Scale-invariant Perturbation of the Friedmann Universe

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We use de Vaucouleurs’ power-law density-distance relation, to study a hierarchical perturbation of the Friedmann universe. We solve the Einstein equation and obtain the density contrast and the amplification factor for the perturbation. It is shown that, scale-invariant inhomogeneities decay in Einstein-de-Sitter universe. On the contrary, in an open universe, the inhomogeneities grow. For low values of Ω, amplification peaks sharply and the fluctuations can grow by up to a factor of $10^{13}$ from the recombination to the present time. Our analysis of the closed universe further confirms that, unlike the common belief that perturbations grow faster with increasing Ω, scale-invariant perturbations amplify with decreasing Ω. This result is consistent with the very low density expected for a hierarchical universe.

It is unanimously agreed that, up to a certain scale, the distribution of matter in the universe is inhomogeneous \cite{1}. However, the scale at which the distribution shifts from a power law to a random behaviour is a subject of strong controversy \cite{2,3}. For scales up to 100 pc, it is well-established that the interstellar medium obeys a scaling law \cite{4}. Further up the scale to 150 Mpc, numerous red shift surveys indicate the existence of large voids and filaments \cite{5}. It is argued that these structures have a fractal distribution \cite{6}. These facts have brought doubts about the hypothesis of the homogeneity, which is at the wellspring of the standard cosmology \cite{7,8}.

It remains an ambitious task to study the dynamics of a fractal universe. However, analytic inhomogeneous universes and also, perturbatively, the evolution of inhomogeneities in a homogeneous background have been studied extensively. De Vaucouleurs was among the first to study a hierarchical distribution and the possibility of a universal density-distance power law in cosmology \cite{9}. The relativistic inhomogeneous cosmological models and specially Tolman spacetime have also been studied widely \cite{10,11}. Tolman’s dust solution has been used to model a relativistic hierarchical (fractal) cosmology compatible with the observational analysis of the redshift surveys \cite{12}. In the perturbative approach, the growth of small seeds of inhomogeneities is studied mainly in the context of the structure formation \cite{13,14,15}. The large scale structure of the universe is believed to have grown, due to gravitational instability, from small primordial density perturbations which are fully characterized by the density contrast.

In this article, we attempt to bring these ideas together. A fractal structure is necessarily scale-invariant, eventhought the reverse statement is not true. We consider a scale-invariant spherically symmetric inhomogeneous, but isotropic, universe which allows a non-vanishing pressure. The metric is the Friedmann-Robertson-Walker metric with radial-dependent scale factor and is contained in the Tolman solution for dust. To solve the Einstein equation, we expand the scale factor around the Friedmann scale factor by a small scale-invariant fluctuation. The choice of a scale-invariant function for the density perturbation is inspired by de Vaucouleurs’ original work on modeling a hierarchical universe \cite{16}. We take his power-law density-distance relation to represent our perturbation and aim at establishing if such a density fluctuation has any chance of growing into large scale structures we observe today. Although the growth of a scale-invariant seed does not imply a fully scale-invariant or a fractal universe, its decay would rule out such distributions as viable cosmological models.

To account for the inhomogeneity of space-time, we use the spherically symmetric metric,

$$\text{diag}(g_{\mu \nu}) = \begin{pmatrix} -1, & R_p^2(t, r) \frac{R_p^2(t, r)}{1 - kr^2}, & R_p^2(t, r)r^2, & R_p^2(t, r)r^2 \sin^2 \theta \end{pmatrix},$$

(1)

where the scale factor $R_p(t, r)$ is a function of both time and coordinate. The function $(1 - kr^2)^{-1}$ multiplying $R_p^2(t, r)$ allows for the Friedman-Robertson-Walker metric to be recovered when $R_p$ is space-coordinate independent. We assume that the scale factor differs from the background homogeneous scale factor $R(t)$ by a scale-invariant perturbation term and write

$$R_p(t, r) = R(t) + \delta R(t)r^{-\gamma},$$

(2)

where the comoving radial coordinate has been used. The value of the fractal codimension $\gamma$ has been recently proposed to be about 1 \cite{16} whereas it was originally believed to be approximately 2 \cite{17}. Similarly, we expand around the homogeneous background matter distribution $\rho$ with the scale-invariant fluctuation to obtain the density function

$$\rho_p(t, r) = \rho(t) + \delta\rho(t)r^{-\gamma},$$

(3)

and the radial pressure
\[ P_r(t, r) = \omega \rho(t) + a \delta \rho(t) r^{-\gamma}, \quad (4) \]

where \( \alpha \) is not necessarily equivalent to \( \omega \). For practical purposes, however, it serves to consider an approximately adiabatic perturbation where \( \alpha \) is almost equal to \( \omega \). In general, \( \alpha = \omega = v_s^2 \), where \( v_s \) is the velocity of sound which is assumed to be unaffected by the perturbation \[ \ref{3} \] \[ \ref{11} \].

