Tully-Fisher Scalings and Boundary Conditions for Wave Dark Matter

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Abstract

We investigate a theory of dark matter called wave dark matter, also known as scalar field dark matter (SFDM) and boson star dark matter or Bose-Einstein condensate (BEC) dark matter (also see axion dark matter), and its relation to the Tully-Fisher relation. We exhibit two boundary conditions that give rise to Tully-Fisher-like relations for spherically symmetric static wave dark matter halos: (BC1) Fixing a length scale at the outer edge of wave dark matter halos gives rise to a Tully-Fisher-like relation of the form \( M / v^4 \) = constant. (BC2) Fixing the density of dark matter at the outer edge of wave dark matter halos gives rise to a Tully-Fisher-like relation of the form \( M / v^{3.4} \) = const. These results extend the work of a previous paper [4].

1 Introduction

The baryonic Tully-Fisher relation (BTFR) is an empirical scaling relation for disk galaxies involving the baryonic (i.e., non-dark) mass \( M_b \) and the rotational velocity \( v \) of stars. It is observed that

\[
\frac{M_b}{v^x} = \text{const.} \tag{1}
\]

where the exponent \( x \) (the slope of the relation in log-log space) is usually found to lie between 3 and 4. There is an extensive literature which uses astronomical data to calibrate the relation and estimate \( x \). See, for example, [19, 1, 29, 16, 6, 8, 18, 27, 17].

In the relation (1), the rotational velocity \( v \) is determined by the total gravitating mass (baryonic and dark) while \( M_b \) accounts for the baryonic matter only. Thus, the BTFR provides a link between the baryonic and dark matter and may have something to say about the viability of various theories of dark matter.

In this paper we discuss the BTFR in the context of a theory of wave dark matter. Wave dark matter [2, 3, 5, 4, 20, 21] has been investigated under other names such as scalar field dark matter (SFDM) [10, 9, 14, 26, 22] and boson star dark matter or Bose-Einstein condensate (BEC) dark matter [23, 25, 12, 13, 15, 24, 28]. The difference in names comes from a difference in motivations, but the underlying equation is the Klein-Gordon wave equation (25) for a scalar field.
In spherical symmetry, there are simple “static states” of wave dark matter (see figs. 6 to 9 at the end of the paper). Sin proposed in [25] that each galactic dark matter halo corresponds (in a first approximation) to one of these static states. If we take this suggestion as a starting point, a natural concomitant idea is that the baryonic Tully-Fisher relation might arise because a corresponding “Tully-Fisher-like” relation holds for the wave dark matter halos. This Tully-Fisher-like relation would hold because of the nature of wave dark matter. The fundamental question we ask in this paper is whether the equations of wave dark matter have the potential to give rise to such a Tully-Fisher-like relation (see Question 1 on page 5). We answer this question in the affirmative and exhibit two boundary conditions, which could be physical conditions imposed at the edge of each wave dark matter halo, that give rise to just such a Tully-Fisher-like relation. The boundary conditions are as follows:

**BC1:** Fixing a length scale at the outer edge of halos implies a Tully-Fisher-like relation with slope 4. This is a more general version of a result from a previous paper [4].

**BC2:** Fixing the dark matter density at the outer edge of halos implies a Tully-Fisher-like relation with slope 3.4.

The remainder of this paper runs as follows: In section 2 we introduce the spherically symmetric static states and their scaling properties. In section 3 we exhibit BC1 and BC2 numerically, and in section 4 we derive BC1 and BC2 theoretically. In section 5 we describe other conditions one might impose which do not lead to Tully-Fisher-like relations. Finally in section 6 we briefly discuss the connection between a Tully-Fisher-like relation for wave dark matter halos and the baryonic Tully-Fisher relation.

## 2 The Spherically Symmetric Static States

The equations underlying the theory of wave dark matter are the Einstein-Klein-Gordon (EKG) equations:

\[
G = 8\pi \left( \frac{df \otimes d\bar{f} + d\bar{f} \otimes df}{\Upsilon^2} - \left( \frac{|df|^2}{\Upsilon^2} + |f|^2 \right) g \right) \tag{2a}
\]

\[
\Box_g f = \Upsilon^2 f. \tag{2b}
\]

(Throughout, we use geometrized units in which the universal gravitation constant and the speed of light are unity: \(G = c = 1\).) Here \(f\) is a complex scalar field representing the dark matter and \(\Upsilon\) is a fundamental constant of nature. The relationship between \(\Upsilon\) and the wave dark matter particle mass \(m\) is

\[
m = \frac{h\Upsilon}{c} = 2.09 \times 10^{-23} \text{eV} \left( \frac{\Upsilon}{1 \text{ly}^{-1}} \right). \tag{3}
\]
In this paper for definiteness we use \( \Upsilon = 10 \text{ly}^{-1} \), which is consistent with constraints from other studies [11, 5, 26].

