Scattering for radial bounded solutions of focusing supercritical wave equations in odd dimensions

Guher Camliyurt and Carlos E. Kenig

Abstract. We consider the wave equation with an energy-supercritical focusing nonlinearity in dimension seven. We prove that any radial solution that remains bounded in the critical Sobolev space is global and scatters to a linear solution.

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1. Introduction

In this paper we consider the Cauchy problem for the focusing wave equation

\[ \partial_t^2 u - \Delta u - |u|^{p-1} u = 0, \quad \text{in } \mathbb{R}^7 \times I, \]

\[ \vec{u}(0) = (u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^7) \]

in the energy-supercritical radial setting. Here, the set \( I \) is an interval around 0, and \( \dot{H}^{s_p} \) denotes the homogeneous \( L^2_x \)-based Sobolev space over \( \mathbb{R}^7 \) with

\[ p \geq 3, \quad s_p = \frac{7}{2} - \frac{2}{p-1}. \]

The class of solutions to the Cauchy problem (1.1) is invariant under the scaling

\[ \vec{u}(t, x) \mapsto (\lambda^{-a_p} u(t/\lambda, x/\lambda), \lambda^{-1-a_p} \partial_t u(t/\lambda, x/\lambda)) \]

(1.3)

where \( a_p = 2/(p-1) \). The scaling in (1.3) also determines the critical regularity space for the initial data: we note that the \( \dot{H}^{s_p} \times \dot{H}^{s_p-1} \) norm of \( (u_0, u_1) \) stays invariant under (1.3). Due to the lower bound on the \( p \) exponent, the space for initial data is equipped with \( s_p > 1 \), which places the Cauchy problem (1.1) in an energy-supercritical regime.

Key words and phrases. Scattering for radial solutions, wave equations, odd dimensions.
Date. January 14, 2022.
We prove that any radial solution of the Cauchy problem (1.1) that is bounded in the critical regularity space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ (throughout its maximal interval of existence) must be global and must scatter to a linear solution. We remark that the analogous assertions were established by Duyckaerts, Kenig, and Merle [16] in three dimensions and by Dodson and Lawrie [11] in five dimensions. While we particularly work in seven dimensions, we expect that our approach will generalize to all higher odd dimensions. Our main result is below.

**Theorem 1.1.** Let $\tilde{u}(t)$ be a radial solution to the equation (1.1) with maximal interval of existence $I_{\text{max}}(\tilde{u}) = (T_-(\tilde{u}), T_+(\tilde{u}))$ such that

$$\sup_{t \in (0, T_+(\tilde{u}))} \| (u(t), \partial_t u(t)) \|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^7)} < \infty.$$  

(1.4)

Then, $I_{\text{max}}(\tilde{u}) \cap (0, \infty) = (0, \infty)$ and $\tilde{u}(t)$ scatters to a free wave as $t \to \infty$.

A direct consequence of Theorem 1.1 is that any finite time blow-up solution must admit a critical Sobolev norm diverging to infinity along a sequence of times.

We establish the local wellposedness theory for the Cauchy problem (1.1) by means of standard techniques based on the Strichartz estimates. More precisely, in Section 2 we show that for every initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$, there is a unique solution $\tilde{u}(t)$, defined on a maximal interval of existence $I_{\text{max}}(\tilde{u})$, which belongs to the class of functions $C^0(I_{\text{max}}; \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^7))$. The Strichartz estimates also yield a norm to define the scattering size of a solution on a time interval $J \subset I_{\text{max}}$. Moreover, by the local theory we deduce that if the initial data is sufficiently small in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, then the corresponding solution $\tilde{u}(t)$ is a global solution and it scatters to free waves in both time directions as $t \to \pm\infty$. Nevertheless, these tools will not be sufficient to analyze global dynamics of solutions with large data. The goal of our main result is to address the asymptotic dynamics of such solutions in the energy-supercritical radial setting.

Power-type nonlinear wave equations have received particular attention in the energy-critical setting

$$\partial_t^2 u - \Delta u = \pm |u|^{\frac{4}{d-2}}u, \quad \text{in } \mathbb{R}^d \times I,$$

(1.5)

where $d \geq 3$ denotes the dimension and the signs $+, -$ correspond to the focusing and defocusing cases, respectively. For the defocusing problem, global existence and scattering results were first obtained in three dimensions by Struwe [53] in the radial setting, and then by Grillakis [26] in the general setting. The results were then generalized to higher dimensions by Grillakis [27], Shatah-Struwe [48, 49, 50], Bahouri-Shatah [2], and Kapitanski [28].

In the energy-critical focusing case, the asymptotic dynamics of solutions with large initial data require a much closer look. In 1974, Levine [44] showed that if $(u_0, u_1) \in \dot{H}^1 \times L^2$ is a non-zero initial data where

$$E(u_0, u_1) = \int \frac{1}{2}(|u_1|^2 + |\nabla u_0|^2) - \frac{d-2}{2d} \int |u_0|^{\frac{2d}{d-2}} < 0$$

then the solution must break down in finite time. Although this work does not provide an answer on the nature of the blow-up, it stimulated the search for subsequent blow-up constructions in the literature.

Firstly, we observe that

$$\varphi(t, x) = \left( \frac{(d-2)d}{4} \right)^{\frac{d+2}{2}} (1 - t)^{-\frac{(d-2)}{2}}$$

is a solution to the ODE, $\partial_t \varphi = |\varphi|^{\frac{4}{d-2}} \varphi$, which fails to be in $\dot{H}^1 \times L^2$. Nevertheless, by truncating the data and using finite speed of propagation, we may find a solution $u(x, t)$ to the focusing problem (1.5) that has unbounded critical Sobolev norm, i.e., $\lim_{t \to t_1} \| u(x, t) \|_{H^1 \times L^2(\mathbb{R}^d)} = \infty$. We refer to this behaviour as type-I blow up.
Additionally, if a nonzero solution \( \vec{u}(t) \) of (1.5) has critical norm that remains bounded on \((0, T_+(\vec{u}))\), namely
\[
\sup_{0 < t < T_+(\vec{u})} \| \vec{u}(t) \|_{H^1 \times L^2(\mathbb{R}^d)} < \infty,
\]
then we call \( \vec{u}(t) \) a type-II solution. There are type-II solutions to the focusing problem that blow-up in finite time, i.e., type-II solutions with \( T_+(\vec{u}) < \infty \). Such behaviour is generally referred to as type-II blow-up. In [43] Krieger, Schlag, and Tataru constructed a radial type-II blow-up solution for the energy critical focusing problem (1.5) in three dimensions using the unique radial ground state solution \( W \) for the underlying elliptic equation. The blow-up occurs at \( T_- = 0 \) and their blow-up solution \( u(t, x) \) has the form
\[
u(t) = \lambda(t)^{-\frac{1}{2}} W(\lambda(t)x) + \eta(t, x),
\]
where as \( t \to 0 \), the scaling parameter \( \lambda(t) = t^{-1-\nu} \) diverges to infinity and the local energy of the term \( \eta \) inside the light cone converges to 0. The latter limiting behaviour is given by
\[
E_{loc}(\eta) = \int_{|x| < t} \left( \eta_t^2 + |\nabla \eta|^2 + \eta^6 \right) \, dx \to 0 \quad \text{as} \quad t \to 0.
\]
Later on, the original condition \( \nu > 1/2 \) was relaxed to \( \nu > 0 \) in [42]. Furthermore, Donninger, Huang, Krieger, and Schlag investigated the question of other rescaling functions which could yield similar type of blow-up solutions. Indeed, their results in [14] exhibit an uncountable family of admissible rates for \( \lambda(t) \) that are not of pure-power type.

In [34, 35], Kenig and Merle developed a program to address the ground state conjecture for critical focusing problems. In particular, for the energy-critical focusing problem (1.5) they established that the energy of the ground state solution \( W \) was a threshold for global existence and scattering. The method behind these results has come to be known as the concentration-compactness/rigidity method has found numerous applications within nonlinear dispersive and wave equations. We refer the reader to [29, Introduction] for more details and further references. Moreover, in a series of articles [19, 21, 22] Duyckaerts, Kenig, and Merle gave a classification of solutions that remain bounded in the three dimensional radial case. Particularly, in [22] the authors established the soliton resolution conjecture in dimension three, which yields that any type-II radial solution asymptotically resolves into a sum of decoupled solitary waves and a radiation term in \( \dot{H}^1 \times L^2 \).

In the energy supercritical regime, global in time well-posedness and scattering results accompanied by the boundedness of the critical Sobolev norm were obtained firstly for the defocusing case. In [33], Kenig and Merle addressed these assertions in the radial setting for dimension three. Killip and Visan generalized these claims to all dimensions in [39] for a range of energy supercritical exponents. Analogous results for the cubic nonlinear wave equation were studied by Bulut [5] in dimension five; see also [4, 6, 40] for results addressing the non-radial setting.

Utilizing the channel of energy method, Duyckaerts, Kenig, and Merle extended the global well-posedness and scattering results of [33] to the focusing case in [16]. Additionally, similar results were obtained in dimension three by Duyckaerts and Roy in [23], and by Duyckaerts and Yang in [24] with an improvement on the uniform boundedness condition. In [11], Dodson and Lawrie studied the focusing cubic wave equation in five dimensions as well as the one-equivariant wave maps equations in three dimensions. We note that the methods in [11] apply to all supercritical exponents, yielding analogous results to [16]. For results addressing the nonradial setting, see [15, 7] and the references cited therein. Also, we refer the reader to [37] for a corresponding result addressing the focusing nonlinear Schrödinger equation.

Analogous to the results in [16, 11], in this article we are concerned with type-II solutions, namely solutions to the problem (1.1) for which
\[
\sup_{t \in (0, T_+(\vec{u}))} \| \vec{u}(t) \|_{H^s \times H^{s-1}(\mathbb{R}^7)} < \infty.
\]
Our main result Theorem 1.1 shows that radial solutions to (1.1) with (1.8) achieve \( T_+ = \infty \) and they scatter.
We also remark that in the energy supercritical regime, there are blow-up constructions for the focusing nonlinear wave equations and focusing NLS under slightly different boundedness conditions; see [8] for a family of blow-up solutions which become singular via concentration of a soliton profile. In [8] solutions break down at finite time even though the norms below critical scaling remain bounded, i.e.,

$$\limsup_{t \to T} \| \tilde{u}(t) \|_{\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)} < \infty$$  \tag{1.9}$$

for \( s \in [1, s_p) \), where \( T \) is the blow-up time (we note that the critical norms of these solutions are unbounded over \([0, T)\)). The blow-up scenario constructed in [8] highlights the large space dimensions, which motivates us to extend the present work in seven dimensions to all odd dimensions \( d \geq 7 \). A related result for the focusing NLS is given in [45] by Merle, Raphael, and Rodnianski. Both of these blow-up scenarios are constructed in large dimensions \( d \geq 11 \), addressing sufficiently large energy supercritical exponents \( p \).

Additionally, Dai and Duyckaerts have recently shown the existence of a countable family of self-similar blow-up solutions to the focusing energy supercritical wave equations under the assumption (1.9) in dimension three and for a range of supercritical exponents in dimensions \( d \geq 4 \), [10].

There are also several works in the energy subcritical regime addressing the asymptotic dynamics of type II solutions. For instance, see [46, 47] for blow-up behaviour of solutions to the focusing nonlinear wave equations. In addition, conditional scattering results that are parallel to Theorem 1.1 may be found in [51, 52, 12, 54, 13] for dimensions three, four, and five.

1.1. Overview of the proof of Theorem 1.1. The general framework for the proof of Theorem 1.1 follows closely the concentration compactness/rigidity method introduced by Kenig and Merle in [34, 35], and extended into the energy supercritical regime in the works [16, 11].

To begin with, we observe that Theorem 1.1 is equivalent to the fact that the claim below holds for all \( A > 0 \).

**Claim.** Let \( \tilde{u}(t) \) be a radial solution of the Cauchy problem (1.1) with \( I_{max} = (T_-, T_+) \) such that

$$\sup_{t \in [0, T_+)} \| (u(t), \partial_t u(t)) \|_{\dot{H}^{sp} \times \dot{H}^{sp-1}(\mathbb{R}^d)} \leq A.$$  \tag{1.10}$$

Then, \( T_+ = \infty \) and \( \tilde{u}(t) \) scatters to a free wave in the positive time direction.

The small data theory guarantees that the claim is true for sufficiently small \( A > 0 \). If Theorem 1.1 failed, this would lead to a critical value \( A_C > 0 \) so that the claim above holds for all \( A < A_C \), and for \( A \geq A_C \) it fails. The profile decomposition results for the wave equations then allow for an extraction of a minimal solution to (1.1), called “critical solution”, which does not scatter. In this context, minimality refers to the size of the solution in the accompanying condition assumed under Theorem 1.1. In the present application of the concentration compactness procedure, we appeal to a profile decomposition result [3] by Bulut, which extends the earlier work of Bahouri-Gerard [1] from \( \dot{H}^1 \times L^2 \) initial data in three dimensions to \( \dot{H}^s \times \dot{H}^{s-1} \) initial data in higher dimensions with \( s \geq 1 \). Such critical solutions are shown to have pre-compact trajectories, up to modulation, in the space \( \dot{H}^{sp} \times \dot{H}^{sp-1} \), which is the main property that under further analysis produces a contradiction.

As noted above, in order to prove Theorem 1.1 we need to show the non-existence of a non-zero critical element. To that end, we follow the rigidity argument developed for the energy-supercritical wave equations in [33, 16, 11]. As a first step in that direction, we define and study solutions that exhibit a compactness property: a solution \( \tilde{u}(t) \) is said to have the compactness property if there exists \( \lambda : I_{max}(\tilde{u}) \to (0, \infty) \) so that the set

$$K = \left\{ \left( \frac{1}{\lambda(t)} \frac{1}{d-2} u \left( t, \frac{x}{\lambda(t)} \right), \frac{1}{\lambda(t)} \frac{1}{d-2} \partial_t u \left( t, \frac{x}{\lambda(t)} \right) \right) : t \in I_{max}(\tilde{u}) \right\}$$
has compact closure in $\dot{H}^{s_p} \times \dot{H}^{s_p-1} (\mathbb{R}^7)$. Such solutions are obtained from critical solutions via convergence: if $\vec{u}(t)$ is a non-scattering solution to (1.1) that satisfies
\[
\sup_{0 < t < T_+ (u)} \left\| (u(t), \partial_t u(t)) \right\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1} (\mathbb{R}^7)} < \infty,
\]
then there exists $t_n \to T_+ (u)$ such that, up to modulation, $(u(t_n), \partial_t u(t_n))$ converges to $(v_0, v_1)$ in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, where the corresponding solution $\vec{v}(t)$ has the compactness property.

In [17], the authors showed that a solution with the compactness property must be global. Their result addresses focusing wave equations both in the energy-critical and energy-supercritical cases (as well as energy-supercritical Schrödinger equations) and it directly precludes the possibility of a self similar solution that remains bounded in critical Sobolev norm and blows up in finite time.

Having eliminated a finite time blow up scenario, our main rigidity result takes the following form.

**Proposition (Proposition 3.13).** Let $\vec{u}(t)$ be a radial solution of (1.1) with $I_{\max}(\vec{u}) = \mathbb{R}$, which has the compactness property. Suppose that we also have $\lambda(t) > A_0 > 0$ for all $t \in \mathbb{R}$. Then, $\vec{u} \equiv (0,0)$.

In order to implement the rigidity argument, we first show that solutions with the compactness property have better decay than we have assumed. More specifically, we prove that $\vec{u}(t) \in \dot{H}^1 \times L^2(\mathbb{R}^7)$ for all $t \in \mathbb{R}$, and in fact the trajectory
\[
\{ \vec{u}(t) : t \in \mathbb{R} \}
\]
has compact closure in $\dot{H}^1 \times L^2(\mathbb{R}^7)$. As a direct consequence, we obtain the following vanishing: For all $R > 0$,
\[
\lim_{t \to -\infty} \sup_t \left\| \vec{u}(t) \right\|_{\dot{H}^1 \times L^2(\mathbb{R}^7)} = \lim_{t \to -\infty} \sup_t \left\| \vec{u}(t) \right\|_{\dot{H}^1 \times L^2(\mathbb{R}^7)} = 0.
\]
The additional decay that lands the solution trajectories in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^7)$ is achieved via the double-Duhamel trick. This technique was introduced by Colliander, Keel, Staffilani, Takaoka, and Tao in [9] and has been extensively utilized (see for instance [37, 38, 40, 41, 4]). It was also employed in [11] for the analogous problem in dimension five.

An essential ingredient of the rigidity argument for super-critical focusing type equations in the radial setting is the so-called channels of energy method. These estimates were first considered for linear radial wave equation in three dimensions in [19], and for the five dimensional case in [30]. Both of these results were then utilized in the adaptation of rigidity arguments to super-critical focusing nonlinearities; see [16, 11] and references therein. Here we rely on the general form of the channel of energy estimates, which were proven in all odd dimensions by Kenig, Lawrie, Liu, and Schlag in [31]: More specifically, considering the solution $V(t,x)$ to the linear wave equation with radial initial data $(f, g) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$, the result in [31] states that in any odd dimension $d$, the radial energy solution $V(t,r)$ satisfies
\[
\max_{\pm} \lim_{t \to \pm \infty} \int_{r \geq |t| + R} |\nabla_{x,r} V(t,r)|^2 r^{d-1} dr \geq \frac{1}{2} \left\| \pi_{P(R)}^\perp (f, g) \right\|^2_{\dot{H}^1 \times L^2(\mathbb{R}^d \cap r \geq |t| + R, \ r^{d-1} dr)}
\]
for all $R > 0$. Similar to [16, 11], the estimates above can be directly employed in our rigidity argument.

The operator $\pi_{P(R)}^\perp$ on the right hand side denotes the orthogonal projection onto the complement of the subspace $P(R)$ in $\dot{H}^1 \times L^2(\mathbb{R}^d \cap r \geq R, \ r^{d-1} dr)$. When $d = 1$, we have $P(R) = \emptyset$, and when $d = 3$, $P(R)$ is the line
\[
P(R) = \{ (k/r, 0) : k \in \mathbb{R} \}.
\]
The formula for the subspace in general odd dimensions is given by
\[
P(R) = \text{span} \left\{ (r^{2k_1-d}, 0) \cdots, \left( r^{2k_1-d}, 0 \right) \right\} : k_1 = 1, 2, \ldots, \left\lfloor \frac{d + 2}{4} \right\rfloor ; k_2 = 1, 2, \ldots, \left\lfloor \frac{d}{4} \right\rfloor \}.
\]
As a result, in order to adapt the rigidity arguments into our setting, where \( d = 7 \), we need to project away from a three dimensional subspace rather than a line as in \([16]\) or a 2-plane as in \([11]\). The change in the level of complexity also manifests at every step of the rigidity argument.

Another tool needed for the rigidity argument is a one-parameter family of solutions to the underlying elliptic equation, whose behaviour near infinity are similar to that of \( u(t, r) \) given in the main rigidity result. Similar to the focusing cubic wave equation in dimension five, this can be done via phase portrait analysis after the equation is written as an autonomous ODE system. This way, we obtain a radial \( C^2 \) solution of the elliptic equation

\[
-\partial_{rr}\varphi - \frac{6}{r}\partial_r\varphi = |\varphi|^{p-1}\varphi, \quad r > 0
\]

which fails to belong in the critical space \( \dot{H}^{sp} \times \dot{H}^{sp-1}(\mathbb{R}^7) \).

Finally, by applying the channel of energy method, we prove the main rigidity result: Let \( \bar{u}(t) \) be as in Proposition 3.13. Then, \( u_0(r) \) must coincide with a singular stationary solution, whose construction is outlined above. This produces a contradiction because such stationary solutions do not lie in \( \dot{H}^{sp} \times \dot{H}^{sp-1}(\mathbb{R}^7) \).

2. The Cauchy problem

In order to study the global dynamics of solutions to the Cauchy problem (1.1), we must first establish a local well-posedness theory. To that end, we review the Strichartz estimates from \([25, 55]\) and develop the theory of the Cauchy problem for the nonlinear wave equation.

First, we recall the Strichartz estimates for the linear wave equation in \( \mathbb{R}^7 \times I \)

\[
\partial_t^2 \omega - \Delta \omega = h,
\]

\( \omega(0) = (\omega_0, \omega_1) \in \dot{H}^{sp} \times \dot{H}^{sp-1}(\mathbb{R}^7) \). \tag{2.1}

The solution operator to (2.1) is given by

\[
\omega(t) = S(t)(\omega_0, \omega_1) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} h(s) \, ds \tag{2.2}
\]

where

\[
S(t)(\omega_0, \omega_1) = \cos(t \sqrt{-\Delta}) \omega_0 + (-\Delta)^{-1/2} \sin(t \sqrt{-\Delta}) \omega_1. \tag{2.3}
\]

The operators in (2.2)–(2.3) are defined via the Fourier transform: namely, we have

\[
\mathcal{F}(\cos(t \sqrt{-\Delta}) f)(\xi) = \cos(t|\xi|) \mathcal{F}(f)(\xi) \tag{2.4}
\]

and

\[
\mathcal{F}((-\Delta)^{-1/2} \sin(t \sqrt{-\Delta}) f)(\xi) = |\xi|^{-1/2} \sin(t|\xi|) \mathcal{F}(f)(\xi). \tag{2.5}
\]

Similarly, we define the fractional differentiation operators as

\[
\mathcal{F}(D^\alpha f)(\xi) = |\xi|^\alpha \mathcal{F}(f)(\xi). \tag{2.6}
\]

In what follows we say that a triple \((q, r, \gamma)\) is admissible if

\[
q, r \geq 2, \quad \frac{1}{q} + \frac{3}{r} \leq \frac{3}{2}, \quad \frac{1}{q} + \frac{7}{r} = \frac{7}{2} - \gamma.
\]

Denote by \( q' \) and \( r' \) the conjugate indices to \( q \) and \( r \), respectively, i.e., we have

\[
\frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1.
\]
LEMMA 2.1 (Strichartz Estimates \([25, 55]\)). Let \((q, r, \gamma)\) and \((a, b, \rho)\) be admissible triples with \(r < \infty\) and \(b < \infty\). Then, any solution \(\omega\) to the linear Cauchy problem (2.1) with initial data \(\tilde{\omega}(0) \in H^s \times H^{s-1}\) satisfies
\[
\|\tilde{\omega}(t)\|_{L_t^q(I; W_x^{s-\gamma, r} \times W_x^{s-\gamma, 1-r})} \lesssim \|\tilde{\omega}(0)\|_{H^s \times H^{s-1}} + \|h\|_{L_t^q(I; W_x^{s+1+\rho, b'})}.
\] (2.7)

REMARK 2.2. Lemma 2.1 may also be stated in homogeneous Besov spaces, in which case the requirement \(r, b < \infty\) is no longer needed in dimensions \(n > 3\). We refer the reader to [55, Corollary 8.3] for further details.

Applying (2.7) with \((a, b, \rho) = (\infty, 2, 0)\), \((q, r, \gamma) = (2(p-1), \frac{14}{5}(p-1), s_p)\), and a certain selection of intermediate admissible triples, we obtain the following Strichartz estimate for solutions to the linear Cauchy problem (2.1):
\[
\sup_{t \in \mathbb{R}} \|\omega(t)\|_{H^s \times H^{s-1}} + \|\omega\|_{S_p(I)} + \|\omega\|_{L_t^{9/4(p-1)} L_x^{12/9(p-1)}} + \|D^{s_p-1}\omega\|_{W(I)} + \|D^{s_p-2}\partial_t \omega\|_{W(I)} \lesssim \|\tilde{\omega}(0)\|_{H^s \times H^{s-1}} + \|D^{s_p-1}h\|_{L_t^1(I; L_x^2)};
\] (2.8)
where
\[
\|\omega\|_{W(I)} = \|\omega\|_{L_t^{4} L_x^{1}}
\] (2.9)
denotes the \(W(I)\) norm. In order to define the \(S_p(I)\) norm, we recall that that \(s_p = \frac{7}{2} - \frac{2}{p-1}\), implying
\[
s_p - 1 = \begin{cases} 
1 + \alpha_0 & \text{if } p \in [3, 5) , \\
2 & \text{if } p = 5 , \\
2 + \alpha & \text{if } p > 5 .
\end{cases}
\] (2.10)
where \(\alpha_0 \in [1/2, 1)\) and \(\alpha \in (0, 1/2)\). By (2.10), the \(S_p(I)\) norm is determined by the value of \(p\).

For \(p > 5\), we set
\[
\|\omega\|_{S_p(I)} = \|\omega\|_{L_t^{2(p-1)} L_x^{\frac{14}{5}(p-1)}} + \|D^{s_p-3}\omega\|_{L_t^{4} L_x^{28}} + \|D\omega\|_{L_t^{8(p-1)/(p+3)} L_x^{56(p-1)/(7p+5)}} + \|D^{s_p-2}\omega\|_{L_t^{8} L_x^{56/9}} + \|D^2\omega\|_{L_t^{4(p-1)/(p+1)} L_x^{28(p-1)/(7p+1)}}.
\] (2.11)

In the \(3 \leq p < 5\) regime, it suffices to include
\[
\|\omega\|_{S_p(I)} = \|\omega\|_{L_t^{2(p-2)} L_x^{\frac{14}{5}(p-1)}} + \|D^{s_p-2}\omega\|_{L_t^{8} L_x^{56/9}} + \|D\omega\|_{L_t^{8(p-1)/(p+3)} L_x^{56(p-1)/(7p+5)}}.
\] (2.12)

Lastly, when \(p = 5\), we define \(S_5(I)\) by
\[
\|\omega\|_{S_5(I)} = \|\omega\|_{L_t^{8} L_x^{56}} + \|D\omega\|_{L_t^{16/5} L_x^{112/19}}.
\] (2.13)

Having defined the \(S_p(I)\) norm as above, we establish the following nonlinear estimates. These estimates along with the Strichartz estimate (2.8) will be required to establish the theory of the local Cauchy problem for the nonlinear wave equation.

LEMMA 2.3. Let \(p \geq 3\). Then, we have
\[
\|D^{s_p-1}(|u|^{p-1}u)\|_{L_t^{1} L_x^{2}} \lesssim \|u\|_{S_p(I)}^{(p-1)} \|D^{s_p-1}u\|_{W(I)}
\] (2.14)
Applying the fractional Leibniz rule, we simply obtain
\begin{equation}
\|D^{s_p-1}(|u|^{p-1}u - |v|^{p-1}v)\|_{L_t^1L_x^2} \lesssim \left( \left\|u\right\|_{S_p(I)}^{(p-1)} + \left\|v\right\|_{S_p(I)}^{(p-1)} \right) \|u - v\|_{S_p(I)} + \left( \left\|u\right\|_{S_p(I)}^{(p-2)} + \left\|v\right\|_{S_p(I)}^{(p-2)} \right) \|u - v\|_{S_p(I)} \times \left( \|D^{s_p-1}u\|_{W(I)} + \|D^{s_p-1}v\|_{W(I)} \right) + \|u\|_{S_p(I)}^{(p-1)} \|D^{s_p-1}(u - v)\|_{W(I)}.
\tag{2.15}
\end{equation}

**Remark 2.4.** The estimates (2.14)–(2.15) rely on the fractional chain rule, as given in [33, Lemma 2.2]. More specifically, if $F \in C^2$ is a power-type function, then for $\alpha \in (0, 1)$ we have
\begin{equation}
\|D^\alpha F(u)\|_{L_t^p} \lesssim \|F'(u)\|_{L_t^{p_1}} \|D^\alpha u\|_{L_t^{p_2}}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.
\tag{2.16}
\end{equation}

Moreover,
\begin{equation}
\|D^\alpha (F(u) - F(v))\|_{L_t^p} \lesssim \left( \|F'(u)\|_{L_t^{p_1}} + \|F'(v)\|_{L_t^{p_1}} \right) \|D^\alpha (u - v)\|_{L_t^{p_2}} + \left( \|F''(u)\|_{L_t^{r_1}} + \|F''(v)\|_{L_t^{r_1}} \right) \left( \|D^\alpha u\|_{L_t^{r_2}} + \|D^\alpha v\|_{L_t^{r_2}} \right) \|u - v\|_{L_t^{r_3}}
\tag{2.17}
\end{equation}
where $1/p = 1/p_1 + 1/p_2$ and $1/p = 1/r_1 + 1/r_2 + 1/r_3$.

