Fast classification rates without standard margin assumptions

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Abstract

We consider the classical problem of learning rates for classes with finite VC dimension. It is well known that fast learning rates up to $O\left(\frac{d}{n}\right)$ are achievable by the empirical risk minimization algorithm (ERM) if one of the low noise/margin assumptions such as Tsybakov’s and Massart’s condition \cite{tsybakov2004optimal, massart2007concentration} is satisfied. In this paper, we consider an alternative way of obtaining fast learning rates in classification if none of these conditions are met.

We first consider Chow’s reject option model \cite{chow1965optimal} and show that by lowering the impact of a small fraction of hard instances, a learning rate of $O\left(\frac{d}{n}\right)$ is achievable in an agnostic model by a specific learning algorithm. Similar results were only known under special versions of margin assumptions \cite{schapire2002improved, bousquet2002stability}. We also show that the learning algorithm achieving these rates is adaptive to standard margin assumptions and always satisfies the risk bounds achieved by ERM.

Based on our results on Chow’s model, we then analyze a particular family of VC classes, namely classes with finite combinatorial diameter. Using their special structure, we show that there is an improper learning algorithm that provides fast (up to) $O\left(\frac{d}{n}\right)$ rates of convergence even in the (poorly understood) situations where ERM is suboptimal. This provides the first setup in which an improper learning algorithm may significantly improve the learning rates for non-convex losses.

Finally, we discuss some implications of our techniques to the analysis of ERM.

1 Introduction

The analysis of agnostic learning of VC classes (sometimes referred to as the agnostic PAC model) is the cornerstone of statistical learning theory originated in the seminal work of Vapnik and Chervonenkis \cite{vapnik2013nature}. Given a class $\mathcal{F}$ of binary functions of VC dimension $d$, based on the i.i.d. training sample $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$, the aim is to construct a classifier $\hat{f}$ such that its excess risk

$$\mathcal{E}(f) = \Pr(\hat{f}(X) \neq Y) - \inf_{f \in \mathcal{F}} \Pr(f(X) \neq Y)$$

is small with high probability, where the risk $R(f) = \Pr(f(X) \neq Y)$ is measured with respect to the same distribution as the training sample.

The canonical and entirely agnostic (no extra assumptions are needed) bound has the following form (see e.g., \cite{bousquet2002stability, koltchinskii2005local}). With probability at least $1 - \delta$,

$$\Pr(\hat{f}(X) \neq Y) - \inf_{f \in \mathcal{F}} \Pr(f(X) \neq Y) \lesssim \sqrt{\frac{d + \log \frac{1}{\delta}}{n}},$$

where $\hat{f}$ is an empirical risk minimizer defined as $\hat{f} = \arg\min_{f \in \mathcal{F}} \sum_{i=1}^{n} \mathbb{1}(f(X_i) \neq Y_i)$ and if there are several empirical risk minimizers we may choose any of them. Over the last two decades,
refinements of these results have relied on the introduction of different noise assumptions, which are sometimes referred to as margin assumptions, which may improve the convergence rates \( \sqrt{n} \) from slow \( \frac{1}{\sqrt{n}} \) convergence up to faster \( \frac{1}{n} \) learning rates. The most standard assumptions are Tsybakov’s \([34]\) and Massart’s \([27]\) noise models, as well as the more general Bernstein assumption \([3]\), which are by now well-studied (we refer to \([35]\) for related discussions). One of the problems of the classical low noise assumptions in \([34, 27]\) is that fast rates are obtained only under the additional assumption that the Bayes-optimal classifier is in the class. A more general Bernstein assumption is difficult to verify in the case of the binary classification and the indicator loss (see some related discussions in Section 1 in \([14]\)). Finally, these assumptions are related to the performance of ERM and are only known to be sufficient conditions for fast rates of classification. Moreover, standard assumptions do not take into account that at least sometimes improper learning algorithms can achieve better performance, which is the case, as will be shown below.

Our aim will be, on the one hand, to explore situations in which favorable learning rates can be obtained without requiring the standard low noise conditions, and on the other hand, to show that the algorithms which achieve these rates have the same performance as ERM in the case of more restrictive noise conditions.

One of the natural extensions of the agnostic learning framework consists in considering classification with a reject option. This framework has numerous practical motivations and many authors have studied its statistical properties (see e.g., \([14, 18, 12, 39, 4, 40]\) and references therein). One of the first reject option models appears in the seminal paper of Chow \([10]\) (see the discussion in Section 9 in \([12]\) and the analysis of this framework in \([18, 4]\)). Consider the \(\{0, 1, *\}\)-valued function \(f\) and define its risk as

\[
R_p(f) = \Pr (f(X) \neq Y \text{ and } f(X) \in \{0, 1\}) + \left(\frac{1}{2} - p\right) \Pr (f(X) = *),
\]

where \(p \in [0, \frac{1}{2}]\) is a fixed parameter. The value * corresponds to the abstention. The intuitive interpretation of this risk function is the following: the learner pays the fixed price \(\frac{1}{2} - p\) for all points where they decided to abstain from the prediction. This price is only slightly lower than the expected risk of the random guess if \(p\) is close to zero. In particular, it may be easily seen (see e.g., the discussions in \([18]\)) that the case \(p = 0\) corresponds to the standard agnostic learning. This is because in that case the price of predicting * is the same as the price of the random choice of 0 or 1. Another natural way to interpret this risk model is as follows: if on the test sample, the price of misclassification is too high, the learner should ask an expert to classify the point. The reason why the learner is not using the expert all the time, and the actual learning is happening is because the expert may predict only marginally better than a random coin-flip. Therefore, if we want \(R_p(\hat{f}) < \frac{1}{2} - p\), then only a small fraction of points can be assigned with *.

Interestingly, the values of \(p\) only marginally greater than zero are sufficient to eliminate possible slow learning rates.

Related reject models attracted attention in the statistical literature as well as in the so-called selective classification framework \([14, 33, 12]\). Although to the best of our knowledge, all the results presented up to date (except the recent paper \([14]\)) are restricted to the realizable case or some other low noise assumptions. We also note that in \([14]\) a model of selective classification for the so-called pointwise competitive classifiers is considered, with risk bounds not depending on the low noise assumptions. Instead, these bounds depend on the so-called disagreement coefficient. We discuss some relations with our results in Section 6.

One of the main technical contributions of this paper is the construction of the classifier with reject option, which has the following nice properties:

1. Our regions of abstention will have a very simple interpretation: the learner refuses to predict and outputs * if, based on the available information, there is nothing better than
a random coin flip to predict the label.

2. By replacing $*$ by a random coin-flip, we recover the well-known learning rates in the agnostic case as well as faster rates if the appropriate low noise assumption holds.

3. By replacing $*$ by a prediction with risk marginally better than $1/\ell$ on average, which corresponds to Chow’s reject option model with $p < 1/\ell$ (3), we prove that our algorithm provides fast learning rates for the excess risk (1) in a completely agnostic scenario.

4. The results in Section 6 will be based on the fact that for some classes, it is possible to use the training sample to specify the prediction on the points of abstention. This will provide new upper bounds in the setting where no abstentions are possible.

The technical part of the paper is inspired mostly by some recent advances in aggregation theory. Statistical aggregation theory was initiated by Nemirovskii [30]. Assume that we are given a finite dictionary $F$ of real-valued functions and denote $M = |F|$. The general problem in model selection aggregation [33] is to construct an estimator $\hat{f}$ such that

$$\mathbb{E}(\hat{f}(X) - Y)^2 - \inf_{f \in F} \mathbb{E}(f - Y)^2 \lesssim \frac{\log M}{n},$$

(4)

based on the training sample provided that $|Y| \leq 1$ and $|f(X)| \leq 1$ for all $f \in F$ and the expectation is also taken with respect to the training sample. Interestingly, since $F$ is finite, and thus not convex, no proper estimator (i.e., such that $\hat{f} \in F$), may achieve the desired bound $\mathbb{E}(\hat{f}(X) - Y)^2 \lesssim \frac{\log M}{n}$. At the same time, there are several ways to construct a class $G$, such that $F \subset G$ and an estimator $\hat{f} \in G$ such that the learning rate (4) is satisfied. In the last decade, several optimal algorithms were introduced, see [1, 2, 24, 25, 26, 28, 29] and references therein. Unfortunately, the results of aggregation theory are very specific to the squared loss and can be extended to strongly convex Lipschitz losses [2, 24], but in the binary loss case none of them are applicable: in the theory of classification general $\frac{1}{\sqrt{n}}$ lower bounds can be provided for any learning algorithm. Our strategy will thus be to take the tools from aggregation theory and provide a simple aggregation result for general strongly convex and Lipschitz losses and make a generalization from finite dictionaries of size $M$, which are usually considered, to infinite VC classes. Our analysis will exploit and further develop the following recent observation of Mendelson [24, 28]: in the model selection aggregation the class $G$ introduced above may be restricted only to the functions of the form $\frac{f + g}{2}$ for some $f, g \in F$, as long as an appropriate learning algorithm is used. Previously the progressive mixture rules [1], as well as the variants of the empirical star-algorithm considered in [1, 24, 27], used significantly more expressive classes of outputs $G$ in order to obtain results of the form (4). Our final trick will be to connect the (locally) strongly convex loss $\ell_q$, with $q \in (1, 2)$ of the $\{0, 1, \frac{1}{2}\}$-valued function to the Chow’s rejection model. For the sake of simplicity and clarity, since the focus of this paper is on classification, some of the results will not be provided in their most general forms. We note, however, that these generalizations are usually straightforward. We also point that similar in spirit results appeared in the work of Freund, Mansour, and Schapire [3]. However, their analysis has several restrictions, which are addressed in our paper. These discussions are also presented in Section 6

Apart from the relations between the strongly convex loss $\ell_q$ and the classification with reject option we show that similar techniques can be used for the analysis of some specific VC classes, namely, the classes with finite combinatorial diameter, first studied by Ben-David and Urner [2]. They noticed that in the case of deterministic labeling, learning rates of order $O(\frac{4}{n})$ are sometimes achievable for these classes. We show, in particular, that for the classes with finite combinatorial diameter, fast learning rates are possible in a wide range of noise assumptions, even if any proper learning algorithm fails to achieve a rate faster than $O(\frac{4}{\sqrt{n}})$. Our results imply the first general setup to studying the learning rates in classification in the situations where standard margin assumptions are not satisfied.

