Effects of Electron Band Structure on Neutrino Pair Bremsstrahlung in Neutron Star Crusts

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(March 21, 2022)

Abstract

We calculate the rate of energy emission by bremsstrahlung of neutrino pairs by electrons moving in the crystalline lattice of ions in dense matter in the crust of a neutron star. Since the periodic potential in the solid gives rise to electronic band gaps which can be as large as about 1 MeV, it is necessary in estimating the bremsstrahlung rate at low temperatures to take into account band structure in detail. We find that, in the densest parts of the inner crust of a neutron star, neutrino emission at temperatures of about $2 \times 10^9$K or less is much suppressed compared with earlier estimates that treated the electron-lattice interaction perturbatively, and conclude that neutrino pair bremsstrahlung by electrons in the crusts of neutron stars is much less impor-
tant for neutron star thermal evolution than was previously thought.
I. INTRODUCTION

The possible importance of neutrino pairs generated in collisions of electrons with nuclei was pointed out by Pontecorvo [1], and the first detailed calculations of the rate of the process were made by Gandel’man and Pineev [2], who considered the case of non-degenerate, non-relativistic electrons scattering from uncorrelated nuclei. This work was extended by Festa and Ruderman [3], who performed calculations for degenerate electrons. They allowed for relativistic effects, and took into account correlations between ions and screening of the ionic Coulomb potential by electrons. Their results were extended by Cazzola, De Zotti and Saggion [4], who performed calculations for uncorrelated ions for a range of conditions. A general many-body formulation of the problem of emission of neutrino pairs by electrons scattering from ions was made by Flowers [5], who applied his results to estimate bremsstrahlung by scattering from the static lattice and also gave expressions for rates of processes in which lattice phonons are emitted or absorbed. Flowers also stressed the importance of the finite nuclear size in reducing the bremsstrahlung rate. The effects of weak neutral currents was investigated by Dicus, Kolb, Schramm, and Tubbs [6]. Further studies were made by Soyeur and Brown [7], who compared energy loss by the bremsstrahlung process with other energy loss mechanisms for various neutron star models. In recent years the process has been studied in detail by Itoh and collaborators [8].

Basic to all the above works is the assumption that the effects of the electron-ion interaction may be treated in lowest-order perturbation theory, and the two perturbation theory diagrams considered are shown in Fig. 1. Since the energy and momentum carried away by the neutrino pair are of order $k_B T$ and $k_B T/c$ respectively, and since the final particles are on mass shell, it is easy to see that the intermediate energy denominator must be of order $k_B T$. The dimensionless parameter in the perturbation expansion is thus of order $V_q/k_B T$, where $V_q$ is the Fourier transform of the potential due to the ions. For a random collection of ions, the Fourier transform of the potential experienced by an electron will be of order $n_{Z}^{\frac{1}{2}} \mathcal{V}_q/\Omega^{\frac{1}{2}}$ where $\mathcal{V}_q$ is the Fourier transform of the potential due to a single ion, $n_{Z}$ is the
number density of ions, and \( \Omega \) is the volume of the system. Thus one does not expect problems with the perturbation expansion in this case. However, the situation is very different if matter is a solid, with ions ordered on a periodic lattice. The Fourier transform of the electron-ion interaction is then of order \( n_Z V_K \), where \( K \) is a reciprocal lattice vector and this will lead to energy splittings in the electron spectrum of this order of magnitude. When the temperature is low compared with this splitting, perturbation theory fails. As we shall argue in detail below, the splittings can be as large as 1 MeV in the inner crust of neutron stars, and therefore at temperatures below about \( 10^{10} \) K the first-order perturbation theory calculations of the rate fail.

In this paper we calculate bremsstrahlung of neutrino pairs by electrons moving in the periodic potential of the ions. Our primary purpose in this paper is to examine elastic scattering of electrons from the lattice, since this process was previously thought to be more important at low temperatures than processes in which lattice phonons are emitted or absorbed \[4\]. Instead of using perturbation theory to treat the electron-ion interaction, we first calculate band electron states, thereby implicitly taking into account the electron-ion interaction to all orders in perturbation theory, and then evaluate the rate of emission of neutrino pairs when an electron makes a transition from one band state to another. What we find is that bremsstrahlung of neutrino pairs is suppressed exponentially at temperatures low compared with the strength of the ionic potential, which determines the splitting between bands. A brief account of part of this work has been given previously \[9\].

The paper is organized as follows. In Sec.II we describe the basic physics and derive an expression for the neutrino bremsstrahlung emissivity. Analytical expressions for the emissivity in the high- and low-temperature limits are derived and numerical calculations are made for intermediate temperatures in Sec.III. Section IV contains a discussion of the results, and a comparison of neutrino emission by bremsstrahlung from the periodic lattice with that from conversion of phonons into neutrino pairs.
II. BASIC CALCULATIONS

The basic process we consider is shown in Fig. 2, where the double lines represent a band electron. If the band electron propagators in Fig. 2 are expanded to first order in the electron-ion interaction, one recovers the two diagrams shown in Fig. 1. To lowest order in the weak interaction, the rate for an electron in state 1 to emit a neutrino pair in a transition to a state 2, assumed to be unoccupied initially, may be found from Fermi’s Golden Rule, and is given by

\[
dR = \frac{2\pi}{\hbar} \delta(E_f - E_i)|H_{fi}|^2 d\Gamma,
\]  

(1)

where \(H_{fi}\) is the matrix element of the weak interaction between the initial electron state \(i\) and the final state \(f\). The energies of all particles in the initial and final states are denoted by \(E_i\) and \(E_f\). We denote the energies of the initial and final electrons by \(E_1\) and \(E_2\) respectively, and those of the outgoing neutrino and antineutrino by \(E_\nu\) and \(E_\bar{\nu}\), and \(d\Gamma\) is an element of the volume of phase space. The total rate of energy loss in neutrino pairs per unit volume is thus given by

\[
\dot{E} = \frac{2\pi}{\hbar \Omega} \sum \delta(E_f - E_i)f(E_1)(1 - f(E_2))|H_{fi}|^2(E_\nu + E_\bar{\nu}).
\]  

(2)

Here \(f(E)\) is the Fermi function, and the factors \(f(E_1)\) and \(1 - f(E_2)\) in Eq. (2) take into account the Pauli principle, since the transition can occur only if the initial electron state is occupied, and the final one empty. There are no blocking factors for the neutrinos, since we assume that neutrinos are able to escape easily, a good assumption for neutron stars except for some seconds immediately following the birth of the star.