The restriction $p \geq 3$ in Lemma 2.3 stems from the fact that the estimates in (2.16)–(2.17) are not directly applicable as $s_p - 1 > 1$. In particular, in the regime $3 \leq p < 5$, we have $s_p = 1 + \alpha_p$, and one of terms that may be estimated by (2.17) has the form
\begin{equation}
\|Du\| \|D^\alpha (|u|^{p-1} - |v|^{p-1})\|. \tag{2.18}
\end{equation}

We note that $F(x) = |x|^{p-1}$ fails to be a $C^2$ function when $p - 1 < 2$. The details of how to obtain (2.14)–(2.15) using the estimates (2.16)–(2.17) is examined below.

**Proof of Lemma 2.3.** In order to simplify the notation, we introduce the following exponents.
\begin{align*}
(q_0, r_0) &= \left(2(p-1), \frac{14}{3}(p-1)\right), \quad (q_1, r_1) = (4, 28), \quad (q_2, r_2) = \left(\frac{4(p-1)}{p+1}, \frac{28(p-1)}{7p-1}\right), \\
(q_3, r_3) &= \left(\frac{8(p-1)}{p+3}, \frac{56(p-1)}{7p+5}\right), \quad (q_4, r_4) = \left(\frac{8}{3}, \frac{56}{9}\right).
\tag{2.19}
\end{align*}

Also, we define $(c(q_i), c(r_i))$ via
\begin{equation*}
\frac{1}{c(q_i)} = 1 - \frac{1}{q_i} \quad \text{and} \quad \frac{1}{c(r_i)} = 1 - \frac{1}{r_i}.
\end{equation*}

Firstly, we verify the estimates for $p > 5$ range. Noting (2.6) and using the summation convention $\Delta f = \partial_x^2 f$, we observe that the following operators are equal under the Fourier transform
\begin{equation*}
D^2(|u|^{p-1}u) = -\Delta (|u|^{p-1}u) = -\partial_{x_i} (p|u|^{p-1}\partial_{x_i} u) = -p|u|^{p-1}\Delta u - p(p-1)|u|^{p-3}u|\nabla u|^2.
\end{equation*}

Applying the fractional Leibniz rule, we simply obtain
\begin{align*}
\|D^{2+\alpha}(|u|^{p-1}u)\|_{L_t^1L_x^2} \lesssim \|u\|_{L_t^1L_x^{q_4}} \|D^{s_p-1}u\|_{W(I)} + \|D^\alpha (|u|^{p-1})\|_{L_t^{q_2}L_x^{r_2}} \|D^2 u\|_{L_t^{p_2}L_x^{r_2}} + \|u\|_{L_t^{p-2}} D u\|_{L_t^{q_4}L_x^{r_4}} \|D^{1+\alpha} u\|_{L_t^{q_4}L_x^{r_4}} + \|D^\alpha (|u|^{p-3} u)\|_{L_t^{q_5}L_x^{r_5}} \|D u\|_{L_t^{q_5}L_x^{r_5}}^2
\end{align*}
\begin{equation*}
=: I_1 + I_2 + I_3 + I_4.
\tag{2.20}
\end{equation*}

We have
\begin{equation*}
I_1 \lesssim \|u\|_{L_t^{q_0}L_x^{r_0}} \|D^{2+\alpha} u\|_{W(I)}
\tag{2.21}
\end{equation*}
and
\[ I_3 \lesssim \| u \|_{L^{p/3}_{x} L_{t}^{q_{1}} L_{x}^{r_{1}}}^{p-3} \| D^{\alpha} u \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}} \| D u \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}}^{2} \lesssim \| u \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}}^{p-1} \| D^{2+\alpha} u \|_{W(I)} . \]  
(2.22)

The second inequality above simply follows from interpolation.

In order to treat \( I_2 \) and \( I_4 \), we apply the chain rule for fractional derivatives, as in (2.16), on the terms \( |u|^{p-1} \) and \( |u|^{p-3} u \), respectively. We then obtain
\[ I_2 \lesssim \| u \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}}^{p-2} \| D^{\alpha} u \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}} \| D^{2} u \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}} \lesssim \| u \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}}^{p-1} \| D^{2+\alpha} u \|_{W(I)} . \]  
(2.23)

and
\[ I_4 \lesssim \| u \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}}^{p-3} \| D^{\alpha} u \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}} \| D u \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}}^{2} \lesssim \| u \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}}^{p-1} \| D^{2+\alpha} u \|_{W(I)} . \]  
(2.24)

Finally, we combine the upper bounds on \( I_1, I_2, I_3, \) and \( I_4 \) to obtain the estimate (2.14) for \( p > 5 \).

In a similar manner, we combine the classical and fractional Leibniz rule to verify (2.15). Let \( p > 5 \). Note that
\[ \| D^{2+\alpha} (|u|^{p-1} u - |v|^{p-1} u) \|_{L^{q}_{t} L_{x}^{r}} \lesssim \| D^{\alpha} (p |u|^{p-1} u - p |v|^{p-1} u) \|_{L^{q}_{t} L_{x}^{r}} + \| D^{\alpha} (\Delta u - \Delta v) \|_{L^{q}_{t} L_{x}^{r}} \]  
(2.25)

\[ = K_1 + K_2. \]

We further split \( K_1 \) and \( K_2 \) by adding and subtracting the mixed terms. Namely, we write
\[ K_1 \lesssim \| D^{\alpha} (|u|^{p-1} (\Delta u - \Delta v)) \|_{L^{q}_{t} L_{x}^{r}} + \| D^{\alpha} (\Delta v (|u|^{p-1} - |v|^{p-1})) \|_{L^{q}_{t} L_{x}^{r}} \]  
(2.26)

Similarly,
\[ K_2 \lesssim \| D^{\alpha} (|u|^{p-3} u (|\nabla u|^2 - |\nabla v|^2)) \|_{L^{q}_{t} L_{x}^{r}} + \| D^{\alpha} (\Delta v (|u|^{p-3} u - |v|^{p-3} u)) \|_{L^{q}_{t} L_{x}^{r}} \]  
(2.27)

We begin by estimating \( K_{22} \) from above. By the fractional Leibniz rule we get
\[ K_{22} \lesssim \| D^{\alpha} (|\nabla v|^2) (|u|^{p-3} u - |v|^{p-3} u) \|_{L^{q}_{t} L_{x}^{r}} + \| (|\nabla v|^2) (D^{\alpha} (|u|^{p-3} u - |v|^{p-3} v)) \|_{L^{q}_{t} L_{x}^{r}} \]  
(2.28)

For the first term on the right hand side above, we apply the mean value theorem to express the difference
\[ |u|^{p-3} u - |v|^{p-3} u = (p - 2) |u - v| + (1 - c) |u|^{p-3} (u - v) \]  
(2.29)

where \( c = c(x, t, u, v) \in (0, 1) \). Therefore, we may estimate the first term on the right hand side in (2.28) by
\[ \| D^{1+\alpha} (u - v) \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} \| u - v \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}} \lesssim \left( \| u \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}}^{p-3} \| v \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} \right) \| u - v \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}} + \| v \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}}^{p-3} \| u - v \|_{L^{q_{1}}_{t} L_{x}^{r_{1}}} . \]  
(2.30)

Going back to (2.28), we treat the second term on the right hand side. Since \( p > 5 \), we may apply (2.17) to control the term
\[ \| D^{\alpha} (|u|^{p-3} u - |v|^{p-3} u) \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} \lesssim \left( \| u \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}}^{p-3} \| v \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} \right) \| D^{\alpha} (u - v) \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} + \| v \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} \left( \| D^{\alpha} u \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} + \| D^{\alpha} v \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} \right) \times \| u - v \|_{L^{q_{0}}_{t} L_{x}^{r_{0}}} . \]
Hence, we get
\[
K_{22} \lesssim \| D^{1+\alpha} v \|_{L_t^{q_4} L_x^{r_4}} \| Dv \|_{L_t^{q_3} L_x^{r_3}} \| u - v \|_{L_t^{q_0} L_x^{r_0}} \left( \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-3} + \| v \|_{L_t^{q_0} L_x^{r_0}}^{p-3} \right) \\
+ \left( \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-3} + \| v \|_{L_t^{q_0} L_x^{r_0}}^{p-3} \right) \| D^\alpha (u - v) \|_{L_t^{q_1} L_x^{r_1}} \| Dv \|_{L_t^{q_3} L_x^{r_3}}^2 \\
+ \left( \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-4} + \| v \|_{L_t^{q_0} L_x^{r_0}}^{p-4} \right) \left( \| D^\alpha u \|_{L_t^{q_1} L_x^{r_1}} + \| D^\alpha v \|_{L_t^{q_1} L_x^{r_1}} \right) \\
\times \| u - v \|_{L_t^{q_0} L_x^{r_0}} \| Dv \|_{L_t^{q_3} L_x^{r_3}}^2.
\]
(2.31)

Next, we estimate $K_{21}$ in (2.27). As demonstrated above, by applying the Leibniz rule and chain rule for fractional derivatives, we obtain
\[
K_{21} \lesssim \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-3} \| D^\alpha u \|_{L_t^{q_1} L_x^{r_1}} \| D(u - v) \|_{L_t^{q_3} L_x^{r_3}} \\
\times \left( \| Du \|_{L_t^{q_3} L_x^{r_3}} + \| Dv \|_{L_t^{q_3} L_x^{r_3}} \right) \\
+ \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-2} \left( \| Du \|_{L_t^{q_3} L_x^{r_3}} \| D^{1+\alpha} u \|_{L_t^{q_4} L_x^{r_4}} + \| D^{1+\alpha} v \|_{L_t^{q_4} L_x^{r_4}} \| D(u - v) \|_{L_t^{q_3} L_x^{r_3}} \right) \\
(2.32)
\]

We then combine the bounds (2.31) and (2.32) for $K_{22}$ and $K_{21}$ respectively. Recalling (2.9)–(2.11), we find that the sum may be controlled by the right hand side of (2.15).

Going back to (2.26), we show that $K_1$ also obeys the right hand side of (2.15). Firstly, by fractional Leibniz and chain rule we find
\[
K_{11} \lesssim \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-1} \| D^{2+\alpha} u \|_{W(I)} + \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-2} \| D^\alpha u \|_{L_t^{q_1} L_x^{r_1}} \| D^2(u - v) \|_{L_t^{q_2} L_x^{r_2}}.
\]
(2.33)

which may be controlled by the right hand side of (2.15). Also, we may bound $K_{12}$ by fractional Leibniz rule. We get
\[
K_{12} \lesssim \| D^{2+\alpha} u \|_{W(I)} \left( \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-1} + \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-1} \right) \\
+ \| D^2 v \|_{L_t^{q_2} L_x^{r_2}} \| D^\alpha (|u|^{p-1} - |v|^{p-1}) \|_{L_t^{q_2} L_x^{r_2}}.
\]
(2.34)

Utilizing (2.17) with $F(u) = |u|^{p-1}$ we obtain
\[
\| D^\alpha (|u|^{p-1} - |v|^{p-1}) \|_{L_t^{q_2} L_x^{r_2}} \lesssim \left( \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-2} + \| v \|_{L_t^{q_0} L_x^{r_0}}^{p-2} \right) \| D^\alpha (u - v) \|_{L_t^{q_1} L_x^{r_1}} \\
+ \left( \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-3} + \| v \|_{L_t^{q_0} L_x^{r_0}}^{p-3} \right) \left( \| D^\alpha u \|_{L_t^{q_1} L_x^{r_1}} + \| D^\alpha v \|_{L_t^{q_1} L_x^{r_1}} \right) \\
\times \| u - v \|_{L_t^{q_0} L_x^{r_0}}.
\]
(2.35)

Using the upper bound above on the right hand side of (2.34), we conclude that $K_{12}$ is also estimated from above by the right hand side of (2.15).

Next, we consider the estimates (2.14)–(2.15) in the regime $3 \leq p < 5$. Note that in this case
\[
s_p - 1 = 1 + \alpha_0
\]
(2.36)

with $\alpha_0 \in [1/2, 1)$, and
\[
\| D^{s_p - 1} u \|_{L_t^1 L_x^2} \lesssim \| D^\alpha (p|u|^{p-1}\nabla u) \|_{L_t^1 L_x^2}.
\]
(2.37)

Similar to (2.20), we estimate
\[
\| D^{s_p - 1} u \|_{L_t^1 L_x^2} \lesssim \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-1} \| D^{1+\alpha_0} u \|_{W(I)} + \| u \|_{L_t^{q_0} L_x^{r_0}}^{p-2} \| D^\alpha u \|_{L_t^{q_4} L_x^{r_4}} \| Du \|_{L_t^{q_3} L_x^{r_3}} \\
\lesssim \| u \|_{S_p(I)} \| D^{s_p - 1} u \|_{W(I)}.
\]
(2.38)
Also, (2.37) implies that
\[
\|D^{s_p-1}|u|^{p-1}u - |v|^{p-1}v\|_{L^1_tL^2_x} \lesssim \|D^{\alpha_0} (|u|^{p-1}(\nabla u - \nabla v))\|_{L^1_tL^2_x} \\
+ \|D^{\alpha_0} (\nabla v(\|u|^{p-1} - |v|^{p-1}))\|_{L^1_tL^2_x} =: K_1 + K_2.
\]

Beginning with $K_2$, we distribute the fractional derivative on the product and we obtain
\[
K_2 \lesssim \|D^{1+\alpha_0}v\|_{W(I)} \|u - v\|_{L^q_tL^r_x} \left( \|u\|_{L^q_tL^r_x} + \|v\|_{L^q_tL^r_x} \right) \\
+ \|Dv\|_{L^q_tL^r_x} \|D^{\alpha_0}(\|u|^{p-1} - |v|^{p-1})\|_{L^q_tL^r_x}. \tag{2.40}
\]

The first term on the right hand side above is estimated by the mean value theorem applied on the difference $(|u|^{p-1} - |v|^{p-1})$. For the second term, we once again invoke (2.17) to obtain the upper bound given in (2.35). We note that (2.17) is only applicable in this case for exponents $p \geq 3$.

Back to (2.39), we use the fractional chain rule as done in (2.33) and we find that $K_1$ is bounded from above by
\[
K_1 \lesssim \|u\|_{L^q_tL^r_x}^{p-2} \|D^{\alpha_0}u\|_{L^q_tL^r_x} \|D(u - v)\|_{L^q_tL^r_x} \\
+ \|u\|_{L^q_tL^r_x}^{p-1} \|D^{1+\alpha_0}(u - v)\|_{W(I)}. \tag{2.41}
\]

Combining the estimates for $K_1$ and $K_2$, we conclude that the right hand side of (2.39) may be controlled by the upper bound in (2.15).

Lastly, when $p = 5$, the left hand side of (2.14) becomes
\[
\|D^2(u^5)\|_{L^1_tL^2_x} \lesssim \|u\|_{L^8_tL^{16/3}_x}^4 \|D^2u\|_{W(I)} + \|u\|_{L^{8}_tL^{16/3}_x}^3 \|D^2u\|_{L^{16/3}_tL^{256/19}_x} \tag{2.42}
\]

which may be estimated by the right hand side of (2.14) after interpolating
\[
\|D^2u\|_{L^{16/5}_tL^{12/19}_x} \lesssim \|u\|_{L^{8}_tL^{16/3}_x} \|D^2u\|_{W(I)}. \]

Similarly, we obtain (2.15) by distributing the two derivatives via classical Leibniz rule.

Using the two estimates in (2.14) and (2.15), we obtain the following local well-posedness result.

**Theorem 2.5.** Assume that $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$, with $p \geq 3$. Let $I \in \mathbb{R}$ such that $0 \in I^0$ and assume that $\|(u_0, u_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \leq A$. Then, there exists $\delta = \delta(A, p) > 0$ such that if
\[
\|S(t)(u_0, u_1)\|_{s_p(I)} < \delta
\]
there exists a unique solution $u$ to (1.1) in $\mathbb{R}^7 \times I$ with $(u, \partial_tu) \in C(I; \dot{H}^{s_p} \times \dot{H}^{s_p-1})$ that satisfies
\[
\|u\|_{s_p(I)} < 2\delta, \quad \|D^{s_p-1}u\|_{W(I)} + \|D^{s_p-2}\partial_tu\|_{W(I)} < \infty.
\]

In addition, we get $\|u\|_{L^{2(p-1)}_tL^{2(p-1)}_x} < \infty$.

Furthermore, if $(u_{0,k}, u_{1,k}) \to (u_0, u_1)$ as $k \to \infty$ in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, then
\[
(u_k, \partial_tu_k) \to (u, \partial_tu) \quad \text{in} \ C(I; \dot{H}^{s_p} \times \dot{H}^{s_p-1})
\]
where $u_k$ is the solution corresponding to $(u_{0,k}, u_{1,k})$.

The proof of Theorem 2.5 follows from standard contraction arguments presented in [35, Theorem 2.7] (see also [34, 11]).

**Remark 2.6.** As noted in analogous results in [33, 35, 34], the proof of Theorem 2.5 implies that there exists $\bar{\delta} > 0$ such that if $\|(u_0, u_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < \bar{\delta}$, the above result holds with $I = \mathbb{R}$.

Next, we consider a perturbation theorem for approximate solutions to (1.1) that will be used in the concentration compactness argument.
THEOREM 2.7. Let \((u_0, u_1) \in \dot{H}^{sp} \times \dot{H}^{sp-1}\) and \(I \subset \mathbb{R}\) be an open interval containing \(t_0\). Assume that \(v\) is a solution to

\[
\partial^2_t v - \Delta v = |v|^{p-1} v + e,
\]

satisfying

(i) \(\sup_{t \in I} \|v\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}} \leq A\)

(ii) \(\|v\|_{S_p(I)} \leq M\) and \(\|D^{sp-1}v\|_{W(I')} < \infty\) for each \(I' \subset I\).

Additionally, assume that

\[
\|(u_0 - v(t_0), u_1 - \partial_t v(t_0))\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}} \leq A'
\]

and

\[
\|D^{sp-1}e\|_{L^p_tL^{2p}_x} + \|S(t-t_0)(u_0 - v(t_0), u_1 - \partial_t v(t_0))\|_{S_p(t)} \leq \epsilon.
\]

Then, there exists \(\epsilon_1 = \epsilon_1(M, A, A') > 0\) and a solution \(u\) of (1.1) in \(I\) with \((u(t_0), \partial_t u(t_0)) = (u_0, u_1)\) such that for every \(0 < \epsilon < \epsilon_1\), we have

\[
\sup_{t \in I} \|(u_0 - v(t_0), u_1 - \partial_t v(t_0))\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}} \leq C(M, A, A')(A' + \epsilon^\alpha), \quad \alpha > 0.
\]

Furthermore, we get

\[
\|u\|_{S_p(I)} \leq C(M, A, A').
\]

The proof of the Theorem 2.7 follows from the analogous version in [33] for \(d = 3\). The only difference arises from the Strichartz estimates we employed in (2.8) and the proof may be adapted by selecting \(\beta\), \((q, r)\), and \((\bar{q}, \bar{r})\) so that we have

\[
\|D^\beta f\|_{L^q_tL^\infty_x} \lesssim \|f\|_{S_p(I)}^{1-\theta}\|D^{sp-1}f\|_{W(I)}^\theta
\]

\[
\|f\|_{L^p_tL^{q'}_x} \lesssim \|f\|_{S_p(I)}^{p-1}\|D^\beta f\|_{L^q_tL^\infty_x}.
\]

In order to guarantee (2.48) we invoke inhomogeneous Strichartz estimates. The version we state below is due to Taggart [55, Corollary 8.7].

LEMMA 2.8. Let \(\beta = \theta(sp - 1)\) with \(0 < \theta < 1\). Define \(q, \bar{q} > 0\) as follows:

\[
\frac{1}{q} = \frac{1}{2} \frac{1 - \theta}{2(p - 1)} + \frac{\theta}{2}, \quad \frac{1}{\bar{q}} = \frac{1}{2} - \frac{1}{q}.
\]

Next, we select \(r_1, \bar{r}_1\) so that

\[
\frac{1}{q} + \frac{1}{\bar{q}} = 3 \left(1 - \frac{1}{r_1} - \frac{1}{\bar{r}_1}\right)
\]

and

\[
\frac{4}{r_1} \leq \frac{6}{r_1}, \quad \frac{4}{\bar{r}_1} \leq \frac{6}{\bar{r}_1}.
\]

Then, by selecting \(r \geq r_1\) and \(\bar{r} \geq \bar{r}_1\) so that

\[
\frac{1}{r} + \frac{1}{\bar{r}} = \frac{11}{14}
\]

we arrive at

\[
\left\|D^\beta \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} g(s) \, ds \right\|_{L^q_tL^\infty_x} \lesssim \|D^\beta g\|_{L^q_tL^{q'}_x}
\]

\[
\left\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} g(s) \, ds \right\|_{S_p(I)} \lesssim \|D^\beta g\|_{L^q_tL^{q'}_x}.
\]
Remark 2.9 (Continuity property). As an application to the Theorem 2.7, we deduce that the flow associated to (1.1) has a continuity property. More precisely, let \((u_0, u_1) \in H_s \times H_{s-1}\) and let \(\vec{u}(t)\) be the solution of (1.1) with maximal interval of existence

\[ I_{\text{max}}(\vec{u}) = (T_-(\vec{u}), T_+(\vec{u})). \]

Assume that \((u_{0,n}, u_{1,n}) \to (u_0, u_1)\) in \(H_s \times H_{s-1}\) and denote by \(\vec{u}_n(t)\) the corresponding solution with

\[ I_{\text{max}}(\vec{u}_n) = (T_-(\vec{u}_n), T_+(\vec{u}_n)). \]

Then,

\[ (T_-(\vec{u}), T_+(\vec{u})) \subset (\lim \inf_{n} T_-(\vec{u}_n), \lim \inf_{n} T_+(\vec{u}_n)). \]

Moreover, for each \(t \in (T_-(\vec{u}), T_+(\vec{u}))\) we have

\[ (u_n(t), \partial_t u_n(t)) \to (u(t), \partial_t u(t)) \text{ in } H_s \times H_{s-1}. \]

In a standard manner, we may obtain the rest of the results from the local Cauchy theory by following the arguments presented in [33, Section 2]. Below we state the finite time blow-up criterion and a scattering result for convenience.

Remark 2.10 (Global Existence and Scattering). Let \(\vec{u}(t)\) be a solution of (1.1) in \((T_-(\vec{u}), T_+(\vec{u}))\). If \(T_+(\vec{u}) < \infty\), then we have

\[ \|u\|_{S_p([0,T_+])} = \infty. \]

Noting the statement above in the contrapositive direction, we recall the equivalence between scattering and boundedness of \(S_p\) norms. More precisely, we have \(\|u\|_{S_p([0,T_+(\vec{u}))} < \infty\) if and only if \(\vec{u}(t)\) scatters to a free wave as \(t \to \infty\), i.e., there exists \((u^+_0, u^+_1) \in H_s \times H_{s-1}\) so that

\[ \lim_{t \to \infty} \|(\vec{u}(t) - S(t)(u^+_0, u^+_1))\|_{H_s \times H_{s-1}} = 0. \]

The same equivalence also holds for solutions \(\vec{u}(t)\) on \((T_-(\vec{u}), 0]\). A finite time blow-up criterion may be stated for \(T_-(\vec{u}) > -\infty\) as well.

3. Concentration compactness procedure

The first component in establishing Theorem 1.1 is a concentration compactness argument. The approach we follow here was introduced by Kenig and Merle [34, 35] and studied further in several works [33, 16, 11, 32].

3.1. Existence and compactness of a critical solution. In order to highlight the essential tools in the proof of Theorem 1.1, we begin with some notation.

Definition 3.1. For \(A > 0\) and \(p \geq 3\), define

\[ \mathcal{B}(A) := \{(u_0, u_1) \in H^s \times H^{s-1} : \sup_{t \in [0, T_+(\vec{u})]} \|\vec{u}(t)\|_{H^s \times H^{s-1}} \leq A\} \]

where \(\vec{u}(t)\) denotes the unique solution to (1.1) in \(H^s \times H^{s-1}\) with initial data \((u_0, u_1)\) and the maximal interval of existence \(I_{\text{max}}(\vec{u}) = (T_-(\vec{u}), T_+(\vec{u}))\).

Definition 3.2. We say that \(\mathcal{SC}(A)\) holds if for each \((u_0, u_1) \in \mathcal{B}(A)\) we have \(T_+(\vec{u}) = \infty\) and

\[ \|\vec{u}\|_{S_p([0,\infty))} < \infty. \]

We also say that \(\mathcal{SC}(A; (u_0, u_1))\) holds if \((u_0, u_1) \in \mathcal{B}(A)\) implies \(T_+(\vec{u}) = \infty\) and

\[ \|\vec{u}\|_{S_p([0,\infty))} < \infty. \]

Analogously, we may define \(\mathcal{B}_{rad}(A)\) and \(\mathcal{SC}_{rad}(A)\) by restricting \((u_0, u_1)\) to radial initial data.

Remark 3.3. By Theorem 2.5 and Remarks 2.6–2.10 there exists \(\delta > 0\) sufficiently small so that \(\mathcal{SC}(\delta; (u_0, u_1))\) holds. Combined with Remark 2.10 we deduce that Theorem 1.1 is equivalent to the statement that \(\mathcal{SC}_{rad}(A)\) holds for all \(A > 0\). Hence, in the event that Theorem 1.1 fails, there exists a critical value \(A_C > 0\) so that for \(A < A_C\), \(\mathcal{SC}(A)\) holds, and for \(A > A_C\), \(\mathcal{SC}(A)\) fails.
The next result states that in the failure of Theorem 1.1, there exists radial initial data \((u_{0,C}, u_{1,C}) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}\) so that \(\mathcal{S}(A_C, (u_{0,C}, u_{1,C}))\) fails. Furthermore, the solution to (1.1) that corresponds to \((u_{0,C}, u_{1,C})\) satisfies a compactness property and plays a crucial role in our discussion.

**Proposition 3.4.** Suppose that Theorem 1.1 is false. Then, there exists \((u_{0,C}, u_{1,C})\) radial such that \(\mathcal{S}(A_C, (u_{0,C}, u_{1,C}))\) fails. Additionally, there exists a continuous function \(\lambda : [0, T_+ (\tilde{u}_C)) \to (0, \infty)\) so that the set

\[
\left\{ \left( \frac{1}{\lambda(t)^{p-1}} u_C \left( t, \frac{x}{\lambda(t)} \right), \frac{1}{\lambda(t)^{p+1}} \partial_t u_C \left( t, \frac{x}{\lambda(t)} \right) \right) : t \in [0, T_+ (\tilde{u}_C)) \right\}
\]

has compact closure in \(\dot{H}^{s_p} \times \dot{H}^{s_p-1}\).

**Definition 3.5.** Let \(\tilde{u}_C(t) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}\) be a radial solution to (1.1). We say that \(\tilde{u}_C(t)\) is a critical solution if it satisfies the conclusions of Proposition 3.4. More precisely, we have

\[
\sup_{t \in [0, T_+ (\tilde{u}_C))} \| \tilde{u}_C(t) \|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} = A_C, \quad \| \tilde{u}_C \|_{\mathcal{S}_p ([0, T_+ (\tilde{u}_C))]} = \infty,
\]

and there exists a continuous function \(\lambda : [0, T_+ (\tilde{u}_C)) \to (0, \infty)\) so that the set given in (3.2) is pre-compact in \(\dot{H}^{s_p} \times \dot{H}^{s_p-1}\).

Once a critical solution \(\tilde{u}_C(t)\) is given, it is possible to construct another critical solution with a corresponding scaling function that is bounded away from zero. We state this result next and refer the reader to [36, Lemma 3.10] for an analogous proof.

**Lemma 3.6.** There is a critical solution \(\tilde{\omega}(t)\) with a corresponding \(\lambda_\omega\) continuously defined on \([0, T_+ (\tilde{\omega}))\) such that

\[
\inf_{t \in [0, T_+ (\tilde{\omega})]} \lambda_\omega(t) \geq A_0 > 0.
\]

Going back to Proposition 3.4, the main ingredient in extracting a critical solution is a profile decomposition theorem for linear solutions. The profile decomposition for the wave equation is introduced by Bahouri–Gerard [1] for initial data belonging to \( \dot{H}^1 \times L^2 \) in three dimensions, and extended to higher dimensions by Bulut [3]. Below, we state a higher dimensional version of the profile decomposition for initial data in \(\dot{H}^{s_p} \times \dot{H}^{s_p-1}\).

