Aspects of $(4 + 4)$-Kaluza–Klein type theory

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Abstract

We develop a type of Kaluza–Klein formalism in $(4 + 4)$-dimensions. In the framework of this formalism, we obtain a new kind of Schwarzschild metric solution that, via Kruskal–Szekeres, can be interpreted as mirror black and white holes. We found that this new type of mirror black and white hole solution in $(3 + 1)$-dimensions supports the idea that original spacetime can be extended to a $(4 + 4)$-signature. Using octonions, we also discuss linearized gravity in $(4 + 4)$-dimensions.

Keywords: Kaluza–Klein, black holes, Kruskal–Szekeres

1. Introduction

Kaluza–Klein type theories have already been considered for non-compact spacetime [1–3] with interesting results in inflationary and cosmological models [4–14]. In this article, we are interested in deriving a Kaluza–Klein type theory for a spacetime of four-time and four space dimensions ($(4 + 4)$-dimensions). Our formalism is based on the analogue of the approach of [15–19].

There are at least three physical motivations for considering $(4 + 4)$-dimensions emerging from different sources. First, considering the splitting of the form $(4 + 4) = (3 + 1) + (1 + 3)$, one may ask if an electron lives in the $(1 + 3)$-world and one wonders what could be the corresponding mirror electron in $(3 + 1)$-dimensions. In [20], it was shown that massless Majorana–Weyl spinors satisfying the Dirac equation in $(4 + 4)$-dimensions can be identified with a massive complex spinor in $(1 + 3)$-dimensions. This means that a Majorana–Weyl massless fermion in $(4 + 4)$-dimensions can be seen as an electron in $(1 + 3)$-dimensions. Second, it is interesting that in $(4 + 4)$-dimensions, one may consider the chain of maximal embeddings and branches,

\[ \text{so}(4, 4) \supset \text{s}(2, R) \oplus \text{so}(2, 3) \supset \text{so}(1, 1) \]
\[ \oplus \text{sl}(2, R) \oplus \text{sl}(2, 2). \]

One may ask the question, why, at the macroscopic level, is our world $(1 + 3)$-dimensional? Surprisingly, until now, there has not been a satisfactory answer. Any answer needs to explain the asymmetry between time and space in the $(1 + 3)$-world. In this sense, it seems to us interesting to start with the more symmetric $(4 + 4)$-world model and to explore its gravitational consequences.

On the other hand, it is well known that Kruskal–Szekeres coordinates give an alternative description of the event horizon of black holes [21, 22]. Traditionally, one starts with the Schwarzschild metric described by the coordinates $t, r$ and the angular coordinates $\theta$ and $\phi$ associated with a unit sphere and, after several algebraic steps, one computes the Kruskal–Szekeres coordinates $T$ and $X$ which become functions of $t$ and $r$. The key result of this process is that, while the Schwarzschild metric is singular at the horizon $r = 2GM = r_s$, the Kruskal–Szekeres metric is not. However, the final transformations between the coordinates $t, r$ and $T$ and $X$ seem to be, in a sense, intriguing because, instead of describing only two regions (2-region), as in the case of Schwarzschild black holes (interior and exterior), one ends up with four regions (4-region) in the Kruskal–Szekeres black hole (see [23] and [24] for details).

In an effort to better understand the above intriguing 2-region and 4-region result, we discover that one may start with a more general Kruskal–Szekeres transformation between $t, r$ and $T, X$ which turns out to describe not only the previous 4-region but, surprisingly, a total of 8-regions. This, of course, may be interpreted as an extra complication. However, in this work, we show that such an 8-region transformation suggests that the $(1 + 3)$-signature of the
original spacetime can be extended to a (4 + 4)-signature. It turns out that the (4 + 4)-signature can be split as (4 + 4) = (1 + 3) + (3 + 1), given the original (1 + 3)-signature and some kind of mirror (3 + 1)-signature. But it has been a subject of much interest to consider invariant theories of reversal signature transformation, such as (1 + 3) \cong (3 + 1) [25]. Moreover, one may ask, assuming that one has a black hole in (1 + 3)-dimensions, what could be the corresponding mirror black hole in (3 + 1)-dimensions? Thus, we argue that our extended Kruskal–Szekeres coordinates suggest that instead of thinking of the (4 + 4)-world as an exotic spacetime, it may considered as a more interesting scenario in which one has not only ordinary black holes in (1 + 3)-dimensions, but a new type of mirror black hole in (3 + 1)-dimensions.

Finally, it is worth mentioning that using a reversal signature, a relation between (3 + 1) and (1 + 3) signatures has already been investigated in the context of string theory [25, 26]. Secondly, in (4 + 4)-dimensions, Majorana–Weyl spinors exist [27–32]. Another source of physical interest emerges from the fact that (4 + 4)-dimensional theory may be obtained from the dimensional reduction of a (5 + 5)-dimensional theory which originates from so-called M’-theory [33, 34] (see also [35–39]) which is defined in (5 + 6)-dimensions. In fact, upon space-like compactification, (5 + 6)-dimensional theory leads to one type II A’ and two type II B’ string theories which ‘live’ in (5 + 5)-dimensions [38]. Further, we believe that our work may be useful in several physical scenarios. In particular, since the (4 + 4)-world contains the same number of time and space coordinates, one has a more symmetric scenario for considering quantum and topological frameworks.

Our work is organized as follows. In section 2, we develop a new kind of Kaluza–Klein theory in (4 + 4)-dimensions. In section 3, we propose extended Kruskal–Szekeres coordinates. In section 4, we interpret the new solutions as mirror black and white holes, thus making clear the necessity of a (4 + 4)-spacetime structure with a unified description of both kind of solutions within the same single gravitational framework. In section 5, as another non-trivial application of our formalism, we describe linearized gravity in (4 + 4)-dimensions. In the appendix, we describe a method to obtain 8-region Kruskal–Szekeres coordinates. Finally, in section 6, we provide some final remarks.