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We sum up our contributions and outline the structure of the paper:
1. In Section 2 we formulate our Theorem 2.2 on classification with reject option. The result is interesting in its own right, and it will be used as a technical tool in Section 3.
2. In Section 3.1 we present several standard tools in empirical process theory. Among the new observations is the fact that in the binary classification, the classical bounds under Massart’s and Tsybakov’s noise assumptions [27, 34] can be recovered using only the original bounds of Vapnik and Chervonenkis [37].
3. In Section 4 we apply Theorem 2.2 to the analysis of VC classes with finite combinatorial diameter. As a result, we develop the first general statistical framework that shows that an improper learning algorithm may significantly improve the learning rates for non-convex losses. We also discuss the distribution-dependent version of the combinatorial diameter and corresponding risk bounds.
4. In Section 5 we prove general ERM bounds that are adaptive to the standard margin conditions. Both results are natural corollaries of the proof of Theorem 2.2.
5. Finally, in Section 6 we prove a simple result that improves the logarithmic factor in the risk bound for VC classes with finite combinatorial diameter in the case of deterministic labeling.

2 Classification with a reject option

2.1 Preliminaries

Finally, we introduce some notation and basic definitions that will be used throughout the text. We define the instance space $\mathcal{X}$ and the label space $\mathcal{Y} = \{0, 1\}$. We assume that the set $\mathcal{X} \times \mathcal{Y}$ is equipped with some $\sigma$-algebra and a probability measure $P = P_{\mathcal{X},\mathcal{Y}}$ on measurable subsets defined. We also assume that we are given a set of classifiers $\mathcal{F}$; these are measurable functions with respect to the introduced $\sigma$-algebra, mapping $\mathcal{X}$ to $\mathcal{Y}$. The risk of a binary classifier $f$ is its probability of error, denoted $R(f) = P(f(X) \neq Y)$. Symbol $\mathbb{1}[A]$ will denote an indicator of the event $A$. The notation $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ will mean that for some universal constant $c > 0$ we have $f(x) \leq cg(x)$ for all $x$ in some domain, which will be clear from the context. Similarly, we introduce $f(x) \asymp g(x)$ to be equivalent to $g(x) \lesssim f(x) \lesssim g(x)$. For $a, b \in \mathbb{R}$ we set $a \land b = \min\{a, b\}$. We use the letters $c, c_1, c_2, \ldots$ to denote constants that can change from line to line.

A learner observes $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$, an i.i.d. training sample from an unknown distribution $P$. Also denote $Z_i = (X_i, Y_i)$ and $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$. By $P_n$ we will denote the expectation with respect to the empirical measure (empirical mean) induced by this sample. We say a set $\{x_1, \ldots, x_k\} \in \mathcal{X}^k$ is shattered by $\mathcal{F}$ if there are $2^k$ distinct classifications of $\{x_1, \ldots, x_k\}$ realized by classifiers in $\mathcal{F}$. The VC dimension of $\mathcal{F}$ is the largest integer $d$ such that there exists a set $\{x_1, \ldots, x_d\}$ shattered by $\mathcal{F}$ [30]. We define the growth function $\mathcal{S}_F(n)$ as the maximum possible number of different classifications of a set of $n$ points realized by classifiers in $\mathcal{F}$ (maximized over the choice of the $n$ points). A famous bound by Vapnik and Chervonenkis [37] implies that

$$\log \mathcal{S}_F(n) \lesssim d \log \frac{n}{d}.$$ 

We will use this result several times not referring to it. Analogously, we define the notion of the growth function for arbitrary classes of functions: in this case it is equal to the maximum possible number of different projections of the set of function on a set of $n$ points. For a real valued function $g$ and $p \geq 1$ we define the $L_p$ norm as $\|g\|_{L_p} = (\mathbb{E}[|g(Z)|^p])^{\frac{1}{p}}$. For a class $\mathcal{G}$ we define $D(\mathcal{G}, L_2) = \sup_{f, g \in \mathcal{G}} \|f - g\|_{L_2}$. In some of our discussions we use the standard $O, \Omega$ notation. We adopt the standard assumption that the events appearing in probability claims are measurable.

We also need the following definition.
Definition 2.1 (Bernstein Assumption for the binary loss). The class of binary functions $F$ together with the distribution $P_{X,Y}$ satisfy the $(\beta, B)$-Bernstein assumption if for all $f \in F$,
\[
\Pr(f(X) \neq f^*(X)) \leq B \left( R(f) - R(f^*) \right)^\beta,
\]
where $\beta \in [0, 1]$ and $B \geq 1$.

It is well known that Tsybakov’s and Massart’s noise assumptions are equivalent to the Bernstein assumption for some particular values of $\beta$ and $B$ (see e.g., [6]).

2.2 Excess risk bounds

Before we present one of our main results, we recall our notation. For a $\{0, 1, *\}$-valued function according to the notation (3) we have
\[
R_p(f) = \Pr \left( \hat{f}(X) \neq Y \text{ and } f(X) \in \{0, 1\} \right) + \left( \frac{1}{2} - p \right) \Pr(f(X) = *).
\]
The case of $p = 0$ and the corresponding value $R_0(f)$ may be interpreted as the situation where any abstention is replaced by a coin flip. This is a particular case of the binary classification (see the discussions in [18]). Observe also that $R_p$ is monotonic in $p$. This means, in particular, that
\[
R_p(f) - R(f^*) \leq R_0(f) - R(f^*).
\]

Theorem 2.2. Fix $p \in [0, \frac{1}{2})$ and consider a class $F$ of binary functions with VC dimension $d$. For any $\delta \in (0, 1)$, there is a learning algorithm with output denoted by $\hat{f}_p$, taking its values in $\{0, 1, *\}$ such that the following holds: with probability at least $1 - \delta$,
\[
R_p(\hat{f}_p) - R(f^*) \lesssim \frac{d \log \frac{d}{\delta} + \log \frac{1}{\delta}}{np}
\]
and
\[
R_0(\hat{f}_p) - R(f^*) \lesssim \sqrt{\frac{d \log \frac{d}{\delta} + \log \frac{1}{\delta}}{n}}.
\]
If the $(\beta, B)$-Bernstein assumption holds for $(F, P_{X,Y})$, then with probability at least $1 - \delta$,
\[
R_0(\hat{f}_p) - R(f^*) \lesssim \left( \frac{B \beta \log \frac{d}{\delta} + B \log \frac{1}{\delta}}{n} \right)^{\frac{1}{\beta}}.
\]
Moreover, in that case we have, with probability at least $1 - \delta$,
\[
\Pr(\hat{f}_p(X) = *) \lesssim \left( \frac{B^2 \beta \log \frac{d}{\delta} + B^2 \log \frac{1}{\delta}}{n} \right)^{\frac{1}{2-\beta}}.
\]

Our Theorem 2.2 shows, in particular, that for $p = 0$, our classifiers mimic the behavior of ERM but simultaneously take advantage of the parameter $p$ being separated from 0.

Before we proceed with the proof and the formal definition of the algorithm, we discuss some related results and important implications. The reader familiar with results in [27] may notice that the parameter $p$ in our bound plays a role similar to the role of the margin parameter $h$ in their work. In [27], the authors assume that $|2\mathbb{E}[Y|X] - 1| \geq h$ almost surely and that the Bayes optimal classifier defined by
\[
f_B^*(x) = 1 \left[ \mathbb{E}[Y|X = x] \geq \frac{1}{2} \right]
\]
belongs to the class $\mathcal{F}$. Under these assumptions, provided that $h \geq \sqrt{\frac{d}{n}}$, any ERM $\hat{f}$ satisfies, with probability at least $1 - \delta$,

$$R(\hat{f}) - R(f_B^*) \lesssim \frac{d \log \frac{n h^2}{d} + \log \frac{1}{\delta}}{n h}.$$  \hfill (9)

Moreover, matching lower bounds can be provided \cite{27, 42}. Finally, if the noise level is high, that is $h < \sqrt{\frac{d}{n}}$, no rates better than $\sqrt{\frac{d}{n}}$ can be obtained. In fact, Massart's noise assumption is limited to the restrictive assumption that the Bayesian optimal classifier $f_B^*$ belongs to $\mathcal{F}$. Kääriäinen \cite{21} and later Ben-David and Urner \cite{5} showed that the general lower bound of order $\frac{1}{\sqrt{n}}$ holds in some cases even if $Y = f_B^*(X)$ almost surely. This happens if there is no noise in the labelling mechanism but the problem is misspecified. This means that $f_B^* \notin \mathcal{F}$.

Therefore, we know at least two different reasons for slow classification rates:

- the labelling mechanism noise corresponding to the situations where $E[Y|X]$ is close to $\frac{1}{2}$,
- the misspecification; this means that $f_B^* \notin \mathcal{F}$.

Our Theorem 2.2 shows that whatever the source of the noise is, from the perspective of the learner, the only reason for slow rates is the impossibility to classify some hard instances better than by a random coin flip based on the information provided by the training sample.

We are ready to present our learning algorithm.

**The learning algorithm of Theorem 2.2**

- Fix $\alpha = \frac{d \log \frac{\sqrt{\delta}}{d} + \log(1/\delta)}{n}$ and let $c > 0$ be an appropriately chosen absolute constant.
- Split the sample of size $2n$ into two parts. Based on $\{(X_i, Y_i)\}_{i=1}^n$ run ERM with respect to the binary loss and denote its output by $\hat{g}$.
- Consider the class of almost empirical minimizers:
  \[ \hat{\mathcal{F}} = \left\{ f \in \mathcal{F} : R_n(f) - R_n(\hat{g}) \leq c \left( \alpha^2 + \alpha \sqrt{P_n[\hat{g} = f]} \right) \right\}. \hfill (10) \]
- Based on $\hat{\mathcal{F}}$ define the set $\hat{\mathcal{G}}$ as follows: this set consists of all $\{0, 1, *\}$-valued functions $g$ parametrized by all $f \in \hat{\mathcal{F}}$ and defined as
  \[ g(x) = \begin{cases} f(x), & \text{if } f(x) = \hat{g}(x), \\ * , & \text{if } f(x) \neq \hat{g}(x). \end{cases} \]
- Based on the second part of the sample for all $p \in [0, \frac{1}{2}]$ define the output of the algorithm as
  \[ \hat{f}_p = \arg \min_{f \in \hat{\mathcal{G}}} \sum_{i=n+1}^{2n} \left( 1[f(X_i) \neq Y_i \text{ and } f(X_i) \in \{0, 1\}] + \left( \frac{1}{2} - p \right) 1[f(X_i) = *] \right), \hfill (11) \]

and if $p \in (\frac{1}{4}, \frac{1}{2}]$, then set $\hat{f}_p = \hat{f}_{1/4}$.

