Recurring nuclear band spectra

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Abstract. We demonstrate that the ground-state band spectra of most heavy even-even elements can all be well described by the same non-parametric equations.

1. Introduction

Close inspection of the data on heavy even-even isotopes showed that the quasi-rotational band spectra of very many rare-Earth and Actinide nuclei possessed some identical internal structures. These structures take the form of recursion relations which depend only on the energies and associated angular momenta of the band levels and are therefore parameter-free.

Remarkably, our proposed difference equations can be interpreted by means of the cluster model and they lead to some interesting physical predictions.

2. The Basic Equation

Writing $E(J)$ and $E(K)$ for the excitation energies of two levels of a band with spins $J > K$ we found empirically that

$$
E(J) - E(K) = \frac{E(J+L) - E(K-L)}{(J-K)} \cdot [E(J) - E(K)]
$$

where $L$ changes in steps of 2 for $K^\pi = 0^+$ and $K^\pi = 0^-$ bands and indeed also, separately, for the alternating even and odd states of other bands.

As the spin values $(J + L)$ and $(K - L)$ change with $L$ we see that their sum remains equal to $(J + K)$. This simple recursive formula can be rearranged (and modified) in several ways for various purposes, e.g. for illustration of its extensive range of application, for prediction of missing band levels and for checking or correcting experimental assignments.

3. Application

To show the accuracy, or otherwise, of the formula, we here consider the ground state $K^\pi = 0^+$ bands in a database of 76 even-even nuclei as presented in Table 1 [1].

We now rewrite the recursion relation in the form:

$$
[E(J + L) - E(K - L)] = \frac{J - K + 2L}{(J - K)} [E(J) - E(K)]
$$

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Table 1. Database of 76 rare-Earth and Actinide nuclei

| Table Position | Z  | A  | Table Position | Z  | A  |
|----------------|----|----|----------------|----|----|
| 01             | 56 | 144| 39             | 72 | 168|
| 02             | 56 | 146| 40             | 72 | 170|
| 03             | 58 | 148| 41             | 72 | 172|
| 04             | 60 | 150| 42             | 72 | 174|
| 05             | 60 | 152| 43             | 72 | 176|
| 06             | 62 | 150| 44             | 72 | 178|
| 07             | 62 | 152| 45             | 72 | 180|
| 08             | 62 | 154| 46             | 74 | 168|
| 09             | 64 | 152| 47             | 74 | 170|
| 10             | 64 | 154| 48             | 74 | 172|
| 11             | 64 | 156| 49             | 74 | 174|
| 12             | 64 | 158| 50             | 74 | 180|
| 13             | 64 | 160| 51             | 74 | 182|
| 14             | 66 | 154| 52             | 74 | 186|
| 15             | 66 | 156| 53             | 76 | 180|
| 16             | 66 | 158| 54             | 76 | 184|
| 17             | 66 | 160| 55             | 76 | 186|
| 18             | 66 | 162| 56             | 76 | 188|
| 19             | 66 | 164| 57             | 76 | 192|
| 20             | 68 | 156| 58             | 88 | 222|
| 21             | 68 | 158| 59             | 88 | 224|
| 22             | 68 | 160| 60             | 88 | 226|
| 23             | 68 | 162| 61             | 90 | 222|
| 24             | 68 | 164| 62             | 90 | 226|
| 25             | 68 | 166| 63             | 90 | 228|
| 26             | 68 | 168| 64             | 90 | 230|
| 27             | 68 | 170| 65             | 90 | 232|
| 28             | 70 | 158| 66             | 90 | 234|
| 29             | 70 | 162| 67             | 92 | 230|
| 30             | 70 | 164| 68             | 92 | 232|
| 31             | 70 | 166| 69             | 92 | 234|
| 32             | 70 | 168| 70             | 92 | 236|
| 33             | 70 | 170| 71             | 92 | 238|
| 34             | 70 | 172| 72             | 94 | 238|
| 35             | 70 | 174| 73             | 94 | 240|
| 36             | 70 | 176| 74             | 94 | 242|
| 37             | 72 | 164| 75             | 94 | 244|
| 38             | 72 | 166| 76             | 96 | 248|

and plot the energy differences on each side against each other. The plotted data points for the various nuclei should then all lie in a straight line with slope:

\[ M = \frac{(J - K + 2L)}{(J - K)}. \]
Figure 1. Plots of excitation energies, \( E(J + L) - E(K - L) \) against \( E(J) - E(K) \), all in keV, for all nuclei listed in Table 1. Each nucleus generates a point for each of the four \( J, K, L \) pairings listed in Table 2. The expected gradients \( \frac{J - K + 2L}{J - K} \) of 5/3, 2, 3 and 5 are indicated by the solid lines through the origin. See discussion of Eq.(2) in text, and Table 2, for details.

From each band we select several pairs of energy differences as shown below in Table 2, together with the predicted slopes \( M \). The resulting fan plot is shown in Figure 1.

