A NOTE ON THE SMOOTHING PROBLEM IN CHOW’S THEOREM

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The authors offer the present paper to Prof. Giorgio Fusco for many years of collaboration and friendship.

Abstract. This paper concerns a solution of the smoothing problem in Chow-Rashevskii’s connectivity theorem proposed in [1].

§1. INTRODUCTION AND OBJECTIVES

Let \( M \) be a finite dimensional paracompact smooth manifold endowed with a smooth linear subbundle \( \mathcal{D} \) of \( TM \). The well-known Chow-Rashevskii’s connectivity theorem (see [2] and generalizations by P. Stefan in [3, 4] and by H. Sussmann in [7]) asserts that, if \( \mathcal{D} \) is bracket-generating, any two points in the same connected component of \( M \) may be connected by a sectionally smooth path tangent to \( \mathcal{D} \). The question of whether or not any two points in \( M \) may be connected by a smooth

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horizontal immersion was posed by R. Bryant and L. Hsu in [1] and
affirmatively answered by M. Gromov in [3], who named the problem
as “the smoothing problem in Chow’s theorem”.

The purpose of this note is to present an alternative approach to
Gromov’s solution by means of a method that, to our taste, seems t o
be more geometrically intuitive. Besides, it conveys some additional
information on the connectivity problem: we prove in theorem 2 and
its corollary 3 that, if the distribution \( D \) is bracket-generating, any
two points in a connected open set \( U \subset M \) may be connected on \( U \)
by a smooth horizontal \( C^1 \)-immersion with arbitrary given initial and
final velocities in \( D \). Our method is quite simple: given \( p,q \in U \),
\( v_p \in D_p \setminus \{0\} \) and \( v_q \in D_q \setminus \{0\} \), we apply the orbit theorem to
show that \( v_p \) and \( v_q \) may be connected on \( (D|_U)^* \) (i.e. \( D|_U \) with the
zero section removed) by means of a sectionally smooth curve whose
smooth arcs are integral curves of second order vector fields on \( D \),
i.e. local smooth sections of \( \tau_D : T \mathcal{D} \to \mathcal{D} \) whose integral curves are
lifts of smooth curves on \( M \). It then follows that the projection on
\( M \) of this sectionally smooth curve is a horizontal \( C^1 \) immersed curve
connecting \( p \) and \( q \) on \( U \), whose initial and final velocities coincide
with \( v_p \) and \( v_q \), respectively. This method may also be applied in case
the linear subbundle \( \mathcal{D} \) is not bracket-generating: we prove in theorem
3 that, if \( \mathcal{D} \) satisfies Sussmann’s necessary and sufficient condition for
reachability given in theorem 7.1 of [7], then any two points in the same
connected component of \( M \) may be connected by smooth horizontal \( C^1 \)-
immersion with arbitrary given initial and final velocities in \( \mathcal{D} \).

§2. PRELIMINARIES AND NOTATION

1. Smooth Distributions

We denote the tangent bundle of a finite dimensional paracompact
smooth manifold \( M \) by \( \tau_M : TM \to M \). Following the notation and
definitions in [7], a distribution \( \mathcal{D} \) on \( M \) is a family \( \{ \mathcal{D}_x \}_{x \in M} \) of linear
subspaces of each fiber of the tangent bundle \( \tau_M : TM \to M \). The
distribution \( \mathcal{D} \) is called smooth if \( \mathcal{D}_x \) varies smoothly with \( x \in M \), in
the sense that there exists a set \( \mathcal{D} \) of locally defined smooth vector fields
on \( M \) such that, for each \( x \in M \), \( \mathcal{D}_x = \text{span} \{ V(x) \mid V \in \mathcal{D}, x \in \text{dom} V \} \)
(with the convention that the linear span of the empty set is \( \{0\} \)). If
that is the case, we say that the smooth distribution \( \mathcal{D} \) is generated
by \( \mathcal{D} \). Equivalently, and perhaps more naturally, the distribution \( \mathcal{D} \)
is smooth if there exists a subsheaf \( \mathcal{D} \) of the sheaf \( \mathcal{C}^\infty_{TM} \) of germs of
smooth sections of \( TM \) (considered as a sheaf of \( \mathcal{C}^\infty(M) \)-modules) such
that, for each \( x \in M \), \( \mathcal{D}_x = \{ V(x) \mid V \in \mathcal{D}_x \} \) (where \( \mathcal{D}_x \) denotes the
stalk of \( \mathcal{D} \) over \( x \)). We avoid, however, the use of sheaves, in order to

\(^1\) smooth in this paper means “\( \mathcal{C}^\infty \)”
keep the notation and formalism compatible with that of [7] and [5], [6]. Note that the rank of $D_x$ depends on $x$, i.e. $D$ need not be a linear subbundle of $TM$ (but we do assume that as a hypothesis for our main results). If $D$ is a set of locally defined smooth vector fields on $M$, we denote by $[D]$ the smooth distribution generated by $D$.

