On Universal Eigenvalues of Casimir Operator

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Abstract. Motivated by the universal knot polynomials in the gauge Chern-Simons theory, we show that the values of the second Casimir operator on an arbitrary power of Cartan product of $X_2$ and adjoint representations of simple Lie algebras can be represented in a universal form. We show that it complies with $N \rightarrow -N$ duality of the same operator for $SO(2n)$ and $Sp(2n)$ algebras (the part of $N \leftrightarrow -N$ duality of gauge $SO(2n)$ and $Sp(2n)$ theories). We discuss the phenomena of non-zero universal values of Casimir operator on zero representations.

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1 Introduction

The universal formulae for simple Lie algebras were first derived by P. Vogel in his Universal Lie Algebra [1, 2]. The main aim was to derive the most general weight system for Vassiliev's finite knot invariants. This program met difficulties, however, as a byproduct there appeared the uniform parameterization of simple Lie algebras by the values of Casimir operators on three representations, appearing in decomposition of the symmetric square of the adjoint representations:

\[ S^2 \mathfrak{g} = 1 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma) \]  

(1)

One denotes the value of the second Casimir operator on the adjoint representation \( \mathfrak{g} \) as \( 2t \), and parameterizes the values of the same operator on representations in (1) as \( 4t - 2\alpha, 4t - 2\beta, 4t - 2\gamma \) correspondingly (hence notation of representations in \([1]\)). It appears that \( \alpha + \beta + \gamma = t \). The values of the parameters for all simple Lie algebras are given in the table 1, and in the table 2 in another form. According to the definitions, the entire theory is invariant with respect to rescaling of the parameters (which corresponds to rescaling of invariant scalar product in algebra), and with respect to the permutation of the universal (=Vogel’s) parameters \( \alpha, \beta, \gamma \). So, effectively they belong to a projective plane, which is factorized w.r.t. its homogeneous coordinates, and is called Vogel’s plane.

| Root system | Lie algebra | \( \alpha \) | \( \beta \) | \( \gamma \) | \( t \) |
|-------------|-------------|-------------|-------------|-------------|-------------|
| \( A_n \)  | \( \mathfrak{sl}_{n+1} \) | \(-2\)       | \(2\)       | \(n+1\)     | \(n+1\)     |
| \( B_n \)  | \( \mathfrak{so}_{2n+1} \) | \(-2\)       | \(4\)       | \(2n-3\)    | \(2n-1\)    |
| \( C_n \)  | \( \mathfrak{sp}_{2n} \)   | \(-2\)       | \(1\)       | \(n+2\)     | \(n+1\)     |
| \( D_n \)  | \( \mathfrak{so}_{2n} \)   | \(-2\)       | \(4\)       | \(2n-4\)    | \(2n-2\)    |
| \( G_2 \)  | \( \mathfrak{g}_2 \)       | \(-2\)       | \(10/3\)    | \(8/3\)     | \(4\)       |
| \( F_4 \)  | \( \mathfrak{f}_4 \)       | \(-2\)       | \(5\)       | \(6\)       | \(9\)       |
| \( E_6 \)  | \( \mathfrak{e}_6 \)       | \(-2\)       | \(6\)       | \(8\)       | \(12\)      |
| \( E_7 \)  | \( \mathfrak{e}_7 \)       | \(-2\)       | \(8\)       | \(12\)      | \(18\)      |
| \( E_8 \)  | \( \mathfrak{e}_8 \)       | \(-2\)       | \(12\)      | \(20\)      | \(30\)      |

Table 1: Vogel’s parameters for simple Lie algebras

| Algebra/Parameters | \( \alpha \) | \( \beta \) | \( \gamma \) | \( t \) | Line |
|--------------------|-------------|-------------|-------------|-------------|-------------|
| \( \mathfrak{sl}_N \) | \(-2\)       | \(2\)       | \(N\)       | \(N\)       | \(\alpha + \beta = 0\) |
| \( \mathfrak{so}_N \) | \(-2\)       | \(4\)       | \(N-4\)     | \(N-2\)     | \(2\alpha + \beta = 0\) |
| \( \mathfrak{sp}_N \) | \(-2\)       | \(1\)       | \(N/2 + 2\) | \(N/2 + 1\) | \(\alpha + 2\beta = 0\) |
| \( \text{Exc}(n) \) | \(-2\)       | \(2n + 4\) | \(n + 4\)   | \(3n + 6\)  | \(\gamma = 2(\alpha + \beta)\) |

For the exceptional line \( n = -2/3, 0, 1, 2, 4, 8 \) for \( \mathfrak{g}_2, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8 \), respectively.

