From nonassociativity to solutions of the KP hierarchy* †

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Abstract

A recently observed relation between ‘weakly nonassociative’ algebras $\mathbb{A}$ (for which the associator $(\mathbb{A}, \mathbb{A}^2, \mathbb{A})$ vanishes) and the KP hierarchy (with dependent variable in the middle nucleus $\mathbb{A}'$ of $\mathbb{A}$) is recalled. For any such algebra there is a nonassociative hierarchy of ODEs, the solutions of which determine solutions of the KP hierarchy. In a special case, and with $\mathbb{A}'$ a matrix algebra, this becomes a matrix Riccati hierarchy which is easily solved. The matrix solution then leads to solutions of the scalar KP hierarchy. We discuss some classes of solutions obtained in this way.

1 Introduction

Let us call an algebra $\mathbb{A}$ (over a commutative ring) weakly nonassociative (WNA) [1] if

$$
(a, b, c, d) = 0 \quad \forall a, b, c, d \in \mathbb{A},
$$

where $(a, b, c) := (ab)c - a(bc)$ is the associator in $\mathbb{A}$. The middle nucleus of $\mathbb{A}$,

$$
\mathbb{A}' := \{ b \in \mathbb{A} \mid (a, b, c) = 0 \quad \forall a, c \in \mathbb{A} \},
$$

is an associative subalgebra and a two-sided ideal. Let us fix $f \in \mathbb{A} \setminus \mathbb{A}'$ and define $a \circ_1 b := ab$ and

$$
\begin{align*}
a \circ_{n+1} b := a (f \circ_n b) - (a f) \circ_n b & \quad n = 1, 2, \ldots .
\end{align*}
$$

The subalgebra $\mathbb{A}(f)$, generated in $\mathbb{A}$ by $f$, is called $\delta$-compatible if, for all $n \in \mathbb{N}$,

$$
\delta_n(f) := f \circ_n f
$$

extends to a derivation of $\mathbb{A}(f)$.

* ©2006 by A. Dimakis and F. Müller-Hoissen
† Presented at the International Colloquium “Integrable Systems and Quantum Symmetries”, Prague, 15-17 June 2006.
1 As a consequence of (1), these products only depend on the equivalence class $[f]$ of $f$ in $\mathbb{A}/\mathbb{A}'$. 
Theorem 1. [1] Let $\mathbb{H}(f)$ be $\delta$-compatible. The derivations $\delta_n$ then commute on $\mathbb{H}(f)$ and satisfy identities which are in correspondence (via $\delta_n \mapsto \partial_n$) with the equations of the potential KP hierarchy (with dependent variable in $\mathbb{H}'$).

Example. The first three derivations are determined by
\[
\delta_1(f) = f^2, \quad \delta_2(f) = f f^2 - f^2 f, \quad \delta_3(f) = f (f f^2) - f f^2 f - f^2 f + (f^2 f) f
\]
and the derivation rule $\delta_n(ab) = \delta_n(a) b + a \delta_n(b)$. They satisfy the identity
\[
\delta_1 \left( 4 \delta_3(f) - \delta_1^3(f) + 6 (\delta_1(f))^2 \right) - 3 \delta_2^2(f) + 6 [\delta_2(f), \delta_1(f)] \equiv 0,
\]
which via $\delta_n \mapsto \partial_n$ becomes the potential KP equation (for $-f$).

The result formulated in theorem 1 provides us with a way to obtain solutions of the KP hierarchy by solving ordinary differential equations (ODEs).

Theorem 2. [1] Let $\mathbb{H}$ be any WNA algebra over the ring of functions of independent variables $t_1, t_2, \ldots$. If $f \in \mathbb{H}$ solves the hierarchy
\[
\partial_{\ell_n}(f) = f \circ_n f \quad n = 1, 2, \ldots
\]
of ODEs, then $-\partial_{\ell_1}(f)$ lies in $\mathbb{H}'$ and solves the KP hierarchy (with dependent variable in $\mathbb{H}'$). If there is a constant $\nu \in \mathbb{H} \setminus \mathbb{H}'$ with $[\nu] = [f] \in \mathbb{H}/\mathbb{H}'$, then
\[
\phi := \nu - f \in \mathbb{H}'
\]
solves the potential KP hierarchy.

