Canonical analysis of the Jackiw–Teitelboim model in the temporal gauge: I. The classical theory

Clisthenis P Constantinidis\textsuperscript{1,2}, José André Lourenço\textsuperscript{1,2,3}, Ivan Morales\textsuperscript{2,3,4}, Olivier Piguet\textsuperscript{1,2} and Alex Rios\textsuperscript{2,3,4}

Departamento de Física, CCE, Universidade Federal do Espírito Santo (UFES), Av. Fernando Ferrari, 514, BR-29075-910, Vitória, ES, Brazil

E-mail: clisthen@cce.ufes.br, quantumlourenco@gmail.com, mblivan@gmail.com, opiguet@yahoo.com and rios_alex@ig.com.br

Received 3 March 2008
Published 2 June 2008
Online at stacks.iop.org/CQG/25/125003

Abstract

As a preparation for its quantization in the loop formalism, the two-dimensional gravitation model of Jackiw and Teitelboim is analysed in the classical canonical formalism. The dynamics is of pure constraints as is well known. A partial gauge fixing of the temporal type being performed, the resulting second class constraints are sorted out and the corresponding Dirac bracket algebra is worked out. Dirac observables of this classical theory are then calculated.

PACS numbers: 04.60.Ds, 04.60.Pp, 04.60.Kz, 11.15.−q

1. Introduction

The full quantization of general relativity remains an open problem despite the very important advances achieved during the last two decades, mainly within the ‘loop quantization’ formalism of Ashtekar, Rovelli, Smolin and others (see, e.g., the books and review [1–3]). It is thus still worthwhile investigating lower dimensional gravitation theories, where the technical difficulties of the four-dimensional theory are somewhat milder [4–12].

The purpose of the present paper is to investigate the canonical formulation of the model of two-dimensional gravity proposed independently by Jackiw and Teitelboim (JT) some time ago [4]. The JT model contains a dilaton-type scalar field, beyond the metric field, in order to have an action which does not reduce to boundary terms. Moreover, a cosmological constant is introduced in order to be able to write a non-degenerate action. The invariance of the theory is thus de Sitter or anti-de Sitter (‘(A)dS’). We shall start from the ‘\textit{BF}’ formulation introduced...
in [7], which explicitly identifies the JT model as a topological gauge theory [13], the gauge
group being (A)dS.

Since most approaches to loop gravity are based—explicitly or implicitly—on a partial
gauge fixing of the temporal type (‘temporal gauge’) [1–3], we shall choose to work with a
two-dimensional version of the temporal gauge. The focus of this paper will be on the classical
theory, the quantization being left for future publication [14].

After recalling the main features of the JT model in section 2, we write down the canonical
formulation of the BF version of the theory in section 3. Using the quantization scheme of
Dirac [15] for theories with constraints, we separate, in section 4, the second class constraints
originated from the temporal gauge fixing, and show that the remaining first class constraints
generate a gauge symmetry which is equivalent—up to field equations—to the invariance
under spacetime diffeomorphisms. The classical Dirac observables are briefly discussed in
section 5, and some brief conclusions are given in section 6.

Part of the material of the present paper has been included by two of the authors [16, 17]
in the thesis presented as a requirement to the obtention of the Master degree.

2. The Jackiw–Teitelboim model

2.1. The Jackiw–Teitelboim action

Pure gravity in two spacetime dimensions cannot be based on the Einstein–Hilbert action
\( \int d^2x \sqrt{-g} R \), which is a surface integral, corresponding to an identically vanishing Einstein
tensor: \( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \) [4]. A simple but nontrivial model has been proposed long ago
independently by Jackiw and by Teitelboim [4]. The model contains, besides the spacetime
metric \( g_{\mu\nu}(x) \), a dilaton-type scalar field \( \psi(x) \). Its action is given by

\[
S_{JT} = \frac{1}{2} \int d^2x \sqrt{-g} \psi (R - 2k),
\]

(2.1)

It is invariant under spacetime diffeomorphisms and leads to the Liouville equation

\[
R - 2k = 0,
\]

(2.2)

and to the equation for \( \psi \) [5],

\[
\nabla_\mu \nabla_\nu \psi + kg_{\mu\nu} \psi = 0,
\]

(2.3)

where \( \nabla_\mu \) is the Levi-Civita covariant derivative associated with the metric \( g_{\mu\nu} \). Equation (2.2)
yields a geometry with constant curvature, the parameter \( k \) playing the role of the cosmological
constant.

A canonical quantization of this model in terms of the variables \( g_{\mu\nu} \) and \( \psi \) has been given
by Henneaux [5].

