TWO APPLICATIONS OF STRONG HYPERBOLICITY

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ABSTRACT. We present two analytic applications of the fact that a hyperbolic group can be endowed with a strongly hyperbolic metric. The first application concerns the crossed-product C*-algebra defined by the action of a hyperbolic group on its boundary. We construct a natural time flow, involving the Busemann cocycle on the boundary. This flow has a natural KMS state, coming from the Hausdorff measure on the boundary, which is furthermore unique when the group is torsion-free. The second application is a short new proof of the fact that a hyperbolic group admits a proper isometric action on an $\ell^p$-space, for large enough $p$.

1. INTRODUCTION

Hyperbolicity, in the sense of Gromov, is a coarse notion of negative curvature for metric spaces. In turn, a hyperbolic group is a group which admits a proper and cocompact isometric action on a geodesic hyperbolic space. Such a space is said to be a geometric model for the group. Hyperbolic groups form a large class of groups, and they have received a lot of attention–usually from an algebraic and geometric perspective. Herein, the aims are mostly analytic.

A sharp notion of negative curvature for metric spaces is captured by the $\text{CAT}(-1)$ condition. This condition implies, and predates, hyperbolicity. Gromov’s Jugendtraum [6, p.193], that every hyperbolic group admits a geometric model which is $\text{CAT}(-1)$, is still wildly open. It is expected to fail, but no counterexamples are known. Let us mention, however, that the past decade has seen great strides in the $\text{CAT}(0)$ direction. We now understand that an extraordinary number of hyperbolic groups act on $\text{CAT}(0)$ cube complexes.

The search for enhanced geometric models of hyperbolic groups is often motivated by analytic needs. We use the term ‘enhanced hyperbolicity’ as a broad and informal way of describing hyperbolicity with additional $\text{CAT}(-1)$ properties. Such desirable properties depend on the specific context. In [13], we introduced the metric notion of strong hyperbolicity. We find this idea satisfactory on two accounts. Firstly, it is an intermediate metric notion between the $\text{CAT}(-1)$ condition and hyperbolicity, which grants the additional $\text{CAT}(-1)$ properties that, so far, have come up in analytic applications. Secondly, it turns out that every hyperbolic group admits a geometric model which is strongly hyperbolic. We briefly discuss strong hyperbolicity in Section 2 below, and we refer to [13] for more details.
The purpose of this note is to further illustrate the use of strong hyperbolicity in studying analytic aspects of hyperbolic groups. The first application concerns the $C^*$-crossed product $C(\partial \Gamma) \rtimes \Gamma$ defined by the action of a hyperbolic group $\Gamma$ on its boundary $\partial \Gamma$. We use strong hyperbolicity to construct a natural $\mathbb{R}$-flow on $C(\partial \Gamma) \rtimes \Gamma$, from the Busemann cocycle on the boundary. We show that the Hausdorff measure on the boundary defines a KMS state for this Busemann flow, with inverse temperature equal to the Hausdorff dimension of the boundary. Furthermore, this is the unique KMS state for the flow when $\Gamma$ is torsion-free. Previously, these facts were known in two particular cases: for free groups [5], respectively for uniform lattices in $\text{SO}(n, 1)$ [10]. Compare also [8].

The second application is a short new proof of the fact that a hyperbolic group admits a proper isometric action on an $\ell^p$-space, for large enough $p \in [1, \infty)$. This result is due to Yu [15], and different proofs have been subsequently offered in [3, 12, 1]. The argument explained in Section 4 provides a link between Haagerup’s original construction for free groups [7], and the boundary construction of [12].

## 2. Strong hyperbolicity

### 2.1. Strongly hyperbolic spaces.

Let $X$ be a metric space. We write $|x, y|$ for the distance between two points $x, y \in X$. Recall that the Gromov product with respect to a basepoint $o$ is defined by the formula

$$\langle x, y \rangle_o = \frac{1}{2}(|o, x| + |o, y| - |x, y|).$$

The metric space $X$ is said to be strongly hyperbolic if the Gromov product satisfies

$$e^{-\langle x, y \rangle_o} \leq e^{-\langle x, z \rangle_o} + e^{-\langle z, y \rangle_o}$$

for all $x, y, z, o \in X$. (Compare with the original definition [13, Def.4.1], involving an additional ‘visual’ parameter $\epsilon > 0$. The present definition is the normalized case when $\epsilon = 1$, and this can always be achieved by rescaling the metric.)

