Creation of a monopole in a spinor condensate

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We propose a method to create a monopole structure in a spin-1 spinor condensate by applying the basic methods used to create vortices and solitons experimentally in single-component condensates. We show, however, that by using a two-component structure for a monopole, we can simplify our proposed experimental approach and apply it also to ferromagnetic spinor condensates. We also discuss the observation and dynamics of such a monopole structure, and note that the dynamics of the two-component monopole differs from the dynamics of the three-component monopole.

Introduction—The experimental realization of spinor Bose-Einstein condensates makes it feasible to extend the study of topological quantum objects into an entirely new field of physics. Ordinary single-component condensates have many topologically interesting properties such as the existence of vortices. But in spinor condensates one can also study phenomena that cannot exist in the single-component systems. One example is a monopole structure in a spinor condensate with antiferromagnetic interactions as proposed by Stoof et al. recently. Other novel possibilities also exist such as skyrmions. With Bose-Einstein condensates a crucial aspect is not only the existence and stability of topological structures, but also the methods for their creation and observation, as well as their dynamics. In this Letter we address all these aspects for a monopole structure in an experimentally relevant case of a multicomponent Bose-Einstein condensate.

A monopole is a topological defect in a vector field. It is characterized by a unit vector that is radial in respect to some unique central point (i.e. the “hedgehog” defect). In spinor condensates the vector quantity could be the local spin of the condensed atoms, but other choices are also possible. Monopoles have been studied theoretically in two-dimensional condensates. Recently some results for the three-dimensional case (such as density distribution, energy and dynamics of such a defect) have been studied by Stoof et al. They described a monopole created in an antiferromagnetic spin-1 condensate such as Na. The monopole was characterized by a spinor

\[
\zeta = \sqrt{n} \begin{pmatrix} -m_x + im_y \\ \sqrt{2m_z} \\ m_x + im_y \end{pmatrix},
\]

where \(n\) is the condensate density and the vector \(m = \pm\hat{r}/r\) is a radial unit vector and has the spherically symmetric “hedgehog” structure. Stoof et al. demonstrated that this particular spin texture is a unique consequence of the unit winding number and minimization of the gradient energy. The monopole can also be displaced from the center of the trap without changing their argument. The spinor in Eq. (1) is non-magnetized and can be achieved from the single-component mean-field ground state, \(\zeta_T = \sqrt{n}(010)\), with local spin-field rotations. Consequently, at each position it resembles the ground state and thus the absence of dynamical instabilities which lead to domain formation is ensured.

Two-component monopole—One is not, however, limited to the antiferromagnetic texture given in Eq. (1) when considering monopoles. We can alternatively map the vector \(m\) into an effective two-component system:

\[
\zeta = \sqrt{n} \begin{pmatrix} -m_x + im_y \\ 0 \end{pmatrix}.
\]

To ensure the stability of this texture against phase-separation, the spin-1 condensate must have ferromagnetic interactions, which makes the \(^{87}\)Rb spinor condensate a potential candidate. In other words, the preparation of a monopole is not limited to the antiferromagnetic \(^{23}\)Na system as expected before, if we accept spinors that do not have the order-parameter space of the ground state. Consequently, it should be noted, that the texture in Eq. (2) cannot be produced by local rotations from the ferromagnetic ground state, \(\zeta_F = \sqrt{n}(100)\).

The static properties of the two-component monopole are similar to those of a three-component spinor (there are differences in dynamics, as we shall discuss later). Because it should also be easier to create experimentally, we focus on the two-component case although our calculations are done for the actual three-component system. We note that as the spinor in Eq. (2) is not the mean-field ground state locally, some relaxation towards the true ground state is to be expected. But this requires spin-changing processes, that are very slow and can thus be ignored at timescales of interest.

