The precision theoretical analysis of neutron radiative beta decay is discussed. In the Appendix, we give the detailed calculation of the amplitude and rate of the neutron radiative β-decay with one-real photon emission by taking into account the contributions of the weak magnetism and proton recoil to order 1/M, where M = (m_n + m_p)/2 is the averaged nucleon mass. The latter is important for a calculation of a robust theoretical background for the experimental analysis of interactions beyond the Standard Model. The paper is organized as follows. In section II, we give the analytical expressions for the amplitude and rate of the neutron radiative β-decay. In section III, we adduce the results of the numerical analysis of the branching ratios of the neutron radiative β-decay for the experimental regions of photons energies. We discuss the obtained results and perspectives of the further theoretical analysis of the neutron radiative β-decay. In the Appendix, we give the detailed calculation of the amplitude and rate of the neutron radiative β-decay in the tree–approximation.
II. RADIATIVE $\beta^–$–DECAY OF NEUTRON IN THE TREE–APPROXIMATION TO NEXT–TO–LEADING ORDER IN THE LARGE M EXPANSION

In the Standard Model the neutron radiative $\beta^–$–decay is described by the following interactions

\[
\mathcal{L}_{\text{int}}(x) = \mathcal{L}_W(x) + \mathcal{L}_{\text{em}}(x). \tag{1}
\]

Here $\mathcal{L}_W(x)$ is the effective Lagrangian of low–energy $V - A$ interactions with a real axial coupling constant $\lambda = -1.2750(9)$ \cite{[17]} and the contribution of the weak magnetism $\mathcal{L}_{\text{em}}(x)$

\[
\mathcal{L}_W(x) = -\frac{G_F}{\sqrt{2}} V_{\text{ud}} \left\{ \bar{\psi}_p(x) \gamma_\mu (1 + \lambda \gamma^5) \psi_n(x) \right\} + \frac{\kappa}{2M} \partial^\nu (\bar{\psi}_p(x) \sigma_{\mu\nu} \psi_n(x)), \tag{2}
\]

invariant under time reversal, where $G_F = 1.1664 \times 10^{-11} \text{ MeV}^{-2}$ is the Fermi coupling constant, $|V_{\text{ud}}| = 0.97417(21)$ is the Cabibbo–Kobayashi–Maskawa matrix element and $\kappa = \kappa_p - \kappa_n = 3.7058$ is the isovector anomalous magnetic moment of the nucleon, defined by the anomalous magnetic moments of the proton $\kappa_p = 1.7928$ and the neutron $\kappa_n = -1.9130$ and measured in nuclear magneton \cite{[18]}. Then, $\bar{\psi}_p(x), \psi_n(x), \psi_e(x)$ and $\psi_\nu(x)$ are the fields operators of the proton, neutron, electron and antineutrino, respectively; $\gamma^\mu, \gamma^5$ and $\sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ are the Dirac matrices \cite{[19]}. Then, $\mathcal{L}_{\text{em}}(x)$ is the Lagrangian of the electromagnetic interaction

\[
\mathcal{L}_{\text{em}}(x) = -e \left\{ [\bar{\psi}_p(x) \gamma_\mu \psi_p(x)] - [\bar{\psi}_e(x) \gamma^\mu \psi_e(x)] \right\} A_\mu(x), \tag{3}
\]

where $e$ is the proton electric charge, related to the fine–structure constant $\alpha$ by $e^2 = 4\pi\alpha$, and $A_\mu(x)$ is the electromagnetic potential \cite{[10]}.

In the tree–approximation the Feynman diagrams of the amplitude of the neutron radiative $\beta^–$–decay are shown in Fig.1. The amplitude of the neutron radiative $\beta^–$–decay, defined by the diagrams in Fig.1 we describe by the expression \cite{[10]}

\[
M(n \rightarrow pe^–\bar{\nu}_e\gamma)_{\lambda'} = e \frac{G_F}{\sqrt{2}} V_{\text{ud}} M(n \rightarrow pe^–\bar{\nu}_e\gamma)_{\lambda'}, \tag{4}
\]

where $M(n \rightarrow pe^–\bar{\nu}_e\gamma)_{\lambda'}$ is equal to

\[
M(n \rightarrow pe^–\bar{\nu}_e\gamma)_{\lambda'} = \left[ \bar{u}_p(k_p, \sigma_p) \varepsilon^\lambda_{\nu}(k) \frac{1}{m_p - k_p - k - i\delta} O_{\mu\nu}(\vec{k}_n, \sigma_n) \right] \left[ \bar{u}_e(\vec{k}_e, \sigma_e) \gamma^\mu (1 - \gamma^5) \gamma^\nu(n, \vec{k} + \frac{1}{2}) \right] - \left[ \bar{u}_p(k_p, \sigma_p) O_{\mu\nu}(\vec{k}_n, \sigma_n) \right] \left[ \bar{u}_e(\vec{k}_e, \sigma_e) \varepsilon^\nu_{\lambda}(k) \frac{1}{m_e - k_e - k - i\delta} \gamma^\mu (1 - \gamma^5) \gamma^\nu(n, \vec{k} + \frac{1}{2}) \right]. \tag{5}
\]

The matrix $O_{\mu\nu}$ is given by \cite{[10]}

\[
O_{\mu\nu} = \gamma_\mu (1 + \lambda \gamma^5) + i \frac{\kappa}{2M} \sigma_{\mu\nu}(k_p - k_n)^\nu. \tag{6}
\]

Then, $\bar{u}_p(k_p, \sigma_p), u_n(\vec{k}_n, \sigma_n), \bar{u}_e(\vec{k}_e, \sigma_e)$ and $\nu_\nu(\vec{k}_\nu, +\frac{1}{2})$ are the Dirac wave functions of the free proton, neutron, electron and electron antineutrino with 3–momenta $k_p, \vec{k}_n = \vec{0}, \vec{k}_e$ and $\vec{k}_\nu$ and polarizations $\sigma_p = \pm 1, \sigma_n = \pm 1, \sigma_e = \pm 1$ and $+\frac{1}{2}$ \cite{[16] [22]}, respectively. $\varepsilon^\nu_{\lambda}(k)$ is the polarization vector of the photon in the polarization state $\lambda' = 1, 2$ with a 4–momentum $\vec{k}$, obeying the constraint $\varepsilon^\nu_{\lambda}(k) \cdot k = 0$. The amplitude Eq.\cite{[5]} is gauge invariant. Indeed, one may show that, replacing $\varepsilon^\nu_{\lambda}(k) \rightarrow k^\alpha$ and using the Dirac equations for the free proton and electron, the amplitude Eq.\cite{[5]} vanishes.
The rate of the neutron radiative $\beta^-$–decay, described by the Feynman diagrams in Fig. 1 with photon from the energy region $\omega_{\text{min}} \leq \omega \leq \omega_{\text{max}}$, is equal to (see [10] and Eq. (A.33) of the Appendix)

$$
\lambda_{\beta\gamma}(\omega_{\text{max}}, \omega_{\text{min}}) = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{2\pi^3} \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \int_{m_n}^{E_0 - \omega} dE E \sqrt{E^2 - m_e^2} (E_0 - E - \omega)^2 F(E, Z = 1) \rho_{\beta\gamma}(E, \omega),
$$

(7)

where $E_0 = (m_n^2 - m_e^2 + m_p^2)/2m_n$ is the end–point energy of the electron energy–spectrum of the neutron $\beta^-$–decay [10] and $F(E, Z = 1)$ is the relativistic Fermi function, describing Coulomb proton–electron final–state interaction. It is equal to [10]

$$
F(E, Z = 1) = \left(1 + \frac{1}{2 \gamma}\right) \frac{4(2\pi m_e \beta \gamma)^2}{\Gamma^2(3 + 2\gamma)} \frac{e^{\pi\alpha/\beta}}{(1 - \beta^2)^{\gamma}} |\Gamma(1 + \gamma + i \alpha/\beta)|^2,
$$