We say that $V$ is a (local) smooth section of a smooth distribution $D$ if it is a smooth (local) section of $\tau_M : TM \to M$ defined on an open set $U \subset M$ such that, for all $x \in U$, $V(x) \in D_x$. We denote the set of such local smooth sections by $\Gamma^\infty_{loc}(D)$; it is clear that the smooth distribution $D$ is generated by $\Gamma^\infty_{loc}(D)$.

Given two locally defined smooth vector fields on $M$, their Lie bracket is a well-defined smooth vector field on the intersection of their domains. We say that a set of locally defined smooth vector fields $D$ on $M$ is involutive if it is closed by the operation of taking Lie brackets. Any set $D$ of locally defined smooth vector fields on $M$ is contained in a smallest involutive set of locally defined smooth vector fields on $M$, which we denote by $D^*$. Indeed, the family $F$ of all involutive sets of locally defined smooth vector fields containing $D$ is nonempty (since $\Gamma^\infty_{loc}(TM)$ is such a set) and $\cap F$ does the work. We say that a smooth distribution $D$ on $M$ is involutive if so is $\Gamma^\infty_{loc}(D)$.

We say that a smooth distribution $D$ on $M$ is bracket-generating if the smooth distribution generated by $\Gamma^\infty_{loc}(D)^*$ coincides with $TM$.

2. Orbits of Local Groups of Diffeomorphisms and Distributions

A local group of diffeomorphisms $G$ on $M$ is a set of smooth diffeomorphisms defined on open subsets of $M$ which is closed under compositions and under taking inverses, i.e. if $\phi : U \to V$ and $\psi : U' \to V'$ belong to $G$, then both $\phi^{-1} : V \to U$ and $\psi \circ \phi : \phi^{-1}(U' \cap V) \to \psi(U' \cap V)$ belong to $G$ (note that the diffeomorphism with empty domain, that is, the empty set, is allowed). If $G$ is a set of locally defined smooth diffeomorphisms on $M$, there exists a smallest local group of diffeomorphisms $G_*$ which contains $G$: we take the intersection $\cap F$ of the family $F$ of all local groups of diffeomorphisms which contain $G$ (note that $F$ is nonempty, since the set of all locally defined diffeomorphisms on $M$ is such a local group). We call $G_*$ the local group of diffeomorphisms generated by $G$.

Let $G$ be a local group of diffeomorphisms on $M$. We define an equivalence relation on $M$ by $x \sim y$ if $x = y$ or if there exists $\phi \in G$ such that $x \in \text{dom } \phi$ and $\phi(x) = y$. The equivalence classes of this relation are called orbits of $G$. Note that, if $x \in M$ and there is no $\phi \in G$ such that $x \in \text{dom } \phi$, the orbit of $x$ is the singleton of $x$. If $G$ is a set of locally defined smooth diffeomorphisms on $M$, we define the orbits of $G$ as the orbits of $G_*$. 
Given a locally defined smooth vector field $X$ on $M$, we denote by $(X_t)_{t \in \mathbb{R}}$ the local one-parameter group of diffeomorphisms associated with $X$. If $\mathcal{D}$ is a set of locally defined smooth vector fields on $M$, we denote by $\Theta \mathcal{D}$ the set of locally defined smooth diffeomorphisms on $M$ given by

$$\Theta \mathcal{D} = \bigcup_{X \in \mathcal{D}, t \in \mathbb{R}} X_t,$$

and by $\Psi \mathcal{D}$ the local group of diffeomorphisms on $M$ generated by $\Theta \mathcal{D}$, i.e. the set of all finite compositions of local diffeomorphisms in $\Theta \mathcal{D}$ (we are borrowing here the notation from [5],[6]). We define the orbits of $\mathcal{D}$ as the orbits of $\Theta \mathcal{D}$. If $\mathcal{D}$ is a smooth distribution on $M$, we define the orbits of $\mathcal{D}$ as the orbits of $\Gamma_\infty^{loc}(\mathcal{D})$.

We say that a smooth distribution $\mathcal{D}$ on $M$ is invariant by a local group of diffeomorphisms $G$ on $M$ if, for each $x \in M$, each $v \in \mathcal{D}_x$ and each $\phi \in G$ such that $x \in \text{dom } \phi$, we have $\phi_* v \in \mathcal{D}_{\phi(x)}$, where $\phi_*$ denotes the tangent map of $\phi$. We say that a smooth distribution $\mathcal{D}$ on $M$ is invariant by a set $\mathcal{G}$ of locally defined smooth diffeomorphisms on $M$ if it is invariant by $\mathcal{G}_*$. We say that $\mathcal{D}$ is invariant by a set $\mathcal{D}$ of locally defined smooth vector fields on $M$ if $\mathcal{D}$ is invariant by $\Psi \mathcal{D}$.

Given $\mathcal{D}$ and $\mathcal{D}'$ distributions on $M$, we say that $\mathcal{D} \subset \mathcal{D}'$ if, for all $x \in M$, $\mathcal{D}_x \subset \mathcal{D}'_x$.

