HOPF MODULES FOR AUTONOMOUS PSEUDOMONOIDS AND THE MONOIDAL CENTRE

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ABSTRACT. In this work we develop some aspects of the theory of Hopf algebras to the context of autonomous map pseudomonoids. We concentrate in the Hopf modules and the Centre or Drinfel’d double. If $A$ is a map pseudomonoid in a monoidal bicategory $\mathcal{M}$, the analogue of the category of Hopf modules for $A$ is an Eilenberg-Moore construction for a certain monad in $\text{Hom}(\mathcal{M}^{\text{op}}, \text{Cat})$. We study the existence of the internalisation of this notion, called the Hopf module construction, by extending the completion under Eilenberg-Moore objects of a 2-category to a endo-homomorphism of tricategories on $\text{Bicat}$.

Our main result is the equivalence between the existence of a left dualization for $A$ (i.e., $A$ is left autonomous) and the validity of an analogue of the structure theorem of Hopf modules. In this case the Hopf module construction for $A$ always exists.

We use these results to study the lax centre of a left autonomous map pseudomonoid. We show that the lax centre is the Eilenberg-Moore construction for a certain monad on $A$ (one existing if the other does). If $A$ is also right autonomous, then the lax centre equals the centre. We look at the examples of the bicategories of $\mathcal{V}$-modules and of comodules in $\mathcal{V}$, and obtain the Drinfel’d double of a coquasi-Hopf algebra $H$ as the centre of $H$.

1. Introduction

This work addresses the problem of extending the basic results of the theory of Hopf algebras to the context of autonomous pseudomonoids. We will focus mainly on two constructions: Hopf modules and the Drinfel’d double of a Hopf algebra.

Left autonomous pseudomonoids, introduced in [4], generalise not only Hopf and (co)quasi-Hopf algebras but also (pro)monoidal enriched categories. In fact, this is the conceptual reason underlying the well-known fact that the category of finite-dimensional (co)representations of a (co)quasi-Hopf algebra is left autonomous.

Our starting point is the so-called theorem of Hopf modules for (co)quasi-Hopf algebras [12, 28], that extends the classical result for ordinary Hopf algebras [21]. A coquasibialgebra $H$, although not associative in $\text{Vect}$, is an associative algebra in the category $\text{Comod}(H, H)$ of $H$-bicomodules and thus we can consider the category of left $H$-modules in $\text{Comod}(H, H)$. This is by definition the category of $H$-Hopf modules. There is a monoidal functor from the category of right $H$-comodules to the category of $H$-Hopf modules sending $M$ to the tensor product bicomodule $H \otimes M$, where $M$ is considered as trivial $H$-comodule on the left. It is shown in [28] that when $H$ is a coquasi-Hopf algebra this functor is an equivalence, and in a dual fashion, that a finite-dimensional quasibialgebra is quasi-Hopf if and only if the module version of this functor is an equivalence.

We prove that an analogous result holds if we replace coquasibialgebras by map pseudomonoids (i.e., pseudomonoids whose multiplication and unit have a right adjoint), Hopf modules by the Eilenberg-Moore construction for certain monad and coquasi-Hopf algebras by left autonomous map pseudomonoids. Moreover, in

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our general setup no finiteness condition is necessary. We take this as an indication that the concept of dualization is more natural than the one of antipode.

When the monoidal bicategory involved is right closed, and in particular when it is right autonomous, our Hopf module construction can be internalised. Naturally, this internalisation need not exist, being an Eilenberg-Moore construction for some monad on the endo-hom object \([A,A]\) of the map pseudomonoid \(A\). However, it does exist when the map pseudomonoid is left autonomous, and its object part is equivalent to \(A\).

The centre of a monoidal category was defined in [16], and more recently generalised to the centre construction for pseudomonoids [31]. A classical result reads that, for a Hopf algebra \(H\), the category of two-sided Hopf modules is monoidally equivalent to the centre of the category of \(H\)-(co)modules; this has been extended to the case of quasi-Hopf algebras in [27]. Both versions use the category of Yetter-Drinfel’d modules as an intermediate stage in proving the equivalence.

In our general context, the lax centre of a left autonomous map pseudomonoid is the Eilenberg-Moore construction for a certain monad on the pseudomonoid, and if this is also right autonomous, then the lax centre is in fact the centre. In this way we reduce the problem of the existence of the lax centre of such pseudomonoids to the problem of the existence of a particular Eilenberg-Moore object. No analogue of the Yetter-Drinfel’d modules appear in our construction. If we think of a Hopf algebra as an autonomous pseudomonoid in the appropriate monoidal bicategory, its centre is equivalent to the Drinfel’d double of the Hopf algebra. This shows that the centre construction generalises the classical quantum double construction for Hopf algebras. When we apply our results to the bicategory of \(\mathcal{V}\)-modules, we are able to show that any left autonomous map pseudomonoid has a lax centre. In particular, any left autonomous monoidal \(\mathcal{V}\)-category has a lax centre in \(\mathcal{V}\)-\text{Mod}.

We shall describe the content of each section.

In Section 3 we introduce the Hopf modules for a map pseudomonoid \(A\) in a monoidal bicategory \(\mathcal{M}\) as the Eilenberg-Moore construction for a certain monad in \([\mathcal{M}^{\text{op}}, \text{Cat}]\), and explain what we mean by the theorem of Hopf modules.

Section 4 surveys some well-known facts about lax actions and opmonoidal morphisms.

When the monad in the definition of Hopf modules is representable by a monad \(t : [A,A] \to [A,A] \in \mathcal{M}\), we call an Eilenberg-Moore construction for it a Hopf module construction for \(A\). This is introduced in Section 5 along with the proof that \(t\) is a opmonoidal monad. A Hopf module construction, of course, need not exist in general.

In Section 6 we use completions under Eilenberg-Moore objects to study the existence of Hopf module constructions. To that end, we extend these completions to a Gray-functor on \(\text{Gray}\), and then to a homomorphism of tricategories on \(\text{Bicat}\).

In Section 7 we prove our main result: a map pseudomonoid \(A\) is left autonomous if and only if the theorem of Hopf modules holds for \(A\). Also, we use the results of the preceding section to show that a map pseudomonoid is left autonomous if and only if it has a Hopf module construction of a particular form, relating the problem of the existence of a dualization with a completeness problem.

In Section 8 we look at the relationship between autonomous and Frobenius pseudomonoids.

Section 9 deals with the relation between the Hopf module construction and the lax centre construction. We show that for a left autonomous map pseudomonoid \(A\) there exists an opmonoidal monad on \(A\) for which an Eilenberg-Moore construction
is exactly the lax centre of $A$. Moreover, if $A$ is also right autonomous then the lax centre coincides with the centre of $A$, and one exists if and only if the other does.

The last two sections are devoted to the applications. In Section 10 we treat the example of the bicategory $\mathcal{V}$-$\text{Mod}$ of $\mathcal{V}$-modules, for a complete and cocomplete closed symmetric monoidal category $\mathcal{V}$. As shown in [4], a left autonomous pseudomonoid in $\mathcal{V}$-$\text{Mod}$ whose multiplication, unit and dualization are representable by $\mathcal{V}$-functors is exactly a left autonomous $\mathcal{V}$-category in the usual sense that every object has a left dual. The bicategory $\mathcal{V}$-$\text{Mod}$ has Eilenberg-Moore objects for monads and therefore any map pseudomonoid in it has a Hopf module construction and any left autonomous map pseudomonoid has a lax centre, of which an explicit description is provided. We also exhibit for a promonoidal $\mathcal{V}$-category $A$, a canonical equivalence of $\mathcal{V}$-categories $[\mathcal{Z}_\ell(A), \mathcal{V}] \simeq [\mathcal{Z}_\ell([A, \mathcal{V}])];$ here the lax centre on the left hand side is the one of $A$ as a pseudomonoid in $\mathcal{V}$-$\text{Mod}$, while on the right hand side we have the lax centre in $\mathcal{V}$-$\text{Cat}$.

The example of the bicategory of comodules over a braided monoidal category is studied in the Section 11. Along with explicit descriptions of the general constructions of the previous sections, we show how our work generalises results in [28] on the theorem of Hopf modules. For example, a coquasibialgebra $C$ has a dualization (which in this case is a left $C^{\text{cop}}$ and right $C$-bicomodule) if and only if the theorem of Hopf modules holds for $C$. Thus, admitting dualizations instead of mere antipodes $C^{\text{cop}} \to C$, we are able to drop the finiteness conditions on $C$ required in [28]. We also show that the Hopf module construction always exists for finite dimensional coquasibialgebras, and that our notion of the theorem of Hopf modules, which implies the one of [28], is in fact equivalent to it when $C$ is finite dimensional. Finally, by means of the results in Section 9 we show that if $C$ is a finite coquasi-Hopf algebra, then the centre of $C$ exists and is equivalent to the Drinfel’d double of $C$.

While preparing a definitive version of this work the author received the preprint [26], where Pastro and Street study what they call doubles of monoidal enriched categories. They are lead to structures in the bicategory of $\mathcal{V}$-modules very similar to some of the ones we describe in Section 10.

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2. Preliminaries on pseudomonoids

In this section we introduce the analogue for a map pseudomonoid of the category of Hopf modules. We call maps the 1-cells in a bicategory with right adjoint.

Recall that a Gray monoid [6] is a monoid in the monoidal category Gray. For a detailed treatment of the category Gray see [11]. As a category, Gray is just the category of 2-categories and 2-functors. However, the monoidal structure we are interested in is not the one given by the cartesian product.

A cubical functor in two variables is a pseudofunctor between 2-categories $F: \mathcal{K} \times \mathcal{L} \to \mathcal{N}$ with the following property: when we fix one of the variables we get a 2-functor, and for any 1-cell $(f, g)$ in $\mathcal{K} \times \mathcal{L}$ the constraint $F(1, g)F(f, 1) \cong F(f, g)$ is equal to the identity. For any pair of 2-categories $\mathcal{K}$ and $\mathcal{L}$, there is a 2-category $\mathcal{K} \Box \mathcal{L}$ equipped with a cubical functor $\mathcal{K} \times \mathcal{L} \to \mathcal{K} \Box \mathcal{L}$ inducing a bijection between cubical functors $\mathcal{K} \times \mathcal{L} \to \mathcal{N}$ and 2-functors $\mathcal{K} \Box \mathcal{L} \to \mathcal{N}$. This defines a monoidal structure on Gray, which moreover is symmetric with the symmetry induced by the usual one for the cartesian product. A Gray monoid is the same as a one-object Gray-category in the sense of enriched category theory, and therefore
it can be thought of as a one-object tricategory, that is, a monoidal bicategory (see [11]). By the coherence theorem in [11], any monoidal bicategory is monoidally biequivalent (that is, triequivalent as a tricategory) to a Gray monoid. This allows us to develop the general theory by using Gray monoids instead of general monoidal bicategories.

Our main examples of monoidal bicategories will be the bicategory of comodules $\text{Comod}(\mathcal{V})$ in a monoidal category $\mathcal{V}$ (see Section [11]) and the bicategory of $\mathcal{V}$-modules $\mathcal{V}\text{-Mod}$ (see Section [11]).

Let $\mathcal{M}$ be a Gray monoid and fix a map pseudomonoid $(A,j,p)$ in $\mathcal{M}$, that is, a pseudomonoid whose unit $j: I \to A$ and multiplication $p: A \otimes A \to A$ are maps. Recall from [6] that a pseudomonoid, in addition to the unit and multiplication, is equipped with isomorphisms $\phi : p(p \otimes A) \Rightarrow p(A \otimes p), p(j \otimes 1) \Rightarrow 1_A$ and $p(1_A \otimes j) \Rightarrow 1_A$ satisfying three axioms which ensure, as shown in [19], that any 2-cell formed by pasting of tensor products of these isomorphisms, 1-cells and pseudonaturality constraints of the Gray monoid is uniquely determined by its domain and codomain 1-cells.

If $(A,j,p)$ is a map pseudomonoid, then $(A,j^*,p^*)$ is a pseudocomonoid, that is, a pseudomonoid in the opposite Gray monoid. By definition the unit isomorphism $(A \otimes j^*)^* \cong 1_A$ of the pseudocomonoid $(A,j^*,p^*)$ is the mate of the constraint $p(A \otimes j) \cong 1_A$, and thus the following equality holds.

\[
\begin{array}{ccc}
A & \cong & 1 \otimes j \\
\downarrow \phi & & \downarrow \phi \\
A^2 & = & 1 \otimes j
\end{array}
\]

\[
\begin{array}{ccc}
A^2 & \cong & p \downarrow \downarrow \\
\downarrow \phi & & \downarrow \\
A & \cong & A
\end{array}
\]

(1)

We mention this because it will be useful in Section [11].

In order to give a concise and conceptual definition of the Hopf modules in the next section, we will need to use the Kleisli bicategory of a pseudocomonad.

One can define a pseudomonad on the 2-category $\mathcal{K}$ as a pseudomonoid in the monoidal 2-category $\text{Hom}(\mathcal{K}, \mathcal{K})$ of pseudofunctors, pseudonatural transformations and modifications. A pseudocomonad is a pseudomonoid in the same monoidal 2-category. As before, if $T$ is a pseudomonad with unit $\eta : 1 \Rightarrow T$ and multiplication $\mu : T^2 \Rightarrow T$ which are maps, then $T$ together with $\eta^*$ and $\mu^*$ have a canonical structure of a pseudocomonad on $\mathcal{K}$.

A lax $T$-algebra is an arrow $a : TA \to A$ in $\mathcal{K}$ equipped with a 2-cell $a(Ta) \Rightarrow a\mu_A : T^2 A \to A$ satisfying the axioms in [23, p. 39] and [19], but without the requirement of the invertibility of the 2-cell.

Let $G$ be a pseudocomonad on the 2-category $\mathcal{K}$, and denote its comultiplication and counit by $\delta$ and $\epsilon$, respectively. The Kleisli bicategory $\text{Kl}(T)$ of $\mathcal{K}$ has the same objects as $\mathcal{K}$, and hom-categories $\text{Kl}(T)(X,Y) = \mathcal{K}(GX,Y)$. We denote the 1-cells of $\text{Kl}(T)$ by $f : X \to Y$. The composition of this $f$ with $g : Y \to Z$ is given by $g(Gf)\delta_X : GX \to Z$, while the identity of the object $X$ is $\epsilon_X : GX \to X$.

The following is a slight generalisation of part of [4, Prop. 4.6].

**Lemma 2.1.** Let $T : \mathcal{K} \to \mathcal{K}$ be a pseudomonad whose unit $\eta$ and multiplication $\mu$ are maps. Then, there exists a bijection between the following structures on an arrow $a : TA \to A$ in $\mathcal{K}$: structures of an lax $T$-algebra and structures of a monad in $\text{Kl}(T)$. 

A structure of a monad in Kl\((T)\) on \(a : A \rightarrow A\) is given by a pair of 2-cells \(\eta_A \Rightarrow a\) and \(\mu_A \Rightarrow a\) in \(\mathcal{K}\). The bijection above is given by

\[
\begin{array}{c}
\begin{array}{ccc}
T^2A & \xrightarrow{Ta} & TA \\
\mu_A & \Downarrow & \Downarrow \\
TA & \xrightarrow{a} & A
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
T^2A & \xrightarrow{Ta} & TA \\
\mu_A & \Downarrow & \Downarrow \\
TA & \xrightarrow{a} & A
\end{array}
\end{array}
\]

### 3. The theorem of Hopf modules

If \((A, j, p)\) is a map pseudomonoid in the Gray monoid \(\mathcal{M}\), the 2-functor \(A \otimes -\) has the structure of a pseudomonad with unit \(j \otimes X : X \rightarrow A \otimes X\) and multiplication \(p \otimes X : A \otimes A \otimes X \rightarrow A \otimes X\), and also the structure of a pseudocomonad with counit \(j^* \otimes X\) and comultiplication \(p^* \otimes X\). The associativity constraint \(p(A \otimes p) \Rightarrow p(p \otimes A)\) endows \(A \otimes A \rightarrow A\) with the structure of a lax \((A \otimes -)\)-algebra, and hence by Lemma 2.1, with the structure of a monad \(p : A \rightarrow A\) in the Kleisli bicategory Kl\((A \otimes -)\).

**Definition 3.1.** We will denote by \(\theta\) the monad Kl\((A \otimes -)(-, p)\) in Kl\((A \otimes -)\). Hence, \(\theta\) is a monad on the 2-functor \(\mathcal{M}(A \otimes -, A)\) in the 2-category \(\text{Hom}(\mathcal{M}^{\text{op}}, \text{Cat})\) of pseudofunctors, pseudonatural transformations and modifications.

Explicitly, \(\theta X(f) = p(A \otimes f)(p^* \otimes X)\) and the multiplication and unit of the monad, depicted in (2) and (3), are induced by the counits of the adjunctions \(p \dashv p^*\) and \(j \dashv j^*\) respectively.

\[
\begin{array}{ccc}
A \otimes X & \xrightarrow{p^* \otimes 1} & A^2 \otimes X \\
\phi^* \cong & \otimes & \phi^* \otimes 1 \\
A^2 \otimes X & \xrightarrow{1 \otimes f} & A^2
\end{array}
\]

(2)

\[
\begin{array}{ccc}
A \otimes X & \xrightarrow{1 \otimes f} & A^2 \\
\cong & \otimes & \cong \\
A \otimes X & \xrightarrow{p^* \otimes 1} & A^2 \otimes X
\end{array}
\]

(3)

Our generalisation of the category of Hopf modules is the Eilenberg-Moore construction \(\nu : \mathcal{M}(A \otimes -, A)^\theta \rightarrow \mathcal{M}(A \otimes -, A)\) for the monad \(\theta\) in \(\text{Hom}(\mathcal{M}^{\text{op}}, \text{Cat})\). We denote by \(\varphi\) the left adjoint of \(\nu\). As we explain in Section 11 when \(A\) is the pseudomonoid in \(\text{Comod}(\text{Vect})\) induced by a coquasi-bialgebra, \(\text{Comod}(\text{Vect})(A, A)^\theta\) is the category of Hopf modules described in [28]. In particular, when \(A\) is induced by a bialgebra, we get the usual category of Hopf modules (simultaneous \(A\)-bicomodules and left \(A\)-modules, plus compatibility between the two structures).

**Definition 3.2.** We say that the theorem of Hopf modules holds for a map pseudomonoid \(A\) if the pseudonatural transformation \(\lambda\) given by

\[
\mathcal{M}(-, A) \xrightarrow{\mathcal{M}(j^* \otimes -, A)} \mathcal{M}(A \otimes -, A) \xrightarrow{\varphi} \mathcal{M}(A \otimes -, A)^\theta
\]

is an equivalence.

**Observation 3.1.** The composition \(\nu X \lambda_X = \theta X\mathcal{M}(j^* \otimes X, A) : \mathcal{M}(X, A) \rightarrow \mathcal{M}(A \otimes X, A)\) is, up to isomorphism, the functor given by

\[
(X \xrightarrow{f} A) \mapsto (A \otimes X \xrightarrow{1 \otimes f} A \otimes A \xrightarrow{p}, A).
\]
Recall that a 1-cell in a bicategory is fully faithful if it is a map and the unit of the adjunction is an isomorphism.

**Proposition 3.2.** The pseudonatural transformation $\lambda$ is fully faithful.

**Proof.** It is clear that $\lambda$ has right adjoint $M(j \otimes - , A)$. The component at $X$ of this adjunction is the composite $1 \to M(j \otimes X, A)\circ (j \otimes X, A) \to M(j \otimes X, A)\circ (j \otimes X, A)$, where each arrow is induced by the corresponding unit. Evaluation of this natural transformation at $f : X \to A$ gives the pasted composite (where the unlabelled 2-cells denote the obvious counits)

![Diagram](attachment:diagram.png)

which by $(1)$ is equal to

![Diagram](attachment:diagram.png)

and thus to

![Diagram](attachment:diagram.png)

This last pasting composite clearly is an isomorphism because of the isomorphism $p(A \otimes j) \cong 1$. □

The following observation will be of use in Section 7.
Observe that $\mathcal{M}(A \otimes X, A)^\theta_X$ is the closure under $\nu_X$-split coequalizers of the full subcategory determined by the image of the functor $\varphi_X$, and these coequalizers are preserved by $\varphi_X \mathcal{M}(j^* \otimes -, A^\nu_X)$, since they become absolute coequalizers after applying $\nu_X$. It follows that $\varepsilon_X$ is an isomorphism if and only if $\varepsilon X \varphi_X$ is an isomorphism. Using the fact that each $\nu_X$ is conservative, we deduce that $\varepsilon$ is an isomorphism if and only if $\nu \varepsilon \varphi$ is so.

Observation 3.4. There is another equivalent way of defining Hopf modules. The category $\mathcal{M}(A, A)$ has a convolution monoidal structure, with tensor product $f \ast g = p(A \otimes g)(f \otimes A)p^*$ and unit $j j^*$. This monoidal category acts on the pseudofunctor $\mathcal{M}(A \otimes -, A) : \mathcal{M}^{\text{op}} \to \text{Cat}$ by sending $h : A \otimes X \to A$ to $p(A \otimes h)(p^* \otimes X)$, in the sense that this defines a monoidal functor from $\mathcal{M}(A, A)$ to $\text{Hom}(\mathcal{M}^{\text{op}}, \text{Cat})(\mathcal{M}(A \otimes -, A), \mathcal{M}(A \otimes -, A))$. Now, $1_A : A \to A$ has a canonical structure of a monoid in $\mathcal{M}(A, A)$, with multiplication $p p^* \Rightarrow 1$ and $j j^* \Rightarrow 1$ the respective counits of the adjunctions. Hence, $1_A$ defines via the action described above, a monad on $\mathcal{M}(A \otimes -, A)$ in $\text{Hom}(\mathcal{M}^{\text{op}}, \text{Cat})$. This monad is just the monad $\theta$ of Definition 3.1.

4. OPMONOIDAL MORPHISMS AND OPLAX ACTIONS

In this section we spell out the relation between opmonoidal morphisms and right oplax actions in a right closed Gray monoid. Everything in this section is well-known, though we have not found the present formulation in the literature. The case when the monoidal 2-category is strict and has certain completeness conditions is studied in [18].

Let $A$ be a pseudomonoid in $\mathcal{M}$. Briefly, a right oplax action of $A$ on an object $B$ is an oplax algebra for the pseudomonad $- \otimes A$ on $\mathcal{M}$. This amounts to a 1-cell $h : B \otimes A \to B$ together with 2-cells

satisfying axioms dual to those in [23, p. 39] or [19] but without the invertibility requirement on the 2-cells. A morphism of right oplax actions on $B$ from $h$ to
Right oplax actions of $A$ on $B$ and their morphisms form a category $\text{Opact}_A(B)$ which comes equipped with a canonical forgetful functor to $\mathcal{M}(B \otimes A, B)$.

For each Gray monoid $\mathcal{M}$ we have a 2-category $\text{Mon}(\mathcal{M})$ whose objects, 1-cells and 2-cells are respectively pseudomonoids in $\mathcal{M}$, lax monoidal morphisms and monoidal 2-cells. See [25] and references therein. Define $\text{Opmon}\mathcal{M}$ by the following data:

- Objects: the objects of $\mathcal{M}$.
- 1-cells: the 1-cells of $\mathcal{M}$.
- 2-cells: the 2-cells of $\mathcal{M}$, called opmonoidal morphisms, are 1-cells $f : A \to B$ of $\mathcal{M}$ equipped with 2-cells

$$A \otimes A \xrightarrow{(B \otimes f)(f \otimes A)} B \otimes B$$

satisfying the obvious equations, and the 2-cells $f \Rightarrow g$ are the 2-cells of $\mathcal{M}$ satisfying compatibility conditions with $f_2$, $g_2$ and $f_0$, $g_0$.

Now suppose that $\mathcal{M}$ is a right closed Gray monoid in the sense of [6], that is, there is a pseudofunctor $[-, -] : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and a pseudonatural equivalence

$$\mathcal{M}(X \otimes [Y, Z]) \simeq \mathcal{M}(X, [Y, Z])$$

Equivalently, for each pair of objects $Y$, $Z$ of $\mathcal{M}$ there is another one denoted by $[Y, Z]$ and an evaluation 1-cell $\text{ev}_{Y,Z} : Y \otimes [Y, Z] \to Z$ inducing $\mathcal{M}$ for each object $X$ of $\mathcal{M}$, the internal hom $[X, X]$ has a canonical structure of a pseudomonoid; namely, there are composition and identity 1-cells $\text{comp} : [X, X] \otimes [X, X] \to [X, X]$ and $\text{id} : I \to [X, X]$ corresponding respectively to

$$X \otimes [X, X] \otimes [X, X] \xrightarrow{\text{ev} \otimes \text{ev}} X \otimes [X, X] \xrightarrow{\text{ev}} X \text{ and } X \xrightarrow{1_X} X.$$

**Proposition 4.1.** For any pseudomonoid $A$ and any object $B$, the closedness equivalence $\mathcal{M}(B \otimes A, B) \simeq \mathcal{M}(A, [B, B])$ lifts to an equivalence

$$\text{Opact}_A(B) \simeq \text{Opmon}\mathcal{M}(A, [B, B]).$$

Moreover, under this equivalence pseudoactions correspond to pseudomonoidal morphisms.

