General Structure of Conformal Anomaly and 4 Dimensional Photon-Dilaton Gravity

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Abstract
The general structure of the conformal anomaly and the dilaton’s effect to the anomaly are analysed. First we give a new formal proof of the statement that the conformal anomaly, in the theory which is conformal invariant at the classical level, is conformal invariant. The heat-kernel regularization and Fujikawa’s method are taken for the analysis. We present a new explicit result of the conformal anomaly in 4 dimensional photon-dilaton gravity. This result is examined from the point of the general structure.

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1 Introduction

The conformal anomaly is a common phenomenon in the quantum field theories. It means the conformal symmetry is inevitably broken by the quantization procedure even if we start from the conformally invariant theory at the classical level. The quantization generally requires some mass scale in order to regularize the theory, which causes the anomaly. Many explicit calculations have been performed and they indicate the existence of some structure in the anomaly. It has been noticed and analysed since a long time ago[1]. Several years ago, Deser and Schwimmer[2] argued that the conformal anomaly are made of three types of terms: A) the Euler term; B) the conformal invariants; C) "trivial" terms. They took the dimensional regularization and analysed the effective action at the 1-loop perturbative calculation. The formal proof, however, does not seem to exist so far. We examine the problem in a different approach. Firstly we adopt the heat-kernel regularization for the ultraviolet divergences. Secondly the anomaly is taken from the Jacobian factor under the change of integration variables (Fujikawa's method[3, 4]). It is known that this method makes the anomaly calculation clear and all known anomalies are succinctly reproduced[5, 9]. Based on these, we give a formal proof about the conformal invariance of the conformal anomaly up to "trivial" terms(defined in Sec.2).

We present an explicit new result of the conformal anomaly in the photon-dilaton gravity. The dilaton and the gravitational fields are treated as the background field. We examine the result from the point of the conformal symmetry. Although the dilaton is a popular field in the string theory, its true role in the field theory still remains obscure. It has the characteristic interaction of the type $e^{c\text{onst} \times \phi}$. Recently, in the 2 dimensional(dim) simple model, some interesting aspect of the dilaton is clarified in relation to the Weyl anomaly and the Hawking radiation [6, 7, 8]. In the present paper, we examine the 4 dim case. It turns out that the dilaton field behaves in good harmony with the gravitational field.
2 Conformal Invariance of the Conformal Anomaly

Let us consider a conformally invariant action $S[\varphi; g]$ in the $n$-dim Euclidean space with the following quadratic structure with respect to (w.r.t.) a matter field.

$$e^{\Gamma[g]} = \int \mathcal{D}\varphi \ e^{S[\varphi; g]} \quad , \quad S[\varphi; g] = \int d^n x \ \varphi \tilde{D}_g \varphi \quad ,$$

where $\varphi$ and $g_{\mu\nu}$ are the quantum matter field (appropriately scaled, we consider the scalar field for simplicity) and the background metric respectively. $\Gamma[g]$ is the Wilsonian effective action. $\tilde{D}_g$ is an elliptic differential operator of the system (we call it the system operator). Free fields (scalar, spinor, vector, etc.), on the curved background space, all have the above structure essentially. $S[\varphi; g]$ is assumed to be invariant under the Weyl (conformal) transformation.

$$\varphi' = W_\alpha(x) \varphi \quad , \quad g_{\mu\nu}' = e^{2\alpha(x)} g_{\mu\nu} \quad ,$$

$$S[\varphi'; g'] = S[\varphi; g] \quad , \quad W_\alpha \tilde{D}_g W_{\alpha} = \tilde{D}_g \quad ,$$

where $\alpha(x)$ is the Weyl transformation parameter. Generally $W_\alpha$ has the form $W_\alpha(x) \propto e^{\text{p} \alpha(x)}$, with a constant $p$ depending on each matter field.  

