HOMOMORPHISMS BETWEEN DIFFERENT QUANTUM TOROIDAL AND AFFINE YANGIAN ALGEBRAS

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Abstract. This paper concerns the relation between the quantum toroidal algebras and the affine Yangians of $\mathfrak{sl}_n$, denoted by $U_{q_1,q_2,q_3}^{(n)}$ and $Y_{h_1,h_2,h_3}^{(n)}$, respectively. Our motivation arises from the milestone work [GTL], where a similar relation between the quantum loop algebra $U_q(Lg)$ and the Yangian $Y_h(g)$ has been established by constructing an isomorphism of $\mathbb{C}[[h]]$-algebras $\Phi : \widehat{U}_{\exp(h)}(Lg) \to \widehat{Y}_h(g)$ (with $\widehat{\cdot}$ standing for the appropriate completions). These two completions model the behavior of the algebras in the formal neighborhood of $h = 0$. The same construction can be applied to the toroidal setting with $U_{q_1,q_2,q_3}^{(n)}$ and the Yangians of $\mathfrak{g}$.

In the current paper, we generalize this construction to the case of the quantum toroidal algebras and the affine Yangians of $\mathfrak{sl}_n$ and $\mathfrak{gl}_1$. To make our notation uniform, we use $U_{q_1,q_2,q_3}^{(n)}$ to denote the quantum toroidal algebra of $\mathfrak{sl}_n$ (if $n \geq 2$) and of $\mathfrak{gl}_1$ (if $n = 1$). This algebra depends on three nonzero parameters $q_1,q_2,q_3$ such that $q_1q_2q_3 = 1$. We also use $Y_{h_1,h_2,h_3}^{(n)}$ to denote the affine Yangian of $\mathfrak{sl}_n$ (if $n \geq 2$) and of $\mathfrak{gl}_1$ (if $n = 1$). This algebra depends on three parameters $h_1,h_2,h_3$.

Introduction

Given a simple Lie algebra $\mathfrak{g}$, one can associate to it two interesting Hopf algebras: the quantum loop algebra $U_q(Lg)$ and the Yangian $Y_h(g)$. Their classical limits, corresponding to the limits $q \to 1$ or $h \to 0$, recover the universal enveloping algebras $U(Lg)$ and $U(g)$, respectively. The representation theories of $U_q(Lg)$ and $Y_h(g)$ have a lot of common features:

- the descriptions of finite dimensional simple representations involve Drinfeld polynomials,
- these algebras act on the equivariant K-theories/cohomologies of Nakajima quiver varieties.

However, there was no explicit justification for that until the recent construction from [GTL]. In the loc. cit., the authors construct a $\mathbb{C}[[h]]$-algebra isomorphism

$$\Phi : \widehat{U}_{\exp(h)}(Lg) \to \widehat{Y}_h(g)$$

of the appropriately completed formal versions of these algebras. Taking the limit $h \to 0$ corresponds to factoring by $(h)$ in the formal setting. The classical limit of the above isomorphism

$$\Phi : U(qz^{-1}) \to U(g),$$

is induced by $\lim \mathbb{C}[z, z^{-1}]/(z - 1)^r \to \lim \mathbb{C}[w]/(w)^r \cong \mathbb{C}[w]$ with $z \mapsto e^w$.

In the current paper, we generalize this construction to the case of the quantum toroidal algebras and the affine Yangians of $\mathfrak{sl}_n$ and $\mathfrak{gl}_1$. To make our notation uniform, we use $U_{q_1,q_2,q_3}^{(n)}$ to denote the quantum toroidal algebra of $\mathfrak{sl}_n$ (if $n \geq 2$) and of $\mathfrak{gl}_1$ (if $n = 1$). This algebra depends on three nonzero parameters $q_1,q_2,q_3$ such that $q_1q_2q_3 = 1$. We also use $Y_{h_1,h_2,h_3}^{(n)}$ to denote the affine Yangian of $\mathfrak{sl}_n$ (if $n \geq 2$) and of $\mathfrak{gl}_1$ (if $n = 1$). This algebra depends on three parameters $h_1,h_2,h_3$. The representation theories of $U_{q_1,q_2,q_3}^{(n)}$ and $Y_{h_1,h_2,h_3}^{(n)}$ explain the similarity between the representation theories of $U_{q_1,q_2,q_3}^{(n)}$ and $Y_{h_1,h_2,h_3}^{(n)}$ with different parameters $n$. 
parameters $h_1, h_2, h_3$ such that $h_1 + h_2 + h_3 = 0$. For $n \geq 2$, these algebras were introduced long time ago by \cite{GKV} \cite{C}. However, the quantum toroidal algebra and the affine Yangian of $\mathfrak{gl}_1$ appeared only recently in the works of different people, see \cite{MFT, SV1, MO, SV2, T}.

In Theorem 3.2 we construct a homomorphism

$$\Phi_{m,n}^{\omega_{m,n}} : \hat{U}_{h_1,h_2}^{(m),\omega_{m,n}} \rightarrow \hat{Y}_{h_1,h_2}^{(m)}$$

from the completion of the formal version of $U_{q_1, q_2, q_3}$ to the completion of the formal version of $y_{h_1, h_2, h_3}^{(m)}$. Formal versions mean that we consider these algebras over the ring $\mathbb{C}[[h_1, h_2]]$ with

$$h_1 = \frac{h_1}{mn}, h_2 = \frac{h_2}{mn}, h_3 = \frac{h_3}{mn} \quad \text{and} \quad q_1 = e^{\frac{h_1}{mn}}, q_2 = e^{\frac{h_2}{mn}}, q_3 = e^{-\frac{h_3}{mn}},$$

where $h_3 = -h_1 - h_2$ and $\omega_N \in \mathbb{C}^\times$ is an $N$th root of unity. For $n = 1 = \omega_m, n$, we recover an analogue of the homomorphism $\Phi$ applied in the toroidal setting (see \cite{T} for $m = n = \omega_m = 1$).

In contrast to \cite{GTL, T}, our new feature is that we construct homomorphisms between formal versions of quantum and Yangian algebras corresponding to different Lie algebras. Another difference is that $q_1$ is in the formal neighborhood of a root of unity, not necessarily equal to 1.

The structure of formulas for $\Phi_{m,n}^{\omega_{m,n}}$ is similar to that in \cite{GTL}. Let $\{e_{i,k}, f_{i,k}, h_{i,k}\}_{0 \leq i \leq m - 1} \in \mathfrak{gl}_n$ be the generators of $U_{q_1, q_2, q_3}$ and $\{x^\pm_{i', r}, \xi_{i', r}\}_{0 \leq i' \leq m, r} \in \mathfrak{gl}_n$ be the generators of $y_{h_1, h_2, h_3}^{(m), 0}$. We use the same notation for the case of formal versions. Let $y_{h_1, h_2, h_3}^{(m), 0} \subset y_{h_1, h_2, h_3}^{(m)}$ be the subalgebra generated by $\xi_{i', r}$. Then, we have:

- the images $\Phi_{m,n}^{\omega_{m,n}}(h_{i,k})$ belong to the completion of $y_{h_1, h_2, h_3}^{(m), 0}$,
- the images $\Phi_{m,n}^{\omega_{m,n}}(e_{i,k})$ are of the form $\sum_{r=0}^{\infty} g_{i', r} (k) x_{i', r}$,
- the images $\Phi_{m,n}^{\omega_{m,n}}(f_{i,k})$ are of the form $\sum_{r=0}^{\infty} g_{i', r} (k) x_{i', r}$,

where $g_{i', r} (k)$ belong to the completion of $y_{h_1, h_2, h_3}^{(m), 0}$.

Our motivation partially comes from \cite{BTT}, where a 4d AGT relation on the ALE space $X_n$ (minimal resolution of $\mathbb{A}_{n-1}$ singularity $\mathbb{C}^2/\mathbb{Z}_n$) was studied. The main tool in \cite{BTT} was the limit of $K$-theoretic (5 dimensional) AGT relation on $\mathbb{C}^2$, where $q_1 \to \omega_m, q_2 \to \omega_m$. Recall that the quantum toroidal algebra $U_{q_1, q_2, q_3}$ acts on the equivariant $K$-theory of the moduli spaces of torsion free sheaves on $\mathbb{C}^2$, while the affine Yangian $y_{h_1, h_2, h_3}^{(m)}$ acts on the equivariant cohomologies of the moduli spaces of torsion free sheaves on $X_n$. Therefore, it was conjectured in \cite{BTT} that the limit of $U_{q_1, q_2, q_3}$ as $q_1 \to \omega_m, q_2 \to \omega_m$ should be related to the affine Yangian of $\mathfrak{sl}_n$. The $m = 1$ case of our Theorem 3.2 can be viewed as a precise formulation of this idea.

We also refer an interested reader to \cite{K} for the related work.

Another new feature of the toroidal setting is an existence of homomorphisms

$$\Psi_{m,n}^{\omega_{m,n}} : \hat{U}_{h_1,h_2}^{(m),\omega_{m,n}} \rightarrow \hat{U}_{h_1,h_2}^{(m),\omega_{m,n}}$$

between completions of the formal versions of different quantum toroidal algebras, see Theorem 3.3. We are not aware of a similar phenomena for classical quantum loop algebras.

To find the formulas for the images of $h_{i,k}, e_{i,k}, f_{i,k}$ under $\Phi_{m,n}^{\omega_{m,n}}$ and $\Psi_{m,n}^{\omega_{m,n}}$ as well as to check all the defining relations, except for the Serre relations, we used the same reasoning as in \cite{T}. However, we are not aware of how to deduce the Serre relations from all others in contrast to \cite{T}. Therefore, we choose a different approach to prove Theorems 3.3 and 3.3. We construct isomorphisms between faithful representations of the algebras in question, which are compatible with the formulas for $\Phi_{m,n}^{\omega_{m,n}}$ and $\Psi_{m,n}^{\omega_{m,n}}$. This immediately proves both theorems.

\footnote{Actually, we will need to modify slightly their construction in the $n = 2$ case.}
HOMOMORPHISMS BETWEEN QUANTUM TOROIDAL AND AFFINE YANGIAN ALGEBRAS

This paper is organized as follows:

- In Section 1, we recall the definition of the quantum toroidal algebra $U_{q_1, q_2, q_3}^{(n)}$ and the affine Yangian $\mathcal{Y}_{h_1, h_2, h_3}$ of $\mathfrak{sl}_n$ (if $n \geq 2$) and $\mathfrak{g}_l$ (if $n = 1$). They depend on $n \in \mathbb{N}$ and continuous parameters $q_1, q_2, q_3 \in \mathbb{C}^\times$ or $h_1, h_2, h_3 \in \mathbb{C}$ satisfying $q_1 q_2 q_3 = 1$ and $h_1 + h_2 + h_3 = 0$. We also explain the way one can view the algebras $U_{q_1, q_2, q_3}^{(n)}$ as “additivizations” of $U_{q_1, q_2, q_3}^{(n)}$. We recall a family of Fock $U_{q_1, q_2, q_3}^{(n)}$-representations $F^p(u)$ ($p \in \mathbb{Z}/n\mathbb{Z}, u \in \mathbb{C}^\times$) from [FJMM1] and introduce a similar class of Fock $U_{h_1, h_2, h_3}$-representations $^\ast F^p(v)$ ($p \in \mathbb{Z}/n\mathbb{Z}, v \in \mathbb{C}$).

- In Section 2, we introduce the formal versions of these algebras and study their classical limits. Let $\mathcal{Y}_{h_1, h_2}$ be an associative algebra over $\mathbb{C}[\{h_1, h_2\}]$ with the same collections of the generators and the defining relations as for $\mathcal{Y}_{h_1, h_2, -h_1 - h_2}$ with $h_1 \rightsquigarrow h_1/n$ and $h_2 \rightsquigarrow h_2/n$.

One can similarly define the formal versions of $U_{q_1, q_2, q_3}^{(m)}$, but this heavily depends on the presentation of $q_1, q_2, q_3 \in \mathbb{C}[\{h_1, h_2\}]$. In this paper, we are interested in the behavior of the algebras $U_{q_1, q_2, q_3}$ and $U_{h_1, h_2}$ as $q_1 \rightarrow \omega N, q_2 \rightarrow 1, q_3 \rightarrow \omega N^{-1}$ and $h_1, h_2, h_3 \rightarrow 0$, respectively. Therefore, we will be mainly concerned with the following relation between $\{h_i\}$ and $\{q_i\}$:

$$q_1 = \omega N \cdot \exp(h_1/m), \quad q_2 = \exp(h_2/m), \quad q_3 = \omega N^{-1} \exp(h_3/m).$$

The formal version of the corresponding $U_{q_1, q_2, q_3}$ will be denoted by $U_{h_1, h_2}^{(m)}$.

Taking the limit $h_2 \rightarrow 0$ corresponds to factoring by $h_2$ in the formal setting. We describe the classical limits $U_{h_1, h_2}^{(m)}, \omega N \cdot \mathcal{Y}_{h_1, h_2}$ and $U_{h_1, h_2}$ as $q_1 \rightarrow \omega N, q_2 \rightarrow 1, q_3 \rightarrow \omega N^{-1}$ and $h_1, h_2, h_3 \rightarrow 0$, respectively. Finally, we show that the algebras $U_{h_1, h_2}$ and $U_{h_1, h_2}^{(m)}$, are flat $\mathbb{C}[\{h_2\}]$-deformations of the corresponding limit algebras $U_{h_1}^{(m)}$ and $U_{h_1}^{(m), \omega N}$, $U_{h_1}^{(m)}$. We also prove that the direct sum of all legible finite tensor products of Fock modules for either $U_{h_1, h_2}$ or $U_{h_1, h_2}^{(m)}$ form a faithful representation of the corresponding algebra.

- In Section 3, we present the main results of this paper. We construct the homomorphisms

$$\Phi_{m,n}^{\omega_n} : U_{h_1, h_2}^{(m), \omega mn} \rightarrow \mathcal{Y}_{h_1, h_2}^{(m)}$$

for any $m, n \in \mathbb{N}$ and an order $mn$ root of unity $\omega mn \in \mathbb{C}^\times$. We compute the classical limit of $\Phi_{m,n}^{\omega_n}$, where we use the aforementioned identification of the limit algebras $U_{h_1}^{(m), \omega mn}$ and $U_{h_1}^{(m), \omega N}$, with matrix algebras over the rings of difference or differential operators on $\mathbb{C}^\times$.

We also construct the homomorphisms

$$\Psi_{m,n}^{\omega, \omega'} : U_{h_1, h_2}^{(m), \omega} \rightarrow U_{h_1, h_2}^{(m), \omega'}$$

for any $m, n \in \mathbb{N}$ and two roots of unity $\omega, \omega'$ such that $\omega'/\omega$ is an $mn$th root of unity.

Finally, we establish the compatibilities between different homomorphisms from above.

- In Section 4, we construct isomorphisms between tensor products of the Fock modules for $U_{h_1, h_2}$ and $U_{h_1, h_2}^{(m)}$ which are compatible with the defining formulas for $\Phi_{m,n}^{\omega_n}$. We also construct isomorphisms between tensor products of the Fock modules for $U_{h_1, h_2}$ and $U_{h_1, h_2}^{(m)}$, which are compatible with the defining formulas for $\Psi_{m,n}^{\omega, \omega'}$. Combining these results with the faithfulness statement from Section 2, we deduce that the homomorphisms $\Phi_{m,n}^{\omega_n}$ and $\Psi_{m,n}^{\omega, \omega'}$ are well-defined (Theorems 3.2, 3.3) and prove compatibilities between them (Theorems 3.4, 3.5).

Finally, we recall the geometric realization of tensor products of the Fock modules for the quantum toroidal algebra and the affine Yangian of $\mathfrak{sl}_n$. We conclude by providing a geometric interpretation of the aforementioned two isomorphisms of tensor products of the Fock modules.
1. Basic definitions and constructions

In this section, we introduce the key objects of interest: the quantum toroidal algebra and the affine Yangian of \( \mathfrak{sl}_n \). We also recall the family of Fock representations for these two algebras.

1.1. Quantum toroidal algebras of \( \mathfrak{sl}_n \) \((n \geq 2)\) and \( \mathfrak{gl}_1 \)

The quantum toroidal algebras of \( \mathfrak{sl}_n \) \((n > 2)\), depending on \( q, d \in \mathbb{C}^\times \), were first introduced in [GKV]. The quantum toroidal algebra of \( \mathfrak{gl}_1 \) was introduced much later in the works of different people, see [M FT SV]. Finally, the correct version of the quantum toroidal of \( \mathfrak{sl}_2 \) was proposed in [FJMM2]. To make our exposition shorter, we use the uniform notation \( \mathcal{U}^{(n)}_{q_1, q_2, q_3} \) for such algebras, where \( n \in \mathbb{N} \) and \( q_1 = d/q, q_2 = q^2, q_3 = 1/(dq) \), so that \( q_1 q_2 q_3 = 1 \). This algebra coincides with the quotient of the algebra \( \mathcal{E}_n \) from [FJMM2] by \( q^c = 1 \). Since the former algebra was called the “quantum toroidal of \( \mathfrak{gl}_n \)”, in the “loc. cit.”, we will refer to \( \mathcal{U}^{(n)}_{q_1, q_2, q_3} \) as the quantum toroidal algebra of \( \mathfrak{sl}_n \) (see the above explanation for the cases of \( n = 1, 2 \)).

