GEOMETRY OF STATISTICAL SUBMANIFOLDS OF
STATISTICAL WARPED PRODUCT MANIFOLDS BY
OPTIMIZATION TECHNIQUES

ALIYA NAAZ SIDDIQUI, FATEMAH MOFARREH, ALI HUSSAIN ALKHALDI,
AND AKRAM ALI

Abstract. This paper deals with the applications of an optimization method
on submanifolds, that is, geometric inequalities can be considered as optimization
problems. In this regard, we obtain optimal Casorati inequalities and
Chen-Ricci inequality for a statistical submanifold in a statistical warped prod-
uct manifold of type $\mathbb{R} \times_{f} \mathcal{M}$ (almost Kenmotsu statistical manifold), where $\mathbb{R}$
and $\mathcal{M}$ are trivial statistical manifold and almost Kaehler statistical manifold,
respectively.

1. Introduction

The concept of warped product manifolds has many applications in physics as
well as in differential geometry. For instance, different models of space-time in gen-
eral relativity are expressed in terms of warped geometry, and the Einstein field
equations and modified field equations have many exact solutions as the warped
products. In 1969, the idea of warped product manifolds has been initiated by R.L.
Bishop and B. O’Neil [3] with manifolds of negative curvature. These manifolds
are the most fruitful and natural generalization of Riemannian product manifolds.

A statistical structure can be considered as a fruitful generalization of a Rie-
mannian structure (a pair of a Riemannian metric and its Levi-Civita connection).
By following this idea in complex geometry, H. Furuhata gave the definition
of a holomorphic statistical manifold as a nice generalization of a pair of a Kaehler
structure and Levi-Civita connection (see [10, 11]). It could be treated as the sta-
tistical counterpart of the notion of complex manifold. Siddiqui et al. [31] studied
and obtained several results on totally real statistical submanifolds of holomor-
phic statistical manifolds. Later on, Furuhata et al. [12] introduced the statistical
counterpart of a Sasakian manifold and defined the notion of a Sasakian statistical
manifold in the contact geometry. Vilcu et al. [39] investigated the existence of the
almost quaternionic structures on the statistical manifolds. They defined the con-
cept of quaternionic Kaehler-like statistical manifolds. By extending some results
derived by K. Takano concerning statistical manifolds endowed with almost complex
[35] and almost contact structures [36], they obtained the main curvature properties
of quaternionic Kaehler-like statistical submersions. L. Todjihownde established a
dualistic structure on the warped product manifold in [37]. In light of his work,
Furuhata et al. [13] studied the statistical counterpart of a Kenmotsu manifold and introduced the notion of Kenmotsu statistical manifolds. In addition, it can be locally treated as the warped product of a holomorphic statistical manifold and a line. They proved that a Kenmotsu statistical manifold of constant $\phi-$sectional curvature is constructed from a holomorphic statistical manifold. They equipped a Kenmotsu manifold with an affine connection and gave a method of how to construct a Kenmotsu statistical manifold of constant $\phi-$sectional curvature as the warped product of a holomorphic statistical manifold and a line. Recently, in light of Todjihounde’s work, C. Murathan and B. Sahin [20] obtained Wintgen-like inequality for statistical submanifolds of statistical warped product manifolds. They also studied how to construct Kenmotsu-like statistical manifolds and cosymplectic-like statistical manifold based on the existence of Kaehler-like statistical manifold. The Einstein statistical warped products were studied in [15].

We have proposed an application to the area of optimization on manifolds, written as a digest of [24] enhanced with references to the most recent literature. We have the most interesting result on optimization on Riemannian manifold as follow:

**Optimizations on submanifolds:** Let $(N, g)$ be a Riemannian submanifold of a Riemannian manifold $(\overline{M}, \overline{g})$ and $f : \overline{M} \to \mathbb{R}$ be a differentiable function.

**Theorem 1.1.** [24] If $x \in N$ is a solution of the constrained extremum problem $\min_{x_0 \in N} f(x_0)$, then

(a) $(\text{grad } f)(x) \in T^\perp_x N$,
(b) the bilinear form $\Theta : T_x N \times T_x N \to \mathbb{R}$,

$$\Theta(E, F) = \text{Hess}_f(E, F) + \overline{g}(h'(E, F), (\text{grad } f)(x))$$

is positive semi-definite, where $h'$ is the second fundamental form of $N$ in $\overline{M}$, $\text{grad } f$ denotes the gradient of $f$.

In summary, optimization on manifolds is about exploiting tools of differential geometry to build optimization schemes on abstract manifolds, then turning these abstract geometric algorithms into practical numerical methods for specific manifolds, with applications to problems that can be rephrased as optimizing a differentiable function over a manifold. This research program has shed new light on existing algorithms and produced novel methods backed by a strong convergence analysis. We close by pointing out that optimization of real-valued functions on manifolds, as formulated in Theorem 1.1, is not the only place where optimization and differential geometry meet and also is the Riemannian geometry of the central path in linear programming. Therefore, in the present paper, we obtain optimal Casorati inequalities and Chen-Ricci inequality for a statistical submanifold in a statistical warped product manifold of type $\mathbb{R} \times_f \overline{M}$, where $\mathbb{R}$ and $\overline{M}$ are trivial statistical manifold and almost Kaehler statistical manifold, respectively. Such inequalities with a pair of conjugate affine connections involving the intrinsic and extrinsic curvature invariants of statistical submanifolds in different ambient spaces are derived (see [2, 9, 13, 19, 28, 32]).
2. Preliminaries

A Riemannian manifold \((M, g)\) with an affine connection \(\nabla\) is said to be a statistical manifold \([10]\) \((M, g, \nabla)\) if \(\nabla\) is a torsion-free connection on \(M\) and the covariant derivative \(\nabla g\) is symmetric. A statistical manifold is a Riemannian manifold \((M, g)\) endowed with a pair of torsion-free affine connections \(\nabla\) and \(\nabla^*\) satisfying

\[
E g(F, G) = g(\nabla E F, G) + g(F, \nabla^* E G),
\]

for any \(E, F, G \in \Gamma(TM)\). The connections \(\nabla\) and \(\nabla^*\) are called dual connections.

The notion of conjugate connection was introduced by Amari \([1]\) into statistics. Later, his studies were developed by Lauritzen \([17]\). Clearly, \((\nabla^*)^* = \nabla\). Also, a dual connection of any torsion free affine connection \(\nabla\) is given by \(2\nabla^0 = \nabla + \nabla^*\), where \(\nabla^0\) is the Levi-Civita connection on \(M\). Moreover, if \((\nabla, g)\) is a statistical structure on \(M\), then \((\nabla^*, g)\) is also a statistical structure.

An almost Hermitian manifold \((M, g, J)\) is said to be an almost Kaehler manifold if its fundamental 2-form \(\omega\), defined by

\[
\omega(E, F) = g(E, JF),
\]

for any \(E, F \in \Gamma(TM)\), is closed, that is, \(d\omega = 0\) (cf. \([41]\)).

**Definition 2.1.** \([42]\) Let \((M, \nabla, g, J)\) be a statistical manifold. If \((M, g, J)\) is an almost Hermitian manifold, then \((M, \nabla, g, J)\) is called an almost Hermitian statistical manifold. If \((M, g, J)\) is a (almost) Kaehler manifold then \((M, \nabla, g, J)\) is called a (almost) Kaehler statistical manifold.

**Definition 2.2.** \([11]\) Let \((M, \nabla, g, J)\) be a Kaehler manifold with an affine connection \(\nabla\) on \(M\). Then \((M, \nabla, g, J)\) is said to be a holomorphic statistical manifold if

1. \((M, \nabla, g)\) is a statistical manifold, and
2. a 2–form \(\omega\) on \(M\) is \(\nabla\)–parallel (means \(\nabla \omega = 0\)).

**Definition 2.3.** \([11]\) A holomorphic statistical manifold \((M, \nabla, g, J)\) is of constant holomorphic curvature \(c \in \mathbb{R}\) if and only if

\[
S(E, F)G = \frac{\gamma}{4} \{ g(F, G)E - g(E, G)F + g(JF, G)JF - g(JG, F)JE + 2g(JF, G)JG \}.
\]

(2.1)

for any \(E, F, G \in \Gamma(TM)\). It is denoted by \((M, \nabla, g)\).