In order to apply this result, we need to know more about WNA algebras. For our purposes, it is sufficient to recall from [1] that any WNA algebra with $\dim(\mathbb{H}/\mathbb{H}') = 1$ is isomorphic to one determined by the following data:

1. an associative algebra $\mathcal{A}$ (e.g. any matrix algebra)
2. a fixed element $g \in \mathcal{A}$
3. linear maps $L, R : \mathcal{A} \to \mathcal{A}$ such that
\[
[L, R] = 0, \quad L(ab) = L(a) b, \quad R(ab) = a R(b).
\]

Augmenting $\mathcal{A}$ with an element $\nu$ such that $\nu \nu = g, \nu a = L(a), a \nu = R(a)$, leads to a WNA algebra $\mathbb{H}$ with $\mathbb{H}' = \mathcal{A}$.

2 A class of WNA algebras and corresponding KP solutions

Let $\mathcal{A} = \mathcal{M}(M, N)$ be the algebra of complex $M \times N$ matrices with the product
\[
A \bullet A' := A K A',
\]

\[\text{Our conventions correspond to ‘KPII’. In section 3 we also consider ‘KPI’ which is obtained from KPII via } t_{2n} \mapsto t_{2n}.\]

For water waves, KPI applies to the case where surface tension dominates over gravity.

\[\text{The flows given by (7) indeed commute [1]. } f \text{ has to be differentiable, of course, which requires a corresponding (e.g. Banach space) structure on } \mathbb{H}. \text{ If } f \text{ solves (7), the algebra } \mathbb{H}(f) \text{ generated by } f \text{ in } \mathbb{H} \text{ over the subring of constants is } \delta-\text{compatible [1].}\]
where $K$ is a fixed $N \times M$ matrix. We define linear maps $L, R$ as multiplication (from left, respectively right) by a constant $M \times M$ matrix $L$, respectively a constant $N \times N$ matrix $R$. Then (9) holds. Furthermore, we introduce the new product

$$A \circ_1 A' := (AR) \bullet A' - A \bullet (LA') = A(RK - KL)A'$$

(11)

and augment $(A, \circ_1)$ by an element $\nu$ such that

$$\nu \circ_1 \nu = 0, \quad \nu \circ_1 A = L A, \quad A \circ_1 \nu = -AR,$$

(12)

to obtain a WNA algebra $(A, \circ_1)$ with $A' = A$. The reason for resolving the WNA product $\circ_1$ in terms of $\bullet$ is the drastic simplification in

$$A \circ_n A' = A(R^n K - K^n)A',$$

(13)

which turns the hierarchy of ODEs (7) into the special matrix Riccati equations

$$\partial_{\tau_n}(\phi) = L^n \phi - \phi R^n + \phi (KL^n - R^n K) \phi \quad n = 1, 2, \ldots .$$

(14)

They are easily solved:

$$\phi = -(I_M + e^{\xi(L)} C e^{-\xi(R)} K)^{-1} e^{\xi(L)} C e^{-\xi(R)} = -e^{\xi(L)} (I_M + B)^{-1} e^{-\xi(R)}$$

(15)

with a constant matrix $C \in M(M, N)$, the $M \times M$ unit matrix $I_M$, and

$$B := C e^{-\xi(R)} K e^{\xi(L)}, \quad \xi(L) := \sum_{n \geq 1} t_n L^n.$$  

(16)

According to theorem 2, $\phi$ solves the matrix potential KP hierarchy in $(A, \circ_1)$, thus

$$\varphi := \phi (RK - KL)$$

(17)

solves the matrix potential KP hierarchy with the ordinary matrix product.\(^5\) If $-C$ is the unit matrix, such a solution appeared in [3] in the context of an operator approach towards solutions of scalar nonlinear equations. The basic idea is to associate with the respective nonlinear (soliton) equation an operator (e.g. matrix) version, to look for exact solutions of the latter and a homomorphism into scalars, which then determines solutions of the scalar nonlinear (soliton) equation (see also [4]). In fact, in the case under consideration such a homomorphism is obtained as described below, if $K, L, R$ are such that $\text{rank}(RK - KL) = 1$ (see also [3,5]). The latter condition means that there is a $v \in \mathbb{C}^M$ and a $w \in \mathbb{C}^N$ with $RK - KL = w v^T$. Then the map from $A$ to smooth functions of $t_1, t_2, \ldots$, defined by