(8)

where $\beta = k_e/E_e = \sqrt{E_0^2 - m_e^2}/E_e$ is the electron velocity, $\gamma = \sqrt{1 - \alpha^2} - 1$, $r_p$ is the electric radius of the proton and $\alpha = 1/137.036$ is the fine–structure constant. In numerical calculations we will use $r_p = 0.841$ fm [22]. The function $\rho_{\beta\gamma}(E, \omega)$ is calculated in the Appendix. It is given by the integral

$$
\rho_{\beta\gamma}(E, \omega) = \int \frac{d\Omega_{\gamma\gamma}}{4\pi} \left[1 + 2 \frac{\omega}{M} \frac{E_e - k_e \cdot \bar{n}}{E_0 - E_e - \omega} + \frac{3}{M} \left(E_e + \omega - \frac{1}{3} E_0\right) + \frac{\lambda^2 - 2(\kappa + 1) + \lambda + 1}{1 + 3\lambda^2} \frac{E_0 - E_e - \omega}{M} \right] \times \left[\left(1 + \frac{\omega}{E_e}\right) \frac{k_e^2 - (k_e \cdot \bar{n})^2}{E_e - k_e \cdot \bar{n}} + \frac{\omega^2}{E_e - k_e \cdot \bar{n}} + 3\lambda^2 - 1 - 1 \frac{E_e}{E_e - k_e \cdot \bar{n}} \left(\frac{k_e^2 + \omega k_e \cdot \bar{n}}{E_e - k_e \cdot \bar{n}} \frac{k_e^2 - (k_e \cdot \bar{n})^2}{E_e - k_e \cdot \bar{n}} \right) + \frac{\omega}{E_e - k_e \cdot \bar{n}} \right]

+ \left(\frac{\omega + k_e \cdot \bar{n}}{E_e - k_e \cdot \bar{n}} \left(1 + \frac{\omega}{E_e - k_e \cdot \bar{n}} - \frac{m_e^2}{E_e (E_e - k_e \cdot \bar{n})} - \frac{\omega}{E_e - k_e \cdot \bar{n}} \right) \right) - \lambda(\lambda - 1) \frac{1}{1 + 3\lambda^2} \frac{E_e}{E_e - k_e \cdot \bar{n}} \left[\frac{k_e^2 + \omega^2 + 2\omega k_e \cdot \bar{n}}{E_e - k_e \cdot \bar{n}} + \frac{\omega^2}{E_e - k_e \cdot \bar{n}} + \frac{\lambda^2 - 2(\kappa + 1) + \lambda + 1}{1 + 3\lambda^2} \frac{E_0 - E_e - \omega}{M} \right] \frac{d\Omega_{\gamma\gamma}}{4\pi}.
$$

(9)

Here $d\Omega_{\gamma\gamma}$ is an infinitesimal solid angle of the electron–photon momentum correlations $\bar{k}_e \cdot \bar{n} = k_e \cos \theta_{\gamma\gamma}$, where $\bar{n} = \vec{k}/\omega$ is a unit vector along the photon 3–momentum [10, 11]. The results of the numerical analysis of the rate of the neutron $\beta^-$–decay, calculated relative to the neutron lifetime $\tau_n = 879.6(1.1)$ s [11, 11], we give and discuss in section III.

### III. DISCUSSION AND CONCLUSION

Recent new experimental measurements of the branching ratio of the neutron radiative $\beta^-$–decay, reported by the RDK II Collaboration Bales et al. [16], have been the impetus for our theoretical analysis of the neutron radiative $\beta^-$–decay. We have performed the calculation of the amplitude of the neutron radiative $\beta^-$–decay in the tree–approximation to next–to–leading order in the large proton mass expansion. We have taken into account the complete set of contributions of the weak magnetism and proton recoil to order $1/M$, where $M$ is the averaged nucleon mass. The obtained results we consider as a first step towards the precision theoretical analysis of the neutron radiative $\beta^-$–decay in the Standard Model to a relative order $10^{-3}$ [11, 11]. The detailed calculations of the amplitude and decay rate we give in the Appendix. The numerical values of the branching ratios of the neutron radiative $\beta^-$–decay, calculated relative to the neutron lifetime $\tau_n = 879.6(1.1)$ s [11, 11], for the experimental regions of photon energies are added in the Table I.

| $\omega$ [keV] | BR$_{\beta\gamma}$ (Experiment) | BR$_{\beta\gamma}$ (Theory) | BR$_{\beta\gamma}$ (Theory) $M \to \infty$ | $\Delta$BR$_{\beta\gamma}$ (Theory) |
|---------------|-------------------------------|---------------------------|-------------------------------|---------------------------|
| 15 $\leq \omega \leq$ 340 | $(3.09 \pm 0.32) \times 10^{-3}$ [13] | 2.89 $\times 10^{-3}$ | 2.87 $\times 10^{-3}$ | 0.70% |
| 14 $\leq \omega \leq$ 782 | $(3.35 \pm 0.05$ stat $\pm 0.15$ syst) $\times 10^{-3}$ [16] | 3.04 $\times 10^{-3}$ | 3.02 $\times 10^{-3}$ | 0.66% |
| 0.4 $\leq \omega \leq$ 14 | $(5.82 \pm 0.23$ stat $\pm 0.62$ syst) $\times 10^{-4}$ [16] | 5.08 $\times 10^{-4}$ | 5.05 $\times 10^{-4}$ | 0.60% |

TABLE I: Branching ratios of the radiative $\beta^-$–decay of the neutron for three photon energy regions, calculated for the lifetime of the neutron $\tau_n = 879.6(1.1)$ s [11]. In the last column we give a relative contribution of the $1/M$ corrections, caused by the weak magnetism and proton recoil.

Of course, the numerical values of the branching ratios should not be practically changed if we would use the world averaged value of the lifetime of the neutron $\tau_n = 880.2(1.0)$ s [15], which agrees perfectly well with the theoretical
one $\tau_n = 879.6(1.1)\,\text{s}$ \cite{10}. From the comparison of the branching ratios of the neutron radiative $\beta^-\text{-decay}$, calculated to leading order in the large proton mass expansion, one may see that the contributions of the weak magnetism and proton recoil make up of about 0.70%, 0.66% and 0.60% for the photon energy regions $15\,\text{keV} \leq \omega \leq 340\,\text{keV}$, $14\,\text{keV} \leq \omega \leq 782\,\text{keV}$ and $0.4\,\text{keV} \leq \omega \leq 14\,\text{keV}$, respectively. Thus, at first glimpse, the contributions of the weak magnetism and proton recoil to the neutron radiative $\beta^-\text{-decays}$ seem to be not very important. Moreover, that such contributions are small compared to the values of the stochastic errors of the new experimental data 1.49% and 3.95% for the photon energy regions $14\,\text{keV} \leq \omega \leq 782\,\text{keV}$ and $0.4\,\text{keV} \leq \omega \leq 14\,\text{keV}$, respectively. The account for the systematic errors, making up of about 4.78% and 11.34% of the new experimental values, measured for the photon energy regions $14\,\text{keV} \leq \omega \leq 782\,\text{keV}$ and $0.4\,\text{keV} \leq \omega \leq 14\,\text{keV}$, respectively, makes the contribution of the weak magnetism and proton recoil fully intangible. Nevertheless, we would like to emphasize that the values of the $1/M$ corrections, caused by the weak magnetism and proton recoil, to the branching ratios of the neutron radiative $\beta^-\text{-decay}$ are large compared by a factor of 4 to the contribution of the weak magnetism and proton recoil of about 0.16% to the neutron lifetime, calculated to order $1/M$ in \cite{10}. Thus, one may conclude that the weak magnetism and proton recoil, taken to order $1/M$, play more important role for the rate of the neutron radiative $\beta^-\text{-decay}$ than for the rate of the neutron $\beta^-\text{-decay}$. On the whole they give a relative corrections of order $10^{-2}$.

