CONLEY CONJECTURE REVISITED

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Dedicated to Dusa McDuff on the occasion of her 70th birthday

ABSTRACT. We show that whenever a closed symplectic manifold admits a Hamiltonian diffeomorphism with finitely many simple periodic orbits, the manifold has a spherical homology class of degree two with positive symplectic area and positive integral of the first Chern class. This theorem encompasses all known cases of the Conley conjecture (symplectic CY and negative monotone manifolds) and also some new ones (e.g., weakly exact symplectic manifolds with non-vanishing first Chern class).

The proof hinges on a general Lusternik–Schnirelmann type result that, under some natural additional conditions, the sequence of mean spectral invariants for the iterations of a Hamiltonian diffeomorphism never stabilizes. We also show that for the iterations of a Hamiltonian diffeomorphism with finitely many periodic orbits the sequence of action gaps between the “largest” and the “smallest” spectral invariants remains bounded and, as a consequence, establish some new cases of the \( C^\infty \)-generic existence of infinitely many simple periodic orbits.

CONTENTS

1. Introduction and main results 2
   1.1. Introduction 2
   1.2. Main results 3

2. Preliminaries 4
   2.1. Conventions and notation 4
   2.2. Floer homology 6
   2.3. The pair-of-pants product 8
   2.4. Spectral invariants and action carriers 10

3. Mean action 12
   3.1. Mean action and the Lusternik–Schnirelmann inequality 12
   3.2. Proof of Theorem 3.1 13

4. Proof of the main theorem 19

5. Perfect Hamiltonians and generic existence 20

References 26

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1. INTRODUCTION AND MAIN RESULTS

1.1. INTRODUCTION. We prove a Conley conjecture type theorem encompassing all known cases of the conjecture and also covering some new ones. An essential new feature of the theorem and its proof is a direct connection between the Conley conjecture and properties of the symplectic form, whereas all previous results utilized either only the first Chern class (the CY case) or the interplay between the two cohomology classes (the negative monotone case). The key to the proof is a general result concerning the behavior of the sequence of the mean spectral invariants for iterations of a Hamiltonian diffeomorphism. Namely, we show that the sequence never stabilizes when the periodic orbits are isolated and none of the orbits is a symplectically degenerate maximum (SDM). We also further investigate Hamiltonian diffeomorphisms with finitely many simple periodic orbits. In particular, we relate mean spectral invariants of such maps to resonance relations for augmented actions and show that the sequence of certain action gaps remains bounded. As a consequence, we prove the generic existence of infinitely many simple periodic orbits in some new cases.

Let us now review the context and background for these results in more detail.

The Conley conjecture asserts that for a broad class of closed symplectic manifolds every Hamiltonian diffeomorphism has infinitely many simple periodic orbits. The conjecture has been established for all symplectic CY manifolds, \cite{GG09a, He}, and all negative monotone symplectic manifolds, \cite{CGG, GG12}; see also \cite{FH, Gi10, Hi, SZ} for some relevant results and \cite{GG15} for a general survey and further references. In this paper, we prove that when a closed symplectic manifold \((M, \omega)\) admits a Hamiltonian diffeomorphism with finitely many periodic orbits, there is a class \(A \in \pi_2(M)\) with \(\omega(A) > 0\) and \(\langle c_1(TM), A \rangle > 0\). This result implies the Conley conjecture for CY and negative monotone manifolds. Furthermore, it also shows that the Conley conjecture holds, for instance, for weakly exact symplectic manifolds \((M, \omega)\) (i.e., such that \(\omega|_{\pi_2(M)} = 0\)) with \(c_1(TM)|_{\pi_2(M)} \neq 0\). (We refer the reader to \cite{Go} for a construction of such manifolds.)

The Conley conjecture fails for \(S^2\), \(\mathbb{CP}^n\), complex Grassmannians and, in fact, for all closed symplectic manifolds admitting Hamiltonian circle or torus actions with isolated fixed points. Indeed, then a generic element of the torus gives rise to a Hamiltonian diffeomorphism with finitely many periodic orbits. In fact, all known manifolds for which the Conley conjecture fails admit Hamiltonian \(S^1\)-actions with isolated fixed points. However, these manifolds may have other types of Hamiltonian diffeomorphisms with finitely many periodic orbits. This is the case when \(M = S^2\) and \(M = (S^2)^n\) and hypothetically for all symplectic manifolds with such \(S^1\)-actions. For \(S^2\) these diffeomorphisms are the so-called pseudo-rotations, which play a prominent role in low-dimensional dynamics (see, e.g., \cite{AK, FK}), and in general one can view Hamiltonian diffeomorphisms with finitely many periodic orbits as higher-dimensional analogs of pseudo-rotations.

Perhaps the most comprehensive conjecture identifying the manifolds for which the Conley conjecture fails is due to Chance and McDuff. It asserts that such a manifold has a non-vanishing GW invariant or even a non-trivially deformed quantum product. This conjecture fits well with the examples discussed above; for every symplectic manifold with Hamiltonian \(S^1\)-action in a certain sense uniruled and thus has a non-vanishing GW invariant; see \cite{McD09}. Our main result provides further evidence supporting the Chance–McDuff conjecture. Indeed, it implies, in
particular, that $\omega|_{\pi_2(M)} \neq 0$ whenever the Conley conjecture fails, and hence the manifold can at least have non-zero GW invariants. Additional evidence comes from the results on the bounded action gap discussed below.

Yet not a single particular case of the Chance–McDuff conjecture has been proved. The difficulty lies in identifying a source of holomorphic curves. Indeed, while the effect of holomorphic curves on Hamiltonian dynamics is well understood, it is completely unknown how to detect the existence of holomorphic curves from the dynamical behavior (e.g., periodic orbits) of Hamiltonians.

The method used in the proof of the main theorem is quite different from the proofs of other Conley conjecture type results. Namely, the main new ingredient is a Lusternik–Schnirelmann type result in the spirit of [GG09a, Prop. 6.2] and [GG16b, Thm. 1.1], which might be of independent interest. To state this result, assume that all periodic orbits of a Hamiltonian diffeomorphism $\varphi_H$ of a rational symplectic manifold $M$ are isolated and none of these orbits is an SDM. Then we show that the sequence of the mean spectral invariants $\hat{c}_k := c([M])(H^k) / k$ associated with the fundamental class of $M$ for the iterations $\varphi^k_H$ never stabilizes and, in fact, $\hat{c}_k > \hat{c}_\infty := \lim \hat{c}_k$. The key point here is that the inequality is strict; the non-strict inequality is an easy consequence of the standard properties of spectral invariants and holds under no additional assumptions on $H$. (Furthermore, one can also use this result to prove the negative monotone case of the Conley conjecture.)

We also investigate Hamiltonian diffeomorphisms with finitely many periodic orbits of monotone symplectic manifolds. We show that for such a Hamiltonian diffeomorphism the sequence of action gaps $c([M])(H^k) - c([pt])(H^k)$ remains bounded as $k \to \infty$. Usually such upper bounds result from non-trivial relations in the quantum homology of $M$. Hence this theorem can also be viewed as further evidence supporting the Chance–McDuff conjecture. We also relate the limit $\hat{c}_\infty$ to the augmented action of periodic orbits, refine the action–index resonance relations from [CGG, GG09a] and, as a consequence, obtain a new $C^\infty$-generic existence result for infinitely many simple periodic orbits.

1.2. Main results. Let us now state the main theorems and outline the organization of the paper.

The conventions and notation used in the paper and the necessary preliminary material on filtered and local Floer homology, the pair-of-pants product, spectral invariants and action carriers are reviewed in Section 2. Most of the definitions and facts stated there are quite standard, although the Lusternik–Schnirelmann inequality for the pair-or-pants product (Proposition 2.2) might be new. Here we only note that we focus exclusively on contractible periodic orbits, i.e., a “periodic orbit” means a “contractible periodic orbit.” Our key Conley conjecture type result is the following.

**Theorem 1.1.** Assume that a closed symplectic manifold $M$ admits a Hamiltonian diffeomorphism $\varphi_H$ with finitely many periodic orbits. Then there exists $A \in \pi_2(M)$ such that $\omega(A) > 0$ and $\langle c_1(TM), A \rangle > 0$.

The main new point of this theorem is the existence of a spherical class $A$ satisfying the first of these two conditions. Theorem 1.1 is proved in Section 4. The proof hinges on the following general fact established in Section 3.

**Theorem 1.2** (Lusternik–Schnirelmann inequality for mean spectral invariants). Assume that $M$ is rational, all periodic orbits of $H$ are isolated and none of the
orbits is an SDM. Then
\[ \hat{c}_k > \hat{c}_\infty \]
for all \( k \), where \( \hat{c}_k = c_{[M]}(H^{2^k})/k \) is the mean spectral invariant associated with the fundamental class and applied to the "k-iterated" Hamiltonian \( H^{2^k} \), and \( \hat{c}_\infty = \lim k \hat{c}_k \).

Finally, in Section 5 we turn to Hamiltonians \( H \) with finitely many simple periodic orbits. By passing to an iteration, we can always assume that every simple periodic orbit of \( H \) is one-periodic, i.e., \( H \) is perfect.

**Theorem 1.3** (Action Gap). Assume that \( H \) is perfect and \( M \) is monotone with monotonicity constant \( \lambda \). Then
\[ c_{[M]}(H^{2^k}) - c_{[pt]}(H^{2^k}) \leq 2\lambda n, \tag{1.1} \]
for all but possibly a finite number of iterations \( k \in \mathbb{N} \).

We also show in Theorem 5.1 that for a perfect Hamiltonian \( H \), the asymptotic mean spectral invariant \( \hat{c}_\infty(H) \) is equal to the augmented action of its action carrier. As a consequence, we refine the action–index resonance relations from [CGG, GG09a] by proving the existence of several geometrically distinct simple orbits with augmented actions equal \( \hat{c}_\infty(H) \); see Corollaries 5.6, 5.9 and 5.10. As another application, in Corollary 5.7 we establish \( C^\infty \)-generic existence of infinitely many simple periodic orbits for essentially all closed monotone symplectic manifolds with a minor hypothetical exception; cf. [GG09b].

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## 2. Preliminaries

The goal of this section is to set notation and conventions and to give a brief review of Floer homology and several other notions used in the paper.

### 2.1. Conventions and notation.

Let \( (M^{2n}, \omega) \) be a closed symplectic manifold. Throughout the paper we will usually assume that \( M \) is rational, i.e., the group \( \langle [\omega] \rangle \subset \mathbb{R} \) formed by the integrals of \( \omega \) over the spheres in \( M \) is discrete. This condition is obviously satisfied when \( M \) is negative monotone or monotone, i.e., \( [\omega]_{[\pi_2(M)]} = \lambda c_1(TM) \mid_{\pi_2(M)} \) for some \( \lambda < 0 \) in the former case or \( \lambda \geq 0 \) in the latter. When \( \lambda > 0 \), we will sometimes say that \( M \) is positive monotone. The positive generator \( N \) of the group \( \langle c_1(TM), \pi_2(M) \rangle \subset \mathbb{Z} \) is called the minimal Chern number of \( M \). (When this group is zero, we set \( N = \infty \).) Recall also that \( M \) is symplectic Calabi–Yau (CY) if \( c_1(TM) \mid_{\pi_2(M)} = 0 \) and \( M \) is called symplectically aspherical when, in addition, \( [\omega]_{[\pi_1(M)]} = 0 \).

