Understanding rigid body motion in arbitrary dimensions

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Abstract

Why would anyone wish to generalize the already unappetizing subject of rigid body motion to an arbitrary number of dimensions? At first sight, the subject seems to be both repellent and superfluous. The author will try to argue that an approach involving no specific three-dimensional constructs is actually easier to grasp than the traditional approach and might thus be generally useful to understand rigid body motion both in three dimensions and in the general case. Specific differences between the viewpoint suggested here and the usual one include the following: here angular velocities are systematically treated as antisymmetric matrices, a symmetric tensor $I$ quite different from the moment of inertia tensor plays a central role, whereas the latter is shown to be a far more complex object, namely a tensor of rank four. A straightforward way to define it is given. The Euler equation is derived and the use of Noether’s theorem to obtain conserved quantities is illustrated. Finally the equations of motion for a heavy top as well as for two bodies linked by a spherical joint are derived to display the simplicity and the power of the method.

Keywords: classical mechanics, rigid body, Lagrangian formalism

1. Introduction

Rigid body motion is one of the jewels of classical mechanics: it gives us a straightforward description of a system which behaves in an unexpected and often counterintuitive manner. Its practical importance is also greater than is sometimes suspected: it merits a thorough treatment by Laplace [1], for example, due to its importance in the description of the Earth’s motion. Its applications, however, are nearly numberless: gyroscopes, robots, computer animation, toys such as the eternally fascinating top, and many more.

The classical presentation of the subject is remarkably beautiful and the accounts of it given in various textbooks on classical mechanics, such as [2–5], are all rather similar, which
surely indicates that this topic has achieved a nearly perfect form: it all starts with kinematics, the definition of angular velocity as an axial vector, and finally the relation between the axial vector of angular momentum and the axial vector of angular velocity through the moment of inertia tensor, from which an equation of motion such as the Euler equation is then derived.

It is an essential feature of the classical treatment that it takes place entirely in three dimensions: the vector product and the concept of the axial vector always play a central role. The moment of inertia tensor also plays a vital role, and, as we shall see, it cannot be defined in the usual fashion for rigid bodies in more than three dimensions. All these concepts derive from the fact that a rotation in three dimensions is always characterized by an axis as well as an angle of rotation around this axis. This is why rotational motion can be characterized—at the infinitesimal level at least—by vectors, displaying the instantaneous axis of the motion.

All of this is false in dimensions higher than three: a rotation in four dimensions, for example, consists of two independent two-dimensional rotations, characterized by two angles $\phi_1$ and $\phi_2$, taking place in two orthogonal two-dimensional spaces. Clearly, no concept of axis survives. Thus, what we are proposing here is a description of rigid body motion that does not use axes. While this might look like an idle exercise, it turns out that the algebra becomes considerably more transparent if we only limit ourselves to concepts that can be extended without difficulty to arbitrary dimensions. While the algebra becomes simpler, it must be admitted that the geometric intuition becomes rather less clear. However, this author, at least, has never found the classical treatment of the general case to be very clear geometrically. Ever since Lagrange [6] proudly stated that his work contained no figures and relied solely on calculation, there has been a fruitful tension between those who emphasize the geometric aspects of mechanics and those more inclined to algebra. This paper, as will become clear, is squarely within the tradition of Lagrange. It may thus be of interest for a certain class of readers.

At this stage, it should of course be emphasized that the results are not new. Certainly Euler’s equation for the $n$-dimensional free rigid body has been stated and analysed previously. The first to propose the problem of the generalization of the Euler equation to $n$ dimensions was Cayley [7]. This problem was then solved by Frahm [8]. Independently, a similar extension was made by Weyl, as an “exercise” in the use of tensor calculus [9]. Finally, a derivation in modern terms was given by Arnold [10]. In contradistinction to the three-dimensional case, these higher-dimension Euler equations are not obviously integrable. This issue has thus attracted substantial mathematical interest. Among the first results proved in this respect was the integrability of the four-dimensional free Euler equations, shown by Schottky [11]. Non-trivial integrals of motion for arbitrary $n$ were derived in [12, 13]. These lead to the result that the $n$-dimensional Euler equation is integrable in arbitrary dimensions. These results, however, are beyond the scope of this paper. It should, however, be pointed out that the subject has generated a large mathematical literature, of which a small, unsystematic selection may get the interested reader started: various results concerning the free $n$-dimensional top are found in [14–16]. Concerning the $n$-dimensional heavy top, the reader will find further work in [17, 18]. It should, however, be emphasized that the works just cited make considerable use of differential geometry and require some significant mathematical background to be understood.

The difference between these works and the present paper is the point of view adopted: I am attempting to show that such an approach can be made quite elementary and shall use no advanced mathematics whatever in the following. Further, I hope to make clear that this approach can be used to provide a better understanding of ordinary three-dimensional rigid body motion. I have recently become aware of another such elementary treatment of this
subject [19, 20], however I believe the present approach is still sufficiently different to stand on its own.

It must, of course, be emphasized that there exist many attempts to clarify rigid body motion in a way entirely different from that pursued here; namely by improving our geometric understanding of the three-dimensional system, specifically of rotations in three-dimensional space. This work is associated with the names of Poinsot [21] and Klein and Sommerfeld [22] and has been pursued by a large number of workers, of which space only allows the author to cite a few [23–32].

