A QUANTITATIVE REGULARITY ESTIMATE FOR NONNEGATIVE SUPERSOLUTIONS OF FULLY NONLINEAR UNIFORMLY PARABOLIC EQUATIONS

JESSICA LIN

Abstract. We establish a quantitative lower bound for nonnegative supersolutions of fully nonlinear, uniformly parabolic equations in a fraction of the original domain. Our result may be interpreted as a nonlinear, quantitative version of Krylov’s growth lemma for nonnegative supersolutions of linear, uniformly parabolic equations in nondivergence form [10], however our approach is different. We obtain our result as a consequence of the Fabes-Stroock estimate [8, 1]. Our work is a parabolic version of an elliptic regularity estimate established in [4].

1. Introduction

We present a regularity estimate which relates a function $u$ with $Lu$, where $L$ denotes a fully nonlinear, uniformly parabolic differential operator. In particular, we consider the case when $u$ is a nonnegative supersolution to $Lu \geq f$, with $0 \leq f \leq 1$. In the time-independent setting, when $L$ is fully nonlinear and uniformly elliptic, this question has been completely addressed. The Alexandroff-Backelman-Pucci estimate [2] yields that $u$ is controlled from above by the $L^{N+1}$ norm of $f$. Moreover, it was shown in [4] that by applying the so-called Fabes-Stroock estimate [8], in a subset of the original domain, $u$ is controlled from below by the $L^{N+1}$ norm of $f$, raised to a large power which depends only on the ellipticity constants and dimension of the space. In the context of fully nonlinear, uniformly parabolic equations in $\mathbb{R}^{N+1}$, the parabolic analogue of the Alexandroff-Backelman-Pucci estimate was established by Kryov and Tso [9, 11], and extended to viscosity solutions by Wang in [12]. The purpose of this paper is to address how $u$ is controlled from below by the $L^{N+1}$-norm of $f$.

We prove that in a fraction of the original domain, nonnegative supersolutions of uniformly parabolic equations are bounded below by $\|f\|_{L^{N+1}}^\alpha$, where $\alpha \sim \frac{1}{\|f\|_{L^{N+1}}}$. Although the result is presented for fully nonlinear equations, it is new, to our knowledge, for linear, nondivergence form equations with bounded measurable coefficients. Our work is the parabolic version of the lower bound established by Caffarelli, Souganidis, and Wang in [4], which was used in the error estimates for stochastic homogenization of uniformly elliptic equations in random media [3]. Although our general approach follows theirs, it is necessary to develop a number of new arguments to handle the parabolic structure of the problem. We will also show that we may recover the elliptic result of [4] from our estimates in the limit as $t \to \infty$.

Before stating our results, we briefly explain the notation and setting. We consider viscosity supersolutions of

$$
\begin{cases}
  u_t - F(D^2 u, x, t) \geq f & \text{in } Q_1,
  
  u \geq 0,
\end{cases}
$$

where $Q_1$ denotes a parabolic cylinder of the form $Q_1 = B_1(0) \times (-1, 0] \subset \mathbb{R}^{N+1}$, and $\partial_p Q_1$ denotes the parabolic boundary $\partial_p Q_1 = (B_1(0) \times \{t = -1\}) \cup (\partial B_1(0) \times [-1, 0))$. In general, we will denote $Q_R(x_0, t_0) = B_R(x_0) \times (-r^2 + t_0, t_0]$ and $\partial_p Q_R(x_0, t_0) = (B_r(x_0) \times \{t = -r^2 + t_0\}) \cup (\partial B_r(x_0) \times [-r^2 + t_0, t_0])$. We assume that $F$ is uniformly elliptic with ellipticity constants $\lambda, \Lambda$, i.e. for every $M, K \in \mathcal{S}^N$ (the space of...
symmetric $N \times N$ matrices), $K \geq 0$, we have
\begin{equation}
\lambda \|K\| \leq F(M + K, x, t) - F(M, x, t) \leq \Lambda \|K\| \quad \text{for all } (x, t) \in Q_1,
\end{equation}
where $\|K\|$ denotes the maximum eigenvalue of $K$. This is equivalent to saying that (1.4) is uniformly parabolic.

The main result we prove is the following:

**Theorem 1.1.** Let $u$ satisfy (1.4) with the hypothesis (1.2). Let $0 \leq f - F(0, \cdot, \cdot) \leq 1$. For every $\kappa \in (0, 1)$, there exists $c, C, \rho, \beta > 0$, depending on $\lambda, \Lambda, N, \kappa$, such that for all $|x| \leq \kappa$, $0 \geq t \geq -\kappa^2 - 1 \|f - F(0, x, t)\|_{L^{N+1}(Q_1)}^{N+1}$,
\begin{equation}
c \|f - F(0, x, t)\|_{L^{N+1}(Q_1)}^{N+1} \leq u(x, t) \leq C \|f - F(0, x, t)\|_{L^{N+1}(Q_1)}^{N+1},
\end{equation}
with $\alpha = \rho + 2\beta \|f - F(0, x, t)\|_{L^{N+1}(Q_1)}^{-N+1}$.

We emphasize that the right inequality follows immediately from the parabolic Aleksandrav-Backelman-Pucci estimate [12], and the focus of this paper is to obtain the left inequality. In order to prove Theorem 1.1, we appeal to the standard “linearized” interpretation of (1.1) using Pucci’s extremal operators (see [2, 12]).

For any domain $D \subset \mathbb{R}^{N+1}$, we define $\mathcal{S}(g, D)$ to be the set of viscosity supersolutions to
\begin{equation}
u_t - \mathcal{M}^-(D^2 u) \geq g \quad \text{in } D,
\end{equation}
and, respectively, $\mathcal{S}(g, D)$ to be the set of viscosity subsolutions to
\begin{equation}
u_t - \mathcal{M}^+(D^2 u) \leq g \quad \text{in } D.
\end{equation}
We will refer to the equations above as Pucci’s equations. We also define $S(g, D) = \mathcal{S}(g, D) \cap \mathcal{S}(g, D)$ and $S^*(g, D) = \mathcal{S}(-\|g\|_{\infty}, D) \cap \mathcal{S}(|g|_{\infty}, D)$.