2.2. BF formulation of the Jackiw–Teitelboim model

The model is equivalent to a BF model based on the gauge group (A)dS, i.e. the two-
dimensional de Sitter or anti-de Sitter group, SO(1, 2) on SO(2, 1), according to the sign of
the cosmological constant \( k \) [6, 7]. The (A)dS gauge connection is written as

\[
A(x) = e^I(x) P_I + \omega(x) \Lambda,
\]

(2.4)
where the operators $P_I$ ($I = 0, 1$) and $\Lambda$ are the ‘translation’ generators and the Lorentz boost generator, respectively, obeying the (A)dS algebra\(^5\)

\[
[\Lambda, P_I] = \epsilon_I^J P_J, \quad [P_I, P_J] = k \epsilon_I^J \Lambda. \tag{2.5}
\]

The coefficients in (2.4) are the zweibein and Lorentz connection forms

\[
e^I = e^I_\mu dx^\mu, \quad \omega = \omega_{\mu \nu} dx^\mu. \tag{2.6}
\]

The spacetime metric is given in terms of the zweibein by

\[
g_{\mu \nu} = \eta_{IJ} e^I_\mu e^J_\nu. \tag{2.7}
\]

Introducing the indices $i, j, \ldots = 0, 1, 2$ and denoting the generators of (A)dS as $J_i$:

\[
\{ J_i \} = \{ J_0, J_1, J_2 \} = \{ P_0, P_1, \Lambda \}, \tag{2.8}
\]

the algebra (2.5) reads

\[
[ J_i, J_j ] = f_{ij}^k J_k = k \epsilon_{ij}^k J_k, \tag{2.9}
\]

where the nonzero structure constants $f_{ij}^k$ are\(^6\)

\[
f_{01}^2 = k, \quad f_{12}^0 = \sigma, \quad f_{20}^1 = 1. \tag{2.10}
\]

(A)dS possesses an invariant nondegenerate quadratic form $\langle J_i, J_j \rangle = k_{ij}$, where $k_{ij}$ is the Killing metric

\[
k_{ij} = - \frac{\sigma}{2} f_{ik}^l f_{jl}^k. \tag{2.11}
\]

This metric and its inverse are used to lower and raise the indices $i, j, \ldots$. Note that a nonvanishing cosmological constant $k$ is necessary in order to ensure the nondegeneracy of the Killing metric.

The ‘$B$-field’ of the theory is a Lie algebra valued scalar field

\[
\phi = \phi^I J_I = : \phi^I P_I + \psi \Lambda. \tag{2.12}
\]

With the Yang–Mills curvature given by\(^7\)

\[
F = F^{IJ} J_I \equiv F^{IJ} P_I + F^2 \Lambda = dA + AA = \frac{1}{2} F_{\mu \nu} dx^\mu dx^\nu J_I, \tag{2.13}
\]

the ‘$BF$’ action reads\(^6, 13\)

\[
S_{BF}[A, \phi] = \int (\phi, F) = \frac{1}{2} \int d^2 x e^\mu \nu k_{ij} \phi^I F_{\mu \nu}^I = - \int dt L_{BF}, \tag{2.14}
\]

where the Lagrangian $L_{BF}$ explicitly reads\(^8\)

\[
L_{BF} = \int dx (\phi \partial_t A^I_\mu + A^I_\mu D_x \phi). \tag{2.15}
\]

---

\(^5\) By convention the antisymmetric tensor $\epsilon_{IJ}$ is defined by $\epsilon_{01} = 1$. The indices $I, J, \ldots = 0, 1$ are lowered and raised by the ‘flat’ metric $\eta_{IJ} = \text{diag}(\sigma, 1)$ or its inverse $\eta^{IJ}$, where $\sigma = \pm 1$ for the Riemannian, resp. Lorentzian theory.

\(^6\) The completely antisymmetric tensor $\epsilon_{ijl}$ is defined by $\epsilon_{012} = 1$.

\(^7\) The wedge symbol $\wedge$ for exterior products of forms is omitted.

\(^8\) The values of the spacetime indices $\mu, \nu, \ldots$ are denoted by $t, x$. The antisymmetric Levi-Civita tensor $\epsilon^{\mu \nu}$ is defined by $\epsilon^{12} = +1$. 

---

3
Note that the curvature components,
\[ F^I = dA^I + f_{jk}^I A^j \wedge A^k = de^I + \omega^I_j \wedge e^j, \]
\[ F^2 = dA^2 + \frac{1}{2} f_{jk}^2 A^j \wedge A^k = de + \frac{k}{2} e^I \wedge e^j \epsilon_{IJ}, \]
represent the torsion \( T^I := F^I \) and the Riemann curvature with cosmological term added, respectively.

The action (2.14), which is invariant under the (A)dS gauge transformations, turns out to be automatically invariant under the diffeomorphisms, on shell, as a general result for topological theories of this type [13].