**Remark 2.3.** Observe that the output of our learning algorithm depends on $\delta, n, d, p$. Although the dependence on $p$ is natural, the dependence on the remaining parameters can be potentially

\[1\] We note that the Bayesian optimal classifier is the best predictor of $Y$ given $X$.
removed. Indeed, our analysis is closely related to the analysis of the empirical-star algorithm of Audibert [1], which is completely parameter-free.

3 Proof of Theorem 2.2

3.1 Some results from empirical process theory

At first, we present several results related to the aggregation of classifiers under \( \ell_q \) loss for \( q \geq 1 \). In this section we assume sometimes that \( \mathcal{F} \) is a class of real valued functions taking their values in \([0,1]\). The loss function we consider is the \( \ell_q \) loss corresponding to the risk \( L_q^\ast(f) = \mathbb{E}|f(X) - Y|^q \). We maintain this notation in this section: the \( \ell_q \) risk will be denoted by \( L_q^\ast(f) \) and the binary risk will be defined by \( R(f) \). We also define the excess loss class by

\[
\mathcal{L}_q = \{(x,y) \rightarrow |f(x) - y| - |f^\ast(x) - y| : f \in F\},
\]

where \( f^\ast = \text{arg min}_f \mathbb{E}|f(X) - Y|^q \). Observe that as long as only \([0,1]\)-valued functions and \( f \in \mathcal{F} \) are considered, the \( \ell_q \) losses are equivalent to the binary loss.

We start with the following standard lemma that holds for arbitrary VC classes.

**Lemma 3.1.** Consider a class of binary valued functions \( \mathcal{F} \) of VC dimension \( d \) and assume that \( Y \in \{0,1\} \). Fix \( \delta \in (0,1) \) and \( \alpha = \sqrt{\frac{d \log \frac{2 + \log(1/\delta)}{\delta}}{n}} \). Then, simultaneously for all \( f, g \in \mathcal{F} \), with probability at least 1 - \( \delta \),

\[
|P_n|f - g| - P|f - g|| \lesssim \alpha \sqrt{P_n|f - g|} + \alpha^2,
\]

as well as

\[
|P_n|f - g| - P|f - g|| \lesssim \alpha \sqrt{P|f - g|} + \alpha^2,
\]

and simultaneously for all \( h \in \mathcal{L}_q \),

\[
|P_h - P_n h| \lesssim \alpha \sqrt{P_n|f - f^\ast|} + \alpha^2.
\]

Because of the general importance of Lemma 3.1 we give some new insights on its proof and applications in Appendix A. As already mentioned, we show there that it is possible to prove optimal risk bounds under Massart’s and Tsybakov’s noise conditions based only on the original ratio-type estimates of Vapnik and Chervonenkis because it is possible to relate the quantities of interest to results on binary-valued functions. The next result is also standard.

**Lemma 3.2.** Consider a class of real valued functions \( \mathcal{F} \) taking their values in \([0,1]\) and assume that \( Y \in [0,1] \) almost surely, \( q \in [1,2] \) and that \( \mathcal{F} \) has a finite growth function. Then for any \( \delta \in (0,1) \), with probability at least 1 - \( \delta \),

\[
\sup_{h \in \mathcal{L}_q} |P_h - P_n h| \lesssim D(\mathcal{F}, L_2) \sqrt{\frac{\log \mathcal{S}_\mathcal{F}(n) + \log(1/\delta)}{n}} + \frac{\log \mathcal{S}_\mathcal{F}(n) + \log(1/\delta)}{n}.
\]

**Proof.** The strategy of the proof is standard (see e.g., Lemma 3.2 in [24], where the analysis is presented in the case of finite classes). At first, we provide an upper bound on \( \mathbb{E} \sup_{h \in \mathcal{L}_q} |P_h - P_n h| \). Observe that since \( q \in [1,2] \) and due to our boundedness assumption we have for any \( f, g \in \mathcal{F} \),

\[
||f(X) - Y|^q - |g(X) - Y|^q| \leq 2|f(X) - g(X)|.
\]
Therefore, by the symmetrization argument and the standard bound for Bernoulli processes \[31\] together with Jensen’s inequality

\[
E \sup_{h \in \mathcal{L}_q} |P h - P_n h| \lesssim E \sup_{h \in \mathcal{L}_q} \sqrt{\log S_{\mathcal{F}}(n) P_n h^2/n} \lesssim \sqrt{\log S_{\mathcal{F}}(n)/n} \sup_{f \in \mathcal{F}} P_n (f - f^*)^2,
\]

where we used the fact that \( \mathcal{L}_q \) has the same growth function as \( \mathcal{F} \) together with \((16)\). Now, using the same lines

\[
E \sup_{f \in \mathcal{F}} P_n (f - f^*)^2 \leq E \sup_{f \in \mathcal{F}} |P_n (f - f^*)^2 - P(f - f^*)^2| + \sup_{f \in \mathcal{F}} E(f - f^*)^2
\]

\[
\lesssim \sqrt{\log S_{\mathcal{F}}(n)/n} + \sup_{f \in \mathcal{F}} E(f - f^*)^2
\]

Combining two bounds together we have

\[
E \sup_{h \in \mathcal{L}_q} |P h - P_n h| \lesssim D(\mathcal{F}, L_2) \sqrt{\log S_{\mathcal{F}}(n)/n} + \log S_{\mathcal{F}}(n)/n.
\]

To prove the high probability version of the bound we apply Talagrand’s concentration inequality for empirical processes (Theorem 12.2 in \[7\]) to the process \( \sup_{h \in \mathcal{L}_q} |P h - P_n h| \) and combine it with \((16)\). The claim follows.

### 3.2 Pairwise Aggregation

Finally, we provide the aggregation algorithm for the \( \ell_q \) loss. The algorithm of interest is the simplified version of the empirical star algorithm \[1\] and the two-step aggregation algorithm of \[24\] adapted for our purposes. Unfortunately, to the best of our knowledge, none of the known aggregation results can be directly applied in our case, since they are usually tuned to finite dictionaries, some of them are specific to the square loss or some special moment assumptions, and the class of output functions \( \mathcal{G} \) is almost always too large. Thus, we provide a simple self-contained analysis.

**Pairwise aggregation for VC classes**

- Let \( \alpha = \sqrt{d \log \frac{2}{\varepsilon} + \log(\frac{1}{\delta})}/n \) and \( c \) be a well-chosen absolute constant.
- Based on the sample \( \{(X_i, Y_i)_{i=1}^n\} \) run ERM with respect to the binary loss and denote its output by \( \hat{g} \).
- Based on \( \{(X_i, Y_i)_{i=1}^n\} \) consider the class of almost empirical minimizers
  \[
  \hat{\mathcal{F}} = \{ f \in \mathcal{F} : R_n(f) - R_n(\hat{g}) \leq c(\alpha^2 + \alpha \sqrt{P_n|\hat{g} - f|}) \}. \tag{17}
  \]
- Consider the class \( \hat{\mathcal{G}} = \{ f + \hat{g} : f \in \hat{\mathcal{F}} \} \).
- Output \( \hat{f} \) defined as \( \ell_q \)-loss ERM over the class \( \hat{\mathcal{G}} \) based on the second half of the sample \( \{(X_i, Y_i)_{i=n+1}^{2n}\} \).
Observe that if we consider the binary valued class $\mathcal{F}$, the output of the pairwise aggregation algorithm is a $\{0, 1, \frac{1}{2}\}$-valued function.

**Theorem 3.3.** Consider the $l_q$-loss for $q \in (1, 2]$, and assume that $Y \in \{0, 1\}$ and that $\mathcal{F}$ is a binary class with VC dimension $d$. Then, for any $\delta \in (0, 1)$, we have for the pairwise aggregation algorithm, with probability at least $1 - \delta$,

$$L_q' (\hat{f}) - L_q' (f^*) \lesssim \frac{d \log \frac{2}{\delta} + \log \frac{1}{\delta}}{n(q - 1)},$$

where $f^*$ minimizes the binary risk $R(f)$ in $\mathcal{F}$.

**Remark 3.4.** Interestingly, according to Theorem 8.4 in [2] the rates of (18) and the dependence on $q$ are minimax optimal in-expectation up to logarithmic factors provided that $q \in \left[1 + c \sqrt{\frac{d}{n}}, 2\right]$ for some $c > 0$.

**Remark 3.5.** We observe that the two-step algorithms of Theorem 2.2 and of Theorem 3.3 are based on splitting the sample into two independent parts. With more effort, it is possible to change both algorithms in a way such that the entire training sample is used twice in both steps of the algorithm. This will make our algorithm more similar to the empirical star algorithm of Audibert [1]. At the same time, the learning rates will not be changed by more than a constant factor. For the sake of brevity, we will not present this analysis.

Finally, we need the following simple result.

**Lemma 3.6.** Under the assumptions introduced above, provided that $\alpha = \sqrt{\frac{d \log \frac{2}{\delta} + \log \frac{1}{\delta}}{n(q - 1)}}$, we have, with probability at least $1 - \delta$, that for any $f \in \hat{\mathcal{F}}$,

$$R(f) - R(f^*) \lesssim \alpha D(\hat{\mathcal{F}}, L_2) + \alpha^2.$$

**Proof.** It is straightforward to see that (14) implies that, with probability at least $1 - \delta$, simultaneously for any $f, g \in \mathcal{F}$,

$$P_n |f - g| \lesssim P|f - g| + \alpha^2.$$  \hfill (19)

For any $f \in \hat{\mathcal{F}}$ and $q = 1$, with probability at least $1 - \delta$,

$$R(f) - R(f^*) \leq \sup_{h \in \mathcal{L}_q} |P h - P_n h| + R_n(f) - R_n(f^*)$$

$$\leq \sup_{h \in \mathcal{L}_q} |P h - P_n h| + R_n(f) - R_n(\hat{g}) \lesssim \alpha D(\hat{\mathcal{F}}, L_2) + \alpha^2,$$

where in the last line we combined Lemma 3.2, (17) and (19).

**Remark 3.7.** Proposition 5.1 below will improve upon Lemma 3.6. However, this weaker version is sufficient for the proof of Theorem 3.3.

