On global attractors for 2D damped driven nonlinear Schrödinger equations

A. I. Komech and E. A. Kopylova

Faculty of Mathematics of Vienna University, Institute for Transmission Information Problems of RAS, Moscow, Russia

ABSTRACT
The well-posedness and global attraction are proved for nonautonomous 2D nonlinear damped Schrödinger equations with almost periodic external source in a bounded region. The proofs rely on a development of the method of energy equation. The research is inspired by the problems of laser and maser coherent radiation.

ARTICLE HISTORY
Received 26 October 2020
Accepted 18 February 2021

CONTACT
A. I. Komech alexander.komech@univie.ac.at

1. Introduction
We consider the weakly damped driven nonlinear Schrödinger equation in a bounded region $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial \Omega$,

$$i\dot{\psi}(x,t) = -\Delta \psi(x,t) + f(\psi(x,t)) - i\gamma \psi(x,t) + p(x,t), \quad x \in \Omega, \quad t \in \mathbb{R},$$

(1)

where the 'friction coefficient' $\gamma > 0$, and the 'pumping' $p(\cdot, t)$ is an almost periodic function. All the derivatives here and below are understood in the sense of distributions. Equations of type (1) arise for instance in plasma physics [1] and in optical fibers models [2].

We impose the Dirichlet boundary condition

$$\psi(x,t) = 0, \quad x \in \partial \Omega, \quad t \in \mathbb{R}.$$  

(2)

Our analysis of the nonautonomous Schrödinger equation is motivated by mathematical problems of laser and maser coherent radiation [3,4]. In particular, the maser action is described by the damped driven nonlinear Maxwell–Schrödinger equations in a bounded cavity, and the Dirichlet boundary condition (2) holds in the case of a solid-state gain medium [3].
We will identify complex numbers \( z \in \mathbb{C} \) with two-dimensional real vectors \( z = (\text{Re} \, z, \text{Im} \, z) \). We assume that the nonlinear term is potential, i.e.

\[
f(\psi) = U'(\psi), \quad \psi \in \mathbb{C} \equiv \mathbb{R}^2.
\]

Here \( U(\psi) \in C^2(\mathbb{R}^2) \) is a real function, and the derivatives mean derivatives with respect to \( \psi_1 = \text{Re} \, \psi \) and \( \psi_2 = \text{Im} \, \psi \), i.e.

\[
f(\psi) := (\partial_1 U(\psi), \partial_2 U(\psi)), \quad \psi \in \mathbb{R}^2.
\]

Accordingly, equation (1) is an identity of \( \mathbb{R}^2 \)-valued distributions, and the multiplication by \( i \) in (1) means the application of the real \( 2 \times 2 \)-matrix

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

Similarly, \( f'(\psi) \) and \( f''(\psi) \) are tensors:

\[
f_{ij}^i(\psi) := \partial_j f_i(\psi_1, \psi_2) = \partial_j \partial_i U(\psi_1, \psi_2), \quad f_{ijk}^i(\psi) := \partial_j \partial_k f_i(\psi_1, \psi_2), \quad i, j, k = 1, 2.
\]

Note that \( f'(\psi) \) is a real symmetric matrix. The Schrödinger equation (1) under the boundary conditions (2) can be written in 'Hamiltonian form'

\[
i \dot{\psi}(t) = DH(\psi(t)) - i \gamma \psi(t) + p(\omega t), \quad t \in \mathbb{R},
\]

with the Hamilton functional

\[
\mathcal{H}(\psi) := \int_{\Omega} \frac{1}{2} |\nabla \psi(x)|^2 \, dx + U(\psi), \quad U(\psi) := \int_{\Omega} U(\psi(x)) \, dx.
\]

Our main results. For the damped driven Schrödinger equation (1) with almost-periodic pumping in a bounded region, we establish

(I). The well-posedness of the Cauchy problem for equation (1) (Theorem 3.1);

(II). Global attraction in the Sobolev norm \( H^1 \) of all finite energy solutions to a compact subset in \( H^1 \) (Theorem 4.2);

Let us comment on previous works in these directions. In the case \( p(x, t) \equiv 0 \) and \( \gamma = 0 \), the well-posedness for nonlinear Schrödinger equations of type (1) was established in [5]. The approach [5] relies essentially on the conservation of charge and energy.

Compact attractors in the energy space \( H^1 \) for autonomous damped nonlinear Schrödinger equations were constructed (i) on the circle \( \Omega = \mathbb{R}/\mathbb{Z} \) or a bounded interval \( \Omega \subset \mathbb{R} \) by Ghidaglia [6], (ii) on a bounded region \( \Omega \subset \mathbb{R}^2 \) by Abounouh and Goubet [7,8] and (iii) on the entire space \( \Omega = \mathbb{R}^N \) with \( N \leq 3 \) by Laurençot [9]. Similar results were obtained by Ghidaglia and Heron [10] for the Ginzburg–Landau equations on a bounded region \( \Omega \subset \mathbb{R}^n \) with \( n = 1, 2 \). The methods of these papers rely on the method of energy equation introduced by Ball [11]. Note that the pumping terms (or 'external force') in [7–10] do not depend on time. In [6], bounded absorbing sets are established for the pumping term depending on time, while the attractor is constructed for the autonomous case and for time-periodic pumping. The papers [12–14] deal with smoothness properties of functions from attractors of 1D nonlinear Schrödinger equations.

Tao [15] established the existence of a global attractor for radial solutions of nonlinear autonomous defocusing Schrödinger equation without damping in \( \mathbb{R}^n \) with \( n \geq 11 \). The paper [16] concerns the well-posedness and decay of solutions to the 2D damped Schrödinger equation in a bounded region.

Recently Cazenave and Han established the long-time behavior for nonlinear autonomous Schrödinger equation in \( \mathbb{R}^n \) with a nonlinear subcritical dissipation [17].

For nonautonomous PDEs, the theory of attractors was started by Haraux [18,19] in the context of nonlinear parabolic and damped wave equations with almost periodic external force. The method
of energy equation [11] was extended to nonautonomous PDEs first in [20] for 1D Schrödinger equations on the circle, and in [21] for nonautonomous KdV and 2D Navier-Stokes equations.

