MM Algorithms for Distance Covariance based Sufficient Dimension Reduction and Sufficient Variable Selection

Runxiong Wu and Xin Chen*

Department of Statistics and Data Science, Southern University of Science and Technology
Shenzhen 518055, China; 11930643@mail.sustech.edu.cn, chenx8@sustech.edu.cn

Abstract
Sufficient dimension reduction (SDR) using distance covariance (DCOV) was recently proposed as an approach to dimension-reduction problems. Compared with other SDR methods, it is model-free without estimating link function and does not require any particular distributions on predictors (see Sheng and Yin, 2013, 2016). However, the DCOV-based SDR method involves optimizing a nonsmooth and nonconvex objective function over the Stiefel manifold. To tackle the numerical challenge, we novelly formulate the original objective function equivalently into a DC (Difference of Convex functions) program and construct an iterative algorithm based on the majorization-minimization (MM) principle. At each step of the MM algorithm, we inexactly solve the quadratic subproblem on the Stiefel manifold by taking one iteration of Riemannian Newton’s method. The algorithm can also be readily extended to sufficient variable selection (SVS) using distance covariance. We establish the convergence property of the proposed algorithm under some regularity conditions. Simulation studies show our algorithm drastically improves the computation efficiency and is robust across various settings compared with the existing method. Supplemental materials for this article are available.

Keywords: Sufficient dimension reduction (SDR); Distance covariance (DCOV); Variable selection; Manifold optimization; Majorization-Minimization (MM); Riemannian Newton’s method;

1 Introduction
In regression analysis, sufficient dimension reduction (SDR) provides a useful statistical framework to analyze a high-dimensional dataset without losing any information. It finds one or the fewest linear combinations of predictors that captures a full regression...
relationship. Let $Y$ be an univariate response and $X = (x_1, \ldots, x_p)\top$ be a $p \times 1$ predictor vector, SDR aims to find a $p \times d$ matrix $\beta$ such that

$$Y \perp X | \beta\top X, \quad (1.1)$$

where $\perp$ denotes the statistical independence. The column space of $\beta$ satisfying (1.1) is called a dimension reduction subspace. Under mild conditions \cite{Cook1996, Yin2008}, the intersection of all the dimension reduction subspaces exists and is unique. In this case, if the intersection itself is also a dimension reduction subspace, we call it the central subspace \cite{Cook1994, Cook1996} for the regression of $Y$ on $X$ and denote it to $S_{Y|X}$. Note that the dimension of $S_{Y|X}$ denoted by $\text{dim}(S_{Y|X})$ is usually much smaller than the original predictor’s dimension $p$. Thus, we reduce the dimensionality of the predictor space. The primary interest of SDR is to find such central subspace $S_{Y|X}$.

Since the introduction of sliced inverse regression (SIR; \cite{Li1991}) and sliced average variance estimation (SAVE; \cite{Cook1991}), many methods have been proposed for estimating the basis of $S_{Y|X}$, including inverse regression (IR; \cite{Cook2005}), directional regression (DR; \cite{Li2007}), minimum average variance estimation method (MAVE; \cite{Xia2002}), sliced regression (SR; \cite{Wang2008}), ensemble approach (\cite{Yin2011}), Fourier transform approach (\cite{Zhu2006}), integral transform method (\cite{Zeng2010}), Kullback-Leibler distance based estimator (\cite{Yin2011}), and semiparametric approach (\cite{Ma2012}), etc.

All of the aforementioned dimension reduction methods require certain conditions on the predictors or complicated smoothing technique. In reality, these conditions are not easy to be verified and the results of these methods may be misleading if the conditions are violated. Recently, \cite{Sheng2013, Sheng2016} proposed a method using distance covariance (DCOV; \cite{Szekely2007, Szekely2009}) for estimating the central subspace $S_{Y|X}$. Distance covariance is an elegant measure that quantifies the dependence strength between two random vectors. Consequently the DCOV-based SDR method requires only mild conditions on the predictors and does not require any link function or nonparametric estimation. It can be also easily extended to handle regression with multivariate responses.

The most challenging part of the DCOV-based SDR method is that it involves solving a nonconvex and nonsmooth optimization problem over the Stiefel manifold. All existing DCOV-based SDR methods are optimized by using sequential quadratic programming (SQP; \cite{Gill1981}). The SQP method works well when the dimension $p$ and the sample $n$ is not too large, but optimization is often computationally difficult for moderately high dimensional settings. Beyond that, the literature on solving this kind of problem is scarce.

In this article, we fill in the blank and propose a new algorithm which presents three major contributions to the literature of sufficient dimension reduction and manifold optimization. First, we novelty write the DCOV objective function of the model as a difference of convex functions equivalently. Therefore we design a highly efficient algorithm for solving the corresponding optimization problem based on the new objective function form. Second, we construct the convergence property of the proposed algorithm over the Stiefel manifold. Third, we extend our method to sufficient variable selection based on distance covariance. Simulation studies show our algorithm is ten to hundred times faster than the methods relying on SQP algorithm.
1.1 Notation and the Stiefel Manifold

The following notations and knowledge about the Stiefel manifold discussed in \cite{Absil2009,Edelman1998} will be used in our exposition. The trace of a matrix $A$ is $\text{tr}(A)$ and the Euclidean inner product of two matrices $A, B$ is $\langle A, B \rangle = \text{tr}(A^\top B)$. We use $\|\cdot\|_2$ and $\|\cdot\|_F$ to denote the Euclidean norm of a vector and the Frobenius norm of a matrix respectively. The notation $\text{St}(d, p) = \{ \gamma \in \mathbb{R}^{p \times d} | \gamma^\top \gamma = I_d \}$ with $d \leq p$ is referred to the Stiefel manifold and $\mathcal{T}_\gamma \text{St}(d, p)$ is the tangent space to $\text{St}(d, p)$ at a point $\gamma \in \text{St}(d, p)$. According to \cite{Edelman1998}, $\mathcal{T}_\gamma \text{St}(d, p) = \{ \gamma U + \gamma_\perp V \mid U \in \text{Skew}(d), V \in \mathbb{R}^{(p-d)\times 1} \}$. Here $\gamma_\perp$ is the orthogonal complement of $\gamma$ and $\text{Skew}(d)$ denotes the set of $d \times d$ skew-symmetric matrices. We use $\text{vec}(W)$ to denote the vector formed by stacking the column vectors of $W$. For a skew-symmetric matrix $W \in \text{Skew}(d)$, $\text{veck}(W)$ denotes a $d(d - 1)/2$-dimensional column vector obtained by stacking the columns of the lower triangular part of $W$. For a square matrix $W$, we use $\text{sym}(W) = (W + W^\top)/2$ and $\text{skew}(W) = (W - W^\top)/2$ to denote the symmetric and skew-symmetric parts of $W$ respectively. Induced from the Euclidean inner product, the Riemannian metric on $\text{St}(d, p)$ we consider here is defined as $\langle \xi_1, \xi_2 \rangle_\gamma = \text{tr}(\xi_1^\top \xi_2)$, for any $\xi_1, \xi_2 \in \mathcal{T}_\gamma \text{St}(d, p)$. Under this metric, the orthogonal projection of $W$ onto the tangent space $\mathcal{T}_\gamma \text{St}(d, p)$ is expressed as $\text{Proj}_{\mathcal{T}_\gamma \text{St}(d, p)}(W) = W - \gamma \text{sym}(\gamma^\top W)$. Let $f$ be a smooth function and $\nabla f$ be the Euclidean gradient, the Riemannian gradient of point $\gamma \in \text{St}(d, p)$ is defined as $\text{grad}(f)(\gamma) = \text{Proj}_{\mathcal{T}_\gamma \text{St}(d, p)}(\nabla f(\gamma))$. Correspondingly, the Riemannian Hessian of point $\gamma \in \text{St}(d, p)$ acting on $\xi \in \mathcal{T}_\gamma \text{St}(d, p)$ is defined as $\text{Hess}(f)(\gamma)[\xi] = \text{Proj}_{\mathcal{T}_\gamma \text{St}(d, p)}(D(\text{grad} f)(\gamma)[\xi])$ and $D(\text{grad} f)(\gamma)[\xi]$ is the directional derivative of $\text{grad} f(\gamma)$ along the direction $\xi$. We use $\text{Retr}$ to denote the retraction operation. For the Stiefel manifold, the QR retraction is used in the article.

1.2 Organization

The rest of the article is organized as follows. Section 2 reviews briefly key knowledge of the DCOV-based SDR method and illustrates our motivation. Section 3 describes the proposed algorithm for solving DCOV-based SDR models in details and Section 4 extends the proposed algorithm to DCOV-based SVS models. In Section 5, we evaluate the superior numeric performance of the proposed algorithm through various simulation studies. Finally, we draw some concluding remarks about the article in Section 6. All proofs and extra simulation studies are given in the Appendix.