It is easily checked that a strongly hyperbolic space is, in particular, hyperbolic in the usual, Gromov sense. On the other hand, a CAT($-1$) space is strongly hyperbolic [13, Thm.5.1]. To put it differently, strong hyperbolicity is a weak CAT($-1$) condition. For the purposes of this paper, a useful consequence of strong hyperbolicity is the following [13, Thm.4.2]:

**Theorem 2.1.** Let $X$ be a strongly hyperbolic space, and let $o \in X$ be a basepoint. Then the Gromov product $\langle \cdot, \cdot \rangle_o$ extends continuously to the bordification $X \cup \partial X$, and $e^{-\langle \cdot, \cdot \rangle_o}$ is a compatible metric on the boundary $\partial X$.

### 2.2. Strongly hyperbolic metrics for hyperbolic groups.

Let $\Gamma$ be a hyperbolic group. To avoid trivialities, we will always assume that $\Gamma$ is non-elementary. A metric on $\Gamma$ is said to be admissible if it enjoys the following properties:

(i) it is equivariant: $|gx, gy| = |x, y|$ for all $g, x, y \in \Gamma$;

(ii) it is roughly geodesic: there is a constant $C \geq 0$, so that for every pair of points $x, y \in \Gamma$ there is a (not necessarily continuous) map $\gamma: [a, b] \to \Gamma$ satisfying $\gamma(a) = x$, $\gamma(b) = y$, and $|s - t| - C \leq |\gamma(s), \gamma(t)| \leq |s - t| + C$ for all $s, t \in [a, b]$;

(iii) it is quasi-isometric to any word metric on $\Gamma$.

An admissible metric on $\Gamma$ is hyperbolic, since hyperbolicity is a quasi-isometry invariant for roughly geodesic spaces.
Admissible metrics naturally arise from geometric models for $\Gamma$. Let $X$ be a geodesic hyperbolic space on which $\Gamma$ acts isometrically, properly and cocompactly, and pick a basepoint $o \in X$. Then the orbit metric on $\Gamma$, given by $|g,h|_o := |go,ho|$, is admissible. (An innocuous issue is that $o$ might have non-trivial stabilizer. This is easily made irrelevant either by language, allowing pseudo-metrics instead of metrics, or by coarse bookkeeping.)

If $\Gamma$ admits a CAT（-1）geometric model, then the induced orbit metrics on $\Gamma$ are strongly hyperbolic. The following theorem is a general statement to that effect, circumventing the delicate question whether a CAT（-1）geometric model is always available.

**Theorem 2.2.** There exist admissible metrics on $\Gamma$ which are strongly hyperbolic.

Implicitly, this was first proved in [11] by an involved construction of combinatorial flavour. In [13] we show that there are, in fact, natural admissible metrics that are strongly hyperbolic. Namely, the Green metric defined by any symmetric and finitely supported random walk on $\Gamma$ is, up to a rescaling, strongly hyperbolic [13, Thm.6.1].

### 3. The Busemann Flow for Boundary Actions of Hyperbolic Groups

#### 3.1. Preliminaries

Let us start with some general facts on cocycles, flows, and KMS states for crossed-products. These matters are well-known, and they go back to Renault’s foundational work [14]. A minor difference is that we choose to work with reduced crossed-products, rather than full crossed-products.

Let $G$ be a discrete countable group acting by homeomorphisms on a compact Hausdorff space $\Omega$. The algebraic crossed-product $C(\Omega) \rtimes_{\text{alg}} G$ consists of finite sums of the form $\sum \phi_g g$, where $\phi_g \in C(\Omega)$ and $g \in G$. This is an algebra for the multiplication whose defining rule is that $(\phi g)(\psi h) = \phi(g,\psi)gh$. The reduced crossed-product $C(\Omega) \rtimes_{\text{r}} G$ is the reduced $C^*$-completion of $C(\Omega) \rtimes_{\text{alg}} G$.