For \( n \in \mathbb{N} \), we set \([n] := \{0, 1, \ldots, n-1\}\) which should be viewed as a set of mod \( n \) residues. Let \((a_{i,j})_{i,j}[n] \) be the Cartan matrix of type \( A^{(1)}_{n-1} \) for \( n \geq 2 \) and a zero matrix for \( n = 1 \).

Consider two collections of constants \( \{d_{i,j}\}_{i,j}[n] \) and \( \{m_{i,j}\}_{i,j}[n] \) defined by

\[
d_{i,j} = \begin{cases} d^{+1} & \text{if } j = i \pm 1 \text{ and } n > 2 \\ -1 & \text{if } j \neq i \text{ and } n = 2 \\ 1 & \text{otherwise} \end{cases}, \quad m_{i,j} = \begin{cases} 1 & \text{if } j = i - 1 \text{ and } n > 2 \\ -1 & \text{if } j = i + 1 \text{ and } n > 2 \\ 0 & \text{otherwise} \end{cases}.
\]

Finally, we define a collection of polynomials \( \{g_{i,j}(z,w)\}_{i,j}[n] \) as follows:

\[
g_{i,j}(z,w) = \begin{cases} z - q^{a_{i,j}}d^{-m_{i,j}}w & \text{if } n > 2 \\ z - q^2w & \text{if } i = j \text{ and } n = 2 \\ (z - q w)(z - q^3w) & \text{if } i \neq j \text{ and } n = 2 \\ (z - q w)(z - q^2w)(z - q^3w) & \text{if } n = 1 \end{cases}.
\]

The algebra \( \mathcal{U}^{(n)}_{q_1, q_2, q_3} \) is an associative unital \( \mathbb{C} \)-algebra generated by \( \{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}\}_{i,k \in \mathbb{Z}} \) with the defining relations (T0–T6) to be given below:

\[
\begin{align*}
\text{(T0)} & \quad \psi_{i,0} \cdot \psi_{i,0}^{-1} = \psi_{i,0}^{-1} \cdot \psi_{i,0} = 1, \quad [\psi_{i,k}^\pm(z), \psi_{j,m}^\pm(w)] = 0, \quad [\psi_{i,k}^+(z), \psi_{j,m}^-(w)] = 0, \\
\text{(T1)} & \quad [e_i(z), f_j(w)] = \delta_{i,j} \cdot \delta(w/z)(\psi_{i,m}^+(w) - \psi_{j,n}^+(z))/(q - q^{-1}), \\
\text{(T2)} & \quad d_{i,j}g_{i,j}(z,w)e_i(z)e_j(w) = -g_{j,i}(w,z)e_j(w)e_i(z), \\
\text{(T3)} & \quad d_{i,j}g_{i,j}(w,z)f_i(z)f_j(w) = -g_{j,i}(z,w)f_j(w)f_i(z), \\
\text{(T4)} & \quad d_{i,j}g_{i,j}(z,w)\psi_{i,k}^\pm(z)e_j(w) = -g_{j,i}(w,z)e_j(w)\psi_{i,k}^\pm(z), \\
\text{(T5)} & \quad d_{i,j}g_{i,j}(w,z)\psi_{i,k}^\pm(z)f_j(w) = -g_{j,i}(z,w)f_j(w)\psi_{i,k}^\pm(z),
\end{align*}
\]

where these generating series are defined as follows:

\[
e_i(z) := \sum_{k=0}^{\infty} e_{i,k}z^{-k}, \quad f_i(z) := \sum_{k=0}^{\infty} f_{i,k}z^{-k}, \quad \psi_{i,k}^\pm(z) := \psi_{i,k,0}^\pm + \sum_{r>0} \psi_{i,k,\pm r}z^{\pm r}, \quad \delta(z) := \sum_{k=\infty}^{\infty} z^k.
\]

Set \([a, b]_x := ab - xa \cdot ba\). Let us now specify (T6), called the Serre relations.

- **Case** \( n > 2 \): we impose

\[
[e_i(z), e_j(w)] = 0, \quad [f_i(z), f_j(w)] = 0 \quad \text{if } a_{i,j} = 0,
\]
For $n > 1$, the relations $(T4, T5)$ are equivalent to the following (we use $q$):

$$\text{Sym}_{z_1, z_2} [e_i(z_1), e_i(z_2), e_i+1(w)]_{q^{-1}} = 0, \quad \text{Sym}_{z_1, z_2} [f_i(z_1), f_i(z_2), f_i+1(w)]_{q^{-1}} = 0.$$ 

**Case $n = 2$:** we impose

$$\text{Sym}_{z_1, z_2, z_3} [e_i(z_1), e_i(z_2), e_i(z_3), e_{i+1}(w)]_{q^{-2}} = 0, \quad \text{Sym}_{z_1, z_2, z_3} [f_i(z_1), f_i(z_2), f_i(z_3), f_{i+1}(w)]_{q^{-2}} = 0.$$

**Case $n = 1$:** we impose

$$\text{Sym}_{z_1, z_2, z_3} [e_0(z_1), e_0(z_2), e_0(z_3)] = 0, \quad \text{Sym}_{z_1, z_2, z_3} [f_0(z_1), f_0(z_2), f_0(z_3)] = 0.$$

**Remark 1.1.** For any $n > 1$ and $\omega = \sqrt[n]{1} \in \mathbb{C}$, there is an isomorphism of algebras $U^{(n)}_{q_1, q_2, q_3} \cong U^{(n)}_{\omega n, q_1, q_2, \omega^{-1}, q_3}$, given by $e_i(z) \mapsto e_i(\omega^{-i}z)$, $f_i(z) \mapsto f_i(\omega^{-i}z)$, $\psi^\pm_i(z) \mapsto \psi^\pm_i(\omega^{-i}z)$.

It is convenient to use the generators $\{h_{i, t}\}_{t \in \mathbb{Z} \setminus \{0\}}$ instead of $\{\psi_{i, t}\}_{t \in \mathbb{Z} \setminus \{0\}}$ defined by

$$\exp\left(\pm (q - q^{-1}) \sum_{r > 0} h_{i, \pm r} z^{\pm r}\right) = \psi^\pm_i(z) := \psi^{\pm 1}_{i, 0} \psi^\pm_i(z), \quad h_{i, \pm r} \in \mathbb{C}[\psi^{\pm 1}_{i, 0}, \psi_{i, \pm 1}, \psi_{i, \pm 2}, \ldots].$$

Then the relations (T4, T5) are equivalent to the following (we use $[m] := (q^m - q^{-m})/(q - q^{-1})$):

(T4) \hspace{1cm} $\psi_{i, 0} e_{j, t} = q^{[m]} e_{j, t} \psi_{i, 0}, \quad [h_{i, k}, e_{j, t}] = b_n(i, j; k) \cdot e_{j, t+k} (k \neq 0),$ 

(T5) \hspace{1cm} $\psi_{i, 0} f_{j, t} = q^{-[m]} f_{j, t} \psi_{i, 0}, \quad [h_{i, k}, f_{j, t}] = -b_n(i, j; k) \cdot f_{j, t+k} (k \neq 0),$

with constants $b_n(i, j; k)$ given explicitly by:

$$b_n(i, j; k) = \begin{cases} \frac{[k \alpha_{i, j}]}{k} \cdot d^{-km_{i, j}} & \text{if } n > 2 \\ \frac{[k \alpha_{i, j}]}{k} \cdot (d^k + d^{-k}) & \text{if } i = j \text{ and } n = 2 \\ \frac{[k \alpha_{i, j}]}{k} \cdot (q^k + q^{-k} - d^k - d^{-k}) & \text{if } i \neq j \text{ and } n = 2 \end{cases}.$$ 

We equip the algebra $U^{(n)}_{q_1, q_2, q_3}$ with the principal $\mathbb{Z}$-grading by assigning

$$\deg(e_{i, k}) = 1, \quad \deg(f_{i, k}) = -1, \quad \deg(\psi_{i, k}) = 0.$$ 

Following [DI], we also equip $U^{(n)}_{q_1, q_2, q_3}$ with a formal coproduct by assigning

$$\Delta(e_{i, k}) = e_{i, k} \otimes 1 + \psi_i^z(1) \otimes e_{i, k},$$

$$\Delta(f_{i, k}) = f_{i, k} \otimes \psi_i^z(1) + 1 \otimes f_{i, k},$$

$$\Delta(\psi_i^z(1)) = \psi_i^z(1) \otimes \psi_i^z(1).$$

1.2. **Affine Yangians of $\mathfrak{sl}_n$ ($n \geq 2$) and $\mathfrak{gl}_1$.**

The affine Yangian of $\mathfrak{sl}_n$ ($n \geq 2$), denoted by $\widehat{U}_{(\mathfrak{sl}_n)} (\chi, \bar{\chi} \in \mathbb{C})$, was first introduced in [G]. For $n = 2$, we will need a slight upgrade of this construction. Finally, the affine Yangian of $\mathfrak{gl}_1$ has recently appeared in the works of Maulik-Okounkov [MO] and Schiffmann-Vasserot [SV2]. However, in the current paper, we will need the loop presentation of this algebra from [T].

To make our exposition shorter, we call such algebras the affine Yangians of $\mathfrak{sl}_n$ and denote them by $\widehat{U}_{(n)}^{(n)}$, where $n \in \mathbb{N}$ and $h_1 = \beta - h$, $h_2 = 2h$, $h_3 = -\beta - h$ ($\beta, h \in \mathbb{C}$), so that $\sum h_i = 0$. The algebra $\widehat{U}_{(n)}^{(n)}$ is an associative unital $\mathbb{C}$-algebra generated by $\{x_{s, t}^\pm, \xi_{s, t}\}_{s \in \mathbb{Z}, t \in [n]}$ (here $\mathbb{Z}_+ := \{s \in \mathbb{Z} | s \geq 0\} = \mathbb{N} \cup \{0\}$) with the defining relations (Y0–Y5) to be given below.
The first two relations are independent of $n \in \mathbb{N}$:

(Y0) $[\xi_{i,r},\xi_{j,s}] = 0$,

(Y1) $[x_{i,r}^+,x_{j,s}^-] = \delta_{i,j} \cdot \xi_{i,r+s}$.

Set $\{a,b\} := ab + ba$. Let us now specify (Y2–Y5) in each of the cases: $n > 2$, $n = 2$, $n = 1$.

- **Case $n > 2$:** we impose

(Y2) $[x_{i,r+1}^+,x_{j,s}^+] - [x_{i,r}^+,x_{j+1,s}^+] = -m_{i,j}\beta[x_{i,r}^+,x_{j,s}^+], a_{i,j}h(x_{i,r}^+,x_{j,s}^+),$  

(Y3) $[\xi_{i,r+1},x_{j,s}^+] - [\xi_{i,r},x_{j+1,s}^+] = -m_{i,j}\beta[\xi_{i,r},x_{j,s}^+] + a_{i,j}h(\xi_{i,r},x_{j,s}^+),$

(Y4) $[\xi_{i,0},x_{j,s}^+] = \pm a_{i,j}x_{j,s}^+,$

(Y5) Sym $\text{Sym}_{r_1,r_2} [x_{i,r_1}^+,x_{i,r_2}^+,x_{i+1,s}] = 0$ and $[x_{i,r}^+,x_{j,s}^+] = 0$ if $a_{i,j} = 0$.

- **Case $n = 2$:** we impose

(Y2.1) $[x_{i,r+1}^+,x_{i,s}^+] - [x_{i,r}^+,x_{i+1,s}^+] = \pm h_2(x_{i,r}^+,x_{i,s}^+),$

(Y2.2) $[x_{i,r+2}^+,x_{j,s}^+] - 2[x_{i+1,r+1}^+,x_{j+1,s}^+] + [x_{i,r}^+,x_{j+s+2}^+] =$  

$-h_1h_3[x_{i,r}^+,x_{j,s}^+] + h_2(\{\xi_{i,r+1},x_{j,s}^+\} - \{x_{i,r}^+,x_{j,s}^+\})$ for $j \neq i,$

(Y3.1) $[\xi_{i,r+1},x_{i,s}^+] - [\xi_{i,r},x_{i+1,s}^+] = \pm h_2(\xi_{i,r},x_{i,s}^+),$

(Y3.2) $[\xi_{i,r+2},x_{j,s}^+\} - 2[\xi_{i+1,r+1},x_{j,s+1}^+] + [\xi_{i,r},x_{j+s+2}^+] =$  

$-h_1h_3[\xi_{i,r},x_{j,s}^+] + h_2(\{\xi_{i,r+1},x_{j,s}^+\} - \{\xi_{i,r},x_{j,s}^+\})$ for $j \neq i,$

(Y4) $[\xi_{i,0},x_{j,s}^+] = \pm a_{i,j}x_{j,s}^+,$  

$[\xi_{i,1},x_{i+1,s}^+] = \mp (2x_{i+1,s}^+ + h_2(\xi_{i,0},x_{i+1,s}^+)),$

(Y5) Sym $\text{Sym}_{r_1,r_2,r_3} [x_{i,r_1}^+,x_{i,r_2}^+,x_{i,r_3}^+,x_{i+1,s}] = 0$.

- **Case $n = 1$:** we impose

(Y2) $[x_{0,r+3}^+,x_{0,s+3}^+] - 3[x_{0,r+2},x_{0,s+2}^+] + 3[x_{0,r+1},x_{0,s+2}^+] - [x_{0,r},x_{0,s+3}^+] =$  

$-\sigma_2(x_{0,r+1}^+,x_{0,s}^+) - [x_{0,r},x_{0,s}^+] + \pm 3\{x_{0,r}^+,x_{0,s}^+\},$

(Y3) $[\xi_{0,r+3},x_{0,s}^+] - 3[\xi_{0,r+2},x_{0,s+1}^+] + 3[\xi_{0,r+1},x_{0,s+2}^+] - [\xi_{0,r},x_{0,s+3}^+] =$  

$-\sigma_2(\xi_{0,r+1},x_{0,s}^+) - [\xi_{0,r},x_{0,s}^+] + \pm 3\{\xi_{0,r},x_{0,s}^+\},$

(Y4) $[\xi_{0,0},x_{0,s}^+] = 0,$  

$[\xi_{0,1},x_{0,s}^+] = 0,$  

$[\xi_{0,2},x_{0,s}^+] = \pm 2h_1h_3x_{0,s}^+,$

(Y5) Sym $\text{Sym}_{r_1,r_2,r_3} [x_{0,r_1}^+,x_{0,r_2}^+,x_{0,r_3}^+,x_{0,s+1}] = 0$.

where $\sigma_1 := h_1 + h_2 + h_3 = 0$, $\sigma_2 := h_1h_2 + h_1h_3 + h_2h_3$, $\sigma_3 := h_1h_2h_3$.

**Remark 1.2.** (i) For $n > 2$, our definition of $\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ reproduces the algebras introduced by Guay in [G]. To be more precise, we have an isomorphism $\mathcal{Y}^{(n)}_{h_1,h_2,-h_1-h_2} \simeq \mathcal{Y}_{h_2,\frac{1}{2}h_2 - \frac{1}{2}(2h_1+h_2)},$

(ii) Our definition of $\mathcal{Y}^{(2)}_{h_1,h_2,h_3}$ coincides with the corrected version of $\mathcal{Y}_{h_3,-h_1}(\hat{\mathfrak{sl}}_2)$ from [K].

(iii) Our definition of $\mathcal{Y}^{(1)}_{h_1,h_2,h_3}$ first appeared in [T] under the name of affine Yangian of $\mathfrak{gl}_1$. 
1.3. Affine Yangians as additivizations of quantum toroidal algebras.