In [22], the theory of attractors of nonautonomous equations was profoundly developed by Chepyzhov and Vishik [22] in the context of Navier–Stokes equations, Ginzburg–Landau equations, damped nonlinear wave equations, and reaction–diffusion equations. A great deal of attention was focused there on the classes of translation-compact external forces. In [23], the theory [22] was extended to more general classes of admissible external forces, such as translation compact, time-regular, space-regular, and normal in the terminology of [23]. Attractors of nonautonomous Schrödinger equations with nonperiodic pumping were not considered previously.

Let us comment on our approach. First, we prove the existence of a weak solution, using the standard Galerkin approximations. Our choice of dimension two is motivated by the Trudinger inequality, which ensures uniqueness of the weak solution by Theorem 3.6.1 of [5]. Further, we develop the method of energy equation [11,20,21] to prove that (i) the weak solution is the strong solution, and (ii) the solution continuously depends on the initial state and the pumping.

To prove the convergence in $H^1$-norm of all finite energy solutions to a compact attractor, we extend Ball’s ideas [11] and the general theory [22] to nonlinear Schrödinger equations.

We consider almost-periodic pumping which is translation-compact in the terminology [22,23]. This condition is physically relevant in the problem of laser and maser coherent radiation, where the pumping is a mixture of harmonics with a discrete set of frequencies [3,4]. We suppose that this condition can be relaxed by using methods [23]. We do not touch the continuous dependence of the attractor on the pumping function, that will be considered elsewhere.

### 2. A priori estimates for smooth solutions

Here we prove a priori estimates for sufficiently smooth solutions to the Schrödinger equation (1). The estimates ensure the existence of bounded absorbing sets in the energy norm. We will get rid of the smoothness assumption in the next section. Our assumptions are as follows.

**Nonlinearity.** We will assume that the potential $U(\psi) \in C^3(\mathbb{C})$, satisfies the following estimates

$$\varphi_1 |\psi|^4 - b_1 \leq U(\psi) \leq C |\psi|^4,$$

$$\Re f(\psi) \overline{\psi} \geq \varphi_2 U(\psi) - b_2,$$

$$|f(\psi)| \leq \varphi_3 (1 + |\psi|^3),$$

$$|f'(\psi)| \leq \varphi_4 (1 + |\psi|^2),$$

$$f'(\psi) \geq -b_3,$$

$$|f''(\psi)| \leq \varphi_5 (1 + |\psi|),$$

with all $\varphi_l > 0$ and all $b_l \in \mathbb{R}$. Let us recall that $f'(\psi)$ in (11) and (12) denotes the matrix (5). For example, the potentials

$$U(\psi) = a_2 |\psi|^4 + a_1 |\psi|^2 + a_0$$

satisfy all these conditions if $a_k \in \mathbb{R}$ and $a_2 > 0$. Any smooth potential which differs from (14) only in a bounded domain $|\psi| \leq C$ also satisfies these conditions.

Denote $H^p := H^p(\Omega)$, $L^p := L^p(\Omega)$. Let us introduce the Hilbert phase space

$$E := H^1_0(\Omega), \quad \|\psi\|_E = \|\nabla \psi\|,$$

where $\| \cdot \|$ stands for the norm in $L^2$. Then $E^* = H^{-1}$.
The pumping. We will assume that
\[ p \in C_b(\mathbb{R}, E) \text{ is a uniform almost periodic function of } t \in \mathbb{R}, \quad p_0 := \|p\|_{C_b(\mathbb{R}, E)} < \infty. \]  
(16)
By definition (see Appendix of [24] for the case of scalar functions), a function \( p \in C_b(\mathbb{R}, E) \) is uniformly almost periodic if for any sequence \( t_k \in \mathbb{R} \) there is a subsequence \( t_{k^*} \) such that the translations \( p_{k^*}(t) := p(t + t_{k^*}) \) are uniformly converging,
\[ p_{k^*} \xrightarrow{C_b(\mathbb{R}, E)} p^*, \quad k^* \to \infty. \]  
(17)

2.1. Energy estimates

First we prove a priori estimates in the norm of the energy space \( E \).

Lemma 2.1: Let conditions (8)–(12) and (16) hold, and \( \psi(x, t) \in C^2(\overline{\Omega} \times \mathbb{R}) \) be a solution to (1) with \( \psi(\cdot, 0) \in E \). Then
\[ \|\psi(t)\|_E^2 \leq C_0 e^{-\alpha \pm t} + D_0, \quad \pm t \geq 0, \]  
(18)
where \( C_0 = C_0(\|\psi(0)\|_E), \alpha_\pm > 0, \) and \( D_0 \) depends on \( p_0 \) only.

Proof: Differentiating the Hamilton functional \( \mathcal{H}(t) := \mathcal{H}(\psi(t)) \) and using (6), we obtain the energy balance
\[ \dot{\mathcal{H}}(t) = \langle D\mathcal{H}(\psi(t)), \dot{\psi}(t) \rangle = \langle D\mathcal{H}(\psi(t)), -iD\mathcal{H}(\psi(t)) \rangle + \langle -\Delta \psi + f(\psi), -\gamma \psi(t) - ip(t) \rangle, \quad t \in \mathbb{R}, \]  
(19)
where the brackets stand for duality, which is the inner product in the real Hilbert space \( L^2(\Omega) \otimes \mathbb{R}^2 \). Hence the first term on the right-hand side vanishes. Therefore,
\[ \dot{\mathcal{H}}(t) = \langle -\Delta \psi(t) + f(\psi(t)), -\gamma \psi(t) - ip(t) \rangle \]  
\[ = -\gamma \langle \nabla \psi(t), \nabla \psi(t) \rangle - \gamma \langle f(\psi(t)), \psi(t) \rangle - \langle \nabla \psi(t), i\nabla p(t) \rangle - \langle f(\psi(t)), ip(t) \rangle, \quad t \in \mathbb{R}. \]  
(20)
According to (9),
\[ \langle f(\psi(t)), \psi(t) \rangle \geq \kappa_2 U(t) - b_2 |\Omega|, \quad U(t) := \int_{\Omega} U(\psi(x, t)) dx. \]  
(21)
Further, (8), (10), and (16) imply that
\[ |\langle f(\psi(t)), ip(t) \rangle| \leq \kappa_3 \int (1 + |\psi(x, t)|^3) |p(x, t)| dx \]  
\[ \leq \kappa_3 \|p(t)\|_{L^1(\Omega)} + \int \left( \frac{\gamma \kappa_1 \kappa_2}{2} |\psi(x, t)|^4 + C|p(x, t)|^4 \right) dx \leq \frac{\gamma \kappa_2}{2} U(t) + C(p_0) \]  
(22)
since \( p(\tau) \in E \subset L^4(\Omega) \) by the Sobolev embedding theorem in dimension two. Moreover, (16) gives
\[ |\langle \nabla \psi(t), i\nabla p(t) \rangle| \leq \frac{\gamma}{4} \|\nabla \psi(t)\|^2 + \frac{1}{\gamma} \|\nabla p(t)\|^2 \leq \frac{\gamma}{4} \|\nabla \psi(t)\|^2 + \frac{1}{\gamma} p_0. \]  
(23)
Now (20)–(23) imply
\[ \dot{\mathcal{H}}(t) \leq -\gamma \|\nabla \psi(t)\|^2 - \gamma \kappa_2 U(t) + \frac{\gamma}{4} \|\nabla \psi(t)\|^2 + \frac{\gamma \kappa_2}{2} U(t) + C_1(p_0). \]
\[ \leq -\alpha_+ \mathcal{H}(t) + C_1(p_0), \quad t \in \mathbb{R}, \] (24)