$$\Psi(A) := v^T A w = \text{tr}(A w v^T) = \text{tr}(A (RK - KL)),$$

(18)

has the homomorphism property\(^6\) $\Psi(A \circ_1 A') = \Psi(A) \Psi(A')$. As a consequence, a solution of the scalar KP hierarchy is given by

$$u := \Psi(\phi)_t = \text{tr}(\varphi) = (\log \tau)_t \quad \text{with} \quad \tau := \det(I_M + B).$$

(19)

Note that $\tau$ is in particular invariant under

$$C \mapsto P C \tilde{P}^{-1}, \quad K \mapsto \tilde{P} K P^{-1}, \quad L \mapsto P L P^{-1}, \quad R \mapsto \tilde{P} R \tilde{P}^{-1},$$

(20)

with any constant invertible $M \times M$ matrix $P$ and $N \times N$ matrix $\tilde{P}$. This can be used to reduce both, $L$ and $R$, to Jordan normal form.

\(^4\)Here we set $g = 0$, which is a restriction of the possibilities in the case under consideration.

\(^5\)This follows e.g. immediately from a functional representation of the potential KP hierarchy [1,2].

\(^6\)More generally, one can construct homomorphisms into matrix algebras in a similar way [1].
Let \( L = \text{diag}(p_1, \ldots, p_M), R = \text{diag}(q_1, \ldots, q_N), \) and
\[
K_{ij} = (q_i - p_j)^{-1},
\]
assuming \( q_i \neq p_j, i = 1, \ldots, N, j = 1, \ldots, M. \) Then \( \text{rank}(RK - KL) = 1 \) and
\[
B = CE \quad \text{where} \quad E_{ij} := e^{\xi(p_i) - \xi(q_i)}/(q_i - p_j).
\]
If \( M = N \) and \( C = \text{diag}(c_1, \ldots, c_N), \) \( u \) becomes an \( N \)-soliton solution of the scalar KP hierarchy [6–8]. For example, with \( M = N = 2, p_1 = -p_2 = \alpha + \beta, q_1 = -q_2 = \alpha - \beta, \)
\[
C = (2\alpha\beta/\sqrt{\alpha^2 - \beta^2}) \text{diag}(-1, 1),
\]
we obtain
\[
\tau = 2e^{4\alpha\beta t_2}[(\alpha/\sqrt{\alpha^2 - \beta^2}) \cosh(2\beta(t_1 + (3\alpha^2 + \beta^2)t_3)) + \cosh(4\alpha\beta t_2)]
\]
for \( t_4, t_5, \ldots = 0. \) We may drop the exponential factor, since it does not influence \( u. \) The corresponding solution of the KPII equation, which is regular if \( \alpha > 0 \) and \( |\alpha| > |\beta|, \) is shown in Fig. 1.\(^7\) For \( M = N = 2 \) and no restriction on \( C, \) we have
\[
\tau = 1 + \frac{C_{11} e^{\xi(p_1) - \xi(q_1)}}{q_1 - p_1} + \frac{C_{12} e^{\xi(p_1) - \xi(q_2)}}{q_2 - p_1} + \frac{C_{21} e^{\xi(p_2) - \xi(q_1)}}{q_1 - p_2} + \frac{C_{22} e^{\xi(p_2) - \xi(q_2)}}{q_2 - p_2}
- \det(C) \frac{(p_2 - p_1)(q_2 - q_1)}{(q_1 - p_1)(q_2 - q_1)(q_1 - p_2)(q_2 - p_2)} e^{\xi(p_1) + \xi(p_2) - \xi(q_1) - \xi(q_2)}.\]
\[
(24)
\]
Let us fix an order: \( p_1 < p_2 < q_1 < q_2. \) Then \( \tau \) is positive for all \( t_1, t_2, \ldots \in \mathbb{R}, \) so that the KP solution \( u \) is regular, if \( C_{ij} > 0 \) and \( \det(C) < 0. \) Let \( 1 \leq n \leq 5 \) be the number of linearly independent exponential terms in this expression. For \( n = 1 \) we have a 1-soliton solution. For \( n = 2, \) this is a Miles resonance [10], a ‘Y junction’. For \( n = 3 \) we have an ordinary 2-soliton solution and for \( n = 4 \) the type of resonance shown in Fig. 2.\(^8\) For \( n = 5 \) the behavior of the solution is shown in Fig. 3. With other values of \( M \) and \( N, \) and real \( L, R, C, \) one obtains further line soliton resonances. For \( M = 1, N = 2, \) we have \( \tau = 1 + (\alpha e^{-\xi(q_1)} + \beta e^{-\xi(q_2)}) e^{\xi(p_1)}, \) where \( \alpha := C_{11}/(q_1 - p_1), \beta := C_{12}/(q_2 - p_1). \) Then \( u, \) which is regular if \( \alpha, \beta > 0, \) is a Miles resonance. More involved examples are easily generated [11,12].\(^9\)

