A construction of the abstract induced subgraph poset of a graph from its abstract edge subgraph poset

Deisiane Lopes Gonçalves * Bhalchandra D. Thatte †

December 2, 2020

Abstract

The abstract induced subgraph poset of a graph is the isomorphism class of the induced subgraph poset of the graph, suitably weighted by subgraph counting numbers. The abstract bond lattice and the abstract edge-subgraph poset are defined similarly by considering the lattice of subgraphs induced by connected partitions and the poset of edge-subgraphs, respectively. Continuing our development of graph reconstruction theory on these structures, we show that if a graph has no isolated vertices, then its abstract bond lattice and the abstract induced subgraph poset can be constructed from the abstract edge-subgraph poset except for the families of graphs that we characterise. The construction of the abstract induced subgraph poset from the abstract edge-subgraph poset generalises a well known result in reconstruction theory that states that the vertex deck of a graph with at least 4 edges and without isolated vertices can be constructed from its edge deck.

Mathematics Subject Classification MSC2020: 05C60

1 Introduction

One of the beautiful conjectures in graph theory, which has been open for more than 70 years, is about vertex reconstruction of graphs [1]. It was proposed by Ulam and Kelly [5], and was reformulated by Harary [3] in the more intuitive language of reconstruction. This conjecture asserts that every finite simple undirected graph on three or more vertices is determined, up to isomorphism, by its collection of vertex-deleted subgraphs (called the deck). Harary [3] also proposed an analogous conjecture, known as the edge reconstruction conjecture. It asserts that every finite simple
undirected graph with four or more edges is determined, up to isomorphism, by its
collection of edge-deleted subgraphs (called the edge-deck).

In a series of papers [9, 10, 11], the second author considered three objects related
to the vertex and the edge decks of a graph \(G\), namely, the abstract induced sub-
graph poset \(\overline{P}(G)\), the abstract bond lattice \(\overline{\Omega}(G)\), and the abstract edge-subgraph
poset \(\overline{Q}(G)\). The induced subgraph poset \(P(G)\) is the poset of distinct (mutually non-
isomorphic) induced non-empty subgraphs of \(G\), where each edge of the poset, say
from \(G_i\) to \(G_j\), is weighted by the number of induced subgraphs of \(G_i\) that are isomor-
phic to \(G_j\). By abstract we mean the isomorphism class of \(P(G)\), and denote it by \(\overline{P}(G)\).

Analogously, the abstract bond lattice \(\overline{\Omega}(G)\) is defined for the distinct spanning sub-
graphs induced by connected partitions of \(G\), and the abstract edge-subgraph poset
\(\overline{Q}(G)\) is defined for the distinct edge-subgraphs of \(G\).

Many old and new results on the reconstruction conjectures were proved for the
problem of reconstructing a graph from its abstract induced subgraph poset or from
its abstract edge-subgraph poset. For example, it was shown in [9] that all graphs
on 3 or more vertices are vertex reconstructible if and only if all graphs except the
empty graphs are \(P\)-reconstructible (reconstructible from the abstract induced sub-
graph poset). On the other hand, even for classes of graphs known to be vertex
reconstructible (e.g., disconnected graphs), the problem of \(P\)-reconstruction may be
prohibitive difficult. Moreover, it was shown in [11] that \(\overline{\Omega}(G)\) and \(\overline{P}(G)\) can be
constructed from each other except in a few cases that were characterised. It was also
shown in [10] that the edge reconstruction conjecture is true if and only if all graphs
(except graphs from a particular family) are \(Q\)-reconstructible (reconstructible from
the abstract edge-subgraph poset). Here we continue to develop reconstruction the-
ory for the three posets.

In Section 2, we define the notation and terminology for graphs and posets, give
formal definitions of the three posets \(\overline{P}(G), \overline{Q}(G)\) and \(\overline{\Omega}(G)\), and state the corre-
sponding reconstruction problems.

In Section 3, we prove the main theorem of the paper, namely, Theorem 3.1, which
states that \(\overline{P}(G)\) can be constructed from \(\overline{Q}(G)\), except when \(G\) belongs to a certain
family \(\mathcal{N}\) of graphs, and that \(\overline{\Omega}(G)\) can be constructed from \(\overline{Q}(G)\), except when \(G\)
belongs to a certain family \(\mathcal{M}\) of graphs. The two families \(\mathcal{M}\) and \(\mathcal{N}\) differ slightly,
and we characterise them completely. The construction of \(\overline{P}(G)\) from \(\overline{Q}(G)\) gen-
eralises a well known result in reconstruction theory that states that the vertex deck of
a graph with at least 4 edges and without isolated vertices can be constructed from
its edge deck [4].

# Notation and terminology

In this section, we summarise the notational conventions and definitions from [10, 11],
which we use with minor changes.
2.1 Graphs

First we clarify our convention about unlabelled graphs. We identify each isomorphism class of graphs (also called an unlabelled graph) with a unique representative of the class. If \( G \) is a graph, then \( \overline{G} \) is the unique representative of the isomorphism class of \( G \). Let \( \mathcal{G} \) be the set consisting of one representative element from each isomorphism class of finite simple graphs. We assume that all newly declared graphs are from \( \mathcal{G} \). On the other hand, if a graph \( H \) is derived from a graph \( G \) (e.g., \( H = G - e \)), then \( H \) may not be in \( \mathcal{G} \). If \( H \in \mathcal{G} \) is isomorphic to a subgraph of \( G \), we say that \( G \) contains \( H \) as a subgraph or \( G \) contains \( H \) or \( H \) is contained in \( G \).

