Second-order corrections to the non-commutative Klein-Gordon equation with a Coulomb potential

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Abstract

We improve the previous study of the Klein-Gordon equation in a non-commutative space-time as applied to the Hydrogen atom to extract the energy levels, by considering the second-order corrections in the non-commutativity parameter. Phenomenologically we show that non-commutativity is the source of lamb shift corrections.

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1 Introduction

The non-commutative field theory has received a wide appreciation as an alternative approach to understanding many physical phenomena such as the ultraviolet and infrared divergences \([1]\), unitarity violation \([2]\), causality \([3]\), and new physics at very short distances of the Planck-length order \([4]\).

The non-commutative field theory is motivated by the natural extension of the usual quantum mechanical commutation relations between position and momentum, by imposing further commutation relations between position coordinates themselves. As in usual quantum mechanics, the non-commutativity of position coordinates immediately implies a set of uncertainty relations between position coordinates analogous to the Heisenberg uncertainty relations between position and momentum; namely:

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu
\nu},
\]

where \(\hat{x}^\mu\) are the coordinate operators and \(\theta^{\mu\nu}\) are the non-commutativity parameters of dimension of area that signify the smallest area in space that can be probed in principle. This idea is similar to the physical meaning of the Planck constant in the relation \([\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\), which as is known is the smallest phase-space in quantum mechanics.

One can study the physical consequences of this theory by making detailed analytical estimates for measurable physical quantities and compare the results with experimental data to find an upper bound on the \(\theta\) parameter. The most obvious natural phenomena to use in hunting for non-commutative effects are simple quantum mechanics systems, such as the hydrogen atom \([5,6,7]\). In the non-commutative space one expects the degeneracy of the initial spectral line to be lifted, thus one may say that non-commutativity plays the role of spin.

In this work we present an important contribution to the non-commutative approach to the hydrogen atom. Our goal is to solve the Klein Gorden equation for the Coulomb potential in a non-commutative space-time up to second-order of the non-commutativity parameter using the Seiberg-Witten maps and the Moyal product. We thus find the non-commutative modification of the energy levels of the hydrogen atom and we show that the non-commutativity is the source of lamb shift corrections.

This paper is organized as follows. In section 2 we derive the corresponding Seiberg-Witten maps up to the second order of \(\theta\) for the various dynamical fields, and we propose an invariant action of the non-commutative charged scalar field in the presence of an electric field. In section 3, using the generalised Euler-Lagrange field equation, we derive the deformed Klein-Gordon (KG) equation. Applying these results to the hydrogen atom, we solve the deformed KG equation and obtain the non-commutative modification of the energy levels. In section 4, we introduce the non-relativistic limit of the NC Klein–Gordon equations and solve them using perturbation theory and deduce that the non-relativistic NC Klein–Gordon equation is the same as Schrödinger equation on NC space. The last section is devoted to a discussion.
2 Seiberg-Witten maps

Here we look for a mapping $\phi^A \rightarrow \hat{\phi}^A$ and $\lambda \rightarrow \hat{\lambda}(\lambda, A_\mu)$, where $\phi^A = (A_\mu, \varphi)$ is a generic field, $A_\mu$ and $\varphi$ are the gauge and charged scalar fields respectively (the Greek and Latin indices denote curved and tangent space-time respectively), and $\lambda$ is the U(1) gauge Lie-valued infinitesimal transformation parameter, such that:

$$\hat{\phi}^A(A) + \hat{\delta}_\lambda \hat{\phi}^A(A) = \hat{\phi}^A(A + \delta_\lambda A),$$

where $\delta_\lambda$ is the ordinary gauge transformation and $\hat{\delta}_\lambda$ is a noncommutative gauge transformation which are defined by:

$$\hat{\delta}_\lambda \hat{\varphi} = i\lambda \hat{\varphi}, \quad \delta_\lambda \varphi = i\lambda \varphi,$$

$$\hat{\delta}_\lambda A_\mu = \partial_\mu \hat{\lambda} + i [\hat{\lambda}, \hat{A}_\mu], \quad \delta_\lambda A_\mu = \partial_\mu \lambda.$$