2 Background Review and Motivation

2.1 DCOV-based SDR Model

Let $(X, Y) = \{(X_i, Y_i) : i = 1, \ldots, n\}$ be a random sample from $(X, Y)$. $X$ denotes a $p \times n$ data matrix and $Y$ denotes a $1 \times n$ response data matrix. We present here an univariate response, however, the method can naturally be extended to multivariate responses without any issue due to the nature of DCOV. The empirical solution of DCOV-based SDR method for these $n$ observations relies on solving the following objective function:

$$\max_{\beta \in \mathbb{R}^{p \times d}} \frac{1}{n^2} \sum_{k,l=1}^{n} A_{kl}(\beta) B_{kl}, \text{ s.t., } \beta^\top \Sigma X \beta = I_d,$$

(2.1)
where $\hat{\Sigma}_X$ is the sample covariance matrix of $X$, $I_d$ is a $d$-dimensional identity matrix and for $k, l = 1, \ldots, n$,

$$A_{kl}(\beta) = a_{kl}(\beta) - \bar{a}_k(\beta) - \bar{a}_l(\beta) + \bar{a}.,(\beta)$$

$$a_{kl}(\beta) = \|\beta^\top X_k - \beta^\top X_l\|_2; \quad \bar{a}_k(\beta) = \frac{1}{n} \sum_{l=1}^n a_{kl}(\beta),$$

$$\bar{a}_l(\beta) = \frac{1}{n} \sum_{k=1}^n a_{kl}(\beta); \quad \bar{a}.,(\beta) = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta).$$

Similarly, define $b_{kl} = \|Y_k - Y_l\|_2$ and $B_{kl} = b_{kl} - \bar{b}_k - \bar{b}_l + \bar{b}..$ Sheng and Yin (2013, 2016) showed that under mild conditions, the solution of the above problem (2.1) is a $\sqrt{n}$-consistent estimator of a basis of $\mathcal{S}_V|X$.

### 2.2 Motivation

In the Appendix of Székely et al. (2007), it was proved that $V^2_n(\beta^\top X, Y)$ has another expression, i.e.,

$$V^2_n(\beta^\top X, Y) = S_1 + S_2 - 2S_3,$$

where

$$S_1 = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)b_{kl},$$

$$S_2 = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta) \frac{1}{n^2} \sum_{l,m=1}^n b_{km} = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\bar{b}..,$$

$$S_3 = \frac{1}{n^3} \sum_{k=1}^n \sum_{l,m=1}^n a_{kl}(\beta)b_{km} = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\bar{b}_k..$$

Notice that $\frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\bar{b}_k.. = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\bar{b}_l$ because for any $k, l = 1, \ldots, n$, $a_{kl}(\beta)\bar{b}_k.. = a_{lk}(\beta)\bar{b}_k..$. Then, we have the following way to express $2S_3$:

$$2S_3 = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\bar{b}_k.. + \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\bar{b}_l = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\left(\bar{b}_k.. + \bar{b}_l\right).$$

Substituting equations (2.3) and (2.4) into (2.2), we obtain

$$V^2_n(\beta^\top X, Y) = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)b_{kl} + \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\bar{b}.. - \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\left(\bar{b}_k.. + \bar{b}_l\right),$$

$$= \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)\left(b_{kl} + \bar{b}.. - \bar{b}_k.. - \bar{b}_l\right),$$

$$= \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta)B_{kl}.$$
convex functions decomposition (DC). Indeed, we can write the function (2.5) into a DC formulation
\[ V^2_n(\beta^\top X, Y) = \left( \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta) B_{kl} I(B_{kl} > 0) \right) - \left( -\frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta) B_{kl} I(B_{kl} < 0) \right) \] (2.6)
through the indicator function \( I(\cdot) \). This equivalent function form (2.6) motivates us to design a highly efficient algorithm from the viewpoint of difference convex algorithm (DCA; Pham Dinh and Le Thi 1997). More details about DCA and some of its recent developments can be found in Le Thi and Pham Dinh (2005, 2018); Pham Dinh and Le Thi (1997, 1998, 2014).

Thus, the objective function (2.1) of the DCOV-based SDR model can be equivalently transformed to
\[ \max_{\beta \in \mathbb{R}^{p \times d}} V^2_n(\beta^\top X, Y) := \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\beta) B_{kl}, \quad \text{s.t.,} \quad \beta^\top \hat{\Sigma} X \beta = I_d. \] (2.7)
Let \( \gamma = \hat{\Sigma}^{1/2} \beta \) and \( Z = \hat{\Sigma}^{-1/2} X \), the above function (2.7) can be rewritten as
\[ \max_{\gamma} V^2_n(\gamma^\top Z, Y) := \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\gamma) B_{kl}, \quad \text{s.t.,} \quad \gamma \in \text{St}(d, p), \] (2.8)
where \( a_{kl}(\gamma) = \|\gamma^\top Z_k - \gamma^\top Z_l\|_2 \). In later sections, we will make full use of the equivalent form (2.8) rather than (2.7).

3 Methodology

3.1 Preliminaries

In fact, DCA is based on MM algorithm which is a principle of designing algorithms. The idea of designing a MM algorithm for finding \( \hat{x} = \arg \max_{x \in \mathcal{X}} f(x) \) where \( \mathcal{X} \) is the constraint region is as follows. At each iterate \( x^{(t)} \), we need to construct a surrogate function \( g(x|x^{(t)}) \) satisfying
\[ f(x^{(t)}) = g(x^{(t)}|x^{(t)}) \]
\[ f(x) \geq g(x|x^{(t)}) \quad \text{for any} \quad x \in \mathcal{X}. \]

Then, MM algorithm updates the estimation with
\[ x^{(t+1)} = \arg \max_{x \in \mathcal{X}} g(x|x^{(t)}). \]

Because
\[ f(x^{(t+1)}) \geq g(x^{(t+1)}|x^{(t)}) \geq g(x^{(t)}|x^{(t)}) = f(x^{(t)}), \]
the iterate estimates generated by MM algorithm drive the objective function uphill. Under mild conditions, MM algorithm generally converges to a stationary point of the objective function.
The most important component of designing a MM algorithm is to find an appropriate surrogate function $g(x|x^{(t)})$. In general, many surrogate functions may be derived from various inequalities stemming from convexity or concavity, see, e.g., [Lange et al. (2000)] or [Hunter and Lange (2004)]. One of the most used inequalities to construct a surrogate function is the supporting hyperplane inequality. Suppose $f(x)$ is convex with gradient $\nabla f(x)$, the supporting hyperplane inequality is

$$f(y) \geq f(x) + \langle \nabla f(x), y-x \rangle.$$  (3.1)

Our derivation of the MM algorithm for the DCOV-based SDR model hinges on the convexity of the two functions mentioned in the next lemma.

**Lemma 1.** (a) The scalar function $f(x) = x^{\frac{1}{2}} - \epsilon \log \left(1 + \frac{x^{\frac{1}{2}}}{\epsilon}\right)$ is concave and differentiable in $x > 0$ where $\epsilon > 0$ is a constant. (b) The matrix function $f(A) = \|A\|_2 - \epsilon \log \left(1 + \frac{\|A\|_2}{\epsilon}\right)$ is convex and differentiable in the $n \times p$ matrix $A$ where $c \in \mathbb{R}^p$ is a constant vector and $\epsilon > 0$ is a constant scalar.

**Proof:** See Appendix A.1.

### 3.2 MM Algorithm

It is often challenging to directly optimize the objective function (2.8) due to the non-smoothness. One way to tackle the difficulty is to perturb objective function slightly to render it differentiable, then to optimize this differentiable function using a MM algorithm ([Hunter and Li, 2005] [Yu et al., 2015]). Motivated by this idea, we introduce a perturbed version $V_{2,n,\epsilon}^2(\gamma^\top Z, Y)$ of the objective function (2.8) for the DCOV-based SDR model:

$$V_{2,n,\epsilon}^2(\gamma^\top Z, Y) = \frac{1}{n^2} \sum_{k,l=1}^{n} \left\{ a_{kl}(\gamma) - \epsilon \log \left(1 + \frac{a_{kl}(\gamma)}{\epsilon}\right) \right\} B_{kl},$$

$$= \frac{1}{n^2} \sum_{k,l=1}^{n} \left\{ \|\gamma^\top(Z_k - Z_l)\|_2 - \epsilon \log \left(1 + \frac{\|\gamma^\top(Z_k - Z_l)\|_2}{\epsilon}\right) \right\} B_{kl}. $$  (3.2)

Below we conclude some properties of the perturbed objective function $V_{2,n,\epsilon}^2(\gamma^\top Z, Y)$.

**Proposition 1.** For $\epsilon > 0$, (i) $V_{2,n,\epsilon}^2(\gamma^\top Z, Y)$ is a continuous and differentiable DC function and a DC decomposition of it is

$$V_{2,n,\epsilon}^2(\gamma^\top Z, Y) = \left( \frac{1}{n^2} \sum_{k,l=1}^{n} \left\{ a_{kl}(\gamma) - \epsilon \log \left(1 + \frac{a_{kl}(\gamma)}{\epsilon}\right) \right\} B_{kl} \right) I(B_{kl} > 0),$$

$$- \left( -\frac{1}{n^2} \sum_{k,l=1}^{n} \left\{ a_{kl}(\gamma) - \epsilon \log \left(1 + \frac{a_{kl}(\gamma)}{\epsilon}\right) \right\} B_{kl} I(B_{kl} < 0) \right),$$  (3.3)

where $I(\cdot)$ is an indicator function, (ii) $V_{2,n,\epsilon}^2(\gamma^\top Z, Y)$ converges to $V_{2,n}^2(\gamma^\top Z, Y)$ uniformly on the Stiefel manifold $\gamma \in \text{St}(d,p)$ as $\epsilon$ approaches to zero.
Proof: See Appendix A.2.

