Decentralized Markov Chain Gradient Descent

Tao Sun, and Dongsheng Li

Abstract—Decentralized stochastic gradient method emerges as a promising solution for solving large-scale machine learning problems. This paper studies the decentralized Markov chain gradient descent (DMGD) algorithm — a variant of the decentralized stochastic gradient methods where the random samples are taken along the trajectory of a Markov chain. This setting is well-motivated when obtaining independent samples is costly or impossible, which excludes the use of the traditional stochastic gradient algorithms. Specifically, we consider the first- and zeroth-order versions of decentralized Markov chain gradient descent over a connected network, where each node only communicates with its neighbors about intermediate results. The nonergodic convergence and the ergodic convergence rate of the proposed algorithms have been rigorously established, and their critical dependences on the network topology and the mixing time of Markov chain have been highlighted. The numerical tests further validate the sample efficiency of our algorithm.

Index Terms—Markov chain sampling, Gradient descent, Decentralization, Distributed machine learning, Convergence.

I. INTRODUCTION

Distributed machine learning is an attractive solution to tackle large-scale learning tasks [1], [2]. In this paper, we consider that $m$ agents represent the nodes of a connected network and collaboratively solve the following stochastic optimization problem

$$
\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \mathbb{E}_{\xi(i)}(F(x; \xi(i))),
$$

(1)

where $\mathbb{E}_{\xi(i)}(F(x; \xi(i))) := \int_{\Xi_i} F(x, \xi(i)) d\Pi_i(\xi(i))$ and $\Xi_i$ is a statistical sample space with probability distribution $\Pi_i$ at node $i$ (we omit the underlying $\sigma$-algebra), and $F(\cdot; \xi(i)) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a closed (possibly nonconvex) function associated with $\xi(i) \in \Xi_i$. This formulation contains various multi-agent machine learning, reinforcement learning and statistical problems. We are particularly interested in cases where obtaining an independent and identically distributed (i.i.d.) sample $\xi(i)$ from $\Xi_i$ is very hard or even impossible at every node $i$; see an example in [3] where the cost of i.i.d. sampling can be very expensive. In statistics, a common method to overcome the issue of difficult sampling is employing a Markov chain whose stationary distribution is $\Pi_i$. Therefore, to solve (1), one can still use the parallel implementation of the widely-used method Stochastic Gradient Descent (SGD) [4]:

$$
x^{k+1} = x^k - \gamma_k \sum_{i=1}^{m} \nabla F(x^k; \xi^k(i)), \quad \xi^k(i) \sim \Pi_i
$$

(2)

where $\gamma_k$ and $x^k$ are the stepsize and parameter at iteration $k$, and $\nabla F(x^k; \xi^k(i))$ is a nearly unbiased stochastic gradient of $\mathbb{E}_{\xi(i)}(F(x; \xi(i)))$ obtained at node $i$. By using the Markov chain, in order to perform one iteration of (2), each node $i$ has to generate a sequence of samples $\xi^1(i), \xi^2(i), \ldots, \xi^T(i)$ and only uses the last one $\xi^T(i) := \xi^T(i)$. According to [3], Theorem 4.9, to get a sample that is nearly i.i.d., one needs to simulate the Markov chain for a sufficiently long time; e.g., a large $T$. For this reason, we call the iteration (2) as SGD-$T$.

In addition, to update the next parameter via (2), all the local gradients need to be collected. Therefore, applying iteration (2) for (1) over the distributed nodes has following two limitations.

Sample inefficiency. Different from standard stochastic optimization settings, when it is difficult to obtain i.i.d. samples $\xi(i)$ from $\Xi_i$, implementing SGD-$T$ for (1) requires regenerating Markov chains at each node per iteration. Nevertheless, this wastes a sizeable amount of variable samples, especially when the Markov chain has a large mixing time.

Communication inefficiency. The presumption of implementing (2) is that there is a fusion center (which can be a designated agent) aggregating the local gradients and carrying out the parameter update. However, this incurs a significant amount of synchronization and communication overhead, especially when the network is large and sparse.

In this context, our goal is to find the near-optimal solution of (1) in a sample- and communication-efficient manner.

A. Prior Art

In this part, we briefly review three lines of related works: decentralized optimization, decentralized stochastic optimization, and Markov chain gradient descent.

Decentralized optimization. Decentralized algorithms have been originally studied in control and signal processing communities, e.g., calculating the mean of data distributed over multiple sensors [6]–[9]. The decentralized (sub)gradient descent (DGD) algorithms for the finite sum optimization have been studied in [10]–[14]. However, DGD converges to an inexact solution due to that it actually minimizes an unconstrained penalty function rather than the original one. To fix this, the dual information is leveraged in recent works such as decentralized ADMMs and primal-dual algorithms [15]–[18]. Although DGD is slower than decentralized ADMMs and primal-dual algorithms in the convex settings, it is much simpler and therefore easier to extend to the nonconvex, online and delay-tolerant settings [19]–[21].

Decentralized stochastic optimization. Generalizing methods for the decentralized deterministic optimization, decentralized SGD (DSGD) has been studied recently. By assuming a local Poisson clock for each agent, asynchronous gossip algorithms is proposed by [22], in which each worker randomly selects
part of its neighbors to communicate with. In fact, these algorithms used random communication graphs. The decentralized algorithm with random communication graph for the constrained problem is introduced by [23], the subgradient version is given by [24]. In recent works [25]–[27], the theoretical convergence complexity analysis of convex and nonconvex DSGD is provided. In [25], the authors presented the complexity analysis for a stochastic decentralized algorithm. In [26], the authors design a kind of stochastic decentralized algorithm by recruiting the dual information, and provide the related computational complexity. In latter paper [27], the authors show the speedup when the number of nodes is increased. And in paper [28], the authors proposed the asynchronous decentralized stochastic gradient descent and presented the theoretical and numerical results.

**Markov chain gradient descent.** While i.i.d. samples are not always available in stochastic optimization, recent focus has been on the analysis of stochastic algorithms following a single trajectory of the Markov chain or other general ergodic processes. The key challenge of analyzing MGD is to deal with the biased expectation of gradients. The ergodic convergence results have been reported in [29], [30]. Specifically, [29], [30] study the conditional expectation with a sufficiently large delay which is sufficiently close to the gradient (but still different). The authors of [31] proved the almost sure convergence under the diminishing stepizes $\gamma_k = 1/k^q$, $2/3 < q \leq 1$. In [32], the authors improved convergence results with larger stepizes $\gamma_k = 1/\sqrt{k}$ in the sense of ergodic convergence. In all these works, the Markov chain is required to be reversible, and the functions have to be convex. In a very recent paper [33], the non-ergodic convergence of MGD has been shown in the nonconvex case with non-reversible Markov chain, but the algorithm needs to be implemented in a centralized fashion.

