A twisted invariant of a compact Riemann surface *

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Abstract

We introduce a twisted version of the Kawazumi-Zhang invariant $a_g(C) = \varphi(C)$ of a compact Riemann surface $C$ of genus $g \geq 1$, and discuss how it is related to the first Mumford-Morita-Milller class $\kappa_1^1$ on the moduli space of compact Riemann surfaces and the original Kawazumi-Zhang invariant.

1 Introduction

A compact Riemann surface is still a fascinating research subject even in the twenty-first century. In fact, various mathematicians and even physicists are now studying the Teichmüller space $T_g$ of genus $g$, a universal covering space of the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g$, from their own various viewpoints. In this paper we study functions on the moduli space $\mathcal{M}_g$, i.e., invariant functions on the Teichmüller space $T_g$ under the mapping class group action. They are regarded also as analytic invariants of a compact Riemann surface. There are no non-constant holomorphic functions on $\mathcal{M}_g$ if $g \geq 3$, while the $3g - 3$-dimensional orbifold $\mathcal{M}_g$ has an enormous amount of $C^\infty$ functions. Hence we would like to find functions which have some geometric and/or arithmetic meanings. Hyperbolic geometry is a highly effective tool for studying Riemann surfaces and Teichmüller spaces. But, in the context of hyperbolic geometry, the author knows no interesting functions on the moduli space $\mathcal{M}_g$ of compact Riemann surfaces except the injective radius of a compact hyperbolic surface.

Other effective tools for studying compact Riemann surfaces are the classical theory of Abelian integrals and its powerful descendant, arithmetic geometry including the Arakelov geometry. Theta functions provide meaningful functions on the moduli space $\mathcal{M}_g$. Moreover, based on his own Arakelov geometry, Faltings [11] introduced an analytic invariant of a compact Riemann surface. Hain and Reed [2] introduced another invariant by the Hodge line bundle and the intermediate Jacobian on the moduli space $\mathcal{M}_g$. Zhang [26] and the author [15, 16] introduced the same invariant $a_g = \varphi$ independently. Zhang’s approach comes from the Arakelov geometry, while the author’s from the Johnson homomorphism similar to that of Hain and Reed, but based on the classical theory of Abelian integrals. Later de Jong [8] proved that the Faltings invariant, the Hain-Reed invariant and the Kawazumi-Zhang invariant are linearly dependent. Moreover he has

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clarified various aspects of the invariant $a_g$ in papers including [9, 10, 11]. In particular, de Jong and Shokrieh [13, Theorem 7.1] gave a complete description of the asymptotic behavior of the invariant $a_g$ along 1-parameter families of compact Riemann surfaces degenerating into a stable curve. D’Hoker and Green [4] found some relation of the genus 2 invariant $a_2$ to high energy physics. D’Hoker, Green, Pioline and Russo [5] proved the function $a_2$ is an eigenfunction with respect to the Laplacian on the moduli space of principally polarized Abelian varieties $A_2$. In §3.2 we will show this does not hold in the case $g = 3$ by using a twisted version of the invariant $a_g$. The obstruction for $a_3$ to be an eigenfunction is the (extended) first Johnson homomorphism [7, 20].

In this paper, we introduce a twisted version of the invariant $a_g$, a map $A$ from the 4-fold tensor product of the first cohomology group of the surface to the complex numbers. One can recover the invariant $a_g$ from the map $A$ by using the intersection pairing on the cohomology group, and can construct new invariants by a similar way to $a_g$. Moreover, from the invariant $A$, we obtain an explicit $(1, 1)$-form on the moduli space $M_g$ representing the first Mumford-Morita-Miller class $e_1 = \kappa_1$ in §3.1.

The invariant $A$ is given as an integral over the 2-fold product of the surface. D’Hoker and Schlotterer [6] generalized our construction to integrals over the $n$-fold product of the surface for $n \geq 3$, and proved highly nontrivial identities among them. It is amazing that their proof of the identities is based only on formal natures of the Arakelov Green function $G(\cdot, \cdot)$.

This paper is organized as follows. The twisted version $A$ of the invariant $a_g$ of a compact Riemann surface $C$ is introduced in §2. In §3 we give two kinds of interpretation of the map $A$ on the moduli space $M_g$. First it defines a $(1, 1)$-form on $M_g$. As is proved in §4 it is closed and represents the first Mumford-Miller-Morita class $e_1 = \kappa_1$. In §3.2 using the second interpretation, we discuss how the invariant $a_g$ behaves in the cases $g = 2, 3$. The proof of Proposition 4, which we need for our consideration in the case $g = 3$, is given in §5.

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A twisted invariant of a compact Riemann surface

Throughout this paper, we assume the genus \( g \) of a compact Riemann surface \( C \) is not smaller than 1, \( g \geq 1 \). Then the vector space \( H' := H^0(\mathcal{C}; \Omega^1_C) \) of holomorphic 1-forms on \( C \) does not vanish. Its complex conjugate \( H'' := H^0(\mathcal{C}; \Omega^{1\bar} \mathcal{C}) \) is the vector space of anti-holomorphic 1-forms on \( C \). The first cohomology group \( H := H^1(\mathcal{C}; \mathbb{C}) \) is naturally identified with the direct sum \( H' \oplus H'' \), the vector space of harmonic 1-forms on the surface \( C \). The intersection pairing

\[
(\phi_1, \phi_2) \in H \times H \mapsto \phi_1 \cdot \phi_2 := \int_C \phi_1 \wedge \phi_2 \in \mathbb{C}, \quad (1)
\]

is non-degenerate. Hence we identify \( H \) with its dual through the pairing. Under this identification, the subspaces \( H' \) and \( H'' \) are dual to each other.