Let us evaluate the variation of $\Gamma[g]$, under the Weyl transformation, as

$$e^{\Gamma[g']} e^{\Delta S[g']} = \int \mathcal{D}\varphi' e^{S'[\varphi'; g'] + \Delta S'[g']} = \int \mathcal{D}\varphi \det \frac{\partial \varphi'}{\partial \varphi} e^{S[\varphi; g] + \Delta S[\varphi; g']} \quad ,$$

where we have introduced the counterterm action $\Delta S[g]$ which depends only on the background metric. Following Fujikawa, we identify the Jacobian $\det \frac{\partial \varphi'}{\partial \varphi} = \exp \text{Tr} \ln \{W_\alpha(x)\delta^n(x - y)\} = \exp \text{Tr} \{p \alpha(x)\delta^n(x - y) + O(\alpha^2)\}$ as the conformal anomaly and regularize it by the heat-kernel for the system operator $\tilde{D}_g$.

$$\Gamma[g'] + \Delta S[g'] - \Gamma[g] - \Delta S[g] = p \text{ Tr} \alpha(x)\delta^n(x - y) + \Delta S[g'] - \Delta S[g] + O(\alpha^2)$$

$$= \lim_{t \rightarrow +0} \{p \text{ Tr} \alpha(x) < x|e^{-t\tilde{D}_g}|y> + \Delta S[g'] - \Delta S[g] \} + O(\alpha^2) \quad ,$$

\[2\] The suffix 'a' of $W_\alpha$ shows its dependence on $\alpha(x)$, not the field index.
where $\alpha(x)$ is considered infinitesimal and $t$ is the proper time which plays the role of the ultraviolet regularization. Finally the conformal anomaly is given by

$$\text{Anomaly} \equiv \frac{\delta}{\delta \alpha(x)} (\Gamma[g'] + \Delta S[g'] - \Gamma[g] - \Delta S[g])|_{\alpha=0}$$

$$= \lim_{t \to +0} \{ p \, \text{tr} \, G_g(x, x; t) + \frac{\delta}{\delta \alpha(x)} \Delta S[g']|_{\alpha=0} \} , \quad G_g(x, y; t) = \langle x | e^{-t\tilde{D}_g} | y \rangle , \quad (5)$$

where $\text{tr} \, G_g(x, x; t)$ has pole terms proportional to $1/t^s$ ($s = n/2, n/2 - 1, \ldots, 1$) as $t \to 0$, which are subtracted by $\Delta S$-terms through $\frac{\delta}{\delta \alpha(x)} \Delta S[g']|_{\alpha=0}$, i.e., the renormalization of some massive parameters such as the cosmological constant and the gravitational constant. The $t^0$-part of $\text{tr} \, G_g(x, x; t)$ is the conformal anomaly. Some terms of it can be subtracted by the finite part of $\Delta S$ through $\frac{\delta}{\delta \alpha(x)} \Delta S[g']|_{\alpha=0}$. They are called ”trivial” (anomaly) terms. (”Trivial” terms are defined by those which can be obtained by the Weyl transformation of local actions.) It is the advantage of the present approach that the anomaly is so succinctly expressed as in (5). We will now prove the conformal invariance of the anomaly using this expression.

In the heat-kernel equation of (5), first we replace $g_{\mu\nu}$ with the Weyl-transformed one $g_{\mu\nu}'$ of (2) and then we multiply it by $W_\alpha(x)$ from ”left” and by $W_\alpha(y)$ from ”right”.

$$(W_\alpha(x)^2 \frac{\partial}{\partial t} + W_\alpha(x) \tilde{D}_g'(x) W_\alpha(x)) W_\alpha(x)^{-1} G_g'(x, y; t) W_\alpha(y) = 0 , \quad (6)$$

where $W_\alpha(x) \tilde{D}_g'(x) W_\alpha(x)$ is the Weyl-transformed operator, therefore the operand $W_\alpha(x)^{-1} G_g'(x, y; t) W_\alpha(y) \equiv G'(x, y; t)$ can be considered to be the Weyl-transformed heat-kernel. In particular the transformed heat-kernel $G'(x, y; t)$ satisfies the boundary condition: $\lim_{t \to +0} G'(x, y; t) = \delta^n(x - y)$. Using the conformal invariance of the system operator $W_\alpha(x) \tilde{D}_g'(x) W_\alpha(x) = \tilde{D}_g(x)$ of (2), $G'$ satisfies

$$(W_\alpha(x)^2 \frac{\partial}{\partial t} + \tilde{D}_g(x)) G'(x, y; t) = 0 . \quad (7)$$
Introducing a new point $z$ near $x$, we can rewrite Eq.(7) as