To calculate the rate at which an electron in one band state makes a transition to another band state we need to evaluate matrix elements of the weak interaction Lagrangian

\[
\mathcal{L} = -\sqrt{2}G\bar{\Psi}_\nu \Gamma^\alpha_L \gamma_\nu \Psi (C_L \Gamma^\alpha_L + C_R \Gamma^\alpha_R) \Psi,
\]  

(3)

using band states for the electrons, rather than the plane waves of most interest in the majority of applications. In Eq.(3), \(G\) is the Fermi coupling constant, \(\Gamma^\alpha_{L,R} = \gamma^\alpha(1 \mp \gamma_5)/2\),
and in terms of the weak mixing angle $\theta_W$, $C_L = 1 + 2 \sin^2 \theta_W$ and $C_R = 2 \sin^2 \theta_W$ for the emission of electron neutrinos. The corresponding couplings for the emission of muon and $\tau$ neutrinos, are $C'_L = -1 + 2 \sin^2 \theta_W$ and $C'_R = 2 \sin^2 \theta_W$.

For the moment, let us be quite general, and consider transitions in which the initial electron state is specified by the wavenumber, $p$, a band index, $n$, and a spin index, $\sigma$, and the final state is specified by similar quantities with primes. The squared modulus of the matrix element, summed over spins of the initial and final electron, neutrino and antineutrino states, is given by

$$
\sum_{\sigma_\nu, \bar{\sigma}_\nu} |H_{fi}|^2 = \frac{1}{\Omega} 2 G^2 R_{\alpha\beta} \langle p_1, \sigma_1, n_1 | J^\alpha_{q_1+q_2} | p_2, \sigma_2, n_2 \rangle \langle p_2, \sigma_2, n_2 | J^\beta_{q_1-q_2} | p_1, \sigma_1, n_1 \rangle ,
$$

where

$$
J^\alpha_k = \frac{1}{\sqrt{\Omega}} \int d^3 x e^{-i k \cdot x} (C_L \Gamma^\alpha_L + C_R \Gamma^\alpha_R) ,
$$

and the neutrino part of the trace is given by

$$
R_{\alpha\beta} = \sum_{\sigma_1, \sigma_2} \bar{u}_{\sigma_1}(q_1) \Gamma^\alpha_{L} v_{\sigma_2}(q_2) \bar{v}_{\sigma_2}(q_2) \Gamma^\beta_{L} u_{\sigma_1}(q_1) .
$$

The neutrino part may be evaluated by using the fact that for ultrarelativistic fermions

$$
u_{\sigma_1}(k) \otimes \bar{v}_{\sigma_2}(k) = \frac{k}{2k^0} ,
$$

and one finds

$$
R_{\alpha\beta} = \frac{2}{2q_1^0 2q_2^0} (q_1^0 q_2^\beta + q_1^\beta q_2^0 - g^{\alpha\beta} q_1 \cdot q_2 + i \epsilon^{\alpha\beta\gamma\delta}(q_1)_\gamma(q_2)_\delta) .
$$

The electronic matrix elements depend on the neutrino momenta only through the total four momentum of the pair, $q = (\omega/c, q) = q_1 + q_2$ and therefore one may sum over neutrino states in the expression for the energy emission rate, Eq.(2), keeping the total four momentum of the pair fixed. We then use the result

$$
\frac{c(2\pi \hbar)^3}{\Omega} \sum_{q_1, q_2} \delta(\omega - E_{\nu} - E_{\bar{\nu}}) \delta_{q_1+q_2} \frac{q_1^0 q_2^\beta}{2q_1^0 2q_2^0} = \frac{\pi}{24} (2q^\alpha q^\beta + g^{\alpha\beta} q^2) \theta(\omega - c|q|) ,
$$
to show that

$$\sum_{q_1,q_2} \delta_{q_1+q_2} \delta(\omega - E_{\nu} - E_{\bar{\nu}}) \mathcal{R}^{\alpha\beta} = \frac{\Omega}{24\pi^2\hbar^3 c} (q^\alpha q^\beta - q^2 g^{\alpha\beta}) \theta(\omega - c|\mathbf{q}|) . \quad (10)$$

Here $\theta(x)$ is the Heaviside step function, which is unity for $x > 0$ and vanishes otherwise. The form of this result, apart from the numerical coefficient, follows directly from the observation that the left hand side of this equation is a second rank Lorentz tensor with vanishing divergence. The vanishing of the divergence is a consequence of neutrino current conservation for free neutrinos, as may be seen by contracting Eq.(6) with $q^\sigma$. Inserting Eq.(10) into Eq.(4), we find

$$\sum |H_{fi}|^2 = \frac{G^2}{12\pi^2\hbar^3 c} \sum_{\mathbf{q},\mathbf{p}_1,\mathbf{p}_2} \sum_{\sigma_1,\sigma_2} \int d\omega \theta(\omega - c|\mathbf{q}|) \left( \langle 1| q \cdot J_{\mathbf{q}} |2 \rangle \langle 2| q \cdot J_{-\mathbf{q}} |1 \rangle - q^2 \langle 1| J_{\mathbf{q}}^\alpha |2 \rangle \langle 2| J_{-\mathbf{a}_\alpha} |1 \rangle \right). \quad (11)$$

This expression is quite general: effects of band structure are included, through the matrix elements, and no assumption has been made regarding how relativistic the electrons are.

First we make a general remark about what transitions are possible. The bremsstrahlung process is kinematically forbidden for initial and final electron states in the same band, n. The argument is essentially identical to the one to demonstrate that (photon) bremsstrahlung by a free electron is impossible. As a consequence of the fact that the electron (group) velocity $\mathbf{\nabla}_p E$ cannot exceed $c$, the energy difference, $E_1 - E_2$ between electron states must be less than $cq$, where $q$ is the total momentum of the neutrino pair. However, this energy difference, which is equal to the energy of the neutrino pair, must exceed $cq$. Consequently, it is impossible simultaneously to conserve energy and momentum for electron states in the same band. This argument does not apply to transitions between electron states in different bands because the energy difference between the two electron states will generally be non-zero for small momentum transfers.

We now turn to the case of the inner crust of neutron stars, where the electrons are highly relativistic, with Lorentz factors of 50 or more. Under these circumstances it is an
excellent approximation to neglect the mass of the electron, since this leads to errors of order $(m_e c^2/\mu_e)^2$, which is less than $10^{-3}$. Here $\mu_e$ is the electron chemical potential (neglecting electrostatic contributions), which in the extremely relativistic limit is simply $p_F c$, with $p_F$ being the electron Fermi momentum. The electron helicity is a good quantum number in this limit, and this leads to several simplifications. First, the left- and right-handed currents satisfy a continuity equation $q \cdot J_q = 0$ for each helicity, and therefore the contributions in Eq. (11) involving such quantities vanish. The energy emission rate is thus

$$
\dot{E} = -\frac{G^2}{6\pi \Omega h^4 c} \sum_{\mathbf{q}, \mathbf{p_1}, \mathbf{p_2}} \int d\omega \, \omega q^2 \langle 1 | J^\alpha_q | 2 \rangle \langle 2 | J^\alpha_{-\mathbf{q}} | 1 \rangle \delta(E_1 - E_2 - \omega) \, f(E_1) \left(1 - f(E_2)\right) \theta(\omega - c|\mathbf{q}|).
$$

(12)

Second, in the term of the form $J \cdot J$ the two helicities decouple, and the net result is proportional to $C_L^2 + C_R^2 = 2(C_V^2 + C_A^2)$, where $C_V = (C_L + C_R)/2$ and $C_A = (C_L - C_R)/2$. The rate of energy emission is thus $(\frac{1}{2} + \sin^2 \theta_W)^2$ times the rate calculated in V-A theory, for which $C_A = C_V = 1$. The rate in muon and $\tau$ neutrinos is obtained in a similar fashion. For them the vector and axial vector couplings are $C'_A = 1 - C_A$ and $C'_V = 1 - C_V \[6\]$. The total rate of neutrino energy emission in all types of neutrinos is thus given by replacing $C_V^2 + C_A^2$ with $C'_V^2 + C'_A^2 + 2((1 - C_A)^2 + (1 - C_V)^2)$.