The algebras $\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ can be considered as natural “additivizations” of the algebras $U^{(n)}_{q_1,q_2,q_3}$ in the same way as $Y_k(\mathfrak{g})$ is an “additivization” of $U_q(\mathfrak{g})$. We explain this by rewriting $(Y0–Y5)$ in a form similar to the defining relations (T0–T6). We also introduce an algebra $D\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$

Let us introduce the generating series:

$$x^+_i(z) := \sum_{r \geq 0} x^+_{i,r} z^{-r-1}, \quad \xi_i(z) := 1 + h_2 \sum_{r \geq 0} \xi_{i,r} z^{-r-1}.$$ 

We also define a collection of polynomials $\{p_{i,j}(z,w)\}_{i \in [n]}$ as follows:

$$p_{i,j}(z,w) = \begin{cases} 
  z - w + m_{i,j} \beta - a_{i,j} h & \text{if } n > 2 \\
  z - w - h_2 & \text{if } i = j \text{ and } n = 2 \\
  (-1)^{\delta_{i,j}} (z - w - h_1)(z - w - h_3) & \text{if } i \neq j \text{ and } n = 2 \\
  (z - w - h_1)(z - w - h_2)(z - w - h_3) & \text{if } n = 1 
\end{cases}$$

Let $\mathcal{Y}^{(n);-,\mathcal{Y}^{(n);0},\mathcal{Y}^{(n);+}$ be the subalgebras of $\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ generated by $\{x^+_{i,r}\}, \{\xi_i\}$ and $\{x^+_{i,r}\}$, respectively. Let $\mathcal{Y}^{(n);2}$ and $\mathcal{Y}^{(n);2}$ be the subalgebras of $\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ generated by $\mathcal{Y}^{(n);0}, \mathcal{Y}^{(n);+}$ and $\mathcal{Y}^{(n);0}$, $\mathcal{Y}^{(n);+}$, respectively. The following result is standard:

**Proposition 1.3.** (a) The subalgebra $\mathcal{Y}^{(n);0}$ is a polynomial algebra in the generators $\{\xi_i\}$. 
(b) The subalgebras $\mathcal{Y}^{(n);0}$ are generated by $\{x^+_{i,r}\}$ with the defining relations $(Y2, Y5)$. 
(c) The subalgebras $\mathcal{Y}^{(n);0}$ are generated by $\{\xi_i, x^+_{i,r}\}$ with the defining relations $(Y0, Y2–Y5)$. 

Consider the homomorphisms $\sigma^+_i : \mathcal{Y}^{(n);+0} \to \mathcal{Y}^{(n);+0}$ defined by $\xi_j \mapsto \xi_{j,k}, x^+_{i,r} \mapsto x^+_{i,k+r+}$. These are well-defined due to Proposition 1.3. Let $\mu : \mathcal{Y}^{(n)}_{h_1,h_2,h_3} \otimes \mathcal{Y}^{(n)}_{h_1,h_2,h_3} \to \mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ be the multiplication map. The following is straightforward:

**Proposition 1.4.** (a) The relation $(Y0)$ is equivalent to $[\xi_i(z), \xi_j(w)] = 0$. 
(b) The relation $(Y1)$ is equivalent to $h_2 \cdot (w - z)[x^+_{i,r}(z), x^-_{j,s}(w)] = \delta_{i,j}(\xi_i(z) - \xi_i(w))$. 
(c) The relations $(Y3)+$ and $(Y4)$ are equivalent to $p_{i,j}(z, \sigma^+_j(z)x^+_{j,s}(z), \sigma^-_j(z)x^-_{j,s}(z)) = -p_{j,i}(z, \sigma^+_j(z,x^-_{j,s}(z), \sigma^-_j(z)x^+_{j,s}(z))$. 
(d) The relation $(Y2)$ is equivalent to $\partial^\delta_{i,j} \mu \left( p_{i,j}(z, \sigma^+_j(z)x^+_{j,s}(z) + p_{j,i}(z, \sigma^-_j(z)x^-_{j,s}(z)) \otimes x^+_{i}(z) \right) = 0$, $\partial^\delta_{i,j} \mu \left( p_{i,j}(z, \sigma^-_j(z)x^-_{j,s}(z) \otimes x^-_{i}(z) \right) = 0$, 

where we set $\delta_{i,j,n} := \deg(p_{i,j}(z,w)), \sigma^+_{j}(a \otimes b) := \sigma^+_{j}(a) \otimes b, \sigma^-_{j}(a \otimes b) := a \otimes \sigma^-_{j}(b)$. 

**Remark 1.5.** Let $D\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ be an associative unitary $\mathbb{C}$-algebra generated by $\{x^\pm_{i,k}, \tilde{\xi}^\pm_{i}\}_{k \in \mathbb{Z}}$ with the defining relations $(Y0–Y5)$. A similar construction for $Y_k(\mathfrak{g})$ was first introduced in [D] (see also [KT]). We equip $D\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ with a formal coproduct by assigning

$$\Delta(\tilde{x}^+_{i}(z)) = \tilde{x}^+_{i}(z) \otimes 1 + \tilde{\xi}^+_{i}(z) \otimes \tilde{x}^+_{i}(z), \quad \Delta(\tilde{\xi}^+_{i}(z)) = \tilde{\xi}^+_{i}(z) \otimes \tilde{\xi}^+_{i}(z) + 1 \otimes \tilde{x}^+_{i}(z),$$

$$\Delta(\tilde{\xi}^-_{i}(z)) = \tilde{\xi}^-_{i}(z) \otimes \tilde{\xi}^-_{i}(z),$$

where $\tilde{x}^+_{i}(z) := \sum_{k \in \mathbb{Z}} x^+_{i,k} z^{-k-1}, \tilde{\xi}^+_{i}(z) := 1 + \sum_{r \geq 0} \xi_{i,r} z^{-r-1}, \tilde{\xi}^-_{i}(z) := 1 - \sum_{s < 0} \xi_{i,s} z^{-s-1}$. 

HOMOMORPHISMS BETWEEN QUANTUM TOROIDAL AND AFFINE YANGIAN ALGEBRAS 7
1.4. Fock representations.

For \( p \in [n] \) and \( u \in \mathbb{C}^* \), let \( F^p(u) \) be a \( \mathbb{C} \)-vector space with the basis \( \{ |\lambda\rangle \} \) labeled by all partitions \( \lambda \) of \( n \) and let \( s \in \mathbb{N} \), \( \lambda \equiv (\lambda_1, \ldots, \lambda_s, 1, \lambda_{s+1}, \ldots) \) be a partition. Define \( c_s(\lambda) = p + s - \lambda_s \in \mathbb{Z} \). Set \( \psi(z) = (q - q^{-1})/z \) and write \( a \equiv b \) if \( a - b \) is divisible by \( n \).

The following result is due to [FJMM 1, Proposition 3.3] (see also [FJMM 2, Section 2.5]).

**Proposition 1.6.** (a) For \( n > 1 \), the following formulas define an action of \( U_{q_1, q_2, q_3}^{(1)} \) on \( F^p(u) \) :

\[
\langle \lambda | e_j(z) | \lambda + 1 \rangle = \delta_{c_1(\lambda) \equiv j+1} \prod_{c_s(\lambda) \equiv j} \psi(q_1^{\lambda_s} - q_3^{1-s}) \cdot \delta(q_1^{1} q_3^{1-s} u/z),
\]

\[
\langle \lambda | f_j(z) | \lambda + 1 \rangle = \delta_{c_1(\lambda) \equiv j+1} \prod_{c_s(\lambda) \equiv j} \psi(q_1^{\lambda_s} - q_3^{1-s}) \cdot \delta(q_1^{1} q_3^{1-s} u/z),
\]

\[
\langle \lambda | \psi^+_j(z) | \lambda \rangle = \prod_{c_s(\lambda) \equiv j} \psi \left( \frac{z q_1^{\lambda_s} - q_3^{1-s}}{z} \right) \cdot \prod_{c_s(\lambda) \equiv j} \psi \left( \frac{z q_3^{1-s} - q_1^{1-s} u}{z} \right),
\]

while all other matrix coefficients are set to be zero. Here \((\cdots)\) denotes decomposition in \( z^\pm \).

(b) For \( n = 1 \), same formulas with the matrix coefficient of \( f_0(z) \) multiplied by \( q(1-q_3)/(1-q_1) \) define an action of \( U_{q_1, q_2, q_3}^{(1)} \) on \( F^0(u) \).

**Remark 1.7.** (a) If \( \lambda + 1 \) (respectively \( \lambda - 1 \)) is not a partition, then the above formulas automatically give \( \langle \lambda + 1 | e_j(z) | \lambda \rangle = 0 \) \( \forall \ j \in [n] \) (respectively \( \langle \lambda - 1 | f_j(z) | \lambda \rangle = 0 \) \( \forall \ j \in [n] \)).

(b) The above infinite products can be simplified to finite products, due to \( \psi(1/z) \psi(q_2 z) = 1 \).

(c) The Fock representations \( F^p(u) \) were originally constructed from the “vector representations” by using the semi-infinite wedge construction and the formal coproduct \( \mathcal{T} \) on \( U_{q_1, q_2, q_3}^{(1)} \)

Let us analogously define Fock representations of \( Y_{h_1, h_2, h_3}^{(n)} \). For \( p \in [n] \) and \( v \in \mathbb{C} \), let \( a F^p(v) \) be a \( \mathbb{C} \)-vector space with the basis \( \{ |\lambda\rangle \} \). We also set \( \phi(z) = \frac{h_1}{z} \) and \( \delta^+(z) = \sum_{x=0}^{\infty} z^x \).

**Proposition 1.8.** (a) For \( n > 1 \), the following formulas define an action of \( Y_{h_1, h_2, h_3}^{(n)} \) on \( a F^0(v) \) :

\[
\langle \lambda + 1 | x_j^+(z) | \lambda \rangle = \delta_{c_1(\lambda) \equiv j+1} \prod_{c_s(\lambda) \equiv j} \phi((\lambda_s - \lambda_{s-1} - 1) h_1 + (s-i) h_3) \cdot \prod_{c_s(\lambda) \equiv j+1} \phi((\lambda_s - \lambda_{s-1}) h_1 + (i-s) h_3)
\]

\[
\times \frac{1}{z} \delta^+(((\lambda_s h_1 + (i-1) h_3 + v)/z),
\]

\[
\langle \lambda | x_j^-(z) | \lambda + 1 \rangle = \delta_{c_1(\lambda) \equiv j+1} \prod_{c_s(\lambda) \equiv j} \phi((\lambda_s - \lambda_{s-1} - 1) h_1 + (s-i) h_3) \cdot \prod_{c_s(\lambda) \equiv j+1} \phi((\lambda_s - \lambda_{s-1}) h_1 + (i-s) h_3)
\]

\[
\times \frac{1}{z} \delta^+(((\lambda_s h_1 + (i-1) h_3 + v)/z),
\]

\[
\langle \lambda | \xi_j(z) | \lambda \rangle = \prod_{c_s(\lambda) \equiv j} \phi((\lambda_s - 1) h_1 + (s-1) h_3 + v + z) \cdot \prod_{c_s(\lambda) \equiv j+1} \phi(z - (\lambda_s h_1 + (s-1) h_3 + v)),
\]

while all other matrix coefficients are set to be zero.

(b) For \( n = 1 \), same formulas with the matrix coefficient of \( x_0^-(z) \) multiplied by \(-h_3/h_1 \) define an action of \( Y_{h_1, h_2, h_3}^{(1)} \) on \( a F^0(v) \) (see [L Proposition 4.4]).

**Remark 1.9.** For \( v \notin \{-a h_1 - b h_3 \} \), we get an action of \( D Y_{h_1, h_2, h_3}^{(n)} \) (from Remark 1.8) on \( a F^p(v) \) by changing \( \delta^+(((\lambda_s h_1 + (i-1) h_3 + v)/z) \).
1.5. Tensor products of Fock representations.

We will need not only Fock modules but also their tensor products. Given \( r \in \mathbb{N} \) and \( p = (p_1, \ldots, p_r) \in [n]^r \), \( u = (u_1, \ldots, u_r) \in \mathbb{C}^r \), consider the Fock modules \( \{ F^p(u_i) \}_{i=1}^r \).

Using the formal coproduct \( \boxdot \) on the algebra \( \mathcal{U}_{q_1,q_2,q_3}^{(n)} \), one can define an action of \( \mathcal{U}_{1,q_2,q_3}^{(n)} \) on \( F^p(\mathbf{u}) := F^{p_1}(u_1) \otimes \cdots \otimes F^{p_r}(u_r) \), but only if \( \{ u_i \} \) are not in resonance, see [FJMM11]. This module has the basis \( \{ \lambda \} \) labeled by \( r \)-tuples of partitions \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \). By \( \lambda + 1^{(a)}_i \) we denote \( (\lambda^{(1)}, \ldots, \lambda^{(a)} + 1, \lambda^{(a+1)}, \ldots) \). For \( 1 \leq a, l \leq r \) and \( s, t \in \mathbb{N} \), we say \( (a, s) \prec (l, i) \) (or \( (l, i) \succ (a, s) \)) if either \( a < l \) or \( a = l, s < i \). We also set \( \chi^{(a)}_s := q_1^{\lambda^{(a)}_s} q_3^{-1} u_a \).

**Proposition 1.10.** (a) For \( n > 1 \), the following formulas define an action of \( \mathcal{U}_{q_1,q_2,q_3}^{(n)} \) on \( F^p(\mathbf{u}) :\)

\[
\langle \lambda + 1^{(l)}_i | e_j(z) | \lambda \rangle = \delta_{c_i, (\lambda^{(l)})} \prod_{a \in 1, c_i(\lambda^{(l)})} \psi \left( \frac{\chi^{(a)}_i}{q_1^{x_a(z)}} \right) \prod_{a \in 1, c_i(\lambda^{(l)})} \psi \left( \frac{\chi^{(l)}_i}{\chi^{(a)}_s} \right) \cdot \delta \left( \frac{x^{(l)}_i}{z} \right),
\]

\[
\langle \lambda | f_j(z) | \lambda + 1^{(l)}_i \rangle = \delta_{c_i, (\lambda^{(l)})} \prod_{a \in 1, c_i(\lambda^{(l)})} \psi \left( \frac{\chi^{(a)}_i}{q_1^{x_a(z)}} \right) \prod_{a \in 1, c_i(\lambda^{(l)})} \psi \left( \frac{\chi^{(l)}_i}{\chi^{(a)}_s} \right) \cdot \delta \left( \frac{x^{(l)}_i}{z} \right),
\]

\[
\langle \lambda | \psi^{\pm}_{j}(z) | \lambda \rangle = \left( \prod_{a=1}^{r} \prod_{1 \leq c_i(\lambda^{(a)})} \psi \left( \frac{\chi^{(a)}_i}{q_1^{x_a(z)}} \right) \right) \prod_{a=1}^{r} \prod_{1 \leq c_i(\lambda^{(a)})} \psi \left( \frac{z}{x^{(a)}_i} \right),
\]

while all other matrix coefficients are set to be zero.

(b) For \( n = 1 \), same formulas with the matrix coefficient of \( f_0(z) \) multiplied by \( q(1-q_3)(1-q_1^{-1}) \) define an action of \( \mathcal{U}_{q_1,q_2,q_3}^{(n)} \) on \( F^0(\mathbf{u}) \).

**Remark 1.11.** Actually, the parameters \( \{ u_i \} \) are not in resonance exactly when the first two formulas are well-defined (do not have zeros in denominators) for any two partitions \( \lambda, \lambda + 1^{(l)}_i \).

Let \( r \in \mathbb{N}, p \in [n]^r, v \in \mathbb{C}^r \) and assume that \( \{ v_i \} \) are not in resonance. Considering the “additive version” of the above proposition, we get an action of \( \mathcal{U}_{1,q_2,q_3}^{(n)} \) on the vector space \( a F^p(v) \) with the basis \( \{ \lambda \} \) labeled by \( r \)-tuples of partitions. Set \( x^{(a)}_s := \chi^{(a)}_s h_1 + (s-1) h_3 + v_a \).

**Proposition 1.12.** (a) For \( n > 1 \), the following formulas define an action of \( \mathcal{U}_{q_1,q_2,q_3}^{(n)} \) on \( a F^p(v) :\)

\[
\langle \lambda + 1^{(l)}_i | x^{(a)}_j(\pm) | \lambda \rangle = \delta_{c_i, (\lambda^{(l)})} \prod_{a \in 1, c_i(\lambda^{(l)})} \psi \left( \frac{\chi^{(a)}_i - x^{(a)} + h_3}{x^{(a)}_i - x^{(a)} - h_2} \right) \cdot \delta \left( \frac{x^{(l)}_i}{z} \right),
\]

\[
\langle \lambda | \psi^{\pm}_{j}(z) | \lambda \rangle = \left( \prod_{a=1}^{r} \prod_{1 \leq c_i(\lambda^{(a)})} \psi \left( \frac{\chi^{(a)}_i - z + h_3}{x^{(a)}_i - z - h_1} \right) \right) \prod_{a=1}^{r} \prod_{1 \leq c_i(\lambda^{(a)})} \psi \left( \frac{z - x^{(a)} - h_2}{z - x^{(a)}_i} \right),
\]

while all other matrix coefficients are set to be zero.

