Chern–Simons superconductor

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Received 19 June 2014, revised 10 October 2014
Accepted for publication 27 October 2014
Published 24 November 2014

Abstract
We study the effect of a bulk Chern–Simons (CS) term on a (3 + 1) dimensional type II superconductor in the context of the AdS/CFT correspondence. We holographically compute the supercurrent and find that it is nonlocal in nature. It receives nontrivial corrections due to the presence of the CS term. Considering a large limit of a parameter \(\lambda\) (we call this limit the long wavelength limit), which is effectively the high temperature limit of the theory, we find that this nonlocal supercurrent boils down to a local quantity. The leading term (without the CS term) of this current matches the result of the Ginzburg–Landau (GL) theory. We compute the effect of the CS term on the GL current and find that the effect is greatly suppressed at high temperature \(T \gg T_c\). Finally, the free energy of the vortex configuration has been calculated. The free energy also receives nontrivial correction on the order of \(1/\lambda^2\) in the long wavelength approximation.

Keywords: AdS/CMT, superconductor, anomaly
PACS numbers: 04.70.-s, 11.25.Tq

1. Introduction and summary

The phenomenon of superconductivity is characterized by two different effects: (i) vanishing of resistivity and (ii) macroscopic diamagnetism below a critical temperature \(T_c\). When the temperature is above \(T_c\), the material behaves like a normal metal. At \(T = T_c\), the sample undergoes a second order phase transition from a normal phase to a superconducting phase.

A phenomenological model proposed by Ginzburg and Landau in the mid-1900s explains this phenomenon qualitatively. According to this model, in the normal phase the free energy of the system is invariant under a global \(U(1)\) symmetry. However, this symmetry is spontaneously broken below the critical temperature \(T_c\) and the vacuum expectation value of a
charged scalar field $\Psi$ (order parameter of phase transition) becomes nonzero (condensation). This spontaneous breakdown of a global $U(1)$ symmetry characterizes a second order phase transition from a normal phase to a superconducting phase. We briefly review the Ginzburg–Landau (GL) model in section 2.

The basic idea of a holographic superconductor came after phenomenal papers by S Gubser [1, 2]. In his work he argued that 'coupling the Abelian Higgs model to gravity with a negative cosmological constant leads to black holes which spontaneously break the global $U(1)$ gauge invariance via a charged scalar condensate slightly outside their horizon. This suggests that black holes can superconduct.' Based on this observation, Hartnoll, Herzog, and Horowitz proposed a gravity dual of a $2+1$ dimensional superconductor [3]. In this paper they showed that below a critical temperature, a charged condensate is formed via a second order phase transition and the (DC) conductivity becomes finite. They considered a complex scalar field $\psi$, minimally coupled with a $U(1)$ gauge field in four-dimensional $AdS$ black brane geometry. Ignoring the effect of the back reaction of the scalar field and the gauge field on the black brane geometry, it was observed that nontrivial coupling between the gauge field and the scalar field destabilizes the $\psi = 0$ solution at some critical temperature $T_c$ and below that temperature, the black hole admits a nontrivial scalar hair solution. The effect of back reaction has been considered in [4].

The effect of an external magnetic field on the charged scalar was first studied holographically in [4]. The authors added a magnetic field passing perpendicularly through the material and obtained the London equation and the magnetic penetration depth. They also showed that these holographic superconductors are of type II. From their analysis it became quite evident that in the presence of a large magnetic field applied at low temperature, the system behaves like a normal metal. As we decrease the magnetic field below a critical value, the material develops superconducting vortices.

The vortex solution was further studied in [5]. The authors studied the bulk equations of motion for a scalar field near the phase transition line and showed that the solutions are a holographic realization of Abrikosov’s vortex lattice. The vortex solution is thermodynamically favored below a critical magnetic field. Holographically calculation of free energy and superconducting current turn out to be nonlocal functions. But these expressions are reduced to the known form of the sGL theory at long wavelengths.

It is worthwhile to mention that all the aforementioned holographic models have successfully described various known features of usual type II superconductors. Apart from those having the aforementioned features, some unconventional superconducting materials exist where one might expect additional interesting properties like the existence of anomalous supercurrent as well as nontrivial temperature dependence of the supercurrent across the (Josephson) junction of superconducting materials [12–17].

The existence of anomalous supercurrent was predicted long ago in [12]. Later on a detailed analysis regarding the anomalous behavior of the supercurrent across the (Josephson) junction in the presence of time reversal symmetry breaking was performed in [13]. In [15] the authors discussed the effect of anomalous current on the Meissner penetration depth. They also discussed the connection between such an anomalous current and that with the quasiparticle picture for unconventional superconductors. Most interestingly it was observed that the supercurrent flowing across the junction of such unconventional materials exhibits nontrivial temperature dependence. For example, in [16] the authors observed that due to the presence of the long-range correlation the supercurrent decreases logarithmically with
temperature. In [17] the authors found that the supercurrent across the junction between the anisotropic superconductors increases rapidly as the temperature is lowered toward $T = 0$.

It is interesting to note that in spite of having so much real-life evidence regarding the anomalous behavior as well as the temperature dependence of the supercurrent, to date no attempts have been made to establish these facts holographically, which motivates us to perform the following analysis of this paper:

In this paper we study the properties of a 3 + 1 dimensional superconductor. The bulk theory is given by a five-dimensional black brane geometry. We add a Chern–Simons term in the bulk Lagrangian$^4$. Our main objective is to see whether the effect of adding a CS term in the bulk description can be realized as a finite temperature effect in the language of the boundary theory. In other words the motivation of our present analysis is to find evidence of temperature dependence of the supercurrent holographically. In this sense, our analysis can be treated as the first step toward the holographic realization of temperature dependence of the supercurrent across the (Josephson) junction of the (unconventional) superconducting materials. The Chern–Simons terms are interesting because they source anomalous effects in the boundary theory even in the absence of the charged scalar field. There are three types of anomaly in a quantum field theory: (1) associated with breakdown of a classical symmetry at the quantum level, (2) associated with breakdown of classical gauge symmetry at quantum level, and (3) associated with breakdown of a genuine symmetry of a quantum theory in the presence of background fields. There are many references to anomaly [24–28] and a recent one is [29], where the three different kinds of anomaly have been explained with examples. In our case, the presence of a bulk CS term actually signifies the breakdown of a global $U(1)$ symmetry of the dual field theory. We are interested in determining whether the CS term introduces some anomalous effect in the superconducting current and vortex solution.

Our observations are as follows:

1. The Chern–Simons term has no direct effect on the holographic vortex lattice solution.
2. The Chern–Simons term modifies the superconducting current. There are two kinds of effects. Holographically computed superconducting current turns out to be nonlocal. But in the low frequency or long wavelength limit, it becomes a local quantity [5]. In the absence of the CS term the current takes the familiar form (in long wavelength approximation) of a GL current

$$ J_j \sim \epsilon^i \partial_i \sigma(x, y) $$

where $\sigma(x, y)$ is $|\psi|^2$. However, there are two different contributions of the CS term to this current. (a) In the long wavelength limit it only modifies the overall coefficient of the GL current. (b) The nontrivial local contribution that comes to the superconducting current due to the presence of the CS term is on the order of $1/\lambda^2$ (with respect to the leading term),

$$ J_j \sim \epsilon^i \partial_i \sigma(x, y) + \frac{\kappa^2}{\lambda^2} \epsilon^i \partial_i (\Delta \sigma(x, y)), $$

where $\kappa$ is the coefficient of the CS coupling and $\lambda$ is defined in (4.24). Later we will see that the $1/\lambda^2$ correction is effectively high temperature correction of the current. This therefore gives us the first nontrivial holographic model for a temperature-dependent supercurrent, which was predicted long before for real-life superconducting materials as mentioned previously.

$^4$ Holographic superconductors in the presence of a $U(1)$ Chern–Simons (CS) term were investigated in [18]. Besides this analysis, the effect of adding CS terms in the five-dimensional dual gravitational description has also been explored extensively in the context of anomalous hydrodynamics in [19–23].
This paper is organized as follows. In section 2 we give a brief summary of the usual GL theory for type II superconductors. The details of the dual gravitational description for a $(3+1)$ dimensional superconductor are provided in section 3. In section 4, using the AdS/CFT prescription [30, 31] we compute the boundary current in the presence of the CS term. In section 5, we compute the free energy for the vortex configuration. Finally we conclude the paper in section 6.

The long appendix consists of details of some formulae that appear in the expression of boundary current.

2. Ginzburg–Landau theory of superconductivity: a quick review

In 1950, V L Ginzburg and L Landau proposed a phenomenological theory of superconductivity [32]. In this model, they introduced a complex scalar field $\psi(\mathbf{r})$ as an order parameter of the system. $|\psi(\mathbf{r})|^2$ represents the local density of superconducting electrons. The Ginzburg–Landau (GL) model was a generalization of London’s theory of superconductors, where the density $n_s$ of superconducting electrons was constant in space. Because of the phenomenological foundation the GL theory did not get much attention before 1959, when Gor’kov [33] showed that GL theory is derivable as a special limiting case of microscopic theory proposed by Bardeen, Cooper, and Schrieffer in 1957 [34].

In this section we intend to discuss important aspects of GL theory, which we need to understand the results of this paper. We will not go into details of the derivations; rather we concentrate on the main results of the GL theory. Interested readers can find the details in [35].