where \( \alpha_+ := \frac{\gamma}{2} \min(3, \frac{2}{\kappa_2}) > 0 \). Hence the Gronwall inequality gives
\[ \mathcal{H}(t) \leq C(\mathcal{H}(0))e^{-\alpha_+ t} + D(p_0), \quad t \geq 0. \] (25)

Now (8) implies (18) for \( t \geq 0 \).

For \( t \leq 0 \), the bound (18) follows by similar arguments applied to the time-reversed equation (1) for the function \( \varphi(t) = \psi(-t) \):
\[ \dot{\varphi}(t) = iD\mathcal{H}(\varphi(t)) + \gamma \varphi(t) + ip(-t), \quad t \in \mathbb{R}. \] (26)

Now for \( \mathcal{H}_-(t) := \mathcal{H}(\varphi(t)) \), similarly to (20), we obtain
\[ \dot{\mathcal{H}}_-(t) = \gamma \langle \nabla \varphi(t), \nabla \varphi(t) \rangle + \gamma \langle f(\varphi(t)), \varphi(t) \rangle + \langle \nabla \varphi(t), i\nabla p(-t) \rangle + \langle f(\varphi(t)), ip(-t) \rangle. \] (27)

In this case, the bound (21) becomes
\[ |\langle f(\varphi(t)), \varphi(t) \rangle| \leq \frac{2\kappa_3}{\kappa_1} \mathcal{U}(t) + C|\Omega|, \] (28)

which follows from (8) and (10). Now (27) together with (22), (23), and (28) imply
\[ \dot{\mathcal{H}}_-(t) \leq \alpha_- \mathcal{H}_-(t) + C, \quad t > 0, \] (29)

with some \( \alpha_- > 0 \). Hence the Gronwall inequality gives \( \mathcal{H}_-(t) \leq C(\mathcal{H}(0))e^{\alpha_- t} + C \) for \( t > 0 \), which implies (18) for \( t < 0 \) by (8).

### 3. Well-posedness

In this section, we prove the well-posedness of the Cauchy problem for the Schrödinger equation (1) in the energy class. The key role is played by the energy equation (20).

**Theorem 3.1:** Let conditions (8)–(12), (16) hold. Then

(i) For any initial state \( \psi(0) \in E \) equation (1) admits a unique solution \( \psi(t) \in C(\mathbb{R}, E) \cap C^1(\mathbb{R}, E^*) \), and the bounds (18) hold.

(ii) The energy equation (20) holds,
\[ \frac{d}{dt} \mathcal{H}(\psi(t)) = \langle -\Delta \psi(t) + f(\psi(t)), -\gamma \psi(t) - ip(t) \rangle, \quad t \in \mathbb{R}. \] (30)

(iii) The map \( S : (\psi(0), p) \mapsto \psi(\cdot) \) is continuous
\[ S : E \times C(\mathbb{R}, E) \to C(\mathbb{R}, E). \] (31)

#### 3.1. Weak solutions

First, we construct ‘weak solutions’.

**Lemma 3.2:** Let the conditions of Theorem 3.1 hold. Then, for any initial state \( \psi(0) \in E \), equation (1) admits a unique solution \( \psi(t) \in C(\mathbb{R}, E_w) \cap L^\infty_{\text{loc}}(\mathbb{R}, E) \cap W_{\text{loc}}^{1,\infty}(\mathbb{R}, E^*), \) and the bounds (18) hold.

**Proof:** The existence of a weak solution \( \psi(t) \in C(\mathbb{R}, E_w) \cap L^\infty_{\text{loc}}(\mathbb{R}, E) \cap W_{\text{loc}}^{1,\infty}(\mathbb{R}, E^*) \) follows by the Galerkin approximations. We recall this construction in Appendix 1, since we will use it in the proof.
of (30). The uniqueness of this solution in the case \( p(x, t) \equiv 0 \) and \( \gamma = 0 \) is deduced in Theorem 3.6.1 of [5] from the Trudinger inequality [25]. The proof of uniqueness for \( \gamma > 0 \) and \( p(x, t) \) satisfying (16) remains almost unchanged: we give the required modifications in Appendix 2. Uniqueness implies that the convergence (A5) actually holds for the entire sequence of Galerkin approximations: for any \( s < 1 \)
\[
\psi_m(t) \xrightarrow{C([-T, T], H^s)} \psi(t), \quad m \to \infty, \quad T > 0.
\] (32)
The uniform bounds (A2) hold for the Galerkin approximations \( \psi_m \). Hence the bounds (18) hold also for their limit \( \psi(t) \).

### 3.2. Energy equation

Let us prove the energy equation (30).

**Proposition 3.3:** Let the conditions of Theorem 3.1 hold. Then the energy equation (30) holds for any solution \( \psi(t) \in C(\mathbb{R}, E_w) \cap L^\infty_{\text{loc}}(\mathbb{R}, E) \cap W^{1, \infty}_{\text{loc}}(\mathbb{R}, E^*) \) to (1).