Let \((\overline{M}, \overline{\nabla}, \overline{g})\) be a statistical manifold and \(N\) be a submanifold of \(\overline{M}\). Then \((N, \nabla, g)\) is also a statistical manifold with the induced statistical structure \((\nabla, g)\) on \(N\) from \((\overline{\nabla}, \overline{g})\) and we call \((N, \nabla, g)\) as a statistical submanifold in \((\overline{M}, \overline{\nabla}, \overline{g})\). The fundamental equations in the geometry of Riemannian submanifolds (see \([41]\)) are the Gauss and Weingarten formulae and the equations of Gauss, Codazzi and Ricci. In the statistical setting, Gauss and Weingarten formulae are respectively defined by \([40]\)

\[
\begin{align*}
\nabla_E F &= \nabla_E F + h(E, F)V, \\
\nabla_E V &= -A_V(E) + \nabla_E V,
\end{align*}
\]

(2.2)
for any $E, F \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$, where $\nabla$ and $\nabla'$ (resp., $\nabla$ and $\nabla^*$) are the dual connections on $\overline{M}$ (resp., on $N$). The symmetric and bilinear imbedding tensor of $N$ in $\overline{M}$ with respect to $\nabla$ and $\nabla'$ are denoted by $h$ and $h^*$, respectively. The relation between $h$ (resp. $h^*$) and $A_V$ (resp. $A_V^*$) is defined by \[ \begin{align*}
abla h(E, F), V) &= g(A_V^* E, F), \\
abla h^*(E, F), V) &= g(A_V E, F), \end{align*} \tag{2.3} \]
for any $E, F \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$.

Let $\overline{R}$ and $R$ be the curvature tensor fields of $\nabla$ and $\nabla$, respectively. The corresponding Gauss, Codazzi and Ricci equations are respectively given by \[ \overline{\nabla}(R(E, F)G, H) = g(R(E, F)G, h^*(F, H)) + \overline{\nabla}(h(E, G), h^*(F, H)) \tag{2.4} \]
\[ \overline{R}(E, F)G) = \nabla^*_E h(F, G) - h(\nabla_E F, G) - h(F, \nabla_E G) - h(E, F, G), \tag{2.5} \]
\[ \overline{\nabla}(R^*(E, F)V, U) = g(R(E, F)V, U) + g([A_V^*, A_V] E, F), \tag{2.6} \]
for any $E, F, G, H \in \Gamma(TN)$ and $V, U \in \Gamma(T^\perp N)$, where $R^\perp$ is the Riemannian curvature tensor on $T^\perp N$. Similarly, $\overline{R}$ and $R^*$ are respectively the curvature tensor fields with respect to $\nabla^*$ and $\nabla^*$. We can obtain the duals of all equations (2.3)-(2.6) with respect to $\nabla^*$ and $\nabla^*$. Then the curvature tensor fields of $\overline{M}$ and $N$ are respectively given by \[ \begin{align*}
\overline{\nabla} \overline{\nabla} = \frac{1}{2}(\overline{R} + \overline{R}^*), \quad \text{and} \quad S = \frac{1}{2}(R + R^*). \tag{2.7} \end{align*} \]

Thus, the sectional curvature $K_{\nabla^*, \nabla^*}$ on $N$ of $\overline{M}$ is given by \[ K_{\nabla^*, \nabla^*}(E \wedge F) = \frac{1}{2} g(\delta(E, F)F, E) + g(R^*(E, F)F, E)), \tag{2.8} \]
for any orthonormal vectors $E, F \in T_x N$, $x \in N$.

The notion of warped product manifolds came up as a nice generalization of Riemannian product manifolds. According to R.L. Bishop and B. O’Neill, these manifolds were defined as follows [3]:

**Definition 2.4.** Let $(N_1, g_1)$ and $(N_2, g_2)$ be two (pseudo)-Riemannian manifolds and $\bar{f} > 0$ be a differentiable function on $N_1$. Consider the product $\rho : N_1 \times N_2 \rightarrow N_1$ and $\delta : N_1 \times N_2 \rightarrow N_2$. Then the warped product $N = N_1 \times \bar{f} N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that \[ \overline{\nabla}(E, F) = g_1(\rho_* E, \rho_* F) + \bar{f}^2(u)g_2(\delta_* E, \delta_* F)), \tag{2.9} \]
for any $E, F \in \Gamma(T_{(u, v)}N)$, $u \in N_1$ and $v \in N_2$, where $*$ is the symbol for the tangent maps. The function $\bar{f}$ is called the warping function of the warped product.

Let $\chi(N_1)$ and $\chi(N_2)$ be the set of all vector fields on $N_1 \times N_2$ which is the horizontal lift of a vector field on $N_1$ and the vector lift of a vector field on $N_2$, respectively. Thus, it can be seen that $\rho_* (\chi(N_1)) = \Gamma(TN_1)$ and $\delta_* (\chi(N_2)) = \Gamma(TN_2)$. So, $\rho_* (X) = E \in \Gamma(TN_1)$, $\rho_* (Y) = F \in \Gamma(TN_1)$, $\delta_* (U) = G \in \Gamma(TN_2)$ and $\delta_* (V) = H \in \Gamma(TN_2)$. We recall a general result for a dualistic structure on the warped product manifold $N_1 \times \bar{f} N_2$ given by L. Todjihounde in [37].
Proposition 2.5. Let \((g_1, \nabla^{N_1}, \nabla^{N_1*})\) and \((g_2, \nabla^{N_2}, \nabla^{N_2*})\) be dualistic structures on \(N_1\) and \(N_2\). For \(X, Y \in \chi(N_1)\) and \(U, V \in \chi(N_2)\), \(\nabla\) on \(N_1 \times N_2\) satisfy

\[
\begin{align*}
(a) \quad & \nabla_X Y = \nabla^E E X F, \\
(b) \quad & \nabla_X U = \nabla^U U X = \frac{E(U)}{G}, \\
(c) \quad & \nabla_U V = \nabla^{N_2}_G H - \frac{F(U, V)}{G} \text{grad} \, f, \\
& \nabla^U V = \nabla^{N_2*}_G H - \frac{F(U, V)}{G} \text{grad} \, f.
\end{align*}
\]

Then \((\mathcal{M}, \nabla, \nabla)\) is a dualistic structure on \(N_1 \times N_2\).

We consider a statistical warped product manifold of type \(M = \mathbb{R} \times \mathcal{M}\) with metric \(\mathcal{g} = g_1 + f^2(z) g_2\), where \(\mathbb{R}\) is a trivial statistical manifold with metric \(g_1 = dz^2\) and \(\mathcal{M}\) is an almost Kenmotsu statistical manifold \([14]\) with metric \(g_2\) and dual affine connections \(\nabla^\mathcal{M}\) and \(\nabla^{\mathcal{M}*}\). The structure vector field on \(M\) is denoted by \(\xi = \partial z\). Any arbitrary vector field on \(M\) is defined by \(Z = \eta(Z) + E\), where \(E\) is any vector field on \(\mathcal{M}\) and \(dz = \eta\). Moreover, a new tensor field \(\phi\) of type \((1,1)\) on \(M\) can be defined by using tensor field \(J\), that is, \(\phi Z = \beta E\). R. Gorunus et al. named this type of statistical warped product as an almost Kenmotsu statistical manifold \([14]\). The statistical curvature tensor \(\mathcal{S}\) of \(M = \mathbb{R} \times \mathcal{M}(\tau)\), where \(\mathcal{M}(\tau)\) is a holomorphic statistical manifold of constant holomorphic sectional curvature \(\tau \in \mathbb{R}\), is given by \([14]\)

\[
\mathcal{S}(E, F, G, H) = \frac{1}{2} \left( \mathcal{R}(E, F, G, H) + \mathcal{R}^*(E, F, G, H) \right)
\]

\[
= \alpha \left( \mathcal{g}(E, H) \mathcal{g}(F, G) - \mathcal{g}(E, G) \mathcal{g}(F, H) \right)
+ \beta \left( \mathcal{g}(E, G) \mathcal{g}(F, \partial z) \mathcal{g}(H, \partial z) - \mathcal{g}(E, F) \mathcal{g}(G, \partial z) \mathcal{g}(H, \partial z) \right)
+ \gamma \left( \mathcal{g}(E, \phi F) \mathcal{g}(\phi G, H) - \mathcal{g}(F, \phi G) \mathcal{g}(\phi E, H) + 2 \mathcal{g}(E, \phi F) \mathcal{g}(\phi G, H) \right),
\]