All Hamiltonians \( H \) considered in this paper are assumed to be \( k \)-periodic in time, i.e., \( H : S^1_k \times M \to \mathbb{R} \), where \( S^1_k = \mathbb{R}/k\mathbb{Z} \) and \( k \in \mathbb{N} \). When the period \( k \) is not specified, it is equal to one and \( S^1 = S^1_1 = \mathbb{R}/\mathbb{Z} \). We set \( H_t = H(t, \cdot) \) for \( t \in S^1_k \). The Hamiltonian vector field \( X_H \) of \( H \) is defined by \( i_{X_H} \omega = -dH \). The
(time-dependent) flow of $X_H$ is denoted by $\varphi'_H$ and its time-one map by $\varphi_H$. Such time-one maps are referred to as Hamiltonian diffeomorphisms. A one-periodic Hamiltonian $H$ can always be treated as $k$-periodic, which we will then denote by $H^{tk}$ and, abusing terminology, call $H^{tk}$ the $k$th iteration of $H$.

Let $H$ and $K$ be one-periodic Hamiltonians such that $H_1 = K_0$ together with $t$-derivatives of all orders. We denote by $H_2K$ the two-periodic Hamiltonian equal to $H_t$ for $t \in [0, 1]$ and $K_{t-1}$ for $t \in [1, 2]$. Thus $H^{tk} = H^2 \ldots ^kH$ ($k$ times). More generally, when $H$ is $l$-periodic and $K$ is $k$-periodic, $H^2K$ is $(l+k)$-periodic. (Strictly speaking, here we need to assume that $H$ is the negative of that in $\{0, 1\}$ and $H^2 \ldots ^kH$ is $k$-periodic.)

Let $x: S^1 \to M$ be a contractible loop. A capping of $x$ is an equivalence class of maps $A: D^2 \to M$ such that $A|_{S^1} = x$. Two cappings $A$ and $A'$ of $x$ are equivalent if the integrals of $\omega$ and $c_1(TM)$ over the sphere obtained by attaching $A$ to $A'$ are equal to zero. A capped closed curve $\bar{x}$ is, by definition, a closed curve $x$ equipped with an equivalence class of cappings. In what follows, the presence of capping is always indicated by a bar.

The action of a Hamiltonian $H$ on a capped closed curve $\bar{x} = (x, A)$ is

$$A_H(\bar{x}) = -\int_A \omega + \int_{S^1} H_t(x(t)) dt.$$

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of $A_H$ on this space are exactly the capped one-periodic orbits of $X_H$. The action spectrum $S(H)$ of $H$ is the set of critical values of $A_H$. This is a zero measure set; see, e.g., [HZ]. When $M$ is rational, $S(H)$ is a closed, and hence nowhere dense, set. Otherwise, $S(H)$ is dense in $\mathbb{R}$.

These definitions extend to $k$-periodic orbits and Hamiltonians in an obvious way. Clearly, the action functional is homogeneous with respect to iteration:

$$A_{H^{tk}}(\bar{x}^k) = kA_H(\bar{x}),$$

where $\bar{x}^k$ is the $k$th iteration of the capped orbit $\bar{x}$.

As mentioned in the introduction, all of our results concern only contractible periodic orbits and throughout the paper a periodic orbit is always assumed to be contractible, even if this is not explicitly stated. We denote the set of $k$-periodic orbits of $H$ by $P_k(H)$. The set of all periodic orbits will be denoted by $P(H)$. Finally, we will write $P_k(H)$ and $P(H)$ for the collections of simple (i.e., not iterated) periodic orbits of $H$.

A periodic orbit $x$ of $H$ is said to be non-degenerate if the linearized return map $d\varphi_H: T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. Following [SZ], we call $x$ weakly non-degenerate if at least one of the eigenvalues is different from one and totally degenerate if all eigenvalues are equal to one. A Hamiltonian $H$ is (weakly) non-degenerate if all its one-periodic orbits are (weakly) non-degenerate and $H$ is strongly non-degenerate if all periodic orbits of $H$ (of all periods) are non-degenerate.

Let $\bar{x} = (x, A)$ be a non-degenerate capped periodic orbit. The Conley–Zehnder index $\mu_{cz}(\bar{x}) \in \mathbb{Z}$ is defined, up to a sign, as in [Sa, SZ]. (Sometimes, we will also use the notation $\mu_{cz}(x, A).$) In this paper, we normalize $\mu_{cz}$ so that $\mu_{cz}(\bar{x}) = n$ when $x$ is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian. With this normalization, the Conley–Zehnder index is the negative of that in [Sa]. The mean index $\hat{\mu}_{cz}(\bar{x}) \in \mathbb{R}$ measures, roughly
speaking, the total angle swept by certain unit eigenvalues of the linearized flow $d\varphi^t_H|_x$ with respect to the trivialization associated with the capping; see [Lo, SZ]. The mean index is defined even when $x$ is degenerate and depends continuously on $H$ and $\bar{x}$ in the obvious sense. Furthermore,
\[ |\hat{\mu}_{cz}(\bar{x}) - \mu_{cz}(\bar{x})| \leq n \]
and the inequality is strict when $x$ is weakly non-degenerate. The mean index is homogeneous with respect to iteration:
\[ \hat{\mu}_{cz}(\bar{x}^k) = k\hat{\mu}_{cz}(\bar{x}). \]

2.2. Floer homology. In this subsection, we very briefly discuss, mainly to set notation, the construction of filtered Floer homology. We refer the reader to, e.g., [FO, GG09a, HS, MS, Sa, SZ] for detailed accounts and additional references. We also recall the definition of the local Floer homology.

2.2.1. Filtered Floer homology. Fix a ground field $\mathbb{F}$. Let $H$ be a non-degenerate Hamiltonian on $M$. Denote by $\text{CF}^*(\alpha, \beta)(H)$, with $\alpha, \beta \in (-\infty, \infty] \setminus S(H)$, the vector space of formal linear combinations
\[ \sigma = \sum_{x \in P(H)} \alpha_x \bar{x}, \]
where $\alpha_x \in \mathbb{F}$ and $\mu_{cz}(\bar{x}) = m$ and $\mathcal{A}_H(\bar{x}) = b$. Furthermore, we require, for every $a \in \mathbb{R}$, the number of terms in this sum with $\alpha_x \neq 0$ and $\mathcal{A}_H(\bar{x}) > a$ to be finite. The graded $\mathbb{F}$-vector space $\text{CF}^*(\alpha, \beta)(H)$ is endowed with the Floer differential counting the $L^2$-anti-gradient trajectories of the action functional. Thus we obtain a filtration of the total Floer complex $\text{CF}_*(H) := \text{CF}^*(-\infty, \infty)(H)$. Furthermore, set
\[ \text{CF}_*(\alpha, \beta)(H) := \text{CF}_*(\alpha, \beta)(H)/\text{CF}_*(\alpha, \alpha)(H), \]
where $-\infty \leq a < b \leq \infty$ are not in $S(H)$. The resulting homology, the filtered Floer homology of $H$, is denoted by $\text{HF}_*(\alpha, \beta)(H)$ and by $\text{HF}_*(H)$ when $(\alpha, \beta) = (-\infty, \infty)$. Working with filtered Floer homology, we will always assume that the end points of the action interval are not in the action spectrum. The degree of a class $v \in \text{HF}_*(\alpha, \beta)(H)$ is denoted by $|v|$.

Over $\mathbb{F} = \mathbb{Z}_2$, we can view a chain $\sigma = \sum \alpha_x \bar{x} \in \text{CF}_*(H)$ as simply a collection of capped one-periodic orbits $\bar{x}$ for which $\alpha_x \neq 0$. In general, we will say that $\bar{x}$ enters the chain $\sigma$ when $\alpha_x \neq 0$. Note also that it is often convenient to view $\text{CF}_*(\alpha, \beta)(H)$ as a subspace, but not in general a subcomplex, of $\text{CF}_*(H)$.

The total Floer complex and homology are modules over the Novikov ring $\Lambda$. In this paper, the latter is defined as follows. Set
\[ I_\omega(A) = -\omega(A) \text{ and } I_{c_1}(A) = -2 \langle c_1(TM), A \rangle, \]
where $A \in \pi_2(M)$. Thus
\[ I_\omega = \frac{\lambda}{2} I_{c_1} \]
when $M$ is monotone or negative monotone.

Let $\Gamma$ be the quotient of $\pi_2(M)$ by the equivalence relation where the two spheres $A$ and $A'$ are considered to be equivalent if $I_\omega(A) = I_\omega(A')$ and $I_{c_1}(A) = I_{c_1}(A')$. In other words,
\[ \Gamma = \frac{\pi_2(M)}{\ker I_\omega \cap \ker I_{c_1}}. \]
For instance, $\Gamma \simeq \mathbb{Z}$ when $M$ is negative monotone or monotone with $\lambda \neq 0$. The homomorphisms $I_{\omega}$ and $I_{c_1}$ descend to $\Gamma$.

The group $\Gamma$ acts on $\text{CF}_*(H)$ and on $\text{HF}_*(H)$ by recapping: an element $A \in \Gamma$ acts on a capped one-periodic orbit $\bar{x}$ of $H$ by attaching the sphere $A$ to the original capping. Denoting the resulting capped orbit by $\bar{x}#A$, we have

$$\mu_{cz}(\bar{x}#A) = \mu_{cz}(\bar{x}) + I_{c_1}(A)$$

when $x$ is non-degenerate. In a similar vein,

$$A_H(\bar{x}#A) = A_H(\bar{x}) + I_{\omega}(A) \quad \text{and} \quad \hat{\mu}_{cz}(\bar{x}#A) = \hat{\mu}_{cz}(\bar{x}) + I_{c_1}(A) \quad (2.4)$$

regardless of whether $x$ is non-degenerate or not.

The Novikov ring $\Lambda$ is a certain completion of the group ring of $\Gamma$ over $\mathbb{F}$. Namely, $\Lambda$ comprises formal linear combinations $\sum \alpha_A e^A$, where $\alpha_A \in \mathbb{F}$ and $A \in \Gamma$, such that for every $a \in \mathbb{R}$ the sum contains only finitely many terms with $I_{\omega}(A) > a$ and $\alpha_A \neq 0$. The Novikov ring $\Lambda$ is graded by setting $|e^A| = I_{c_1}(A)$. The action of $\Gamma$ turns $\text{CF}_*(H)$ and $\text{HF}_*(H)$ into $\Lambda$-modules.

When $M$ is rational, the definition of Floer homology extends to all, not necessarily non-degenerate, Hamiltonians by continuity. Namely, let $H$ be an arbitrary (one-periodic in time) Hamiltonian on $M$ and let the end-points $a$ and $b$ of the action interval be outside $S(H)$. By definition, we set

$$\text{HF}^{(a,b)}_*(H) = \text{HF}^{(a,b)}_* (\tilde{H}), \quad (2.5)$$

where $\tilde{H}$ is a non-degenerate, small perturbation of $H$. It is easy to see that the right hand side of (2.5) is independent of $\tilde{H}$ once $\tilde{H}$ is sufficiently close to $H$. (The assumption that $M$ is rational is essential at this point; for, otherwise, the right hand side of (2.5) depends on the perturbation $\tilde{H}$ even when $\tilde{H}$ is arbitrarily close to $H$. We refer the reader to [He] and also to [GG09a, Remark 2.3] for the definition in the irrational case.)

The total Floer homology is independent of the Hamiltonian and, up to a shift of the grading and the effect of recapping, is isomorphic to the homology of $M$. More precisely, we have

$$\text{HF}_*(H) \cong H_{*+n}(M; \mathbb{F}) \otimes \Lambda$$

as graded $\Lambda$-modules.