The field equations are
\[
\frac{\delta S_{BF}[A, \phi]}{\delta \Phi} = 0, \quad \Phi = \phi_i, A^I, \tag{2.17}
\]
where
\[
\frac{\delta S_{BF}[A, \phi]}{\delta \phi_i} = F^i = 0, \quad \frac{\delta S_{BF}[A, \phi]}{\delta A^I} = D\phi_i.
\]

In components, with the notation
\[
(\phi_i) = (\phi_0, \phi_1, \phi_2) = (\varphi_0, \varphi_1, \psi),
\]
\[
(A^I_i) = (e^0_i, e^1_i, \omega_i) = (\chi, e^1_i, \omega_i), \tag{2.18}
\]

the functional derivatives read
\[
\frac{\delta S_{BF}[A, \phi]}{\delta \varphi_0} = \partial_t \chi - \partial_x \varphi_1 = \sigma (e^1_i \omega_i - \omega_i N^1),
\]
\[
\frac{\delta S_{BF}[A, \phi]}{\delta \varphi_1} = \partial_x e^1_i - \partial_x N^1 - \omega_i N + \chi N^1,
\]
\[
\frac{\delta S_{BF}[A, \phi]}{\delta \psi} = \partial_t \omega_i - \partial_t \omega_i - k(\chi N^1 - e^1_i N),
\]
\[
\frac{\delta S_{BF}[A, \phi]}{\delta \chi} = -D_t \varphi_0 = -(\partial_t \varphi_0 + kN^1 \psi - \omega_i \varphi_1),
\]
\[
\frac{\delta S_{BF}[A, \phi]}{\delta e^1_i} = -D_t \varphi_1 = -(\partial_t \varphi_1 + \sigma \omega_i \varphi_0 - kN \psi), \tag{2.19}
\]
\[
\frac{\delta S_{BF}[A, \phi]}{\delta \omega_i} = -D_t \psi = -(\partial_t \psi + N \varphi_1 - \sigma N^1 \varphi_0).
\]
\[
\frac{\delta S_{BF}[A, \phi]}{\delta N} = D_t \varphi_0 = \partial_t \varphi_0 + k e^1_i \psi - \omega_i \varphi_1,
\]
\[
\frac{\delta S_{BF}[A, \phi]}{\delta N^1} = D_t \varphi_1 = \partial_t \varphi_1 + \sigma \omega_i \varphi_0 - k \chi \psi,
\]
\[
\frac{\delta S_{BF}[A, \phi]}{\delta \omega_i} = D_t \psi = \partial_t \psi + \chi \varphi_1 - \sigma \omega_i \varphi_0.
\]

The two equations which correspond to the variation of the scalar fields \( \psi \) lead to the conditions of zero torsion. Solving them for \( \omega_i \) and \( \omega_i \) in terms of the zweibein components \( e^I_i \) shows the equivalence of the BF theory with the Jackiw–Teitelboim theory [6, 7].
3. Canonical formalism

As usual in the canonical formalism (see, e.g., [18]), we assume for the spacetime a topological structure of the form \( \mathcal{M} = \mathbb{R} \times \Sigma \), where the real line \( \mathbb{R} \) represents ‘time’, and \( \Sigma \) is a one-dimensional manifold of arbitrary but fixed topology, representing ‘space’. Choosing the components \( A_i^\mu(x, t) \) of the connection as the generalized coordinates, the corresponding momenta will be

\[
\pi_i^{\mathcal{A}}(x) = \frac{\delta L_{BF}}{\delta \partial_x A_i^\mu(x)} = \phi_i, \quad (3.1)
\]

\[
\pi_i^{\mathcal{A}_t}(x) = \frac{\delta L_{BF}}{\delta \partial_t A_i^\mu(x)} = 0, \quad (3.2)
\]

where \( L_{BF} \) is the Lagrangian (2.15). The last equation indicates that we have a singular Lagrangian and must appeal to Dirac’s formalism [15, 19]. This equation amounts to the presence of three primary constraints

\[
\pi_i^{\mathcal{A}_t}(x) \approx 0, \quad i = 0, 1, 2, \quad (3.3)
\]

where the symbol \( \approx \) means ‘weak’ equality according to the terminology of Dirac: such constraints will be solved only after all Poisson algebra calculations have been performed.

A Legendre transformation yields the Hamiltonian

\[
H = -\int d^4x A_i^\mu(x) D_x \phi_i(x). \quad (3.4)
\]

The Poisson bracket algebra is defined by the brackets of the generalized coordinates and their conjugate momenta. The nonvanishing ones are

\[
\{ A_i^\mu(x), \phi_j(y) \} = \delta^\mu_j \delta(x - y) = \{ A_i^\mu(x), \pi_j^{\mathcal{A}_t}(y) \}. \quad (3.5)
\]

The consistency of the dynamics requires that the primary constraints do not evolve, and hence must have (weakly) vanishing Poisson brackets with the Hamiltonian:

\[
\{ \pi_i^{\mathcal{A}_t}(t, x), H \} \approx 0. \quad (3.6)
\]

It results from

\[
\{ \pi_i^{\mathcal{A}_t}(t, x), H \} = \partial_t \phi_i + f_{ij}^k A_j^\mu(x) \phi_k = D_x \phi_i(x)
\]

that we must impose the secondary constraints

\[
\mathcal{G}_i(t, x) := D_x \phi_i(x) \approx 0. \quad (3.7)
\]

We observe that the Hamiltonian (3.4) is made of constraints only, which is expected in a generally covariant theory [19, 20]. The fields \( A_i^\mu \), which are not dynamical, play the role of Lagrange multipliers.

