Dynamics of quantum entanglement in matter field models

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Abstract. A review of the linear canonical transformations and its unitary representation is presented and its importance in the mathematical generation of a special two photon coherent state of light is demonstrated. The dynamical behaviour of the entanglement of the matter and field sectors of an initial composite system under the action of two effective model Hamiltonians is shown. The quantumness and delocalization properties of the corresponding electromagnetic field sectors are established. Finally for the two-photon process interacting with a two-level system, the violation of the Cauchy-Schwarz inequality is exhibited.

1. Introduction
The canonical transformations and its unitary representations have played a very important role in classical and quantum mechanics. The first applications were mainly concerned with the group of point transformations [1, 2]. More general canonical transformation involving position and momentum were used by the founders of quantum mechanics when the quantum mechanics was being developed [3]. The simplest example of linear canonical transformations was provided by the unitary group of symmetries of the one dimensional harmonic oscillator [4]. The extension to the three dimensional case leads to a representation of the U(3) group whose implicit use was made in its applications to nuclear physics [5] and elementary particles [6].

The manuscript is divided in two parts:
First a review of the role of canonical transformations and its unitary representation in quantum mechanics is presented [7, 8]. This is based mainly in the work of Marcos Moshinsky and his collaborators made at the beginning of the seventies ranging from the calculation of radial matrix elements [9], to complex extensions of the canonical transformations to determine accidental degeneracies of simple quantum mechanical systems [10, 11]. This review of the unitary representations of linear canonical transformations is motivated by its applications to describe and mathematically generate a special type of two-photon coherent states of the radiation field [12]. These states have applications in quantum optics because constitute a generalization of the minimum-uncertainty wave packets, that is, the standard coherent states of light [13].

In the second part we investigate two effective model Hamiltonians describing the interaction of a two-level system with a one- or two-mode electromagnetic field in a cavity. These Hamiltonians are called in the literature the Jaynes-Cummings model (one-mode field) and a generalized Jaynes-Cummings model, in which the transition is mediated by two different
modes of photons. In both cases one can get analytical solutions of the eigenvalue equations together with the construction of the corresponding evolution unitary operators. By means of these evolution operators, it is straightforward to study the dynamic entanglement between the matter and field sectors together with the quantumness and delocalization properties of the light for several types of initial states. To measure the entanglement we use the linear entropy concept of quantum information theory, which is associated to the reduced density matrix of the matter or field subsystems. To establish the quantumness and delocalization properties of light we calculate the Wigner quasi-probability distribution function and the second moment of the Husimi function, respectively. Finally we determine also the temporal behaviour of the Cauchy-Schwartz inequality, whose violation is a signature of having strong quantum correlations [16].

2. Canonical transformations and its unitary representation

In classical mechanics a canonical transformation is a transformation in phase space leaving invariant the Poisson brackets. Thus if \{z_\alpha\} and \{\bar{z}_\alpha\} are 2n dimensional vector of the phase space, transformation \{z_\alpha\} \rightarrow \{\bar{z}_\alpha\} satisfies that

\[ \sum_{\gamma,\delta} \frac{\partial \bar{z}_\alpha}{\partial z_\gamma} \Sigma_{\gamma\delta} \frac{\partial \bar{z}_\beta}{\partial z_\delta} = \Sigma_{\alpha\beta}. \] (1)

If additionally the transformation between the new \bar{z}_\alpha and the old z_\beta vectors in phase space is linear,

\[ \bar{z}_\alpha = \sum_{\beta} S_{\alpha,\beta} z_\beta, \]

the transformation will be canonical if

\[ S \Sigma \bar{S} = \Sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \]

where \bar{S} denotes the transpose of the real symplectic matrix S. Therefore a classical linear canonical transformation belongs to the real symplectic group of 2n dimensions, Sp(2n, \mathbb{R}).

The appearance of this group in the Hilbert space of quantum mechanics was discussed since the early days of the formulation of quantum mechanics. A theorem of von Neumann established the following statement: Every canonical transformation can be represented by a unitary transformation in quantum mechanics.

The construction of the unitary representation associated to the real canonical transformation \((x_i, p_i) \rightarrow (\bar{x}_i, \bar{p}_i)\) is as follows:

In the position representation the matrix elements of the operators \(x_i\) and \(p_i\) are given by

\[ \langle x' | x_i | x'' \rangle = x'_i \delta(x' - x''), \]

\[ \langle x' | p_i | x'' \rangle = -\frac{1}{i} \frac{\partial}{\partial x'_i} \delta(x' - x''), \] (2)

where we take \(\hbar = 1\). For the new canonical set of operators \(\bar{x}_i = \bar{x}_i(x, p)\) and \(\bar{p}_i = \bar{p}_i(x, p)\), their matrix elements must have the same form with respect to the new basis denoted by the round kets \(|\bar{x}\rangle\).

From here on, to simplify the notation, we consider only one degree of freedom, thus by means of the expansion

\[ |\bar{x}'\rangle = \int |x'\rangle \, dx' \langle x'|\bar{x}\rangle, \]
with the scalar product $\langle x'|\vec{x} \rangle$ denoting the transformation bracket (TB) one has the connection between the two basis states. The TB is determined in unique form by establishing the eigenvalue equation and multiplying by the state $\langle x'|$,

$$\vec{x} \left( x', \frac{1}{i} \frac{\partial}{\partial x'} \right) \langle x'| \vec{y} \rangle = \vec{y} \langle x'| \vec{x}' \rangle,$$

together with the normalization and matrix element of the momentum operator,

$$\int \langle \vec{x}'| x' \rangle dx' \langle x'| \vec{y} \rangle = \delta(\vec{x}' - \vec{y}) \delta(\vec{x}' - \bar{\vec{x}}).$$

