MULTIPLICATIVE GROUPS OF FIELDS AND HEREDITARILY IRREDUCIBLE POLYNOMIALS

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Abstract. In this paper we explore the concept of good heredity for fields from a group theoretic perspective. Extending results from [9], we show that several natural families of fields are of good heredity, and some others are not. We also construct several examples to show that various wishful thinking expectations are not true.

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1. Introduction

In this paper we investigate a few properties of fields closely related to the freeness of their multiplicative groups. The first author stumbled upon one of them in her investigation of divisibility of quasiendomorphisms of abelian varieties in [9]. This property of a field \( F \), which we call good heredity, deals with (ir)reducibility of \( P(x^n) \) for a polynomial \( P(x) \) over \( F \), as \( n \) varies. The other two are properties of the multiplicative group of the field, probably slightly weaker than freeness. We were surprised to discover that the two are not equivalent to each other, nor to good heredity (see Examples 17 and 22 and Section 6.2). One advantage of our good heredity over all these related properties of the multiplicative group is that good heredity passes to finite extensions of fields (see Theorem 30), while the various properties of the multiplicative group do not (see example in §7.3 and §7.1).

Definition 1. For a field \( F \),
- let \( F^\times \) denote the multiplicative group of \( F \), an abelian group;
- let \( \mu(F) \) denote the group of roots of unity on \( F \), i.e. the torsion subgroup of \( F^\times \);
- and let \( F^\times/\mu(F) \) denote their quotient, a torsion-free abelian group.

May [6] shows that any locally cyclic abelian group can show up as a direct summand of \( F^\times \). May [8] proves that for many interesting fields, \( F^\times/\mu(F) \) is a free abelian group; and constructs an example showing that this property (freeness of \( F^\times/\mu(F) \)) does not pass to finite extensions, even in the very tame setting of algebraic extensions of \( \mathbb{Q} \).

Definition 2. An abelian group \( G \) is rootless if for any non-torsion element \( a \in G \), the divisible hull inside \( G \) of the subgroup of \( G \) generated by \( a \) is free. We call \( G \) free modulo torsion if the quotient of \( G \) by its torsion subgroup is free abelian.

A field \( F \) is rootless if the group \( F^\times \) is rootless.

A field \( F \) is rootless modulo torsion if the group \( F^\times/\mu(F) \) is rootless.

It is easy to see that if \( F^\times/\mu(F) \) is free, then \( F \) is rootless modulo torsion; and if \( F \) is rootless modulo torsion, then \( F \) is rootless. We show that the converses of both statements are false with Example Sections 7.2 and 7.4.

We show that, like freeness of \( F^\times/\mu(F) \), these two properties of fields do not pass to finite extensions (see Example 6.1). However, they shed light on the third, more complicated property we call “good
heredity” (see Definition 25), which does pass up to finite extensions (Theorem 30). This good heredity was the original motivation for this investigation. We show that good heredity of $F$ implies that every finite extension of $F$ is rootless (Theorem 30 and Proposition 27), and is implied by every finite extension of $F$ being rootless modulo torsion (Proposition 27). We do not know whether converses of these hold; if the following is true, all three are equivalent.

**Question 3.** Suppose that every finite extension of a field $F$ is rootless; does it follow that $F$ is rootless modulo torsion? In particular, does the rootless but not rootless modulo torsion field constructed in Section 6.2 have a finite extension that is not rootless?

In [9], the first author needs to understand finite-to-finite group correspondences (i.e. quasiendomorphisms) of cartesian powers of an elliptic curve $E$. These can be encoded by matrices over $F$, the field of fractions of the ring of endomorphisms of $E$. Good heredity of the number field $F$ is used to analyze the characteristic polynomials of these matrices. While the model-theoretic goals of [9] are beyond the scope of this paper, they do suggest a desirable generalization, from elliptic curves to arbitrary simple abelian varieties.

**Question 4.** Do quasiendomorphism division rings of higher dimensional abelian varieties have good heredity? and what does that even mean when multiplication is not commutative?

This paper includes a few model-theoretic side comments for the initiated, without defining or explaining any of the terminology; rest assured that none of these side comments are used in the main flow of logic.

Our results go well beyond the model-theoretic motivations of [9]. We show that the following fields have good heredity:

- $\mathbb{F}^{\text{alg}}_p$ for any prime $p$;
- any finitely generated field (Corollary 33);
- the maximal abelian extension of any number field (Corollary 36, leveraging May’s results from [8]);
- any extension of $\mathbb{Q}$ by a set of algebraic elements of bounded degree (Corollary 36, leveraging May’s results from [8]);
- any subfield of a field of good heredity (Remark 26); and
- any finitely generated field extension of a field of good heredity (Proposition 32).

All of these except for $\mathbb{F}^{\text{alg}}_p$ are very far from being algebraically closed. Indeed, the multiplicative group of an algebraically closed field is divisible, so with the exception of $\mathbb{F}^{\text{alg}}_p$ where all elements are roots
of unity, no algebraically closed field is rootless, so none have good heredity. More generally, we show that no local field is rootless, so none of those have good heredity (Corollary 34).

**Question 5.** Are there any pseudofinite fields of good heredity? More generally, any PAC fields of good heredity? Morally, how far does a field have to be from ACF to have a chance of good heredity?

The last section of this paper contains nasty counterexamples to two things we tried very hard to prove: that rootless and rootless modulo torsion are the same, and that this property passes to finite field extensions.

**Terminology Warning 1.** Beware that the term “hereditarily irreducible polynomial” has been used to mean something else (though related) in [10, 1, 2]. In these references a polynomial $P(y_1,\ldots,y_n)$ is called hereditarily irreducible, if for all non-constant single variable polynomials $f_1, f_n$ the polynomial $P(f_1(y_1),\ldots,f_n(y_n))$ is irreducible. This is a far more restrictive condition, and much harder to check for specific polynomials, or classes thereof. It is, for example, an already non-trivial result from [10] that if $f(x)$ is any square free polynomial of degree exceeding one, then the two variable polynomial $P(x,y) = yf(x) + 1$ is hereditarily irreducible.

**Terminology Warning 2:** In this article roots refers to roots of elements of a field, e.g. roots of unity, or the second root of 2. When talking about polynomial, we will use zero to mean the root of the polynomial.

## 2. Polynomials

**Definition 6.** A hereditary factor of a polynomial $P \in F[x]$ over a field $F$ is any factor of $P(x^n)$ for some $n \in \mathbb{N}$.

A polynomial $P \in F[x]$ over a field $F$ is hereditarily irreducible over $F$ if for every $n \in \mathbb{N}$, the polynomial $P(x^n)$ is irreducible over $F$.