In accordance with the general method of gauge theories, in the non-commutative space, using these transformations one can get at second order in the non-commutative parameter $\theta^{\mu\nu}$ (or equivalently $\theta$) the following Seiberg–Witten maps [3]:

$$\hat{\varphi} = \varphi + \theta \varphi^1 + \theta^2 \varphi^2 + \mathcal{O} \left( \theta^3 \right),$$

$$\hat{\lambda} = \lambda + \theta \lambda^1 (\lambda, A_\mu) + \theta^2 \lambda^2 (\lambda, A_\mu) + \mathcal{O} \left( \theta^3 \right),$$

$$\hat{A}_\xi = A_\xi + \theta A_\xi^1 (A_\xi) + \theta^2 A_\xi^2 (A_\xi) + \mathcal{O} \left( \theta^3 \right),$$

$$\hat{F}_{\mu\xi} = F_{\mu\xi} (A_\xi) + \theta F_{\mu\xi}^1 (A_\xi) + \theta^2 F_{\mu\xi}^2 (A_\xi) + \mathcal{O} \left( \theta^3 \right),$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu].$$

To begin, we consider a non-commutative field theory with a charged scalar particle in the presence of an electrodynamic gauge field in a Minkowski space-time. We can write the action as:

$$S = \int d^4 x \left( \eta^{\mu\nu} \left( \hat{D}_\mu \hat{\varphi} \right)^\dagger \right. \hat{D}_\nu \hat{\varphi} + m^2 \hat{\varphi}^\dagger \hat{\varphi} - 1 \left. \frac{\hat{F}_{\mu\nu} \hat{F}^{\mu\nu}}{4} \right),$$

where the gauge covariant derivative is defined as: $\hat{D}_\mu \hat{\varphi} = (\partial_\mu + ie \hat{A}_\mu) \hat{\varphi}$.

Next we use the generic-field infinitesimal transformations (3) and (4) and the star-product tensor relations to prove that the action in eq. (10) is invariant. By varying the scalar density under the gauge transformation and from the generalised field equation and the Noether theorem we obtain [9]:

$$\frac{\partial L}{\partial \hat{\varphi}} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \hat{\varphi})} + \partial_\mu \partial_\nu \frac{\partial L}{\partial (\partial_\mu \partial_\nu \hat{\varphi})} - \partial_\mu \partial_\nu \partial_\sigma \frac{\partial L}{\partial (\partial_\mu \partial_\nu \partial_\sigma \hat{\varphi})} + \mathcal{O} \left( \theta^3 \right) = 0.$$
3 Non-commutative Klein-Gordon equation

In this section we study the Klein-Gordon equation for a Coulomb interaction \((-e/r)\) in the free non-commutative space. This means that we will deal with solutions of the U(1) gauge-free non-commutative field equations \([10]\). For this we use the modified field equations in eq. \((11)\) and the generic field \(\hat{A}_\mu\) so that:

\[
\delta \hat{A}_\mu = \partial_\mu \lambda - ie \hat{A}_\mu \ast \lambda + ie \lambda \ast \hat{A}_\mu, \tag{12}
\]

and the free non-commutative field equations:

\[
\partial^\mu \hat{F}_{\mu\nu} - ie \left[ \hat{A}_\mu, \hat{F}_{\mu\nu} \right] = 0. \tag{13}
\]

Using the Seiberg-Witten maps \([7]–[8]\) and the choice \((13)\), we can obtain the following deformed Coulomb potential \([10]\):

\[
\hat{a}_0 = -\frac{e}{r} + \frac{e^5}{20 r^5} (\theta^{ij})^2 + O (\theta^3), \tag{14}
\]

\[
\hat{a}_i = \frac{e^3}{4 r^4} \theta^{ij} x_j + O (\theta^3). \tag{15}
\]