**Proof:** The identity (30) for smooth solutions can be rewritten as
\[
\frac{1}{2} \partial_t \| \nabla \psi(t) \|^2 + \gamma \| \psi(t) \|^2 + \partial_t U(\psi(t)) = \langle \nabla \psi(t), -i \nabla p(t) \rangle + \langle f(\psi(t)), -\gamma \psi(t) - i p(t) \rangle, \quad t \in \mathbb{R}.
\]
Equivalently,
\[
\frac{1}{2} \left[ e^{2\gamma t} \langle \nabla \psi(t), \nabla \psi(t) \rangle \right]_0^T = -[e^{2\gamma t} U(\psi(t))]_0^T + 2\gamma \int_0^T e^{2\gamma t} U(\psi(t)) \, dt - \int_0^T e^{2\gamma t} \langle f(\psi(t)), \gamma \psi(t) + i p(t) \rangle \, dt - \int_0^T e^{2\gamma t} \langle \nabla \psi(t), i \nabla p(t) \rangle \, dt, \quad T \in \mathbb{R}. \quad (33)
\]
We will deduce (33) by passage to the the limit in the corresponding equation for the Galerkin approximations. Namely, the Galerkin equation (A1) imply that for any \( T \in \mathbb{R} \)
\[
\frac{1}{2} \left[ e^{2\gamma t} \langle \nabla \psi_m(t), \nabla \psi_m(t) \rangle \right]_0^T = -[e^{2\gamma t} U(\psi_m(t))]_0^T + 2\gamma \int_0^T e^{2\gamma t} U(\psi_m(t)) \, dt - \int_0^T e^{2\gamma t} \langle f(\psi_m(t)), \gamma \psi_m(t) + i p_m(t) \rangle \, dt - \int_0^T e^{2\gamma t} \langle \nabla \psi_m(t), i \nabla p_m(t) \rangle \, dt. \quad (34)
\]

**Lemma 3.4:** (a) The right-hand side of (34) converges to the same of (33).
(b) The left-hand side of (33) is dominated by the limit of the left-hand side of (34).

**Proof:** It suffices to consider the case \( T > 0 \).
(a) The convergence (32) of the Galerkin approximations with any \( s < 1 \) implies that
\[
\psi_m(t) \xrightarrow{C([0, T], L^r)} \psi(t), \quad m \to \infty.
\] (35) with any \( r < \infty \) by the Sobolev embedding theorem in dimension two. Hence the convergence of the first three terms on the right-hand side of (34) to the same of (33) follows.
The last term of (34) converges to the last term of (33) by the Lebesgue theorem due to (1) the uniform bounds (A2), (2) the strong convergence \( p_m(t) \to p(t) \) and (3) the weak convergence \( \psi_m(t) \to \psi(t) \) for all \( t \geq 0 \). This weak convergence follows from (32) and (A2).

(b) The second assertion of the lemma follows from the weak convergence \( \psi_m(T) \to \psi(T) \), and the strong convergence \( \psi_m(0) := p_m \psi(0) \to \psi(0) \).

As a result, we obtain in the limit the energy inequality: the left-hand side of (33) is dominated by its right-hand side for every \( T \in \mathbb{R} \).

On the other hand, \( \psi(t) \in C(\mathbb{R}, E_w) \) by Lemma 3.2, and hence, \( \psi(T) \in E \). Therefore, the same lemma implies the existence of a solution \( \tilde{\psi}(t) \in C(\mathbb{R}, E_w) \cap L^\infty_{\text{loc}}(\mathbb{R}, E) \cap W^{1,\infty}_{\text{loc}}(\mathbb{R}, E^*) \) starting from the initial state \( \tilde{\psi}(T) = \psi(T) \in E \). The same arguments of Lemma 3.4 show that the right-hand side of (33), with \( \tilde{\psi} \) instead of \( \psi \), is dominated by its left-hand side.

Finally, the uniqueness of a solution in Lemma 3.2 implies that \( \tilde{\psi}(t) = \psi(t) \) for \( t \in \mathbb{R} \). Hence the integral energy equations (33) holds for \( t \in \mathbb{R} \). We conclude that (30) holds for a.a. \( t \in \mathbb{R} \) and in the sense of distributions. The proposition is proved. □

3.3. Strong solutions

Now we can prove that \( \psi(t) \in C(\mathbb{R}, E) \cap C^1(\mathbb{R}, E^*) \). Indeed, \( \psi(t) \in C(\mathbb{R}, E_w) \cap L^\infty_{\text{loc}}(\mathbb{R}, E) \) by Lemma 3.2. On the other hand, the energy equation (33) holds for all \( T \in \mathbb{R} \), and the right-hand side of (33) is a continuous function of \( T \in \mathbb{R} \) by the same arguments as in the proof of Lemma 3.4. Hence (33) implies that the norm \( \| \psi(t) \|_E \) is also a continuous function of \( t \in \mathbb{R} \). Therefore, \( \psi(t) \in C(\mathbb{R}, E) \), and the equation (1) together with Lemma A.1, (ii) and condition (16) imply that \( \tilde{\psi}(t) \in C(\mathbb{R}, E^*) \).

3.4. Continuous dependence

The continuity of the map (31) follows by similar arguments. Namely, let \( \psi(0), \psi^n(0) \in E, p, p^n \in C(\mathbb{R}, E) \) and

\[
\sup_n \| \psi^n(0) \|_E < \infty, \quad \psi^n(0) \to E \psi(0); \quad p^n \to p, \quad n \to \infty. \tag{36}
\]

Consider the corresponding unique solutions \( \psi^n(t) \in C(\mathbb{R}, E) \cap C^1(\mathbb{R}, E^*) \) to (1). Similarly to (32), we obtain for any \( s < 1 \) and any \( T > 0 \)

\[
\psi^n(t) \to C([-T,T], H^s) \psi(t), \quad n \to \infty. \tag{37}
\]

Hence the Banach theorem on weak compactness implies that

\[
\psi^n(t) \to E \psi(t), \quad n \to \infty, \quad t \in \mathbb{R}. \tag{38}
\]

On the other hand, the energy equation (33) implies the norm-convergence

\[
\| \psi^n(t) \|_E \to C(\mathbb{R}) \| \psi(t) \|_E, \quad n \to \infty
\]

by the same arguments as in the proof of Lemma 3.4, using the uniform bounds (18) and the uniform convergence (37). Hence

\[
\psi^n(t) \to C(\mathbb{R}, E) \psi(t), \quad n \to \infty.
\]

Theorem 3.1 is proved.
3.5. Weak continuity of the process

In conclusion, we prove a technical lemma which will be important in the next section.