The expression \[ \ref{2} \] for the scale factor is chosen because it subsequently implies \[ \ref{3} \] for the density, which has the same form as that suggested by de Vaucouleurs \[ \ref{12} \]. The radial dependence of the scale factor leads to the same radial dependence for the density through Einstein equation. This scale-invariant behaviour is transferred to the pressure through the equation of state. Furthermore, the scale factor and the density perturbations are required, by the energy-momentum conservation equation, to have the same fractal codimension.

The perturbation of the scale factor, \[ \ref{3} \], is a genuine perturbation of the metric and not just a gauge mode. It is not difficult to see that a solution \( \xi_\mu \) of \( \delta g_{\mu\nu} = \xi_\mu \xi_\nu + \xi_\nu \xi_\mu \), such that \( \delta g_{0r} = 0 \) and \( \delta g_{r\rho} \sim t^\alpha r^{-\gamma} \) for a general \( a \), cannot be constructed \[ \ref{10} \].

Our perturbation is only valid for comoving distances larger than the correlation length, where the distribution is homogeneous and an average density is defined. The correlation length has the controversial minimum value of 5 Mpc and maximum value of 25 Mpc for some of the 3D galaxy catalogues \[ \ref{4} \] \[ \ref{3} \] \[ \ref{3} \] \[ \ref{3} \]. For distances smaller than the correlation length, the fluctuation is large and the density can be non-analytic with no specific average value \[ \ref{2} \]. We are also assuming that crossover to homogeneity is reached at some stage. Otherwise, the notion of correlation length is meaningless and the present analytic approach is not applicable \[ \ref{3} \].

The Einstein equation is written for the metric \[ \ref{1} \] and the perturbations \[ \ref{2} \] \[ \ref{3} \] \[ \ref{4} \]. The zeroth-order contributions, i.e. those with vanishing fractal codimension, to the Einstein equation give the Friedmann equations. The first-order contributions lead to the second-order differential equation

\[ \ddot{Y} + \left( -6\pi G(\alpha + 1)(\alpha \rho - P) - A_3 \frac{k}{R^2} \right) Y = 0 \quad (5) \]

where

\[ Y(R) = R^{-(3\alpha + 1)/2} \delta R, \]

\[ A_3 = (1 + \gamma)(1 - \alpha(\gamma - 3)) - \frac{3}{4}(\alpha + 1)(3\alpha + 1). \quad (6) \]

In the Einstein-de Sitter model, i.e. for \( k = 0 \), the differential equation \[ \ref{3} \] has exact solutions which simplify to

\[ \delta R \sim t^x, \quad x = -\frac{3\alpha + 1}{3(1 + \omega)} + \frac{1}{2} \pm \frac{1}{2} \left| \frac{\omega - 2\alpha - 1}{\omega + 1} \right| \quad (7) \]

at large times. The scale factor fluctuation grows when \( \alpha + 1 \) and \( \omega + 1 \) have the same sign and \( 2|\alpha + 1| \geq |\omega + 1| \).

The solutions \[ \ref{7} \] can be substituted back in the time component of the Einstein equation to find the perturbation in the density. The density contrast is subsequently given by

\[ \frac{\delta(t)}{\rho} = t^{-\alpha} \quad (8) \]

which shows that our perturbation does not grow. Specifically, in matter or radiation-dominated eras the perturbation decays as \( t^{-1} \) which is the same as the standard decaying mode \[ \ref{3} \] \[ \ref{3} \]. Indeed, a simple calculation of the amplification factor, \( \frac{\delta(t)}{\delta(t)} \), shows that the perturbation decays by a factor of \( 10^{-5} \) in the pressureless era (i.e., for \( t/t_0 \approx 10^5 \)). The second solution for the density contrast is just a constant. The standard growing mode of \( t^{2/3} \) in the matter-dominated era and \( t \) in the radiation-dominated era \[ \ref{4} \] \[ \ref{11} \] do not arise in our scheme. In order to recover these results, we need go to higher perturbative orders and expand the scale factor (and subsequently density and pressure) in series of inverse powers of \( r \), i.e. as,

\[ R_p(r, t) = \sum_{n=0}^{\infty} \frac{R_n}{r^n}, \quad (9) \]

where \( n = 1 \) term is equivalent to our perturbation for \( \gamma \approx 1 \). Had we used this perturbation instead of \[ \ref{3} \] we would have obtained the standard growing modes at the third order of the perturbation (i.e., at \( n = 3 \) \[ \ref{8} \]). Therefore, our analysis shows that, in spite of the growing modes, the perturbation cannot grow to scales much larger than the correlation length, in a flat universe. On the contrary, we shall see that in an open universe a growing mode exists for these scales.