From $\langle x'| \vec{x} |\vec{x}_n \rangle$ together with the relations

$$\langle x'| U |x_n \rangle = \langle x'| \vec{x}_n \rangle, \quad \langle x'| U^{-1} |x_n \rangle = \langle x'| \vec{x}_n \rangle.$$

It is immediate to find

$$\vec{x} = U x U^{-1}, \quad \vec{p} = U p U^{-1}.$$

For a single particle moving in one dimension, the most general transformation in phase space is given by

$$\left( \begin{array}{c} \vec{x} \\ \vec{p} \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} x \\ p \end{array} \right),$$

where $a, b, c, d$ are real numbers, which is canonical if $ad - bc = 1$. For the linear canonical transformations we have found the quantum mechanical equivalent of the classical canonical transformations and a ray representation of the two dimensional symplectic group, i.e.,

$$\langle x'| U |x_n \rangle = \frac{1}{\sqrt{2\pi b}} \exp \left\{ -\frac{i}{2b} \left( a x_n^2 - 2 b x' x_n + d x_n' \right) \right\}. \quad (3)$$

2.1. Unitary representation in the oscillator basis:

For a single mode of radiation, a linear canonical transformation is given by

$$b = U_L a U_L^\dagger = u a + v a^\dagger, \quad b^\dagger = U_L a^\dagger U_L^\dagger = u^* a^\dagger + v^* a,$$

where $U_L$ is the associated unitary representation and the condition $|u|^2 - |v|^2 = 1$. The general transformation can always be factorized in terms of rotation and dilation subgroups as,

$$\left( \begin{array}{cc} u & v \\ v^* & u^* \end{array} \right) = \left( \begin{array}{cc} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{array} \right) \left( \begin{array}{cc} \cosh \frac{\theta}{2} & \sinh \frac{\theta}{2} \\ \sinh \frac{\theta}{2} & \cosh \frac{\theta}{2} \end{array} \right) \left( \begin{array}{cc} e^{-i\chi/2} & 0 \\ 0 & e^{i\chi/2} \end{array} \right),$$

where $0 \leq \phi, \chi < 2\pi$ and $\theta$ an arbitrary real number.

The eigenstate of the one dimensional harmonic oscillator is given by

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle,$$

with the expression $a|0\rangle = 0$ defining the vacuum state.

Under the transformation of O(2) by an angle $\phi/2$ in phase space, one gets

$$b^\dagger = e^{-i\phi/2} a^\dagger.$$
Therefore the unitary representation of O(2) that take us from $|n\rangle \to |n\rangle$ is given by

$$\langle n'| U |n''\rangle = e^{-i\theta'/2} \delta_{n',n''}. \quad (5)$$

For the dilatation subgroup of Sp(2) one has the transformation in phase space,

$$\bar{x} = e^{\theta/2} x, \quad \bar{p} = e^{-\theta/2} p,$$

or equivalently in the harmonic oscillator representation

$$\begin{pmatrix} b^\dagger \\ b \end{pmatrix} = \begin{pmatrix} \cosh \frac{\theta}{2} & \sinh \frac{\theta}{2} \\ \sinh \frac{\theta}{2} & \cosh \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} a^\dagger \\ a \end{pmatrix}.$$  

The unitary representation of the dilation subgroup is given by

$$\langle n'| U |n''\rangle = \left( \frac{n'! n''!}{2^n + n'' \cosh \frac{\theta}{2}} \right)^{1/2} \left( 1 + \frac{(-1)^{n'-n''}}{4} \right) \left( \tanh \frac{\theta}{2} \right)^{\frac{n'-n''}{2}} \times \sum_{r=0}^{\min[n',n'']} \left( 1 + (-1)^{n'-r} \right) 2^r (-1)^{n'-r} \left( \sinh \frac{\theta}{2} \right)^{-r} \frac{r!}{(\frac{n'-2}{2})! (\frac{n''-2}{2})! r!}. \quad (6)$$

3. Two photon coherent states (TCS)

The concept of TCS is introduced for applications in quantum optics. They are generated by unitary operators associated to quadratic Hamiltonians and constitute a generalization of the wave packets of minimum uncertainty [12]. In terms of $b$, $b^\dagger$ one can define: Fock states $|n\rangle_q$, coherent states, $|\beta\rangle_q$, a differential realization, and so on. The TCS are defined by

$$b|\beta\rangle_q = \beta |\beta\rangle_q = U_L |\beta\rangle,$$  

where one has the linear canonical transformation $b = u^* a + v^* a^\dagger$. Notice that, for the parameters $u = 1, v = 0$ the TCS reduces to the standard definition of a coherent state of light introduced by Glauber [13].

A coherent state of light is an eigenstate of the annihilation operator, $a$,

$$a|\alpha\rangle = \alpha |\alpha\rangle,$$

with $a|0\rangle = 0$. It can be also obtained via the action of the displacement unitary operator

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

on the vacuum state.

There is another construction of the coherent state called the generating function definition, that is, by the action of the operator $e^{\alpha a^\dagger}$ on the vacuum state, which allows to calculate the following representation of the creation and annihilation operators

$$\langle \alpha|a^\dagger|\psi\rangle = \alpha^* \langle \alpha|\psi\rangle, \quad \langle \alpha|a|\psi\rangle = \frac{\partial}{\partial \alpha^*} \langle \alpha|\psi\rangle,$$  

for an arbitrary state $|\psi\rangle$.

1 Let us expand eigenfunctions of a harmonic oscillator with frequency $\omega$ in terms of those with frequency 1, which can be used to determine Frank-Condon factors in molecules [14, 15].
A coherent state has a Poisson distribution function for the statistics of the photons,

\[ |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}, \tag{9} \]

where the average number of photons is given by the expression \( \langle n \rangle = |\alpha|^2 \).