Given a smooth distribution $\mathcal{D}$ on $M$ and a local group of diffeomorphisms $G$ on $M$, there exists a smallest smooth distribution $\mathcal{D}^G$ on $M$ which contains $\mathcal{D}$ and is invariant by $G$: if $\mathcal{D}$ is generated by the set of locally defined smooth vector fields $\mathcal{D}$, $\mathcal{D}^G$ is the distribution generated by the set of locally defined smooth vector fields $\{\phi_* X \mid \phi \in G, X \in \mathcal{D}\}$, where $\phi_* X$ denotes the pushforward of $X$ by $\phi$ (which is a locally defined smooth vector field on $M$). Consequently, if $\mathcal{D}$ is a set of locally defined smooth vector fields on $M$, there exists a smallest smooth distribution $P_\mathcal{D}$ (this time we are borrowing the notation from [7]) on $M$ which contains the distribution $[\mathcal{D}]$ generated by $\mathcal{D}$ and which is invariant by $\mathcal{D}$, i.e. it is invariant by $\Psi \mathcal{D}$. The smooth distribution $P_\mathcal{D}$ is generated by $\{\phi_* X \mid \phi \in \Psi \mathcal{D}, X \in \mathcal{D}\}$.

We can finally enunciate a version of the so-called orbit theorem. The following statement is a subset of the more general statements contained in [7] (Theorem 4.1) and [3] (Theorems 1 and 5).

**Theorem 1 (orbit theorem).** Let $M$ be a finite dimensional paracompact smooth manifold and $\mathcal{D}$ a set of locally defined smooth vector fields on $M$. Then each orbit $S$ of $\mathcal{D}$ is an immersed smooth submanifold of $M$ such that, for each $x \in S$, the tangent space of $S$ at $x$ coincides with $P_\mathcal{D}(x)$.

It was actually proved in [5] that each orbit $S$ of $\mathcal{D}$ admits a unique smooth manifold structure which turns it into a leaf of $M$, i.e. a smooth immersed submanifold with the property that, for each locally connected topological space $N$ and each continuous map $f : N \to M$ with
image contained in \( S \), the induced map \( f : N \to S \) is continuous. Besides, the partition of \( M \) determined by the orbits of \( S \) is a *foliation with singularities* (cf. definition on page 700 of \([5]\)). In particular, \( P_D \) is an involutive distribution (that was also proved in \([7]\)). It then follows that (recall that \( D_* \) denotes the smallest involutive subset of locally defined smooth vector fields on \( M \) containing \( D \)) we have inclusions

\[
[D] \subset [D_*] \subset P_D.
\]

Indeed, the first inclusion is clear, and the second inclusion follows from the inclusion \( D_* \subset \Gamma^\infty_{\text{loc}}(P_D) \) (since, by the involutiveness of the distribution \( P_D \), \( \Gamma^\infty_{\text{loc}}(P_D) \) is an involutive set of locally defined smooth vector fields containing \( D \), hence it must contain \( D_* \)) and from the fact that \( P_D \) is generated by \( \Gamma^\infty_{\text{loc}}(P_D) \). We therefore conclude that, if \( \mathcal{D} \) is a smooth bracket-generating distribution on \( M \) and \( D = \Gamma^\infty_{\text{loc}}(\mathcal{D}) \), then

\[
[D_*] = P_D = TM.
\]

In particular, if \( M \) is connected, \( \mathcal{D} \) admits a unique orbit which coincides with \( M \). We have thus proved the following version of Chow-Rashevskii’s connectivity theorem. We say that a sectionally smooth curve on \( M \) is *horizontal* with respect to \( \mathcal{D} \) if all of its tangent vectors belong to \( \mathcal{D} \).

**Corollary 1 (Chow-Rashevskii).** Let \( M \) be a finite dimensional paracompact connected smooth manifold and \( \mathcal{D} \) a smooth bracket-generating distribution on \( M \). Then \( M \) is \( \mathcal{D} \)-connected, i.e. any two points in \( M \) may be connected by a sectionally smooth curve on \( M \) horizontal with respect to \( \mathcal{D} \).

The converse to Chow-Rashevskii’s theorem fails, i.e. the bracket-generating condition is not necessary for \( \mathcal{D} \)-connectivity (see \([4]\), page 24).

A necessary and sufficient condition for \( \mathcal{D} \)-connectivity may be obtained as a direct consequence of the following corollary of theorem \([1]\) (cf. theorem 7.1 in \([7]\)).

**Corollary 2 (Sussmann’s condition for \( \mathcal{D} \)-connectivity).** Let \( M \) be a finite dimensional paracompact connected smooth manifold and \( \mathcal{D} \) a set of locally defined smooth vector fields on \( M \). Then \( M \) is \( \mathcal{D} \)-connected (i.e. \( M \) is an orbit of \( \mathcal{D} \)) if, and only if,

\[
P_D = TM.
\]

3. Fiber and Parallel Derivatives

Our last ingredient is a computational tool. Given a smooth linear subbundle \( \mathcal{D} \) of \( TM \), we shall need to compute Lie brackets of vector fields in \( \mathfrak{x}(\mathcal{D}) \). That could be accomplished by means of local charts on \( M \) and local trivializations of the vector bundle \( \pi_\mathcal{D} : \mathcal{D} \to M \), but in that case the computations we need to perform become rapidly
messy. Instead, we compute by means of a method introduced in [8] and summarized below.