2. Kaluza–Klein type formalism

As is usual in the light of Kaluza–Klein theories, our starting point is the metric ansatz

$$\gamma_{AB}(x^{C}) = \left(\begin{array}{c}
g_{\mu\nu} + A_{\mu}^{\alpha} A_{\nu}^{\beta} g_{\beta\gamma} + g_{\mu\nu} B_{\gamma}^{\nu} + g_{\mu\nu} B_{\gamma}^{\nu}
A_{\mu}^{\alpha} B_{\gamma}^{\nu} + g_{\mu\nu} B_{\gamma}^{\nu} + B_{\mu}^{\alpha} B_{\gamma}^{\nu} g_{\beta\gamma}
A_{\mu}^{\alpha} B_{\gamma}^{\nu} + g_{\mu\nu} B_{\gamma}^{\nu} + B_{\mu}^{\alpha} B_{\gamma}^{\nu} g_{\beta\gamma}
\end{array}\right),$$

(1)

where $x^{A} = x^{A}(x^{\lambda}, x^{\nu})$, $g_{\mu\nu} = g_{\mu\nu}(x^{\lambda})$ and $g_{\mu\nu}(x^{\nu})$ are the metrics associated with the (1 + 3)-dimensional world (l + 1 -world) and the (3 + 1)-dimensional world ((3 + 1)-world), respectively. The fields $A_{\mu}^{\alpha}$ and $B_{\mu}^{\alpha}$ are types of gauge fields associated with the (1 + 3)-world and (3 + 1)-world respectively.

Now, if we assume that the gauge fields $A_{\mu}^{\alpha}$ and $B_{\mu}^{\alpha}$ obey

$$A_{\mu}^{\alpha} B_{\mu}^{\alpha} = 0,$$

(2)

and

$$A_{\mu}^{\alpha} B_{\mu}^{\alpha} = 0,$$

(3)

then we find that the inverse $\gamma^{AB}$ of the original metric $\gamma_{AB}$ is given by

$$\gamma^{AB}(x^{\lambda}, x^{\nu}) = \left\{ \begin{array}{l}
\left( g^{\mu\nu} + A_{\mu}^{\alpha} B_{\nu}^{\alpha} - B_{\nu}^{\alpha} A_{\mu}^{\alpha} \right) \left( g^{\mu\nu} + B_{\mu}^{\alpha} B_{\nu}^{\alpha} \right)
\end{array} \right.$$

(4)

Note that our assumptions (2) and (3) can be interpreted in some sense as requiring that the gauge fields $A_{\mu}^{\alpha}$ and $B_{\mu}^{\alpha}$ are orthogonal.

With the help of (1), it can be easily seen that the differential line element in (4 + 4)-space can be written as

$$\mathrm{d}x^{2} = (\mathrm{d}x^{\mu} + A_{\mu}^{\alpha} \mathrm{d}x^{\alpha})(\mathrm{d}x^{\nu} + B_{\nu}^{\alpha} \mathrm{d}x^{\alpha}) g_{\mu\nu} + (\mathrm{d}x^{\nu} + A_{\nu}^{\alpha} \mathrm{d}x^{\alpha})(\mathrm{d}x^{\mu} + B_{\mu}^{\alpha} \mathrm{d}x^{\alpha}) g_{\mu\nu}.$$  

(5)

Thus, it follows that the basis 1-forms $\omega^{\mu}$ and $\omega^{\nu}$ read

$$\omega^{\mu} = \mathrm{d}x^{\mu} + A_{\mu}^{\alpha} \mathrm{d}x^{\alpha},$$  

(6)

and

$$\omega^{\nu} = \mathrm{d}x^{\nu} + A_{\nu}^{\alpha} \mathrm{d}x^{\alpha}.$$  

(7)

Therefore, employing (6) and (7), equation (5) becomes

$$\mathrm{d}x^{2} = \omega^{\mu} \omega^{\nu} g_{\mu\nu} + \omega^{\nu} \omega^{\mu} g_{\mu\nu}.$$  

(8)

Now, using (2) and (3), it is obtained from (6) and (7) that the dual basis can be written as

$$D_{\mu} = \partial_{\mu} - A_{j}^{\mu} \partial_{k}$$  

(9)

and

$$D_{\nu} = \partial_{\nu} - B_{j}^{\nu} \partial_{k}.$$  

(10)

Of course, it is not difficult to see that both derivatives $D_{\mu}$ and $D_{\nu}$ can be interpreted as covariant derivatives with connections $A_{\mu}^{\alpha}$ and $B_{\mu}^{\alpha}$, respectively.

Let us now analyze the ananzs (1) from another point of view. If we write the metric $\gamma_{AB}$ in terms of a vielbein field $E_{A}^{(C)}$ and the flat metric $\eta_{CD}^{(CD)} = \text{diag}(-1, 1, 1, 1, -1, -1, -1, 1)$ in the form

$$\gamma_{AB} = E_{A}^{(C)} E_{B}^{(D)} \eta_{CD},$$  

(11)

then we can split (11) as

$$\gamma_{\mu\nu} = e_{A}^{(\alpha)} e_{B}^{(\beta)} \eta_{AB},$$  

(12)

$$\gamma_{\mu\nu} = e_{A}^{(\alpha)} e_{B}^{(\beta)} \eta_{AB},$$  

(13)

$$\gamma_{\mu\nu} = e_{A}^{(\alpha)} e_{B}^{(\beta)} \eta_{AB},$$  

(14)

$$\gamma_{\mu\nu} = e_{A}^{(\alpha)} e_{B}^{(\beta)} \eta_{AB},$$  

(15)
Thus, making the identifications $e^{(c)}_\mu \equiv E^{(c)}_\mu$, $e^{(c)}_i \equiv A^{(c)}_i$, $g^{\alpha\beta} = \eta^{(c)}_{\alpha\beta}$, $g_{ij} = \epsilon^{(c)}_{ij} \eta^{(c)}_{\alpha\beta}$, $A^{(c)}_\mu = e^{(c)}_\mu A^{(c)}_\mu$ and $B^{(c)}_{ij} = e^{(c)}_{ii} D^{(c)}_{ij}$ from (12) we can obtain (1).

