Welfare and Distributional Effects of Joint Intervention in Networks*

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Abstract

We study a planner’s optimal interventions in both the standalone marginal utilities of players on a network and weights on the links that connect players. The welfare-maximizing joint intervention exhibits the following properties: (a) when the planner’s budget is moderate (so that optimal interventions are interior), the change in weight on any link connecting a pair of players is proportional to the product of eigen-centralities of the pair; (b) when the budget is sufficiently large, the optimal network takes a simple form: It is either a complete network under strategic complements or a complete balanced bipartite network under strategic substitutes. We show that the welfare effect of joint intervention is shaped by the principal eigenvalue, while the distributional effect is captured by the dispersion of the corresponding eigenvalues, i.e., the eigen-centralities. Comparing joint intervention in our setting with single intervention solely on the standalone marginal utilities, as studied by Galeotti et al. (2020), we find that joint intervention always yields a higher aggregate welfare, but may lead to greater inequality, which highlights a possible trade-off between the welfare and distributional impacts of joint intervention.

JEL Classification: D21; D29; D82.

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1 Introduction

Network models play an important role in the analysis of markets with spillover effects. A significant part of the literature focuses on identifying the optimal actions of a planner to benefit society; a key paper in this field is that of Ballester, Calvó-Armengal, and Zenou (2006). In practice, a planner can intervene with the market in various ways, such as by

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changing incentives or modifying the network structure. The planner could also consider different objectives, such as increasing utility and/or decreasing inequality. While many papers have analyzed such interventions individually, few studies have combined and compared the various outcomes. We aim to shed light on the interactions and trade-offs between various intervention schemes.

This paper studies a model in which a benevolent social planner maximizes the total welfare of players embedded in a weighted network. Given that players’ incentives are driven by private returns and peer effects, we allow the planner to simultaneously intervene in two dimensions. First, the planner can perform a characteristic intervention on the players’ standalone marginal utilities, in the same way as Galeotti, Golub, and Goyal (2020). This models the ability of the planner to create incentives or disincentives for specific players to influence their actions directly. Second, we also allow the planner to modify the network structure itself by changing the intensity of the links in the network, which indirectly influence players’ actions. Since intervening in either dimension is costly to the planner, the optimal way to allocate a fixed amount of budget in this joint intervention problem is the main focus of this study.

Our model reveals complex interactions between two types of interventions, since the marginal returns to characteristic interventions clearly depend on the strength of peer effects. In contrast, the marginal returns to intervening in network weights hinge on the centralities of players through their private incentives and social links. We summarize these interactions in a set of necessary conditions for optimality in Lemma 1 using variational methods.

Further specialising to the case in which the original standalone marginal utilities are negligible, we obtain in Theorem 1 that the optimal change in the intensity of a link is proportional to the product of eigen-centralities of the pair of players involved. Here the eigen-centralities are the entries of the eigenvector associated with the first principal eigenvalue when activities are strategic complements or the last principal eigenvalue when activities are strategic substitutes. In the former case with strategic complements, the eigen-centralities are of the same sign, which implies that network weights increase under the optimal intervention. Furthermore, players who are central in terms of eigen-centrality experience the largest increase in the intensity of their links. In the latter case with strategic substitutes, eigen-centralities are of mixed signs. Thus, we can use their signs to obtain a natural partition of the network into two subsets and show that the weights of links across both subsets increase, while the weights within subsets decrease.

Theorem 1 also provides a clue to the topology of the optimal network for large budgets: either a complete network in the former case, or a bipartite shape in the latter case. As one of our key results, Theorem 2 formally shows that if the planner’s budget is suffi-
ciently large, the optimal network structure takes on one of two forms: If the activity levels chosen are strategic complements for the players, the planner will choose to create a complete graph. On the other hand, if the activity levels chosen are strategic substitutes, the planner will instead opt for a complete bipartite graph of balanced sizes, i.e., with half of the players on each side. This provides a simple solution for the planner’s structural intervention, which holds regardless of the initial state of the network. This result is driven by the observation, already noted by Galeotti, Golub, and Goyal (2020), that this shadow price of budget is approximately an increasing function of the first principal eigenvalue with strategic complements or a decreasing function of the last principal eigenvalue with strategic complements. Thus, the optimal network, under large budgets, must either maximize the first principal eigenvalue in the former case or minimize the last principal eigenvalue in the latter. The optimal network in either setting is identified in Lemma 2. We also study the optimal configuration of the complete balanced bipartite graph in the case of strategic substitutes and briefly discuss the computational complexity in Proposition 1 by linking the configuration problem to the well-known maximum cut problem.

In Section 4, we analyze the welfare and distributional effects of joint intervention using spectral decompositions of the network. In Proposition 2, we show that the welfare effects of joint intervention is monotonic in the principal eigenvalue, evaluated at the post-intervention network. The same is true in the setting of Galeotti, Golub, and Goyal (2020), with the difference that the network structure is exogenous in their setting but endogenously given in our setting. Regarding distributional effects, we adopt the Theil T index, a statistic primarily used to measure payoff inequality. In Proposition 3 we show that payoff inequality is related to the entropy of the eigen-centralities of the post-intervention network when the planner’s budget is large. The reason is that with large budgets, the relative actions in equilibrium equal the relative standalone marginal utilities, which equal the relative eigen-centralities. Thus, greater dispersion of eigen-centralities leads to larger payoff inequality.

Combining Propositions 2 and 3, we perform a comparison of our results obtained under joint intervention with the optimal intervention without network design derived by Galeotti, Golub, and Goyal (2020). By comparing the eigenvalues in the post-intervention network in our setting and the initial network in Galeotti, Golub, and Goyal (2020), we characterize the additional welfare gained from network design in Theorem 3. We find that there is a significant benefit to having network design, and the gain can be arbitrarily large under certain choices of parameters. We also analyze the differences in terms of payoff distribution. When budgets are large, payoff inequality can be fully eliminated in joint intervention, since the corresponding optimal network (either the complete network or the complete balanced bipartite network) features equal eigen-centralities. On
the other hand, payoff inequality can persistently stay above zero under single intervention, as the network structure (and its eigen-centralities) cannot be adjusted.

However, when budgets are moderate, we show in Theorem 4 that joint intervention can lead to a larger inequality compared with single intervention. This theorem points out a possible trade-off to be considered by the planner between the goals of maximizing aggregate welfare and minimizing inequality. We also briefly discuss the welfare cost of equality in Proposition 4 in a modified setting in which the planner is forced to induce zero payoff inequality.

Related Literature

Our paper adopts the linear quadratic model of Ballester, Calvó-Armengal, and Zenou (2006) and Bramoullé, Kranton, and D’Amours (2014) to determine players’ activity levels and welfare and contributes to the growing literature on both incentive targeting and structural interventions. Optimal targeting can take the form of network-based discriminatory pricing, as found in Candogan, Bimpikis, and Ozdaglar (2012) and Bloch and Quérou (2013), in which players are offered personalized prices depending on their centrality in the network. Demange (2017) considers more general targeting frameworks and functional forms, while Bimpikis, Ozdaglar, and Yildiz (2016) instead consider competitive targeting through advertising and the spreading of information, and observe the possibility of asymmetric equilibria. Liu (2019) studies targeting in terms of industrial policies in production networks, while King, Tarbush, and Teytelboym (2019) make use of sectoral targeting to obtain efficient carbon tax reforms. Furthermore, Galeotti, Golub, and Goyal (2020) evaluate the optimal targeting strategy by using principal component analysis on the network graph. Our work with joint intervention builds on their insights.

Interventions on the network structure have been studied in many papers since Ballester et al. (2006) and Golub and Lever (2010), who focused on identifying the key players and key links. These issues are particularly pertinent in criminal networks, as studied by Mastrobuoni and Patacchini (2012). Subsequent papers considered more general changes in the network. Belhaj, Bervoets, and Deroïan (2016) demonstrate that optimal unweighted undirected networks with a fixed number of total links are nested split graphs in general, while Li (2021) shows that the concept can also be extended to weighted and directed graphs, in which the optimal networks are considered to be generalized nested split graphs. König, Tessone, and Zenou (2014) allow for stochastic decay in the network and also find convergence to nested split graphs. Sciabolazza, Vacca, and McCarty (2020) provide empirical evidence on a collaborative network to demonstrate the importance of such structural interventions.

Another related area of the literature concerns a class of models in which the network
structure is either jointly determined by or influenced by players’ actions. For instance, Cabrales, Calvó-Armengol, and Zenou (2011) analyze a situation in which players decide on both the network structure by their socialization levels and activity levels, and show the possibility of multiple equilibria in this case. More recently, Sadler and Golub (2021) study a more general formulation with endogenous networks and show that the equilibrium graphs are nested split graphs and overlapping cliques.

Rogers and Ye (2021) compare the outcomes of an endogenous equilibrium network with the socially efficient network chosen by the planner, and show that under certain assumptions, the two networks coincide when players receive appropriately chosen benefits from the links they create. Baumann (2021) obtains a similar result for “reciprocal” equilibria in which the investment in each link is equal for the two players involved. Focusing on the structural properties of the equilibrium networks under a weighted link formation, Bloch and Dutta (2009) show that stable and efficient networks are stars, while Kinateder and Merlino (2022) identify the complete core-periphery network as the equilibrium outcome in a public good setting. Ding (2020) considers a general model that contains link investment substitutability and finds a variety of possible equilibrium networks that depend on the initial parameters of the model. In a two-sided platform setting, Carlson (2020) studies the optimal design of a bipartite network.

While an equivalence between characteristic and structural interventions has previously been studied by Sun, Zhao, and Zhou (2021), the presence of a budget constraint in our model reveals richer interaction between these two types of interventions. Relatedly, Hendricks, Piccione, and Tan (1995) analyze a monopoly’s problem in simultaneously choosing both the structure of an airline network and pricing decisions for routes, and attribute the optimality of hub-and-spoke structures to economies of traffic density. Our paper offers a different welfare-maximizing objective and also broadens the strategic interactions to allow for both complements and substitutes. Due to different underlying economic forces, the structure of the optimal network in our setting departs from that of Hendricks, Piccione, and Tan (1995).

