Double diffusion structure of logarithmically damped wave equations with a small parameter

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Abstract

We consider a wave equation with a nonlocal logarithmic damping depending on a small parameter \(0 < \theta < \frac{1}{2}\). This research is a counterpart of that was initiated by Charão-D’Abbicco-Ikehata considered in [5] for the large parameter case \(\theta > \frac{1}{2}\). We study the Cauchy problem for this model in \(\mathbb{R}^n\) to the case \(\theta \in (0, \frac{1}{2})\), and we obtain an asymptotic profile and optimal estimates in time of solutions as \(t \to \infty\) in \(L^2\)-sense. An important discovery in this research is that in the case when \(n = 1\), we can present a threshold \(\theta^* = \frac{1}{4}\) of the parameter \(\theta \in (0, \frac{1}{2})\) such that the solution of the Cauchy problem decays with some optimal rate for \(\theta \in (0, \theta^*)\) as \(t \to \infty\), while the \(L^2\)-norm of the corresponding solution never decays for \(\theta \in [\theta^*, \frac{1}{2})\), and in particular, in the case \(\theta \in [\theta^*, \frac{1}{2})\) it shows an infinite time \(L^2\)-blow up of the corresponding solutions. The former (i.e., \(\theta \in (0, \theta^*)\) case) indicates an usual diffusion phenomenon, while the latter (i.e., \(\theta \in [\theta^*, \frac{1}{2})\) case) implies, so to speak, a singular diffusion phenomenon. Such a singular diffusion in the one dimensional case is a quite novel phenomenon discovered through our new model produced by logarithmic damping with a small parameter \(\theta\). It might be already prepared in the usual structural damping case such as \((-\Delta)^\theta u_t\) with \(\theta \in (0, 1/2)\), however unfortunately nobody has ever just pointed out even in the structural damping case.

1 Introduction

We consider in this work the dissipative wave equation based on an operator \(L_\theta\), that combines the composition of logarithm function with the Laplace operator, as follows:

\[
\begin{align*}
  u_{tt} - \Delta u + L_\theta u &= 0, & (t, x) &\in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) &= u_0(x), & u_t(0, x) &= u_1(x), & x &\in \mathbb{R}^n.
\end{align*}
\]

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where the linear operator

\[ L_\theta : D(L) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \theta > 0, \]

is defined as follows:

\[ D(L_\theta) := \left\{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} (\log(1 + |\xi|^{2\theta}))^2 |\hat{f}(\xi)|^2 d\xi < +\infty \right\}, \]

and for \( f \in D(L_\theta), \)

\[ (L_\theta f)(x) := \mathcal{F}_{\xi \to x}^{-1} \left( \log(1 + |\xi|^{2\theta})\hat{f}(\xi) \right)(x). \]

Here, one has just denoted the Fourier transform \( \mathcal{F}_{x \to \xi} \) of \( f(x) \) by

\[ \mathcal{F}_{x \to \xi}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx, \quad \xi \in \mathbb{R}^n, \]

as usual with \( i := \sqrt{-1} \), and \( \mathcal{F}_{\xi \to x}^{-1} \) expresses its inverse Fourier transform. Since the operator \( L_\theta \) is non-negative and self-adjoint in \( L^2(\mathbb{R}^n) \) (see [3]), the square root

\[ L_\theta^{1/2} : D(L_\theta^{1/2}) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \]

can be defined, and is also nonnegative and self-adjoint with its domain

\[ D(L_\theta^{1/2}) = \left\{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} (\log(1 + |\xi|^{2\theta}))|\hat{f}(\xi)|^2 d\xi < +\infty \right\}. \]

Note that \( D(L_\theta^{1/2}) \) becomes Hilbert space with its graph norm

\[ \| v \|_{D(L_\theta^{1/2})} := \left( \| v \|^2 + \| L_\theta^{1/2} v \|^2 \right)^{1/2}, \]

where to simplify the notation we define the \( L^2(\mathbb{R}^n) \)-norm by

\[ \| \cdot \| := \| \cdot \|_{L^2(\mathbb{R}^n)}. \]

We also note that

\[ H^s(\mathbb{R}^n) \hookrightarrow D(L_\theta^{1/2}) \hookrightarrow L^2(\mathbb{R}^n) \]

for any \( s > 0 \). Symbolically writing, one can see

\[ L_\theta = \log(I + (-\Delta)^\theta), \]

where \( \Delta \) is the usual Laplace operator defined on \( H^2(\mathbb{R}^n) \). Since \( L_\theta \) is constructed by a nonnegative-valued multiplication operator, it is nonnegative and self-adjoint in \( L^2(\mathbb{R}^n) \). Then, by a similar argument to [25, Proposition 2.1] based on the Lumer-Phillips Theorem one can find that the problem (1.1)-(1.2) has a unique mild solution

\[ u \in C([0, \infty); H^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n)) \]

for each \( [u_0, u_1] \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), and the associated energy identity holds

\[ E_u(t) + \int_0^t \| L_\theta^{1/2} u(s, \cdot) \|^2 ds = E_u(0), \quad t > 0, \quad (1.3) \]

where

\[ E_u(t) := \frac{1}{2} \left( \| u(t, \cdot) \|^2_{L^2} + \| \nabla u(t, \cdot) \|^2_{L^2} \right). \]

The identity (1.3) implies that the total energy is a non-increasing function in time because of the existence of the dissipative term \( L_\theta u_t \).
We first try to review some known historical results (basically restricting to linear equations).

On the strongly damped wave equation such that

\[ u_{tt} - \Delta u - \Delta u_t = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad (1.4) \]

one can mention to the celebrated papers [37] and [35] concerning the \( L^p-L^q \) estimates of solutions to the equation (1.4), and all related researches have their origin in there. After [37] and [35], a next important topic concerning (1.4) is extensively studied by [29], [23] and [24], that is, they investigate asymptotic profiles, and optimal estimates of the \( L^2 \)-norm of solutions as \( t \to \infty \). By measuring the solution in terms of \( L^2 \)-norm, a singularity near 0-frequency region of the solution to problem (1.4) can be captured precisely. For a higher order asymptotic expansion as \( t \to \infty \) of the solution to (1.4) it can be derived by [31] recently. A higher order asymptotic expansion in time of the square of \( L^2 \)-norm of solutions to the equation (1.4) can be precisely obtained by [2] [8]. In this connection, it should be mentioned that a critical exponent problem to the semi-linear equation:

\[ u_{tt} - \Delta u - \Delta u_t = |u|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n \]

can be studied by [17] based on the \( L^p-L^q \) estimates due to [37]. Unfortunately, it seems still open to determine the critical exponent \( p^* \) of the power \( p > 1 \) of the nonlinearity. As for small data global existence and blow-up results to the strongly damped waves with two different kinds of nonlinearity \( |u|^p + |u_t|^q \) or (simply) \( |u|^q \), one can cite [5], [10]. The problems [5], [10] are considered in an exterior domain.

On the other hand, recently a more general fractionally damped wave equations such that

\[ u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\theta u_t = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad (1.5) \]

are studied intermittently, however, a research in the case of \( \theta = 0 \) has initiated to the paper by [30] in 2000, there the author has captured a self-similar profile in asymptotic sense as \( t \to \infty \) (in fact, the author treats more general semi-linear problems). A generalized version of [30] has been just published recently in [21]. One can also cite [13], [14], [17], [18], [23], [34] with \( \sigma = 1 \) about the study for precise asymptotic profiles and/or critical exponent of the nonlinear problems, \( L^p-L^q \) estimates and/or asymptotic profiles in the case of \( \frac{1}{2} \leq \theta < 1 \) to the linear equation (1.5) with \( \sigma = 1 \) can be considered by [16], [31], [28] and [33].

Under these observations on the equation (1.5), quite recently the authors in [6] have presented a new type of wave equations with a non-local logarithmic damping:

\[ u_{tt} - \Delta u + \log(I - \Delta) u_t = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad (1.6) \]

where \( I \) is an identity operator, and the authors investigated asymptotic profiles and optimal decay rates of solutions to problem (1.6). In this connection, one can understand that the study by [6] has a close relation to some properties of hypergeometric functions. As consequence, one can know that the solution to (1.6) has a similar property to that of (1.4). As a next natural problem to (1.6), the authors in [5] study the following generalized equation with parameter \( \theta > 0 \):

\[ u_{tt} - \Delta u + \log(I + (-\Delta)^\theta) u_t = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n. \quad (1.7) \]

In fact, the authors in [5] treat the case of \( \frac{1}{2} < \theta < 1 \), and investigate asymptotic profiles and optimal decay rates for the solution itself and the total energy. A general theory on hypergeometric functions can be also effectively used in the proof. The profiles include an oscillation property coming from the condition \( \theta > \frac{1}{3} \). In this connection, even if \( \theta \) is large enough, say \( \theta > 1 \), the equation (1.7) does not have any regularity-loss structures, while in the case when \( \theta > 1 \) the equation (1.5) with \( \sigma = 1 \) has such a regularity-loss structure as was pointed out in [26], [20]. A research in [26] has been motivated by an interesting paper by [22]. By the way, such a regularity loss structure has been first discovered by S. Kawashima through the research on Timoshenko systems (e.g., [23]). Additionally, a similar study on asymptotic wave-like property of the solution to the equation

\[ u_{tt} + \log(I - \Delta) u + \log(I - \Delta) u_t = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n. \]
can be deeply investigated in [7].

Here, we mention an answer to the question: why do we study the wave equation with a log-damping term in a series of our papers? This comes from an observation below.

At first, notice that the properties of the equation such that
\[
u_{tt} - \Delta u + (-\Delta)^\theta u_t = 0
\] (1.8)
can be divided into two parts:
(a) \(0 \leq \theta \leq 1\) \(\Rightarrow\) the equation (1.8) does not have a regularity-loss structure, and this fact is well-studied.
(b) \(\theta > 1\) \(\Rightarrow\) the equation (1.8) has a regularity-loss structure coming from a priori estimates in the high frequency region (cf., [26]) as is already mentioned.

By comparing the equations (1.8) with (1.1) can be included in the result below. This part will be essential in our paper.

By the way, it should be noticed that general theories including semigroup approaches concerning the logarithmic Laplacian can be studied in detail by a series of papers and books due to Amann [1, Chapter III, p. 152], Nollau [32], Reed-Simon [36, p. 317], and Weilenmann [40], Chen-Weth [11], and in particular, one can refer to an interesting recent published paper [4], which studies the logarithmic wave case. As a result, the equation (1.1) does not have any regularity-loss structures for all \(\theta > 0\) (even if \(\theta\) is big enough). This is one of our merit to study the equation (1.1) with \(0 < \theta < \theta^*\) for any \(\theta^* > 0.3\) this implies that the asymptotic behavior in the high frequency region of the solution to (1.1)-(1.2) in the case when (ideally speaking) \(0 < \theta < 1\) is only from a pure mathematical point of view, and without loss of generality we can assume that the

Our main goal in this paper is to find an asymptotic profile of solutions in the \(L^2\)-framework to problem (1.1)-(1.2) in the case when (ideally speaking) \(0 < \theta < 1\), and is to apply them to investigate the optimal decay rates, depending on the dimension \(n\) and the parameter \(\theta\), of solutions to problem (1.1)-(1.2). The case \(0 < \theta < \frac{1}{2}\) is the missing one in the previous researches. Our interest to study the equation (1.1) is only from a pure mathematical point of view, and without loss of generality we can assume that the initial amplitude \(u_0 = 0\) when one concentrates only on capturing the leading term as time goes to infinity.

Now, we introduce the asymptotic profile as \(t \to \infty\) of the solutions to problem (1.1)-(1.2) with \(u_0 = 0\): \[
\varphi(t, \xi) := e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t} - e^{-\log(1+|\xi|^{2\theta})t} \frac{\partial}{\partial t} \left( P_1 (\varphi_1(t, \xi) - \varphi_2(t, \xi)) \right),
\] where the 0th-moment of the initial velocity \(P_1 \in \mathbb{R}\) is defined by
\[
P_1 := \int_{\mathbb{R}^n} u_1(x)dx.
\]

Then, our main result reads as follows. It should be strongly mentioned that the case of \(n = 1\) and \(\theta \geq \frac{1}{4}\) can be included in the result below. This part will be essential in our paper.

**Theorem 1.1** (I). Let \(n = 1\), \(0 < \theta \leq \frac{1}{3}\), and \(u_1 \in L^{1,2\theta}(\mathbb{R}) \cap L^2(\mathbb{R})\). Then it holds that
\[
\|u(t, \cdot) - F_{\xi \to x}^{-1}(\varphi(t, \xi)) \|_{L^2} \leq \begin{cases} C(\|u_1\|_1 + \|u_1\|_{L^{1,2\theta}}) \left( t^{-\frac{1}{\theta(1-\theta)} - \frac{1}{\sqrt{\theta}}} + \frac{1}{\sqrt{\theta}} t^{\frac{1}{\theta}} \right), & \text{if } 0 < \theta \leq \frac{1}{5}, \smallskip \phantom{b} \begin{cases} C(\|u_1\|_1 + \|u_1\|_{L^{1,2\theta}}) \left( t^{-\frac{1}{\theta(1-\theta)} - \frac{1}{\sqrt{\theta}}} + \frac{1}{\sqrt{\theta}} t^{\frac{2-4\theta}{4\theta}} \right), & \text{if } \frac{1}{6} < \theta \leq \frac{1}{3}.
\end{cases}
\end{cases}
\]
for \(t \gg 1\), where \(u(t, x)\) is a unique solution to problem (1.1)-(1.2) with \(u_0 = 0\).
(II). Let \( n \geq 2 \), \( 0 < \theta \leq \frac{5}{12} \), and \( u_1 \in L^{1,2\theta}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). Then it holds that

\[
\| u(t, \cdot) - \mathcal{F}_{-\xi}^{-1}(\varphi(t, \xi)) (\cdot) \|_{L^2} \leq \left\{ \begin{array}{ll}
C(\| u_1 \|_1 + \| u_1 \|_{L^{1,2\theta}}) \left( t^{-\frac{n-1}{2\theta}} + \frac{1}{\sqrt{\theta}} t^{-\frac{5}{26}} \right), & \text{if } 0 < \theta \leq \frac{1}{6}, \\
C(\| u_1 \|_1 + \| u_1 \|_{L^{1,2\theta}}) \left( t^{-\frac{n-1}{2\theta}} + \frac{1}{\sqrt{\theta}} t^{-\frac{n-1}{26}} \right), & \text{if } \frac{1}{6} < \theta \leq \frac{1}{12}, \\
C(\| u_1 \|_1 + \| u_1 \|_{L^{1,2\theta}}) \left( t^{-\frac{n-1}{2\theta}} + \frac{1}{\sqrt{\theta}} t^{-\frac{n-1}{26}} \right), & \text{if } \frac{1}{12} < \theta \leq \frac{5}{72}
\end{array} \right.
\]

for \( t \gg 1 \), where \( u(t, x) \) is a unique solution to problem \((1.1)-(1.2)\) with \( u_0 = 0 \).

