The $SL(2,\mathbb{R})$ totally constrained model: three quantization approaches.

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We provide a detailed comparison of the different approaches available for the quantization of a totally constrained system with a constraint algebra generating the non-compact $SL(2,\mathbb{R})$ group. In particular, we consider three schemes: the Refined Algebraic Quantization, the Master Constraint Programme and the Uniform Discretizations approach. For the latter, we provide a quantum description where we identify semiclassical sectors of the kinematical Hilbert space. We study the quantum dynamics of the system in order to show that it is compatible with the classical continuum evolution. Among these quantization approaches, the Uniform Discretizations provides the simpler description in agreement with the classical theory of this particular model, and it is expected to give new insights about the quantum dynamics of more realistic totally constrained models such as canonical general relativity.

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I. INTRODUCTION

Fully constrained models are a class of settings that successfully describe the physics of systems largely characterized by certain symmetries. However, the quantization of fully constrained systems encounters several obstacles undermining the validity of the resulting microscopical description. The usual strategy for first class systems, proposed originally by Dirac [1], consists of representing the constraints as quantum operators on a given Hilbert space, identifying the quantum observables and states invariant under the symmetries gen-
erated by the constrains and, if they give rise to a large enough set, endow them with a Hilbert space structure. One of the most prominent examples is gravity, which turns out to be diffeomorphism invariant. In particular, if one tries to carry out a quantum description of general relativity, the implementation of the constraints seems necessary if one is interested in studying the full quantum dynamics. For instance, the lack of a consistent implementation of the quantum constraints that reproduce in the classical limit general relativity is one of the reasons because the quantization programme of Loop Quantum Gravity \cite{2-4} is still incomplete. This is usually known as the problem of dynamics and it is the main problem that we address in this paper for the particular case of the $SL(2, \mathbb{R})$ model.

In the last years, several approaches have emerged attempting to shed light on the fundamental description of this kind of models. One of the quantization programmes that will be considered in this manuscript is the so-called Algebraic Quantization \cite{5} (and its more modern version known as Refined Algebraic Quantization \cite{6, 7}). In particular, in this approach one assumes that given a kinematical Hilbert space $H_{\text{kin}}$ and a $*$-algebra $A_{\text{obs}}$ of observables, the latter are represented on a dense, linear subspace $\Phi \subset H_{\text{kin}}$, while the constraints have solutions belonging to the algebraic dual $\Phi^*$ of $\Phi$. The final step consists of endowing with a suitable Hilbert space structure this space of solutions, following on the physical Hilbert space, and looking for a suitable representation of Dirac observables on it. In many cases this last step is accomplished by applying, e.g., the group averaging technique \cite{8}. This method can be applied without major difficulties if one knows an invariant domain of the quantum constraints. But in occasions such a domain is not known or pathological in so far as it either prevents the application of the group averaging technique or introduces ambiguities that require additional inputs in order to achieve a consistent quantum theory. A complementary procedure is available when a complete set of observables is known, and requires reality condition on them. This requirement turns out to uniquely select an inner product \cite{9}, and the completion of the space of solutions with gives the physical Hilbert space.

A second possibility was put forward in Refs. \cite{10, 11} with the Master Constraint Quantization. It arises with the purpose of treating constraints that lead to quantum anomalies but allows to treat other pathological situations as we shall see. The Master Constraint Programme tries to replace the (possibly complicated) algebra by a much simpler Master Constraint Algebra given by the single Master Constraint $M$, defined basically as a quadratic
form of the original constraints and which commutes with itself. Although this quantization scheme fails to detect weak Dirac observables by means of linear conditions on them, this problem is avoided by considering second order conditions. This approach has been successfully tested in many situations \[12,13\].

Finally, a recent approach, known as the Uniform Discretizations \[14\], has emerged in parallel following in part the lines of the Master Constraint Quantization and attempting to improve other discrete quantization procedures known as consistent discretizations \[15\]. In fact, this new paradigm essentially consists in recovering the original, continuum theory from a set of discrete theories, just like, e.g., continuum QCD can be recovered from lattice QCD. The advantage that this new approach presents is that one starts with a discrete version of the continuum theory that is under control, free of drawbacks, and where a consistent quantum description is available, a priori, since one is working off-shell within this approach, having no further constraints to be imposed. In consequence, the discrete theory contains a higher number of degrees of freedom with respect to the continuum one, but keeping in mind that the latter can be systematically obtained from its discrete version thanks to the existence of certain conserved quantities (with respect to the evolution) characterizing the continuum limit. Additionally, it succeeds in identifying both discrete and continuum Dirac observables. That is, given a discrete constant of motion one can identify its corresponding perennial in the continuum limit, and viceversa, for a given perennial in the continuum model there are in general many discrete constants of the motion associated with.

The main purpose of this manuscript is to confront these different quantization schemes in a simple but non-trivial, totally constrained system, which is characterized by two Hamiltonians and one diffeomorphism constraints satisfying an $sl(2,\mathbb{R})$ Lie algebra. This toy model was originally introduced in Ref. \[16\] testing so the Dirac approach, together with a deeper analysis regarding its dynamics carried out in Ref. \[17\]. Besides, a considerable number of publications \[18–20\] has appeared within the Algebraic Quantization, and also when testing the Master Constraint Programme \[13\]. In both schemes, the constraints can be imposed simultaneously at the quantum level, but the final physical Hilbert space requires additional inputs for achieving a semiclassical limit compatible with the classical theory. More specifically, in Algebraic Quantization approach \[16,18,20\] the group averaging technique cannot be suitably applied since the symmetry group is non-amenable. Therefore, in these cases, one has to appeal to the reality conditions of a given family of Dirac observables in
order to determine the inner product of the physical Hilbert space. Similarly, in the Master Constraint Programme [13], the physical Hilbert space with the correct semiclassical limit is obtained after incorporating in the Master Constraint additional information about the Dirac observables or an additional condition tantamount to certain specific linear combination of constraints with non trivial coefficients dependent on the kinematical variables.

In conclusion, all this proposals incorporate at their fundamental level information about the observables. But in more realistic settings, as gravity, this requirement can defeat the completion of the quantum description, essentially because of the difficulty that entails the identification of Dirac observables. For just such an eventuality, it seems natural to investigate different possibilities, like the one we are proposing in this manuscript. In particular, we adhere to the Uniform Discretizations approach for the quantization of a fully constrained $SL(2, \mathbb{R})$ model. The Hamiltonian which defines the discrete (off-shell) evolution coincides in form with the Master Constraint [13]. After quantization, we study its spectrum, paying special attention to the lowest eigenvalues, since they will provide the best candidates for semiclassical states. There, we identify sectors (whose states have finite norm) that naturally induce restrictions to the accessible expectation values of a given subset of observables. We consider these subspaces as the semiclassical sectors of the theory. At this level, we follow two strategies. One consists of strictly restricting the study to one of these subspaces, which requires to modify the observable algebra quantum-mechanically, such that the modified algebra leave invariant these sectors while reproduces a semiclassical limit compatible with the continuum theory. Among these subspaces, the best candidate corresponds to the lowest possible eigenvalue, providing a similar description to the one of the Master Constraint proposal [13]. The alternative strategy deals with the study of the quantum dynamics in the context of the Uniform Discretizations. It is well known that any quantum mechanical system evolves (non trivially) whenever there are available several (at least two) energy states. With this in mind, we consider all the subspaces corresponding to the lowest eigenvalues of the Hamiltonian. Among them, we discuss about possible semiclassical sectors where the quantum discrete dynamics would be compatible with the classical one and, moreover, with the classical constrained model, providing robustness to the physical predictions of the Uniform Discretizations.

Hence our prescription requires that at the end of the day the theory provides a suitable semiclassical limit, with the observables playing a secondary role since the fundamental
structure of the final quantum theory does not depend drastically on them, but just on the requirement of compatibility with the continuous classical version of the theory. Let us remark that this is no longer the case for the previous proposals since they necessarily incorporate the observables at different levels of the construction, in order to achieve a description compatible with the classical theory. With all this in mind, we find the Uniform Discretizations as the most promising quantization scheme (among the ones considered in this manuscript) since it provides a simple and robust arena for dealing with the problem of the quantum dynamics of this particular toy model, but with possible applications to more general settings such as gravity.

The manuscript is organized as follows. In Sec. III we provide a basic description about the classical system. The quantum kinematical framework is introduced in Sec. IV. We provide a detailed description of the Algebraic Quantization and the Master Constraint Programme in Secs. V and VI respectively. In Sec. VII we present the Uniform Discretizations approach. Finally, the main conclusions can be found in Sec. VIII. We have also included the Appendixes A and B with additional technical details.

II. CLASSICAL SYSTEM: KINEMATICS, CONSTRAINTS AND OBSERVABLES

The phase space of our model is composed by four configuration variables $u_1, u_2, v_1$ and $v_2$, and their corresponding momenta $p_i$ and $\pi_i$, with $i = 1, 2$. This model is endowed with three constraints

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2 - v_1^2 - v_2^2), \quad H_2 = \frac{1}{2}(\pi_1^2 + \pi_2^2 - u_1^2 - u_2^2),$$

$$D = u_1p_1 + u_2p_2 - v_1\pi_1 - v_2\pi_2,$$

whose corresponding algebra is given by

$$\{H_1, H_2\} = D, \quad \{H_1, D\} = -2H_1, \quad \{H_2, D\} = 2H_2. \quad (2)$$

The total Hamiltonian of this classical theory is

$$H_T = NH_1 + MH_2 + \lambda D, \quad (3)$$

with $N$, $M$ and $\lambda$ playing the role of Lagrange multipliers, i.e., they do not correspond to dynamical variables. The equations of motion of the phase space variables can be easily
computed, yielding

\[ \dot{u}_i = Mp_i + \lambda u_i, \quad \dot{v}_i = M\pi_i - \lambda v_i, \]
\[ \dot{p}_i = Mu_i - \lambda p_i, \quad \dot{\pi}_i = Mv_i + \lambda \pi_i, \]

(4)

for \( i = 1, 2 \), and where the dot indicates time derivation.

The classical dynamics of this model can be studied by solving this set of equations or, equivalently, by means of the parametrized observables or evolving constants [16, 22], as we are going to discuss in what follows. The constants of motion (or Dirac observables) of this system

\[ O_{12} = u_1p_2 - p_1u_2, \quad O_{23} = u_2v_1 - p_2\pi_1, \]
\[ O_{13} = u_1v_1 - p_1\pi_1, \quad O_{24} = u_2v_2 - p_2\pi_2, \]
\[ O_{14} = u_1v_2 - p_1\pi_2, \quad O_{34} = \pi_1v_2 - v_1\pi_2, \]

(5)

commute with the three constraints and reflect the global \( O(2, 2) \)-symmetry codified in the model. They constitute the \( so(2, 2) \) Lie algebra which is isomorphic to the \( so(2, 1) \times so(2, 1) \) algebra

\[ Q_1 = \frac{1}{2}(O_{23} + O_{14}), \quad P_1 = \frac{1}{2}(O_{23} - O_{14}), \]
\[ Q_2 = \frac{1}{2}(-O_{13} + O_{24}), \quad P_2 = \frac{1}{2}(-O_{13} - O_{24}), \]
\[ Q_3 = \frac{1}{2}(O_{12} - O_{34}), \quad P_3 = \frac{1}{2}(O_{12} + O_{34}). \]

(6)

The Poisson brackets of these observables are

\[ \{Q_i, Q_j\} = \epsilon_{ij}^k Q_k, \quad \{P_i, P_j\} = \epsilon_{ij}^k P_k, \quad \{Q_i, P_j\} = 0 \]

(7)

where \( \epsilon_{ij}^k = g^{lk}\epsilon_{ijl} \), with \( g^{lk} \) being the inverse of the metric \( g_{lk} = \text{diag}(1, 1, -1) \). The Levi-Civita symbol \( \epsilon_{ijk} \) is totally antisymmetric with \( \epsilon_{123} = 1 \). Besides, repeated indexes indicates sum on them.

Together with this observables there is a reflection operator that commutes with the constraints. It is defined as

\[ R_{\epsilon_1, \epsilon_2} : (u_1, u_2, v_1, v_2, p_1, p_2, \pi_1, \pi_2) \to (u_1, \epsilon_1 u_2, \epsilon_2 v_1, \epsilon_1 \epsilon_2 v_2, p_1, \epsilon_1 p_2, \pi_1, \epsilon_1 \epsilon_2 \pi_2), \]

(8)

where \( \epsilon_i = \{1, -1\} \). Its action on the observables \( Q_i \) produces reflections of the type \( Q_i \to -Q_i \) for \( i = 1, 3 \) (and the very same for the \( P_i \)), and also exchanges of the type \( Q_i \leftrightarrow P_i \).
for \( i = 1, 2, 3 \). The classical observable algebra consists of \( Q_i, P_i \) and \( R_{\epsilon_1, \epsilon_2} \), together with the commutation relations (7).