We describe the spinor condensate with a multicomponent wave function and label the components as \(\psi_m\) with the spin projection quantum number \(m\) (\(m = 0, \pm 1\)). The mean-field Gross-Pitaevskii (GP) equations are
\[
\begin{align*}
    \hbar \frac{\partial \psi}{\partial t} &= \mathcal{L} \psi + \lambda_n \left( \psi_0^* \psi_1 + |\psi_0|^2 \psi_1 + |\psi_1|^2 \psi_0 - |\psi_1|^2 \psi_0 \right) \\
    \hbar \frac{\partial \psi_0}{\partial t} &= \mathcal{L} \psi_0 + \lambda_n \left( 2 \psi_1^* \psi_0 |\psi_0|^2 + |\psi_1|^2 \psi_0 + |\psi_1|^2 \psi_0 \right) \\
    \hbar \frac{\partial \psi_1}{\partial t} &= \mathcal{L} \psi_1 + \lambda_n \left( \psi_0^* \psi_{-1} + |\psi_1|^2 \psi_1 + |\psi_0|^2 \psi_1 - |\psi_1|^2 \psi_0 \right),
\end{align*}
\]

where \( \mathcal{L} = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} + \lambda_s \left( |\psi_{-1}|^2 + |\psi_0|^2 + |\psi_1|^2 \right) \), \( \lambda_s = \frac{\mathcal{B}}{B} (a_0 + 2a_2) \), \( \lambda_n = \frac{\mathcal{B}}{B} (a_0 - a_0) \), and \( a_F \) is the s-wave scattering length in the total hyperfine two-atom \( F \)-channel. For a cylindrically symmetric trapping potential we use \( V_{\text{trap}} = m\omega_{\text{z}}^2 (x^2 + y^2)/2 + m\omega_{\text{r}}^2 z^2/2 \).

Monopole structure and stability.—To understand and test our approach for monopole creation we study the monopole structure by solving Eq. (3) numerically. In order to create the monopole as an initial state of our numerical study we force the texture given in Eq. (2) (or Eq. (1)) into the order parameter (this should not be confused with the actual proposal for experimental creation of the monopole, to come later). The imprinted spinor is then propagated in imaginary time until sufficient convergence is reached. If the monopole is at the center of the condensate the imprint has to be done only once at the beginning of the iteration, otherwise the imprint must be repeated in the course of the iteration to prevent the monopole from drifting away from the intended location, to a location with lower energy. In Fig. 1 we show the typical density distribution of the spin-1 monopole located at the center of a trap.

By looking at the individual condensate components we gain relevant insight into the structure of the monopole. Here \( \psi_{-1} \) has a vortex at \( z = 0 \) with a core size that is a function of \( z \). On the other hand, \( \psi_0 \) goes through a \( \pi \) phase shift as we move from positive to negative \( z \) values; consequently, this component relates to a soliton in a single-component condensate. The \( m = 0 \) component atoms fill the vortex line everywhere else except at the origin, where the density of \( m = 0 \) component also vanishes. Therefore the intersection of the vortex line with the soliton plane gives rise to a monopole core. Thus, if we can experimentally create a vortex and a soliton in a two-component system, we can obtain a monopole.

If we displace the monopole from the center of the spherically symmetric trap, the energy of system decreases as function of the displacement (in qualitative agreement with the results in Ref. [3]). This indicates that in a dissipative environment the monopole is expected to drift to the edge of the condensate and vanish. The estimate for the timescale of this process goes beyond the model used in this paper. But in single-component condensates the vortex lifetimes can be several seconds [4] and this timescale is also, presumably, indicative of the monopole lifetime. On the other hand, the dissipative processes might be masked by the topological instability of the monopole. The unit winding number of the spin texture is not sufficient to protect the topological stability of the system and instead of drifting smoothly to the condensate edge, the monopole might decay into other excitations.

Creation of monopoles.—The separate look into each spin component of the monopole structure suggests a possible way to create it. For example, we can prepare a spinor condensate with \( 2/3 \) population at the \( m = 1 \) state and the rest at \( m = 0 \) state, e.g. with an rf-pulse [14]. A “blueprint” of the monopole is achieved by creating a vortex into the \( m = 1 \) component and a soliton (with \( \pi \) phase discontinuity) into the \( m = 0 \) component, both at the trap center. The vortex line should lie along the phase discontinuity in the \( m = 0 \) wavefunction. In Fig. 2 we demonstrate the time-evolution of such a mixture in a cigar shaped trap in real time, when the soliton and vortex were created using phase-imprint method [6]. With other excitations abound, it is clear that we nevertheless have a monopole inside the condensate.