For \( \phi'_1, \phi'_2 \in H' = H^0(\mathcal{C}; \Omega^1_C) \) we can define a positive definite Hermitian pairing

\[
(\phi'_1, \phi'_2) \in H' \otimes H' \mapsto \frac{-1}{2} \int_C \phi'_1 \wedge \overline{\phi'_2} \in \mathbb{C}. \quad (2)
\]

In fact, any holomorphic 1-form \( \phi' \) locally given by \( \phi' = f(z) dz \) in terms of a local complex coordinate \( z = x + i y \) satisfies \( \frac{-1}{2} \phi' \wedge \overline{\phi'} = |f(z)|^2 dx \wedge dy \). We take an orthonormal basis \( \{\psi_i\}_{i=1}^g \subset H' \) with respect to the Hermitian pairing

\[
\frac{-1}{2} \int_C \psi_i \wedge \overline{\psi}_j = \delta_{ij}, \quad (1 \leq i, j \leq g).
\]

Then the element

\[
\hat{\omega} := \sum_{i=1}^g \psi_i \otimes \overline{\psi}_i \in H' \otimes H'' \subset H \otimes H \quad (3)
\]

is independent of the choice of the basis \( \{\psi_i\}_{i=1}^g \). In fact, the set \( \{\psi_i, \frac{-1}{2} \psi_i\}_{1 \leq i \leq g} \) is a symplectic basis of \( H = H^1(\mathcal{C}; \mathbb{C}) \) with respect to the pairing \( \Omega \). Hence the symplectic form on \( H \), which depends only on the pairing \( \Omega \), equals

\[
\frac{-1}{2} \sum_{i=1}^g \psi_i \otimes \overline{\psi}_i - \frac{-1}{2} \sum_{i=1}^g \overline{\psi}_i \otimes \psi_i \in H^1(\mathcal{C}; \mathbb{C})^\otimes 2,
\]

whose component in \( H' \otimes H'' \) equals \( \frac{-1}{2} \hat{\omega} \). In particular, a \((1, 1)\)-form \( B \) on the surface \( C \) defined by

\[
B := \frac{-1}{2g} \sum_{i=1}^g \psi_i \wedge \overline{\psi}_i,
\]

is independent of the choice of the basis \( \{\psi_i\}_{i=1}^g \). We have \( \int_C B = 1 \). Moreover the form \( B \) is a real volume form on the surface \( C \), since the 1-forms \( \psi_i \)'s have no common zeroes because of the Riemann-Roch formula.

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We denote by $\Omega^p(C)$ the vector space of $\mathbb{C}$-valued $p$-forms on $C$ for $p = 0, 1, 2$. Then the Hodge $*$-operator $*: \Omega^1(C) \to \Omega^1(C)$ depends only on the complex structure of the surface $C$. In fact, in terms of a local complex coordinate $z$, we have $*dz = -\sqrt{-1}d\bar{z}$ and $*d\bar{z} = \sqrt{-1}dz$. Then, since the operator $d * d$ is equivalent to the Laplacian by means of the volume form $B$, we have an exact sequence

$$0 \to \mathbb{C} \to \Omega^0(C) \xrightarrow{d \omega} \Omega^2(C) \xrightarrow{\int_C} \mathbb{C} \to 0,$$

where the left $\mathbb{C}$ means the constant functions. Hence we have a unique linear map $\Phi : \Omega^2(C) \to \Omega^0(C)$ such that

$$d * d\Phi(\Omega) = \Omega - \left( \int_C \Omega \right) B \quad \text{and} \quad \int_C \Phi(\Omega) B = 0$$

for any $\Omega \in \Omega^2(C)$. It is nothing but the Green operator associated with the volume form $B$. The operator $\hat{\Phi}$ is real, $\Phi = \hat{\Phi}$, and the induced map

$$\hat{\Phi}|_{\text{Ker}(\omega)} : \text{Ker} \left( \int_C : \Omega^2(C) \to \mathbb{C} \right) \to \Omega^0(C)/\mathbb{C}$$

depends only on the complex structure on the surface $C$. Moreover we have

$$\int_C \Phi(\Omega) \Omega' = \int_C \Omega \hat{\Phi}(\Omega') = \frac{1}{4\pi} \int_{(P_1, P_2) \in C \times C} (\Omega) P_1 (\log G(P_1, P_2))(\Omega') P_2$$

for any $\Omega, \Omega' \in \Omega^2(C)$, where we denote by $G(\cdot, \cdot)$ the Arakelov Green function. Here we remark $\hat{\Phi}(B) = 0$. In fact, the function $\hat{\Phi}(B)$ is constant since $d * d\hat{\Phi}(B) = 0$, and so $\hat{\Phi}(B) = \int_C \hat{\Phi}(B) B = 0$.

Now we define a map $A$ by

$$A : H' \otimes H'' \otimes H' \otimes H'' \to \mathbb{C}, \quad \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4 \mapsto \int_C \phi_1 \wedge \phi_2 \hat{\Phi}(\phi_3 \wedge \phi_4),$$

which is independent of the choice of the basis $\{\psi_i\}_{i=1}^g$, and so is a twisted invariant of the compact Riemann surface $C$. One can extend it to a map $H \otimes H \otimes H \otimes H \to \mathbb{C}$ by the same formula (4), but with no additional information. In order to compute the map $A$ explicitly, it is convenient to introduce a complex number $A_{ijkl}$ defined by

$$A_{ijkl} := A(\psi_i \otimes \psi_j \otimes \psi_k \otimes \psi_l) = \int_C \psi_i \wedge \psi_j \hat{\Phi}(\psi_k \wedge \psi_l)$$

$$= \frac{1}{4\pi} \int_{(P_1, P_2) \in C \times C} (\psi_i \wedge \psi_j) P_1 (\log G(P_1, P_2))(\psi_k \wedge \psi_l) P_2$$

for $1 \leq i, j, k, l \leq g$ by using the orthonormal basis $\{\psi_i\}_{i=1}^g$. Then we have

$$\overline{A_{ijkl}} = \int_C \psi_l \wedge \psi_i \hat{\Phi}(\psi_k \wedge \psi_j) = A_{jikl},$$

$$A_{ijkl} = \int_C \hat{\Phi}(\psi_i \wedge \psi_j) \psi_k \wedge \psi_l = A_{klij}.$$
The latter follows from (4). We can contract the indices $i, k, \ldots$ and the ones $\bar{j}, \bar{l}, \ldots$ to get a new complex number. For example, the Kawazumi-Zhang invariant\footnote{The author uses the symbol $a_g = a_g(C)$, while Zhang and others use the symbol $\varphi = \varphi(C)$.} $a_g = a_g(C)$ is obtained by

$$a_g = \sum_{i,j=1}^g A_{ij} = A((24)(\hat{\omega} \otimes \hat{\omega})), \quad \text{which is a real number by the formula (7).}$$

Here (24) : $H' \otimes H'' \otimes H' \otimes H'' \rightarrow H' \otimes H'' \otimes H' \otimes H' \otimes H''$ is the switch map of the 2nd and the 4th components of the tensor product $H' \otimes H'' \otimes H' \otimes H''$.