A polynomial $P \in F[x]$ has good heredity if $P(x^n)$ factors into hereditarily irreducible factors over $F$, for some positive integer $n$.

One interesting question, intimately related to the arithmetic of the field $F$, is the classification of hereditarily irreducible polynomials in $F[x]$. The following question seems like an interesting question:

**Question 7.** Fix a field $F$. What polynomials over $F$ are hereditarily irreducible over $F$?
Remark 8. Hereditarily irreducible polynomials are in particular irreducible. The inverse is clearly false: polynomials $x$ and $x - 1$ are not hereditarily irreducible over any field; more generally, no polynomial whose zeroes are roots of unity is hereditarily irreducible.

It is not clear what criteria one can formulate to guarantee a polynomial be hereditarily irreducible, but the following is an interesting example.

Lemma 9. If $R$ is a UFD, Eisenstein polynomials are hereditarily irreducible over the fraction field of $R$.

Proof. Let $R$ be an integral domain, and $P \in R[x]$ an Eisenstein polynomial with respect to a prime ideal $p$. Then for any $n \in \mathbb{N}$, $P(x^n)$ is Eisenstein with respect to $p$. If $R$ is also a UFD, then Eisenstein polynomials are irreducible over $R$ and, by Gauss’s Lemma, over its field of fractions. \qed

It can be enlightening to arrange hereditary factors of a polynomial into a tree. To avoid spurious branching due to constant factors, we require polynomials to be monic—an entirely harmless assumption over a field. We use this construction in the proof of Theorem 24.

Definition 10. For a monic irreducible polynomial $P \in F[x]$, let $T_0(P, F)$ be the following tree. The nodes on the $n$-th level of $T_0(P, F)$ are the monic irreducible factors of $P(x^n)$ over $F$. The partial order of the tree is given by divisibility: a factor $R(x)$ of $P(x^{(n+1)}1)$ lies above a factor $Q(x)$ of $P(x^n)$ whenever $R(x)$ divides $Q(x^{n+1})$.

If $P$ is not irreducible, the same construction yields a forest instead of a tree: $T_0(P, F)$ has several nodes on the lowest level.

To construct a similar object for a non-monic polynomial $P$, let $a$ be the leading coefficient of $P$, and let $\tilde{P} := \frac{1}{a}P$, a monic polynomial. Finally, pick one branch of $T_0(\tilde{P}, F)$ and multiply all nodes on that branch by $a$. The arbitrary choice of the branch is the reason we stick to monic polynomials in the definition. Alternately, we could take the nodes to be equivalence classes of irreducible polynomials under the equivalence relation of “non-zero multiple”.

Since $P(x)$ is irreducible, it has no repeated roots in $F^{\text{alg}}$. As long as $P(x) \neq x$, it follows that $P(x^n)$ also has no repeated roots in $F^{\text{alg}}$; and then $P(x^n)$ has no repeated factors in $F[x]$. Thus, by unique factorization in $F[x]$, the product of all the nodes on $n$-th level of $T_0(P, F)$ is precisely $P(x^n)$. The polynomial $P(x) = x$ is ignored in this paper, as it is clearly not hereditarily irreducible; the particularly
A factor $Q(x)$ of $P(x^{n!})$ is hereditarily irreducible if and only if there are no splits above it in this tree $T_0(P, F)$; the sole exception is $P = x$, for which every level of the tree contains nothing but $x$.

**Definition 11.** Let $T(P, F)$ be the tree obtained from $T_0(P, F)$ by trimming all nodes above hereditarily irreducible factors.

**Lemma 12.** The polynomial $P \in F[x]$ has good heredity if and only if $T(P, F)$ is finite.

**Proof.** This finitely-branching tree is infinite if and only if it has an infinite branch. That is, if there are integers $1 = n_0 < n_1 < n_2 < n_3 < \ldots$ (all factorials, with $n_i$ dividing $n_{i+1}$ for each $i$) and irreducible factors $Q_i(x) \in F[x]$ of $P(x^{n_i})$ such that $Q_{i+1}(x)$ properly divides $Q_i(x^{n_{i+1}/n_i})$ for each $i$.

**Lemma 13.** If the polynomial $P \in F[x]$ has good heredity, then for any $M \in \mathbb{N}$ there are only finitely many polynomials of degree less than or equal to $M$ in $T_0(P, F)$.

**Proof.** The only polynomials lying above a hereditarily irreducible factor $Q(x)$ of $P(x^{n!})$ in $T_0(P, F)$ are of the form $Q(x^m)$ for increasing $m$, so only finitely many of them will have degree less than $M$.

### 3. (in)Divisibility in the Multiplicative Group

#### 3.1. Roots

In this section we explore the connections between hereditary (ir)reducibility in $F[x]$ and (in)divisibility in the multiplicative group of $F$, that is, lack of roots. The problem of understanding the structure of the multiplicative group of an arbitrary field, or the classification of those Abelian groups with locally cyclic torsion subgroups which happen to be (isomorphic to) multiplicative groups of fields is very non-trivial, and despite non-trivial contributions by many mathematicians over the last half a century, there are many open questions in this area. For an old survey of results and review of the history of these results, see [5], Chapter 4.

**Definition 14.** An element $f$ of a field $F$ is **very rootless** if there is no $g \in F$ and $n \in \mathbb{N}$ with $n \geq 2$ and $f = g^n$. An element $f$ of a field $F$ is **very rootless modulo torsion** if there is no root of unity $\zeta \in F$, element $g \in F$ and integer $n \geq 2$ such that $f = \zeta g^n$.

For example in the field $\mathbb{Q}$, the element $-4$ is very rootless, but not very rootless modulo torsion. An element $f$ of the field $F$ is very
rootless modulo torsion if the image of $f$ in the quotient of the multiplicative group of $F$ by its torsion (i.e. by the group of roots of unity) has no proper roots.

Note that $a$ is very rootless (modulo torsion) if and only if polynomials $x^n - a$ ($x^n - \zeta a$ for roots of unity $\zeta$) have no zeros in $F$ for $n \geq 2$.

**Lemma 15.** If $F$ contains all roots of unity then an element $f \in F$ is rootless, resp. very rootless, if and only if it is rootless modulo torsion, resp. very rootless modulo torsion.

**Proof.** If $F$ contains all roots of unity, then any root of unity $\zeta$ has an $n$-th root $\xi \in F$ for any $n$, so that $b \in F$ is a solution of $x^n = a$ if and only if $\xi b$ is a solution of $x^n = \zeta a$. \hfill \Box

**Proposition 16.** If an element $a$ is very rootless modulo torsion, then $(x - a)$ is hereditarily irreducible over $F$. If $(x - a)$ is hereditarily irreducible over $F$, then $a$ is very rootless.