In section 2 we shall first review the kinematics of rigid body motion in the terms best adapted to the general n-dimensional framework we are interested in. We then display a general Lagrangian for the rigid body, without using specific coordinates such as the Euler angles. We proceed to derive a quite general equation which might be called Newton’s equation for rigid body motion. In section 3 we obtain from the equation of motion a generalized version of Euler’s equations and define the n-dimensional version of the moment of inertia tensor. In section 4 we apply Noether’s theorem to obtain expressions for the angular momentum of a rigid body, as well as the conservation laws that follow from the symmetry of the rigid body. In section 5 we first analyse the two-dimensional case. We then present two non-trivial examples of our formalism: first, we derive the equations of motion for a heavy top, that is, a rigid body suspended at a point different from its centre of mass and subjected to a constant force. Second we consider the case of two rigid bodies linked at one point by a frictionless spherical joint. In section 6 we present some conclusions.

2. Newton’s equations for the rigid body

The goal of this section is to arrive at an equation of motion for a rigid body under the influence of an arbitrary potential, which is the equivalent of Newton’s equation for a particle: it is a set of equations of second order involving and determining uniquely all those parameters which describe the orientation of the rigid body. We first begin with some elementary definitions, then define a Lagrangian, and finally derive the equation of motion.

2.1. Kinematics: rotations as coordinates

The first difficulty in rigid body motion is to characterize the orientation of our system. A rigid body consists of an arbitrary number of particles linked by the constraint that all interparticle distances remain constant. A naive description would therefore involve a possibly large number of particle coordinates with a comparably large number of constraints.

To bypass this difficulty, we define a reference body at rest, and describe the configuration of the moving body by applying to the reference body a time-dependent rotation \( R(t) \) followed by a time-dependent translation \( \vec{X}(t) \). Thus, if the body consists of \( N \) bodies of masses \( m_i \), with \( 1 \leq i \leq N \) in the positions \( \vec{x}_i(t) \), then there exist fixed positions \( \vec{a}_i \) as well as a rotation \( R(t) \) and a vector \( \vec{X}(t) \) such that

\[
\vec{x}_i(t) = R(t)\vec{a}_i + \vec{X}(t). \tag{1}
\]

While this certainly appears intuitively clear, a rigorous proof is not obvious: the interested reader is referred to [25, 27]. It follows that the kinetic energy is given by
which can also trivially be reformulated in the case of a continuous mass distribution. In the following, for simplicity’s sake, we shall always assume either that $\ddot{X}(t) = 0$ or that the origin of the rotations is taken at the centre of mass of the body. We are thus assuming that the origin is always taken at the centre of mass, except occasionally, when one point of the rigid body is fixed at some point, which we then take as the origin of rotations. This leads us to the two following expressions for the kinetic energy

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \left( \dot{\ddot{x}}_i + \ddot{X}(t), \dddot{R}(t) \dddot{a}_i \right),$$  \hspace{1cm} (2)

Finally, let us quickly state explicitly a few elementary properties of rotations which we shall need in the following. The definition of a rotation $R$ is that, for every $\vec{x}$ and $\vec{y}$, one has

$$R \vec{x}, R \vec{y} = (\vec{x}, \vec{y}),$$  \hspace{1cm} (4)

which follows from the definition of a rotation as a linear map that leaves distances—and hence angles—invariant. From equation (4) follows that, for an arbitrary rotation $R$

$$R^T R = I$$  \hspace{1cm} (5)

where $I$ is the identity matrix and $R^T$ denotes the transpose matrix. Let us now consider an arbitrary time-dependent rotation $R(t)$. Differentiating equation (5) for $R(t)$ with respect to $t$, yields

$$R^T(t) \dot{R}(t) + R^T(t) R(t) = 0,$$  \hspace{1cm} (6)

from which immediately follows that the matrices

$$\Omega_b(t) = R^{-1}(t) \dot{R}(t)$$  \hspace{1cm} (7a)

$$\Omega_l(t) = \dot{R}(t) R^{-1}(t)$$  \hspace{1cm} (7b)

are both antisymmetric.

These antisymmetric matrices have a very remarkable significance when $R(t)$ represents the motion of a rigid body according to equation (1) and both will play a crucial role in all of what follows. To understand their physical meaning, first note that $R(t)$ maps the reference body on the moving body. We may therefore say that $\dot{R}(t)$ maps the positions $\dddot{a}_i$ of the reference body to the velocity of the corresponding point $\dddot{x}_i(t)$. The matrix $\Omega_b$ therefore maps the positions $\dddot{a}_i$ to the velocity which the point $\dddot{a}_i$ would have, if the reference body moved similarly to the physical body. Similarly, the matrix $\Omega_l$ maps the points $\dddot{x}_i(t)$ to the velocity of $\dddot{x}_i(t)$. The matrix $\Omega_b$ is thus called the angular velocity in the body frame, whereas the matrix $\Omega_l$ is called the angular velocity in the laboratory frame.
2.2. The rigid body Lagrangian

We now derive a Lagrangian for a rigid body in a general potential. The coordinate describing the orientation of the rigid body is the rotation $R(t)$. We assume the potential energy to depend solely on $R$, so that we need only focus on the expression for the kinetic energy. We have already written an expression for it, see equation (3). We now rewrite it as follows:

$$ T = \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^{N} m_{\alpha} \sum_{i=1}^{N} R_{\alpha\beta} a_{i,\alpha} R_{\gamma\beta} a_{i,\gamma}. $$

At this stage, we may disregard the term describing centre of mass motion in (3a). Let us make here the following remarks concerning notation, which we shall stick to throughout the paper: Latin indices refer to particles and run from 1 to $N$, whereas Greek indices from the beginning of the alphabet refer to coordinates in the $n$-dimensional space in which the motion takes place, and thus run from 1 to $n$. We shall always use the notation $a_{i,\alpha}$ to refer to the component $\alpha$ of the vector $\vec{a}_i$. After these remarks, equation (8) should be a straightforward consequence of equation (3).