A flow on a $C^*$-algebra $A$ is a strongly continuous group homomorphism $\sigma : \mathbb{R} \to \text{Aut}(A)$. On crossed-products, cocycles give rise to flows, as follows. Consider a cocycle $c : G \to C(\Omega,\mathbb{R})$, the real-valued continuous maps on $\Omega$. (Throughout this paper, the cocycle property is in the additive sense: $c(gh) = c(g) + g.c(h)$ for all $g,h \in G$.) Then there is a flow $\sigma^c$ on $C(\Omega) \rtimes_{\text{r}} G$, defined by the formula

$$\sigma^c_t \left( \sum \phi_g g \right) = \sum e^{itc(g)} \phi_g g$$

on $C(\Omega) \rtimes_{\text{alg}} G$.

Let $\sigma$ be a flow on a $C^*$-algebra $A$, and $\beta \in \mathbb{R}$. A state $\omega$ on $A$ is said to be a $\beta$-KMS state for $\sigma$ if

$$\omega(b \sigma_{t,\beta}(a)) = \omega(ab)$$

for all $a, b$ in a dense subalgebra of $\sigma$-entire elements of $A$. We refrain from defining the notion of $\sigma$-entire elements of $A$, except to mention that the $\sigma$-entire elements form a dense $*$-subalgebra of $A$. The parameter $\beta$ is called inverse temperature.

Now consider the flow $\sigma^c$ on $C(\Omega) \rtimes_{\text{r}} G$, induced by a cocycle $c$ as above. Then all elements in $C(\Omega) \rtimes_{\text{alg}} G$ are entire. Let $\omega$ be a $\beta$-KMS state for $\sigma^c$. As for any state on $C(\Omega) \rtimes_{\text{r}} G$, the restriction of $\omega$ to $C(\Omega)$ defines a probability $\mu$ on $\Omega$. (Here, and in what follows, we use the term ‘probability’ as a shorthand for ‘regular Borel probability
The KMS condition means that \( \mu \) is \( e^{\beta c} \)-conformal, in the sense that
\[
\frac{d(g \ast \mu)}{d\mu} = e^{\beta c(g)}
\]
for each \( g \in G \). Conversely, let \( \mu \) be an \( e^{\beta c} \)-conformal probability on \( \Omega \). Consider the state \( \omega_\mu \) on \( C(\Omega) \rtimes_r G \), defined by
\[
\omega_\mu\left( \sum \phi_\gamma g \right) = \int \phi_1 \, d\mu
\]
on \( C(\Omega) \rtimes_{\text{alg}} G \). In other words, \( \omega_\mu \) is the composition of the standard expectation \( C(\Omega) \rtimes_r G \to C(\Omega) \) with \( \mu \), viewed as a state on \( C(\Omega) \). Then \( \omega_\mu \) is a \( \beta \)-KMS state for the cocycle flow \( \sigma_c \).

The following result says that the previous construction is the only source of KMS states for \( \sigma_c \), whenever \( c \) satisfies a certain non-vanishing condition.

**Theorem 3.1** (Kumjian - Renault [9]). Consider a cocycle flow \( \sigma_c \) on \( C(\Omega) \rtimes_r G \). Assume that, for all non-trivial \( g \in G \), \( c(g) \) is non-zero at each fixed point of \( g \). Then every \( \beta \)-KMS state for \( \sigma_c \) is of the form \( \omega_\mu \) for some \( e^{\beta c} \)-conformal probability \( \mu \) on \( \Omega \).

The non-vanishing condition could be thought of as a strong cohomological non-triviality. For if the cocycle \( c : G \to C(\Omega, \mathbb{R}) \) is of the form \( c(g) = g.\theta - \theta \), then \( c(g) \) vanishes at each fixed point of \( g \), for all \( g \in G \).

### 3.2. The boundary crossed product of a hyperbolic group.

Now let \( \Gamma \) be a hyperbolic group and consider the reduced crossed-product \( C(\partial \Gamma) \rtimes_r \Gamma \), defined by the action of \( \Gamma \) on its boundary \( \partial \Gamma \). Endow \( \Gamma \) with a strongly hyperbolic, admissible metric.

A remarkable cocycle on \( \Gamma \) is the Busemann cocycle. To begin, there is the group Busemann cocycle, given by
\[
b(g)(x) = 2\langle g, x \rangle - |g| \quad (x \in \Gamma)
\]
for each \( g \in \Gamma \). Here, and in all that follows, the Gromov product is based at the identity, and we write \( |g| \) for \( |1, g| \), the distance from \( g \) to the identity. The cocycle property for \( b \) is easily checked. In fact, writing \( b(g)(x) = |x| - |g^{-1}x| \) exhibits \( b \) as a coboundary.

