1. Introduction

Let \((X, d)\) be a compact metric space and \(f : X \rightarrow X\) be a continuous map, then the pair \((X, f)\) is called a dynamical system. The main concern to study a dynamical system is to understand the dynamics of the orbit \(\{f^n(x) \mid n \in \mathbb{N}\}\), for each \(x \in X\) where \(f^n\) denote the \(n\)-times composition of \(f\). Consequently, the idea is to study the discrete dynamical system

\[ x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots \]  

(1)

Let \(\kappa(X)\) denote the set of compact subsets of \(X\). Anologously, define induced discrete dynamical system on \(\kappa(X)\) as

\[ E_{n+1} = \tilde{f}(E_n), \quad n = 0, 1, 2, \ldots \]  

(2)

where \(\tilde{f} : \kappa(X) \rightarrow \kappa(X)\) is defined as \(\tilde{f}(E) = f(E)\) for \(E \in \kappa(X)\).

To deal with non-deterministic problems (see [1]) such as demographic fuzziness, environmental fuzziness and life expectancy crisp dynamical systems (1) and (2) are not enough to model such systems accurately. In that case, we consider the discrete fuzzy system

\[ u_{n+1} = \hat{f}(u_n), \quad n = 0, 1, 2, \ldots \]  

(3)

where \(\hat{f} : \mathcal{F}(X) \rightarrow \mathcal{F}(X)\) is the Zadeh’s extension of \(f\) to \(\mathcal{F}(X)\) and \(\mathcal{F}(X)\) denote the space of all non-empty compact fuzzy sets on \(X\).
The fundamental question here is to find the chaotic dynamical relations between \( f \) and \( \hat{f} \). In this direction, Flores and Cano [2], Kupka [3–6], Wu et al. [7] and Flores et al. [8] have investigated some chaotic dynamics between \( f \) and \( \hat{f} \).

In recent years, the concept of sensitive dependence on initial conditions has attracted significant attention. In [9], an investigation has been done to find the interrelation between sensitivity, asymptotic sensitivity and strong sensitivity and their induction between \( f \) and \( \hat{f} \). Recently, Wu et al. [10] and Zhao et al. [11] have done similar kind of investigation for sensitivity and its various forms on the generalised version of Zadeh extension called g-fuzzification (given by Kupka [5]), and obtained satisfactory results. In this paper, we study some more stronger forms of sensitivity associated to the fuzzy dynamical system (given by Zadeh’s extension principle) (3) via system (2) or otherwise.

This paper is organised as follows. Section 2 contains basic notations and results used in this paper. In Section 3, we give the definition of sensitivity, strong sensitivity, multisensitivity, asymptotic sensitivity, syndetic sensitivity, and cofinite sensitivity. An investigation has been done to find the interrelation between these forms of sensitivity. In the later part of this section we study the relation of these stronger forms of sensitivity for \( f \) and \( \hat{f} \). Section 4 consists of conclusion.

2. Preliminaries

Let \((X, d)\) be a metric space with metric \(d\). Let \(\kappa(X)\) be the collection of all non-empty compact subsets of a metric space \(X\). If \(A \in \kappa(X)\), then \(\epsilon\)-neighbourhood of \(A\) is defined as the set \(U(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\}\).

The Hausdorff metric (distance) \(\mathcal{H}\) on \(\kappa(X)\) is defined by

\[
\mathcal{H}(A, B) = \inf\{\epsilon > 0 \mid A \subseteq U(B, \epsilon) \text{ and } B \subseteq U(A, \epsilon)\}, \text{ for } A, B \in \kappa(X).
\]

It is well known that \((\kappa(X), \mathcal{H})\) is compact (complete, separable, respectively), if and only if \((X, d)\) is compact (complete, separable, respectively).

A fuzzy set \(u\) on \(X\) is a function \(u : X \rightarrow [0, 1]\). A fuzzy set \(u\) is upper semi-continuous (u.s.c.) if for any sequence \(\{x_n \mid n \in \mathbb{N}\}\), in \((X, d)\), converging to a point \(x \in X\), then \(x\) is at least as much in \(u\) as the \(x_n\), i.e. \(u(x) \geq \limsup_{n \in \mathbb{N}} u(x_n)\).