Lemma 3.5: Let conditions (8)–(11) and (16) hold, and let 1/3 ≤ s < 1. Then, for any T > 0, the map $S(T, 0): (ψ(0), p) \mapsto ψ(T)$ is continuous from $H^s \times C([0, T], E)$ to $H^s$.

Proof: Let $ψ(0), ψ^n(0) \in E, p, p^n \in C([0, T], E)$ and

$$\sup_n \|ψ^n(0)\|_E < \infty, \quad ψ^n(0) \xrightarrow{H^s} ψ(0); \quad p^n \xrightarrow{C([0, T], E)→ p, \ n → ∞}. \tag{38}$$

Consider the corresponding unique solutions $ψ^n(t) \in C(\mathbb{R}, E) \cap C^1(\mathbb{R}, E^s)$ to

$$iψ^n(x, t) = -Δψ^n(x, t) - iγψ^n(x, t) + f(ψ^n(x, t)) + p^n(x, t), \quad x \in Ω, \quad t ∈ \mathbb{R}. \tag{39}$$

These trajectories are equicontinuous on $[0, T]$ with values in $H^s$. This follows from the interpolation inequality (A4) since $\sup_n \sup_{t∈[0, T]} \|ψ^n(t)\|_{H^s} < ∞$ by (18), while $\sup_n \sup_{t∈[0, T]} \|ψ^n(t)\|_{E^s} < ∞$ by equations (39) together with Lemma A.1, (ii) and (16).

Now the Arzelà–Ascoli theorem implies that there exists a subsequence $ψⁿ’$ converging in $C([0, T], H^s)$:

$$ψⁿ’ \xrightarrow{C([0, T], H^s)} \tilde{ψ}, \quad n’ → ∞. \tag{40}$$

The limit function $\tilde{ψ}(t) \in C([0, T], H^s)$ satisfies the Schrödinger equation (1). This follows by the passage to the limit in the equation (39), taking into account Lemma A.1, i) and the uniform convergence of $p^n$ in (38). Moreover, $\tilde{ψ}(0) = ψ(0)$ by the convergence of $ψ^n(0)$ in (38). Hence the uniqueness arguments as above imply that $\tilde{ψ}(t) ≡ ψ(t)$ for all $t ∈ [0, T]$. Therefore, the continuity of $S(T, 0)$ follows from (40) since the limit does not depend on the subsequence $n’$.

4. Global attractor

In this section, we prove the convergence in the $H^1$-norm of all finite energy solutions to a compact attractor. Proposition 3.1,(i) allows us to define the continuous process $S_p(t, τ) \in E$ acting by

$$S_p(t, τ) : ψ(τ) ↦ ψ(t), \tag{41}$$

where $ψ(t) \in C(\mathbb{R}, E) \cap C^1(\mathbb{R}, E^s)$ is a solution to (1). The bounds (18) imply that the process admits a bounded uniform absorbing set in $H^1$. Indeed, let $B_R$ denote the ball $\{ψ ∈ E : \|ψ\|_E ≤ R\}$. Denote $\hat{B} := B_{\sqrt{D+1}}$, where $D$ is the constant from (18). Then (18) implies that

$$S_p(t, τ)B_R ⊂ \hat{B}, \quad t - τ > \frac{\log A(R)}{α}. \tag{42}$$

Now we prove the existence of a uniform compact attractor in $H^1$. Let us recall the definition of uniform attractor for nonautonomous equations, see, e.g. [19], [26, Definition A2.3], [27, Definition (7)].

Definition 4.1: A set $A ⊂ E$ is called a uniform attractor of the process $S_p(t, τ)$ if it is a uniformly attracting set and if any other closed attracting set contains $A$.

The term ‘uniformly attracting’ means that, for any bounded subset $B ⊂ E$,

$$\text{dist}_E(S_p(t, τ)B, A) → 0, \quad t - τ → ∞. \tag{42}$$
**Theorem 4.2:** Let the assumptions of Theorem 3.1 hold, \( \gamma > 0 \), and let the pumping \( p(t) \) be an almost periodic function with values in \( E \). Then the process \( S_p(t, \tau) \) admits a compact uniform attractor in \( E \).

For the proof, we develop the strategy [11] for a nonautonomous equation. Namely, we define the functionals (cf. (2.3) and (2.4) of [9] and (2.12), (2.25) of [20])

\[
\Phi(\psi, t) := \mathcal{H}(\psi) + \langle p(t), \psi \rangle, \quad \Psi(\psi, t) := \mathcal{U}(\psi) - \frac{1}{2} \langle f(\psi), \psi \rangle + \frac{1}{2} \langle p(t), \psi \rangle, \quad \psi \in E, \quad t \geq 0.
\]

(43)

**Lemma 4.3:** For any \( s \geq 1/2 \) and any \( t \geq 0 \), the functionals \( \mathcal{U}(\cdot) \) and \( \Psi(\cdot, t) \) are continuous on \( H^s \) and bounded on bounded subsets.

**Proof:** Using (10) and (11), we obtain

\[
|\mathcal{U}(\psi_1) - \mathcal{U}(\psi_2)| \leq C \int_{\Omega} (1 + |\psi_1(x)|^3 + |\psi_2(x)|^3)|\psi_1(x) - \psi_2(x)|dx,
\]

\[
|\langle f(\psi_1), \psi_1 \rangle - \langle f(\psi_2), \psi_2 \rangle| \leq \int_{\Omega} (1 + |\psi_1(x)|^2 + |\psi_2(x)|^2)|\psi_1(x) - \psi_2(x)||\psi_1|dx
\]

\[
\leq C \left( \int_{\Omega} (1 + |\psi_1(x)|^3 + |\psi_2(x)|^3)|\psi_1(x) - \psi_2(x)|dx \right)
\]

\[
\leq C_1 \int_{\Omega} (1 + |\psi_1(x)|^3 + |\psi_2(x)|^3)|\psi_1(x) - \psi_2(x)|dx
\]