(b) For \( n = 1 \), same formulas with the matrix coefficient of \( x^{(a)}_0(\pm) \) multiplied by \( -h_3/h_1 \) define an action of \( \mathcal{U}_{q_1,q_2,q_3}^{(n)} \).
2. Limit algebras

In this section, we introduce the formal versions of our two algebras of interest and relate their classical limits to the well-known algebras of difference and differential operators on $\mathbb{C}^\times$.

We use the notation $'$ when working in the formal setting, i.e. over $\mathbb{C}[[h]]$ or $\mathbb{C}[[h_1, h_2]]$, where $h, h_i$-formal variables. Our notation differ from that in [11], where we treated the $n = 1$ case.

2.1. Algebras $\mathfrak{d}_q(n)'$ and $\mathfrak{d}_q(n)^\prime$. For any $q \in \mathbb{C}[[h]]^\times$, we define the algebra of $q$-difference operators on $\mathbb{C}^\times$, denoted by $\mathfrak{d}_q'$, to be an associative algebra over $\mathbb{C}[[h]]$ topologically generated by $Z^\pm 1, D^\pm 1$ subject to:

$$Z \cdot Z^{-1} = Z^{-1} \cdot Z = 1, \ D \cdot D^{-1} = D^{-1} \cdot D = 1, \ D \cdot Z = qZ \cdot D.$$  

Define the associative algebra $\mathfrak{d}_q(n)' = M_n \otimes \mathfrak{d}_q'$, where $M_n$ stands for the algebra of $n \times n$ matrices (so that $\mathfrak{d}_q(n)'$ is the algebra of $n \times n$ matrices with values in $\mathfrak{d}_q'$). We will view $\mathfrak{d}_q(n)'$ as a Lie algebra with the natural commutator-Lie bracket $[\cdot, \cdot]$. It is easy to check that the following formula defines a 2-cocycle $\phi_{\mathfrak{d}_q(n)'} \in C^2(\mathfrak{d}_q(n)'\),\mathbb{C}[[h]])):

$$\phi_{\mathfrak{d}_q(n)'}(M_1 \otimes D^k Z^l, M_2 \otimes D^{k'} Z^{-l'}) = \left\{ \begin{array}{ll} \text{tr}(M_1 M_2) & \text{if } l = l' \\ 0 & \text{otherwise} \end{array} \right.$$  

for any $M_1, M_2 \in M_n$ and $k, k', l, l' \in \mathbb{Z}$. Here $\frac{1-q^{l(l+k)}}{1-q^{l+k}}$ is understood in the sense of evaluating $\frac{1-x^l}{1-x}$ at $x = q^{k+k'}$. In particular, $\frac{1-q^{l(l+k)}}{1-q^{l+k}} \in \mathbb{C}[[h]]$ and $\frac{1-q^{l(l+k)}}{1-q^{l+k}} = l$ if $k + k' = 0$.

This endows $\mathfrak{d}_q(n)' = \mathfrak{d}_q(n)' \oplus \mathbb{C}[[h]] \cdot c_0$ with the Lie algebra structure.

2.2. Algebras $\mathfrak{d}_h(n)'$ and $\mathfrak{d}_h(n)^\prime$. Define the algebra of $h$-differential operators on $\mathbb{C}^\times$, denoted by $\mathfrak{d}_h'$, to be an associative algebra over $\mathbb{C}[[h]]$ topologically generated by $\partial, x^\pm 1$ subject to the following defining relations:

$$x \cdot x^{-1} = x^{-1} \cdot x = 1, \ \partial \cdot x = x \cdot (\partial + h).$$  

Define the associative algebra $\mathfrak{d}_h(n)' = M_n \otimes \mathfrak{d}_h'$ (so that $\mathfrak{d}_h(n)'$ is the algebra of $n \times n$ matrices with values in $\mathfrak{d}_h'$). We will view $\mathfrak{d}_h(n)'$ as a Lie algebra with the natural commutator-Lie bracket $[\cdot, \cdot]$. Following [11], consider a 2-cocycle $\phi_{\mathfrak{d}_h(n)'} \in C^2(\mathfrak{d}_h(n)'\),\mathbb{C}[[h]])):

$$\phi_{\mathfrak{d}_h(n)'}(M_1 \otimes f_1(\partial)x^l, M_2 \otimes f_2(\partial)x^{-l'}) = \left\{ \begin{array}{ll} \text{tr}(M_1 M_2) \cdot \sum_{a=0}^{l-1} f_1(ah)f_2((a-l)h) & \text{if } l > l' > 0 \\ -\text{tr}(M_1 M_2) \cdot \sum_{a=0}^{l-1} f_2(ah)f_1((a+l)h) & \text{if } l' > l > 0 \\ 0 & \text{otherwise} \end{array} \right.$$  

for arbitrary polynomials $f_1, f_2$ and any $M_1, M_2 \in M_n$, $l, l' \in \mathbb{Z}$.

This endows $\mathfrak{d}_h(n)' = \mathfrak{d}_h(n)' \oplus \mathbb{C}[[h]] \cdot c_\partial$ with the Lie algebra structure.

2.3. Homomorphism $\mathfrak{T}^\omega_{m,n}$. In this section, we assume that $q = \omega_N \in h\mathbb{C}[[h]]^\times$ for a certain order $N$ root of unity $\omega_N = \sqrt[N]{1} \in \mathbb{C}^\times$. Let us consider the completions of $\mathfrak{d}_q(n)'$, $\mathfrak{d}_q(n)'$, and $\mathfrak{d}_h(n)'$, $\mathfrak{d}_h(n)'$ with respect to the ideals $J_{\mathfrak{d}_q(n)'} = M_n \otimes (D^N - 1, q - \omega_N)$ and $J_{\mathfrak{d}_h(n)'} = M_n \otimes (\partial, h)$:

$$\mathfrak{d}_q(n)' := \lim_{\longrightarrow} \mathfrak{d}_q(n)' / \mathfrak{d}_q(n)' \cdot (D^N - 1, q - \omega_N)'$$  

and $\mathfrak{d}_h(n)' := \lim_{\longrightarrow} \mathfrak{d}_h(n)' / \mathfrak{d}_h(n)' \cdot (D^N - 1, q - \omega_N)'$.

$$\mathfrak{d}_h(n)' := \lim_{\longrightarrow} \mathfrak{d}_h(n)' / \mathfrak{d}_h(n)' \cdot (\partial, h)'$$  

and $\mathfrak{d}_h(n)' := \lim_{\longrightarrow} \mathfrak{d}_h(n)' / \mathfrak{d}_h(n)' \cdot (\partial, h)'$.
Remark 2.1. (i) Taking completions of $\mathfrak{d}_q^{(n)}$ and $\mathfrak{D}_h^{(n)}$ with respect to the ideals $J_{\mathfrak{d}_q^{(n)}}$ and $J_{\mathfrak{D}_h^{(n)}}$, commutes with taking central extensions with respect to the 2-cocycles $\phi_{\mathfrak{d}_q^{(n)}}$ and $\phi_{\mathfrak{D}_h^{(n)}}$.
(ii) Specializing $h$ or $q$ to complex parameters $h_0 \in \mathbb{C}$ or $q_0 \in \mathbb{C}^\times$, we get the matrix algebras $\mathfrak{d}_{h_0}^{(n)}$ and $\mathfrak{D}_{h_0}^{(n)}$ with values in the classical $\mathbb{C}$-algebras of difference/differential operators on $\mathbb{C}^\times$ as well as their one-dimensional central extensions. In other words, we consider the $\mathbb{C}$-algebras given by the same definitions of the generators and the defining relations. However, one can not define their completions as above. This explains why we choose to work in the formal setting.

For any $m, n \in \mathbb{N}$, we identify $M_m \otimes M_n \simeq M_{mn}$ via $E_{i,j} \otimes E_{k,l} \mapsto E_{m(k-1)+i,m(l-1)+j}$ for any $1 \leq i,j \leq m, 1 \leq k,l \leq n$. Our next result relates those different families of completions.

Theorem 2.2. (a) For any $n$ and $q = \omega_n \exp(h)$, there exists a $C[[h]]$-algebra homomorphism
$$\Upsilon^{\omega_n}_{1,n} : \mathfrak{d}^{(1),\prime}_{\omega_n \exp(h)} \longrightarrow \mathfrak{D}^{(n),\prime}_h,$$
such that
$$D \mapsto \left( \begin{array}{cccc}
q^{n-1}e^{-n\theta} & 0 & 0 & \cdots & 0 \\
0 & q^{n-2}e^{-n\theta} & 0 & \cdots & 0 \\
0 & 0 & q^{n-3}e^{-n\theta} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & qe^{-n\theta} \\
0 & 0 & 0 & \cdots & e^{-n\theta}
\end{array} \right),
Z \mapsto \left( \begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
x & 0 & \cdots & 0
\end{array} \right).$$

(b) Combining the homomorphism $\Upsilon^{\omega_n}_{1,n}$ from part (a) with the identification $M_m \otimes M_n \simeq M_{mn}$ from above, we get a $C[[h]]$-algebra homomorphism $\Upsilon^{\omega_n}_{m,n} : \mathfrak{d}^{(m),\prime}_{\omega_n \exp(h)} \longrightarrow \mathfrak{D}^{(mn),\prime}_h$, for any $m, n$.

(c) There exists a homomorphism of $C[[h]]$-algebras
$$\bar{\Upsilon}^{\omega_n}_{m,n} : \mathfrak{d}^{(m),\prime}_{\omega_n \exp(h)} \longrightarrow \mathfrak{D}^{(mn),\prime}_h$$
such that $\bar{\Upsilon}^{\omega_n}_{m,n}(c_D) = c_D$ and $\bar{\Upsilon}^{\omega_n}_{m,n}(A) = \Upsilon^{\omega_n}_{m,n}(A)$ for any $A \in \mathfrak{d}^{(m),\prime}_{\omega_n \exp(h)}$.

(d) If $\omega_n$ is a primitive nth root of unity, then $\Upsilon^{\omega_n}_{m,n}$ and $\bar{\Upsilon}^{\omega_n}_{m,n}$ are isomorphisms.

Proof.
(a) Let us denote the above $n \times n$ matrices by $X$ and $Y$. These two matrices are invertible and satisfy the identity $XY = qYX$ (which follows from $e^{n\theta}xe^{-n\theta} = e^{n\theta}x = q^n x$). Hence, there exists a $C[[h]]$-algebra homomorphism $\Upsilon^{\omega_n}_{1,n} : \mathfrak{d}^{(1),\prime}_{\omega_n \exp(h)} \longrightarrow \mathfrak{D}^{(n),\prime}_h$, such that $\Upsilon^{\omega_n}_{1,n}(D) = X$ and $\Upsilon^{\omega_n}_{1,n}(Z) = Y$. Since $q - \omega_n \in hC[[h]]$ and $\Upsilon^{\omega_n}_{1,n}(D^n - 1) \in \mathfrak{D}^{(n),\prime}_h$, the above homomorphism induces a homomorphism $\hat{\Upsilon}^{(1)\prime}_{\omega_n \exp(h)} : \mathfrak{D}^{(n),\prime}_h$, which we denote by the same symbol $\Upsilon^{\omega_n}_{1,n}$.

(b) Follows immediately from (a).

(c) It suffices to check that
$$\left( M_1 \otimes D^k Z^l, M_2 \otimes D^{k'} Z^{l'} \right) = \phi_{\mathfrak{d}_h^{(n)}}(\Upsilon^{\omega_n}_{m,n}(M_1 \otimes D^k Z^l), \Upsilon^{\omega_n}_{m,n}(M_2 \otimes D^{k'} Z^{l'}))$$
for any $M_1, M_2 \in M_m$ and $k, k', l, l' \in \mathbb{Z}$. It is a straightforward verification.

(d) Assume $\omega_n$ is a primitive nth root of unity. Let us now prove that $\Upsilon^{\omega_n}_{m,n}$ is an isomorphism. It suffices to show that the linear map
$$\Upsilon^{\omega_n}_{m,n} : \mathfrak{d}^{(m),\prime}_{\omega_n \exp(h)} / (D^n - 1, h)^r \longrightarrow \mathfrak{D}^{(mn),\prime}_h / (\partial, h)^r,$$induced by $\Upsilon^{\omega_n}_{m,n}$, is an isomorphism for any $r \in \mathbb{N}$. This condition follows from $\Upsilon^{\omega_n}_{1,n}$ being an isomorphism, due to our definition of $\Upsilon^{\omega_n}_{m,n}$. 

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For any $r \in \mathbb{N}$, it is easy to see that:

- $\{h^{tn}D^{s}Z^{k}| k \in \mathbb{Z}, s_{1}, s_{2} \in \mathbb{Z}_{+}, ns_{1} + s_{2} < nr\}$ is a $\mathbb{C}$-basis of $\mathfrak{g}^{(1),\prime}_{\omega_{n}}/((D^{n} - 1, \hbar))$.
- $\{E_{a,b} \otimes h^{a}D^{s}Z^{k}| 1 \leq a, b \leq n, k \in \mathbb{Z}, s_{1}, s_{2} \in \mathbb{Z}_{+}, s_{1} + s_{2} < r\}$ is a $\mathbb{C}$-basis of $\mathfrak{d}^{(n),\prime}_{\hbar}/(\partial, \hbar)^{r}$.

Combining this with part (c), we also see that $\bar{\Psi}$ is not well-defined as we have

$$\text{dim}(\text{span}_C h^{s} \cdot (D^{n} - 1)^{r-s-1}D^{i}0 \leq i \leq n-1)$$

maps isomorphically onto $\text{span}_C \{E_{a,b} \otimes h^{a}D^{s}Z^{k}| 1 \leq a, b \leq n, s_{1}, s_{2} \in \mathbb{Z}_{+}, s_{1} + s_{2} < r\}$.

For any $0 \leq s \leq r-1$, the restriction of $\Upsilon_{n}^{\omega^{r},0}$ to $\text{span}_C h^{s} \cdot (D^{n} - 1)^{r-s-1}D^{i}0 \leq i \leq n-1$ maps isomorphically onto $\text{span}_C \{E_{a,b} \otimes h^{a}D^{s}Z^{k}| 1 \leq a, b \leq n, s_{1}, s_{2} \in \mathbb{Z}_{+}, s_{1} + s_{2} < r\}$.

For any $0 \leq s \leq r-1$, the restriction of $\Upsilon_{n}^{\omega^{r},0}$ to $\text{span}_C h^{s} \cdot (D^{n} - 1)^{r-s-2}D^{i}0 \leq i \leq n-1$ maps isomorphically onto $\text{span}_C \{E_{a,b} \otimes h^{a}D^{s}Z^{k}| 1 \leq a, b \leq n, s_{1}, s_{2} \in \mathbb{Z}_{+}, s_{1} + s_{2} < r\}$.

Proceeding further by induction, we see that $\Upsilon_{n}^{\omega^{r},0}$ is surjective. Therefore:

$$\Upsilon_{n}^{\omega^{r},0} - \text{isomorphism} \Rightarrow \Upsilon_{m}^{\omega^{r},0} - \text{isomorphism} \Rightarrow \Upsilon_{m}^{\omega^{r},0} - \text{isomorphism} \Rightarrow \Upsilon_{m}^{\omega^{r},0} - \text{isomorphism}$$

Combining this with part (c), we also see that $\Upsilon_{m}^{\omega^{r},0}$ is a $\mathbb{C}[\mathfrak{g}]$-algebra isomorphism.