Let us define \(\mathcal{F}(X)\) as the system of all u.s.c. fuzzy sets on \(X\). An empty fuzzy set \(\phi_X\) is defined as \(\{x \in X \mid \phi_X(x) = 0\}\). Let \(\mathcal{F}_0(X)\) denotes the set of all non-empty fuzzy sets on \(X\). The levelwise metric \(D\) on \(\mathcal{F}_0(X)\) is defined by

\[
D(u, v) = \sup_{\alpha \in [0, 1]} \mathcal{H}(L_\alpha u, L_\alpha v),
\]

where \(L_\alpha u = \{x \in X \mid u(x) \geq \alpha\}\) for each \(\alpha \in (0, 1]\) and \(L_0 u = \{x \in X \mid u(x) > 0\}\) (\(\bar{A}\) denotes the closure of \(A\)). This metric holds for non-empty fuzzy sets \(u, v \in \mathcal{F}_0(X)\) whose maximal values are identical. Since the Hausdorff distance \(\mathcal{H}\) is only defined for non-empty closed
subsets of the space $X$, therefore, an extension (cf. [5]) is considered as follows

$$
\mathcal{H}(\phi, \phi) = 0 \text{ and } \mathcal{H}(\phi, C) = \text{diam}(X) \text{ for any } C \in \kappa(X),
$$

which implies

$$
D(\phi x, \phi x) = 0 \text{ and } D(\phi x, v) = \text{diam}(X) \text{ for any } v \in \mathcal{F}_0(X),
$$

where diam($A$) = $\sup\{d(x, y) | x, y \in A\}$. With this extension (4) correctly defines the levelwise metric on $\mathcal{F}(X)$.

**Remark 2.1:** It is known that if $(X, d)$ is complete, compact and separable then $(\mathcal{F}(X), D)$ is complete but fails to be compact and separable (see [12]), and if $u$ is an u.s.c. fuzzy set on $X$ then $L_\alpha u$ is closed in $X$ for all $\alpha \in [0, 1]$.

Every continuous map $f : X \to X$ induces a continuous extension $\tilde{f} : \kappa(X) \to \kappa(X)$, by letting $\tilde{f}(A) = f(A)$ for every $A \in \kappa(X)$.

A fuzzification (or Zadeh’s extension) of the dynamical system $(X, f)$ is the map $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ defined by

$$
\hat{f} u(x) = \sup_{y \in f^{-1}(x)} \{u(y)\}, \text{ for any } u \in \mathcal{F}(X).
$$

If $X$ is a compact metric space then $f : (X, d) \to (X, d)$ is continuous if and only if $\hat{f} : (\mathcal{F}(X), D) \to (\mathcal{F}(X), D)$ is continuous (cf. [2]). Continuity of $f$ and $\hat{f}$ is equivalent even if $X$ is locally compact metric space (cf. [5]).

Further, we define $\mathcal{F}^1(X)$ as the class of all the normal fuzzy sets on $X$, as

$$
\mathcal{F}^1(X) = \{u \in \mathcal{F}(X) | u(x) = 1, \text{ for some } x \in X\}.
$$

It can be seen that $\mathcal{F}^1(X)$ with levelwise metric is a subspace of $\mathcal{F}(X)$. Also, it has been proved that with this metric $\mathcal{F}^1(X)$ is complete, but not compact and is not separable, refer [5, 12].

**Proposition 2.1 ([2]):** For any fuzzy set $u$, the family $\{L_\alpha u \mid \alpha \in (0, 1]\}$, satisfies the following properties:

1. $L_0 u \supseteq L_\alpha u \supseteq L_\beta u$, whenever $0 \leq \alpha \leq \beta$.
2. $u = v \iff L_\alpha u = L_\alpha v$, for all $\alpha \in [0, 1]$, $u, v \in \mathcal{F}(X)$.
3. $L_\alpha \hat{f}(u) = f(L_\alpha u)$, for all $\alpha \in [0, 1]$.

A u.s.c. map $f$ is piecewise constant if there is a finite number of sets $D_i \subset X$ such that $\bigcup D_i = X$ and $f|_{\text{int}(D_i)}$ is constant. A fuzzy set $u$ is piecewise constant if there exists a strictly decreasing sequence $(C_1, C_2, \ldots, C_n)$ of closed subsets of $X$ and strictly increasing sequence of reals $(a_1, a_2, \ldots, a_n) \subset (0, 1]$ such that,

$$
L_\alpha u = C_{i+1}, \text{ whenever } \alpha \in (a_i, a_{i+1}].
$$

**Lemma 2.2 ([4]):** For any $u \in \mathcal{F}(X)$ and $\epsilon > 0$ there exists a piecewise constant fuzzy set $v \in \mathcal{F}(X)$ such that $D(u, v) < \epsilon$, i.e. the set of piecewise constant fuzzy sets is dense in $\mathcal{F}(X)$. 
3. Sensitivity for Induced Fuzzified Map

A continuos map $f : X \to X$ is said to be

- **sensitive dependence on initial conditions (or sensitive)**, if there is $\delta > 0$ (sensitivity constant) such that for every point $x \in X$ and for each $\epsilon > 0$ there is $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) \geq \delta$.

