RANDOM ATTRACTORS FOR STOCHASTIC TIME-DEPENDENT DAMPED WAVE EQUATION WITH CRITICAL EXPONENTS

QINGQUAN CHANG, DANDAN LI AND CHUNYOU SUN

School of Mathematics and Statistics, Lanzhou University
Lanzhou 730000, China

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Abstract. We study the asymptotic behavior of solutions of a stochastic time-dependent damped wave equation. With the critical growth restrictions on the nonlinearity $f$ and the time-dependent damped term, we prove the global existence of solutions and characterize their long-time behavior. We show the existence of random attractors with finite fractal dimension in $H_0^1(U) \times L^2(U)$. In particular, the periodicity of random attractors is also obtained with periodic force term and coefficient function. Furthermore, we construct the pullback random exponential attractors.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where

$$\Omega = \{ \omega = (\omega_1, \omega_2, \cdots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0 \},$$

the Borel $\sigma$-algebra $\mathcal{F}$ on $\Omega$ is generated by the compact open topology (see [1]) and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$. In this paper, we deal with the existence of the random attractor and exponential attractor for a stochastic wave equation defined on a bounded open set $U \subseteq \mathbb{R}^3$ with smooth boundary $\partial U$:

$$u_{tt} + \beta(t)u_t - \Delta u + f(u) - g(x,t) = \sum_{j=1}^m h_j(x) \circ \frac{dW_j}{dt}, \quad (1.1)$$

with the initial and boundary conditions

$$u(x,\tau) = u_\tau(x), \quad u_t(x,\tau) = u_{t,\tau}(x), \quad x \in U, t > \tau, \tau \in \mathbb{R},$$

$$u|_{\partial U} = 0, \quad (1.2)$$

where $\Delta$ denotes the Laplacian operator.

We assume $\beta(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous and differentiable function with $0 < \beta_0 < \beta(t) < \beta_1$, for some $\beta_1 > \beta_0 \in \mathbb{R}$ (see Chapter 15 in [8]). Suppose $g \in C_b(\mathbb{R}; L^2(U))$ and $\{h_j\}_{j=1}^m$ are given in $H_0^1(U), \{W_j\}_{j=1}^m$ are independent two-side real-valued Wiener process $(\Omega, \mathcal{F}, \mathbb{P})$ with path $\omega(\cdot)$ in $C(\mathbb{R}, \mathbb{R}^m)$ and we denote $\circ$ as Stratonovich integral. We identify $\omega(t)$ with $(W_1(t), W_2(t), \ldots, W_m(t))$, i.e. $\omega(t) = (W_1(t), W_2(t), \ldots, W_m(t)), t \in \mathbb{R}$ (cf. [28, 29]).

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* Corresponding author: Chunyou Sun.
We assume that the nonlinearity \( f \) is a \( C^1 \) function on \( \mathbb{R} \) and there exist positive constants \( C_1 \) and \( C_2 \) such that, for any \( s \in \mathbb{R} \),
\[
|f(s)| \leq C_1(1 + |s|^{q+1}), \quad (1.4)
\]
\[
|f'(s)| \leq C_2(1 + |s|^q), \quad (1.5)
\]
\[
\lim_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1, \quad (1.6)
\]
where \( 0 \leq q \leq 2 \), \( \lambda_1 \) is the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition. Denote \( F(u) = \int_0^u f(s)ds \). Actually one could derive (1.4) from (1.5), we list (1.4) here for the convenience of the proofs.

Equations formulated as (1.1) are applied in many branches of physics, engineering, biology and geography [11, 21, 24, 27]. For instance, the solution \( u(x, t) \) is the voltage inside a piece of telegraph or transmission wire (the telegraph equation [23]); the dynamics of a Josephson junction driven by a current source is modeled by the Sine-Gordon equation of the deterministic form of (1.1) with \( f(u) = \beta \sin u \). When \( f(u) = |u|^q u \), the damped wave equations are encountered in relativistic quantum mechanics [27].

The random attractor and random exponential attractor are very useful tools to analyze the long-time behavior of random dynamical systems, which have been investigated by many scholars such as [17, 18, 19, 31, 32]. In order to develop the attractor theory for (1.1)-(1.3), one needs to derive the dissipation and the compactness of the cocycle generated by (1.1)-(1.3). When dealing with random exponential attractors, the random system should also satisfy the quasi-differentiability (see [14] for the definition).

Usually the dissipation estimate of wave equations formulated as (1.1)-(1.3) is derived from the Gronwall’s inequality or the so-called barrier method. For deterministic equations, when the nonlinearity endowed with (1.4) and (1.6), the barrier method is applied to show the dissipation. There are many novel and profound abstract conclusions and inspiring examples proposed in [3, 12, 15, 16, 27]. In [12] and [15], the authors developed a decomposing on nonlinearity, especially the technique on dominating the quartic term is very heuristic and impressive. The terms with high powers from the nonlinearity were proved vanishing through an approximation by stationary solutions of semilinear wave equations, which further showed the dissipation. In [16], due to (1.6), the nonlinearity could be controlled by \( \|u\|_{L^2}^2 \) and some positive constants, then the dissipation of deterministic version of (1.1)-(1.3) came from the barrier method. While for stochastic cases, to our best knowledge, the dissipation usually comes from the application of the Gronwall’s Lemma with nonlinearity endowed with following restrictions: for \( s \in \mathbb{R} \),
\[
\begin{cases}
  f(s)s \geq C \int_0^s f(\xi)d\xi + \zeta_1; \\
  F(s) = \int_0^s f(\xi)d\xi \geq C|s|^{q+2} - \zeta_2,
\end{cases}
\]
where \( \zeta_1, \zeta_2 \) are positive constants (see [28, 29, 30, 31, 33, 34] and references therein). During the process for proving a priori estimate there would usually be a product of the nonlinearity and the random term, namely \((f(u),\mu(\theta_\omega))_{L^2}\). In order to dominate this product through the nonlinearity, scholars derived a estimate with \( |u|^{q+2} \) on the right hand of the estimate by using the Hölder inequality. Then by taking virtue of the second inequality in (1.7), one could derive a differential inequality which satisfies Gronwall’s Lemma’s condition and further the dissipation.
As in our paper, we endow the nonlinearity with the assumption (1.6), which is weaker than the assumption (1.7) since there is no direct dominating inequalities and we can not deduce a differential inequality for the Gronwall's inequality. The approximation in [14] we mentioned above derived the vanishing of terms with high powers. But the same approximation is hard to be applied here directly because it is not known whether solutions to the stationary equation of (1.1) are vanishing. Therefore the main difficulty caused by the sign condition (1.6) is to dominate the product \((f(u), \mu(\theta t \omega))_{L^2}\) in proving the existence of random attractor. In order to overcome the difficulty mentioned above, a new decomposing technique on nonlinearity is proposed in this paper to control the product. With the help of this new technique, \((f(u), \mu(\theta t \omega))_{L^2}\) could be dominated by \(\|u\|_{L^2}^2\) together with \(\|\mu(\theta t \omega)\|_{H^1}\) and some positive constants.

The asymptotic compactness is an important part of the attractor theory, which is also a main difficulty in proving the existence of the attractor. When the nonlinearity is critical, one could not derive the compactness through the Sobolev embedding. Many researchers have invested this problem, see [3, 12, 15, 16, 31] and references therein. In [3], an energy method was firstly proposed to overcome the lack of compactness for the wave equation equipped with critical nonlinearity. In [12], I. Chueshov et al proved the asymptotic compactness for semilinear wave equation through an application of a decomposing technique. In [16], the compactness for deterministic equation formulated as (1.1) was obtained from taking advantages of asymptotically smoothness and Lyapunov functions. For equations similar to our case, in [8, 30], the existence of random attractor was proved but with subcritical nonlinearity. As for our critical case, we apply the energy method in [3] to overcome the lack of compactness, which proves the asymptotic compactness of the system.

In this paper, we not only show the existence of random attractor, but also prove the finiteness of fractal dimension for the random attractor. The finiteness of the attractor is a very important tool to analyze the property of the random system. Our proof mainly obeys the criterions in [14, 16, 27, 33]. The finite fractal dimension was showed in [14] and [16] for deterministic wave equation, in which the system should satisfy the quasi-differentiability, i.e. the system could be dominated by a contraction map (the squeezing property) and a compact map (the smoothness property), here refer readers to [16] for details. In [33], a criterion was established for the finiteness of fractal dimension for random dynamical systems. The random systems should also be dominated by a contraction map and a compact map. Differently from deterministic system, the coefficients for contraction and compact map are random with finite expectation.

Lastly, we construct a random exponential attractor to (1.1)-(1.3). To our best knowledge, there is only a few results on random exponential attractors, they are [26, 34, 7]. The random exponential attractor was firstly proposed in [26], in which the definition, a criterion and an application to a stochastic parabolic equation were given. In [34], another criterion was showed in Hilbert spaces. Different from being constants in [26], the coefficients in the squeezing property and the smoothness property were random parameters with finite expectation. For the Banach space, a criterion for the existence of random exponential attractors was presented in [7], in which the restrictions on random dynamical system were generalized. Inspired by these works, we could derive the squeezing and smoothness properties for the random dynamical system generated by (1.1)-(1.3) since the tempered absorbing set
and finiteness of fractal dimension are derived. Therefore we construct a random exponential attractor for (1.1)-(1.3).

This paper is organized as follows: in the next section, we recall some basic concepts related to random attractors for general random dynamical systems. In Section 3, we define a continuous random dynamical system for problem (1.1)-(1.3). The absorbing set is obtained in Section 4. In Section 5, we prove the pullback asymptotic compactness and the existence of random attractors for the stochastic wave equation on \( \mathbb{R}^3 \). In Section 6, we give an upper bound of the fractal dimension of the random attractor obtained in Section 5. In Section 7, the pullback random exponential attractor is constructed.

2. Preliminaries. In this section, we recall some basic concepts related to random attractors for stochastic dynamical systems. Let \((X, \| \cdot \|_X)\) be a separable Hilbert space with Borel \( \sigma \)-algebra \( B \).

Definition 2.1 ([1]). \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\) is called a metric dynamical system if \( \theta : \mathbb{R} \times \Omega \to \Omega \) is \((\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})\)-measurable, \( \theta_0 \) is the identity on \( \Omega \), \( \theta_{s+t} = \theta_t \circ \theta_s \) for all \( s, t \in \mathbb{R} \) and \( \theta_t P = P \) for all \( t \in \mathbb{R} \).

Definition 2.2 ([29]). A continuous random dynamical system (RDS) on \( X \) over a metric dynamical system \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\) is a mapping
\[
\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X, (t, \tau, \omega, x) \mapsto \Phi(t, \tau, \omega, x),
\]
which is \((\mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))\)-measurable and satisfies, for \( P \)-a.e. \( \omega \in \Omega \),

1: \( \Phi(\cdot, \tau, \cdot, \cdot) \) is \((\mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))\)-measurable;

2: \( \Phi(0, \tau, \omega, \cdot) \) is the identity on \( X \);

3: \( \Phi(t+s, \tau, \omega, \cdot) = \Phi(t, \tau, \theta_s \omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot) \) for all \( s, t \in \mathbb{R}^+ \);

4: \( \Phi(t, \tau, \omega, \cdot) : X \to X \) is continuous for all \( t \in \mathbb{R}^+ \).

Hereafter, we always assume that \( \Phi \) is a continuous RDS on \( X \) over \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\).
**Definition 2.7** ([29]). Let \( \mathcal{D} \) be a collection of random subsets of \( X \). Then \( \Phi \) is said to be an \( \mathcal{D} \)-pullback asymptotically compact in \( X \) if for \( P \)-a.e. \( \omega \in \Omega \), \( \{ \Phi(t_n, \tau - t_n, \theta_{-t_n}, x_n) \}_{n=1}^{\infty} \) has a convergent subsequence in \( X \) whenever \( t_n \to \infty \), and \( x_n \in B(\tau - t_n, \theta_{-t_n}) \) with \( \{ B(\tau, \omega) \}_{\omega \in \Omega} \in \mathcal{D} \).

**Definition 2.8** ([29]). Let \( \mathcal{D} \) be a collection of random subsets of \( X \) and \( \{ \mathcal{A}(\tau, \omega) \}_{\omega \in \Omega} \in \mathcal{D} \). Then \( \{ \mathcal{A}(\tau, \omega) \}_{\omega \in \Omega} \) is called a \( \mathcal{D} \)-random attractor (or \( \mathcal{D} \)-pullback attractor) for \( \Phi \) if the following conditions are satisfied, for \( P \)-a.e. \( \omega \in \Omega \):

1. \( \mathcal{A}(\tau, \omega) \) is compact, and \( \omega \mapsto d(x, \mathcal{A}(\tau, \omega)) \) is measurable for every \( x \in X \);
2. \( \{ \mathcal{A}(\tau, \omega) \}_{\omega \in \Omega} \) is invariant, that is for every \( \tau \in \mathbb{R} \),
   \[ \Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_{t} \omega), \quad \forall \ t \geq 0; \]
3. \( \{ \mathcal{A}(\tau, \omega) \}_{\omega \in \Omega} \) attracts every set in \( \mathcal{D} \), that is, for every \( B = \{ B(\tau, \omega) \}_{\omega \in \Omega} \in \mathcal{D} \),
   \[ \lim_{t \to \infty} \text{dist}(\Phi(t, \tau - t, \omega, B(\tau - t, \theta_{-t} \omega)), \mathcal{A}(\tau, \omega)) = 0, \]
   where \( \text{dist} \) is the Hausdorff semi-metric given by \( \text{dist}(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} \| y - z \|_{X} \) for any \( Y \subseteq X \) and \( Z \subseteq X \).

**Definition 2.9** ([7]). A random set \( \mathcal{M} \) is a random pullback exponential \( \mathcal{D} \)-attractor for the random dynamical system \( (\theta, \varphi) \) on \( V \) (Banach space), if the sections \( M(\omega) \neq 0 \) are compact and \( \mathcal{M} \) is positively \( \varphi \)-invariant, i.e.
\[ \varphi(t, \omega, M(\omega)) \subseteq M(\theta_{t} \omega) \quad \forall \ t \in \mathbb{R}^{+}. \]
Moreover, the fractal dimension of \( \mathcal{M}(\omega) \) is finite, i.e. there exists \( \alpha > 0 \) such that
\[ \dim_{f}(\mathcal{M}(\omega)) \leq \infty, \]
and \( \mathcal{M}(\omega) \) is pullback \( \mathcal{D} \)-attracting at an exponential rate, i.e. there exists \( \alpha > 0 \) such that
\[ \lim_{t \to \infty} e^{\alpha t} \text{dist}_{H}(\varphi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), \mathcal{M}(\omega)) = 0, \quad \forall D \in \mathcal{D}. \]

**Theorem 2.10** ([29]). Let \( \mathcal{D} \) be an inclusion-closed collection of random subsets of \( X \) and \( \Phi \) be a continuous RDS on \( X \) over \( (\Omega, \mathcal{F}, P, (\theta_{t})_{t \in \mathbb{R}}) \). Suppose that \( \{ K(\omega) \}_{\omega \in \Omega} \) is a closed absorbing set of \( \Phi \) in \( \mathcal{D} \) and \( \Phi \) is \( \mathcal{D} \)-pullback asymptotically compact in \( X \). Then \( \Phi \) has a unique \( \mathcal{D} \)-random attractor \( \{ \mathcal{A}(\tau, \omega) \}_{\omega \in \Omega} \) which is given by
\[ \mathcal{A}(\tau, \omega) = \bigcap_{\tau > 0} \bigcup_{t \geq \tau} \Phi(t, \tau - t, \theta_{-t} \omega, K(\theta_{-t} \omega)). \]

In this paper, we will denote the collection of all bounded tempered random sets of \( H_{0}^{1}(U) \times L^{2}(U) \) by \( \mathcal{D} \) and prove that problem (1.1)-(1.3) has a \( \mathcal{D} \)-random attractor.