Our inclusion of network design can also be viewed as a follow-up of Galeotti, Golub, and Goyal (2020) to allow for long-run interventions. While a targeted intervention on individual players can be performed by simple pricing or advertising mechanisms (Candogan et al. 2012), an extensive intervention in the network’s design can require significant structural and technological investment that cannot easily be changed, yet can offer large improvements in total welfare (O’Connor et al. 2020). Therefore, in such applications, we can view the network to be fixed in the short run. While Galeotti, Golub, and Goyal (2020) demonstrate the advantage of using principal component analysis in shaping the optimal standalone marginal utilities in a single intervention setting with an exogenous network, our paper further shows that in a joint intervention setting, the
eigen-centralities, associated with the principal eigenvector, determine the degree of intervention in the weight of each link.

Furthermore, we identify the critical role of the principal eigen-centralities in shaping the payoff inequality under optimal interventions. Similarly, Elliot and Golub (2019) make use of spectral properties to analyze a public good in a network, and show that Pareto-efficient outcomes are precisely those in which the largest eigenvalue of the network is one. In a recent paper that identifies an interesting connection between implementation and the network literature, Ollár and Penta (2021) show that robust implementation depends crucially on the spectral radius of a network matrix that describes payoff externalities among agents, and analyze a related unweighted network design problem to minimize the spectral radius while preserving the row sums. Using a model of perceived competition, Bochet, Faure, Long, and Zenou (2021), among other things, present an axiomatic foundation of eigen-centrality in networks. While Ballester, Calvó-Armengal, and Zenou (2006) give a micro-foundation of the Katz-Bonacich centralities in a specific network game with quadratic payoffs, Sadler (2022) characterizes an ordinal ranking of nodes to develop a more robust comparative prediction of agents’ equilibrium actions in all network games of strategic complements. Together, these papers reveal the importance of various centrality measures in network economics.

The remainder of the paper is as follows. Section 2 introduces the model and the definitions and notations used throughout. Section 3 provides a characterization of the optimal intervention. Section 4 analyzes the resulting welfare and distributional effects and provides a comparison with the literature without structural interventions. Section 5 discusses some generalizations and concludes the paper. Finally, Appendix A contains proofs that are omitted in the main text.

2 Model

2.1 Setup

Consider a game on a weighted network $g$ over a set of players $N = \{1, \cdots, n\}$. Each player $i \in N$ chooses an action $x_i \in \mathbb{R}$ and receives payoff

$$\pi_i(x_i; x_{-i}) = a_ix_i - \frac{1}{2}x_i^2 + \phi \sum_{j=1}^n g_{ij}x_i x_j,$$

where $a_i$ represents player $i$’s standalone marginal utility, $g_{ij}$ denotes the weight of the link between $i$ and $j$, and $\phi$ captures the strategic interactions between players. The case $\phi > 0$ corresponds to strategic complements, while the case $\phi < 0$ corresponds

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1 See Bloch, Jackson, and Tebaldi (2021) for a comprehensive survey of commonly used centralities.
2 See, for instance, Ballester et al. (2006); Bramoullé et al. (2014); Galeotti et al. (2020).
to strategic substitutes. We use the matrix \( g = (g_{ij})_{1 \leq i, j \leq n} \) to summarize the network structure. We suppose that \( g \) is symmetric and has no self-loops, and that there exists an exogenous \( \bar{w} \) such that \( g_{ij} \in [0, \bar{w}] \) for all \( i, j \). That is, \( g \) lies in the space\(^3\)

\[
\mathcal{G}_n = \{ g \in \mathbb{R}^{n \times n} : g_{ij} = g_{ji} \in [0, \bar{w}] \text{ for all } i, j, \text{ and } g_{kk} = 0 \text{ for all } k \}.
\]

Let

\[
x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad g = \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix}.
\]

It is known that the players’ equilibrium choices of \( x^* \) in this game satisfy

\[
x^*(a, g) = (x^*_1(a, g), \cdots, x^*_n(a, g))^T = [I - \phi g]^{-1} a,
\]

subject to the regularity condition whereby the largest eigenvalue of \( \phi g \) is less than 1 (see Ballester et al. (2006); Bramoullé et al. (2014)). Furthermore, each player’s equilibrium payoff equals

\[
\pi_i(x^*(a, g)) = \frac{1}{2}(x^*_i(a, g))^2, \quad i \in N.
\]

We will later show in Remark 2 that this regularity condition is satisfied for any \( g \in \mathcal{G}_n \) whenever the following assumption holds:

**Assumption 1.**

\[
\bar{w} < \begin{cases} 
\frac{1}{\phi(n-1)} & \phi > 0; \\
-\frac{2}{\phi n} & \phi < 0 \text{ and } n \text{ is even}; \\
-\frac{2}{\phi \sqrt{n^2 - 1}} & \phi < 0 \text{ and } n \text{ is odd}.
\end{cases}
\]

The planner intervenes in both standalone marginal utilities \( a \) and weights on links \( g \) to maximize twice the total payoff:

\[
V(a, g) := 2 \sum_{i=1}^n \pi_i(x^*(a, g)) = \sum_{i=1}^n (x^*_i(a, g))^2 = a^T[I - \phi g]^{-2} a,
\]

where we use (3) in the second equality and (2) in the last equality. We assume that this intervention comes at a quadratic cost to the planner, so the planner solves

\[
\max_{a \in \mathbb{R}^n, g \in \mathcal{G}_n} V(a, g; \hat{g}, \hat{a}, C) = a^T[I - \phi g]^{-2} a
\]

subject to \( \kappa \| g - \hat{g} \|^2 + \| a - \hat{a} \|^2 \leq C \).

\( ^3 \)We discuss the case with unweighted networks in the conclusion section.
Here \( \mathbf{a} \) and \( \mathbf{g} \) are the pre-intervention standalone marginal utilities and network, respectively. \( C \) is the total budget. \( \kappa \in (0, \infty] \) is a parameter that measures the relative cost of intervening in \( \mathbf{g} \) compared with \( \mathbf{a} \). The quadratic form of intervention cost greatly simplifies computation (see Galeotti, Golub, and Goyal (2020) for related discussion), though we expect qualitatively similar results hold with alternative convex costs.\(^4\)

In the limiting case in which \( \kappa = \infty \), we recover the setting of Galeotti et al. (2020), in which the planner cannot intervene in the network design; thus \( \mathbf{g} = \mathbf{\hat{g}} \). We will refer to the intervention with exogenous \( \mathbf{g} \) in the setting of Galeotti et al. (2020) as the single intervention and the intervention with endogenous \( \mathbf{g} \) in (4) as the joint intervention.

Note that \( \| \cdot \| \) refers to the \( L_2 \) norm for vectors and the Frobenius norm for matrices—for any vector \( \mathbf{v} \in \mathbb{R}^m \) and any matrix \( \mathbf{h} \in \mathbb{R}^{m \times n} \),

\[
\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{m} v_i^2} = \sqrt{\mathbf{v}^T \mathbf{v}}, \quad \|\mathbf{h}\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} h_{ij}^2} = \sqrt{\text{Tr}(\mathbf{h}^T \mathbf{h})}.
\]

Here \( \text{Tr}(\cdot) \) refers to the trace of a matrix.

### 2.2 Notation

In this paper, for any \( p, q \in \mathbb{Z}^+ \), we write \( \mathbf{1}_p \) as the length \( p \) vector of ones, \( \mathbf{I}_p \) as the \( p \times p \) identity matrix, \( \mathbf{J}_{pq} \) as the \( p \times q \) matrix of ones, and \( \mathbf{0}_p \) as the \( p \times p \) matrix of zeroes. If subscripts are omitted, we assume the matrices to be of size \( n \times n \). Also, write \( \mathbf{K}_p \) as the complete graph represented by the adjacency matrix \( \mathbf{J}_{pp} - \mathbf{1}_p \), and \( \mathbf{K}_{pq} \) as the complete bipartite graph represented by the adjacency matrix \( \begin{pmatrix} \mathbf{0}_p & \mathbf{J}_{pq} \\ \mathbf{J}_{qp} & \mathbf{0}_q \end{pmatrix} \).

We define the inner product of two column vectors \( \mathbf{v}_1, \mathbf{v}_2 \) as \( \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1^T \mathbf{v}_2 \), and the inner product of two matrices \( \mathbf{m}_1, \mathbf{m}_2 \) of the same size as \( \langle \mathbf{m}_1, \mathbf{m}_2 \rangle = \text{Tr}(\mathbf{m}_1 \mathbf{m}_2^T) \).

Finally, for any \( p \times p \) symmetric matrix \( \mathbf{m} \) write \( \lambda_1(\mathbf{m}) \) and \( \lambda_p(\mathbf{m}) \) as the largest and smallest eigenvalues of \( \mathbf{m} \), respectively. Further, write \( \mathbf{u}^1(\mathbf{m}) \) and \( \mathbf{u}^p(\mathbf{m}) \) as representative unit eigenvectors corresponding to \( \lambda_1(\mathbf{m}) \) and \( \lambda_p(\mathbf{m}) \), respectively.\(^5\) We state the following well-known results about the complete graph and the complete bipartite graph.

**Fact 1.** (a) The largest eigenvalue of \( \mathbf{K}_p \) is \( \lambda_1(\mathbf{K}_p) = p - 1, \)\(^6\) with corresponding eigenspace span\( \{1, 1, \cdots, 1\} \). (b) The smallest eigenvalue of \( \mathbf{K}_{pq} \) is \( \lambda_{p+q}(\mathbf{K}_{pq}) = -\sqrt{pq} \), with corresponding eigenspace \( \{-\sqrt{p}, \cdots, -\sqrt{p}, \sqrt{q}, \cdots, \sqrt{q}\} \).

\(^4\)See the conclusion section for detailed discussion.

\(^5\)Pick the eigenvector arbitrarily if \( \lambda_1(\mathbf{m}) \) or \( \lambda_p(\mathbf{m}) \) occur with multiplicity larger than 1.

\(^6\)To simplify notation, we identify a graph with its adjacency matrix.
## 3 Analysis

In this section, we provide two complementary approaches to characterize the planner’s program (4). The first approach uses standard variational analysis to pin down the necessary optimality conditions in any candidate solution. In the second approach, we reformulate program (4) as a two-stage program, in which in the first stage the planner implements a post-intervention network \( g \), then in the second stage selects the optimal post-intervention standalone marginal utilities \( a \) subject to the adjusted budget (after subtracting the intervening cost of implementing \( g \)). Exploiting several key results in Galeotti, Golub, and Goyal (2020) in the second stage regarding the optimal \( a^* \) with an exogenous network \( g \) and the shadow price of the budget,\(^7\) we are able to gain insights into the optimal network endogenously selected by the planner in the first stage.