- **strongly sensitive** if there is a $\delta > 0$ such that for each $x \in X$ and for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\sup_{y \in B_d(x, \epsilon)}\{d(f^n(x), f^n(y))\} > \delta$, where $B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$.

- **multi-sensitive** if there is $\delta > 0$ such that for every $k \geq 1$ and any non-empty open subsets $U_1, U_2, \ldots, U_k \subset X$, the set $\bigcap_{1 \leq i \leq k} S_f(U, \delta)$ is non-empty, where $S_f(U, \delta) = \{n \in \mathbb{N} \mid \exists x, y \in U \text{ with } d(f^n(x), f^n(y)) > \delta\}$.

An infinite subset $A$ of $\mathbb{N}$ is said be syndetic if it has bounded gaps, i.e. if $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ then there exist $M \in \mathbb{N}$ such that $a_{i+1} - a_i < M$ for every $i \in \mathbb{N}$, and $A$ is said be cofinite if $\mathbb{N} \setminus A$ is finite.

Now, we define, some more forms of sensitivity depending upon the ‘largeness’ of the set of all $n \in \mathbb{N}$ where this sensitivity happens. We say that,

- $f$ is asymptotically sensitive if there is $\delta > 0$ such that for each open set $U$ in $X$, we have that $S_f(U, \delta)$ is infinite.

- $f$ is syndetically sensitive if there is $\delta > 0$ such that for every non-empty open subset $U \subset X$, the set $S_f(U, \delta)$ is syndetic.

- $f$ is cofinitely sensitive if there is $\delta > 0$ such that for every non-empty open subset $U \subset X$, the set $S_f(U, \delta)$ is cofinite.

It is easy to see that,

cofinite sensitivity $\Rightarrow$ syndetic sensitivity $\Rightarrow$ asymptotic sensitivity

$\Rightarrow$ sensitivity.

cofinite sensitivity $\Rightarrow$ strong sensitivity $\Rightarrow$ multi-sensitivity $\Rightarrow$ sensitivity.

Clearly, cofinite sensitivity implies all the other forms of sensitivities, and some of them are not related to each other in any way.

Consider a one-sided symbolic dynamical system $(\Sigma_2^+, \sigma)$, where

$$\Sigma_2^+ = \{(s_0, s_1, s_2, \ldots) \mid s_i \in \{0, 1\}\},$$

and $\sigma$ is a shift map on $\Sigma_2^+$, defined as

$$(\sigma(s))_n = s_{n+1},$$

which is continuous (cf. [13]). It is known that $(\Sigma_2^+, \sigma)$ is a compact metric space with the metric $d$, defined as

$$d(s, t) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{2^n},$$

where $s = (s_0, s_1, s_2, \ldots)$ and $t = (t_0, t_1, t_2, \ldots)$. 
Now, consider a subspace $S$ of the symbolic dynamical system $(\Sigma^+, \sigma)$, consisting of all the sequences which are eventually zero. Clearly, the restriction of $\sigma$ on $S$ is strongly sensitive and multi-sensitive but not asymptotically sensitive (also not, cofinitely sensitive and syndetic sensitive).

For a continuous map $f : X \to X$ on a compact metric space, asymptotic sensitivity is equivalent to sensitivity (cf. [14]). It has been proved that there is no relation between strong sensitivity and asymptotic sensitivity, even on compact metric space (refer to [9]).

**Proposition 3.1 ([9]):** If $(\kappa(X), \bar{f})$ is sensitive, then $(X, f)$ is also sensitive.

The following example shows that, in general, converse of the above proposition is not true.

**Example 1:** Consider the one-sided shift space $\Sigma^+_2$ on two symbols, let $T$ be the irrational rotation on the circle $S^1$ given by $T(\theta) = \theta + \alpha$, where $\alpha$ is a very small irrational multiple of $2\pi$. By dividing $S^1$ into two hemispheres, define a sequence $\bar{x} = (x_n) \in \Sigma^+_2$ as

$$x_n = \begin{cases} 0, & 0 \leq T^n(0) < \pi \\ 1, & \pi \leq T^n(0) < 2\pi, \end{cases}$$

$n \in \mathbb{N}$.

Define $X = \{\sigma^n(\bar{x}) \mid n \geq 0\}$, then $(X, \sigma)$ is sensitive but $(\kappa(X), \sigma)$ in not sensitive, refer [9].

In the following proofs, by $\chi_x$ we mean the *characteristic function* defined as

$$\chi_x(y) = \begin{cases} 1, & y = x \\ 0, & \text{otherwise} \end{cases}$$

**Theorem 3.2:** Let $f : X \to X$ be a continuous function. Then, sensitivity of $\hat{f}$ implies sensitivity of $\bar{f}$.