An empty graph is a graph with empty edge set. A null graph \( \Phi \) is a graph with no vertices.

Let \( G \) be a graph. We denote the number of vertices of \( G \) by \( v(G) \), the number of edges of \( G \) by \( e(G) \), and the number of components of \( G \) by \( k(G) \), the maximum degree in \( G \) by \( \Delta(G) \), and the minimum degree in \( G \) by \( \delta(G) \).

For \( X \subseteq V(G) \), we denote the subgraph induced by \( X \) by \( G[X] \), the subgraph induced by \( V(G) \setminus X \) by \( G - X \), or \( G - u \) when \( X = \{u\} \). For \( E \subseteq E(G) \), we denote the subgraph induced by \( E \) by \( G[E] \), it is called an edge-subgraph. The spanning subgraph of \( G \) with edge set \( E(G) \setminus E \) by \( G - E \), or \( G - e \) if \( E = \{e\} \).

We denote the number of subgraphs (induced subgraphs, edge-subgraphs) of \( G \) that are isomorphic to \( H \) by \( s(H,G) \) (\( p(H,G) \), \( q(H,G) \), respectively).

A path on \( n \) vertices is denoted by \( P_n \) and a cycle on \( n \) vertices is denoted by \( C_n \). We denote a complete graph on \( n \) vertices by \( K_n \), a complete bipartite graph with \( n \) and \( m \) vertices in two partitions by \( K_{n,m} \), and the graph \( K_4 \) minus an edge by \( K_4 \setminus e \).

We refer to the following graphs in some proofs.

\[
\begin{align*}
S_m, m \geq 3 & \quad m - 3 \\
T_m, m \geq 3 & \quad m - 2 \\
B_1 & \quad B_2 \\
B_3 & \quad B_4
\end{align*}
\]

Figure 1: Some graphs referred to in proofs.

For terminology and notation about graphs not defined here, we refer to Bondy and Murty [2].

2.2 Partially ordered sets

Let \((S, \leq)\) be a partially ordered set. If \(x, y \in S\), \(x \leq y\), \(x \neq y\) and there is no \(z \in S \setminus \{x, y\}\) such that \(x \leq z \leq y\), then we say \(y\) covers \(x\). We say that \(\rho : S \to \mathbb{Z}\) is a rank function if for all \(x, y \in S\), \(y\) covers \(x\) implies \(\rho(y) = \rho(x) + 1\).

A weighted poset is a poset \((S, \leq)\) with a compatible weight function \(\omega : S \times S \to \mathbb{Z}\), where compatible means \(\omega(x, y) = 0\) unless \(x \leq y\). We say that weighted posets \((S, \leq, \omega)\) and \((S', \leq', \omega')\) are isomorphic if there is a bijection \(f : S \to S'\) such that for all \(x, y \in S\), we have \(x \leq y\) if and only if \(f(x) \leq' f(y)\) and \(\omega(x, y) = \omega'(f(x), f(y))\).

For terminology and notation about partially ordered sets, we refer to Stanley [8].
2.3 Graph posets

Definition 2.1 (Edge-subgraph poset). Let $Q := (\mathcal{G}, \leq_e, q)$ be a weighted poset, where for all $H, G \in \mathcal{G}$, we have $H \leq_e G$ if $H$ is isomorphic to an edge subgraph of $G$, and $q : \mathcal{G} \times \mathcal{G} \to \mathbb{Z}$ is a weight function such that for all $H, G \in \mathcal{G}$, $q(H, G)$ is the number of edge-subgraphs of $G$ that are isomorphic to $H$. For $G \in \mathcal{G}$, let $Q(G) := \{H \in \mathcal{G} \mid H \leq_e G \text{ and } e(H) > 0\}$. The concrete edge-subgraph poset of $G$ is the restriction of $\mathcal{Q}$ to $Q(G)$; it is denote by just $Q(G)$. The abstract edge-subgraph poset of $G$ is the isomorphism class of $Q(G)$, which we identify with a weighted poset $\overline{Q}(G) := (\{g_1, \ldots, g_m\}, \leq_e, q)$, which is a representative concrete edge-subgraph poset in the isomorphism class of $Q(G)$. We assume that $g_1$ and $g_m$ are the minimal and the maximal elements in $\overline{Q}(G)$, respectively. We define a rank function $e : \overline{Q}(G) \to \mathbb{Z}$ such that $e(g_1) = 1$ (so that $e(g_i)$ is the number of edges of $g_i$).

Definition 2.2. A graph $G$ is $Q$-reconstructible if it is determined up to isomorphism by its abstract edge-subgraph poset. An invariant of $G$ is said to be $Q$-reconstructible if it is determined by $\overline{Q}(G)$. A class of graphs is $Q$-reconstructible if each element of the class is $Q$-reconstructible.

Remark 2.3. For all graph $G$, we have $\overline{Q}(G) = \overline{Q}(G + K_1)$. Therefore, we understand $Q$-reconstructibility to mean $Q$-reconstructibility modulo isolated vertices.

Definition 2.4 (Induced subgraph poset). Let $P := (\mathcal{G}, \leq_v, p)$ be a weighted poset, where for all $H, G \in \mathcal{G}$, we have $H \leq_v G$ if $H$ is isomorphic to an induced subgraph of $G$, and $p : \mathcal{G} \times \mathcal{G} \to \mathbb{Z}$ is a weight function such that for all $H, G \in \mathcal{G}$, $p(H, G)$ is the number of induced subgraphs of $G$ that are isomorphic to $H$. For $G \in \mathcal{G}$, let $P(G) := \{H \in \mathcal{G} \mid H \leq_v G \text{ and } e(H) > 0\} \cup \{K_1\}$. The concrete induced subgraph poset of $G$ is the restriction of $P$ to $P(G)$; it is denote by just $P(G)$. The abstract induced subgraph poset of $G$ is the isomorphism class of $P(G)$, which we identify with a weighted poset $\overline{P}(G) := (\{g_1, \ldots, g_m\}, \leq_v, p)$, which is a representative concrete induced subgraph poset in the isomorphism class of $P(G)$.