(2.10)

where \(\partial z = \frac{\partial}{\partial z}\) denotes the unit tangent vector field on \(\mathbb{R}\) and \(\alpha\), \(\beta\) and \(\gamma\) are given by

\[
\alpha = \frac{\mathcal{\tau}}{4 f^2} - \frac{(\mathcal{f})^2}{f^2}, \quad \beta = \frac{\mathcal{\tau}}{4 f^2} - \frac{(\mathcal{f})^2}{f^2} + \frac{\mathcal{f}''}{\mathcal{f}}, \quad \gamma = \frac{\mathcal{\tau}}{4 f^2}.
\]

3. A Family of Optimal Casorati Inequalities

F. Casorati \([5]\) defined an extrinsic invariant, called Casorati curvature of a sub-manifold of a Riemannian manifold, as the normalized square length of the second fundamental form. This concept extended the study of the principal direction of a hypersurface of a Riemannian manifold. The geometrical aspects and the significance of the Casorati curvatures discussed by some eminent geometers \([8, 16, 38]\). It showed an increasing development in pure Riemannian geometry. Therefore, geometers were curious to obtain the optimal inequalities for the Casorati curvatures of a sub-manifold of different ambient spaces.
Let $N$ be any $m$–dimensional submanifold of a $(2n + 1)$–dimensional almost
Kenmotsu statistical manifold $M^{2n+1} = \mathbb{R} \times J\mathcal{T}^{2n}$. If \{\(e_1, \ldots, e_m\)\} and \{\(\xi_1, \ldots, \xi_p\)\} are respectively orthonormal basis of \(T\varphi N^m\) and \(T^\perp \varphi N^m\) for \(\varphi \in M\). Then the mean
curvature vectors \(\mathcal{H}\) and \(\mathcal{H}^*\) of \(N\) are
\[
\mathcal{H} = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i), \quad \text{and} \quad \mathcal{H}^* = \frac{1}{m} \sum_{i=1}^{m} h^*(e_i, e_i).
\]
Also, we set
\[
h_{ij}^k = g(h(e_i, e_j), e_k), \quad \text{and} \quad h_{ij}^{*k} = g(h^*(e_i, e_j), e_k),
\]
for \(i, j \in \{1, \ldots, m\}, \ k \in \{1, \ldots, p\}\).

The squared norm of second fundamental forms \(h\) and \(h^*\) are denoted by \(\mathcal{C}\) and \(\mathcal{C}^*\), respectively, called the Casorati curvatures of \(N\) in \(M\).

\[
(3.1) \quad \mathcal{C} = \frac{1}{m} ||h||^2, \quad \text{(resp. } \mathcal{C}^* = \frac{1}{m} ||h^*||^2),
\]
where
\[
||h||^2 = \sum_{k=1}^{p} \sum_{i,j=1}^{m} (h_{ij}^k)^2, \quad \text{(resp. } ||h^*||^2 = \sum_{k=1}^{p} \sum_{i,j=1}^{m} (h_{ij}^{*k})^2).
\]

If we consider a \(s\)–dimensional subspace \(W\) of \(TN\), \(s \geq 2\), and an orthonormal basis \{\(e_1, \ldots, e_s\)\} of \(W\). Then the scalar curvature of the \(s\)–plane section \(W\) is defined as
\[
\tau(W) = \sum_{1 \leq i < j \leq s} \xi(e_i, e_j, e_j, e_i),
\]
and the Casorati curvatures of the subspace \(W\) are as follows:
\[
\mathcal{C}(W) = \frac{1}{s} \sum_{k=1}^{p} \sum_{i,j=1}^{s} (h_{ij}^k)^2, \quad \text{(resp. } \mathcal{C}^*(W) = \frac{1}{s} \sum_{k=1}^{p} \sum_{i,j=1}^{s} (h_{ij}^{*k})^2).
\]

(1) The normalized Casorati curvatures \(\delta_\mathcal{C}(m - 1)\) and \(\delta^*_\mathcal{C}(m - 1)\) are defined as
\[
[\delta_\mathcal{C}(m - 1)]_\varphi = \frac{1}{2} \mathcal{C}_\varphi + (\frac{m + 1}{2m}) \inf \{\mathcal{C}(W)\} \mid W: \text{a hyperplane of } T\varphi N^m\},
\]
and \(\delta^*_\mathcal{C}(m - 1)]_\varphi = \frac{1}{2} \mathcal{C}^*_\varphi + (\frac{m + 1}{2m}) \inf \{\mathcal{C}^*(W)\} \mid W: \text{a hyperplane of } T\varphi N^m\}.

(2) The normalized Casorati curvatures \(\hat{\delta}_\mathcal{C}(m - 1)\) and \(\hat{\delta}^*_\mathcal{C}(m - 1)\) are defined as
\[
[\hat{\delta}_\mathcal{C}(m - 1)]_\varphi = 2\mathcal{C}_\varphi - (\frac{2m - 1}{2m}) \sup \{\mathcal{C}(W)\} \mid W: \text{a hyperplane of } T\varphi N^m\},
\]
and \(\hat{\delta}^*_\mathcal{C}(m - 1)]_\varphi = 2\mathcal{C}^*_\varphi - (\frac{2m - 1}{2m}) \sup \{\mathcal{C}^*(W)\} \mid W: \text{a hyperplane of } T\varphi N^m\}.

**Theorem 3.1.** Let \(\mathbb{R}(d\xi, \nabla^\phi)\) be a trivial statistical manifold and \((\overline{M}, \overline{\nabla}, \overline{\xi}, \overline{\xi})\) be a holomorphic statistical manifold of constant holomorphic sectional curvature \(\overline{\tau}\).
If \(N\) is an \(m\)–dimensional statistical submanifold of a statistical warped product
manifold of type $\mathbb{R} \times \overline{M}(\tau)$, then the normalized Casorati curvatures $\delta_c(m-1)$ and $\delta_{c}^*(m-1)$ satisfy

\[ \rho^{\nabla,\nabla^*} \leq 2\delta_c^0 + \frac{1}{m-1}\delta^0 + \left( \frac{\tau}{4\ell^2} - \frac{(f')^2}{\ell^2} \right) + \frac{3\sigma}{4m(m-1)^2}||\phi||^2 \]

\[ (3.2) \quad - \frac{2}{m} \left( \frac{\tau}{4\ell^2} - \frac{(f')^2}{\ell^2} + \frac{\ell''}{f} \right)||T||^2 - \frac{m}{2(m-1)} \left( ||\mathcal{G}||^2 + ||\mathcal{H}^*||^2 \right), \]

where $2\delta_c^0 = \mathcal{C} + \mathcal{C}^*$ and $2\delta_c^0(m-1) = \delta_c(m-1) + \delta_{c}^*(m-1)$.

**Proof.** The scalar curvature $\tau^{\nabla,\nabla^*}$ of $N$ is given by

\[ 2\tau^{\nabla,\nabla^*} = \sum_{1 \leq i < j \leq m} S(e_i, e_j, e_j, e_i) = \frac{1}{2} \sum_{1 \leq i < j \leq m} \left\{ R(e_i, e_j, e_j, e_i) + R^*(e_i, e_j, e_j, e_i) \right\} \]

\[ = \sum_{1 \leq i < j \leq m} R(e_i, e_j, e_j, e_i) \]

\[ = m(m-1) \left( \frac{\tau}{4\ell^2} - \frac{(f')^2}{\ell^2} \right) - 2(m-1) \left( \frac{\tau}{4\ell^2} - \frac{(f')^2}{\ell^2} + \frac{\ell''}{f} \right)||T||^2 + \frac{3\sigma}{4m}||\phi||^2 \]

\[ + \frac{1}{2} \sum_{i,j} \left\{ \mathcal{G}(h^*(e_i, e_i), h(e_j, e_j)) + \mathcal{H}(h(e_i, e_i), h^*(e_j, e_j)) \right\} \]

\[ (3.3) \quad - \frac{1}{2} \sum_{i,j} \mathcal{G}(h(e_i, e_i), h^*(e_i, e_j)), \]

where $T = \partial z - \sum_{k=1}^p \lambda_k \xi_k$ is a vector field tangent to $N$ and $\lambda_1, \ldots, \lambda_p$ are smooth functions over $N$. Now we define a polynomial $\mathcal{Q}$ in terms of the components of the second fundamental form $h^0$ (with respect to $\nabla^0$) of $N$.