Because we are interested in studying galactic dark matter halos, most of which are thought to be approximately spherically symmetric, we study the EKG equations in spherical symmetry. The spacetime metric we will use is

\[
g = -e^{2V(t,r)} \, dt^2 + \Phi(t,r)^{-1} \, dr^2 + r^2 (d\theta^2 + \sin^2 \phi \, d\phi^2) \quad (4)
\]

where \( \Phi(t,r) = 1 - \frac{2M(t,r)}{r} \). In the non-relativistic limit, the functions \( V(t,r) \) and \( M(t,r) \) have natural interpretations as the gravitational potential at time \( t \) and radius \( r \) and the total mass contained inside a ball of radius \( r \) at time \( t \). The simplest solutions to eq. (2) in the metric (4) are the spherically symmetric static states, which are of the form

\[
f(t,r) = F(r) e^{i\omega t}. \quad (5)
\]

Substituting the ansatz (5) into eq. (2) and using eq. (4), we obtain (see [21]) the relatively simple system of ODEs

\[
\begin{align*}
M_r &= 4\pi r^2 \cdot \frac{1}{\Upsilon^2} \left[ (\Upsilon^2 + \omega^2 e^{-2V}) \, F^2 - \Phi F_r^2 \right] \quad (6a) \\
\Phi V_r &= \frac{M}{r^2} - 4\pi r \cdot \frac{1}{\Upsilon^2} \left[ (\Upsilon^2 - \omega^2 e^{-2V}) \, F^2 - \Phi F_r^2 \right] \quad (6b) \\
F_{rr} + \frac{2}{r} F_r + V_r F_r + \frac{\Phi_r}{\Phi} F_r &= \Phi^{-1} \left( \Upsilon^2 - \omega^2 e^{-2V} \right) F \quad (6c)
\end{align*}
\]

which can be integrated on a computer. Here the functions \( M(r), V(r), F(r) \) are functions of \( r \) only, and the only dependence on \( t \) appears in eq. (5).

In the non-relativistic limit the solutions of (6) are indistinguishable from the solutions of the even simpler system of ODEs

\[
\begin{align*}
M_r &= 4\pi r^2 \cdot 2 F^2 \quad (7a) \\
V_r &= \frac{M}{r^2} \quad (7b) \\
\frac{1}{2\Upsilon} \left( F_{rr} + \frac{2}{r} F_r \right) &= (\Upsilon - \omega + \Upsilon V) F. \quad (7c)
\end{align*}
\]

This system is the Poisson-Schrödinger (PS) system written in spherical symmetry. Here we have a special case of the well-known fact that the PS system is the non-relativistic limit of the EKG system—see [7]. From here on in this paper we work exclusively with the simpler PS system (7) because we are interested in wave dark matter in galaxies which are, considered as an aggregate, non-relativistic systems.

Finite-mass solutions to eq. (7) come in the form of ground states and excited states. It was proposed in [25] that each galactic dark matter halo corresponds (in a first approximation) to one of these states. In figs. 6 to 9 we display some representative graphs of solutions so that the reader can get a sense of what they look like, and refer
the reader to a previous paper [4] for more examples. We assign each solution a non-negative integer \( n \) which counts the number of zeros (nodes) of \( F \). A ground state has \( n = 0 \); excited states have \( n \geq 1 \). Any solution can be given as a quadruple \((\omega; M, V, F)\) consisting of the constant \( \omega \) followed by the three functions \( M(r), V(r), \) and \( F(r) \). It is easily verified that if \((\omega; M, V, F)\) is a solution, then so is \((\omega + \Upsilon V; M, V + \bar{V}, F)\). We use this freedom to shift every potential function \( V(r) \) so that it approaches zero at infinity.

For each static state there is a special value of \( r \) which we refer to as \( R_{\text{DM}} \) and which we think of as the edge of the dark matter halo. Mathematically, \( R_{\text{DM}} \) is the particular value of \( r \) at which the function \( F(r) \) switches from oscillatory to exponentially decreasing behavior. Looking at eq. (7c), we see that \( R_{\text{DM}} \) satisfies the equation \( \Upsilon - \omega + \Upsilon V(R_{\text{DM}}) = 0 \). Thus we make the following definition:

**Definition 1.** Given any spherically symmetric static state we define \( R_{\text{DM}} \) to be the radius at which the function \( F \) switches from oscillatory to exponential behavior. It satisfies the equation

\[
\Upsilon - \omega + \Upsilon V(R_{\text{DM}}) = 0. \tag{8}
\]

We regard \( r = R_{\text{DM}} \) as the edge of the dark matter halo because almost all the dark matter is contained in the region \( r \leq R_{\text{DM}} \). See figs. 6 to 9.