The primary constraints (3.3) being trivially solved, we are left with the secondary constraints (3.7). These constraints are first class according to Dirac’s terminology, since they form a closed Poisson bracket algebra. This algebra is best expressed in terms of the ‘smeared’ constraints

\[
\mathcal{G}(\epsilon) = \int dx \epsilon^i(x) \mathcal{G}_i(x), \quad (3.8)
\]

where \( \epsilon^i \) are arbitrary smooth functions. Then

\[
\{ \mathcal{G}(\epsilon), \mathcal{G}(\eta) \} = \mathcal{G}(\epsilon \times \eta), \quad (3.9)
\]

9 In the following, only the dependence on the spatial coordinate, denoted by \( x, y \), etc, will be written explicitly.
where \((\epsilon \times \eta)^k = f_i^j \epsilon^i \eta^j\): this is the local version of the Lie algebra of the group \((A)dS\). These constraints generate the \((A)dS\) gauge transformations:

\[
\{G(\chi), A_i^\rho(x)\} = \partial_\chi \epsilon^\rho + f_\mu^\rho A_i^\mu \epsilon^i = D_\chi \epsilon^\rho,
\]

\[
\{G(\chi), \phi_p(x)\} = -f_p^k \epsilon^i \phi_k = (\phi \times \epsilon)_p.
\] (3.10)

4. Partial gauge fixing

4.1. The temporal gauge

Following an approach commonly used for the four-dimensional theory, as described, e.g., in the review [1], we introduce a partial gauge fixing, the ‘temporal gauge’, which consists in making the component \(\chi := e^0_\chi\) of the zweibein vanish. This condition is implemented as a new constraint,

\[
\chi \approx 0, \tag{4.1}
\]

by modifying the action (2.14) as

\[
S = \int d^2x (\phi_i F^i + B \chi) = \int d^2x \left( (\partial_t A_i^\rho(x)) \phi_i + A_i^\rho(x) \partial_\rho \phi_i + B \chi \right), \tag{4.2}
\]

where \(B\) is a Lagrange multiplier field. This will introduce second class constraints which will be treated using Dirac’s formalism [15, 19].

The conjugate momenta and the nonvanishing Poisson brackets are now

\[
\pi_i A_i(x) = \phi_i(x), \quad \pi_i \approx 0, \quad \pi^\rho \approx 0, \tag{4.3}
\]

and

\[
\{A_i^\rho(x), \phi_j(y)\} = \delta_i^j \delta(x - y),
\]

\[
\{B(x), \pi^\rho(y)\} = \delta(x - y),
\]

\[
\{\pi_i A_i(x), A_j^\rho(y)\} = \delta_i^j \delta(x - y).
\]

We have now four primary constraints (the last two weak equalities in (4.3). Repeating the argument of the preceding section, we arrive at the Hamiltonian

\[
H = -\int dx (A_i^\rho(x) \partial_\rho \phi_i + B \chi), \tag{4.4}
\]

and four secondary constraints:

\[
\mathcal{G}_i := D_\chi \phi_i \approx 0 \quad (i = 0, 1, 2) \tag{4.5}
\]

\[
\mathcal{G}_3 := \chi \approx 0.
\]

The Poisson brackets of these constraints read, in matrix notation and up to constraints,

\[
\{G_\alpha(x), G_\beta(y)\} \approx C_{\alpha \beta}(x, y) \approx \begin{pmatrix}
0 & 0 & 0 & -\partial_\chi \\
0 & 0 & 0 & -\sigma \omega_x \\
0 & 0 & 0 & \sigma e^1_x \\
-\partial_\chi & \sigma \omega_x & -\sigma e^1_x & 0
\end{pmatrix} \delta(x - y) \tag{4.6}
\]

with \(\alpha, \beta = 0, 1, 2, 3\). The rank of the matrix \(C_{\alpha \beta}\) is equal to 2, which means that we have two second class constraints. In order to separate them from the first class ones, we proceed to a redefinition

\[
G_0(x) = (e^1_x) G_0(x) - \sigma \left( \partial_\chi e^1_x \right) G_2(x) + \sigma e^1_x \partial_\chi G_2(x),
\]

\[
G_1(x) = e^1_x G_1(x) + \omega_x(x) G_2(x),
\]

\[
G_2(x) = G_2(x),
\]

\[
G_3(x) = G_3(x). \tag{4.7}
\]
The new Poisson bracket matrix is
\[ C_{\alpha\beta}^\prime(x, y) \approx \begin{pmatrix} 0 & 0 \\ 0 & C_{ab}^\prime(x, y) \end{pmatrix} \delta(x - y), \] (4.8)
with the $2 \times 2$ submatrix $(a, b = 2, 3)$ given by
\[ C_{ab}^\prime(x, y) = \begin{pmatrix} 0 & \sigma e_1^1 \\ -\sigma e_1^1 & 0 \end{pmatrix} \delta(x - y). \] (4.9)
The latter has an inverse,
\[ C_{ab}^\prime(x, y):= (C_{ab}^\prime(x, y))^{-1} = \begin{pmatrix} 0 & -\sigma/e_1 \\ \sigma/e_1 & 0 \end{pmatrix} \delta(x - y), \] (4.10)
in the convolution sense, i.e.
\[ \int dz C_{ab}^\prime(x, z) C_{bc}^\prime(z, y) = \delta^a_c \delta(x - y). \]
We conclude that the constraints $G_0^\prime$ and $G_1^\prime$ are first class, whereas $G_2^\prime$ and $G_3^\prime$ are second class.