It is shown in [12] Lemma 3.12 that $u$ satisfying (1.1) also satisfies $u \in \mathcal{S}(f - F(0, \cdot, \cdot), Q_1)$. Therefore, in order to prove Theorem 1.1, it is enough to prove

**Theorem 1.2.** Let $0 \leq f \leq 1$, and let $u$ be nonnegative in $\mathcal{S}(f, Q_1)$. For every $\kappa \in (0, 1)$, there exists $c, C, \rho, \beta > 0$ which only depend on $\lambda, \Lambda, N, \kappa$, so that for all $|x| \leq \kappa$ and $0 \geq t \geq -\kappa^2 - 1 \|f\|_{L^{N+1}(Q_1)}^{N+1}$,
\begin{equation}
c \|f\|_{L^{N+1}(Q_1)}^{N+1} \leq u(x, t) \leq C \|f\|_{L^{N+1}(Q_1)}^{N+1},
\end{equation}
with $\alpha = \rho + 2\beta \|f\|_{L^{N+1}(Q_1)}^{-N+1}$.

For comparison, we also state the elliptic version of this result from [4]:

**Theorem 1.3** (Caffarelli, Souganidis, Wang, [4]). Let $u \geq 0$ solve $-\mathcal{M}^-(D^2 u) = f$ in $B_1$, with $0 \leq f \leq 1$. For every $\kappa \in (0, 1)$, there exists $c, C, \alpha > 0$ depending only on $\lambda, \Lambda, N, \kappa$ such that for all $|x| \leq \kappa$,
\begin{equation}
c \|f\|_{L^\infty(B_1)} \leq u(x) \leq C \|f\|_{L^\infty(B_1)}.
\end{equation}
We note that in the parabolic setting, the domain where \([1.3]\) holds depends on \(\|f\|_{L^{N+1}(Q_1)}\). This is a consequence of the causality inherent in parabolic equations. However, if we sacrifice using \(\|f\|_{L^{N+1}(Q_1)}\), and replace this by \(\|f\|_{L^{N+1}(Q_{1-\kappa,1-\kappa})}\), where \(Q_{1-\kappa} = B_1-\kappa(0) \times (-1,-\kappa)\), we can obtain a sharper estimate in a domain which only depends on \(\kappa\).

**Corollary 1.4.** Let \(0 \leq f \leq 1\), and let \(u\) be nonnegative in \(\mathcal{S}(f,Q_1)\). For every \(\kappa \in (0,1)\), there exists \(c,C,\beta > 0\) which only depend on \(\lambda,\Lambda,N,\kappa\) so that for all \(|x| \leq \kappa\) and \(0 \geq t \geq -\kappa\),

\[
\begin{equation}
c \|f\|_{L^{N+1}(Q_{1-\kappa,1-\kappa})}^\beta \leq u(x,t).
\end{equation}
\]

Theorem 1.2 follows relatively easily from:

**Theorem 1.5.** Let \(u\) be nonnegative in \(\mathcal{S}(f,Q_1)\). Set \(\Gamma = \{ f > \alpha \}\) and \(m = |\Gamma|\). For every \(\kappa \in (0,1)\), there exists \(c,\rho,\beta > 0\) depending only \(\lambda,\Lambda,N,\kappa\) so that for all \(|x| \leq \kappa\), \(0 \geq t \geq -\kappa m\),

\[
\begin{equation}
u(x,t) \geq cm^\rho + \beta/m \alpha.
\end{equation}
\]

In order to prove Theorem 1.5, we compare \(u \in \mathcal{S}(f,Q_1)\) to \(w(x,t) = w(x,t;\Gamma)\), the “fundamental solution corresponding to the domain \(\Gamma\),” which solves

\[
\begin{equation}
\begin{cases}
w_t - \mathcal{M}(D^2w) = \chi_\Gamma & \text{in } Q_1, \\
w = 0 & \text{on } \partial_0 Q_1.
\end{cases}
\end{equation}
\]

We first prove (1.5) for the solution \(w\). Theorem 1.5 may be interpreted as a quantitative version of Krylov’s growth lemma (see [10], Theorem 4.2.1), where the lower bound is given in terms of an unknown function \(\varphi(\Gamma)\) satisfying \(\varphi(m) > 0\) for \(m > 0\). Our approach differs from the original strategy of Krylov, where the author uses classical covering arguments to cover \(\Gamma\) with cylinders which contain a significant proportion of \(\partial_0 Q_1\). They are then able to quantify the general approach of Krylov, and obtain Corollary 1.4 with an independent proof from ours here. We take an alternative strategy by invoking the Fabes-Stroock estimate [8, 1], which quantitatively compares fundamental solutions corresponding to different domains. This allows us to work with a covering of \(\Gamma\) which is completely contained inside of \(Q_1\), and we obtain a quantitative lower bound under Krylov’s more general hypotheses.

We will refer to the following corollary as the Fabes-Stroock estimate. Combining the results of [11, Corollary 2.10], [6, Lemma 5], and [5], we have:

**Corollary 1.6.** Let \(E \subset Q_1(x_0,t_0)\) such that \(\tilde{Q}_{3r}(x_0,t_0) = B_{3r}(x_0) \times (t_0 - 9r^2, t_0 + 9r^2) \subset Q_1\). For every \(\kappa \in (0,1)\), there exists \(\sigma, C_{cfs} > 0\) depending on \(\lambda,\Lambda,N\), such that for all \((x,t) \notin \tilde{Q}_{3r}(x_0,t_0), t \geq t_0 + 9r^2\),

\[
\begin{equation}
\frac{w(x,t;E)}{w(x,t;Q_{\kappa})} \geq C_{cfs} \left( \frac{|E|}{|Q_{\kappa}|} \right)^\sigma.
\end{equation}
\]

We point out that although (1.7) is based on results for linear, uniformly parabolic operators with smooth coefficients, the constants \(C_{cfs}\) and \(\sigma\) are independent of the smoothness of the coefficients. Therefore, by standard approximation techniques, we may extend the estimates to solutions of uniformly parabolic operators with bounded, measurable coefficients. It is well-established that Pucci’s equations serve as a “linearization” for fully nonlinear equations [2, 12]; any estimates which hold true for solutions of linear equations with bounded, measurable coefficients also hold for functions in the class \(\mathcal{S}(g,D), \mathcal{S}(g,D)_L\) and \(\mathcal{S}(g,D)\) appropriately. This allows us to apply (1.7) in the case where \(w(x,t;\Gamma)\) solves (1.6).