### 3.1 A variational approach

The Lagrangian of the planner’s intervention problem (4) is given by

\[
L(a, g) = a^T[I - \phi g]^{-2}a + \mu(C - \kappa\|g - \hat{g}\|^2 - \|a - \hat{a}\|^2),
\]

where \( \mu > 0 \) is the Lagrangian multiplier of the budget constraint. For brevity, we have omitted the variables \( \hat{g}, \hat{a}, C \) in \( L \). We first present the following necessary conditions for optimality.

**Lemma 1.** Let \((a^*, g^*)\) be the optimal intervention.\(^8\) Then there exists a multiplier \( \mu^* > 0 \) so that the following conditions must hold:

(i) \([I - \phi g^*]^{-2}a^* = \mu^*(a^* - \hat{a})\).

(ii) For all \( i \neq j \),

\[
\phi([I - \phi g^*]^{-1}a^* a^{*T}[I - \phi g^*]^{-2})_{ij} + \phi([I - \phi g^*]^{-1}a^* a^{*T}[I - \phi g^*]^{-2})_{ji} \begin{cases} 
= 2\mu^*\kappa(g^* - \hat{g})_{ij}, & \text{if } g^*_{ij} \in (0, \bar{w}); \\
\leq 2\mu^*\kappa(g^* - \hat{g})_{ij}, & \text{if } g^*_{ij} = 0; \\
\geq 2\mu^*\kappa(g^* - \hat{g})_{ij}, & \text{if } g^*_{ij} = \bar{w}.
\end{cases}
\]

Both parts of Lemma 1 follow from a standard variational argument by employing some matrix calculus results reported in Lemma 4 in the Appendix. Intuitively, in both equilibrium conditions, the left-hand side represents the marginal utility of an intervention

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\(^7\)When \( g \) is given, the problem in the second step is precisely the optimal targeting intervention problem as in Galeotti, Golub, and Goyal (2020).

\(^8\)The program (4) clearly has at least one solution, since \( V \) is continuous and the feasible set is compact.
in \( a \) or \( g \), and the right-hand side represents the marginal costs. Inequalities in Lemma 1 (ii) arise when the weight \( g_{ij}^\ast \) is in the corner.

The multiplier \( \mu^\ast \), representing the shadow value of budget, appears in both parts of the optimality conditions in Lemma 1 because the planner faces a key trade-off in allocating its budget between intervening in \( a^\ast \) and \( g^\ast \). Furthermore, \( \mu^\ast \) is endogenous and it depends on the budget \( C \).

Before we proceed, we will highlight the difference between positive and negative \( \phi \) with a simple example.

**Example 1.** Let \( \kappa = 0.5, \bar{w} = 1, \)

\[
\hat{g} = \begin{pmatrix} 0 & 0.3 & 0.5 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0.7 & 0 \end{pmatrix}, \hat{a} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.6 \end{pmatrix}.
\]

As \( C \) ranges from 0 to 5, the optimal interventions \( g^\ast, a^\ast \) and the maximum \( V^\ast \) are shown in Figure 1, in which the left and right columns show complements (\( \phi = 0.2 \)) and substitutes (\( \phi = -0.2 \)), respectively.

In Example 1, we see that for a sufficiently large \( C \), when \( \phi > 0 \), the optimal graph \( g^\ast \) goes to the complete graph, while when \( \phi < 0 \), the optimal graph \( g^\ast \) becomes a star \( K_{1,2} \). Also, we find that the optimal standalone marginal utilities are all increasing in the budget when \( \phi > 0 \), but \( a_2^\ast \) becomes negative when \( \phi < 0 \). Finally, welfare grows asymptotically linearly in the total budget, but the gradient is smaller for the case of negative \( \phi \). We will generalize and formally prove these results in the remainder of the paper.

### 3.2 The optimal intervention

We begin by showing that the optimal intervention in a link is closely related to the eigen-centralities of the players involved. We consider the planner’s optimization problem as a sequential one—first choosing \( g \), then choosing \( a \) subject to the remaining budget. The second stage of the problem is the focus of Galeotti, Golub, and Goyal (2020), who use principal component analysis of \( g \) to characterize the optimal intervention \( a^\ast \) under any pre-intervention standalone marginal utilities vector \( \hat{a} \). In other words, \( g \) is exogenous in Galeotti, Golub, and Goyal (2020), but endogenous in our setting. Recall that we refer to the intervention with exogenous \( g \) as the single intervention and the intervention with endogenous \( g \) as the joint intervention.
Figure 1: Optimal interventions against budget
In particular, Galeotti, Golub, and Goyal (2020) solve the problem
\[
V_{\text{single}}^*(g, \hat{a}, C) := \max_{a \in \mathbb{R}^n} \ a^T [I - \phi g]^{-2} a,
\]
s.t. \( \|a - \hat{a}\|^2 \leq C. \) (5)

Note, however, that their principal component analysis approach is not directly applicable in our setting, since the network can be also changed in the first stage of the problem. Using the solution \( V_{\text{single}}^* \) to problem (5), we select the optimal choice of \( g \) from the first-stage network design problem
\[
\begin{align*}
V_{\text{joint}}^*(\hat{g}, \hat{a}, C) &= \max_{g \in G_n} \ V_{\text{single}}^*(g, \hat{a}, C - \kappa \|g - \hat{g}\|^2).
\end{align*}
\] (6)

The following useful proposition is directly adapted from Galeotti, Golub, and Goyal (2020) to follow our notation.

**Proposition 0.** (Galeotti et al.) At the solution to (5), the shadow price of the planner’s budget \( \mu^* \) satisfies the following expression:
\[
\lim_{C \to \infty} \mu^* = \frac{1}{(1 - \lambda_1(\phi g))^2} = \begin{cases} 
\frac{1}{(1 - \phi \lambda_1(\hat{g}))^2}, & \phi > 0; \\
\frac{1}{(1 - \phi \lambda_n(\hat{g}))^2}, & \phi < 0.
\end{cases}
\] (7)

Furthermore, when \( \hat{a} = 0 \), then (7) holds for all \( C \). Also, the solution to (5) satisfies
\[
\lim_{C \to \infty} \frac{\|a^* \cdot u^1(\phi g)\|}{\|a^*\|} = 1,
\] (8)

with the value function satisfying
\[
\lim_{C \to \infty} \frac{V_{\text{single}}^*(g, \hat{a}, C)}{C} = \frac{1}{(1 - \lambda_1(\phi g))^2}.
\] (9)

Proposition 0 provides results for the second-stage problem, in which \( g \) is fixed by using the principal components of \( g \), and agrees with our result in Lemma 1(i). From (8), when \( \phi \) is positive, the optimal choice \( a^* \) must be approximately in the direction of the first eigenvector of \( g^* \). On the other hand, when \( \phi \) is negative, the optimal choice \( a^* \) is approximately in a direction corresponding to the last eigenvalue of \( g^* \).

The setting with \( \hat{a} = 0 \) is special. First, we see from Proposition 0 that the multiplier \( \mu^* \) does not depend on the budget of the planner. Second, \( a^* \) will be an eigenvector of \( g^* \) and \( [I - \phi g^*]^{-1} \), so \( a^* a^T \) commutes with \( [I - \phi g^*]^{-1} \). Thus, we can simplify the results of the first-stage problem of selecting the optimal \( g^* \).
**Theorem 1.** Assume \( \dot{a} = 0 \). Then there exists a unit eigenvector \( u^1_1(\phi g^\ast) = (u^1_1, \ldots, u^1_n)^T \) corresponding to \( \lambda_1(\phi g^\ast) \) such that for all \( i \neq j \),

\[
\begin{align*}
g_{ij}^\ast - \hat{g}_{ij} & = \begin{cases} 
\frac{\phi |a^*|^2}{\kappa(1-\lambda_1(\phi g^\ast))} u^1_i u^1_j, & g_{ij}^\ast \in (0, \bar{w}); \\
\frac{\phi |a^*|^2}{\kappa(1-\lambda_1(\phi g^\ast))} u^1_i u^1_j, & g_{ij}^\ast = 0; \\
\frac{\phi |a^*|^2}{\kappa(1-\lambda_1(\phi g^\ast))} u^1_i u^1_j, & g_{ij}^\ast = \bar{w},
\end{cases}
\end{align*}
\]

where \( ||a^*||^2 = C - \kappa ||g^\ast - \hat{g}||^2 \) from the budget constraint.

Therefore, if \( g_{ij}^\ast \in (0, \bar{w}) \) for all \( i \neq j \), the total intervention cost on the network structure satisfies the equation

\[
\kappa ||g^\ast - \hat{g}||^2 = \frac{\phi^2 (C - \kappa ||g^\ast - \hat{g}||^2)^2}{\kappa(1 - \lambda_1(\phi g^\ast))^2} \left( 1 - \sum_{i=1}^n (u^1_i)^4 \right).
\]

**Theorem 1** follows from substituting the optimal choice of \( a^* \) obtained above into the conditions derived in Lemma 1 to study the intervention on each link.

Focusing on the first case with \( g_{ij}^\ast \in (0, \bar{w}) \), Theorem 1 shows that the degree of intervention in the strength of the link between two players is proportional to the product of the eigenvector weights of both players. We make our result clearer by taking ratios:

\[
\frac{g_{ij}^\ast - \hat{g}_{ij}}{g_{ik}^\ast - \hat{g}_{ik}} = \frac{u^1_i}{u^1_k}
\]

for any \( g_{ij}^\ast, g_{ik}^\ast \in (0, \bar{w}) \) that satisfies \( u^1_k \neq 0 \). (12) shows that the ratio of interventions only depends on the relative components of the eigenvector of \( g^\ast \).

Next, note that the eigenvector \( u^1(\phi g^\ast) \) corresponds to the largest eigenvalue \( \lambda_1(g^\ast) \) under strategic complementarity \( (\phi > 0) \), while it corresponds to the smallest eigenvalue \( \lambda_n(g^\ast) \) under strategic substitution \( (\phi < 0) \). Optimal interventions in the network differ dramatically, depending on the sign of \( \phi \).

(i) When \( \phi > 0 \), by the Perron-Frobenius theorem the signs of \( u^1_i \) are the same for all \( i \), so \( g_{ij}^\ast > \hat{g}_{ij} \) for all \( i, j \) by (10). That is, the planner does not decrease the weight of any link. Intuitively, this hints that as the budget increases, the optimal graph would tend toward the complete graph (we will formalize this observation later).