**Theorem 2.11** ([33]). Let \( (\Omega, \mathcal{F}, P, (\theta_{t})_{t \in \mathbb{R}}) \) be an ergodic metric dynamical system.
Let \( \{ \Phi(t, \tau, \omega) \}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega} \) be a random dynamical system on a separable Banach space \( X \) driven by \( (\Omega, \mathcal{F}, P, (\theta_{t})_{t \in \mathbb{R}}) \). Suppose the following conditions hold for a family of bounded closed random subsets \( B(\tau, \omega) \) for any \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \):

**H1:** There exists a tempered random variable \( R_{\omega} \) (independent of \( \tau \)) such that the diameter \( \| B(\tau, \omega) \| \) of \( B(\tau, \omega) \) is bounded by \( R_{\omega} \), i.e. \( \sup_{\tau \in \mathbb{R}} \sup_{\omega \in \Omega} \sup_{u \in B(\tau, \omega)} \| u \| \leq R_{\omega} < \infty \) and \( R_{\theta_{t} \omega} \) is continuous in \( t \) for all \( t \in \mathbb{R} \).
H2: invariance: \( B(t + \tau, \theta \omega) = \Phi(t, \tau, \omega)B(\tau, \omega) \) for all \( t \geq 0 \);

H3: there exist positive numbers \( \lambda, \delta, t_0 \), random variable \( C_0(\omega) \geq 0 \) and \( m \)-dimensional projector \( P_m: H \to P_mH(\dim(P_mH = m)) \) such that for any \( u, v \in B(\tau, \omega) \)

\[
\|P_m\Phi(t_0, \tau, \omega)u - P_m\Phi(t_0, \tau, \omega)v\|_X \leq e^{\int_0^{t_0} C_0(\theta \omega)ds}\|u - v\|_X \tag{2.1}
\]

and

\[
\|(I - P_m)\Phi(t_0, \tau, \omega)u - (I - P_m)\Phi(t_0, \tau, \omega)v\|_X \leq (e^{-\lambda t_0} + \delta e^{\int_0^{t_0} C_0(\theta \omega)ds})\|u - v\|_X, \tag{2.2}
\]

where \( \lambda, \delta, t_0, m \) are independent of \( \tau \) and \( \omega \).

If \( \mathbb{E}(C_0^2) \leq \infty, t \geq \frac{1}{\lambda}, 0 < \delta \leq \frac{1}{\lambda}e^{-e^{-\frac{t}{\lambda}}} \mathbb{E}(C_0^2), \) then for any \( \tau \in \mathbb{R}, \omega \in \Omega, \)

\[
\dim_f B(\tau, \omega) \leq \frac{2m \ln \left( \frac{\sqrt{\pi\tau}}{\lambda} + 1 \right)}{\ln \frac{1}{\delta}}
\]

Throughout this paper, we denote \( \| \cdot \| \) and \((.,.)\) as the norm and the inner product of \( L^2(U) \) respectively. We also use the notation \( \| \cdot \|_p \) as the norm of \( L^p(U) \). The letters \( c \) and \( C_i, i = 1, 2, \ldots \) are positive constants and may change from line to line.

3. Random dynamical systems. In this section, we define a continuous random dynamical system for problem (1.1)-(1.3). Take \( z = u_t + \delta u \), where \( \delta \) is a positive number that small enough and will be determined later. Substituting \( u_t = z - \delta u \) into (1.1) obtain

\[
\frac{du}{dt} + \delta u = z, \tag{3.1}
\]

\[
\frac{dz}{dt} + (\beta(t) - \delta)z + (\delta^2 - \beta(t))u - \Delta u + f(u) = g(x, t) + \sum_{j=1}^{m} h_j(x) \circ dW, \tag{3.2}
\]

with the initial conditions:

\[
u(x, \tau - t) = u_{\tau - t}(x), \quad z(x, \tau - t) = z_{\tau - t}(x), \quad \tag{3.3}
\]

where \( z_{\tau - t}(x) = u_{\tau - t}(x) + \delta u_{\tau - t}(x), \quad x \in U, t > \tau \quad \text{with} \ \tau \in \mathbb{R}. \) Let \( \{t_\theta\}_{\theta \in \mathbb{R}} \) be a family of measure-preserving shift operator given by

\[
\theta \omega(\cdot) = \omega(\cdot + t) - \omega(t), \forall \omega \in \Omega \text{ and } t \in \mathbb{R}.
\]

Then \( (\Omega, \mathcal{F}, P, \{t_\theta\}_{\theta \in \mathbb{R}}) \) forms a metric dynamical system.

By (1.6) we can deduce that it is equivalent to that: for some \( 0 < \gamma < \lambda_1 \), there exists a \( C_\gamma > 0 \) such that for \( s \in \mathbb{R}, \)

\[
f(s)s \geq - (\gamma s^2 + C_\gamma). \tag{3.4}
\]

In order to study the equations (3.1)-(3.2), we need to convert the stochastic systems into a deterministic one with a random parameter. For \( j = 1, 2, \ldots, m \), consider the one-dimensional Ornstein-Uhlenbeck equation

\[
d\mu_j + \mu_j dt = dW_j(t).
\]

The unique solution of the above equation is given by \( \mu_j(\theta \omega) = - \int_{t-\infty}^t e^s(\theta s \omega_j) (s)ds, t \in \mathbb{R}. \) Note that the random variable \( \mu_j(\omega_j) \) is tempered, and there is a \( \theta \)-invariant \( \Omega_0 \subset \Omega \) with \( P(\Omega_0) = 1 \) such that \( t \mapsto \mu_j(\theta \omega) \) is continuous for \( \omega \in \Omega_0 \).
and $j = 1, 2, \ldots, m$. From Proposition 4.3.3 in [1], for any $\epsilon > 0$, there is a tempered function $\gamma(\omega) > 0$ such that
\[
\sum_{j=1}^{m} ((\mu_j(\omega_j))^2 + |\mu_j(\omega_j)|^{q+2}) \leq r(\omega),
\] (3.5)
where $r(\omega)$ satisfies for $P$-a.e. $\omega \in \Omega$,
\[
r(\omega, t) \leq e^{\epsilon|t|r(\omega)}, \quad t \in \mathbb{R}.
\] (3.6)
Take $\mu(\theta, \omega) = \sum_{j=1}^{m} h_j \mu_j(\theta, \omega_j)$, which solves $d\mu + \mu dt = \sum_{j=1}^{m} h_j \frac{dW_j}{dt}$. From [1, 33, 34] we know that for almost every $\omega \in \Omega$, $\forall \epsilon > 0, t \in \mathbb{R},$
\[
E \left[ \left| \mu_j(\theta, \omega_j) \right|^r \right] = \frac{\Gamma\left(\frac{1+r}{2}\right)}{\sqrt{\pi}},
\] (3.7)
where $E[\cdot]$ denotes the function of taking expectation, $\Gamma(\cdot)$ means the Gamma function.

We set $v(\tau, t) = u(\tau, \tau - t, \theta, \omega, \theta, \omega)$, which satisfies
\[
du + \delta u - v = \mu(\theta, \omega),
\] (3.8)
with initial condition $u(x, \tau - t) = u_{\tau - t}(x), v(x, \tau - t) = v_{\tau - t}(x) = k_{\tau - t}(x) - \mu(\theta, \omega)$. (3.9)

And (3.9) could also be written as follows:
\[
\frac{dv}{dt} + (\beta(t) - \delta) v + (\delta^2 - \delta\beta(t)) u - \Delta u + f(u) = g(x, t) + (\delta - \beta(t))\mu(\theta, \omega).
\] (3.11)

**Theorem 3.1.** The equations (3.8)-(3.11) has a unique weak solution $(u, v)$ satisfies $u(x, \tau - t) = u_{\tau - t}(x), v(x, \tau - t) = v_{\tau - t}(x) + \delta u_{\tau - t}(x) - \mu(\theta, \omega)$. (3.12)

**Proof.** Through a standard Galerkin’s approximation, one can derive the existence and uniqueness of solutions for (3.8)-(3.11). We refer readers to Chapter 5.1 in [16] and Chapter 7 in [20] for details. We omit the proof here. \qed

We now define a random dynamical system for the stochastic wave equation. Let $\Phi$ be a mapping, $\Phi : \mathbb{R} \times \mathbb{R} \times \Omega \times H^1_0(U) \times L^2(U) \to H^1_0(U) \times L^2(U)$ given by
\[
\Phi(t, \tau - t, \theta, \omega, u_{\tau - t}, v_{\tau - t}) = (u(t, \tau - t, \theta, \omega, v_{\tau - t}), v(t + \tau, \theta, \omega, v_{\tau - t} + \mu(\theta, \omega)))
\] (3.12)
for every $(t, \tau, \omega, u_{\tau - t}, v_{\tau - t}) \in \mathbb{R} \times \Omega \times H^1_0(U) \times L^2(U)$, where $v_{\tau - t} = z_{\tau - t}$. We denote the initial data as $\Phi_{\tau - t} = (u_{\tau - t}, v_{\tau - t})$. Then $\Phi$ is a continuous random dynamical system over $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ on $H^1_0(U) \times L^2(U)$. By (3.1) and (3.8), we could find that
\[
\Phi(t, \tau, \omega, (u_{\tau}, v_{\tau})) = (u(t + \tau, \theta, \omega, u_{\tau}), v(t + \tau, \theta, \omega, v_{\tau} + \mu(\theta, \omega))).
\] (3.13)

Next we will give a lemma on the weak dependence on the initial data for the above weak solution, which will be important in the proof of asymptotic compactness.
Lemma 3.2. Assume that $g$, $h$ and $f$ satisfy the assumptions in Theorem 3.1, then the solution $(u, v)$ of (3.8)-(3.10) is weak-continuously dependent on the initial date $(u_\tau, v_\tau)$.

Proof. We denote $\varphi(t) = (u(t), u(t))^T$. By virtue of the theorem above and Theorem 6.27 in [8], we could find that

$$ \varphi(t) = e^{-A(t-\tau)}\varphi(\tau) + \int_\tau^t e^{-A(t-s)}\dot{F}(\varphi(s))ds, \tag{3.14} $$

where $A = \begin{pmatrix} 0 & I \\ -\Delta & 0 \end{pmatrix}$, $\dot{F} = \begin{pmatrix} -\beta(t)u_t + f(u) + g + d\mu(\theta_t\omega) \\ 0 \end{pmatrix}$.

Now let $\varphi^m(t) = (u^m(t), u^m(t))^T$ with initial weak convergence condition $\varphi^m(\tau) \rightharpoonup \varphi(\tau)$ in $H_0^1(U) \times L^2(U)$. Let any $T > \tau, \rho \in (H_0^1(U) \times L^2(U))^*$. Firstly, we will show that $\langle \varphi^m, \rho \rangle$ is equi-continuous on $[\tau, T]$. Take $\tau \leq t \leq t + \nu \leq T, \nu \geq 0$,

$$ \langle \varphi^m(t), \varphi^m(t) - \varphi^m(t) + \varphi^m(t), \rho \rangle 
= \left\langle \left( e^{-A(t+\nu)} - e^{-A(t)} \right) \varphi^m(\tau), \rho \right\rangle + \int_t^{t+\nu} \left\langle e^{A(t+\nu-s)}\dot{F}(\varphi^m(s)), \rho \right\rangle ds + \int_t^\nu \left\langle e^{-A(t+\nu-s)} - e^{-A(t-s)} \right\rangle \dot{F}(\varphi^m(s)), \rho \right\rangle. \tag{3.15} $$

From the theory of semigroup and the properties of $A$, if $\zeta_n \rightharpoonup \zeta$ in $H_0^1(U) \times L^2(U), t_n \rightarrow t$ in $[\tau, T]$, then $e^{-At_n} \zeta_n \rightharpoonup e^{-At} \zeta$. Thus we could obtain that $\forall \epsilon > 0$, given $M > 0$, $\exists \delta > 0$ such that

$$ \left| \left\langle \left( e^{-A(t+\nu)} - e^{-A(t)} \right) \zeta, \rho \right\rangle \leq \epsilon, $$

whenever $\|\zeta\|_{H_0^1(U) \times L^2(U)} \leq M$ and $0 \leq \nu \leq \delta$. By Theorem 3.1 and the definition of $F(\varphi^m)$, we can obtain that $\|\varphi^m\|_{H_0^1(U) \times L^2(U)}$ and $\|F(\varphi^m)\|_{H_0^1(U) \times L^2(U)}$ are uniformly bounded for $s \in [\tau, T]$ independently of $m$. Thus we can deduce that $\langle \varphi^m, \rho \rangle$ is equi-continuous on $[\tau, T]$.

By Lemma 5.12 in [2], there is a subsequence $\varphi^m_n$ of $\varphi^m$ and a weakly continuous map $\varphi^* : [\tau, T] \rightarrow H_0^1(U) \times L^2(U)$ with $\varphi^*(\tau) = \varphi(\tau)$ such that $\varphi_n^* \rightharpoonup \varphi^*$. Next we will show that $\varphi^* = \varphi$. By the definition of $F$, we just need to show that the weak continuity of $f$ and $\beta(t)u_t$. Since $u^m_n \rightharpoonup u^* \in H_0^1(U), u^m_n \rightharpoonup u^* \in L^2(U)$. Thus

$$ |f(u^*) - f(u^m_n)| \leq \max f'(u)\|u^*(t) - u^m_n(t)\| \leq c\|u^*(t) - u^m_n(t)\| \rightarrow 0. $$

Together with the definition of $\varphi^*$, we can deduce that $\beta(t)u^m_{n,1,\tau} \rightharpoonup \beta(t)u_{\tau, \tau}$ in $L^2(U)$. For $\forall \rho \in (H_0^1(U) \times L^2(U))^*$,

$$ \langle \varphi_n^m(t), \rho \rangle = \left\langle e^{-At} \varphi_n^m(\tau), \rho \right\rangle + \int_{\tau}^t \left\langle e^{-A(t-s)}\dot{F}(\varphi_n^m(s)), \rho \right\rangle ds. $$

By taking the limit of above equation with respect to $n$ and the uniqueness of weak solution, we could show that $\varphi^*$ is a weak solution of (3.14), which indicates that $\varphi^* = \varphi$. Since $T$ is arbitrary, the weak continuity of solutions to (3.8)-(3.10) follows from the definition of $v$ and $v_1$. 

\qed
4. **Absorbing set.** Let \( \delta > 0 \) be small enough such that

\[
\beta(t) - \delta > 0, \quad \delta^2 - \delta \beta(t) + \lambda_1 > 0,
\]

where \( \lambda_1 \) is the first eigenvalue of \(-\Delta\). We denote

\[
\sigma = \min \left\{ \frac{1}{4} \beta(t) - \delta, \frac{\delta}{2 \lambda_1} (\lambda_1 - \gamma) \right\}.
\]

**Theorem 4.1.** Assume that \( h(x) \in H^1_0(U) \), \( g(x,t) \in C_b(\mathbb{R}; L^2(U)) \) and \( f(u) \) satisfies (1.4)-(1.6). Let \( D = \{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D} \). Then for \( P \)-a.e. \( \omega \in \Omega \), there is a \( T = T(\tau, D, \omega) > 0 \) such that for all \( t > T > 0 \), the solution \( (u,v) \) of problem (3.8)-(3.10) with \((u_{\tau-t}, v_{\tau-t}) \in D(\theta_{\tau-t}, \omega)\), satisfies,

\[
\|u(\tau, t - \tau, \theta_{\tau\omega}, \Phi_{\tau})\|_{H^1_0(U)}^2 + \|v(\tau, t - \tau, \theta_{\tau\omega}, \Phi_{\tau})\|_{L^2(U)}^2 \leq R(\tau, \omega),
\]

where \( R(\tau, \omega) \) satisfies \( \lim_{t \to \infty} e^{-\sigma t} R(\tau, \omega) = 0 \).