To describe the spatial non-uniformity of the order parameter $\psi(\mathbf{r})$, Ginzburg and Landau had to go beyond the idea of the constant order parameter (mean-field approximation), which can qualitatively describe the phenomenon of phase transition. They introduced a nonlocal free energy of the system written in powers of $\psi(\mathbf{r})$ and $\nabla \psi(\mathbf{r})$,

$$ F(\psi, A) = \int d^3r \left[ \alpha |\psi(\mathbf{r})|^2 + \frac{\beta}{2} |\psi(\mathbf{r})|^4 + \frac{1}{2m^*} \left( \frac{1}{i} \nabla - qA \right) \psi(\mathbf{r}) \right]^2 + \frac{B^2}{8\pi}. $$  

(2.1)

Thermodynamical equilibrium is attained, minimizing the free energy. Minimizing $F$ with respect to $\psi(\mathbf{r})$, we get

$$ -\frac{1}{2m^*} \left( \frac{1}{i} \nabla - qA \right)^2 \psi(\mathbf{r}) + \alpha \psi(\mathbf{r}) + \beta |\psi(\mathbf{r})|^2 \psi(\mathbf{r}) = 0. $$  

(2.2)

However, minimization with respect to the gauge field $A$ gives rise to the following equation:

$$ -i\frac{q}{2m^*} \left( \psi(\mathbf{r}) \nabla \psi^*(\mathbf{r}) - \psi^*(\mathbf{r}) \nabla \psi(\mathbf{r}) \right) + \frac{q^2}{m^*} |\psi(\mathbf{r})|^2 A + \frac{1}{4\pi} \nabla \times B = 0. $$  

(2.3)

Using Ampère’s law one can find a local supercurrent $j_s(\mathbf{r})$ given by

$$ j_s(\mathbf{r}) = \frac{1}{4\pi} \nabla \times B $$

$$ = i\frac{q}{2m^*} \left( \psi(\mathbf{r}) \nabla \psi^*(\mathbf{r}) - \psi^*(\mathbf{r}) \nabla \psi(\mathbf{r}) \right) - \frac{q^2}{m^*} |\psi(\mathbf{r})|^2 A. $$  

(2.4)

As a consistency check, we find that in the limit of the constant order parameter ($\psi(\mathbf{r})$) we obtain
\[ \psi = -q m j_\text{rr} A \] (2.5)

In this limit we expect that this supercurrent should reproduce the London equation

\[ \psi = -n m q n r e^* . (2.6) \]

which requires

\[ |\psi(r)|^2 = \frac{e^2 m^*}{q^2 m} n_i. \] (2.7)

We introduce two intrinsic length scales of this theory:

- \( \tilde{\lambda} \): determines the penetration strength of the external magnetic field
- \( \tilde{\xi} \) (Ginzburg–Landau coherence length): measures the length scale of variation of the order parameter in the surface area and deep inside the superconductor

In the mean field approximation one can show that both the order parameters scale as

\[ \tilde{\lambda}, \tilde{\xi} \sim \frac{1}{\sqrt{T_c - T}} \]

close to \( T_c \). Therefore, we define a dimensionless quantity

\[ \tilde{\kappa} = \frac{\tilde{\lambda}(T)}{\tilde{\xi}(T)} \] (2.8)

which is a function of temperature. \( \tilde{\kappa} \) is called the Ginzburg–Landau parameter. Depending on the value of \( \tilde{\kappa} \) we classify two different kinds of superconductors: (i) type \( \text{I} \): \( \tilde{\kappa} < 1/\sqrt{2} \) and (ii) type \( \text{II} \): \( \tilde{\kappa} > 1/\sqrt{2} \) (see figure 1).

In a type \( \text{I} \) superconductor the superconducting state disappears if the applied magnetic field is greater than a critical value \( H_c \). The equilibrium state in the bulk of the superconductor is uniform. The correct thermodynamic potential in the presence of a magnetic field is given by the Gibbs free energy density (\( B \) is magnetic induction inside the material),

\[ G/V = F/V - \frac{1}{V} \int d^3 r \frac{H \cdot B}{4\pi}. \] (2.9)
In the superconducting phase the magnetic field inside the superconductor is zero; hence the Gibbs potential is given by

\[ G_s/V = \alpha \left| \psi \right|^2 + \frac{\beta}{2} \left| \psi \right|^4 = -\frac{\alpha^2}{2\beta} \] (2.10)

whereas, in the normal phase, \( \psi \) vanishes, and the magnetic field penetrates the material. Hence, free energy for the normal state is given by

\[ G_N/V = F_N/V = -\frac{H^2}{4\pi} \] (2.11)

Therefore, there is competition between these two phases, and at \( H = H_c \) these two free energies become equal:

\[ H_c = \sqrt{\frac{\alpha^2}{4\pi\beta}}. \] (2.12)

If the applied magnetic field \( H < H_c \), the superconducting state dominates the thermodynamics, and for \( H > H_c \), the normal state takes over.

If the sample is very large (for example, consider a thin slab) and the applied magnetic field passes perpendicularly through the sample, then even for \( H < H_c \) the flux penetrates through the sample (since going around the large sample would cost much energy), forming small normal strips. Therefore, in a type I superconductor one can see such ‘intermediate states’.

### 2.1. Type II superconductor: vortex solution

A type II superconductor is characterized by the formation of magnetic vortices in an applied magnetic field. Vortices form in the superconducting material when \( \kappa > 1/\sqrt{2} \).

Consider an area \( S \) with boundary \( \partial S \). We assume that at least the boundary \( \partial S \) lies inside the superconductor. Therefore, the net flux passing through this area is given by

\[ \Phi = \oint_{\partial S} \mathbf{A} \cdot ds. \] (2.13)

Using equation (2.4) one can write

\[ \Phi = -\frac{m}{e^2 n_s} \oint_{\partial S} \mathbf{j}_s \cdot ds - \frac{1}{2e} \oint_{\partial S} \mathbf{A} \cdot \nabla \phi \] (2.14)

where \( \phi \) is the phase of the complex field \( \psi \). The last term is an integer multiple of \( 2\pi \),

\[ \oint_{\partial S} \mathbf{A} \cdot \nabla \phi = -2\pi n, \quad n \in \mathbb{Z}. \] (2.15)

Thus we see that

\[ \Phi' = \Phi + \frac{m}{e^2 n_s} \oint_{\partial S} \mathbf{j}_s \cdot ds = n\Phi_0, \quad \Phi_0 = \frac{\pi}{e} \] (2.16)

is quantized. The quantity \( \Phi' \) is called the fluxoid. Therefore, the magnetic fields penetrate the superconductor in a quantized way, i.e., the magnetic fluxoid is an integral multiple of the minimum fluxoid \( \Phi_0 \). The penetration takes place in terms of vortices or vortex lines. We consider that the direction of a vortex is along the direction of the magnetic field. As we go around a vortex anticlockwise, the phase of \( \psi (\mathbf{r}) \) changes by an amount \( -2\pi \). At the center of a vortex the phase is not defined. Therefore, for a consistent solution for \( \psi (\mathbf{r}) \) we consider...
ψ(r) = 0 at the core. In a cylindrical coordinate system (ρ, φ, z) (the vortex is along the z direction), we find that

\[ ψ(ρ = 0) = 0 \]
\[ |ψ(ρ → ∞)| = ψ_0 \]
\[ B(ρ → ∞) = 0. \] (2.17)

Since the system has cylindrical symmetry, we can choose

\[ ψ(r) = ψ_{arg}(ρ)e^{-iφ} \]
\[ j_ρ(r) = ϕ_i(ρ) \]
\[ B(r) = zB(ρ) \] (2.18)

Choosing the vector potential along the \( \hat{φ} \) direction one can solve the Landau–Ginzburg equation in cylindrical coordinates along with the expression for supercurrent (Ampère’s law) under the boundary conditions given in (2.17). The equations are highly nonlinear. One can solve them numerically.

In a type II superconductor, nothing interesting happens unless the applied magnetic field is greater than the lower critical magnetic field \( H_{c1} \). For \( H < H_{c1} \) the sample exhibits the Meisner phase. The first vortex enters the superconductor at \( H = H_{c1} \). For a type II superconductor with \( κ >> 1 \) the lower critical magnetic field is given by

\[ H_{c1} = \frac{H_c \ln \tilde{k}}{\sqrt{2}}. \] (2.19)

2.1.1. Vortex lattice. The flux in a superconductor should enter as a periodic triangular lattice to minimize the free energy of the vortex state. However, the vortex state disappears as we increase the external magnetic field beyond the upper critical value \( H_{c2} \). For \( H > H_{c2} \), the sample becomes normal metal.

Based on the Ginzburg–Landau theory Abrikosov proposed a solution for this vortex lattice. Abrikosov’s results are quantitatively valid only near \( H_{c2} \). In his construction he assumed that the magnetic flux density is uniform, which is valid only near \( H_{c2} \).

For a constant B along the \( \hat{z} \) direction, we write \( A_y = H_0 \). The linearized Ginzburg–Landau equation becomes

\[ \frac{1}{2m^*}
\left[
\mathbf{\hat{y}}^2 + 2eH\hat{y}x
\right]^2 ψ + α \psi = 0
\]
\[ -\nabla^2 - \frac{4\pi i}{\Phi_0} H_0 \frac{∂}{∂y} + \left( \frac{2πH}{\Phi_0} \right)^2 x^2 \]
\[ \psi = \frac{1}{ξ^2} \psi \] (2.20)

where

\[ ξ^2(T) = \frac{1}{2m^*α(T)} \] (2.21)

This equation looks like the Schrödinger equation with nontrivial potential along the \( x \) direction. The particle behaves like a free particle in the \( y \) and \( z \) directions. Therefore we consider the following ansatz for ψ:
\[ \psi = e^{ik_x y + ik_y x} f(x). \quad (2.22) \]

Therefore equation (2.20) becomes
\[ -f''(x) + \left( \frac{2\pi H}{\Phi_0} \right)^2 (x - x_0)^2 f(x) = \left( \frac{1}{\xi^2} - k_z^2 \right) f(x) \quad (2.23) \]

where
\[ x_0 = \frac{k_y \Phi_0}{2\pi H}. \quad (2.24) \]

This equation is exactly the same as for a one-dimensional quantum harmonic oscillator with frequency \( \omega_c = 2He/m^*c \) and the minimum of the potential shifted by an amount \( x_0 \). This is an eigenvalue equation, and the solution exists if
\[ H = \frac{\Phi_0}{2\pi (2n + 1)} \left( \frac{1}{\xi^2} - k_z^2 \right). \quad (2.25) \]

This equation says that there exists an upper limit of the magnetic field which corresponds to \( n = 0 \) and \( k_z = 0 \). This is the upper critical magnetic field \( H_{c2} \). Therefore, we find that
\[ H_{c2} = \frac{\Phi_0}{2\pi \xi^2(T)}. \quad (2.26) \]

One can also write the relation between the upper critical magnetic field and the thermodynamic magnetic field:
\[ H_{c2} = \sqrt{2} \kappa H_c. \quad (2.27) \]

We consider the lowest energy solution \( k_z = 0, \; n = 0 \):
\[ \psi_0(x, y) = e^{ik_x y + k_y x} f_0(x). \quad (2.28) \]

Abrikosov assumed that the solution for \( H \lesssim H_{c2} \) is periodic in the \( x \) and \( y \) directions. The \( y \) direction has a period \( a_y \); hence,
\[ k_y = \frac{2\pi}{a_y}, \; l \in \mathbb{Z}. \quad (2.29) \]

