CHARACTER SERIES AND QUATERNIONIC SKLYANIN ALGEBRAS

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Abstract. This paper has 2 goals: to prove certain properties of character series of graded algebras on which a finite group works as algebra automorphisms and a detailed analysis of representations of 5-dimensional quaternionic Sklyanin algebras. We also prove that for any odd n, the Sklyanin algebras associated to points of order 2 are graded Clifford algebras.

1. Introduction

In [8] Odesskii and Feigin constructed the so-called elliptic Sklyanin algebras, which are noncommutative graded algebras depending on an elliptic curve E and a point $\tau \in E$ with n generators and $\binom{n}{2}$ relations for each $n \geq 3$. By construction, these algebras are deformations of the polynomial ring in n variables and the Heisenberg group of order $n^3$ works on these algebras as gradation preserving automorphisms. In particular for $n = 3$, one gets in this way almost all quadratic Artin-Schelter regular algebras.

However, in [6] the author showed that for each $p \geq 3$ prime there is a $\frac{p-1}{2}$-dimensional family of graded Clifford algebras on which $H_p$, the finite Heisenberg group of order p, works as gradation preserving automorphisms. Moreover, when $p = 5$, these graded Clifford algebras are not Sklyanins. It was conjectured that Sklyanins associated to points of order 2 are indeed graded Clifford algebras for any odd prime global dimension $p$.

This paper is divided in 2 parts: at first we show some properties of algebras on which a finite group acts as gradation preserving automorphisms and we show how such algebras are constructed. We show how Koszul duality can be used to find character series, which decode the decomposition of an algebra in simple $G$-representations. Next, we prove the following theorem

Theorem 1.1. Let $Z$ be an irreducible variety parametrizing graded algebras $A_x$, $x \in Z$ such that $H_{A_x}(t) = H_{A_y}(t) \forall x, y \in Z$. Then we also have for the character series that $Ch_{A_x}(g,t) = Ch_{A_y}(g,t) \forall g \in G, x, y \in Z$.

We apply this theorem for $G = H_p$ to Sklyanin algebras and homogeneous coordinate rings of elliptic curves embedded in $\mathbb{P}^{p-1}$.

The second part concerns Sklyanin algebras associated to points of order 2. First of all, we prove

Theorem 1.2. The n-dimensional Sklyanin algebras with $n \geq 3$ odd associated to points of order 2 are graded Clifford algebras.

Using this fact, we completely determine the simple representations of the 5-dimensional Sklyanin algebras associated to points of order 2 and the associated $\text{Proj}$ of these algebras.
1.1. Notation. In this article, we use the following notations:
- \( V(I) \) for \( I \subset \mathbb{C}[a_1, \ldots, a_n] \) an ideal is the Zariski-closed subset of \( \mathbb{A}^n \) or \( \mathbb{P}^{n-1} \) determined by \( I \), it will be clear from the context if the projective or affine variety is used.
- \( D(I) \) for \( I \subset \mathbb{C}[a_1, \ldots, a_n] \) is the open subset \( \mathbb{A}^n \setminus V(I) \) or \( \mathbb{P}^{n-1} \setminus V(I) \), it will be clear from the context if it is an open subset of affine space or of projective space. If \( I = (a) \), then we write \( D(a) \) for \( D(I) \).
- \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \) for \( n \in \mathbb{N} \).
- \( \text{Grass}(m, n) \) will be the projective variety parametrizing \( m \)-dimensional vector spaces in \( \mathbb{C}^n \).
- For an algebra \( A \) and elements \( x, y \in A \), \( \{x, y\} = xy + yx \).
- The algebra \( C(a_0 : \ldots, a_{p^2-1}) \) with \( (a_0 : \ldots : a_{p^2-1}) \in \mathbb{F}_{p^2} \) stands for the algebra with generators \( x_0, \ldots, x_{p-1} \) and \( H_p \)-representatives as relations
  \[
  \begin{aligned}
  &a_0 \{x_i, x_{i-1}\} = a_i x_0^2, 1 \leq i \leq \frac{p^2-1}{2}, \\
  &a_i+1 \{x_i x_{i-1}, x_{i-1} x_i\} = a_i \{x_{(i+1)} x_{-(i+1)}, x_{-(i+1)} x_{(i+1)}\}, 1 \leq i \leq \frac{p^2-3}{2}.
  \end{aligned}
  \]
- For \( V \) a \( n \)-dimensional vector space, we set \( T(V) = \bigoplus_{k=0}^{\infty} V^\otimes k \).
- Every graded algebra \( A \) will be positively graded, finitely generated over \( \mathbb{C} \) and connected, that is \( A_0 = \mathbb{C} \).

2. The finite Heisenberg group

Let \( p \geq 3 \) be any prime number.

Definition 2.1. The Heisenberg group of order \( p^3 \) is the finite group given by the generators and relations

\[
H_p = \langle e_1, e_2, z | e_1^p = e_2^p = z^p, [e_1, e_2] = z, e_1 z = ze_1, e_2 z = ze_2 \rangle
\]

and it is a central extension of the group \( \mathbb{Z}_p \times \mathbb{Z}_p \), so we have the exact sequence

\[
\begin{array}{c}
1 \longrightarrow \mathbb{Z}_p \longrightarrow H_p \longrightarrow \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow 1.
\end{array}
\]

All the 1-dimensional simple representations of \( H_p \) are induced by the characters of \( \mathbb{Z}_p \times \mathbb{Z}_p \). The other simple representations are \( p \)-dimensional and are determined by a primitive \( p \)th root of unity. They are defined in the following way: choose a primitive \( p \)th root of unity \( \omega \), then define the following action of \( H_p \) on the vector space \( V = \mathbb{C} x_0 + \ldots + \mathbb{C} x_{p-1} \)

\[
e_1 \cdot x_i = x_{i-1}, e_2 \cdot x_i = \omega^i x_i,
\]

indices taken in \( \mathbb{Z}_p \). Taking another primitive root gives you another simple representation. This means that there are \( p^2 \) 1-dimensional and \( p - 1 \) \( p \)-dimensional irreducible representations, which are all the simple ones. There are \( p^2 + p - 1 \) conjugacy classes, 1 for each central element and the other \( p^2 - 1 \) classes contain a unique element of the form \( e_1^a e_2^b \), \( a, b \in \mathbb{Z}_p, (a, b) \neq (0, 0) \).