Definition 2.5. A graph $G$ is $P$-reconstructible if it is determined up to isomorphism by its abstract induced subgraph poset. An invariant of $G$ is said to be $P$-reconstructible if it is determined by $\overline{P}(G)$. A class of graphs is $P$-reconstructible if each element of the class is $P$-reconstructible.

Definition 2.6. Let $G$ be a graph. We say that a partition $\pi := \{X_1, \ldots, X_n\}$ of $V(G)$ is a connected partition of $V(G)$, if the induced subgraphs $G[X_1], \ldots, G[X_n]$ are connected; we write $\pi \vdash_c V(G)$.

Definition 2.7. For $H_i, H_j \in \mathcal{G}$, if there exists $U \subseteq V(H_j)$ and $\pi \vdash_c U$ such that $H_j|\pi| \cong H_i$, we denote $H_i \leq_{\pi} H_j$. Let $\Omega := (\mathcal{G}, \leq_{\pi}, \omega)$ be a weighted poset, where the weight function $\omega : \mathcal{G} \times \mathcal{G} \to \mathbb{Z}$ is such that for all $H, G \in \mathcal{G}$, we have $\omega(H, G) := |\{(\pi, U) \mid U \subseteq V(G), \pi \vdash_c U \text{ and } G[\pi] \cong H\}|$. For $G \in \mathcal{G}$, let $\Omega(G) := \{H \in \mathcal{G} \mid H \leq_{\pi} G, \omega(H) = \omega(G)\}$. The concrete bond lattice of $G$ is the restriction of $\Omega$ to $\Omega(G)$; it is denote by just $\Omega(G)$. The abstract bond lattice of $G$ is the isomorphism class of $\Omega(G)$, which we identify with a weighted poset $\overline{\Omega}(G) := (\{h_1, \ldots, h_m\}, \leq_{\pi}, \omega)$, which is a representative concrete bond lattice in the isomorphism class of $\Omega(G)$. 

4
We assume that \( h_1 \) and \( h_m \) are the minimal and the maximal elements in \( \overline{\Omega}(G) \), respectively. We define a rank function \( \gamma \) on \( \overline{\Omega}(G) \) such that \( \gamma(h_1) = 0 \); hence \( k(h_i) = v(G) - \gamma(h_i) \).

Let \( G \) be a graph. We consider elements of \( \overline{Q}(G) \) as unknown graphs in \( G \). A weight preserving isomorphism \( \ell : \overline{Q}(G) \to Q(H) \) is called a legitimate labelling of \( \overline{Q}(G) \), and \( H \) is called a \( Q \)-reconstruction of \( G \). (Note that \( Q(H) \) is a concrete poset.) A graph invariant \( f \) of \( G \) is \( Q \)-reconstructible if \( f(H) = f(G) \) for all \( Q \)-reconstructions \( H \) of \( G \). Let \( x \in \overline{Q}(G) \). We say that \( x \) is reconstructible or uniquely labelled (or \( x \) satisfies property \( P \)), or a graph invariant \( f \) of \( x \) is reconstructible) if there is a graph \( F \) such that for all legitimate labelling maps \( \ell \) from \( \overline{Q}(G) \), we have \( \ell(x) = F \) (or \( \ell(x) \) satisfies property \( P \), or \( f(\ell(x)) \) is identical, respectively). Similarly, a graph \( F \) is a distinguished subgraph of \( G \) (or simply \( F \) is distinguished) if there is an element \( x \) such that for all legitimate labelling maps \( \ell \) from \( \overline{Q}(G) \), we have \( \ell(x) = F \). Analogous terminology may be used for \( \overline{P}(G) \) and \( \overline{\Omega}(G) \).

3 Constructing \( \overline{P}(G) \) and \( \overline{\Omega}(G) \) from \( \overline{Q}(G) \)

Let

\[
\mathcal{F}_0 := \{3K_2, K_3, K_{1,3}\}, \\
\mathcal{F}_1 := \{P_4, K_{1,2} + K_2, P_4 + K_2, T_4\}, \\
\mathcal{F}_2 := \{C_4, 2K_{1,2}, C_4 + K_2, B_1, P_6, B_2, B_3, B_4\}, \\
\mathcal{F}_3 := \{K_{1,m} \mid m > 1 \text{ and } m \neq 3\} \cup \{mK_2 \mid m > 1 \text{ and } m \neq 3\},
\]

\[
\mathcal{F}_4 := \{rK_3 + sK_{1,3} + F \mid r \neq s \text{ and } F \in \mathbb{N}^X\} \setminus \{K_3, K_{1,3}\}, \quad \text{where} \\
X := \{P_n \mid n \geq 2\} \cup \{C_n \mid n \geq 4\} \cup \{S_4, K_4 \setminus e, K_4\},
\]

and \( \mathbb{N}^X \) is the set of all unlabelled finite graphs (including the null graph) with components from \( X \).

The following result is the main theorem of this paper.