By means of the differential realization of the creation and annihilation operators, the coherent state can be written in the quadrature representation \( X \),

\[ \langle X|\alpha\rangle = \left(\frac{2}{\pi}\right)^{1/4} e^{-(X-\alpha)^2+\alpha(\alpha^*-\alpha)}. \tag{10} \]

In a similar form by taking into account the linear canonical transformation, the coherent state representation of the TCS can be calculated, that is,

\[ \langle \alpha|\beta \rangle_q = \frac{1}{\sqrt{u}} \exp\left\{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \frac{v^* \beta^2}{2u} - \frac{v \alpha^*}{2u} + \frac{v \alpha^*}{u}\right\}, \tag{10} \]

which satisfies the normalization condition,

\[ \int |\langle \alpha|\beta \rangle_q|^2 \frac{1}{\pi} d \alpha d \text{Im} \alpha = 1. \tag{11} \]

In the photon number representations, \( a^\dagger a \), the TCS can be written in the form

\[ \langle n|\beta \rangle_q = \frac{1}{\sqrt{n!u}} \left(\frac{v}{2u}\right)^{n/2} \exp\left\{-\frac{|\beta|^2}{2} + \frac{v \beta^2}{2u} - \frac{v \alpha^*}{2u}ight\} H_n\left(\frac{\beta}{\sqrt{2v}u}\right). \tag{12} \]

while in terms of a quadrature of the electromagnetic field \( X \) is given by the expression,

\[ \langle X|\beta \rangle_q = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{|u-v|} \exp\left\{-\frac{|\beta|^2}{2} + \frac{2 X \beta}{u-v} - \frac{(u-v)^2 \beta^2}{2(u-v)} - \frac{(u+v)X^2}{u-v}\right\}. \tag{13} \]

4. Quadratic Hamiltonian

The TCS can be generated by the quadratic Hamiltonian,

\[ H = f_1(t)a^\dagger a + f_2^2(t)a^2 + f_2(t)a^\dagger a^2 + f_3^2(t)a + f_3(t)a^\dagger, \]

where the coefficient \( f_1 \) represents the free radiation energy of the mode, \( f_2 \) describes the two photon mechanism and \( f_3 \) the corresponding one photon process or driving mechanism.

By making an inhomogeneous symplectic transformation, the quadratic hamiltonian can be written in the form

\[ H_q = c_1(t)b^\dagger b + c_2(t) + c_0(t), \tag{14} \]

where the time dependent functions are given by

\[ c_1(t) = \frac{f_1^2}{f_0} - \frac{2 (f_2 + f_2^2)}{f_0}, \quad c_2(t) = \frac{f_1 - f_0}{2f_0} - \frac{(f_2 + f_2^2)}{f_0}, \]

\[ c_0(t) = \frac{1}{f_0^2} \left( f_2 f_3^2 + f_2^2 f_3^2 - f_1 f_3^2 \right), \]

The two-photon coherent states can be generated in degenerate four wave mixing experiments [17].
with the definition \( f_0 = \sqrt{f_1^2 - 4|f_2|^2} \). Establishing the positivity condition \( f_1^2 - 4|f_2|^2 \geq 0 \) to have an Hermitian Hamiltonian.

One can propose the evolution operator associated to the quadratic Hamiltonian in the form
\[
U(t) = e^{A(t)} e^{C(t) a^\dagger} e^{D(t)(a^\dagger) + 1} e^{E(t) a} e^{B(t) a^\dagger + E(t) a},
\]
and by taking the matrix elements with respect to coherent states, it is straightforward to find,
\[
\langle \alpha | U(t) | \beta \rangle = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + A + B \beta^* + C \alpha^* + (D+1) \alpha^* \beta + E \beta + F \alpha^*},
\]
where the time dependent functions satisfy the differential equations,
\[
\begin{align*}
\frac{dA}{dt} &= 2C f_2^* + f_3^* F + f^* F^2, \\
\frac{dB}{dt} &= f_2^* (D + 1)^2, \\
\frac{dC}{dt} &= f^* C^2 - f, \\
\frac{dD}{dt} &= (f_1 + 4C f_2^*) (D + 1), \\
\frac{dE}{dt} &= (2 f_2^* + f_3^*) (D + 1), \\
\frac{dF}{dt} &= f_1 F + 4C F f_2^* + 2C f_3^* + f_3 .
\end{align*}
\]
with the initial conditions \( A(0) = B(0) = C(0) = D(0) = E(0) = F(0) = 0 \), and we have made the following definitions: \( C(t) = \frac{1}{2} C e^{-2i \int_0^t f_1(\tau) d\tau} \), and \( f(t) = 2i f_2 e^{-2i \int_0^t f_1(\tau) d\tau} \). Notice that,
\[
\frac{dC}{dt} = f^* C^2 - f
\]
is a Riccati nonlinear equation. Furthermore, from expressions in Eq.(17) one may see that given the solution of the Riccati equation the parameters \( \{ A, B, D, E, F \} \) may be found by direct integration.