$$z \to x, \quad (W_\alpha(z)^2 \frac{\partial}{\partial t} + \bar{\mathcal{D}}_g(x)) G'(x, y; t)$$

$$= (W_\alpha(z)^2 - W_\alpha(x)^2) \frac{\partial}{\partial t} G'(x, y; t) = f(z, x) \frac{\partial}{\partial t} G'(x, y; t),$$

where $f(z, x) \equiv W_\alpha(z)^2 - W_\alpha(x)^2$, $f(x, x) = 0$. Instead of (7), we solve (8) for $z \to x$. Because $\lim_{z \to x} f(z, x) = 0$, we can treat the term of $f(z, x) \frac{\partial}{\partial t} G'(x, y; t)$ as a small perturbation so far as $x \neq y$. Comparing the heat-kernel equations in (5) and (8), we see

$$G'(x, y; t) = \lim_{z \to x} \{ G_g(x, y; \frac{t}{W_\alpha(z)^2}) + O(f) \}$$

where $O(f)$ means the contribution from $f(z, x) \frac{\partial}{\partial t} G'(x, y; t)$. If we take the limit $|z - x| \to +0$ before the limit $|x - y| \to +0$, the $O(f)$ term vanish.

In this case, this equation shows that $t^0$-part of $\text{tr} G_g(x, x; t)$, which is the conformal anomaly up to ”trivial” terms, is conformal-invariant. It is well-established that $t^0$-part of $\text{tr} G_g(x, x; t)$ corresponds to the log-divergent part of the quantum system of $\bar{\mathcal{D}}$, say, in the dimensional regularization, so this result is consistent with Ref.[2]'s characterization of B-type terms. So far we have analysed only local properties of $\text{tr} G_g(x, x; t)$. We can not exclude the topological term (Euler term) in it, because it does not depend on any local quantity. This completes a formal proof of the general structure of Weyl anomaly stated in Sec.1. The explicit calculations, such as the scalar-gravity theory, show its validity.

In the case of the gauge theory on the curved background, the above proof can not be applied directly because we have to introduce the gauge-fixing term and which breaks the conformal invariance of the action. We notice, however, the gauge-fixed action combined with the ghost term is conformal invariant up to a ”BRST-trivial” term (see Sec.3). Therefore we

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3 We can regard this limiting procedure as a sort of (ultraviolet) regularization.

4 This proof is contrasting with the case of ”gauge-invariance” of the heat-kernel where the invariance is valid for all powers of $t$.

5 From the point of conformal group symmetry on the flat space ( SO(4,2)-symmetry ) , it was already noted that the gauge-fixing procedure requires some nonlocal terms in the BRST transformation if the gauge theory is BRST-quantized keeping the conformal symmetry of the action at the classical level.
can still expect that the conformal invariance is kept for the case of the gauge theory. Explicit calculations of some theories show it is indeed the case. The conformal anomaly of the 4 dim free photon on the curved background was obtained by [14, 15] in the zeta-function regularization as

\[
\text{Anomaly(photino,no dilaton) = } \frac{1}{(4\pi)^2} \frac{1}{180}(-31\sqrt{g}E + 18I_0 + 18J_0) .
\] (10)

It is made of the Euler term \(\sqrt{g}E\) with

\[
E = \frac{1}{4} R_{\mu\nu\alpha\beta} R_{\lambda\sigma\gamma\delta} \epsilon^{\mu\nu\lambda\sigma} \epsilon^{\alpha\beta\gamma\delta} = R^2 + R_{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R_{\mu\nu} ,
\] (11)

and a conformal invariants \(I_0\) defined by

\[
I_0 \equiv \sqrt{g} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = \sqrt{g}(-2R_{\mu\nu} R_{\mu\nu} + R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \frac{1}{3} R^2) , I'_0 = I_0 .
\] (12)

where \(C_{\mu\nu\alpha\beta}\) is the Weyl tensor, and a "trivial" term \(J_0\) defined by

\[
J_0 \equiv \sqrt{g} \nabla^2 R , K_0 \equiv \sqrt{g} R^2 , g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}(x)} \int d^4 x K_0 = 6J_0 .
\] (13)