**Proof.** If $Q(x)$ is an irreducible factor of $x^n - a$ of degree $m \leq n$, then every zero $b$ of $Q$ has $b^n = a$, and the product $c$ of all $m$ of them is in $F$ and satisfies $c^n = a^m$. Let $k := \gcd(m, n)$ so that $(c^{m'})^k = (a^{m'})^k$, so $(c^{m'}) = \zeta (a^{m'})$ for some $k$th root of unity $\zeta$. Since $m \leq n$, it must be that $n' \geq 1$. Since $m'$ and $n'$ are relatively prime, there are integers $\alpha$ and $\beta$ such that $\alpha m' + \beta n' = 1$. Now

$$a = a^\alpha m' + \beta n' = a^{\alpha m'} a^{\beta n'} = \zeta^{-\alpha} (c^{m'})^\alpha a^{\beta n'} = \zeta (c^\alpha a^\beta)^{n'}$$

makes $a$ a non-trivial power modulo torsion, so $a$ cannot be very rootless modulo torsion.

If $a = c^\alpha$, then $x - c$ divides $x^n - a$. \hfill \Box

**Example 17.** The converses of the two statements can fail. On one hand, $x^n + 4$ is irreducible over $\mathbb{Q}$ for all $n$, but $-4$ is not very rootless modulo torsion. On the other hand, $-2$ is very rootless in $F := \mathbb{Q}(\sqrt{2})$, but $x^4 + 2$ factors over $F$ as $(x^2 + \alpha^2 - \alpha^3 x)(x^2 + \alpha^2 + \alpha^3 x)$ where $\alpha^4 = 2$.

**Corollary 18.** If $F$ contains all roots of unity and $a \in F$, the following are equivalent.

- $a$ is very rootless modulo torsion.
- $(x - a)$ is hereditarily irreducible.
- $a$ is very rootless.

Also, none of those hold if $a$ is a root of unity.
Proof. If \( F \) contains all roots of unity, then any root of unity has any root in \( F \). Thus, very rootless and very rootless modulo torsion are the same notion. \( \square \)

3.2. Descending chains.

**Definition 19.** A nonzero element \( a \in F \) that is not a root of unity is rootless if only finitely many of the equations \( x^n = a \) have a solution in \( F \).

A nonzero element \( a \in F \) that is not a root of unity is rootless modulo torsion if for all but finitely many \( n \), the equation \( x^n = \zeta a \) has no solutions in \( F \) for any root of unity \( \zeta \in F \).

**Remark 20.** An element \( a \in F \) is rootless if and only if the set \( \{ g \in F : g^n = a \text{ for some } n \in \mathbb{N} \} \) of roots of \( a \) is finite.

An element \( a \) of the field \( F \) is rootless modulo torsion if the image of \( a \) in the quotient \( F^\times / \mu(F) \) of the multiplicative group of \( F \) by its torsion has only finitely many proper roots. Equivalently, the subgroup \( R(a, F) \) of \((\mathbb{Q}, +)\) defined by

\[
R(a, F) := \{ q \in \mathbb{Q} \mid \bar{a}^q \in F^\times / \mu(F) \}
\]

is cyclic. (Here, \( \bar{a} \) is the image of \( a \) in \( F^\times / \mu(F) \).) Since \( F^\times / \mu(F) \) is torsion-free, roots in it are well defined when they exist, so \( \bar{a}^q \) is well-defined.

**Lemma 21.** Fix a field \( F \) and a nonzero element \( a \in F \) that is not a root of unity.

If \( a \) is rootless modulo torsion, then \( P(x) := x - a \) has good heredity.

If \( P(x) \) has good heredity, then \( a \) is rootless.

**Proof.** The contrapositive of the second claim is easiest to prove. Suppose that there is a sequence of natural numbers \( \ell_1 < \ell_2 < \ell_3 < \ldots \), and elements \( \beta_i \in F^\times \) with

\[
a = \beta_i^{\ell_i}
\]

for each \( i \). Suppose that \( P(x^n) \) is a product of hereditarily irreducible polynomials \( P_i \) for some \( n \). Next,

\[
P(x^{n\ell_i}) = x^{n\ell_i} - a = x^{n\ell_i} - \beta_i^{\ell_i} = (x^n - \beta_i)P_i(x).
\]

Since \( a \) is not a root of unity, all \( \beta_i \) are distinct, and so \( (x^n - \beta_i) \) are infinitely many distinct polynomials of the same degree in the finitely many trees \( T_0(P, F) \), contradicting Lemma 13 in at least one of the trees.

Now, for the contrapositive of the first claim, Suppose that \( (x - a) \) does not have good heredity over \( F \), so there is an infinite branch in \( T(P, F) \):
(1) integers $1 = n_0 < n_1 < n_2 < n_3 < \ldots$ and
(2) irreducible polynomials $Q_i(x) \in F[x]$ of degree $k_i$, such that:
(3) all $n_i$ are factorials, and $n_i$ divides $n_{i+1}$ for each $i$;
(4) $Q_i(x)$ divides $P(x^{n_i})$; and
(5) $Q_{i+1}(x)$ properly divides $Q_i(x^{n_{i+1}/n_i})$ for each $i$.

From (4), $Q_i(x)$ divides $(x^{n_i} - a)$, so the $n_i$th power of any zero of $Q_i$ is $a$. Let $a_i$ be the product of the $k_i$ zeros of $Q_i$; then
$$a_i^{n_i} = a^{k_i}.$$ 

This $a_i$ is in $F$ because $(-1)^{k_i}a_i$ is the constant coefficient of $Q_i$. Let $b_i$ and $b$ be images of $a_i$ and $a$ in the quotient $F/\mu$ of the multiplicative group of $F$ by its torsion, i.e. roots of unity. Of course, we still have
$$b_i^{n_i} = b^{k_i}.$$ 

In the terminology of Remark 20, we get $k_i/n_i \in R(a, F)$ for each $i$. Since (5) implies that $k_{i+1}/n_{i+1} \leq k_i/n_i$ for each $i$, this makes $R(a, F)$ not cyclic, and, therefore, $a$ not rootless modulo torsion. \qed

Example 22. Consider $F_n := \mathbb{Q}(\sqrt[n]{17})$ and let $a := -17$. Note that, $a$ is even very rootless: as $F_n$ is embeddable in the reals, $a$ has no even-degree roots; and for any odd $p$, the degree of $\mathbb{Q}(\sqrt[p]{-17})$ over $\mathbb{Q}$ is odd, so $\mathbb{Q}(\sqrt[p]{-17})$ cannot be a subfield of $F_n$. Of course, $(-1)(-17) = 17$ is not rootless in $F$. We suspect that $x + 17$ is hereditarily irreducible over $F_n$, but we do not know how to prove this.