(ii) On the other hand, when \( \phi < 0 \), we can identify two subsets of players\(^9\)

\[
S^+ = \{i : u^1_i > 0\}, \quad S^- = \{j : u^1_j < 0\}.
\]

Thus, the planner increases the links across both sets, while decreasing the links.

---

\(^9\)Here we ignore the nodes with \( u^1_i = 0 \), since their weights do not change by Theorem 1.
within each set, since by (10),

\[ g_{ij}^* - \hat{g}_{ij} \begin{cases} 
> 0 & i \in S^+ \& j \in S^- \text{ or } i \in S^- \& j \in S^+; \\
< 0 & i, j \in S^+ \text{ or } i, j \in S^-.
\end{cases} \]

We visualize this in Figure 2 below. This suggests that as the budget grows large, the links across the sets \( S^+ \) and \( S^- \) increase to \( \bar{w} \), while the links within the sets decrease to 0. This reveals a tendency for the network to take a complete bipartite structure as \( C \) increases.

![Figure 2: Changes in edge weights](image)

The following example illustrates Theorem 1.

**Example 2.** Let \( \kappa = 0.5, \phi = -0.2, \bar{w} = 1, \hat{a} = 0, C = 1.5, \text{ and} \)

\[ \hat{g} = \begin{pmatrix} 
0 & 0.6 & 0.7 & 0.7 \\
0.6 & 0 & 0.7 & 0.3 \\
0.7 & 0.7 & 0 & 0.3 \\
0.7 & 0.3 & 0.3 & 0
\end{pmatrix}. \]

Then the optimal network can be calculated to be

\[ g^* = \begin{pmatrix} 
0 & 0.493 & 0.962 & 0.913 \\
0.493 & 0 & 0.795 & 0.377 \\
0.962 & 0.795 & 0 & 0.112 \\
0.913 & 0.377 & 0.112 & 0
\end{pmatrix}, \]

with unit eigenvector \( u^1(\phi g^*) = u^n(g^*) = (0.642, 0.232, -0.567, -0.461)^T \), since \( \phi \) is negative.\(^{10}\) We see that \( S^+ = \{1, 2\} \) and \( S^- = \{3, 4\} \). Consistent with Theorem 1, the strengths of the edges \( g_{13} \) and \( g_{14} \) have increased while the strength of the edge between \( g_{12} \) has decreased, i.e., \( g_{13}^* > \hat{g}_{13}, g_{14}^* > \hat{g}_{14}, \text{ and } g_{12}^* < \hat{g}_{12}. \) Additionally, we can check that the changes in \( g_{ij} \) follow

\(^{10}\)Values given are rounded to 3 decimal places.
(12); for instance,
\[
\begin{align*}
\frac{g_{12}^* - \hat{g}_{12}}{g_{13}^* - \hat{g}_{13}} &= -0.107 \quad \frac{0.262}{0.262} = -0.409 = \frac{0.232}{-0.567} = \frac{u_1^1}{u_3^2}, \\
\frac{g_{14}^* - \hat{g}_{14}}{g_{13}^* - \hat{g}_{13}} &= 0.213 \quad \frac{0.262}{0.262} = 0.813 = \frac{-0.461}{-0.567} = \frac{u_1^1}{u_3^2}.
\end{align*}
\]

Remark 1. The assumption \( \hat{a} = 0 \) in Theorem 1 seems restrictive. When the budget \( C \) is sufficiently large, \( \hat{a} \) plays a diminishing role and the results in Theorem 1 should hold approximately. To see that, we note that (7) holds approximately as \( C \to \infty \), regardless of \( \hat{a} \). Thus, the optimal \( a^* \) will also be approximately parallel to the corresponding eigenvector of \( g^* \).

We also note that the two methods we propose are complementary. In particular, the first variational method provides the conditions when the optimal interventions are interior and the second method gives intuitions for optimal interventions when \( \hat{a} \) is small compared with the size of the budget.

We next turn our attention to situations in which the budget is near zero. Suppose that \( \hat{g} \) is strictly interior—i.e., \( g_{ij} \in (0, 1) \) for all \( i \neq j \)—and further assume that \( \lambda_1(\phi \hat{g}) \) is an eigenvalue with multiplicity 1. If \( C \) is small, we have that \( g^* \approx \hat{g} \), so \( \lambda_1(\phi g^*) \approx \lambda_1(\phi \hat{g}) \) and \( u^1(\phi g^*) \approx u^1(\phi \hat{g}) \) by continuity. Consequently, we can perform an asymptotic analysis of Theorem 1, characterizing the order of growth of the link interventions as well as the budget allocation in terms of the primitives. Letting \( u^1(\phi \hat{g}) = (\hat{u}_1^1, \ldots, \hat{u}_n^1) \), we have
\[
\lim_{C \to 0} \frac{g_{ij}^* - \hat{g}_{ij}}{C} = \frac{\phi}{\kappa(1 - \lambda_1(\phi \hat{g}))} \hat{u}_i^1 \hat{u}_j^1,
\]
(13)
\[
\lim_{C \to 0} \frac{\kappa \|g^* - \hat{g}\|^2}{C^2} = \frac{\gamma}{\kappa}, \quad \lim_{C \to 0} \frac{\|a^*\|}{C} = 1,
\]
(14)
where
\[
\gamma = \frac{\phi^2}{(1 - \lambda_1(\phi \hat{g}))^2} \left( 1 - \sum_{i=1}^n (\hat{u}_i^1)^4 \right)
\]
is independent of \( C \). Proofs of the above statements can be found in the Appendix. When the budget is small, (13) implies that the change in the link intensity \( g_{ij} \) is increasing in the magnitude of the network effects of \( \phi \), but is inversely proportional to the cost of network design \( \kappa \). However, (14) shows that the planner allocates most of its budget toward intervening on the standalone marginal utilities \( a^* \), and thus the optimal joint intervention will be similar to the single intervention.
3.3 The case of large budgets

We now formally analyze the case in which the planner’s budget is large. Here, significant differences can be found from including joint intervention. We begin by showing that the optimal network always takes a simple form in this case.

**Theorem 2.** Suppose \( \bar{\omega} \) satisfies Assumption 1.

(a) If \( \phi > 0 \), then there exists \( \bar{C} \) such that for all \( C > \bar{C} \),

\[
g^*(C) = \bar{\omega}K_n.
\]

(b) If \( \phi < 0 \), then there exists \( \bar{C} \) such that for all \( C > \bar{C} \),

\[
g^*(C) \approx \bar{\omega}K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}.\]

Theorem 2 characterizes the optimal network structure when the planner’s budget is sufficiently large. Intuitively, if we fix a choice of \( g \), then \((9)\) tells us that welfare increases in \( C \) at a rate of approximately \( \frac{1}{(1 - \lambda_1(\phi g))^2} \). We can thus write

\[
V_{\text{single}}^*(g, \hat{a}, C) = \frac{1}{(1 - \lambda_1(\phi g))^2}(C - \kappa \|g - \hat{g}\|^2) + o(C),
\]

where \( o(C) \) is the little-o notation.\(^{12}\) Therefore, the dominant term in the above expression is governed by a social multiplier \( \frac{1}{(1 - \lambda_1(\phi g))^2} \). If \( g' \) is another network such that \( \lambda_1(\phi g') > \lambda_1(\phi g) \), then we must have \( V_{\text{single}}^*(g', \hat{a}, C) > V_{\text{single}}^*(g, \hat{a}, C) \) for sufficiently large \( C \). Consequently, as the budget goes to infinity, the eigenvalue of the optimal network, \( \lambda_1(\phi g) \), must approach the largest possible values among all possible \( g \). Depending on the sign of \( \phi \), we either seek the network with the largest first eigenvalue \( \lambda_1 \) or the network with the smallest last eigenvalue \( \lambda_n \). The rest of the argument uses the following lemma on eigenvalue-maximizing graphs.\(^{13}\)

**Lemma 2.** Let \( g \in \mathcal{G}_n \).

(i) \[
\lambda_1(g) \leq \bar{\omega}(n - 1),
\]

with equality if and only if \( g \) is the complete graph \( \bar{\omega}K_n \).

\(^{11}\)Given two graphs \( H \) and \( H' \) on \( p \) vertices, we say \( H \) is isomorphic to \( H' (H \cong H') \) if there exists a permutation \( \sigma \) on \( \{1, \cdots, p\} \) such that \( h_{ij} = h'_{\sigma(i)\sigma(j)} \) for all \( i, j \).

\(^{12}\)We write \( f(x) = o(x) \) if for any \( \epsilon > 0 \) there exists \( x_0 \) such that \( |f(x)| < \epsilon x \) for all \( x > x_0 \).

\(^{13}\)Furthermore, we can tighten the above analysis to show that the optimal network must be equal to the complete or complete balanced bipartite network for large enough \( C \), as in the theorem statement, but the technical details are left to the Appendix.
(ii) \[ \lambda_n(g) \geq -\bar{\omega} \sqrt{\frac{\lfloor n/2 \rfloor}{\lceil n/2 \rceil}}, \]

with equality if and only if \( g \) is isomorphic to the complete bipartite graph \( \bar{\omega}K_{\lfloor n/2 \rfloor,\lceil n/2 \rceil} \).

Lemma 2 part (i) is well known because the largest eigenvalue of a nonnegative matrix is monotone in its entries (see, for instance, the Perron–Frobenius theorem). The problem of finding the smallest possible \( \lambda_n \) for the case of unweighted graphs has previously been studied by Bramoullé et al. (2014).\(^ {\text{14}} \) In particular, Bramoullé et al. (2014) show that for any unweighted graph \( g \) over \( n \) vertices, \( \lambda_n(g) \geq \lambda_n(K_{\lfloor n/2 \rfloor,\lceil n/2 \rceil}) \). In the Appendix we show that a similar argument can be used to extend the result to general weighted graphs in Lemma 2.

As a by-product, Lemma 2 justifies our choice of bounds in Assumption 1.

Remark 2. \( \lambda_1(\phi g) < 1 \) for all \( g \in G_n \) whenever Assumption 1 holds.

That is, the regularity condition \( \lambda_1(\phi g) < 1 \) is satisfied for any choice of intervention by the planner.

