Generic chaining and the $\ell_1$-penalty
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Abstract We address the choice of the tuning parameter $\lambda$ in $\ell_1$-penalized M-estimation. Our main concern is models which are highly nonlinear, such as the Gaussian mixture model. The number of parameters $p$ is moreover large, possibly larger than the number of observations $n$. The generic chaining technique of Talagrand [2005] is tailored for this problem. It leads to the choice $\lambda \approx \sqrt{\log p/n}$, as in the standard Lasso procedure (which concerns the linear model and least squares loss).

1 Introduction

Let $X_1, \ldots, X_n$ be independent observations with values in some observation space $\mathcal{X}$, and let for $\theta$ in a parameter space $\Theta \subset \mathbb{R}^p$ be given a loss function $\rho_\theta : \mathcal{X} \rightarrow \mathbb{R}$. The parameter $\theta$ is potentially high-dimensional, i.e. possibly $p \gg n$. In this article we study the $\ell_1$-regularized M-estimator
\[ \hat{\theta} := \arg \min_{\theta \in \Theta} \left\{ P_n \rho_\theta + \lambda \| \theta \|_1 \right\}. \]

Here, we use the notation $P_n \rho_\theta := \frac{1}{n} \sum_{i=1}^n \rho_\theta(X_i)$, i.e., it is the empirical measure of the loss function $\rho_\theta$, often referred to as the empirical risk. Moreover, $\lambda > 0$ is a tuning parameter and $\| \theta \|_1 := \sum_{j=1}^p |\theta_j|$ is the $\ell_1$-norm of $\theta$.

A special case is the Lasso (Tibshirani [1996]), which has quadratic loss:
\[ \rho_\theta(X) := (Y - \theta^T Z)^2, \quad X = (Y, Z), \]
where $Y \in \mathbb{R}$ is the response variable and $Z \in \mathbb{R}^p$ are covariables. There are many papers on the Lasso, see for example van de Geer [2001], Bunea et al. [2006], Bunea et al. [2007a], Bunea et al. [2007b], van de Geer [2008], Koltchinskii [2009a], Bickel et al. [2009], For an overview and further results, see also Bühlmann and van de Geer [2011]. It is known that generally the choice $\lambda \approx \sqrt{\log p/n}$ is appropriate. Under some distributional assumptions. this choice leads to favorable theoretical properties of the Lasso, such as good oracle bounds for the estimation and prediction error.

In this note we address the following question: is the choice $\lambda \approx \sqrt{\log p/n}$ also appropriate for non-linear situations? The example described above is a linear situation. More generally, we call the situation linear if for some $\psi : \mathcal{X} \rightarrow \mathbb{R}^p$
\[ (P_n - P)(\rho_\theta - \rho_\hat{\theta}) = (\theta - \hat{\theta})^T (P_n - P)\psi, \quad \forall \theta, \hat{\theta}, \]
where $P \rho_\theta := \frac{1}{n} \sum_{i=1}^n \mathbb{E} \rho_\theta(X_i)$ is the theoretical risk. Any generalized linear model (GLM) loss function with canonical link function and fixed design is a linear situation. Also density estimation using an exponential family is a linear situation. A non-linear situation occurs for instance in linear least squares regression with random design. Our focus is more on other examples,
such as mixture models (Städler and van de Geer [2010]) or mixed effect models (Schelldorfer et al. [2011]), etc.

Let us define the “true” parameter

\[ \theta^0 := \arg \min_{\theta \in \Theta} P_{\rho_{\theta}}, \]

where we assume \( \rho_{\theta} \) is defined for all \( \theta \) in the possibly extended space \( \tilde{\Theta} \supset \Theta \).

Let \( \theta^* \in \Theta \) be some “approximation” of \( \theta^0 \). Here, we have in mind the best approximation within \( \Theta \) (in the case of misspecified models), and possibly the best “sparse” approximation (see Remark 2.1 for a definition). Our choice for the tuning parameter is governed by the behavior over \( \ell_1 \)-balls \( \Theta_M(\theta^*) := \{ \theta \in \Theta_* : \| \theta - \theta^* \|_1 \leq M \} \) of the empirical process \( (P_n - P)(\rho_{\theta} - \rho_{\theta^*}) \), where \( \Theta_* = \Theta \) or, in the case of Theorem 2.2 (convex loss) \( \Theta_* \) is the smallest convex set containing \( \Theta \).

In the linear case, the supremum of the empirical process can be easily bounded using the dual norm inequality

\[ \sup_{\theta \in \Theta_M(\theta^*)} |(P_n - P)(\rho_{\theta} - \rho_{\theta^*})| \leq \|(P_n - P)\psi\|_{\infty} M, \tag{1} \]

where for a vector \( v \in \mathbb{R}^p, \|v\|_{\infty} := \max_{1 \leq j \leq p} |v_j| \) is the uniform norm. Moreover, for example for \( \mathcal{N}(0, 1/n) \)-random variables \( \{V_j\}_{j=1}^p \) (say), it holds that

\[ \max_{1 \leq j \leq p} |V_j| = \mathcal{O}_P \left( \sqrt{\frac{\log p}{n}} \right). \]

We show in this paper that in many non-linear cases, one still has

\[ \sup_{\theta \in \Theta_M(\theta^*)} |(P_n - P)(\rho_{\theta} - \rho_{\theta^*})| = \mathcal{O}_P \left( \sqrt{\frac{\log p}{n}} \right) M. \tag{2} \]

This follows rather easily from a generic chaining (Talagrand [1996]) and Sudakov minoration argument. We will use the book [Talagrand 2005].

In the case of regression with robust GLM loss (robust quasi-likelihood loss functions, quantile functions), we have

\[ \rho_{\theta}(X) = \rho(Y, \theta^T Z), \]

with \( \rho(y, \cdot) \) Lipschitz for all \( y \). In that situation, one may apply the contraction inequality (Ledoux and Talagrand [1991]) to arrive at (2). We will explain this in Subsection 3.2.