We now define the following basic object, namely the tensor of second moments:

$$ I_{\alpha\beta} = \sum_{i=1}^{N} m_i a_{i,\alpha} a_{i,\beta}. $$

An important remark: this is quite different from the moment of inertia tensor in the traditional approach. Thus $I_{11}$ is large when the body is extended in the direction of the $x_1$ axis. In contradistinction to this, for the 11 component of the tensor of inertia to be large, the body must be extended in the 23-plane. Another obvious difference between $I$ and the moment of inertia tensor is the fact that, for $n = 2$, $I$ is a 2 × 2 matrix, not a number.

We can now use the tensor $I$, see equation (9), to simplify (8) considerably:

$$ T = \frac{1}{2} \text{Tr} \left( R I R^T \right). $$

For a definition of the trace as well as some useful elementary properties, see appendix A. This, together with the potential term $V(R)$ defines the Lagrangian, up to an important issue: we have not specified explicitly that the matrices $R$ must be rotations. The easiest way to do this is by imposing a Lagrange multiplier $\Lambda$, which is an $n \times n$ matrix. The condition for $R$ to be a rotation can be stated via equation (6), so that the final Lagrangian describing the rotational motion is given by

$$ L(\dot{R}, R) = \frac{1}{2} \text{Tr} \left( \dot{R} I R^T \right) + \text{Tr} \left[ \Lambda \left( R^T \dot{R} + \dot{R} R \right) \right] - V(R). $$

A very important observation should be made at this stage: since the matrix $R^T \dot{R} + \dot{R} R$ is automatically symmetric, $\Lambda$ can, without loss of generality, be assumed symmetric also.

2.3. Equations of motion

We now proceed to derive the equations of motion from the Lagrangian (11). Using the various tricks described in appendix A, we readily obtain

$$ \frac{\partial L(\dot{R}, R)}{\partial R} = \dot{R} I + 2RA. $$

(12a)
\[ \frac{\partial L(\dot{\hat{R}}, \hat{R})}{\partial \hat{R}} = 2\Lambda - \frac{\partial V(\hat{R})}{\partial \hat{R}}. \]  

(12b)

Writing down the Euler–Lagrange equations thus yields

\[ \dot{RI} + 2\Lambda = -\frac{\partial V(\hat{R})}{\partial \hat{R}}. \]

(13)

It now remains to eliminate the \( \Lambda \). This is done via the observation made immediately after equation (11), that \( \Lambda = \Lambda^T \). Multiplying (13) on the left by \( -R^{-1} \) and rearranging, one finds

\[ R^{-1}\dot{RI} + R^{-1}\left( \frac{\partial V(\hat{R})}{\partial \hat{R}} \right) = -2\Lambda. \]

(14)

From this it follows that the left-hand side of (14) is symmetric, that is, using the fact that \( R^{-1} = R^T \) and antisymmetrizing:

\[ R^T\dot{RI} - IR^T R = \left( \frac{\partial V(\hat{R})}{\partial \hat{R}} \right)^T R - R^T \left( \frac{\partial V(\hat{R})}{\partial \hat{R}} \right). \]

(15)

At this point we should pause to ask the meaning of such quantities as \( \frac{\partial V}{\partial \hat{R}} \) and in particular the right-hand side of (15). Since \( \text{Tr}(AB^T) \) defines a scalar product among matrices (see (A.7)), we can define \( \frac{\partial V}{\partial \hat{R}} \) by the relation

\[ V(R + \delta R) - V(R) \approx \text{Tr}\left[ \left( \frac{\partial V(\hat{R})}{\partial \hat{R}} \right)(\delta R)^T \right]. \]

(16)

where \( \delta R \) is a small matrix with the property that \( R + \delta R \) is still a rotation. It is thus in first order of the form \( R\delta A \), where \( \delta A \) is an infinitesimal antisymmetric matrix. We may thus rewrite (16) as

\[ V(R + R\delta A) - V(R) \approx \text{Tr}\left[ \left( R^T \frac{\partial V(\hat{R})}{\partial \hat{R}} \right)(\delta A)^T \right] = \frac{1}{2} \text{Tr}\left[ \left( R^T \frac{\partial V(\hat{R})}{\partial \hat{R}} - R \frac{\partial V(\hat{R})}{\partial \hat{R}}^T \right)(\delta A)^T \right]. \]

(17)

where in the final step we use the antisymmetry of \( \delta A \). The right-hand side of (15) is thus—up to a sign and a factor of two—the change in energy caused by an infinitesimal rotation in the body frame.

We now claim that equation (15) can justly be viewed as ‘Newton’s equations for a rigid body’. They are a set of second-order differential equations, the number of which is exactly sufficient to describe the dynamics of the rotation \( R(t) \). Indeed, equation (15) states that two antisymmetric matrices are equal. The number of independent equations in equation (15) is therefore \( n(n-1)/2 \). That rotations are described by the same number of independent parameters follows, for example, from the fact that rotations near the identity \( I \) are in first order equal to \( I + \Lambda \), where \( \Lambda \) is an arbitrary antisymmetric matrix. For applications, we might now describe \( R \) by our favourite parametrization—whether Euler angles, quaternions, or any other—and obtain equations of motion for these without further ado.\(^1\)

\(^1\) One might even consider integrating (15) numerically in Cartesian coordinates, that is, using all nine entries of the matrix \( R(t) \) as coordinates and trusting to the nature of the equation to keep \( R(t) \) a rotation. This is in principle possible, but numerical instabilities might be a problem.
3. The Euler equation

