Quantum affine algebras at roots of unity
and equivariant K-theory

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March 5, 2018

Abstract

We show that the algebra homomorphism $U_q(\mathfrak{gl}_n) \to K^{GL_d \times C^*}(Z) \otimes C(q)$ constructed by Ginzburg and Vasserot between the quantum affine algebra of type $\mathfrak{gl}_n$ and the equivariant K-theory group of the Steinberg variety (of incomplete flags), specializes to a surjective homomorphism $U^t_{\text{eq}}(\mathfrak{gl}_n) \to K^{GL_d \times C^*}(Z)$. In particular, this shows that the parametrization of irreducible $U^t_{\text{eq}}(\mathfrak{gl}_n)$-modules and the multiplicity formulas of [7],[9] are still valid when $\epsilon$ is a root of unity.

We will use the following notations: let $n, d \in \mathbb{N}$. We set $V_d = \{(v_1, v_2, \ldots, v_n) \in \mathbb{N}^n \mid \sum_i v_i = d\}$; for $v \in V_d$ put $\bar{v} = v_1 + \cdots + v_i$ and identify $v$ with the point $\sum_i v_i e_i \in \mathbb{C}^n$ where $(e_i)$ is the canonical basis of $\mathbb{C}^n$. For $v, w \in V_d$, let

$$M(v, w) = \{ A = (a_{ij})_{i,j=1}^n \in \mathbb{N}^{n^2} \mid \sum_j a_{ij} = v_i, \sum_i a_{ij} = w_j \}$$

and $M = \bigsqcup_{v,w} M(v, w)$. For $l \geq 1$, we denote by $\mathcal{G}^l$ the symmetric group in $l$ variables and we put $\mathcal{G}^{(v)} = \mathcal{G}^{v_1} \times \cdots \times \mathcal{G}^{v_n} \subset \mathcal{G}^d$ for $v \in V_d$ and $\mathcal{G}^{(A)} = \mathcal{G}^{a_{11}} \times \mathcal{G}^{a_{21}} \times \cdots \times \mathcal{G}^{a_{nn}} \subset \mathcal{G}^d$ for $A \in M$. For all $v, w \in V_d$ the set $M(v, w)$ can be identified with $\mathcal{G}^{(v)} \backslash \mathcal{G}^d / \mathcal{G}^{(w)}$ (cf [7]). We will write $\leq$ for the Bruhat order on $\mathcal{G}^d$ and again $\leq$ for the induced order on $M(v, w)$. We set $R = \mathbb{C}[x_1^\pm \ldots, x_d^\pm, q^\pm]$ and $R^{(v)} = R\mathcal{G}^{(v)}$ and $R^{(A)} = R\mathcal{G}^{(A)}$. Finally, $\bar{X}$ will denote the Zariski closure of a subset $X$ of an algebraic variety.

1 K-theory of the Steinberg variety

Let $F$ be the variety of $n$-step flags in $\mathbb{C}^d$:

$$F = \{(D_i) \mid 0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n = \mathbb{C}^d\}$$

Then $F = \bigsqcup_{v \in V_d} F_v$ where $F_v$ is the connected component of $F$ consisting of flags $(D_i)$ satisfying dim $(D_i/D_{i-1}) = v_i$. The group $GL_d(\mathbb{C})$ acts diagonally on $F \times F = \bigsqcup_{v,w} F_v \times F_w$ and the corresponding orbits are parametrized by $M$: to $A \in M(v, w)$ corresponds

$$O_A = \{ ((D_i), (D'_j)) \mid \dim (D_i \cap D'_j) = \sum_{h \leq i, k \leq j} a_{hk} \} \subset F_v \times F_w$$