Table 2: Vogel’s parameters for simple Lie algebras: lines
As an example of application of this parametrization universal formulae [1, 3] for dimensions of representations from (1) are presented below:

\[
\dim g = \frac{(2t - \alpha)(2t - \beta)(2t - \gamma)}{\alpha\beta\gamma} \tag{2}
\]

\[
\dim Y_2(\alpha) = \frac{(2t - 3\alpha)(\beta - 2t)(\gamma - 2t)t(\beta + t)(\gamma + t)}{\alpha^2(\alpha - \beta)(\beta - \alpha)(\alpha - \gamma)\gamma} \tag{3}
\]

and other two representations which are obtained by permutations of the parameters. These are typical universal formulae for dimensions: ratios of products of linear homogeneous functions of universal parameters.

There are a number of universal formulae for different objects in the theory and applications of simple Lie algebras. E.g. Vogel [1] found complete decomposition of third power of the adjoint representation in terms of Universal Lie Algebra, defined by himself, and universal dimension formulae for all representations involved. Landsberg and Manivel [3] present a method which allows derivation of certain universal dimension formulae for simple Lie algebras and derive those for Cartan powers of the adjoint, \(Y_2(\cdot)\), and their Cartan products. Sergeev, Veselov and Mkrtchyan derived [4] a universal formula for generating function for the eigenvalues of higher Casimir operators on the adjoint representation.

In subsequent works applications to physics were developed, particularly the universality of the partition function of Chern-Simons theory on a sphere [18, 3, 7], and its connection with q-dimension of \(k\Lambda_0\) representation of affine Kac-Moody algebras [8] were shown.

Moreover, the universal knot polynomials for 2- and 3-strand torus knots [9, 10, 11, 12] were calculated. This is a partial realization of initial Vogel’s program.

The main motivation for present paper is the extension of construction of invariant knot polynomials of [10] to higher-strand torus knots. The construction of [10] was based on the Rosso-Jones formula [13]

\[
P_{R}^{[m,n]}(q) = q^{\mu_n^{\mu_R}} \sum_{Q} q^{-\mu_Q} \varphi_Y(\sigma^{[m,n]}) D_Q(q) \tag{4}
\]

Two main ingredients of this formula are eigenvalues \(\lambda_Q\) of second Casimir operators on representation \(Q\) and quantum dimensions \(D_Q(q)\) of the same representation. \(Q\) is one of the irreps, appearing in the decomposition of the \(m\)-th power of representation \(R\) with symmetry of a young diagram \(Y\) with \(m\) boxes, appearing in the decomposition of powers of the adjoint representation. If one wants to get a universal answer, he has to present these two quantities in the universal form. The other elements of the formula such as the character of the symmetric group \(\varphi_Y(\cdot)\) are ”universal” in the sense that they do not depend on group, and we shall not consider them here.

We do not consider the universal form of quantum dimensions, and in this paper will focus on the first quantity - Casimir eigenvalues. We present a universal expression for its eigenvalues on certain irreps described below, and appearing in Rosso-Jones formula for the case when \(R\) is the adjoint representation.

The antisymmetric square of the adjoint representation of semisimple Lie algebras is known to be decomposed in the following universal form:

\[
\Lambda^2 g = g \oplus X_2
\]

First of all, let’s suppose that for each algebra the square of the long root is equal to 2. This corresponds to the set of Vogel’s parameters with \(\alpha = -2\). Having this normalization in mind, we calculate the eigenvalues of the Casimir operator on arbitrary powers of Cartan product of \(X_2\) and \(g\). The eigenvalue of the Casimir operator on an irrep with \(\lambda\) highest weight is equal to \((\lambda + 2\rho, \lambda)\). The corresponding weights are given in the Table 3, where the same labeling of Dynkin diagrams as in [14] is used.

