [H_{AE}, E[\rho]]] ,$$

(30a)

or when the non-Schrödinger term is omitted, as in Zurek’s analysis,

$$\frac{dE[\rho]}{dt} = -i[H_{AE}, E[\rho]] .$$

(30b)

Zurek points out that the evolution of Eq. (30b) introduces correlations between the apparatus and the environment, which select as the “pointer basis” of the apparatus, that registers the measurement, the eigenstates $|A_p\rangle$ of a “pointer observable” $\hat{\Pi}$ that commutes with $H_{AE}$; in other words, the pointer basis projectors $\Pi_p = |A_p\rangle\langle A_p|$ must satisfy

$$[\Pi_p, H_{AE}] = 0 .$$

(30c)

Returning to the full evolution equation of Eq. (30a), with the non-Schrödinger terms included, we see that the argument of Sec. III, when applied to this equation using Eq. (30c), implies state vector collapse to the eigenstates of the Zurek pointer basis. Thus an energy-based stochastic reduction equation, when analyzed within the framework of Zurek’s approximations, is consistent with, and adds further support to, the picture of the measurement process that Zurek proposes in [18].

In addition to the issues just discussed, there are further questions that must be addressed in an energy-based approach, such as whether Hughston’s estimated $\sigma$ gives sufficiently rapid (but also not too rapid) reduction of state vectors for all classes of experiments that have been carried out. Answering this question is beyond the scope of the present paper, but is an important issue for future study. Ultimately, the decision between an energy-based or localization-based approach (or yet some other choice of the operator $A$ driving the
stochastic terms) may depend on which form of the modified Schrödinger equation can be derived as an approximation to relativistically invariant physics at a deeper level.

To summarize, we have shown that Hughston’s stochastic extension of the Schrödinger equation has properties that make it a viable physical model for state vector reduction. This opens the challenge of seeing whether it can be derived as a phenomenological approximation to a fundamental pre-quantum dynamics, along the lines of existing work on open dynamical systems [19]. Specifically, we suggest that since Adler and Millard [20] have argued that quantum mechanics can emerge as the thermodynamics of an underlying non-commutative operator dynamics, and since the corrections to the thermodynamic approximation in this dynamics are driven by the trace of the energy operator multiplied by a coefficient parameter with dimensions of inverse mass, it may be possible to show that Hughston’s stochastic process is the leading statistical fluctuation correction to this thermodynamics.

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\[
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\]
which when \( V_j^\dagger = V_j \) reduces to
\[
\frac{d\rho}{dt} = -i[H, \rho] - \frac{1}{2} \sum_j [V_j, [V_j, \rho]]
\]
corresponding to the structure of Eq. (22a). The positivity results of (vi) and (vii) below are special to the case of self-adjoint \( V_j \), and do not extend to the general Lindblad-type evolution with \( V_j^\dagger \neq V_j \).

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