Now, let us assume that the commutators of $e^{(c)}_\mu$ and $e^{(c)}_i$ are given by

$$[e^{(c)}_\mu, e^{(c)}_\nu] = C^{(c)}_{\mu\nu} e^{(c)}_\sigma,$$

and

$$[e^{(c)}_i, e^{(c)}_j] = 0,$$

where $C^{(c)}_{\mu\nu}$ and $C^{(c)}_{i} \equiv C^{(c)}_{ij} \equiv C^{(c)}_{i\alpha\beta}$ are structure constants associated with the $(1 + 3)$-world and the $(3 + 1)$-world, respectively. Thus, it is not difficult to see that $D_\mu = \partial_\mu$ and (10) can now be written as

$$D_\mu = \partial_\mu - A^{(c)}_\mu A^{(c)}_\mu, \quad D_\alpha \equiv D^{(c)}_\alpha,$$

where

$$F^{(c)}_{\mu\nu} = \partial_\mu A^{(c)}_\nu - \partial_\nu A^{(c)}_\mu + C^{(c)}_{(\mu\nu)} A^{(c)}_\sigma,$$

and

$$G^{(c)}_{ij} = \partial_i B^{(c)}_j - \partial_j B^{(c)}_i + C^{(c)}_{ij} B^{(c)}_k,$$

are the corresponding field strengths.

In a non-coordinate basis, the metric $\gamma^{(c)}_{AB} \rightarrow {\tilde \gamma}^{(c)}_{AB}$ becomes

$$\tilde \gamma^{(c)}_{AB}(x^C) = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix},$$

while the connection $\Gamma^{(c)}_{ABC}$ results in

$$\Gamma^{(c)}_{ABC} = \frac{1}{2} (D_\alpha \tilde \gamma^{(c)}_{AB} + D_B \tilde \gamma^{(c)}_{AC} - D_A \tilde \gamma^{(c)}_{BC}),$$

$$+ \frac{1}{2} (\partial \tilde \gamma^{(c)}_{ABC} + \tilde \gamma^{(c)}_{ABC} - \tilde \gamma^{(c)}_{BCA}).$$

Here, $D_\alpha$ is the covariant derivative associated with $\tilde \gamma^{(c)}_{AB}$. In this manner, since $C^{(c)}_{\mu\nu} = 0$ and $\tilde \gamma^{(c)}_{AB} = g_{\mu\nu}$, we find that

$$\Gamma^{(c)}_{\mu\nu\alpha} = \frac{1}{2} (g^{\mu\nu\alpha} + g^{\mu\alpha\nu} - g^{\nu\alpha\mu}) = \{\mu\nu\alpha\}.\quad (25)$$

Similarly, because $C^{(c)}_{i\alpha\beta} = -F^{(c)}_{i\alpha\beta}$, $G^{(c)}_{ij} = -G^{(c)}_{ji}$ we arrive at the expressions

$$\Gamma^{(c)}_{i\mu\nu} = -F^{(c)}_{i\mu\nu}, \quad \Gamma^{(c)}_{ij} = -G^{(c)}_{ij},$$

Finally, since $C^{(c)}_{ikj} = 0$ and $\tilde \gamma^{(c)}_{ij} = g^{(c)}_{ij}$ we obtain that

$$\Gamma^{(c)}_{ijk} = \frac{1}{2} (g_{ijk} + g_{kji} - g_{ikj}) = \{ijk\}.\quad (28)$$

Now, employing the formulæ for the non-coordinate components of the Riemann tensor

$$R^{(c)}_{ABCD} = \partial_\mu \Gamma^{(c)}_{ABCD} + \partial_\nu \Gamma^{(c)}_{ABC} + \Gamma^{(c)}_{EBC} \Gamma^{(c)}_{ABE} - \Gamma^{(c)}_{EDC} \Gamma^{(c)}_{ABD},$$

we can calculate the scalar curvature $\mathcal{R} = C^{BD} R^{(c)}_{ABCD}$ with this idea in mind, straightforward computations lead to

$$\mathcal{R} = \mathcal{R} + \frac{1}{2} F^{(c)}_{\mu\nu} F^{(c)}_{\mu\nu} + \frac{1}{2} G^{(c)}_{ij} G^{(c)}_{ij},$$

where the action corresponding to the $(1 + 3)$-space is

$$S_1 = a \int d^{(1+3)} \sqrt{-g} (\mathcal{R} + \frac{1}{2} F^{(c)}_{\mu\nu} F^{(c)}_{\mu\nu}),$$

while for the $(3 + 1)$ we have

$$S_2 = b \int d^{(3+1)} \sqrt{-g} (\mathcal{R} + \frac{1}{2} G^{(c)}_{ij} G^{(c)}_{ij}),$$

Here, $a$ and $b$ are volume constants and $\tilde g$ and $g$ denote the determinant of $g_{\mu\nu}$ and $g_{ij}$, respectively. Besides, $\mathcal{R}$ is the scalar curvature associated with $g_{\mu\nu}$ while $\mathcal{R}$ is the scalar curvature associated with $g_{ij}$. Moreover, we defined the quantities \( \tilde g_{ij} = \int d^{(1+3)} g_{ij} \). An interesting feature is that it seems that $S_1$ admits an interpretation of gravitational Yang–Mills theory in a $(1 + 3)$-world, with $\tilde g_{ij}$ as a metric in a group manifold. Similarly, $S_2$ admits an interpretation of gravitational Yang–Mills theory in a $(3 + 1)$-world, with $g^{(c)}_{ij}$ as a metric in a group manifold.

