On the Explicit Asymptotic $W_5$ Symmetry of 3D Chern-Simons Higher Spin $AdS_3$ Gravity

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ABSTRACT: In this paper, we explicitly construct a higher spin asymptotic $W_5$ symmetry algebra of the three-dimensional anti-de Sitter($AdS_3$) gravity. We use a $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Lie algebra valued Chern-Simons gauge theory coupled to three-dimensional Einstein gravity with a negative cosmological constant and its asymptotic symmetry algebra is explicitly calculated as two copies of the classical $W_5$ algebra with central charge $c$. Our results can be interpreted as a spin 5 extension of the $BTZ$ solution and a proof of how the higher spin Ward identities and as well as the asymptotic $W_5$ symmetry algebra are derived from the higher spin bulk field equations of motion. This higher spin asymptotic $W_5$ symmetry algebra contains a finite number of conformal primary spins: $s = 2, 3, 4, 5$. 

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1 Introduction

Holographic dualities, which have found many remarkable applications that range from mathematics to modern theoretical physics during the last twenty years, claim an equivalence between gravitational theories defined in the bulk of some region of space-time and field theories on the boundary of that region. By now it is studied in many diverse subfields, and the literature on the subject has become enormous. The most famous realization of the holographic duality is the \textit{AdS/CFT} correspondence that was first conjectured by Juan Maldacena in 1997 \cite{Maldacena:1997re}. To put it simply, the correspondence says that a theory of quantum gravity with a negative cosmological constant defined in an \textit{AdS}$_5$ is equivalent to a certain \textit{CFT}$_4$ living on its boundary.

In the case of three dimensions, the space \textit{AdS}$_3$ has the two-dimensional conformal group \textit{SO}(2, 2) as isometry group. This group acts on the two-dimensional boundary of \textit{AdS}$_3$ as the symmetry of the two-dimensional conformal field theory \textit{CFT}$_2$. It is well-known that \textit{CFT}$_2$ has a fundamental role in modern physics, therefore it has always been of interest to see to what degree a Virasoro algebra, which is its underlying symmetry algebra, may be extended. In this context, \textit{W} algebras are extensions of the Virasoro algebra by currents of higher spin \cite{Fateev:1989db,Feigin:1992au,Feigin:1992us,Feigin:1993bs,Feigin:1993uy}. These algebras are available for arbitrary values of central charge, \( c \).
Another intriguing $W$ algebra which is very closely linked to $W_N$ algebra is $W_\infty$ family. $W$ algebra can then be commented in the context of a $W_\infty$ algebra [7–10]. Such an algebra is based on a higher spin operators with spin $s \geq 1$ in an infinite numbers. It should be emphasized that the $W_\infty$ has been studied for special values of parameters where the algebra is linearly realized. The first marks of connection between $W_\infty$ and $W_N$ is given in the pioneering Ref. [10]. Based on the contemporary point of view, one can say that $W_\infty$ is an algebra with two parameters [11–13]. The connection between $W_N$ algebra with one parameter and $W_\infty$ algebra with two parameters is similar to the construction of the relevant higher spin algebra in Vasiliev theory [14]. In an exactly related context of the extended $W$ algebras, one can say that the $AdS_3$ higher spin gravity [15–17] is an interesting extension of the pure Einstein gravity. In the case of the pure Einstein gravity on $AdS_3$, Einstein gravity theory is a lot easier to handle in three dimensions than in higher dimensions, because it allows a reformulation in terms of a Chern-Simons gauge theory [18, 19]. It has also been shown by Brown and Henneaux in their seminal work [20] that the asymptotic symmetry algebra of $AdS_3$ boundary is given by two copies of a classical Virasoro algebra. This result can be interpreted as a pioneer application of the $AdS/CFT$ correspondence. Then application was generalized by Henneaux and Rey [21] and Campoleoni at al. [22, 23] to higher spin extensions on $AdS_3$.

Another route is $W_N$ minimal models [24]. The quantum asymptotic symmetry algebra of the $AdS_3$ higher spin gravity can be seen in the context of the minimal model holographic dualities. In particular, the $W_N$ minimal models, which describe a family of two-dimensional $CFT_2$ with finite values of parameter $N$ and the central charge $c$ are dual to higher spin theories in $AdS_3$ [25]. More importantly, the quantization problems are partially solved by Gaberdiel and Gopakumar in [26], by the direct construction of the most general quantum $W_N$ algebra depending on asymptotic symmetry algebra with one parameter instead of asymptotic symmetry algebra with two parameters.

The impressive progress in the classification of the asymptotic symmetry algebras in recent years indicates that it is necessary to describe the $AdS_3$ higher spin gravity theories for sufficiently large conformal spin values. Therefore, the higher spin extensions of the $AdS_3$ higher spin gravity theories is still an open problem because it is not yet known at least whether to be consistent with both small and large central charges. The reason for this is fundamentally the appearance of the composite or nonlinear terms in the related asymptotic symmetry algebras. We emphasize here that the appearance of these composite terms in the related asymptotic symmetry algebras have some contradictions from the proper quantum field theory point of view. To this end, one straightforward way is to replace the usual semi-classical asymptotic symmetry algebra with a quantum version one. Finally, a deformed version of usual semi-classical asymptotic symmetry algebra with new arbitrary structure constants can be obtained. This will be the final form of the quantum asymptotic symmetry algebra of the $AdS_3$ higher spin gravity [2, 27, 28, 31].

The higher spin gravity theories have also come to play recently for its remarkable role in the context of $AdS/CFT$ correspondence. In three dimensions, within the setting of $AdS_3/CFT_2$ duality, the difficulty of the higher spin gravity theories can be simplified as it is possible to truncate [21] from an infinite number of spin fields to a set of the finite
spin fields within the scope of $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ Chern-Simons theory. Specifically, in the $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern-Simons theory, the higher spin $AdS_3$ gravity has asymptotic classical $W_5$ symmetry algebra, which is the focus of the present paper. We emphasize that the main idea of this type of works is that of Hamiltonian reduction through AdS-type boundary conditions, therefore $W_N$ symmetry algebras should follow straightforwardly for sufficiently large values of $N$, within the scope of $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ Chern-Simons theory. Therefore, this work is concerned with two-dimensional $CFT_2$ that appear in the holographic duality with higher spin theory on three dimensional $AdS_3$ space. The dual $CFT_2$ can be formulated as asymptotic $W_5$ symmetry algebra which is higher-spin generalisation of the Virasoro algebra describing conformal symmetry.

This paper has the following outline. In Section 2, we give a fundamental formulation of higher spin gravity within $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern-Simons theory in three-dimensions. Section 3 is particularly devoted to the case of spin 5 where we showed in this section the principal embedding of $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{sl}(5, \mathbb{R})$, and also demonstrated how $W_5$ symmetry and higher spin Ward identities arise from the bulk equations of motion coupled to spin $s$, $(s = 3, 4, 5)$ currents. Finally, classical $W_5$ symmetry algebra as asymptotic spin 5 symmetry algebra is obtained and the results are checked with the quantum $W_5$ algebra. We conclude with a summary for our results and a few suggestions for the future work. Appendix $A$ and $B$ collect our conventions for the $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra generators, and also the classical $W_5$ symmetry algebra is shown in Appendix $C$.

2 Gravity in Three Dimensions, a review:

Three dimensions is a good candidate because of its topological nature, which is a consequence of the lack of degrees of freedom. Chern-Simons theory is a quantum theory in three dimensions that computes only topological invariants. It can be defined on any manifold, and the metric does not need to be specified as it is a topological theory. Thus the physical quantities do not depend on the local geometry. Chern-Simons gravity is also a gauge theory and used as an interesting playground for investigating the $AdS_3/CFT_2$ correspondence by Brown and Henneaux in their seminal work [20] in 1986. They showed that any quantum gravity theory with asymptotically $AdS_3$ boundary conditions in three-dimensions must be dual to $CFT_2$ at the meaning that asymptotic symmetries of $AdS_3$ are given by two duplicate copies of the Virasoro algebra and related the central charge $c$ of the $CFT_2$ to the corresponding $AdS_3$ radius $\ell$ in the following way

$$c = \frac{3\ell}{2G},$$

(2.1)

This Brown Henneaux – formula reveals that the appropriate limit corresponds to small Newton’s constant $G$ (or large $\ell$) and so large $c$, known as the semi-classical limit.

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1 in this convention, the $\mathfrak{sl}(N, \mathbb{R})$ Lie algebra belongs to $SL(N, \mathbb{R})$ group
2.1 Connection to Chern-Simons Theory

It is a striking fact that the vacuum Einstein $AdS_3$ gravity in three dimensions with a negative cosmological constant can be formulated as a Chern-Simons gauge theory, as it was first proposed by Achucarro and Townsend in [18] and developed by Witten in [19]. One can start by defining 1-forms $(A, \bar{A})$ taking values in the gauge group’s $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra, and the trace is taken over the group generators. The Chern-Simons action can be written in the form,

$$ S = S_{CS}[A] - S_{CS}[\bar{A}] $$

(2.2)

where

$$ S_{CS}[A] = \frac{k}{4\pi} \int \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). $$

(2.3)

Here $k = \frac{\ell^3}{4G} = \frac{c}{6}$ is the level of the Chern-Simons theory depending on the $AdS$ radius $\ell$ and the Newton’s constant $G$ with the related central charge $c$ of the $CFT_2$. Trace shows a metric on the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra. The mark of the cosmological constant determines a convenient gauge group, therefore, for a negative cosmological constant $A$ and $\bar{A}$ are $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra valued one-form, and they depend on the vielbein $e^\mu_a$ and dual spin connection $\omega^a_\mu = \epsilon^{abc} \omega_{bc\mu}$ as follows

$$ A = \left( \omega^a_\mu + \frac{e^a_\mu}{\ell} \right) L_a dx^\mu, \quad \bar{A} = \left( \omega^a_\mu - \frac{e^a_\mu}{\ell} \right) L_a dx^\mu. $$

(2.4)

The equations of motion for the Chern-Simons gauge theory gives the flatness condition $F = \bar{F} = 0$ where

$$ F = dA + A \wedge A = 0, $$

(2.5)

is the same as the Einstein’s equation. If $L_a, (a = \pm 1, 0)$ are the generators of $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra,

$$ [L_a, L_b] = (a - b) L_{a+b} $$

(2.6)

by defining the invariant bilinear form

$$ \text{tr}(L_a L_b) = \frac{1}{2} \eta_{ab}. $$

(2.7)

$A$ and $\bar{A}$ are related to the metric $g_{\mu\nu}$ through the vielbein $e = \frac{\ell}{2} (A - \bar{A})$

$$ g_{\mu\nu} = \frac{1}{2} \text{tr}(e_\mu e_\nu) $$

(2.8)
2.2 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ Chern-Simons Theory

