ASYMPTOTICS OF THE SELF-DUAL DEFORMATION COMPLEX

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Abstract. We analyze the indicial roots of the self-dual deformation complex on a cylinder \((\mathbb{R} \times Y^3, dt^2 + g_Y)\), where \(Y^3\) is a space of constant curvature. An application is the optimal decay rate of solutions on a self-dual manifold with cylindrical ends having cross-section \(Y^3\). We also resolve a conjecture of Kovalev-Singer in the case where \(Y^3\) is a hyperbolic rational homology 3-sphere, and show that there are infinitely many examples for which the conjecture is true, and infinitely many examples for which the conjecture is false. Applications to gluing theorems are also discussed.

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1. Introduction

Let \((M^4, g)\) be a four-dimensional Riemannian manifold and let \(Rm\) denote the Riemannian curvature tensor of \(g\). Recall that \(Rm\) admits an orthogonal decomposition of the form

\[
Rm = W + \frac{1}{2} E \otimes g + \frac{1}{24} Rg \otimes g,
\]

where \(W\) is the Weyl tensor, \(E\) is the traceless Ricci tensor of \(g\), \(R\) is the scalar curvature, and \(\otimes\) is the Kulkarni-Nomizu product. If \((M^4, g)\) is oriented, there is a further decomposition of \((\mathbb{R} \times Y^3, dt^2 + g_Y)\). The Hodge-* operator associated to \(g\) acting on...
2-forms is a mapping $\ast : \Lambda^2 \mapsto \Lambda^2$ satisfying $\ast^2 = Id$, and $\Lambda^2$ admits a decomposition of the form
\begin{equation}
\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{1.2}
\end{equation}
where $\Lambda^2_\pm$ are the $\pm 1$ eigenspaces of $\ast|_{\Lambda^2}$. Sections of $\Lambda^2_+$ and $\Lambda^2_-$ are called self-dual and anti-self-dual 2-forms, respectively. The curvature tensor can be viewed as an operator $\mathcal{R} : \Lambda^2 \mapsto \Lambda^2$, and we let $\mathcal{W}$ and $\mathcal{E}$ denote the operators associated to the Weyl and traceless Ricci tensors, respectively. With respect to the decomposition $(1.2)$, the full curvature operator decomposes as
\begin{equation}
\mathcal{R} = \begin{pmatrix}
\mathcal{W}^+ + \frac{R}{12} I & \frac{1}{2} \mathcal{E} \pi_- \\
\frac{1}{2} \mathcal{E} \pi_+ & \mathcal{W}^- + \frac{R}{12} I
\end{pmatrix}, \tag{1.3}
\end{equation}
where $\pi_\pm$ is the projection onto $\Lambda^2_\pm$, and the self-dual and anti-self-dual Weyl tensors are defined by $\mathcal{W}^\pm = \pi_\pm \mathcal{W} \pi_\pm$.

**Definition 1.1.** Let $(M^4, g)$ be an oriented four-manifold. Then $g$ is called **self-dual** if $\mathcal{W}^- = 0$, and $g$ is called **anti-self-dual** if $\mathcal{W}^+ = 0$. In either case $g$ is said to be **half-conformally-flat**.

By reversing orientation, a self-dual metric becomes an anti-self-dual metric, so without loss of generality, we will only consider self-dual metrics.

Since Poon’s example of a 1-parameter family of self-dual metrics on $\mathbb{C}P^2 \# \mathbb{C}P^2$ in 1988 [Poo88], there has been an explosion of examples of self-dual metrics on various four-manifolds. We do not attempt to give a complete history here, but only mention a few results closely related to our results in this paper. In 1989, Donaldson and Friedman developed a twistor space gluing procedure which invoked many non-trivial results in algebraic geometry [DF89]. In 1991, Floer produced examples on $n \# \mathbb{C}P^2$ by an analytic gluing procedure [Flo91]. Then in 2001, Kovalev and Singer generalized Floer’s analytic gluing result to cover the case of gluing orbifold self-dual metrics [KS01]. We will describe the relation of our work with these prior works in more detail below, but first will state our main results.

Since the SD condition is conformally invariant, we are free to conformally change an end to obtain different types of asymptotics. For simplifying computations, the most useful type of geometry is that of cylindrical ends:

**Definition 1.2.** Let $(Y^3, g_Y)$ be a compact 3-manifold with constant curvature. A complete Riemannian manifold $(M^4, g)$ is called **asymptotically cylindrical** or **AC** with cross-section $Y$ of order $\tau$ if there exists a diffeomorphism $\psi : M \setminus K \to \mathbb{R}_+ \times Y$ where $K$ is a subset of $M$ containing all other ends, satisfying
\begin{align}
(\psi_* g)_{ij} &= (g_C)_{ij} + O(e^{-\tau t}), \tag{1.4} \\
\partial^k (\psi_* g)_{ij} &= O(e^{-\tau t}), \tag{1.5}
\end{align}
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for any partial derivative of order \( k \), as \( t \to \infty \), where \( g_C = dt^2 + g_Y \) is the product cylindrical metric.

Self-dual metrics have a rich obstruction theory. If \( (M, g) \) is a self-dual four-manifold, the deformation complex is given by

\[
\Gamma(T^*M) \xrightarrow{\mathcal{K}_g} \Gamma(S_0^2(T^*M)) \twoheadrightarrow \Gamma(S_0^2(\Lambda_2)),
\]

where \( \mathcal{K}_g \) is the conformal Killing operator defined by

\[
(\mathcal{K}_g(\omega))_{ij} = \nabla_i \omega_j + \nabla_j \omega_i - \frac{1}{2} (\delta \omega) g,
\]

with \( \delta \omega = \nabla^i \omega_i \), and \( D = (\nabla^-)'_g \) is the linearized anti-self-dual Weyl curvature operator. This complex is then wrapped-up into a single operator

\[
F : \Gamma(S_0^2(T^*M)) \longrightarrow \Gamma(S_0^2(\Lambda_2)) \oplus \Gamma(S_0^2(T^*M)),
\]

defined by

\[
F(h) = (Dh, 2\delta h),
\]

and \( (\delta h)_j = \nabla^i h_{ij} \). This operator is mixed order elliptic of order \((2, (0, 1))\) in the sense of Douglis-Nirenberg [DN55], with formal \( L^2 \)-adjoint

\[
F^* : \Gamma(S_0^2(\Lambda_2)) \oplus \Gamma(S_0^2(T^*M)) \longrightarrow \Gamma(S_0^2(T^*M)),
\]

given by

\[
F^*(Z, \omega) = D^*Z - \mathcal{K}_g \omega.
\]

**Definition 1.3.** The indicial roots of \( F \) are those complex numbers \( \lambda \) for which there is a solution \( h \) of \( F(h) = 0 \) such that the components of \( h \) have the form \( e^{\lambda t}p(y, t) \) where \( p \) is a polynomial in \( t \) with coefficients in \( C^\infty(Y) \). The indicial roots of \( F^* \) are defined analogously for pairs \( (Z, \omega) \).

We will first determine the indicial roots of \( F^* \). The indicial roots of \( F \) can then be obtained by using an index theorem, as we will show below. One could equivalently first analyze the indicial roots of \( F \), however, for purposes of computation it turns out to be somewhat easier to completely analyze the cokernel (although the computations are in principle equivalent).

**1.1. Spherical cross-section.** Our first result deals with cross-section \( Y \) having constant positive curvature. In Theorem 7.1 below, we determine all indicial roots of \( F^* \), but for simplicity we only state the following here in the introduction:

**Theorem 1.4.** Let \( M = \mathbb{R} \times S^3/\Gamma \) with product metric \( g = dt^2 + g_{S^3/\Gamma} \), where \( g_{S^3/\Gamma} \) is a metric of constant curvature 1. Let \( \mathcal{I}^* \) denote the set of indicial roots of \( F^* \). If \( \beta \in \mathcal{I}^* \) satisfies \( |\text{Re}(\beta)| < 2 \) then \( \beta = 0 \) or \( \beta = \pm 1 \). In these cases, the corresponding solutions are of the form \((0, \omega)\), where \( \omega \) is dual to a conformal Killing field (that is, \( \mathcal{K}_g \omega = 0 \)). Consequently,

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\[\text{Our definition of indicial roots differs from that in [LM85] by a factor of } \sqrt{-1}.\]
• Case (0): \( 0 \in \mathcal{I}^* \), and the corresponding solutions are given by \((0, dt)\), or \((0, \omega_0)\) for \(\omega_0\) dual to a Killing field on \(S^3/\Gamma\).

• Case (1): \( \pm 1 \in \mathcal{I}^* \) if and only if \(\Gamma\) is trivial. In this case, the corresponding solutions are given by \((0, \omega)\), where \(\omega\) is given by \(e^{\pm t}(\phi dt \mp d\phi)\) where \(\phi\) is a lowest nontrivial eigenfunction of \(\Delta_{S^3}\) with eigenvalue 3.

**Remark 1.5.** The indicial roots \(\beta \in \mathcal{I}^*\) satisfying \(|\text{Re}(\beta)| \geq 2\) fall into two classes. The indicial roots in the first class are integers and the corresponding solutions are of the form \((Z, 0)\); these are Cases (2) and (3) in Theorem 7.1. The indicial roots in the other class have non-zero imaginary part, and the corresponding solutions are of the form \((Z, \omega)\) with \(Z\) nontrivial and \(K_g(\omega) \neq 0\); these are Cases (4) and (5) in Theorem 7.1.

We can also completely characterize the indicial roots of the forward operator \(F\). This follows from the above determination of the cokernel indicial roots, together with the index theorem of Lockhart and McOwen; it turns out that these are the same. We will describe all kernel elements explicitly below in Theorem 7.3, but for purposes of brevity in the introduction we only state here the following theorem which generalizes a well-known result of Floer [Flo91]. In order to state the theorem, we define the symmetric product of 1-forms \(\omega_1\) and \(\omega_2\) by

\[
\omega_1 \circ \omega_2 = \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1.
\]

**Theorem 1.6.** Let \(M = \mathbb{R} \times S^3/\Gamma\) with product metric \(g = dt^2 + g_{S^3/\Gamma}\), where \(g_{S^3/\Gamma}\) is a metric of constant curvature 1, and let \(\mathcal{I}\) denote the set of indicial roots of \(F\). Then \(\mathcal{I} = \mathcal{I}^*\). For \(\beta = 0 \in \mathcal{I}\), the corresponding solutions of \(F(h) = 0\) are given by

\[
\text{span}\{3dt \otimes dt - g_{S^3}, dt \otimes \omega_0\},
\]
where \(\omega_0\) is a dual to a Killing field on \(S^3/\Gamma\).

Next, \(\beta = \pm 1 \in \mathcal{I}\) if and only if \(\Gamma\) is trivial. In this case, the corresponding kernel elements are given by

\[
h_\phi = p(t)\phi(3dt \otimes dt - g_{S^3}) + q(t)(dt \circ d\phi),
\]
where \(p(t) = C_3e^t - C_4e^{-t}\) and \(q(t) = C_3e^t + C_4e^{-t}\), for some constants \(C_3\) and \(C_4\), and \(\phi\) is a lowest nonconstant eigenfunction of \(\Delta_{S^3}\).

Moreover, solutions in (1.13) and (1.14) are in the image of the conformal Killing operator. All other indicial roots \(\beta \in \mathcal{I}\) satisfy \(|\text{Re}(\beta)| \geq 2\).

**Remark 1.7.** As in Remark 1.5, the indicial roots \(\beta \in \mathcal{I}\) satisfying \(|\text{Re}(\beta)| \geq 2\) fall into two classes. The indicial roots in the first class are integers and the corresponding solutions are not in the image of the conformal Killing operator; these are Cases (2) and (3) in Theorem 7.3. The indicial roots in the other class have non-zero imaginary part, and the corresponding solutions are in the image of the conformal Killing operator; these are Cases (4) and (5) in Theorem 7.3.

A corollary is the optimal result:
Corollary 1.8. Let \((M^4, g)\) be the the cylinder \(\mathbb{R} \times Y^3\), where \(Y^3 = S^3/\Gamma\) with \(\Gamma \subset SO(4)\) a finite subgroup acting freely on \(S^3\) with product metric \(g = dt^2 + g_Y\), where \(g_Y\) is a metric of constant curvature 1.

(a) Let \((Z, \omega)\) be a solution of \(D^*Z = K_\omega \omega\). If \(Z = o(e^{2|t|})\) and \(\omega = o(e^{2|t|})\) as \(|t| \to \infty\) then \(Z = 0\), and \(\omega\) is dual to a conformal Killing field.
(b) Let \(h\) be a solution of \(Dh = 0\) and \(\delta h = 0\). If \(h = o(e^{2|t|})\) as \(|t| \to \infty\) then \(h\) can be written as a linear combination of elements in (1.13) and (1.14).

Standard analysis in weighted spaces then implies the following corollary for AC manifolds with spherical cross-section:

Corollary 1.9. Let \((M^4, g)\) be self-dual and asymptotically cylindrical with cross-section \(Y^3 = S^3/\Gamma, g_Y\) with \(\Gamma \subset SO(4)\) a finite subgroup acting freely on \(S^3\), where \(g_Y\) is a metric of constant curvature 1.

(a) Let \((Z, \omega)\) be a solution of \(D^*Z = K_\omega \omega\). If \(Z = o(e^{2t})\) and \(\omega = o(e^{2t})\) then \(\omega\) is dual to a conformal Killing field, and \(Z = O(e^{-2t})\) as \(t \to \infty\).
(b) Let \(h\) be a solution of \(Dh = 0\) and \(\delta h = 0\). If \(h = o(e^{2t})\) as \(|t| \to \infty\) then \(h\) has an asymptotic expansion with leading term as in (1.13) or (1.14).

In Section 8, we apply Corollary 1.9 to fix a gap in the proof of a key step in the main gluing result in [KS01].

To state the next result, we require the following definition.

Definition 1.10. A complete Riemannian manifold \((M^4, g)\) is called asymptotically locally Euclidean or ALE of order \(\tau\) if it has finitely many ends, and for each end there exists a finite subgroup \(\Gamma \subset SO(4)\) acting freely on \(S^3\) and a diffeomorphism \(\psi : M \setminus K \to (\mathbb{R}^4 \setminus B(0, R))/\Gamma\) where \(K\) is a subset of \(M\) containing all other ends, and such that under this identification,

\[
(\psi_* g)_{ij} = \delta_{ij} + O(r^{-\tau}),
\]

\[
\partial^k (\psi_* g)_{ij} = O(r^{-\tau-k}),
\]

for any partial derivative of order \(k\), as \(r \to \infty\), where \(r\) is the distance to some fixed basepoint.

It is known that any self-dual ALE metric is ALE of order 2, after a possible change of coordinates at infinity. ALE of any order \(\tau < 2\) was first shown by [TV05], while ALE of order exactly 2 was shown in [Str10], see also [Che09, AV13]. This order is optimal, so without loss of generality we will assume that all ALE spaces are ALE of order 2.

We also have the following optimal decay result for self-dual ALE spaces:

Theorem 1.11. Let \((M, g)\) be self-dual and asymptotically locally Euclidean.

(a) Any solution of \(D^*Z = K_\omega \omega\) satisfying \(Z = o(1)\) and \(\omega = o(r^{-1})\) must satisfy \(\omega = 0\) and \(Z = O(r^{-4})\) as \(r \to \infty\).
(b) Any solution of \(Dh = 0\) and \(\delta h = 0\) satisfying \(h = o(1)\) must satisfy \(Z = O(r^{-2})\) as \(r \to \infty\).
1.2. **Hyperbolic cross-section.** We first define

\begin{equation}
H_C^1(Y) = \{ B \in S^2_0(T^*Y) \mid d^V B = 0, \text{tr}(B) = 0 \},
\end{equation}

where \((d^V B)_{kl} \) is given by \((d^V B)_{kl} = \nabla_k B_{lj} - \nabla_l B_{kj} \), to be the vector space of traceless Codazzi tensor fields. For the case of hyperbolic cross-section, we have the following:

**Theorem 1.12.** Let \((M^4, g)\) be the the cylinder \(\mathbb{R} \times Y^3\), where \((Y^3, g_Y)\) is compact and hyperbolic with constant curvature \(-1\), with product metric \(g = dt^2 + g_Y\), and let \(I^*\) denote the set of indicial roots of \(F^*\). Then there exists an \(\epsilon > 0\) such that if \(\beta \in I^*\) with \(|\text{Re}(\beta)| < \epsilon\) then \(\beta = 0\). The corresponding kernel of \(F^*\) has dimension

\begin{equation}
1 + b_1(Y) + 2 \dim(H_C^1(Y)).
\end{equation}

The corresponding kernel of \(F\) has the same dimension and is spanned by

\begin{equation}
\{ 3 dt \otimes dt - g_Y, dt \circ \omega, \cos(t) \cdot B, \sin(t) \cdot B \},
\end{equation}

where \(\omega\) is any harmonic 1-form \(\omega\), and \(B\) is an any traceless Codazzi tensor on \(Y^3\).

**Remark 1.13.** The element \((0, dt)\) is in the cokernel, which accounts for the 1 in \((1.17)\). The other cokernel elements in case \(b_1(Y) \neq 0\) arise from non-trivial harmonic 1-forms, and those in case \(H_C^1(Y) \neq \{0\}\) of course arise from non-trivial traceless Codazzi tensor fields. These elements are written down explicitly in Section 5, see Propositions 5.5(b) and 5.9(b). We only note here that the nontrivial solutions in this case satisfy \(Z = O(1)\) as \(|t| \to \infty\) or are periodic in \(t\).

We define\(^2\)

\[ H_C^2(\mathbb{R} \times Y^3) = \{ Z \in S^2_0(\Lambda^2) \mid D^* Z = 0 \text{ and } Z = O(e^{\epsilon|t|}) \text{ as } |t| \to \infty \text{ for every } \epsilon > 0 \}. \]

In [KS01], Conjecture 4.11, it was conjectured that \(H_C^2(\mathbb{R} \times Y^3) = \{0\}\) for any hyperbolic rational homology 3-sphere. Theorem 1.12 shows that this is true if and only if \(Y^3\) does not admit any non-trivial traceless Codazzi tensor field. Using this, and some examples of certain hyperbolic 3-manifolds of [Kap94, DeB06], we obtain infinitely many examples for which the conjecture is true, and infinitely many examples for which the conjecture is false.

**Theorem 1.14.** Let \((Y^3, g_Y)\) be a hyperbolic rational homology 3-sphere, with \(g_Y\) of constant curvature \(-1\), and \(M = \mathbb{R} \times Y^3\) with the product metric \(g = dt^2 + g_Y\). Then \(H_C^2(\mathbb{R} \times Y^3) = \{0\}\) if and only if \(Y^3\) admits no non-trivial traceless Codazzi tensor fields. Furthermore, there are infinitely many hyperbolic rational homology 3-spheres satisfying \(H_C^2(\mathbb{R} \times Y^3) = \{0\}\), and infinitely many satisfying \(H_C^2(\mathbb{R} \times Y^3) \neq \{0\}\).