Remark 2.1. In general, in order for the Floer homology to be defined, certain regularity conditions must be satisfied generically. To ensure this, one has to either require $M$ to be weakly monotone (see [HS, MS, On, Sa]) or utilize the machinery of virtual cycles (see [FO, FO$^3$, LT] or, for the polyfold approach, [HWZ10, HWZ11] and references therein). In the latter case, the ground field $\mathbb{F}$ is required to have zero characteristic. Most of the proofs in this paper do not rely on the machinery of virtual cycles. To be more specific, the proof of Theorem 1.1 comprises known and new cases of the Conley conjecture. In the proof of the new cases (Proposition 4.2), arguing by contradiction, one can assume that $I_{\omega} = 0$ and hence the Floer homology can be defined without virtual cycles.

2.2.2. Local Floer homology. The notion of local Floer homology goes back to the original work of Floer and it has been revisited a number of times since then. Here we only briefly recall the definition following mainly [Gi10, GG09a, GG10] where the reader can find a much more thorough discussion and further references.
Let $\bar{x} = (x, A)$ be a capped isolated one-periodic orbit of a Hamiltonian $H : S^1 \times M \to \mathbb{R}$. Pick a sufficiently small tubular neighborhood $U$ of $x$ and consider a non-degenerate $C^2$-small perturbation $\tilde{H}$ of $H$. (Strictly speaking $U$ should be a neighborhood of the graph of the orbit in the extended phase space $S^1 \times M$.) The orbit $x$ splits into non-degenerate one-periodic orbits $x_j$ of $\tilde{H}$, which are $C^1$-close to $x$. The capping of $\bar{x}$ gives rise to a capping of $x_j$ and $A_{\tilde{H}}(\bar{x}_j)$ is close to $A_H(\bar{x})$.

Every Floer trajectory between the orbits $\bar{x}_j$ is contained in $U$, provided that $\|\tilde{H} - H\|_{C^2}$ is small enough. Thus, by the compactness and gluing theorems, every broken anti-gradient trajectory connecting two such orbits also lies entirely in $U$. Similarly to the definition of the ordinary Floer homology, consider the complex $CF_*(\tilde{H}, \bar{x})$ over $\mathbb{F}$ generated by the capped orbits $\bar{x}_j$, graded by the Conley–Zehnder index and equipped with the Floer differential defined in the standard way. The continuation argument shows that the homology of this complex is independent of the choice of $\tilde{H}$ and of other auxiliary data (e.g., an almost complex structure). We refer to the resulting homology group, denoted by $HF_*(\bar{x})$ or $HF_*(x, A)$, as the local Floer homology of $\bar{x}$. For instance, if $x$ is non-degenerate and $\mu_{CZ}(\bar{x}) = m$, we have $HF_l(\bar{x}) = \mathbb{F}$ when $l = m$ and $HF_l(\bar{x}) = 0$ otherwise.

The above construction is local: it requires $H$ to be defined only on a neighborhood of $x$ and the capping of $x$ is used only to fix a trivialization of $TM|_x$ and hence an absolute $\mathbb{Z}$-grading of $HF_*(\bar{x})$.

By definition, the support of $HF_*(\bar{x})$ is the collection of integers $m$ such that $HF_m(\bar{x}) \neq 0$. By (2.1) and continuity of the mean index, $HF_*(\bar{x})$ is supported in the interval $[\hat{\mu}_{CZ}(\bar{x}) - n, \hat{\mu}_{CZ}(\bar{x}) + n]$. Moreover, when $x$ is weakly non-degenerate, the closed interval can be replaced by an open interval.

Recall that a capped isolated periodic orbit $\bar{x}$ is called a symplectically degenerate maximum (SDM) if $\hat{\mu}_{CZ}(\bar{x}) + n(\bar{x}) \neq 0$, where we set $HF_*(\bar{x}) = 0$ when $* \notin \mathbb{Z}$. This property is independent of the capping. An SDM orbit is necessarily totally degenerate and an iteration of an SDM is again an SDM; see, e.g., [Gi10, GG10]. By [GG09a, Thm. 1.18], a Hamiltonian diffeomorphism of a rational symplectic manifold with an SDM orbit has infinitely many periodic orbits.

2.3. The pair-of-pants product. In this section, we briefly recall several properties of the pair-of-pants product in Floer homology, referring the reader to, e.g., [AS, MS, PSS] for more detailed accounts.

The filtered Floer homology carries the so-called pair-of-pants product. On the level of complexes, this product, which we denote by $\langle * \rangle$, is a map

$$\text{CF}^{(-\infty, a)}_m(H) \otimes \text{CF}^{(-\infty, b)}_l(K) \to \text{CF}^{(-\infty, a+b)}_{m+l-n}(H^2 K) \quad (2.6)$$

giving rise on the level of homology to an associative, graded-commutative product

$$HF^{(-\infty, a)}_m(H) \otimes HF^{(-\infty, b)}_l(K) \to HF^{(-\infty, a+b)}_{m+l-n}(H^2 K). \quad (2.7)$$

The product turns the direct sum of the total Floer homology

$$\bigoplus_{k \geq 0} HF_* (H^{2k})$$

into an associative and graded-commutative unital algebra, where $HF_* (H^{20})$ is by definition the quantum homology $HQ_*(M)$ of $M$. This direct sum is isomorphic, as an algebra, to the algebra of polynomials with coefficients in $HQ_*(M)$. The unit in this algebra is the fundamental class of $M$ in $HQ_*(M)$. 
On the level of complexes the product (2.6) is not associative in any sense. Furthermore, in order for the product to be defined the Hamiltonians \( H \) and \( K \) must meet certain generic regularity conditions. With this in mind, the product on the level of homology is defined “by continuity” for all Hamiltonians, at least when \( M \) is rational.

Note that the multiplication by the fundamental class \([M] \in HF_n(H)\) is a grading-preserving (but not action-preserving) isomorphism

\[
[M]: \, \text{HF}_n(H^{\text{tk}}) \xrightarrow{\cong} \text{HF}_n(H^{\text{z}(k+1)}).
\]

A word of warning is due regarding the behavior of the action filtration with respect to the pair-of-pants product. The original definition of the product as in, e.g., [MS, PSS] relies on cutting off the Hamiltonians. With this definition the action filtration is not preserved in the sense of (2.6) and (2.7). Cutting off can be avoided when \( H_t \) and \( K_t \) vanish for \( t \) close to \( 0 \in S^1 \), which can always be achieved for closed manifolds by simply reparametrizing the Hamiltonians. Under this extra assumption, (2.6) and (2.7) hold as stated. However, a more elegant definition of the pair-of-pants product is given in [AS], where the domain \( \Sigma \) of a pair-of-pants curve \( u \) is treated as a double cover of the cylinder, branching at one point. The domain \( \Sigma \) naturally carries the “coordinates” \((s, t)\) lifted from the cylinder, which are true coordinates on the three open half-cylindrical parts of the domain, and on each of these parts \( u \) satisfies the Floer equation for the corresponding Hamiltonian \( H \) or \( K \) or \( H_k \). For a pair-of-pants curve \( u \) connecting \( x \) and \( y \) to \( z \), we have

\[
A_H(x) + A_K(y) - A_{H \# K}(z) = E(u), \quad \text{where } E(u) := \int_\Sigma \|\partial_s u\|^2 \, ds \, dt \geq 0,
\]

which, in particular, implies (2.6) and (2.7); see [AS, Eq. (3-18)]. Note that here the capping of \( z \) is obtained by attaching \( u \) to the cappings of \( x \) and \( y \).

Clearly, \( A_H(x) + A_K(y) = A_{H \# K}(z) \) if and only if \( E(u) = 0 \), and if and only if \( u \) is a “trivial” pair-of-pants curve. More specifically, then \( \partial_s u \equiv 0 \) and \( x(0) = y(0) \), and thus \( u \) is the projection from \( \Sigma \) to the figure-8 curve formed by \( x \) and \( y \).

As a consequence of (2.9), there is a variant of the Lusternik-Schnirelmann inequality in the spirit of [GG09a, Prop. 6.2] and [GG16b, Thm. 1.1] for the pair-of-pants product, which plays an important role in the proof of Theorem 1.1. For the sake of brevity, we state here only a simple version of this inequality sufficient for our purposes and concerning iterated Hamiltonians.

Let \( \hat{H} \) be a one-periodic Hamiltonian such that all its one-periodic orbits \( x_i \) are isolated. Let us fix small disjoint neighborhoods \( U_i \) of these orbits. It is convenient to assume that the orbits \( x_i \) are constant and hence the neighborhoods \( U_i \) are simply small disjoint balls in \( M \). Note that the orbits can be made constant by composing the flow of \( H \) with contractible loops in the group of Hamiltonian diffeomorphisms; see [Gi10, Sect. 5.1]. (This is not necessary and we can take a small neighborhood of the image of \( x_i \) as \( U_i \). However, in this case we also need to require the images of \( x_i \) to be disjoint.) Assume, furthermore, that no \( k \)- or \((k + 1)\)-periodic orbit of \( H \), other than \( x_i^k \) or \( x_i^{k+1} \), enters any of the neighborhoods \( U_i \).

Let \( \tilde{H} \) and \( K \) be small non-degenerate perturbations of \( H \) and \( H^{\text{tk}} \) such that the regularity conditions for the pair-of-pants product between \( \tilde{H} \) and \( K \) are satisfied. (Here we think of \( K \) as a \( k \)-periodic Hamiltonian, and hence \( \tilde{H} \# K \) is \((k + 1)\)-periodic.) Let \( u \) be a pair-of-pants curve from a capped one-periodic orbit \( \tilde{x} \) of \( \tilde{H} \) and a capped \( k \)-periodic orbit \( \tilde{y} \) of \( K \) to a capped \((k + 1)\)-periodic orbit \( \tilde{z} \) of \( \tilde{H} \# K \).
Outline of the proof of Proposition 2.2. If u does not lie entirely in the neighborhood $U_i$ containing $x$, it has to pass through a neighborhood $W$ of the boundary $\partial U_i = U_i \setminus U_i$. When the perturbations $\tilde{H}$ and $K$ are sufficiently close to $H$ and, respectively, $H^{2k}$, no $k$-periodic orbit of $K$ and $(k+1)$-periodic orbit of $\tilde{H}^{2k}$, other than the orbits which $x^k$ and $x^{k+1}$ split into, enters $U_i$. Thus there are no periodic orbits of $H$, $K$ and $\tilde{H}^{2k}$ with periods respectively 1, $k$ and $k+1$ passing through $W$. By (2.9), it is enough to obtain a lower bound $\epsilon > 0$ on the energy $E(u)$. When $E(u)$ is below a certain threshold, $\|\partial_u u\|_{L^\infty}$ is small and, in fact, $\|\partial_u u\|_{L^\infty} = o(1)$ as $E(u) \to 0$. Then $\partial_u u$ is close to the Hamiltonian vector field, since in half-cylindrical parts of its domain $\Sigma$ the map $u$ is governed by the corresponding Floer equations. Now arguing as in [Sa, Sect. 1.5] or [FO, Lemma 19.8] it is not hard to see that $u$ has to acquire a certain amount of energy, a priori bounded from below, while passing through $W$. As a consequence, we obtain a lower bound on $E(u)$. \qed

2.4. Spectral invariants and action carriers. The theory of Hamiltonian spectral invariants was developed in its present Floer-theoretic form in [Oh, Scw], although the first versions of the theory go back to [HZ, Vi]. Here we briefly recall some elements of this theory essential for what follows, mainly following [GG09a]. We refer the reader to [U11] for a detailed treatment of spectral invariants.

Let $M$ be a closed rational symplectic manifold and let $H$ be a Hamiltonian on $M$. The spectral invariant or action selector $c_w$ associated with a class $w \in \text{HF}_*(H) = \text{HQ}_*(M)$ is defined as

$$c_w(H) = \inf\{a \in \mathbb{R} \setminus \mathcal{S}(H) \mid w \in \text{im}(i^a)\} = \inf\{a \in \mathbb{R} \setminus \mathcal{S}(H) \mid j^a(w) = 0\},$$

where $i^a: \text{HF}_{(-\infty, a]}(H) \to \text{HF}_*(H)$ and $j^a: \text{HF}_*(H) \to \text{HF}_*[a, \infty)(H)$ are the natural “inclusion” and “quotient” maps. It is easy to see that $c_w(H) > -\infty$.