Let \( \pi_E : E \to M \) be a smooth vector bundle over \( M \) endowed with a connection \( \nabla^E : \mathfrak{X}(M) \times \Gamma^\infty(E) \to \Gamma^\infty(E) \). The connection \( \nabla^E \) defines a horizontal subbundle \( \text{Hor}(E) \) of \( TE \), where \( (\forall v_q \in E) \text{Hor}_v_q(E) \) is the image of the horizontal lift at \( v_q \), \( \text{Hor}_v_q : T_qM \to T_vqE \), defined by \( w_q \mapsto TV \cdot w_q \), where \( T \) denotes the tangent map and \( V \) is any smooth local section of \( \pi_E : E \to M \) defined on an open neighborhood of \( q \) such that \( V(q) = v_q \) and \( \nabla_{\text{Hor}}^E V = 0 \). The horizontal lift \( \text{Hor}_v_q : T_qM \to T_vqE \) is therefore a linear isomorphism onto \( \text{Hor}_{v_q}(E) \) whose inverse is the restriction of the tangent map \( T\pi_E \) to \( \text{Hor}_{v_q}(E) \). Denoting by \( \text{Ver}(E) \) the vertical subbundle of the tangent bundle \( TE \), we thus have a Whitney sum decomposition
\[
TE = \text{Hor}(E) \oplus \text{Ver}(E).
\]

The connector \( \kappa_E : TE \to E \) associated to the connection is given by \( X_{v_q} \in T_{v_q}E \mapsto P_V(X_{v_q}) \in \text{Ver}_{v_q}(E) \) (where \( P_V \) is the projection on the vertical subbundle induced by the Whitney sum decomposition above) followed by the inverse of the vertical lift \( \lambda_{v_q} : E_q \to \text{Ver}_{v_q}(E) \) at \( v_q \) (which is the canonical linear isomorphism \( E_q \equiv T_{v_q}(E_q) = \text{Ver}_{v_q}(E) \)). Note that, with these definitions:

1) for all \( X_{v_q} \in TE \), \( X_{v_q} = \text{Hor}_{v_q}(T\pi_E \cdot X_{v_q}) + \lambda_{v_q}(\kappa_{v_q} \cdot X_{v_q}) \);

2) for all \( w_q \in TM \), for all \( V \) smooth local section of \( \pi_E : E \to M \) defined on an open neighborhood of \( q \), we have \( \nabla^E_{w_q} V = \kappa_{v_q} \cdot TX \cdot w_q \in E_q \).

Next, we consider two smooth vector bundles \( \pi_E : E \to M \) and \( \pi_F : F \to N \) over paracompact smooth manifolds \( M \) and \( N \), respectively, and \( b : E \to F \) be a morphism of smooth fiber bundles (i.e. it preserves fibers and is smooth) over \( b : M \to N \). We denote by \( \mathbb{F}b : E \to L(E,b^*F) \) the fiber derivative of \( b \), i.e. the morphism of smooth fiber bundles defined by, for all \( v_q, w_q \in E_q \), \( \mathbb{F}b(v_q) \cdot w_q = \kappa^V_F \cdot Tb \cdot \lambda_{v_q}(w_q) \in F_{b(q)} \), where \( \kappa^V_F \) denotes the restriction of the connector \( \kappa_F \) to the vertical subbundle (that is, \( \kappa^V_F \) is the inverse of the vertical lift). We don’t need the connections to define the fiber derivative; what we need them for is to define the parallel derivative \( \mathbb{P}b : E \to L(TM,b^*F) \). That is a smooth fiber bundle morphism defined by, for all \( v_q \in E \) and all \( z_q \in T_qM \),
\[
\mathbb{P}b(v_q) \cdot z_q = \kappa_F \cdot Tb \cdot H_{v_q}(z_q) \in F_{b(q)}.
\]

The idea in considering these fiber and parallel derivatives is to provide a coordinate-free technique to compute the tangent map of \( b \), allowing its computation at a given element of \( TE \) in terms of its vertical and horizontal components, so that they play a role of “partial derivatives”. That is to say, for all \( X_{v_q} \in TE \), the following formulæ
hold:
\[ T\pi_F \cdot T\tilde{b} \cdot X_{v_q} = T\tilde{b} \cdot T\pi_E \cdot X_{v_q} \]
\[ \kappa_F \cdot T\tilde{b} \cdot X_{v_q} = \mathbb{P}b(v_q) \cdot \kappa_E \cdot X_{v_q} + \mathbb{P}b(v_q) \cdot T\pi_E \cdot X_{v_q}. \]