Spin 2 case is given by reviewing asymptotically $AdS_3$ boundary conditions for a $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ Chern-Simons theory, and how to determine the asymptotic symmetry algebra using the methods described in [29]. By using the Fefferman-Graham expansion method, the most general solution of the Einstein’s equation that is asymptotically $AdS$, is given by with a flat boundary metric [30]

$$ds^2 = l^2 \left\{ d\rho^2 - \frac{8\pi G}{l^2} \left( \mathcal{L} (dx^+)^2 + \tilde{\mathcal{L}} (dx^-)^2 \right) - \left( e^{2\rho} + \frac{64\pi^2 G^2}{l^2} \mathcal{L} \tilde{\mathcal{L}} e^{-2\rho} \right) dx^+ dx^- \right\}$$  \hspace{1cm} (2.9)

where $(\rho, x^\pm \equiv \frac{l}{l} \pm \phi)$ shows the solid cylinder as the light-like coordinates and $\mathcal{L} \equiv \mathcal{L} (x^+), \tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}} (x^-)$ are arbitrary functions of $x^\pm$. Therefore, one can write the light-like components of the gauge fields by using the principle embedding of $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra generators

$$A = b^{-1}a(x^+)b + b^{-1}db, \quad \bar{A} = b\bar{a}(x^-)b^{-1} + bdb^{-1},$$  \hspace{1cm} (2.10)

with $b = e^{\rho L_0}$, we obtain

$$a = \left( L_1 + \frac{2\pi}{k} L_{-1} \right) dx^+, \quad \bar{a} = -\left( L_{-1} + \frac{2\pi}{k} \tilde{L}_1 \right) dx^-.$$  \hspace{1cm} (2.11)

A very important point in this description is that this theory has to be asymptotically $AdS_3$, as required in [22] the boundary conditions have to be defined in a similar conditions

$$(A - A_{AdS_3}) \bigg|_{\text{boundary}} = O(1).$$  \hspace{1cm} (2.12)

We will carry out such an analysis for a connection of the principal embedding including a higher spin charge, finding that the asymptotic symmetries can be identified with a classical $W_2$-algebra. After that, we will only work on the positive chiral components having $x^+$, although the same can be done in the other one as well. If we expand $\lambda(x^+)$ in the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra, as we did also the connection, then

$$\lambda = \sum_{i=-1}^{1} \epsilon_i L_i.$$  \hspace{1cm} (2.13)

We are now interested in the transformation that preserve the structure of (2.10). Under an infinitesimal gauge transformation with gauge parameter $\lambda$, $a$ which is equivalent to $A$ transforms as the flatness condition:

$$\delta a = d\lambda + [a, \lambda] = 0.$$  \hspace{1cm} (2.14)

Thus we have to impose that all terms proportional to $L_0, L_1$ vanish. These constraints can be solved to find $\epsilon_0$ and $\epsilon_{-1}$ in terms of $\epsilon_1$ and their derivatives. Writing $\epsilon_1 \equiv \epsilon$ which is
called the gauge parameter related to $\mathfrak{sl}(2, \mathbb{R})$ and superscripted primes denote $\partial_{x^+}$, one finds

$$
\epsilon_0 = -\epsilon', \\
\epsilon_{-1} = \frac{1}{2} \epsilon'' - \frac{2\pi}{k} \epsilon \mathcal{L}.
$$

(2.15)

Now, one can also determine how the function $\mathcal{L}$ transforms under this gauge transformation. This is given by

$$
\mathcal{L} \rightarrow \mathcal{L} + \delta \epsilon \mathcal{L},
$$

(2.16)

where

$$
\delta \epsilon \mathcal{L} = 2 \mathcal{L} \epsilon' + \mathcal{L}' \epsilon + \frac{k}{4\pi} \epsilon'''.
$$

(2.17)

As a final step, one now has to determine the canonical boundary charge $Q[e]$ that generates this transformation. Therefore, the corresponding variation of the boundary charge $Q[e]$ can be integrated which reads

$$
Q[e] = \int dx^+ \epsilon(x^+) \mathcal{L}(x^+).
$$

(2.18)

This leads to Dirac bracket algebra by using $\delta F = \{F, Q[\epsilon]\}$

$$
\{ \mathcal{L}(x^+), \mathcal{L}(y^+) \} = 2 \mathcal{L}(y^+) \delta'(x^+ - y^+) + \mathcal{L}'(y^+) \delta(x^+ - y^+) + \frac{k}{4\pi} \delta'''(x^+ - y^+).
$$

(2.19)

One can also expand $\mathcal{L}(x^+)$ into Fourier modes $\mathcal{L}(x^+) = \sum_n L_n e^{-inx^+}$, and replacing $i\{\cdot, \cdot\} \rightarrow [\cdot, \cdot]$. A Virasoro algebra can then be defined as:

$$
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}.
$$

(2.20)

Performing a Wick rotation, in Sec.(2.2) $t \rightarrow -it$, $x^\pm \equiv \frac{t}{\ell} \pm \phi \rightarrow -i(\frac{t}{\ell} \pm i\phi)$, the coordinates $z \equiv \frac{t}{\ell} - i\phi$ and $\bar{z} \equiv \frac{t}{\ell} + i\phi$ describe the Wick rotated complex cylinder. Therefore, the Virasoro algebra in these space is equivalent to operator product algebra

$$
\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{\mathcal{L}}{z_{12}^3} + \frac{2 \mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}}
$$

(2.21)

where the central charge $c$ is again (2.1) related to the level of the Chern-Simons theory as

$$
c = 6k = \frac{3\ell}{2G}.
$$

(2.22)

The same algebra is also realized as the asymptotic symmetry algebra of the anti-chiral connection $\tilde{\mathcal{A}}$ having $x^-$ in terms of the second copy of $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra.

### 3 Higher spin Chern-Simons theory

The Chern-Simons formalism in three dimensions lends itself to generalization to higher spin theories coupled to gravity by raising the gauge group to $\mathfrak{sl}(N, \mathbb{R})$. An $\mathfrak{sl}(N, \mathbb{R})$ Chern-Simons theory describes $AdS_3$ gravity theories coupled to a finite tower of integer spin-s
The principal embedding of \( \text{can be found in Appendix B}. \)

find 11 expressions for the structure constants \( \text{can be achieved by the arranging of the left and right hand sides of the commutators with} \)

Our next goal is to find concrete expressions for the structure constants \( \text{can be represented in terms of the commutation relations for} \)

3.1 A different approach to the calculation of \( \text{sl}(5, \mathbb{R}) \) commutators

The principal embedding of \( \text{sl}(2, \mathbb{R}) \) Lie algebra to \( \text{sl}(5, \mathbb{R}) \) Lie algebra contains \( \text{sl}(2, \mathbb{R}) \) the spin 2 triplet \( L_n, (n = \pm 1, 0) \), \( \text{sl}(3, \mathbb{R}) \) the spin 3 triplet \( W_n^{(3)}, (n = \pm 2, \pm 1, 0) \), \( \text{sl}(4, \mathbb{R}) \) the spin 4 quartet \( W_n^{(4)}, (n = \pm 3, \pm 2, \pm 1, 0) \) and \( \text{sl}(5, \mathbb{R}) \) the spin 5 quintet \( W_n^{(5)}, (n = \pm 4, \pm 3, \pm 2, \pm 1, 0) \) can be represented in terms of the commutation relations for \( L_n \equiv W_n^{(2)}. \)

Let us now explain how \( \text{sl}(5, \mathbb{R}) \) Lie algebra can be constructed. We proceed in two steps. First, we make the most general ansatz for the commutators between all the Lie algebra generators \( W_n^{(s_1)}, W_m^{(s_2)} \) as follows

\[
[L_n, W_m^{(s)}] = ((s - 1)n - m)W_{n+m}^{(s)}, (s = 2, 3, 4, 5) \tag{3.1a}
\]

\[
[W_n^{(3)}, W_m^{(3)}] = (n - m)f_{33}^2(n, m)L_{n+m} + (n - m)f_{33}^4(n, m)W_{n+m}^{(4)} \tag{3.1b}
\]

\[
[W_n^{(3)}, W_m^{(4)}] = f_{34}^3(n, m)W_{n+m}^{(3)} + f_{34}^5(n, m)W_{n+m}^{(5)} \tag{3.1c}
\]

\[
[W_n^{(4)}, W_m^{(4)}] = (n - m)f_{44}^2(n, m)L_{n+m} + (n - m)f_{44}^4(n, m)W_{n+m}^{(4)} \tag{3.1d}
\]

\[
[W_n^{(3)}, W_m^{(5)}] = f_{35}^3(n, m)W_{n+m}^{(4)} \tag{3.1e}
\]

\[
[W_n^{(4)}, W_m^{(5)}] = f_{45}^3(n, m)W_{n+m}^{(3)} + f_{45}^5(n, m)W_{n+m}^{(5)} \tag{3.1f}
\]

\[
[W_n^{(5)}, W_m^{(5)}] = (n - m)f_{55}^2(n, m)L_{n+m} + (n - m)f_{55}^4(n, m)W_{n+m}^{(4)} \tag{3.1g}
\]

Our next goal is to find concrete expressions for the structure constants \( f_{s_1, s_2}(n, m) \). This can be achieved by the arranging of the left and right hand sides of the commutators with the spin gradation. Then, we used the associativity of the \( \text{sl}(5, \mathbb{R}) \) Lie algebra with Jacobi identities. Since this associativity condition is of central importance for this section, we will find 11 expressions for the structure constants \( f_{s_1, s_2}(n, m) \). (Further details of our conventions can be found in Appendix B).

\[
f_{33}^2(n, m) = \sigma_1 (2m^2 - nm + 2n^2 - 8) \tag{3.2a}
\]

\[
f_{33}^4(n, m) = \sigma_2 \tag{3.2b}
\]

\[
f_{34}^3(n, m) = \sigma_3 (m^3 - 3m^2n + 5mn^2 - 9m + 5n^3 = 17n) \tag{3.2c}
\]

\[
f_{34}^5(n, m) = \sigma_4 (3n - 2m) \tag{3.2d}
\]

\[
f_{44}^2(n, m) = \sigma_5 (3m^3 - 3m^2n + 4m^2n^2 - 39m^2 - 2mn^3 + 20mn + 3n^4 - 39n^2 + 108) \tag{3.2e}
\]

\[
\leq N. \text{Therefore, an interesting extension of pure AdS gravity can be constructed by adding massless higher spin fields to the spectrum, which proves to be particularly easy in Chern-Simons formulation. The first nontrivial spin 3 extensions of higher spin theories coupled to the pure gravity can be seen in the Refs.}\ [21, 22], \text{where a consistent set of asymptotic conditions for the theory is described. It can be shown that, instead of describing difficult nonlinear interactions of the higher spin fields, promoting the sl}(2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) \text{symmetry group of the Chern-Simons action to sl}(N, \mathbb{R}) \oplus \text{sl}(N, \mathbb{R}) \text{with} N > 2, \text{is sufficient to successfully describe a higher spin theory.}