Until this point, in order to motivate the subject, we have focused on a $(4 + 4)$-world. However, our calculations are also valid for any $(n + n)$-world. Now, to derive an application of the formalism, we will study solutions of field equations with static fields and spherical symmetry. This can be done in the simplest way: considering the gauge fields $B^{(c)}_{ij}$ and $A^{(c)}_\mu$ as null, hence, the line element (5) acquires the form

$$ds^2 = g_{\mu\nu} x^\mu x^\nu + g_{ij} (x^i x^j),$$

where Greek indices from now on run from $0$–$3$ and represent the coordinates of $(1 + 3)$ spacetime, whereas Latin indices run from $4$–$7$ and denote the coordinates of $(3 + 1)$ spacetime. Note that both spaces are of complementary signature. Hence, the Levi–Civita connection can be split up into

$$\Gamma^{(4+4)}_{\mu\nu} = \begin{pmatrix} \Gamma^{(c)}_{\alpha\beta} & 0 \\ 0 & \Gamma^{(c)}_{ij} \end{pmatrix}.$$

Consequently, according to action (31), the Einstein field equations in vacuum lead to $R^{(c)}_{ABCD} = 0$ and consequently one obtains

$$R^{(c)}_{\mu\nu} = 0, \quad R^{(c)}_{ij} = 0.$$
In what follows, we shall show that this split is not necessarily a trivial case.

3. The extended Kruskal–Szekeres coordinates

Now, in order to study spherically symmetric solutions of (36), we introduce the $(4+4)$ line element with a spherical symmetry as

$$\text{d}s^2_{(4+4)} = -e^{2\phi'(t)}\text{d}t^2 + e^{2h(r)}\text{d}r^2 + r^2\text{d}\Omega^2 + e^{2\phi'(t)}\text{d}t^2 - e^{2h(r)}\text{d}r^2 - r^2\text{d}\Omega^2,$$

(37)

where $(t, r, \theta, \phi)$ are the $(1+3)$-coordinates, $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$ denote the $(3+1)$-coordinates, $\text{d}\Omega^2 = \text{d}\theta^2 + \sin^2\theta\text{d}\phi^2$ and $\text{d}\tilde{\Omega}^2 = \text{d}\tilde{\theta}^2 + \sin^2\tilde{\theta}\text{d}\tilde{\phi}^2$.

It follows from (36) and (37) that the Schwarzschild metric solution for a $(4+4)$-black hole reads

$$\text{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\text{d}t^2 + \frac{\text{d}r^2}{1 - \frac{r_s}{r}} + r^2\text{d}\Omega^2 + \left(1 - \frac{r_s}{r}\right)\text{d}t^2 - \frac{\text{d}r^2}{1 - \frac{r_s}{r}} - r^2\text{d}\Omega^2,$$

(38)

where $r_s = 2GM$ is the Schwarzschild radius, $M$ is associated with the mass of the black hole and $(r, \theta, \phi)$ are the spherical polar coordinates (here, we use units such that the light velocity $c = 1$). Notice that we have set $\tilde{r} = r_s$, but in general, of course, one may have other cases in which $\tilde{r} \neq r_s$. The solution corresponding to a $(1+3)$-signature then is given by

$$\text{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\text{d}t^2 + \frac{\text{d}r^2}{1 - \frac{r_s}{r}} + r^2\text{d}\Omega^2.$$

(39)

Now, we introduce the extended Kruskal–Szekeres coordinates in the form

$$X = \epsilon \left[\eta \left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \cos h \left(\frac{t}{2r_s}\right),$$

$$T = \epsilon \left[\eta \left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \sin h \left(\frac{t}{2r_s}\right),$$

(40)

where the new quantities $\epsilon$ and $\eta$ are parameters taking values in the set $\{\pm 1\}$. Using the coordinate transformations (40) one obtains the expression

$$X^2 - T^2 = \eta \left(\frac{r}{r_s} - 1\right) e^{\frac{\eta}{2}}$$

(41)

and using the line element (39) we arrive at

$$\frac{4r^3}{r} e^{-\frac{\eta}{2}} (-\text{d}T^2 + \text{d}X^2) = \eta \left[-\left(1 - \frac{r_s}{r}\right)\text{d}t^2 + \frac{\text{d}r^2}{1 - \frac{r_s}{r}}\right].$$

(42)

Similarly, if we now write the extended Kruskal–Szekeres coordinates in the alternative form

$$X = \epsilon \left[\eta \left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \sin h \left(\frac{t}{2r_s}\right),$$

$$T = \epsilon \left[\eta \left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \cos h \left(\frac{t}{2r_s}\right),$$

(43)

it is not difficult to show that the analogous of (41) reads

$$T^2 - X^2 = \eta \left(\frac{r}{r_s} - 1\right) e^{\frac{\eta}{2}}$$

(44)

and considering the line element (39) we get the expression

$$\frac{4r^3}{r} e^{-\frac{\eta}{2}} (-\text{d}T^2 + \text{d}X^2) = \eta \left[\left(1 - \frac{r_s}{r}\right)\text{d}t^2 - \frac{\text{d}r^2}{1 - \frac{r_s}{r}}\right].$$

(45)

Notice that equations (40) and (43) contain both cases: $\epsilon = \pm 1$ and $\eta = \pm 1$. This means that we have eight different ways of defining Kruskal–Szekeres coordinates, and every case is valid in different regions of spacetime. We shall denote these cases as an $8$-region approach.