**Proof.** Taking the inner product of (3.9) with \( v \) in \( L^2(U) \), we get

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + (\beta(t) - \delta) (u_t, v) - (\triangle u, v) + (f(u), v) = (g, v).
\]

From (3.8), we have

\[
(u_t, v) = \|u_t\|^2 + \delta (u_t, u) - (u_t, \mu(\theta_t \omega)),
\]

\[
- (\triangle u, v) = \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla u\|^2 - (\nabla u, \nabla \mu(\theta_t \omega)),
\]

\[
(f(u), v) = \frac{d}{dt} \int_U F(u) dx + \delta (f(u), u) - (f(u), \mu(\theta_t \omega)).
\]

From (4.5)-(4.7), we find that

\[
\frac{1}{2} \frac{d}{dt} \left( \|v\|^2 + \|\nabla u\|^2 + 2 \int_U F(u) dx \right)
\]

\[
= (g, v) - (\beta(t) - \delta) \|u_t\|^2 + (\beta(t) - \delta) (u_t, \mu(\theta_t \omega)) - \delta \|\nabla u\|^2
\]

\[
+ (\nabla u, \nabla \mu(\theta_t \omega)) - \delta (f(u), u) + (f(u), \mu(\theta_t \omega)) - \delta (\beta(t) - \delta) (u_t, u).
\]

Now take

\[
V(u, v) = \|v\|^2 + \|\nabla u\|^2 + 2 \int_U F(u) dx,
\]

and

\[
\mathcal{R}(u, v) = (\beta(t) - \delta) \|u_t\|^2 - (\beta(t) - \delta) (u_t, \mu(\theta_t \omega)) + \delta (\beta(t) - \delta) (u_t, u)
\]

\[
- (\nabla u, \nabla \mu(\theta_t \omega)) + \delta \|\nabla u\|^2 + \delta (f(u), u) - (f(u), \mu(\theta_t \omega)) - (g, v).
\]

Thus we could deduce that from (4.8)

\[
\frac{1}{2} \frac{d}{dt} V(u, v) + \mathcal{R}(u, v) \leq 0.
\]

Integrate the above inequality over \([\tau, t]\), we end up with

\[
V(u(t), v(t)) \leq V(u(\tau), v(\tau)) - \int_{\tau}^{t} 2 \mathcal{R}(u, v) ds.
\]
Now we will give some estimates on $V$ and $\mathcal{R}$. From (1.6) we know that for some $\lambda_1 > \gamma > 0$, there exists a $k_\gamma > 0$ such that
\[
F(u) = \int_0^u f(s)ds \geq \int_0^u -(\gamma s + k_\gamma)ds = -\frac{1}{2} \gamma u^2 - k_\gamma u \geq -\gamma u^2 - C_\gamma', \quad (4.13)
\]
Choosing appropriate $\delta$ such that $\frac{\delta(\beta(t)-\delta)}{\lambda_1-\gamma} \leq \frac{1}{4}$, then we have
\[
V(u, v) = \|v\|^2 + \|\nabla u\|^2 + 2 \int_U F(u)dx
\geq \|v\|^2 + \|\nabla u\|^2 - \gamma \|u\|^2 - C_\gamma'|U|
= C_{4,1}(\|v\|^2 + \|\nabla u\|^2) - C_{4,2},
\]
where $C_{4,1}, C_{4,2} > 0$ are constants that independent of $t$ and the initial data.

Next we will estimate $\mathcal{R}(u, v)$ term by term. For the second, third, fourth and eighth terms:
\[
(\beta(t) - \delta)(u_t, \mu(\theta_t\omega)) \leq \frac{1}{4}(\beta(t) - \delta)\|u_t\|^2 + c\|\mu(\theta_t\omega)\|^2,
\]
where $\beta_1$ is a positive constant that depends on $\beta_1, \delta$ and $\gamma_1$. For the seventh term, we set
\[
f_0(s) = f(s) \cdot \chi_{[-\infty, -m)} \cup (m, \infty)](s) + \gamma s,
\]
\[
f_1(s) = f(s) \cdot \chi_{[-m, m]}(s) - \gamma s.
\]
Then we could derive that $f(s) = f_0(s) + f_1(s)$ and
\[
f_0(s) \cdot s \geq 0, \quad \forall s \in \mathbb{R}.
\]
From (1.4),
\[
|f_0(s)| \leq C_{1,1}(1 + |s|^{q+1}). \quad (4.18)
\]
And
\[
|f_1(s)| \leq C_{1,2} + \gamma|s|, \quad \forall s \in \mathbb{R}. \quad (4.19)
\]
Now for the nonlinear function $f_0(\cdot)$, we have
\[
|f_0(s)| \leq C_{1,3}|s|^{q+1}, \quad \text{as } |s| \geq 1,
\]
since $q \leq 2$. Thus
\[
|f_0(s)|^{q+2} \leq C_{1,3}|f_0(s)|^{q+1} \cdot |s|^{q+1} = C_{1,3}|f_0(s)s|^{q+1},
\]
which implies that as $|s| \geq 1$,
\[
|f_0(s)| \leq C_{1,3}(f_0(s)s)^{\frac{q+1}{2}}. \quad (4.20)
\]
From (4.15) and (4.16),
\[
(f(u), \mu(\theta_t\omega)) = (f_0 + f_1, \mu(\theta_t\omega)). \quad (4.21)
\]
Then
\[ |(f_1(u), \mu(\theta_t \omega))| \leq C_{1,2} \|\mu(\theta_t \omega)\|_{L^1} + \gamma \|u\| \cdot \|\mu(\theta_t \omega)\|, \]
and from (4.20)
\[ (f_0(u), \mu(\theta_t \omega)) \leq \|\mu(\theta_t \omega)\|_{L^{q+2}} \cdot \left( \int_U |f_0(u)|^q \frac{q+1}{q} \, dx \right) \frac{q+1}{q} \]
\[ \leq c \|\mu(\theta_t \omega)\|_{H^q_0} \cdot C_{1,3} \left( \int_U f_0(u) \cdot udx \right) \frac{q+1}{q+2} \]
\[ \leq c \|\mu(\theta_t \omega)\|_{H^q_0} + \delta \left( \int_U f_0(u) \cdot udx \right). \]
Thus from (4.21), we could obtain that
\[ (f(u), \mu(\theta_t \omega)) \leq C_{1,2} \|\mu(\theta_t \omega)\|_{L^1} + \frac{\delta}{8} (\lambda_1 - \gamma) \|u\|^2 + c \|\mu(\theta_t \omega)\|^2 
+ c \|\mu(\theta_t \omega)\|_{H^q_0} + \delta (f_0(u) \cdot u). \]
Then
\[ \delta (f(u), u) - (f, \mu(\theta_t \omega)) \geq \delta (f_1, u) - C_{1,2} \|\mu(\theta_t \omega)\|_{L^1} - \frac{\delta}{8} (\lambda_1 - \gamma) \|u\|^2 
- c \|\mu(\theta_t \omega)\|^2 - c \|\mu(\theta_t \omega)\|_{H^q_0}. \] (4.22)
From (4.16) and (4.19) we have
\[ (f_1, u) \geq -(C_{1,2} + \gamma |u|, |u|) \geq -C_{1,2} - \frac{1}{8} (\lambda_1 - \gamma) \|u\|^2 - \gamma \|u\|^2 \]
Hence we submitting the above inequality into (4.22), then
\[ \delta (f(u), u) - (f, \mu(\theta_t \omega)) \geq \delta (-C_{1,2} - \frac{1}{8} (\lambda_1 - \gamma) \|u\|^2 - \gamma \|u\|^2) - C_{1,2} \|\mu(\theta_t \omega)\|_{L^1} 
- \frac{\delta}{8} (\lambda_1 - \gamma) \|u\|^2 - c \|\mu(\theta_t \omega)\|^2 - c \|\mu(\theta_t \omega)\|_{H^q_0} 
= - \delta \left( \frac{1}{4} (\lambda_1 - \gamma) + \gamma \right) \|u\|^2 - \delta C_{1,2} - c \|\mu(\theta_t \omega)\|^2 
- c \|\mu(\theta_t \omega)\|_{H^q_0}. \]
Since (3.8) holds, we have
\[ \|v\|^2 = \|u_t\|^2 + \delta \|\mu(\theta_t \omega)\|^2 + \|\mu(\theta_t \omega)\|^2 + 2 \delta (v, u) - 2 (v, \mu(\theta_t \omega)) - 2 \delta (u, \mu(\theta_t \omega)) 
\leq (2 + \delta) \|u_t\|^2 + \frac{\delta (2 + \delta)}{\lambda_1} \|\nabla u\|^2 + (2 + \delta) \|\mu(\theta_t \omega)\|^2. \]
Consequently, there exists a $C_3 > 0$ such that
\[ \|v\|^2 + \|\nabla u\|^2 \leq C_3 (\|u_t\|^2 + \|\nabla u\| + \|\mu(\theta_t \omega)\|^2). \]
Thus we have
\[ \mathcal{R}(u, v) \geq \frac{1}{4} (\beta(t) - \delta) \|u_t\|^2 - \frac{1}{2} (\beta(t) - \delta) \|\mu(\theta_t \omega)\|^2 + \delta \frac{7 \lambda_1 + \gamma}{8 \lambda_1} \|\nabla u\|^2 
- c \|\nabla \mu(\theta_t \omega)\|^2 - C_3 g \|g\|^2 + \delta \left( \frac{5}{8} (\lambda_1 - \gamma) + \gamma \right) \|u\|^2 - c. \]
\[ -c\|\mu(\theta_1\omega)\|^2 - c\|\mu(\theta_1\omega)\|_{H^2_0}^{q+2} \]
\[ \geq \frac{1}{4}(\beta(t) - \delta)\|u_t\|^2 + \frac{\delta}{4\lambda_1}(\lambda_1 - \gamma)\|\nabla u\|^2 - C_g\|g\|^2 \]
\[ - c \left( 1 + \|\mu(\theta_1\omega)\|^2 + \|\nabla \mu(\theta_1\omega)\|^2 + \|\mu(\theta_1\omega)\|_{H^2_0}^{q+2} \right) \]
\[ \geq \sigma_1 \|(u_t)^2 + \|\nabla u\|^2\) - c \left( 1 + \|\mu(\theta_1\omega)\|^2 + \|\nabla \mu(\theta_1\omega)\|^2 + \|\mu(\theta_1\omega)\|_{H^2_0}^{q+2} \right) \]
\[ \geq \frac{\sigma_1}{C_3} \|(u_t)^2 + \|\nabla u\|^2\] 

where \(c_1 = 2c\) and \(\sigma_1 = \frac{\sigma}{C_3}\) in the above inequality. Thus from (4.8), (4.14) and replacing \(\omega\) with \(\theta_{-\tau}\omega\) we could deduce that 
\[ C_{4,1} \|(u_t)^2 + \|\nabla u\|^2\] 
\[ \leq V(u(t - \tau), v(t - \tau)) - 2 \int_{t-\tau}^t \sigma_1 (\|u_t\|^2 + \|\nabla u\|^2) ds \]
\[ + \int_{t-\tau}^t c_1 \left( 1 + \|\mu(\theta_{-\tau}\omega)\|^2 + \|\nabla \mu(\theta_{-\tau}\omega)\|^2 + \|\mu(\theta_{-\tau}\omega)\|_{H^2_0}^{q+2} \right) \]
\[ = V(u(t - \tau), v(t - \tau)) - 2 \int_{t-\tau}^t \sigma_1 (\|u_t\|^2 + \|\nabla u\|^2) ds \]
\[ + \int_{t-\tau}^t c_1 \left( 1 + \|\mu(\theta_{-\tau}\omega)\|^2 + \|\nabla \mu(\theta_{-\tau}\omega)\|^2 + \|\mu(\theta_{-\tau}\omega)\|_{H^2_0}^{q+2} \right) ds. \]

Since \(\mu(\theta_{-\tau}\omega) = \sum_{j=1}^m h_j\mu_j(\theta_{-\tau}\omega_j)\) and \(h_j \in H^1_0(U), j = 1, 2, ... m\) from (3.5) and (3.6) with \(\epsilon = \frac{\sigma}{2}\), we could obtain 
\[ 1 + \|\mu(\theta_{-\tau}\omega)\|^2 + \|\nabla \mu(\theta_{-\tau}\omega)\|^2 + \|\mu(\theta_{-\tau}\omega)\|_{H^2_0}^{q+2} \]
\[ \leq c \left( 1 + e^{\frac{\sigma t}{\tau}} r(\omega) \right). \]

Choosing \(r_0(\omega) = \left[ c \left( 1 + e^{\frac{\sigma t}{\tau}} r(\omega) \right) \right] / 2\sigma_1\) and using (4.14), we have 
\[ C_{4,1} \|(u_t)^2 + \|\nabla u\|^2\] 
\[ - C_{4,2} \leq - (t - \tau) + V(u(t), v(t)) \]

as long as \(\|u_t\|^2 + \|\nabla u\|_1^2 \geq r_0(\theta_{-\tau}\omega).\)

As a result, we could show that if \(t - \tau > V(u(t), v(t))\), then 
\[ \|u_t\|^2 + \|\nabla u\|^2 \leq \min \left\{ r_0(\omega), \frac{C_{4,2}}{C_{4,1}} \right\}. \]

It follows that we may take 
\[ R(t, \omega) = c \left( 1 + e^{\frac{\sigma t}{\tau}} r(\omega) \right) + \frac{C_{4,2}}{C_{4,1}}. \]

Therefore there is a \(T = T(\tau, U, \omega) > 0\) such that for all \(t \geq T\), we have 
\[ \|u(t, \tau - t, \theta_{-\tau}\omega, \Phi_{\tau - t})\|^2 + \|\nabla u(t, \tau - t, \theta_{-\tau}\omega, \Phi_{\tau - t})\|^2 \leq R(t, \omega). \]

Lastly we need to prove \(R(t, \omega)\) satisfies \(\lim_{t \to \infty} e^{-\beta t} R(t, \omega) = 0\). Notice that \(r(\omega)\) is tempered. We now take \(\beta = \sigma\). Then we have 
\[ e^{-\beta t} c \left( 1 + e^{\frac{\sigma t}{\tau}} r(\theta_{-\tau}\omega) \right) = ce^{-\beta t} + ce^{-\beta t} \left( 1 + e^{\frac{\sigma t}{\tau}} r(\theta_{-\tau}\omega) \right) \]
\[ \leq ce^{-\beta t} + ce^{-\beta t} e^{\frac{\sigma t}{\tau}} r(\theta_{-\tau}\omega). \]
Since \( r(\omega) \) is tempered, now taking limits for both sides of (4.26) with respect to \( t \to \infty \),
\[
\lim_{t \to \infty} e^{-\beta t} c \left( 1 + e^{-\frac{\beta t}{h^2}} r(\theta_{t_\omega}) \right) \leq \lim_{t \to \infty} c e^{-\beta t} + \lim_{t \to \infty} c e^{-\beta t} e^{-\frac{\beta t}{h^2}} r(\theta_{t_\omega}) = 0.
\]

Thus we can conclude that \( R(\tau, \omega) \) satisfies \( \lim_{t \to \infty} e^{-\alpha t} R(\tau, \omega) = 0 \). Hence we complete the proof.