2.4. Algebras $\mathcal{U}_{h_{1}, h_{2}}^{(m),\omega,\prime}$ and $\mathcal{U}_{h_{2}}^{(m),\omega,\prime}$.

Throughout this section, we let $h_{1}, h_{2}$ be formal variables and set $h_{3} := -h_{1} - h_{2}$. We also fix any root of unity $\omega \in \mathbb{C}^{\times}$. We would like to consider the formal version of the quantum toroidal algebra $\mathcal{U}_{q_{1}, q_{2}, q_{3}}^{h_{1}/m}$ with $q_{1} = \omega^{e_{1}/m}, q_{2} = e_{2}/m, q_{3} = \omega^{-1} e^{-(h_{1} + h_{2})/m}$. Let

$$q_{1} = \omega \exp(h_{1}/m), q_{2} = \exp(h_{2}/m), q_{3} = \omega^{-1} \exp(h_{3}/m) \in \mathbb{C}[h_{1}, h_{2}]^{\times}.$$ 

Note that replacing $q_{2}$ with $q_{1}$, the relations $(T0, T2–T6)$ are defined over $\mathbb{C}[h_{1}, h_{2}]$, while $(T1)$ is not well-defined as we have $q - q^{-1}$ in denominator. To fix this, we modify the relation $(T1)$ in an appropriate way. We present $\psi^{\pm 1}_{i,0}$ in the form $\psi^{\pm 1}_{i,0} = \exp(\pm h_{2}/2m \cdot h_{i,0})$, so that

$$\psi^{\pm}_{i,0}(z) = \exp \left( \pm \frac{h_{2}}{2m} \cdot h_{i,0} \right) \exp \left( \pm (q - q^{-1}) \sum_{r > 0} h_{i,\pm r} x^{r} \right) \text{ with } q = \sqrt{q_{2}} = \exp \left( \frac{h_{2}}{2m} \right).$$

Switching from $\{\psi_{i,k}, \psi_{k}^{-1} \}_{k \in \mathbb{Z}}$ to $\{h_{i,k}, h_{k}^{-1} \}_{k \in \mathbb{Z}}$, the relations $(T4, T5)$ get modified to:

$$\{h_{i,k}, e_{j,l} \}_{i \neq j, l \in [m]} = b_{m}(i, j; k) \cdot e_{j+k} \cdot f_{j+k} \cdot f_{j+k}^{-1}, \quad b_{m}(i, j; k) = -b_{m}(i, j; k), \quad f_{j,l} \cdot f_{j,l}^{-1} \text{ for all } i, j \in [m], \ k, l \in \mathbb{Z},$$

where $b_{m}(i, j; 0) = a_{i,j},$ in the formal setting since $k_{q} = \mathbb{C}[h_{1}, h_{2}]$. Note also that the right-hand side of $(T1)$ is a series in $z^{k,1}, \omega^{k,1}$ with coefficients in $\mathbb{C}[h_{1}, h_{2}]$.

**Definition 2.3.** $\mathcal{U}_{h_{1}, h_{2}}^{(m),\omega,\prime}$ is an associative unital $\mathbb{C}[h_{1}, h_{2}]$-algebra topologically generated by $\{e_{i,k}, f_{i,k}, h_{i,k} \}_{k \in \mathbb{Z}}$ with the defining relations $(T0, T1, T2, T3, H, T6)$ (where $q_{i} \rightarrow q_{i}, n \rightarrow m$).
Finally, we define \( U^{(m),\omega'}_{h_1} \) by
\[
U^{(m),\omega'}_{h_1} := U^{(m),\omega'}_{h_1}/(h_2).
\]
It is an associative unital \( \mathbb{C}[[h_1]] \)-algebra topologically generated by \( \{ e_{i,k}, f_{i,k}, h_{i,k} \}_{i \in [m]} \) subject to the relations (T2,T3,T6) (where \( q_1 \sim q_1, q_2 \sim 1, q_3 \sim q_1^{-1}, q \sim 1, d \sim q_1, n \sim m \) and
\[
[h_{i,k}, h_{j,l}] = 0,
\]
(T0L)
\[
[e_{i,k}, f_{j,l}] = \delta_{i,j} \cdot h_{i,l+k},
\]
(T1L)
\[
h_{i,k}, e_{j,l} = b'_m(i,j ; k) \cdot e_{j,l+k},
\]
(T4L)
\[
h_{i,k}, f_{j,l} = -b'_m(i,j ; k) \cdot f_{j,l+k},
\]
for all \( i, j \in [m], k, l \in \mathbb{Z} \), where \( b'_m(i,j ; k) \in \mathbb{C}[h_1] \) is the image of \( b_m(i,j ; k) \in \mathbb{C}[[h_1,h_2]] \), i.e.,
\[
b'_m(i,j ; k) = \begin{cases} a_{i,j} \cdot q_1^{-km_{i,j}} & \text{if } m > 2 \\ 2\delta_{i,j} - (q_1^k + q_1^{-k})\delta_{i+1,j} & \text{if } m = 2 \\ 2 - q_1^k - q_1^{-k} & \text{if } m = 1 \end{cases}
\]

**Remark 2.4.** For \( h_1 \in \mathbb{C} \), set \( q_1 = \omega^2 e^{h_1/m} \in \mathbb{C}^\times \), and define \( U^{(m),\omega'}_{h_1} \) as the \( \mathbb{C} \)-algebra generated by \( \{ e_{i,k}, f_{i,k}, h_{i,k} \}_{i \in [m]} \) with the same defining relations (T0L,T1L,T2,T3,T4L,T5L,T6).

The following result is straightforward:

**Proposition 2.5.** There exists a homomorphism \( \theta^{(m)} : U^{(m),\omega'}_{h_1} \rightarrow U^{(m),\omega'}_{\tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'}} \) such that
\[
\begin{align*}
\theta^{(m)}(e_{0,k}) &= E_{m,1} \otimes D^k Z \\
\theta^{(m)}(f_{0,k}) &= E_{1,m} \otimes Z^{-1} D^k \\
\theta^{(m)}(h_{0,k}) &= E_{m,m} \otimes D^k - E_{1,1} \otimes (q_1^{m} D^k) + c_0
\end{align*}
\]
\( \theta^{(m)}(h_{i,k}) = (E_{i,i} - E_{i+1,i+1}) \otimes (q_1^{m} - D^k) \) for \( i \neq 0 \).

Define a free \( \mathbb{C}[[h_1]] \)-submodule \( \tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'} \subset \tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'} \) as follows:

- For \( m \geq 2 \), \( \tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'} \) is spanned by
  \[ \{ c_2, A_{k,l} \otimes D^k Z^l | k, l \in \mathbb{Z}, A_{k,l} \in \mathbb{M}_m \otimes \mathbb{C}[[h_1]] \} \]
- For \( m = 1 \), \( \tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'} \) is spanned by
  \[ \{ c_2, h_1 D^{\pm s}, h_1^{-1} D^k Z^{\pm s} | k, s \in \mathbb{N} \} \]

**Lemma 2.6.** \( \tilde{\mathcal{H}}_{q_1^{-m}}^{(m),0} \) is a Lie subalgebra of \( \tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'} \) and \( \text{Im}(\theta^{(m)}) \subset U^{(m),\omega'}_{\tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'}} \).

In fact, we have the following result:

**Theorem 2.7.** The homomorphism \( \theta^{(m)} \) provides an isomorphism \( U^{(m),\omega'}_{h_1} \cong U^{(m),\omega'}_{\tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'}} \).

Since all the defining relations of \( U^{(m),\omega'}_{h_1} \) are of Lie-type, it is an enveloping algebra of the Lie algebra generated by \( e_{i,k}, f_{i,k}, h_{i,k} \) with the aforementioned defining relations. Thus, Theorem 2.7 provides a presentation of the Lie algebra \( \tilde{\mathcal{H}}_{q_1^{-m}}^{(m),\omega'} \) by generators and relations.

Actually, a more general result holds:

**Theorem 2.8.** If \( h_1 \in \mathbb{C}\setminus\{ q \cdot \pi \sqrt{-1} \} \), then \( \theta^{(m)} \) induces an isomorphism of the \( \mathbb{C} \)-algebras:
\[ U^{(m),\omega'}_{h_1} \cong U^{(m),0}_{\tilde{\mathcal{H}}_{q_1^{-m}}^{(m),0}}, \]
where \( \tilde{\mathcal{H}}_{q_1^{-m}}^{(m),0} \) is a Lie subalgebra spanned by \( \{ c_0 \} \cup \{ A_{k,l} \otimes D^k Z^l | k, l \in \mathbb{Z}, A_{k,l} \in \mathbb{M}_m, \text{tr}(A_{0,0}) = 0 \} \).
2.5. Algebras $\mathcal{Y}^{(n),\prime}_{h_1,h_2}$ and $\mathcal{Y}^{(n),\prime}_{h_1}$.

Analogously to the previous section, we let $h_1, h_2$ be formal variables and set $h_3 := -h_1 - h_2$.

**Definition 2.9.** The algebra $\mathcal{Y}^{(n),\prime}_{h_1,h_2}$ is an associative unital $\mathbb{C}[[h_1,h_2]]$-algebra topologically generated by $\{x^\pm_{i,r}, \xi_{i,r}\}_{r \in \mathbb{Z}^+}$ subject to the defining relations (Y0–Y5) with $h_i = h_i/n$.

We equip the algebra $\mathcal{Y}^{(n),\prime}_{h_1,h_2}$ with the $\mathbb{Z}_+$-grading via $\deg(x^\pm_{i,r}) = \deg(\xi_{i,r}) = r$, $\deg(h_3) = 1$.

We define $\mathcal{Y}^{(n),\prime}_{h_1}$ (a formal version of the $\mathcal{C}$-algebra $\mathcal{Y}^{(n),\prime}_{h_1} := \mathcal{Y}^{(n),\prime}_{h_1,n,0,-h_1/n}$ with $h_1 \in \mathbb{C}$) by

$$\mathcal{Y}^{(n),\prime}_{h_1} := \mathcal{Y}^{(n),\prime}_{h_1,h_2}/(h_2).$$

It is an associative algebra over $\mathbb{C}[[h_1]]$. The following result is straightforward:

**Proposition 2.10.** There exists a homomorphism $\vartheta^{(n)} : \mathcal{Y}^{(n),\prime}_{h_1,h_2} \to U(\widehat{D}^{(n),\prime}_{h_1})$ such that

$$\vartheta^{(n)}(x^+_{0,r},) = E_{n,1} \otimes \partial^r x^r \text{ and } \vartheta^{(n)}(x^-_{0,r}) = E_{1,n} \otimes \partial^{(1-i/n)}(\partial + (1-i/n)h_1)^r \text{ for } i \neq 0,$$

$$\vartheta^{(n)}(x^+_{0,r}) = E_{n,1} \otimes x^{(1)} \partial^r \text{ and } \vartheta^{(n)}(x^-_{0,r}) = E_{1,n} \otimes (\partial + (1-i/n)h_1)^r \text{ for } i \neq 0,$$

$$\vartheta^{(n)}(\xi_{0,r}) = E_{n,1} \otimes \partial^r - E_{1,1} \otimes (\partial + h_1)^r + \delta_{0,r}c\partial,$$

$$\vartheta^{(n)}(\xi_{i,r}) = (E_{i,i} - E_{1,i+1})(\partial + 1/(i/n)h_1)^r \text{ for } i \neq 0.$$

Define a free $\mathbb{C}[[h_1]]$-submodule $\widehat{D}^{(n),\prime}_{h_1}$ of $\mathcal{Y}^{(n),\prime}_{h_1}$ as follows:

- For $n \geq 2$, $\widehat{D}^{(n),0}_{h_1}$ is spanned by $\{cA, A_{r,l} \otimes \partial^r x^s | r \in \mathbb{Z}_+, l \in \mathbb{Z}, A_{r,l} \in M_n \otimes \mathbb{C}[[h_1]] \text{ such that } \text{tr}(A_{r,l}) \in h_3 \mathbb{C}[[h_1]]\}$.

- For $n = 1$, $\widehat{D}^{(n),0}_{h_1}$ is spanned by $\{c\partial, h_1 \cdot \partial^r, h_1^{(1)} \cdot \partial^r x^l \cdot r \in \mathbb{Z}_+, s \in \mathbb{N}\}$.

**Lemma 2.11.** $\widehat{D}^{(n),0}_{h_1}$ is a Lie subalgebra of $\mathcal{Y}^{(n),\prime}_{h_1}$ and $\text{Im}(\vartheta^{(n)}) \subseteq U(\widehat{D}^{(n),0}_{h_1}).$

In fact, we have the following result:

**Theorem 2.12.** The homomorphism $\vartheta^{(n)}$ provides an isomorphism $\vartheta^{(n)} : \mathcal{Y}^{(n),\prime}_{h_1,h_2} \sim U(\widehat{D}^{(n),0}_{h_1})$.

Note that all the defining relations of $\mathcal{Y}^{(n),\prime}_{h_1,h_2}$ are of Lie-type. Hence, $\mathcal{Y}^{(n),\prime}_{h_1}$ is an enveloping algebra of the Lie algebra generated by $x^\pm_{i,r}, \xi_{i,r}$ with the aforementioned defining relations. Thus, Theorem 2.12 provides a presentation of the Lie algebra $\widehat{D}^{(n),0}_{h_1}$ by generators and relations.

Actually, a more general result holds:

**Theorem 2.13.** For $h_1 \in \mathbb{C}^\times$, $\vartheta^{(n)}$ induces an isomorphism of $\mathbb{C}$-algebras $\vartheta^{(n)} : \mathcal{Y}^{(n)}_{h_1} \sim U(\widehat{D}^{(n),0}_{h_1})$.

2.6. Flatness and faithfulness.

The main result of this section is

**Theorem 2.14.** (a) The algebra $\mathcal{U}^{(n),\omega,\prime}_{h_1,h_2}$ is a flat $\mathbb{C}[[h_2]]$-deformation of $\mathcal{U}^{(n),\omega,\prime}_{h_1}$.

(b) The algebra $\mathcal{Y}^{(n),\prime}_{h_1,h_2}$ is a flat $\mathbb{C}[[h_2]]$-deformation of $\mathcal{Y}^{(n),\prime}_{h_1}$.

**Proof.**

To prove Theorem 2.14 it suffices to provide a faithful $U(\widehat{D}^{(n),0,\prime}_{h_1})$-representation (respectively $U(\widehat{D}^{(n),0,\prime}_{h_1})$-representation) which admits a flat deformation to a representation of $\mathcal{U}^{(n),\omega,\prime}_{h_1,h_2}$ (respectively $\mathcal{Y}^{(n),\prime}_{h_1,h_2}$). To make use of the representations from Sections 1.4, 1.5, we will need to work not over $\mathbb{C}[[h_1,h_2]]$, but rather over the ring $R$, defined as a localization of $\mathbb{C}[[h_1,h_2]]$.
Corollary 2.15. (a) The following is a faithful $\mathcal{U}_R^{(m),\omega,\cdot}$-representation:

$$F_R := \bigoplus_{p_1,\ldots,p_r \in [m]} \bigoplus_{\substack{r \in \mathbb{N} \backslash \{0\} \ldots \cup \{0\} \ldots \cup \{0\}} \bigoplus \mathcal{F}_{\mathcal{R}_R^{(m),\omega,\cdot}}^{p_1,\ldots,p_r} (u_1) \otimes \cdots \otimes \mathcal{F}_{\mathcal{R}_R^{(m),\omega,\cdot}}^{p_1,\ldots,p_r} (u_r).$$

(b) The following is a faithful $\mathcal{Y}_R^{(n),\cdot}$-representation:

$$aF_R := \bigoplus_{p_1,\ldots,p_r \in [m]} \bigoplus_{\substack{r \in \mathbb{N} \backslash \{0\} \ldots \cup \{0\} \ldots \cup \{0\}} \bigoplus \mathcal{F}_{\mathcal{R}_R^{(m),\omega,\cdot}}^{p_1,\ldots,p_r} (v_1) \otimes \cdots \otimes \mathcal{F}_{\mathcal{R}_R^{(m),\omega,\cdot}}^{p_1,\ldots,p_r} (v_r).$$
3. Main Results

Following the notation and ideas of [GTL], we construct a $\mathbb{C}[[h_1, h_2]]$-algebra homomorphism

$$\Phi^{\omega_{m,n}} : \hat{U}_{h_1, h_2}^{(m), \omega_{m,n}} \to \hat{U}_{h_1, h_2}^{(mn),'}$$

between the appropriate completions of the two algebras of interest for any $m, n \in \mathbb{N}$ and $\omega_{mn} = \sqrt[n]{1} \in \mathbb{C}^\times$. Motivated by this result, we also construct a homomorphism

$$\Psi^{\omega, \omega'} : \hat{U}_{h_1, h_2}^{(m), \omega} \to \hat{U}_{h_1, h_2}^{(mn), \omega'}$$

for any two roots of unity $\omega, \omega'$ such that $\omega'/\omega^n = \omega_{mn}$ is an $m$th root of unity.

We establish compatibilities between these homomorphisms and evaluate their classical limits, i.e., the induced homomorphisms on the factor algebras by $(h_2)$.

3.1. Homomorphism $\Phi^{\omega_{m,n}}$.

To state our main result, we introduce the following notation (compare with [GTL]):

- Let $\hat{Y}_{h_1, h_2}^{(m),\omega_{mn}}$ be the completion of $Y_{h_1, h_2}^{(m),\omega_{mn}}$ with respect to the $\mathbb{Z}_+$-grading from Section 2.5.
- Let $\hat{J} \subset \hat{U}_{h_1, h_2}^{(m),\omega_{mn}}$ be the kernel of the composition
  
  $$Y_{h_1, h_2}^{(m),\omega_{mn}} \to U_{q_1^n}^{(m),\omega_{mn}} \to U_{q_1^n, J/(q_1^n)}^{(m),\omega_{mn}}$$

and let

$$\hat{U}_{h_1, h_2}^{(m),\omega_{mn}} := \lim_{\leftarrow \hat{J}} Y_{h_1, h_2}^{(m),\omega_{mn}}.$$

be the completion of $U_{h_1, h_2}^{(m),\omega_{mn}}$ with respect to the ideal $J$.