Our emphasis is however on cases that go beyond GLM loss. An example is the Gaussian mixture model

\[ \rho_{\theta}(Y, Z) = \log \left( \sum_{k=1}^{p} \pi_k \phi_{\sigma_k} (Y - \beta_k^T Z_k) \right), \]
where $\phi_\sigma = \phi(\cdot/\sigma)/\sigma$ is the density of the $\mathcal{N}(0,\sigma^2)$-distribution, $\{\pi_k\}_{k=1}^r$ are mixing coefficients ($\sum_{k=1}^r \pi_k = 1$), $\beta_k$ and $Z_k$ are vectors in $\mathbb{R}^{p_k}$, $k = 1,\ldots,r$, and where $Y$ is again a response variable and $Z^T := (Z_1^T,\ldots,Z_r^T)$ a covariable. This model has been studied in St"adler and van de Geer [2010]. The tuning parameter is there taken of order $\lambda \asymp \sqrt{\log 3 n \log(p \lor n)}/n$. The parameters in this model are $\theta := (\pi,\sigma,\beta_1,\ldots,\beta_r)$ (and in St"adler and van de Geer [2010], the penalty is $\lambda \|\beta\|_1 = \lambda \sum_{k=1}^r \|\beta_k\|_1$, i.e., it does not include the parameters $\pi$ and $\sigma$). The model is definitely non-linear. It is essentially a GLM albeit that there are $r$ linear functions involved instead of just one, and there are the further parameters $\pi$ and $\sigma$. We call such a model an extended GLM. The contraction inequality will not help us anymore in this case, but as we will see in Subsection 4.4 the generic chaining argument gives a multivariate version of the contraction theorem. This leads to the reduced choice $\lambda \asymp \sqrt{\log p/n}$.

Another situation is where $\rho_\theta$ is a general non-linear function. In that case, we will restrict ourselves to the medium-dimensional situation with $p$ sufficiently smaller than $n$. Again the generic chaining bound can be used.

Our results rely on the following condition.

**Condition 1.1 (Componentwise Lipschitz condition)** There exist functions $\{\psi_j\}$ ($\psi_j : \mathcal{X} \times \{1,\ldots,n\} \to \mathbb{R}$) and constants $\{c_{i,\theta}\}$ such that for all $\theta$ and $\tilde{\theta}$ in $\Theta_*$

$$\left| [\rho_\theta(X_i) - c_{i,\theta}] - [\rho_{\tilde{\theta}}(X_i) - c_{i,\tilde{\theta}}] \right| \leq \sum_{j=1}^p |\theta_j - \tilde{\theta}_j| \psi_j(X_i,i), \ \forall \ i.$$ 

The constants $\{c_{i,\theta}\}$ will generally be either all zero, or equal to the expectation $c_{i,\theta} = \mathbb{E}[\rho_\theta(X_i)]$.

Generic chaining gives a bound $\gamma_2$ (following the notation of Talagrand [2005]) for the supremum of stochastic processes (see Theorem 5.1). This bound $\gamma_2$ is defined by the geometry of the index set of the process. By Sudakov’s minorization $\gamma_2$ is also a lower bound in the case of Gaussian processes. This is the argument we will use. It means that we need not directly calculate $\gamma_2$ but instead obtain an upper bound for free. Nevertheless, it would be of interest to directly bound $\gamma_2$ using geometric arguments (Talagrand’s research problem 2.1.9 in Talagrand [2005]). The Dudley bound (see Dudley [1967] or Dudley [2010]) results in additional (and hence superfluous) $(\log n)$-factors (see Section 5).

We remark that the bounds are based on arguments for Gaussian processes, and in fact on the behavior of maxima of i.i.d. Gaussians. This is so to speak the worst case: the bounds are here the largest. In particular for random variables which are highly dependent, one may have smaller bounds. Moreover, in the statistical application of $\ell_1$-regularized estimation, strong dependencies may lead to choosing the tuning parameter $\lambda$ of much smaller order than $\sqrt{\log p/n}$. This is explained in van de Geer and Lederer [2012] for the case of the Lasso.
It means that even when the result
\[
\sup_{\theta \in \Theta_M(\theta^*)} |(P_n - P)(\rho_\theta - \rho_{\theta^*})| = O_p\left(\sqrt{\frac{\log p}{n}}\right)M,
\]
leaves no room for improvement, there are situations where the choice \(\lambda \asymp \sqrt{\log p/n}\) is much too large. We will not address this issue here but refer to van de Geer and Lederer [2012].

That generic chaining arguments can be used to theoretically show that \(\lambda \asymp \sqrt{\log p/n}\) is appropriate is perhaps of little practical value. One may argue for example that cross-validation will rather be used in practice, instead of a theoretical value. Our finding is primarily interesting from a theoretical point of view.

Generic chaining plays an important role in the statistics literature, for example to empirical risk minimization [Bartlett and Mendelson [2006]], PAC-Bayesian learning [Audibert and Bousquet [2007]], and the Lasso with random design [Bartlett et al. [2009]]. We believe the application in this paper, addressing the choice of the tuning parameter \(\lambda\) in \(\ell_1\)-regularization for M-estimators, is an nice opportunity to clearly demonstrate the elegance of Talagrand’s approach.

1.1 Organization of the paper

In Section 2 we review the basic oracle inequality for the \(\ell_1\)-penalized M-estimator. This purpose of this section is to highlight the role of the supremum
\[
\sup_{\theta \in \Theta_M(\theta^*)} |(P_n - P)(\rho_\theta - \rho_{\theta^*})|.
\]
The proofs of Theorems 2.1 and 2.2 follow closely Bühlmann and van de Geer [2011], and are given for completeness in Section 7. In Section 4 we show that
\[
E\left(\sup_{\theta \in \Theta_M(\theta^*)} |P_n^\varepsilon(\rho_\theta - \rho_{\theta^*})|\bigg| X\right) = O_p\left(\sqrt{\frac{\log p}{n}}\right)M.
\]
Here, \(P_n^\varepsilon\) is the symmetrized measure defined in Section 3 and \(X := (X_1, \ldots, X_n)\). Moreover, \(\rho_\theta(X_i, i) = \rho_\theta(X_i) - c_{i,\theta}\), with the constants \(c_{i,\theta}\) as in Condition 1.1. Section 3 summarizes why bounds on the conditional mean of the symmetrized process suffice: they lead to exponential probability inequalities using a deviation inequality of Massart [2000a]. Section 5 gives the details concerning generic chaining and a consequence concerning the geometry of \(\ell_1\)-balls. It summarizes some results in Talagrand [2005] and makes a comparison with Dudley’s entropy bound.

2 The oracle inequality

We let for \(\theta\) and \(\theta^*\) in \(\Theta\),
\[
Y(\theta, \theta^*) := (P_n - P)(\rho_\theta - \rho_{\theta^*}).
\]
In this section, we show why bounds for \( \sup_{\theta \in \Theta_M(\theta^*)} |Y(\theta, \theta^*)| \) can be used to choose the tuning parameter \( \lambda \) and arrive at an oracle inequality for the \( \ell_1 \)-regularized M-estimator \( \hat{\beta} \). The line of reasoning is as in Bühlmann and van de Geer [2011]. Define the excess risk

\[
\mathcal{E}(\theta; \theta_0) := P(\rho_0 - \rho_{\theta_0}).
\]

The following condition quantifies the curvature of \( \mathcal{E}(\theta; \theta_0) \) around its minimizer \( \theta_0 \).