In our numerical studies we have used a certain amount of smoothing to reduce the amount of noise that would be created if a phase-imprint is too abrupt. Smoothing for a vortex was done by assuming that not only do we have a phase-mask, but also a narrow (on the order of the coherence length) laser beam that bores a hole through the \( m = 1 \) component along the vortex line. For a soliton the \( \pi \) phase jump was done at the distance of the order of the coherence length. Without such smoothing our numerical approach becomes unstable. It is not clear how much is this smoothing required in actual experiments, although some amount would be always present. To obtain this combined phase-imprint experimentally is

FIG. 1. The total density \( (l = \sqrt{\hbar/m\omega}) \) of the spin-1 condensate with \( 5 \cdot 10^4 \text{ } ^{87}\text{Rb} \) atoms when \( y = z = 0.05 \) (origin is not present in our discretization scheme). The monopole is at the center of a spherically symmetric trap with a trap frequency \( \omega = \omega_r = \omega_z = (2\pi) 50 \text{ Hz.} \)
probably complicated, but at least the two main ingredients, namely experimental creation of vortices \[14\] and solitons \[17\] has already been achieved.

\[
\begin{align*}
\text{\(\omega t = 0.00\)} & \quad \text{\(\omega t = 0.57\)} & \quad \text{\(\omega t = 2.83\)} \\
\end{align*}
\]

FIG. 2. The time evolution of the total density of a condensate with \(5 \cdot 10^4\) \(^{87}\)Rb atoms in a cylindrically symmetric trap with frequencies \(\omega = \omega_r = (2\pi) 250\) Hz and \(\omega_z = (2\pi) 50\) Hz. Initially there is a vortex at the \(m = -1\) component (perpendicular line) and a soliton at the \(m = 0\) component (horizontal line). The vortex and the soliton are imprinted at \(t = 0\) and the figures show the cut of the total density in \(y = 0.09\) plane. \(x\)-axis is along the horizontal direction.

**Observation of monopoles.**—The monopole core has roughly the size of the healing length, and it is inside the condensate. Thus its direct observation is difficult. But the same methods used to observe vortices \[14\] and vortex rings \[18\] can be applied to observe monopoles as well. One should first let the condensate expand and then image the 3D structure of the different \(m\)-states using two orthogonal probe beams. As for separating the different \(m\)-states, one can use an appropriate Stern-Gerlach apparatus \[16\].

We have studied the behavior of a freely expanding monopole using the time-dependent generalization of the monopole GP-equation in Ref. \[3\]. This approximation assumes equal scattering lengths, but at the timescales of interest the role of differing scattering lengths is in fact negligible. In the limit that \(a_2 = a_0\) all the components feel the same spherically symmetric potential (external potential and the mean field terms) and are not sensitive to the phase of the other components. Therefore a single-component GP-equation is sufficient for modeling the expanding monopole.

In Fig. \[3\] we show an example of the time-evolution of the ratio of the monopole core size to the system radius once the trapping potential is turned off. Expansion is qualitatively similar to vortex expansion in a scalar condensate. At small times the monopole core size, \(\xi\), adjusts (almost) instantaneously to the local density \[14\] and one expects the size of the core to scale as the healing length, \(\xi_0 = 1/\sqrt{4\pi a n}\). If we model the wave function as

\[
\Psi(r) = A \exp \left( -\frac{r^2}{2R^2} \right) \exp \left( i\beta r^2 \right),
\]

where \(A\) is the normalization factor, the monopole grows faster than the expanding condensate, or more precisely

\[
\frac{\xi_0}{R} = \frac{\pi^{1/4} R^{1/2}}{\sqrt{4\pi a N}}.
\]

This happens as long as the characteristic time for adjustment of the core size, \(\tau_{ad} \sim \hbar/n\lambda_s\), is much less than the expansion time \(\tau_{ex} \sim R/c_s\), where \(c_s\) is the sound velocity. The parameters in Fig. \[3\] imply that in this regime the condensate can expand by an order of magnitude, and in the end of this regime the monopole size compared to the condensate size has greatly increased. At later times the atoms in the cloud will evolve as free particles and \(\xi_0/R\) will settle to some constant value. We also compared the results obtained with a single component GP-equation (at small times) against the solution of the multi-component GP-equations \[3\] and found that the two approaches give essentially the same results.