The character of a simple \( p \)-dimensional representation \( V \) is given by

\[
\chi(z^k) = p \omega^k, \quad \chi(e_1^a e_2^b) = 0, (a, b) \neq (0, 0).
\]
Such a representation $V$ also defines an antisymmetric bilinear form on the $\mathbb{Z}_p$-vector space $\mathbb{Z}_p \times \mathbb{Z}_p$. Identifying $e_1$ and $e_2$ with their images in $\mathbb{Z}_p \times \mathbb{Z}_p$, we get this form by setting $\langle e_1, e_2 \rangle = \omega$ and extending it $\mathbb{Z}_p$-linearly to $\mathbb{Z}_p \times \mathbb{Z}_p$, thus
\[
\langle ae_1 + be_2, ce_1 + de_2 \rangle = \omega^{ad-bc}.
\]
If we define a group morphism $(z) \xrightarrow{\phi} \mu_p$ by $\phi(z) = \omega$ (written multiplicatively in $\mu_p$), then we have a commutative diagram

\[
\begin{array}{ccc}
H_p \times H_p & \xrightarrow{\phi} & \mathbb{Z}_p \times \mathbb{Z}_p \\
\downarrow \phi & & \downarrow \phi \\
\{z\} & & \mu_p
\end{array}
\]

Since every $p$-dimensional representation is determined by the image of $z$, every nontrivial antisymmetric bilinear form on $\mathbb{Z}_p \times \mathbb{Z}_p$ uniquely defines a simple representation of $H_p$. Conversely, every simple $p$-dimensional representation of $H_p$ defines a unique nontrivial antisymmetric bilinear form on $\mathbb{Z}_p \times \mathbb{Z}_p$ by extending linearly $\langle e_1, e_2 \rangle = \chi_p(z)$.

For the remainder of this paper, we will fix a primitive $p$th root of unity $\omega$ and the unique simple $p$-dimensional representation of $H_p$ by $V$. By the representation $V_i$ we will then imply the $p$-dimensional simple representation associated to $\omega^i$ for $1 \leq i \leq p-1$.

When $p$ is not prime, $H_p$ is still a central extension of $\mathbb{Z}_p \times \mathbb{Z}_p$ with $\mathbb{Z}_p$ and the simple $p$-dimensional representations are also uniquely determined by a primitive $p$th root of unity, but there are other simple representations of dimension $1 < k < p$ to consider (cfr. [13]).

3. Shioda's modular surface $S(5)$

The following section is a summary of chapter V, section 5 of [11].

**Theorem 3.1.** Every elliptic curve $(E, O)$ can be embedded in $\mathbb{P}^4$ in such a way that $O$ is sent to $O_a = (0 : 1 : a : -a : -1)$ for some $a \in \mathbb{C}$ and such that the action of $H_5$ on $\mathbb{P}^4$ is an extension of the action of $\mathbb{Z}_5 \times \mathbb{Z}_5$ on $E$, call this curve $C_a$ with $O = O_a$. The relations of $C_a$ are given by $x_i^2 + ax_{i+1}x_{i-1} - \frac{1}{a}x_{i+2}x_{i-2}$, $0 \leq i \leq 4$, indices taken in $\mathbb{Z}_5$.

The surface $S(5) \subset \mathbb{P}^4_{[x_0 : \ldots : x_4]} \times \mathbb{P}^1_{[A : B]}$ defined by the relations $ABx_i^2 + A^2x_{i+1}x_{i-1} - B^2x_{i+2}x_{i-2}$, $0 \leq i \leq 4$, indices taken in $\mathbb{Z}_5$ is an elliptic surface over $X(5)$, the modular curve which is a compactification of the curve $X'(5)$. $X'(5)$ parametrizes elliptic curves with level 5 structure, that is, elliptic curves $E$ and embeddings $\mathbb{Z}_5 \times \mathbb{Z}_5 \rightarrow E$ such that the following diagram commutes

\[
\begin{array}{ccc}
\mathbb{Z}_5 \times \mathbb{Z}_5 & \xrightarrow{\cdot} & \mathbb{Z}_5 \times \mathbb{Z}_5 \\
\downarrow & & \downarrow \\
\mathbb{Z}_5 \cong \mu_5 & & \mu_5
\end{array}
\]

where $[-, -]$ is an antisymmetric $\mathbb{Z}_5$-bilinear form defined by $[(1, 0), (0, 1)] = \omega$ and $\langle - , - \rangle$ is the Weil pairing on $E[5]$ (cfr. [13], III.8).
$X(5) \cong \mathbb{P}^1$ and the map between $S(5)$ and $\mathbb{P}^1$ is just given by projection on the second factor. PSL$_2(5)$ acts on $\mathbb{P}^1$ such that points belong to the same orbit if and only if the corresponding fibers are isomorphic as varieties. For every point $a \in X(5)$ the fiber $C_a$ is an elliptic curve, except for the PSL$_2(5)$-orbit of 0. For the PSL$_2(5)$-orbit of 0, the fiber is a cycle of 5 lines intersecting 2 by 2.

**Theorem 3.2.** The projection of $S(5)$ to $\mathbb{P}^4$ is a determinantal surface $S_{15}$, defined by taking the $3 \times 3$-minors of the matrix

$$
\begin{bmatrix}
  x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \\
x_2x_3 & x_3x_4 & x_4x_0 & x_0x_1 & x_1x_2 \\
x_1x_4 & x_2x_0 & x_3x_1 & x_4x_2 & x_0x_3
\end{bmatrix}.
$$

This projection is 1-to-1 except for the 30 points of $\mathbb{P}^4$ with a non-trivial stabilizer in $H_5$, where this map is 2-to-1. These 30 points are the singular points of $S_{15}$ and the projection map $S(5) \longrightarrow S_{15}$ is a desingularization of these points.

On $\mathbb{P}^4$ one has the involution $\phi$ defined by $x_i \mapsto x_{-i}$ which fixes the equation for $C_a$ for each $a \in \mathbb{P}^1$. As this involution fixes $O_a$ for each elliptic curve $C_a$, it follows that $\phi$ is the automorphism $[-1]$ on $C_a$ whenever $C_a$ is an elliptic curve. The other fixed points by this involution are the points of order 2 and they are found by intersecting $C_a$ with the plane $V(x_1 - x_4, x_2 - x_3)$.

**Proposition 3.3.** The curve in $\mathbb{P}^4$ which intersects $S_{15}$ in the 2-torsion points of $C_a$ whenever $C_a$ is an elliptic curve is determined by the equations

$$
\begin{align*}
x_1 - x_4, \\
x_2 - x_3, \\
x_0^4x_1x_2 - x_0^2x_1^2x_2^2 - x_0(x_1^5 + x_2^5) + 2x_1^3x_2^3.
\end{align*}
$$

**Proof.** See amongst others [II]. Alternatively, one can check by computer that the intersection of the plane $V(x_1 - x_4, x_2 - x_3)$ with $S_{15}$ is indeed the claimed curve.