An analog of Corollary 18 holds: if $F$ has enough roots of unity in it, then $a$ is rootless if and only if it is rootless modulo torsion, so also if and only if $x - a$ has good heredity.

3.3. Higher-degree polynomials. We start this subsection with a remark.

Remark 23. A polynomial $P(x)$ has good heredity if and only if $P(x^n)$ has good heredity for some/all $n$.

By Unique Factorization in polynomial rings over fields, any product of hereditarily irreducible polynomials has good heredity.

It is easy to see that if $F \leq E$ and $P \in F[x]$ has good heredity over $E$, then $P$ also has good heredity over $F$. (See Section 5 below for an overly technical explanation of this.)

The next result was the original motivation for this work, Lemma 4.15 of [9]. Much of its original proof has already been presented in bits and pieces earlier in this paper.
Theorem 24. Let $F$ be a field and let $P(x) \in F[x]$ be a separable irreducible polynomial whose zeros in $F^{\text{alg}}$ are not roots of unity nor zero. Suppose for every root $a \in F^{\text{alg}}$ of $P$, $a$ is rootless modulo torsion in $F(a)$. Then $P$ has good heredity over $F$.

Proof. By Lemma 12, it suffices to show that the tree $T(P, F)$ is finite. Suppose towards contradiction that it is not. This infinite, finitely-branching tree $T$ must have an infinite chain: integers $1 = n_0 < n_1 < n_2 < n_3 < \ldots$ (all factorials, with $n_i$ dividing $n_{i+1}$ for each $i$) and irreducible factors $Q_i(x) \in F[x]$ of $P(x^{n_i})$ such that $Q_{i+1}(x)$ properly divides $Q_i(x^{\frac{n_{i+1}}{n_i}})$ for each $i$.

Since $P$ is irreducible, any two roots of $P$ in $F^{\text{alg}}$ are conjugate by an automorphism $\rho$ of $F^{\text{alg}}$ over $F$. Since $\rho$ fixes the coefficients of $Q_i$, each root of $P$ has the same number $k_i \leq n_i$ of $n_i$th roots that are also roots of $Q_i$. Since $Q_{i+1}(x)$ properly divides $Q_i(x^{\frac{n_{i+1}}{n_i}})$ for each $i$, we must have $k_{i+1} \frac{n_{i+1}}{n_i} \leq k_i$ for each $i$.

Fix a root $a \in F^{\text{alg}}$ of $P$, let $\tilde{F} := F(a)$, and let $\tilde{P}(x) := x - a$, an irreducible polynomial in $\tilde{F}[x]$. One of the hypotheses of this theorem is that $a$ is rootless modulo torsion in $\tilde{F}$, and it follows by Proposition 16 that $\tilde{P}$ is hereditarily irreducible over $\tilde{F}$. The $k_i n_i$th roots of $a$ which are also roots of $Q_i$ form a Galois orbit over $\tilde{F}$ and thus correspond to a factor $\tilde{Q}_i \in \tilde{F}[x]$ of $\tilde{P}(x^{n_i}) = x^{n_i} - a$, of degree $k_i$. Now these $\tilde{Q}_i$ form an infinite branch in $T(\tilde{P}, \tilde{F})$ contradicting hereditary irreducibility of $\tilde{P}$ over $\tilde{F}$ via Lemma 12 again. \\[ \Box \]

4. Fields of good heredity: Basic properties

It is clear that a polynomial which is divisible by $x$ cannot be hereditarily irreducible.

Definition 25. A field $F$ has good heredity if every polynomial $P \in F[x]$ none of whose roots in $F^{\text{alg}}$ are roots of unity or zero has good heredity over $F$.

A field $F$ is rootless if any nonzero $f \in F$ that is not a root of unity is rootless in $F$.

A field $F$ is rootless modulo torsion if any nonzero $f \in F$ that is not a root of unity is rootless modulo torsion in $F$.

Remark 26. These three properties of fields (rootless, rootless modulo torsion, and good heredity) pass to subfields.

Proof. The failure of each of these properties passes up from subfields because it is witnessed by an infinite collection of elements of the field
satisfying some equations and failing to satisfy others. The witnesses are still there in the bigger field, and the two fields agree on whether the equations are satisfied. □

We have already shown the following.

**Proposition 27.** If $F$ is rootless modulo torsion, then all linear polynomials in $F$ have good heredity over $F$. If all linear polynomials in $F$ have good heredity over $F$, then $F$ is rootless.

If $F$ contains all roots of unity, or only finitely many roots of unity, then these three properties are equivalent.

If all finite extensions of $F$ are rootless modulo torsion, then $F$ has good heredity.

**Proof.** The first statement is an immediate consequence of Lemma 21.

The second statement follows directly from Lemma 15 for fields with all roots of unity. If $F$ only has finitely many roots of unity and $f \in F$ is not rootless modulo torsion, then infinitely many equations $x^n = \zeta_n f$ have solutions in $F$ - but then, as infinitely many integers $n$ try to fly into finitely many roots of unity $\zeta$, some $\zeta$ must appear infinitely often, making $\zeta f$ not rootless in $F$.

The last statement is an immediate consequence of Theorem 24. □

**Corollary 28.** If the multiplicative group of every finite extension of the field $F$ is free modulo torsion, then $F$ is of good heredity.

**Proof.** The first sentence after Definition 2. □

**Proposition 29.** A purely transcendental extension of a rootless (resp. rootless modulo torsion) field is rootless (resp. rootless modulo torsion).

**Proof.** Lemma 10.2 on page 503 of [5] states that for any purely transcendental extension $E$ of $F$, there is some free Abelian group $A$ such that $E^\times \cong F^\times \times A$. □

The situation for algebraic extensions is far more complicated, as demonstrated by the existence of examples of the sort presented in §7.3. However, good heredity passes to finite extensions, making it a better property than rootlessness.

**Theorem 30.** If a field $F$ has good heredity and $E \supseteq F$ is a finite field extension of $F$, then $E$ also has good heredity.