The action selector $c_w$ is a symplectic invariant of $H$ with the following properties:

(A1) Continuity: $c_w$ is Lipschitz in $H$ in the $C^0$-topology.

(A2) Monotonicity: $c_w(H) \geq c_w(K)$ whenever $H \geq K$ pointwise.

(A3) Hamiltonian shift:

$$c_w(H + a(t)) = c_w(H) + \int_{S^1} a(t) \, dt,$$

where $a: S^1 \to \mathbb{R}$. 

\textit{Remark 2.3.} If the image of $u$ is in $U_i$, the orbits $x$, $y$ and $z$ also lie in $U_i$. Note also that the condition that the neighborhoods $U_i$ are disjoint can be omitted but then the inequality holds when $u$ is not contained in a connected component of $\cup U_i$. The proof of the proposition is rather standard and we only spell out the main idea.
(AS4) Homotopy invariance: \( c_w(H) = c_w(K) \) when \( \varphi_H = \varphi_K \) in the universal covering of the group of Hamiltonian diffeomorphisms and \( H \) and \( K \) have the same mean value.

(AS5) Triangle inequality or sub-additivity: \( c_{w_1 \ast w_2}(H_2 K) \leq c_{w_1}(H) + c_{w_2}(K) \).

(AS6) Spectrality: \( c_w(H) \in S(H) \). More specifically, there exists a capped one-periodic orbit \( \bar{x} \) of \( H \) such that \( c_w(H) = A_H(\bar{x}) \).

The above list of properties of \( c_w \) is far from exhaustive, but it is more than sufficient for our purposes. Most of these properties are rather direct consequences of the definition. However, the sub-additivity, (AS5), relies on (2.7). It is worth emphasizing that the rationality assumption plays an important role in the proofs of the homotopy invariance and spectrality; see [Oh, Scw]. (The latter property also holds in general for non-degenerate Hamiltonians. This is a non-trivial result; see [U08].)

When \( H \) is non-degenerate, the action selector can also be evaluated as

\[
c_w(H) = \inf_{[\sigma] = w} A_H(\sigma),
\]

where we set

\[
A_H(\sigma) = \max \left\{ A_H(\bar{x}) \mid \alpha \bar{x} \neq 0 \right\} \text{ for } \sigma = \sum \alpha \bar{x} \in \text{CF}_w(H).
\]

The infimum here is obviously attained, since \( M \) is rational and thus \( S(H) \) is closed. Hence there exists a cycle \( \sigma = \sum \alpha \bar{x} \in \text{CF}_w(H) \), representing the class \( w \), such that \( c_w(H) = A_H(\bar{x}) \) for an orbit \( \bar{x} \) entering \( \sigma \). In other words, \( \bar{x} \) maximizes the action on \( \sigma \) and the cycle \( \sigma \) minimizes the action over all cycles in the homology class \( w \). We call such an orbit \( \bar{x} \) a carrier of the action selector. This is a stronger requirement than just that \( c_w(H) = A_H(\bar{x}) \) and \( \mu_{\text{CZ}}(\bar{x}) = |w| \). When \( H \) is possibly degenerate, a capped one-periodic orbit \( \bar{x} \) of \( H \) is a carrier of the action selector if there exists a sequence of \( C^2 \)-small, non-degenerate perturbations \( H_i \to H \) such that one of the capped orbits \( \bar{x} \) splits into a carrier for \( H_i \). An orbit (without capping) is said to be a carrier if it turns into one for a suitable choice of capping.

It is easy to see that a carrier necessarily exists, provided that \( M \) is rational. A carrier is not in general unique, but it becomes unique when all one-periodic orbits of \( H \) have distinct action values.

As an immediate consequence of the definition of the carrier and continuity of the action and the mean index, we have

\[
c_w(H) = A_H(\bar{x}) \quad \text{and} \quad |\mu_{\text{CZ}}(\bar{x}) - |w|| \leq n,
\]

and the inequality is strict when \( x \) is weakly non-degenerate.

Here we will be mainly interested in the spectral invariant associated with the fundamental class \( w = [M] \in H_{2n}(M;\mathbb{F}) \subset HF_*(H) = HQ_*(M) \) and set for the sake of brevity

\[
c(H) := c_{[M]}(H).
\]

Then \( c(H) = \max H \) when \( H \) is autonomous and \( C^2 \)-small, and (AS5) takes a simpler form:

\[
c(H_2 K) \leq c(H) + c(K).
\]

As an immediate consequence of (AS5), we see that the multiplication map \( \ast [M] \) from (2.8) shifts all spectral invariants by at most \( c(H) \) upward. A carrier \( \bar{x} \) for \( c \) is in some sense homologically essential. Namely, \( HF_n(\bar{x}) \neq 0 \), provided that all
one-periodic orbits of $H$ are isolated; [GG12, Lemma 3.2]. (In fact, this is true for all spectral invariants $c_w$, where now $HF_{|w|}(\bar{x}) \neq 0$.)

3. Mean action

3.1. Mean action and the Lusternik–Schnirelmann inequality. Consider the sequence $c_k := c(H^{\#k})$. By (AS5), the sequence $c_k$ is sub-additive, i.e.,

$$c_{k+l} \leq c_k + c_l,$$

and the normalized sequence $\hat{c}_k = \hat{c}_k(H) = c_k/k$ converges. In fact, we have a slightly more precise result. Namely,

$$\hat{c}_\infty := \lim_{k \to \infty} \hat{c}_k = \inf_k \hat{c}_k.$$  \hspace{1cm} \text{(3.1)}

This is an easy consequence of the sub-additivity of $c_k$ and of the obvious fact that $\hat{c}_\infty > -\infty$; see, e.g., [PS, p. 37, Problem 98]. In particular,

$$\hat{c}_k \geq \hat{c}_\infty.$$  

Moreover, under certain natural additional conditions we have a strict inequality along the lines of Lusternik-Schnirelmann theory as interpreted in [GG09a] and [GG16b]. This inequality, Theorem 1.2, plays a key role in the proof of Theorem 1.1. For the reader’s convenience we state the result again.

**Theorem 3.1** (Lusternik–Schnirelmann inequality for mean spectral invariants). Assume that $M$ is rational, all periodic orbits of $H$ are isolated and none of the orbits is an SDM. Then, for all $k$,

$$\hat{c}_k > \hat{c}_\infty.$$  

We give a self-contained and detailed proof of the theorem in the next section.

**Remark 3.2.** The underlying principle behind this theorem is that it is very easy for the action to go down in the triangle inequality, (AS5), and the equality in (AS5) imposes strong restrictions on the behavior of periodic orbits. For instance, one can infer from the equality that $w_1$ and $w_2$ admit carriers $\bar{x}$ and $\bar{y}$ with common initial condition, i.e., $x(0) = y(0)$, provided that all one-periodic orbits of $H$ and $K$ are isolated. This can be proved similarly to Theorem 3.1 with suitable modifications and, in fact, simplifications.

**Remark 3.3.** In Theorem 3.1 the assumption that the periodic orbits are isolated is certainly necessary. (For instance, when $H = \text{const}$, we have $\hat{c}_k = \hat{c}_\infty$ for all $k$.

There are also “less trivial” examples.) It is also likely that the condition that none of the orbits is an SDM is essential, although we do not have an example readily showing this.

**Remark 3.4.** In general, the proof of Theorem 3.1 relies on the machinery of virtual cycles (and hence, in particular, we must set $F = \mathbb{Q}$), unless the manifold $M$ is assumed to be weakly monotone. In this paper we apply Theorem 3.1 in the setting of Proposition 4.2 when $I_\omega = 0$, and hence the latter requirement is automatically satisfied.
Finally, denote by \( \hat{S}(H) \) the \textit{normalized or mean action spectrum} of \( H \), i.e., the union of the increasing sequence of the nested sets \( \hat{S}(H^{2k})/k \):

\[
\hat{S}(H) = \bigcup_k \frac{1}{k} S(H^{2k}).
\]

This set arises naturally in the proofs of some of the Conley conjecture type results; see [GG09a]. Clearly, just because \( \hat{c}_k \to \hat{c}_\infty \) and \( \hat{c}_k \in \hat{S}(H) \), we have

\[
\hat{c}_\infty(H) \geq \inf \hat{S}(H).
\]

The invariant \( \hat{c}_\infty(H) \) is closely related to Calabi quasi-morphisms; see [EP, McD10, Os, U11]. For us, however, \( \hat{c}_\infty \) is of interest because of its role in the proofs of the main theorems.

Remark 3.5. The connection between \( \hat{c}_\infty \) and the Calabi invariant, established in [EP], along with (3.1), (3.2) and Theorem 3.1 seem to suggest that one might be able to reprove and extend to higher dimensions recent results of Hutchings from [Hu] by using “conventional” symplectic topological techniques, not relying on the ECH machinery. However, we have not been able to do this.

3.2. \textbf{Proof of Theorem 3.1.} The proof of the theorem is carried out in several steps.

\textit{Step 1: Preparation and action stabilization.} First, note that it is sufficient to show that \( \hat{c}_l \neq \hat{c}_\infty \) for any \( l \); for \( \hat{c}_\infty = \inf \hat{c}_l \) by (3.1). Arguing by contradiction, assume that \( \hat{c}_l = \hat{c}_\infty \) for some \( l \). Then, since the sequence \( c(H^{2k}) \) is sub-additive by (AS5) and again by (3.1), we have \( \hat{c}_1 = \hat{c}_2 = \hat{c}_3 = \ldots = \hat{c}_\infty \). Replacing \( H \) by \( \hat{c}_\infty \), we obtain a Hamiltonian for which the sequence \( \hat{c}_k \) stabilizes in the first term:

\[
\hat{c}_1 = \hat{c}_2 = \hat{c}_3 = \ldots = \hat{c}_\infty.
\]

Furthermore, replacing \( H \) by \( H - \hat{c}_\infty \), we can also ensure that \( \hat{c}_\infty = 0 \). To summarize, we now have

\[
c(H^{2k}) = 0
\]

for all \( k \in \mathbb{N} \).

It will also be convenient to assume that all one-periodic orbits of \( H \) are constant. This can always be achieved by composing the flow \( \varphi^t_H \) with contractible loops in the group of Hamiltonian diffeomorphisms; see, e.g., [Gi10, Sect. 5.1].

Finally, we will focus on the case where \( N < \infty \), i.e., \( L_1 \neq 0 \). For symplectic CY manifolds, the proof is simpler, but the wording requires superficial modifications.

\textit{Idea of the proof.} With Step 1 completed, before turning to the actual proof of the theorem, let us outline the idea of the argument. To this end, we need to make several simplifying assumptions. Namely, let us assume that \( H \) is (strongly) non-degenerate, all one-periodic orbits have distinct action, and that the regularity conditions for the pair-of-paints product are satisfied for the Hamiltonians \( H \) and \( H^{2k} \) for all \( k \). (Note that the strong non-degeneracy assumption supersedes the condition that none of the periodic orbits of \( H \) is an SDM.) Let \( \bar{x} \) be the unique action carrier for \( c(H) \). We claim that \( \bar{x}^k \) is an action carrier for \( c(H^{2k}) \). This would be obvious by (3.3) if we assumed in addition that all \( k \)-periodic orbits have distinct actions since then \( \bar{x}^k \) would be the only orbit with action equal to \( c(H^{2k}) \).