4. Constructing $G$-algebras

**Definition 4.1.** Let $G$ be a reductive group. We call a graded connected algebra $A$, finitely generated in degree 1, a $G$-algebra if $G$ acts on it by gradation preserving automorphisms.

This implies that there exists a representation $V$ of $G$ such that $T(V)/R \cong A$ with $R$ a graded ideal of $T(V)$, which is itself a $G$-subrepresentation of $T(V)$.

The general construction of quadratic $G$-algebras with relations is as follows: let $V$ be a $G$-representation. Then $V \otimes V$ is also a $G$-representation which decomposes as a summation of simple representations, say $V \otimes V \cong \oplus_{i=1}^m S_i^{a_i}$ with the $S_i$ different simple representations and all $a_i \geq 0$. Then all algebras are constructed by taking embeddings of the $S_i$ in $V \otimes V$ and taking these as relations of the algebra. One can of course do the same for all graded algebras by taking relations in $T(V)_i = V^\otimes i$ and take different embeddings of the simple representations of $G$ in this vector space as relations.
Definition 4.2. Let $A$ be a G-algebra with a minimal set of relations given by a finite number of G-representations $R_2, \ldots, R_k$, where $R_i \subset V^\otimes i$. We call $B$ a G-deformation of $A$ if $B$ is also a quotient of $T(V)$ with a minimal set of relations $R'_2, \ldots, R'_k$ such that $R'_i \subset V^\otimes i$ and $R'_i \cong R_i$ as G-representations.

If the relations for $A$ are all of the same degree, then all G-deformations of $A$ depend on a product of grassmannians. For example, if $A$ is a quadratic algebra, if $R = \bigoplus_{i=1}^m S^i \subset V \otimes V = \bigoplus_{i=1}^m S^i$, then all the G-deformations are parametrized by the product $\prod_{i=1}^m \text{Grass}(e_i, a_i)$.

Definition 4.3. We say that a variety $Z$ parametrizes G-deformations of a quadratic G-algebra $A$ if $Z$ can be embedded in a product of grassmannians $\prod_{i=1}^m \text{Grass}(e_i, a_i)$, which is the moduli space of G-deformations of $A$.

The leading example of G-deformations of an algebra $A$ will be where $G = H_p$ and $A = \mathbb{C}[V]$ with $V$ a simple $p$-dimensional representation of $H_p$.

Of course, one can also take relations in higher degree than 2 as generators of $R$, on the condition that $A$ also has relations in the same degree. In that case one has to look at a larger product of grassmannians, but the construction stays the same. However, in this case the G-deformations will not necessarily be parametrized by a product of grassmannians, but rather a Zariski closed subset of a product of grassmannians.

Example 4.4. Let $G = H_3$ and $V = \mathbb{C}x_0 + \mathbb{C}x_1 + \mathbb{C}x_2$ with the action of $H_3$ defined by $e_1 \cdot x_i = x_{i-1}$ and $e_2 \cdot x_i = \omega^i$. Then $V \otimes V \cong (V^*)^\otimes 3$ with generators $x_1x_2 + x_2x_1, x_1x_2 - x_2x_1$ and $x_i^2$ for each copy. $\mathbb{C}(x_1x_2 - x_2x_1) + \mathbb{C}(x_1x_0 - x_0x_1 - x_1x_0) = V \wedge V$, which are the relations for the polynomial algebra $\mathbb{C}[x_0, x_1, x_2]$. In order to find $H_3$-deformations of $\mathbb{C}[x_0, x_1, x_2]$, we need to find the embeddings of $V^*$ into $(V^*)^\otimes 3$, which is just given by a vector by Schur’s lemma. Such an embedding is completely determined by $A(x_1x_2 + x_2x_1) + B(x_1x_2 - x_2x_1) + Cx_0^2$ with $(A : B : C) \in \text{Grass}(1, 3) \cong \mathbb{P}^2$. Putting $a = A + B, b = A - B, c = C$, one gets the familiar relations for the 3-dimensional Sklyanin algebras:

\[
\begin{align*}
ax_1x_2 + bx_2x_1 + cx_0^2 \\
ax_2x_0 + bx_0x_2 + cx_1^2 \\
ax_0x_1 + bx_1x_0 + cx_2^2.
\end{align*}
\]

Apart from the 3-dimensional Sklyanin algebras, one also has the twisted coordinate ring $\mathcal{O}_*(E)$, which is a quotient of the Sklyanin algebra $\mathcal{A}_*(E)$ by a central element of degree 3. One can easily check that this element is fixed by the Heisenberg action. Therefore, the twisted coordinate rings are $H_3$-deformations of the graded coordinate ring $\mathcal{O}(E)$, where $E$ is embedded in $\mathbb{P}^2$ in Hesse normal form.

5. Koszul Algebras

Definition 5.1. Given a quadratic algebra $A = T(V)/I$ with generators $V = \mathbb{C}x_0 + \cdots + \mathbb{C}x_n$ and relations given by $I_2$, we define the Koszul dual $A^!$ to be the quadratic algebra $T(V^*)/J$, with $J_2$ defined as the subspace of $V^* \otimes V^*$ such that $\forall w \in J_2, \forall v \in I_2 : w(v) = 0$.

We say that $A$ is Koszul iff $A^! \cong \text{Ext}_A(\mathbb{C}, \mathbb{C})$. The standard properties of Koszul algebras we will need are that there is a relation between the Hilbert series of $A$
and $A^t$, given by

$$H_A(t)H_{A^t}(-t) = 1$$

and that $A$ is Koszul iff $A^t$ is Koszul.

An important fact concerning Koszul algebras is that the Koszul complex associated to these algebras is of the form

$$\cdots \rightarrow A \otimes (A^t)^* \xrightarrow{(d_K)_n} A \otimes (A^t)^* \rightarrow \cdots$$

and that

$$(A^t)^*_n = V^\otimes n^{-2} \otimes I_2 \cap \ldots \cap I_2 \otimes V^\otimes n^{-2}.$$  

$(d_K)_n$ is given by taking the first component of $(A^t)^*_n$ and absorbing it in $A$, for example $(d_K)_1(a \otimes x) = ax \in A$. It follows from this description that each $(d_K)_n$ is a $G$-morphism, whenever $G$ acts on $A$ as gradation preserving algebra automorphisms. This is useful for finding the character series when $G$ is a finite (or more generally, reductive) group. (Everything about Koszul algebras and character series has been observed in [12], but the focus there was more about finding the ring of invariants than really calculating character series).