It is well-known that solutions to the PS system (7) have certain scaling properties. Specifically, for \( \alpha, \beta > 0 \), we can scale any given solution as follows:

\[
\begin{align*}
\bar{r} &= \alpha^{-1} \beta^{-1} r \\
\bar{M} &= \alpha \beta^{-3} M \\
\bar{V} &= \alpha^2 \beta^{-2} V \\
\bar{F} &= \alpha^2 F \\
\bar{\Upsilon} &= \beta^2 \Upsilon \\
(\bar{\Upsilon} - \bar{\omega}) &= \alpha^2 (\Upsilon - \omega)
\end{align*}
\]

and obtain a new solution. These scalings can be derived immediately from eqs. (7a) to (7c). Note that \( \beta \) effectively controls the fundamental constant of nature \( \Upsilon \) and \( \alpha \) can be used to control the “peakedness” of the solution. States which are more compact have higher mass; states which are more spread out have lower mass. In fig. 1 we show the effect of scaling a ground state using values of \( \alpha \) near 1. It follows from these scalings that having fixed \( \Upsilon \), there is a one-parameter family of static states of order \( n \). Once we fix a further characteristic of the solution, a static state is uniquely determined for each \( n \), giving us a sequence of static states. We call this “fixing a scaling” or “imposing a scaling condition”.
Figure 1: Five scalings of the $n = 1$ excited state from fig. 7 (shown in blue) using $\alpha = 1.1, 1.2, 1.3, 1.4, 1.5$ in the scaling formulas (9). Here we only show the dark matter profile function $F(r)$.

3 Tully-Fisher Scalings and Boundary Conditions: Numerical Evidence

The basic question we are asking in this paper is:

**Question 1.** Are there any scaling conditions for which the sequence of static states $n = 0, 1, 2, \ldots$ obeys a Tully-Fisher-like relation

$$\frac{M}{v^x} = \text{constant}$$

for some $3 \leq x \leq 4$?

We need to make this question more precise, as it is unclear above what $M$ and $v$ are, exactly, given a particular static state. Astrophysicists attempting to calibrate the baryonic Tully-Fisher relation are faced with a similar problem. In that case, $M$ is an estimate for the total baryonic mass. For $v$, some take it to be the maximum observed velocity; some take it to be an average velocity computed in some well-defined way; some take it to be the velocity observed at a particular (arbitrarily chosen) radius. It is well-known that the rotation curves of many spiral galaxies are somewhat flat, and in this case any of these three choices will produce similar values for $v$.

We have chosen to take $M = M(R_{DM})$ and $v = v(R_{DM})$. This latter quantity is the circular velocity which would be observed for objects orbiting at $r = R_{DM}$. This is computed in the standard Newtonian way:

$$\frac{v(r)^2}{r} = \frac{M(r)}{r^2} \quad \implies \quad v(r) = \sqrt{\frac{M(r)}{r}}. \quad (11)$$
Perhaps a more natural choice for $M$ would be $M_x = \lim_{r \to \infty} M(r)$, i.e. the total mass of the static state. But $M(R_{DM}) \approx M_x$ since $R_{DM}$ is by definition the “outer edge” of the halo (see figs. 6 to 9), so it does not really matter which we choose. Similarly, another natural choice for $v$ would be $v_{max}$, the maximum value of the function $v(r)$. But again, by examining figs. 6 to 9, one can see that $v(R_{DM}) \approx v_{max}$. Thus, the following more precise version of Question 1 seems reasonable:

**Question 2.** Are there any scaling conditions for which the sequence of static states $n = 0, 1, 2, \ldots$ obeys a Tully-Fisher-like relation

$$\frac{M(R_{DM})}{v(R_{DM})^x} = \text{constant} \quad (12)$$

for some $3 \leq x \leq 4$?

Now, in one sense the answer to Question 2 is a trivial “yes”, for we can just pick any $x$ we like and let eq. (12) itself be the scaling condition. However, this is obviously cheating; we are interested in scaling conditions which might correspond to something physical. In fact, we have found two conditions that we consider to be more physical. These were given in section 1 along with the Tully-Fisher-like relations they imply. We restate them here more precisely:

**BC1:** Fixing a length scale at $R_{DM}$ implies a Tully-Fisher-like relation with slope 4.

**BC2:** Fixing $|F(R_{DM})|$ implies a Tully-Fisher-like relation with slope 3.4.

With regards to **BC1**, we should explain what “fixing a length scale” means. We postpone a full explanation to section 4 and limit ourselves here to an explanation which involves what we term the “halflength” of a state, defined as follows:

**Definition 2.** The $F(r)$ function for any static state has an exponentially decreasing “tail” lying to the right of $R_{DM}$. There is some $R > R_{DM}$ such that $F(R) = \frac{1}{2}F(R_{DM})$; we refer to $R - R_{DM}$ as the halflength of the state.