3 Trace anomaly of 4D free photon on the background dilaton-gravity

It is very interesting to know how the dilaton affects the structure of the conformal anomaly. The non-gauge case was already analysed in Ref.\[7\] where the scalar-dilaton-gravity theory, \(S_S = \frac{1}{2} \int d^4 x \sqrt{g} e^{P(\phi)} \varphi(-\nabla^2 - \frac{1}{6} R) \varphi,\) \((\phi: \text{dilaton}; \varphi: \text{scalar}; P(\phi) \text{ is an arbitrary function of } \phi)\) is taken. Using another conformal invariant \(I_4\) defined by

\[
I_4 \equiv \sqrt{g}(P_{\mu}P^{\mu})^2 , \quad I'_4 = I_4 ,
\] (14)

and another "trivial" term \(J_{2a}\) defined by

\[
J_{2a} \equiv \sqrt{g} \nabla^2 (P_{\mu}P^{\mu}) , \quad K_{2a} \equiv \sqrt{g} R P_{\mu}P^{\mu} , \quad g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}(x)} \int d^4 x K_{2a} = +3J_{2a} ,
\] (15)
besides \( \sqrt{g} E, I_0, J_0 \) defined in Sec.2, the anomaly is given by

\[
\text{Anomaly(scalar)} = \frac{1}{(4\pi)^2}\left\{ \frac{1}{360}(3I_0 - \sqrt{g} E) + \frac{1}{32}I_4 - \frac{1}{180}J_0 - \frac{1}{24}J_{2a} \right\} . \tag{16}
\]

The general structure given in Sec.2 is strictly obeyed in this case. In the notation \( I_0, I_4, \cdots \), in the above and in Sec.2, the numbers in the lower suffixes show the power numbers of \( P \).

Let us consider the photon coupled system, instead of the scalar field, and examine whether the anomaly keeps this general structure or not. This is the gauge theory and we expect a new aspect will appear due to the gauge-fixing procedure. The action is given by

\[
S_V = \int d^4x \sqrt{g}\left(-\frac{1}{4}e^{P(\phi)}F_{\mu\nu}F^{\mu\nu}\right) , \quad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{17}
\]

Besides the 4 dim general coordinate symmetry, this theory has the local Weyl symmetry,

\[
g'_{\mu\nu} = e^{2\alpha(x)}g_{\mu\nu} , \quad A_\mu : \text{fixed} , \quad \phi : \text{fixed} , \tag{18}
\]

and has the Abelian local gauge symmetry.

\[
A'_\mu = A_\mu + \nabla_\mu \Lambda(x) , \quad g_{\mu\nu} : \text{fixed} , \quad \phi : \text{fixed} . \tag{19}
\]

The gauge-fixing term and the corresponding ghost lagrangian are

\[
S_{gf} = \int d^4x \sqrt{g}\left(-\frac{1}{2}e^{P(\phi)}(\nabla_\mu A^\mu)^2\right) , \quad S_{gh} = \int d^4x \sqrt{g}e^{P(\phi)}\bar{c}\nabla_\mu \nabla^\mu c , \tag{20}
\]

where \( \bar{c} \) and \( c \) are the anti-ghost and ghost fields respectively. They are hermitian scalar fields with Fermi statistics (Grassmann variables). The total action \( S[A, \bar{c}, c; g, \phi] = S_V + S_{gf} + S_{gh} \), is invariant for the BRST symmetry.

\[
\begin{align*}
\delta A_\mu &= \xi \partial_\mu c \equiv \xi \hat{s}A_\mu , \\
\delta \bar{c} &= \xi \nabla_\mu A_\mu \equiv \xi \hat{s}\bar{c} , \\
\delta c &= 0 \equiv \xi \hat{s}c , \\
\delta \phi &= 0 \equiv \xi \hat{s}\phi , \\
\delta g_{\mu\nu} &= 0 \equiv \xi \hat{s}g_{\mu\nu} ,
\end{align*} \tag{21}
\]

where \( \xi \) is the BRST parameter (Grassmann,global) and \( \hat{s} \) is introduced as the BRST operator. \( S_{gf} \) breaks the Weyl symmetry (18), besides the gauge symmetry (19). Therefore we must take close care to define the conformal
anomaly generally in the gauge theories. It is non-trivial whether the general structure suggested by Ref. [2] holds true.