Let us consider first the case $\eta = +1$. Then (40) and (43) lead to the Kruskal–Szekeres coordinates

$$X = \epsilon \left[\left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \cos h \left(\frac{t}{2r_s}\right),$$

$$T = \epsilon \left[\left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \sin h \left(\frac{t}{2r_s}\right),$$

(46)

and

$$X = \epsilon \left[\left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \sin h \left(\frac{t}{2r_s}\right),$$

$$T = \epsilon \left[\left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \cos h \left(\frac{t}{2r_s}\right),$$

(47)

respectively. These transformations correspond to values of $r$ such that $r > r_s$.

While if we consider $\eta = -1$ in (40) and (43), the corresponding Kruskal–Szekeres coordinates are given by

$$X = \epsilon \left[-\left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \cos h \left(\frac{t}{2r_s}\right),$$

$$T = \epsilon \left[-\left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \sin h \left(\frac{t}{2r_s}\right),$$

(48)

and

$$X = \epsilon \left[-\left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \sin h \left(\frac{t}{2r_s}\right),$$

$$T = \epsilon \left[-\left(\frac{r}{r_s} - 1\right)\right]^{1/2} e^{\frac{\eta}{2}} \cos h \left(\frac{t}{2r_s}\right),$$

(49)

respectively. Now, these transformations correspond to values of the radius $r$ such that $r < r_s$. 

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If we now regard $\epsilon = +1$, it is easy to see that (46)–(49) yield

$$X = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$T = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

(50)

$$X = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$T = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

(51)

$$X = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$T = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

(52)

$$X = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$T = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

respectively. While if in (46)–(49) we choose $\epsilon = -1$, we obtain

$$X = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$T = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

(53)

$$X = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$T = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$X = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$T = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

(54)

$$X = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$T = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$X = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$T = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$X = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$T = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

(55)

$$X = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$T = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$X = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$T = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$X = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$T = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

(56)

$$X = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$T = \left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$X = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

$$T = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$X = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \sin \left( \frac{t}{2r_t} \right),$$

$$T = -\left( \frac{r}{r_t} - 1 \right)^{3/2} e^{\frac{\pi}{\alpha}} \cos \left( \frac{t}{2r_t} \right),$$

(57)

Considering all eight transformations (50)–(57), an 8-region spacetime structure is obtained (see the appendix). In the four expressions (50), (53), (54) and (57) we recognize the traditional 4-region spacetime described, for instance, in [23]. What seems to be new are transformations (51), (52), (55) and (56). These give another 4-region. So if one adds the traditional and the new 4-region, one gets an 8-region spacetime structure. However, one may ask: what are the key results that distinguish these two 4-regions? In the following section, we shall show that the traditional 4-region can be associated with a spacetime of (1 + 3)-signature, while the new 4-region must be associated with a (3 + 1)-signature. Our final conclusion will be that the 8-region Kruskal–Szekeres transformation corresponds to a world with a (4 + 4)-signature.

4. (4 + 4)-spacetime structure and mirror black and white holes

In order to obtain information about the physical and geometrical meaning of the new 4-region, we proceed as follows.

First, it is important to note that in the process of all the Kruskal–Szekeres transformations, the angular part of (39), which is given by

$$d\mathbf{s}_{1\mathbf{3}}^2 \equiv r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(58)

has not been considered. In fact, the traditional Kruskal–Szekeres method focuses only on the first part of (39), namely

$$d\mathbf{s}_{1\mathbf{1}}^2 \equiv -\left( 1 - \frac{r_t}{r} \right) dr^2 + \frac{dr^2}{1 - \frac{r_t}{r}}.$$  

(59)

Of course, it is not difficult to see that when $r \to \infty$, equations (58) and (59) lead to a flat world of (1 + 3)-signature. It turns out that the 4-region transformations (50), (53), (54) and (57) are compatible with the (1 + 3)-signature when we set $\eta = +1$ in (42) and $\eta = -1$ in (45).

On the other hand, the angular part of the (3 + 1)-part of the metric (38) is given by

$$d\mathbf{l}_{1\mathbf{2}}^2 \equiv -r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(60)

while its radial part reads

$$d\mathbf{l}_{1\mathbf{1}}^2 \equiv -\left( 1 - \frac{r_t}{r} \right) dr^2 + \frac{dr^2}{1 - \frac{r_t}{r}}.$$  

(61)

