Geometric Structures and Loop Variables in (2+1)-Dimensional Gravity

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Ashtekar’s connection representation for general relativity and the closely related loop variable approach have generated a good deal of excitement over the past few years. While it is too early to make firm predictions, there seems to be some real hope that these new variables will allow the construction of a consistent nonperturbative quantum theory of gravity. Some important progress has been made: large classes of observables have been found, a number of quantum states have been identified, and the first steps have been taken towards establishing a reasonable weak field perturbation theory.

Progress has been hampered, however, by the absence of a clear physical interpretation for the observables built out of Ashtekar’s new variables. In part, the problem is simply one of unfamiliarity — physicists accustomed to metrics and their associated connections can find it difficult to make the transition to densitized triads and self-dual connections. But there is a deeper problem as well, inherent in almost any canonical formulation of general relativity. To define Ashtekar’s variables, one must choose a time slicing, an arbitrary splitting of spacetime into spacelike hypersurfaces. But real geometry and physics cannot depend on such a choice; the true physical observables must somehow forget any details of the time slicing, and refer only to the invariant underlying geometry. To a certain extent, this is already a source of trouble in classical general relativity, where one must take care to separate physical phenomena from artifacts of coordinate choices. In canonical quantization, however, the problem becomes much sharper — all observables must be diffeomorphism invariants, and the need to reconstruct geometry and physics from such quantities becomes unavoidable.

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In a sense, the Ashtekar program is a victim of its own success. For the first time, we can actually write down a large set of diffeomorphism-invariant observables, the loop variables of Rovelli and Smolin [3]. But although some progress has been made in defining area and volume operators in terms of these variables [5], the goal of reconstructing spacetime geometry from such invariant quantities remains out of reach.

The purpose of this article is to demonstrate that such a reconstruction is possible in the simple model of (2+1)-dimensional gravity, general relativity in two spatial dimensions plus time. A reduction in the number of dimensions greatly simplifies general relativity, allowing the use of powerful techniques not readily available in the realistic (3+1)-dimensional theory. As a consequence, many of the specific results presented here will not readily generalize to higher dimensions. But the success of (2+1)-dimensional gravity can be viewed as an “existence proof” for canonical quantum gravity, and one may hope that at least some of the technical results have extensions to our physical spacetime.

1. (2+1)-Dimensional Gravity: From Geometry to Holonomies

Let us begin with a brief review of (2+1)-dimensional general relativity in first order formalism. As our spacetime we take a three-manifold $M$, which we shall often assume to have a topology $\mathbb{R} \times \Sigma$, where $\Sigma$ is a closed orientable surface. The fundamental variables are a triad $e^{\mu a}$ — a section of the bundle of orthonormal frames — and a connection on the same bundle, which can be specified by a connection one-form $\omega^{\mu ab}$. The Einstein-Hilbert action can be written as

$$I_{\text{grav}} = \int_M e^a \wedge \left( d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \right), \quad (1.1)$$

where $e^a = e^\mu_a dx^\mu$ and $\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{d}^{bc} dx^d$. The action is invariant under local SO(2,1) transformations,

$$\delta e^a = \epsilon^{abc} e^b \tau_c$$
$$\delta \omega^a = d\tau^a + \epsilon^{abc} \omega^b \tau_c, \quad (1.2)$$

as well as “local translations,”

$$\delta e^a = d\rho^a + \epsilon^{abc} \omega^b \rho_c$$
$$\delta \omega^a = 0. \quad (1.3)$$

$I_{\text{grav}}$ is also invariant under diffeomorphisms of $M$, of course, but this is not an independent symmetry: Witten has shown [3] that when the triad $e^\mu_a$ is invertible, diffeomorphisms in

*Indices $\mu, \nu, \rho, \ldots$ are spacetime coordinate indices; $i, j, k, \ldots$ are spatial coordinate indices; and $a, b, c, \ldots$ are “Lorentz indices,” labeling vectors in an orthonormal basis. Lorentz indices are raised and lowered with the Minkowski metric $\eta_{ab}$. This notation is standard in papers in (2+1)-dimensional gravity, but differs from the usual conventions for Ashtekar variables, so readers should be careful in translation.
the connected component of the identity are equivalent to transformations of the form (1.2)–(1.3). We therefore need only worry about equivalence classes of diffeomorphisms that are not isotopic to the identity, that is, elements of the mapping class group of $M$.