Let

\[ \Sigma = \bar{x} + \ldots, \]
be an action minimizing cycle, where here and below the dots stand for the terms with action less than zero. Set

\[ C_k := \Sigma^{*k}. \]

The homology class \([C_k]\) is the fundamental class \([M]\). Clearly, \(A_{\mathcal{H}}(C_k) \leq 0\) and, since \(c \left( H^{*k} \right) = 0\), the cycle \(C_k\) is also action minimizing and

\[ A_{\mathcal{H}}(C_k) = 0. \]

Next, it is not hard to see from Proposition 2.2 that the only term in \(C_k\) with zero action is \(\beta \hat{x}^k\) for some \(\beta \in \mathbb{F}^\ast\):

\[ C_k = \beta \hat{x}^k + \ldots. \]

By (3.3), \(\beta \neq 0\) and \(\hat{x}^k\) is an action carrier for \(c \left( H^{*k} \right)\).

Finally, we have \(\mu_{CZ}(\bar{x}) = n\) and hence, since \(x\) is non-degenerate, \(\mu_{CZ}(\bar{x}) > 0\) by (2.1). Therefore, by (2.2) and again (2.1), \(\mu_{CZ}(\hat{x}^k) \to \infty\) which is impossible because \(\hat{x}^k\) is an action carrier and thus \(\mu_{CZ}(\hat{x}^k) = n\).

None of our simplifying assumptions are satisfied in general, and one difficulty that arises in the proof is that to meet the regularity conditions for the pair-of-pants product one has to perturb \(H\) and \(H^{*k}\) independently. Furthermore, even if regularity could be achieved by a perturbation of \(H\) only, we would not be able to establish the strict inequality \(\hat{c}_k(H) > \hat{c}_\infty(H)\) by passing to the limit over perturbations.

**Step 2: Orbit stabilization.** Let us fix a large positive integer \(k_0\) to be specified later. Consider a sequence of non-degenerate perturbations \(H_k, k = 1, \ldots, k_0,\) of \(H\). More specifically, we pick a (strongly) non-degenerate perturbation \(H_1\) of \(H\) and let \(H_k\) with \(k = 2, \ldots, k_0\) be small perturbations of \(H_1\). In what follows, we will need to repeatedly require \(H_1\) to be close to \(H\) and \(H_k\) close to \(H_1\), and we will not keep track of how small these perturbations must actually be.

Denote by \(\{\bar{x}_1, \ldots, \bar{x}_r\}\) the collection of all capped one-periodic orbits of \(H\) with \(A_H(\bar{x}_i) = 0\) and \(0 \leq \mu_{CZ}(\bar{x}_i) \leq 2n\). (Note that when \(\omega_{|x(M)} = 0\) an orbit \(x_i\) can enter this list several times. Later on we will discard the capped orbits with \(H^\ast_n(\bar{x}_i) = 0\) from this list but at the moment it is convenient to consider all orbits with zero action.) Under the perturbation \(H_k\) each of the orbits \(x_i\) breaks down into non-degenerate orbits \(x_{ij}(k)\). When \(H_k\) is close to \(H_1\), each of the orbits \(x_{ij}(k)\) is a small perturbation of \(x_{ij}(1)\). In particular, there is a one-to-one correspondence between the orbits \(x_{ij}(1)\) and \(x_{ij}(k)\). Hence, in what follows, we denote these orbits by \(x_{ij}\) suppressing \(k\) in the notation. This one-to-one correspondence extends to capped orbits in a natural way with orbits inheriting the capping from \(\bar{x}_i\). Then \(\mu_{CZ}(\bar{x}_{ij}(k)) = \mu_{CZ}(\bar{x}_{ij}(1))\). Thus we can just use the notation \(\mu_{CZ}(\bar{x}_{ij})\). Furthermore, \(A_{H_k}(\bar{x}_{ij})\) is close to \(A_H(\bar{x}_i)\).

More generally, when all \(H_k\) are close to \(H_1\), there is a natural isomorphism between the Floer complexes \(\mathcal{C}\) for a suitable choice of an almost complex structure. With this in mind, we can identify the complexes \(\mathcal{C}\) with one complex which we denote by \(\mathcal{C}\). In particular, \(\bar{x}_{ij}\) is the image of \(\bar{x}_{ij}(k)\) under this identification. Note, however, that this identification preserves the action filtration only up to an error bounded by the Hofer distance between the Hamiltonians.

Recall that the orbits \(x_i\) are assumed to be constant. Fix small neighborhoods \(U_i\) of \(x_i\) such that these neighborhoods are disjoint from each other and from other
periodic orbits of $H$ of period up to $k_0$. Let $\epsilon_0$ be the minimum of $\epsilon$ from Proposition 2.2 as $k$ ranges from 1 to $k_0 - 1$. Set

$$
\eta = \min \{ \delta, \epsilon_0/2 \},
$$

where $\delta$ is half of the distance from 0 to other points of $S(H)$. In other words,

$$
\delta = \frac{1}{2} \inf \{|a| \mid 0 \neq a \in S(H)\}.
$$

When $H_k$ is sufficiently close to $H$, we have, using the notation from (2.10),

$$
|c(H_k)| < \eta \quad \text{and} \quad |\mathcal{A}_H(x_{ij})| < \eta,
$$

and the orbits $\bar{x}_{ij}$ are the only orbits of $H_k$ with action in the interval $[-\delta, \delta]$.

Pick a cycle $\Sigma_k \in \text{CF}_n(H_k)$ over $\mathbb{F}$ representing the fundamental class $[M]$ and such that $\mathcal{A}_H(\Sigma_k) = c(H_k)$.

We write

$$
\Sigma_k = \sum_{i,A} \sum_j \alpha_{ij,A} \cdot (x_{ij}, A) + \ldots,
$$

where henceforth the dots stand for the terms with action less than $-\eta$ and in fact less than $-\delta$. The sum extends only over the capped orbits with index $n$. Note that no orbits with action greater than $-\eta$ other than $x_{ij}$ can enter $\Sigma_k$. (Otherwise, we would have $c(H_k) > \eta$ since $\Sigma_k$ is an action minimizing cycle.) When the Hamiltonians $H_k$ are close to $H_1$, these cycles can be chosen so that the coefficients $\alpha_{ij}$ are independent of $k$. In other words, we can think of $\Sigma_k$ as one cycle $\Sigma$ in the complex $\text{CF}_*(H_1)$ obtained by identifying the complexes $\text{CF}_*(H_k)$.

The capping of the orbit $\bar{x}_{ij}$ comes from the capping of $\bar{x}_i$. Thus the same orbit $x_{ij}$ may enter (3.5) several times with different cappings. (This can happen when $\omega|_{\pi_1(M)} = 0$ and $N \leq n$, and, in particular, $x_i$ can contribute to the list of capped orbits with action zero more than once.) To account for this, it is convenient to rewrite (3.5) as the sum

$$
\Sigma_k = \sum_{i,A} \sum_j \alpha_{ij,A} \cdot (x_{ij}, A) + \ldots,
$$

where $A$ runs over all cappings of $x_{ij}$ or equivalently of $x_i$ and we have set $\alpha_{ij,A} = 0$ when $\mu_{CZ}(x_{ij}, A) \neq n$. For a fixed $i$ we can have only a finite number of different cappings $A$ occurring in this sum with $\alpha_{ij,A} \neq 0$.

Before proceeding with the argument, we need to “trim” the cycle $\Sigma_k$ to guarantee that it involves only homologically essential orbits $x_i$. Consider the (unordered) collection

$$
\alpha_{i,A} = \{ \alpha_{ij,A} \}.
$$

We say that $\alpha_{i,A} = 0$ when $\alpha_{ij,A} = 0$ for all $j$.

**Lemma 3.6.** The cycles $\Sigma_k$ can be chosen so that

$$
\alpha_{i,A} \neq 0 \quad \Rightarrow \quad \text{HF}_n(x_i, A) \neq 0.
$$

In other words, constructing the cycle $\Sigma_k$ we can discard from the collection of the orbits $\bar{x}_i = (x_i, A)$ the orbits with $\text{HF}_n(\bar{x}_i) = 0$.

Note that $\alpha_{i,A} \neq 0$ for at least one $i$ for any choice of $\Sigma_k$; for otherwise we would have $c(H_k) < -\eta$. 
Proof. Due to an isomorphism between the complexes $\text{CF}_*(H_k)$, it is enough to do this for one value of $k$. Thus we drop $k$ from the notation of the cycle $\Sigma_k$. The cycle $\Sigma$ lies in $\text{CF}_*^{-\infty,n}(H_k)$. Hence it can be projected to $\text{CF}_*^{-\infty,n}(H_k)$. When the Hamiltonian $H_k$ is close to $H$, we have the direct sum decomposition of complexes

$$\text{CF}_*^{-\infty,n}(H_k) = \bigoplus_{i,A} \text{CF}_*(x_i, A),$$

(3.9)

where the complex $\text{CF}_*(x_i, A)$ is spanned by the capped periodic orbits $(x_{ij}, A)$ which $(x_i, A)$ splits into. (Here we are using the fact that a Floer trajectory which is not entirely contained in one of the neighborhoods $U_i$ must have energy bounded from below by some constant independent of the perturbations $H_k$; see [Sa, Sect. 1.5] or [FO, Lemma 19.8].) The homology of this complex is the local homology $\text{HF}_*(x_i, A)$. Let $\Sigma_{i,A}$ be the projection $\Sigma$ to $\text{CF}_*(x_i, A)$.

Clearly, $\text{HF}_n(x_i, A) \neq 0$ when $[\Sigma_{i,A}] \neq 0$ in $\text{HF}_*(x_i, A)$. Thus it is sufficient to eliminate the entries $\alpha_{ij,A}$ for all $i$ such that $[\Sigma_{i,A}] = 0$. Pick one such $i$ and let $\beta$ be a primitive of $\Sigma_{i,A}$ in $\text{CF}_*(x_i, A)$. We can view $\text{CF}_*(x_i, A)$ as a subspace of $\text{CF}_*(H_k)$ and thus $\beta \in \text{CF}_*(H_k)$. Consider the cycle $\Sigma' = \Sigma - \partial \beta$, where $\partial$ is the differential in the total complex $\text{CF}_*(H_k)$. This cycle still represents $[M]$ and, as is easy to see, $A_{H_k}(\Sigma') = c(H_k)$. Furthermore, $\tilde{\alpha}_{i,A} = 0$ for $\Sigma'$, while other groups $\tilde{\alpha}_{i',A'}$, where $i' \neq i$ or $A' \neq A$, remain the same as for $\Sigma$. (However, the “lower order terms”, i.e., the terms with action below $-\eta$, can be effected by this change.) Applying this procedure to all $i$ and $A$ with $[\Sigma_{i,A}] = 0$, we obtain a new cycle, which we still denote by $\Sigma$ or $\Sigma_k$, satisfying (3.8).

Note that $\mu_{c_2}(x_i, A) > 0$ for every orbit $(x_i, A)$ with $\text{HF}_n(x_i, A) \neq 0$ and hence, by (3.8), for every orbit with $\tilde{\alpha}_{i,A} \neq 0$. Indeed, $\mu_{c_2}(x_i, A) \geq 0$ by (2.1). If we had $\tilde{\alpha}_{c_2}(x_i, A) = 0$, the orbit $(x_i, A)$ would be an SDM, which would contradict the conditions of the theorem. Thus $\tilde{\alpha}_{c_2}(x_i, A) > 0$ for all orbits $(x_i, A)$ that enter the cycle $\Sigma_k$ with $\tilde{\alpha}_{i,A} \neq 0$.