Using the modified field equations in eq. \((11)\) and the generic field \(\hat{\varphi}\) so that:

\[
\delta \lambda \hat{\varphi} = i \lambda \ast \hat{\varphi}, \tag{16}
\]

the Klein-Gordon equation in a non-commutative space-time in the presence of the vector potential \(\hat{A}_\mu\) can be cast into:

\[
\left( \eta^{\mu\nu} \partial_\mu \partial_\nu - m^2 \right) \hat{\varphi} + \left( i e \eta^{\mu\nu} \partial_\mu \hat{A}_\nu - e^2 \eta^{\mu\nu} \hat{A}_\mu \ast \hat{A}_\nu + 2 i e \eta^{\mu\nu} \hat{A}_\mu \partial_\nu \right) \hat{\varphi} = 0. \tag{17}
\]

Now using the fact that:

\[
\eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial_0^2 + \Delta, \tag{18}
\]

and

\[
2 i e \eta^{\mu\nu} \hat{A}_\mu \partial_\nu = i \frac{e^2}{r} \partial_0 - i \frac{2 e^6}{20 r^5} (\theta^{ij})^2 \partial_0 - e^4 \frac{e^2}{2 r^4} \tilde{\theta} \tilde{L}, \tag{19}
\]

and

\[
- e^2 \eta^{\mu\nu} \hat{A}_\mu \ast \hat{A}_\nu = \frac{e^4}{r^2} - \frac{2 e^8}{20 r^6} (\theta^{ij})^2 - \frac{e^8}{16 r^8} (\theta^{ij} x_j)^2, \tag{20}
\]

where \(\tilde{L} = r \times p\) and \(\tilde{\theta} = (\theta_1, \theta_2, \theta_3)\). Setting that \(\theta^{ij} = \epsilon^{ijk} \theta_k\) \((\epsilon^{ijk}\) is the Levi-Civita simbol, which is sub-tensor of the third level, the lachting symmetry completely) then the Klein-Gordon equation \((17)\) up to \(O (\theta^3)\) takes the form:

\[
\left[ -\partial_0^2 + \Delta - m^2 - \frac{e^4}{r^2} + \frac{i e^2}{r} \partial_0 - \frac{e^4}{2 r^4} \tilde{\theta} \tilde{L} - \frac{e^8}{16 r^8} (\theta^{ij} x_j)^2 - i \frac{4 e^6}{20 r^5} \theta^2 \partial_0 - \frac{4 e^8}{20 r^6} \theta^2 \right] \hat{\varphi} = 0. \tag{21}
\]
The solution to eq. (21) in spherical polar coordinates \((r, \theta, \phi)\) takes the separable form [11]:

\[
\hat{\phi}(r, \theta, \phi, t) = \frac{1}{r} R(r) \tilde{Y}(\theta, \phi) \exp(-iEt).
\] (22)

Then eq. (21) reduces to the radial equation:

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1) - e^4}{r^2} + \frac{2Ee^2}{r} + E^2 - m_e^2 - \frac{e^4}{2r^4} \vec{\theta} \cdot \vec{L} - \frac{e^6}{5r^5} E\theta^2 - \frac{e^8}{16r^8} (\theta^{ij} x_j)^2 - \frac{e^8}{5r^6} \theta^2 \right] R(r) = 0. \tag{23}
\]

In eq. (23) the coulomb potential in non-commutative space appears within the perturbation terms:

\[
- \frac{e^4}{2r^4} \vec{\theta} \cdot \vec{\tilde{L}} - \frac{e^6}{5r^5} E\theta^2 - \frac{e^8}{16r^8} (\theta^{ij} x_j)^2 - \frac{e^8}{5r^6} \theta^2,
\] (24)

The first term is obtained in [10], the second, third and last terms are new corrections obtained from the seiberg-witten maps at second orders, and induce new splittings in the 1S state [10], which the non-commutative parameter \(\theta\) plays the role of the spin.