The equations of motion coming from the action (1.1) are easily derived:

$$de^a + \epsilon^{abc} \omega_b \wedge e_c = 0$$

and

$$d\omega^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge \omega_c = 0.$$  

These equations have four useful interpretations, which will form the basis for our analysis:

1. We can solve (1.4) for $\omega$ as a function of $e$, and rewrite (1.5) as an equation for $\omega[e]$. The result is equivalent to the ordinary vacuum Einstein field equations,

$$R_{\mu\nu}[g] = 0,$$  

for the Lorentzian (that is, pseudo-Riemannian) metric $g_{\mu\nu} = \epsilon^a_\mu \epsilon^b_\nu \eta_{ab}$. In 2+1 dimensions, these field equations are much more powerful than they are in 3+1 dimensions: the full Riemann curvature tensor is linearly dependent on the Ricci tensor,

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\rho} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} - \frac{1}{2} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) R,$$

so (1.6) actually implies that the metric $g_{\mu\nu}$ is flat. The space of solutions of the field equations can thus be identified with the set of flat Lorentzian metrics on $M$.

2. As a second alternative, note that equation (1.5) depends only on $\omega$ and not $e$. In fact, (1.5) is simply the requirement that the curvature of $\omega$ vanish, that is, that $\omega$ be a flat $SO(2,1)$ connection. Moreover, equation (1.4) can be interpreted as the statement that $e$ is a cotangent vector to the space of flat $SO(2,1)$ connections; indeed, if $\omega(s)$ is a curve in the space of flat connections, the derivative of (1.3) gives

$$d \left( \frac{d\omega^a}{ds} \right) + \epsilon^{abc} \omega_b \wedge \left( \frac{d\omega_c}{ds} \right) = 0,$$

which can be identified with (1.4) with

$$e^a = \frac{d\omega^a}{ds}.  \tag{1.9}$$

To determine the physically inequivalent solutions of the field equations, we must still factor out the gauge transformations (1.2)–(1.3). The local Lorentz transformations (1.2) act on $\omega$ as ordinary $SO(2,1)$ gauge transformations, and tell us that only gauge equivalence classes of flat $SO(2,1)$ connections are relevant. Let us denote the space of such equivalence classes as $\tilde{N}$. The transformations of $e$ can once again be
interpreted as statements about the cotangent space: if we consider a curve \( \tau(s) \) of \( \text{SO}(2,1) \) transformations, it is easy to check that the first equations of (1.2) and (1.3) follow from differentiating the second equation of (1.2), with
\[
\rho^a = \frac{d\tau^a}{ds} \quad (1.10)
\]
and \( e^a \) as in (1.9). A solution of the field equations is thus determined by a point in the cotangent bundle \( T^*\tilde{N} \).

Now, a flat connection on the frame bundle of \( M \) is determined by its holonomies, that is, by a homomorphism
\[
\rho : \pi_1(M) \to \text{SO}(2,1), \quad (1.11)
\]
and gauge transformations act on \( \rho \) by conjugation. We can therefore write
\[
\tilde{N} = \text{Hom}(\pi_1(M), \text{SO}(2,1))/\sim, \quad (1.12)
\]
\[
\rho_1 \sim \rho_2 \quad \text{if} \quad \rho_2 = h \cdot \rho_1 \cdot h^{-1}, \quad h \in \text{SO}(2,1).
\]
It remains for us to factor out the diffeomorphisms that are not in the component of the identity, the mapping class group. These transformations act on \( \tilde{N} \) through their action as a group of automorphisms of \( \pi_1(M) \), and in many interesting cases — for example, when \( M \) has the topology \( \mathbb{R} \times \Sigma \) — this action comprises the entire set of outer automorphisms of \( \pi_1(M) \) \[7\]. If we denote equivalence under this action by \( \sim' \), and let \( \tilde{N}/\sim' = N \), we can express the space of solutions of the field equations (1.4)–(1.5) as
\[ T^*N. \]

When \( M \) has the topology \( \mathbb{R} \times \Sigma \), this description can be further refined. In that case, the space \( \tilde{N} \) — or at least the physically relevant connected component of \( \tilde{N} \) — is homeomorphic to the Teichmüller space of \( \Sigma \), and \( N \) is the corresponding moduli space \[8,12\]. The set of vacuum spacetimes can thus be identified with the cotangent bundle of the moduli space of \( \Sigma \), and many powerful results from Riemann surface theory become applicable.