**Proposition 4.2.**

1. For any map $f : X \to Y$ the 1-cell $[f^*, f]$ from $[X, X]$ to $[Y, Y]$ has a canonical structure of an opmonoidal morphism. If $\tau : f \Rightarrow g$ is an invertible 2-cell then $([\tau^{-1}]^*, \tau) : [f^*, f] \Rightarrow [g^*, g]$ is an invertible monoidal 2-cell.

2. For any pair of objects $X, Y$ of $\mathcal{M}$, the 1-cell $i_Y^X : [X, X] \to [Y \otimes X, Y \otimes X]$ corresponding to $\text{ev} : Y \otimes X \otimes [X, X] \to Y \otimes X$ has a canonical structure of a strong monoidal morphism. Moreover, there are canonical monoidal isomorphisms $(i^W_Y \otimes X)(i^Y_X) \cong i^W_X \otimes Y$.
(3) For any map \( f : X \to Z \) and any object \( Y \) there exists a canonical monoidal isomorphism

\[
[X,X] \xrightarrow{\iota^X_Y} [Y \otimes X, Y \otimes X]
\]

\[
[f^* \otimes 1, 1 \otimes f] \xrightarrow{\simeq} [Y \otimes Z, Y \otimes Z]
\]

(4) Given a map \( f : Y \to Z \) and an object \( X \), the counit of \( f \vdash f^* \) induces a monoidal 2-cell

\[
[X,X] \xrightarrow{\iota^X_Y} [Y \otimes X, Y \otimes X]
\]

\[
[f^* \otimes 1, f \otimes 1] \xrightarrow{\simeq} [Y \otimes Z, Y \otimes Z]
\]

\[\tau_{f^* \otimes 1, f \otimes 1}\]

Prove. (1) It is not hard to show that the 2-cells \( \otimes1 \) and \( \otimes \) equip

\[Y \otimes [X,X] \xrightarrow{f^* \otimes 1} X \otimes [X,X] \xrightarrow{ev} X \xrightarrow{f} Y\]

with a structure of right oplax action of \([X,X]\) on \( Y \), and that

\[Y \otimes [X,X] \xrightarrow{f^* \otimes 1} X \otimes [X,X] \xrightarrow{ev} X \xrightarrow{\tau_{f^* \otimes 1, f \otimes 1}} Y\]

is a morphism of right oplax actions on \( Y \).

(2) The evaluation \( ev : X \otimes [X,X] \to X \) has a canonical structure of right oplax action (in fact, pseudoaction) and it is obvious that any 2-functor \( Y \otimes - \) preserves right oplax actions. This shows that \( \iota^X_Y \) has a canonical opmonoidal structure. The
existence of the isomorphism \((i^W_{Y \otimes X})(i^Y_X) \cong i^{W \otimes Y}_X\) follows from the fact that both 1-cells correspond to the right pseudoaction \(W \otimes Y \otimes \text{ev} : W \otimes Y \otimes X \otimes [X, X] \to W \otimes Y \otimes X\).

(3) The two legs of the rectangle correspond, up to isomorphism, to the 1-cell
\[
Y \otimes Z \otimes [X, X] \xrightarrow{1 \otimes f' \otimes 1} Y \otimes X \otimes [X, X] \xrightarrow{1 \otimes \text{ev}} Y \otimes X \xrightarrow{1 \otimes f} Y \otimes Z
\]
and therefore there exists an isomorphism as claimed. Moreover, this isomorphism is monoidal by Proposition 4.1.

(4) The 2-cell corresponds under the closedness equivalence to
\[
Z \otimes X \otimes [X, X] \xrightarrow{1 \otimes \text{ev}} Z \otimes X \xrightarrow{f' \otimes 1} Y \otimes X \xrightarrow{f \otimes 1} Y \otimes Z
\]
This 2-cell is readily shown to be a morphism of right \([X, X]-\)actions on \(Z \otimes X\). □

5. The object of Hopf modules

In this section we shall assume that \(A\) is a map pseudomonoid in a Gray monoid \(\mathcal{M}\) such that the 2-functor \(A \otimes -\) has right biadjoint \([A, -]\). This is true, for instance, when \(\mathcal{M}\) is closed; see Section 4. Under these assumptions the monad \(\theta\) on \(\mathcal{M}(A \otimes -)\) is representable by a monad \(t : [A, A] \to [A, A]\); that is, there is an isomorphism
\[
\mathcal{M}(A \otimes X, A) \xrightarrow{\theta_X} \mathcal{M}(A \otimes X, A)
\]
\[
\cong \mathcal{M}(X, [A, A]) \xrightarrow{\mathcal{M}(X, t)} \mathcal{M}(X, [A, A])
\]
pseudonatural in \(X\). More explicitly, \(t\) is the 1-cell
\[
[A, A] \xrightarrow{i_A^2} [A \otimes A, A \otimes A] \xrightarrow{[p^*, p]} [A, A]
\]
(9) where \(i_A^2\) was defined in Proposition 4.2. The multiplication and unit of \(t\) are respectively

\[
[A, A] \xrightarrow{i_A^2} [A^2, A^2] \xrightarrow{[p^*, p]} [A, A]
\]
\[
[A^3, A^3] \xrightarrow{[1 \otimes p^*, 1 \otimes p]} [A^2, A^2] \cong [A^2, A^2] \xrightarrow{[p^*, p]} [A, A]
\]
\[
[A^2, A^2] \xrightarrow{[p^*, p]} [A, A]
\]
where the unlabelled 2-cells are ones defined in Proposition 4.2.4. Recall that an opmonoidal monad is a monad in \(\text{Opmon}(\mathcal{M})\) (see Section 4).

**Proposition 5.1.** The monad \(t : [A, A] \to [A, A]\) is opmonoidal.
\textbf{Proof.} It is consequence of Proposition 4.2 and the description of the multiplication and unit of \( t \) above. \hfill \Box

Recall that a (bicategorical) Eilenberg-Moore construction for a monad \( s : B \to B \) in a bicategory \( \mathcal{B} \) is a birepresentation of the pseudofunctor \( \mathcal{B}(-, B)\mathcal{B}(-, s) : \mathcal{B}^{\text{op}} \to \text{Cat} \), or equivalently, the unit \( u : B^s \to B \) of that birepresentation. Op-

monoidal monads \( s : B \to B \) have the property that if they have an Eilenberg-

Moore construction \( u : B^s \to B \) in \( \mathcal{M} \), then this construction lifts to \( \text{Opmon} (\mathcal{M}) \); in other words, the forgetful 2-functor \( \text{Opmon} (\mathcal{M}) \to \mathcal{M} \) creates Eilenberg-Moore objects. Moreover, \( u : B^s \to B \) is strong monoidal and an arrow \( q : C \to B^s \) is opmonoidal (strong monoidal) if and only if \( uq \) is so. The case of \( \mathcal{B} = \text{Cat} \) can be

found in \cite{24}, while the general case is in \cite{2} Lemma 3.2.

\textbf{Definition 5.1.} Suppose that the monad \( t \) has an Eilenberg-Moore construction \( u : [A, A]^t \to [A, A] \), with \( f \circ u \). So, \( [A, A]^t \) has a unique (up to isomorphism) structure of a pseudomonoid such that \( u \) is strong monoidal. The Eilenberg-Moore construction \( u : [A, A]^t \to [A, A] \) is called a \textit{Hopf module construction on} \( A \).

For a justification for the name see Section \ref{section1} below. The Hopf module con-

struction, of course, need not exist in general, and this problem is addressed in the subsequent sections.

\textbf{Observation 5.2.} When \( A \) has a Hopf module construction the pseudonatural transformation \( \lambda \) in Definition \ref{def5.1} is representable by

\[ \ell : A \xrightarrow{[j^*, 1]} [A, A] \xrightarrow{f} [A, A]^t. \]

There exist isomorphisms as depicted below, where \( w \) is the 1-cell corresponding to \( 1_{A^t} \) under the closedness equivalence \( \mathcal{M} (A, [A, A]^t) \cong \mathcal{M} ([A, A], A^t) \).

\begin{equation}
\begin{array}{ccc}
A & \xrightarrow{[j^*, 1]} & [A, A] \\
\downarrow{w} & \cong & \downarrow{t^*_A} \\
[A, A^2] & \cong & [A^2, A^2] \\
\downarrow{[1, p]} & \cong & \downarrow{[p^*, p]} \\
[1, p] & \cong & [A, A]
\end{array}
\end{equation}

\text{The isomorphism on the right hand side is the isomorphism of \( t \)-algebras} \( uf \cong t \) induced by the universal property of \( u \). We consider \( [A, p]w \) as equipped with the unique \( t \)-algebra structure such that \( tA \) is a morphism of \( t \)-algebras. Explicitly, this \( t \)-algebra structure is \( t([A, p]w) \equiv t[p^*, p]j^*_A[A, A] = t[j^*, A] \to t[j^*, A] \cong [A, p]w \), where the non-isomorphic arrow induced by the multiplication of \( t \).

\textbf{Proposition 5.3.} Suppose that \( A \) has a Hopf module construction. The 1-cell \( \ell \) in \( 10 \) is fully faithful and strong monoidal. Moreover, \( \ell \) is an equivalence if and only if the theorem of Hopf modules holds for \( A \) (see Definition \ref{def5.2}).

\text{Proof.} The first and last assertions follow trivially from Proposition \ref{prop5.2} and Definition \ref{def5.1} so we only have to prove that \( \ell \) is strong monoidal, or equivalently, that \( ut \cong t[j^*, A] \) is strong monoidal. This 1-cell is isomorphic to \( [A, p]w \) as in Observation \ref{obs5.2} The 1-cell \( [A, p]w : A \to [A, A] \) corresponds up to isomorphism under \( \mathcal{M} (A, [A, A]) \cong \mathcal{M} (A \otimes A, A) \) to \( p : A \otimes A \to A \), which is obviously a right pseudoaction of \( A \) on \( A \), and hence \( [A, p]w \) is strong monoidal by Proposition \ref{prop4.1}. This endows \( \ell \) with the structure of a strong monoidal morphism, by transport of structure. \hfill \Box
Corollary 5.4. The theorem of Hopf modules holds for $A$ if and only if the 1-cell
\[ A \xrightarrow{w} [A, A^2] \xrightarrow{[A, \eta]} [A, A] \] (12)
provides a Hopf module construction for $A$.

Proof. The pseudonatural transformation $\lambda$ in Definition 3.2 is an equivalence if and only if the composition
\[ \nu \lambda : \mathcal{M}(\cdot, A) \to \mathcal{M}(A \otimes \cdot, A)^\theta \to \mathcal{M}(A \otimes \cdot, A) \]
is an Eilenberg-Moore construction for the monad $\theta$ in $[\mathcal{M}^{op}, \mathbf{Cat}]$. But $\nu \lambda$ is represented by the 1-cell (12) and $\theta$ is represented by $t$, and the result follows. □

Corollary 5.5. (1) Suppose that the monad $t$ has an Eilenberg-Moore construction $f \dashv u : [A, A]^t \to [A, A]$. If the theorem of Hopf modules holds for $A$ then $f$ is a Kleisli construction for $t$.

(2) Suppose that the monad $t$ has a Kleisli construction $k : [A, A] \to [A, A]_t$. If the theorem of Hopf modules holds for $A$ then $k^*$ is an Eilenberg-Moore construction for $t$.

Proof. Let $\mathcal{C} \subset \mathcal{M}(A \otimes X, A)^{\theta_X}$ be the full image of the free $\theta_X$-algebra functor $\varphi_X : \mathcal{M}(A \otimes X, A) \to \mathcal{M}(A \otimes X, A)^{\theta_X}$. When thought of as with codomain $\mathcal{C}$, $\varphi_X$ provides a Kleisli construction for $\theta_X$. The theorem of Hopf modules holds if and only if $\lambda_X = \varphi_X \mathcal{M}(j^* \otimes X, A)$ is an essentially surjective on objects, since it is always fully faithful by Proposition 3.2. Hence, the theorem of Hopf modules holds if and only if the inclusion of $\mathcal{C}$ into $\mathcal{M}(A \otimes X, A)^{\theta_X}$ is an equivalence, which is equivalent to saying that $\varphi_X$ is a (bicategorical) Kleisli construction for $\theta$. This proves (1) since $t$ and $f$ represent $\theta$ and $\varphi$ respectively. To show (2), since $\varphi_X : \mathcal{M}(A \otimes X, A) \to \mathcal{C}$ is a Kleisli construction for $\theta_X$, the 1-cell $k^*$ is an Eilenberg-Moore construction for $t$ if and only if the right adjoint of $\varphi_X$, $\mathcal{C} \hookrightarrow \mathcal{M}(A \otimes X, A)^{\theta_X} \to \mathcal{M}(A \otimes X, A)$, is an Eilenberg-Moore construction for $\theta_X$ and this happens only if the inclusion $\mathcal{C} \hookrightarrow \mathcal{M}(A \otimes X, A)^{\theta_X}$ is an equivalence. □

6. On the existence of hopf modules

In this section we study the existence of the Hopf module construction for an arbitrary map pseudomonoid. Since this construction is an Eilenberg-Moore construction for a certain monad, it is natural to embed $\mathcal{M}$ into a 2-category where this exists, and the obvious choice is the completion of $\mathcal{M}$ under $\mathbf{(Cat}$-enriched) Eilenberg-Moore objects. This is a 2-category $\mathbf{EM}(\mathcal{M})$ with a fully faithful universal 2-functor $E : \mathcal{M} \to \mathbf{EM}(\mathcal{M})$. However, in order to speak of the Hopf module construction for a map pseudomonoid $B$ in $\mathbf{EM}(\mathcal{M})$ we need $\mathbf{EM}(\mathcal{M})$ to be a monoidal 2-category and the pseudofunctor $B \otimes -$ to have right biadjoint.

We prove that when $\mathcal{M}$ is a Gray monoid there exists a model of its completion under Eilenberg-Moore objects which is also a Gray monoid and such that the 2-functor $E : \mathcal{M} \to \mathbf{EM}(\mathcal{M})$ is strict monoidal; this model is the 2-category explicitly described in [20]. In fact, we prove this by extending the assignment $\mathcal{M} \hookrightarrow \mathbf{EM}(\mathcal{M})$ to a monoidal functor on the monoidal category $\mathbf{Gray}$, which turns out to be a Gray-functor. In order to show that if $A \otimes -$ : $\mathcal{M} \to \mathcal{M}$ has right biadjoint then the same is true for $E(A)$ in $\mathbf{EM}(\mathcal{M})$ we have to move from $\mathbf{Gray}$, where the 1-cells are 2-functors, to $\mathbf{Bicat}$, where 1-cells are pseudofunctors. For this we extend $\mathbf{EM}$ to a homomorphism of tricategories on $\mathbf{Bicat}$. So far we have only considered bicategorical Eilenberg-Moore constructions. However, in this section we will use the completion of a 2-category under $\mathbf{Cat}$-enriched Eilenberg-Moore objects. Recall that a $\mathbf{Cat}$-enriched Eilenberg-Moore construction on a monad $s : Y \to Y$ in a 2-category $\mathcal{X}$ is a representation of
the 2-functor \( \mathcal{H}(-, Y) : \mathcal{K} \to \mathbf{Cat} \). Any 2-categorical Eilenberg-Moore construction is also a bicategorical one because 2-natural isomorphisms are pseudonatural equivalences.

From [20] we know that \( \mathbf{EM}(\mathcal{K}) \), the completion under Eilenberg-Moore objects of the 2-category \( \mathcal{K} \), may be described as the 2-category with objects the monads in \( \mathcal{K} \), 1-cells from \((X, r)\) to \((Y, s)\) monad morphisms, i.e., a 1-cells \( f : X \to Y \) equipped with a 2-cell \( \psi : sf \Rightarrow ft \) satisfying

\[
\begin{array}{ccc}
X & \xrightarrow{t} & X \\
\downarrow_{f} & & \downarrow_{f} \\
Y & \xrightarrow{s} & Y
\end{array}
= \begin{array}{ccc}
X & \xrightarrow{t} & X \\
\downarrow_{\psi} & & \downarrow_{\psi} \\
Y & \xrightarrow{s} & Y
\end{array}
\]

and

\[
\begin{array}{ccc}
X & \xrightarrow{t} & X \\
\downarrow_{f} & & \downarrow_{f} \\
Y & \xrightarrow{s} & Y
\end{array}
= \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{\eta} & & \downarrow_{\eta} \\
Y & \xrightarrow{s} & Y
\end{array}
\]

and 2-cells \((f, \psi) \Rightarrow (g, \chi)\) 2-cells \( \rho : sf \Rightarrow gt \) in \( \mathcal{K} \) such that

\[
\begin{array}{ccc}
X & \xrightarrow{t} & X \\
\downarrow_{f} & & \downarrow_{f} \\
Y & \xrightarrow{s} & Y
\end{array}
= \begin{array}{ccc}
X & \xrightarrow{t} & X \\
\downarrow_{\mu} & & \downarrow_{\mu} \\
Y & \xrightarrow{s} & Y
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{t} & X \\
\downarrow_{f} & & \downarrow_{f} \\
Y & \xrightarrow{s} & Y
\end{array}
= \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{\rho} & & \downarrow_{\rho} \\
Y & \xrightarrow{s} & Y
\end{array}
\]

This is called the unReduced form of the 2-cells in [20].

The completion comes equipped with a fully faithful 2-functor \( E : \mathcal{K} \to \mathbf{EM}(\mathcal{K}) \) given on objects by \( X \mapsto (X, 1_X) \). This 2-functor has a universal property: for any 2-category with Eilenberg-Moore objects \( \mathcal{L} \), \( E \) induces an isomorphism of categories \( [\mathbf{EM}(\mathcal{K}), \mathcal{L}] \cong [\mathcal{K}, \mathcal{L}] \), where \([\mathbf{EM}(\mathcal{K}), \mathcal{L}]_{EM} \subset [\mathbf{EM}(\mathcal{K}), \mathcal{L}]\) is the full sub 2-category of Eilenberg-Moore object-preserving 2-functors. Moreover, any object of \( \mathbf{EM}(\mathcal{K}) \) is the Eilenberg-Moore construction on some monad in the image of \( E \).

Denote by \( \mathbf{Hom} \) the category whose objects are 2-categories and whose arrows are pseudofunctors. This category is monoidal under the cartesian product.

**Proposition 6.1.** Completion under Eilenberg-Moore objects defines a strong monoidal functor \( \mathbf{EM} : \mathbf{Hom} \to \mathbf{Hom} \).

**Proof.** We use the explicit description of the Eilenberg-Moore completion given in [20]. Define \( \mathbf{EM} \) on a pseudofunctor \( F : \mathcal{K} \to \mathcal{L} \) as sending an object \((X, r)\) to...
the monad $(FX,Fr)$ in $\mathcal{L}$, a 1-cell $(f,\psi)$ to $(Ff,F\psi)$ and a 2-cell $\rho$ to $F\rho$. The comparison 2-cell $(\text{EM}(g,\chi))(\text{EM}(f,\psi)) \to \text{EM}(g,\chi)(f,\psi)$ is defined to be
\[
(Fr)(Fg)(Ff) \cong F(rgf) \xrightarrow{F((g\circ f)\cdot \chi))} F(gft) \cong F(gf)(Ft)
\]
or what is the same thing
\[
(Fr)(Fg)(Ff) \cong F(rg)(Ff) \xrightarrow{(Fg)(Ff)} (Fg)F(sf) \xrightarrow{(Fg)F(\rho)} F(gf)(Ft) \cong F(gf)(Ft)
\]
where the unlabelled isomorphisms are (the unique possible) compositions of the structural constraints of the pseudofunctor $F$. The axioms of a 2-cell in $\text{EM}(\mathcal{L})$ follow from the fact that $(Fg)(Ff)$ and $F(gf)$ are monad morphisms. Similarly, the identity constraint of $1_{\text{EM}(\mathcal{X})} \to (\text{EM}(F))(1_X)$ is defined as
\[
(Ft)_{1_{FX}} (Ft)(F1_{FX}) \cong (F1_{FX})(Ft) \xrightarrow{(Ft)_{Ft}} (F1_{FX})(Ft)
\]
where $Ft$ is the identity constraint of $F$.

It is clear that this defines a functor $\text{EM}$. It is also clear that it is strong monoidal, with constraints the evident isomorphisms $\text{EM}(\mathcal{X}) \times \text{EM}(\mathcal{L}) \cong \text{EM}(\mathcal{X} \times \mathcal{L})$ and $E_1 : 1 \cong \text{EM}(1)$.

Observation 6.2. If $F : \mathcal{X} \to \mathcal{L}$ is a biequivalence between 2-categories, then $\text{EM}$ is a biequivalence too. This is straightforward from the definition of $\text{EM}$ on pseudofunctors in the proof of Proposition 6.1 above.

Recall from Section 2 the notion of cubical functor.

Corollary 6.3. The pseudofunctor below is a cubical functor whenever $F : \mathcal{X} \times \mathcal{L} \to \mathcal{J}$ is one.
\[
\text{EM}(\mathcal{X}) \times \text{EM}(\mathcal{L}) \xrightarrow{\cong} \text{EM}(\mathcal{X} \times \mathcal{L}) \xrightarrow{\text{EM}(F)} \text{EM}(\mathcal{J})
\]

Proof. Consider 1-cells in $\text{EM}(\mathcal{X}) \times \text{EM}(\mathcal{L})$
\[
((X',t'),(X,t)) \xrightarrow{(f',\psi'),(f,\psi)} ((Y',s'),(Y,s)) \xrightarrow{(g',\chi'),(g,\chi)} ((Z',r'),(Z,r)).
\]
If $(X,t) = (Y,s)$ and $(f,\psi)$ is the identity 1-cell of $(X,t)$, then $(f,\psi) = (1_X,1_t)$, then the constraint defined in 13 above is
\[
F(r',r)F(g',g)F(f',1) = F(r',r)F(g'f',g) \xrightarrow{F(g'f',g)} F(r'f',r)F(g',g)\]
which is exactly the identity 2-cell of the 1-cell $\text{EM}F((g',\chi')(f',\psi), (g,\chi))$ in the 2-category $\text{EM}(\mathcal{J})$. The rest of the proof is similar.

Recall from Section 2 the Gray tensor product of 2-categories. If $\mathcal{X}, \mathcal{L}$ are 2-categories, its Gray tensor product $\mathcal{X} \boxtimes \mathcal{L}$ is a 2-category classifying cubical functors out of $\mathcal{X} \times \mathcal{L}$.

Corollary 6.4. Completion under Eilenberg-Moore objects induces a monoidal functor $\text{EM}$ from $\text{Gray}$ to itself. Furthermore, the 2-functors $E_{\mathcal{X}} : \mathcal{X} \to \text{EM}(\mathcal{X})$ are the components of a monoidal natural transformation.

Proof. Define the structural arrow $\text{EM}(\mathcal{X}) \boxtimes \text{EM}(\mathcal{L}) \to \text{EM}(\mathcal{X} \boxtimes \mathcal{L})$ as corresponding to $\text{EM}(\mathcal{X}) \times \text{EM}(\mathcal{L}) \cong \text{EM}(\mathcal{X} \times \mathcal{L}) \to \text{EM}(\mathcal{X} \boxtimes \mathcal{L})$, which is a cubical functor by Corollary 6.3 and the arrow $1 \to \text{EM}(1)$ as the universal $E_1$. Here the symbol $\boxtimes$ denotes the Gray tensor product. The axioms of lax monoidal functor follow from the fact that $\text{EM}$ is monoidal with respect to the cartesian product.
The naturality of the arrows $E_{\mathcal{K}}$ follows from the universal property of the completions under Eilenberg-Moore objects. We only have to prove that the resulting natural transformation is monoidal. Consider the diagram

One of the two axioms we have to check is the commutativity of the exterior diagram. This commutativity can be proven by observing that each one of the four internal diagrams commute and then applying the universal property of $\mathcal{K} \times \mathcal{L} \to \mathcal{K} \Box \mathcal{L}$. The other axiom, involving $E_1 : 1 \to EM(1)$ is trivial, since $E_1$ itself is the unit constraint.

Corollary 6.5. $EM(\mathcal{M})$ is a Gray monoid whenever $\mathcal{M}$ is a Gray monoid. Moreover, the 2-functor $E_{\mathcal{M}} : \mathcal{M} \to EM(\mathcal{M})$ is strict monoidal, so that $\mathcal{M}$ can be identified with a full monoidal sub 2-category of $EM(\mathcal{M})$.

