Utility Function from Maximum Entropy Principle

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(Dated: March 22, 2022)

Abstract

We apply the maximum entropy principle to economic systems in equilibrium and find the density function for the market’s wealth. This is the same as price density which is used for insurance pricing. The risk aversion parameter of the agent then it’s utility function with respect to this density is derived.

PACS numbers: 89.65.Gh, 05.20.-y

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I. INTRODUCTION AND SUMMARY

Utility functions one of the most important and useful concepts in the economics which show the tendency of an economic agent for acquiring the more benefit. It is originated from the concept of potential energy in the physics [1]. There is no direct way for finding the utility function of an agent with respect to it’s financial condition, namely its wealth. Different companies find their utility function from analysis of trading data.

Recently the insurance market which is one of the important branch of economy have attracted the attention of physicists [2, 3, 4, 5, 6, 7]. The statistical mechanics concepts specially the maximum entropy principle is used for pricing the insurance [6, 7]. The well known results on economic premium calculation [8, 9] are retrieved.

In the next section we follow the work of Darooneh [7] to obtain the price density based on the maximum entropy principle then in last section we apply it for multi agents model of insurance market [7, 8]. Finally the utility function will be derived. The main assumption here is coincidence of concept of the equilibrium in physics and economics. This is not strange because both approaches to equilibrium have the same results [6, 7, 8]. However the problem of equilibrium for multi part systems is under the investigation by author.

The power of statistical mechanics enables us to extend and apply this method for calculation of utility function in other cases such as finite markets and take into the account the effects of other constraint in the market.

II. THE MAXIMUM ENTROPY METHOD IN ECONOMICS

The risky events affect the financial market. The randomness in the market as a consequence of the risks will be increased when the times goes forward. The state of the market with maximum randomness is called equilibrium. In equilibrium state we lose the most information about the status of economic agents in the market, in this respect the adoption of a strategy for trading becomes more cumbersome. The important task of a trader, and also an economist, is prediction state of the market. This is carried out by calculation of distribution function for falling of the market into a possible equilibrium state. The maximum entropy principle appears as the best way when we make inference about an unknown distribution based only on the incomplete information. The Jaynes entropy may be written
as 

\[ H[\varphi] = -\int_{\Omega} \varphi(\omega) \ln \varphi(\omega) d\Pi(\omega). \]  

(1)

Where \( \omega \) is an element of the risk’s probability space \( \Omega \). The measure of the integral demonstrates the weight for occurrence a random event (risk).

The distribution function \( \varphi(\omega) \) is normalized function.

\[ \int_{\Omega} \varphi(\omega) d\Pi(\omega) = 1. \]  

(2)

It is assumed that the average of the market’s wealth in the equilibrium state should be constant.

\[ < W > = \int_{\Omega} \varphi(\omega) W(\omega) d\Pi(\omega) = \text{Const.} \]  

(3)

This is not surprising assumption because money exchanging between different agents (market’s constituents) doesn’t alter the market’s wealth totally. That money which enter or leave the market is controlled by trading strategies.

The eqs. 2 and 3 should be satisfied simultaneously when we attempt to maximize the entropy 1.

\[ \delta H[\varphi] + \lambda \int_{\Omega} \varphi(\omega) d\Pi(\omega) + \beta \delta \int_{\Omega} \varphi(\omega) W(\omega) d\Pi(\omega) = 0. \]  

(4)

The canonical distribution is the well known solution to above equation 11.

\[ \varphi(\omega) = \frac{e^{-\beta W(\omega)}}{\int_{\Omega} e^{-\beta W(\omega)} d\Pi(\omega)}. \]  

(5)

The parameter \( \beta \) has important roll in price density, it can be calculated on basis of the method that is introduced in previous works 4, 6, but our intuition from similar case in thermal physics tell us 11,

\[ \beta \approx \frac{1}{< W >}. \]  

(6)

In the case of insurance market the wealth of a typical agent; insurer or insurant, is given by,

\[ W_i(\omega) = W_{0i} - X_i(\omega) + Y_i(\omega) - \int_{\Omega} \varphi(\omega) Y_i(\omega) d\Pi(\omega). \]  

(7)

The index \( i \) indicates the different agents.

Each agent in the market will be incurred \( X_i(\omega) \) if \( \omega \) is happening. He insured himself for the price \( \int_{\Omega} \varphi(\omega) Y_i(\omega) d\Pi(\omega) \) and receives \( Y_i(\omega) \) upon occurrence of this event.
As we mentioned before the money is only exchanged between agents and doesn’t alter the total money in the market. It is what is called the clear condition.

\[ \sum_i Y_i(\omega) = 0. \]  
(8)

The market’s wealth is sum of the individual wealth of the agents.

\[ W(\omega) = \sum_i W_i(\omega) = W_0 - Z(\omega). \]  
(9)

It is also true for initial wealth \( W_0 \) and aggregate risk \( Z(\omega) \),

\[ W_0 = \sum_i W_{0i}, \]
\[ Z(\omega) = \sum_i X_i(\omega). \]  
(10)

With the aid of eq. (9) we can rewrite eq. (6) to retrieve the Bühlmann result on economic premium calculation [8, 9].

\[ \varphi(\omega) = \frac{e^{\beta Z(\omega)}}{\int_{\Omega} e^{\beta Z(\omega)}d\Pi(\omega)}. \]  
(11)

In the next section we try to obtain the utility function from the results that have been fund in this section.