We can fix the Weyl transformation of $\bar{c}$ and $c$ \[13\] as

$$c' = e^{-2\alpha(x)}\bar{c} \quad , \quad c' = c,$$

from two requirements: (i) the ghost action $S_{gh}$, \[20\], is invariant for the global Weyl transformation ($\alpha(x)$ is independent of $x$); (ii) For the infinitesimal Weyl transformation, $S_{gf} + S_{gh}$ is conformal invariant up to "BRST-trivial" terms.

$$(S_{gf} + S_{gh})' - (S_{gf} + S_{gh}) = \int d^4x \sqrt{g} e^{P(\phi)} \{ -2\hat{s}(\delta^\lambda \alpha \cdot A_\lambda) + O(\alpha^2) \}.$$ \(23\)

After rescaling the vector and ghost fields (for normalizing the kinetic term): $e^{P(\phi)}A_\mu = B_\mu$, $e^{P(\phi)}\bar{c} = \bar{b}$, the actions can be expressed as

$$S_V + S_{gf} = \int d^4x \left[ \frac{1}{2} \sqrt{g} B_\mu \{ g_{\mu\nu} \nabla^2 + (N^\lambda)_\mu \nabla_\lambda + M_{\mu\nu} \} B^\nu + \text{total deri.} \right],$$

$$(N^\lambda)_{\mu\nu} = P_{\mu\nu}\delta^\lambda - P_{\nu\mu}\delta^\lambda, \quad M_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}(P_{\lambda\nu}P^{\lambda} + 2\nabla^2P),$$

$$S_{gh} = \int d^4x \sqrt{|\delta|} \nabla^2 c.$$ \(24\)

As shown by Fujikawa [16], we should adopt the BRST(w.r.t. 4 dim general coordinate transformation)-invariant measure by taking $\tilde{B}_\mu, \tilde{\bar{b}}$ and $\tilde{c}$, instead of $B_\mu, \bar{b}$ and $c$ respectively, as the path-integral variables. \[\]

\[\tilde{B}_\mu \equiv g^{1/8}B_\mu \quad (\tilde{B}^\mu \equiv g^{3/8}B^\mu) \quad \tilde{\bar{b}} \equiv \sqrt{\frac{3}{2}} \bar{b}\quad \tilde{c} \equiv \sqrt{\frac{3}{2}} c,\]

$$S_V + S_{gf} = \int d^4x \left[ -\frac{1}{2} \tilde{B}_\mu \tilde{\nabla}^\mu \tilde{B}_\nu + \text{total deri.} \right],$$

$$\tilde{B}_\nu = -g^{1/8}\left\{ \delta_{\mu}^\nu \nabla^2 + (N^\lambda)_\mu \nabla_\lambda + M_{\mu\nu} \right\} g^{-1/8},$$

$$S_{gh} = -\int d^4x \tilde{\bar{b}} \tilde{\nabla} \tilde{c} \quad , \quad \tilde{d} \equiv -\sqrt{\frac{3}{2}} \nabla^2 \frac{1}{\sqrt{g}},$$

$$\exp\{\Gamma[g, \phi]\} = \int \mathcal{D}\tilde{B} \mathcal{D}\tilde{\bar{b}} \mathcal{D}\tilde{c} \exp(\tilde{S}[\tilde{B}, \tilde{\bar{b}}, \tilde{c}; g, \phi]) \quad ,$$

$$\tilde{S}[\tilde{B}, \tilde{\bar{b}}, \tilde{c}; g, \phi] = S[A, \bar{c}, c; g, \phi],$$ \(25\)

\[\text{The measure is also invariant under the BRST transformation w.r.t. the Abelian gauge symmetry: } \delta \tilde{B}_\mu = \xi e^{P(\phi)} \frac{1}{g} (\nabla^2 \tilde{B}_\mu - \frac{1}{8}g^{1/8}(\tilde{\nabla}^2 \tilde{B}_\mu), \delta \tilde{c} = 0, \det\{\partial(\tilde{p}^{\mu}B_\mu, \tilde{\bar{b}'}, \tilde{c}'\} / \partial(\tilde{p}^{\mu}B_\mu, \tilde{\bar{b}'}, \tilde{c}') = 1.\]
where \( \Gamma[g, \phi] \) is the (Wilsonian) effective action.