Combining the statistic and systematic errors the experimental values for the branching ratios we obtain $\text{BR}^{\text{exp}}_{\beta^-\gamma} = 3.35(16) \times 10^{-3}$ and $\text{BR}^{\text{exp}}_{\beta^-\gamma} = 5.82(66) \times 10^{-3}$ for the photon energy regions $14\,\text{keV} \leq \omega \leq 782\,\text{keV}$ and $0.4\,\text{keV} \leq \omega \leq 14\,\text{keV}$, respectively. One may see that the theoretical values of the branching ratios may agree with the experimental ones only within $2$ and $1.2$ standard deviations, respectively. The theoretical values of the branching ratios are below the experimental mean-values about by $9\%$ and $13\%$. This leaves room for the analysis of other contributions to the neutron radiative $\beta^-\text{-decay}$. Following Bernard et al. \cite{9} these contributions in the tree–approximation can be collected from the baryon resonances \cite{18}. For example, the contribution of the $\Delta(1232)$–resonance with spin and parity $J^\pi = \frac{3}{2}^+$, the mass of which $m_\Delta \simeq 1232\,\text{MeV}$ is not far from the proton mass \cite{13}, has been calculated by Bernard et al. \cite{9}. According to Cooper et al. \cite{13}, the contribution of the $\Delta(1232)$–resonance makes up only of about 0.5% of the branching ratio, measured for the photon energy region $15\,\text{keV} \leq \omega \leq 340\,\text{keV}$ \cite{13}. This implies that other baryon resonances with heavier masses \cite{18} should give contributions to the rate of the neutron radiative $\beta^-\text{-decay}$, which are small compared even to that by the $\Delta(1232)$–resonance.

Thus, one may expect that the theoretical analysis of the neutron radiative $\beta^-\text{-decay}$, which can be performed in the Standard Model and in the tree–approximation, may in principle change the rate of the neutron radiative $\beta^-\text{-decay}$ not stronger than by about 1.5% or even smaller. Hence, as a next step towards a precision theoretical analysis of the neutron radiative $\beta^-\text{-decay}$, which can be performed in the Standard Model, we may relate only to the analysis beyond the tree–approximation. To our point of view this should concern the radiative corrections to order $O(\alpha/\pi)$, calculated to leading order in the large proton mass expansion \cite{10}. It is well–known \cite{2} that the radiative corrections, calculated to order $O(\alpha/\pi)$ and to leading order in the large proton mass expansion, change the rate of the neutron $\beta^-\text{-decay}$ by about 3.75% \cite{10}. Taking into account that the corrections of the weak magnetism and proton recoil of order $1/M$ to the neutron radiative $\beta^-\text{-decay}$ rate are by a factor 4 large compared to the $1/M$ corrections to the rate of the neutron $\beta^-\text{-decay}$ one may expect that the relative contribution of the radiative corrections of order $O(\alpha/\pi)$ to the rate of the neutron radiative $\beta^-\text{-decay}$ can be also substantially enhanced. The first step in the direction of the account for the radiative corrections of order $O(\alpha/\pi)$ to the neutron radiative $\beta^-\text{-decay}$ has been made by Gardner and He \cite{14}, \cite{15}. However, the results, obtained by Gardner and He \cite{14}, \cite{15}, concern only the radiative corrections of order $O(\alpha/\pi)$ to T–odd momentum correlations in the neutron radiative $\beta^-\text{-decay}$. We are planning to give a detailed analysis of the radiative corrections of order $O(\alpha/\pi)$, allowing to describe the rate of the neutron radiative $\beta^-\text{-decay}$ to order $O(\alpha^2/\pi^2)$, in our forthcoming publication.

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Appendix A: Amplitude of radiative $\beta^-$-decay of neutron, described by Feynman diagrams in Fig. 1, with "weak magnetism" and proton recoil corrections to order $1/M$

The amplitude of the neutron radiative $\beta^-$-decay, given by Eq. (5), we transcribe into the form

$$\mathcal{M}(n \to p e^- \bar{\nu}_e \gamma) = \left[ \bar{u}_p(k_p, \sigma_p) O_0 u_n(k_n, \sigma_n) \right] \left[ \bar{u}_e(k_e, \sigma_e) \frac{1}{2k_e \cdot k} Q_e \gamma^\mu (1 - \gamma^5) v_\nu(k_\nu, +\frac{1}{2}) \right]$$

where $\bar{u}_p(k_p, \sigma_p)$, $u_n(k_n, \sigma_n)$, $\bar{u}_e(k_e, \sigma_e)$, and $v_\nu(k_\nu, +\frac{1}{2})$ are the Dirac bispinor wave functions of the free proton, neutron, electron and electron antineutrino with 3-momenta $k_p$, $k_n = \vec{0}$, $k_e$ and $k_\nu$ and polarizations $\sigma_p = \pm 1$, $\sigma_n = \pm 1$, $\sigma_e = \pm 1$ and $+\frac{1}{2}$, respectively, $Q_e$ and $Q_p$ are defined by

$$Q_e = 2\varepsilon^\mu_\nu \cdot k_e + \varepsilon^\mu_\nu \cdot \vec{k}_e$$

$$Q_p = 2\varepsilon^\mu_{\nu p} \cdot k_p + \varepsilon^\mu_{\nu p} \cdot \vec{k}_p$$

For the derivation of Eq. (A-1) we have used the Dirac equations for the free proton and electron. Replacing $\varepsilon^\mu_\nu \to k$ and using $k^2 = 0$ we get $\mathcal{M}(n \to p e^- \bar{\nu}_e \gamma)|_{\varepsilon^\mu_\nu \to k} = 0$. The matrix $Q_p$ is given by Eq. (4). For the analysis of the neutron radiative $\beta^-$-decay in the non–relativistic approximation we have to rewrite the amplitude Eq. (A-1) in terms of time and space components of the matrix $O_\mu = (\mathcal{O}_0, -\hat{O})$.

The time $O_0$ and spatial $\hat{O}$ components of the matrix $O_\mu$ we determine to order $1/M$ in the large $M$ expansion

$$O_0 = \left( \begin{array}{c} \lambda + \frac{\kappa}{2M} (\vec{\sigma} \cdot \vec{k}_p) \\ -\lambda \end{array} \right)$$

and

$$\hat{O} = \left( \begin{array}{c} \lambda \vec{\sigma} + i \frac{\kappa}{2M} (\vec{\sigma} \times \vec{k}_p) \\ -\vec{\sigma} \left( 1 + \frac{\kappa}{2M} (E_e + E_\nu + \omega) \right) \\ -\vec{\sigma} \left( \frac{\kappa}{2M} \lambda \end{array} \right)$$

For the derivation of Eq. (A-1) and Eq. (5) we have kept only the terms of order $1/M$ and used the energy conservation $m_n = E_p + E_e + E_\nu + \omega$, where $m_n$, $E_p = \sqrt{m^2_n + \vec{k}^2_p}$, $E_e$, $E_\nu$ and $\omega$ are the neutron mass and the proton, electron, antineutrino and photon energies, respectively. Then, $\vec{\sigma}$ are the Pauli $2 \times 2$ matrices.