**Remark 1.1** In the results of Theorem 1.1, one can notice the coefficient \( 1/\sqrt{\theta} \) in front of each final estimates. By observing this coefficient, one may conclude that we have captured the unique nature for the log-damping (or fractional damping) with parameter \( \theta > 0 \). This property can be found by searching the leading term more precisely than previous researches.

**Remark 1.2** It follows from Theorem 1.1 that \( \hat{u}(t, \xi) \sim P_1 (\varphi_1(t, \xi) - \varphi_2(t, \xi)) \) in \( L^2(\mathbb{R}^n) \) as \( t \to \infty \). It is important to notice that \( \varphi_1(t, \xi) \) and \( \varphi_2(t, \xi) \) are exact solutions of the first order in time equations in the Fourier space, respectively:

\[
-\Delta v + L_\theta v_t = 0,
\]

and

\[
L_\theta v + v_t = 0.
\]

In some sense, the solution to problem \((1.1)-(1.2)\) with small parameters \( \theta \in (0, 1/2) \) has a double diffusion phenomenon. This kind of important double diffusion phenomenon has been first discovered by D’Abbicco-Ebert \cite{13} to the equation \( (1.8) \) with \( \theta \in (0, 1/2) \). Theorem 1.1 corresponds to that of \cite{13} Theorem 2. We find that \((1.1)-(1.2)\) has a similar property to it. While, in the case when \( n \geq 2 \) and \( \theta \in (0, 1/2) \) an asymptotic profile of the solution to \((1.8)\) is captured as

\[
e^{-t|\xi|^{2(1-\theta)} |\xi|^{2\theta}}
\]

in [28, Theorem 1.5]. In some sense, \((1.8)\) is similar to \( \varphi_1(t, \xi) \) because of \( \log(1 + \nu^{2\theta}) \sim \nu^{2\theta} \) for small \( \nu > 0 \).

**Remark 1.3** A restriction \( \theta \in (0, \frac{1}{3}) \) or \( \theta \in (0, \frac{5}{12}] \) is just a technical condition, however, in the course of proof of Theorem 1.1 one has frequently used the following fact

\[
\lim_{r \to +0} \frac{\log(1 + r^{2\theta})}{r} = \infty.
\]

\((1.10)\) is also true in a more wider range \( \theta \in (0, \frac{1}{2}) \). So, reconsidering, the case of \( \theta \in (\frac{1}{3}, \frac{1}{2}) \) for \( n = 1 \) or \( \theta \in (\frac{5}{72}, \frac{1}{2}) \) for \( n \geq 2 \) is still open.

**Remark 1.4** The condition \( u_1 \in L^2(\mathbb{R}^n) \) in Theorem 1.1 is used to make sure the unique existence of the mild solution \( u(t, x) \). However, it does not affect directly on the \( L^2 \)-estimate of the solution, even in the high-frequency estimates although the estimate \((3.42)\) in the high frequency zone can be easily estimated in terms of \( \| u_1 \|_1 \) instead of \( \| u_1 \|_1 \).

As an application of Theorem 1.1 one can derive the following sharp decay estimates, which imply the optimal decay rates of the \( L^2 \)-norm of the solution to problem \((1.1)-(1.2)\).

**Theorem 1.2** Let \( n = 1 \) with \( 0 < \theta < \frac{1}{2} \) and \( n \geq 2 \) with \( 0 < \theta \leq \frac{5}{72} \). For \( u_1 \in L^{1,2\theta}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), it holds that

\[
K_1 |P_1| t^{-\frac{n-4\theta}{4n-1}} \leq \| u(t, \cdot) \| \leq K_2 (|P_1| + \| u_1 \|_{L^{1,2\theta}}) \left( t^{-\frac{n-4\theta}{4n-1}} + \frac{1}{\sqrt{\theta}} t^{-\frac{n-4\theta}{4n-1}} \right), \quad t \gg 1
\]

with some constant \( K_1, K_2 > 0 \) depending only on \( n \) and \( \theta \), where \( u(t, x) \) is a unique solution to problem \((1.1)-(1.2)\) with \( u_0 = 0 \).
Finally, we denote the surface area of the relation $f$ as a result of Theorem 1.2, one can observe that $\|u(t,\cdot)\| \sim t^{-\frac{n-\theta}{2(1-\theta)}}$ ($t \to \infty$). Thus, as for an ultimate situation when $\theta \to 0^+$ formally, the optimal decay order will approach $t^{-\frac{n}{2}}$, which is the Gauss kernel. This is quite natural because in the case when $\theta = 0$, the equation corresponds to the frequently studied damped wave equation. In this sense, all results in this paper reflect a diffusive aspect of the equation (1.1) with small $\theta$. This property is quite different from those studied in [4] for large $\theta \geq \frac{1}{2}$. In [5], a wave like property is captured.

**Remark 1.5** A similar $L^p$-$L^q$ type ”decay” estimates only from above has been already studied precisely in [14] and [15 Corollary 2.2] to the solution of the equation (1.8) for $n = 1$ and $0 < \theta < 1/4$, or $n \geq 2$ and $0 < \theta < 1/2$. The lower bound itself in Theorem 1.2 seems new.

**Remark 1.6** As a result of Theorem 1.2 one can observe that $∥·∥_θ$ for the case $n$ in [14] and [15, Corollary 2.2] to the solution of the equation (1.8) for $\theta$ for an ultimate situation when $\theta > 1/4$ and $0 = \theta < 1/2$. The optimal decay order $1$ for $\theta$ while the other is the infinite time blowup results in the case of $\theta$ of the equation (1.1) with small $\theta$. In [15], the optimal decay rate of the “energy” and $L^2$-norm of the solutions are studied by developing a new energy method in the Fourier space in [8]. So, the structure of the equation (1.1) is quite similar to (1.8) with $\theta \in (0,1/2)$.

Contrary to the decay results as in Theorem 1.2 one can observe the following surprising property, which shows infinite time blowup results of the solution to problem (1.1)-(1.2) in the one dimensional case. We believe this is the first discovery in the damped wave equation community. In [17] and [14], they apply the decay estimates of the solution for the equation (1.8) to the nonlinear problems, they necessarily avoid to treat the case of $n = 1$, and $1/4 \leq \theta < 1/2$. The following crucial result makes their mechanism clear because of $\log(1 + r^2) \sim r^2$ for small $r > 0$.

**Theorem 1.3** Let $n = 1$ with $\frac{1}{4} \leq \theta \leq \frac{1}{3}$. For $u_1 \in L^{1,2\theta}(\mathbb{R}) \cap L^2(\mathbb{R})$, there exists positive constants $K_1, K_2$, which depend only on $\theta$, such that

$$K_1|P_i|t^{\frac{4\theta-1}{4\theta}} \leq ∥u(t,\cdot)∥ \leq K_2\left(\frac{1}{\sqrt{4\theta - 1}}|P_i| + ∥u_1∥_{L^{1,2\theta}}\right)t^{-\frac{4\theta-1}{4\theta}}, \quad t \gg 1 \tag{1.11}$$

for $\frac{1}{4} < \theta \leq \frac{1}{3}$ and

$$K_1|P_i|\sqrt{\log t} \leq ∥u(t,\cdot)∥ \leq K_2\left(|P_i| + ∥u_1∥_{L^{1,2\theta}}\right)\sqrt{\log t}, \quad t \gg 1 \tag{1.12}$$

for the case $\theta = \frac{1}{4}$.

**Remark 1.8** We find that the number $\theta^* = \frac{1}{4}$ is critical in the one dimensional case because $\theta^*$ divides the structure of the corresponding solution $u(t,x)$ into two parts: one is decay property for $0 < \theta < \theta^*$, while the other is the infinite time blowup results in the case of $\theta^* \leq \theta < \frac{1}{2}$. Moreover, we note that in Theorem 1.3 there is not a contradiction between the estimate (1.11) when $\theta \to (1/4)^+$ and the estimate (1.12) for $\theta = 1/4$ because of the singularity $\sqrt{\log t}$ at $\theta = 1/4$.

**Notation.** Throughout this paper, $∥·∥_g$ stands for the usual $L^g(\mathbb{R}^n)$-norm. For simplicity of notation, in particular, we use $∥·∥$ instead of $∥·∥_2$. Furthermore, we denote $∥·∥_{H^j}$ as the usual $H^j$-norm. We also define a relation $f(t) \sim g(t)$ as $t \to \infty$ by: there exist constant $C_j > 0$ ($j = 1, 2$) such that

$$C_1g(t) \leq f(t) \leq C_2g(t) \quad (t \gg 1).$$

For $\Omega \subset \mathbb{R}^n$ we denote $f \approx g$ on $\Omega$, if and only if there are constants $K_1, K_2$ such that

$$K_1f(y) \leq g(y) \leq K_2f(y), \quad \text{for all } y \in \Omega.$$

We also introduce the following weighted functional spaces for $\gamma > 0$:

$$L^{1,\gamma}(\mathbb{R}^n) := \left\{ f \in L^1(\mathbb{R}^n) \mid ∥f∥_{L^{1,\gamma}} := \int_{\mathbb{R}^n} (1 + |x|^\gamma)|f(x)|dx < +\infty \right\}.$$

Finally, we denote the surface area of the $n$-dimensional unit ball by $\omega_n := \int_{|\omega| = 1} d\omega$. 

6
2 Basic preliminary results

In this section we shall collect important lemmas to derive precise estimates of the several quantities related to the solution to problem (1.1)-(1.2). These are already studied and developed in our previous works (see [6, 5]).

The following estimate for the function

\[ I_p(t) = \int_0^1 (1 + r^2)^{-t} r^p \, dr \]

is a direct consequence of the cases \( p \geq 0 \) in Charão-Ikehata [6] and \(-1 < p < 0 \) in Charão-D’Abbicco-Ikehata [5].

**Lemma 2.1** Let \( p > -1 \) be a real number. Then

\[ I_p(t) \sim t^{-\frac{p+1}{2}}, \quad t \gg 1. \]

In order to deal with the high frequency part of estimates, one defines a function again

\[ J_p(t) = \int_1^\infty (1 + r^2)^{-t} r^p \, dr \]

for \( p \in \mathbb{R} \).

Then the next lemma is important to get estimates on the zone of high frequency to the solutions of the problem (1.1)-(1.2). The proof appears in Charão-Ikehata [6].

**Lemma 2.2** Let \( p \in \mathbb{R} \). Then it holds that

\[ J_p(t) \sim \frac{2^{-t}}{t-1}, \quad t \gg 1. \]

For later use we prepare the following simple lemma, which implies the exponential decay estimates of the middle frequency part.

**Lemma 2.3** Let \( p \in \mathbb{R} \), and \( \eta \in (0, 1] \). Then there is a constant \( C > 0 \) such that

\[ \int_\eta^1 (1 + r^2)^{-t} r^p \, dr \leq C(1 + \eta^2)^{-t}, \quad t \geq 0. \]

**Remark 2.1** We note that the proof of Lemma 2.1 is done by using simple differential calculus and the theory from hypergeometric functions (see Watson [39]). These are already developed in [6] and [5].

**Lemma 2.4** There exists a constant \( K > 0 \) such that

\[ \frac{\sinh x}{x} \leq K \cdot e^x \]

for \( x > 0 \).

We will need the following decomposition for the Fourier transform of a function \( f \) in \( L^1(\mathbb{R}^n) \) as follows

\[ \hat{f}(\xi) = A_f(\xi) - iB_f(\xi) + P_f, \quad (2.1) \]

for all \( \xi \in \mathbb{R}^n \), where

- \( A_f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (\cos(x \cdot \xi) - 1)f(x) \, dx \),
- \( B_f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \sin(x \cdot \xi)f(x) \, dx \),
- \( P_f = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \, dx \).
Then, the next lemma has been already prepared in [24] (see Notation for the definition of $L^{1,\kappa}(\mathbb{R}^n)$).

**Lemma 2.5**

i) If $f \in L^1(\mathbb{R}^n)$ then for all $\xi \in \mathbb{R}^n$ it is true that

$$|A_f(\xi)| \leq L\|f\|_{L^1} \quad \text{and} \quad |B_f(\xi)| \leq N\|f\|_{L^1}.$$  

ii) If $0 < \kappa \leq 1$ and $f \in L^{1,\kappa}(\mathbb{R}^n)$ then for all $\xi \in \mathbb{R}^n$ it is true that

$$|A_f(\xi)| \leq K\|\xi\|^{\kappa}\|f\|_{L^{1,\kappa}} \quad \text{and} \quad |B_f(\xi)| \leq M\|\xi\|^{\kappa}\|f\|_{L^{1,\kappa}}$$

with $L, N, K$ and $M$ positive constants depending only on the dimension $n$ or $n$ and $\kappa$.

**Lemma 2.6** Let $0 \leq \theta < 1$ and $q > -1$. Then

$$\int_0^1 (1 + r^{2-2\theta})^{-qr}dr \sim \frac{1}{1 - \theta}t^{-\frac{q+1}{q+2}}, \quad t \gg 1.$$  

In particular, for $0 \leq \theta \leq 1/2$ and $q > -1$ it holds that

$$\int_0^1 (1 + r^{2-2\theta})^{-qr}dr \sim t^{-\frac{q+1}{q+2}}, \quad t \gg 1.$$  

**Proof.** Let $s = r^{1-\theta}$. Then

$$\int_0^1 (1 + r^{2-2\theta})^{-qr}dr = \frac{1}{1 - \theta} \int_0^1 (1 + s^2)^{-qt} \sim t^{-\frac{q+1}{q+2}}ds.$$  

Since $0 \leq \theta < 1$ and $q > -1$, we have $\frac{q+\theta}{1-\theta} > -1$. Thus, we can apply the Lemma 2.1 to obtain the result. \(\square\)

**Remark 2.2** Actually, for $\eta > 0$, $0 \leq \theta \leq 1/2$ and $q > -1$, it holds that

$$\int_0^\eta (1 + r^{2-2\theta})^{-qr}dr \geq C \frac{1}{t^{(q+1)/(1-\eta)}} \quad t \gg 1$$

for some constant $C > 0$ depending on each $\eta > 0$.