The Weyl transformation of \((18)\) and that of the integration variables 
\((\tilde{B}^\prime = e^\alpha \tilde{B}_\mu, \tilde{b}' = \tilde{b}, \tilde{c}' = e^{2\alpha} \tilde{c})\) give us the Ward-Takahashi identity for the \(\) Weyl transformation.

\[
\exp \Gamma[g', \phi] = \int \mathcal{D}\tilde{B}' \mathcal{D}\tilde{b} \mathcal{D}\tilde{c}' \exp \tilde{S}[\tilde{B}', \tilde{b}, \tilde{c}'; g', \phi]
\]
\[
= \int \mathcal{D}\tilde{B} \cdot \det \frac{\partial \tilde{B}'}{\partial \tilde{B}} \cdot \mathcal{D}\tilde{b} \mathcal{D}\tilde{c} \cdot \det \frac{\partial \tilde{c}'}{\partial \tilde{c}} \exp \tilde{S}[\tilde{B}', \tilde{b}, \tilde{c}'; g', \phi]
\]
\[
= \int \mathcal{D}\tilde{B} \mathcal{D}\tilde{b} \mathcal{D}\tilde{c} \cdot \det(e^{\alpha} \delta_{\mu}^{\nu} \delta^4(x - y)) \cdot \det(e^{2\alpha} \delta^4(x - y))
\]
\[
\times \exp \left[ \tilde{S}[\tilde{B}, \tilde{b}, \tilde{c}; g, \phi] + \delta \tilde{B}_\mu \frac{\delta \tilde{S}}{\delta \tilde{B}_\mu} + \delta \tilde{c} \frac{\delta \tilde{S}}{\delta \tilde{c}} + \delta g_{\mu\nu} \frac{\delta \tilde{S}}{\delta g_{\mu\nu}} + O(\alpha^2) \right] . \quad (26)
\]

Considering the infinitesimal transformation \((\delta g_{\mu\nu} = 2\alpha g_{\mu\nu}, \delta \tilde{B}_\mu = \alpha \tilde{B}_\mu, \delta \tilde{c} = 2\alpha \tilde{c})\) in the above, and regularizing the space delta-functions \(\delta^4(x - y)\) in terms of the heat-kernels, we obtain

\[
\Gamma[g', \phi] - \Gamma[g, \phi] = 2\alpha g_{\mu\nu} \frac{\delta \Gamma}{\delta g_{\mu\nu}} + O(\alpha^2)
\]

\[
= \alpha \left< \tilde{B}_\mu \frac{\delta \tilde{S}}{\delta \tilde{B}_\mu} \right> + 2\alpha \left< \tilde{c} \frac{\delta \tilde{S}}{\delta \tilde{c}} \right> + 2\alpha \left< g_{\mu\nu} \frac{\delta \tilde{S}}{\delta g_{\mu\nu}} \right>
\]
\[
+ \text{Tr} \ln\left\{ e^{\alpha} < x|e^{-t\tilde{D}_\mu}|y > \right\} - \text{Tr} \ln\left\{ e^{2\alpha} < x|e^{-d\tilde{d}}|y > \right\} + O(\alpha^2) \quad , \quad (27)
\]

where \( t \) is the regularization parameter: \( t \to +0 \). The variation part of \( \tilde{S} \) above gives the "naive" Ward-Takahashi identity, while the remaining two terms (terms with "Tr") give the deviation from it. Therefore the last two terms can be regarded as the Weyl anomaly in this theory.

\[
T_1 \equiv \frac{\partial}{\partial \alpha} \text{Tr} \ln\left\{ e^{\alpha} < x|e^{-t\tilde{D}_\mu}|y > \right\} \bigg|_{\alpha=0,t=0} ,
\]
\[
T_2 \equiv - \frac{\partial}{\partial \alpha} \text{Tr} \ln\left\{ e^{2\alpha} < x|e^{-d\tilde{d}}|y > \right\} \bigg|_{\alpha=0,t=0} . \quad (28)
\]