Applying the Hölder inequality and the Sobolev embedding theorem, we obtain for any \( s \geq 1/2 \)

\[
|\mathcal{U}(\psi_1) - \mathcal{U}(\psi_2)| + |\langle f(\psi_1), \psi_1 \rangle - \langle f(\psi_2), \psi_2 \rangle| \leq C(1 + \|\psi_1\|_{L^4}^3 + \|\psi_2\|_{L^4}^3)\|\psi_1 - \psi_2\|_{L^4}
\]

\[
\leq C(1 + \|\psi_1\|_{H^s}^3 + \|\psi_2\|_{H^s}^3)\|\psi_1 - \psi_2\|_{H^s}.
\]

Similarly,

\[
|\langle p(t), \psi_1 - \psi_2 \rangle| \leq \|p(t)\|\|\psi_1 - \psi_2\| \leq C_0 \|\psi_1 - \psi_2\|_{H^s},
\]

for any \( s \geq 0 \).

Let \( \psi(t) \in C(\mathbb{R}, E) \cap C^1(\mathbb{R}, E^*) \) be a solution to (1). Now the energy equation (30) implies

\[
\frac{d}{dt} \Phi(\psi(t), t) = -2\gamma \Phi(\psi(t), t) + 2\gamma \Psi(\psi(t), t) + \langle \dot{p}(t), \psi(t) \rangle, \quad t > 0,
\]

(44)

where the last term is well defined in the sense of distributions: for any test function \( \varphi \in C^\infty(\mathbb{R}_+) \)

\[
\int_0^t \langle \dot{p}(s), \psi(s) \rangle \varphi(s) ds := [\langle p(s), \psi(s) \rangle \varphi(t)]_0^t - \int_0^t \langle p(s), \psi(s) \rangle \varphi(s) + \langle p(s), \psi(s) \rangle \dot{\varphi}(s) ds.
\]

(45)

To prove (44), we differentiate using (1) and (30),

\[
\frac{d}{dt} \Phi(\psi(t), t) - \langle \dot{p}(t), \psi(t) \rangle
\]

\[
= \mathcal{H}(\psi(t)) + \langle \dot{\psi}(t), p(t) \rangle
\]
In particular, for $E$ inclusion and an element $\phi(\psi)$ strongly converging in $H^s$ strongly converging in $Hs$.

Proposition 4.4: For any sequences $t_k, \tau_k \to \infty$ with $t_k - \tau_k \to \infty$ and $\phi_k \in \hat{B}$, there is a subsequence $k^*$ and $\phi \in E$ such that

$$S_p(t_k^*, \tau_k^*)\phi_k^* \to E \phi, \quad k^* \to \infty.$$ \hspace{1cm} (47)

Proof: The sequence $S_p(t_k, \tau_k)\phi_k$ is bounded in $E$ by (18). Let us fix an arbitrary $s \in [1/2, 1)$. The inclusion $E \subset H^s$ is compact by the Sobolev embedding theorem. Hence there is a subsequence strongly converging in $H^s$:

$$S_p(t_k', \tau_k')\phi_{k'} \to H^s \phi, \quad k' \to \infty.$$ \hspace{1cm} (48)

Now to prove (47) it suffices to check that for a subsequence $\{k^*\} \subset \{k'\}$, we have

$$\lim_{k^* \to \infty} \|S_p(t_{k*}, \tau_{k*})\phi_{k*}\|_E \leq \|\phi\|_E.$$ \hspace{1cm} (49)

Recall that $t_k - \tau_k \to \infty$. Hence similarly to (48), for any $T > 0$, there exists a subsequence $\{k''\} \subset \{k'\}$ and an element $\phi(-T) \in E$ such that

$$S_p(t_{k''} - T, \tau_{k''})\phi_{k''} \to H^s \phi(-T), \quad k'' \to \infty.$$ \hspace{1cm} (50)

We set $p_{k''}(t) := p(t_{k''} - T + t)$ and denote

$$\psi_{k''}(t) := S_p(t_{k''} - T + t, \tau_{k''})\phi_{k''} = S_{p_k^{'}}(t, 0)S_p(t_{k''} - T, \tau_{k''})\phi_{k''}, \quad t \in [0, T].$$ \hspace{1cm} (51)

In particular, for $t = T$ the convergence (48) implies that

$$\psi_{k''}(T) := S_p(t_{k''}, \tau_{k''})\phi_{k''} \to H^s \phi, \quad k'' \to \infty.$$ \hspace{1cm} (52)

Now we are going to apply Lemma 3.5(iv). At this step, we need the almost-periodicity (17) of the pumping $p(t)$. It implies that for a subsequence $\{k^*\} \subset \{k''\}$, we have

$$p_{k^*} \to C_p(\mathbb{R}, E) \quad p^*, \quad k^* \to \infty.$$ \hspace{1cm} (53)

Hence (50), (51) and the continuity of the map (31) imply the convergence for $t \in [0, T]$,

$$\psi_{k^*}(t) = S_{p_{k^*}}(t, 0)S_p(t_{k^*} - T, \tau_{k^*})\phi_{k^*} \to H^s \psi^*(t) := S_{p^*}(t, 0)\phi(-T), \quad k^* \to \infty.$$ \hspace{1cm} (54)
In particular, for $t = T$ the convergence (52) gives that

$$
\psi^*(T) := S_p(T, 0) \phi(-T) = \phi.
$$

(55)

Now we apply the integral identity (46) to solutions $\psi_{k^*}(t)$ and $\psi^*(t)$ of Equation (1) with the pumping $p_{k^*}(t)$ and $p^*(t)$ respectively. We obtain

$$
\Phi(\psi_{k^*}(T), T) = e^{-2\gamma T} \Phi(\psi_{k^*}(0), 0) + \int_0^T e^{2\gamma(T-\tau)} [2\gamma \Psi(\psi_{k^*}(\tau), \tau) + \langle \dot{p}_{k^*}(\tau), \psi_{k^*}(\tau) \rangle]d\tau
$$

(56)

and

$$
\Phi(\psi^*(T), T) = e^{-2\gamma T} \Phi(\psi^*(0), 0) + \int_0^T e^{2\gamma(T-\tau)} [2\gamma \Psi(\psi^*(\tau), \tau) + \langle \dot{p}^*(\tau), \psi^*(\tau) \rangle]d\tau,
$$