Next, consider the collection of capped $k$-periodic orbits of $H$ with zero action. Among these are the iterated orbits $\bar{x}_i^k$ where $\text{HF}_n(\bar{x}_i) \neq 0$, but in general this collection may include some other orbits. Moreover, when $L_0 = 0$, an orbit $\bar{x}_i^k$ may occur in this collection with a capping different from that of $\bar{x}_i^k$.

Denote by $y^{(k)}_i$ the $k$-periodic orbits of $H$. For a generic choice of the perturbations $H_k$, the Hamiltonian

$$F_k = H_{1k} \cdots \bar{z} H_k$$

is a small perturbation of the non-degenerate Hamiltonian $H_{1k}^k$ and hence of $H_{2k}$. (At this point it is essential that the range of $k$ is fixed and finite.) Under this perturbation, the orbits $y^{(k)}_i$ split into non-degenerate orbits which we denote by $y^{(k)}_{ij}$. For instance, when $y^{(k)}_i = x_i^k$, these are the orbits $x_{ij}^k$ or, to be more precise, small perturbations $x_{ij}^{k'}$ of these orbits, and perhaps some other orbits. The capping of $y^{(k)}_i$ gives rise to cappings of the orbits it splits into. In particular, when $\bar{y}^{(k)}_i$ has zero action, we denote the resulting capped orbits by $\bar{y}^{(k)}_{ij}$. This collection of orbits of $F_k$ includes the orbits $\bar{x}^{(k)}_{ij}$ when $\bar{z}_i$ has zero action, possibly the orbits $x^{(k)}_{ij}$ with other cappings, and finally some other orbits.
When the Hamiltonians $H_k$ are close to $H_1$, there is again a natural isomorphism
between the Floer complexes $CF_*(F_k)$ and $CF_*(H_1^{2k})$. Identifying these complexes,
we can view the orbits $\tilde{y}_{ij}^{(k)}$ as elements of $CF_*(H_1^{2k})$.

Similarly to (3.4), we can ensure that

$$|c(F_k)| < \eta \quad \text{and} \quad |A_{F_k}(\tilde{y}_{ij}^{(k)})| < \eta,$$

and $\tilde{y}_{ij}^{(k)}$ are the only orbits of $F_k$ with action in $[-\delta, \delta]$ by taking $H_k$ close to $H$.
(At this point again it is essential that $k$ is taken within a finite range $1, \ldots, k_0$.)

Consider now the cycle

$$C_k = \Sigma_1 * \cdots * \Sigma_k \in CF_*(F_k)$$

representing the fundamental class $[M]$ in $HF_*(F_k)$. Since the pair-of-pants product
is not associative on the level of complexes, the placing of parentheses matters here.
Thus, to be more precise, the cycle $C_k$ is defined inductively by

$$C_{k+1} = C_k \ast \Sigma_{k+1}$$

with $C_1 = \Sigma_1$. Alternatively, we can view the multiplication on the right by $\Sigma_{k+1}$
as a map of complexes

$$\Phi_k : CF_*(F_k) \to CF_*(F_{k+1})$$

and

$$C_{k+1} = \Phi_k \circ \cdots \circ \Phi_1(\Sigma_1).$$

It is easy to see that

$$|A_{F_k}(C_k)| < \eta.$$ 

Indeed, $A_{H_k}(\Sigma_k) = c(H_k)$ and

$$A_{F_k}(C_k) \leq \sum_{j=1}^{k} A_{H_j}(\Sigma_j) = \sum_{j=1}^{k} c(H_j).$$

When all $H_k$ are sufficiently close to $H$, the spectral invariants $c(H_k)$ are close to $c(H) = 0$. In particular, we can ensure that $A_{F_k}(C_k) < \eta$. Furthermore,
$A_{F_k}(C_k) > -\eta$ because $c(F_k) > -\eta$. In other words, $C_k \in CF_{(-\infty, \eta]}(F_k)$.

Let us denote by $C_k'$ the natural projection of $C_k$ to the quotient complex

$$CF_{(-\eta, \eta]}(F_k).$$

In other words, $C_k'$ is the leading term in the expression

$$C_k = \sum \beta_{ij}^{(k)} \tilde{y}_{ij}^{(k)} + \ldots,$$ 

(3.10)

where, as before, the dots stand for the terms with action less than $-\eta$, and the sum
extends only over the capped orbits with index $n$. Similarly to (3.9), when the Hamiltonians $H_k$ are sufficiently close to $H$, the complex $CF_{(-\eta, \eta]}(F_k)$ decomposes
into a direct sum of complexes $CF_*(y_{i}^{(k)})$ formed by the orbits which $y_{i}^{(k)}$ with all possible cappings breaks into:

$$CF_{(-\eta, \eta]}(F_k) = \bigoplus_{i} CF_*(y_{i}^{(k)}).$$ 

(3.11)

We denote by $C_{k,i}$ the projection of $C_k'$ to $CF_*(y_{i}^{(k)})$.

By Proposition 2.2 and the choice of $\eta$, $\tilde{y}_{i,j}^{(k)} * \bar{x}_{ij} = \ldots$ when $y_{i}^{(k)} \neq x_i^k$. 

---
where again the dots denote the terms with action less than $-\eta$. Hence the operators $\Phi_k$ block-diagonalize with respect to the decomposition (3.11):

$$\Phi_k = \bigoplus_i \Phi_{k,i}.$$ 

Moreover, $\Phi_{k,i} = 0$ when $y_i^{(k)}$ is not one of the orbits $x_i^{(k)}$. In other words, the leading term $C_k^i$ in (3.10) involves only the orbits $x_i^{(k)}$ with some cappings.

The key result of this step is the following.

**Lemma 3.7.** For some $i$ independent of $k \leq k_0$ we have $C_{k,i} \neq 0$, i.e., there exists a sequence of capped orbits $(x_{ij}^{(k)}, B_k)$, indexed by $k$, with $B_k$ and $j$ possibly depending on $k$, entering the cycles $C_k$ for $1 \leq k \leq k_0$ with non-zero coefficients.

**Proof.** Since the operators $\Phi_k$ block-diagonalize, we have $C_{k,i} = \Phi_k^{-1}C_{k-1,i}$. As a consequence, $C_{k,i} = 0 \implies C_{k+1,i} = 0, \ldots, C_{k_0,i} = 0$.

Clearly, $C_k \neq 0$ since $[C_k] = [M]$ and hence $C_{k,i} \neq 0$ for all $k \leq k_0$ for some $i$ independent of $k$. $\square$

In the setting of Lemma 3.7, $C_{1,i}$ is the sum of the cycles $\Sigma_{i,A}$ for all suitable cappings $A$, and hence $\Sigma_{i,A} \neq 0$ for some $A$. Then for $\bar{x}_i = (x_i, A)$ we also have $HF_n(\bar{x}_i) \neq 0$ and $\hat{\mu}_{CZ}(\bar{x}_i) > 0$ (3.12) by Lemma 3.6. Here the second inequality is a consequence of the first one and the fact that $H$ does not have SDM orbits.

**Remark 3.8.** The role of Lemma 3.6 in this argument is to ensure that (3.12) is satisfied for the orbit $(x_i, A)$. There is a different way to do this. First, note that for $i$ as in Lemma 3.7, some orbit $\bar{x}_{ij}^{(1)} = \bar{x}_{ij}$ is a carrier for $H_1$. (In contrast, the orbits $\bar{x}_{ij}^{(k)}$ with $k \geq 2$ may fail to be action carriers, although they are carriers up to a small error $\eta$.) Taking a sequence of Hamiltonians $H_1 \to H$ and also the Hamiltonians $H_k \to H$, choosing the cycles $\Sigma_k$ for them, and applying Lemma 3.7, we obtain a sequence of carriers $\bar{x}_{ij}$ for $H_1$. Passing to a subsequence, we can guarantee that $i$ and the inherited capping $A$ of $x_i$ are independent of $k$. Then $(x_i, A)$ is an action carrier and thus $HF_n(x_i, A) \neq 0$; [GG12, Lemma 3.2].

**Step 3: Index growth.** First, let us specify the value of $k_0$. Consider all capped one-periodic orbits $(x_i, A)$ entering the cycle $\Sigma$ with $\bar{\alpha}_{i,A} \neq 0$ or more generally all capped one-periodic orbits with $HF_n(x_i, A) \neq 0$. As has been pointed out in Step 2, $\hat{\mu}_{CZ}(x_i, A) > 0$ for each such orbit. Therefore, since $\mathcal{P}_1(H)$ is finite,

$$\Delta := \min_{(x_i, A)} \hat{\mu}_{CZ}(x_i, A) > 0.$$ 

We set

$$k_0 := \left\lceil \frac{2n + 2}{\Delta} \right\rceil. \tag{3.13}$$
Next, consider the orbits \( x_i \) and \( x_{ij}^{(1)} \) from Lemma 3.7. For the sake of brevity, set
\[
z := x_i, \quad z_j := x_{ij} \quad \text{and} \quad z_j^{(k)} := (\tilde{x}_{ij}^{(k)}, B_k).
\]
The orbit \( z_j^{(k)} \), with capping ignored, is a small perturbation of \( z_j^k \) and we may simply identify these orbits. Since \( z \) is a constant orbit (see Step 1), we can take the trivial (i.e., constant) capping of \( z \) as a reference. The orbits \( z_j^k \) need not be constant. However, each of these orbits is contained in a small ball \( U \) centered at \( z \) and we can fix a capping contained in \( U \) as a reference capping for \( z_j^k \). With this in mind, the collections of all cappings of \( z \) or \( z_j^k \) can be identified with the group \( \Gamma = \pi_2(M)/\ker I_c \cap \ker I_w \). In particular, we have a one-to-one correspondence between the cappings of these orbits.

When the Hamiltonians \( H_k \) are sufficiently close to \( H \), every pair-of-pants curve connecting \( z_{j'}^{(1)} \) and \( z_{j''}^{(k-1)} \) to \( z_j^{(k)} \) must be trivial, i.e., contained in \( U \). For, otherwise, we would have \( A_{H_k} (\tilde{z}_j^{(k)}) < -\eta \) by Proposition 2.2 and the assumption that the perturbations \( H_k \) are sufficiently close to \( H \). Then, as follows from the definition of the cycle \( C_k \),
\[
\tilde{z}_j^{(k)} = (z_j^k, A_1 + \cdots + A_k),
\]
where each \( A_l \) is one of the cappings of \( z \) with \( \text{HF}_n(z, A_l) \neq 0 \).

Let \( A_0 \) be the capping of \( z \) such that \( \Delta_0 := \tilde{\mu}_{cz}(z, A_0) \) is the smallest possible value of \( \tilde{\mu}_{cz}(z, A) \) when \( \text{HF}_n(z, A) \neq 0 \). Clearly, \( \Delta_0 \geq \Delta > 0 \) and
\[
I_c (A_l - A_0) \geq 0.
\]
Hence,
\[
n = \mu_{cz} (\tilde{z}_j^{(k)}) = \mu_{cz} (z_j^k, kA_0) + \sum_l I_{cz} (A_l - A_0) \geq \mu_{cz} (z_j^k, kA_0)
\]
for all \( k \leq k_0 \). On the other hand, \( \tilde{\mu}_{cz}(z_j, A) \) can be made arbitrarily close to \( \tilde{\mu}_{cz}(z, A) \) uniformly in \( A \) by taking the Hamiltonians \( H_k \) close to \( H \). Therefore,
\[
\mu_{cz} (z_j^k, kA_0) \geq k\Delta_0 - n - 1 \geq k\Delta - n - 1,
\]
where we subtracted \( n + 1 \) rather than \( n \) to account for the discrepancy between \( \tilde{\mu}_{cz}(z_j, A_0) \) and \( \tilde{\mu}_{cz}(z, A_0) \). Setting \( k = k_0 \) and using (3.13), we arrive at a contradiction:
\[
n = \mu_{cz} (z_j^{(k)}) \geq k_0\Delta - n - 1 \geq n + 1.
\]
\[\square\]

4. Proof of the main theorem

For the reader’s convenience we begin with restating the main theorem (Theorem 1.1) of the paper.

