The renormalization group and Weyl invariance

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Abstract
We consider matter fields conformally coupled to a background metric and dilaton and describe in detail a quantization procedure and related renormalization group flow that preserve Weyl invariance. Even though the resulting effective action is Weyl-invariant, the trace anomaly is still present, with all its physical consequences. We discuss first the case of free matter and then extend the result to interacting matter. We also consider the case when the metric and dilaton are dynamical and gravitons enter the loops.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The definition of a quantum field theory generally begins with a classical field theory with a bare action $S$, which is then quantized by defining a functional integral. Even if $S$ is scale- or (in curved spacetime) Weyl-invariant, the resulting quantum effective action in general is not, because in the definition of the functional integral one necessarily introduces a mass scale. This is the origin of the celebrated trace anomaly [1].

It has been known since early on that when a dilaton is present, there is a way of perturbatively quantizing the theory which preserves Weyl invariance [2]. This has been rediscovered several times in the literature [3–8], and has been used recently in a proof of the $a$-theorem [9]. In this paper, we will discuss mainly the implications of this type of quantization procedure for the renormalization group (RG). Actually, we will adopt a point of view that puts the RG first, and views the quantum effective action as the result of following the RG flow all the way to the infrared (IR). We will use a nonperturbative, ‘Wilsonian’ definition of the RG which is seen as the dependence of the effective action on a cutoff that is introduced by hand in the definition of the functional integral. By using this definition, we can extend the validity of the preceding statement to any theory, independent of its renormalizability properties.

The discussion will be pedagogical and self-contained. In the second section, we discuss scale and Weyl invariance, and the notion of Weyl geometry. We recall that when a dilaton is
present, one can make any action Weyl-invariant by replacing all dimensionful couplings by dimensionless couplings multiplied by the powers of the dilaton. This is a gravitational version of the ‘Stückelberg trick’. In the subsequent sections, we show, in increasingly complicated cases, that Weyl invariance can be preserved in the quantum theory. We begin with massless, free matter fields conformally coupled to a background metric and dilaton. Since in this case the functional integral is Gaussian, one can prove directly that the effective action is Weyl-invariant. In order to extend this statement to more complicated, interacting theories, it turns out to be better to view the effective action as the IR endpoint of a Wilsonian RG flow. One can then show quite generally that there exists a way of constructing the RG flow which preserves Weyl invariance, so if the initial point of the flow (in the UV) is Weyl-invariant, the IR endpoint will also be. For clarity, we present this logic first in the case of free massless fields, confirming the result obtained from the direct evaluation of the effective action. In these cases, one can actually construct explicit one-parameter families of Weyl-invariant actions that interpolate continuously between the bare action in the UV and the effective action in the IR. The construction is complete in two dimensions, where the effective action is (the Weyl-invariant version of) the Polyakov action, and limited to the first terms in a curvature expansion in four dimensions. We then observe that the condition of masslessness can be easily relaxed, since a mass can be viewed as a coupling of the field to the dilaton. Since any dimensionful coupling can be traded for a dimensionless coupling times some power of the dilaton, the proof can be extended to the case of interacting matter coupled to a background metric and dilaton and finally to the case when gravity itself is quantized, by which we mean that gravitons and dilatons are allowed to circulate in the loops. In this way, one has a fully nonperturbative proof that there exists a Weyl-invariant definition of the effective action.

Normal physical theories are neither conformal- nor scale-invariant. The RG running describes the dependence of couplings on one dimensionful scale and the theory becomes conformally invariant only at a fixed point. If we now reformulate an arbitrary theory in a Weyl-invariant way, then several obvious questions arise: What is the meaning of a cutoff in a Weyl-invariant theory? What distinguishes a fixed point from any other point? Are these Weyl-invariant quantum theories physically equivalent to ordinary non-Weyl-invariant ones? We will address these questions in the course of our derivations and summarize the state of our understanding in the conclusions.

2. Weyl invariance

A global scale transformation is a rescaling of all lengths by a fixed, constant factor $\Omega$. In flat space, scale transformations are usually interpreted as the map $x \rightarrow \Omega x$. As such, they form a particular subgroup of diffeomorphisms. Alternatively, one can think of rescaling the metric $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$. The two points of view are completely equivalent, since lengths are given by integrating the line element $ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$. For our purposes, it will be convenient to adopt the second point of view.

Let us now define the scaling dimension of a quantity. Consider a theory with fields $\psi_a$, parameters $g_i$ (which include masses, couplings, wavefunction renormalizations, etc) and action $S(g_{\mu\nu}, \psi_a, g_i)$. There is a unique choice of numbers $w_a$ (one per field) and $w_i$ (one per parameter) such that $S$ is invariant:

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a} \psi_a, \Omega^{w_i} g_i).$$

(1)

(It does not matter here whether the metric is fixed or dynamical.) The numbers $w_a, w_i$ are called the scaling dimensions, or the weights, of $\psi_a$ and $g_i$. In this paper, we will assume that the spacetime coordinates are dimensionless and we use natural units where $c = 1, \hbar = 1$. Then,
the scaling dimensions are equal to the ordinary length dimensions of \( \psi \), and \( g \), in the sense of dimensional analysis. Since in particle physics it is customary to use mass dimensions, when we talk of ‘dimensions’ without further specification we will refer to the mass dimensions \( d_a = -w_a \) and \( d_i = -w_i \). In \( d \) spacetime dimensions, the dimensions of scalar, spinor and vector fields are \((d - 2)/2\), \((d - 1)/2\) and \((d - 4)/2\), respectively. One can easily convince oneself that the dimensions of all parameters in the Lagrangian, such as masses and couplings, are the same as in the more familiar case when coordinates have a dimension of length.

Changing couplings is usually interpreted as changing theory, so in general the transformations (1) are not symmetries of a theory but rather maps from one theory to another. In the case when all the \( w_i \) are equal to zero, we have

\[
S(g_{\mu \nu}, \psi_\alpha, g_i) = S(\Omega^2 g_{\mu \nu}, \Omega^{\mu \nu} \psi_\alpha, g_i).
\]

Since these are transformations that map a theory to itself, a theory of this type is said to be globally scale-invariant.

Scale transformations with \( \Omega \) being a positive real function of \( x \) are called Weyl transformations. They act on the metric and the fields exactly as in (1). What about the parameters? They are supposed to be \( x \)-independent, so the transformation \( g_i \mapsto \Omega(x)^{w_i} g_i \) would not make much sense. One can overcome this difficulty by promoting the dimensionful parameters to fields. One can then meaningfully ask whether (1) holds. In general, the answer will be negative, but there is a simple procedure that allows one to make a scale-invariant theory also Weyl-invariant: it is called Weyl gauging and it was the earliest incarnation of the notion of gauge theory. In this paper, we will restrict ourselves to a special case of Weyl gauging, namely the case when the connection is flat. We pick a mass parameter of the theory, let us call it \( \mu \), and promote it to a function that we shall denote \( \chi \). We can write

\[
\chi(x) = \mu e^{\sigma(x)},
\]

where \( \mu \) is constant. The function \( \chi \), or sometimes \( \sigma \), is called the dilaton. Note that unlike an ordinary scalar field, it has dimension 1 independent of the spacetime dimensionality. Thus, it transforms under the Weyl transformation as \( \chi \mapsto \Omega^{-1} \chi \). Now we can take any other dimensionful coupling of the theory and write

\[
g_i = \chi^{-w_i} \tilde{g}_i = \chi^{d_i} \tilde{g}_i,
\]

where \( \tilde{g}_i \) is dimensionless (and therefore Weyl-invariant). In general, a caret over a symbol denotes the same quantity measured in units of the dilaton. In principle, one could promote more than one dimensionful parameter, or even all dimensionful parameters, to independent dilatons. This may have interesting applications, but for the sake of simplicity, in this paper we shall restrict ourselves to the case when there is a single dilaton.

With the dilaton, we construct a pure-gauge Abelian gauge field \( b_\mu = -\chi^{-1} \partial_\mu \chi \), transforming under (1) as \( b_\mu \mapsto b_\mu + \Omega^{-1} \partial_\mu \Omega \). Let \( \tilde{\nabla}_\mu \) be the covariant derivative with respect to the Levi-Civita connection of the metric \( g \). Define a new (non-metric) connection

\[
\tilde{\Gamma}_\mu^{\lambda \nu} = \Gamma_\mu^{\lambda \nu} - \delta^\lambda_\mu b_\nu - \delta^\nu_\mu b_\lambda + g_{\mu \nu} b^\lambda,
\]

where \( \Gamma_\mu^{\lambda \nu} \) are the Christoffel symbols of \( g \). The corresponding covariant derivative is denoted \( \tilde{\nabla} \). The connection coefficients \( \tilde{\Gamma} \) are invariant under (1). For any tensor \( t \) of weight \( w \), define the covariant derivative \( \tilde{D}t \) to be

\[
\tilde{D}_\mu t = \tilde{\nabla}_\mu t - w b_\mu t,
\]

where all indices have been suppressed. We see that the weight (or the dimension) acts like the Weyl charge of the field. The tensor \( \tilde{D}t \) is covariant under diffeomorphisms and under Weyl transformations. The curvature of \( D \) is defined by

\[
[D_\mu, D_\nu]u^\rho = R^{\rho \sigma}_{\mu \nu} u^\sigma.
\]
The tensor $\mathcal{R}_{\mu\nu,\rho\sigma} = R_{\mu\nu,\rho\sigma} + g_{\mu\rho}(\nabla_{\nu}b_{\sigma} + b_{\nu}b_{\sigma}) - g_{\mu\sigma}(\nabla_{\nu}b_{\rho} + b_{\nu}b_{\rho}) - g_{\nu\rho}(\nabla_{\mu}b_{\sigma} + b_{\mu}b_{\sigma}) + g_{\nu\sigma}(\nabla_{\mu}b_{\rho} + b_{\mu}b_{\rho}) - (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})b^{2}$. (8)

From here, one finds the analogues of the Ricci tensor and Ricci scalar

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + (d - 2)b_{\mu}b_{\nu} + (d - 2)\nabla_{\mu}b_{\nu} - (d - 2)b^{2}g_{\mu\nu} + \nabla^{\rho}b_{\rho}g_{\mu\nu},$$

and

$$\mathcal{R} = R + 2(d - 1)\nabla^{\mu}b_{\mu} - (d - 1)(d - 2)b^{2}.$$ (10)

It is also possible to define the tensor $C^{\mu}_{\nu\alpha\beta}$ which is related to $\mathcal{R}^{\mu}_{\nu\alpha\beta}$ by the same formula that relates $C^{\mu}_{\nu\alpha\beta}$ to $R^{\mu}_{\nu\alpha\beta}$, and therefore reduces to the standard Weyl tensor in a gauge where $\chi$ is constant.