- For $i', j' \in [mn]$, we write $i' \equiv j'$ if $i' - j'$ is divisible by $m$.
- For $i \in [m], i' \in [mn]$, we write $i' \equiv i$ if $i = i' \mod m$.
- For $i' \in [mn]$, we define $\xi_{i'}(z)$ as in Section 1.3:

$$\xi_{i'}(z) := 1 + \frac{h_2}{mn} \sum_{r \geq 0} \xi_{i', r} z^{-r-1} \in Y_{h_1, h_2}^{(mn),'} \left[[z^{-1}]\right].$$

- For $i' \in [mn], r \in \mathbb{Z}_+$, we define $t_{i', r} \in Y_{h_1, h_2}^{(mn),'}$ by

$$\sum_{r \geq 0} t_{i', r} z^{-r-1} = t_{i'}(z) := \ln(\xi_{i'}(z)).$$

- Define the inverse Borel transform

$$B : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[w]]$$

by

$$\sum_{i=0}^{\infty} \frac{a_i}{z^{i+1}} \mapsto \sum_{i=0}^{\infty} \frac{a_i}{i!} w^i.$$

- For $i' \in [mn]$, we define $B_{i'}(w) \in \mathbb{C}[[w]]$ to be the inverse Borel transform of $t_{i'}(z)$.
- For $i', j' \in [mn]$, we define $H_{i', j'}(v) \in 1 + v\mathbb{C}[[v]]$ by

$$H_{i', j'}(v) = \begin{cases} e^{\frac{v}{mg} - \frac{v}{mg'}} & \text{if } \omega_{mn} = \omega_{m,n}' \\
\frac{e^{-\frac{v}{mg} - \frac{v}{mg'}} - \omega_{m,n}' - \omega_{m,n}}{\omega_{m,n} e^{-\frac{v}{mg'}} - \omega_{m,n} e^{-\frac{v}{mg'}}} & \text{if } \omega_{mn} \neq \omega_{m,n}'. \end{cases}$$

- For $i', j' \in [mn]$, we define a function

$$G_{i', j'}(v) := \log(H_{i', j'}(v)) \in v\mathbb{C}[[v]].$$

- For $i', j' \in [mn]$, we define

$$\gamma_{i', j'}(v) := -B_{j'}(-\partial_v) \partial_v G_{i', j'}(v) \in \hat{Y}_{h_1, h_2}^{(mn),'} [[v]].$$
For $i' \in [mn]$, we define $g_{i'}(v) := \sum_{r \geq 0} g_{i',v^r} \in \hat{g}_{h_1, h_2}^{(mn),i}[v]$ by

$$g_{i'}(v) := \left( \frac{h_2}{m(q-q^{-1})} \right)^{1/2} \cdot \exp \left( \frac{1}{2} \sum_{j' \neq i'} \gamma_{i',j'}(v) \right) \cdot \prod_{j' \neq i'} (\omega_{i'n}^j = \omega_{i'n}^j)^{1/2} \cdot \xi_{i'}(v)^{1/2}.$$ 

The identity $B(\log (1 - \frac{1}{q})) = \frac{1}{m} \omega_m^{w_0}$ immediately implies the following result:

**Corollary 3.1.** The equalities from Proposition 1.4(c) applied to $\hat{g}_{h_1, h_2}^{(mn),i}$ are equivalent to

$$[B_{i'}(v), x_{j',s}^\pm] = \pm \delta_{j',i'} (e^\frac{h_2 v_{i,n}}{m(q-q^{-1})} - e^{-\frac{h_2 v_{i,n}}{m(q-q^{-1})}}) + \delta_{j',i'} (e^\frac{h_2 v_{i,n}}{m(q-q^{-1})} - e^{-\frac{h_2 v_{i,n}}{m(q-q^{-1})}}) \delta_{j',i'} \delta_{j',i'} 1 \cdot \frac{h_2}{m(q-q^{-1})} e^{\frac{h_2}{m(q-q^{-1})}} x_{j',s}^\pm$$

for any $i', j' \in [mn]$ and $s \in \mathbb{Z}_+$.

We are ready to state our first main result:

**Theorem 3.2.** For any $m, n \in \mathbb{N}$, $\omega_{mn} = m \sqrt{i} \in \mathbb{C}^\times$, there exists an algebra homomorphism

$$\Phi_{m,n}^{\omega_{mn}} : \hat{U}_{h_1, h_2}^{(m),\omega_{mn},i} \rightarrow \hat{g}_{h_1, h_2}^{(mn),i}$$

defined on the generators by

(0) \quad h_{i,0} \mapsto \sum_{i' \equiv i} \xi_{i',0},

(1) \quad h_{i,l} \mapsto \frac{n}{q-q^{-1}} \sum_{i' \equiv i} \omega_{i,n}^{-l} B_{i'}(ln),

(2) \quad e_{i,k} \mapsto \sum_{i' \equiv i} \omega_{i,n}^{-ki} e^{kn\sigma^+} g_{i'}(\sigma^+) x_{i',0}^+, \quad e_{i,k} \mapsto \sum_{i' \equiv i} \omega_{i,n}^{-ki} e^{kn\sigma^-} g_{i'}(\sigma^-) x_{i',0}^-,

for all $i \in [m], k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\}$.

### 3.2. Homomorphism $\Psi_{m,n}^{\omega,\omega'}$.

Our next construction establishes homomorphisms between completions $\hat{U}_{h_1, h_2}^{(n),\omega'}$ for different $n$. This result has no counterpart in the classical setting of quantum loop algebras and was solely motivated by our construction of $\Phi_{m,n}^{\omega_{mn}}$.

To state the main result, we will need some more notation:

- For any $N \in \mathbb{N}$ and a root of unity $\omega$, we denote the generators of $U^{(N),\omega'}_{h_1, h_2}$ by $h_{i,k}, e_{i,k}, f_{i,k}$.
- For $N \in \mathbb{N}$, we define $q_N \in \mathbb{C}[[h_1, h_2]]$ via $q_N = \exp \left( \frac{h_2}{m} \right)$.
- Define $U^{(N),\omega'}_{h_1, h_2}$ and $U^{(N),\omega'}_{h_1, h_2}$ as the subalgebras of $U^{(N),\omega}_{h_1, h_2}$ generated by $\{h_{i,k}, e_{i,k}, f_{i,k}\}_{i \in [N]}$ and $\{h_{i,k}, f_{i,k}\}_{i \in [N]}$, respectively. Let us consider the homomorphisms

$$\tau^+_{i} : U^{(N),\omega'}_{h_1, h_2} \rightarrow U^{(N),\omega'}_{h_1, h_2} \quad \text{defined by} \quad h_{j,k} \mapsto h_{j,k}, \quad e_{j,k} \mapsto e_{j,k+\delta_{i,j}}, \quad f_{j,k} \mapsto f_{j,k+\delta_{i,j}}$$

(compare with the definition of $\omega_{i,n}^\pm$ from Section 1.3).
For $m, n \in \mathbb{N}, i' \in [mn], \omega$ – root of unity, define $R_{\iota}^{\pm}(v) = \sum_{r \geq 0} R_{\iota}^{\pm, v^r} \in U_{h_1, h_2}^{(mn, \omega)}([v])$ via

$$R_{\iota}^{\pm}(v) = \left(\frac{n(q_{mn} - q_{m}^{-1})}{q_{m} - q_{m}^{-1}}\right)^{1/2} \cdot \prod_{j' \equiv i'} \prod_{c=0}^{n-1} \psi_{j'}^{(mn), +} \left(\frac{e^{2\pi i j' c}}{\omega_{mn}^c} \cdot \psi_{j'}^{(mn), +}(v)^{1/2}\right).$$

**Theorem 3.3.** For any $m, n \in \mathbb{N}$ and roots of unity $\omega', \omega$ such that $\omega'/\omega = \sqrt[2n]{\omega}$, there exists an algebra homomorphism

$$\Psi_{m,n}^{\omega', \omega} : \hat{U}_{h_1, h_2}^{(mn, \omega')} \rightarrow \hat{U}_{h_1, h_2}^{(mn, \omega)}$$

defined on the generators by

**(ψ0)**

$$h_{i,0}^{(m)} \mapsto \sum_{i' \equiv i} h_{i',0}^{(mn)}.$$ (Psi)

**(ψ1)**

$$h_{i,l}^{(m)} \mapsto \frac{n(q_{mn} - q_{m}^{-1})}{q_{m} - q_{m}^{-1}} \cdot \sum_{i' \equiv i} \omega_{mn}^{-l' i'} h_{i',ln}^{(mn)}.$$ (Psi)

**(ψ2)**

$$e_{i,k}^{(m)} \mapsto \sum_{i' \equiv i} \omega_{mn}^{-k' i'} R_{i'}^{+}(\tau_{i'}^{+}) e_{i,kn}^{(mn)}.$$ (Psi)

**(ψ3)**

$$f_{i,k}^{(m)} \mapsto \sum_{i' \equiv i} \omega_{mn}^{-k' i'} R_{i'}^{-(\tau_{i'}^{+})} f_{i,kn}^{(mn)}.$$ for all $i \in [m], k \in \mathbb{Z}, l \in \mathbb{Z}\setminus\{0\}.$

The following result is straightforward:

**Proposition 3.4.** The action of $\Psi_{m,n}^{\omega', \omega}$ on the currents $\psi_{i}^{(m), \pm}(z), e_{i}^{(m)}(z), f_{i}^{(m)}(z)$ is given by

$$\Psi_{m,n}^{\omega', \omega}(\psi_{i}^{(m), \pm}(z)) = \prod_{i' \equiv i} \prod_{c=0}^{n-1} \psi_{i'}^{(mn), \pm} \left(\frac{e^{2\pi i c}}{\omega_{mn}^c} \cdot (\omega_{mn}^c)^{\pm}\right),$$

$$\Psi_{m,n}^{\omega', \omega}(e_{i}^{(m)}(z)) = \frac{1}{n} \sum_{i' \equiv i} \sum_{c=0}^{n-1} \left\{ R_{i'}^{+}(u) \cdot e_{i,kn}^{(mn)}(u) \right\}_{u = -\frac{2\pi i c}{n} \cdot (\omega_{mn}^c)^{\pm}}$$

$$\Psi_{m,n}^{\omega', \omega}(f_{i}^{(m)}(z)) = \frac{1}{n} \sum_{i' \equiv i} \sum_{c=0}^{n-1} \left\{ R_{i'}^{-(\tau_{i'}^{+})}(u) \cdot f_{i,kn}^{(mn)}(u) \right\}_{u = -\frac{2\pi i c}{n} \cdot (\omega_{mn}^c)^{\pm}}.$$

**Proof.**

(i) Note that $\Psi_{m,n}^{\omega', \omega}(h_{i,0}^{(mn)}) = \sum_{i' \equiv i} h_{i',0}^{(mn)} \cdot q_{m} = q_{mn}^{n} \Rightarrow \Psi_{m,n}^{\omega', \omega}(\psi_{i}^{(mn)}) = \left(\prod_{i' \equiv i} \psi_{i'}^{(mn)}\right)^{n}.$

Let us now compute the images of $\psi_{i}^{(m), \pm}(z) := (\psi_{i}^{(m)})^{\mp 1} \cdot \psi_{i}^{(m), \pm}(z)$ under $\Psi_{m,n}^{\omega', \omega}$.

$$\Psi_{m,n}^{\omega', \omega}(\psi_{i}^{(m), \pm}(z)) = \exp \left(\frac{n(q_{mn} - q_{m}^{-1})}{q_{m} - q_{m}^{-1}} \sum_{i' \equiv i} \sum_{r=1}^{\infty} h_{i', \pm mn}^{(rn)}(\omega_{mn}^r z)^{\mp r}\right)$$

$$\prod_{i' \equiv i} \exp \left(\frac{n(q_{mn} - q_{m}^{-1})}{q_{m} - q_{m}^{-1}} \sum_{r=1}^{\infty} h_{i', \pm mn}^{(rn)}(\omega_{mn}^r z)^{\mp r}\right)$$

$$\left(\frac{e^{2\pi i c}}{\omega_{mn}^c} \cdot (\omega_{mn}^c)^{\pm}\right).$$
Combining these two computations, we get the first formula of the proposition.

(ii) By construction, 
\[ \Psi_{\omega_n, \omega}^{\omega'}(e^{(m)}(z)) = \sum_{k \in \mathbb{Z}} \sum_{r' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} z^{-k} \omega_{mn}^{k'} \cdot R_{k', r}^{+} e^{(mn)}_{r, r', r+k}, \]
where we set \( R_{k', r}^{+} = 0 \) for \( r < 0 \). The second formula follows from
\[ \sum_{r, s \in \mathbb{Z}} e^{2\pi i r/s} = n \delta_{n=0} \]
and \( \{ R_{k', r}^{+} u, e^{(mn)}_{r, r', r+s} \} \)

(iii) Same arguments as those used in (ii) imply the above formula for \( \Psi_{\omega_n, \omega}^{\omega'}(f^{(m)}(z)) \).

\[ \square \]

3.3. Limit homomorphisms.

Recall the established isomorphisms
\[ \theta^{(m)} : U_{q_1}^{(m)}(h_2) \xrightarrow{\sim} U_{\varphi_1}^{(m)}(h_2), \]
\[ \vartheta^{(m)} : U_{q_1}^{(m)}(h_2) \xrightarrow{\sim} U_{\varphi_1}^{(m)}(h_2), \]
where \( q_1 = \omega_{mn} \exp(h_1/m) \Rightarrow q_1^m = \omega_n \exp(h_1) \) with \( \omega_n = \omega_{mn}^m \).

Considering factors by \( (h_2) \), we get the classical limit of \( \Psi_{\omega_n, \omega}^{(mn)} \), viewed as
\[ \tilde{\Phi}_{\omega_n, \omega}^{(mn)} : U_{\varphi_1}^{(m)}(h_1) \longrightarrow U_{\varphi_1}^{(m)}(h_1). \]

**Proposition 3.5.** The limit homomorphism \( \tilde{\Phi}_{\omega_n, \omega}^{(mn)} \) is induced by \( \tilde{\gamma}_{\omega_n} \).

**Proof.**

Note that \( \frac{h_2}{m q_1} \equiv 1 \pmod{h_2}, \frac{m n q_1}{h_2} B(v) \equiv \sum_{r=0}^{\infty} \frac{\xi_r \zeta_r}{m} \pmod{h_2} \Rightarrow g_t(v) \equiv 1 \pmod{h_2} \).

Together with the identity \( \tilde{\Phi}_{\omega_n, \omega}^{(mn)}(h_{i,k}) \equiv \sum_{l'=l}^{\omega_n^{k'}-k'} \sum_{r=0}^{\infty} \frac{(k')^r}{r!} \xi_{l', r} \equiv \sum_{l=l}^{n} \sum_{r=0}^{\omega_n^{k'-k}} \frac{(k')^r}{r!} \xi_{l,n-i+1,r} \pmod{h_2} \),

\[ \tilde{\Phi}_{\omega_n, \omega}^{(mn)}(e_{i,k}) \equiv \sum_{l'=l}^{\omega_n^{k'}-k'} \sum_{r=0}^{\infty} \frac{(k')^r}{r!} x_{l', r} \equiv \sum_{l=l}^{n} \sum_{r=0}^{\omega_n^{k-1-n+i}} \frac{(k')^r}{r!} x_{l,n-i+1,r} \pmod{h_2} \]

Recalling the formulas for the action of \( \theta^{(m)} \) and \( \vartheta^{(mn)} \) on the generators \( \{ e_{i,k}, f_{i,k}, h_{i,k} \}_{i \in \mathbb{Z}} \)
and \( \{ x_{l', r}, \xi_{l', r} \}_{l', r \in \mathbb{Z}} \) (see Propositions 2.5, 2.10), we get the result. \[ \square \]

Combining this result with Theorems 2.2, 2.1, we get:

**Corollary 3.6.** If \( \omega_n = \omega_{mn}^m \) is a primitive \( n \)th root of unity, then \( \tilde{\Phi}_{\omega_n, \omega}^{(mn)} \) is injective. In particular, the homomorphisms \( \tilde{\Phi}_{\omega_n}^{(N,1)} \) are injective for any \( N \in \mathbb{N} \) and \( \omega_N = \sqrt[N]{1} \in \mathbb{C}^\times \).

Analogously, one can consider the classical limit of \( \Psi_{\omega_n, \omega}^{\omega'} \), viewed as
\[ \Psi_{\omega_n, \omega}^{\omega'} : U_{\varphi_1}^{(m)}(h_1) \longrightarrow U_{\varphi_1}^{(m)}(h_1). \]

Set \( Q_1 := (\omega')^m \exp(h_1), Q_2 := \omega^{mn} \exp(h_1) \in \mathbb{C}[[h_1]]^\times \), so that \( Q_1 = \omega_n, Q_2 = \omega_n \).