For the calculation of the amplitude of the neutron radiative $\beta^-$-decay we define the Dirac bispinor wave functions of the neutron and the proton as follows

$$u_n(\vec{0}, \sigma_n) = \sqrt{2m_n} \left( \begin{array}{c} \varphi_n \\ 0 \end{array} \right), \quad u_p(k_p, \sigma_p) = \sqrt{E_p + m_p} \left( \begin{array}{c} \varphi_p \\ \vec{\sigma} \cdot \vec{k}_p \end{array} \right)$$

where the Pauli spinorial wave functions $\varphi_n$ and $\varphi_p$ depend on the polarizations $\sigma_n$ and $\sigma_p$, respectively. The matrix elements $[\bar{u}_p(k_p, \sigma_p) O_0 u_n(\vec{0}, \sigma_n)]$ and $[\bar{u}_p(k_p, \sigma_p) \hat{O} u_n(\vec{0}, \sigma_n)]$ are equal to

$$[\bar{u}_p(k_p, \sigma_p) O_0 u_n(\vec{0}, \sigma_n)] = \sqrt{2m_n (E_p + m_p)} \left( [\varphi^+_p \varphi_n] + \frac{\lambda}{2M} [\varphi_p (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] \right)$$

$$[\bar{u}_p(k_p, \sigma_p) \hat{O} u_n(\vec{0}, \sigma_n)] = \sqrt{2m_n (E_p + m_p)} \left( [\varphi^+_p \varphi_n] + \frac{\lambda}{2M} [\varphi_p (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] \right)$$

A-5
and
\[ [\bar{u}_p(\vec{k}, \sigma_p)\vec{O}u_n(\vec{0}, \sigma_n)] = \sqrt{2m_n(E_p + m_p)} \left\{ \lambda[\vec{\varphi}_{\sigma}^\dagger \varphi_n] + i \frac{\kappa}{2M} [\varphi_{\sigma}^\dagger (\vec{\varphi} \times \vec{k}) \varphi_n] + \frac{1}{2M} [\varphi_{\sigma}^\dagger (\vec{\varphi} \cdot \vec{k}) \vec{\varphi} \varphi_n] \right\}. \] \tag{A-8}

For the calculation of the matrix elements
\[ [\bar{u}_p(\vec{k}, \sigma_p)Q_p \frac{1}{2k_p \cdot k} \vec{O}0u_n(\vec{0}, \sigma_n)] , \ [u_p(\vec{k}, \sigma_p)Q_p \frac{1}{2k_p \cdot k} \vec{O}u_n(\vec{0}, \sigma_n)] \] \tag{A-9}
we have to define the products \( Q_p(1/2k_p \cdot k) \vec{O} \) and \( Q_p(1/2k_p \cdot k) \vec{O} \). For this aim we define the matrix \( Q_p(1/2k_p \cdot k) \vec{O} \) as follows
\[ O_p \frac{1}{2k_p \cdot k} = \left( \begin{array}{c c}
\frac{2\varepsilon_{\lambda'}^* \cdot k_p + i \vec{\varepsilon}_{\lambda'} \cdot (\vec{\varepsilon}_{\lambda'}^* \times \vec{k})}{2k_p \cdot k} & -\frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M} \\
-\frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M} & \frac{2\varepsilon_{\lambda'}^* \cdot k_p + i \vec{\varepsilon}_{\lambda'} \cdot (\vec{\varepsilon}_{\lambda'}^* \times \vec{k})}{2k_p \cdot k}
\end{array} \right), \] \tag{A-10}
where we have denoted \( \varepsilon_{\lambda'}^*(\vec{\varepsilon}_{\lambda'}^*, k) = (\omega, \vec{k}) \). Replacing \( \varepsilon_{\lambda'}^*(\vec{\varepsilon}_{\lambda'}^*, k) \to k^\alpha = (\omega, \vec{k}) \) we get the matrix \( Q_p(1/2k_p \cdot k) \vec{O} \) equal to a unit matrix \( Q_p(1/2k_p \cdot k)|_{\varepsilon_{\lambda'}^* \to k} = 1 \). Thus, the product \( Q_p(1/2k_p \cdot k) \vec{O} \) is given by
\[ O_p \frac{1}{2k_p \cdot k} Q_0 = \left( \begin{array}{c c}
(O_p \frac{1}{2k_p \cdot k} Q_0)_{11} & (O_p \frac{1}{2k_p \cdot k} Q_0)_{12} \\
(O_p \frac{1}{2k_p \cdot k} Q_0)_{21} & (O_p \frac{1}{2k_p \cdot k} Q_0)_{22}
\end{array} \right), \] \tag{A-11}
where we have denoted
\[ (O_p \frac{1}{2k_p \cdot k} Q_0)_{11} = \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega} + \lambda \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega}, \]
\[ (O_p \frac{1}{2k_p \cdot k} Q_0)_{12} = \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega} + \lambda \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega}, \]
\[ (O_p \frac{1}{2k_p \cdot k} Q_0)_{21} = \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega} + \lambda \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega}, \]
\[ (O_p \frac{1}{2k_p \cdot k} Q_0)_{22} = \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega} + \lambda \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega}. \] \tag{A-12}
In turn, the product \( Q_p(1/2k_p \cdot k) \vec{O} \) reads
\[ O_p \frac{1}{2k_p \cdot k} \vec{O} = \left( \begin{array}{c c}
(O_p \frac{1}{2k_p \cdot k} \vec{Q})_{11} & (O_p \frac{1}{2k_p \cdot k} \vec{Q})_{12} \\
(O_p \frac{1}{2k_p \cdot k} \vec{Q})_{21} & (O_p \frac{1}{2k_p \cdot k} \vec{Q})_{22}
\end{array} \right), \] \tag{A-13}
where we have denoted
\[ (O_p \frac{1}{2k_p \cdot k} \vec{Q})_{11} = \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega} + \lambda \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega}, \]
\[ (O_p \frac{1}{2k_p \cdot k} \vec{Q})_{12} = \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega} + \lambda \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega}, \]
\[ (O_p \frac{1}{2k_p \cdot k} \vec{Q})_{21} = \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega} + \lambda \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega}, \]
\[ (O_p \frac{1}{2k_p \cdot k} \vec{Q})_{22} = \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega} + \lambda \frac{\varepsilon_{\lambda'}^0(\vec{\varphi} \cdot \vec{k}) + \omega(\varepsilon_{\lambda'}^* \cdot \vec{\varphi})}{2M \omega}. \] \tag{A-14}
We would like to remind that because of the relation \( \varepsilon_{\lambda'}^* \cdot k = 0 \) the time-component of the polarization vector is equal to \( \varepsilon_{\lambda'}^* = (\varepsilon_{\lambda'}^* \cdot \vec{k})/\omega \). Replacing \( \varepsilon_{\lambda'}^* \to \vec{k} \) and using \( |\vec{k}| = \omega \) we get \( Q_p(1/2k_p \cdot k) \vec{Q}_0|_{\varepsilon_{\lambda'}^* \to \vec{k}} = Q_0 \) and
The hermitian conjugate amplitude is equal to

\[
\beta^+ + p \left( 1 - \frac{\mathbf{M}}{2\omega} \right) \phi_n = \frac{\phi^+_p \left( O_{p,2k_p,k} \right) \phi_n}{\sqrt{2m_n (E_p + m_p)}} \left[ \phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n \right]
\]

and

\[
\left[ \mathbf{K} \phi_n \right] = \frac{\phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n}{\sqrt{2m_n (E_p + m_p)}} \left[ \phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n \right] = \lambda \frac{\phi^+_p \left( \mathbf{M} \phi_n \right)}{\sqrt{2m_n (E_p + m_p)}} \left[ \phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n \right].
\]