Therefore, the harmonic oscillator is centered at
\[ x_0 = \frac{\Phi_0}{H_a} l \quad (2.30) \]

and the solution is given by
\[ \psi_0(x, y) = \mathcal{N} \exp \left( \frac{2\pi y}{a_y} \right) \exp \left( -\frac{1}{2} m^* \omega_c \left[ x - \frac{\Phi_0}{H_a} l \right]^2 \right). \quad (2.31) \]

In the vicinity of \( H_{c2} \), we can write
\[ \omega_c m^* = \frac{2He}{c} \approx \frac{2H_{c2}}{c} e = \frac{1}{\xi(T)^2}. \quad (2.32) \]
Thus,
\[ \psi_0(x, y) = N \exp\left( \frac{2\pi y}{a_y} l \right) \exp\left( -\frac{1}{2\xi(T)^2} \left[ x - \frac{\Phi_0}{H_2 a_y} \right]^2 \right). \] (2.33)

The most general solution, therefore, is given by linear superposition for different \( l \):
\[ \psi_0(x, y) = \sum_{l=-\infty}^{\infty} c_l \exp\left( \frac{2\pi y}{a_y} l \right) \exp\left( -\frac{1}{2\xi(T)^2} \left[ x - \frac{2\pi l}{\xi(T)^2} \right]^2 \right) \] (2.34)

We can write this solution in terms of the elliptic theta function,
\[ \psi_0(x, y) = e^{-\frac{1}{2\xi(T)^2}} \theta_3(v, \tau). \] (2.35)

The theta function is defined by
\[ \theta_3 = \sum_{l=-\infty}^{\infty} q^{l^2} z^{l^2}, \quad q = e^{i\pi \tau}, \quad z = e^{i\pi v} \] (2.36)

where
\[ v = \frac{-ix + y}{a_y}, \quad \tau = \frac{2\pi i - a_y^2 \xi(T)^2}. \] (2.37)

Here we have written \( c_l \) in terms of a new constant \( a_x \),
\[ c_l = \exp\left( -\frac{i\pi a_t}{a_y} \xi(T)^2 l^2 \right) \] (2.38)

From the periodicity of the theta function,
\[ \theta_3(v + 1, \tau) = \theta_3(v, \tau) \]
\[ \theta_3(v + \tau, \tau) = e^{-2\pi i (v + \tau/2)} \theta_3(v, \tau) \] (2.39)

one can verify that
\[ \psi_0 \left( x, y + a_y \right) = \psi_0(x, y) \]
\[ \psi_0 \left( x + \frac{2\pi x^2}{a_y}, y + \frac{a_y^2 \xi^2}{a_y} \right) = \exp \left[ 2\pi i \left( \frac{v}{a_y} + \frac{a_y^2 \xi^2}{2a_y^2} \right) \right] \psi_0(x, y). \] (2.40)

Thus \( \sigma(x, y) = \lambda \psi_0^2 \) represents a lattice in which the fundamental region is spanned by two vectors, \( b_1 = a_y \partial_y \) and \( b_2 = \frac{2a_y^2}{a_x} \partial_x + \frac{a_x^2}{a_y} \partial_y \). The area of the fundamental region is given by \( 2\pi \xi^2 \). Therefore the total flux passing through the fundamental region is given by,
\[ xH_2 \times \text{Area} = 2\pi \xi^2 H_2 = \Phi_0. \] (2.41)

The current evaluated for this vortex solution is given by
\[ j_i = e^i \partial_j \sigma(x, y). \] (2.42)
3. Holographic model for a 3+1 dimensional superconductor

We consider the following five-dimensional action with a Chern–Simons term:

\[ S = \frac{1}{2l_p^2} \int \sqrt{-g} \left( 12 + R - \frac{1}{4} F^2 + \frac{\kappa}{3} \epsilon^{abcde} A_d F_{be} F_{de} \right) \]

\[ - \left| \nabla \psi \right|^2 - m^2 |\psi|^2 \right) \]  

(3.1)

We consider the gauge field and the scalar field in the probe approximation and treat the effect of the Chern–Simons term perturbatively (\( \kappa \) small). The background (asymptotically) AdS black brane solution is given by\(^5\)

\[ ds^2 = -f(u) dt^2 + \frac{r_0^2}{4 u f^2(u)} du^2 + \frac{r_0^2}{u} \left( dx^2 + dy^2 + dz^2 \right), \quad f(u) = \frac{r_0^2}{u} \left( 1 - u^2 \right). \]  

(3.2)

The horizon is located at \( u = 1 \) and the asymptotic boundary is at \( u \to 0 \). The temperature of this black brane is given by

\[ T = \frac{r_0}{\pi}. \]  

(3.3)

We consider the following ansatz for the gauge field and the scalar field:

\[ A = \left( A_x(\phi(u), 0, 0, A_y(x), P(u)) \right) \]

\[ \psi = \psi(u, x, y). \]  

(3.4)

Using this ansatz, the equation of motion for the gauge field turns out to be

\[ \phi''(u) - \frac{r_0^2}{2u f(u)} \phi'(u)(u, x, y)^2 + \frac{12 \kappa A_x'(x) P'(u)}{r_0^2} = 0 \]

\[ u A_x''(x) - 2 r_0^2 A_y(x) \psi (u, x, y)^2 = 0 \]

\[ P''(u) + \frac{P'(u)}{u} + \frac{f'(u) P'(u)}{f(u)} = \frac{r_0^2}{2u f(u)} \psi (u, x, y)^2 + \frac{12 \kappa A_x'(x) \psi'(u)}{u f(u)} = 0. \]  

(3.5)

Likewise, the equation for the scalar field is given by

\[ \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial f(u) \partial u} + \frac{r_0^2 \phi(u)^2 - f(u) \left( m^2 r_0^2 + u P(u)^2 \right)}{4 u f(u)^2} \right) \psi \]

\[ + \frac{1}{4 u f(u)} \left( \Delta - 2i A_x(x) - A_y(x)^2 \right) \psi = 0. \]  

(3.6)

Our goal is to find a consistent solution to these equations. We impose the following boundary conditions in order to solve these equations.

- We consider the background temperature to be stable. We neglect the effect of the scalar field and the gauge field on the background. Therefore, the black brane temperature is kept fixed in our calculation.

\(^5\) Our choice of coordinate is special so that \( u f(u) \) is finite at the boundary \( u \to 0 \). This choice differs from the usual AdS metric on the Poincare patch (which we usually use to define the dimensions of various fields) along the radial direction as \( u = r_0^2 \).
• The asymptotic value of the time component of the gauge field \( \mu = A_t(0) \), which behaves as though a chemical potential of the boundary field theory is kept fixed in our study. This means we are considering the boundary field theory in a grand canonical ensemble.
• We also demand that the norm of the gauge field be finite at the horizon. This imposes a second boundary condition on \( A_u(0) = A_u(1) = 0 \).
• The non-normalizable mode of the gauge field, i.e., \( xy \) component of the field strength, deforms the boundary. The asymptotic value of \( F_{xy} \) gives the external magnetic field \( H \).
• The scalar field has different falloff at the \( \text{AdS} \) boundary for different values of \( m^2 \). We consider the mass of the scalar field to satisfy the \( BF \) bound in five dimensions, i.e., \( m^2 = -3 \). In this case the scalar field goes as

\[
\psi \sim \alpha_1 \sqrt{u} + \alpha_2 u^{3/2}.
\] (3.7)

We see that both modes are normalizable. Therefore, the coefficients \( \alpha_1 \) and \( \alpha_2 \) correspond to the source and the vacuum expectation value of some dual operators with dimensions \( \Delta = 3 \) respectively.

The external magnetic field \( H \) is the tuning parameter of our system. We will control the value of this magnetic field from outside and study the effect on condensation and currents in the boundary theory. We expect that the scalar field begins to condensate below a critical magnetic field. We call this magnetic field \( H_c^2 \) and define a parameter \( \epsilon = \frac{H - H_c}{H_c^2} \) with \( \epsilon \ll 1 \). Our goal is to find solution to the foregoing equations of motion in a power series of \( \epsilon \). We expand the fields in the following way:

\[
\psi(u, x, y) = e^{1/2} \psi_1(u, x, y) + e^{3/2} \psi_2(u, x, y) + \cdots
\]

\[
A_u(u, x, y) = A_u^{(0)} + \epsilon A_u^{(1)}(u, x, y) + \cdots.
\] (3.8)

The solution in the \( \epsilon \to 0 \) limit is given by

\[
A_t^{(0)} = \phi(u) = \mu(1 - u) - \frac{144 \kappa^2 \mu H_{c}^2 (2u \log 2 - (u + 1) \log (u + 1))}{r_0^4} + \mathcal{O}(\kappa^4),
\]

\[
A_y^{(0)}(x) = H_{c} x,
\]

\[
A_z^{(0)}(u) = P(u) = \kappa \left( C_2 - \frac{12 \mu H_{c} \log (u + 1)}{r_0^2} \right) + \mathcal{O}(\kappa^3).
\] (3.9)

The preceding solution is perturbative in the CS coefficient \( \kappa \). Since we are treating the gauge (and scalar field) as a probe, we begin with the \( \text{AdS} \) Schwarzschild solution as the feed and solve for the gauge fields perturbatively in \( \kappa \) up to the first nontrivial order, setting the scalar to zero. The \( \kappa = 0 \) solution for \( A_z \) is a constant, and we set that to zero.\(^6\)

We substitute these leading order solutions to the equation for the scalar field (3.6) to solve \( \psi_1 \). We write the scalar field as a product of two independent parts, and since the \( y \) direction is a free direction we take the following ansatz for \( \psi_1 \):

\[
\psi_1 = \rho(u) X(x) e^{i \nu y}.
\]
Substituting this in the scalar field equation we get

\[ \frac{f'(u)\rho'(u)}{f(u)} - \frac{m^2 n_2^2 \rho(u)}{4u^2 f(u)} - \frac{P(u)\rho^2(u)}{4u^2 f(u)} + \frac{n_2^2 \rho(u)\phi(u)^2}{4u^2 f(u)} + \rho''(u) = -\lambda H \frac{\rho(u)}{4u^2 f(u)} \]  

(3.10)

\[ X^*(x) - H^2 \left( x - \frac{\rho}{H} \right)^2 X(x) = -\lambda X(x), \]  

(3.11)

where \( \lambda \) is an arbitrary constant. Comparing equation (3.11) with equation (2.23) we find that in our case \( \Phi = 2\pi, \lambda = \frac{1}{\epsilon} (k_z = 0, \text{in our case}). \) The maximum magnetic field below which a vortex solution exists is given by \( H_{c2} = \lambda \). Equation (3.10) has a nontrivial solution only when \( \lambda \) (or \( H_{c2} \)) is a nontrivial function of temperature \( T \). A critical temperature \( T_c \) exists at which the solution \( \rho = 0 \) becomes marginally stable. Below that temperature, the scalar field condenses and we get a nontrivial profile satisfying the corresponding boundary conditions. Thus we see that the CS term does not have any effect on the vortex equation, but it has a nontrivial effect on condensation of the scalar mode.