From (1.7), we obtain a lower bound for \(w(x,t;E)\) by controlling \(w(x,t,Q_{\kappa})\) from below. In order to do so, we introduce an iterative method to prove regularity estimates in larger space domains for later times. Our proof is inspired by some of the ideas from Krylov’s argument [10], however the constructions are used quite differently in our work. This is the most delicate part of the argument in the parabolic setting, whereas it follows relatively easily in the elliptic case.
This paper is organized as follows. Section 2 is devoted to establishing a quantitative lower bound for \( w(x, t; Q_\tau) \). We revisit some of the results in Krylov’s original work [10], however we present them for Pucci’s extremal operators. We also describe the iterative construction, which is completely contained inside of \( Q_1 \), and allows us to obtain regularity estimates in larger space domains at later times. In Section 3 we use a covering argument to complete the proof of Theorem 1.5. We consequently obtain Theorem 1.2 and we make several remarks about how to recover the aforementioned related results.

2. Quantitative Lower Bounds on Fundamental Solutions for Subcylinders

We prove a quantitative lower bound for fundamental solutions of subcylinders, \( w(x, t; Q_\tau) \). We note that in the elliptic setting, one can generally obtain a lower bound for \( w(x; B_r) \) by iteratively applying the weak Harnack inequality. However, in the parabolic setting, the argument becomes more delicate in order to account for the time shifts of the parabolic Harnack inequality [12]. Our proof is inspired by the iterative approach in the elliptic setting, however we do not employ the parabolic Harnack inequality. Instead, we use some of the ideas and constructions originally found in [10] to compare supersolutions the iterative approach in the elliptic setting, however we do not employ the parabolic Harnack inequality.

We first present an important comparison lemma found in [10]. This lemma allows us to obtain a “weak” comparison principle for oblique cylinders. We generalize the proof in [10] by proving the statement for the weak Harnack inequality. However, in the parabolic setting, the argument becomes more delicate in

\[ u(x, t_1) \geq u(x, t_2) \quad \text{for } t_1 > t_2, \]

without loss of generality, we may perform a transformation to assume that \( Q_1 \) is a right cylinder. We generalize the proof in [10] by proving the statement for Pucci operators, relaxing the hypotheses of the original lemma, and keeping track of the dependencies of the constants.

**Lemma 2.1.** Let \( Q \) be a cylinder whose base is given by \( B_R(x_1) \) in \( \{t = t_1\} \) and \( B_R(x_2) \) in \( \{t = t_2\} \), with \( Q \subset Q_1 \). Let \( h = t_2 - t_1 > 0 \), and let \( d = |x_2 - x_1| \). Suppose that there exists \( \eta, \tau_1, \tau_2 \) so that \( \frac{d}{R} \leq \eta \), and \( \tau_1 \leq \frac{h}{\tau_2} \leq \tau_2 \). Let \( u \geq 0 \) solve

\[ u_t - \mathcal{M}^{-}(D^2u) \geq 0 \quad \text{in } Q. \]

Let \( \theta \in (0, 1) \). Suppose that \( u(x, t_1) \geq 1 \) for all \( x \) such that \( |x - x_1| \leq \delta R \). Then there exists \( \gamma(\theta, \delta) \) and \( \alpha = \alpha(\lambda, \theta, \eta, \tau_1, \tau_2, \Lambda, N) > 0 \), so that for all \( |x - x_2| \leq (1 - \theta)R \),

\[ u(x, t_2) \geq \gamma \delta^\alpha. \]

**Proof.** Without loss of generality, we may perform a transformation to assume that \( Q \) is a right cylinder. Suppose we are working on an oblique cylinder. If we let \( u^1(x, t) = u(x + bt, t) \), then

\[ u_t^1 - \mathcal{M}^{-}(D^2u^1) - b \cdot Du^1 = u_t - \mathcal{M}^{-}(D^2u) \geq f(x, t) \quad \text{for } \bar{Q} = \{(x, t) : (x + bt, t) \in Q\}. \]

We note that for any oblique cylinder \( Q \), there exists a choice of \( b \in \mathbb{R}^N \) so that \( \bar{Q} \) is a right cylinder. Moreover, we have that the magnitude of \( |b| = \frac{d}{h} \).

By scaling and adjusting the operator, we may also assume that \( Q = Q_1 \). We let \( u^2(x, t) = u^1(Rx, ht) \).

We have

\[ \frac{u_t^2}{h} - \frac{1}{R^2} \mathcal{M}^{-}(D^2u^2) - \frac{1}{R} b \cdot Du^2 \geq 0. \]

This yields

\[ u_t^2 - \frac{h}{R^2} \mathcal{M}^{-}(D^2u^2) - \frac{h}{R} b \cdot Du^2 \geq 0 \quad \text{in } Q_1. \]

Now that we are working in \( Q_1 \), our objective is to obtain an estimate in \( B_{(1-\theta)}(0) \) when \( t = 0 \). We fix \( x_0 \in B_{(1-\theta)}(0) \). Without loss of generality, we may choose \( \delta \) so that \( \delta \leq \frac{1}{2} \theta \). We examine the cylinder \( \bar{Q}_{\theta, 1} \), which is a cylinder with base \( B_0(0, -1) \) and top \( B_0(x_0, 0) \). By our choices, \( \bar{Q}_{\theta, 1} \) fits inside of \( Q_1 \). Inside of \( \bar{Q}_{\theta, 1} \), \( u^2(x, t) \) is a supersolution to \( (2.2) \), \( u^3(x, t) = \geq 0 \), and \( u^2(x, -1) \geq 1 \) for all \( |x| \leq \delta \). We perform yet another change of coordinates to straighten this cylinder as before. We set \( u^3(x, t) = u^2(x + ct, t) \), where \( |c| \leq 1 - \theta \). \( u^3(x, t) \) solves

\[ u_t^3 - \frac{h}{R^2} \mathcal{M}^{-}(D^2u^3) - \left( \frac{h}{R} b + c \right) \cdot Du^3 \geq 0 \quad \text{in } Q_{\theta, 1} = B_0(0) \times [-1, 0]. \]
The problem reduces to showing that $u^3(0,0) \geq \gamma \delta^\alpha$ for some choice of $\gamma, \delta, \alpha$.