### 3.1 The solution

Equation (23) has not yet been solved exactly in the presence of the perturbation terms (24), whereas in their absence its exact solution is available in ref. [13]. To obtain the solution we choose \(\theta = 0\) and arrive at:

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1) - e^4}{r^2} + \frac{2Ee^2}{r} + E^2 - m_e^2 \right] R(r) = 0. \tag{25}
\]

This equation is a generalized equation of hypergeometric type [14],

\[
R'' + \frac{\tilde{\tau}(r)}{\sigma(r)} R' + \frac{\tilde{\sigma}(r)}{\sigma^2(r)} R = 0,
\] (26)

where \(\sigma(r) = r, \tilde{\tau} = 0, \tilde{\sigma} = (Er + e^2)^2 - r^2 m_e^2 - l(l+1)\). Using the transformation:

\[
R(r) = \phi(r) y(r),
\] (27)

equation (25) reduces to an equation of hypergeometric type:

\[
\sigma(r)y'' + \tau(r)y' + \lambda y = 0,
\] (28)

and \(\phi(r)\) is the solution of the equation:

\[
\frac{\phi'(r)}{\phi(r)} = \frac{\pi(r)}{\sigma(r)},
\] (29)
where the polynomial $\pi(r)$ is defined so that:

$$
\pi(r) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}
$$

$$
= \frac{1}{2} \pm \sqrt{\left(l + \frac{1}{2}\right)^2 - e^4 - 2e^2Er + (m_e^2 - E^2)r^2 + kr}.
$$

(30)

According to the Nikiforov-Uvarov (NU) method [14], the expression in the square–root must be the square of a polynomial, and so one can find new possible functions for each $k$ as:

$$
\pi(r) = \frac{1}{2} \pm \begin{cases}
\sqrt{m_e^2 - E^2r + (l + \frac{1}{2})^2 - e^4}, & \text{for } k = 2e^2E + 2\sqrt{m_e^2 - E^2}\sqrt{(l + \frac{1}{2})^2 - e^4} \\
\sqrt{m_e^2 - E^2r - (l + \frac{1}{2})^2 - e^4}, & \text{for } k = 2e^2E - 2\sqrt{m_e^2 - E^2}\sqrt{(l + \frac{1}{2})^2 - e^4}.
\end{cases}
$$

(31)

For all possible forms of $\pi(r)$ we must choose the one for which the function $\tau(r) = \tilde{\tau}(r) + 2\pi(r)$ has roots on the interval $[0, +\infty]$ and a negative derivative. These conditions are satisfied by the function:

$$
\tau(r) = 1 + 2\left(\sqrt{(l + \frac{1}{2})^2 - e^4} - \sqrt{m_e^2 - E^2r}\right),
$$

(32)

which corresponds to:

$$
\pi(r) = \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - e^4 - \sqrt{m_e^2 - E^2r}}
$$

$$
\lambda = 2\left[e^2E - \left(\frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - e^4}\right)\sqrt{m_e^2 - E^2}\right],
$$

(33)

and

$$
\phi(r) = r^{\frac{1}{2}} + \sqrt{(l + \frac{1}{2})^2 - e^4} \exp\left(-\sqrt{m_e^2 - E^2r}\right).
$$

(34)

The exact energy eigenvalues of the radial part of the Klein-Gordon equation with a Coulomb potential can be found from the equation:

$$
\lambda + n\tau' + \frac{n(n - 1)}{2}\sigma'' = 0,
$$

(35)

We obtain

$$
E = E_{n,l}^0 = \frac{m_e\left(n + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - \alpha^2}\right)}{\left[(n + \frac{1}{2})^2 + (l + \frac{1}{2})^2 + 2(n + \frac{1}{2})\sqrt{(l + \frac{1}{2})^2 - \alpha^2}\right]^{\frac{1}{2}}}, \quad \alpha = e^2.
$$