**Markov chain trajectory sampling reduces to the uniform random sampling.** The numerical results are presented to demonstrate that DMGD performs better than DSGD in terms of sample efficiency.

Although this paper’s proofs also need to use the delay expectation techniques used in [31], the significant difference remains. This is because the DMGD introduced in this paper has no objective function to minimize. The proof of DMGD is built on the average of all nodes’ parameters, which is also quite different from MCGD. To this end, several techniques are developed to characterize the difference between the average and nodes’ parameters under Markov chain sampling. Compared with [3], this paper also investigates the zeroth-order case that has never been studied in MCGD.

**Notation:** Let $\lambda_i(\cdot)$ denote the i-th eigenvalue of a matrix. Let $x(i) \in \mathbb{R}^N$ denote the local copy of $x$ at node $i$. For a matrix $A = (a_{i,j})_{m \times n}$, $\|A\|_2 := \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a^2_{i,j}}$ and $\|A\|_{\infty} := \max_{i,j} |a_{i,j}|$. For a positive semidefinite matrix $B$, $\|A\|_B := \|B^{1/2}A\|_2$.

## II. DECENTRALIZED MARKOV CHAIN GRADIENT DESCENT

### A. Preliminaries

We first consider the discrete case of our problem (1), that is, all the distributions $(\Pi_1, \Pi_2, \ldots, \Pi_m)$ are supported on a set of $M$ points $y_1, y_2, \ldots, y_M$ (for $\Pi_j$). We define the functions as $f_j(x) := M \cdot \text{Prob}(\xi = y_j) \cdot F(x, y_j)$, and thus problem (1) becomes the following finite-sum formulation

$$\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{m} \sum_{j=1}^{m} f_j(x), \quad (3)$$

where $f_j(x) := \frac{1}{M} \sum_{i=1}^{M} f_i(x)$ is the loss function in the jth node. The consensus optimization is the main workhorse to distribute the training algorithm in machine learning tasks. There are just a few non-consensus methods; as far as we know, it seems only block coordinate gradient descent [33]–[35]. Mathematically, it applies the coordinate descent to minimize function $f$ or its dual problem. Thus, we can get a centralized system whose work computes the gradient for one block for algorithmic distribution. However, block coordinate gradient descent has still not been widely used for distributed training tasks due to the vast data may cause random access memory overflow even for one block; on the other hand, the large batch size implies indecent generalization property [36] that further limits the use of block coordinate gradient descent. Hence, this paper still follows the consensus optimization routine.

Denote $(j_{i,k})_{k \geq 0} \subseteq \{1, 2, \ldots, M\}$ as the trajectory of the Markov chain in the i-th node and k-th iteration. We use a connected graph $G = (V, E)$ with vertex set $V = \{1, \ldots, M\}$

1For notational brevity, we assume the same cardinal number for different distribution support sets.
and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Any edge $(i, l) \in \mathcal{E}$ represents a communication link between nodes $i$ and $l$. And more, let

$$
j_k := \begin{bmatrix} j_{1,k} \\ j_{2,k} \\ \vdots \\ j_{m,k} \end{bmatrix}, \quad x := \begin{bmatrix} x(1)^T \\ x(2)^T \\ \vdots \\ x(m)^T \end{bmatrix}, \quad u^k := \begin{bmatrix} \nabla f_{1,k}^j(x(k)^{(1)}) \\ \nabla f_{2,k}^j(x(k)^{(2)}) \\ \vdots \\ \nabla f_{m,k}^j(x(k)^{(m)}) \end{bmatrix}.
$$

**Mixing matrix**: The mixing matrix is frequently used in decentralized optimization. In many cases, it can be designed by the users according to the given graph. Formally, it is defined as follows.

**Definition 1**: The mixing matrix $W = [w_{ij}] \in \mathbb{R}^{m \times m}$ is assumed to have the following properties:
1. (Graph) If $i \neq j$ and $(i, j) \notin \mathcal{E}$, then $w_{ij} = 0$, otherwise, $w_{ij} > 0$;
2. (Symmetry) $W = W^T$;
3. (Null space property) $\text{null}\{I - W\} = \text{span}\{1\}$;
4. (Spectral property) $I \geq W > -I$.

With the symmetricity of $W$, its eigenvalues are real and can be sorted in the nonincreasing order. Thus, let $\lambda_i(W)$ denote the $i$th largest eigenvalue of $W$; then, it holds that $\lambda_1(W) = 1 > \lambda_2(W) \geq \cdots \geq \lambda_m(W) > -1$.

**Markov chain**: We recall several definitions, properties, and existing results of the finite-state time-homogeneous Markov chain, which will be used in the proposed algorithms.

**Definition 2**: Let $H$ be an $n \times n$-matrix with real-valued elements. A stochastic process $\pi$ is a row vector $\{\pi_1, \pi_2, \ldots, \pi_n\}$ specific to an existing results of the finite-state time-homogeneous Markov chain $H$. Denote the probability distribution of $\pi$ as $\sum_{i=1}^n \pi_i = 1$, and $\pi$ satisfies $\sum_{i=1}^n \pi_i^k = 1$. For the time-homogeneous Markov chain, it holds $\pi^k = \pi^{k-1}H$, and $\pi_1, \pi_2, \ldots, \pi_n$ are the stationary distribution. The largest eigenvalue of $H$ is 1, and the corresponding eigenvector is $\pi^*$. The largest eigenvalue of $H$ is 1, and the corresponding eigenvector is $\pi^*$.

**Mixing time**: An important concept which describes how long a Markov chain evolves until its current state has a distribution very close to its stationary distribution. The literature studies about various kinds of mixing times, whose majority, however, is about reversible Markov chains (i.e., $\pi_i H_{i,j} = \pi_j H_{j,i}$). With basic matrix analysis, the mixing time introduced in [3] provides a direct relationship between $k$ and the deviation of the distribution of the current state from the stationary distribution (Lemma 4 in the Appendix).