**Theorem 3.7 ([3, Theorem 1.3]).** Let \(s \geq 1\) be given and let \((u_{0,n}, u_{1,n})_{n \in \mathbb{N}}\) be a bounded sequence in \(\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)\) with \(d \geq 3\). Then there exists a subsequence of \((u_{0,n}, u_{1,n})\) (still denoted \((u_{0,n}, u_{1,n})\)), a sequence \((V_{0,j}^j, V_{1,j}^j)_{j \in \mathbb{N}} \subset \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)\), and a sequence of triples \((\epsilon_n^j, x_n^j, t_n^j) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}\) such that, for every \(j \neq j'\),

\[
\frac{\epsilon_n^j}{\epsilon_n^{j'}} + \frac{\epsilon_n^{j'}}{\epsilon_n^j} + \frac{|t_n^j - t_n^{j'}|}{\epsilon_n^j} + \frac{|x_n^j - x_n^{j'}|}{\epsilon_n^j} \to \infty, \quad n \to \infty
\]

and for every \(l \geq 1\), if

\[
V_{j}^j = S(t) (V_{0,j}^j, V_{1,j}^j) \quad \text{and} \quad V_{j}^j(t, x) = \frac{1}{(\epsilon_n^j)^{2-s}} V_{j}^j \left( \frac{t - t_n^j}{\epsilon_n^j}, \frac{x - x_n^j}{\epsilon_n^j} \right)
\]

then

\[
\begin{align*}
\begin{aligned}
& u_{0,n}(x) = \sum_{j=1}^l V_{n,j}^j(0, x) + \omega_{0,n}^j(x) \\
& u_{1,n}(x) = \sum_{j=1}^l \partial_t V_{n,j}^j(0, x) + \omega_{1,n}^j(x)
\end{aligned}
\]

for every \(n \in \mathbb{N}\).
follows from

\[ S(t;\alpha_{0,n},\omega_{1,n}) \in L^q_{L^r} \rightarrow 0, \quad l \rightarrow \infty \]

for every pair \((q,r)\) with \(q, r \in (2, \infty)\) which satisfies

\[
\frac{1}{q} + \frac{d - 1}{2r} \leq \frac{d - 1}{4}, \quad \frac{1}{q} + \frac{d}{r} = \frac{d - s}{2}.
\]

For every \(l \geq 1\), we also have

\[
\|u_{0,n}\|_{H^s}^2 + \|u_{1,n}\|_{H^{s-1}}^2 = \sum_{j=1}^l \left( \|V^j_0\|_{H^s}^2 + \|V^j_1\|_{H^{s-1}}^2 \right) + \|\omega_{0,n}\|_{H^s}^2 + \|\omega_{1,n}\|_{H^{s-1}}^2 + O(1), \quad n \rightarrow \infty.
\]

**Remark 3.8.** When the sequence \((u_{0,n}, u_{1,n})\) is radial, we may select \((V^j_0, V^j_1)\) radial with \(x^j_t \equiv 0\).

Invoking the profile decomposition theorem as stated above, the proof of Proposition 3.4 follows from the procedure developed in [36, Section 3]. In a broad manner, the failure of Theorem 1.1 along with the profile decomposition result above leads to a minimizing sequence of non-scattering solutions to (1.1) in \(L^\infty_t (H^{sp} \times H^{sp-1})\) norm. Through further analysis, a critical solution \(\bar{u}_C\) which possesses additional compactness properties may be extracted. The continuity property of \(\lambda(t)\) on \([0, T_+(\bar{u}_C))\) follows from the continuity of \(\bar{u}_C(t)\) on \([0, T_+(\bar{u}_C))\) in \(H^{sp} \times H^{sp-1}\). For a detailed treatment, please see Remark 5.4 of [34].

### 3.2. The compactness property

In view of the properties of a critical solution from Lemma 3.6, we deduce that Theorem 1.1 follows from the next result.

**Theorem 3.9.** Let \(\bar{u}(t)\) be a radial solution of (1.1) with \(p \geq 3\). Assume that there exists a continuous function \(\lambda : [0, T_+(\bar{u})) \rightarrow (0, \infty)\) so that

\[
K_+ := \left\{ \left( \frac{1}{\lambda(t)^{p-1}} u \left( t, \frac{x}{\lambda(t)} \right), \frac{1}{\lambda(t)^{p-1}+1} \partial_t u \left( t, \frac{x}{\lambda(t)} \right) \right) : t \in [0, T_+(\bar{u})) \right\}
\]

has compact closure in \(H^{sp} \times H^{sp-1}\) and we have

\[
\inf_{t \in [0, T_+(\bar{u}))} \lambda(t) > 0.
\]

Then, \(\bar{u} \equiv (0, 0)\).

Before we approach the proof of Theorem 3.9, we consider two separate scenarios. Let \(\bar{u}(t)\) be a solution as in Theorem 3.9. First, we eliminate the case with a finite time blow-up, i.e., we cannot have \(T_+ (\bar{u}) < \infty\). Secondly, we consider the case where \(I_{\text{max}}(\bar{u}) = \mathbb{R}\) and

\[
\inf_{t \in I_{\text{max}}(\bar{u})} \lambda(t) > 0.
\]

We argue that in this case \(\bar{u}(t)\) must be the zero solution. The proof of Theorem 3.9 then follows from studying the properties of the scaling parameter \(\lambda(t)\). To begin with, we introduce the following definition.

**Definition 3.10.** Let \(\bar{u}(t)\) be a solution of (1.1) defined on its maximal interval of existence \(I_{\text{max}}(\bar{u}) = [T_-(\bar{u}), T_+(\bar{u}))\). We say that \(\bar{u}(t)\) has the compactness property if there exists \(\lambda : I_{\text{max}}(\bar{u}) \rightarrow (0, \infty)\) so that the set

\[
K = \left\{ \left( \frac{1}{\lambda(t)^{p-1}} u \left( t, \frac{x}{\lambda(t)} \right), \frac{1}{\lambda(t)^{p-1}+1} \partial_t u \left( t, \frac{x}{\lambda(t)} \right) \right) : t \in I_{\text{max}}(\bar{u}) \right\}
\]

has compact closure in \(H^{sp} \times H^{sp-1}\).
Note that in the definition above the scaling function $\lambda(t)$ is defined on the entire interval $I_{\text{max}}$ as opposed to the half-open interval $[0, T_+ (\tilde{u}))$. The fact that pre-compactness is preserved when we pass from $K_+$ to $K$ is depicted in the next lemma.

**Lemma 3.11.** Let $\tilde{u}(t)$ be a solution of (1.1) as in Theorem 3.9. Let $\{t_n\}_n$ be a sequence of times in $[0, T_+ (\tilde{u}))$ such that $\lim_n t_n = T_+ (\tilde{u})$. Assume that there exists $(v_0, v_1) \in \dot{H}^s \times \dot{H}^{s-1}$ such that

$$
\left( \frac{1}{\lambda(t_n)^{p-1}} u \left( t_n, \frac{x}{\lambda(t_n)} \right) , \frac{1}{\lambda(t_n)^{p-1}} \partial_t u \left( t_n, \frac{x}{\lambda(t_n)} \right) \right) \to (v_0, v_1) \quad \text{as } n \to \infty
$$

for $\dot{H}^s \times \dot{H}^{s-1}$. Let $\tilde{v}(t)$ be the solution of (1.1) with initial data $(v_0, v_1)$ at $t = 0$. Then, $\tilde{v} \not \equiv (0, 0)$ provided that $\tilde{u} \not \equiv (0, 0)$. Additionally, $\tilde{v}(t)$ has the compactness property.

By the hypothesis of Theorem 3.9, the sequence on the left hand side of (3.6) belongs to the pre-compact set $K_+$. As a result, after passing to a subsequence, we deduce that there exists $(v_0, v_1) \in \overline{K}_+$ so that the limit in (3.6) holds.

**Proof.** The proof of Lemma 3.11 is very similar to those in [33, Lemma 6.1] and [18, Claim C.1]. We will show the changes below. First, we may assume that $(u_0, u_1) \not \equiv (0, 0)$ since the result becomes trivial with $(0, 0)$ initial data. Below, we denote by $a = 2/(p - 1)$.

Since $(u_0, u_1) \not \equiv (0, 0)$, we have

$$
\inf_{I_{\text{max}}(\tilde{u})} \| \tilde{u}(t) \|_{\dot{H}^s \times \dot{H}^{s-1}} > 0
$$

by the local Cauchy theory. As a result, for any sequence $\{t_n\}_n \subset I_{\text{max}}$ we deduce

$$
\inf_n \left\| \left( \frac{1}{\lambda(t_n)^{a}} u \left( t_n, \frac{x}{\lambda(t_n)} \right) , \frac{1}{\lambda(t_n)^{a+1}} \partial_t u \left( t_n, \frac{x}{\lambda(t_n)} \right) \right) \right\|_{\dot{H}^s \times \dot{H}^{s-1}} > 0.
$$

Hence, the limit $(0, 0) \not \in K_+$, which further implies that $(v_0, v_1) \not \equiv (0, 0)$. This proves the first claim.

**Step 1.** We claim that for each $s \in (T_-(\tilde{v}), T_+(\tilde{v}))$ we have

$$
t_n + s/\lambda(t_n) \geq 0
$$

for large $n$. As $\{t_n\}$ and $\{\lambda(t_n)\}$ are non-negative sequences, the inequality above holds for $s \in [0, T_+ (\tilde{v}))$. We assume for a contradiction that (3.8) fails for some $s \in (T_-(\tilde{v}), 0)$. We may then extract a subsequence so that

$$
t_n \lambda(t_n) + s < 0
$$

for every $n$. Set $s_n = -t_n \lambda(t_n)$. Note that $s_n \in [s, 0]$. By passing into a subsequence if necessary, we have $\lim_n s_n = \theta \in [s, 0] \subset (T_-(\tilde{v}), T_+(\tilde{v}))$. Noting Remark 2.9 and the fact that $s_n + t_n \lambda(t_n) = 0$, we get

$$
\left( \frac{1}{\lambda(t_n)^a} u \left( 0, \frac{x}{\lambda(t_n)} \right) , \frac{1}{\lambda(t_n)^{a+1}} \partial_t u \left( 0, \frac{x}{\lambda(t_n)} \right) \right) \to (v(\theta, x), \partial_t v(\theta, x))
$$

in $\dot{H}^s \times \dot{H}^{s-1}$. Since $(v_0, v_1) \not \equiv (0, 0)$, we also get $\tilde{v}(\theta) \not \equiv (0, 0)$. Therefore, we obtain

$$
\frac{1}{C} \leq \lambda(t_n) \leq C \quad \text{for every } n
$$

for some constant $C > 0$, which will then yield a contradiction. If $T_+ (\tilde{u}) = \infty$, (3.11) contradicts with (3.9). If $T_+ (\tilde{u}) < \infty$, by Proposition 5.3 in [33] (see also [34, Prop. 5.3]) we obtain

$$
\lambda(t_n) \geq \frac{C_0}{T_+ (\tilde{u}) - t_n}
$$

which implies that $\lambda(t_n) \to \infty$, contradicting (3.11).

**Step 2.** We aim to show that for every $s \in (T_-(\tilde{v}), T_+(\tilde{v}))$ there exists $\tilde{\lambda}(s) > 0$ so that

$$
\left( \frac{1}{\tilde{\lambda}(s)^a} v \left( s, \frac{x}{\tilde{\lambda}(s)} \right) , \frac{1}{\tilde{\lambda}(s)^{a+1}} \partial_t v \left( s, \frac{x}{\tilde{\lambda}(s)} \right) \right) \in \overline{K}_+.
$$
Setting \( \tau_n = t_n + s/\lambda(t_n) \), we note that
\[
\left( \frac{1}{\lambda(\tau_n)^a} u \left( \tau_n, \frac{x}{\lambda(\tau_n)} \right), \frac{1}{\lambda(\tau_n)^{a+1}} \partial_t u \left( \tau_n, \frac{x}{\lambda(\tau_n)} \right) \right) \in K_+ \quad (3.14)
\]
as \( t_n + s/\lambda(t_n) \geq 0 \) for \( n \) sufficiently large. By passing to a subsequence, we find \( (\omega_0(s), \omega_1(s)) \in \overline{K}_+ \) such that
\[
\left( \frac{1}{\lambda(t_n)^a} u \left( \tau_n, \frac{x}{\lambda(t_n)} \right), \frac{1}{\lambda(t_n)^{a+1}} \partial_t u \left( \tau_n, \frac{x}{\lambda(t_n)} \right) \right) \to (\omega_0(s,x), \omega_1(s,x)) \quad (3.15)
\]
in \( \dot{H}^{s_p} \times \dot{H}^{s_p-1} \). At the same time, combining (3.6) and the continuity property of the flow as stated in Remark 2.9 we also get
\[
\left( \frac{1}{\lambda(t_n)^a} u \left( t_n + \frac{s}{\lambda(t_n)}, \frac{x}{\lambda(t_n)} \right), \frac{1}{\lambda(t_n)^{a+1}} \partial_t u \left( t_n + \frac{s}{\lambda(t_n)}, \frac{x}{\lambda(t_n)} \right) \right) \to (v(s,x), \partial_s v(s,x)) \quad (3.16)
\]
in \( \dot{H}^{s_p} \times \dot{H}^{s_p-1} \). We rescale (3.16) in \( x \) by \( \lambda(t_n)/\lambda(\tau_n) \) so that the convergence in (3.15) may be utilized. We then find that
\[
\left( \frac{\lambda(t_n)^a}{\lambda(\tau_n)^a} v \left( s, \frac{x\lambda(t_n)}{\lambda(\tau_n)} \right), \frac{\lambda(t_n)^{a+1}}{\lambda(\tau_n)^{a+1}} \partial_s v \left( s, \frac{x\lambda(t_n)}{\lambda(\tau_n)} \right) \right) \to (\omega_0(s,x), \omega_1(s,x)) \quad (3.17)
\]
in \( \dot{H}^{s_p} \times \dot{H}^{s_p-1} \). Since \( (\omega_0(s), \omega_1(s)) \neq (0, 0) \) as it belongs to the compact set \( \overline{K}_+ \), we deduce that
\[
0 < \frac{1}{C(s)} \leq \frac{\lambda(t_n)}{\lambda(t_n + s/\lambda(t_n))} \leq \tilde{C}(s) < \infty \quad (3.18)
\]
for every \( n \). Therefore, we may find a further subsequence so that
\[
\lim_{n \to \infty} \frac{\lambda(t_n)}{\lambda(t_n + s/\lambda(t_n))} =: \tilde{X}(s) \in (0, \infty) \quad (3.19)
\]
and
\[
\left( \frac{1}{\lambda(s)^a} v \left( s, \frac{x}{\lambda(s)} \right), \frac{1}{\lambda(s)^{a+1}} \partial_s v \left( s, \frac{x}{\lambda(s)} \right) \right) \in \overline{K}_+ \text{ for every } s \in (T_-(\tilde{v}), T_+^-(\tilde{v})) \), which completes the proof.

Next, we show that there is no solution \( \tilde{u}(t) \) to (1.1) as in Theorem 3.9 with \( T_+(\tilde{u}) < \infty \). In [17, Section 3], the authors consider the equation (1.1) under the hypothesis that \( p \) is an odd integer or large enough so that the local well-posedness theory holds. Proposition 3.1 in [17] shows that a solution of the equation (1.1) which has the compactness property on its maximal interval of existence is global. For exponents \( p \) that are not odd integers, the range \( p > N/2 \) is provided as a sufficient condition in which \( N \) denotes the dimension. The local well-posedness theory in Section 2 lets us carry through the proof of Proposition 3.1 in [17] and eliminate the possibility of a self similar solution that blows up in finite time. For convenience, we will provide the details below.

**Proposition 3.12 ([17, Proposition 3.1]).** Let \( p \geq 3 \) and let \( \tilde{u}(t) \) be a solution of (1.1) with the compactness property. Then, \( \tilde{u}(t) \) is global.

**Sketch of the Proof.** Let \( \tilde{u}(t) \) be a solution of (1.1) on its maximal interval of existence \( I_{\max}(\tilde{u}) = (T_-(\tilde{u}), T_+(\tilde{u})) \) which has the compactness property as defined in Definition 3.10. Since we are concerned with the radial case, we will assume that \( \tilde{u}(t) \) is a radial solution. Fixing a non-zero radial solution \( \tilde{u}(t) \), we simplify the notation and write \( I_{\max} = (T_-, T_+) \). For a contradiction, suppose that \( T_- \) is finite.

**Step 1.** We claim that for every \( t \in I_{\max} \)
\[
\text{supp } \tilde{u}(t) \subset \{|x| \leq |T_- - t|\} \quad (3.20)
\]
The proof of this step may be completed by following the same strategy in [33, Section 4], which relies on finite speed of propagation, the small data theory, and the perturbation result we established in Section 2. In particular, since Theorem 2.5 and Theorem 2.7 hold for $p \geq 3$, we may proceed with the rest of the proof as outlined below. Firstly, we obtain that for all $t \in I_{\text{max}}$

$$\lambda(t) \geq \frac{C(K)}{t - T_-} > 0$$

as done in [33, Lemma 4.14] (cf. [35, Lemma 4.7] for the details). In particular, we deduce

$$\lim_{t \to T_-} \lambda(t) = \infty.$$  

(3.22)

Then, by following the same arguments in [33, Lemma 4.15] we prove the statement (3.20).

Step 2. We conclude the proof by finding a monotone function in time in terms of the solution $\tilde{u}(t)$. We simply provide the proof of Step. 2 in [17, Proposition 3.1] for the sake of completeness. Let

$$y(t) = \int u^2(t, x) \, dx.$$  

(3.23)

By Step 1, $y(t)$ is well-defined for all $t \in I_{\text{max}}$, and furthermore $\tilde{u}(t) \in \dot{H}^1 \times L^2$. Using the equation (1.1), we obtain

$$y'(t) = 2 \int u(t, x) \partial_t u(t, x) \, dx$$  

(3.24)

and

$$y''(t) = 2 \int (\partial_t u(t, x))^2 \, dx - 2 \int |\nabla u(t, x)|^2 \, dx + 2 \int |u(t, x)|^{p+1} \, dx.$$  

Next, we recall that the conserved energy for the flow is given by

$$E(\tilde{u}(t)) = \int \left( \frac{1}{2} |\partial_t u(t, x)|^2 + \frac{1}{2} |\nabla u(t, x)|^2 - \frac{1}{p+1} |u(t, x)|^{p+1} \right) \, dx.$$  

Noting that $\tilde{u}(t)$ is uniformly bounded in $\dot{H}^s \times \dot{H}^{s-1}$ by the compactness property with $s_p > 1$, we deduce that the condition (3.20) on the support of $\tilde{u}(t)$ leads to

$$\lim_{t \uparrow T_-} E(\tilde{u}(t)) = 0 \quad \text{and} \quad \lim_{t \uparrow T_-} y(t) = \lim_{t \uparrow T_-} y'(t) = 0.$$  

(3.25)

We then obtain by conservation of the energy that

$$E(\tilde{u}(t)) = 0$$  

(3.26)

for all $t \in I_{\text{max}}$, and we rewrite

$$y''(t) = (p + 3) \int (\partial_t u(t, x))^2 \, dx + (p - 1) \int |\nabla u(t, x)|^2 \, dx > 0.$$  

(3.27)

Thus, we must have $y'(t) > 0$ for all $t \in I_{\text{max}}$. We note that in the case $T_+ < \infty$ we also obtain

$$\lim_{t \uparrow T_+} y'(t) = 0.$$  

(3.28)

which contradicts with the fact that $y(t)$ is a strictly convex function with the limit (3.25). We then deduce that $T_+ = \infty$. Combining (3.24) with (3.27) we obtain

$$y'(t)^2 \leq 4 \left( \int u^2(t, x) \, dx \right) \left( \int (\partial_t u(t, x))^2 \, dx \right) \leq \frac{4}{p + 3} y(t) y''(t).$$  

(3.29)

Using (3.29) and the fact that $y'(t) > 0$ for all $t \in I_{\text{max}}$, we claim that $y^{-(p-1)/4}$ is strictly decreasing and concave down. To see this, note that

$$\frac{d}{dt} \left( y^{-(p-1)/4} (t) \right) = -\frac{(p - 1)}{4} y^{-(p+3)/4} (t) y'(t) < 0.$$  


and
\[
\frac{d^2}{dt^2} \left( y^{-(p-1)/4}(t) \right) = \frac{(p-1)}{4} y^{-(p+7)/4}(t) \left( \frac{p+3}{4} (y'(t))^2 - y(t) y''(t) \right) \leq 0.
\]
This however contradicts the fact that \( T_+ = \infty \).

We apply the above proposition for the solution \( \vec{v}(t) \) constructed in Lemma 3.11, which satisfies the compactness property on its maximal interval of existence \( (T_-(\vec{v}), T_+(-\vec{v})) \). Note that Remark 2.9 implies that if \( \vec{v}(t) \) is a global, then \( \vec{u}(t) \) as in Theorem 3.9 must be a global solution as well. Having eliminated the case \( T_+(\vec{u}) < \infty \), we focus on the following result.

**Proposition 3.13.** Let \( \vec{u}(t) \) be a radial solution of (1.1) with \( I_{\max}(\vec{u}) = \mathbb{R} \), which has the compactness property on \( \mathbb{R} \). Suppose that we have
\[
\inf_{t \in (-\infty, \infty)} \lambda(t) > 0. \tag{3.30}
\]
Then, \( \vec{u} \equiv (0, 0) \).

The proof of Theorem 3.9 may be completed by proceeding as in [16, Section 6]. More specifically, letting \( \vec{u} \) be as in Theorem 3.9, we follow the arguments in Lemma 6.3–6.6 and employ Proposition 3.13 to examine the further properties of the corresponding scaling function \( \lambda(t) \), and we arrive at the conclusion that \( \vec{u} \equiv (0, 0) \).

The remainder of the article deals with the proof of Proposition 3.13. Firstly, we focus on showing that solutions of (1.1) with the compactness property enjoy additional spacial decay, which yields the fact that the trajectory of \( \vec{u}(t) \in \dot{H}^1 \times L^2 \). Next, we highlight a family of singular stationary solutions with asymptotic properties similar to those of solutions given in the hypothesis of the Proposition 3.13 yet these singular stationary solutions fail to belong to the critical space \( \dot{H}_{\text{cr}}^p \). Finally, using the exterior energy estimates from [31] we may go through the rigidity method in three main steps to show that a non-zero radial solution of (1.1) with the compactness property has to coincide with a singular stationary solution. As a result, we obtain the desired conclusion of Proposition 3.13 that \( \vec{u} \equiv (0, 0) \).

### 4. Decay results for solutions with the compactness property

In this section, we apply the double Duhamel method to study the decay rates of solutions to the Cauchy problem (1.1) which has the compactness property. The methods in this section are analogous to the discussion in [11, Section 4] for the focusing cubic wave equation in \( \mathbb{R}^5 \).

First, we recall some preliminary facts from harmonic analysis which will be frequently used throughout the section. We begin with a radial Sobolev inequality, quoted verbatim from [58, Corollary A.3] for convenience of readers.

**Lemma 4.1 (Radial Sobolev inequality).** Let \( 1 \leq p, q \leq \infty \), \( 0 < s < 7 \), and \( \beta \in \mathbb{R} \) obey the conditions
\[
\beta > \frac{7}{q'}, \quad 1 \leq \frac{1}{p} + \frac{1}{q} \leq 1 + s
\]
and the scaling condition
\[
7 - \beta - s = \frac{7}{p'} + \frac{7}{q'}
\]
with at most one of the equalities
\[
p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 + s
\]
holding. Then, for any radial function \( f \in \dot{W}^{s,p}(\mathbb{R}^7) \), we have
\[
\| |x|^{\beta} f \|_{L^{p'}(\mathbb{R}^7)} \leq C \| D^s f \|_{L^p(\mathbb{R}^7)}.
\]

We also recall the Bernstein inequalities for dimension \( d \geq 1 \). The version stated below is included in the book [56, Appendix A].
LEMMA 4.2 (Bernstein’s inequalities). Let $s \geq 0$ and $1 \leq p \leq q \leq \infty$. For $f : \mathbb{R}^d \to \mathbb{R}$, we have
\[
\| P_{\geq N} f \|_{L^p(\mathbb{R}^d)} \leq N^{-s} \| D^s P_{\geq N} f \|_{L^p(\mathbb{R}^d)} \\
\| P_{\leq N} D^s f \|_{L^p(\mathbb{R}^d)} \leq N^s \| P_{\leq N} f \|_{L^p(\mathbb{R}^d)} \\
\| P_N D^{\pm s} f \|_{L^p(\mathbb{R}^d)} \leq N^{\pm s} \| P_N f \|_{L^p(\mathbb{R}^d)} \\
\| P_{\leq N} f \|_{L^q(\mathbb{R}^d)} \leq N^q \| P_{\leq N} f \|_{L^q(\mathbb{R}^d)} \\
\| P_N f \|_{L^q(\mathbb{R}^d)} \leq N^q \| P_N f \|_{L^q(\mathbb{R}^d)}
\]
where the implicit constants depend on $p, s, d$ in the first three inequalities and on $p, q, d$ in the latter two inequalities.

REMARK 4.3. For the remainder of this section, we may assume that $p \geq 3$ and $\bar{u}(t)$ is a solution of (1.1) as in Proposition 3.13. Namely, $\bar{u}(t)$ is a radial solution of 1.1 with $I_{\max}(\bar{u}) = \mathbb{R}$. Additionally, $\bar{u}(t)$ has the compactness property on $\mathbb{R}$ and the corresponding scaling parameter $\lambda$ satisfies
\[
\inf_{t \in (-\infty, \infty)} \lambda(t) > 0. \tag{4.1}
\]
Nevertheless, the results in this section continue to hold when we waive our assumption that $\bar{u}(t)$ is global. Also, we may allow $p > 2$.

Next, we state a quantitative result for solutions with the compactness property. By the Arzela-Ascoli theorem, we may simply obtain the following uniform estimates on the $H^{s_p} \times H^{s_p-1}$ norm of a solution that has the compactness property. For similar estimates, see [11, Remark 4]

LEMMA 4.4 (Uniformly Small Tails). Let $\bar{u}(t)$ be a solution of the equation (1.1) as in Remark 4.3. Then for any $\eta > 0$ there are $0 < c(\eta) < C(\eta) < \infty$ such that for all $t \in \mathbb{R}$ we have
\[
\left| D^{s_p} u(t, x) \right|^2 \, dx + \int_{|\xi| \geq c(\eta)\lambda(t)} |\xi|^{2s_p} |\hat{u}(t, \xi)|^2 \, d\xi \leq \eta \tag{4.2}
\]
\[
\left| D^{s_p-1} u(t, x) \right|^2 \, dx + \int_{|\xi| \geq c(\eta)\lambda(t)} |\xi|^{2s_p} |\hat{u}_t(t, \xi)|^2 \, d\xi \leq \eta
\]
\[
\left| D^{s_p-1} u_t(t, x) \right|^2 \, dx + \int_{|\xi| \geq c(\eta)\lambda(t)} |\xi|^{2s_p-2} |\hat{u}_t(t, \xi)|^2 \, d\xi \leq \eta
\]
\[
\left| D^{s_p-1} u_t(t, x) \right|^2 \, dx + \int_{|\xi| \geq c(\eta)\lambda(t)} |\xi|^{2s_p-2} |\hat{u}_t(t, \xi)|^2 \, d\xi \leq \eta.
\]

We will also utilize the following version of Duhamel’s formula for solutions to (1.1) with the compactness property. The standard Duhamel formula combined with the fact that the linear part of the evolution vanishes weakly in $H^{s_p} \times H^{s_p-1}$ yields the following lemma. Analogous results on weak limits are proved in [57, Section 6] and [51, Proposition 3.8].