3.1 A different approach to the calculation of sl}(5, \mathbb{R}) \text{commutators}

The principal embedding of \( \text{sl}(2, \mathbb{R}) \) Lie algebra to \( \text{sl}(5, \mathbb{R}) \) Lie algebra contains \( \text{sl}(2, \mathbb{R}) \) the spin 2 triplet \( L_n, (n = \pm 1, 0) \), \( \text{sl}(3, \mathbb{R}) \) the spin 3 triplet \( W_n^{(3)}, (n = \pm 2, \pm 1, 0) \), \( \text{sl}(4, \mathbb{R}) \) the spin 4 quartet \( W_n^{(4)}, (n = \pm 3, \pm 2, \pm 1, 0) \) and \( \text{sl}(5, \mathbb{R}) \) the spin 5 quintet \( W_n^{(5)}, (n = \pm 4, \pm 3, \pm 2, \pm 1, 0) \) can be represented in terms of the commutation relations for \( L_n \equiv W_n^{(2)} \).
We now proceed to construct where the flat connection condition (2.14),

\[ f_{A}^{(n,m)} = \sigma_{6}(m^{2} + n^{2} - mn - 7) \]  

\[ f_{B}^{(n,m)} = \sigma_{7}(2m^{3} - 9m^{2}n + 21mn^{2} - 32m - 28n^{3} + 88n) \]  

\[ f_{C}^{(n,m)} = \sigma_{8}(3m^{5} - 10m^{4}n + 20m^{3}n^{2} - 75m^{3} - 30m^{2}n^{3} + 220m^{2}n + 35mn^{4} - 355mn^{2} + 432m - 28n^{5} + 340n^{3} - 792n) \]  

\[ f_{D}^{(n,m)} = \sigma_{9}(5m^{3} - 15m^{2}n + 21mn^{2} - 59m - 14n^{3} + 86n) \]  

\[ f_{E}^{(n,m)} = \sigma_{10}(4m^{6} - 3m^{5}n + 6m^{4}n^{2} - 116m^{4} - 4m^{3}n^{3} + 79m^{3}n + 6m^{2}n^{4} - 156m^{2}n^{2} + 976m^{2} - 3mn^{5} + 79mn^{3} - 508mn + 4n^{6} - 116n^{4} + 976n^{2} - 2304) \]  

\[ f_{F}^{(n,m)} = \sigma_{11}(14m^{4} - 21m^{3}n + 29m^{2}n^{2} - 310m^{2} - 21mn^{3} + 315mn + 14n^{4} - 310n^{2} + 1376) \] (3.3f)

where

\[ \sigma_{3} = \frac{-64\sigma_{1}}{49\sigma_{2}}, \sigma_{5} = \frac{64\sigma_{1}^{2}}{49\sigma_{2}^{2}}, \sigma_{6} = \frac{8\sigma_{1}}{7\sigma_{2}}, \sigma_{7} = \frac{-5\sigma_{1}}{49\sigma_{4}}, \sigma_{8} = \frac{-320\sigma_{1}^{2}}{2401\sigma_{2}^{2}\sigma_{4}} \]

\[ \sigma_{9} = \frac{8\sigma_{1}}{49\sigma_{2}}, \sigma_{10} = \frac{320\sigma_{1}^{2}}{2401\sigma_{2}^{2}\sigma_{4}}, \sigma_{11} = \frac{-40\sigma_{1}^{2}}{2401\sigma_{2}^{2}\sigma_{4}} \] (4.4a)

where \(\sigma_{i}\)'s are the 11 arbitrary parameters, depending on three arbitrary parameters, mainly \(\sigma_{1}, \sigma_{2}\) and \(\sigma_{4}\), which can be changed by rescaling \(W_{n}^{(i)}\).

### 3.2 Spin 5 \(AdS_{3}\) solution and conformal Ward identities

We now proceed to construct \(\mathfrak{sl}(5,\mathbb{R})\) connections which describe Spin 5 \(AdS_{3}\) solution with non-zero higher spin charge. We consider the class of connection using the same gauge and notation as in Section(2.2), one can write down the spin 5 extension of (2.11) as [22]. In this Part, we will now make a similar calculation for Spin 5. We begin with the ansatz

\[ a = \left( L_{1} + \alpha L_{-1} + \beta W_{3}W_{-2}^{(3)} + \gamma W_{4}W_{-3}^{(4)} + \delta W_{5}W_{-4}^{(5)} \right) dx^{+} \] (3.5)

\[ \bar{a} = -\left( L_{-1} + \alpha \bar{L}_{1} + \beta \bar{W}_{3}W_{3}^{(3)} + \gamma \bar{W}_{4}W_{3}^{(4)} + \delta \bar{W}_{5}W_{3}^{(5)} \right) dx^{-} \] (3.6)

where \(\alpha, \beta, \gamma\) and \(\delta\) are some scaling parameters to be determined later. After this, we analyze the holomorphic connection \(a^{2}\).

If we expand \(\lambda(x^{+})\) in the \(\mathfrak{sl}(5,\mathbb{R})\) Lie algebra, as we did also for the connection which is compatible with (3.5), then

\[ \lambda = \sum_{i=-1}^{1} \epsilon_{i}L_{i} + \sum_{i=-2}^{2} \chi_{i}W_{i}^{(3)} + \sum_{i=-3}^{3} f_{i}W_{i}^{(4)} + \sum_{i=-4}^{4} \eta_{i}W_{i}^{(5)}. \] (3.7)

The flat connection condition (2.14), \(d\lambda + [a, \lambda] = 0\), emerges the following constraints for 24 undetermined functions of \(x^{\pm}\). There are 24 equations with 4 parameters \(\epsilon_{1}, \chi_{2}, f_{3}, \eta_{4}\) that

\(^{2}\)All resulting expressions involving the \(A\) sector can be obtained similar as in the \(A\) sector.
can be chosen arbitrarily. For notational convenience, we write $\epsilon_1 \equiv \epsilon, \chi_2 \equiv \chi, f_3 \equiv f, \eta_4 \equiv \eta$, and show derivatives with respect to $x^+$ as primes for convenience, one finds

\[
\begin{align*}
\epsilon_0 &= -\epsilon' \\
\epsilon_1 &= -\epsilon'' + \frac{69120\gamma f_0^2 W_4}{49\sigma_2^2} + 24\beta\sigma_1\chi W_3 + \frac{368640\delta \eta \sigma^3 W_5}{343\sigma_2^2 \sigma_4^2} + \alpha \mathcal{L} \epsilon + \frac{\epsilon''}{2} \\
\chi_1 &= -\chi' \\
\chi_0 &= \frac{38400 f \sigma_1 W_3}{49\sigma_2^2} + \frac{\chi''}{2} + \frac{460800 \gamma \sigma^2 W_4}{343 \sigma_2^2 \sigma_4} + 2\alpha \chi \mathcal{L} \\
\chi_{-1} &= -\frac{2560 \beta \sigma_1 W_3 f'}{49\sigma_2} - \frac{1280 \beta f \sigma_1 W_3'}{49\sigma_2} - \frac{\chi^{(3)}}{6} - \frac{38400 \gamma \sigma^2 W_4 \eta'}{49\sigma_2^2 \sigma_4} - \frac{153600 \gamma \sigma^2 W_4'}{343 \sigma_2^2 \sigma_4} \\
&\quad - \frac{2}{3} \alpha \chi \mathcal{L}' - \frac{5}{3} \alpha \mathcal{L} \chi' \\
f_2 &= -f' \\
f_1 &= \frac{f'}{2} + 3\alpha f \mathcal{L} + \frac{240 \beta \eta \sigma_1 W_3}{7 \sigma_4} \\
fv &= -\frac{f^{(3)}}{6} - \frac{8}{3} \alpha \mathcal{L} f' - \frac{\alpha f \mathcal{L}'}{7 \sigma_4} - \frac{180 \beta \sigma_1 W_3 \eta'}{7 \sigma_4} - \frac{80 \beta \eta \sigma_1 W_3'}{7 \sigma_4} \\
fv &= \frac{f^{(4)}}{24} + \frac{7}{6} \alpha f \mathcal{L}'' + \frac{11}{12} \alpha f' \mathcal{L}' + \frac{240 \gamma f \sigma_1 W_4}{7 \sigma_2} + 3\alpha f \mathcal{L}^2 + \frac{1}{4} \alpha f \mathcal{L}'' + \frac{465 \beta \sigma_1 W_3 \eta''}{49 \sigma_4} \\
&\quad + \frac{65 \beta \sigma_1 \eta' W_3'}{7 \sigma_4} + \frac{20 \beta \eta \sigma_1 W_3''}{7 \sigma_4} + \beta \sigma_2 \chi W_3 - \frac{19200 \delta \eta \sigma^2 W_5}{49 \sigma_2^2 \sigma_4} + \frac{2880 \alpha \beta \eta \sigma_1 W_3 \mathcal{L}}{49 \sigma_4} \\
fv &= \frac{f^{(5)}}{120} - \frac{1}{3} \alpha f^{(3)} \mathcal{L} - \frac{5}{12} \alpha f \mathcal{L}'' - \frac{144 \gamma \sigma_1 W_4 f'}{7 \sigma_2} - \frac{11}{5} \alpha^2 f \mathcal{L}^2 f' - \frac{7}{30} \alpha f \mathcal{L}'' - \frac{48 \gamma f \sigma_1 W_4'}{7 \sigma_2} \\
&\quad - \frac{1}{20} \alpha f \mathcal{L}^{(3)} - \frac{9}{5} \alpha^2 f \mathcal{L} f' - \frac{112 \beta \eta \sigma_1 W_3}{49 \sigma_4} - \frac{184 \beta \sigma_1 \eta'' W_3'}{49 \sigma_4} - \frac{17 \beta \sigma_1 \eta' W_3''}{7 \sigma_4} - \frac{4 \beta \eta \sigma_1 W_3 (3)}{7 \sigma_4} \\
&\quad + \frac{4}{5} \beta \sigma_2 W_3 \chi' - \frac{1}{5} \beta \sigma_2 \chi W_3' + \frac{9600 \delta \sigma_2 \sigma_4 W_5 \eta'}{49 \sigma_2 \sigma_4^2} + \frac{3840 \delta \eta \sigma_1^2 W_5'}{49 \sigma_2 \sigma_4^2} - \frac{736 \alpha \beta \eta \sigma_1 W_3 \mathcal{L}'}{49 \sigma_4} \\
&\quad - \frac{1772 \alpha \beta \sigma_1 W_3 \eta'}{49 \sigma_4} - \frac{912 \alpha \beta \sigma_1 \mathcal{L} W_3'}{49 \sigma_4} \\
&\quad (3.8j) \\
&\quad (3.8k)
\end{align*}
\]
\[ f_3 = \frac{f^{(6)}}{720} + \frac{5}{72} \alpha f^{(4)} \mathcal{L} + \frac{1}{8} \alpha f^{(3)} \mathcal{L}' + \frac{40 \gamma_1 \sigma_1 W_4 f''}{7 \sigma_2} + \frac{34}{45} \alpha^2 \mathcal{L}' f'' + \frac{13}{120} \alpha f'' \mathcal{L}'' + \frac{32 \gamma_1 f' W_4'}{7 \sigma_2} \]
\[ + \frac{17}{360} \alpha \mathcal{L}^{(3)} f' + \frac{241}{180} \alpha^2 \mathcal{L} f' \mathcal{L}' + \frac{200}{7} \beta f \sigma_1 W_3^2 + \frac{8 \gamma f \sigma_1 W_4'}{7 \sigma_2} + \frac{1}{120} \alpha f \mathcal{L}^{(4)} + \alpha^3 f \mathcal{L}^3 \]
\[ + \frac{23}{60} \alpha^2 f \mathcal{L} \mathcal{L}'' + \frac{3}{10} \alpha^2 f (\mathcal{L}')^2 + \frac{176 \alpha \gamma f \sigma_1 W_4 \mathcal{L}}{7 \sigma_2} + \frac{384 \beta \gamma \sigma_2^2 W_3 W_3}{7 \sigma_2 \sigma_1} + \frac{41 \beta \eta^{(4)} \sigma_1 W_3}{98 \sigma_4} \]
\[ + \frac{99 \beta \eta^{(3)} \sigma_1 W_3'}{98 \sigma_4} + \frac{101 \beta \sigma_1 \eta'' W_3'}{98 \sigma_4} + \frac{\beta \sigma_1 W_3^{(3)} \eta'}{2 \sigma_1} + \frac{2 \beta \sigma_1 W_3^{(4)}}{21 \sigma_4} + \frac{3}{10} \beta \sigma_2 W_3 \mathcal{L}'' \]
\[ + \frac{1}{6} \beta \sigma_2 \chi' W_3 + \frac{1}{30} \beta \sigma_2 \chi W_3' + \frac{1}{6} \beta \sigma_1 \mathcal{L} \mathcal{L} + \frac{1600 \delta \mathcal{L}^2 W_3 \eta''}{343 \sigma_2^2 \sigma_4^2} - \frac{320 \delta \mathcal{L}^2 \eta' W_3'}{7 \sigma_2 \sigma_4^2} - \frac{640 \delta \sigma_2^2 W_3''}{49 \sigma_2 \sigma_4^2} \]
\[ + \frac{820 \alpha \beta \sigma_1 W_5}{7 \sigma_4} + \frac{1200 \alpha^2 \beta \sigma_1 W_3 \mathcal{L}}{49 \sigma_4} + \frac{136 \alpha \beta \sigma_1 W_3 \eta''}{49 \sigma_4} + \frac{468 \alpha \beta \sigma_1 W_3 \eta' \mathcal{L}'}{49 \sigma_4} \]
\[ + \frac{824 \alpha \beta \sigma_1 W_3' \mathcal{L}'}{147 \sigma_4} + \frac{517 \alpha \beta \sigma_1 W_3 \eta''}{49 \sigma_4} + \frac{599 \alpha \beta \sigma_1 W_3' \eta''}{49 \sigma_4} + \frac{599 \alpha \beta \sigma_1 \mathcal{L} \mathcal{L}''}{147 \sigma_4} \]
\[ + \alpha \beta \sigma_2 \chi W_3 \mathcal{L} = \frac{83200 \alpha \delta \eta^2 W_5 \mathcal{L}}{343 \sigma_2 \sigma_4^2} \]