\[\text{Note, that for } G_2 \text{ algebra the } \omega_2 \text{ weight corresponds to the long root.}\]
Table 3: Highest weights of $X_2$ and $g$ representations

|    | $\lambda_{X_2}$                          | $\lambda_g$ |
|----|------------------------------------------|-------------|
| $A_n, n \geq 3$ | $(2\omega_1 + \omega_{n-1}) \oplus (\omega_2 + 2\omega_N)$ | $\omega_1 + \omega_N$ |
| $B_n, n \geq 4$ | $\omega_1 + \omega_3$ | $\omega_2$ |
| $C_n, n \geq 3$ | $2\omega_1 + \omega_2$ | $2\omega_1$ |
| $D_n, n \geq 5$ | $\omega_1 + \omega_3$ | $\omega_2$ |
| $G_2$ | $3\omega_1$ | $\omega_2$ |
| $F_4$ | $\omega_2$ | $\omega_3$ |
| $E_6$ | $\omega_3$ | $\omega_6$ |
| $E_7$ | $\omega_2$ | $\omega_1$ |
| $E_8$ | $\omega_6$ | $\omega_7$ |
2 Universal Casimir Eigenvalues for $X_2$ Representation

Taking into account the above mentioned data, we carry out the direct calculations for $X_2$ first.

$$A_N$$

$$\alpha = -2, \beta = 2, \gamma = N + 1, \lambda_1 = 2\omega_1 + \omega_{N-1}, \lambda_2 = \omega_2 + 2\omega_N,$$

For $\lambda_1 = 2\omega_1 + \omega_{N-1}$ case

$$C = (2\omega_1 + \omega_{N-1}, 2\omega_1 + \omega_{N-1}) + 2(\omega_1 + \cdots + \omega_N, 2\omega_1 + \omega_{N-1}) =$$

$$\frac{6N + 6}{N + 1} + \frac{2}{N + 1} (N(N + 1) + (N - 1)(N + 1)) = 6 + 2N + 2N - 2 = 4N + 4$$

For $\lambda_2 = \omega_2 + 2\omega_N$ irrep the Casimir eigenvalue coincides with the one calculated above, so the eigenvalue on the direct sum of these two irreps will be $C$.

$$B_N$$

$$\alpha = -2, \beta = 4, \gamma = 2N - 3, \lambda = \omega_1 + \omega_3$$

$$C = (\omega_1 + \omega_3, \omega_1 + \omega_3) + 2(\omega_1 + \cdots + \omega_N, \omega_1 + \omega_3) =$$

$$F_{11} + F_{31} + F_{13} + F_{33} + 2((F_{11} + F_{12} + \cdots + F_{1N}) + F_{31} + \cdots + F_{3N}) =$$

$$6 + 2N - 1 + 9 + 6(N - 3) = 2N + 14 + 6N - 18 = 8N - 4$$

Where $F_{i,k} = (\omega_i, \omega_k)$.

$$C_N$$

$$\alpha = -2, \beta = 1, \gamma = N + 2, \lambda = 2\omega_1 + \omega_2$$

$$C = (2\omega_1 + \omega_2, 2\omega_1 + \omega_2) + 2(\omega_1 + \cdots + \omega_N, 2\omega_1 + \omega_2) =$$

$$4F_{11} + 4F_{12} + F_{22} + F_{33} + 2(2(F_{11} + F_{12} + \cdots + F_{1N}) + F_{21} + \cdots + F_{2N}) =$$

$$2 + 2 + 1 + 2(N + 1/2 + N - 1) = 5 + 2N + 1 + 2N - 2 = 4N + 4$$

$$D_N$$

$$\alpha = -2, \beta = 4, \gamma = 2N - 4, \lambda = \omega_1 + \omega_3$$

$$C = (\omega_1 + \omega_3, \omega_1 + \omega_3) + 2(\omega_1 + \cdots + \omega_N, \omega_1 + \omega_3) = 6 + 2(N - 1 + 6 + 3N - 12) = 8N - 8$$

$$G_2$$

$$\alpha = -2, \beta = 10/3, \gamma = 8/3, \lambda = 3\omega_1$$

$$C = (3\omega_1, 3\omega_1) + 2(\omega_1 + \omega_2, 3\omega_1) = 9F_{11} + 2(3F_{11} + 3F_{12}) = 6 + 4 + 6 = 16$$
\[ F_4 \]
\[
\alpha = -2, \beta = 5, \gamma = 6, \lambda = \omega_2
\]
\[ C = (\omega_2, \omega_2) + 2(\omega_1 + \omega_2 + \omega_3 + \omega_4, \omega_2) = 6 + 30 = 36 \]