The evolution operator given in (16) can be written in the form
\[
\langle \alpha | U(t) | \beta \rangle \equiv \langle \alpha | \beta + \zeta(t); u, v \rangle e^{i \phi(t)},
\]
with a real phase factor given by
\[
\phi(t) = \Theta - i(\beta^* \zeta - \beta \zeta^*).
\]
Notice that for \( t = 0 \), the function \( \zeta(0) = 0 \) to reproduce that the evolution operator is equivalent to the overlap of coherent states \( \langle \alpha | \beta \rangle \) (see Eq. 16). By comparing the expression for the two photon coherent state Eq. (10), making the replacement \( \beta \rightarrow \beta + \zeta(t) \), with the expression (17), one gets the relations between the complex functions \( u \) and \( v \) with \( A, B, C, D, F \) is determined by
\[
\begin{align*}
B &= \frac{v^*}{2u}, & C &= -\frac{v}{2u}, & D + 1 &= \frac{1}{u}, & E &= -\zeta^* + \frac{v^*}{u} \zeta, & F &= \frac{\zeta}{u},
\end{align*}
\]
and one has
\[
A(t) = -\frac{1}{2} \ln u + \zeta^* \frac{v^*}{2u} - \frac{1}{2} |\zeta(t)|^2 + i \Theta.
\]
The initial conditions are given by \( u(0) = 1, \ v(0) = 0, \) and \( \zeta(0) = 0. \)

For an arbitrary state \(|\psi(t_0)\rangle\), its temporal evolution is determined by
\[
|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle = \frac{1}{\pi} \int U(t, t_0)|\beta\rangle \langle \beta|\psi(t_0)\rangle \ d \text{Re}(\beta) \ d \text{Im}(\beta),
\]
where we have introduced the decomposition of the identity operator in the coherent state representation.

While for a TCS \(|\beta_0; u_0 v_0\rangle\), one has immediately that
\[
U(t, t_0)|\beta_0; u_0 v_0\rangle = |\beta_0 + \zeta(t, t_0); u(t, t_0), v(t, t_0)\rangle e^{i\phi(t, t_0)},
\]
where we have used the group property
\[
u(t, t_0) = u(t) \ u_0 + v^*(t) \ v_0, \quad v(t, t_0) = u_0 \ v(t) + u^*(t) \ v_0.
\]

5. Entanglement, quantumness, and delocalization of matter-field Hamiltonians

We are going to study the temporal evolution of the entanglement between matter and field, of the quantumness of the light, and of the delocalization properties of light in phase space for several initial wave packets. The temporal evolution is determined by two Hamiltonian models describing the interaction in a cavity of one atom or molecule (2-level systems) with an electromagnetic field of one or two photon processes.

We are going to consider the temporal evolution of a pure bipartite composite state,
\[
|\psi\rangle = \sum_{i,\mu} c_{i\mu} |i\rangle |\mu\rangle,
\]
with \( \sum_{i,\mu} |c_{i\mu}|^2 = 1 \). Thus one can construct the density matrix of the composite system as follows: \( \rho_{i\mu;j\nu} = c_{i\mu} c_{j\nu}^* \), which satisfies that \( \text{Tr} \rho_{i\mu;j\mu} = 1 \). The corresponding reduced density matrices of the subsystems are given by
\[
(\rho_1)_{i\mu} = \sum_{j\mu} c_{i\mu} c_{j\mu}^*, \quad (\rho_2)_{\mu;\nu} = \sum_{i} c_{i\mu} c_{i\nu}^*.
\]

For a bipartite pure state it is well known that any of the reduced density matrices can be used to measure the quantum entanglement. This measures can be done through the concepts of linear entropy \( S_L \) or von Neumann Entropy \( S_{VN} \), that is,
\[
S_L = \text{Tr}(\rho_1)^2, \quad S_{VN} = -\sum_{i} \lambda_i \ln \lambda_i,
\]
where \( \rho_1 \) is the reduced density operator of the light or matter subsystem and \( \lambda_i \) are their corresponding eigenvalues.

A visualization of the behavior of light in the cavities can be done through the calculation of the Wigner and Husimi quasi-probability distribution functions. The inverse of the second moment of the Husimi function is a measure of the localization properties of light while the negative values of the Wigner function indicate the quantumness of the light. Additionally we determine the Cauchy-Schwartz inequality (CSI) to establish the quantum correlations of the pure system and see if the matter-field Hamiltonians can be used to study the Einstein-Podolski-Rosen (EPR) type of quantum correlations.
Figure 1. At left the occupation probability of the excited state $P_e$ and the linear entropy $S_L$ are plotted as function of time. At right the Husimi function is displayed in phase space for the adimensional time $\tau = 10$. In the cavity an average number of photons $n_0 = 16$ was considered.

Notice that, the CSI simply states that the absolute value of the inner product of two vectors cannot be larger than the product of their lengths. In the probability theory and classical physics the CSI can be applied to fluctuating quantities and states that

$$|\langle I_1 I_2 \rangle| \leq \sqrt{\langle I_1^2 \rangle \langle I_2^2 \rangle}.$$ 

In quantum mechanics correlations can be stronger than those allowed by the CSI, then its violation implies: (i) strong non-classical correlations and (ii) the possibility of formation of quantum states that exhibit EPR correlations.

The second order correlation function is defined by $G_{ij}^{(2)} = \langle : n_i n_j : \rangle$. Then the CSI takes the form:

$$|G_{12}^{(2)}| \leq |G_{11}^{(2)} G_{22}^{(2)}|^{1/2},$$

which can be written as follows

$$|\langle N_1 N_2 \rangle| \leq \sqrt{\langle a_1^\dagger a_2 \rangle \langle a_2^\dagger a_1 \rangle}.$$ 

The normal order ensures the correct quantum mechanical interpretation of the process of detection of pairs of photons that contribute to the measurement of the second order correlation function [13].

5.1. Hamiltonian of a two-level system interacting with a one photon process

The Hamiltonian of a two level system (atom or molecule) in a cavity interacting with a one-mode electromagnetic radiation is given by [18, 19]

$$H/\hbar = \Omega(M - I/2) + \Delta \sigma_z + \lambda (\sigma_- a^\dagger + \sigma_+ a),$$

where $\sigma_z, \sigma_\pm = 1/2(\sigma_x \pm i \sigma_y)$ are given in terms of the Pauli matrices, $M = a^\dagger a + 1/2(\sigma_z + 1)$ is the total number of excitations, $\Delta = (\omega_0 - \Omega)/2$ the detuning parameter and $\lambda$ the coupling strength between the two-level system and electromagnetic field. Notice that $M$ commutes with the Hamiltonian and thus it is a constant of motion.