3. A third approach is available if \( M \) has the topology \( \mathbb{R} \times \Sigma \). For such a topology, it is useful to split the field equations into spatial and temporal components. Let us write
\[
\begin{align*}
\tau &= \bar{\tau} + dt \partial_0, \\
e^a &= \bar{e}^a + e^a_0 dt, \\
\omega^a &= \bar{\omega}^a + \omega^a_0 dt.
\end{align*}
\]

†Strictly speaking, one more subtlety remains. The space of homomorphisms \( \text{Hom}(\pi_1(M), \text{SO}(2,1)) \) is not always connected, and it is often the case that only one connected component corresponds to physically admissible spacetimes. See \[8,9\] for the mathematical structure and \[6,9,10,11\] for physical implications.
(This decomposition can be made in a less explicitly coordinate-dependent manner — see, for example, [13] — but the final results are unchanged.) The spatial projections of (1.4)–(1.5) take the same form as the original equations, with all quantities replaced by their “tilded” spatial equivalents. As above, solutions may therefore be labeled by classes of homomorphisms, now from \( \pi_1(\Sigma) \) to \( \text{SO}(2,1) \), and the corresponding cotangent vectors. The temporal components of the field equations, on the other hand, now become

\[
\begin{align*}
\partial_0 \tilde{e}^a &= \tilde{d}e_0^a + \epsilon^{abc}\tilde{\omega}_b e_0^c + \epsilon^{abc}\tilde{e}_b \omega_0^c, \\
\partial_0 \tilde{\omega}^a &= \tilde{d}\omega_0^a + \epsilon^{abc}\tilde{\omega}_b \omega_0^c.
\end{align*}
\] (1.14)

Comparing to (1.2)–(1.3), we see that the time development of \((\tilde{e}, \tilde{\omega})\) is entirely described by a gauge transformation, with \(\tau^a = \omega_0^a\) and \(\rho^a = e_0^a\).

This is consistent with our previous results, of course. For a topology of the form \( \mathbb{R} \times \Sigma \), the fundamental group is simply that of \( \Sigma \), and an invariant description in terms of holonomies should not be able to detect a particular choice of spacelike slice. Equation (1.14) shows in detail how this occurs: motion in coordinate time is merely a gauge transformation, and is therefore invisible to the holonomies. But the central dilemma described in the introduction now stands out sharply. For despite equation (1.14), solutions of the (2+1)-dimensional field equations are certainly not static as geometries — they do not, in general, admit timelike Killing vectors. The real physical dynamics has somehow been hidden by this analysis, and must be uncovered if we are to find a sensible physical interpretation of our solutions. This puzzle is an example of the notorious “problem of time” in gravity [14], and exemplifies one of the basic issues that must be resolved in order to construct a sensible quantum theory.

4. A final approach to the field equations (1.4)–(1.5) was suggested by Witten [6], who observed that the triad \( e \) and the connection \( \omega \) could be combined to form a single connection on an ISO(2,1) bundle. ISO(2,1), the three-dimensional Poincaré group, has a Lie algebra with generators \( \mathcal{J}^a \) and \( \mathcal{P}^b \) and commutation relations

\[
[\mathcal{J}^a, \mathcal{J}^b] = \epsilon^{abc}\mathcal{J}_c, \quad [\mathcal{J}^a, \mathcal{P}^b] = \epsilon^{abc}\mathcal{P}_c, \quad [\mathcal{P}^a, \mathcal{P}^b] = 0.
\] (1.15)

If we write a single connection one-form

\[
A = e^a\mathcal{P}_a + \omega^a\mathcal{J}_a
\] (1.16)

and define a “trace,” an invariant inner product on the Lie algebra, by

\[
\text{Tr} (\mathcal{J}^a \mathcal{P}^b) = \eta^{ab}, \quad \text{Tr} (\mathcal{J}^a \mathcal{J}^b) = \text{Tr} (\mathcal{P}^a \mathcal{P}^b) = 0,
\] (1.17)

then it is easy to verify that the action (1.1) is simply the Chern-Simons action [15] for \( A \),

\[
I_{CS} = \frac{1}{2} \int_M \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\}.
\] (1.18)
The field equations now reduce to the requirement that $A$ be a flat ISO(2,1) connection, and the gauge transformations (1.2)–(1.3) can be identified with standard ISO(2,1) gauge transformations. Imitating the arguments of our second interpretation, we should therefore expect solutions of the field equations to be characterized by gauge equivalence classes of flat ISO(2,1) connections, that is, by homomorphisms in the space

$$\tilde{\mathcal{M}} = \text{Hom}(\pi_1(M), \text{ISO}(2,1))/\sim,$$

$$\rho_1 \sim \rho_2 \text{ if } \rho_2 = h \cdot \rho_1 \cdot h^{-1}, \quad h \in \text{ISO}(2,1).$$