Thus, it seems evident from (60) and (61) that when $r \to \infty$, a flat world of a (3 + 1)-signature is obtained. Remarkably, transformations (51), (52), (55) and (56) can be made compatible with (60) and (61) if we substitute $\eta = -1$ in (42) and $\eta = +1$ in expression (45). Hence, in order to distinguish the black-hole solution (60) and (61) from the usual one, we shall call it a mirror black hole. This name can be justified because, while ordinary black holes live in (1 + 3)-dimensions, the mirror black hole lives in a mirror world of (3 + 1)-dimensions.
Let us now analyze, from the perspective of Kruskal–Szekeres coordinates, geometrical implications when \( r = 0 \) in both cases: black and mirror black holes. Observe first that when \( r = 0 \), equations (41) and (44) lead to
\[
X^2 - T^2 = -\eta, \tag{62}
\]
and
\[
T^2 - X^2 = \eta, \tag{63}
\]
respectively. In the traditional steps of Kruskal–Szekeres coordinates, transformations (51), (52), (55) and (56) are considered in such a way that in (62) and (63), the value \( \eta = +1 \) is taken. It is evident that in this case, the two expressions (62) and (63) lead to exactly the same equation, namely
\[
T^2 - X^2 = +1. \tag{64}
\]
Of course, this expression corresponds to a hyperbola in the plane \( X \) and \( T \). The branch of this hyperbola in the positive values of \( T \) is identified with the true singularity of the black hole. The other branch of the hyperbola for negative values of \( T \) is associated with the singularity of a possible white hole. On the other hand, if in equations (62) and (63) we use \( \eta = -1 \), and considering the transformations (51), (52), (55) and (56), we obtain, instead of (64), the formula
\[
T^2 - X^2 = -1. \tag{65}
\]
It is easy to verify that the hyperbola in this case will correspond to positive and negative values of \( X \). This means that we shall not only have a singularity associated with the mirror black hole but also a kind of mirror white hole. Hence, we have mirror black and white holes in a \( (3+1) \)-signature spacetime, from which we can infer that, from a global point of view, in order to describe black/white and mirror black/white holes in a single gravitational setting, a theory with a \( (4+4) \)-spacetime structure would be necessary.

5. Linearized gravity in \( (4+4) \)-dimensions

Another non-trivial aspect of general relativity in \( (4+4) \)-dimensions emerges from linearized gravity. It turns out that in 2003, Nishino and Rajpoot [40] presented a self-dual \( N = (1, 0) \) supergravity in Euclidean 8-dimensions. Their method consisted of considering the self-duality concept of the curvature in terms of four-form octonion structure constants \( \eta^{ABCD} \). This motivates us to consider the Rarita–Schwinger field equation in eight dimensions
\[
\eta^{ABCD} \partial_B \psi_D = 0, \tag{66}
\]
in complete analogy to the case of 4-dimensions
\[
\epsilon^{\mu\nu\sigma\alpha} \partial_\mu \psi_\nu = 0. \tag{67}
\]

On the other hand, a classical description of (67)
\[
S_\nu^{\alpha} = \epsilon^{\mu\nu\alpha} \theta_\mu R_\nu = 0 \tag{68}
\]
in terms of anticommuting Grassmann variables \( \theta_\nu \) leads to the remarkable result that \( S_\nu^{\alpha} \) can be understood as the ‘square root’ of linearized gravity in the sense that
\[
\{ S_\nu^{\alpha}, S_\rho^{\beta} \} = H_{\mu}^{\alpha\beta}, \tag{69}
\]
where the expression
\[
H_{\mu}^{\alpha\beta} = -\epsilon^{\alpha} \epsilon^{\beta} \epsilon^\mu \epsilon^\nu p_\mu - \eta^{\alpha\beta} \eta_\mu p_\nu + \epsilon^{\alpha} \epsilon^\mu \eta_\beta p_\nu + \epsilon^{\beta} \epsilon^\nu \eta_\alpha p_\mu - \eta_\mu p_\nu \epsilon^{\alpha\beta} + \epsilon_\mu \eta^{\alpha\beta} \eta_\nu
\]
\[
\epsilon^{\alpha\beta} p_\alpha p_\beta, \tag{70}
\]
corresponds to a first-class constraint associated with linearized gravity (see [41] for details). Of course, this result is inspired in the supersymmetric \( \frac{1}{2} \)-spin formalism. In this case, the Dirac equation
\[
(\gamma^\mu p_{\mu} + m) \psi = 0 \tag{71}
\]
is considered as a first-class constraint
\[
S = \theta^\mu p_{\mu} + m \approx 0 \tag{72}
\]
and in this way one proves the relation
\[
\{ S, S \} = H, \tag{73}
\]
meaning that \( S \) is the square root of the constraint
\[
H = p^\mu p_{\mu} + m^2 \approx 0. \tag{74}
\]
Of course, at the quantum level, this constraint can be associated with the Klein–Gordon equation
\[
(\hat{p}^\mu \hat{p}_{\mu} + m^2) \varphi = 0. \tag{75}
\]
One of our motivations is to see whether one can follow similar steps to the case of the gravitino field equation (66). For this purpose, in analogy to (68), let us associate (66) with the following constraint
\[
S_A^E = \eta_A^{DEFG} \theta_D P_F \approx 0. \tag{76}
\]
Our goal is to determine
\[
\{ S_A^E, S_B^F \} = \mathcal{H}_{AB}^{EF}. \tag{77}
\]
First, one observes that
\[
\mathcal{H}_{AB}^{EF} = \eta_A^{GHE} p_H \eta_B^{QF} P_Q. \tag{78}
\]
This expression can be rewritten as
\[
\mathcal{H}_{AB}^{EF} = \eta_A^{QHE} p_H \eta_B^{QF} P_Q. \tag{79}
\]
Now, it is known that octonionic structure constants \( \eta^{ABCD} \) satisfy [42, 43],
\[
\eta^{ABCD} \epsilon_{EFGD} = \delta_{EFG}^{ABC} + \Sigma_{EFG}^{ABC},
\]
where \( \Sigma_{EFG}^{ABC} \) is given by
\[
\Sigma_{EFG}^{ABC} = \eta_{EF} \epsilon_{G}^{A} + \eta_{EF} \epsilon_{G}^{A} + \eta_{EF} \epsilon^{G}.
\]

Hence, substituting (80) and (81) into (79) yields
\[
\mathcal{H}_{AB}^{FE} = H_{AB}^{FE} + \Omega_{AB}^{FE}.
\]
Here, \( H_{AB}^{FE} \) has exactly the same form as (70) but in 8-dimensions, and \( \Omega_{AB}^{FE} \) is given by
\[
\Omega_{AB}^{FE} = \eta_{AB} \eta^{FRS} EOG \Sigma_{RBB}^{OG} P_{G} P_{H},
\]
which, by virtue of (81), leads to
\[
\Omega_{AB}^{FE} = -\eta_{EF} \epsilon_{P}^{D} P_{O} - \eta_{AB} \epsilon_{P}^{D} P_{O} + \eta_{EF} \epsilon_{P}^{D} P_{A} + \eta_{AB} \epsilon_{P}^{D} P_{A}.
\]

Observe that
\[
\Omega_{AB}^{FE} = -\Omega_{AB}^{FE} = -\Omega_{BA}^{FE}.
\]
Thus, since \( H_{AB}^{FE} \) is obtained from the Riemann tensor, we see that result (85) implies a non-trivial additional term \( \Omega_{AB}^{FE} \) in the Einstein field equations.