We finally come back to our initial setting, i.e. take \( M \) a finite dimensional paracompact smooth manifold endowed with a smooth linear subbundle \( \mathcal{D} \) of \( TM \). We fix an auxiliary Riemannian metric tensor \( g \) on \( M \), which induces a Whitney sum decomposition \( TM = \mathcal{D} \oplus_M \mathcal{D}^\perp \). We denote by \( P : TM \rightarrow \mathcal{D} \) the projection on the first factor determined by this Whitney sum, and by \( \nabla^{\mathcal{D}} : \mathfrak{X}(M) \times \Gamma^\infty(\mathcal{D}) \rightarrow \Gamma^\infty(\mathcal{D}) \) the connection on the vector bundle \( \pi_{\mathcal{D}} : \mathcal{D} \rightarrow M \) given by

\[ \nabla^{\mathcal{D}}_X Y := P\nabla_X Y, \]

where \( \nabla \) is the Levi-Civita connection of \((M, g)\). Thus, both vector bundles \( \tau_M : TM \rightarrow M \) and \( \pi_{\mathcal{D}} : \mathcal{D} \rightarrow M \) are endowed with connections \( \nabla \) (Levi-Civita) and \( \nabla^{\mathcal{D}} \), with respective connectors and horizontal lifts denoted by \( \kappa, H_v \) and \( \kappa_{\mathcal{D}}, H^v_{\mathcal{D}} \). With respect to these connections, the Lie bracket \([X, Y]\) of (possibly locally defined) smooth vector fields, \( X, Y \in \mathfrak{X}(\mathcal{D}) \) was computed in proposition 1 of [S] by means of the following formulae, given \( v_q \in \text{dom} \ X \cap \text{dom} \ Y \):

\[
\kappa_{\mathcal{D}} \cdot [X, Y](v_q) = \mathbb{F}(\kappa_{\mathcal{D}} \circ Y)(v_q) \cdot \kappa_{\mathcal{D}} \cdot X(v_q) + \mathbb{F}(\kappa_{\mathcal{D}} \circ Y)(v_q) \cdot T\pi_{\mathcal{D}} \cdot X(v_q) -
\]
\[
- \mathbb{F}(\kappa_{\mathcal{D}} \circ X)(v_q) \cdot \kappa_{\mathcal{D}} \cdot Y(v_q) - \mathbb{P}(\kappa_{\mathcal{D}} \circ X)(v_q) \cdot T\pi_{\mathcal{D}} \cdot Y(v_q) +
\]
\[
+ \mathbb{R}^{\mathcal{D}}(T\pi_{\mathcal{D}} \cdot Y(v_q), T\pi_{\mathcal{D}} \cdot X(v_q)) \cdot v_q,
\]
\[
T\pi_{\mathcal{D}} \cdot [X, Y](v_q) = \mathbb{F}(T\pi_{\mathcal{D}} \circ Y)(v_q) \cdot \kappa_{\mathcal{D}} \cdot X(v_q) + \mathbb{P}(T\pi_{\mathcal{D}} \circ Y)(v_q) \cdot T\pi_{\mathcal{D}} \cdot X(v_q) -
\]
\[
- \mathbb{F}(T\pi_{\mathcal{D}} \circ X)(v_q) \cdot \kappa_{\mathcal{D}} \cdot Y(v_q) - \mathbb{P}(T\pi_{\mathcal{D}} \circ X)(v_q) \cdot T\pi_{\mathcal{D}} \cdot Y(v_q),
\]

where \( \mathbb{R}^{\mathcal{D}} \) is the curvature tensor of \( \nabla^{\mathcal{D}} \).

We shall need the formulae above in the particular case in which: 1) \( X \) is the nonholonomic vector field \( X_{\mathcal{D}} \) of \((M, g, \mathcal{D})\), i.e. the vector field given by

\[ X_{\mathcal{D}}(v_q) = H^v_{\mathcal{D}}(v_q) = TP \cdot S(v_q), \]

where \( S \) is the geodesic spray of \((M, g)\); 2) \( Y \) is an arbitrary (locally defined) smooth vertical vector field. In this case, the above formulae simplify to, for all \( v_q \in \text{dom} \ Y \):

\[ \kappa_{\mathcal{D}} \cdot [X_{\mathcal{D}}, Y](v_q) = \mathbb{P}(\kappa_{\mathcal{D}} \circ Y)(v_q) \cdot v_q \]
\[ T\pi_{\mathcal{D}} \cdot [X_{\mathcal{D}}, Y](v_q) = -\kappa_{\mathcal{D}} \cdot Y(v_q). \]

\section{Statement and Proof of the Main Results}

\textbf{Theorem 2 (Smoothing in Chow’s Theorem).} \textit{Let} \( M \) \textit{be a finite dimensional paracompact connected smooth manifold endowed with a smooth linear subbundle} \( \mathcal{D} \) \textit{of} \( TM \). \textit{If} \( \mathcal{D} \) \textit{is bracket-generating, then any two points in} \( M \) \textit{may be connected by a horizontal curve which is both a}
**C^1 immersion and sectionally smooth, with arbitrary given initial and final velocities in \( D \).**