4. Boundary current

In this section, using the AdS/CFT prescription [28], we compute the boundary current (\( J^\mu \)) for the four-dimensional chiral \( U(1) \) gauge theory living on the boundary of the five-dimensional bulk AdS space-time. To compute the current coupled to the \( U(1) \) gauge field, we evaluate the onshell action to the first order in the gauge fluctuations, and finally, using the AdS/CFT dictionary we obtain

\[ J^\mu = \frac{\partial S}{\partial A_{\mu}} \bigg|_{u=0} = \frac{\sqrt{-g}}{16\pi G} \left( F^{au} + \frac{4\kappa}{3\sqrt{-g}} \epsilon_{abcd} A_b F_{cd} \right) \bigg|_{u=0}. \]  

(4.1)

Setting \( a = i = (x, y, z) \), this finally yields

\[ J_i = \frac{\sqrt{-g} e}{16\pi G} \left( F^{(1)i} g^{iu} g_{bu} + \frac{4\kappa}{3\sqrt{-g}} \epsilon_{abcd} \left( A_b^{(1)} F_{cd}^{(0)} + A_b^{(0)} F_{cd}^{(1)} \right) \right) \bigg|_{u=0} + \mathcal{O} \left( e^2 \right). \]  

(4.2)

Our next task is to explicitly compute (4.2) for the first order gauge fluctuations. This implies that we need to solve the Maxwell equations for the first order, which is given by

\[ \nabla_i F^{(1)ab} = 2 \frac{\kappa}{\sqrt{-g}} \epsilon^{abcd} F_{bc}^{(0)} F_{de}^{(1)} = (J^1)_a. \]  

(4.3)

To solve (4.3), we express it explicitly for the temporal as well as spatial parts of the gauge field. Writing (4.3) explicitly for the temporal part we find

\[ \nabla_t F^{(1)ab}_t = 2 \frac{\kappa}{\sqrt{-g}} \epsilon^{abcd} F_{bc}^{(0)} F_{de}^{(1)} = (J^1)_a. \]  

(4.3)
On the other hand, for the spatial components we enumerate the following equations\(^\text{10}\)

\[
L_i A_{i}^{(1)} - 16\kappa u \left( \frac{\nabla A_{i}^{(0)} \cdot \nabla A_{i}^{(1)}}{r_{0}^2} + H_{2} \partial_{\rho} A_{i}^{(1)} \right) = \frac{2\kappa^2}{u} A_{i}^{(0)} \left| \psi_{1} \right|^2. \tag{4.4}
\]

where we have defined the two following linear operators as

\[
L_i = \left( 4f(u)u^2 \partial_{\rho}^2 + \Delta \right), \quad L_s = \left( 4 u \partial_{\rho}^{(1)} u \partial_{\rho} \right) \tag{4.6}
\]

Note that in the preceding we have denoted the first order fluctuations in the time as well as as in the spatial components of the gauge field as \(A_{i}^{(1)}\) and \(A_{i}^{(1)} (i = x, y, z)\) respectively. Remember that these fluctuations arise due solely to the presence of a nontrivial profile of scalar hair, which is basically a perturbation that one can identify as the order parameter.

Since our bulk theory contains a nontrivial coupling between the scalar field and the \(U(1)\) gauge sector, the reader might guess that turning on fluctuations both in the scalar and in the gauge sector eventually makes it quite difficult to solve these equations exactly. This fact is indeed quite evident from the foregoing set of equations (4.4) and (4.5). Therefore in our computations we aim to solve these equations perturbatively in \(\kappa\) and we keep terms up to the quadratic order in \(\kappa\)\(^{11}\). In order to solve (4.4) and (4.5) perturbatively in \(\kappa\), we consider the following expansion of the gauge field:

\[
A_{n}^{(m)} = A_{n}^{(m)}(\kappa^{(0)}) + \kappa A_{n}^{(m)}(\kappa^{(1)}) + \kappa^2 A_{n}^{(m)}(\kappa^{(2)}) + \mathcal{O}(\kappa^3). \tag{4.7}
\]

Also, from the structure of the radial equation (3.10), it is indeed evident that we may express the solution perturbatively in \(\kappa\) as

\[
\rho_{0} = \rho_{0}^{(0)}(\kappa^{(0)}) + \kappa^2 \rho_{0}^{(0)}(\kappa^{(2)}) + \mathcal{O}(\kappa^3). \tag{4.8}
\]

Note that we have used two different indices in the superscript, both of which correspond to a different order of perturbations. Index \((m)\) corresponds to fluctuations at a different order due to the presence of a nontrivial value of the order parameter. This is basically the perturbation in \(\varepsilon\) that we mentioned earlier in (3.8). On the other hand, the superscript \((\kappa^{(n)})\) stands for different terms in the perturbative expansion in \(\kappa\).

We now substitute (4.7) and (4.8) into (4.4) and (4.5) to solve these equations order by order in \(\kappa\). Let us first consider the zeroth order equation.

**Equations in the zeroth order in \(\kappa\)**

\(^{10}\) To arrive at these equations we choose the gauge \(A_{0} = 0\). Also, we have exploited the residual gauge symmetry \(A_{i}^{(1)} \rightarrow A_{i}^{(1)} - \partial_{\rho} A_{i}(x, y)\) in order to fix the gauge \(\partial_{\rho} A_{i}^{(1)} + \partial_{\rho} A_{i}^{(1)} = 0\).

\(^{11}\) The reason for keeping terms up to quadratic order in \(\kappa\) follows from the fact that in the preceding section we already retained our solutions for the gauge fields up to the quadratic order in the parameter \(\kappa\) (see (3.9)).
We first write (4.4) and (4.5) for the zeroth order, which take the following form:

\[
\begin{align*}
L_x A_x^{(1)}(\kappa^{(m)}) &= \frac{2}{\kappa^{0} \rho_0} \frac{2(\kappa^{(0)})}{u} \sigma(x) A_x^{(0)}(\kappa^{(m)}) \\
L_y A_y^{(1)}(\kappa^{(m)}) &= \frac{\kappa^2}{\kappa^0 \rho_0} \frac{2(\kappa^{(0)})}{u} \epsilon_{\rho z} \partial_{\rho} \sigma(x) \\
L_z A_z^{(1)}(\kappa^{(m)}) &= \frac{\kappa^2}{\kappa^0 \rho_0} \frac{2(\kappa^{(0)})}{u} \epsilon_{\rho z} \partial_{\rho} \sigma(x) \\
L_{\kappa^1} A_{\kappa^1}^{(1)}(\kappa^{(m)}) &= 0
\end{align*}
\]

where we have used the fact that \( \epsilon_{ij} \) is anti-symmetric, satisfying \( \epsilon_{12} = -\epsilon_{21} = 1 \). Note that on the rhs of the preceding set of equations (4.9), we have expressed the source \( j_i^{(1)} \) (associated with the first order fluctuations of the nonzero condensate) in terms of \( \sigma \psi = x(|\psi|^2) \) that corresponds to the vortex solution (corresponding to a triangular lattice) in the \((x, y)\) plane.

Since these are inhomogeneous differential equations with a source term on the rhs, the solutions of (4.9) can be written in terms of Green’s functions, which satisfy the appropriate boundary conditions at the AdS boundary. The solutions at the zeroth order are given here:\(^{12}\):

\[
\begin{align*}
A_x^{(1)}(\kappa^{(m)}) &= -\left(2\kappa^0 \right) \int_0^1 du \frac{2(\kappa^{(0)})}{u'} A_x^{(0)}(\kappa^{(m)}) (u') \times \int dx' G_x(u, u'; x, x') \sigma(x') \\
A_y^{(1)}(\kappa^{(m)}) &= a_i(x) - \kappa^0 \epsilon_{ij} \int_0^1 du \frac{2(\kappa^{(0)})}{u'} \times \int dx' G_x(u, u'; x, x') \partial_{x_j} \sigma(x') \\
A_z^{(1)}(\kappa^{(m)}) &= 0.
\end{align*}
\]

Note that \( a_i(x) \) corresponds to a homogeneous solution of (4.5) which is solely responsible for giving rise to the critical magnetic field \( (H_c = \epsilon_{ij} \partial_i a_j) \) at the boundary of the AdS. \( G_x(u, u'; x, x') \) and \( G_y(u, u'; x, x') \) are Green’s functions of the preceding set of equations (4.9) that satisfy the following equations of motion:

\[
\begin{align*}
L_x G_x(u, u'; x, x') &= -\delta(u - u') \delta(x - x') \\
L_y G_y(u, u'; x, x') &= -\delta(u - u') \delta(x - x')
\end{align*}
\]

along with the following (Dirichlet) boundary conditions at the AdS boundary:

\[
\begin{align*}
G_x(u, u'; x, x')|_{\kappa^0=0} = G_x(u, u'; x, x')|_{\kappa^0=1} = 0 \\
G_y(u, u'; x, x')|_{\kappa^0=0} = u \delta (u) \partial_{x_i} G_i(u, u'; x, x')|_{\kappa^0=1} = 0.
\end{align*}
\]

These boundary conditions also ensure the following two things.

1. The source \( \mu(= A_i (u = 0)) \) is kept fixed at the boundary.
2. We have a uniform magnetic field \( (H_{z2}) \) at the boundary of the AdS.

\(^{12}\) Note that at the zeroth order in \( \kappa \) the \( z \) component of the gauge field does not possess any solution, which is consistent with the findings of the preceding section.
Next we follow almost the same procedure to find solutions to (4.5) for the leading order as well as the next-to-leading order in $\kappa$. Let us first note the equations in the leading order in $\kappa$.