**Theorem 3.1** (Main Theorem). Let \( G \) be a graph with no isolated vertices. Then

1. \( \overline{\Omega}(G) \) can be constructed from \( \overline{Q}(G) \) if and only if \( G \) does not belong to \( \mathcal{M} := \mathcal{F}_0 \cup \mathcal{F}_2 \cup \mathcal{F}_4 \).

2. \( \overline{P}(G) \) can be constructed from \( \overline{Q}(G) \) if and only if \( G \) does not belong to \( \mathcal{N} := \bigcup_{i=0}^{4} \mathcal{F}_i \).

Throughout this section we assume that \( G \) is a graph without isolated vertices, and that we are given \( \overline{Q}(G) \). The main idea in proving Theorem 3.1 is the following lemma. When it is applicable, it allows us to recognise which elements of \( \overline{Q}(G) \) must be in \( \overline{\Omega}(G) \) and to calculate the weight function in the definition of \( \overline{\Omega}(G) \).

**Lemma 3.2.** Let \( g_i, g_k \in \overline{Q}(G) \). Then \( \omega(g_i, g_k) \) may be expressed as a polynomial in \( q(g_r, g_s) \), where \( g_1 \leq g_r < g_s \leq g_k \) and, for all \( g_r \), we have \( v(g_i) = v(g_r) \) and \( k(g_i) = k(g_r) \), and for all \( g_s \neq g_k \), we have \( v(g_i) = v(g_s) \) and \( k(g_i) = k(g_s) \).
Proof. If $(v(g_i), k(g_i)) = (v(g_k), k(g_k))$, then
\[
\omega(g_i, g_k) = \begin{cases} 1 & \text{if } g_i = g_k, \\ 0 & \text{otherwise}. \end{cases}
\]

Also, if $g_i \not< e g_k$, then $\omega(g_i, g_k) = 0$. Hence in the following calculation we assume that $(v(g_i), k(g_i)) \neq (v(g_k), k(g_k))$ and $g_i < e g_k$. We have
\[
q(g_i, g_k) = \sum_{g_j|v(g_i) = v(g_j), k(g_j) = k(g_i)} q(g_i, g_j) \omega(g_j, g_k).
\]

We rewrite Equation (1) as
\[
\omega(g_i, g_k) = q(g_i, g_k) - \sum_{g_j|g_i < e g_j} q(g_i, g_j) \omega(g_j, g_k),
\]

and repeatedly expand the factors $\omega(g_i, g_k)$ in each term on the right hand side, with the condition that $\omega(g_j, g_k) = q(g_j, g_k)$ if there is no $g_r$ such that $g_j < e g_r < e g_k$ and $v(g_r) = v(g_j)$ and $k(g_r) = k(g_j)$. Thus we obtain the required polynomial.\hfill\Box

Lemma 3.2 can be used to calculate $\omega(g_i, g_k)$ only if we know the number of vertices and the number of components of all elements that appear in the computation. Hence most of the following lemmas are meant to either Q-reconstruct G or to show that the number of vertices and the number of components of all elements that appear in the computation of $\omega(g_i, g_k)$ can be reconstructed.

We use several results from [10, 11] in the proof.

**Theorem 3.3** (Theorem 2.1 [10]). The graphs in $\mathcal{N}$ are not Q-reconstructible. The edge reconstruction conjecture is true if and only if all graphs, except the graphs in $\mathcal{N}$, are Q-reconstructible.

**Lemma 3.4** (Corollary 2.11 [10]). Acyclic graphs (i.e., trees and forests) that are not in $\mathcal{N}$ are Q-reconstructible.

**Lemma 3.5** (Lemma 2.7 [10]). For all $m \geq 4$, if $G$ belongs to the class $\{K_{1,m}^+\} \setminus \{K_{1,m+1}\}$, then $G$ is Q-reconstructible, and $K_{1,3}$ and $K_3$ (if $G$ contains $K_3$) are distinguished.

**Lemma 3.6** (Lemma 2.8 [10]). For all $m \geq 4$, if $G$ belongs to the class $\{(mK_2)^+\} \setminus \{(m + 1)K_2\}$, then $G$ is Q-reconstructible, and $3K_2$ is distinguished.

**Lemma 3.7** (Proposition 2.9 [10]). Graphs with at most 7 edges, except the ones in $\mathcal{N}$, are Q-reconstructible.

**Theorem 3.8** (Theorem 3.7 [11]). If $G$ does not belong to $\{K_{1,n}(n > 1), nK_2(n > 1), P_4, K_{1,2} + K_2, T_4, P_4 + K_2\}$, then $\overline{P}(G)$ can be constructed from $\overline{\Omega}(G)$.

Next we prove several propositions and lemmas that eventually imply the main theorem. We will first prove part (1) of the theorem, and then use Theorem 3.8 to prove the second part.
Proposition 3.9 (The “only if” parts of the main theorem).

1. If \( G \) belongs to \( \mathcal{M} \), then \( \overline{\Omega}(G) \) cannot be constructed from \( \overline{Q}(G) \).

2. If \( G \) belongs to \( \mathcal{N} \), then \( \overline{P}(G) \) cannot be constructed from \( \overline{Q}(G) \).

Proof. 1. The graphs \( 3K_2, K_{1,3}, K_3 \) have the same \( \overline{Q}(.) \), and the graphs \( 3K_2, K_{1,3} \) have the same \( \overline{\Omega}(.) \), but \( \overline{\Omega}(K_3) \) is different.

If \( G \in \mathcal{F}_4 \), then suppose that \( G = rK_3 + sK_{1,3} + F \) and \( H = sK_3 + rK_{1,3} + F \), where \( r \neq s \). Now \( \overline{\Omega}(G) \neq \overline{\Omega}(H) \): this follows from the fact that the rank of the maximal element in \( \overline{\Omega}(.) \) is \( v(.) - k(.) \), hence the two posets have different heights.