The anomaly formula for the differential operator \( \tilde{D}_\mu \nu = -g^{1/8}(\delta_\mu^{\nu} \nabla^2 + (N^\lambda)_\mu^{\nu} \nabla_\lambda + M^\mu_\nu)g^{-1/8} \) with arbitrary general covariants of \( N^\lambda \) and \( M \) is ,
taking the heat-kernel regularization and the Fujikawa method, given as

\[ T = \frac{1}{(4\pi)^2} \sqrt{g} \left\{ \text{Tr} \left[ \frac{1}{6} D^2 X + \frac{1}{2} X^2 + \frac{1}{12} Y_{\alpha\beta} Y^{\alpha\beta} \right] \right. \]
\[ + 4 \times \frac{1}{180} \left( R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - R_{\mu\nu} R^{\mu\nu} + 0 \times R^2 - \nabla^2 R \right) \}
\]
\[ X_{\mu}^{\nu} = M_{\mu}^{\nu} - \frac{1}{2} (\nabla_\alpha N^\alpha)_{\mu}^{\nu} - \frac{1}{4} (N^\alpha N_\alpha)_{\mu}^{\nu} - \frac{1}{6} \delta_{\mu}^{\nu} R , \]
\[ (Y_{\alpha\beta})_{\mu}^{\nu} = \left\{ \frac{1}{2} (\nabla_\alpha N_\beta)_{\mu}^{\nu} + \frac{1}{4} (N_\alpha N_\beta)_{\mu}^{\nu} - \alpha \leftrightarrow \beta \right\} + R_{\alpha\beta\mu}^{\nu} , \]
\[ (D_\alpha X)_{\mu}^{\nu} = \nabla_\alpha X_{\mu}^{\nu} + \frac{1}{2} [N_\alpha, X]_{\mu}^{\nu} , \]
\[ (D^2 X)_{\mu}^{\nu} = \nabla^2 X_{\mu}^{\nu} + \frac{1}{2} [N^\alpha, D_\alpha X]_{\mu}^{\nu} \]

This formula is the same as the counter-term formula of [17, 18] which was derived in the dimensional regularization. Putting all explicit equations into the formula, the vector contribution to the Weyl anomaly, \( T_1 \), is finally given by

\[ T_1 = \frac{1}{(4\pi)^2} \sqrt{g} \left\{ \frac{1}{180} (-11 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + 86 R_{\mu\nu} R^{\mu\nu} - 20 R^2 + 6 \nabla^2 R) \right. \]
\[ + \left\{ \frac{1}{16} (P_{\mu} P^{\mu})^2 + \frac{1}{12} \nabla^2 (P_{\mu} P^{\mu}) + \frac{5}{12} (\nabla^2 P)^2 - \frac{1}{6} P_{\mu\nu} P^{\mu\nu} \
\quad + \frac{2}{3} R_{\mu\nu} P_{\mu} P_{\nu} - \frac{1}{6} R P_{\mu} P^{\mu} \right\} \]
\[ \left. + \left\{ -\frac{1}{3} \nabla^4 P - \frac{1}{6} (P_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \nabla^2 P) \cdot P^{\mu\nu} + \frac{1}{3} (R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) P_{\mu\nu} \right\} \right\} . \tag{30} \]

Terms in the first big bracket (\{\}) show the even power part of \( P(\phi) \), while those in the second show the odd power part. The ghost contribution to the Weyl anomaly is obtained, by evaluating the heat-kernel with \( \tilde{d} \equiv -\sqrt{g} \nabla^2 1/\sqrt{g} \), as

\[ T_2 = \frac{1}{(4\pi)^2} \sqrt{g} \left\{ -2 \times \frac{1}{180} (-6 \nabla^2 R + R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - R_{\mu\nu} R^{\mu\nu} + \frac{5}{2} R^2) \right\} . \tag{31} \]

There are no P-terms in the ghost part. \( T_2 \) is the same as eq.(B.11) of [19]. \( T_1 + T_2 \) gives the final result of the Weyl anomaly. It is given by
the Euler term($\sqrt{gE}$), five Weyl invariants ($I_4, I_2, I_{1a}, I_{1b}, I_0$) and four trivial terms ($J_3, J_{2a}, J_{2b}, J_0$) which are defined and explained in the following.

