THE WEAK SOLUTION TO A BOLTZMANN TYPE EQUATION AND ITS ENERGY CONSERVATION

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Abstract. In this paper, we study the initial value problem of a Boltzmann type equation with a nonlinear degenerate damping. We prove the existence of global weak solutions with large initial data, in three dimensional space. We rely on a variant version of the Gronwall inequality and $L^p$ regularity of average velocities to derive the compactness of solutions to a suitable approximation. This allows us to recover a weak solution by passing to the limits. After the existence result, we also prove energy conservation for the weak solution under some certain condition.

1. Introduction

Kinetic approach plays a critical role in many variant fields of mathematical physics and applied sciences, from micro- and nano-physics to continuum mechanics, and from social science to biological science. It is an important tool in the modeling and simulation of phenomena across length and time scales, from the atomistic to the continuum. Thus, it has diverse applications in gas dynamics, engineering, medicine, geophysics and astrophysics, and has attracted numerous mathematical interests in modeling and analysis, see the references [1, 2, 3, 7, 8, 9, 12, 14, 15, 18, 19]. Naturally, one of the fundamental problems is to study the existence of weak solution and its energy conservation.

In this paper we consider the following nonlinear partial differential equation [1, 2, 9, 15]:

$$f_t + \xi \cdot \nabla_x f + \text{div}_\xi (fF) = \Omega(f), \quad (1.1)$$

where $f(t, x, \xi)$ is the density function for individuals at time $t \in \mathbb{R}^+$, physical position of an individual $x \in \mathbb{R}^3$, with the velocity $\xi \in \mathbb{R}^3$. Thus, $f(t, x, \xi) \, dx \, d\xi$ is the number density of individuals at position between $x$ and $x + dx$ and with velocity between $\xi$ and $\xi + d\xi$. The function

$$\int_{\mathbb{R}^3} f(t, x, \xi) \, d\xi$$

denotes the number density of individuals at the physical position $x$ at time $t$. The evolution of density distribution $f(t, x, \xi)$ is described by equation (1.1). Note that, $F$ is the external distribution $f(t, x, \xi)$ is described by equation (1.1).
\( \Omega(f) \) denotes the rate of change of \( f \) due to reaction, random choice of velocity, etc. We assume that the external force \( F \equiv 0 \) in this paper.

Let us give some backgrounds on the operator \( \Omega(f) \). Assume that two different processes contribute to \( \Omega(f) \), that is

\[
\Omega = \Omega_1 + \Omega_2,
\]

where \( \Omega_1 \) denotes a birth-death process, \( \Omega_2 \) stands for a process that generates random velocity changes. The birth-death process can be described as follows

\[
\Omega_1(f) = -\mu(n)f,
\]

where \( \mu(n) \geq 0 \) is the birth-death rate which only depends on the number density of individuals \( n(t, x) \). For more details on this operator, we refer the reader to [7, 15]. The number density of individuals is given by

\[
n(t, x) = \int_{\mathbb{R}^3} f(t, x, \xi) \, d\xi.
\]

It is also called the particles density (or zeroth moment) in the kinetic theory. The stochastic process is modeled by the following operator (see [15]),

\[
\Omega_2(f) = -\lambda f(t, x, \xi) + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi) \, d\xi',
\]

(1.2)

where \( \lambda > 0 \) is the break-up frequency. The kernel \( T(\xi, \xi') \) is the probability of a change with respect to velocity from \( \xi' \) to \( \xi \), and \( \xi \) is the velocity of individuals before the collision while \( \xi' \) is the velocity immediately after the collision. Thus, equation (1.1) becomes to the following

\[
f_t + \xi \cdot \nabla_x f = -\mu(n)f - \lambda f + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \, d\xi'.
\]

Furthermore, from the conservation of kinetic energy it follows,

\[
|\xi|^2 = |\xi'|^2,
\]

from which, we deduce

\[
|\xi| = |\xi'|.
\]

(1.3)

Thus, of particular interest in this paper is the case in which the speed does not change with reorientation. Given that a reorientation occurs, the probability function \( T(\xi, \xi') \) is a non-negative function and after normalization we may have

\[
\int_{\mathbb{R}^3} T(\xi, \xi') \, d\xi = 1.
\]

(1.4)

Next, following the work of [9], we assume that \( T(\xi, \xi') \) satisfies a self-similarity property, namely

\[
T(\xi, \xi') = H(|\xi'|) T\left( \frac{\xi}{|\xi'|}, \frac{\xi'}{|\xi'|} \right), \quad \text{for some function } H(\cdot).
\]

(1.5)

In this current paper, we assume that (1.3), (1.4) and (1.5) hold.
The objective of our current work is to investigate the issue of existence of global weak solutions to

$$f_t + \xi \cdot \nabla_x f = -\mu(n)f - \lambda f + \lambda \int_{\mathbb{R}^3} T(\xi, \xi')f(t, x, \xi')\, d\xi'$$  \hspace{0.5cm} (1.6)

with the following initial data:

$$f(0, x, \xi) = f^0(x, \xi),$$  \hspace{0.5cm} (1.7)

where \( n(t, x) = \int_{\mathbb{R}^3} f \, d\xi \), and \((t, x, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3\). The first goal is to prove the global existence of weak solutions with large initial data. After the existence result, we shall prove the energy conservation for such a weak solution. Note that, the energy conservation is a fundamental problem in the physical theory and the mathematical study of kinetic equation.

To the best of our knowledge, the first existence result related to (1.6) goes back to the work of Leger-Vasseur [9] where they established the existence of solutions to the coupled system by Navier-Stokes and (1.6) with \( \mu(n) = 0 \). In [9] they constructed the weak solutions to the associated kinetic equation for \((x, \xi)\) in bounded domains. Roughly speaking, they build a sequence of nonnegative solutions \( \{f_n\}_{n=1}^{\infty} \) to a suitable approximation. Because the energy inequality does not hold at the approximation level, the uniform estimates cannot be derived directly. They proved this sequence is an increasing sequence in \( n \) by induction, and applied the Monotone Convergence Theorem to \( \{f_n\}_{n=1}^{\infty} \) to derive its strong convergence. Note, the energy inequality does not hold at the approximated iteration. Combined the increasing of \( \{f_n\}_{n=1}^{\infty} \), the bad terms in energy inequality can be bounded by means of \( L^p \) estimates in bounded domain. We point out that the Monotone Convergence Theorem is a crucial tool to handle the kinetic equation in [9]. Our main goal of this paper is to extend the existence result of kinetic equation in [9] to (1.6) with \( \mu(n) \) in the setting \( \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \). The nonlinear term \( \mu(n)f \) leads to the loss of the monotonicity for the sequences of solutions to the suitable approximation, the arguments of Leger-Vasseur [9] cannot apply for the convergence. For this reason we need to develop new argument to guarantee the convergence of solutions to approximation.