The differential equation \[ \ref{3} \] for a non-flat universe, \( k \neq 0 \), is more complicated. To start with, we consider a matter-dominated universe where both the homogeneous background pressure and the perturbative contributions from the inhomogeneities vanish (i.e. \( \alpha = 0, \omega = 0 \)). The Friedmann scale factor is given parametrically by

\[ R = \frac{4t}{d\psi}; \quad t = \frac{\Omega}{2H(1 - \Omega)^{3/2}}(\psi - \sin \psi), \quad (10) \]

where \( \psi \) is real for an open universe and imaginary, \( \psi = -i\theta \), for a closed universe, and \( \Omega \) is the ratio of the present to the critical density \[ \ref{4} \] \[ \ref{4} \]. The differential equation \[ \ref{6} \] is now a hypergeometric equation which can be solved exactly. The density contrast is given by the growing mode

\[ \delta_+(\psi) \sim \frac{\sin \left( \sqrt{3} \psi \right)}{\sqrt{3}} \left[ \frac{-3 \sinh \psi}{\cosh \psi - 1} + \frac{2 \sinh \psi}{\cosh 1} \right] \]

\[ + \cos \left( \sqrt{3} \psi \right) \left( \frac{5 + \cosh \psi}{\cosh \psi - 1} \right), \quad (11) \]
and the decaying mode
\[
\delta_-(\psi) \sim \cos(\sqrt{\gamma}\psi) \left[ \frac{-3\sinh\psi}{(\cosh\psi - 1)^2} + \frac{2\gamma\sinh\psi}{\cosh\psi - 1} \right] - \sqrt{\gamma}\sin(\sqrt{\gamma}\psi) \left[ 5 + \cosh\psi \right] \left[ \frac{\cosh\psi - 1}{\cosh\psi - 1} \right].
\] (12)

We note that for \( \gamma = 0 \), the above modes coincide with the usual results in the literature \([9,11]\). Contrary to the decay of fluctuations in the flat universe, scale-invariant inhomogeneities can grow in an open matter-dominated universe. In fact, unlike the standard open universe density contrast which remains constant at a redshift of approximately \( 2/5\Omega \) (i.e., \( \cosh\psi \leq 5 \), Fig. 1), our density contrast oscillates with \( \psi \) (Fig. 2).

FIG. 1. The graph of standard density contrast \((\gamma = 0)\) versus \(\psi\) for an open universe.

FIG. 2. The graph of density contrast, at \(\gamma = 1\) versus \(\psi\), for an open universe.

For the amplification factor, \(\delta(t_0)/\delta(t_r)\), from the time of recombination to now, once again we have the standard results for \(\gamma = 0\) in which the amplification rises monotonically with \(\Omega\) (Fig. 3). However, for \(\gamma = 1\), we see that the amplification varies with \(\Omega\) in an oscillatory manner (Fig. 4). Furthermore, going to a very high resolution, at small values of Omega, we see a distinct peak at \(\Omega \approx 0.001\) where an amplification by up to a factor of \(10^{13}\) is obtained (Fig. 5). The amplification is finely tuned to the value of \(\Omega\). For example an amplification of \(10^5\) occurs for \(\Omega \approx 0.00101629\). At higher values of \(\Omega\) no peaks and at much lower values of \(\Omega\) many more peaks are observed. This remarkable phenomenon does not occur in the standard analysis at \(\gamma = 0\). Indeed, a self-similar or hierarchical universe is expected to have a very low density which is why our perturbation decays in a flat universe \([3]\). Here, we have shown that for the growth of a scale-invariant inhomogeneity to be in accordance with the anisotropy of the microwave background radiation, an upper limit of \(\Omega \approx 0.001\) is required for the density of a hierarchical universe.