To complete our analysis of the optimal joint intervention, we state the optimal choice of \( a^* \) below, which is the main concern of Galeotti et al. (2020). The following Remark 3 is obtained by directly applying the results derived by Galeotti et al. (2020) (or see our Proposition 0) on the post-intervention network characterized in Theorem 2.

Remark 3. Suppose \( \bar{\omega} \) satisfies Assumption 1.

(a) If \( \phi > 0 \), then there exists some \( \xi \in \{1, -1\} \) such that

\[ \lim_{C \to \infty} \frac{a^*(C) - \hat{a}}{\sqrt{C}} = \frac{\xi}{\sqrt{n}} 1_n. \]

(b) If \( \phi < 0 \), then there exists a sequence \( (c_i) \) with \( c_i \to \infty \), a permutation \( \sigma \), and some \( \xi \in \{1, -1\} \) such that

\[ \lim_{i \to \infty} \frac{a^*(c_i) - \hat{a}}{\sqrt{c_i}} = \frac{\xi}{\sqrt{n}} \sigma(u_n(K_{\lfloor n/2 \rfloor,\lceil n/2 \rceil})). \]

We illustrate the above results by continuing from Example 2.

Example 3. Using the parameters given in Example 2, we numerically obtain the optimal \( g^* \) for varying \( C \in [0, 8] \) to plot the following graph:

\(^{\text{14}}\)See also Constantine (1985).
We illustrate the network more clearly in the diagrams below, in which a thicker connection represents a larger value of \( g_{ij}^* \):

![Diagrams showing network for different budgets](image)

As predicted, for sufficiently large \( C \), the graph \( g^* \) goes toward the complete bipartite graph, in which within-partition links \( g_{12}^* \) and \( g_{34}^* \) are 0, while the rest of the links are \( \bar{w} \).

Thus far, we know that when \( \phi \) is negative, \( g^* (C) \) is a complete balanced bipartite graph for large \( C \). The next issue is to identify the optimal partition into two balanced subsets—i.e., of almost equal size. When the standalone marginal utilities are all equal—that is, \( \hat{a}_i = \hat{a}_j \) for all \( i, j \)—the optimal partition is the one that minimizes the cost of intervention in the network weights, since all nodes are otherwise equal. However, we show that this problem is computationally difficult even under this special case.

**Proposition 1.** When \( \hat{a}_i = \hat{a}_j \) for all \( i, j \), the orientation problem of choosing \( g^* \) that maximizes total payoffs is NP-hard.

Given a sufficiently large budget \( C \), Theorem 2 tells us that \( g^* \) must be isomorphic to \( K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \), with value function

\[
V_{\text{joint}}^*(\hat{g}, \hat{a}, C) = \mu \|a^*\|^2 = \frac{1}{(1 - \phi \lambda_n(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}))^2} (\sqrt{C} - \kappa \|g^* - \hat{g}\|^2 + \|\hat{a}\|)^2.
\]
Therefore, \( V^*_\text{joint}(\hat{g}, \hat{a}, C) \) is maximized when \( \kappa \| g^* - \hat{g} \|^2 \) is minimized. Letting \( S \) be a part of the partition of \( \mathcal{N} \) induced by \( g^* \) with size \( \lfloor \frac{n}{2} \rfloor \), we see that \( g^* \) minimizes

\[
\| g^* - \hat{g} \|^2 = \sum_{i,j \in S} \hat{g}_{ij}^2 + \sum_{i,j \not\in S} \hat{g}_{ij}^2 + \sum_{i \in S} \sum_{j \not\in S} (\bar{w} - \hat{g}_{ij})^2 = \| \hat{g} \|^2 + 2 \left[ \frac{n}{2} \right] \bar{w}^2 - 2 \bar{w} \sum_{i \in S, j \not\in S} \hat{g}_{ij}.
\]

Recall the definition of the weight of a cut \( S \subset \mathcal{N} \) as

\[
\text{Cut}(S) = \sum_{i \in S, j \not\in S} \hat{g}_{ij},
\]

so that the orientation \( g^* \) of \( K_{\lfloor \frac{n}{2}, \lceil \frac{n}{2} \rceil} \) is the one that maximizes \( \text{Cut}(S) \). Aside from the constraint on the size of \( S \), this is similar to the nonnegative weighted maximum cut problem (MAX-CUT), which is known to be NP-hard (Karp 1972). In the Appendix, we complete the proof of NP-hardness of the orientation problem by showing reducibility from the constrained version.

We illustrate by using \( \hat{g} \) as given in Example 2. The total weight of each cut of \( \hat{g} \) into equal parts is given in the following table:

| \( S \)     | \{1,2\} | \{1,3\} | \{1,4\} |
|------------|--------|--------|--------|
| \( \text{Cut}(S) \) | 2.4    | 2.3    | 1.9    |

Table 1: Cut weights of \( \hat{g} \) for each partition \( S \).

Therefore, the choice \( S = \{1,2\} \) maximizes the cut weight and provides the optimal orientation for large \( C \), as seen in Example 3.

### 4 Welfare and distributional effects

In this section, we analyze the effects of joint intervention on welfare and inequality. Given that single intervention is a special case and an important benchmark for subsequent comparison, we also present the corresponding analysis for the case of single intervention.

We first summarize the welfare effect of intervention in the following proposition:

**Proposition 2.** Under single intervention, the optimal welfare satisfies

\[
\lim_{C \to \infty} \frac{V^*_\text{single}(\hat{g}, \hat{a}, C)}{C} = \frac{1}{(1 - \lambda_1(\Phi \hat{g}))^2}.
\]
On the other hand, under joint intervention, we have

$$
\lim_{C \to \infty} \frac{V^*_j(g, \hat{a}, C)}{C} = \begin{cases} 
\frac{1}{(1-(n-1)\phi \bar{w})^2} & \phi > 0; \\
\frac{1}{(1+\phi \bar{w})\sqrt{\left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil}} & \phi < 0.
\end{cases}
$$

Proposition 2 shows that the welfare effects of interventions are determined by the principal eigenvalues of the post-intervention network. The first part is given by Galeotti et al. (2020) (also see our Proposition 0). The second part follows from the first part, since the network $g^*$ takes the form given in Theorem 2 under large budgets.

What determines the distributional effects of interventions? To answer this, we first introduce a measure of payoff inequality. Consider the entropy function of a vector $(v_1, \ldots, v_n) \in \mathbb{R}^n_+$, given by

$$
f(v_1, \ldots, v_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{v_i}{\bar{v}} \ln \left( \frac{v_i}{\bar{v}} \right) \in [0, \infty),
$$

where $\bar{v} = \frac{1}{n} \sum_{i=1}^{n} v_i$. Applying this function to the equilibrium payoff vector,$^{15}$ we obtain the following measure of inequality:$^{16}$

$$
T(g, \hat{a}, C) = f(\pi^*_1, \ldots, \pi^*_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{\pi^*_i}{\bar{\pi}^*} \ln \left( \frac{\pi^*_i}{\bar{\pi}^*} \right) \in [0, \infty),
$$

where $\bar{\pi}^* = \frac{1}{n} \sum_{i=1}^{n} \pi^*_i$ is the average equilibrium payoff of the players. Note that an entropy index of 0 occurs when all players receive the same payoff, while larger values of the entropy index represent more inequality of payoffs.

As shown below, the distributional effects of intervention are shaped by the eigen-centralities of the post-intervention network.

**Proposition 3.** Let $u^1(\phi g) = (\hat{u}^1_1, \ldots, \hat{u}^1_n)$. Under single intervention, the entropy index satisfies

$$
\lim_{C \to \infty} T_{\text{single}}(g, \hat{a}, C) = \sum_{i=1}^{n} (\hat{u}^1_i)^2 \ln(n(\hat{u}^1_i)^2).
$$

On the other hand, under joint intervention, we have

$$
\lim_{C \to \infty} T_{\text{joint}}(g, \hat{a}, C) = \begin{cases} 
0, & \phi > 0; \\
\ln(n) - \ln \left( 2\sqrt{\left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil} \right) & \phi < 0.
\end{cases}
$$

$^{15}$We assume that $\lambda_1(\phi g^*)$ is an eigenvalue with multiplicity 1, so that the optimal payoff is unique for sufficiently large $C$, and $T$ is well defined. This follows from the Perron-Frobenius theorem when $\phi > 0$ and holds generically when $\phi < 0$.

$^{16}$Also known as the Theil T index.
Since each player’s equilibrium payoff equals half of the square of equilibrium effort, the relative payoff of two players is equal to the square of their relative Katz-Bonacich centralities. When \( C \to \infty \), \( a^* \) —hence, \( x^* \) —is approximately a principal eigenvector of \( g^* \) by Proposition 0. In other words, the relative standalone marginal utilities approximately equal the relative equilibrium efforts, which approximately equal the relative eigen-centralities—i.e., \( x_i^* \approx \frac{a_i^*}{a_j^*} \approx \frac{u_i^1}{u_j^1} \) for all \( i, j \). Here \( u^1(\phi g^*) = (u^1_1, \ldots, u^1_n) \) is a unit eigenvector corresponding to \( \lambda_1(\phi g^*) \). In combination, we obtain that

\[
\lim_{C \to \infty} \frac{\pi_i^*}{\pi_j^*} = \lim_{C \to \infty} \frac{x_i^{*2}}{x_j^{*2}} = \lim_{C \to \infty} \frac{a_i^{*2}}{a_j^{*2}} = \frac{(u_1^1)^2}{(u_j^1)^2}.
\]

Since \( u^1 \) is of length 1, we establish the following limit of the inequality measure

\[
\lim_{C \to \infty} T(g, \hat{a}, C) = \lim_{C \to \infty} \sum_{i=1}^{n} (u_i^1)^2 \ln(n(u_i^1)^2),
\]

which only depends on the principal eigenvector \( u^1(\phi g^*) \). This formula is applicable to both single intervention and joint intervention, in which in the former case the network \( g^* \) equals the initial network \( \hat{g} \) by definition, and in the latter case the network \( g^* \) is found in Theorem 2 with the corresponding eigenvector stated in Fact 1.

Observe that in our joint intervention setting, by allowing for endogenous networks, we can completely eliminate payoff inequality for large \( C \) when either \( \phi > 0 \) or \( \phi < 0 \) and \( n \) is even. Furthermore, inequality can be small even when \( \phi < 0 \) and \( n \) is odd as \( \lim_{n \to \infty} \ln(n) - \ln \left(2\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil}\right) = 0 \).