Now start from a generic action for matter and gravity of the form $S(g_{\mu\nu}, \psi_{a}, g_{i})$. Express every parameter $g_{i}$ as in (4). Replace all covariant derivatives $\nabla$ by Weyl-covariant derivatives $\hat{\nabla}$ and all curvatures $R$ by the Weyl-covariant curvatures $\hat{\mathcal{R}}$. Now all the terms appearing in the action are products of Weyl-covariant objects, and local Weyl invariance just follows from the fact that the action is dimensionless. In this way, we have defined an action $\hat{S}(g_{\mu\nu}, \chi, \psi_{a}, \hat{g}_{i})$. It contains only dimensionless couplings $\hat{g}_{i}$, and is Weyl-invariant by construction. One can choose a gauge where $\chi = \mu$ is constant (equivalently, $\sigma = 0$), and in this gauge the action $\hat{S}(g_{\mu\nu}, \chi, \psi_{a}, \hat{g}_{i})$ reduces to the original one.

The above construction defines an ‘integrable Weyl geometry’, since the curvature of the Weyl gauge field $b_{\mu}$ is zero. In this integrable case, there is also another way of defining a Weyl-invariant action from a non-invariant one, namely to replace all the arguments in $S$ by the corresponding dimensionless quantities $\hat{g}_{\mu\nu} = \chi^{2}g_{\mu\nu}, \hat{\psi}_{a} = \chi^{m}\psi_{a}$ and $\hat{g}_{i} = \chi^{i}g_{i}$, and subsequently re-express the action in terms of the original fields

$$\hat{S}(g_{\mu\nu}, \chi, \psi_{a}, \hat{g}_{i}) = S(\hat{g}_{\mu\nu}, \hat{\psi}_{a}, \hat{g}_{i}).$$ (11)

It is easy to see that this construction gives the same result as the preceding one. This follows from the fact that (S) are the Christoffel symbols of $\hat{g}_{\mu\nu}, \nabla_{\mu}\hat{\psi}_{a} = \chi^{m}D_{\mu}\psi_{a}$ and the curvature tensor of $\Gamma$ is $\mathcal{R}_{\mu\nu,\rho\sigma}$.

The above procedure can be used to rewrite any theory in a Weyl-invariant form. Not all Weyl-invariant theories are of this type: there are also theories that are Weyl-invariant without containing a Weyl gauge field $b_{\mu}$ (or a dilaton). In such theories, the terms generated by a Weyl transformation that contain the derivatives of the transformation parameter are compensated by terms generated by the variations of Ricci tensors. Since Weyl invariance can be viewed as a gauged version of global scale invariance, this has been called ‘Ricci gauging’ in [10]. It was also shown that such Ricci-gauged theories correspond (under mild additional assumptions) to theories that are conformal-invariant, as opposed to merely scale-invariant, in flat space. The existence of well-behaved theories that are scale- —but not conformal-invariant in flat space has been reexamined recently [11–13].

3. The effective action of free matter fields coupled to an external gravitational field

3.1. The standard measure

In this section, we review the evaluation of the effective action for free, massless matter fields conformally coupled to a metric. This will provide the basis for different quantization

3 If we call $\hat{R}_{\mu\nu,\rho\sigma}$ the Riemann tensor of $\hat{g}_{\mu\nu}$, then we have $\hat{R}^{\mu}_{\nu\rho} = \mathcal{R}^{\mu}_{\nu\rho}$ and $\hat{R}_{\mu\nu,\rho\sigma} = \chi^{2}\mathcal{R}_{\mu\nu,\rho\sigma}$. 

4
procedures to be described in the following. Much of the discussion can be carried out in arbitrary even dimension \( d \).

For definiteness, let us consider first a single conformally coupled scalar field, with the equation of motion \( \Delta(0) \phi = 0 \), where \( \Delta(0) = -\nabla^2 + \frac{d-2}{4(d-1)} \mathcal{R} \). Functional integration over \( \phi \) in the presence of a source \( j \) leads to a generating functional \( \Gamma(g_{\mu\nu}, j) \), whose Legendre transform \( \Gamma(g_{\mu\nu}, \phi) = W(g_{\mu\nu}, j) - \int j \phi \) is the effective action. For the definition of the functional integral, one needs a metric (more precisely an inner product) in the space of the fields. We choose

\[
\mathcal{G}(\phi, \phi') = \mu^2 \int \text{d}x \sqrt{g} \phi \phi',
\]

where \( \mu \) is an arbitrary mass that has to be introduced for dimensional reasons. The action can be written as

\[
S_\delta(g_{\mu\nu}, \phi) = \frac{1}{2} \int \text{d}x \sqrt{g} \Delta(0) \phi = \frac{1}{2} \mathcal{G}(\phi, \frac{\Delta(0)}{\mu^2} \phi) = \frac{1}{2} \sum_n a_n^2 \lambda_n / \mu^2,
\]

where \( \lambda_n \) are the eigenvalues of \( \Delta(0) \), \( \phi_n \) are the corresponding eigenfunctions and \( a_n \) are the (dimensionless) coefficients of the expansion of \( \phi \) on the basis of the eigenfunctions

\[
\Delta(0) \phi_n = \lambda_n \phi_n; \quad \mathcal{G}(\phi_n, \phi_m) = \delta_{nm}; \quad \phi = \sum_n a_n \phi_n; \quad a_n = \mathcal{G}(\phi, \phi_n).
\]

(For simplicity, we assume that the manifold is compact and without boundary, so that the spectrum of the Laplacian is discrete.) Weyl-covariance means that under a Weyl transformation the operator \( \Delta(0) \) transforms as

\[
\Delta(0)_{\Omega^2 g} = \Omega^{-1 - \frac{d}{2}} \Delta(0)_g \Omega^{\frac{d}{2} - 1},
\]

where we have made the dependence of the metric explicit. For an infinitesimal transformation \( \Omega = 1 + \omega \),

\[
\delta\omega \Delta(0) = -2 \omega \Delta(0) + \left( \frac{d}{2} - 1 \right) [\Delta(0), \omega].
\]

The functional measure is \( (d\phi) = \prod_n \text{d}a_n \), so the Gaussian integral can be evaluated as

\[
e^{-W(g_{\mu\nu}, j)} = \prod_n \left( \int \text{d}a_n \: e^{-\frac{1}{2} \lambda_n^2 a_n^2 / \mu^2} \right) = \det \left( \frac{\Delta(0)}{\mu^2} \right)^{-1/2} \: e^{\frac{1}{2} \int j \Delta^{-1} j}
\]

up to a field-independent multiplicative constant. From here, one obtains (using the same notation for the vacuum expectation value (VEV) as for the field) \( \phi = -\Delta(0)^{-1} j \), so finally the Legendre transform gives

\[
\Gamma(\phi, g_{\mu\nu}) = S_\delta(\phi, g_{\mu\nu}) + \frac{1}{2} \text{Tr} \log \left( \frac{\Delta(0)}{\mu^2} \right).
\]

UV regularization is needed to define this trace properly. We see that the scale \( \mu \), which has been introduced in the definition of the measure, has made its way into the functional determinant.

Things work much in the same way for the fermion field, which contributes to the effective action a term

\[
S_D(\bar{\psi}, \psi, g_{\mu\nu}) = \frac{1}{2} \text{Tr} \log \left( \frac{\Delta^{(1/2)}}{\mu^2} \right),
\]

where \( S_D \) is the classical action and \( \Delta^{(1/2)} = -\nabla^2 + \frac{R}{4} \) is the square of the Dirac operator.
The Maxwell action is Weyl-invariant only in $d = 4$. With our conventions, the field $A_\mu$ is dimensionless and the Weyl-invariant inner product in field space is

$$\mathcal{G}(A, A') = \mu^2 \int d^4x \sqrt{g} \delta^{\mu\nu} A_\mu A_\nu.$$  \hfill (20)

Using the standard Faddeev–Popov procedure, we add gauge fixing and ghost actions

$$S_{\text{GF}} = \frac{1}{2\alpha} \int d^4x \sqrt{g} (\nabla_\mu A^\mu)^2; \quad S_{gh} = \int d^4x \sqrt{g} \mathcal{C} \Delta^{(gh)} C,$$  \hfill (21)

with $\Delta^{(gh)} = -\nabla^2$. Then, in the gauge $\alpha = 1$, the gauge-fixed action becomes

$$S_M + S_{\text{GF}} = \frac{1}{2} \int d^4x \sqrt{g} A^{\mu} \Delta^{(1)} A_{\nu} = \frac{1}{2} \mathcal{G}(A, \frac{\Delta^{(1)}}{\mu^2} A),$$  \hfill (22)

where $\Delta^{(1)} = -\nabla^2 \delta^{\mu\nu} + R^{\mu\nu}$ is the Laplacian on 1-forms. Following the same steps as for the scalar field, we obtain a contribution to the effective action equal to

$$S_{\text{M}}(A_\mu, g_{\mu\nu}) + \frac{1}{2} \text{Tr} \log \left( \frac{\Delta^{(1)}}{\mu^2} \right) - \text{Tr} \log \left( \frac{\Delta^{(gh)}}{\mu^2} \right).$$  \hfill (23)

Note that even though the Maxwell action $S_M$ is Weyl-invariant, the gauge fixing action is not, nor is the ghost action. As a result, the operators $\Delta^{(1)}$ and $\Delta^{(gh)}$ are not Weyl-covariant. Instead of an equation like (16), they satisfy (in four dimensions)

$$\delta_\omega \Delta^{(gh)} = -2\omega \Delta^{(b)} - 2\nabla^i \omega \nabla_i;$$  \hfill (24)

$$\delta_\omega \Delta^{(1)} = -2\omega \Delta^{(1)} - 2\nabla^i \omega \nabla_i - 2\nabla^i \omega \nabla_i - 2\nabla^i \omega \nabla_i.$$  \hfill (25)

We shall see in the following section how these non-invariances compensate each other in the effective action, so that the breaking of Weyl invariance is only due to the presence of the scale $\mu$ which was introduced in the inner product.