We also have the following application to AC manifolds with hyperbolic cross-section:

\(^2\)Note that this definition is more general than the definition in [KS01] Section 4.2.1] in that we allow solutions which have polynomial growth in \(t\).
Corollary 1.15. Let \((M^4, g)\) be self-dual and asymptotically cylindrical with cross-section \((Y^3, g_Y)\) a hyperbolic rational homology 3-sphere with \(g_Y\) of constant curvature \(-1\), satisfying \(H_C^1(Y) = \{0\}\).

(a) Let \((Z, \omega)\) be a solution of \(D^*Z = \mathcal{K}_g\omega\). Then there exists a constant \(\epsilon > 0\), such that if \((Z, \omega)\) solves \(D^*Z = \mathcal{K}_g\omega\) and satisfies \(Z = o(e^{\epsilon|t|})\) and \(\omega = o(e^{\epsilon|t|})\) as \(t \to \infty\) then \(\omega\) is dual to a conformal Killing field and \(Z = o(e^{-\epsilon|t|})\) as \(t \to \infty\).

(b) Let \(h\) be a solution of \(Dh = 0\) and \(\delta h = 0\). Then there exists a constant \(\epsilon > 0\), such that if \(h = o(e^{\epsilon|t|})\) as \(|t| \to \infty\), then \(h\) admits an expansion
\[
h = c \cdot (dt \otimes dt - 3g_Y) + O(e^{-\epsilon|t|})
\]
for some constant \(c\) as \(|t| \to \infty\).

1.3. Flat cross-section. Finally, in the case that \((Y^3, g)\) is a flat torus, we have the following:

Theorem 1.16. Let \((M^4, g)\) be the the cylinder \(\mathbb{R} \times Y^3\), where \((Y^3, g_Y)\) is compact and flat, with product metric \(g = dt^2 + g_Y\), and let \(I\) denote the set of indicial roots of \(F\). Then there exists an \(\epsilon > 0\) such that if \(\beta \in I\) with \(|\text{Re}(\beta)| < \epsilon\) then \(\beta = 0\). The corresponding kernel of \(F\) has dimension 14 and is spanned by
\[
\{3dt \otimes dt - g_Y, dt \otimes \omega, B, tB\},
\]
where \(\omega\) is any parallel 1-form and \(B\) is any parallel traceless symmetric 2-tensor on \(Y^3\).

The corresponding cokernel of \(F\) has dimension 14 and is spanned by
\[
\{(0, dt), (0, \omega_0), (Z, 0), (tZ, 0)\},
\]
where \(\omega_0\) is a parallel 1-form, and \(Z\) is any parallel section of \(S_0^2(\Lambda^2)\).

Remark 1.17. One can easily use our computations to explicitly determine all indicial roots in the case of flat cross-section \(Y^3 = T^3\). However, in the interest of brevity this is omitted.

1.4. Remarks and outline of the paper. We next give a brief outline of the paper. Sections 2 and 3 will be concerned with the derivation of the linearized anti-self-dual Weyl tensor in separated variables. In these sections, there is overlap with computations in Floer’s paper \cite{Flo91}. However, the main formula given in Floer for \((W^-)'\) at a cylindrical metric is incorrect \cite[Proposition 5.1]{Flo91} (in addition to mistakes in the coefficients, Floer’s formula omits crucial terms involving the trace component \(h_{00}\)). The correct formula (which moreover holds for any cross-section \(Y^3\) with constant curvature) is given in Theorem 3.3. Section 4 contains required formulas for a Dirac-type operator, as well as some necessary eigenvalue computations. Section 5 contains the core analysis of the kernel of \(D^*\). The analysis in Section 6 is necessary to determine the possibilities for the 1-form \(\omega\) appearing in the adjoint equation. The proofs of all the main theorems are then completed in Section 7. In Section 8 we discuss the application of our results to gluing theorems.
Finally, the Appendix contains the derivation of a crucial formula relating the square of the Dirac operator to the linearized Einstein equation on the cross-section. In the case of spherical cross-section, Floer writes down such a formula [Flo91, Lemma 5.1], but which has errors in the coefficients. The correct formula (which moreover holds for general cross-section $Y$) is given in Corollary A.1.

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2. The anti self-dual part of the curvature tensor

Let $(M^4, g)$ be an orientable 4-manifold. As mentioned in the introduction, according to the decomposition

$$Rm = W^+ + W^- + \frac{1}{2}E \otimes g + \frac{1}{24}R_g g \otimes g,$$

we have the associated curvature operators are written as

$$\mathcal{R} = \mathcal{W}^+ + \mathcal{W}^- + \mathcal{E} + \frac{1}{24}S,$$

where

$$\left( \mathcal{W}^\pm + \frac{1}{24}S \right) : \Lambda^2_\pm (T^*M) \mapsto \Lambda^2_\pm (T^*M),$$

and

$$\mathcal{E} : \Lambda^2_\pm (T^*M) \mapsto \Lambda^2_\mp (T^*M).$$

Some basic properties of the tensors $W^\pm$ and of the curvature operators $\mathcal{W}^\pm$ are the following

1. Viewed as a $(1, 3)$ tensor, for any $C^2$ function $f$, $W^\pm(e^{-2f}g) = W^\pm(g)$.
2. Letting $\mathcal{C} : S^2(\Lambda^2 (T^*M)) \mapsto S^2 (T^*M)$ be the Ricci Contraction Map defined by $(\mathcal{C}U)_{ab} = g^{cd}U_{acbd}$, then

$$\mathcal{C}W^\pm = 0.$$

3. Both $\mathcal{W}^+$ and $\mathcal{W}^-$ are traceless.

We note that our convention is that if $P_{ijkl}$ is a tensor satisfying $P_{ijkl} = -P_{jikl} = -P_{ijlk} = P_{klij}$, then the associated operator $\mathcal{P} : \Lambda^2 \to \Lambda^2$ is given by

$$(\mathcal{P}\omega)_{ij} = \frac{1}{2} \sum_{k,l} P_{ijkl}\omega_{kl}.$$
2.1. The anti self-dual part of the Weyl tensor as a bilinear form. Consider a warped product metric on $M = \mathbb{R} \times Y^3$ of the form
\begin{equation}
    g = dt^2 + g_Y,
\end{equation}
Where $g_Y$ is a smooth metric on $Y$, possibly depending on $t$. Our ultimate goal is a formula for the linearized anti-self-dual Weyl curvature $D$, which maps from
\begin{equation}
    D : S^2_0(T^*M) \rightarrow S^2_0(\Lambda^2_-).
\end{equation}
Using the decomposition $T^*M = \{dt\} \oplus T^*Y$, we have
\begin{equation}
    S^2(T^*M) = S^2(dt) \oplus (dt \odot T^*Y) \oplus S^2(T^*Y),
\end{equation}
which we will write as
\begin{equation}
    \tilde{h} = h_{00} dt \otimes dt + (\alpha \otimes dt + dt \otimes \alpha) + h.
\end{equation}
Next, we have
\begin{equation}
    \Lambda^2(dt \oplus T^*Y) = (\Lambda^1(dt) \otimes \Lambda^1(T^*Y)) \oplus \Lambda^2(T^*Y).
\end{equation}
Given an orientation, we then have
\begin{equation}
    \Lambda^2(dt \oplus T^*Y) = (\Lambda^1(dt) \otimes \Lambda^1(T^*Y)) \oplus \Lambda^1(T^*Y).
\end{equation}
Under this decomposition, the self-dual forms correspond to
\begin{equation}
    dt \wedge \alpha + \tilde{*} \alpha,
\end{equation}
while the anti-self-dual forms correspond to
\begin{equation}
    dt \wedge \alpha - \tilde{*} \alpha,
\end{equation}
where $\tilde{*}$ is the Hodge-$*$ operator on $Y$. Consequently, we have the identification
\begin{equation}
    S^2_0(\Lambda^2_-) = S^2_0(T^*Y),
\end{equation}
and we can therefore view $D$ as a mapping
\begin{equation}
    D : S^2_0(T^*M) \rightarrow S^2_0(T^*Y).
\end{equation}
In order to proceed, we must first write down $\mathcal{W}^-$, considered as an element of $S^2_0(T^*Y)$.

**Proposition 2.1.**
\begin{equation}
    (\mathcal{W}^-)_{ij} = \Phi_{ij} - \Psi_{ij} + \Omega_{ij},
\end{equation}
where
\begin{equation}
    \Phi_{ij} = \text{tf}(R_{00ij}(g)),
\end{equation}
\begin{equation}
    \Psi_{ij} = \text{tf} \left( \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{ikl} R_{0jkl}(g) \right) \right),
\end{equation}
\begin{equation}
    \Omega_{ij} = \frac{1}{4} \text{tf} \left( \sum_{k,l} \sum_{p,q} \epsilon_{ikl} \epsilon_{jlp} R_{klpq}(g) \right),
\end{equation}
where the symbols $\epsilon_{ijk}$ are the components of the volume element defined by
\[ e_i \wedge e_j \wedge e_k = \epsilon_{ijk} e_1 \wedge e_2 \wedge e_3, \]
Sym in (2.19) denotes the Symmetrization Operator given by
\[ \text{Sym}_{ij}(F) = \frac{1}{2} (F_{ij} + F_{ji}), \]
and for $h \in S^2(T^*Y)$, $\text{tf}(h)$ denotes the traceless component of $h$ with respect to the metric $g_Y$.

**Proof.** Given $g$ as in (2.7) and considering the decomposition (2.2), we conclude from (2.2), (2.3) and (2.4) that for any $\omega, \omega' \in \Lambda^2(M)$
\[ \langle R\omega, \omega' \rangle = \langle R\omega, \omega' \rangle. \]
In order to compute $\langle R\omega, \omega' \rangle$ we use the isomorphism between $\Omega^1(Y) \oplus \Omega^1(Y)$ and $\Lambda^2(\mathbb{R} \times Y)$ given as follows: any 2-form $\omega$ in $\Lambda^2(\mathbb{R} \times Y)$ can be written uniquely as
\[ \omega = dt \wedge \pi^* (\xi) + \pi^* (\tilde{\eta}) , \]
where $\pi$ is the projection map $\pi : \mathbb{R} \times Y \mapsto Y$, $\xi$ and $\eta$ are 1-forms in $\Omega^1(Y)$ and $\tilde{\eta}$ is the Hodge-$*$ operator with respect to the metric $g_Y$. Given a local orthonormal oriented basis $\{e_1, e_2, e_3\}$ of $\Gamma(TY)$, the operator $\tilde{*} : \Omega^1(Y) \mapsto \Omega^2(Y)$ takes the form
\[ \tilde{*}(\xi)_{ij} = \sum_{k=1}^{3} \epsilon_{ijk} \xi_k. \]
Using $i, k, l$ to denote indices in $\{1, 2, 3\}$ we see that the 1-forms $\xi, \eta$ in (2.23) can be written in coordinates as
\[ \xi_k = \omega_{0k}, \quad \eta_i = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} \omega_{jk} = \sum_{j<k} \epsilon_{ijk} \omega_{jk}, \]
Moreover, in these coordinates, the Hodge-$*$ operator can be computed as
\[ \ast \omega = dt \wedge \pi^* (\eta) + \pi^* (\tilde{\ast} \eta) , \]
therefore, $\omega \in \Lambda^2_\pm (T^*Y)$ if and only if $\xi = \pm \eta$. Given 2-forms $\omega, \omega' \in \Lambda^2(M)$, let us write $\omega = dt \wedge \pi^* (\xi) + \pi^* (\tilde{\eta})$ and $\omega' = dt \wedge \xi' + \pi^* (\tilde{\eta'})$ , then we can express $\langle R\omega, \omega' \rangle$ as
\[ \langle R\omega, \omega' \rangle = \alpha_{ij} \xi_i \xi'_j + \beta_{ij} \xi_i \eta'_j + \beta_{ij} \eta_i \xi'_j + \gamma_{ij} \eta_i \eta'_j \]
where clearly
\[ \alpha_{ij} = R_{0i0j}(g), \quad \beta_{ij} = \frac{1}{2} \sum_{k,l} \epsilon_{jkl} R_{0ikl}(g), \quad \gamma_{ij} = \frac{1}{4} \sum_{k,l} \sum_{p,q} \epsilon_{ikl} \epsilon_{jpq} R_{klmn}(g). \]
If now $\ast \omega = -\omega$ and $\ast \omega' = -\omega'$, from (2.26) and (2.25) we obtain
\[ \langle R\omega, \omega' \rangle = (\alpha_{ij} - (\beta_{ij} + \beta_{ji}) + \gamma_{ij}) \xi_i \xi'_j. \]
This shows that with the isomorphism defined by (2.23), we can identify the map 
\[ (W^− + \frac{1}{24}S) : \Lambda^2_+ (T^*M) \mapsto \Lambda^2_+ (T^*M) \]
with a bilinear form in \( S^2 (T^*Y) \) such that in the local orthonormal basis \( \{ e_1, e_2, e_3 \} \) has components
\[ (2.29) \quad \alpha_{ij} - (\beta_{ij} + \beta_{ji}) + \gamma_{ij}. \]
with \( \alpha_{ij}, \beta_{ij}, \gamma_{ij} \) given by (2.27). Since the scalar curvature operator \( S \) contributes a pure-trace term, we are done. □

We next give a more detailed description of some of the terms appearing in (2.17) and for that purpose we will make use of the following notation:
- All letters \( i, j, k, l, \ldots \) will denote non-zero indices.
- Given \( H \in S^2 (\Lambda^2 (M)) \), by \( c_Y H \) we will mean the map defined as 
  \[ (c_Y H)_{jk} = g^{il} Y_{H}^i jlk, \]
- By \( \text{tr}_Y (c_Y H) \) we will mean \( g^{ij} (c_Y H)_{ij} \),
- We will use \( \dot{g}_Y \ast \dot{g}_Y \) to denote linear combinations of contractions of \( \dot{g}_Y \otimes \dot{g}_Y \) using the metric \( g_Y \).

**Proposition 2.2.** We have the identity
\[ \frac{1}{4} \sum_{k,l,u,v} \epsilon_{ikl} \epsilon_{juv} H_{kluv} = - \left( c_Y H - \frac{1}{2} \text{tr}_Y (c_Y H) g_Y \right)_{ij}. \]

**Proof.** Suppose \( i = j \), and let \( p, q \) with \( p < q \) be indices such that \( \{1, 2, 3\} = \{i, p, q\} \)
\[ \frac{1}{4} \sum_{k,l} \sum_{u,v} \epsilon_{ikl} \epsilon_{juv} H_{kluv} = \frac{1}{4} \sum_{k,l} \sum_{u,v} \epsilon_{ikl} \epsilon_{juv} H_{klmn} = H_{ppqq}. \]
Note that the trace of \( c_Y H \) is given by
\[ \frac{1}{2} \text{tr}_Y (c_Y H) = H_{ppqq} + H_{iipi} + H_{iiqiq}. \]
Therefore
\[ H_{ppqq} = \frac{1}{2} \text{tr}_Y (c_Y H) - (H_{iipi} + H_{iiqiq}) = \frac{1}{2} \text{tr}_Y (c_Y H) - (c_Y H)_{ii} \]
\[ = \frac{1}{2} \text{tr}_Y (c_Y H) \delta_{ii} - (c_Y H)_{ii} = - \left( c_Y H - \frac{\text{tr}_Y (c_Y H)}{2} g_Y \right)_{ii}. \]
If now \( i \neq j \) and \( p \) is such that \( \{1, 2, 3\} = \{i, j, p\} \), we have
\[ \frac{1}{4} \sum_{k,l} \sum_{u,v} \epsilon_{ikl} \epsilon_{juv} H_{kluv} = \epsilon_{ijp} \epsilon_{jip} H_{jpjp} \]
\[ = - H_{jpjp} = - (c_Y H)_{ij} = - \left( c_Y H - \frac{\text{tr}_Y (c_Y H)}{2} g_Y \right)_{ij}, \]
and the claim follows. □

We will also need to compute the Christoffel symbols and components of the curvature tensor of \( g \) in terms of the metric \( g_Y \):
Proposition 2.3. The Christoffel symbols of the metric \( g = dt^2 + g_Y \) are given by
\[
\begin{align*}
\Gamma^k_{i0}(g) &= \frac{1}{2} g^{kl}(\dot{g}_Y)_{il}, \\
\Gamma^0_{ij}(g) &= -\frac{1}{2} (\ddot{g}_Y)_{ij}, \\
\Gamma^k_{ij}(g) &= \Gamma^k_{ij}(g_Y), \\
\Gamma^k_{00}(g) &= \Gamma^0_{00}(g) = 0.
\end{align*}
\]
For the components of the curvature tensor we have
\[
\begin{align*}
R^0_{0ij} &= -\frac{1}{2} (\dddot{g}_Y)_{ij} + (\dot{g}_Y \ast \dot{g}_Y)_{ij}, \\
R^k_{ijl}(g) &= R^k_{ijl}(g_Y) + (\dot{g}_Y \ast \dot{g}_Y)_{ijl}.
\end{align*}
\]
In particular, if \( g_Y \) is independent of \( t \), then
\[
\Gamma^\gamma_{\alpha\beta}(g) = \begin{cases} 0 & \text{if any } \alpha, \beta, \gamma \text{ equals } 0 \\ \Gamma^\gamma_{\alpha\beta}(g_Y) & \text{otherwise} \end{cases},
\]
and consequently
\[
R^\nu_{\alpha\beta\mu} = \begin{cases} 0 & \text{if any } \alpha, \beta, \mu, \nu \text{ equals } 0 \\ R^\nu_{\alpha\beta\mu}(g_Y) & \text{otherwise} \end{cases}.
\]

Proof. The proof follows from a straightforward computation.

We can now write out a more convenient expression for \( \Omega_{ij} \) in (2.17).

Proposition 2.4. The term \( \Omega_{ij} \) in \((2.17)\) has the form
\[
\Omega_{ij} = (-E(g_Y) + \dot{g}_Y \ast \dot{g}_Y)_{ij},
\]
where \( E(g_Y) \) is the traceless Ricci tensor of \( g_Y \).