We now rewrite (15) as two equations of first order. As the first, we take the definition of $\Omega_b$ given by (7a). We then use (7a) to express $\Omega_b$:

$$\dot{R} = R\Omega_b$$

$$\dot{R} = R\Omega_b + R\Omega_b = R\left(\Omega_b^2 + \dot{\Omega}_b\right)$$

Putting equation (18b) in the equation of motion (15), one obtains

$$I\dot{\Omega}_b + \dot{\Omega}_b I + \left(\Omega_b^2 I - I\Omega_b^2\right) = \left(\frac{\partial V(R)}{\partial R}\right)^T R - R^T \left(\frac{\partial V(R)}{\partial R}\right).$$

This equation (19) can be rewritten as:

$$\{I, \dot{\Omega}_b\} + \left[\Omega_b^2, I\right] = \left(\frac{\partial V(R)}{\partial R}\right)^T R - R^T \left(\frac{\partial V(R)}{\partial R}\right).$$

It is readily verified that the various commutators and anticommutators involved are all antisymmetric as is the right-hand side of (21). Equation (21) is, as we shall see, the Euler equation with a torque term. We proceed to show that it can be written in a form reminiscent of the usual one.

We thus look for an analogue of the moment of inertia tensor for equation (19). Let us define the superoperator $\hat{\Theta}_I$ which maps every antisymmetric matrix $A$ to another such in the following manner:

$$\hat{\Theta}_I(A) = \{I, A\} = IA + AI$$

$\hat{\Theta}_I$ is therefore an operator defined on the space of all $n \times n$ antisymmetric matrices, that is, on a space of dimension $n(n - 1)/2$. It is hence a tensor of fourth rank on the original space. Using this definition, one finds that (19) can be rewritten as

$$\hat{\Theta}_I(\dot{\Omega}_b) + \left[\Omega_b^2, \hat{\Theta}_I(\Omega_b)\right] = \left(\frac{\partial V(R)}{\partial R}\right)^T R - R^T \left(\frac{\partial V(R)}{\partial R}\right).$$

which now looks quite similar to the usual form of the Euler equation where the right-hand side is found to correspond to the torque—expressed in the body frame—exerted on the body by $V(R)$. That equation (23) does in fact reduce to the ordinary Euler equation in three dimensions for the case in which $V(R) = 0$ is shown in Appendix B.

Note that (21) is not in any way more simple than the original Newton equation (15). But in the particular case that $V = 0$, a simplification arises: equation (21) becomes closed in $\Omega_b$, that is, we may, by solving (21), obtain $\Omega_b(t)$ from the initial angular velocities $\Omega_b(0)$. This is the Euler equation for free rigid body motion.

This equation yields the same kind of information as the usual Euler equation: for example, we see that there are permanent rotations, that is $\dot{\Omega}_b = 0$, if and only if $I$ commutes with $\Omega_b^2$. As follows from elementary linear algebra, all the eigenspaces of $\Omega_b^2$ are two-dimensional, except possibly for a one-dimensional null zero eigenspace. A rotation is
therefore only permanent when all these eigenspaces—including the null space if it exists—are so chosen that the eigenvectors of \( I \) lie in them. Such a statement is, of course, well known to hold in three dimensions, though it is usually formulated somewhat differently.

Finally, we point out that solving the Euler equations (23) for \( V = 0 \) does not mean that the motion of the system is known: to this end one needs to solve additionally the first-order equation

\[
\dot{R}(t) = R(t) \Omega_b(t).
\]

Given \( \Omega_b(t) \), this is a time-dependent system of ordinary linear differential equations, which cannot be solved, save in exceptional cases, using the matrix exponential [33], since in general

\[
[\Omega_b(t), \Omega_b(t')] \neq 0 \quad (t \neq t').
\]

However, it frequently happens that knowledge of \( \Omega_b \) is sufficient. A geometric way of obtaining \( R(t) \) for the three-dimensional case, is given by the celebrated Poinsot construction, which we shall not discuss further, though it can be generalized to arbitrary dimensions, see for example [15].

4. Conserved quantities, Noether’s theorem and angular momentum

It is a standard theorem of mechanics that any symmetry of a Lagrangian is associated to the presence of a conserved quantity associated to that symmetry. Let us briefly state the theorem, referring to [2–5] for a proof. A symmetry of a system described by the generalized coordinates \( q_1, \ldots, q_f \) is defined as follows: let us consider a continuous transformation of the \( q_k \) depending on parameter \( \lambda \) and inducing a transformation on the velocity variables given by

\[
Q_k(q_1, \ldots, q_f; \lambda) = \Phi_k(q_1, \ldots, q_f; \lambda)
\]

\[
\dot{Q}_k(q_1, \ldots, q_f; \lambda) = \sum_{l=1}^f \frac{\partial \Phi_l(q_1, \ldots, q_f; \lambda)}{\partial q_l} \dot{q}_l,
\]

where \( 1 \leq k \leq f \). Such a transformation is called a symmetry if it leaves the Lagrangian invariant, that is, if

\[
L\left( (Q_k(q_1, \ldots, q_f; \lambda))_{k=1}^f; (Q_k(q_1, \ldots, q_f; \lambda))_{k=1}^f \right) = L\left( (\dot{q}_k)_{k=1}^f; (\dot{q}_k)_{k=1}^f \right).
\]

In the presence of the symmetry defined by (26), it can be shown that the following quantity is conserved:

\[
S = \sum_{l=1}^f \left( \frac{\partial L}{\partial q_l} \frac{\partial \Phi_l(q_1, \ldots, q_f; \lambda)}{\partial \lambda} \right)_{\lambda=0}.
\]

In the following, we shall differentiate between scalar- and matrix-conserved quantities by denoting the former with lowercase and the latter with capitalized Latin letters.