LEMMA 4.5 (Weak Limits). Let $\bar{u}(t)$ be a solution of the equation (1.1) as in Remark 4.3. Then, for any $t_0 \in \mathbb{R}$ we have
\[
u(t_0) = \lim_{T \to \infty} \int_{t_0}^{T} \frac{\sin((t_0 - \tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} |u|^p \, d\tau \quad \text{weakly in } H^{s_p}(\mathbb{R}^7)
\]
\[
u_t(t_0) = \lim_{T \to \infty} \int_{t_0}^{T} \cos((t_0 - \tau)\sqrt{-\Delta}) |u|^p \, d\tau \quad \text{weakly in } H^{s_p-1}(\mathbb{R}^7)
\]
\[
u(t_0) = \lim_{T \to -\infty} \int_{T}^{t_0} \frac{\sin((t_0 - \tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} |u|^p \, d\tau \quad \text{weakly in } H^{s_p}(\mathbb{R}^7)
\]
\[ u_t(t_0) = \lim_{T \to -\infty} \int_T^{t_0} \cos((t_0 - \tau)\sqrt{-\Delta}) |u|^{p-1} u \, d\tau \quad \text{weakly in } \dot{H}^{s_p-1}(\mathbb{R}^7). \]

The following is the main result of this section on the decay of compact solutions to the equation (1.1).

**PROPOSITION 4.6.** Let \( \bar{u}(t) \) be a solution to (1.1) as in Remark 4.3. Then, for all \( t \in \mathbb{R} \)
\[
\|\bar{u}(t)\|_{\dot{H}^{3/4} \times \dot{H}^{-1/4}(\mathbb{R}^7)} \leq C_p
\]
where the constant \( C \) is uniform in time.

The proof of Proposition 4.6 follows from a double Duhamel technique as shown in [11] and [57]. Following the procedure introduced in [11, Section 4.2], we define
\[
v = u + \frac{i}{\sqrt{-\Delta}} u_t.
\]

As \( \bar{u} \) solves (1.1), we get
\[
v_t = u_t + \frac{i}{\sqrt{-\Delta}} (\Delta u + |u|^{p-1} u)
\]
\[
= -i\sqrt{-\Delta} v + \frac{i}{\sqrt{-\Delta}} |u|^{p-1} u.
\]

Duhamel’s formula then gives us
\[
v(t) = e^{-it\sqrt{-\Delta}} v(0) + i \int_0^t e^{-i(t-\tau)\sqrt{-\Delta}} |u|^{p-1} u(\tau) \, d\tau.
\]

Assuming that \( \bar{u}(t) \) has the compactness property, we deduce by Lemma 4.5 that for any \( t_0 \in \mathbb{R} \)
\[
\int_{t_0}^T e^{-i(t_0-\tau)\sqrt{-\Delta}} |u|^{p-1}(\tau) \, d\tau \to iv(t_0), \quad \text{as } T \to \pm\infty
\]
weakly in \( \dot{H}^{s_p} \). Moreover,
\[
\|\bar{u}(t)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \cong \|v(t)\|_{\dot{H}^{s_p}}.
\]

We may now begin the proof of Proposition 4.6.

**PROOF OF PROPOSITION 4.6.** Our goal is to find a sequence \( \beta = \{\beta_k\} \) of positive numbers such that
\[
\sup_{t \in \mathbb{R}} \| (P_k u(t), P_k u_t(t)) \|_{\dot{H}^{3/4} \times \dot{H}^{-1/4}} \lesssim 2^{-\frac{k}{4}} \beta_k
\]
for all \( k \in \mathbb{Z} \), and
\[
\| \{2^{-\frac{k}{4}} \beta_k \} \|_2 \lesssim 1.
\]

A sequence \( \beta \in \ell^2 \) that satisfies the above properties is called a frequency envelope. In this section, \( P_k \) denotes the Littlewood-Paley projection corresponding to the dyadic number \( 2^k \), equivalently, \( P_k f \) is given by convolution
\[
P_k f := 2^k \tilde{\phi}(2^k \cdot) * f
\]
where \( \tilde{\phi} \) belongs to the Schwartz class.

**CLAIM 4.7.** A frequency envelope that satisfies (4.10)–(4.11) may be defined as below: we take
\[
\beta_k := 1 \quad \text{for } k \geq 0
\]
and for \( k < 0 \), we set
\[
\beta_k := \sum_j 2^{-|j-k|} a_j
\]
where

$$a_j = 2^{qj} \| P_j u \|_{L^\infty_t L^2} + 2^{(q-1)j} \| P_j u_t \|_{L^\infty_t L^2} \quad \text{for } j \in \mathbb{Z}. \quad (4.15)$$

Recalling the definition of $v$ in (4.7) we observe that

$$\| P_j v \|_{L^\infty_t \dot{H}^s} \approx a_j. \quad (4.16)$$

In order to verify (4.10)–(4.11) we first estimate $\| P_j v \|_{L^\infty_t \dot{H}^s}$. Throughout the proof of Claim 4.7, the estimates will be uniform in $t$. For that reason, it suffices to estimate the term $\| P_j v(0) \|_{\dot{H}^s}$, i.e. we set $t = 0$. We also note that the implicit constants carried through the computations below are allowed to depend on the norm $\| v \|_{L^\infty_t \dot{H}^s}$.

Let $M > 0$ be an arbitrary frequency. By (4.7),

$$\langle P_M v(0), P_M v(0) \rangle_{\dot{H}^s} = \left\langle P_M \left( e^{it\sqrt{-\Delta}} v(T_1) - i \int_0^{T_1} e^{i\tau \sqrt{-\Delta}} |u|^{p-1} u(\tau) \ d\tau \right), P_M v(0) \right\rangle_{\dot{H}^s}$$

for any $T_1 > 0$. We then take the limit $T_1 \to \infty$, which yields

$$\langle P_M v(0), P_M v(0) \rangle_{\dot{H}^s} = \lim_{T_1 \to \infty} \left\langle P_M \left( e^{it\sqrt{-\Delta}} v(T_1) - i \int_0^{T_1} e^{i\tau \sqrt{-\Delta}} |u|^{p-1} u(\tau) \ d\tau \right), P_M v(0) \right\rangle_{\dot{H}^s} \quad (4.17)$$

On the last line in (4.17), we used Lemma 4.5 to have the weak limit

$$\lim_{t \to \infty} e^{it\sqrt{-\Delta}} v(t) = 0$$

in $\dot{H}^s$. We also observe that

$$\lim_{t \to -\infty} e^{it\sqrt{-\Delta}} v(t) = 0.$$

weakly in $\dot{H}^s$. Similarly, we use the formula (4.7) on the second term in (4.17), and take the weak limit $T \to -\infty$ to obtain the reduction

$$\langle P_M v(0), P_M v(0) \rangle_{\dot{H}^s} = \left\langle P_M \int_0^\infty e^{it\sqrt{-\Delta}} |u|^{p-1} u(\tau) \ d\tau, P_M \int_{-\infty}^0 e^{i\tau \sqrt{-\Delta}} |u|^{p-1} u(\tau) \ d\tau \right\rangle_{\dot{H}^s}. \quad (4.18)$$

Next, we take a non-increasing bump function $\chi \in C^\infty_0(\mathbb{R}^T)$, which satisfies

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Also, let $c > 0$ be a small fixed constant, say $c = 1/4$. We then express the $\dot{H}^s$ inner product (4.18) as

$$\langle A + B, \tilde{A} + \tilde{B} \rangle_{\dot{H}^s} = \langle A, \tilde{A} + \tilde{B} \rangle_{\dot{H}^s} + \langle A + B, \tilde{A} \rangle_{\dot{H}^s} + \langle A, \tilde{A} \rangle_{\dot{H}^s} + \langle B, \tilde{B} \rangle_{\dot{H}^s}$$

where

$$A := \int_0^\Lambda M e^{it\sqrt{-\Delta}} P_M |u|^{p-1} u(\tau) \ d\tau + \int_{\Lambda/2}^\infty e^{i\tau \sqrt{-\Delta}} (1 - \chi)(x/c\tau) P_M |u|^{p-1} u(\tau) \ d\tau \quad (4.19)$$

$$= A_1 + A_2$$
and
\[ B := \int_{-\infty}^{\infty} e^{i\tau\sqrt{-\Delta}} \chi(x/ct) P_M |u|^{p-1} u(\tau) \, d\tau. \] (4.20)

The constant \( \Lambda > 0 \) will be determined below.

Note that the terms \( \tilde{A} \) and \( \tilde{B} \) are defined analogously in the negative time direction. First, we treat the term \( \langle A, \tilde{A} \rangle \) by estimating \( \| A \|_{H^{s_p}} \) and \( \| \tilde{A} \|_{H^{s_p}} \).

**Claim 4.8.** Let \( \eta > 0 \) be arbitrary. There is \( N_0 > 0 \) so that
\[ \| A_1 \|_{H^{s_p}} \lesssim \Lambda M^{s_p} \eta^{p-1} \| P_{> M^4} u \|_{L^\infty_t L^2} + \Lambda M^{s_p} N_0^{-s_p}. \] (4.21)

First, we note that
\[ \left\| P_M \left( \int_{-\infty}^{\Lambda/M} e^{i\tau\sqrt{-\Delta}} |u|^{p-1} u(\tau) \, d\tau \right) \right\|_{H^{s_p}} \leq M^{s_p-1} \left\| \int_{-\infty}^{\Lambda/M} e^{i\tau\sqrt{-\Delta}} P_M(|u|^{p-1}u)(\tau) \, d\tau \right\|_{L^2} \]
\[ \lesssim \Lambda M^{s_p-2} \left\| P_M(|u|^{p-1}) \right\|_{L_t^\infty L^2}. \] (4.22)

Let \( \eta > 0 \) be a small positive number. Since \( u \) has the compactness property, we have \( \inf_{t \in \mathbb{R}} \lambda(t) > A_0 \) (cf. (4.1)) for some positive constant \( A_0 \), and so Lemma 4.4 yields that there is a positive number \( N_0 = N_0(\eta) \) such that
\[ \| P_{\leq N_0} u \|_{H^{s_p}} \lesssim \eta \]
which then leads to
\[ \| P_{\leq N_0} u \|^2_{L^2((p-1)/2)} \lesssim \eta \] (4.23)
by the Sobolev embedding.

In order to estimate the term \( \| P_M(|u|^{p-1}) \|_{L_t^\infty L^2} \) in (4.22), we start with the following decomposition
\[ \| P_M(|u|^{p-1}) \|_{L^2} = \| P_M(u|P_{\leq N_0} u|^{p-1} - (u|u|^{p-1} - u|P_{\leq N_0} u|^{p-1}) \|_{L^2} \]
\[ \leq \| P_M(u|P_{\leq N_0} u|^{p-1}) \|_{L^2} + \| P_M(u|u|^{p-1} - u|P_{\leq N_0} u|^{p-1}) \|_{L^2} \]
\[ = I + \tilde{I}. \] (4.24)

We then write the term \( u|P_{\leq N_0} u|^{p-1} \) in \( I \) as a product of two factors decomposed into high-low frequencies around \( M/4 \) and \( N_0 \). In other words, we get
\[ I = \| P_M((P_{\leq M/4} u + P_{> M/4} u)|P_{\leq N_0} u|^{p-1}) \|_{L^2} \]
\[ \leq \| P_M(P_{\leq M/4} u|P_{\leq N_0} u|^{p-1}) \|_{L^2} + \| P_M(P_{> M/4} u|P_{\leq N_0} u|^{p-1}) \|_{L^2} \]
\[ = I_1 + I_2. \] (4.25)

We begin with
\[ I_1 = \| P_M((P_{\leq M/4} u)|P_{\leq N_0} u|^{p-1}) \|_{L^2}. \] (4.26)

Note that if \( N_0 \leq M/4 \), then we get \( I_1 = 0 \). We simply assume that \( N_0 > M/4 \), and split \( I_1 \) in two parts
\[ I_1 \lesssim \| P_M((P_{\leq M/4} u)|P_{\leq M/4} u|^{p-1}) \|_{L^2} \]
\[ + \| P_M((P_{\leq M/4} u)(|P_{\leq N_0} u|^{p-1} - |P_{M/4} u|^{p-1})) \|_{L^2} \]
\[ = I_{11} + I_{12}. \] (4.27)

As noted above, we deduce that \( I_{11} = 0 \). By the mean value theorem, we get
\[ \| P_{\leq N_0} u|^{p-1} - |P_{M/4} u|^{p-1} \| \approx |cP_{\leq N_0} u + (1 - c)P_{M/4} u|^{p-2} |P_{M/4} u|N_0 u \]
\[ (4.28) \]
for some \( c = c(x, t, u) \in (0, 1) \). Recalling (4.12) we apply Young’s and Holder’s inequality to get

\[
I_{12} \lesssim \| M^7 \tilde{\phi}(M\cdot) \|_{L^{7/5}} \| P_{M/4} u \|_{L^{7/2}} \| P_{M/4 < N_0} u \|_{L^2}
\]

\[
\lesssim \| M^7 \tilde{\phi}(M\cdot) \|_{L^{7/5}} \| P_{N_0} u \|_{L^{7/2}} \| P_{M/4 < N_0} u \|_{L^2}
\]

where we used (4.23) on the last line.

Similarly, we estimate

\[
\begin{aligned}
I_2 & \lesssim \| M^7 \tilde{\phi}(M\cdot) \|_{L^{7/5}} \| P_{N_0} u \|_{L^{7/2}} \| P_{M/4} u \|_{L^2} \\
& \lesssim M^2 \| P_{M/4} u \|_{L^2}.
\end{aligned}
\]

Combining (4.27), (4.29), and (4.30), we obtain

\[
I \lesssim M^2 \| P_{M/4} u \|_{L^2}.
\]

Next, we estimate \( \tilde{I} \) in (4.24). Similarly, we may express the difference

\[
\| u \|_{L^2} - \| P_{N_0} u \|_{L^2} \approx | \tilde{c} u + (1 - \tilde{c}) P_{N_0} u |_{L^2}
\]

with \( \tilde{c} = \tilde{c}(x, t, u) \in (0, 1) \). Using the Young’s and Holder’s inequality followed by the Sobolev embedding and Bernstein’s inequality at the last step, we obtain

\[
\begin{aligned}
\tilde{I} & \lesssim \| M^7 \tilde{\phi}(M\cdot) \|_{L^{7/5}} \| u \|_{L^{7/2}} \| \tilde{c} u + (1 - \tilde{c}) P_{N_0} u \|_{L^{7/2}} \| P_{N_0} u \|_{L^2} \\
& \lesssim M^2 \| u \|_{L^{7/2}} \| P_{N_0} u \|_{L^2} \\
& \lesssim M^2 N_0^{-sp} \| u \|_{L^{7/2}}.
\end{aligned}
\]

By (4.24), (4.31), and (4.32) we have

\[
\| P_{M} u \|_{L^2} \lesssim M^2 \| P_{M/4} u \|_{L^2} + M^2 N_0^{-sp}
\]

which yields by (4.22)

\[
\begin{aligned}
\left\| P_{M} \left( \int_0^M e^{it\sqrt{-\Delta}} |u|^{p-1} u(\tau) \, d\tau \right) \right\|_{H^{sp}} \lesssim M^{sp-2} \| P_{M} |u|^{p-1} u \|_{L^{7/2} L^2} \\
& \lesssim M^{sp} \| P_{M/4} u \|_{L^{7/2} L^2} + M^{sp} N_0^{-sp}.
\end{aligned}
\]

This completes the proof of Claim 4.8.

Next, we consider \( A_2 \) in (4.19). Recall that

\[
A_2 = \left\| \int_{\Lambda/M}^\infty e^{-it\sqrt{-\Delta}} (1 - \chi)(x/ct) P_{M} \left( |u|^{p-1} u(t) \right) \, dt \right\|_{H^{sp}}.
\]

First, we move the spacial norm inside the integral and obtain

\[
A_2 \lesssim \int_{\Lambda/M}^\infty \| (1 - \chi)(x/ct) P_{M} \left( |u|^{p-1} u(t) \right) \|_{H^{sp}} \, dt.
\]

Denote by

\[
\tilde{A}_2(t, x) := (1 - \chi)(x/ct) P_{M} \left( |u|^{p-1} u(t) \right).
\]

Noting that \( s_p - 1 = 5/2 - 2/(p - 1) \in [3/2, 5/2) \) we then estimate by interpolation

\[
\| \tilde{A}_2(t, x) \|_{H^{sp}} < \| \tilde{A}_2(t, x) \|_{H^{sp}}^{\theta} \| \tilde{A}_2(t, x) \|_{H^{sp}}^{1-\theta}
\]

where \( \theta = 2/(p - 1) \).
By another application of interpolation and Leibniz rule we write
\[
\left\| \tilde{A}_2(t, x) \right\|_{H^{3/2}} \lesssim \left\| \frac{1}{ct} \chi'(x/ct) P_M (|u|^{p-1} u) \right\|_{H^{1/2}} \\
+ \left\| (1 - \chi(x/ct)) \nabla P_M (|u|^{p-1}) \right\|_{H^{1/2}} \\
\lesssim \frac{1}{|ct|} \left\| \chi'(x/ct) P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} \left\| \chi'(x/ct) P_M (|u|^{p-1} u) \right\|_{H^1}^{1/2} \\
+ \left\| (1 - \chi(x/ct)) \nabla P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} \left\| (1 - \chi(x/ct)) \nabla P_M (|u|^{p-1} u) \right\|_{H^1}^{1/2} \\
=: J_1 + J_2.
\]

We begin with $J_1$. By Leibniz rule we further split $J_1$ into two terms
\[
J_1 \leq \left( \frac{1}{|ct|^{1/2}} \left\| \chi''(x/ct) P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} + \left\| \chi'(x/ct) \nabla P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} \right) \\
\times \frac{1}{|ct|} \left\| \chi'(x/ct) P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2}
\]
\[
=: J_{11} + J_{12}.
\]

Since $\chi \in C_0^\infty (\mathbb{R}^7)$, we obtain
\[
J_{11} = \left( \frac{1}{|ct|^{3/2}} \left\| \chi'(x/ct) P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} \left\| \chi''(x/ct) P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} \right)
\]
\[
\lesssim \frac{1}{|ct|^{3/2}} \left\| P_M (|u|^{p-1} u) \right\|_{L^2}.
\]

For $J_{12}$, we observe that $\text{supp}(\chi'(x/ct)) \subset \{ x : ct \leq |x| \leq 2ct \}$. Since $\chi'$ is also a bounded function, we get
\[
J_{12} = \left( \frac{1}{|ct|^{3/2}} \left\| \chi'(x/ct) P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} \left\| \chi'(x/ct) \nabla P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} \right)
\]
\[
\lesssim \frac{1}{|ct|^{3/2}} \left\| r P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2} \left\| P_M (|u|^{p-1} u) \right\|_{H^1}^{1/2}.
\]

Using the radial Sobolev inequality followed by Bernstein’s inequality we bound the right hand side in (4.42) from above by
\[
J_{12} \lesssim \frac{1}{|ct|^{3/2}} \left\| DP_M (|u|^{p-1} u) \right\|_{L^{4/11}}^{1/2} \left\| DP_M (|u|^{p-1} u) \right\|_{L^2}^{1/2}
\]
\[
\lesssim \frac{M}{|ct|^{3/2}} \left\| P_M (|u|^{p-1} u) \right\|_{L^{4/11}}^{1/2} \left\| P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2}
\]
\[
(4.43)
\]

Thus, by (4.41) and (4.42) we have
\[
J_1 \lesssim \frac{1}{|ct|^{3/2}} \left\| P_M (|u|^{p-1} u) \right\|_{L^2} + \frac{M}{|ct|^{3/2}} \left\| P_M (|u|^{p-1} u) \right\|_{L^{4/11}}^{1/2} \left\| P_M (|u|^{p-1} u) \right\|_{L^2}^{1/2}.
\]
\[
(4.44)
\]

Next we estimate
\[
J_2 = \left\| (1 - \chi) (x/ct) \nabla P_M (|u|^{p-1} u(t)) \right\|_{H^1}^{1/2} \left\| (1 - \chi) (x/ct) \nabla P_M (|u|^{p-1} u(t)) \right\|_{H^1}^{1/2}
\]
\[
(4.45)
\]
in (4.39). Since $\text{supp}(1 - \chi) \subset \{ x : |x| \geq |ct| \}$, we bound the first factor by
\[
\left\| (1 - \chi) (x/ct) \nabla P_M (|u|^{p-1} u(t)) \right\|_{L^2}^{1/2} \lesssim \left\| \frac{r^2}{|ct|^2} \nabla P_M (|u|^{p-1} u(t)) \right\|_{L^2}^{1/2}.
\]
Once again, utilizing the radial Sobolev inequality and Bernstein’s inequality we may estimate the right hand side above

\[ \frac{1}{|ct|} \left\| DP_M \left( |u|^{p-1}u \right) \right\|_{L^{4/11}}^{1/2} \lesssim \frac{M^{1/2}}{|ct|} \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^{4/11}}^{1/2} \]  \hspace{1cm} (4.46)

The second factor in (4.45) may be dealt with in the same way. Firstly, we distribute the derivative by Leibniz rule, and then apply the radial Sobolev and Bernstein’s inequalities as demonstrated above. We get

\[ \left\| (1 - \chi) \left( \frac{x}{ct} \right) \nabla P_M \left( |u|^{p-1}u(t) \right) \right\|_{H^1}^{1/2} \lesssim \frac{1}{|ct|^{1/2}} \left\| \chi' \left( \frac{x}{ct} \right) \nabla P_M \left( |u|^{p-1}u(t) \right) \right\|_{L^2}^{1/2} \]
\[ + \left\| (1 - \chi) \left( \frac{x}{ct} \right) \Delta P_M \left( |u|^{p-1}u(t) \right) \right\|_{L^2}^{1/2} \]  \hspace{1cm} (4.47)

Combining the upper bounds obtained in (4.44), (4.46), and (4.47) we estimate

\[ \left\| \tilde{A}_2(t, x) \right\|_{H^{3/2}} \lesssim J_1 + J_2 \]
\[ \lesssim \frac{M}{|ct|^{3/2}} \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^{4/11}}^{1/2} \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^2}^{1/2} \]
\[ + \frac{M^{3/2}}{|ct|^2} \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^{4/11}} + \frac{1}{|ct|^{3/2}} \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^2} . \]  \hspace{1cm} (4.48)

We may further simplify the upper bound in (4.48). First, we apply Young’s inequality on the first term and combine it with the other two terms. Next, we check the balance of prefactors involving \( M \) and \( |ct| \). In (4.37) we have \( |ct| > \Lambda/M \) with \( \Lambda \geq 1 \), which implies that

\[ \left\| \tilde{A}_2(t, x) \right\|_{H^{3/2}} \lesssim \left( \frac{M^{3/2}}{|ct|^2} + \frac{M^2}{|ct|^{3/2}} \right) \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^{4/11}} + \frac{1}{|ct|^{3/2}} \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^2} \]
\[ \lesssim \frac{M^2}{|ct|^{3/2}} \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^{4/11}} + \frac{1}{|ct|^{3/2}} \left\| P_M \left( |u|^{p-1}u \right) \right\|_{L^2} . \]  \hspace{1cm} (4.49)

Back to (4.38) we estimate

\[ \left\| \tilde{A}_2(t, x) \right\|_{H^{5/2}} = \left\| (1 - \chi)(x/ct)P_M \left( |u|^{p-1}u \right) \right\|_{H^{5/2}} . \]  \hspace{1cm} (4.50)

By Leibniz rule, we split the right hand side above into three main terms:

\[ \left\| \tilde{A}_2(t, x) \right\|_{H^{5/2}} \leq \frac{1}{|ct|^2} \left\| \left( \chi''(x/ct)P_M \left( |u|^{p-1}u \right) \right) \right\|_{H^{1/2}} \]
\[ + \frac{1}{|ct|} \left\| \left( \chi'(x/ct)\nabla P_M(|u|^{p-1}u) \right) \right\|_{H^{1/2}} \]
\[ + \left\| (1 - \chi)(x/ct)\Delta P_M(|u|^{p-1}u) \right\|_{H^{1/2}} =: K_1 + K_2 + K_3 . \]  \hspace{1cm} (4.51)
We follow similar arguments to treat the three terms in (4.51). Starting with $K_1$, we apply interpolation and use the fact that $\chi \in C_0^\infty(\mathbb{R}^7)$. We get
\begin{align}
K_1 & \lesssim \frac{1}{|ct|^2} \| \chi''(x/ct) P_M(|u|^{p-1}u) \|^{1/2}_{H^1} \| \chi''(x/ct) P_M(|u|^{p-1}u) \|^{1/2}_{L^2} \\
& \lesssim \frac{1}{|ct|^2} \left( \frac{1}{|ct|^{1/2}} \| P_M(|u|^{p-1}u) \|^{1/2}_{L^2} + M^{1/2} \| P_M(|u|^{p-1}u) \|^{1/2}_{L^2} \right) \\
& \times \| P_M(|u|^{p-1}u) \|^{1/2}_{L^2} \\
& \lesssim \left( \frac{1}{|ct|^{5/2}} + \frac{M^{1/2}}{|ct|^2} \right) \| P_M(|u|^{p-1}u) \|_{L^2}.
\end{align}
(4.52)

Now, for $K_2$ we have
\begin{align}
K_2 & \lesssim \frac{1}{|ct|} \| \chi'(x/ct) \nabla P_M(|u|^{p-1}u) \|^{1/2}_{H^1} \| \chi'(x/ct) \nabla P_M(|u|^{p-1}u) \|^{1/2}_{L^2}.
\end{align}
(4.53)

The second factor may be estimated by Bernstein’s inequality
\begin{align}
\| \chi'(x/ct) \nabla P_M(|u|^{p-1}u) \|^{1/2}_{L^2} & \lesssim \| DP_M(|u|^{p-1}u) \|^{1/2}_{L^2} \lesssim M^{1/2} \| P_M(|u|^{p-1}u) \|^{1/2}_{L^2}.
\end{align}
(4.54)

To control the first factor we follow the arguments used to bound the term $J_1$ above.
\begin{align}
\| \chi'(x/ct) \nabla P_M(|u|^{p-1}u) \|_{H^1} & \lesssim \frac{1}{|ct|} \| DP_M(|u|^{p-1}u) \|_{L^2} + \| \chi'(x/ct) \Delta P_M(|u|^{p-1}u) \|_{L^2} \\
& \lesssim \frac{M}{|ct|} \| P_M(|u|^{p-1}u) \|_{L^2} + \frac{1}{|ct|} \| r \Delta P_M(|u|^{p-1}u) \|_{L^2} \\
& \lesssim \frac{M}{|ct|} \| P_M(|u|^{p-1}u) \|_{L^2} + \frac{M^3}{|ct|} \| P_M(|u|^{p-1}u) \|_{L^{14/11}}.
\end{align}

We then obtain
\begin{align}
K_2 & \lesssim \frac{M}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|_{L^2} + \frac{M^2}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|^{1/2}_{L^{14/11}} \| P_M(|u|^{p-1}u) \|^{1/2}_{L^2}.
\end{align}
(4.55)

Next, we estimate $K_3$ in (4.51). Similarly, by interpolation, we factor $K_3$ into two components
\begin{align}
K_3 & \lesssim \| (1 - \chi)(x/ct) \Delta P_M(|u|^{p-1}u) \|^{1/2}_{H^1} \| (1 - \chi)(x/ct) \Delta P_M(|u|^{p-1}u) \|^{1/2}_{L^2}.
\end{align}
(4.56)

We treat both factors by the radial Sobolev and Bernstein’s inequalities as demonstrated above. First, we recall that $\text{supp}(1 - \chi) \subset \{ x : |x| \geq |ct| \}$, and estimate the second factor by
\begin{align}
\| (1 - \chi)(x/ct) \Delta P_M(|u|^{p-1}u) \|^{1/2}_{L^2} & \lesssim \frac{1}{|ct|} \| r^2 \Delta P_M(|u|^{p-1}u) \|^{1/2}_{L^2} \\
& \lesssim \frac{1}{|ct|} \| D^2 P_M(|u|^{p-1}u) \|^{1/2}_{L^{14/11}} \\
& \lesssim \frac{M}{|ct|} \| P_M(|u|^{p-1}u) \|^{1/2}_{L^{14/11}}.
\end{align}
(4.57)

In the same fashion, the first factor in (4.56) may be bounded from above by
\begin{align}
\frac{1}{|ct|^{1/2}} \| \Delta P_M(|u|^{p-1}u) \|^{1/2}_{L^2} + \frac{1}{|ct|^{1/2}} \| r \nabla P_M(|u|^{p-1}u) \|^{1/2}_{L^2} \\
& \lesssim \frac{M}{|ct|^{1/2}} \| P_M(|u|^{p-1}u) \|^{1/2}_{L^2} + \frac{M^2}{|ct|^{1/2}} \| P_M(|u|^{p-1}u) \|^{1/2}_{L^{14/11}}.
\end{align}
(4.58)
Multiplying the bounds in (4.57) and (4.58) we get
\[
K_3 \lesssim \frac{M^2}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|_{L^{14/11}}^{1/2} \| P_M(|u|^{p-1}u) \|_{L^2}^{1/2} + \frac{M^3}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|_{L^{14/11}}.
\]  
(4.59)
Lastly, we add up the bounds in (4.52), (4.55), and (4.59) for \( K_1, K_2, K_3 \), apply Young's inequality on the terms with \( L^{14/11}, L^2 \) norms, and simplify the pre-factors to arrive at the estimate
\[
\| \tilde{A}_2(t,x) \|_{H^{5/2}} \lesssim \left( \frac{1}{|ct|^{5/2}} + \frac{M^{1/2}}{|ct|^{3/2}} + \frac{M}{|ct|^{3/2}} \right) \| P_M(|u|^{p-1}u) \|_{L^2} + \frac{M^3}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|_{L^{14/11}}
\]  
(4.60)

Before we go back to the interpolation inequality in (4.38), we address how to estimate the term \( \| P_M(|u|^{p-1}u) \|_{L^{14/11}} \). Following the same argument as shown in the proof of Claim 4.8, we begin with decomposing \( u = P_{\leq M/4}u + P_{> M/4}u \) and we write
\[
\| P_M(|u|^{p-1}u) \|_{L^{14/11}} \leq \| P_M((P_{\leq M/4}u + P_{> M/4}u)|u|^{p-1}) \|_{L^{14/11}}
\]  

\[
\leq \| P_M((P_{\leq M/4}u)|P_{\leq M/4}u|^{p-1}) \|_{L^{14/11}} + \| P_M((P_{\leq M/4}u)(|u|^{p-1} - |P_{\leq M/4}u|^{p-1})) \|_{L^{14/11}}
\]  

\[
+ \| P_M((P_{> M/4}u)|u|^{p-1}) \|_{L^{14/11}}
\]
Note that the first term \( \| P_M(P_{\leq M/4}u)|P_{\leq M/4}u|^{p-1} \|_{L^{14/11}} = 0 \). In order to control the last two terms above, we apply Young's and Holder's inequalities, as shown in the proof of Claim 4.8, and we obtain
\[
\| P_M(|u|^{p-1}u) \|_{L^{14/11}} \lesssim \| M^7 \hat{\varphi}(M) \|_{L^1} \| P_{> M/4}u \|_{L^2} \| u \|_{L^{7(p-1)/2}}^{p-1}
\]  
(4.61)
On the last line above we once again used the Sobolev embedding to get
\[
\| u \|_{L^{7(p-1)/2}} \lesssim \| u \|_{\dot{H}^{sp}}
\]
and absorbed the \( \dot{H}^{sp} \) norm of \( u \) in the implicit constant.