**B. Algorithmic Development of DMGD**

The local scheme of DMGD at the $i$th node is

$$x^{k+1}(i) = \sum_{l \in \mathcal{N}(i)} w_{il} x^k(l) - \gamma_k \nabla f_{j,k}^i(x^k(i)).$$

In each iteration, each node calculates the local gradient on the Markov chain trajectory $(j_k)_{k \geq 0}$, and then communicates with its neighbors $\mathcal{N}(i)$ with a weighted average $\sum_{l \in \mathcal{N}(i)} w_{il} x^k(l)$ to update the iteration. Here, $w_{il}$ is the $(i, l)$-element of the mixing matrix. It is easy to see that if the Markov chain trajectory is the uniform sampling, [3] reduces to the DSGD algorithm. The parameter $\gamma_k$ will go to zero. The settings are to guarantee the convergence of the algorithm; the diminishing stepsizes are used to reduce the variances cost by the gradients samplings. We present the comparison of DSGD-T (Algorithm 1) and DMGD (Algorithm 2). Our work is the first to distribute the consensus optimization under the Markov sampling assumption in the decentralized case. In our algorithm (8), if $\mathcal{N}(i) = \{i\}$, it is then the centralized Markov chain gradient descent. Thus, our method contains the previous distributed method for Markov chain samplings.

**Algorithm 1 DSGD-T**

**Require**: parameters $(\gamma_k)_{k \geq 0}$

- **Initialization**: $x(1) = x(1) = \ldots = x(m) = x^0$
  - for $k = 1, 2, \ldots$
  - for $i = 1, 2, \ldots, m$
  1. Resample a Markov chain $j_0(i), \ldots, j_T(i)$
  2. Collect $x^k(i)$ from its neighbors
  3. Update $x^{k+1}(i)$ via [5] with $j_i, k \leftarrow j_T(i)$

**end for**

**end for**

**Algorithm 2 DMGD**

**Require**: parameters $(\gamma_k)_{k \geq 0}$

- **Initialization**: $x(1) = x(1) = \ldots = x(m) = x^0$
  - for $k = 1, 2, \ldots$
  - for $i = 1, 2, \ldots, m$
  1. Sample $j_{i,k}$ via a Markov chain
  2. Collect $x^k(i)$ from its neighbors
  3. Update $x^{k+1}(i)$ via [5] with $j_{i,k}$

**end for**

**end for**

The global scheme can be described as

$$x^{k+1} = W x^k - \gamma_k u^k$$

where $u^k$ has been given before. This iterative formulation can help us to understand the convergence of the algorithm. Suppose that the Markov chains all reduce to the uniform sampling. By defining the $\sigma$-algebra as $\chi^k := \sigma(x^0, x^1, \ldots, x^k)$,
we can see that \( \mathbb{E}(u^k | x^k) = \sum_{i=1}^m \nabla f_i[x^k(i)] \) in this condition; that means DMGD actually admits the DSGD with converge guarantees in the perspective of expectation. For general Markov chains, the analysis is much more complicated for the biased conditional expectation.

C. Key Challenge of Analyzing DMGD

Markov chain sampling is neither cyclic nor i.i.d. stochastic. For any large \( K \), it is still possible that a sample is never visited during some \( k+1, \ldots, k+K \) iterations. For a fixed node \( i \), unless the graph \( G_i \) is a complete graph (i.e., all elements are directly connected), there are elements \( l, h \) without an edge connecting them, i.e., \( (l, h) \notin E_i \). Hence, given \( j, k-1 = l_i \), it is impossible to have \( j, k = h_i \). So, no matter how one selects the sampling probability and stepsize \( \gamma_k \), we generally cannot have \( \mathbb{E}(\gamma_k u^k | j_{k-1} = 1) = C(\sum_{i=1}^m \nabla f_i[x^k(i)]) \) for any constant \( C \), where \( 1 = (l_1, \ldots, l_m) \). This fact, unfortunately, breaks down all the existing analyses of stochastic decentralized optimization since they all need a non-vanishing probability for every sample in each node can be selected.

III. CONVERGENCE ANALYSIS OF DMGD

In this part, we present the theoretical results of DMGD with finite-state Markov chains. Our analysis builds on the following assumptions.

**Assumption 1:** Function \( f_i \) is lower bounded, that is \( f_i(x) \geq -\infty \), \( \forall i, \forall x \).

**Assumption 2:** The gradient of \( f_i \) is uniformly bounded, that is, there exists a constant \( B > 0 \) such that \( \|\nabla f_i(x)\| \leq B, \forall i \in \{1, 2, \ldots, M\}, \forall j \in \{1, 2, \ldots, m\} \).

**Assumption 3:** The gradient of \( f_i \) is Lipschitz continuous with \( L^j_i \), i.e., \( \|\nabla f_i(x) - \nabla f_j(x)\| \leq L^j_i \|x - y\|, \forall i \in \{1, 2, \ldots, M\}, \forall j \in \{1, 2, \ldots, m\} \).

**Assumption 4:** The Markov chains in all nodes are time-homogeneous, irreducible, and aperiodic, which have the same transition matrix \( H \) and the same stationary distribution.

Following the routines in stochastic decentralized community, the convergence of the algorithm is described by

\[
\overline{x}^k := \frac{1}{m} \sum_{j=1}^m x^k(j).
\]

**Theorem 1:** Suppose Assumptions \( \text{II} \) hold and the stepizes are selected as

\[
\gamma_k = \frac{1}{(k+1)^\theta}, \quad \frac{1}{2} < \theta < 1.
\]

For \((x^k)_{k \geq 0}\) generated by DMGD, we have the following nonegordic convergence result

\[
\lim_{k \to \infty} \mathbb{E}[\|\nabla f(\overline{x}^k)\|] = 0.
\]

And the ergodic convergence rate is

\[
\min_{1 \leq i \leq k} \mathbb{E}[\|\nabla f(\overline{x}^k)\|] = O\left(\frac{1}{\ln(1/\lambda(H)) \cdot \frac{1}{k^{\frac{1-\theta}{2}}}}\right), \tag{9}
\]

Theorem 1}

where \( \lambda(H) := \max\{\lambda_2(H) \mid \lambda_{\min}(H)\} + 1 \in [0, 1) \), and \( \lambda_2(H) \) and \( \lambda_{\min}(H) \) denote the second and smallest eigenvalue of \( H \), respectively.

In Theorem 1, the functions are not necessary to be convex. In fact, it is more difficult to prove (8) than to prove (9). The descent on a Lyapunov function and the Schwarz inequality can directly derive (8), while (9) requires a technical lemma, which first given in [19] and generalized by [3]. An extreme case is that \( m = 1 \) and \( W = 1 \); DMGD will reduce to the classical MGD. But Theorem 1 cannot cover the existing convergence results of MGD. In [3], the authors estimated the convergence of MGD with the stepsizes requirements

\[
\sum_{k=1}^{+\infty} \gamma_k = +\infty, \quad \sum_{k=1}^{+\infty} \ln^2 k \cdot \gamma_k^2 < +\infty. \tag{10}
\]

The stepsize (7) can satisfy (10) but not vice versa.