Indeed, it suffices to check the case of $0 < \eta < 1$. In this case, one notices

$$\int_0^\eta (1 + r^{2-2\theta})^{-qr}dr = \int_0^1 (1 + r^{2-2\theta})^{-qr}dr - \int_\eta^1 (1 + r^{2-2\theta})^{-qr}dr,$$

and one has

$$\int_\eta^1 (1 + r^{2-2\theta})^{-qr}dr \leq \frac{1}{1 + q}(1 - \eta^{q+1})(1 + \eta^{2-2\theta})^{-t}.$$  

Since the last term implies the exponential decay, the desired estimate can be derived soon via Lemma 2.6. \(\square\)

**Lemma 2.7** Let $\theta > 0$ and $q > -1$. Then

$$\int_0^1 (1 + r^{2\theta})^{-qr}dr \sim \frac{1}{\theta}t^{-\frac{q+1}{\theta}}, \quad t \gg 1.$$  

**Proof.** We consider the change of variable $s = r^{\theta}$. Then

$$\int_0^1 (1 + r^{2\theta})^{-qr}dr = \frac{1}{\theta} \int_0^1 (1 + s^2)^{-qr}ds \sim \frac{1}{\theta} t^{-\frac{q+1}{\theta}}ds$$

for $t \geq 0$. Finally, $\frac{q+1-\theta}{\theta} > -1$ because of $q > -1$. From Lemma 2.7 the desired result follows. \(\square\)
Lemma 2.8 Let $\theta > 0$ and $q \in \mathbb{R}$. Then
\[
\int_1^\infty (1 + r^{2\theta})^{-t}r^q dr \sim \frac{2^{-t}}{\theta t - 1}, \quad t \gg 1.
\]

Lemma 2.9 Let $0 \leq \theta < 1$ and $q \in \mathbb{R}$. Then
\[
\int_1^\infty (1 + r^{2-2\theta})^{-t}r^q dr \sim \frac{2^{-t}}{t - 1}, \quad t \gg 1.
\]

Proof of Lemmas 2.8 and 2.9 From lemma 2.2 and the change of variables as in Lemmas 2.7 and 2.6, the result now follows. □

3 Asymptotic profile

As mentioned in the introduction, our main interest in this work is to investigate the problem (1.1)-(1.2) for the case of $0 < \theta < \frac{1}{2}$ in order to compensate the research in [5] studying the case of $\theta > \frac{1}{2}$.

The associated problem to (1.1)-(1.2) in Fourier space is the following
\[
\hat{u}_{tt} + \log(1 + |\xi|^{2\theta})\hat{u}_t + |\xi|^2 \hat{u} = 0, \quad t > 0, \quad \xi \in \mathbb{R}^n, \tag{3.1}
\]
\[
\hat{u}(0, \xi) = 0, \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \quad \xi \in \mathbb{R}^n, \tag{3.2}
\]
where the associated characteristic polynomial is
\[
\lambda^2 + \log(1 + |\xi|^{2\theta})\lambda + |\xi|^2 = 0.
\]

The characteristics roots are expressed as
\[
\lambda_{\pm} = -\frac{\log(1 + |\xi|^{2\theta}) \pm \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2}}{2}, \quad \xi \in \mathbb{R}^n. \tag{3.3}
\]

Lemma 3.1 There exists $\delta = \delta(\theta), \, 0 < \delta < 1$ such that
\[
\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2 \geq 0 \text{ for } |\xi| \leq \delta, \tag{3.4}
\]
\[
\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2 < 0 \text{ for } |\xi| > \delta. \tag{3.5}
\]

Proof. Working with $r = |\xi|$, we first observe that $\log(1 + r^2) < 2r$ for all $r > 0$. Also, $r^{2\theta} \leq r^2$ for $r \geq 1$, since $\theta \leq 1$. Therefore, in the case $\theta < \frac{1}{2}$, one has
\[
\log(1 + r^{2\theta}) \leq \log(1 + r^2) < 2r \tag{3.6}
\]
for all $r \geq 1$. Thus we may conclude that the function $f(r) := \log(1 + r^{2\theta}) - 2r$ is negative for all $r \geq 1$. However, the similar phenomena does not happen near the origin. In fact, we first notice that
\[
\lim_{r \to 0} \frac{\log(1 + r^{2\theta})}{r} = \infty,
\]
for $\theta \in (0, \frac{1}{2})$. Therefore, there exists $r_0 = r_0(\theta) < 1$ such that
\[
\frac{\log(1 + r^{2\theta})}{r} > 2
\]
for all $r \in \mathbb{R}^n$ satisfying $0 < r < r_0$. Then, $f(r) = \log(1 + r^{2\theta}) - 2r \geq 0$ for $0 \leq r < r_0$. Furthermore, for $r \geq 0$ one can get
\[
f''(r) = \frac{2\theta r^{2\theta - 2} [(2\theta - 1)(1 + r^{2\theta}) - 2\theta r^{2\theta}]}{(1 + r^{2\theta})^2} = \frac{2\theta r^{2\theta - 2} [2\theta - 1 - r^{2\theta}]}{(1 + r^{2\theta})^2}.
\]
Since $0 < \theta < \frac{1}{2}$, the function $f : [0, \infty) \to \mathbf{R}$ satisfies $f''(r) < 0$. Due to $f(0) = 0$ and (3.8) one can conclude that there exists a unique number $\delta = \delta(\theta)$, $0 < \delta < 1$ such that $f(r) \geq 0$ for all $0 \leq r \leq \delta$ and $f(r) \leq 0$ for all $r \geq \delta$. Finally, one can write

$$\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2 = f(|\xi|) \left( \log(1 + |\xi|^{2\theta}) + 2|\xi| \right).$$

Therefore, using the properties of the function $f(r) = f(|\xi|)$ one can obtain the desired statement. □

By Lemma 3.1 we see that the characteristics roots (3.3) are real-valued for $|\xi| \leq \delta$, and complex-valued for $|\xi| > \delta$. This is a crucial different point from that observed in the case of $1/2 \leq \theta$.

### 3.1 Estimates on the region $|\xi| \leq \delta$

First part of this section we analyze the behavior of the characteristics roots near the origin $\xi = 0$. To do that we need some remarks and lemmas.

**Remark 3.1** For $q \geq 0$ it is easy to check the inequality $\frac{1}{2} r^q \leq \log(1 + r^q) \leq r^q$ for $r \in [0, 1]$.

In particular, for $0 < \theta < \frac{1}{2}$ we have

$$\frac{1}{2} |\xi|^{2\theta} \leq \log(1 + |\xi|^{2\theta}) \leq \frac{3}{2} |\xi|^{2\theta}, \quad (3.7)$$

$$\frac{1}{2} |\xi|^2 \leq \log(1 + |\xi|^2) \leq \frac{3}{2} |\xi|^2, \quad (3.8)$$

$$\frac{1}{2} |\xi|^2 - 2^\theta \leq \log(1 + |\xi|^{2-2\theta}) \leq \frac{3}{2} |\xi|^{2-2\theta} \quad (3.9)$$

for $|\xi| \leq 1$.

We note that for $0 \leq \theta < 1/2$ it holds that

$$\lim_{r \to +0} \frac{r^{4-4\theta}}{r^2} = 0.$$

Thus, there exists $\delta_1 = \delta_1(\theta)$, $0 < \delta_1 < 1$ that satisfies

$$\frac{|\xi|^{4-4\theta}}{q^2} \leq \frac{1}{25} \quad (3.10)$$

whenever $0 < |\xi| \leq \delta_1$. Moreover, one can choose $\delta_1 \in (0, 1)$ such as $\delta_1 < \delta$. In fact, from (3.10) one has

$$25|\xi|^2 \leq |\xi|^{4\theta}$$

for $0 \leq |\xi| \leq \delta_1$. In this region it also holds $|\xi|^{4\theta} \leq 4 \log^2(1 + |\xi|^2)$, due to (3.7). Thus

$$\log^2(1 + |\xi|^2) \geq \frac{25}{4} |\xi|^2 \geq \frac{16}{3} |\xi|^2 \geq 4|\xi|^2 \quad (3.11)$$

for $|\xi| \leq \delta_1$. Comparing (3.11) with (3.3), we may conclude that $\delta_1 < \delta$. From (3.11) we also obtain

$$\log^2(1 + |\xi|^{2\theta}) \geq \frac{16}{3} |\xi|^2$$

whenever $|\xi| \leq \delta_1$.

Now we define a new number:

$$\eta := \sup\{\alpha > 0 : \frac{|\xi|^{4-4\theta}}{q^2} \leq \frac{1}{25} \text{ for } 0 < |\xi| \leq \alpha \}. \quad (3.12)$$

We note that $\eta$ is positive and is well defined, because the set $\{\alpha > 0 : \frac{|\xi|^{4-4\theta}}{q^2} \leq \frac{1}{25} \text{ for } |\xi| \leq \alpha \}$ is not empty ($\delta_1$ is a member of this set) and is bounded from above. In fact, for example, 1 is an upper bound for this set, and $\eta < \delta < 1$ with $\delta$ defined in Lemma 3.1. In particular, the following two properties are true for $|\xi| \leq \eta$:

$$\frac{3}{4} \log^2(1 + |\xi|^{2\theta}) \geq 4|\xi|^2, \quad (3.13)$$

$$25|\xi|^{4-4\theta} \leq |\xi|^2. \quad (3.14)$$
Lemma 3.2 Let \( \eta \) be the number defined by (3.12). Then, for \( |\xi| \leq \eta \) it holds that

(i) \( \lambda_+ - \lambda_- \approx \log(1 + |\xi|^{2\theta}) \);

(ii) \( \lambda_+ \approx \log(1 + |\xi|^{2 - 2\theta}) \approx -|\xi|^{2 - 2\theta} \);

(iii) \( \lambda_- \approx \log(1 + |\xi|^{2\theta}) \).

Proof.
(i) The upper estimate is simple because for \( |\xi| \leq \eta < \delta \) it holds that

\[
\lambda_+ - \lambda_- = \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2} \leq \sqrt{\log^2(1 + |\xi|^{2\theta})} = \log(1 + |\xi|^{2\theta}).
\]

On the other hand, by (3.13) we have

\[
\frac{1}{4} \log^2(1 + |\xi|^{2\theta}) \leq \log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2, \quad |\xi| \leq \eta.
\]

For this reason, in the zone \( |\xi| \leq \eta \) it holds that

\[
\frac{1}{2} \log(1 + |\xi|^{2\theta}) \leq \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2}.
\]

(ii) The inequality (3.14) provides us to get

\[
0 \geq 25|\xi|^{4 - 4\theta} - 5|\xi|^2 + 4|\xi|^2 = 25|\xi|^{4 - 4\theta} - 5|\xi|^{2 - 2\theta} + 4|\xi|^2. \tag{3.15}
\]

The lower inequality in (3.7) implies that \( -10 \log(1 + |\xi|^{2\theta}) \leq -5|\xi|^{2\theta} \) for \( |\xi| \leq 1 \) and in particular for \( |\xi| \leq \eta \). By combining this fact with (3.15), we obtain

\[
25|\xi|^{4 - 4\theta} - 10 \log(1 + |\xi|^{2\theta})|\xi|^{2 - 2\theta} + 4|\xi|^2 \leq 0, \quad |\xi| \leq \eta.
\]

Adding \( \log^2(1 + |\xi|^{2\theta}) \) on both sides we may obtain

\[
(\log(1 + |\xi|^{2\theta}) - 5|\xi|^{2 - 2\theta})^2 = \log^2(1 + |\xi|^{2\theta}) - 10 \log(1 + |\xi|^{2\theta})|\xi|^{2 - 2\theta} + 25|\xi|^{4 - 4\theta} \leq \log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2.
\]

Hence, for \( |\xi| \leq \eta \), \( \log(1 + |\xi|^{2\theta}) - 5|\xi|^{2 - 2\theta} \leq \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2} \) and

\[
- \frac{5}{2}|\xi|^{2 - 2\theta} \leq - \frac{\log(1 + |\xi|^{2\theta}) + \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2}}{2} = \lambda_+.
\]

Furthermore, we also concludes that

\[
- 5 \log(1 + |\xi|^{2 - 2\theta}) \leq - \frac{5}{2}|\xi|^{2 - 2\theta} \leq \lambda_+ \tag{3.16}
\]

on the zone \( |\xi| \leq \eta \), due to (3.9).

In order to prove the upper estimate part of (ii) we first observe that

\[
0 \leq |\xi|^2 + |\xi|^{4 - 4\theta} = |\xi|^2 + |\xi|^{4 - 4\theta} - 3|\xi|^{2\theta}|\xi|^{2 - 2\theta}.
\]

In the zone \( |\xi| \leq \eta \) it holds that \( -3|\xi|^{2\theta} \leq -2 \log(1 + |\xi|^{2\theta}) \) by (3.7), which implies that

\[
-3|\xi|^{2\theta}|\xi|^{2 - 2\theta} \leq -2 \log(1 + |\xi|^{2\theta})|\xi|^{2 - 2\theta}.
\]

By using the inequality just above we may obtain that

\[
0 \leq 4|\xi|^2 + |\xi|^{4 - 4\theta} - 2 \log(1 + |\xi|^{2\theta})|\xi|^{2 - 2\theta}. \tag{3.17}
\]

We add \( \log^2(1 + |\xi|^{2\theta}) \) in both side of (3.17) in order to get the following estimate:

\[
\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2 \leq \log^2(1 + |\xi|^{2\theta}) - 2 \log(1 + |\xi|^{2\theta})|\xi|^{2 - 2\theta} + |\xi|^{4 - 4\theta} = (\log(1 + |\xi|^{2\theta}) - |\xi|^{2 - 2\theta})^2.
\]
This implies
\[
\lambda_+ = \frac{-\log(1 + |\xi|^{2\theta}) + \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2}}{2} \leq -\frac{1}{2}|\xi|^{2-2\theta}.
\] (3.18)

When one derives (3.18), one must check the fact that \( \log(1 + |\xi|^{2\theta}) - |\xi|^{2-2\theta} \geq 0 \) on \( |\xi| \leq \eta \). Indeed, this can be easily observed by a combination of (3.13) and (3.14).