and we have $\bar{O}_A = \bigsqcup_{B \leq A} O_B$. For $A \in M$, denote by $Z_A = T^*_A(F \times F) \subset T^*F^2$ the conormal bundle, set $Z_{v,w} = \bigcup_{A \in M(v,w)} Z_A$, and $Z = \bigcup_{v,w} Z_{v,w}$ (the Steinberg variety). The group $G = GL_d \times \mathbb{C}^*$ acts on $Z$: $GL_d$ acts diagonally and $z \in \mathbb{C}^*$ acts by rescaling by $z^{-2}$ along the fibers of $T^*F^2$. Let $K^G(X)$ be the Grothendieck group of coherent $G$-equivariant sheaves on a $G$-variety $X$ and let $[\mathcal{H}]$ denote the class of a sheaf $\mathcal{H}$. Then $K^G(F_v) \simeq R^{(v)}$, $K^G(O_A) \simeq R^{(A)}$. For all $A \in M$, set $Z \leq A = \bigcup_{B \leq A} Z_B$ and $Z \leq A = \bigcup_{B \leq A} Z_B$. This filtration induces a filtration on $K^G(Z)$, whose associated graded is $\text{gr}(K^G(Z)) = \bigoplus_A K^G(Z_A) \simeq \bigoplus_A R^{(A)}$ (cf [7]).

The convolution product in equivariant K-theory $* : K^G(Z) \otimes K^G(Z) \to K^G(Z)$ equips $K^G(Z)$ with a $\mathbb{C}[q, q^{-1}]$-associative algebra structure which is compatible with the filtration (cf [7], chap.5).
Fix $a \in \mathbb{N}$ and let $v \in V_{d-a}$ be a composition of $d - a$ in $n$ parts. Set $E_{ij}(v; a) = \text{diag}(v) + aE_{ij} \in M(v + ae_i, v + ae_j)$. The orbits $O_{E_{i,\pm 1}(v; a)}$ are closed and the projection $p_1 : O_{E_{i,\pm 1}(v; 1)} \to F_{v+e_i}$ are smooth and proper, with fibers isomorphic to $\mathbb{P}^n$. Denote by $O_{p_1}(k)$ and $T^*p_1$ the $k$th Serre twist and the cotangent sheaf relative to the fibers of $p_1$. Finally let $E_{i,v,k}$ (resp. $F_{i,v,k}$) be the pullback of $\text{Det}(T^*p_1) \otimes O_{p_1}(k)$ on $Z_{E_{i,\pm 1}(v; 1)}$ (resp. on $Z_{E_{i,\pm 1}(v; 3)}$) which we view as $G$-equivariant sheaves on $Z$.

2 The quantum affine algebra

Let $U_q(\mathfrak{g}_n)$ be the quantum loop algebra of $\mathfrak{g}_n$ over $\mathbb{C}(q)$, with generators $E_{i,k}, F_{i,k}, K_j^{\pm 1}, K_{ij}$ for $i = 1, \ldots, n - 1$, $j = 1, \ldots, n$, $k \in \mathbb{Z}$ and $j \in \mathbb{Z}^n$, with the relations of Drinfeld’s new presentation (cf [8], we use the same notations as in [9]).

For $\lambda = \sum_{i=1}^n \lambda_i e_i \in \mathbb{Z}^n$ let $\pi_\lambda$ denote the projector on the space of weight $\lambda$: the operators $\pi_\lambda$ satisfy the following relations:

$$\pi_\lambda \pi_\mu = \delta_{\mu,\lambda} \pi_\mu, \quad K_i \pi_\lambda = \pi_\lambda K_i = K_{ij} \pi_\lambda = \pi_\lambda K_{ij} = q^{\lambda_i} \pi_\lambda,$$

$$E_{i,k} \pi_\lambda = \pi_{\lambda + e_i - e_{i+1}} E_{i,k}, \quad F_{i,k} \pi_\lambda = \pi_{\lambda - e_i + e_{i+1}} F_{i,k}. \tag{2}$$

By definition, the modified quantum loop algebra is $\tilde{U}_q(\mathfrak{g}_n) = \bigoplus_{\lambda} U_q(\mathfrak{g}_n)\pi_\lambda$ (cf [8]). The algebras $U_q(\mathfrak{g}_n)$ and $\tilde{U}_q(\mathfrak{g}_n)$ share the same finite-dimensional representation theory.