The next task is to evaluate the weak interaction matrix elements, and to do this we require a knowledge of the electronic states. Since splittings between bands are small compared with a typical electron energy, it is a very good approximation for all but a few electron states to use the almost-free-electron approximation, in which one takes into account just one Fourier component of the periodic potential at a time. The component of the lattice potential with wavenumber $\mathbf{K}$ mixes free electron states with wavenumber $\mathbf{k}$ with those with wavenumbers $\mathbf{k} \pm \mathbf{K}$. In the nearly-free electron approximation (see e.g., Ashcroft and Mermin \[11\]), one considers only the mixing of the two electron states that are closest in energy, and for definiteness we denote the wavenumbers of the two states by $\mathbf{k}$ and $\mathbf{k} - \mathbf{K}$. The energy eigenvalues are given by
\[ E^\pm(k) = \frac{\epsilon_k + \epsilon_{k-K}}{2} \pm \sqrt{\left(\frac{\epsilon_k - \epsilon_{k-K}}{2}\right)^2 + V_k^2}, \]

where the plus sign corresponds to the upper band, and the minus sign to the lower band. Here \( \epsilon_k \) is the energy of a free electron, and \( V_K \) is the matrix element of the static part of the electron-ion interaction between the two free-electron states that are mixed. Since the helicity of the electrons is conserved in the ultrarelativistic limit that we are treating, the matrix element is simply the Fourier transform of the potential for wavenumber \( K \) acting on an electron, multiplied by a factor giving the overlap between the states with wavenumbers \( k \) and \( k - K \), but of the same helicity. The helicity factor is, apart from a phase, just \( \cos\left(\frac{1}{2} \theta k, k - K\right) \), where \( \theta k, k - K \) is the angle between the two electron wave vectors. In our calculations the band states of greatest interest are those close to the Fermi surface, and with a component of the wavevector along \( K \) close to \( \pm K/2 \). For this case the modulus of the helicity factor is then simply \( v_\perp = (1 - (K/(2k_F))^2)^{1/2} \). The Fourier component of the electrostatic potential due to the ions is given by \( 4\pi e/K^2 \langle \rho^i_k \rangle \), where \( \langle \rho^i_k \rangle \), is the statistical average of the Fourier transform of the charge density of the ions. As a consequence of screening by electrons, the total electrostatic potential due to both ions and electrons is given by the expression for ions alone, reduced by a factor of the electronic dielectric function, \( \epsilon_k \). In the standard Thomas-Fermi approximation this is given by \( \epsilon_k = 1 + (k_{TF}/K)^2 \), where for the screening wavenumber we use the result for an ultrarelativistic electron gas, \( k_{TF}^2 = 4\alpha k_F^2/\pi \), with \( \alpha = e^2/(\hbar c) \). In summary, the matrix elements of the lattice potential are given by

\[ V_K = -v_\perp \frac{4\pi e \langle \rho^i_k \rangle}{K^2} = -v_\perp \frac{4\pi e^2 n_Z Z F_k e^{-W}}{K^2 \epsilon_k}, \]

where \( n_Z \) and \( Z \) are, respectively, the number density and proton number of the ions, \( F_k \) is the form factor reflecting the shape of the nuclear charge distribution and and \( W \) is the Debye-Waller factor describing thermal vibrations.

The evaluation of the electron trace is more complicated because of the electron band structure. First of all, a remark about bookkeeping: the electron spectrum is shown in Fig. 4.
as a function of $k_\parallel$ the component of the electron wavenumber parallel to $\mathbf{K}$ for a fixed value of $k_\perp$, the component of the electron wavenumber perpendicular to $\mathbf{K}$. For simplicity we take into account only the effects of the lattice potential at a single reciprocal lattice vector. At low temperatures the electron states that can participate in neutrino pair emission are those close to the maxima and minima of the electron spectrum. With the normal choice of Brillouin zone boundaries, these states would lie close to the zone boundary, and one could, for example, have processes in which an electron on branch 1 made a transition to a state on branch 2, or to a state on branch 4. The fact that the wavenumbers of states on branches 2 and 4 differ by $\mathbf{K}$ complicates the calculations, and it is convenient to redefine the zone boundary so that the branches $3'$ and $4'$ occur in the same zone as branches 1 and 2. All states with $k_\parallel \simeq K/2$ are then given by a single expression, with no discontinuities. For definiteness the wavenumbers of the two plane wave states which are strongly mixed will be denoted by $\mathbf{k}$ and $\mathbf{k} - \mathbf{K}$.

The expression for the electronic part of the matrix element may be simplified by making use of the fact that we are interested in processes at low temperatures, and therefore the energy and momentum of the neutrino pair are small compared with the Fermi momentum of the electrons, and the reciprocal lattice vectors. In particular, the small momentum of the neutrino pair means that the only Fourier transforms of the current operators of interest are ones with long wavelengths, and therefore $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$. Matrix elements of the current operator between the two states will possess components with wavenumbers $\mathbf{p}_1 - \mathbf{p}_2 \pm \hbar\mathbf{K}$, Umklapp terms, but these will not enter in our calculation as a consequence of the fact that the low temperature restricts possible neutrino momenta to values much less than $\hbar K$. We note that had we made the conventional choice of position of the zone boundary, we would have been forced to consider processes in which $\mathbf{p}_1 - \mathbf{p}_2$ is comparable in magnitude to $\hbar K$. This reflects the fact that what are normal processes for one choice of the zone boundary may be Umklapp processes for some other choice. The states in the upper and lower electron bands are
\[ \Psi_{p,\sigma}^+(r) = \frac{1}{\sqrt{\Omega}} (u_p e^{i p \cdot r} u_\sigma(p) + v_p e^{i (p - hK) \cdot r} u_\sigma(p - hK)) \]

\[ \Psi_{p,\sigma}^-(r) = \frac{1}{\sqrt{\Omega}} (v_p e^{i p \cdot r} u_\sigma(p) - u_p e^{i (p - hK) \cdot r} u_\sigma(p - hK)) \]  

(15)

The expressions for the “coherence factors” \( u_p \) and \( v_p \) will be given below.