We can describe the numbers $A_{i,j,k,l}$ and $a_g$ in the diagrams

$$A_{i,j,k,l} = \quad a_g = \quad \text{Here the circle with indices } i \text{ and } j \text{ means } \psi_i \wedge \overline{\psi_j}, \text{ the double line without arrows } \log G(-, -), \text{ and a line with an arrow the contraction. For example, since } \sqrt{-1} \frac{1}{2g} \hat{\Phi}(\sum_{i=1}^g \psi_i \wedge \overline{\psi_i}) = \hat{\Phi}(B) = 0, \text{ we have } \sum_{i,j,l=1}^g A_{i,j,l} = \sum_{j,k,l=1}^g A_{j,k,l} = 0, \text{ that is,}$$

$$\sum_{i,j,l=1}^g A_{i,j,l} = \sum_{j,k,l=1}^g A_{j,k,l} = 0$$

Similarly we can introduce complex numbers associated with the Riemann surface $C$, i.e., functions on the moduli space $M_g$ of compact Riemann surfaces. For example, we can consider a function on $M_g$ defined by the diagram

The number $A_{i,j,k,l}$ is given by an integral over the 2-fold direct product $C^{\times 2} = C \times C$. We can generalize this construction to the 3-fold direct product and more. For example,
we can consider
\[
\int_C \Phi(\psi_i \wedge \psi_j) \psi_m \wedge \psi_n \Phi(\psi_k \wedge \psi_l) = \int_C \psi_i \wedge \psi_j \Phi((\psi_m \wedge \psi_n) \Phi(\psi_k \wedge \psi_l))
\]
\[
= \frac{1}{(4\pi)^2} \int\int\int_{(P_1, P_2, P_3) \in C \times C \times C} (\psi_i \wedge \psi_j) P_1 (\log G(P_1, P_2))(\psi_m \wedge \psi_n) P_2 (\log G(P_2, P_3))(\psi_k \wedge \psi_l) P_3
\]
for \(1 \leq i, j, k, l, m, n \leq g\). This can be described in the diagram

\[
\begin{array}{c}
  i \\
  \downarrow \\
  j \\
  \downarrow \\
  m \\
  \downarrow \\
  l \\
  \downarrow \\
  k
\end{array}
\]

D’Hoker and Schlotterer [6] gave a systematic generalization of \(A_{ijk} \) where they constructed modular graph tensors using integrals on the direct product \(C \times C \times C\) for any \(n \geq 1\), and proved highly nontrivial identities among them. It is amazing that their proof of the identities is based only on formal natures of the Arakelov Green function \(G(\cdot, \cdot, \cdot)\).

3 Tensor fields on the moduli space \(\mathbb{M}_g\)

When the Riemann surface \(C\) runs over the moduli space \(\mathbb{M}_g\) of compact Riemann surfaces of genus \(g\), the transpose \(^t A\) of the linear map \(A\) in (5) can be regarded as some tensor fields on the moduli space \(\mathbb{M}_g\). Now recall the vector spaces \(H'\) and \(H''\) are dual to each other through the pairing (11). In particular, the basis \(\{\psi_i\}_{i=1}^g\) and \(\{\sqrt{-1} \psi_i\}_{i=1}^g\) are dual to each other. Then the transpose \(^t A : C \to H'' \otimes H' \otimes H' \otimes H'\) is determined by its own value at \(1 \in C\)

\[
^t A(1) = \frac{1}{16} \sum_{i,j,k,l=1}^g A_{ijkl} \psi_i \otimes \psi_j \otimes \psi_k \otimes \psi_l \in H'' \otimes H' \otimes H' \otimes H'.
\]

When the surface \(C\) runs over the moduli space \(\mathbb{M}_g\), the collections of \(H'\) and \(H''\) are holomorphic and anti-holomorphic vector bundles, respectively. Then \(^t A(1)\) is a \(C^\infty\) section of a tensor product of these vector bundles.

3.1 The first Mumford-Morita-Miller class \(e_1 = \kappa_1\)

Here we recall the holomorphic cotangent space \(T^*_{[C]} \mathbb{M}_g\) of the moduli space \(\mathbb{M}_g\) at the equivalence class \([C]\) of the Riemann surface \(C\) is canonically isomorphic to the space of holomorphic quadratic differentials

\[
T^*_{[C]} \mathbb{M}_g = H^0(C; (\Omega^1_C)^{\otimes 2}).
\]
For example, we may regard \( \psi \psi_j = \psi_j \psi_l \in T^*_C[M_g] \) and \( \overline{\psi_1 \psi_k} = \overline{\psi_k} \psi_1 \in T^*_C[M_g] \). Hence we can define a \((1, 1)\)-form \( e^g_1 \) on the moduli space \( M_g \) by

\[
e^g_1|_C := -3\sqrt{-1} \sum_{i,j,k,l=1}^g (\psi_j \psi_l \overline{A_{\hat{J}kT}\overline{\psi_k} \psi_i} - \psi_j \psi_l \overline{A_{\hat{J}kT}\overline{\psi_k} \psi_i}).
\]

In fact, from (5), the RHS is independent of the choice of the basis \( \{\psi_i\}_{i=1}^g \). Then we have

**Theorem 1.** The \((1, 1)\)-form \( e^g_1 \) is closed on the moduli space \( M_g \), and represents the first Mumford-Morita-Miller class \( e_1 \in H^2(M_g; \mathbb{C}) \).

The theorem was stated implicitly in the equation (5.5) of the preprint [15] without proof. We will give its proof in the next section [4].