Next, we plug (4.49) and (4.60) into the estimate (4.38) to get the following upper bound for the \( \dot{H}^{sp-1} \) norm of \( \tilde{A}_2(t,x) \):
\[
\left( \frac{M^2}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|_{L^{14/11}} + \frac{1}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|_{L^2} \right)^\theta \times \left( \frac{M}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|_{L^2} + \frac{M^3}{|ct|^{3/2}} \| P_M(|u|^{p-1}u) \|_{L^{14/11}} \right)^{1-\theta}.
\]  
(4.62)
Utilizing the bounds in (4.33) and (4.61), we control the preceding estimate by
\[
\frac{1}{|ct|^{3/2}} \left( M^2 N_0^{-sp} + M^2 \| P_{> M/4}u \|_{L^2} \right)^\theta \left( M^3 N_0^{-sp} + M^3 \| P_{> M/4}u \|_{L^2} \right)^{1-\theta}
\]  
\[
\lesssim \frac{1}{|ct|^{3/2}} \left( M^{3-\theta} N_0^{-sp} + M^{3-\theta} \| P_{> M/4}u \|_{L^2} \right).
\]  
(4.63)
The upper bound on the second line above follows from Young's inequality combined with the fact that \( \theta = 2/(p-1) \in (0,1] \).
Having estimated the integrand in (4.36), we now integrate in $t$ to obtain
\[
A_2 \lesssim M^{\frac{7}{2}} - \frac{\theta}{2} A^{-\frac{1}{2}} N_0^{-s} + M^{\frac{7}{2}} - \frac{\theta}{2} \| P_{> M/4} u \|_{L_t^\infty L^2}. \tag{4.64}
\]
Combining the estimates for $A_1$ and $A_2$ from Claim 4.8 and (4.64), and setting $\Lambda := \eta^{-\frac{1}{2}}$ for $\eta \in (0, 1)$ to be fixed below, we have
\[
\| A \|_{\dot{H}^{s'}} \lesssim A_1 + A_2 \\
\lesssim \Lambda M^{s'} p^{p-1} \| P_{> M/4} u \|_{L_t^\infty L^2} + \Lambda M^{s'} N_0^{-s} \\
+ \frac{M^{s'}}{\Lambda^{1/2}} \left( N_0^{-s} + \| P_{> M/4} u \|_{L_t^\infty L^2} \right) \tag{4.65}
\lesssim \eta^{\frac{1}{2}} M^{s'} \| P_{> M/4} u \|_{L_t^\infty L^2} + \eta^{-\frac{1}{2}} M^{s'} N_0^{-s}.
\]
Note that the right hand side of (4.65) also controls the term $\| \tilde{A} \|_{\dot{H}^{s'}}$.

Next, we consider $\langle A, \tilde{A} + \tilde{B} \rangle_{\dot{H}^{s'}}$ and $\langle A + B, \tilde{A} \rangle_{\dot{H}^{s'}}$. We recall once again that $e^{it\sqrt{-\Delta}} v(t) \to 0$ weakly in $\dot{H}^{s'}$ as $t \to \pm \infty$ by Lemma 4.5. Therefore,
\[
A + B \to P_M v(0) \quad \text{as } t \to \infty \tag{4.66}
\]
and
\[
\tilde{A} + \tilde{B} \to P_M v(0) \quad \text{as } t \to -\infty \tag{4.67}
\]
weakly in $\dot{H}^{s'}$. We may then estimate
\[
\| \langle A, \tilde{A} + \tilde{B} \rangle_{\dot{H}^{s'}} \| \lesssim \| A \|_{\dot{H}^{s'}} \| P_M v \|_{L_t^\infty \dot{H}^{s'}} \\
\lesssim \left( \eta^{\frac{1}{2}} M^{s'} \| P_{> M/4} u \|_{L_t^\infty L^2} + \eta^{-\frac{1}{2}} M^{s'} N_0^{-s} \right) M^{s'} \| P_M v \|_{L_t^\infty L^2}. \tag{4.68}
\]
The same estimate holds for the term $\| \langle A + B, \tilde{A} \rangle_{\dot{H}^{s'}} \|$ as well.

Lastly, we show that $\langle B, \tilde{B} \rangle_{\dot{H}^{s'}} = 0$. Note that
\[
\langle B, \tilde{B} \rangle_{\dot{H}^{s'}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle e^{i(t-\tau)\sqrt{-\Delta}} \chi(x/ct) P_M (|u|^{p-1} u(t)), e^{i\tau\sqrt{-\Delta}} \chi(x/ct) P_M (|u|^{p-1} u(\tau)) \right\rangle_{\dot{H}^{s'}} d\tau dt \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \chi(x/ct) P_M (|u|^{p-1} u(t)), D^{2p-2} e^{i(t-\tau)\sqrt{-\Delta}} \chi(x/ct) P_M (|u|^{p-1} u(\tau)) \right\rangle_{L^2} d\tau dt. \tag{4.69}
\]
Due to the Huygens Principle, when $c = 1/4$, we have
\[
\text{supp} \left( e^{i(t-\tau)\sqrt{-\Delta}} \chi(x/ct) P_M (|u|^{p-1} u(\tau)) \right) \subset \{ x : |x| \geq \frac{3}{4}|t - \tau| \}.
\]
Since $t > \Lambda/M > 0$ and $\tau < -\Lambda/M < 0$, the support of the function on the right side of the bracket in (4.69) is included in the set $|x| > \frac{3}{4} t$, whereas that of the function $\chi(x/ct) P_M (|u|^{p-1} u(t))$ is in the set $|x| < t/4$. Therefore, we get
\[
\langle B, \tilde{B} \rangle_{\dot{H}^{s'}} = 0. \tag{4.70}
\]
Combining (4.65), (4.68), and (4.70), we arrive at the conclusion that
\[
\| \langle P_M v(0), P_M v(0) \rangle_{\dot{H}^{s'}} \| \lesssim \left( \eta^{\frac{1}{2}} M^{s'} \| P_{> M/4} u \|_{L_t^\infty L^2} + \eta^{-\frac{1}{2}} M^{s'} N_0^{-s} \right)^2 \\
+ \left( \eta^{\frac{1}{2}} M^{s'} \| P_{> M/4} u \|_{L_t^\infty L^2} + \eta^{-\frac{1}{2}} M^{s'} N_0^{-s} \right) M^{s'} \| P_M v \|_{L_t^\infty L^2}. \tag{4.71}
\]
As noted at the beginning of the proof, we may utilize the same logic and arguments to estimate the term \(\|P_M v(t)\|_{\dot{H}^{s_p}}\) and therefore control it uniformly in \(t\) with the upper bound in (4.71). Namely, we have

\[
|\langle P_M v(t), P_M v(t) \rangle_{\dot{H}^{s_p}}| \lesssim \left( \eta \frac{1}{2} M^{s_p} \|P_{> M/4} u\|_{L_t^{\infty} L_x^2} + \eta \frac{1}{2} M^{s_p} N_0^{-s_p} \right)^2 \\
+ \left( \eta \frac{1}{2} M^{s_p} \|P_{> M/4} u\|_{L_t^{\infty} L_x^2} + \eta \frac{1}{2} M^{s_p} N_0^{-s_p} \right) M^{s_p} \|P_M v\|_{L_t^{\infty} L_x^2}.
\]

(4.72)

We now go back to the proof of Claim 4.7. Setting \(M = 2^j\) for \(j \in \mathbb{Z}^-\) and recalling (4.16), we get for \(j < 0\)

\[
a_j^2 \lesssim \left( \eta \frac{1}{2} M^{s_p} \|P_{> M/4} u\|_{L_t^{\infty} L_x^2} + \eta \frac{1}{2} M^{s_p} N_0^{-s_p} \right)^2 \\
+ \left( \eta \frac{1}{2} M^{s_p} \|P_{> M/4} u\|_{L_t^{\infty} L_x^2} + \eta \frac{1}{2} M^{s_p} N_0^{-s_p} \right) M^{s_p} \|P_M v\|_{L_t^{\infty} L_x^2} \\
\lesssim \left( \eta \frac{1}{2} \sum_{i>|j-2|} 2^{s_p(i-j)} a_i + 2^{s_p} \eta \frac{1}{2} N_0^{-s_p} \right)^2 \\
+ a_j \left( \eta \frac{1}{2} \sum_{i>|j-2|} 2^{s_p(i-j)} a_i + 2^{s_p} \eta \frac{1}{2} N_0^{-s_p} \right)
\]

which implies that

\[
a_j \lesssim \eta \frac{1}{2} \sum_{i>|j-2|} 2^{s_p(i-j)} a_i + 2^{s_p} \eta \frac{1}{2} N_0^{-s_p}
\]

for \(j < 0\).

For \(j > 0\), it suffices to use the estimate

\[
a_j \lesssim \|P_j v\|_{L_t^{\infty} \dot{H}^{s_p}} \lesssim 1.
\]

(4.73)

Recalling the definition of \(\beta_k\) in (4.13)-(4.14), we then obtain for \(k < 0\)

\[
\beta_k \lesssim \sum_{j>0} 2^{-|j-k|} + \frac{1}{2} \left( \sum_{j<0} 2^{-|j-k|} \right) \left( \sum_{i|j-2} 2^{-s_p|j-i|} a_i \right) \\
+ \eta \frac{1}{2} N_0^{-s_p} \sum_{j<0} 2^{-|j-k|} 2^{s_p} j \\
\lesssim \eta \frac{1}{2} \beta_k + \sum_{j>0} 2^{-|j-k|} + \eta \frac{1}{2} N_0^{-s_p} \sum_{j<0} 2^{s_p} j.
\]

(4.74)

Selecting \(\eta = 1/4\) in (4.74), we absorb the first term on the last line above into the left hand side, and we obtain

\[
\beta_k \lesssim \sum_{j} 2^{-|j-k|} \min(1, 2^{s_p} j)
\]

which yields

\[
\beta_k \lesssim 2^k \quad \text{for} \ k < 0.
\]

As we set \(\beta_k = 1\) for \(k \geq 0\), we conclude that \(\left\{2^{-3k/4} \beta_k\right\}_{k \in \mathbb{Z}} \in \ell^2\), which completes the proof of Proposition 4.6.
5. Channels of energy for the linear radial wave equation

A key ingredient in the rigidity argument is the exterior energy estimates for radial solutions of the wave equation in odd dimensions. We state the following linear estimates in dimension seven. A proof of this result for general odd dimensions may be found in [31].

**Theorem 5.1.** Let \( V \) be a radial solution of
\[
\partial_t^2 V - \Delta V = 0, \quad x \in \mathbb{R}^7, t \in \mathbb{R}
\]
\[
\dot{V}(0) = (V_0, V_1) \in \dot{H}^1 \times L^2(\mathbb{R}^7).
\]
For every \( R > 0 \),
\[
\max_{\pm} \lim_{t \to \pm \infty} \int_{r \geq |t| + R} |\nabla_{x,t} V(r,t)|^2 r^6 \, dr
\]
\[
\geq \frac{1}{2} \|\pi_R(V_0, V_1)\|^2_{H^1 \times L^2(r \geq R, r^6 \, dr)}
\]
where \( \pi_R = \text{Id} - \pi_R \) is the orthogonal projection onto the plane
\[
P(R) := \text{span} \{ (1/r^5, 0), (1/r^3, 0), (0, 1/r^5) \}
\]
in the space \( \dot{H}^1 \times L^2(r \geq R, r^6 \, dr) \).

The left hand side of (5.2) vanishes for all data in \( P(R) \). Moreover, in (5.2) equality holds for data of the form \((0, V_1)\) and \((V_0, 0)\).

**5.1. Algebraic identities for the projection.** In this part, we discuss the orthogonal projection onto the plane \( P(R) \) in \( \dot{H}^1 \times L^2(r \geq R, r^6 \, dr) \). Similar to Theorem 5.1, the results of this section in general odd dimensions may be found in [32, Section 4]. For convenience, we review the case for dimension seven below. We also introduce the notation \( \mathcal{H} = \dot{H}^1 \times L^2(\mathbb{R}^7 \setminus \{0\}) \) that will be commonly used for the rest of the discussion.

First we derive explicit formulas for the projection coefficients using the linear algebra techniques, and then point out some algebraic identities that highlight the relationship between the exterior energy of the projected solutions \( \|\pi_R \tilde{u}(t)\|_{\mathcal{H}(r \geq R, r^6 \, dr)} \) and the projection coefficients.

Note that, fixing \( R > 0 \) in (5.2)-(5.3), the orthogonal projections will be of the form
\[
\pi_R \tilde{u}(t, r) = (\lambda_1(t, R)r^{-5} + \lambda_2(t, R)r^{-3}, \mu(t, R)r^{-5})
\]
and
\[
\pi_R^{-1} \tilde{u}(t, r) = (u(t, r) - \lambda_1(t, R)r^{-5} - \lambda_2(t, R)r^{-3}, u_t(t, R) - \mu(t, R)r^{-5}) .
\]
Here, \( \lambda_1(t, R), \lambda_2(t, R), \) and \( \mu(t, R) \) denote the coefficients of the orthogonal projections of \( \tilde{u}(t) \) onto the subspace \( P(R) \).

Denote by \( \mathcal{V} \) the inner product space \( L^2(r \geq R, r^6 \, dr) \) and consider the line
\[
W := \{ c/r^5 : c \in \mathbb{R} \}
\]
in the space \( \mathcal{V} \). Then, for \( g \in \mathcal{V} \) we have
\[
\langle \text{Proj}_{W^\perp} g, 1/r^5 \rangle = \langle g, 1/r^5 \rangle - \mu(R) \langle 1/r^5, 1/r^5 \rangle = 0
\]
where the inner product \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2(r \geq R, r^6 \, dr) \). Solving the equation above with \( u_t(t, r) \), we find
\[
\mu(t, R) = \frac{\int_R^\infty u_t(t, r) r \, dr}{\int_R^\infty \frac{1}{r} \, d r} = 3R^3 \int_R^\infty u_t(t, r) r \, dr.
\]
Similarly, denote by
\[
\tilde{W} := \{d_1/r^5 + d_2/r^3 : d_1, d_2 \in \mathbb{R}\}
\]
the subspace in the inner product space \(\tilde{V} = H^1(\mathbb{R} \geq R, \ r^6 dr)\).

For any \(u \in \tilde{V}\), the orthogonal projection onto \(\tilde{W}^\perp\) may be given as
\[
\text{Proj}_{\tilde{W}^\perp} u = u - (\lambda_1 r^{-5} + \lambda_2 r^{-3})
\]
where the coefficients satisfy the following two formulas. We have
\[
\langle \text{Proj}_{\tilde{W}^\perp} u, 1/r^5 \rangle_{\tilde{V}} = \langle u, 1/r^5 \rangle_{\tilde{V}} - \lambda_1 \langle 1/r^5, 1/r^5 \rangle_{\tilde{V}} - \lambda_2 \langle 1/r^3, 1/r^5 \rangle_{\tilde{V}} = 0
\]  
(5.8)

and
\[
\langle \text{Proj}_{\tilde{W}^\perp} u, 1/r^3 \rangle_{\tilde{V}} = \langle u, 1/r^3 \rangle_{\tilde{V}} - \lambda_1 \langle 1/r^5, 1/r^3 \rangle_{\tilde{V}} - \lambda_2 \langle 1/r^3, 1/r^3 \rangle_{\tilde{V}} = 0.
\]  
(5.9)

Note that the inner product \(\langle \cdot, \cdot \rangle_{\tilde{V}}\) denotes the inner product in the space \(\tilde{H}^1(\mathbb{R} \geq R, \ r^6 dr)\). Solving (5.8)–(5.9), we obtain
\[
\lambda_1(t, R) = -\frac{9}{4} R^5 \int_R^\infty u_r(t, r) \ dr + \frac{3}{4} R^3 \int_R^\infty u_r(t, r)r^2 \ dr
\]
\[
\lambda_2(t, R) = \frac{5}{4} R^3 \int_R^\infty u_r(t, r) \ dr + \frac{3}{4} R \int_R^\infty u_r(t, r)r^2 \ dr
\]  
(5.10)

for \(R > 0\) fixed above.

Next, we derive several algebraic identities that will help us understand the relationship between \(\|\pi_R \tilde{u}(t)\|_{\mathcal{H}(r \geq R)}\) and the projection coefficients \(\lambda_1(t, R), \lambda_2(t, R), \mu(t, R)\). Note that the formulas in (5.7) and (5.10) can be used to express \(\int_R^\infty u_r(t, r) \ dr, \int_R^\infty u_r(t, r)r^2 \ dr, \text{ and } \int_R^\infty u_r(t, r)r \ dr\). Additionally, we may rewrite the formulas for \(\lambda_1\) and \(\lambda_2\) by using integration by parts. We state these formulas below.

**Lemma 5.2.** For \(R > 0\), we have
\[
\lambda_1(t, R) = 3R^5 u(t, R) - \frac{3}{2} R^3 \int_R^\infty u(t, r)r \ dr
\]
\[
\lambda_2(t, R) = -2R^3 u(t, R) + \frac{3}{2} R \int_R^\infty u(t, r)r \ dr
\]  
(5.11)

We may then find the explicit formulas for the norms of the orthogonal projections \(\pi_R\) and \(\pi_{\tilde{R}}\).

**Lemma 5.3.** Given \(\tilde{u}(t) \in \mathcal{H}\), let \(\mu(t, R), \lambda_1(t, R), \text{ and } \lambda_2(t, R)\) be defined as in (5.7) and (5.10), respectively. Then, we have
\[
\|\pi_R \tilde{u}(t)\|^2_{\mathcal{H}(r \geq R)} = \frac{5 \lambda_1^2(t, R)}{R^5} + \frac{9 \lambda_2^2(t, R)}{R^3} + \frac{10 \lambda_1(t, R) \lambda_2(t, R)}{R^3} + \frac{\mu^2(t, R)}{3R^3}
\]  
(5.12)

and
\[
\|\pi_{\tilde{R}} \tilde{u}(t)\|^2_{\mathcal{H}(r \geq R)} \cong \int_R^\infty \left( (\partial_r \lambda_1(t, r)r^{-2})^2 + (\partial_r \lambda_2(t, r))^2 + (\partial_r \mu(t, r)r^{-1})^2 \right) \ dr.
\]  
(5.13)

**Proof.** Using (5.4), we express \(\|\pi_R \tilde{u}(t)\|^2_{\mathcal{H}(r \geq R)}\) as the sum of two inner products, namely
\[
\|\pi_R \tilde{u}(t)\|^2_{\mathcal{H}(r \geq R)} = \langle \lambda_1(t, R)/r^5 + \lambda_2(t, R)/r^3, \lambda_1(t, R)/r^5 + \lambda_2(t, R)/r^3 \rangle_{\tilde{V}} + \mu(t, R) \langle 1/r^5, 1/r^5 \rangle_{\tilde{V}}.
\]
Recalling that $\langle \cdot, \cdot \rangle_{\tilde{V}}$ denotes the inner product in $\tilde{H}^1(r \geq R, r^6dr)$ and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(r \geq R, r^6dr)$, we compute the right hand side above as

$$
\|\pi_R \tilde{u}(t)\|_{\tilde{H}(r \geq R)}^2 = \lambda_1^2(1/r^5, 1/r^5)_{\tilde{V}} + 2\lambda_1(t, R)\lambda_2(t, R) (1/r^5, 1/r^3)_{\tilde{V}} + \lambda_2^2(t, R) (1/r^3, 1/r^3)_{\tilde{V}} + \mu^2(t, R) (1/r^5, 1/r^3)
$$

which then gives us the formula in (5.12). For (5.13), we first utilize the orthogonality of the projections. Omitting the dependence on $t$ and $R$ for brevity, we write

$$
\|\pi_R \tilde{u}(t)\|_{\tilde{H}(r \geq R)}^2 = \int_R^\infty (u_r(t, r))^2 r^6dr - \left( \frac{5\lambda_1^2}{R^6} + \frac{9\lambda_2}{R} + \frac{10\lambda_1\lambda_2}{R^3} \right) - \frac{\mu^2}{3R^3}.
$$

We then differentiate the equation above with respect to $R$. The right hand side of (5.14) becomes

$$
- (\partial_R u(t, R))^2 R^6 - \left( \frac{10\lambda_1 \partial_R \lambda_1}{R^6} + \frac{18\lambda_2 \partial_R \lambda_2}{R^3} + \frac{10(\partial_R \lambda_1 \lambda_2 + \lambda_1 \partial_R \lambda_2)}{R^3} \right) + \left( \frac{25\lambda_1}{R^6} + \frac{9\lambda_2}{R^2} + \frac{30\lambda_1 \lambda_2}{R^4} \right) - \left( u_r(t, R) \right)^2 R^6 - \frac{2\mu \partial_R \mu}{3R^3} + \frac{\mu^2}{R^4}.
$$

Next, we replace $(\partial_R u(t, R))^2 R^6$ and $(u_r(t, R))^2 R^6$ with expressions involving $\lambda_1, \lambda_2,$ and $\mu$. We find such expressions by differentiating (5.7) and (5.10) with respect to $R$. To be more precise, we obtain

$$
\partial_R u(t, R) = \frac{\partial_R \lambda_1}{R^5} + \frac{\partial_R \lambda_2}{R^3} - \frac{5\lambda_1}{R^6} - \frac{3\lambda_2}{R^4} = \frac{5\partial_R \lambda_1}{R^6} + \frac{3\partial_R \lambda_2}{R^3} - \frac{5\lambda_1}{R^6} - \frac{3\lambda_2}{R^4},
$$

$$
u_r(t, R) = -\frac{\partial_R \mu}{3R^4} + \frac{\mu}{R^5}.
$$

We plug these expressions into (5.15), and after cancellations, we find that

$$
-\partial_R \|\pi_R \tilde{u}(t)\|_{\tilde{H}(r \geq R)}^2 \simeq \left( \frac{\partial_R \lambda_1}{R^2} \right)^2 + (\partial_R \lambda_2)^2 + \left( \frac{\partial_R \mu}{R^{-1}} \right)^2.
$$

Finally we integrate the formula above from $R$ to $\infty$ to get (5.13).

\hfill \square

6. Singular stationary solutions

In this section, we cook up a one-parameter family of singular stationary solutions to the equation (1.1) whose asymptotic behaviour resemble that of a nonzero solution to (1.1) with the compactness property. By construction, these singular stationary solutions do not lie in the critical Sobolev space $H^{s_p} \times H^{s_p-1}(\mathbb{R}^7)$. We will utilize this fact to close the contradiction argument in the next section.

**Proposition 6.1.** Let $p \geq 3$. For any $l \in \mathbb{R}\setminus\{0\}$ there exists a radial $C^2$ solution $Z_l$ of

$$
\Delta Z_l + |Z_l|^{p-1} Z_l = 0 \quad \text{in} \quad \mathbb{R}^7\setminus\{0\}
$$

with the asymptotic behaviour

$$
r^5 Z_l(r) = l + O\left(r^{-5p+7}\right) \quad \text{as} \quad r \to \infty.
$$

Furthermore, $Z_l \notin L^q_{\nu}(\mathbb{R}^7)$, where $q_{\nu} := \frac{7(p-1)}{2}$ is the critical Sobolev exponent corresponding to $H^{s_p}$. This implies that $Z_l \notin H^{s_p}(\mathbb{R}^7)$.

**Proof.** Let $\varphi \in C^2(\mathbb{R}^7\setminus\{0\})$ be a radial function that solves the equation (6.1), i.e.,

$$
-\partial_{rr} \varphi - \frac{6}{r} \partial_r \varphi = |\varphi|^{p-1} \varphi, \quad r > 0.
$$

\hfill (6.3)
Setting $\omega(r) = r \varphi(r)$, we note that (6.3) is equivalent to
\[-\partial_{rr} \omega - \frac{4}{r} \partial_r \omega + \frac{4r^{p-3} \omega(r) - |\omega|^{p-1} \omega}{r^{p-1}} = 0.\]

In order to guarantee that $\varphi$ satisfies (6.2) with $l \in \mathbb{R} \setminus \{0\}$, we impose the condition
\[\lim_{r \to \infty} \omega(r) = 0.\] (6.4)

Next, we introduce the new variables $s = \log(r)$ and $\phi(s) = \omega(r)$ and obtain a non-autonomous differential equation for $\phi$. We get
\[\dot{\phi} + 3 \ddot{\phi} - 4 \dot{\phi} + |\phi|^{p-1} \phi e^{-(p-3)s} = 0.\] (6.5)

We may rewrite the equation above as a $2 \times 2$ system by setting $x(s) = \phi(s)$ and $y(s) = \dot{\phi}(s)$. We then obtain
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
-3y + 4x - |x|^{p-1} x e^{-(p-3)s} \\
-y
\end{pmatrix} =: F(x, y).
\] (6.6)

We note that $(0, 0)$ is the only equilibrium point of the above system. Let $\Phi_s$ denote the flow associated to this system. Checking the linearized system associated to (6.6) at $(0, 0)$, we find that
\[DF((0, 0)) = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}\]
with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -4$. Denote by $E_{(1)}$ and $E_{(-4)}$ the corresponding eigenspaces. More precisely, we have
\[E_{(1)} = \left\{ ce^{1} : c \in \mathbb{R} \right\}\]
and
\[E_{(-4)} = \left\{ ce^{-4} : c \in \mathbb{R} \right\}.\]

We then write the formula for solutions to the linearized system
\[
\begin{pmatrix}
x_L \\
y_L
\end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^s + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4s}, \quad s \in \mathbb{R}
\] (6.7)

Note that $E_{(-4)}$ denotes the stable subspace of the space of solutions to the linear system given in (6.7).