In calculating long-wavelength components of the current matrix elements it is convenient to invoke the identity

\[ \bar{u}_{\sigma_1}(k) \gamma^\mu u_{\sigma_2}(k) = \frac{k^\mu}{k_0} \delta_{\sigma_1,\sigma_2} \]  

(16)

We find

\[ \langle p_2, \sigma_2, n_2 | J_\alpha^a | p_1, \sigma_1, n_1 \rangle \]

\[ = \frac{1}{\sqrt{\Omega} p_F} \delta_{q,p_1-p_2}\delta_{n_1,n_2+1} (u_1 v_2 P^\alpha - v_1 u_2 P^\alpha) (C_L \delta_{-,\sigma_1,\sigma_2} + C_R \delta_{+,\sigma_1,\sigma_2}) + Umklapp \]  

(17)

where we have neglected terms of order \( \Delta p/p_F \), where \( \Delta p \) is the difference between the electron (pseudo)momentum and the Fermi momentum. In Eq.\((17)\), \( u_{1,2} = u_{p_{1,2}}, v_{1,2} = v_{p_{1,2}} \) and the electron four momenta \( p_{1,2} = (E_{1,2}/c, p_{1,2}) \) occur in the combinations \( P = (p_1 + p_2)/2 \), \( P' = P - (0, hK) \). We thus find for the electron trace

\[ \sum_{\sigma_1,\sigma_2} \langle 1 | J_\alpha^a q | 2 \rangle \langle 2 | J_{-\alpha} | 1 \rangle = -\frac{16 v_\parallel^2 C_A^2 + C_R^2}{\Omega} u_1 v_1 u_2 v_2 \delta_{q,p_1-p_2} \delta_{n_1,n_2+1} + Umklapp \]  

(18)

where \( v_\parallel = \sqrt{1 - v_\perp^2} = K/(2k_F) \). From Eq.\((18)\) follows that the energy emission rate in neutrinos is given by

\[ \dot{E} = \frac{G^2}{48 \pi^7 \hbar^3} \frac{C_A^2 + C_R^2}{2} \sum_{\mathbf{K}} v_\parallel^2 \int d\omega dq dp_1 dp_2 \omega q^2 u_1 v_1 u_2 \]

\[ \delta(E_1 - E_2 - \omega) \delta^{(3)}(p_1 - p_2 - q) f(E_1) (1 - f(E_2)) \theta(\omega - c |q|) \]  

(19)

The sum over electron bands is contained in the sum over reciprocal lattice vectors \( \mathbf{K} \) in Eq.\((19)\). Note that since electron states were defined within a given Brillouin zone, Eq.\((19)\) includes a factor of \( 1/2 \) so that there is no overcounting when summing over equal but opposite reciprocal lattice vectors. From Eq.\((17)\) one also finds that \( \langle 1 | q \cdot J_\alpha | 2 \rangle \langle 2 | q \cdot J_{-\alpha} | 1 \rangle \propto \)}
\[(u_1 v_2 q \cdot P - v_1 u_2 q \cdot P')^2\]. It may be verified that this vanishes identically, as argued earlier on the basis of current conservation.

We now perform the energy and momentum integrations. To do this we require expressions for the “coherence factors”, \(u_p\) and \(v_p\). They are identical in form to those found in the standard treatment of nonrelativistic electrons in a weak periodic potential and are given by

\[
u_p = \frac{V_K}{\sqrt{2 \mathcal{E}_p (\mathcal{E}_p - \xi_p)}}, \quad v_p = \sqrt{\frac{\mathcal{E}_p - \xi_p}{2 \mathcal{E}_p}},
\]

\[
u^2_p = \frac{1}{2} (1 + \frac{\xi_p}{\mathcal{E}_p}), \quad v^2_p = \frac{1}{2} (1 - \frac{\xi_p}{\mathcal{E}_p}), \quad \text{and} \quad u_p v_p = \frac{V_K}{2 \mathcal{E}_p},
\]

with \(\xi_p = (\epsilon_p - \epsilon_{p\to n}\mathbf{K})/2\) and \(\mathcal{E}_p = \sqrt{\mathcal{E}_p^2 + V_K^2}\). We also note that, in terms of the variables \(\xi^2 + V_K^2\), the energies of the upper and lower electron states are

\[E^\pm(\Delta p^\perp, \Delta p^\parallel) = p_F c + \Delta p^\perp v^\perp c \pm \sqrt{(\Delta p^\parallel v^\parallel c)^2 + V_K^2},\]

from which we find \(\omega = cv^\perp q^\perp + E_1 + E_2\).

It is convenient to express momenta in cylindrical coordinates, with respect to \(\mathbf{K}\). Also, since the electrons participating in neutrino pair emission are close the maxima and minima of the electron spectrum, we define the shifted variables

\[
\Delta p^\parallel = p^\parallel - v^\parallel p_F \text{ and } \Delta p^\perp = p^\perp - v^\perp p_F.
\]

We also have

\[
q^\perp = p^\perp_1 - p^\perp_2 = \Delta p^\perp_1 - \Delta p^\perp_2
\]

\[
q^\parallel = p^\parallel_1 - p^\parallel_2 = \Delta p^\parallel_1 - \Delta p^\parallel_2
\]

\[
q_{\phi} = p_F v^\perp (\phi_1 - \phi_2),
\]

where in the last expression, we have discarded curvature terms, which contribute only to next-to-leading order in \(k_B T/p_F c\). The \(p_2\) integration is now trivially performed, with the aid of momentum conserving delta function, and the \(p_1\) angular integration simply gives the
constant $2\pi$. Since the coherence factors do not depend on $p^\perp$, the integration over $p^\perp_1$ is straightforward and yields the result

$$\int p^\perp_1 dp^\perp_1 f(E_1)(1 - f(E_2)) = v^\perp_{PF} \int d\Delta p^\perp_1 \frac{f(E_1 - \omega) - f(E_1)}{e^{\beta \omega} - 1}$$

$$= p^\perp_{PF} \frac{\omega}{c e^{\beta \omega} - 1},$$  (24)

where we have again utilized the fact that characteristic temperatures are much smaller than the Fermi energy. Performing the remaining energy integration finally yields the expression for the energy emission rate per unit volume,

$$\dot{E} = \frac{G^2}{24 \pi^6 h^{10} c^3} \frac{C_A^2 + C_V^2}{2} \mu_e \sum_{\mathbf{k}} v^3_{\parallel} \int d\Delta p^\parallel_1 \alpha q^2 q^2 u_1 v_1 u_2 v_2 \frac{1}{e^{\beta \omega} - 1} \theta(\omega - c|q|),$$  (25)

with $\omega = E_1 - E_2$.