The convergence results in Theorem 1 can be extended to general Markov chains with extra assumptions given in existing works. In [31], the time non-homogeneous Markov chain but with extra assumptions (Assumptions 4 and 5, in Section 4 of [31]) is proposed. These two assumptions involve with many details; several majors are doubly stochastic, uniformly bounded away from zero, diagonals of the transition matrices are positive, and strong connections of some edges. In paper [32], the authors use more general Markov chain but also with an assumption (Assumption C, in Section 2 of [32]), which can be satisfied by finite-state time-homogeneous Markov chain.

To reach the error as \( \mathbb{E}[\|\nabla f(\overline{x}^k)\|^2] \leq \epsilon \), we need the learning rate as \( \gamma = \Theta(\epsilon) \) and \( K = O(1/\epsilon^2) \). This result is almost the same as the diminishing learning rate case. However, the constant learning rate case cannot prove the non-ergodic convergence result (11). Thus, this paper uses diminishing ones.

IV. ZERO-ORDER DMGD

This section presents the zero-order version of DMGD with two-points feedback strategy [37]–[40]. This paper employs the method given in [37], [38]. Specifically, it uses the estimator of the gradient of \( f \) by querying at \( x + \delta h \) and \( x \) with returning \( \hat{f}(f_i(x + \delta h) - f(x)) \) where \( h \) is a random unit vector and \( \delta > 0 \) is a small parameter. In the zeroth-DMGD, we use the two-points feedback to replace the local gradients and obtain following iteration at the \( i \)th node

\[
x^{k+1}(i) = \sum_{i \in N(i)} w_{i,j} x^k(l) - \gamma_k n(f_{j,i}(x^k(i) + \delta_k h^{i,k}) - f_{j,i}(x^k(i))) \delta_k h^{i,k}, \tag{11}
\]

where \((j_{i,k})_{k \geq 0}\) still denotes the Markov chain trajectory in the \( i \)th node, and \( h^{i,k} \) is uniformly sampled for the unite sphere in \( \mathbb{R}^n \), and \( \delta_k \) is the parameter in the \( k \)th iteration. In this algorithm, we just use the function values information rather
where (23). Mathematically, this fact can be presented as

\[
\text{homogeneous and reversible infinite-state Markov chains are continuums, and turn back to the problem (1). Time-
}
\]

\text{considered in this case. With the results in [Theorem 4.9, [5],}

\]

\text{stant factor}

\]

\text{Compared with the first-order DMGD in Theorem 1, a con-
}

\]

\text{order DMGD.}

\]

\text{the speed might be slowed down.}

\]

\text{Theorem 2: Suppose Assumptions [1] and [2, 3, and 4 hold, and}

\]

\text{h, i, k is a random unit vector in the kth iteration in ith node.
}

\text{If the stepsizes are selected as}

\[
\gamma_k = \frac{1}{(k + 1)^\theta}, \quad 1 < \theta < 1, \quad \sum_k \gamma_k \delta_k < +\infty
\]

\text{then for (x^k)_{k \geq 0} generated by zeroth-order DMGD, we have the nonergodic convergence as}

\[
\lim_k \mathbb{E}[\|\nabla f(x^k)\|] = 0
\]

\text{and the ergodic convergence rate is}

\[
\min_{1 \leq k \leq K} \mathbb{E}[\|\nabla f(x^k)\|] = O\left(\frac{1}{\ln(1/\lambda(H))} \cdot \frac{1}{1 - \lambda(\mathbb{W})} + n^2 k^{-\frac{1}{2}}\right).
\]

\text{Compared with the first-order DMGD in Theorem [1] a con-
}

\text{stant factor n^2 k^{-\frac{1}{2}} degrades the convergence rates. Such difference}

\text{comes from the two-points estimation errors of gradient which is dimension-dependent. This result indicates that in low-
}

\text{dimension case, the zeroth-order version can work well as DMGD, as could be expected; but for high-dimension case, the speed might be slowed down.}

\]

\text{V. ANALYSIS ON CONTINUOUS STATE SPACE}

\text{In this part, we consider the case that \Pi_1, \Pi_2, \ldots, \Pi_m are continuums, and turn back to the problem [1]. Time-
}

\text{homogeneous and reversible infinite-state Markov chains are considered in this case. With the results in [Theorem 4.9, [5],}

\text{the mixing time in this case still enjoys geometric decrease like [23]. Mathematically, this fact can be presented as}

\[
\|\delta^k\|_\infty \leq C \cdot \lambda^k, \quad k \geq 0,
\]

\text{where \delta^k still denotes the deviation matrix \Pi^* - H^k. Here C}

\text{and \lambda are constants determined by the Markov chain. Here, we use notation C and \lambda to give the difference to CP and λ(P) in Lemma [1].}

\text{Let \xi(i), \xi(i), \ldots be the Markov chain trajectory in ith node. We first present the local scheme:}

\[
x^{k+1}(i) = \sum_{l \in N(i)} w_{ij} x^k(l) - \gamma_k \nabla F(x^k(i); \xi^k(i)).
\]

\text{By defining}

\[
d^k := [\nabla F(x^k(1); \xi^k(1)); \ldots; \nabla F(x^k(m); \xi^k(m))]^T,
\]

\text{the global scheme is then of the following form}

\[
x^{k+1} = W x^k - \gamma_k d^k.
\]

\text{The convergence is proved for a possibly nonconvex objective function F(·; ξ) and time-homogeneous and reversible chains, which obey the following assumption:}

\text{Assumption 5: For any ξ ∈ Ξ, it holds that (1)\|\nabla F(x; ξ) - \nabla F(y; ξ)\| \leq L \|x - y\|, \forall x, y \in \mathbb{R}^n; (2) \}

\text{sup_{x \in \mathbb{R}^n, \xi \in \Xi} \{\|\nabla F(x; ξ)\|\} < +\infty; (3) \}

\text{E_ξ (\nabla F(x; ξ)) = \nabla (E_ξ F(x; ξ)), \forall x \in \mathbb{R}^n; (4) \inf_{x \in \mathbb{R}^n} (E_ξ F(x; ξ)) > -\infty,}

\text{for high-dimension case, the speed might be slowed down.}

\text{VI. NUMERICAL TESTS}

\text{In this section, we compare the performance of our al-
}

\text{gorithm with the decentralized parallel SGD (DSGD) on an autoregressive model, which closely resembles the first experiment in [22]. Assume that there are m autoregressive processes distributed on a graph of m nodes. We attempt to recover a consensus vector u from the multiple processes. On each node j, set matrix A^j as a subdiagonal matrix with random entries A_{i,l-1(i)} \sim U[0,8,0,99]. Randomly sample a vector u ∈ \mathbb{R}^n, with the unit 2-norm. In our experiments, we tested with m = 10, n = 50 and m = 20, n = 100. The data (ξ^2(i), ξ^2(i))_{i=1}^\infty are generated by the following auto regressive process:}

\[
\xi^1(i) = A(i)\xi^1_{i-1} + e_1 W^t, \quad W \sim N(0,1)
\]