\[ \eta_3 = -\eta' \]
\[ \eta_2 = \eta'' + 4 \alpha \eta \mathcal{L} \]
\[ \eta_1 = -\frac{\eta^{(3)}}{6} - 4 \frac{\alpha \eta \mathcal{L}}{3} - 11 \frac{\alpha \mathcal{L} \eta'}{3} \]
\[ \eta_0 = 3 \beta f \mathcal{L} \mathcal{L} + \frac{\eta^{(4)}}{24} - \frac{480 \gamma_1 \sigma_1 W_4}{7 \sigma_2} + \frac{6 \alpha^2 \mathcal{L}^2}{3} + 6 \alpha \eta \mathcal{L}'' + \frac{5}{3} \alpha \eta' \mathcal{L}' + \frac{5}{3} \alpha \eta'' \]
\[ \eta_{-1} = -\frac{13}{5} \beta \sigma_4 W_3 f' - \frac{3}{5} \beta \sigma_4 W_3 - \frac{\eta^{(5)}}{120} + \frac{48 \gamma_1 W_4}{7 \sigma_2} + \frac{96 \gamma \sigma_1 W_4'}{7 \sigma_2} - \frac{1}{15} \alpha \eta \mathcal{L} (3) \]
\[ - \frac{73}{15} \alpha^2 \mathcal{L}^2 \eta' - \frac{19}{60} \alpha \eta' \mathcal{L}' - \frac{56}{15} \alpha^2 \mathcal{L} \mathcal{L}' - \frac{7}{12} \alpha \eta'' \mathcal{L}' - \frac{1}{2} \alpha \eta^{(3)} \mathcal{L} \]
\[ \eta_{-2} = \frac{11}{10} \beta \mathcal{L} f' - \frac{8}{15} \beta \mathcal{L} f' W_3 - \frac{1}{10} \beta \mathcal{L} f' W_3' - \frac{320 \delta f \mathcal{L} W_5}{7 \sigma_2} + 6 \alpha \beta f \mathcal{L} \mathcal{L} + \frac{\eta^{(6)}}{720} \]
\[ + \frac{320}{7} \beta \eta \sigma_1 W_3' - \frac{72 \gamma \sigma_1 W_4}{9 \sigma_2} - \frac{72 \gamma \sigma_1 W_4'}{7 \sigma_2} - \frac{16 \gamma \sigma_1 W_4''}{7 \sigma_2} + 2 \gamma \sigma_1 \mathcal{L} W_4 + \frac{1}{90} \alpha \eta \mathcal{L} (4) \]
\[ + 4 \alpha^3 \mathcal{L}^3 + \frac{23}{360} \alpha \mathcal{L} (3) \eta' + \frac{173}{90} \alpha^2 \mathcal{L}^2 \eta'' + \frac{38}{45} \alpha^2 \mathcal{L} \mathcal{L}'' + \frac{3}{20} \alpha \eta' \mathcal{L}' + \frac{277}{90} \alpha^2 \mathcal{L} \eta' \mathcal{L}' \]
\[ + \frac{28}{45} \alpha^2 \mathcal{L} (3) \eta' + \frac{13}{72} \alpha \mathcal{L} (3) \eta' + \frac{1}{9} \alpha \eta^{(4)} \mathcal{L} + \frac{5440 \alpha \gamma \sigma_1 W_4 \mathcal{L}}{49 \sigma_2} \]

(3.8m)

(3.8n)

(3.8o)

(3.8p)

(3.8q)

(3.8r)
\begin{align}
\eta_{-3} &= - \frac{3}{10} \beta f^{(3)} \sigma_4 W_3 - \frac{7}{30} \beta \sigma_4 f'' W'_3 - \frac{19}{210} \beta \sigma_4 f' W''_3 + \frac{1280 \delta \sigma_1 W_3 f'}{49 \sigma_2} - \frac{149}{35} \alpha \beta \sigma_4 W_3 L f' \\
&\quad - \frac{1}{70} \beta f \sigma_4 W_3^{(3)} + \frac{320 \delta \sigma_1 W_3^2}{49 \sigma_2} - \frac{12}{7} \alpha \beta f \sigma_4 W_3 L - \frac{39}{35} \alpha \beta \sigma_4 L W_3 - \eta^{(7)} - \frac{200}{7} \beta^2 \sigma_1 W_3^2 \eta' \\
&\quad - \frac{160}{7} \beta^2 \eta_1 W_3 W'_3 + \frac{1192 \gamma \eta^{(3)} \sigma_1 W_4}{343 \sigma_2} + \frac{1296 \gamma \sigma_1 \eta'' W'_4}{49 \sigma_2} + \frac{88 \gamma \eta_1 \eta'' W'_4}{49 \sigma_2} + \frac{16 \gamma \eta_1 W_4^{(3)}}{49 \sigma_2} \\
&\quad - \frac{11}{7} \gamma \sigma_4 W_4 W' \gamma - \frac{2}{7} \gamma \sigma_4 \chi W_4 - \frac{1}{630} \alpha \eta \gamma \theta^{(5)} - \frac{3}{280} \alpha \mathcal{L}(4) \eta' - \frac{93}{35} \alpha^3 \mathcal{L}^2 \eta' - \frac{47}{315} \alpha^2 \eta \mathcal{L}(3) \\
&\quad - \frac{11}{360} \alpha \mathcal{L}(3) \eta'' - \frac{22}{45} \alpha^2 \eta^{(3)} \mathcal{L} - \frac{877 \alpha^2 \mathcal{L} \eta' \mathcal{L}''}{1260} - \frac{17}{360} \alpha^2 \mathcal{L}^2 \mathcal{L}'' - \frac{223}{180} \alpha^2 \mathcal{L} \eta'' \mathcal{L}' - \frac{37}{70} \alpha^2 \mathcal{L}^2 \mathcal{L}' \\
&\quad - \frac{1}{24} \alpha \mathcal{L}(4) \mathcal{L} + \frac{8640 \alpha \gamma \eta \sigma_1 W_4 \mathcal{L}}{343 \sigma_2} + \frac{116}{35} \alpha^3 \eta \mathcal{L}^2 \mathcal{L}' - \frac{94}{315} \alpha^2 \mathcal{L} \mathcal{L}'' \mathcal{L}' - \frac{7}{360} \alpha \eta^{(5)} \mathcal{L} \\
+ \frac{21296 \alpha \gamma \sigma_1 W_4 \eta' \mathcal{L}}{343 \sigma_2} + \frac{7456 \alpha \gamma \sigma_1 \mathcal{L} W_4}{343 \sigma_2} + \frac{1}{343 \sigma_2} (3.8s)
\end{align}