In general, the need for an inner product in field space can also be seen in a more geometrical way as follows. The classical action, being quadratic in the fields, has the form $\mathcal{H}(\phi, \phi)$, where $\mathcal{H} = \frac{\delta^2}{\delta\phi\delta\phi}$ can be viewed as a covariant symmetric tensor in field space: when contracted with a field (a vector in field space), it produces a 1-form in field space. Now, the determinant of a covariant symmetric tensor is not a basis-independent quantity. One can only define in a basis-independent way the determinant of an operator mapping a space into itself, i.e. a mixed tensor. One can transform the covariant tensor $\mathcal{H}$ to a mixed tensor $\mathcal{O}$ by ‘raising an index’ with a metric:\footnote{In de Witt’s condensed notation, where an index $i$ stands both for a point $x$ in spacetime and whatever tensor or spinor indices the field may be carrying, this equation reads $\mathcal{O}_i = \mathcal{H}_{ik}\partial^k$.}

$$\mathcal{H}(\phi, \phi') = \mathcal{G}(\phi, \mathcal{O}_i).$$  \hfill (26)

It is the determinant of the operator $\mathcal{O}$ that appears in the effective action. Again, we see that the scale $\mu$ appears through the metric $\mathcal{G}$, which is needed to define the determinant. Note that since $\mathcal{O}_i\phi$ is another field of the same type as $\phi$, $\mathcal{O}$ must necessarily be dimensionless, and this is guaranteed by the factors of $\mu$ contained in $\mathcal{G}$. For example, in the scalar case, $\mathcal{O} = \frac{1}{\mu^2} \Delta^{(0)}$.

3.2. Trace anomaly

Under an infinitesimal Weyl transformation, the variation of the effective action is

$$\delta_\omega \Gamma = \int dx \frac{\delta \Gamma}{\delta g_{\mu\nu}} 2\omega g_{\mu\nu} = - \int dx \sqrt{g} \omega (\mathcal{T}^a_{\mu\nu}).$$  \hfill (27)

The trace of the energy–momentum tensor vanishes for a Weyl-invariant action, so the appearance of a nonzero trace is the physical manifestation of the anomaly.
For non-interacting fields, the one-loop effective action is exact. We use the proper time representation [14]

$$\Gamma = S - \frac{1}{2} \int_{\mu^2}^{\infty} dt \, \text{Tr} \, e^{-i\Delta},$$

(28)

where $\epsilon$ is a dimensionless UV regulator. We also need the general formula stating the Weyl covariance of an operator $\Delta$ acting on fields of weight $w$ (see (16)):

$$\delta_\omega \Delta = -2\omega \Delta + w[\Delta, \omega].$$

(29)

Varying (28) and using the fact that the commutator cancels under the trace, one finds

$$\delta_\omega \Gamma = \frac{1}{2} \int_{\mu^2}^{\infty} dt \, \text{Tr} \, \delta_\omega \Delta e^{-i\Delta} = -\int_{\mu^2}^{\infty} dt \, \text{Tr} (\omega \Delta e^{-i\Delta}) = \int_{\mu^2}^{\infty} dt \frac{d}{dt} \text{Tr} (\omega e^{-i\Delta})$$

$$= -\text{Tr} [\omega e^{-i\Delta/\mu^2}].$$

For $\epsilon \to 0$, one has from the asymptotic expansion of the heat kernel

$$\text{Tr} [\omega e^{-i\Delta/\mu^2}] = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{\rho_0} \left[ \frac{\mu^d}{e^{d/2} 2^d b_0(\Delta)} + \frac{\mu^{d-2}}{e^{d/2-1} 2 b_2(\Delta)} + \cdots + b_2(\Delta) + \cdots \right],$$

(30)

where $b_i$ are scalars constructed with $i$ derivatives of the metric. All terms $b_i$ with $i > d$ tend to zero in the limit, so assuming that the power divergences (for $i < d$) are removed by renormalization, there remains a universal, finite limit

$$\delta_\omega \Gamma = -\frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{\rho_0} b_2(\Delta),$$

(31)

which implies that

$$\langle T^{\mu \nu}_\mu \rangle = \frac{1}{(4\pi)^{d/2}} b_2(\Delta).$$

(32)

Aside from the different prefactor, the calculation follows the same steps in the case of massless spinors. The Maxwell field, however, requires some additional considerations, because the operators $\Delta^{(1)}$ and $\Delta^{(gh)}$ that appear in (23) are not covariant. (We restrict ourselves now to $d = 4$.) The first two steps of the preceding calculation give

$$\delta_\omega \Gamma = \frac{1}{2} \int_{\mu^2}^{\infty} dt \, \text{Tr} \, \delta_\omega \Delta^{(1)} e^{-i\Delta^{(1)}} - \int_{\mu^2}^{\infty} dt \, \text{Tr} \, \delta_\omega \Delta^{(gh)} e^{-i\Delta^{(gh)}}$$

$$= \frac{1}{2} \int_{\mu^2}^{\infty} dt \, \text{Tr} (-2\omega \Delta^{(1)} + \rho^{(1)} \epsilon^{-i\Delta^{(1)}} - \int_{\mu^2}^{\infty} dt \, \text{Tr} (-2\omega \Delta^{(gh)} + \rho^{(gh)} \epsilon^{-i\Delta^{(gh)}}),$$

(33)

where the violation of Weyl covariance is due to

$$\rho^{(gh)} = -2\nabla^\nu \nabla_\nu; \quad \rho^{(1)}_{\mu} = 2\nabla_\mu \omega \nabla^\nu - 2\nabla^\nu \omega \nabla_\mu - 2\nabla_\mu \nabla^\nu \omega.$$

(34)

Since $\Delta^{(1)}$ maps longitudinal fields to longitudinal fields and transverse fields to transverse fields, $\rho^{(1)} \epsilon^{-i\Delta^{(1)}}$ has vanishing matrix elements between transverse gauge fields. Therefore, the trace containing $\rho^{(1)}$ can be restricted to the subspace of longitudinal gauge potentials. Let $\phi_\nu$ be a basis of the eigenfunctions of $\Delta^{(gh)}$ satisfying an orthonormality condition as in (14). Then, a basis in the space of longitudinal potentials satisfying a similar orthonormality

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5 Note that in four dimensions the term in $b_2$ proportional to $\Box R$, which is a total derivative, can be renormalized at will by adding a local term to the effective action.
condition with respect to the inner product (20) is given by the fields $A^L_n = \frac{1}{\sqrt{\lambda_n}} \nabla_\mu \phi_n$. The traces of the terms violating Weyl covariance are therefore

$$\frac{1}{2} \text{Tr} \rho^{(1)} e^{-t\Delta^{(1)}} - \text{Tr} \rho^{(gb)} e^{-t\Delta^{(gb)}} = \frac{1}{2} \sum_n G(A^L_n, \rho^{(1)} e^{-t\Delta^{(1)}} A^L_n) - \frac{1}{2} \sum_n G(\phi_n, \rho^{(gb)} e^{-t\Delta^{(gb)}} \phi_n).$$

(35)

Noting that

$$\Delta^{(1)} A^L_n = \frac{1}{\sqrt{\lambda_n}} \Delta^{(1)} \nabla_\mu \phi_n = \frac{1}{\sqrt{\lambda_n}} \nabla_\mu \Delta^{(gb)} \phi_n = \lambda_n A^L_n,$$

we can evaluate the matrix elements

$$G(A^L_n, \rho^{(1)} e^{-t\Delta^{(1)}} A^L_n) = -4e^{-t\lambda_n} G(\phi_n, \nabla_\nu \nabla_\nu \phi_n),$$

whereas in the ghost trace we have

$$G(\phi_n, \rho^{(gb)} e^{-t\Delta^{(gb)}} \phi_n) = -2e^{-t\lambda_n} G(\phi_n, \nabla_\nu \nabla_\nu \phi_n).$$

We see that the sums in (35) cancel mode by mode. As a result, only the first term remains in each of the traces in (33). From this point onwards, the calculation proceeds as in the case of the scalar and finally gives

$$\delta \omega \Gamma = \frac{1}{(4\pi)^2} \int d^4 x \sqrt{g} [b_4(\Delta^{(1)}) - 2b_4(\Delta^{(gb)})].$$

(36)

The coefficients of the expansion of the heat kernel for Laplace-type operators are well known. If there are $n_S$ scalars and $n_D$ spinors, then one has in two dimensions

$$\langle T^{\mu \mu} \rangle = \frac{c}{24\pi} R,$$

(37)

with $c = n_S + n_D$, whereas in four dimensions (assuming also the existence of $n_M$ Maxwell fields),

$$\langle T^{\mu \mu} \rangle = cC^2 - aE,$$

(38)

where $E = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 4 R_{\mu \nu} R^{\mu \nu} + R^2$ is the integrand of the Euler invariant, $C^2 = C_{\mu \nu \rho \sigma} C^{\mu \nu \rho \sigma}$ is the square of the Weyl tensor and the anomaly coefficients are $^6$

$$a = \frac{1}{360(4\pi)^2} (n_S + 11n_D + 62n_M); \quad c = \frac{1}{120(4\pi)^2} (n_S + 6n_D + 12n_M).$$

(39)