\text{δ^2(i) = \begin{cases} 1, & \text{if } \langle u, \xi^2(i) \rangle > 0, \\
0, & \text{otherwise};
\end{cases}
\]

\text{ξ^2(i) = \begin{cases} \xi^2(i), & \text{with probability 0.8,} \\
1 - \xi^2(i), & \text{with probability 0.2.}
\end{cases}
\]
Clearly, for any $i \in \{1, 2, \ldots, M\}$, $(\xi^i_1(i), \xi^2(i))_{i=1}^\infty$ forms a Markov chain. Let $\Pi_i$ denote the stationary distribution of the Markov chain on the $i$-th node. By defining the loss function as $\ell(x; \xi^1(i), \xi^2(i)) = -\xi^2(i) \log(\sigma(x, \xi^1(i))) - (1 - \xi^3(i)) \log(1 - \sigma(x, \xi^1(i)))$ with $\sigma(t) = \frac{1}{1 + \exp(-t)}$, we reconstruct $u$ as the solution to the following problem:

$$\min_x \sum_{i=1}^m E(\xi^1(i), \xi^2(i)) \sim \Pi_i \ell(x; \xi^1(i), \xi^2(i)). \tag{22}$$

We choose $\gamma_k = \frac{1}{m}$ as our stepsize, where $q = 0.51$. This choice is consistently with our theory below. Specifically we compare:

**DMGD**, where $(\xi^1(i), \xi^2(k))$ is from one trajectory of the Markov chain on the $i$-th node;

**MCGD**, (i.e., the centralized Markov chain gradient descent), where $(\xi^1(i), \xi^2(k))$ is from one trajectory of the Markov chain on the $i$-th node;

**DSGD-T**, for $T = 1, 2, 4, 8, 16$, where each $(\xi^1(i), \xi^2(k))$ is the $T$-th sample of an independent trajectory on the $i$-th node. All trajectories are generated by starting from the same initial state.

To compute $T$ gradients, DSGD-T uses $T$ times as many samples as DMGD. We did not try to adapt $T$ as $k$ increases because there lacks a theoretical guidance. The numerical comparison results are reported in Figure 1, which show that DMGD outperforms the DSGD-T with $T = 1, 2, 4, 8, 16$. The numerical results in Figure 1 are quite positive on DMGD. As expected, DMGD used significantly fewer total samples than DSGD on each T. Surprisingly, DMGD did not cost even more gradient computations. It is important to find that DSGD1 and DSGD2, as well as DSGD4, stagnate at noticeably lower accuracies due to that their $T$ values are too small. On the other hand, we observe that DMGD may not beat the centralized in the gradient computation and sampling costs, but it can significantly reduce the busiest node’s communications.

VII. PROOFS

This section contains the proofs of the main results.

A. Technical lemmas

**Lemma 1:** Let Assumption 4 hold and let $\lambda_i(H) \in \mathbb{C}$ be the $i$th largest eigenvalue of $H$, and

$$\lambda(H) := \max\{|\lambda_2(H)|, |\lambda_M(H)|\} + \frac{1}{2} \in [0, 1).$$

Then, we can bound the largest entry-wise absolute value of the deviation matrix $\delta_k := \Pi^t - H^k \in \mathbb{R}^{m \times m}$ as

$$||\delta_k||_{\infty} \leq C_H \cdot \lambda^k(H) \tag{23}$$

for $k \geq K$, where $C_H$ is a constant that also depends on the Jordan canonical form of $H$ and $K$ is a constant that depends on $\lambda(H)$ and $\lambda_2(H)$.

First, we define $1 := [11 \cdots 1]^\top$. And the projection matrix is given as

$$P := \frac{11^\top}{m} \in \mathbb{R}^{m \times m}. \tag{24}$$

It is easy to see

$$WP = PW = P. \tag{25}$$

**Lemma 2** (Corollary 1.14., [5]): Let $P \in \mathbb{R}^{m \times m}$ be the matrix whose elements are all $1/m$. Given any $k \in \mathbb{Z}^+$, the mixing matrix $W \in \mathbb{R}^{m \times m}$ satisfies

$$||W^k - P|| \leq \lambda_2(W)^k,$$

where $0 \leq \lambda_2(W) < 1$ is the absolute value of the second eigenvalue of $W$. 
Lemma 3: Let \((x^k)_{k \geq 0}\) be generated by Algorithm 1 and Assumption 2 hold, then we have
\[
\frac{1}{m} \sum_{i=1}^{m} \|x^k(i) - x^k\| \leq \sqrt{m} B \sum_{j=0}^{k} \gamma_j \lambda_2(W)^{k-j}.
\]
for any \(k \geq 0\). Further if \(\gamma_j = \frac{1}{(j+1)^{1/2}}\) with \(\frac{1}{2} \leq \theta < 1\), it follows
\[
\frac{1}{m} \sum_{i=1}^{m} \|x^k(i) - x^k\| \leq \sqrt{m} B C_W (k+1)^{\theta/2}.
\]
and \(C_W\) is a positive constant dependent on \(W\) and \(C_W = \mathcal{O}(\frac{1}{\lambda_2(W)})\).

Lemma 4: Consider two nonnegative sequences \((\beta_k)_{k \geq 0}\) and \((h_k)_{k \geq 0}\) that satisfy
1. \(\lim_k h_k = 0\) and \(\sum_k h_k = +\infty\), and
2. \(\sum_k \beta_k h_k < +\infty\), and
3. \(|\beta_{k+1} - \beta_k| \leq ch_k\) for some \(c > 0\) and \(k = 0, 1, \ldots\).
Then, we have \(\lim \beta_k = 0\).