We apply this result to the Lagrangian (11). The following transformation is a symmetry if \( V(R) = 0 \):
\[ \Phi(R; \lambda) = e^{i\lambda \Omega_0} R \]  
(29a)

\[ \frac{\partial \Phi(R; \lambda)}{\partial \lambda} = \Omega_0 e^{i\lambda \Omega_0} \dot{R}, \]  
(29b)

where \( \Omega_0 \) is a fixed antisymmetric matrix. Here we use the usual definition of the matrix exponential and remind the reader that the exponential of an antisymmetric matrix is always a rotation, as follows, say, by integrating any of the forms of (7). This can be done without problems, since all matrices of the form \( e^{i\lambda \Omega} \) commute among each other.

Since the \( q_i \) in the preceding formulae correspond to \( R_{a\beta} \), we see that the indices \( l \) correspond to double indices \( \alpha \beta \) in our problem. One gets

\[ s(\Omega_0) = \sum_{a\beta=1}^n (RI)_{a\beta} (\Omega_0 R)_{a\beta} = \text{Tr} \left( RIR^T \Omega_0^T \right). \]  
(30)

We may now rewrite this as

\[ s(\Omega_0) = -\text{Tr} \left( RIR^T \Omega_0 \right). \]  
(31)

Since \( \Omega_0 \) is an arbitrary antisymmetric matrix, it follows that the antisymmetric part of \( RIR^T \) is a (matrix) conserved quantity:

\[ S = \dot{RIR}^T - RIR^T. \]  
(32)

Using the definitions of \( \Omega_0 \) and \( \Omega_i \), we can give two interesting expressions for \( S \):

\[ S = R \left( \Omega_b I + I \Omega_b \right) R^{-1} \]  
(33a)

\[ = \Omega_b RIR^{-1} + RIR^{-1} \Omega_i. \]  
(33b)

\( S \) can thus finally be expressed in terms of \( \Omega_b \) or \( \Omega_i \) and an appropriate moment of inertia:

\[ S = R \hat{\Theta}_I(\Omega_b) R^{-1} \]  
(34a)

\[ = \hat{\Theta}_{RIR^{-1}}(\Omega_i). \]  
(34b)

Since \( S \) is obtained from rotational invariance, it is identified as the angular momentum of the system. We have shown that all its components are conserved in the free case. If we have a potential \( V(R) \), this will generally not be true anymore. If \( V(R) \) is symmetric under some group of rotations, however, say the rotations generated by a given \( \Omega_0 \), then \( \text{Tr} \left( S \Omega_0 \right) \), which might be called the ‘\( \Omega_0 \) component’ of the angular momentum tensor \( S \), is conserved. Note further that the tensor \( S \) is an object that maps points belonging to the moving body to points belonging to the moving body again, that is, it is an object defined in the laboratory frame. Of course, using the techniques described in appendix B, we obtain the usual expression for the angular momentum in either frame for three-dimensional systems.

From equation (34a) we can rederive the Euler equation by writing out the conservation of angular momentum in terms of \( \Omega_b \). This is nothing else than the usual derivation presented in textbooks. For the sake of completeness, we show it:

\[ 0 = \frac{dS}{dt} \]  
\[ = R \dot{\hat{\Theta}}_I(\Omega_b) R^{-1} + R \hat{\Theta}_I(\Omega_b) R^{-1} + R \hat{\Theta}_I(\Omega_b) R^T \]  
\[ = R \left( \hat{\Theta}_I(\Omega_b) + \left[ \Omega_b, \hat{\Theta}_I(\Omega_b) \right] \right) R^{-1}, \]  
(35)

from which Euler’s equation (23) follows.
We have defined the symmetry (29) by premultiplying $R$ by a constant rotation. This is essential: if we instead attempt the transformation

$$R(\lambda) = Re^{i\Omega_0}$$

$$\tilde{R}(\lambda) = \tilde{R}e^{i\Omega_0},$$

(36a)

(36b)

it is quite easy to check that this does not, in general, leave the Lagrangian (11) invariant. It does so only if $I$ commutes with $\Omega_0$. If this happens, as is the case when the rigid body has some symmetry, then we can indeed derive a conservation law from this symmetry. By an exactly analogous computation, we see that the corresponding conservation law is given by

$$\tilde{s}(\Omega_0) = -\text{Tr} \left( R^T R \Omega_0 \right).$$

(37)

If we define $\tilde{S}$ by

$$\tilde{S} = R^T \tilde{R} I - I R^T R = \Omega_b I + I \Omega_b = \hat{\Theta}_b(\Omega_b),$$

(38)

then we see that the expression

$$\text{Tr} \left( \tilde{S} \Omega_0 \right)$$

is conserved whenever $\Omega_0$ commutes with $I$. This is, of course, the angular momentum defined in the body frame, which is conserved for a symmetric free body, though not otherwise.

5. Two examples

To show how the formalism described preceding works, let us first do a routine exercise: we look at the case of a two-dimensional rigid body, for the description of which only one angle is needed. The formula for the rotation as a function of the angle is

$$R(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$$

(40)

and hence

$$\Omega_b = \begin{pmatrix} 0 & \dot{\phi} \\ -\dot{\phi} & 0 \end{pmatrix} = \Omega_b.$$ 

(41)

Since we are dealing with antisymmetric $2 \times 2$ matrices, we may characterize them uniquely by their upper right matrix element. Note that $\Omega_b$ and $\Omega_b$ are still different conceptually, and cannot really be compared, as they act on different spaces. They are, however, numerically equal.