**Remark 4.2.** Since each \( h_j \in H^1_0 \), from Lemma 4.1 in [6], one may further prove\[(1 + \|\mu(\theta_{t_\omega})\|^2 + \|\nabla\mu(\theta_{t_\omega})\|^2 + \|\mu(\theta_{t_\omega})\|_{H^1_0}^2) \text{ is actually tempered.}\]

5. **Asymptotic compactness.** From Theorem 4.1 we obtain that for every \( D = \{D(\tau, \omega)\}_{\omega \in \Omega} \in \mathcal{D} \) and \( P \)-a.e. \( \omega \in \Omega \), there is \( T = T(\tau, U, \omega) > 0 \) such that for all \( t > T \), the solution \((u, v)\) of problem (3.8)-(3.10) with \( \Phi_{\tau-t} = (u_{\tau-t}, v_{\tau-t}) \in B(\tau - t, \theta_{-\tau_\omega}) \) satisfies\[(1.4) \quad \|\Phi(t, \tau-t, \theta_{-\tau_\omega}, \Phi_{\tau-t})\|_{H^1_0(U)}^2 + \|v(t, \tau-t, \theta_{-\tau_\omega}, \Phi_{\tau-t})\|_{L^2(U)}^2 \leq R(\tau, \omega),
\]
where \( R(\tau, \omega) \) is the radius of the absorbing set. Thus with (3.13) we could deduce that for all \( t \geq T, \omega \in \Omega \),
\[
\|\Phi(t, \tau-t, \theta_{-\tau_\omega}, (u_{\tau-t}, v_{\tau-t}))\|_{H^1_0(U) \times L^2(U)}^2 \leq R(\tau, \omega).
\]
Take
\[
B(\tau, \omega) = \left\{(u, v) \in H^1_0(U) \times L^2(U) : \|u\|_{H^1_0(U)}^2 + \|v\|_{L^2(U)}^2 \leq R(\tau, \omega)\right\},
\]
which shows that \( B(\tau, \omega) \) is a closed random absorbing set for \( \Phi \) in \( \mathcal{D} \). In order to prove the existence of random attractor for problem (3.8)-(3.10), we need to show the asymptotic compactness of \( \Phi \).

**Lemma 5.1.** Assume \( g \in C_b(\mathbb{R}; L^2(U)) \), \( h_j \in H^1_0(U), j = 1, 2, \ldots, m \) and (1.4)-(1.6) are satisfied. Then for \( P \)-a.e. \( \omega \in \Omega \), the sequence
\[
\{u(\tau, \tau-t_n, \theta_{-\tau_\omega}, \Phi_{\tau-t_n}), v(\tau, \tau-t_n, \theta_{-\tau_\omega}, \Phi_{\tau-t_n})\}_{n \in \mathbb{N}^+}
\]
has a convergent subsequence in \( H^1_0(U) \times L^2(U) \), provided that \( t_n \to \infty \) and \( \Phi_{\tau-t_n} = (u_{\tau-t_n}, v_{\tau-t_n}) \in D(\theta_{-t_n} \omega) \) with \( D = \{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D} \).

**Proof.** Since \( t_n \to \infty \), it follows from (5.1) that there is \( N_1 = N_1(B, \omega) > 0 \) such that for all \( n \geq N_1 \),
\[
\|u(\tau, \tau-t_n, \theta_{-\tau_\omega}, \Phi_{\tau-t_n})\|_{H^1_0(U)}^2 + \|v(\tau, \tau-t_n, \theta_{-\tau_\omega}, \Phi_{\tau-t_n})\|_{L^2(U)}^2 \leq R(\tau, \omega).
\]

Since \( H^1_0(U) \times L^2(U) \) is reflexive, we could obtain that there is \((\tilde{u}, \tilde{v}) \in H^1_0(U) \times L^2(U) \) such that there is subsequence satisfies
\[
(u(\tau, \tau-t_n, \theta_{-\tau_\omega}, \Phi_{\tau-t_n}), v(\tau, \tau-t_n, \theta_{-\tau_\omega}, \Phi_{\tau-t_n})) \to (\tilde{u}, \tilde{v})
\]
weakly in \( H^1_0(U) \times L^2(U) \), which implies that
\[
\liminf_{n \to \infty} \|u(\tau, \tau-t_n, \theta_{-\tau_\omega}, \Phi_{\tau-t_n})\|_{H^1_0(U)}^2 + \|v(\tau, \tau-t_n, \theta_{-\tau_\omega}, \Phi_{\tau-t_n})\|_{L^2(U)}^2
\geq \|(\tilde{u}, \tilde{v})\|^2_{H^1_0(U) \times L^2(U)}.
\]
Next we want to prove (5.4) is actually a strong convergence. In order to prove this, we just need to show that
\[
\limsup_{n \to \infty} \|u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n})\|_{H^1_0(U)}^2 + \|v(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n})\|_{H^1_0(U)}^2 \leq \|(\tilde{u}, \tilde{v})\|_{H^1_0(U) \times L^2(U)}^2.
\] (5.5)

We have following three steps to prove the above inequality.

**Step One.** Weak continuity. Here we analysis the weak continuity of $u$ and $v$, which is a basic result for applying the energy method in [3] and [28].

From Theorem 4.1, there exits $N_2 = N_2(D, \omega) > 0$ such that for all $n \geq N_2$,
\[
\|u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n})\|_{H^1_0(U)}^2 + \|v(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n})\|_{H^1_0(U)}^2 \leq R(\tau, \omega),
\] (5.6)
where $-t_n \leq t \leq 0$. Given $m > 0$, let $N_3 = N_3(m) > 0$ be large enough such that $t_n \geq m$ for all $n \geq N_3$. Denote by $N_4 = \max\{N_2, N_3\}$. Then for all $n \geq N_4$, we have
\[
\|u(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n})\|_{H^1_0(U)}^2 + \|v(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n})\|_{H^1_0(U)}^2 \leq R(\tau, \omega).
\] (5.7)

By a diagonal procedure, we could conclude from (5.7) that there exists a sequence $\{\tilde{u}_m, \tilde{v}_m\}_{m=1}^\infty$ in $H^1_0(U) \times L^2(U)$ and a subsequence of $\{u_n, \Phi_{\tau-t,n}\}_{n=1}^\infty$ (not relabel) such that for every positive integer $m$, when $n \to \infty$,
\[
\Phi_m := (u(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n}), v(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n})) \rightarrow (\tilde{u}_m, \tilde{v}_m)
\] (5.8)
weakly in $H^1_0(U) \times L^2(U)$. Notice that
\[
(u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n}), v(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n})) = (u(\tau - m, \theta_{-\tau} \omega, \Phi_m), v(\tau - m, \theta_{-\tau} \omega, \Phi_m)).
\] (5.9)

Thus we could deduce that for every positive integer $m$, when $n \to \infty$
\[
u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n}) \rightarrow \nu(\tau, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m))
\] weakly in $H^1_0(U)$
and
\[v(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau-t,n}) \rightarrow v(\tau, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m))
\] weakly in $L^2(U)$, since Lemma 3.2 holds. Hence we know
\[
\tilde{u} = u(\tau, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m)) \quad \text{and} \quad \tilde{v} = v(\tau, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m)).
\]

**Step Two.** Energy inequality. Here we mainly analysis the energy functional. By virtue of Fatou’s Lemma, Sobolev’s embedding and Hölder inequality, an energy inequality has been obtained.

Take
\[
E(u, v) = \|v\|^2 + (\delta^2 - \delta \beta(t))\|u\|^2 + \|\nabla u\|^2 + 2 \int U F(u)dx
\] (5.10)
and
\[
\Psi(u, v) = -2(\beta(t) - \delta - 2\sigma)\|v\|^2 - 2(\delta - 2\sigma)(\delta^2 - \delta^2(\beta(t)))\|u\|^2 \\
- 2\delta\beta'(t)\|u\|^2 - 2(\delta - 2\sigma)\|\nabla u\| + 8\sigma \int_U F(U) dx - 2\delta \int_U f(u) \cdot u dx \\
+ 2 \big(\delta^2 - \delta^2(\beta(t))\big)(u, \mu(\theta_\omega)) + 2(\nabla u, \nabla \mu(\theta_\omega)) + 2\int_U f(u) \mu(\theta_\omega) dx \\
+ 2(g, v) + 2(\delta - \beta(t))(v, \mu(\theta_\omega)).
\]  

By virtue of (5.10) and (5.11), we could obtain
\[
\frac{dE}{dt} + 4\sigma E = \Psi.
\]  

Integrating (5.12) on $(\tau, t)$ we get
\[
E(u(t, \tau, \omega, \Phi_\tau), v(t, \tau, \omega, \Phi_\tau)) \\
= e^{-4\sigma(t-\tau)}E(u_\tau, v_\tau) + \int_\tau^t e^{4\sigma(t-\tau)}\Psi(u(\xi, \tau, \omega, \Phi_\tau), v(\xi, \tau, \omega, \Phi_\tau))d\xi.
\]  

Applying (5.13) to \( (u(\tau, \tau - m, \theta_\tau, \omega, (\bar{u}_m, \bar{v}_m)), v(\tau, \tau - m, \theta_\tau, \omega, (\bar{u}_m, \bar{v}_m))) \), by (5.12) we get
\[
E(\bar{u}, \bar{v}) \\
= e^{-4\sigma m}E(\bar{u}_m, \bar{v}_m) + \int_{\tau-m}^\tau e^{4\sigma(\xi-\tau)} \\
\quad \cdot \Psi(u(\xi, \tau - m, \theta_\tau, \omega, (\bar{u}_m, \bar{v}_m)), v(\xi, \tau - m, \theta_\tau, \omega, (\bar{u}_m, \bar{v}_m + m))d\xi.
\]  

Applying (5.13) to
\[
(u(\tau, \tau - m, \theta_\tau, \omega, \Phi_m), v(\tau, \tau - m, \theta_\tau, \omega, \Phi_m)),
\]  

by (5.12) and (5.9) we get
\[
E(u(\tau - t_n, \theta_\tau, \omega, \Phi_t), v(\tau - t_n, \theta_\tau, \omega, \Phi_t)) \\
= e^{-4\sigma m}E(u(\tau - m, \tau - t_n, \theta_\tau, \omega, \Phi_{t-n}), v(\tau - m, \tau - t_n, \theta_\tau, \omega, \Phi_{t-n})) \\
+ \int_{\tau-n}^{\tau} e^{4\sigma(\xi-\tau)}\Psi(u(\xi, \tau - m, \theta_\tau, \omega, \Phi_m), v(\xi, \tau - m, \theta_\tau, \omega, \Phi_m))d\xi \\
= e^{-4\sigma m}E(u(\tau - m, \tau - t_n, \theta_\tau, \omega, \Phi_{t-n}), v(\tau - m, \tau - t_n, \theta_\tau, \omega, \Phi_{t-n})) \\
- 2\int_{\tau-n}^{\tau} (\beta(t) - \delta - 2\sigma)e^{4\sigma\xi}\|v(\xi, \tau - m, \theta_\tau, \omega, \Phi_m))\|^2 d\xi \\
- 2\int_{\tau-n}^{\tau} (\delta - 2\sigma)(\delta^2 - \delta^2(\beta(t)))e^{4\sigma\xi}\|u(\xi, \tau - m, \theta_\tau, \omega, \Phi_m))\|^2 d\xi \\
- 2\int_{\tau-n}^{\tau} \delta\beta'(\xi)e^{4\sigma\xi}\|u(\xi, \tau - m, \theta_\tau, \omega, \Phi_m))\|^2 d\xi \\
- 2\int_{\tau-n}^{\tau} (\delta - 2\sigma)e^{4\sigma\xi}\|\nabla u(\xi, \tau - m, \theta_\tau, \omega, \Phi_m))\|^2 d\xi \\
+ 8\sigma \int_{\tau-n}^{\tau} e^{4\sigma\xi} \int_U F(u(\xi, \tau - m, \theta_\tau, \omega, \Phi_m)) dx d\xi \\. 
\[ -2\delta \int_{\tau-m}^{\tau} e^{4\sigma \xi} \int_{U} f(u(\xi, \tau-m, \theta_{-\tau}\omega, \Phi_{m})) \cdot u(\xi, \tau-m, \theta_{-\tau}\omega, \Phi_{m}) \, dx \, d\xi \]
\[ + 2 \int_{\tau-m}^{\tau} (\delta^2 - \delta \beta(\xi)) e^{4\sigma \xi} \int_{U} u(\xi, \tau-m, \theta_{-\tau}\omega, \Phi_{m}) \cdot \mu(\theta_{\xi}\omega) \, dx \, d\xi \]
\[ + 2 \int_{\tau-m}^{\tau} e^{4\sigma \xi} \int_{U} \nabla u(\xi, \tau-m, \theta_{-\tau}\omega, \Phi_{m}) \cdot \nabla \mu(\theta_{\xi}\omega) \, dx \, d\xi \]
\[ + 2 \int_{\tau-m}^{\tau} e^{4\sigma \xi} \int_{U} f(u(\xi, \tau-m, \theta_{-\tau}\omega, \Phi_{m})) \cdot \mu(\theta_{\xi}\omega) \, dx \, d\xi \]
\[ + 2 \int_{\tau-m}^{\tau} e^{4\sigma \xi} \int_{U} g(x, \xi) \cdot v(\xi, \tau-m, \theta_{-\tau}\omega, \Phi_{m}) \, dx \, d\xi \]
\[ + 2 \int_{\tau-m}^{\tau} e^{4\sigma \xi} (\delta - \beta(\xi)) \int_{U} v(\xi, \tau-m, \theta_{-\tau}\omega, \Phi_{m}) \cdot \mu(\theta_{\xi}\omega) \, dx \, d\xi. \]  

(5.15)

Now we need to deal with every term on the right-hand side of (5.15). For the first term, by the definition of \( E \), we have

\[ e^{-4\sigma m} E\left(u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}), v(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n})\right) \]
\[ + e^{-4\sigma m} \left(\left\| v(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}) \right\|^2 + (\delta^2 - \delta \beta(t)) \left\| u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}) \right\|^2 \right) \]
\[ + e^{-4\sigma m} \left(\left\| \nabla u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}) \right\|^2 \right. \]
\[ + 2 \int_{U} F(u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n})) \, dx \right) \]

which along with (5.7) shows that for all \( n \leq N_4 \)

\[ e^{-4\sigma m} E\left(u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}), v(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n})\right) \]
\[ \leq c e^{-4\sigma m} R(\omega) + 2 e^{-4\sigma m} \int_{U} F(u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n})) \, dx. \]  

(5.16)

Since \( f \) satisfies (1.4) and (1.6), thus we could obtain that

\[ F(u) \leq C_1 |u| + \frac{C_1}{q+2} |u|^q+2 \]
\[ \leq c |u| + |u|^q+2 \]  

(5.17)

Thus with Hölder inequality we have the estimate for the second term on the right-hand side of (5.16):

\[ \int_{U} F(u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n})) \, dx \]
\[ \leq c \left( \left\| u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}) \right\|_{H^q(U)}^{q+2} + \left\| u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}) \right\|^{q+2} \right) \]
\[ \leq c \left( \left\| u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}) \right\|_{H^q(U)}^{q+2} + \left\| u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n}) \right\|^{q+2} \right), \]

which along with (5.7) indicates that for \( n \geq N_4 \),

\[ \int_{U} F(u(\tau-m, \theta_{-\tau}\omega, \Phi_{-t,n})) \leq c \left( R(\tau-m, \theta_{-\tau}\omega) + R^2(\tau-m, \theta_{-\tau}\omega) + 1 \right). \]  