**Proof of Theorem 3.3**

To analyze the properties of the $\ell_q$ loss we need some basic properties of the function $|x|^q$ for $q \in (1, 2]$ and $|x| \leq 1$. By Taylor’s formula, for $x, y \neq 0$ we have

$$|y|^q - |x|^q = q |x|^{q-1} (y - x) + \frac{q(q - 1)}{2} |x|^{q-2} (y - x)^2,$$
where $\xi$ is some midpoint. Since $q \leq 2$, we have $|\xi|^{q-2} \geq 1$, therefore

$$|y|^q - |x|^q \geq q|x|^{q-1}(y-x) + \frac{q(q-1)}{2}(y-x)^2,$$

The parameter $q(q-1)$ plays a role of the modulus of strong convexity in this case. In particular, we have in our range (see e.g. [8]) that for any $t \in [0,1]$,

$$|tx + (1-t)y|^q \leq t|x|^q + (1-t)|y|^q - \frac{1}{2}q(q-1)t(1-t)(x-y)^2.$$  \hfill (20)

We note that the logarithm of the growth function of $\hat{\mathcal{G}}$ is controlled up to an absolute constant by the logarithm of the growth function of $\mathcal{F}$. Moreover, the $L_2$-diameter of the random set $\hat{\mathcal{G}}$ is the same as the diameter of $\mathcal{F}$. Assume that $g^*$ minimizes $L'_q(g)$ over $g \in \hat{\mathcal{G}}$. Since we work with the independent second part of the sample, we may assume that $g^*$ is fixed. Then for $g \in \hat{\mathcal{G}}$, defined by $g = (\hat{g} + h)/2$, where $h \in \mathcal{F}$ is a function with $\|\hat{g} - h\|_{L_2} \geq D(\mathcal{F}, L_2)/2$ (it exists by the definition), we have

$$L'_q(g) \leq L'_q(g) \leq \frac{1}{2}P|Y - \hat{g}|^q + \frac{1}{2}P|Y - h|^q - \frac{1}{32}(q-1)P|\hat{g} - h|^2$$

$$= \frac{1}{2}P|Y - \hat{g}| + \frac{1}{2}P|Y - h| - \frac{1}{32}(q-1)P|\hat{g} - h|$$

$$\leq P|Y - f^*|^q + 2c(D(\hat{\mathcal{F}}, L_2)\alpha + \alpha^2) - \frac{1}{128}(q-1)D(\hat{\mathcal{F}}, L_2)^2,$$

where we used (20), Lemma 3.6 and that $\hat{g} \in \mathcal{F}$. Finally, by Lemma 3.2 and the fact that $\hat{f}$ minimizes the empirical risk we have for some $c_1 > 0$,

$$L'_q(\hat{f}) - L'_q(g^*) \leq c_1 \left( D(\hat{\mathcal{F}}, L_2)\alpha + \alpha^2 \right).$$

This implies for some $c_2 > 0$,

$$L'_q(\hat{f}) - L'_q(f^*) \leq c_2 \left( D(\hat{\mathcal{F}}, L_2)\alpha + \alpha^2 \right) = \frac{1}{2}P|Y - \hat{g}| + \frac{1}{2}P|Y - h| - \frac{1}{32}(q-1)P|\hat{g} - h|$$

$$\leq P|Y - f^*|^q + 2c(D(\hat{\mathcal{F}}, L_2)\alpha + \alpha^2) - \frac{1}{128}(q-1)D(\hat{\mathcal{F}}, L_2)^2 \lesssim \frac{d \log \frac{n}{d} + \log \frac{1}{\delta}}{(q-1)n},$$

where we used that $D(\hat{\mathcal{F}}, L_2) \geq 0$.

### 3.3 From aggregation to Chow’s reject option model

In this section, we finish the first part of the proof of Theorem 2.2. We need some elementary inequalities. Observe that for $f$, such that $f(X) \in \{0,1,\frac{1}{2}\}$ and $Y \in \{0,1\}$, we have

$$L'_q(f) = \Pr(f(X) \neq Y \text{ and } f(X) \in \{0,1\}) + \frac{1}{2} \Pr \left( f(X) = \frac{1}{2} \right).$$

**Case** $p \in [0, \frac{1}{4}]$. We define $\frac{1}{2} - p = \frac{1}{2}p$. Since we consider $q \in [1,2]$, our analysis will cover only the case $p \in [0, \frac{1}{4}]$. An elementary inequality for $\frac{1}{2} = \frac{1}{2} \exp(-(q-1) \log 2)$ implies,

$$\frac{1}{2} - \frac{1}{2}(q-1) \log 2 \leq \frac{1}{2p} \leq \frac{1}{2} - \frac{1}{4}(q-1) \log 2,$$

provided that $q \in [1,2]$. This implies $\frac{1}{q-1} \leq \frac{\log 2}{2p}$. Now using the result of Theorem 3.3 we have, with probability at least $1 - \delta$,

$$R_p(\hat{f}_p) - R(f^*) \lesssim \frac{d \log \frac{n}{d} + \log \frac{1}{\delta}}{np}.$$
where \( \hat{f}_p \) is defined as \( \hat{f} \) in Theorem 3.3 by replacing the outputs \( \frac{1}{2} \) with *. Indeed, in our case \( L'_q(\hat{f}) = R_p(\hat{f}_p) \) and \( L'_q(f^*) = R(f^*) \).

Case \( p \in (\frac{1}{2}, \frac{3}{4}] \). Observe that the bound of Theorem 3.2 changes by at most the factor of two if \( p \in [\frac{1}{2}, \frac{3}{4}] \). Therefore, to prove the result, we may redefine \( \hat{f}_p \) in a way such that if \( p \in (\frac{1}{2}, \frac{3}{4}] \), the output function is \( \hat{f}_{1/4} \) defined by (14).

Proof of (5)

We prove both (5) and (6) for an arbitrary function in the class \( \hat{G} \). Therefore, it does not matter which function in \( \hat{G} \) will be chosen by \( \hat{f}_p \). At first, by the boundedness assumption we have

\[
\hat{F} \subseteq \{ f \in F : R_n(f) - R_n(\hat{g}) \leq 2c \alpha \},
\]

provided that \( \alpha \leq 1 \). From the standard uniform convergence result [32] we have, with probability at least 1 − \( \delta \),

\[
R(f) - R(\hat{g}) - R_n(f) + R_n(\hat{g}) \lesssim \sqrt{\frac{d \log \frac{\alpha}{\delta} + \log \frac{1}{\delta}}{n}}.
\]

This implies that for any \( f \in \hat{F} \), with probability at least 1 − \( \delta \),

\[
R(f) - R(\hat{g}) \lesssim \sqrt{\frac{d \log \frac{\alpha}{\delta} + \log \frac{1}{\delta}}{n}} \quad \text{and} \quad R(\hat{g}) - R(f^*) \lesssim \sqrt{\frac{d \log \frac{\alpha}{\delta} + \log \frac{1}{\delta}}{n}}.
\]

Observe that for any \( f \in \hat{F} \),

\[
R(f) + R(\hat{g}) = \mathbb{E}(\mathbb{I}[f \neq Y] + \mathbb{I}[\hat{g} \neq Y]) = 2R_0(f'),
\]

where

\[
f'(x) = \begin{cases} f(x), & \text{if } f(x) = \hat{g}(x), \\ * , & \text{if } f(x) \neq \hat{g}(x). \end{cases}
\]

At the same time, using (21) we have, with probability at least 1 − \( \delta \),

\[
R_0(f') - R(f^*) = \frac{1}{2}(R(f) - R(\hat{g})) + R(\hat{g}) - R(f^*) \lesssim \sqrt{\frac{d \log \frac{\alpha}{\delta} + \log \frac{1}{\delta}}{n}}.
\]

Since the construction of \( \hat{G} \) does not depend on \( p \) and \( f' \) is an arbitrary function in \( \hat{G} \), we prove the claim.

Proof of (6)

At first, we prove that for some absolute \( c' > 0 \), with probability at least 1 − \( \delta \), for any \( f \in \hat{F} \),

\[
R(f) - R(f^*) \lesssim \left( \frac{B \log \frac{\alpha}{\delta} + B \log \frac{1}{\delta}}{n} \right)^{1/m}.
\]

We first recall that \( \alpha = \sqrt{\frac{d \log \frac{\alpha}{\delta} + \log \frac{1}{\delta}}{n}} \). By (15) and (13) and the definition of \( \hat{F} \) for \( f \in \hat{F} \) with probability at least 1 − \( \delta \),

\[
R(f) - R(f^*) \lesssim R_n(f) - R_n(\hat{g}) + R_n(\hat{g}) - R_n(f^*) + \alpha^2 + \alpha \sqrt{P|f - f^*|} \lesssim R_n(f) - R_n(\hat{g}) + \alpha^2 + \alpha \sqrt{P|f - f^*|} \lesssim \alpha^2 + \alpha \sqrt{P|\hat{g} - f|} + \alpha \sqrt{P|f - f^*|} \lesssim \alpha^2 + \alpha B^{\frac{1}{m}}(R(\hat{g}) - R(f^*))^{\frac{m}{2}} + \alpha B^{\frac{1}{m}}(R(f) - R(f^*))^{\frac{m}{2}}.
\]
Since $\hat{g}$ is ERM it follows from [27] (see also the proof in Appendix A or equation 22 in [15]) that, with probability at least $1 - \delta$,

$$R(\hat{g}) - R(f^*) \lesssim \left( \frac{Bd \log \frac{n}{d} + B \log \frac{1}{\delta}}{n} \right)^{1/\beta}.$$  

This implies,

$$R(f) - R(f^*) \lesssim \alpha^2 + \alpha B^{1/2} \left( \frac{Bd \log \frac{n}{d} + B \log \frac{1}{\delta}}{n} \right)^{\beta} \Bigg( R(\hat{g}) - R(f^*) \Bigg)^{1/2} \lesssim \alpha^2 + \alpha B^{1/2} \left( \frac{Bd \log \frac{n}{d} + B \log \frac{1}{\delta}}{n} \right)^{1/\beta}.$$  

Solving this inequality with respect to $R(f) - R(f^*)$ we prove [23]. Thus, for any $f \in \hat{F}$,

$$\frac{1}{2}(R(f) + R(\hat{g}) - 2R(f^*)) = R_0(f') - R(f^*),$$

where $f'$ is defined by [22]. At the same time,

$$R_0(f') - R(f^*) = \frac{1}{2}(R(f) + R(\hat{g}) - 2R(f^*)) \lesssim \left( \frac{Bd \log \frac{n}{d} + B \log \frac{1}{\delta}}{n} \right)^{1/\beta}.$$  

Finally, we have

$$\Pr(f'(X) = *) = \Pr(f(X) \neq \hat{g}(X)) \leq \Pr(f(X) \neq f^*(X)) + \Pr(\hat{g}(X) \neq f^*(X))$$

$$\leq B(R(f) - R(f^*))^\beta + B(\hat{g} - R(f^*))^\beta \lesssim B \left( \frac{Bd \log \frac{n}{d} + B \log \frac{1}{\delta}}{n} \right)^{\beta/10}.$$  

The claim follows.