Requiring the halflength of all states to be the same is one way to fix a length scale at $R_{DM}$. We refer the reader to section 4 for full details.

With regards to **BC2** note that, by eq. (7a), $|F|$ determines the density of the wave dark matter. This explains why in section 1 we referred to “fixing the dark matter density” in **BC2**.

It remains in this section to show numerically that **BC1** and **BC2** really do give Tully-Fisher-like relations with the stated slopes. We used a computer to derive particular static state solutions for $n = 0, 1, \ldots, 800$ and then scaled these solutions in accordance with eq. (9) so that, first, they all had the same halflength, and second, so they all had the same value for $|F(R_{DM})|$. (The precise values for the halflength and $|F(R_{DM})|$ were chosen so that the resulting static states would be of reasonable sizes to be representing galactic halos.) The results are shown in fig. 2 in which we have plotted $M(R_{DM})$ vs. $v(R_{DM})$ in log-log space. For large $n$, the static states form a line with the stated slope. (For small $n$, the states deviate slightly from the correct slope. This will be explained in section 4.)
4 Tully-Fisher Scalings and Boundary Conditions: Theoretical Derivations

In this section we wish to explain theoretically why the statements made in BC1 and BC2 are true. We have already exhibited their truth numerically in fig. 2.

Both statements involve boundary conditions which fix some aspect of the solutions to the PS system around the point \( r = R_{DM} \). We can approximate the solutions around \( R_{DM} \) by using a linear approximation for \( V \):

\[
V(r) \approx V(R_{DM}) + V'(R_{DM})(r - R_{DM})
\]

\[
= V(R_{DM}) + \frac{M(R_{DM})}{R_{DM}^2}(r - R_{DM})
\]

for \( r \approx R_{DM} \). Substituting this approximation for \( V \) into eq. (7c) and using eq. (8), we find that for \( r \approx R_{DM} \), \( F(r) \) approximately solves the differential equation

\[
F_{rr} + \frac{2}{r}F_r = \frac{2\Upsilon^2 M(R_{DM})}{R_{DM}^2}(r - R_{DM})F.
\]
Multiplying through by $r$, we can write this differential equation as
\[
(rF)_{rr} = \frac{2\Upsilon^2 M(R_{DM})}{R_{DM}^2}(r - R_{DM})(rF). \tag{14}
\]

Compare this to the Airy differential equation $y'' = xy$ whose general solution is a linear combination of the two standard functions $\text{Ai}(x)$ and $\text{Bi}(x)$. Both $\text{Ai}(x)$ and $\text{Bi}(x)$ oscillate for negative $x$ and have exponential behavior for positive $x$. The function $\text{Ai}(x)$ is shown in fig. 3. It exponentially decreases to zero for $x > 0$, whereas the function $\text{Bi}(x)$ (not shown) exponentially increases for $x > 0$.

![Figure 3: In black, the standard solution $\text{Ai}(x)$ to the Airy differential equation $y'' = xy$. In brown and dashed, the approximation (21) to $\text{Ai}(x)$ for $x < 0$.](image)

Looking at eq. (14), we see that for $r \approx R_{DM}$, $rF$ approximately solves an equation of the form $y'' = a(x - x_0)y$. The general solution to this differential equation is $A \cdot \text{Ai}(a^{1/3}(x - x_0)) + B \cdot \text{Bi}(a^{1/3}(x - x_0))$. Since $F$ exponentially decreases for $r > R_{DM}$, we must have, therefore,
\[
F(r) \propto \frac{1}{r} \text{Ai} \left( \left( \frac{2\Upsilon^2 M(R_{DM})}{R_{DM}^2} \right)^{1/3} (r - R_{DM}) \right) \tag{15}
\]
for $r \approx R_{DM}$. If we choose the proportionality constant in eq. (15) so that the approximation has the correct value at $r = R_{DM}$, then we get
\[
F(r) \approx F(R_{DM}) \frac{R_{DM}}{r} \frac{1}{\text{Ai}(0)} \text{Ai} \left( \left( \frac{2\Upsilon^2 M(R_{DM})}{R_{DM}^2} \right)^{1/3} (r - R_{DM}) \right) \tag{16}
\]
for $r \approx R_{DM}$.