**Proof:** Suppose $\hat{f}$ is sensitive on $\mathcal{F}(X)$ (with sensitivity constant $\delta$). Let $A \in \kappa(X)$ and $\epsilon > 0$. As $\chi_A \in \mathcal{F}(X)$, there exists $\nu \in \mathcal{F}(X)$ and $m \in \mathbb{N}$ such that $D(\chi_A, \nu) < \epsilon$ and $D(\hat{f}^m(\chi_A), \hat{f}^m(\nu)) > \delta$. Now,

$$D(\hat{f}^m(\chi_A), \hat{f}^m(\nu)) = \sup_{\alpha \in [0,1]} \mathcal{H}(L_\alpha \hat{f}^m(\chi_A), L_\alpha \hat{f}^m(\nu))$$

$$= \sup_{\alpha \in [0,1]} \mathcal{H}(f^m(A), f^m(L_\alpha \nu)) > \delta.$$

We can find $\alpha_0 \in [0, 1]$ such that

$$\mathcal{H}(f^m(A), f^m(L_{\alpha_0} \nu)) = \mathcal{H}(\bar{f}^m(A), \bar{f}^m(L_{\alpha_0} \nu)) > \delta.$$

$D(\chi_A, \nu) < \epsilon$, implies $\mathcal{H}(A, L_\alpha \nu) < \epsilon$ for all $\alpha \in [0, 1]$. Hence, $\bar{f}$ is sensitive on $\kappa(X)$. ■

**Theorem 3.3 ([2]):** Let $f : X \to X$ be a continuous function and $(\mathcal{F}(X), \hat{f})$ is sensitive, then $(X, f)$ is sensitive.
Remark 3.1: Converse of the above theorem does not hold, e.g. the subsystem of one-sided shift space taken in Example 1 is sensitive but its set-valued counterpart is deprived of sensitivity. Therefore, by Theorem 3.2, $(\mathcal{F}(X), \hat{f})$ cannot be sensitive.

Proposition 3.4 ([9]): Let $f : X \to X$ be a continuous function. Then $(X, f)$ is strongly sensitive if and only if $(\kappa(X), \hat{f})$ is strongly sensitive.

Now, we prove the same relation between $(X, f)$ and $(\mathcal{F}(X), \hat{f})$, i.e.

Theorem 3.5: Let $f : X \to X$ be a continuous function. Then $\hat{f}$ is strongly sensitive on $\mathcal{F}(X)$ if and only if $f$ is strongly sensitive on $X$.

Proof: Let $\hat{f}$ be strongly sensitive on $\mathcal{F}(X)$ with sensitivity constant $\delta$. Let $x \in X$ and $\epsilon > 0$ be given. For $x \in \mathcal{F}(X)$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{u \in B_{\delta}(x, \epsilon)} |D(f^n(x), \hat{f}_n(u))| > \delta,$$ for every $n \geq n_0$.

Choose any $m \geq n_0$, there exists $v \in B_{\delta}(x, \epsilon)$ such that $D(\hat{f}^m(x), \hat{f}^m(v)) > \delta$.

Now,

$$D(\hat{f}^m(x), \hat{f}^m(v)) = \sup_{\alpha \in [0,1]} \mathcal{H}(L_{\alpha} \hat{f}^m(x), L_{\alpha} \hat{f}^m(v))$$

$$= \sup_{\alpha \in [0,1]} \mathcal{H}(f^m(x), f^m(L_{\alpha} v))$$

$$= \sup_{\alpha \in [0,1]} \{ \sup_{y \in L_{\alpha} v} d(f^m(x), f^m(y)) \}$$

$$= \sup_{y \in L_{\alpha} v} d(f^m(x), f^m(y)) > \delta.$$

We can find $x_0 \in L_{\alpha} v$ such that $d(f^m(x), f^m(x_0)) > \delta$. Consequently, $\sup_{y \in B_{\delta}(x, \epsilon)} d(f^n(x), f^n(y)) > \delta$, for every $n \geq n_0$.

Conversely, let $f$ be strongly sensitive with sensitivity constant $\delta$. Let $u \in \mathcal{F}(X)$ and $\epsilon > 0$ be given. There exists a piecewise constant fuzzy set $v$, such that $D(u, v) < \frac{\delta}{2}$, represented by some strictly decreasing sequence of closed subsets $\{C_1, C_2, \ldots, C_p\}$ of $X$ and strictly increasing sequence of reals $\{a_1, a_2, \ldots, a_p\} \subset (0, 1]$ such that $L_{a_i} v = C_{i+1}$, whenever $\alpha \in (a_i, a_{i+1}]$.