Proof. We know that $EM$ is a monoidal functor, and as such it preserves monoids. Moreover, $E_{\mathcal{M}}$ is strict monoidal, that is, a morphism of monoids in Gray, since $E$ is a monoidal natural transformation (see Corollary 6.4). □

The tensor product in $EM(\mathcal{M})$ is induced by the one of $\mathcal{M}$; for instance, the tensor product of $(X,r)$ with $(Y,s)$, denoted by $(X,r) \otimes (Y,s)$, is $(X \otimes Y, r \otimes s)$.

In order to show that $EM$ is in fact a Gray-functor we state the following easy result.

Lemma 6.6. Let $\mathcal{V}$ be a symmetric monoidal closed category and $F : \mathcal{V} \to \mathcal{V}$ be a lax monoidal functor. Then, any monoidal natural transformation $\eta : 1_{\mathcal{V}} \Rightarrow F$ induces on $F$ a structure of a $\mathcal{V}$-functor.

Proof. Define $F$ on enriched homs as

$$F_{X,Y} : [X,Y] \xrightarrow{\eta_{[X,Y]}} F([X,Y]) \xrightarrow{\vartheta_{X,Y}} [FX,FY]$$

where $\vartheta_{X,Y}$ is the arrow corresponding to $F[X,Y] \otimes FX \to F([X,Y] \otimes X) \xrightarrow{Fev} FY$. □

Corollary 6.7. $EM : \text{Gray} \to \text{Gray}$ has a canonical structure of Gray-functor.

Proof. Let $\mathcal{V}$ in the lemma above be Gray and $\eta$ be the transformation defined by the inclusions $E_{\mathcal{M}} : \mathcal{K} \to EM(\mathcal{K})$, which is easily shown to be a monoidal transformation. Now apply the lemma. □

Let $Ps(\mathcal{K},\mathcal{L})$ denote the 2-category of pseudofunctors from $\mathcal{K}$ to $\mathcal{L}$, pseudonatural transformations between them and modifications between these.
**Observation 6.8.** In the case of EM, the transformation \( \vartheta_{\mathcal{X},\mathcal{L}} \) is defined by the commutativity of the following diagram

\[
\begin{array}{c}
\text{Ps}(\text{EM}(\mathcal{X}), \text{EM}(\mathcal{L})) \otimes \text{EM}(\mathcal{X}) \\
\text{ev} \\
\text{EMev}
\end{array}
\]

that is,

\[
(\vartheta_{\mathcal{X},\mathcal{L}}(F, \tau))(X, t) = (\text{EMev})(F, \tau) = (ev(F, X), ev(\tau, t)) = (FX, (F\tau_X),
\]

and then EM is defined on homs by the 2-functor

\[
\vartheta_{\mathcal{X},\mathcal{L}} : \text{Ps}(\mathcal{X}, \mathcal{L}) \rightarrow \text{Ps}(\text{EM}(\mathcal{X}), \text{EM}(\mathcal{L}))
\]

whose value on a 2-functor \( F \) is the 2-functor sending a monad \((X, t)\) to \((FX, Ft)\).

Then we see that our Gray-functor has as underlying ordinary functor just the restriction to Gray of the functor in Proposition 6.1.

Denote by Bicat the tricategory of bicategories, pseudofunctors, pseudonatural transformations and modifications as defined in [11 5.6]. (There is another canonical choice for a tricategory structure on Bicat, as explained in that paper.)

We shall describe an extension of the Gray-functor EM to a homomorphism of tricategories \( \overline{\text{EM}} : \text{Bicat} \rightarrow \text{Bicat} \). In order to do this we will use the construction of a homomorphism of tricategories \( \text{Bicat} \rightarrow \text{Gray} \) given in [11, of which we recall some aspects. For each bicategory \( \mathcal{B} \) there is a 2-category \( \text{st}\mathcal{B} \) and a pseudofunctor \( \xi_{\mathcal{B}} : \mathcal{B} \rightarrow \text{st}\mathcal{B} \) inducing for each 2-category \( \mathcal{X} \) an isomorphism of 2-categories \( \text{Bicat}(\mathcal{B}, \mathcal{X}) \cong \text{Ps}(\text{st}\mathcal{B}, \mathcal{X}) \). Moreover, \( \xi_{\mathcal{B}} \) is a biequivalence of bicategories. As usual, we get a pseudofunctor

\[
\text{st}_{\mathcal{A},\mathcal{B}} : \text{Bicat}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Ps}(\text{st}\mathcal{A}, \text{st}\mathcal{B})
\]

which turns out to be a biequivalence. Finally, the object part of the homomorphism of tricategories \( \text{Bicat} \rightarrow \text{Gray} \) is given by \( \mathcal{B} \mapsto \text{st}\mathcal{B} \) while on hom-bicategories it is given by the biequivalence \( \text{st}_{\mathcal{A},\mathcal{B}} \).

Define a homomorphism of tricategories \( \overline{\text{EM}} \) by

\[
\begin{array}{c}
\text{Bicat} \\
\overline{\text{EM}} \\
\text{Gray}
\end{array}
\]

It is given on objects by \( \mathcal{B} \mapsto \text{EM}(\text{st}\mathcal{B}) \) and on homs by

\[
\text{Bicat}(\mathcal{A}, \mathcal{B}) \xrightarrow{\text{st}} \text{Ps}(\text{st}\mathcal{A}, \text{st}\mathcal{B}) \xrightarrow{F} \text{EMPs}(\text{st}\mathcal{A}, \text{st}\mathcal{B}) \xrightarrow{\vartheta} \text{Ps}(\text{EM}(\text{st}\mathcal{A}), \text{EM}(\text{st}\mathcal{B}))
\]

which by the Observation 6.8 sends a pseudofunctor \( F : \mathcal{A} \rightarrow \mathcal{B} \) to the 2-functor \( \text{EM}(\text{st}F) \) defined in Proposition 6.1.

**Proposition 6.9.** Every biadjunction between pseudofunctors \( F \dashv_h G : \mathcal{L} \rightarrow \mathcal{X} \), where \( \mathcal{X} \) and \( \mathcal{L} \) are 2-categories, induces a biadjunction \( \overline{\text{EM}}F \dashv_h \overline{\text{EM}}G \).

**Proof.** Since \( \overline{\text{EM}} \) is a homomorphism of tricategories on Bicat, \( \overline{\text{EM}}F = \text{EM}(\text{st}F) \) is left biadjoint to \( \text{EM}G = \text{EM}(\text{st}G) \). The 2-functor \( \text{st}F \) is defined as the unique 2-functor such that \((\text{st}F)\xi_{\mathcal{X}} = \xi_{\mathcal{L}}F\), and similarly for \( G \). It follows, by functoriality
of \( EM \) with respect to pseudofunctors (Proposition 6.1), that
\[
EM(stF)EM\xi = EM\xi EMF \quad \text{and} \quad EM(stG)EM\xi = EM\xi EMG.
\]
Since each component of \( \xi \) is a biequivalence and these are preserved by \( EM \) (see Observation 6.2), we have
\[
EMF \simeq (EM\xi)^*EM(stF)EM\xi \quad \text{and} \quad EMG \simeq EM(stG)EM\xi.
\]

**Corollary 6.10.** If \( X \) is an object in a Gray monoid \( \mathcal{M} \) such that \( X \otimes - \) has right biadjoint \( [X, -] \), then \( (EX \otimes -) : EM(\mathcal{M}) \to EM(\mathcal{M}) \) has right biadjoint \( \langle EX, - \rangle \) given by \( \langle EX, (Y, s) \rangle = ([X, Y], [X, s]) \).

**Proof.** The 2-functor \( (EX \otimes -) \) is just \( EM(X \otimes -) \), and then by the proposition above it has right biadjoint \( EM([X, -]) \). This is given by the stated formula as a consequence of the description of the effect of \( EM \) on pseudofunctors in the proof of Proposition 6.1. \( \square \)

**Theorem 6.11.** For any closed Gray monoid \( \mathcal{M} \) there exists another Gray monoid \( \mathcal{N} \) and a fully faithful strict monoidal 2-functor \( \mathcal{M} \to \mathcal{N} \) such that any map pseudomonoid in \( \mathcal{M} \) has a Hopf module construction in \( \mathcal{N} \). Moreover, \( \mathcal{N} \) can be taken to be \( EM(\mathcal{M}) \).

**Proof.** The proof is only a matter of putting Corollaries 6.9 and 6.10 together with the definition of the object of Hopf modules. \( \square \)

**Proposition 6.12.** Let \( A \) be a map pseudomonoid in a Gray monoid \( \mathcal{M} \) such that \( A \otimes - \) has right biadjoint. Suppose that the theorem of Hopf modules holds for \( E(A) \in ob EM(\mathcal{M}) \); then it also holds for \( A \). Moreover, in this case \( A \) has a Hopf module construction provided by
\[
A \xrightarrow{\mu} [A, A \otimes A] \xrightarrow{[A, p]} [A, A]
\]
(14)
as in Corollary 6.4.

**Proof.** Consider the image of the monad \( t \) under the 2-functor \( E : \mathcal{M} \to EM(\mathcal{M}) \). Denote by \( \hat{\theta} \) the monad \( EM(\mathcal{M})(-, Et) \) on \( EM(\mathcal{M})(-, E[A,A]) \) and \( \hat{\varphi} = \hat{\upsilon} \) the adjunction arising from its Eilenberg-Moore construction in \( Hom(EM(\mathcal{M})^{op}, \text{Cat}) \). Observe that by the fully faithfulness of \( E \), the monad \( \hat{\theta}_{EMMP} \) can be identified with the monad \( \theta_X \) of Definition 3.1 and the adjunction \( \hat{\varphi}_{EMMP} = \hat{\upsilon}_{EMMP} \) with the adjunction \( \varphi_X = \upsilon_X \) corresponding to \( \theta \).

If the theorem of Hopf modules holds for \( E(A) \) then in particular for each object \( X \) of \( \mathcal{M} \) the functor
\[
EM(\mathcal{M})(E(X), E(A)) \xrightarrow{EM(\mathcal{M})(1, E([j^*, A]))} EM(\mathcal{M})(E(X), E[A, A]) \xrightarrow{\hat{\varphi}_{E(X)}} EM(\mathcal{M})(E(X), E[A, A])\hat{\theta}_{E(X)}
\]
(15)
is an equivalence (Definition 3.2). But by the fully faithfulness of the 2-functor \( E \) this is, up to composing with suitable isomorphisms, just the functor \( \lambda_X \) in Definition 3.2 and then the theorem of Hopf modules holds for \( A \).

The last assertion follows directly from Corollary 6.4. \( \square \)
7. LEFT AUTONOMOUS PSEUDOMONOIDS

In this section we specialise to a special kind of pseudomonoid, central to our work, namely the autonomous pseudomonoids. We begin by recalling the necessary background.

A bidual pair in a Gray monoid \( \mathcal{M} \) is a pseudoadjunction (see for example [19]) in the one-object Gray-category \( \mathcal{M} \). Explicitly, it consists of a pair of 1-cells \( e : X \otimes Y \to I \) and \( n : I \to Y \otimes X \) together with invertible 2-cells

\[
\begin{array}{c}
\begin{array}{ccc}
Y & \xrightarrow{1} & Y \\
\downarrow n & & \downarrow \eta \\
X \otimes Y & & \downarrow \epsilon \\
\end{array} & \quad & \begin{array}{ccc}
X \otimes Y \otimes Y & \xrightarrow{1 \otimes n} & X \otimes Y \\
\downarrow e \otimes 1 & & \downarrow \epsilon \\
X & & X \\
\end{array}
\end{array}
\]

(16)

such that the following composites are identities.

\[
\begin{array}{c}
\begin{array}{ccc}
f \otimes u & \xrightarrow{1 \otimes n \otimes 1} & f \otimes u \otimes f \otimes u \\
\downarrow \psi & & \downarrow \psi \otimes 1 \\
f \otimes u & \xrightarrow{1} & f \otimes u \\
\end{array} & \quad & \begin{array}{ccc}
u \otimes f & \xrightarrow{n \otimes 1} & \nu \otimes f \otimes f \otimes \nu \\
\downarrow \psi & & \downarrow \psi \otimes 1 \\
u \otimes f & \xrightarrow{1} & \nu \otimes f \\
\end{array}
\end{array}
\]

(17)

\[
\begin{array}{c}
\begin{array}{ccc}
u & \xrightarrow{n} & \nu \otimes f \\
\downarrow \psi \otimes 1 & & \downarrow \psi \otimes 1 \\
\nu & \xrightarrow{1} & \nu \\
\end{array}
\end{array}
\]

(18)

The object \( X \) is called a right bidual of \( Y \), denoted by \( Y^\circ \), and \( Y \) is called a left bidual of \( X \), denoted by \( X^\circ \).

A Gray monoid in which every object has a right (left) bidual is called right (left) autonomous.

If \( X \) has a right bidual \( X^\circ \), then the 2-functor \( X \otimes - \) has a right biadjoint \( X^\circ \otimes - \), and \( - \otimes X \) has a left biadjoint \( - \otimes X^\circ \), and dually for left biadjoints. In particular, any right (left) autonomous Gray monoid is right (left) closed with internal hom [18].

Recall from [4] that a left dualization for a pseudomonoid \( (A,j,p) \) in \( \mathcal{M} \) is a 1-cell \( d : A^\circ \to A \) equipped with two 2-cells \( \alpha : p(d \otimes A) \Rightarrow j \) and \( \beta : j \Rightarrow p(A \otimes d) \) satisfying two axioms. Left dualization structures on \( d : A^\circ \to A \) are in bijection with adjunctions \( p(d \otimes A) \dashv (A^\circ \otimes p)(n \otimes A) \) and with adjunctions

\[
p \dashv (p \otimes A)(A \odot d \otimes A)(A \otimes n).
\]

(19)
For example, given $\alpha$ and $\beta$ the counit of the corresponding adjunction (19) is

\[ A \otimes A^\circ \otimes A \xymatrix@=15px{\ar[r]<20pt>^{1 \otimes d \otimes 1} & A^3 \ar[r]<20pt>_{p \otimes 1} & A^2} \]

\[ A \xymatrix@=15px{\ar[r]<20pt>_{1 \otimes n} & A \ar[r]<20pt>_{1 \otimes j} & A^2 \ar[r]<20pt>_{p} & A} \]

(To be precise, in [4] the authors define left dualization in a right autonomous Gray monoid, i.e., a Gray monoid where any object has a right bidual, but the only really necessary condition is that the pseudomonoid itself have a right bidual).

A pseudomonoid equipped with a left dualization is called left autonomous. If a left dualization exists, then it is isomorphic to $(A \otimes e)(p^* \otimes A^\circ)(j \otimes A)$ [4, Proposition 1.2]. Examples of this structure are the left autonomous (pro)monoidal $\mathcal{V}$-categories (for a good monoidal category $\mathcal{V}$) and (co)quasi-Hopf algebras. See [4] or Section 10 and 11.

Given a left autonomous pseudomonoid $A$ define the following 2-cell.

\[ \gamma := \xymatrix{A^2 \ar[r]<20pt>_{1 \otimes p^*} & A^3 \ar[r]<20pt>_{p \otimes 1} & A^2} \]

\[ A \xymatrix@=15px{\ar[r]<20pt>_{1 \otimes n} & A \ar[r]<20pt>_{1 \otimes j} & A^2 \ar[r]<20pt>_{p} & A} \]

In the lemma below we show that this 2-cell $\gamma$ is invertible, and in fact this property will turn out to be equivalent to the existence of a left dualization.

**Lemma 7.1.** For a left autonomous pseudomonoid $A$ the following equality holds.

\[ \gamma = \xymatrix{A^2 \otimes A^\circ \otimes A \ar[r]<20pt>_{1 \otimes d \otimes 1} & A^3} \]

\[ A \xymatrix@=15px{\ar[r]<20pt>_{1 \otimes n} & A \otimes A^\circ \otimes A \ar[r]<20pt>_{1 \otimes d \otimes 1} & A^3} \]

In particular, $\gamma$ is invertible.

**Proof.** The 2-cell on the right hand of (21) pasted with the counit of the adjunction (19) gives the following 2-cell.
which itself is equal to

\[
\begin{array}{c}
A^2 \xrightarrow{A^2 \otimes n} A^2 \otimes A^0 \otimes A \xrightarrow{A^2 \otimes d \otimes 1} A^4 \xrightarrow{1 \otimes p \otimes 1} A^3 \\
A \xrightarrow{1 \otimes j} A^2 \xrightarrow{p} A
\end{array}
\]

The result follows. \[\square\]

Define the 2-cell \(\omega\) as

\[
\begin{array}{c}
A^2 \xrightarrow{1 \otimes p^*} A^3 \\
A \xrightarrow{p} A^2
\end{array}
\]

(22)

Now we state the basic result of this work.

**Theorem 7.2.** Let \((A, j, p)\) be a map pseudomonoid in a Gray monoid \(\mathcal{M}\) and suppose that \(A\) has a right bidual. Then, the following assertions are equivalent.

1. \(A\) is left autonomous.
2. The 2-cell \(\gamma\) in (20) is invertible.
(3) The 2-cell \( \omega \) in (22) is invertible.
(4) The theorem of Hopf modules holds for \( A \).
(5) The functor \( \lambda_{A^e} : \mathcal{M}(A^e, A) \xrightarrow{\mathcal{M}(j^* \otimes 1, 1)} \mathcal{M}(A \otimes A^e, A) \xrightarrow{\varphi_{A^e}} \mathcal{M}(A \otimes A^e, A)^{\theta_{A^e}} \) is an equivalence.

Proof. (1) implies (2) by Lemma 7.1.3, and (3) follows trivially from (2) as (5) does from (4). By Observation 5.3 to prove that (3) implies (4) it is enough to show that for each object \( X \) the natural transformation \( \nu_X \) \( \phi_X \) is an isomorphism. For \( g \in \mathcal{M}(A \otimes X, A)^{\theta_X} \), the component \( \nu_X(\phi_X)_g \) is the pasting

\[
\begin{array}{c}
A \otimes X \xrightarrow{p^* \otimes 1} A \otimes X^2 \xrightarrow{1 \otimes j^* \otimes 1} A \otimes X \xrightarrow{1 \otimes g} A^2 \otimes X \xrightarrow{p} A
\end{array}
\]


where \( \nu \) is the action of \( \theta_X \) on \( g \) and the unlabelled arrow is induced by the counit of \( j \) \( j^* \). This 2-cell pasted with \( 1_{A \otimes X} \cong (A \otimes j^* \otimes X)(p^* \otimes X) \) gives, by the equality (1),

\[
\begin{array}{c}
A \otimes X \xrightarrow{1 \otimes j^* \otimes 1} A^2 \otimes X \xrightarrow{\nu \otimes 1} A^2 \otimes X \xrightarrow{1 \otimes g} A^2 \otimes X \xrightarrow{p} A
\end{array}
\]


Then (23) is equal to the pasting of \( \phi^{-1} : p(A \otimes p) \Rightarrow p(p \otimes A) \) with the following 2-cell

\[
\begin{array}{c}
A \otimes X \xrightarrow{1 \otimes j \otimes 1} A \otimes X \xrightarrow{p^* \otimes 1} A \otimes X \xrightarrow{1 \otimes p^* \otimes 1} A \otimes X \xrightarrow{A^2 \otimes h} A^3 \xrightarrow{p \otimes 1} A^2 \otimes X \xrightarrow{1 \otimes h} A^2
\end{array}
\]


which is nothing but \( \omega \otimes X \) pasted on the right with an isomorphism, and so it is itself an isomorphism.

Now we show that (5) implies (1). Suppose that \( \lambda_{A^e} \) is an equivalence. Define a 1-cell \( b = A \otimes A^e \xrightarrow{p^* \otimes 1} A^2 \otimes A^e \xrightarrow{1 \otimes e} A \); it has a structure of \( \theta_{A^e} \)-algebra \( \theta_{A^e}(b) \Rightarrow b \) given by

\[
\begin{array}{c}
A \otimes A^e \xrightarrow{p^* \otimes 1} A^2 \otimes A^e \xrightarrow{1 \otimes p^* \otimes 1} A^3 \otimes A^e \xrightarrow{A^2 \otimes e} A^2 \otimes A^e \xrightarrow{p} A
\end{array}
\]
Denote by \( d : A^\circ \to A \) a 2-cell corresponding (up to isomorphism) to \( b \); that is, \( b \cong p(A \otimes d) \) in \( \mathcal{M}(A \otimes A^\circ, A)^{\theta_{A^\circ}} \). Then we have

\[
(p \otimes A)(A \otimes d \otimes A)(A \otimes n) \cong (A \otimes e \otimes A)(p^* \otimes A^\circ \otimes A)(A \otimes n)
\]

\[
\cong (A \otimes e \otimes A)(A \otimes A \otimes n)p^*
\]

\[
\cong p^*
\]

showing that \( d \) is a left dualization for \( A \).

If \( A \) has a right bidual the 2-functor \( A \otimes - \) has right biadjoint given by \([A, -] = A^\circ \otimes -\) (see the discussion on biduals at the beginning of the section). In this case, the monad \( t \) of \( (12) \) can be expressed as

\[
t : A^\circ \otimes A \xrightarrow{1 \otimes n \otimes 1} A^\circ \otimes A^\circ \otimes A \xrightarrow{(p^*)^\circ \otimes 1 \otimes 1} A^\circ \otimes A \otimes A \xrightarrow{1 \otimes p} A^\circ \otimes A
\]

or

\[
A^\circ \otimes A \xrightarrow{n \otimes 1 \otimes 1} A^\circ \otimes A \otimes A^\circ \otimes A \xrightarrow{1 \otimes p^* \otimes 1 \otimes 1} A^\circ \otimes A \otimes A^\circ \otimes A \xrightarrow{1 \otimes 1 \otimes e \otimes 1} A^\circ \otimes A \otimes A \xrightarrow{1 \otimes p} A^\circ \otimes A
\]  

(24)

(we omitted the canonical equivalence \( A^\circ \otimes A^\circ \cong (A \otimes A)^\circ \)), and the 1-cell \( \ell \) in \( (10) \) as

\[
A \xrightarrow{1 \otimes 1 \otimes e \otimes 1} A^\circ \otimes A \otimes A \xrightarrow{f} (A^\circ \otimes A)^f.
\]

The 1-cell \( (12) \) can be expressed as \((A^\circ \otimes p)(n \otimes A) : A \to A^\circ \otimes A \otimes A \to A \otimes A \).

Recall that this 1-cell has a canonical \( t \)-algebra structure, described in Observation 5.2.

**Theorem 7.3.** For any map pseudomonoid \( A \) with right bidual the following are equivalent.

1. \( A \) is left autonomous.
2. \( A \) has a Hopf module construction provided by

\[
A \xrightarrow{n \otimes 1} A^\circ \otimes A \otimes A \xrightarrow{1 \otimes p} A^\circ \otimes A.
\]

Moreover, in this case the dualization is given by \( A^\circ \xrightarrow{1 \otimes j} A^\circ \otimes A \xrightarrow{f} A \), where \( f \) is left adjoint to \( (25) \) and thus, by Corollary 7.4, a Kleisli construction for the monad \( t \).

**Proof.** By Corollary 5.4 \( (25) \) is a Hopf module construction for \( A \) if and only if the theorem of Hopf modules holds for \( A \), and this is equivalent to the existence of a left dualization by Theorem 7.2.

As we already mentioned at the beginning of the section, to provide a 1-cell \( d : A^\circ \to A \) with a structure of a dualization is to provide an adjunction \( p(d \otimes A) \dashv (A^\circ \otimes p)(n \otimes A) \).

**Proposition 7.4.** For a left autonomous map pseudomonoid \( A \) the adjunction \( p(d \otimes A) \dashv (A^\circ \otimes p)(n \otimes A) \) induces the monad \( t \). Moreover, this adjunction is monadic.

**Proof.** By Corollary 5.4 we know that \((A^\circ \otimes p)(n \otimes A) : A \to A^\circ \otimes A \) provides an Eilenberg-Moore construction for \( t \).

By definition [11], a right dualization \( d' : A^\vee \to A \) for a pseudomonoid \( A \) in \( \mathcal{M} \) is a left dualization for \( A \) in \( \mathcal{M}^{rev} \), where \( \mathcal{M} \) with the reverse tensor product (or the opposite tricategory, when we think of \( \mathcal{M} \) as a one-object tricategory). In particular, \( A^\vee \) is a left bidual for \( A \). A pseudomonoid equipped with a right dualization is called **right autonomous** and a left and right autonomous pseudomonoid is simply
called *autonomous*. A left autonomous *map* pseudomonoid with dualization \(d\) is autonomous if and only if \(d\) is an equivalence \([4]\) Propositions 1.4 and 1.5.

Recall from \([29]\) that, given monads \(s\) on \(X\) and \(s'\) on \(X'\), a morphism of monads is a pair \((f, \phi)\) where \(f: X \to X'\) is a 1-cell and \(\phi: s'f \Rightarrow fs\) is a 2-cell compatible with the multiplications and units; these compatibility conditions can be found in Section 6. With the obvious definition for the 2-cells, we have a 2-category \(\mathcal{X}\), denoted by \(\text{Mnd}(\mathcal{X})\); a morphism of monads \((f, \phi)\) is an equivalence in \(\text{Mnd}(\mathcal{X})\) precisely when \(f\) is an equivalence and \(\phi\) is invertible.