\[ E_6 \]
\[
\alpha = -2, \beta = 6, \gamma = 8, \lambda = \omega_3
\]
\[ C = F_{33} + 2 \sum_{k=1}^{6} F_{3k} = 6 + 2(6 + 15) = 48 \]

\[ E_7 \]
\[
\alpha = -2, \beta = 8, \gamma = 12, \lambda = \omega_2
\]
\[ C = F_{22} + 2 \sum_{k=1}^{6} F_{2k} = 6 + 2(9 + 14 + 10) = 72 \]

\[ E_8 \]
\[
\alpha = -2, \beta = 12, \gamma = 20, \lambda = \omega_6
\]
\[ C = F_{66} + 2 \sum_{k=1}^{6} F_{6k} = 6 + 2 \cdot 57 = 120 \]

It can be easily noticed, that for each of the algebra the obtained value can be expressed as \[ C = 4t = 4(\alpha + \beta + \gamma). \]

In the work of M. Cohen and R. de Man ([15]) the Casimir eigenvalues on each of the irrep appearing in the decomposition of up to 4th power of the adjoint representation for the exceptional Lie algebras have been computed. So we can check the correspondence between our formula and their calculated one for the exceptional algebras. If we scale the Casimir eigenvalue to be equal to 1 on the adjoint representation, as it is done in [15], our formula will be \[ C_c = C/2t = 4t/2t = 2, \]
which coincides with the value, presented in that work. Below we shall make similar check for other representations, also.
3 Universal Casimir Eigenvalues on Cartan Product of Powers of $X_2$ and $g$

Now we turn to the Cartan product case. The highest weights are now $k\lambda_{X_2}$ and $n\lambda_g$ correspondingly. It is easy to check, that

$$C_{k,n} = C_{k\lambda_{X_2}} + C_{n\lambda_g} + 2kn(\lambda_{X_2}, \lambda_g)$$

Substituting the highest weights in the expression, written above for the Casimir eigenvalue, one obtains the expressions shown in the following table:

| $A_N, N \geq 3$ | $B_N, N \geq 4$ | $C_N, N \geq 3$ | $D_N, N \geq 5$ |
|------------------|------------------|------------------|------------------|
| $A_N, N \geq 3$ | $6k^2 + k(4N - 2)$ | $6kn$ | $6k^2 + k(4N - 2) + 2n(n + N) + 6kn$ |
| $B_N, N \geq 4$ | $6k^2 + k(8N - 10)$ | $2n(n + 2N - 2)$ | $6kn$ | $6k^2 + k(8N - 10) + 2n(n + 2N - 2) + 6kn$ |
| $C_N, N \geq 3$ | $5k^2 + k(4N - 1)$ | $2n(n + N)$ | $6kn$ | $5k^2 + k(4N - 1) + 2n(n + N) + 6kn$ |
| $D_N, N \geq 5$ | $6k^2 + k(8N - 14)$ | $2n(n + 2N - 3)$ | $6kn$ | $6k^2 + k(8N - 14) + 2n(n + 2N - 3) + 6kn$ |

For $G_2$

| $G_2$ | $6k^2 + 10k$ | $2n(n + 3)$ | $6kn$ | $6k^2 + 10k + 2n(n + 3) + 6kn$ |

Universal Form

| $3\alpha(k - k^2) + 4tk$ | $\alpha(n - n^2) + 2tn$ | $-3\alpha kn$ | $\alpha(3k - 3k^2 + n - n^2 - 3kn) + t(4k + 2n)$ |

One can easily check, that for each of these cases (except for the $C_N$) the $C_{k\lambda_{X_2}}$ eigenvalue can be expressed as

$$C_{k\lambda_{X_2}} = -3\alpha k^2 + (4t + 3\alpha)k = 3\alpha(k - k^2) + 4tk,$$

for $C_{n\lambda_g}$

$$C_{n\lambda_g} = -\alpha n^2 + 2n(t - 1) = \alpha(n - n^2) + 2tn$$

and

$$2kn(\lambda_{X_2}, \lambda_g) = -3k\alpha n$$

Notice, that when $k = 1$, $C_X = 4t$ and when $n = 1$, $C_g = 2t$, as expected.

