Local Search for Fast Matrix Multiplication

Marijn Heule, Manuel Kauers, and Martina Seidl

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Matrix Multiplication: Introduction

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} \\
a_{2,1} & a_{2,2}
\end{pmatrix}
\begin{pmatrix}
b_{1,1} & b_{1,2} \\
b_{2,1} & b_{2,2}
\end{pmatrix}
= 
\begin{pmatrix}
c_{1,1} & c_{1,2} \\
c_{2,1} & c_{2,2}
\end{pmatrix}
\]

\[
c_{1,1} = a_{1,1} \cdot b_{1,1} + a_{1,2} \cdot b_{2,1}
\]
\[
c_{1,2} = a_{1,1} \cdot b_{1,2} + a_{1,2} \cdot b_{2,2}
\]
\[
c_{2,1} = a_{2,1} \cdot b_{1,1} + a_{2,2} \cdot b_{2,1}
\]
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\end{pmatrix}
\]

\[
c_{1,1} = M_1 + M_4 - M_5 + M_7
\]
\[
c_{1,2} = M_3 + M_5
\]
\[
c_{2,1} = M_2 + M_4
\]
\[
c_{2,2} = M_1 - M_2 + M_3 + M_6
\]
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\]

... where

\[
M_1 = (a_{1,1} + a_{2,2}) \cdot (b_{1,1} + b_{2,2})
\]
\[
M_2 = (a_{2,1} + a_{2,2}) \cdot b_{1,1}
\]
\[
M_3 = a_{1,1} \cdot (b_{1,2} - b_{2,2})
\]
\[
M_4 = a_{2,2} \cdot (b_{2,1} - b_{1,1})
\]
\[
M_5 = (a_{1,1} + a_{1,2}) \cdot b_{2,2}
\]
\[
M_6 = (a_{2,1} - a_{1,1}) \cdot (b_{1,1} + b_{1,2})
\]
\[
M_7 = (a_{1,2} - a_{2,2}) \cdot (b_{2,1} + b_{2,2})
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- This scheme needs 7 multiplications instead of 8.
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- This scheme needs 7 multiplications instead of 8.
- Recursive application allows to multiply $n \times n$ matrices with $O(n^{\log_2 7})$ operations in the ground ring.
(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}) (\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}) = (\begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix})

- This scheme needs 7 multiplications instead of 8.
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- Let $\omega$ be the smallest number so that $n \times n$ matrices can be multiplied using $O(n^\omega)$ operations in the ground domain.
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- Let $\omega$ be the smallest number so that $n \times n$ matrices can be multiplied using $O(n^\omega)$ operations in the ground domain.
- Then $2 \leq \omega < 3$. What is the exact value?
Efficient Matrix Multiplication: Theory

- Strassen 1969: \( \omega \leq \log_2 7 \leq 2.807 \)
Efficient Matrix Multiplication: Theory

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- Pan 1978: \( \omega \leq 2.796 \)
- Bini et al. 1979: \( \omega \leq 2.7799 \)
- Schönhage 1981: \( \omega \leq 2.522 \)
- Romani 1982: \( \omega \leq 2.517 \)
- Coppersmith/Winograd 1981: \( \omega \leq 2.496 \)
- Strassen 1986: \( \omega \leq 2.479 \)
- Coppersmith/Winograd 1990: \( \omega \leq 2.376 \)
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- Coppersmith/Winograd 1990: \( \omega \leq 2.376 \)
- Stothers 2010: \( \omega \leq 2.374 \)
- Williams 2011: \( \omega \leq 2.3728642 \)
- Le Gall 2014: \( \omega \leq 2.3728639 \)
Efficient Matrix Multiplication: Practice

- Only Strassen’s algorithm beats the classical algorithm for reasonable problem sizes.
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- Want: a matrix multiplication algorithm that beats Strassen’s algorithm for matrices of moderate size.

- Question: What’s the minimal number of multiplications needed to multiply two $3 \times 3$ matrices?

- Answer: Nobody knows.
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The 3x3 Case is Still Open

Question: What’s the minimal number of multiplications needed to multiply two 3 × 3 matrices?
The 3×3 Case is Still Open

Question: What’s the minimal number of multiplications needed to multiply two $3 \times 3$ matrices?

- naive algorithm: 27

- padd with zeros, use Strassen twice, cleanup: 25

- best known upper bound: 23 (Laderman 1976)

- best known lower bound: 19 (Bläser 2003)

- maximal number of multiplications allowed if we want to beat Strassen: 21 (because $\log_3 21 < \log_2 7 < \log_3 22$).
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Laderman’s scheme from 1976