Equation (15) demonstrates mathematically the truth of BC1. For up to a vertical scaling, the shape of $F(r)$ around $r = R_{DM}$ is given by the expression on the right side of (15). Thus, fixing a length scale around $r = R_{DM}$ evidently amounts to fixing the value of the constant
\[
\frac{2\Upsilon^2 M(R_{DM})}{R_{DM}^2}. \tag{17}
\]
This argument explains why fixing a length scale is basically equivalent to fixing a value for the half-length (as we did in section 3) or to any of a number of other conditions which fix some length around $R_{DM}$ (for another example, see our previous paper [4]). All such conditions are fixing a particular horizontal scaling of the expression in eq. (16).

Now, the rotation curve calculation (11) shows that

$$\frac{R_{DM}^2}{M(R_{DM})} = \frac{M(R_{DM})}{v(R_{DM})^4};$$

(18)

since $\Upsilon$ is a constant (in this paper, $\Upsilon = 10 \text{ly}^{-1}$), we see that fixing a length scale is equivalent to fixing the constant in eq. (18). This explains why we get the Tully-Fisher-like relation shown in fig. 2 with slope 4. The approximation (15) gets better for larger $n$ (see figs. 6 to 9), which explains why in fig. 2 the lower order static states deviate from the slope 4 line.

To show the truth of BC2, we begin with the approximation (16). This expression contains the four numbers $\Upsilon$, $R_{DM}$, $M_{p}R_{DM}^{q}$, and $F_{p}R_{DM}^{q}$. These cannot all be independently chosen, for the static states effectively live in a 3-parameter space. (Recall the 2-parameter family of scalings (9). A third parameter hiding there is the discrete parameter $n$, the order of the static state.) Thus, we should be able to find some relation between these four numbers. In fact, we will argue that

$$|F_{p}R_{DM}^{q}| \approx 2^{-17/12} \text{Ai}(0) \cdot \Upsilon^{1/6} M(R_{DM})^{7/12} R_{DM}^{-17/12}$$

for large $n$ (19)

where

$$2^{-17/12} \text{Ai}(0) \approx 0.133$$

(20)

This explains BC2, for then we see that fixing $|F(R_{DM})|$ amounts to fixing $M(R_{DM})^{7} R_{DM}^{-17}$. By eq. (11), $R_{DM} = M(R_{DM})v(R_{DM})^{-2}$, so fixing $M(R_{DM})^{7} R_{DM}^{-17}$ is equivalent to fixing $M(R_{DM})^{-10} v(R_{DM})^{34} = (M(R_{DM})/v(R_{DM})^{3.4})^{-10}$, i.e. equivalent to fixing

$$\frac{M(R_{DM})}{v(R_{DM})^{3.4}}.$$

Our goal now is to derive eq. (19) beginning with the approximation (16). We will utilize a well-known approximation for the Airy function $\text{Ai}(x)$ for negative inputs:

$$\text{Ai}(-x) \approx \frac{\sin \left( \frac{2}{3} x^{3/2} + \frac{\pi}{4} \right)}{\sqrt{x^{1/4}}}$$

for $x > 0$. (21)

This approximation can be seen in fig. 3. It is extremely good for $x > 1$. We will also use the following definite integral, which is easily obtained:

$$\int_{0}^{L} \sin^{2} \left( \frac{2}{3} x^{3/2} + \frac{\pi}{4} \right) \frac{dx}{x^{1/2}} = L^{1/2} + O(1).$$

(22)

The notation $O(1)$ is mathematical shorthand for a bounded function of $L$. 

9
Using eqs. (7a) and (16), we have

\[
M(R_{DM}) = \int_0^{R_{DM}} 8\pi r^2 F^2 \, dr
\]

\[
\approx \int_0^{R_{DM}} 8\pi F(R_{DM})^2 R_{DM}^2 \text{Ai} \left[-\left(\frac{2\gamma^2 M(R_{DM})}{R_{DM}^2}\right)^{1/3} (R_{DM} - r)\right]^2 \, dr
\]

We will temporarily abbreviate \( R_{DM} = R, M(R_{DM}) = M, F(R_{DM}) = F, \text{Ai}(0) = A \).