III. UTILITY FUNCTION

The utility function demonstrates what an agent is interested for making specified amounts of profit in a competitive market. The utility function should depend on financial status of an economic agent which is frequently described by its wealth, \( u(W) \). It is assumed that the utility function has positive first derivative, \( u'(W) > 0 \), to guarantee that the agent is willing the profit and negative second derivative, \( u''(W) < 0 \), to restrict it’s avarice. The risk aversion parameter, \( \beta(W) = -u''(W)/u'(W) \), is also involved in the utility function to scale the agent’s will in the market with respect to it’s wealth.

The market reach at equilibrium upon the agents are satisfied from their trade. In this respect the agent’s utility function should be the most in the equilibrium state. Because of the risks in the market which induce the randomness, the equilibrium condition may be expressed as an average form.

\[ \int_{\Omega} u_i(W_i(\omega))d\Pi(\omega) = max. \]  
(12)
The only function that can be changed upon our request is $Y_i(\omega)$ hence maximizing of the utility function means,

$$\frac{\delta}{\delta Y_i(\omega)} \int_{\Omega} u_i(W_i(\omega))d\Pi(\omega) = 0. \quad (13)$$

It is not so hard to derive the following result by using variational technique.

$$u'_i(W_i(\omega)) = \varphi(\omega) \int u'_i(W_i(v))d\Pi(v)$$

$$u'_i(W_i(\omega)) = C_i\varphi(\omega). \quad (14)$$

Where the derivative are respect to argument of the functions.

We have freedom to rescale the risk exchange function $Y_i(\omega)$, up to a constant number since the agent’s wealth remains unaltered.

$$Y_{New}^i(\omega) = Y_{Old}^i(\omega) - \int_{\Omega} \varphi(\omega)Y_{Old}^i(\omega)d\Pi(\omega) \quad (15)$$

The risk function is also renewed to absorb the risk exchange function.

$$X_{New}^i(\omega) = X_{Old}^i(\omega) - Y_{New}^i(\omega). \quad (16)$$

Often a loss event is incurred the market totally and the agent’s risk function depends on $\omega$ through $\zeta = Z(\omega)$ hence it is suitable and very often necessary to deal with $\zeta$ instead of $\omega$.\[9\]

$$\sum X_i(\zeta) = \zeta$$

$$\varphi(\zeta) \sim e^{\beta\zeta}. \quad (17)$$

Taking the logarithmic derivative of eq. [14] we obtain,

$$- \frac{u''_i(W_i(\zeta))}{u'_i(W_i(\zeta))} X'_i(\zeta) = \frac{\varphi'(\zeta)}{\varphi(\zeta)}. \quad (18)$$

From definition of the risk aversion parameter we have,

$$\beta_i = \frac{\beta}{X'_i(\zeta)}. \quad (19)$$

If the risk function for an agent be specified then the utility function is given by the following relation.

$$u_i(\xi) = \int_0^\xi e^{-\int_0^\eta \beta_i d\eta} d\eta. \quad (20)$$

In the above equation we standardize the utility function by assuming $u_i(0) = 0$ and $u'_i(0) = 1$. For constant risk aversion parameter the exponential form for utility function is obtained.
Acknowledgments

The author acknowledge Dr. Saeed for reading the manuscript and his valuable comments. This work has been supported by the Zanjan university research program on Non-Life Insurance Pricing No. 8243.

[1] Z. Burda, J. Jurkiewicz and M. A. Nowak, Is Econophysics a Solid Science?, arXiv:cond-mat/0301096. Will be appeared in Physica Polonica B.
[2] M. E. Fouladvand and A. H. Darooneh, Premium Forecasting for an Insurance Company, Proc. of 2-nd Nikkei Econophys. Symp. Springer Verlag, New York, 303 (2003).
[3] A. H. Darooneh, Non Life Insurance Pricing, Proc. 18-th Ann. Iranian Phys. Conf., 162 (2003).
[4] A. H. Darooneh, Non Life Insurance Pricing: Statistical Mechanics Viewpoint, arXiv:cond-mat/0305062
[5] A. H. Darooneh, Physics of Insurance. Will be published in Proc. First Int. Phys. Conf. (2004).
[6] A. H. Darooneh, Premium Calculation Based on Physical Principles, arXiv:cond-mat/0401308
[7] A. H. Darooneh, Non Life Insurance Pricing: Multi Agents Model, arXiv:cond-mat/0401308
[8] H. Buhlmann, An Economic Premium Principle, ASTIN Bulletin,11, 52 (1980).
[9] H. Buhlmann, General Economic Premium Principle, ASTIN Bulletin,14, 13 (1984).
[10] E.T. Jaynes, Information Theory and Statistical Mechanics, Phy. Rev.,106, 620 (1957).
[11] R. K. Pathria, Statistical Mechanics, Pergamon Press, Oxford, (1972).