6. Final remarks

In this work, we have introduced a new type of (4+4)-Kaluza–Klein theory of gravity. As an application of the theory, we have studied spherical symmetric solutions in vacuum, in particular where the gauge fields \( A_{i}^{A} \) and \( B_{i}^{A} \) are null. Hence, considering generalized Kruskal–Szekeres coordinate transformations, we obtain two 4-regions representing black/white hole solutions in a (1+3)-spacetime signature and mirror black/white hole solutions in a (3+1)-spacetime signature, respectively.

The present developments clearly point towards a more general theory which combines black/white-holes and mirror-black/white-holes. Since the corresponding signatures are (1+3) and (3+1), the natural choice for such a generalized theory seems to be a framework in (4+4)-dimensions. This is in part due to the fact that the interesting splitting \((4+4) = (1+3) + (3+1)\) is valid. Surprisingly, \((4+4)\)-dimensional theory can be understood as a particular case of so-called double field theories [39]. It may be interesting for future research to explore this possible connection.

Moreover, in [20], it was shown that the Dirac equation in \((4+4)\)-dimensions leads to the surprising result that a complex spinor associated with a \(\frac{1}{2}\)-spinor in \((1+3)\)-dimensions can be understood as a Majorana–Weyl spinor in \((4+4)\)-dimensions. So, one would expect that the Majorana–Weyl vector-spinor \(\Psi_{\mu} \) of the Rarita–Schwinger field equation in \((4+4)\)-dimensions (66) can be associated with a complex spinor vector-spinor in \((1+3)\)-dimensions. Furthermore, using a Cayley hyperdeterminant, it was shown that black hole/qubit correspondence exists in \((4+4)\)-dimensions [44], which in turn can be linked to oriented matroid theory (see [45–49] and references therein). Finally, it is worth mentioning that an Ashtekar formalism in 8-dimensions has been developed in [50]. Therefore, one may expect further work to emerge when the present formalism of general relativity in \((4+4)\)-dimensions becomes connected with such fascinating developments.

Let us discuss an additional number of issues concerning our proposed formalism of a (4+4)-world. First, in [51], following a unified spinor field theory, a pre-inflationary universe which makes a global topological phase transition was considered. It seems interesting to link our work with this reference in order to consider a kind of topological phase transition in a (4+4)-universe. Similarly, in [52, 53], a SO(5, 5)-duality is considered. Since the relevant group of the (4+4)-universe with a cosmological constant is precisely the group SO(5, 5) it may be interesting for future research to consider the duality concept in our formalism as in such references.

Second, it is natural to ask why we should select the splitting \((4+4) = (1+3) + (3+1)\) instead of other possibilities such as \((4+4) = (0+4) + (4+0)\). In general, one may consider the space \(R^{4+4}\). So there is not a particular reason why \(R^{4+4} = R^{3+3} \oplus R^{3+1}\) should be chosen. Something similar happens in twistor theory [54] when the space \(R^{3+3}\) is generalized to a complex space \(C^{3}\). There seems to be no particular reason why complex \(R^{3+3}\) space should be chosen out of many possibilities offered by \(C^{4}\). It seems to us that in the lack of any particular mathematical criterion, one makes a choice for physical reasons. In the case of \(C^{4}\), one selects the complex Minkowski hyperplane space and, in our case, one chooses to derive the physical Minkowski space of our \((1+3)\)-world from a \((4+4)\)-world and therefore the splitting \(R^{4+4} = R^{3+3} \oplus R^{3+1}\) seems to be the right choice.

Third, at first sight, one may think that our black-hole solution corresponds to two copies of general relativity in 4-dimensions since in such a case the theory is invariant under the change of signature \((1+3) \leftrightarrow (3+1)\) [25]. However, this is not the case in higher dimensional general relativity and when matter \(\frac{1}{2}\)-spin fields are included (see [25] for details). Furthermore, it must be clear from our formalism that our discovery of 8-regions of Kruskal–Szekeres coordinates is not trivial and it seems difficult to imagine how to derive such an 8-region structure considering the two copies of black holes in the usual separate context of 4-dimensions. Of course, further work is required in order to find a nontrivial charged black-hole solution in \((4+4)\)-dimensions, with the gauge fields \(A_{i}^{A} = 0 \) and \(B_{i}^{A} = 0\) considered in section 2, but at least we are providing the first necessary steps in such a direction. In particular, in section 5, we prove that linearized gravity in 8-dimensions with \(A_{i}^{A} = 0 \) and \(B_{i}^{A} = 0\) leads to non-trivial development of general relativity.