Secondly, and more importantly for the purposes of this section, there is a boundary Busemann cocycle. By Theorem 2.1, the group Busemann cocycle extends, by continuity and as a continuous function, to the boundary. The boundary Busemann cocycle is given, for each \( g \in \Gamma \), by
\[
b(g)(\xi) = 2\langle g, \xi \rangle - |g| \quad (\xi \in \partial \Gamma).
\]
The boundary Busemann cocycle \( b \) takes values in \( C(\partial \Gamma, \mathbb{R}) \), so it defines a flow \( \sigma^b \) on \( C(\partial \Gamma) \rtimes_r \Gamma \).

On the other hand, by Theorem 2.1 once again, the Gromov product based at the identity induces a compatible metric
\[
d(\xi_1, \xi_2) = e^{-(\xi_1, \xi_2)}
\]
on \( \partial \Gamma \). Let \( \mu \) be the probability on \( \partial \Gamma \) defined by normalizing the Hausdorff measure, and let \( D \) denote the Hausdorff dimension of \( \partial \Gamma \).

**Theorem 3.2.** Consider the Busemann cocycle flow \( \sigma^b \) on \( C(\partial \Gamma) \rtimes_r \Gamma \). Then the probability \( \mu \) induces a KMS state \( \omega_\mu \) for \( \sigma^b \), at inverse temperature \( D \). If \( \Gamma \) is torsion-free, then \( \omega_\mu \) is the unique KMS state for \( \sigma^b \).
Proof. In order for \( \omega_\mu \) to be a KMS state for \( \sigma^b \) at inverse temperature \( D \), we need to know that the probability \( \mu \) is \( e^{Db^{-1}} \)-conformal. Fix \( g \in \Gamma \). We have

\[
-2\langle gx, gy \rangle = b(g^{-1})(x) + b(g^{-1})(y) - 2\langle x, y \rangle
\]

for all \( x, y \in \Gamma \). This identity extends by continuity to the boundary, leading to

\[
d(g\xi, g\eta)^2 = e^{b(g^{-1})(\xi)} e^{b(g^{-1})(\eta)} d(\xi, \eta)^2
\]

for all \( \xi, \eta \in \partial \Gamma \). It follows, see [12, Lem. 8], that

\[
\frac{d(g^* \mu)}{d\mu} = e^{Db^{-1}}
\]

for each \( g \in G \). Up to replacing \( g \) by \( g^{-1} \), this is means that \( \mu \) is \( e^{Db^{-1}} \)-conformal, as desired.

Now let us turn to the uniqueness statement, in which \( \Gamma \) is assumed to be torsion-free. We wish to apply the Kumjian - Renault criterion, so let us check that \( b \) satisfies the non-vanishing condition of Theorem 3.1. Let \( g \) be a non-trivial element of \( \Gamma \). Then the following properties hold. Firstly, the infinite cyclic subgroup generated by \( g \) is quasi-isometrically embedded in \( \Gamma \). Secondly, there are two distinct points \( g^+, g^- \in \partial \Gamma \) such that \( g^n \to g^+ \) and \( g^{-n} \to g^- \) as \( n \to \infty \). Thirdly, the points fixed by \( g \) on the boundary are precisely \( g^+ \) and \( g^- \).

For the group Busemann cocycle, we have

\[
b(g)(g^n) = |g^n| - |g^{n-1}|, \quad b(g)(g^{-n}) = |g^{-n}| - |g^{-(n+1)}| = -b(g)(g^{n+1}).
\]

Letting \( n \to \infty \), the second relation yields

\[
b(g)(g^-) = -b(g)(g^+),
\]

while the first leads to

\[
b(g)(g^+) = \lim_{n \to \infty} \left( |g^n| - |g^{n-1}| \right) = \lim_{n \to \infty} \frac{|g^n|}{n}
\]

by the discrete l’Hospital rule. But the right-hand limit is positive, as \( g \) is undistorted, and we conclude that \( b(g)(g^+) > 0 \) and \( b(g)(g^-) < 0 \).