Now let $\gamma^{(t)}$ denote the current estimate, we plan to construct the minorization $g_t(\gamma|\gamma^{(t)})$ for the perturbed objective function $V_{n,\epsilon}^{B}(\gamma^\top Z, Y)$ based on the DC decomposition (3.3). The convexity of the function $A \mapsto \|Ac\|_2 - \epsilon \log \left(1 + \frac{\|Ac\|_2}{\epsilon}\right)$ implies that

$$a_{kl}(\gamma) - \epsilon \log \left(1 + \frac{a_{kl}(\gamma)}{\epsilon}\right) = \|\gamma^\top(Z_k - Z_l)\|_2 - \epsilon \log \left(1 + \frac{\|\gamma^\top(Z_k - Z_l)\|_2}{\epsilon}\right) \geq \|\gamma^{(t)}(Z_k - Z_l)\|_2 - \epsilon \log \left(1 + \frac{\|\gamma^{(t)}(Z_k - Z_l)\|_2}{\epsilon}\right)$$

$$+ \langle(Z_k - Z_l)(Z_k - Z_l)^\top \gamma^{(t)}\|\gamma^{(t)}(Z_k - Z_l)\|_2 + \epsilon, \gamma - \gamma^{(t)}\rangle.$$

Multiplying both sides by a nonnegative term $B_{kl}I(B_{kl} > 0)$ and averaging over all pairs $(k, l)$ leads to the minorization

$$\frac{1}{n^2} \sum_{k,l=1}^n \left\{ a_{kl}(\gamma) - \epsilon \log \left(1 + \frac{a_{kl}(\gamma)}{\epsilon}\right) \right\} B_{kl}I(B_{kl} > 0) \geq \frac{1}{n^2} \sum_{k,l=1}^n \left\{ a_{kl}(\gamma^{(t)}) - \epsilon \log \left(1 + \frac{a_{kl}(\gamma^{(t)})}{\epsilon}\right) \right\} B_{kl}I(B_{kl} > 0) \geq \frac{1}{n^2} \sum_{k,l=1}^n \langle(Z_k - Z_l)(Z_k - Z_l)^\top \gamma^{(t)}\|\gamma^{(t)}(Z_k - Z_l)\|_2 + \epsilon, \gamma - \gamma^{(t)}\rangle B_{kl}I(B_{kl} > 0). \quad (3.4)$$

Next focusing on the term $a_{kl}(\gamma) - \epsilon \log \left(1 + \frac{a_{kl}(\gamma)}{\epsilon}\right) B_{kl}I(B_{kl} < 0)$, we use the fact that $f(x) = x^{\frac{1}{2}} - \epsilon \log \left(1 + \frac{x^{\frac{1}{2}}}{\epsilon}\right)$ is concave in $x > 0$ to show

$$x^{\frac{1}{2}} - \epsilon \log \left(1 + \frac{x^{\frac{1}{2}}}{\epsilon}\right) \leq x^{(t)}^{\frac{1}{2}} - \epsilon \log \left(1 + \frac{x^{(t)}^{\frac{1}{2}}}{\epsilon}\right) + \frac{x - x^{(t)}}{2 \left(x^{(t)}^{\frac{1}{2}} + \epsilon\right)}.$$

Then, we take $x = \|\gamma^\top(Z_k - Z_l)\|_2$ and $x^{(t)} = \|\gamma^{(t)}(Z_k - Z_l)\|_2$, the above inequality becomes

$$\|\gamma^\top(Z_k - Z_l)\|_2 - \epsilon \log \left(1 + \frac{\|\gamma^\top(Z_k - Z_l)\|_2}{\epsilon}\right) \leq \|\gamma^{(t)}(Z_k - Z_l)\|_2 - \epsilon \log \left(1 + \frac{\|\gamma^{(t)}(Z_k - Z_l)\|_2}{\epsilon}\right)$$

$$+ \frac{\|\gamma^\top(Z_k - Z_l)\|_2^2 - \|\gamma^{(t)}(Z_k - Z_l)\|_2^2}{2 \left(\|\gamma^{(t)}(Z_k - Z_l)\|_2 + \epsilon\right)}.$$

Multiplying both sides by a nonpositive term $B_{kl}I(B_{kl} < 0)$ and averaging over all pairs
(k, l), we obtain the minimization

\[
\frac{1}{n^2} \sum_{k,l=1}^{n} \left\{ a_{kl}(\gamma) - \epsilon \log \left( 1 + \frac{a_{kl}(\gamma)}{\epsilon} \right) \right\} B_{kl} I(B_{kl} < 0)
\]

\[
\geq \frac{1}{n^2} \sum_{k,l=1}^{n} \left\{ a_{kl}(\gamma^{(t)}) - \epsilon \log \left( 1 + \frac{a_{kl}(\gamma^{(t)})}{\epsilon} \right) \right\} B_{kl} I(B_{kl} < 0)
\]

\[
+ \frac{1}{n^2} \sum_{k,l=1}^{n} \frac{\|\gamma^\top (Z_k - Z_l)\|_2^2 - \|\gamma^{(t)} (Z_k - Z_l)\|_2^2}{2 \left( \|\gamma^{(t)} (Z_k - Z_l)\|_2 + \epsilon \right)} B_{kl} I(B_{kl} < 0).
\]

Combination of the minorizations (3.4) and (3.5) gives the overall minorization

\[
g_c(\gamma|\gamma^{(t)}) = \frac{1}{n^2} \sum_{k,l=1}^{n} \frac{B_{kl} I(B_{kl} < 0)}{\|\gamma^{(t)} (Z_k - Z_l)\|_2 + \epsilon} \|\gamma^\top (Z_k - Z_l)\|_2^2
\]

\[
+ \frac{1}{n^2} \sum_{k,l=1}^{n} \langle D_{kl}(Z_k - Z_l)(Z_k - Z_l)^\top \gamma^{(t)} , \gamma \rangle B_{kl} I(B_{kl} > 0) + c^{(t)},
\]

where \(c^{(t)}\) is an irrelevant constant. To make clear of the surrogate function, we write it in a matrix form. Let \(C\) be a \(n \times n\) matrix with every entry \(C_{kl} = \frac{B_{kl} I(B_{kl} < 0)}{\|\gamma^{(t)} (Z_k - Z_l)\|_2 + \epsilon}\) and \(D\) be a \(n \times n\) matrix with every entry \(D_{kl} = \frac{B_{kl} I(B_{kl} > 0)}{\|\gamma^{(t)} (Z_k - Z_l)\|_2 + \epsilon}\), then the surrogate function (3.6) becomes

\[
g_c(\gamma|\gamma^{(t)}) = \frac{1}{n^2} \sum_{k,l=1}^{n} \frac{C_{kl}}{2} \|\gamma^{(t)} (Z_k - Z_l)\|_2^2
\]

\[
+ \frac{1}{n^2} \sum_{k,l=1}^{n} \langle D_{kl}(Z_k - Z_l)(Z_k - Z_l)^\top \gamma^{(t)} , \gamma \rangle + c^{(t)}.
\]

After some algebraic manipulation, we have

\[
g_c(\gamma|\gamma^{(t)}) = \frac{1}{2} \text{tr} \left( \gamma^\top Z \frac{2(\text{diag}(\text{sum}(C, 2)) - C)}{n^2} Z^\top \gamma \right)
\]

\[
+ \text{tr} \left( \gamma^{(t)} Z \frac{2(\text{diag}(\text{sum}(D, 2)) - D)}{n^2} Z^\top \gamma \right) + c^{(t)},
\]

where \(\text{sum}(A, 2)\) is a column vector of the sums of each row in the matrix \(A\) and \(\text{diag}(a)\) is the \(n \times n\) diagonal matrix whose entries are the \(n\) elements of the vector \(a\). Let \(Q = Z \frac{2(\text{diag}(\text{sum}(C, 2)) - C)}{n^2} Z^\top\) and \(L = Z \frac{2(\text{diag}(\text{sum}(D, 2)) - D)}{n^2} Z^\top\), the surrogate function \(g_c(\gamma|\gamma^{(t)})\) finally has the form

\[
g_c(\gamma|\gamma^{(t)}) = \frac{1}{2} \text{tr} \left( \gamma^\top Q \gamma \right) + \text{tr} \left( \gamma^\top L \right),
\]

subject to \(\gamma \in \text{St}(d, p)\). Maximizing the surrogate function \(g_c(\gamma|\gamma^{(t)})\) under the constraint drives the loss function uphill. However, due to the existence of the manifold constraint,
it is still difficult to accurately solve the subproblem (3.7) although the objective function is only a quadratic function. In fact, the validity of the ascent property depends only on increasing $g_e(\gamma|\gamma(t))$ over the Stiefel manifold $\text{St}(d,p)$, not on maximizing $g_e(\gamma|\gamma(t))$. Similar to Lange (1995) and Xu et al. (2018), we propose inexactly maximizing the surrogate function $g_e(\gamma|\gamma(t))$ by taking a single Newton’s step but over the Stiefel manifold $\text{St}(d,p)$. At each iterate $\gamma(t)$, we need to solve the following Newton’s equation of the problem (3.7)

$$\text{Hess } g_e(\gamma(t))[\xi] = -\text{grad } g_e(\gamma(t)),$$

subject to $\xi \in T_{\gamma(t)}\text{St}(d,p)$. After obtaining the Newton’s direction $\xi$ at the current estimate $\gamma(t)$, we can update estimate by

$$\gamma(t+1) = \text{Retr}_{\gamma(t)}(\xi) = qf(\gamma(t) + \xi),$$

where $qf(\cdot)$ denotes the Q factor of the QR decomposition of the matrix. To safeguard the MM algorithm preserving the ascent property, we can take step-halving strategy at every iterate. We call this MM algorithm for solving the DCOV-based SDR model MMRN algorithm and the following Algorithm 1 summarizes the MMRN algorithm using step-halving based on satisfying the Armijo condition.