**Proof.** It suffices to consider the case \( \dim M \geq 2 \), otherwise the thesis is trivial. Then, since \( D \) is bracket-generating, we must have \( \text{rk } D \geq 2 \); it then follows that the slit bundle \( D^* \) (i.e. \( D \) with the zero section removed) is a connected open submanifold of \( D \) (the fact that it is connected is a consequence of being the total space of a fiber bundle with fibers and base connected). We may apply the orbit theorem \(^{11}\) to the paracompact connected smooth manifold \( D^* \) endowed with the set \( D \) of locally defined smooth second order vector fields on \( D^* \), i.e. (noting that \( T(D^*) = T|_{D^*} \))

\[
D = \{ X \in \Gamma^\infty_{\text{loc}}(T|_{D^*}) \mid \forall v_q \in \text{dom } X, \pi_{D^*} \cdot X(v_q) = v_q \}.
\]

We contend that \( P_D = T|_{D^*} \). Once we prove this contention, we conclude that each orbit of \( D \) is a connected open submanifold of \( D^* \), which implies, due to the connectedness of \( D^* \), that \( D^* \) is the only orbit of \( D \). That is to say, given \( p, q \in M \) and \( v_p \in D \setminus \{0\} \), \( v_q \in D^* \setminus \{0\} \), there exists a sectionally smooth curve in \( D^* \) connecting \( v_p \) to \( v_q \), whose smooth arcs are integral curves of vector fields in \( D \), i.e. of second order vector fields. The projection on \( M \) of this sectionally smooth curve connects \( p \) to \( q \), with initial velocity \( v_p \) and final velocity \( v_q \), and it is both a sectionally smooth and a \( C^1 \)-immersed horizontal curve on \( M \). By the arbitrariness of \( p, q \) taken in \( M \) and of the initial and final velocities in \( D^* \), we have thus reached the thesis.

It remains, therefore, to prove our contention, i.e. that \( P_D = T|_{D^*} \). Given \( v_q \in D^* \), we must prove that \( P_D(v_q) = T(v_q) \), which will be done along the steps below. We fix an auxiliary Riemannian metric tensor \( g \) on \( M \) and use the notation from subsection \(^3\) of the preliminaries.

1) Since any local smooth vertical vector field in \( \mathfrak{X}(D^*) \) may be written as a difference of two smooth second order vector fields, i.e. of two vector fields in \( D \subset \Gamma^\infty_{\text{loc}}(P_D) \), and since \( P_D \) is a smooth distribution, we conclude that any local smooth vertical vector field in \( \mathfrak{X}(D^*) \) is a smooth local section of \( P_D \), which implies that the vertical space \( \text{Ver}_{v_q}(D) \) is contained in \( P_D(v_q) \).

2) Let \( X_{\varphi} \) be the nonholonomic vector field of \( (M, g, D) \) (which is a second order vector field in \( \mathfrak{X}(D) \), so that its restriction to the open submanifold \( D^* \) belongs to \( D \)) and \( Y \) an arbitrary vertical smooth vector field in \( \mathfrak{X}(D^*) \) defined on an open neighborhood of \( v_q \). Then both \( X_{\varphi}|_{D^*} \) and \( Y \) are sections of \( P_D \); since the latter smooth distribution is involutive, we conclude that the Lie bracket \([X_{\varphi}, Y]\) is a section of \( P_D \). But, as we have computed in \(^{11}\),

\[
\pi_{D^*} \cdot [X_{\varphi}, Y](v_q) = -\kappa_{\varphi} \cdot Y(v_q).
\]

It then follows that the vector

\[
H_{v_q}(-\kappa_{\varphi} \cdot Y(v_q)) = [X_{\varphi}, Y](v_q) - \lambda_{v_q}(\kappa_{\varphi} \cdot [X_{\varphi}, Y](v_q)).
\]
It follows from the previous step and from the arbitrariness of the
fixed \( v_q \in \mathcal{D}^* \) that, for any smooth locally defined vector field \( X \in \Gamma^\infty_{\text{loc}}(\mathcal{D}) \), the horizontal lift \( \pi_{\text{Hor}}(w_q) = X(q) = X \cdot \pi_{\mathcal{D}}(w_q) \), i.e., the vector fields \( \pi_{\text{Hor}} \) and \( X \) are \( \pi_{\mathcal{D}} \)-related. Then so are the Lie brackets of vector fields of this form, i.e., if \( Y \) is another smooth locally defined vector field in \( \Gamma^\infty_{\text{loc}}(\mathcal{D}) \), the locally defined vector fields \([X_{\text{Hor}},Y_{\text{Hor}}]\) is a linear isomorphism onto \( \mathcal{D}^* \), and since the smooth vertical vector field \( Y \) in \( X(\mathcal{D}^*) \) on a neighborhood of \( v_q \) was arbitrarily taken, we conclude that

\[
H^\mathcal{D}_{v_q}(\mathcal{D}_q) \subset \mathcal{P}_D(v_q).
\]