(1.19)

To relate this description to our previous results, note that ISO(2,1) is itself a cotangent bundle with base space SO(2,1). Indeed, a cotangent vector at the point $\Lambda_1 \in \text{SO}(2,1)$ can be written in the form $d\Lambda_1 \Lambda_1^{-1}$, and the multiplication law

$$(\Lambda_1, d\Lambda_1 \Lambda_1^{-1}) \cdot (\Lambda_2, d\Lambda_2 \Lambda_2^{-1}) = (\Lambda_1 \Lambda_2, d(\Lambda_1 \Lambda_2)(\Lambda_1 \Lambda_2)^{-1})$$

$$= (\Lambda_1 \Lambda_2, d\Lambda_1 \Lambda_1^{-1} + \Lambda_1 (d\Lambda_2 \Lambda_2^{-1}) \Lambda_1^{-1})$$

(1.20)

may be recognized as the standard semidirect product composition law for Poincaré transformations. The space of homomorphisms from $\pi_1(M)$ to ISO(2,1) inherits this cotangent bundle structure in an obvious way, leading to the identification $\tilde{\mathcal{M}} \approx T^*\tilde{\mathcal{N}}$, where $\tilde{\mathcal{N}}$ is the space of homomorphisms (1.12). It remains for us to factor out the mapping class group. But this group acts in (1.12) and (1.19) as the same group of automorphisms of $\pi_1(M)$; writing the quotient as $\tilde{\mathcal{M}}/\sim' = \mathcal{M}$, we thus see that $\mathcal{M} \approx T^*\mathcal{N}$.

Of these four approaches to the (2+1)-dimensional field equations, only the first corresponds directly to our usual picture of spacetime physics. Trajectories of physical particles, for instance, are geodesics in the flat manifolds of this description. The second approach, on the other hand, is the one that is closest to the loop variable picture in (3+1)-dimensional gravity. The loop variables of Rovelli and Smolin [2,3,16,17] may be expressed as follows. Let

$$U[\gamma, x] = P \exp \left\{ \int_{\gamma} \omega^a J_a \right\}$$

(1.21)

be the holonomy of the connection one-form $\omega^a$ around a closed path $\gamma(t)$ based at $\gamma(0) = x$. (Here, $P$ denotes path ordering, and the basepoint $x$ specifies the point at which the path ordering begins.) We then define

$$\mathcal{T}^0[\gamma] = \text{Tr } U[\gamma, x]$$

(1.22)

and

$$\mathcal{T}^1[\gamma] = \int_{\gamma} dt \text{ Tr } \left\{ U[\gamma, x(t)] e_{\mu}^a(\gamma(t)) \frac{dx^\mu}{dt}(\gamma(t)) J_a \right\}.$$

(1.23)
\( \mathcal{T}^0[\gamma] \) is thus the trace of the SO(2,1) holonomy around \( \gamma \), while \( \mathcal{T}^1[\gamma] \) is essentially a cotangent vector to \( \mathcal{T}^0[\gamma] \): indeed, given a curve \( \omega(s) \) in the space of flat SO(2,1) connections, we can differentiate (1.22) to obtain

\[
\frac{d}{ds} \mathcal{T}^0[\gamma] = \int_\gamma \text{Tr} \left( U[\gamma, x(t)] \frac{d\omega^a}{ds}(\gamma(t)) J_a \right),
\]

(1.24)

and we have already seen that the derivative \( d\omega^a/ds \) can be identified with the triad \( e^a \).