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix}
\begin{pmatrix}
b_{1,1} & b_{1,2} & b_{1,3} \\
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\end{pmatrix}
= 
\begin{pmatrix}
c_{1,1} & c_{1,2} & c_{1,3} \\
c_{2,1} & c_{2,2} & c_{2,3} \\
c_{3,1} & c_{3,2} & c_{3,3}
\end{pmatrix}
\]

\[c_{1,1} = -M_6 + M_{14} + M_{19}\]
\[c_{2,1} = M_2 + M_3 + M_4 + M_6 + M_{14} + M_{16} + M_{17}\]
\[c_{3,1} = M_6 + M_7 - M_8 + M_{11} + M_{12} + M_{13} - M_{14}\]
\[c_{1,2} = M_1 - M_4 + M_5 - M_6 - M_{12} + M_{14} + M_{15}\]
\[c_{2,2} = M_2 + M_4 - M_5 + M_6 + M_{20}\]
\[c_{3,2} = M_{12} + M_{13} - M_{14} - M_{15} + M_{22}\]
\[c_{1,3} = -M_6 - M_7 + M_9 + M_{10} + M_{14} + M_{16} + M_{18}\]
\[c_{2,3} = M_{14} + M_{16} + M_{17} + M_{18} + M_{21}\]
\[c_{3,3} = M_6 + M_7 - M_8 - M_9 + M_{23}\]
Laderman’s scheme from 1976

\[
\begin{pmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} \\
  a_{2,1} & a_{2,2} & a_{2,3} \\
  a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix}
\begin{pmatrix}
  b_{1,1} & b_{1,2} & b_{1,3} \\
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\end{pmatrix}
=\begin{pmatrix}
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  c_{3,1} & c_{3,2} & c_{3,3}
\end{pmatrix}
\]

where . . .

\[
M_1 = (-a_{1,1} + a_{1,2} + a_{1,3} - a_{2,1} + a_{2,2} + a_{3,2} + a_{3,3}) \cdot b_{2,2}
\]

\[
M_2 = (a_{1,1} + a_{2,1}) \cdot (b_{1,2} + b_{2,2})
\]

\[
M_3 = a_{2,2} \cdot (b_{1,1} - b_{1,2} + b_{2,1} - b_{2,2} - b_{3,2} + b_{3,1} - b_{3,3})
\]

\[
M_4 = (-a_{1,1} - a_{2,1} + a_{2,2}) \cdot (-b_{1,1} + b_{1,2} + b_{2,2})
\]

\[
M_5 = (-a_{2,1} + a_{2,2}) \cdot (-b_{1,1} + b_{1,2})
\]

\[
M_6 = -a_{1,1} \cdot b_{1,1}
\]

\[
M_7 = (a_{1,1} + a_{3,1} + a_{3,2}) \cdot (b_{1,1} - b_{1,3} + b_{2,3})
\]

\[
M_8 = (a_{1,1} + a_{3,1}) \cdot (-b_{1,3} + b_{2,3})
\]

\[
M_9 = (a_{3,1} + a_{3,2}) \cdot (b_{1,1} - b_{1,3})
\]
Laderman's scheme from 1976

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
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\end{pmatrix}
\begin{pmatrix}
b_{1,1} & b_{1,2} & b_{1,3} \\
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= \begin{pmatrix}
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\end{pmatrix}
\]

where . . .

\[M_{10} = (a_{1,1} + a_{1,2} - a_{1,3} - a_{2,2} + a_{2,3} + a_{3,1} + a_{3,2}) \cdot b_{2,3}\]
\[M_{11} = (a_{3,2}) \cdot (-b_{1,1} + b_{1,3} + b_{2,1} - b_{2,2} - b_{2,3} - b_{3,1} + b_{3,2})\]
\[M_{12} = (a_{1,3} + a_{3,2} + a_{3,3}) \cdot (b_{2,2} + b_{3,1} - b_{3,2})\]
\[M_{13} = (a_{1,3} + a_{3,3}) \cdot (-b_{2,2} + b_{3,2})\]
\[M_{14} = a_{1,3} \cdot b_{3,1}\]
\[M_{15} = (-a_{3,2} - a_{3,3}) \cdot (-b_{3,1} + b_{3,2})\]
\[M_{16} = (a_{1,3} + a_{2,2} - a_{2,3}) \cdot (b_{2,3} - b_{3,1} + b_{3,3})\]
\[M_{17} = (-a_{1,3} + a_{2,3}) \cdot (b_{2,3} + b_{3,3})\]
\[M_{18} = (a_{2,2} - a_{2,3}) \cdot (b_{3,1} - b_{3,3})\]
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$$
\begin{pmatrix}
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  a_{2,1} & a_{2,2} & a_{2,3} \\
  a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix}
\begin{pmatrix}
  b_{1,1} & b_{1,2} & b_{1,3} \\
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\end{pmatrix}
$$

where . . .

\[ M_{19} = a_{1,2} \cdot b_{2,1} \]
\[ M_{20} = a_{2,3} \cdot b_{3,2} \]
\[ M_{21} = a_{2,1} \cdot b_{1,3} \]
\[ M_{22} = a_{3,1} \cdot b_{1,2} \]
\[ M_{23} = a_{3,3} \cdot b_{3,3} \]
Other schemes with 23 multiplications

- While Strassen’s scheme is essentially the only way to do the $2 \times 2$ case with 7 multiplications, there are several distinct schemes for $3 \times 3$ matrices using 23 multiplications.
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If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for $3 \times 3$ matrices and 23 multiplications.
Other schemes with 23 multiplications