Furthermore, the following identities between observables and constraints

\[
Q_1^2 + Q_2^2 - Q_3^2 = P_1^2 + P_2^2 - P_3^2 = \frac{1}{4}(D^2 + 4H_1H_2) =: C, 
\]

(9)

\[
4Q_3P_3 = (u_1^2 + u_2^2)H_1 - (u_1p_1 + u_2p_2 + v_1\pi_1 + v_2\pi_2)D - (v_1^2 + v_2^2)H_2, 
\]

(10)

will be useful from now on. We have also defined \( C \) as the Casimir which commutes with all the constraints, and with the Dirac observables (it is a combination of constraints).

The condition the three constraints vanish implies that the system has two physical degrees of freedom. The solution space has the topology of four cones joined in the origin:

a) \( P_3 = 0 \) and \( Q_3 \in \mathbb{R} \), with \( Q_1^2 + Q_2^2 = Q_3^2 \),

b) \( Q_3 = 0 \) and \( P_3 \in \mathbb{R} \), with \( P_1^2 + P_2^2 = P_3^2 \),

c) \( Q_3 = 0 \) and \( P_3 = 0 \).

Finally, and following the ideas of Ref. [16] (see also Ref. [17]), the kinematical variables can be expressed in terms of two independent continuous Dirac observables plus two discrete ones. Let us recall that the Dirac observables in Eq. (6) fulfill the identities (9) on the constraint surface. For instance, if we define

\[
\epsilon = \frac{O_{12}}{|O_{12}|}, \quad \epsilon' = \frac{O_{34}}{|O_{34}|}, \quad J = |O_{12}|, \quad \tan \phi = \frac{O_{14}}{O_{13}}, 
\]

(11)

any configuration variable can be solved in terms of these observables and the three remaining configuration variables (as well as for the momenta). More concretely, the identity

\[
u_i v_j \epsilon^{ik} \epsilon'^{ji} (u_k v_l - p_k \pi_l) = O_{12} O_{34},
\]

(12)

allows us to solve \( u_1 \) as

\[
 u_1 = \frac{-\epsilon' u_2 (v_2 \cos \phi - v_1 \sin \phi) + \epsilon J}{\epsilon (v_1 \cos \phi + v_2 \sin \phi)}. 
\]

(13)

Therefore, one can define an evolving constant \( U_1 \) that takes the value \( u_1 \) when the remaining variables take the values \( u_2 = x, v_1 = y \) and \( v_2 = z \), i.e.,

\[
 U_1 = \frac{-\epsilon' x (z \cos \phi - y \sin \phi) + \epsilon J}{\epsilon (y \cos \phi + z \sin \phi)}, 
\]

(14)

as well for the remaining phase space variables of the model. Concluding, the gauge invariant evolution of the model is totally captured in this description.
III. QUANTIZATION: KINEMATICAL HILBERT SPACE

First of all, we will introduce the kinematical Hilbert space where the canonical commutation relations of the phase space variables will be defined. In particular, we will adopt a standard Schrödinger representation of square integrable functions $\mathcal{H}_{\text{kin}} = L^2(\mathbb{R}^4)$, and $\hbar = 1$. For a given $\psi(u,v) \in \mathcal{H}_{\text{kin}}$, the phase space variables are promoted to the operator representation

\begin{align*}
\hat{p}_i \psi(u,v) &= -i\partial_u \psi(u,v), \quad \hat{\pi}_i \psi(u,v) = -i\partial_v \psi(u,v), \\
\hat{u}_i \psi(u,v) &= u_i \psi(u,v), \quad \hat{v}_i \psi(u,v) = v_i \psi(u,v).
\end{align*}

(15)

Within this representation, the quantum operators corresponding to the constraints (1) are given by

\begin{align*}
\hat{H}_1 &= -\frac{1}{2}(\partial^2_{u_1} + \partial^2_{u_2} + v_1^2 + v_2^2), \\
\hat{H}_2 &= -\frac{1}{2}(\partial^2_{v_1} + \partial^2_{v_2} + u_1^2 + u_2^2), \\
\hat{D} &= -i(u_1\partial_{u_1} + u_2\partial_{u_2} - v_1\partial_{v_1} - v_2\partial_{v_2}),
\end{align*}

(16)

where the factor ordering of $\hat{D}$ is selected such that the quantum commutation relations agree with the classical constraint algebra (2), that is

\begin{align*}
[\hat{H}_1, \hat{H}_2] &= i\hat{D}, \\
[\hat{H}_1, \hat{D}] &= -2i\hat{H}_1, \\
[\hat{H}_2, \hat{D}] &= 2i\hat{H}_2.
\end{align*}

(17)

Regarding the observable algebra (6), since their definition involves product of commuting phase space variables, the corresponding operators will be free of factor ordering ambiguities. Their commutation relations are given by

\begin{align*}
[\hat{Q}_i, \hat{Q}_j] &= i\epsilon_{ij}^k \hat{Q}_k, \\
[\hat{P}_i, \hat{P}_j] &= i\epsilon_{ij}^k \hat{P}_k, \\
[\hat{Q}_i, \hat{P}_j] &= 0.
\end{align*}

(18)

Finally, the quantum analogues to the classical identities (9) and (10) can be obtained directly just by replacing the classical elements by their quantum versions.

IV. ALGEBRAIC QUANTIZATION AND REFINED ALGEBRAIC QUANTIZATION

Here we will detail the Algebraic Quantization adopted in Ref. [16], together with a brief description of the Refined Algebraic Quantization [19] at the end of this section.
A. Algebraic Quantization

Following the results of [16, 19], one may look for the solutions to the quantum constraints (16). After a transformation to polar coordinates
\[ u_1 = u \cos \alpha, \quad u_2 = u \sin \alpha, \]
\[ v_1 = v \cos \beta, \quad v_2 = v \sin \beta, \]
one finds that the corresponding solutions to the three constraints are given by
\[ \Psi_{m, \epsilon} := e^{im(\alpha + \epsilon \beta)} J_m(uv), \]  
where \( m \in \mathbb{Z}, \ \epsilon \in \{1, -1\} \), and the functions \( J_m(uv) \) are the Bessel functions of first kind [21]. With the exception of the identity \( \Psi_{0,1} = \Psi_{0,-1} \), the remaining solutions are linearly independent [19].

Regarding the observable algebra (6), let us introduce the more convenient basis
\[ \hat{Q}_+ = \frac{1}{\sqrt{2}} (\hat{Q}_1 + i\hat{Q}_2), \quad \hat{Q}_- = \frac{1}{\sqrt{2}} (\hat{Q}_1 - i\hat{Q}_2), \]
\[ \hat{P}_+ = \frac{1}{\sqrt{2}} (\hat{P}_1 + i\hat{P}_2), \quad \hat{P}_- = \frac{1}{\sqrt{2}} (\hat{P}_1 - i\hat{P}_2), \]  
with \( \hat{Q}_3 \) and \( \hat{P}_3 \) unaltered. Their commutation relations are
\[ [\hat{Q}_3, \hat{Q}_\pm] = \mp \hat{Q}_\pm, \quad [\hat{Q}_+, \hat{Q}_-] = -\hat{Q}_3, \]
\[ [\hat{P}_3, \hat{P}_\pm] = \pm \hat{P}_\pm, \quad [\hat{P}_+, \hat{P}_-] = -\hat{P}_3. \]  
The action of these operators on the solutions (19) is
\[ \hat{Q}_3 \Psi_{m, \epsilon} = \delta_{1, \epsilon} m \Psi_{m, \epsilon}, \quad \hat{Q}_\pm \Psi_{m, \epsilon} = \pm i\sqrt{2} \delta_{1, \epsilon} m \Psi_{m\pm 1, \epsilon}, \]
\[ \hat{P}_3 \Psi_{m, \epsilon} = \delta_{-1, \epsilon} m \Psi_{m, \epsilon}, \quad \hat{P}_\pm \Psi_{m, \epsilon} = \pm i\sqrt{2} \delta_{-1, \epsilon} m \Psi_{m\pm 1, \epsilon}. \]  
Now, we can identify the sectors of this solution space providing an irreducible representation of this observable algebra, and endow them with a suitable inner product according to the adjoint relations
\[ (\hat{Q}_3)^\dagger = \hat{Q}_3, \quad (\hat{Q}_\pm)^\dagger = \hat{Q}_\mp; \quad (\hat{P}_3)^\dagger = \hat{P}_3, \quad (\hat{P}_\pm)^\dagger = \hat{P}_\mp. \]  
More specifically, the inner product is such that
\[ 2(m \pm 1)^2 (\Psi_{m, \epsilon_1=1}, \Psi_{m, \epsilon_2=1}) = - (\hat{Q}_+ \Psi_{m\pm 1, \epsilon_1=1}, \hat{Q}_+ \Psi_{m\pm 1, \epsilon_2=1}) \]
\[ = - (\hat{Q}_+ \Psi_{m\pm 1, \epsilon_1=1}, \hat{Q}_+ \Psi_{m\pm 1, \epsilon_2=1}) = 2m(m \pm 1) (\Psi_{m\pm 1, \epsilon_1=1}, \Psi_{m\pm 1, \epsilon_2=1}). \]
This condition together with the corresponding ones associated with the remaining values of \( \epsilon_1 \) and \( \epsilon_2 \) (recalling that the states \( \Psi_{m,\epsilon} \) are orthogonal) are fulfilled if

\[
(\Psi_{m,\epsilon}, \Psi_{m',\epsilon'}) = a_{\epsilon_1,\epsilon_2} |m\rangle \delta_{m,m'} \delta_{\epsilon_1,\epsilon_2},
\]

(26)

where the four constants \( a_{\epsilon_1,\epsilon_2} \) do not depend on the label \( m \). Therefore, the physical Hilbert space is endowed with a basis of normalizable states \( \Psi_{m,\epsilon}/\sqrt{|m|} \) that provides four irreducible representations of the observable algebra. Each of them corresponds the completion of the subspaces

\[
V_{\epsilon_1,\epsilon_2} := \text{span}\{\Psi_{m,\epsilon_2} | \epsilon_1 m > 0\},
\]

(27)

with the inner product (26). In the following, we will refer to each of those Hilbert spaces as \( \mathcal{H}_{\epsilon_1,\epsilon_2} \). The states \( \Psi_{0,\epsilon} \) have zero norm. This is one of the main handicaps of the Algebraic Quantization, since its presence in the solution space prevents the construction of any Hilbert space. The remedy in this case is dropping the troublesome states, something attainable since they are annihilated by the whole observable algebra and then they can be decoupled from the physical Hilbert space.

Finally, we have four different constants \( a_{\epsilon_1,\epsilon_2} \) in the inner product (26). Fortunately, we can take advantage of the reflection observable defined in Eq. (8) and represent it as an operator in the quantum theory. Its action on the solution space is

\[
\hat{R}_{\epsilon_1,\epsilon_2} : V_{\epsilon_1,\epsilon_2} \mapsto V_{\epsilon_1',\epsilon_2'},
\]

\[
\psi(u_1, u_2, v_1, v_2) \mapsto \psi(u_1, \epsilon_1 u_2, v_1, \epsilon_1 \epsilon_2 v_2),
\]

(28)

and it fulfills the adjoint relation \( (\hat{R}_{\epsilon_1,\epsilon_2})^\dagger = \hat{R}_{\epsilon_1,\epsilon_2} \). This last requirement restricts the previous inner product (26) and imposes that the constants \( a_{\epsilon_1,\epsilon_2} \) coincide [19]. One ends with a Hilbert space provided by the direct sum of the four spaces \( \mathcal{H}_{\epsilon_1,\epsilon_2} \), but now equipped with the same inner product.