Now, by combining inequalities (3.13) and (3.14) one obtain
\[
\lambda_+ \leq -\frac{1}{2}|\xi|^{2-2\theta} \leq -\frac{1}{3}\log(1 + |\xi|^{2\theta})
\]
because of \( |\xi| \leq \eta \). The inequalities just above and (3.16) imply the desired statement of item (ii).

(iii). In the course of the proof of item (i) in the region \( |\xi| \leq \eta \), we also have
\[
-\log(1 + |\xi|^{2\theta}) \leq -\sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2} \leq -\frac{1}{2}\log(1 + |\xi|^{2\theta}).
\]
Therefore, one can easily conclude that
\[
-\log(1 + |\xi|^{2\theta}) \leq -\frac{\log(1 + |\xi|^{2\theta}) - \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2}}{2} \leq -\frac{3}{4}\log(1 + |\xi|^{2\theta}), \quad |\xi| \leq \eta.
\]
This implies the desired statement of item (iii). \( \square \)

### 3.1.1 Estimates on the low-frequency zone \( |\xi| \leq \eta^3 \)

Throughout this paper we assume the initial amplitude \( u_0 \) satisfies \( u_0 = 0 \), without loss of generality in order to investigate the asymptotic profiles of solutions.

We first remember the number \( \eta \in (0, \delta) \) defined in (3.7)-(3.14). Also, since \( 0 < \eta < 1 \), we have \( \eta^3 < \eta \). In the zone of low frequency \( |\xi| \leq \eta^3 \), the characteristics roots \( \lambda_\pm \) are real, and the solution of (3.11)-(3.2) is explicitly given by
\[
\hat{u}(t, \xi) = \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-} \hat{u}_1(\xi).
\] (3.19)

The purpose in this section is to get an asymptotic profile to the solution \( \hat{u}(t, \xi) \), and in order to do that we need to obtain useful estimates. For this reason, we defined a function \( g : [0, \delta] \to \mathbb{R} \) inspired by an idea from [19], as follows. A discovery of this function \( g(s) \) is one of decisive points in our proof.

\[
g(s) := \begin{cases} 
1 + \sqrt{1 - \frac{4s^6}{\log^2(1 + s^{6\theta})}} & \text{if } 0 < s \leq \delta \\
0 & \text{if } s = 0.
\end{cases}
\] (3.20)

Note that for \( 0 < \theta < 1/2 \),
\[
\lim_{s \to 0^+} \frac{s^6}{\log^2(1 + s^{6\theta})} = 0.
\]

**Remark 3.2** Let \( t > 0 \) and \( \xi \in \mathbb{R}^n, 0 < |\xi| \leq \eta \), be fixed. We recall that \( \eta < \delta < 1 \). Let us consider the function \( h(s) \) defined on \( [0, \eta] \) as follows:
\[
h(s) := e^{-\frac{t\log(1 + |\xi|^{2\theta})}{2}g(s)}.
\]

We see that \( h(s) \) is differentiable on \( (0, \eta) \). Then, it should be noted that one can apply the mean value theorem in the interval \( [0, s] \) for each \( s \in (0, \eta] \) to get
\[
\frac{h(s) - h(0)}{s} = \frac{e^{-\frac{t\log(1 + |\xi|^{2\theta})}{2}g(s)} - e^{-\frac{t\log(1 + |\xi|^{2\theta})}{2}g(0)}}{s} = -\frac{t\log(1 + |\xi|^{2\theta})}{2}e^{-\frac{t\log(1 + |\xi|^{2\theta})}{2}g(\alpha s)}g'(\alpha s)
\] (3.21)

with some \( \alpha = \alpha(s, t, |\xi|) \in (0, 1) \).
We observe that on the low frequency zone \(0 \leq \xi \leq \eta^3\) it holds that

\[
\lambda_+ = -\frac{\log(1 + |\xi|^{2\theta})}{2} g(\sqrt[3]{\xi}).
\]

By applying (3.21) for \(t > 0\) and \(s = \frac{1}{\sqrt[3]{\xi}}, 0 < |\xi| \leq \eta^3\), we have

\[
e^{t \lambda_+} = e^{-t \log(1 + |\xi|^{2\theta})} - \frac{t}{2} \log(1 + |\xi|^{2\theta}) \sqrt[3]{\xi} e^{-t \log(1 + |\xi|^{2\theta})} g\left(\alpha \sqrt[3]{\xi} \right) g'\left(\alpha \sqrt[3]{\xi} \right) \tag{3.22}
\]

with \(\alpha := \alpha(s, t, |\xi|) = \alpha(t, |\xi|) \in (0, 1)\).

From the Chill-Haraux [12] idea, we also observe that

\[
\lambda_+ = -\frac{\lambda_+^2 + |\xi|^2}{\log(1 + |\xi|^{2\theta})},
\]

so that one has

\[
e^{t \lambda_+} = e^{-\frac{|\xi|^2}{\log(1 + |\xi|^2)}} e^{-\frac{\lambda_+^2}{\log(1 + |\xi|^2)}} t. \tag{3.23}
\]

On the other hand, because of (3.3) we see that

\[
\frac{1}{\lambda_+ - \lambda_-} = \frac{1}{\log(1 + |\xi|^{2\theta})} + R(|\xi|)
\]

where

\[
R(r) = \frac{4r^2}{\log^3(1 + r^{2\theta})} \sqrt{1 - \frac{4r^2}{\log^2(1 + r^{2\theta})}} \left(1 + \sqrt{1 - \frac{4r^2}{\log^2(1 + r^{2\theta})}}\right). \tag{3.24}
\]

By combining (3.22), (3.23) and (3.24) with the decomposition of initial data

\[
\dot{u}_1(\xi) = A_{u_1}(\xi) - iB_{u_1}(\xi) + P_{u_1} =: A_1(\xi) - iB_1(\xi) + P_1
\]

as in (2.1), we can write the solution of (3.1)-(3.2) given by (3.19), for \(|\xi| \leq \eta^3\), as follows

\[
\dot{u}(t, \xi) = e^{-t \log(1 + |\xi|^{2\theta})} P_1 - e^{-t \log(1 + |\xi|^{2\theta})} P_1 + R(|\xi|) e^{-t \log(1 + |\xi|^{2\theta})} \dot{u}_1(\xi) - R(|\xi|) e^{-t \log(1 + |\xi|^{2\theta})} \dot{u}_1(\xi)
\]

\[
+ e^{-t \log(1 + |\xi|^{2\theta})} \left(A_1(\xi) - iB_1(\xi)\right) + e^{-t \log(1 + |\xi|^{2\theta})} \left(1 - \frac{\lambda_+}{\lambda_+ - \lambda_-}\right) \dot{u}_1(\xi)
\]

\[
- e^{-t \log(1 + |\xi|^{2\theta})} \left(A_1(\xi) - iB_1(\xi)\right) + \frac{t}{2(\lambda_+ - \lambda_-)} \log(1 + |\xi|^{2\theta}) \sqrt[3]{\xi} e^{-t \log(1 + |\xi|^{2\theta})} g\left(\alpha \sqrt[3]{\xi} \right) g'\left(\alpha \sqrt[3]{\xi} \right) \dot{u}_1(\xi). \tag{3.25}
\]

Our main goal in this subsection is to introduce an asymptotic profile as \(t \to +\infty\) of the solution \(\dot{u}(t, \xi)\) in the low frequency region \(|\xi| \leq \eta^3 < \eta\) in the simple form

\[
\varphi(t, \xi) := e^{-\frac{|\xi|^2}{\log(1 + |\xi|^2)} t} P_1 - e^{-\frac{\lambda_+^2}{\log(1 + |\xi|^2)} t} P_1. \tag{3.26}
\]

Thus, we need to prove that

\[
||\dot{u}(t, \cdot) - \varphi(t, \cdot)|| \to 0, \quad t \to \infty
\]
much faster than the components in the right hand side of (3.26). For this purpose, we consider the following six remainder functions

\[
F_1(t, \xi) = R(|\xi|) e^{-\frac{|\xi|^2}{\log(1+|\xi|^{20})}} \hat{u}_1(\xi)
\]

\[
F_2(t, \xi) = -R(|\xi|) e^{-t \log(1+|\xi|^{20})} \hat{u}_1(\xi)
\]

\[
F_3(t, \xi) = \frac{e^{-|\xi|^2 t}}{\log(1 + |\xi|^{20})} (A_1(\xi) - iB_1(\xi))
\]

\[
F_4(t, \xi) = e^{-|\xi|^2 t} \left( e^{\frac{\lambda_2^2}{\log(1+|\xi|^{20})}} - 1 \right) \hat{u}_1(\xi)
\]

\[
F_5(t, \xi) = \frac{e^{-t \log(1+|\xi|^{20})}}{\log(1 + |\xi|^{20})} (A_1(\xi) - iB_1(\xi))
\]

\[
F_6(t, \xi) = t \frac{\log(1 + |\xi|^{20})}{2(\lambda_+ - \lambda_-)} \left( \frac{\log(1+|\xi|^{20})}{\log(1+|\xi|^{20})} \right) g' \left( \frac{\alpha}{\sqrt{|\xi|}} \right) \hat{u}_1(\xi).
\]

From (3.25) and (3.26), for $|\xi| \leq \eta$, we have

\[
\hat{u}(t, \xi) - \varphi(t, \xi) = \sum_{j=1}^{6} F_j(t, \xi).
\]

In order to obtain decay rates in time to these functions we assume the additional condition on the initial data such that

\[
u_1 \in L^{1,20}(\mathbb{R}^n), \quad 0 < \theta < 1/2.
\]

To begin with, we estimate the function $F_3(t, \xi)$. Indeed, by using Lemma 3.2 and Lemma 2.5 with $\kappa := \theta \in (0, 1/2)$ (this is our crucial idea), and the inequality (3.37) and (3.39) one can estimate

\[
\int_{|\xi| \leq \eta^3 < 1} |F_3(t, \xi)|^2 d\xi = \int_{|\xi| \leq \eta^3} \frac{e^{-\frac{|\xi|^2 t}{\log(1+|\xi|^{20})}}}{\log^2(1 + |\xi|^{20})} |A_1(\xi) - iB_1(\xi)|^2 d\xi
\]

\[
\leq \int_{|\xi| \leq \eta^3} \frac{e^{-\frac{4\theta |\xi|^2}{\log(1+|\xi|^{20})}}}{\log^2(1 + |\xi|^{20})} |A_1(\xi) - iB_1(\xi)|^2 d\xi
\]

\[
\leq (M + K)^2 \|u_1\|_{L^{1,20}}^2 \int_{|\xi| \leq \eta^3} \frac{e^{-\frac{4\theta |\xi|^2}{\log(1+|\xi|^{20})}}}{\log^2(1 + |\xi|^{20})} |\xi|^2 d\xi
\]

\[
\leq C(M + K)^2 \|u_1\|_{L^{1,20}}^2 \int_{|\xi| \leq \eta^3} (1 + |\xi|^{2-20})^{-\frac{\eta}{2}} d\xi
\]

\[
\leq C\|u_1\|_{L^{1,20}}^2 \int_{0}^{\eta^3} (1 + r^{2-20})^{-\frac{\eta}{2}} r^{n-1} dr
\]

\[
\leq C\|u_1\|_{L^{1,20}}^2 t^{-\frac{n}{2(1-\theta)}}, \quad t \gg 1.
\]

(3.27)

with a generous constant $C > 0$ depending only on $n$, and the last inequality is due to Lemma 2.6 where one has just used the fact that

\[
\lim_{\sigma \to +0} \frac{\sigma}{\log(1 + \sigma)} = 1.
\]
Similarly, we can also estimate
\[
\int_{|\xi| \leq \eta^3} |F_5(t, \xi)|^2 \, d\xi = \int_{|\xi| \leq \eta^3} e^{-2 \log(1+|\xi|^{2\theta}) t} |A_1(\xi) - iB_1(\xi)|^2 d\xi \\
= \int_{|\xi| \leq \eta^3} \frac{(1 + |\xi|^{2\theta} - 2t)}{\log^2(1 + |\xi|^{2\theta})} |A_1(\xi) - iB_1(\xi)|^2 d\xi \\
\leq C(K + M)^2 \|u_1\|^2_{L^1_{t,2\theta}} \int_{|\xi| \leq \eta^3} (1 + |\xi|^{2\theta} - 2t)^{-1} d\xi \\
= C\omega_n (K + M)^2 \|u_1\|^2_{L^1_{t,2\theta}} \int_0^\eta (1 + r^{2\theta} - 2t - r^{n-1}) \, dr \\
\leq \frac{C}{\theta} \|u_1\|^2_{L^1_{t,2\theta}} t^{-\frac{n}{2\theta}} \log(1+\theta), \quad t \gg 1, \quad (3.28)
\]
where the last inequality is due to Lemma 2.7.

On the next estimates to the functions $F_j(t, \xi)$ we also rely on Lemma 2.6 or Lemma 3.2.

In order to estimate $F_4(t, \xi)$ we use the fact that $|e^{-a} - 1| \leq a$ for all $a \geq 0$. Then, Lemma 3.2 and
inequality (3.10) imply the existence of a constant $C > 0$ such that
\[
\int_{|\xi| \leq \eta^3} |F_4(t, \xi)|^2 \, d\xi = \int_{|\xi| \leq \eta^3} \left( e^{rac{\lambda_+^2}{\log(1 + |\xi|^{2\theta}) t}} - 1 \right)^2 e^{-2 \log(1 + |\xi|^{2\theta}) t} |\hat{u}_1(\xi)|^2 \, d\xi \\
\leq t^2 \|u_1\|^2_{L^1_{t,2\theta}} \int_{|\xi| \leq \eta^3} \frac{\lambda_+^2}{\log^2(1 + |\xi|^{2\theta})} e^{-2 \log(1 + |\xi|^{2\theta}) t} \, d\xi \\
\leq Ct^2 \|u_1\|^2_{L^1_{t,2\theta}} \int_{|\xi| \leq \eta^3} \frac{|\xi|^{8 - 8\theta}}{\log^4(1 + |\xi|^{2\theta})} e^{-2 \log(1 + |\xi|^{2\theta}) t} \, d\xi \\
\leq Ct^2 \|u_1\|^2_{L^1_{t,2\theta}} \int_{|\xi| \leq \eta^3} (1 + |\xi|^{2\theta} - 2t)^{-\frac{3}{2}} |\xi|^{8 - 16\theta} \, d\xi \\
= C\omega_n t^2 \|u_1\|^2_{L^1_{t,2\theta}} \int_0^\eta (1 + r^{2\theta} - 2t)^{-\frac{3}{2}} r^{7 - 16\theta + n} \, dr \\
= C\|u_1\|^2_{L^1_{t,2\theta}} t^{-\frac{4 - 12\theta + n}{2\theta}} \log(1+\theta), \quad t \gg 1. \quad (3.29)
\]

Remark 3.3 Note that in the above estimate (3.29) to apply Lemma 2.6 it is necessary to check $7 - 16\theta + n > -1$, but this holds for $0 \leq \theta < 1/2$. Moreover, according to our computations above, we have to prove that all $L^2$-norm of the six functions $F_1(t, \xi), \cdots, F_6(t, \xi)$ decay to zero in time. However, to get such decay estimates in (3.29), we need additional restriction such that $0 \leq \theta < \frac{\lambda}{2} < \frac{1}{2}$ in the case $n = 1$. For $n \geq 2$ this restriction is not necessary because $\frac{4 - 12\theta + n}{2\theta} \to 0$ when $t \to \infty$, for any $\theta \in (0, \frac{1}{2})$.