4 Learning VC classes with finite diameter

In this section, we demonstrate that fast learning rates are possible for some VC classes in the standard statistical learning framework, even if none of the known margin assumptions hold. To the best of our knowledge, this phenomenon went almost unnoticed in the literature. A rare exception is the case of deterministic labeling studied by Ben David and Urner [5]. Our general results will be obtained by a modification of the algorithm of our Theorem 2.2.

4.1 Massart’s noise and Misspecification

**Definition 4.1** (Massart’s margin parameter). Given a distribution $P$ of the pair $(X, Y)$ (with $Y \in \{0, 1\}$), we define the margin parameter $h$ as the smallest $h \geq 0$ such that almost surely

$$|2\mathbb{E}[Y|X] - 1| \geq h.$$  

Notice that $h = 1$ corresponds to the case where $Y$ is a deterministic function of $X$ (no noise) while $h = 0$ corresponds to the case where $\mathbb{E}[Y|X]$ can be arbitrarily close to 1/2.

Let us recall some discussions of Section 2. The parameter $h$ in the definition [4.1] was introduced by Massart and Nédélec [27] who proven that whenever $h > 0$, i.e. the noise is not
arbitrarily large, and additionally \( f_B^* \in \mathcal{F} \), then the ERM achieves fast rates of convergence of the form (9).

However, if the learning problem of interest is misspecified, i.e., if \( f_B^* \notin \mathcal{F} \), even with favorable noise conditions (i.e., \( h > 0 \)), such fast rates cannot be achieved by a proper learning algorithm. Indeed, Theorem 13 in [5] shows that for the class of at most \( d \) ones there is a joint distribution of \( X \) and \( Y \), satisfying \( h = 1 \) (deterministic labelling) such that any proper learning algorithm will have a learning rate lower bounded by \( \Omega \left( \sqrt{\frac{2}{n}} \right) \).

Similarly, if the condition \( h > 0 \) is not satisfied, fast learning rates cannot also be achieved. Indeed, if there is \( x \in \mathcal{X} \) such that \( |2\mathbb{E}[Y|X = x] - 1| \leq \sqrt{\frac{2}{n}} \) and there are \( f, g \in \mathcal{F} \) such that \( f(x) \neq g(x) \), then the lower bound \( \Omega \left( \sqrt{\frac{1}{n}} \right) \) will immediately follow from the lower bound in [27]. Moreover, the same is true even in the case of the learning problem is not misspecified (i.e. if \( f_B^* \in \mathcal{F} \)).

Remark 4.2. It is well known that Massart’s noise condition implies the Bernstein assumption with \( B = \frac{1}{h} \) and \( \beta = 1 \) (see e.g., Section 5.2 in [6]), which is a sufficient condition for fast ERM rates. The proof of this fact is based on the strong assumption that \( f_B^* \in \mathcal{F} \) and this is consistent with lower bound \( \Omega \left( \sqrt{\frac{2}{n}} \right) \) that holds for any proper learning algorithm if \( f_B^* \notin \mathcal{F} \).

In the remainder of this section, we will further investigate the possibility of getting fast learning rates with improper learning algorithms in the case of misspecified learning problems under non-trivial margin (\( h > 0 \)) conditions.

4.2 Combinatorial Diameter

Definition 4.3 ([5]). The combinatorial diameter of \( \mathcal{F} \) is defined as

\[
D = \max_{f,g \in \mathcal{F}} |\{x \in \mathcal{X} : f(x) \neq g(x)\}|.
\]

It is easy to see that the VC dimension is upper bounded by the diameter (i.e., \( d \leq D \)). Although \( D \) is infinite for many natural classes, it is still an important measure of complexity. Indeed, almost all existing lower bounds in statistical learning are obtained by classes with a finite diameter [27]. Therefore, the classes with finite diameter are expressive enough to show the tightness of the known upper bounds in different scenarios. One particular example of such classes is the class of binary functions having at most \( \mathcal{D} \) ones. This class satisfies \( D = d \).

Let us first introduce the only known upper bound for the classes with finite combinatorial diameter. Theorem 9 in [5] immediately implies that in the case of \( Y = f_B^*(X) \) (deterministic labeling), but \( f_B^*(X) \) is not necessary in the class \( \mathcal{F} \), the following holds: there is a learning algorithm with output denoted by \( \hat{f} \) such that, with probability at least \( 1 - \delta \),

\[
R(\hat{f}) - R(f^*) \lesssim \frac{D \log n + \log \frac{1}{\delta}}{n},
\]

where \( D \) is the combinatorial diameter of the class \( \mathcal{F} \). The problem is that the analysis of Ben-David and Urner is very specific to the assumption that there is no noise in the labeling mechanism. Their algorithm actually memorizes the training sample and uses the predictions of a fixed function in \( \mathcal{F} \) on all the points that are not in the training sample. Nevertheless, their result is the first indication that it is possible to get fast classification rates for misspecified learning problems.
Additionally, as shown in [21, 5], the fact that the class has finite diameter is necessary. Indeed, even if $h = 1$, but $D = \infty$ the lower bound $\Omega \left( \sqrt{\frac{1}{n}} \right)$ holds for an arbitrary learning algorithm for some particular distributions.

We now show that Theorem 2.2 can be used to analyze the classes with finite combinatorial diameter in full generality. Thus, we obtain an almost complete characterization of the phenomenon that for some VC classes the improper learning algorithms may significantly outperform the naive ERM strategy.

**Theorem 4.4.** Consider a VC class $\mathcal{F}$ with finite diameter $D$. If Massart’s margin parameter $h$ is greater than zero, then there is a learning algorithm with output denoted by $\hat{f}$ such that, with probability at least $1 - \delta$,

$$R(\hat{f}) - R(f^*) \lesssim \frac{d \log \frac{n}{d} + D + \log \frac{1}{\delta}}{nh}.$$ 

We provide the algorithm achieving these fast learning rates. For the sake of simplicity, we assume that the noise parameter $h$ is known to the learner. We note that the analysis may be generalized to the case where the precise value of $h$ is not known. In particular, the analysis holds whenever only a reasonable lower bound on $h$ is available. We need the following definition: given a sample $S = \{(X_i, Y_i)_{i=1}^n\}$ define the majority vote function $\text{maj}_S$ as follows

$$\text{maj}_S(x) = \mathbb{1} \left[ \sum_{i=1}^n \mathbb{1}[Y_i = 1 \text{ and } X_i = x] > \sum_{i=1}^n \mathbb{1}[Y_i = 0 \text{ and } X_i = x] \right].$$

Given the sample $S$ this function counts the number of ones corresponding to $x$ and the number of zeros corresponding to $x$ and outputs the value according to the majority. If there is a tie, which may happen, in particular, if the instance $x$ does not belong to $X_1, \ldots, X_n$, then $\text{maj}_S$ outputs zero.

One well-known observation is that $|2\mathbb{E}[Y|X] - 1| \geq h$ implies

$$\Pr(f_B(X) \neq Y|X) \leq \frac{1}{2}(1 - h). \quad (25)$$

Let us highlight some informal intuition behind the algorithm of Theorem 4.4. Theorem 2.2 suggests that the reason for potential slow learning rates is a pair of almost ERMs such that their disagreement set is relatively large. However, under the assumption that $D < \infty$ we can actually use the labelled points to specify our predictions on these problematic instances.

To keep the logic of Theorem 2.2 we formulate the algorithm in a way similar to our pairwise aggregation algorithm.

**The learning algorithm of Theorem 4.4**

- Split the sample of size $3n$ into three parts.
- Let $\alpha = \frac{d \log \frac{n}{d} + \log \frac{1}{\delta}}{n}$ and $c$ be a well-chosen absolute constant.
- Based on the sample $\{(X_i, Y_i)_{i=1}^n\}$ run ERM with respect to the binary loss and denote its output by $\hat{g}$.
- Based on $\{(X_i, Y_i)_{i=1}^n\}$ consider the class of almost minimizers

$$\hat{\mathcal{F}} = \{f \in \mathcal{F} : R_n(f) - R_n(\hat{g}) \leq c(\alpha^2 + \alpha \sqrt{P_n[|\hat{g} - f|]})\}. $$
Based on \( \hat{F} \) define the set \( \hat{G} \) as follows. This set consists of all \{0, 1, *\}-valued functions \( g \) parametrized by \( f \in \hat{F} \) defined by
\[
g(x) = \begin{cases} f(x), & \text{if } f(x) = \hat{g}(x), \\ *, & \text{if } f(x) \neq \hat{g}(x). \end{cases}
\]

Based on the second part of the sample define the output of the algorithm as \( \hat{f}_{h/2} = \arg \min_{f \in \hat{G}} \sum_{i=n+1}^{2n} \left( \mathbb{1}[f(X_i) \neq Y_i \text{ and } f(X_i) \in \{0, 1\}] + \frac{1}{2} (1-h) \mathbb{1}[f(X_i) = *) \right) \).

Use the third part of the sample \( S_3 = \{(X_i, Y_i)_{i=2n+1}^{3n}\} \) to convert the \{0, 1, *\}-valued function \( \hat{f}_{h/2} \) to the \{0, 1\}-valued function \( \hat{f} \) using the following rule:
\[
\hat{f}(x) = \begin{cases} \hat{f}_{h/2}(x), & \text{if } \hat{f}_{h/2}(x) \in \{0, 1\}, \\ \text{maj}_{S_3}(x), & \text{if } \hat{f}_{h/2}(x) = *. \end{cases}
\]

Output \( \hat{f} \).

Remark 4.5. Observe that in the case of \( h = 1 \) our general algorithm gives a bound better than (24). Moreover, our algorithm does not require to remember the entire training sample, which is the case in (24).

Proof. At first, Theorem 2.2 implies that with probability \( 1 - \delta \) we have
\[
R_{h/2}(\hat{f}_{h/2}) - R(f^*) \leq c_1 \frac{d \log \frac{d}{\delta}}{\frac{1}{2} n h} + \log \frac{1}{\delta},
\]
where \( c_1 > 0 \) is an absolute constant. Since the diameter of the class is finite, there are at most \( D \) points in the domain \( \mathcal{X} \) such that \( \hat{f}_{h/2} = * \). This follows from the way we construct the classifier in Theorem 2.2. We want to use the third part of the sample to specify the prediction on these points. From now on, we work conditionally on the first and the second parts of the sample and therefore the \( D \) points of interest are assumed to be fixed. This set of points will be denoted by \{\( x_1, \ldots, x_D \)\} without loss of generality.