\begin{align}
\eta_{-4} &= \mathcal{L}^4 \eta \alpha^4 + \frac{62}{63} \mathcal{L} \eta (\mathcal{L}')^2 \alpha^3 + \frac{2747 \mathcal{L}^2 \mathcal{L}' \eta' \alpha^3}{1260} + \frac{197}{315} \mathcal{L}^2 \eta \mathcal{L}'' \alpha^3 + \frac{256 \eta \mathcal{L}^3 \eta'' \alpha^3}{315} \\
&\quad + \frac{47 \eta \mathcal{L}'' \mathcal{L}' \alpha^2}{1260} + \frac{3 f \eta \beta W_3 \sigma_4 \alpha^2}{5040} + \frac{517 \mathcal{L} \eta' \mathcal{L}'' \alpha^2}{2016} + \frac{99}{280} f \beta \sigma_4 f' \mathcal{L}' \alpha \\
&\quad + \frac{17 \beta^2 \eta W_3 \sigma_1 \alpha}{210} + \frac{2000 \beta^2 \eta W_3 \sigma_1 \alpha}{49 \sigma_2} + \frac{613 \mathcal{L} \eta \mathcal{L}'' \alpha^2}{315} + \frac{99}{140} \mathcal{L} \eta \mathcal{L} \alpha \\
&\quad + \frac{84 \beta \mathcal{L} \mathcal{L} \alpha}{49 \sigma_2} + \frac{169 \beta \mathcal{L} \beta \sigma_4 f' W'_3 \alpha}{210} + \frac{99}{280} f \beta \sigma_4 f' \mathcal{L} \alpha \\
&\quad + \frac{17 \beta^2 \eta W_3 \sigma_1 \alpha}{343 \sigma_2} + \frac{4476 \gamma \sigma_1 \eta' W_3 \alpha}{343 \sigma_2} + \frac{146}{105} \mathcal{L} \beta W_3 \sigma_4 f'' \alpha + \frac{19}{56} f \beta W_3 \sigma_4 \mathcal{L} \alpha \\
&\quad - \frac{5448 \gamma \mathcal{L} \mathcal{L} W_4 \sigma_1 \eta'' \alpha}{343 \sigma_2} - \frac{23}{140} f \mathcal{L} \beta \sigma_4 W'_3 \alpha - \frac{1128 \gamma \mathcal{L} \mathcal{L} W_4 \sigma_1 \eta'' \alpha}{343 \sigma_2} - \frac{7}{210} \mathcal{L} \alpha^{(3)} \alpha + \frac{13 \eta'' \mathcal{L}^{(4)} \alpha}{2520} \\
&\quad + \frac{1}{90} \mathcal{L}' \eta \alpha^{(4)} \alpha + \frac{31 \eta'' \mathcal{L}^{(5)} \alpha}{20160} + \frac{11 \mathcal{L}^{(5)} \alpha}{1440} + \frac{\eta \mathcal{L}^{(6)} \alpha}{5040} + \frac{1}{360} \mathcal{L} \eta \alpha \\
&\quad + \frac{20}{7} \beta^2 \sigma_1 (W'_3)^2 + \delta \chi W_5 + \frac{1}{2} \beta^2 \chi W_3^2 \sigma_1 \sigma_4 + \frac{480 \beta \gamma W_3 W_4 \mathcal{L}}{7 \sigma_2} + \frac{205}{14} \beta^2 W_3 \sigma_1 \mathcal{L} W'_3 \\
&\quad + \frac{13}{56} \gamma \sigma_4 \chi W'_4 - \frac{200 \delta \sigma_1 f' W'_3}{49 \sigma_2} - \frac{3288 \delta \sigma_1 f' W''_3}{49 \sigma_2} + \frac{815}{98} \beta^2 W_3 \mathcal{L} \eta'' \alpha + \frac{4}{7} \gamma W_4 \sigma_1 \chi'' \\
&\quad + \frac{30}{7} \beta^2 \eta W_3 \sigma_1 W'_3 + \frac{17}{420} \beta \sigma_4 f'' W'_3 + \frac{1}{28} \beta \sigma_4 f'' W'_3 + \frac{239 \gamma \mathcal{L} \sigma_1 \eta'' W''_4}{343 \sigma_2} - \frac{40 \delta \sigma_1 W'_5}{49 \sigma_2} \\
&\quad + \frac{1}{15} \beta \sigma_4 W''_3 \alpha^{(3)} - \frac{311 \gamma \sigma_1 \eta'' W''_4}{343 \sigma_2} + \frac{11}{840} \beta \sigma_4 f' W'_3 \alpha^{(3)} - \frac{13 \gamma \sigma_1 \mathcal{L} \mathcal{L} \alpha^{(3)}}{49 \sigma_2} - \frac{7}{120} \beta W_3 \sigma_4 f'(4) \\
&\quad - \frac{184 \gamma W_4 \sigma_1 \eta'' W''_4}{343 \sigma_2} + \frac{1}{560} f \beta \sigma_4 W_3 \alpha^{(4)} - \frac{2 \gamma \mathcal{L} \mathcal{L} W_4}{49 \sigma_2} + \frac{\eta^{(8)}}{40320} - \frac{21120 \delta \eta_1 W_3 W_5 \sigma_2}{49 \sigma_2} \\
+ \frac{57600 \gamma \mathcal{L} \sigma_1 \mathcal{L} W'_3}{49 \sigma_2} + \frac{1}{343 \sigma_2} (3.8t)
\end{align}

which we will refer to as the twenty auxiliary equations. One can now also determine how...
the functions $\mathcal{L}, W_s (s = 3, 4, 5)$ transform under the gauge transformation in terms of $\epsilon, \chi, f, \eta, \mathcal{L}, W_s (s = 3, 4, 5)$ and their derivatives. The transformations of $\mathcal{L}, W_s (s = 3, 4, 5)$ are given by

$$\mathcal{L} \rightarrow \mathcal{L} + \delta_\epsilon \mathcal{L} + \delta_\chi \mathcal{L},$$  

$$W_s \rightarrow W_s + \delta_\epsilon W_s + \delta_\chi W_s + \delta_f W_s + \delta_\eta W_s, \ (s = 3, 4, 5)$$  

where

$$\delta_\epsilon \mathcal{L} = 2 \mathcal{L} \epsilon' + \mathcal{L}' \epsilon + \frac{\epsilon''}{2\alpha}$$  

(3.10)

here $\epsilon$ is the gauge parameter related to $\mathfrak{sl}(2, \mathbb{R})$ subgroup in $\mathfrak{sl}(5, \mathbb{R})$, which generate conformal transformations. Thus, we see that $\mathcal{L}$ can be identified with the $CFT_2$ stress tensor if we relate the Chern-Simons level and central charge as $c = 6k$. Now, first observe that if we set from (2.17) and (2.22), we can write the first scaling parameter $\alpha$ as

$$\alpha = \frac{6}{c}$$  

(3.11)

which is compatible with the central charge of the Virasoro algebra $c = 6k$. Finally we will set $\sigma_1 = \frac{1}{36}, \sigma_2 = \frac{16}{\sqrt{105}}$, $\sigma_4 = \frac{\sqrt{15}}{7}$, and also $\beta = \frac{15}{7}, \gamma = \frac{26}{7}$ and $\delta = \frac{45}{7}$. In the light of these results, for the later calculations from now on we generalize the connection $(3.5)$ for $\mathfrak{sl}(N, \mathbb{R})$ and $\mathcal{L}_n \equiv \mathcal{W}_n^{(2)}$ as

$$a = \left( W_1^{(2)} + \sum_{s=2}^{N} \frac{s(2s-1)}{c} W_s W_{s+1} \right) dx^+,$$  

(3.12)

in order to match $AdS_3$ connection parameters to the classical $\mathcal{W}_5$ asymptotic symmetry algebra as in Appendix C. Then, the following variations
define, that the each conformal field $W_s$ has the conformal spin $s$, with $s = 3, 4, 5$.

$$\delta_\chi W_s = s W_s \epsilon' + \epsilon \epsilon', \ (s = 3, 4, 5)$$  

(3.13a)

$$\delta_\chi W_3 = \frac{c}{360} \chi^{(5)} + \frac{\chi'' \mathcal{L}'}{2} + \frac{1}{3} \chi^{(3)} \mathcal{L} + \chi' \left( \frac{32 \mathcal{L}^2}{5c} + \frac{32 W_4}{\sqrt{105}} + \frac{3 \mathcal{L}''}{10} \right)$$

$$+ \chi \left( \frac{32 \mathcal{L} \mathcal{L}'}{5c} + \frac{16 W_4^2}{\sqrt{105}} + \frac{\mathcal{L}^{(3)}}{15} \right)$$  

(3.14a)

where $\chi$ is the gauge parameter related to $\mathfrak{sl}(3, \mathbb{R})$ subgroup in $\mathfrak{sl}(5, \mathbb{R})$, which defines spin 3 conformal Ward identity.

$$\delta_\chi W_4 = + \frac{4 \chi^{(3)} W_3}{\sqrt{105}} + \frac{4 \chi'' W_3}{\sqrt{105}} + \chi' \left( \frac{416}{7c} \sqrt{\frac{3}{35}} \mathcal{L} W_3 + \frac{4}{7} \sqrt{\frac{3}{35}} W_3^2 + \frac{5 \sqrt{15}}{7} W_3 \right)$$

$$+ \chi \left( \frac{40}{7c} \sqrt{\frac{15}{7}} \mathcal{L} \mathcal{L}' + \frac{144}{7c} \sqrt{\frac{3}{35}} W_3^2 + \frac{2}{7} \sqrt{15} W_3^3 + \frac{2 W_3^{(3)}}{7 \sqrt{105}} \right)$$  

(3.15a)
\begin{align}
\delta f \mathcal{W}_4 &= \frac{cf^{(7)}}{20160} + \frac{1}{60}f^{(5)\mathcal{L}} + \frac{1}{24}f^{(4)\mathcal{L}'} + f^{(3)} \left( \frac{7\mathcal{L}^2}{5c} + \frac{1}{2} \sqrt{\frac{3}{35}} \mathcal{W}_4 + \frac{\mathcal{L}''}{20} \right) \\
&\quad + f'' \left( \frac{21\mathcal{L}\mathcal{L}'}{5c} + \frac{3}{4} \sqrt{\frac{3}{35}} \mathcal{W}_4 + \frac{\mathcal{L}'}{30} \right) + f' \left( \frac{864\mathcal{L}^3}{35c^2} + \frac{45\mathcal{W}_3^2}{2c} + \frac{88\mathcal{L}\mathcal{L}''}{35c} + \frac{59(\mathcal{L}')^2}{28c} \right) \\
&\quad + \frac{4\mathcal{W}_4\sqrt{\frac{21}{5}}\mathcal{L}}{c} + \frac{1}{4} \sqrt{\frac{5}{21}} \mathcal{W}_4 + \frac{\mathcal{L}''}{84} \right) + f \left( \frac{1296\mathcal{L}^2\mathcal{L}'}{35c^2} + \frac{45\mathcal{W}_3\mathcal{W}_4}{2c} + \frac{39\mathcal{L}''}{70c} \right) \\
&\quad + \frac{2}{c} \frac{\sqrt{21}}{3} \mathcal{W}_4\mathcal{L}' \right) + \frac{177\mathcal{L}\mathcal{L}''}{140c} + \frac{2}{14} \frac{\sqrt{\frac{21}{5}}\mathcal{W}_4}{c} + \frac{\mathcal{W}_4^{(3)}}{4\sqrt{105}} + \frac{\mathcal{L}^{(5)}}{560} \right) \\
&= (3.16a)
\end{align}

where \( f \) is the gauge parameter related to \( \mathfrak{sl}(4, \mathbb{R}) \) subgroup in \( \mathfrak{sl}(5, \mathbb{R}) \), which defines spin 4 conformal Ward identity.