On the other hand, there is a homomorphism of \( \mathbb{C}[[h_1]]\)-algebras
\[ \tilde{\Gamma}_{\omega_n, \omega} : \tilde{\phi}_{Q_1}^{(m)}, \tilde{\phi}_{Q_2}^{(mn)} \]
constructed analogously to \( \tilde{\gamma}_{\omega_n} \) from Theorem 2.2 with \( e^{\omega n} \) being replaced by \( D^n \). The proof of the following result is straightforward (compare to Proposition 3.3):

**Proposition 3.7.** The limit homomorphism \( \Psi_{\omega_n, \omega}^{\omega'} \) is induced by \( \tilde{\Gamma}_{\omega_n, \omega} \).
3.4. Compatibility between homomorphisms.

- **Compatibility between $\Phi$ and $\Psi$ homomorphisms.**
  For any $m, n, l \in \mathbb{N}$ and $\omega_{mn} = \sqrt[n]{\sqrt[m]{T}}$, $\omega_{mnl} = \sqrt[n]{\sqrt[m]{\sqrt[l]{T}}}$, let us consider the following algebras:
  
  $$\widehat{U}_{h_1, h_2}^{(m)l}, \widehat{U}_{h_1, h_2}^{(mm)l}, \widehat{U}_{h_1, h_2}^{(mnl)};$$

  where we define $\omega'_{mnl} = \omega_{mn} \omega_{ml}^{n}$. So far, we have constructed three homomorphisms
  
  $$\Phi_{m, n, l}^{\omega_{mnl}}, \Phi_{m, n, l}^{\omega_{mnl}'}, \Phi_{m, n, l}^{\omega_{mnl}''}: \widehat{U}_{h_1, h_2}^{(m)l} \rightarrow \widehat{U}_{h_1, h_2}^{(mm)l} \rightarrow \widehat{U}_{h_1, h_2}^{(mnl)}.$$

  These homomorphisms are compatible in the following sense:

  **Theorem 3.8.** We have $\Phi_{m, n, l}^{\omega_{mnl}'} = \Phi_{m, n, l}^{\omega_{mnl}''} \circ \Phi_{m, n, l}^{\omega_{mnl}'}$.

  In other words, we have the following commutative diagram:

  ![Diagram](https://via.placeholder.com/150)

  - **Compatibility between $\Psi$ homomorphisms.**
    For any $m, n, l \in \mathbb{N}$ and roots of unity $\omega''', \omega', \omega$ such that $\omega'''/(\omega')^{n} = \omega_{mn}, \omega'/\omega'' = \omega_{mnl}$, consider the following formal quantum toroidal algebras:
    
    $$\widehat{U}_{h_1, h_2}^{(m)l}, \widehat{U}_{h_1, h_2}^{(mm)l}, \widehat{U}_{h_1, h_2}^{(mnl)}.$$

    In Theorem 3.3 we have constructed the following homomorphisms between those
    
    $$\Psi_{m, n}^{\omega_{mnl}'}, \Psi_{m, n}^{(m)l}, \Psi_{m, n}^{(mm)l}, \Psi_{m, n}^{(mnl)};$$

    These homomorphisms are compatible in the following sense:

    **Theorem 3.9.** We have $\Psi_{m, n}^{\omega_{mnl}'} = \Psi_{m, n}^{\omega_{mnl}''} \circ \Psi_{m, n}^{\omega_{mnl}'}$.

    In other words, we have the following commutative diagram:

    ![Diagram](https://via.placeholder.com/150)

    The proofs of Theorems 3.2, 3.3, 3.8, 3.9 are based on the similar arguments and will be presented in the next section.
4.1. Compatible isomorphisms of representations.

Given \( m,n \) and an order \( mn \) root of unity \( \omega_{mn} \), we consider the two algebras of interest:
\[ U^m_{h_1,h_2}, \quad \text{and} \quad Y^m_{h_1,h_2}. \]
To proceed further, choose \( r \in \mathbb{N} \) and the following \( r \)-tuples:
\[ p = (p_1, \ldots, p_r) \in [mn]^r, \quad v = (v_1, \ldots, v_r) \in ((h_1, h_2)\mathbb{C}[[h_1, h_2]])^r, \]
\[ p' = (p'_1, \ldots, p'_r) \in [m]^r, \quad u' = (u'_1, \ldots, u'_r) \in (\mathbb{C}[[h_1, h_2]])^r. \]

Associated to this data, we have a collection of Fock \( U^m_{h_1,h_2} \)-representations \( \{ F^p_R(u_i) \}_{i=1}^{r} \) and a collection of Fock \( Y^m_{h_1,h_2} \)-representations \( \{ a F^p_R(v_i) \}_{i=1}^{r} \). Following Section 1.5, we consider
\[ F^p_R(u') := F^p_R(u'_1) \otimes F^p_R(u'_2) \cdots \otimes F^p_R(u'_r) \]
and
\[ a F^p_R(v) := a F^p_R(v_1) \otimes a F^p_R(v_2) \cdots \otimes a F^p_R(v_r) \]
whenever those representations are well-defined, i.e. \( \{ u'_i \}_{i=1}^{r} \) and \( \{ v_i \}_{i=1}^{r} \) are not in resonance. Both of these tensor products have natural bases \( \{ \lambda \} \) labeled by \( r \)-tuples of partitions
\[ \lambda = (\lambda(1), \ldots, \lambda(r)) \]

The action of the generators \( \{ h_{i,k}, e_{i,k}, f_{i,k} \}_{i \in [m]} \) and \( \{ \xi_{r,s}, x_{r,s} \}_{r \in \mathbb{Z}^+} \) in these bases is given by explicit formulas from Propositions 1.10, 1.12, where we replace \( Z \) and the corresponding \( s \) satisfies the property
\[ (\Pi) \quad F^p_R(u') \sim \Phi^\omega_{mn} \Phi^\omega_{mn} = \Phi^\omega_{mn} \Phi^\omega_{mn} \]
whenever those representations are well-defined, i.e. \( \{ u'_i \}_{i=1}^{r} \) and \( \{ v_i \}_{i=1}^{r} \) are not in resonance. Both of these tensor products have natural bases \( \{ \lambda \} \) labeled by \( r \)-tuples of partitions
\[ \lambda = (\lambda(1), \ldots, \lambda(r)) \]

Theorem 4.1. For any \( m,n,\omega_{mn},p,v \), define \( p'_i, u'_i \) via \( p'_i = p_i \mod m \) and \( u'_i = \frac{\omega_{mn}^i}{\omega_{mn}} \).
There exists a unique collection of constants \( c^\Phi_{m,n,\omega_{mn}} \in R \) such that \( c^\Phi_{m,n,\omega_{mn}} = 1 \) and the corresponding \( R \)-linear isomorphism of vector spaces
\[ I^m_{m,n,\omega_{mn}} : F^p_R(u') \sim \Phi^\omega_{mn} \Phi^\omega_{mn} \]

satisfies the property
\[ (\Pi) \quad I^m_{m,n,\omega_{mn}}(X(w)) = \Phi^\omega_{mn} \Phi^\omega_{mn} \]

If \( (\Pi) \) holds, we say that \( I^m_{m,n,\omega_{mn}} \) is compatible with \( \Phi^\omega_{mn} \).

Proof.
By straightforward computation, one can see that \( (\Pi) \) holds for any \( w = |\lambda|, X = h_{i,k} \) and an arbitrary choice of \( c^\Phi_{m,n,\omega_{mn}} \). On the other hand, the equalities
\[ I^m_{m,n,\omega_{mn}}(e_{i,k}) = \Phi^\omega_{mn} \Phi^\omega_{mn} \]
are both equivalent to
\[ \frac{c^\Phi_{m,n,\omega_{mn}}}{c^\Phi_{m,n,\omega_{mn}}} = d_{m,n,\omega_{mn}} \]
where

\[
\frac{d_{\lambda,1}^{\Phi}(m, n; \omega_{mn})}{d_{\lambda,1}^{\Phi}(m, n; \omega_{mn})} := \prod_{a=1}^{r} c_a(\lambda^{(a)}) \prod_{m,n} \psi(q_1^{-1} \lambda_s^{(a)} / \lambda_i^{(l)}) \cdot \prod_{a=1}^{r} c_a(\lambda^{(a)}) \prod_{m,n} \psi(\lambda_i^{(l)} / \lambda_s^{(a)}) \times
\]

\[
\prod_{a=1}^{r} \prod_{m,n} \psi(q_1^{-1} \lambda_s^{(a)} / \lambda_i^{(l)}) \cdot \prod_{a=1}^{r} c_a(\lambda^{(a)}) \prod_{m,n} \psi(\lambda_i^{(l)} / \lambda_s^{(a)}) \times
\]

\[
\prod_{a=1}^{r} \prod_{m,n} \left( \frac{x_s^{(a)} - x_i^{(l)}}{x_s^{(a)} - x_i^{(l)} + \frac{b_n}{m_n}} \right) ^{-\epsilon(a,i)} \cdot \prod_{a=1}^{r} \prod_{m,n} \left( \frac{x_i^{(l)} - x_s^{(a)}}{x_i^{(l)} - x_s^{(a)} + \frac{b_m}{m_n}} \right) ^{-\epsilon(a,i)}
\]

In this formula, we use the following notation:

\[
\lambda_s^{(a)} = q_1^{(a)} q_3^{s-1} u_a, \quad x_s^{(a)} = \lambda_i^{(a)} \cdot \frac{h_1}{m_n} + (s-1) \cdot \frac{h_3}{m_n} + u_a, \quad \epsilon(a,i) := \begin{cases} 
1/2 & \text{if } (a,s) \succ (l,i) \\
-1/2 & \text{if } (a,s) \prec (l,i) \\
0 & \text{if } a = l, s = i
\end{cases}
\]

The uniqueness of \( c_{\lambda}^{\Phi}(m, n; \omega_{mn}) \in R \) satisfying the relation with the initial condition \( c_{\lambda}^{\Phi}(m, n; \omega_{mn}) = 1 \) is obvious. The existence of such \( c_{\lambda}^{\Phi}(m, n; \omega_{mn}) \) is equivalent to

\[
d_{\lambda,1}^{\Phi}(m, n; \omega_{mn}) \cdot d_{\lambda,1}^{\Phi}(m, n; \omega_{mn}) = d_{\lambda,1}^{\Phi}(m, n; \omega_{mn}) \cdot d_{\lambda,1}^{\Phi}(m, n; \omega_{mn})
\]

for all possible \( \lambda, 1^{(l)}, 1^{(k)} \). The verification of this identity is straightforward.

Let us now consider the setup from Section 3.2: \( m, n \in \mathbb{N} \) and \( \omega', \omega \) are roots of unity such that \( \omega'/\omega = \omega_{mn} = \sqrt{m} \). For any \( r \in \mathbb{N} \), we consider the following \( r \)-tuples:

\[
p = (p_1, \ldots, p_r) \in [mn]^r, \quad u = (u_1, \ldots, u_r) \in (\mathbb{C}[h_1, h_2])^r,
\]

\[
p' = (p_1', \ldots, p_r') \in [m]^r, \quad u' = (u_1', \ldots, u_r') \in (\mathbb{C}[h_1, h_2])^r.
\]

We further assume that \( \{u_i\}_{i=1}^r \) and \( \{u'_i\}_{i=1}^r \) are not in resonance, so that we can define

\[
F_R^p(u) := \prod_{k=1}^r F_R^{p_k}(u_k) \otimes \cdots \otimes F_R^{p_r}(u_r) - \text{representation of } U_{m,n}^{\omega'},
\]

\[
F_R^{p'}(u') := \prod_{k=1}^r F_R^{p_k'}(u_k') \otimes \cdots \otimes F_R^{p_r'}(u_r') - \text{representation of } U_{m,n}^{\omega'}.
\]

Our next result establishes an isomorphism of these tensor products, compatible with \( \Psi_{\omega_{mn}, \omega} \).

**Theorem 4.2.** For any \( m, n, \omega', \omega, r, p, u \) as above, define \( p_i := p_i \mod m \) and \( u_i' := \frac{u_i}{m} \).

There exists a unique collection of constants \( c_{\lambda}^{\Phi}(m, n; \omega', \omega) \in R \) such that \( c_{\lambda}^{\Phi}(m, n; \omega', \omega) = 1 \) and the corresponding \( R \)-linear isomorphism of vector spaces

\[
J_{m,n,\omega',\omega}^{p,u}(X(w)) := \Psi_{m,n}^{\omega',\omega}(X(J_{m,n,\omega',\omega}^{p,u}(w))) \text{ for any } w \in F_R^{p'}(u'), X \in \{h_i,k,e_i,k,f_i,k\}_{i \in [m]}.
\]

If \( \dagger \) holds, we say that \( J_{m,n,\omega',\omega}^{p,u}(X(w)) \) is compatible with \( \Psi_{m,n}^{\omega',\omega} \).

**Proof.**

It is straightforward to see that \( \dagger \) holds for all \( w = |\lambda| \), \( X = h_i,k \) under any choice of \( c_{\lambda}^{\Phi}(m, n; \omega', \omega) \). On the other hand, the equalities

\[
J_{m,n,\omega',\omega}^{p,u}(e_i,k(\lambda)) = \Psi_{m,n}^{\omega',\omega}(e_i,k(J_{m,n,\omega',\omega}^{p,u}(\lambda))) \forall \lambda, \ i \in [m], \ k \in \mathbb{Z},
\]

\[
J_{m,n,\omega',\omega}^{p,u}(f_i,k(\lambda)) = \Psi_{m,n}^{\omega',\omega}(f_i,k(J_{m,n,\omega',\omega}^{p,u}(\lambda))) \forall \lambda, \ i \in [m], \ k \in \mathbb{Z}
\]


are both equivalent to
\begin{equation}
\psi^\lambda_{\omega+1}(m, n; \omega', \omega) = \psi^\lambda_{\omega}(m, n; \omega', \omega),
\end{equation}
where
\begin{equation}
d^\psi_{\lambda+1}(m, n; \omega', \omega) :=
\prod_{a=1}^{r} \prod_{s \in \mathbb{N}} \psi'(q_{a}^{-1} x^{(a)} / \chi_{i})^{(a,s)} \prod_{a=1}^{s} \prod_{m,n} \psi'(x_{i}^{(l)} / \chi_{s})^{(m,n)}. \times
\end{equation}

In this formula, we use the following notation:
\begin{equation}
q_{1}' = \omega' \exp(h_{1}/m), \quad q' = \exp(h_{2}/m), \quad q_{2}' = q' = \omega' \exp(h_{3}/m), \quad \chi_{s}^{(a)} = q_{1}^{A_{s}} q_{3}^{s-1} u_{a},
\end{equation}
\begin{equation}
q_{1} = \omega \exp(h_{1}/m), \quad q = \exp(h_{2}/m), \quad q_{2} = q = \omega \exp(h_{3}/m), \quad \chi_{s}^{(a)} = q_{1}^{A_{s}} q_{3}^{s-1} u_{a},
\end{equation}
\begin{equation}
\psi'(z) = (q' - q^{-1} z)/(1 - z), \quad \psi'(z) = (q - q^{-1} z)/(1 - z).
\end{equation}

The rest of the arguments are the same as in the proof of Theorem 4.1. □

4.2. Proofs of Main Results.

Now we are ready to provide short proofs of all theorems from Section 3.

- **Proof of Theorem 3.3**
  Recall the faithful $U^{(m), \omega_{mn}, \omega'}_{R}$-representation $F_{R} := \bigoplus_{r}^{p} F^{p}_{R}(u')$ and the faithful $Y^{(m), \omega}_{R}$-representation $\Phi^{(m), \omega}_{R}$ from Corollary 2.15. According to Theorem 4.1, we have an $R$-linear isomorphism $I : F_{R} \xrightarrow{\sim} \Phi^{(m), \omega}_{R}$ compatible with $\Phi^{(m), \omega}_{m,n}$ in the following sense:
\begin{equation}
I(X(w)) = \Phi^{(m), \omega}_{m,n}(X(I(w))) \quad \text{for any} \quad w \in F_{R}, \quad X \in \{h_{i,k}, e_{i,k}, f_{i,k}^{1} \in \mathbb{Z} \}.
\end{equation}

For any $X \in \{h_{i,k}, e_{i,k}, f_{i,k}^{1} \in [m] \}$, consider the assignment $X \mapsto \Phi^{(m), \omega}_{m,n}(X)$ with the right-hand side defined via (4.3). Then Theorem 3.2 is equivalent to saying that this assignment preserves all the defining relations of $U^{(m), \omega}_{m,n}$. The latter follows immediately from the faithfulness of $\Phi^{(m), \omega}_{R}$ combined with an existence of the compatible isomorphism $I$.