Finally, the operators \( \hat{Q}_\pm, \hat{Q}_3, \hat{P}_\pm \) and \( \hat{P}_3 \) acting on the physical Hilbert space are

\[
\hat{Q}_3 \Psi_{m,\epsilon} = \delta_{1,\epsilon} m \Psi_{m,\epsilon}, \quad \hat{P}_3 \Psi_{m,\epsilon} = \delta_{-1,\epsilon} m \Psi_{m,\epsilon},
\]

\[
\hat{Q}_\pm \Psi_{m,\epsilon} = \pm i \sqrt{2} \delta_{1,\epsilon} \sqrt{|m(m \pm 1)|} \Psi_{m\pm 1,\epsilon},
\]

\[
\hat{P}_\pm \Psi_{m,\epsilon} = \pm i \sqrt{2} \delta_{-1,\epsilon} \sqrt{|m(m \pm 1)|} \Psi_{m\pm 1,\epsilon}.
\]

(29)
B. Refined Algebraic Quantization

Regarding the Refined Algebraic Quantization of this model \[19\], one starts with the representation \( (16) \) and assumes that their solutions belong to the algebraic dual \( \Phi^* \) of a dense subspace \( \Phi \subset \mathcal{H}_{\text{kin}} \). The latter is usually selected as an invariant, dense domain of the constraints \( (16) \). The observable algebra \( \mathcal{A}_{\text{obs}} \) of the model is automatically determined by these requirements, and does not need to be included explicitly. The final step consists in introducing the so-called rigging map \[7\] between the spaces \( \Phi \) and \( \Phi^* \), and which induces an inner product in the solutions space, and then the physical Hilbert space can be constructed out of. As was pointed out in Ref. \[19\], this map can be suitably defined once a convenient choice of \( \Phi \) is made. In other words, the rigging map depends on the specific choice of test states \( \Phi \), and so the observable algebra \( \mathcal{A}_{\text{obs}} \) and the physical Hilbert space. In particular, the overcompleted observable algebra considered in the Algebraic Quantization turns out to be a subalgebra of \( \mathcal{A}_{\text{obs}} \), well defined on the corresponding physical Hilbert space. Therefore, the Refined Algebraic Quantization can be seen as a generalization of the Algebraic Quantization.

Now, we would like to emphasize that the natural choice of test space \( \Phi \) includes zero norm vectors that impede the construction of a consistent rigging map. Then, the solution proposed in Ref. \[19\] is to ensure that at the end of the day the troublesome subspace is dropped by selecting a suitable test space \( \Phi \). The latter is carefully identified by means of the observable algebra explicitly introduced in the Algebraic Quantization, taking care that the observable algebra \( \mathcal{A}_{\text{obs}} \) remain large enough. Then, the rigging map can be consistently constructed, and the quantum description completed.

In summary, both quantization schemes require additional information at different levels of the construction, providing final results that depend on it.

V. MASTER CONSTRAINT PROGRAMME

Now, we will continue with the quantum description within the Master Constraint Programme \[10\] \[13\]. To this end, and for convenience, we will introduce the set of constraints \( \hat{H}_\pm = \hat{H}_1 \pm \hat{H}_2 \) with commutation relations

\[
[\hat{H}_+, \hat{H}_-] = -2i\hat{D}, \quad [\hat{H}_+, \hat{D}] = -2i\hat{H}_-, \quad [\hat{H}_-, \hat{D}] = -2i\hat{H}_+.
\] (30)
The Master Constraint will be defined as
\[ \mathcal{M} = \frac{1}{2}(\hat{H}_+^2 + \hat{H}_-^2 + \hat{D}^2) = 2\hat{C} + \hat{H}_-^2, \] (31)
where we have employed the identity \( 4\hat{C} = \hat{H}_+^2 + \hat{D}^2 - \hat{H}_-^2 \) in the previous expression.¹

In addition, we will introduce an equivalent formulation known as polarized Fock basis (see Ref. [13]). This basis is provided by the operators
\[ \hat{A}_\pm := \frac{1}{\sqrt{2}}(\hat{a}_1 \mp i\hat{a}_2), \quad \hat{A}^\dagger_\pm := \frac{1}{\sqrt{2}}(\hat{a}^\dagger_1 \pm i\hat{a}^\dagger_2), \] (32)
and the corresponding ones for the v-coordinates
\[ \hat{B}_\pm := \frac{1}{\sqrt{2}}(\hat{b}_1 \mp i\hat{b}_2), \quad \hat{B}^\dagger_\pm := \frac{1}{\sqrt{2}}(\hat{b}^\dagger_1 \pm i\hat{b}^\dagger_2), \] (33)
where
\[ \hat{a}_i := \frac{1}{\sqrt{2}}(\hat{u}_i + i\hat{p}_i), \quad \hat{b}_i := \frac{1}{\sqrt{2}}(\hat{v}_i + i\hat{\pi}_i), \] (34)
and their adjoints \( \hat{a}^\dagger_i \) and \( \hat{b}^\dagger_i \), are the standard creation-annihilation variables. A Fock state with respect to the annihilation operators \( \hat{A}_\pm \) and \( \hat{B}_\pm \) is given by \( |k_+, k_-, k'_+, k'_-\rangle \). They are defined by means of
\[ |k_+, k_-, k'_+, k'_-\rangle = \frac{(\hat{A}^\dagger_+)^{k_+} (\hat{A}^\dagger_-)^{k_-} (\hat{B}^\dagger_+)^{k'_+} (\hat{B}^\dagger_-)^{k'_-}}{\sqrt{k_+!} \sqrt{k_-!} \sqrt{k'_+!} \sqrt{k'_-!}} |0, 0, 0, 0\rangle, \] (35)
where \( |0, 0, 0, 0\rangle \) is the state which is annihilated by all four annihilation operators (in the same way that it is the vacuum state compatible with \( \hat{a}_i \) and \( \hat{b}_i \) with \( i = 1, 2 \)).

We will carry out a spectral decomposition of several quantities in our model. In particular, the observables \( \hat{C}, \hat{H}_-, \hat{Q}_3 \) and \( \hat{P}_3 \) can be simultaneously diagonalized together with the Master Constraint \( \mathcal{M} \).

First of all, in the polarized Fock basis the observables are given by
\[ \hat{Q}_\pm = \mp \frac{i}{\sqrt{2}}(\hat{A}_+ \hat{B}_+ + \hat{A}_-^\dagger \hat{B}_-^\dagger), \]
\[ \hat{Q}_3 = \frac{1}{2}(\hat{A}_+^\dagger \hat{A}_+ - \hat{A}_-^\dagger \hat{A}_- + \hat{B}_+^\dagger \hat{B}_+ - \hat{B}_-^\dagger \hat{B}_-), \]
\[ \hat{P}_\pm = \mp \frac{i}{\sqrt{2}}(\hat{A}_-^\dagger \hat{B}_+ + \hat{A}_+ \hat{B}_-^\dagger), \]
\[ \hat{P}_3 = \frac{1}{2}(\hat{A}_+^\dagger \hat{A}_+ - \hat{A}_-^\dagger \hat{A}_- - \hat{B}_+^\dagger \hat{B}_+ - \hat{B}_-^\dagger \hat{B}_-). \] (36)

¹ For convenience, we have selected a Master Constraint with a factor two with respect to the one adopted in Ref. [13].
Since $\hat{Q}_3$ and $\hat{P}_3$ commutes with $\hat{C}$ (i.e. with $\hat{M}$), we can diagonalize them simultaneously, and similarly with $\hat{H}_-$. This four observables are sufficient to identify any state of the system.

The action of $\hat{Q}_3$ and $\hat{P}_3$ is given by

$$\hat{Q}_3 |k_+, k_-, k'_+, k'_-\rangle = q_3 |k_+, k_-, k'_+, k'_-\rangle, \quad q_3 := \frac{1}{2} (j - j'),$$

$$\hat{P}_3 |k_+, k_-, k'_+, k'_-\rangle = p_3 |k_+, k_-, k'_+, k'_-\rangle, \quad p_3 := \frac{1}{2} (j + j'), \quad (37)$$

where $j := k_+ - k_-$ and $j' := -k'_+ + k'_-$. Regarding the constraints $\hat{H}_\pm$ and $\hat{D}$, it is not difficult to realize that

$$\hat{H}_- = \hat{A}_+^\dagger \hat{A}_+ + \hat{A}_-^\dagger \hat{A}_- - \hat{B}_+^\dagger \hat{B}_+ - \hat{B}_-^\dagger \hat{B}_-, \quad (38)$$

with the spectrum of $\hat{H}_-$

$$\hat{H}_- |k_+, k_-, k'_+, k'_-\rangle = k |k_+, k_-, k'_+, k'_-\rangle, \quad (39)$$

and $k := k_+ + k_- - k'_+ - k'_-$. Finally, we will deal with the spectral decomposition of $\hat{M}$, which is determined by the spectral properties of $\hat{H}_-$ and $\hat{C}$ (the Casimir). Here we will sketch the main properties that will be necessary in our study (for a more detailed description see [13] and the references therein). Additional details can be found in [A].

On the one hand, the spectrum of $\hat{C}$ possesses both discrete and continuous counterparts. The discrete counterpart of $\hat{M}$ is only for $k > 0$ and $|j| - |j'| \geq 2$, and for $k < 0$ and $|j| - |j'| \leq 2$:

$$\lambda_{\text{discr}} = 2t(1 - t) + k^2,$$

with $t = 1, 2, \ldots, \frac{1}{2} \min(|k|, |j| - |j'|)$ for even $k,$

and $t = \frac{3}{2}, \frac{5}{2}, \ldots, \frac{1}{2} \min(|k|, |j| - |j'|)$ for odd $k. \quad (40)$

Otherwise, the continuous part is

$$\lambda_{\text{cont}} = \frac{1}{2} + \frac{1}{2} x^2 + k^2 > 0, \quad x \in [0, \infty), \quad (41)$$

where $x$ is independent of the particular values of $k$, $j$ and $j'$. 
The normalized eigenfunctions $|j,j'\rangle_{t,k}$ corresponding to the discrete part of the spectrum are calculated in A, while the continuous ones were determined explicitly in Ref. [13]. We will use the notation $|x,k,j,j'\rangle$ for them, which will facilitate the distinction between normalizable and generalized eigenstates.

As one can see, the spectrum of the Master Constraint never vanishes. Its minimum value is in fact of the order of the square of the Planck constant (the reader must remind that we have set it to the unity), and belongs to the continuous part of the spectrum, corresponding to $x = 0$ and $k = 0$, i.e., to the eigenvalue $\lambda_{\text{cont}} = 1/2$. The prescription suggested in Ref. [13] modifies this observable by subtracting the corresponding contribution, yielding a new Master Constraint with a vanishing minimum eigenvalue. From now on we will refer to this eigenspace $|x = 0, k = 0, j, j'\rangle$ as the physical Hilbert space (for the Master Constraint Programme). Additionally, the restriction to this space of solutions of the observable algebra

$$\hat{Q}_3 |x = 0, k = 0, j, j'\rangle = q_3 |x = 0, k = 0, j, j'\rangle,$$
$$\hat{P}_3 |x = 0, k = 0, j, j'\rangle = p_3 |x = 0, k = 0, j, j'\rangle,$$

(42)

where $q_3$ and $p_3$ are arbitrary (semi)integers –see Eqs. (37)–, indicates that the spectrum of these observables can simultaneously achieve arbitrary large values. This is in contradiction with the classical theory, where the condition $Q_3 = 0$ or $P_3 = 0$ (or both) must be recovered somehow.

Again, the proposal in Ref. [13] consists in suitably reducing the quantum degrees of freedom by adding to the Master Constraint the condition $Q_3 P_3 = 0$. Consequently, the final Master Constraint would be

$$\hat{M}'' = \hat{M} - \frac{1}{2} \hat{j} + \frac{1}{2} (\hat{Q}_3 \hat{P}_3)^2.$$  (43)

The spectral decomposition of $\hat{M}''$ is already known, since $\hat{Q}_3$ and $\hat{P}_3$ are diagonal in the basis $|x, k, j, j'\rangle$. The restriction to the eigenspace corresponding to the minimum eigenvalue of $\hat{M}''$ yields $|x = 0, k = 0, j, j'\rangle$ but now with the condition $|j| = |j'|$, that is the requirement for a suitable semiclassical limit. Following Ref. [13], we will call this space of solutions SOL$''$.

However, the observables (36), while they obey the relations (9), they do not leave invariant SOL$''$. Nevertheless, one can find an alternative set of observable carrying out the relevant physical information. In fact, the classical observables of the type $p_1(Q_i)Q_3$ and
\[ p_2(P_i)P_3 \text{ with } p_1(y) \text{ and } p_2(y) \text{ being polynomial functions of } y, \text{ commutes weakly with the Master Constraint } M'' \]. In consequence, any observable \( p_1(Q_i)|\text{sgn}(Q_3)| \text{ and } p_2(P_i)|\text{sgn}(P_3)| \) (with \( \text{sgn}(x) = \{1, 0, -1\} \) for \( x > 0, x = 0 \) and \( x < 0 \), respectively) are also Dirac observables in the space of solutions, and then, they leave SOL'' invariant.