Inserting into Eq. (25) the expressions for the coherence factors (20) and using Eq. (21) we find the energy emission rate,

$$\dot{E} = \frac{G^2}{96 \pi^6 h^{10} c^3} \frac{C_A^2 + C_V^2}{2} \mu_e \sum_{\mathbf{k}} v^3_{\parallel} V^2_k \int dQ^\parallel q^2 q^2 u_1 v_1 u_2 v_2 \frac{1}{e^{\beta \omega} - 1} \theta(\omega - c|q|),$$  (26)

with $P^\parallel = \Delta p^\parallel_1 - \frac{1}{2} q_0$ is an average crystal momentum variable for the initial and final electron states. The $q_0$ integral in Eq. (26) may now be performed, leaving an integral over the remaining momenta, which we express in terms of the dimensionless variables

$$x_1 = \frac{q^\parallel_{\mathbf{k}}}{k_B T}, \quad x_2 = \frac{q^\parallel_{\mathbf{k}}}{k_B T}, \quad \text{and} \quad x_3 = \frac{q^\perp_{\mathbf{k}}}{k_B T}.$$  (27)

The energy emission rate may thus be written as

$$\dot{E} = \frac{4 \pi G^2}{567 \pi^6 h^{10} c^3} \frac{C_A^2 + C_V^2}{2} \mu_e (k_B T)^6 \sum_{\mathbf{k}} v^3_{\parallel} V^2_k I(v^\parallel_{\mathbf{k}}, \frac{k_B T}{|V|}),$$  (28)

where $I$ is the dimensionless integral

$$I(v^\parallel_{\mathbf{k}}, t) = \frac{63}{8 \pi^7} \int_0^\infty dx_1 \int_0^\infty dx_2 \int_{x_3^{\min}}^{x_3^{\max}} dx_3 \frac{w^2 (w^2 - x_2^2 - x_3^2)^2}{e_1 e_2 (e^w - 1)}.$$  (29)

The integral is to be carried out over the region where $w^2 - x_2^2 - x_3^2$ is positive. We have defined the dimensionless energies

13
\[ e_{1,2} = \sqrt{\frac{1}{t^2} + v_\parallel^2(x_1 \pm \frac{1}{2}x_2)^2} \text{ and } w = v_\perp x_3 + e_1 + e_2, \tag{30} \]

in terms of the variable \( t = k_B T / V_K \). The limits of the \( x_3 \) integration in Eq.(29) are

\[ x_3^{\text{max, min}} = \frac{v_\perp}{v_\parallel}(e_1 + e_2) \pm \frac{1}{v_\parallel} \sqrt{(e_1 + e_2)^2 - v_\parallel^2 x_2^2}. \tag{31} \]

The limits may be put in a more symmetric form by defining the variable \( y = x_3 - v_\perp (e_1 + e_2) / v_\parallel \), leading to

\[ I = \frac{63}{8\pi^2 v_\parallel^3} \int_0^\infty dx_1 \int_0^\infty dx_2 \int_{-a}^{+a} dy \frac{1}{e_1 e_2} \left( a^2 - y^2 \right)^{3/2} \frac{\left( v_\perp y + \frac{1}{v_\parallel}(e_1 + e_2) \right)^2}{v_\perp + \frac{1}{v_\parallel}(e_1 + e_2)} - 1, \tag{32} \]

with \( a = \frac{1}{v_\parallel} \sqrt{(e_1 + e_2)^2 - v_\parallel^2 x_2^2} \).

**III. EVALUATION OF THE NEUTRINO EMISSION RATE**

In this section we evaluate our basic result for a number of different situations, beginning with the low- and high-temperature limits, and then we describe results for more general cases.

**A. High-and Low-Temperature Limits**

Analytical results may be obtained in limiting cases. We first consider temperatures small compared with \( |V_K| \), where our results differ dramatically from earlier ones. This limit corresponds to \( t \ll 1 \). In this case, the denominator in Eq.(29) is essentially \( e^w \), and we simply expand the integrand around the state of lowest “energy” \( w \), given by

\[ x_1 = x_2 = 0 \text{ and } x_3 = x_3^{\text{min}} = -\frac{1}{1 + v_\perp} \frac{2}{t}. \tag{33} \]

All three integrals are thus reduced to gaussian ones, with the result

\[ I = \frac{189}{2\pi^{11/2}(1 - v_\perp)^{1/2}(1 + v_\perp)^2 v_\perp^2} t^{-\frac{5}{2}} e^{x_3^{\text{min}}}. \tag{34} \]

The rate of energy emission is therefore
\[
\hat{E}_\vartriangleleft = \frac{2G^2}{3\pi^2\hbar^9e^9} \frac{C_A^2 + C_Y^2}{2} \mu_e(k_BT)^\frac{7}{2} \sum_K \frac{(1 - v_\bot)^{1/2}}{v_\bot^{1/2}(1 + v_\bot)} |V_K|^2 e^{-\frac{|V_K|}{k_BT}(1 + v_\bot^2)}, \quad k_BT \ll |V_K|.
\] (35)

The exponential dependence reflects the fact that the minimum energy of the neutrino pair is \(2|V_K|/(1 + v_\bot)\). This is easily seen by observing that the energy of the neutrino and antineutrino is given by \(\omega = E_1 - E_2 \geq 2|V_K| + q_\bot v_\bot\), and that, in addition, the four-momentum of the neutrino pair must be time-like, \(\omega \geq cq \geq c|q_\bot|\).

We next consider the limit of temperatures much greater than \(V_K\), corresponding to \(t \gg 1\). In this limit, the range of \(y\) integration in Eq.(32) goes to zero in the region \(2x_1 < x_2\). It is furthermore convenient to perform the \(x_1\) integration last, since for fixed \(x_1\), \(x_2\) and \(y\) lie on a disk of radius \(2x_1\). We write

\[
I = \frac{63v_\parallel}{2\pi^2} \int_0^\infty dx_1 \int_{-2x_1}^{2x_1} dy \frac{(v_\parallel y + 2x_1)^2}{e^{v_\parallel + 2x_1} - 1} I_1,
\] (36)

where

\[
I_1 = \int_0^{\sqrt{4x_1^2 - y^2}} dx_2 \frac{(4x_1^2 - x_2^2 - y^2)^{1/2}}{4x_1^2 - x_2^2} = \frac{\pi}{4}(4x_1^2 - 3y^2 + \frac{|y|^3}{x_1}).
\] (37)

The remaining integrals are performed by defining the variables \(\eta_1 = 2x_1 + v_\bot y\), associated with energy, and the orthogonal variable \(\eta_2 = -v_\bot x_1 + 2y\). The \(\eta_2\) integral is elementary, and the final, \(\eta_1\), integral is evaluated using standard techniques for Bose integrals

\[
\int_0^\infty d\eta_1 \frac{\eta_1^5}{e^{\eta_1} - 1} = \frac{8\pi^6}{63}.
\] (38)