3 The map $\tilde{U}_q(\mathfrak{g}_n) \to K^{G_{L_d} \times \mathbb{C}^*}(Z) \otimes \mathbb{C}(q)$

**Theorem 1** (Ginzburg-Vasserot) The assignments

$$E_{i,k} \pi_{v+e_i} \mapsto (-q)^{-v} [E_{i,v,k}], \quad F_{i,k} \pi_{v+e_i} \mapsto (-q)^{1-v} [F_{i,v,k}] \tag{3}$$

for $v \in V_{d-1}$ and $E_{i,k} \pi_{v} \mapsto 0, F_{i,k} \pi_{v} \mapsto 0$ otherwise, and

$$(K_j^{\pm 1} + \sum_{l>0} K_{j,l+z^{-l}}^\pm) \pi_v \mapsto \prod_{m=1}^{v-1} \frac{q^2 z - x_m}{q^2 z - q x_m} \cdot \prod_{m=v}^{d} \frac{z - q^2 x_m}{q z - q x_m} \in K^G(F_v[[z^{-1}]]) \subset K^G(F)[[z^{-1}]] \tag{4}$$

for $v \in V_{d}$, extend to a surjective algebra homomorphism $\Phi : \tilde{U}_q(\mathfrak{g}_n) \to K^G(Z) \otimes \mathbb{C}(q)$.

In the last line, we identify $T^*F$ with the conormal bundle of the diagonal of $F \times F$ in $T^*F^2$ and we consider $K^G(F) \simeq K^G(T^*F) \subset K^G(Z)$. This theorem gives a geometric construction of all irreducible, finite-dimensional $U_q(\mathfrak{sl}_n)$-modules when $q$ is not a root of unity.

4 Restriction to integral forms

Let $U_q^{\text{res}}(\mathfrak{g}_n)$ denote the restricted integral form of $U_q(\mathfrak{g}_n)$ (cf [8]): $U_q^{\text{res}}(\mathfrak{g}_n)$ is the $\mathbb{C}[q,q^{-1}]$-algebra generated by $E_{ik}^{(m)} = \frac{e_i^m}{[m]!}, F_{ik}^{(m)} = \frac{e_i^m}{[m]!}, K_j^{\pm 1}$ and $H_j^{tr}$ where $H_j^{tr}$ is determined by the relation

$$\sum_{k \geq 1} H_{j, \pm k} z^\mp k = (q - q^{-1}) \log(1 + \sum_{l>0} K_j^{\mp 1} K_{j,\pm l} z^\pm l) \tag{5}$$

By definition, $\tilde{U}_q^{\text{res}}(\mathfrak{g}_n) = \bigoplus_{\lambda} U_q^{\text{res}}(\mathfrak{g}_n)\pi_\lambda$.

**Theorem 2** The map $\Phi : \tilde{U}_q(\mathfrak{g}_n) \to K^G(Z) \otimes \mathbb{C}(q)$ restricts to a surjective map $\tilde{\Phi} : \tilde{U}_q^{\text{res}}(\mathfrak{g}_n) \to K^G(Z)$.

The theorem is a consequence of the following two lemmas.

**Lemma 1** The map $\Phi$ restricts to a map $\tilde{\Phi} : \tilde{U}_q^{\text{res}}(\mathfrak{g}_n) \to K^G(Z)$.
Proof: it is clear that \( \Phi(K_{j}^{\pm 1} \pi_{v}) \in K^{G}(Z) \) for all \( v \in V \); a direct computation shows that

\[
\Phi\left( \frac{H_{j,r}}{\nu} \pi_{v} \right) = - \frac{1}{r} (q^{\mp r} \sum_{t=1}^{m-1} x_{t}^{\pm r} + q^{\pm r} \sum_{t=1}^{d} x_{t}^{\pm r}) \in K^{G}(Z) \tag{6}
\]

For \( v \) a composition of \( d - 1 \) in \( n \) parts, set \( v^{(l)} = v + l(e_{i+1} - e_{i}) \). Write \( \prod_{i=1}^{r} u_{i} \) for the ordered product \( u_{1}u_{2} \ldots u_{r} \). We have

\[
\Phi\left( E_{i,k}^{m} \pi_{v(m-1)+e_{i+1}} \right) = \Phi\left( E_{i,k} \pi_{v(0)}+e_{i} \right) \star \ldots \star \Phi\left( E_{i,k} \pi_{v(m-1)+e_{i+1}} \right) = \prod_{l=0}^{m-1} (-q)^{v_{l}-1}[\pi_{v(l),k}] .
\]