Therefore, the basic quantum algebra will be determined by the self-adjoint operators \( \hat{Q}_i' := |\text{sgn}(\hat{Q}_3)|\hat{Q}_i|\text{sgn}(\hat{Q}_3)| \) and \( \hat{P}_i' := |\text{sgn}(\hat{P}_3)|\hat{P}_i|\text{sgn}(\hat{P}_3)| \), defined by means of the spectral decomposition of \( \hat{Q}_3 \) and \( \hat{P}_3 \). These operators superselect five sectors in SOL'', one corresponding to each semiaxis of the coordinates \( q_3 \) and \( p_3 \) satisfying the condition \( q_3p_3 = 0 \), together with the origin \( q_3 = 0 \) and \( p_3 = 0 \). Additionally, in Ref. [13] are considered combinations of operators of the form \( |\text{sgn}(\hat{Q}_3)|p_1(\hat{Q}_i)|\text{sgn}(\hat{Q}_3)| \) and \( |\text{sgn}(\hat{P}_3)|p_2(\hat{P}_i)|\text{sgn}(\hat{P}_3)| \), breaking the mentioned superselection of the operators \( \hat{Q}_i' \) and \( \hat{P}_i' \) regarding the coordinates \( q_3 \) and \( p_3 \), and the resulting physical Hilbert space is given by the joint of the five previous subspaces.

We would like to comment that, from our point of view, the explicit inclusion of the observable \( (\hat{Q}_3\hat{P}_3) \) in \( \hat{M}'' \) might not be a good prescription to be adopted for the systematic quantization of fully constrained models owing to the difficulty of recognizing Dirac observables in more complicated settings. Notice that even though the additional term is a combination of constraints, there is no way of identifying its specific form without the help of the observables.

\section*{VI. UNIFORM DISCRETIZATIONS}

\subsection*{A. Classical description}

In this scheme [14], we start by considering a discrete version of a classical (continuum) theory, such that one works off-shell (but close to) with respect to the latter. On it, there is a clear notion of discrete evolution of any phase space function \( F \), that is dictated by

\[ F_{n+1} = e^{\{\cdot, H\}}F_n := F_n + \{F_n, H\} + \frac{1}{2!}\{\{F_n, H\}, H\} + \ldots , \] (44)

with \( H := f(H_1, H_2, D) \) a well defined functional of the constraints such that \( f(x_1, x_2, x_3) \) is any non-negative function that only vanishes at the origin, it is non-linear in the coordinates
\( x_1, x_2 \) and \( x_3 \) and the second derivatives satisfy
\[
\frac{\partial^2 f}{\partial x_i \partial x_j} \neq 0, \quad \forall x_i. \tag{45}
\]

One of the key ideas of this approach is that if one chooses initial data such that \( H = \delta^2/2 \), with \( \delta \) an arbitrary parameter, the constraints will remain bounded throughout the evolution, and they will approach the constraint surface in the limit \( \delta \to 0 \). One can easily realize this fact since \( H \) is itself preserved by the evolution dictated by Eq. \( \text{(44)} \).

In order to follow a similar analysis like the one proposed by the Master Constraint approach, we will identify \( H \) with the Master Constraint \( M \), i.e.,
\[
H := 2C + H^2, \tag{46}
\]
recalling that the Casimir \( C \) was already defined in \( \text{(9)} \). One can straightforwardly prove that \( H \) satisfies all the previous requirements as a function of the constraints. In order to analyze the classical discrete evolution, we will consider as initial data \( H = \delta^2/2 \). Let us start by noticing that the observables \( \text{(6)} \) commutes with \( H \), since they commute with the three constraints. Consequently, their discrete evolution is given by
\[
(Q_i)_{n+1} = (Q_i)_n, \quad (P_i)_{n+1} = (P_i)_n, \quad \forall i = 1, 2, 3; \quad n \in \mathbb{N}. \tag{47}
\]
They are, in consequence, constants of the motion, as well as \( C, H_- \) and \( H \) itself. However, the constraints \( H_+ \) and \( D \) do not commute with \( H \). In fact, one can prove that they oscillate around the surface constraint. Their evolution is dictated by
\[
(H_+)_{n+1} = (H_+)_n \cos(4H_-) + D_n \sin(4H_-),
\]
\[
D_{n+1} = D_n \cos(4H_-) + (H_+)_n \sin(4H_-). \tag{48}
\]
The transition matrix from an instant \( n \) to \( n + 1 \) is an \( SO(2) \) rotation, of angle \( \alpha = 4H_- \). From the relation of \( H \) with the constraints and the initial data condition, one can easily see that \( |\alpha| \leq 4\delta \). Recursively, one obtains
\[
(H_+)_n = (H_+)_0 \cos(n\alpha) - D_0 \sin(n\alpha),
\]
\[
D_n = D_0 \cos(n\alpha) + (H_+)_0 \sin(n\alpha), \tag{49}
\]
where \( D_0 \) and \( (H_+)_0 \) are the initial data corresponding to each discrete trajectory. The amplitude of the oscillations is bounded since it is given by
\[
0 < D_0^2 + (H_+)_0^2 = 2H - H_-^2 \leq 2H = \delta^2. \tag{50}
\]
In fact, at any other instant \( n \), the quantity \( D_n^2 + (H_n^+)^2 \) is a constant of motion, so the previous condition holds anytime. Clearly, in the limit \( \delta \to 0 \), one recovers the continuum theory.

Let us, however, see this in more detail, by analyzing the classical discrete dynamics of any arbitrary phase space function \( F \). In the original continuum theory, the time evolution of any space function can be computed by means of the Poisson brackets of this phase space function with the classical Hamiltonian \( H_T \) given in Eq. (3). Therefore, we find that

\[
\dot{F} = N\{F, H_1\} + M\{F, H_2\} + \lambda\{F, D\}. \tag{51}
\]

Within the Uniform Discretizations, the evolution is dictated by the discrete version of this equation, i.e., Eq. (44). If we initially choose \( H = \delta^2/2 \), with \( \delta \) close enough to the surface constraint, Eq. (44) simplifies since one can disregard high order contributions on \( \delta \). More concretely, close to the constraint surface one has

\[
F_{n+1} = F_n + \{F_n, H\} + O(\delta^2), \tag{52}
\]

with

\[
\{F_n, H\} = H_1\{F_n, H_1\} + H_2\{F_n, H_2\} + D\{F_n, D\}, \tag{53}
\]

and where we have omitted the label \( n \) in the constraints for simplicity. Now, let us write the constraints in a more convenient form

\[
H_1 = c\epsilon \cos \beta \sin \gamma, \quad H_2 = c\epsilon \sin \beta \sin \gamma, \quad D = c\epsilon \cos \gamma, \tag{54}
\]

where \( c \in \mathbb{R}^+ \) is an arbitrary positive parameter, \( \epsilon > 0 \), and the angles are \( \beta \in [0, \pi) \) and \( \gamma \in [0, 2\pi) \). This particular form of the constraints (up to the points where the spherical coordinates are ill defined) allows one to realize that \( \delta = c\epsilon \). Therefore, the limit \( \delta \to 0 \) now amounts to \( \epsilon \to 0 \) for constant and finite values of \( c \).

Keeping these considerations in mind, let us come back to Eq. (52) and write it in the more convenient form

\[
\frac{F_{n+1} - F_n}{\epsilon} = c \cos \beta \sin \gamma \{F_n, H_1\} + c \sin \beta \sin \gamma \{F_n, H_2\} + c \cos \gamma \{F_n, D\} + O(\epsilon). \tag{55}
\]

If we take the limit \( \epsilon \to 0 \) in the previous expression (which corresponds to the limit \( \delta \to 0 \)), the right hand side of this equation is well defined, and the left one is identified with \( \dot{F} \). We
then recover Eq. (51), as we wanted to show, but with the Lagrange multipliers determined by the initial conditions for the constraints off-shell, i.e.,

\[ N = c \cos \beta \sin \gamma, \quad M = c \sin \beta \sin \gamma, \quad \lambda = c \cos \gamma. \]  

(56)

Therefore, at a given time, the discrete dynamics reproduces in a very good approximation the continuum classical one since any choice of the Lagrange multipliers in the continuum theory corresponds to a suitable choice of initial data off-shell for the discrete classical theory, in particular for the constraints.

B. Quantization

Now, we will study the corresponding quantization of the classical discrete theory. In the following, we will concentrate mainly on aspects concerning the compatibility of the discrete quantum theory with the continuum classical model, and the advantages it presents with respect to previous approaches. We will leave as a matter of future research the analysis of the genuine quantum discrete dynamics involving parameterized Dirac observables also known as evolving constants of motion [17, 22]. For the quantum description of the model, we will adhere to the kinematical Hilbert space explained in Sec. III. We represent the operators \( \hat{u}_i, \hat{p}_i, \hat{v}_i \) and \( \hat{\pi}_i \) with \( i = 1, 2 \) in \( \mathcal{H}_{\text{kin}} = L^2(\mathbb{R}^4) \), as was done in Eq. (15). Hence, the Hamiltonian (16) is promoted to the operator \( \hat{H} \). Its spectral decomposition is the same as the one carried out for the Master Constraint \( \hat{M} \). Then, the spectrum of \( \hat{H} \) is

\[ \lambda^H_{\text{discr}} = 2t(1 - t) + k^2, \]

with \( t = 1, 2, \ldots, \frac{1}{2}\min(|k|, ||j| - |j'||) \) for even \( k \), and

\[ t = \frac{3}{2}, \frac{5}{2}, \ldots, \frac{1}{2}\min(|k|, ||j| - |j'||) \) for odd \( k \),

(57)

for the discrete counterpart, with \( k > 0 \) and \( |j| - |j'| \geq 2 \) or \( k < 0 \) for \( |j| - |j'| \leq 2 \), and

\[ \lambda^H_{\text{cont}} = \frac{1}{2} + \frac{1}{2}x^2 + k^2 > 0, \quad x \in [0, \infty), \]

(58)

being its continuous spectrum.

The minimum eigenvalue of \( \hat{H} \) is provided by \( k = 0 \) and the minimum of the spectrum of \( \hat{C} \), which is in its continuous counterpart. The restriction to it, however, does not introduce
any condition to \(j\) and \(j'\), as we saw in Sec. In consequence, the spectrum of both observables \(\hat{Q}_3\) and \(\hat{P}_3\) on this space can achieve any arbitrarily large value simultaneously.

This is a fundamental aspect that force us to consider alternative possibilities, like extending our study to other eigenspaces of \(\hat{H}\). For instance, in its discrete spectrum, the minimum is provided by \(t = 1\) and \(k = \pm 2\), with no obvious restrictions for \(|j| - |j'|\) —see (57). Then, on this subspace, the observables \(\hat{Q}_3\) and \(\hat{P}_3\) are not compatible with the continuum theory, like in the subspace related to the minimum of \(\lambda^H_{\text{cont}}\). Nevertheless, we can consider instead the whole infrared spectrum of \(\hat{H}\) —i.e. those eigenvalues (non-negative real numbers) lower or of the order of \(\hbar^2\) — that is compatible with, at least, certain subalgebra of observables.

Specifically, any state satisfying \(2t < |k| < \lambda^H_{\text{discr}}\) provides in fact satisfactory restrictions to the possible values of \(|j| - |j'|\), compatible with the continuum theory (up to quantum corrections). For a given

\[
\lambda^H_{\text{discr}} = 2t(1 - t) + k^2, \quad \text{with} \quad 2t < |k| < \lambda^H_{\text{discr}},
\]

which implies \(||j| - |j'|| = 2t\), as one can deduce from the definition of the discrete eigenvalues \(\lambda^H_{\text{discr}}\) in Eq. (57). Consequently, we have two subspaces labeled by \(\pm k\). Each of them, in turn, can be split in

\[
|j| - |j'| = \pm 2t.
\]

If we recall the definition of \(q_3\) and \(p_3\) (the eigenvalues of \(\hat{Q}_3\) and \(\hat{P}_3\), respectively) given in Eq. (37), the previous condition (60) is equivalent to

\[
|q_3 + p_3| - |p_3 - q_3| = \pm 2t.
\]

A simple inspection yields

\[
q_3 = \pm t \quad \text{and} \quad |p_3| \geq t, \quad \text{or} \quad p_3 = \pm t \quad \text{and} \quad |q_3| \geq t.
\]

From now on, we will call \(|q_3, p_3\rangle_{t,k}\) and \(|x, k, q_3, p_3\rangle\) the normalizable and generalized eigenstates of \(\hat{H}\), respectively, where we employ the labels \(q_3\) and \(p_3\) instead of \(j\) and \(j'\), in order to distinguish between our approach and the Master Constrain one. The subspace \(|q_3, p_3\rangle_{t,k}\) with \(q_3\) and \(p_3\) fulfilling (59) —and consequently (62)— is 2-fold degenerated since \(k > 2t\) if \(k > 0\) and \(k < -2t\) if \(k < 0\) (the specific expressions for these states can be found in [A]).
Among them, the states with the lowest eigenvalue of the Hamiltonian operator compatible with condition \(2t < |k| < \lambda_{\text{discr}}^H\) correspond to \(\lambda_{\text{discr}}^H = 16\), and consequently \(k = \pm 4\) and \(t = 1\). These states yield the best approximation to the classical theory. In this sense, this subspace is singled out from a physical point of view among the remaining ones. The first quantum description that we propose consists in restricting the study to this subspace of states. As we will see later, this proposal is similar to the one provided by the Master Constraint \[13\]. Let us also comment that any other eigenvalue \(\lambda_{\text{discr}}^H\) with the previous restriction and close to the lowest eigenvalue of \(\hat{H}\) would give a suitable quantum description.