\[ \mathcal{Q} = \frac{1}{2} m(m-1)\delta^0 + \frac{1}{2} (m-1)(m+1)\mathcal{C}^0(W) + \frac{m}{2} (\mathcal{C} + \mathcal{C}^*) + \frac{3\sigma}{4m}||\phi||^2 \]

\[ + \frac{1}{m} \left( ||\mathcal{G}||^2 + ||\mathcal{H}^*||^2 \right) - 2\tau^{\nabla,\nabla^*} \]

\[ (3.4) \quad - m^2 \left( ||\mathcal{G}||^2 + ||\mathcal{H}^*||^2 \right) - 2\tau^{\nabla,\nabla^*}, \]

where $W$ is a hyperplane of $T_v N$. Without loss of generality, we assume that $W$ is spanned by $\{e_1, \ldots, e_m\}$. Then, by (3.3) and (3.4), we derive

\[ \mathcal{Q} = \sum_{k=1}^p \left( \sum_{i,j=1}^{m} \frac{m+3}{2} (h_{ij}^{0k})^2 + \frac{m+1}{2} \sum_{i,j=1}^{m-1} (h_{ij}^{0k})^2 - 2 \sum_{i=1}^{m-1} (h_{ii}^{0k})^2 \right) \]

\[ = \sum_{k=1}^p \left( 2(m+2) \sum_{1 \leq i < j \leq m-1} (h_{ij}^{0k})^2 + (m+3) \sum_{i=1}^{m-1} (h_{ii}^{0k})^2 \right) \]

\[ + m \sum_{i=1}^{m-1} (h_{ii}^{0k})^2 - 4 \sum_{1 \leq i < j \leq m} (h_{ij}^{0k} h_{jj}^{0k}) + \frac{m-1}{2} (h_{mm}^{0k})^2 \]

\[ \geq \sum_{k=1}^{m} \sum_{i=1}^{m-1} m(h_{ii}^{0k})^2 + \frac{m-1}{2} (h_{mm}^{0k})^2 - 4 \sum_{1 \leq i < j \leq m} h_{ij}^{0k} h_{jj}^{0k}. \]
For any $k \in \{1, \ldots, p\}$, we define a quadratic form $P_k : \mathbb{R}^m \to \mathbb{R}$ by

$$
P_k(h_{11}^0, h_{22}^0, \ldots, h_{m-1,m-1}^0, h_{mm}^0) = \sum_{i=1}^{m-1} m(h_{ii}^0)^2 + \frac{m-1}{2}(h_{mm}^0)^2 - 4 \sum_{1 \leq i < j \leq m} h_{ii}^0 h_{jj}^0,
$$

(3.5)

Further, we consider the constrained extremum problem $\min P_k$ subject to

$$\overline{N} : \sum_{i=1}^{m} h_{ii}^0 = a^k,$$

where $a^k$ is a real constant. From (3.5), we find that the critical points $h^0_{ii} = (h_{11}^0, h_{22}^0, \ldots, h_{m-1,m-1}^0, h_{mm}^0)$ of $\overline{N}$ are the solutions of the following system of linear homogeneous equations.

$$
\begin{align*}
\frac{\partial P_k}{\partial h_{ii}^0} &= 2(m+2)(h_{ii}^0) - 4 \sum_{r=1}^{m} h_{rr}^0 = 0, \\
\frac{\partial P_k}{\partial h_{mm}^0} &= (m-1)h_{mm}^0 - 4 \sum_{r=1}^{m-1} h_{rr}^0 = 0,
\end{align*}
$$

(3.6)

for $i \in \{1, 2, \ldots, m-1\}$ and $k \in \{1, \ldots, p\}$. Hence, every solution $h^0_{ii}$ has

$$
h_{ii}^0 = \frac{1}{m+1} a^k, \quad h_{mm}^0 = \frac{4}{m+3} a^k,
$$

for $i \in \{1, 2, \ldots, m-1\}$ and $k \in \{1, \ldots, p\}$. Now, we fix $x \in \overline{N}$. The bilinear form $\Theta : T_x \overline{N} \times T_x \overline{N} \to \mathbb{R}$ has the following expression:

$$\Theta(E, F) = Hess_{P_k}(E, F) + < h'(E, F), (\text{grad } P_k)(x) >,$$

where $h'$ denotes the second fundamental form of $\overline{N}$ in $\mathbb{R}^m$ and $< \cdot, \cdot >$ denotes the standard inner product on $\mathbb{R}^m$. The Hessian matrix of $P_k$ is given by

$$Hess_{P_k} = \begin{pmatrix}
2(m+2) & -4 & \ldots & -4 & -4 \\
-4 & 2(m+2) & \ldots & -4 & -4 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-4 & -4 & \ldots & 2(m+2) & -4 \\
-4 & -4 & \ldots & -4 & (m-1)
\end{pmatrix}.$$
However, the point $h^{0c}$ is the only optimal solution, that is, the global minimum point of problem and reaches a minimum $Q(h^{0c}) = 0$ for the solution $h^{0c}$ of the system \[ \text{(3.6)}. \] It follows that $Q \geq 0$.

\[
2\tau \nabla^* \nabla^* \leq \frac{1}{2} m(m-1)\mathcal{E}^0 + \frac{1}{2} (m-1)(m+1)\mathcal{E}^0(W) + \frac{m}{2} (\mathcal{E} + \mathcal{E}^*) + \frac{3\rho}{4m}\|\phi\|^2
\]
\[
+ m(m-1)\left(\frac{\tau}{4f^2} - \left(\frac{\phi}{f}\right)^2\right) - 2(m-1)\left(\frac{\tau}{4f^2} - \left(\frac{\phi}{f}\right)^2 + \frac{\phi^\prime}{f}\right)\|T\|^2
\]
\[
- \frac{m^2}{2} \left(||\mathcal{E}||^2 + ||\mathcal{E}^*||^2\right).
\]

The normalized scalar curvature $\rho$ of $N$ is defined as

\[
(3.7) \hspace{1cm} \rho \nabla^* \nabla^* = \frac{2\tau \nabla^* \nabla^*}{m(m-1)}.
\]

Hence, we get

\[
\rho \nabla^* \nabla^* \leq \frac{1}{2} \mathcal{E}^0 + \frac{m+1}{2m} \mathcal{E}^0(W) + \frac{1}{2(m-1)} (\mathcal{E} + \mathcal{E}^*) + \frac{3\rho}{4m(m-1)f^2} ||\phi||^2
\]
\[
+ \left(\frac{\tau}{4f^2} - \left(\frac{\phi}{f}\right)^2\right) - 2 \left(\frac{\tau}{4f^2} - \left(\frac{\phi}{f}\right)^2 + \frac{\phi^\prime}{f}\right)\|T\|^2
\]
\[
- \frac{m}{2(m-1)} \left(||\mathcal{E}||^2 + ||\mathcal{E}^*||^2\right),
\]

for every tangent hyperplane $W$ of $N$. If we take the infimum over all tangent hyperplanes $W$. Therefore, our assertion \[ \text{(3.2)} \] follows. This completes the proof of Theorem \[ \text{(3.1)}. \]

One can prove that the normalized scalar curvature is bounded above by the normalized Casorati curvatures $\delta_{\mathcal{E}}(m-1)$ and $\delta_{\mathcal{E}}^*(m-1)$, that is,

**Corollary 3.2.** Let $(\mathbb{R}, dz, \nabla^\mathbb{R})$ be a trivial statistical manifold and $(\nabla, \mathcal{E}, \mathcal{E}, \nabla, \mathcal{J})$ be a holomorphic statistical manifold of constant holomorphic sectional curvature $\tau$. If $N$ is an $m$-dimensional statistical submanifold of a statistical warped product manifold of type $\mathbb{R} \times_f \nabla M(\tau)$, then the normalized Casorati curvatures $\delta_{\mathcal{E}}(m-1)$ and $\delta_{\mathcal{E}}^*(m-1)$ are given by

\[
(3.8) \hspace{1cm} \rho \nabla^* \nabla^* \leq 2\delta_{\mathcal{E}}^0 + \frac{1}{m-1} \mathcal{E}^0 + \left(\frac{\tau}{4f^2} - \left(\frac{\phi}{f}\right)^2\right) + \frac{3\rho}{4m(m-1)f^2} ||\phi||^2
\]
\[
- \frac{2m}{2(m-1)} \left(||\mathcal{E}||^2 + ||\mathcal{E}^*||^2\right).
\]

Proof. For this, consider the following polynomial

\[
Q = 2m(m-1)\mathcal{E}^0 + \frac{1}{2} (m-1)(m+1)\mathcal{E}^0(W) + \frac{m}{2} (\mathcal{E} + \mathcal{E}^*) + \frac{3\rho}{4f^2}\|\phi\|^2
\]
\[
+ m(m-1)\left(\frac{\tau}{4f^2} - \left(\frac{\phi}{f}\right)^2\right) - 2(m-1)\left(\frac{\tau}{4f^2} - \left(\frac{\phi}{f}\right)^2 + \frac{\phi^\prime}{f}\right)\|T\|^2
\]
\[
- \frac{m^2}{2} \left(||\mathcal{E}||^2 + ||\mathcal{E}^*||^2\right) - 2\tau \nabla^* \nabla^*,
\]
where \( W \) is a hyperplane of \( T_pN \). The rest proof is similar to the proof of Theorem 3.1.