### 4.2. Dirac brackets

In order to take care of the second class constraints, continuing to follow Dirac, we define the Dirac bracket between two fields $A$ and $B$—local functionals of the fields $e_1^1, \omega_x, \varphi_0, \varphi_1, \psi$ as $[A(x), B(y)]_D = [A(x), B(y)]$
\[ = \int d^3z_1 d^3z_2 (A(x), G_{ab}^\prime(z_1)) e_{abc}^\prime(z_1, z_2) [G_{bc}^\prime(z_2), B(y)], \] (4.11)
where $C_{ab}$ is the matrix (4.10). For $A$ and $B = e_1^1, \omega_x, \varphi_1, \psi$ (but not $\varphi_0$), the result is simply
\[ [A(x), G_a^\prime(y)]_D = 0, \quad a = 2, 3, \quad \forall A(x), \] (4.12)
Moreover, the Dirac bracket of any field $A$ with a second class constraint is vanishing:
\[ [A(x), G_a^\prime(y)]_D = 0, \quad a = 2, 3, \quad \forall A(x), \] (4.13)
which allows us to impose the second class constraints as strong equalities:
\[ G_2^\prime = 0, \quad G_3^\prime = \chi = 0. \] (4.14)
The second equality is just the temporal gauge condition, and the first one allows us to express $\varphi_0$ as a function of the other basic fields
\[ \varphi_0 = \sigma \frac{\partial_x \psi}{e_1^1}. \] (4.15)
With this, the first class constraints $G_0^\prime$ and $G_1^\prime$ become
\[ G_0^\prime(x) = (e_1^1) G_0^\prime(x) = \sigma e_1^1 \partial_x \left( \frac{\partial_x \psi}{e_1^1} \right) + k(e_1^1)^2 \psi - e_1^1 \omega_x \varphi_1, \] (4.16)
\[ G_1^\prime(x) = e_1^1 G_1^\prime(x) = e_1^1 \partial_x \varphi_1 + \omega_x \partial_x \psi. \] (4.17)
The Dirac bracket algebra of these constraints is closed:
\[ [G_0^\prime(\epsilon), G_0^\prime(\eta)]_D = \sigma G_1^\prime(\epsilon \partial_x \eta - \eta \partial_x \epsilon), \]
\[ [G_0^\prime(\epsilon), G_1^\prime(\eta)]_D = -G_0^\prime(\epsilon \partial_x \eta - \eta \partial_x \epsilon), \]
\[ [G_1^\prime(\epsilon), G_1^\prime(\eta)]_D = -G_1^\prime(\epsilon \partial_x \eta - \eta \partial_x \epsilon), \] (4.18)
where $[\epsilon, \eta] = (\epsilon \partial_x \eta - \eta \partial_x \epsilon)$, which confirms that $G_0^\prime$ and $G_1^\prime$ are first class.
4.3. Gauge symmetry and invariance under the diffeomorphisms

The independent dynamical variables are now the fields $\epsilon^1_x$, $\omega_x$, $\varphi_1$ and $\psi$. Their nonvanishing Dirac brackets are

$$\{\epsilon^1_x(x), \varphi_1(y)\}_D = \delta(x - y) = \{\omega_x(x), \psi(y)\}_D.$$  \hfill (4.19)

The constraints $G'_0$ and $G'_1$ generate the following gauge transformations, which are symmetries of the theory:

$$\{G'_0(\varepsilon), \varphi_1(y)\}_D = \frac{1}{\epsilon^1_x} \partial_x (\varepsilon \partial_x \psi) + 2k \epsilon \epsilon^1_x \psi - \varepsilon \omega_x \varphi_1,$$
$$\{G'_0(\varepsilon), \psi(y)\}_D = -\varepsilon \epsilon^1_x \varphi_1,$$
$$\{G'_0(\varepsilon), \epsilon^1_x(y)\}_D = \varepsilon \epsilon^1_x \omega_x,$$
$$\{G'_0(\varepsilon), \omega_x(y)\}_D = -\sigma \partial_x \left( \frac{1}{\epsilon^1_x} \partial_x \left( \epsilon \epsilon^1_x \right) \right) - k \epsilon (\epsilon^1_x)^2,$$ \hfill (4.20)

and, for $G'_1$:

$$\{G'_1(\eta), \varphi_1(y)\}_D = \eta(y) \partial_y \varphi_1(y),$$
$$\{G'_1(\eta), \psi(y)\}_D = \eta(y) \partial_y \psi(y),$$
$$\{G'_1(\eta), \epsilon^1_x(y)\}_D = \partial_y (\eta(y) \epsilon^1_x(y)),$$
$$\{G'_1(\eta), \omega_x(y)\}_D = \partial_y (\eta(y) \omega_x(y)).$$ \hfill (4.21)