**Table 2.** Selected \( J, K, L \) values and predicted slopes \( M \).

| \( J + L \) | \( K - L \) | \( L \) | \( J \) | \( K \) | \( M \) |
|---|---|---|---|---|---|
| 12 | 2 | 2 | 10 | 4 | 5/3 |
| 10 | 2 | 2 | 8 | 4 | 2 |
| 10 | 4 | 2 | 8 | 6 | 3 |
| 12 | 2 | 4 | 8 | 6 | 5 |

4. Refining the Equation

Clearly, the plotted points in Figure 1 produce fairly good straight lines. But, the slopes of these lines are not *quite* equal to the predicted ones. So we now indicate a revised version of the basic formula. After some algebra, which involves a (parametrised) solution of the recursion relation [2], we were able to show that, if we define: \( e(J) = E(J)/J \), etc., then these new quantities obey the same form of recurrences as before, i.e.,

\[
[e(J + L) - e(K - L)] = \frac{(J - K + 2L)}{(J - K)} [e(J) - e(K)].
\] (4)
Figure 2. Plots of modified excitation energies, $e(J+L) - e(K-L)$ against $e(J) - e(K)$, where $e(J) = E(J)/J$, all in keV/ℏ, for all nuclei listed in Table 1. Each nucleus generates a point for each of the four $J$, $K$, $L$ pairings listed in Table 2. The expected gradients $\frac{J-K+2L}{J-K}$ of $5/3$, $2$, $3$ and $5$ are indicated by the solid lines through the origin. See discussion of Eq.(4) in text, and Table 2, for details.

The result of plotting this equation for the same data as before is now in better agreement with predictions, as is evident in Figure 2. Further, the above relation can be slightly modified so as to apply to excited bands in both even-even and even-odd nuclei.

5. Clues to the Physics

A possible approach to an explanation of these regularities is provided by the core-cluster model of heavy elements, which has proved very useful in accounting for many other observed properties of nuclei [3,4]. One form of the radial equation for an energy level $E(J)$ of a binary cluster-core system of spin $J$ can be written as

$$
\frac{h^2}{2\mu} \left[ -\frac{d^2}{dr^2} + \frac{J(J+1)}{r^2} \right] \phi(J) + V(r)\phi(J) = E(J)\phi(J),
$$

(5)$\mu$ being the reduced mass and $\phi(J)$ the radial wave function. Considering also the similar equation for $\phi(K)$ and assuming that the potential is unchanged, it is easy to show that:

$$
\frac{h^2}{2\mu} \int_0^\infty \frac{\phi(J)\phi(K)}{r^2} dr [J(J+1) - K(K+1)] = [E(J) - E(K)] \int_0^\infty \phi(J)\phi(K)dr,
$$

(6)and that $[J(J+1) - K(K+1)] = (J-K)(J+K+1)$. This equation has been used in the past to show that two degenerate energy levels imply that the matrix element $\langle J|\frac{1}{r^2}|K \rangle = 0$ [5]. However, we have found other uses for this interesting result, which give clues to the physical
meaning of our observations. On rewriting Eq.(6) for \( E(J) - E(K) \), and the corresponding equation for \( E(J + L) - E(K - L) \), as

\[
\frac{\hbar^2}{2\mu} \langle J | \frac{1}{r^2} | K \rangle (J + K + 1) = \frac{E(J) - E(K)}{(J - K)}
\]

and

\[
\frac{\hbar^2}{2\mu} \langle J + L | \frac{1}{r^2} | K - L \rangle [(J + L) + (K - L) + 1] = \frac{E(J + L) - E(K - L)}{(J + L) - (K - L)}
\]

we see, by dividing Eq.(7) by Eq.(8), that recovery of our basic recursion equation

\[
\frac{E(J) - E(K)}{J - K} = \frac{E(J + L) - E(K - L)}{(J + L) - (K - L)}
\]

implies that the matrix element ratios in Eqs.(7) and (8) must be equal.

This equality of physical matrix elements is not entirely mysterious since it can be shown that in at least one realistic example it must be true. Let us suppose that there is a potential \( V(r) \) which yields something close to a pure rotational spectrum (such potentials exist [6]). Then Eq.(6) could be specialised to give the result:

\[
\frac{\hbar^2}{2\mu} \langle J | \frac{1}{r^2} | K \rangle [J(J + 1) - K(K + 1)] = C[J(J + 1) - K(K + 1)]
\]

where \( C = 1/2I \), \( I \) being the constant moment of inertia. It follows at once that all the matrix element ratios are equal in this case. It is still, perhaps, a little puzzling that such equalities persist even when the spectra are only quasi-rotational.

6. Conclusions
We have shown that ground-state bands in many even-even nuclei have a recursive parameter-free structure. We believe that this type of result will hold also for numerous other bands in both even-even and even-odd nuclei. One possible physical explanation for these observations involves the cluster model, but more research is required before this can be further verified.

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