Note that, the smooth solution of (1.6)-(1.7), satisfies the energy equality. In particular, we have energy inequality

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m)f \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)(1 + |\xi|^m)f \, d\xi \, dx \, dt$$

$$\leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m)f^0 \, d\xi \, dx,$$  \hspace{0.5cm} (1.8)

for any \( T > 0 \). Energy inequality (1.8) will derive in Section 2. Thus, it is natural to ask the initial data satisfies the following ones

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m)f^0 \, d\xi \, dx < +\infty.$$  \hspace{0.5cm} (1.9)
In fact, we will prove the stronger version of (1.8) in Section 5, for a weak solution under some certain condition,

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)(1 + |\xi|^m) f \, d\xi \, dx \, dt
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0 \, d\xi \, dx,
\]

for any \( t \in [0, T] \). This implies that such a weak solution preserves the energy conservation.

Based on energy inequality (1.8) and related estimates, we give the definition of weak solutions in the following sense.

**Definition 1.1.** The function \( f \) is a global weak solution to the initial value problem (1.6)-(1.7) if, for any \( T > 0 \), the following properties hold,

- The function \( f(t, x, \xi) \) has the following regularities

\[
\begin{cases}
  & f(t, x, \xi) \geq 0 \quad \text{for any } (t, x, \xi) \in (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3; \\
  & f \in C(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3))), \\
  & |\xi|^m f \in L^\infty(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \quad \text{for any } m \in [0, m_0] \text{ for some } m_0 \geq 3;
\end{cases}
\]

- The initial value problem (1.6)-(1.7) holds in the sense of distribution, that is, for any test function \( \varphi \in C_\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]) \), the following weak formulation holds

\[
\begin{align*}
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f^0 \varphi(0, x, \xi) \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} (f \varphi_t + \xi f \cdot \nabla_x \varphi) \, d\xi \, dx \, dt + \\
\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n) f \varphi \, d\xi \, dx \, dt = -\lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \varphi \, d\xi \, dx \, dt \\
+ \lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \varphi \, d\xi' \, d\xi \, dx \, dt.
\end{align*}
\]

- Energy inequality holds for almost every where \( t > 0 \):

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0 \, d\xi \, dx.
\]

As our main result, we proved the following theorem on the existence of global weak solutions to the initial value problem (1.6)-(1.7).

**Theorem 1.1.** If \( f^0 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3) \), and \( f^0(1 + |\xi|^m) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \) for any \( m \in [0, m_0] \) for some \( m_0 \geq 3 \); the probability function \( T(\xi, \xi') \) satisfies (1.4) and (1.5); the function \( \mu(\cdot) \) is a Lipschitz continues function; then there exists a global weak solution to the initial value problem (1.6)-(1.7).

We first follow the approximation introduced in [9], which is equation (3.1). As the same in [9], the energy inequality does not hold at the approximation.
level. However, the sequence of solutions to (3.1) is no longer increasing in $n$ due to a nonlinear term $\mu(n_k)f_k$. It is necessary to develop new arguments to facilitate the compactness of solutions to approximation. The crucial step here is to show the strong convergence of the zeroth moment and the first moment in $L^p$ space. By means of the $L^p$ regularity of average velocities, the strong convergence can be obtained if they are bounded uniformly in $L^p$ space. In particular, we shall rely on a variant of the Gronwall inequality (see Lemma 3.1) to show that, for any $0 \leq t \leq T$, there exists a constant $K > 0$, such that

$$\int_{\mathbb{R}^3} |\xi|^m f_k d\xi \leq Ke^{KT},$$

for all $k > 0$. It allows us to control the zeroth moment and first moment

$$\int_{\mathbb{R}^3} \xi f_k d\xi \quad \text{and} \quad \int_{\mathbb{R}^3} f_k d\xi$$

in $L^p$ space for some $p > 1$. In order to assure the convergence $\mu(n_{k-1})n_k$, we need to prove the strong convergence of $n_k$ in $L^p$ space. This can be done by the $L^p$ regularity of average velocities. Thus

$$\int_{\mathbb{R}^3} \mu(n_{k-1})f_k d\xi \rightarrow \int_{\mathbb{R}^3} \mu(n)f d\xi$$

in $L^\infty(0, T; L^1_{loc}(\mathbb{R}^3))$ as $k \rightarrow \infty$. With a suitable approximation and the weak stability, the existence of weak solutions can be done.

Formally, the classical solutions of the physical PDEs always keep the energy conservation, but not a weak solution, at least for fluid equations. Naturally, the question is how badly behaved the solution can be in order that a weak solution still can preserve the energy. In fluid equations, there are a lot of study related to this question. For example, Serrin [16] showed if a weak solution $u$ to incompressible Navier-Stokes equations with additional condition, then a weak solution $u$ holds the energy equality for any $0 \leq t \leq T$. Shinbrot [17] proved the same conclusion under some different conditions. For the Euler’s equation, E-Constantin-Titi [4] proved the energy equality for a weak solution, which is the answer to the first part of Onsager’s conjecture [13]. As before that we mentioned, the energy conservation is a fundamental problem in physics and related mathematical analysis. Thus, we are interested in studying this question for a weak solution to (1.6). Our second main result is on the energy conservation for a weak solution constructed in Theorem 1.1.

**Theorem 1.2.** Let $f$ be a weak solution constructed in Theorem 1.1, in addition,

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |\xi|^{3+m} f d\xi dx \leq K < \infty,$$

(1.11)
for any $t \in [0, T]$, then it preserves the energy conservation, that is, $f$ satisfies

$$
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)(1 + |\xi|^m) f \, d\xi \, dx \, dt = \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0 \, d\xi \, dx,
$$

for any $t \in [0, T]$.

Remark 1.1. The condition (1.11) can ensure the first moment $n$ is bounded in $L^\infty(0, T; L^{\frac{6+4m}{3}}(\mathbb{R}^3))$ and ensure the $m$ moments

$$
\int_{\mathbb{R}^3} |\xi|^m f \, d\xi
$$

is bounded in $L^\infty(0, T; L^{\frac{6+4m}{3+m}}(\mathbb{R}^3))$. Thus, we find that

$$
\int_{\mathbb{R}^3} \mu(n) f(1 + |\xi|^m) \, d\xi
$$

is bounded in $L^\infty(0, T; L^1(\mathbb{R}^3))$. This is a crucial estimate in showing (1.12).

The rest of the paper is organized as follows. In Section 2, we derive the energy inequality and a crucial estimate on the probability function $T(\xi, \xi')$. In Section 3, we construct a sequence of smooth solutions to an approximation with some uniformly estimates. In Section 4, we show the weak stability of the sequence of solutions and study the limiting process to obtain the existence result of global weak solutions. In Section 5, we will show the energy equality (1.12).

2. A PRIORI ESTIMATES

In this section, we aim at deriving a priori estimates of (1.6)-(1.7), which will help us to get the weak stability of the solutions. Firstly, we derive the energy inequality for any smooth solutions of (1.6)-(1.7), which is the estimate in the following Lemma.

**Lemma 2.1.** For any smooth solutions of (1.6)-(1.7), they satisfy the following energy inequality

$$
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)(1 + |\xi|^m) f \, d\xi \, dx \, dt \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0 \, d\xi \, dx,
$$

for any $T > 0$. 