Now we want to obtain an estimate for $F_1(t, \cdot)$ on the region $|\xi| \leq \eta^3$. Initially, from (3.7) we may see that
\[
\int_{|\xi| \leq \eta^3} |F_1(t, \xi)|^2 \, d\xi = \int_{|\xi| \leq \eta^3} e^{-2 \log(1 + |\xi|^{2\theta}) t} |R(\xi)|^2 \, d\xi \approx \int_{|\xi| \leq \eta^3} e^{-\log(1 + |\xi|^{2\theta}) t} |R(\xi)|^2 \, d\xi \\
\leq \|u_1\|^2_{L^1_{t,2\theta}} \int_{|\xi| \leq \eta^3} e^{-\log(1 + |\xi|^{2\theta}) t} |R(\xi)|^2 \, d\xi \quad (3.30)
\]

Here, the function $R(r)$ is bounded on the low frequency zone for $0 < \theta \leq \frac{1}{3}$, because of
\[
\lim_{r \to 0+} R(r) = \begin{cases} 0 & \text{for } 0 < \theta < \frac{1}{3}, \\
\frac{1}{4} & \text{for } \theta = \frac{1}{3}.
\end{cases} \quad (3.31)
\]
Therefore, for $0 < \theta \leq \frac{1}{3}$ and $n \geq 1$, from (3.30) and (3.31) we may conclude the existence of a positive constant $C$ such that
\[
\int_{|\xi| \leq \eta^3} |F_1(t, \xi)|^2 d\xi \leq C \|u_1\|^2 \int_{|\xi| \leq \eta^3} e^{-\log(1+|\xi|^2 \theta) t} d\xi
\]
\[
= C \|u_1\|^2 \omega_n \int_0^{\eta^3} (1 + r^2 - 29) - \xi, dr
\]
\[
\sim \|u_1\|^2 t^{-\frac{\eta^3}{2}}, \quad t \gg 1. \tag{3.32}
\]

However, in the case of $0 \leq \theta \leq \frac{5}{12}$ we also notice that the function $R(r) \sqrt{r}$ in the region $|\xi| \leq \eta^3$ is bounded, because
\[
\lim_{r \to 0} \sqrt{r} R(r) = \begin{cases} 0 & \text{for } 0 < \theta < \frac{5}{12}, \\
\frac{4}{3} & \text{for } \theta = \frac{5}{12}. \end{cases}
\]

In particular, $R(r) \sqrt{r}$ in the region $r = |\xi| \leq \eta^3$ is bounded for $\frac{1}{4} < \theta \leq \frac{5}{12}$. Thus, from (3.30), in the case of $\frac{1}{4} < \theta \leq \frac{5}{12}$ and $n \geq 2$, one can obtain
\[
\int_{|\xi| \leq \eta^3} |F_1(t, \xi)|^2 d\xi \leq \omega_n \|u_1\|^2 \int_0^{\eta^3} (1 + r^2 - 29) - \xi, dr
\]
\[
\leq C \|u_1\|^2 \int_0^{\eta^3} (1 + r^2 - 29) - \xi, dr
\]
\[
\leq C \|u_1\|^2 t^{-\frac{\eta^3}{2}}, \quad t \gg 1. \tag{3.33}
\]

Similarly to the way used to obtain estimates for $F_1(t, \cdot)$ one can arrive at the following estimates for $F_2(t, \cdot)$:
\[
\int_{|\xi| \leq \eta^3} |F_2(t, \xi)|^2 d\xi \leq \begin{cases} \frac{C}{\theta} \|u_1\|^2 t^{-\frac{\eta^3}{2}} & \text{for } n \geq 1 \text{ and } 0 < \theta \leq \frac{1}{3}, \quad t \gg 1, \\
\frac{C}{\eta^3} \|u_1\|^2 t^{-\frac{\eta^3}{2}} & \text{for } n \geq 2 \text{ and } \frac{1}{4} < \theta \leq \frac{5}{12}, \quad t \gg 1. \end{cases} \tag{3.34}
\]

Let us estimate the $L^2$-norm of $F_0(t, \xi)$ at the final stage in this subsection 3.1.1. To do that we need to analyze the function $g(s)$ given by (3.20). Note that it is easy to see that
\[
1 \leq g(s) \leq 2 \tag{3.35}
\]
for $s \leq \delta$, and its derivative is given by
\[
g'(s) = \frac{1}{2 \sqrt{1 - \log^4 (1 + s^6)}} \left( \frac{48 \theta^5 s^{60} + 5}{(1 + s^6)^3 (1 + s^6)^2} - \frac{24 s^5}{\log^2 (1 + s^6)} \right) \).
\]

Then, for $\theta \in [0, \frac{5}{12}]$, the function $g'(s)$ is bounded on the interval $0 < s \leq \eta$. In fact, the limits
\[
\lim_{s \to \theta} s^{60} (1 + s^6)^3 (1 + s^6)^2 \quad \text{and} \quad \lim_{s \to 0} \frac{s^5}{\log^2 (1 + s^6)}
\]
are finite because of $0 \leq \theta \leq \frac{5}{12}$. It should be mentioned that the same does not happen on the zone $\eta < s < \delta$ because
\[
\lim_{s \to \delta} \left( 1 - \frac{4 s^6}{\log^2 (1 + s^6)} \right)^{-1} = +\infty
\]
(See (3.34)-(3.35)). Recall again that for $\theta \in (0, 1/2)$
\[
\lim_{s \to 0} \frac{s^6}{\log^2 (1 + s^6)} = 0.
\]

By summarizing above facts, there exists a constant $K > 0$ depending on $\theta \in [0, \frac{12}{5}]$ and $\eta > 0$ such that for all $s \in [0, \eta]$ it holds that
\[
|g'(s)| \leq K.
\]
In particular, for $|\xi| \in [0, \eta^3]$, we have $\sqrt[3]{|\xi|} \in [0, \eta]$ and $\alpha(t, \xi) \sqrt[3]{|\xi|} \in [0, \eta]$. Thus

$$|g'(\alpha \sqrt[3]{|\xi|})| \leq K, \quad |\xi| \leq \eta^3. \tag{3.36}$$

From (3.35) and (3.36), for $0 < \theta \leq \frac{5}{12}$ and $n \geq 1$ we can estimate the $L^2$-norm of $F_0(t, \cdot)$ as follows:

$$\int_{|\xi| \leq \eta^3} |F_0(t, \xi)|^2 d\xi = \frac{4}{3} t^2 \int_{|\xi| \leq \eta^3} e^{-t \log(1+|\xi|^{2\theta})} |\xi|^{\frac{2}{3}} (\alpha \sqrt[3]{|\xi|}) \|f_0\|_2 \|\tilde{u}_1(\xi)|^2 d\xi$$

$$\leq Ct^2 \|u_1\|_1^2 \int_{|\xi| \leq \eta^3} e^{-t \log(1+|\xi|^{2\theta})} |\xi|^{\frac{2}{3}} d\xi$$

$$= C\omega n^2 \|u_1\|_1^2 \int_0^\eta (1 + r^{2\theta})^{-t} r^{1-n-2} dr$$

$$\sim \frac{1}{\theta} t^2 \|u_1\|_1^2 t^{-\frac{n-4\theta}{2}}, \quad t \gg 1. \tag{3.37}$$

As a result one can conclude the following Propositions. In that case, it is essential whether the factor $1/\theta$ can be included or not in the final estimates as the coefficient.

**Proposition 3.1** Let $n = 1, 0 < \theta \leq \frac{1}{6}$, and $\varphi(t, \xi)$ be given by (3.26). If $u_1 \in L^{1,2\theta}(\mathbb{R})$, then

$$\int_{|\xi| \leq \eta^3} |\tilde{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq \left\{ \begin{array}{ll}
C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2) \left( t^{-\frac{n-4\theta}{2}} + \frac{1}{\theta} t^{-\frac{2\theta}{3}} \right), & \text{if } 0 < \theta \leq \frac{1}{6}, \\
C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2) \left( t^{-\frac{n-4\theta}{2}} + \frac{1}{\theta} t^{-\frac{2\theta}{3}} \right), & \text{if } \frac{1}{6} < \theta \leq \frac{1}{3},
\end{array} \right.$$}

for $t \gg 1$.

**Proof.** The proof is obtained by choosing the slowest estimates as $t \to \infty$ among (3.27), (3.28), (3.29), (3.32), (3.34) and (3.37). Note that the case $1/6 < \theta \leq 1/3$ is coming from the relation such that $\frac{9-4\theta}{2\theta} \leq \frac{n}{2\theta}$ with $n = 1$. \qed

**Proposition 3.2** Let $n \geq 2, 0 < \theta \leq \frac{5}{12}$ and $\varphi(t, \xi)$ be given by (3.26). If $u_1 \in L^{1,2\theta}(\mathbb{R}^n)$, then

$$\int_{|\xi| \leq \eta^3} |\tilde{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq \left\{ \begin{array}{ll}
C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2) \left( t^{-\frac{n-4\theta}{2\theta}} + \frac{1}{\theta} t^{-\frac{2\theta}{3}} \right), & \text{if } 0 < \theta \leq \frac{1}{6}, \\
C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2) \left( t^{-\frac{n-4\theta}{2\theta}} + \frac{1}{\theta} t^{-\frac{2\theta}{3}} \right), & \text{if } \frac{1}{6} < \theta \leq \frac{1}{3}, \\
C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2) \left( t^{-\frac{n-4\theta}{2\theta}} + \frac{1}{\theta} t^{-\frac{2\theta}{3}} \right), & \text{if } \frac{1}{4} < \theta \leq \frac{5}{12},
\end{array} \right.$$}

for $t \gg 1$.

**Proof.** We may conclude this result by comparing the estimates (3.27), (3.28), (3.29), (3.32), (3.34) and (3.37). Note that the case $1/6 < \theta \leq 1/3$ is coming from the relation such that $\frac{n-4\theta}{2\theta} \leq \frac{n}{2\theta}$ with $n \geq 2$. \qed

**3.1.2 Estimates on the middle-frequency zone** $\eta^3 \leq |\xi| \leq \delta$

We call the zone $\eta^3 \leq |\xi| \leq \delta$ the middle-frequency because the characteristics roots given by (3.35) are real on this zone, and therefore the solution of (3.31), (3.32) is given by

$$\tilde{u}(t, \xi) = e^{-t \log(1+|\xi|^{2\theta})} \sinh(C(\xi) t) \frac{2C(\xi)}{2C(\xi)} \tilde{u}_1(\xi),$$

where

$$C(\xi) = \frac{\sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2}}{2}.$$
We remember that \( \eta \) is defined in (3.12). Since the function \(|\xi| \mapsto \frac{|\xi|^{4-4\theta}}{|\xi|^2}\) is increasing for \( 0 < \theta < \frac{1}{2} \), we may observe that
\[
\eta^3 = \sup\{\alpha > 0; \frac{|\xi|^{4-4\theta}}{|\xi|^2} \leq \frac{1}{25^3} \text{ for } 0 < |\xi| \leq \alpha\}. \tag{3.38}
\]

**Lemma 3.3** There exists \( \beta = \beta(\theta), \, 0 < \beta \leq \eta^3 \), such that
\[
\frac{2}{25^3} \log^2(1 + |\xi|^{2\theta}) \geq 4|\xi|^2 \text{ for } |\xi| \leq \beta,
\]
\[
\frac{2}{25^3} \log^2(1 + |\xi|^{2\theta}) \leq 4|\xi|^2 \text{ for } |\xi| \geq \beta.
\]

**Proof.** The argument used to prove the existence of \( \beta = \beta(\theta) \in (0, 1) \), which satisfies both conclusions of this lemma. So, it suffices to check that \( \beta \leq \eta^3 \).

From Remark 3.1 we know that \( \log^2(1 + |\xi|^{2\theta}) \leq |\xi|^{4\theta} \), for \( |\xi| \leq 1 \). Thus, if \( |\xi| \leq \beta \), we have
\[
\frac{2}{25^3} |\xi|^{4\theta} \geq \frac{2}{25^3} \log^2(1 + |\xi|^{2\theta}) \geq 4|\xi|^2.
\]
This implies
\[
2 \times 25^3 |\xi|^{4\theta} \leq |\xi|^2, \quad |\xi| \leq \beta.
\]
and the condition \( \frac{|\xi|^{4-4\theta}}{|\xi|^2} \leq \frac{1}{25^3} \) is satisfied for \( |\xi| \leq \beta \). Therefore, one has \( \beta \leq \eta^3 \) from (3.38).

In other words, Lemma 3.3 tells us that
\[
\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2 < \frac{25^3 - 2}{25^3} \log^2(1 + |\xi|^{2\theta}) \text{ for } |\xi| \geq \beta
\]
and in particular, the definition of \( C(\xi) \) implies that
\[
0 < 2C(\xi) = \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2} < \frac{\sqrt{25^3 - 2}}{\sqrt{25^3}} \log(1 + |\xi|^{2\theta}) \text{ for } \beta \leq |\xi| < \delta.
\]
Therefore, if \( \eta^3 \leq |\xi| < \delta \) one has
\[
- \log(1 + |\xi|^{2\theta}) + 2C(\xi) < \left( \frac{\sqrt{25^3 - 2}}{25^3} - 1 \right) \log(1 + |\xi|^{2\theta}) = -c \log(1 + |\xi|^{2\theta}) \tag{3.39}
\]
with \( 0 < c < 1 \) a constant, due to the fact that \( \beta \leq \eta^3 \).