**Condition 2.1 (Margin condition)** We say that the margin condition holds for all \( \theta \in \Theta_M(\theta^*) \) if for some norm \( \tau \) on \( \Theta \), and some strictly convex non-negative function \( G \), satisfying \( G(0) = 0 \),

\[
\mathcal{E}(\theta; \theta_0) \geq G(\tau(\theta - \theta_0)), \quad \forall \theta \in \Theta_M(\theta^*).
\]

**Definition 2.1 (Convex conjugate)** Let \( G \) be a strictly convex non-negative function with \( G(0) = 0 \). The convex conjugate of \( G \) is

\[
H(v) := \sup_{u \geq 0} \left\{ uv - G(u) \right\}, \quad v \geq 0.
\]

For sets \( S \) and vectors \( \theta \in \mathbb{R}^p \) we let

\[
\theta_{j,S} := \theta_j \mathbb{1}\{j \in S\}, \quad j = 1, \ldots, p.
\]

**Definition 2.2 (Effective sparsity)** Let

\[
\delta(L, S) := \min\{\tau(\theta) : \|\theta_S\|_1 = 1, \|\theta_{S^c}\|_1 \leq L\}.
\]

Then \( \Gamma^2(L, S) := 1/\delta^2(L, S) \) is called the effective sparsity (of the set \( S \)).

Following van de Geer [2007], we call \( \phi^2(L, S) := |S|\delta^2(L, S) \) the compatibility constant (for the set \( S \)). If it is not too small, the norms \( \tau \) and the \( \ell_1 \)-norm \( \| \cdot \|_1 \) are “compatible” with each other.

We define for some constant \( \lambda_0 \), the set

\[
T_M(\theta^*) := \{|Y(\theta, \theta^*)| \leq \lambda_0\|\theta - \theta^*\|_1 \vee \lambda_0^2, \quad \forall \theta \in \Theta_M(\theta^*)\},
\]

and let \( T(\theta^*) := T_\infty(\theta^*) \) and \( \Theta_\infty(\theta^*) = \Theta \).

Our task in Sections 3 and 4 is to show that with \( \lambda_0 \simeq \sqrt{\log p/n} \), the set \( T_M(\theta^*) \) has large probability (for any \( \theta^* \) and suitable \( M \)).

We first give in Theorem 2.1 a result where the margin assumption is assumed to hold ”globally”. We then refine this in Theorem 2.2 to local conditions for the convex case.
Theorem 2.1 Let $\lambda > \lambda_0$. Assume Condition 2.1 (the margin condition) for all $\theta \in \Theta$. Let $H$ be the convex conjugate of $G$. If $\theta^0 \in \Theta$, we have on $T(\theta_0)$, for all $0 < \delta < 1$,

$$
(1 - \delta)E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta} - \theta^0\|_1 \leq \delta H\left(\frac{2\lambda G(L, S_0)}{\delta}\right) \lor 2\lambda^2,
$$

with $L = (\lambda + \lambda_0)/(\lambda - \lambda_0)$. Moreover, for all $0 < \delta < 1$ and all $\theta^* \in \Theta$, on $T(\theta^*)$,

$$
(1 - \delta)E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta} - \theta^*\|_1 \leq 2\delta H\left(\frac{4(1 + \delta)\lambda G(L_\delta, S_0)}{\delta^2}\right) \lor 2\lambda^2 + (1 + \delta)E(\theta^*; \theta_0),
$$

with $L_\delta = 2((1 + \delta)/\delta)((\lambda + \lambda_0)/(\lambda - \lambda_0))$. Here $S_{\theta^*} := \{j : \theta^*_j \neq 0\}$ is the support set of $\theta^*$.

The proof of Theorem 2.1 is given in Section 7.

Remark 2.1 With the above result, one can define the best sparse approximation as a solution $\theta^*$ of the minimization

$$
\min_{\theta \in \Theta} \left\{2\delta H\left(\frac{4(1 + \delta)\lambda G(L_\delta, S_0)}{\delta^2}\right) \lor 2\lambda^2 + (1 + \delta)E(\theta; \theta_0)\right\},
$$

where $S_{\theta^*} := \{j : \theta_j^* \neq 0\}$.

The next theorem assumes convexity and then needs the margin condition only in a neighborhood of $\theta^*$.

Theorem 2.2 Let $\lambda > \lambda_0$. Let $\theta_\ast$ be the smallest set containing $\Theta$ and suppose that the map $\theta \mapsto \rho_\theta$, $\theta \in \Theta_\ast$, is convex. Let $(\lambda - \lambda_0)M_0$ and $(\lambda - \lambda_0)M_\ast$ be the bounds given in the right hand side of (3) and (4) respectively, i.e.,

$$
M_0 := \frac{\delta}{\lambda - \lambda_0} \left\{H\left(\frac{2\lambda G(L, S_0)}{\delta}\right) \lor 2\lambda^2\right\},
$$

with $L = (\lambda + \lambda_0)/(\lambda - \lambda_0)$, and

$$
M_\ast := \frac{1}{\lambda - \lambda_0} \left\{2\delta H\left(\frac{4(1 + \delta)\lambda G(L_\delta, S_\ast)}{\delta^2}\right) \lor 2\lambda^2 + (1 + \delta)E(\theta^*; \theta_0)\right\},
$$

with $L_\delta = 2((1 + \delta)/\delta)((\lambda + \lambda_0)/(\lambda - \lambda_0))$. Here, $H$ is a strictly convex increasing function with $H(0) = 0$.

If $\theta^0 \in \Theta$ and the margin condition holds for all $\theta \in \Theta_{2M_0}(\theta_0)$, with $G$ the convex conjugate of $H$, then again on $T_{2M_0}(\theta_0)$,

$$
(1 - \delta)E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta} - \theta^0\|_1 \leq (\lambda - \lambda_0)M_0
$$

For general $\theta^*$, if the margin condition holds for all $\theta \in \Theta_{2M_\ast}(\theta^*)$, with $G$ the convex conjugate of $H$, then again on $T_{2M_\ast}(\theta^*)$,

$$
(1 - 2\delta)E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta} - \theta^\ast\|_1 \leq (\lambda - \lambda_0)M_\ast.
$$
The proof of Theorem 2.2 is also in Section 7.

**Remark 2.2** To handle the set \( T_M(\theta^*) \) we prove in Sections 3 and 4 that with \( \lambda_0 \approx \sqrt{\log p/n} \), with large probability

\[
\sup_{\theta \in \Theta_M(\theta^*)} |Y(\theta, \theta^*)| \leq \lambda_0 M
\]

for all \( M \leq \text{const.} \) and then apply the peeling device (the latter being detailed in Subsection 3.4). However, as is clear from the proof of Theorem 2.2, one can refrain from peeling in the convex case, because one already places oneself in a suitable neighborhood of \( \theta^* \) (see also van de Geer [2007] and van de Geer [2008]).

**Remark 2.3** The models considered in Städler and van de Geer [2010] and Schelldorfer et al. [2011] are not convex. There, the margin condition holds in a bounded neighborhood and these bounds are imposed on the parameters. Then the peeling device is invoked.