Graphs in \( \mathcal{F}_2 \) form pairs so that graphs in each pair have the same abstract edge subgraph poset but different abstract bond lattice.

2. If \( G, H \in \mathcal{N} \) such that \( \overline{Q}(G) \cong \overline{Q}(H) \) and \( G \not\cong H \), then there are only two cases in which \( v(G) = v(H) \), which are \( \{G, H\} = \{C_4 + K_2, B_1\} \) and \( \{G, H\} = \{B_2, B_3\} \). In these cases, we verify that \( \overline{P}(G) \neq \overline{P}(H) \). In all other cases, \( v(G) \neq v(H) \), hence \( G \) and \( H \) cannot have the same abstract induced subgraph poset. Hence if \( G \in \mathcal{N} \), then \( \overline{P}(G) \) cannot be constructed from \( \overline{Q}(G) \). \( \square \)

Proposition 3.10. If \( G \in \mathcal{F}_1 \cup \mathcal{F}_3 \), then \( \overline{\Omega}(G) \) can be constructed from \( \overline{Q}(G) \).

Proof. Graphs in each pair \( \{K_{1,m}, mK_2\}, m \neq 3 \) have the same \( \overline{Q}(.) \) and the same \( \overline{\Omega}(.) \). We prove that no other graph has the same \( \overline{Q}(.) \) as one of these graphs by induction on the number of edges of \( G \in \{K_{1,m}, mK_2\} \). If \( e(G) = 2 \) or \( e(G) = 4 \) then the result is true. Suppose that the result is true for all \( G \in \{K_{1,m}, mK_2\} \) such that \( m \geq 4 \). If \( e(G) = m + 1 \), then by induction hypothesis, there is an element \( x \in \overline{Q}(G) \) such that \( x \) represents \( K_{1,m} \) or \( mK_2 \). By Lemmas 3.5 and 3.6, we have \( G \in \{K_{1,m+1}, (m+1)K_2\} \).

Graphs in each pair \( \{P_4, K_{1,2} + K_2\}, \{P_4 + K_2, T_4\} \) have the same \( \overline{Q}(.) \) and the same \( \overline{\Omega}(.) \). By Lemma 3.7, no other graph has the same \( \overline{Q}(.) \) as one of these graphs. \( \square \)

Lemma 3.11. If \( G \not\in \mathcal{N} \), then \( v(G) \) and \( k(G) \) are \( Q \)-reconstructible.

Proof. We prove the result by induction on the number of edges of \( G \). All graphs on at most 7 edges that are not in \( \mathcal{N} \) are \( Q \)-reconstructible (Lemma 3.7). Hence we take \( e(G) = 7 \) as the base case, for which the result is true. Suppose now that the result is true for all \( G \not\in \mathcal{N} \) such that \( 7 \leq e(G) \leq m \). Let \( G \not\in \mathcal{N} \) be a graph on \( m + 1 \) edges.

If \( G \) is an acyclic graph, then \( G \) is \( Q \)-reconstructible (Lemma 3.4), thus \( k(G), v(G) \) are known. So assume that \( G \) contains a cycle. Hence each edge \( e \) in \( G \) that is on a cycle is such that \( k(G - e) = k(G) \) and \( v(G - e) = v(G) \) and \( G - e \) has no isolated vertices (since we have assumed that \( G \) has no isolated vertices).

Now consider an arbitrary element \( x \) of rank \( m \) in \( \overline{Q}(G) \). Suppose that \( x := \overline{G} - e \) (minus the resulting isolated vertices). We claim that \( x \) cannot be in \( \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \) since \( m \geq 7 \) and all graphs in \( \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \) have at most 6 edges. We also assume that \( x \) cannot be in \( \mathcal{F}_3 \) since in that case \( G \) would be \( Q \)-reconstructible (by Lemma 3.5 and 3.6). If \( x \not\in \mathcal{N} \), then we know \( v(x) \) and \( k(x) \) by induction hypothesis; and we annotate \( x \) with this information. If it is known from \( \overline{Q}(G) \) that \( G \) has an edge \( e \) such
that $e$ is on a cycle and $G - e$ is not in $\mathcal{F}_4$, then $v(G)$ is the maximum $v(H)$ among all graphs of rank $m$ in $\overline{Q}(G)$ that are not in $\mathcal{N}$. Next we show that such an edge must exist.

Consider a graph $H \in \mathcal{F}_4$. Suppose that $G = \overline{H} + uv$, where $u$ and $v$ are non-adjacent vertices in the same component of $H$ (so that $uv$ is on a cycle in $H + uv$). Each component of $H$ is in $\mathcal{X} \cup \{K_{1,3}, K_3\}$. We verify that for all possible ways of adding an edge $uv$ between two vertices of the same component of $H$, we either obtain a graph in $\mathcal{N}$ or obtain a graph $G$ which contains an edge $e$ on a cycle such that $G - e$ is not in $\mathcal{N}$. Since $G \not\in \mathcal{N}$, it is the latter case. Thus $v(G)$ is reconstructible.

Now, $k(G)$ is equal to the minimum number of components among graphs not in $\mathcal{N}$ that correspond to elements of rank $m$ and that have the same number of vertices as $G$. \hfill \Box

**Lemma 3.12** (Lemma 2.5 [10]). The graph $K_{1,2} + 2K_2$ is $Q$-reconstructible, and all elements of $\overline{Q}(K_{1,2} + 2K_2)$ are uniquely labelled.