Very recently, de Jong and van der Lugt [12, 19] introduced a subalgebra \( \mathcal{R}(M_g) \) of the de Rham complex \( \Omega^*(M_g) \), called the ring of tautological forms on \( M_g \). It consists entirely of closed forms, and its image in the cohomology algebra \( H^*(M_g; \mathbb{C}) \) is generated by all the Mumford-Morita-Miller classes, i.e., equals the image of the tautological ring of \( M_g \). It is remarkable that the ring \( \mathcal{R}(M_g) \) is finite-dimensional, which suggests us that study of differential forms on \( M_g \) could help us to understand the tautological ring of \( M_g \). As was shown in [12, 19], the degree 2 part of the ring \( \mathcal{R}(M_g) \) is closely related to the invariant \( a_g \). So it would be very interesting if we could find further relations of the ring \( \mathcal{R}(M_g) \) with the complex numbers \( A_{\hat{J}kT}'s \) or generalizations by D’Hoker and Schlotterer [6].

### 3.2 The Jacobian map \( \text{Jac} : M_g \rightarrow A_g \)

The transpose \( ^tA(1) \) in (5) has another interpretation: it can be regarded as a \( C^\infty \) section of the pullback \( \text{Jac}^*\left(T^*A_g \otimes T^*\mathbb{A}_g\right) \) over the moduli space \( M_g \). Here \( \mathbb{A}_g \) is the moduli space of principally polarized Abelian varieties of genus \( g \), and \( \text{Jac} : M_g \rightarrow \mathbb{A}_g, [C] \mapsto [\text{Jac}(C)] \), the Jacobian map. The moduli space \( \mathbb{A}_g \) is the quotient space of the Siegel upper half space \( \mathcal{H}_g \) by the Siegel modular group \( \text{Sp}_{2g}(\mathbb{Z}) \). The space \( \mathcal{H}_g = \text{Sp}_{2g}(\mathbb{R})/U_g \) is the space of all complex structures \( J \) on the standard symplectic vector space \( \mathbb{R}^{2g} \) with positivity condition. The tautological vector bundle \( E_g := \bigsqcup_{J \in \mathcal{H}_g}(\mathbb{R}^{2g}, J) \) over \( \mathcal{H}_g \) is an \( \text{Sp}_{2g}(\mathbb{R}) \)-equivariant holomorphic vector bundle with canonical Hermitian metric, and so descends to the moduli space \( \mathbb{A}_g \). Through a canonical isomorphism \( T\mathbb{A}_g = \text{Sym}^2(E_g) \subset E_g^{\otimes 2} \), one can define a Kähler metric on the moduli space \( \mathbb{A}_g \). See, for example, [22] Chapter II. In particular, we may consider the Laplacian \( \Delta \) on the moduli space \( \mathbb{A}_g \). The pullback \( \text{Jac}^*E_g^* \) of the Hermitian vector bundle \( E_g \) under the Jacobian map \( \text{Jac} \) at the equivalence class \( [C] \in M_g \) is isometric to the holomorphic 1-forms \( H^1 = H^0(C; \Omega^1_C) \) with metric given in (2). Hence the transpose \( ^tA(1) \) in (5) can be regarded as a \( C^\infty \) section of the pullback \( \text{Jac}^*(T^*\mathbb{A}_g \otimes T^*\mathbb{A}_g) \).

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3In [17, 18] we denote \( e^g_i \) by \( e^P_i \), whose coefficients was \( 48\sqrt{-1} \). It is not true. This mistake comes from computation of the intersection number \( Y_i \cdot Y_i \).
By Rauch’s formula \[23\], the differential dJac of the Jacobian map is given by the commutative diagram

\[
\begin{array}{ccc}
T_{\text{Jac}(C)}M_g & \xrightarrow{(\text{dJac})^*} & T_{\text{Jac}(C)^*}\mathbb{A}_g \\
\| & & \| \\
H^0(C; (\Omega_C^1)^{\otimes 2}) & \xleftarrow{\text{multiplication}} & \text{Sym}^2 H^0(C; \Omega_C^1).
\end{array}
\]

In particular, from M. Noether’s theorem, the differential dJac[C] is injective if and only if \( g = 2 \) or \((g \geq 3 \text{ and } C \text{ is non-hyperelliptic})\). On the other hand, \( \dim M_g = \dim \mathbb{A}_g \) if and only if \( g = 2 \) or 3. Hence the Laplacian \( \Delta \) on \( \mathbb{A}_g \) acts on functions on \( M_g^{\text{open}} \subset \mathbb{A}_2 \) and \( M_3^{\text{open}} \subset \mathbb{A}_3 \). Here we denote by \( \mathbb{H}_g \subset M_g \) the locus of hyperelliptic Riemann surfaces of genus \( g \geq 2 \).

D’Hoker, Green, Pioline and Russo \[5\] proved the following remarkable theorem.

**Theorem 2** (\[23\](4.23)). \( \Delta a_2 = 5a_2 \). In particular, the invariant \( a_2 \) is an eigenfunction of the Laplacian \( \Delta \).

Based on the theorem, Pioline \[23\] gave an explicit formula for \( a_2 \) in terms of theta functions. Later, for any \( g \geq 1 \), Wilms \[25\] gave an explicit formula for \( a_g \) in terms of a function introduced by de Jong as well as theta functions. But the methods of Wilms \[25\] seem to be different from those discussed in this paper.

There remains the case \( g = 3 \). In this paper we prove

**Theorem 3.** In the case \( g = 3 \), the invariant \( a_3 \) is not an eigenfunction of the Laplacian \( \Delta \).

The obstruction for \( a_3 \) to be an eigenfunction is the first Johnson homomorphism \[7\] [20], which is represented by a twisted 1-form on \( M_g \) as follows. We denote by \( U = U(C) \) the kernel of the contraction map \( \Lambda^3 H \to H \), \( \phi_1 \wedge \phi_2 \wedge \phi_3 \mapsto (\phi_2 \cdot \phi_3)\phi_1 + (\phi_3 \cdot \phi_1)\phi_2 + (\phi_1 \cdot \phi_2)\phi_3 \). It should be remarked that \( U \) vanishes if and only if \( g \leq 2 \). For \( \phi_i = \phi_i^0 + \phi_i^1 \in H = H' \oplus H'' \), \( 1 \leq i \leq 3 \), we define \( Q(\phi_1, \phi_2, \phi_3) \in C\infty(C; (T^*C)^{\otimes 2}) \) by

\[
Q(\phi_1, \phi_2, \phi_3) := -\sqrt{-1}\phi_1^0 \partial \bar{\partial} \Phi(\phi_2 \wedge \phi_3) - \sqrt{-1}\phi_2^0 \partial \bar{\partial} \Phi(\phi_3 \wedge \phi_1) - \sqrt{-1}\phi_3^0 \partial \bar{\partial} \Phi(\phi_1 \wedge \phi_2). \tag{10}
\]