Proof. Recall that \( \Omega_{ij} \) is given by
\[
\Omega_{ij} = tf \left( \frac{1}{4} \sum_{k,l,m,n} \epsilon_{ikl} \epsilon_{jmn} R_{klmn}(g) \right),
\]
and from Proposition 2.2 we must have
\[
\Omega_{ij} = -tf \left( c_Y Rm(g) - \frac{1}{2} \text{tr}_Y(c_Y Rm(g)) g_Y \right).
\]

With the expressions obtained for the components of \( Rm \) in Proposition 2.3, we have
\[
- (c_Y Rm(g))_{ij} = (-\text{Ric}(g_Y) + \dot{g}_Y \ast \dot{g}_Y)_{ij},
\]
and then
\[
\text{tr}_Y (c_Y Rm(g)) = R_{gY} + \dot{g}_Y \ast \dot{g}_Y,
\]
which implies (2.32).
3. **Linearization of $W^-$ at a cylindrical metric**

Consider the cylindrical metric

\[ g = dt^2 + g_Y, \]

defined on $M = \mathbb{R} \times Y$, where $g_Y$ is a fixed metric of constant curvature $\kappa = +1, 0,$ or $-1$. We note that $g$ is locally conformally flat, and therefore is self-dual. We are interested in studying the linearization of $W^-$ at $g$. Given $\tilde{h} \in S^2(M)$, we consider a path of metrics $g(\epsilon)$ with $\epsilon \in (-\delta, \delta)$ for some $\delta > 0$ satisfying $g(0) = g$ and $g'(0) = \tilde{h}$. The linearization of $W^-$ at $g$ in the direction of $\tilde{h}$ is the map

\[ (W^-)'_g (\tilde{h}) = \frac{\partial}{\partial \epsilon} W^-(g_\epsilon)|_{\epsilon=0}. \]

We next define a Dirac-type operator:

**Definition 3.1.** Let \{e_1, e_2, e_3\} be a local orthonormal basis of $\Gamma (TY)$. Then, for any $h \in S^2(T^*Y)$ the operator $\phi h$ is given in these coordinates by

\[ (\phi h)_{ij} = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{ijkl} d^\nabla h_{klj} \right), \]

where $(d^\nabla h)_{klj}$ is given by $(d^\nabla h)_{klj} = \nabla_k h_{lj} - \nabla_l h_{kj}$.

We also recall the conformal Killing operator:

**Definition 3.2.** For an $n$-dimensional manifold $(M^n, g)$, the **conformal Killing operator** with respect to the metric $g$ is the map $K_g : \Lambda^1(T^*M) \to S^2_0(T^*M)$, given by

\[ K_g(\tilde{\omega}) = \mathcal{L}_g(\tilde{\omega}) - \frac{2}{n} (\delta \tilde{\omega}) g, \]

where $\mathcal{L}_g$ is the Lie derivative operator.

In cylindrical coordinates, a tensor $\tilde{h} \in S^2(M)$ can be decomposed as

\[ \tilde{h} = h_{00} dt \otimes dt + \alpha \otimes dt + h, \]

where $h_{00} \in \Lambda^0(M)$, $\alpha \in \Lambda^1(T^*Y)$ and $h \in S^2(T^*Y)$, so we will use the notation $\tilde{h} = \{h_{00}, \alpha, h\}$. The main result of this section is the following

**Theorem 3.3.** For the cylindrical metric given by $g = dt^2 + g_Y$, the linearization $(W^-)'_g (\tilde{h})$ with $\text{tr}_g(\tilde{h}) = 0$, is given by

\[ (W^-)'_g (h_{00}, \alpha, h) = \frac{1}{2} K_{g_Y} \left( -\frac{1}{2} dh_{00} + \ast d\alpha - \ast d\alpha \right) - \frac{1}{2} tf(\tilde{h}) + \frac{1}{2} \tilde{h} - E'(h), \]
where $E'(h)$ is the linearization of the traceless Ricci tensor at $g_Y$. Equivalently, after computing $E'(h)$ explicitly, $(W^-)'_{gY}(\tilde{h})$ is given by

$$
(W^-)'(h_{00}, \alpha, h) = \frac{1}{2} K_{gY} \left( -\frac{1}{2} dh_{00} - \delta_Y h + \alpha - *d\alpha + \frac{1}{2} dtr_Y(h) \right) - \frac{1}{2} tf(\tilde{h}) - \kappa \cdot tf(h) + \frac{1}{2} \delta_Y h + \frac{1}{2} \Delta_{gY} tf(h).
$$

where $\Delta_{gY}$ is the rough laplacian on $S^2(T^*Y)$, and $(\delta_Y h)_j = \nabla^j_Y h_{ij}$ is the divergence.

The remainder of the section will be concerned with the proof of Theorem 3.3.

3.1. Conformal Killing operator and $\partial$. The operator $\partial$ enjoys the following properties:

**Proposition 3.4.** For the operator $\partial$, we have

(3.5) $\partial : S^2(T^*Y) \to S^2_0(T^*Y)$,

(3.6) $\partial(ug_Y) = 0$ for any $u \in C^2(Y)$,

(3.7) $\partial : S^2_0(T^*Y) \to S^2_0(T^*Y)$ is formally self-adjoint.

**Proof.** For the first property, in an orthonormal basis

$$
\text{tr}_Y(\partial h) = \sum_{i,j} \sum_{k,l} \delta_{ij} \text{Sym}_{ij} (\epsilon_{ikl} d^\nabla h_{klj}) = \sum_{i=1}^3 \sum_{k,l \neq i} \epsilon_{ikl} (\nabla_k h_{li} - \nabla_l h_{ki}) = \sum_{i=1}^3 \sum_{k,l \neq i} \epsilon_{ikl} \nabla_k h_{li} - \sum_{i=1}^3 \sum_{k,l \neq i} \epsilon_{ikl} \nabla_l h_{ki} = 0.
$$

For (3.6), in an orthonormal basis and using that $g_Y$ is parallel we have

$$
\partial(ug_Y)_{ij} = \sum_{k,l} \text{Sym}_{ij} (\epsilon_{ikl} d^\nabla (ug_Y)_{klj}) = \text{Sym}_{ij} \left( \sum_{k,l} (\epsilon_{ikl} (\nabla_k u)(g_Y)_{lj} - \epsilon_{ikl} (\nabla_l u)(g_Y)_{kj}) \right) = \text{Sym}_{ij} \left( \sum_{k=1}^3 \epsilon_{ijk} \nabla_k u - \sum_{l=1}^3 \epsilon_{ijl} \nabla_l u \right) = -2\text{Sym}_{ij} \left( \sum_{k=1}^3 \epsilon_{ijk} \nabla_k u \right),
$$

and since $\epsilon_{ijk} \nabla_k u$ is skew-symmetric in $i,j$, it follows that $\partial(ug_Y) = 0$. 


Finally, let $h, h'$ be elements in $S^2(T^*Y)$, then in an orthonormal basis we have

$$
\int_Y \langle dh, h' \rangle dV_g = \frac{1}{2} \sum_{i,j} \sum_{k,l} \int_Y \epsilon_{ikl} (\nabla_k h_{ij} - \nabla_l h_{kj}) h'_{ij} dV_g \\
= \sum_{i,j} \sum_{k,l} \int_Y \epsilon_{ikl} \nabla_k h_{ij} h'_{ij} dV_g = - \sum_{i,j} \sum_{k,l} \int_Y \epsilon_{ikl} h_{ij} \nabla_k h'_{ij} dV_g \\
= \sum_{i,j} \sum_{k,l} \int_Y \epsilon_{ikl} \nabla_k h'_{ij} h_{ij} dV_g = \int_Y \langle h, dh' \rangle dV_g.
$$

\[\Box\]

For the operators $\partial, \mathcal{K}_g$ and $\mathcal{D}$ we have

**Proposition 3.5.** The operators $\partial$ and $\mathcal{K}_g$ satisfy the following identities

\begin{align*}
(3.8) \quad & \partial \mathcal{L}_{gy}(\omega) = \mathcal{K}_{gy}(\ast d\omega), \\
(3.9) \quad & \mathcal{D}(\mathcal{K}_{gy}(\tilde{\omega})) = 0 \text{ for any } \tilde{\omega} \in \Lambda^1(T^*M).
\end{align*}

**Proof.** Identity (3.8) is a consequence of the following computation: let $h = \mathcal{L}_{gy}(\omega)$ then

\[(\partial h)_{ij} = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{ikl} (\nabla_k h_{ij} - \nabla_l h_{kj}) \right).\]

(3.10) \\

\[(\partial h)_{ij} = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{ikl} (\nabla_k \nabla_l \omega_j + \nabla_k \nabla_j \omega_l - \nabla_l \nabla_k \omega_j - \nabla_l \nabla_j \omega_k) \right).\]

Commuting covariant derivatives in (3.10) we obtain

\[(3.11) \quad (\partial h)_{ij} = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{ikl} (\nabla_j \nabla_k \omega_l - \nabla_j \nabla_l \omega_k - R^p_{klj} \omega_p - R^p_{kjl} \omega_p + R^p_{ljk} \omega_p) \right).\]

Note that $-R^p_{klj} - R^p_{kjl} + R^p_{ljk} = -2R^p_{klj}$ by the algebraic Bianchi identity, so (3.11) becomes

\[(3.12) \quad (\partial h)_{ij} = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{ikl} (\nabla_j \nabla_k \omega_l - \nabla_j \nabla_l \omega_k - 2 \sum_{k,l} \epsilon_{ikl} R^p_{klj} \omega_p) \right).\]

Since $gy$ has constant sectional curvature equal to $\kappa$

\[-2 \sum_{k,l} \epsilon_{ikl} R^p_{klj} \omega_p = -2\kappa \sum_{k,l} \epsilon_{ikl} (\delta^p_{kj} - \delta^p_{lj}) \omega_p \]

\[= -2\kappa \left( \sum_{k,l} \epsilon_{ikl} \omega_k \delta_{lj} - \sum_{k,l} \epsilon_{ikl} \omega_l \delta_{kj} \right) = -2\kappa \left( \sum_k (\epsilon_{ikj} - \epsilon_{ijk}) \omega_k \right).\]
Since $\epsilon_{ikj} - \epsilon_{ijk}$ is skew-symmetric in $i, j$ we obtain
\[
(dh)_{ij} = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{ikl} (\nabla_j \nabla_k \omega_l - \nabla_j \nabla_l \omega_k) \right)
\]
\[
= \text{Sym}_{ij} \left( \nabla_j \left( \sum_{k,l} \epsilon_{ikl} (\nabla_k \omega_l - \nabla_l \omega_k) \right) \right)
\]
\[
= 2\text{Sym}_{ij} (\nabla_j (\tilde{\ast}d\omega)_i) = \nabla_j (\tilde{\ast}d\omega)_i + \nabla_i (\tilde{\ast}d\omega)_j
\]
\[
= (L_g \gamma)(\tilde{\ast}d\omega))_{ij}.
\]
Since $dh$ is traceless, we actually obtain $dL_g \gamma(\omega) = K_g \gamma(\tilde{\ast}d\omega)$ as needed. For proving (3.9), we note that by diffeomorphism invariance of $W^-$ and since $g$ is locally conformally flat we have $D(L_g(\tilde{\omega})) = 0$ for any 1-form $\tilde{\omega} \in \Lambda^1(T^*M)$. By the conformal invariance of $W^-$, we have $D(fg) = 0$ for any $f \in C^\infty(M)$, therefore the composition of $D$ and $K_g$ is zero. \hfill \square

3.2. The case of no radial components. We first compute $(W^-)'(\tilde{h})$ assuming that $\tilde{h}$ has no radial components, i.e. $\tilde{h}$ has the form $\tilde{h} = \{0, 0, h\}$.

**Proposition 3.6.** The linearization of $W^-$ at $g = dt^2 + g_Y$ in the direction $\tilde{h} = \{0, 0, h\}$ is
\[
(W^-)'(\tilde{h}) = -\frac{1}{2} tf\gamma(\tilde{h}) + \frac{1}{2} \left( dh \right) - \text{E}'_{gY}(h).
\]

**Proof.** We start by linearizing the component $\Omega_{ij}$ in (2.17). Note that
\[
\frac{\partial}{\partial \epsilon} (\dot{g}_Y(\epsilon) * \dot{g}_Y(\epsilon)) |_{\epsilon=0} = 0,
\]
for any variation which is purely spherical, that is, a variation which only deforms the cross-section metric on $Y$. From Proposition 2.3 and (3.14) it is clear that for $\tilde{h} = \{0, 0, h\}$ we have
\[
\Omega_{ij}(\tilde{h}) = -\text{E}'(h).
\]
For the term $\Phi_{ij}$ in (2.17), we consider a purely spherical deformation $g_\epsilon$ of $g$ in the direction of $h$ so that from (2.18) we have
\[
\Phi_{ij}(g_\epsilon) = tf_{gY(\epsilon)} \left( R_{0\alpha\beta}(dt^2 + g_Y(\epsilon)) \right),
\]
and from Proposition 2.3
\[
R_{0\alpha\beta}(g_\epsilon) = \left( -\frac{1}{2} \dot{g}_Y(\epsilon) + \dot{g}_Y(\epsilon) \ast \dot{g}_Y(\epsilon) \right)_{ij},
\]
then
\[
(\Phi'_g)_{ij}(\tilde{h}) = \frac{\partial}{\partial \epsilon} \left( -tf_{gY(\epsilon)}(\dot{g}_Y(\epsilon)) + \dot{g}_Y(\epsilon) \ast \dot{g}_Y(\epsilon) \right) |_{\epsilon=0} = -\frac{1}{2} tf_{gY} \tilde{h}.
\]
Finally, for the components Ψ_{ij} we recall that we can express Ψ_{ij}(dt^2 + g_Y) as

\[ Ψ_{ij} = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{jkl}(g_Y) R_{0ikl}(dt^2 + g_Y) \right). \]

Note that taking the tracefree part is not necessary, see Proposition 3.4. Before linearizing \( \epsilon_{jkl}(g_Y)R_{0ikl}(dt^2 + g_Y) \), we note that if we evaluate Ψ_{ij} along a purely spherical deformation \( g_\epsilon \) of \( g \) in the direction of \( h \), the symbol \( \epsilon_{jkl} \) may depend on \( g_Y(\epsilon) \) and so we must write

\[ Ψ_{ij}(g_\epsilon) = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{jkl}(g_Y(\epsilon)) R_{0ikl}(g_\epsilon) \right), \]

however, since \( R_{0ijk}(dt^2 + g_Y) = 0 \) for all choices of \( i, j, k \) as seen in (2.31), we conclude that the linearization of \( Ψ_{ij} \) in the direction \( \tilde{h} = \{0,0,h\} \) is

\[ \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{jkl}(R'_g)_{0ikl}(\tilde{h}) \right). \]

Linearizing \( Rm \) at \( g \) in the direction of \( \tilde{h} \), and using Proposition 2.3 we obtain

\[ (R'_g)_{0ikl}(\tilde{h}) = \frac{1}{2} \left( \nabla_0 \nabla_i \tilde{h}_{kl} - \nabla_0 \nabla_k \tilde{h}_{il} - \nabla_l \nabla_i \tilde{h}_{0k} + \nabla_i \nabla_k \tilde{h}_{0l} \right). \]

It is easy to see that

\[ \nabla_0 \nabla_k \tilde{h}_{il} = \nabla_k \tilde{h}_{il}, \text{ and } \nabla_i \nabla_k \tilde{h}_{0l} = \nabla_i \nabla_l \tilde{h}_{0j} = 0, \]

so we have proved

\[ Ψ'_{ij}(\tilde{h}) = \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{jkl}(R'_g)_{0ikl}(\tilde{h}) \right) \]

\[ = -\frac{1}{2} \text{Sym}_{ij} \left( \sum_{k,l} \epsilon_{jkl} \left( \nabla_k \tilde{h}_{il} - \nabla_l \tilde{h}_{ik} \right) \right) = -\frac{1}{2} (\delta h)_{ij}. \]

The proposition follows from combining (3.15), (3.17) and (3.19). □

3.3. The case of conformal variations. Using conformal invariance, we next extend the formula in Proposition 3.6 to tensors of the form \( \{h_{00}, 0, h\} \).

**Proposition 3.7.** The linearization of \( W^- \) at \( g \) in the direction \( \tilde{h} = \{h_{00}, 0, h\} \) is

\[ D(h_{00}dt \otimes dt + h) = D(h) - \frac{1}{2} \left( \nabla_Y^2 h_{00} - \frac{1}{3}(\Delta g_Y h_{00})g_Y \right). \]

**Proof.** Since the cylinder is locally conformally flat, for any \( C^2 \) function \( v \) we have

\[ D \left( v (dt^2 + g_Y) \right) = 0, \]
therefore
\[ D(h_00 dt \otimes dt + h) = D(h_00 (dt^2 + g_Y) - h_00 g_Y + h) \]
\[ = D(h - h_00 g_Y) = D(h) - D(h_00 g_Y). \]

Since \( h_00 g_Y \) is a scalar tensor we have by Corollary 3.6 and (3.6)
\[ D(h_00 g_Y) = -E'_{g_Y} (h_00 g_Y) = -E'_{g_Y} (h_00 g_Y). \]

Next, consider a path \( \{g_s\} \) of metric on \( Y \) given by \( g_s = e^{su} g_Y \), then \( g_0 = g_Y \) and \( \partial_s g_s|_{s=0} = u g_Y \). Since \( g_Y \) is Einstein, a standard formula for conformal changes gives
\[ E(g_s) = -\frac{s}{2} \left( \nabla^2_{g_Y} u - \frac{1}{3}(\Delta_{g_Y} u)g_Y \right) + \frac{s^2}{4} \left( du \otimes du - \frac{1}{3}|\nabla_{g_Y} u|^2g_Y \right). \]
Differentiating at \( s = 0 \), we obtain
\[ E'_{g_Y} (u g_Y) = -\frac{1}{2} \left( \nabla^2_{g_Y} u - \frac{1}{3}(\Delta_{g_Y} u)g_Y \right), \]
and the proposition follows.

3.4. Completion of proof of Theorem 3.3. Consider now a variation \( \tilde{h} \) of the form \( \tilde{h} = \{0, \alpha, 0\} \).