### 3 Symmetrization, contraction and deviation inequalities, and the peeling device

We write the sample as \( X := (X_1, \ldots, X_n) \). Let \( \varepsilon_1, \ldots, \varepsilon_n \) be a Rademacher sequence independent of \( X \). For constants \( \{c_{i,\theta}\} \) (which we will choose as in Condition [1.3]), we define

\[
\rho_{\theta}^c(X_i, i) = \rho_{\theta}(X_i) - c_{i,\theta}, \quad i = 1, \ldots, n,
\]

and the symmetrized empirical process

\[
P_n^\varepsilon \rho_{\theta}^c := \frac{1}{n} \sum_{i=1}^{n} [\rho_{\theta}(X_i) - c_{i,\theta}] \varepsilon_i, \quad \theta \in \Theta_*,
\]

and we let

\[
Y^\varepsilon(\theta, \theta^*) := P_n^\varepsilon (\rho_{\theta} - \rho_{\theta^*}), \quad \theta \in \Theta_*.
\]

For a function \( g : \mathcal{X} \times \{1, \ldots, n\} \), we use the notation

\[
\|g\|_{n}^2 := \frac{1}{n} \sum_{i=1}^{n} g^2(X_i, i), \quad \|g\|^2 := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}g^2(X_i, i).
\]

In this section, we summarize the arguments that show that up to constants, one can reduce the problem of deriving probability inequalities for the process \( Y(\theta, \theta^*) \) to studying the symmetrized process \( Y^\varepsilon(\theta, \theta^*) \). In fact, we only need bounds for the conditional expectation

\[
E_n := \mathbb{E}\left( \sup_{\theta \in \Theta_M(\theta^*)} |Y^\varepsilon(\theta, \theta^*)| \bigg| X \right).
\]
Alternatively, one can use direct arguments in certain regression problems (with sub-Gaussian errors) or invoking or example Bernstein’s inequality (but then one has to adjust Sudakov’s minoration argument to the case of independent Gamma-distributed variables). We also discuss the peeling device (but as noted in Remark 2.2 this device is not always needed).

3.1 Symmetrization

We cite the following result (see Pollard [1984]).

**Lemma 3.1** Let $R := \sup_{\theta \in \Theta_M(\theta^*)} \| \rho^e_{\theta} - \rho^e_{\theta^*} \|$ and let $t \geq 4$. Then

$$\mathbb{P}\left( \sup_{\theta \in \Theta_M(\theta^*)} |Y(\theta, \theta^*)| > 4R \sqrt{\frac{2t}{n}} \right) \leq 4\mathbb{P}\left( \sup_{\theta \in \Theta_M(\theta^*)} |Y^e(\theta, \theta^*)| > R \sqrt{\frac{2t}{n}} \right).$$

3.2 Contraction

Suppose that for all $\theta, \tilde{\theta} \in \Theta$,

$$|\rho^e_{\tilde{\theta}}(X_i, i) - \rho^e_{\theta}(X_i, i)| \leq |f_\theta(X_i, i) - f_{\tilde{\theta}}(X_i, i)|, \quad \forall \ i,$$

for some functions $f_\theta : X \times \{1, \ldots, n\} \to \mathbb{R}, \theta \in \Theta$.

By the contraction inequality of Ledoux and Talagrand [1991],

$$E_n := \mathbb{E}\left( \left[ \sup_{\theta \in \Theta_M(\theta^*)} |Y^e(\theta, \theta^*)| \right] |X\right) \leq 2\mathbb{E}\left( \left[ \sup_{\theta \in \Theta_M(\theta^*)} |X^e(\theta, \theta^*)| \right] |X\right),$$

with

$$X^e(\theta, \theta^*) := P^e_n(f_\theta - f_{\theta^*}) := \frac{1}{n} \sum_{i=1}^n \varepsilon_i(f_\theta(X_i, i) - f_{\tilde{\theta}}(X_i, i)).$$

3.3 A deviation inequality

Write

$$R_n := \sup_{\theta \in \Theta_M(\theta^*)} \| \rho^e_{\theta} - \rho^e_{\theta^*} \|_n.$$

We have for all $t > 0$ (see Massart [2000a]),

$$\mathbb{P}\left( \left[ \sup_{\theta \in \Theta_M(\theta^*)} |Y^e(\theta, \theta^*)| \right] \geq E_n + R_n \sqrt{\frac{2t}{n}} \right) \leq \exp[-t].$$

Combining this with the symmetrization result of Section 3.1 we obtain the following corollary.
Corollary 3.1 Let for some $\bar{R}$
\[
\sup_{\theta \in \Theta_M(\theta^*)} \| \rho_\theta^c - \rho_{\theta^*}^c \| \leq \bar{R},
\]
and let $t \geq 4$. Then for any $\bar{E}$,
\[
\mathbb{P}\left( \left\lceil \sup_{\theta \in \Theta_M} |Y(\theta, \theta^*)| \right\rceil \geq 8\bar{E} + 4\bar{R}\sqrt{\frac{2t}{n}} \right) \leq 4\exp[-t] + 4\mathbb{P}(R_n > \bar{R} \lor E_n > \bar{E}).
\]

As for the random variables $E_n$ and $R_n$, in our context we use Condition 1.1. Consider first $R_n$. Condition 1.1 yields by the triangle inequality
\[
\sup_{\theta \in \Theta_M(\theta^*)} \| \rho_\theta^c - \rho_{\theta^*}^c \| \leq MK_n,
\]
where
\[
K_n := \max_{1 \leq j \leq p} \| \psi_j \|_n.
\]
Thus, on the set
\[
T_0 := \left\{ \max_{1 \leq j \leq p} \| \psi_j \|_n \leq \bar{K} \right\},
\]
(5)

(where $\bar{K}$ is some constant) we can bound the random radii $R_n$ by $MK_n$. We will see in Section 4 that a bound for the conditional expectation $E_n$ also only involves $K_n$:
\[
E_n \leq \lambda_0 MK_n,
\]
for some constant $\lambda_0 \asymp \sqrt{\log p/n}$.