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**Algorithm 1**: MMRN Algorithm for (2.7)

**Input**: $X \in \mathbb{R}^{p \times n}, Y \in \mathbb{R}^{1 \times n}$, perturbation constant $\epsilon$

1. Initialize $\gamma^{(0)} \in \text{St}(d,p)$, $\alpha \in (0, 1)$, $\sigma \in (0, 1)$, $t = 0$

2. Precompute $\hat{\Sigma}^{-\frac{1}{2}}X$, $B = (b_{kl})$, $Z = \hat{\Sigma}^{-\frac{1}{2}}X$

3. repeat

4. $C_{kl} \leftarrow \frac{B_{kl}I(B_{kl} < 0)}{\|\gamma(t)\|_2 (Z_k - Z_l)\|_2 + \epsilon}$, $D_{kl} \leftarrow \frac{B_{kl}I(B_{kl} > 0)}{\|\gamma(t)\|_2 (Z_k - Z_l)\|_2 + \epsilon}$, for any $k, l = 1, \ldots, n$

5. $Q \leftarrow Z \frac{2(\text{diag}(\text{sum}(C, 2)) - C)}{n^2}Z^\top$, $L \leftarrow Z \frac{2(\text{diag}(\text{sum}(D, 2)) - D)}{n^2}Z^\top \gamma(t)$

6. Solve the Newton’s equation

$$\text{Hess } g_e(\gamma(t))[\xi] = -\text{grad } g_e(\gamma(t)),$$

for unknown $\xi \in T_{\gamma(t)}\text{St}(d,p)$

7. $s \leftarrow 1$

8. repeat

9. $s \leftarrow \sigma s$

10. until $V_{n, \epsilon}^2(\text{Retr}_{\gamma(t)}(s\xi)^\top Z, Y) \geq V_{n, \epsilon}^2(\gamma(t)^\top Z, Y) + \alpha s\|\xi\|_F^2$

11. $\gamma(t+1) \leftarrow \text{Retr}_{\gamma(t)}(s\xi)$

12. $t \leftarrow t + 1$

13. until objective value converges;

**Output**: $\hat{\gamma}_e = \gamma^{(t+1)}$, $\hat{\beta}_e = \hat{\Sigma}^{-\frac{1}{2}}\hat{\gamma}_e$
3.3 Solving the Riemannian Newton’s equation (3.8)

The MM algorithm is a well-applicable and simple algorithmic framework for solving DC problems. The key challenge in making the proposed algorithm efficient numerically lies in solving the equation \( (3.8) \). Aihara and Sato (2017) and Sato (2017) recently proposed an effective way of solving Newton’s equation on the Stiefel manifold. The idea of the method is to rewrite original Newton’s equation expressed by a system of matrix equations into a standard linear system through the Kronecker product and the vec and veck operators. The resultant linear system can be effectively solved while reducing the dimension of the equation to that of the Stiefel manifold.

Before applying their method to solve the Newton’s equation of our subproblem \((3.8)\) formally, we introduce some useful properties of Kronecker, vec, and veck operators.

1. For any \( A \in \mathbb{R}^{m \times p}, X \in \mathbb{R}^{p \times q}, \) and \( B \in \mathbb{R}^{q \times n}, \) we have
   \[
   \text{vec}(AXB) = (B^\top \otimes A) \text{vec}(X).
   \]

2. For any matrix \( U \in \text{Skew}(d), \) we have
   \[
   \text{vec}(U) = D_d \text{veck}(U),
   \]
   and
   \[
   \text{veck}(U) = \frac{1}{2} D_d^\top \text{vec}(U).
   \]

Here \( D_d \) is a \( d^2 \times d(d-1)/2 \) matrix defined by
   \[
   D_d = \sum_{d \geq i > j \geq 1} \left( E_{(d^2 \times d(d-1)/2)}^{(d-1)+i, j(d-(j+1)/2)-d+i} - E_{d(i-1)+j, j(d-(j+1)/2)-d+i}\right),
   \]
   where \( E_{(p \times q)}^{i,j} \) denotes the \( p \times q \) matrix that has the \((i, j)\)-component equal to 1 and all other components equal to 0.

3. There exists an \( n^2 \times n^2 \) permutation matrix \( T_n \) such that
   \[
   \text{vec}(W^\top) = T_n \text{vec}(W), \quad W \in \mathbb{R}^{n \times n},
   \]
   where \( T_n = \sum_{i,j=1}^n E_{ij}^{(n \times n)} \otimes E_{ji}^{(n \times n)}. \)

From the above properties, we can easily derive that
   \[
   \text{vec}(\text{skew}(W)) = \frac{1}{2} (I_{n^2} - T_n) \text{vec}(W), \quad \text{for any } W \in \mathbb{R}^{n \times n}.
   \]

After these preparations, we begin to solve the Newton’s equation \((3.8)\). For a given \( \hat{\gamma} \in \text{St}(d, p), \) the Newton’s equation \((3.8)\) is equivalent to
   \[
   \text{Hess} g_\epsilon(\hat{\gamma})[\xi] = -\text{grad} g_\epsilon(\hat{\gamma}), \quad (3.9)
   \]
   subject to \( \xi \in T_{\hat{\gamma}}\text{St}(d, p). \) Specifically, the gradient of \( g_\epsilon \) at a point \( \hat{\gamma} \in \text{St}(d, p) \) is expressed as
   \[
   \text{grad} g_\epsilon(\hat{\gamma}) = Q\hat{\gamma} + L - \hat{\gamma}S, \quad (3.10)
   \]
and the Hessian acts on $\xi \in T_{\gamma} \text{St}(d, p)$ as

$$\text{Hess} g_{c}(\gamma)[\xi] = Q\xi - \xi S - \text{sym} \left( \gamma^T Q\xi - \gamma^T \xi S \right),$$

where $S \equiv \text{sym}(\gamma^T Q\gamma + \gamma^T L)$. $\xi \in T_{\gamma} \text{St}(d, p)$ can be expressed as

$$\xi = \tilde{\gamma} U + \tilde{\gamma}_L V, \quad U \in \text{Skew}(d), \; V \in \mathbb{R}^{(p-d) \times d}. \quad (3.12)$$

Hess $g_{c}(\gamma)[\xi] \in T_{\gamma} \text{St}(d, p)$ can also be written as

$$\text{Hess} g_{c}(\gamma)[\xi] = \tilde{\gamma} U_H + \tilde{\gamma}_L V_H, \quad U_H \in \text{Skew}(d), \; V_H \in \mathbb{R}^{(p-d) \times d}. \quad (3.13)$$

Substituting the equation (3.12) into the equation (3.11) and combining the resultant equation with the equation (3.13), we can obtain a relationship between $U_H, V_H$ and $U, V$. The following proposition gives the relationship.

**Proposition 2.** Let $\tilde{\gamma} \in \text{St}(d, p)$ and $\tilde{\gamma}_L$ be its orthonormal complement. If a tangent vector $\xi \in T_{\gamma} \text{St}(d, p)$ is expressed as (3.12), then the Hessian Hess $g_{c}(\gamma)[\xi]$ of the function (3.7) acts on $\xi$ as Hess $g_{c}(\gamma)[\xi] = \tilde{\gamma} U_H + \tilde{\gamma}_L V_H$ with

$$U_H = \text{skew} \left( \gamma^T Q\gamma U + \gamma^T Q\tilde{\gamma}_L V - US \right), \quad (3.14)$$

and

$$V_H = \tilde{\gamma}_L Q\gamma U + \tilde{\gamma}_L Q\tilde{\gamma}_L V - VS. \quad (3.15)$$

**Proof:** See Appendix A.3.

From Equation (3.14) and (3.15), we know the Hessian Hess $g_{c}(\gamma)$ at $\tilde{\gamma} \in \text{St}(d, p)$ is a linear transformation $H$ on $\mathbb{R}^K$ that transforms a $K$-dimensional vector $(\text{veck}(U)^T, \text{veck}(V)^T)^T$ into $(\text{veck}(U_H)^T, \text{veck}(V_H)^T)^T$. A goal of the method is to obtain the linear transformation $H$.

**Proposition 3.** Let $K = \text{dim}(\text{St}(d, p)) = d(d-1)/2 + (p-d)d$, there exists a linear transformation $H$ on $\mathbb{R}^K$ such that

$$H \begin{pmatrix} \text{veck}(U) \\ \text{veck}(V) \end{pmatrix} = \begin{pmatrix} \text{veck}(U_H) \\ \text{veck}(V_H) \end{pmatrix},$$

and the linear transformation $H$ is given by

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where

$$\begin{align*}
H_{11} &= \frac{1}{4} D_d^T \left[ I_d \otimes (\gamma^T Q\gamma - S) + (\gamma^T Q\gamma - S) \otimes I_d \right] D_d, \\
H_{12} &= \frac{1}{4} D_d^T (I_{d^2} - T_d) (I_d \otimes \gamma^T \gamma_\perp), \\
H_{21} &= (I_d \otimes \gamma^T_\perp \gamma_\perp) D_d, \\
H_{22} &= I_d \otimes \gamma^T_\perp \gamma_\perp - S \otimes I_d.
\end{align*}$$

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Proof: See Appendix A.4.