In the sequel, we deduce some properties when equality case hold in Theorem 3.1 and Corollary 3.2. Therefore, we can announce the following result to analogous of Theorem 3.1 and Corollary 3.2.

**Theorem 3.3.** Let \((\mathbb{R}, dz, \nabla)\) be a trivial statistical manifold and \((M, g, \nabla, J)\) be a holomorphic statistical manifold of constant holomorphic sectional curvature \( c \). We suppose that \( N \) is an \( m \)-dimensional statistical submanifold of a statistical warped product manifold of type \( \mathbb{R} \times_1 M^m(\tau) \). Then the equality holds in the inequalities (3.2) and (3.8) if and only if the symmetric and bilinear imbedding tensors of \( N \) with respect to \( \nabla \) and \( \nabla^* \) satisfy 
\[
h = -h^*.
\]

**Proof.** The equality holds in the inequalities (3.2) and (3.8) if and only if the second fundamental form with respect to \( \nabla^0 \) is zero, that is, \( h^0 = 0 \). This further gives \( h = -h^* \). Partially, we say that \( h \) and \( h^* \) are linearly dependent. This completes the proof of Theorem 3.3.

If \( \phi(TN) = T^\perp N \), then \( N \) is called a Legendrian submanifold. In particular case, \( n = m \). As an application, we can directly give the following result based on Theorem 3.1.

**Corollary 3.4.** Let \((\mathbb{R}, dz, \nabla)\) be a trivial statistical manifold and \((M, g, \nabla, J)\) be a holomorphic statistical manifold of constant holomorphic sectional curvature \( c \). If \( N \) is an \( m \)-dimensional Legendrian submanifold of a statistical warped product manifold of type \( M^{2m+1} = \mathbb{R} \times_1 M^m(\tau) \), then the normalized Casorati curvatures \( \delta_c(m - 1) \) and \( \delta^*_c(m - 1) \) satisfy 
\[
r^{\nabla, \nabla^*} \leq 2\delta^0_c + \frac{1}{m - 1} \delta^0_c + \left( \frac{\tau}{4f^2} - \frac{(f')^2}{f^2} \right) - \frac{m}{2(m - 1)} \left( ||\mathcal{C}||^2 + ||\mathcal{C}^*||^2 \right),
\]
where \( 2\delta^0_c = \mathcal{C} + \mathcal{C}^* \) and \( 2\delta^0_c(m - 1) = \delta_c(m - 1) + \delta^*_c(m - 1) \). Moreover, the equality holds in the inequality if and only if the symmetric and bilinear imbedding tensors of \( N \) with respect to \( \nabla \) and \( \nabla^* \) satisfy \( h = -h^* \).

We note that the submanifolds for which the equality case of inequalities for the Casorati curvatures holds at every point are called *Casorati ideal submanifolds* [8]. Thus, we can state the following:

**Theorem 3.5.** Let \( N \) be an \( m \)-dimensional Legendrian Casorati ideal submanifold of a statistical warped product manifold of type \( M^{2m+1} = \mathbb{R} \times_1 M^m(\tau) \) for (3.10). Then it is a totally geodesic (with respect to Levi-Civita connection) Legendrian submanifold.

4. **Chen-Ricci Inequality for Statistical Submanifolds**

In the initial paper, Chen established inequalities between the scalar curvature, the sectional curvature and the squared norm of the mean curvature of a submanifold in a real space form. He also obtained the inequalities between \( k \)-Ricci curvature, the squared mean curvature and the shape operator for the submanifolds in the real space form with arbitrary codimension [5]. Since then different geometeters obtained similar inequalities for different submanifolds and ambient spaces. By
means of optimization techniques which applied in the setup of Riemannian geometry, T. Oprea [25] derived Chen-Ricci inequality for a submanifold of real space form. Recently, A.N. Siddiqui et al. [33] studied Chen-Ricci inequality for a submanifold of a Kenmotsu statistical manifold of constant \( \phi \)-sectional curvature by adopting optimization technique. Therefore, an important application of Theorem 3.1 is the following.

**Theorem 4.1.** Let \( (\mathbb{R}, dz, \nabla^\mathbb{R}) \) be a trivial statistical manifold and \( (\overline{M}, \overline{g}, \overline{\nabla}, \overline{J}) \) be a holomorphic statistical manifold of constant holomorphic sectional curvature \( c \).

If \( N \) is an \( m \)-dimensional statistical submanifold of a statistical warped product manifold of type \( \mathbb{R} \times_f \overline{M}(\tau) \), then the following inequality holds for each unit vector \( E \in T_\varphi N \), \( \varphi \in N \):

\[
\text{Ric}^N \nabla^* (E) \geq 2\text{Ric}^\varphi (E) - \frac{m}{8} [||\mathcal{H}||^2 + ||\mathcal{H}^*||^2] - \left[ \left( \frac{\tau}{4f^2} - \frac{(f')^2}{f^2} \right) (m - 1) \right.
\]
\[
+ \frac{3\pi}{4f^2} ||\phi E||^2 + \left( \frac{\tau}{4f^2} - \frac{(f')^2}{f^2} + \frac{f''}{f} \right) \left( (2 - m) g^2(E, T) - ||T||^2 \right) \],
\]

where \( \text{Ric}^\varphi \) denotes the Ricci curvature with respect to Levi-Civita connection. Moreover, the equality holds in the inequality (4.1) if and only if

\[
h(E, E) = \frac{m}{2} \mathcal{H}(\varphi) \text{, and } h(E, F) = 0,
\]

\[
h^*(E, E) = \frac{m}{2} \mathcal{H}^*(\varphi) \text{, and } h^*(E, F) = 0,
\]

for all \( F \in T_\varphi N^m \) orthogonal to \( E \).

**Proof.** Let \( N \) be an \( m \)-dimensional statistical submanifold of a statistical warped product manifold of type \( \mathbb{R} \times_f \overline{M}(\tau) \). We choose \( \{e_1, \ldots, e_m\} \) as the orthonormal frame of \( T_\varphi N \) such that \( e_1 = E \) and \( ||E|| = 1 \), and \( \{\tilde{e}_1, \ldots, \tilde{e}_p\} \) as the the orthonormal frame of \( T_\varphi N \). Then by (2.10) and (2.7), we have

\[
\sum_{i=2}^{m} s(e_1, e_i, e_1, e_i) = \left( \frac{\tau}{4f^2} - \frac{(f')^2}{f^2} \right) (m - 1) - \left( \frac{\tau}{4f^2} - \frac{(f')^2}{f^2} \frac{f''}{f} \right)
\]
\[
+ \left( \sum_{i=2}^{m} g(e_i, e_i) g^2(E, T) + \sum_{i=1}^{m} g^2(T, e_i) - g^2(E, T) \right) \left( \frac{3\pi}{4f^2} ||\phi E||^2 \right)
\]
\[
= \left( \frac{\tau}{4f^2} - \frac{(f')^2}{f^2} \right) (m - 1) + \frac{3\pi}{4f^2} ||\phi E||^2
\]
\[
+ \left( \frac{\tau}{4f^2} - \frac{(f')^2}{f^2} + \frac{f''}{f} \right) \left( (2 - m) g^2(E, T) - ||T||^2 \right) \].
\]
From (2.4), dual of (2.3) and (2.4), we get