**Proof.** Since good heredity of polynomials passes to subfields (Remark 23), it suffices to consider the case where $E/F$ is Galois. Let $\tilde{P}(x) \in E[x]$ be an irreducible polynomial whose zeros are not roots of unity; we need to show that $\tilde{P}(x)$ has good heredity over $E$. 
Suppose that $\tilde{P}$ does not have good heredity over $E$; then, as in the proof of Lemma 21, there are

1. integers $1 = n_0 < n_1 < n_2 < n_3 < \ldots$ and
2. irreducible polynomials $\tilde{Q}_i(x) \in F[x]$ of degree $k_i$, such that:
3. all $n_i$ are factorials, and $n_i$ divides $n_{i+1}$ for each $i$;
4. $\tilde{Q}_i(x)$ divides $\tilde{P}(x^{n_i})$; and
5. $\tilde{Q}_{i+1}(x)$ properly divides $\tilde{Q}_i(x^{\frac{n_{i+1}}{n_i}})$ for each $i$.

Let $G$ be the Galois group of $E$ over $F$, and let

$$P(x) := \prod_{\sigma \in G} \sigma(\tilde{P}) \in F[x],$$

and

$$Q_i(x) := \prod_{\sigma \in G} \sigma(\tilde{Q}_i) \in F[x].$$

Now the same integers $n_i$ together with polynomials $Q_i$ witness that $P$ does not have good heredity over $F$. Since any zero $b \in F^{\text{alg}}$ of $P$ is a Galois conjugate of some zero $\tilde{b}$ of $\tilde{P}$, none of the zeros of $P$ are roots of unity. A few more steps as in Remark 23 obtain irreducible $Q_i$ finishing the proof. □

5. A model-theorist’s musings.

Here the first author collects a few observation on the logical complexity of rootlessness and heredity; they are mostly irrelevant to the rest of the paper.

For a subset $S \subset \mathbb{N}$, let $\text{rooty}_S(x)$ be the type

$$\text{rooty}_S(x) := \{\exists y \ y^\ell = x \ | \ \ell \in S\} \cup \{x^\ell \neq 1 \ | \ \ell \in \mathbb{N}^+\} \cup \{x \neq 0\}.$$

An element $a$ of a field $F$ is rootless if and only if it does not realize $\text{rooty}_S$ for any infinite $S$. In particular, being rootless is an $L_{\kappa,\omega}$-universal property (for $\kappa = 2^\omega$), so it passes to subfields of $F$ containing $a$. Similarly, A field $F$ is rootless if and only if it omits the types $\text{rooty}_S$ for all infinite $S$; so this property passes to all subfields of $F$.

Similarly, failure to be rootless modulo torsion is witnessed by realizing the type

$$\{\exists y \exists z \ y^m = zx \text{ and } z^n = 1 \ | \ (m, n) \in S\} \cup \{x^\ell \neq 1 \ | \ \ell \in \mathbb{N}^+\} \cup \{x \neq 0\}$$

for some function $S \subset \mathbb{N} \times \mathbb{N}$ with infinite domain.

It is clear, but cumbersome to write down, that hereditary irreducibility and good heredity are also negations of large disjunctions of first-order existential types, both for individual polynomials and for entire fields; they again pass to subfields.
All these properties are very far from being first order, as witnessed by the following proposition.

**Proposition 31.** Any infinite field has an elementary extension that is not rootless and, therefore, is not of good heredity.

**Proof.** Consider the type

\[ p(x) := \{\exists y y^\ell = x \mid \ell \in \mathbb{N}^+\} \cup \{x^\ell \neq 1 \mid \ell \in \mathbb{N}^+\} \cup \{x \neq 0\}. \]

It suffices to produce an elementary extension of the original field \( F \) with a realization of \( p \). That is, it suffices to show that \( p \) is consistent with the theory of \( F \). That is, it suffices to show that any finite subset of \( p \) is realized in \( F \), or in some elementary extension of \( F \). Any finite subset of \( p \) is contained in

\[ p_N(x) := \{\exists y y^\ell = x \mid 1 \leq \ell \leq N\} \cup \{x^\ell \neq 1 \mid 1 \leq \ell \leq N\} \cup \{x \neq 0\} \]

for some \( N \). Since \( F \) is infinite, it has an elementary extension \( F^+ \) containing a nonzero element \( a \) that is not a root of unity. Now \( a^{N!} \) realizes \( p_N \). \( \square \)

6. **Fields of good heredity: Examples**

Here we use other people's work to find fields with good heredity, mostly by finding fields all of whose finite extensions are rootless modulo torsion. We start with some easy observations we could've made long ago.

**Proposition 32.**

1. The field \( \mathbb{F}_p^{alg} \) is of good heredity. These fields are the only algebraic closed fields of good heredity.

2. All global fields in any characteristic are rootless, rootless modulo torsion, and of good heredity.

3. A finitely generated extension of a field of good heredity is of good heredity.

**Proof.** The field \( \mathbb{F}_p^{alg} \) is vacuously of good heredity, as every non-zero element is a root of unity. The multiplicative group of an algebraically closed field is divisible. By Dirichlet's Unit Theorem, the multiplicative group of any global field and all of its finite extensions are free modulo torsion. Corollary 28 implies the statement about global fields. Since every finitely generated extension of a field is an algebraic extension of a purely transcendental extension, Theorem 30 implies that in order to prove statement 3 we may assume that the extension is purely transcendental. But for a purely transcendental extension \( F(S) \) of \( F \), a reducibility statement in \( F(S)[x] \) gives rise to a reducibility statement in \( F[x] \) by specialization. \( \square \)
Corollary 33. Any field finitely generated over $\mathbb{Q}$ or over any subfield of $\mathbb{F}_{p}^{alg}$ has good heredity.

Proof. By Proposition 32, $\mathbb{Q}$ and subfields of $\mathbb{F}_{p}^{alg}$ have good heredity. Now induct on the number of generators, using Lemma 30 deals with algebraic extensions and Remark 32 again to deal with transcendental ones. □

Corollary 34. Local fields are not rootless and, therefore, are not of good heredity.

Proof. Archimedean local fields $\mathbb{R}$ and $\mathbb{C}$ are clearly not of good heredity. So, let $F$ be a non-archimedean local field of residue characteristic $p$. Let $\mathcal{O}$ be the ring of integers of $F$, and $p$ the prime ideal. Then let $\alpha \in \mathcal{O}$ be an integer satisfying

$$\alpha \equiv 1 \mod p.$$  

We also assume that $\alpha$ is not a root of unity. Let $r$ be a natural number not divisible by $p$. Now consider the equation $x^r = \alpha$. This equation reduces to $x^r \equiv 1 \mod p$, which is clearly solvable. Furthermore, $(x^r)' = rx^{r-1}$ evaluated at $\alpha$ is congruent to $r$ modulo $p$ which is by assumption non-zero. Hensel’s lemma for non-archimedean local fields shows that the equation $x^r = \alpha$ is solvable in $F$. Now Lemma 21 implies that $F$ is not of good heredity. □

There are many results in the literature about fields such that $K^\times/\mu(K)$ is free Abelian. Here are some non-trivial results we use.