As $\hat{f}$ is strongly sensitive (by Proposition 3.4) there exists a sensitivity constant $\lambda_\delta > 0$. Therefore, for each $C_i \in \kappa(X)$ there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$\sup_{Y \in B_{\mathcal{H}(\kappa)}(C_i, \epsilon)} \mathcal{H}(\hat{f}^m(C_i), \hat{f}^m(Y)) > \lambda_\delta.$$

Let $n_0 = \max\{n_i | 1 \leq i \leq p\}$. Take $m \geq n_0$. For each $1 \leq i \leq p$ there exists $E_i \in \kappa(X)$ such that $\mathcal{H}(C_i, E_i) < \frac{\delta}{2}, \mathcal{H}(\hat{f}^m(C_i), \hat{f}^m(E_i)) > \lambda_\delta$ and $E_i \supset E_{i+1}$.

We get a piecewise constant fuzzy set $\eta$, by defining it as $L_{a_i} \eta = E_{i+1}$, whenever $\alpha \in (a_i, a_{i+1}]$. 
there exist strictly decreasing sequence 

increasing sequence of reals 

wise constant fuzzy sets are dense in 

\( D \) and 

\( B \)

Theorem 3.7 ([15]):

\( \text{Theorem 3.6:} \)

\( \text{Proof:} \) Let \( \hat{f} \) is multi-sensitive with sensitivity constant \( \delta > 0 \). Since the set of all the piecewise constant fuzzy sets are dense in \( \mathcal{F}(X) \), it is enough to prove that \( \hat{f} \) is multi-sensitive on this set. Consider, piecewise constant fuzzy sets \( \nu_1, \nu_2, \ldots, \nu_k \) in \( \mathcal{F}(X) \). For each \( 1 \leq i \leq k \) there exist strictly decreasing sequence \( \{C_{i1}, C_{i2}, \ldots, C_{ik}\} \) of closed subsets in \( X \) and strictly increasing sequence of reals \( \{a_{i1}, a_{i2}, \ldots, a_{ik}\} \subset (0, 1] \) such that 

\[
L_a \nu_i = C_{ij}, \text{ whenever } \alpha \in (a_{ij}, a_{ij+1}].
\]

Since \( \hat{f} \) is multi-sensitive, for each \( 1 \leq i \leq k \) and for every \( 1 \leq j \leq k_i \), we can choose \( S_{ij} \in B_\epsilon(C_{ij}, \delta) \) such that 

\[
\mathcal{H}(\hat{f}_i(C_{ij}), \hat{f}_i(A_{ij})) < \epsilon \text{ and } A_{ij} \supset A_{ij+1}.
\]

For each \( 1 \leq i \leq k \), define a piecewise constant fuzzy set \( u_i \) as 

\[
L_a u_i = A_{ij}, \text{ whenever } \alpha \in (a_{ij}, a_{ij+1}].
\]

Hence for each \( 1 \leq i \leq k \), we get 

\[
D(\nu_i, u_i) < \epsilon \text{ and } D(\hat{f}_i(\nu_i), \hat{f}_i(u_i)) > \delta.
\]

Consequently, 

\[
\bigcap_{1 \leq i \leq k} S_{ij}(B_\epsilon(u_i, \delta), \delta) \neq \emptyset.
\]

Conversely, let \( \hat{f} \) is multi-sensitive with sensitivity constant \( \delta_0 \). Consider \( A_1, A_2, \ldots, A_p \in \kappa(X) \) and \( \epsilon > 0 \). Then, \( \chi_{A_1}, \chi_{A_2}, \ldots, \chi_{A_p} \) are fuzzy sets in \( \mathcal{F}(X) \), there exist \( n \in \mathbb{N} \) such that 

\[
\sup_{u \in B_\delta(\chi_{A_i}, \epsilon)} D(\hat{f}_i(\chi_{A_i}), \hat{f}_i(u)) = \sup_{u \in B_\delta(\chi_{A_i}, \epsilon)} \mathcal{H}(L_a \hat{f}_i(\chi_{A_i}), L_a \hat{f}_i(u)) = \sup_{u \in B_\delta(\chi_{A_i}, \epsilon)} \mathcal{H}(f^n(A_i), f^n(u)) = \sup_{u \in B_\delta(\chi_{A_i}, \epsilon)} \mathcal{H}(\hat{f}_i^n(A_i), \hat{f}_i^n(u)) > \delta_0.
\]

Hence, for each \( 1 \leq i \leq p \) there exist \( u_i \in \mathcal{F}(X) \) and \( \alpha_i \in [0, 1] \) such that 

\[
\mathcal{H}(A_i, L_a(u_i)) < \epsilon \text{ and } \mathcal{H}(\hat{f}_i^n(A_i), \hat{f}_i^n(L_a(u_i))) > \delta_0.
\]

So, we can conclude that 

\[
\bigcap_{1 \leq i \leq p} S_{ij}(B_\epsilon(A_i, \delta), \delta_0) \neq \emptyset
\]

and consequently, \( \hat{f} \) is multi-sensitive. 

\[\blacksquare\]

**Theorem 3.7 ([15]):** \( (\kappa(X), \hat{f}) \) is multi-sensitive if and only if \( (X, f) \) is so.
From the above two theorems, we have the following result.