\[ 2\mathbf{R}(e_1, e_i, e_1, e_i) = 2\mathbf{R}(e_1, e_i, e_1, e_i) - g(h(e_1, e_1), h^*(e_i, e_i)) - g(h^*(e_1, e_i), h(e_i, e_i)) \]
\[ + 2g(h(e_1, e_i), h^*(e_1, e_i)) \]
\[ = 2\mathbf{R}(e_1, e_i, e_1, e_i) - \left\{ 4g(h^0(e_1, e_1), h^0(e_i, e_i)) - g(h(e_1, e_1), h(e_i, e_i)) \right. \]
\[ - g(h^*(e_1, e_i), h^*(e_1, e_i)) - 4g(h^0(e_1, e_i), h^0(e_i, e_i)) \]
\[ + g(h(e_1, e_i), h(e_i, e_i)) + g(h^*(e_1, e_i), h^*(e_1, e_i)) \right\} \]
\[ = 2\mathbf{R}(e_1, e_i, e_1, e_i) - 4\sum_{k=1}^{p} (h_{11}^{0k} h_{ii}^{0k} - (h_{11}^{0k})^2) \]
\[ + \sum_{k=1}^{p} (h_{11}^{k} h_{ii}^{k} - (h_{11}^{k})^2) + \sum_{k=1}^{p} (h_{11}^{*k} h_{ii}^{*k} - (h_{11}^{*k})^2). \]

Summing over \(2 \leq i \leq m\) and using (1.2), we derive

\[ 2 \left[ \left( \frac{\gamma}{4f^2} - \frac{(f')^2}{f^2} \right)(m - 1) + \frac{3\pi}{4f^2} \|\phi E\|^2 \right] \]
\[ + \left( \frac{\gamma}{4f^2} - \frac{(f')^2}{f^2} + \frac{\gamma''}{f} \right)((2 - m)g^2(E, T) - ||T||^2) \]
\[ = 2\text{Ric}^\nabla, \nabla^\ast(E) - 4\sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11}^{0k} h_{ii}^{0k} - (h_{11}^{0k})^2) \]
\[ + \sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11}^{k} h_{ii}^{k} - (h_{11}^{k})^2) + \sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11}^{*k} h_{ii}^{*k} - (h_{11}^{*k})^2), \]

where \(\text{Ric}^\nabla, \nabla^\ast(E)\) denotes the Ricci curvature of \(N\) with respect to \(\nabla\) and \(\nabla^\ast\) at \(\phi\). Further, we derive

\[ 2\text{Ric}^\nabla, \nabla^\ast(E) - 2 \left[ \left( \frac{\gamma}{4f^2} - \frac{(f')^2}{f^2} \right)(m - 1) + \frac{3\pi}{4f^2} \|\phi E\|^2 \right] \]
\[ + \left( \frac{\gamma}{4f^2} - \frac{(f')^2}{f^2} + \frac{\gamma''}{f} \right)((2 - m)g^2(E, T) - ||T||^2) \]
\[ = 4\sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11}^{0k} h_{ii}^{0k} - (h_{11}^{0k})^2) - \sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11}^{k} h_{ii}^{k} - (h_{11}^{k})^2) \]
\[ + \sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11}^{*k} h_{ii}^{*k} - (h_{11}^{*k})^2). \]

(4.5)

By Gauss equation with respect to Levi-Civita connection, it follows that

\[ \text{Ric}^0(E) - \left[ \left( \frac{\gamma}{4f^2} - \frac{(f')^2}{f^2} \right)(m - 1) + \frac{3\pi}{4f^2} \|\phi E\|^2 \right] \]
\[ + \left( \frac{\gamma}{4f^2} - \frac{(f')^2}{f^2} + \frac{\gamma''}{f} \right)((2 - m)g^2(E, T) - ||T||^2) \]
\[ = \sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11}^{0k} h_{ii}^{0k} - (h_{11}^{0k})^2). \]
Now, we substitute it into (4.5), we arrive at

\[
(4.6) \quad -2 \text{Ric} \nabla^* (E) - \frac{1}{2} \left( \frac{\tau}{4f^2} - \frac{\left( \frac{\tau}{f} \right)^2}{4f^2} \right) \left( m - 1 \right) + \frac{3\tau}{4f^2} \| \phi E \|^2 \\
+ \left( \frac{\tau}{4f^2} - \frac{\left( \frac{\tau}{f} \right)^2}{4f^2} + \frac{\nu}{f} \right) \left( (2 - m) g^2 (E, T) - \| T \|^2 \right) \\
+ 4 \text{Ric}^0 (E) \\
= \sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11} h_{ii}^k - (h_{1i}^k)^2) + \sum_{k=1}^{p} \sum_{i=2}^{m} (h_{11}^k h_{ii}^k - (h_{1i}^k)^2) \\
\leq \sum_{k=1}^{p} \sum_{i=2}^{m} h_{11} h_{ii}^k + \sum_{k=1}^{p} \sum_{i=2}^{m} h_{11}^k h_{ii}^k. 
\]

Let us define the quadratic form \( \mathcal{P}_k, \mathcal{P}_k^* : \mathbb{R}^m \to \mathbb{R} \) by

\[
\mathcal{P}_k (h_{11}^k, h_{22}^k, \ldots, h_{mm}^k) = \sum_{k=1}^{p} \sum_{i=2}^{m} h_{11} h_{ii}^k, \\
\mathcal{P}_k^* (h_{11}^k, h_{22}^k, \ldots, h_{mm}^k) = \sum_{k=1}^{p} \sum_{i=2}^{m} h_{11}^k h_{ii}^k, 
\]

We consider the constrained extremum problem \( \max \mathcal{P}_k \) subject to

\[
\mathcal{N} : \sum_{i=1}^{m} h_{ii}^k = a^k, 
\]

where \( a^k \) is a real constant. The gradient vector field of the function \( \mathcal{P}_k \) is given by

\[
\text{grad} \mathcal{P}_k = \left( \sum_{i=2}^{m} h_{ii}^k, h_{11}^k, h_{11}^k, \ldots, h_{11}^k \right). 
\]

For an optimal solution \( h^{oc} = (h_{11}^k, h_{22}^k, \ldots, h_{mm}^k) \) of the problem in question, the vector \( \text{grad} \mathcal{P}_k \) is normal to \( \mathcal{N} \) at the point \( h^{oc} \). It follows that

\[
h_{11}^k = \frac{a^k}{2}. 
\]

Now, we fix \( x \in \mathcal{N} \). The bilinear form \( \Theta : T_x \mathcal{N} \times T_x \mathcal{N} \to \mathbb{R} \) has the following expression:

\[
\Theta (E, F) = \text{Hess}_{P_k} (E, F) + \langle h' (E, F), (\text{grad} P_k)(x) \rangle, 
\]

where \( h' \) denotes the second fundamental form of \( \mathcal{N} \) in \( \mathbb{R}^m \) and \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( \mathbb{R}^m \). The Hessian matrix of \( P_k \) is given by

\[
\text{Hess}_{P_k} = \begin{pmatrix}
0 & 1 & \ldots & 1 \\
1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
1 & 0 & \ldots & 0
\end{pmatrix}. 
\]
We consider a vector $E \in T_\mathcal{N}$, which satisfies a relation $\sum_{i=2}^{m} E_i = -E_1$. As $h' = 0$ in $\mathbb{R}^m$, we get

$$\Theta(E, E) = Hess_{\mathcal{P}_k}(E, E) = 2 \sum_{i=2}^{m} E_1 E_i = (E_1 + \sum_{i=2}^{m} E_i)^2 - (E_1)^2 - (\sum_{i=2}^{m} E_i)^2$$

$$= -2(E_1)^2 \leq 0.$$ 

However, the point $h^{\mathcal{N}}$ is the only optimal solution, that is, the global maximum point of problem. Thus, we obtain

$$\mathcal{P}_k \leq \frac{1}{4}(\sum_{i=1}^{m} h_{i}^{k})^2 = \frac{m^2}{4}(\mathcal{H}^{k})^2. \quad (4.8)$$