(5.18)
By virtue of (5.16) and (5.18) we get that for all \( n \geq N_4 \),
\[
e^{-4\sigma m} E(u(\tau - m, \tau - t_n, \theta_{-\tau}, \Phi_{\tau - t_n}), v(\tau - m, \tau - t_n, \theta_{-\tau}, \Phi_{\tau - t_n})) \leq c e^{-4\sigma m} (1 + R^2(\tau - m, \theta_{-\tau})) .
\] (5.19)

Next we deal with the second term on the right-hand side of (5.15). By Theorem 3.1 and (5.8), we find that for every \( \xi \in [\tau - m, \tau] \), when \( n \to \infty \),
\[
v(\xi, \tau - m, \theta_{-\tau}, \Phi_m) \to v(\xi, \tau - m, \theta_{-\tau}, (\bar{u}_m, \bar{v}_m)) \text{ weakly in } L^2(U),
\]
which implies that
\[
\liminf_{n \to \infty} \|v(\xi, \tau - m, \theta_{-\tau}, \Phi_m)\|^2 \geq \|v(\xi, \tau - m, \theta_{-\tau}, (\bar{u}_m, \bar{v}_m))\|^2 .
\]

Thus the inequality above and Fatou’s Lemma we obtain
\[
\liminf_{n \to \infty} \int_{\tau - m}^{\tau} e^{4\sigma \xi} \|v(\xi, \tau - m, \theta_{-\tau}, \Phi_m)\|^2 d\xi
\geq \int_{\tau - m}^{\tau} e^{4\sigma \xi} \liminf_{n \to \infty} \|v(\xi, \tau - m, \theta_{-\tau}, \Phi_m)\|^2 d\xi
\geq \int_{\tau - m}^{\tau} e^{4\sigma \xi} \|v(\xi, \tau - m, \theta_{-\tau}, (\bar{u}_m, \bar{v}_m))\|^2 d\xi .
\]

By the definition of \( \delta \), we have
\[
\limsup_{n \to \infty} -2 \int_{\tau - m}^{\tau} (\beta(\xi) - \delta - 2\sigma) e^{4\sigma \xi} \|v(\xi, \tau - m, \theta_{-\tau}, \Phi_m)\|^2 d\xi
\leq -2 \int_{\tau - m}^{\tau} (\beta(\xi) - \delta - 2\sigma) e^{4\sigma \xi} \|v(\xi, \tau - m, \theta_{-\tau}, (\bar{u}_m, \bar{v}_m))\|^2 d\xi .
\] (5.20)

By the definition of \( \delta \), (5.15) and Fatou’s Lemma, we could also prove that
\[
\limsup_{n \to \infty} -2(\delta - 2\sigma) \int_{\tau - m}^{\tau} e^{4\sigma \xi} \|\nabla u(\xi, \tau - m, \theta_{-\tau}, \Phi_m)\|^2 d\xi
\leq -2(\delta - 2\sigma) \int_{\tau - m}^{\tau} e^{4\sigma \xi} \|\nabla u(\xi, \tau - m, \theta_{-\tau}, (\bar{u}_m, \bar{v}_m))\|^2 d\xi .
\] (5.21)

Next, we prove the convergence of the sixth term on the right-hand side of (5.15), which is a nonlinear term. We claim that
\[
\lim_{n \to \infty} - \int_{\tau - m}^{\tau} \left[ 2(\delta - 2\sigma)(\delta^2 - \delta \beta(\xi)) + 2\delta \beta'(\xi) \right] e^{4\sigma \xi} \|u(\xi, \tau - m, \theta_{-\tau}, \Phi_m)\|^2 d\xi
= - \int_{\tau - m}^{\tau} \left[ 2(\delta - 2\sigma)(\delta^2 - \delta \beta(\xi)) + 2\delta \beta'(\xi) \right] e^{4\sigma \xi} \|u(\xi, \tau - m, \theta_{-\tau}, (\bar{u}_m, \bar{v}_m))\|^2 d\xi
\] (5.22)

and
\[
\lim_{n \to \infty} \int_{\tau - m}^{\tau} e^{4\sigma \xi} \int_{U} F(u(\xi, \tau - m, \theta_{-\tau}, \Phi_m)) dx d\xi
= \int_{\tau - m}^{\tau} e^{4\sigma \xi} \int_{U} F(u(\xi, \tau - m, \theta_{-\tau}, (\bar{u}_m, \bar{v}_m))) dx d\xi .
\] (5.23)

By Theorem 3.1 and (5.8), we obtain that
\[
u(\xi, \tau - m, \theta_{-\tau}, \Phi_m) \to u(\xi, \tau - m, \theta_{-\tau}, (\bar{u}_m, \bar{v}_m)) \text{ weakly in } H^1_0(U)
\] (5.24)
for $\xi \in [\tau - m, \tau]$. By (5.24) and the compactness of embedding $H^1_0(U) \hookrightarrow L^2(U)$, we find that for $\xi \in [\tau - m, \tau]$,

$$u(\xi, \tau - m, \theta_{-\tau}\omega, \Phi_m) \to u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m))$$

strongly in $L^2(U)$ \hspace{1cm} (5.25)

Thus (5.22) holds obviously. By the mean value theorem for integrals, (1.4) and (5.7), we obtain

$$\left| \int_U F(u(\xi, \tau - m, \theta_{-\tau}\omega, \Phi_m)) - F(u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m))) \right| \dx$$

$$\leq \int_U f(\bar{u})(u(\xi, \tau - m, \theta_{-\tau}\omega, \Phi_m) - u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m))) \dx$$

$$\leq \left( \int_U |f(\bar{u})|^2 \dx \right)^{\frac{1}{2}} \left| u(\xi, \tau - m, \theta_{-\tau}\omega, \Phi_m) - u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m)) \right|$$

$$\leq C_1(1 + |\bar{u}|^{q+1}) \left| u(\xi, \tau - m, \theta_{-\tau}\omega, \Phi_m) - u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m)) \right|.$$ \hspace{1cm} (5.26)

Since $\bar{u} = \mu u(\xi, \tau - m, \theta_{-\tau}\omega, \Phi_m) + (1 - \mu)u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m))$, which along with (5.6) show that

$$(1 + |\bar{u}|^{q+1}) \leq C \left( \left| u(\xi, \tau - m, \theta_{-\tau}\omega, \Phi_m) \right|_{H^1_0(U)}^{q+1} + \left| u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m)) \right|^{q+1} \right)$$

$$\leq C \left( e^{-\sigma_0(\xi^{q+1})} R_{\alpha(\xi)}^{q+1} \omega + \left| u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m)) \right|^{q+1} \right).$$ \hspace{1cm} (5.27)

Thus from (5.25)-(5.27), we could deduce that as $n \to \infty$

$$\int_U F(u(\xi, \tau - m, \theta_{-\tau}\omega, \Phi_m)) \dx$$

$$\to \int_U F(u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m))) \dx.$$ \hspace{1cm} (5.28)

It follows from (5.6), (5.28) and the Dominated Convergence Theorem that there exists $N_3 \geq N_4$ such that for all $n \leq N_3$, we have (5.23) holds. Thus we complete the proof of the claim. By an argument similarly to the proof of (5.23), we can also show the convergence of the remaining terms on the right-hand side of (5.15). Now, taking the limit of (5.15) as $n \to \infty$, by (5.19), (5.20)-(5.23) and the convergence of the rest terms of (5.15) we find that

$$\limsup_{n \to \infty} E(u(\tau, \tau - t_n, \theta_{-\tau}\omega, \Phi_{\tau - t_n, n}), v(\tau, \tau - t_n, \theta_{-\tau}\omega, \Phi_{\tau - t_n, n}))$$

$$\leq Ce^{-2\sigma m}(1 + R^2_\omega) - 2 \int_{\tau - m}^{\tau} (\beta(\xi) - \delta - 2\sigma) e^{4\xi} \left\| u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m)) \right\|^2 d\xi$$

$$- \int_{\tau - m}^{\tau} 2(\delta - 2\sigma)(\delta^2 - \delta \beta(\xi)) + 2\delta \beta(\xi)) e^{4\xi} \left\| u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m)) \right\|^2 d\xi$$

$$- \int_{\tau - m}^{\tau} 2(\delta - 2\sigma) e^{4\xi} \left\| \nabla u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m)) \right\|^2 d\xi$$

$$+ 8\sigma \int_{\tau - m}^{\tau} e^{4\xi} \int_U F(u(\xi, \tau - m, \theta_{-\tau}\omega, (\tilde{u}_m, \tilde{v}_m))) \dx d\xi.$$
\[-2\delta \int_{\tau_m}^{\tau} e^{4\sigma \xi} \int_U u(\xi, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m)) f(u(\xi, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m))) d\xi d\tau + 2 \int_{\tau_m}^{\tau} (\delta^2 - \delta \beta(\xi)) e^{4\sigma \xi} \int_U u(\xi, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m)) \mu(\theta_{-\tau} \omega) d\xi d\tau + 2 \int_{\tau_m}^{\tau} e^{4\sigma \xi} \int_U \nabla u(\xi, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m)) \nabla \mu(\theta_{-\tau} \omega) d\xi d\tau + 2 \int_{\tau_m}^{\tau} e^{4\sigma \xi} \int_U f(u(\xi, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m))) \mu(\theta_{-\tau} \omega) d\xi d\tau + 2 \int_{\tau_m}^{\tau} e^{4\sigma \xi} \int_U g(x, \xi) v(\xi, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m)) \mu(\theta_{-\tau} \omega) d\xi d\tau + 2 \int_{\tau_m}^{\tau} e^{4\sigma \xi}(\delta - \beta(\xi)) \int_U v(\xi, \tau - m, \theta_{-\tau} \omega, (\tilde{u}_m, \tilde{v}_m)) \mu(\theta_{-\tau} \omega) d\xi d\tau.\]

From the inequality above and (5.11), we could deduce that
\[
\limsup_{n \to \infty} E(u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n}), v(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n})) \leq e^{-4\sigma m} (1 + R^2(\tau, \omega)) + e^{-4\sigma m} E(\tilde{u}_m, \tilde{v}_m) + E(\tilde{u}, \tilde{v}).
\]

**Step Three.** Strong convergence. For the second term on the right-hand side of (5.29), by virtue of (1.6) and (5.10) we have
\[
-e^{-4\sigma m} E(\tilde{u}_m, \tilde{v}_m) \leq 2e^{-4\sigma m} C_\gamma |U| + 2e^{-4\sigma m} \gamma \|u\|^2.
\]

Thus it follows that
\[
\limsup_{n \to \infty} E(u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n}), v(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n})) \leq e^{-4\sigma m} (1 + R^2(\tau, \omega)) + 2e^{-4\sigma m} |U| + E(\tilde{u}, \tilde{v}).
\]

Let \(m \to \infty\), we could deduce that
\[
\limsup_{n \to \infty} E(u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n}), v(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n})) \leq E(\tilde{u}, \tilde{v}).
\]

At the same time, take \(\xi = \tau\), we have
\[
\lim_{\tau \to \infty} \int_U F(u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n})) dx = \int_U F(\tilde{u}) dx,
\]
along with (5.10) implies that
\[
\limsup_{n \to \infty} E(u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n}), v(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n})) = \limsup_{n \to \infty} \left(\|u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n})\|^2 + (\delta^2 - \delta \beta(t)) \|u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n})\|^2 + \|\nabla u(\tau, \tau - t_n, \theta_{-\tau} \omega, \Phi_{\tau - t_n, n})\|^2\right) + 2 \int_U F(\tilde{u}) dx.
\]
this case, we generalize the conditions on nonlinearity. Actually, if one only want to
consider the finite fractal dimension and the existence of random pullback attractor. Under
these assumptions, we assume that

And if the system $\Phi$ is pullback asymptotically compact in $H^1(U)$, then we can apply
Lemma 5.2.

Remark 5.5. In this paper, we assume $g \in C_b(\mathbb{R}; L^2(U))$ in order to guarantee
the finite fractal dimension and the existence of random pullback attractor. Under
this case, we generalize the conditions on nonlinearity. Actually, if one only want to
consider the finite fractal dimension and the existence of random pullback attractor. Under
these assumptions, we assume that

$$\limsup_{n \to \infty} \left( \|v(\tau, \tau-t_n, \theta_{-\tau} \omega, \Phi_{\tau-t_n})\|^2 + \|\nabla u(\tau, \tau-t_n, \theta_{-\tau} \omega, \Phi_{\tau-t_n})\|^2 
+ (\delta^2 - \delta \beta(t))\|u(\tau, \tau-t_n, \theta_{-\tau} \omega, \Phi_{\tau-t_n})\|^2 \right)$$

$$\leq \|\tilde{v}\|^2 + (\delta^2 - \delta \beta(t))\|\tilde{u}\|^2 + \|\nabla \tilde{u}\|^2.$$  

(5.31)

Notice that the left and the right expressions in the above inequality are equivalent norms of $H^1(U) \times L^2(U)$. Therefore, we find that

$$\limsup_{n \to \infty} \left( \|u(\tau, \tau-t_n, \theta_{-\tau} \omega, \Phi_{\tau-t_n})\|^2_{H^1(U)} + \|v(\tau, \tau-t_n, \theta_{-\tau} \omega, \Phi_{\tau-t_n})\|^2 \right)$$

$$\leq \|\tilde{u}\|^2_{H^1(U)} + \|\tilde{v}\|^2.$$

Thus we have (5.5) holds. Finally, we get the strong convergence:

$$(u(\tau, \tau-t_n, \theta_{-\tau} \omega, \Phi_{\tau-t_n}), v(\tau, \tau-t_n, \theta_{-\tau} \omega, \Phi_{\tau-t_n})) \to (\tilde{u}, \tilde{v})$$ strongly in $H^1(U) \times L^2(U)$.

Hence we completes the proof. ∎

Easily from the consequence of Lemma 5.1, we obtain that the random dynamical system $\Phi$ is pullback asymptotically compact in $H^1(U) \times L^2(U)$.

Lemma 5.2. Assume that $g \in C_b(\mathbb{R}; L^2(U))$, $h_j \in H^1_0(U)$, $j = 1, 2, \ldots, m$ and

$$\text{(1.4)} - \text{(1.6)} \text{ are satisfied. Then the random dynamical system } \Phi \text{ is } \mathcal{D}-\text{pullback asymptotically compact in } H^1_0(U) \times L^2(U).$$

Consequently, we have the existence of the random attractor for the stochastic wave equation:

Theorem 5.3. Assume that $g \in C_b(\mathbb{R}; L^2(U))$, $h_j \in H^1_0(U)$, $j = 1, 2, \ldots, m$ and

$$\text{(1.4)} - \text{(1.6)} \text{ are satisfied. Then the random dynamical system } \Phi \text{ has a unique } \mathcal{D}-\text{random attractor } \{A(\tau, \omega)\}_{\omega \in \Omega} \text{ in } H^1_0(U) \times L^2(U).$$

Corollary 5.4. Assume that $h_j \in H^1_0(U)$, $j = 1, 2, \ldots, m$, (1.4)-(1.6) are satisfied,

$\beta(t)$ and $g \in C_b(\mathbb{R}; L^2(U))$ are periodic with periodicity $\tilde{T}$. Then the random dynamical system $\Phi$ is $\tilde{T}$-periodic and the $\mathcal{D}$-random attractor $\{A(\tau, \omega)\}_{\omega \in \Omega}$ is also $
\tilde{T}$-periodic in $H^1_0(U) \times L^2(U)$.