We deduce that a KMS state for \( \sigma^b \) at inverse temperature \( D' \) must be induced by a probability \( \mu' \) on \( \partial \Gamma \) which is \( e^{D' b^{-1}} \)-conformal. Results of Coornaert [4], and their generalizations to the roughly geodesic context by Blachère, Haïssinsky, and Mathieu [2], imply that \( D' = D \) and \( \mu' = \mu \).

4. The Haagerup cocycle for hyperbolic groups

4.1. The Haagerup cocycle for free groups. Let \( \mathbb{F} \) be a non-abelian free group. Then \( \mathbb{F} \) admits a proper isometric action on a Hilbert space. This is due to Haagerup [7], up to a slight reinterpretation, and his elegant construction runs as follows.

Consider the standard Cayley graph of \( \mathbb{F} \) with respect to the free generators and their inverses. This is a regular undirected tree. Let \( \tilde{E} \) be the set of its oriented edges. Then \( \mathbb{F} \) acts on \( \tilde{E} \) in a natural way, and we may consider the corresponding orthogonal representation of \( \mathbb{F} \) on \( \ell^2(\tilde{E}) \). Next, we perturb this linear isometric action by a cocycle \( c : \mathbb{F} \to \ell^2(\tilde{E}) \). Given \( g \in \mathbb{F} \), let \( c_g \) be the following function on \( \tilde{E} \): \( c_g \) is supported on the geodesic path joining \( g \) to the identity 1, and for an oriented edge \( e \) lying on this path
we value $c_g(e)$ to be $+1$ or $-1$ according to whether $e$ points towards or away from $g$. In short:

$$c_g = \sum_{e \in \{1 \to g\}} \delta_e - \sum_{e \in \{g \to 1\}} \delta_e$$

The cocycle property, $c_{gh} = c_g + g.c_h$ for all $g, h \in F$, can be seen by drawing the geodesic tripod defined by $1, g$, and $gh$, and noting that the oriented edges lying on the leg towards $g$ cancel out. Clearly, $c_g \in \ell^2(\vec{E})$ and

$$\|c_g\|_2^2 = 2|g|.$$ 

In particular, the cocycle $c$ is proper: $\|c_g\|_2 \to \infty$ as $g \to \infty$ in $F$. It follows that the affine isometric action of $F$ on $\ell^2(\vec{E})$ given by $(g, \phi) \mapsto g.\phi + c_g$ is proper. Note that this construction applies, in fact, to any space $\ell^p(\vec{E})$ for $p \in [1, \infty)$.

We wish to adapt Haagerup’s construction to a general hyperbolic context, and we start by recasting the above cocycle in a more convenient form. Firstly, we think of the oriented edge-set $\vec{E}$ as the set $\{(x, y) \in F \times F : |x| = 1\}$. Second, we note that the cocycle $c$ can be described by in metric terms by the following formula:

$$c_g(x, y) = \langle g, x \rangle - \langle g, y \rangle$$

Recall that $\langle \cdot, \cdot \rangle$ denotes the Gromov product based at the identity. In this form, the cocycle property is even more transparent: writing

$$c_g(x, y) = \frac{1}{2}(|x| - |g^{-1}x|) - \frac{1}{2}(|y| - |g^{-1}y|)$$

we obtain the coboundary formula $c_g = F - g.F$, for $F(x, y) = \frac{1}{2}(|x| - |y|)$.

### 4.2. The Haagerup cocycle for hyperbolic groups.

Let $\Gamma$ be a hyperbolic group, which we may assume to be non-elementary. Endow $\Gamma$ with a strongly hyperbolic admissible metric. We also consider a coarse relative of the underlying set we have used in the free group case. Namely, let

$$\Delta = \{(x, y) \in \Gamma \times \Gamma : K - C \leq |x| \leq K + C\}$$

where $C \geq 0$ is a rough geodesic constant, and $K > 0$ is another constant. For the purposes of the following theorem, we ask that $K > 2C$. Note that $\Delta$ is non-empty. This can be seen by choosing a convenient point along a rough geodesic from the identity to some sufficiently remote group element.

The group $\Gamma$ acts on $\Delta$, by $g.(x, y) = (gx, gy)$. Let $c_g$ be defined on $\Delta$ by the metric formula $[\text{[}]$. Then $c$ is a cocycle for $\Gamma$, for the same reasons as explained above.