3.3. The Weyl-invariant measure

Let us now assume that the theory contains also a dilaton $\chi$. For the purposes of this section, it will be considered as part of the gravitational sector and treated as an external field. The crucial observation is that we can now construct Weyl-invariant metrics in the spaces of scalar, Dirac and Maxwell fields, replacing the fixed scale $\mu$ by the dilaton $^7$:

$$G_S(\phi, \phi') = \int d^4 x \sqrt{g} \chi^{\phi \phi'},$$

(40)

$$G_D(\bar{\psi}, \psi') = \int d^4 x \sqrt{g} \chi [\bar{\psi} \psi' + \bar{\psi}' \psi],$$

(41)

$$G_M(A, A') = \int d^4 x \sqrt{g} \chi^2 A_\mu \bar{g}^{\mu \nu} A'_\nu.$$
One can follow step by step the calculation in section 3.2, the only change being the replacement of $\mu$ by $\chi$. The final result for the one-loop contribution to the effective action can be written as

$$\frac{R_g}{2} \text{Tr} \log O_S - \frac{n_D}{2} \text{Tr} \log O_D + \frac{n_M}{2} \text{Tr} \log O_M - n_M \text{Tr} \log O_{gh},$$

where now

$$O_S = \chi^{-2} \Delta^{(0)}; \quad O_D = \chi^{-2} \Delta^{(1/2)}; \quad O_{M\mu} = \chi^{-2} g_{\mu\nu} (\Delta^{(1)})^{\sigma\nu}; \quad O_{gh} = \chi^{-2} \Delta^{(gh)}.$$

One can then verify that

$$O_\Omega^I (\Omega^{-1} \phi) = \Omega^{-1} O_\Omega \phi; \quad O_\Omega^D (\Omega^{-3/2} \psi) = \Omega^{-3/2} O_D \psi,$$

$$O_{M\mu}^i A_i = O_{M\mu} A_i; \quad O_{gh}^\Omega (\Omega^{-1} c) = \Omega^{-1} O_{gh} c,$$

where the notation $O_\Omega$ stands for the operator $O$ constructed with the transformed metric $g^\Omega = \Omega^2 g$ and dilaton $\chi^\Omega = \Omega^{-1} \chi$. These operators map fields into fields transforming in the same way. (As observed earlier, they are dimensionless.) This implies that the eigenvalues of the operators $O$ are Weyl-invariant and therefore, their determinants are also invariant.

We conclude that in the presence of a dilaton there exists a quantization procedure for noninteracting matter fields that respects Weyl invariance.

### 3.4. The Wess–Zumino action

We have seen that in the presence of a dilaton one has a choice between different quantization procedures, which can be understood as different functional measures: one of them breaks Weyl invariance while the other maintains it. Let us denote by $\Gamma^I$ the effective action obtained with the standard measure and by $\Gamma^{II}$ the one obtained with the Weyl-invariant measure. The former is anomalous:

$$\delta_w \Gamma^I = \int dx \frac{\delta \Gamma^I}{\delta g_{\mu\nu}} g_{\mu\nu} = - \int dx \sqrt{g_0} (T^\mu_{\mu})^I \neq 0,$$

whereas the second is Weyl-invariant: $\Gamma^{II}(g^\Omega, \chi^\Omega) = \Gamma^{II}(g, \chi)$, or in an infinitesimal form:

$$0 = \delta_w \Gamma^{II} = \int dx \sqrt{g_0} \left( 2 \frac{\delta \Gamma^{II}}{\delta g_{\mu\nu}} g_{\mu\nu} - \frac{\delta \Gamma^{II}}{\delta \chi} \chi \right).$$

The Weyl-invariant measure differs from the standard one simply by the replacement of the fixed mass $\mu$ by the dilaton $\chi$; therefore, we have

$$\Gamma^{II}(g_{\mu\nu}, \mu) = \Gamma^I(g_{\mu\nu}).$$

We see that $\Gamma^{II}$ can be obtained from $\Gamma^I$ by applying the Stäckelberg trick after quantization, i.e. to the mass parameter $\mu$ that has been introduced by the functional measure.

Another useful point of view is the following. Noting that $\Omega = \chi/\mu$ can be interpreted as the parameter of a Weyl transformation, the variation of $\Gamma^I$ under a finite Weyl transformation defines a functional $\Gamma_{WZ}(g, \chi)$, the so-called Wess–Zumino (WZ) action, by

$$\Gamma^I(g^\Omega) - \Gamma^I(g) = \Gamma_{WZ}(g, \mu \Omega).$$

It satisfies the so-called WZ consistency condition, which can be written in the form

$$\Gamma_{WZ}(g^\Omega, \chi^\Omega) - \Gamma_{WZ}(g, \chi) = - \Gamma_{WZ}(g, \mu \Omega),$$

8 Here, we view the WZ action as a functional of a metric and a dilaton, two dimensionful fields. Sometimes, one may prefer to think of it as a functional of a metric and a Weyl transformation, the latter being a dimensionless function. The two points of view are related by some factors of $\mu$. 

9
where $g^2 = \Omega^2 g$, $\chi^2 = \Omega^{-1} \chi$. This shows that the variation of the WZ action under a Weyl transformation is exactly opposite to that of the action $\Gamma^I$. From the Weyl invariance of $\Gamma^{II}$ and equation (49), one finds that $\Gamma^{II}(g_{\mu\nu}, \Omega\mu) = \Gamma^{II}(g^2_{\mu\nu}, \mu) = \Gamma^I(g^2_{\mu\nu})$. Thus, replacing $\mu \Omega$ by $\chi$ and using (50), we see that the $\chi$-dependence of the Weyl-invariant action is entirely contained in a WZ term:

$$\Gamma^{II}(g, \chi) = \Gamma^I(g) + \Gamma_{WZ}(g, \chi).$$  \hspace{1cm} (52)

In section 4.2.1, these statements will be verified by direct calculation in $d = 2$, where all these functionals can be written explicitly. We can think of the Weyl-invariant effective action as the ordinary effective action to which a WZ term has been added, with the effect of canceling the Weyl anomaly$^9$.

In the case of non-interacting, massless, conformal matter fields, the WZ actions can be computed explicitly by integrating the trace anomaly. Let $\Omega_0$ be a one-parameter family of Weyl transformations with $\Omega_0 = 1$ and $\Omega_1 = \Omega$, and let $g(t)_{\mu\nu} = \Omega(t)^{\mu\nu}_{\sigma\tau} g_{\sigma\tau}$:

$$\Gamma_{WZ}(g_{\mu\nu}, \Omega) = \int_0^1 dt \int dx \frac{\delta \Gamma}{\delta g_{\mu\nu}(g(t))} \delta g(t)_{\mu\nu} = -\int_0^1 dt \int dx \sqrt{g(t)} T^\mu_{\mu\nu}(g(t), \Omega(t))^{-1} \frac{d\Omega}{dt}. \hspace{1cm} (53)$$

In two dimensions, integrating the anomaly (37) and using the parametrization (3), one finds

$$\Gamma_{WZ}(g_{\mu\nu}, \mu \sigma) = -\frac{c}{24\pi} \int d^2 x \sqrt{g}(R \sigma - \nabla^2 \sigma). \hspace{1cm} (54)$$

A similar procedure in four dimensions using (38) leads to

$$\Gamma_{WZ}(g_{\mu\nu}, \mu \sigma) = -\int dx \sqrt{g} \left\{ c E^2 \sigma - a \left[ \left( E - \frac{2}{3} \Box R \right) \sigma + 2 \sigma \Delta_k \sigma \right] \right\}, \hspace{1cm} (55)$$

where

$$\Delta_k = \Box^2 + 2 R^\mu_{\nu\rho} \nabla_{\mu} \nabla_{\rho} - \frac{2}{3} R \Box + \frac{1}{3} \nabla^\mu R \nabla_{\mu}. \hspace{1cm} (56)$$

At this point the reader will wonder whether the two procedures described above lead to different physical predictions or not. If the metric and dilaton are treated as classical external fields, but we allow them to be transformed, the two quantization procedures yield equivalent physics. In the Weyl-invariant procedure, one has the freedom of choosing a gauge where $\chi = \mu$, and in this gauge all the results reduce to those of the standard procedure. In particular, we observe that the traces of the energy–momentum tensor derived from the two actions $\Gamma^I$ and $\Gamma^{II}$ are the same. This follows from the fact that

$$\int dx \sqrt{g} \left| \frac{\delta \Gamma_{WZ}}{\delta g_{\mu\nu}} \right|_{(g, \chi = \mu)} 2 \omega g_{\mu\nu} = 0, \hspace{1cm} (57)$$

which in turn follows from (50). On the other hand, if we assume that the metric (and dilaton) are going to be quantized too, the answer hinges on the choice of their functional measure. We defer a discussion of this point to section 7.

4. The effective average action of free matter fields coupled to an external gravitational field

In this section, we introduce a generalization of the effective action, called effective average action (EAA), which depends on a scale $k$ having the meaning of IR cutoff. The main virtue of this definition is that there exists a simple formula for the derivative of the EAA with respect to $k$, called the functional RG equation (FRGE) or the Wetterich equation [16]. Its generalization

$^9$ This is completely analogous to what happens with gauge invariance in chiral theories [15].
to the context of gravity has been presented in [17]. The full power of the FRGE, which is an exact equation, manifests when one considers interacting fields. In this section, we shall familiarize ourselves with the FRGE in the context of free matter fields coupled to an external metric and dilaton, where the one-loop approximation is exact. We leave the discussion of interacting matter to section 5.

4.1. The EAA and its flow at one loop

The definition of the EAA follows the same steps of the definition of the ordinary effective action, except that one modifies the bare action by adding to it a cutoff term $\Delta S_k(\phi)$ that is quadratic in the fields and therefore modifies the propagator without affecting the interactions. Using the notation of (13), the cutoff term is

$$\Delta S_k(g_{\mu\nu}, \phi) = \frac{1}{2} \partial^\mu \phi \left( \frac{R_\Delta}{\mu^2} \phi \right) + \frac{1}{2} \frac{k^2}{\mu^2} \sum_a q_a^2 \left( \frac{\lambda_a}{k^2} \right), \quad (58)$$

where we have written the cutoff (which has the dimension of mass squared) as $R_k(z) = k^2 r(z/k^2)$. The kernel $R_k(z)$ is arbitrary, except for the general requirements of being a monotonically decreasing function both in $z$ and $k$, tending fast to zero for $z \gg k^2$ and to $k^2$ for $z \to 0$. These conditions ensure that the contributions to the functional integral of field modes with momenta $q^2 \ll k^2$ are suppressed, while that of modes with momenta $q^2 \gg k^2$ are unaffected.