Next, we deal with the constrained extremum problem $\max \mathcal{P}_k^*$ subject to $\mathcal{N}^*$:

$$\sum_{i=1}^{m} h_{i}^{*k} = a^{*k},$$

where $a^{*k}$ is a real constant. By similar arguments as above, we find

$$\mathcal{P}_k^* \leq \frac{1}{4}(\sum_{i=1}^{m} h_{i}^{*k})^2 = \frac{m^2}{4}(\mathcal{H}^{*k})^2. \quad (4.9)$$

On combining $(4.6)$, $(4.8)$ and $(4.9)$, we get our desired inequality $(4.1)$. Moreover, the vector field $E$ satisfies the equality case if and only if

$$h_{i}^{k} = 0, \quad h_{1}^{*k} = 0, \quad i \in \{2, \ldots, m\},$$

and

$$h_{11}^{k} = \frac{m}{2} \mathcal{H}, \quad h_{11}^{*k} = \frac{m}{2} \mathcal{H}^*.$$

This completes the proof of Theorem 4.1. $\Box$

Some immediate consequences of Theorem 4.1

**Corollary 4.2.** Let $N$ be an $m$–dimensional anti-invariant submanifold of a statistical warped product manifold of type $\mathbb{R} \times \mathcal{M}(\sigma)$. Then the following inequality holds for each unit vector $E \in T_{\psi}N$, $\psi \in N$:

$$Ric_{\mathcal{N}}^{\mathcal{N}}(E) \geq 2Ric^0(E) - \frac{m^2}{8} (||\mathcal{H}||^2 + ||\mathcal{H}^*||^2) - \left[\left(\frac{\sigma}{4f^2} - \frac{(f')^2}{f^2}\right)(m - 1) + \left(\frac{\sigma^*}{4f^2} - \frac{(f^*)^2}{f^2} + \frac{f''}{f}\right)\right] (2 - m) g^2(E, T) - ||T||^2\right].$$

Furthermore, the equality holds in this inequality if and only if

$$h(E, E) = \frac{m}{2} \mathcal{H}(\psi), \quad h(E, F) = 0,$$

$$h^*(E, E) = \frac{m}{2} \mathcal{H}^*(\psi), \quad h^*(E, F) = 0,$$

for all $F \in T_{\psi}N^m$ orthogonal to $E$. 

Corollary 4.3. Let \( N \) be an \( m \)-dimensional invariant submanifold of a statistical warped product manifold of type \( \mathbb{R} \times M(\bar{\tau}) \). Then the following inequality holds for each unit vector \( E \in T_\varphi N, \varphi \in N \):

\[
Ric^N,\xi^*(E) \geq 2\text{Ric}^0(E) - \frac{m^2}{8} (\|\xi\|^2 + \|\xi^*\|^2) - \left( \left( \frac{\tau}{4f^2} - \frac{(\bar{\xi})^2}{f^2} \right)(m-1) + \frac{3\tau}{4f^2} \right)
\]

\[
+ \left( \frac{\tau}{4f^2} - \frac{(\bar{\xi})^2}{f^2} + \frac{\bar{\xi}'^2}{f} \right)(2 - m)g^2(E, T) - \|T\|^2.
\]

In addition, the equality holds in this inequality if and only if

\[
(4.12) \quad h(E, E) = \frac{m}{2} \xi(\varphi), \quad \text{and} \quad h(E, F) = 0,
\]

\[
(4.13) \quad h^*(E, E) = \frac{m}{2} \xi^*(\varphi), \quad \text{and} \quad h^*(E, F) = 0,
\]

for all \( F \in T_\varphi N^m \) orthogonal to \( E \).

5. Examples

We provide some non-trivial examples of statistical immersions into the statistical warped product manifolds, illustrating the main results stated above.

Example 1. Following [30], we give a non-trivial example satisfying Theorem 4.3.
We consider a statistical manifold \( (M^3, g_{\mathbb{H}^3}, D^{(-1)}, D^{(+1)}) \) of constant statistical sectional curvature \( \bar{\tau} = r^2 - 1, \ r \in \mathbb{R} \) and the translation surface

\[
\left( M^2, g_M \right) = \left( \frac{1}{y^2}((p^2 + 1)dx^2 + dy^2), D^{(-1)}, D^{(+1)} \right), \ p \in \mathbb{R}, \ (x, y) \in \mathbb{R}^2, \ y > 0
\]

which is a statistical submanifold of

\[
\left( \mathbb{H}^3 = \{(u, v, t) \in \mathbb{R}^3|t > 0\}, g_{\mathbb{H}^3} = \frac{1}{t^2}(du^2 + dv^2 + dt^2), D^{(-1)}, D^{(+1)} \right).
\]

Then it is easy to prove that under the following isometric immersion, the second fundamental forms \( h \) and \( h^* \) vanish

\[
f : \left( N = \mathbb{R} \times \cosh(z)M^2, dz^2 + \cosh^2(z)g_M \right) \rightarrow \left( \mathbb{R} \times \cosh(z)\mathbb{H}^3, g = dz^2 + \cosh^2(z)g_{\mathbb{H}^3} \right),
\]

\[
f(z, x, y) = (z, x, ax + by), \ a, b \in \mathbb{R}.
\]

This means that \( N \) is a totally geodesic submanifold of \( \mathbb{R} \times \cosh(z)\mathbb{H}^3 \) with respect to the induced Levi-Civita connection.

Example 2. In [26], some non-trivial examples are discussed for statistical warped product manifolds. For this reason, we use Example 4.9 of [26] to give a non-trivial example satisfying the equality of Theorems 4.1. First we consider the standard statistical warped product manifold

\[
M = \mathbb{R} \times \cosh(\lambda t)\mathbb{H}^{m+p-1}(\tau) = \{t, y^1, \ldots, y^{m+p-1} \in \mathbb{R}^{m+p}|y^{m+p-1} > 0\},
\]

where

\[
\bar{\tau} = \kappa(r^2 - 1), \ \lambda, r \in \mathbb{R}, \ \kappa > 0, \ m \geq 2, \ p \geq 1.
\]

For constants \( (a^1, \ldots, a^p) \in \mathbb{R}^p \), we have the submanifold \( N \) of \( M \) as

\[
N = \{(a^1, \ldots, a^p, x^1, \ldots, x^m) \in M(x^1, \ldots, x^m) \in \mathbb{R}^{m-1} \times \mathbb{R}^+\}.
\]
The second fundamental forms with respect to dual connections are given by
\[ h(e_i, e_j) = h^*(e_i, e_j) = -\delta_{ij} \lambda \tanh(\lambda a^1) \tilde{e}_1, \]

where
\[ \{ e_i = \sqrt{\kappa x^m \cosh^{-1}(\lambda a^1)} \frac{\partial}{\partial y^{i+p-1}} | i = 1, \ldots, m \} \]

and
\[ \{ \tilde{e}_1 = \frac{\partial}{\partial t}, \tilde{e}_k = \sqrt{\kappa x^m \cosh^{-1}(\lambda a^1)} \frac{\partial}{\partial y^{k-1}} | k = 2, \ldots, p \} \]

are the adopted orthonormal frames. Thus, \( N \) satisfies the equality condition of Theorem 4.1 at each point if and only if either \( m \geq 3 \) and \( \lambda a^1 = 0 \) or \( m = 2 \).

6. Conclusions and Remarks

As we know that the curvature invariants are widely used in the field of differential geometry and in physics also. The Ricci curvature is the essential term in the Einstein field equations, which plays a key role in general relativity. It is immensely studied in differential geometry which gives a way of measuring the degree to which the geometry determined by a given Riemannian metric might differ from the ordinary Euclidean \( n \)-space. Ricci curvature measures the amount by which the volume of a small geodesic ball deviates from the volume of a ball in Euclidean space, small geodesic balls will have no volume deviation, but their shape may vary from the shape of the standard ball in Euclidean space. Since it is essential to be able to control the extrinsic quantities relative to intrinsic ones. In modern Riemannian geometry, obtaining elementary relationships between extrinsic and intrinsic curvature invariants is a fundamental problem. For example, the lower bounds on the Ricci tensor on a Riemannian manifold enable one to find global geometric and topological information by comparison with the geometry of a constant curvature space form. In this paper, we have proved various optimal inequalities involving basic curvature invariants for statistical submanifold of a statistical warped product manifold of type \( \mathbb{R} \times_{f} \mathbb{M}^{7} \) and also discussed the equality case of the obtained inequalities. By carefully analyzing the nature of the terms in the main inequalities proved above, it is easy to deduce that the simplest intrinsic curvature invariant has an upper bound expressed in terms of some basic extrinsic curvature invariants. Finally, we have used examples from [26, 30] to show that the equality cases of the main inequalities can be attained. For further research, it would be interesting to obtain new or similar kind of inequalities for various statistical submersions (see [27, 29, 34]).