The tensor \( Q(\phi_1, \phi_2, \phi_3) \) is alternating in \( \phi_1, \phi_2, \phi_3 \), but is not holomorphic in general. Since \( d \ast d = -2\sqrt{-1}\partial \bar{\partial} \), we have

\[
2\partial \bar{\partial} Q
\]

\[
= \phi_1^0(\phi_2 \wedge \phi_3) - \left( \int_C \phi_2 \wedge \phi_3 \right) B + \phi_2^0(\phi_3 \wedge \phi_1) - \left( \int_C \phi_3 \wedge \phi_1 \right) B
\]

\[
+ \phi_3^0(\phi_1 \wedge \phi_2) - \left( \int_C \phi_1 \wedge \phi_2 \right) B
\]

\[
= \phi_1^0 \phi_2^0 \phi_3^0 - \phi_1 \phi_2^0 \phi_3^0 + \phi_2^0 \phi_3 \phi_1^0 - \phi_2 \phi_3^0 \phi_1^0 + \phi_3^0 \phi_1 \phi_2^0 - \phi_3 \phi_2^0 \phi_1^0 - (\phi_1^0(\phi_2 \cdot \phi_3) + \phi_2^0(\phi_3 \cdot \phi_1) + \phi_3^0(\phi_1 \cdot \phi_2)) B
\]

\[
= - (\phi_1^0(\phi_2 \cdot \phi_3) + \phi_2^0(\phi_3 \cdot \phi_1) + \phi_3^0(\phi_1 \cdot \phi_2)) B.
\]
This implies the restriction of \( Q \) to the subspace \( U \) has its value in the space \( H^0(C; (\Omega^1_C)^{\otimes 2}) \) of holomorphic quadratic differentials

\[
Q : U = U(C) \to H^0(C; (\Omega^1_C)^{\otimes 2}) = T^*_C M_g.
\]

We remark the map \( Q \) is equivariant under the action of the holomorphic automorphism group \( \text{Aut}(C) \). In particular, if the Riemann surface \( C \) is hyperelliptic, we have

\[
Q(U(C)) \subset (\text{the } (-1)\text{-eigenspace of the hyperelliptic involution in } T^*_C M_g),
\]

since the involution acts on the space \( U(C) \) by \(-1\).

We may identify the map \( Q \) with its transpose \( \tilde{Q} : T_C M_g \to (U(C))^* \), which can be regarded as a twisted \((1,0)\) form on the moduli space \( M_g \). Here the pairing

\[
\langle\cdot,\cdot\rangle : \Lambda^3 H \times \Lambda^3 H \to \mathbb{C},
\]

\[
(\phi_1 \wedge \phi_2 \wedge \phi_3, \psi_1 \wedge \psi_2 \wedge \psi_3) \mapsto \sum_{\sigma, \tau \in \mathfrak{S}_3} (\phi_{\sigma(1)} \cdot \phi_{\tau(1)})(\phi_{\sigma(2)} \cdot \phi_{\tau(2)})(\phi_{\sigma(3)} \cdot \phi_{\tau(3)})
\]

is non-degenerate on \( U = U(C) \), and so we identify \( U(C) \) with its dual \((U(C))^*\) through the pairing \((12)\). We denote by the same symbol \( U \) the flat vector bundle \( \coprod_{[C] \in M_g} U(C) \) over the moduli space \( M_g \). The tensor \( Q \) equals the \((1,0)\)-part of the first variation of the harmonic volumes introduced by Harris \([3\text{, Theorem 5.8}]\). In particular, the sum \( Q + \overline{Q} \)

\[
\text{is a twisted closed 1-form with values in the flat vector bundle } U.
\]

As was proved in \([14\text{, }\S8]\), the cohomology class of \(-Q + \overline{Q}\) equals the extended first Johnson homomorphism \( \tilde{k} \) \([20]\)

\[
\tilde{k} = -[Q + \overline{Q}] \in H^1(M_g; U).
\]

The cohomology group \( H^1(M_g; U) \) is canonically isomorphic to the first group cohomology of the mapping class group of a genus \( g \) Riemann surface with values in the vector space \( U = U(C) \). The restriction of \( k \) to the Torelli group equals the first Johnson homomorphism \([17\text{, 11}]\). As was proved by Morita \([21]\), the first Mumford-Morita-Miller class \( e_1 \) equals the contraction \( \langle \hat{k}, \hat{k} \rangle \in H^2(M_g; \mathbb{C}) \) of the cup product \( \hat{k} \cup \hat{k} \in H^2(M_g; U^{\otimes 2}) \) up to non-zero constant factor. By the same recipe using the pairing in \((12)\), we obtain a \((1,1)\)-form \( e_1^I \) on \( M_g \) representing the class \( e_1 \), which equals \( \langle Q + \overline{Q}, Q + \overline{Q} \rangle \) up to non-zero constant factor. From \([15\text{, Theorem 6.1}]\), we have

\[
\frac{1}{12} e_1^I = -\frac{2\sqrt{-1}g}{2(2g + 1)}(\partial \overline{\partial} a_g) + \frac{1}{12} e_1^I.
\]

An alternative proof of the formula \((13)\) using the Deligne pairing is given by de Jong \([11]\). If \( g = 2 \), then \( U = 0 \), so that \( e_1^I = 0 \). But \( e_1^I \neq 0 \) for \( g \geq 3 \).