**Definition 5.2.** Let $G$ be a finite group. The character series for an element $g \in G$ and for a graded algebra $A$ on which $G$ acts as gradation preserving automorphisms is a formal sum

$$Ch_A(g,t) = \sum_{n \in \mathbb{Z}} \chi_{A_n}(g)t^n.$$  

For example, if $g = 1$, $Ch_A(1,t) = H_A(t)$, the Hilbert series of $A$. As a character of a representation is constant on conjugacy classes, we can represent the decomposition of $A$ in simple $G$-representations as a vector of length equal to the number of conjugacy classes and on the $i$th place the character series $Ch_A(g,t)$ with $g \in C_i$, the $i$th conjugacy class.

**Lemma 5.3.** Let $V$ be a simple representation of $G$ and let $A$ be a $G$-algebra constructed from $T(V)$. For every element of the center, we have that $Ch_A(z,t) = H_A(\lambda t)$, where $z$ works on $V$ by multiplication with $\lambda$.

**Proof.** It follows that in degree $k$ the action of $z$ on $A_k$ is given by multiplication with $\lambda^k$, so the character series in this case is given by $\sum_{k=0}^{\infty} \lambda^k \dim A_k t^k = H_A(\lambda t)$.  

Suppose now that $A$ is a Koszul algebra and that a finite group $G$ acts on it as gradation preserving automorphisms. Because the Koszul complex is a free resolution of the trivial module $\mathbb{C}$, which is isomorphic as $G$-representation to the trivial representation and because the Koszul complex consists of $G$-morphisms, we have a similar formula for finding the character series of the Koszul dual as we have for the Hilbert series. More precisely, we have

$$Ch_A(g,t)Ch_{A^t}(g,-t) = 1.$$  

This allows us to compute $Ch_A(g,t)$ whenever we know $Ch_A(g,t)$. To know the character series of $A^t$, we have to take the complex conjugates of the coefficients of $Ch_{(A^t)^*}(g,t)$. In short, for a Koszul algebra $A$, the character series associated to $A$ is completely determined by the character series of $A^t$. 
5.1. Application to polynomial rings. For polynomial rings $\mathbb{C}[V]$, one has the advantage that $\mathbb{C}[V] = \wedge V^*$, which is a finite dimensional algebra and therefore easier to decompose in simple $G$-representations. In [4] the author calculated the character series of $\mathbb{C}[V]$ using this technique where $V$ was a simple representation of $H_p$. There it was shown that

$$Ch_{\mathbb{C}[V]}(1, t) = \frac{1}{(1 - t)^p}$$
$$Ch_{\mathbb{C}[V]}(z^k, t) = \frac{1}{(1 - \omega^k t)^p}$$
$$Ch_{\mathbb{C}[V]}(e^p, t) = \frac{1}{1 - t^p}, (k, l) \neq (0, 0).$$

6. Character series are constant

**Lemma 6.1.** Let $A$ be a $G$-algebra with $T(V) \xrightarrow{p} A$ the natural projection map. Decompose $A_k = \oplus_{i=1}^m S_i^{\otimes e_i}$ into simple $G$-representations and similarly $T(V)_k = \oplus_{i=1}^m S_i^{\otimes a_i}$ with naturally $a_i \geq e_i$. Then there exists a subspace $W \subset T(V)_k$ such that $W \cong A_k$ as $G$-representations and $p|_W$ is an isomorphism of $G$-representations.

**Proof.** It follows from Schur’s lemma that the map $T(V)_k \xrightarrow{p_k} A_k$ is a surjective element of $\oplus_{i=1}^m \text{Hom}(S_i^{\otimes a_i}, S_i^{\otimes e_i}) \cong \oplus_{i=1}^m \text{Hom}(A_i^{\otimes a_i}, A_i^{\otimes e_i})$. Therefore it reduces to a statement of linear maps, which follows from standard linear algebra.

More importantly, this means that if $A_k = \oplus_{i=1}^m S_i^{\otimes e_i}$, we can choose $G$-generators $v_{i,j} \in A_k, 1 \leq i \leq m, 0 \leq j \leq e_i$ and elements $w_{i,j}, 1 \leq i \leq m, 0 \leq j \leq e_i$ of $T(V)_k$ such that $G \cdot w_{i,j} \cong S_i \cong G \cdot v_{i,j}$ and $p(w_{i,j}) = v_{i,j}$.

**Theorem 6.2.** Let $Z$ be an irreducible variety parametrizing graded algebras $A_x$, $x \in Z$ such that $H_{A_x}(t) = H_{A_y}(t) \forall x, y \in Z$. Then we also have for the character series that $Ch_{A_x}(g, t) = Ch_{A_y}(g, t) \forall g \in G, x, y \in Z$.

**Proof.** For $x \in Z$, let $T(V) \xrightarrow{p_x} A_x$ denote the natural homomorphism. Suppose that the character series of $A_x$ and $A_y$ are not the same. There exists a minimal $k \geq 3$ such that $(A_x)_k \cong (A_y)_k$ as $G$-modules. According to the lemma we can find a subspace $W$ of $T(V)_k$ such that $p_x(W) = (A_x)_k$ and $W \cong (A_x)_k$ as $G$-representations. Then there exists an open subset $U$ with $x \in U$ of $Z$ such that the images of the chosen generators of $W$ are linearly independent $\forall z \in U$. This implies that $W \cap \ker(p_x)_k = 0$. As we know that all algebras have the same Hilbert series, this automatically implies that $W \cong (A_x)_k$ as $G$-representations. Similarly there exists an open subset $U'$ with $y \in U'$ and a subspace $W'$ of $T(V)_k$ such that $W' \cong (A_y)_k$ and $((p_z)_k)|_{W'}$ a $G$-isomorphism $\forall z \in U'$. As $Z$ was irreducible, there exists a point $a \in U \cap U'$. But then it follows that

$$(A_x)_k \cong W \cong (A_a)_k \cong W' \cong (A_y)_k$$

which is a contradiction.

From this we immediately deduce

**Corollary 6.3.** Let $Z$ be a connected variety that parametrizes graded algebras $A_z$ with constant Hilbert series. Then the character series is also constant.
6.1. Application to elliptic curves. It is well-known that any elliptic curve $E$ can be embedded in $\mathbb{P}^{p-1}$ with $p$ prime such that the finite Heisenberg group acts on $\mathbb{P}^{p-1}$ by a projectivication of one of its irreducible $p$-dimensional representations. $E$ is stable under this action and the action on $E$ reduces to translation with $p$-torsion points. In particular, this means that the relations of $E$ in $\mathbb{C}[x_0, \ldots, x_{p-1}]$ form an $H_p$-orbit in $\mathbb{C}[V]$ and that the graded coordinate ring $\mathcal{O}(E)$ has an action of $H_p$ as gradation preserving automorphisms. The relations for $\mathcal{O}(E)$ are quadratic and form an $H_p$-orbit in $\mathbb{C}[x_0, \ldots, x_{p-1}]$. It is also known that for $p \geq 5$ these families of elliptic curves degenerate to a cycle of $p$ lines, determined by the equations $x_{i+k}x_{i-k}, 1 \leq i \leq \frac{p-3}{2}, 0 \leq k \leq p-1$. Denote this cycle of lines by $S$. Therefore, in order to calculate the character series of $\mathcal{O}(E)$, it is sufficient to calculate the character series of $\mathcal{O}(S)$.