- **Proof of Theorem 3.5**
  The result of Theorem 3.3 is equivalent to saying that the assignment
\begin{equation}
X \mapsto \Phi^{(m), \omega}_{m,n}(X) \quad \text{for} \quad X \in \{h_{i,k}, e_{i,k}, f_{i,k}^{1} \in \mathbb{Z} \},
\end{equation}
defined via (4.3), preserves all the defining relations of $U^{(m), \omega'}_{R}$. The latter follows immediately by combining Corollary 2.15(a) with Theorem 4.2 (as in our proof of Theorem 3.2).

- **Proof of Theorem 3.6**
  Let $m,n,l \in \mathbb{N}$ and $\omega_{mn}, \omega_{mnl}, \omega'_{mnl}$ be as in Section 3.4. For any $r \in \mathbb{N}$ and $r$-tuples
\begin{equation}
p = (p_{1}, \ldots, p_{r}) \in [mn]^{r}, \quad v = (v_{1}, \ldots, v_{r}) \in ((h_{1}, h_{2})C[[h_{1}, h_{2}]]^{r}),
\end{equation}
we define $p' \in [mn]^{r}$, $p'' \in [mn]^{r}$, $u' \in (C[[h_{1}, h_{2}]]^{r})$, $u'' \in (C[[h_{1}, h_{2}]]^{r})^{r}$ via
\begin{equation}
p'_{1} := p_{1} \mod mn, \quad p'_{i} := p_{i} \mod m, \quad u'_{i} := e_{p'_{i}}^{l_{v_{i}}} / \omega_{mnl}^{l_{p'_{i}}}, \quad u''_{i} := e_{p'_{i}}^{l_{v_{i}}} / \omega_{mn}^{l_{p'_{i}}},
\end{equation}
Assume that \( \{v_i\}_{i=1}^r, \{v_i'\}_{i=1}^r, \{v_i''\}_{i=1}^r \) are not in resonance, so that we can consider the representations \( V'' := F_R(u''), V' := F_R(u'), V := a F_R(u) \) of \( \mathcal{U}_m(m,\omega_{mnl}') \). Consider the isomorphisms between these vector spaces, constructed in Theorems 4.1 and 4.2:

\[
J_{m,n,l;\omega_{mnl}}^{p,v} : V'' \longrightarrow V, \quad J_{m,n,l;\omega_{mnl}}^{u,v} : V'' \longrightarrow V', \quad J_{m,n,l;\omega_{mnl}}^{u,v'} : V'' \longrightarrow V'.
\]

Lemma 4.3. We have \( J_{m,n,l;\omega_{mnl}}^{p,v} = J_{m,n,l;\omega_{mnl}}^{u,v} \circ J_{m,n,l;\omega_{mnl}}^{u,v'} \).

Proof.

All three isomorphisms are diagonal in the bases \( \{|\lambda\} \). Therefore, it suffices to show that

\[
c_{\lambda}^\Phi(m,n;l;\omega_{mnl}') = c_{\lambda}^\Phi(m,n;l;\omega_{mnl}) \cdot c_{\lambda}^\Phi(m,n;l;\omega_{mnl}), \quad \forall \lambda.
\]

By the definition of \( c_{\lambda}^\Phi(m,n;l;\omega_{mnl}) \), this reduces to

\[
d_{\lambda,1_i}^{\Phi}(m,n;l;\omega_{mnl}') = d_{\lambda,1_i}^{\Phi}(m,n;l;\omega_{mnl}) \cdot d_{\lambda,1_i}^{\Phi}(m,n;l;\omega_{mnl}), \quad \forall \lambda,1_i.
\]

which follows immediately from the formulas (6) and (12).

Combining this lemma, the compatibilities (i) and (1), and faithfulness of the representation \( ^aF_R \) of \( \mathcal{U}_m(m,\omega_{mnl}') \), we get the proof of Theorem 3.8.

Proof of Theorem 3.8:

Let \( m,n,l \in \mathbb{N} \) and \( \omega', \omega, \omega_{mnl}, \omega_{mnl}' \) be as in Section 3.4. For any \( r \in \mathbb{N} \) and \( r \)-tuples

\[
p = (p_1, \ldots, p_r) \in [mn]^r, \quad u = (u_1, \ldots, u_r) \in (\mathbb{C}[[h_1, h_2]]^\times)^r,
\]

we define the other four \( r \)-tuples

\[
p' \in [mn]^r, \quad u' \in (\mathbb{C}[[h_1, h_2]]^\times)^r,
\]

and

\[
u'' \in (\mathbb{C}[[h_1, h_2]]^\times)^r.
\]

Assume that \( \{u_i''\}_{i=1}^r, \{u_i'\}_{i=1}^r, \{u_i\}_{i=1}^r \) are not in resonance, so that we can consider the representations \( F_R(u''), F_R(u'), F_R(u) \) of \( \mathcal{U}_m(m,\omega', \omega) \), \( \mathcal{U}_m(m,\omega, \omega') \), \( \mathcal{U}_m(m,\omega_{mnl}) \), respectively.

Lemma 4.4. We have \( J_{m,n,l;\omega', \omega}^{p,v} = J_{m,n,l;\omega, \omega'}^{p,u} \circ J_{m,n,l;\omega_{mnl}}^{u,v'} \).

Proof.

Analogously to the previous lemma, this result boils down to a verification of

\[
d_{\lambda,1_i}^{\Phi}(m,n;l;\omega', \omega) = d_{\lambda,1_i}^{\Phi}(m,n;l;\omega) \cdot d_{\lambda,1_i}^{\Phi}(m,n;l;\omega), \quad \forall \lambda,1_i.
\]

This identity follows immediately from the structure of the formula (6).

Theorem 3.8 follows from this lemma and faithfulness of the \( \mathcal{U}_m(m,\omega_{mnl}) \)-representation \( F_R \).

Remark 4.5. (a) One can check that \( \Phi_{m,n}^{\omega_{mnl}} \) defined on the generators via (40)–(43) preserves all the defining relations of \( \mathcal{U}_m(m,\omega_{mnl}) \), except for the Serre relations, by direct computations (compare with [GTL]). We also used this approach to determine (40)–(43). However, we are not aware of any direct verification of the Serre relations (the arguments of [GTL] heavily rely on the existence of subalgebras \( U_q(L\mathfrak{sl}_2) \subset U_q(L\mathfrak{g}) \) for which there are no Serre relations).

(b) One can similarly check that \( \Phi_{m,n}^{\omega'} \) defined on the generators via (40)–(43) preserves all the defining relations of \( \mathcal{U}_m(m,\omega') \), except for the Serre relations, by direct computations.

(c) One can also prove Theorems 3.6, 3.8 by direct computations (that was our original proof). However, we find the above arguments more elegant and uniform in the spirit.
4.3. Geometric interpretation.

The goal of this section is to provide the geometric realization for

- the Fock modules $F^p(u)$ and $^aF^p(v)$ from Section 1.4,
- the tensor products of Fock modules $FP(u)$ and $^aFP(v)$ from Section 1.5,
- the intertwining isomorphisms $J_{m,n}^{p,v}$ and $J_{m,n}^{p,u}$ from Section 4.1.

Given a quiver $Q$ and dimension vectors $v, w \in Z^\text{vert}(Q)$ (vert($Q$)—the set of vertices of $Q$), one can define the associated Nakajima quiver variety $\mathfrak{M}^Q(v, w)$. These varieties play a crucial role in the geometric representation theory of quantum and Yangian algebras. For the purposes of our paper, we will be interested only in the following set of quivers $Q$ (labeled by $n \in \mathbb{N}$):

- $Q_1$ is the Jordan quiver with one vertex (vert($Q_1$) = $\{1\}$) and one loop,
- $Q_n$ (with $n > 1$) is the cyclic quiver with vert($Q_n$) = $\{n\}$.

For any $Q_n$ as above and $v, w \in Z^n$, let us consider $[n]$-graded vectors spaces $V = \bigoplus_{i \in [n]} V_i$ and $W = \bigoplus_{i \in [n]} W_i$ such that dim($V_i$) = $v_i$ and dim($W_i$) = $w_i$. Define:

$$M(v, w) = \bigoplus_{i \in [n]} \text{Hom}(V_i, V_{i+1}) \oplus \bigoplus_{i \in [n]} \text{Hom}(V_i, V_{i-1}) \oplus \bigoplus_{i \in [n]} \text{Hom}(W_i, V_i) \oplus \bigoplus_{i \in [n]} \text{Hom}(V_i, W_i).$$

Elements of $M(v, w)$ can be written as $(B = \{B_i\}, \bar{B} = \{\bar{B}_i\}, a = \{a_i\}, b = \{b_i\})_{i \in [n]}$. Consider the moment map $\mu : M(v, w) \to \bigoplus_{i \in [n]} \text{End}(V_i)$ defined by

$$\mu(B, \bar{B}, a, b) = \sum_{i \in [n]} (B_{i-1} \bar{B}_i - B_{i+1} B_i + a_i b_i).$$

An element $(B, \bar{B}, a, b) \in \mu^{-1}(0)$ is called stable if there is no non-zero $(B, \bar{B})$-invariant subspace of $V$ contained in Ker$(b)$. Let $\mu^{-1}(0)^s$ denote the subset of all stable elements of $\mu^{-1}(0)$. An important property of $\mu^{-1}(0)^s$ is that the group $G_v = \prod_{i \in [n]} GL(V_i)$ acts freely on $\mu^{-1}(0)^s$.

The Nakajima quiver variety $\mathfrak{M}(v, w)$ (associated to $Q_n$ and $v, w \in Z^n$) is defined as

$$\mathfrak{M}(v, w) = \mu^{-1}(0)^s / G_v.$$

There is a natural action of the maximal torus $T_w := \mathbb{C}^* \times \mathbb{C}^* \times \prod_{i \in [n]} (\mathbb{C}^*)^{u_i}$ on $\mathfrak{M}(v, w)$ for any $v$. Moreover, it is known that the set of $T_w$-fixed points is parametrized by the tuples of Young diagrams $\lambda = (\lambda(i, j))_{1 \leq i, j \leq u_i}$ satisfying the following requirement. For any $i, j$ as above, let us color the boxes of $\lambda(i, j)$ into $n$ colors $[n]$, so that the box staying in the $a$th row and $b$th column has color $i + a - b$. Our requirement is that the total number of color $i$ boxes equals $v_i$.

For $w \in Z^n$, consider the direct sum of equivariant cohomology $H(w) = \bigoplus_{i \in \mathbb{N}} H^i_T(w)$. It is a module over $H_T^*(pt) = \mathbb{C}[t_w] = \mathbb{C}[s_1, s_2, \{x_{i,j}\}_{i \in [n]}]$, where $t_w = \text{Lie}(T_w)$. Define $H^i_T(w)_{\text{loc}} := H^i_T(w) \otimes H^k_T(pt) \text{Frac}(H^k_T(pt))$. Let $[\lambda]$ be the direct image of the fundamental cycle of the $T_w$-fixed point, corresponding to $\lambda$. The set $\{[\lambda]\}$ forms a basis of $H_T(w)_{\text{loc}}$.

Let us consider an analogous direct sum of equivariant K-groups $K(w) = \bigoplus_{i \in \mathbb{N}} K^i_T(w)$. It is a module over $K_T^*(pt) = \mathbb{C}[T_w] = \mathbb{C}[t_{1,1}^{\pm 1}, t_{2,1}^{\pm 1}, \{x_{i,j}\}_{i \in [n]}]$. Define the localized version $K^i_T(w)_{\text{loc}} := K^i_T(w) \otimes K_T^k(pt) \text{Frac}(K^k_T(pt))$. Let $[\lambda]$ be the direct image of the structure sheaf of the $T_w$-fixed point, corresponding to $\lambda$. The set $\{[\lambda]\}$ forms a basis of $K_T(w)_{\text{loc}}$.

The following result goes back to [Na, V] (for $n > 1$) and [SV1, SV2, T] (for $n = 1$):

**Theorem 4.6.** (a) For any $w \in Z^n$, there is a natural action of $U^{(n)}_{s_1, -s_1, -s_2} H(w)_{\text{loc}}$.

(b) For any $w \in Z^n$, there is a natural action of $U^{(n)}_{t_1, -t_1, t_2, -t_2} K(w)_{\text{loc}}$.

In what follows, we set $h_1 = s_1$, $h_2 = -s_1 - s_2$, $h_3 = s_2$ and $q_1 = t_1$, $q_2 = t_1^{-1} t_2^{-1}$, $q_3 = t_2$. 
Proposition 4.7. For $p \in [n]$, define $w^{(p)} = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}[n]$ with 1 at the $p$th place.
(a) There is an isomorphism of $U_{q_1,q_2,q_3}^{(n)}$-representations $\alpha : F^p(x_{p,1}) \mapsto K(w^{(p)})_{\text{loc}}$.
(b) There is an isomorphism of $Y_{h_1,h_2,h_3}^{(n)}$-representations $\alpha : a F^p(x_{p,1}) \mapsto H(w^{(p)})_{\text{loc}}$.
(c) Both isomorphisms are given by diagonal matrices in the bases $\{ \lambda \}$ and $\{ \lambda \}$.

Proof.
The $n = 1$ case of this result was treated in [11 Section 4], while the general case can be easily deduced from the former by the standard procedure of “taking a $\mathbb{Z}/n\mathbb{Z}$-invariant part”. □

The higher-rank generalization of this result is straightforward.

Proposition 4.8. For any $n \in \mathbb{N}$ and $w = (w_0, \ldots, w_{n-1}) \in \mathbb{Z}^n$, the following holds:
(a) There is an isomorphism of $U_{q_1,q_2,q_3}^{(n)}$-representations $\alpha : \bigotimes_{i=0}^{n-1} \bigotimes_{j=1}^{i} F^i(x_{i,j}) \mapsto K(w)_{\text{loc}}$.
(b) There is an isomorphism of $Y_{h_1,h_2,h_3}^{(n)}$-representations $\alpha : \bigotimes_{i=0}^{n-1} \bigotimes_{j=1}^{i} a F^i(x_{i,j}) \mapsto H(w)_{\text{loc}}$.
(c) Both isomorphisms are given by diagonal matrices in the bases $\{ \lambda \}$ and $\{ \lambda \}$.
(d) Parts (a, b) still hold for an arbitrary reordering of tensor products from the left-hand sides.

There exists a well-known relation between the Nakajima quiver varieties associated to the quivers $Q_m$ and $Q_{mn}$. Let $w = \sum_{i=1}^r w^{(p_i)}$, $w' = \sum_{i=1}^r w'^{(p'_i)}$ with $p_i \in [m], p'_i \in [m]$, where $p'_i \equiv p_i \pmod{m}$. Then there is an action of the group $\mathbb{Z}/mn\mathbb{Z}$ on $\bigcup_{v} \mathfrak{M}_{Q_{mn}}(v, w')$, such that the variety of fixed points is isomorphic to $\bigcup_{v} \mathfrak{M}_{Q_{mn}}(v, w)$.

We therefore have an inclusion $\bigcup_{v} \mathfrak{M}_{Q_{mn}}(v, w) \hookrightarrow \bigcup_{v'} \mathfrak{M}_{Q_{mn}}(v', w')$. Let $\mathcal{J}_{m,n} : K(w')_{\text{loc}} \to K(w)_{\text{loc}}$ be the corresponding pull-back in localized equivariant $K$-theory. This map is diagonal in the bases of fixed points, hence, it is an isomorphism. In a similar way, define $\mathcal{J}_{m,n} : K(w')_{\text{loc}} \to H(w)_{\text{loc}}$ as a composition of an equivariant Chern character map and a pull-back in localized equivariant cohomology.

Our main result of this subsection reveals a geometric realization of $I_{r,m,n,w_{/\omega}}^{p,v}$ and $J_{r,m,n,w_{/\omega}}^{p,u}$.

Theorem 4.9. The following two diagrams are commutative:

\[\begin{array}{ccc}
F^p(u') & \xrightarrow{J_{r,m,n,w_{/\omega}}^{p,v}} & a F^p(v) \\
\alpha \downarrow & & \alpha \\
K(w')_{\text{loc}} & \xrightarrow{\mathcal{J}_{m,n}} & H(w)_{\text{loc}}
\end{array}\]

\[\begin{array}{ccc}
F^p(u') & \xrightarrow{J_{r,m,n,w_{/\omega}}^{p,u}} & F^p(u) \\
\alpha \downarrow & & \alpha \\
K(w')_{\text{loc}} & \xrightarrow{\mathcal{J}_{m,n}} & K(w)_{\text{loc}}
\end{array}\]

where parameters $u, v, p$ are specified in Proposition 4.8.

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