**Proposition 3.8.** The linearization of \( W^- \) at \( g \) in the direction \( \{0, \alpha, 0\} \) is given by
\[ D(\{0, 0, 0\}) = \frac{1}{2} K_{g_Y} (\dot{\alpha} - \ddot{\alpha} \alpha). \]

**Proof.** Choose \( \omega \) so that \( \dot{\omega} = \alpha \). In this case the conformal Killing operator equals
\[ K_g(\omega) = \left\{ -\frac{1}{2} \delta_Y \omega, \alpha, K_{g_Y} (\omega) + \left( \frac{1}{6} \delta_Y \omega \right) g_Y \right\}. \]

We write
\[ D(\{0, 0, 0\}) = D(\{0, 0, 0\} - K_g(\omega)) \]
\[ = D \left( \{0, 0, 0\} - \left\{-\frac{1}{2} \delta_Y \omega, \alpha, K_{g_Y} (\omega) + \left( \frac{1}{6} \delta_Y \omega \right) g_Y \right\} \right) \]
\[ = D \left( \frac{1}{2} \delta_Y \omega, 0, -K_{g_Y} (\omega) - \left( \frac{1}{6} \delta_Y \omega \right) g_Y \right). \]
Recall that for any \( C^2 \) function \( u \) we have \( D(udt^2) = D(-ug_Y) \), using (3.20) we obtain
\[ D(\{0, 0, 0\}) = D \left( -K_{g_Y} (\omega) - \left( \frac{1}{6} \delta_Y \omega \right) g_Y - \left( \frac{1}{2} \delta_Y \omega \right) g_Y \right) = D(-L_{g_Y} (\omega)) \]
From Corollary 3.6 and from (3.8) and (3.22) we obtain
\[ D(\{0, 0, 0\}) = K_{g_Y} \left( \frac{1}{2} \dot{\omega} \right) - \frac{1}{2} \ddot{\omega} L_{g_Y} (\omega), \]
and since $\dot{\omega} = \alpha$, we have
\begin{align}
\dot{\omega} &= \dot{\alpha}, \\
\dot{\mathcal{L}}_{g_Y}(\dot{\omega}) &= K_{g_Y}(\ddot{\omega} \ddot{\alpha}) = K_{g_Y}(\dddot{\alpha}),
\end{align}
so from (3.23), (3.24) and (3.25) we obtain (3.21). \square

With (3.21) we are ready to prove Theorem 3.3.

\textbf{Proof of Theorem 3.3.} Combining Corollary 3.6, Proposition 3.7 and (3.21) we obtain (3.3). In order to prove (3.4) we linearize $E$ at $g_Y$ in the direction of $h$
\begin{align}
E'(h) &= \left(Ric(g) - \frac{1}{3} R_{gY} g\right)'_{g_Y}(h) = Ric'_{g_Y}(h) - \frac{1}{3} R'_{g_Y}(h) g_Y - \frac{1}{3} R_{gY} h.
\end{align}
The linearization of $Ric$ is
\begin{align}
(Ric'_{g_Y}(h))_{ij} &= -\frac{1}{2} \Delta_L h_{ij} - \frac{1}{2} \nabla^2_{ij} \operatorname{tr}(h) + \frac{1}{2} \left(\nabla_i \delta_j h + \nabla_j \delta_i h\right),
\end{align}
where $\Delta_L h$ is the Lichnerowicz Laplacian given by
\begin{align}
\Delta_L h_{ij} &= \Delta_{g_Y} h_{ij} + 2 R_{i j p} h^{l p} - R^p_i h_{j p} - R^p_j h_{i p}.
\end{align}
Since $g_Y$ has constant sectional curvature $\kappa$, $\Delta_L$ can be computed as
\begin{align}
\Delta_{g_Y} h_{ij} &= \Delta_{g_Y} h_{ij} + 2 \kappa \left( (g_Y)_{ij} (g_Y)_{ip} - (g_Y)_{ip} (g_Y)_{ij} \right) h^{lp} - 2 \kappa \delta^p_i h_{jp} - 2 \kappa \delta^p_j h_{ip} \\
&= \Delta_{g_Y} h_{ij} + 2 \kappa \operatorname{tr}(h) (g_Y)_{ij} - 2 \kappa h_{ij} - 4 \kappa h_{ij} \\
(3.28) &= \Delta_{g_Y} h_{ij} - 6 \kappa \operatorname{tf}(h)_{ij}.
\end{align}
On the other hand, the linearization of $Rg$ is
\begin{align}
R'(h) &= -\Delta_{g_Y} \operatorname{tr}(h) + \delta_Y \delta_Y h - (Ric(g_Y), h)_{g_Y}.
\end{align}
and
\begin{align}
\frac{1}{3} (R_{gY} h - (Ric(g_Y), h)_{g_Y} g_Y) &= 2 \kappa \operatorname{tf}(h).
\end{align}
Combining (3.26) and (3.29), we conclude that $E'_{g_Y}(h)$ is given by
\begin{align}
E'_{g_Y}(h) &= -\frac{1}{2} \left( \Delta_L h + \nabla^2 \operatorname{tr}(h) - \frac{2}{3} \Delta_{g_Y} \operatorname{tr}(h) g_Y \right) \\
(3.31) \quad + \frac{1}{2} \left( \mathcal{L}_{g_Y}(\delta_Y h) - \frac{1}{3} (\delta_Y \delta_Y h) g_Y \right) - \frac{1}{3} (R_{gY} h - (Ric(g_Y), h)_{g_Y}),
\end{align}
and using (3.28) and (3.30), we finally obtain
\begin{align}
E'_{g_Y}(h) &= -\frac{1}{2} \left( \Delta_{g_Y} \operatorname{tf}(h) + \nabla^2 \operatorname{tr}(h) \right) + \frac{1}{2} K_{g_Y} (\delta_Y h) + \kappa \cdot \operatorname{tf}(h),
\end{align}
where $\nabla^2$ denotes the traceless Hessian operator. From (3.3) and (3.32), (3.4) follows easily. \square
4. Some properties of $\mathcal{D}$

In this section we derive several useful identities for the operator $\mathcal{D}$ introduced in Section 3 apart from those proved in Subsection 3.1. First, we have a crucial formula for the square of $\mathcal{D}$:

**Proposition 4.1.** The operator $\mathcal{D}^2 : S^2(T^*Y) \mapsto S^2_0(T^*Y)$ is given by

$$\mathcal{D}^2 h = -4\Delta_{g_Y} tf(h) - 2 \nabla^2 \text{tr}_{Y}(h) + 3\mathcal{K}_{g_Y}(\delta_Y h) + 12\kappa \cdot \Phi(h).$$

**Proof.** The proof is moved to Appendix A. □

Next, we have

**Proposition 4.2.** For any $h \in S^2(T^*Y)$ we have $\delta_Y (\mathcal{D} h) = \mathcal{D} \delta_Y h$.

**Proof.** In a local orthonormal basis we have

$$(\delta_Y (\mathcal{D} h))_i = \sum_{j=1}^{3} \nabla_j (\mathcal{D} h)_{ij}$$

(4.2)

$$= \sum_{j=1}^{3} \sum_{k,l} \epsilon_{ikl} \nabla_j \nabla_k h_{lj} + \sum_{j=1}^{3} \sum_{p,q} \epsilon_{jpq} \nabla_j \nabla_p h_{qi}.$$  

Commuting covariant derivatives we have

$$\sum_{j=1}^{3} \sum_{k,l} \epsilon_{ikl} \nabla_j \nabla_k h_{lj} = \sum_{j=1}^{3} \sum_{k,l} \epsilon_{ikl} \left( \nabla_k \nabla_j h_{lj} - R^s_{jkli} h_{sj} - R^s_{jkli} h_{ls} \right)$$

$$= \sum_{j=1}^{3} \sum_{k,l} \epsilon_{ikl} \left( \nabla_k \nabla_j h_{lj} + \kappa \left( - h_{jij} g_{kl} + h_{kjg_{jl}} - h_{tij} g_{kj} + h_{ukg_{jj}} \right) \right)$$

$$= (\mathcal{D} \delta_Y h)_i + \kappa \sum_{k,l} \left( \sum_{j=1}^{3} \left\{ - h_{jij} \epsilon_{ikl} g_{kl} + \epsilon_{ikl} h_{kl} - \epsilon_{ikl} h_{lk} + 3\epsilon_{ikl} h_{lk} \right\} \right).$$

Since all terms in the sum consist of a term skew-symmetric in $k$ and $l$ times a term symmetric in $k$ and $l$, the sum is zero, so we obtain

(4.3) 

$$\sum_{j=1}^{3} \sum_{k,l} \epsilon_{ikl} \nabla_j \nabla_k h_{lj} = (\mathcal{D} \delta_Y h)_i.$$ 

We also have

$$\sum_{j=1}^{3} \sum_{p,q} \epsilon_{jpq} \nabla_j \nabla_p h_{qi} = \sum_{j=1}^{3} \sum_{p,q} \epsilon_{jpq} \left( \nabla_p \nabla_j h_{qi} - R^s_{jpqi} h_{si} - R^s_{jpqi} h_{qs} \right)$$

$$= \sum_{j=1}^{3} \sum_{p,q} \epsilon_{jpq} \left( \nabla_p \nabla_j h_{qi} + \kappa \left( - h_{ijq} g_{pq} + h_{pqg_{qj}} + h_{qjg_{pi}} - h_{pqg_{ji}} \right) \right),$$
and clearly the last 4 terms sum to zero. So we have
\[
\sum_{j=1}^{3} \sum_{p,q} \epsilon_{jpq} \nabla_j \nabla_p h_{qi} = \sum_{j=1}^{3} \sum_{p,q} \epsilon_{jpq} \nabla_p \nabla_j h_{qi}
\]
By reindexing \(j\) and \(p\) on the right hand side, we obtain
\[
\sum_{j=1}^{3} \sum_{p,q} \epsilon_{jpq} \nabla_j \nabla_p h_{qi} = \sum_{j=1}^{3} \sum_{p,q} \epsilon_{pjq} \nabla_p \nabla_j h_{qi} = - \sum_{j=1}^{3} \sum_{p,q} \epsilon_{jpq} \nabla_j \nabla_p h_{qi},
\]
so this sum vanishes. Combining this with (4.2) and (4.3), the proposition then follows. □

**Corollary 4.3.** For any \(h \in S^2(T^*Y)\) we have
\[
\delta \Delta_{g_Y} h = \Delta_{g_Y} \delta h.
\]

**Proof.** From (4.1) we have
\[
\delta^3 h = -4\delta \Delta_{g_Y} tf h - 2\delta^2 \nabla^2 tr_Y(h) + 3\delta \mathcal{K}_{g_Y} (\delta_Y h) + 12\kappa \cdot \delta(\delta_Y h),
\]
and clearly
\[
-4\delta \Delta_{g_Y} tf h = -4\delta \Delta_{g_Y} h,
\]
\[
\delta^2 \nabla^2 tr_Y(h) = \frac{1}{2} \delta \mathcal{K}_{g_Y} (dtr_{g_Y} h) = 0,
\]
\[
3\delta \mathcal{K}_{g_Y} (\delta_Y h) = 3\mathcal{K}_{g_Y} (d \delta_Y h) = 3\mathcal{K}_{g_Y} (\delta_Y \delta h),
\]
\[
\delta(\delta_Y h) = \delta h.
\]
On the other hand we have
\[
\delta^3 h = \delta^2 \delta h = -4\Delta_{g_Y} \delta h - 2\nabla^2 tr_Y(\delta h) + 3\mathcal{K}_{g_Y} (\delta_Y \delta h) + 12\kappa \cdot \delta h
\]
\[
= -4\Delta_{g_Y} \delta h + 3\mathcal{K}_{g_Y} (\delta_Y \delta h) + 12\kappa \cdot \delta h,
\]
and this proves the claim. □

For the next lemma we will use \(\Delta_H\) to denote the Hodge-Laplacian on \(\Lambda^1(T^*Y)\) which is given by
\[
\Delta_H \omega = -d \delta_Y \omega - \delta_Y d \omega
\]
\[
= d \bar{\delta} \bar{d} \omega - \bar{d} \delta \omega,
\]
which is related to the rough Laplacian on 1-forms by the Weitzenböck formula
\[
\Delta_{g_Y} = -\Delta_H + 2\kappa.
\]

**Lemma 4.4.** The operator \(\mathcal{K}_{g_Y}(\omega)\) satisfies
\[
\delta_Y \mathcal{K}_{g_Y}(\omega) = \Delta_{g_Y} \omega + \frac{1}{3} \cdot d \delta_Y \omega + 2\kappa \omega
\]
\[
= -\Delta_H \omega + \frac{1}{3} \cdot d \delta_Y \omega + 4\kappa \omega,
\]
and also
\[
\Delta_{gy} \mathcal{K}_{gy} (\omega) = \mathcal{K}_{gy} \left( (\Delta_{gy} + 4\kappa) \omega \right) \\
= \mathcal{K}_{gy} \left( -\Delta_H + 6\kappa \right) \omega.
\]

Proof. Both identities follow from straightforward computations, see for example [Str10 Appendix]. □

Corollary 4.5. If \( h \) is a divergence-free eigentensor of \( \Delta_{gy} \) with eigenvalue \( -\lambda \) in \( S^2_0(T^*Y) \) which satisfies \( \Delta_{gy}(h) = -\lambda \cdot h \) then \( \varphi h = \pm 2\sqrt{\lambda + 3\kappa} \cdot h \) and if \( h \) has the form \( h = \mathcal{K}_{gy}(\omega) \) with \( \Delta_H \omega = \nu \omega \) and \( \delta_Y \omega = 0 \) then \( \varphi h = \pm \sqrt{\nu} \cdot h \). Both signs occur if \((Y^3, g_Y)\) admits an orientation-reversing isometry (which is always true for \( S^3 \) or \( \kappa = 0 \)).

Proof. Note that in either of the above cases we must have \( \varphi^2 h = c^2 \cdot h \) for some constant \( c \). To see this, if \( \Delta_{gy} h = -\lambda h \) and \( \delta_Y h = 0 \) with \( \lambda > 0 \) then by Proposition 4.1 we have
\[
\varphi^2 h = -4 \Delta_{gy} h + 12\kappa \cdot h = (4\lambda + 12\kappa) \cdot h.
\]
In this case \( c^2 = (4\lambda + 12\kappa) \). For the second case, from Lemma 4.4 we have the identities
\[
\Delta_{gy} h = (6\kappa - \nu) \cdot \mathcal{K}_{gy}(\omega) = (6\kappa - \nu) \cdot h,
\]
and
\[
\delta_{gy} \mathcal{K}_{gy}(\omega) = (4\kappa - \nu) \cdot \mathcal{K}_{gy}(\omega) = (4\kappa - \nu) \cdot h.
\]
Since we also have
\[
\text{tr}_Y(h) = 0,
\]
we easily obtain from Proposition 4.1
\[
\varphi^2 h = \nu \cdot h,
\]
so in this case \( c^2 = \nu \).

In both cases, observe that if we fix an eigenvalue \( \nu \) of \( \Delta_{gy} \) on \( S^2_0(T^*Y) \) then the eigenspace \( E_{\nu} = \{ h \in S^2_0(T^*Y) : \Delta_{gy} h = -\nu \cdot h \} \) is not necessarily \( SO(3) \)-irreducible, and decomposes into \( E_{\nu} = A^+_c \oplus A^-_c \), where \( A^+_c = \{ h \in S^2_0(T^*Y) : \varphi h = \pm c \cdot h \} \) and are \( SO(3) \)-invariant. To see this, given any eigentensor, writing the equation \( \varphi^2 h = c^2 \cdot h \) as
\[
(\varphi + c \cdot I) (\varphi - c \cdot I) h = 0,
\]
we conclude that either \(+c\) or \(-c\) occurs as an eigenvalue of \( \varphi \). If \( Y^3 \) admits an orientation-reversing isometry, then since the operator \( \varphi \) changes sign under reversal of orientation, pulling an eigentensor back along an orientation-reversing isometry shows that both \( A^+_c \) and \( A^-_c \) are nontrivial and of the same dimension. □
**Corollary 4.6.** If \( \omega \) is an eigenform of the Hodge Laplacian on 1-forms with eigenvalue \( \nu \) satisfying \( \delta_Y \omega = 0 \) then \( \tilde{\ast} d\omega = \pm \sqrt{\nu} \cdot \omega \). Both signs occur if \((Y^3, g_Y)\) admits an orientation-reversing isometry (which is always true for \( S^3 \) or \( \kappa = 0 \)).

**Proof.** Obviously, since \( \delta_Y \omega = 0 \), then

\[
(\tilde{\ast} d)^2 = \tilde{\ast} d \tilde{\ast} d = -\delta_Y d\omega = \Delta_H \omega = \nu \cdot \omega.
\]

Using a similar argument as in Corollary 4.5 we conclude that \( \tilde{\ast} d\omega = \pm \sqrt{\nu} \cdot \omega \), with both signs occurring on \( S^3 \). \( \square \)

**5. The Adjoint of \( D \)**

The adjoint operator will map from

\[
D^* : S_0^2(\Lambda_2^\perp) \to S_0^2(T^*M),
\]

and using the decompositions in Subsection 2.1 we will think of this as

\[
D^* : S_0^2(T^*Y) \to S^2(\nu) \oplus (\nu \odot T^*(Y)) \oplus S^2(T^*Y).
\]

**Proposition 5.1.** The adjoint operator is given by

\[
D^* Z = \left\{ -\frac{1}{2} \delta_Y^2 Z, \frac{1}{2} \delta_Y \dot{Z} + \frac{1}{2} \delta_H \tilde{\ast} \delta_Y Z, -\frac{1}{2} \dot{Z} - \kappa Z - \frac{1}{2} d\dot{Z} + \frac{1}{2} \Delta_{g_Y} Z \right\}
\]

\[
-\frac{1}{2} \mathcal{L}_{g_Y}(\delta_Y Z) + \frac{1}{2} (\delta_Y^2 Z) g_Y \right\}. \tag{5.3}
\]

Where \( \delta_H \) is the Hodge divergence on forms given by \( \delta_H = d^* \).