\(^7\)The plots in this work were generated with Mathematica [9].

\(^8\)Using instead of \( t_3 \) any other ‘evolution time’ from \( t_4, t_5, \ldots \) only means a difference in the velocity.

\(^9\)The relation with the expression for \( \tau \) functions in [12] is as follows. First we write \( \tau = \det(I_M + B) = \tilde{\tau} \det(e^{\xi(L)}) \) with \( \tilde{\tau} := \det(e^{-\xi(L)} + C e^{-\xi(R)} K) = \det(C \Theta K^{T}), \) where \( C := (I_M, C), \) \( K := (I_M, K) \) and
\[
\Theta := \begin{pmatrix}
    e^{-\xi(L)} & 0_{M \times N} \\
    0_{N \times M} & e^{-\xi(R)}
\end{pmatrix}.
\]
Since \( u = (\log(\tau))_{t_1, t_1} = (\log(\tilde{\tau}))_{t_1, t_1}, \) we may replace \( \tau \) by \( \tilde{\tau}. \) Multiplication of \( \tilde{C}, \) and independently \( \tilde{K}, \) from the left by any constant invertible \( M \times M \) matrix does not change \( u. \)
where $\gamma$ is positive if $\alpha, \beta > 0$. Some plots of corresponding KPI solutions are shown in Fig. 4. In the limit $\gamma \to \infty$, a KPI lump solution [7] is recovered. Multi-lump generalizations [7, 14, 15] can be obtained from $N \times N$ Jordan type matrices $L, R$. 

Figure 2: Contour plot of a 2-soliton resonance with $p_1 = -2, p_2 = -\frac{1}{2}, q_1 = 1, q_2 = \frac{3}{2}, C_11 = C_12 = C_{21} = 1, C_{22} = 0$ at $t_3 = -200, 0, 200$, respectively.

Figure 3: Contour plot of a 2-soliton resonance with $p_1 = -2, p_2 = -\frac{1}{2}, q_1 = 1, q_2 = \frac{3}{2}, C_11 = C_12 = C_{22} = 1, C_{21} = 2$ at $t_3 = -200, 0, 200$, respectively.
Figure 4: Snapshots at \( t_3 = 0 \) of KPI solutions determined by (26). For \( \alpha = \frac{1}{4}, \beta = -\frac{1}{2}, \gamma = 300 \), a lump moves between two line solitons (left plot). For \( \alpha = 2, \beta = -\frac{1}{2}, \gamma = 100 \), the right line soliton develops a lump, shortly after swallowed by the left one (right plot).

4 Final remarks

So far we only looked at the special case where \( \mathbb{A}' \) is a matrix algebra and \( g = 0 \) (see the end of section 1 and the first of eqs (12)). If \( g \neq 0 \), the nonassociative hierarchy of ODEs leads to more complicated Riccati equations which, however, can also be solved. We plan to present a corresponding analysis elsewhere. Theorem 2 offers still further possibilities to obtain solutions of KP hierarchies.

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