It is straightforward to solve the eigenvalue equation and determine the dressed states with a fixed total number of excitations $M = n + 1$:

$$|\phi_+(n)\rangle = \frac{1}{\sqrt{2\omega_+(\omega_+ - \Delta)}} \left( -\lambda \sqrt{n+1} |n e\rangle + (\Delta - \omega_+) |n + 1 g\rangle \right),$$

$$|\phi_-(n)\rangle = \frac{1}{\sqrt{2\omega_+(\omega_+ + \Delta)}} \left( -\lambda \sqrt{n+1} |n e\rangle + (\Delta + \omega_+) |n + 1 g\rangle \right),$$

where $\omega_\pm = (\omega_0 \pm \Omega)/2$. The normalized form of these dressed states is $\langle \phi_{\pm}(n) | \phi_{\mp}(n) \rangle = \delta_{\pm \mp}$. The eigenvalues of $M$ are $M = n + 1$. As $M$ is a constant of motion, the eigenstates $|\phi_{\pm}(n)\rangle$ of $H/\hbar$ are eigenstates of $M$. The Hamiltonian $H/\hbar$ commutes with $M$ and therefore $H/\hbar |\phi_{\pm}(n)\rangle = E_n |\phi_{\pm}(n)\rangle$ for a given eigenvalue $E_n$.
whose eigen-energies are given by
\[ \omega_{\pm} = \pm \sqrt{\Delta^2 + \lambda^2(n+1)}, \quad E_{\pm} = \Omega(n+1) + \omega_{\pm}. \]

Notice that the energy spectrum is constituted: (i) for \( M = 0 \) by a single state given by \( |0g\rangle \) with energy \( E_0 = -\Delta \) and (ii) for \( M \neq 0 \) by a stair of two energy levels.

The evolution operator takes the form,
\[ U(t) = e^{-i\omega t(M-I/2)} U_I(t), \]
where the operator in the interaction picture is defined by
\[ U_I(t) \equiv e^{-it\bar{H}}, \]
with
\[ \bar{H} = \Delta \sigma_z + \lambda (\sigma_- a + \sigma_+ a^\dagger). \]

Making the Taylor series expansion of \( U_I(t) \), one has
\[ U_I(t) = G(n, t) |e\rangle\langle e| - i\Delta \sin\left(\frac{\sqrt{\Delta^2 + \lambda^2} n t}{\sqrt{\Delta^2 + \lambda^2} n}\right), \]
where we make the definitions
\[ F(n, t) = \cos\left(\frac{\sqrt{\Delta^2 + \lambda^2} n t}{\sqrt{\Delta^2 + \lambda^2} n}\right) - i\frac{\Delta}{\sqrt{\Delta^2 + \lambda^2} n} \sin\left(\frac{\sqrt{\Delta^2 + \lambda^2} n t}{\sqrt{\Delta^2 + \lambda^2} n}\right), \]
\[ G(n, t) = \frac{\sin\left(\frac{\sqrt{\Delta^2 + \lambda^2} n t}{\sqrt{\Delta^2 + \lambda^2} n}\right)}{\sqrt{\Delta^2 + \lambda^2} n}. \]

We study the evolution of the system in a cavity for the following initial wave packets:

- \( |\psi(0)\rangle = |e\rangle \otimes |\alpha\rangle \), with an average number of photons in the cavity \( n_0 = |\alpha|^2 = 16 \). The probability of finding the atom or molecule in the excited state as function of time is given by the expression
\[ P_e(\tau) = \sum_{k=0}^{k_{\text{max}}} \frac{e^{-n_0}}{k!} n_0^k \cos^2 \sqrt{k+1} \tau, \quad (25) \]
where \( k_{\text{max}} = 40 \). This expression together with calculation of the linear entropy are plotted in Fig.1 as function of time. The Rabi oscillations collapse and revival in a cavity can be observed, note also that at time \( \tau \approx 12 \) there is not correlations between matter and field.

For the study of the Rabi oscillations one is interested in the reduced density matrix of the matter, that is tracing over the field variables. The behaviour of the light is obtained by calculating the Husimi function for a fixed time, this is, \( \tau = 10 \), the Husimi function is displayed in phase space, at the right side of Fig. 1, one notice a cat type structure for the Husimi function.

- \( |\psi(0)\rangle = (\cos \theta |\nu\rangle + \sin \theta |\nu+3\rangle) \otimes |g\rangle \). This linear combination generates states of light with discrete symmetries [20] associated to the point symmetry group \( C_{M_2-M_1} \). In Fig. 2 the Husimi Function with \( C_3 \) symmetry is displayed in contour and three dimensional plots in phase space for a fixed time \( \tau = 5 \), an initial state with \( \nu = 4 \), and \( \theta = \pi/4 \), and use \( \lambda = 1 \) and \( \Delta = 0 \) for the parameters of the Hamiltonian.
Figure 2. The Husimi function with $C_3$ point symmetry in phase space is plotted. At left a contour plot and at right a three dimensional plot. We take the parameters: $\theta = \pi/4$, $\Delta = 0$, $\lambda = 1$, $\nu = 4$, $\tau = 5$.

5.2. Hamiltonian of a two photon process interacting with a two-level system

The two photon quantum optics has been introduced with the purpose of analysing instruments in which the output modes create or destroy 2 photons at the same time. Examples of this type of instruments are the parametric amplifier and mirrors of conjugate phases. Applications are given in optical communications and in high precision interferometric measurements.