**Proof.** Let \( Z \in S_0^2(T^*Y) \), from the decomposition (2.9) we can see \( S_0^2(T^*Y) \) as embedded in \( S_2(T^*M) \), so taking \langle , \rangle to be the inner product induced by the cylindrical metric \( g_Y \) on \( S_2(T^*M) \) we observe that since \( Z \) is traceless with respect to \( g_Y \) we have for any \( \hat{h} = h_{00} dt \otimes dt + dt \odot \alpha + h \),

\[
\langle D \hat{h}, Z \rangle = \frac{1}{2} \mathcal{L}_{g_Y} \left( -\frac{1}{2} dh_{00} - \delta_Y h + \dot{\alpha} - \tilde{\ast} d\alpha + \frac{1}{2} dtr_Y(h) \right), Z \rangle
\]

\[
+ \left\langle -\frac{1}{2} \hat{h} - \kappa h + \frac{1}{2} \mathcal{L}_{g_Y}(\delta_Y Z), Z \right\rangle + \left\langle \frac{1}{2} \Delta_{g_Y} h, Z \right\rangle \tag{5.4}
\]

Formal integration by parts then yields

\[
\int_0^\infty \int_Y \langle D \hat{h}, Z \rangle dt dV_{g_Y} = -\frac{1}{2} \int_0^\infty \left( \langle h_{00} \cdot \delta_Y^2 Z \rangle + \langle \alpha, \delta_Y \dot{Z} + \delta_H \tilde{\ast} \delta_Y Z \rangle \right) dt dV_{g_Y}
\]

\[
+ \int_0^\infty \int_Y \langle h, \frac{1}{2} (\delta_Y^2 Z) g_Y \rangle dt dV_{g_Y}
\]

\[
- \int_0^\infty \int_Y \left\langle h, \frac{1}{2} \mathcal{L}_{g_Y}(\delta_Y Z) - \frac{1}{2} \dot{Z} - \kappa Z \right\rangle + \frac{1}{2} \langle \mathcal{L}_{g_Y}(\tilde{\ast} \delta_Y Z), Z \rangle \right\rangle dt dV_{g_Y}.
\]

Note that by the inner product

\[
\langle \alpha, \delta_Y \dot{Z} + \delta_H \tilde{\ast} \delta_Y Z \rangle,
\]

\[
\Delta_{g_Y} h, Z \right\rangle
\]

\[
\mathcal{L}_{g_Y}(\delta_Y Z) - \frac{1}{2} \dot{Z} - \kappa Z \right\rangle + \frac{1}{2} \langle \mathcal{L}_{g_Y}(\tilde{\ast} \delta_Y Z), Z \rangle \right\rangle dt dV_{g_Y}.
\]
we mean the usual inner product on 1-forms, however, using the decomposition in (5.2), we identify a 1-form $\xi$ with the tensor
\[
\{0, \xi, 0\} = \xi \otimes dt + dt \otimes \xi,
\]
so we obtain
\[
\langle \alpha, \delta Y \dot{Z} + \delta H^* \delta Y Z \rangle = \frac{1}{2} \langle \{0, \alpha, 0\}, \{0, \delta Y \dot{Z} + \delta H^* \delta Y Z, 0\} \rangle.
\]
Finally, the proposition follows using that $d$ is formally self-adjoint. $\square$

**Proposition 5.2.** We have the decompositions
\[
S^2_0(T^*Y) = \text{Ker}(\delta Y) \oplus \text{Im}(K_{gy}),
\]
and
\[
\Lambda^1(T^*Y) = \text{Im}(d) \oplus \text{Ker}(d^*).
\]

**Proof.** Since $\delta Y$ is the formal adjoint of $-\frac{1}{2} K_{gy}$, (5.5) follows from standard Fredholm theory. The Hodge decomposition theorem says that
\[
\Lambda^1(T^*Y) = \mathcal{H}^1(T^*Y) \oplus (d\Lambda^0(T^*Y)) \oplus (d^* \Lambda^2(T^*Y))
\]
where $\mathcal{H}^1(T^*Y)$ is the space of harmonic 1-forms in $\Lambda^1(T^*Y)$, and (5.6) follows easily from this since $\mathcal{H}^1(T^*Y)$ and $d^* \Lambda^2(T^*Y)$ are both contained in $\text{Ker}(d^*)$. $\square$

Using this decomposition we obtain:

**Corollary 5.3.** Any time dependent $Z \in S^2_0(T^*Y)$ can be written uniquely as an infinite linear combination of elements of three types, namely

1. **Elements of type I:**
\[
f(t) \cdot K_{gy}(d\phi),
\]
where $\phi$ is an eigenfunction of $\Delta_H$ on $\Lambda^0(T^*Y)$,

2. **Elements of type II:**
\[
f(t) \cdot K_{gy} (\omega),
\]
where $\omega$ is an eigenform of $\Delta_H$ on $\Lambda^1(T^*Y)$ satisfying $\delta Y \omega = 0$,

3. **Elements of type III:**
\[
f(t) \cdot B,
\]
where $B$ is an eigentensor of $\Delta_{gy}$ on $S^2_0(T^*Y)$ satisfying $\delta Y B = 0$.

In all of the three above cases $f(t)$ denotes a real-valued function.

From Propositions 4.2, 4.4, and 4.5 we observe that the image of $D^*$ on an element of type I has the form
\[
D^* (f(t) \cdot K_{gy}(d\phi)) = a_1(t) \cdot \phi dt \otimes dt + a_2(t) \cdot d\phi \otimes dt + a_3(t) \cdot K_{gy}(d\phi) + a_4(t) \cdot g_Y,
\]
where each coefficient \( a_i \) depends on \( f \) and the eigenvalue of \( \Delta_H \) corresponding to \( \phi \). On elements of type II the image of \( \mathcal{D}^* \) is

\[
\mathcal{D}^* (f(t) \cdot \mathcal{K}_{gy} (\omega)) = b_1(t) \cdot \omega \otimes dt + b_2(t) \cdot \mathcal{K}_{gy} (\omega),
\]

where each \( b_i \) depends on \( f \) and the eigenvalue of \( \Delta_H \) on divergence-free 1-forms corresponding to \( \eta \). Finally, on elements of type III we have

\[
\mathcal{D}^* (f(t) \cdot B) = \tilde{f}(t) \cdot B,
\]

where \( \tilde{f} \) is determined by \( f \) and the eigenvalue of \( \Delta_{gy} \) corresponding to \( B \). In a similar way to Corollary \[5.3\] one can prove that all elements in \( S^2_0 (T^* M) \) can be written uniquely as an infinite sum of elements as in the right hand sides of (5.11), (5.12) and (5.13), so it follows that in order to find the general solution of \( \mathcal{D}^* Z = 0 \) it suffices to consider solutions \( Z \) of types I, II and III separately. For example, if \( Z \) has the form (5.8) then writing \( \mathcal{D}^* Z \) as in (5.11) one sees that in order to obtain \( \mathcal{D}^* Z = 0 \) one must solve for \( f \) in (5.8) so that in (5.12) one has \( a_1 = a_2 = a_3 = a_4 = 0 \) and in general this amounts to solving an ordinary differential equation on \( f \). We start by considering solutions of type III. For that purpose we use the following:

**Lemma 5.4.** If \( \lambda \) is an eigenvalue of \( -\Delta_{gy} \) on divergence-free sections of \( S^2_0 (T^* Y) \), then

(a) if \( \kappa = 1 \), \( \lambda \geq 6 \),

(b) if \( \kappa = -1 \), \( \lambda \geq 3 \) with equality achieved only for nontrivial Codazzi tensors \( h \in S^2_0 (T^* Y) \), that is, \( \nabla^2 h = 0 \).

(c) if \( \kappa = 0 \), \( \lambda \geq 0 \) with equality for parallel sections in \( S^2_0 (T^* Y) \).

**Proof.** These are due to Koiso, we only give a brief argument \[Koi78\]. For (a), the inequality

\[
\int_{S^3} |\nabla_i h_{jk} + \nabla_j h_{ki} + \nabla_k h_{ij}|^2 dV \geq 0,
\]

easily implies that \( \lambda \geq 6 \). For (b), the inequality

\[
\int_Y |\nabla_i h_{jk} - \nabla_j h_{ik}|^2 dV \geq 0,
\]

implies that \( \lambda \geq 3 \), with equality exactly for Codazzi tensors. Finally, the \( \kappa = 0 \) case is trivial. \( \square \)

The classification of type III solutions is given by the following.

**Proposition 5.5.** Let \( 0 \leq \lambda_1 < \lambda_2 < \ldots \) be the eigenvalues of \( -\Delta_{gy} \) on divergence-free tensors in \( S^2_0 (T^* Y) \) and let \( \beta_j = \sqrt{\lambda_j + 3 \kappa} \). For each eigenvalue \( \lambda_j \) there exist trace-free and divergence-free eigentensors \( B_j^\pm \) and \( C_j^\pm \) satisfying

\[
\mathcal{D} B_j^\pm = \pm \beta_j B_j^\pm, \quad \mathcal{D} C_j^\pm = \pm \beta_j C_j^\pm,
\]

\[
\phi B_j^\pm = \pm \beta_j B_j^\pm, \quad \phi C_j^\pm = \pm \beta_j C_j^\pm,
\]

where each coefficient \( a_i \) depends on \( f \) and the eigenvalue of \( \Delta_H \) corresponding to \( \phi \). On elements of type II the image of \( \mathcal{D}^* \) is

\[
\mathcal{D}^* (f(t) \cdot \mathcal{K}_{gy} (\omega)) = b_1(t) \cdot \omega \otimes dt + b_2(t) \cdot \mathcal{K}_{gy} (\omega),
\]

where each \( b_i \) depends on \( f \) and the eigenvalue of \( \Delta_H \) on divergence-free 1-forms corresponding to \( \eta \). Finally, on elements of type III we have

\[
\mathcal{D}^* (f(t) \cdot B) = \tilde{f}(t) \cdot B,
\]

where \( \tilde{f} \) is determined by \( f \) and the eigenvalue of \( \Delta_{gy} \) corresponding to \( B \). In a similar way to Corollary \[5.3\] one can prove that all elements in \( S^2_0 (T^* M) \) can be written uniquely as an infinite sum of elements as in the right hand sides of (5.11), (5.12) and (5.13), so it follows that in order to find the general solution of \( \mathcal{D}^* Z = 0 \) it suffices to consider solutions \( Z \) of types I, II and III separately. For example, if \( Z \) has the form (5.8) then writing \( \mathcal{D}^* Z \) as in (5.11) one sees that in order to obtain \( \mathcal{D}^* Z = 0 \) one must solve for \( f \) in (5.8) so that in (5.12) one has \( a_1 = a_2 = a_3 = a_4 = 0 \) and in general this amounts to solving an ordinary differential equation on \( f \). We start by considering solutions of type III. For that purpose we use the following:

**Lemma 5.4.** If \( \lambda \) is an eigenvalue of \( -\Delta_{gy} \) on divergence-free sections of \( S^2_0 (T^* Y) \), then

(a) if \( \kappa = 1 \), \( \lambda \geq 6 \),

(b) if \( \kappa = -1 \), \( \lambda \geq 3 \) with equality achieved only for nontrivial Codazzi tensors \( h \in S^2_0 (T^* Y) \), that is, \( \nabla^2 h = 0 \).

(c) if \( \kappa = 0 \), \( \lambda \geq 0 \) with equality for parallel sections in \( S^2_0 (T^* Y) \).

**Proof.** These are due to Koiso, we only give a brief argument \[Koi78\]. For (a), the inequality

\[
\int_{S^3} |\nabla_i h_{jk} + \nabla_j h_{ki} + \nabla_k h_{ij}|^2 dV \geq 0,
\]

easily implies that \( \lambda \geq 6 \). For (b), the inequality

\[
\int_Y |\nabla_i h_{jk} - \nabla_j h_{ik}|^2 dV \geq 0,
\]

implies that \( \lambda \geq 3 \), with equality exactly for Codazzi tensors. Finally, the \( \kappa = 0 \) case is trivial. \( \square \)

The classification of type III solutions is given by the following.

**Proposition 5.5.** Let \( 0 \leq \lambda_1 < \lambda_2 < \ldots \) be the eigenvalues of \( -\Delta_{gy} \) on divergence-free tensors in \( S^2_0 (T^* Y) \) and let \( \beta_j = \sqrt{\lambda_j + 3 \kappa} \). For each eigenvalue \( \lambda_j \) there exist trace-free and divergence-free eigentensors \( B_j^\pm \) and \( C_j^\pm \) satisfying

\[
\mathcal{D} B_j^\pm = \pm \beta_j B_j^\pm, \quad \mathcal{D} C_j^\pm = \pm \beta_j C_j^\pm,
\]
such that the general solution of $D^*Z = 0$ with $Z$ satisfying
\begin{align}
\delta_Y Z &= 0, \\
\text{tr}_Y Z &= 0,
\end{align}

can be written in in the following way:

(a) If $\kappa = 1$ then
\[
Z = \sum_{j=1}^{\infty} \left( e^{(\beta_j + 1)t} B_j^+ + e^{(\beta_j - 1)t} C_j^+ + e^{(-\beta_j + 1)t} B_j^- + e^{(-\beta_j - 1)t} C_j^- \right).
\]

Letting $\alpha_j^\pm = \beta_j \pm 1$, we have $0 < |\alpha_1^\pm| < |\alpha_2^\pm| < \ldots$, and $|\alpha_1^\pm| = 2$.

(b) If $\kappa = -1$
\[
Z = \sum_{j=1}^{\infty} \left\{ e^{\beta_j t} \left( B_j^+ \cos(t) + C_j^+ \sin(t) \right) + e^{-\beta_j t} \left( B_j^- \cos(t) + C_j^- \sin(t) \right) \right\},
\]

with $\beta_1 = 0$ and where $B_1^\pm$ and $C_1^\pm$ are trace-free Codazzi tensors.

(c) If $\kappa = 0$
\[
Z = B_1 + t C_1 + \sum_{j=2}^{\infty} \left( e^{\beta_j t} B_j^+ + t e^{\beta_j t} C_j^+ + e^{-\beta_j t} B_j^- + t e^{-\beta_j t} C_j^- \right),
\]

where $B_1$ and $C_1$ are parallel sections of $S^2_0(T^*Y)$.

Proof. Let $Z$ be a solution of $D^*Z = 0$ of type III, that is, $Z = fB$ with $B$ satisfying
\begin{align}
(5.17) \quad &\delta_Y Z = 0, \\
(5.18) \quad &\text{tr}_Y Z = 0,
\end{align}

and $\Delta_{g_Y} B = -\lambda \cdot B$, then $f$ and $B$ satisfy the equation
\[
0 = D^*(fB) = \left\{ 0,0,-\frac{1}{2} \dot{f}B - \frac{1}{2} \dot{f} \delta B - \left( \kappa + \frac{\lambda}{2} \right) fB \right\},
\]

from (4.35) we have
\[
(5.19) \quad \delta \dot{Z} = \delta \left( \dot{f} \cdot B \right) = \pm \dot{f} \left( 2\sqrt{\lambda + 3\kappa} \cdot B \right).
\]

It follows that $f$ is a solution of the ordinary differential equation
\[
(5.20) \quad -\frac{1}{2} \dot{f} \pm \sqrt{\lambda + 3\kappa} \cdot \dot{f} - \left( \kappa + \frac{\lambda}{2} \right) f = 0.
\]

Letting $\beta = \sqrt{\lambda + 3\kappa}$, then the characteristic roots of (5.20) are
\[
\pm \beta \pm \sqrt{\kappa}.
\]

The expansions follow from considering the different solutions obtained for $\kappa \in \{1,-1,0\}$, and Lemma 5.4. \qed

We now turn to solutions of $D^*(Z) = 0$ with $Z$ of types I and II. We will need to introduce the operator
\[
\Box_K : \Lambda^1(T^*Y) \mapsto \Lambda^1(T^*Y),
\]
given by
\[ \Box_{K} \eta = \delta_{Y} K_{g_Y}(\eta). \]

We have the following

**Lemma 5.6.** Let \( \eta \in \Lambda^{1}(T^{*}Y) \), then
\begin{equation}
\Box_{K} \eta = (\delta_{Y} d + \frac{4}{3} d \delta_{Y} + 4 \kappa) \eta. \tag{5.21}
\end{equation}

Also, if \( \eta \) is an eigenform of \( \Delta_{H} \) on 1-forms then
\[ \Box_{K} \eta = c \cdot \eta, \]
where \( c = (-\frac{4}{3} \mu + 4 \kappa) \) if \( \eta = d \phi \) and \( \Delta_{H} \phi = \mu \phi \) or \( c = (4 \kappa - \nu) \) if \( \Delta_{H} \eta = \nu \eta \) and \( \delta_{Y} \eta = 0 \). Moreover, in either case the constant \( c \) is nonzero unless \( Z = 0 \).

**Proof.** The expression (5.21) for \( K_{g_Y} \) is a direct consequence of Lemma 4.4. Suppose now that \( \Delta_{H} \eta = \lambda \cdot \eta \). If \( \eta = d \phi \) with \( \Delta_{H} \phi = \nu \phi \), then observe that
\[ \Box_{K} \eta = \left( \delta_{Y} d + \frac{4}{3} d \delta_{Y} + 4 \kappa \right) \eta = \left( \frac{4}{3} d \delta + 4 \kappa \right) \eta. \]

If \( \delta_{Y} \eta = 0 \), then
\[ \Box_{K} \eta = \left( \delta_{Y} d + \frac{4}{3} d \delta_{Y} + 4 \kappa \right) \eta = (\delta_{Y} d + 4 \kappa) \eta = -\Delta_{H} \eta + 4 \kappa \eta = (-\nu + 4 \kappa) \eta. \]

Finally, in order to show that \( c = 0 \) does not occur we note that when \( \kappa = 1 \), there are eigenforms of \( \Delta_{H} \) corresponding to the eigenvalue \( \mu = 3 \) on closed forms and to \( \nu = 4 \) on co-closed forms and in these cases \( c = 0 \). However, for any of these eigenvalues, the corresponding eigenforms are conformally Killing. In the hyperbolic case \( \kappa = -1 \), the constant \( c \) is strictly negative for either closed or co-closed eigenforms of the Hodge Laplacian. In the flat case \( \kappa = 0 \), the constant \( c \) equals zero only for parallel forms, but in this case \( Z = 0 \).

Next, assume that \( Z \) is a non-trivial solution of \( \mathcal{D}^{*}Z = 0 \) with \( Z \) of type I or II and \( c \neq 0 \), where \( c \) is the constant in Lemma 5.6. The first component of (5.3) yields
\begin{equation}
0 = \delta_{Y}(\delta_{Y} Z) = f(t) \cdot \delta_{Y} \Box_{K} \eta = f(t) \cdot \delta_{Y}(c \eta). \tag{5.22}
\end{equation}

Since \( Z \) is non-trivial and \( c \neq 0 \), we conclude that \( \delta_{Y} \omega = 0 \) and hence, solutions of type I do not occur. Furthermore, we can prove

**Proposition 5.7.** We have
\begin{equation}
\dot{f} * d \omega = -\nu f \omega. \tag{5.23}
\end{equation}
Proof. The second component of (5.3) yields
\[ 0 = \dot{f} \cdot \delta_Y K_{gy}(\omega) - f \delta_Y \tilde{*} \delta_Y K_{gy}(\omega), \]
which by Lemma 4.4 is equivalent to
\[ \dot{f} (-\nu + 4 \kappa) \omega + f \delta_H \tilde{*} (-\nu + 4 \kappa) \omega = 0 \]
and writing \( \delta_H \) on 2-forms as \( \tilde{*} d \tilde{*} \) we obtain
\[ \dot{f} \omega = -f \tilde{*} d \omega, \]
and after taking \( \tilde{*} d \) on both sides we conclude that
\[ \dot{f} \tilde{*} d \omega = -f \tilde{*} d \tilde{*} d \omega = -f d^* d \omega = -\nu f \omega, \]
as needed. \( \square \)

Proposition 5.8. Let \( Z = f(t) \cdot K_{gy} \omega \) satisfy \( D^* Z = 0 \), where \( \omega \) satisfies \( \delta_Y \omega = 0 \) and
\[ \Delta_H \omega = \nu \cdot \omega. \]
Then
\[ f(t) = e^{\alpha t}, \]
with
\[ \alpha = \pm (\sqrt{\nu}), \]
or
\[ f(t) = c_0, \]
for a constant \( c_0 \), if \( \omega \) is a non-trivial harmonic 1-form.