Proof. For any smooth solutions of (1.6)-(1.7), multiplying \((1 + |\xi|^m)\) on both sides of (1.6), one obtains that
\[
\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)(1 + |\xi|^m)f \, d\xi \, dx = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Omega_2(f)(1 + |\xi|^m) \, d\xi \, dx. \tag{2.1}
\]
We calculate the right side term of (2.1) as follows:
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \Omega_2(f)(1 + |\xi|^m) \, d\xi \, dx = -\lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx + \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \, d\xi' (1 + |\xi'|^m) \, d\xi \, dx. \tag{2.2}
\]
Thanks to Fubini’s theorem, (1.3) and (1.4), the second term of right side on (2.2) gives us
\[
\lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \, d\xi' (1 + |\xi'|^m) \, d\xi \, dx = \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \, d\xi' (1 + |\xi'|^m) \, d\xi \, dx \tag{2.3}
\]
This, with (2.2), implies
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \Omega_2(f)(1 + |\xi|^m) \, d\xi \, dx = 0. \tag{2.4}
\]
Combining (2.1) and (2.4), one obtains
\[
\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)(1 + |\xi|^m)f \, d\xi \, dx = 0,
\]
which yields
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)(1 + |\xi|^m)f \, d\xi \, dx \, dt \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m)f^0 \, d\xi \, dx, \tag{2.5}
\]
for any \(T > 0\), at least for the smooth solutions. Thus, it is natural to ask the initial data satisfies the following ones
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m)f^0 \, d\xi \, dx < +\infty. \tag{2.6}
\]
To develop further estimates, we will rely on the following lemma. Under the assumption (1.5), we can show the following Lemma on the probability function \(T(\xi, \xi')\).
Lemma 2.2. Assume that $T(\xi, \xi')$ satisfies (1.4) and (1.5), and $T$ is invariant under rotations of the pair $(\xi, \xi')$, then

$$\int_{\mathbb{R}^3} T(\xi, \xi') \, d\xi' \leq \bar{K} < \infty. \quad (2.7)$$

Proof. The proof is motivated by the work of [9]. From (1.3), we deduce

$$T(\xi, \xi') = 0 \quad \text{if } |\xi| > |\xi'|. \quad (2.8)$$

By (1.4) and (2.8), we find

$$1 = \int_{\mathbb{R}^3} T(\xi, \xi') \, d\xi = \int_{B(0, |\xi'|)} T(\xi, \xi') \, d\xi + \int_{|\xi|>|\xi'|} T(\xi, \xi') \, d\xi$$

$$= \int_{B(0, |\xi'|)} T(\xi, \xi') \, d\xi, \quad (2.9)$$

where $B(0, |\xi'|)$ is the set of points in the interior of a sphere of radius $|\xi'|$, centered at 0. Note that, (1.5) and (2.9), one obtains

$$1 = H(|\xi'|) \int_{B(0, |\xi'|)} T(z, \frac{\xi'}{|\xi'|}) \frac{d\xi}{|\xi'|^3}$$

$$= H(|\xi'|) \int_{B(0, 1)} T(z, \frac{\xi'}{|\xi'|}) \frac{d\xi}{|\xi'|^3} \int_{B(0, 1)} T(z, \frac{\xi'}{|\xi'|}) \frac{d\xi}{|\xi'|^3}$$

$$= H(|\xi'|) \int_{B(0, 1)} T(z, \frac{\xi'}{|\xi'|}) \frac{d\xi}{|\xi'|^3} \int_{B(0, 1)} T(z, \frac{\xi'}{|\xi'|}) \frac{d\xi}{|\xi'|^3}$$

where $z = \frac{\xi'}{|\xi'|}$. This gives us

$$H(|\xi'|) = \frac{1}{|\xi'|^3}, \quad (2.10)$$

and hence

$$\int_{\mathbb{R}^3} T(\xi, \xi') \, d\xi' = \int_{B(0, |\xi'|)} H(|\xi'|) T(z, \frac{\xi'}{|\xi'|}) \, d\xi'$$

$$= \frac{1}{|\xi'|^3} \int_{B(0, |\xi'|)} T(z, \frac{\xi'}{|\xi'|}) \, d\xi'$$

$$= \frac{1}{|\xi'|^3} \int_{B(0, 1)} T(z, \frac{\xi'}{|\xi'|}) \, d\xi'$$

$$\leq \bar{K},$$

where we used (1.3) and (2.10), $\bar{K} > 0$ is a fixed number. \qed

With these two lemmas, we are ready to construct smooth solutions of a suitable approximation and pass to the limits to recover the weak solutions.
3. Regularized equation

The subjective of this section is to construct a sequence of smooth solutions to a regularized equation and to derive some uniformly estimates on them. In particular, we construct a sequence of solutions verifying the following regularized equation

$$\begin{cases}
(f_k)_t + \xi \cdot \nabla f_k = -\mu (n_{k-1}) f_k - \lambda f_k + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_{k-1}(t, x, \xi') d\xi', \\
f_k(0, x, \xi) = f_0^k(x, \xi), \\
n_{k-1} = \int_{\mathbb{R}^3} f_{k-1} d\xi, \\
f_0(t, x, \xi) = 0,
\end{cases}$$

(3.1)

where $k \geq 0$ are integers, $h_{\epsilon}(x) = h \ast \eta_{\epsilon}(x)$. In fact, a similar approximation in [9] motivated us to propose the above ones. For any given $f_{k-1}$, we can solve regularized equation (3.1) by the characteristic method. In particular, the smooth solution of the following ODE:

$$\begin{cases}
\frac{dx}{dt} = \xi; \\
x(0) = x
\end{cases}$$

(3.2)

is given by $x(t) = x + t\xi$. Thus, along the characteristic line

$$x(t) = x + t\xi,$$

the initial value problem (3.1) corresponds to the following ODE:

$$\begin{cases}
\frac{df_k}{dt}(t, x(t), \xi) = -\mu (n_{k-1}) f_k(t, x(t), \xi) - \lambda f_k(t, x(t), \xi) \\
+ \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_{k-1}(t, x(t), \xi') d\xi', \\
n_{k-1} = \int_{\mathbb{R}^3} f_{k-1} d\xi, \\
f_k(0, x, \xi) = f_0^k(x, \xi).
\end{cases}$$

(3.3)

Our task is to solve $f_k$ for any given $f_{k-1}$. By means of the classical theory of ODE, there exists a smooth solution to (3.1)

$$f_k(t, x, \xi) = e^{-\int_0^t (\mu (n_{k-1})(s) + \lambda) ds} f_0^k(x + t\xi, \xi)$$

$$+ \int_0^t \int_{\mathbb{R}^3} e^{-\int_\tau^t (\mu (n_{k-1})(s) + \lambda) ds} T(\xi, \xi') f_{k-1}(\tau, x(\tau), \xi') d\xi' d\tau.$$  

(3.4)

This yields

$$f_k \geq 0 \quad \text{for all } k \geq 0.$$

The following lemma gives us a general version of Gronwall inequality for a sequence of the nonnegative continuous functions. It allows us to derive further uniformly estimates of $f_k$. This Gronwall inequality was proved by induction in the paper of Boudin-Desvillettes-Grandmont-Moussa [9]. We rely on it to deduce our several key estimates in this current paper.
Lemma 3.1. Let $T > 0$, a sequence $\{a_n\}_{n=0}^\infty$ of nonnegative continuous function on $[0, T]$, for any $n \geq 0$, if

$$a_n(t) \leq A + B \int_0^t a_{n-1}(s) \, ds, \quad \text{for any } 0 \leq t \leq T,$$

then, there exists a constant $K \geq 0$ such that

$$a_n(t) \leq K e^{Kt}, \quad \text{for any } 0 \leq t \leq T,$$

where

$$K = \max\{A, B, \sup_{0 \leq t \leq T} a_0\}.$$

In (3.4), the value of $f_k$ depends on $f_{k-1}$, thus we apply Lemma 3.1 to get the following lemma on the estimates of $f_k$.