The amplitude of the neutron radiative $\beta^-$-decay, calculated to order $1/M$, caused by the weak magnetism and proton recoil, is equal to

\[
\mathcal{M}(n \to p e^- \bar{\nu}_e) = \sqrt{2m_n (E_p + m_p)} \left[ \phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n \right] \left[ \frac{\phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n}{\sqrt{2m_n (E_p + m_p)}} \right]
\]

Since the factor $\sqrt{2m_n (E_p + m_p)}$ to order $1/M$ is equal to

\[
\sqrt{2m_n (E_p + m_p)} = 2m_n \left( 1 - \frac{E_e + E_\nu + \omega}{2M} \right),
\]

the amplitude Eq. (A-17) can be transcribed into the form

\[
\mathcal{M}(n \to p e^- \bar{\nu}_e) = \frac{\phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n}{\sqrt{2m_n (E_p + m_p)}} \left[ \phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n \right]
\]

The hermitian conjugate amplitude is equal to

\[
\mathcal{M}^\dagger(n \to p e^- \bar{\nu}_e) = \frac{\phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n}{\sqrt{2m_n (E_p + m_p)}} \left[ \phi^+_p \left( \frac{\mathbf{M}}{2\omega} \right) + \frac{\lambda}{2M \omega} \phi^+_p \left( \mathbf{K} \right) \phi_n \right]
\]
where $\tilde{Q}_c = 2k_c \cdot k + 2k\\bar{e}_X$. The squared absolute value of the amplitude Eq.(A-19) summed over the polarizations of the proton and electron and antineutrino degrees of freedom. Using the following relations \[10\] (see Appendix B of Ref.\[10\])

\begin{align}
\phi_n^\dagger \frac{\epsilon_0^\nu}{\omega} (\lambda \bar{\sigma} - \lambda \frac{E_c + E_w + \omega}{2M} \bar{\sigma} - i \frac{\kappa}{2M} (\bar{\sigma} \times \vec{k}_p) - i \lambda \bar{\sigma} \cdot (\vec{\varepsilon}_\lambda \times \vec{k}) ) \left\{ \bar{\nu}_n(\vec{k}, +\frac{1}{2}) \gamma (1 - \gamma^5) u_c(\vec{k}_c, \sigma_c) \right\} + \frac{1}{(k_c \cdot k)^2} \left\{ \bar{\nu}_n(\vec{k}, +\frac{1}{2}) \gamma (1 - \gamma^5) u_c(\vec{k}_c, \sigma_c) \right\},
\end{align}

(A-20)

One may show that the expression in Eq.(A-21) is gauge invariant. For this aim one has to replace $\epsilon_0^\nu$ with $\epsilon_0^\nu$ $\bar{\nu}_n^\dagger \frac{\epsilon_0^\nu}{\omega} (\lambda \bar{\sigma} - \lambda \frac{E_c + E_w + \omega}{2M} \bar{\sigma} - i \frac{\kappa}{2M} (\bar{\sigma} \times \vec{k}_p) - i \lambda \bar{\sigma} \cdot (\vec{\varepsilon}_\lambda \times \vec{k}) )$ and $\epsilon_0^\nu$ with $\epsilon_0^\nu$ $\bar{\nu}_n^\dagger \frac{\epsilon_0^\nu}{\omega} (\lambda \bar{\sigma} - \lambda \frac{E_c + E_w + \omega}{2M} \bar{\sigma} - i \frac{\kappa}{2M} (\bar{\sigma} \times \vec{k}_p) - i \lambda \bar{\sigma} \cdot (\vec{\varepsilon}_\lambda \times \vec{k}) )$. Now we may proceed to the calculation of the traces over electron and antineutrino degrees of freedom. Using the following relations \[10\] (see Appendix B of Ref.\[10\])

\begin{align}
\phi_n^\dagger \frac{\epsilon_0^\nu}{\omega} (\lambda \bar{\sigma} - \lambda \frac{E_c + E_w + \omega}{2M} \bar{\sigma} - i \frac{\kappa}{2M} (\bar{\sigma} \times \vec{k}_p) - i \lambda \bar{\sigma} \cdot (\vec{\varepsilon}_\lambda \times \vec{k}) ) \left\{ \bar{\nu}_n(\vec{k}, +\frac{1}{2}) \gamma (1 - \gamma^5) u_c(\vec{k}_c, \sigma_c) \right\} + \frac{1}{(k_c \cdot k)^2} \left\{ \bar{\nu}_n(\vec{k}, +\frac{1}{2}) \gamma (1 - \gamma^5) u_c(\vec{k}_c, \sigma_c) \right\},
\end{align}

(A-21)

One may show that the expression in Eq.(A-21) is gauge invariant. For this aim one has to replace $\epsilon_0^\nu \rightarrow k$ and $\epsilon_\lambda \rightarrow k$ and collect the coefficients in font of the traces $\text{tr}\{\bar{\nu}_n(\vec{k}, +\frac{1}{2}) \gamma (1 - \gamma^5) u_c(\vec{k}_c, \sigma_c)\}$. Now we may proceed to the calculation of the traces over electron and antineutrino degrees of freedom. Using the following relations \[10\] (see Appendix B of Ref.\[10\])

\begin{align}
\gamma_0 \gamma \gamma_\mu = \gamma_0 \gamma_\mu, \quad \gamma_0 \gamma_\mu = \gamma_0 \gamma_\mu + i \alpha_\nu \gamma_\mu, \quad \gamma_0 \gamma_\mu \gamma_5 = \gamma_0 \gamma_\mu \gamma_5,
\end{align}

(A-22)

where $\eta^{\alpha \beta}$ is the Minkowski metric tensor and $\alpha_\nu \gamma_\mu$ is the Levi–Civita tensor defined by $\epsilon^{0123} = 1$ and $\epsilon_{\alpha \mu \nu \beta} = -\epsilon^{\alpha \mu \nu \beta}$ \[10\] and \[10\] (see Appendix B of Ref.\[10\]), we obtain

\begin{align}
\frac{1}{4} \text{tr}\{\bar{\nu}_n(\vec{k}, +\frac{1}{2}) \gamma (1 - \gamma^5) u_c(\vec{k}_c, \sigma_c)\} = 4(\epsilon_0^\nu \cdot \vec{k}_c)(\epsilon_\lambda \cdot \vec{k}_c)(k_c + k)^{\mu} - 2(\epsilon_0^\nu \cdot \epsilon_\lambda)(k_c \cdot k) k^{\mu}
\end{align}
Using Eq. (A-23) and Eq. (A-24) for the traces over the electron and antineutrino degrees of freedom, containing both $k^2 + 2i\omega\epsilon$ for a photon on–mass shell $k^2 = 0$. Then, we get

\[
\begin{align*}
\frac{1}{4} \text{tr}\{k_e \epsilon\gamma^\mu(1 - \gamma^5)\} &= (\epsilon^\ast \cdot k_e) (2k_e + k)\mu - (k_e \cdot k)\epsilon^\ast \cdot k_e + i\epsilon\sigma^{\alpha\beta\nu} \epsilon^\ast_{\chi\alpha} \epsilon_{\chi\beta} k_{\rho} k_{e\nu} \tag{A-23} \\
\frac{1}{4} \text{tr}\{k_e \epsilon^\ast\gamma^\mu(1 - \gamma^5)\} &= (\epsilon^\ast \cdot k_e) (2k_e + k)\mu - (k_e \cdot k)\epsilon^\ast \cdot k_e + i\epsilon\sigma^{\alpha\beta\nu} \epsilon^\ast_{\chi\alpha} \epsilon_{\chi\beta} k_{\rho} k_{e\nu} \tag{A-24}
\end{align*}
\]