In some cases (regression with fixed design) $K_n$ is not random, and the assumption $\max_{1 \leq j \leq p} \| \psi_j \|_n \leq \bar{K}$ is a matter of normalization. In other situations, one can for example apply Bernstein’s inequality (Bennet [1962]):

Lemma 3.2 Suppose that the $\psi_j(X_i)$ are uniformly sub-Gaussian, that is, for some positive constants $L$ and $\tau$, it holds for all $j$,
\[
\frac{2L^2}{n} \sum_{i=1}^{n} \mathbb{E}\left[ \exp[\psi_j^2(X_i)/L^2] - 1 \right] \leq \tau^2.
\]

Then for all $t > 0$,
\[
\mathbb{P}\left( \max_{1 \leq j \leq p} \| \psi_j \|_n^2 - \| \psi_j' \|_n^2 \right) \geq 2\tau L \sqrt{\frac{2(t + \log p)}{n} + \frac{2L(t + \log p)}{n}} \leq 2\exp[-t].
\]

Proof. The sub-Gaussianity implies that for all $m \in \{1, 2, 3, \ldots\}$,
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\psi_j^2(X_i)]^m/n \leq \frac{L^{2m}m!}{n} \sum_{i=1}^{n} \left[ \mathbb{E}\exp[\psi_j^2(X_i)/L^2] - 1 \right] \leq \frac{m!}{2} L^{2(m-1)} \tau^2.
\]
But then
\[ \frac{1}{n} \sum_{i=1}^{n} |E|\psi_{j}^{2}(X_{i}) - E|\psi_{j}^{2}(X_{i})| = \frac{m!}{2} 2^{m} L^{2(m-1)} \tau^{2}. \]

By Bernstein’s inequality (Bennet [1962]), for all \( t > 0 \),
\[ P \left( \left| (P_{n} - P)\psi_{j}^{2} \right| \geq 2 \tau L \sqrt{\frac{2t}{n} + \frac{2Lt}{n}} \right) \leq 2 \exp[-t], \]
and hence, by the union bound, for all \( t > 0 \),
\[ P \left( \max_{1 \leq j \leq p} \left| (P_{n} - P)\psi_{j}^{2} \right| \geq 2 \tau L \sqrt{\frac{2(t + \log p)}{n} + \frac{2L(t + \log p)}{n}} \right) \leq 2 \exp[-t]. \]

The assumption of sub-Gaussianity is not a necessary condition. One may replace it by an \( m \)-th order moment condition, with \( p^{2}/n^{m} \) sufficiently small. However, we then will no longer have exponential probability inequalities.

### 3.4 The peeling device

The peeling device goes back to Alexander [1985], the terminology being introduced in van de Geer [2000]. In the present context we can use it in the following form.

We show in the next section that under certain conditions
\[ E_{n} \leq \lambda_{0} MK_{n}, \tag{6} \]
where \( \lambda_{0} \asymp \sqrt{\log p/n} \), and \( K_{n} := \max_{1 \leq j \leq p} \|\psi_{j}\|_{n} \). Then, under Condition 1.1 and the sub-Gaussianity assumption of Lemma 3.2 we have for all \( M \), and all \( t > 0 \),
\[ P \left( \sup_{\theta \in \Theta_{M}(\theta^{*})} |Y(\theta, \theta^{*})| \geq \frac{\lambda_{*} M}{e} \left( 1 + K_{*} \left[ \sqrt{\frac{t}{\log p}} + \frac{t}{n} \right] \right) \right) \leq 6 \exp[-t]. \tag{7} \]

with \( K_{*} \) depending on \( L \) and \( \tau \) but not on \( n \) and \( p \), and \( \lambda_{*} \asymp \sqrt{\log p/n} \). This follows from (6) and from Subsection 3.3. The constant 6 in the right hand side of inequality (7) comes from a 4 from the symmetrization plus a 2 from Lemma 3.2 (we actually may replace 2 by 1 here because we only need a one-sided version).

Once (7) is established, we can invoke the peeling device as follows. Let \( \bar{M} \) be fixed, and let \( M_{j} := e^{-j} \bar{M}, j = 0, \ldots, p \). Then for all \( t > 0 \),
\[ P \left( \sup_{\theta \in \Theta_{M}(\theta^{*})} \frac{|Y(\theta, \theta^{*})|}{\|\theta - \theta^{*}\|_{1} \vee e^{-p-1} \bar{M}} \geq \frac{\lambda_{*} M_{j}}{e} \left( 1 + K_{*} \left[ \sqrt{\frac{t + \log p}{\log p}} + \frac{t + \log p}{n} \right] \right) \right) \leq \sum_{j=1}^{p} P \left( \sup_{\theta \in \Theta_{M_{j-1}}(\theta^{*})} |Y(\theta, \theta^{*})| > \lambda_{*} M_{j} \left( 1 + K_{*} \left[ \sqrt{\frac{t + \log p}{\log p}} + \frac{t + \log p}{n} \right] \right) \right) \leq 6 \exp[\log p - (\log p + t)] \leq 6 \exp[-t]. \]
4 Bounds for the symmetrized process

In the previous section we argued that the main task is to establish bounds for the expectation of the symmetrized process, i.e., for

$$\mathbb{E}\left(\sup_{\theta \in \Theta_M(\theta^*)} |Y^\varepsilon(\theta, \theta^*)| \bigg| X\right).$$

We are looking for bounds of the type (6). One can then derive deviation inequalities as shown in Section 3, and hence (as shown in Theorems 2.1 and 2.2) theoretical bounds for the tuning parameter of the $\ell_1$-regularized M-estimator.

4.1 Linear functions

Let us briefly recall the linear case. Let $[\rho_\theta(X_i) - c_{i,\theta}] - [\rho_{\tilde{\theta}}(X_i) - c_{i,\tilde{\theta}}]$ be linear:

$$[\rho_\theta(X_i) - c_{i,\theta}] - [\rho_{\tilde{\theta}}(X_i) - c_{i,\tilde{\theta}}] = \sum_{j=1}^{p} (\theta_j - \tilde{\theta}_j) \psi_j(X_i, i), \ i = 1, \ldots, n.$$

One then clearly has

$$\mathbb{E}\left(\sup_{\theta \in \Theta_M(\theta^*)} |Y^\varepsilon(\theta, \theta^*)| \bigg| X\right) \leq M \|\varepsilon^T \psi/n\|_\infty.$$

Moreover, by Hoeffding’s inequality \cite[see also Lemma 14.14 in B"uhlmann and van de Geer 2011]{Hoeffding1963},

$$\mathbb{E}\left(\|\varepsilon^T \psi/n\|_\infty \bigg| X\right) \leq \sqrt{\frac{2 \log(2p)}{n} K_n},$$

where $K_n := \max_{1 \leq j \leq p} \|\psi_j\|_n$.