In the Euler equation, the commutator term vanishes, so we are left with the term $\hat{\Theta}_b(\Omega_b)$, which is simply the antisymmetric matrix corresponding to $I\dot{\phi}$. We may now introduce a scalar potential $V'_b(\phi)$ defined as $V[R(\phi)]$, where $R(\phi)$ is defined via (40). From the discussion leading to (17), we see that the right-hand side of (15) is simply the antisymmetric matrix corresponding to $-V'_b(\phi)$. Euler’s equation thus reduces to

$$I\ddot{\phi} = -V'_b(\phi).$$

(42)

Deriving the corresponding expressions for energy and angular momentum is an easy exercise, best left to the reader.
We now proceed to work out two less trivial examples in which the method described here leads straightforwardly both to compact expressions for the equations of motion as well as for the conservation laws. First, let us consider the heavy top, suspended at an arbitrary point. We make no assumption of an axis or symmetry, nor any further assumption on the location of the centre of mass on some principal axis.

As coordinates we take only rotations $R$, assuming the top’s point of suspension to be the origin, so that no displacement $\vec{X}$ is needed. The tensor $I$ is thus computed from the suspension point. Defining $\vec{a}$ as the coordinate of the centre of mass in the reference body and $\vec{g}$ to be the direction of the acceleration gravity, we have for the potential

$$V(R) = -m(\vec{g}, R\vec{a}) = -m \text{Tr} \left[ (\vec{a} \otimes \vec{g}) R \right],$$

where $m$ is the mass of the body. Obviously, both $\vec{a}$ and $\vec{g}$ can be chosen to be in the $z$-direction by an appropriate choice of orientation in both the reference bodies and the laboratory coordinate system. However we shall not do this, as the gain in clarity resulting from clearly separating body-fixed quantities such as $\vec{a}$ from laboratory quantities such as $\vec{g}$ outweighs any advantage in having slightly shorter formulae.

The equations of motion are thus:

$$\dot{\Omega}_b = R \Omega_b$$

If $\Omega_0$ is any antisymmetric matrix such that $\Omega_0 \vec{g} = 0$, then the quantity

$$s(\Omega_0) = -\text{Tr} \left( R I^T \Omega_0 \right) = -\frac{1}{2} \text{Tr} \left[ (RIR^T - RIR^T) \Omega_0 \right] = -\frac{1}{2} \text{Tr} \left[ \hat{\Omega} RIR^{-1}(\Omega_0) \Omega_0 \right]$$

is conserved. If further the vector $\vec{a}$ is an eigenvector of $I$ and additionally an antisymmetric matrix $\Omega_1$ exists such that

$$\Omega_0 \vec{a} = 0, \quad [\Omega_1, I] = 0,$$

then the quantity

$$s(\Omega_1) = -\text{Tr} \left( R I \Omega_1 \right) = -\frac{1}{2} \text{Tr} \left( \dot{\hat{\Omega}}(\Omega_0) \Omega_1 \right)$$

is also conserved. This corresponds, of course, to the situation in the integrable Lagrange top in three dimensions. Indeed, in three dimensions, this gives two integrals of motion, which together with the energy yield enough integrals of motion, which in the Hamiltonian formalism turn out to be in involution, to give the complete integrability of the system. Note that, in this respect, the higher-dimensional systems are truly more complicated: first, there are several degrees of symmetry, depending on how many eigenvalues of $I$ are degenerate. Second, if one counts the number of conserved quantities obtained in this way, one generally does not have enough to guarantee integrability. In fact, even the integrability of the Euler equation in $n$ dimensions is by no means obvious [13] and certainly does not follow from rotational invariance alone.

At first sight, this requires using some parametrization of the rotations, such as Euler angles. Such an approach does indeed yield the usual equations, as described in [2–4]. This is, however, not necessary, as the following easy observation shows: define

$$\vec{g}(t) = m R(t)^T \vec{g}$$

(48)
The equations then read
\[ \Theta_i(\Omega_b) + \left[ \Omega_b, \Theta_i(\Omega_b) \right] = (\vec{a} \otimes \vec{\gamma})^\top - \vec{a} \otimes \vec{\gamma} \]
(49a)
\[ \dot{\vec{\gamma}} = -\Omega_b \vec{\gamma}. \]
(49b)

In three dimensions this is readily rewritten as
\[ \Theta \dot{\vec{\omega}}_b + \vec{\omega} \land \Theta \vec{\omega}_b = -\vec{a} \land \vec{\gamma} \]
(50a)
\[ \dot{\vec{\gamma}} = -\vec{a} \land \vec{\gamma}. \]
(50b)

Here, of course, \( \vec{\omega}_b \) is the vector corresponding to \( \Omega_b \) and \( \Theta \) is the \( 3 \times 3 \) matrix corresponding to the superoperator \( \Theta_i \). This form for the equations of the heavy top is not usually given in the classical textbooks [2–5], but it appears, for example, in [6], the second part, section 6, paragraph 3, number 52. It is also the form used by Kowalevski [34] to derive the integrable case of the heavy top named after her.