Using Eq. (A-23) and Eq. (A-24) for the traces over the electron and antineutrino degrees of freedom, containing both $Q_e$ and $Q_\nu$ matrices, we obtain the following expressions:
For the traces over the electron and antineutrino degrees of freedom, containing either $\nu_e$ or $\bar{\nu}_e$, we get

$$\frac{1}{4} \text{tr}\{(\hat{k} + m_e) Q_e \gamma^0 \hat{k}_\nu \gamma^0 (1 - \gamma^5)\} = E_\nu \left[ (\epsilon_{\nu_e} \cdot k_e)(2E_e + \omega) - (k_e \cdot k)\epsilon_{\nu_e}^\alpha - i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^\alpha k_\beta k_{e\nu} \right]$$

$$+ \hat{k}_\nu^i \left[ (\epsilon_{\nu_e} \cdot k_e)(2\hat{k}_\nu + \hat{k})^i - (k_e \cdot k)\epsilon_{\nu_e}^i - i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^i k_\beta k_{e\nu} \right],$$

$$\frac{1}{4} \text{tr}\{(\hat{k} + m_e) Q_e \gamma^0 \hat{k}_\nu \gamma^0 (1 - \gamma^5)\} = E_\nu \left[ (\epsilon_{\nu_e} \cdot k_e)(2\hat{k}_\nu + \hat{k})^i - (k_e \cdot k)\epsilon_{\nu_e}^i - i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^i k_\beta k_{e\nu} \right]$$

$$+ \hat{k}_\nu^i \left[ (\epsilon_{\nu_e} \cdot k_e)(2E_e + \omega) - (k_e \cdot k)\epsilon_{\nu_e}^i - i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^i k_\beta k_{e\nu} \right]$$

$$+ i \epsilon^{ij} \hat{k}_\nu^i \left[ (\epsilon_{\nu_e} \cdot k_e)(2\hat{k}_\nu + \hat{k})^j - (k_e \cdot k)\epsilon_{\nu_e}^j - i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^j k_\beta k_{e\nu} \right],$$

$$\frac{1}{4} \text{tr}\{(\hat{k} + m_e) Q_e \gamma^0 \hat{k}_\nu \gamma^0 (1 - \gamma^5)\} = E_\nu \left[ (\epsilon_{\nu_e} \cdot k_e)(2\hat{k}_\nu + \hat{k})^i - (k_e \cdot k)\epsilon_{\nu_e}^i - i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^i k_\beta k_{e\nu} \right]$$

$$- i \epsilon^{ij} \hat{k}_\nu^i \left[ (\epsilon_{\nu_e} \cdot k_e)(2\hat{k}_\nu + \hat{k})^j - (k_e \cdot k)\epsilon_{\nu_e}^j - i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^j k_\beta k_{e\nu} \right],$$

and

$$\frac{1}{4} \text{tr}\{(\hat{k} + m_e) Q_e \gamma^0 \hat{k}_\nu \gamma^0 (1 - \gamma^5)\} = E_\nu \left[ (\epsilon_{\nu_e} \cdot k_e)(2E_e + \omega) - (k_e \cdot k)\epsilon_{\nu_e}^0 - i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^0 k_\beta k_{e\nu} \right]$$

$$+ \hat{k}_\nu^i \left[ (\epsilon_{\nu_e} \cdot k_e)(2\hat{k}_\nu + \hat{k})^i - (k_e \cdot k)\epsilon_{\nu_e}^i + i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^i k_\beta k_{e\nu} \right],$$

$$\frac{1}{4} \text{tr}\{(\hat{k} + m_e) Q_e \gamma^0 \hat{k}_\nu \gamma^0 (1 - \gamma^5)\} = E_\nu \left[ (\epsilon_{\nu_e} \cdot k_e)(2\hat{k}_\nu + \hat{k})^i - (k_e \cdot k)\epsilon_{\nu_e}^i + i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^i k_\beta k_{e\nu} \right]$$

$$+ \hat{k}_\nu^i \left[ (\epsilon_{\nu_e} \cdot k_e)(2E_e + \omega) - (k_e \cdot k)\epsilon_{\nu_e}^0 + i \epsilon^{0\alpha\beta\nu} \epsilon_{\nu_e}^0 k_\beta k_{e\nu} \right].$$
In turn, for the traces without $Q$, however, it is obvious that one has to sum over only physical degrees of freedom of real photons, which are defined by

$$\sum_{\nu} \varepsilon^{ij}_{\nu} k_{ij} = 0,$$

has the following properties

$$\sum_{\lambda'} \varepsilon^{\lambda'}_\nu = 0, \sum_{\lambda'} \varepsilon^{ij}_{\lambda'} = 0,$$

where $\bar{n} = \bar{k}/\omega$. For the summation over only physical degrees of freedom of a real photon it is convenient to remove from Eq. (A-21) all terms proportional to the time-component of the photon polarization vector. This gives

$$\delta^{ij} \left( 1 - \frac{E_c + E_\nu + \omega}{M} \right) \varepsilon^{ij}_\nu.$$

Now we may sum over the photon polarizations. Because of gauge invariance of Eq. (A-21) one may use any gauge. However, it is obvious that one has to sum over only physical degrees of freedom of real photons, which are defined by the polarization vector $\varepsilon^{ij}_\lambda = (0, \varepsilon^{ij}_\lambda)$ [21] (see also [20]). The polarization vector $\varepsilon^{ij}_\lambda$ has the following properties

$$\sum_{\lambda'} \varepsilon^{\lambda'}_\nu = 0, \sum_{\lambda'} \varepsilon^{ij}_{\lambda'} = 0,$$

where $\bar{n} = \bar{k}/\omega$. For the summation over only physical degrees of freedom of a real photon it is convenient to remove from Eq. (A-21) all terms proportional to the time-component of the photon polarization vector. This gives

$$\delta^{ij} \left( 1 - \frac{E_c + E_\nu + \omega}{M} \right) \varepsilon^{ij}_\nu.$$
Summing over the photon polarizations we obtain

\[
\sum_{\lambda'} \text{tr}\{(\mathbf{k}_e + m_e) \cdot \mathbf{Q}_{e \gamma^0} \mathbf{Q}_{e}(1 - \gamma^5)\} = E_e E_{\nu'} \left[ \left( 1 + \frac{\gamma^5}{E_e} \right) \left( k^2_e - (\mathbf{k}_e \cdot \mathbf{n})^2 \right) + \frac{\omega^2}{E_e} (E_e - \mathbf{k}_e \cdot \mathbf{n}) - \frac{m^2_e}{E_e} \right],
\]

\[
\sum_{\lambda'} \text{tr}\{(\mathbf{k}_e + m_e) \cdot \mathbf{Q}_{e \gamma^1} \mathbf{Q}_{e}(1 - \gamma^5)\} = E_e E_{\nu'} \left[ \left( 1 + \frac{\gamma^5}{E_e} \right) \left( k^2_e - (\mathbf{k}_e \cdot \mathbf{n})^2 \right) + \frac{\omega^2}{E_e} (E_e - \mathbf{k}_e \cdot \mathbf{n}) - \frac{m^2_e}{E_e} \right],
\]