**Remark 4.** When \( \hat{a} = 0 \), \( T_{\text{single}}(\hat{g}, \hat{a}, C) \) does not vary with \( C \), since it equals \( \sum_{i=1}^{n} (\hat{a}_i^1)^2 \ln(n(\hat{a}_i^1)^2) \) for any \( C \). See Example 4 and Table 2 below.

**Remark 5.** When \( \phi > 0 \), \( \lim_{C \to \infty} T_{\text{single}}(\hat{g}, \hat{a}, C) = 0 \) if and only if \( \hat{g} \) is regular.

**Remark 6.** Since we find that inequality under joint intervention goes to 0 for large \( C \), our analysis here is robust to the choice of inequality measure.

### 4.1 Differences in welfare and inequality

In this section, we will use our previous results to compare welfare and inequality under joint intervention with that under single intervention. We show that for a payoff-maximizing planner, including joint intervention increases welfare and decreases inequality for large budgets, but inequality can increase when budgets are moderate.

We first discuss welfare. When we allow for endogenous networks, the growth rate of

\[17\text{In this case, for any } C, \frac{x_i^2}{x_j^2} = \frac{a_i^2}{a_j^2} = \frac{u_i^1}{u_j^1} \text{ for all } i, j.\]
welfare with respect to the planner’s budget remains asymptotically linear by Proposition 2, but the gradient increases in general.

**Theorem 3.** \( V_{\text{joint}}^*(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C) \geq V_{\text{single}}^*(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C) \) for any \( C \). Furthermore,

(a) If \( \phi > 0 \), then

\[
\lim_{C \to \infty} \frac{V_{\text{joint}}^*(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C)}{V_{\text{single}}^*(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C)} = \left( \frac{1 - \phi \lambda_1(\hat{\mathbf{g}})}{1 - (n-1)\phi \bar{w}} \right)^2.
\]

(b) If \( \phi < 0 \), then

\[
\lim_{C \to \infty} \frac{V_{\text{joint}}^*(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C)}{V_{\text{single}}^*(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C)} = \left( \frac{1 - \phi \lambda_n(\hat{\mathbf{g}})}{1 + \phi \bar{w} \sqrt{\frac{n}{2} \frac{n}{2}}} \right)^2.
\]

Clearly, \( V_{\text{joint}}^*(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C) \geq V_{\text{single}}^*(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C) \), since the planner’s feasible choice set is larger under joint intervention. Theorem 3 further quantifies the value of the additional intervention in the network design for large budgets by comparing the social multipliers under the initial network \( \hat{\mathbf{g}} \) and the optimal network \( \mathbf{g}^* \). As a simple example, when \( \bar{w} = 1, \phi = 0.2, n = 4, \hat{\mathbf{g}} = K_{2,2} \), we obtain an asymptotic welfare ratio of 2.25. That is, allowing the planner to add two links to the network more than doubles the resulting welfare. In fact, we see that when \( \hat{\mathbf{g}} \neq K_n \), as \( \phi \to \frac{1}{(n-1)\bar{w}} \), the limit in Theorem 3(a) above goes to infinity, so the gain in welfare from allowing an intervention on the links can be arbitrarily large. A similar result occurs in the situation in which \( \phi < 0 \), which demonstrates the benefits of joint intervention in all cases.

We next compare the resulting inequalities under the two intervention schemes. We have seen that for sufficiently large \( C \), if \( \phi > 0 \) or \( \phi < 0 \) with \( n \) even, then \( T_{\text{joint}}(\hat{\mathbf{g}}, \hat{\mathbf{a}}, C) \) goes to zero. In contrast, single intervention can often lead to nonzero inequality, unless the initial network \( \hat{\mathbf{g}} \) is regular (see Remark 5). Therefore, allowing for changes in network weights in our setting with joint intervention is also beneficial for reducing inequality when the budget is sufficiently large.

However, despite the eventual decrease in inequality, we find that it is possible for network changes to increase inequality for intermediate budgets. Considering the situation in which \( \phi > 0 \), such an increase in inequality is particularly stark when \( \hat{\mathbf{g}} \) is sufficiently close to a graph that is regular but not vertex-transitive.\(^{18}\) In this case, inequality at \( \hat{\mathbf{g}} \) is near 0, but there is a range of \( C \) in which the payoff-maximizing choice of \( \mathbf{g}^* \) moves away from being regular as it transforms into the optimal network for large budgets \( K_n \).

**Theorem 4.** The welfare-maximizing joint intervention can induce a larger payoff inequality compared with single intervention. That is, there exists a choice of parameters \( \hat{\mathbf{a}}, \hat{\mathbf{g}}, C \) such that

\(^{18}\)A graph \( \mathbf{g} \) is vertex-transitive if, for any two vertices \( v \) and \( v' \), there exists an automorphism \( \Psi \) on \( \mathbf{g} \) such that \( \Psi(v) = v' \).
$T_{\text{joint}}(\hat{g}, \hat{a}, C) > T_{\text{single}}(\hat{g}, \hat{a}, C)$.

We prove Theorem 4 using the example below.

**Example 4.** Let $\kappa = 0.5$, $\phi = 0.15$, $\bar{w} = 1$, $\hat{a} = 0$, and

\[
\hat{g} = \begin{pmatrix}
0 & 0.14 & 0.23 & 0.63 & 0.05 \\
0.14 & 0 & 0.25 & 0.14 & 0.46 \\
0.23 & 0.25 & 0 & 0.09 & 0.39 \\
0.63 & 0.14 & 0.09 & 0 & 0.11 \\
0.05 & 0.46 & 0.39 & 0.11 & 0
\end{pmatrix}.
\]

We note that $\hat{g}$ is almost regular—the row sums range from 0.96 to 1.05. We list the entropy indices for several values of $C$ in the table below:

| $C$ | 0     | 4     | 8     |
|-----|-------|-------|-------|
| $T_{\text{joint}}(\hat{g}, \hat{a}, C)$ | 0.00114 | 0.00185 | 0     |
| $T_{\text{single}}(\hat{g}, \hat{a}, C)$ | 0.00114 | 0.00114 | 0.00114 |

Table 2: Inequality under joint and single interventions

From Table 2, we find that $T_{\text{joint}}(\hat{g}, 4) > T_{\text{single}}(\hat{g}, 4)$, which is where the graph intervention causes a dispersion in the degree distribution.\(^{19}\) However, inequality vanishes at $C = 8$, where the budget is large enough for the planner to choose the $K_4$ network under joint intervention. We also note that the inequality under single intervention is independent of $C$, since the condition $\hat{a} = 0$ implies that $a^*$ is always an eigenvector of $g^*$ (see Remark 4).

**Remark 7.** We note that for large budgets, since $\pi^*_i / \pi^*_j \approx a^*_i / a^*_j$, joint intervention also results in similar degrees of intervention in the standalone marginal utilities $a$ across agents, while large variation in intervention is possible in single intervention, depending on the principal components of $\hat{g}$.

### 4.2 The welfare cost of equality

Previously, we identified the possible adverse effect of joint intervention on payoff inequality for a welfare-maximizing planner. Now we will study the welfare cost of imposing zero payoff inequality under single and joint interventions. To analyze this trade-off,
we consider the related problem in which the planner prioritizes minimizing inequality over maximizing total welfare. That is, the planner solves (4) subject to the additional constraint $\pi^*_i = \pi^*_j$ for all $i, j$.

We show that this welfare loss is small under joint intervention for large $n$, but can be significant under single intervention. Indeed, we know that for large $C$, if $\phi > 0$ or $\phi < 0$ with $n$ even, then $T_{\text{joint}}(\hat{g}, \hat{a}, C)$ approaches 0, so the welfare loss with this additional constraint will also be asymptotically 0. We analyze the final case, in which $\phi < 0$ and $n$ is odd in the following lemma.

**Lemma 3.** Let $n \geq 5$ be odd, and $g \in \mathcal{G}_n$ such that $|u^n_i(g)| = |u^n_j(g)|$ for all $i, j$. Then

$$\lambda_n(g) \geq -\bar{w} \left( \frac{n-1}{2} \right),$$

with equality when

$$g_{ij} = \begin{cases} 
0, & i, j \leq \frac{n+1}{2}; \\
1, & i \leq \frac{n+1}{2} < j \text{ or } j \leq \frac{n+1}{2} < i; \\
\frac{2}{k-3}, & i, j > \frac{n+1}{2} \text{ and } i \neq j.
\end{cases}$$

Lemma 3 is analogous to Lemma 2, restricted to network structures whose last eigenvector has entries all of the same magnitude in absolute values. Such choices of $g^*$ will result in asymptotically zero inequality for large $C$ by (15). From Proposition 2, we know that optimal welfare when $\phi < 0$ depends critically on the lower bound of $\lambda_n(g)$ when $C$ is large; thus a comparison of the two bounds of $\lambda_n(g)$ in Lemmas 2 and 3 yields insight into the welfare cost of removing inequality. Furthermore, we note that the ratio of lower bounds satisfies

$$\lim_{n \to \infty} \frac{-\bar{w} \left( \frac{n-1}{2} \right)}{-\bar{w} \sqrt{\frac{n^2}{2}-1}} = 1,$$

so the cost of providing equality is insignificant when $n$ is large.

On the other hand, under single intervention, inequality is strongly linked to the initial network $\hat{g}$, and thus the planner could incur a much larger cost of welfare to be able to reduce inequality.

**Proposition 4.** Suppose Assumption 1 holds, $\phi > 0$ and $\hat{a} = 0$.\(^{20}\) Then for any choice of single intervention such that $T_{\text{single,eq}}(\hat{g}, \hat{a}, C) = 0$, the total welfare satisfies

$$V_{\text{single,eq}}(\hat{g}, \hat{a}, C) \leq \frac{1}{\|I - \phi \hat{g} z\|^2}.$$

\(^{20}\)For general $\hat{a}$, similar results hold for $C \to \infty$. 

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where \( z = \frac{1}{\sqrt{n}} \mathbf{1}_n \) is the normalized vector of ones.

In particular, we note that if \( \hat{g} \) is not regular, then \( z \) is not an eigenvector of \( \hat{g} \), and \( V_{single,eq}(\hat{g}, \hat{a}, C) / V_{single}(\hat{g}, \hat{a}, C) \) is strictly less than 1. Therefore, we find that including joint intervention can also reduce the trade-off between inequality and payoffs, and lead to greater improvement in welfare when the planner chooses to take social inequality into account.