We now give another non-trivial example, for which the method here discussed is remarkably straightforward. Consider two rigid bodies free to move arbitrarily in space, except for the constraint that they be freely linked at a joint. This is described as follows: we denote by \( \sigma \) an index taking the values 1 and 2 and referring to the two rigid bodies. Each body is described by a given rotation \( R_\sigma(t) \) with respect to its centre of mass and a translation \( \vec{X}_\sigma(t) \). Both bodies are characterized by a tensor of second moments \( I_\sigma \). Finally, the fact that they are linked is expressed by the fact that there exist constant vectors \( \vec{A}_\sigma \) in the two bodies of reference such that
\[ \vec{R}_\sigma A_\sigma + \vec{X}_\sigma(t) - \vec{A}_\sigma = \vec{R}_\sigma \vec{X}_\sigma(t) - \vec{A}_\sigma, \]
(51)
for all \( t \). The Lagrangian is hence given by
\[
L \left( R_1, R_2, \vec{X}_1, \vec{X}_2; R_1, R_2, \vec{X}_1, \vec{X}_2 \right) = \sum_{\sigma=1,2} \left\{ \frac{1}{2} \text{Tr} \left( R_\sigma I_\sigma R_\sigma^\top \right) + \text{Tr} \left[ \Lambda_\sigma \left( R_\sigma^\top R_\sigma + R_\sigma R_\sigma^\top \right) \right] \right\} + \sum_{\sigma=1,2} \left[ \frac{M_\sigma}{2} \left( \vec{X}_\sigma - \vec{A}_\sigma \right) + \vec{\gamma} \cdot \left( R_1(t) \vec{X}_1(t) - R_2(t) \vec{X}_2(t) - \vec{A}_1 - \vec{A}_2 \right) \right].
\]
(52)

Here all the notation is familiar, except for the vector \( \vec{\lambda} \), which is the Lagrange multiplier imposing the constraint (51). From (A.9) of appendix A we can express the scalar product in a more convenient way:
\[ \vec{\lambda} \cdot (R \vec{A}_\sigma) = \text{Tr} \left[ \left( \vec{\lambda} \otimes \vec{A}_\sigma \right) R_\sigma^\top \right]. \]
(53)

which allows us to use the techniques given in appendix A for these terms as well.

The Euler–Lagrange equations are obtained in just the same way as in section 2 for the rotational part. The part involving translations requires no further comment:
\[ M_1 \ddot{\vec{X}}_1 = \vec{\lambda} \]
(54a)
\[ M_2 \ddot{\vec{X}}_2 = -\vec{\lambda} \]
(54b)
The right-hand sides of equations (54c) and (54d) can be expressed in terms of $\Omega_b$ in the usual manner

$$\Theta_{\ell_1}(\Omega_{b,\sigma}) + \Theta_{\ell_2}(\Omega_{b,\sigma})$$

Introducing relative and centre of mass variables, and taking the centre of mass to be at rest, which is possible due to Galilean invariance, we have

$$0 = M_1\vec{X}_1 + M_2\vec{X}_2$$

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We may now eliminate $\vec{x}$ using (54a), (54b), and (56), obtaining, after some straightforward algebra

$$\Theta_{\ell_1}(\Omega_{b,\sigma}) + \Theta_{\ell_2}(\Omega_{b,\sigma}) = \mu \left[ R^T_1 \left( \vec{x} \times \vec{A}_1 \right) - \left( \vec{x} \times \vec{A}_1 \right)^T R_1 \right]$$

$$\Theta_{\ell_2}(\Omega_{b,\sigma}) + \Theta_{\ell_2}(\Omega_{b,\sigma}) = \mu \left[ \left( \vec{x} \times \vec{A}_2 \right)^T R_2 - R_2^T \left( \vec{x} \times \vec{A}_2 \right) \right]$$

$$\vec{x} = R_2(t)\vec{A}_2 - R_1(t)\vec{A}_1$$

$$\mu = \frac{M_1M_2}{M_1 + M_2}$$

After substituting (57c) into (57a), and (57b) one gets—for the three-dimensional case—a set of six equations for the twelve unknowns $R_\sigma$ and $\Omega_{b,\sigma}$. Together with the equations that define $\Omega_{b,\sigma}$ in terms of $R_\sigma$ and $R_\sigma$, given by (7a), one obtains a closed set of equations. These remarks extend trivially to the general $n$-dimensional case.

The physical meaning of these equations is clear: the left-hand sides of equations (57a) and (57b) are the same as that of the Euler equation with torque, see equation (23). Their right-hand sides, on the other hand, express the torque which acts on the body $\sigma$ due to the action of the joint, caused by the relative acceleration between both bodies.

6. Conclusions

In this paper I have primarily focused on free rigid body motion. We have seen how to derive both a simple equation of motion, namely equation (15), valid quite generally, as well as Euler’s equation for a free top and the relation between the angular velocity matrix and the conserved angular momentum via the (generalized) moment of inertia tensor. This way of obtaining an equation of motion for a system involving one or many rigid bodies is quite general and flexible, as we have seen in the example of section 5. Extensions to other groups than rotations are also possible, as well as to the description of systems such as approximately rigid bodies, for which one introduces coordinates involving a translation, a rotation, and deviations from the reference positions. In all these cases, computations quite similar to those described previously straightforwardly yield an equation of motion.
A significant issue with the method developed so far is the absence of a canonical formalism. This means that we cannot say which conservation laws are in involution and which are not. This severely limits our ability to identify integrable systems. A Hamiltonian formalism can, in fact, be developed, but it is by no means as elementary as the Lagrangian formalism presented here. Such developments are reserved for a future publication.

Once the equation of motion has been obtained, we may still proceed to study its conservation laws without further algebraic difficulties, as we have shown by the application of Noether’s theorem to the free and the symmetrical top. In fact, we see that the technique described previously leads to new insights: the fact that the conservation of an appropriate component of the angular momentum in the body frame follows from the body’s symmetry with respect to the corresponding set of rotations, is ordinarily not derived in this fashion. On the other hand, solving the equation of motion usually requires going to specific coordinates. This can be arduous, and it may often be simpler to do so directly at the level of the Lagrangian. Nevertheless, working on the problem at the abstract level tells us a great deal about its structure, as I hope to have made clear in the examples presented previously. Concerning the true usefulness of this approach, however, we might aptly quote one of the fathers of analytical mechanics [35]: ‘It may happen to me, as to others, that a meditation which has long been dwelt on shall assume an unreal importance; and that a method which has for a long time been practised shall acquire an only seeming facility.’