Proof. From (5.23) we have
\[ -\frac{1}{2} d \dot{Z} = -\frac{1}{2} f d K_{gy}(\omega) = -\frac{1}{2} f K_{gy}(\tilde{*} d \omega) = \frac{\nu}{2} f K_{gy}(\omega), \]
we also have from Lemma 4.4
\[ \delta_Y Z = f(t) \cdot \delta_Y K_{gy}(\omega) \]
\[ = f(t) \cdot (-\nu + 4 \kappa) \cdot \omega, \]
\[ \Delta_{gy} Z = f(t) \cdot \Delta_{gy} K_{gy}(\omega) \]
\[ = f(t) \cdot (6 \kappa - \nu) \cdot K_{gy}(\omega), \]
\[ \delta_Y^2 Z = f(t) \cdot \delta_Y^2 K_{gy}(\omega) \]
\[ = f(t) \cdot (-\nu + 4 \kappa) \cdot \delta_Y \omega = 0. \]
The equation on the purely spherical component of \( D^* (Z) \) is
\[ 0 = -\frac{1}{2} \ddot{Z} - \kappa Z - \frac{1}{2} d \dot{Z} + \frac{1}{2} \Delta_{gy} Z - \frac{1}{2} \mathcal{L}_{gy}(\delta_Y Z) + \frac{1}{2} (\delta_Y^2 Z) g_Y, \]
which by (5.33), (5.34), (5.35) and (5.32) simplifies to

$$0 = \left( \frac{1}{2} \ddot{f} - \kappa f + \frac{1}{2} (6 \kappa - \nu) f - \frac{1}{2} (4 \kappa - \nu) \right) \mathcal{K}_{g_Y}(\omega) - \frac{1}{2} \dot{f} \mathcal{K}_{g_Y}(\tilde{\omega} d\omega)$$

(5.37)

$$\frac{1}{2} \ddot{f} \mathcal{K}_{g_Y}(\omega) + \frac{1}{2} \nu f \mathcal{K}_{g_Y}(\omega),$$

which we write as

(5.38)

$$\left( \ddot{f} - \nu \right) \mathcal{K}_{g_Y}(\omega) = 0.$$

and for 1-forms $\omega$ that are not dual to Killing fields we obtain solutions

(5.39)

$$f(t) = e^{\pm \sqrt{\nu} t}.$$  

The solutions with $f$ as in (5.31) correspond to $\nu = 0$ which is the case of a harmonic 1-form. In this case, the tensor $Z = t \cdot \mathcal{K}_{g_Y}(\omega)$ is ruled out by (5.26) above, and only the solution $Z = c_0 \cdot \mathcal{K}_{g_Y}(\omega)$ occurs. □

Let $0 = \nu_0 < \nu_1 < \ldots$, be all the eigenvalues of $\Delta_H$ on co-closed forms in $\Lambda^1(T^*Y)$. Note that there are non-trivial eigenforms corresponding to $\nu_0$ if and only if $b_1(T^*Y) \neq 0$ since such eigenforms are harmonic 1-forms. In particular, for $\kappa = 1$ there are no nontrivial 1-forms in $\Lambda^1(T^*Y)$. We close this section with the following.

**Proposition 5.9.** Let $Z \in S_0^2(T^*Y)$ be a solution of $D^*Z = 0$. Then $Z$ can be written as

$$Z = \mathcal{K}_{g_Y}(\omega) + Z_0,$$

where $Z_0$ is divergence-free and has an expansion as in Proposition 5.5. Also for each eigenvalue $\nu_j$, $j = 0, 1, \ldots$, there are eigenforms $\omega_j^\pm$ such that $\mathcal{K}_{g_Y}(\omega)$ can be written uniquely as an infinite sum of the following form

(a) If $\kappa = 1$ then

$$\mathcal{K}_{g_Y}(\omega) = \sum_{j=2}^{\infty} (e^{\sqrt{\nu_j} t} \mathcal{K}_{g_Y}(\omega_j^+) + e^{-\sqrt{\nu_j} t} \mathcal{K}_{g_Y}(\omega_j^-)),$$

where $c_j^\pm$ are constants. In the case $Y = S^3$, $\nu_j = (j + 1)^2$.

(b) If $\kappa = -1$ then

$$\mathcal{K}_{g_Y}(\omega) = \mathcal{K}_{g_Y}(\omega_0) + \sum_{j=1}^{\infty} \left( e^{\sqrt{\nu_j} t} \mathcal{K}_{g_Y}(\omega_j^+) + e^{-\sqrt{\nu_j} t} \mathcal{K}_{g_Y}(\omega_j^-) \right),$$

where $\omega_0$ is a harmonic 1-form.

(c) If $\kappa = 0$ then

$$\mathcal{K}_{g_Y}(\omega) = \sum_{j=1}^{\infty} (e^{\sqrt{\nu_j} t} \mathcal{K}_{g_Y}(\omega_j^+) + e^{-\sqrt{\nu_j} t} \mathcal{K}_{g_Y}(\omega_j^-)).$$
Proof. From Proposition 5.8 we can write the 1-form $\omega$ as an infinite sum of the form

$$\omega = \omega_0 + \sum_{j=1}^{\infty} \left( e^{\sqrt{\nu_j} t} \omega_j^+ + c_j^- e^{-\sqrt{\nu_j} t} \omega_j^- \right).$$

\[ \text{(5.40)} \]

In case $\kappa = 1$, there are no harmonic 1-forms, and all eigenforms corresponding to $\nu_1 = 4$ are dual to Killing fields, so the sum starts at $j = 2$ in this case. The form of the eigenvalues $\nu_j$ in the case $\kappa = 1$ follows from \[\text{[Fol89]}\]. The $\kappa = -1$ case follows directly from (5.40). In the case $\kappa = 0$, any harmonic 1-form is parallel. \[ \square \]

6. Mixed solutions

Returning to the full system

$$D^*Z = \mathcal{K}_g(\check{\omega}),$$

we note that since $D^*Z$ is divergence-free, the 1-form $\check{\omega}$ automatically satisfies the equation

$$\delta_g \mathcal{K}_g(\check{\omega}) \equiv \square_{\mathcal{K}_g}(\check{\omega}) = 0,$$

so we next analyze solutions of (6.2) at a cylindrical metric $g = dt^2 + g_Y$. The conformal Killing operator on a 1-form $\check{\omega} = f dt + \omega$ is

$$\mathcal{K}_g(\check{\omega}) = \left( \frac{3}{2} \dot{f} - \frac{1}{2} \delta_Y \omega \right) dt \otimes dt + (\check{\omega} + df) \otimes dt + \mathcal{L}_{g_Y}(\omega) - \frac{1}{2} \left( \dot{f} + \delta_Y \omega \right) g_Y.$$

The divergence of a traceless symmetric 2-tensor

$$\check{h} = h_{00} dt \otimes dt + (\alpha \otimes dt + dt \otimes \alpha) + h$$

is given by

$$\delta \check{h} = (\check{h}_{00} + \delta_{S3} \alpha) dt + \dot{\alpha} + \delta_{S3} h.$$

Combining (6.3) and (6.5), we obtain

$$\square_{\mathcal{K}_g}\check{\omega} = \left( \frac{3}{2} \check{f} + \frac{1}{2} \delta_Y \check{\omega} - \Delta_H f \right) dt + \check{\omega} + \delta_Y \mathcal{L}_{g_Y}(\omega) - \frac{1}{2} d \delta_Y \omega.$$

Commuting covariant derivatives as in Lemma 4.4, we have

$$\delta_Y \mathcal{L}_{g_Y}(\omega) = -\Delta_H \omega + d \delta_Y \omega + 4 \kappa \omega,$$

so $\square_{\mathcal{K}_g}\check{\omega}$ takes the form

$$\square_{\mathcal{K}_g}(\check{\omega}) = \left( \frac{3}{2} \check{f} + \frac{1}{2} \delta_Y \check{\omega} - \Delta_H f \right) dt + \check{\omega} - \Delta_H \omega + \frac{1}{2} d \delta_Y \omega + 4 \kappa \omega + \frac{1}{2} d \dot{f}.$$

Any 1-form $\check{\omega} \in \Lambda^1(T^*M)$ can be written as an infinite sum of 1-forms of two types, namely
(i) Forms of type (a)

\[ c(t)\phi dt + k(t)d\phi, \]

where \( \phi \) is an eigenfunction of the Hodge laplacian \( \tilde{\Delta}_H \) on \( \Lambda^0(T^*Y) \) with eigenvalue \( \mu \) and \( c = c(t), \ k = k(t) \) are functions of \( t \),

(ii) Forms of type (b)

\[ m(t)\eta, \]

where \( \eta \) is an eigenform of the Hodge Laplacian \( \Delta_H \) on \( \Lambda^1(T^*Y) \) satisfying \( \delta Y \eta = 0 \), and \( m = m(t) \).

Let us start by solving \( \Box_{K,\tilde{\omega}} = 0 \) assuming that \( \tilde{\omega} \) is of type (a). In this case, from (6.6) we conclude that the functions \( l \) and \( m \) satisfy the following system of ordinary differential equations

\[ \ddot{l} = \frac{2}{3}\mu l + \frac{\mu}{3}\dot{m}, \]
\[ \ddot{m} = -\frac{1}{2}l + \left(\frac{3}{2}\mu - 4\kappa\right)m. \]

If we let \( l_1 = l, \ l_2 = \dot{l}_1, \ m_1 = m \) and \( m_2 = \dot{m}_1 \), the system (6.8) is equivalent to the first order linear system

\[ \dot{X} = AX, \]

where \( X \) and \( A \) are given by

\[ X = \begin{pmatrix} l_1 \\ l_2 \\ m_1 \\ m_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3}\mu & 0 & 0 & \frac{\mu}{3} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \left(\frac{3}{2}\mu - 4\kappa\right) & 0 \end{pmatrix}. \]

The characteristic roots of the matrix \( A \) are \( \pm \alpha^{\pm}(\mu) \) where \( \alpha^{\pm}(\mu) \) is given by

\[ \alpha^{\pm} = \alpha^{\pm}(\mu) = \sqrt{\mu - 2\kappa \pm 2\sqrt{\kappa^2 - \frac{\mu^3}{3\kappa}}}, \]

We now consider solutions of \( \Box_{K,\tilde{\omega}} = 0 \) with \( \tilde{\omega} \) of type (b). If \( \tilde{\omega} \) is as in (6.7), the system \( \Box_{K,\tilde{\omega}} = 0 \) takes the form

\[ \ddot{m} - \nu m + 4\kappa m = 0, \]

and the characteristic roots of this equation are

\[ \pm \sqrt{\nu - 4\kappa}. \]

Let \( 0 = \mu_0 < \mu_1 < \ldots \) be all the eigenvalues of \( \Delta_H \) on \( \Lambda^0(T^*Y) \) and let \( \nu_j \) for \( j = 0, 1, \ldots \), denote all the eigenvalues of \( \Delta_H \) on co-closed forms in \( \Lambda^1(T^*Y) \). In particular if \( \kappa = 1 \) and \( \Gamma = \{ e \}, \mu_j = j(j+2) \). We have
Proposition 6.1. Let \((Z, \tilde{\omega})\) be a solution of (6.1). Then up to addition of 1-forms which are dual to conformal Killing fields, the 1-form \(\tilde{\omega}\) can be written as follows.

(a) If \(\kappa = 1\) and \(\Gamma = \{e\}\), \(\tilde{\omega}\) is an infinite sum of the form
\[
\sum_{j=2}^{\infty} e^{\pm \beta_j t} \left( \{ \phi_{1j}^\pm \cos(\gamma_j t) + \phi_{2j}^\pm \sin(\gamma_j t) \} dt + \{ c_{1j}^\pm \cos(\gamma_j t) d\phi_{1j}^\pm + c_{2j}^\pm \sin(\gamma_j t) d\phi_{2j}^\pm \} \right)
\]
\[
+ \sum_{j=2}^{\infty} \left( e^{\delta_j t} \omega_j^+ + e^{-\delta_j t} \omega_j^- \right),
\]
where \(\phi_{1j}^\pm\) and \(\phi_{2j}^\pm\) are eigenfunctions of \(\Delta_H\) corresponding to \(\mu_j = j(j + 2)\), and the coefficients \(c_{1j}^\pm\) and \(c_{2j}^\pm\) are constants, for \(j \geq 2\). The rates \(\beta_j\) satisfy \(\sqrt{6} < \beta_j\) for \(j \geq 2\) and are given by \(\beta_j = \text{Re}(\alpha_j^\pm), \gamma_j = \text{Im}(\alpha_j^\pm)\), where
\[
\alpha_j^\pm = \sqrt{j(j + 2) - 2 \pm \frac{2}{3} \sqrt{-1} \cdot \sqrt{3}(j - 1)(j + 3)}.
\]
Also, \(\omega_j^\pm\) are eigenforms corresponding to the eigenvalues \(\nu_j = (j + 1)^2\) of \(\Delta_H\) on co-closed forms, and \(\delta_j = \sqrt{\nu_j - 4}\).

If \(\Gamma \neq \{e\}\) then \(\pm \alpha_j^\pm\) or \(\pm \delta_j\) will occur as indicial roots if and only if the corresponding eigenfunction or eigenform descends to the quotient \(S^3/\Gamma\), respectively.

(b) If \(\kappa = -1\), then
\[
\tilde{\omega} = \sum_{j=1}^{\infty} \left( \{ e^{\pm \sigma_j^+ t} \phi_{1j}^\pm + e^{\pm \sigma_j^- t} \phi_{2j}^\pm \} dt + c_{1j}^\pm e^{\pm \sigma_j^+ t} d\phi_{1j}^\pm + c_{2j}^\pm e^{\pm \sigma_j^- t} d\phi_{2j}^\pm \right)
\]
\[
+ \sum_{j=0}^{\infty} \left( e^{\tau_j t} \omega_j^+ + e^{-\tau_j t} \omega_j^- \right),
\]
where \(\omega_j^\pm\) are harmonic 1-forms in \(\Lambda^1(T^*Y)\). The numbers \(\sigma_j^\pm\) for \(j \geq 1\) are real and are given by
\[
(6.11) \quad \sigma_j^\pm = \sqrt{\mu_j + 2 \pm 2 \sqrt{1 + \frac{\mu_j}{3}}},
\]
where \(\mu_j\) are the eigenvalues with respect to the hyperbolic metric. The numbers \(\tau_j\) are also real and are given by \(\tau_j = \sqrt{\nu_j + 4}\), where \(\nu_j\) are the eigenvalues with respect to the hyperbolic metric. The coefficients \(c_{1j}^\pm\) and \(c_{2j}^\pm\) for \(j \geq 1\) are constants.

(c) If \(\kappa = 0\),
\[
\tilde{\omega} = \sum_{j=1}^{\infty} e^{\pm t \sqrt{\mu_j}} \left( \{ \phi_{1j}^\pm + t \phi_{2j}^\pm \} dt + \{ c_{1j}^\pm d\phi_{1j}^\pm + t c_{2j}^\pm d\phi_{2j}^\pm \} \right) + \sum_{j=1}^{\infty} \left( e^{t \sqrt{\nu_j}} \omega_j^+ + e^{-t \sqrt{\nu_j}} \omega_j^- \right),
\]
where the notation is as above, but with eigenvalues \(\mu_j\) and \(\nu_j\) corresponding to the metric on \(T^3\).
Proof. For the case $\kappa = 1$, the only real roots in (6.9) correspond to the eigenvalues $\mu = 0, 3$ of $\Delta_H$ on $\Lambda^0(T^*Y)$, however, we see in either case that for the solution $\tilde{\omega}$ of (6.2) obtained, $\mathcal{K}_g(\tilde{\omega})$ is not in the image of $\mathcal{D}^*$. To clarify this observation, we note that for $\mu = 0$, $\mathcal{K}_g(\tilde{\omega})$ has the form

$$l(t)dt \otimes dt + k(t)g_Y,$$

i.e., $l$ and $k$ only depend on $t$ and for $\mu = 3$, $\mathcal{K}_g(\tilde{\omega})$ has the form

$$c_1(t)\phi dt \otimes dt + c_2(t)d\phi \otimes dt + c_3(t)\phi g_Y,$$

where $\phi$ is a spherical harmonic of order 1 (and hence $d\phi$ is conformally Killing with respect to the metric $g_Y$). From (5.11), (5.12) and (5.13), we conclude that elements of the form (6.12) or (6.13) in the image of $\mathcal{D}^*$ can only arise from evaluating $\mathcal{D}^*$ at an element of the form $f(t) \cdot \mathcal{K}_gY(d\psi)$, but in either case $\mathcal{K}_gY(d\psi) = 0$ and therefore $\tilde{\omega}$ must be conformally Killing with respect to the metric $g$. Similarly, for forms of type (b) in the case $\kappa = 1$, we see that the solution of (6.2) obtained for the least positive eigenvalue of $\Delta_H$ on co-closed forms in $\Lambda^1(T^*Y)$ (which is $\nu = 4$), is dual to a Killing field in $Y$, and in this case $\mathcal{K}_g(\tilde{\omega})$ is not in the image of $\mathcal{D}^*$ either. It is also easy to see from (6.9) that for $\mu > 3$ all the rates $\alpha^\pm(\mu)$ satisfy $|Re(\alpha^\pm(\mu))| > \sqrt{6}$. The rest of the Proposition follows from straightforward computations. $\square$

We are now ready to describe the general solution of (6.1). If $(Z, \tilde{\omega})$ is a solution of (6.1), then we can write $Z$ as

$$Z = Z_0 + \tilde{Z},$$

where $Z_0$ satisfies $\mathcal{D}^*Z_0 = 0$ and $\tilde{Z}$ is a non-zero solution of (6.1). We now prove that this solution $\tilde{Z}$ indeed exists.

**Proposition 6.2.** Let $\tilde{\omega} \in \Lambda^1(T^*M)$ be a solution of (6.2) of type (a) or type (b) which is not conformally Killing. There exists a nonzero $\tilde{Z} \in \Lambda^2_0(T^*M)$ such that $\mathcal{D}^*\tilde{Z} = \mathcal{K}_g\tilde{\omega}$.