**Theorem 6.4.** The character series of $\mathcal{O}(S)$ is given by

$$
Ch(1, t) = \frac{1 + (p-2)t + t^2}{(1-t)^2}
$$

$$
Ch(z^k, t) = \frac{1 + (p-2)\omega^k t + (\omega^k t)^2}{(1-\omega^k t)^2}
$$

$$
Ch(e_1^a e_2^b z^k, t) = 1 \text{ if } (a, b) \neq (0, 0)
$$

**Proof.** All this follows if we can construct an easy basis of $\mathcal{O}(S)_k$ for each $k \geq 2$. In degree 2, a basis is given by $x_i^2, x_i x_{i+1}, 0 \leq i \leq p-1$. Using this, we can make in degree $k$ the basis

$$
x_i^l x_{i+1}^{k-l}, 1 \leq l \leq k-1, 0 \leq i \leq p-1
$$

$$
x_i^k \text{ if } 0 \leq i \leq p-1
$$

From this it follows that the Hilbert series is given by $1 + \sum_{j=1}^{\infty} jpt^j = \frac{1 + (p-2)t + t^2}{(1-t)^2}$ and for the center of $H_p$, it follows from lemma 6.3. For the other elements of $H_p$, we see that the action of $e_1$ is given by a permutation on these elements without fixed elements and for $e_2$, the action on $x_i^k$ is given by multiplication by $\omega^k$ and on $x_i^l x_{i+1}^{k-l}$ by $\omega^l \omega^{(l+1)(k-l)} = \omega^k$. If $k \neq 0 \mod p$, then the character $\chi_{\mathcal{O}(S)_k}(e_2)$ is 0 as we then have $\mathcal{O}(S)_k \cong V^\otimes k$ with $V$ a simple $p$-dimensional representation of $H_p$. If $k \equiv 0 \mod p$ and $k \neq 0$, we get the summation of the $p$th roots of unity $\frac{k}{p}$ times, which is 0. As the characters of $e_1, e_2$ and $z$ determine the character series for $H_p$, we are done. \hfill \Box

From theorem 6.4 it follows that

**Corollary 6.5.** The character series of the coordinate ring of an elliptic curve embedded in $\mathbb{P}^{p-1}$ with $p \geq 5$ prime such that $H_p$ works on it by translation with $p$-torsion points has as character series

$$
Ch(1, t) = \frac{1 + (p-2)t + t^2}{(1-t)^2},
$$

$$
Ch(z^k, t) = \frac{1 + (p-2)\omega^k t + (\omega^k t)^2}{(1-\omega^k t)^2},
$$

$$
Ch(e_1^a e_2^b z^k, t) = 1 \text{ if } (a, b) \neq (0, 0).
$$
7. Sklyanin algebras

In [8] Odesskii and Feigin constructed Sklyanin elliptic algebras using $\theta$-functions on lattices for every dimension $n$. By construction, $H_n$ works as gradation preserving automorphisms on these algebras. In the notation of [8], these algebras $Q_{n,1}(E, \tau)$ degenerate to the polynomial ring when $\tau \to 0$. In particular, as these algebras are parametrized by (a blowdown of) an open subset of Shioda’s modular surface $S(n)$, which is an irreducible variety, we find

**Theorem 7.1.** The character series of the Sklyanin algebras is the same as the character series of the polynomial ring in $n$ variables.

In [9], the author calculated for regular Clifford algebras which are $H_n$-deformations of the polynomial ring $\mathbb{C}[V]$ the character series using the fact that such Clifford algebras are free modules over a polynomial ring. It was remarked that these algebras have the same character series as $\mathbb{C}[V]$. However, we can now give a new proof using theorem 6.2 which not only works for $p$ prime, but for all $n$ odd.

**Theorem 7.2.** The $n$-dimensional Sklyanin algebras with $n \geq 3$ odd associated to points of order 2 are graded Clifford algebras.

**Proof.** According to Odesskii and Feigin (cfr. [8]), the relations for the $n$-dimensional Sklyanin algebra associated to the elliptic curve $E$ and the point $\tau \in E$ are given by

$$\sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i}(0)}{\theta_{i-r}(-\tau)\theta_{r}(\tau)} x_{j-r} x_{i+r} = 0$$

with $i,j \in \mathbb{Z}_n, i \neq j$. Now, for $n$ odd, we know that these relations form an $H_n$-stable subspace of $V \otimes V \cong V_2^\otimes n$ with $V_2$ simple (according to [13]), so it is sufficient to consider a basechange for $i = -j$ to see if we get relations of the form $x_i x_{-i} + x_{-i} x_i = a_i x_0^2$. As $\tau = -\tau$, we find that the relations are given by $H_n$-orbits of

$$\sum_{r \in \mathbb{Z}_n} \frac{\theta_{2j}(0)}{\theta_{2j-r}(-\tau)\theta_{r}(\tau)} x_{j-r} x_{-j+r} = 0$$

with $1 \leq j \leq \frac{n-1}{2}$. Fix $j$ and $r$, then in the same equation, we get for the coefficient for $x_{-j+r} x_{j-r} = x_{j-(2j-r)} x_{-j+(2j-r)}$

$$\frac{\theta_{2j}(0)}{\theta_{2j-(2j-r)}(-\tau)\theta_{2j-r}(\tau)} = \frac{\theta_{2j}(0)}{\theta_{r}(\tau)\theta_{2j-r}(\tau)}$$

which is exactly the same as the coefficient for $x_{j-r} x_{-j+r}$. This means that every relation belongs to the vector space generated by the $\{x_j, x_{-j}\}, 1 \leq j \leq \frac{n-1}{2}$, and $x_0^2$. This vector space is $\frac{n-1}{2}$-dimensional in which we need to find a $\frac{n-1}{2}$-dimensional subspace, call this space $R$. Let $A$ be the $\frac{n-1}{2} \times \frac{n-1}{2}$-matrix with on place $(i,j)$ the coefficient from $\{x_j, x_{-j}\}$ coming from the $i$th equation and let $b$ be the column vector with on the $i$th place the coefficient of $x_0^2$ of the $i$th equation. Let $M$ be the matrix $[A, b]$, then the rank of this matrix is $\frac{n-1}{2}$. There are now 2 options:

(a) The rank of $A$ is $\frac{n-1}{2}$, then we are done as this means that up to basechange in $R$ we have relations of the form $\{x_1, x_{-1}\} + a_i x_0^2 = 0$. 