Consider the class \( \mathcal{C} \) which consists of functions that shatter the set \{\( x_1, \ldots, x_D \)\} and are equal to zero everywhere else. There are exactly \( 2^D \) functions and they form a class of VC dimension \( D \). Consider the following joint distribution of \( X \) and \( Y \) (denoted by \( \tilde{P} \) in what follows):

- The distribution of \( X \), that is \( P_X \), remains unchanged.
- For \( x \in \{x_1, \ldots, x_D\} \) the distribution \( Y | x \) is also unchanged.
- For \( x \notin \{x_1, \ldots, x_D\} \) we set \( Y | x = 0 \).

It is easy to see that \( S_3 \) can be used to learn the class \( \mathcal{C} \) with respect to the distribution \( \tilde{P} \). Observe that for \( \tilde{P} \) the Bayes optimal rule \( f^*_{B, \tilde{P}} \) is in \( \mathcal{C} \). Moreover, in this case \( \text{maj}_{S_3} \) is ERM over \( \mathcal{C} \). Finally, we have, with probability at least \( 1 - \delta \),
\[
R(\text{maj}_{S_3}) - R(f^*_{B, \tilde{P}}) \leq c_2 \frac{D + \log \frac{1}{\delta}}{n h},
\]
(27)
where the risk is defined with respect to $\tilde{P}$ and $c_2 > 0$ is an absolute constant and (27) is the inequality 19 in [3]. Observe that

$$R(\text{maj}_{S_n}) = \Pr(\text{maj}_{S_n}(X) \neq Y \text{ and } X \in \{x_1, \ldots, x_D\}),$$

and by (25), conditioned on first two parts of the sample, we have

$$R(f^*_{B,h}) \leq \frac{1}{2}(1 - h) \Pr(\{x_1, \ldots, x_D\}).$$

Using the union bound and combining (26) and (27) we have

$$R(f) = \Pr(\hat{f}(X) \neq Y \text{ and } X \notin \{x_1, \ldots, x_D\}) + \Pr(\text{maj}_{S_n}(X) \neq Y \text{ and } X \in \{x_1, \ldots, x_D\})$$

$$\leq \Pr(\hat{f}_{h/2}(X) \neq Y \text{ and } \hat{f}_{h/2}(X) \in \{0,1\}) + \frac{1}{2}(1 - h) \Pr(\{x_1, \ldots, x_D\}) + c_2 \frac{D + \log \frac{1}{\delta}}{nh}$$

$$= R_{h/2}(\hat{f}_{h/2}) + c_2 \frac{D + \log \frac{1}{\delta}}{nh} \leq R(f^*) + c_2 \frac{D + \log \frac{1}{\delta}}{nh} + c_1 \frac{d \log \frac{1}{\delta} + \log \frac{1}{\delta}}{nh}.$$

The claim follows.

\[\square\]

### 4.3 $P_X$-dependent generalizations of diameter

In this paragraph, we present an important generalization of Theorem 4.4 and provide a distribution-dependent version of the combinatorial diameter. In order to simplify the presentation of this more involved case, we make two assumptions:

- We consider the case of the deterministic labelling, that is $h = 1$,
- We assume that the marginal distribution $P_X$ is known to the learner.

As already mentioned, if the diameter $D$ is infinite, then it is impossible to improve the $\Omega(\frac{1}{\sqrt{n}})$ lower bound even in the case of $h = 1$ (see (5)). This lower bound is demonstrated by the worst-case marginal distribution $P_X$. Therefore, in order to generalize our bound, we need to introduce a distribution-dependent analog of the combinatorial diameter.

The proof of Theorem 4.4 is based on the following fact: for any two functions $f, g \in F$ the boolean cube supported on the set $\{x \in \mathcal{X} : f(x) \neq g(x)\}$ is finite, this implies, in particular, that the corresponding boolean cube is learnable with respect to any distribution. This fact was explicitly used in the proof of Theorem 4.4. This is, of course, the case since we consider the class $\mathcal{F}$ with finite combinatorial diameter $D$. However, in order to make the same argument applicable for some specific distributions $P_X$ in the case of $D = \infty$, we first need the following definition. For $\mathcal{H} \subseteq \{0,1\}^X$ and $\varepsilon > 0$ let $\mathcal{N}(\mathcal{H}, \varepsilon, L_1)$ denote the covering number (see e.g., [2]) with respect to $L_1(\mathcal{P}_X)$ metric. Denote the corresponding covering set by $\mathcal{N}(\mathcal{H}, \varepsilon, L_1)$. This means, in particular, that $|\mathcal{N}(\mathcal{H}, \varepsilon, L_1)| = \mathcal{N}(\mathcal{H}, \varepsilon, L_1)$. For a pair of binary functions $f, g$ denote by $\mathcal{C}_{f,g}$ the set of binary classifiers that shatters the set $\mathcal{X}' = \{x \in \mathcal{X} : f(x) \neq g(x)\}$. These functions are set to be equal to zero on $\mathcal{X} \setminus \mathcal{X}'$. The simplest distribution-dependent analog of the combinatorial diameter we found in our context is the following fixed point.

**Definition 4.6** (A $P_X$-dependent analog of the combinatorial diameter). Let $c_1 > 0$ be an absolute constant to be specified later. Given a class of classifiers $\mathcal{F}$, a marginal distribution $P_X$ and a sample size $n$, we define

$$D_{P_X}(n) := \sup_{f,g \in \mathcal{F}} \max \{\gamma \geq 0 : c_1 n \gamma \leq \log_2 \mathcal{N}(\mathcal{C}_{f,g}, \gamma, L_1)\}.$$
Remark 4.7. It is straightforward to see that $D_{P_X}(n) \leq \frac{D}{c_1}$, since for any $\gamma > 0$, 
\[ \sup_{f,g \in F} \log_2 \mathcal{N}(C_{f,g}, \gamma, L_1) \leq D. \]

However, for some classes $\mathcal{F}$ and some marginal distributions $P_X$ the value of $D_{P_X}(n)$ can be finite even if $D$ is infinite. The simplest example is the situation where for any $\varepsilon \in [0,1]$ and any two $f, g \in F$ there is a finite set $X'_{f,g}(\varepsilon) \subseteq \{ x \in X : f(x) \neq g(x) \}$, such that 
\[ \Pr (\{ x \in X : f(x) \neq g(x) \} \setminus X'(\varepsilon)) \leq \varepsilon. \]

Since the binary cube supported on $X'_{f,g}(\varepsilon)$ forms an $\varepsilon$-cover of the set $C_{f,g}$, we have 
\[ D_{P_X}(n) \leq n \sup_{f,g \in F} \max \{ \gamma \geq 0 : c_1 n \gamma \leq |X'_{f,g}(\gamma)| \}. \]

The main result of this paragraph is the following analog of Theorem 4.4.

Corollary 4.8. If 4.1 is satisfied with $h = 1$, then there is a $P_X$-dependent learning algorithm with output denoted by $\hat{f}$ such that, with probability at least $1 - \delta$, 
\[ R(\hat{f}) - R(f^*) \lesssim \frac{d \log rac{n}{\delta} + D_{P_X}(n) + \log \frac{1}{\delta}}{n}. \]

We present the learning algorithm.

**The learning algorithm of Corollary 4.8**

- Fix $h = 1$ and use the first two parts of the sample $(S_1$ and $S_2$) in the same way as in the learning algorithm of Theorem 4.4.

- Use the third part of the sample $S_3 = \{(X_i, Y_i)\}_{i=2n+1}^{3n}$ to convert the $\{0, 1, *\}$-valued function $\hat{f}_{1/2}$ to the $\{0, 1\}$-valued function $\hat{f}$. To do so, recall that $\hat{g}$, is the ERM based on the first part of the sample and let $\hat{g}_1 \in \mathcal{F}$ be the classifier that together with $\hat{g}$ defines $\hat{f}_{1/2}$. For $c_2 > 0$, which is a tuned absolute constant, consider the value 
\[ r = c_2 \left( \frac{D_{P_X}(n)}{n} + \frac{\log \frac{1}{\delta}}{n} \right). \] (28)

Given the $(S_1, S_2$-dependent) set $\mathcal{C}' = \mathcal{N}(\mathcal{C}_{\hat{g}, \hat{g}_1}, r, L_1)$, define 
\[ \tilde{g} = \arg \min_{g \in \mathcal{C}'} \sum_{i=2n+1}^{3n} \mathbb{1} [g(X_i) \neq Y_i \text{ and } \hat{f}_{1/2}(X_i) = *]. \] (29)

Finally, set 
\[ \hat{f}(x) = \begin{cases} \hat{f}_{1/2}(x), & \text{if } \hat{f}_{1/2}(x) \in \{0, 1\}, \\ \tilde{g}(x), & \text{if } \hat{f}_{1/2}(x) = *. \end{cases} \]

- Output $\hat{f}$.

**Proof.** We highlight only the steps that are different from the proof of Theorem 4.4. Using the notation above we have 
\[ R(\hat{f}) = \Pr (\hat{f}_{1/2}(X) \neq Y \text{ and } \hat{f}_{1/2}(X) \neq *) + \Pr (\tilde{g}(X) \neq Y \text{ and } \hat{f}_{1/2}(X) = *). \]
By Theorem 2.2 we have, with probability at least $1 - \delta$,
\[
\Pr(\hat{f}_{1/2}(x) \neq Y \text{ and } \hat{f}_{1/2}(X) \neq *) - R(f^*) \lesssim \frac{d \log \frac{n}{\delta} + \log \frac{1}{\delta}}{n}
\]
It is only left to prove that, with probability at least $1 - \delta$,
\[
\Pr(\hat{g}(X) \neq Y \text{ and } \hat{f}_{1/2}(X) = *) \lesssim \frac{D_{P_X}(n) + \log \frac{1}{\delta}}{n}.
\]
Recall the notation of the proof of Theorem 4.4. Observe that for $\hat{P}$ the Bayes optimal rule $f^*_{B, \hat{P}}$ is in $C_{\hat{g}, \hat{B}}$. Moreover, since $h = 1$ for this distribution we have $Y = f^*_{B, \hat{P}}$ almost surely. Therefore, our problem corresponds to the realizable (the Bayes optimal rule is in the class and there is no noise) case classification. To analyze this problem, we fix $\varepsilon > 0$ and consider the $\varepsilon$-net of $C_{\hat{g}, \hat{B}}$ with respect to the $L_1(P_X)$ metric. Observe that $\hat{g}$ corresponds to ERM over this set for $\varepsilon = r$ defined by (28). The corresponding risk bound is well known and is immediately implied by Theorem 5 in [9]. This result claims, in particular, that if
\[
n \geq 64 \log N(C_{\hat{g}, \hat{B}}, \varepsilon, L_1) + 32 \log \frac{1}{\varepsilon},
\]
then, with probability at least $1 - \delta$, we have
\[
R_{\hat{P}}(\hat{g}) = \Pr(\hat{g}(X) \neq Y \text{ and } \hat{f}_{1/2}(X) = *) \leq \varepsilon.
\]
The claim follows once we set $\varepsilon = r$ and tune $c_1, c_2 > 0$ in a way such that (30) is satisfied.