Proof. As the argument in [29], and according to (3.12) and the well-posedness of

$$\text{(1.1)} - \text{(1.3)}, \text{ one could easily figure out that}$$

$$\Phi(t, \tau + T, \omega, \cdot)$$

$$=(u(\tau + T + t, \tau + T, \theta_{-\tau} \omega, \cdot), z(\tau + T + t, \tau + T, \theta_{-\tau} \omega, \cdot))$$

$$=(u(\tau + t, \tau, \theta_{-\tau} \omega, \cdot), z(\tau + t, \tau, \theta_{-\tau} \omega, \cdot))$$

$$=\Phi(t, \tau, \omega, \cdot).$$

By (4.25), the absorbing set is $\tilde{T}$-periodic since the constant $c$ in (4.25) depends
on $g$. Therefore by Theorem 5.1, the random attractor $A(\tau, \omega)$ is also periodic. ∎
prove the existence of random attractor, the force term could be \( g \in L^2_{ loc}(\mathbb{R}; L^2(U)) \) and \( \sup_{s \leq \tau_0} \int_{-\infty}^{t_0} e^{\alpha t} \|g(\sigma + s)\|^2 \, ds < \infty \) for some \( \alpha > 0 \) and \( \tau_0 \in \mathbb{R} \). But as a price for this restriction on \( g \), the sign condition on the nonlinearity should be for some positive constants \( \zeta_1, \zeta_2 \),

\[
\begin{cases}
  f(s)s \geq C \int_{\tau_0}^{t_0} f(s) \, ds + \zeta_1; \\
  \int_{\tau_0}^{t_0} f(s) \, ds \geq C |s|^{q+2} - \zeta_2.
\end{cases}
\]

With the conditions above, we could also prove the existence of the random pullback attractor.

6. Fractal dimension of random attractor. In this section, we obtain an upper bound of the fractal dimension of \( A(\tau, \omega) \) for \( \Phi \) by checking that \( A(\tau, \omega) \) satisfies the conditions of Theorem 2.11 one by one.

It is easy to deduce that \( A(\tau, \omega) \) satisfies conditions (H1) and (H2). We just need to verify \( A(\tau, \omega) \) satisfies (H3).

For any \( \tau \in \mathbb{R}, \omega \in \Omega \), and any \( (u^{(1)}, u^{(1)}_t), (u^{(2)}, u^{(2)}_t) \in A(\tau, \omega) \). Note \( w = u^{(1)} - u^{(2)} \) and \( w_t = u^{(1)}_t - u^{(2)}_t \).

**Lemma 6.1.** There exists a tempered random variable \( C_5(\omega) > 0 \) such that for any \( \tau \in \mathbb{R}, t \geq 0, \omega \in \Omega \) and any \( (u^{(1)}, u^{(1)}_t), (u^{(2)}, u^{(2)}_t) \in A(\tau, \omega) \), it holds that

\[
\|u^{(1)}_t - u^{(2)}_t\| + \|u^{(1)} - u^{(2)}\| \leq e^{t_0 C_5(\theta, \omega)} \left( \|u^{(1)}_{\tau, \tau} - u^{(2)}_{\tau, \tau}\| + \|u^{(1)} - u^{(2)}\|_{H^1_t} \right),
\]

where \( u^{(1)} = u^{(1)}(t + \tau, \tau, \theta_{-\tau}, \omega, (u^{(1)}_{\tau, \tau}, u^{(1)})) \).

**Proof.** From (1.1), we have

\[
u^{(1)}_t - u^{(2)}_t + \beta(t)(u^{(1)}_t - u^{(2)}_t) - \Delta(u^{(1)} - u^{(2)}) = f(u^{(2)}) - f(u^{(1)}).
\]

Taking inner product of the above equation with \( (u^{(1)}_t - u^{(2)}_t) \), we find that

\[
\frac{1}{2} \frac{d}{dt} \|u^{(1)}_t - u^{(2)}_t\|^2 + \beta(t) \|u^{(1)}_t - u^{(2)}_t\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla(u^{(1)} - u^{(2)})\|^2 \\
= (f(u^{(2)}) - f(u^{(1)}), u^{(1)}_t - u^{(2)}_t)
\]

\[
\leq C \|f(u^{(2)}) - f(u^{(1)})\|^2 + \beta_0 \|u^{(1)}_t - u^{(2)}_t\|^2.
\]

Then

\[
\frac{1}{2} \frac{d}{dt} \|u^{(1)}_t - u^{(2)}_t\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla(u^{(1)} - u^{(2)})\|^2 \leq \|f(u^{(2)}) - f(u^{(1)})\|^2.
\]

From (1.4), we could easily find that

\[
\|f(u^{(2)}) - f(u^{(1)})\|^2 \leq C \int_{\mathbb{R}} \left( 1 + |u^{(1)}|^{2q} + |u^{(2)}|^{2q} \right) \|u^{(1)}_t - u^{(2)}_t\|^2 \, dx
\]

\[
\leq C \left( 1 + R^2(\theta, \omega) \right) \|u^{(1)} - u^{(2)}\|^2.
\]
Thus we can find
\[
\frac{d}{dt} \left\| u_t^{(1)} - u_t^{(2)} \right\|^2 + \frac{d}{dt} \left\| \nabla (u_t^{(1)} - u_t^{(2)}) \right\|^2 \\
\leq c \left( 1 + R^{2q}(\theta, \omega) \right) \left( \left\| \nabla (u_t^{(1)} - u_t^{(2)}) \right\|^2 + \left\| u_t^{(1)} - u_t^{(2)} \right\|^2 \right). \tag{6.6}
\]
From Gronwall’s inequality, we have
\[
\left\| u_t^{(1)} - u_t^{(2)} \right\|^2 + \left\| \nabla (u_t^{(1)} - u_t^{(2)}) \right\|^2 \\
\leq e^{c \int_0^1 (1 + R^{2q}(\theta, \omega)) ds} \left( \left\| u_{t, \tau}^{(1)} - u_{t, \tau}^{(2)} \right\|^2 + \left\| \nabla (u_{t, \tau}^{(1)} - u_{t, \tau}^{(2)}) \right\|^2 \right) \\
\leq e^{c \int_0^1 (1 + R^{2q}(\theta, \omega)) ds} \left( \left\| u_t^{(1)} - u_t^{(2)} \right\|^2 + \left\| \nabla (u_t^{(1)} - u_t^{(2)}) \right\|^2 \right). \tag{6.7}
\]
By \( \omega \to \theta - \tau \) and \( r \to t + \tau \), we have
\[
\left\| u_t^{(1)}(t + \tau, \tau, \theta - \tau, \varphi(1)_{\tau}) - u_t^{(2)}(t + \tau, \tau, \theta - \tau, \varphi(2)_{\tau}) \right\|^2 \\
+ \left\| \nabla (u_t^{(1)}(t + \tau, \tau, \theta - \tau, \varphi(1)_{\tau}) - u_t^{(2)}(t + \tau, \tau, \theta - \tau, \varphi(2)_{\tau})) \right\|^2 \\
= e^{c \int_0^1 (1 + R^{2q}(\theta, \omega)) ds} \left( \left\| u_{t, \tau}^{(1)} - u_{t, \tau}^{(2)} \right\|^2 + \left\| \nabla (u_{t, \tau}^{(1)} - u_{t, \tau}^{(2)}) \right\|^2 \right) \\
\leq e^{c \int_0^1 (1 + R^{2q}(\theta, \omega)) ds} \left( \left\| u_t^{(1)} - u_t^{(2)} \right\|^2 + \left\| \nabla (u_t^{(1)} - u_t^{(2)}) \right\|^2 \right). \tag{6.8}
\]
Take \( C_5(\theta) = 1 + R^{2q}(\theta, \omega) \). The proof is completed. \( \square \)

Let \( H_n(U) = \text{span}\{e_1, e_2, \ldots, e_n\}, H_n^\perp = \text{span}\{e_{n+1}, e_{n+2}, \ldots\} \), \( n \in \mathbb{N} \), then \( H_n(U) \times H_n(U) \) is a 2n-dimensional subspace of \( H^1_0(U) \times L^2(U) \). Let \( P_n : H^1_0(U) \times L^2(U) \to H_n(U) \times H_n(U) \),
\[
P_n = I - P_n : H^1_0(U) \times L^2(U) \to H^1_n(U) \times H^1_n(U)
\]
be the orthogonal projectors. For \((u, u_t) \in H^1_0(U) \times L^2(U)\), let
\[
Q_n(u, u_t) = (u_{nq}, u_{t,nq}) \in H^1_n(U) \times H^1_n(U).
\]

In the following part of this section, we assume that \( f \in C^2(\mathbb{R}, \mathbb{R}) \) and there exists \( \overline{C} \) such that for \( \forall s \in \mathbb{R} \),
\[
\left| f''(s) \right| \leq \overline{C}(1 + |s|^{q-1}). \tag{6.9}
\]

**Lemma 6.2.** For any \( \tau \in \mathbb{R}, t \geq 0, \omega \in \Omega \), there exists a random variable \( C_6(\omega) > 0 \) and a 2n-dimensional finite dimensional projector \( P_n : H^1_0(U) \times L^2(U) \to H_n(U) \times H_n(U) \) such that for any \( (u^{(1)}, u_t^{(1)}), (u^{(2)}, u_t^{(2)}) \in \mathcal{A}(\tau, \omega) \), it holds that
\[
\left\| (I - P_n)u^{(1)} - (I - P_n)u^{(2)} \right\|_{H^1_0} + \left\| (I - P_n)u_t^{(1)} - (I - P_n)u_t^{(2)} \right\|_{L^2(U)} \\
\leq \left( c_{\gamma+1} e^{c_{\theta}\omega} \right) \left( \left\| u^{(1)} - u^{(2)} \right\|_{H^1_0} + \left\| u_t^{(1)} - u_t^{(2)} \right\|_{L^2(U)} \right). \tag{6.10}
\]
and
\[
\| P_n u_t^{(1)} - P_n u_t^{(2)} \|_{H_0^1} + \| P_n u_t^{(1)} - P_n u_t^{(2)} \|_{H_0^1} \\
\leq e^{\int_0^t C_0(t, \omega) ds} \left( \| u_t^{(1)} - u_t^{(2)} \|_{H_0^1} + \| u_t^{(1)} - u_t^{(2)} \|_{H_0^1} \right),
\]
where \( E[C_0(\omega)] < \infty \), \( \gamma_{n+1} = \frac{1}{\sqrt{\lambda_{n+1}}} \) as \( n \to \infty \).

**Proof.** Taking the inner product of (6.2) with \( Q_n (u_t^{(1)} - u_t^{(2)}) \), we have
\[
\frac{1}{2} \frac{d}{dt} \| Q_n (u_t^{(1)} - u_t^{(2)}) \|^2 + \beta(t) \| Q_n (u_t^{(1)} - u_t^{(2)}) \|^2 + \frac{1}{2} \frac{d}{dt} \| \nabla Q_n (u_t^{(1)} - u_t^{(2)}) \|^2 \\
= \left( f(u_t^{(2)}) - f(u_t^{(1)}), Q_n (u_t^{(1)} - u_t^{(2)}) \right)
\]
Since \( \sqrt{\lambda_{n+1}} \| Q_n u \| \leq \| Q_n u \|_{H_1^0} \), it is easy to deduce that
\[
\lambda_{n+1} \| Q_n u \|_{H_1^0} \leq \| Q_n u \|^2.
\]
Thus by (6.1) and (6.9), we have
\[
\left( f(u_t^{(2)}) - f(u_t^{(1)}), Q_n (u_t^{(1)} - u_t^{(2)}) \right)
\]
\[
\leq \left( 1 + R(\tau, \omega) \right)^{2q} \| \nabla u_t^{(2)} - \nabla u_t^{(1)} \|^2.
\]
Then we have
\[
\left( f(u_t^{(2)}) - f(u_t^{(1)}), Q_n (u_t^{(1)} - u_t^{(2)}) \right)
\]
\[
\leq \| f(u_t^{(2)}) - f(u_t^{(1)}) \|_{H_0^1} \cdot \| Q_n (u_t^{(1)} - u_t^{(2)}) \|_{H_1^0} \\
\leq \gamma_{n+1}^2 c \left( 1 + R(\tau, \omega) \right)^{2q} e^{\int_0^t C_0(t, \omega) ds} \left( \| u_t^{(1)} - u_t^{(2)} \|^2 + \| u_t^{(1)} - u_t^{(2)} \|^2_{H_0^1} \right)
\]
\[
+ \beta(t) \| Q_n (u_t^{(1)} - u_t^{(2)}) \|^2.
\]
Hence we have that
\[
\frac{1}{2} \frac{d}{dt} \| Q_n (u_t^{(1)} - u_t^{(2)}) \|^2 + \frac{1}{2} \frac{d}{dt} \| \nabla Q_n (u_t^{(1)} - u_t^{(2)}) \|^2 \\
\leq \gamma_{n+1}^2 c \left( 1 + R(\tau, \omega) \right)^{2q} e^{\int_0^t C_0(t, \omega) ds} \left( \| u_t^{(1)} - u_t^{(2)} \|^2 + \| u_t^{(1)} - u_t^{(2)} \|^2_{H_0^1} \right).
\]
Integrate the above inequality on \( [\tau, t+\tau] \), we have
\[
\left( \| Q_n (u_{t+\tau}^{(1)} - u_{t+\tau}^{(2)}) \|^2 + \| \nabla Q_n (u_{t+\tau}^{(1)} - u_{t+\tau}^{(2)}) \|^2 \right. \\
\leq \gamma_{n+1}^2 c \left( \| u_{t+\tau}^{(1)} - u_{t+\tau}^{(2)} \|^2_{H_0^1} + \| u_{t+\tau}^{(1)} - u_{t+\tau}^{(2)} \|^2 \right)
\]
\[
\cdot \int_\tau^{t+\tau} \left( 1 + R(\tau, \omega) \right)^{2q} e^{\int_\tau^r C_0(\theta, \omega) ds} dr.
\]
Since $\sqrt{x} \leq e^x$ for any $x \geq 0$, it follows that

$$\int_\tau^{t+\tau} (1 + R(\tau, \omega)^{2p}) \, dr \leq c e^\int_\tau^{t+\tau} (1 + R(\tau, \omega)^{4p}) \, ds \tag{6.18}$$

Thus we have

$$\|Q_n(u_1^{(1)} - u_1^{(2)})\|^2 + \|\nabla Q_n(u_1^{(1)} - u_1^{(2)})\|^2$$

$$\leq \gamma_{n+1}^2 e^{\left(\|u_1^{(1)} - u_1^{(2)}\|^2_{H^1_0} + \|u_1^{(1)} - u_1^{(2)}\|^2_{H^1_0}\right) \int_\tau^{t+\tau} (1 + R(\tau, \omega)^{2p}) e^\int_\tau^{t+\tau} C_5(\theta_s - \tau \omega) \, ds \, dr}$$

$$\leq (\gamma_{n+1} c_0 e^\int_0^{t_0} C_6(\theta_{s\tau}) \, ds)^2 \left(\|u_1^{(1)} - u_1^{(2)}\|^2_{H^1_0} + \|u_1^{(1)} - u_1^{(2)}\|^2_{H^1_0}\right). \tag{6.19}$$

where $C_6(\theta_{s\tau}) = \frac{1}{2} C_5(\theta_{s\tau}) + (1 + R(\tau, \omega)^{4p}).$ \hfill \Box