**Corollary 7.5.** Suppose that \(A\) is an autonomous map pseudomonoid. Then there exists an equivalence of monads

\[
\begin{array}{ccc}
A^\circ \otimes A & \overset{t}{\longrightarrow} & A^\circ \otimes A \\
\downarrow d \otimes 1 & \cong & \downarrow d \otimes 1 \\
A \otimes A & \overset{p^* p}{\longrightarrow} & A \otimes A
\end{array}
\]

and, moreover, \(p^* : A \to A \otimes A\) is monadic.

**Proof.** The first assertion is clear since \(d\) is an equivalence and \(t\) is induced by \(p(d \otimes A) \dashv (d^* \otimes A)p^*\); see Proposition 7.5. By the same theorem, \((d^* \otimes A)p^*\) is monadic, and then so is \(p^*\) since \(d\) is an equivalence. \(\square\)

**Proposition 7.6.** For any left autonomous map pseudomonoid \(A\) the dual duality \(d: A^\circ \to A\) has the structure of a strong monoidal morphism from \((A^\circ, (j^*), (p^*)^\circ)\) to \((A, j, p)\).

**Proof.** It is enough to show that

\[
A^\circ \xrightarrow{d} A \xrightarrow{n \otimes 1} A^\circ \otimes A \otimes A \xrightarrow{1 \otimes p} A^\circ \otimes A
\]

(26)

is strong monoidal, since \((A^\circ \otimes p)(n \otimes A)\) is an Eilenberg-Moore object in the 2-category \(\text{Opmon}(\mathcal{M})\). In the proof of Theorem 7.2 we saw that \(p(A \otimes d) \cong (A \otimes e)(p^* \otimes A^\circ)\), so we have to show that \((A^\circ \otimes A \otimes e)(A^\circ \otimes p^* \otimes A^\circ)(n \otimes A^\circ)\) is a strong monoidal morphism, or equivalently, by Proposition 7.1, that \((A \otimes e)(p^* \otimes A^\circ) : A \otimes A^\circ \to A\) is a right pseudoaction of \(A^\circ\) on \(A\) (i.e., a \((- \otimes A^\circ)\)-pseudoalgebra structure on \(A\)). This itself turns to be equivalent to say that \(p^* : A \to A \otimes A\) is a right pseudoaction of \(A\) on \(A\) (i.e., a \((- \otimes A)\)-pseudoalgebra structure on \(A\)), which is obviously true. \(\square\)

We finish this section with some comments on autonomous monoidal lax functors. The notion of *right autonomous monoidal lax functor* was introduced in \([4]\), and it consists of a monoidal lax functor equipped with structure necessary to ensure that it preserves, in some lax sense, right biduals. Another way of looking at this concept is as a monoidal lax functor with extra structure such that when we take the domain Gray monoid as the unit Gray monoid (i.e., the Gray monoid whose only cells are identities), then we get a *left* autonomous pseudomonoid. More explicitly, a right autonomous monoidal lax functor is a monoidal lax functor \(F\) equipped with a pseudonatural transformation \(\kappa_X : (FX)^\circ \to F(X^\circ)\) and modifications

\[
\begin{align*}
F(X) \otimes (F(X))^\circ & \xrightarrow{\kappa_X} F(X) \otimes F(X^\circ) \\
\downarrow \xi \hat{\phi} & \quad \downarrow \xi X.X^\circ & \quad \downarrow \xi X^\circ.X & \quad \downarrow \xi X^\circ.X \\
F(I) & \quad F(X \otimes X^\circ) & \quad F(I) & \quad F(X^\circ \otimes X)
\end{align*}
\]

\[
\begin{align*}
F(X) \otimes F(X)^\circ & \xrightarrow{1 \otimes \kappa_X} F(X) \otimes F(X^\circ) \\
\downarrow \xi \hat{\phi} & \quad \downarrow \xi X.X^\circ & \quad \downarrow \xi X^\circ.X & \quad \downarrow \xi X^\circ.X \\
F(I) & \quad F(X \otimes X^\circ) & \quad F(I) & \quad F(X^\circ \otimes X)
\end{align*}
\]
Proposition 7.7. Let $F : \mathcal{M} \rightarrow \mathcal{N}$ be a monoidal special lax functor between right autonomous Gray monoids and $A$ be a left autonomous map pseudomonoid in $\mathcal{M}$. Assume $F$ has the following two properties: the monoidal constraints $\iota : I \rightarrow FA$ and $\chi_{A,A} : FA \otimes FA \rightarrow FA \otimes A$ are maps, and the 2-cell below is invertible.

Then, the map pseudomonoid $FA$ is left autonomous with left dualization

$$F(A)^c \xrightarrow{(Fj)\otimes \iota} F(A) \otimes F(A)^c \xrightarrow{\chi(Fp)^c \otimes \iota} F(A)^2 \otimes F(A)^c \xrightarrow{1 \otimes \iota \otimes Fp} F(A)^2 \xrightarrow{\eta} F(A).$$

(27)

Proof. Recall that $F(A)$ has multiplication $F(p)\chi : FA \otimes FA \rightarrow FA$ and unit $F(\iota) : I \rightarrow FA$, so that it is a map pseudomonoid. Using the conditions above plus the fact that (28) is invertible, it can be shown that the corresponding 2-cell for $FA$ is invertible, and hence $FA$ is left autonomous. The formula for the left dualization is just the general expression of any left dualization in terms of the product, unit and evaluation. □

If $F$ is strong monoidal (sometimes called weak monoidal) in the sense that $\iota$ and $\chi$ are equivalences, then $F$ preserves biduals; more explicitly, there exists $\kappa : FA \rightarrow FA^c$, unique up to isomorphism, such that

$$(F(A) \otimes FA^c) \xrightarrow{1 \otimes \kappa} F(A) \otimes FA^c \xrightarrow{\chi_{A,A}^c} F(A \otimes A^c) \xrightarrow{Fe} FI \cong \iota e,$$

(28)

and $\kappa$ is a fortiori an equivalence.

Proposition 7.8. Suppose $F : \mathcal{M} \rightarrow \mathcal{N}$ is a strong monoidal special lax functor between Gray monoids and $A$ is a left autonomous map pseudomonoid in $\mathcal{M}$ with left dualization $d$. Then $FA$ is a left autonomous map pseudomonoid too, with left dualization $(Fd)\kappa : FA \rightarrow FA$.

Proof. The fact that (21) is invertible and that $\chi : FA \otimes FA \rightarrow FA \otimes A$ is an equivalence ensures that the hypotheses of Proposition 7.7 are satisfied, and hence $FA$ is left autonomous. The formula for the dualization follows from (27) using (28) and the fact that $\chi$ is an equivalence. □

Note that although losing some generality, we gain in simplicity by restricting to the case of left autonomous map pseudomonoids, in what our proofs are not based on big diagrams but on the theory of Hopf modules.

The following is a simple corollary about left autonomous map pseudomonoids in a braided Gray monoid. For background about braidings in a Gray monoid see Section 9.1 and references therein.
Corollary 7.9. If $A$ and $B$ are left autonomous map pseudomonoids, with left dualizations $d_A$ and $d_B$ respectively, in a braided Gray monoid $\mathcal{M}$, then $A \otimes B$ is a left autonomous map pseudomonoid too, with left dualization

$$B^\circ \otimes A^\circ \xrightarrow{c_{B^\circ A^\circ}} A^\circ \otimes B^\circ \xrightarrow{(d_A \otimes 1)(1 \otimes d_B)} A \otimes B.$$ 

Proof. As pointed out in [6], the tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to M$ is a strong monoidal pseudofunctor with $\chi_{(X,Y),(Z,W)} = 1 \otimes c_{Y,Z} \otimes 1 : X \otimes Y \otimes Z \otimes W \to X \otimes Z \otimes Y \otimes W$, and $e$ equal to the identity. Then, it is easy to show that, if we take $B^\circ \otimes A^\circ$ as right bidual for $A \otimes B$, the corresponding 1-cell $\kappa$ is just $c_{B^\circ A^\circ}$. □

8. Frobenius and autonomous map pseudomonoids

In this section we study the relationship between autonomous pseudomonoids, the condition [2] in Theorem [7,2] and Frobenius pseudomonoids. In [7] it is shown that any autonomous pseudomonoid is Frobenius, and we showed in Theorem [2] that autonomy is equivalent to the invertibility of the 2-cell $\gamma$ in (20) and its dual, i.e., the corresponding 2-cell $\gamma'$ in $\mathcal{M}^{\text{rev}}$. We show a converse in absence of biduals, namely: if $\gamma$ and $\gamma'$ are invertible, then $A$ is Frobenius, and as such it has right and left bidual, and moreover $A$ is autonomous.

A Frobenius structure for a pseudomonoid $A$ is a 1-cell $\varepsilon : A \to I$ such that $\varepsilon p : A \otimes A \to I$ is the evaluation of a bidual pair;

Lemma 8.1. Let $A$ be a pseudomonoid whose multiplication $p$ is a map, and call $\gamma$ and $\gamma'$, respectively, the following 2-cells.

Then the following equalities hold

Proof. The proof is a standard calculation involving mates and the axioms of a pseudomonoid. □

Proposition 8.2. Suppose $A$ is a map pseudomonoid and that the 2-cells $\gamma$ and $\gamma'$ in Lemma 8.1 are invertible. Then $j^* p : A \otimes A \to I$ and $p^* j : I \to A \otimes A$ have the
structure of a bidual pair. In particular, $A$ is a Frobenius pseudomonoid and given a choice of right and left bifurcations, $A$ is autonomous.

Proof. The 2-cells

$$(j^* \otimes A)(p \otimes A)(A \otimes p^*)(A \otimes j) \xrightarrow{(j^* \otimes A)\gamma(A \otimes j)} (j^* \otimes A)p^*(A \otimes j) \cong 1_A$$

$(A \otimes j^*)(A \otimes p)(p^* \otimes A)(j \otimes A) \xrightarrow{(A \otimes j^*)\gamma'(j \otimes A)} (A \otimes j^*)p^*(j \otimes A) \cong 1_A$

endow $j^*p$ and $p^*j$ with the structure of a bidual pair. The axioms of a bidual pair follow form Lemma 8.1.

Observation 8.3. In the hypothesis of the proposition above, different choices of a bifurcated object for $A$ give rise to different dualizations. For example, when we take the bifurcated pair $j^*p, p^*j$, so that $A$ is right and left bifurcated of itself, the resulting left and right dualizations are just the identity $1_A$. Slightly more generally, given any equivalence $f : B \rightarrow A$, $B$ has a canonical structure of a bifurcated dual of $A$ such that the corresponding left dualization is (isomorphic) to $f$. To see this just consider the evaluation $j^*p(A \otimes f) : A \otimes B \rightarrow I$ and the coevaluation $(f^* \otimes A)p^*j : I \rightarrow B \otimes A$.

9. Hopf modules and the centre construction

The most classical notion of the centre of an algebraic structure is the centre of a monoid. If $M$ is a monoid, its centre is the set of elements of $M$ with the property of commuting with every element of $M$. Anyone would agree if we slightly change our point of view and said that the centre of $M$ is the set whose elements are pairs $(x, (x \cdot -) = (- \cdot x))$: elements of $x \in M$ equipped with the extra structure of an equality between the multiplication with $x$ on the left and on the right. The centre of a monoidal category, defined in 10, follows the spirit of the latter: from the algebraic structure of a monoidal category $\mathcal{C}$ one forms a new algebraic structure $Z\mathcal{C}$, called the centre of $\mathcal{C}$. What we actually have is a functor $Z\mathcal{C} \rightarrow \mathcal{C}$, and $Z\mathcal{C}$ has a monoidal structure such that this functor is strong monoidal. Moreover, $Z\mathcal{C}$ has a canonical braiding. The objects of $Z\mathcal{C}$ are pairs $(x, \gamma_x)$ where $\gamma_x : (- \otimes x) \Rightarrow (x \otimes -)$ is an invertible natural transformation. In this context one can also consider the lax centre of $\mathcal{C}$, simply by dropping the requirement of the invertibility of $\gamma_x$.

See Example 9.1. The functor $Z\mathcal{C} \rightarrow \mathcal{C}$ is the universal one satisfying certain commutation properties, as we shall see later.

Another centre-like object classically considered is the Drinfeld double of a finite-dimensional Hopf algebra, or, more recently, of a (co)quasi-Hopf algebra. See 22. Here the concept is not the one of the object classifying maps with certain commutation properties, but it is a representational one. Roughly speaking, the Drinfeld double of a finite dimensional Hopf algebra $H$ is a Hopf algebra $D(H)$ such that the category of representations of $D(H)$ is monoidally equivalent to the centre of the category of representations of $H$.

In this section we study centres and lax centres of autonomous pseudomonoids by means of the theory of Hopf modules developed in the previous sections. When applied to the bicategory of comodules, this approach proves the existence of the centre of a finite dimensional coquasi-Hopf algebra (considered as a pseudomonoid) and, moreover, this centre is equivalent to the Drinfeld’s double (see Section 11). When applied to the bicategory of $\mathcal{V}$-modules, we see that left autonomous promonoidal $\mathcal{V}$-categories always have lax centres (see Section 10).

9.1. The lax centre. We shall work in a braided Gray monoid, in the sense of 9. A braided Gray monoid is a Gray monoid equipped with pseudonatural equivalences $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ and invertible 2-cells

$$(X \otimes c_{W \otimes Y, Z})(c_{W \otimes X}Y \otimes Z) \cong (c_{W, X \otimes Z} \otimes Y)(W \otimes X \otimes c_{Y, Z}).$$
satisfying three axioms. These axioms imply that the tensor pseudofunctor \( \otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) is (strong) monoidal.

The centre of a pseudomonoid was defined in [31]. Here we will be interested in the lax version of the centre, called the lax centre of a pseudomonoid. The definition is exactly the same as that of the centre but for the fact that we drop the requirement of the invertibility of certain 2-cells.

**Definition 9.1.** Given a pseudomonoid in a braided Gray monoid \( \mathcal{M} \) define for each object \( X \) a category \( CP_\ell(X, A) \). The objects, called lax centre pieces, are pairs \((f, \gamma)\) where \( f : X \to A \) is a 1-cell and \( \gamma \) is a 2-cell

\[
\begin{array}{ccc}
A \otimes X & \xleftarrow{c_{X, A}} & X \otimes A \\
\downarrow \downarrow & & \downarrow \downarrow \\
A \otimes A & \xleftarrow{z} & A \otimes A \\
\downarrow & & \downarrow \\
A & \xleftarrow{p} & A
\end{array}
\]

satisfying axioms [30] and [31] in Figure 2. The arrows \((f, \gamma) \to (f', \gamma')\) are the 2-cells \( f \Rightarrow g \) which are compatible with \( \gamma \) and \( \gamma' \) in the obvious sense.

This is the object part of a pseudofunctor \( CP_\ell(-, A) : \mathcal{M}^{op} \to \mathbf{Cat} \), that is defined on 1-cells and 2-cells just by precomposition. When \( CP_\ell \) is birepresentable we call a birepresentation \( z_\ell : Z_\ell A \to A \) a lax centre of the pseudomonoid \( A \).

A centre piece is a lax centre piece \((f, \gamma)\) such that \( \gamma \) is invertible. The full subcategories \( CP(X, A) \subset CP_\ell(X, A) \) objects the centre pieces define a pseudofunctor \( CP(-, A) : \mathcal{M}^{op} \to \mathbf{Cat} \), and we call a birepresentation of it a centre of \( A \), denoted by \( z : Z A \to A \).

**Definition 9.2.** The inclusion \( CP(-, A) \hookrightarrow CP_\ell(-, A) \) induces a 1-cell \( z_\ell : Z A \to Z_\ell A \), unique up to isomorphism such that \( z_\ell z_\ell \cong z \) as centre pieces. When \( z_\ell \) is an equivalence we will say that the centre of \( A \) coincides with the lax centre.

**Example 9.1.** The centre of a pseudomonoid in \( \mathbf{Cat} \), that is, of a monoidal category, is the usual centre defined in [16]. In fact, lax centres and centres of pseudomonoids in \( \mathcal{Y} \text{-Cat} \) exist and are given by the constructions in [5]. If \( \mathcal{A} \) is a monoidal \( \mathcal{Y} \)-category, its lax centre \( Z_\ell \mathcal{C} \) has objects pairs \((x, \gamma)\) where \( x \) is an object of \( \mathcal{C} \) and \( \gamma : (- \otimes x) \Rightarrow (x \otimes -) \) is a \( \mathcal{Y} \)-natural transformation. The \( \mathcal{Y} \)-enriched hom \( Z_\ell \mathcal{C}((x, \gamma), (y, \delta)) \) is the equalizer of the pair of arrows

\[
\begin{array}{ccc}
\mathcal{C}(x, y) & \xrightarrow{\mathcal{C}(\text{id}, \text{id})} & \mathcal{C}((- \otimes x, - \otimes y) \\
\downarrow & & \downarrow \\
\mathcal{C}(x \otimes -, y \otimes -) & \xrightarrow{\mathcal{C}(\text{id}, \text{id})} & \mathcal{C}(x \otimes -, - \otimes y)
\end{array}
\]

**Observation 9.1.** By [31], in a bicategory with finite products, iso-inserters and cotensoring with the arrow category any pseudomonoid has a centre.

We would like to exhibit an equivalence \( \mathcal{M}(I, Z_\ell A) \simeq Z_\ell \mathcal{M}(I, A) \). Our leading example is the one of the bicategory \( \mathcal{Y} \text{-Mod} \) of \( \mathcal{Y} \)-categories and \( \mathcal{Y} \)-modules. For details about this bicategory see Section 10. Henceforth, we shall assume our Gray monoid \( \mathcal{M} \) satisfies additional properties, which we explain below.
Recall that a 2-cell
\[
\begin{array}{c}
Y \\
\xrightarrow{g} \\
X \\
\xleftarrow{f}
\end{array}
\]
in a bicategory \( \mathcal{B} \) is said to exhibit \( f \circ g \) as the right lifting of \( g \) through \( f \) if it induces a bijection \( \mathcal{B}(Y, X)(k, f \circ g) \cong \mathcal{B}(Y, Z)(k, f) \), natural in \( k \). Clearly, right liftings are unique up to compatible isomorphisms. See [32].

We shall assume that our braided Gray monoid \( \mathcal{M} \) is closed (see Section 4 and references therein) and has right liftings of arrows out of \( I \) through arrows out of \( I \). As explained in [6], this endows each \( \mathcal{M}(X, Y) \) with the structure of a \( \mathcal{V} \)-category. Here \( \mathcal{V} = \mathcal{M}(I, I) \) is a symmetric monoidal closed category whose tensor product is given by composition. The \( \mathcal{V} \)-enriched hom \( \mathcal{M}(X, Y)(f, g) \) is \( f \circ g \), the right lifting
Proof. We give only a sketch of a proof; the details are an exercise in the universal property of right liftings. Given two lax centre pieces \((f, \alpha)\) and \((g, \beta)\) of \(\hat{\mathcal{A}}\) under postcomposition with the arrows where the unlabelled arrows are induced by the universal property of the right liftings. Observe that the underlying category of the \(\mathcal{Y}\)-category \(\mathcal{M}(X,Y)\) is the hom-category \(\mathcal{M}(X,Y)\). For, \(\mathcal{Y}(1_1, \mathcal{M}(X,Y)(f,g)) = \mathcal{Y}(1_1, \hat{f} \hat{g}) \equiv \mathcal{M}(I,[X,Y])(\hat{f}, \hat{g}) \equiv \mathcal{M}(X,Y)(\hat{f}, \hat{g}).\)

One can define composition \(\mathcal{Y}\)-functors \(\mathcal{M}(Y,Z) \otimes \mathcal{M}(X,Y) \to \mathcal{M}(X,Z)\) on objects just by composition in \(\mathcal{M}\) and on \(\mathcal{Y}\)-enriched homs in the following way. Given \(f,h : Y \to Z\) and \(g,k : X \to Y\), define an arrow \(\mathcal{M}(I,[Y,Z])(\hat{f}, \hat{h}) \otimes \mathcal{M}(I,[X,Z])(\hat{g}, \hat{k}) \to \mathcal{M}(I,[X,Z])(\hat{f} \hat{g}, \hat{hk})\) as the 2-cell in \(\mathcal{M}\) corresponding to the following pasting.

![Diagram](image)

There are also identity \(\mathcal{Y}\)-functors from the trivial \(\mathcal{Y}\)-category to \(\mathcal{M}(X,X)\). On objects they just pick the identity 1-cells \(1_X\) and homs they are given by the arrows \(1_I \to (1_X)_{\hat{I}}\) corresponding to the identity 2-cells \(1_X \Rightarrow 1_X\). These composition and identity \(\mathcal{Y}\)-functors endow \(\mathcal{M}\) with the structure of a category weakly enriched in \(\mathcal{Y}\text{-Cat}\), in the sense that the category axioms hold only up to specified \(\mathcal{Y}\)-natural isomorphisms (e.g. when \(\mathcal{Y}\) is the category of sets, we get a bicategory with locally small hom-categories).

Now we shall further suppose that the category \(\mathcal{Y} = \mathcal{M}(I,I)\) is complete. This allows us to consider functor \(\mathcal{Y}\)-categories. In this situation, the composition \(\mathcal{Y}\)-functors induce \(\mathcal{Y}\)-functors \(\mathcal{M}(X,-)_{\mathcal{Y},Z} : \mathcal{M}(Y,Z) \to [\mathcal{M}(X,Y),\mathcal{M}(X,Z)]\) making the pseudofunctor \(\mathcal{M}(X,-) : \mathcal{M} \to \mathcal{Y}\text{-Cat}\) locally a \(\mathcal{Y}\)-functor.

**Lemma 9.2.** Under the hypothesis above, if \(A\) is a pseudomonoid in \(\mathcal{M}\), \(CP_{\mathcal{Y}}(I,A)\) has a canonical structure of a \(\mathcal{Y}\)-category such that the forgetful functor \(CP_{\mathcal{Y}}(I,A) \to \mathcal{M}(I,A)\) is the underlying functor of a \(\mathcal{Y}\)-functor. Moreover, \(CP(I,A)\) is a full sub-\(\mathcal{Y}\)-category of \(CP_{\mathcal{Y}}(I,A)\).

**Proof.** We give only a sketch of a proof; the details are an exercise in the universal property of right liftings. Given two lax centre pieces \((f, \alpha)\) and \((g, \beta)\), define the \(\mathcal{Y}\)-enriched hom \(CP_{\mathcal{Y}}(I,A)((f, \alpha), (g, \beta))\) as the equalizer in \(\mathcal{Y}\) of the pair

\[
\mathcal{M}(I,A)(f,g) \xrightarrow{\mathcal{M}(A,A)(p(A \otimes f), p(A \otimes g))} \mathcal{M}(A,A)(p(A \otimes f), p(A \otimes g)) \xrightarrow{\mathcal{M}(A,A)(\alpha,1)} \mathcal{M}(A,A)(p(A \otimes f \otimes A), p(A \otimes g)) = \mathcal{M}(A,A)(p(f \otimes A), p(g \otimes A)) \xrightarrow{\mathcal{M}(A,A)(1,\beta)} \mathcal{M}(A,A)(p(f \otimes A), p(A \otimes g))
\]

where the unlabelled arrows are induced by the universal property of right liftings under postcomposition with the arrows \(A \to [A,A]\) corresponding to \(p\) and \(pc_{A,A}\).
With this definition, an arrow $1_f \to CP_{\ell}(I, A)((f, \alpha), (g, \beta))$ in $\mathcal{V} = \mathcal{M}(I, I)$ corresponds to an arrow $(f, \alpha) \to (g, \beta)$ in the ordinary category $CP_{\ell}(I, A)$. The composition $CP_{\ell}(I, A)((g, \beta), (h, \gamma)) \otimes CP_{\ell}(I, A)((f, \alpha), (g, \beta)) \to CP_{\ell}(I, A)((f, \alpha), (h, \gamma))$ is induced by the composition $\mathcal{M}(I, A)(g, h) \otimes \mathcal{M}(I, A)(f, g) \to \mathcal{M}(I, A)(f, h)$ and the universal property of the equalizers, and likewise for the identities.

Proposition 9.3. Assume the lax centre of $A$ exists, with universal centre piece $(z_{\ell}, \gamma)$. Under the hypothesis above, $(z_{\ell}, \gamma)$ induces a $\mathcal{V}$-enriched equivalence $U$ making the following diagram commute.

\[
\begin{array}{ccc}
\mathcal{M}(I, Z_{\ell}A) & \xrightarrow{U} & CP_{\ell}(I, A) \\
\downarrow & & \downarrow \\
\mathcal{M}(I, A) & \xrightarrow{\mathcal{M}(I, z_{\ell})} & CP_{\ell}(I, A)
\end{array}
\]

Moreover, the same is true if the centre of $A$ exists and we use $CP(I, A)$ instead of $CP_{\ell}(I, A)$.