Theorem 35 (May, 1980). Assume $F$ is a field such that for every finite extension of $E$, $E^\times$ is free modulo torsion, e.g. if $F$ is finitely generated.

(1) If $K$ is any field generated over $F$ by algebraic elements whose degree over $F$ are bounded, then $K^\times$ is free modulo torsion.

(2) Suppose, additionally, for every finite extension $E$ of $F$, $\mu(E)$ is finite. Then if $K$ is any Abelian extension of $F$, $K^\times$ is free modulo torsion.

Proof. [8], or [5], Theorems 10.12 and 10.21. □

Corollary 36. The maximal Abelian extension of $\mathbb{Q}$ and all of its finite extensions are rootless modulo torsion. The same statement holds for the maximal Abelian extension of any number field. These fields are all of good heredity.

Corollary 37. For every natural number $k$, let $\mathbb{Q}_k$ be the field obtained by adding the roots of every polynomial with rational coefficients of degree less than or equal to $k$. Then $\mathbb{Q}_k$ and all of its finite extensions are...
rootless. The same statement holds for the field obtained by adjoining the k-th roots of any set of prime numbers to \( \mathbb{Q} \). These fields are all of good heredity.

7. Non-examples

7.1. Free modulo torsion doesn’t climb. We note that Warren May has constructed an algebraic extension \( F \) of \( \mathbb{Q} \) and a quadratic extension \( K/F \) with the following properties:

- \( F^\times \) is free modulo torsion, but \( K^\times \) is not free modulo torsion;
- every finite extension of \( F \) contains only finitely many roots of unity.

The construction is as follows: Let \( \alpha_0 = (2 + i)(2 - i)^{-1} \), and define a sequence \( \alpha_n \) of complex numbers, \( n \geq 1 \), by

\[
\alpha_4^n = \alpha_{n-1}.
\]

Put \( K_0 = \mathbb{Q}(i) \), and \( K_n = K_0(\alpha_n) \). We then have \( K_0 \subset K_1 \subset K_2 \subset \cdots \). We let

\[
K = \bigcup_{n=0}^{\infty} K_n.
\]

Complex conjugation stabilizes each \( K_n \). We let \( F = K \cap \mathbb{R} \). Proving that \( K \) and \( F \) have the desired properties is hard. For the proof, see Theorem 10.18, page 510 of [5], or the paper [6].

7.2. Rootless modulo torsion, but not free modulo torsion. One may be tempted to conjecture that fields that are rootless modulo torsion have multiplicative groups that are free modulo torsion. Here we observe that this is not true. First we recall a result due to Fuchs and Loonstra 1:

Theorem 38 ([3], Lemma 2). There exists a torsion free abelian group \( G \) of rank 2 such that every rank 1 subgroup is cyclic, and every rank 1 torsion free factor group is divisible.

In particular this group \( G \) is not free. By Theorem 12.3, page 520 of [5] (Theorem 39 below), there is a field \( F \) such that

\[
F^\times \simeq G \times \mathbb{Z}/2\mathbb{Z} \times A
\]

with \( A \) a free abelian group. It is clear that the torsion subgroup of \( F^\times \) is \( \mathbb{Z}/2\mathbb{Z} \). Since \( F^\times/t(F^\times) \) is isomorphic to \( G \times A \), it is clear that \( F^\times \) is rootless modulo torsion, and certainly not free modulo torsion.

---

1We learned about this result from a post by Andreas Blass on mathoverflow on March 8, 2012. We hereby acknowledge this.
7.3. Extensions of rootless (modulo torsion) are not necessarily rootless (modulo torsion). In this subsection we construct an example of a finite extension of a rootless (modulo torsion) field which is not rootless (modulo torsion).

Let

\[ F_n := \mathbb{Q}(x_n) \]

with

\[ x_n := \cos\left(\frac{\alpha}{2^n}\right) \]

for a real number \( \alpha \). Since there are only countably many reals \( \alpha \) such that \( \alpha/\pi \) is irrational and/or \( \cos \alpha \) is transcendental, we may and do fix some \( \alpha \) such that \( \alpha/\pi \) is irrational and \( \cos \alpha \) is transcendental. As an abstract field, each \( F_n \) is a purely transcendental extension \( \mathbb{Q}(y) \) of \( \mathbb{Q} \) of transcendence degree 1.

Let \( T(x) := 2x^2 - 1 \), so that \( \cos(2\theta) = T(\cos \theta) \). Now \( T(x_{n+1}) = x_n \), so the fields \( F_n \) form an increasing chain \( F_1 \leq F_2 \leq \ldots \leq F_n \leq \ldots \). As extensions of abstract fields, all pairs \( F_{n+1}/F_n \) are isomorphic to \( \mathbb{Q}(y)/\mathbb{Q}(T(y)) \). Any two tails \( F_\ell \leq F_{\ell+1} \leq \ldots \) and \( F_m \leq F_{m+1} \leq \ldots \) of our chain are isomorphic as chains of abstract fields. We finally set

\[ F := \bigcup_n F_n. \]

Claim 1: The field \( F \) is rootless.

Proof of Claim 1. Suppose towards contradiction that \( f \in F \) is not a root of unity and has infinitely deep roots in \( F \). We may assume without loss of generality that \( f \in F_1 \), since all tails of the chain are the same. Since all algebraic elements of \( F \) are rational, \( f \notin \mathbb{Q} \). Thus, \( f(x_1) \) is a rational function over \( \mathbb{Q} \). Since \( F_1 = \mathbb{Q}(x_1) \) is rootless, we may and do assume without loss of generality that \( f \) has no roots in \( F_1 \). Since the degree of the field extension \( F_n/F_1 \) is \( 2^n \), the infinite chain of roots of \( f \) must consist of an infinite chain of square roots. That is, for each \( m \) there exists some \( n \) and some rational function \( g_m \) over \( \mathbb{Q} \) such that \( (g_m(x_n))^{2^m} = f(x_1) \) in \( F_n \). That is,

\[ (g_m(y))^{2^m} = f(T^{m}(y)) \]

as rational functions in \( y \) over \( \mathbb{Q} \). Here, \( T^m \) is the \( m \)th compositional power of \( T \). All zeros of \( (g_m(y))^{2^m} \) on \( \mathbb{P}^1 \) have multiplicity \( r_m2^m \) for some integer \( r_m \). We obtain our contradiction by getting an upper bound on the multiplicity of any root \( a \) of \( f(T^{m}(y)) \) independent of \( n \). Indeed, \( f \) can only contribute its degree towards the multiplicity of \( a \), so we only need an upper bound on the multiplicity of a root \( a \) of
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\[ T^n(y) - T^n(a). \]  

An upper bound on the multiplicities of roots of its derivative suffices. Note that \( T^n(0) \neq 0 \) for any \( n \): indeed, \( T(0) = -1 \), \( T(-1) = 1 = T(1) \). Thus the derivative

\[ (T^n)'(y) = 2T^{n-1}(y)((T^{n-1})'(y)) = 2^nT^{n-1}(y)T^{n-2}(y)\ldots T^2(y)T(y)y \]

has roots of multiplicity at most two, as no two \( T^k(y) \) share roots because \( T^n(0) \neq 0 \) for any \( n \).