Finally, the universal formula for the Casimir eigenvalues on the Cartan powers of $X_2$ and $g$ representations is

$$C_{k,n} = 3\alpha(k - k^2) + 4tk + \alpha(n - n^2) + 2tn = \alpha(3k - 3k^2 + n - n^2 - 3kn) + t(4k + 2n).$$

3.1 Conformity Check

Now we turn to the comparison of our universal expression with the values presented in [15]. The representations on which the Casimir eigenvalues are to be compared are those defined with the following highest weights: $2\lambda_{X_2}, \lambda_{X_2} + \lambda_g$ and $\lambda_{X_2} + 2\lambda_g$. So, we calculate $\gamma(H), \gamma(C), \gamma(G)$ (i.e. Casimirs in [15] notation) and compare them with $C_{2,0}, C_{1,1}, C_{1,2}$, written in the corresponding scaling.

For the $k = 2$ and $n = 0$ case our formula in the corresponding scaling gives:

$$C_{2,0} = \frac{3\alpha(k - k^2) + 4tk}{2t} = \frac{-3\alpha + 4t}{t} = \frac{6 + 4t}{t}.$$ 

For $k = 1, n = 1$

$$C_{1,1} = \frac{6t - 3\alpha}{2t} = \frac{3(t - 1)}{t}$$

Finally, for $k = 1, n = 2$

$$C_{1,2} = \frac{-8\alpha + 8t}{2t} = \frac{8 + 4t}{t}$$

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In the following table the corresponding Casimir eigenvalues calculated in [15] and those obtained by our formula are shown.

| $a$  | $\gamma(H) = 4 + 6a$ | $\gamma(C) = 3 + 3a$ | $\gamma(G) = 4 + 8a$ | $t$ | $C_{2,0} = (6 + 4t)/t$ | $C_{1,1} = 3(t-1)/t$ | $C_{1,2} = (8 + 4t)/t$ |
|------|------------------|------------------|------------------|---|-----------------|-----------------|-----------------|
| $A_1$ | 1/2             | 7               | 9/2              | 8 | 2               | 7               | 9/2             | 8               |
| $A_2$ | 1/3             | 6               | 20/3             | 3 | 6               | 4               | 20/3            |
| $G_2$ | 1/4             | 11/2            | 15/4             | 8 | 4               | 11/2            | 15/4            |
| $D_4$ | 1/6             | 5               | 7/2              | 16/3 | 6 | 5               | 7/2             | 16/3            |
| $F_4$ | 1/9             | 14/3            | 10/3             | 14/3 | 9 | 14/3            | 10/3            |
| $E_6$ | 1/12            | 9/2             | 13/4             | 14/3 | 12 | 9/2             | 13/4            |
| $E_7$ | 1/18            | 13/3            | 19/6             | 40/9 | 18 | 13/3            | 19/6            |
| $E_8$ | 1/30            | 21/5            | 31/10            | 64/15 | 30 | 21/5            | 31/10           |

Thus, we see that the Casimir eigenvalues coincide.

4 Non-zero Universal Values of Casimir on Zero Representations

In the recent work of M. Avetisyan and R. Mkrtchyan ([16]) a universal formula for dimensions of the $k$-th Cartan power of the $X_2$ representation has been obtained. A notable quality of the $X_2(k, \alpha, \beta, \gamma)$ formula is that for the parameters, corresponding to the $C_N$ algebra it gives 0 for any $k \geq 2$, while we see in the Section 2 that the Casimir eigenvalues on those irreps are not 0.

A similar situation regarding $A_2$ algebra takes place. The universal decomposition of the symmetric square of the adjoint representation writes as follows:

$$S^2g = 1 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma)$$

The $Y_2(\beta)$ for $A_2$ is 0, whilst the Casimir eigenvalue on the same representation is $4t - 2\beta$. At first glance it seems natural to expect, that the Casimir eigenvalues on that representations should be equal to 0, while we see, that they are not. If one thinks deeper, it is easy to understand, that the Casimir eigenvalue does not have to be equal to 0 on a zero-dimensional representation. Indeed, for the points close to the (-2,2,3) on the Vogel plane the Casimir operator acting on the symmetric square of the adjoint representation of $A_2$ has three eigenvalues, so in an appropriate basis it has a block-diagonal form. At (-2,2,3) point all that happens is that $Y_2(\gamma)$ becomes zero for that particular combination of parameters, and the corresponding block of the Casimir operator acts on a zero-dimensional vector subspace. Thus we do not see anything that dictates that block to be a zero-matrice at that particular point.