**Equations in the leading order in $\kappa$**

\[
L_t A^{(1)}_t(x^{(1)}) = \frac{16u^2 f(u)}{r_0^2} \left( \partial_\mu A^{(0)}_z(x^{(0)}) F^{(1)}_{xy}(x^{(0)}) + H_{2z} \partial_\mu A^{(1)}_z(x^{(0)}) \right)
\]
\[
= \frac{2 \rho_0^2 u}{r_0^2} \sigma(x) A^{(0)}_t(x^{(1)})
\]
\[
L_t A^{(1)}_x(x^{(1)}) = 16u \left( \partial_\mu A^{(0)}_t(x^{(0)}) \partial_\nu A^{(1)}_x(x^{(0)}) - \partial_\mu A^{(0)}_z(x^{(0)}) \partial_\nu A^{(1)}_t(x^{(0)}) \right) = 0
\]
\[
L_t A^{(1)}_y(x^{(1)}) = 16u \left( \partial_\mu A^{(0)}_t(x^{(0)}) \partial_\nu A^{(1)}_y(x^{(0)}) - \partial_\mu A^{(0)}_z(x^{(0)}) \partial_\nu A^{(1)}_t(x^{(0)}) \right) = 0
\]
\[
L_t A^{(1)}_z(x^{(1)}) = 16u \left( \partial_\mu A^{(0)}_t(x^{(0)}) F^{(1)}_{xy}(x^{(0)}) + H_{2z} \partial_\mu A^{(1)}_t(x^{(0)}) \right)
\]
\[
= \frac{2 \rho_0^2 u}{r_0^2} \sigma(x) A^{(0)}_t(x^{(1)}).
\]

These are again a set of inhomogeneous, nonlinear differential equations. Therefore, considering the rhs as the source, we may express solutions in terms of Green’s functions as usual. From our earlier solutions (3.9) and (4.10) for gauge fields, it is quite evident that only the last equation corresponding to $A^{(1)}_z(x^{(0)})$ will have a nontrivial solution at the leading order in $\kappa$. In the following we enumerate the solutions at the leading order as

\[
A^{(1)}_t(x^{(1)}) = 0
\]
\[
A^{(1)}_x(x^{(1)}) = 0
\]
\[
A^{(1)}_z(x^{(1)}) = -\left(2 \rho_0^2 \right) \int_0^1 \frac{du}{u} \int d^4x' G_t(u, u'; x, x')
\]

where $J(u, x)$ is the source for $A^{(1)}_z(x^{(1)})$ and is given by

\[
J(u, x) = 2 \rho_0^2 \sigma(x) A^{(0)}_t(x^{(0)}) + 8u^2 \frac{1}{r_0^2} \left( \partial_\mu A^{(0)}_t(x^{(0)}) F^{(1)}_{xy}(x^{(0)}) + H_{2z} \partial_\mu A^{(1)}_t(x^{(0)}) \right).
\]

Finally we write the first order Maxwell equation (4.5) at the quadratic order in $\kappa$, which are given as follows.

**Equations in the second order in $\kappa$**

\[
L_t A^{(1)}_t(x^{(1)}) - \frac{16u^2 f(u)}{r_0^2} \left( \partial_\mu A^{(0)}_z(x^{(0)}) F^{(1)}_{xy}(x^{(0)}) + H_{2z} \partial_\mu A^{(1)}_z(x^{(0)}) \right)
\]
\[
= \frac{2 \rho_0^2 \sigma(x)}{u} \left( 2A^{(0)}_t(x^{(0)}) \rho_0^2 (x^{(2)}) + A^{(1)}_t(x^{(0)}) \rho_0^2 (x^{(2)}) \right).
\]

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Using our standard technique, we can again express these solutions in terms of Green’s function, which take the following form:

\[
L_\nu A^{(1)}_{\nu}(k^{(2)}) - 16\alpha \left( \partial_\nu A^{(0)}_{\nu}(k^{(0)}) \partial_\nu A^{(1)}_{\nu}(k^{(1)}) - \partial_\nu A^{(0)}_{\nu}(k^{(0)}) \partial_\nu A^{(1)}_{\nu}(k^{(0)}) \right) \\
= \frac{2\eta^2 \rho_0^2(k^{(0)})}{\mu} \epsilon^{\nu}_\tau \partial_\nu \sigma(x)
\]

\[
L_\nu A^{(1)}_{\nu}(k^{(2)}) - 16\alpha \left( \partial_\nu A^{(0)}_{\nu}(k^{(0)}) \partial_\nu A^{(1)}_{\nu}(k^{(1)}) - \partial_\nu A^{(0)}_{\nu}(k^{(0)}) \partial_\nu A^{(1)}_{\nu}(k^{(0)}) \right) \\
= \frac{2\eta^2 \rho_0^2(k^{(0)})}{\mu} \epsilon^{\nu}_\tau \partial_\nu \sigma(x)
\]

\[
L_\nu A^{(1)}_{\nu}(k^{(2)}) - 16\alpha \left( \partial_\nu A^{(0)}_{\nu}(k^{(0)}) \partial_\nu A^{(1)}_{\nu}(k^{(1)}) - \partial_\nu A^{(0)}_{\nu}(k^{(0)}) \partial_\nu A^{(1)}_{\nu}(k^{(0)}) \right) \\
= \frac{2\eta^2 \rho_0^2(k^{(0)})}{\mu} \epsilon^{\nu}_\tau \partial_\nu \sigma(x)
\]

(4.16)

It is really remarkable to note at this stage that even in the presence of anomaly we can express the source as a total derivative of some function \( \chi(u, x) \) which is given by

\[
\chi(u, x) = \rho_0^{(1)}(k^{(1)}) \rho_0^{(2)}(k^{(2)}) \sigma(x) + \frac{8\alpha^2}{\eta^2} \mathcal{H}(u, x)
\]

(4.19)

where \( \mathcal{H} \) is some function that depends on the gauge fields and its derivatives:

\[
\mathcal{H} = \partial_\nu A^{(0)}_{\nu}(k^{(0)}) A^{(1)}_{\nu}(k^{(1)}) - \partial_\nu A^{(0)}_{\nu}(k^{(0)}) A^{(1)}_{\nu}(k^{(0)}).
\]

(4.20)

This is the exact expression for the anomalous current in the presence of a vortex lattice up to the quadratic order in \( \kappa \), where we have combined both the entities \( \Xi \) and \( \Theta \) into a single entity \( \Gamma \). Note that the quantities \( \Gamma \) and \( \Sigma \) are explicit functions of gauge fields and their derivatives. In the following we give the full (analytic) expression for these

\[ \text{We have used the fact that } \epsilon_i \text{ is anti-symmetric, } \eta_i = 1, \text{ and } \frac{\Sigma}{m} = -\frac{\Theta}{m} \]
quantities:

\[ \mathcal{Z} = \int \! \! dx' \sigma(x') \partial_d \int_0^1 \! \! du' P_0^{2(k_0)}(u') G_s(u, u'; x, x') + 2 \kappa^2 \int \! \! dx' \partial_d \]

\[ \times \int_0^1 \! \! du' \gamma(u', x') G_s(u, u'; x, x') \big|_{u=0} \]

\[ \Theta = \kappa^2 \left( A_{i(0)}^{(0)}(u') A_{j(1)}^{(1)}(u) - A_{i(0)}^{(1)}(u') A_{j(1)}^{(1)}(v) \right) \big|_{u=0} \]

\[ \Sigma = \int \! \! dx' \partial_d \int_0^1 \! \! du' \frac{J(u', x')}{u'} G_s(u, u'; x, x') \big|_{u=0} \]

(4.22)

Note that the quantity \( \mathcal{Z} \) has two parts. The term that is proportional to \( \kappa^2 \) is arises due purely to the presence of the CS term, whereas on the other hand, the first term is always there even if we do not have any anomaly in our theory. Also note that the quantity \( \Theta \) arises due solely to the presence of the CS term. Therefore, this term will be present in the current only if the anomaly is there.

From the preceding expressions (4.22) it is quite evident that the current thus obtained is nonlocal. On the other hand, in the usual GL theory the current is expressed as a local function of the order parameter. Therefore we are still a few steps away from the standard GL theory. However, one may arrive at the local expression for the current in a certain limit which we discuss hereafter.

From (4.22) the reader might notice that nonlocalities are associated with both the radial \((u)\) direction and the \((x, y)\) plane. To make these expressions local, as a first step we extract the \(u\) dependence of Green’s functions. We define a complete set of orthonormal basis such that

\[ G_i(u, u'; x, x') = \sum_\zeta \xi_\zeta(u) \xi_\zeta^\dagger(u') \tilde{G}_i(x, \zeta) \]

\[ G_s(u, u'; x, x') = \sum_\lambda \eta_\lambda(u) \eta_\lambda^\dagger(u') \tilde{G}_s(x, \lambda) \]

(4.23)

with

\[ \mathcal{L}_{i \xi}(u) = \xi_\zeta^\dagger(u); \quad \sum_\zeta \xi_\zeta(u) \xi_\zeta^\dagger(u') = \delta(u - u'); \quad \langle \xi_\zeta | \xi_\zeta^\dagger \rangle = \delta_{\zeta \zeta}^\dagger \]

\[ \mathcal{L}_{i \eta}(u) = \lambda \eta_\lambda(u); \quad \sum_\lambda \eta_\lambda(u) \eta_\lambda^\dagger(u') = \delta(u - u'); \quad \langle \eta_\lambda | \eta_\lambda^\dagger \rangle = \delta_{\lambda \lambda}. \]

(4.24)

Here \( \tilde{G}_i(x, \lambda) \) and \( \tilde{G}_s(x, \lambda) \) are Green’s functions defined in the \((x, y)\) plane that basically satisfy the equation of motion of the form

\[ \left( \Delta - \varphi^2 \right) \tilde{G}(x, \varphi^2) = -\delta(x) \]

(4.25)

for any real positive value of \( \varphi^2 \).