We define a $k$-dependent generating functional $W_k$ in the same way as the functional $W$, except for the replacement of $S$ by $S + \Delta S_k$. The EAA is obtained by Legendre transforming, and then subtracting the cutoff:

$$\Gamma_k(g_{\mu\nu}, \phi) = W_k(g_{\mu\nu}, j) - \int dx J\phi - \Delta S_k(g_{\mu\nu}, \phi). \quad (59)$$

Since $R_k \to 0$ when $k \to 0$, the EAA becomes the ordinary effective action in this limit.

The evaluation of the EAA for Gaussian matter fields, conformally coupled to a metric, follows the same steps that led to (18). The only differences are the replacement of $S$ by $S + \Delta S_k$ and hence of the ‘inverse propagator’ $\Delta$ by the ‘cutoff inverse propagator’ $P_k(\Delta) = \Delta + R_k(\Delta)$, and in the end, the subtraction of $\Delta S_k$. The result is

$$\Gamma_k^I(g_{\mu\nu}, \phi) = S(g_{\mu\nu}, \phi) + \frac{1}{2} \text{Tr} \log \left( \frac{P_k(\Delta)}{\mu^2} \right). \quad (60)$$

We used here the superscript I to denote that this EAA has been obtained by using the standard measure and reduces to $\Gamma^I$ for $k = 0$. We would like to now define a Weyl-invariant form of EAA, to be called $\Gamma_k^{II}$ in analogy to the effective action $\Gamma^{II}$ discussed previously.

The first step is to clarify the meaning of the cutoff $k$ in this context. In the usual treatment, i.e. in a non-gravitational context, $k$ is a constant with a dimension of mass. In the present context, these two properties are contradictory. A quantity that has a nonzero dimension cannot generally be a constant: it can only be constant in some special gauge. This means that the cutoff must be allowed to be a generic non-negative function on spacetime.

Now we must give meaning to the notion that the couplings depend on the cutoff. In a Weyl-invariant theory, all couplings are dimensionless, and the only way they can depend on $k$ is via the dimensionless combination $u = k/\chi$. Note that by definition, the dilaton cannot vanish anywhere, whereas the cutoff should be allowed to go to zero. So $u$ is a non-negative dimensionless function on spacetime. This raises the question of the meaning of a running coupling whose argument is itself a function on spacetime. In order to avoid such issues, we will restrict ourselves to the case when $u$ is a constant; in other words, the cutoff and the dilaton are proportional.
With this point understood, the evaluation of the EAA with the Weyl-invariant measure is very simple: as in section 3.3, we just have to replace $\mu$ by $\chi$

$$\Gamma_u^{\mu}(g_{\mu\nu}, \phi) = S(g_{\mu\nu}, \phi) + \frac{1}{2} \text{Tr} \log \left( \frac{\Delta + R_k(\Delta)}{\chi^2} \right)$$  (61)

$$= S(g_{\mu\nu}, \phi) + \frac{1}{2} \text{Tr} \log (O + u^2 r(u^2 O)).$$  (62)

In the second line, we have re-expressed the EAA as a function of the Weyl-covariant operator $O = \chi^{-2} \Delta$, the Weyl-invariant cutoff parameter $u$ and the dimensionless function $r(z/k^2) = R_k(z)/k^2$. It manifests that all dependence on $k$ is via $u$ and that $\Gamma_u^{\mu}$ is Weyl-invariant.

### 4.2. Calculating the effective action with the FRGE

It can be shown that the EAA satisfies the following FRGE:

$$\frac{k}{d} \frac{d\Gamma_k}{dk} = \frac{12}{2} \text{Tr} \left[ \frac{\delta^2 (\Gamma_k + \Delta S_k)}{\delta \phi^2} \right] \frac{d}{dk} \frac{\delta^2 \Delta S_k}{\delta \phi^2},$$  (63)

which is an exact equation holding for any theory [16]10. We will not give the general proof of this equation but we shall derive it in the special case of free matter coupled to an external gravitational field. Before doing this, however, it is convenient to discuss the use of this equation as a tool to calculate the effective action.

The rhs of the FRGE (63) can be regarded as the ‘beta functional’ of the theory, giving the $k$-dependence of all the couplings. To see this, let us assume that $\Gamma_k$ admits a derivative expansion of the form

$$\Gamma_k(\phi, g_i) = \sum_{n=0}^{\infty} \sum_{i} g_i^{(n)}(k) O_i^{(n)}(\phi),$$  (64)

where $g_i^{(n)}(k)$ are coupling constants and $O_i^{(n)}$ are all possible operators constructed with the field $\phi$ and $n$ derivatives, which are compatible with the symmetries of the theory. We have

$$\frac{k}{d} \frac{d\Gamma_k}{dk} = \sum_{n=0}^{\infty} \sum_{i} \beta_i^{(n)}(g_i, k),$$  (65)

where $\beta_i^{(n)}(g_i, k) = \frac{d g_i^{(n)}(k)}{dk} = \frac{d g_i^{(n)}(k)}{d \Delta}$ are the beta functions of the couplings. Here, we have introduced $t = \log(k/k_0)$, with $k_0$ being an arbitrary initial value. If we expand the trace on the rhs of (63) in operators $O_i^{(n)}$ and compare with (65), then we can read off the beta functions of the individual couplings.

The most remarkable property of the FRGE is that the trace on the rhs is free of UV and IR divergences. This is because the derivative of the cutoff kernel goes rapidly to zero for $\sqrt{q^2} > k^2$, and $k$ also acts effectively as a mass. So, even though the EAA defined above is as ill-defined as the usual effective action, its $t$-derivative is well-defined. Given a ‘theory space’ which consists of a class of functionals of the fields, one can define on it a flow without having to worry about UV regularizations. All the beta functions are finite. This can be done for any theory, whether renormalizable or not.

10 Note that the structure of (63) in field space is the trace of a contravariant 2-tensor times a covariant 2-tensor (in de Wett notation, $(r(\gamma^{(2)} + \Delta S_k^{(2)})^{-1}) / (\delta \Delta S_k^{(2)})_{\mu},$ where superscript (2) denotes a second-functional derivative and $t = \log k$) and is therefore an invariant expression. In passing from (63) to (66), one uses the field space metric $G$ to raise and lower indices and transform the covariant and contravariant tensors into mixed tensors, each of which can be seen as a function of $\Delta$. In practice, this amounts to canceling all factors of $\sqrt{2}$ and $\mu$. 

12
Then, one can pick an initial point in theory space, which can be identified with the bare action at some UV scale \( \Lambda \), and study the trajectory passing through it in either direction. The EAA can be obtained by solving the first-order differential equation (63) and taking the limit \( k \to 0 \).

The issue of the divergences presents itself, in this formulation, when one tries to move \( \Lambda \) to higher energies, which is equivalent to solving the RG equation for growing \( k \). If the trajectory is renormalizable, then all dimensionless couplings remain finite in the limit \( k \to \infty \). This implies that only the relevant dimensionful couplings diverge, and one expects only a finite number of these. The ambiguities that correspond to these divergences are fixed by the choice of RG trajectory, because the IR limit (i.e. the renormalized couplings) is kept fixed. On the other hand, if some dimensionless coupling diverges (e.g. at a Landau pole), the theory ceases to make sense there and the trajectory describes an effective low-energy field theory.

Let us now return to the case of free matter in an external gravitational field. Taking the derivative of (60) with respect to \( \mu \) and using the definition \( R_k(\Delta) = k^2 r(\Delta/k^2) \), we obtain

\[
\frac{d \Gamma^I_k}{dk} = \text{Tr} \frac{r(\Delta/k^2) - (\Delta/k^2) r'(\Delta/k^2)}{(\Delta/k^2) + r(\Delta/k^2)}.
\]

This is a special case of the FRGE (63), and the fall-off properties of the function \( r \) guarantee that the trace on the rhs is finite.

One can repeat this argument in the case of the Weyl-invariant EAA with little changes, and the flow equation reads

\[
\frac{d \Gamma^{II}_u}{du} = \text{Tr} \frac{r(O/\mu^2) - (O/\mu^2) r'(O/\mu^2)}{(O/\mu^2) + r(O/\mu^2)}.
\]

In this form, the rhs of the FRGE is manifestly Weyl-invariant, since \( u \) is Weyl-invariant and one has the trace of a function of a Weyl-covariant operator.

If one starts from a given Weyl-invariant classical matter action at scale \( \Lambda \) and integrates the flow of \( \frac{d \Gamma^I_k}{dk} (\mu \frac{d \Gamma^{II}_u}{du}) \) down to \( k = 0 \) (\( u = 0 \)), then one obtains exactly the effective action \( \Gamma^I \) (\( \Gamma^{II} \)). Furthermore, at each \( u \), \( \Gamma^{II}_u \) is obtained from \( \Gamma^I_k \) by the St"uckelberg trick. It is instructive to explicitly illustrate these statements in the case of \( d = 2 \) and, for the \( c \)-anomaly, also in the case \( d = 4 \).

4.2.1. \( d = 2 \): the Polyakov action. As an example, consider the effective action \( \Gamma^I \) of a scalar field coupled to gravity in two dimensions. Expanding in powers of curvature, one can make the ansatz

\[
\Gamma^I_k = \int d^2 x \sqrt{g} [a_k + b_k R + R c_k(\Delta) R] + O(R^2),
\]

where \( c_k(\Delta) \) is a nonlocal form-factor which, for dimensional reasons, can be written in the form \( c_k(\Delta) = \frac{1}{\chi^2} c(\Delta) \) and \( \Delta = \Delta/k^2 \). The FRGE for the function \( c_k \) has been derived in [19] using nonlocal heat kernel techniques [21]. Integrating the FRGE down to \( k = 0 \), one reobtains the Polyakov action [18]. The running of \( c_k \) for three different cutoff types is shown in figure 1.