(57)

where the integrals with $\dot{p}_{k^*}(\tau)$ and $\dot{p}(\tau)$ are definite according to (45). Letting $k^* \to \infty$ in (56), we obtain

$$
\limsup_{k^* \to \infty} \Phi(\psi_{k^*}(T), T) \leq Ce^{-2\gamma T} + \int_0^T e^{2\gamma(T-\tau)} [2\gamma \Psi(\psi^*(\tau), \tau) + \langle \dot{p}^*(\tau), \psi^*(\tau) \rangle]d\tau
$$

(58)

by the following arguments:

(i) $\sup_{k^*} \|\psi_{k^*}(0)\|_E < \infty$, while the functionals $\Phi(\cdot, 0)$ are bounded on bounded subsets of $E$,

(ii) $\psi_{k^*}(\tau) \rightharpoonup \psi^*(\tau)$ by (54), while the functionals $\Psi(\cdot, \tau)$ are continuous on $H^s$ by Lemma 4.3, since $s \in [1/2, 1)$, and

(iii) (53) implies the convergence of the integrals with $\dot{p}_{k^*}(\tau)$ by their definition (45).

Now eliminating the integral term from (57) and (58), we obtain

$$
\limsup_{k^* \to \infty} \Phi(\psi_{k^*}(T), T) \leq Ce^{-2\gamma T} + \Phi(\psi^*(T), T) - e^{-2\gamma T} \Phi(\psi^*(0), 0).
$$

(59)

By definitions (7) and (43),

$$
\Phi(\psi_{k^*}(T), T) = \frac{1}{2} \|\psi_{k^*}(T)\|_E + \mathcal{U}(\psi_{k^*}(T)) + \langle p_{k^*}(T), \psi_{k^*}(T) \rangle,
$$

$$
\Phi(\psi^*(T), T) = \frac{1}{2} \|\psi^*(T)\|_E + \mathcal{U}(\psi^*(T)) + \langle p^*(T), \psi^*(T) \rangle,
$$

Now Lemma 4.3 together with (48), (52), and (55) imply that

$$
\lim_{k^* \to \infty} \mathcal{U}(\psi_{k^*}(T)) = \mathcal{U}(\phi) = \mathcal{U}(\psi^*(T)), \quad \lim_{k^* \to \infty} \langle p_{k^*}(T), \psi_{k^*}(T) \rangle = \langle p^*(T), \phi \rangle = \langle p^*(T), \psi^*(T) \rangle.
$$

Hence (59) becomes

$$
\frac{1}{2} \limsup_{k^* \to \infty} \|\psi_{k^*}(T)\|_E \leq Ce^{-2\gamma T} + \frac{1}{2} \|\phi\|_E - e^{-2\gamma T} \Phi(\psi^*(0), 0).
$$

According to the definition (52), we can replace $\psi_{k^*}(T)$ by $S_p(t_{k^*}, \tau_{k^*})\phi_{k^*}$. Now letting $T \to \infty$, we obtain (49) since $\gamma > 0$. ■
4.2. Compactness

Let us denote by \( A \) the set of all points \( \phi \) from (47). This set is obviously closed. Now Theorem 4.2 will follow from the next lemma.

**Lemma 4.5:** (i) The set \( A \) is compact in \( E \), and (ii) the set \( A \) is uniformly attracting in \( E \).

**Proof:** (i) Let us consider a sequence \( \phi_n \in A \). Then, for each \( n \),

\[
S_p(t_{nk}, \tau_{nk})\phi_{nk} \xrightarrow{E} \phi_n, \quad t_{nk} - \tau_{nk} \to \infty
\]

as \( k \to \infty \), where \( \phi_{nk} \in \hat{B} \). Hence there exists a sequence \( k(n) \) such that

\[
\|S_p(t_{nk(n)}, \tau_{nk(n)})\phi_{nk(n)} - \phi_n\|_E \to 0, \quad t_{nk(n)} - \tau_{nk(n)} \to \infty
\]

as \( n \to \infty \). However, Proposition 4.4 implies that for a subsequence \( n' \), we have

\[
S_p(t_{n'k(n')}, \tau_{n'k(n')})\phi_{n'k(n')} \xrightarrow{E} \overline{\phi}, \quad n' \to \infty.
\]

Hence \( \overline{\phi} \in A \), and (61) implies that \( \phi_n' \xrightarrow{E} \overline{\phi} \). Therefore, the first assertion of the lemma is proved.

(ii) Let us assume the contrary. Then there exists a sequence \( S(t_k, \tau_k)\phi_k \) such that

\[
\text{dist}_E(S(t_k, \tau_k)\phi_k, A) \geq \varepsilon > 0, \quad t_k - \tau_k \to \infty.
\]

On the other hand, Proposition 4.4 implies that for a subsequence \( k' \), we have

\[
S(t_{k'}, \tau_{k'})\phi_{k'} \xrightarrow{E} \phi \in A, \quad k' \to \infty.
\]

This contradiction proves the second assertion of the lemma.

|Disclosure statement|

No potential conflict of interest was reported by the author(s).

**Funding**

The research is supported by the Austrian Science Fund (FWF) [grant numbers P28152–N35 and P34177–N].

**References**

[1] Nozaki K, Bekki N. Low-dimensional chaos in a driven damped nonlinear Schrödinger equation. Phys D. 1986;21:381–393.
[2] Blow KJ, Doran NJ. Global and local chaos in the pumped nonlinear Schrödinger equation. Phys Rev Lett. 1984;52:526–529.
[3] Haken H. Laser theory. Berlin: Springer; 1984.
[4] Nussenzveig H. Introduction to quantum optics. London: Gordon and Breach; 1973.
[5] Cazenave T. Semilinear Schrödinger equations. New York: AMS; 2003.
[6] Ghidaglia JM. Finite-dimensional behaviour for weakly damped driven Schrödinger equations. Ann Inst Henri Poincaré. 1988;5:365–405.
[7] Abounouh M. Asymptotic behavior for a weakly damped Schrödinger equation in dimension two. Appl Math Lett. 1993;6:29–32.
[8] Abounouh M, Goubet O. Attractor for a damped cubic Schrödinger equation on a two-dimensional thin domain. Differ Integral Equ. 2000;13(1–3):311–340.
Appendices