Finally, we note that the generalization of this result for $h \neq 1$ can be done using the techniques in [41].

5 Some implications for ERM

As a corollary of the proof of Theorem 2.2 we have the following result for ERM (compare it with the widely used Corollary 5.3 in [6]). The result shows that the learning rate of ERM in classification is controlled by the empirical diameter of the set of almost empirical risk minimizers.

Proposition 5.1. Consider a VC class $\mathcal{F}$ and let $\hat{F}$ be defined by (17). For any ERM $\hat{g}$ we have, with probability at least $1 - \delta$,
\[
R(\hat{g}) - R(f^*) \lesssim \sqrt{\sup_{f \in \hat{F}} P_n|f - \hat{g}|} \sqrt{\frac{d \log \frac{n}{\delta} + \log \frac{1}{\delta}}{n} + \frac{d \log \frac{n}{\delta} + \log \frac{1}{\delta}}{n}}.
\]

Observe that the bound (31) can be computed based on the data: it does not depend on the unknown parameters that appear in the margin condition. Moreover, the bound adapts to the Bernstein condition 2.1. Indeed, if (2.1) is satisfied, then, with probability at least $1 - \delta$, the right-hand side of (31) is upper bounded by \( (B d \log \frac{n}{\delta} + B \log \frac{1}{\delta}) \frac{1}{\delta} \) up to an absolute constant factor. The proof of the this fact repeats the lines of the proof of (6) and we omit it.

Proof. By the definition of $f^*$ we have $R(f^*) - R(\hat{g}) \leq 0$. By (15) we have
\[
R_n(f^*) - R_n(\hat{g}) \lesssim R(f^*) - R(\hat{g}) + \sqrt{P_n|f^* - \hat{g}|} \sqrt{\frac{d \log \frac{n}{\delta} + \log \frac{1}{\delta}}{n} + \frac{d \log \frac{n}{\delta} + \log \frac{1}{\delta}}{n}}
\]
\[
\lesssim \sqrt{P_n|f^* - \hat{g}|} \sqrt{\frac{d \log \frac{n}{\delta} + \log \frac{1}{\delta}}{n} + \frac{d \log \frac{n}{\delta} + \log \frac{1}{\delta}}{n}}.
\]
Therefore, with probability at least $1 - \delta$, we have $f^* \in \hat{F}$ if the absolute constant $c$ is chosen properly. Finally, we use (14) again. By the definition of $\hat{g}$ we have $R_n(f^*) - R_n(\hat{g}) \geq 0$. Therefore, using the union bound we have, with probability at least $1 - \delta$,

$$R(\hat{g}) - R(f^*) \lesssim \sqrt{\sup_{f \in \mathcal{F}} P_n|f - \hat{g}|} \sqrt{\frac{d \log \frac{n}{d} + \log(\frac{1}{\delta})}{n} + \frac{d \log \frac{n}{d} + \log(\frac{1}{\delta})}{n}}.$$  

By Lemma 3.1 we can rewrite the bound (31) using the real diameter instead of the empirical diameter. That is, with probability at least $1 - \delta$,

$$R(\hat{g}) - R(f^*) \lesssim \sqrt{\sup_{f \in \mathcal{F}} P_n[f - \hat{g}]} \sqrt{\frac{d \log \frac{n}{d} + \log(\frac{1}{\delta})}{n} + \frac{d \log \frac{n}{d} + \log(\frac{1}{\delta})}{n}}.$$  

An even more interesting result follows from Theorem 2.2. At first, we provide a heuristic argument. Theorem 2.2 implies that, with probability at least $1 - \delta$,

$$R_0(\hat{f}_p) - R(f^*) \leq p \Pr(\hat{f}_p = \ast) + c \frac{d \log \frac{n}{d} + \log(\frac{1}{\delta})}{np}.$$  

where $c > 0$ is an absolute constant. This inequality shows that there is a tradeoff between the size of the region of abstention and the learning rate. Assume that there is a constant $p^* \in [0, \frac{1}{2})$ such that, with probability at least $1 - \delta/2$,

$$p^* \Pr(\hat{f}_p = \ast) \simeq \frac{d \log \frac{n}{d} + \log(\frac{1}{\delta})}{np^*}.$$  

(32)

This implies that, with probability at least $1 - \delta$,

$$R_0(\hat{f}_{p^*}) - R(f^*) \lesssim \sqrt{\frac{d \log \frac{n}{d} + \log(\frac{1}{\delta})}{n} \Pr(\hat{f}_p = \ast)},$$

which is better than (31). Indeed, the rate of convergence of randomized $\{0, 1\}$-valued classifier is defined as a measure of disagreement of two particular almost ERM and not by the empirical diameter of the entire set $\tilde{F}$. A more formal argument will be presented in Appendix B.

6 Concluding remarks and comparisons

It is interesting to compare our analysis with some recent results presented in the literature. In what follows, we discuss mainly the implications of Theorem 2.2. One of the related directions is the selective classification framework studied extensively in [12, 39, 14]. The idea of selective classification is roughly the following: if we decide to classify an instance $x$, then we should always output $f^*(x)$. In particular, in the realizable case, one should always output the true label. Simultaneously, we want to minimize the total mass of the region of the abstention. As can be easily seen from our results, our aim is slightly less ambitious, and therefore our results may not be directly compared with the results in selective classification. However, our analysis does not require any assumptions presented in the selective classification literature such as
realizability, low noise assumptions, or control over the so-called disagreement coefficient \[14\]. As a final remark, we emphasize that the points that we classify always give at most \(O(d/n)\) contribution to the excess risk, and according to \([7]\), our region of abstention shrinks, once we have some favorable margin assumptions. This is somewhat similar to the goals in the selective classification.

The statistical analysis of the Chow’s model was presented in \(18, 4\). Notice that there are some important differences: for example, in \(18\) the authors assume that we are given the class \(\mathcal{F}'\) of \(\{0, 1, \ast\}\)-valued functions and are interested in the analysis of ERM over \(\mathcal{F}'\) with respect to Chow’s risk \([10]\). To get the fast learning rates with respect to this risk a special version of Tsyabkov’s assumption was introduced for classes of \(\{0, 1, \ast\}\)-valued functions. Our results are of a slightly different flavour: we start with a class \(\mathcal{F}\) of \(\{0, 1\}\)-valued functions and build a class \(\mathcal{G}\) of \(\{0, 1, \ast\}\)-valued functions in order to avoid any margin assumptions.

Finally, we discuss the application of aggregation theory in the context of classification with reject option studied by Freund, Mansour and Shapire in \([13]\). The authors consider the weighted majority type algorithm. A direct comparison with our results seems difficult, because the analysis in \([13]\) is tuned to finite classes and, more importantly, the risk bounds relate the risk of the algorithm to twice the risk of the best function in the class. It is by now well-known that in this case ERM always has \(O(d/n)\) rates of convergence (this result follows from Corollary 5.3 in \([8]\)). Finally, their results are not adaptive to various low noise assumptions. We refer to section 10 in \([12]\) for detailed discussions.

One of the directions of future research is in improving the logarithmic factors appearing in our bounds. This direction has taken some attention recently in related scenarios \([15, 22, 41, 42]\). Here we add a particularly simple result that removes the superfluous logarithmic factor if \(h = 1\), which corresponds to the case of deterministic labelling. Thus, we are improving the in-expectation version of the upper bound of Theorem \(4.4\) as well as the bound of Theorem 9 in \([5]\). The form of the result is similar to the upper bound of the one inclusion graph algorithm in \([6]\), but in our case, \(D\) plays the role of the VC dimension.

**Proposition 6.1.** Consider a VC class \(\mathcal{F}\) with finite diameter \(D\). If \(4.1\) holds with \(h = 1\), then there is a learning algorithm with output denoted by \(\hat{f}\) such that, with probability at least \(1 - \delta\),

\[
ER(\hat{f}) - R(f^*) \leq \frac{D}{n + 1}.
\]

**Proof.** At first we recall that \(h = 1\) corresponds to the case of \(Y = f_0(X)\) almost surely. Assume without loss of generality that \(f = 0\) belongs to \(\mathcal{F}\). We are given a sample \(S = ((X_1, Y_1), \ldots, (X_{n+1}, Y_{n+1}))\). Denote by \(S^{(i)}\) the sample \(S\) with hidden \(i\)-th label, that is

\[
S^{(i)} = ((X_1, Y_1), \ldots, (X_{i-1}, Y_{i-1}), (X_i, \ast), (X_{i+1}, Y_{i+1}), \ldots, (X_{n+1}, Y_{n+1})).
\]

Let \(\hat{f}_{S^{(i)}}\) denote the classifier \(\hat{f}\), which was trained on the sample \(S^{(i)}\). The leave-one-out error is defined by

\[
LOO = \frac{1}{n + 1} \sum_{i=1}^{n+1} \mathbf{1}[\hat{f}_{S^{(i)}}(X_i) \neq Y_i].
\]

It is well-known \([13]\) that \(ELOO = ER(\hat{f})\). We consider the following learning algorithm. Given the sample \(S\) it outputs \(\hat{f}\) defined as follows

\[
\hat{f}(x) = \begin{cases} Y_i, & \text{if } x = X_i \text{ for some } X_i \in S, \\ 0, & \text{otherwise.} \end{cases}
\]

Since there is no noise in the labelling, our algorithm is correctly defined. Let \(S^n\) be a subset of \(S\) such that any \((X_i, Y_i) \in S\) is presented only ones. This set is obtained from \(S\) by removing
all excess copies. We have

$$(n + 1)\text{LOO} = \sum_{i=1}^{n+1} 1[\hat{f}_{S(i)}(X_i) \neq Y_i] = \sum_{(X_i, Y_i) \in S} 1[\hat{f}_{S(i)}(X_i) \neq Y_i] = \sum_{(X_i, Y_i) \in S} 1[Y_i = 1]$$

$$\leq \left( D + \inf_{f \in F} \sum_{(X_i, Y_i) \in S} 1[f(X_i) \neq Y_i] \right) \leq \left( D + \inf_{f \in F} \sum_{i=1}^{n+1} 1[f(X_i) \neq Y_i] \right)$$

$$\leq \left( D + \sum_{i=1}^{n+1} 1[f^*(X_i) \neq Y_i] \right).$$

By taking the expectation with respect to both sides of the last inequality and using $E \text{LOO} = E R(f)$ we prove the claim.