Lemma 3.2. If $f_k(t, x, \xi)$ is given by (3.4), for any $k \geq 0$, then $f_k(t, x, \xi)$ is bounded in

$$L^\infty(0, T; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3)),
$$

and hence

$$f_k(t, x, \xi) \text{ is bounded in } L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \quad \text{for any } p \geq 1. \quad (3.5)$$

Proof. Using (3.4) and (2.7), we deduce

$$\|f_k(t, x, \xi)\|_{L^\infty} \leq \|f^0_k\|_{L^\infty} + \lambda \overline{K} \int_0^t \|f_{k-1}(\tau, x, \xi)\|_{L^\infty} \, d\tau. \quad (3.6)$$

Applying Lemma 3.1 to (3.6), one obtains

$$\|f_k\|_{L^\infty} \leq \lambda K \overline{K} e^{Kt} \quad \text{for any } 0 \leq t \leq T,$$

where

$$K = \max\{\|f^0_k\|_{L^\infty}, 1, \|f_0\|_{L^\infty}\}.$$ 

Thus, $f_k(t, x, \xi)$ is bounded in $L^\infty(0, T; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ for any $k \geq 0$.

By (3.3), for any $0 \leq t \leq T$, we find

$$\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_k(t, x(t), \xi) \, d\xi \, dx + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n_{k-1}) f_k(t, x(t), \xi) \, dx \, d\xi

+ \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_k(t, x(t), \xi) \, dx \, d\xi

= \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} T(\xi, \xi') f_{k-1}(t, x(t), \xi') \, d\xi' \, dx. \quad (3.7)$$

The term on the right side of (3.7) is given by

$$\lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_{k-1}(t, x(t), \xi) \, d\xi \, dx.$$
Thus, (3.7) gives us
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_k \, d\xi \, dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} f^0 \, d\xi \, dx + \lambda \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_{k-1}(\tau, x(\tau), \xi) \, d\xi \, d\tau.
\]
Thanks to Lemma 3.1, there exists a constant \( K > 0 \), for any \( 0 \leq t \leq T \), such that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |f_k| \, d\xi \, dx \leq \lambda Ke^{Kt}.
\]
Thus, \( f_k(t, x, \xi) \) is bounded in \( L^\infty(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \). □

The estimates in the following Lemma 3.3 are crucial to control the bounds of the kinetic density and the kinetic current. The proof is also based on Lemma 3.1 again.

**Lemma 3.3.** If \( f_k(t, x, \xi) \) is given by (3.4), and
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0 \, d\xi \, dx < +\infty
\]
for some \( m \geq 1 \), then, there exists a constant \( K > 0 \), such that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k \, d\xi \, dx \leq \lambda Ke^{Kt},
\]
(3.8)
for any \( 0 \leq t \leq T \).

**Proof.** For some \( m \geq 1 \), using \( 1 + |\xi|^m \) to multiply on the both sides of (3.3), one obtains the following energy law
\[
\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k \, d\xi \, dx + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n_{k-1})(1 + |\xi|^m) f_k \, d\xi \, dx
\]
\[
+ \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k \, d\xi \, dx = \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_{k-1}(t, x(\xi), \xi) \, d\xi \, d\xi.
\]
(3.9)
Thanks to (1.3), (1.4), and the Fubini’s theorem, the right side term on (3.9) is given by
\[
\lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_{k-1}(t, x, \xi)(1 + |\xi|^m) \, d\xi \, dx.
\]
This, with (3.9), yields
\[
\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k \, d\xi \, dx + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n_{k-1})(1 + |\xi|^m) f_k \, d\xi \, dx
\]
\[
+ \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k \, d\xi \, dx = \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_{k-1}(t, x, \xi)(1 + |\xi|^m) \, d\xi \, dx.
\]
(3.10)
Integrating on both sides of (3.11) with respect to $t$, one obtains
\begin{equation}
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k \, d\xi \, dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k^0 \, d\xi \, dx \\
+ \lambda \int_{0}^{t} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_{k-1}(\tau, x(\tau, \xi), \xi)(1 + |\xi|^m) \, d\xi \, d\tau.
\end{equation}
(3.11)
Thus, applying Lemma 3.1 to (3.11), there exists a constant $K > 0$, such that
\begin{equation}
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k \, d\xi \, dx \leq \lambda Ke^{Kt}
\end{equation}
(3.12)
for any $0 \leq t \leq T$ and any $k \geq 0$, where
\[ K = \max \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k^0 \, d\xi \, dx, \lambda, \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_0 \, d\xi \, dx \right\}. \]
\[ \square \]

**Remark 3.1.** Lemma 3.3 allows us to deduce that the estimates on the kinetic density and the kinetic current. Those estimates, with the help of $L^p$ regularity of average velocities, yield the strong convergence of them in $L^p$ space.

To our convenience, we introduce the kinetic density (zero moment)
\[ n_k(t, x) = \int_{\mathbb{R}^N} f_k \, d\xi, \]
and the kinetic current (first moment)
\[ j_k(t, x) = \int_{\mathbb{R}^N} \xi f_k \, d\xi \]
in the space $\mathbb{R}^N$ with respect to $\xi$.

We estimate these quantities in the following lemma 3.4 that may be similar to the variation of the classical regularity of moments, see [11]. The estimates of the kinetic density and the kinetic current help us to get the weak stability of kinetic equation.

**Lemma 3.4.** For any $p \geq 1$, $0 \leq t \leq T$, if $f_k$ is bounded in $L^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times [0, T])$, we have
\[ \|n_k(t, x)\|_{L^\infty(0, T; L^{\frac{N+p}{N}}(\mathbb{R}^N))} \leq C_{N,T}(\|f_k\|_{L^\infty} + 1) \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^p f_k \, d\xi \, dx \right)^{\frac{N}{N+p}}, \]
\[ \|j_k\|_{L^\infty(0, T; L^{\frac{N+p}{N+1}}(\mathbb{R}^N))} \leq C_{N,T}(\|f_k\|_{L^\infty} + 1) \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^p f_k \, d\xi \, dx \right)^{\frac{N+1}{N+p}}. \]
Proof. The proof is following the same line in the work of Hamdache [5]. For any $R > 0$, we estimate $n_k$ as follows
\[
 n_k(t, x) = \int_{\mathbb{R}^N} f_k \, d\xi = \int_{|\xi| \leq R} f_k \, d\xi + \int_{|\xi| \geq R} f_k \, d\xi 
 \leq C_N R^N \|f_k\|_{L^\infty} + \frac{1}{R^p} \int_{|\xi| \geq R} |\xi|^p f_k \, d\xi.
\]
(3.13)

Taking
\[
 R = \left( \int_{\mathbb{R}^N} |\xi|^p f_k \, d\xi \right)^{\frac{1}{N+p}},
\]
one obtains
\[
 n_k(t, x) \leq C_N (\|f_k\|_{L^\infty} + 1) \left( \int_{\mathbb{R}^N} |\xi|^p f_k \, d\xi \right)^{\frac{N}{N+p}}.
\]
This yields
\[
 \|n_k(t, x)\|_{L^\infty(0, T; L^\frac{N+p}{N}(\mathbb{R}^N))} \leq C_N (\|f_k\|_{L^\infty} + 1) \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^p f_k \, d\xi \, dx \right)^{\frac{N}{N+p}},
\]
where $f_k$ is bounded in $L^\infty$ due to (3.5).