To complete the quantization, we need to identify the observables that leave invariant each of these spaces, and particularly the one for \(\lambda_{\text{discr}}^H = 16\).

### C. Observable algebra

The action of the observables \[36\] will be easily deduced from their commutation relations (up to a global phase), instead of a direct calculation involving a considerable number of algebraic manipulations. The phase will be then straightforwardly deduced.

Let us restrict the study to the observables \(\hat{Q}_\pm\), since the analysis applies directly to the \(\hat{P}_\pm\) ones. Recalling that the commutation relations of these observables are

\[
[\hat{Q}_3, \hat{Q}_\pm] = \pm \hat{Q}_\pm, \quad [\hat{Q}_+, \hat{Q}_-] = -\hat{Q}_3,
\]

and that the Casimir operator is

\[
\hat{C} = \hat{Q}_+ \hat{Q}_- + \hat{Q}_- \hat{Q}_+ - \hat{Q}_3^2,
\]

one can easily solve, thanks to the commutation relations \[63\],

\[
2\hat{Q}_+ \hat{Q}_- = \hat{Q}_3^2 - \hat{Q}_3 + \hat{C}, \quad 2\hat{Q}_- \hat{Q}_+ = \hat{Q}_3^2 + \hat{Q}_3 + \hat{C}.
\]

Having said that, and recalling that the states \(|q_3, p_3\rangle_{t,k}\) are normalized eigenfunctions of \(\hat{Q}_3\) with eigenvalue \(q_3\), from the commutation relations \[63\] we deduce that \(\hat{Q}_\pm |q_3, p_3\rangle_{t,k}\) is either zero or proportional to \(|q_3 \pm 1, p_3\rangle_{t,k}\), respectively. From the relations \[65\], we get

\[
t_t, k \langle q_3 \pm 1, p_3 | q_3 \pm 1, p_3 \rangle_{t,k} = t_t, k \langle q_3, p_3 | (\hat{Q}_\pm)^\dagger \hat{Q}_\pm | q_3, p_3 \rangle_{t,k} = t_t, k \langle q_3, p_3 | \hat{Q}_\pm \hat{Q}_\pm | q_3, p_3 \rangle_{t,k} = t_t, k \langle q_3, p_3 | \hat{Q}_3^2 \pm \hat{Q}_3 + \hat{C} | q_3, p_3 \rangle_{t,k} = \frac{1}{2} q_3^2 \pm \frac{1}{2} q_3 + 2t(1 - t).
\]
Now, let assume that
\[ \hat{Q}_\pm |q_3, p_3\rangle_{t,k} = q_\pm (q_3) |q_3 \pm 1, p_3\rangle_{t,k}. \]  
(66)

Hence
\[ \hat{Q}_\mp \hat{Q}_\pm |q_3, p_3\rangle_{t,k} = q_\mp (q_3 \pm 1) q_\pm (q_3) |q_3, p_3\rangle_{t,k} = \left[ \frac{1}{2} q_3^2 \pm \frac{1}{2} q_3 + 2t(1 - t) \right] |q_3, p_3\rangle_{t,k}. \]  
(67)

Furthermore
\[ q_+(q_3) = _{t,k} \langle q_3 + 1, p_3 | \hat{Q}_+ |q_3, p_3\rangle_{t,k} = _{t,k} \langle q_3, p_3 | \hat{Q}_- |q_3 + 1, p_3\rangle_{t,k} = q_-(q_3 + 1). \]  
(68)

The solution to these equations is given by
\[ q_\pm (q_3) = \frac{z_\pm (q_3)}{\sqrt{2}} (q_3 \pm t), \quad \text{and} \quad z_+(q_3) z_-(q_3 + 1) = 1, \]
with \( |z_\pm| = 1. \) Consequently, the states are determined up to a global phase.

The last step consists in determining this phase. The observables defined in Eqs. (36), up to the global factor \( i, \) are a linear combination of products (second order polynomial) of the operators \( \hat{A}_\pm^\dagger, \hat{A}_\pm, \hat{B}_\pm^\dagger \) and \( \hat{B}_\pm. \) Now, consider the basis elements \( |k_+, k_-', k_+', k'_-\rangle \) of the polarized Fock quantization. The states \( |q_3, p_3\rangle_{t,k} \) are linear combinations of these basis elements, with real coefficients. Besides, the action of the previous operators on a given state of the polarized basis turns out to be a linear combination of the elements of the basis, with also real coefficients. This allows us to conclude that, up to a global irrelevant sign, \( z_\pm = \mp i. \)

Finally, in the basis \( |q_3, p_3\rangle_{t,k}, \)
\[ \hat{Q}_+ |q_3, p_3\rangle_{t,k} = \frac{-i}{\sqrt{2}} [q_3 + t] |q_3 + 1, p_3\rangle_{t,k} \]
\[ \hat{Q}_- |q_3, p_3\rangle_{t,k} = \frac{i}{\sqrt{2}} [q_3 - t] |q_3 - 1, p_3\rangle_{t,k} \]
\[ \hat{Q}_3 |q_3, p_3\rangle_{t,k} = q_3 |q_3, p_3\rangle_{t,k} \]
\[ \hat{P}_+ |q_3, p_3\rangle_{t,k} = \frac{-i}{\sqrt{2}} [p_3 + t] |q_3, p_3 + 1\rangle_{t,k} \]
\[ \hat{P}_- |q_3, p_3\rangle_{t,k} = \frac{i}{2 \sqrt{2}} [p_3 - t] |q_3, p_3 - 1\rangle_{t,k} \]
\[ \hat{P}_3 |q_3, p_3\rangle_{t,k} = p_3 |q_3, p_3\rangle_{t,k} \]  
(69)

Therefore, even if one starts with an state fulfilling (59) —the one that reproduces a good semiclassical limit for \( \hat{Q}_3 \) and \( \hat{P}_3 \)—, the repeated action of the observables \( \hat{Q}_\pm \) and \( \hat{P}_\pm \) would turn out in a state that is not compatible with condition (59) —unless we consider on this subspace states with arbitrary large values of \( k, \) then losing the semiclassical limit—.
D. Modified observable algebra

In order to overcome this drawback, we will present here a prescription of a modified observable algebra, based partially on the new observables introduced at the end of Sec. V.

Let us define, appealing to the spectral theorem, the following operator

\[ \hat{t} = \frac{1}{2} \hat{I} + \sqrt{\frac{1}{4} \hat{I} - \hat{C}_{\text{disc}}} \]  

(70)

with \( \hat{C}_{\text{disc}} \) the restriction of the Casimir to its discrete spectrum and \( \hat{I} \) the identity on \( \mathcal{H}_{\text{kin}} = L^2(\mathbb{R}^4) \). The operator \( \hat{t} \) has a discrete spectrum that equals the values of the parameter \( t \) in Eq. (40).

We will also define

\[ \hat{\varepsilon}_q := \hat{I} - \delta_{|Q_3|,t}. \]  

(71)

The spectrum of this operator is equal to the unity when \( q_3 \neq \pm t \), and zero if \( q_3 = \pm t \). Similarly, we define the operator

\[ \hat{\varepsilon}_p := \hat{I} - \delta_{|P_3|,t}. \]  

(72)

which are the identity when \( p_3 \neq \pm t \), and zero in the subspaces \( p_3 = \pm t \). These operators mimic the action of the operators \( |\text{sgn}(\hat{Q}_3)| \) and \( |\text{sgn}(\hat{P}_3)| \), respectively, employed in the definitions of \( \hat{Q}_3' \) and \( \hat{P}_3' \) (see the end of Sec. V).

Our new modified algebra will consist in the original \( \hat{Q}_3 \) and \( \hat{P}_3 \), and the modified ladder operators

\[ \hat{Q}_\pm := \hat{\varepsilon}_q \hat{Q}_\pm \hat{\varepsilon}_q, \quad \hat{P}_\pm := \hat{\varepsilon}_p \hat{P}_\pm \hat{\varepsilon}_p. \]  

(73)

Their action in a given space \( |q_3, p_3\rangle_{t,k} \) is

\[
\begin{align*}
\hat{Q}_+ |q_3, p_3\rangle_{t,k} &= (1 - \delta_{|q_3|,t})(1 - \delta_{|q_3+1|,t}) \frac{-i}{\sqrt{2}} [q_3 + t] |q_3 + 1, p_3\rangle_{t,k}, \\
\hat{Q}_- |q_3, p_3\rangle_{t,k} &= (1 - \delta_{|q_3|,t})(1 - \delta_{|q_3-1|,t}) \frac{i}{\sqrt{2}} [q_3 - t] |q_3 - 1, p_3\rangle_{t,k}, \\
\hat{Q}_3 |q_3, p_3\rangle_{t,k} &= q_3 |q_3, p_3\rangle_{t,k}, \\
\hat{P}_+ |q_3, p_3\rangle_{t,k} &= (1 - \delta_{|p_3|,t})(1 - \delta_{|p_3+1|,t}) \frac{-i}{\sqrt{2}} [p_3 + t] |q_3, p_3 + 1\rangle_{t,k}, \\
\hat{P}_- |q_3, p_3\rangle_{t,k} &= (1 - \delta_{|p_3|,t})(1 - \delta_{|p_3-1|,t}) \frac{i}{\sqrt{2}} [p_3 - t] |q_3, p_3 - 1\rangle_{t,k}, \\
\hat{P}_3 |q_3, p_3\rangle_{t,k} &= p_3 |q_3, p_3\rangle_{t,k}. 
\end{align*}
\]  

(74)
From this observable algebra, we deduce that i) the four states $|q_3 = \pm t, p_3 = \pm t\rangle_{t,k}$ remain invariant under the action of all the previous observables, and ii) the subspaces

$$\{ |q_3 = \pm t, p_3 > t\rangle_{t,k} \}, \quad \{ |q_3 = \pm t, p_3 < -t\rangle_{t,k} \},$$

$$\{ |q_3 > t, p_3 = \pm t\rangle_{t,k} \} \quad \text{and} \quad \{ |q_3 < -t, p_3 = \pm t\rangle_{t,k} \},$$

are also left invariant under this modified observable algebra. Besides, these new observables together with the previous subspaces provide a semiclassical limit in agreement with the Algebraic Quantization and the Master Constraint Programme.

Eventually, the classical reflection observables defined in Eq. (8) can be represented as a discrete quantum operator. It preserves the sectors associated with each pair of quantum numbers $(t, k)$ but maps each of the previous subspaces (in which these sectors can be divided) among them.\(^2\)

Therefore, this quantum operator plus the modified observable algebra (74), together with the restriction to the sector corresponding to the lowest admissible eigenvalue $\lambda^H_{\text{discr}} = 16$ and $t = 1$ provide the final physical Hilbert space. In particular, it is the direct sum

$$\bigoplus_{q_3, p_3} \left[ \bigoplus_{\epsilon_1 = \pm 1} \bigoplus_{\epsilon_2 = \pm 1} \left( |\epsilon_1 q_3 = 1, \epsilon_2 p_3 = 1\rangle_{1,k} \oplus |\epsilon_1 q_3 > 1, \epsilon_2 p_3 = 1\rangle_{1,k} \oplus |\epsilon_1 q_3 = 1, \epsilon_2 p_3 > 1\rangle_{1,k} \right) \right],$$

and is 2-fold degenerated since $k = \pm 4$.