In 3+1 dimensions, the variables \( \mathcal{T}^0 \) and \( \mathcal{T}^1 \) depend on particular loops \( \gamma \), and considerable work is still needed to construct diffeomorphism-invariant observables that depend only on knot classes. In 2+1 dimensions, on the other hand, the loop variables are already invariant, at least under diffeomorphisms isotopic to the identity. The key difference is that in 2+1 dimensions the connection \( \omega \) is flat, so the holonomy \( U[\gamma, x] \) depends only on the homotopy class of \( \gamma \). Some care must be taken in handling the mapping class group, which acts nontrivially on \( \mathcal{T}^0 \) and \( \mathcal{T}^1 \); this issue has been investigated in a slightly different context by Nelson and Regge [18].

Of course, \( \mathcal{T}^0 \) and \( \mathcal{T}^1 \) are not quite the equivalence classes of holonomies of our interpretation number two above: \( \mathcal{T}^0[\gamma] \) is not a holonomy, but only the trace of a holonomy. But knowledge of \( \mathcal{T}^0[\gamma] \) for a large enough set of homotopically inequivalent curves may be used to reconstruct a point in the space \( \mathcal{N} \) of equation (1.12), and indeed, the loop variables can serve as local coordinates on \( \mathcal{N} \) [19, 20, 21].

2. Geometric Structures: From Holonomies to Geometry

The central problem described in the introduction can now be made explicit. Spacetimes in 2+1 dimensions can be characterized à la Ashtekar, Rovelli, and Smolin as points in the cotangent bundle \( T^*\mathcal{N} \), our description number two of the last section. Such a description is fully diffeomorphism invariant, and provides a natural starting point for quantization. But our intuitive geometric picture of a (2+1)-dimensional spacetime is that of a manifold \( M \) with a flat metric — description number one — and only in this representation do we know how to connect the mathematics with ordinary physics. Our goal is therefore to provide a translation between these two descriptions.

To proceed, let us investigate the space of flat spacetimes in a bit more detail. If \( M \) is topologically trivial, the vanishing of the curvature tensor implies that \((M, g)\) is simply ordinary Minkowski space \((V^{2,1}, \eta)\), or at least to some subset of \((V^{2,1}, \eta)\) that can be extended to the whole of Minkowski space. If the spacetime topology is nontrivial, \( M \) can still be covered by contractible coordinate patches \( U_i \) that are each isometric to \( V^{2,1} \), with the standard Minkowski metric \( \eta_{\mu \nu} \) on each patch. The geometry is then encoded in the transition functions \( \gamma_{ij} \) on the intersections \( U_i \cap U_j \), which determine how these patches are glued together. Moreover, since the metrics in \( U_i \) and \( U_j \) are identical, these transition functions must be isometries of \( \eta_{\mu \nu} \), that is, elements of the Poincaré group ISO(2,1).

Such a construction is an example of what Thurston calls a geometric structure [22, 23, 24, 25], in this case a Lorentzian or \((\text{ISO}(2,1), V^{2,1})\) structure. In general, a \((G, X)\) manifold
is one that is locally modeled on $X$, just as an ordinary $n$-dimensional manifold is modeled on $\mathbb{R}^n$. More precisely, let $G$ be a Lie group that acts analytically on some $n$-manifold $X$, the model space, and let $M$ be another $n$-manifold. A $(G, X)$ structure on $M$ is then a set of coordinate patches $U_i$ covering $M$ with “coordinates” $\phi_i : U_i \to X$ taking their values in the model space and with transition functions $\gamma_{ij} = \phi_i \circ \phi_j^{-1} | U_i \cap U_j$ in $G$. While this general formulation may not be widely known, specific examples are familiar: for example, the uniformization theorem for Riemann surfaces implies that any surface of genus $g > 1$ admits an $(\mathbb{H}^2, \text{PSL}(2, \mathbb{R}))$ structure.

A fundamental ingredient in the description of a $(G, X)$ structure is its holonomy group, which can be viewed as a measure of the failure of a single coordinate patch to extend around a closed curve. Let $M$ be a $(G, X)$ manifold containing a closed path $\gamma$. We can cover $\gamma$ with coordinate charts

$$\phi_i : U_i \to X, \quad i = 1, \ldots, n$$

with constant transition functions $g_i \in G$ between $U_i$ and $U_{i+1}$, i.e.,

$$\phi_i | U_i \cap U_{i+1} = g_i \circ \phi_{i+1} | U_i \cap U_{i+1}$$

$$\phi_n | U_n \cap U_1 = g_n \circ \phi_1 | U_n \cap U_1.$$  

(2.2)