Following the same arguments, we can show
\[
 \|j_k(t, x)\|_{L^\infty(0, T; L^\frac{3+m}{3}(\mathbb{R}^3))} \leq C_N (\|f_k\|_{L^\infty} + 1) \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\xi|^m f_k \, d\xi \, dx \right)^{\frac{3}{3+m}},
\]
(3.14)
\[
 \|j_k(t, x)\|_{L^\infty(0, T; L^\frac{3+m}{4}(\mathbb{R}^3))} \leq C_N (\|f_k\|_{L^\infty} + 1) \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\xi|^m f_k \, d\xi \, dx \right)^{\frac{3}{3+m}}.
\]
(3.15)

On one hand, in three dimensional space, Lemma 3.4 implies
\[
 \|n_k(t, x)\|_{L^\infty(0, T; L^\frac{3+m}{3}(\mathbb{R}^3))} \leq C_N (\|f_k\|_{L^\infty} + 1) \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k \, d\xi \, dx \right)^{\frac{3}{3+m}},
\]
(3.16)
for any $0 \leq t \leq T$, where $K > 0$ only depends on the initial data.

Thus, the right sides of (3.14)-(3.15) can be controlled by the initial data, thanks to (3.16). This yields the kinetic density
\[
 n_k(t, x) \text{ is bounded in } L^\infty(0, T; L^r(\mathbb{R}^3)) \text{ for any } 1 \leq r \leq \frac{3 + m}{3},
\]
(3.17)
and the kinetic current
\[
 j_k(t, x) \text{ is bounded in } L^\infty(0, T; L^s(\mathbb{R}^3)) \text{ for any } 1 \leq s \leq \frac{3 + m}{4}.
\]
(3.18)
Thus, we have proved the following proposition on the existence of solution to initial value problem (3.1) in this section.

**Proposition 3.1.** For any given \( \varepsilon > 0 \), \( k \geq 0 \) and \( T > 0 \), under the assumption of Theorem 1.1, there exists a smooth solution to initial value problem (3.1) which is given by (3.4). Moreover, the solution satisfies the energy equality (3.9), and the estimates of (3.5), (3.8), (3.17), and (3.18).

**Remark 3.2.** The solution constructed in above Proposition 3.1 is a smooth solution, it obeys the energy equality (3.10).

4. Recover the weak solutions

The proof of Theorem 1.1 will be developed in this current section. It relies on the introduction of a regularized equation (3.1) (for which the existence of a solutions is given in Proposition 3.1 in fact, it is a smooth solution for any given \( k > 0 \) and \( \varepsilon > 0 \)). In order to recover the weak solution to (1.6)-(1.7), we shall pass to the limits as \( k \) goes to large and \( \varepsilon \) tends to zero, show that the limit function is a weak solution to initial value problem (1.6)-(1.7). Thus, we need some convergence on the function \( f_{k,\varepsilon}(t, x, \xi) \), the particles density (zero moment) \( n_{k,\varepsilon}(t, x) \) and the kinetic current (first moment) \( j_{k,\varepsilon}(t, x) \) in \( L^p \) space for some \( p > 1 \). In the following subsections, we will handle the limits with respect to \( k \) in Subsection 4.1 and pass to the limits with respect to \( \varepsilon \) in Subsection 4.2.

4.1. **Passing to the limits as \( k \to \infty \).** In this subsection, we use \( \{f_k\}_{k=0}^{\infty} \) to denote the sequence of solutions to (3.1) that constructed in Proposition 3.1 for any fixed \( \varepsilon > 0 \). Here we aim at passing to the limits as \( k \) goes to infinity for any given \( \varepsilon > 0 \).

The estimates of Lemma 3.2 and Lemma 3.3 are crucial ones in this subsection. In fact, these solutions satisfy, for all \( T > 0 \),

\[
\begin{align*}
\|f_k\|_{L^\infty(0,T;L^p(\mathbb{R}^3 \times \mathbb{R}^3))} &\leq C, \quad \text{for any } 1 \leq p \leq +\infty, \\
\|n_k(t,x)\|_{L^\infty(0,T;L^r(\mathbb{R}^3))} &\leq C, \quad \text{for any } 1 \leq r \leq \frac{3+m}{4}, \\
\|j_k(t,x)\|_{L^\infty(0,T;L^s(\mathbb{R}^3))} &\leq C, \quad \text{for any } 1 \leq s \leq \frac{3+m}{4},
\end{align*}
\]

(4.1)

where all \( C > 0 \) only depend on the initial data.

First of all, by (4.1), it follows that there exists a function \( f(t, x, \xi) \) such that

\[
f_k \rightharpoonup f \quad \text{weakly in } L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)), \quad \text{for any } p > 1.
\]

(4.2)

In particular, this limit function \( f(t, x, \xi) \) is bounded in \( L^\infty(0, T; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)) \).

Due to nonlinear term \( \mu(n)f \) in (1.6), or the nonlinear term \( \mu(n_{k-1})f_k \) in (3.4), we shall study the strong convergence of \( n_k \) and \( j_k \) in some \( L^p \) space in the following Lemma 4.4.
Lemma 4.1. Let $f_k$ be a solution to (3.1) constructed in Proposition 3.1, then

\[
\begin{align*}
n_k & \to n \text{ strongly in } L^\infty(0,T; L^r_{\text{loc}}(\mathbb{R}^3)), \\
j_k & \to j \text{ strongly in } L^\infty(0,T; L^1_{\text{loc}}(\mathbb{R}^3)),
\end{align*}
\] (4.3)

\[
\int_{\mathbb{R}^3} (1 + |\xi|^m) f_k d\xi \to \int_{\mathbb{R}^3} (1 + |\xi|^m) f d\xi \text{ strongly in } L^\infty(0,T; L^1_{\text{loc}}(\mathbb{R}^3)).
\] (4.5)

Proof. Thanks to (3.8) with $m = 2$, $f_k$ is bounded in $L^\infty(0,T; L^1(\mathbb{R}^3 \times \mathbb{R}^3), 1 + |\xi|^2)$.

By (3.8) and (3.10) for any $m \geq 1$, one obtains

\[
\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) \mu(n_{k-1}) f_k d\xi dx dt \leq C.
\] (4.6)

This yields

\[
\mu(n_{k-1}) f_k \text{ is bounded in } L^1(0,T; L^1(\mathbb{R}^3 \times \mathbb{R}^3)).
\] (4.7)

Meanwhile, the Fubini’s Theorem gives us

\[- \lambda f_k + \lambda \int T(\xi, \xi') f_{k-1}(t,x,\xi') d\xi' \text{ is bounded in } L^\infty(0,T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)),\]

thanks to (4.1).