The solution of (4.25) may be expressed in terms of a modified Bessel function

\[ \tilde{G}(x, \varphi^2) = \frac{1}{2\pi} K_0(|\varphi|x) \]

(4.26)

which satisfies the condition \( \lim_{|x|\to\infty} |\tilde{G}(x)| < \infty \).
The eigen functions \( \{ \xi \} \) and \( \{ \eta \} \) that form a complete orthonormal basis satisfy the following boundary conditions\(^{14}\):

\[
\begin{align*}
\xi(0) &= \xi(\ell) = 0 \\
\eta(0) &= \eta(\ell) = 0.
\end{align*}
\]

(4.27)

To remove nonlocalities in the \((x, y)\) plane\(^{15}\) we consider the long wavelength approximation, which comes with the following idea: Green’s function varies on a length scale \(\lambda \sim 1\) that is small compared with that of the vortex solution, whose size can be determined by the length scale \(\xi\) that is fixed by the vortex lattice. Therefore, in this long wavelength approximation, we may take our condensate to be uniform over the length scale on which Green’s function fluctuates\(^{16}\). Mathematically we may state this long wavelength approximation as

\[
\frac{1}{\sqrt{\lambda}} \ll \xi.
\]

(4.28)

This is indeed a vital assumption which finally removes the nonlocalities associated with \(x\) and helps us to express our anomalous current locally as a function of the order parameter \(\sigma(x)\).

In this long wavelength approximation, we define the convolution (*) of Green’s function in the \((x, y)\) plane as

\[
\left[ \tilde{G}_i(\lambda) \ast \sigma \right](x) = \int \! \! \int \! \! dx' \tilde{G}_i(x - x', \lambda) \sigma(x') = \frac{\sigma(x)}{\lambda}
\]

(4.29)

where \(\tilde{G}_i\) is Green’s function satisfying

\[
\int \! \! \int \! \! dx \tilde{G}_i(x) = \frac{1}{\lambda}.
\]

(4.30)

Therefore, using the foregoing approximations and definitions we finally write down the current as a local function of the condensate \(\sigma(x)\). We give our results as follows:

\[
\begin{align*}
\Xi &= \tilde{N}_1 \sigma(x) + 2\kappa^2 \mathcal{N}_{(2)}(\Delta) \\
\Theta &= M_1 \sigma(x) + M_2 + M_3 \Delta \sigma(x) + M_4 \sigma(x) + M_5 \sigma(x) \\
\Sigma &= P_1 \sigma(x) + P_2 + P_3 \Delta \sigma(x) + \sum \sigma(x)
\end{align*}
\]

(4.31)

where \(\mathcal{N}, M, \) and \(P\) are all constants which can be expressed as summation over \(\lambda\) whose details are given in the appendix. In the following we will concentrate only on \(\Xi\) and \(\Theta\) because we are interested in finding the current \(J_i\).

\(^{14}\) Note that these boundary conditions are nothing but the artifact of our previously defined boundary conditions in (4.12).

\(^{15}\) We are interested in removing the nonlocalities associated with the \((x, y)\) plane because the nontrivial condensate \((-\sigma(x))\) which forms below a critical magnetic field \((-H_c)\) is basically spanned in the \((x, y)\) plane.

\(^{16}\) According to the BCS theory, the usual superconductors are associated with small length scales characterized by the BCS coherence length and the mean free path. In the presence of these small length scales the condensate does not remain uniform over the region where the gauge field fluctuates, and as a result the current takes the nonlocal form.
4.1. Large $\lambda$ limit

In this section we further simplify the expressions for the constants $\mathcal{N}$ and $\mathcal{M}$ taking into account the large $\lambda$ limit, which essentially means that we replace $\lambda$ with $\lambda_{\text{min}}$ in the sum and are concerned only with those terms that are mostly dominant in the series expansion of $\lambda$. Considering this fact we finally obtain,

$$
\Xi = \frac{\eta_{\lambda_{\text{min}}}(0)}{\lambda_{\text{min}}} \mathcal{F} \left( \lambda_{\text{min}}, \zeta_{\text{min}} \right) \sigma(x) + \kappa^2 H_{22} C
$$

$$
\Theta = \frac{\eta_{\lambda_{\text{min}}}(0)}{\lambda_{\text{min}}} \left[ \sum_{i=7,8} C_i + \frac{C_0}{\lambda_{\text{min}}} \Delta + \frac{H_{22} C_{10}}{\zeta_{\text{min}}} \sigma(x) + H_{22} C_{11} \right]
$$

(4.32)

where the function $\mathcal{F} \left( \lambda_{\text{min}}, \zeta_{\text{min}} \right) \sigma(x)$ takes the following form:

$$
\mathcal{F} \left( \lambda_{\text{min}}, \zeta_{\text{min}} \right) = C_1 + \kappa^2 \left( C_2 + \frac{C_3}{\lambda_{\text{min}}} + \frac{C_4}{\zeta_{\text{min}}} \Delta + \frac{H_{22} C_{10}}{\lambda_{\text{min}} \zeta_{\text{min}}} \right).
$$

(4.33)

Here $C_i$s are all constants that can be estimated by knowing the behavior of the eigen functions $\{\xi_i\}$ and $\{\eta_j\}$ as well as the radial function $(\rho_0(u))$ near the horizon and the boundary of the AdS.

Our next task is to further simplify the expressions given in (4.32). From our earlier discussions in section 2 one can note that the vortex solution can be expressed as $\sigma(x) = |\psi_0|^2$ where $\psi_0$ is given by (2.35). Using this relation, we finally express $\Delta \sigma(x)$ in terms of the lowest energy solution $(\psi_0)$ of the GL equation, which essentially turns out to be

$$
\Delta \sigma(x) = \left( \partial_x^2 + \partial_y^2 \right) e^{-2i\theta} \psi_0^2
$$

$$
= \left( -2H_{22} + 4A_y^2 \right)|\psi_0|^2 - \frac{4ix}{d^2 \xi^2} e^{-2i\theta} \frac{\partial}{\partial \psi^*} \left( \frac{\partial}{\partial \psi} \right) e^{2i\theta} |\psi_0|^2
$$

$$
+ \frac{4}{d^2 \xi^2} e^{-2i\theta} \frac{\partial^2}{\partial \psi^* \partial \psi} e^{2i\theta} |\psi_0|^2
$$

(4.34)

where we have used the fact that

$$
A_y = H_{22} = \frac{x}{\xi^2}. \tag{4.35}
$$

Also, from the boundary conditions (4.27) it is quite evident that all the terms that are proportional to $n_{\text{nec}}(0)$ will ultimately vanish. In this sense the term $\theta$ does not exist at all, and therefore we are left solely with $\Xi$, which finally determines the GL current in the presence of anomaly. Using all these facts one can finally express $\Xi$ as

$$
\Xi = A_1 |\psi_0|^2 + A_2 |\psi_0|^2 + A_3
$$

(4.36)
where the coefficients $A_1$, $A_2$, and $A_3$ are given as follows:

$$
A_1 = \frac{\eta'_{\text{hase}}(0)}{\lambda_{\text{min}}} \left[ C_1 + \kappa^2 \left( C_2 + \frac{C_3}{\xi_{\text{min}}} + \frac{C_4}{\xi_{\text{min}}^2} + \frac{H_{12} C_6}{\xi_{\text{min}}^2} \right) \right]
$$

$$
A_2 = \frac{\kappa^2 \eta'_{\text{hase}}(0) C_5}{\lambda_{\text{min}}^3} \left[ \left( -2H_{12} + 4A_2^2 \right) + \left( -\frac{4ix}{\kappa^2} e^{-\frac{x}{\kappa}} \left( \frac{\partial}{\partial \mu} - \frac{\partial}{\partial v} \right) \right) \right]
$$

$$
+ \frac{4}{\alpha^2 \xi^2} e^{-\frac{x}{\kappa}} \left( \frac{\partial^2}{\partial \mu^2} \right) e^{\frac{x}{\kappa}}
$$

$$
A_3 = \frac{\kappa^2}{\lambda_{\text{min}}^2} \eta'_{\text{hase}}(0) H_{12} C.
$$

From the expressions (4.34), (4.36), and (4.37) and using the AdS/CFT prescription, it is now quite trivial to compute the boundary current (4.21) (which is basically the GL current for the present case) in the long wavelength approximation, which takes the following form:

$$
J_i = A_1 e^{i/\lambda} \partial_j \sigma(x) + \frac{\kappa^2 \eta'_{\text{hase}}(0) C_5}{\lambda_{\text{min}}^3} e^{i/\lambda} \partial_j (\Delta \sigma(x)).
$$

This is the final expression for the GL current in the presence of the global $U(1)$ anomaly. From the preceding expression (4.38) it can be easily noticed that in the presence of the anomaly the usual GL current receives a nontrivial correction that arises at the quadratic order in $\lambda$ w.r.t the leading term. By nontrivial correction we explicitly mean the term associated with the derivative of the local function $\Delta \sigma(x)$. The coefficient for this nontrivial correction term can be estimated by knowing the constant $C_5$, which is of the following form:

$$
C_5 = \int_{0}^{1} du' u'' \partial_{\mu'} A_i^{0}(x) (u') \eta'_{\text{hase}}(u') \eta'_{\text{hase}}^{-1}(u')
$$

$$
\times \int_{0}^{1} du'' u''' \partial_{\mu''} A_i^{0}(x) (u'') \eta'_{\text{hase}}(u'') \eta'_{\text{hase}}^{-1}(u'') \int_{0}^{1} du'''' \partial_{\mu''''} A_i^{0}(x) (u''') \eta'_{\text{hase}}(u''') \eta'_{\text{hase}}^{-1}(u''').
$$

Before we proceed further, some important issues must be discussed. First, considering the boundary behavior (3.7) of the scalar field $\psi$ (which basically tells us about the nature of the radial solution $\rho_0(u)$ as $u \to 0$), we note that all the integrals (over the radial coordinate $u$) that appear in (4.39) are finite, and therefore the coefficient $C_5$ is a finite number.

Second and most importantly, looking at (4.6) and (4.24) one can easily figure out that $\sqrt{\lambda} \sim T$, where $T$ is the temperature of our system. Thus the nontrivial term on the rhs of equation (4.38) essentially corresponds to a finite temperature correction to the usual GL current. This is the correction to the current (due to the presence of the anomaly) that goes as $\sim 1/T^2$ w.r.t the leading term and thereby greatly suppressed at high temperatures. Thus, following the AdS/CFT prescription, for real-life superconductors one should realize similar effects at finite but nonzero temperatures. This further suggests that the usual GL current would be modified at finite nonzero temperatures.