An effective Hamiltonian describing the interaction of two photon process with a two-level system is given by [21, 22, 23]

$$H / \hbar = \Omega \Lambda_1 + \epsilon \Lambda_2 + W_2$$

where $\Lambda_k$ with $k = 1, 2$ are constants of motion, i.e., $\Lambda_1 = K_0 + \frac{1}{2} \sigma_z$ and $\Lambda_2 = a_1^\dagger a_1 - a_2^\dagger a_2$ while the remaining interaction is defined by

$$W_2 = \frac{\delta}{2} \sigma_z + g(\sigma_- K_+ + \sigma_+ K_-).$$

We define the parameters $\Omega = \Omega_1 + \Omega_2$, $\epsilon = (\Omega_1 - \Omega_2)/2$, and $\omega_0 = \Omega_1 + \Omega_2 + \delta$. Together with $\{K_+, K_0, K_-\}$ the generators of $\text{su}(1,1)$, that is,

$$K_0 = \frac{1}{2} \left( a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \right), \quad K_- = a_1 a_2, \quad K_+ = a_1^\dagger a_2^\dagger.$$

The eigenvalue equation for the Hamiltonian can be solved and then its corresponding eigenvectors for the constants of motion taking the fixed values,

$$(\Lambda_1, \Lambda_2) \rightarrow (\lambda_1, \lambda_2) = \left( \frac{n_1 + n_2}{2} + 1, n_1 - n_2 \right),$$

are denoted in the form

$$|\chi_+(n_1, n_2)\rangle = \frac{1}{\sqrt{\omega_+ (2 \omega_+ - \delta)}} \times \left( -g \sqrt{(n_1 + 1)(n_2 + 1)} |n_1 n_2 \epsilon\rangle + (\delta / 2 - \omega_+) |n_1 + 1 n_2 + 1 \epsilon\rangle \right)$$

and

$$|\chi_-(n_1, n_2)\rangle = \frac{1}{\sqrt{\omega_+ (2 \omega_+ + \delta)}} \times \left( -g \sqrt{(n_1 + 1)(n_2 + 1)} |n_1 n_2 \epsilon\rangle + (\delta / 2 + \omega_+) |n_1 + 1 n_2 + 1 \epsilon\rangle \right).$$
The corresponding eigenvalues are given by \( E_{\pm} = \Omega \lambda_1 + \epsilon \lambda_2 + \omega_{\pm} \), with

\[
\omega_{\pm} = \pm \sqrt{\left( \frac{\delta}{2} \right)^2 + g^2(n_1 + 1)(n_2 + 1)}.
\]

For \((\lambda_1, \lambda_2) = (0, 0)\), one has \(|00\rangle\) with energy \( E_0 = -\frac{\delta}{2} \).

The evolution operator is written in the form \( U(t) = e^{-i(2\omega A_1 + \epsilon A_2) t} e^{-i W_2 t} \). Making the Taylor series expansion of \( U_2(t) \equiv e^{-i W_2 t} \),

\[
U_2(t) = F(\nu, t)|e\rangle \langle e| - i K_- \overline{G}(\nu', t)|e\rangle \langle g| - i K_+ \overline{G}(\nu, t)|g\rangle \langle e| + \overline{F}(\nu', t)|g\rangle \langle g|,
\]

with the definitions

\[
F(\nu, t) = \cos \left( gt \sqrt{\nu} \right) - i \frac{\delta \sin \left( gt \sqrt{\nu} \right)}{2g \sqrt{\nu}}, \quad \overline{G}(\nu, t) = \frac{\sin \left( gt \sqrt{\nu} \right)}{\sqrt{\nu}}.
\]

In the last expressions, the operators \( \nu, \nu' \) are given by

\[
\nu = \frac{\delta^2}{4g^2} I + N_1 N_2, \quad \nu' = \frac{\delta^2}{4g^2} I + N_1 N_2,
\]

where \( N_k = a_k^\dagger a_k \), with \( k = 1, 2 \).

For the two photon case, one needs to define two dimensional Fock states: \(|n + q, n\rangle\), and the action of the SU(1, 1) generators on the states is given by

\[
K_0 |n + q, n\rangle = \frac{1}{2}(q + 2n + 1) |n + q, n\rangle, \quad K_+ |n + q, n\rangle = \sqrt{(n + q + 1)(n + 1)} |n + q, n + 1, n + 1\rangle, \quad K_- |n + q, n\rangle = \sqrt{(n + q - 1)(n - 1)} |n + q - 1, n - 1\rangle,
\]

where \( n + q, n \) denote the number of \( a_1, a_2 \) photons, respectively. The irreducible representation (IRREP) of the SU(1, 1) group is denoted by: \( W = \frac{q+1}{2} \), where we consider bigger the number of photons \( a_1 \) than \( a_2 \).

The associated coherent state can be written in the form

\[
|\eta, q\rangle = (1 - |\eta|^2)^{q+1} \sum_{n=0}^{\infty} \binom{n + q}{n}^2 \eta^n |n + q, n\rangle.
\]

In this case, the real part of \( \eta \) can be written in terms of the average number of photons in the cavity, i.e., \( |\eta|^2 = \frac{n_0}{q+q_0+1} \).

We study the evolution of the matter-field quantum entanglement in a cavity of the following initial wave packets:

- \(|\psi(0)\rangle = |q\rangle \otimes |\eta, q\rangle\).