\begin{align}
\delta f \mathcal{W}_5 &= \frac{2}{7} \sqrt{\frac{5}{3}} f^{(5)\mathcal{W}_4} + \frac{1}{14} \sqrt{15} \mathcal{W}_4^{(3)} + \mathcal{L}'' \mathcal{W}_4 + \chi' \left( \frac{24\mathcal{W}_3^2}{\sqrt{3c}} + \frac{32\sqrt{15} \mathcal{W}_4\mathcal{L}}{7c} + \frac{1}{14} \sqrt{\frac{5}{3}} \mathcal{W}''_4 \right) \\
&\quad + \chi \left( \frac{16\mathcal{W}_3 \mathcal{W}_4}{\sqrt{7c}} + \frac{32\mathcal{W}_4\mathcal{L}'}{\sqrt{15c}} + \frac{128\mathcal{L}\mathcal{W}_4}{7\sqrt{15c}} + \mathcal{W}_4^{(3)} \right) \\
&= (3.17a)
\end{align}

\begin{align}
\delta f \mathcal{W}_5 &= \frac{1}{56} \sqrt{\frac{3}{5}} f^{(5)\mathcal{W}_3} + \frac{1}{56} \sqrt{\frac{5}{3}} f^{(4)\mathcal{W}_3} + f^{(3)} \left( \frac{33\sqrt{15} \mathcal{W}_3\mathcal{L}}{49c} + \frac{1}{196} \sqrt{15} \mathcal{W}_3^{(3)} - \frac{11}{28} \sqrt{\frac{5}{21}} \mathcal{W}_5 \right) \\
&\quad + f'' \left( \frac{48\sqrt{15} \mathcal{W}_3\mathcal{L}'}{49c} + \frac{34\sqrt{15} \mathcal{W}_3\mathcal{L}''}{49c} - \frac{1}{14} \sqrt{\frac{3}{35}} \mathcal{W}_5 + \frac{1}{392} \sqrt{15} \mathcal{W}_3^{(3)} \right) \\
&\quad + f' \left( \frac{4896}{49c^2} \sqrt{\frac{3}{5}} \mathcal{W}_3\mathcal{L}' + \frac{74\mathcal{W}_3 \mathcal{W}_4}{\sqrt{7c}} + \frac{563}{196c} \sqrt{\frac{3}{5}} \mathcal{W}_3\mathcal{L}'' + \frac{397}{196c} \sqrt{\frac{5}{3}} \mathcal{W}_3\mathcal{L}' + \frac{89}{98c} \sqrt{\frac{5}{3}} \mathcal{L}\mathcal{W}_3'' \right) \\
&\quad - \frac{58}{7c} \sqrt{\frac{15}{7}} \mathcal{W}_5 \mathcal{L} - \frac{1}{14} \sqrt{\frac{15}{7}} \mathcal{W}_5'' + \frac{5}{2352} \sqrt{\frac{5}{3}} \mathcal{W}_3^{(4)} \right) + f \left( \frac{1728}{49c^2} \sqrt{\frac{3}{5}} \mathcal{L}_2 \mathcal{W}_3 + \frac{4752}{49c^2} \sqrt{\frac{3}{5}} \mathcal{L}\mathcal{W}_3 + \frac{29\sqrt{15} \mathcal{W}_3^{(3)}}{196c} \right) \\
&\quad + \frac{6\sqrt{7} \mathcal{W}_4 \mathcal{W}_5}{c} + \frac{24\mathcal{W}_3 \mathcal{W}_4}{\sqrt{7c}} + \frac{123}{196c} \sqrt{\frac{3}{5}} \mathcal{W}_3\mathcal{L}^{(3)} + \frac{97}{98c} \sqrt{\frac{3}{5}} \mathcal{L}_2 \mathcal{W}_3^{(3)} + \frac{29\sqrt{15} \mathcal{W}_3^{(3)}}{196c} \mathcal{L}' \right) \\
&\quad - \frac{4}{c} \sqrt{\frac{15}{7}} \mathcal{W}_5 \mathcal{L}' - \frac{118}{7c} \sqrt{\frac{3}{35}} \mathcal{L}_2 \mathcal{W}_5 + \frac{5\sqrt{15} \mathcal{W}_3^{(3)}\mathcal{L}}{98c} - \frac{1}{28} \sqrt{\frac{5}{21}} \mathcal{W}_5^{(3)} + \frac{\mathcal{W}_3^{(5)}}{784\sqrt{15}} \right) \\
&= (3.18a)
\end{align}
\[ \delta_\gamma W_5 = \frac{\eta^{(9)}}{1814400} + \frac{\eta^{(7)}}{2520} + \frac{\eta^{(6)}}{720} \eta' L' + \eta^{(5)} \left( \frac{13 \mathcal{L}^2}{150c} - \frac{11 W_3}{140 \sqrt{105}} + \frac{\mathcal{L}''}{400} \right) + \eta^{(4)} \left( \frac{13 \mathcal{L} L'}{30c} - \frac{11 W_4'}{56 \sqrt{105}} + \frac{\mathcal{L}^{(3)}}{360} \right) \]
\[ + \eta^{(3)} \left( \frac{656 \mathcal{L}^3}{105 c^2} + \frac{181 W_3^2}{84c} + \frac{109 \mathcal{L}'''}{210c} + \frac{73 (\mathcal{L}')^2}{168c} - \frac{374 W_4 \mathcal{L}}{21 \sqrt{105} c} - \frac{11 \sqrt{\frac{5}{21}} W_4'}{252} \right) \]
\[ + \frac{L^{(4)}}{504} + \eta'' \left( \frac{984 \mathcal{L}^2 L'}{35 c^2} + \frac{181 W_3 W_3'}{28c} + \frac{29 \mathcal{L} (3) \mathcal{L}}{84c} - \frac{187 W_4 L'}{7 \sqrt{105} c} + \frac{219 \mathcal{L} L'}{280c} \right) \]
\[ - \frac{187 \mathcal{L} W_4'}{7 \sqrt{105} c} - \frac{11 W_4 (3)}{84 \sqrt{105}} + \frac{L^{(5)}}{1120} + \eta'' \left( \frac{36864 \mathcal{L}^3 L'}{175 c^2} - \frac{11008 \mathcal{L}^2 W_4'}{35 \sqrt{105 c^2}} - \frac{22016 \mathcal{L} W_4 L'}{35 \sqrt{105 c^2}} \right) \]
\[ + \frac{496 W_3 W_3'}{7 c^2} + \frac{496 (\mathcal{L}')^3}{105 c^2} + \frac{1952 (3) \mathcal{L}^2}{525 c^2} + \frac{4864 \mathcal{L} L' L''}{72c} + \frac{992 W_3 L W_3'}{3c} + \frac{5 W_3 W_3'}{35 \sqrt{105 c}} \]
\[ - \frac{30 W_3 W_3'}{7 \sqrt{7} c} + \frac{448 \mathcal{W}_4 W_4'}{15c} - \frac{30 W_3 W_5'}{\sqrt{7} c} + \frac{19 W_3 W_3'}{21c} + \frac{35 \mathcal{W}_5 (3) \mathcal{L}}{1575c} - \frac{368 W_4 L (3)}{105 \sqrt{105 c}} - \frac{281 W_4 L''}{35 \sqrt{105 c}} \]
\[ - \frac{157 W_5 L'}{21 \sqrt{105} c} + \frac{13 L (4) L'}{210 c} + \frac{47 \mathcal{L} (3) L''}{450c} - \frac{62 W_4 (3) \mathcal{L}}{21 \sqrt{105 c}} - \frac{W_4 (5)}{180 \sqrt{105}} + \frac{\mathcal{L} (7)}{37800} \right) (3.19a) \]

where \( \eta \) is the gauge parameter related to \( \mathfrak{sl}(5, \mathbb{R}) \), which defines spin 5 conformal Ward identity.

As in the spin 2 case in (2.18), one can now determine the corresponding canonical boundary charges as

\[ Q[\epsilon] = \int dx^+ (\epsilon(x^+) L(x^+) + \chi(x^+) W_3(x^+) + f(x^+) W_4(x^+) + \eta(x^+) W_5(x^+)) \] (3.20)

In fact, from these Ward identities, that is, from the variation of these fields one can derive their corresponding classical asymptotic \( W_5 \) symmetry algebra as in Appendix C.