We are thus led to the following result for the basic phase space integral

\[
I = \frac{1}{v_\bot^2 v_\parallel^2} (1 - \frac{v_\bot^2}{v_\parallel^2} \log \frac{1}{v_\parallel^2}),
\] (39)

with the corresponding energy emission rate given by

\[
\hat{E}_\triangleright = \frac{4\pi G^2}{576\pi h^9 e^9} \frac{C_A^2 + C_Y^2}{2} \mu_e(k_BT)^6 \sum_K \frac{v_\parallel^2}{v_\bot^2} (1 - \frac{v_\bot^2}{v_\parallel^2} \log \frac{1}{v_\parallel^2}) |V_K|^2, \quad k_BT \gg |V_K|.
\] (40)

We have checked that Eq.(40) is consistent with earlier calculations in which the electron-lattice interaction was treated perturbatively (Fig.1). Flowers \[\text{[5]}\] has given the energy...
emission in integral form, valid for nonvanishing electron mass. By evaluating these integrals in the limit of vanishing electron mass we were able to arrive at the analytical result (40).

B. Results for Arbitrary Temperatures

The general expression for the neutrino emission rate is rather complicated, so we begin by discussing a number of its ingredients.

In our discussions we shall focus on the highest density regions in a neutron star crust since most of the mass of the crust is at densities close to that at the inner boundary of the crust, and consequently such matter is expected to be responsible for most of the neutrino pairs produced by electron bremsstrahlung. An important quantity is the strength of the periodic potential, given by Eq. (14). First of all, we note that the effects of electron screening are negligible, since for the smallest reciprocal lattice vector for a bcc lattice, \( K = 2\sqrt{2}\pi/a \), and therefore the dielectric function is \( \varepsilon = 1 + (9/(2\pi^5))^{1/3}Z^{2/3}\alpha \approx 1 + 1.79 \times 10^{-3}Z^{2/3} \), which differs from unity by less than 3% for the largest Z's encountered in neutron star crusts. Next, we turn to the nuclear form factors. At the highest densities, their effects are significant, as we now show. According to the calculations of Ref. [12], spherical nuclei have a radius of \( r_N = 8.7 \) fm when spherical nuclei become unstable with respect to formation of rod-like nuclei. The interface between nuclear matter and the neutron liquid outside nuclei is rather diffuse, so we assume that the nuclear charge distribution may be approximated by a gaussian \( \exp(-3r^2/(2r_{rms}^2)) \), and for \( r_{rms} \) we take the value \( \sqrt{3/5}r_N \), corresponding to a liquid drop with uniform proton density. The charge density in coordinate space is thus proportional to \( \exp(-5r^2/(2r_N^2)) \), and the form factor is given by \( F_q = \exp(-q^2r_N^2/10) \). For matter at the highest density at which nuclei are roughly spherical, for which \( \mu_e = 78\text{MeV} \) and \( Z = 48 \) [12], the form factor is 0.675 for the smallest reciprocal lattice vector, and

\[ \text{We draw attention to the fact that in Ref. [12] the cell radius at the phase transition between spherical nuclei and rod-shaped ones the cell radius is given as 19.2 fm. This is a misprint, and the} \]
0.009 for the largest one, approximately $2p_F$.

Finally, we consider the Debye-Waller factor. This is more difficult to estimate since, to the best of our knowledge, there exists no treatment of lattice vibrations in dense matter that takes into account the effects of the neutrons between nuclei. To make estimates we shall use the results for the pure Coulomb lattice given by Yakovlev and Kaminker [15]. Again we assume conditions appropriate for the highest density at which spherical nuclei survive. The total number of nucleons per unit cell is $A' \approx 1466$, and the number of nucleons within the nucleus is $A \approx 427$. Thus the total number of nucleons per proton, $x$ is 0.033. (Note that, because there are neutrons outside nuclei, $x \neq Z/A$.) The total mass density is $1.07 \times 10^{14} \text{g cm}^{-3}$. The Debye-Waller factor is

$$W(q) \approx 0.078 \frac{50}{Z} \left( \frac{x}{0.1} \right)^{1/6} \left( \frac{Z}{0.1A} \right)^{1/2} \rho_{14}^{1/3} \left( 1.4 e^{-9.1t} + 12.995t \right) \left( \frac{q}{2p_F} \right)^2.$$  (41)

Here $t = T/T_p$, where the “plasma” temperature is given by

$$T_p = \frac{\hbar \omega_p}{k_B} \approx \frac{\hbar}{k_B} \left( \frac{4 \pi e^2 n_Z m_{\text{ion}}}{m_{\text{ion}}} \right)^{1/2} \approx 7.8 \times 10^9 \left( \rho_{14} \frac{x}{0.1} \frac{Z}{0.1A} \right)^{1/2} \text{K},$$  (42)

where $\omega_p$ is the plasma frequency for the ions alone, neglecting the presence of the outside neutrons. The quantity $\rho_{14}$ denotes the mass density in units of $10^{14} \text{g cm}^{-3}$. For a temperature of $10^9 \text{K}$, $e^{-W(q)}$ is 0.98 for the smallest reciprocal lattice vector, and 0.81 for the largest one. At $10^{10} \text{K}$, the corresponding values are 0.83 and 0.14. This demonstrates that the effects of Debye-Waller factors can be significant for temperatures close to the melting temperature, which is about $1.2 \times 10^{10} \text{K}$, but the effects of the periodic potential are still substantial for the lowest reciprocal lattice vectors. We emphasize that these estimates are based on the assumption that the lattice dynamics is that of a pure Coulomb crystal, and more realistic estimates can be made only after a thorough investigation of lattice dynamics that allows for the neutrons between nuclei.

For conditions under which nuclear form factors and Debye-Waller factors may be neglected, the dependence of emission rate on the $\mu_e$, $T$, and $Z$ is of the general form
\[ \dot{E} \sim T^6 \mu_e^3 g(Z, T/\mu_e), \]  

(43)

where \( g \) approaches a constant for temperatures large compared with \( Z^{2/3} \alpha \mu_e \), the magnitude of the largest band gaps.

Before presenting the results of detailed calculations we describe qualitative features of the emission rate. As one can see from the asymptotic low- and high-temperature expressions Eqs. (35) and (40), the crossover between the two sorts of behavior occurs at a temperature \( \sim 2|V_K|/(1 + v_\perp)/k_B \). Thus the crossover temperature for large wavevectors, \( \sim 2p_F \), is a factor \( \sim Z^{2/3} \) smaller than for the smallest reciprocal lattice vectors, and thus there is a large temperature range in which neither of the limiting expressions is a good approximation. We note that reciprocal lattice vectors close to \( 2p_F \) give very small contributions because helicity conservation ensures that electrons cannot be scattered by ions for \( K = 2p_F \), as reflected by the \( v_\perp \) factor in the expression for the ionic potential, Eq. (14). At high temperatures, a small reciprocal lattice vector gives a larger contribution than a large one, because the emission rate is proportional to the square of the Coulomb matrix element. As the temperature is lowered, the first reciprocal lattice vectors for which bremsstrahlung is suppressed are the smallest ones, because they give rise to the largest band gaps. Consequently, at lower temperatures, the reciprocal lattice vectors that give the largest contributions become larger and larger, until in the low-temperature limit, the dominant ones have wavevectors close to \( 2p_F \).