**Lemma 3.13.** If $G \not\in \mathcal{N}$ and contains both $K_{1,2} + 2K_2$ and $T_4$ as subgraphs, then $v(x)$ and $k(x)$ are reconstructible for all $x$ in $\overline{Q}(G)$.

**Proof.** Let $x$ be an element of $\overline{Q}(G)$. If $x \not\in \mathcal{N}$, then $v(x)$ and $k(x)$ are known by Lemma 3.11. Hence assume that $x \in \mathcal{N}$.

We claim that all elements of $\overline{Q}(G)$ of rank 3 are uniquely labelled. Indeed, all elements of $\overline{Q}(K_{1,2} + 2K_2)$ are uniquely labelled (by Lemma 3.12), $\overline{Q}(T_4) \cong \overline{Q}(P_4 + K_2)$, and there is no other graph $H$ such that $Q(H) \cong Q(T_4)$, we distinguish $T_4$. Now $K_{1,3}$ is distinguished since $T_4$ contains $K_{1,3}$, but does not contain $K_3$ or $3K_2$. Furthermore, $3K_2$ is distinguished. Thus $K_3$ is distinguished also. Other subgraphs with 3 edges are distinguished since they are edge reconstructible, and graphs with 2 edges are distinguished. Thus, the graphs in $\mathcal{F}_0$, $P_4 + K_2$ and $T_4$ are distinguished.

Graphs $P_4$ and $K_{1,2} + K_2$ are distinguished since $K_{1,2} + 2K_2$ contains $K_{1,2} + K_2$, but does not contain $P_4$.

Pairs of graphs in $\mathcal{F}_2$ are distinguished since $P_4$ and $K_{1,3}$ are distinguished.

Graphs $K_{1,m}$ and $mK_2$ are distinguished since $K_{1,m}$ contains $K_{1,3}$ as a subgraph, while $mK_2$ does not.

Graphs $rK_3 + sK_{1,3} + F$ and $sK_3 + rK_{1,3} + F$, where $r \neq s$, are distinguished since $K_{1,3}$ and $K_3$ are distinguished. \hfill \Box

**Lemma 3.14.** If $G \not\in \mathcal{N}$, $e(G) \geq 5$, and $\Delta(G) \geq 4$, then $v(x)$ and $k(x)$ are reconstructible for all $x$ in $\overline{Q}(G)$.

**Proof.** Let $x \in \overline{Q}(G)$. If $x \not\in \mathcal{N}$, then $v(x)$ and $k(x)$ are known by Lemma 3.11. Hence assume that $x \in \mathcal{N}$. By Lemma 3.5, $K_{1,3}$ is distinguished. This allows distinguishing subgraphs isomorphic to $P_4 + K_2, T_4, B_1, C_4 + K_2, B_2, P_6, B_3, B_4$ and all subgraphs in $\mathcal{F}_3 \cup \mathcal{F}_4$. Hence for such elements, $v(x)$ and $k(x)$ are known. The only elements in $\mathcal{N}$ that are not yet distinguished are $P_4, K_{1,2} + K_2, C_4, 2K_{1,2}$.

We have $\Delta(G) \geq 4$ and $G$ itself is not $K_{1,\Delta(G)}$, hence $G$ contains one of the graphs in $\{K_{1,4}^+, \ldots, K_{1,5}\}$ (i.e., one of $K_{1,4} + K_2, S_5, T_5$) as a subgraph. We use such a subgraph to distinguish $P_4$ and $K_{1,2} + K_2$. 

8
If \( G \) contains \( K_{1,4} + K_2 \), then \( K_{1,4} + K_2 \) is distinguished (by Lemma 3.5). Now \( K_{1,2} + K_2 \) and \( P_4 \) are distinguished since \( K_{1,4} + K_2 \) does not contain \( P_4 \) but contains \( K_{1,2} + K_2 \).

If \( G \) contains \( S_5 \), then it also contains \( S_4 \), and \( S_4 \) is \( Q \)-reconstructible. Hence \( S_4 \) is distinguished. Now \( K_{1,2} + K_2 \) and \( P_4 \) are distinguished since \( S_4 \) does not contain \( K_{1,2} + K_2 \) but contains 2 subgraphs isomorphic to \( P_4 \).

If \( G \) does not contain \( S_4 \) but contains \( T_5 \), then \( G \) contains \( T_4 \), which is distinguished (as noted above). Now \( P_4 \) and \( K_{1,2} + K_2 \) are distinguished since \( T_4 \) contains one subgraph isomorphic to \( K_{1,2} + K_2 \) and two subgraphs isomorphic to \( P_4 \).

Once \( P_4 \) is distinguished, we also distinguish \( C_4 \) and \( 2K_{1,2} \) since only \( C_4 \) contains \( P_4 \) as a subgraph.

Now \( v(x) \) and \( k(x) \) are known for all \( x \) in \( \overline{Q}(G) \).

**Lemma 3.15.** If \( G \notin \mathcal{N} \), \( \Delta(G) \leq 3 \), and \( G \) contains \( K_{1,2} + 2K_2 \) but does not contain \( T_4 \), then \( G \) is \( Q \)-reconstructible.