By (4.7) and (4.8), we deduce from Diperna-Lions-Meyer’ $L^p$ regularity of average velocities, (see [6]), for each $\phi(\xi) \in \mathcal{D}(\mathbb{R}^3)$,

\[
\left\{ \int_{\mathbb{R}^3} f^k \phi(\xi) d\xi \right\}_k
\]

is relatively compact in $L^1(0,T; L^1(B_R))$, for any $R < +\infty$. In particular, we find

\[
n_k = \int_{\mathbb{R}^3} f^k d\xi \to n = \int_{\mathbb{R}^3} f d\xi, \text{ a.e.}
\]

\[
j_k = \int_{\mathbb{R}^3} \xi f^k d\xi \to j = \int_{\mathbb{R}^3} \xi f d\xi, \text{ a.e.}
\]

and

\[
\int_{\mathbb{R}^3} (1 + |\xi|^m) f_k d\xi \to \int_{\mathbb{R}^3} (1 + |\xi|^m) f d\xi \text{ a.e.}
\] (4.9)

Thanks to the estimates of (4.1) for $n_k$ and $j_k$, we strengthen the above convergence of $n_k$ and $j_k$ as follows

\[
n_k \to n \text{ strongly in } L^\infty(0,T; L^r_{\text{loc}}(\mathbb{R}^3)),
\]

and

\[
j_k \to j \text{ strongly in } L^\infty(0,T; L^1_{\text{loc}}(\mathbb{R}^3)).
\]

Using (3.16), we strengthen the above convergence of $\int_{\mathbb{R}^3} (1 + |\xi|^m) f_k d\xi$ as follows

\[
\int_{\mathbb{R}^3} (1 + |\xi|^m) f_k d\xi \to \int_{\mathbb{R}^3} (1 + |\xi|^m) f d\xi \text{ strongly in } L^\infty(0,T; L^1_{\text{loc}}(\mathbb{R}^3)).
\]
With (4.1) at hand, we are ready to show the convergence of \( \nu(n-k-1)n_k \) in the sense of distributions on \((0, T) \times \mathbb{R}^3\). We address it in the following lemma.

**Lemma 4.2.** If \( f_k \) is given by (3.4), then

\[
\int_{\mathbb{R}^3} \mu(n-k-1)f_k \, d\xi \to \int_{\mathbb{R}^3} \mu(n) \, d\xi
\]

in \( L^\infty(0, T; L^p_{loc}(\mathbb{R}^3)) \) as \( k \to \infty \). Moreover, if \( m \geq 3 \), then \( \frac{p}{2} \geq 1 \).

**Proof.** Firstly,

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n-k-1)f_k \, d\xi \, dx = \int_{\mathbb{R}^3} \mu(n-k-1)n_k \, dx,
\]

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)f \, d\xi \, dx = \int_{\mathbb{R}^3} \mu(n)n \, dx.
\]  \( \text{(4.11)} \)

Since \( \mu(\cdot) \) is a Lipschitz function and (4.3), then

\[
\mu(n-k-1) \to \mu(n) \text{ strongly in } L^\infty(0, T; L^p_{loc}(\mathbb{R}^3)).
\]  \( \text{(4.12)} \)

Thanks to (4.3), (4.11) and (4.12), we find, for any \( m \geq 3 \),

\[
\mu(n-k-1)n_k \to \mu(n)n
\]

in \( L^\infty(0, T; L^p_{loc}(\mathbb{R}^3)) \) as \( k \to \infty \), and \( \frac{p}{2} \geq 1 \). \( \square \)

To pass to the limits with respect to \( k \), we still need the following lemma on the convergence of

\[
\mathcal{Q}_2(f_k) = -\lambda f_k + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f_{k-1}(t, x, \xi') \, d\xi'
\]

in some \( L^p \) space.

**Lemma 4.3.** Let \( f_k \) be given by (3.4), then \( \mathcal{Q}_2(f_k) \) is bounded in

\[
L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3))
\]

for any \( p \geq 1 \), and

\[
\mathcal{Q}_2(f_k) \to \mathcal{Q}_2(f) \text{ weakly in } L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \text{ for any } p > 1,
\]  \( \text{(4.13)} \)

as \( k \to \infty \).

**Proof.** The proof follows the same line as in [9] and we sketch it just for sake of completeness. We estimate

\[
\|\mathcal{Q}_2(f_k)\|_{L^\infty} \leq \lambda\|f_k\|_{L^\infty} + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') \, d\xi' \|f_{k-1}\|_{L^\infty} \\
\leq \lambda\|f_k\|_{L^\infty} + \lambda K\|f_{k-1}\|_{L^\infty},
\]  \( \text{(4.14)} \)
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |\Omega_2(f_k)| \, d\xi \, dx = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left| -\lambda f_k + \lambda \int T(\xi, \xi') f_{k-1}(t, x, \xi') \, d\xi' \right| \, d\xi \, dx \\
\leq \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_k \, d\xi \, dx + \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} T(\xi, \xi') f_{k-1}(t, x, \xi') \, d\xi' \, d\xi \, dx \\
\leq \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_k \, d\xi \, dx + \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_{k-1} \, d\xi \, dx,
\]
where we used \( f_k(t, x, \xi) \geq 0 \) for all \( k \) and (1.4). Thus,

\[
\left\| \Omega_2(f_k) \right\|_{L^\infty(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3))} \leq C \left\| f_k \right\|_{L^\infty(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3))}.
\]

(4.15)

Using (4.14) and (4.15), \( \Omega_2(f_k) \) is bounded in \( L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \) for any \( p \geq 1 \).

For any \( \varphi(x) \in L^p(0, T; L^q(\mathbb{R}^3)) \), we consider

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( \int T(\xi, \xi') f_{k-1}(t, x, \xi') \, d\xi' - \int T(\xi, \xi') f(t, x, \xi') \, d\xi' \right) \varphi(x) \, d\xi \, dx \\
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} T(\xi, \xi') (f_{k-1}(t, x, \xi') - f(t, x, \xi')) \varphi(x) \, d\xi' \, d\xi \, dx \\
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} (f_{k-1}(t, x, \xi) - f(t, x, \xi)) \varphi(x) \, d\xi \, dx \rightarrow 0
\]

(4.16)
as \( k \rightarrow \infty \), thanks to (4.12).

With the help of (4.12), (4.15) and (4.16), \( \Omega_2(f_k) \) converges to \( \Omega(f) \) weakly in \( L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \) for any \( p > 1 \).