The rank of \( A \) is \( \frac{n-3}{2} \): this means that there is a column which is a linear combination of the other ones. As the rank of \( M \) is \( \frac{n-1}{2} \), this necessarily means that \( \{x_i, x_{i-1}\} = a_i \{x_k, x_{k-1}\} \) and \( x_i^2 = a_0 \{x_k, x_{k-1}\} \) for some \( 1 \leq k \leq \frac{n-1}{2} \). As \( x_i^2 \neq 0 \), \( a_0 \neq 0 \). But this means that \( A \) should have rank \( \frac{n-1}{2} \), a contradiction. We have proved that there exists a basis of the relations of the form \( \{x_i, x_{i-1}\} + a_i x_i^2 = 0 \), which are indeed graded Clifford algebras as we know that these Sklyanin algebras are regular and have Hilbert series \( \frac{1}{(1-t)^2} \). □

This theorem is false if \( n \) is even: in this case, there can be a coefficient before \( x_i^2 \) which is not 0. Therefore, the proof does not work. Also, there are too many elements in the center for \( Q_{n,1}(E, \tau) \) to be a Clifford algebra when \( n \) is even.

**Corollary 7.3.** The graded regular Clifford algebras with relations defined by \( H_n \)-orbits of \( \{x_i, x_{i-1}\} = a_i x_i^2, 1 \leq i \leq \frac{n-1}{2} \) have the same character series as the polynomial ring in \( n \) variables.

**Proof.** The moduli space of graded regular \( H_n \)-Clifford algebras is an open subset \( U \) of \( \mathbb{P}^{\frac{n-1}{2}} \) which intersects the moduli space of Sklyanin algebras in an infinite number of points. As we already know that the character series of all the Sklyanin algebras are the same and \( U \) is irreducible, we know from corollary 6.3 that the character series are the same. □

Let \( p \) be prime. In [6] it was shown that the \( H_p \)-Clifford algebras form a \( \mathbb{P}^{\frac{p-1}{2}} \)-dimensional family of \( \text{Grass}(\mathbb{P}^{\frac{p-1}{2}}, p) \). On the other hand, the \( p \)-dimensional Sklyanin algebras are 2-dimensional family. Then theorem 7.2 implies that in \( \text{Grass}(\mathbb{P}^{\frac{p-1}{2}}, p) \) the moduli space of Sklyanin algebras and the moduli space of \( H_p \)-Clifford algebras intersect in a curve.

### 8. Sklyanin algebras of global dimension 5 and points of order 2

When the global dimension is 5, we can find the equations for the Sklyanin algebras \( Q_{5,1}(E, \tau) \) with \( \tau \) of order 2. We know that the relations can be written as

\[
\{x_{1+k}, x_{4+k}\} = ax_k^2, \quad 0 \leq k \leq 4 \\
\{x_{2+k}, x_{3+k}\} = bx_k^2, \quad 0 \leq k \leq 4
\]

with associated quadratic form

\[
Q = \begin{bmatrix}
2x_0^2 & bx_0^2 & ax_0^2 & ax_0^2 & bx_0^2 \\
ax_1^2 & bx_1^2 & ax_1^2 & ax_1^2 & bx_1^2 \\
bx_2^2 & bx_2^2 & bx_2^2 & bx_2^2 & bx_2^2 \\
ax_3^2 & bx_3^2 & ax_3^2 & ax_3^2 & bx_3^2 \\
bx_4^2 & bx_4^2 & bx_4^2 & bx_4^2 & bx_4^2
\end{bmatrix}
\]

over the polynomial ring \( \mathbb{C}[x_0, x_1, x_2, x_3, x_4^2] \).

**Theorem 8.1.** If the Clifford algebra \( C(1 : a : b) \) determines a Sklyanin algebra, then it lies on the affine curve

\[
C' = V(-a^3b^3 + a^5 + b^5 + 2a^2b^2 - 8ab) \subset \mathbb{A}^2_{(a,b)}.
\]
The elliptic curve \( E' = E/\langle \tau \rangle \) is given by the curve \( C_t \) with

\[
t = \frac{a^3b - b^3 - 2a^2}{a^4 - ab^2 - 4b}
\]

**Proof.** For a graded Clifford algebra \( C \) with associated quadratic form \( Q \) over a polynomial ring \( R \), the point modules are determined by the graded prime ideals where \( Q \) has rank \( \leq 2 \) after specialisation (that is, the rank of the Clifford algebra \( C/P \) over \( R/P \) is \( \leq 2 \) where \( P \) is a non-trivial maximal graded prime ideal of \( R \)). For a Sklyanin algebra, the variety parametrizing point modules is equal to the elliptic curve \( E \), as proved by Smith in [19]. The corresponding variety in \( \text{Proj}(R) \) is \( E' = E/\langle \tau \rangle \).

In any case, this means that one has to find \( t \in \mathbb{C} \) such that the \( 3 \times 3 \)-minors are all 0 on a curve of the form \( C_t \). In particular, all the \( 3 \times 3 \)-minors in the point \((0 : 1 : t : -t : -1)\) should be 0. Using Macaulay2, one sees that 2 of all the \( 3 \times 3 \)-minors are given by \(-2b^2t^3 + 2ab^2t - 2a^2 \) and \(-a^2bt^2 - ab^2t + 2a^2 \). Then one can eliminate the variable \( t \) to get the following equation

\[
a(-a^3b^3 + a^5 + b^5 + 2a^2b^2 - 8ab) = 0.
\]

The projective closure of the line \( V(a) \) is not closed under the action of \( \text{PSL}_2(5) \) on \( \mathbb{P}^2 \) which was constructed in [6], so we necessarily see that the curve \( C' \) parametrizing 5-dimensional Sklyanin algebras associated to points of order 2 is given by the equation

\[
-a^3b^3 + a^5 + b^5 + 2a^2b^2 - 8ab = 0.
\]

From this one deduces that

\[
t = -\frac{(a^3 - 2b^2)a}{a^4b^2 + b^4 + 2a^2b^2}.
\]

However, in the fraction field of \( C' \), \( t \) is equal to \( \frac{a^3b^3 - 4a^2}{a^4 - ab^2 - 4b} \). As \( t \) is not constant, neither are \( a \) and \( b \).