**Proof.** Let \( \mathcal{I} \) be the class of graphs satisfying the conditions in the statement of the lemma.

First we show that we can recognise if \( G \) is in \( \mathcal{I} \). Graphs not in \( \mathcal{N} \) with at most 7 edges are \( Q \)-reconstructible (Lemma 3.7), so we assume that \( e(G) > 7 \). Since \( G \) is not \( K_{1,m} \) for any \( m \), we can assume that \( \Delta(G) < e(G) \). By Lemma 3.5, we can recognise if \( \Delta(G) \geq 4 \). Since \( K_{1,2} + 2K_2 \) is \( Q \)-reconstructible, we can recognise if \( G \) contains \( K_{1,2} + 2K_2 \) as a subgraph. We can recognise if \( G \) contains \( T_4 \) as a subgraph as follows: if \( G \) contains \( T_4 \), then \( G \) has a 7-edge subgraph \( F \) that contains \( T_4 \), and thus cannot be in \( \mathcal{N} \). Hence \( F \) is \( Q \)-reconstructible (by Lemma 3.7). Thus we know that \( G \) contains \( T_4 \) as a subgraph. Hence we now assume that \( G \) and all its \( Q \)-reconstructions are in \( \mathcal{I} \).

Next we make some observations about the structure of any graph \( G \) in class \( \mathcal{I} \). Graphs \( S_4, K_4 \setminus e, K_4 \) are in \( \mathcal{I} \), and since \( G \) does not contain \( T_4 \), these graphs can only occur as subgraphs of 4-vertex components. Also, we can verify that any subgraph isomorphic to \( K_3 \) or \( K_{1,3} \) is either a component of \( G \) or a subgraph of a component isomorphic to \( S_4, K_4 \setminus e \) or \( K_4 \). All other components of \( G \) are paths or cycles. Hence \( G \) is of the form \( rK_3 + sK_{1,3} + F \), where components of \( F \) are in \( \mathcal{F} \). But \( G \) is not in \( \mathcal{N} \), hence \( r = s \). Since \( q(K_{1,3}, H) = q(K_3, H) \) for all \( H \in \{ S_4, K_4 \setminus e, K_4 \} \), we have \( k(K_3, G) = k(K_{1,3}, G) \) and \( q(K_3, G) = q(K_{1,3}, G) \) for all graphs in \( \mathcal{I} \).

Graphs \( S_4, K_4 \setminus e, K_4 \) are all \( Q \)-reconstructible, hence \( q(S_4, G), q(K_4 \setminus e, G) \) and \( q(K_4, G) \) are known. Therefore, \( k(S_4, G), k(K_4 \setminus e, G), k(K_4, G), k(K_3, G) \) and \( k(K_{1,3}, G) \) are all reconstructible by the following sequence of calculations:

\[
\begin{align*}
k(K_4, G) &= q(K_4, G) \\
k(K_4 \setminus e, G) &= q(K_4 \setminus e, G) - q(K_4 \setminus e, K_4)k(K_4, G) \\
k(S_4, G) &= q(S_4, G) - q(S_4, K_4 \setminus e)k(K_4 \setminus e, G) - q(S_4, K_4)k(K_4, G), \\
k(K_3, G) &= k(K_{1,3}, G) = q(K_3, G) - k(S_4, G) - 2k(K_4 \setminus e, G) - 4k(K_4, G).
\end{align*}
\]

The graph \( K_{1,2} + 2K_2 \) is contained in \( G \), and by Lemma 3.12, the graphs \( K_2, P_3, 2K_2, K_{1,2} + K_2, 3K_2 \), and \( K_{1,2} + 2K_2 \) are distinguished. Now \( P_4 \) is distinguished because it is edge reconstructible and all edge-deleted subgraphs of \( P_4 \) are distinguished. The
argument extends to $C_4$, $P_5$, $C_5$, $P_6$ and $C_6$ - they are all distinguished. (Of these graphs, $C_5$, $C_6$ and $P_5$ are $Q$-reconstructible.) Thus we know $q(P_i, G), 2 \leq i \leq 6$ and $q(C_i, G), 4 \leq i \leq 6$.

Graphs with at most 7 edges that are not in $\mathcal{N}$ are $Q$-reconstructible, hence we prove the lemma by induction on $e(G)$, with $e(G) = 7$ as the base case. Suppose that the result is true for all $G \in \mathcal{I}$ such that $7 \leq e(G) \leq m$. Let $G$ be a graph with $e(G) = m + 1$, and suppose that we are given $\overline{Q}(G)$.

Paths and cycles with $i$ edges, for $7 \leq i \leq m$ are distinguished by induction hypothesis. Hence if $G$ itself is a cycle or path, then its edge-deck is known, and by edge reconstructibility of cycles and paths, $G$ is $Q$-reconstructible. Hence we assume that $G$ is not a path or cycle.

Cycles of length 5 or more can only be components (since $G$ does not contain $T_4$). Now, we calculate $k(P_i, G), 2 \leq i \leq m$ and $k(C_i, G), 4 \leq i \leq m$ by solving the following equations (in that order):

$$k(C_4, G) = q(C_4, G) - q(C_4, K_4 \setminus e)k(K_4 \setminus e, G) - q(C_4, K_4)k(K_4, G),$$
$$k(C_i, G) = q(C_i, G) \text{ for } 5 \leq i \leq m,$$
$$k(P_i, G) = q(P_i, G) - \sum_{H} q(P_i, H)k(H, G) \text{ for } i = m, m - 1, \ldots, 2,$$

where the summation in the last equation is over $H \in \{K_{1,3}, K_3, S_4, K_4 \setminus e, K_4, P_r, r > i, C_s, s \geq 4\}$.

Now all components of $G$ are known along with their multiplicities, completing the induction step and the proof.

**Lemma 3.16.** If $G \notin \mathcal{N}$, $\Delta(G) \leq 3$, and $G$ contains $T_4$ but does not contain $K_{1,2} + 2K_2$, then $G$ is $Q$-reconstructible.