It is worth commenting that, nevertheless, the observable $\hat{Q}_3 \hat{P}_3$ is not bounded on the physical space. This is one of the differences with the Algebraic Quantization and the Master Constraint Programme, where the previous quantity identically vanishes on physical solutions. Clearly, the semiclassical condition $q_3 p_3 \simeq 0 + O(t^2)$ for the eigenvalues of $\hat{Q}_3$ and $\hat{P}_3$ is more restrictive than $q_3 \simeq 0 + O(t)$ and/or $p_3 \simeq 0 + O(t)$, where the symbol $O(t^n)$ indicates contributions of the order of $t^n$ and higher. In our proposal, the latter is satisfied while the former do not. Nevertheless, in the limit $\hbar \to 0$, both conditions are equivalent and the continuum classical theory is always recovered. Besides, the considerations explained before seem to be the best one can do within the Uniform Discretizations, as well as in the Master Constraint Programme by direct application, in order to achieve a suitable

\(^2\) This reflection operator plays a similar role than the one of the observables $|\text{sgn}(\hat{Q}_3)|p_1(\hat{Q}_3)|\text{sgn}(\hat{Q}_3)|$ and $|\text{sgn}(\hat{P}_3)|p_2(\hat{P}_1)|\text{sgn}(\hat{P}_3)|$ that break the superselection sectors within the Master Constraint [13].
semiclassical description without including non-trivial contributions of the type \((\hat{Q}_3 \hat{P}_3)^2\). Concretely, the classical counterpart of this quantity fulfills the identity \((10)\). On the one hand, the left hand side of this relation is a function of (some of) the observables. The addition of this contribution involves that one needs to incorporate at least some Dirac observables of the system at the fundamental level of the approach. On the other hand, the right hand side of Eq. \((10)\) is a linear combination of constraints that involves coefficients depending on phase space. It is therefore legitimate its inclusion, for instance, within the Master Constraint approach \([13]\), since it is just a constraint. However, it is unclear how this specific condition can be inferred without the previous knowledge of some of the observables of the model. In this context, as we have pointed out before, such type of considerations would make extremely difficult to extend this approach to more general situations like gravity.

E. Discrete quantum dynamics

Let us now study the dynamics of this particular model at the quantum level. It can be analyzed in two different ways. One of them is by means of parametrized observables, as we mentioned at the end of Sec. \([11]\) but this time restricted to the subspace \(\lambda^H_{\text{discr}} = 16\) and \(t = 1\). However, we will leave this analysis for a future research, and concentrate in the second perspective that we mentioned at the beginning of this manuscript. It concerns the genuine quantum discrete evolution of the Uniform Discretizations \([14]\). In the Heisenberg picture, the quantum dynamics is dictated by the unitary operator

\[
\hat{U} = e^{-i\hat{H}},
\]

(keeping in mind that we have chosen \(\hbar = 1\)). The quantum version of Eq. \((44)\) is given by

\[
\hat{F}_n = \hat{U}^{-1} \hat{F}_{n-1} \hat{U} = \hat{U}^{-n} \hat{F}_0 \hat{U}^n,
\]

for any quantum observable \(\hat{F}_0\) defined on the initial time section. From this point of view, the evolution becomes more interesting since the Hamiltonian possesses non-vanishing eigenvalues. Therefore, we will not restrict the study to a subspace associated with a particular eigenvalue of the Hamiltonian, but instead we will consider all its lowest eigenvalues. As we will see in a particular example, states peaked around the subspaces fulfilling \((59)\) will provide a good semiclassical description. Within this picture, it is expected that those observables that do not commute with the Hamiltonian will show a non-trivial discrete dynamics.
Once a particular state $|\psi\rangle$ in the kinematical Hilbert space has been chosen, the expectation value of the quantum analog to Eq. (78) is given by

$$\langle \hat{F}_{n+1} \rangle_\psi = \langle \hat{U}^{-1} \hat{F}_n \hat{U} \rangle_\psi = \langle \hat{U}^{-n} \hat{F}_0 \hat{U}^n \rangle_\psi$$

(79)

Now, since we assume that the Hamiltonian is a selfadjoint operator, its eigenstates provide a complete basis on the kinematical Hilbert space. For simplicity, we will denote $\lambda_m$ and $\lambda$ as the eigenvalues belonging to the discrete and continuous parts of the spectrum of this operator, respectively. Therefore, the state $|\psi\rangle$ can be decomposed as

$$|\psi\rangle = \sum_m \psi_m |\lambda_m\rangle + \int d\lambda \psi(\lambda) |\lambda\rangle,$$

(80)

where we have also omitted the degeneration labels of each eigenvalue for simplicity. If we introduce this in Eq. (79), we find that

$$\langle \hat{F}_{n+1} \rangle_\psi = \sum_{m,m'} e^{-in(\lambda_m - \lambda_{m'})} \psi_{m'}^* \psi_m \langle \lambda_{m'} | \hat{F}_0 | \lambda_m \rangle + \int d\lambda d\tilde{\lambda} e^{-in(\lambda - \tilde{\lambda})} \psi(\tilde{\lambda})^* \psi(\lambda) \langle \tilde{\lambda} | \hat{F}_0 | \lambda \rangle$$

$$+ 2\mathbb{R} \left[ \sum_m \int d\lambda e^{-in(\lambda_m - \lambda)} \langle \lambda | \hat{F}_0 | \lambda_m \rangle \right] + \ldots$$

(81)

where the symbol $\mathbb{R}$ denotes the real part. Then, the discrete quantum evolution is essentially a linear combination of oscillatory functions in the discrete time $n$ multiplied by the matrix elements of $\hat{F}_0$ on the basis of eigenstates of $\hat{H}$. If the state $|\psi\rangle$ is peaked around a given $\lambda_0$, the previous equation can be simplified

$$\langle \hat{F}_{n+1} \rangle_\psi - \langle \hat{F}_n \rangle_\psi = \sum_{m,m'} (-i(\lambda_m - \lambda_{m'})) \psi_{m'}^* \psi_m \langle \lambda_{m'} | \hat{F}_n | \lambda_m \rangle$$

$$+ \int d\lambda d\tilde{\lambda} \left( -i(\lambda - \tilde{\lambda}) \right) \psi(\tilde{\lambda})^* \psi(\lambda) \langle \tilde{\lambda} | \hat{F}_n | \lambda \rangle$$

$$+ 2\mathbb{R} \left[ \sum_m \int d\lambda (-i(\lambda_m - \lambda)) \langle \lambda | \hat{F}_n | \lambda_m \rangle \right] + \ldots$$

(82)

since in the sum (81) only those eigenvalues close to $\lambda_0$ will not be suppressed by the wave functions. In consequence, the differences between two eigenvalues like $(\lambda - \lambda')^a$ with $a > 1$ are supposed to be negligible with respect to the corresponding linear terms (both $\lambda$ and $\lambda'$ must be similar to $\lambda_0$ or they will be strongly suppressed). The dots in these expressions account for those subdominant contributions. Therefore, in this approximation, the previous equation can be written as

$$\langle \hat{F}_{n+1} \rangle_\psi - \langle \hat{F}_n \rangle_\psi = -i \langle [\hat{F}_n, \hat{H}] \rangle_\psi + \ldots$$

(83)
Since the Hamiltonian is a quadratic form of the constraints, the previous expression is analogous to
\[
\langle \hat{F}_{n+1} \rangle - \langle \hat{F}_n \rangle = -\frac{i}{2} \langle ([\hat{H}_+ [\hat{F}_n, \hat{H}_+] + [\hat{F}_n, \hat{H}_+] \hat{H}_+]) \rangle - \frac{i}{2} \langle ([\hat{H}_- [\hat{F}_n, \hat{H}_-] + [\hat{F}_n, \hat{H}_-] \hat{H}_-]) \rangle - \frac{i}{2} \langle ([\hat{D} [\hat{F}_n, \hat{D}] + [\hat{F}_n, \hat{D}] \hat{D}) \rangle \langle \psi \rangle + \ldots
\]
(84)

Now, if the state \( \ket{\psi} \) fulfills in a good approximation \( \hat{H}_\pm \ket{\psi} \simeq h_\pm \ket{\psi} \) and \( \hat{D} \ket{\psi} \simeq d \ket{\psi} \), with \( h_\pm \) and \( d \) some real coefficients of the order of \( \sqrt{\lambda_0} \), we would get
\[
\langle \hat{F}_{n+1} \rangle - \langle \hat{F}_n \rangle = -ih_- \langle ([\hat{F}_n, \hat{H}_-]) \rangle - ih_+ \langle ([\hat{F}_n, \hat{H}_+]) \rangle - id \langle ([\hat{F}_n, \hat{D}] \rangle \langle \psi \rangle + \ldots
\]
(85)

In this circumstances, it is expected that this equation will allow us to get a good approximation of Eq. (55), and therefore, of Eq. (51), whenever \( \lambda_0 \to 0 \). In the case in which this limit cannot be strictly taken (as in our particular model), it is natural to introduce an external energy scale which would allow one to distinguish between the discrete theory and the continuum limit. For a moment, let us recover the Planck constant \( \hbar \) as well as \( c \), the speed of light. Besides, if we compare our model with the harmonic oscillator, the frequencies and the masses involved are \( \omega = 1 \) and \( m = 1 \). Now, since the constraints \( H_\pm \) and \( D \) must have units of energy, the Hamiltonian of the Uniform Discretizations must have units of action. Therefore, we must keep in mind that our Hamiltonian is in fact normalized by \( H/K \), with \( K = mc^2 \omega \) a suitable constant (with \( m = 1 \) and \( \omega = 1 \)) determined by the constants of our theory and with units of energy over time. Besides, the eigenvalues of \( \hat{H} \) have units of energy square, i.e., \( \lambda = h^2 \omega^2 \tilde{\lambda} \) where \( \tilde{\lambda} \) is a nonvanishing constant (and with \( \omega = 1 \)). One can easily realize that the time step of the discrete evolution will be \( \Delta t = \sqrt{\tilde{\lambda}} h/(mc^2) \).

Whenever \( \sqrt{\tilde{\lambda}} \) is of the order of the unit, which corresponds to the lowest eigenvalues of the Hamiltonian \( \hat{H} \), the time step will be of the order of \( h/(mc^2) \) with \( m = 1 \). Since it is a really small physical time, the discrete evolution for few steps will give a very good approximation of the continuous evolution.

Let us also comment that the conditions required to the previous states \( \ket{\psi} \) are not enough since there could be observables, like \( \hat{Q}_3 \) and \( \hat{P}_3 \), with expectation values taking any arbitrary value simultaneously. In this case, we propose an additional requirement for the semiclassical states: they must be peaked around the subspaces restricted by condition (59). With this final remark, it is expected to achieve a good semiclassical description of the model. In summary, we require to the semiclassical states to be peaked around states
fulfilling Eq. (59) and such that the expectation values of \( \hat{H} \) be of the order of its smallest eigenvalues.

We will now study an example where the main aspects of the discrete quantum dynamics of the model will be discussed. Let us consider, for instance, the evolution of the operators corresponding to the constraints of the classical theory \( \hat{H}_+ \) and \( \hat{D} \). They are two unconstrained phase space functions, which do not commute with the Hamiltonian \( \hat{H} \). The classical discrete dynamics of the classical analogues of these two observables is given by Eq. (49). These operators, as it is shown by Eqs. (B5) and (B6) of App. B, have a well defined action on every eigenstate of the Hamiltonian (see App. B for comments). For the sake of simplicity, let us consider the semiclassical state

\[
|\psi\rangle = \frac{1}{\sqrt{2}}(|q_3, p_3\rangle_{t(k+1)} + |q_3, p_3\rangle_{t(k-1)}),
\]

with \( k - 1 > 2t + 4 \). We will comment later the situation in which such inequality is not fulfilled. This state is then a linear combination of two different normalizable eigenfunctions of the Hamiltonian \( \hat{H} \), as well as they fulfill the condition (59), i.e., they are compatible with the classical theory. Besides, we have chosen this particular superposition because the transition amplitude of \( |\psi\rangle \) with itself by means of \( \hat{H}_+ \) and/or \( \hat{D} \) is non-vanishing. This can be easily seen, since the expectation values of the constraints on this state are

\[
\langle \hat{H}_+ \rangle_{\psi} = \frac{\sqrt{k^2 - (2t - 1)^2}}{2} \left[ (-1)^{r_-} (q_3, p_3, k+1, t) + (-1)^{r_+} (q_3, p_3, k-1, t) \right]
\]

\[
\langle \hat{D} \rangle_{\psi} = i \frac{\sqrt{k^2 - (2t - 1)^2}}{2} \left[ (-1)^{r_-} (q_3, p_3, k+1, t) - (-1)^{r_+} (q_3, p_3, k-1, t) \right],
\]

with the exponents \( r_\pm \) some integers depending on the quantum numbers of the eigenstates. These expectation values can be identified with the initial data section. Besides, they are proportional to Planck constant. Let us consider now any arbitrary time section \( n \) and the corresponding operators \( \hat{H}_{+,n} \) and \( \hat{D}_n \) defined by means of the unitary operator \( \hat{U} \) and Eq. (78). Their expectation values are given now by

\[
\langle \hat{H}_{+,n} \rangle_{\psi} = \langle \hat{H}_+ \rangle_{\psi} \cos(4kn) - \langle \hat{D} \rangle_{\psi} \sin(4kn),
\]

\[
\langle \hat{D}_n \rangle_{\psi} = \langle \hat{D} \rangle_{\psi} \cos(4kn) + \langle \hat{H}_+ \rangle_{\psi} \sin(4kn).
\]