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### A On Lemma 3.1

In this section we discuss various proofs of Lemma 3.1. One of the implications of our analysis, which to the best of our knowledge was not observed in the previous literature, is that there is a short way of proving almost optimal risk bounds under various margin conditions using only on the original ratio-type estimates of Vapnik and Chervonenkis [37] which apply to binary-valued functions. This is remarkable, because the existing proofs in the binary case [3, 22, 27, 34] exploit Talagrand’s inequality and technical results from empirical process theory. The trick will be to always go back to binary-valued functions.

At first, the class $H = \{x \rightarrow 1[f(x) \neq g(x)] : f, g \in \mathcal{F}\}$ has VC dimension at most $10d$ [38] (we may alternatively control the growth function). Now by the classical ratio-type estimates of Vapnik and Chervonenkis [37] we have, with probability at least $1 - \delta$,

\[
\frac{P|f - g| - P_n|f - g|}{\sqrt{P|f - g|}} \lesssim \sqrt{\frac{d \log \frac{n}{d} + \log \frac{1}{\delta}}{n}},
\]

and

\[
\frac{P_n|f - g| - P|f - g|}{\sqrt{P_n|f - g|}} \lesssim \sqrt{\frac{d \log \frac{n}{d} + \log \frac{1}{\delta}}{n}}.
\]

To prove (13) and (14) we use that for $a, b, c \geq 0$ the inequality $a \leq b \sqrt{a} + c$ implies $a \leq b^2 + b \sqrt{c} + c$ (see [6]). Both inequalities now follow from simple algebra.
To the best of our knowledge the simplest proof of (15) follows from the idea in [11,19] and is essentially presented there. We reproduce this short argument for the sake of completeness. Consider the class of $\{0,1\}$-valued functions

$$\mathcal{F}_1 = \{(x,y) \to 1[f(x) \neq y \land f^*(x) = y] : f \in \mathcal{F}\}.$$

and

$$\mathcal{F}_2 = \{(x,y) \to 1[f(x) = y \land f^*(x) \neq y] : f \in \mathcal{F}\}.$$  

We have for $h_f \in \mathcal{F}_1$ and $g_f \in \mathcal{F}_2$ that $h_f(x,y) + g_f(x,y) = 1[f(x) \neq f^*(x)]$ and the VC dimension of both classes is at most $d$. Moreover, $h_f(x,y) - g_f(x,y) = 1[f(x) \neq y] - 1[f^*(x) \neq y]$. 

Applying the ratio-type estimates of [37,6] to $\mathcal{F}_h$ we have for $f^*$ and $\delta$ that

$$\sup_{h \in \mathcal{F}} |(P - P_n)(f^* - Y)(f - f^*)| \leq \frac{B \log \frac{n}{\delta}}{n} + \frac{B \log \frac{1}{\delta}}{n}.$$

where we used $\sqrt{P_n h_f} + \sqrt{P_n g_f} \leq \sqrt{2P_n(h_f + g_f)}$. The inequality (15) follows.

In particular, combining (15) with (14) we have for any ERM (denoted by $\hat{g}$) over $\mathcal{F}$ we have, with probability at least $1 - \delta$,

$$R(\hat{g}) - R(f^*) \lesssim \alpha^2 + \alpha \sqrt{P|\hat{g} - f^*|}.$$

Under Assumption 2.1 this implies immediately that, with probability at least $1 - \delta$,

$$R(\hat{g}) - R(f^*) \lesssim \left(\frac{Bd \log \frac{n}{\delta} + B \log \frac{1}{\delta}}{n}\right) \xrightarrow{\frac{1}{n}} 0.$$

An alternative and self-contained proof of Lemma 3.1 follows from general techniques in empirical process theory. We give a simplified version of the argument. In what follows, we will not use the ratio-type estimates of [37,6]. Recall the definition (12). At first, observe that the following inequality always holds for all $h \in \mathcal{L}_1$,

$$|Ph - P_n h| \leq |(P - P_n)|f - f^*|| + 2|(P - P_n)(f^* - Y)(f - f^*)|$$

(34)

The first term (34) may be analyzed completely similar to the second term. Moreover, the proof of (13) and (14) follows from the same analysis for the term $|(P - P_n)(f - g)^2|$. For the sake of brevity, we focus only on the analysis of $|(P - P_n)(f^* - Y)(f - f^*)|$.

Consider the class of real valued functions $\mathcal{H} = \text{star}(\mathcal{F} - f^*,0)$, that is the star-shaped hull of $\mathcal{F} - f^*$ centred around zero (see the definition in [2]). Consider the following fixed point

$$\gamma(\eta,\delta) = \inf \left\{ s \geq 0 : \Pr \left( \sup_{h \in \mathcal{H}, \|h\|_{L_2} \leq s} |(P - P_n)(f^* - Y)h| \leq \eta s^2 \right) \geq 1 - \delta \right\},$$

where $\eta$ is an absolute constant to be specified later. By the definition we have, with probability at least $1 - \delta$,

$$\sup_{h \in \mathcal{H}, \|h\|_{L_2} \leq \gamma(\eta,\delta)} |(P - P_n)(f^* - Y)h| \leq \eta \gamma(\eta,\delta)^2.$$

Now consider any $h \in \mathcal{H}$ such that $\|h\|_{L_2} = r \geq \gamma(\eta,\delta)$. By the star-shapedness we have for any $h' \in \mathcal{H}$ defined by $\frac{\text{sign}(h')}{r}$ that

$$|(P - P_n)(f^* - Y)h'| \leq \eta \gamma(\eta,\delta)^2.$$
which implies for any \( f \in \mathcal{F} \),
\[
|(P - P_n)(f^* - Y)(f - f^*)| \leq \eta \gamma(\eta, \delta) \sqrt{P|f - f^*|} + \eta \gamma(\eta, \delta)^2.
\] (35)

Finally, we provide an upper bound on \( \gamma(\eta, \delta) \) and choose the value of \( \eta \). In what follows, we prove that, with probability at least \( 1 - \delta \),
\[
\sup_{h \in \mathcal{H}, \|h\|_{L^2} \leq s} |(P - P_n)(f^* - Y)h| \lesssim s \sqrt{\frac{d \log \frac{1}{\varepsilon}}{n}} + s \sqrt{\frac{\log \frac{1}{\varepsilon}}{n}}.
\] (36)

By the symmetrization and contraction arguments (see e.g., [31]) we have
\[
\mathbb{E} \sup_{h \in \mathcal{H}, \|h\|_{L^2} \leq s} \left| \frac{1}{n} \sum_{i=1}^{n} \xi_i h(X_i) \right| \lesssim s \sqrt{\frac{\log \frac{1}{\varepsilon}}{n}},
\]
provided that \( s \gtrsim \sqrt{\frac{d \log \frac{1}{\varepsilon}}{n}} \). Applying Talagrand’s concentration inequality, we have in this regime, with probability at least \( 1 - \delta \),
\[
\sup_{h \in \mathcal{H}, \|h\|_{L^2} \leq s} |(P - P_n)(f^* - Y)h| \lesssim s \sqrt{\frac{d \log \frac{1}{\varepsilon}}{n}} + s \sqrt{\frac{\log \frac{1}{\varepsilon}}{n}} + \frac{\log \frac{1}{\varepsilon}}{n}.
\]
Fixing an appropriate absolute constant \( \eta > 0 \), we have
\[
\gamma(\eta, \delta) \lesssim \sqrt{\frac{d \log \frac{1}{\varepsilon}}{n}} + \frac{\log \frac{1}{\varepsilon}}{n}.
\]
The claim follows immediately by (35) and (13). We recall that (13) can be proven by repeating the same arguments.

**B On implications for ERM**

In this section we sketch the argument that may be used to find \( p^* \), defined by (32), in a data-dependent manner. For any \( p \in [0, \frac{1}{2}] \) we have, with probability at least \( 1 - \delta \),
\[
R_0(\hat{f}_p) - R(f^*) \leq \left( p \Pr(\hat{f}_p = *) + c \frac{d \log \frac{1}{\varepsilon}}{np} + \frac{\log \frac{1}{\varepsilon}}{n} \right) \wedge c \sqrt{\frac{d \log \frac{1}{\varepsilon}}{n} + \frac{\log \frac{1}{\varepsilon}}{n}},
\]
\[
25
\]
where $c > 0$ is an absolute constant. The values of $p \leq \sqrt{\frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n}}$ lead to the standard bound $R_0(\hat{f}_p) - R(f^*) \leq \sqrt{\frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n}}$. Thus, we may focus on $p \geq \sqrt{\frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n}}$. Define the sequence $p_1, \ldots, p_k$ by

$$p_i = 2^{i-1} \sqrt{\frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n}} \quad (37)$$

and let $k$ be chosen in a way such that $p_k \leq \frac{1}{2}$, but $p_{k+1} > \frac{1}{2}$. A simple computation shows that $k \leq \log \frac{d}{n}$. Using (13) and an independent sample of size $n$, we may guarantee that for any $p \in (0, \frac{1}{2}]$ we have

$$p \Pr(\hat{f}_p = \ast) \leq p P_n [\hat{f}_p(X) = \ast] + \gamma,$$

where $\gamma = p \frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n}$. Using the union bound and since $k \leq \log \frac{n}{d}$, we have simultaneously for all $\hat{p} \in \{1, \ldots, k\}$, with probability at least $1 - \delta$,

$$R_0(\hat{f}_{\hat{p}}) - R(f^*) \leq \left( p_i P_n [\hat{f}_{p_i} = \ast] + c \frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n p_i} \right) \wedge c \sqrt{\frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n}},$$

where $c'$ is an absolute constant. Observe that the bound $p_i P_n [\hat{f}_{p_i} = \ast] + c \frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n p_i}$ is data dependent, so we may choose the parameter $i$ that minimizes this quantity. We denote the corresponding value of $p_i$ by $\hat{p}$. Therefore, we have, with probability at least $1 - \delta$,

$$R_0(\hat{f}_{\hat{p}}) - R(f^*) \leq \left( \hat{p} P_n [\hat{f}_{\hat{p}} = \ast] + \frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n \hat{p}} \right) \wedge \sqrt{\frac{d \log \frac{d}{2} + \log \frac{1}{\delta}}{n}}, \quad (38)$$