This calculation can be repeated for the action \( \Gamma^{II} \). Using the Weyl-invariant measure, the effective action is given by the determinant of the dimensionless operator \( O = \Delta = \frac{1}{\tilde{\chi}^2} \Delta \), which can be identified with \( \Delta_k \), the operator constructed with the dimensionless, Weyl-invariant metric \( \tilde{g}_{\mu\nu} = \chi^2 g_{\mu\nu} \). Therefore, as already discussed, \( \Gamma^{II} \) differs from \( \Gamma^I \) just in the replacement of \( \mu^2 g_{\mu\nu} \) by \( \chi^2 g_{\mu\nu} \). We have to generlize this for finite \( k \neq 0 \). As discussed
above, we assume that the cutoff is a constant multiple of the dilaton: \( k = u \chi \). Neglecting the \( a \) and \( b \) terms, the EAA can then be written in the manifestly Weyl-invariant form:

\[
\Gamma_{\mu}^{\Pi}(g_{\mu\nu}, \chi) = \int d^2 x \sqrt{g} R \frac{1}{\chi^2} c \left( \frac{O}{u^2} \right) R.
\]

with the same function \( c \) appearing in [19]. In particular, the Weyl-invariant version of the Polyakov action is obtained in the limit \( u \to 0 \):

\[
\Gamma_{\mu}^{\Pi} = -\frac{1}{96 \pi} \int d^2 x \sqrt{g} R \frac{1}{\chi^2} R.
\]

It is now easy to check explicitly equation (52): for \( c = 1 \), using \( R = R + 2 \Delta \sigma \), one finds

\[
\Gamma_{\mu}^{\Pi} = -\frac{1}{96 \pi} \int d^2 x \sqrt{g} R \frac{1}{\chi^2} R = \Gamma_{\mu}^{\Pi} + \Gamma_{WZ}.\]

We have claimed in the end of section 3.4 that the traces of the energy–momentum tensors computed from \( \Gamma_{\mu}^{\Pi} \) and \( \Gamma_{\mu}^{I} \) coincide in the gauge \( \chi = \mu \). This statement actually holds also for \( k \neq 0 \). A direct calculation yields

\[
[T_{\mu
u}, \mu] = -\frac{2}{\sqrt{g}} g^{\mu
u} \frac{\delta \Gamma_{\mu}^{\Pi}}{\delta g_{\mu\nu}} = -4c \left( \frac{O}{u^2} \right) R - \frac{2}{u^2} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} c_n \left( \frac{u^{2k}}{O^k} R \right) \left( \frac{u^{2(n-k)}}{O^{n-k}} R \right).
\]

One can verify that this is also equal to \( \frac{1}{\sqrt{g}} \frac{\delta \Gamma_{\mu}^{\Pi}}{\delta \chi} \), thereby obtaining an explicit check of the general statement (48). It is also interesting to observe that if we think of \( \Gamma_{\mu}^{\Pi} \) as a function of \( k \), \( \chi \) and \( g_{\mu\nu} \), and vary each keeping the other two fixed, then the metric variation is again given by equation (71), the \( \chi \) variation gives the first term on the rhs of (71) and the \( k \) variation gives the second term. We also note that the ‘beta functional’ can be written in general as

\[
u \frac{d \Gamma_{\mu}^{\Pi}}{du} = -\int d\chi \sqrt{g} \frac{2}{u^2} R c \left( \frac{O}{u^2} \right) R.
\]

\[d = 4: \text{the } c\text{-anomaly action.} \] One would like to repeat the analysis of the previous section in \( d = 4 \) as much as possible. The main difference is that while in \( d = 2 \), the Polyakov action is the full effective action, in \( d = 4 \) there are terms with higher powers of curvature.
We limit our analysis to second order in the curvatures. The EAA $\Gamma^1_I$ has been computed in [22] and when $k \to 0$, after changing the basis expansion from powers of $(R, C_{\mu\nu\rho\sigma})$ and their derivatives to powers of $(R, R_{\mu\nu})$ and their derivatives, the result can be compared to those found in [20]. The computation shows that the first term of the effective action has the form suggested by Deser and Schwimmer [23] as the source of the $c$-anomaly, namely the terms proportional to $C^2_{\mu\nu\rho\sigma}$ in (38). This action (in contrast to the Riegert action discussed below) also produces the correct flat spacetime limit for the correlation functions of the energy–momentum tensor $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ [24]; see also [25].

In the basis of the tensors $(R, R_{\mu\nu})$, the terms cubic in curvature are known explicitly [26]. When the Riemann squared term in the anomaly is expanded in an infinite series in $(R, R_{\mu\nu})$, the action of [26] correctly reproduces the first terms of this expansion [27]. In order to reproduce the full anomaly (both $c$ and $a$ terms), one would also need terms in the effective action of order higher than 3.

It is possible to write closed form actions that generate the full anomaly. A functional that generates both $c$- and $a$-anomaly is the Riegert action [28]

$$W(g_{\mu\nu}) = \int dx \sqrt{\frac{1}{8} \left( E - \frac{2}{3} \Box R \right) \Delta^{-1}_4 \left[ 2c C^2 - a \left( E - \frac{2}{3} \Box R \right) \right] + \frac{a}{18} R^2}.$$  

(73)

It has the drawback that it gives zero for the flat spacetime limit of the correlator of two energy–momentum tensors. Since the Deser–Schwimmer action can be written as the Riegert action plus Weyl-invariant terms, one can still write the full effective action as the sum of the Riegert action and Weyl-invariant terms. In this case, the energy–momentum correlator comes from the Weyl-invariant terms.

The relation between the WZ term (55) and the Riegert action (73) is very similar to the one between the two-dimensional WZ action (54) and the Polyakov action: using the Riegert action in (50), one recovers the WZ action (55). Unlike the two-dimensional case, however, the converse procedure is not unique. The general idea is to replace the dilaton $\chi = \mu e^\sigma$, which in the WZ action is treated as an independent variable, by a functional of the metric $g_{\mu\nu}$ having the right transformation properties. One choice, which has been proposed in [3, 30], is

$$\sigma(g_{\mu\nu}) = \log \left( 1 - \frac{1}{\Delta + R/6} \right).$$  

(74)

Another possibility is

$$\sigma(g_{\mu\nu}) = -\frac{1}{4} \frac{1}{\Delta_4} \left( E + \frac{2}{3} \Delta R + b C^2 \right),$$  

(75)

where $b$ is an arbitrary constant. In both cases, $\sigma(g_{\mu\nu}) \to \sigma(g_{\mu\nu}) - \log \Omega$ under a Weyl transformation. Note that (75), for $b = c$, is the equation of motion for the dilaton coming from the WZ action (55), while for $b = 0$, it is the equation of motion coming from the $a$ term of the WZ action. The latter choice reproduces (73); other choices of $b$ give the Riegert action plus Weyl-invariant terms, while (74) gives another form of the anomaly functional.

In $d = 2$, knowing the explicit form of the effective action, we were able to check equation (52). In $d = 4$, we have only limited knowledge of the effective action. Instead of trying to check equation (52), we can use it to obtain some additional information on the effective action $\Gamma^1$:

$$\Gamma^1(g_{\mu\nu}) = \Gamma^\Pi(g_{\mu\nu}, \chi) - \Gamma_{WZ}(g_{\mu\nu}, \chi),$$  

(76)

where the first term on the rhs is Weyl-invariant by construction and the anomaly comes entirely from the second term. For example, if we use (75) with $b = 0$, the second term exactly
reproduces the Riegert action and the correlator of two energy–momentum tensors must come from the first term. One finds
\[ \Gamma_{II}(g_{\mu\nu}, \mu e^{O(g_{\mu\nu})}) = -\frac{1}{2} \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left( \frac{N_0 + 6N_{1/2} + 12N_1}{120} C_{\mu\nu\rho\sigma} \log \left( \frac{O}{u^2} \right) C^{\mu\nu\rho\sigma} + \cdots \right), \]
(77)
where \( C_{\mu\nu\rho\sigma} \) is the Weyl tensor constructed with the metric \( e^{2\sigma(g_{\mu\nu})} g_{\mu\nu} \). Expanding this to second order in the curvature of \( g_{\mu\nu} \), one re-obtains as a leading term the one proposed by Deser. The Weyl non-invariance of this term is compensated by higher terms in the expansion. This shows that there is no contradiction between the presence of the Riegert and the Deser–Schwimmer terms in the effective action \( \Gamma_{I} \), and the flat space limit of energy–momentum tensor correlators. Thus, there is also no disagreement with [29] and [30].

Finally, following [22], we can write the explicit form of the interpolating EAA \( \Gamma_{II}^{u} \). For a scalar field, we have
\[ \Gamma_{II}^{u}(g_{\mu\nu}, \chi) = -\frac{1}{2} \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} C_{\mu\nu\rho\sigma} \left\{ \frac{1}{120} \log u^2 + \theta \left( \frac{O}{u^2} - 4 \right) \left[ -\frac{1}{120} \log u^2 + \frac{4\mu^2 \sqrt{O}}{75O^2} - \frac{11\mu^4 \sqrt{O}}{225O^2} - \frac{23}{900} \sqrt{1 - \frac{4\mu^2}{O}} \right] + \frac{1}{120} \log \left( \frac{O}{2} \left( \sqrt{1 - \frac{4\mu^2}{O}} + 1 \right) - u^2 \right) \right\} C^{\mu\nu\rho\sigma} + \cdots \]  
(78)

The function \( f_C \) given by the curly bracket is plotted in figure 2. The same computation can be repeated in the case of fermions and vectors and a different interpolating function can be found. When \( u \rightarrow 0 \), we get back equation (77).