By (3.7), it follows that $\forall s \in \mathbb{R}, k \geq 0,$

$$\mathbb{E}[C_6(\omega)] = \frac{1}{2} \mathbb{E}[C_5(\theta_{s\tau}) + 1 + R(\tau, \omega)^{4p}] < \infty,$$

(independent of $s$ and $\omega$). Hence we have the following main result of this section

**Theorem 6.3.** Suppose (1.4)-(1.6) hold. Then for any $\tau \in \mathbb{R}, \omega \in \Omega$, $A(\tau, \omega)$ has a finite fractal dimension:

$$\dim_f A(\tau, \omega) \leq 4n_0 \ln \left(\frac{\sqrt{2n_0 \lambda_{n_0+1}}}{c_0} + 1\right) \leq \frac{4n_0 \ln \left(\frac{\sqrt{2n_0 \lambda_{n_0+1}}}{c_0} + 1\right)}{\ln \frac{4}{3}} < \infty, \tag{6.20}$$

where

$$n_0 = \min \left\{ n : c_0 \gamma_{n+1} \leq \frac{1}{8} e^{-\frac{2(\ln 2)^3}{\pi^2 \ln 3} \mathbb{E}[C_6(\omega)]} \right\} < \infty. \tag{6.21}$$

**Proof.** Comparing (2.2) and (6.11), we find that

$$0 < \delta_{n_0} = c_0 \gamma_{n_0+1} = c_0 \frac{1}{\sqrt{\lambda_{n_0+1}}}.$$

By (6.21),

$$0 < \delta_{n_0} \leq \frac{1}{8} e^{-\frac{2(\ln 2)^3}{\pi^2 \ln 3} \mathbb{E}[C_6(\omega)]} = \frac{1}{8} e^{-\frac{2}{\ln \frac{4}{3} \left(\frac{4\ln 2}{3}\right)^2 \mathbb{E}[C_6(\omega)]}}.$$

Taking $t = t_0 = \frac{4\ln 2}{3}$, then by Theorem 2.11, for any $\tau \in \mathbb{R}, \omega \in \Omega$, it follows that

$$\dim_f A \leq \frac{4n_0 \ln \left(\frac{\sqrt{2n_0 \lambda_{n_0+1}}}{c_0} + 1\right)}{\ln \frac{4}{3}} \leq \frac{4n_0 \ln \left(\frac{\sqrt{2n_0 \lambda_{n_0+1}}}{c_0} + 1\right)}{\ln \frac{4}{3}} < \infty.$$

Hence we complete the proof. \hfill \Box
7. **Exponential attractors.** In this section, we will establish the random pullback exponential attractors for (1.1)-(1.3). We also need to assume that the nonlinearity $f \in C^{2}(\mathbb{R}, \mathbb{R})$ as in Section 6. Projection operator are defined as in Section 6 as well. Denote $w = (u^{(1)} - u^{(2)})$, for the initial data $w_{t,0} = (u_{t,0}^{(1)} - u_{t,0}^{(2)})$, $w_{0} = (u_{0}^{(1)} - u_{0}^{(2)})$, \{B(\tau, \omega)\} is the pullback absorbing set obtained in Section 4, which is tempered. Before constructing the pullback exponential attractor, we firstly prove the contraction property for the projection operator $Q_{N}$, where $N > 0$ will be determined later.

Taking inner product of (6.2) with $Q_{n}w_{t} + \delta Q_{n}w$, we have

$$
\frac{1}{2} \frac{d}{dt} \| Q_{n}w_{t} \|^{2} + \beta(t) \| Q_{n}w_{t} \|^{2} + \frac{1}{2} \frac{d}{dt} \| \nabla Q_{n}w \|^{2} + \delta \frac{d}{dt} (w_{t}, Q_{n}w) - \delta \| Q_{n}w \|^{2} \\
+ \delta \beta(t)(w_{t}, Q_{n}w) + \| \nabla Q_{n}w \|^{2} \\
= (f(u^{(2)}) - f(u^{(1)}), Q_{n}w_{t} + \delta Q_{n}w).
$$

(7.1)

Thus by (6.1), (6.9), Hölder inequality, Sobolev embedding inequality and Theorem 4.1, we have when $t > T$, $T$ is defined as in Section Absorbing Set,

$$
\frac{d}{dt} \left( \frac{1}{2} \| Q_{n}w_{t} \|^{2} + \frac{1}{2} \| \nabla Q_{n}w \|^{2} + \delta(Q_{n}w_{t}, Q_{n}w) \right) + (\beta(t) - \delta - \frac{\delta^{2}}{\lambda_{1}}) \| Q_{n}w_{t} \|^{2} \\
+ (\delta - \frac{1}{4})\| \nabla Q_{n}w \|^{2} \\
\leq \left( Q_{n}f(u^{(2)}) - Q_{n}f(u^{(1)}), Q_{n}w_{t} + \delta Q_{n}w \right).
$$

(7.2)

By Theorem 4.1, when $t > T(\tau, \omega)$,

$$
\left\| Q_{n}f(u^{(2)}) - Q_{n}f(u^{(1)}) \right\|_{H_{0}^{1}} \\
= \left( \int_{\mathcal{U}} |\nabla Q_{n}f(u^{(2)}) - \nabla Q_{n}f(u^{(1)})|^{2} \, dx \right)^{\frac{1}{2}} \\
\leq \left( \int_{\mathcal{U}} \left| C_{2}(1 + |u^{(2)}|^q)Q_{n}\nabla(u^{(2)} - u^{(1)}) + f''(\xi)Q_{n}(u^{(2)} - u^{(1)})\nabla u^{(1)} \right|^{2} \, dx \right)^{\frac{1}{2}} \\
\leq C_{C_{2}, \lambda_{1}}R^{q}(\tau, \omega)\| Q_{n}\nabla w \|,
$$

denote $C_{7} = C_{C_{2}, \lambda_{1}}$ in the above estimate, then

$$
(f(u^{(2)}) - f(u^{(1)}), Q_{n}w_{t}) = \| f(u^{(2)}) - f(u^{(1)}) \|_{H_{0}^{1}} \| Q_{n}w_{t} \|_{H^{-1}} \\
\leq C_{7}R^{q}(\tau, \omega) \frac{1}{\lambda_{n}} \| Q_{n}\nabla w \| \cdot \| Q_{n}w_{t} \| \\
\leq \frac{\delta_{0}}{4} \| Q_{n}w_{t} \|^{2} + C_{7}^{2}R^{2q}(\tau, \omega) \frac{1}{\lambda_{n}} \| Q_{n}\nabla w \|^{2}.
$$

(7.3)

For the second term in the right hand of (7.2),

$$
(Q_{n}f(u^{(2)}) - Q_{n}f(u^{(1)}), \delta Q_{n}w) = (f'(\xi)(Q_{n}u^{(2)} - Q_{n}u^{(1)}), \delta Q_{n}w) \\
\leq \delta_{C_{1}|U|, q, C_{2}} (1 + \| \xi \|^{2q}) \| Q_{n}w \|^{2} = \delta_{C_{8}} (1 + R(\tau, \omega)^{2q}) \frac{1}{\lambda_{n}} \| \nabla Q_{n}w \|^{2}.
$$

(7.4)
Since \( \lambda_n \to \infty \) as \( n \to \infty \), there is a \( N \) such that for every \( \delta > 0 \), for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \)
\[
\frac{1}{\lambda_N} C_9 \left( 1 + R^{2q}(\tau, \omega) \right) \leq \frac{\delta}{4},
\]
where \( C_9 = \max \{ C_7^2, C_8 \} \). Substituting (7.3) and (7.4) into (7.2), one could obtain that
\[
\frac{d}{dt} \left( \frac{1}{2} \| Q_n w_t \|^2 + \frac{1}{2} \| \nabla Q_n w \|^2 + \delta (Q_n w_t, Q_n w) \right)
+ \left( (\beta(t) - \delta - \delta \frac{\beta^2}{\lambda_1} \right) \right) \| Q_n w_t \|^2
+ \left( \frac{\delta}{4} - \delta \frac{\beta^2}{4} \right) \| \nabla Q_n w \|^2 \leq 0.
\]

Take \( 0 < \delta < 1 \) small enough such that \( (\beta(t) - \delta - \delta \frac{\beta^2}{\lambda_1}) > 0 \). Then denote
\[
\sigma_1 = \min \left\{ \beta(t) - \delta - \delta \frac{\beta^2}{\lambda_1}, \frac{\beta_0}{4}, \frac{\delta}{4} - \delta \frac{\beta^2}{4} \right\}.
\]

Hence
\[
\frac{d}{dt} \left( \frac{1}{2} \| Q_n w_t \|^2 + \frac{1}{2} \| \nabla Q_n w \|^2 + \delta (Q_n w_t, Q_n w) \right)
+ \frac{\sigma_1}{2} \left( \| Q_n w_t \|^2 + \| \nabla Q_n w \|^2 + \sigma_1 \delta (Q_n w_t, \nabla Q_n w) \right)
\leq \frac{d}{dt} \left( \frac{1}{2} \| Q_n w_t \|^2 + \frac{1}{2} \| \nabla Q_n w \|^2 + \delta (Q_n w_t, Q_n w) \right)
+ \sigma_1 \delta (Q_n w_t, \nabla Q_n w) - \sigma_1 \| Q_n w_t \|, \| Q_n \nabla w \| \leq 0.
\]

Through the Gronwall’s inequality, it yields that the following contraction estimate holds:
\[
\| Q_n w_t \|_{\tau, \tau - t, \theta_{-\tau} \omega, B(\tau - t, \theta_{-\tau} \omega)}^2
+ \| Q_n \nabla w \|_{\tau, \tau - t, \theta_{-\tau} \omega, B(\tau - t, \theta_{-\tau} \omega)}^2 \leq ce^{-\sigma_1 t} \| Q_n w_t \|_{\tau, \tau - t, \theta_{-\tau} \omega, B(\tau - t, \theta_{-\tau} \omega)}^2
\]
\[
+ ce^{-\sigma_1 t} \| Q_n \nabla w \|_{\tau, \tau - t, \theta_{-\tau} \omega, B(\tau - t, \theta_{-\tau} \omega)}^2,
\]
which indicates that there is \( 0 < \eta < \frac{1}{4} \) such that \( ce^{-\sigma_1 t} < \eta \) when \( t > T \). Assume integer \( N \in \mathbb{N}^* \) is the number such that (7.5)-(7.6) are valid.

Next we will prove the existence of pullback random exponential attractors. In order to construct the pullback random exponential attractor for stochastic dynamical system \( \Phi \), we firstly construct the pullback random exponential attractor for a discrete stochastic dynamical system.
7.1. Discrete random dynamical system. Let \( \{ \Phi(m\hat{T}, n - m\hat{T}, \theta_{-m\hat{T}}\omega, \Phi_{n-m\hat{T}}) \} \) be a discrete cocycle in \( H^1_0(U) \times L^2(U) \), \( \hat{T} = [T(n, \omega)] + 1 \). \( \lceil \cdot \rceil \) denotes the integer function and \( T(n, \omega) \) is the minimum time such that \( (7.6) \) holds. For simplification, denote \( \hat{\Phi}(n)(\cdot) = \{ \Phi(T, n - mT, \theta_{-mT}\omega, \cdot) \} \) and \( \hat{V} = H^1_0(U) \times L^2(U) \)

**Theorem 7.1.** Let the assumptions in Section 6 hold, where initial data satisfies \( \Phi_{n-m\hat{T}} \in B(n - m\hat{T}, \theta_{-m\hat{T}}\omega) \), then for any \( \nu \in (0, \frac{1}{2} - \eta) \), the discrete stochastic dynamical system \( \{ \Phi(m\hat{T}, n - m\hat{T}, \theta_{-m\hat{T}}\omega, \Phi_{n-m\hat{T}}) \} \) possesses a pullback random exponential attractor \( \{ M(n, \omega) | n \in \mathbb{Z} \} \) and the fractal dimension of its sections can be estimated by

\[
\dim_f(M(n, \omega)) \leq \log \frac{\nu}{n^{\eta \nu}} (N_\nu(B^\nu_1(\hat{V}(0))),
\]

where \( B^\nu_1(\hat{V}(0)) \) is the unit ball in \( P_N(H^1_0(U) \times L^2(U)) \).

**Proof.** Covering of \( \Phi(k\hat{T}, n - k\hat{T}, \theta_{-k\hat{T}}\omega, B(n - k\hat{T}, \theta_{-k\hat{T}}\omega)) \)

Since the absorbing set is obtained, we could let \( \nu \in (0, \frac{1}{2} - \eta) \) be fixed, \( R_n > 0 \) and \( v_n \in B(n, \omega) \) such that \( B(n, \omega) \subset B_{R_n}(v_n) \) for all \( n \in \mathbb{N} \), where \( B(n, \omega) \) is the absorbing set. Moreover choose \( y_1, y_2, \ldots, y_N \in H^1_0(U) \times L^2(U) \) such that

\[
B^\nu_1(\hat{V}(0)) \subset \bigcup_{i=1}^{\tilde{N}} B^\nu_\theta(y_i),
\]

since \( P_N \) is a finite dimensional projector. Define \( W^0(n) = \{ v_n \} \) and construct by induction in \( k \in \mathbb{N} \) sets \( W^k(n) \), \( k \in \mathbb{N} \) that depend on the time instant \( n \) and satisfy for all \( n \in \mathbb{Z} \)

(W1): \( W^k(n) \subset \Phi(k, n - k, \theta_{-k}\omega, B(n - k, \theta_{-k}\omega)) \subset B(n, \omega) \)

(W2): \( \text{Card}(W^k(n)) \leq N^k \)

(W3): \( \Phi(k, n - k, \theta_{-k}\omega, B(n - k, \theta_{-k}\omega)) \subset \bigcup_{\xi \in W^{k+1}(n)} B^\nu_{2(\nu + \eta) R_{n+1}}(\xi) \)

The detail construction of \( W^k(n) \) will contain two steps:

**Step1.** to construct a covering of the image \( \hat{\Phi}(n)B(n - \bar{T}, \theta_{-\bar{T}}\omega) \), \( n \in \mathbb{Z} \). Note \( v \in B_{R_{n-\bar{T}}} \) implies that

\[
\frac{1}{R_{n-\bar{T}}} P_{R_{n-\bar{T}}} v - v_n \in B^\nu_1(\hat{V}(0)) \subset \bigcup_{i=1}^{\tilde{N}} B^\nu_\theta(y_i)
\]

and consequently,

\[
B_{R_{n-\bar{T}}}^\nu (v_n - \bar{T}) \subset \bigcup_{i=1}^{\tilde{N}} B_{R_{n-\bar{T}}}^\nu (R_{n-\bar{T}}y_i + v_n - 1).
\]