Due to the hyperbolic nature of the matrix $DF((0, 0))$, the stable curve theorem yields a one-dimensional manifold $S$ tangent to the stable subspace $E_{(-4)}$ at the origin with the following property: there is a neighborhood $B$ of the origin such that $B \cap S$ is positively invariant, i.e.,
\[\Phi_t(B \cap S) \subset B \cap S, \quad t \geq 0\]
and for all $(x_0, y_0) \in B \cap S$ we have
\[|\Phi_t((x_0, y_0)) - (x_0, -4x_0)e^{-4t}| = O \left( e^{-(5p+3)t} \right).\] (6.8)

Furthermore, the flow $\Phi_t((x_0, y_0))$ is as smooth as the nonlinear term in (6.6). In particular, we need to have $p > 2$ to guarantee a $C^2$ solution. Note that for any $(x_0, y_0) \in B \cap S \setminus \{(0, 0)\}$ the nonlinear flow $\Phi_t((x_0, y_0))$ never passes through the origin, i.e., there is no $t \in (-\infty, \infty)$ such that $\Phi_t((x_0, y_0)) = (\Phi_{1,t}((x_0, y_0)), \Phi_{2,t}((x_0, y_0))) = (0, 0)$.

This can easily be seen from (6.8) for $t \geq 0$. For the negative time direction, it follows from the fact that one may trace the flow $\Phi_t((x_0, y_0))$ that belongs to the stable manifold in the negative direction and for each $((x_0, y_0)) \in B \cap S$, there is a unique entry point to the neighborhood $B \cap S$. We also remark that with the choice of $x_0 = 0$, any solution on the stable manifold $S$ vanishes to the equilibrium solution $(0, 0)$ with higher order terms as given on the right hand side of (6.8) turning identically zero.
Changing back to $\omega(r) = \phi(s)$ and $r = e^s$ we obtain
\[ \omega(r) = \Phi_{1, \log t}(x_0, y_0) \quad \omega(1) = x_0. \] (6.9)
with the asymptotic estimate
\[ |\omega(r) - x_0 r^{-4}| = O \left( r^{-5p+3} \right). \] (6.10)

Moreover, we have
\[ \lim_{r \to 0} \omega(r) = \lim_{t \to -\infty} \Phi_{1, \log t}(x_0, y_0) \neq 0 \] (6.11)
if and only if $x_0$ is nonzero. This follows from the observation that the nonlinear flow $\Phi_{1}(x_0, y_0)$ with $x_0 \neq 0$ only approaches $(0, 0)$ as $t \to \infty$.

We switch back to the initial setting by $\varphi(r) = \omega(r)/r$. By (6.11), we deduce that $\varphi \notin L^\frac{7}{2}(\mu-1)(\mathbb{R}^7)$. Also by (6.10), our solution $\varphi(r)$ to (6.3) satisfies (6.2) with $x_0 \neq 0$. As $x_0$ might vary in a bounded neighborhood around 0, we may scale our solutions by $\varphi = \lambda^{-\frac{2}{p-1}} \varphi(\frac{x}{\lambda})$. This way, we obtain a solution of the equation (6.1) that satisfies (6.2) with $l = x_0 \lambda^{5-\frac{2}{p-1}}$.

\[ \square \]

7. Rigidity argument

In this section, we prove Proposition 3.13. The proof proceeds in three main steps and follows the line of arguments presented in [32] for solutions to exterior wave maps in all equivariance classes.

First, we state an important outcome of the decay results obtained in Section 4. We show that boundedness in $\dot{H}^{3/4} \times \dot{H}^{-1/4}(\mathbb{R}^7)$ combined with the pre-compactness in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^7)$ yields pre-compactness in the energy space.

**Corollary 7.1.** Let $\tilde{u}(t)$ be a solution to (1.1) as in Proposition 3.13. Then, we have $\tilde{u}(t) \in \dot{H}^{1} \times L^2(\mathbb{R}^7)$ for all $t \in \mathbb{R}$. Moreover, the trajectory
\[ K_1 = \{ \tilde{u}(t) : t \in \mathbb{R} \} \] (7.1)
is pre-compact in $\dot{H}^{1} \times L^2(\mathbb{R}^7)$. As a result, we have for all $R > 0$\n\[ \lim_{t \to +\infty} \sup_{t \geq |R|} \| \tilde{u}(t) \|_{\mathcal{H}(R \geq |R|)} = 0 \] (7.2)

**Proof.** The proof of Corollary 7.1 is similar to the proof of Lemma 6.1 in [11]. We first prove that the trajectory $K_1$ is pre-compact in $\dot{H}^{1} \times L^2(\mathbb{R}^7)$. We take a sequence $\{t_n\} \subset \mathbb{R}$ and show that $\{\tilde{u}(t_n)\}$ has a convergent sequence. The argument below shows that it suffices to consider $t_n \to \pm \infty$. Without loss of generality, we let $t_n \to \infty$.

Firstly, we consider the case $\{\lambda(t_n)\}$ remains bounded, which implies that $\{\lambda(t_n)\}$ is a pre-compact sequence. Note that in this case the sequence $\{\tilde{u}(t_n)\}$ is pre-compact in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ if and only if the sequence
\[ \left\{ \left( \frac{1}{\lambda(t_n)} \frac{x}{\lambda(t_n)}, t_n \right), \frac{1}{\lambda(t_n)} \frac{x}{\lambda(t_n)} \partial_x u \left( \frac{x}{\lambda(t_n)}, t_n \right) \right\} \] (7.3)
is pre-compact in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, where the latter fact is guaranteed by hypothesis.

Using interpolation we control the norm in energy space by\n\[ \| \tilde{u}(t_n) - \tilde{u}(t_m) \|_{\dot{H}^{1} \times L^2} \lesssim \| \tilde{u}(t_n) - \tilde{u}(t_m) \|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}}^{\alpha_p} \| \tilde{u}(t_n) - \tilde{u}(t_m) \|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}}^{1-\alpha_p} \] (7.3)
where $\alpha_p \in (0, 1)$. Then, by Proposition 4.6 we get
\[ \| \tilde{u}(t_n) - \tilde{u}(t_m) \|_{\dot{H}^{1} \times L^2} \lesssim \| \tilde{u}(t_n) - \tilde{u}(t_m) \|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}}^{1-\alpha_p}. \]
Since the sequence on the right hand side is precompact as discussed above, so is the sequence on the left hand side.

Next, we consider the case $\lambda(t_n) \to \infty$. We will show that in this case
\[
\bar{u}(t_n) \to 0 \quad \text{in } H^1 \times L^2.
\] (7.4)
Let $\eta > 0$ be given. By Lemma 4.4 there is $c(\eta) > 0$ so that
\[
\int_{|\xi| \leq c(\eta)\lambda(t_n)} |\xi|^{2p} |\hat{u}(\xi, t_n)|^2 d\xi \leq \eta.
\] (7.5)
Then we get
\[
\|u(t_n)\|_{H^1}^2 = \int_{|\xi| \leq c(\eta)\lambda(t_n)} |\xi|^2 |\hat{u}(\xi, t_n)|^2 d\xi + \int_{|\xi| \geq c(\eta)\lambda(t_n)} |\xi|^2 |\hat{u}(\xi, t_n)|^2 d\xi
\leq \int_{|\xi| \leq c(\eta)\lambda(t_n)} |\xi|^2 |\hat{u}(\xi, t_n)|^2 d\xi + (c(\eta)\lambda(t_n))^{2p-2} \|u(t_n)\|_{H^{sp}}^2.
\] (7.6)
Recall that the norm $\|u(t_n)\|_{H^{sp}}$ is invariant when scaled by $\lambda(t_n)$ as in (7.3), and therefore is bounded in $n$. Combined with our assumption that $\lambda(t_n) \to \infty$, we find that the second term in (7.6) approaches zero as $n \to \infty$. The first term is controlled by interpolation as done above. We get
\[
\int_{|\xi| \leq c(\eta)\lambda(t_n)} |\xi|^2 |\hat{u}(\xi, t_n)|^2 d\xi
\leq \left( \int_{|\xi| \leq c(\eta)\lambda(t_n)} |\xi|^2 |\hat{u}(\xi, t_n)|^2 d\xi \right)^{\alpha_p} \left( \int_{|\xi| \geq c(\eta)\lambda(t_n)} |\xi|^{2p} |\hat{u}(\xi, t_n)|^2 d\xi \right)^{1-\alpha_p}
\leq \eta^{1-\alpha_p} \|u(t_n)\|_{H^{1/4}}^{\alpha_p} \lesssim \eta^{1-\alpha_p}
\]
where we used (7.5) and Proposition 4.6 in the last line. As a result, $u(t_n)$ tends to zero in $H^1$. Using the same line of arguments, we may also get $\partial_t u(t_n) \to 0$ in $L^2$, which completes the proof of the first claim that the set $K_1$ is precompact in $H^1 \times L^2$.

We note that the pre-compactness of $K_1$ implies that
\[
\|u(t)\|_{H^1 \times L^2(r \geq R)} \to 0 \quad \text{as } R \to \infty
\]
uniformly in $t \in \mathbb{R}$. Therefore, it leads to the fact that the energy of $\bar{u}(t)$ on the exterior cone $\{r > R + |t|\}$ vanishes as $t \to \pm \infty$. \hfill \Box

7.1. Step 1. Let $\bar{u}(t)$ satisfy the assumptions of Proposition 3.13. The goal of this part is to estimate $\pi^\perp_R \bar{u}(t)$ in $\mathcal{H}(r \geq R)$. We combine the linear estimates in Theorem 5.1 with Corollary 7.1 to obtain the following result.

**Lemma 7.2.** There exists $R_0 > 0$ such that for all $R \geq R_0$ and for all $t \in \mathbb{R}$ we have
\[
\|\pi^\perp_R \bar{u}(t)\|_{\mathcal{H}(r \geq R)}^2 \lesssim R^{-5(p-\frac{4}{p})} \|\pi_R \bar{u}(t)\|_{\mathcal{H}(r \geq R)}^{2p}. \tag{7.7}
\]
where the projections $\pi$ and $\pi^\perp$ are as in Section 5.

First, we prove a preliminary result concerning a Cauchy problem for finite energy solutions away from the origin.

**Notation 7.3.** Let $I \subset \mathbb{R}$ be an interval with $0 \in I$. For $q \in [1, \infty]$, denote by $L^q_I := L^q(\mathbb{R}^7 \times I)$. 

First, we take a radial cut-off function \( \chi \in C^\infty(\mathbb{R}^7) \) so that
\[
\chi(r) = \begin{cases} 
1 & \text{if } r \geq 1/2, \\
0 & \text{if } r \leq 1/4.
\end{cases} \tag{7.8}
\]

For \( r_0 > 0 \), denote by \( \chi_{r_0}(r) = \chi(r/r_0) \) and consider the Cauchy problem
\[
\begin{align*}
\partial_t^2 h - \Delta h &= |V + \chi_{r_0}h|^p - 1(V + \chi_{r_0}h) - |V|^{p-1}V \quad \text{in } \mathbb{R}^7 \times I, \\
(h, \partial_t h)\big|_{t=0} &= (h_0, h_1) \in \mathcal{H}.
\end{align*} \tag{7.9}
\]

**Lemma 7.4.** There exists \( \delta_0 > 0 \) satisfying the following property: let \( V \in L^4_{I}(p-1) \) be radial in the \( x \) variable satisfying
\[
\|D^{1/2}V\|_{L^8_I} \leq \delta_0 r_0^\beta \quad \text{and} \quad \|V\|_{L^4_{I}(p-1)} \leq \delta_0.
\tag{7.10}
\]
where \( \beta = \frac{5}{2} \left( \frac{p-2}{p-1} \right) \). Furthermore, \( (h_0, h_1) \in \mathcal{H} \) be radial functions with
\[
\|(h_0, h_1)\|_{\mathcal{H}} \leq \delta_0 r_0^\beta.
\tag{7.11}
\]
Then, the Cauchy problem (7.9) is well-posed on the interval \( I \), and we have
\[
\sup_{t \in I} \|h(t) - S(t)(h_0, h_1)\|_{\mathcal{H}} \leq \frac{1}{100} \|(h_0, h_1)\|_{\mathcal{H}}. \tag{7.12}
\]
Moreover, if \( V = 0 \), we may take \( I = \mathbb{R} \) and we obtain
\[
\sup_{t \in \mathbb{R}} \|h(t) - S(t)(h_0, h_1)\|_{\mathcal{H}} \leq \frac{1}{2} \left( \frac{5}{2} \right) \|(h_0, h_1)\|_{\mathcal{H}}. \tag{7.13}
\]

**Proof.** Let \( F_V(h) = |V + \chi_{r_0}h|^{p-1}(V + \chi_{r_0}h) - |V|^{p-1}V \). We apply a fixed point argument to show that the formula
\[
h(t) = S(t)(h_0, h_1) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_V(h(s)) \, ds
\]
holds for \( t \in I \). We define the norm
\[
\|h\|_S := \|h\|_{L^{16/5}_I} + \|D^{1/2}h\|_{L^8_I} + \sup_{t \in I} \|(h, h_t)\|_{\mathcal{H}}
\]
and for \( \alpha > 0 \) we denote by
\[
B_\alpha := \{ h \in L^{16/5}_I : h \text{ is radial, } \|h\|_S \leq \alpha \}.
\]
Now, for \( v \in B_\alpha \) we set
\[
\Phi(v)(t) := S(t)(h_0, h_1) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_V(h(s)) \, ds.
\]
We will show that if (7.10)–(7.11) hold, we can set \( \alpha > 0 \) small enough that \( \Phi \) is a contraction on \( B_\alpha \).

By Strichartz estimates,
\[
\|\Phi(v)\|_S \leq C(\|(h_0, h_1)\|_{\mathcal{H}} + \|D^{1/2}F_V(v)\|_{L^8_I}) \tag{7.14}
\]
We estimate the second term above using the chain rule for fractional derivatives (cf. (2.16)). First, we let \( G(h) = |h|^{p-1} h \) so that we may write \( F_V(v) = G(V + \chi_{r_0} v) - G(V) \). We get

\[
\|D^{1/2} F_V(v)\|_{L_t^8 L_x^3} = \|D^{1/2} (G(V + \chi_{r_0} v) - G(V))\|_{L_t^8 L_x^3} \\
\leq C \left( \|G'(V + \chi_{r_0} v)\|_{L_t^4} + \|G'(V)\|_{L_t^4} \right) \|D^{1/2} (\chi_{r_0} v)\|_{L_t^8 L_x^3} \\
+ C \left( \|G''(V + \chi_{r_0} v)\|_{L_t^4(p-1)/(p-2)} + \|G''(V)\|_{L_t^4(p-1)/(p-2)} \right) \|\chi_{r_0} v\|_{L_t^4 L_x^{p-1}} \\
\times \left( \|D^{1/2} (V + \chi_{r_0} v)\|_{L_t^8 L_x^3} + \|D^{1/2} (V)\|_{L_t^8 L_x^3} \right) .
\]

(7.15)

Note that

\[
\|\chi_{r_0} v\|_{L_t^{4(p-1)-1} L_x^{p-1}} = \left( \int_{I} \int_{J(r_0)} |v(r)|^{4(p-1)-1} dr dt \right)^{1/4} \\
= \left( \int_{I} \int_{J(r_0)} \frac{|r|^{5/2} v(r)|^{4(p-1)-1} \frac{1}{r^5}}{r^{10p-18}} |v(r)|^{16/5} r^6 dr dt \right)^{1/4} .
\]

(7.16)

Recalling the Sobolev inequality for radial functions \( f \in \dot{H}^1(\mathbb{R}^7) \)

\[
\|r^{5/2} f\|_{L^{\infty}(\mathbb{R}^7)} \lesssim \|f\|_{\dot{H}^1(\mathbb{R}^7)},
\]

we estimate (7.16) from above by

\[
\|\chi_{r_0} v\|_{L_t^{4(p-1)-1} L_x^{p-1}} \lesssim r_0^{-\frac{2}{5}(p-1)+2} \|v\|_{L^{(p-9)/5}} \|v\|_{L_t^{4}} L_x^{16/5} \\
\lesssim r_0^{-\frac{2}{5}(p-1)+2} \|v\|_{S}^{p-1} .
\]

(7.17)

Therefore, we may control the right hand side of (7.15) by

\[
C \left( \|V\|_{L_t^{4(p-1)+1}} + r_0^{-\frac{2}{5}(p-1)+2} \|v\|_{S}^{p-1} \right) \|v\|_{S} \\
+ C \left( \|V\|_{L_t^{4(p-1)+1}} + r_0^{-\frac{2}{5}(p-2)+2} \|v\|_{S}^{p-1} \right) r_0^{-5/2+2/(p-1)} \|v\|_{S} \\
\times \left( \|D^{1/2} V\|_{L_t^8 L_x^3} + \|v\|_{S} \right) \\
\leq C \|v\|_{S} \left( \|V\|_{L_t^{4(p-1)-1}} + r_0^{-5/2(p-1)+2} \|D^{1/2} V\|_{L_t^8 L_x^3} + r_0^{-5/2(p-1)+2} \|v\|_{S}^{p-1} \right)
\]

(7.19)

where we applied Young’s inequality on the second term in order to obtain the upper bound on the last line.

Combining the bound in (7.19) with (7.14) we get

\[
\|\Phi(v)\|_{S} \leq C_0 \|(h_0, h_1)\|_{\mathcal{H}} + C_0 \alpha \left( \|V\|_{L_t^{4(p-1)-1}} + r_0^{-5/2(p-1)+2} \|D^{1/2} V\|_{L_t^8 L_x^3} + r_0^{-5/2(p-1)+2} \|v\|_{S}^{p-1} \right)
\]

(7.20)

for some \( C_0 > 0 \). We set

\[
\alpha = 2C_0 \|(h_0, h_1)\|_{\mathcal{H}} \leq 2C_0 \delta_0 r_0^\beta .
\]

(7.21)

By (7.10)–(7.11), we then obtain

\[
\|\Phi(v)\|_{S} \leq C_0 \|(h_0, h_1)\|_{\mathcal{H}} + 6\delta_0^p C_0^2 \|(h_0, h_1)\|_{\mathcal{H}}.
\]

(7.22)

Selecting \( \delta_0 > 0 \) sufficiently small we guarantee that \( \Phi(v) \in B_\alpha \) for every \( v \in B_\alpha \).

The contraction property may be proved using the same arguments. For each \( v, \omega \in B_\alpha \) the difference

\[
\|D^{1/2} (F_V(V + \chi_{r_0} v) - F_V(V + \chi_{r_0} \omega))\|_{L_t^8 L_x^3}
\]
is estimated by using once again the chain rule for fractional derivatives. Namely, we have
\[ ||D^{1/2} (F_V(V + \chi_{\theta_0}v) - F_V(V + \chi_{\theta_0}\omega))||_{L^s_t} = ||D^{1/2}(G(V + \chi_{\theta_0}v) - G(V + \chi_{\theta_0}\omega))||_{L^s_t} \]
\[ \leq C \left( ||G'(V + \chi_{\theta_0}v)||_{L^4_t} + ||G'(V + \chi_{\theta_0}\omega)||_{L^4_t} \right) ||D^{1/2}(\chi_{\theta_0}v - \chi_{\theta_0}\omega)||_{L^s_t} \]
\[ + C \left( ||G''(V + \chi_{\theta_0}v)||_{L^{4(p-1)/(p-2)}_t} + ||G''(V + \chi_{\theta_0}\omega)||_{L^{4(p-1)/(p-2)}_t} \right) ||\chi_{\theta_0}(v - \omega)||_{L^{s(p-1)}_t} \]
\[ \times \left( ||D^{1/2}(V + \chi_{\theta_0}v)||_{L^{s/3}_t} + ||D^{1/2}(V + \chi_{\theta_0}\omega)||_{L^{s/3}_t} \right) \]
\[ \leq C(||v - \omega||_S \left( ||V||_{L^{p-1}_t} + r_0^{-5(2p-1)/2} \right) \right) \]  \( (7.23) \)

Therefore, we obtain \( \Phi(h(t)) = h(t) \). Moreover, by Strichartz estimates and (7.19) we get
\[ ||h - S(t)(h_0, h_1)||_S \leq C_0 \alpha \left( ||V||_{L^{p-1}_t} + r_0^{-5(2p-1)/2} \right) \]  \( (7.24) \)

which implies (7.12) with our choice of \( \alpha \) in (7.21) provided that \( \delta > 0 \) is sufficiently small. Similarly, in the case \( V = 0 \), the inequality (7.24) yields (7.13).

Going back to the proof of Lemma 7.2, we follow the ideas demonstrated in [32, Prop. 5.3]

**Proof of Lemma 7.2.** First we prove the inequality (7.7) for \( t = 0 \). We take \( R > 0 \) and denote the truncated initial data by
\[ \bar{u}_R(0) := (u_0, u_1, u_R) \]  \( (7.25) \)

where
\[ u_0, R := \begin{cases} u_0(r) & \text{for } r \geq R \\ u_0(R) & \text{for } r \leq R \end{cases} \] 
and
\[ u_1, R := \begin{cases} u_1(r) & \text{for } r \geq R \\ 0 & \text{for } r \leq R. \end{cases} \]

Observe that
\[ ||\bar{u}_R(0)||_H \leq ||\bar{u}(0)||_{H(\delta \geq R)} \]  \( (7.26) \)

which implies that we may select \( R_0 > 0 \) sufficiently large so that for all \( R \geq R_0 \) the truncated initial data is small in \( \dot{H}^1 \times L^2 \). In particular, fixing \( \delta < \min(\delta_0, 1) \), where \( \delta_0 \) denotes the positive constant given in Lemma 7.4, we may guarantee that
\[ ||\bar{u}_R(0)||_H \leq \delta \]
for all \( R > R_0 \).

Let \( \bar{u}_R(t) \) denote the solution to the equation
\[ \partial^2_t h - \Delta h = \chi_R |h|^{p-1} h \quad \text{in } \mathbb{R}^7 \times I, \]
\[ (h, \partial_t h)|_{t=0} = (h_0, h_1) \in \mathcal{H} \]
given by (7.9) in the case \( V = 0 \). Note that in this case the solution \( \bar{u}_R(t) \) exists for all \( t \in \mathbb{R} \). Moreover, by finite speed of propagation,
\[ \bar{u}_R(t, r) = \bar{u}(t, r) \]  \( (7.27) \)
for all \( t \in \mathbb{R} \) and \( r \geq R + |t| \).
Next, we define \( \tilde{u}_{R,L}(t) = S(t)\tilde{u}_R(0) \) and note that
\[
\|\tilde{u}(t)\|_{H(t \geq R+|t|)} = \|\tilde{u}_R(t)\|_{H(t \geq R+|t|)} \\
\geq \|\tilde{u}_{R,L}(t)\|_{H(t \geq R+|t|)} - \|\tilde{u}_R(t) - \tilde{u}_{R,L}(t)\|_{H(t \geq R+|t|)}.
\] (7.28)

By Lemma 7.4,
\[
\sup_{t \in \mathbb{R}} \|\tilde{u}_R(t) - \tilde{u}_{R,L}(t)\|_H \leq \frac{C_0}{R^{2(p-\frac{5}{2})}} \|\tilde{u}_R(0)\|_H^p.
\]

Combining this estimate with (7.28) we have
\[
\|\tilde{u}(t)\|_{H(t \geq R+|t|)} \geq \|\tilde{u}_{R,L}(t)\|_{H(t \geq R+|t|)} - \frac{C_0}{R^{2(p-\frac{5}{2})}} \|\tilde{u}_R(0)\|_H^p.
\] (7.29)

Recall that the linear estimates in Theorem 5.1 yields a lower bound for the term \( \|\tilde{u}_{R,L}(t)\|_{H(t \geq R+|t|)} \), namely we have
\[
\|\pi_{R}^{\frac{1}{p}}\tilde{u}_R(0)\|_H^2 \leq \max_{\pm} \lim_{t \to \pm \infty} \|\tilde{u}_{R,L}\|_{H(t \geq R+|t|)}^2.
\]

We then let \(|t| \to \infty\) according to the choice of sign dictated by Theorem 5.1, which leads to the vanishing of the left hand side in (7.29). Therefore we have
\[
\|\pi_{R}^{\frac{1}{p}}\tilde{u}_R(0)\|_H^2 \leq \frac{C_0}{R^{5(p-\frac{5}{2})}} \|\tilde{u}(0)\|_{H(t \geq R)}^{2p}.
\]

Once again using (7.27) we note that \( \|\pi_{R}^{\frac{1}{p}}\tilde{u}_R(0)\|_H^2 = \|\pi_{R}^{\frac{1}{p}}\tilde{u}(0)\|_{H(t \geq R)}^2 \), which gives us
\[
\|\pi_{R}^{\frac{1}{p}}\tilde{u}(0)\|_H^2 \leq \frac{C_0}{R^{5(p-\frac{5}{2})}} \|\tilde{u}(0)\|_{H(t \geq R)}^{2p} \\
\leq \frac{C_0}{R^{5(p-\frac{5}{2})}} (\|\pi_{R}^{\frac{1}{p}}\tilde{u}(0)\|_{H(t \geq R)} + \|\pi_{R}^{\frac{1}{p}}\tilde{u}(0)\|_{H(t \geq R)})^{2p}.
\]

Then, we choose \( R_0 \) large enough to absorb \( C_0 R^{-(p-\frac{5}{2})} \|\pi_{R}^{\frac{1}{p}}\tilde{u}(0)\|_{H(t \geq R)}^{10} \) on the left side, which completes the proof of Lemma 7.2 for \( t = 0 \).

We utilize Corollary 7.1 to prove the inequality (7.7) for all \( t \in \mathbb{R} \). By the pre-compactness of \( K_1 \) we may select \( R_0 = R_0(\delta_0) \) such that
\[
\|\tilde{u}(t)\|_{H(t \geq R)} \leq \min(\delta_0, 1)
\]
uniformly in \( t \in \mathbb{R} \).

Therefore, for fixed \( t_0 \in \mathbb{R} \), we take
\[
\tilde{u}_{t_0,R}(r) := \begin{cases} 
    u(t_0, r) & \text{for } r \geq R \\
    u(t_0, R) & \text{for } r \leq R
\end{cases}
\]
and
\[
\tilde{u}_{t_0,R}(r) := \begin{cases} 
    u(t_0, r) & \text{for } r \geq R \\
    0 & \text{for } r \leq R
\end{cases}
\]
as the truncated initial data and repeat the same steps to obtain (7.7) for \( t = t_0 \). \[\square\]
7.2. Step 2. Next, we aim to investigate the asymptotic behaviour of \((u_0(r), u_1(r))\) as \(r \to \infty\). Our goal is to establish the asymptotic rates given in the following proposition.

**Proposition 7.5.** Let \(\vec{u}(t)\) be as in Proposition 3.13 with \(\vec{u}(0) = (u_0, u_1)\). Then, there exists \(\ell \in \mathbb{R}\) so that

\[
r^5 u_0(r) = \ell + O(r^{-5p+7}) \quad \text{as} \quad r \to \infty
\]

\[
\int_0^\infty u_1(s) s \, ds = O(r^{-5p+3}) \quad \text{as} \quad r \to \infty.
\]

First, we recall the bounds on the norms \(\|\pi_R \vec{u}(t)\|_{\mathcal{H}(t)}\) and \(\|\pi_R^1 \vec{u}(t)\|_{\mathcal{H}(t)}\) given in Lemma 5.3 and rewrite Lemma 7.2 in terms of \(\lambda_1(t, r), \lambda_2(t, r),\) and \(\mu(t, r)\).