From the Newton’s equation \((3.9)\) together with Equation \((3.13)\), we have

\[
\begin{aligned}
U_H &= -\hat{\gamma}^\top \text{grad} \, g_e(\hat{\gamma}), \\
V_H &= -\hat{\gamma}_\perp^\top \text{grad} \, g_e(\hat{\gamma}).
\end{aligned}
\]

\( (3.16) \)

Applying the veck and vec operators to the equations \((3.16)\) respectively and using equation \((3.10)\), we immediately obtain

\[
\begin{aligned}
\text{veck}(U_H) &= -\text{veck} \left( \text{skew}(\hat{\gamma}^\top Q\hat{\gamma} + \hat{\gamma}^\top L) \right), \\
\text{vec}(V_H) &= -\text{vec}(\hat{\gamma}_\perp^\top Q\hat{\gamma} + \hat{\gamma}_\perp^\top L).
\end{aligned}
\]

By Proposition 3, we have a standard linear system

\[
H \begin{pmatrix} \text{veck}(U) \\ \text{vec}(V) \end{pmatrix} = - \begin{pmatrix} \text{veck} \left( \text{skew}(\hat{\gamma}^\top Q\hat{\gamma} + \hat{\gamma}^\top L) \right) \\ \text{vec}(\hat{\gamma}_\perp^\top Q\hat{\gamma} + \hat{\gamma}_\perp^\top L) \end{pmatrix}.
\]

If \( H \) is invertible, we can solve the above linear equation as

\[
\begin{pmatrix} \text{veck}(U) \\ \text{vec}(V) \end{pmatrix} = -H^{-1} \begin{pmatrix} \text{veck} \left( \text{skew}(\hat{\gamma}^\top Q\hat{\gamma} + \hat{\gamma}^\top L) \right) \\ \text{vec}(\hat{\gamma}_\perp^\top Q\hat{\gamma} + \hat{\gamma}_\perp^\top L) \end{pmatrix}.
\]

In our numerical studies, we have not noticed the case \( H \) is not invertible. After \( \text{veck}(U) \) and \( \text{vec}(V) \) are obtained, we can easily reshape \( U \in \text{Skew}(d) \) and \( V \in \mathbb{R}^{(p-d) \times d} \). Therefore, we can calculate the solution of Newton’s equation \((3.9)\) by \( \xi = \hat{\gamma} U + \hat{\gamma}_\perp V \). Detailed information can be seen in Algorithm 2.
Algorithm 2: Solving the Riemannian Newton’s equation (3.8)

Input: $Q \in \mathbb{R}^{p \times p}$, $L \in \mathbb{R}^{p \times d}$, $\gamma^{(t)} \in \mathbb{R}^{p \times d}$, $D_d \in \mathbb{R}^{d^2 \times \frac{d(d+1)}{2}}$, and $T_d \in \mathbb{R}^{d^2 \times d^2}$

1. Compute $\gamma^{(t)}_\perp$ such that $\gamma^{(t)}_\perp \gamma^{(t)}_\perp = 0$ and $\gamma^{(t)}_\perp \gamma^{(t)}_\perp = I_{p-d}$
2. Compute $S = \text{sym}(\gamma^{(t)}_\perp Q \gamma^{(t)} + \gamma^{(t)}_\perp L)$
3. Compute the linear transformation $H \in \mathbb{R}^{K \times K}$ by

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where

$$H_{11} = \frac{1}{4} D_d^T \left[ I_d \otimes (\gamma^{(t)}_\perp Q \gamma^{(t)} - S) + (\gamma^{(t)}_\perp Q \gamma^{(t)} - S) \otimes I_d \right] D_d,$$

$$H_{12} = \frac{1}{4} D_d^T (I_d^2 - T_d) \left( I_d \otimes \gamma^{(t)}_\perp Q \gamma^{(t)}_\perp \right),$$

$$H_{21} = (I_d \otimes \gamma^{(t)}_\perp Q \gamma^{(t)}_\perp) D_d,$$

$$H_{22} = I_d \otimes \gamma^{(t)}_\perp Q \gamma^{(t)}_\perp - S \otimes I_d.$$

4. Compute $\text{veck}(U)$ and $\text{vec}(V)$ using

$$\begin{pmatrix} \text{veck}(U) \\ \text{vec}(V) \end{pmatrix} = -H^{-1} \begin{pmatrix} \text{vec} \left( \text{skew} \left( \gamma^{(t)}_\perp Q \gamma^{(t)} + \gamma^{(t)}_\perp L \right) \right) \\ \text{vec} \left( \gamma^{(t)}_\perp Q \gamma^{(t)} + \gamma^{(t)}_\perp L \right) \end{pmatrix}.$$

5. Construct $U \in \text{Skew}(d)$ and $V \in \mathbb{R}^{(p-d) \times d}$ from $\text{veck}(U)$ and $\text{vec}(V)$

6. Compute $\xi = \gamma^{(t)} U + \gamma^{(t)}_\perp V$

Output: $\xi \in T_{\gamma^{(t)}} \text{St}(d, p)$
3.4 Convergence Analysis

In this section, we construct the convergence property of the proposed algorithm for solving the DCOV-based SDR model. We first show that the sequence \( \{ \hat{\gamma}^{(t)} \}_{t \geq 0} \) generated by the MMRN algorithm converge to a stationary point of the perturbed function (3.2). Then, we show that a maximizer \( \hat{\gamma}_\epsilon \) of the perturbed objective function (3.2) exhibits a minimal difference from a maximizer \( \hat{\gamma} \) of the true objective (2.8) for sufficiently small \( \epsilon \).

**Proposition 4.** Let \( \gamma \in \text{St}(d, p) \), \( \alpha \in (0, 1) \), and \( \sigma \in (0, 1) \), there exists an integer \( t > 0 \) such that
\[
V^2_n(\text{Retr}_\gamma(\sigma^\epsilon \xi) \top Z, Y) \geq V^2_n(\gamma \top Z, Y) + \alpha \sigma^t \| \xi \|^2_F,
\]
where \( \xi \) is a solution of \( \text{Hess}_\gamma(g_\epsilon(\gamma))\{\xi\} = -\text{grad} g_\epsilon(\gamma) \).

**Proof:** See Appendix A.5.

We now prove the convergence of our perturbed MM algorithm safeguarded by the Armijo step-halving strategy.

**Proposition 5.** For any \( \epsilon > 0 \), the limit point \( \hat{\gamma}_\epsilon \) generated by the Algorithm 1 is a stationary point of \( V^2_n(\gamma \top Z, Y) \), that is \( \text{grad} V^2_n(\hat{\gamma}_\epsilon \top Z, Y) = 0 \).

**Proof:** See Appendix A.6.

**Proposition 6.** Consider an arbitrary decreasing sequence \( \{ \epsilon_m, m = 1, 2, \ldots, \infty \} \) that converges to 0. Then, any limit point of \( \hat{\gamma}_{\epsilon_m} \) is a maximizer of \( V^2_n(\gamma \top Z, Y) \) over the Stiefel manifold, provided that \( \gamma \mid V^2_n(\gamma \top Z, Y) = V^2_n(\hat{\gamma}_\epsilon \top Z, Y) \) and \( \gamma \top \gamma = I_d \) is nonempty.

**Proof:** See Appendix A.7.

Combining Proposition 5 and 6, it is straightforward to see that the MM algorithm generates solutions that converge to a stationary point of \( V^2_n(\gamma \top Z, Y) \) as \( \epsilon \) decreases to zero.

**Theorem 1.** The sequence of the solutions \( \{ \hat{\gamma}^{(t)} \}_{t \geq 0} \) generated by the proposed perturbed MM algorithm converges to a maximizer of \( V^2_n(\gamma \top Z, Y) \) over the Stiefel manifold. Moreover, the sequence of functionals \( \{ V^2_n(\gamma \top Z, Y) \}_{t \geq 0} \) converges to the maximum value of \( V^2_n(\gamma \top Z, Y) \).

**Proof:** See Appendix A.8.

4 Extension

In this section, we will extend the above proposed method to solve sufficient variable selection (SVS) using distance covariance. The DCOV-based SVS method is developed by Chen et al. (2018) through combining DCOV-based SDR with penalty terms, such as LASSO type penalty terms (Tibshirani, 1996; Yuan and Lin, 2006; Chen et al., 2010) or adaptive LASSO (Zou, 2006), to achieve a sparse solution. Specifically, the model is to solve the following problem

\[
\max_{\beta} \ V^2_n(\beta \top X, Y) - \lambda \sum_{i=1}^p \theta_i \| \beta_i \|_2,
\]  

(4.1)
subject to \( \beta^T \Sigma_X \beta = \mathbf{I}_d \), where \( \beta_i \) denotes the \( i \)-th row vector of \( \beta \), \( \theta_i \geq 0 \) serves as the \( i \)-th penalty weight and \( \lambda > 0 \) is a tuning parameter. Plugging \( \gamma = \frac{1}{2} \Sigma_X^{-\frac{1}{2}} \beta \) and \( Z = \frac{1}{2} \Sigma_X^{-\frac{1}{2}} X \) into the equation (4.1) together with using equivalent expression (2.5) for \( \nabla_{\beta}^2 (\beta^T X, Y) \), we can transform the objective function (4.1)

\[
\phi_\lambda(\gamma) = \frac{1}{n^2} \sum_{k,l=1}^n a_{kl}(\gamma) B_{kl} - \lambda \sum_{i=1}^p \theta_i \rho_i(\gamma),
\]