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Appendix A. Some helpful formulae for calculations with traces

The trace of an \( n \times n \) matrix is defined as

\[
\text{Tr} (A) = \sum_{\alpha=1}^{n} A_{\alpha\alpha}. \tag{A.1}
\]

It is readily verified that

\[
\text{Tr} (AB) = \sum_{\alpha,\beta=1}^{n} A_{\alpha\beta} B_{\beta\alpha} = \text{Tr} (BA), \tag{A.2}
\]

from which straightforwardly follows

\[
\text{Tr} (ABC) = \text{Tr} (BCA) = \text{Tr} (CAB). \tag{A.3}
\]

It follows immediately from (A.2) that

\[
\frac{\partial}{\partial X_{\alpha\beta}} \text{Tr} (XY) = Y_{\beta\alpha} \tag{A.4}
\]

which can be symbolically rewritten as

\[
\frac{\partial}{\partial X} \text{Tr} (XY) = Y^{T}. \tag{A.5}
\]
Throughout the text we shall often combine (A.3) and (A.5) to obtain such results as
\[
\frac{\partial}{\partial y} \text{Tr} \left( XYZ^T \right) = X^T Z. \tag{A.6}
\]

The trace can also be used for other purposes. For example, note that
\[
\text{Tr} \left( AB^T \right) = \sum_{\alpha, \beta} A_{\alpha \beta} B_{\alpha \beta} \tag{A.7}
\]
defines a scalar product on the set of matrices. In fact, it defines the standard scalar product and we shall often use this fact.

We also sometimes need to reduce matrix elements of operators to trace form. This can be done using the concept of a tensor product: given two vectors \( \vec{x} \) and \( \vec{y} \), we define \( \vec{x} \otimes \vec{y} \) as the matrix given by
\[
(\vec{x} \otimes \vec{y})_{\alpha \beta} = x_\alpha y_\beta. \tag{A.8}
\]
From this it follows that, for any matrix \( A \)
\[
(\vec{x}, A \vec{y}) = \text{Tr} \left[ (\vec{y} \otimes \vec{x}) A \right] \tag{A.9}
\]

Appendix B. Deriving the usual form of the Euler equations

It is standard [5] that one can, to each \( 3 \times 3 \) antisymmetric matrix \( \Omega \), assign a vector \( \vec{\omega} \in \mathbb{R}^3 \) such that for all \( \vec{x} \in \mathbb{R}^3 \)
\[
\Omega \vec{x} = \vec{\omega} \times \vec{x}, \tag{B.1}
\]
where \( \vec{x} \times \vec{y} \) denotes the usual vector product between \( \vec{x} \) and \( \vec{y} \). Using well-known properties of the vector product, we can show the following very useful equalities:
\[
(\Omega_1 + \Omega_2) \vec{x} = (\vec{\omega}_1 + \vec{\omega}_2) \times \vec{x} \tag{B.2a}
\]
\[
\Omega_1 \Omega_2 \vec{x} = (\vec{\omega}_1 \times \vec{\omega}_2) \times \vec{x} \tag{B.2b}
\]
\[
R \Omega_1 R^{-1} \vec{x} = (R \vec{\omega}_1) \times \vec{x} \tag{B.2c}
\]
for all \( \vec{x} \), where we have assumed
\[
\Omega_i \vec{x} = \vec{\omega}_i \times \vec{x} \tag{B.3}
\]
for all \( \vec{x} \). We therefore see that addition and commutation of matrices translate into addition and vector product of vectors, whereas a change of coordinates will change the matrix and the vector in compatible ways, see equation (B.2c).

The superoperator \( \hat{\Theta}_I \) assigns linearly to every antisymmetric matrix \( \Omega \) the matrix \( I \Omega + \Omega I \). It thus translates into a linear operator \( \Theta \) on the vectors \( \vec{\omega} \). To determine it, start by considering the case in which the basis is chosen in such a way as to make \( I \) diagonal (principal axes). In this case, one sees easily that
\[
(\hat{\Theta}_I \Omega)_{\alpha \beta} = (I_\alpha + I_\beta) \Omega_{\alpha \beta}, \tag{B.4}
\]
where there is no summation over repeated indices. Using the explicit form of the transformation of \( \Omega \) to \( \vec{\omega} \), we find
\[ \Theta = \begin{pmatrix} I_2 + I_1 & 0 & 0 \\ 0 & I_1 + I_3 & 0 \\ 0 & 0 & I_1 + I_2 \end{pmatrix} \]  \hspace{1cm} (B.5)

This can be expressed in the form

\[ \Theta = \text{Tr} (I) \cdot I - I. \]  \hspace{1cm} (B.6)

Since this is an expression which transforms under rotations in the same way as \( \Theta \), namely as a tensor, equation (B.6) is generally true. The matrix \( \Theta \) which acts on vectors \( \vec{\omega} \) in the same way as the superoperator \( \hat{\Theta} \) does on antisymmetric matrices, is thus given by the usual expression for the moment of inertia tensor.

All we now need to do is to use equations (B.2) to translate the Euler equations (23) derived in section 3 into an equation for vectors \( \vec{\omega} \). One obtains:

\[ \Theta \dot{\vec{\omega}} + \vec{\omega} \wedge (\Theta \vec{\omega}) = 0. \]  \hspace{1cm} (B.7)

It goes without saying that the vector \( \vec{\omega} \) refers to the antisymmetric matrix \( \Omega_b \), that is, to the angular velocity in the body system.

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