We consider

$$\psi(x,t) = ((\theta^2 - \delta^2)(1 + t) - |x|^2 + \delta^2)^2((\theta^2 - \delta^2)(1 + t) + \delta^2)^{-\alpha}$$

where we will choose $\alpha$ later in the proof. Let $\hat{Q} = \{(x,t) : (\theta^2 - \delta^2)(1 + t) - |x|^2 + \delta^2 > 0 \text{ with } -1 < t < 0\}$. We note that $\hat{Q} \subset Q_{\delta,1}$, and for $(x,t) \in \partial_p \hat{Q} \cap \{ t > -1 \}$ (the lateral boundary), $\psi(x,t) = 0$. Moreover, when $t = -1$, we see that $\hat{Q}(t = -1) \subset B_\delta(t = -1)$, and $\psi(x,-1) \leq \delta^{1-2\alpha}$. Therefore,

$$\delta^{2\alpha-4}\psi \leq u^3 \text{ on } \partial_p \hat{Q}.$$

Now we are ready to understand the solution properties of $\psi$. We let $\rho = \rho(t) = (\theta^2 - \delta^2)(1 + t) + \delta^2$, and $\varphi = \varphi(x,t) = \rho(t) - |x|^2$, so that $\psi(x,t) = \varphi^2 \rho^{-\alpha}$. On $\hat{Q}$, we have

$$\rho^\alpha\psi_t - \frac{h}{R^2}\rho^\alpha M^- (D^2\psi) - \rho^\alpha \left( \frac{h}{R} b + c \right) \cdot D\psi$$

$$= -\frac{\alpha}{\rho}(\theta^2 - \delta^2)\varphi^2 + 2\varphi(\theta^2 - \delta^2) - \frac{h}{R^2}M^- (8\theta^2 x - 4\varphi Id) + 4\varphi \left( \frac{h}{R} b + c \right) \cdot x$$

$$\leq -\frac{\alpha}{\rho}(\theta^2 - \delta^2)\varphi^2 + \left( 2(\theta^2 - \delta^2) + 4 \left( \frac{h}{R} b + c \right) \cdot x + 4\frac{\Lambda \lambda}{R^2} \right) \varphi - \frac{8h}{R^2} \lambda \delta^\alpha \leq 0$$

where $C_0 = C_0(\theta, \delta, \eta, \tau_2, N, \lambda) = 2(\theta^2 - \delta^2) + 4(\eta \theta + (1 - \theta)\theta) + 4\Lambda \lambda \tau_2$.

For certain, if $8\lambda \tau_1 |x|^2 \geq C_0 \varphi$, then $\psi$ is a subsolution with 0 right hand side. If we are in the case where $8\lambda \tau_1 |x|^2 < C_0 \varphi$, then we must have

$$8\lambda \tau_1 (\rho - \varphi) < C_0 \varphi$$

$$\frac{8\lambda \tau_1}{C_0 + 8\lambda \tau_1} \varphi < \rho^{-1} \varphi^2.$$

This gives us that

$$\rho^\alpha\psi_t - \frac{h}{R^2}\rho^\alpha M^- (D^2\psi) - \rho^\alpha \left( \frac{h}{R} b + c \right) \cdot D\psi \leq -\alpha(\theta^2 - \delta^2)\frac{8\lambda \tau_1}{C_0 + 8\lambda \tau_1} \varphi - 8\lambda \tau_1 \lambda \delta^\alpha \leq 0$$

using the fact that $\delta \leq \frac{\theta}{2}$. Therefore, if

$$\alpha > C_0 \frac{C_0 + 8\lambda \tau_1}{6\theta^2 \lambda \tau_1}$$

then $\rho^\alpha\psi(x,t)$ is subsolution everywhere in $\hat{Q}$. This is how we will choose $\alpha$.

Since $M^-(\cdot)$ is uniformly elliptic, $\frac{d}{R}$ is bounded, by the comparison principle we must have that

$$\delta^{2\alpha-4}\psi \leq u.$$
everywhere inside $\hat{Q}$. In particular, we obtain that
\[ u(0,0) \geq \delta^2 \alpha^{-4} \psi(0,0) = \delta^2 \alpha^{-4} \theta^4 \theta^{-2\alpha} \geq \theta^4 \delta^{2\alpha-4} \]
and this completes the proof. We note that this construction holds for all $x \in B_{1-\theta}(0,0)$ as desired.

\[ \blacksquare \]

**Remark 2.2.** In future constructions, we will vary the values of $\tau_1$, while all of the other constants in Lemma 2.1 will become independent of $R, h$. We should keep in mind that $\alpha$ is a large constant. From (2.5), we see that asymptotically, as $\tau_1$ gets small, $\alpha$ is controlled from below by $\alpha > \frac{C_1}{\tau_1}$ for $C_q = C_q(\theta, \delta, \Lambda, N, \tau_2, \eta)$. Therefore, we will think of $\alpha \sim \frac{1}{\tau_1}$.

**Remark 2.3.** If we assume the cylinder $Q = Q_r$, a standard right cylinder with parabolic scaling (radius $r$ and height $r^2$), then by (2.5), $\alpha = \alpha(\Lambda, \Lambda, N, \theta, \delta)$.

Next, we state a lemma which will be useful for obtaining local lower bounds of $u(x, t)$. The linear analogue of this lemma and its proof can be found in [10].

**Lemma 2.4.** Suppose that
\begin{equation}
\begin{cases}
v_t - M^-(D^2v) \geq 1 & \text{ in } Q_1, \\
v(x, t) = 0 & \text{ on } \partial_p Q_1.
\end{cases}
\end{equation}
Then, there exists a constant $C_k$ depending on $\kappa, N, \Lambda, \Lambda$ so that for all $|x| \leq \kappa$,
\begin{equation}
\min v(x, 0) \geq C_k.
\end{equation}