Lemma 5: Let \(a > 0\), and \(x > 0\) be a enough large real number. If
\[
y - a \ln y + c = x.
\]
Then, it holds
\[
y - x \leq 2a \ln x.
\]

B. Proof of Lemma 3
With direct calculating,
\[
x^k = W^k x^0 - \sum_{j=0}^{k-1} \gamma_j W^{k-j} u^j.
\]
Recall (25), we have
\[
(I - W)x^k = (I - W)(I - P)x^k.
\]
Thus, we need to bound \(\| (I - P)x^k \| \) and use the fact
\[
(W - P)(I - P) = W - 2P + WP = W - P = W - PW.
\]
Using (25), we have
\[
\| (I - P)x^k \| = \| (W - P)x^0 - \sum_{j=0}^{k-1} \gamma_j (W^{k-j} - P) u^j \|
\leq \| W^k - P \| \cdot \| x^0 \| + \sum_{j=0}^{k-1} \gamma_j \| W^{k-j} - P \| \cdot \| u^j \|
\leq B \sum_{j=0}^{k} \gamma_j \lambda_2(W)^{k-j}.
\]
Noticing that \(\| (I - P)x^k \|^2 = \sum_{i=1}^{m} \| x^k(i) - x^k \|^2 \),
\[
\frac{1}{m} \sum_{i=1}^{m} \| x^k(i) - x^k \| \leq \sqrt{m} \sum_{i=1}^{m} \| x^k(i) - x^k \|^2 m
\leq \sqrt{m} \| (I - P)x^k \| \leq \sqrt{m} B \sum_{j=0}^{k} \gamma_j \lambda_2(W)^{k-j}.
\]
If \(\gamma_k = \frac{1}{(k+1)^{1/2}}\), we have
\[
\sum_{j=0}^{k} \gamma_j \lambda_2(W)^{k-j} = \sum_{j=0}^{k} \gamma_j \lambda_2(W)^{k-j}
+ \sum_{j=0}^{k} \gamma_j \lambda_2(W)^{k-j}\leq k \lambda_2(W)^{k/2}
+ \frac{2^\theta}{(k+1)^{\theta/2}} \sum_{j=0}^{k} \lambda_2(W)^{k-j}\leq k \lambda_2(W)^{k/2}
+ \frac{2^\theta}{(k+1)^{\theta/2}} 1 \leq \lambda_2(W) \leq C_W (k+1)^{\theta/2},
\]
where \(C_W := \frac{2^\theta}{1 - \lambda_2(W)} + \sup_k \{(k + 1)^{1+\theta} \lambda_2(W)^{k/2}\}\) and \(C_W = \mathcal{O}(\frac{1}{\lambda_2(W)})\).

C. Proof of Theorem 1
Multiplying both sides of (26) with \(P\),
\[
P x^{k+1} = PW x^k - \gamma_k P u^k = P x^k - \gamma_k P u^k.
\]
With direct computations, we get
\[
\|x^{k+1} - x^k\| = \frac{\|P x^{k+1} - P x^k\|}{\sqrt{m}} = \frac{\|\gamma_k P u^k\|}{\sqrt{m}} \leq B \cdot \gamma_k.
\]
For integer \(k \geq 1\), denote the integer \(T_k\) as
\[
T_k := \min \{ \max \left\{ \left[ \ln \left( \frac{k}{2C_H B^2} \right) / \ln \left( \frac{1}{\lambda(H)} \right) \right], K_H \right\}, k \}.
\]
By using Lemma 4 we then get
\[
\|H^{T_k}_{i,j} - \frac{1}{M} \| \leq \frac{1}{k} \cdot \frac{1}{2B^2},
\]
for any \(i, j \in \{1, 2, \ldots, M\}\). The remaining of the proof consists of two major steps:
1. in first step, we will prove \(\sum_k \gamma_k E[\| \nabla f(x^{k-T_k}) \|^2] \leq \mathcal{O}(\max \{1, \frac{1}{\ln(1/H)}\})\),
2. in second step, we will show \(\sum_k \gamma_k E[\| \nabla f(x^{k-T_k}) \|^2] \leq \gamma_k E[\| \nabla f(x^k) \|^2] \leq \mathcal{O}(\max \{1, \frac{1}{\ln(1/H)}\})\).
Summing them together, we derive
\[
\sum_k \gamma_k E[\| \nabla f(x_k) \|^2] \leq \mathcal{O}(\frac{1}{\ln(1/H)} \cdot \frac{1}{1 - \lambda_2(W)}).
\]
Then, we are led to
\[
\left( \sum_{i=1}^{k} \gamma_i \right) \cdot E \left( \min_{1 \leq i \leq k} \{ \| \nabla f(x^i) \|^2 \} \right) \leq \sum_{i=1}^{k} \gamma_i E[\| \nabla f(x^i) \|^2] \leq \mathcal{O}(\frac{1}{\ln(1/H)} \cdot \frac{1}{1 - \lambda_2(W)}).
\]
Rearrangement of (34) together with Schwarz inequality then gives us (29).

Step 1. Denote the shorthand notation \(\tilde{u}^k := \frac{\sum_{h=1}^{m} \nabla f_{j,h,k}(x^k(h))}{m}, \tilde{u}^{k-T_k} := \frac{\sum_{h=1}^{m} \nabla f_{j,h,k}(x^{k-T_k}(h))}{m}\),
we calculate the lower bound for following inner product:

\[
\mathbb{E}_{j_k} \left( \langle \nabla f(x^k - \tau_k), \tilde{u}^k - \tau_k \rangle \mid \chi^k - \tau_k \right) = \left\langle \nabla f(x^k - \tau_k), \frac{1}{m} \sum_{h=1}^{M} \nabla f_{j_h,k}(x^k - \tau_k(h)) \cdot \mathbb{P}(j_{h,k} = i \mid \chi^k - \tau_k) \right\rangle
\]

\[
\geq \left\langle \nabla f(x^k - \tau_k), \frac{1}{m} \sum_{h=1}^{M} \nabla f_{j_h,k}(x^k - \tau_k(h)) \cdot \mathbb{P}(j_{h,k} = i \mid j_{h,k} - \tau_k) \right\rangle
\]

\[
\geq \left\langle \nabla f(x^k - \tau_k), \frac{1}{m} \sum_{i=1}^{m} \nabla f_i(x^k - \tau_k(i)) \right\rangle - \frac{1}{2k},
\]

(35)

where \(a)\) is due to the conditional expectation, and \(b)\) uses the property of Markov chain, and \(c)\) is the matrix form of the probability, and \(d)\) is from (32). Direct calculations yield

\[
\|\nabla f(x^k - \tau_k)\|^2 = \left\langle \nabla f(x^k - \tau_k), \frac{1}{m} \sum_{i=1}^{m} \nabla f_i(x^k - \tau_k(i)) \right\rangle
\]

\[
= \|\nabla f(x^k - \tau_k)\|^2 + \langle \nabla f(x^k - \tau_k), \frac{1}{m} \sum_{i=1}^{m} \nabla f_i(x^k - \tau_k(i)) \rangle - \nabla f(x^k - \tau_k) \leq \|\nabla f(x^k - \tau_k)\|^2
\]

\[
+ \frac{L}{m} \sum_{i=1}^{m} \|x^k - \tau_k(i) - x^k - \tau_k\|.
\]