To see that \( \Phi(E_{i,k}^{m} \pi_{v(m-1)+e_{i+1}}) \in K^{G}(Z) \), it is enough to treat the case \( n = 2 \). Let \( v = (v_{1}, v_{2}) \), \( v_{1} + v_{2} = d - 1 \). Recall that orbits of type \( O_{E_{12}(v;a)} \) are closed. Using \([9]\), Lemme 12 et Exemple 4, we have

\[
[E_{1,v(l),k}] = x_{1-l+v_{1}}^{k} \prod_{t=1}^{v_{1}} \frac{x_{t}}{x_{1-l+v_{1}}} \in \mathbb{R}(E_{12}(v^{(l)};1)) = K^{G}(Z_{E_{12}(v^{(l)};1)})
\]

and the convolution product

\[
\star: K^{G}(Z_{E_{12}(v-l;1)}) \otimes K^{G}(Z_{E_{12}(v^{0};1)}) \to K^{G}(Z_{E_{12}(v-l+1;1)})
\]

can be written

\[
f \otimes g \to \mathcal{S}\left( I_{X} \right) \left( J_{I} \prod \frac{1-q^{2}x_{j}/x_{i}}{1-x_{i}/x_{j}} \right)
\]

where \( I = \{ v_{1} - l \} \), \( J = \{ v_{1} - l + 1, v_{1} + 1 \} \) and \( \mathcal{S}_{I_{X} \otimes J} \) is the symmetrization map \( \mathcal{R}^{\mathbb{S}_{1} \otimes \mathbb{S}_{j}} \to \mathcal{R}^{\mathbb{S}_{I} \otimes \mathbb{S}_{J}} \). It follows by induction that

\[
\prod_{l=0}^{m-1} [E_{1,v(l),k}] = q^{m(m-1)/2} \frac{[m]!x_{2-m+v_{1}}^{k} \ldots x_{1+v_{1}}^{k}}{1-m+v_{1}} \prod_{t=1}^{1-m+v_{1}} \frac{x_{t}^{m}}{x_{2-m+v_{1}}^{k} \ldots x_{1+v_{1}}} .
\]

Hence \( \Phi(E_{i,k}^{m} \pi_{v(m-1)+e_{i+1}}) \in K^{G}(Z) \). The case of \( E_{i,k}^{m} \pi_{\lambda} \) is similar.

We now show that \( \Phi \) is surjective. The algebra \( K^{G}(Z) \) is generated by sheaves supported on \( \overline{Z_{A}} \), where \( A \in M \) is diagonal or of the type \( E_{i,i+1}(v;a) \), \( a \in \mathbb{N} \) (cf. \([9]\), proof of proposition 11). From \([9]\), we deduce that \( K^{G}(Z_{A}) \subset \text{Im} \Phi \) for all diagonal \( A \). It thus remains to show the following:

**Lemma 2** We have \( K^{G}(Z_{A}) \subset \text{Im} \Phi \) for \( A = E_{i,i+1}(v;a) \).

Proof: We treat the case \( A = E_{i,i+1}(v;a) \) and proceed by induction on \( a \). We have \( K^{G}(Z_{E_{i,i+1}(v;a)}) \cong \mathcal{R}^{E_{i,i} \times \ldots \times E_{v_{1}} \times \ldots \times E_{v_{n}}} \). The algebra \( K^{G}(Z_{\text{diag}(v+a)}) \cong \mathcal{R}^{E_{v_{1}} \times \ldots \times E_{v_{a}} \times \ldots \times E_{v_{n}}} \) acts (by convolution) on \( K^{G}(Z_{E_{i,i+1}(v;a)}) \). For \( a = 1 \), the elements \( [E_{i,v}, k] \), \( k \in \mathbb{Z} \) form a generating system of the \( K^{G}(Z_{\text{diag}(v+1)}) \)-module \( K^{G}(Z_{E_{i,i+1}(v+1)}) \) (cf. \([9]\)). Now let \( a > 1 \) and suppose that \( K^{G}(Z_{E_{i,i+1}(v+1)}) \subset \text{Im} \Phi \) for all \( v \) and all \( b < a \). The \( K^{G}(Z_{\text{diag}(v+a)}) \)-module \( K^{G}(Z_{E_{i,i+1}(v+a)}) \) is generated by the subspace \( \mathbb{C}[y_{1}^{a+1} \ldots y_{a}^{a+1}] \mathcal{S}\alpha \) where we set \( y_{1} = x_{1+v_{1}} \). Let us denote by \( \mathcal{S}: \mathbb{C}[y_{1}^{a+1} \ldots y_{a}^{a+1}] \to \mathbb{C}[y_{1}^{a+1} \ldots y_{a}^{a+1}] \mathcal{S}\alpha \), the operator of complete symmetrization, so that we have \( \mathbb{C}[y_{1}^{a+1} \ldots y_{a}^{a+1}] \mathcal{S}\alpha = \mathbb{C}[y_{1}^{a+1} \ldots y_{a}^{a+1}] \mathcal{S}\alpha \) for \( \alpha = (\alpha_{i}) \subset \mathbb{Z}^{a} \). We show, by induction on \( \| \alpha \| = \sum_{i}(\alpha_{i} - \text{inf} \alpha)^{2} \), that \( \mathcal{S}(y_{1}^{a+1} \ldots y_{a}^{a+1}) \in \text{Im} \Phi \). Using \([8]\) with \( m = a \) we obtain

\[
\Phi(E_{i,v}^{(a)} \pi_{v(a-1)(a-1)+v_{1}+1}) = (-1)^{a} q^{a(a-1+1)} y_{1}^{a+1} \ldots y_{a}^{a+1} \prod_{l=1}^{a} \frac{x_{l}^{a+1}}{y_{1}^{a+1} \ldots y_{a}^{a+1}} .
\]

It follows that \( y_{1}^{a+1} \ldots y_{a}^{a+1} \in \bigcap_{k} K^{G}(Z_{\text{diag}(v+a)}) \Phi(E_{i,v}^{(a)} \pi_{v(a-1)(a-1)+v_{1}+1}) \subset \text{Im} \Phi \) for all \( k \in \mathbb{Z} \). Now let \( h > 0 \) and suppose that \( \mathcal{S}(y_{1}^{a+1} \ldots y_{a}^{a+1}) \in \text{Im} \Phi \) for all \( \alpha \) such that \( \| \alpha \| < h \). Fix some \( \alpha = (\alpha_{1} \ldots = \ldots \alpha_{a}) \subset \mathbb{Z}^{a} \)
\[ \alpha_s > \alpha_{s+1} \geq \ldots \geq \alpha_a, \| \alpha \| = h. \]

Denote by \( \mathcal{G}' : \mathbb{C}[\hat{y}^{\pm 1}_{a+1}, \ldots, \hat{y}^{\pm 1}_a] \to \mathbb{C}[y^{\pm 1}_{a+1}, \ldots, y^{\pm 1}_a]\) the operator of symmetrization and set

\[
P = \mathcal{G}'(y^{\alpha_{a+1}}_{a+1} \ldots y^{\alpha_a}_a) \prod_{u=a+1}^a y_u \in \mathbb{C}[y^{\pm 1}_{a+1}, \ldots, y^{\pm 1}_a]^{\mathcal{G}'-}\) \subset K^G(Z_{E_{a+1}(v+ae_i-M_a)}),
\]

\[
Q = y^{\alpha_1}_{1} \ldots y^{\alpha_s}_s \prod_{t=1}^s \frac{1}{y_t} \in \mathbb{C}[y^{\pm 1}_{1}, \ldots, y^{\pm 1}_s]\) \in K^G(Z_{E_{a+1}(v+(a-s)e_i-M_a)}).
\]

Then

\[
P \ast Q = \mathcal{G}(y^{\alpha_1}_{1} \ldots y^{\alpha_s}_s) \prod_{t=1}^s \prod_{u=a+1}^a (1 + (1 - q^2) \frac{y_u}{y_t - y_u}) \quad (9)
\]