Making the substitution \( x = \left(\frac{2\gamma^2 M}{R^2}\right)^{1/3} r \) and using the approximation (21), we obtain

\[
M \approx 8\pi F^2 R^2 \frac{\text{Ai}}{A^2} \cdot \left(\frac{2\gamma^2 M}{R^2}\right)^{-1/3} \int_0^{(2\gamma^2 MR)^{1/3}} \text{Ai}(-x)^2 \, dx
\]

\[
\approx \frac{2^{8/3}}{A^2} \gamma^{-2/3} F^2 R^{8/3} M^{-1/3} \int_0^{(2\gamma^2 MR)^{1/3}} \sin^2 \left(\frac{2x^3/2 + \pi}{4}\right) \, dx.
\]

Using the definite integral (22), we obtain

\[
M \approx \frac{2^{8/3}}{A^2} \gamma^{-2/3} F^2 R^{8/3} M^{-1/3} \left[\left(\frac{2\gamma^2 MR}{R^2}\right)^{1/6} + O(1)\right]
\]

\[
\approx \frac{2^{17/6}}{A^2} \gamma^{-1/3} F^2 R^{17/6} M^{-1/6}.
\]

(We dropped the \( O(1) \) term because the other term dominates for large \( n \).) Rearranging, we get

\[
F^2 \approx 2^{-17/6} A^2 \gamma^{1/3} M^{7/6} R^{-17/6}.
\]

Taking square roots gives us eq. (19), which completes the argument.

One might wonder how good the approximation (19) is for small \( n \). In fig. 4 we have graphed the ratio

\[
\frac{|F(R_{DM})|}{2^{-17/12} \text{Ai}(0) \cdot \gamma^{1/6} M(R_{DM})^{7/12} R_{DM}^{-17/12} R_{DM}^{17/12}}
\]

as a function of \( n \). For the ground state \((n = 0)\), the ratio is approximately 1.15, which is reasonably close to 1, and for large \( n \) the ratio is extremely close to 1.

Combining eqs. (16) and (19) gives the following approximation for the \( F(r) \) function of a static state of order \( n \). (By convention \( F(0) \) is positive so the factor \( (-1)^n \) is necessary.)

\[
F(r) \approx (-1)^n 2^{-17/12} \gamma^{2/12} M(R_{DM})^{7/12} R_{DM}^{-5/12} \cdot \frac{1}{r} \text{Ai} \left[\left(\frac{2\gamma^2 M}{R_{DM}^2}\right)^{1/3} (r - R_{DM})\right]
\]

This approximation is shown in figs. 6 to 9.
5 Other Scaling Conditions

The scaling conditions given in BC1 and BC2 are boundary conditions imposed at the edge of dark halos. The first imposes a horizontal scale on $F(r)$ and the second imposes a vertical scale. One might wonder whether there are any other scaling conditions which lead to Tully-Fisher-like relations. For example, what happens when we try the same thing at the center of the halo? To impose a horizontal scale on $F(r)$ at the origin, we can fix the location of the first node (for $n \geq 1$). To impose a vertical scaling, we can fix $|F(0)|$. In fig. 5 we show the results of imposing these two scaling conditions. Neither leads to a Tully-Fisher-like relation. In fact, besides BC1 and BC2 no other scaling condition we have tried leads to a Tully-Fisher-like relation.

6 Conclusion

We have exhibited two boundary conditions, BC1 and BC2, which we restate here once more in their more colloquial form from section 1:

BC1: Fixing a length scale at the outer edge of spherically symmetric wave dark matter halos implies a Tully-Fisher-like relation with slope 4.

BC2: Fixing the dark matter density at the outer edge of spherically symmetric wave dark matter halos implies a Tully-Fisher-like relation with slope 3.4.

We gave both numerical evidence and theoretical arguments illustrating the truth of BC1 and BC2. Our main goal in this paper has been to demonstrate that a Tully-Fisher-like relation can arise naturally out of the equations of wave dark matter.

To conclude we comment briefly on the connection between a Tully-Fisher-like relation for wave dark matter halos and the observed baryonic Tully-Fisher relation. Define the baryon fraction of a galaxy to be the fraction of the galaxy’s mass which is baryonic. If the baryon fraction were the same for every galaxy, then the baryonic Tully-Fisher relation would have the same slope as a Tully-Fisher-like relation for dark halos. However, it is observed that the baryon fraction of galaxies decreases with size. Small galaxies contain mostly dark matter, and only very large galaxies contain baryons in sufficient quantities to approach the cosmic baryon fraction, which is around 1/6. This is known as the missing baryon problem (see [18]).

In $M$ vs. $v$ log-log space, when switching from plotting dark mass to baryonic mass, all galaxies shift vertically downward. The missing baryon problem implies that large galaxies shift less than small galaxies. Therefore, if the baryonic Tully-Fisher relation is related to a Tully-Fisher-like relation for dark halos, the former relation should have a steeper slope than the latter.
Figure 4: The ratio given in eq. (23) plotted as a function of $n$ ($n = 0, 1, \ldots, 800$). This graph shows that the approximation (19) is very good for large $n$ and not too bad for small $n$.