**Proof.** For the proof, we consider a solution of (6.1) with $\tilde{\omega}$ of type (a), that is,

$$\tilde{\omega} = l(t)\phi dt + md\phi,$$

where $\phi$ is an eigenfunction of $\Delta_H$ on $\Lambda^0(T^*Y)$ with eigenvalue $\mu$ and $l, m$ are solutions of (6.8). On the other hand, for the element of type I, $f(t) \cdot \mathcal{K}_gY(d\phi)$, the operator $\mathcal{D}^*$ can be computed explicitly following the results in Section 5 as

$$\mathcal{D}^*(f\mathcal{K}_gY(d\phi)) = \left\{ \mu(2\kappa - \frac{2}{3}\mu)f\phi, \mathcal{L}_{g_Y}(d\phi), \frac{-1}{2}\left( \mathcal{L}_{g_Y}(d\phi) - \frac{\mu}{3}\mathcal{L}_{g_Y}(d\phi) \right) \right\}.$$

Suppose that $d\phi$ is not conformally Killing with respect to $g_Y$, then $\mathcal{L}_{g_Y}(d\phi)$ and $\phi g_Y$ are linearly independent. If we write $Z = f\mathcal{K}_gY(d\phi)$, then we can solve for $f$ such
that $D^* Z = \mathcal{K}_g(\tilde{\omega})$ by considering the system

\[
\begin{align*}
\mu(2\kappa - \frac{2}{3}\mu) f &= \frac{3}{2} \dot{l} + \frac{\mu}{2} m, \\
(2\kappa - \frac{2}{3}\mu) \dot{f} &= l + \dot{m}, \\
-\frac{1}{2} \left( \ddot{f} - \frac{\mu}{3} f \right) &= m, \\
-\frac{\mu}{3} \left( \ddot{f} - (\mu - 2\kappa) f \right) &= -\frac{1}{2}(\dot{l} - \mu m).
\end{align*}
\]

(6.14)

Note that from the condition that $d\phi$ is not conformally Killing we see that in the cases $\kappa = \pm 1$ we have $\mu(2\kappa - \frac{2}{3}\mu) \neq 0$ and then, from (6.8) we see that if we set

\[
f = \left. \frac{1}{\mu(2\kappa - \frac{2}{3}\mu)} \right( \frac{3}{2} \dot{l} + \frac{\mu}{2} m ),
\]

then $f$ is a nontrivial solution of the system (6.14) unless $\tilde{\omega}$ is conformally Killing with respect to the metric $g = dt^2 + g_Y$ and hence $Z = f\mathcal{K}_{g_Y}(d\phi)$ is a nontrivial solution of (6.1). The case $\kappa = 0$ is similar.

If now $\tilde{\omega}$ is of type (b), we can write $\tilde{\omega} = m\eta$ where $\eta \in \Lambda^1(T^*Y)$ is a co-closed eigenform of $\Delta_H$ with eigenvalue $\nu$. Let us also consider an element of type II written as $Z = f\mathcal{K}_{g_Y}(\eta)$ where $f$ is a function. Assuming that $\eta$ is not conformally Killing, we have

\[
D^* Z = \left\{ 0, \frac{1}{2}(4\kappa - \nu) \left( \dot{f} \pm \sqrt{\nu} \dot{f} \right) \eta, \frac{1}{2} \left( -\ddot{f} \mp \sqrt{\nu} \ddot{f} \right) \mathcal{L}_{g_Y}(\eta) \right\},
\]

where the sign of $\pm \sqrt{\nu}$ arises from Corollary 4.6. In order to solve $D^* Z = \mathcal{K}_g(\tilde{\omega})$, we consider $\tilde{\omega} = m\eta$, where $m$ solves (6.10) so (6.1) reduces to the system

\[
\begin{align*}
\frac{1}{2}(4\kappa - \nu) \left( \dot{f} \pm \sqrt{\nu} \dot{f} \right) &= \dot{m}, \\
-\frac{1}{2} \left( \ddot{f} \mp \sqrt{\nu} \ddot{f} \right) &= m.
\end{align*}
\]

(6.15)

and again since $\eta$ is not conformally Killing with respect to $g_Y$ it follows that $4\kappa - \nu$ is non-zero and if we find $f$ satisfying

\[
\dot{f} \pm \sqrt{\nu} \dot{f} = \frac{2\dot{m}}{4\kappa - \nu},
\]

then $f$ is a solution of (6.15) and $f\mathcal{K}_{g_Y}(\eta)$ is a solution of (6.1) with $\tilde{\omega} = m\eta$. We can choose $f$ to be

\[
f(t) = e^{\pm \sqrt{\nu} t} f_0 + \frac{2e^{\pm \sqrt{\nu} t}}{4\kappa - \nu} \int_0^t \dot{m}(s)e^{\mp \sqrt{\nu} s} ds,
\]

where $f_0$ is a constant. It is clear that we can choose the constant $f_0$ so that $f$ is a solution of (6.10). The case $\kappa = 0$ is similar. \[\square\]
7. Completion of proofs

We first state the following which determines all indicial roots of $F^*$ in the spherical case:

**Theorem 7.1.** Let $M = \mathbb{R} \times S^3/\Gamma$ with product metric $g = dt^2 + g_{S^3/\Gamma}$, where $g_{S^3/\Gamma}$ is a metric of constant curvature 1. Let $\mathcal{I}^*$ denote the set of indicial roots of $F^*$.

- Case (0): $0 \in \mathcal{I}^*$.
- Case (1): If $\Gamma = \{e\}$ then $j = \pm 1 \in \mathcal{I}^*$. If $\Gamma$ is non-trivial, then $j = \pm 1 \notin \mathcal{I}^*$.

All solutions in Case (0) and Case (1) are of the form $(0, \omega)$, where $\omega$ is dual to a conformal Killing field (that is, $K_g \omega = 0$).

- Case (2): If $B$ is a nontrivial eigentensor of $\Delta_{S^3/\Gamma}$ on divergence-free symmetric 2-tensors, with eigenvalue $j^2 + 2j - 2$ with $j \geq 2$, then $\{\pm j, \pm (j + 2)\} \in \mathcal{I}^*$.
- Case (3): If $\omega$ is an eigenform of $\Delta_{S^3/\Gamma}$ on divergence-free 1-forms with eigenvalue $(j + 1)^2$, with $j \geq 2$, then $\pm (j + 1) \in \mathcal{I}^*$.

All solutions in Case (2) and Case (3) are of the form $(Z, 0)$.

- Case (4) If $u$ is an eigenfunction of $\Delta_{S^3/\Gamma}$ with eigenvalue $j(j + 2)$, $j \geq 2$ then
  \[
  \pm \alpha_j^\pm = \pm \sqrt{j(j + 2) - 2 \pm \frac{2}{9}} \sqrt{-1} \sqrt{(j + 3)(j - 1)} \in \mathcal{I}^*.
  \]
- Case (5) If $\omega$ is an eigenform of $\Delta_{S^3/\Gamma}$ on divergence-free 1-forms with eigenvalue $(j + 1)^2$, with $j \geq 2$, then
  \[
  \pm \delta_j = \pm \sqrt{(j + 1)^2 - 4} \in \mathcal{I}^*.
  \]

All solutions in Case (4) and Case (5) are of the form $(Z, \omega)$ with both $Z$ and $\omega$ nontrivial and $K_g \omega \neq 0$.

**Remark 7.2.** If $\Gamma = \{e\}$, then all of the above indicial roots do in fact occur. For nontrivial $\Gamma$, exactly which roots occur depends on which eigentensors descend from $S^3$ to the quotient $S^3/\Gamma$.

**Proof of Theorem 7.1.** This follows from combining Propositions 5.9 and 6.2 for the case $\kappa = 1$. \hfill $\Box$

**Proof of Theorem 1.4.** This follows immediately from Theorem 7.1 since Cases (2) and (3) obviously have real part larger than 2, and it is easy to see that $|Re(\alpha_j^\pm)| > \sqrt{6}$ and $|Re(\delta_j)| \geq \sqrt{5}$ for all $j \geq 2$. The determination of the conformal Killing fields follows easily from Section 6. \hfill $\Box$

Next we state the following Theorem, which immediately implies Theorem 1.6.

**Theorem 7.3.** Let $M = \mathbb{R} \times S^3/\Gamma$ with product metric $dt^2 + g_{S^3/\Gamma}$, where $g_{S^3/\Gamma}$ is a metric of constant curvature 1. Let $\mathcal{I}$ denote the set of indicial roots of $F$.

- Case (0): $0 \in \mathcal{I}$. The corresponding kernel is
  \[
  \text{span}\{3dt \otimes dt - g_{S^3}, dt \otimes \omega_0\}
  \]
  where $\omega_0$ is dual to a Killing field on $S^3/\Gamma$. 

  \[
  (7.3)
  \]
• Case (1): If \( \Gamma = \{ e \} \) then \( j = \pm 1 \in \mathcal{I} \). If \( \Gamma \) is non-trivial, then \( j = \pm 1 \not\in \mathcal{I} \).

The corresponding kernel elements are given by

\[
h_\phi = p(t)\phi(3dt \otimes dt - g_{S^3}) + q(t)(dt \otimes d\phi),
\]

where \( p(t) = C_3e^t - C_4e^{-t} \) and \( q(t) = C_3e^t + C_4e^{-t} \), for some constants \( C_3 \) and \( C_4 \), and \( \phi \) is a lowest nonconstant eigenfunction of \( \Delta_{S^3/\Gamma} \). In particular, if \( \Gamma \) is nontrivial, then \( j = \pm 1 \) are not indicial roots.

All solutions in Case (0) and Case (1) are in the image of the conformal Killing operator. More precisely,

\[
3dt \otimes dt - g_{S^3} = K_g(2tdt) \quad \text{and} \quad dt \otimes \omega_0 = K_g(t\omega_0),
\]

and

\[
h_\phi = K_g \left\{ \frac{1}{2} \left( C_3(t + 3)e^t - C_4(t - 3)e^{-t} \right) \phi dt + \frac{1}{2} \left( -C_3te^t - C_4te^{-t} \right) d\phi \right\}.
\]

• Case (2): If \( B \) is a nontrivial eigentensor of \( \Delta_{S^3/\Gamma} \) on divergence-free symmetric 2-tensors, with eigenvalue \( j^2 + 2j - 2 \) with \( j \geq 2 \), then \( \{ \pm j, \pm (j + 2) \} \in \mathcal{I} \).

• Case (3): If \( \omega \) is an eigenform of \( \Delta_{S^3/\Gamma} \) on divergence-free 1-forms with eigenvalue \( (j + 1)^2 \), with \( j \geq 2 \), then \( \pm (j + 1) \in \mathcal{I} \).

The kernel elements in Case (2) are of the form \( h = f(t)B \), and in Case (3) are of the form \( h = f_0(t) \cdot \omega \otimes dt + f_1(t) \cdot K_{S^3}(\omega) \). Neither of these are in the image of the conformal Killing operator \( K_g \) of the cylinder.

• Case (4) If \( u \) is an eigenfunction of \( \Delta_{S^3/\Gamma} \) with eigenvalue \( j(j + 2) \), \( j \geq 2 \) then \( \pm \alpha_j^\pm \in \mathcal{I} \), where \( \alpha_j^\pm \) were defined in (7.1).

• Case (5) If \( \omega \) is an eigenform of \( \Delta_{S^3/\Gamma} \) on divergence-free 1-forms with eigenvalue \( (j + 1)^2 \), with \( j \geq 2 \), then \( \pm \delta_j \in \mathcal{I} \), where \( \delta_j \) were defined in (7.2).

All solutions in Case (4) and Case (5) are in the image of the conformal Killing operator of the cylinder. More precisely, they are exactly those solutions of \( \Box_{K,g} \omega = 0 \) which are not conformally Killing.

**Remark 7.4.** As before, if \( \Gamma = \{ e \} \), then all of the above indicial roots do in fact occur. For nontrivial \( \Gamma \), exactly which roots occur depends on which eigentensors descend to the quotient.

**Proof of Theorem 7.3.** From the index theorem of Lockhart-McOwen, it follows that the real parts of indicial roots of \( F \) are the same as those of \( F^* \) and the dimensions of the space of solutions of the form \( e^{M}p(y,t) \) where \( p \) is a polynomial in \( t \) with coefficients in \( C^{\infty}(Y) \) are the same for all indicial roots with the same real part. We consider Cases (0) – (5) in order.

For Case (0), the corresponding kernel of \( F^* \) is of the form \( (0, \omega) \), where \( \omega \) is dual to a bounded conformal Killing field on the cylinder. By direct calculation, elements in (7.3) form the corresponding space of kernel elements.

For Case (1), the corresponding kernel of \( F^* \) is of the form \( (0, \omega) \), where \( \omega \) is dual to a conformal Killing field which grows like \( e^t \) on one end. For \( S^3 \), this is an 8-dimensional space, while if \( \Gamma \) is nontrivial, this space is empty. Again, by direct
calculation, elements in \((7.4)\) form the corresponding 8-dimensional space of kernel elements in the case of the sphere. The formulas in \((7.5)\) and \((7.6)\) can also easily be verified by direct calculation, which we omit.

For Case (2), we consider solutions of the form

\[ f(t) \cdot B, \]

where \(f\) is a function, and \(B \in S^2(T^*Y)\) is an eigentensor of \(\Delta_{gy}\) with eigenvalue \(-\lambda\) satisfying \(\text{tr}_Y(B) = 0\) and \(\delta_Y B = 0\). In this case the equation \(F = 0\) takes the form

\[-\frac{1}{2} \ddot{f} \cdot B - f \cdot B - \frac{\lambda}{2} \dot{f} \cdot B + \dot{f} \cdot \delta B = 0.\]

Case (2) follows as in the proof of Proposition \[5.5\], and the index theorem.

For Case (3), we consider solutions of the form

\[ \tilde{h} = f_0(t) \cdot \omega \odot dt + f_1(t) \cdot K_{gy}(\omega), \]

which are not in the image of \(K_{gy}\), where \(\omega \in \Lambda^1(T^*Y)\) is an eigenform of \(\Delta_H\) with eigenvalue \(\nu > 0\) satisfying \(\delta_Y \omega = 0\). Case (3) then follows as in Proposition \[5.9\], and the index theorem.

For Cases (4) and (5), we consider \(\tilde{h}\) of the form

\[ \tilde{h} = K_{gy}(\tilde{\omega}), \]

where \(\tilde{\omega} \in \Lambda^1(T^*M)\). The equation \(\delta \tilde{h} = 0\) says that \(\omega\) is a solution of \(\Box_{K_{gy}} \omega = 0\), and the solutions of these equations were completely classified in Section \[6\] into those of types (a) and (b). Cases (4) and (5) then follow from Proposition \[6.2\] and the index theorem.

**Proof of Corollaries \[1.8\] and \[1.9\]** Corollary \[1.8\] follows immediately from Theorem \[1.4\]. Corollary \[1.9\] then follows using a standard argument that solutions of elliptic equations in weighted spaces admit asymptotic expansions with leading terms solutions on the cylinder corresponding to indicial roots \[LM85\].

**Proof of Theorem \[1.11\]** Applying the divergence operator to the equation \(D^*Z = K_{gy}\omega\), we see that \(\omega\) satisfies \(\Box_K \omega = 0\). An integration by parts shows that \(K_{gy}\omega = 0\), which implies that \(\omega = 0\) since there are no nontrivial decaying conformal Killing fields. We next convert \((M, g)\) into a manifold with a cylindrical end, using the conformal factor \(u^{-2}\) which is smooth and positive and equal to \(r^{-2}\) outside of some compact set, and let \(\hat{g} = r^{-2}g\). From conformal invariance of \(D^*\), we have that \(D^*_g Z = 0\). Using Corollary \[1.9\] we conclude that \(|Z|_{\hat{g}} = O(e^{-2t})\), where \(t = \log(u)\) as \(t \to \infty\). This implies that \(|Z|_{g} = O(r^{-4})\) as \(r \to \infty\).

Next, if \(h\) solves \(Dh = 0\) and \(\delta h = 0\), then \(B'(h) = D^* Dh = 0\), where \(B'\) is is the linearized Bach tensor \[Ito95\]. Since \(B'\) is asymptotic to \(\Delta^2\) as \(r \to \infty\), \[AViS\ Proposition 2.2\], implies that there is no \(O(r^{-1})\) term in the asymptotic expansion of \(h\) and therefore \(h = O(r^{-2})\) as \(r \to \infty\).
Proof of Theorem 1.12. The cokernel statements follow from combining Propositions 5.9 and 6.2 for the case \( \kappa = -1 \). The kernel statements follow from an analysis similar to the one outlined in the proof of Theorem 7.3, using the index theorem. For the indicial root of 0, the corresponding kernel of \( F^* \) is of dimension \( 1 + b_1(Y) + 2 \text{dim}(H^1_C(Y)) \). From Theorem 3.3 we see that \( 3dt \otimes dt - gy \) is in the kernel of \( F \). For a harmonic 1-form \( \omega \), from Theorem 3.3 we also see that \( \omega \otimes dt \) is also in the kernel of \( F \). For a traceless Codazzi tensor \( B \) on \( Y^3 \), from Theorem 3.3 it follows that

\[
(c_1 \cos(t) + c_2 \sin(t))B,
\]

is in the kernel of \( F \) for any constants \( c_1 \) and \( c_2 \). By counting dimensions and using the index theorem, this accounts for all kernel elements of \( F \) corresponding to the indicial root 0.

Proof of Theorem 1.14. A compact hyperbolic 3-manifold \((Y, g_Y)\) corresponds to a discrete cocompact subgroup \( \Gamma \subset O_0(3,1) \) without torsion. The space of locally conformally flat deformations of \( Y \) is given by \( H^1(\Gamma, g) \), where \( g \) is the lie algebra to \( O(4,1) \) viewed as a \( \Gamma \)-module under the adjoint representation. If \( Y^3 \) is a hyperbolic rational homology 3-sphere, then by assumption \( b_1(Y) = 0 \), so Theorem 1.12 implies that \( H^2_\ast(\mathbb{R} \times Y^3) = \{0\} \) if and only if \( H^1_\ast(Y) = \{0\} \).

In [Kap94] it was shown that infinitely many \((p,q)\)-surgeries on a hyperbolic 2-bridge knot satisfy \( H^1(\Gamma, g) = \{0\} \) (a 2-bridge knot is any knot that my be embedded in \( \mathbb{R}^3 \) with only 2 local maxima, and the figure 8 knot is an example of a hyperbolic 2-bridge knot). These have \( p \geq 2 \) and are therefore rational homology 3-spheres, and all but finitely many are hyperbolic by Thurston’s hyperbolic Dehn surgery Theorem (see, for example, [HK05] or [PP00]). By [Laf83, Lemma 6], there is an injection \( H^1_C(Y) \hookrightarrow H^1(\Gamma, g) \), so these examples are therefore an infinite family of hyperbolic rational homology 3-spheres satisfying \( H^2_\ast(\mathbb{R} \times Y^3) \neq \{0\} \).

Next, it was shown by DeBlois that there are infinitely many hyperbolic rational homology 3-spheres containing closed embedded totally geodesic surfaces [DeB06] (these examples are \( n \)-fold cyclic branched covers of \( S^3 \) branched along a certain \( 2 \)-component link). By [Laf83, Theorem 2], such a surface yields a non-trivial traceless Codazzi tensor field on \( Y \). Thus by Theorem 1.12, the examples of DeBlois are an infinite family of examples of hyperbolic rational homology 3-spheres satisfying \( H^2_\ast(\mathbb{R} \times Y^3) \neq \{0\} \).