If we compare these expectation values with the classical evolution (49) of the constraints at different time instants, we see that both descriptions share several similarities. In both
cases, the classical constraints and their corresponding expectation values in the quantum theory simply oscillate around a constant initial data, that in the latter is provided by the expectation value of the constraints on the state $|\psi\rangle$. Let us recall that the amplitude of the oscillations will be of the order of $\hbar \omega \sqrt{k^2 - (2t - 1)^2}$ with $\omega = 1$. Besides, one can easily realize that the frequency of the oscillations corresponding to the discrete time $n$ of the classical and the quantum descriptions agree (though this is a consequence of the particular state $|\psi\rangle$ under consideration), where in the latter it will be proportional to $(\hbar \omega^2 k/K) = \hbar \omega k/mc^2$, with $\omega = 1$, $m = 1$ and $k$ the eigenvalue of $\hat{H}$. Let us recall that we have considered a linear combination of states in Eq. (86) such that they satisfy $k - 1 > 2t + 4$. This requirement has been adopted in order to the states $|q_3, p_3\rangle_{t(k+1)}$ and $|q_3, p_3\rangle_{t(k-1)}$ fulfill condition (59). If this is not true for both states, we still would require that at least one of them belong to the subspace compatible with (59). For instance, let us consider that $|q_3, p_3\rangle_{t(k+1)}$ belongs to such subspace by requiring $k = 2t + 1$, and any arbitrary state $|\tilde{q}_3, \tilde{p}_3\rangle_{\tilde{t}k}$. In order to have a non-vanishing transition amplitude of $|\psi\rangle$ with itself by means of the constraints, the states $|\tilde{q}_3, \tilde{p}_3\rangle_{\tilde{t}k}$ should be suitably selected. This condition restricts the possible choices to $|q_3, p_3\rangle_{t(k-1)}$, which is still a state that does not fulfill condition (59). However, the expectation values of the constraints computed with $|\psi\rangle$ still give a good semiclassical description for these two particular unconstrained observables.

This simple example indicates that the existence of semiclassical sectors in the kinematical Hilbert space would be sufficient in order to describe in a good approximation the classical continuum theory. Let us remark that we have selected a particular semiclassical state and we have shown how to construct it out of the subspace of states fulfilling Eq. (59). Although alternative choices of semiclassical states in agreement with the classical theory would be admissible, the previous subspace seems to be a good starting point since it provides natural restrictions like condition (62). Besides, this description with semiclassical states does not require the restriction to any modified observable algebra, like the one proposed in Sec. VII.D being possible to work with the full algebra of observables. This modified observable algebra is just required whenever the study is restricted to a particular eigenspace of the Hamiltonian $\hat{H}$, like in the case of $\lambda_{\text{disc}} = 16$.

Let us add a final comment. The previous semiclassical states give a good semiclassical description just for the constraint observables $\hat{H}_\pm$ and $\hat{D}$. In the case of more general phase space functions, these states must be generalized. However, it is well known that the
identification of general semiclassical states in general quantum systems is not a trivial task. We believe that the existence of the sectors fulfilling (59) provides an additional ingredient that could facilitate the identification of general semiclassical states of the theory.

VII. CONCLUSIONS

We have considered a totally constrained system with an $SL(2, \mathbb{R})$ gauge group. This system is sufficiently simple and manageable while carries difficulties that could be found in more sophisticated, totally constrained theories, like general relativity. We have reviewed different approaches for the quantization of this model, with special emphasis in the different advantages and handicaps they present. In particular, the Algebraic Quantization (and its more sophisticated version, known as Refined Algebraic Quantization) is able to provide a quantization where the physical Hilbert space is constructed from a subspace of the algebraic dual of a dense set of the kinematical Hilbert space, once it has been equipped with a suitable inner product. The main inconvenient, within this approach, is that at the end of the day one appeals to certain observables, requiring reality conditions in order to pick out the physical inner product. But in general models, the identification of such an observables could be a non-trivial task. Within the Refined Algebraic Quantization, one applies the group averaging techniques. But this approach requires averaging within a non-amenable group, introducing additional difficulties in order to achieve well defined integrations. Again, this question can be overcome by selecting a suitable family of test states [19]. As we have already seen, an alternate approach that is free of some of these drawbacks is the Master Constraint Programme. The Master Constraint possesses a minimum, non-vanishing eigenvalue where the corresponding infinite dimensional eigenspace is not entirely compatible with a suitable semiclassical limit. Hence, the proposed solution to this problem is to include a modified Master Constraint, which is explicitly dependent on the Dirac observables, allowing one to restrict the study to a particular subspace where a suitable semiclassical theory is recovered. Therefore, one again appeals to the Dirac observables as a fundamental ingredient in the quantum description. Let us also comment on the fact that one could even be tempted to consider a reduced phase space quantization (adopting gauge fixing conditions and avoiding so the implementation of the constraints at the quantum level). However, in this situation, one would not have a kinematical Hilbert space structure, and one would be far from giving
a suitable answer to the inconveniences found in more realistic situations. In fact, it is well known that in many realistic situations gauge fixings are not able to describe the complete constrained surface.

We suggest an alternative prescription, partially based on the Master Constraint Programme, within the Uniform Discretizations scheme. We identify the discrete Hamiltonian (the generator of the discrete evolution) with the original Master Constraint—a quadratic form in the $sl(2, \mathbb{R})$ constraints. After quantization, we propose relaxing the restriction to the minimum eigenvalue adopted in the Master Constraint, and considering instead all the subspaces associated to the lowest eigenvalues of the Hamiltonian. There we have seen that neither the set of generalized eigenstates nor some subsets of normalizable eigenfunctions reproduce by themselves a correct semiclassical limit. Nevertheless, there is a subfamily of finite norm eigenstates carrying out an inherent cut off, the ones fulfilling $2t < |k| < \lambda_{\text{discr}}^H$, compatible with a semiclassical description when certain subalgebra of observables is considered. Whether this cut off is just accidental or not is something that must be understood studying alternative systems with non-amenable, gauge groups. At this level, we can follow two strategies. One consists of restricting the study to any of those subspaces, in particular to the lowest admissible eigenvalue, i.e., $\lambda_{\text{discr}}^H = 16$, $t = 1$ and $k = \pm 4$. Nevertheless, these solution spaces are not invariant under the whole $so(2,1) \times so(2,1)$ algebra, losing the compatibility with the classical theory. In order to overcome this inconvenient, we modify this observable algebra at the quantum level, in such a way they have a well defined action and leave invariant these subspaces while reproduce a suitable semiclassical limit. This family of states (together with the modified observable algebra) gives the best description in agreement with the classical theory, and is analogous to the description proposed in the Master Constraint. In this situation, the dynamics of the model can be studied by means of the so-called evolving constants (or parametrized observables) \cite{16, 17, 22}. The second strategy consists in the study of the genuine quantum discrete dynamics of the Uniform Discretizations. In this case we consider all the kinematical Hilbert space and we identify there semiclassical states. We have seen that there is a suitable family of states, those fulfilling $2t < |k| < \lambda_{\text{discr}}^H$, of which it would be possible to obtain a suitable semiclassical description of the unconstrained model with a non-trivial quantum discrete dynamics compatible with the classical (discrete and continuous) theory. More specifically, we have studied a particular example for a couple of dynamical variables, providing a (partial but) successful
semiclassical description.

Our proposal can obviously be adopted by the Master Constraint Programme in the first situation. These two approaches, in comparison with the Algebraic Quantization, possess a kinematical structure well adapted to the physical one, while in the latter the physical states belong to a larger functional space where, in particular, the state \( q_3 = 0 = p_3 \) is excluded by the quantum theory. Let us emphasize that the Uniform Discretizations seems to carry all the relevant information about a suitable semiclassical description at the quantum level, without requiring any additional input. We understand, from a conceptual point of view, that our proposal provides a radically different perspective with respect to the other two quantization approaches, even more if one is interested in the application of these quantization techniques to more general totally constrained models like, e.g., general relativity.

Finally, let us give some final remarks about the quantum dynamics of the system. In the continuum theory, an usual strategy has been making use of the so-called evolving constants [22]. This was the original point of view adopted in Ref. [16], where the reality conditions required to the quantum evolving constants considerably restrict the possible choices of these quantum observables [17]. The Uniform Discretizations, as well as the Master Constraint proposal of Ref. [13], admits a description in these lines, as we have already mentioned. In particular, it would be interesting to compare them with the continuous quantization provided in Ref. [17]. Additionally, within the Uniform Discretizations, the freedom that we have introduced by considering arbitrary states but, obviously, giving expectation values of \( \hat{H} \) of the order of its lowest eigenvalues, turns out into a non-trivial, discrete quantum dynamics, where the system can evolve since there are many “energy” states available. From this point of view, one can also study the relational dynamics analyzing the conditional probabilities [14, 23] without selecting any particular variable as a time parameter and its consequent treatment as a classical variable. Here, one considers the probability that a given observable have a particular value when we make a measurement on another one. This point of view seems to be more natural since all the variables in the kinematical space of the system can be treated quantum-mechanically. Then, it would be interesting to check under which approximations this relational dynamics coincides with the one resulting from the use of the evolving constants technique [16, 17]. We will study all these aspects in the future.
Appendices

Appendix A: Physical states: normalizable solutions

In this Appendix, we will describe the spectral resolution of $\hat{H}$ adopting the treatment of Ref. [13]. Essentially, one starts with a representation of the positive and negative discrete series of $sl(2,\mathbb{R})$. Each representation is associated with the corresponding Hilbert spaces of holomorphic and anti-holomorphic functions on the open unit disc in $\mathbb{C}$, respectively, endowed with the scalar product, in both cases,

$$\langle f, h \rangle_l = \frac{l-1}{\pi} \int_D f(z) \overline{h(z)} (1-|z|^2)^{l-2} dx \, dy$$  \hspace{1cm} (A1)

where $D$ is the unit disc and $dx \, dy$ is the Lebesgue measure on $\mathbb{C}$. If $l = 1$ one simply considers the limit $l \to 1$ in the previous expression.

For the positive series, an orthonormal basis is given by

$$f^{l}_n := \left[ \mu_l(n) \right]^{-\frac{1}{2}} z^n \quad (n \in \mathbb{N}) \quad \text{with} \quad \mu_l(n) = \frac{\Gamma(n+1)\Gamma(l)}{\Gamma(l+n)},$$  \hspace{1cm} (A2)

while for the negative series, the corresponding basis is given by the complex conjugated of $f^{l}_n$. There exists also a unitary map between the polarized basis $\{|k_+, k_-, k'_+, k'_-\rangle\}$ and the basis provided by $f^{l}_n \otimes (f^{l'}_n)^*$, given by

$$U : f^{|j|+1}_n \otimes (f^{|j'|+1}_{n'})^* \mapsto (-1)^{n'} |k_+, k_-, k'_+, k'_-\rangle \quad \text{where}$$

$$2n = k_+ + k_- - |j|, \quad j = k_+ - k_-, \quad 2n' = k'_+ + k'_- - |j'|, \quad j' = -k'_+ + k'_-.$$  \hspace{1cm} (A3)

In this representation the Master Constraint is a differential operator [13], whose eigenfunctions are of the form

$$f_{k,j,j'}(z_1, \overline{z}_2, t) = f_{k,j,j'}(z_1 \overline{z}_2, t) z_1^{\frac{1}{2}(k-|j|+|j'|)} z_2^{\frac{1}{2}(k-|j|+|j'|)},$$  \hspace{1cm} (A4)

where the solutions that are regular at $z = 0$, with $z := z_1 \overline{z}_2$, are

$$f_{k,j,j'}(z, t) = (1-z)^{1-t-\frac{1}{2}(|j|+|j'|+2)} \times F \left( 1-t+\frac{1}{2}(-|j|+|j'|), 1-t+\frac{1}{2}k, 1+\frac{1}{2}(k-|j|+|j'|); z \right),$$  \hspace{1cm} (A5)
for $k - |j| + |j'| \geq 0$, and

$$f_{k,j,j'}(z,t) = (1 - z)^{1-t-\frac{1}{2}(|j|+|j'|+2)} z^{\frac{1}{2}(-k+|j|-|j'|)}$$

$$\times F\left(1 - t - \frac{1}{2} k; 1 - t + \frac{1}{2} (|j| - |j'|), 1 + \frac{1}{2} (-k + |j| - |j'|); z\right),$$

(A6)

for $k - |j| + |j'| \leq 0$, being $t = \frac{1}{2} (1 + \sqrt{1 - \lambda + 2k^2})$, Re(t) $\geq \frac{1}{2}$ and $F(a, b; c; z)$ the hypergeometric function [21].