□

Some remarks about \( C' \):

- Let \( C \) be the projective closure of \( C' \) in \( \mathbb{P}^2_{[A:B:C]} \). In [6] the author constructed a \( \text{PSL}_2(5) \)-action on \( \mathbb{P}^2_{[A:B:C]} \) such that points lying in the same orbit gave isomorphic algebras. The curve \( C \) is stable under this action.
- \( C \) has 6 singularities: the \( \text{PSL}_2(5) \)-orbit of the point \([1 : 0 : 0]\), whose points give the algebras isomorphic to the quantum space with relations \( x_i x_j + x_j x_i = 0, i \neq j \). One of course expects these 6 points to be singular if one looks at the degeneration of the point modules: for each 5-dimensional Sklyanin algebra, the point modules are given by the elliptic curve \( E \). Such a family of elliptic curves degenerates to a cycle of 5 lines, so if \( C \) was smooth, one expects the quantum space to have as point modules 1 cycle of 5 lines. But according to the answer of [15], a quantum \( \mathbb{P}^n \) has at least in it’s point scheme the full graph on \( n + 1 \) points.
- There are 12 points on \( C \), which are smooth points, but the corresponding algebras are all isomorphic to the algebra with relations \( x_i^2 = 0, x_i x_{i+1} + x_{i+1} x_i = 0, 0 \leq i \leq 4 \) (that is, they lie in the same \( \text{PSL}_2(5) \)-orbit as the point \([0 : 0 : 1]\)). These 12 points form the intersection of \( C \) with the curve \( V(AB + C^2) \), which parametrizes the Koszul dual of the graded coordinate rings of all elliptic curves with level-5 structure. However, these 12 points
do not give the Koszul dual of graded coordinate rings of elliptic curves, but of a cycle of 5 lines.

8.1. **Representations.** Before we start the description of representations of 5-dimensional Sklyanin algebras associated to points of order 2, we prove a lemma regarding the possible 1-dimensional representations of $H_p$-Clifford algebras.

**Lemma 8.2.** The only $H_p$-Clifford algebras with non-trivial 1-dimensional representations are the algebras isomorphic to the quantum algebra $\mathbb{C}[-1][x_0, \ldots, x_{p-1}]$.

**Proof.** The algebras isomorphic to the quantum algebra are given by the $\text{PSL}_2(p)$-orbit of the point $(1 : 0 : \ldots : 0)$, which we know consists of $p+1$ elements (see [6]). Apart from this point, the other points are given by the action of the element $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ on the point $(1 : 2 : \ldots : 2)$. These other points are elements of the open subset $D(a_1 \cdots a_{\frac{p+1}{2}}) \subset \bigcup_{i=1}^{\frac{p-1}{2}} D(a_i)$ and there are $p$ of them. We will prove that the number of points in $\bigcup_{i=1}^{\frac{p-1}{2}} D(a_i)$ for which there exists a 1-dimensional non-trivial representation is equal to $p$. Suppose that $a_i \neq 0$ and that there exists a non-trivial 1-dimensional representation. Using the Heisenberg action, we may assume that $x_0$ is not sent to 0. Using the $\mathbb{C}^*$-action, we may also assume that the image of $x_0$ is 1. Let $y_i \in \mathbb{C}$ be the image of $x_i$ in $\mathbb{C}$ under this representation. We then have

$$2y_i y_{-i} = a_i, 2y_{(k+1)i} y_{(k-1)i} = a_i y_{ki}^2$$

As $a_i \neq 0$, we necessarily have $y_i \neq 0$.

**Lemma 8.3.** We have the formula $y_{ki} = \frac{a_i^{\left(\frac{r}{2}\right)} y_i^p}{2^{\left(\frac{r}{2}\right)}}$.

**Proof.** It is trivially true for $k = 0, 1$. Then it is an easy induction proof. \qed

In particular, for $k = p$ we get $y_0 = \frac{a_i^{\left(\frac{r}{2}\right)} y_i^p}{2^{\left(\frac{r}{2}\right)}} = 1$. For $k = p+1$ we find

$$y_i = \frac{a_i^{\left(\frac{r+1}{2}\right)} y_i^{p+1}}{2^{\left(\frac{r+1}{2}\right)}}.$$  

This implies

$$y_i^p = \frac{2^{\left(\frac{r}{2}\right)}}{a_i^{\left(\frac{r}{2}\right)}} = \frac{2^{\left(\frac{r+1}{2}\right)}}{a_i^{\left(\frac{r+1}{2}\right)}}$$

So we have $a_i^p = 2^p$. It follows that $y_i$ is a $p$th root of unity. From the lemma follows that $y_{ki}$ is uniquely determined for all $k$, but as $i \neq 0$, this means that all $y_i$ are uniquely determined by $y_i$. But the $a_j$ are determined by $2y_j y_{-j} = a_j$. So the number of points in $\bigcup_{i=1}^{\frac{p-1}{2}} D(a_i)$ with non-trivial 1-dimensional representations is less or equal to $p$. As we know that there are certainly $p$ points in this set, we are done. \qed

As $A = Q_{5,1}(E, \tau)$ is a graded Clifford algebra with quadratic form $Q$ over $\mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2, x_4^2]$, it follows that the center $Z(A)$ of $A$ is given by the 5 elements $x_i^2, 0 \leq i \leq 4$ of degree 2 and the square root of the determinant of $Q$, call this $c_5$ of degree 5. These 6 elements satisfy one relation of degree 10

$$\phi(x_0^2, x_1^2, x_2^2, x_3^2, x_4^2) = \det Q = c_5^2.$$
with \( \phi \) an \( H_2 \)-invariant polynomial of degree 5 (this follows from [6]). As \( A \) is a free module of rank \( 2^5 \) over the polynomial ring \( \mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2, x_4^2] \), it follows that the PI-degree of \( A \) is \( 2^4 = 4 \), that is, generically all the simple representations are 4-dimensional. The dimension of a simple representation lying above a point in \( \text{Spec}(Z(A)) \) is determined by the rank of the quadratic form \( Q \) after specialization. Let \( V = \text{Spec}(Z(A)), A^5 = \text{Spec}(\mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2, x_4^2]) \) and denote \( \psi \) for the double cover \( V \xrightarrow{\psi} A^5 \) coming from the inclusion \( \mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2, x_4^2] \hookrightarrow Z(A) \). This cover is ramified over \( V(\det Q) \). We will use the degree of the central elements coming from its inclusion in \( A \), so the \( x_i^4 \) have degree 2.

**Theorem 8.4.** Above any point of \( V \) lies a 4-dimensional simple representation of \( A \) with exception of the cone above the elliptic curve \( E' \). For points on the cone above the elliptic curve \( E' \) there exists a unique 2-dimensional simple representation, except for the trivial representation lying above the maximal ideal \((x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, c_5)\).