Appendix 1. Galerkin approximations

Denote by $E^m$, $m = 1, 2, \ldots$, the linear span of the first $m$ eigenfunctions of the Laplacian $\Delta$ on the region $\Omega$ with the Dirichlet boundary conditions (2). We define the Galerkin approximations $\psi_m(t)$ as solutions to the following finite-dimensional dissipative equations in $E^m$

$$i\dot{\psi}_m(t) = -\Delta \psi_m(t) - i\gamma \psi_m(t) + P_m f(\psi_m(t)) + p_m(t), \quad t \in \mathbb{R}; \quad \psi_m(0) = P_m \psi(0).$$

(A1)

Here $P_m$ stands for the orthogonal projection of $L^2$ onto $E^m$, and $p_m(t) := P_m p(t)$. Applying the calculations from the proof of Lemma 2.1 to Equation (A1), we obtain the uniform estimates of type (18) for $\psi_m$ with the same constants:

$$\|\psi_m(t)\|_{E^m} \leq C(\|\psi(0)\|_E)e^{-\alpha t} + D, \quad t \in \mathbb{R}.$$  

(A2)

Hence, the Galerkin approximations $\psi_m(t)$ exist globally in time.

Let us show that these uniform estimates imply the existence of a limiting function $\psi(t)$ of the Galerkin approximations. For this purpose, we will use the known continuity property of the nonlinear term:

**Lemma A.1:** Let nonlinear function $f(\psi)$ satisfy condition (11). Then

(i) For any $r \geq 1$ and $s \geq 1 - \frac{2}{5r}$, the nonlinearity $N : \psi(\cdot) \mapsto f(\psi(\cdot))$ is a continuous map $H^r \to L^r$.

(ii) $N : \psi(\cdot) \mapsto f(\psi(\cdot))$ is a continuous map $E \to E^*$.
(iii) $N$ is a continuous map $E_1 \rightarrow L^2$.

**Proof:** (i) Condition (11) implies that
\[
|f(\psi_1(x)) - f(\psi_2(x))| \leq C(1 + |\psi_1(x)|^2 + |\psi_2(x)|^2)|\psi_1(x) - \psi_2(x)|.
\]
Hence, the Hölder inequality and the Sobolev embedding theorem give, for $r \geq 1$,
\[
\|f(\psi_1) - f(\psi_2)\|_{L^r} \leq C\left(1 + \|\psi_1\|_{L^{2r}}^2 + \|\psi_2\|_{L^{2r}}^2\right)\|\psi_1 - \psi_2\|_{L^{2r}}
\leq C(1 + \|\psi_1\|_{H^r}^2 + \|\psi_2\|_{H^r}^2)\|\psi_1 - \psi_2\|_{H^r}.
\] (A3)

(ii) In particular, (A3) implies that $N$ is continuous $H^{2/3} \rightarrow L^2$. It remains to note that $E \subset H^{2/3}$, while $L^2 \subset E^\ast$. (iii) The map $N : E_1 \rightarrow C(\Omega)$ is continuous by the Sobolev embedding theorem.

Further, the Galerkin approximations are equicontinuous in $H^s$ with any $s < 1$ by the Dubinsky 'Theorem on Three Spaces' ([28, Theorem 5.1]). Namely, this equicontinuity follows from the interpolation inequality: for any $C$ convergence (A5) with $\|\psi\|_{E^\ast} \leq \|\psi\|_{H^s}$ and $\|\psi\|_{H^s} \leq C\|\psi\|_{E^\ast}$.

The limit function $\psi(t) \in C(\Omega, H^s)$ satisfies the estimates (18) by the uniform bounds (A2), and hence, $\psi(t) \in C(\Omega, E^\ast) \cap L^{\infty}_{loc}(\Omega, E)$.

The limit function $\psi(t)$ satisfies the Schrödinger equation (1) in the sense of distributions. This follows from the convergence (A5) with $s < 1$ close to 1 by passage to the limit in the Galerkin equation (A1) taking into account Lemma A.1. Finally, the convergence $\psi_m(0) \rightarrow \psi(0)$ as $m \rightarrow \infty$ implies that $\psi(0) = \psi(0)$.

Finally, the equation (1) implies that $\psi(t) \in W^{1,\infty}_{loc}(\Omega, E^\ast)$ by Lemma A.1, (ii).

**Appendix 2. Uniqueness**

Here we reduce the proof of the uniqueness of the solution to (1) with $\gamma > 0$ and fixed $\psi(0) \in E$ to the case $\gamma = 0$ considered in Section 3.6 of [5]. Namely, let $\psi_1, \psi_2 \in L^{\infty}_{loc}(\Omega, E) \cap W^{1,\infty}_{loc}(\Omega, E^\ast)$ be two solutions to (1) with the same initial state $\psi_1(0) = \psi_2(0)$. Then the difference $z = \psi_1 - \psi_2$ satisfies the equation
\[
iz(x, t) = -\Delta z(x, t) - i\gamma z(x, t) + f(\psi_1(x, t)) - f(\psi_2(x, t)), \quad x \in \Omega, \quad t \in \mathbb{R}.
\]
This equation implies
\[
\frac{d}{dt}\|z(t)\|_{L^2}^2 + 2\gamma \|z(t)\|_{L^2}^2 = 2\left(f(\psi_1(t)) - f(\psi_2(t)), z(t)\right), \quad t \in \mathbb{R}.
\] (A6)

Equivalently, we have the equation
\[
\frac{d}{dt}\|z(t)\|_{L^2}^2 = 2e^{2\gamma t}\left(f(\psi_1(t)) - f(\psi_2(t)), z(t)\right), \quad t \in \mathbb{R}.
\] (A7)

Finally, denoting $h(t) = |\psi_1(t)| + |\psi_2(t)|$, $w(t) = e^{\gamma t}z(t)$ and using (11), we obtain the inequality
\[
\frac{d}{dt}\|w(t)\|_{L^2}^2 \leq C \int_{\Omega} (1 + h^2(t))|w(x, t)|^2 dx, \quad t \in \mathbb{R},
\] (A8)

which coincides with the corresponding inequality in Section 3.6 of [5]. Further, using the Trudinger inequality as in [5], we obtain $w(t) \equiv 0$.