Let us now try to analytically continue the coordinate $\phi_1$ from the patch $U_1$ to the whole of $\gamma$. We can begin with a coordinate transformation in $U_2$ that replaces $\phi_2$ by $\phi_2' = g_1 \circ \phi_2$, thus extending $\phi_1$ to $U_1 \cup U_2$. Continuing this process along the curve, with $\phi_j' = g_1 \circ \ldots \circ g_{j-1} \circ \phi_j$, we will eventually reach the final patch $U_n$, which again overlaps $U_1$. If the new coordinate function $\phi_n' = g_1 \circ \ldots \circ g_n \circ \phi_n$ happens to agree with $\phi_1$ on $U_n \cap U_1$, we will have succeeded in covering $\gamma$ with a single patch. Otherwise, the holonomy $H$, defined as

$$H(\gamma) = g_1 \circ \ldots \circ g_n,$$ 

measures the obstruction to such a covering.

It may be shown that the holonomy of a curve $\gamma$ depends only on its homotopy class $[\gamma]$. In fact, the holonomy defines a homomorphism

$$H : \pi_1(M) \to G.$$  

(2.3)

Note that if we pass from $M$ to its universal covering space $\hat{M}$, we will no longer have noncontractible closed paths, and $\phi_1$ will be extendable to all of $\hat{M}$. The resulting map $D : \hat{M} \to X$ is called the developing map of the $(G, X)$ structure. At least in simple examples, $D$ embodies the classical geometric picture of development as “unrolling” — for instance, the unwrapping of a cylinder into an infinite strip.

The homomorphism $H$ is not quite uniquely determined by the geometric structure, since we are free to act on the model space $X$ by a fixed element $h \in G$, thus changing the transition functions $g_i$ without altering the $(G, X)$ structure of $M$. It is easy to see that such a transformation has the effect of conjugating $H$ by $h$, and it is not hard to prove that $H$ is in fact unique up to such conjugation $[22]$. For the case of a Lorentzian structure, where $G = \text{ISO}(2,1)$, we are thus led to a space of holonomies of precisely the form $[1, 19]$.

This identification is not a coincidence. Given a $(G, X)$ structure on a manifold $M$, it is straightforward to define a corresponding flat $G$ bundle $[24]$. To do so, we simply form
the product $G \times U_i$ in each patch — giving the local structure of a $G$ bundle — and use the transition functions $\gamma_{ij}$ of the geometric structure to glue together the fibers on the overlaps. It is then easy to verify that the flat connection on the resulting bundle has a holonomy group isomorphic to the holonomy group of the geometric structure.

We can now try to reverse this process, and use one the holonomy groups of equation (1.19) — approach number three to the field equations — to define a Lorentzian structure on $M$, reproducing approach number one. In general, this step may fail: the holonomy group of a $(G, X)$ structure is not necessarily sufficient to determine the full geometry. For spacetimes, it is easy to see what can go wrong. If we start with a flat three-manifold $M$ and simply cut out a ball, we can obtain a new flat manifold without affecting the holonomy of the geometric structure. This is a rather trivial change, however, and we would like to show that nothing worse can go wrong.

Mess [9] has investigated this question for the case of spacetimes with topologies of the form $\mathbb{R} \times \Sigma$. He shows that the holonomy group determines a unique “maximal” spacetime $M$ — specifically, a spacetime constructed as a domain of dependence of a spacelike surface $\Sigma$. Mess also demonstrates that the holonomy group $H$ acts properly discontinuously on a region $W \subset V^{2,1}$ of Minkowski space, and that $M$ can be obtained as the quotient space $W/H$. This quotient construction can be a powerful tool for obtaining a description of $M$ in reasonably standard coordinates, for instance in a time slicing by surfaces of constant mean curvature.

For topologies more complicated than $\mathbb{R} \times \Sigma$, I know of very few general results. But again, a theorem of Mess is relevant: if $M$ is a compact three-manifold with a flat, non-degenerate, time-orientable Lorentzian metric and a strictly spacelike boundary, then $M$ necessarily has the topology $\mathbb{R} \times \Sigma$, where $\Sigma$ is a closed surface homeomorphic to one of the boundary components of $M$. This means that for spatially closed three-dimensional universes, topology change is classically forbidden, and the full topology is uniquely fixed by that of an initial spacelike slice. Hence, although more exotic topologies may occur in some approaches to quantum gravity, it is not physically unreasonable to restrict our attention to spacetimes $\mathbb{R} \times \Sigma$.