**Proof.** The fact that on the cone above \( E' \) there are only simple representations of dimension \( \leq 2 \) follows from the fact these representations are representations of the twisted coordinate ring \( \mathcal{O}_z(E) \), which is of PI-degree 2. Also, as none of the Sklyanin algebras are isomorphic to the quantum space, all these representations are necessarily of dimension 2. Let \( J \subset \mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2] \) be the ideal associated to the cone above \( E' \), which is generated by 5 elements of degree 4. Let \( J_k \) be the ideal of \( \mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2, x_4^2] \) generated by all the \( k \times k \) minors of the quadratic form \( Q \). Using Macaulay2, one checks that \( (J_0)_6 = I_6 \), that is, the degree 6 elements in \( J_3 \) and \( I \) are the same. For \( J_4 \), one checks that all the \( 4 \times 4 \)-minors of \( Q \) are the generators for \( I^2 \).

For the open set \( D(J_1) \) the rank of \( Q \) is 4 or 5, in both cases the corresponding simple representation is 4-dimensional. \( \square \)

One even has the following description of \( \phi \).

**Proposition 8.5.** Let \( \text{Sec}^2 E \subset \mathbb{P}^4 \) be the secant variety of an elliptic curve \( E \) embedded in \( \mathbb{P}^4 \), that is, the Zariski-closure of all lines in \( \mathbb{P}^4 \) that intersect \( E \) in 2 points. We have

\[
V(\phi) = \text{Sec}^2 E' \subset \mathbb{P}^4 \setminus [x_0^2, \ldots, x_7^2]
\]

**Proof.** According to proposition VIII.2.5, if \( E \) is an elliptic curve given as the intersection of 5 quadrics \( Q_i = z_i^2 + t z_{i+1} z_{i+4} - \frac{1}{4} z_{2+3} z_{3+3}, 0 \leq i \leq 4 \) in \( \mathbb{P}^4 \), then the defining equation for \( \text{Sec}^2 E \) is given by

\[
\det \left( \frac{\partial Q_i}{\partial z_j} \right) = 0.
\]

One checks by a computer computation that \( \det \left( \frac{\partial Q_i}{\partial z_j} \right) = 0 \) in the quotient ring \( R/(\det Q) \). As both \( \det \left( \frac{\partial Q_i}{\partial z_j} \right) \) and \( \det Q \) are of the same degree, this implies that they are equal to each other up to a scalar. \( \square \)

The (cone over) the secant variety is singular along (the cone over) \( E' \), which is equal to the ramification locus of \( A \). This follows also from proposition 5 of [17], as the codimension of the ramification locus is 3 and the global dimension of these algebras is finite.
8.2. From repr\(_n\)(A) to Proj(A). We have found that there are 3 different types of non-trivial simple representations:

- Representations of dimension 4 where \(c_5\) does not act trivial.
- Representations of dimension 4 where \(c_5\) does act trivial.
- Representations of dimension 2.

These representations are determined by the rank of the quadratic form \(Q\). Let \(Y = \mathbb{P}^4[x_0^2,...,x_4^2]\). We set

\[Y_k = \{p \in Y | \text{rank}(Q(P)) = k\}\]

where \(Q(P)\) is the matrix in \(M_5(\mathbb{C}[t])\) with \(t\) of degree 2 one gets after taking the quotient of \(Q\) by the graded ideal determined by \(P\). We have found that

- \(Y_5 = Y \setminus V(\phi)\),
- \(Y_4 = V(\phi) \setminus E'\),
- \(Y_2 = E'\).

Recall that Proj\((A)\) is the quotient category of all graded left \(A\)-modules modulo the subcategory of torsion modules. In noncommutative algebraic geometry one studies linear modules in Proj\((A)\), that is, left graded \(A\)-modules with Hilbert series \(1(1-t)^n\) for some \(n\). This \(n\) is called the dimension of the module. If \(n = 1\), one speaks of point modules, \(n = 2\) are line modules, and so on.

However, in some cases there are other modules to consider: fat point modules. This are modules with in their class in Proj\((A)\) a representative module with Hilbert series \(e(1-t)^n\) with \(e > 1\). This \(e\) is called the multiplicity of the corresponding module.

These fat point modules and point modules are determined by \(\mathbb{C}^* \times \text{PGL}_n(\mathbb{C})\)-orbits in repr\(_1\)(A) and repr\(_2\)(A), cfr. [16]. Here for a graded algebra \(A\) which is a finite module over its center, we define

\[\text{repr}^\text{ff}_n(A) = \{ A \phi \mathfrak{m}_n(\mathbb{C}) | \exists z \in Z(A)k, k > 0 : z(\phi) \neq 0\},\]

see [16] and [2] for more information.

**Theorem 8.6.** The fat points and point modules of Proj\((A)\) are determined by:

- For each point on Proj\((Z(A)) \setminus V(\phi)\), we have one corresponding fat point module of multiplicity 4 in Proj\((A)\).
- For each point on \(V(\phi) \setminus E'\), there are 2 corresponding fat point modules of multiplicity 2.
- For each point on \(E'\), there are 2 point modules in Proj\((A)\). The point modules of \(A\) are given by \(E\) and the map between \(E\) and \(E'\) is the natural isogeny \(E \longrightarrow E'\).

**Proof.** All this follows from proposition 8 and proposition 9 of [17].

In [18] Odesskii and Feigin proved that for an elliptic curve \(E\) and \(\tau\) a torsion point of order \(m\), the center of the Sklyanin algebra \(Q_{n,1}(E,\tau)\) for \(n\) odd is generated by a central element \(c_n\) of degree \(n\) and \(n\) algebraically independent elements of degree \(m\), with one relation of the form

\[\phi(u_0, u_1, \ldots, u_{n-1}) = c_n^m.\]

Moreover, \(Q_{n,1}(E,\tau)\) is a finite module over its center. For \(n = 5\) and assuming \((3,m) = (5,m) = 1\), these results with respect to points of order 2 might suggest the following regarding the Proj of these algebras:
The last part is an easy consequence from the fact that these point modules are point modules of the twisted coordinate ring \( O_{[3]}(E) \). The other conjectures will be the subject of future work.

Another interesting question is the following: from [6] it follows that we get a morphism from a Zariski-open subset of \( \mathbb{P}^2 \) to the projective space over the \( H_5 \)-invariant polynomials in \( \mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2] \), by taking the determinant of the associated quadratic form \( Q \). \( \mathbb{C}[x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2 H_n] \) is 6-dimensional, so this map can not be surjective. A natural question is then to ask what the image of this map is, what are the singularities in the image and is this map defined over the entire \( \mathbb{P}^2 \)?

Hopefully this will have something to do with the Horrocks-Mumford bundle (cfr. [11]), but only time will tell.

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