5. Interacting matter fields

In the preceding sections, we have shown that there exists a quantization procedure such that the effective action which is obtained by integrating out free (Gaussian) matter fields remains Weyl-invariant. The proof was simple because the integration over matter was Gaussian. Here, we discuss the generalization to the case when there are matter interactions.

As in the preceding section, we begin by considering the case when the initial matter action is Weyl-invariant even without invoking a coupling to the dilaton. This is the case for massless,
renormalizable quantum field theories such as $\phi^4$, Yang–Mills theory and fermions with Yukawa couplings in $d = 4$. The interactions are of the form $S_{\text{int}}(g_{\mu\nu}, \Psi) = \lambda \int dx \sqrt{g} \mathcal{L}_{\text{int}}$, where $\mathcal{L}_{\text{int}}$ is a dimension $d$ operator and $\lambda$ is dimensionless. Interactions generate new anomalous terms over and above those that we have already considered for Gaussian matter. The trace anomaly of free matter vanishes in the limit of flat space, but this is not true for interacting fields: the trace is then proportional to the beta function. For the interaction term given above, one has in flat space

$$\int dx \omega(T^\mu_\mu) = -\delta_w S_{\text{int}} = \int dx \omega \beta_\lambda \mathcal{L}_{\text{int}}, \quad (79)$$

where $\beta_\lambda = k \frac{\delta \omega}{\delta \lambda}$. We want to study the effective action which is obtained by integrating out the matter fields. In order to be able to make non-perturbative statements, we will use the FRGE as a machine for calculating the effective action and use it to construct a Weyl-invariant EAA $\Gamma_{\text{E}}^\Pi_k$.

This goal is seemingly in contrast with (79), which implies that Weyl invariance can only be achieved when all beta functions are zero. How can one maintain Weyl invariance along a flow? As we saw in section 4, the trick is to consider the flow as the dependence of $\lambda$ on the dimensionless parameter $u = k/\chi$. In the spirit of Weyl’s geometry, the dilaton is interpreted physically as the unit of mass and $u$ is the cutoff measured in the chosen units. We assume $u$ to be constant to avoid issues related to the interpretation of a coupling depending on a function\(^{11}\). Since $u$ is Weyl-invariant, also $\lambda(u)$ is. We now see that with this definition of RG, the running of couplings does not in itself break Weyl invariance: $\delta_w S_{\text{int}} = 0$ even when the beta function is not zero.

It is important to stress that this should not be interpreted as a vanishing trace of the energy–momentum tensor. We argued in section 3.4 that the energy-momentum tensor is the same whether one uses the standard or the Weyl-invariant measure. That argument applies here too. So, the physical content of the Weyl-invariant theory is exactly the same as in the usual formalism. The recovery of Weyl invariance is due to additional terms that involve the variation of the action with respect to the dilaton.

Let us return now to the issue of the Weyl invariance of the flow. As in section 4, in order to have Weyl invariance in the presence of the cutoff $k$, the latter must be transformed as a field of dimension 1. Then, one can construct a Weyl-invariant cutoff action. Since the cutoff action is always quadratic in the quantum fields, one can use exactly the same procedure that we followed in the case of free fields. The rhs of the FRGE is given in (63) as a trace of a function involving the Hessian and the $k$-derivative of the cutoff. If the field $\phi$ has weight $w$, then the two factors on the rhs of (63) have the transformation properties:

$$\frac{\delta^2 (\Gamma_k + \Delta S_k)}{\delta \phi \delta \phi} \mapsto \Omega^{-w} \frac{\delta^2 (\Gamma_k + \Delta S_k)}{\delta \phi \delta \phi}, \quad k \frac{d}{dk} \frac{\delta^2 \Delta S_k}{\delta \phi \delta \phi} \mapsto \Omega^{-w} k \frac{d}{dk} \frac{\delta^2 \Delta S_k}{\delta \phi \delta \phi} \Omega^{-w}.$$ 

As a consequence, the trace on the rhs of (63) is invariant. Since the beta functional is Weyl-invariant, if we start from some initial condition that is Weyl-invariant, we will remain within the subspace of theories that are Weyl-invariant. The effective action $\Gamma^\Pi_k$, which is obtained as the limit of the flow for $k \to 0$, will also be Weyl-invariant.

The advantage of the calculation based on the FRGE is that it extends easily to arbitrary theories. Let us relax all constraints on the functional form of the action and suppose that we know the form of the EAA $\Gamma_k$ at some (constant) UV scale $k = u\mu$. It gives rise, via its flow, to an effective action $\Gamma^\Pi$. Using the Stückelberg trick, we can rewrite the UV action in a Weyl-invariant form and take it as the initial point of the flow at cutoff $k = u\chi$ (where $\chi = \mu$

\(^{11}\) Note that in this way the couplings will remain constant in spacetime. In this sense, our approach differs from those in [31, 9, 13], where the couplings are allowed to become functions on spacetime.
is some special gauge). Flowing toward the IR from this starting point leads to an effective action $\Gamma_{\text{II}}$ that is still Weyl-invariant. When $\Gamma_{\text{II}}$ is evaluated at a constant $\chi = \mu$, it agrees with the effective action $\Gamma_{\text{I}}$ evaluated in the Weyl-non-invariant flow. In other words, $\Gamma_{\text{II}}$ could be obtained from $\Gamma_{\text{I}}$ using the Stückelberg trick. We thus see that quantization commutes with the Stückelberg trick\textsuperscript{12}.

As mentioned earlier, renormalizability is not required for these arguments, because the FRGE is UV finite. Divergences manifest themselves when one tries to solve for the flow toward large $k$. The question whether this theory has a sensible UV limit can be answered by studying the flow for increasing $u$. If the trajectory tends to a UV fixed point, then it is called a ‘renormalizable’ or ‘asymptotically safe’ trajectory. If instead the trajectory diverges in the UV, then it describes an effective field theory with a UV cutoff scale. We will address in section 7 the meaning of a fixed point in a theory space consisting of Weyl-invariant actions.

### 6. Dynamical gravity

To treat also the case in which gravity is dynamical, we have to be able to ‘quantize gravity’. If one does not insist on UV completeness, using the background field method, quantum general relativity can be seen as a (perturbatively nonrenormalizable) effective field theory of a spin-2 field propagating on a curved manifold, not unlike the general theories discussed in the previous sections\textsuperscript{13}. As discussed for example in [34], the background field method actually guarantees that the theory is ‘background independent’ in the sense that no background plays a special role. As long as one stays below the UV cutoff of the theory, a finite number of loops is sufficient to describe the data with a predefined precision, and only finitely many divergences are encountered (i.e. the theory is predictive).

One way in which such a theory could be UV complete is if the RG has a nontrivial fixed point and the world corresponds to a renormalizable trajectory. The advantage of such a situation is that if the attraction basin of the fixed point is finite dimensional, it places infinitely many constraints on experiments at any energy scale, and is therefore highly predictive\textsuperscript{14}. This possibility, however, is not essential for our main result.

The background field method can be used, in the presence of a dilaton, to write an EAA that is Weyl-invariant [35].\textsuperscript{15} This general conclusion can be illustrated explicitly by a one-loop calculation in the Einstein–Hilbert truncation. Applying the Weyl covariantization (Stückelberg) procedure described in section 3, the Einstein–Hilbert action can be rewritten as

$$\Gamma_{\text{II}}(g, \chi) = \int d^4x \sqrt{g} \left[ \lambda Z^2 \chi^4 - \frac{1}{12} Z \chi^2 R \right].$$

The running of the couplings $Z$ and $\lambda$ has been computed in [35]. In the background gauge $\chi = \mu$, one finds the same EAA one would have obtained by integrating with the Weyl-non-invariant measure where $\chi$ is replaced by $\mu$ and with the action written in the same gauge. Thus, also in the case of dynamical gravity, one sees that the Stückelberg procedure of Weyl covariantizing an action commutes with quantization.

\textsuperscript{12} The relation between $\Gamma_{\text{I}}$ and $\Gamma_{\text{II}}$ will always be as in (52), but in the general case $\Gamma_{\text{I}}$, and consequently also the WZ action, will contain infinitely many Weyl-non-invariant terms.

\textsuperscript{13} Ordinary perturbation theory is a special case where the background is flat.

\textsuperscript{14} This is the logic that led to the standard model of particle physics.

\textsuperscript{15} For the coupling of spinor fields one should use a frame field rather than a metric. We refer to [32, 33] for a recent discussion.
The solution of the one-loop flow equations in the presence of minimally coupled matter [38–40], together with the IR boundary conditions $Z(0) = Z_0$, $\lambda(0) = \lambda_0$, is

$$Z(u) = Z_0 + \frac{23 + 2n_M - n_D}{8\pi^2} u^2, \quad (81)$$

$$\lambda(u) = \frac{\pi^2 ((2 + n_S + 2n_M - 4n_D)u^4 + 128\pi^2 Z_0^2 \lambda_0)}{2(8\pi^2 Z_0 + (23 + 2n_M - n_D)u^2)^2}. \quad (82)$$

The gravitational fixed point that one finds in the Einstein–Hilbert truncation [17, 36, 37] corresponds to the limit of large $u$, where $\lambda(u) \rightarrow \lambda_* = \frac{\pi^2 ((2 + n_S + 2n_M - 4n_D)u^4 + 128\pi^2 Z_0^2 \lambda_0)}{2(8\pi^2 Z_0 + (23 + 2n_M - n_D)u^2)^2}$ and $Z(u) \rightarrow Z_* u^2$, with $Z_* = \frac{23 + 2n_M - n_D}{8\pi^2}$. This is consistent with the notion of a fixed point because the wavefunction renormalization $Z$ is a redundant coupling [41].

One can compute the one-loop contribution of gravitons to the effective action by using non-local heat kernel techniques. The first term in the curvature expansion is similar to the one given in section 4.2.2 for a scalar field.