Due to the definition of projector \( P_N \),

\[
\| P_N \hat{\Phi}(n) \bar{u} - P_N \hat{\Phi}(n) \bar{v} \|_{H^1_0 \times L^2} \leq 2\nu R_{n-\bar{T}}
\]

for all \( \bar{u}, \bar{v} \in B_{R_{n-\bar{T}}}^\nu (R_{n-1}y_i + v_n - 1) \), which yields that

\[
P_N \hat{\Phi}(n)B_{R_{n-\bar{T}}}^\nu (v_n - \bar{T}) \cap B(n - \bar{T}, \theta_{-\bar{T}}\omega) \subset \bigcup_{i=1}^{\tilde{N}} B_{2\nu R_{n-\bar{T}}}^\nu (y_i)
\]
for some $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{\tilde{N}} \in P_N \hat{\Phi}(n) B(\hat{T}, \theta_{-\hat{T}\omega})$. Particularly, one could choose $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{\tilde{N}} \in B(n - \hat{T}, \theta_{-\hat{T}\omega})$ such that $\hat{y}_i = P_N \hat{\Phi}(n) \hat{y}_i, i = 1, 2, \ldots, \tilde{N}$. For any $\varphi \in B(n - \hat{T}, \theta_{-\hat{T}\omega})$, from (7.6) one could deduce that
\[
\|Q_N \hat{\Phi}(n) \varphi - Q_N \hat{\Phi}(n) \hat{y}_i\| \leq \eta \|\varphi - \hat{y}_i\| < 2\eta R_{n_{-\hat{T}}},
\]
for $i = 1, 2, \ldots, \tilde{N}$, thus the following covering holds:
\[
Q_N \hat{\Phi}(n) B(n - \hat{T}, \theta_{-\hat{T}\omega}) \subset B_{2\eta R_{n_{-\hat{T}}}}(Q_N \hat{\Phi}(\hat{y}_i)).
\]
Finally, we could obtain the covering
\[
\hat{\Phi}(n) B(n - \hat{T}, \theta_{-\hat{T}\omega}) = (P_N + Q_N)(\hat{\Phi}(n) B(n - \hat{T}, \theta_{-\hat{T}\omega}))
\subset \bigcup_{i=1}^{\tilde{N}} B_{2\nu R_{n_{-\hat{T}}}} \left( P_N \hat{\Phi}(n) \hat{y}_i \cup B_{2\eta R_{n_{-\hat{T}}}}(Q_N \hat{\Phi}(n) \hat{y}_i) \right)\quad (7.8)
\]
\[
\subset \bigcup_{i=1}^{\tilde{N}} B_{2(\nu + \eta) R_{n_{-\hat{T}}}} (\hat{\Phi}(n) \hat{y}_i),
\]
where $\hat{\Phi}(n) \hat{y}_i \in \hat{\Phi}(n) B(n - \hat{T}, \theta_{-\hat{T}\omega}), i = 1, 2, \ldots, \tilde{N}$. Denote the new set of centers by $W^1(n)$ and it follows that
\[
\hat{\Phi}(n) B(n - \hat{T}, \theta_{-\hat{T}\omega}) \subset \bigcup_{\phi \in W^1(n)} B_{2(\nu + \eta) R_{n_{-\hat{T}}}} (\phi)
\]
with $W^1(n) \subset \hat{\Phi}(n) B(n - \hat{T}, \theta_{-\hat{T}\omega}) \subset B(n, \omega)$ and $Card(W^1(n)) \leq \tilde{N}$.

**Step 2.** Assume $W^l(n)$ are already constructed for all $l \leq k$ and $k \in \mathbb{Z}$, which yields the covering
\[
\Phi \left( k\hat{T}, n - k\hat{T}, \theta_{-k\hat{T}\omega}, B(n - k\hat{T}, \theta_{-k\hat{T}\omega}) \right) \subset \bigcup_{\phi \in W^k(n)} B_{2(\nu + \eta)^k R_{n_{-k\hat{T}}}} (\phi)
\]
Let $\xi \in \Phi \left( k\hat{T}, n - k\hat{T}, \theta_{-k\hat{T}\omega}, B(n - k\hat{T}, \theta_{-k\hat{T}\omega}) \right)$. We using the covering of unit ball $B(0,1)$ by $\nu$-balls to conclude
\[
B_{2(\nu + \eta)^k R_{n_{-(k+1)\hat{T}}}} (u) \subset \bigcup_{i=1}^{\tilde{N}} B_{2(\nu + \eta)^k R_{n_{-(k+1)\hat{T}}}} \left( (2(\nu + \eta))^k R_{n_{-(k+1)\hat{T}}} \hat{y}_i + u \right).
\]
Then from the definition of finite-dimensional projector $P_N$, one could obtain
\[
P_N \hat{\Phi}(n) \left( \Phi(k)(B(n - k\hat{T}, \theta_{-(k+1)\hat{T}\omega}) \cap B_{2(\nu + \eta)^k R_{n_{-(k+1)\hat{T}}}} (u)) \right)
\subset \bigcup_{i=1}^{\tilde{N}} B_{2(\nu + \eta)^k R_{n_{-(k+1)\hat{T}}}} (P_N \hat{\Phi}(n) \xi_i)
\]
for some $\xi_1, \xi_2, \ldots, \xi_{\tilde{N}} \in \Phi \left( k\hat{T}, n - (k+1)\hat{T}, \theta_{-(k+1)\hat{T}\omega}, B(n - (k+1)\hat{T}, \theta_{-(k+1)\hat{T}\omega}) \right)$.

From (7.6), one could prove that
\[
Q_N \hat{\Phi}(n) \left( \Phi(k)B(n - (k+1)\hat{T}, \theta_{-(k+1)\hat{T}\omega}) \cap B_{2(\nu + \eta)^k R_{n_{-(k+1)\hat{T}}}} (u) \right)
\subset \bigcup_{i=1}^{\tilde{N}} B_{2(\nu + \eta)^k R_{n_{-(k+1)\hat{T}}}} (P_N \hat{\Phi}(n) \xi_i),
\]
where \( \Phi(k)(\cdot) = \Phi(k\hat{T}, n - (k + 1)\hat{T}, \theta_{-(k+1)\hat{T}\omega}, \cdot) \). Hence we could deduce the following covering

\[
\tilde{\Phi}(n) \left( \Phi(k)B(n - (k + 1)\hat{T}, \theta_{-(k+1)\hat{T}\omega}) \cap \bigcap_{i=1}^{\tilde{N}} B_{(2(\nu + \eta))^{k+1}R_{n-(k+1)\hat{T}}}(u) \right) \\
\subset \bigcup_{i=1}^{\tilde{N}} B_{(2(\nu + \eta))^{k+1}R_{n-(k+1)\hat{T}}}(P_N \tilde{\Phi}(n)\xi_i) \\
\subset \bigcup_{i=1}^{\tilde{N}} B_{(2(\nu + \eta))^{k+1}R_{n-(k+1)\hat{T}}}(P_N \tilde{\Phi}(n)\xi_i),
\]

where \( \tilde{\Phi}(n)\xi_i \in \tilde{\Phi}\left((k + 1)\hat{T}, n - (k + 1)\hat{T}, \theta_{-(k+1)\hat{T}\omega}, B(n - (k + 1)\hat{T}, \theta_{-(k+1)\hat{T}\omega})\right) \) for \( i = 1, 2, \ldots, \tilde{N} \). By the same way, one could establish a covering with radius \((2(\nu + \eta))^{k+1}R_{n-(k+1)\hat{T}}\) that covers \( \tilde{\Phi}\left((k + 1)\hat{T}, n - (k + 1)\hat{T}, \theta_{-(k+1)\hat{T}\omega}, B(n - (k + 1)\hat{T}, \theta_{-(k+1)\hat{T}\omega})\right) \) in \( \tilde{V} \). Denote this new set of centers by \( W^{k+1}(n) \). This yields that

\[
\text{Card}(W^{k+1}(n)) \leq N\text{Card}(W^k(n - 1)) \leq \tilde{N}^{k+1}
\]

and

\[
\Phi\left((k + 1)\hat{T}, n - (k + 1)\hat{T}, \theta_{-(k+1)\hat{T}\omega}, B(n - (k + 1)\hat{T}, \theta_{-(k+1)\hat{T}\omega})\right) \\
\subset \bigcup_{\xi \in W^{k+1}(n)} B_{(2(\nu + \eta))^{k+1}R_{n-(k+1)\hat{T}}}^{\nu}(\xi), \tag{7.9}
\]

which conclude the proof of the properties (W1)-(W3).

Now we could construct the exponential attractor for the discrete system.

**Definition of the pullback random exponential attractor** Define \( E^n(\omega) = W^0(\omega) \) for all \( n \in \mathbb{Z} \) and \( E^m(\omega) := W^m(\omega) \bigcup \Phi(\omega)(E^{m-1}(n - \hat{T})) \), \( m \in \mathbb{N} \). Then from the properties of \( W^k(n) \) and the semi-invariance of absorbing sets \( \{ B(n, \omega) \} \), we could prove the following properties for \( E^m \):

\( E^1 \): \( \Phi(\hat{T}, n - \hat{T}, \theta_{-\hat{T}\omega}, E^n(n - \hat{T})) \subset E^{n+1}(n) \), \( E^m(n) \subset \Phi(m\hat{T}, n - m\hat{T}, \theta_{-m\hat{T}\omega}, B(n - m\hat{T}, \theta_{-m\hat{T}\omega})) \subset B(n, \omega) \);

\( E^2 \): \( E^m(n) = \bigcup_{1=0}^m \Phi(l\hat{T}, n - l\hat{T}, \theta_{-l\hat{T}\omega}, W^{m-1}(n - l\hat{T})) \) and \( \text{Card}(E^m(n)) \leq \sum_{l=0}^m \tilde{N}^l \);

\( E^3 \): \( \Phi(m\hat{T}, n - m\hat{T}, \theta_{-m\hat{T}\omega}, B(n - m\hat{T}, \theta_{-m\hat{T}\omega})) \subset \bigcup_{\xi \in E^m(n)} B_{(2(\nu + \eta))^{m}R_{n-m\hat{T}}}^{\nu}(\xi) \).

Then define \( M(\tau, \omega) := \bigcup_{m \in \mathbb{N}^*} E^m(\tau) \) for all \( \tau \in \mathbb{Z} \). We claim that \( \{ M(\tau, \omega) \} \) yields an exponential attractor for the random dynamical system \( \Phi \).

**Semi-invariance of the Exponential attractor**

For all \( l \in \mathbb{N} \), \( \tau \in \mathbb{Z} \), taking advantage of (E1),

\[
\Phi \left( l\hat{T}, \tau - l\hat{T}, \theta_{-l\hat{T}\omega}, \bigcup_{m \in \mathbb{N}^*} E^m(\tau - l\hat{T}) \right) \\
= \bigcup_{m \in \mathbb{N}^*} \Phi \left( l\hat{T}, \tau - l\hat{T}, \theta_{-l\hat{T}\omega}, E^m(\tau - l\hat{T}) \right) \subset \bigcup_{m \in \mathbb{N}^*} E^{m+1}(\tau) \subset \bigcup_{m \in \mathbb{N}^*} E^m(\tau),
\]
According to the continuity of $\Phi$, we could deduce
\[
\Phi \left( \mathcal{I}^T, \tau - \Delta T^\epsilon, \theta_{-mT^\epsilon}^{\omega}, \bigcup_{m \in \mathbb{N}^*} E_m(\tau - \Delta T^\epsilon) \right) \subset \Phi \left( \mathcal{I}^T, \tau - \Delta T, \theta_{-mT}^{\omega}, \bigcup_{m \in \mathbb{N}^*} E_m(\tau - \Delta T) \right)
\]
\[\subset \bigcup_{m \in \mathbb{N}^*} E^{m+1}(\tau) = M(\tau, \omega). \tag{7.10}\]

**Finite Dimension of the exponential attractor** For all $m \in \mathbb{N}$,
\[
\bigcup_{m \in \mathbb{N}^*} E_m(n) = \bigcup_{k=0}^m E_k(n) \cup \bigcup_{k=m+1}^\infty E_k(n)
\subset \bigcup_{k=0}^m E_k(n) \cup \Phi \left( mT, n - mT, \theta_{-mT}^{\omega}, B(n - mT, \theta_{-mT}^{\omega}) \right).
\]

Let $\epsilon > 0$ and $m \in \mathbb{N}$ be large enough such that $(2(\nu + \eta))^{m-1} R_{n-m\mathcal{T}} \leq \epsilon \leq (2(\nu + \eta))^{m-1} R_{n-(m-1)\mathcal{T}}$, then
\[
\Phi \left( mT, n - mT, \theta_{-mT}^{\omega}, B(n - mT, \theta_{-mT}^{\omega}) \right) \subset \bigcup_{\xi \in W^m(n)} B_{\tilde{\epsilon}^m}(\xi).
\]

As a consequence,
\[
N_\epsilon(M(n)) \leq \text{Card}(\bigcup_{k=0}^m E_k(n)) + \text{Card}(W^m(n)) \leq (m + 1) \tilde{N}^m + \tilde{N}^m = (m + 2) \tilde{N}^m,
\]
then
\[
\dim_f(M(n)) = \limsup_{\epsilon \to 0} \frac{-\ln N_\epsilon(M(n))}{-\ln \epsilon} \leq \limsup_{\epsilon \to 0} \frac{-\ln \left( (m + 2) \tilde{N}^m \right)}{-\ln \epsilon} \leq \frac{\ln \tilde{N}}{-\ln(2\eta + 2\nu)} \tag{7.11}
\]

**Exponential attraction** Let $k_0 > 0$ be large enough such that for any bounded subset $D \in \mathcal{D}$, $\Phi(kT, n - kT, \theta_{-kT}^{\omega}, D) \subset B(n, \omega)$ for all $k \geq k_0$. If $k = k_0 + k_1$, $k, k_0, k_1$ are all positive integer, then
\[
\text{dist}_H\left( \Phi \left( kT, n - kT, \theta_{-kT}^{\omega}, D \right), M(n, \omega) \right) \leq \text{dist}_H\left( \Phi \left( k_1T, n - k_0T, \theta_{-k_0T}^{\omega}, \cdot \right) \left( B \left( n - k_0T, \theta_{-k_0T}^{\omega} \right), E^{k_0}(n) \right) \right) \leq (2(\nu + \eta))^{k_0} R_{n-k_0T}. \tag{7.12}
\]

Since $\nu + \eta < \frac{1}{2}$, the pullback exponential attraction follows naturally. Due to Theorem 5.1, (7.10),(7.11) and (7.12), we can conclude that $M(\tau, \omega)$ is a random pullback exponential attractor for the discrete stochastic dynamical system $\left\{ \Phi(mT, n - mT, \theta_{-mT}^{\omega}, \Phi_{n-mT}) \right\}$. \[
\]

**7.2. Continuous random dynamical system.** We conclude this section by constructing the random pullback exponential attractor for the continuous dynamical system by taking advantages of the theorem above.
Theorem 7.2. Let the assumptions in Section 6 hold. Then the stochastic dynamical system $\Phi$ possess a random pullback exponential attractor $M^c(\tau, \omega)$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Proof. Denote $M^d(n, \omega)$ as the random pullback exponential attractor deduced Theorem 7.1. Define

$$M^c(\tau, \omega) := \Phi((\tau - n, n, \omega, M^d(n, \theta_{-n}\omega))),$$

(7.13)

where $0 \leq \tau - n \leq \tilde{T}$. Then taking a similar procedure to [9], one could prove $M^c(\tau, \omega)$ defined as (7.13) is a random exponential attractor for the stochastic dynamical system $\Phi$.

Remark 7.3. The temperedness of the absorbing set plays an important role during the construction of an exponential attractor. Without this property, the exponential attraction may not hold.

Remark 7.4. In this paper, we discuss the asymptotic behavior of solutions in the framework of random attractors with critical growing nonlinearity. The stability of the attractor is an interesting question. As presented in [4, 5, 8], the pullback attractor is remains gradient-like under some perturbations when the autonomous semigroup is gradient, where perturbations is caused by $\beta(t)$. The pullback attractor is gradient-like even the equation is not a small perturbation of an autonomous one. These works inspires us to verify the process associated to (1.1)-(1.2) is gradient-like and then analysis the structure of the attractor.

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E-mail address: changqq12@lzu.edu.cn
E-mail address: lidd2008@lzu.edu.cn
E-mail address: sunchy@lzu.edu.cn