(36)
The proper function \( y(r) \) in eq. (28) is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation:

\[
y_n(r) = \frac{B_{nl}}{\alpha^2 e^{-2 \sqrt{m_e^2 - E^2} r}} \int_0^{\infty} t^{n+2 \sqrt{(l+\frac{1}{2})^2 - \alpha^2} - 2 \sqrt{m_e^2 - E^2} r} \, dt,
\]

where \( B_{nl} \) is a normalizing constant. Merging this with the Laguerre polynomials for \( x = 2 \sqrt{m_e^2 - E^2} r \), the radial functions are written as:

\[
R_{nl}(r) = C_{nl} \frac{\alpha}{n + \nu + 1} \left( \frac{n!}{\Gamma(n + 2 \nu + 2)} \right)^{1/2} x^{\nu + 1} e^{-x/2} L_n^{2\nu+1}(x),
\]

where

\[
\nu = -\frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 - \alpha^2},
\]

and \( a = \sqrt{m_e^2 - E^2} \).

Now, to obtain the modification of the energy levels as a result of the terms (24) due to the non-commutativity of space-time, we use the perturbation theory. To simplify, we take \( \theta_3 = \theta \) and assume that the other components are all zero, such that \( \bar{\theta} \bar{L} = \theta L_z \) and \((\bar{\theta} x_j)^2 = \theta^2 [(r^2 - z^2) - 2xy] \). In addition we use:

\[
\langle nlm \mid L_z \mid nlm' \rangle = m_l \delta_{mm'} \quad -l \leq m_l \leq l,
\]

and also the fact that in the first-order perturbation theory the expectation value of \( 1/r^4 \), \( 1/r^5 \) and \( 1/r^6 \) with respect to the exact solution of eq. (25), are given by:

\[
\langle nlm \mid r^{-k} \mid nlm' \rangle = \int_0^\infty R_{nl}(r) r^{-k} dr \delta_{mm'}
\]

\[
= \frac{2^k \alpha^k n!}{2 (n + \nu + 1) \Gamma(n + 2 \nu + 2)} \int_0^\infty x^{2\nu + 2 - k} e^{-x} \left[ L_n^{2\nu+1}(x) \right]^2 dx \delta_{mm'}
\]

\[
= f(k) \quad k = 3, 4, 5, 6.
\]

We use the relation between the confluent hypergeometric function \( F(-n; \nu + 1; x) \) and the associated Laguerre polynomials \( L_n^\nu(x) \), namely:

\[
L_n^\nu(x) = \frac{\Gamma(n + \nu + 1)}{\Gamma(n + 1) \Gamma(\nu + 1)} F(-n; \nu + 1; x),
\]

\[
\int_0^\infty x^{\nu-1} e^{-x} \left[ F(-n; \gamma; x) \right]^2 dx = \frac{n! \Gamma(\nu)}{\gamma (\gamma + 1) \cdots (\gamma + n - 1)} \left\{ 1 + \frac{n(\gamma - \nu - 1)(\gamma - \nu)}{12\gamma} + \frac{n(n - 1)(\gamma - \nu - 2)(\gamma - \nu - 1)(\gamma - \nu)(\gamma - \nu + 1)}{12 \cdot 2 \cdot \gamma (\gamma + 1)} + \cdots \right. \left. + \frac{n(n - 1) \cdots 1(\gamma - \nu - n)(\gamma - \nu - n + 1) \cdots (\gamma + n - 1)}{12 \cdot 2 \cdot \cdot \cdot n^2 \gamma (\gamma + 1) \cdots (\gamma + n - 1)} \right\}.
\]
Equation (42) becomes:

\[ \langle nlm \mid r^{-4} \mid nlm' \rangle = \int_0^\infty R_{nlm}^2(r)r^{-4}dr\delta_{mm'} \]

\[ = \frac{16a^4 n!}{2(n + \nu + 1)\Gamma(n + 2\nu + 2)} \int_0^\infty x^{2\nu+1-1}e^{-x} \left[ L_{n+1}^{2\nu+1}(x) \right]^2 dx\delta_{mm'} \]

\[ = \frac{8a^4 n!}{(n + \nu + 1)\Gamma(n + 2\nu + 2)} \left[ \frac{\Gamma(n + 2\nu + 2)}{\Gamma(n + 1)\Gamma(2\nu + 2)} \right]^2 \times \]