Acknowledgments: The second author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project No. (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. The third author, Ali Hussain Alkhalidi, extends his appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under grant number: R.G.P.2/429/44.

References

[1] Amari, S., Differential-Geometrical Methods in Statistics. Lecture Notes in Statistics, Springer-Verlag: New York, USA, 28 (1985).
[2] Aydin, M.E., Mihai, A., Mihai, I., Some Inequalities on Submanifolds in Statistical Manifolds of Constant Curvature. Filomat 29(3), 465-477 (2015).
[3] Bishop, R. L., O’Neill, B., Manifolds of negative curvature. Trans. Amer. Math. Soc. 145, 1–49 (1969).
[4] Bahadir, O., Tripathi, M. M., Geometry of lightlike hypersurfaces of a statistical manifold, arXiv:1901.09251 [math.DG] (2019).
[5] Casorati, F., Mesure de la courbure des surfaces suivant l’idée commune, Acta Math. 14, 95–110 (1890).
[6] Chen, B.-Y., Relationship between Ricci curvature and shape operator for submanifolds with arbitrary codimensions. Glasg. Math. J. 41, 33-41 (1999).
[7] Chen, B.-Y., Differential geometry of warped product manifolds and submanifolds, Worlds Scientific, Hackensack, New Jersey, (2017).
[8] Decu, S., Haesen, S., Verstraelen, L., Optimal inequalities involving Casorati curvatures. Bull. Transylv. Univ. Brasov, Ser B. 49 85-93 (2007).
[9] Decu, S., Haesen, S., Verstraelen, L., Vilcu, G.E., Curvature Invariants of Statistical Submanifolds in Kenmotsu Statistical Manifolds of Constant −φ Sectional Curvature. Entropy 20, 529; DOI:10.3390/e20070529 (2018).
[10] Furuhata, H., Hypersurfaces in statistical manifolds. Diff. Geom. Appl. 27, 420-429 (2009).
[11] Furuhata, H., Hasegawa, I., Submanifold theory in holomorphic statistical manifolds. In: Dragomir, S., Shahid, M.H., Al-Solamy, F.R. (eds.) Geometry of Cauchy-Riemann Submanifolds, pp. 179–215, Springer, Singapore (2010).
[12] Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K., Shahid, M. H., Sasakian statistical manifolds, J. Geom. Phys., 117, 179-186 (2017).
[13] Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K.: Kenmotsu statistical manifolds and warped product. J. Geom. DOI 10.1007/s00022-017-0403-1 (2017).
[14] Gorunus, R., Erken, I.K., Yazla, A., Murathan, C., Generalized Wintgen inequality for Legendrian submanifolds in almost Kenmotsu manifolds. Inter. Elec. J. Geom., 12(1), 43-56 (2019).
[15] Hulya, A., Cihan, O., Einstein statistical warped product manifolds. Filomat 32 (11), 3891-3897 (2018).
[16] Kowalczyk, D.: Casorati curvatures. Bull. Transilvania Univ. Brasov Ser. III. 50(1), 2009-2013 (2008).
[17] Lauritzen, S.: In statistical manifolds. In: Amari, S., Barndorff-Nielsen, O., Kass, R., Lauritzen, S., Rao, C.R. (eds.) Differential Geometry in Statistical Inference, 10, pp. 163–216. IMS Lecture Notes Institute of Mathematical Statistics, Hayward (1987).
[18] Lee, C. W., Yoon, D. W., Lee, J. W., A pinching theorem for statistical manifolds with Casorati curvatures. J. Non. Sci. Appl., 10, 4908–4914 (2017).
[19] Mihai, A., Mihai, I., Curvature Invariants for Statistical Submanifolds of Hessian Manifolds of Constant Hessian Curvature. Mathematics 44(6) (2018).
[20] Murathan, C., Sahin, B., A study of Wintgen like inequality for submanifolds in statistical warped product manifolds. J. Geom. 109(30) (2018).
[21] Nguiffo Boyom, M., Siddiqui, A.N., Mior Othman, W.A., Shahid, M.H., Classification of totally umbilical CR-statistical submanifolds in holomorphic statistical manifolds with constant holomorphic curvature. In: Nielsen F., Barbaresco F. (eds) Geometric Science of Information. GSI 2017. Lecture Notes in Computer Science, 10589. Springer, Cham
[22] Opozda, B., Bochner’s technique for statistical structures. Ann. Glob. Anal. Geom. 48(4), 357-395 (2015).
[23] Opozda, B., A sectional curvature for statistical structures. Linear Algebra Appl. 497, 134-161 (2016).
[24] Oprea, T., Chen’s inequality in the Lagrangian case. Colloq. Math. 108 163-169 (2007).
[25] Oprea, T., On a geometric inequality. arXiv:math/0511088v1 [math.DG] 3 Nov 2005.
[26] Satoh, N., Furuhata, H., Hasegawa, I., Nakane, T., Okuyama, Y., Sato, K., Shahid, M.H., Siddiqui, A.N., Statistical submanifolds from a viewpoint of the Euler inequality, Info. Geo. 4, 189-213 (2021).
[27] Siddiqui, A.N., Alkhaldi, A.H., Siddiqi, M.D., Ali, A., Lower Bounds on Statistical Submersions with Vertical Casorati Curvatures. Intern. J. Geom. Meth. Mod. Phys. 19(3) (2022).
[28] Siddiqui, A.N., Al-Solamy, F.R., Shahid, M.H., Mihai, I., On CR-Statistical Submanifolds of Holorpmic Statistical Manifolds. Filomat 35(11), 3571–3584 (2021).
[29] Siddiqui, A.N., Chen, B.-Y., Siddiqi, M.D., *Chen inequalities for statistical submersions between statistical manifolds*, Intern. J. Geom. Meth. Mod. Phys. **18**(40), pp. 17 (2021).
[30] Siddiqui, A.N., Murathan, C., Siddiqi, M.D., *The Chen’s first inequality for submanifolds of statistical warped product manifolds*, J. Geom. Phys. **169**, 1-13 (2021).
[31] Siddiqui, A.N., Shahid, M.H., *On totally real statistical submanifolds*, Filomat **32**(13), 4473-4483 (2018).
[32] Siddiqui, A.N., Shahid, M.H., *Optimizations on statistical hypersurfaces with Casorati curvatures*, Kragujevac J. Math. **45**(3), 449–463 (2021).
[33] Siddiqui, A.N., Suh, Y.J., Bahadir, O., *Extremities for statistical submanifolds in Kenmotsu statistical manifolds*, Filomat **35**(2), 591-603 (2021).
[34] Siddiqui, A.N., Uddin, S., Shahid, M.H., *B.-Y. Chen’s inequalities for Kähler-like statistical submersions*, Intern. Elec. J. Geom. **15**(2), 278-287 (2022).
[35] Takano, K., *Statistical manifolds with almost complex structures and its statistical submersions*, Tensor N.S. **65**, 128-142 (2004).
[36] Takano, K., *Statistical manifolds with almost contact structures and its statistical submersions*, J. Geom. **85**, 171-187 (2006).
[37] Todjihounde, L., *Dualistic structures on warped product manifolds*, Diff. Geom. Dyn. Syst. **8**, 278-284 (2006).
[38] Verstraelen, L., *Geometry of submanifolds I, The first Casorati curvature indicatrices*, Kragujevac J. Math. **37** 5-23 (2013).
[39] Vilcu, A.D., Vilcu, G.E., *Statistical manifolds with almost quaternionic structures and quaternionic Kaehler-like statistical submersions*, Entropy **17**, 6213-6228 (2015).
[40] Vos, P. W., *Fundamental equations for statistical submanifolds with applications to the Bartlett correction*, Ann. Inst. Stat. Math. **41**(3) 429-450 (1989).
[41] Yano, K., Kon, M., *Structures on manifolds*, World Scientific, Singapore (1984).
[42] Erken, I. K., Murathan, C., Yazla, A. *Almost cosymplectic statistical manifolds*, Quaestiones Mathematicae, **43**(2), 265-282 (2020).