Turning to the next issue, the smooth solution \( f_k \) of (3.1) satisfies the following weak formulation

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f^0 \varphi(0, x, \xi) \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_k \varphi_t \, d\xi \, dx \, dt + \xi f_k \cdot \nabla_x \varphi \, d\xi \, dx \, dt + \\
\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n_{k-1}) f_k \varphi \, d\xi \, dx \, dt - \lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_k \varphi \, d\xi \, dx \, dt \\
+ \lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]} T(\xi, \xi') f_{k-1}(t, x, \xi') \varphi \, d\xi \, dx \, dt,
\]

(4.17)

where

\[
n_{k-1} = \int_{\mathbb{R}^3} f_{k-1} \, d\xi,
\]
and \( \varphi \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]) \) is any test function.
Letting $k$ tends to infinity in (4.17), using the above convergence, in particularly, by (4.2), (4.3), (4.4), (4.10), (4.13), one obtains
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0 \varphi(0, x, \xi) d\xi dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \varphi_t + \xi f \cdot \nabla_x \varphi d\xi dx dt + \\
\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n) f \varphi d\xi dx dt = -\lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \varphi d\xi dx dt + \\
+ \lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \varphi d\xi dx dt,
\]
and
\[
n = \int_{\mathbb{R}^3} f d\xi.
\]

Concerning the energy inequality, it is reasonable to expect that a priori estimate we present can be further developed. Taking integration on both sides of (3.10) with respect to $t$, one obtains
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k d\xi dx + \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n_{k-1})(1 + |\xi|^m) f_k d\xi dx dt \\
= -\lambda \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k d\xi dx dt + \\
+ \lambda \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_{k-1}(1 + |\xi|^m) d\xi dx dt + \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_0^\varepsilon d\xi dx.
\]
Thanks to (4.5) and (4.9), the Fatou’s Lemma yields
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f d\xi dx \leq \liminf_k \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_k d\xi dx.
\]
By (3.8) and (4.9), the sum of the first two terms on the right side of (4.19) is zero.

Letting $k$ goes to infinite in (4.19), we have
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f d\xi dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_0^\varepsilon d\xi dx,
\]
for any $0 \leq t \leq T$.

Thus, we have proved the following existence of weak solutions in this subsection by letting $k$ tends to infinity.

**Proposition 4.1.** For any given $\varepsilon > 0$, under assumption of Theorem 1.1, there exists a weak solution for any $T > 0$ to the following initial value problem
\[
\begin{cases}
    f_t + \xi \cdot \nabla f = -\mu(n)f - \lambda f + \lambda \int T(\xi, \xi') f(t, x, \xi') d\xi', \\
    f(0, x, \xi) = f_0^\varepsilon(x, \xi), \\
    n = \int_{\mathbb{R}^3} f d\xi,
\end{cases}
\] (4.20)
where \( h_\varepsilon(x) = h * \eta_\varepsilon(x) \). Moreover, the weak solution \( f(t, x, \xi) \) has the following energy inequality

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0_\varepsilon \, d\xi \, dx,
\]

for any \( 0 \leq t \leq T \), and

\[
\| f \|_{L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3))} \leq C,
\]

for any \( 1 \leq p \leq +\infty \), where \( C \) only depends on the initial data.

### 4.2. Passing to the limits as \( \varepsilon \to 0 \)

In this subsection, we use \( \{f_\varepsilon\}_{\varepsilon > 0} \) to denote a sequence of solutions to initial value problem (4.20) that constructed in Proposition 4.1 for any \( \varepsilon > 0 \). Here we aim to pass to the limits for recovering the weak solutions to (1.6)-(1.7) as \( \varepsilon \) goes to zero.

On one hand, by Proposition 4.1, these solutions have the following estimate

\[
\| f_\varepsilon \|_{L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3))} \leq C,
\]

for any \( 1 \leq p \leq +\infty \), where \( C \) only depends on the initial data. This yields,

\[
f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^\infty(0, T; L^p(\mathbb{R}^3 \times \mathbb{R}^3))
\]

for any \( 1 < p < \infty \).

On the other hand, the solutions obey the following energy inequality,

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_\varepsilon \, d\xi \, dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0_\varepsilon \, d\xi \, dx,
\]

for all \( \varepsilon > 0 \).

By the definition of \( f^0_\varepsilon \), thus

\[
f^0_\varepsilon \to f^0 \quad \text{a.e. as } \varepsilon \to 0, \quad \text{and } f^0_\varepsilon \to f^0 \quad \text{in } L^p(\mathbb{R}^3 \times \mathbb{R}^3)
\]

for any \( p > 1 \). Let us to denote \( \overline{g}(t, x, \xi) = g * \eta_\varepsilon(x) \), where \( \{\eta_\varepsilon\}_{\varepsilon > 0} \) is a suitable family of regularizing kernels with respect to the space variable \( x \). We state the following lemma:

**Lemma 4.4.** Let \( \xi \) be the third variable, then, for any function \( h = h(\xi) \), we have

\[
\int_{\mathbb{R}^3} \overline{g} h(\xi) \, d\xi = \int_{\mathbb{R}^3} g h \, d\xi.
\]

**Proof.** Let us to calculate the left hand side,

\[
\int_{\mathbb{R}^3} \overline{g} h(\xi) \, d\xi = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta_\varepsilon(y - x) g(t, y, \xi) h(\xi) \, dy \, d\xi,
\]

and the right hand side is as follows

\[
\int_{\mathbb{R}^3} g(t, x, \xi) h(\xi) \, d\xi = \int_{\mathbb{R}^3} \eta_\varepsilon(y - x) \left( \int_{\mathbb{R}^3} g(y, \xi) h(\xi) \, d\xi \right) \, dy.
\]

By the Fubini’s theorem, we proved this lemma. \( \square \)
We use \( f_0 = f_0^\varepsilon \), by Lemma \( 4.3 \), we find that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_0^\varepsilon d\xi dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |\xi|^m) f^0 d\xi dx.
\]
Note that,
\[
\int_{\mathbb{R}^3} (1 + |\xi|^m) f^0 d\xi
\]
is uniformly bounded in \( L^1(\mathbb{R}^3) \), thus, for any \( R \in (0, \infty) \),
\[
\int_{B_R} \int_{\mathbb{R}^3} (1 + |\xi|^m) f^0 d\xi dx \to \int_{B_R} \int_{\mathbb{R}^3} (1 + |\xi|^m) f^0 d\xi dx,
\]
where \( B_R \) is a ball with radius \( R > 0 \) and center at zero in \( \mathbb{R}^3 \). \( (4.25) \) is true for any \( R > 0 \), thus Letting \( R \) go to \(+\infty\) in \( (4.25) \), we have
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |\xi|^m) f_0^\varepsilon d\xi dx \to \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |\xi|^m) f_0^0 d\xi dx,
\]
and hence, as \( \varepsilon \) tends to zero,
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_0^\varepsilon d\xi dx \to \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_0^0 d\xi dx
\]
as \( \varepsilon \to 0 \). Using \( (4.26) \), we deduce
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f_\varepsilon d\xi dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0 d\xi dx < +\infty
\]
for all \( \varepsilon > 0 \).