These infinitesimal gauge transformations can be rewritten as\textsuperscript{10}

$$\{G'_0(\varepsilon), \varphi_1(y)\}_D = \frac{\varepsilon}{\epsilon^1_x} G'_0(y) + \frac{\xi^t}{\epsilon^1_x} G'_1(y) + \frac{\xi}{\epsilon^1_x} \frac{\delta S_{BF}[A, \phi]}{\delta \epsilon^1_x} - \frac{\sigma}{\epsilon^1_x} \lambda \partial_x \psi + L_{(\xi^t, \xi^t)} \varphi_1(y),$$ \hfill (4.22)

$$\{G'_0(\varepsilon), \psi(y)\}_D = \frac{\xi^t}{\epsilon^1_x} \frac{\delta S_{BF}[A, \phi]}{\delta \epsilon^1_x} + L_{(\xi^t, \xi^t)} \psi(y),$$ \hfill (4.23)

$$\{G'_0(\varepsilon), \epsilon^1_x(y)\}_D = -\xi^t \frac{\delta S_{BF}[A, \phi]}{\delta \varphi_1} + L_{(\xi^t, \xi^t)} \epsilon^1_x(y),$$ \hfill (4.24)

$$\{G'_0(\varepsilon), \omega_x(y)\}_D = -\xi^t \frac{\delta S_{BF}[A, \phi]}{\delta \psi} + \sigma \partial_y \lambda + L_{(\xi^t, \xi^t)} \omega_x(y),$$ \hfill (4.25)

where

$$\lambda = \frac{\varepsilon}{N} \partial_y N - \frac{\varepsilon}{\epsilon^1_x} \partial_x \epsilon^1_x - \partial_x \varepsilon, \quad \xi^t = \frac{\varepsilon \epsilon^1_x}{N}, \quad \xi^t = \frac{\epsilon N^1}{N},$$ \hfill (4.26)

and

$$\{G'_1(\eta), \varphi_1(y)\}_D = L_{(\xi^t, \xi^t)} \varphi_1(y)$$
$$\{G'_1(\eta), \psi(y)\}_D = L_{(\xi^t, \xi^t)} \psi(y)$$
$$\{G'_1(\eta), \epsilon^1_x(y)\}_D = L_{(\xi^t, \xi^t)} \epsilon^1_x(y)$$
$$\{G'_1(\eta), \omega_x(y)\}_D = L_{(\xi^t, \xi^t)} \omega_x(y).$$ \hfill (4.27)

In expressions (4.22)–(4.25) and (4.27), the symbol $L_{(v^t, v^t)}$ represents the Lie derivative in the direction of the vector $(v^t, v^t)$, which generates the time and space diffeomorphisms.

\textsuperscript{10} For the transformations generated by $G'_0$, the field equations (2.19) are used and some heavy algebraic manipulations are necessary.
The interpretation of this result is as follows. The time gauge condition (4.1), which breaks gauge invariance, leaves two residual symmetries unbroken. The first one is that of time diffeomorphisms, generated by \( G'_0 \), up to constraints, up to field equations (‘on-shell realization’), and up to a compensating local Lorentz transformation of parameter \( \lambda \) which takes care of the time gauge condition. The second unbroken invariance is that of space diffeomorphisms, generated by \( G'_1 \).

The definition of \( G'_0 \) and \( G'_1 \) in (4.7) has been chosen in order to be scalar densities of weight 1. This indeed ensures that they form a Lie algebra (4.18) which is closed, in contrast to gravity in higher dimensions where the algebra closes with field-dependent structure ‘constants’ [1–3]. Such a feature is a characteristic of two-dimensional theories with general covariance, such as the bosonic string in the approach of [22].

A new redefinition
\[
C_+ = \frac{\sqrt{-\sigma}}{2} G'_0 - \frac{1}{2} G'_1, \tag{4.28}
\]
\[
C_- = -\frac{\sqrt{-\sigma}}{2} G'_0 - \frac{1}{2} G'_1, \tag{4.29}
\]
leads to the algebra
\[
\{C_+(\epsilon), C_+(\eta)\}_D = C_+(\{\epsilon, \eta\}),
\]
\[
\{C_-(\epsilon), C_- (\eta)\}_D = C_- (\{\epsilon, \eta\}),
\]
\[
\{C_+(\epsilon), C_- (\eta)\}_D = 0,
\]
which shows a factorization in two classical Virasoro algebras.