**Lemma 7.6.** There exists \(R_0 > 0\) so that for all \(R > R_0\) we have

\[
\int_R^\infty \left( (\partial_r \lambda_1(t, r) r^{-2})^2 + (\partial_r \lambda_2(t, r))^2 + (\partial_r \mu(t, r) r^{-1})^2 \right) \, dr
\]

\[
\lesssim \frac{1}{R^{6(p- \frac{2}{3})}} \left( \frac{\lambda_1^2(t, R)}{R^{5p}} + \frac{\lambda_2^2(t, R)}{R^p} + \frac{\mu^2(t, R)}{R^{3p}} \right),
\]

where the implicit constant on the right hand side is uniform in \(t \in \mathbb{R}\).

**Remark 7.7.** Lemma 7.6 yields uniform in time estimates on the projection coefficients, which then leads to difference estimates. Let \(\delta_0\) and \(R_0\) denote the constants introduced in the proof of Lemma 7.2. We take \(\delta_1 \in (0, \delta_0)\) to be determined below. By the pre-compactness of the set \(K_1\) in (7.1), we may find \(R_1 > R_0\) such that for all \(R \geq R_1\),

\[
\|\vec{u}(t)\|_{\mathcal{H}(t)}^{p-1} \leq \delta_1, \quad t \in \mathbb{R}
\]

and

\[
1/R_1 \leq \min(\delta_1, 1).
\]

Consequently, we obtain the following estimates that hold uniformly in time: for every \(r \geq R_1\) and for all \(t \in \mathbb{R}\)

\[
\frac{|\lambda_1(t, r)|^{p-1}}{r^{\frac{5p}{2}(p-1)}} \leq \delta_1, \quad \frac{|\lambda_2(t, r)|^{p-1}}{r^{\frac{5p}{2}(p-1)}} \leq \delta_1, \quad \frac{|\mu(t, r)|^{p-1}}{r^{\frac{5p}{2}(p-1)}} \leq \delta_1.
\]

**Lemma 7.8.** Let \(R_1\) be as in (7.34). For all \(r, r'\) such that \(R_1 \leq r \leq r' \leq 2r\), the following difference estimates hold uniformly. We have for all \(t \in \mathbb{R}\),

\[
|\lambda_1(t, r) - \lambda_1(t, r')| \leq r^{-\frac{5p}{2}(p-1)} \left( r^{-5p/2} |\lambda_1(t, r)|^p + r^{-p/2} |\lambda_2(t, r)|^p + r^{-3p/2} |\mu(t, r)|^p \right)
\]

and

\[
|\lambda_2(t, r) - \lambda_2(t, r')| \leq r^{-\frac{5p}{2}(p-2)} \left( r^{-5p/2} |\lambda_1(t, r)|^p + r^{-p/2} |\lambda_2(t, r)|^p + r^{-3p/2} |\mu(t, r)|^p \right).
\]

Similarly, for all \(t \in \mathbb{R}\),

\[
|\mu(t, r) - \mu(t, r')| \leq r^{-\frac{5p}{2}(p-1)} \left( r^{-5p/2} |\lambda_1(t, r)|^p + r^{-p/2} |\lambda_2(t, r)|^p + r^{-3p/2} |\mu(t, r)|^p \right).
\]

**Proof.** The inequalities (7.36)–(7.38) follow directly from Lemma 7.6. First, we consider (7.36). We express difference on the left hand side as an integral from \(r\) to \(r'\) and apply the inequality (7.32) in Lemma 7.6.

\[
|\lambda_1(t, r) - \lambda_1(t, r')|^2 = \left| \int_r^{r'} \partial_s \lambda_1(t, s) \, ds \right|^2 \lesssim \left( \int_r^{r'} (\partial_s \lambda_1(t, s) s^{-2})^2 \, ds \right) \left( \int_r^{r'} s^4 \, ds \right)
\]

\[
\lesssim r^{-5(p- \frac{14}{5})} \left( r^{-5p} |\lambda_1(t, r)|^{2p} + r^{-p} |\lambda_2(t, r)|^{2p} + r^{-3p} |\mu(t, r)|^{2p} \right).
\]
In the same fashion, we obtain
\[
|\lambda_2(t,r) - \lambda_2(t,r')|^2 = \left| \int_r^{r'} \partial_s \lambda_2(t,s) \, ds \right|^2 \lesssim r \left( \int_r^{r'} (\partial_s \lambda_2(t,s))^2 \, ds \right)
\lesssim r^{-5(p-2)} \left( r^{-5p} |\lambda_1(t,r)|^{2p} + r^{-p} |\lambda_2(t,r)|^{2p} + r^{-3p} |\mu(t,r)|^{2p} \right)
\]
and
\[
|\mu(t,r) - \mu(t,r')|^2 = \left| \int_r^{r'} \partial_s \mu(t,s) \, ds \right|^2 \lesssim \left( \int_r^{r'} (\partial_s \mu(t,s)s^{-1})^2 \, ds \right) \left( \int_r^{r'} s^2 \, ds \right)
\lesssim r^{-5\left(p-\frac{12}{5}\right)} \left( r^{-5p} |\lambda_1(t,r)|^{2p} + r^{-p} |\lambda_2(t,r)|^{2p} + r^{-3p} |\mu(t,r)|^{2p} \right).
\]

Recalling the setting in (7.33)–(7.35), we state a direct consequence of the difference estimates above.

**Corollary 7.9.** Let $R_1$ and $\delta_1$ be defined as in (7.34). Then, for all $r$ and $r'$ with $R_1 < r < r' < 2r$ and for all $t \in \mathbb{R}$, we have
\[
|\lambda_1(t,r) - \lambda_1(t,r')| \lesssim r^{-\frac{5}{2}(p-1)} \delta_1 \left( r^{-5/2} |\lambda_1(t,r)| + r^{-1/2} |\lambda_2(t,r)| + r^{-3/2} |\mu(t,r)| \right)
\]
\[
|\lambda_2(t,r) - \lambda_2(t,r')| \lesssim r^{-\frac{5}{2}(p-2)} \delta_1 \left( r^{-5/2} |\lambda_1(t,r)| + r^{-1/2} |\lambda_2(t,r)| + r^{-3/2} |\mu(t,r)| \right)
\]
\[
|\mu(t,r) - \mu(t,r')| \lesssim r^{-\frac{5}{2}(p-3)} \delta_1 \left( r^{-5/2} |\lambda_1(t,r)| + r^{-1/2} |\lambda_2(t,r)| + r^{-3/2} |\mu(t,r)| \right). \tag{7.39}
\]

Next, we recall the equations obtained in Lemma 5.2: we have
\[
\lambda_1(t,r) = 3r^5 u(t,r) - \frac{3}{2} r^3 \int_r^\infty u(t,s) \, ds \tag{7.40}
\]
\[
\lambda_2(t,r) = -2r^3 u(t,r) + \frac{3}{2} r \int_r^\infty u(t,s) \, ds
\]
for all $(t,r) \in \Omega_r = \{ r \geq R + |t| \}$. Moreover, adding the formulas for $\lambda_1$ and $\lambda_2$ we may express $u(t,r)$ as
\[
u(t,r) = r^{-5} \left( \lambda_1(t,r) + r^2 \lambda_2(t,r) \right). \tag{7.41}
\]
As a result of (7.41)–(7.40), we obtain the following formula that relates the difference in $\lambda_1(t,r)$ at different times to that of $\lambda_2(t,r)$.

**Lemma 7.10.** For every $t_1 \neq t_2$, we have
\[
(\lambda_1(t_1,r) - \lambda_1(t_2,r)) = -\frac{3}{2} r^2 (\lambda_2(t_1,r) - \lambda_2(t_2,r)) + \frac{5}{2} \int_{t_1}^{t_2} \mu(t,r) \, dt \tag{7.42}
\]
provided that $(t_1,r), (t_2,r) \in \Omega_R$.

**Proof.** Using (7.40), we write
\[
\frac{1}{3r^5} (\lambda_1(t_1,r) - \lambda_1(t_2,r)) = u(t_1,r) - u(t_2,r) - \frac{1}{2r^2} \int_r^\infty (u(t_1,s) - u(t_2,s)) \, s \, ds
\]
\[
= u(t_1,r) - u(t_2,r) - \frac{1}{2r^2} \int_{t_2}^{t_1} \int_r^\infty u_1(t,s) s \, ds \, dt
\]
\[
= u(t_1,r) - u(t_2,r) - \frac{1}{6r^5} \int_{t_2}^{t_1} \mu(t,r) \, dt.
\]
Similarly, we get
\[-\frac{1}{2r^3} (\lambda_2(t_1, r) - \lambda_2(t_2, r)) = u(t_1, r) - u(t_2, r) - \frac{3}{r^2} \int_r^\infty (u(t_1, s) - u(t_2, s)) \, ds \]
\[= u(t_1, r) - u(t_2, r) - \frac{1}{r^2} \int_{t_2}^{t_1} \mu(t, r) \, dt.\]
The equation (7.42) then follows from combining the last two equations. \[\square\]

**Lemma 7.11.** Let $\epsilon > 0$ be a small number. For all $t \in \mathbb{R}$, we have
\[|\lambda_1(t, r)| \lesssim r^{1+\epsilon}, \quad |\lambda_2(t, r)| \lesssim 1, \quad |\mu(t, r)| \lesssim r^\epsilon \quad (7.43)\]
where the implicit constant $C$ depends on $\epsilon$.

**Proof.** Let $\epsilon > 0$ be given. We fix $r_0 > R_1$ and set
\[r = 2^n r_0, \quad r' = 2^{n+1} r_0, \quad n \in \mathbb{N} \setminus \{0\}\]
in Corollary 7.9. We then obtain for all $t \in \mathbb{R}$,
\[|\lambda_1(t, 2^{n+1} r_0)| \leq \left( 1 + \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{7}{2})}} \right) |\lambda_1(t, 2^n r_0)| + \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{7}{2})}} |\lambda_2(t, 2^n r_0)| + \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{5}{2})}} |\mu(t, 2^n r_0)|\]
\[|\lambda_2(t, 2^{n+1} r_0)| \leq \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-1)}} |\lambda_1(t, 2^n r_0)| + \left( 1 + \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{7}{2})}} \right) |\lambda_2(t, 2^n r_0)| + \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{5}{2})}} |\mu(t, 2^n r_0)|\]
and
\[|\mu(t, 2^{n+1} r_0)| \leq \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{7}{2})}} |\lambda_1(t, 2^n r_0)| + \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{5}{2})}} |\lambda_2(t, 2^n r_0)| + \left( 1 + \frac{C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{5}{2})}} \right) |\mu(t, 2^n r_0)|.\]

Setting
\[H_n := \frac{|\lambda_1(t, 2^n r_0)|}{(2^n r_0)^2} + |\lambda_2(t, 2^n r_0)| + \frac{|\mu(t, 2^n r_0)|}{2^n r_0}\]
we deduce that
\[H_n \leq \left( 1 + \frac{3C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{7}{2})}} \right) H_n \]
\[\leq \left( 1 + \frac{3C \delta_1}{(2^n r_0)^{\frac{3}{2}(p-\frac{5}{2})}} \right)^{n+1} H_0. \quad (7.44)\]

We then pick $\delta_1 > 0$ sufficiently small that
\[1 + 3C \delta_1 < 2^\epsilon\]
and the inequality (7.44) above yields that
\[H_n \leq C (2^n r_0)^\epsilon.\]
Note that the constant $C$ depends only on $H_0$, which is uniformly bounded for all $t \in \mathbb{R}$ by (7.35) once $r_0 > R_1$ is fixed. Keeping this in mind, we get

$$\frac{|\lambda_1(t, 2^n r_0)|}{(2^n r_0)^2} + \frac{|\lambda_2(t, 2^n r_0)|}{2^n r_0} + \frac{|\mu(t, 2^n r_0)|}{2^n r_0} \lesssim (2^n r_0)^\epsilon. \quad (7.45)$$

Next we use these estimates in difference inequalities (7.36)–(7.38) individually. Starting with $\mu(t, r)$, we plug in the estimates for $\lambda_1(t, r), \lambda_2(t, r)$ above into (7.38). Recalling (7.35) and that $p \geq 3$ we get

$$|\mu(t, 2^{n+1} r_0) - \mu(t, 2^n r_0)| \lesssim \frac{(2^n r_0)^{2p+pe}}{(2^n r_0)^5 p - 6} + \frac{(2^n r_0)^{pe}}{(2^n r_0)^3 p - 6} + \frac{|\mu(t, 2^n r_0)|^p}{(2^n r_0)^{4p - 6}}$$

$$\lesssim (2^n r_0)^{pe - 3(p - 2)} + C \delta_1 |\mu(t, 2^n r_0)| \left(\frac{2^n r_0}{2^n r_0} \right)^{2(p - \frac{2}{3})}$$

which yields

$$|\mu(t, 2^{n+1} r_0)| \lesssim (1 + C \delta_1)|\mu(t, 2^n r_0)| + (2^n r_0)^{pe - 3(p - 2)}$$

$$\lesssim 2^\epsilon |\mu(t, 2^n r_0)| + (2^n r_0)^{pe - 3(p - 2)} \quad (7.46)$$

Iterating (7.46) we obtain the improved bound

$$|\mu(t, 2^n r_0)| \lesssim (2^n r_0)^\epsilon. \quad (7.47)$$

Next, we refine the growth rate of $\lambda_2(t, r)$. The difference estimate (7.37) combined with (7.45) yields

$$|\lambda_2(t, 2^{n+1} r_0) - \lambda_2(t, 2^n r_0)| \lesssim \frac{|\lambda_1(t, 2^n r_0)|^p}{(2^n r_0)^5 p - 5} + \frac{|\lambda_2(t, 2^n r_0)|^p}{(2^n r_0)^3 p - 5} + \frac{|\mu(t, 2^n r_0)|^p}{(2^n r_0)^{4p - 5}}$$

$$\lesssim \frac{(2^n r_0)^{pe}}{(2^n r_0)^{3p - 5}}$$

which can be iterated as above

$$|\lambda_2(t, 2^{n+1} r_0)| \lesssim |\lambda_2(t, r_0)| + \sum_{k=0}^n \frac{1}{(2^k r_0)^{1+\eta}} \quad (7.48)$$

for some positive number $\eta$. As the right hand side of (7.48) is uniformly bounded in $t$ and $n$, we deduce that

$$|\lambda_2(t, 2^n r_0)| = O(1) \quad (7.49)$$

where the implicit constant may depend on the fixed radius $r_0$.

Using (7.47) and (7.49) we may also improve the growth rate of $\lambda_1(t, r)$. Once again revisiting the difference inequality (7.36) we write

$$\frac{|\lambda_1(t, 2^{n+1} r_0)|}{(2^{n+1} r_0)} \lesssim \frac{|\lambda_1(t, 2^n r_0)|}{(2^n r_0)} + C \delta_1 \frac{|\lambda_1(t, 2^n r_0)|}{(2^n r_0)^{2(p - \frac{2}{3})}} + \frac{1}{(2^n r_0)^{3p - 6}}$$

Similarly, we iterate the inequality above to obtain

$$|\lambda_1(t, 2^n r_0)| \lesssim (2^n r_0)^{1+\epsilon}.$$ 

Finally, combining these growth estimates with the difference estimates in Lemma 7.8, we obtain the result for arbitrary $r > R_1$.

\begin{lemma}
There exist a uniformly bounded function $\ell_2(t)$ such that

$$|\lambda_2(t, r) - \ell_2(t)| = O \left( r^{-3p+5} \right) \quad \text{as } r \to \infty \quad (7.50)$$

uniformly in $t \in \mathbb{R}$.
\end{lemma}
Moreover, we remark that both \( \lim_{n \to \infty} \lambda_2(t, 2^n r_0) \) exists for every \( t \in \mathbb{R} \). Let
\[
\ell_2(t) := \lim_{n \to \infty} \lambda_2(t, 2^n r_0).
\]

This implies that
\[
\sum_n |\lambda_2(t, 2^{n+1} r_0) - \lambda_2(t, 2^n r_0)| < \infty
\]
as a result, we deduce that \( \lim_{n \to \infty} \lambda_2(t, 2^n r_0) \) exists for every \( t \in \mathbb{R} \). Let
\[
\ell_2(t) := \lim_{n \to \infty} \lambda_2(t, 2^n r_0).
\]

Moreover,
\[
|\ell_2(t) - \lambda_2(t, r_0)| = \lim_{n \to \infty} |\lambda_2(t, 2^n r_0) - \lambda_2(t, r_0)|
\leq \lim_{n \to \infty} \sum_{k=1}^n |\lambda_2(t, 2^k r_0) - \lambda_2(t, 2^{k+1} r_0)|
\lesssim \frac{1}{r_0^{3p-5}} \sum_{k=1}^\infty \left(2^k\right)^{-3p+5}.
\]

One more application of difference estimate for \( \lambda_2(t, r) \) results in the asymptotic estimate
\[
|\ell_2(t) - \lambda_2(t, r)| = O\left(r^{-3p+5}\right) \quad \text{as} \quad r \to \infty.
\]

We also remark that both \( \ell_2(t) \) and \( \lambda_2(t, r) \) are uniformly bounded in \( t \).

Combining the \( \epsilon \)-growth estimates for \( \lambda_1(t, r) \) and \( \lambda_2(t, r) \) with the expansion formula for \( u \) as given in (7.41), we obtain the following result.

**Lemma 7.13.** The following holds uniformly in time:
\[
r^3u(t, r) = \ell_2(t) + O\left(r^{-1+\epsilon}\right).
\]

**Lemma 7.14.** The limit \( \ell_2(t) \) is independent of time.

**Proof.** The result follows from the equality (7.42) in Lemma 7.10. We take \( t_1 \) and \( t_2 \neq t_1 \) and check the difference of \( \ell_2(t_1) \) and \( \ell_2(t_2) \). By (7.50) and (7.42),
\[
|\ell_2(t_1) - \ell_2(t_2)| = |\lambda_2(t_1, r) - \lambda_2(t_2, r)| + O\left(r^{-3p+5}\right)
\leq \frac{2}{3} r^{-2} |\lambda_1(t_1, r) - \lambda_1(t_2, r)| + \frac{5}{3} r^{-2} \int_{t_2}^{t_1} |\mu(t, r)| \, dt
+ O\left(r^{-3p+5}\right)
\lesssim |t_1 - t_2| O\left(r^{-2+\epsilon}\right) + O\left(r^{-1+\epsilon}\right) + +O\left(r^{-3p+5}\right).
\]
The last step above follows from the \( \epsilon \)-growth estimates in Lemma 7.11. Letting \( r \to \infty \), we arrive at the conclusion that \( \ell_2(t_1) = \ell_2(t_2) \).

From here on, we will denote the limit by \( \ell_2 \).

**Lemma 7.15.** The limit \( \ell_2 = 0 \).
PROOF. We consider the term
\[
\frac{1}{R} \int_R^{2R} (\mu(t_1, r) - \mu(t_2, r)) dr.
\] (7.55)
Using the equation (1.1), we rewrite this term and split it into two parts.
\[
\frac{1}{R} \int_R^{2R} (\mu(t_1, r) - \mu(t_2, r)) dr = \frac{1}{R} \int_R^{2R} 3r^3 \int_r^{\infty} (u(t_1, s) - u(t_2, s)) ds dr
\]
\[
= \frac{1}{R} \int_R^{2R} 3r^3 \int_r^{\infty} \int_{t_2}^{t_1} u_{tt}(t, s) s ds dr
\]
\[
= \frac{1}{R} \int_R^{2R} 3r^3 \int_r^{\infty} \int_{t_2}^{t_1} \left( u_{ss}(t, s) + \frac{6}{s} u_s(t, s) + |u|^{p-1} u(t, s) \right) s dt ds dr
\]
\[
= \frac{1}{R} \int_{t_2}^{t_1} \int_R^{2R} 3r^3 \int_r^{\infty} \left( u_{ss}(t, s) + \frac{6}{s} u_s(t, s) \right) s dt ds dr
\]
\[
+ \frac{1}{R} \int_{t_2}^{t_1} \int_R^{2R} 3r^3 \int_r^{\infty} |u|^{p-1} u(t, s) ds dr dt
\]
\[
= I + II.
\]
By integration by parts, we may write \(I\) as
\[
I = - \int_{t_2}^{t_1} \frac{1}{R} \int_R^{2R} 18u(t, r)r^3 dr dt - \int_{t_2}^{t_1} \frac{1}{R} \int_R^{2R} 3r^3 \int_r^{\infty} u_s(t, s) ds dr dt
\]
\[
+ \int_{t_2}^{t_1} \frac{1}{R} \int_R^{2R} 3r^3 (u_r(t, r) r) dr dt
\]
\[
= \int_{t_2}^{t_1} \frac{27}{R} \int_R^{2R} u(t, r)r^3 dr dt + \int_{t_2}^{t_1} \frac{3}{R} (u(t, 2R)(2R)^4 - u(t, R)R^4) dt
\]
In the calculation above, we used the fact that \(\lim_{r \to \infty} u(t, r) = 0\) and \(\lim_{r \to \infty} u_r(t, r) r = 0\). Using the pointwise bounds in Lemma 7.13, we see that
\[
|I| = 27|t_1 - t_2||\|L\|_2| + |t_1 - t_2|O(R^{-1+\epsilon}) - 2|t_1 - t_2||\|L\|_2| + |t_1 - t_2|O(R^{-1+\epsilon})
\]
\[
= 25|t_1 - t_2||\|L\|_2| + |t_1 - t_2|O(R^{-1+\epsilon}) \quad (7.56)
\]
Similarly, we may employ the bounds in Lemma 7.14 to obtain
\[
|II| = |t_1 - t_2|O(R^{-10}) \quad (7.57)
\]
Adding the estimates (7.56) and (7.57), we control the difference in (7.55) by
\[
\frac{1}{R} \int_R^{2R} (\mu(t_1, r) - \mu(t_2, r)) dr = 25|t_1 - t_2||\|L\|_2| + |t_1 - t_2|O(R^{-1+\epsilon}) \quad (7.58)
\]
We take a closer look at the equation above. Since the \(c\)-growth rate for \(\mu(t, r)\) holds uniformly in time, we deduce that
\[
\frac{1}{R} \int_R^{2R} (\mu(t_1, r) - \mu(t_2, r)) dr = O(R^c)
\]
for all \(t \in \mathbb{R}\) uniformly. Assuming that \(\ell_2 \neq 0\), we may select and fix \(R \geq r_0\) large enough so that
\[
|t_1 - t_2||\|L\|_2| \leq CR^c \quad (7.59)
\]
for some \(C > 0\). Letting \(|t_1 - t_2| \to \infty\) we obtain a contradiction. Therefore, we must have \(\ell_2 = 0\). \(\square\)

Equation (7.58) may be used to derive further conclusions on \(\mu(t, r)\). Firstly, we will study the asymptotic behaviour of \(\mu(t, r)\).
Lemma 7.16. There exists a number $\rho$ such that
\[
|\mu(t, r) - \rho| = O\left(r^{-4p+6+pe}\right) \quad \text{as} \quad r \to \infty. \tag{7.60}
\]
uniformly in $t \in \mathbb{R}$.

Proof. The proof of this lemma is very similar to that of Lemma 7.12. Fixing $\epsilon > 0$ and $r_0$ as in the proof of Lemma 7.11, we recall the growth rates for $\lambda_1(t, r)$ and $\mu(t, r)$ given in (7.43) and note the improved asymptotics for $\lambda_2(t, r)$ below: by Lemma 7.15 we have
\[
|\lambda_2(t, r)| = O\left(r^{-3p+5}\right) \quad \text{as} \quad r \to \infty
\]
uniformly for all $t \in \mathbb{R}$. We then select $r_1 > r_0$ if necessary, and apply these estimates to the difference inequality for $\mu(t, r)$ to get
\[
|\mu(t, 2^{n+1}r_1) - \mu(t, 2^n r_1)| \lesssim \frac{(2^n r_1)^{p+pe}}{(2^n r_1)^{p-6}} + \frac{(2^n r_1)^{p(3p-5)}}{(2^n r_1)^{3p-6}} + \frac{(2^n r_1)^{pe}}{(2^n r_1)^{4p-6}} \tag{7.61}
\]
Repeating the same strategy as done in the proof of Lemma 7.12 we deduce that $\lim_{n \to \infty} \mu(t, 2^n r_1) =: \rho(t)$ exists for all $t \in \mathbb{R}$. Moreover, the upper bound on the right hand side of (7.61) combined with the difference inequality (7.38) yields the asymptotic rates
\[
|\mu(t, r) - \rho(t)| = O\left(r^{-4p+6+pe}\right).
\]
Lastly, as demonstrated in Lemma 7.12 for $\lambda_2(t, r)$ the inequality (7.61) implies that $\mu(t, r)$ as well as $\rho(t)$ are uniformly bounded in $t \in \mathbb{R}$.

Now, we go back to the equation (7.58) and take a second look with the fact that $\ell_2 = 0$. We find that for $t_1 \neq t_2$
\[
|\rho(t_1) - \rho(t_2)| = \frac{1}{R} \left| \int_{R}^{2R} \left( \mu(t_1, r) - \mu(t_2, r) \right) dr \right| + O\left(R^{-4p+6+pe}\right) \tag{7.62}
\]
We take the limit $R \to \infty$ in (7.62) and obtain $\rho(t_1) = \rho(t_2)$ for $t_1 \neq t_2$. \qed

Lemma 7.17. The limit $\rho = 0$.

Proof. Using (7.60) and recalling the definition of $\mu(t, r)$ as given in (5.7), we express
\[
3R^3 \int_{R}^{\infty} \int_{0}^{\infty} u_t(t, r) rdr = \rho + O\left(R^{-4p+6+pe}\right). \tag{7.63}
\]
Selecting $R > 0$ sufficiently large we may guarantee that $3R^3 \int_{R}^{\infty} \int_{0}^{\infty} u_t(t, r) rdr$ and $\rho$ share the same sign, and obtain
\[
\left| 3R^3 \int_{R}^{\infty} u_t(t, r) rdr \right| > \frac{1}{2} |\rho|.
\]
Next, we integrate the equation (7.63) from 0 to $T$, which yields
\[
\left| \int_{0}^{T} 3R^3 \int_{R}^{\infty} u_t(t, r) rdr \right| > \frac{T}{2} |\rho|. \tag{7.64}
\]
Changing the order of the integral on the left hand side we use the asymptotic estimate in Lemma 7.13 once again. We note the change due to $\ell_2 = 0$. As a result, we obtain a uniform in time control of the left hand side of (7.64)
\[
\left| \int_{0}^{T} 3R^3 \int_{R}^{\infty} u_t(t, r) rdr \right| = \left| 3R^3 \int_{R}^{\infty} (u(T) - u(0, )) s ds dt \right| \lesssim R^{1+\epsilon}
\]
which yields

\[ \frac{T}{2} |\rho| \lesssim R^{1+\epsilon}. \]

As we may take \( T \to \infty \), we find that \( \rho \) must be zero.

Recall that by Lemma 7.15 and Lemma 7.17, the asymptotic decay rates given in Lemma 7.12 became

\[
|\lambda_2(t, r)| = O \left( r^{-3p+5} \right) \quad \text{as} \quad r \to \infty
\]

\[
|\mu(t, r)| = O \left( r^{-4p+6+p\epsilon} \right) \quad \text{as} \quad r \to \infty.
\]

(7.65)

Lastly, we check the asymptotic decay rate of the leading coefficient \( \lambda_1(t, r) \). As we can see from the statement of Proposition 7.5, it will be sufficient to obtain this result at time \( t = 0 \). For that reason, we simplify the notation and denote by

\[
\lambda_1(r) := \lambda_1(0, r)
\]

\[
\lambda_2(r) := \lambda_2(0, r)
\]

\[
\mu(r) := \mu(0, r).
\]

(7.66)

**Lemma 7.18.** There exists \( \ell \in \mathbb{R} \) so that

\[
|\lambda_1(r) - \ell| = O \left( r^{-5p+7} \right) \quad \text{as} \quad r \to \infty.
\]

(7.67)

**Proof.** Going back to the difference equation (7.36) for \( \lambda_1 \), and utilizing the decay rates in (7.65) and the \( r^{1+\epsilon} \)-growth of \( \lambda_1(r) \) in (7.43), we obtain

\[
|\lambda_1(2^{n+1}r_0) - \lambda_1(2^nr_0)| \lesssim (2^nr_0)^{-4p+7+p\epsilon} + (2^nr_0)^{-3p^2+2p+7} + (2^nr_0)^{-4p^2+2p+7+p\epsilon}
\]

(7.68)

where \( n \) is any positive integer, and \( r_0 \) is a fixed positive integer selected as in the proof of Lemma 7.11. Following the arguments in the proof of Lemma 7.12 we deduce that

\[
\sum_{n=0}^{\infty} |\lambda_1(2^{n+1}r_0) - \lambda_1(2^nr_0)| < \infty
\]

which leads to the limit \( \ell := \lim_{n \to \infty} \lambda_1(2^nr_0) \). Using the boundedness of the term \( |\lambda_1(2^nr_0)| \) we update the right hand side of (7.68). Improving those bounds, we arrive at the conclusion (7.67) where the asymptotic decay rate is denoted by the exponent \( \alpha_1 \).