\[ \times \int_0^\infty x^{2\nu-2}e^{-x} \left[ F(-n; 2\nu + 2; x) \right]^2 dx\delta_{mm'} \]

\[ = \frac{4a^4}{(2\nu - 1)\nu(2\nu + 1)(n + \nu + 1)} \left[ 1 + \frac{3n}{(\nu + 1)} \right. \]

\[ + \frac{3n(n - 1)}{(\nu + 1)(2\nu + 3)} \left] \delta_{mm'} \right) \]

\[ = f(4), \quad (45) \]

\[ \langle nlm \mid r^{-5} \mid nlm' \rangle = \frac{4a^5}{(2\nu - 1)\nu(2\nu + 1)(n + \nu + 1)} \left[ 1 + \frac{6n}{(\nu + 1)} \right. \]

\[ + \frac{15n(n - 1)}{(\nu + 1)(2\nu + 3)} + \frac{5n(n - 1)(n - 2)}{(\nu + 1)(2\nu + 3)(\nu + 2)} \left] \delta_{mm'} \right) \]

\[ = f(5), \quad (47) \]

\[ \langle nlm \mid r^{-6} \mid nlm' \rangle = \frac{4a^5}{(2\nu - 1)\nu(2\nu + 1)(n + \nu + 1)} \left[ 1 + \frac{6n}{(\nu + 1)} \right. \]

\[ + \frac{15n(n - 1)}{(\nu + 1)(2\nu + 3)} + \frac{5n(n - 1)(n - 2)}{(\nu + 1)(2\nu + 3)(\nu + 2)} \]

\[ + \frac{15n(n - 1)}{(\nu + 1)(2\nu + 3)} + \frac{5n(n - 1)(n - 2)}{(\nu + 1)(2\nu + 3)(\nu + 2)} \left] \delta_{mm'} \right) \]

\[ = f(6). \quad (51) \]

Putting these results together one gets

\[ \Delta E_{\text{nc}} = -\frac{\alpha^2 ml}{2} f(4)\theta - \frac{\alpha^3}{5} \left( E_{n,l}^0 f(5) + \frac{29}{24}\alpha f(6) \right) \theta^2. \quad (53) \]

The energy shift is due to the terms (24). In addition, the first term of order \( \theta \) multiplied by magnetic quantum number indicates the splitting of states with the same orbital angular momentum into the corresponding components. This behavior is similar to the Zeeman effect without spin. The rest of the terms of second order in \( \theta \) are independent of magnetic quantum number, which clearly reflects the existence of spin. Furthermore it is worth noting that the correction terms containing \( \theta^2 \) are very similar to the spin–spin coupling, thus the non-commutative parameter \( \theta \) plays the role of spin and thus the degeneracy of levels is completely removed.
The energy levels of the hydrogen atom in the framework of the non-commutative Klein-Gordon equation are:

\[
\hat{E} = E_{n,l}^0 - \frac{\alpha^2 m_e}{2} f(4) \theta - \frac{\alpha^3}{5} \left( E_{n,l}^0 f(5) + \frac{29}{24} \alpha f(6) \right) \theta^2.
\] (54)

4 Non-relativistic limit of NC Klein-Gordon equation

Now we consider the non-relativistic limit of the non-commutative Klein-Gordon equation (21). To do this we take the wave function in the new form:

\[
\hat{\varphi}(r, \theta, \phi, t) = \frac{1}{r} R(r) Y(\theta, \phi) \exp(-i (\varepsilon + m_e) t),
\] (55)

where \(\varepsilon\) is the non-relativistic energy for which the conditions \(|eA_0| \ll m_e |R(r)/r Y(\theta, \phi)|\) and \(\varepsilon \ll m_e\) are valid [15]. Then the non-relativistic limit of non-commutative Klein-Gordon equation (21) is:

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} + \frac{2m_e e^2}{r} + \varepsilon - \frac{e^4}{2r^4} \theta \cdot \tilde{L} - \frac{e^8}{16r^8} (\theta^{ij} x_j)^2 - \frac{2m_e e^6}{5r^5} \theta^2 - \frac{e^8}{5r^6} \theta^2 \right] R(r) = 0,
\] (56)

where the perturbation terms in this equation are the same as those in equation (23) (explicitly given in (24)) with the replacement \(E \rightarrow m_e\). Hence in eq. (56) the NC terms are similar to the NC Hamiltonian of hyperfine splitting in NC space [12]. In addition, the new term is similar to the spin-spin coupling in which the non-commutativity plays the role of the spin. Thus the energy spectrum and radial wave functions corresponding to \(\theta = 0\) in equation (56) are given by:

\[
\varepsilon_n = -\frac{m_e \alpha^2}{2\hbar^2 n^2}
\] (57)

and

\[
R_{nl}(r) = \frac{1}{n} \left( \frac{(n - l - 1)!}{a (n + l)!} \right)^{1/2} x^{l+1} e^{-x/2} L_{n-l-1}^{2l+1}(x), \quad x = \frac{2}{an} r.
\] (58)

where \(a = \hbar^2/(m_e \alpha)\), the Bohr radius of the Hydrogen atom.

Now to obtain the modification of energy levels as a result of the non-commutative terms in eq. (56), we use the first-order perturbation theory. The expectation value of \(r^{-4}, r^{-5}\) and
\( r^{-6} \) with respect to the solution in eq. \( (58) \) are given by:

\[
\langle nlm | r^{-4} | nlm' \rangle_{l>0} = \frac{4}{a^4 n^5 l(l+1)(2l-1)(2l+1)(2l+3)} [3n^2 - l(l+1)] \delta_{mm'},
\]

\( \equiv f(4) \) (59)

\[
\langle nlm | r^{-5} | nlm' \rangle_{l>1} = \frac{4}{3a^5 n^5 (l-l+1)(l+1)(2l+1)} \times
\]

\[
\times \left[ -1 + \frac{5(3n^2 - l(l+1))}{(2l-1)(2l+3)} \right] \delta_{mm'},
\]

\( \equiv f(5) \) (60)

\[
\langle nlm | r^{-6} | nlm' \rangle_{l>1} = \frac{4}{a^6 n^5 l(l+1)(2l-3)(2l+1)(2l+5)} \times
\]

\[
\times \left[ -\frac{7}{3(l-1)(l+2)} \delta_{mm'} - \frac{3(3n^2 - l(l+1))}{n^2(2l-1)(2l+3)} + \frac{35(3n^2 - l(l+1))}{3(l-1)(l+2)(2l-1)(2l+1)(2l+3)} \right] \delta_{mm'}.
\]

\( \equiv f(6) \) (61)

So the total modification of the energy levels \( \Delta E_{nlm}^{\text{NC}} \) is given by:

\[
\Delta E_{nlm}^{\text{NC}} = -\frac{\alpha^2}{2} m_e \theta f(4) - \frac{\alpha^3}{5} \theta^2 \left( 2m_e f(5) + \frac{29}{24} \alpha f(6) \right)
\]

(63)

This energy shift is due to the additional noncommutative terms of eq.(58). The first term is the first-order correction in \( \theta \) and represents the Lamb shift correction for \( l \neq 0 \) states, where the energy-levels \( l \) split to \( 2l+1 \) sublevels. The second term is the second order correction in \( \theta \) which represents Lamb shift corrections for \( l = 0 \) states.