After the discussion of this situation one concludes, that the universal description sheds a light on the fact, that it is not just only reasonable, but turns out to be necessary to consider some non-zero eigenvalues of Casimir operators on non-existing, i.e. zero-dimensional representations. Thus, it seems natural to believe, that the universal formulae "take care" of the "invisibility" of that sort of Casimirs. In other words, we expect that in the universal formulae the Casimir eigenvalues appear in the product with the universal dimensions, or, more generally, with expressions, which are necessarily zero, if the dimension is zero.

In support of this idea we bring a formula, presented by Deligne in [17]:

$$\text{Tr}(C_2, [R]|V) = \frac{1}{n!} \sum_{\sigma} \chi(\sigma)m(\sigma)(\text{dim}V)^{n(\sigma)}^{-1}\text{Tr}(C_2, V)$$

where $V$ is a representation of the algebra, $R$ is a representation of the $S_n$ group, $[R]|V := \text{Hom}_{S_n}(R, \otimes^n V)$, $\sigma$ is an element of $S_n$, $\chi(\sigma)$ is the character on that element, $m(\sigma)$ is the sum of the squares of the lengths of cycles of $\sigma$, $n(\sigma)$ is the number of cycles of $\sigma$. 

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For the symmetric square of the adjoint representation, we rewrite this formula explicitly:

\[ 1 \cdot C_2(\mathbb{1}) + \dim Y_2(\alpha)C_2(Y_2(\alpha)) + \dim Y_2(\beta)C_2(Y_2(\beta)) + \dim Y_2(\gamma)C_2(Y_2(\gamma)) = (2 + \dim g) \cdot \dim g C_2(g), \]

where \( g \) is the adjoint representation.

Substituting the corresponding universal formulae, one can check, that for \( A_2 \) algebra this formula is true.
5 Conformity With \( sp(-2n) = so(2n) \) Duality

In ([IS] R.Mkrtychyan and A.Veselov have discussed the duality of higher-order Casimir operators for \( SO(2n) \) and \( Sp(2n) \) groups. Using the Perelomov and Popov ([19]) formula for the generating function for the Casimir spectra and parametrizing the Young diagrams in a different way ([18]), they have explicitly shown the \( C_{Sp(2n)}(\lambda, z) = -C_{SO(-2n)}(\lambda, -z) \) duality for rectangular Young diagrams.

Here we write the expressions for the corresponding eigenvalues of the second Casimir operator \( (C_2) \) for \( so(2n) \) and \( sp(2n) \) algebras, in the \( A, B \) parametrization, used in [IS].

\( so(2n) \)

For \( so(2n) \) the Casimir spectra writes as follows

\[
C_{so(2n)}(z, A, B) = \sum_{p=0}^{\infty} C_{p, so(2n)} z^p = \frac{(1 - zn)(2 - z(4n - 3))}{z(1 - z(n - 1))(2 - z(4n - 2))} \times \prod_{i=0}^{k} \frac{1 - z(-A_{k-i} + B_i + 2n - 1)}{1 - z(A_{k-i} - B_i)} \times \prod_{i=1}^{k} \frac{1 - z(-A_{i} + B_i + 2n - 1)}{1 - z(-A_{i} + B_i + 2n - 1)}
\]

After a proper expansion of \( C_{so(2n)}(z, A, B) \) into series in the vicinity of the \( z_0 = 0 \) point, one can check, that the coefficient of \( z^2 \), i.e. \( C_{2, so(2n)} \) can be expressed as follows:

\[
C_{2, so(2n)}(A, B) = \sum_{i=1}^{k} \left(4nA_i(B_{-i+k+1} - B_{k-i}) + 2A_i^2(B_{k-i} - B_{-i+k+1}) + 2A_i(A_{k-i} - B_{i}) + 2B_i^2(A_{-i+k+1} - A_{k-i}) - 4nA_0B_k + A_0^2(2B_k + 4B_0) + 2A_0(B_k - B_0) - B_0^2(2A_k + 4A_0) - n(A_0 - B_0) + 2n(A_0^2 + B_0^2) + 2(B_0^3 - A_0^3) + 1/2(A_0 - B_0).
\]

\( sp(2n) \)