5 Concluding remarks

We have shown that for sufficiently large budgets, the welfare-maximizing planner will choose to create either a complete network or a complete bipartite network. Although we performed our analysis for the case of weighted networks, the same arguments in Theorem 2 apply in the unweighted version. However, since the set of possible graphs is now discrete, the analytic results in Lemma 1(ii) are no longer relevant for studying the optimal intervention for small budgets. Nevertheless, since there are finitely many possible network structures in the unweighted case, the two-stage approach can still be applied to each graph to determine the optimal intervention.

We also note that the results in Theorem 2 are robust to specification of the cost function. As long as (i) the cost of intervening in \( g \) is continuous and bounded and (ii) the cost of intervening in \( a \) is convex, the optimal network will be of the given form for large budgets. This is because we know that the eigenvalue \( \lambda_1(\phi g) \) acts as a multiplier to the total welfare, so we can use a similar argument as the one given in this paper to show that the eigenvalue has to be maximized under such constraints on the cost. Therefore, even in general contexts we expect to have complete and complete balanced bipartite networks when a social planner is able to choose the network structure. However, as shown in Proposition 1, choosing the optimal partition when \( \phi < 0 \) is NP-hard, and further research could identify approximation algorithms to reduce computational difficulty. Finally, since our analysis of the distributional effects in Section 4 largely focuses on comparisons with the maximally equitable outcome in which all players obtain the same payoff, our results are also independent of the choice of inequality measure.

As a counterpart to Galeotti, Golub, and Goyal (2020), it may also be of interest to study the problem in which the standalone marginal utilities are fixed, so the planner is only able to intervene in the network structure. In this case, we find that Lemma 1(ii) will still be valid and can be used to study the optimal network. Another direction of research could follow from Sadler and Golub (2021), in which the network is endogenous but links are chosen by the players instead of the planner. A planner can then intervene by choosing to subsidize or tax link formation. Analyzing the optimal solution to such a formulation would offer another perspective on targeted interventions.
Appendix

A Omitted Proofs

Proof of Fact 1. (a) It is easy to check that \((1, 1, \cdots, 1)\) is an eigenvector of \(K_p\). By the Perron-Frobenius theorem, it must also be a basis of the eigenspace of \(\lambda_1(K_p)\).

(b) We note that
\[
K_{p,q} = \begin{pmatrix}
0_p & J_{p,q} \\
J_{q,p} & 0_q
\end{pmatrix}
\]
is of rank two and has zero trace, so it has a unique eigenvector that corresponds to a negative eigenvalue. We can verify that the given vector is the desired eigenvector of \(\lambda_{p+q}(K_{p,q})\). □

Proof of Lemma 1. Part (i) is just the FOC of \(L\) with respect to \(a\) (recall that \(g\) is symmetric):
\[
2[I - \phi g]^{-2}a = 2\mu(a - \hat{a}). \quad (16)
\]

For Part (ii), we first observe that \(L(a^*, g^*) \geq L(a^*, (1 - t)g^* + tg')\) for any \(t \in [0, 1]\) and \(g' \in G_n\). Thus the directional directive of \(L(a^*, \cdot)\), in the direction of \(g' - g^*\) must be nonpositive. We evaluate them in the following Lemma:

Lemma 4. (Some matrix calculus results) Define
\[
\mathcal{H} = \{ h \in \mathbb{R}^{n \times n} | h_{ij} = h_{ji} \text{ and } h_{ii} = 0 \text{ for all } i, j, \}.
\]

(a) As a function of the network \(g\), the directional derivative of \(a^T[I - \phi g]^{-2}a\) in the direction of \(h \in \mathcal{H}\) equals
\[
\lim_{\epsilon \to 0} \frac{a^T[I - \phi(g + \epsilon h)]^{-2}a - a^T[I - \phi g]^{-2}a}{\epsilon} = 2Tr(\phi[I - \phi g]^{-1}a)[I - \phi g]^{-2}h). \quad (17)
\]

(b) As a function of the network \(g\), the directional derivative of \(\|g - \hat{g}\|^2\) in the direction of \(h \in \mathcal{H}\) equals
\[
\lim_{\epsilon \to 0} \frac{\|g + \epsilon h - \hat{g}\|^2 - \|g - \hat{g}\|^2}{\epsilon} = 2Tr((g - \hat{g})h). \quad (18)
\]

Proof of Lemma 4. The proof follows from straightforward matrix operations.

(a)
\[
\lim_{\epsilon \to 0} \frac{a^T[I - \phi(g + \epsilon h)]^{-2}a - a^T[I - \phi g]^{-2}a}{\epsilon} = \lim_{\epsilon \to 0} \frac{\|I - \phi(g + \epsilon h)]^{-1}a\|^2 - \|I - \phi g[^{-1}a\|^2}{\epsilon}
\]
\[= \lim_{\epsilon \to 0} \langle ([I - \phi(g + \epsilon h)]^{-1} + [I - \phi g]^{-1})a, ([I - \phi(g + \epsilon h)]^{-1} - [I - \phi g]^{-1})a \rangle \epsilon \]

\[= \langle 2[I - \phi g]^{-1}a, \lim_{\epsilon \to 0} \frac{([I - \phi(g + \epsilon h)]^{-1} - [I - \phi g]^{-1})a}{\epsilon} \rangle \]

\[= 2\langle [I - \phi g]^{-1}a, [I - \phi g]^{-1}ph[I - \phi g]^{-1}a \rangle = 2Tr(\phi[I - \phi g]^{-1}aa^T[I - \phi g]^{-2}h). \]

(b)

\[\lim_{\epsilon \to 0} \frac{\|g + \epsilon h - \hat{g}\|^2 - \|g - \hat{g}\|^2}{\epsilon} = \lim_{\epsilon \to 0} \frac{2g + \epsilon h - 2\hat{g}, \epsilon h}{\epsilon} = \langle (g - \hat{g}), h \rangle = 2Tr((g - \hat{g})h). \]

Applying equations (17) and (18) in Lemma 4, we obtain that for any \(g' \in G_n\),

\[\langle \left\{ \phi[I - \phi g^*]^{-1}a^*a^T[I - \phi g^*]^{-2} - \mu^*\kappa(g^* - \hat{g}) \right\}, g' - g^* \rangle \leq 0. \]

Define \(e_{ij}\) to be a matrix with 1 on the \((i, j)\) and \((j, i)\) entries and 0 elsewhere. Whenever \(g^*_{ij} \in (0, \bar{w})\), we can choose sufficiently small \(\eta > 0\) so that \(g' = g^* \pm \eta e_{ij}\) are in \(G_n\). Since

\[\langle \left\{ \phi[I - \phi g^*]^{-1}a^*a^T[I - \phi g^*]^{-2} - \mu^*\kappa(g^* - \hat{g}) \right\}, \eta e_{ij} \rangle \]

\[= -\langle \left\{ \phi[I - \phi g^*]^{-1}a^*a^T[I - \phi g^*]^{-2} - \mu^*\kappa(g^* - \hat{g}) \right\}, -\eta e_{ij} \rangle, \]

we must have

\[\langle \left\{ \phi[I - \phi g^*]^{-1}a^*a^T[I - \phi g^*]^{-2} - \mu^*\kappa(g^* - \hat{g}) \right\}, e_{ij} \rangle = 0. \]

Expanding the inner product gives the first case of Lemma 1(ii) and similar arguments give the rest.

\[\Box\]

Proof of Lemma 2.

(i) Let \(g \in G_n\). Let \(u = u_1(g)\).\(^{21}\) Pick any \(u_k = \max_i u_i > 0\). Then

\[\lambda_1(g)u_k = (gu)_k = \sum_{i=1}^n g_{ki}u_i \leq \bar{w}(n - 1)u_k, \]

with equality only if \(g_{ki} = \bar{w}\) and \(u_i = u_k\) for all \(i \neq k\). The latter implies that our choice of \(k\) can be replaced by any other \(j\), so we have \(g_{ji} = \bar{w}\) for all \(i \neq j\). Hence \(g\) represents \(\bar{w}K_n\).

\(^{21}\)Since \(g\) is nonnegative, such a nonnegative eigenvector exists by the Perron-Frobenius theorem.
(ii) We begin by stating Proposition 7 of Bramoullé, Kranton, and D’Amours (2014):

**Proposition (Bramoullé, Kranton, and D’Amours (2014)).** Let \( g \) be a simple graph. Let \( u \) be an eigenvector for \( \lambda_n(g) \) and let \( R = \{ i : u_i \geq 0 \} \), \( S = \{ j : u_j < 0 \} \). Construct \( g' \) by removing links within \( R \) and \( S \), and adding links between \( R \) and \( S \). Then \( \lambda_n(g') \leq \lambda_n(g) \).

**Proof.** We have

\[
\lambda_n(g) = \sum_{i,j \in R} u_i u_j g_{ij} + \sum_{i,j \in S} u_i u_j g_{ij} + 2 \sum_{i \in R, j \in S} u_i u_j g_{ij} \geq \sum_{i,j \in R} u_i u_j g'_{ij} + \sum_{i,j \in S} u_i u_j g'_{ij} + 2 \sum_{i \in R, j \in S} u_i u_j g'_{ij} = \lambda_n(g'),
\]

(19)

Clearly, the same argument applies even if \( g \) is allowed to be a weighted graph, so a complete bipartite graph is optimal. Furthermore, among the set of complete bipartite graphs, the smallest eigenvalue occurs when the vertices are partitioned into sets of size \( \lfloor \frac{n}{2} \rfloor \) and \( \lceil \frac{n}{2} \rceil \). It remains to show that \( wK_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \) is the unique graph (up to isomorphism) that minimizes \( \lambda_n(g) \), with

\[
\lambda_n(wK_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) = -w \sqrt{\frac{n-1}{2}} \cdot \sqrt{\frac{n-1}{2}}.
\]

Let \( g \) be a network that is not isomorphic to \( wK_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \). First suppose that \( u_i \neq 0 \) for all \( i \). Then the inequality in (19) holds strictly, so there exists \( g' \) with \( \lambda_n(g') < \lambda_n(g) \), thus \( g \) cannot be optimal.