### 7. Discussion

In the discussions of conformal invariance, misunderstandings frequently arise due to the different physical interpretations of the transformations that are used by different authors. In particle physics language, a theory that contains dimensionful parameters is obviously not conformal. Thus, conformal invariance is a property of a very restricted class of theories. In particular, in quantum field theory, the definition of the path integral generally requires the use of dimensionful parameters (cutoffs, renormalization points) which break conformality even if it was present in the original classical theory. True conformality is only achieved at a fixed point of the RG. Let us call this the point of view I.

On the other hand, in Weyl’s geometry and its subsequent ramifications, conformal (Weyl) transformations are usually interpreted as relating different local choices of units. Since the choice of units is arbitrary and cannot affect the physics, it follows that essentially any physical theory can be formulated in a Weyl-invariant way. This point of view is more common among relativists. Let us call it the point of view II.

The way in which a generic theory containing dimensionful parameters can be made Weyl-invariant is by allowing those parameters to become functions on spacetime, i.e. to become fields. This is the step that the adherents of the interpretation I are generally unwilling to make, since then one would have to ask whether these fields have a dynamics of their own or not, and, in the quantum case, whether they have to be functionally integrated over or not. It can be unnatural to have fields in the theory that do not obey some specific dynamical equation, and it is clear that in general, if one allows all the dimensionful couplings to become dynamical fields, the theory is physically distinct from the original one.

There is however one way in which Weyl invariance can be introduced in any theory without altering its physical content, and that is to introduce a single scalar field, which we called a dilaton (sometimes also called a ‘Stückelberg’ or ‘Weyl compensator’ or ‘spurion’ field) and to assume that all dimensionful parameters are proportional to it. This field carries a nonlinear realization of the Weyl group, since it is not allowed to become zero anywhere. Even though the new field obeys dynamical equations, it does not modify the physical content of the theory because it is exactly neutralized by the enlarged gauge invariance. In practice, it can be eliminated by choosing the Weyl gauge such that it becomes constant.

All this is well known in the classical case. It had already been observed both in a perturbative and nonperturbative context that the above considerations can be generalized to the context of quantum field theory by treating the cutoff or the renormalization point in the
same way as the mass or dimensionful parameters that are present in the action. In this paper, we have discussed in particular the formulation of the RG using the point of view II. It has proven convenient to adopt a non-perturbative definition of the RG, where one considers the dependence of the effective action on an externally prescribed smooth cutoff $k$. The advantage of this procedure is that the resulting ‘beta functional’ is both UV and IR finite and one can use it to define a first-order differential equation whose solution, for $k \to 0$, is the effective action. It can therefore be viewed as a non-perturbative way of defining (and calculating) the effective action. Using this method, we have shown in complete generality that one can define a flow of Weyl-invariant actions whose IR endpoint is a Weyl-invariant effective action. It is important to emphasize that this holds also when the dilaton and metric field are quantized, at least in the background field method. This is our main result.

This provides an answer to the following question. Suppose we start from a theory that contains dimensionful parameters, and recast it in a Weyl-invariant form by the Stückelberg trick of introducing a dilaton field. If we quantize this Weyl-invariant theory, is the result equivalent to the one we would have obtained by quantizing the original theory? The answer is affirmative, if we use for all fields the Weyl-invariant measure\textsuperscript{16}. Thus, there is a quantization procedure that commutes with the Stückelberg trick.

It is important to understand that although Weyl invariance is not anomalous, there is still a trace anomaly, in the sense that the trace of the energy–momentum, which is classically zero, is not zero in the quantum theory. This can be easily understood from the fact that in the Weyl-invariant quantization, one obtains an effective action that depends not only on the metric but also on the dilaton. The Weyl invariance of the effective action is compatible with a nonvanishing trace, because the latter cancels out against the variation of the dilaton. (We have provided fully explicit examples of this phenomenon in section 4.2.1.) The physics of the Weyl-invariant quantization procedure is completely equivalent to the standard one. In particular, all the proposed physical applications of the trace anomaly remain valid\textsuperscript{[42–44]}. In order not to modify the dynamics, we have used the ‘only one dilaton’ prescription, with the consequence that the cutoff and the dilaton are proportional. The proportionality factor, which we have called $u$, is the RG parameter in this formulation. It is dimensionless (since it expresses the cutoff relative to the unit of mass), Weyl-invariant and constant on spacetime. Thus, running couplings are functions $g(u)$. In this, we differ from other approaches to dilaton dynamics where the couplings depend on a mass scale $k$ that is allowed to be a function on spacetime. In practice, the difference is not so important, because the variation of $g(u)$ with respect to $k$, keeping $\chi$ constant, is the same that one would obtain if one assumed that $g$ is a function of $k$.

Given that in this formalism all theories are conformally invariant, one can also ask what is special about conformal field theories (in the standard sense of quantum field theory), and in particular about fixed points of the RG. The answer is that for generic theories, conformal invariance is only achieved at the price of having a dilaton in the effective action. True conformal field theories are conformal even without the dilaton, so one must expect that as the RG flow approaches a fixed point, the dilaton must decouple.

Weyl invariance is the statement of conformal invariance in a general relativistic setting, so we expect the formalism developed here to be especially relevant in the discussion of a possible

\textsuperscript{16} By contrast, suppose that after having quantized the matter fields we also quantize the metric and dilaton, using the standard, Weyl-non-invariant measure I. (One does not need to have a full quantum gravity for this argument, it is enough to think of a one-loop calculation in the context of an effective field theory). The integration over metric and dilaton will now proceed with total actions $S_M + I^\text{I}$ and $S_M + I^\text{II}$, depending on whether we used for the matter measure I or II. Clearly, the resulting theories are physically inequivalent. In the first case, the action is not Weyl-invariant, so the dilaton field is physical, and in the second case, the action is Weyl-invariant and the dilaton can be gauged away. So, all else being equal, quantizing matter with measures I and II leads to physically different theories.
fixed point for gravity. Barring exotic phenomena, one would expect that a gravitational fixed point must correspond to a Weyl-invariant, as opposed to merely globally scale-invariant theory. We have illustrated in section 6.1 the appearance of a gravitational fixed point in the case of the Einstein–Hilbert truncation. It would be interesting to extend the discussion to higher derivative terms. In particular, it has been observed in [45] that in an $f(R)$-truncation, a fixed point would correspond to an effective action that is proportional to $R^2$. As such, this would only be globally scale-invariant. We conjecture that the fixed point is conformal and that this term should be interpreted as a piece of the Euler term, which is the only local, Weyl-invariant combination of curvatures besides the square of the Weyl tensor. Consistent with this, we observe that the RG trajectory that corresponds to the fixed point discussed in section 6.1 corresponds to putting $Z_0 = 0$ in (81). The corresponding EAA is given by (80) with $Z(u) = Z_u u^2$ and $\lambda(u) = \lambda_u$. When we take the $u \to 0$ limit, to obtain the effective action, we find zero.

We conclude by mentioning some possible extensions and applications of this work. Recalling that here we have analyzed the case of integrable Weyl geometry, it is natural to extend our results to the non-integrable case. This work is in progress. The example in section 6.2 deals with a truncation of the gravitational action to terms involving at most two derivatives. In four dimensions, such terms necessarily have dimensionful couplings. In view of the remarks in the preceding paragraph, it seems interesting to re-analyze the terms with four derivatives, in particular the Weyl-squared and the Euler term. Perhaps the most important applications of Weyl geometry is to cosmology, where it is often useful to change a conformal frame. Even at a classical level, there has been some controversy on the issue whether such frames should be interpreted as defining physically equivalent situations. Our point of view agrees with that of [46]. The question is, however, much more delicate in the quantum theory. For explicit quantum calculations where different conformal frames can be seen to yield equivalent physics, see [47]. The present work provides a general proof that with a suitable quantization procedure, the equivalence between conformal frames can also be maintained in the quantum theory. One could use this to study the relation between $f(R)$ and scalar–tensor theories at the quantum level.

References

[1] Capper D M and Duff M J 1974 Nuovo Cimento A 23 173
Capper D M and Duff M J 1975 Phys. Lett. A 53 361
Deser S, Duff M J and Isham C J 1976 Nucl. Phys. B 111 45
Duff M J 1977 Nucl. Phys. B 125 334
Duff M J 1994 Class. Quantum Grav. 11 1387 (arXiv:hep-th/9308075)

[2] Englert F, Truffin C and Gastmans R 1976 Nucl. Phys. B 117 407

[3] Fradkin E S and Vilkovilsky G A 1978 Phys. Lett. B 73 209

[4] Floreanini R and Percacci R 1995 Nucl. Phys. B 436 141 (arXiv:hep-th/9305172)
Floreanini R and Percacci R 1995 Phys. Rev. D 52 830 (arXiv:hep-th/9412181)

[5] Reuter M and Wetterich C 1997 Nucl. Phys. B 506 483–520 (arXiv:hep-th/9605039)
Reuter M 1996 arXiv:hep-th/9612188

[6] Reuter M and Weyers H 2009 Phys. Rev. D 79 105005 (arXiv:0801.3287 [hep-th])
Reuter M and Weyers H 2009 Phys. Rev. D 80 025001 (arXiv:0804.1475 [hep-th])

[7] Machado P F and Percacci R 2009 Phys. Rev. D 80 024020 (arXiv:0904.2510 [hep-th])

[8] Shaposhnikov M and Zenhausern D 2009 Phys. Lett. B 671 162 (arXiv:1104.1392 [hep-th])
Shaposhnikov M and Tkachev I 2009 Phys. Lett. B 675 403 (arXiv:0811.1967 [hep-th])

[9] Komargodski Z and Schwimmer A 2011 J. High Energy Phys. JHEP12(2011)099 (arXiv:1107.3987)
Komargodski Z 2012 J. High Energy Phys. JHEP12(2012)069 (arXiv:1112.4538)

[10] Iorio A, O’Raifeartaigh L, Sachs I and Wiesendanger C 1997 Nucl. Phys. B 495 433–50

[11] Jackiw R and Pi S-Y 2011 J. Phys. A: Math. Theor. 44 223001 (arXiv:1101.4886 [math-ph])
El-Showk S, Nakayama Yu and Rychkov S 2011 Nucl. Phys. B 848 578–93 (arXiv:1101.5385 [hep-th])