FIG. 3. Time evolution of the ratio of the monopole core size \(R_m\) to the condensate size \(R_c\). These sizes were determined from locations where the density was one half of the maximum density. Initially we assumed \(5 \cdot 10^4\) condensed rubidium atoms in a spherically symmetric trap with a trap frequency \(\omega = (2\pi) 50\) Hz.

**Monopole dynamics.**—The dynamics of a monopole is quite interesting. As expected, the monopole at the origin is stable and stationary. A displaced monopole, on the other hand, behaves differently \[20\]. The monopole precesses around the trap center inside the condensate, just like a displaced vortex line does. It returns to its initial location after \(T = 2\pi/\Omega_P\). For a vortex close to the center the precession frequency \(\Omega_P\) coincides with the frequency of the anomalous mode \(\omega_a\). For a disc-shape trap an analytic result is available \[21\] and is given by
\[ \omega_n = \frac{3 \hbar}{2 m R^2} \ln \left( \frac{R}{\xi} \right), \]  

where \( R \) is the radius of the system and \( \xi \) is the healing length. Even though the trap geometry in our example is nowhere near the disk-shape we expect that Eq. (6) gives a reasonable order of magnitude estimate. Especially so since the \( m = -1 \) atoms are "squeezed" between lobes of \( m = 0 \) atoms, thus making the disc-shaped approximation rather justified.

As a test case we take a spherically symmetric trap with trap frequency \( \omega = (2\pi) 50 \text{ Hz} \) and a \(^{87}\text{Rb} \) condensate with \( 5 \times 10^4 \) atoms. Setting \( R \) to the Thomas-Fermi radius of the system and calculating \( \xi \) from the Thomas-Fermi result at the trap center we get an estimate \( \omega \tau \approx 35 \). This value is fairly close to the value actually seen in our 3D simulation of the monopole dynamics. If the monopole was displaced by one-fifth of the Thomas-Fermi radius our numerical result is \( \omega \tau \approx 38 \). The above estimate is surprisingly accurate. In particular because the precessing vortex should feel the mean field of the other component.

The dynamical behavior of the three-component monopole of Eq. (1) is different from above. In this case one has a vortex at the \( m = -1 \) state and an anti-vortex at the \( m = 1 \) state. As a displaced vortex and an anti-vortex precess in opposite directions, the monopole core will vanish only to reappear at the opposite side as soon as the vortices have precessed that far. This recurrence is almost perfect \([20]\). Partial revival of a monopole has also been predicted in case of a 2D monopole \([8]\). As the vortices precess in opposite directions the order parameter becomes magnetized and can no longer be represented in the form given by Eq. (1). Therefore, the order parameter is no longer in the order parameter space of the ground state. Obviously, our proposed approach to create monopoles experimentally applies to the three-component case as well. But then one needs to create a vortex/antivortex in the \( m = \pm 1 \) state, respectively, in addition to the soliton in the \( m = 0 \) state.

In Ref. \([8]\) the dynamics of the monopole were due to the two different scattering lengths. In our inhomogeneous spinor condensate the dynamics are not intimately connected with differing scattering lengths. Setting \( a_2 = a_0 \) does not change our results qualitatively and even the quantitative changes are small. At longer times some population dynamics can be observed, but the dynamics of the total density is almost unaffected. Therefore it seems clear that in an inhomogeneous spinor-condensate the dynamical behavior of the monopole goes beyond the model suggested by Stoof \( et \ al. \) in Ref. \([4]\).

To summarise, we have proposed and demonstrated numerically a method to create monopoles in three-dimensional Bose-Einstein condensates, and shown that monopole creation is not limited to the antiferromagnetic spinor condensates. In addition we have studied the detection by expansion of such monopoles, and also the dynamics of displaced monopoles in a trap. For the creation of two-dimensional monopoles, an off-resonant Raman beam has been suggested \([7]\), but so far there has not been any suggestions for creating a three-dimensional monopole in a realistic experiment. Also, by replacing a single vortex at one component with a lattice of vortices in our approach could lead to creation of multiple monopoles, and allow one to study interactions between monopoles.

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