**Theorem 3.8:** \((F(X), \hat{f})\) is multi-sensitive if and only if \((X, f)\) is so.

**Theorem 3.9:** If \((F(X), \hat{f})\) is asymptotically sensitive, then \((X, f)\) is asymptotically sensitive.

**Proof:** Let \((F(X), \hat{f})\) be asymptotically sensitive with sensitivity constant \(\delta > 0\). Suppose \(x \in X\) and \(\epsilon > 0\) be given. As \(\chi_x \in F(X)\), we can find \(\nu_1 \in F(X)\) such that \(D(\chi_x, \nu_1) < \epsilon\) and 
\[
\limsup_{n \to \infty} D(\hat{f}^n(\chi_x), \hat{f}^n(\nu_1)) > \delta.
\]

Now,
\[
\limsup_{n \to \infty} D(\hat{f}^n(\chi_x), \hat{f}^n(\nu_1)) = \limsup_{n \to \infty} \sup_{\alpha \in [0,1]} H(L_\alpha \hat{f}^n(\chi_x), L_\alpha \hat{f}^n(\nu_1))
\]
\[
= \limsup_{n \to \infty} \sup_{\alpha \in [0,1]} H(f^n(x)), f^n(L_\alpha \nu_1))
\]
\[
= \limsup_{n \to \infty} \sup_{y \in L_\alpha \nu_1} d(f^n(x), f^n(y)) > \delta.
\]

We can find \(n_1 \in \mathbb{N}\) and \(x_1 \in L_0 \nu_1\) such that \(d(f^{n_1}(x), f^{n_1}(x_1)) > \delta\), if \((x, x_1)\) form an asymptotic sensitive pair, then we are done. If not, then we can find \(t_1 > n_1\) such that 
\[
d(f^{n_1}(x), f^{n_1}(x_1)) < \delta/2 \text{ for all } n \geq t_1.
\]

Also, since \(f^{n_1}\) is continuous we can find a neighbourhood \(V_1\) of \(x_1\) such that 
\[
V_1 \subset B_d(x, \epsilon) \text{ and } d(f^{n_1}(x), f^{n_1}(y)) > \delta \text{ for all } y \in V_1.
\]

We can find an \(\epsilon_1 > 0\) such that \(B_d(x_1, \epsilon_1) \subset V_1\).

Again for \(\epsilon_1 > 0\) we can find \(\nu_2 \in F(X)\) such that \(D(\chi_{x_1}, \nu_2) < \epsilon_1\) and 
\[
\limsup_{n \to \infty} D(\hat{f}^n(\chi_{x_1}), \hat{f}^n(\nu_2)) > \delta.
\]

We can find \(n_2 \in \mathbb{N}\) and \(x_2 \in L_0 \nu_2\) such that 
\[
d(f^{n_2}(x_1), f^{n_2}(x_2)) > \delta.
\]

Consequently, \(d(f^{n_2}(x), f^{n_2}(x_2)) > \delta/2\).

If \((x, x_2)\) form an asymptotic sensitive pair, then we are done. If not, then we can find \(t_2 > n_2\) such that 
\[
d(f^{n_2}(x), f^{n_2}(x_2)) < \delta/2 \text{ for all } n \geq t_2.
\]

Also, since \(f^{n_2}\) is continuous we can find a neighbourhood \(V_2\) of \(x_2\) such that 
\[
V_2 \subset B_d(x_1, \epsilon_1) \text{ and } d(f^{n_2}(x), f^{n_2}(y)) > \delta \text{ for all } y \in V_2.
\]

We can find an \(\epsilon_2 > 0\) such that \(B_d(x_2, \epsilon_2) \subset V_2\).

Continuing like this, we either get required asymptotic sensitive pair \((x, x_0)\) or a sequence \(\{x_n\} \subset B_d(x, \epsilon)\). Let \(l\) be the limit point of this sequence, then 
\[
d(f^{n_i}(x), f^{n_i}(l)) > \delta \text{ for each } i \in \mathbb{N}, \text{ which implies } \limsup_{n \to \infty} d(f^n(x), f^n(l)) > \delta.
\]

Consequently, \(f\) is asymptotically sensitive.