17 Note that the coefficients $A_1$ and $A_3$ arise solely due to the presence of the anomaly.
18 Here we have re-scaled the current $J_i$ by the factor $\frac{\pi}{\alpha^2}$.
19 The anomaly corrections that appear in $A_1$ are trivial since they are associated with the vortex solution $\sigma(x)$.
20 See the corresponding term $\mathcal{N}_{\text{lg}}$ in the appendix.
4.2. Modifying the GL theory

Inspired by the entire analysis done so far, we are now in a position to make the following comments regarding the usual GL theory for real-life superconductors: According to the AdS/CFT duality, the usual GL current \( J_{GL} \) would receive a highly nontrivial correction once we incorporate the effect of anomaly in the theory, and the effect of such a correction could be measured only at finite (low) temperatures. As the temperature of the sample is increased we would get back to the usual GL current with some trivial correction due to anomaly, due to the presence of \( C_2 \) in \( A_1 \) as is evident from equations (4.37) and (4.38). Therefore, based on this observation we propose the following modification to the usual GL current for ordinary superconductors:

\[
J_{GL} \rightarrow \epsilon_i \partial_j \sigma(x) + \kappa^2 \epsilon_i \partial_j (\Delta \sigma(x))
\]  

(4.40)

where the first term stands for the usual GL current that is already familiar to us, whereas the second term on the rhs of (4.40) stands for the nontrivial correction that appears at finite but low temperatures.

5. Free energy

Computation of free energy is always important in order to describe the thermodynamic stability of a given system. A particular configuration is stable if it possesses the minimum free energy. The phase transition that we describe throughout this paper is basically a second order transition between a normal and a superconducting phase in the presence of an external magnetic field.

In the usual GL theory, the thermodynamically most favorable configuration corresponds to a triangular lattice solution that minimizes the free energy. Since our boundary theory possesses a global \( U(1) \) anomaly which arises due to the presence of the CS term in the bulk action, it is naturally expected that in the present case, the free energy would receive some nontrivial corrections due to the presence of this CS term. From the computations of this section we find that this is indeed the case, where considering the large \( \lambda \) approximation we give the precise expression of the free energy up to the quadratic order in \( \kappa \).

The free energy of a particular system is defined as the onshell action evaluated with appropriate counter terms,

\[
F = -S_{onshell}.
\]

(5.1)

Let us first evaluate the (onshell) action \( S_{onshell} \) (3.1) for the scalar field. Note that the equation of motion for the scalar field may be written as

\[
D_a^2 \psi - m^2 \psi = 0
\]

(5.2)

where

\[
D_a = \nabla_a - iA_a.
\]

(5.3)

Using this we find that the action (3.1) evaluated onshell turns out to be

\[
S_{\psi} \bigg|_{onshell} = -\frac{1}{2} \int_{\partial M} d\Sigma_a \sqrt{-g} \left( \nabla^a - iA^a \right) |\psi|^2.
\]

(5.4)

Here \( d\Sigma_a \) is the volume of the hypersurface \( \partial M \) which is the boundary of the AdS space \( M \). In the following we consider various cases regarding the choice of \( \partial M \).
• If we take \( a = t \), then \( \partial M \) will correspond to (past and future) spacelike surfaces where according to our choice of \textit{stationary} field configurations the time derivative of the fields vanishes.

• If we choose \( a = u \), then this integral vanishes at the horizon (as \( g^{uu} = 0 \)) and also at the boundary \( (u \to 0) \) due to the rapid falloff of the scalar field.

• For the case \( a = i \), we choose \( x = \text{constant} \) hyper surfaces in such a way that the condensate is highly suppressed due to the presence of the exponentially decaying factor\(^{21} \).

Thus, considering all these facts the action for the scalar field evaluated \textit{onshell} turns out to be

\[
S_{\psi_0} = 0. \tag{5.5}
\]

Therefore we are finally left with the \textit{onshell} action for the gauge fields only:

\[
S_{os} = \int d^5x \sqrt{-g} \left( -\frac{1}{4} F^2 + \frac{\kappa}{\sqrt{-g}} \epsilon^{abcde} A_a F_{bc} F_{de} \right). \tag{5.6}
\]

To evaluate (5.6), we expand it perturbatively in \( \epsilon \):

\[
S_{os} = S_{os}^{(0)} + \epsilon S_{os}^{(1)} + \epsilon^2 S_{os}^{(2)} + \mathcal{O}(\epsilon^3) \tag{5.7}
\]

where various terms on the rhs of (5.6) correspond to perturbations of the action at different orders in \( \epsilon \). Our aim is to evaluate various terms in (5.7) using the zeroth as well as the first order Maxwell equations,

\[
V_b F^{(0)ab} - \frac{\kappa}{\sqrt{-g}} \epsilon^{abcde} F^{(0)bc} F^{(0)de} = 0
\]

\[
V_b F^{(1)ab} - 2\frac{\kappa}{\sqrt{-g}} \epsilon^{abcde} F^{(0)bc} F^{(1)de} = j^{(1)a}
\]

(5.8)

along with the orthogonality condition

\[
\int d^5x \sqrt{-g} j^{(1)a} A_a^{(1)} = 0. \tag{5.9}
\]

First, we may drop \( S_{os}^{(0)} \) because we are interested in computing the free energy corresponding to a nonzero value of the condensate.

Next, using (5.8) and (5.9), we compute the first order correction to the action, which turns out to be

\[
S_{os}^{(1)} = -\int_{\partial M} d\Sigma \sqrt{-g} F^{(0)ab} A_b^{(1)} \bigg|_{u=0} = 0 \tag{5.10}
\]

where we have used the fact that the sources corresponding to \( A_t \) and \( A_z \) are kept fixed at the boundary. As a result both \( A_t^{(i)} \) and \( A_z^{(i)} \) \((i = 1, 2, \ldots)\) vanish as we approach the boundary of the AdS \((u \to 0)\).

\(^{21}\) Note that since \( \varphi(x, y) \approx \epsilon^{ik} \varphi_0(x) \), \( \varphi \mid^2 \) will depend only on \( x \) (see (2.28)).
Finally we compute the second order correction to the action \( S_{os}^{(2)} \), which turns out to be
\[
S_{os}^{(2)} = -\int d\Sigma u \sqrt{-g} F^{(0)ab} A_b^{(2)} - \frac{1}{2} \int d\Sigma u \sqrt{-g} F^{(1)ab} A_b^{(1)} - 2\kappa \int d\Sigma u e^{abdef} A_b^{(0)} F_{cd}^{(0)} A_c^{(2)} - 2\kappa \int d\Sigma u e^{abdef} A_b^{(0)} F_{cd}^{(1)} A_c^{(1)}. \tag{5.11}
\]

Following our previous arguments, one can note that the first and the third term on the rhs of (5.11) vanish identically. On the other hand, using (4.2) the second and the fourth term may be combined as
\[
S_{os}^{(2)} = \frac{1}{2\epsilon} \int d\Sigma u J^{(1)} A_i^{(1)} \bigg|_{u=0}. \tag{5.12}
\]

Finally, using (4.21), we arrive at the following expression for the onshell action\(^{22}\):
\[
S_{os} = \frac{\kappa^2}{2} \int d\Sigma u e_{ij} \partial_i \Xi A_i^{(1)} \bigg|_{u=0} = -\frac{\epsilon^2}{2} \int d\Sigma u \Xi e_{ij} \partial_i \partial_j \bigg|_{u=0} = \frac{H_2 \epsilon^2}{2} \int d\Sigma \Xi(x). \tag{5.13}
\]

For any function \( f(x) \), let us define the following quantity:
\[
\bar{f} = \frac{1}{V} \int d\Sigma f(x) \tag{5.14}
\]
which represents the average of \( f(x) \) over the volume \( V \) in the \((x, y)\) plane.

Substituting (5.13) into (5.1) and using the preceding definition (5.14), the free energy \( F \) per unit volume \( V \) in the \((x, y)\) plane turns out to be
\[
F/V = -\frac{H_2 \epsilon^2}{2} \Xi \tag{5.15}
\]
where
\[
\Xi = A_i \sigma_i(x) + \frac{\kappa^2 \eta^{(0)}_{\text{max}}}{\lambda_{\text{min}}^3} \Xi(x). \tag{5.16}
\]

This is the final expression for the GL free energy in the presence of a global \( U(1) \) anomaly. From the expressions (5.15) and (5.16) it should be clear by now that in the presence of a global \( U(1) \) anomaly, the quantity \( F/V \) would be modified by a nontrivial factor. This effect is quite similar to that which we have found in the preceding section while computing the GL current in the presence of a \( U(1) \) anomaly. As in the preceding case we note that this nontrivial effect appears as \( O(\frac{1}{T^4}) \) w.r.t. the leading term in the expression and therefore would modify the original GL free energy only at finite (low) temperatures.

6. Conclusions

We conclude our discussion with a few remarks.

The holographic computation of supercurrent turns out to be a nonlocal quantity. However, in the large \( \lambda \) limit, which we considered in section 4.1, the expression takes the

\(^{22}\) Note that \( \Theta \) is zero at the boundary \( u = 0 \).
form of a local quantity. The large \( \lambda \) correction essentially says that
\[
\sqrt{\frac{B_{1/2}}{T}} \ll 1.
\] (6.1)

Therefore the effect of the CS term on the supercurrent is suppressed at high temperatures with respect to the leading term.

An important fact about the final expression of supercurrent (equation (4.40)) is that it is not only the CS term that produces \( 1/\lambda^2 \) correction in the current. There are other terms which also give \( 1/\lambda^2 \) corrections. The source for these terms comes from the large \( \lambda \) correction of \( \sigma(x) \):
\[
\sigma(x') \sim \sigma(x) + \frac{\eta}{\sqrt{\lambda}} \sigma'(x) + \cdots
\] (6.2)

When we substitute the expansion of \( \sigma(x') \) in equation (4.29) we get a correction on the order of \( 1/\lambda, 1/\lambda^2 \), and so on. However, our goal in this paper is not to find a large \( \lambda \) correction of the supercurrent but to compute the effect of the CS term on holographic supercurrent. Therefore, we did not consider these terms in our final result.

Acknowledgments

We would like to thank Suhas Gangadharaiah and Krishnendu Sengupta for valuable discussions. SD acknowledges the hospitality of NIKHEF, Amsterdam, where part of the work was done. NB would like to thank the hospitality of IISER Bhopal at the final stage of this work. Research of NB is supported by a DST Ramanujan Grant. Finally we are thankful to the people of India for their support of science.

Appendix A. Calculation of \( \Xi, \Theta, \) and \( \Sigma \)

The appendix consists of all important formulae that appear while computing the boundary current. The expressions are extremely important, and hence we provide them here. In equation (4.21), the two quantities that appear in the \( i \)th component of the current \( J_i \) are \( \Xi \) and \( \Theta \), and the one that appears in \( J_z \) is \( \Sigma \). Hereafter, we outline the steps to calculate these three quantities.