**Theorem 4.1.** For large enough $p \in [1, \infty)$, the affine isometric action of $\Gamma$ on $\ell^p(\Delta)$ given by $(g, \phi) \mapsto g.\phi + c_g$ is well-defined and proper.

**Proof.** For the action to be well-defined, we need to have $c_g \in \ell^p(\Delta)$ for each $g \in \Gamma$. An application of the mean value theorem to the function $t \mapsto e^{-t}$ yields

$$|e^{-\langle g, x \rangle} - e^{-\langle g, y \rangle}| \geq e^{-\max\{\langle g, x\rangle, \langle g, y \rangle\}} |\langle g, x \rangle - \langle g, y \rangle|.$$ 

The left-hand side is at most $e^{-\langle x, y \rangle}$, thanks to strong hyperbolicity. On the right-hand side, both $\langle g, x \rangle$ and $\langle g, y \rangle$ are at most $|g|$. It follows that

$$|c_g(x, y)| \leq e|g| e^{-\langle x, y \rangle}.$$
We complete the argument by showing that $e^{-n} \in L^p(\Delta)$ for large enough $p \in [1, \infty)$. If $(x, y) \in \Delta$ then $|x - y| > |x|$, we deduce that
\[
\sum_{(x, y) \in \Delta} e^{-p (x, y)} \leq C_1 \sum_{(x, y) \in \Delta} e^{-p |x|} \leq C_2 \sum_{x \in \Gamma} e^{-p |x|}
\]
and the latter sum converges when $p$ is large enough.

For the action to be proper, we need to argue that $\|c_g\|^p \to \infty$ as $g \to \infty$ in $\Gamma$. In fact, we show that there are constants $C', C'' > 0$, depending only on $K, C,$ and $p$, such that
\[
\|c_g\|^p \geq C' |g| - C''
\]
for each $g \in \Gamma$.

Let $\gamma : [a, b] \to \Gamma$ be a rough geodesic joining the identity to $g$. The basic idea is that $|c_g(x, y)|$ is roughly $|x, y|$ whenever $x$ and $y$ lie on $\gamma$, and that we can find about $|g|/K$ pairs of points on $\gamma$ that belong to $\Delta$. Now let us be precise.

Consider the elements $\gamma(t_i) \in \Gamma$ arising from a partition $a = t_0 < \ldots < t_n \leq b$ into $n$ intervals of length $K$, and a remainder of length less than $K$. Then $|\gamma(t_i), \gamma(t_{i+1})|$ is within $C$ of $|t_i - t_{i+1}| = K$, so $(\gamma(t_1), \gamma(t_{1+1})) \in \Delta$. Also, $c_g(\gamma(t_i), \gamma(t_{i+1}))$ can be written as
\[
\frac{1}{2}(|\gamma(a), \gamma(t_i)| - |\gamma(b), \gamma(t_i)|) - \frac{1}{2}(|\gamma(a), \gamma(t_{i+1})| - |\gamma(b), \gamma(t_{i+1})|)
\]
which is within $2C$ of
\[
\frac{1}{2}(|a - t_i| - |b - t_i|) - \frac{1}{2}(|a - t_{i+1}| - |b - t_{i+1}|) = t_{i+1} - t_i = K.
\]
In particular, $c_g(\gamma(t_i), \gamma(t_{i+1})) \geq K - 2C > 0$, according to our assumption on $K$. Hence
\[
\|c_g\|^p = \sum_{(x, y) \in \Delta} |c_g(x, y)|^p \geq \sum_{i=0}^{n-1} |c_g(\gamma(t_i), \gamma(t_{i+1}))|^p \geq (K - 2C)^p n.
\]

On the other hand, we can relate $n$ and $|g|$. The way we defined the partition implies that $K(n + 1) > b - a$, and $b - a \geq |g| - C$ by using the rough geodesic property at the endpoints. Therefore $n \geq (|g| - (K + C))/K$, and the desired claim follows. \qed

We end by pointing out that the cocycle used in [12] is the boundary analogue of [3], namely $c_g(\xi, \eta) = \langle g, \xi \rangle - \langle g, \eta \rangle$ for $\xi, \eta \in \partial \Gamma$.

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