(4.2)

subject to \( \gamma \in \text{St}(d,p) \), where \( \rho_i(\gamma) = \|1_i^T \Sigma_X^{-\frac{1}{2}} \gamma\|_2^2 \) and \( 1_i \) denotes a column vector with one in the \( i \)-th position and zero in the others. Correspondingly, a perturbed version \( \phi_{\lambda, \epsilon}(\gamma) \) of the objective function (4.2) is given by

\[
\phi_{\lambda, \epsilon}(\gamma) = \frac{1}{n^2} \sum_{k,l=1}^n \left\{ a_{kl}(\gamma) - \epsilon \log \left( 1 + \frac{a_{kl}(\gamma)}{\epsilon} \right) \right\} B_{kl} - \lambda \sum_{i=1}^p \theta_i \left\{ \rho_i(\gamma) - \epsilon \log \left( 1 + \frac{\rho_i(\gamma)}{\epsilon} \right) \right\},
\]

(4.3)

Due to the minorization (3.7) for the first term, it only needs to minorize the penalty function in the equation (4.3) to obtain a surrogate function of \( \phi_{\lambda, \epsilon}(\gamma) \). The supporting hyperplane minorization for \(-\lambda \theta_i \left\{ x^\frac{1}{2} - \epsilon \log \left( 1 + \frac{x^\frac{1}{2}}{\epsilon} \right) \right\}\) is

\[
- \lambda \theta_i \left\{ x^\frac{1}{2} - \epsilon \log \left( 1 + \frac{x^\frac{1}{2}}{\epsilon} \right) \right\} \geq - \lambda \theta_i \left\{ (x^{(t)})^\frac{1}{2} - \epsilon \log \left( 1 + \frac{(x^{(t)})^\frac{1}{2}}{\epsilon} \right) \right\} + \frac{-\lambda \theta_i (x - x^{(t)})}{2 (x^{(t)})^\frac{1}{2} + \epsilon}.
\]

(4.4)

Taking \( x = \|1_i^T \Sigma_X^{-\frac{1}{2}} \gamma\|_2^2 \) and \( x^{(t)} = \|1_i^T \Sigma_X^{-\frac{1}{2}} \gamma^{(t)}\|_2^2 \), and summing over \( i = 1, \ldots, p \) give the minoration for penalty function \(-\lambda \sum_{i=1}^p \theta_i \rho_i(\gamma)\)

\[
- \lambda \sum_{i=1}^p \theta_i \rho_i(\gamma) \geq - \lambda \sum_{i=1}^p \frac{-\lambda \theta_i \|1_i^T \Sigma_X^{-\frac{1}{2}} \gamma\|_2^2}{2 (\|1_i^T \Sigma_X^{-\frac{1}{2}} \gamma^{(t)}\|_2 + \epsilon)},
\]

(4.5)

where \( c \) is an irrelevant constant. After some algebraic manipulation, we have

\[
\sum_{i=1}^p -\lambda \theta_i \|1_i^T \Sigma_X^{-\frac{1}{2}} \gamma\|_2^2 = \frac{1}{2} \text{tr} \left( \gamma^T \Sigma_X^{-\frac{1}{2}} \text{diag}(a) \Sigma_X^{-\frac{1}{2}} \gamma \right),
\]

(4.6)

where \( a = \left( -\lambda \theta_1, \ldots, -\lambda \theta_p \right)^T \) is a \( p \times 1 \) column vector. Combining the minoration (3.7) and (4.6) gives the overall minoration

\[
g_{\lambda, \epsilon}(\gamma^{(t)}) = \frac{1}{2} \text{tr} \left( \gamma^T \left[ Q + \Sigma_X^{-\frac{1}{2}} \text{diag}(a) \Sigma_X^{-\frac{1}{2}} \right] \gamma \right) + \text{tr}(\gamma^T L).
\]

(4.7)
Note that the form of surrogate function (4.7) for the DCOV-based SVS model is the same as the surrogate function (3.7) for the DCOV-based SDR model. Thus, we can use the same method for solving the DCOV-based SVS model.

5 Numerical Studies

In this section, we compare our proposed unified algorithm for solving both DCOV-based SDR and DCOV-based SVS to their corresponding existing algorithms, focusing on computational cost. Since the method in Chen et al. (2018) solving DCOV-based SVS combines SQP and local quadratic approximation (LQA; Fan and Li, 2001), we denote it to SQP+LQA for convenience. SQP and SQP+LQA in all of the simulation studies use the default setups in the original work to guarantee accuracy. In the MMRN, we set the stepsize multiplicative factor $\sigma = 0.5$ and perturbation constant $\epsilon = 10^{-10}$ to avoid machine precision error. Besides, we set $\alpha = 10^{-20}$ to lead fewer number of line search steps. The MMRN algorithm terminates at the $r$-th step when the relative error of the objective function at the $r$-th step computed by $|f(\gamma^{(r)}) - f(\gamma^{(r-1)})|/|f(\gamma^{(r-1)})|$ becomes smaller than $10^{-7}$ or the iteration number $r$ exceeds 1000. Here the function $f$ denotes the objective functions in DCOV-based SDR and SVS. All algorithms use the solutions from existing dimension reduction methods such as SIR or DR as the initial value. All codes are implemented in MATLAB and run on a standard PC (Intel Core i9-8950HK CPU (2.90 GHz) and 32 GB RAM).

5.1 Simulation for DCOV-based SDR

We use the same simulation settings as in Sheng and Yin (2016) to illustrate the performance comparison of the MMRN algorithm and the SQP algorithm in solving DCOV-based SDR models. There are three different models and two sample size configurations $(n, p) = (100, 6)$ and $(500, 20)$. Let $\epsilon$, $\epsilon_1$, and $\epsilon_2$ be independent standard normal random variables, the three models are:

(A) \[ Y = (\beta_1^\top X)^2 + (\beta_2^\top X) + 0.1\epsilon, \]

(B) \[ Y = \text{sign}(2\beta_1^\top X + \epsilon_1) \times \log |2\beta_2^\top X + 4 + \epsilon_2|, \]

(C) \[ Y = \exp(\beta_3^\top X)\epsilon, \]

where $\beta_1, \beta_2$, and $\beta_3$ are $p$-dimensional vectors with their first six components being $(1, 0, 0, 0, 0, 0)^\top$, $(0, 1, 0, 0, 0, 0)^\top$, and $(1, 0.5, 1, 0, 0, 0)^\top$ and the last $p - 6$ components being 0 if $p > 6$. Each model has three different kinds of $X = (x_1, \ldots, x_p)^\top$: Part (1), standard normal predictors $X \sim N(0, I_p)$; Part (2), nonnormal predictors; and Part (3), discrete predictors. Specific predictors setups for Part (2) and Part (3) in each model are summarized in Table 1.

Each simulation scenario repeats 100 times. At each time, we use the following distance to measure the accuracy of the estimator $\hat{\beta}$

\[ \Delta_m(P_{\hat{\beta}}, P_\beta) = \| P_{\hat{\beta}} - P_\beta \|, \]

where $\beta$ is a basis of the true central subspace, $P_{\hat{\beta}}$ and $P_\beta$ are the respective projections of $\hat{\beta}$ and $\beta$, and $\| \cdot \|$ is the maximum singular value of a matrix. The smaller the $\Delta_m$
Table 1: Setups for Part (2) and Part (3)

| Part (2) | Part (3) |
|----------|----------|
| Model A  | Model A  |
| \(\{x_i + 2\over 5\}_{i=1}^p \overset{\text{iid}}{\sim} \text{Beta}(0.75, 1)\) | \(\{x_i\}_{i=1}^p \overset{\text{iid}}{\sim} \text{Poisson}(1)\) |
| Model B  | Model B  |
| \(\{x_i\}_{i=1}^p \overset{\text{iid}}{\sim} \text{Uniform}(-2, 2)\) | \(\{x_i\}_{i=1}^p \overset{\text{iid}}{\sim} \text{Binomial}(10, 0.1)\) |
| Model C  | Model C  |
| \(\{x_i + 1\over 2\}_{i=1}^p \overset{\text{iid}}{\sim} \text{Beta}(1.5, 1)\) | \(\{x_i\}_{i \neq 6} \overset{\text{iid}}{\sim} \text{Poisson}(1)\) and \(x_6 \sim \text{Binomial}(10, 0.3)\) |

NOTE: iid means independent identically distributed.