Proof. Let $(z_{\ell}, \gamma) \in CP_{\ell}(Z_{\ell}A, A)$ be the universal lax centre piece. On objects, $U$ is equal to the usual functor, that is, it sends $f : I \to Z_{\ell}A$ to the lax centre piece $(z_{\ell}f, \gamma(f \otimes A))$. Next we describe our $\mathcal{V}$-functor $U$ on homs. Define $\varphi$ by the following equality, where $\pi$ exhibits $\ell_k$ as a right lifting of $k$ through $h$ and $\varpi$ exhibits $(z_{\ell}h)(z_{\ell}k)$ as a right lifting of $z_{\ell}k$ through $z_{\ell}h$.

\[
\begin{array}{ccc}
I & \xrightarrow{h_k} & Z_{\ell}A \\
\downarrow \pi & & \downarrow z_{\ell} \\
A & \xrightarrow{\varphi} & Z_{\ell}A \\
\downarrow \varpi & & \downarrow z_{\ell}
\end{array}
\]

This pasted composite is trivially a morphism of lax centre pieces $U(h^\ell k) \to U(k)$, and this means exactly that $\varphi$ factors through the equalizer

\[
CP_{\ell}(I, A)(U(h), U(k)) \to (z_{\ell}h)(z_{\ell}k) = \mathcal{M}(I, A)(z_{\ell}h, z_{\ell}k);
\]

in defining $CP_{\ell}(I, A)(U(h), U(k))$ on $\mathcal{V}$-enriched homs. Denote by $\tilde{\varphi} : h_k = \mathcal{M}(I, A)(h, k) \to CP_{\ell}(I, A)(U(h), U(k))$ the resulting arrow in $\mathcal{V}$. This is by definition the effect of $U$ on enriched homs.

Observe that the underlying ordinary functor of $U$ is the usual equivalence given by the universal property of the lax centre. Hence, $U$ is essentially surjective on objects as a $\mathcal{V}$-functor. It is sufficient, then, to show that $U$ is fully faithful, or, in other words, that $\tilde{\varphi}$ is invertible. To do this, we shall show that $\varphi$ has the universal property of the equalizer defining $CP_{\ell}(I, A)(U(h), U(k))$. 


Suppose $\nu : v \to (z_l h)(z_l k)$ is an arrow in $\mathcal{V}$ equalizing the pair of arrows $(z_l h)(z_l k) \to \mathcal{M}(A, A)(p(z_l h \otimes A), p(A \otimes z_l k))$ analogues to (32). If one unravels this condition, one gets the following equality.

\[
\begin{array}{c}
A \\
\downarrow h \otimes 1 \\
Z_l A \otimes A \\
\downarrow z_l \otimes 1 \\
A^2 \\
\downarrow p \\
A
\end{array}
\quad \cong \quad
\begin{array}{c}
A \\
\downarrow h \otimes v \\
A \otimes Z_l A \\
\downarrow (z_l h)(z_l k) \otimes 1 \\
A^2 \\
\downarrow p \\
A
\end{array}
\]

This means that the 2-cell $\varpi(z_l h v)$ is an arrow in the ordinary category $\mathcal{CP}_1(I, A)$ from $U(hv) = (z_l h v, \gamma((hv) \otimes A))$ to $U(k) = (z_l k, \gamma(k \otimes A))$, and therefore there exists a unique 2-cell $\tau : hv \Rightarrow k : I \to Z_l A$ such that $z_l \tau = \varpi(z_l h v)$. From the universal property of right liftings, we deduce the existence of a unique $\tau' : v \Rightarrow b k$ such that $\pi(h \tau') = \tau$. In order to show that $\varrho : b k \Rightarrow (z_l h)(z_l k)$ has the universal property of the equalizer as explained above, we have to show that $\varrho \tau' = \nu$. But the pasting of $\varrho \tau'$ with $\varpi$, $\varpi(z_l h (\varrho \tau'))$, is equal, by definition of $\varrho$, to $z_l (\pi(h \tau')) = z_l \tau = \varpi(z_l h v)$. It follows that $\varrho \tau' = \nu$.

The case of the centre is completely analogous to the case of the lax centre. The $\mathcal{V}$-functor $U$ is defined on objects by sending $f : I \to Z A$ to the centre piece $(z f, \gamma(f \otimes A))$, where $(z, \gamma)$ is the universal centre piece. The definition of $U$ on $\mathcal{V}$-enriched homs is the same as in the case of the lax centre above. 

In order to exhibit the desired equivalence $\mathcal{M}(I, Z_l A) \simeq Z_l(\mathcal{M}(I, A))$, we shall require of our closed braided Gray monoid $\mathcal{M}$ three further properties.

Firstly, we require the monoidal closed category $\mathcal{V}$ be complete. This allows us to talk about functor $\mathcal{V}$-categories.

Secondly, the pseudofunctor $\mathcal{M}(I, -) : \mathcal{M} \to \mathcal{V}$-\textbf{Cat} must be locally faithful. In other words, for every pair of 1-cells $f, g$, the following must be a monic arrow in $\mathcal{V}$.

\[
\mathcal{M}(X, Y)(f, g) \rightarrow [\mathcal{M}(I, X), \mathcal{M}(I, Y)](\mathcal{M}(I, f), \mathcal{M}(I, g)) \tag{34}
\]

Finally, for any $f, g : X \to Y$, the image of the arrow under $\mathcal{V}(I, -) : \mathcal{V} \to \textbf{Set}$ must be surjective. This condition is saying that every $\mathcal{V}$-natural transformation $\mathcal{M}(I, f) \Rightarrow \mathcal{M}(I, g)$ is induced by a 2-cell $f \Rightarrow g$; this 2-cell is unique by the condition in the previous paragraph.
All these properties are satisfied by our main example of \( \mathcal{V} \)-Mod, as we shall see later.

**Theorem 9.4.** Under the hypothesis above, if \( A \) has a lax centre then there exists a \( \mathcal{V} \)-enriched equivalence making the following diagram commutes up to a canonical isomorphism.

\[
\begin{array}{ccc}
\mathcal{M}(I, Z_I A) & \cong & \mathcal{M}(I, Z_I (I, A)) \\
\downarrow \mathcal{M}(I, Z_I) & & \downarrow \mathcal{M}(I, Z_I) \\
\mathcal{M}(I, A) & \rightarrow & \mathcal{V}
\end{array}
\]

Here the \( \mathcal{V} \)-category on the right hand side is a lax centre in \( \mathcal{V} \)-Cat and \( V \) is the forgetful \( \mathcal{V} \)-functor. Furthermore, the result remains true if we write centres in place of lax centres.

**Proof.** By Proposition 9.3 it is enough to exhibit a \( \mathcal{V} \)-enriched equivalence between \( CP_I(I, A) \) and \( Z_I(\mathcal{M}(I, A)) \) commuting with the forgetful functors.

Define a \( \mathcal{V} \)-functor \( \Phi : CP_I(I, A) \rightarrow Z_I(\mathcal{M}(I, A)) \) as follows. On objects \( \Phi(f, \alpha) = (f, \Phi_1(\alpha)) \) where

\[
\Phi_1(\alpha) : h \ast f \cong p(A \otimes f)h \xrightarrow{\alpha h} p(f \otimes A)h \cong f \ast h.
\]

Recall that the \( \mathcal{V} \)-enriched hom \( CP_I(I, A)((f, \alpha), (g, \beta)) \) is the equalizer of \( \Phi \) and \( Z_I(\mathcal{M}(I, A))(\Phi(f, \alpha), \Phi(g, \beta)) \) is the equalizer of the diagram in Example 9.1 where \( \mathcal{C} = \mathcal{M}(I, A) \), \( x = f, y = g, \gamma = \Phi_1(\alpha) \) and \( \delta = \Phi_1(\beta) \). We can draw a diagram

\[
\begin{array}{ccc}
CP_I(I, A)((f, \alpha), (g, \beta)) & \rightarrow & \mathcal{M}(I, A)(f, g) \\
\downarrow \mathcal{M}(I, A)((f, \alpha), (g, \beta)) & & \downarrow \mathcal{M}(I, A)(f, g) \\
\mathcal{M}(A, A)(p(f \otimes A), p(A \otimes g)) & \rightarrow & \mathcal{M}(I, A) \circ \mathcal{M}(I, A)((f \ast -, -, \ast g)
\end{array}
\]

where \( CP_I(I, A)((f, \alpha), (g, \beta)) \) is the equalizer of the pair of arrows in the top row and \( Z_I(\mathcal{M}(I, A))(\Phi(f, \alpha), \Phi(g, \beta)) \) is the equalizer of the other diagonal pair of arrows. Moreover, the diagram serially commutes, as the vertical arrow is induced by the effect of the pseudofunctor \( \mathcal{M}(I, -) : \mathcal{M} \rightarrow \mathcal{V} \)-Cat on \( \mathcal{V} \)-enriched homs, and hence monic by hypothesis. It follows that there exists an isomorphism \( CP_I(I, A)((f, \alpha), (g, \beta)) \rightarrow Z_I(\mathcal{M}(I, A))(\Phi(f, \alpha), \Phi(g, \beta)) \). One can check that these isomorphisms are part of a \( \mathcal{V} \)-functor \( \Phi \), which, obviously, is fully faithful.

It only rests to prove that \( \Phi \) is essentially surjective on objects. Here is where the hypothesis on \( \mathcal{M}(I, -) : \mathcal{M} \rightarrow \mathcal{V} \)-Cat come into play. An object \( (f, \gamma) \) of \( Z_I(\mathcal{M}(I, A)) \) gives rise to a \( \mathcal{V} \)-natural transformation

\[
\gamma_h : p(A \otimes f)h \cong h \ast f \xrightarrow{\gamma h} \ast h \cong p(f \otimes A)h.
\]

By hypothesis, \( \gamma' \) is induced by a unique \( \alpha : p(A \otimes f) \Rightarrow p(f \otimes A) \). The equalities \( \beta \) and \( \gamma \) for the 2-cell \( \alpha \) follow from the fact that \( (f, \gamma) \) is an object in the lax centre of \( \mathcal{M}(I, A) \) and the fact that \( \mathcal{M}(A^2, A) \rightarrow \mathcal{M}(I, A) \) is fully faithful. Now observe that \( \Phi(f, \alpha) = (f, \gamma) \). Finally, \( \alpha \) is invertible if and only if \( \gamma \) is invertible, so that proof also applies to centres. \( \square \)

Recall from [4] that for a right autonomous pseudomonoid \( A \), with right dualization \( d : A^\vee \rightarrow A \), every map \( f : I \rightarrow A \) has a right dual in the monoidal \( \mathcal{V} \)-category \( \mathcal{M}(I, A) \). A right dual of \( f \) is given by \( d(f^*)^\vee \), where \( f^* \) is a right adjoint to \( f \). Then the full subcategory \( \text{Map}(\mathcal{M}(I, A)) \) of \( \mathcal{M}(I, A) \) is right autonomous (in the classical sense that it has right duals).
Theorem 9.5. In addition to the hypothesis above, assume the following: \( \mathcal{V} \) is complete and cocomplete monoidal closed category, \( \mathcal{M} \) has all right liftings, the inclusion \( \mathcal{V} \)-functor \( \mathcal{M}(I,A) \to \mathcal{M}(I,A) \) is dense and \( \mathcal{M}(I,-) : \mathcal{M} \to \text{Cat} \) reflects equivalences. If \( A \) is left autonomous, then the centre of \( A \) coincides with the lax centre whenever both exist.

Proof. By Theorem 9.4 there exists an isomorphism as depicted below.

\[
\begin{align*}
\mathcal{M}(I,ZA) & \xrightarrow{\cong} Z(\mathcal{M}(I,A)) \\
\mathcal{M}(I,zc) & \cong \\
\mathcal{M}(I,ZtA) & \xrightarrow{\cong} Zt(\mathcal{M}(I,A))
\end{align*}
\]

A straightforward modification of [6, Prop. 6] (using the property of the right liftings with respect to composition dual to [32, Prop. 1]) shows that the monoidal \( \mathcal{V} \)-category \( \mathcal{M}(I,A) \) is closed as a \( \mathcal{V} \)-category. It follows that the \( \mathcal{V} \)-functors \( (f \circ -) = p(f \otimes A) - : \mathcal{M}(I,A) \to \mathcal{M}(I,A) \) given by tensoring with an object \( f \) are cocontinuous. As \( \mathcal{M}(I,A) \) has a dense sub \( \mathcal{V} \)-category of which every object has a right dual, the hypotheses of [5, Theorem 3.4] are satisfied, and we deduce that the inclusion \( Z(\mathcal{M}(I,A)) \hookrightarrow Zt(\mathcal{M}(I,A)) \) is the identity. It follows that \( \mathcal{M}(I,zc) \) is an equivalence and hence \( z_c \) is an equivalence. \( \square \)

9.2. Lax centres of autonomous pseudomonoids. The lax centre of a pseudomonoid was defined as a birepresentation of the pseudofunctor \( CP \). As \( \mathcal{V} \)-category \( \mathcal{V} \) is an equivalence and hence \( \alpha \) is an equivalence, if \( A \) is left autonomous, then the centre of \( A \) coincides with the lax centre whenever both exist.

Definition 9.3. Given a map pseudomonoid \( A \) in a braided Gray monoid \( \mathcal{M} \) define a pseudonatural transformation \( \sigma : \mathcal{M}(A \otimes -, A) \Rightarrow \mathcal{M}(A \otimes -, A) \) with components

\[
\sigma_X(g) = (A \otimes X \xrightarrow{p^* \otimes 1} A^2 \otimes X \xrightarrow{1 \otimes c_{X,A}^*} A \otimes X \otimes A \xrightarrow{g \otimes 1} A^2 \xrightarrow{p^*} A).
\]

Lemma 9.6. The pseudonatural transformation \( \sigma \) has a canonical structure of a monad.

Proof. Just note that \( \sigma \) is isomorphic to the monad \( \theta \) of Definition 8.1 for the map pseudomonoid \( (A, j, p, c_{A,A}^1) \).

Explicitly, the multiplication of \( \sigma \) is given by components

\[
\begin{align*}
A \otimes X & \xrightarrow{p^* \otimes 1} A^2 \otimes X \\
\xrightarrow{1 \otimes c_{X,A}^*} & \xrightarrow{\cong} A \otimes X \otimes A \\
\xrightarrow{g \otimes 1} & \xrightarrow{\cong} A^2 \\
\xrightarrow{p \circ 1} & A
\end{align*}
\]
and the unit by

\[
\begin{align*}
A \otimes X & \xrightarrow{g} A \\
A \otimes X & \xrightarrow{g} A \\
A \otimes X & \xrightarrow{g} A \\
A \otimes X & \xrightarrow{g} A
\end{align*}
\]

When \( A \otimes - \) has right biadjoint the monads \( \theta \) and \( \sigma \) are represented by monads \( t \) and \( s : [A, A] \to [A, A] \). The monad \( s \) is

\[
[A, A] \xrightarrow{\text{id}} [A \otimes A, A \otimes A] \xrightarrow{[c, A, A, c^*_{A,A}]} [A \otimes A, A \otimes A] \xrightarrow{[p, p]} [A, A],
\]

which is the monad \( t \) for the opposite pseudomonoid of \( A \) with respect to \( c^* \), in other words, \((A, j, p c^*_{A,A})\). Alternatively, \( t \) and \( s \) can be taken respectively as

\[
[A, A] \xrightarrow{\text{id} \otimes \text{id}} [A, A] \otimes [A, A] \xrightarrow{[p, p]} [A, A]
\]

and

\[
[A, A] \xrightarrow{1 \otimes \text{id}} [A, A] \otimes [A, A] \xrightarrow{[p, p]} [A, A]
\]

where \( \text{id} : I \to [A, A] \) is the 1-cell corresponding to \( 1_A \) under the equivalence \( \mathcal{M}(A, A) \simeq \mathcal{M}(I, [A, A]) \).

**Observation 9.7.** At this point we should remark that for a map pseudomonoid \( A, [A, A] \) has two pseudomonoid structures. The one we have considered so far is the **composition** pseudomonoid structure, but we also have the **convolution** pseudomonoid structure.

If \((C, \varepsilon, \delta)\) is a pseudocomonoid in the braided Gray monoid \( \mathcal{M} \) such that the 2-functor \( C \otimes - \) has a right biadjoint \( [C, -] \), this is lax monoidal in the standard way. The unit constraint \( I \to [C, I] \) corresponds under the closedness equivalence to the counit \( \varepsilon : C \to I \) and the 1-cells \([C, X] \otimes [C, Y] \to [C, X \otimes Y] \) correspond

\[
C \otimes [C, X] \otimes [C, Y] \xrightarrow{\text{ev}} C^2 \otimes [C, X] \otimes [C, Y] \xrightarrow{1 \otimes \text{ev}} (C \otimes [C, X])^2 \xrightarrow{\text{ev} \otimes 1} X \otimes Y.
\]

In particular, for a pseudomonoid \( A, [C, A] \) has a canonical **convolution** pseudomonoid structure. This structure corresponds to the usual convolution tensor product in \( \mathcal{M}(C, A) \) given by \( f \ast g = p(A \otimes g)(f \otimes C)\delta \) with unit \( j \). As we saw in Observation 9.7 for a map pseudomonoid \( A \), the identity \( 1_A \) has a canonical structure of a monoid in the convolution monoidal category \( \mathcal{M}(A, A) \). It follows that the corresponding 1-cell \( \text{id} : I \to [A, A] \) is a monoid in \( \mathcal{M}(I, [A, A]) \).

**Observation 9.8.** Let \( B \) be a pseudomonoid in \( \mathcal{M} \) and consider \( \mathcal{M}(I, B) \) and \( \mathcal{M}(B, B) \) as monoidal categories with the convolution and the composition tensor product respectively. We have monoidal functors \( L, R : \mathcal{M}(I, B) \to \mathcal{M}(B, B) \) given by \( L(f) = p(f \otimes B) \) and \( R(f) = p(B \otimes f) \). The associativity constraint of \( B \) induces isomorphisms \( L(f)R(g) \cong R(g)L(f) \), natural in \( f \) and \( g \). If \( m \) and \( n \) are monoids in \( \mathcal{M}(I, B) \), then these isomorphisms form an invertible distributive law between the monads \( L(m) \) and \( R(n) \).

The monoidal functors \( L, R \) are compatible with monoidal pseudofunctors: if \( F : \mathcal{M} \to \mathcal{N} \) is a monoidal pseudofunctor, then there is a monoidal isomorphism
of monoidal functors

\[
\begin{array}{ccc}
\mathcal{M}(I,B) & \xrightarrow{L,R} & \mathcal{M}(B,B) \\
F_{1,B} & \cong & F_{B,B} \\
\mathcal{N}(FI,FB) & \cong & \mathcal{N}(FB,FB)
\end{array}
\]

In particular, if \(m\) is a monoid in \(M(I,B)\), we have an isomorphisms \(F(L(m)) \cong L(Fm)\) and \(F(R(m)) \cong R(Fm)\) of monoids in \(\mathcal{M}(B,B)\).

**Proposition 9.9.** There exists an invertible distributive law between the monads \(t\) and \(s\), and hence between the monads \(\theta\) and \(\sigma\).

**Proof.** Apply Observation 9.8 above to the convolution pseudomonoid \(B = [A,A]\) and the monoid \(m = n = \text{id} : I \to [A,A]\), noting that \(t = L(\text{id})\) and \(s = R(\text{id})\).

The 1-cell \(\text{id}\) is a monoid with the structure given by Observation 9.7. \(\square\)

If \(t\) has an Eilenberg-Moore construction \(u : [A,A] \to [A,A]\) the monad \(\hat{s}\) is represented by some \(\hat{s} : [A,A] \to [A,A]\).

**Proposition 9.10.** The monads \(s\) and \(\hat{s}\) are opmonoidal monads.

**Proof.** As we noted, \(s\) is the monad \(t\) for the pseudomonoid \((A,j,pc_A,A)\). It can also be regarded as the corresponding monad \(t\) for the pseudomonoid \((A,j,p)\) in \(\mathcal{M}^{\text{rev}}\), and thus it is opmonoidal in \(\mathcal{M}^{\text{rev}}\), and hence in \(\mathcal{M}\). The monad \(\hat{s}\) is opmonoidal since \([A,A]^t\) is an Eilenberg-Moore construction in \(\text{Opmon}(\mathcal{M})\). \(\square\)

Denote by \(\hat{\sigma}\) the induced monad on \(\mathcal{M}(A \otimes -, A)^\theta\) such that

\[
\begin{array}{ccc}
\mathcal{M}(A \otimes -, A)^\theta & \xrightarrow{\hat{\sigma}} & \mathcal{M}(A \otimes -, A)^\theta \\
\nu & \cong & \nu \\
\mathcal{M}(A \otimes -, A) & \xrightarrow{\sigma} & \mathcal{M}(A \otimes -, A)
\end{array}
\]

commutes. There exists an equivalence \((\mathcal{M}(A \otimes -, A)^\theta)^\sigma \simeq \mathcal{M}(A \otimes -, A)^{\sigma \theta}\).

Suppose that there exists a pseudonatural transformation \(\hat{\sigma} : \mathcal{M}(-,A) \to \mathcal{M}(-,A)\) such that \(\lambda \hat{\sigma} \cong \hat{\sigma} \lambda\); since \(\lambda\) is fully faithful (see Proposition 3.2), this is equivalent to saying that for each \(X\) the monad \(\hat{\sigma}_X\) restricts to a monad on the replete image of \(\lambda_X\) in \(\mathcal{M}(A \otimes X, A)_{\sigma \theta}\), and in this case \(\hat{\sigma} = \lambda^\ast \hat{\sigma} \lambda\). Moreover, \(\hat{\sigma}\) carries the structure of a monad induced by the one of \(\hat{\sigma}\). Such a monad \(\hat{\sigma}\) clearly exists if the theorem of Hopf modules holds for \(A\), i.e., if \(\lambda\) is an equivalence.

**Theorem 9.11.** There exists an equivalence in the 2-category \([\mathcal{M}^{\text{op}}, \text{Cat}]\) between \(\mathcal{M}( -, A)^\sigma\) and \(\mathcal{C}P_l(-,A)\) whenever the monad \(\hat{\sigma}\) exists. Moreover, this equivalence commutes with the corresponding forgetful pseudonatural transformations.

**Proof.** We shall consider the restriction of \(\hat{\sigma}_X\) to the replete image of \(\lambda_X\) instead of \(\hat{\sigma}_X\). Take \(f : X \to A\) and assume that \(\lambda_X(f : X \to A)\) has a structure \(\nu\) of
\(\sigma\)-algebra. This means that the action \(\nu\) is a 2-cell

\[
\begin{array}{c}
\begin{array}{ccc}
A \otimes A \otimes X & \xrightarrow{1 \otimes c_{X,A}} & A \otimes X \otimes A \\
p^* \otimes 1 & \Downarrow & 1 \otimes f \otimes 1 \\
A \otimes X & \leftrightsquigarrow & A \otimes A \otimes A \\
p \otimes 1 & \Downarrow & p \otimes 1 \\
A \otimes A & \Downarrow & A \\
p & \Downarrow & p 
\end{array}
\end{array}
\]

(40)

which is a morphism of \(\theta_X\)-algebras from \(\hat{\sigma}_X \lambda_X(f)\) to \(\lambda_X(f)\). Furthermore, the pasting

\[
\begin{array}{c}
\begin{array}{ccc}
A^2 \otimes X \otimes A & \xrightarrow{1 \otimes c^* \otimes 1} & A \otimes X \otimes A^2 \\
p^* \otimes 1 \otimes 1 & \Downarrow & 1 \otimes f \otimes 1 \\
A^2 \otimes X & \xrightarrow{1 \otimes f} & A^2 \\
p \otimes 1 & \Downarrow & p \otimes 1 \\
A & \Downarrow & p 
\end{array}
\end{array}
\]

should be equal to the composition \(\sigma_X \sigma_X \lambda_X(f) \xrightarrow{\nu} \lambda_X(f)\) of the multiplication of \(\sigma_X\) (35) and \(\nu\), and the composition \(\lambda_X(f) \xrightarrow{\nu} \lambda_X(f)\) of the unit of \(\sigma\) (36) and \(\nu\) is the identity. The 2-cells (40) correspond, under pasting with \(\phi^{-1} : p(A \otimes p) \cong p(p \otimes A)\), to 2-cells \(p(A \otimes (p(f \otimes A)c_{X,A})) \xrightarrow{(p^* \otimes X)} p(A \otimes f)\), and then to 2-cells \(p(A \otimes (p(f \otimes A)c_{X,A})) \xrightarrow{p(A \otimes f)(p \otimes A) \cong p(A \otimes p)(A \otimes A \otimes f)}\). Since \(\lambda_X\) is fully faithful, and \(\hat{\sigma}\) restricts to its replete image, it follows that the 2-cells \(\nu\) correspond to the 2-cells \(\gamma\) (20). The axiom of associativity for the action \(\nu\) translates into the axiom (23) for \(\gamma\) and the axiom of unit for \(\nu\) into the axiom (31) for \(\gamma\). This shows that the composition of the forgetful functor \(\hat{V} : CP(X, A) \rightarrow \mathcal{M}(X, A)\) with \(\lambda_X\) factors as \(G\) followed by \(U\), as depicted below.