**Proof.** We would like to apply Lemma 2.1 in the case where $Q = Q_1$. However, we cannot apply the result directly to $v$, because $v$ does not have a natural lower bound when $t = -1$. Instead, we build an auxiliary function which satisfies Lemma 2.1 and apply a comparison argument. Let $\varphi(x) = 1 - |x|^2$, and let $\psi(x, t)$ solve
\begin{equation}
\begin{cases}
\psi_t - M^+(D^2\psi) = -M^-(D^2\varphi) & \text{ in } Q_1, \\
\psi(x, t) = 0 & \text{ on } \partial_p Q_1.
\end{cases}
\end{equation}
By the comparison principle, we have that $\varphi - \psi \leq 1$. If we look at $\eta(x, t) = (1 + t)[\varphi(x) - \psi(x, t)]$, this solves
\[ \eta_t - M^-(D^2\eta) = \varphi(x) - \psi(x, t) - (1 + t)\psi_t - (1 + t)M^-(D^2\varphi - D^2\psi) \]
\[ \leq (\varphi - \psi) + (1 + t)(\varphi_t - \psi_t) - (1 + t)M^-(D^2\varphi) + (1 + t)M^+(D^2\psi) \]
\[ \leq 1 \]
with $\eta(x, t)|_{\partial_p Q_1} = 0$. Therefore, by the comparison principle, $v(x, t) \geq \eta(x, t) = (1 + t)[\varphi(x) - \psi(x, t)]$. We note that $(\varphi - \psi) - M^-(D^2(\varphi - \psi)) \geq 0$, and for $|x| \leq \frac{1}{2}$, $\varphi(x) - \psi(x, -1) = \varphi(x) \geq \frac{1}{2}$. Thus, we may apply Lemma 2.1 to the function $\varphi - \psi$ to obtain that there exists a $\gamma, \alpha > 0$ depending only on $\Lambda, \Lambda, N, \kappa$ such that for all $|x| \leq \kappa$,
\[ \varphi(x) - \psi(x, 0) \geq \gamma \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{\alpha} \]
This implies that $v(x, 0) \geq \eta(x, 0) = (1 + (1 + 0)[\varphi(x) - \psi(x, 0)] \geq \gamma \frac{1}{2} \left( \frac{1}{2} \right)^{\alpha} = C_k$ for all $|x| \leq \kappa$.

By scaling, we see that if
\begin{equation}
\begin{cases}
v_t - M^-(D^2v) \geq \sigma & \text{ in } Q_r(x_0, t_0), \\
v = 0 & \text{ on } \partial_p Q_r(x_0, t_0),
\end{cases}
\end{equation}
then $v(x, t_0) \geq C_k \sigma r^2$ for all $|x - x_0| \leq kr$.

Equipped with these results, we are now ready to prove the lower bound for $w(x, t; Q_r(x_0, t_0))$. 
Proposition 2.5. Let $Q_r(x_0,t_0) \subset Q_1$, $\kappa \in (0,1)$, and let $w(x,t)$ solve
\begin{align}
\begin{cases}
  w_t - \mathcal{M}^{-}(D^2w) \geq \eta & \text{in } Q_1 \\
  w = 0 & \text{on } \partial_\delta Q_1.
\end{cases}
\end{align}
Then there exists $C, \rho, \beta > 0$ depending only on $\lambda, \Lambda, N, \kappa$ so that for all $|x| \leq \kappa$, $0 \geq t \geq t_0$,
\begin{align}
w(x,t) \geq C_{r,\rho,\beta} \log(1/r).
\end{align}

Proof. We first prove (2.9) in the case when $t_0 = t = 0$. Let $t_0' = -\left(\frac{3\rho^2}{r}\right)$. We consider the cylinder $Q_{r/2}(x_0, t_0') \subset Q_r(x_0,0)$. We let $\theta \leq 1$, so that $1 - \theta \geq \frac{1}{2}$. By scaling Lemma 2.4 there exists $c_0$ so that for all $|x - x_0| \leq \frac{\kappa}{4}$,
\begin{align}
w(x,t_0') \geq c_0 r^2.
\end{align}
Using the information in this disc, we build our way to gaining information in $B_\kappa(0)$ at $t = 0$. We note that since $Q_r(x_0,0) \subset Q_1$, this implies $|x_0| \leq 1 - r$, and $|t_0'| \geq \frac{3\rho^2}{r}$.

If $r > \kappa$, we draw an oblique cylinder with base $B_r(x_0, t_0')$, and top $B_r(0,0) \supset B_\kappa(0,0)$. We apply Lemma 2.1 with the choices $R = r$, $h \geq \frac{3\rho^2}{r}$, $\eta = \frac{1}{2} - 1$, $\delta = \frac{1}{4}$, and $\tau_1 = \frac{3}{4}$, and $\tau_2 = \frac{1}{2\kappa}$. There exists a universal constant $C_{\kappa} = C_{r}(\kappa, \Lambda, \lambda, N)$ so that for all $|x| \leq \kappa$,
\begin{align}
w(x,0) \geq C_{r}\kappa^2,
\end{align}
and we are done.

If we are in the case where $r \leq \kappa$, then we need to perform an iterative construction. We consider a sequence of stacked oblique cylinders with expanding radii, and repeatedly apply Lemma 2.1 in order to obtain a lower bound for $w(x,0)$ in $B_\kappa(0)$. We proceed inductively by first considering what happens at the base of the construction. We build an oblique cylinder with base $B_r(x_0, t_0')$, and top $B_r(x_1,t_1)$ with $x_1, t_1$ to be chosen later. By Lemma 2.1 since $w(x,t_0') \geq C_0 r^2$ for all $|x - x_0| \leq \frac{\kappa}{4}$, we obtain that
\begin{align}
w(x,t_1) \geq C_1 r^2 \left(\frac{1}{4}\right)^\alpha
\end{align}
for all $|x - x_1| \leq \frac{\kappa}{4}$, (since $1 - \theta \geq \frac{1}{2}$) with $C, \alpha$ sharing the same dependencies as in Lemma 2.1 (we will trace the exact dependencies later in the proof).

Now we have gained information in a larger disc, at a higher time. In general, if we have a lower bound for $w(x,t_{j-1})$ for all $|x - x_{j-1}| \leq \frac{\kappa}{4} R_j$, then by building another oblique cylinder with top $B_{R_j}(x_j,t_j)$, and applying Lemma 2.1 we have
\begin{align}
w(x,t_j) \geq (\text{previous lower bound}) \gamma_j \left(\frac{1}{4}\right)^\alpha_j
\end{align}
for all $|x - x_j| \leq \frac{1}{4} R_j$, where $\gamma_j, \alpha_j$ depend on the construction of the cylinder.