Table 2: Simulation results under the same settings as in [Sheng and Yin (2016)]

| (n, p) | Model | Part | SQP | MMRN |
|--------|-------|------|-----|------|
|        |       |      | \(\bar{\Delta}_m\) | Time (sec) | \(\bar{\Delta}_m\) | Time (sec) |
| \(n = 100, p = 6\) | A   | (1)  | 0.1903(0.0617) | 0.5223(0.1576) | 0.1904(0.0617) | 0.0770(0.0315) |
| |     | (2)  | 0.1876(0.0613) | 0.5462(0.0922) | 0.1876(0.0613) | 0.0719(0.0201) |
| |     | (3)  | 0.0041(0.0141) | 1.1816(0.2509) | 0.0039(0.0142) | 0.1241(0.0766) |
| | B   | (1)  | 0.2863(0.0977) | 0.4872(0.2042) | 0.2862(0.0976) | 0.1841(0.0938) |
| |     | (2)  | 0.2173(0.0666) | 0.4425(0.0816) | 0.2171(0.0667) | 0.1002(0.0306) |
| |     | (3)  | 0.2805(0.1798) | 0.4837(0.1668) | 0.2696(0.1813) | 0.1332(0.1015) |
| | C   | (1)  | 0.1996(0.0664) | 0.3825(0.1862) | 0.1983(0.0662) | 0.1570(0.0550) |
| |     | (2)  | 0.3087(0.1170) | 0.3254(0.0755) | 0.2972(0.1041) | 0.2499(0.1278) |
| |     | (3)  | 0.2163(0.0981) | 0.3883(0.1113) | 0.2154(0.0983) | 0.1139(0.0472) |
| \(n = 500, p = 20\) | A   | (1)  | 0.1622(0.0230) | 11.4133(1.8403) | 0.1622(0.0230) | 1.2690(0.1202) |
| |     | (2)  | 0.1680(0.0277) | 13.4738(1.9631) | 0.1680(0.0277) | 1.3125(0.1430) |
| |     | (3)  | 0.0005(0.0002) | 53.6144(4.8351) | 0.0000(0.0000) | 2.0206(0.5792) |
| | B   | (1)  | 0.2377(0.0389) | 10.2649(1.6321) | 0.2376(0.0389) | 10.5633(2.4016) |
| |     | (2)  | 0.1908(0.0275) | 10.5633(2.4016) | 0.1908(0.0275) | 1.9242(0.1953) |
| |     | (3)  | 0.1849(0.0722) | 14.7220(3.6393) | 0.1853(0.0720) | 2.2362(0.4736) |
| | C   | (1)  | 0.1530(0.0267) | 9.6444(0.9580) | 0.1528(0.0267) | 4.1263(0.6654) |
| |     | (2)  | 0.2355(0.0409) | 11.1993(1.1581) | 0.2352(0.0411) | 10.5883(3.1583) |
| |     | (3)  | 0.1407(0.0264) | 12.2877(1.3704) | 0.1406(0.0264) | 3.3439(0.5474) |
is, the more accuracy the estimator is. We report the mean and the standard error of $\Delta_m$’s and CPU times in Table 2. We can observe that both the SQP algorithm and the MMRN algorithm have satisfactory performance in terms of estimation accuracy, but the MMRN algorithm takes less time than the SQP algorithm. For part (3) of model A at $n = 500$ and $p = 20$, the MMRN algorithm takes about 2 seconds on average while the SQP algorithm averages more than 50 seconds. It is approximately 25 times faster. Also, the MMRN algorithm is more stable than the SQP algorithm since the standard deviation of the running time is less. Overall, the MMRN algorithm has almost the same performance as the SQP algorithm across various models, but with less time.

Figure 1: Computational performance comparison on large problem size for three different models with standard normal predictors. There was no significant difference of the two methods in the estimation accuracy. Therefore, estimation accuracy is not displayed the graph.

To test the performance of our proposed MMRN algorithm in large datasets, we use four different levels for sample size configuration, $(n, p)$: $(500, 50)$, $(1000, 100)$, $(2000, 200)$, and $(3000, 300)$. Here, we only consider the cases with the standard predictors and generate 20 datasets for each study. Figure 1 displays a graph of the average runtime for each algorithm under the different problem sizes considered. We can see that our proposed algorithm can outperform the SQP algorithm even in large datasets. Note that we did
not run the SQP algorithm on sample size \((n, p) = (3000, 300)\) for model C with standard predictors since it will take much time (> 7 hours once) to solve the problem.

### 5.2 Simulation for DCOV-based SVS

This subsection compares the performance of our proposed MMRN algorithm and the SQP+LQA algorithm in solving DCOV-based SVS models. We consider two sample size configurations \((n, p) = (60, 24)\) and \((120, 24)\) and generate 100 datasets for each simulation. To assess how well the algorithms select variables, we define the true positive rate TPR as the proportion of correctly identified active predictors, and the false positive rate FPR as the proportion of irrelevant predictors that are incorrectly identified to be active. When computing the TPR and FPR in practice, the estimate obtained by the MMRN algorithm is truncated by zeroing out its entries whose magnitude is smaller than \(10^{-7}\). In addition, we use the Bayesian information criterion (BIC) to select the tuning parameters, see., Chen et al. (2018).

We conduct the following simulation studies with the same model settings as the scenarios \(n > p\) in Chen et al. (2018).

**Study 1.** A nonlinear regression model with four active predictors:

\[
Y = (\beta_1^\top X + 0.5)^2 + 0.5 \epsilon, \\
\text{where } \epsilon \sim N(0, 1) \text{ and } X = (x_1, \ldots, x_{24})^\top \sim N(0, \Sigma) \text{ with } \Sigma_{ij} = 0.5^{|i-j|} \text{ for } 1 \leq i, j \leq 24. \text{ The central subspace is spanned by the vectors } \beta_1 = (0.5, 0.5, 0.5, 0.5, 0, \ldots, 0)^\top.
\]

**Study 2.** A nonlinear regression model with two active predictors:

\[
Y = \frac{\beta_1^\top X}{0.5 + (\beta_2^\top X + 1.5)^2} + 0.2 \epsilon, \\
\text{where } \epsilon \sim N(0, 1) \text{ and } X = (x_1, \ldots, x_{24})^\top \sim N(0, \Sigma) \text{ with } \Sigma_{ij} = 0.5^{|i-j|} \text{ for } 1 \leq i, j \leq 24. \text{ The central subspace is spanned by the vectors } \beta_1 = (1, 0, \ldots, 0)^\top \text{ and } \beta_2 = (0, 1, 0, \ldots, 0)^\top.
\]

**Study 3.** A nonlinear regression model with four active predictors:

\[
Y = (\beta_1^\top X)^2 + |\beta_2^\top X| + 0.5 \epsilon, \\
\text{where } \epsilon \sim N(0, 1). \text{ The predictor } X = (x_1, \ldots, x_{24})^\top \text{ is defined as follows: the last 23 components } (x_2, \ldots, x_{24})^\top \sim N(0, \Sigma) \text{ with } \Sigma_{ij} = 0.5^{|i-j|} \text{ for } 1 \leq i, j \leq 23 \text{ and the first component } x_1 = |x_2 + x_3| + \xi, \text{ where } \xi \sim N(0, 1). \text{ The central subspace is spanned by the vectors } \beta_1 = (0.5, 0.5, 0.5, 0.5, 0, \ldots, 0)^\top \text{ and } \beta_2 = (0.5, -0.5, 0.5, -0.5, 0, \ldots, 0)^\top.
\]

**Study 4.** A multivariate response model with four active predictors:

\[
\begin{aligned}
Y_1 &= \beta_1^\top X + \epsilon_1, \\
Y_2 &= (\beta_2^\top X + 0.5)^2 + \epsilon_2,
\end{aligned}
\]

\text{where } \epsilon_1, \epsilon_2 \overset{\text{iid}}{\sim} N(0, 1) \text{ and } X = (x_1, \ldots, x_{24})^\top \sim N(0, \Sigma) \text{ with } \Sigma_{ij} = 0.5^{|i-j|} \text{ for } 1 \leq i, j \leq 24. \text{ The central subspace is spanned by the vectors } \beta_1 = (0.5, 0.5, 0.5, 0.5, 0, \ldots, 0)^\top \text{ and } \beta_2 = (0.5, -0.5, 0.5, -0.5, 0, \ldots, 0)^\top.
Table 3 gives the simulation results. The MMRN algorithm is much less time-consuming than the SQP+LQA algorithm to achieve the same or even slightly better effect in terms of TPR and FPR. Especially in Study 2 and Study 4, we can observe that the performance of MMRN algorithm in TPR and FPR is better than SQP + LQA, and its speed is nearly 100 times faster.

Table 3: Performance comparison

|        | SQP+LQA          | MMRN         |
|--------|------------------|--------------|
|        | TPR   | FPR   | Time (sec) | TPR   | FPR   | Time (sec) |
| Study 1 |       |       |           |       |       |           |
| n = 60  | 0.695 | 0.063 | 385.5     | 0.685 | 0.077 | 12.1      |
| n = 120 | 0.990 | 0.004 | 532.9     | 0.988 | 0.002 | 27.6      |
| Study 2 |       |       |           |       |       |           |
| n = 60  | 0.770 | 0.031 | 1051.2    | 0.870 | 0.016 | 5.4       |
| n = 120 | 0.930 | 0.010 | 1518.3    | 0.975 | 0.004 | 9.4       |
| Study 3 |       |       |           |       |       |           |
| n = 60  | 0.715 | 0.010 | 1122.4    | 0.725 | 0.002 | 11.9      |
| n = 120 | 0.785 | 0.002 | 1746.3    | 0.785 | 0.001 | 26.8      |
| Study 4 |       |       |           |       |       |           |
| n = 60  | 0.655 | 0.029 | 1293.8    | 0.700 | 0.011 | 12.9      |
| n = 120 | 0.905 | 0.009 | 1778.4    | 0.930 | 0.007 | 30.0      |

6 Conclusion

In the article, we notice that the empirical distance covariance can have a difference of convex functions decomposition. Based on this observation, we leverage the MM principle to design powerful and versatile algorithms uniformly for DCOV-based SDR and DCOV-based SVS models. The proposed algorithms take one single Riemannian Newton’s step at each iterate to tackle the Manifold constraints. The simulation studies show our proposed algorithms are highly efficient and very stable even in large \( n \) and large \( p \) scenarios. Furthermore, we establish the convergence property of our proposed algorithms under mild conditions.

As a possible future work, we plan to design a new algorithm with the aim to handle the large \( p \) small \( n \) scenarios directly rather than incorporate it in the framework of sequential SDR (Yin and Hilafu, 2015).