The essential dynamics of the initial wave packet is due to the evolution operator \( U_2(t) \), then we have compared the properties of the evolution of the initial states with \( q = 0 \) with \( q = 4 \). We consider the interaction Hamiltonian \( W_2 \) with the parameters \( g = 1 \), and \( \delta = 0 \). In both cases we have taken \( \eta = 0.8 \), which yields an average number of photons \( \bar{n}_{20} = 1.78 \) and \( \bar{n}_{10} = q + 1.78 \) for the modes in the cavity. In Fig. 3 we have plotted the occupation probabilities of the ground and excited states of the two-level system for the IRREPs \( q = 0 \) and \( q = 4 \) of SU(1, 1). Notice that for the case \( q = 4 \) at the time \( \tau \approx 3 \) there is an
Figure 3. At the top row the occupation probabilities of the ground (red dashed line) and excited states (blue line) as function of time are displayed. At the middle row the linear entropy is plotted as function of time. At the bottom the autocorrelation function is displayed as function of time. In all the rows we have plotted the functions at left for $q = 0$ while at right for $q = 4$. Parameters $g = 1$, $\delta = 0$, $\eta = 0.8$.

inversion of population from the ground to the excited state. In the same figure we show also the entanglement properties of the matter and field sectors for the mentioned IRREPs of SU(1, 1). One can see that for the times $\tau = 0, \pi, 2\pi$ there is not entanglement between the matter and radiation field. Finally we displayed the autocorrelation function of the considered initial states as function of time, again for the mentioned IRREPs of SU(1, 1), one can notice in both cases the collapses and revivals of the function.

- $|\psi(0)\rangle = (\cos\theta|\nu_1\nu_2\rangle + \sin\theta|\nu_1+3,\nu_2\rangle) \otimes |g\rangle$.

For both cases we are considering an average number of photons equal to $n_0$ in the cavity. In the second initial case, we are considering a combination of states with constants of motion with values equal to: $\lambda_{(1)} = (\nu_1 + \nu_2)/2$ and $\lambda_{(2)} = \nu_1 - \nu_2$, which has a discrete symmetry $C_{\lambda_{(2)}}$.

In Fig. 4 the Husimi Function with $C_3$ symmetry is displayed in contour and three dimensional plots in phase space for a fixed time $\tau = 1$, an initial state with $\nu_1 = \nu_2 = 2$, and $\theta = \pi/4$, and for the Hamiltonian we take the parameters $g = 1$ and $\delta = 0$.

As an example about the construction of reduced density matrix, we calculate explicitly the reduced density matrix of the first mode of light, that is,

$$\rho_{F1}(t) = \sum_{\sigma, n_2} \langle \sigma n_2 | \rho(t) | \sigma n_2 \rangle.$$ 

By substituting the density matrix of the system $\rho(t) = U_2(t) \rho(0) U_2^\dagger(t)$, one gets
Figure 4. The Husimi function with $C_3$ point symmetry in phase space is plotted. At left a contour plot and at right a three dimensional plot. We take the parameters: $\theta = \pi/4$, $\delta = 0$; $g = 1$; $\nu_1 = 2$, $\nu_2 = 2$; $\tau = 1$.

The Husimi function is defined by the expression

$$Q(x,y,t) = (1 - |\eta|^2)^{q+1} e^{-(x^2+y^2)} \frac{(x^2 + y^2)^q}{q!} \sum_k \frac{(q+k)!}{q!k!} |\eta|^2^k \left\{ |G(k)|^2 |q+k\rangle \langle q+k| + (q+k+1)(k+1) F^2(k) |q+k+1\rangle \langle q+k+1| \right\} .$$

The second moment of the Husimi function is defined by the expression

$$M(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q^2(x,y,t) \, dx \, dy ,$$

whose inverse determines the localization properties of the Husimi function in phase space. In Fig. 5 the second moment of the Husimi function is plotted as function of time for the SU(1,1) IRREPs $q = 0$ and $q = 4$. Notice that the state of light for the first mode of the system is more localized for $q = 0$ than for $q = 4$. 

$$\rho_{F_1}(t) = (1 - |\eta|^2)^{q+1} \sum_k \frac{(q+k)!}{q!k!} |\eta|^2^k \left( |G(k)|^2 |q+k\rangle \langle q+k| + (q+k+1)(k+1) F^2(k) |q+k+1\rangle \langle q+k+1| \right) ,$$

where we have defined two function $F$ and $G$, i.e.,

$$F^2(k) = \sin^2 \left\{ gt \sqrt{\frac{\delta^2}{4 g^2} + (q+k+1)(k+1)} \right\} ,$$

$$|G^2(k)|^2 = \cos^2 \left\{ gt \sqrt{\frac{\delta^2}{4 g^2} + (q+k+1)(k+1)} \right\} + \frac{\delta^2}{4 g^2} \sin^2 \left\{ gt \sqrt{\frac{\delta^2}{4 g^2} + (q+k+1)(k+1)} \right\} .$$

In the last expressions, I remind you the meaning of the parameters: $\delta$ is the detuning, $g$ the coupling constant, $q$ is denoting the IRREP of the SU(1,1), and $\eta$ the coherent state complex parameter. By taking the expectation value of the reduced density matrix of the field with respect to the corresponding coherent state, one gets the Husimi function for the first mode of light.
5.3. Wigner Function for one and two photon process.

For any linear combination of Fock states $|\psi\rangle = \sum_n c_n |n\rangle$ the density matrix is given by
$$\rho = \sum_{n,m} c_n c_m^* |n\rangle \langle m|.$$

The Wigner function is given by
$$W(x,p) = \sum_{n,m} c_n c_m^* W_{|n\rangle \langle m|}(x,p),$$
where the Moyal function is defined by
$$W_{|n\rangle \langle m|}(x,p) = \frac{(-1)^m}{\pi} 2^{-n-m} \frac{n!}{m!} (x-ip)^{n-m} e^{-(x^2+p^2)} L_n^{m-n}(2(x^2+p^2)).$$

for $n \leq m$ for the opposite case, change $x-ip \leftrightarrow x+ip$ and $n \leftrightarrow m$.