Using the relation \((y^{\alpha_i}_i y^{\beta_i+1}_i y^{\beta_i}_i)/(y_t - y_u) = y^{\alpha_i-1}_i y^{\beta_i+1}_i + \ldots + y^{\beta_i+1}_i y^{\alpha_i-1}_i, \) noticing that \(((a-i)^2 + (\beta + i)^2 < \alpha^2 + \beta^2, \) if \( i > \alpha - \beta, \) \) it is easy to see that the r.h.s of (9) is equal to \( (a-s)\mathcal{G}(y^{\alpha_1}_{1} \ldots y^{\alpha_s}_s) + T \) where \( T \) is a linear combination of polynomials \( \mathcal{G}(y^{\alpha_1}_{1} \ldots y^{\alpha_s}_s) \) with \( \sigma_2(\beta) = \sigma_2(\alpha) = h. \)

Since \( a-s < a, \) it follows by the induction hypothesis that \( \mathcal{G}(y^{\alpha_1}_{1} \ldots y^{\alpha_s}_s) \in \text{Im } \Phi. \) Hence, \( \mathbb{C}[y^{\pm 1}_{1}, \ldots, y^{\pm 1}_s]\) \subset \text{Im } \Phi and \( K^G(Z_{E_{a+1}(v+a-M_a)}) \subset \text{Im } \Phi. \) This concludes the induction and the proof of the theorem. \( \blacksquare \)

There is a surjective morphism \( \Psi : U_q(\hat{\mathfrak{g}}_n) \to K^G(Z) \otimes \mathbb{C}(q) \) analogous to \( \Phi, \) (defined by \( \Psi(u) = \sum_{\lambda \in \mathbb{V}_q} \Phi(u \pi_\lambda) \) for \( u \in U_q(\hat{\mathfrak{g}}_n). \)) For \( \epsilon \in \mathbb{C}^*, \) let us set \( U_{\epsilon, q}^\text{res}(\hat{\mathfrak{g}}_n) = U_{\epsilon, q}^\text{res}(\hat{\mathfrak{g}}_n)|_{q=e} \) and \( K^\epsilon G(Z) = K^G(Z)_{|q=e}. \) Lemma 1 shows that \( \Psi \) restricts to a map \( \tilde{\Psi} : U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n) \to K^G(Z), \) and specializes to a morphism \( \tilde{\Psi} : U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n) \to K^G(Z). \)

**Corollary 1** For all \( \epsilon \in \mathbb{C}^*, \) the map \( \tilde{\Psi} : U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n) \to K^G(Z) \) is surjective.

Indeed, for all \( \epsilon \in \mathbb{C}^*, \) the projectors on weight subspaces \( \pi_\lambda \) can be realized inside \( U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n), \) i.e there exists \( \pi_\lambda \in U^\epsilon_{\epsilon, q}(\hat{\mathfrak{g}}_n) \) satisfying (2). Let us denote by \( i_\epsilon : U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n) \to U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n) \) the algebra map defined by \( i_\epsilon(u \pi_\lambda) = u \pi_\lambda \) for \( u \in U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n). \) We have \( \Phi_{\epsilon} = \tilde{\Psi} \circ i_\epsilon \) and the surjectivity of \( \Phi_{\epsilon} \) now follows from the surjectivity of \( \tilde{\Psi}. \) \( \blacksquare \)

Theorem 2 justifies the constructions of \( U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n) \)-modules and the Kazhdan-Lusztig multiplicity formulas of (2, 3) when \( \epsilon \) is a root of unity:

**Corollary 2** The parametrization of irreducible, finite-dimensional \( U^\text{res}_{\epsilon}(\hat{\mathfrak{g}}_n) \)-modules \( L_O \) (resp. standard modules \( M_O \)) in terms of graded nilpotent orbits \( O \) (cf (2)) and the multiplicity formula

\[
[M_O : L_O] = \sum_i \dim \mathcal{H}(IC_O)|_O
\]

are valid when \( \epsilon \) is a root of unity.

**Acknowledgments:** I would like to thank Eric Vasserot for his patience and precious advice.

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