Proof of Corollary 1.15. This follows from Theorem 1.12, again using a standard argument that solutions of elliptic equations in weighted spaces admit asymptotic expansions with leading terms solutions on the cylinder corresponding to indicial roots [LM85].

Proof of Theorem 1.16. The cokernel statements follow from combining Propositions 5.9 and 6.2 for the case \( \kappa = 0 \). The kernel statements follow from an analysis similar to the one outlined in the proof of Theorem 7.3, using the index theorem.
8. The gluing problem

We will next describe the setup to the gluing theorem of Kovalev-Singer. A brief statement of the theorem is as follows.

**Theorem 8.1** (Floer, Kovalev-Singer, Donaldson-Friedman [Flo91, KS01, DF89]). Let \((X_1, [g_1])\) and \((X_2, [g_2])\) be self-dual conformal structures on compact 4-manifolds \(X_i\) satisfying \(H^2_+(X_i, [g_i]) = 0\) for \(i = 1, 2\). Then the connect sum \(X_1 \# X_2\) admits self-dual conformal structures.

Donaldson-Friedman proved this using twistor theory, using methods from the deformation theory of singular complex 3-folds. The proofs of Floer and Kovalev-Singer are analytic, and thus generalize more easily to the setting of orbifolds. Consequently, the gluing can be performed at isolated orbifold points \(p_i \in X_i, i = 1, 2\), provided they are compatible. This means that there is an orientation-reversing intertwining map between the actions of the respective orbifold groups \(\Gamma_i \subset SO(4)\) at the gluing points.

We will next outline the idea of the analytic proof. Let \(r_i(x) = d(p_i, x)\) in sufficiently small neighborhoods of \(p_i\), and extend to smooth positive functions on each \(X_i\). Consider the conformal cylindrification of \(X_i\), which is \(\tilde{X}_i = X_i \setminus \{p_i\}\) with metric \(\tilde{g}_i = r_i^{-2} g_i\), and let \(t_i = -\log r_i\). These metrics are then “glued” together with a cylindrical region in between using cutoff functions, we refer the reader to [KS01, Section 2.3] for the exact formulas. We only need to remark here that the main argument of [KS01] is to reduce the gluing problem to the study of the deformation complex on the component cylindrified spaces. A weight \(\delta > 0\) and a weight function are chosen so that in the limit, the weight function is \(e^{\delta t}\) in the middle cylindrical region, and \(e^{\delta t_1}\) on \(\tilde{X}_1\) and \(e^{-\delta t_2}\) on \(\tilde{X}_2\). One next considers the operators

\[
F_i : e^{\delta t_1} C^{k, \alpha}(A_1) \xrightarrow{D^{2\delta \delta_1}} e^{\delta t_1} C^{k, -2, \alpha}(B_1) \oplus e^{\delta t_1} C^{k, -1, \alpha}(C_1),
\]

\[
F_0 : e^{\delta t} C^{k, \alpha}(A_0) \xrightarrow{D^{2\delta \delta_0}} e^{\delta t} C^{k, -2, \alpha}(B_0) \oplus e^{\delta t} C^{k, -1, \alpha}(C_0),
\]

\[
F_2 : e^{-\delta t_2} C^{k, \alpha}(A_2) \xrightarrow{D^{2\delta \delta_2}} e^{-\delta t_2} C^{k, -2, \alpha}(B_2) \oplus e^{-\delta t_2} C^{k, -1, \alpha}(C_2),
\]

where \(A_i = T^* \tilde{X}_i, B_i = S^2_0(T^* \tilde{X}_i), \) and \(C_i = S^2_0(\Lambda^2_-)(T^* \tilde{X}_i)\). The adjoints of these operators are maps

\[
F_i^* : e^{-\delta t_1} C^{k, \alpha}(B_1) \oplus e^{-\delta t_1} C^{k, -1, \alpha}(C_1) \rightarrow e^{-\delta t_1} C^{k, -2, \alpha}(A_1)
\]

\[
F_0^* : e^{-\delta t} C^{k, \alpha}(B_0) \oplus e^{-\delta t} C^{k, -1, \alpha}(C_0) \rightarrow e^{-\delta t} C^{k, -2, \alpha}(A_0)
\]

\[
F_2^* : e^{\delta t_2} C^{k, \alpha}(B_2) \oplus e^{\delta t_2} C^{k, -1, \alpha}(C_2) \rightarrow e^{\delta t_2} C^{k, -2, \alpha}(A_2),
\]

given by

\[
F_i^*(Z, \omega) = D^* Z - K_{\tilde{g}_i} \omega.
\]

Note the duals of the Hölder spaces are not Hölder spaces, but we are only interested in the kernel and cokernel, which will be smooth by elliptic regularity, so this slight abuse of notation does not matter.
On the middle cylindrical region, Corollary 1.8 shows that $F_0$ is an isomorphism for $0 < \delta < 2$. On $\tilde{X}_1$, we consider solutions of $F^*_1(Z, \omega) = 0$ with both $Z = O(e^{-\delta t_1})$ and $\omega = O(e^{-\delta t_1})$ as $t_1 \to \infty$. Corollary 1.9 implies that $\omega$ is a conformal Killing field, and $Z = O(e^{-2t_1})$. The conformal transformation formula $D^*_g Z = r^2 D^*_g Z$ shows that $Z$ is a solution of $D^*_g Z = 0$ on $X_2 \setminus \{p_2\}$ satisfying $Z = O(1)$ as $r_1 \to 0$. The operator $D^*D^*$ is an elliptic operator with leading term $\Delta^2$ (AVis, Str10). Consequently, the singularity is removable, so $Z$ extends to a smooth solution of $D^*Z = 0$ on $X_1$, and $Z$ then vanishes by the assumption that $H^2_c(X_1, [g_1]) = 0$.

On $\tilde{X}_2$, we consider solutions of $F^*_2(Z, \omega) = 0$ with both $Z = O(e^{\delta t_2})$ and $\omega = O(e^{\delta t_2})$ as $t_2 \to \infty$. Vanishing of $Z$ again follows from Corollary 1.9 and the assumption that $H^2_c(X_2, [g_2]) = 0$.

**Remark 8.2.** The argument given on [KS01] page 1259-1260 to handle the case of $\tilde{X}_2$ is incorrect, because there was a mistake in the order of growth given there. Namely, the growth rate given on the bottom on page 1258 for $H^{2\pm}$ should be $|\Psi|_0 = O(r^{-2\pm\delta})$, and not $|\Psi|_0 = O(r^{-2+\delta})$ as written there and then applied incorrectly in the subsequent argument. Indeed, on $X_2$ the weight function is $e^{-\delta t}$, while the argument given there to remove the singularity (quoting Biquard’s Theorem from Biq91) requires $\delta > 0$. The above argument fixes this gap.

The remainder of the proof then proceeds as in [KS01].

**Remark 8.3.** We note that there can be asymptotic cokernel arising from conformal Killing fields on the factors. Namely, on $\tilde{X}_1$ there are conformal Killing fields in the cokernel satisfying $\omega = O(e^{-\delta t_1})$ as $t_1 \to \infty$. The conformal transformation formula $\mathcal{K}_g(\omega) = r^2_1 \mathcal{K}_g(r^2_1 \omega)$ shows that $r^2_1 \omega$ is a conformal Killing field on $X_1$ satisfying $|r^2_1 \omega|_{g_1} = O(r^{1+\delta}_1)$ as $r_1 \to 0$. Thus the asymptotic cokernel contains conformal Killing fields on $X_1$ which vanish at $p_1$ and whose first derivatives vanish at $p$. Similarly, on $X_2$ there are conformal Killing fields in the cokernel satisfying $\omega = O(e^{\delta t_2})$ as $t_2 \to \infty$. Then $r^2_2 \omega$ is a conformal Killing field on $X_2$ satisfying $|r^2 \omega|_{g_2} = O(r^{1-\delta}_2)$ as $r_2 \to 0$. Thus the asymptotic cokernel also contains conformal Killing fields on $X_2$ which vanish at $p_2$. However, the existence of this cokernel does not affect finding a self-dual metric, since we only need to find a zero of the first component of $F$ (and do not necessarily need to find a zero of the divergence map).

**Appendix A. The square of $\phi$**

In this appendix, we give the proof of Proposition 4.1.

**Proof of Proposition 4.1.** In a local orthonormal basis we have

$$
\phi^2 h_{ij} = \sum_{a,b} \epsilon_{iab} \nabla_a (\phi h)_{bj} + \sum_{c,d} \epsilon_{jcd} \nabla_c (\phi h)_{di}.
$$
Expanding the right hand side, we obtain
\[ \delta^2 h_{ij} = \sum_{a,b} \sum_{k,l} \epsilon_{iab} \epsilon_{bkd} \nabla_a \nabla_k h_{lj} + \sum_{a,b} \sum_{m,n} \epsilon_{iab} \epsilon_{jmn} \nabla_a \nabla_m h_{nb} + \sum_{c,d} \sum_{p,q} \epsilon_{jcd} \epsilon_{dpq} \nabla_c \nabla_p h_{qj} + \sum_{c,d} \sum_{u,v} \epsilon_{jcd} \epsilon_{iuv} \nabla_c \nabla_u h_{vd} \]
\[ = I + II + III + IV. \]

Note that \( I + III \) is twice the symmetrization of \( I \) and \( II + IV \) is twice the symmetrization of \( II \), so it will suffice to compute \( I \) and \( II \). A straightforward computation shows that if we let \( a, b \) be indices such that \( \{i, a, b\} = \{1, 2, 3\} \) then
\[ I = \nabla_i \nabla_j h_{aj} - \nabla_i \nabla_a h_{ij} + \nabla_b \nabla_i h_{bj} - \nabla_b \nabla_b h_{ij}. \]

Commuting covariant derivatives we have
\[ \nabla_a \nabla_i h_{aj} = \nabla_i \nabla_a h_{aj} - R_{aij}^p h_{pj} - R_{aij}^p h_{ap} \]
\[ = \nabla_i \nabla_a h_{aj} - \kappa (\delta^p_a (g_Y)_{ia} - \delta^p_i (g_Y)_{aa}) h_{pj} \]
\[ - \kappa (\delta^p_a (g_Y)_{ij} - \delta^p_i (g_Y)_{aj}) h_{ap} \]
\[ = \nabla_i \nabla_a h_{aj} + \kappa (h_{ij} - h_{aa} (g_Y)_{ij} - (g_Y)_{aj} h_{ai}). \]

Similarly,
\[ \nabla_b \nabla_i h_{bj} = \nabla_i \nabla_b h_{bj} + \kappa (h_{ij} - h_{bb} (g_Y)_{ij} - (g_Y)_{bj} h_{bi}). \]

It follows that
\[ \nabla_a \nabla_i h_{aj} + \nabla_b \nabla_i h_{bj} = \nabla_i (\delta_Y)_j - \nabla_i \nabla_i h_{ij} \]
\[ + 2 \kappa h_{ij} + \kappa (h_{aa} + h_{bb} + h_{ii}) (g_Y)_{ij} \]
\[ - \kappa (h_{ii} (g_Y)_{ij} + h_{ai} (g_Y)_{aj} + h_{bi} (g_Y)_{bj}) \]
\[ = \nabla_i (\delta_Y)_j - \nabla_i \nabla_i h_{ij} + 3 \kappa \text{tf}(h), \]

and clearly
\[ I = \nabla_i (\delta_Y)_j - \Delta h_{ij} + 3 \kappa \text{tf}(h). \]

We conclude that
\[ I + III = \mathcal{L}_{g_Y} (\delta_Y h) - 2 \Delta h + 6 \kappa \text{tf}(h). \]

For \( II \) we consider two cases. If \( i \neq j \) we let \( l \) be an index such that \( \{i, j, l\} \) such that \( \{i, j, l\} = \{1, 2, 3\} \), then it is easy to see that \( II \) equals
\[ II = -\nabla_j \nabla_i h_{il} + \nabla_j \nabla_i h_{il} + \nabla_j \nabla_i h_{lj} - \nabla_j \nabla_j h_{ij} \]
\[ = A_1 + A_2 + A_3 + A_4. \]

For the terms in \( (A.1) \) we have
\[ A_1 = -\nabla_j \nabla_i h_{il} = -\nabla_j \nabla_i \text{tr}_Y (h) + \nabla_j \nabla_i h_{ii} + \nabla_j \nabla_i h_{jj}, \]
\[ A_2 = \nabla_j \nabla_i h_{il} = \nabla_j (\delta_Y h)_i - \nabla_j \nabla_i h_{ii} - \nabla_j \nabla_j h_{ij}, \]
For $A_3$ we commute covariant derivatives

$$A_3 = \nabla_i \nabla_i h_{ij} = \nabla_i \nabla_i h_{ij} - R^p_{ij}h_{pj} - R^p_{ip}h_{jp}$$

$$= \nabla_i \nabla_i h_{ij} - \kappa (\delta^p_i g_{il} - \delta^p_j g_{ii}) h_{pj} - \kappa (\delta^p_i g_{ij} - \delta^p_j g_{jj}) h_{jp}$$

$$= \nabla_i \nabla_i h_{ij} + \kappa h_{ij}$$

and finally

$$A_4 = -\nabla_i \nabla_i h_{ij} = -\Delta_{gy} h_{ij} + \nabla_i \nabla_i h_{ij} + \nabla_j \nabla_j h_{ij},$$

We then have

$$A_1 + A_2 = -\nabla_j \nabla_i \text{tr}_Y (h) + \nabla_j (\delta_Y h)_i + \nabla_j \nabla_i h_{jj} - \nabla_j \nabla_j h_{ij},$$

and

$$A_3 + A_4 = \nabla_i (\delta_Y h)_j - \nabla_i \nabla_i h_{ij} - \nabla_i \nabla_j h_{ij} + \kappa h_{ij}$$

$$- \Delta_{gy} h_{ij} + \nabla_i \nabla_i h_{ij} + \nabla_j \nabla_j h_{ij}$$

$$= \nabla_i (\delta_Y h)_j + \kappa h_{ij} - \Delta_{gy} h_{ij}$$

$$- \nabla_i \nabla_j h_{jj} + \nabla_j \nabla_j h_{ij},$$

so then

$$II = A_1 + A_2 + A_3 + A_4$$

$$= -\nabla_j \nabla_i \text{tr}_Y (h) - \Delta_{gy} h_{ij}$$

$$+ \mathcal{L}_{gy} (\delta_Y h) + \kappa h_{ij} + \nabla_j \nabla_i h_{jj} - \nabla_i \nabla_j h_{jj}.$$ 

Commuting covariant derivatives we have

$$\nabla_j \nabla_i h_{jj} - \nabla_i \nabla_j h_{jj} = -R^p_{jj}h_{pj} - R^p_{ij}h_{jp} = -2R^p_{jj}h_{pj}$$

$$= -2\kappa (\delta^p_j g_{ij} - \delta^p_i g_{jj}) h_{jp}$$

$$= 2\kappa h_{ij},$$

so we have shown

$$II = -\Delta_{gy} h_{ij} + \mathcal{L}_{gy} (\delta_Y h) + 3\kappa h_{ii},$$

so then $II + IV$ is given by

(A.3) $$II + IV = -2\Delta_{gy} h - 2\nabla_i \nabla_j \text{tr}_Y (h) + 2\mathcal{L}_{gy} (\delta_Y h) + 6\kappa h_{ij}.$$ 

If now $i = j$ we let $a, b$ be indices such that $\{i, a, b\} = \{1, 2, 3\}$ so that we have

$$II = \nabla_a \nabla_a h_{bb} - \nabla_a \nabla_b h_{ab} - \nabla_b \nabla_a h_{ab} + \nabla_b \nabla_b h_{aa},$$

which simplifies to

$$II = \Delta_{gy} \text{tr}_Y (h) - \nabla_i \nabla_i \text{tr}_Y (h) - (\delta_Y \delta_Y h) + \nabla_i (\delta h)_i - \Delta_{gy} h_{ii}$$

$$+ \nabla_i \nabla_i h_{ii} + \nabla_a \nabla_i h_{ai} + \nabla_b \nabla_i h_{ib}.$$
Commuting covariant derivatives we obtain
\[
II = \Delta_{g_Y} \text{tr}_Y(h) - \nabla_i \nabla_i \text{tr}_Y(h) - (\delta_Y \delta_Y h) - \Delta_{g_Y} h_{ii} + \nabla_i (\delta_Y h)_i - R^p_{aii} h_{pi} - R^p_{api} h_{pi} - R^p_{bpi} h_{ip},
\]
and it is easy to see from this expression that
\[
(A.4)\quad II = \Delta_{g_Y} \text{tr}_Y(h) - \nabla_i \nabla_i \text{tr}_Y(h) - (\delta_Y \delta_Y h) - \nabla_i (\delta_Y h)_i + 2\nabla_i (\delta_Y h)_i + 3\kappa f(h),
\]
so then
\[
II + IV = 2\Delta_{g_Y} \text{tr}_Y(h) - 2\nabla_i \nabla_i \text{tr}_Y(h) - 2(\delta_Y \delta_Y h) - 2\nabla_i (\delta_Y h)_i + 6\kappa f(h).
\]
Now, since we are using a local orthonormal basis to compute $\partial^2 h$, it is clear that in either case $i = j$ or $i \neq j$ the results in (A.3) and (A.4) are equivalent to
\[
II + IV = -2\Delta_{g_Y} \text{tr}_Y(h) - 2\nabla^2 h + 2\mathcal{L}_{g_Y} (\delta_Y h) - 2(\delta_Y \delta_Y g_Y) - \frac{2}{3}(\Delta_{g_Y} \text{tr}_Y(h)) g_Y + \frac{2}{3} \kappa f(h).
\]
Expressing $I + III$ as
\[
I + III = -2\Delta_{g_Y} \text{tr}_Y(h) - \frac{2}{3}\Delta_{g_Y} \text{tr}_Y(h) g_Y + \mathcal{L}_{g_Y} (\delta_Y h) + 6\kappa f(h),
\]
we conclude that
\[
\partial^2 h = -4\Delta_{g_Y} \text{tr}_Y(h) - 2\nabla^2 \text{tr}_Y(h) + 3\mathcal{K}_{g_Y}(\delta_Y h) + 12\kappa f(h),
\]
as needed. □

The following corollary should be compared with [Flo91, Lemma 5.1]:

**Corollary A.1.** For any $h \in S^2(T^*Y)$ we have
\[
(A.5)\quad E'_g(h) = \frac{1}{8} \partial^2 h - \frac{1}{4} \nabla^2 \text{tr}_Y(h) + \frac{1}{8} \mathcal{K}_{g_Y}(\delta_Y h) - \frac{\kappa}{2} f(h).
\]

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