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Appendix A: Some technical derivations

A.1 Proof of Lemma 1:

For part (a), for any \( x > 0 \), we have the second derivative \( f''(x) = -\frac{1}{4(\epsilon + \sqrt{x})^2 \sqrt{x}} \leq 0 \) and immediately obtain that \( f(x) \) is concave in \( x > 0 \).

For part (b), recall that \( g(x) = x - \epsilon \log \left(1 + \frac{x}{\epsilon}\right) \) is convex and increasing in \( x > 0 \), and \( h(A) = \|Ac\|_2 \) is convex in \( A \in \mathbb{R}^{n \times p} \). By the composition property we know function \( g(h(A)) \) is convex. Thus, we complete our proof.

A.2 Proof of Proposition 1:

For part (i), it only needs to prove that \( a_{kl}(\gamma) - \epsilon \log \left(1 + \frac{a_{kl}(\gamma)}{\epsilon}\right) \) is convex with respect to \( \gamma \). This proof follows immediately from the part (b) of Lemma 1 when you take \( A = \gamma^T \) and \( c = Z_k - Z_l \).

For part (ii), recall that
\[
0 \leq V^2_n(\gamma^T Z, Y) - V^2_{n,\epsilon}(\gamma^T Z, Y) = \frac{1}{n^2} \sum_{k,l=1}^{n} \epsilon \log \left(1 + \frac{\|\gamma^T (Z_k - Z_l)\|_2}{\epsilon}\right)
\]
\[
\leq \frac{1}{n^2} \sum_{k,l=1}^{n} \epsilon \log \left(1 + \sup_{\gamma \in \text{St}(d,p)} \frac{\|\gamma^T (Z_k - Z_l)\|_2}{\epsilon}\right)
\]
The suprema in the rightmost side are achieved and finite because \( \text{St}(d, p) \) is bounded. Then, the rightmost side monotonically decreases to 0 as \( \epsilon \) goes to 0.

A.3 Proof of Proposition 2:

Multiplying the equation (3.11) by \( \tilde{\gamma}^T \) from the left and using the relations \( \tilde{\gamma}^T \tilde{\gamma} = I_d \) and \( \tilde{\gamma}^T \tilde{\gamma}_\perp = 0 \) yield
\[
U_H = \tilde{\gamma}^T \text{Hess } g_c(\tilde{\gamma})[\xi],
\]
\[
= \tilde{\gamma}^T Q\xi - \tilde{\gamma}^T \xi S - \text{sym}(\tilde{\gamma}^T Q\xi - \tilde{\gamma}^T \xi S), \quad (A.1)
\]
\[
= \text{skew}(\tilde{\gamma}^T Q\xi - \tilde{\gamma}^T \xi S).
\]
Similarly, we multiply the equation (3.11) by \( \tilde{\gamma}^T \) from the left to obtain
\[
V_H = \tilde{\gamma}^T \text{Hess } g_c(\tilde{\gamma})[\xi],
\]
\[
= \tilde{\gamma}^T Q\xi - \tilde{\gamma}^T \xi S. \quad (A.2)
\]
Substituting the expression (3.12) of \( \gamma \) into (A.1) and (A.2), we can immediately obtain equations (3.14) and (3.15).
A.4 Proof of Proposition 3:

From equations (3.14) and (3.15) together with the properties of these operators and $U^T = -U$, vec($U_H$) and vec($V_H$) are calculated as follows:

$$\text{vec}(U_H) = \frac{1}{2} D_d^T \text{vec}(U_H)$$

$$= \frac{1}{2} D_d^T \text{vec} \left( \text{skew}(\gamma^T Q \gamma U + \gamma^T Q \gamma U - US) \right)$$

$$= \frac{1}{4} D_d^T (I_d - T_d) \text{vec} \left( \gamma^T Q \gamma U + \gamma^T Q \gamma U - US \right)$$

$$= \frac{1}{4} D_d^T \text{vec} \left( \gamma^T Q \gamma U - US \right) + \frac{1}{4} D_d^T (I_d - T_d) \text{vec} \left( \gamma^T Q \gamma U \right)$$

$$= \frac{1}{4} D_d^T \text{vec} \left( \gamma^T Q \gamma U + U \gamma^T \gamma - US - SU \right) + \frac{1}{4} D_d^T (I_d - T_d) \text{vec} \left( \gamma^T Q \gamma U \right)$$

$$= \frac{1}{4} D_d^T \left[ I_d \otimes (\gamma^T Q \gamma - S) + (\gamma^T Q \gamma - S) \otimes I_d \right] D_d \text{vec}(U)$$

$$+ \frac{1}{4} D_d^T (I_d - T_d) \left( I_d \otimes \gamma^T Q \gamma \right) \text{vec}(V)$$

$$= H_{11} \text{vec}(U) + H_{12} \text{vec}(V),$$

and

$$\text{vec}(V_H) = \text{vec} \left( \gamma^T Q \gamma U + \gamma^T Q \gamma U - VS \right)$$

$$= (I_d \otimes \gamma^T Q \gamma) \text{vec}(U) + (I_d \otimes \gamma^T Q \gamma - S \otimes I_d) \text{vec}(V)$$

$$= (I_d \otimes \gamma^T Q \gamma) D_d \text{vec}(U) + (I_d \otimes \gamma^T Q \gamma - S \otimes I_d) \text{vec}(V)$$

$$= H_{21} \text{vec}(U) + H_{22} \text{vec}(V).$$

This completes our proof.

A.5 Proof of Proposition 4:

When the Riemannian Newton’s vector $\xi$ is an ascent direction of $g_\epsilon(\gamma)$, we assert that there exists an integer $t > 0$ satisfying

$$g_\epsilon(\text{Retr}_\gamma(\sigma^t \xi)) \geq g_\epsilon(\gamma) + \alpha \sigma^t \|\xi\|^2_F.$$

The assertion could be proved by applying the standard argument for Armijo condition in vector spaces, see [Nocedal and Wright (2006) Lemma 3.1]. Combining the property of the surrogate function, we then immediately obtain

$$V_{n,\epsilon}^2(\text{Retr}_\gamma(\sigma^t \xi) ^T Z, Y) \geq g_\epsilon(\text{Retr}_\gamma(\sigma^t \xi)) \geq g_\epsilon(\gamma) + \alpha \sigma^t \|\xi\|^2_F = V_{n,\epsilon}^2(\gamma^T Z, Y) + \alpha \sigma^t \|\xi\|^2_F.$$

Thus, we complete the proof.

A.6 Proof of Proposition 5:

Since the sequence $\left\{ V_{n,\epsilon}^2(\gamma^{(t)} ^T Z, Y) \right\}$ is increasing and bounded above, $V_{n,\epsilon}^2(\gamma^{(t+1)} ^T Z, Y) - V_{n,\epsilon}^2(\gamma^{(t)} ^T Z, Y)$ converges to 0. According to the Proposition 4, there exists an integer
The first and third term in the right-hand side vanish respectively by the continuity of \( \sigma \). For the second part, we have sufficiently large \( t \) of \( \{ \gamma_0 \} \).

The limit point \( \hat{\gamma} \) where \( \hat{\gamma} \) is a limit point \( \gamma \) of \( \{ \gamma_\epsilon \} \) with \( \epsilon \downarrow 0 \), we have

\[
\lim_{m \to \infty} \mathcal{V}_n^2(\gamma_{\epsilon m}^\top Z, Y) = \mathcal{V}_n^2(\gamma^\top Z, Y) = \mathcal{V}_n^2(\gamma^\top Z, Y) = \max_{\gamma \in St(p, d)} \mathcal{V}_n^2(\gamma^\top Z, Y).
\]

by the continuity of \( \mathcal{V}_n^2(\gamma^\top Z, Y) \). Thus, we complete our proof.

**A.8 Proof of Theorem 1:**

For the first part, we have

\[
\| \hat{\gamma}_\epsilon^{(t)} - \hat{\gamma} \|_F = \| \hat{\gamma}_\epsilon^{(t)} \|_F + \| \hat{\gamma}_\epsilon - \hat{\gamma} \|_F,
\]

where \( \hat{\gamma}_\epsilon \) is a maximizer of \( \mathcal{V}_n^2(\gamma^\top Z, Y) \) over the Stiefel manifold, and \( \hat{\gamma} \) is a limit point of \( \{ \gamma_\epsilon \}_{\epsilon > 0} \) as \( \epsilon \downarrow 0 \). By Proposition 5, we know the first term becomes arbitrarily small for sufficiently large \( t \), whereas the second term does so for sufficiently small \( \epsilon \) by Proposition 6. The limit point \( \hat{\gamma} \) is a maximizer of \( \mathcal{V}_n^2(\gamma^\top Z, Y) \) over the Stiefel manifold by Proposition 6. For the second part, we have

\[
| \mathcal{V}_n^2(\gamma_{\epsilon}^{(t)} Z, Y) - \mathcal{V}_n^2(\gamma^\top Z, Y) | \leq \mathcal{V}_n^2(\gamma_{\epsilon}^{(t)} Z, Y) - \mathcal{V}_n^2(\gamma_{\epsilon} Z, Y) |
\]

The first and third term in the right-hand side vanish respectively by the continuity of \( \mathcal{V}_n^2(\gamma_{\epsilon}^{(t)} Z, Y) \) and \( \mathcal{V}_n^2(\gamma_{\epsilon} Z, Y) \); the second term by the uniform convergence of \( \mathcal{V}_n^2(\gamma_{\epsilon}^{(t)} Z, Y) \) to \( \mathcal{V}_n^2(\gamma^\top Z, Y) \), as shown in the proof of Proposition 1. Thus, we have completed our proof.