We now present results of a numerical evaluation of the the rate of energy emission from neutrino pair bremsstrahlung. We consider matter at the highest density at which nuclei are roughly spherical, for which \( \mu_e = 78 \text{MeV} \) and \( Z = 48 \), according to the recent calculations of Ref. [12]. We shall neglect Debye-Waller factors.

In Fig. 5 we plot energy emission rates, for several values of \( v_\parallel = K/(2k_F) \), compared with the corresponding quantities in the high temperature limit. The solid lines in Fig. 5 are for nuclear form factor \( F_q = 1 \), and the dashed lines are for \( F_q = \exp(-q^2 r_N^2/10) \), where \( r_N = 8.7 \text{fm} \).
The total energy emission rate is obtained by evaluating the basic phase space integral, Eq. (32) for all reciprocal lattice vectors entering in the sum of Eq. (26). The relevant reciprocal lattice vectors are those of length $K < 2k_F$, where $k_F$ is the electron Fermi momentum. The Fermi sphere is of radius $R = (3Z/(32\pi))^{1/3}$ in units of $4\pi/a$, the side length of the conventional cubic cell of the reciprocal (fcc) lattice. Here, $a = 2/n_Z$ is the length of a side in the conventional cubic cell of the direct (bcc) lattice. For $Z = 48$ we find that $R = 1.13$, and that there are 200 reciprocal lattice vectors contributing to the sum.

The numerical evaluation may be streamlined by exploiting the symmetry of the lattice when evaluating phase space integrals. This becomes especially relevant when there is a large number of reciprocal lattice vectors contributing to the energy emission rate. A given reciprocal lattice vector has 47 partners of equal length, if the three vector components differ in magnitude, and none are zero. The degeneracy is reduced if two or more components are equal, or if one or more components are zero. To give an idea what is to be gained from such considerations, we note that the 200 reciprocal lattice vectors contributing to neutrino emission have only 11 distinct lengths.

The three dimensional integral (32) was evaluated using the extended trapezoidal rule [16]. We checked that both the low temperature (34) and high temperature (39) limits were reproduced to within better than 1% for a range of values of $v_\parallel$, corresponding to different choices for reciprocal lattice vectors.

In Fig.6 we show the rate of energy emission from neutrino pair bremsstrahlung, divided by the high temperature rate, Eq.(44), for the conditions we have described. The solid line shows this ratio for nuclear form factor $F_q = 1$. The short dashed line shows the corresponding ratio for $F_q = \exp(-q^2r_N^2/10)$, with $r_N = 8.7$ fm. The rate from the 12 smallest reciprocal lattice vectors alone divided by the total high temperature rate is given by the line with alternating short and long dashes, for $F_q = 1$. The long dashed line gives the emission rate due to conversion of thermal phonons into neutrino pairs, Eq.(44), as compared with Eq.(40).
IV. DISCUSSION OF RESULTS

An unexpected feature of the results is that the ratio $\dot{E}/T^6$ is not a monotonic function of temperature, and it has a maximum. We do not believe this to be an artifact of the numerical calculations for two reasons. First, the numerical calculations agree with the analytical results. This is a non-trivial check since the regions of integration in the high- and low-temperature limits are very different. Second, we have performed the numerical integrations by two different methods, with identical results. It is interesting to note that non-monotonic temperature dependence was predicted theoretically and subsequently observed in a quite different physical context, the relaxation of nuclear spins in superconductors. We surmise that the origin of the non-monotonic behavior in the case of neutrino emission may be a similar effect, because the excitation spectrum of electrons near a band gap is similar to that for the excitations in a superconductor. However, the neutrino emission process we have considered is an interband process, and thus its analog in a superconductor would be the annihilation of two positive-energy excitations. However, the relaxation of a nuclear spin in a superconductor takes place by scattering of one positive energy excitation to another positive energy state, which is analogous to an intraband process in band theory. It is desirable to confirm this analytically, by, for example, calculating the leading corrections to the high-temperature limit.

While the suppression of the bremsstrahlung rate for processes with a particular reciprocal lattice vector is rather abrupt, the suppression of the total rate from all reciprocal lattice vectors is less abrupt, as a consequence of the fact that the characteristic temperatures for suppression span a range of order $Z^{2/3}$.

From Fig.6 we see that the smallest reciprocal lattice vectors give almost $2/3$ of the total rate at high temperature. As the temperature is lowered, this fraction drops rapidly, since the emission from the smallest reciprocal lattice vectors is the first to be suppressed.

Figs.5 and 6 also illustrate the effect of nontrivial nuclear form factors. For a given reciprocal lattice vector $\mathbf{K}$, the crossover temperature is lower if the nuclear form factor
$F_K$ differs from unity than if $F_K = 1$, since the matrix element of the lattice potential is smaller. For large $K$, where $F_K$ can be very small, this effect is much more pronounced than for small $K$. However, the contribution of large reciprocal lattice vectors to the total rate is small, due to the $v_\perp$ factor in the ionic potential. Consequently, the magnitude of the change resulting from nontrivial nuclear form factors is closer to that given by the smallest reciprocal lattice vectors.

In our calculations we have assumed that the periodic potential mixes only pairs of free electron states. However, for points of high symmetry in the Brillouin zone, it is possible for more than two free-electron states to be degenerate. At high temperatures such points will be of little significance because the phase space for transitions to states in their vicinities is small compared with the total. However, at low temperatures they could be significant if it happened that two bands could cross, even when the effects of the periodic potential are taken into account, since this would give the possibility of the neutrino emissivity at low temperatures behaving as a power of the temperature, rather than exponentially. With this in mind we have investigated the degeneracy of bands at points of high symmetry. The standard discussion of this problem for terrestrial solids [13,14] cannot be carried over to case of dense matter because electrons are highly relativistic, and therefore spin-orbit coupling is strong. However, by generalizing the standard treatment to take into account electron spin and spin-orbit coupling by employing double groups, one can demonstrate that for the bcc structure, bands cannot be degenerate, and therefore the neutrino emissivity will behave exponentially at low temperatures. We conclude that points of high symmetry will not play an overwhelming role at low temperatures, and consequently our calculations that allowed for mixing of only two wavenumbers at a time will give a reliable estimate.