The Casimir spectra for this case is

\[
C_{sp(2n)}(z, A, B) = \sum_{p=0}^{\infty} C_{p, sp(2n)} z^p = \frac{(1 - zn)(2 - z(4n + 3))}{z(1 - z(n + 1))(2 - z(4n + 2))} \times \prod_{i=0}^{k} \frac{1 - z(B_{k-i} - A_i + 2n + 1)}{1 - z(-B_{k-i} + A_i)} \times \prod_{i=1}^{k} \frac{1 - z(-B_{-i+k+1} + A_i)}{1 - z(B_{i} + 2n + 1)}
\]

And for \( C_{2, sp(2n)} \) one has

\[
C_{2, sp(2n)}(A, B) = \sum_{i=1}^{k} \left(-4nB_i(A_{-i+k+1} - A_{k-i}) + 2A_i^2(B_{i+k} - B_{-i+k}) + 2B_i^2(A_{k-i} - A_{-i+k+1}) + 4nA_iB_k + A_i^3(2B_k + 4B_0) - 2B_0(A_k - A_0) - B_0^2(2A_k + 4A_0) - n(A_0 - B_0) + 2n(A_0^2 + B_0^2) + 1/2(A_0 - B_0) - 2(A_0^3 - B_0^3).
\]

Therefore, we have obtained formulae for second Casimir eigenvalues on irreps of \( sp(2n) \) and \( so(2n) \) algebras, corresponding to any Young diagram (any \( (A, B) \) set).

It can be checked, that

\[
C_{2, so(2n)}(A, B) = -C_{2, sp(-2n)}(B, A)
\]
i.e. the Casimir duality for the second Casimir holds for any Young diagram (for any $A, B$ set). In particular, for $X_2$ one has the values, shown in the Table 4. It can be observed, that $C_{2, so(2n)} = 2C_{2, sp(2n)} = 1/2C_{2, so(2n)}(A, B)$, which indicates the difference of the definition of the Killing form in \cite{[18]}.

In \cite{[16]} it has been shown, that when permuting the Vogel parameters corresponding to the $so(2n)$ algebra in this way: $(\alpha, \beta, \gamma) \rightarrow (\beta, \alpha, \gamma)$, the $X_2(k)$ formula gives dimensions for some representations of the $sp(2n)$ algebra. More precisely, that permutation specifies a correspondence between $\lambda_{so(2n)} = k(\omega_1 + \omega_3)$ and $\lambda_{sp(2n)} = 2\omega_k + \omega_{2k}$ representations. One can notice, that the Young diagrams, associated with these representations are conjugate with each other. Indeed, in $A, B$ parametrization the associated sets are

$$\lambda_{so(2n)} \leftrightarrow A_0 = B_0 = 0, A_1 = 1, B_1 = k, A_2 = 3, B_2 = 2k,$$

$$\lambda_{sp(2n)} \leftrightarrow A_0 = B_0 = 0, A_1 = k, B_1 = 1, A_2 = 2k, B_2 = 3.$$

Therefore, it is reasonable to check the Casimir duality for these representations. Substituting the corresponding $(A, B)$ sets into the expressions for $C_2(A, B)$ written above, one gets

$$C_{2, so(2n)}(A, B) = 12k^2 + k(16n - 28),$$

$$C_{2, sp(2n)}(B, A) = -12k^2 + k(16n + 28) = -(12k^2 + k(16(-n) - 28) = -C_{2, so(2n)}(A, B).$$

So, the Casimir duality holds for representations, associated with the $X_2(k, -2, 4, 2n-4) \leftrightarrow X_2(k, 4, -2, 2n-4)$ transformation of the $X_2(k, \alpha, \beta, \gamma)$ universal formula \cite{[16]}. For the same representations in the Cartan-Killing normalization we have

$$C_{2, so(2n)} = 6k^2 + k(8n - 14),$$

$$C_{2, sp(2n)} = -3k^2 + k(4n + 7),$$

i.e.

$$C_{2, so(2n)}(\lambda) = -2C_{2, sp(-2n)}(\lambda'),$$

as expected.

\footnote{in \cite{[16]} the Killing form is defined as $Tr(X^a, X^b)$ in the fundamental representation, while our normalization (so called Cartan-Killing normalization) corresponds to the Killing form, defined as $Tr(adX^a, adX^b)$, i.e. in the adjoint representation.}
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