This result is very important: as a possible means of introducing electron spin we replace \( l \rightarrow \pm (j + \frac{1}{2}) \) and \( n \rightarrow n - j - \frac{1}{2} \), where \( j \) is the quantum number associated to the total angular momentum, then the \( l = 0 \) state have the same total quantum number \( j = \frac{1}{2} \). In this case the noncommutative value of the energy levels indicates the splitting of \( 1s \) states. These results show that the non-commutative non-relativistic (relativistic) degeneracy is completely removed. As we have explicitly shown the non-commutativity contributes to the correction of the Lamb shift between these levels. Thus, one can use the data on the Lamb shift to impose some bounds on the value of the noncommutativity parameter, \( \theta \). According to \[16\] the current theoretical accuracy on the \( 1s \) Lamb shift is about 14 kHz. From the splitting (63), this then gives the bound

\[
\theta \lesssim (2 \times 10^3 \text{GeV})^{-2}
\]

This is in agreement with other results presented in the ref \[3, 12\].
5 Conclusions

In this work we started from quantum field theory in a canonical non-commutative space and used the relativistic charged scalar particle in the Minkowski space-time to find the action which is invariant under the generalised infinitesimal gauge transformation. By using the Seiberg-Witten maps and the Moyal product up to second order in the non-commutativity parameter $\theta$, we generalised the equations of motion and derived the deformed Klein-Gordon equation for non-commutative Coulomb potential. By solving the deformed KG equation we found the energy shift up to the second order of $\theta$, where the first term is proportional to the magnetic quantum number. This behavior is similar to the Zeeman effect that is applied in the magnetic field of the system without spin and the second term is proportional to $\theta^2$, thus we explicitly accounted for a spin effects in this space. Hence we can say that the Klein-Gordon equation in non-commutative space at the second order of $\theta$ describes particles with spin. After that we have obtained the non-relativistic limit of the deformed Klein-Gordon equation for a non-commutative Coulomb potential. We showed that the non-relativistic Klein-Gordon equation is the same the Schrödinger equation for a Coulomb potential in non-commutative space. Thus we came to the conclusion that the non-commutative non-relativistic theory degeneracy is completely removed and induced the Lamb shift where the energy spectrum of the hydrogen atom depends on the second-order on the non-commutative parameter $\theta$.

References

[1] S.Minwalla, M. van Raamsdonk and N.Seiberg, JHEP 0002 (2000) 020, [hep-th/9912072].

[2] J. Gomis, and T. Mehen, Nucl. Phys. B591, 178 (2000).

[3] N. Seiberg, L. Susskind and N. oumbas, JHEP 0006 (2000) 044, [hep-th/0005015].

[4] S. Doplicher, K. Fredenhagen and J.E. Roberts, Spacetime Quantization Induced by-Classical Gravity, Phys. Lett. B331 (1994) 39;
The Quantum Structure of Spacetime at the Planck Scale and Quantum Fields, Commun. Math. Phys. 172 (1995) 187;
T. Yoneya, String Theory and Spacetime Uncertainty Principle, Progr. Theor. Phys.103 (2000) 1081 [hep-th/0004074].

[5] M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. 86(2001) 2716.

[6] T. C. Adorno, M. C. Baldiotti, M. Chaichian, D. M. Gitman, A. Tureanu, Phys. Lett. B682:235-239(2009).

[7] H. Motavalli, A. R. Akbarieh, Mod. Phys. Lett. A25:2523-2528(2010)

[8] N. Seiberg and E. Witten, JHEP 032 (1999) 9909.

[9] N. Mebarki, S. Zaim, L. Khodja and H. Aissaoui, Phys. Scripta 78 (2008) 045101.

[10] A. Stern, phy. Rev. Lett. 100, 061601(2008).
[11] J. Naudts, arXiv:physics/0507193v2.

[12] S.A. Alavi, Phys. Scr. 78(2008)015005.

[13] I. I. Gol’dman and D. V. Krivchenkov, Problems in quantum mechanics (Pergamon, London, 1961).

[14] A. F. Nikiforov, V. B. Uvarov, Special Functions of Mathematical Physics (Birkhauser, Basel, 1988).

[15] M. R. Setare and O. Hatami, Commun. Theor. Phys. (Beijing, China) 51(2009) pp. 1000-1002.

[16] M. I. Eides, H. Grotch and V. A. Shelyuto, Phys. Rept. 342, 63 (2001).