Finally, we would like to raise the following question: is it possible that a \((4+4)\)-world theory may be useful in the
quest for quantum gravity? This is, in a sense, the hidden motivation for our interest in developing the present (4 + 4)-world formalism. As is known, quantum theory of the gravitational field is still an unsolved problem (see [55, 56] and references therein). One of the main candidates is loop quantum gravity theory [57, 58], but it turns out that although this is one of the most promising proposals for quantum gravity, it contains many difficulties. Perhaps the most serious problem lies in specifying a meaning for the time evolution parameter of loop quantum states. There are attempts to make sense of loop quantum gravity in higher dimensions, but the central duality concept in 4-dimensions seems to be lost. Surprisingly, by considering octonions, the duality concept in 4-dimensions may still be recast in 8-dimensions [50] (see also [59, 60]) and in particular in the case of the (4 + 4)-world.

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Appendix

In order to obtain the complete 8-region Kruskal–Szekeres transformation, let us consider the Schwarzschild metric without the angular part. This is given by

$$ds^2 = -(1 - \frac{r_s}{r})dr^2 + \frac{dr^2}{(1 - \frac{r_s}{r})}, \quad (A1)$$

Then, the first step is to consider photon trajectories in the Schwarzschild metric. Therefore, we need to equate (A1) to zero, that is

$$0 = -(1 - \frac{r_s}{r})dr^2 + \frac{dr^2}{(1 - \frac{r_s}{r})}, \quad (A2)$$

where we have defined a new coordinate $r^*$. From (2) one can obtain the relation

$$dr^* = \frac{\eta dr}{(1 - \frac{r_s}{r})}, \quad (A3)$$

where $\eta = \pm 1$ emerges since we have taken the ‘square root’. Integrating (A3) leads to

$$r^* = \eta r + \eta_0 \ln \left[ \eta \left( \frac{r}{r_s} - 1 \right) \right] + \eta \ln \varepsilon^2. \quad (A4)$$

In principle, $\varepsilon$ is an arbitrary constant but by convenience we set $\varepsilon$ as $\varepsilon = \pm 1$. It is not difficult to see that from (A4) one obtains

$$e^{r^*/\eta} = \eta e^{\varepsilon^2 \left[ \eta \left( \frac{r}{r_s} - 1 \right) \right]^{1/2}}. \quad (A5)$$

Furthermore, it is important to note that using (A4), we can rewrite (A1) as

$$ds^2 = \left( 1 - \frac{r_s}{r} \right) (-dr^2 + dr^{*2}). \quad (A6)$$

Let us now introduce the coordinates $u$ and $v$ given by

$$u = t + \eta^* \quad (A7)$$

and

$$v = t - \eta^*. \quad (A8)$$

Then, with these new coordinates, expression (A6) becomes

$$ds^2 = -\frac{r_s}{r} \left( \frac{r}{r_s} - 1 \right) dv^2. \quad (A9)$$

Moreover, using (A5), the previous line element acquires the form

$$ds^2 = -v \frac{r_s}{r} e^{-\xi} e^{\varepsilon^2 \left[ \eta \left( \frac{r}{r_s} - 1 \right) \right]^{1/2}} dv^2. \quad (A10)$$

It is important to mention that we have used the result $\eta^* = \frac{1}{2}(u - v)$ obtained from (A7) and (A8).

The next step is to define new coordinates $U$ and $V$ in the form

$$U = e^{\frac{r}{\eta}} \quad (A11)$$

and

$$V = e^{-\xi}. \quad (A12)$$

With these new variables, the line element in (A10) becomes

$$ds^2 = 4\eta \frac{r_s}{r} e^{-\xi} dU dV. \quad (A13)$$

Then, we define another two variables $X$ and $T$, which are related to $U$ and $V$, by

$$X = \frac{1}{2}(U + \xi V) \quad (A14)$$

and

$$T = \frac{1}{2}(U - \xi V) \quad (A15)$$

where $\xi = \pm 1$. Consequently, the Schwarzschild line element acquires its final form, given by

$$ds^2 = 4\eta \frac{r_s}{r} e^{-\xi} (-dT^2 + dX^2), \quad (A16)$$

where we have used that $U = X + T$ and $V = \xi (X - T)$.

On the other hand, in order to obtain the explicit form of $X$ and $T$ in terms of $r$ and $t$, it is important to come back to the first previous steps. Thus, we first substitute (A11) and (A12) into (A14) and (A15). One obtains

$$X = \frac{1}{2}(e^{\frac{r}{\eta}} + \xi e^{-\frac{r}{\eta}}). \quad (A17)$$
and

\[ T = \frac{1}{2} (e^{\frac{\pi}{\eta}} - \xi e^{\frac{\pi}{\eta}}). \quad \text{(A18)} \]

Now, using (A7) and (A8), one learns that (A17) and (A18) become

\[ X = \frac{1}{2} e^{\frac{\pi}{\eta}} (e^{\frac{\pi}{\eta}} + \xi e^{\frac{\pi}{\eta}}) \quad \text{(A19)} \]

and

\[ T = \frac{1}{2} e^{\frac{\pi}{\eta}} (e^{\frac{\pi}{\eta}} - \xi e^{\frac{\pi}{\eta}}) \quad \text{(A20)} \]

respectively. Finally, substituting (A5) into (A19) and (A20) leads to

\[ X = \frac{\xi}{2} e^{\frac{\pi}{\eta}} \left[ \left( \frac{r}{r_1} - 1 \right) \right]^{\eta/2} (e^{\frac{\pi}{\eta}} + \xi e^{\frac{\pi}{\eta}}) \quad \text{(A21)} \]

and

\[ T = \frac{\xi}{2} e^{\frac{\pi}{\eta}} \left[ \left( \frac{r}{r_1} - 1 \right) \right]^{\eta/2} (e^{\frac{\pi}{\eta}} - \xi e^{\frac{\pi}{\eta}}). \quad \text{(A22)} \]

Hence, choosing all possible values for \( \eta, \xi, \) and \( \epsilon, \) transformations (A21) and (A22) determine 8-region Kruskal–Szekeres space.

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