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{M}(X, A) & \xrightarrow{\lambda_X} & \mathcal{M}(A \otimes X, A)^{\theta_X} \\
\xrightarrow{G_X} & \xrightarrow{\hat{U}_X} & \xrightarrow{\hat{I}_X} \mathcal{M}(X, A)^{\theta_X} \\
\xrightarrow{V_X} & \xrightarrow{H_X} \mathcal{M}(X, A)^{\theta_X} \\
CP(X, A) & \xrightarrow{\hat{U}_X} & \mathcal{M}(X, A)^{\theta_X} \\
\xrightarrow{\hat{I}_X} \mathcal{M}(X, A)^{\theta_X} \\
\end{array}
\end{array}
\]

Moreover, \(G_X\) factors through the image of \(\hat{\lambda}_X\), since \(\hat{U}_X G_X\) factors through \(\lambda_X\), and in fact \(G_X\) is an equivalence into the image of \(\hat{\lambda}_X\). Here \(\hat{\lambda}_X\) is the functor induced on Eilenberg-Moore constructions by \(\lambda_X\); in particular, \(\hat{\lambda}_X\) is fully faithful since \(\lambda_X\) is fully faithful. Therefore we have an equivalence \(H_X\) as in the diagram,
such that \( \hat{\lambda}_X H_X = G_X \). Hence, \( \lambda_X \hat{U}_X H_X = \hat{U}_X \hat{\lambda}_X H_X = \hat{U}_X G_X = \lambda_X V_X \), and \( \hat{U}_X H_X = V_X \). The equivalences \( H_X \) are clearly pseudonatural in \( X \).

□

**Corollary 9.12.** If the theorem of Hopf modules holds for a map pseudomonoid \( A \) then there exists an equivalence \( CP_\ell(-, A) \simeq \mathcal{M}(A \otimes -, A)^{\sigma^\theta} \).

**Proof.** \( \lambda_X \) is an equivalence and then the monad \( \hat{\sigma} \) exists and

\[
\mathcal{M}(-, A)^\theta \simeq (\mathcal{M}(A \otimes -, A)^{\sigma^\theta}) \hat{\sigma} \simeq \mathcal{M}(A \otimes -, A)^{\sigma^\theta}.
\]

□

**Corollary 9.13.** Suppose that the theorem of Hopf modules holds for the map pseudomonoid \( A \) and that it has a Hopf module construction. Then the lax centre of \( A \) is the Eilenberg-Moore construction for the opmonoidal monad

\[
\hat{s} := \ell^* \hat{s} \ell = A \to A
\]

one of them existing if the other does. Moreover,

\[
\hat{s} \cong \left( A \xrightarrow{j} A \otimes A \xrightarrow{p^* \otimes 1} A \otimes A \otimes A \xrightarrow{1 \otimes c_{A,A}} A \otimes A \otimes A \xrightarrow{p \otimes A} A \otimes A \xrightarrow{P} A \right).
\]

**Proof.** The monad \( \hat{s} \) exists and is opmonoidal since \( t : [A, A] \to [A, A] \) has an Eilenberg-Moore construction in \( \text{Opmon}(\mathcal{M}) \). Hence, \( \hat{s} \) has a canonical opmonoidal monad structure induced by the one of \( s \). The Theorem 9.11 implies that the lax centre of \( A \) exists, that is, \( CP(-, A) \) is birepresentable, if and only if the monad \( \hat{s} \) has an Eilenberg-Moore construction.

To obtain an expression for the 1-cell \( \hat{s} \) recall that, by definition, \( \mathcal{M}(-, \hat{s}) \) is isomorphic to \( \lambda^* \hat{\sigma} \lambda \). It is easy to show that

\[
\lambda_X \hat{\sigma}_X \lambda_X (f : X \to A) = p(p \otimes A)(A \otimes f \otimes A)(A \otimes c_{X,A})(p^* \otimes X)(j \otimes X)
\]

\[
\cong p(p \otimes A)(A \otimes c_{A,A})(p^* \otimes A)(j \otimes A) f;
\]

see Definitions 3.2 and 9.3. It follows that the expression for \( \hat{s} \) of the statement holds.

□

**Observation 9.14.** The thesis of Corollary 9.13 above holds under the sole hypothesis of that \( A \) be left autonomous. This is so because every left autonomous map pseudomonoid has a Hopf module construction (Theorem 7.3).

**Observation 9.15.** Suppose that the theorem of Hopf modules holds for \( A \) and that \( A \) has a Hopf module construction (e.g., \( A \) is left autonomous). Then, when the lax centre \( Z_\ell(A) \) of \( A \) exists, it has a canonical structure of a pseudomonoid such that the universal \( Z_\ell(A) \to A \) is strong monoidal.

Now we concentrate in the case of autonomous map pseudomonoids. Let \( A \) be a such pseudomonoid. The internal hom \( [A, A] \) is given by \( A^\otimes \otimes A \), where \( A^\circ \) is a right bidual of \( A \). The 1-cell \( \text{id} \) is just the coevaluation \( n : I \to A^\otimes \otimes A \). The convolution

**Corollary 9.16.** Suppose \( F : \mathcal{M} \to \mathcal{N} \) is a pseudofunctor between Gray monoids with the following properties: \( F \) preserves Eilenberg-Moore objects, is braided and strong monoidal. Then, \( F \) preserves lax centres of left autonomous map pseudomonoids.

**Proof.** Let \( A \) be a left autonomous map pseudomonoid in \( \mathcal{M} \). By Observation 9.14, the lax centre of \( A \) is the Eilenberg-Moore construction for the opmonoidal monad \( \hat{s} : A \to A \), one existing if the other does. On the other hand, \( FA \) is also a left autonomous map pseudomonoid by Proposition 7.8. Therefore, it is enough
to show that \( F \) preserves the monad \( \hat{s} \), in the sense that \( F\hat{s} \) is isomorphic to the correponding monad \( \hat{s} \) for \( FA \).

Since \( \hat{s} \) is the lifting of the monad \( s \) on \( A^p \otimes A \) to the Eilenberg-Moore construcion \( (A^p \otimes p)(n \otimes A) : A \rightarrow A^p \otimes A \) of the monad \( t \) (see Theorem 7.3), it suffices to prove that \( F \) preserves the monads \( t \) and \( s \). We only work with \( t \), the proof for the monad \( s \) being completely analogous. Now, we know from the proof of Proposition 9.9 that \( t = L(n) \) and \( s = R(n) \), where \( L, R : \mathcal{M}(I, A^p \otimes A) \rightarrow \mathcal{M}(A^p \otimes A, A^p \otimes A) \) are the functors defined in Observation 9.3. Therefore, \( Ft = F(L(n)) \cong L(I) \cong FI \xrightarrow{F^n A} F(A^p \otimes A) \cong L(n_{FA}) \), which is the monad \( t \) corresponding to the pseudomonoid \( FA \).

\[\square\]

**Theorem 9.17.** For a (left and right) autonomous map pseudomonoid the centre equals the lax centre, either existing if the other does.

**Proof.** Consider the commutative diagram

\[
\begin{array}{c}
\mathcal{M}(A \otimes X, A)^\eta_X \longrightarrow \mathcal{M}(A \otimes X, A)^\nu_X \\
\downarrow \downarrow \\
\mathcal{M}(A \otimes X, A)^\sigma_X \longrightarrow \mathcal{M}(A \otimes X, A)^\sigma_X \\
A \otimes A \otimes X \longrightarrow A \otimes A \otimes X \\
\downarrow \downarrow \downarrow \\
A \otimes X \longrightarrow A \otimes A \\
\downarrow \downarrow \downarrow \\
A \\
\end{array}
\]

In Theorem 9.11 we proved that any lax centre piece arises as

\[
\begin{array}{c}
A \otimes A \otimes X \longrightarrow A \otimes A \otimes X \xrightarrow{1 \otimes c_{A,X}} A \otimes X \otimes A \\
\downarrow \downarrow \downarrow \\
A \otimes X \longleftarrow A \otimes A \\
\downarrow \downarrow \downarrow \\
A \\
\end{array}
\]

for some \( \hat{\sigma}_X \)-algebra \( \nu : \hat{\sigma}_X(h) \rightarrow h \), so we have to prove that \( (41) \) is invertible. Consider the canonical split coequalizer \( \hat{\sigma}_X^2(h) \rightrightarrows \hat{\sigma}_X(h) \twoheadrightarrow h \) in \( \mathcal{M}(A \otimes X, A)^\theta_X \), and its image \( \nu : \sigma_X(h) \rightarrow h \) in \( \mathcal{M}(A \otimes X, A) \). The arrow \( \nu \) is a morphism of \( \sigma_X \)-algebras. This implies that the lower rectangle in the diagram below commutes.

\[
\begin{array}{c}
p(\sigma_X(h) \otimes A)(A \otimes c_{A,X}) \longrightarrow p(h \otimes A)(A \otimes c_{A,X}) \\
p(\sigma_X(h) \otimes A)(A \otimes c_{A,X})(\eta \otimes X) \longrightarrow p(h \otimes A)(A \otimes c_{A,X})(\eta \otimes X) \\
p(\sigma_X(h) \otimes A)(A \otimes c_{A,X})(p^*p \otimes X) \longrightarrow p(h \otimes A)(A \otimes c_{A,X})(p^*p \otimes X) \\
\sigma_X^2(h)(p \otimes X) \longrightarrow \sigma_X(h)(p \otimes X) \\
(\mu_X)(p \otimes X) \longrightarrow \nu(p \otimes X) \\
\sigma_X(h)(p \otimes X) \longrightarrow h(p \otimes X) \\
\end{array}
\]

The upper rectangle commutes by naturality of composition. Here \( \eta \) denotes the unit of the adjunction \( p \dashv p^* \) and \( \mu \) the multiplication of the monad \( \sigma \). Observe that the rows are coequalizers and the right-hand column is just \( (41) \). Then, to show that this last arrow is invertible it suffices to show that the left-hand side column, which is the pasting of \( \eta \) with the multiplication of \( \sigma \), is so. But this
2-cell is invertible because $A$ is right autonomous and by the dual of Theorem 9.2.2, the 2-cell below is invertible. This completes the proof.

\[
\begin{array}{c}
\begin{tikzcd}
A^2 & A^2 & A^3 \\
\downarrow & \downarrow_{p'} & \downarrow_{1 \otimes p'}
\end{tikzcd}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzcd}
A & A^2 & A^2 \\
\downarrow_{p} & \downarrow & \downarrow_{p'}
\end{tikzcd}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzcd}
A^2 & A^2 & A^2 \\
\downarrow & \downarrow_{1 \otimes p} & \downarrow_{p \otimes 1}
\end{tikzcd}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzcd}
A^2 & A^2 & A^2 \\
\downarrow & \downarrow_{p} & \downarrow_{p'}
\end{tikzcd}
\end{array}
\]

Corollary 9.18. Any autonomous map pseudomonoid in a braided monoidal bicategory with Eilenberg-Moore objects has both a centre and a lax centre, and the two coincide.

10. \(\mathcal{V}\)-modules

In this section we interpret the results of the previous section in the particular context of the bicategory of \(\mathcal{V}\)-modules.

10.1. Review of the bicategory of \(\mathcal{V}\)-modules. Throughout this section \(\mathcal{V}\) will be a complete and cocomplete closed symmetric monoidal category. There is a bicategory \(\mathcal{V}\)-\textbf{Mod} whose objects are the small \(\mathcal{V}\)-categories and hom-categories \(\mathcal{V}\)-\textbf{Mod}(\mathcal{A}, \mathcal{B}) = [\mathcal{A}^{\text{op}} \otimes \mathcal{B}, \mathcal{V}]_0\), the category of \(\mathcal{V}\)-functors from the tensor product of the \(\mathcal{V}\)-categories \(\mathcal{A}^{\text{op}}\) and \(\mathcal{B}\) to \(\mathcal{V}\), and \(\mathcal{V}\)-natural transformations between them. Objects of this category are called \(\mathcal{V}\)-modules and arrows morphisms of \(\mathcal{V}\)-modules. The composition of two \(\mathcal{V}\)-modules \(M : \mathcal{A} \to \mathcal{B}\) and \(N : \mathcal{B} \to \mathcal{C}\) is given by \((NM)(a, c) = \int^x N(x, c) \otimes M(a, x)\). This coend exists because \(\mathcal{V}\) is complete. The identity module \(1_{\mathcal{A}}\) is given by \(1_{\mathcal{A}}(a, a') = \mathcal{A}(a, a')\). A \(\mathcal{V}\)-module \(M : \mathcal{A} \to \mathcal{B}\) can also be thought as a \((\text{ob} \mathcal{A} \times \text{ob} \mathcal{B})\)-indexed family of objects \(M(a, b)\) of \(\mathcal{V}\) with compatible actions of \(\mathcal{A}\) on the right and of \(\mathcal{B}\) on the left; this is, actions \(\mathcal{B}(b, b') \otimes M(a, b) \to M(a, b')\) and \(M(a, b) \otimes \mathcal{A}(a', a) \to M(a', b)\) subject to compatibility conditions.

Our convention is that a \(\mathcal{V}\)-module from \(\mathcal{A}\) to \(\mathcal{B}\) as a \(\mathcal{V}\)-functor \(\mathcal{A}^{\text{op}} \otimes \mathcal{B} \to \mathcal{V}\), but some authors prefer to use \(\mathcal{V}\)-functors \(\mathcal{A} \otimes \mathcal{B}^{\text{op}} \to \mathcal{V}\). A different approach was taken in [1], where the bicategory of \(\mathcal{V}\)-matrices was used to define \(\mathcal{V}\)-modules for a bicategory \(\mathcal{V}\).

There is a pseudofunctor \((-)_{*} : \mathcal{V}\text{-Cat}^{\text{co}} \to \mathcal{V}\text{-Mod}\) which is the identity on objects and on hom-categories \([\mathcal{A}, \mathcal{B}]_{\mathcal{V}}^{\text{op}} \to [\mathcal{A}^{\text{op}} \otimes \mathcal{B}, \mathcal{V}]_{\mathcal{V}}\) sends a \(\mathcal{V}\)-functor \(F\) to the \(\mathcal{V}\)-functor \(F_{*}(a, b) = \mathcal{B}(F(a), b)\). Moreover, the \(\mathcal{V}\)-module \(F_{*}\) has right adjoint \(F^{*}\) given by \(F^{*}(b, a) = \mathcal{B}(b, F(a))\). The pseudofunctor \((-)_{*}\) is easily shown to be strong monoidal and symmetry-preserving.

The tensor product of \(\mathcal{V}\)-categories induces a structure of a monoidal bicategory on \(\mathcal{V}\text{-Mod}\), which on hom-categories \([\mathcal{A}^{\text{op}} \otimes \mathcal{B}, \mathcal{V}]_{\mathcal{V}} \otimes [\mathcal{A'}^{\text{op}} \otimes \mathcal{B'}, \mathcal{V}]_{\mathcal{V}} \to [\mathcal{A'}^{\text{op}} \otimes \mathcal{B} \otimes \mathcal{B'}, \mathcal{V}]_{\mathcal{V}}\) is given by point-wise tensor product in \(\mathcal{V}\). Moreover, the usual symmetry of \(\mathcal{V}\text{-Cat}\) together with the symmetry of \(\mathcal{V}\) induce a structure of symmetric monoidal bicategory on \(\mathcal{V}\text{-Mod}\), or rather, induce a symmetry in the sense of [2] in any Gray monoid monoidally equivalent to \(\mathcal{V}\text{-Mod}\). The natural isomorphisms \(\mathcal{V}\text{-Mod}(\mathcal{B}, \mathcal{A}^{\text{op}} \otimes \mathcal{C}) \cong [\mathcal{B}^{\text{op}} \otimes \mathcal{A}^{\text{op}} \otimes \mathcal{C}, \mathcal{V}]_{\mathcal{V}} \cong \mathcal{V}\text{-Mod}(\mathcal{A} \otimes \mathcal{B}^{\text{op}} \otimes \mathcal{C})\).
show that our monoidal bicategory is also autonomous with (right and left) biduals given by the opposite $\mathcal{V}$-category. The coevaluation $n : I \rightarrow \mathcal{A}^{\text{op}} \otimes \mathcal{A}$ and coevaluation $e : \mathcal{A} \otimes \mathcal{A}^{\text{op}} \rightarrow I$ modules are given respectively by $n(a, b) = a(a, b)$ and $e(a, b) = \mathcal{A}(b, a)$, and the bidual of a $\mathcal{V}$-module $M : \mathcal{A} \rightarrow \mathcal{B}$ can be taken as $M^{\ast}(b, a) = M(a, b)$. (Note that $e$ and $n$ do not induce the isomorphisms above, but pseudonatural equivalences isomorphic to these).

One of the many pleasant properties of $\mathcal{V}$-Mod is that it has right liftings. If $M : \mathcal{B} \rightarrow \mathcal{C}$ and $N : \mathcal{A} \rightarrow \mathcal{C}$ are $\mathcal{V}$-modules, a right lifting of $N$ through $M$ is given by the formula $MN(a, b) = \int_{c \in \mathcal{C}} [M(b, c), N(a, c)]$. As explained in Section 10.2, the existence of right liftings endows each hom-category $\mathcal{V}$-Mod$(I, \mathcal{A})$ with a canonical structure of a $\mathcal{V}$-Mod$(I, I)$-category, where $I$ is the trivial $\mathcal{V}$-category. Henceforth, each $\mathcal{V}$-Mod$(I, \mathcal{A})$ is canonically a $\mathcal{V}$-category via the monoidal isomorphism $\mathcal{V}$-Mod$(I, I) \cong \mathcal{V}$. The $\mathcal{V}$-enriched hom $\mathcal{V}$-Mod$(I, \mathcal{A})(M, N)$ is given by $MN$, or explicitly by the object $\int_{a \in \mathcal{A}} [M(a), N(a)]$. This is exactly the usual $\mathcal{V}$-category structure of $\mathcal{A}$, $\mathcal{V}$. Henceforth, each hom-category $\mathcal{V}$-Mod$(\mathcal{A}, \mathcal{B})$ is canonically a $\mathcal{V}$-category, in a way such that the equivalence $\mathcal{V}$-Mod$(\mathcal{A}, \mathcal{B}) \simeq \mathcal{V}$-Mod$(\mathcal{A}, \mathcal{A}^{\text{op}} \otimes \mathcal{B})$ is a $\mathcal{V}$-functor.

Another feature of $\mathcal{V}$-Mod we will need is the existence of Kleisli and Eilenberg-Moore constructions for monads. The existence of the former was shown in \cite{34}. If $(M, \eta, \mu)$ is a monad in $\mathcal{V}$-Mod on $\mathcal{A}$, $\text{Kl}(M)$ has the same objects as $\mathcal{A}$ and homs $\text{Kl}(M)(a, b) = M(a, b)$. Composition is given by

$$M(b, c) \otimes M(a, b) \rightarrow \int^{b \in \mathcal{A}} M(b, c) \otimes M(a, b) \xrightarrow{\mu_{b,c}} M(a, c)$$

and the units by $I \xrightarrow{\text{id}} \mathcal{A}(a, a) \xrightarrow{\eta_{a,a}} M(a, a)$.

One can verify that the $\mathcal{V}$-module $K_{s}$ induced by the $\mathcal{V}$-functor $K : \mathcal{A} \rightarrow \text{Kl}(M)$ given by the identity on objects and by $\eta_{a,b} : \mathcal{A}(a, b) \rightarrow M(a, b)$ on homs has the universal property of the Kleisli construction. With regard to Eilenberg-Moore constructions, $K^{*}$ induces, for each $\mathcal{X}$, a functor $(K^{*} \circ -) : \mathcal{V}$-Mod$(\mathcal{X}, \text{Kl}(M)) \rightarrow \mathcal{V}$-Mod$(\mathcal{X}, \mathcal{A})$. This functor is isomorphic to the one sending $M : \mathcal{X}^{\text{op}} \otimes \text{Kl}(M) \rightarrow \mathcal{V}$ to the $M(\mathcal{X}^{\text{op}} \otimes K) : \mathcal{X}^{\text{op}} \otimes \mathcal{A} \rightarrow \mathcal{V}$. Therefore, $(K^{*} \circ -)$ is conservative because $K$ is the identity on objects: if $\alpha : M \Rightarrow N$ is a $\mathcal{V}$-module morphism such that $\alpha(\mathcal{X}^{\text{op}} \otimes K)$ is an isomorphism, we have $\alpha_{x,a} = \alpha_{x,K(a)}$ is an isomorphism for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$. It follows that $(K^{*} \circ -)$ is monadic, as it is clearly cocontinuous and has a left adjoint. This being true for all $\mathcal{X}$, we deduce $K^{*}$ is monadic in $\mathcal{V}$-Mod, and hence it is an Eilenberg-Moore construction for $M$.

10.2. Left autonomous pseudomonoids and left autonomous $\mathcal{V}$-categories. A pseudomonoid in $\mathcal{V}$-Mod is a promonoidal $\mathcal{V}$-category \cite{34}. The pseudomonoid structure amounts to a multiplication and a unit $\mathcal{V}$-functors $P : \mathcal{A}^{\text{op}} \otimes \mathcal{A}^{\text{op}} \otimes \mathcal{A} \rightarrow \mathcal{V}$ and $J : \mathcal{A} \rightarrow \mathcal{V}$ together with associativity and unit $\mathcal{V}$-natural constraints satisfying axioms. Any monoidal $\mathcal{V}$-category can be thought of as a promonoidal $\mathcal{V}$-category, in fact a map pseudomonoid, by using the monoidal pseudofunctor $(-)_{*} : \mathcal{V}$-Cat$^{\text{op}} \rightarrow \mathcal{V}$-Mod; explicitly, if $\mathcal{A}$ is a monoidal $\mathcal{V}$-category, then the induced promonoidal structure is given by $P(a, b; c) = \mathcal{A}(b \otimes a, c)$ and $J(a) = \mathcal{A}(I, a)$.

Next we show how the results on Hopf modules specialise to the bicategory of $\mathcal{V}$-modules, and give explicit descriptions of the main constructions. Although these descriptions carry over to arbitrary left autonomous map pseudomonoids, here we will concentrate on the simpler case of the left autonomous monoidal $\mathcal{V}$-categories $\mathcal{A}$. The monoidal monad $T : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \rightarrow \mathcal{A}^{\text{op}} \otimes \mathcal{A}$ defined in Section 9.3 is given as a $\mathcal{V}$-module by $T(a, b; c, d) = \int x \mathcal{A}(b \otimes x, d) \otimes \mathcal{A}(c, a \otimes x)$. The multiplication
is has components

\[ T^2(a, b; c, d) = \int^x \int^y \mathcal{A}(v \otimes x, d) \otimes \mathcal{A}(c, u \otimes x) \otimes \int^y \mathcal{A}(b \otimes y, v) \otimes \mathcal{A}(u, a \otimes y) \]

\[ \cong \int^x \int^y \mathcal{A}((b \otimes y) \otimes x, d) \otimes \mathcal{A}(c, (a \otimes y) \otimes x) \]

\[ \cong \int^x \int^y \mathcal{A}(b \otimes (y \otimes x), d) \otimes \mathcal{A}(c, a \otimes (y \otimes x)) \rightarrow T(a, b; c, d) \]

where the last arrow is induced by the obvious arrows \( \mathcal{A}(b \otimes (y \otimes x), d) \otimes \mathcal{A}(c, a \otimes (y \otimes x)) \rightarrow \int^x \mathcal{A}(b \otimes x, d) \otimes \mathcal{A}(c, a \otimes x) \). The unit has components

\[ (\mathcal{A}^{op} \otimes \mathcal{A})(a, b; c, d) = \mathcal{A}(b, d) \otimes \mathcal{A}(c, a) \xrightarrow{\eta} \int^x \mathcal{A}(b \otimes x, d) \otimes \mathcal{A}(c, a \otimes x), \]

the component corresponding to \( I \in \text{obj} \mathcal{A} \).

The existence of Eilenberg-Moore constructions in \( \mathcal{V}-\text{Mod} \) implies the following.