**Remark 3.2:** Converse of the above theorem is not true. Since a sensitive map is asymptotically sensitive on a compact metric space, and the dynamical system \((X, \sigma)\) considered in Example 1 is compact and sensitive, hence asymptotically sensitive. Since the hyperspace \((\kappa(X), \sigma)\) is not sensitive hence cannot be asymptotically sensitive. Therefore, \((F(X), \hat{f})\) cannot be asymptotically sensitive (Theorem 3.2).

In the presence of dense set of periodic points sensitivity imply asymptotic sensitivity (see [16]). Using this fact and our theorem 3.5 we give the following corollary.

**Corollary 3.10:** If \((X, f)\) has dense set of periodic points and strongly sensitive, then \((F(X), \hat{f})\) is asymptotically sensitive.
**Proof:** By Theorem 5 of [2], periodic density of \( f \) implies periodic density of \( \hat{f} \), and strong sensitivity of \( f \) implies the same for \( \hat{f} \) (Theorem 3.5). Consequently, \( \hat{f} \) is asymptotically sensitive. □

**Lemma 3.11 ([15]):** Let \( a, b, c, d \) be real numbers with \( a < b \) and \( c < d \). If there is \( L > 0 \) such that \( b - a < L \) and \( d - c < L \), then \( \min\{b, d\} - \min\{a, c\} < L \).

**Theorem 3.12:** \((\kappa(X), \hat{T})\) is syndetically sensitive if and only if \((\mathcal{F}(X), \hat{T})\) is syndetically sensitive.

**Proof:** Let \( \hat{T} \) is syndetically sensitive with sensitivity constant \( \delta > 0 \). We do the proof for \( \hat{T} \) on the set of piecewise constant fuzzy sets in \( \mathcal{F}(X) \), as it is dense in \( \mathcal{F}(X) \). Let \( u \) be any piecewise constant fuzzy set and let \( \epsilon > 0 \). There exists a strictly decreasing sequence \( \{A_1, A_2, \ldots, A_k\} \) of closed subsets in \( X \) and strictly increasing sequence of reals \( \{a_1, a_2, \ldots, a_k\} \subset (0, 1) \) such that

\[
L_\alpha u_i = A_i, \text{ whenever } \alpha \in (a_i, a_{i+1}].
\]

For each \( 1 \leq i \leq k \), we can choose \( C_i \in B_\gamma(A_i, \epsilon) \) such that

\[
M_i = \{n \in \mathbb{N} \mid \hat{\mathcal{H}}(\hat{f}_n(A_i), \hat{f}_n(C_i)) > \delta\}
\]

is syndetic and \( C_i \supset C_{i+1} \).

For each \( 1 \leq i \leq k \) lets rewrite the set \( M_i = \{n_{ij} \mid n_{ij+1} > n_{ij} \text{ for all } j \geq 1\} \).

By the hypothesis, for each \( 1 \leq i \leq k \), there is \( L_i > 0 \) such that

\[
n_{ij+1} - n_{ij} < L_i \text{ for all } j \geq 1 \text{. Take } L = \max\{L_i \mid 1 \leq i \leq k\}. \text{ Define,}
\]

\[
M = \{n_j = \min\{n_{ij} \mid 1 \leq i \leq k\} \text{ for } j \geq 1\}.
\]

By Lemma 3.11, \( M \) is syndetic with \( n_{j+1} - n_j < L \) and

\[
\sup_{1 \leq i \leq k} \{\mathcal{H}(\hat{f}_n(A_i), \hat{f}_n(C_i))\} > \delta, \text{ for all } j \geq 1.
\]

Define a piecewise constant fuzzy set \( v \) as,

\[
L_\alpha v = C_{i+1} \text{ whenever, } \alpha \in (a_i, a_{i+1}].
\]

Clearly, \( D(u, v) < \epsilon \) and

\[
D(\hat{f}_n(u), \hat{f}_n(v)) = \sup_{\alpha \in [0, 1]} \mathcal{H}(L_\alpha \hat{f}_n(u), L_\alpha \hat{f}_n(v))
\]

\[
= \sup_{\alpha \in [0, 1]} \mathcal{H}(f_n(L_\alpha u), f_n(L_\alpha v))
\]

\[
= \sup_{1 \leq i \leq k} \{\mathcal{H}(\hat{f}_n(A_i), \hat{f}_n(C_i))\} > \delta,
\]

for all \( j \geq 1 \), which completes the proof.