Now, from Lemma 2.4 and inequality (3.39) we can prove the exponential decay for the \( L^2 \)-norm of \( \hat{u}(t, \cdot) \) on the middle frequency zone as follows:
\[
\int_{\eta^3 \leq |\xi| < \delta} |\hat{u}(t, \xi)|^2 d\xi = \int_{\eta^3 \leq |\xi| < \delta} e^{-t \log(1 + |\xi|^{2\theta})} \frac{\sinh^2(C(\xi) t)}{4(C(\xi))^2} |\hat{u}_1(\xi)|^2 d\xi
\]
\[
\leq \frac{K^2}{4} t^2 \int_{\eta^3 \leq |\xi| \leq \delta} e^{-t \log(1 + |\xi|^{2\theta}) + 2C(\xi) |\hat{u}_1(\xi)|^2} d\xi
\]
\[
\leq K^2 t^2 \int_{\eta^3 \leq |\xi| \leq \delta} e^{-c t \log(1 + |\xi|^{2\theta})} |\hat{u}_1(\xi)|^2 d\xi \quad (c > 0)
\]
\[
= K^2 t^2 \int_{\eta^3 \leq |\xi| \leq \delta} (1 + |\xi|^2) - c t |\hat{u}_1(\xi)|^2 d\xi
\]
\[
= K^2 \omega_n t^2 \|u_0\|_1^2 \int_{\eta^3}^{\delta} (1 + r^2) - c t r^{n-1} dr
\]
\[
\leq C t^2 (1 + \eta^{2\theta}) - c t \|u_1\|_1^2, \quad t \gg 1, \tag{3.40}
\]
with \( C \) a positive constant depending on the space dimension \( n \) and \( c > 0 \) a constant given in (3.39).
3.2 Estimates on the high-frequency zone \(|\xi| \geq \delta\)

On the high frequency zone \(|\xi| > \delta\) the characteristics roots are complex and the solution of (3.1)-(3.2) is given by

\[ \hat{u}(t, \xi) = \frac{1}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t) \hat{u}_1(\xi) \]

where

\[ a(\xi) = \frac{\log(1 + |\xi|^2)}{2}, \quad b(\xi) = \sqrt{4|\xi|^2 - \log^2(1 + |\xi|^2)} \]

We know that \(|\sin a| \leq a\) for all \(a \geq 0\). Then \(\frac{|\sin(b(\xi)t)|}{b(\xi)} \leq t\) for all \(t \geq 0\), and so one has

\[ \int_{|\xi| > \delta} |\hat{u}(t, \xi)|^2 d\xi = \int_{|\xi| > \delta} (1 + |\xi|^2)^{-t} \frac{\sin^2(b(\xi)t)}{b(\xi)^2} |\hat{u}_1(\xi)|^2 d\xi \]

\[ \leq t^2 \|u_1\|^2 \int_{|\xi| > \delta} (1 + |\xi|^2)^{-t} d\xi \]

\[ = \omega_n t^2 \|u_1\|^2 \int_{\delta}^1 (1 + r^{2\theta})^{-t} r^{n-1} dr + \omega_n t^2 \|u_1\|^2 \int_{1}^\infty (1 + r^{2\theta})^{-t} r^{n-1} dr \]

\[ \sim \|u_1\|^2 t^2 \left( (1 + \delta^{2\theta})^{-t} + \frac{2^{-t}}{t-1} \right), \quad t \gg 1. \quad (3.41) \]

The last inequality is obtained by using Lemma 2.2

3.3 Proof of Theorem 1.1

Now by combining Propositions 3.1-3.2, (3.40), and (3.41) one can prove our main Theorem 1.1

**Proof of Theorem 1.1** We first note that \(\log(1 + |\xi|^2) \leq |\xi|^2\) for all \(\xi \in \mathbb{R}^n\), which implies \(\frac{|\xi|^2}{\log(1 + |\xi|^2)} \geq 1\). Then, one can get the next estimate for \(t \gg 1\) on the zone of high frequency \(|\xi| \geq \eta^2\) as follows:

\[ \int_{|\xi| \geq \eta^2} |\varphi(t, \xi)|^2 d\xi \leq P_1 \int_{|\xi| \geq \eta^2} e^{-\frac{2|\xi|^2}{\log(1 + |\xi|^2)} t} \log^2(1 + |\xi|^2) d\xi + P_1 \int_{|\xi| \geq \eta^2} e^{-\frac{2|\xi|^2}{\log(1 + |\xi|^2)} t} \log^2(1 + |\xi|^2) d\xi \]

\[ = P_1 \int_{|\xi| \geq \eta^2} e^{-\frac{2|\xi|^2}{\log(1 + |\xi|^2)} t} \log^2(1 + |\xi|^2) d\xi + P_1 \int_{|\xi| \geq \eta^2} (1 + |\xi|^2)^{-t} \log^2(1 + |\xi|^2) d\xi \]

\[ \leq P_1 \int_{|\xi| \geq \eta^2} e^{-2\theta |\xi|^2} \log^2(1 + |\xi|^2) d\xi + P_1 \int_{|\xi| \geq \eta^2} (1 + |\xi|^2)^{-t} \log^2(1 + |\xi|^2) d\xi \]

\[ \leq \frac{P_1^2 \omega_n}{(\log 2)^2} \int_{|\xi| \geq \eta^3} e^{-2\theta |\xi|^2} \log^2(1 + |\xi|^2) d\xi + \frac{P_1^2 \omega_n}{(\log 2)^2} \int_{\eta^2}^\infty \frac{(1 + r^{2\theta})^{-t}}{r^{n-1}} dr \]

\[ + \frac{P_1^2 \omega_n}{(\log 2)^2} \int_{\eta^2}^\infty \frac{(1 + r^{2\theta})^{-t}}{r^{n-1}} dr \]

\[ \leq CP_1^2 \left( e^{-t \eta_6^6} + (1 + \eta_6^6)^{-t} + \frac{2^{-t}}{t-1} \right), \quad t \gg 1. \quad (3.42) \]
Now, it follows from the Plancherel Theorem that
\[
\int_{\mathbb{R}^n} |u(t, x) - F_{\xi=0}^{-1}(\varphi(t, \xi))(x)|^2 dx = \int_{\mathbb{R}^n} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi
\]
for \( t \geq 0 \). Furthermore, one has
\[
\int_{\mathbb{R}^n} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq \int_{|\xi| \leq \eta^3} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi + \int_{|\xi| \geq \eta^3} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi + \int_{|\xi| \geq \eta^3} |\varphi(t, \xi)|^2 d\xi \quad (3.43)
\]
for \( t > 0 \).

From (3.40) and (3.41), we know that the \( L^2 \)-estimates on the zone \( |\xi| \geq \eta^3 \) to \( \hat{u}(t, \xi) \) are of exponential type. The estimate to \( \varphi(t, \xi) \) on \( |\xi| \geq \eta^3 \) obtained in (3.42) is also faster than those obtained in Propositions 3.1 and 3.2. The result of Theorem 1.1 follows by combining Propositions 3.1 and 3.2 with inequalities (3.40), (3.41), (3.42) and (3.43).

\[ \square \]

4 Optimality of the decay rates

Our goal in this section is to prove Theorem 1.2, which shows the optimal decay rates in time depending on the dimension \( n \) to the solution of the problem (1.1)-(1.2). From (3.20), we have
\[
\varphi(t, \xi) = \varphi_1(t, \xi) - \varphi_2(t, \xi), \quad t \geq 0, \quad \xi \in \mathbb{R}^n,
\]
where
\[
\varphi_1(t, \xi) := \frac{e^{-\frac{1}{\log(1 + |\xi|^2)^2} t}}{\log(1 + |\xi|^2)^a} P_1, \quad \varphi_2(t, \xi) := \frac{e^{-\frac{1}{\log(1 + |\xi|^2)^2} t}}{\log(1 + |\xi|^2)^a} P_1.
\]

Lemma 4.1 Let \( n = 1 \) with \( 0 < \theta < \frac{1}{4} \) and \( n \geq 2 \) with \( 0 < \theta \leq \frac{5}{12} \). If \( u_1 \in L^1(\mathbb{R}^n) \), then
\[
C_1 P_1^2 t^{-\frac{n-4\theta}{2(n-\theta)}} \leq \int_{\mathbb{R}^n} |\varphi(t, \xi)|^2 d\xi \leq C_2 P_1^2 \left( t^{-\frac{n-4\theta}{2(n-\theta)}} + \frac{1}{\theta} t^{-\frac{n-4\theta}{2\theta}} \right), \quad t \gg 1,
\]
where the constants \( C_1, C_2 \) depend only on \( \theta \) and \( n \).

Proof. We first note that
\[
\int_{\mathbb{R}^n} |\varphi(t, \xi)|^2 d\xi \leq 2 \int_{\mathbb{R}^n} |\varphi_1(t, \xi)|^2 d\xi + 2 \int_{\mathbb{R}^n} |\varphi_2(t, \xi)|^2 d\xi = 2 \int_{|\xi| \leq \eta} |\varphi_1(t, \xi)|^2 d\xi + 2 \int_{|\xi| \geq \eta} |\varphi_2(t, \xi)|^2 d\xi + 2 \int_{|\xi| \geq \eta} |\varphi_1(t, \xi)|^2 d\xi, \quad t > 0. \quad (4.2)
\]
By using the equivalences obtained in Remark 3.1 we have
\[
\int_{|\xi| \leq \eta} |\varphi_1(t, \xi)|^2 d\xi = \omega_n P_1^2 \int_0^\eta \frac{e^{-\frac{1}{\log(1 + |\xi|^2)^2} t}}{\log(1 + |\xi|^2)^a} d\xi \leq P_1^2 \int_0^\eta \frac{1 + |\xi|^{2\theta}}{\log^2(1 + |\xi|^2)^a} d\xi = \omega_n P_1^2 \int_0^\eta \frac{(1 + |\xi|^{2\theta})^{-t}}{\log^2(1 + |\xi|^2)^a} r^{-n-1-4\theta} r^{4\theta} dr
\]
\[
\leq 4\omega_n P_1^2 \int_0^\eta \frac{(1 + r^{2\theta})^{-t}}{\log^2(1 + r^{2\theta})^{a}} r^{-n-1-4\theta} r^{4\theta} dr = 4\omega_n P_1^2 \int_0^\eta (1 + r^{2\theta})^{-t} r^{-n-1-4\theta} r^{4\theta} dr \leq CP_1^2 t^{-\frac{n-4\theta}{2(n-\theta)}}, \quad t \gg 1. \quad (4.3)
\]
The last decay estimate is obtained from Lemma 2.6 since \(n - 4\theta > 0\). In the same way, by using Lemma 2.7 for \(n - 4\theta > 0\), we have the next estimate.

\[
\int_{|\xi| \leq \eta} |\varphi_2(t, \xi)|^2 d\xi = P_2 t\int_{|\xi| \leq \eta} \frac{e^{-2t \log(1+|\xi|^2)}}{(1 + |\xi|^2)} d\xi \leq \omega_n P_2 \int_0^\eta \frac{(1 + r^{2\theta})^{-t}}{\log^2(1 + r^{2\theta})} r^{n-1} dr \\
\leq 4\omega_n P_2 \int_0^\eta (1 + r^{2\theta})^{-t} r^{n-1-4\theta} dr \leq C\frac{1}{\theta} t^2 t^{-\frac{n-4\theta}{2\theta}}, \quad t \gg 1. \quad (4.4)
\]

Further, from (4.2), the \(L^2\)-estimate to \(\varphi(t, \xi)\) on the zone \(|\xi| \geq \eta\) is of exponential type, because \(|\xi| \geq \eta\) implies that \(|\xi| \geq \eta^3\). Therefore, there exists a constant \(C > 0\) such that

\[
\int_{R^n} |\varphi(t, \xi)|^2 d\xi \leq CP_1^2 \left( t^{-\frac{n-4\theta}{2\theta}} + \frac{1}{\theta} t^{-\frac{n-4\theta}{2\theta}} \right), \quad t \gg 1,
\]
due to (4.2), (1.3) and (1.4).