**Proposition 10.1.** Any map pseudomonoid in \( \mathcal{V}-\text{Mod} \) has a Hopf module construction.

Following the remarks at the end of the previous section, one can give an explicit description of the Hopf module construction for a map pseudomonoid \( \mathcal{A} \). The \( \mathcal{V} \)-category \( (\mathcal{A}^{op} \otimes \mathcal{A})^T = (\mathcal{A}^{op} \otimes \mathcal{A})_T \) has the same objects as \( \mathcal{A}^{op} \otimes \mathcal{A} \), homs \( (\mathcal{A}^{op} \otimes \mathcal{A})(a, b; c, d) = T(a, b; c, d) \) and composition and identities induced by the multiplication and unit of \( T \). The unit of the monad \( T \) defines a \( \mathcal{V} \)-functor \( \eta : \mathcal{A}^{op} \otimes \mathcal{A} \rightarrow (\mathcal{A}^{op} \otimes \mathcal{A})^T \); the Kleisli construction for \( T \) is just the module \( \eta_* \) and the Eilenberg-Moore construction is \( \eta^* \). The module \( L : \mathcal{A} \rightarrow (\mathcal{A}^{op} \otimes \mathcal{A})^T \) in (I) is then

\[ L = (\mathcal{A} \xrightarrow{(J^*)^* \otimes \mathcal{A}} \mathcal{A}^{op} \otimes \mathcal{A} \xrightarrow{\eta_*} (\mathcal{A}^{op} \otimes \mathcal{A})^T) \]

(42)

When the promonoidal structure is induced by a monoidal structure on \( \mathcal{A} \), i.e., \( P(a, b; c) = \mathcal{A}(b \otimes a, c) \) and \( J(a) = \mathcal{A}(I, a) \), we can compute \( L \) explicitly. Firstly note that for any \( \mathcal{V} \)-functor \( F : \mathcal{B} \rightarrow \mathcal{C} \) there exists a canonical isomorphism of \( \mathcal{V} \)-modules \( (F^*)^* \cong (F^{op})_* : \mathcal{B}^{op} \rightarrow \mathcal{C}^{op} \), where \( F^{op} : \mathcal{B}^{op} \rightarrow \mathcal{C}^{op} \) is the usual opposite functor. Then

\[ L \cong \eta_*((J^{op})_* \otimes \mathcal{A}) \cong (\eta(J^{op} \otimes \mathcal{A}))* \]

In components,

\[ L(a; b, c) \cong (\mathcal{A}^{op} \otimes \mathcal{A})^T(\eta(I, a), (b, c)) = T(I, a; b, c) \cong \mathcal{A}(a \otimes b, c) \]

with right \( \mathcal{A} \)-action and left \( (\mathcal{A}^{op} \otimes \mathcal{A})^T \)-action. The latter is given by the composition of \( (\mathcal{A}^{op} \otimes \mathcal{A})^T \), while the \( \mathcal{A} \)-action can be shown to be given as

\[ \mathcal{A}(a \otimes b, c) \otimes \mathcal{A}(a', a) \xrightarrow{1 \otimes (- \otimes b)} \mathcal{A}(a \otimes b, c) \otimes \mathcal{A}(a' \otimes b, a \otimes b) \xrightarrow{\text{comp}} \mathcal{A}(a' \otimes b, c). \]

The fact that \( L \) is a faithful \( \mathcal{V} \)-module (Proposition 10.3) means exactly that the \( \mathcal{V} \)-functor \( \eta(J^{op} \otimes \mathcal{A}) \) is fully faithful. This can be also verified directly, for the effect of this \( \mathcal{V} \)-functor on homs is

\[ \mathcal{A}(b, d) \xrightarrow{1 \otimes L} \mathcal{A}(b, d) \otimes \mathcal{A}(I, I) \xrightarrow{\eta} \int^x \mathcal{A}(b \otimes x, d) \otimes \mathcal{A}(I, I \otimes x) \cong \mathcal{A}(b, d) \]

sending an arrow \( f \) to \( (b \xrightarrow{f} b \otimes I \xrightarrow{1 \otimes L} d \otimes I \xrightarrow{\eta} d) \).

Consider for a moment a general pronomoidal \( \mathcal{V} \)-category \( \mathcal{A} \). It is a left autonomous pseudomonoid in \( \mathcal{V} \)-Mod when there exists a \( \mathcal{V} \)-module \( D : \mathcal{A}^{op} \rightarrow \mathcal{A} \)
such that \((P \otimes \mathcal{A})(\mathcal{A} \otimes D \otimes \mathcal{A})(\mathcal{A} \otimes N)\) is right adjoint to the multiplication \(P : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}\). The former \(\mathcal{V}\)-module is given in components by

\[
(P \otimes \mathcal{A})(\mathcal{A} \otimes D \otimes \mathcal{A})(\mathcal{A} \otimes N)(a; b, c) \cong P(D \otimes \mathcal{A})(a, c; d) \cong \int^x P(a, x; b) \otimes D(c, x).
\]

When \(\mathcal{A}\) is a monoidal \(\mathcal{V}\)-category and the \(D\) is induced by a \(\mathcal{V}\)-functor, denoted by \(\langle - \rangle^\nu : \mathcal{A}^{\text{op}} \to \mathcal{A}\), then the expression above reduces to

\[
(P \otimes \mathcal{A})(\mathcal{A} \otimes D \otimes \mathcal{A})(\mathcal{A} \otimes N)(a; b, c) \cong \mathcal{A}(c^\nu \otimes a, b)
\]

and we obtain a \(\mathcal{V}\)-natural isomorphism \(\mathcal{A}(c^\nu \otimes a, b) \cong P^*(a; b, c) = \mathcal{A}(a, c \otimes b)\).

We see, thus, that a monoidal \(\mathcal{V}\)-category is a left autonomous pseudomonoid in \(\mathcal{V}\text{-Mod}\) with representable dualization if and only if it is left autonomous in the classical sense. This was first shown in \([4]\).

By the Theorem \([4, 1]\) and the definition of \(L\) in \([10]\) we have

**Proposition 10.2.** Every promonoidal \(\mathcal{V}\)-category \(\mathcal{A}\) for which \(P\) and \(J\) have right adjoints has a structure (and a fortiori unique up to isomorphism) of left autonomous pseudomonoid in \(\mathcal{V}\text{-Mod}\) if and only if the \(\mathcal{V}\)-module \(L\) \([12]\) is an equivalence. In particular, this is true for any monoidal \(\mathcal{V}\)-category.

10.3. **Lax centres in \(\mathcal{V}\text{-Mod}\).** In this section we study the centre and lax centre of pseudomonoid in the monoidal bicategory of \(\mathcal{V}\)-modules by means of the theory developed in Section 9. In the way, we compare our work with \([5, 8]\).

First we consider lax centres of arbitrary pseudomonoids. We shall show that the results in Section 9.1 apply to \(\mathcal{V}\text{-Mod}\). To this aim, we have to verify all the hypothesis required in that section.

We already saw in Section 10.1 that liftings of arrows out of \(I\) through arrows out of \(I\) exist. In order to show \(\mathcal{V}\text{-Mod}\) satisfies the other two hypothesis required in Section 9.1 it is enough to prove that the arrow \([\mathcal{A}]\) is an isomorphism for \(\mathcal{M}\) the bicategory of \(\mathcal{V}\)-modules. In this case \([\mathcal{A}]\) becomes

\[
[\mathcal{A}^{\text{op}} \otimes \mathcal{B}, \mathcal{V}](M, N) \to [[\mathcal{A}, \mathcal{V}], [\mathcal{B}, \mathcal{V}]')(\mathcal{M} \circ -, (N \circ -)),
\]

(43)

where \((M \circ -)\) is the \(\mathcal{V}\)-functor given by composition with \(M\). To show that \(43\) is an isomorphism, recall that the \(\mathcal{V}\)-functor

\[
[\mathcal{A}^{\text{op}} \otimes \mathcal{B}, \mathcal{V}] \cong [\mathcal{A}^{\text{op}}, [\mathcal{B}, \mathcal{V}')] \to \text{Cocts}[[\mathcal{B}^{\text{op}}, \mathcal{V}], [\mathcal{A}^{\text{op}}, \mathcal{V}]]
\]

(44)

into the sub-\(\mathcal{V}\)-category of cocontinuous \(\mathcal{V}\)-functors is an equivalence by \([17, \text{Theorem 4.51}]\). This \(\mathcal{V}\)-functor sends \(R : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \to \mathcal{V}\) to the left extension of the corresponding \(R' : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathcal{V}]\) along the Yoneda embedding \(\gamma : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathcal{V}]\), \(\text{Lan}_\gamma R'\), which is exactly \((R \circ -)\).

Theorem \([9, 4]\) gives:

**Corollary 10.3.** Suppose the lax centre of the promonoidal \(\mathcal{V}\)-category \(\mathcal{A}\) exists. Then there exists an equivalence of \(\mathcal{V}\)-categories \(Z_\mathcal{L}\mathcal{A}, \mathcal{V} \simeq Z_\mathcal{L}[\mathcal{A}, \mathcal{V}]\), where on the left hand side appears the lax centre in \(\mathcal{V}\text{-Mod}\) and on the right hand side the lax centre in \(\mathcal{V}\text{-Cat}\). The composition of this equivalence with the forgetful \(\mathcal{V}\)-functor \(Z_\mathcal{L}[\mathcal{A}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}]\) is canonically isomorphic to the \(\mathcal{V}\)-functor given by composing with the universal \(\mathcal{V}\)-module \(Z_\mathcal{L}\mathcal{A} \to \mathcal{A}\). If the centre of \(\mathcal{A}\), rather than the lax centre, exists, then the above holds substituting lax centres by centres throughout.

Now we turn our attention to autonomous pseudomonoids.

The existence of Eilenberg-Moore constructions in \(\mathcal{V}\text{-Mod}\) together with Corollary \([9, 13]\) and Theorem \([9, 4]\) implies

**Proposition 10.4.** Any left autonomous map pseudomonoid in \(\mathcal{V}\text{-Mod}\) has a lax centre. Moreover, if the pseudomonoid is also right autonomous then the lax centre is the centre.
where $x$ is a (bicategorical) Eilenberg-Moore construction for the monad $V \to I$. Hence, for a general small promonoid algebra $A$, we will suppose $A$ is a left autonomous monoidal $\mathcal{V}$-category, and not merely a monoidal one. However, all the following description carries over to the map pseudomonoid case with little modification.

By Corollary 6.13, the lax centre of $A$ in $\mathcal{V}$-$\text{Mod}$ is the Eilenberg-Moore construction for the monad $S$ given by

$$A \xrightarrow{\otimes 1} A \otimes A \xrightarrow{P \otimes 1} \mathcal{V} \otimes A \otimes A \xrightarrow{\mathcal{V} \otimes \text{sw}} A \otimes A \otimes A \xrightarrow{P \otimes 1} A \otimes A \xrightarrow{P} A \quad (45)$$

where $\text{sw}$ denotes the usual symmetry in $\mathcal{V}$-$\text{Cat}$ that switches the two factors. Explicitly,

$$\tilde{S}(a; b) \cong \int^{x,y} A(y \otimes (a \otimes x), b) \otimes A(I, y \otimes x) \cong \int^y A(y \otimes (a \otimes y^\vee), b),$$

where $y^\vee$ denotes the left dual of $y$ in $A$. The multiplication of this monad is given by

$$\tilde{S}^2(a; b) \cong \int^{u,y,z} A(y \otimes (u \otimes y^\vee), b) \otimes A(z \otimes (a \otimes z^\vee), u)$$

$$\cong \int^{y,z} A(y \otimes (z \otimes (a \otimes z^\vee)) \otimes y^\vee, b) \cong \int^{y,z} A((y \otimes z) \otimes (a \otimes (y \otimes z)^\vee), b)$$

$$\quad \quad \quad \rightarrow \int^x A(x \otimes (a \otimes x^\vee), b) \cong \tilde{S}(a; b)$$

where the last arrow is induced by the components $\zeta^{a,b}_{y,z} : A((y \otimes z) \otimes (a \otimes (y \otimes z)^\vee), b) \rightarrow \int^x A(x \otimes (a \otimes x^\vee), b)$ of the universal dinatural transformation defining the latter coend. The unit of $\tilde{S}$ is given by components

$$A(a, b) \xrightarrow{\zeta^a_{a,b}} \int^x A(x \otimes (a \otimes x^\vee), b)$$

of the same dinatural transformation corresponding to $x = I$. Now we have all the ingredients to describe the lax centre $Z_\ell(A)$, that is, a Kleisli construction for $\tilde{S}$. It has the same objects as $A$, enriched homs $Z_\ell(A)(a, b) = \tilde{S}(a; b)$, composition given by the multiplication and unit given by

$$I \rightarrow A(a, a) \xrightarrow{\zeta^a_{a,a}} \tilde{S}(a; a),$$

where the first arrow denotes the identity of $a$ in $A$. The arrows $\zeta^a_{I,b} : A(a, b) \rightarrow \tilde{S}(a, b)$ define a $\mathcal{V}$-functor, which we also call $\zeta$, and the universal $Z_\ell(A) \rightarrow A$ is none but $\zeta^*$. 

**Observation 10.5.** The monad $\tilde{S}$ is closely related to the monad $\tilde{M}$ in [8 Section 5]. There the authors show that for a general small monoidal $\mathcal{V}$-category $C$ to exist a monad $M$ on $C$ in $\mathcal{V}$-$\text{Mod}$ with the following property. Whenever $[C, \mathcal{V}]$ has a small dense sub-$\mathcal{V}$-category of objects with left duals (it is right-dual controlled, in the terminology of [8]), the forgetful $\mathcal{V}$-functor $Z_\ell[C, \mathcal{V}] \rightarrow [C, \mathcal{V}]$ is a (bicategorical) Eilenberg-Moore construction for the monad $M$ on $[C, \mathcal{V}]$ in $\mathcal{V}$-$\text{Cat}$ given by composition with $\tilde{M}$. The monad $\tilde{M}$ is given by

$$\tilde{M}(a, b) = \int^{x,y} P(P \otimes C)(y, a, x, b) \otimes x^\wedge(y),$$

where $x^\wedge$ is the internal hom $[C(x, -), J] \in [C, \mathcal{V}]$ ($J : I \rightarrow C$ is the unit of the monoidal structure).

When $C$ is equipped with a left dualization $D : C^{\text{op}} \rightarrow C$, each $\mathcal{V}$-module $I \rightarrow C$ with right adjoint in $\mathcal{V}$-$\text{Mod}$ has a left dual in the monoidal $\mathcal{V}$-category $\mathcal{V}$-$\text{Mod}(I, C) = [C, \mathcal{V}]$. This was first shown in [4]. Explicitly, a left dual for
$M : I \to C$ is given by $D(M^*) \cong (C \otimes M^*)P^*J$. In particular, $C(x, -)$, which is the $\mathcal{V}$-module induced by the $\mathcal{V}$-functor $I \to C$ constant on $x$, has left dual $(C \otimes C(-, x))P^*J \cong [C(x, -), J]$. It follows that $[C, \mathcal{V}]$ has a small dense sub-$\mathcal{V}$-category with left duals, and the results of [8] mentioned above apply.

In this situation, if we assume $J$ is a map, so that $\mathcal{S}$ exists, we claim that the monads $\tilde{M}$ and $\tilde{S}$ are isomorphic, or more precisely, that both are isomorphic as monoids in the monoidal $\mathcal{V}$-category $\mathcal{V}$-$\mathcal{M}od(C, \mathcal{V}) = [C^{\mathcal{opp}} \otimes C, \mathcal{V}]$. To show this, it is enough to prove that the monads $(\tilde{M} \circ -)$ and $(\tilde{S} \circ -)$ on $\mathcal{V}$-$\mathcal{M}od(I, C) = [C, \mathcal{V}]$ given by composition with $\tilde{M}$ and $\tilde{S}$ respectively are isomorphic. For, the $\mathcal{V}$-functor $[C^{\mathcal{opp}} \otimes C, \mathcal{V}] \cong [C, [C^{\mathcal{opp}}, \mathcal{V}]] \to \text{Cocts}[[C^{\mathcal{opp}}, \mathcal{V}],[C^{\mathcal{opp}}, \mathcal{V}]]$ into the sub-$\mathcal{V}$-category of cointerminally $\mathcal{V}$-functors in [14] is an equivalence by [17] Theorem 4.51). This $\mathcal{V}$-functor is monoidal and sends $R : C^{\mathcal{opp}} \otimes C \to \mathcal{V}$ to $(R \circ -)$.

Now, the monad $(\tilde{S} \circ -)$ is $\mathcal{V}$-$\mathcal{M}od(I, \tilde{S})$, and then it has the forgetful $\mathcal{V}$-functor $Z[I, \mathcal{V}] \to [C, \mathcal{V}]$ as an (bicategorical) Eilenberg-Moore construction by Corollary 10.3 and Proposition 10.4. Then, $(\tilde{S} \circ -)$ and $M = (\tilde{M} \circ -)$ have the same Eilenberg-Moore construction in $\mathcal{V}$-$\mathcal{C}at$ and it follows that both monads are isomorphic as required.

More explicitly, by the description of the left dual of a $\mathcal{V}$-module $I \to C$, we have $[C(x, -), J](y) \cong \int_{u,v} C(u, y) \otimes C(v, x) \otimes (P^*J)(u, v) = (P^*J)(y, x)$ and then $\tilde{M}(a, b) \cong \int_{y\in\mathcal{V}} P(P \otimes C)(y, a, x, b) \otimes P^*J(y, x)$. In other words, $\tilde{M}(a, b) \cong \tilde{S}(a, b)$; see [15].

In conclusion, for a left autonomous map pseudomonoid in $\mathcal{V}$-$\mathcal{M}od$, the monads $\tilde{M}$ and $\tilde{S}$ are isomorphic.

Example 10.1. Let $\mathcal{G}$ be a groupoid. Write $\Delta : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ for the diagonal functor and $E : \mathcal{G} \to 1$ the only possible functor. These give $\mathcal{G}$ a structure of comonoid in $\mathcal{C}at$ and thus $P = \Delta^*$ and $J = E^*$ is a promonoidal structure on $\mathcal{G}$. Explicitly, $P(a, b, c) = \mathcal{G}(a,c) \times \mathcal{G}(b,c)$ and $J(a) = 1$; the monoidal structure induced in $[\mathcal{G}, \mathcal{S}et]$ is given by the point-wise cartesian product. Define a functor $D : \mathcal{G}^{\mathcal{opp}} \to \mathcal{G}$ as the identity on objects and $D(f) = f^{-1}$ on arrows. In [6] Example 10] was essentially shown that $D$ is a left and right dualization for $\mathcal{G}$. Then, by Corollary 11.11 $\mathcal{G}$ has centre and lax centre in $\text{Set}$-$\mathcal{M}od$ and both coincide. On the other hand, $[Z(\mathcal{G}), \mathcal{S}et] \cong [Z(\mathcal{G}, \mathcal{S}et)]$ by Theorem 9.4 which together with [5] Theorem 4.5 shows that the centre of $\mathcal{G}$ in $\text{Set}$-$\mathcal{M}od$ is equivalent to the category called (lax)centre of $\mathcal{G}$ in the latter article.

11. Comodules

This section deals with the case of the monoidal bicategories of comodules $\text{Comod}(\mathcal{V})$. In general, $\mathcal{V}$ will be a braided monoidal category with a certain completeness condition. However, when we consider the lax centre of pseudomonoids the braiding will be a symmetry. Our aim is to show how the general theory developed in previous sections specialises to some of the most basic results of the theory of Hopf algebras.

Throughout the section we will use string diagrams to denote arrows in $\mathcal{V}$. Our convention is that arrows go downwards: the domain of the arrow is the top of the string while the codomain is the bottom string. Arrows are depicted as nodes with the exception of the comultiplication of a comonoid, which is pictured as the bifurcation of one string into two. For background on string diagrams see [15].
Given a monoidal category \( \mathcal{V} \), there is a monoidal 2-category \( \text{Comon}(\mathcal{V}) \) called the 2-category of comonoids. Its objects are comonoids in \( \mathcal{V} \), its 1-cells comonoid morphisms and 2-cells \( \sigma : f \Rightarrow g : C \to D \) are arrows \( \sigma : C \to I \) in \( \mathcal{V} \) such that

\[
\begin{array}{ccc}
C & \overset{f}{\longrightarrow} & D \\
\sigma \downarrow & & \downarrow \sigma' \\
C & \overset{g}{\longrightarrow} & D
\end{array}
\]

Vertical composition of 2-cells is the usual convolution product: \( \sigma \circ \sigma' = (\sigma \otimes \sigma') \Delta \), where \( \Delta \) denotes the comultiplication. The horizontal compositions

\[
A \overset{g}{\downarrow} \sigma B \overset{h}{\longrightarrow} C \quad \text{and} \quad D \overset{k}{\Downarrow} \sigma A \overset{f}{\longrightarrow} B
\]

are \( A \xrightarrow{\sigma} I \) and \( D \xleftarrow{k} A \xrightarrow{\sigma} I \) respectively.

**Observation 11.1.** \( \text{Comon}(\mathcal{V}) \) is the full sub 2-category of \( \mathcal{V}^{\text{op}.} \cdot \text{Cat}^{\text{op}} \) consisting of those \( \mathcal{V}^{\text{op}} \)-categories with just one object. In particular, it is triequivalent to a strict 3-category.

A pseudomonoid \((C, j, p)\) in \( \text{Comon}(\mathcal{V}) \) amounts to a comonoid \( C \) with two comonoid morphisms \( j : I \to C \) and \( p : C \otimes C \to C \) and the invertible 2-cells \( \phi : p(p \otimes C) \Rightarrow p(C \otimes p), \lambda : p(j \otimes C) \Rightarrow 1 \) and \( \rho : p(C \otimes j) \Rightarrow 1 \) satisfying axioms. These 2-cells are convolution-invertible arrows \( \phi : C \otimes C \otimes C \to I \) and \( \lambda, \rho : C \to I \).

**Example 11.1.** Normal pseudomonoids, that is, pseudomonoids whose unit constraints \( \lambda, \rho \) are identities, in the monoidal bicategory \( \text{Comod}(\text{Vect}) \) are coquasibialgebras. The dual of this algebraic structure, called quasibialgebra, was first defined in \cite{quasibialgebra} where also were defined the quasi-Hopf algebras. Then, a coquasibialgebra amounts to a coalgebra \((C, \epsilon, \Delta)\) with a multiplication \( p : C \otimes C \to C \), denoted by \( p(x \otimes y) = x \cdot y \), a unit \( j \in C \), where \( k \) is the field, and an additional functional \( \phi : C \otimes C \otimes C \to k \) satisfying \( p(j \otimes x) = x = p(x \otimes j) \),

\[
\sum \phi(x_1 \otimes y_1 \otimes z_1)(x_2 \cdot y_2) \cdot z_2 = \sum x_1 \cdot (y_1 \cdot z_1)\phi(x_2 \otimes y_2 \otimes z_2)
\]

\((\phi \otimes \epsilon) \star (1 \otimes p \otimes 1) \star (\epsilon \otimes \phi) = \phi(p \otimes 1 \otimes 1) \star \phi(1 \otimes 1 \otimes p)\)

where \( \star \) denotes the convolution product in the dual of \( C \otimes C \otimes C \otimes C \). We used Sweedler’s notation \( \Delta(x) = \sum x_1 \otimes x_2 \in C \otimes C \), as is usual in the theory of Hopf algebras.

Now suppose further that \( \mathcal{V} \) has equalizers of reflexive pairs and each functor \( X \otimes - \) preserves them. Then we can construct the bicategory of comodules over \( \mathcal{V} \), denoted by \( \text{Comod}(\mathcal{V}) \). It has comonoids in \( \mathcal{V} \) as objects and homs \( \text{Comod}(\mathcal{V})(C, D) \) the category of \( C \cdot D \)-bicomodules; this is the category of Eilenberg-Moore algebras for the comonad \( C \otimes - \otimes D \) on \( \mathcal{V} \). The composition of two comodules \( M : C \to D \) and \( N : D \to E \) is given by the equalizer of the following reflexive pair

\[
M \boxtimes_D N \longrightarrow M \otimes N \overset{\chi_{M \otimes N}}{\longrightarrow} M \otimes D \otimes N
\]

where the various \( \chi \) denote the obvious coactions, and with \( C \cdot E \)-comodule structure induced by the structures of \( M \) and \( N \). The comodule \( M \boxtimes_D N \) is sometimes called the cotensor product of \( M \) and \( N \) over \( D \). The identity 1-cell corresponding to a
comonoid $C$ is the regular comodule $C$, i.e. it is $C$ with coaction $(\Delta \otimes 1)\Delta : C \to C \otimes C \otimes C$.