We now show how to explicitly construct this stack of cylinders which will allow us to obtain information for all $|x| \leq \kappa$ at $t = 0$. The key difficulty will be to make sure that the construction stays within $Q_1$. We first focus exclusively on the spatial direction. The number of iterates $l$ to reach $\kappa$ can be estimated by
\begin{align}
2l \frac{r}{4} > \kappa & \geq 2^{l-1} \frac{r}{4} \\
8 \kappa \frac{r}{4} > 2l & \geq \frac{4 \kappa}{r} \\
\log_2 \frac{8 \kappa}{r} = \log_2 \frac{8 \kappa}{r} & \geq l > \log_2 \left(\frac{4 \kappa}{r}\right) = \log \left(\frac{4 \kappa}{r}\right)
\end{align}
This gives us a lower bound on the number of iterates needed in order to reach the desired domain in space. The next difficulty is to make sure that we can fit this construction in the height direction without spilling out of $Q_1$. Recall that $|t_0'| \geq \frac{3\rho^2}{r}$. Therefore, we will choose the heights of every subcylinder to be $\frac{3\rho^2}{r}$. Since we fix the heights, but the radii continue to grow, we see that in the notation of Lemma 2.1 we may choose $\tau_2 = 1$. Moreover, at the $j$-th step of each iteration, $\frac{h}{R^2} \geq \frac{3\rho^2}{4(2^j r^2)}$, so that $\tau_{1,j} = \frac{\beta}{2^j \log(8\kappa/r)}$ for some
\[ \beta = \beta(\lambda, \Lambda, N, \kappa). \] By fixing the heights in this way, we can ensure that the total number of iterates will fit in \( Q_1 \) height-wise.

Another difficulty we face is making sure at every step, we choose \( x_j \) so that we can fit \( B_{R_j}(x_j, t_j) \subset Q_1 \) where \( R_j = 2^{j+1} \frac{R}{4} \). In order to resolve this difficulty, we begin by choosing \( x_j = x_{j-1}(1 - R_j) \). Once \( x_j \) “passes” the spatial origin, then we can build the cylinders upright. In the notation of Lemma 2.1, we see that \( \eta \leq 2 \) in every step.

This completely characterizes this construction. Now we take a closer look at how the estimate depends on these choices. We recall from Lemma 2.1 that at each step, \( w(x, t_j) \) is bounded below by \( \gamma_j \delta^\alpha \) where \( \gamma_j(\theta, \delta) \). Since \( \theta \) and \( \delta \) are absolute numbers in our construction, \( \gamma_j = \gamma \) is independent of \( j \). By Remark 2.2 we have that at each step, for \( |x - x_j| \leq R_j \),

\[
(2.12) \quad w(x, t_j) \geq (\text{previous lower bound}) \gamma \left( \frac{1}{4} \right)^{\beta/\gamma_1}
\]

\[
(2.13) \quad \geq (\text{previous lower bound}) \gamma \left( \frac{1}{4} \right)^{\beta_2 \log(4\kappa/\rho)}.
\]

Therefore, relabeling constants as necessary, after \( \ell \) iterations, we have for all \( |x| \leq \kappa \),

\[
(2.14) \quad w(x, 0) \geq C\rho^2 \gamma \left( \frac{1}{4} \right)^{\sum \beta_2 \log(8\kappa/\rho)} = C\rho^\beta \left( \frac{1}{4} \right)^{\beta_2 (4\kappa/\rho)} = C\rho^\beta \left( \frac{1}{4} \right)^{\beta_2 \log(8\kappa/\rho)} = C\rho^\beta \kappa^{\beta/\tau},
\]

where \( C, \rho, \beta \) depend on \( \lambda, \Lambda, N, \kappa \).

We note that if \( t_0 < 0 \), this argument gives us a lower bound for \( w(x, t_0) \), for \( |x| \leq \kappa \). Moreover, if \( t \geq t_0 \), the lower bound still holds because we can always increase the height of the final cylinder without affecting our choice of constants. This completes the argument for all \( t \geq t_0 \).

\[ \blacksquare \]

3. Quantitative Lower Bounds for Nonnegative Supersolutions

We use the quantitative lower bound on \( w(x, t; Q_r) \) and the Fabes-Stroock estimate to obtain a lower bound on \( w(x, t; \{ f > \alpha \}) \). We then compare that to \( u(x, t) \) to obtain Theorem 1.2.

Before we begin, we elaborate on how we obtain Corollary 1.6. The authors of [1] formulate a parabolic analogue of the original elliptic result of Fabes and Stroock [8] for linear parabolic equations with smooth coefficients. They prove a reverse Holder inequality for Green’s functions:

**Theorem 3.1** (Amar and Norando, Corollary 2.10, [1]). Let \( g(x, y, t, s) \) denote the Green’s function on \( Q_1 \) corresponding to the operator \( D_t - \sum c_{ij}(x) D_{x_i}^2 \), with \( a_{ij} \) smooth in \( x \). There exists a positive constant \( K = K(N, \lambda, \Lambda) \) such that for every cylinder \( Q_r(x_0) \subset Q_3r(x_0, t_0) = B_{3r}(x_0) \times (t_0 - 9r^2, t_0 + 9r^2) \subset Q_1 \), we have for all \( (x, t) \notin Q_{3r}(x_0, t_0) \), with \( t \geq t_0 + 9r^2 \)

\[
(3.1) \quad \left[ r^{-N+2} \int_{Q_r(x_0, t_0)} g(x, t, y, s)^{(N+1)/N} dyds \right]^{N/(N+1)} \leq Kr^{N+2} \int_{Q_r(x_0, t_0)} g(x, t, y, s)dyds.
\]

In the linear setting, if we consider \( w(x, t; E) = \int_E g(x, y, t, s)dyds \), then \( w(x, t) = w(x, t; E) \) solves

\[
\begin{align*}
wt - \sum a_{ij}(x)w_{x_i,x_j} &= \chi_E & \text{in } Q_1, \\
\omega = 0 & \text{ on } \partial_x Q_1.
\end{align*}
\]

Applying the observations of Calderon [5], and combining this with the result of Coifman and Fefferman [6] Lemma 5, we are able to obtain Corollary 1.6. We point out that although the these results are for linear equations with smooth coefficients, we may obtain them for nonlinear equations by standard approximation arguments.