\[
\sum_{\lambda'} \text{tr}\{(\mathbf{k}_e + m_e) \cdot \mathbf{Q}_{e \gamma^5} \mathbf{Q}_{e}(1 - \gamma^5)\} = E_e E_{\nu'} \left[ \left( 1 + \frac{\gamma^5}{E_e} \right) \left( k^2_e - (\mathbf{k}_e \cdot \mathbf{n})^2 \right) + \frac{\omega^2}{E_e} (E_e - \mathbf{k}_e \cdot \mathbf{n}) - \frac{m^2_e}{E_e} \right].
\]
\[ \begin{align*}
\times & \left( E_e - \vec{k}_e \cdot \vec{n} \right) + \omega \vec{n} \cdot \vec{n} \cdot i \xi_{Ja} \vec{n} \cdot \vec{k}_c^a + \omega \xi_{Ja} \vec{n} \cdot \vec{k}_c^a - \omega E_e \xi_{Ja} \vec{n} \cdot \vec{n}^\dagger + \vec{k}_f^j - \left( \vec{k}_c^j - \vec{n}^\dagger (\vec{k}_e \cdot \vec{n}) \right) \\
\times & \left( 2E_e + \omega \right) + \omega \xi_{Ja} \vec{n} \cdot \vec{k}_c^a + i \xi_{Ja} \vec{k}_c^a - \left( \vec{k}_c^i - \vec{n}^\dagger (\vec{k}_e \cdot \vec{n}) \right) (2\vec{k}_e + \omega \vec{n})^\dagger - \left( \delta_i^j - \vec{n}^\dagger \vec{n}^\dagger \right) \\
\times & \omega \left( E_e - \vec{k}_e \cdot \vec{n} \right) + \omega \vec{n} \cdot i \xi_{jab} \vec{n} \cdot \vec{k}_c^b + - \omega \xi_{jab} \vec{k}_c^b - \omega E_e \xi_{jab} \vec{n}^a \\
\frac{1}{4} \sum_{\lambda'} \varepsilon_{\lambda' \lambda} \text{tr} \left\{ (\vec{k}_e + m_e) \gamma_5 \vec{k}_e^\dagger \gamma_0 \vec{Q}_e(1 - \gamma^5) \right\} = & \left[ - \left( \vec{k}_c^i - \vec{n}^\dagger (\vec{k}_e \cdot \vec{n}) \right) (2\vec{k}_e + \omega \vec{n})^\dagger - \left( \delta_i^j - \vec{n}^\dagger \vec{n}^\dagger \right) \\
& \omega \left( E_e - \vec{k}_e \cdot \vec{n} \right) + \omega \vec{n} \cdot i \xi_{Ja} \vec{n} \cdot \vec{k}_c^a + \omega \xi_{Ja} \vec{k}_c^a - \left( \vec{k}_c^i - \vec{n}^\dagger (\vec{k}_e \cdot \vec{n}) \right) (2E_e + \omega) \\
& - \omega i \xi_{Ja} \vec{n} \cdot \vec{k}_c^a - i \xi_{Ja} \vec{k}_c^a - \left( \vec{k}_c^i - \vec{n}^\dagger (\vec{k}_e \cdot \vec{n}) \right) (2\vec{k}_e + \omega \vec{n})^\dagger - \left( \delta_i^j - \vec{n}^\dagger \vec{n}^\dagger \right) \omega \left( E_e - \vec{k}_e \cdot \vec{n} \right) + \omega \vec{n} \cdot i \xi_{jab} \vec{n} \cdot \vec{k}_c^b \\
& + \omega i \xi_{jab} \vec{k}_c^b + \omega E_e \xi_{jab} \vec{n}^a + \frac{1}{4} \sum_{\lambda'} \varepsilon_{\lambda' \lambda} \text{tr} \left\{ (\vec{k}_e + m_e) \gamma_5 \vec{k}_e^\dagger \gamma_0 \vec{Q}_e(1 - \gamma^5) \right\} = & \delta_i^j \varepsilon_{\lambda' \lambda} \left[ - \left( \vec{k}_c^i - \vec{n}^\dagger (\vec{k}_e \cdot \vec{n}) \right) (2E_e + \omega) - \omega i \xi_{jab} \vec{n} \cdot \vec{k}_c^b \right] \\
& - i \xi_{Ja} \varepsilon_{\lambda' \lambda} \left[ - \left( \vec{k}_c^i - \vec{n}^\dagger (\vec{k}_e \cdot \vec{n}) \right) (2\vec{k}_e + \omega \vec{n})^\dagger - \omega \left( \delta_i^j - \vec{n}^\dagger \vec{n}^\dagger \right) \left( E_e - \vec{k}_e \cdot \vec{n} \right) - \omega \vec{n} \cdot i \xi_{jab} \vec{n} \cdot \vec{k}_c^b \\
& - \omega i \xi_{jab} \vec{k}_c^b + \omega E_e \xi_{jab} \vec{n}^a \right] \right].
\end{align*} \]

The rate of the neutron radiative $\beta^-$-decay is given by

\[ \lambda_{\beta^-} = \pi \alpha G_F^2 |V_{ud}|^2 \frac{1}{m_n} \int \frac{1}{2} \sum_{\lambda', \lambda} |\mathcal{M}(n \rightarrow p e^- \nu_e \gamma_\lambda)|^2 F(E_e, Z = 1) \]

\[ \times (2\pi)^4 \delta(4)(k_n - k_p - k_e - k_\nu - k) \frac{d^3k_p}{(2\pi)^3 2E_p} \frac{d^3k_e}{(2\pi)^3 2E_e} \frac{d^3k_\nu}{(2\pi)^3 2E_\nu} \frac{d^3k}{(2\pi)^3 2\omega}, \]

where $F(E_e, Z = 1)$ is the relativistic Fermi function, taking into account the final-state proton–electron Coulomb interaction $[11, 11]$ (see also Eq. [5]). After the integration over the proton momentum $k_p$, the directions of the
momenta of the particles in the final state we are left with the result [10]

\[
\lambda \beta_\gamma(\omega_{\text{max}}, \omega_{\text{min}}) = (1 + 3\lambda^2) \frac{1}{\pi} \frac{G_F^2 |V_{\text{ud}}|^2}{2\pi^3} \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \frac{d\omega}{\omega} \int_{m_n}^{E_c - \omega} dE_c \sqrt{E_c^2 - m_n^2} \frac{1}{(E_c - E_e - \omega)^2} F(E_c, Z = 1) \rho_{\beta_\gamma}(E_c, \omega),
\]

where the rate of the neutron radiative $\beta^{-}$-decay is defined for the photon energy region $\omega_{\text{min}} \leq \omega \leq \omega_{\text{max}}$ and $E_0 = (m_n^2 + m_e^2 + m_\nu^2)/2m_n$ is the end–point energy of the electron energy–spectrum of the neutron $\beta^{-}$-decay [11]. The function $\rho_{\beta_\gamma}(E_c, \omega)$ is given by

\[
\rho_{\beta_\gamma}(E_c, \omega) = \int \frac{d\Omega_{\nu}}{4\pi} \int \frac{d\Omega_{\nu}}{4\pi} \frac{1}{\lambda', \text{pol}} \sum \frac{\omega^2|M(n \rightarrow pe^{-}\bar{\nu}_e\gamma)\lambda|^2}{(1 + 3\lambda^2)32m_n^2E_cE_\nu}\Phi_{\beta_\gamma}(\vec{k}_e, \vec{k}, \vec{k}_\nu),
\]

where $d\Omega_{\nu} = \sin \theta_{\nu} d\theta_{\nu} d\phi_{\nu}$ is an infinitesimal solid angle of the electron–photon momentum correlations $\vec{k}_e \cdot \vec{n} = k_e \cos \theta_{\nu}$, defined by the polar angle $0 \leq \theta_{\nu} \leq \pi$ and the azimuthal angle $0 \leq \varphi_{\nu} \leq 2\pi$, and $d\Omega_{\nu}$ is an infinitesimal solid angle of the antineutrino momentum $\vec{k}_\nu$. The function $\Phi_{\beta_\gamma}(\vec{k}_e, \vec{k}, \vec{k}_\nu)$ is defined by the integral [11][11].