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- Using altogether about 35 years of computation time, we found more than 13000 new schemes for $3 \times 3$ and 23, and we expect that there are many others.
Other schemes with 23 multiplications

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- If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for $3 \times 3$ matrices and 23 multiplications.
- Using altogether about 35 years of computation time, we found more than 13000 new schemes for $3 \times 3$ and 23, and we expect that there are many others.
- Unfortunately we found no scheme with only 22 multiplications.
How to Search for a Matrix Multiplication Scheme? (1)

\[ M_1 = (\alpha_{1,1}^{(1)} a_{1,1} + \alpha_{1,2}^{(1)} a_{1,2} + \cdots)(\beta_{1,1}^{(1)} b_{1,1} + \cdots) \]
\[ M_2 = (\alpha_{1,1}^{(2)} a_{1,1} + \alpha_{1,2}^{(2)} a_{1,2} + \cdots)(\beta_{1,1}^{(2)} b_{1,1} + \cdots) \]
\[ \vdots \]
\[ c_{1,1} = \gamma_{1,1}^{(1)} M_1 + \gamma_{1,1}^{(2)} M_2 + \cdots \]
\[ \vdots \]

Set \( c_{i,j} = \sum_k a_{i,k} b_{k,j} \) for all \( i, j \) and compare coefficients.
How to Search for a Matrix Multiplication Scheme? (2)

This gives the **Brent equations** (for $3 \times 3$ with 23 multiplications)

\[
\forall i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}
\]

The $\delta_{u,v}$ on the right refer to the Kronecker-delta, i.e.,

$\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ otherwise.

$3^6 = 729$ cubic equations

$23 \cdot 9 \cdot 3 = 621$ variables
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$$3^6 = 729 \text{ cubic equations}$$

$$23 \cdot 9 \cdot 3 = 621 \text{ variables}$$

Laderman claims that he solved this system by hand, but he doesn’t say exactly how.
How to Search for a Matrix Multiplication Scheme? (3)

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The search space of the $3 \times 3$ case is enormous, even if $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ are restricted to the values in $\{-1, 0, 1\}$
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Solution: Solve this system in $\mathbb{Z}_2$.

Reading $\alpha_{i,j}^{(q)}, \beta_{k,l}^{(q)}, \gamma_{m,n}^{(q)}$ as boolean variables and $+$ as XOR, the problem becomes a SAT problem.

Notice that solutions in $\mathbb{Z}_2$ may not be solutions in $\mathbb{Z}$.
Lifting

Remember the Brent equations:

\[ \forall i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n} \]

- Suppose we know a solution in \( \mathbb{Z}_2 \).
- Assume it came from a solution in \( \mathbb{Z} \) with coefficients in \( \{-1, 0, +1\} \).
- Then each \( 0 \in \mathbb{Z}_2 \) was \( 0 \in \mathbb{Z} \) and each \( 1 \in \mathbb{Z}_2 \) was \( -1 \in \mathbb{Z} \) or \( +1 \in \mathbb{Z} \).
- Plug the 0s of the \( \mathbb{Z}_2 \)-solution into the Brent equations.
- Solve the resulting equations.

Can every \( \mathbb{Z}_2 \)-solution be lifted to a \( \mathbb{Z} \)-solution in this way?

- No, and we found some which don't admit a lifting.
- But they are very rare. In almost all cases, the lifting succeeds.
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- Plug the 0s of the $\mathbb{Z}_2$-solution into the Brent equations.
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How to Search for a Matrix Multiplication Scheme? (4)

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Another solution: Solve this system by restricting equations with a zero righthand side to zero or two.

Still treat $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables.

Notice that this restriction removes solutions, but it even works for Laderman.
How to Search for a Matrix Multiplication Scheme? (4)

This gives the **Brent equations** (for $3 \times 3$ with 23 multiplications)

$$\forall i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

Another solution: Solve this system by restricting equations with a zero righthand side to zero or two.

Still treat $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables.

Notice that this restriction removes solutions, but it even works for Laderman.

Important challenge: how to break the symmetries?

Most effective approach so far: sort the $\delta_{j,k} \delta_{i,m} \delta_{l,n} = 1$ terms
Neighborhood Search
Neighborhood Search Results
So what?

- Okay, so there are many more matrix multiplication methods for $3 \times 3$ matrices with 23 coefficient multiplications than previously known.

- In fact, we have shown that the dimension of the algebraic set defined by the Brent equation is much larger than was previously known.

- But none of this has any immediate implications on the complexity of matrix multiplication, neither theoretically nor practically.

- In particular, it remains open whether there is a multiplication method for $3 \times 3$ matrices with 22 coefficient multiplications. If you find one, let us know.
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Scheme Database

Check out our website for browsing through the schemes and families we found:

http://www.algebra.uni-linz.ac.at/research/matrix-multiplication/
Local Search for Fast Matrix Multiplication

Marijn Heule, Manuel Kauers, and Martina Seidl

Starting at Carnegie Mellon University in August

SAT 2019 Conference, Lisbon July 9, 2019