Now, we run the difference inequalities (7.35)–(7.38) as many times as needed to obtain the maximal decay rates for the coefficients \( \lambda_1(r), \lambda_2(r), \) and \( \mu(r) \).

Starting with \( \lambda_2(r) \), by (7.37) we see that

\[
|\lambda_2(r)| = O \left( r^{-5p+5} \right) \quad \text{as} \quad r \to \infty.
\]

(7.69)

Similarly, the inequality (7.38) yields

\[
|\mu(r)| = O \left( r^{-5p+6} \right) \quad \text{as} \quad r \to \infty.
\]

(7.70)

Finally, using these improved decay rates in (7.36) we get

\[
|\lambda_1(r) - \ell| = O \left( r^{-5p+7} \right) \quad \text{as} \quad r \to \infty.
\]

(7.71)

**Proof of Proposition 7.5.** Having refined the decay rates for the projection coefficients at \( t = 0 \), we complete the proof of Proposition 7.5 by combining Lemma 7.18 with the identities (7.41) and (5.7). \( \square \)
7.3. Step 3. Here, we show that $\vec{u}(t) \equiv (0,0)$ and close the proof of Proposition 3.13. Recall that in the previous step, we established the asymptotic rates
\[ r^n u_0(r) = \ell + O(r^{-5p+7}) \quad \text{as} \quad r \to \infty \]
\[ \int_r^\infty u_1(s) \, ds = O(r^{-5p+3}) \quad \text{as} \quad r \to \infty. \]

We consider the cases $\ell = 0$ and $\ell \neq 0$ separately.

**Lemma 7.19.** Let $\vec{u}(t)$ and $\ell$ be as in Proposition 7.5. Suppose $\ell = 0$. Then $\vec{u}(0) = (u_0,u_1)$ is compactly supported.

**Proof.** Assume that $\ell = 0$. Then, we get
\[ |\lambda_1(r)| \lesssim \frac{1}{r^{5p-7}}, \quad |\lambda_2(r)| \lesssim \frac{1}{r^{5p-5}}, \quad |\mu(r)| \lesssim \frac{1}{r^{5p-6}} \quad (7.72) \]
for $r \geq R_1$. Taking $r_0 \geq R_1$, we have
\[ |\lambda_1(2^n r_0)| + |\lambda_2(2^n r_0)| + |\mu(2^n r_0)| \lesssim (2^n r_0)^{-5p+7} \quad (7.73) \]
for every $n$. On the other hand, the difference estimates in Corollary 7.9 yield
\[ |\lambda_1(2^{n+1} r_0)| \geq \left( 1 - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} \right) |\lambda_1(2^n r_0)| \]
\[ - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} |\lambda_2(2^n r_0)| \]
\[ - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} |\mu(2^n r_0)| \quad (7.74) \]
\[ |\lambda_2(2^{n+1} r_0)| \geq \left( 1 - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} \right) |\lambda_2(2^n r_0)| \]
\[ - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} |\lambda_1(2^n r_0)| \]
\[ - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} |\mu(2^n r_0)| \]
\[ |\mu(2^{n+1} r_0)| \geq \left( 1 - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} \right) |\mu(2^n r_0)| \]
\[ - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} |\lambda_1(2^n r_0)| \]
\[ - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} |\lambda_2(2^n r_0)|. \]

Then, setting $\delta_1 > 0$ small enough that $C_1 \delta_1 / r_0^8 < 1/4$, we iterate the lower bounds above to get
\[ (|\lambda_1(2^{n+1} r_0)| + |\lambda_2(2^{n+1} r_0)| + |\mu(2^{n+1} r_0)|) \]
\[ \geq \left( 1 - C_1 \delta_1 (2^n r_0)^{-\frac{5}{2}(p-\frac{2}{5})} \right) (|\lambda_1(2^n r_0)| + |\lambda_2(2^n r_0)| + |\mu(2^n r_0)|) \]
\[ \geq (3/4)^{n+1} (|\lambda_1(r_0)| + |\lambda_2(r_0)| + |\mu(r_0)|). \]

Combining (7.73) and (7.77) yields
\[ (|\lambda_1(r_0)| + |\lambda_2(r_0)| + |\mu(r_0)|) \lesssim \frac{4^n}{(3 \cdot 2^n)^{(p-\frac{2}{5})} r_0^\frac{5}{2}(p-\frac{2}{5})} \]
for every $n \in \mathbb{N}$, which leads to
\[ |\lambda_1(r_0)| = |\lambda_2(r_0)| = |\mu(r_0)| = 0 \]
as $p \geq 3$. It then follows from (5.12) and Lemma 7.2 that
\[ \| \pi_{r_0} \vec{u}(0) \|_{H(r \geq r_0)}^2 \lesssim r_0^{-5(p-\frac{2}{5})} \| \pi_{r_0} \vec{u}(0) \|_{H^1(r \geq r_0)}^{10} = 0. \]
Therefore,
\[ \| \bar{u}(0) \|_{H(r \geq r_0)} = 0. \]
In other words \((\partial_r u_0, u_1)\) is compactly supported. Since we have
\[ \lim_{r \to \infty} u_0(r) = 0 \]
we may conclude that \(\bar{u}(0)\) is compactly supported. \(\square\)

**Lemma 7.20.** Let \(\bar{u}(t)\) and \(\ell\) be as in Proposition 7.5. Suppose \(\ell = 0\). Then \(\bar{u}(0) = (0, 0)\).

**Proof.** Assuming \(\ell = 0\), we deduce from Lemma 7.19 that the initial data \((u_0, u_1)\) must be compactly supported. Furthermore, if \((u_0, u_1) \neq (0, 0)\), then there exists a positive radius \(\rho_0\) such that
\[ \rho_0 := \inf \{ \rho : \| \bar{u}(0) \|_{H(r \geq \rho)} = 0 \}. \]
Let \(\delta_1\) be as in (7.33)–(7.35). Additionally, we take a small number \(\epsilon > 0\), to be determined below, and find \(\rho_1 = \rho_1(\epsilon)\) with \(\frac{1}{2} \rho_0 < \rho_1 < \rho_0\) such that
\[ 0 < \| \bar{u}(0) \|_{H(r \geq \rho_1)} < \epsilon \leq \delta_1. \]
By Lemma 5.3, we have
\[ \| \bar{u}(0) \|_{H(r \geq \rho)}^2 \lesssim \frac{5\lambda_2^3(R)}{R^5} + \frac{9\lambda_2^3(R)}{R} + \frac{10\lambda_1(R)\lambda_2(R)}{R^3} + \frac{\mu^2(R)}{3R^3} \]
(7.78)
and
\[ \int_R^\infty \left( (\partial_r \lambda_1(r)r^{-2})^2 + (\partial_r \lambda_2(r))^2 + (\partial_r \mu(r)r^{-1})^2 \right) \, dr. \]
Note that by setting \(R = \rho_0\) above we get
\[ \lambda_1(\rho_0) = \lambda_2(\rho_0) = \mu(\rho_0) = 0. \]
Also, by Lemma 7.6 we may bound the integral on the right hand side above as follows:
\[ \int_R^\infty (\partial_r \lambda_1(r)r^{-2})^2 + (\partial_r \lambda_2(r))^2 + (\partial_r \mu(r)r^{-1})^2 \, dr \]
\[ \lesssim \frac{1}{R^{6(p-\frac{1}{2})}} \left( \frac{\lambda_2(\rho)}{R^p} + \frac{\lambda_2(\rho)}{R^p} + \frac{\mu^2(\rho)}{R^p} \right). \]
(7.80)
We then argue as in the proofs of Lemma 7.8 and Corollary 7.9, and estimate the differences
\[ |\lambda_1(\rho_1) - \lambda_1(\rho_0)| \lesssim \rho_1^{-\frac{1}{2}(p-2)} \epsilon \left( \rho_1^{-5/2} |\lambda_1(\rho_1)| + \rho_1^{-1/2} |\lambda_2(\rho_1)| + \rho_1^{-3/2} |\mu(\rho_1)| \right) \]
\[ |\lambda_2(\rho_1) - \lambda_2(\rho_0)| \lesssim \rho_1^{-\frac{1}{2}(p-2)} \epsilon \left( \rho_1^{-5/2} |\lambda_1(\rho_1)| + \rho_1^{-1/2} |\lambda_2(\rho_1)| + \rho_1^{-3/2} |\mu(\rho_1)| \right) \]
\[ |\mu(\rho_1) - \mu(\rho_0)| \lesssim \rho_1^{-\frac{1}{2}(p-2)} \epsilon \left( \rho_1^{-5/2} |\lambda_1(\rho_1)| + \rho_1^{-1/2} |\lambda_2(\rho_1)| + \rho_1^{-3/2} |\mu(\rho_1)| \right). \]
(7.81)
Next, we set
\[ H = |\lambda_1(\rho_1)| + |\lambda_2(\rho_1)| + |\mu(\rho_1)|. \]
Recalling (7.79) and the fact that \(\frac{1}{2} \rho_0 < \rho_1 < \rho_0\) we may rewrite equation (7.81) as
\[ H \leq C \epsilon H \]
where the constant \(C\) depends only on \(\rho_0\) and the uniform implicit constant in (7.81) due to \(\lesssim\). As \(\rho_0\) is fixed, we may select \(\epsilon \in (0, C^{-1})\) and deduce that \(H = 0\). By setting \(R = \rho_1\) in (7.78) and (7.80) we find that
\[ \| \bar{u}(0) \|_{H(r \geq \rho_1)} = 0. \]
However, this leads to a contradiction as $\rho_1 < \rho_0$. \hfill $\square$

**Lemma 7.21.** Let $\mathcal{V}(t)$ and $\ell$ be as in Proposition 7.5. Then, $\ell = 0$.

In order to prove Lemma 7.21 we show that the case $\ell \neq 0$ leads to a contradiction. If the limit $\ell$ is nonzero, then we may consider the difference $\mathcal{V}(t) - (Z_\ell, 0)$, where $Z_\ell$ is the corresponding stationary solution constructed in Proposition 6.1. Below, we will argue that the results we obtained in Step 2 leads to $u(t, r) = Z_\ell(r)$, which gives us contradiction since $Z_\ell \notin \dot{H}^s(\mathbb{R}^7)$.

Recalling (6.1)–(6.2), we define $\omega_\ell(0) = (\omega_{\ell,0}, \omega_{\ell,1})$ by

$$
\omega_{\ell,0} := u_0(r) - Z_\ell(r) \\
\omega_{\ell,1} := u_1(r)
$$

(7.82)

and we consider for all $t \in \mathbb{R}$

$$
\mathcal{V}_\ell(t) = (\omega_\ell(t, r), \partial_r \omega_\ell(t, r)) \\
:= (u(t, r) - Z_\ell(r), \partial_r u(t, r)).
$$

(7.83)

Note that we may directly utilize the asymptotic decay rates obtained for $\mathcal{V}(0)$ and $Z_\ell$ and estimate

$$
r^5 \omega_{\ell,0}(r) = O(r^{-5p+\gamma}) \quad \text{as} \quad r \to \infty
$$

$$
\int_r^\infty \omega_{\ell,1}(\rho)\rho d\rho = O(r^{-5p+\gamma}) \quad \text{as} \quad r \to \infty.
$$

(7.84)

Next, we check the equation for $\mathcal{V}_\ell(t, r)$. Since $u$ and $Z_\ell$ are solutions to (1.1) and (6.1) respectively, we get

$$
\partial_t \omega_\ell - \partial_{rr} \omega_\ell - \frac{6}{r} \partial_r \omega_\ell = |\omega_\ell + Z_\ell|^{p-1}(\omega_\ell + Z_\ell) - |Z_\ell|^{p-1}Z_\ell
$$

(7.85)

As $Z_\ell$ is stationary, $\mathcal{V}_\ell$ verifies the latter conclusion of Corollary 7.1, i.e., we simply get

$$
\lim_{t \to +\infty} \|\mathcal{V}_\ell(t)\|_{\mathcal{H}(r \geq R + |t|)} = \lim_{t \to -\infty} \|\mathcal{V}_\ell(t)\|_{\mathcal{H}(r \geq R + |t|)} = 0.
$$

(7.86)

**Lemma 7.22.** Suppose $\ell \neq 0$, and let $\mathcal{V}_\ell(t)$ be defined as in (7.83). Then, we must have $\mathcal{V}_\ell(0) \equiv (0, 0)$.

The proof of Lemma 7.22 follows from the same line of arguments presented in the first two steps. Firstly, as done in Step 1, we will obtain an analogous version of Lemma 7.2 and express that in terms of projection coefficients of $\mathcal{V}_\ell(t)$, which will then lead to corresponding difference estimates. As we already established the asymptotic decay of $\mathcal{V}_\ell(0)$ in (7.84), we will close the proof by showing that $\mathcal{V}_\ell(0)$ must be compactly supported. Below, we will outline how to adapt the results of Step 1 and Step 2 for $\mathcal{V}_\ell(t)$.

In order to prove a version of the estimate (7.7), we take a second look at the Cauchy problem in Lemma 7.4. Following the set-up in (7.8), we define $V(t, x) = \chi\left(\frac{x}{R_0 + |t|}\right)Z_\ell(x)$ for some large $\tilde{R}_0 > 0$.

Then, $V$ satisfies the assumptions of Lemma 7.4 with $I = \mathbb{R}$ and $r_0 = \tilde{R}_0$. Letting $r_1 > \tilde{R}_0$ such that

$$
\|\mathcal{V}_\ell(0)\|_{\mathcal{H}(r \geq r_1)} \leq 0
$$

we obtain

$$
\sup_{t \in \mathbb{R}} \|\mathcal{V}_\ell(t) - S(t)(\omega_{\ell,0}, \omega_{\ell,1})\|_{\mathcal{H}} \leq \frac{1}{100} \|\omega_{\ell,0}, \omega_{\ell,1}\|_{\mathcal{H}}.
$$

(7.87)

Having obtained the estimate (7.87) above, we proceed to adjust the result of Lemma 7.2.

**Lemma 7.23.** There exists $\tilde{R}_0 > 0$ such that for all $R > \tilde{R}_0$ we have

$$
\|\pi_R\mathcal{V}_\ell(t)\|_{\mathcal{H}(r \geq R)} \leq \frac{1}{100} \|\pi_R\mathcal{V}_\ell(t)\|_{\mathcal{H}(r \geq R)}.
$$

(7.88)
We omit the proof of Lemma 7.23 since it is identical to the proof of Lemma 7.2. Namely, we follow the same procedure and use the estimate (7.87) instead of (7.13), which leads to the power change on the right hand side of (7.88).

Let us remind that the orthogonal projections in (7.88) will be of the following form:

\[
\pi_R \tilde{\omega}_\ell(t, r) = (\lambda_{\ell,1}(t, R)r^{-5} + \lambda_{\ell,2}(t, R)r^{-3}, \mu_{\ell}(t, R)r^{-5})
\]

\[
\pi_R^{(1)} \tilde{\omega}_\ell(t, r) = (\omega_{\ell}(t, r) - \lambda_{\ell,1}(t, R)r^{-5} - \lambda_{\ell,2}(t, R)r^{-3}, \partial_t \omega_{\ell}(t, R) - \mu_{\ell}(t, R)r^{-5}) .
\]

(7.89)

We define \( \lambda_{\ell,1}(t, R) \) and \( \lambda_{\ell,2}(t, R) \) using the formulas in (5.10). Note that these projection coefficients must be adapted to \( \tilde{\omega}_\ell(t) \). However, since \( \partial_t \omega_{\ell}(t, r) = \partial_t u(t, r) \), the formula (5.7) gives us \( \mu_{\ell}(t, R) = \mu(t, R) \). We refer the reader to Section 5.1 for a comparison.

Recalling the decay rates of \((\omega_{\ell,0}(r), \omega_{\ell,1}(r))\) in (7.84), we immediately deduce the asymptotics for \( \lambda_{\ell,1}(r) \) and \( \lambda_{\ell,2}(r) \). Namely, we get

\[
|\lambda_{\ell,1}(r)| = O(r^{-5p+7}) \quad \text{as} \quad r \to \infty
\]

\[
|\lambda_{\ell,2}(r)| = O(r^{-5p+5}) \quad \text{as} \quad r \to \infty.
\]

(7.90)

Also, we have

\[
\mu_{\ell}(r) = \mu(r) = O(r^{-5p+6}) \quad \text{as} \quad r \to \infty.
\]

(7.91)

Next, we apply the exact same arguments in the proof of Lemma 7.19 to prove that \( \tilde{\omega}_\ell(0) \) is compactly supported.

**Lemma 7.24.** Let \((\omega_{\ell,0}, \omega_{\ell,1})\) be as in (7.83). Then \((\partial_t \omega_{\ell,0}, \omega_{\ell,1})\) is compactly supported.

**Proof of Lemma 7.24.** First, we rewrite the estimate (7.88) at \( t = 0 \) in terms of \( \lambda_{\ell,1}(r), \lambda_{\ell,2}(r), \) and \( \mu(r) \). For all \( R > \tilde{R}_0 \), we get

\[
\int_R^\infty (\partial_t \lambda_{\ell,1}(r)r^{-2})^2 + (\partial_t \lambda_{\ell,2}(r))^2 + (\partial_t \mu(r)r^{-1})^2 \, dr
\]

\[
\leq \frac{1}{10^4} \left( \frac{5\lambda_{\ell,1}^2(R)}{R^5} + \frac{9\lambda_{\ell,2}^2(R)}{R} + \frac{10\lambda_{\ell,1}(R)\lambda_{\ell,2}(R)}{R^3} + \frac{\mu^2(R)}{3R^3} \right) .
\]

(7.92)

We argue exactly as in the proof of Lemma 7.8 to obtain the difference estimates from (7.92). For all \( \tilde{R}_0 \leq r \leq r' \leq 2r \),

\[
|\lambda_{\ell,1}(r) - \lambda_{\ell,1}(r')|^2 \leq \frac{r^5}{10^4} \left( \frac{10|\lambda_{\ell,1}(r)|^2}{r^5} + \frac{14|\lambda_{\ell,2}(r)|^2}{r} + \frac{(|\mu(r)|^2)}{3r^3} \right)
\]

(7.93)

and

\[
|\lambda_{\ell,2}(r) - \lambda_{\ell,2}(r')|^2 \leq \frac{r}{10^4} \left( \frac{10|\lambda_{\ell,1}(r)|^2}{r^5} + \frac{14|\lambda_{\ell,2}(r)|^2}{r} + \frac{(|\mu(r)|^2)}{3r^3} \right).
\]

(7.94)

Similarly, for all \( \tilde{R}_0 \leq r \leq r' \leq 2r \) we have

\[
|\mu(r) - \mu(r')|^2 \leq \frac{r^3}{10^4} \left( \frac{10|\lambda_{\ell,1}(r)|^2}{r^5} + \frac{14|\lambda_{\ell,2}(r)|^2}{r} + \frac{(|\mu(r)|^2)}{3r^3} \right).
\]

(7.95)

Next, we define the vector \( H(r) = (\lambda_{\ell,1}(r), \lambda_{\ell,2}(r), \mu(r)) \). Selecting \( r_0 > \tilde{R}_0 \) we combine the inequalities (7.93)–(7.95) to obtain

\[
|H(2^{n+1}r_0) - H(2^nr_0)| \leq \frac{1}{4}|H(2^nr_0)| .
\]
This implies that
\[ |H(2^{n+1}r_0)| \geq \frac{3}{4} |H(2^n r_0)|. \]

Via iteration on \( n \) we deduce
\[ |H(2^n r_0)| \geq \left( \frac{3}{4} \right)^n |H(r_0)|. \] (7.96)

However, the asymptotic decay rates in (7.90)–(7.91) yield
\[ |H(2^n r_0)| \lesssim (2^n r_0)^{-5p+7}. \] (7.97)

By (7.96)–(7.97) we get
\[ 3^n |H(r_0)| \leq 4^n |H(2^n r_0)| \lesssim r_0^{-5p+7} 4^n 4^{-\frac{5n}{2}(p-\frac{7}{5})}. \]

Letting \( n \to \infty \) we deduce that \( H(r_0) = (0,0,0) \). Going back to (7.92) and using the fact that \( \lambda_{\ell,1}(r_0) = \lambda_{\ell,2}(r_0) = \mu(r_0) = 0 \), we obtain
\[ \int_{r_0}^{\infty} \left( \partial_r \lambda_{\ell,1}(r)r^{-2} \right)^2 + \left( \partial_r \lambda_{\ell,2}(r)r^{-1} \right)^2 \, dr = 0. \]

Hence,
\[ \|\tilde{\omega}_\ell(0)\|^2_{\mathcal{H}(r \geq r_0)} = \|\pi_{r_0} \tilde{\omega}_\ell(0)\|^2_{\mathcal{H}(r \geq r_0)} + \|\pi_{r_0} \tilde{\omega}_\ell(0)\|^2_{\mathcal{H}(r \geq r_0)} = 0 \]
which proves that \((\partial_r \omega_{\ell,0}, \omega_{\ell,1})\) is compactly supported. \( \square \)

Finally we proceed with the proof of Lemma 7.22.

**PROOF OF LEMMA 7.22.** We follow the same argument used in the proof of Lemma 7.20. By the way of contradiction, we assume that \((\partial_r \omega_{\ell,0}, \omega_{\ell,1}) \not= (0,0)\), and define
\[ \rho_0 := \inf \{ \rho : \|\tilde{\omega}_\ell(0)\|_{\mathcal{H}(r \geq \rho)} = 0 \}. \] (7.98)

By hypothesis, we get \( \rho_0 > 0 \) and we deduce
\[ \lambda_{\ell,1}(\rho_0) = \lambda_{\ell,2}(\rho_0) = \mu(\rho_0) = 0. \] (7.99)

We then take \( \rho_1 \in (\frac{\rho_0}{2}, \rho_0) \) such that
\[ \|\tilde{\omega}_\ell(0)\|_{\mathcal{H}(r \geq \rho_1)} < \delta_2 < \delta_0 \rho_1^3. \] (7.100)

Above, we select \( \delta_2 \) sufficiently small that (7.87) holds. Thus, the second inequality in (7.100) guarantees that Lemma 7.23 holds with \( R = \rho_1 \). Reformulating that in terms of the projection coefficients, we get
\[ \int_{\rho_1}^{\infty} \left( \partial_r \lambda_{\ell,1}(r)r^{-2}\right)^2 + \left( \partial_r \lambda_{\ell,2}(r)r^{-1}\right)^2 \, dr \]
\[ \leq \frac{1}{10^4} \left( \frac{5 \lambda_{\ell,1}^2(\rho_1)}{\rho_1^5} + \frac{9 \lambda_{\ell,2}^2(\rho_1)}{\rho_1} + \frac{9 \lambda_{\ell,1}(\rho_1) \lambda_{\ell,2}(\rho_1)}{\rho_1^3} + \frac{\mu^2(\rho_1)}{3 \rho_1^3} \right). \] (7.101)

Once again, we use fundamental theorem of calculus to express the difference \( |\lambda_{\ell,i}(\rho_1) - \lambda_{\ell,i}(\rho_0)| \) and \( |\mu(\rho_1) - \mu(\rho_0)| \) in terms of (7.90). We get
\[ |\lambda_{\ell,1}(\rho_1) - \lambda_{\ell,1}(\rho_0)|^2 \leq \frac{(\rho_0^5 - \rho_1^5)}{10^4} \left( \frac{10 |\lambda_{\ell,1}(\rho_1)|^2}{\rho_1^5} + \frac{14 |\lambda_{\ell,2}(\rho_1)|^2}{\rho_1} + \frac{\mu^2(\rho_1)}{3 \rho_1^3} \right) \] (7.102)
\[ |\lambda_{\ell,2}(\rho_1) - \lambda_{\ell,2}(\rho_0)|^2 \leq \frac{(\rho_0 - \rho_1)}{10^4} \left( \frac{10 |\lambda_{\ell,1}(\rho_1)|^2}{\rho_1^5} + \frac{14 |\lambda_{\ell,2}(\rho_1)|^2}{\rho_1} + \frac{\mu^2(\rho_1)}{3 \rho_1^3} \right) \]
and
\[ |\mu_p(r) - \mu_{p_0}|^2 \leq \frac{(\rho_0^3 - \rho_1^3)}{10^4} \left( \frac{10 |\lambda_{\ell,1}(\rho_1)|^2}{\rho_1^5} + \frac{14 |\lambda_{\ell,2}(\rho_1)|^2}{\rho_1} + \frac{3|\mu(\rho_1)|^2}{\rho_1^3} \right). \] (7.103)

Combining (7.102)-(7.103), and noting (7.99) we estimate
\[ |\lambda_{\ell,1}(\rho_1)|^2 + |\lambda_{\ell,2}(\rho_1)|^2 + |\mu(\rho_1)|^2 \leq (\rho_0 - \rho_1)\tilde{C} \left( |\lambda_{\ell,1}(\rho_1)|^2 + |\lambda_{\ell,2}(\rho_1)|^2 + |\mu(\rho_1)|^2 \right) \]
where \( \tilde{C} > 0 \) depends only on \( \rho_0 \) as we have \( \rho_1 \in (\frac{\rho_0}{2}, \rho_0) \). Finally, selecting \( \rho_1 \) so that
\[ 0 < (\rho_0 - \rho_1) \leq \tilde{C}/2 \]
we arrive at the conclusion that
\[ \lambda_{\ell,1}(\rho_1) = \lambda_{\ell,2}(\rho_1) = \mu(\rho_1) = 0. \]

By (7.101) and (7.89), we then have
\[ \|\tilde{\omega}_\ell(0)\|_{\mathcal{H}(\rho \geq \rho_1)} = 0 \]
which contradicts the definition of \( \rho_0 \) since \( \rho_1 < \rho_0 \). Therefore, \( (\partial_r, \omega_{\ell,0}, \omega_{\ell,1}) \equiv (0, 0) \). Since \( \omega_{\ell,0}(r) \to 0 \) as \( r \to \infty \), we must have \( (\omega_{\ell,0}, \omega_{\ell,1}) \equiv (0, 0) \). \( \square \)

**Proof of Proposition 3.13.** We may now close the proof of Proposition 3.13 by tracing our steps in Section 7. Let \( \tilde{u}(t) \) be a solution of (1.1) as in Proposition 3.13. By Proposition 7.5, there exists \( \ell \in \mathbb{R} \) so that
\[ r^5 u_0(r) = \ell + O(r^{-5p+7}) \quad \text{as} \quad r \to \infty \]
\[ \int_r^\infty u_1(s)s \, ds = O(r^{-5p+3}) \quad \text{as} \quad r \to \infty. \]
If \( \ell \) is zero, then Lemma 7.20 shows that \( \tilde{u}(0) = (0, 0) \) and in turn verifies Proposition 3.13. On the other hand, if \( \ell \) is nonzero, by Lemma 7.22 we get \( \tilde{u}(0) = (Z_\ell, 0) \), where \( Z_\ell \) is the singular stationary solution constructed in Section 6. Finally, this yields the desired contradiction eliminating the case \( \ell \neq 0 \) since \( Z_\ell \) is a nonzero solution to (6.1) with \( Z_\ell \notin \dot{H}^{s_0} (\mathbb{R}^7) \) and \( \tilde{u}(0) \in \dot{H}^{s_0} \times \dot{H}^{s_0-1} (\mathbb{R}^7) \). \( \square \)

**Acknowledgments**

The second author was supported in part by the NSF grant DMS–1800082.

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G. CAMLIYURT AND C. E. KENIG

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637

Email address: camliyurt@math.uchicago.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637

Email address: cek@math.uchicago.edu