To complete this section, let us write the final Hamiltonian
\[
H_F = -\int d\gamma (\xi^0(\gamma) G'_0(\gamma) + \xi^1(\gamma) G'_1(\gamma)), \tag{4.31}
\]
where \( \xi^0 \) and \( \xi^1 \) are scalar densities of weight \(-1\) in one-dimensional space. The equations of the dynamical fields generated by this Hamiltonian are
\[
\partial_t e^i_1(x) = \left\{ e^i_1(x), H_F \right\}_D = \xi^0(x) e^i_1(x) \omega_1(x) + \partial_x (\xi^1(x) e^i_1(x)),
\]
\[
\partial_t \omega_1(x) = \left\{ \omega_1(x), H_F \right\}_D = -\sigma \partial_x \left( \xi^0(x) + \frac{e^0_1(x)}{e^i_1(x)} \partial_x e^i_1(x) \right) - k \xi^0(x) \left( e^i_1(x) \right)^2 + \partial_x (\xi^1(x) \omega_1(x)),
\]
\[
\partial_t \varphi_1(x) = \left\{ \varphi_1(x), H_F \right\}_D = \sigma \partial_x \left( \xi^0 e^i_1 \right) \frac{\partial_x \psi}{(e^i_1)^2} + 2k \xi^0 e^i_1 \psi - \xi^0 \omega_1 \varphi_1 + \xi^1 \partial_x \varphi_1,
\]
\[
\partial_t \psi(x) = \left\{ \psi(x), H_F \right\}_D = -\xi^0 e^i_1 \varphi_1 + \xi^1 \partial_x \psi.
\]
They are equivalent, modulo the constraints, to the field equations (2.19) for the fields \( e^i_1, \omega_1, \varphi_1 \) and \( \psi \).

5. Observables

5.1. In the BF formalism

Classical observables are gauge-invariant functions in phase space. In Dirac’s formalism, this means that they are functions \( \mathcal{O} \) which have vanishing Dirac bracket with the constraints (4.16), (4.17):
\[
\{\mathcal{O}, G_m\}_D \approx 0, \quad m = 0, 1. \tag{5.1}
\]
We shall consider the space manifold $\Sigma_1$ to be compact, homeomorphic to the circle $S^1$. The coordinate $x$ will be denoted by $\theta$, with range $(0, 2\pi)$. The nonvanishing Dirac brackets of the basic fields read
\[
\{e^i(\theta), \varphi_i(\theta')\}_D = \delta(\theta - \theta') = \{\omega_i(\theta), \psi(\theta')\}_D.
\]
The two independent observables present in the theory, denoted by $T$ and $L$, are defined, prior to the time gauge fixing, by\(^{11}\)
\[
T = \text{Tr} P e^{\Lambda A} = \text{Tr} P e^{\Lambda J A'} = \text{Tr} \left( \sum_{n=0}^{\infty} \frac{1}{n!} P \oint A \oint A \cdots \oint A \right)
\]
\[
L = \langle \phi(\theta), \phi(\theta) \rangle = k_{ij} \phi_i(\theta) \phi_j(\theta)
\]
where $A$ is the (A)dS connection and $\phi$ is the scalar field in the adjoint representations as defined in subsection 2.2. $T$, defined by (5.2), is known as a Wilson loop, where $P$ denotes the path ordering in the $\theta$ coordinate, and $J_i$ ($i = 0, 1, 2$) are the generators of (A)dS.

The observable $L$ defined by (5.3) is actually global, too, since it is independent of $\theta$ as a consequence of the field equations.

For explicit calculations in terms of the component fields $e^i$, etc, defined by (2.4) and (2.12), it is useful to take the generators $J_i$ in the fundamental representation as
\[
J_0 = P_0 = -\frac{i}{\sqrt{2}} \sqrt{\kappa} \tau_3, \quad J_1 = P_1 = -\frac{i}{2} \sqrt{\sigma \kappa} \tau_1, \quad J_2 = \Lambda = -\frac{i}{2} \sqrt{\sigma} \tau_2,
\]
where $\tau_1$, $\tau_2$ and $\tau_3$ are the Pauli matrices, the Killing form $\langle , \rangle$ being represented by the trace.

Some useful formulae are
\[
J_i J_j = \frac{1}{2} f_{ij}^k J_k - \frac{\sigma}{4} k_{ij},
\]
\[
\text{Tr}(J_i J_j) = -\frac{\sigma}{2} k_{ij}, \quad \text{Tr}(J_i J_j J_k) = -\frac{\sigma}{4} f_{ij} k_{kj}, \text{ etc.}
\]

5.2. In the time gauge formalism

Let us now compute $T$ and $L$ for the time gauge fixed theory and check that the resulting expressions have vanishing Dirac bracket with the constraints $G_0$ and $G_1$. The calculations for $G_0$ will be performed to the first nontrivial order of the expansion (5.2).