There is a pseudofunctor $(-)_* : \mathbf{Comod}(\mathcal{V}) \to \mathbf{Comod}(\mathcal{V})$ acting as the identity on objects, sending a comonoid morphism $f : C \to D$ to the comodule, denoted by $f_* : C \to D$, with underlying object $C$ and coaction

$$\xymatrix{ C \ar@{..}[d] \ar@{..}[r] & C \otimes C \otimes C \ar@{..}[d] \ar@{..}[r] & C \otimes \Delta \ar@{..}[d] \ar[r] & \eta \ar@{..}[d] \ar[r] & C \ar@{..}[d] \ar[r] & C \otimes C \ar@{..}[d] \ar[r] & C \otimes C \ar[d] \ar[r] & C \ar[d] \ar[r] & C \ar[l] \ar[r] & \eta \ar[l] \ar[r] & 1 \ar[l] }$$

and sending a 2-cell $\sigma : f \Rightarrow g$ to the comodule morphism $\sigma_* : f_* \Rightarrow g_*$ given by

$$\xymatrix{ C \ar@{..}[d] \ar@{..}[r] & C \otimes C \ar@{..}[d] \ar@{..}[r] & C \ar@{..}[d] \ar[r] & \eta \ar@{..}[d] \ar[r] & C \ar@{..}[d] \ar[r] & C \ar@{..}[d] \ar[r] & C \ar[d] \ar[r] & C \ar[d] \ar[r] & C \ar[l] \ar[r] & \eta \ar[l] \ar[r] & 1 \ar[l] }$$

The axioms of coaction and of comodule morphism follow from the ones of comodule morphism and 2-cell in $\mathbf{Comon}(\mathcal{V})$ respectively. It is easy to show that the pseudofunctor $(-)_*$ is locally fully faithful (in fact, locally it can be viewed as a $\mathcal{V}^{\text{op}}$-enriched Yoneda embedding).

An important property of $(-)_*$ is that it sends any 1-cell in $\mathbf{Comon}(\mathcal{V})$ to a map in $\mathbf{Comod}(\mathcal{V})$. For, if $f : C \to D$ is a comonoid morphism, then $f_*$ has a right adjoint, denoted by $f^*$, with underlying object $C$ and coaction

$$\xymatrix{ C \ar@{..}[d] \ar@{..}[r] & C \otimes C \ar@{..}[d] \ar@{..}[r] & C \ar@{..}[d] \ar[r] & \eta \ar@{..}[d] \ar[r] & C \ar@{..}[d] \ar[r] & C \ar@{..}[d] \ar[r] & C \ar[d] \ar[r] & C \ar[d] \ar[r] & C \ar[l] \ar[r] & \eta \ar[l] \ar[r] & 1 \ar[l] }$$

The composition $f_* f^*$ is the comodule with object $C$ and coaction

$$\xymatrix{ C \ar@{..}[d] \ar@{..}[r] & C \otimes C \ar@{..}[d] \ar@{..}[r] & C \ar@{..}[d] \ar[r] & \eta \ar@{..}[d] \ar[r] & C \ar@{..}[d] \ar[r] & C \ar@{..}[d] \ar[r] & C \ar[d] \ar[r] & C \ar[d] \ar[r] & C \ar[l] \ar[r] & \eta \ar[l] \ar[r] & 1 \ar[l] }$$

and the counit of the adjunction is just the arrow $f : C \to D$, which turns out to be a comodule morphism; the unit is the unique map such that

$$f^* f_* = f_\delta D f^* \to C \otimes C$$

where the horizontal arrow is the defining equalizer of $f_\delta D f^*$.

**Observation 11.2.** The bicategory $\mathbf{Comod}(\mathcal{V})$ has Eilenberg-Moore objects for comonads. If $G$ is a comonad on the comonoid $C$ with comultiplication $\delta : G \to G \Box C G$ and counit $\epsilon : G \to C$, its Eilenberg-Moore object admits the following description (which is dual to the description of Kleisli objects for monads in $\mathcal{V}$-$\mathbf{Cat}$ in [30]). As a comonoid, it is $G$ equipped with comultiplication and counit

$$G \xrightarrow{\delta} G \Box C G \to G \otimes G \quad \text{and} \quad G \xrightarrow{\epsilon} C \xrightarrow{\eta} 1.$$

Note that the arrow $\epsilon : G \to C$ in $\mathcal{V}$ becomes a morphism of comonoids. The universal 1-cell is just the comodule $\epsilon_* : G \to C$. 
Observation 11.3. The bicategory \( \text{Comod}(\mathcal{V}) \) can be viewed as the full sub bicategory of \( \mathcal{V}^{\text{op}}\text{-Mod}^{\text{coop}} \) determined by the \( \mathcal{V}^{\text{op}} \)-categories with one object. However, for \( \mathcal{V}^{\text{op}}\text{-Mod} \) to exist further completeness assumptions on \( \mathcal{V} \) have to be made.

When \( \mathcal{V} \) is braided, \( \text{Comon}(\mathcal{V}) \) and \( \text{Comod}(\mathcal{V}) \) have the structure of monoidal 2-categories with tensor product given by the tensor product of \( \mathcal{V} \); note that the braiding is used in defining the comultiplication and coactions on the tensor product of comonoids and comodules. The pseudofunctor \( (\_)^* \) is strong monoidal, so that through \( (\_)^* \) we can think of \( \text{Comon}(\mathcal{V}) \) as a monoidal sub bicategory of \( \text{Comod}(\mathcal{V}) \). Since its tensor product is a 2-functor, by [11], \( \text{Comod}(\mathcal{V}) \) is triequivalent to a strict 3-category.

The bicategory \( \text{Comod}(\mathcal{V}) \) is not just monoidal but it is also left and right autonomous. The right bidual of a comonoid \( C \) is the opposite comonoid \( C^\circ \). The braiding provides pseudonatural equivalences

\[
\text{Comod}(\mathcal{V})(C \otimes D, E) \simeq \text{Comod}(\mathcal{V})(D, C^\circ \otimes E).
\]

The coevaluation \( n : I \to C^\circ \otimes C \) and evaluation \( e : C \otimes C^\circ \to I \) comodules are the object \( C \) with coactions depicted in Figure 11. The left bidual is defined by using the inverse of the braiding.

Example 11.2. As shown in [4], Coquasi-Hopf algebras are exactly the left autonomous normal pseudomonoids in \( \text{Comod(\text{Vect})} \) whose unit, multiplication and dualization are representable by coalgebra morphisms.

11.1. Hopf modules. From now on, \( \mathcal{V} \) will not only have equalizers of reflexive pairs, but all equalizers. The reason for this is that we want the following proposition to hold. Equalizers are necessary as the proof uses the Adjoint Triangle Theorem [10].

Proposition 11.4 ([4]). A comodule \( M : C \to D \) has a right adjoint if and only if its composition with \( \varepsilon_* : D \to I \) has a right adjoint.

We shall describe the monad \( \theta \) for a map pseudomonoid \( (C,j,p) \), which for simplicity we will suppose arising from a pseudomonoid in \( \text{Comon}(\mathcal{V}) \).

Recall from Definition 3.1 that the monad \( \theta_D \) on \( \text{Comod}(C \otimes D, C) \) is just the monad \( \text{KL}(C \otimes -)(D,p) \) on \( \text{KL}(C \otimes -)(D,C) \). In terms of comodules, \( \theta_D(M) \) has underlying object \( C \otimes M \) and coaction the arrow depicted in Figure 11.1. The multiplication (2) and the unit (3) become in this case

\[
C \otimes C \otimes M \to C^{\otimes 3} \otimes D \otimes C \otimes M \otimes C^{\otimes 3} \xrightarrow{\phi^{-1} \otimes p \otimes 1 \otimes 1 \otimes \phi} C \otimes C \otimes M \xrightarrow{\rho_1} C \otimes M
\]

and

\[
M \to C \otimes D \otimes M \otimes C \xrightarrow{\lambda \otimes p \otimes 1 \otimes \lambda^{-1}} M \xrightarrow{j \otimes 1} C \otimes M
\]

where \( \phi^{-1} \) and \( \lambda^{-1} \) are the convolution inverses of \( \phi : C \otimes C \otimes C \to I \) and \( \lambda : C \to I \).
In view of Observation 3.1, the functor \( \psi_D \lambda_D \) (see Definition 3.2) is isomorphic to the one sending a comodule \( M : D \to C \) to \( C \otimes M \) with coaction the arrow in Figure 11.1. The theorem of Hopf modules holds for \( C \) exactly when every \( \theta_D \)-algebra is isomorphic to one of this form.

Now we shall describe for a pseudomonoid \( C \) in \( \text{Comon}(\mathcal{V}) \) the underlying comodule of the monad \( t \) on \( C^\circ \otimes C \) representing \( \theta \). Recall from (24) that

\[
t \cong (C^\circ \otimes p_*)(C^\circ \otimes C \otimes e \otimes C)(C^\circ \otimes p^* \otimes C^* \otimes C)(n \otimes C^* \otimes C)
\]

and so it has underlying object \( C \otimes C \otimes C \) with coaction depicted in Figure 11.1.

The Hopf module construction for a map pseudomonoid in \( \text{Comod}(\mathcal{V}) \) may not exist, as this bicategory does not have an Eilenberg-Moore objects for monads. However, it does have Eilenberg-Moore construction for comonads (Observation 11.2).
Proposition 11.5. Given a map pseudomonoid $C$ in $\text{Comod}(\mathcal{Y})$, if the monad $t : C^\circ \otimes C \to C^\circ \otimes C$ has right adjoint, then $C$ has a Hopf module construction. In particular, this holds if $C \in \text{ob} \mathcal{Y}$ has a dual.

Proof. The 1-cell $t^*$ has a canonical structure of a right adjoint comonad to the monad $t$. It well-known that the Eilenberg-Moore construction for the comonad $t^*$ is an Eilenberg-Moore construction for the monad $t$. To finish, we show that if $C$ has a dual in $\mathcal{Y}$ then $t \cong ((p^\circ)^* \otimes p)((C^\circ \otimes n \otimes C)$ has a right adjoint, and for that it suffices to prove that $n$ does. But by Proposition 11.4, $n$ is a map if and only if $C$ has a dual.

When $\mathcal{Y}$ is the category of vector spaces and $C$ is a coquasi-bialgebra, the assertion that the functor $\lambda_I$ from $\text{Comod}(\mathcal{Y})(I, C)$ to the category of Hopf modules is an equivalence is what Schauenburg [28] calls the theorem of Hopf modules. We shall show that when $C$ has a Hopf module construction both notions are equivalent.

Let $\mathcal{W}$ be a braided monoidal replete full subcategory of $\mathcal{Y}$ closed under equalizers of reflexive pairs. There is an inclusion monoidal pseudofunctor $\text{Comod}(\mathcal{W}) \to \text{Comod}(\mathcal{Y})$. This inclusion, being monoidal, preserves biduals.

Corollary 11.6. Let $\mathcal{W}$ and $\mathcal{Y}$ be as above. Suppose $C$ is a map pseudomonoid in $\text{Comod}(\mathcal{W})$ such that $C$ has a dual in $\mathcal{W}$. Then, the theorem of Hopf modules holds for $C$ in $\text{Comod}(\mathcal{W})$ if and only if it holds for $C$ in $\text{Comod}(\mathcal{Y})$.

Proof. We begin by observing that since $C$ has dual in $\mathcal{W}$, and hence in $\mathcal{Y}$, by Proposition 11.5, $C$ has a Hopf module construction both in $\text{Comod}(\mathcal{W})$ and in $\text{Comod}(\mathcal{Y})$. Moreover, the two coincide. To see this, observe that the monad $t$ is given by (24) and each of the 1-cells in the composition lies in $\text{Comod}(\mathcal{W})$. Since $C$ has a dual, $t$ has a right adjoint comonad, whose Eilenberg-Moore construction, described in Observation 11.2, is the Hopf module construction for $C$. By the description of this Eilenberg-Moore construction, one sees that it lies in $\text{Comod}(\mathcal{W})$.

Hence, we have to prove that the 1-cell $\ell : C \to (C^\circ \otimes C)^t$ (see Proposition 5.3) is an equivalence in $\text{Comod}(\mathcal{W})$ if and only if it is one in $\text{Comod}(\mathcal{Y})$. One direction is trivial, so we shall suppose $\ell$ is an equivalence in $\text{Comod}(\mathcal{Y})$. We have, then, an adjoint equivalence $\ell \dashv \ell^*$; as $\ell$ is always a map (by Proposition 5.3), this adjoint equivalence lifts to $\text{Comod}(\mathcal{W})$.

In the particular case when $\mathcal{Y}$ is the category of vector spaces and $\mathcal{W}$ is the subcategory of finite-dimensional vector spaces, we have:

Corollary 11.7. For any finite-dimensional coquasi-bialgebra $C$ there exists a map pseudomonoid $D$ in $\text{Comod}(\text{Vect})$ such that the category of Hopf modules for $C$ (as defined in [28]) is monoidally equivalent to the category of right $D$-comodules $\text{comod}(\text{Vect})(I, D)$. Moreover, $D$ can be taken to be the Hopf module construction for $C$, and in particular, finite-dimensional.

Note that, in general, the forgetful functor $\text{Comod}(\text{Vect})(I, D) \to \text{Vect}$ is not monoidal.

By Observation 11.2, the Hopf module construction $(C^\circ \otimes C)^t \to C^\circ \otimes C$ can be taken to be of the form $\epsilon_*$, where $\epsilon : (C^\circ \otimes C)^t \to C^\circ \otimes C$ is a comonoid morphism.

Corollary 11.8. Suppose that $C$ is a map pseudomonoid in $\text{Comod}(\text{Vect})$. If $C$ is finite-dimensional, the theorem of Hopf modules holds for $C$ if and only if the functor

$$\lambda_I : \text{Comod}(\text{Vect})(I, C) \to \text{Comod}(\text{Vect})(C, C)^{p_1}$$

(see Definition 5.3) is an equivalence.
Figure 7.

Proof. Only the converse is non trivial. Write \( \mathcal{C} \) for \( \text{Vect} \). By Proposition 11.6, it is enough to show that the theorem of Hopf modules holds for \( C \) in \( \text{Comod}(\mathcal{C}) \).

The functor \( \lambda_I \) is represented by the 1-cell \( \ell : C \rightarrow (C^\circ \otimes C)^\ell \). We have that the functor \( \text{Comod}(\mathcal{C})(I, \ell) \) is an equivalence, and the result follows from the following lemma. □

Lemma 11.9. (1) The functor \( \text{Comod}(\text{Vect}_f)(I, -) \) reflects equivalences.

(2) Any finitely continuous functor from the category \( \text{Comod}(\text{Vect}_f)(I, D) \) to \( \text{Comod}(\text{Vect}_f)(I, E) \) is isomorphic to a functor given by composition with a comodule \( M : D \rightarrow E \).

Proof. (1) Assume that \( \text{Comod}(\text{Vect}_f)(I, M) \) is an equivalence, for \( M : D \rightarrow E \). Taking duals, we see that the functor from \( D^\ast\text{-Mod}_f \) to \( E^\ast\text{-Mod}_f \) given by tensoring with \( M^\ast \) is an equivalence; this implies that \( M^\ast \) is an equivalence in \( \text{Mod}(\text{Vect}_f) \) (a Morita equivalence), and hence, taking duals, \( M \) is an equivalence.

(2) Taking duals, the result follows from the fact that any finitely cocontinuous functor between categories of finite-dimensional modules over a finite-dimensional algebra, is isomorphic to a functor induced by tensoring with a bimodule. □

We obtain the following generalisation of [28, Thm. 3.1].

Corollary 11.10. Let \( C \) be a map pseudomonoid in \( \text{Comod}(\text{Vect}) \) whose underlying space is finite-dimensional. Then \( C \) has a left dualization if and only if the functor \( \lambda_I : \text{Comod}(\text{Vect})(I, C) \rightarrow \text{Comod}(\text{Vect})(C, C)^\theta_I \) is an equivalence.

Proof. By the corollary above, the theorem of Hopf modules holds for \( C \); hence, \( C \) has a left dualization by Theorem 7.2. □

Now suppose that \( C \) is a left autonomous map pseudomonoid in \( \text{Comod}(\mathcal{C}) \). The existence of a left dualization forces the multiplication to be a map [4 Prop. 1.2]. On the other hand, the unit of \( C \) is a map because its underlying object \( I \in \mathcal{C} \) has (right) dual by Proposition 11.4. It follows that any left autonomous pseudomonoid in \( \text{Comod}(\mathcal{C}) \) is a map pseudomonoid. A Hopf module construction for \( C \) is provided by \( (C^\circ \otimes p)(n \otimes C) \cong (p(d \otimes C))^\ast : C \rightarrow C^\circ \otimes C \). In the case when \( C \) is a coquasibialgebra, the comodule \( (C^\circ \otimes p^\ast)(n \otimes C) \) is \( C \otimes C \) with coaction depicted in Figure 11.1.

11.2. Centre and Drinfel’d double. We now consider the results of Section 9 on the lax centre in the context of comodules. We suppose the underlying monoidal category \( \mathcal{C} \) is symmetric, and thus \( \text{Comon}(\mathcal{C}) \) is a symmetric monoidal \( \text{Cat} \)-enriched category. Via the monoidal pseudofunctor \( (-)_* \), we obtain comodules \( c_{M,N} : M \otimes N \rightarrow N \otimes M \) making the usual diagrams commute up to canonical isomorphisms.
Proposition 11.11. Any left autonomous pseudomonoid in \( \text{Comod}(\mathcal{V}) \) whose underlying object in \( \mathcal{V} \) has dual has a lax centre. If the pseudomonoid is also right autonomous then the lax centre equals the centre.

Proof. We have already mention that any left autonomous pseudomonoid \( C \) in \( \text{Comod}(\mathcal{V}) \) is a map pseudomonoid. By Corollary [9, 13] we have to show that the monad \( s : A \to A \) has an Eilenberg-Moore construction, and for that it is enough to show that it has a right adjoint, since \( \text{Comod}(\mathcal{V}) \) has Eilenberg-Moore objects for comonads. Again by Corollary [9, 13] we have \( \tilde{s} \cong p(p \otimes C)(C \otimes C,A)(p^* \otimes C)(j \otimes C) \) and therefore \( \tilde{s} \) has a right adjoint if \( p^*j : I \to C \otimes C \) has one; but \( C \) being left autonomous, this 1-cell is isomorphic to \((d \otimes C)n\) which is a composition of maps: \( d \) by [4, Prop. 1.2] and \( n \) by Proposition [11, 4].

Observation 11.12. In the proposition above, suppose that the full subcategory \( \mathcal{V}_f \) of objects with a dual in \( \mathcal{V} \) is closed under equalizers of reflexive pairs. Then the lax centre \( Z(C) \to C \) lies in \( \text{Comod}(\mathcal{V}_f) \), and it is a lax centre in it.

To prove this observe that \( t : C^* \otimes C \to C \otimes C \) and its Eilenberg-Moore construction \( C \to C^* \otimes C \) lie in \( \text{Comod}(\mathcal{V}_f) \), and the monad \( s \) and the distributive law between \( t \) and \( s \) do so too. It follows that the induced monad \( \tilde{s} \) on \( C \) lies in \( \text{Comod}(\mathcal{V}_f) \), and it has right adjoint in this bicategory, as shown in the proof above, and it is necessarily the same as in \( \text{Comod}(\mathcal{V}) \). It follows from Observation [11, 2] that \( \tilde{s}^* \) has an Eilenberg-Moore construction in \( \text{Comod}(\mathcal{V}_f) \) and coincides with the respective construction in \( \text{Comod}(\mathcal{V}) \). Moreover, this construction is given by \( \epsilon_\sigma : C \to C^{\sigma^*} \), where \( \epsilon \) is the counit of the comonad \( \tilde{s}^* \). Therefore, the lax centre of \( C \) in both bicategories coincide.

The Drinfel’d double or quantum double of a finite-dimensional Hopf algebra is a finite-dimensional braided (also called quasitriangular) Hopf algebra \( D(H) \) with underlying vector space \( H^* \otimes H \) (one can also take \( H \otimes H \)) and suitably defined structure. It is a classical result that the category of left \( D(H) \)-modules is monoidally equivalent to the category of (two-sided) \( H \) Hopf modules and to the centre of the category of \( H \)-modules. The Drinfel’d double of a finite-dimensional quasi-Hopf algebra was defined in [22] using a reconstruction theorem, and explicit constructions were given in [13, 24]. This last paper shows that the category of \( D(H) \)-modules is monoidally equivalent to the centre of the category of \( H \)-modules, via a generalisation of the Yetter-Drinfel’d modules. The quantum double of a coquasi-Hopf algebra was described in [2]. Alternatively, it can be described by dualising the explicit constructions for the quasi-Hopf case. Then the Drinfel’d or quantum double \( D(H) \) of a finite-dimensional coquasi-Hopf algebra \( H \) is finite-dimensional and has the property that the category of \( D(H) \)-comodules \( \text{Comod}(D(H)) \) is monoidally equivalent to the centre of \( \text{Comod}(H) \), and the equivalence commutes with the forgetful functors.

Given a finite-dimensional coquasi-Hopf algebra \( H \), we would like to study the relationship between the centre \( Z(H) \) in \( \text{Comod} (\text{Vect}) \) and the Drinfel’d double \( D(H) \). To this aim we will need some of the machinery of Tannakian reconstruction, of which we give the most basic aspects following [23].

Let \( \mathcal{V} \) be a monoidal category and \( \mathcal{V}_f \) the full sub-monoidal category with objects with left duals. We denote by \( \mathcal{V}_f-\text{Act} \) the 2-category of pseudoalgebras for the pseudomonoid \( (\mathcal{V}_f \times -) \) on \( \text{Cat} \). Objects of this 2-category are pseudoactions of \( \mathcal{V}_f \) and 1-cells are pseudomorphisms of pseudoactions. Observe that \( \mathcal{V}_f \) has a canonical \( \mathcal{V}_f \)-pseudoaction given by the tensor product. We form the 2-category \( \mathcal{V}_f-\text{Alg} / \mathcal{V}_f \) with objects 1-cells \( \sigma : \mathcal{A} \to \mathcal{V}_f \) in \( \mathcal{V}_f-\text{Act} \). The 1-cells are pairs \((F, \phi) : \sigma \to \sigma' \) where \( F : \mathcal{A} \to \mathcal{A}' \) is a 1-cell in \( \mathcal{V}_f-\text{Act} \) and \( \phi : \sigma' F \cong \sigma \) is a 2-cell in \( \mathcal{V}_f-\text{Act} \). 2-cells \((F, \phi) \Rightarrow (F', \phi') \) are just 2-cells \( F \Rightarrow F' \) in \( \mathcal{V}_f-\text{Act} \). There is a 2-functor...
Comod$_f : \text{Comon}(\mathcal{V}) \to \mathcal{V}_f\text{-Act}/\mathcal{V}_f$ sending a comonoid C to the forgetful functor $\omega_C : \text{Comod}_f(C) \to \mathcal{V}_f$; here $\text{Comod}_f(C)$ is the category of right coactions of $C$ with underlying object in $\mathcal{V}_f$. This category has a canonical $\mathcal{V}_f$-pseudoaction such that $\omega$ is an object of $\mathcal{V}_f\text{-Act}/\mathcal{V}_f$. The definition of Comod$_f$ on 1-cells and 2-cells should be more or less obvious; see [24].

Under certain hypothesis on $\mathcal{V}$, the 2-functor Comod$_f$ is bi-fully faithful. Here is the case we will need.

**Proposition 11.13** ([25]). The 2-functor

$$\text{Comod}_f : \text{Comon}(\text{Vect}) \to \text{Vect}_f\text{-Act}/\text{Vect}_f$$

is bi-fully faithful.

**Theorem 11.14.** For any finite-dimensional coquasi-Hopf algebra $H$, the coalgebras $H^\natural$ and $D(H)$ are equivalent coquasibialgebras. Moreover, they are isomorphic as coalgebras.

**Proof.** By Observation 11.12 $H^\natural$ is a centre for the pseudomonoid $H$ in $\text{Comod}(\text{Vect}_f)$. Hence we have an equivalence in $\text{Vect}_f\text{-Act}/\text{Vect}_f$ from the forgetful functor $\text{Comod}_f(H^\natural) \to \text{Vect}_f$ to the forgetful functor $Z(\text{Comod}_f(H)) \to \text{Vect}_f$. On the other hand, there is an equivalence from the latter to $\text{Comod}_f(D(H)) \to \text{Vect}_f$. In this way we get an equivalence from $\text{Comod}_f(H^\natural)$ to $\text{Comod}_f(D(H))$ in $\text{Vect}_f\text{-Act}/\text{Vect}_f$. By Proposition 11.13 we have an equivalence $f : H^\natural \to D(H)$ in $\text{Comod}(\text{Vect})$. That is, both coquasibialgebras are equivalent.

Note that the equivalence $\text{Comod}_f(f)$, given by corestriction along $f$, is just the functor $\text{Comod}(\text{Vect}_f)(I, f_*)$, the composition with the comodule $f_* : H^\natural \to D(H)$. Then $f_*$ is an equivalence in $\text{Comod}(\text{Vect}_f)$ by Lemma 11.10. But this implies that $f$ is an isomorphism in $\text{Vect}_f$, as the counit of the adjunction $f_* - f^*$ is given by $f$ itself. Hence, $f$ is an isomorphism of coalgebras. \hfill $\square$

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