**Theorem 4.1.** Assume that a closed symplectic manifold \( M \) admits a Hamiltonian diffeomorphism \( \varphi_H \) with finitely many periodic orbits. Then there exists \( A \in \pi_2(M) \) such that \( \omega(A) > 0 \) and \( \langle c_1(TM), A \rangle > 0 \).

This result is a formal consequence of already known cases of the Conley conjecture and the following proposition.
Proposition 4.2. Assume that a closed symplectic manifold \( M \) admits a Hamiltonian diffeomorphism \( \varphi_H \) with finitely many periodic orbits. Then \( \omega |_{\pi_2(M)} \neq 0 \), i.e., \( I_\omega \neq 0 \).

Proof of Theorem 4.1. Set \( V_\mathbb{R} := \pi_2(M) \otimes \mathbb{R} \). Clearly, the homomorphisms \( I_\omega \) and \( I_{c_1} \) extend to \( V_\mathbb{R} \). Since the Conley conjecture is known to hold for symplectic Calabi-Yau manifolds (see [GG09a, He]), we have \( I_{c_1} \neq 0 \) on \( V_\mathbb{R} \) under the assumptions of the theorem. Likewise, \( I_\omega \neq 0 \) by Proposition 4.2.

If \( \ker I_{c_1} = \ker I_\omega \) on \( V_\mathbb{R} \), the manifold \( M \) is either negative monotone or strictly positive monotone. The former case is ruled out since the Conley conjecture holds for negative monotone manifolds as is shown in [GG12]. In the latter case, there is \( A \in \pi_2(M) \) such that \( \omega(A) > 0 \) and \( \langle c_1(TM), A \rangle > 0 \) and the proof is finished.

When \( \ker I_{c_1} \neq \ker I_\omega \) on \( V_\mathbb{R} \), there exists \( A' \in V_\mathbb{R} \) such that \( I_{c_1}(A') < 0 \) and \( I_\omega(A') < 0 \). The space \( V_\mathbb{Q} := \pi_2(M) \otimes \mathbb{Q} \) is dense in \( V_\mathbb{R} \) and hence there is \( A'' \in V_\mathbb{Q} \) with similar properties. Finally, \( \omega(A) > 0 \) and \( \langle c_1(TM), A \rangle > 0 \) where \( A = mA'' \in \pi_2(M) \) for a suitably chosen \( m \in \mathbb{Z} \).

Proof of Proposition 4.2. Arguing by contradiction, assume that \( \omega |_{\pi_2(M)} = 0 \). Note that, as a consequence, the Floer homology of \( H \) is defined without virtual cycles and \( (M, \omega) \) is automatically rational with \( \lambda_0 = \infty \). None of the periodic orbits of \( H \) is an SDM, for otherwise \( H \) would have infinitely many periodic orbits. Furthermore, \( S(H) \) is finite and hence the sequence \( \hat{c}_k \) necessarily stabilizes since it is converging, which is impossible by Theorem 3.1.

Remark 4.3. The proof of Proposition 4.2 essentially comprises two cases: the “non-degenerate” case and the “degenerate”, i.e., SDM case. Interestingly, this structure is common to all symplectic topological proofs of the Conley conjecture type results, even though the arguments in the non-degenerate case are quite different. We also note that Theorem 3.1, again in conjunction with the SDM case, can also be used to prove the Conley conjecture for negative monotone symplectic manifolds, thus bypassing a reference to the results from [CGG, GG12] in the proof of Theorem 4.1.

Remark 4.4. Theorem 3.1 is closely related to and can be proved using [GG10, Prop. 5.3] asserting that the pair-of-pants product in the local Floer homology of an isolated non-SDM orbit is nilpotent. However, such an argument, when detailed, would not be much different from or much simpler than the self-contained proof in the previous section. On the other hand, one can directly prove Theorem 4.1 by establishing Proposition 4.2 as a consequence of [GG10, Prop. 5.3] through purely algebraic means. The essence of the argument is that if we had \( \omega \) aspherical and simultaneously \( \mathcal{P}(H) \) finite, the algebra \( \bigoplus_{k \geq 1} \text{HF}_*(H^k) \) would necessarily be nilpotent, which is, of course, impossible by (2.8).

5. Perfect Hamiltonians and generic existence

The notion of the augmented action and the asymptotic spectral invariant \( \hat{c}_\infty \) naturally arise in the study of Hamiltonians with finitely many periodic orbits and, more specifically, perfect Hamiltonians. Recall that a Hamiltonian \( H \) is said to be perfect if \( \mathcal{P}(H) \) is finite and \( \mathcal{P}_1(H) = \mathcal{P}(H) \), i.e., when \( H \) has only finitely many simple periodic orbits and every such orbit is one-periodic. The latter condition is automatically satisfied in all known examples of Hamiltonians with finitely many periodic orbits. Furthermore, all known perfect Hamiltonians are non-degenerate.
Clearly, a suitable iteration of a Hamiltonian with finitely many simple periodic orbits is perfect.

One class of examples of perfect Hamiltonians is given by generic elements in Hamiltonian circle or torus actions with isolated fixed points. There are, however, other examples, e.g., pseudo-rotations, with extremely non-trivial dynamics; see, e.g., [AK, FK].

To see the connection between perfect Hamiltonians and asymptotic spectral invariants, note first that the actual value of $\hat{c}_\infty$ is not completely trivial to calculate directly by definition even for such a Hamiltonian as a quadratic form on $\mathbb{C}P^n$.

However, it has a simple interpretation as the so-called augmented action. Namely, assume that $M$ is strictly monotone (or negative monotone) with monotonicity constant $\lambda \neq 0$; see Section 2.1. For $x \in P_1(H)$, the augmented action of $x$ is

$$\tilde{A}_H(x) = A_H(\bar{x}) - \frac{\lambda}{2} \mu_{cz}(\bar{x}),$$

where on the right hand side $x$ is equipped with an arbitrary capping. By (2.3), the left hand side is well defined, i.e., independent of the capping. Augmented action was introduced in [GG09a] and then applied in the circle of questions related to the Conley conjecture in [CGG, GG16a]. Under certain conditions it behaves similarly to the ordinary action while being capping-independent. In particular, the augmented action is homogeneous with respect to iterations.

**Theorem 5.1.** Assume that $M$ is strictly positive monotone and $H$ is perfect. Let $x$ be a one-periodic orbit whose iterations occur infinitely many times as carriers for $c(H^{\#k})$. Then

$$\hat{c}_\infty = \tilde{A}_H(x). \quad (5.1)$$

A similar result holds when $M$ is negative monotone, but then the assertion is void; for negative monotone symplectic manifolds admit no Hamiltonians with finitely many periodic orbits; [GG12]. Note also that an orbit $x$ satisfying the hypothesis of Theorem 5.1 always exists since $P_1(H) = \hat{P}(H)$ is finite. The theorem has an analog for Reeb flows; see [GG16b, Sect. 6.1.2].

**Proof.** Let $\bar{x}_k$ be a carrier for $c(H^{\#k})$. To prove (5.1), it is enough to show that

$$\hat{c}_\infty = \lim_{k \to \infty} \frac{1}{k} \tilde{A}_{H^{1k}}(x_k)$$

since the augmented action is homogeneous. This readily follows from that

$$|\tilde{A}_{H^{1k}}(x_k) - A_{H^{1k}}(\bar{x}_k)| \leq \frac{\lambda}{2} \cdot 2n = \lambda n,$$

which is, in turn, an immediate consequence of the fact that $HF_n(\bar{x}_k) \neq 0$, and hence $0 \leq \mu_{cz}(\bar{x}_k) \leq 2n$. \qed

**Example 5.2.** Consider a quadratic Hamiltonian $H(z) = \pi(\lambda_0|z_0|^2 + \cdots + \lambda_n|z_n|^2)$ on $\mathbb{C}P^n$, where the coefficients $\lambda_0, \ldots, \lambda_n$ are linearly independent over $\mathbb{Q}$. (Here, we identify $\mathbb{C}P^n$ with the quotient of the unit sphere in $\mathbb{C}^{n+1}$ and hence $\sum |z_i|^2 = 1$.) The Hamiltonian $H$ is perfect and has exactly $n+1$ fixed points, the coordinate axes. A simple calculation shows that their augmented actions are equal to $\pi \sum \lambda_i/(n+1)$, cf. [CGG, Exam. 1.2]. Hence $\hat{c}_\infty = \pi \sum \lambda_i/(n+1)$.
Now we are in a position to prove Theorem 1.3 showing that the gap between $c_{[M]}(H)_{\#k}$ and $c_{[pt]}(H)_{\#k}$, where $[pt] \in \text{HF}_{-n}(H)$ is the homology class of a point, is eventually \textit{a priori} bounded for perfect Hamiltonians. Let us state the theorem again.

**Theorem 5.3 (Action Gap).** Assume that $H$ is perfect and $M$ is positive monotone with monotonicity constant $\lambda > 0$. Then

$$c_{[M]}(H)_{\#k} - c_{[pt]}(H)_{\#k} \leq 2\lambda n,$$

(5.2)

for all but possibly a finite number of iterations $k \in \mathbb{N}$.

In this theorem and in the rest of the section, the choice of the coefficient ring $F$ is immaterial and the ring is suppressed in the notation. In what follows, for the sake of brevity, we also set

$$c^+_k := c_{[M]}(H)_{\#k} \quad \text{and} \quad c^-_k := c_{[pt]}(H)_{\#k}$$

and

$$\hat{c}^\pm_{\pm} = \lim_{k \to \infty} c^\pm_k / k.$$ 

Thus, in particular, $c^+_k = c_k$ and $c^+_k = \hat{c}_k$ in the notation of the previous sections. The difference $c^+ - c^-$ has played a prominent role in some aspects of symplectic topology and is sometimes referred to as the $\gamma$-norm or $\gamma$-metric. It was introduced in [Vi, HZ] and further studied in, e.g., [Oh, Scw]. It is easy to see from (AS5) in Section 2.4 that

$$c^+_k \geq c^-_k$$

and, moreover, the inequality is strict unless $\varphi_H = \text{id}$. The latter assertion is non-trivial and usually proved by comparing $c^+ - c^-$ with the capacity of a small displaced ball; see, e.g., [Scw]. (One can also argue, when $M$ is monotone, as in the proof of [GG09a, Prop. 6.2].)

Upper bounds on the difference between spectral invariants as in (5.2) usually result from non-vanishing of certain GW invariants or relations in the quantum product; see, e.g., [EP, GG09a]. Thus Theorem 5.3 provides further evidence supporting the Chance–McDuff conjecture discussed in the introduction.

**Proof of Theorem 5.3.** By rescaling $\omega$, we may assume that $\lambda = 2$, i.e., $I_\omega = I_{c_1}$, and then (5.2) turns into

$$c^+_k - c^-_k \leq 4n.$$ (5.3)

Furthermore, it is enough to show that every sequence $k_i \to \infty$ contains an infinite subsequence for which (5.3) holds.

Thus let $k_i \to \infty$. Since $H$ is perfect, there is a one-periodic orbit $x$ and an infinite subsequence, which we still denote by $k_i$, such that the iterations $x^{k_i}$ with some cappings are action carriers for $c^+_k$. Next, in a similar vein, there is a one-periodic orbit $y$ and again an infinite subsequence $k_{i_j}$, such that the iterations $y^{k_{i_j}}$ with some cappings are action carriers for $c^-_{k_{i_j}}$. Let us denote the sequence $k_{i_j}$ by $k_i$ again, write $k = k_i$, and let $\bar{x}_k$ and, respectively, $\bar{y}_k$ be the resulting capped periodic orbits.