We will first prove Theorem 1.5 and then show how we may conclude Theorem 1.2.
Proof of Theorem 1.5. We first prove the estimate for $t = 0$. We denote $Q_r = B_r(0) \times (-1, -1 + r^2)$. There exists a choice of $c_1$, independent of $m$, such that

$$|Q_1 \setminus Q_{1-c_1 m}| = 1 - (1 - c_1 m)^{N+2} \leq \kappa m.$$  

Therefore,

$$|\Gamma \cap Q_{1-c_1 m}| \geq |\Gamma \cap Q_1| - |Q_1 \setminus Q_{1-c_1 m}| \geq (1 - \kappa)m.$$  

We note that dist(top($Q_1$), top($Q_{1-c_1 m}$)) = $1 - (1 - c_1 m)^2 \geq (c_1 m)^2$. Next, we cover $Q_{1-c_1 m}$ with cylinders $Q_{c_1 m/4}$. We claim that there exists at least one smaller cylinder $Q_{c_1 m/4}$ such that

$$|\Gamma \cap Q_{(c_1 m)/4}| \geq \frac{(1 - \kappa)m}{2(1 - c_1 m)^{N+2}} |Q_{(c_1 m)/4}| \geq (1 - \kappa)m|Q_{(c_1 m)/4}|.$$  

We note that the number of small cylinders it would take to cover $Q_{1-c_1 m/4}$ is at least $2^{((1-c_1 m)^{N+2})}$. If (3.2) did not hold, then we would have

$$(1 - \kappa)m \leq |\Gamma \cap Q_{1-c_1 m}| \leq \frac{2(1 - c_1 m)^{N+2}}{(c_1 m/4)^{N+2}} \sum |\Gamma \cap Q_{(c_1 m)/4}| < (1 - \kappa)m$$

which is a contradiction. Therefore, we must have that (3.2) holds in some cylinder $Q_{(c_1 m)/4}$, and by construction, $Q_{(c_1 m)/4} \subset \{t \leq -(c_1 m)^2\}$.

By Lemma 2.5 and relabeling constants as necessary,

$$w(x,t;Q_{(c_1 m)/4}^*) \geq C[(c_1 m)/4]^\rho E[(c_1 m)/4]^\beta/m \geq C m^\rho m^{\beta/m}.$$  

Since dist(top($Q_1$),top($Q_{1-c_1 m}$)) = $1 - (1 - c_1 m)^2 \geq (c_1 m)^2$, we may apply Theorem 1.6 (3.2), (3.3), and the comparison principle so that

$$\frac{u(x,0)}{\alpha} \geq w(x,0;E) \geq C m^\rho m^{\beta/m} [(1 - \kappa)m]^{\sigma} = C m^\rho m^{\beta/m}$$

for all $|x| \leq \kappa$. This establishes the estimate for $t = 0$.

We point out that all of the constants in the estimates above are independent of the choice of $m$. Therefore, if we extend $u = 0$ outside of $Q_1$, $f = 0$ outside of $Q_1$, then $u \in S(f, Q_1(0,t_0))$, for any $-\kappa m \leq t_0 \leq 0$. Also, we have

$$|\Gamma \cap Q_1(0,t_0)| \geq |\Gamma \cap Q_1| - |Q_1 \setminus (Q_1(0,t_0) \cap Q_1)| \geq m - \kappa m.$$  

Thus, replacing $m$ by $(1 - \kappa)m$, we obtain that for all $-\kappa m \leq t_0 \leq 0, |x| \leq \kappa$,

$$u(x,t_0) \geq C m^\rho m^{\beta/m} \alpha.$$  

We now complete the Proof of Theorem 1.2 by comparing $|\{f > \alpha\}|$ to $\|f\|_{L^{N+1}(Q_1)}$.

Proof of Thm. 1.2. We note that the right hand side of the estimate is nothing more than the ABP inequality. For the left hand side, we apply Theorem 1.5. Let $\|f\|_{L^{N+1}} = \eta < 1$ (assume $f \neq 1$ identically), then we claim that

$$\left|\left\{f > \frac{\eta}{2}\right\}\right| \geq \frac{\eta^{N+1}}{2}.$$
To see why, suppose this were not the case. Then we have that 
\[
\eta^{N+1} = \int |f|^{N+1} dx = \int_{\{f \leq \eta/2\}} |f|^{N+1} dx + \int_{\{f > \eta/2\}} |f|^{N+1} dx 
\leq \left( \frac{\eta}{2} \right)^{N+1} + |\{f > \eta/2\}| < \eta^{N+1}.
\]

Therefore, by applying Theorem 1.5 since \(\beta/\eta^{N+1} \geq \beta/|\{ f > \eta/2 \}|\), we have 
\[
(3.6) \quad u(x, t) \geq c \left| \left\{ f > \eta/2 \right\} \right|^\beta/|\{ f > \eta/2 \}| \geq c \eta^{\rho+\beta/\eta^{N+1}}
\]
for all \(|x| \leq \kappa, 0 \geq t \geq -\kappa \|f\|^{N+1}_{L^{N+1}(Q_1)} |\{ f > \eta/2 \}| \geq -\kappa \left| \left\{ f > \frac{\eta}{2} \right\} \right| \). Relabeling our constants as necessary, this gives us the desired result.

We now explain how we will obtain Corollary 1.4. This follows from two observations regarding the proofs of Proposition 2.5 and Theorem 1.5. First, we show that in the proof of Proposition 2.5 if \(Q_r(x_0, t_0) \subset \{ (x, t) : |x| \leq 1 - \kappa, -1 < t \leq 0 \}\), then there exists \(\beta = \beta(\lambda, N, \kappa)\) such that \(w(x, t; Q_r) \geq Cr^\beta\) for all \(|x| \leq \kappa, 0 \geq t \geq t_0\). Second, if we re-examine the proof of Theorem 1.5 if \(\Gamma \subset \{ t \leq a \} \subset Q_1\), then we can obtain information for all \(|x| \leq \kappa, a \leq t \leq 0\). This will be enough to complete the proof of Corollary 1.4.

Proof of Corollary 1.4. We justify the two claims we have made above. If \(Q_r \subset \{ (x, t) : |x| \leq 1 - \kappa \}\), then after one step, one can take \(R_1 \geq \kappa\) in the construction, and by (2.10), we obtain an “algebraic” lower bound. Therefore, if \(Q_r \subset Q_{1-\kappa}\), then (2.3) reduces to an algebraic lower bound, for all \(|x| \leq \kappa, 0 \geq t \geq t_0\).

For the second claim, we note that the domain where Theorem 1.5 applies is controlled by the location of the special cylinder, \(Q_{c^m/4}\). If \(\Gamma \subset Q_{1-\kappa}\), then \(Q_{c^m/4} \subset Q_{1-\kappa}\), and thus we obtain information for all \(|x| \leq \kappa, 0 \geq t \geq t_0\).