To summarize: for any \( n \) and any \( a \), the multiplicity of roots of \( T^n(y) - T^n(a) \) is at most 3 because the multiplicity of roots of its derivative is at most 2. Therefore, for any \( n \), the roots of \( f(T^n(y)) \) have multiplicity at most \( 3\deg(f) \). Therefore, for any \( m \geq 3\deg(f) \), there is no rational function over \( \mathbb{Q} \) with \( (g_m(y))^{2m} = f(T^n(y)) \). That is, there is no \( 2^m \)th root of \( f(x_1) \) in \( F_n \) for any \( n \), which means that \( f(x_1) \) has no \( 2^m \)th root anywhere in \( F \).

Claim 1.1: The field \( F \) is rootless modulo torsion.

Proof of Claim 1. Since \( F \subseteq \mathbb{R} \), the only roots of unity in \( F \) are \( \pm 1 \), so the second statement in Corollary 27 makes \( F \) rootless modulo torsion.

Claim 2: The quadratic extension \( E := F(i \sin(\alpha)) \) of the field \( F \) is not rootless.

Proof of Claim 2. Since

\[ \sin\left(\frac{\theta}{2}\right) = \frac{\sin \theta}{2 \cos(\theta/2)}, \]

inducting on \( n \) shows that \( i \sin\left(\frac{\alpha}{2^n}\right) \in E \) for each \( n \). Now \( E \) contains complex numbers \( c_n := \cos\left(\frac{\alpha}{2^n}\right) + i \sin\left(\frac{\alpha}{2^n}\right) \) with the property that \( c_{n+1}^2 = c_n \). Since we chose \( \alpha/\pi \) to be irrational at the beginning, these \( c_n \) are not roots of unity. Thus, \( E \) is not rootless.

Claim 2.1: A fortiori, the quadratic extension \( E := F(i \sin(\alpha)) \) of the field \( F \) is not rootless modulo torsion.

Claim 3: The field \( F \) does not have good heredity.

Proof of Claim 3. Let \( P(x) := x^2 - 2(\cos \alpha)x + 1 \) be the minimal polynomial of \( c_0 = \cos(\alpha) + i \sin(\alpha) \) over \( F \). Over \( E \), this polynomial factors as \( P(x) = (x + c_0)(x + \overline{c_0}) \). Since \( F \subseteq \mathbb{R} \), the complex conjugate \( \overline{c_0} \) of \( c_0 \) is the other root of \( P \). Over \( E \), the polynomial \( x - c_0 \) doesn’t have good heredity, as \( x - c_0 \) is a linear factor of \( x^{2^n} - c_0 \). Thus, \( Q_n(x) := (x + c_n)(x + \overline{c_n}) \in F[x] \) are distinct quadratic factors of \( P(x^{2^n}) \), showing that \( P \) does not have good heredity over \( F \).
7.4. Rootless, but not rootless modulo torsion. In this subsection we will construct a field which is rootless but not rootless modulo torsion. We note that this is a group theoretic statement about the multiplicative group of the field. Let us recall a theorem of W. May [6]. We call an Abelian group $G$ locally cyclic if every finitely generated subgroup of $G$ is cyclic.

**Theorem 39** (May, 1972). Let $G$ be an abelian group such that the torsion subgroup is locally cyclic. Then there is a field $L$ and a group $H$ such that $L^* \cong G \times H$, where $H$ is a free abelian group if $t(G)$ has non-trivial 2-component and $H$ is the direct product of a free abelian group with a cyclic group of order 2 if the torsion subgroup of $G$ has trivial 2-component.

This is Theorem 12.3, page 520 of [5]. Since an Abelian group is rootless, respectively rootless modulo torsion, if and only if its direct product with a free Abelian group is rootless, resp. rootless modulo torsion, May’s theorem reduces the problem to finding an Abelian group $G$ with locally cyclic torsion subgroup $t(G)$ such that $G$ is rootless, but $G/t(G)$ is not. Here, however, instead of construct a concrete field with is rootless, but not rootless modulo torsion.

The field $K$ we construct has transcendence degree 1; its algebraic part $A$ is generated by (some but not all) roots of unity. This field $K$ is an infinite radical extension of $A(x)$ for the transcendental $x$; and this $x$ is not rootless modulo torsion in $K$.

First, we build some scaffolding.

**choices:** Let $\{p_i : i \in \mathbb{N}\}$ be an infinite list of distinct primes; and for each $i$, let $\alpha_i$ be a primitive $p_i$th root of unity.

**non-choices, products:** For each $i$, let $n_i := \prod_{j=0}^{i} p_j$, a square-free integer; and let $\zeta_i := \prod_{j=0}^{i} \alpha_j$, a primitive $n_i$th root of unity.

**non-choice, roots:** Since $\gcd(p_{i+1}, n_i) = 1$, there are $x, y \in \mathbb{Z}$ such that $xp_{i+1} + yn_i = 1$. Letting $\beta_{i+1} := \alpha_{i+1}^y$, we get $\beta_{i+1}^{n_i} = \alpha_{i+1}$. Also, let $\beta_0 = \alpha_0$.

Let $A := \mathbb{Q}(\{\alpha_i : i \in \mathbb{N}\})$; by Theorem 35 and Corollary 36 and Remark 26, $A$ is rootless, rootless modulo torsion, and good heredity. We note here for later use that in $A$, a primitive $\ell$th root of unity
cannot have an $m$th root unless $\gcd(\ell, m) = 1$: there are no $p^2$th roots of unity for any prime $p$.

We build an $\omega$-chain of fields $A(x) \leq A_0 \leq A_1 \leq \ldots$ with each $A_i = A(t_i)$ for transcendentals $t_i$. The goal of the construction is to make $x$ not rootless modulo torsion in $K := \cup_i A_i$, while keeping everything rootless.

We embed $A(x)$ into $A(t_0)$ over $A$ by sending $x$ to $\beta_0 t_0^{\rho_0}$. We embed $A_i$ into $A_{i+1}$ over $A$ by sending $t_i$ to $\beta_i t_{i+1}^{\rho_{i+1}^{i+1}}$.