### 3.3 Determining the 1/c-dependence of the structure constants

So far, the asymptotic symmetry algebra we obtained is not the quantum algebra because of the appearance of the nonlinear terms in the \( W_5 \) algebra, but rather one semi-classical algebra. We know that this fits with the fact that classical limit takes \( c \to \infty \). For this, we are interested in the quantum mechanical version of the asymptotic symmetry algebra, hence we take into account normal ordering effects. We also emphasize here that we know that the normal ordered version of the algebra then becomes inconsistent for finite values of central charge \( c \) and thus has to be modified. So we have to lift the semi-classical results to quantum results and finally end up with the quantum \( W_5 \) asymptotic symmetry algebra. Finally, a deformed version of usual semi-classical asymptotic symmetry algebra with new arbitrary structure constants can be obtained[31]. This will be the final form of the quantum asymptotic symmetry algebra of the \( \text{AdS}_3 \) higher spin gravity.
In the shorthand notation, all important structure constants $c_{s1s2}$'s appearing in the classical asymptotic $W_5$ symmetry algebra as in Appendix C are given by schematically

$$W_3 \ast W_3 \sim \frac{c}{3} \mathbf{1} + c_{33}^4 W_4$$
$$W_3 \ast W_4 \sim c_{34}^3 W_3 + c_{34}^5 W_5$$
$$W_4 \ast W_4 \sim \frac{c}{4} \mathbf{1} + c_{44}^4 W_4 + c_{44}^{[33]} [W_3 W_3]$$
$$W_3 \ast W_5 \sim c_{35}^4 W_4 + c_{35}^{[33]} [W_3 W_3]$$
$$W_4 \ast W_5 \sim c_{45}^3 W_3 + c_{45}^5 W_5 + c_{45}^{[34]} [W_3 W_4] + c_{45}^{[34]} [W_3 W_3]$$
$$W_5 \ast W_5 \sim \frac{c}{5} \mathbf{1} + c_{55}^4 W_4 + c_{55}^{[33]} [W_3 W_3] + c_{55}^{[35]} [W_3 W_5]$$

(3.21)

where some important structure constants $c_{s1s2}$'s for classical and quantum asymptotic $W_5$ symmetry algebra are given by

$$c_{34}^3 = \frac{3}{4} c_{33}^4, \quad c_{35}^4 = \frac{4}{5} c_{34}^5$$
$$c_{45}^3 = \frac{3}{5} c_{34}^5 = \frac{3}{4} c_{35}^4, \quad c_{55}^4 = \frac{4}{5} c_{45}^5$$

(3.22)

| classical algebra | quantum algebra |
|-------------------|-----------------|
| $c_{33}^4$        | $\frac{32}{\sqrt{105}}$ |
| $c_{34}^5$        | $\sqrt{\frac{1024(c+2)(c+23)}{3(5c+22)(1c+68)}}$ |
| $c_{45}^3$        | $\frac{45}{2c}$ |
| $c_{45}^{[34]}$   | $\frac{9(5c+22)}{2(c+2)(c+23)}$ |
| $c_{35}^4$        | $\frac{24}{\sqrt{7c}}$ |
| $c_{45}^{[34]}$   | $\sqrt{\frac{432(2c-1)^2}{(c+2)(c+23)(3c+116)(7c+114)}}$ |
| $c_{55}^4$        | $\frac{74}{\sqrt{7c}}$ |
| $c_{55}^{[34]}$   | $\sqrt{\frac{12(37c+334)^2}{(c+2)(c+23)(3c+116)(7c+114)}}$ |
| $c_{55}^{[35]}$   | $\frac{66}{\sqrt{7c}}$ |
| $c_{55}^{[34]}$   | $\sqrt{\frac{108(37c+116)}{c+2)(c+23)(7c+114)}}$ |
| $c_{55}^{[44]}$   | $\frac{181}{13c}$ |
| $c_{55}^{[44]}$   | $\frac{2(181c+14886c^2+24894c+1507924)}{(c+2)(c+23)(3c+116)(7c+114)}$ |
| $c_{55}^{[44]}$   | $\frac{448}{15c}$ |
| $c_{55}^{[44]}$   | $\frac{64(7c+114)}{(3c+116)(5c+22)}$ |

Table-1: some important structure constants for classical and quantum asymptotic $W_5$ symmetry algebra with $1/c$ correction.

One can check as in the Table-1 that if it can be taken into account the normal ordering effects in the quantum asymptotic $W_5$ symmetry algebra, then the classical asymptotic $W_5$ symmetry algebra can be obtained under $c \to \infty$ classical limit with $1/c$ correction as in the second column of the Table-1, i.e $c_{33}^4$

$$\left(c_{33}^4\right)^2 = \frac{1024(c+2)(c+23)}{3(5c+22)(1c+68)} = \frac{1024 + 25600 + 47104}{105 + \frac{1452}{c} + \frac{448}{c^2}}$$

(3.23)
under $c \to \infty$ classical limit with $1/c$ correction

\[
C_{33}^4 = \frac{32}{\sqrt{105}} + \mathcal{O}\left(\frac{1}{c}\right), \tag{3.24}
\]

which is compatible with Table-1.

Finally, one can calculate the Operator Product Expansions of the quantum $W_N$ asymptotic symmetry algebra by starting with the classical one of Chern-Simons theory based on $\mathfrak{sl}(N, \mathbb{R})$ Lie algebra. The corrections for finite $c$ in the nonlinear terms can also be determined recursively by solving some constraints for the Jacobi identities given in the Appendix C and Ref.[31], especially for $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra.

4 Summary and Conclusion

In this work, we first reviewed a relation between $AdS_3$ and $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern-Simons theory. The Chern-Simons formulation of $AdS_3$ allows for a straightforward generalization to a higher spin theory. The higher spin gauge fields have no propagating degree of freedom, but we noted that there are a large class of interesting non-trivial solutions. Specifically, $AdS_3$ in the presence of a tower of higher spin fields up to spin 5 is obtained by enlarging $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ to $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$. Finally, classical $W_5$ symmetry algebra as spin 5 asymptotic symmetry algebra is obtained.

As in the section 3, the asymptotic $W_5$ symmetry algebra we obtained is not the final form of the related algebra of the $AdS_3$ spin 5 gravity since it is just effectively for the Virasoro central charge $c$ in the large values. One can proceed entirely in the same way for $AdS_3$ spin 5 gravity and thus determine the asymptotic symmetry algebra as quantum $W_5$ algebra instead of rather than semiclassical one. Therefore, we can take into account normal ordering effects of the quantum $W_5$ algebra as in Ref.[31]). So far we only discussed the classical $W_5$ symmetry. One may wonder, whether the quantum case also admits a description in terms of an classical $W_N$ symmetry, or a quantum version of it. These questions seem to fairly non-trivial, but it seems that the procedure does go through straightforwardly.

5 Acknowledgments

We would like to thank Marc Henneaux for both his encouragement and constructive discussions.
\[ A \quad \mathfrak{sl}(2, \mathbb{R}) \text{ generators} \]

Below, we denote the three generators of \( \mathfrak{sl}(2, \mathbb{R}) \) Lie algebra, which will be assumed to be described by a set of matrices \( \mathbf{L}_a \) with \( a = \pm 1, 0 \) given by

\[
\mathbf{L}_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{L}_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

which admit an invariant bilinear form

\[
\eta_{ab} = \begin{pmatrix}
\mathbf{L}_{-1} & \mathbf{L}_0 & \mathbf{L}_1 \\
\mathbf{L}_{-1} & 0 & 0 & -1 \\
\mathbf{L}_0 & 0 & \frac{1}{2} & 0 \\
\mathbf{L}_1 & -1 & 0 & 0
\end{pmatrix},
\]

\[ B \quad \mathfrak{sl}(5, \mathbb{R}) \text{ generators as the principal embedding of } \mathfrak{sl}(2, \mathbb{R}) \]

Below, we denote the five generators of \( \mathfrak{sl}(5, \mathbb{R}) \) Lie algebra, which will be assumed to be described by a set of matrices. The principal embedding of \( \mathfrak{sl}(2, \mathbb{R}) \) Lie algebra to \( \mathfrak{sl}(5, \mathbb{R}) \) Lie algebra contains \( \mathfrak{sl}(2, \mathbb{R}) \) Lie algebra the spin 2 triplet \( \mathbf{L}_n \) with \( n = \pm 1, 0 \), \( \mathfrak{sl}(3, \mathbb{R}) \) Lie algebra the spin 3 triplet \( \mathbf{W}_n^{(3)} \) with \( n = \pm 2, \pm 1, 0 \), \( \mathfrak{sl}(4, \mathbb{R}) \) Lie algebra the spin 4 quartet \( \mathbf{W}_n^{(4)} \) with \( n = \pm 3, \pm 2, \pm 1, 0 \) and \( \mathfrak{sl}(5, \mathbb{R}) \) Lie algebra the spin 5 quintet \( \mathbf{W}_n^{(5)} \) with \( n = \pm 4, \pm 3, \pm 2, \pm 1, 0 \) and can be represented in terms of the commutation relations for \( \mathbf{L}_n \equiv \mathbf{W}_n^{(2)} \). Below, we collect the twenty-four \( \mathfrak{sl}(5, \mathbb{R}) \) generators:

\[
\mathbf{L}_0 = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{L}_{-1} = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
\mathbf{W}_n^{(3)} = \gamma_1, \quad \mathbf{W}_n^{(3)} = \frac{\gamma_1}{2}, \quad \mathbf{W}_n^{(3)} = 2\gamma_1, \quad \mathbf{W}_n^{(3)} = 2\gamma_1, \quad \mathbf{W}_n^{(3)} = 2\gamma_1, \quad \mathbf{W}_n^{(3)} = 2\gamma_1, \quad \mathbf{W}_n^{(3)} = 2\gamma_1, \quad \mathbf{W}_n^{(3)} = 2\gamma_1, \quad \mathbf{W}_n^{(3)} = 2\gamma_1
\]

\[
\mathbf{W}_n^{(4)} = \gamma_2, \quad \mathbf{W}_n^{(4)} = \frac{2\gamma_2}{3}, \quad \mathbf{W}_n^{(4)} = \frac{2\gamma_2}{3}, \quad \mathbf{W}_n^{(4)} = \frac{5\gamma_2}{3}, \quad \mathbf{W}_n^{(4)} = \frac{5\gamma_2}{3}, \quad \mathbf{W}_n^{(4)} = \frac{5\gamma_2}{3}, \quad \mathbf{W}_n^{(4)} = \frac{5\gamma_2}{3}, \quad \mathbf{W}_n^{(4)} = \frac{5\gamma_2}{3}
\]
admit an invariant bilinear form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
\frac{5\gamma_3}{2} \\
\sqrt{6} & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{5\gamma_3}{2} \\
\sqrt{6} & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\]
C Classical $W_5$ symmetry OPEs from the conformal Ward identities

\begin{equation}
\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{\xi}{z_{12}^4} + \frac{2\xi}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}} \tag{C.1a}
\end{equation}

\begin{equation}
\mathcal{L}(z_1)W_s(z_2) \sim \frac{sW_s}{z_{12}^2} + \frac{W_s'}{z_{12}}, \quad (s = 3, 4, 5) \tag{C.1b}
\end{equation}

\begin{equation}
W_3(z_1)W_3(z_2) \sim \frac{\xi}{z_{12}^3} + \frac{2\xi}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}} + \frac{1}{z_{12}^2} \left( \frac{32\mathcal{L}^2}{5c} + \frac{32W_4}{\sqrt{105}} + \frac{3\mathcal{L}''}{10} \right) + \frac{1}{z_{12}} \left( \frac{32\mathcal{L}\mathcal{L}'}{5c} + \frac{16W_4'}{\sqrt{105}} + \frac{\mathcal{L}^{(3)}}{15} \right) \tag{C.1c}
\end{equation}