Remark 3.2. In [7], it was shown that for \(u \in S(\chi_r, B_r \times (-\infty, \infty))\) for \(r \geq 1\), if \(\Gamma \subset B_{1/2} \times [0, 1]\), then \(u(0, 2) \geq c|\Gamma|^m\) for \(c, m\) depending on \(\lambda, N, \kappa\). From Corollary 1.4, we can immediately recover this result.

Remark 3.3. We may obtain Theorem 1.3 from our results. We note that in the limit as \(t \to \infty\), we may consider a cylinder of infinite height and \(f\) to be constant in time. Moreover, since we do not worry about fitting our construction in heightwise, we may always choose right cylinders in the proof of Proposition 2.5, and in light of Remark 2.3, the lower bound may decrease by a universal constant in each iteration. This implies that \(w(x, t) \geq Cr^\beta\), where \(\beta = \beta(\lambda, N, \kappa)\), for all \(|x| \leq \kappa\). Once Proposition 2.5 has an algebraic lower bound, this yields that Theorem 1.2 is also an algebraic lower bound. This recovers Theorem 1.3.

For convenience, we state the unscaled versions of Theorem 1.1 and Theorem 1.2. We also remove the hypothesis that \(0 \leq f \leq 1\), and consider the case for general nonnegative \(f \in L^\infty\). The proofs of these statements follow by standard scaling arguments, which we omit here.

**Theorem 3.4.** Let \(u \) satisfy (1.1) with the hypothesis (1.2). Let \(0 \leq f \) satisfy \(F(0, \cdot, \cdot) \leq \|f - F(0, \cdot, \cdot)\|_{L^\infty(Q_R)}\). Let \(\kappa \in (0, 1)\). Then, there exists \(c, C, \rho, \beta > 0\), depending on \(\lambda, \Lambda, N, \kappa\), such that for all \(|x| \leq \kappa R, 0 \geq t \geq -\kappa^{-1} \|f - F(0, x, t)\|_{L^{N+1}(Q_R)} R^{-N} \), 
\[
cR^2-(N+2)\alpha \|f - F(0, x, t)\|_{L^\infty(Q_R)}^{1-(N+1)\alpha} \|f - F(0, x, t)\|_{L^{N+1}(Q_R)}^{(N+1)\alpha} \leq u(x, t) \leq CR^N/N+1 \|f - F(0, x, t)\|_{L^{N+1}(Q_R)}
\]
with \(\alpha = \rho + 2\beta R^{N+2} \|f - F(0, x, t)\|_{L^{N+1}(Q_R)}\).

**Theorem 3.5.** Let \(0 \leq f \leq \|f\|_{L^\infty(Q_R)}\), and let \(u\) be a nonnegative function in \(\mathcal{S}(f, Q_R)\). For every \(\kappa \in (0, 1)\), there exists \(c, C, \rho, \beta > 0\), depending on \(\lambda, \Lambda, N, \kappa\), such that for all \(|x| \leq \kappa R, 0 \geq t \geq -\kappa \|f\|_{L^{N+1}(Q_R)}^{N+1} \|f\|_{L^{N+1}(Q_R)} R^{2-(N+2)} \), 
\[
cR^2-(N+2)\alpha \|f\|_{L^\infty(Q_R)}^{1-(N+1)\alpha} \|f\|_{L^{N+1}(Q_R)}^{(N+1)\alpha} \leq u(x, t) \leq CR^N/N+1 \|f\|_{L^{N+1}(Q_R)}
\]
with \(\alpha = \rho + 2\beta R^N R^{2-(N+1)} \|f\|_{L^{N+1}(Q_R)}\).
Acknowledgements

This work was completed as a part of the author’s doctoral thesis. The author would like to thank her thesis advisor, Takis Souganidis, for his patient guidance and many helpful discussions. Also, the author would like to thank Carlos Kenig, for referring her to [5] in order to obtain the Fabes-Stroock estimate for parabolic cylinders. The author was supported as a graduate student on NSF grant DGE-1144082.

References

1. Micol Amar and Tullia Norando, On the Green’s function for parabolic equations in nondivergence form, Boll. Un. Mat. Ital. B (7) 6 (1992), no. 4, 703–731. MR 1200733 (93k:35123)
2. Luis A. Caffarelli and Xavier Cabré, Fully nonlinear elliptic equations, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995. MR 1351007 (96h:35046)
3. Luis A. Caffarelli and Panagiotis E. Souganidis, Rates of convergence for the homogenization of fully nonlinear uniformly elliptic pde in random media, Invent. Math. 180 (2010), no. 2, 301–360. MR 2609244 (2011c:35041)
4. Luis A. Caffarelli, Panagiotis E. Souganidis, and L. Wang, Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media, Comm. Pure Appl. Math. 58 (2005), no. 3, 319–361. MR 2116617 (2006b:35016)
5. A.-P. Calderón, Inequalities for the maximal function relative to a metric, Studia Math. 57 (1976), no. 3, 297–306. MR 0442579 (56 #960)
6. R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974), 241–250. MR 0358205 (50 #10670)
7. E. Fabes, N. Garofalo, and S. Salsa, A control on the set where a Green’s function vanishes, Colloq. Math. 60/61 (1990), no. 2, 637–647. MR 1096402 (92a:35074)
8. E. B. Fabes and D. W. Stroock, The Lp-integrability of Green’s functions and fundamental solutions for elliptic and parabolic equations, Duke Math. J. 51 (1984), no. 4, 997–1016. MR 771392 (86g:35057)
9. N. V. Krylov, Sequences of convex functions, and estimates of the maximum of the solution of a parabolic equation, Sibirsk. Mat. Ž. 17 (1976), no. 2, 290–303, 478. MR 0420016 (54 #8033)
10. , Nonlinear elliptic and parabolic equations of the second order, Mathematics and its Applications (Soviet Series), vol. 7, D. Reidel Publishing Co., Dordrecht, 1987, Translated from the Russian by P. L. Buzytsky [P. L. Buzytski]. MR 901759 (88d:35005)
11. Kaising Tso, On an Aleksandrov-Bakel’man type maximum principle for second-order parabolic equations, Comm. Partial Differential Equations 10 (1985), no. 5, 543–553. MR 790223 (87f:35031)
12. Lihe Wang, On the regularity theory of fully nonlinear parabolic equations. I, Comm. Pure Appl. Math. 45 (1992), no. 1, 27–76. MR 1135923 (92m:35126)

University of Chicago, Department of Mathematics, Chicago, IL 60637
E-mail address, Jessica Lin: jessica@math.uchicago.edu