To summarize, we now have a procedure — valid at least for spacetimes of the form $\mathbb{R} \times \Sigma$ — for obtaining a flat geometry from the invariant data given by Ashtekar-Rovelli-Smolin loop variables. First, we use the loop variables determine a point in the cotangent bundle $T^*\mathcal{N}$, establishing a connection to our second approach to the field equations. Next, we associate that point with an ISO(2,1) holonomy group $H \in \mathcal{M}$, as in our approach number four. Finally, we identify the group $H$ with the holonomy group of a Lorentzian structure on $M$, thus determining a flat spacetime of approach number one. In particular, if we can solve the (difficult) technical problem of finding an appropriate fundamental region $W \subset V^{2,1}$ for the action of $H$, we can write $M$ as a quotient space $W/H$.

This procedure has been investigated in detail for the case of a torus universe, $\mathbb{R} \times T^2$, in references [26] and [14]. For a universe containing point particles, it is implicit in the early descriptions of Deser et al. [27], and is explored in some detail in [10]. For the (2+1)-dimensional black hole, the geometric structure can be read off from references [28].
3. Quantization and Geometrical Observables

Our discussion so far has been strictly classical. I would like to conclude by briefly describing some of the issues that arise if we attempt to quantize (2+1)-dimensional gravity.

The canonical quantization of a classical system is by no means uniquely defined, but most approaches have some basic features in common. A classical system is characterized by its phase space, a \(2N\)-dimensional symplectic manifold \(\Gamma\), with local coordinates consisting of \(N\) position variables and \(N\) conjugate momenta. Classical observables are functions of the positions and momenta, that is, maps \(f, g, \ldots\) from \(\Gamma\) to \(\mathbb{R}\). The symplectic form \(\Omega\) on \(\Gamma\) determines a set of Poisson brackets \(\{f, g\}\) among observables, and hence induces a Lie algebra structure on the space of observables. To quantize such a system, we are instructed to replace the classical observables with operators and the Poisson brackets with commutators; that is, we are to look for an irreducible representation of this Lie algebra as an algebra of operators acting on some (normally \(N\)-dimensional) Hilbert space.

As stated, this program cannot be carried out: Van Hove showed in 1951 that in general, no such irreducible representation of the full Poisson algebra of classical observables exists \[33\]. In practice, we must therefore choose a subalgebra of “preferred” observables to quantize, one that must be small enough to permit a consistent representation and yet big enough to generate a large class of classical observables \[34\]. Ordinarily, the resulting quantum theory will depend on this choice of preferred observables, and we will have to look hard for physical and mathematical justifications for our selection.

In simple classical systems, there is often an obvious set of preferred observables — the positions and momenta of point particles, for instance, or the fields and their canonical momenta in a free field theory. For gravity, on the other hand, such a natural choice seems difficult to find. In 2+1 dimensions, where a number of approaches to quantization can be carried out explicitly, it is known that different choices of variables lead to genuinely different quantum theories \[35, 36, 37\].

In particular, each of the four interpretations of the field equations discussed above suggests its own set of fundamental observables. In the first interpretation — solutions as flat spacetimes — the natural candidates are the metric and its canonical momentum on some spacelike surface. But these quantities are not diffeomorphism invariant, and it seems that the best we can do is to define a quantum theory in some particular, fixed time slicing \[26, 38, 39\]. This is a rather undesirable situation, however, since the choice of such a slicing is arbitrary, and there is no reason to expect the quantum theories coming from different slicings to be equivalent.

\footnote{For the black hole, a cosmological constant must be added to the field equations. Instead of being flat, the resulting spacetime has constant negative curvature, and the geometric structure becomes an \((\mathbb{H}^{2,1}, \text{SO}(2,2))\) structure. A related result for the torus will appear in \[30\].}
In our second interpretation — solutions as classes of flat connections and their cotangents — the natural observables are points in the bundle $T^*\mathcal{N}$. These are diffeomorphism invariant, and quantization is relatively straightforward; in particular, the appropriate symplectic structure for quantization is just the natural symplectic structure of $T^*\mathcal{N}$ as a cotangent bundle. The procedure for quantizing such a cotangent bundle is well-established [2], and there seem to be no fundamental difficulties in constructing the quantum theory. But now, just as in the classical theory, the physical interpretation of the quantum observables is obscure.