**Proof.** Graphs with at most 7 edges that are not in $\mathcal{N}$ are $Q$-reconstructible, hence we assume that $e(G) > 7$.

If a graph contains $T_4$, and has 2 or more non-trivial components, then it also contains $K_{1,2} + 2K_2$. Therefore we assume that $G$ is connected. The conditions on $G$ also imply that $\nu(G) \in \{5, 6\}$. If $\nu(G) = 5, \Delta(G) \leq 3$, then $e(G) \leq 7$ and the result is true. Therefore, we can assume that $\nu(G) = 6$. Now $\Delta(G) \leq 3$ and $e(G) > 7$ imply that have $e(G) \in \{8, 9\}$.

For any graph $H$, if $4 \leq e(H) \leq 9$, then $\nu(H) \leq 10$ or $H$ is disconnected. In either case, $H$ is vertex reconstructible, and hence edge reconstructible (see [4, 6, 7]). If $e(G) = 8$, the degree sequence of $G$ is $3, 3, 3, 3, 2, 2$ or $3, 3, 3, 3, 3, 1$. Now a 7-edge subgraph of $G$ cannot have a component isomorphic to $K_3$ or $K_{1,3}$ (since the degree sequence of $K_3$ and $K_{1,3}$ is $2, 2, 2$ and $3, 1, 1, 1$, respectively), hence $G$ does not have a 7-edge subgraph in $\mathcal{N}$. Hence all edge-deleted subgraphs of $G$ are $Q$-reconstructible, and since $G$ is edge reconstructible, it is also $Q$-reconstructible.

If $e(G) = 9$, then the degree sequence of $G$ is $3, 3, 3, 3, 3, 3$, by a similar argument as in the case $e(G) = 8$, we note that no edge deleted subgraph of $G$ is in $\mathcal{N}$, hence using the case $e(G) = 8$, we can construct the edge deck of $G$, and then reconstruct $G$ since $G$ is edge reconstructible. 

\[\square\]
Proof. of Theorem 3.1 (the ‘if’ part) Let $G$ be a graph such that $G \not\in \mathcal{M}$.

If $G \not\in \mathcal{N}$ and $e(G) \leq 7$, then $G$ is $Q$-reconstructible by Lemma 3.7. Hence $\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$. If $G \not\in \mathcal{N}$, and contains $K_{1,2} + 2K_2$ or $T_4$, and $\Delta(G) \leq 3$, then $G$ is $Q$-reconstructible by Lemmas 3.15 and 3.16. Hence $\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$. If $G$ contains neither $T_4$ nor $K_{1,2} + 2K_2$ as a subgraph, then $e(G) \leq 7$. If $G \in \mathcal{F}_3 \cup \{P_4, K_{1,2} + K_2, P_4 + K_2, T_4\}$, then $\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$ by Propositions 3.10. In all other cases, by Lemmas 3.13 and 3.14, we can reconstruct $v(x)$ and $k(x)$ for all $x \in \overline{Q}(G)$.

Now we apply Lemmas 3.2 to $g_i$ and $g_k$, with $g_k$ as the maximal element, to recognise if $g_i$ must be $\overline{\Omega}(G)$. Then we again apply Lemma 3.2 for all $g_i, g_k$ that are marked as elements of $\overline{\Omega}(G)$.

Let $G$ be a graph such that $G \not\in \mathcal{N}$. Given $\overline{Q}(G)$, we construct $\overline{\Omega}(G)$. Now, by Theorem 3.8, we construct $\overline{P}(G)$ from $\overline{\Omega}(G)$. □

The following corollary generalises the result that edge reconstruction conjecture is weaker than the vertex reconstruction conjecture.

**Corollary 3.17.** If a graph $G$ not in $\mathcal{N}$ is $P$-reconstructible, then it is $Q$-reconstructible.

**Acknowledgements**

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001. The first author would like to thank CAPES for the doctoral scholarship at UFMG.

**References**

[1] J. A. Bondy, *Beautiful conjectures in graph theory*, European Journal of Combinatorics 37 (2014), 4–23.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008. MR 2368647 (2009c:05001)

[3] F. Harary, *On the reconstruction of a graph from a collection of subgraphs*, Theory of Graphs and its applications (M. Fiedler, ed) (1964), 47–52.

[4] Robert L. Hemminger, *On reconstructing a graph*, Proc. Amer. Math. Soc. 20 (1969), 185–187. MR MR0232696 (38 #1019)

[5] P. J. Kelly, *On isometric transformations*, Ph.D. thesis, University of Wisconsin. (1942).

[6] Paul J. Kelly, *A congruence theorem for trees*, Pacific J. Math. 7 (1957), 961–968. MR 0087949 (19,442c)

[7] B. D. McKay, *Computer reconstruction of small graphs*, J. Graph Theory 1 (1977), no. 3, 281–283. MR MR0463023 (57 #2987)
[8] Richard P. Stanley, *Enumerative combinatorics: Volume 1*, 2nd ed., Cambridge University Press, USA, 2011.

[9] Bhalchandra D. Thatte, *Kocay’s lemma, Whitney’s theorem, and some polynomial invariant reconstruction problems*, Electron. J. Combin. 12 (2005), no. 4, R63, 30 pp. (electronic). MR 2180800 (2006f:05121)

[10] ———, *The edge-subgraph poset of a graph and the edge reconstruction conjecture*, J. Graph Theory 92 (2019), no. 3, 287–303. MR 4009306

[11] ———, *The connected partition lattice of a graph and the reconstruction conjecture*, J. Graph Theory 93 (2020), no. 2, 181–202. MR 4043753