(36)

Rearrangement of (35) together with (36) gives us

\[
\gamma_k \mathbb{E}\|\nabla f(x^k - \tau_k)\|^2 \leq \gamma_k \mathbb{E}(\langle \nabla f(x^k - \tau_k), \tilde{u}^k - \tau_k \rangle)
\]

\[
+ \frac{B}{m} \sum_{i=1}^{m} \gamma_k \mathbb{E}\|x^k - \tau_k(i) - x^k - \tau_k\| + \frac{\gamma_k}{2k}.
\]

(37)

We offer the bound for \(f(x^{k+1}) - f(x^k)\) as

\[
f(x^{k+1}) - f(x^k) \stackrel{a)}{=} \langle \nabla f(x^k), x^{k+1} - x^k \rangle
\]

\[
+ \frac{L}{2} \|x^{k+1} - x^k\|^2 \stackrel{b)}{=} \langle \nabla f(x^k - \tau_k), x^{k+1} - x^k \rangle
\]

\[
+ \langle \nabla f(x^k) - \nabla f(x^k - \tau_k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2
\]

\[
\leq \langle \nabla f(x^k - \tau_k), x^{k+1} - x^k \rangle + \frac{(L + 1)}{2} \|x^{k+1} - x^k\|^2
\]

\[
+ \frac{L^2}{2} \|x^k - x^{k+1} - \tau_k\|^2,
\]

(38)

where \(a)\) depends on the continuity of \(\nabla f\), and \(b)\) is the basic algebra computation, \(c)\) uses the Schwarz in-
and
\[ \begin{align*}
\text{(III)}: & \mathbb{E}(\gamma_k \|x^k - x^{k-\gamma_k}\|) \leq \gamma_k \sum_{d=k-T_k}^{k-1} \mathbb{E}\|x^{d+1} - x^d\| \\
\leq & B \sum_{d=k-T_k}^{k-1} \gamma_d \gamma_k \leq B \frac{1}{2} \sum_{d=k-T_k}^{k-1} (\gamma_d^2 + \gamma_k^2) \\
= & \frac{\mathcal{T}_k B}{2} \gamma_k^2 + \frac{B}{2} \sum_{d=k-T_k}^{k-1} \gamma_d^2.
\end{align*} \]

and
\[ \begin{align*}
\text{(IV)}: & \mathbb{E}(\|x^{k+1} - x^{k-\gamma_k}\|^2) \leq (\mathcal{T}_k + 1) \sum_{d=k-T_k}^{k} \mathbb{E}\|x^{d+1} - x^d\|^2 \\
\leq & B^2 (\mathcal{T}_k + 1) \sum_{d=k-T_k}^{k} \gamma_d^2.
\end{align*} \]

\[ \begin{align*}
\text{(V)}: & B I \frac{m}{2} \sum_{j=1}^{m} \gamma_k \mathbb{E}\|x^{k-\gamma_k}(i) - x^{k-\gamma_k}\| \\
\leq & B \frac{L C W}{2m} \gamma_k \sum_{d=k-T_k}^{k} \gamma_d^2. \\
\leq & B \frac{L C W}{2m} (\mathcal{T}_k + \gamma_k^2).
\end{align*} \]

It is easy to see if \((\mathcal{T}_k \sum_{d=k-T_k}^{k} \gamma_d^2)_{k \geq 0}\) is summable, (II), (III) and (IV) are all summable. In fact, the proof of summability of \((\mathcal{T}_k \sum_{d=k-T_k}^{k} \gamma_d^2)_{k \geq 0}\) can direct follow the proofs in [3], which is presented here for completeness.

We consider a large enough integer \(K\) which lets Lemma[1] and Lemma[5] be active, and \(\mathcal{T}_k = \ln(\frac{k}{2 \epsilon_n B^2})/\ln(1/\lambda(H))\) when \(k \geq K\). Noting that the summability of sequence is free of finite items, then we can consider studying \((\mathcal{T}_k \sum_{d=k-T_k}^{k} \gamma_d^2)_{k \geq K}\). For any fixed integer \(t \geq K\), \(\gamma_d^2\) only appears at index \(k \geq K\) satisfying \(S_t := \{k \in \mathbb{Z}^+ \mid k - \mathcal{T}_k \leq t - k, k \geq K\}\) in the inner summation. Let \(k(t)\) be the solution of \(k - \mathcal{T}_k = t\). The direct computation gives us \(\#(S_t) \leq k(t) - t \leq 2 \ln(t + 1/\lambda(H)), \) where the last inequality comes from Lemma[5] if \(K\) is large enough, \(\mathcal{T}_k \leq \frac{k}{2},\) and then \(k \leq 2t, \forall k \in S_t\). Noting that \(\mathcal{T}_k\) increases respect to \(k\), then we get \(\mathcal{T}_k \leq 2t_k, \forall k \in S_t\). That means in \(\sum_{k=K}^{t} \mathcal{T}_k \sum_{d=k-T_k}^{k} \gamma_d^2,\) \(\gamma_k^2\) appears at most \(2t\) \(\#(S_t) = O(\ln^2 t)\). The direct calculations then give us
\[ \begin{align*}
\sum_{k=K}^{t} \sum_{d=k-T_k}^{k-1} \gamma_d^2 \\
= O\left(\sum_{d=k}^{\infty} \frac{\ln^2 d}{\ln(1/\lambda(H))}\right) = O(\ln(1/\lambda(H))).
\end{align*} \]

Turning back to (42) and Lemma[3] we are then led to
\[ \begin{align*}
\sum_k \gamma_k \mathbb{E} \|\nabla f(x^{k-\gamma_k})\|^2 = O\left(\frac{1}{\ln(1/\lambda(H))} \cdot \frac{1}{1 - \lambda_2(W)}\right).
\end{align*} \]

\textbf{Step 2:} With Lipschitz of \(\nabla f\), it holds that
\[ \begin{align*}
\gamma_k \|\nabla f(x^{k+1})\|^2 - \gamma_k \|\nabla f(x^{k-\gamma_k})\|^2 & \leq \gamma_k (\nabla f(x^k) - \nabla f(x^{k-\gamma_k}), \nabla f(x^{k-\gamma_k}) + \nabla f(x^{k-\gamma_k})) \\
& \leq \gamma_k \|\nabla f(x^k) - \nabla f(x^{k-\gamma_k})\| \cdot \|\nabla f(x^k) + \nabla f(x^{k-\gamma_k})\| \\
& \leq 2BL \gamma_k \|x^k - x^{k-\gamma_k}\|^2 \\
& \leq BL \gamma_k^2 + BL \|x^k - x^{k-\gamma_k}\|^2.
\end{align*} \]