It is therefore natural to ask whether we can extend the classical relationships between these approaches to the quantum theories. At least for simple topologies, the answer is positive. The basic strategy is as follows.

We begin by choosing a set of physically interesting classical observables of flat space-times. For example, it is often possible to uniquely foliate a spacetime by spacelike hypersurfaces of constant mean extrinsic curvature $\text{Tr}K$; the intrinsic and extrinsic geometries of such slices are useful observables with clear physical interpretations. Let us denote these variables generically as $Q(T)$, where the parameter $T$ labels the time slice on which the $Q$ are defined (for instance, $T = \text{Tr}K$).

Classically, such observables can be determined — at least in principle — as functions of the geometric structure, and thus of the $\text{ISO}(2,1)$ holonomies $\rho$,

$$Q = Q[\rho, T].$$

(3.1)

We now adopt these holonomies as our preferred observables for quantization, obtaining a Hilbert space $L^2(\mathcal{N})$ and a set of operators $\hat{\rho}$. Finally, we translate (3.1) into an operator equation,

$$\hat{Q} = \hat{Q}[\hat{\rho}, T],$$

(3.2)

thus obtaining a set of diffeomorphism-invariant but “time-dependent” quantum observables to represent the variables $Q$. Some ambiguity will remain, since the operator ordering in (3.2) is rarely unique, but in the examples studied so far, the requirement of mapping class group invariance seems to place major restrictions on the possible orderings [15].

This program has been investigated in some detail for the simplest nontrivial topology, $M = \mathbb{R} \times T^2$ [26, 35]. There, a natural set of “geometric” variables are the modulus $\tau$ of a toroidal slice of constant mean curvature $\text{Tr}K = T$ and its conjugate momentum $p_{\tau}$. These can be expressed explicitly in terms of a set of loop variables that characterize the $\text{ISO}(2,1)$ holonomies of the spacetime. Following the program outlined above, one obtains a one-parameter family of diffeomorphism-invariant operators $\hat{\tau}(T)$ and $\hat{p}_{\tau}(T)$ that describe the physical evolution of a spacelike slice. The $T$-dependence of these operators can be described by a set of Heisenberg equations of motion,

$$\frac{d\hat{\tau}}{dT} = i \left[ \hat{H}, \hat{\tau} \right], \quad \frac{d\hat{p}_{\tau}}{dT} = i \left[ \hat{H}, \hat{p}_{\tau} \right],$$

(3.3)

with a Hamiltonian $\hat{H}[\hat{\tau}, \hat{p}_{\tau}, T]$ that can again be calculated explicitly.
Let me stress that despite the familiar appearance of (3.3), the parameter $T$ is \emph{not} a time coordinate in the ordinary sense; the operators $\hat{\tau}(T)$ and $\hat{p}_r(T)$ are fully diffeomorphism invariant. We thus have a kind of “time dependence without time dependence,” an expression of dynamics in terms of operators that are individually constants of motion. For more complicated topologies, such an explicit construction seems quite difficult, although Unruh and Newbury have taken some interesting steps in that direction \cite{40}. Ideally, one would like to find some kind of perturbation theory for geometrical variables like $\hat{Q}$, but little progress has yet been made in this direction.

The specific constructions I have described here are unique to 2+1 dimensions, of course. But I believe that some of the basic features are likely to extend to realistic (3+1)-dimensional gravity. The quantization of holonomies is a form of “covariant canonical quantization” \cite{41,42}, or quantization of the space of classical solutions. We do not yet understand the classical solutions of the (3+1)-dimensional field equations well enough to duplicate such a strategy, but a similar approach may be useful in minisuperspace models. The invariant but $T$-dependent operators $\hat{\tau}(T)$ and $\hat{p}_r(T)$ are examples of Rovelli’s “evolving constants of motion” \cite{43}, whose use has also been suggested in (3+1)-dimensional gravity. Finally, our simple model has strikingly confirmed the power of the Rovelli-Smolin loop variables. A full extension to 3+1 dimensions undoubtedly remains a distant goal, but for the first time in years, there seems to be some real cause for optimism.

Acknowledgements

This work was supported in part by U.S. Department of Energy grant DE-FG03-91ER40674.

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