3) It follows from the previous step and from the arbitrariness of the
fixed \( v_q \in \mathcal{D}^* \) that, for any smooth locally defined vector field \( X \in \Gamma^\infty_{\text{loc}}(\mathcal{D}) \), the horizontal lift \( \pi_{\text{Hor}}(w_q) = X(q) = X \cdot \pi_{\mathcal{D}}(w_q) \), i.e., the vector fields \( \pi_{\text{Hor}} \) and \( X \) are \( \pi_{\mathcal{D}} \)-related. Then so are the Lie brackets of vector fields of this form, i.e., if \( Y \) is another smooth locally defined vector field in \( \Gamma^\infty_{\text{loc}}(\mathcal{D}) \), the locally defined vector fields \([X_{\text{Hor}},Y_{\text{Hor}}]\) is a smooth local section of \( \mathcal{P}_D \). Moreover, for all \( w_q \in \mathcal{D}^* \), we have \( \mathcal{T}_\pi_{\mathcal{D}} \cdot X_{\text{Hor}}(w_q) = X(q) = X \cdot \pi_{\mathcal{D}}(w_q) \), i.e., the vector fields \( X_{\text{Hor}} \) and \( X \) are \( \pi_{\mathcal{D}} \)-related. Then so are the Lie brackets of vector fields of this form, i.e., if \( Y \) is another smooth locally defined vector field in \( \Gamma^\infty_{\text{loc}}(\mathcal{D}) \), the locally defined vector fields \([X_{\text{Hor}},X_{\text{Hor}}]\) is a smooth local section of \( \mathcal{P}_D \) defined on a neighborhood of \( v_q \) and the locally defined vector fields

\[
[\cdots[[X_{\text{Hor}},X_{\text{Hor}}],\cdots]X_{k-1},X_k]\text{ and }[\cdots[[X_1,X_2],\cdots]X_{k-1},X_k]
\]

are \( \pi_{\mathcal{D}} \)-related. It then follows that the vector

\[
H_{v_q}^\mathcal{D}([\cdots[[X_1,X_2],\cdots]X_{k-1},X_k])(q) = [\cdots[[X_1_{\text{Hor}},X_2_{\text{Hor}}],\cdots]X_{k-1}^{\text{Hor}},X_k^{\text{Hor}}](v_q) - \lambda_{v_q}(\pi_{\mathcal{D}} \cdot [\cdots[[X_1^{\text{Hor}},X_2^{\text{Hor}}],\cdots]X_{k-1}^{\text{Hor}},X_k^{\text{Hor}}](v_q))
\]

belongs to \( \mathcal{P}_D(v_q) \), since both vectors on the second member of the previous equality belong to that space. But, since \( \mathcal{D} \) is a bracket-generating distribution, we have

\[
\mathcal{T}_qM = \text{span}\{[\cdots[[X_1,X_2],\cdots]X_{k-1},X_k](q) \mid k \in \mathbb{N}, X_1, \ldots, X_k \in \Gamma^\infty_{\text{loc}}(\mathcal{D})\}.
\]

We finally conclude that \( \text{Hor}_{v_q}(\mathcal{D}) = H_{v_q}^\mathcal{D}(\mathcal{T}_qM) \subset \mathcal{P}_D(v_q) \). Thus, in view of step 1, we have

\[
\mathcal{T}_{v_q}\mathcal{D} = \text{Hor}_{v_q}(\mathcal{D}) \oplus \text{Ver}_{v_q}(\mathcal{D}) \subset \mathcal{P}_D(v_q),
\]

hence the equality holds in the above inclusion and our contention is proved.

\[\square\]
Corollary 3. Let $\mathcal{M}$ be a finite dimensional paracompact smooth manifold endowed with a smooth linear subbundle $\mathcal{D}$ of $\mathcal{T}\mathcal{M}$. If $\mathcal{D}$ is bracket-generating, then any two points belonging to a connected open subset $\mathcal{U} \subset \mathcal{M}$ may be connected by a horizontal curve in $\mathcal{U}$ which is both a $C^1$ immersion and sectionally smooth, with arbitrary given initial and final velocities in $\mathcal{D}$.

Proof. Apply the previous theorem with $\mathcal{U}$ in place of $\mathcal{M}$ and $\mathcal{D}|\mathcal{U}$ in place of $\mathcal{D}$. \hfill $\square$

We finally prove that the same smoothness property holds under Sussmann’s condition for $\mathcal{D}$-connectivity (corollary 2).

Theorem 3 (smoothness in Sussmann’s condition for $\mathcal{D}$-connectivity). Let $\mathcal{M}$ be a finite dimensional paracompact connected smooth manifold endowed with a smooth linear subbundle $\mathcal{D}$ of $\mathcal{T}\mathcal{M}$ such that $P_{\Gamma^\infty_{\text{loc}}}(\mathcal{D}) = \mathcal{T}\mathcal{M}$. Then any two points in $\mathcal{M}$ may be connected by a horizontal curve which is both a $C^1$ immersion and sectionally smooth, with arbitrary given initial and final velocities in $\mathcal{D}$.