Otherwise, without loss of generality suppose that \( u_n = 0 \). Let \( g_{n-1} \) be the \( (n-1) \)-th principal minor of \( g \), and \( u_{1:n-1} \) be the first \( n-1 \) components of \( u \). Then

\[
g u = \lambda_n(g) u \implies g_{n-1} u_{1:n-1} = \lambda_n(g) u_{1:n-1},
\]

so \( \lambda_n(g) \) is also an eigenvalue of \( g_{n-1} \). This implies that

\[
\lambda_n(g) \geq \lambda_{n-1}(g_{n-1}) \geq -w \sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil} > -w \sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil},
\]

so \( g \) also cannot be optimal. Hence the only minimizers of \( \lambda_n(g) \) are isomorphic to \( wK_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \). \[\Box\]

**Proof of Remark 2.** From the bounds in Lemma 2,

(a) If \( \phi > 0 \), then \( \lambda_1(\phi g) \leq \phi w(n-1) < 1 \).
(b) If $\phi < 0$ and $2 \mid n$, then
\[
\lambda_1(\phi g) = \phi \lambda_n(\phi g) \leq -\phi \bar{\omega} \sqrt{\frac{n}{2}} \left\lceil \frac{n}{2} \right\rceil = -\phi \bar{\omega} \frac{n}{2} < 1.
\]

(c) If $\phi < 0$ and $2 \nmid n$, then
\[
\lambda_1(\phi g) = \phi \lambda_n(\phi g) \leq -\phi \bar{\omega} \sqrt{\frac{n^2 - 1}{4}} < 1.
\]

\[\square\]

Proof of Equations (13) and (14). We first derive (14). Let
\[
\gamma^* = \frac{\phi^2}{(1 - \lambda_1(\phi g^*))^2} \left( 1 - \sum_{i=1}^{n} (u_i^1(\phi g^*))^4 \right).
\]

By (11),
\[
\kappa \|g^* - \hat{g}\|^2 = \frac{\gamma^*}{\kappa} (C - \kappa \|g^* - \hat{g}\|^2)^2 \Rightarrow \kappa \|g^* - \hat{g}\| = \sqrt{\gamma^*} (C - \kappa \|g^* - \hat{g}\|^2) \Rightarrow \sqrt{\kappa} \|g^* - \hat{g}\| = \frac{-\sqrt{\kappa} + \sqrt{\kappa + 4 \gamma^* C}}{2\sqrt{\gamma^*}} = \frac{-\sqrt{\kappa} + \sqrt{\kappa + 2 \gamma^* C + O(C^2)}}{2\sqrt{\gamma^*}} = \sqrt{\frac{\gamma^*}{\kappa}} C + O(C^2)
\]

Since we have assumed that $\lambda_1(\phi g)$ occurs with multiplicity 1, then $\lambda_1(\phi g)$ and $u^1(\phi g)$ are continuous at $\hat{g}$. Thus $\lim_{C \to 0} \gamma^* = \gamma$, so
\[
\lim_{C \to 0} \frac{\kappa \|g^* - \hat{g}\|^2}{C^2} = \frac{\gamma^*}{\kappa} = \frac{\gamma}{\kappa}.
\]

Next, by the budget constraint,
\[
\|a^*\|^2 = C - \kappa \|g^* - \hat{g}\|^2 = C + O(C^2) \Rightarrow \lim_{C \to 0} \frac{\|a^*\|^2}{C} = 1.
\]

Finally, applying the above result into (10) gives (13). \[\square\]

Proof of Theorem 2. To prove the existence of a cutoff $\bar{C}$, we first take limits of (16):
\[
\lim_{C \to \infty} 2[I - \phi g^*]^{-2} \frac{a^*}{\sqrt{C}} = \lim_{C \to \infty} 2\mu \frac{a^* - \hat{a}}{\sqrt{C}} \Rightarrow \lim_{C \to \infty} [I - \phi g]^{-2} \frac{a^*}{\sqrt{C}} = \lim_{C \to \infty} \mu \frac{a^*}{\sqrt{C}}.
\]
\[
\lim_{C \to \infty} \mu = \frac{1}{(1 - \lambda)^2}.
\]

Similar to Galeotti et al. (2020), \( a^* / \sqrt{C} \) goes to the corresponding eigenvector \( u(\tilde{g}) \). From Lemma 1(ii), if there exists arbitrary large \( C \) such that \( g^*_{kl} \in (0, \tilde{w}) \), we have

\[
0 = \lim_{C \to \infty} \frac{2\kappa(g^* - \tilde{g})_{kl}}{C} = \lim_{C \to \infty} \frac{1}{\mu C} (\phi[I - \phi g^*]^{-1} a^* a^{*T} [I - \phi g^*]^{-2} + \phi[I - \phi g^*]^{-2} a^* a^{*T} [I - \phi g^*]^{-1})_{kl} = \phi(1 - \lambda)^2 (u(\tilde{g}) u(\tilde{g}))^T [I - \phi g^*]^{-2} + [I - \phi g^*]^{-2} u(\tilde{g}) u(\tilde{g})^T [I - \phi g^*]^{-1})_{kl} = \frac{2\phi}{1 - \lambda} u_k(\tilde{g}) u_l(\tilde{g}) \\
\neq 0,
\]

with the last inequality because \( u_k(\tilde{g}) \neq 0 \) for all \( k \). Therefore, there cannot be interior \( g^*_{ij} \) for sufficiently large \( C \), so \( g^* \) must be either complete or complete bipartite from Lemma 2. \( \square \)

**Proof of Proposition 1.** Call the constrained version of MAX-CUT with \( |S| = \lceil n/2 \rceil \) the balanced maximum cut (BAL-MAX-CUT) problem, and call a partition of \( N \) into parts of sizes \( \lceil n/2 \rceil \) and \( \lceil n/2 \rceil \) a balanced cut.

**MAX-CUT \( \leq_p \) BAL-MAX-CUT:**\(^{22}\) Given an instance \( G \) of MAX-CUT with adjacency matrix \( m_{p \times p} \), consider the instance \( G' \) of BAL-MAX-CUT with adjacency matrix \( \begin{pmatrix} m & 0_p \\ 0_p & 0_p \end{pmatrix} \).

Then every cut of \( G \) can be extended to a balanced cut of \( G' \) by a suitable assignment of the independent vertices, without changing the total cut weight. Similarly, every balanced cut of \( G' \) can be restricted to a cut of \( G \) without changing the cut weight by removing the additional vertices. Thus the instance \( G' \) of BAL-MAX-CUT solves the MAX-CUT problem.

**BAL-MAX-CUT \( \leq_p \) MAX-CUT:** Given an instance \( H \) of BAL-MAX-CUT with adjacency matrix \( m_{p \times p} \), consider an instance \( H' \) of BAL-MAX-CUT with adjacency matrix \( m + \alpha (J_p - I_p) \), where \( \alpha > 1^T_p m 1_p \) is sufficiently large.

Let \( k = \lceil n/2 \rceil \lceil n/2 \rceil \) be the number of edges in a half-cut of \( H' \). Then the weight of any balanced cut is at least \( ak \), while any other cut has at most \( k - 1 \) edges so has weight at most \( \alpha (k - 1) + 1^T_p m 1_p < ak \). Therefore, the maximal cut is the maximal balanced cut and the instance \( H' \) of MAX-CUT solves the BAL-MAX-CUT problem.

\(^{22}\)We write \( X \leq_p Y \) if problem \( X \) is reducible to problem \( Y \) in polynomial time.
Therefore, BAL-MAX-CUT, and hence the orientation problem, is in the same computational class as the MAX-CUT problem and is NP-hard (Karp 1972).

**Proof of Proposition 2.** Proposition 2 follows from applying the optimal eigenvalues in Lemma 2 to the solution of the second stage problem in (9).

**Proof of Proposition 3.** The proof largely follows from the main text. It remains to justify that
\[
\frac{x_i^*}{x_j^*} \approx \frac{a_i^*}{a_j^*} \approx \frac{u_1^i}{u_1^j}
\]
for all \(i, j\) when \(C\) is large. By (8), and possibly multiplying \(u^1(\phi g^*)\) by \(-1\), we have the relation
\[
\lim_{C \to \infty} \frac{a^*}{\|a^*\|} = \lim_{C \to \infty} u^1(\phi g^*).
\]
Therefore, by (2) and the above,
\[
\lim_{C \to \infty} x^* \approx \lim_{C \to \infty} \frac{[I - \phi \hat{g}]^{-1} a^*}{\|a^*\|} = \lim_{C \to \infty} [I - \phi g]^{-1} u^1(\phi g^*) = \lim_{C \to \infty} \frac{1}{1 - \lambda_1(\phi g)} u^1(\phi g^*).
\]
Consequently, \(x^*, a^*, u^1(\phi g^*)\) are approximately proportional vectors when \(C\) is large and the desired equation holds.

**Proof of Lemma 3.** Since \(|u_i^n| = |u_j^n|\) for all \(i, j\), then \(|u_i^n| = \frac{1}{\sqrt{n}}\) for all \(i\). By a relabelling of the indices and possibly multiplying by \(-1\), without loss of generality let \(u_i^n = \frac{1}{\sqrt{n}}\) if \(i \in \{1, \cdots, k\}\), and \(u_i^n = -\frac{1}{\sqrt{n}}\) otherwise. Also let \(k > \frac{n}{2}\). By definition,
\[
\lambda_n(g) u_i^n = \sum_{i=1}^{n} s_{i_1} u_i^n = \sum_{i=k+1}^{n} s_{i_1} u_i^n - \sum_{i=1}^{k} s_{i_1} u_i^n \geq -\bar{\omega}(n-k) u_i^n \geq -\bar{\omega}\left(\frac{n-1}{2}\right) u_i^n,
\]
so \(\lambda_n(g) \geq -\bar{\omega}\left(\frac{n-1}{2}\right)\). Finally, it is easily verified that equality holds under the given choice of \(g\).

**Proof of Proposition 4.** For zero inequality, we must have \(kz = x^* = [I - \phi \hat{g}]^{-1} a^*\) for some \(k \in \mathbb{R}\). Thus \(a^* = k[I - \phi \hat{g}] z\). By the budget constraint, \(\|a^*\|^2 = C = k^2 \|[I - \phi \hat{g}] z\|^2\), so
\[
V_{\text{single}, eq}^* = (a^*)^T [I - \phi \hat{g}]^{-2} a = k^2 \frac{C}{\|[I - \phi \hat{g}] z\|^2}.
\]
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