Using explicitly the time gauge condition and the expression of $\varphi_0$ given from the second class constraints (see equations (4.14), (4.15)), with the help of (5.4), (5.5), we can rewrite (5.2) as
\[
T = \text{Tr} \left( \frac{1}{2} \oint A + \frac{1}{2!} P \oint A \oint A + \cdots \right)
\]
\[
= 2 - \frac{\sigma}{2} \int_0^{2\pi} d\theta_1 \int_0^{\theta_1} d\theta_2 \left( e^i(\theta_1) e^j(\theta_2) k + \omega_j(\theta_1) \omega_i(\theta_2) \right) + O(4),
\]
where $O(4)$ means up to terms of order 4 in the basic fields. One can then check, up to this order, that $T$ is an observable:
\[
\{G_0'(y), T\}_D \approx \{G_1'(y), T\}_D \approx 0 + O(4).
\]

\(^{11}\) They were calculated by the authors of [11] in the case of the compact gauge group SU(2)—corresponding to (A)dS with $\sigma = k = 1$. 

For the quantity \( L \) given by (5.3), we obtain
\[
L = k^{ij} \phi_i \phi_j = \frac{\sigma}{k} (\phi_0)^2 + \frac{1}{k} (\phi_1)^2 + (\psi)^2,
\] (5.8)
with \( \phi_0 = \sigma \theta / e^1 \). It is easy to check that \( L \) has weakly vanishing Dirac brackets with the constraints:
\[
\{ G'_0(\epsilon), L \}_D = -2 \frac{\epsilon \phi_0}{ke^1} G'_1(x) + 2 \frac{\epsilon \phi_1}{ke^1} G'_0(x) \approx 0, \tag{5.9}
\]
\[
\{ G'_1(\epsilon), L \}_D = 2 \frac{\sigma \epsilon \phi_0}{ke^1} G'_0(x) + 2 \frac{\epsilon \phi_1}{ke^1} G'_1(x) \approx 0. \tag{5.10}
\]
Hence \( L \) defines an observable, too.

6. Conclusion

The canonical construction of the classical theory in the time gauge has been completed in the Dirac formalism, including the discussion of the observables.

This represents a first step towards the construction of the corresponding quantum theory using the loop quantization techniques [14, 21].

Acknowledgment

We thank Alejandro Perez for very useful discussions.

References

[1] Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report Class. Quantum Grav. 21 R53 (Preprint gr-qc/0404018)
[2] Rovelli C 2004 Quantum Gravity (Cambridge Monography on Mathematical Physics) (Cambridge: Cambridge University Press)
[3] Thiemann T 2007 Modern Canonical Quantum General Relativity (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)
[4] Jackiw R 1984 Quantum Theory of Gravity ed S Christensen (Bristol: Hilger)
Teitelboim C 1983 Phys. Lett. B 126 41
Teitelboim C 1984 Quantum Theory of Gravity ed S Christensen (Bristol: Hilger)
[5] Henneaux M 1985 Quantum gravity in two dimension: exact solution of Jackiw model Phys. Rev. Lett. 54 959
[6] Fukuyama T and Kamimura K 1985 Gauge theory of two-dimensional gravities Phys. Lett. B 160 259
[7] Isler K and Trugenberger C A 1989 Gauge theory of two-dimensional quantum gravity Phys. Rev. Lett. 63 834
[8] Kummer W, Liebl H and Vassilevich D V 1997 Exact path integral quantization of generic 2-D dilaton gravity Nucl. Phys. B 493 491
[9] Kummer W, Liebl H and Vassilevich D V 1999 Integrating geometry in general 2D dilaton gravity with matter Nucl. Phys. B 544 403
[10] Grumiller D, Kummer W and Vassilevich D V 2002 Phys. Rep. 369 327
[11] Livine E R, Perez A and Rovelli C 2003 2D manifold-independent spinfoam theory Class. Quantum Grav. 20 4425 (Preprint gr-qc/0102051)
[12] Birmingham D, Blau M, Rakowski M and Thompson G T 1991 Topological field theory Phys. Rep. 209 129–340
Blau M and Thompson G 1991 Topological gauge theories of antisymmetric tensor field Ann. Phys. 205 130–72
Constantinidis C P, Lourenço J A and Piguet O, work in progress
[14] Lourenço J A and Piguet O, work in progress
[15] Diaz Alex Rios 2007 Uma revisão da gravitação bidimensional do ponto de vista da gravitação quântica de loops Master Degree Thesis Universidade Federal do Espírito Santo, Brazil
[17] Bautista Luis Ivan Morales 2007 Formalismo Hamiltoniano do modelo de Jackiw-Teitelboim no calibre temporal
Master Degree Thesis Universidade Federal do Espírito Santo, Brazil

[18] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)

[19] Henneaux M and Teitelboim C 1991 Quantization of Gauge Systems (Chicago, IL: University of Chicago Press)

[20] Gitman D M and Tyutin I V 1990 Quantization of Fields with Constraints (Springer Series in Nuclear and Particle Physics) (Berlin: Springer)

[21] Constantinidis C P, Lourenço J A, Piguet O and Spalenza W 2006 Quantização da gravidade em duas dimensões via o formalismo de laços Poster Presented at the XXVII Encontro Nacional de Física de Partículas e Campos (Aguas de Lindóia, SP)

[22] Thiemann T 2006 The LQG string: loop quantum gravity quantization of string theory: I. Flat target space
Class. Quantum Grav. 23 1923 (Preprint hep-th/0401172)