The above results can be verified further by studying the density contrast and amplification factor in a closed universe. In the standard analysis, the amplification factor increases with increasing \(\Omega\) \([9,11]\). Using equations (11) and (12) for a closed universe (i.e., \(\psi = -i\theta\)), we see that the decaying mode (12) is imaginary for \(\Omega > 1\) and the growing mode (11) now decays. The graph of amplification factor shows that the perturbation decays significantly from the recombination to the present time.
Asymptotically, we obtain a closed universe, $\Omega$ where $a = \sqrt{1 - 4A_3}$ and $\theta = -i\psi$. The second solution is obtained by replacing $\alpha$ with $-1 - \alpha$ in the above. Asymptotically, we obtain

$$ \delta R \sim t^x ; \quad x = -\frac{3}{2}\alpha + \frac{1}{2}\sqrt{1 - 4A_3}, $$

$$ \delta \sim t^{-\frac{3\alpha}{2} + \frac{1}{2}\sqrt{1 - 4A_3}}, $$

For complex values of the square-root term in the density contrast, above, there is no growing mode since $\alpha$ is always positive. For real values of the square root, which occur for $\alpha > 0.8$ or $\alpha < -0.4$ with $\gamma = 1$, the situation is more complex. The terms $1 - 4A_3$ can now be written as $9\alpha^2 - 4\alpha - 4$ whose square root is always smaller than $3\alpha$. We, therefore, conclude that asymptotically inhomogeneities cannot grow in an open universe with a vanishing background and a non-vanishing perturbation pressure.

Unlike the preceding cases, an exact solution cannot be obtained for a universe with non-vanishing pressure. In this case, the Friedmann scale factor is given by the expression

$$ t = -\frac{2C_1R^{\frac{2}{3}(1+\alpha)}}{3(\omega + 1)}F \left( \frac{1}{2}; \frac{3(\omega + 1)}{2(3\omega + 1)}, \frac{9\omega + 5}{2(3\omega + 1)}; C_1^2R^{1+3\omega} \right) $$

where $C_1$ is a constant. Although, the exact form of the above hypergeometric function is not unknown, it can be easily shown that it reduces to $(\delta)$ for a matter-dominated universe and, asymptotically, to a linearly-increasing scale factor. In the asymptotic limit, the differential equation $(\delta)$, for an open universe with $\omega > -1/3$, reads

$$ \frac{d^2Y}{dt^2} + C t^{-3\omega - 3} Y + A_3 t^{-2} Y = 0, $$

where $C$ is a constant. This expression reduces to a confluent hypergeometric equation for $F(1/\tau)$ after the redefinition:

$$ Y = t^{\frac{1}{3}(1+\sqrt{1-4A_3})}e^{\frac{x^2}{2}}F \left( \frac{1}{\tau} \right), $$

where $t = C^{-\frac{1}{3}}\tau^{-\frac{1}{3}}$. In the limit of large times, $F(1/\tau)$ approaches a constant value $(\delta)$ and we recover expression $(14)$ whose behaviour does not depend on $\omega$.

For the density contrast, we obtain

$$ \delta \sim t^{-\frac{3\alpha}{2} + \frac{1}{2}\sqrt{1 - 4A_3}}, $$

which shows that the perturbation grows for all positive values of $3\omega - 3\alpha/2$ or for $\omega > -0.3$ and positive values of $\alpha$. The growth rate of the perturbation is significant in the radiation-dominated era ($t/t_0 \approx 10^8$). In this period, the perturbation amplifies by a factor of $10^4$ for $\alpha \approx 1/3$. However, since the baryonic matter cannot grow in this era, due to its coupling to the radiation, this result can only be relevant for non-baryonic dark matter.

The main results are summarized in table 1.
We have evaluated the density contrast for a scale-invariant perturbation in flat, open and closed universes. We have found that, in this specific perturbation scheme, fluctuations decay in the Einstein-de-Sitter universe whereas they grow in an open universe. The standard results for the growing and decaying modes of the density contrast in open and closed universes are contained in our results. The decaying mode for a flat universe is also reproduced by our method. However, the absence of the standard growing mode in the flat universe shows that the perturbation grows to larger scales in an open universe than in a flat universe. In addition, in our analysis, the perturbation does not simply amplify with increasing $\Omega$. It has an oscillatory behaviour. A remarkable feature of our perturbation, which is not observed in the standard analysis, is the existence of sharp peaks at $\Omega \leq 0.001$. At $\Omega \approx 0.001$ the perturbation can be amplified by up to a factor of $10^{13}$, from the recombination to the present time. We further confirm our results by showing that the perturbation decays even faster in a closed universe. We, therefore, conclude that for scale-invariant density perturbations the standard belief that the density contrast grows faster with increasing $\Omega$ is not valid. Our results are fully consistent with the low density expected for a hierarchical universe.

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Table 1: The table of density contrast for flat and open universes in matter and radiation-dominated eras.

| Equation of State | Flat $k = 0$ | Open $k = -1$ |
|------------------|-------------|---------------|
| Matter-dominated $\omega = 0$ | decays | grows |
| Radiation-dominated $\omega = \frac{1}{3}$ | decays for large times | grows |

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