Figure 5: At left, the static states $n = 1, \ldots, 800$ with $\Upsilon = 10 \text{ ly}^{-1}$ and the first node fixed at 1500 ly. At right, the static states $n = 0, 1, \ldots, 800$ with $\Upsilon = 10 \text{ ly}^{-1}$ and $|F(0)| = 8 \times 10^{-7}$. The dashed lines have slope 4. Neither scaling condition gives a Tully-Fisher-like relation.
Figure 6: A ground state \((n = 0)\) with \(\Upsilon = 10 \text{ ly}^{-1}\). Units are geometrized and given in light-years. The first three functions graphed are the solutions \(M(r)\), \(V(r)\), \(F(r)\) to eq. (7). They are, respectively, the total mass profile, the Newtonian potential, and the dark matter profile. The fourth function is a rotation curve \(v(r)\) calculated in accordance with eq. (11). The vertical gray line marks the location of \(R_{\text{DM}}\). The dashed curves show the approximations described in section 4 in eqs. (13) and (24).
Figure 7: An excited state \((n = 1)\). See the caption to fig. 6 for a fuller description.
Figure 8: An excited state \((n = 10)\). See the caption to fig. 6 for a fuller description.
Figure 9: An excited state ($n = 100$). See the caption to fig. 6 for a fuller description. The top and bottom of the $F(r)$ function have been cropped out to show more detail.
References

[1] Eric F. Bell and Roelof S. de Jong. “Stellar Mass-to-Light Ratios and the Tully-Fisher Relation”. In: *Astrophys. J.* 550.1 (2001), pp. 212–229. DOI: 10.1086/319728

[2] Hubert L. Bray. “On Dark Matter, Spiral Galaxies, and the Axioms of General Relativity”. In: *Geometric Analysis, Mathematical Relativity, and Nonlinear Partial Differential Equations*. Vol. 599. Contemp. Math. Amer. Math. Soc., Providence, RI, 2013, pp. 1–64. DOI: 10.1090/conm/599/11945 arXiv: 1004.4016 [physics.gen-ph]

[3] Hubert L. Bray. *On Wave Dark Matter, Shells in Elliptical Galaxies, and the Axioms of General Relativity*. Dec. 2012. arXiv: 1212.5745 [physics.gen-ph]

[4] Hubert L. Bray and Andrew S. Goetz. *Wave Dark Matter and the Tully-Fisher Relation*. Sept. 2014. arXiv: 1409.7347 [astro-ph.GA]

[5] Hubert L. Bray and Alan R. Parry. *Modeling Wave Dark Matter in Dwarf Spheroidal Galaxies*. Jan. 2013. arXiv: 1301.0255 [astro-ph.GA]

[6] M. Geha et al. “The Baryon Content of Extremely Low Mass Dwarf Galaxies”. In: *Astrophys. J.* 653 (Dec. 2006), pp. 240–254. DOI: 10.1086/508604 eprint: astro-ph/0608295.

[7] Domenico Giulini and André Großardt. “The Schrödinger-Newton equation as a Non-Relativistic Limit of Self-Gravitating Klein-Gordon and Dirac Fields”. In: *Class. Quantum Grav.* 29.21 (2012), p. 215010. DOI: 10.1088/0264-9381/29/21/215010

[8] Sebastián Gurovich et al. “The Slope of the Baryonic Tully-Fisher Relation”. In: *Astron. J.* 140.3 (2010), pp. 663–676. DOI: 10.1088/0004-6256/140/3/663

[9] F. Siddhartha Guzmán and L. Arturo Ureña López. “Evolution of the Schrödinger-Newton system for a Self-Gravitating Scalar Field”. In: *Phys. Rev. D* 69.12 (June 2004), p. 124033. DOI: 10.1103/PhysRevD.69.124033

[10] F Siddhartha Guzmán and Tonatiuh Matos. “Scalar Fields as Dark Matter in Spiral Galaxies”. In: *Class. Quantum Grav.* 17.1 (2000), p. L9. DOI: 10.1088/0264-9381/17/1/102

[11] Wayne Hu, Rennan Barkana, and Andrei Gruzinov. “Fuzzy Cold Dark Matter: The Wave Properties of Ultralight Particles”. In: *Phys. Rev. Lett.* 85.6 (Aug. 2000), pp. 1158–1161. DOI: 10.1103/PhysRevLett.85.1158

[12] S. U. Ji and S. J. Sin. “Late-time Phase Transition and the Galactic Halo as a Bose Liquid. II. The Effect of Visible Matter”. In: *Phys. Rev. D* 50.6 (Sept. 1994), pp. 3655–3659. DOI: 10.1103/PhysRevD.50.3655