Another process that can generate neutrino pairs in the crusts of neutron stars is the conversion of thermal phonons into pairs. This has been considered by many authors [5,8], and most recently by Yakovlev and Kaminker [15]. One important characteristic temperature is the “plasma” temperature defined in Eq.(42), in the approximation that the effects of neutrons on the lattice dynamics are neglected, except insofar as they affect the relationship
between the total mass density and the electron density. Another important temperature is the melting temperature, \( T_m = \frac{Z^2 e^2}{(r_c k_B \Gamma_m)} \approx 2.6 \times 10^{10} (Z/60)^{5/3} (\rho_{14} x/0.1)^{1/3} \) K, where for the characteristic coupling parameter at melting, \( \Gamma_m \) we have taken the value 172 obtained by Nagara et al. [17]. The quantity \( r_c = (4\pi n Z/3)^{-1/3} = (8\pi/3)^{1/3} a \), where the latter result is for a bcc lattice. For temperatures larger than about \( T_p/16 \), the energy emission is given approximately by

\[
\dot{E}_{\text{phonon}} \approx 4.2 \times 10^4 (x/0.1)^{2/3} \rho_{14}^{1/3} T_9^7 \text{erg g}^{-1} \text{s}^{-1}.
\] (44)

Thus at the melting temperature, the neutrino emission due to the phonon process is comparable in magnitude to that from bremsstrahlung from the static lattice.

The phonon processes, which at high temperatures are primarily Umklapp ones, begin to be cut off at a temperature for which the wavenumber of a thermal phonon is comparable with the range of wavenumbers for which the electron spectrum is significantly distorted from the free-electron form in the vicinity of a band gap. This amounts to the condition that the thermal energy must be less than or of order \( c_s/c \) times the band gap, where \( c_s \) is a typical sound speed [18]. Thus phonon-induced bremsstrahlung will be suppressed at temperatures which are smaller by a factor of order \( c_s/c \) than the temperatures at which bremsstrahlung from a static lattice is suppressed. For the lowest reciprocal lattice vectors, the corresponding temperature is of order \( T_p Z^{1/3} \alpha \), while for the largest reciprocal lattice vectors it is of order \( T_p Z^{-1/3} \alpha \). At the lowest temperatures phonon-induced bremsstrahlung of neutrinos will take place by normal processes, as opposed to Umklapp ones, and the emission will vary as \( T^{11} \) [3]. Thus as the temperature is lowered, phonon-induced processes will become more important than bremsstrahlung from the static lattice. However, as one can see from Fig. 6, where the rate for emission of neutrino pairs by phonons is shown, calculated from Eq.(44), the phonon rate remains below the rate for the static lattice at temperatures above about \( 10^8 \) K. We remark that if one were to use an expression more accurate than Eq.(44) for the rate of the phonon process, this conclusion would not be altered. In a future paper we shall make a more detailed comparison of the two processes
V. CONCLUSION

The most important lesson to be learned from these calculations is that, even though the electron-ion interaction has a small effect on the energy per electron, since it is of order $Z^{2/3} \alpha \mu_e$ per electron, it can have a major effect on kinetic processes at temperatures low compared with band gaps. For the process we have considered in this paper, the effects are large because they involve interband transitions. For processes which are mediated by intraband transitions the effects will generally be milder because the electron states in the vicinity of band gaps will be relatively less important.

Central to our calculations is the nature of the electronic band structure for ultra-relativistic electrons. This is very similar to the non-relativistic case treated in standard texts on condensed matter physics, but with the important difference that for the ultra-relativistic case the matrix elements of the potential contain the familiar helicity factor which cuts down the amplitude for large-angle scattering. This factor has previously been considered in the scattering of relativistic electrons by phonons [18].

The basic conclusion of our calculations is that the rate of emission of electron pairs by electrons is strongly suppressed at temperatures below the band gap. For the inner crust of neutron stars, the suppression is substantial at temperatures of order $10^9$ K, or precisely in the temperature range where it was earlier argued that it would be the dominant neutrino emission process should the processes in the core be suppressed by superfluidity of the neutrons and protons, which are estimated to have transition temperatures that can range up to about $10^{10}$ K. Of course, estimates of superfluid transition temperatures are uncertain, but our results suggest that the range of temperatures where neutrino bremsstrahlung by electrons could play a significant role in neutron star cooling are extremely limited.

There are a number of issues that need to be addressed in order to arrive at better estimates of the rate of neutrino bremsstrahlung emission by electrons. A major one is the
nature of the collective oscillations of matter in the inner crust which enter the estimates of Debye-Waller factors. At the lowest densities, the density of neutrons outside nuclei is low and they have little effect on the collective modes, which are just the vibrations of the ionic lattice, with a neutralizing background of electrons. At higher densities, the role of the neutrons becomes more important, and one must consider a lattice of nuclei coupled to a neutron fluid, a problem that has been addressed by Sedrakian [20]. However, the problem is not a trivial one, as may be seen from the fact that the most straightforward estimate of the effective mass of a nucleus moving through a normal Fermi liquid leads to a divergent answer [21]. The collective mode structure of the liquid-crystal-like phases with non-spherical nuclei is a completely open problem, but these modes will come to resemble the corresponding giant resonances in nuclear matter as the transition to the uniform phase is approached.

We wish to acknowledge helpful correspondence with D. Yakovlev. The research on which this article is based was partially supported by U. S. National Science Foundation grant NSF AST 93-15133, and by NASA grant NAGW-1583. Part of the work carried out while one of the authors (CJP) enjoyed the hospitality of the Institute for Nuclear Theory at the University of Washington, and support from the Department of Energy grant to the Institute.
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FIG. 1. The basic bremsstrahlung process in first-order perturbation theory. The cross denotes an electron-lattice interaction, and the propagators are free ones.

FIG. 2. The basic bremsstrahlung process. The double line is the propagator for a band electron.
FIG. 3. Electron energy $E$ as a function of $p^\parallel$ for two values of $p^\perp$. The arrow shows a possible transition.
FIG. 4. Brillouin zone boundary. The zone boundary is defined such that the branches of the electron spectrum labelled 1, 2, 3' and 4' occur in the same zone.
FIG. 5. Energy emission rates, for several values of $v_\parallel = K/(2k_F)$, compared with the corresponding quantities in the high temperature limit. Solid lines: Form factor $F_q = 1$; Dashed lines: Form factor $F_q = \exp(-q^2r_N^2/10)$, $r_N = 8.7$ fm.
FIG. 6. Solid line: Energy emission rate, Eq.(28), compared with the high temperature limit, Eq.(40), as a function of temperature, with form factor $F_q = 1$. Alternating short and long dashes: Same as above, but taking only the contribution of the smallest reciprocal lattice vectors to Eq.(28). Short dashes: Same as solid line, but with form factor $F_q = \exp(-q^2r_N^2/10)$, $r_N = 8.7$fm. Long dashes: The emission rate due to conversion of thermal phonons into neutrino pairs, Eq.(44), as compared with Eq.(40).