In order to prove the estimate from below, from Remark 3.1 we have

\[
\int_{R^n} |\varphi_1(t, \xi)|^2 d\xi \geq \int_{|\xi| \leq \eta} |\varphi_1(t, \xi)|^2 d\xi \approx P_2 t\int_{|\xi| \leq \eta} \frac{e^{-t \log(1+|\xi|^2)}}{\log^2(1 + |\xi|^2)} d\xi \\
= \omega_n P_2 \int_0^\eta \frac{e^{-t \log(1+r^{2\theta})}}{\log^2(1 + r^{2\theta})} r^{n-1} dr \geq C\omega_n P_2 \int_0^\eta \frac{e^{-t \log(1+r^{2\theta})}}{r^{4\theta}} r^{n-1} dr \\
= C\omega_n P_2 \int_0^\eta (1 + r^{2\theta})^{-t} r^{n-1-4\theta} dr \\
\geq CP_1^2 t^{-\frac{n-4\theta}{2\theta}}, \quad (4.5)
\]

because of \(n - 4\theta > 0\), due to Remark 2.2 where \(C > 0\) is a generous constant. We also notice that

\[|\varphi_1(t, \xi)| \leq |\varphi(t, \xi)| + |\varphi_2(t, \xi)|\]

and, from Young’s inequality, \(|\varphi_1(t, \xi)|^2 \leq 2|\varphi(t, \xi)|^2 + 2|\varphi_2(t, \xi)|^2\). Thus,

\[|\varphi(t, \xi)|^2 \geq \frac{1}{2} |\varphi_1(t, \xi)|^2 - |\varphi_2(t, \xi)|^2 \leq 0, \quad \xi \in R^n.
\]

Then, from (4.5) and (4.4), we have

\[
\int_{|\xi| \leq \eta} |\varphi(t, \xi)|^2 d\xi \geq \frac{1}{2} \int_{|\xi| \leq \eta} |\varphi_1(t, \xi)|^2 d\xi - \int_{|\xi| \leq \eta} |\varphi_2(t, \xi)|^2 d\xi \\
\geq K_1 P_1^2 t^{-\frac{n-4\theta}{2\theta}} - K_2 \frac{1}{\theta} P_1^2 t^{-\frac{n-4\theta}{2\theta}} \\
= P_1^2 t^{-\frac{n-4\theta}{2\theta}} \left( K_1 - K_2 \frac{1}{\theta} \frac{8\theta^2 - 2\theta n + n - 4\theta}{2\theta} \right). \quad (4.6)
\]

Since \(0 < \theta < \frac{1}{2}\) and \(n - 4\theta > 0\), one can conclude that \(8\theta^2 - 2\theta n + n - 4\theta > 0\). Therefore, it follows from (4.6) that

\[
\int_{R^n} |\varphi(t, \xi)|^2 d\xi \geq \int_{|\xi| \leq \eta} |\varphi(t, \xi)|^2 d\xi \geq \frac{K_1}{2} P_1^2 t^{-\frac{n-4\theta}{2\theta}}, \quad t \gg 1.
\]

These arguments imply the desired estimate for \(\varphi(t, \xi)\).

\[\square\]

The above arguments do not hold for \(n = 1\) and \(\frac{1}{4} \leq \theta \leq \frac{1}{3}\), because the integrals

\[
\int_0^\eta \frac{(1 + r^{2\theta})^{-t}}{\log^2(1 + r^{2\theta})} r^{n-1} dr, \quad \int_0^\eta \frac{(1 + r^{2\theta})^{-t}}{\log^2(1 + r^{2\theta})} r^{n-1} dr
\]

are divergent for all \(t > 0\). For this reason, we need to estimate the \(L^2\)-norm of the function \(\varphi(t, \xi)\) itself:

\[
\varphi(t, \xi) = \frac{e^{-|\xi|^2 \log(1 + |\xi|^2)}_{P_1}}{\log(1 + |\xi|^2)} P_1 - \frac{e^{-(\log(1 + |\xi|^2)_{P_1}}}{\log(1 + |\xi|^2)} P_1.
\]

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Lemma 4.2 Let $n=1$ and $\theta > \frac{1}{4}$. If $u_1 \in L^1(\mathbb{R})$, there exist constants $C_1, C_2 > 0$ such that

$$C_1 P_1^4 t^{\frac{4-\alpha}{4\alpha}} \leq \int_{\mathbb{R}} |\varphi(t, \xi)|^2 d\xi \leq C_2 \frac{1}{4\theta - 1} P_1^4 t^{\frac{4-1}{4\alpha}}, \quad t \gg 1.$$  

Proof. We first note that from (3.42) the $L^2$-norm of $\varphi(t, \xi)$ decays exponentially on the high frequency region $|\xi| \geq \eta > \eta^3$. So, in this proof it suffices to consider the integral only in the low frequency zone $0 < |\xi| \leq \eta$.

Now, we notice that

$$-\log(1 + r^{2\theta}) = \frac{r^2 - \log^2(1 + r^{2\theta})}{\log(1 + r^{2\theta})} - \frac{r^2}{\log(1 + r^{2\theta})},$$

so that one has

$$\log(1 + |\xi|^{2\theta}) \varphi(t, \xi) = P_1 \left(e^{-\frac{|\xi|^2}{\log(1 + |\xi|^{2\theta})}t} - e^{-\log(1 + |\xi|^{2\theta})t} \right)$$

$$= P_1 \left(e^{-\frac{|\xi|^2}{\log(1 + |\xi|^{2\theta})}t} - e^{-\left(\frac{\log(1 + |\xi|^{2\theta})}{\log(1 + |\xi|^{2\theta})}\right) |\xi|^2} \right)$$

$$= P_1 e^{-\frac{|\xi|^2}{\log(1 + |\xi|^{2\theta})}t} \left(1 - e^{-\frac{\log^2(1 + |\xi|^{2\theta}) - |\xi|^2}{\log(1 + |\xi|^{2\theta})}t} \right). \quad (4.7)$$

Due to the fact that for $0 < r < 1$ we have $\frac{1}{2} r^{2\theta} \leq \log(1 + r^{2\theta}) \leq r^{2\theta}$, thus one has

$$r^{2\theta} \frac{(1 - 4r^{2-4\theta})}{4} \leq \frac{\log^2(1 + r^{2\theta}) - r^2}{\log(1 + r^{2\theta})} \leq 2r^{2\theta} \left(1 - r^{2-4\theta} \right). \quad (4.8)$$

Moreover, since $\theta < \frac{1}{4}$ we have $2 - 4\theta > 0$. Therefore, there exists $\beta = \beta(\theta) > 0$, with $\beta < \eta$ such that

$$1 - 4r^{2-4\theta} \geq \frac{1}{2},$$

for $0 < r \leq \beta$. Thus,

$$\frac{1}{2} \leq 1 - 4r^{2-4\theta} \leq 1 - r^{2-4\theta} \leq 1 \quad (4.9)$$

for $0 < r \leq \beta$. From (4.8) and (4.9) one can get

$$\frac{1}{8} r^{2\theta} \leq \frac{\log^2(1 + r^{2\theta}) - r^2}{\log(1 + r^{2\theta})} \leq 2r^{2\theta},$$

for $0 < r \leq \beta$. This implies

$$1 - e^{-\frac{1}{8} tr^{2\theta}} \leq 1 - e^{-\frac{\log^2(1 + r^{2\theta}) - r^2}{\log(1 + r^{2\theta})}} \leq 1 - e^{-2tr^{2\theta}}$$

and

$$\frac{1 - e^{-\frac{1}{8} tr^{2\theta}}}{r^{2\theta}} \leq \frac{1 - e^{-\frac{\log^2(1 + r^{2\theta}) - r^2}{\log(1 + r^{2\theta})}}}{r^{2\theta}} \leq \frac{2}{r^{2\theta}} \frac{1 - e^{-2tr^{2\theta}}}{r^{2\theta}}, \quad (4.10)$$

for $0 < r \leq \beta$. Since

$$\lim_{\sigma \to 0} \frac{1 - e^{-\sigma}}{\sigma} = 1,$$

there exists $\alpha > 0$ such that $\alpha \leq \beta$ and

$$\frac{1}{2} \leq \frac{1 - e^{-\sigma}}{\sigma} \leq \frac{3}{2}, \quad (4.11)$$

for all $0 < \sigma \leq \alpha$. Based on these preparations let us prove the desired estimate for $\varphi(t, \xi)$.
(1) The lower estimate of Lemma:

For $0 < r \leq \left( \frac{8\alpha}{T} \right)^{\frac{1}{20}}$ it holds that $0 < \sigma = \frac{1}{8} tr^{2\theta} \leq \alpha$. Applying estimate (4.11) we get

$$\frac{1}{2} \leq 1 - e^{-\frac{1}{8} tr^{2\theta}} \leq \frac{3}{2}$$  \hspace{1cm} (4.12)

From (4.10) and (4.12), for $0 < r \leq \left( \frac{8\alpha}{T} \right)^{\frac{1}{20}}$, it holds that

$$1 - e^{-t \frac{\log^2(1+r^{2\theta})}{\log(1+r^{2\theta})}} \geq \frac{t}{16}.$$  \hspace{1cm} (4.13)

Let $t_0 > 0$ be such that $\left( \frac{8\alpha}{t_0} \right)^{\frac{1}{20}} \leq \alpha$, and consider $t \geq t_0$. By combining (4.11) with (4.7) and (4.13), since $\alpha \leq \beta \leq \eta$, we obtain

$$\int_{|\xi| \leq \eta} |\varphi(t, \xi)|^2 d\xi = P_1^2 \int_{|\xi| \leq \eta} \left( e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})} t} - e^{-\log(1+|\xi|^{2\theta}) t} \right)^2 \log(1+|\xi|^{2\theta}) d\xi$$

$$= \omega_1 P_1^2 \int_0^\eta e^{-\frac{2r^2}{\log(1+r^{2\theta})} t} \left( 1 - e^{-t \frac{\log^2(1+r^{2\theta})}{\log(1+r^{2\theta})}} \right)^2 d\eta$$

$$\geq \frac{\omega_1}{16} P_1^2 t^2 \int_0^\eta (\frac{4\alpha}{t})^{\frac{1}{20}} e^{-\frac{2r^2}{\log(1+r^{2\theta})} t} dr, \hspace{1cm} t \geq t_0.$$  \hspace{1cm} (4.14)

We also notice that

$$2 \log(1+r^{2-2\theta}) \leq 2r^{2-2\theta} \leq \frac{2r^2}{\log(1+r^{2\theta})} \leq 4r^2-2\theta \leq 8 \log(1+r^{2-2\theta}) \hspace{1cm} 0 < r \leq 1.$$  \hspace{1cm} (4.15)

Thus, from (4.14) and (4.15) one has

$$\int_{|\xi| \leq \eta} |\varphi(t, \xi)|^2 d\xi \geq \frac{\omega_1}{16} P_1^2 t^2 \int_0^\eta (\frac{4\alpha}{t})^{\frac{1}{20}} e^{-8t \log(1+r^{2-2\theta})} dr$$

$$= \frac{\omega_1}{16} P_1^2 t^2 \int_0^\eta (1+r^{2-2\theta})^{-8t} dr$$

$$\geq \frac{\omega_1}{16} P_1^2 t^2 \left( 1 + \left( \frac{8\alpha}{t} \right)^{\frac{2-2\theta}{2\theta}} \right)^{-8t} \int_0^\eta \left( \frac{4\alpha}{t} \right)^{\frac{1}{20}} dr, \hspace{1cm} t \geq t_0.$$  \hspace{1cm} (4.16)

Now we observe that $1 < \frac{2-2\theta}{2\theta} < 3$ for $\frac{1}{4} < \theta < \frac{1}{2}$. Then there exists $T \geq t_0$ such that

$$\frac{1}{2} \leq \left( 1 + \left( \frac{8\alpha}{t} \right)^{\frac{2-2\theta}{2\theta}} \right)^{-t} \leq \frac{3}{2}$$  \hspace{1cm} (4.17)

for all $t \geq T$, because of the fact that

$$\lim_{t \to +\infty} \left( 1 + \frac{1}{t^q} \right)^{-t} = 1$$

provided that $q > 1$. By combining estimates (4.16) and (4.17) one can arrive at the desired estimate.
from below such that
\[
\int_{|\xi| \leq \eta} |\varphi(t, \xi)|^2 d\xi \geq \frac{\omega_n}{16^2} P_1^2 t^2 \left( 1 + \left( \frac{8\alpha}{t} \right)^{2+2\beta} \right)^{-1} \int_0^{(\frac{4\beta}{2\alpha})} dr
\]
\[
\geq \frac{\omega_n}{2 \times 16^2} P_1^2 t^2 \int_0^{(\frac{4\beta}{2\alpha})} dr
\]
\[
= \frac{\omega_n}{2 \times 16^2} P_1^2 \left( \frac{8\alpha}{t} \right)^{\frac{1}{2\alpha}}
\]
\[
= C P_1^2 t^2 e^{-\frac{1}{2\alpha}}, \quad t \geq T
\]

with some constant \( C = C_\theta > 0 \).

(2) The upper estimate of Lemma:

From (4.7), we have
\[
\int_{|\xi| \leq \eta} |\varphi(t, \xi)|^2 d\xi = \omega_1 P_1^2 \int_0^{\eta} e^{-\frac{2\beta^2}{2\log(1+r^{2\beta})}} \left( 1 - e^{-t \frac{\log^2(1+r^{2\theta})}{\log(1+r^{2\theta})} - r^2} \right)^2 \log(1 + r^{2\theta}) \ dr = A_1(t, \theta) + A_2(t, \theta),
\]

where
\[
A_1(t, \theta) := \omega_1 P_1^2 \int_0^{(\frac{\alpha}{2t})} e^{-\frac{2\beta^2}{2\log(1+r^{2\beta})}} \left( 1 - e^{-t \frac{\log^2(1+r^{2\theta})}{\log(1+r^{2\theta})} - r^2} \right)^2 \log(1 + r^{2\theta}) \ dr,
\]
\[
A_2(t, \theta) := \omega_1 P_1^2 \int_{(\frac{\alpha}{2t})}^{\eta} e^{-\frac{2\beta^2}{2\log(1+r^{2\beta})}} \left( 1 - e^{-t \frac{\log^2(1+r^{2\theta})}{\log(1+r^{2\theta})} - r^2} \right)^2 \log(1 + r^{2\theta}) \ dr,
\]

which holds for \( t \geq t_0 \).

Now, for \( 0 < r \leq \left( \frac{\alpha}{2t} \right)^{\frac{1}{2\beta}} \) by using inequality (4.11) we have that
\[
1 - e^{-2t r^{2\theta}} \leq 3t.
\]

Thus for \( 0 < r \leq \left( \frac{\alpha}{2t} \right)^{\frac{1}{2\beta}} \), by combining (4.10) with (4.20) it holds that
\[
1 - e^{-t \frac{\log^2(1+r^{2\theta})}{\log(1+r^{2\theta})} - r^2} \leq 6t, \quad t \geq t_0.
\]

The definition of \( A_1(t, \theta) \) and the inequality (4.21) imply that
\[
A_1(t, \theta) = \omega_1 P_1^2 \int_0^{(\frac{\alpha}{2t})} e^{-\frac{2\beta^2}{2\log(1+r^{2\beta})}} t \left( 1 - e^{-t \frac{\log^2(1+r^{2\theta})}{\log(1+r^{2\theta})} - r^2} \right)^2 \ log(1 + r^{2\theta}) \ dr
\]
\[
\leq 36t^2 \omega_1 P_1^2 \int_0^{(\frac{\alpha}{2t})} e^{-\frac{2\beta^2}{2\log(1+r^{2\beta})}} t \ dr \leq 36t^2 \omega_1 P_1^2 \int_0^{(\frac{\alpha}{2t})} \ dr
\]
\[
= 36t^2 \omega_1 P_1^2 \left( \frac{\alpha}{2t} \right)^{\frac{1}{2\beta}} = C P_1^2 t^2 e^{-\frac{1}{2\alpha}}
\]
\[
= C P_1^2 t^{\frac{4\beta-1}{2\alpha}}, \quad t \gg 1
\]

with some generous constant \( C > 0 \). Note that the estimate given by (4.22) is also holds for \( \theta = \frac{1}{t} \).
In order to estimate $A_2(t, \theta)$, we use (4.10) for $\theta > 1/4$ to get the following estimate

$$A_2(t, \theta) = \omega_1 P^2 \int_{(\frac{\eta}{\pi})^2}^\eta e^{-\frac{2r^2}{\log(1+r^{2\theta})}} \left(1 - e^{-\frac{(1-\log 2\log(1+r^{2\theta}))}{\log(1+r^{2\theta})}}\right)^2 dr$$

$$\leq 4\omega_1 P^2 \int_{(\frac{\eta}{\pi})^2}^\eta e^{-\frac{2r^2}{\log(1+r^{2\theta})}} \left(1 - e^{-2t^{2\theta}}\right)^2 dr$$

$$\leq 4\omega_1 P^2 \int_{(\frac{\eta}{\pi})^2}^\eta \frac{1}{r^{4\theta}} dr$$

$$= 4\omega_1 P^2 \frac{1}{1 - 4\theta} \left(\eta^{1-4\theta} - \left(\frac{\alpha}{2t}\right)^{\frac{1-4\theta}{\theta}}\right)$$

$$\leq CP^2 \frac{1}{4\theta - 1 - \frac{1-4\theta}{\theta}}, \quad t \geq t_0$$

(4.23)

with $C = 4\omega_1 \left(\frac{\pi}{2}\right)^{\frac{1-4\theta}{\theta}}$. It is important to emphasize that the above estimate holds only for $\theta \neq \frac{1}{4}$ and we have just used it for $1 - 4\theta < 0$ to obtain (4.23). The estimates for $A_1$ and $A_2$ prove the desired estimate from below of lemma.