4.2 Generalized linear functions

Suppose that for all $\theta, \tilde{\theta} \in \Theta_*$,

$$\left|\rho_\theta(X_i) - \mathbb{E}_{\rho_\theta}(X_i)\right| - \left|\rho_{\tilde{\theta}}(X_i) - \mathbb{E}_{\rho_{\tilde{\theta}}}(X_i)\right| \leq |f_\theta(X_i, i) - f_{\tilde{\theta}}(X_i, i)|, \ \forall i,$$

where $f_\theta(X_i, i) = \sum_{j=1}^{p} \theta_j \psi_j(X_i, i), \ \theta \in \Theta_*$. Then by the contraction inequality of Subsection 3.2 and the arguments of Subsection 4.1 for the linear case

$$\mathbb{E}\left(\sup_{\theta \in \Theta_M(\theta^*)} |Y^\varepsilon(\theta, \theta^*)| \bigg| X\right) \leq 2M \sqrt{\frac{2 \log(2p)}{n} K_n},$$

with $K_n := \max_{1 \leq j \leq p} \|\psi_j\|_n$.
4.3 Extended generalized linear functions

**Condition 4.1 (Extended GLM condition)** The exist non-negative functions \( \{\psi_{j,k} : j = 1, \ldots, p_k, \; k = 1, \ldots, r\} \) (with \( \sum_{k=1}^{r} p_k = p \)) such that for all \( \theta \) and \( \tilde{\theta} \) in \( \Theta^* \), it holds that

\[
||\rho_{\theta}(X_i) - c_{i,\theta}|| - ||\rho_{\tilde{\theta}}(X_i) - c_{i,\tilde{\theta}}|| \leq \sum_{k=1}^{r} \sum_{j=1}^{p_k} (\theta_{j,k} - \tilde{\theta}_{j,k})\psi_{j,k}(X_i)|i, i = 1, \ldots, n.
\]

**Theorem 4.1 (Multivariate contraction theorem)** Assume Condition 4.1. Let \( \xi_{1,k}, \ldots, \xi_{n,k}, k = 1, \ldots, r, \) be independent \( \mathcal{N}(0,1) \)-distributed random variables, independent of \( X_1, \ldots, X_n \). Let

\[
X_k(\theta, \theta^*) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{p_k} (\theta_{j,k} - \theta_{j,k}^{*})\psi_{j,k}(X_i)\xi_{i,k},
\]

and

\[
X(\theta, \theta^*) := \sum_{k=1}^{r} X_k(\theta, \theta^*) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{r} \sum_{j=1}^{p_k} (\theta_{j,k} - \theta_{j,k}^{*})\psi_{j,k}(X_i)\xi_{i,k}.
\]

Then for a universal constant \( C \),

\[
\mathbb{E} \left( \left[ \sup_{\theta \in \Theta_M(\theta^*)} |Y^\varepsilon(\theta, \theta^*)| \right] X \right) \leq C2^{r-1}\mathbb{E} \left( \left[ \sup_{\theta \in \Theta_M(\theta^*)} X(\theta, \theta^*) \right] X \right).
\]

**Proof.** We apply Theorem 2.1.1 in [Talagrand 2005], cited in the present paper as Theorem 5.1. Note first that

\[
\mathbb{E} \left( |X(\theta, \theta^*) - X(\tilde{\theta}, \theta^*)|^2 |X \right) = \sum_{k=1}^{r} \sum_{j=1}^{p_k} (\theta_{j,k} - \tilde{\theta}_{j,k})\psi_{j,k}^2 \leq \sum_{k=1}^{r} \sum_{j=1}^{p_k} (\theta_{j,k} - \tilde{\theta}_{j,k})\psi_{j,k}^2 \leq 2^{r-1} \sum_{k=1}^{r} \sum_{j=1}^{p_k} (\theta_{j,k} - \tilde{\theta}_{j,k})\psi_{j,k}^2.
\]

For all \( \theta \) and \( \tilde{\theta} \) we have

\[
||\rho_{\theta} - \rho_{\tilde{\theta}}||^2 \leq \sum_{k=1}^{r} \sum_{j=1}^{p_k} (\theta_{j,k} - \tilde{\theta}_{j,k})\psi_{j,k}^2 \leq 2^{r-1} \sum_{k=1}^{r} \sum_{j=1}^{p_k} (\theta_{j,k} - \tilde{\theta}_{j,k})\psi_{j,k}^2.
\]

By Hoeffding’s inequality [Hoeffding 1963]

\[
\mathbb{P} \left( |Y_{\varepsilon}(\theta, \theta^*) - Y_{\varepsilon}(\tilde{\theta}, \theta^*)| \geq ||\rho_{\theta} - \rho_{\tilde{\theta}}|| \sqrt{2t} \right) \leq 2 \exp[-t].
\]

Hence, using Theorem 2.1.5 in Talagrand’s book ([Talagrand 2005]) (see Section 5, Theorem 5.2), we get for a universal constant \( C \),

\[
\mathbb{E} \left( \left[ \sup_{\theta \in \Theta_M(\theta^*)} |Y^\varepsilon(\theta, \theta^*)| \right] X \right) \leq C2^{r-1}\mathbb{E} \left( \left[ \sup_{\theta \in \Theta_M(\theta^*)} X(\theta, \theta^*) \right] X \right).
\]

\( \square \)

As a direct consequence (i.e., by bounding the right hand side in Theorem 4.1), we obtain the bounds of interest for our problem.
Theorem 4.2  Assume Condition 4.1 and let $K_n := \max_{j,k} \|\psi_{j,k}\|_n$. We have for a universal constant $C$,

$$
\mathbb{E} \left( \left[ \sup_{\theta \in \Theta_M(\theta^\star)} |Y^\varepsilon(\theta, \theta^\star)| \right] \bigg| X \right) \leq C 2^{r-1} \sqrt{\frac{2 \log(2p)}{n}} K_n.
$$

Proof. Let $X(\theta, \theta^\star)$ be defined as in Theorem 4.1. As in Subsection 4.1, but now for Gaussians instead of a Rademacher sequence, conditionally on $X := (X_1, \ldots, X_n)$, we have

$$
\mathbb{E} \left( \left[ \sup_{\theta \in \Theta_M(\theta^\star)} X(\theta, \theta^\star) \right] \bigg| X \right) \leq M \sqrt{\frac{2 \log(2p)}{n}} K_n.
$$

\[\Box\]

4.4 Non-linear functions

We now consider the case where the loss $\rho_\theta$ is possibly not extended GLM, that is, its dependence on $\theta$ is strictly non-linear. However, we do assume that it is component-wise Lipschitz in $\theta$, i.e., that Condition 1.1 holds.

Define for $\psi = (\psi_1, \ldots, \psi_p)^T$,

$$
\Sigma_n := \frac{1}{n} \sum_{i=1}^n \psi(X_i, i)\psi^T(X_i, i).
$$

Let $\Lambda_n^2$ be the smallest eigenvalue of $\Sigma_n$ and $\bar{\Lambda}_n^2$ be its largest eigenvalue. We assume that $\Lambda_n > 0$, thus excluding the case $p > n$.