Thanks to \( (4.27) \), we employ the same argument as in last subsection, to have, for any \( \varepsilon \to 0 \),
\[
n_\varepsilon \to n \quad \text{strongly in } L^\infty(0, T ; L^r_{loc}(\mathbb{R}^3)),
\]
where \( n = \int_{\mathbb{R}^3} f d\xi \);
\[
j_\varepsilon \to j \quad \text{strongly in } L^\infty(0, T ; L^s_{loc}(\mathbb{R}^3)),
\]
where \( j = \int_{\mathbb{R}^3} f \xi d\xi \); and
\[
\int_{\mathbb{R}^3} \mu(n_\varepsilon) f_\varepsilon d\xi \to \int_{\mathbb{R}^3} \mu(n) f d\xi
\]
in \( L^\infty(0, T ; L^1_{loc}(\mathbb{R}^3)) \). From \( (1.23) \), as \( \varepsilon \to 0 \), one obtains
\[
\Omega_2(f_\varepsilon) = -\lambda f_\varepsilon + \lambda \int T(\xi, \xi') f_\varepsilon(t, x, \xi') d\xi' \to \\
\Omega_2(f) = -\lambda f + \lambda \int T(\xi, \xi') f(t, x, \xi') d\xi'
\]
weakly in \( L^\infty(0, T ; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \) for any \( p \geq 1 \). Thus, we can pass to the limits for recovering the weak solutions to \( (1.6)-(1.7) \) as \( \varepsilon \) tends to zero. In fact, by
(4.23), (4.28)-(4.31), taking the limits as \( \varepsilon \) tends to zero, in the following weak formulation,

\[
\begin{align*}
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0^\varepsilon \varphi(0, x, \xi) \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_\varepsilon \varphi_t + \xi f_\varepsilon \cdot \nabla_x \varphi \, d\xi \, dx \, dt + \\
\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n_\varepsilon) f_\varepsilon \varphi \, d\xi \, dx \, dt &= -\lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_\varepsilon \varphi \, d\xi \, dx \, dt \\
&\quad + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( \lambda \int_{\mathbb{R}^3} T(\xi', \xi') f_\varepsilon(t, x, \xi') \, d\xi' \right) \varphi \, d\xi \, dx \, dt,
\end{align*}
\]

where

\[ n_\varepsilon = \int_{\mathbb{R}^3} f_\varepsilon \, d\xi; \]

then

\[
\begin{align*}
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0^0 \varphi(0, x, \xi) \, d\xi \, dx + \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \varphi_t + \xi f \cdot \nabla_x \varphi \, d\xi \, dx \, dt + \\
\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n) f \varphi \, d\xi \, dx \, dt &= -\lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \varphi \, d\xi \, dx \, dt \\
&\quad + \lambda \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( \int_{\mathbb{R}^3} T(\xi', \xi') f(t, x, \xi') \, d\xi' \right) \varphi \, d\xi \, dx \, dt,
\end{align*}
\]

where

\[ n = \int_{\mathbb{R}^3} f \, d\xi. \]

Moreover, same to last subsection, letting \( \varepsilon \to 0 \) in (4.27), we have

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \, dx \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |\xi|^m) f^0 \, d\xi \, dx.
\]

Thus, we proved Theorem 1.1.

5. Energy conservation

In this section, we will prove our second main result on the energy conservation. We can use the following quantities

\[ \Phi(t, x, \xi) = \eta_\varepsilon(y - x) \phi(\xi) \]

as a test function in (1.10), where \( \{\eta_\varepsilon\}_{\varepsilon > 0} \) is a suitable family of regularizing kernels with respect to the space variable \( x \). Using \( \overline{g} = g \ast \eta_\varepsilon \), we deduce

\[
\overline{f_t} + \overline{\xi \cdot \nabla f} + \overline{\mu(n)f} = -\lambda f + \lambda \int_{\mathbb{R}^3} T(\xi', \xi') f(t, x, \xi') \, d\xi'.
\]
We use $1 + |\xi|^m$ to multiply the above equality, then
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(f_t + \xi \cdot \nabla_x f + \mu(n)f)(1 + |\xi|^m)}{1 + |\xi|^m} \, dx \, d\xi
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( -\lambda f + \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \, d\xi' \right) (1 + |\xi|^m) \, dx \, d\xi,
\]
which in turn gives us
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \overline{f}(1 + |\xi|^m) \, dx \, d\xi + \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)f(1 + |\xi|^m) \, dx \, d\xi \, dt
= -\lambda \int_0^T \int_{\mathbb{R}^3} f(t, x, \xi)(1 + |\xi|^m) \, dx \, d\xi \, dt
+ \int_0^t \int_{\mathbb{R}^3} \left( \lambda \int_{\mathbb{R}^3} T(\xi, \xi') f(t, x, \xi') \, d\xi' \right) (1 + |\xi|^m) \, dx \, d\xi \, dt
= 0,
\]
and hence, (5.1) reduces to the following one
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \overline{f}(1 + |\xi|^m) \, dx \, d\xi + \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n)f(1 + |\xi|^m) \, dx \, d\xi \, dt
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \overline{f}_0(1 + |\xi|^m) \, dx \, d\xi. \tag{5.2}
\]
Meanwhile, Lemma 4.4 gives us
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \overline{f}(1 + |\xi|^m) \, d\xi \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(1 + |\xi|^m) \, d\xi \, dx \tag{5.3}
\]
To handle the estimate on $\int_{\mathbb{R}^3} f(1 + |\xi|^m) \, d\xi$, we need additional condition and the following lemma. An argument similar to that given above in the proof to Lemma 3.4 shows that,

**Lemma 5.1.** For any $p \geq 1$, $0 \leq t \leq T$, if $f$ is bounded in $L^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times [0, T])$, we have
\[
\| \int_{\mathbb{R}^3} |\xi|^m f \, d\xi \|_{L^\infty(0, T; L^{3+p/(3+p)}(\mathbb{R}^3))} \leq C_T(\|f\|_{L^\infty} + 1) \left( \int_{\mathbb{R}^3} \|f\|^p \, dx \right)^{\frac{m + 3}{3 + p}}.
\]
Thus, under the additional condition (1.11), Lemma 5.1 gives us
\[ \int_{\mathbb{R}^3} (1 + |\xi|^m) f \, d\xi \]
is uniformly bounded in \( L^\infty(0, T; L^\alpha(\mathbb{R}^3)) \) for some \( \alpha > 1 \). This, with the help of (5.3),
\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \overline{f}(1 + |\xi|^m) \, dx \, d\xi \to \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(1 + |\xi|^m) \, dx \, d\xi, \]
where we adopted the same argument of showing (4.25) and (4.26). Similarly, we obtain
\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(1 + |\xi|^m) \, dx \, d\xi \to \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(1 + |\xi|^m) \, dx \, d\xi. \]
Applying Lemma 4.4 one obtains
\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n) f(1 + |\xi|^m) \, dx \, d\xi = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mu(n) f(1 + |\xi|^m) \, d\xi \, dx. \tag{5.4} \]
By Lemma 3.3 and Lemma 5.1 and the additional condition (1.11), we are able to give a uniform bound on
\[ \int_{\mathbb{R}^3} \mu(n) f(1 + |\xi|^m) \, d\xi \]
in \( L^\infty(0, T; L^\alpha(\mathbb{R}^3)) \) for some \( \alpha \geq 1 \). And hence, using the same argument of showing (4.25) and (4.26) again, as \( \varepsilon \) goes to zero,
\[ \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n) f(1 + |\xi|^m) \, dx \, d\xi \, dt \to \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n) f(1 + |\xi|^m) \, dx \, d\xi \, dt. \]
Finally, letting \( \varepsilon \) go to zero in (5.2), we obtain
\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(1 + |\xi|^m) \, dx \, d\xi + \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mu(n) f(1 + |\xi|^m) \, dx \, d\xi \, dt = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(1 + |\xi|^m) \, dx \, d\xi. \]

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