$$\text{Anomaly(photons)} = \frac{1}{(4\pi)^2} \left\{ \frac{1}{180} (-31\sqrt{gE} + 18I_0 + 18J_0) ight\} + \frac{1}{16} I_4 + I_2 + \frac{1}{3} (I_{1a} - I_{1b}) - \frac{1}{12} J_3 + \frac{1}{12} J_{2a} + \frac{1}{3} J_{2b} \right\} , \quad (32)$$

The Weyl anomaly for the ordinary (without the dilaton) photon-graviton theory (Sec.2) is correctly given by the case of $P(\phi) = 0$.

In the above, we have new conformal invariants $I_2, I_{1a}, I_{1b}$ defined by

$$I_2 \equiv \sqrt{g}(R^{\mu\nu}P_{\mu}P_{\nu} - \frac{1}{6} R P_{\mu}P^{\mu} + \frac{3}{4} (\nabla^2 P)^2 - \frac{1}{2} P^{\mu\nu}P_{\mu\nu}) ,$$

$$I_{1a} \equiv \sqrt{g}(R^{\mu\nu}P_{\mu}P_{\nu} - \frac{1}{2} g^{\mu\nu} R)P_{\mu\nu} , \quad I_{1b} \equiv \sqrt{g}\nabla^2 P (= \sqrt{g}\nabla^4 P) . \quad (33)$$

They transforms under the finite Weyl transformation as

$$g'_{\mu\nu} = e^{2\alpha(x)} g_{\mu\nu} , \quad I_2' - I_2 = 2\sqrt{g}\nabla^\mu(\alpha^\nu P_{\mu}P_{\nu}) ,$$

$$I_{1a}' - I_{1a} = \sqrt{g}\nabla^\nu \{ 2(P_{\mu\nu}\alpha^\mu - \nabla^2 P \cdot \nabla_\nu \alpha) - (2\alpha^\mu \alpha_{,\nu} P_{\mu} + \alpha_{,\mu} \alpha^{,\mu} P_{\nu}) \} ,$$

$$I_{1b}' - I_{1b} = 2\sqrt{g} \{ \nabla^2 (\alpha^\mu P_{\mu} \cdot \nabla_\mu P_{\nu}) - \nabla_\nu (\alpha^\mu \nabla^2 P) \} - 4\sqrt{g}\nabla^\mu (\alpha_{,\mu} \alpha^{,\nu} P_{\nu}) . \quad (34)$$

$I_2, I_{1a}, I_{1b}$ are invariant up to the total derivative terms which should be compared with the exactly conformal invariants $I_0$ of (12) and $I_4$ of (14). This result implies that, even in the presence of the dilaton, the structure of the conformal anomaly given in Sec.2 can be valid if we generalize the conformal invariants up to the total derivative terms. In eq.(32), new "trivial" terms $J_3, J_{2b}$ also appear.

$$J_{2b} \equiv \sqrt{g}(R^{\mu\nu} P_{,\mu\nu} - P_{\mu\nu}P^{\mu\nu} - (\nabla^2 P)^2) ,$$

$$K_{2b} \equiv \sqrt{g}(P^{\mu\nu}P_{,\mu\nu} + \frac{1}{2}(\nabla^2 P)^2) , \quad g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}(x)} \int d^4x K_{2b} = +2J_{2b} ,$$

$$J_3 \equiv \sqrt{g}(\nabla^2 P \cdot P_{\mu}P_{,\mu} + 2P_{,\mu}^P P_{\mu}P_{,\mu}) ,$$

$$K_3 \equiv \sqrt{g} P^{\mu\nu}P_{,\mu\nu} , \quad g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}(x)} \int d^4x K_3 = \frac{1}{2} J_3 . \quad (35)$$
4 Conclusion

We have presented a formal proof of the conformal invariance of the conformal anomaly for the non-gauge case. An argument is made about its validity for the gauge case, based on the fact that the variation of the gauge fixed action, under the conformal transformation, is a "BRST-trivial" term. We have examined the dilaton’s effect on the above structure by explicitly calculating the conformal anomaly of the free photon on the dilaton-gravity background. In this case, the conformal anomaly is conformally invariant up to total derivative terms.

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