Calculation of \( \Xi \)

\[
\Xi = \Xi^{(e_0)} + 2\lambda^2 \Xi^{(e_2)}
\] (A.1)

\[
\Xi^{(e_0)} = N_{\frac{1}{2}}(e_0) \sigma(x)
\] (A.2)

where
\[
N_{\frac{1}{2}}(e_0) = \sum_i \frac{\eta_i^{(0)}}{\lambda} \int_0^1 du' \frac{\rho_0^{2(e_0)}(u')}{u'} \eta_i^{(0)}(u')
\] (A.3)

Note
\[
\Xi^{(e_2)} = \Xi^{(e_2)}_{(1)} + \Xi^{(e_2)}_{(2)}
\] (A.4)
Let us calculate the following:

\[
\text{First term}
\]

\[
\mathcal{N}_{\lambda(2)}^{(1)}(\epsilon^{(2)})\sigma(x) = -16 \sum_{\lambda} \frac{\eta^{(0)}_\lambda}{\lambda} \int_0^1 du \int_0^1 du' \partial_{u'} A_i^{(0)}(\epsilon^{(0)}) (u') \partial_{\lambda'} \mathcal{J}(u', x') \times \mathcal{G}_\lambda(u, u'; x, x')
\]  

(A.7)

It has three terms.

Let us calculate the following:

\[
\text{Second term}
\]

\[
\mathcal{N}_{\lambda(2)}^{(2)}(\epsilon^{(2)})\sigma(x) = -H_{\epsilon,2} + \kappa^2 \sum_\lambda \eta^{(0)}_\lambda \int_0^1 du \int_0^1 du' \partial_{u'} A_i^{(0)}(\epsilon^{(0)}) (u') \partial_{\lambda'} \mathcal{J}(u', x') \times \mathcal{G}_\lambda(u, u'; x, x')
\]  

(A.10)

where

\[
\mathcal{N}_{\lambda(2)}^{(2)}(\epsilon^{(2)}) = -128 \sum_{\lambda, \lambda', \lambda''} \frac{\eta^{(0)}_\lambda}{\lambda} \int_0^1 du \int_0^1 du' \partial_{u'} A_i^{(0)}(\epsilon^{(0)}) (u') \partial_{\lambda'} \mathcal{J}(u', x') \times \int_0^1 du'' \partial_{u''} A_i^{(0)}(\epsilon^{(0)}) (u'') \partial_{\lambda''} \mathcal{J}(u'', x'') \times \mathcal{G}_\lambda(u, u'; x, x')
\]  

(A.11)

Third term

\[
\mathcal{N}_{\lambda(2)}^{(3)}(\epsilon^{(2)})\sigma(x) = 
\]  

(A.13)
where
\[ \mathcal{N}_{\lambda(2)}^{(3)}(\varepsilon^{(2)}) = 16^2H_2 \sum_{\lambda,\xi,\zeta} \frac{\eta_\xi(0)}{\lambda} \int_0^1 du' u' \partial_\nu A_{\zeta}(0)(\varepsilon^{(0)})(u') \frac{\eta_\lambda(u')}{\lambda'} \eta_\lambda(u') \]
\[ \times \int_0^1 du'' u'' \frac{\xi(u''\lambda_\zeta)}{\zeta} \eta_\lambda(u''\lambda_\zeta) \int_0^1 du''' u''' \frac{2(\varepsilon^{(0)})}{u'''}(u''')A_{\zeta}(0)(\varepsilon^{(0)})(u''')\xi_\lambda(u''')(u''') \] (A.14)

where

**Last Term**

\[ \mathcal{N}_{\lambda(2)}^{(4)}(\varepsilon^{(2)}) \sigma(x) \] (A.15)

where

\[ \mathcal{N}_{\lambda(2)}^{(4)}(\varepsilon^{(2)}) = -16 \sum_{\lambda,\xi,\zeta} \frac{\eta_\xi(0)}{\lambda} \int_0^1 du' u' \partial_\nu A_{\zeta}(0)(\varepsilon^{(1)})(u') \frac{\xi(u')}{\zeta} \eta_\lambda(u') \]
\[ \times \int_0^1 du'' u'' \frac{2(\varepsilon^{(0)})}{u''}(u'')A_{\zeta}(0)(\varepsilon^{(0)})(u'')\xi_\lambda(u'') \] (A.16)

Finally, adding all these terms, we find

\[ \Xi(\varepsilon^{(2)}) = \mathcal{N}_{\lambda(2)}^{(1)}(\varepsilon^{(2)}) \sigma(x) + \mathcal{N}_{\lambda(2)}^{(0)}(\varepsilon^{(2)}) \] (A.17)

where

\[ \mathcal{N}_{\lambda(2)}^{(1)}(\varepsilon^{(2)}) = \mathcal{N}_{\lambda(2)}^{(1)}(\varepsilon^{(2)}) + \mathcal{N}_{\lambda(2)}^{(2)}(\varepsilon^{(2)}) + \Delta + \mathcal{N}_{\lambda(2)}^{(3)}(\varepsilon^{(2)}) + \mathcal{N}_{\lambda(2)}^{(4)}(\varepsilon^{(2)}) \] (A.18)

and

\[ \mathcal{N}_{\lambda(2)}^{(0)}(\varepsilon^{(2)}) = \frac{128H_2}{\iota_0^2} \sum_{\lambda,\lambda'} \frac{\eta_\lambda(0)}{\lambda} \int_0^1 du' u' \partial_\nu A_{\lambda'}(0)(\varepsilon^{(0)})(u') \frac{\eta_{\lambda'}(u')}{\lambda'} \eta_\lambda(u') \]
\[ \times \int_0^1 du'' u'' \partial_\nu A_{\lambda'}(0)(\varepsilon^{(0)})(u'')\eta_\lambda(u'') \] (A.19)

**Final result**

\[ \Xi = \mathcal{N}_{\lambda(2)}^{(2)} \sigma(x) + 2\varepsilon^2 \mathcal{N}_{\lambda(2)}^{(0)}(\varepsilon^{(2)}) \] (A.20)

where

\[ \mathcal{N}_{\lambda(2)}^{(2)} = \mathcal{N}_{\lambda(2)}^{(2)}(\varepsilon^{(2)}) + 2\varepsilon^2 \mathcal{N}_{\lambda(2)}^{(0)}(\varepsilon^{(2)}) \] (A.21)

**Calculation of \( \Theta \)**

\[ A_t^{(0)}(\varepsilon^{(0)}) \bigg|_{u=0} = q \] (A.22)
\[ A_z^{(0)}(\varepsilon^{(0)}) \bigg|_{u=0} = C \] (A.23)
First term:
\[ \mathcal{M}_1 \sigma(x) \quad (A.24) \]
where
\[ \mathcal{M}_1 = \frac{\kappa^2 qC}{3\pi G} \sum_{\lambda} \eta_{\lambda}(0) \int_0^1 \frac{du'}{u'} \rho_0 z^2(x) (u') \eta_{\lambda}^+(u') \quad (A.25) \]

Second term
\[ \mathcal{M}_2 + \mathcal{M}_3 \Delta \sigma(x) \quad (A.26) \]
where
\[ \mathcal{M}_2 = \frac{8k^2 H_\text{r} q}{3\pi G} \sum_{\lambda} \eta_{\lambda}(0) \int_0^1 du' u' \partial_u A_i^{(0)}(x) (u') \eta_{\lambda}^+(u') \quad (A.27) \]
\[ \mathcal{M}_3 = \frac{8k^2 g}{3\pi G} \sum_{\lambda, \zeta} \eta_{\lambda}(0) \int_0^1 du' u' \partial_u A_i^{(0)}(x) (u') \eta_{\lambda}^+(u') \eta_{\zeta}(u') \quad (A.28) \]

Third term
\[ \mathcal{M}_4 \sigma(x) \quad (A.29) \]
where
\[ \mathcal{M}_4 = \frac{16k^2 H_\text{r} q}{3\pi G} \sum_{\lambda, \zeta} \eta_{\lambda}(0) \int_0^1 du' u' \partial_u \xi_{\zeta}^+(u') \eta_{\lambda}^+(u') \quad (A.30) \]

Last term
\[ \mathcal{M}_5 \sigma(x) \quad (A.31) \]
where
\[ \mathcal{M}_5 = \frac{\kappa^2 C}{3\pi G} \sum_{\lambda} \eta_{\lambda}(0) \int_0^1 du' u' \rho_0 z^2(x) (u') A_i^{(0)}(x) (u') \eta_{\lambda}^+(u') \quad (A.32) \]

Calculation of \( \Sigma \)
First term
\[ P_1 \sigma(x) \quad (A.33) \]
where
\[ P_1 = \sum_{\lambda} \eta_{\lambda}(0) \int_0^1 du' u' \rho_0 z^2(x) (u') A_i^{(0)}(x) (u') \eta_{\lambda}^+(u') \quad (A.34) \]

Second term
\[ P_2 + P_3 \Delta \sigma(x) \quad (A.35) \]
where
\[ P_2 = -4Hc_2 \sum \frac{\eta_j^l(0)}{\lambda} \int_0^1 \frac{du'}{u'^4} \partial_u A_1^{(0)}(u') \eta_j^l(u') \]
\[ P_3 = 4 \sum \frac{\eta_j^l(0)}{\lambda} \int_0^1 \frac{du'}{u'^4} \partial_u A_1^{(0)}(u') \eta_j^l(u') \frac{\eta_j^l(u')}{\lambda'} \int_0^1 \frac{du''}{u''^2} \rho_0^2 \left( \frac{\lambda''}{\lambda''^2} \right) (u') \eta_j^l(u'') \] (A.36)

Third term
\[ P_2 \sigma(x) \] (A.37)

where
\[ P_4 = -8 \sum \frac{\eta_j^l(0)}{\lambda} \int_0^1 \frac{du'}{u'^4} \frac{\eta_j^l(u')}{\lambda'} \eta_j^l(u') \]
\[ \times \int_0^1 \frac{du''}{u''^2} \rho_0^2 \left( \frac{\lambda''}{\lambda''^2} \right) (u') A_1^{(0)}(u'') (u') \eta_j^l(u'') \sigma(x) \] (A.38)

Using the preceding expressions, we can get the explicit form of the boundary current.

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