The Wigner Function for the two-photon process can be obtained through the following procedure: first one determines the one-mode reduced density matrix reduced, that is,
$$\rho_{F1}(t) = \sum_k \{ C(k,t) |q+k\rangle \langle q+k| + D(k,t) |q+k+1\rangle \langle q+k+1| \},$$
where
$$C(k,t) = (1-|\eta|^2)^{q+1} \frac{(q+k)!}{q! k!} |\eta|^{2k} |G(k)|^2$$
$$D(k,t) = (1-|\eta|^2)^{q+1} \frac{(q+k)!}{q! k!} |\eta|^{2k} (q+k+1)(k+1) F^2(k).$$

Then the Wigner function can be calculated
$$W(x,p) = \sum_k \{ C(k,t) W_{|q+k\rangle \langle q+k|}(x,p) + D(k,t) W_{|q+k+1\rangle \langle q+k+1|}(x,p) \},$$
with the Moyal function
$$W_{|n\rangle \langle m|}(x,p) = \frac{(-1)^n}{\pi} e^{-(x^2+p^2)} L_n(2(x^2+p^2)).$$

In Fig. 6 we have displayed the Wigner function in phase space for three times $\tau = 0, \pi/2, 2\pi$ and for the IRREP $q = 4$, in all the selected times one can appreciate that the Wigner function
Figure 6. The Wigner function is displayed in phase space for $\tau = 0$ (left), $\tau = \pi/2$ (middle), $\tau = 2\pi$ (right). Parameters $g = 1$, $\delta = 0$, $\eta = 0.8$, $q = 4$.

takes negative values implying that one mode states of light exhibit strong quantumness properties.

The dynamics of field statistics of the two-mode electromagnetic fields can be done mainly by two aspects: the Cauchy-Schwartz inequality and the squeezing phenomena. In this contribution we calculate only the Cauchy-Schwartz inequality

$$I_0 = \sqrt{G_{11}^{(2)} G_{22}^{(2)}} - 1.$$ (39)

Notice that a violation of the (CS) inequality implies the possibility of formation of quantum states that exhibit EPR correlations or violate a Bell’s inequality [16].

For this Hamiltonian model (5.2), one has $N_1 = K_0 + \frac{1}{2}(q - 1) I$ and $N_2 = K_0 - \frac{1}{2}(q + 1) I$, form which it is straightforward to determine the second order correlation functions, that is,

$$G_{11}^{(2)} = \langle K_0^2 \rangle - (q - 2) \langle K_0 \rangle + \frac{1}{4}(q - 1)(q - 3),$$
$$G_{22}^{(2)} = \langle K_0^2 \rangle - (q + 2) \langle K_0 \rangle + \frac{1}{4}(q + 1)(q + 3),$$
$$G_{12}^{(2)} = \langle K_0^2 \rangle - \langle K_0 \rangle - \frac{1}{4}(q^2 - 1).$$ (40)

In Fig. 7 we displayed the calculation of the CS inequality for the IRREPs $q=0$ and $q=12$ of the group SU(1, 1) as functions of time. Although there are fluctuations at the times $\tau = \pi, 2\pi$ one can see all the time a violation of the inequality, which is stronger for the smaller values of $q$. We use the parameters of the Hamiltonian $g = 1$ and $\delta = 0$, while for the initial state $\eta = 0.8$, which is equivalent to an average number of photons $n_0 = 1.78$ for the second mode.

6. Summary and conclusiones
A review of the unitary representation of linear canonical transformation has been done following a set of papers established by Marcos Moshinsky and collaborators at the beginning of the seventies. These contributions have been very useful to study many body systems. We have established that any linear canonical transformation can be generated by the action of a product of two rotations plus a scaling unitary transformation.

The two photon coherent states have been described as a unitary transformation acting on a coherent state, where it is enhanced that the unitary transformation is associated to a linear canonical transformation. Additionally we have shown that the TCS can be generated by quadratic in position and momentum Hamiltonians and the corresponding evolution operator can be determined analytically in terms of the solutions of a nonlinear Riccati equation (17).
Figure 7. The Cauchy-Schwartz inequality is plotted as function of time. At top-left for the IRREP $q = 0$ and at top-right for $q = 12$. Parameters $g = 1$, $\delta = 0$, $\eta = 0.8$.

Finally we have study the entanglement, quantumness, and delocalization properties of two effective matter-field Hamiltonians. For both effective Hamiltonians the evolution operators can be obtained in analytic form, which have been used to study the dynamic properties of two initial wave packets. By establishing special initial wave packets, both one- and two-photon process Hamiltonians can be used to generate states which carry the irreducible representations of cyclic point groups $C_{|M_2-M_1|}$ and $C_q$, respectively.

For the effective one-photon process Hamiltonian we have exhibited the correlated behaviour of the occupation probability of excited state, and the linear entropy as function of time. Additionally the Husimi function of light in phase space has been displayed, which is similar a cat state solution.

The dynamics of the initial wavepacket with the two-photon process Hamiltonian shows correlations between the occupation probabilities, the linear entropies, and the autocorrelation function, that is, the inversion of population is anti-correlated with respect to the linear entropy together with a series of collapses and revivals. For this case we have shown the delocalization properties of light in phase space through the calculation of the second moment of the Husimi function plus the behaviour of the Wigner function exhibiting negative values. In both cases we have considered the IRREPs $q = 0$ and $q = 4$ of the group SU(1, 1). The Cauchy-Schwartz inequality is violated for any small value of the IRREP $q$ of SU(1, 1), although it is stronger for larger values of $q$. Thus the two-photon process Hamiltonian is a good tool to study EPR type of violations.

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