As a special case one can introduce the following log-order blowup result for the case of $\theta = \frac{1}{4}$.

**Lemma 4.3** Let $n = 1$ and $\theta = \frac{1}{4}$. For $u_1 \in L^1(\mathbf{R})$ the following optimal estimate holds.

$$C_1 P^2 \log t \leq \int_{\mathbf{R}} |\varphi(t, \xi)|^2 d\xi \leq C_2 P^2 \log t, \quad t \gg 1,$$

with some constants $C_1, C_2 > 0$.

**Proof.** We consider the functions $A_1(t, \frac{1}{4})$ and $A_2(t, \frac{1}{4})$ given by (4.18) and (4.19) with $\theta = \frac{1}{4}$. The estimate (4.22) also holds for $\theta = \frac{1}{4}$ and it tells us the fact that

$$A_1(t, \frac{1}{4}) \leq CP^2, \quad t \gg 1.$$  

(4.24)

While, by definition (4.19) and (4.10) we have

$$A_2(t, \frac{1}{4}) = \omega_1 P^2 \int_{(\frac{\eta}{\pi})^2}^\eta e^{-\frac{2r^2}{\log(1+r^{2\theta})}} \left(1 - e^{-\frac{(1-\log 2\log(1+r^{2\theta}))}{\log(1+r^{2\theta})}}\right)^2 dr$$

$$\leq 4\omega_1 P^2 \int_{(\frac{\eta}{\pi})^2}^\eta e^{-\frac{2r^2}{\log(1+r^{2\theta})}} \left(1 - e^{-2t^{2\theta}}\right)^2 dr$$

$$\leq 4\omega_1 P^2 \int_{(\frac{\eta}{\pi})^2}^\eta \frac{1}{r^{4\theta}} dr$$

$$= 4\omega_1 P^2 \left(\log \eta - \log \left(\frac{\alpha}{2t}\right)^{\frac{1}{\theta}}\right)$$

$$= 4\omega_1 P^2 \left(\log \eta - 2 \log \alpha + 2 \log 2 + 2 \log t\right)$$

$$\leq CP^2 \log t, \quad t \gg 1.$$  

(4.25)

The estimates (4.24) and (4.25) allow us to conclude the upper estimate

$$\int_{\mathbf{R}} |\varphi(t, \xi)|^2 d\xi \leq C_2 \log t, \quad t \gg 1,$$

(4.26)

with some constant $C_2 > 0$.

On the other hand, by (4.11) one can get

$$|\varphi(t, \xi)|^2 \geq \frac{1}{2} |\varphi_1(t, \xi)|^2 - |\varphi_2(t, \xi)|^2, \quad t > 0, \quad \xi \in \mathbf{R}.$$
Thus, for $t > 0$,
\[
\int_{\mathbb{R}} |\varphi(t, \xi)|^2 d\xi \geq \int_{t^{-1}}^{t^{-\frac{1}{2}}} |\varphi(t, \xi)|^2 d\xi \geq \frac{1}{2} \int_{t^{-1}}^{t^{-\frac{1}{2}}} |\varphi_1(t, \xi)|^2 d\xi - \int_{t^{-1}}^{t^{-\frac{1}{2}}} |\varphi_2(t, \xi)|^2 d\xi
\]
\[
= P_1^2 \left( \frac{1}{2} K_1(t) - K_2(t) \right),
\]
where
\[
K_1(t) := \int_{t^{-1}}^{t^{-\frac{1}{2}}} \frac{e^{-\frac{2|\xi|^2}{\log(1 + \sqrt{1/\xi})}}}{\log^2(1 + \sqrt{|\xi|})} d\xi,
\]
\[
K_2(t) := \int_{t^{-1}}^{t^{-\frac{1}{2}}} \frac{e^{-2\log(1 + \sqrt{1/\xi})t}}{\log^2(1 + \sqrt{|\xi|})} d\xi.
\]

We remember that
\[
\frac{1}{2} \sqrt{|\xi|} \leq \log(1 + \sqrt{|\xi|}) \leq \sqrt{|\xi|}, \quad |\xi| \leq 1.
\]
Thus by applying Remark 3.1 one has
\[
K_1(t) = \int_{t^{-1}}^{t^{-\frac{1}{2}}} \frac{e^{-\frac{2|\xi|^2}{\log(1 + \sqrt{1/\xi})}}}{\log^2(1 + \sqrt{|\xi|})} d\xi \geq \int_{t^{-1}}^{t^{-\frac{1}{2}}} \frac{e^{-\frac{2|\xi|^2}{\log(1 + \sqrt{1/\xi})}}}{|\xi|} d\xi
\]
\[
= \omega_1 \int_{t^{-1}}^{t^{-\frac{1}{2}}} e^{-\frac{4\sqrt{t}}{3} - \frac{4}{3}\log t} d\xi \geq \omega_1 e^{-\frac{4}{3}\log t + \log t} \int_{t^{-1}}^{t^{-\frac{1}{2}}} \frac{1}{r} dr
\]
\[
= \omega_1 e^{-\frac{4}{3}\log t}, \quad t \geq 1.
\]
Similarly, in the case when large $t > 1$ such that $t^{-\frac{1}{2}} < 1$ it follows from (4.28) that
\[
K_2(t) = \int_{t^{-1}}^{t^{-\frac{1}{2}}} \frac{e^{-2\log(1 + \sqrt{1/\xi})t}}{\log^2(1 + \sqrt{|\xi|})} d\xi \leq 4 \int_{t^{-1}}^{t^{-\frac{1}{2}}} \frac{e^{-\sqrt{t}|\xi|}}{|\xi|} d\xi
\]
\[
= 4\omega_1 \int_{t^{-1}}^{t^{-\frac{1}{2}}} e^{-\sqrt{t} - \frac{4}{3}\log t} d\xi \leq 4\omega_1 e^{-\sqrt{t}} \int_{t^{-1}}^{t^{-\frac{1}{2}}} \frac{1}{r} dr
\]
\[
\leq \frac{4\omega_1}{3} e^{-\sqrt{t} \log t}, \quad t \gg 1.
\]

Therefore, from (4.27), (4.29) and (4.30) one has
\[
\int_{\mathbb{R}} |\varphi(t, \xi)|^2 d\xi \geq P_1^2 \left( \frac{1}{2} K_1(t) - K_2(t) \right)
\]
\[
\geq P_1^2 \omega_1 \left( \frac{e^{-\frac{4}{3}\log t}}{6} - \frac{4}{3} e^{-\sqrt{t} \log t} \right)
\]
\[
= P_1^2 \omega_1 \log t \left( \frac{e^{-\frac{4}{3}\log t}}{6} - \frac{4}{3} e^{-\sqrt{t} \log t} \right), \quad t \gg 1
\]
which implies the desired estimate from below to the case $\theta = 1/4$ and $n = 1$:
\[
\int_{\mathbb{R}} |\varphi(t, \xi)|^2 d\xi \geq CP_1^2 \log t, \quad t \gg 1
\]
with some constant $C > 0$. The estimates (4.26) and (4.31) complete the proof of lemma.
Now, let us prove Theorem 1.2 at a stroke.

**Proof of Theorem 1.2** One first observes that

$$\int_{\mathbb{R}^n} |\hat{u}(t, \xi)|^2 d\xi \leq \int_{\mathbb{R}^n} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi + \int_{\mathbb{R}^n} |\varphi(t, \xi)|^2 d\xi. \quad (4.32)$$

By combining Lemma 4.1 and Theorem 1.1 with (4.32), we have

$$\int_{\mathbb{R}^n} |\hat{u}(t, \xi)|^2 d\xi \leq C(P_2^2 + \|u_1\|_{L^2}^2)\left( t^{-\frac{n-4\theta}{2(n-\theta)} + \frac{1}{\theta}t^{-\frac{n-4\theta}{2(n-\theta)}}} \right), \quad t \gg 1. \quad (4.33)$$

We can also observe that for $0 < \theta < 1/2$ it holds that $2\theta \leq 2 - 2\theta$. Therefore $\frac{n-4\theta}{2(n-\theta)} \leq \frac{n-4\theta}{2n}$. Thus the decay rate $t^{-\frac{n-4\theta}{2(n-\theta)}}$ is faster than $t^{-\frac{n-4\theta}{2n}}$. It results the following upper bound to the $L^2$-norm of the Fourier transformed solution $\hat{u}$ such that

$$\int_{\mathbb{R}^n} |\hat{u}(t, \xi)|^2 d\xi \leq C(P_1^2 + \|u_1\|_{L^2}^2)(1 + \frac{1}{\theta})t^{-\frac{n-4\theta}{2(n-\theta)}}, \quad t \gg 1.$$

By the Plancherel Theorem and from (4.33), the upper bound estimate of the statement of Theorem 1.2 follows with a generous constant $C > 0$.

In order to obtain the lower bound, we observe that

$$|\varphi(t, \xi)| \leq |\hat{u}(t, \xi) - \varphi(t, \xi)| + |\hat{u}(t, \xi)|.$$

By Young’s inequality, we may obtain

$$|\varphi(t, \xi)|^2 \leq 2|\hat{u}(t, \xi) - \varphi(t, \xi)|^2 + 2|\hat{u}(t, \xi)|^2.$$

Therefore,

$$|\hat{u}(t, \xi)|^2 \geq \frac{1}{2}|\varphi(t, \xi)|^2 - |\hat{u}(t, \xi) - \varphi(t, \xi)|^2, \quad t \geq 0, \quad \xi \in \mathbb{R}^n.$$

Thus,

$$\int_{\mathbb{R}^n} |\hat{u}(t, \xi)|^2 d\xi \geq \frac{1}{2} \int_{\mathbb{R}^n} |\varphi(t, \xi)|^2 d\xi - \int_{\mathbb{R}^n} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \quad (4.34)$$

First, we consider the case $n \geq 1$ and $0 < \theta \leq \frac{1}{3}$. By combining (4.34) with the lower estimate of Lemma 4.1 and estimate of Theorem 1.1 we obtain

$$\|\hat{u}(t, \cdot)\|^2 \geq \frac{C_1}{2}P_1^2 t^{-\frac{n-4\theta}{2(n-\theta)}} - C_2(\|u_1\|^2_{L^1} + \|u_1\|^2_{L^{1,2}}) \left( t^{-\frac{n-4\theta}{2(n-\theta)} + \frac{1}{\theta}t^{-\frac{n-4\theta}{2(n-\theta)}}} \right)$$

$$= t^{-\frac{n-4\theta}{2(n-\theta)}} \left( \frac{C_1}{2}P_1^2 - 2C_2(\|u_1\|^2_{L^1} + \|u_1\|^2_{L^{1,2}}) \left( t^{-\frac{4\theta}{2(n-\theta)} + \frac{1}{\theta}t^{-\frac{4\theta}{2(n-\theta)}}} \right) \right), \quad (4.35)$$

for $t \gg 1$ and positive constants $C_1, C_2$. But for $0 < \theta \leq \frac{1}{3}$ we notice that $n + 4\theta^2 - 2n\theta > 0$, so that one can get

$$\lim_{t \to \infty} \left( \frac{C_1}{2}P_1^2 - 2C_2(\|u_1\|^2_{L^1} + \|u_1\|^2_{L^{1,2}}) \left( t^{-\frac{4\theta}{2(n-\theta)} + \frac{1}{\theta}t^{-\frac{4\theta}{2(n-\theta)}}} \right) \right) = \frac{C_1}{2}P_1^2.$$

Therefore, there exists $t_1 > 0$ such that

$$\frac{C_1}{4}P_2^2 \leq \frac{C_1}{2}P_1^2 - 2C_2(\|u_1\|^2_{L^1} + \|u_1\|^2_{L^{1,2}}) \left( t^{-\frac{4\theta}{2(n-\theta)} + \frac{1}{\theta}t^{-\frac{4\theta}{2(n-\theta)}}} \right) \leq C_1P_1^2, \quad t \gg t_1.$$

From (4.35) it follows that

$$\|\hat{u}(t, \cdot)\|^2 \geq \frac{C_1}{4}P_2^2 t^{-\frac{n-4\theta}{2(n-\theta)}}, \quad (4.36)$$

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for $t \gg 1$.

The estimate (4.36) implies the desired estimate for lower bound in $t$ in the case when $n \geq 1$ and $0 < \theta \leq \frac{1}{6}$.

Analogously, we may obtain the results for $n = 1$ with $\frac{1}{6} < \theta < \frac{1}{4}$, and for $n \geq 2$ with $\frac{1}{6} < \theta \leq \frac{5}{12}$ based on the results of Theorem 1.1 for these values of $\theta$ and Lemma 4.1.

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. The proof of Theorem 1.3 can be obtained in the same way as in Theorem 1.2, but using Lemmas 4.2 and 4.3 instead of Lemma 4.1 and observing that the estimates to $\|\varphi(t, \cdot)\|^2$ in Lemmas 4.2 and 4.3 are also worse than the estimates to $\|\hat{u}(t, \cdot) - \varphi(t, \cdot)\|^2$ in Theorem 1.1. □

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