We have

$$c^+_k - c^-_k = A_{H_{\#k}}(\bar{x}_k) - A_{H_{\#k}}(\bar{y}_k).$$

By Theorem 5.1,

$$A_{H_{\#k}}(\bar{x}_k) = \tilde{A}_{H_{\#k}}(\bar{x}_k) + \tilde{\mu}_{CZ}(\bar{x}_k) = \hat{c}^+_{\infty} + \tilde{\mu}_{CZ}(\bar{x}_k).$$
To evaluate the second term, recall that by Poincaré duality
\[ c_{[pt]}(H) = -c_{[M]}(H^{-1}), \]
where \( H^{-1} \) is the Hamiltonian generating the inverse time-dependent flow \( \varphi_t^{-1} \varphi_{H}^{1-t} \).
Therefore,
\[ \hat{c}^+_{\infty}(H^{-1}) = -\hat{c}^-_{\infty}(H) = -\hat{c}^-_{\infty}. \]
Furthermore, let us denote by \( \bar{y}_{-k} \) the orbit \( y_k \) traversed in the opposite direction and equipped with the capping obtained from \( \bar{y}_k \) by reversing the orientation. Then, setting \( H^{-\sharp k} := (H^{-1})^{\sharp k} \), we have
\[ \mathcal{A}_{H^{\sharp k}}(\bar{y}_k) = -\mathcal{A}_{H^{\sharp k}}(\bar{y}_{-k}) = -[\hat{c}^+_{\infty}(H^{-1}) - \mu_{CZ}(\bar{y}_k)] = -[\hat{c}^-_{\infty} - \mu_{CZ}(\bar{y}_k)]. \]
Therefore,
\[ c^+_k - c^-_k = \hat{c}^+_{\infty} - \hat{c}^-_{\infty} + \mu_{CZ}(\bar{x}_k) - \mu_{CZ}(\bar{y}_k). \tag{5.4} \]
Since \( \bar{x}_k \) and \( \bar{y}_k \) are action carriers for \( c^+_k \), we have
\[ |\hat{\mu}_{CZ}(\bar{x}_k) - n| \leq n \text{ and } |\hat{\mu}_{CZ}(\bar{y}_k) + n| \leq n. \]
Furthermore, recall that as is well-known
\[ c^+_k - c^-_k \leq \|H^{\sharp k}\|, \]
where \( \|H^{\sharp k}\| \) stands for the Hofer norm of \( H^{\sharp k} \), and thus
\[ \hat{c}^+_{\infty} - \hat{c}^-_{\infty} \leq \|H\|. \]
As a consequence, by (5.4),
\[ 0 \leq c^+_k - c^-_k \leq \|H\| + 4n. \]
Dividing by \( k \) and passing to the limit as \( k \to \infty \) in the sequence \( k_i \), we conclude that
\[ \hat{c}^+_{\infty} = \hat{c}^-_{\infty}. \]
Returning to (5.4), we now arrive at
\[ c^+_k - c^-_k = \hat{\mu}_{CZ}(\bar{x}_k) - \hat{\mu}_{CZ}(\bar{y}_k) \leq 4n, \tag{5.5} \]
proving (5.3).
\[ \square \]

Remark 5.4. We conjecture that in the setting of Theorem 5.3,
\[ c^+_k - c^-_k \to 0 \]
for some sequence \( k_i \to \infty \). A direct calculation shows that this is indeed true for rotations of \( S^2 \) or quadratic Hamiltonians on \( \mathbb{C}\mathbb{P}^n \). This is not entirely obvious; for while it is easy to see that some action gap goes to zero for a subsequence, it is not immediately clear that this is so for the specific action gap from (5.2).

To further elaborate on the conjecture note that when \( H \) is non-degenerate the first part of (5.5) can be re-written as
\[ c^+_k - c^-_k = 2n + \left( \mu_{CZ}(\bar{x}_k) - \mu_{CZ}(\bar{x}_k) \right) + \left( \mu_{CZ}(\bar{y}_k) - \mu_{CZ}(\bar{y}_k) \right). \tag{5.6} \]
The last two terms on the right hand side are independent of the cappings of $x^k$ and $y^k$. It is easy to see that the right hand side of (5.6) is bounded away from 0 by at least 1 unless both $x$ and $y$ are elliptic. When this is the case, it looks very plausible that indeed $c^+_{k_i} - c^-_{k_i} \to 0$ for some subsequence. (This fact is non-obvious and closely related to the results in [DLW] and [GG16b].) However, there seems to be no way of ensuring that this subsequence has an infinite overlap with the subsequence for which both $x^k$ and $y^k$ are action carriers.

As an immediate application of Theorem 5.3, we obtain a similar upper bound for other homology classes. For $w \in H_*(M)$ set

$$c^w_k := c_w (H^z_k) \text{ and } \hat{c}^w_k := \lim_{k \to \infty} c^w_k / k.$$  

Clearly, $c^-_k \leq c^w_k \leq c^+_k$ when $w \neq 0$. The following result readily follows from Theorem 5.3.

**Corollary 5.5.** Assume that $H$ is a perfect Hamiltonian on a positive monotone symplectic manifold $M$ with monotonicity constant $\lambda > 0$. Then for any two non-zero classes $w_0$ and $w_1$ in $H_*(M)$ we have

$$|c^w_{k_1} - c^w_{k_2}| \leq 2\lambda n \text{ and } \hat{c}^w_{\infty} = \hat{c}^{w_0} = \hat{c}^{w_1},$$

where the first inequality holds for all but possibly a finite number of $k \in \mathbb{N}$.

One important feature of perfect Hamiltonians is the action-index resonance relations, e.g., the fact that certain periodic orbits have the same augmented action; see [CGG] and [GG09a, Cor. 1.11 and Thm. 1.12]. Theorems 5.1 and 5.3 enable us to refine these results identifying the common augmented action value for these orbits with $\hat{c}_\infty$. We start by stating a general result along these lines.

**Corollary 5.6 (Resonance Relations, I).** Assume that $M^{2n}$ is a strictly positive monotone closed symplectic manifold such that $N \geq 2$ or $\dim H_*(M) > n + 1$. Let $H$ be a perfect non-degenerate Hamiltonian on $M$. Then $H$ has two geometrically distinct one-periodic orbits $x$ and $y$ with

$$\hat{A}_H(x) = \hat{A}_H(y) = \hat{c}_\infty.$$  

**Proof.** By our assumptions on $M$, there exists two distinct homology classes with $0 \leq |w_1| - |w_0| < 2N$. (If $N > 1$ we can take two classes with $|w_1| - |w_0| = 2$ and if $N = 1$ but $\dim H_*(M) > n + 1$ we can take two classes with $|w_1| = |w_0|$.)

Let $\tilde{x}$ and $\tilde{y}$ be carriers for $c^{w_1}$ and $c^{w_0}$. Then the orbits $x$ and $y$ are geometrically distinct since

$$|\mu_{cz}(\tilde{x}) - \mu_{cz}(\tilde{y})| = |w_1| - |w_0| < 2N.$$  

Now it remains to apply Corollary 5.5 and Theorem 5.1. \qed

This corollary is a (partial) refinement of [GG09a, Cor. 1.11]. An essential feature of this result is that we make no assumption about the structure of the quantum product in $H_{\ast}(M)$. Also note that in the setting of the proof of the corollary, (5.6) takes the form

$$c^+_k - c^-_k = |w_1| - |w_0| + (\mu_{cz}(\tilde{x}_k) - \mu_{cz}(\tilde{x}_k)) + (\mu_{cz}(\tilde{y}_k) - \mu_{cz}(\tilde{y}_k)),$$

where $\tilde{x}_k$ and $\tilde{y}_k$ are carriers for $c^{w_1}_k$ and, respectively, $c^{w_0}_k$. As in [GG09b], we can infer from Corollary 5.6 the generic existence of infinitely many periodic orbits.
Corollary 5.7 (Generic Existence). Assume that $M^{2n}$ is a strictly monotone closed symplectic manifold and $N \geq 2$ or $\dim H_s(M) > n + 1$. Then the collection of strongly non-degenerate Hamiltonians with infinitely many geometrically distinct simple periodic orbits is of second Baire category in the space of all Hamiltonians with respect to the $C^\infty$-topology.

Proof. First, note that we can assume that $M$ is strictly positive monotone, for otherwise the Conley conjecture holds, [GG12]. Then the argument relies on the fact that the resonance relations, (5.7), can be broken by a $C^\infty$-small perturbation. To be more precise, consider the set $H$ of strongly non-degenerate Hamiltonians $H$ such that for each $k$ all $k$-periodic orbits of $H$ have distinct augmented actions. This set is of second Baire category in the space of all Hamiltonians with respect to the $C^\infty$-topology; see [GG09b]. Furthermore, $H$ is closed under iterations, i.e., $H^\#k \in H$ whenever $H \in H$. Therefore, every $H \in H$ has infinitely many geometrically distinct simple periodic orbits. Indeed, if some $H \in H$ had only finitely many simple periodic orbits, a suitable iteration $H^\#k \in H$ would be perfect, which is impossible by Corollary 5.6. □

Remark 5.8. We are not aware of any example of a closed monotone symplectic manifold $M^{2n}$ not meeting the requirements of Corollaries 5.6 and 5.7. Such a manifold would have $N = 1$ and $H_s(M) = H_s(\mathbb{CP}^n)$.

Corollary 5.7 considerably broadens the class of symplectic manifolds for which $C^\infty$-generic existence of infinitely many periodic orbits is known. Further, it is clear that the corollary can be extended to some rational symplectic manifolds meeting similar requirements. However, the conjecture that this is true for all closed symplectic manifolds still remains completely open.

With more information about the structure of the quantum product one is sometimes able to find several simple periodic orbits with augmented action equal to $\hat{c}_\infty$.

Corollary 5.9 (Resonance Relations, II). Let $H$ be a perfect Hamiltonian on a strictly positive monotone symplectic manifold $M^{2n}$. Assume that

$$w_0 \ast w_1 \ast \cdots \ast w_\ell = q^\nu w_0 \text{ in } HQ_*(M),$$

where $\nu > 0$, for some classes $w_0 \in H_*(M)$, and $w_1, \ldots, w_\ell \in H_{<2n}(M)$, and

$$2n(\ell - 1) - |w_1| - \cdots - |w_\ell - 1| < 2N.$$

Assume furthermore that $\nu = 1$ or $H$ is non-degenerate. Then there exist $\ell$ distinct one-periodic orbits $x_0, \ldots, x_\ell$ such that

$$\tilde{A}_H(x_i) = \hat{c}_\infty.$$

We emphasize that the main point of this result is not the existence of $\ell$ distinct periodic orbits, which is well known, but the fact that all these orbits have augmented action equal to $\hat{c}_\infty$.

The corollary is a consequence of [CGG, Thm. 1.1] and its proof and of Theorem 5.1. Among the manifolds the corollary applies to (with $F = \mathbb{Q}$) are, e.g., complex Grassmannians and their monotone products. Moreover, the conditions of the theorem are satisfied for $M \times V$, where $V$ is symplectically aspherical, once they are satisfied for $M$. (We refer the reader to [CGG] for a more detailed discussion.)

For $M = \mathbb{CP}^n$, Corollary 5.9 takes the following particularly simple form, refining [GG09a, Thm. 1.12].
Corollary 5.10. Let \( H \) be a perfect Hamiltonian on \( \mathbb{C}P^n \). Then \( H \) has \( n + 1 \) distinct one-periodic orbits with augmented action equal to \( \hat{c}_\infty \).

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