Theorem 4.3  Assume Condition 1.1. For a universal constant $C$, it holds that

$$
\mathbb{E} \left( \left[ \sup_{\theta \in \Theta_M(\theta^\star)} |Y^\varepsilon(\theta, \theta^\star)| \right] \bigg| X \right) \leq C M \sqrt{\frac{2 \log(2p)}{n}} \left( \Lambda_n / \bar{\Lambda}_n \right).
$$

Proof. Use that

$$
\| \sum_{j=1}^k \theta_j \psi_j \|_n^2 \geq \Lambda_n^2 \|\theta\|_2^2,
$$

and

$$
\sum_{j=1}^k |\theta_j| \psi_j \|_n^2 \leq \bar{\Lambda}_n^2 \|\theta\|_2^2 = \|\theta\|_2^4.
$$

Then apply the same arguments as in Theorem 4.2. \[\Box\]
5 The geometry of $\ell_1$-balls

We first describe here the generic chaining bound, specialized to our context and with a notation adjusted to our setting. Let $\xi_1, \ldots, \xi_n$ be independent $\mathcal{N}(0,1)$-distributed random variables and $\mathcal{V}$ be a subset of $\mathbb{R}^n$. Define

$$X_v := \frac{1}{n} \sum_{i=1}^{n} v_i \xi_i, \ v \in \mathcal{V}.$$ 

Moreover, write

$$\|v\|_n^2 := \frac{1}{n} \sum_{i=1}^{n} v_i^2, \ v \in \mathbb{R}^n.$$ 

Talagrand ([Talagrand 2005], Definition 1.2.3) calls a sequence of partitions $\{A_s\}_{s=0}^{\infty}$ of $\mathcal{V}$ admissible if it is an increasing sequence (i.e., $A_{s+1}$ contains $A_s$ for all $s \geq 1$), and $|A_s| \leq 2^{2s}$ for all $s$. He defines for each $v \in \mathcal{V}$ and each $s$, the set $A_s(v)$ as the unique element of $A_s$ that contains $v$, and $\Delta(A_s(v))$ as the diameter of $A_s(v)$. He writes

$$\gamma_2(\mathcal{V}, \| \cdot \|_n) := \inf \sup_{v \in \mathcal{V}} \sum_{s \geq 0} 2^{s/2} \Delta(A_s(v)),$$

where the infimum is taken over all admissible partitions.

**Theorem 5.1** (The majorizing measure theorem, see [Talagrand 2005], Theorem 2.1.1) For some universal constant $C$, we have

$$\frac{1}{C} \gamma_2(\mathcal{V}, \| \cdot \|_n) \leq \mathbb{E} \left[ \sup_{v \in \mathcal{V}} X_v \right] \leq C \gamma_2(\mathcal{V}, \| \cdot \|_n).$$

Talagrand derives the lower bound in the above theorem from Sudakov’s minoration argument. As a consequence, Talagrand presents the following result.

**Theorem 5.2** ([Talagrand 2005], Theorem 2.1.5) Let $\{Y_v : v \in \mathcal{V}\}$ be a stochastic process that satisfies for all $t > 0$

$$\mathbb{P} \left( |Y_v - Y_{\tilde{v}}| \geq \sqrt{t} \right) \leq 2 \exp \left[ -\frac{t}{\|v - \tilde{v}\|_n^2} \right], \ \forall \ v, \tilde{v} \in \mathcal{V}.$$ 

Then for a universal constant $C$, we have

$$\mathbb{E} \left[ \sup_{v, \tilde{v} \in \mathcal{V}} |Y_v - Y_{\tilde{v}}| \right] \leq C \mathbb{E} \left[ \sup_{v, \tilde{v} \in \mathcal{V}} |X_v - X_{\tilde{v}}| \right].$$

Let us compare here the situation with Dudley’s entropy bound ([Dudley 1967]). We formulate it using chaining along a tree, as in [Bühlmann and van de Geer 2011], Subsection 14.12.4, or [van de Geer and Lederer 2011]. Define $R_n := \sup_{v \in \mathcal{V}} \|v\|_n$. Let for each $s \in \{0, 1, \ldots, S\}$, $\{v_j^s\}_{j=1}^{N_s} \subset \mathcal{V}$ be a minimal $2^{-s}R_n$-covering set of $\mathcal{V}$, that is, for all $v \in \mathcal{V}$ and all $s$ there is a $v_j^s$ such that
∥v − v_s∥_n ≤ 2^{−s} R_n. Then for all v, we can find an end node v^S ∈ \{v^S\} such that ∥v − v^S∥_n ≤ 2^{S} R_n, and for each end node v^S ∈ \{v^S\} one can find a branch \{v^0, \ldots, v^S\} such that ∥v^s − v^{s−1}∥_n ≤ 2^{−s−1} R_n for all s = 1, \ldots, S. Moreover, we can write (with X_{v,0} = 0)

\[ X_v = \sum_{s=0}^{S} (X_{v^s} − X_{v^{s−1}}) + X_v − X_{v^S}. \]

Invoking

\[ |X_v − X_{v^S}| ≤ 2^{−S} R_n \sqrt{\sum_{i=1}^{n} \xi_i^2 / n}, \]

one arrives at Dudley’s bound

\[ \mathbb{E}\left[ \sup_{v \in \mathcal{V}} X_v \right] \leq \sum_{s=0}^{S} 2^{−(s−1)} R_n \sqrt{\frac{2 \log(2N_s)}{n}} + 2^{−S} R_n. \] (8)

Consider now a special case. We let \{ψ_j\}_{j=1}^p be p vectors in \mathbb{R}^n, and let

\[ \mathcal{V} := \{ \sum_{j=1}^{p} \theta_j \psi_j : ∥\theta∥_1 ≤ 1 \}. \]

Let \( K_n := \max_{1 ≤ j ≤ p} ∥\psi_j∥_n. \)

The following lemma rephrases the first part of Theorem 2.1.6 in [Talagrand 2005]. We present a short proof to show that it is again based on the dual norm inequality (1).

**Lemma 5.1** It holds for some universal constant \( C \) that

\[ \gamma_2(\mathcal{V}, ∥·∥_n) ≤ C \sqrt{\frac{2 \log(2p)}{n}} K_n. \]

**Indirect Proof.** Clearly, by the dual norm inequality

\[ \sup_{v \in \mathcal{V}} X_v = \sup_{∥\theta∥_1 ≤ 1} \frac{1}{n} ∑_{i=1}^{n} ∑_{j=1}^{p} \theta_j \psi_{i,j} \xi_i = \max_{1 ≤ j ≤ p} \left| \frac{1}{n} ∑_{i=1}^{n} \psi_{i,j} \xi_i \right|. \]

Hence,

\[ \mathbb{E}\left[ \sup_{v \in \mathcal{V}} X_v \right] ≤ \mathbb{E}\max_{1 ≤ j ≤ p} \left| \frac{1}{n} ∑_{i=1}^{n} \psi_{i,j} \xi_i \right| ≤ \sqrt{\frac{2 \log(2p)}{n}} K_n. \]

The result now follows from Theorem 5.1.\[\square\]