Proof. As in the proof of theorem 2 it suffices to consider the case $\dim \mathcal{M} \geq 2$, otherwise the thesis is trivial. Then, since $P_{\Gamma^\infty_{\text{loc}}}(\mathcal{D}) = \mathcal{T}\mathcal{M}$, we must have $\operatorname{rk} \mathcal{D} \geq 2$, so that the slit bundle $\mathcal{D}^*$ is a connected open submanifold of $\mathcal{D}$. Once more we consider the paracompact connected smooth manifold $\mathcal{D}^*$ endowed with the set $\mathcal{D}$ of locally defined smooth second order vector fields on $\mathcal{D}^*$, i.e.

$$\mathcal{D} = \{ X \in \Gamma^\infty_{\text{loc}}(\mathcal{T}\mathcal{D}|_{\mathcal{D}^*}) \mid \forall v_q \in \text{dom} \ X, T\pi_{\mathcal{D}^*} \cdot X(v_q) = v_q \}. $$

We contend that $P_{\mathcal{D}} = T\mathcal{D}|_{\mathcal{D}^*}$. Once we prove this contention, the thesis follows from Sussmann’s condition 2.

Given $v_q \in \mathcal{D}^*$, we must prove that $P_{\mathcal{D}}(v_q) = T_{v_q}\mathcal{D}$, which will be done along the steps below.

1) We fix an auxiliary Riemannian metric tensor $g$ on $\mathcal{M}$. Steps 1) and 2) in the proof of theorem 2 apply \textit{ipsis litteris}, so that both the vertical subspace $\text{Vert}_{v_q}(\mathcal{D})$ and the horizontal lift $H_{v_q}(\mathcal{D}^*)$ are linear subspaces of $P_{\mathcal{D}}(v_q)$. Hence, for any smooth locally defined vector field $X \in \Gamma^\infty_{\text{loc}}(\mathcal{D})$, the horizontal lift $X^{\text{Hor}} \in \Gamma^\infty_{\text{loc}}(T\mathcal{D}|_{\mathcal{D}^*})$ is a smooth local section of $P_{\mathcal{D}}$.

2) Since $P_{\mathcal{D}}$ is generated by $\Gamma^\infty_{\text{loc}}(P_{\mathcal{D}})$, it follows from theorems 4.1 and 4.2 in [7] that $P_{\mathcal{D}}$ is $\Gamma^\infty_{\text{loc}}(P_{\mathcal{D}})$-invariant. Hence, for each $X \in \Gamma^\infty_{\text{loc}}(\mathcal{D})$, we conclude from the previous step that $(X^{\text{Hor}})_t \in \mathbb{R}$ preserves $P_{\mathcal{D}}$.

3) Let $w_q \in T_q\mathcal{M}$. Since $T_q\mathcal{M} = P_{\Gamma^\infty_{\text{loc}}(\mathcal{D})}(q)$, we may take $z_p \in \mathcal{D}$ and finite families $(X_i)_{1 \leq i \leq k}$ of smooth local sections of $\mathcal{D}$ and $(t_i)_{1 \leq i \leq k}$ of real numbers such that $(X_{k,t_k} \circ \cdots \circ X_{1,t_1})_t z_p = w_q$. But, for any for any smooth locally defined vector field $X \in \Gamma^\infty_{\text{loc}}(\mathcal{D})$, the horizontal lift $X^{\text{Hor}} \in \Gamma^\infty_{\text{loc}}(T\mathcal{D}|_{\mathcal{D}^*})$ is $\pi_{\mathcal{D}^*}$-related to $X$; it then follows, recalling
that $X_\mathcal{D}$ denotes the nonholonomic vector field of $(\mathcal{M}, g, \mathcal{D})$, that

$$\mathcal{T} \pi_\mathcal{D} \circ (X_{k,t_k}^{\text{Hor}} \circ \cdots \circ X_{1,t_1}^{\text{Hor}})_{\mathcal{D}}(z_p) =$$

$$= (X_{k,t_k} \circ \cdots \circ X_{1,t_1})_\ast \circ \mathcal{T} \pi_\mathcal{D} \cdot X_\mathcal{D}(z_p) = w_q.$$  

We therefore conclude that

$$H_{v_q}^\mathcal{D}(w_q) = (X_{k,t_k}^{\text{Hor}} \circ \cdots \circ X_{1,t_1}^{\text{Hor}})_{\mathcal{D}}(z_p) -$$

$$- \lambda_{v_q} \left( \mathcal{K}_\mathcal{D} \cdot (X_{k,t_k}^{\text{Hor}} \circ \cdots \circ X_{1,t_1}^{\text{Hor}})_{\mathcal{D}}(z_p) \right).$$

Hence, $H_{v_q}^\mathcal{D}(w_q)$ belongs to $P_D(v_q)$, since both vectors on the second member of the previous equality belong to that space, in view of steps 1 and 2. Since $w_q \in T_q M$ was arbitrarily taken, we conclude that $\text{Hor}_{v_q}(\mathcal{D}) = H_{v_q}^\mathcal{D}(T_q M) \subset P_D(v_q)$. Thus, $T_{v_q} \mathcal{D} = \text{Hor}_{v_q}(\mathcal{D}) \oplus \text{Ver}_{v_q}(\mathcal{D}) \subset P_D(v_q)$, hence the equality holds in the above inclusion and our contention is proved.

□

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