**Claim 1:** In $K$, $x = \zeta_i t_i^{n_i}$ for all $i$.

*Proof of Claim 1.* The base case for the induction on $i$ is our choice of the embedding of $A(x)$ into $A(t_0)$ and our choice of $\beta_0$. For the induction step,

$$\zeta_i t_i^{n_i} = \zeta_i (\beta_{i+1} t_{i+1}^{\rho_{i+1}^{i+1}})^{n_i} = (\zeta_i \beta_i^{n_i}) t_{i+1}^{n_i} = (\zeta_i \alpha_{i+1}) t_{i+1}^{n_i} = \zeta_{i+1} t_{i+1}^{n_{i+1}}.$$ 

In particular, $x$ is an $n_i$th power modulo roots of unity, for every $i$, so it is not rootless modulo torsion; so the field $K$ is not rootless modulo torsion.

**Claim 2:** The field $K$ is rootless.

*Proof of Claim 2.* As with §7.3, it suffices to show that elements of $A(x)$ are rootless in $K$. While it is not true that all tails of the chain $A_i$ are isomorphic, the only difference is the particular primes $p_i$ and a few extra roots of unity coprime to everything we care about.

As in §7.3, we take an element $f(x) \in A(x)$ and find a bound on $m$ such that $f(x) \in A_i^m$ independent of $i$.

**Case “constant”:** If $f \in A$ is not a root of unity, then $f$ is rootless in $A$ as we noted above. Since each $A_i = A(t_i)$ is a purely transcendental extension of $A$, $f$ gains no new roots in any $A_i$.

**Case “power”:** Suppose that $f = x^k$ for some nonzero $k \in \mathbb{Z}$. In $A_i = A(t_i)$ we have from Claim 1 that $x = \zeta_i t_i^{n_i}$, so

$$f(x) = f(\zeta_i t_i^{n_i}) = \zeta_i^k t_i^{kn_i} =: g(t_i).$$

Factors of monomials must be monomials, so if $g = h^m$ for some $m$ and some $h \in A_i$, then $h(t_i) = b t_i^r$ and $rm = kn_i$ and $b^m = \zeta_i^k$. We are now entirely inside the torsion group of $A$.

Recall that $\zeta_i$ is a primitive $n_i$th root of unity, so $\eta_i := \zeta_i^k$ is a primitive $\ell_i$th root of unity where $\ell_i := (n_i/\gcd(n_i, k))$. In order for $\eta_i$
to have an \( m \)th root, we must have \( \gcd(\ell_i, m) = 1 \). Now we are just solving divisibility-and-gcd relations in integers.

Let \( \gamma := \gcd(k, n_i) \) and \( n_i = \gamma \ell_i \) and \( k = \gamma k' \) with \( \gcd(k', \ell_i) = 1 \). We also know that \( m \) divides \( kn_i = \gamma^2 k'\ell_i \) and that \( \gcd(\ell_i, m) = 1 \). Thus, \( m \) must divide \( \gamma^2 k' \); so \( m \) must divide \( k^2 \). Actually, \( m \) must also be square-free, so it must actually divide \( k \). In any case, \( m \) is bounded independently of \( i \).

**Case “monomial”:** Suppose that \( f = ax^k \) for some \( a \in A \) and some nonzero \( k \in \mathbb{Z} \).

We first show that we may assume without loss of generality that \( a = 1 \). In \( A_i = A(t_i) \) we have from Claim 1 that \( x = \zeta t_i^{n_i} \), so

\[
f(x) = f(\zeta t_i^{n_i}) = a\zeta^k t_i^{kn_i} =: g(t_i).
\]

Factors of monomials must be monomials, so if \( g = h^m \) for some \( m \) and some \( h \in A_i \), then \( h(t_i) = b t_i^r \) and \( rm = kn_i \) and \( b^m = a\zeta^k \). Unless \( a \) is a root of unity, the fact that \( A \) is rootless modulo torsion gives the desired bound on \( m \). So suppose that \( a^q = 1 \); now it suffices to show that \( f(x)^q = x^{kd} \) is rootless, which was done in the previous Case.

**Case “other”:** Otherwise, some nonzero \( \alpha \in A_{alg} \) is a zero or a pole of \( f \). Since the rootlessness of \( f \) is equivalent to the rootlessness of \( 1/f \), we may and do assume without loss of generality that \( f(\alpha) = 0 \). Let \( r \) be the multiplicity of \( \alpha \) as a zero of \( f \). Now in \( A_i = A(t_i) \) we have from Claim 1 that \( x = \zeta t_i^{n_i} \), so \( f(x) = f(\zeta t_i^{n_i}) =: g(t_i) \). So now for each of the \( n_i \) distinct \( n_i \)th roots \( \beta \) of \( \alpha/\zeta \) in \( A_{alg} \) is a zero of \( g \) with multiplicity exactly \( r \). On the other hand, if \( f(x) = g(t_i) = (h(t_i))^m \) for some \( h \in A_i \), then \( m \) divides the multiplicity of all zeros of \( g \). Thus, \( r \) is an upper bound on \( m \), independent of \( i \).

**References**

[1] Abhyankar, Shreeram S.; Rubel, Lee A., *Every difference polynomial has a connected zero-set*. (English) J. Indian Math. Soc., New Ser. 43, 69-78 (1979). ISSN 0019-5839

[2] G. Angermuller, *A Generalization of Ehrenfeucht’s Irreducibility Criterion*, J. Number Th., 36, 80-84 (1990).

[3] L. Fuchs and F. Loonstra, *On the cancellation of modules in direct sums over Dedekind domains*, Indag. Math. 33 (1971) 163-169).

[4] Hsia and Tucker

[5] G. Karpilovsky, *Field theory. Classical foundations and multiplicative groups*, Monographs and Textbooks in Pure and Applied Mathematics, 120. Marcel Dekker, Inc., New York, 1988. x+551 pp.
[6] W. May, Multiplicative groups of fields, Proc. London Math. Soc. (3), 24 (1972), 295-306.
[7] W. May, Multiplicative groups under field extension, Canad. J. Math. Vol XXXI, no. 2, 436-440.
[8] W. May, Fields with free multiplicative group modulo torsion, Rocky Mountain J. Math., Vol. 10, no. 3, 599-604.
[9] A. Medvedev, QACFA, preprint, arXiv:1508.06007.
[10] L. A. Rubel, A. Schinzel, H. Tverber, On Difference Polynomials and Hereditarily Irreducible Polynomials, J. Number Th., 12, 230-235 (1980)

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