We have proved \(\sum_{k=K}^{\infty} \mathbb{E}\|x^{k+1} - x^{k-\gamma_k}\|^2 = O(1/\lambda(H))\) it is same way to prove \(\sum_{k=K}^{\infty} \mathbb{E}\|x^k - x^{k-\gamma_k}\|^2 = O(1/\lambda(H))\). Thus, we can get
\[ \sum_k (\gamma_k \mathbb{E}\|\nabla f(x^k)\|^2 - \gamma_k \mathbb{E}\|\nabla f(x^{k-\gamma_k})\|^2) = O(1/\lambda(H)). \]

On the other hand, with (50), we can get
\[ \|\nabla f(x^{k+1})\|^2 - \|\nabla f(x^k)\|^2 = O(\gamma_k). \]

Thus, we derive
\[ \|\nabla f(x^{k+1})\|^2 - \|\nabla f(x^k)\|^2 \leq \mathbb{E}\|\nabla f(x^{k+1})\|^2 - \|\nabla f(x^k)\|^2 = O(\gamma_k). \]

With (33) and (47), Lemma[4] then gives us (3).
Now, we are prepared to prove the theorem. This proof is very similar to the proof of Theorem 1. The main difference is to modify (40) as

\[ E(\langle \nabla f(x_h^k), x_h^k - x_{h+1}^k \rangle | \mathcal{G}^k) \]

\[ = a_k \gamma_k E(\langle \nabla f(x_h^k), u_h^k - e^k \rangle | \chi_h^k) \]

\[ = a_k \gamma_k E(\langle \nabla f(x_h^k), u_h^k - e^k \rangle | \chi_h^k) \]

\[ + a_k \gamma_k E(\langle \nabla f(x_h^k), e^k \rangle | \chi_h^k) \]

\[ + a_k \gamma_k E(\langle \nabla f(x_h^k), e^k \rangle | \chi_h^k) \]

\[ \geq a_k \gamma_k E(\langle \nabla f(x_h^k), u_h^k - e^k \rangle | \chi_h^k) - B \gamma_k \| e^k \| \]

\[ - B \cdot L \cdot E(\gamma_k \| x_h^k - x_{h+1}^k \| | \chi_h^k), \] (50)

where \( a_k \) is due to (43), and \( b \) depends on the Cauchy-Schwarz inequality. Taking total expectations on (50), we have

\[ \gamma_k E(\langle \nabla f(x_h^k), u_h^k - e^k \rangle) \]

\[ \leq (I) + (II) + (III) + (IV) + B \gamma_k \| e^k \|, \] (51)

where (I), (II), (III) and (IV) are given by (42). By using (49), we can prove

\[ \gamma_k E(\langle \nabla f(x_h^k), u_h^k - e^k \rangle) \]

\[ = O\left( \frac{1}{\ln(1/\lambda(H))} \cdot \frac{1}{\ln(1/\lambda(H))} + n^{2} \right). \] (52)

The following is the same as the proof for Theorem 1.

E. Proof of Proposition 1

The proofs of Proposition 1 is similar to the proof of Theorem 1. We first present a lemma as follows.

**Lemma 6.** Let \( (x^k)_{k \geq 0} \) be generated by scheme (16) and Assumption 5 hold, if the stepsizes are selected as (7), then we have

\[ \frac{1}{m} \sum_{i=1}^{m} \|x^k(i) - x^k\| = O\left( \frac{1}{1 - \lambda_2(W)} \cdot \frac{1}{(k + 1)^{d}} \right). \]

Like previous methods, the proof also consists of two parts:

1. In the first one, we prove \( \sum_{k} \gamma_k E\| \nabla F(x_h^k) \|^2 \]

\[ = O\left( \frac{1}{\ln(1/\lambda(H))} \cdot \frac{1}{\ln(1/\lambda(H))} \right), \]

2. In the second one, we focus on proving

\[ \sum_{k} \left( \gamma_k E\| \nabla F(x_h^k) \|^2 \right) - \gamma_k E\| \nabla F(x_h^k) \|^2 \]

\[ = O\left( \frac{1}{\ln(1/\lambda(H))} \cdot \frac{1}{\ln(1/\lambda(H))} \right). \]

**Step 1.** Assume that \( C \) and \( \lambda \) are the factors in (15) for Markov chain in the \( n \)th node. Let \( C := \max\{C^1, C^2, \ldots, C^m\} \) and \( \lambda := \max\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \). For integer \( k \geq 1 \), we consider the integer \( \mathcal{H}_k \) as

\[ \mathcal{H}_k := \min\left\{ \left[ \ln \left( \frac{k}{2C \cdot B^2} \right) / \ln(1/\lambda) \right], k \right\}. \] (53)

It is easy to see \( \mathcal{H}_k \leq k \). With [Theorem 4.9, 5], we have the following relation

\[ \int_{\xi} p_{x + \mathcal{H}_k}(\xi) - \pi(\xi) d\mu(\xi) \leq \frac{1}{2 \cdot B \cdot k} \cdot \forall s \in \mathbb{Z}^+ \] (54)

where \( p_{x + \mathcal{H}_k}(\xi) \) denotes the transition p.d.f. from \( s \)

to \( s + \mathcal{H}_k \) with respect to \( \xi \). The property of time-homogeneous of the Markov chain directly gives that

\[ s \] (55)

\[ \| F(x_h^k) \|^2 \leq \gamma_k E\| \nabla F(x_h^k) \|^2 \]

\[ + \gamma_k B \sum_{i=1}^{m} E\| x^k(i) - x^k \|. \] (56)

Then we need to bound \( F(x_h^k) - F(x^k) \). With Assumption

\[ \nabla F(\cdot) \] is Lipschitz continuous, the rest part is almost identical to the one of previous proof and will not be repeated.

**Step 2.** This step is very similar to the second step in the proofs of Theorem 1 and will not be repeated, either.

VIII. CONCLUSIONS

In this paper, we proposed the decentralized Markov chain gradient descent (DMGD) algorithm, where the samples are taken along a trajectory of Markov chain over the network. Our algorithms can be used when it is impossible or very expensive to sample directly from a distribution, or the distribution is even unknown, but sampling via a Markov chain is possible and relatively cheap. The convergence analysis is proved in possibly nonconvex cases.

Building upon the current work, several promising future directions can be pursued. The first one is to extend DMGD to the asynchronous setting, which can further reduce the synchronization overhead. The second one is to reduce the communications cost in DMGD by using quantization or sparsification techniques. Designing Markov chain primal-dual algorithms is also worth investigating.

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