In \([5\text{, Appendices B and C}]\), D’Hoker, Green, Pioline and Russo gave suggestive computations for any genus \( g \geq 2 \). In the equations \((B.4)\) and \((C.1)\), they divide the exact \((1,1)\) form \( \partial \overline{\partial} a_g \) into two parts \( \psi^1_A + \psi_B \) and \( \psi^2_A + \psi_C \). Their formulae \((C.3)\) and

\footnote{For simplicity, we use the same symbol \( Q \), not \( 'Q \).}
(C.5) say that $\psi_A^1 + \psi_B$ and $\psi_A^2 + \psi_C$ equal $e_1^g$ and $e_1^g$ up to non-zero constant factor, respectively. These formulae are equivalent to our equation (13). The formula (C.9) says that $\psi_A^2 + \psi_C$ vanishes if $g = 2$. In order to understand the formula (C.4)

$$\Delta a_g|_{\psi_A^1+\psi_B} = (2g + 1)a_g,$$

we need to introduce a lift of the differential form $e_1^g$ to an element $\tilde{E}_1$ of $C^\infty(\mathcal{M}_g; \text{Jac}^*(T^*\mathbb{A}_g \otimes T^*\mathbb{A}_g))$ defined by

$$\tilde{E}_1|_C := -3\sqrt{-1}\sum_{i,j,k,l=1}^g ((\psi_j \cdot \psi_l)A_{ijkl}(\psi_k \cdot \psi_l) - (\psi_j \cdot \psi_l)A_{ilk}(\psi_k \cdot \psi_l)).$$

Here $\psi_j \cdot \psi_l := \frac{1}{2}(\psi_j \otimes \psi_l + \psi_l \otimes \psi_l) \in \text{Sym}^2H'$ and so on. Now we recall the Laplacian $\Delta$ on any Kähler manifold $X$ is given by

$$\Delta f = 2\sqrt{-1}\Lambda \partial \bar{\partial} f$$

for any $f \in C^\infty(X; \mathbb{C})$. Here $\Lambda : T^*X \otimes T^*X \to X \times \mathbb{C}$ is the Kähler contraction. In our case $X = \mathbb{A}_g$, we have

$$\Lambda \tilde{E}_1|_C = 6 \left( \sum_{j,l=1}^g A_{jll} + \sum_{j,l=1}^g A_{jll} - g \sum_{i,j=1}^g A_{ijkl} - \sum_{i,l=1}^g A_{il} \right) = -6ga_g,$$

since $\int_C \psi_i \wedge \psi_j = -2\sqrt{-1}\delta_{ij}$ and $\sum_{j,k,l=1}^g A_{ijkl} = 0$. Applying the Kähler contraction to the equation (13), we obtain $-\frac{g}{2}a_g = -\frac{g}{12}\Delta a_g + \frac{1}{12}Ae_1^g$, and so

$$\Delta a_g = (2g + 1)a_g + \frac{g}{6}Ae_1^g,$$

if $g = 2, 3$. In the case $g = 3$, the Kähler contraction $|\Lambda e_1^g|$ with respect to the metric of $\mathbb{A}_3$ satisfies the following.

**Proposition 4.** There is a hyperelliptic Riemann surface $C_0$ of genus 3 such that

$$|\Lambda e_1^g| \to +\infty$$

as a non-hyperelliptic Riemann surface $C$ of genus 3 goes to the surface $C_0$.

The proof will be given in [4] and is based on a theorem of Harris [3, Theorem 6.5].

Now the invariant $a_3$ is $C^\infty$ near $[C_0] \in \mathcal{M}_3$, so is locally bounded near $[C_0]$. Proposition [4] together with (13) implies that $\Delta a_3$ is not bounded near $[C_0]$. Hence the function $a_3$ is not an eigenfunction of the Laplacian $\Delta$. This completes the proof of Theorem [5] modulo that of Proposition [4].

Since the Jacobian map $\text{Jac}|_{\mathbb{H}_g} : \mathbb{H}_g \to \mathbb{A}_g$ restricted to the hyperelliptic locus $\mathbb{H}_g$ is an embedding, the Kähler metric on $\mathbb{A}_g$ induces a Kähler metric on $\mathbb{H}_g$ and its Laplacian $\Delta_{\mathbb{H}_g}$. Since the $(1,1)$-form $e_1^g$ vanishes along $\mathbb{H}_g$ from (11), one can clarify $\Delta_{\mathbb{H}_g}(a_g|_{\mathbb{H}_g})$ if the Kähler contraction $\Lambda$ on $\mathbb{H}_g$ can be computed.
4 A proof of Theorem 1

In this section we prove Theorem 1 based on computations given in [15]. There is introduced a closed $(1,1)$-form $E_1^D$ on the moduli space $M_g$ representing $\frac{1}{2}e_1$ in the equation (5.3). We will prove the equality $E_1^D = \frac{1}{12}e_1$ to obtain the theorem.

Our starting point is Lemma 5.5 in [15] which says that

$$E_1^D(\lambda, \mu) = 4M \int_C \mathcal{H}(\omega_1', \lambda) \wedge \omega_1'' \hat{d} \ast (\omega_1'' \mu - \omega_1' \nu)$$

(15)

for any Beltrami differentials $\lambda, \mu \in C^\infty(C; TC \otimes T^*C)$.

First of all we need to explain the notation in the equation (15): $M, \omega_1', \omega_1'', \mathcal{H}$ and $\ell^\nu$. The first complex homology group $H_1(C; \mathbb{C})$ admits the intersection pairing which we denote by $\cdot : H_1(C; \mathbb{C}) \times H_1(C; \mathbb{C}) \rightarrow \mathbb{C}$. The integrand of the RHS is an $H_1(C; \mathbb{C})^\otimes 4$-valued $(1,1)$-form. The symbol $M : H_1(C; \mathbb{C})^\otimes 4 \rightarrow \mathbb{C}$ is given by $M(Z_1Z_2Z_3Z_4) := (Z_2 \cdot Z_3)(Z_4 \cdot Z_1)$ for any $Z_i \in H_1(C; \mathbb{C})$. Here and throughout this section we omit the symbol $\otimes$.

The first complex cohomology group $H = H^1(C; \mathbb{C})$, is embedded into the space of 1-forms $\Omega^1(C)$ as the subspace of harmonic 1-forms. An $H_1(C; \mathbb{C})$-valued real harmonic 1-form $\omega_1(1) \in \Omega^1(C) \otimes H_1(C; \mathbb{C})$ is defined to be the embedding $\in \text{Hom}(H^1(C; \mathbb{C}), \Omega^1(C)) = \Omega^1(C) \otimes H_1(C; \mathbb{C})$. We denote by $\omega_1'$ and $\omega_1''$ its $(1,0)$- and $(0,1)$- parts, respectively.