In his book, Talagrand now poses the research question to prove Lemma 5.1 directly [Talagrand 2005, Research problem 2.1.9]. We claim that this cannot
be done by applying Dudley’s bound. Our reasoning is as follows. Using Theorem 6.2 in Pollard [1990] (see also van der Vaart and Wellner [1996], Lemma 2.6.11, or Bühlmann and van de Geer [2011], Lemma 14.28), we see that
\[
\log(2N_s) \leq 2^{2s} \log(4p), \forall s.
\] (9)
Insert this in (8) with the bound
\[
R_n \leq Kn,
\]
to find that
\[
\mathbb{E} \left[ \sup_{v \in V} X_v \right] \leq 2(S + 1)Kn \sqrt{\frac{2\log(4p)}{n}} + 2^{-S}Kn.
\]
Minimizing this over $S$ gives a bound of order $(\log n)\sqrt{\log p/n}Kn$. In other words (assuming the entropy bound (9) is up to constants tight, which we believe it is) invoking Dudley’s bound instead of generic chaining leads to a redundant $(\log n)$-factor. Apparently, Dudley’s bound does not fully capture the geometry of $\ell_1$-balls.

6 Concluding remarks

This paper combines results in literature concerning symmetrization, contraction, deviation inequalities and chaining. Their application in statistical theory has been highlighted by Massart [2000b]. We have added now a new application, where generic chaining allows one to remove additional $\log n$ factors. For example, we have improved the choice $\lambda \asymp \sqrt{\log^3 n \log(p \vee n)/n}$ in Städler and van de Geer [2010] to $\lambda \asymp \sqrt{\log p/n}$. The geometric arguments to bound $\gamma_2$ in the case of convex hulls are still to be developed. Somehow, the generic chaining bound $\gamma_2$ better exploits the impossibility to play cat and mouse.

7 Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. The Basic Inequality says that
\[
\mathcal{E}(\hat{\theta}; \theta_0) + \lambda \|\hat{\theta}\|_1 \leq Y(\hat{\theta}, \theta^*) + \lambda \|\theta^*\|_1 + \mathcal{E}(\theta^*; \theta_0).
\]
Hence on $\mathcal{T}(\theta^*)$,
\[
\mathcal{E}(\hat{\theta}; \theta_0) + \lambda \|\hat{\theta}\|_1 \leq \lambda_0 \|\hat{\theta} - \theta^*\|_1 \vee \lambda^2 + \lambda \|\theta^*\|_1 + \mathcal{E}(\theta^*; \theta_0).
\]
If $\|\hat{\theta} - \theta^*\|_1 \leq \lambda_0$, we get
\[
\mathcal{E}(\hat{\theta}; \theta^0) + (\lambda - \lambda_0) \|\hat{\theta} - \theta^*\|_1 \leq 2\lambda^2 + \mathcal{E}(\theta^*; \theta^0).
\]
Hence in the rest of the proof, we can assume $\|\hat{\theta} - \theta^*\|_1 \geq \lambda_0$. 

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For $\theta^* = \theta^0$, we get
\[
E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta}_S^0\|_1 \leq (\lambda + \lambda_0)\|\hat{\theta}_S - \theta^0\|_1,
\]
which gives for any $0 < \delta < 1$,
\[
E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta} - \theta^0\|_1 \leq 2\lambda\|\hat{\theta}_S - \theta^0\|_1
\leq 2\lambda \Gamma(L, S_0) \tau(\hat{\theta} - \theta^0)
\leq \delta E(\hat{\theta}; \theta^0) + \delta H\left(\frac{2\lambda \Gamma(L, S_0)}{\delta}\right).
\]

For general $\theta^*$, we get
\[
E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta}_S^0\|_1 \leq (\lambda + \lambda_0)\|\hat{\theta}_S - \theta^*\|_1 + E(\theta^*; \theta_0).
\]
If $$(\lambda + \lambda_0)\|\hat{\theta}_S - \theta^*\|_1 \leq \delta E(\theta^*; \theta_0),$$
we obtain
\[
E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta}_S^0\|_1 \leq (1 + \delta) E(\theta^*; \theta_0).
\]
And then, using $\lambda - \lambda_0 \leq \lambda + \lambda_0$,
\[
E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta} - \theta^*\|_1 \leq (1 + 2\delta) E(\theta^*; \theta_0).
\]
If $(\lambda + \lambda_0)\|\hat{\theta}_S - \theta^*\|_1 \geq \delta E(\theta^*; \theta_0)$, we obtain
\[
E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta}_S^0\|_1 \leq \frac{1 + \delta}{\delta} (\lambda + \lambda_0)\|\hat{\theta}_S - \theta^*\|_1,
\]
and hence
\[
E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta} - \theta^*\|_1 \leq \frac{1 + 2\delta}{\delta} (\lambda + \lambda_0)\|\hat{\theta}_S - \theta^*\|_1,
\]
\[
\leq \frac{1 + 2\delta}{\delta} (\lambda + \lambda_0) \Gamma(L, S_0) \tau(\hat{\theta} - \theta^*) + E(\theta^*; \theta^0)
\leq 4\delta H\left(\frac{(1 + 2\delta)(\lambda + \lambda_0) \Gamma(L, S_0)}{2\delta^2}\right) + \delta E(\hat{\theta}; \theta^0) + (1 + \delta) E(\theta^*; \theta^0).
\]
It follows that
\[
(1 - 2\delta) E(\hat{\theta}; \theta_0) + (\lambda - \lambda_0)\|\hat{\theta} - \theta^*\|_1 \leq 4\delta H\left(\frac{(1 + 2\delta)(\lambda + \lambda_0) \Gamma(L, S_0)}{2\delta^2}\right)
\]
\[
+ (1 + 2\delta) E(\theta^*; \theta_0).
\]
Finally simplify the expression using $\lambda + \lambda_0 \leq 2\lambda$, and replacing $2\delta$ by $\delta$.

$$\square$$

**Proof of Theorem 2.2.** We only describe the case $\theta^* = \theta^0$, the case $\theta^* \neq \theta^0$
following by the same arguments. Repeat the proof of Theorem 2.1 with $\hat{\theta}$ replaced by $\hat{\theta} := t\hat{\theta} + (1 - t)\theta^0$, where
\[
t := \frac{2M_0}{2M_0 + \|\hat{\theta} - \theta^0\|_1}.
\]
Note that $\|\hat{\theta} - \theta^0\|_1 \leq 2M_0$. By the proof of Theorem 2.1, we obtain that actually $\|\hat{\theta} - \theta^0\|_1 \leq M_0$ on $T_{2M_0}(\theta_0)$. But this implies $\|\hat{\theta} - \theta_0\|_1 \leq 2M_0$. Now, repeat the proof again, knowing that $\|\hat{\theta} - \theta^0\|_1 \leq 2M_0$.

$$\square$$
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