For the converse, if \( A \in \kappa(X) \) is a non-empty set, then for \( \chi_A \in \mathcal{F}(X) \) there exists \( u \in \mathcal{F}(X) \) such that \( D(\chi_A, u) < \epsilon \) and the set \( \Omega = \{n \in \mathbb{N} \mid D(\hat{f}_n(\chi_A), f_n(u)) > \delta\} \) is syndetic. Now, for \( n \in \Omega \), we have

\[
\delta < D(\hat{f}_n(\chi_A), \hat{f}_n(u)) = \sup_{\alpha \in [0, 1]} \mathcal{H}(L_\alpha \hat{f}_n(\chi_A), L_\alpha \hat{f}_n(u))
\]

\[
= \sup_{\alpha \in [0, 1]} \mathcal{H}(f_n(L_\alpha \chi_A), f_n(L_\alpha u))
\]

\[
= \sup_{\alpha \in [0, 1]} \mathcal{H}(f_n(A), f_n(L_0 u)) = \mathcal{H}(f_n(A), f_n(L_0 u)).
\]

Clearly, \( \mathcal{H}(A, L_0 u) < \epsilon \). Hence, the proof. □
Theorem 3.13 ([15]): \((\kappa(X), \tilde{f})\) is syndetically sensitive if and only if \((X, f)\) is so.

From the above two theorems, we have the following result.

Theorem 3.14: \((\mathcal{F}(X), \hat{f})\) is syndetically sensitive if and only if \((X, f)\) is so.

Theorem 3.15: \((\kappa(X), \tilde{f})\) is cofinitely sensitive if and only if \((\mathcal{F}(X), \hat{f})\) is cofinitely sensitive.

Proof: Proof is similar to the proof of Theorem 3.12, with slight modifications.

Consequently, we can have the following result.

Theorem 3.16: \((\mathcal{F}(X), \hat{f})\) is cofinitely sensitive if and only if \((X, f)\) is cofinitely sensitive.

4. Conclusion

Let \((X, f)\) a dynamical system, where \(f : X \to X\) be a continuous map on a compact metric space \(X\), and \((\mathcal{F}(X), \hat{f})\) be its fuzzified extension given by Zadeh’s extension principle, where \(\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)\). Our investigation for finding the chaotic dynamical relation between \(f\) and \(\tilde{f}\) in the related dynamical properties of sensitivity and its stronger forms reveal (Theorems 3.3, 3.5, 3.6, 3.9, 3.12, 3.16) that if \(\hat{f}\) is sensitive, strongly sensitive, multisensitive, asymptotically sensitive, syndetically sensitive and cofinitely sensitive, respectively then the same holds for \((X, f)\). For the converse, we prove that if \((X, f)\) is strongly sensitive (multi-sensitive, syndetic sensitive, cofinitely sensitive) then \((\mathcal{F}(X), \hat{f})\) is so (Theorems 3.5, 3.6, 3.12, 3.16), but sensitivity and asymptotic sensitivity, respectively of \(f\) does not imply sensitivity and asymptotic sensitivity, respectively for \(\hat{f}\). It can be clearly noted that we reveal similar relation of sensitivity and its stronger forms for \(f\) and \(\tilde{f}\).

We get Corollary 3.10 as a consequence of Theorems 3.5, where we establish that strong sensitivity of \((X, f)\) implies asymptotic sensitivity for \((\mathcal{F}(X), \hat{f})\), in the presence of periodic density.

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Notes on contributors

Praveen Kumar was born in Delhi, India. He received his B.Sc. Degree from Ramjas College, Delhi, India, in 2003 and M.Sc. in mathematics from IIT Delhi, India. He got his Ph.D Degree in Chaotic Dynamics and Induced Maps on Hyperspaces from University of Delhi, India, in 2017. He is currently an Assistant Professor in the department of Mathematics, Ramjas college, Delhi, India and having a teaching experience of more than 15 year. His research interest is to study topological dynamics.

Ayub Khan was born in Shaharanpur, India. He received his B.Sc. Degree from Ramjas college, Delhi, India, in 1979 and M.Sc, M. Phil degree in Linear and Non-Linear Stability of Dynamical Systems and
Ph.D degree in Chaos in Non-Linear Planar Oscillations of a Satellite in Elliptic Orbits from University of Delhi, India in 1981, 1983 and 1995, respectively. He is currently a Professor and the Head of the Mathematics department of Jamia Millia Islamia University, Delhi, India. He has an teaching experience of more than 34 years. So far he has guided more than 13 Ph.D. students and 4 M.Phil students and have more than 150 research paper. He delivered more than 32 talks in various national and international Seminar/conferences. His research interest are analysis of chaos and synchronization for Non-linear dynamical system and Topological dynamics.

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