\begin{equation}
W_3(z_1)W_4(z_2) \sim \frac{8}{\sqrt{105}} \left( \frac{3W_5}{z_{12}^2} + \frac{W_3'}{z_{12}} \right) + \frac{1}{z_{12}^2} \left( \frac{416}{7c} \sqrt{\frac{3}{35}} \mathcal{L}W_3 + \frac{4}{7} \sqrt{\frac{3}{35}} W_5' + \frac{5}{7} \sqrt{15} W_5 \right) + \frac{1}{z_{12}} \left( \frac{40}{7c} \sqrt{\frac{15}{7}} W_3\mathcal{L}' + \frac{144}{7c} \sqrt{\frac{3}{35}} W_3' + \frac{2}{7} \sqrt{15} W_5' + \frac{2W_3^{(3)}}{7\sqrt{105}} \right) \tag{C.1d}
\end{equation}

\begin{equation}
W_4(z_1)W_4(z_2) \sim \frac{\xi}{z_{12}^3} + \frac{2\xi}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}} + \frac{1}{z_{12}^2} \left( \frac{3\sqrt{3}}{35} W_4 + \frac{42}{5c} \mathcal{L}^2 + \frac{3}{10} \mathcal{L}'' \right) + \frac{1}{z_{12}} \left( \frac{3}{2} \sqrt{\frac{3}{35}} W_4' + \frac{42}{5c} \mathcal{L}\mathcal{L}' + \frac{\mathcal{L}^{(3)}}{15} \right) + \frac{1}{z_{12}^2} \left( \frac{2}{3} \sqrt{\frac{3}{35}} W_4^2 + \frac{59}{28c} (\mathcal{L}')^2 + \frac{88}{35c} \mathcal{L}\mathcal{L}'' + \frac{45}{2c} W_3^2 + \frac{4}{c} \sqrt{\frac{21}{5}} \mathcal{L}W_4 + \frac{1}{4} \sqrt{\frac{5}{21}} W_4' + \frac{\mathcal{L}^{(4)}}{84} \right) + \frac{1}{z_{12}} \left( \frac{1296}{35c^2} \mathcal{L}^3 \mathcal{L}' + \frac{177}{140c} \mathcal{L}^2 \mathcal{L}'' + \frac{45}{2c} \mathcal{L}\mathcal{L}' + \frac{4}{c} \sqrt{\frac{21}{5}} \mathcal{L}W_4 + \frac{2}{c} \sqrt{\frac{21}{5}} W_4 W_3 \right) + \frac{39}{70c} \mathcal{L}^{(3)} + \frac{W_4^{(3)}}{4\sqrt{105}} + \frac{\mathcal{L}^{(5)}}{560} \tag{C.1e}
\end{equation}

\begin{equation}
W_3(z_1)W_5(z_2) \sim \frac{\sqrt{15}}{7} \left( \frac{4W_4}{z_{12}^2} + \frac{W_4'}{z_{12}} \right) + \frac{1}{z_{12}^2} \left( \frac{24W_3^2}{\sqrt{7c}} + \frac{32\sqrt{15} \mathcal{L} W_4}{7c} + \frac{1}{14} \sqrt{\frac{5}{3}} W_4' \right) + \frac{1}{z_{12}} \left( \frac{32W_4\mathcal{L}'}{\sqrt{15c}} + \frac{16W_3 W_5'}{\sqrt{7c}} + \frac{128\mathcal{L} W_4}{7\sqrt{15c}} + \frac{W_4^{(3)}}{21\sqrt{15}} \right) \tag{C.1f}
\end{equation}
\[ W_4(z_1)W_5(z_2) \sim \]
\[ \frac{\sqrt{15}}{7} \left( + \frac{3W_3}{z_{12}^6} + \frac{W_5^c}{z_{12}^5} \right) \]
\[ + \frac{1}{z_{12}^2} \left( \frac{198\sqrt{15}W_3 \mathcal{L}'}{49c} + \frac{3}{98} \sqrt{15}W_3^4 - \frac{11}{14} \frac{\sqrt{15}W_5}{7} \right) \]
\[ + \frac{1}{z_{12}} \left( \frac{96\sqrt{15}W_3 \mathcal{L}'}{49c} + \frac{68\sqrt{15}W_3 \mathcal{L}'}{49c} - \frac{11}{7} \frac{3}{35} W_5 - \frac{1}{196} \sqrt{15}W_3^{(3)} \right) \]
\[ + \frac{1}{z_{12}} \left( \frac{4896}{49c^2} \frac{\sqrt{3}}{5} W_3 \mathcal{L}^2 + \frac{397}{196c} \frac{\sqrt{3}}{5} \mathcal{L}' W_3^4 + \frac{89}{98c} \frac{\sqrt{3}}{5} \mathcal{L}'' W_3^4 + \frac{563}{196c} \frac{\sqrt{3}}{5} W_3 \mathcal{L}'' \right) \]
\[ - \frac{58}{7c} \frac{\sqrt{15}W_3 \mathcal{L}}{7c} + \frac{74W_3W_4}{\sqrt{7}c} - \frac{1}{14} \frac{\sqrt{15}}{7} W_5^4 + \frac{5}{2352} \frac{\sqrt{3}}{5} W_3^{(4)} \]
\[ + \frac{1}{z_{12}^2} \left( \frac{1728}{49c^2} \frac{\sqrt{3}}{5} W_3 \mathcal{L}^2 + \frac{4752}{49c^2} \frac{\sqrt{3}}{5} W_3 \mathcal{L}' \right) + \frac{97}{98c} \frac{\sqrt{3}}{5} W_3^{(3)} \mathcal{L}'' + \frac{29}{196c} \frac{\sqrt{15} \mathcal{L}' \mathcal{L}''}{196c} \]
\[ - \frac{118}{7c} \frac{\sqrt{3}}{35} \mathcal{L} W_3^4 - \frac{1}{c} \frac{\sqrt{15}}{7} \mathcal{L}_1 W_5^4 + \frac{24}{196c} \frac{W_3W_4}{7c} + \frac{5}{98c} \frac{\sqrt{3}}{5} W_3^{(3)} \mathcal{L} \]
\[ + \frac{123}{196c} \frac{\sqrt{3}}{5} W_3 \mathcal{L}^{(3)} - \frac{1}{28} \frac{5}{21} \frac{W_3^{(5)}}{W_5^{(3)}} \]  

\[ W_5(z_1)W_5(z_2) \sim \]
\[ \frac{c}{z_{12}^{10}} + \frac{2L}{z_{12}^7} + \frac{L'}{z_{12}^{12}} \]
\[ + \frac{1}{z_{12}^2} \left( \frac{52L^2}{5c} - \frac{22}{7} \frac{\sqrt{3}}{35} W_4 + \frac{3L''}{10} \right) \]
\[ + \frac{1}{z_{12}} \left( \frac{52L \mathcal{L}'}{5c} - \frac{11}{7} \frac{3}{35} W_4 + \frac{L^{(3)}}{15} \right) \]
\[ + \frac{1}{z_{12}^2} \left( \frac{1312L^3}{35c^3} - \frac{73L \mathcal{L}'}{28c} + \frac{109L \mathcal{L}''}{35c} - \frac{748W_4 \mathcal{L}}{7\sqrt{105}c} + \frac{181W_3^2}{14c} - \frac{11}{42} \frac{\sqrt{5} W_4'}{21} + \frac{L^{(4)}}{84} \right) \]
\[ + \frac{1}{140c} \left( \frac{1968L^2 \mathcal{L}'}{35c^2} + \frac{219L \mathcal{L}''}{140c} - \frac{374W_4 \mathcal{L}'}{7\sqrt{105}c} - \frac{374W_4 \mathcal{L}'}{7\sqrt{105}c} + \frac{181W_4 W_3}{14c} \right) \]
\[ + \frac{29L^{(3)}L}{42c} - \frac{11W_4}{2} \frac{L^{(3)}}{2105} + \frac{L^{(5)}}{560} \]
\[ + \frac{1}{z_{12}^2} \left( \frac{18432L^4}{175c^3} + \frac{592L \mathcal{L}'}{21c^2} + \frac{8824L^2 \mathcal{L}''}{525c^2} - \frac{2201W_4 \mathcal{L}^2}{35\sqrt{105}c^2} + \frac{992W_3^2 \mathcal{L}}{7c^2} \right) \]
\[ + \frac{35(W_4')^2}{12c} - \frac{188L' W_4'}{7\sqrt{105}c} + \frac{25L^{(3)}L'}{72c} - \frac{31L W_4' \mathcal{L}'}{21\sqrt{105}c} + \frac{47(L\mathcal{L}')^2}{200c} - \frac{557W_4 \mathcal{L}''}{35\sqrt{105}c} \]
\[ + \frac{111W_3 W_3'}{28c} + \frac{31 L^{(4)} L}{252c} + \frac{448W_4^2}{15c} - \frac{10W_4}{24\sqrt{105}c} - \frac{W_4}{4320} \]
$$\begin{align*}
&+ \frac{1}{z_{12}} \left( 36864 \mathcal{L}^3 \mathcal{L}' - \frac{11008 \mathcal{L}^2 \mathcal{W}_4'}{35 \sqrt{105} c^2} - \frac{22016 \mathcal{L} \mathcal{L}'^2}{35 \sqrt{105} c^2} + \frac{496 \mathcal{W}_2 \mathcal{L}'^2}{7 c^2} + \frac{496 (\mathcal{L}')^3}{105 c^2} \right) \\
&+ \frac{1952 \mathcal{L}^2}{525 c^2} + \frac{8864 \mathcal{L}' \mathcal{L}''}{525 c^2} + \frac{992 \mathcal{W}_2 \mathcal{W}_3'}{7 c^2} + \frac{5 \mathcal{W}_2 \mathcal{W}_3'}{3 c} - \frac{30 \mathcal{W}_3 \mathcal{W}_3'}{\sqrt{7} c} + \frac{448 \mathcal{W}_4 \mathcal{W}_4'}{15 c} \\
&- \frac{30 \mathcal{W}_3 \mathcal{W}_3'}{\sqrt{7} c} + \frac{19 \mathcal{W}_3 \mathcal{W}_3'}{21 c} + \frac{29 \mathcal{L}' \mathcal{L}''}{1575 c} - \frac{308 \mathcal{W}_4 \mathcal{L}^{(3)}}{105 \sqrt{105} c} - \frac{281 \mathcal{W}_4 \mathcal{L}'^{(3)}}{35 \sqrt{105} c} - \frac{157 \mathcal{W}_4 \mathcal{L}'}{21 \sqrt{105} c} \\
&+ \frac{13 \mathcal{L}^{(4)} \mathcal{L}'}{210 c} + \frac{47 \mathcal{L}^{(3)} \mathcal{L}''}{450 c} - \frac{62 \mathcal{W}_4^{(3)} \mathcal{L}}{21 \sqrt{105} c} - \frac{\mathcal{W}_4^{(5)}}{180 \sqrt{105}} + \frac{\mathcal{L}^{(7)}}{37800} \right) \quad (C.1h)
\end{align*}$$

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