[13] Chi Wai Lai and Matthew W. Choptuik. “Final Fate of Subcritical Evolutions of Boson Stars”. In: (Sept. 2007). arXiv: 0709.0324 [gr-qc]
[14] Juan Magaña and Tonatiuh Matos. “A Brief Review of the Scalar Field Dark Matter Model”. In: Journal of Physics: Conference Series 378.1 (2012). doi:10.1088/1742-6596/378/1/012012 arXiv:1201.6107 [astro-ph]

[15] Tonatiuh Matos and L. Arturo Ureña-López. “Flat Rotation Curves in Scalar Field Galaxy Halos”. In: General Relativity and Gravitation 39.8 (2007), pp. 1279–1286. issn: 0001-7701. doi:10.1007/s10714-007-0470-y

[16] Stacy S. McGaugh. “The Baryonic Tully-Fisher Relation of Galaxies with Extended Rotation Curves and the Stellar Mass of Rotating Galaxies”. In: Astrophys. J. 632.2 (2005), pp. 859–871. issn: 0004-637X. doi:10.1086/432968

[17] Stacy S. McGaugh. “The Baryonic Tully-Fisher Relation of Gas-rich Galaxies as a Test of ΛCDM and MOND”. In: Astron. J. 143.2 (2012), pp. 40–44. doi:10.1088/0004-6256/143/2/40

[18] Stacy S. McGaugh et al. “The Baryon Content of Cosmic Structures”. In: Astrophys. J. Lett. 708.1 (Jan. 2010), pp. L14–L17. doi:10.1088/2041-8205/708/1/L14 arXiv:0911.2700 [astro-ph.CO]

[19] Stacy S. McGaugh et al. “The Baryonic Tully-Fisher Relation”. In: Astrophys. J. Lett. 533.2 (Apr. 2000), p. L99. doi:10.1086/312628 arXiv:astro-ph/0003001

[20] Alan R. Parry. Spherically Symmetric Static States of Wave Dark Matter. Dec. 2012. arXiv:1212.6426 [gr-qc]

[21] Alan R. Parry. “Wave Dark Matter and Dwarf Spheroidal Galaxies”. Ph.D. Duke University, 2012. arXiv:1311.6087 [gr-qc]

[22] Varun Sahni and Limin Wang. “New Cosmological Model of Quintessence and Dark Matter”. In: Phys. Rev. D 62.10 (Oct. 2000), p. 103517. doi:10.1103/PhysRevD.62.103517

[23] Edward Seidel and Wai-Mo Suen. “Dynamical Evolution of Boson Stars: Perturbing the Ground State”. In: Phys. Rev. D 42.2 (July 1990), pp. 384–403. doi:10.1103/PhysRevD.42.384

[24] R. Sharma, S. Karmakar, and S. Mukherjee. “Boson star and dark matter”. In: (Dec. 2008). arXiv:0812.3470 [gr-qc]

[25] Sang-Jin Sin. “Late-time Phase Transition and the Galactic Halo as a Bose Liquid”. In: Phys. Rev. D 50.6 (Sept. 1994), pp. 3650–3654. doi:10.1103/PhysRevD.50.3650 arXiv:hep-ph/9205208

[26] Abril Suárez, Victor H. Robles, and Tonatiuh Matos. “A Review on the Scalar Field/Bose-Einstein Condensate Dark Matter Model”. English. In: Accelerated Cosmic Expansion. Ed. by Claudia Moreno Gonzalez, Jos Edgar Madriz Aguilar, and Luz Marina Reyes Barrera. Vol. 38. Astrophysics and Space Science Proceedings. Springer International Publishing, 2014, pp. 107–142. isbn: 978-3-319-02062-4. doi:10.1007/978-3-319-02063-1_9
[27] S. Torres-Flores et al. “GHASP: an Hα kinematic survey of spiral and irregular galaxies - IX. The near-infrared, stellar and baryonic Tully-Fisher relations”. In: *Mon. Not. R. Astron. Soc.* 416 (Sept. 2011), pp. 1936–1948. DOI: [10.1111/j.1365-2966.2011.19169.x](10.1111/j.1365-2966.2011.19169.x).

[28] L. Arturo Ureña-López and Argelia Bernal. “Bosonic Gas as a Galactic Dark Matter Halo”. In: *Phys. Rev. D* 82.12 (Dec. 2010), p. 123535. DOI: [10.1103/PhysRevD.82.123535](10.1103/PhysRevD.82.123535).

[29] Marc A. W. Verheijen. “The Ursa Major Cluster of Galaxies. V. H I Rotation Curve Shapes and the Tully-Fisher Relations”. In: *Astrophys. J.* 563.2 (2001), pp. 694–715. DOI: [10.1086/323887](10.1086/323887).