If homology classes $Y_i \in H_1 \leq i \leq g$, satisfy

$$\int_{Y_i} Y_j = \delta_{i,j}, \quad \text{and} \quad \int_{Y_i} Y_j = 0$$

for $1 \leq i, j \leq g$, in other words, $\{Y_i, -2\sqrt{-1}Y_i\}_{1 \leq i \leq g}$ is the dual basis of the symplectic basis $\{\psi_i, \sqrt{-1}\psi_i\}_{1 \leq i \leq g}$, then we have

$$\omega_1(1) = \omega_1' + \omega_1'', \quad \omega_1'(1) = \sum_{i=1}^{g} \psi_i Y_i, \quad \text{and} \quad \omega_1''(1) = \sum_{i=1}^{g} \psi_i Y_i.$$ (16)

Here we remark $Y_i \cdot Y_j = \sqrt{-1} \delta_{i,j}$. The Hodge decomposition on the 1-forms on $C$ is given by

$$\varphi = \mathcal{H} \varphi + \ast d \hat{\varphi} d \varphi + d \hat{\varphi} d \ast \varphi$$

for any $\varphi \in \Omega^1(C)$. Here $\mathcal{H}$ is the harmonic projection, which is written by

$$\mathcal{H}(\varphi) = \frac{\sqrt{-1}}{2} \sum_{j=1}^{g} \left( \left( \int_C \varphi \wedge \psi_j \right) \psi_j - \left( \int_C \varphi \wedge \psi_j \right) \psi_j \right).$$

For example, $\omega_1(1)\lambda$ can be regarded as an $H_1(C; \mathbb{C})$-valued $(0,1)$-form by contracting $T^*C$ in $\omega_1(1)$ and $TC$ in $\lambda$, so that we have

$$\mathcal{H}(\omega_1(1)) = \sum_{l=1}^{g} \mathcal{H}(\psi_l \lambda) Y_l = \frac{\sqrt{-1}}{2} \sum_{l,j=1}^{g} \left( \int_C \psi_j \psi_l \lambda \right) \psi_j Y_l,$$ (17)
since $\psi_1 \wedge \psi_j = -\psi_j \wedge \psi_1 = -\psi_j \psi_1 \lambda$. On the other hand, we have
\[
\varphi'' - \mathcal{H}(\varphi'') = *d\hat{\Phi} d\varphi'' + d\hat{\Phi} d * \varphi'' = 2d\hat{\Phi} d * \varphi''
\] (18)
for any $(0,1)$-current $\varphi''$.

Finally $\ell^\mu \in \Omega^0(C) \otimes H_1(C; \mathbb{C})$ is defined by
\[
\ell^\mu := 2\hat{\Phi} d * (\omega'_1 \mu).
\]

From (18) we have
\[
d * (\omega'_1 \ell^\mu - \ell^\mu \omega'_1) = -\sqrt{-1}d(\omega'_1 \ell^\mu - \ell^\mu \omega'_1) = \sqrt{-1}(\omega'_1 - \overline{\omega'}_1) \wedge \omega'_1 \mu - (\overline{\omega'}_1 \mu) \wedge \omega'_1
\]
\[
= 2\sqrt{-1}\omega'_1 \mu \wedge \overline{\Phi} d * (\omega'_1 \mu) + 2\sqrt{-1}d * (\omega'_1 \mu) \wedge \omega'_1
\]
\[
= \sqrt{-1}(\omega'_1 \mu - \mathcal{H}(\omega'_1 \mu)) + \sqrt{-1}(\omega'_1 \mu - \mathcal{H}(\omega'_1 \mu)) \wedge \omega'_1
\]
\[
= -\sqrt{-1}(\omega'_1 \mu - \mathcal{H}(\omega'_1 \mu)) \wedge \omega'_1.
\]
The last equality follows from
\[
\omega'_1 \wedge \omega'_1 \mu + \omega'_1 \mu \wedge \omega'_1 = \sum_{i,j=1}^g (\psi_i \wedge \psi_j \mu + \psi_i \mu \wedge \psi_j) Y_i Y_j
\]
\[
= \sum_{i,j=1}^g (\psi_i \psi_j - \psi_j \psi_i) \mu Y_i Y_j = 0.
\]

Hence, from (15), we have
\[
\mathcal{E}_1^M(\lambda, \mu) = 4\sqrt{-1}M \int_C \mathcal{H}(\omega'_1 \lambda) \wedge \omega'_1 \mu \hat{\Phi} \left( \mathcal{H}(\omega''_1 \mu) \wedge \omega'_1 + \omega'_1 \wedge \mathcal{H}(\omega''_1 \mu) \right),
\]
which is exactly the equation (5.5) in [15].

From (17) we have
\[
4\sqrt{-1}M \int_C \mathcal{H}(\omega'_1 \lambda) \wedge \omega'_1 \mu \hat{\Phi} (\mathcal{H}(\omega''_1 \mu) \wedge \omega'_1)
\]
\[
= \sqrt{-1}\sum_{j,l,i,k,m,n=1}^g \left( \int_C \psi_j \psi_l \lambda \right) \left( \int_C \overline{\psi}_k \overline{\psi}_l \mu \right) \int_C \overline{\psi}_j \wedge \overline{\psi}_m \hat{\Phi} \left( \overline{\psi}_k \wedge \overline{\psi}_n \right) Z(Y_i Y_m \overline{Y}_i \overline{Y}_m)
\]
\[
= -\frac{\sqrt{-1}}{4} \sum_{j,l,i,k,m,n=1}^g \left( \int_C \psi_j \psi_l \lambda \right) \left( \int_C \overline{\psi}_k \overline{\psi}_l \mu \right) A_{mjk}\delta_{mi}\delta_{nl}
\]
\[
= -\frac{\sqrt{-1}}{4} \sum_{j,l,i,k=1}^g \left( \int_C \psi_j \psi_l \lambda \right) \left( \int_C \overline{\psi}_k \overline{\psi}_j \mu \right) A_{ljk}.
\]
On the other hand, we have
\[
4\sqrt{-1}M \int C H(\omega(1) \wedge \omega(1)) \wedge H(\omega(1) \wedge \omega(1))
\]
\[
= \sqrt{-1} \sum_{j,l,i,k,m,n=1}^{g} (\int C \omega_j \omega_l \lambda (\int C \omega_k \omega_m \delta \Phi(\omega_n \wedge H(\omega_n \wedge \omega_i)))
\]
\[
= \sqrt{-1} \frac{1}{4} \sum_{j,l,i,k,m,n=1}^{g} (\int C \omega_j \omega_l \lambda (\int C \omega_k \omega_m \delta \Phi(\omega_n \wedge H(\omega_n \wedge \omega_i)))
\]
Consequently we obtain
\[
E_1^D = -\frac{1}{4} \sum_{i,j,k,l=1}^{g} (\omega_i \omega_l \Lambda \omega_k \omega_l \Lambda \omega_i \delta \Lambda \omega_k \omega_l \Lambda \omega_i) = \frac{1}{12} \hat{\Phi},
\]
as was to be shown.