\[
\Phi_{\beta_\gamma}(\vec{k}_e, \vec{k}, \vec{k}_\nu) = \int_0^\infty \delta(m_n - \sqrt{m_p^2 + (\vec{k}_e + \vec{k} + \vec{k}_\nu)^2}) - E_e - E_\nu - \omega \frac{m_n}{E_\nu} \frac{E_\nu^2 dE_\nu}{(E_0 - E_\nu - \omega)^2}.
\]

The result of the integration over $E_\nu$ is equal to

\[
\Phi_{\beta_\gamma}(\vec{k}_e, \vec{k}, \vec{k}_\nu) = \frac{1}{(1 - \frac{1}{m_n}(E_e + (\vec{k}_\nu + \vec{k}) \cdot \vec{n})\cos \theta_{\nu})^2}.
\]

where the antineutrino energy is defined by

\[
E_\nu = \frac{E_0 - E_e - \omega + \frac{k_e \cdot k}{m_n}}{1 - \frac{1}{m_n}(E_e + \omega - (\vec{k}_e + \vec{k}) \cdot \vec{n})\cos \theta_{\nu}).
\]

Here $\theta_{\nu}$ is an angle between the momenta $\vec{k}_e + \vec{k}$ and $\vec{k}_\nu$. To order $1/M$ the function $\Phi_{\beta_\gamma}(\vec{k}_e, \vec{k}, \vec{k}_\nu)$ is given by

\[
\Phi_{\beta_\gamma}(\vec{k}_e, \vec{k}, \vec{k}_\nu) = 1 + \frac{2}{3M} \frac{\omega}{E_\nu} \left( \frac{E_e - \vec{k}_e \cdot \vec{n}}{E_e - \vec{k}_e \cdot \vec{n}} + \frac{3}{3} \left( \frac{E_e + \omega - \frac{1}{3} \frac{E_0}{\lambda}}{E_e - \vec{k}_e \cdot \vec{n}} \right) \right),
\]

where $E_\nu = E_0 - E_e - \omega$ [11]. Setting $\omega = 0$ we arrive at the corresponding function of the neutron $\beta^{-}$-decay [11]. For the calculation of the function $\rho_{\beta_\gamma}(E_c, \omega)$ we propose, first, to integrate over the directions of the antineutrino 3-momentum. Now making the integration in Eq. [A-34] over the antineutrino momentum solid angle we arrive at the expression

\[
\int \frac{d\Omega_{\nu}}{4\pi} \frac{1}{\lambda', \text{pol}} \sum \frac{\omega^2|M(n \rightarrow pe^{-}\bar{\nu}_e\gamma)\lambda|^2}{(1 + 3\lambda^2)32m_n^2E_cE_\nu}\Phi_{\beta_\gamma}(\vec{k}_e, \vec{k}, \vec{k}_\nu) = \frac{1 + 2}{M} \frac{\omega}{E_\nu} \left( \frac{E_e - \vec{k}_e \cdot \vec{n}}{E_e - \vec{k}_e \cdot \vec{n}} + \frac{3}{3} \left( \frac{E_e + \omega - \frac{1}{3} \frac{E_0}{\lambda}}{E_e - \vec{k}_e \cdot \vec{n}} \right) \right) + \frac{\lambda^2 - 2(k + 1) + \lambda}{3M} \left( \frac{k_e^2 + \omega \vec{k}_e \cdot \vec{n}}{E_e} \left( \frac{k_e^2 - (\vec{k}_e + \vec{n})^2}{(E_e - \vec{k}_e \cdot \vec{n})^2} + \frac{\omega}{E_e - \vec{k}_e \cdot \vec{n}} \right) + (\omega + \vec{k}_e \cdot \vec{n}) \left( \frac{1}{E_e - \vec{k}_e \cdot \vec{n}} - \frac{m_n^2}{E_e - \vec{k}_e \cdot \vec{n}} \right) \right) \right) \}

\[
\times \frac{1}{1 + 3\lambda^2} \frac{1}{M} \left[ \frac{k_e^2 + \omega \vec{k}_e \cdot \vec{n}}{E_e} \left( \frac{k_e^2 - (\vec{k}_e + \vec{n})^2}{(E_e - \vec{k}_e \cdot \vec{n})^2} + \frac{\omega}{E_e - \vec{k}_e \cdot \vec{n}} \right) + (\omega + \vec{k}_e \cdot \vec{n}) \left( \frac{1}{E_e - \vec{k}_e \cdot \vec{n}} - \frac{m_n^2}{E_e - \vec{k}_e \cdot \vec{n}} \right) \right] \}

\]

\[
\frac{1}{1 + 3\lambda^2} \frac{1}{M} \left[ \frac{k_e^2 + \omega \vec{k}_e \cdot \vec{n}}{E_e} \left( \frac{k_e^2 - (\vec{k}_e + \vec{n})^2}{(E_e - \vec{k}_e \cdot \vec{n})^2} + \frac{\omega}{E_e - \vec{k}_e \cdot \vec{n}} \right) + (\omega + \vec{k}_e \cdot \vec{n}) \left( \frac{1}{E_e - \vec{k}_e \cdot \vec{n}} - \frac{m_n^2}{E_e - \vec{k}_e \cdot \vec{n}} \right) \right] \}.
\]

(A-39)
Plugging Eq. (A-39) into Eq. (A-34) we define the function $\rho_{\beta\gamma}(E_e, \omega)$ by the integral over electron–photon correlation angles only. It is given by

$$
\rho_{\beta\gamma}(E_e, \omega) = \int \frac{d\Omega_{\gamma\gamma}}{4\pi} \left[ 1 + \frac{2 \omega}{M} \frac{E_e - k_e \cdot \bar{n}}{E_0 - E_e - \omega} + \frac{3}{M} \left( E_e + \omega - \frac{1}{3} E_0 \right) + \frac{\lambda^2 - 2(\kappa + 1)\lambda + 1}{1 + 3\lambda^2} \frac{E_0 - E_e - \omega}{M} \right]
\]

$$(1 + \frac{\omega}{E_e}) \frac{k_e^2 - (k_e \cdot \bar{n})^2}{(E_e - k_e \cdot \bar{n})^2} + \frac{\omega^2}{E_e E_e - k_e \cdot \bar{n}} \left[ \frac{k_e^2 + \omega \bar{k}_e \cdot \bar{n}}{E_e} (k_e^2 - (k_e \cdot \bar{n})^2) + \frac{\omega}{(E_e - k_e \cdot \bar{n})^2} + \frac{\omega}{E_e - k_e \cdot \bar{n}} \right] \frac{\lambda(\lambda - 1)}{1 + 3\lambda^2} \frac{M}{E_e} \left[ \frac{\omega^2}{E_e E_e - k_e \cdot \bar{n}} + \frac{3 \omega^2}{E_e} \right].$$

To leading order in large proton mass expansion, i.e. at $M \to \infty$, Eq. (A-40) reduces to the expression, calculated in [10].

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