The present proof is based on direct computations given in \[15\]. It is desirable to have a conceptual proof of Theorem \[1\].

5 Proof of Proposition \[4\]

In this section we prove Proposition \[4\] based on the following theorem of Harris.

**Theorem 5** (\[3\], Theorem 6.5). For any \( g \geq 3 \) there exists a hyperelliptic Riemann surface \( C_0 \) such that the inclusion \([11]\) is the equality
\[
Q(U(C_0)) = (the \ (-1)\ -eigenspace \ of \ the \ hyperelliptic \ involution \ in \ \gamma^g_{[C_0]}M_g).
\]
In other words, the extended Johnson homomorphism \( \tilde{k} = -[Q + \tilde{Q}] \) at \( [C_0] \) is non-degenerate in the normal direction to \( \gamma^g_{[C_0]} \) in \( M_g \).

The subspace \( U(C) \subset A^3(H' \oplus H'') = \bigoplus_{p+q=3} A^pH' \otimes A^qH'' \) is decomposed into
\[
U(C) = \bigoplus_{p+q=3} U(C)^{p,q}, \quad U(C)^{p,q} = U(C) \cap (A^pH' \otimes A^qH'').
\]
From the definition of the map \( Q \) \([10]\), we have
\[
Q|_{U(C)^{p,q}} = 0 \quad (19)
\]
if \( (p,q) \neq (2,1) \). Through the pairing \( \langle -, - \rangle \) in \([12]\), \( U(C)^{p,q} \) and \( U(C)^{3-p,3-q} \) are dual to each other. Hence the image of the transpose \( Q : T_{[C]}M_g \to U(C) \) is included in
\((U(C)^2)^* = U(C)^{1,2}\), i.e., we have \(Q(T_C)M_g \subset U(C)^{1,2}\). From the positivity of the Hermitian pairing \(\{\cdot,\cdot\}\), we have
\[
\forall v \in U(C)^{1,2} \setminus \{0\}, \quad -\sqrt{-1}\langle v, \overline{v} \rangle > 0.
\] (20)

In fact, \(\{\psi_i \wedge \psi_j \wedge \psi_k; 1 \leq i, j, k \leq g, j < k\}\) is a orthogonal basis of \(H^\prime \otimes \Lambda^2 H^\prime (\supset U(C)^{1,2})\).

Now we go back to the hyperelliptic Riemann surface \([C_0] \in M_3\) given by Theorem 5 of Harris. We choose a marking \(m_0\) of \(C_0\) to get a point \([C_0, m_0]\) in the Teichmüller space \(T_3\) of genus 3. Its Jacobian \(\text{Jac}(C_0)\) with the marking \(m_0\) defines a point \([\text{Jac}(C_0), m_0]\) in \(\mathcal{H}_3\). Then there are complex coordinates \((z_1, z_2, z_3)\) of \(T_3\) centered at \([C_0, m_0]\), and \((w_1, w_2, w_3)\) of the Siegel upper half space \(\mathcal{H}_3\) centered at \([\text{Jac}(C_0), m_0]\) such that the locus \(\{z_1 = 0\}\) coincides with the inverse image of \(\mathbb{H}_3\) locally, and that
\[
\text{Jac}(z_1, z_2, z_3) = (z_1^2, z_2, z_3) = (w_1, w_2, w_3),
\]
since the Jacobian map \(\text{Jac}\) is a 2 to 1 map in the normal direction of the submanifold \(\mathbb{H}_3\). There are \(U(C)^{1,2}\)-valued functions \(v_1, v_2, v_3\) defined around \([C_0, m_0] \in T_3\) such that \(\dot{k} = v_1 dz_1 + v_2 dz_2 + v_3 dz_3\) on \(T_3\). Then we have \(v_1(0, \ast, \ast) \neq 0\) from Theorem 5, \(v_2(0, \ast, \ast) = v_3(0, \ast, \ast) = 0\) from (11), and the differential form \(e_1^i\) equals
\[
\sum_{i,j=1}^3 (v_i, \overline{v}_j) dz_i \wedge d\overline{z}_j
\]
up to non-zero constant factor. Since \(\frac{\partial}{\partial w_i} = \frac{1}{z_1} \frac{\partial}{\partial z_1}\), the amount \(z_1 \cdot \Lambda e_1^i\) is a combination of the amounts
\[
\frac{1}{z_1} \langle v_1, \overline{v}\rangle, \langle v_1, \overline{v}_j\rangle, \langle v_i, \overline{v}\rangle \text{ and } z_1 \langle v_i, \overline{v}_j\rangle \quad (i, j = 2, 3)
\]
with coefficients in non-zero locally bounded functions near \([C_0, m_0]\). The amounts \(\langle v_1, \overline{v}\rangle, \langle v_i, \overline{v}\rangle\) and \(z_1 \langle v_i, \overline{v}_j\rangle\) go to 0 as \(z_1 \to 0\). Together with (20), this implies
\[
|z_1| \cdot |\Lambda e_1^i| \geq \frac{\text{(positive constant)}}{|z_1|}|(v_1, \overline{v}_1)| \to +\infty
\]
as \(z_1 \to 0\). This completes the proof of Proposition 4 and that of Theorem 3.

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