Finally, we can use the map $U$ in (A3) to transfer these results to the original kinematical Hilbert space $L^2(\mathbb{R}^4)$. To this end we rewrite (A4) into a power series in $z_1$ and $\tau_2$ using the definition of the hypergeometric function

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} z^n,$$

(A7)

and

$$(1 - z)^{1-d} = \sum_{n=0}^{\infty} \frac{\Gamma(d+n-1)}{\Gamma(d-1)\Gamma(n+1)} z^n.$$  

(A8)

For $k - |j| + |j'| \geq 0$ we obtain

$$f(t; k, j, j') = U\left( f_{k,j,j'}(z_1, \tau_2, t) \right) = \sum_{m=0}^{\infty} a_m |k_+(m), k_-(m), k'_+(m), k'_-(m)|,$$

(A9)

where

$$k_+(m) = m + \frac{1}{2} (k + j + |j'|), \quad k_-(m) = m + \frac{1}{2} (k - j + |j'|),$$

$$k'_+(m) = m + \frac{1}{2} (|j'| - j'), \quad k'_-(m) = m + \frac{1}{2} (|j'| + j'),$$

(A10)

and

$$a_m = (-1)^m \left[ \mu_{(|j|+1)} \left( m + \frac{1}{2} (k - |j| + |j'|) \right) \right]^{\frac{1}{2}} \left[ \mu_{(|j'|+1)}(m) \right]^{\frac{1}{2}}$$

$$\times \frac{\Gamma\left(1 + \frac{1}{2}(k - |j| + |j'|)\right)}{\Gamma\left(1 - t + \frac{1}{2} (-|j| + |j'|)\right) \Gamma\left(1 - t + \frac{1}{2} k\right)}$$

$$\times \sum_{l=0}^{m} \frac{\Gamma\left(1 - t + \frac{1}{2} (-|j| + |j'|) + l\right) \Gamma\left(1 - t + \frac{1}{2} k + l\right)}{\Gamma\left(1 + \frac{1}{2} (k - |j| + |j'|) + l\right) \Gamma\left(1 + l\right)}$$

$$\times \frac{\Gamma\left(t + \frac{1}{2} (|j| + |j'|) + (m-l)\right)}{\Gamma\left(m-l+1\right) \Gamma\left(t + \frac{1}{2} (|j| + |j'|)\right)}.$$ 

(A11)
It is worth comment that replacing \( k \) with \(-k\), switching \(|j|\) and \(|j'|\) and multiplying with 
\((-1)^\frac{1}{2} (-|j|+|j'|)\), we obtain the coefficient \( a_m \) for the solution corresponding to \( k-|j|+|j'| \leq 0 \).

1. Normalizable eigenfunctions of \( \hat{H} \):

Let us focus on the normalizable eigenfunctions of \( \hat{H} \) fulfilling the condition \( 2t < |k| < \lambda_{\text{discr}}^H \). More precisely, we will start with those such that \( k-|j|+|j'| \geq 0 \). Since \(|j|-|j'| = \pm 2t\) and \(|k| > 2t\), i.e. \( k \pm 2t > 0\), the only possibility is that of \( k > 0 \). If we substitute this in (A5) we get

\[
\begin{align*}
    f_{\pm}(z) &= (1-z)^{-t\mp|j'|} F\left(1-t \mp t, 1-t + \frac{1}{2}k, 1 \mp t + \frac{1}{2}k; z\right) , \\
    f_+(z) &= (1-z)^{-2t-|j'|} F\left(1-2t, 1-t + \frac{1}{2}k, 1-t + \frac{1}{2}k; z\right) \\
    &= (1-z)^{-|j'|-1} \sum_{l=0}^{\infty} \frac{\Gamma(|j'|+l+1)}{\Gamma(|j'|+1) \Gamma(l+1)} z^l ,
\end{align*}
\]

where

\[
\begin{align*}
    f_+(z) &= (1-z)^{-2t-|j'|} F\left(1-2t, 1-t + \frac{1}{2}k, 1-t + \frac{1}{2}k; z\right) \\
    &= \frac{\Gamma(1+t+\frac{1}{2}k)}{\Gamma(1-t+\frac{1}{2}k)} \sum_{n,l=0}^{\infty} \frac{\Gamma(1-t+\frac{1}{2}k+n)}{\Gamma(1+t+\frac{1}{2}k+n) \Gamma(|j'|) \Gamma(l+1)} z^n \bar{z}^{l+n} .
\end{align*}
\]

In each case, the corresponding eigenfunction (A4) is

\[
\begin{align*}
    f_+(z_1, \bar{z}_2) &= f_+(z_1 \bar{z}_2) z_1^{\frac{1}{2}k-t} = \sum_{l=0}^{\infty} \frac{\Gamma(|j'|+l+1)}{\Gamma(|j'|+1) \Gamma(l+1)} z_1^{l-t+\frac{1}{2}k} , \quad f_+(z_1, \bar{z}_2) z_1^{\frac{1}{2}k-t} = \sum_{l=0}^{\infty} \frac{\Gamma(|j'|+l+1)}{\Gamma(|j'|+1) \Gamma(l+1)} z_1^{l-t+\frac{1}{2}k} ,
\end{align*}
\]

and

\[
\begin{align*}
    f_-(z_1, \bar{z}_2) &= f_-(z_1 \bar{z}_2) z_1^{\frac{1}{2}k+t} = \frac{\Gamma(1+t+\frac{1}{2}k)}{\Gamma(1-t+\frac{1}{2}k)} \times \sum_{n,l=0}^{\infty} \frac{\Gamma(1-t+\frac{1}{2}k+n)}{\Gamma(1+t+\frac{1}{2}k+n) \Gamma(|j'|) \Gamma(l+1)} z_2^n \bar{z}_2^{l+n} z_1^{n+t+\frac{1}{2}k} .
\end{align*}
\]
Let us now consider the case in which \( k - |j| + |j'| \leq 0 \). Again, \( |j| - |j'| = \pm 2t \) and \(|k| > 2t\). Since \( k \pm 2t < 0 \), we conclude that \( k < 0 \). Implementing all this in (A6)

\[
f_{\pm}(z) = (1 - z)^{-t - |j'|} z^{\pm t - \frac{1}{2} k} F \left( 1 - t - \frac{1}{2} k, 1 - t \mp t - \frac{1}{2} k; z \right)
\]

or more specifically

\[
f_{+}(z) = (1 - z)^{-2t - |j'|} z^{t + \frac{1}{2}|k|} F \left( 1 - t + \frac{1}{2} |k|, 1, 1 + t + \frac{1}{2} |k|; z \right)
\]

\[
= \frac{\Gamma(1 + t + \frac{1}{2}|k|)}{\Gamma(1 - t + \frac{1}{2}|k|)} \sum_{n,l=0} \frac{\Gamma(1 - t + \frac{1}{2}|k| + n)}{\Gamma(1 + t + \frac{1}{2}|k| + n)} \frac{\Gamma(2t + |j'| + l)}{\Gamma(2t + |j'|) \Gamma(l + 1)} z^{n + t + \frac{1}{2}|k|}, \quad (A18)
\]

and

\[
f_{-}(z) = (1 - z)^{2t - |j'|} z^{-t + \frac{1}{2}|k|} F \left( 1 - t + \frac{1}{2} |k|, 1 - 2t, 1 - t + \frac{1}{2} |k|; z \right)
\]

\[
= (1 - z)^{2t - |j'| - 1} z^{-t + \frac{1}{2}|k|} = \sum_{l=0} \frac{\Gamma(|j'| - 2t + l + 1)}{\Gamma(|j'| - 2t + 1) \Gamma(l + 1)} z^{l + \frac{1}{2}|k| - t}. \quad (A19)
\]

The corresponding eigenfunctions –see (A4)– are

\[
f_{+}(z_1, \bar{z}_2) = f_{+}(z_1 \bar{z}_2) \bar{z}_2^{-\frac{1}{2}|k| - t} = \frac{\Gamma(1 + t + \frac{1}{2}|k|)}{\Gamma(1 - t + \frac{1}{2}|k|)} \times \sum_{n,l=0} \frac{\Gamma(1 - t + \frac{1}{2}|k| + n)}{\Gamma(1 + t + \frac{1}{2}|k| + n)} \frac{\Gamma(2t + |j'| + l)}{\Gamma(2t + |j'|) \Gamma(l + 1)} z_1^{n + t} \bar{z}_2^{n + t + \frac{1}{2}|k|}, \quad (A20)
\]

and

\[
f_{-}(z_1, \bar{z}_2) = f_{-}(z_1 \bar{z}_2) z_1^{-\frac{1}{2}|k| + t} = \sum_{l=0} \frac{\Gamma(|j'| - 2t + l + 1)}{\Gamma(|j'| - 2t + 1) \Gamma(l + 1)} z_1^{l - t + \frac{1}{2}|k|}. \quad (A21)
\]

The last step in our calculations consists of applying the unitary transformation (A3) to the previous functions, and normalize them. The resulting eigenfunctions now read

\[
f_{\pm}(t, k, j, j') = \sum_{m=0} a_{\pm, m} |k_+(m), k_-(m), k'_+(m), k'_-(m)\rangle, \quad (A22)
\]

with the coefficients \( a_m \) given by

\[
a_{\pm, m} = (-1)^m \left[ \mu_{(|j'|+2t+1)} \left( m - t + \frac{1}{2} k \right) \right]^{\frac{1}{2}} \left[ \mu_{(|j'|+1)}(m) \right]^{\frac{1}{2}} \frac{\Gamma(|j'| + m + 1)}{\Gamma(|j'| + 1) \Gamma(m + 1)}. \quad (A23)
\]
b) \( k - |j| + |j'| \geq 0 \) and \( |j| - |j'| = -2t \),

\[
a_{-m} = (-1)^m \left[ \mu_{(|j'|-2t+1)} \left( m + t + \frac{1}{2}k \right) \right]^{\frac{1}{2}} \left[ \mu_{(|j'|+1)}(m) \right]^{\frac{1}{2}} \\
\times \frac{\Gamma (1 + t + \frac{1}{2}k)}{\Gamma (1 - t + \frac{1}{2}k)} \frac{\sum_{l=0}^{m} \Gamma (1 - t + \frac{1}{2}k + l) \Gamma (|j'| + m - l)}{\Gamma (1 + t + \frac{1}{2}k + l)}.
\]

\[ \tag{A24} \]

\[
= \frac{\Gamma (1 + t + \frac{1}{2}k)}{\Gamma (1 - t + \frac{1}{2}k)} \frac{\sum_{l=0}^{m} \Gamma (1 - t + \frac{1}{2}k + l) \Gamma (|j'| + m - l)}{\Gamma (1 + t + \frac{1}{2}k + l)} \Gamma (m - l + 1).
\]

\[ \tag{A24} \]

\[
c) \ k - |j| + |j'| \leq 0 \) and \( |j| - |j'| = 2t \),

\[
a_{+m} = (-1)^m \left[ \mu_{(|j'|+2t+1)}(m) \right]^{\frac{1}{2}} \left[ \mu_{(|j'|+1)}(m + t + \frac{1}{2}k) \right]^{\frac{1}{2}} \\
\times \frac{\Gamma (1 + t + \frac{1}{2}k)}{\Gamma (1 - t + \frac{1}{2}k)} \frac{\sum_{l=0}^{m} \Gamma (1 - t + \frac{1}{2}k + l) \Gamma (|j'| + m - l)}{\Gamma (1 + t + \frac{1}{2}k + l)} \Gamma (2t + \frac{1}{2}k + l) \Gamma (|j'| + m - l) \Gamma (m - l + 1).
\]

\[ \tag{A25} \]

\[
d) \ k - |j| + |j'| \leq 0 \) and \( |j| - |j'| = -2t \),

\[
a_{-m} = (-1)^m \left[ \mu_{(|j'|+2t+1)}(m) \right]^{\frac{1}{2}} \left[ \mu_{(|j'|+1)}(m - t + \frac{1}{2}k) \right]^{\frac{1}{2}} \\
\times \frac{\Gamma (|j'| - 2t + m + 1)}{\Gamma (|j'| - 2t + 1) \Gamma (m + 1)}.
\]

\[ \tag{A26} \]

The normalized eigenfunctions are finally defined as

\[
|j, j\rangle_{t, k} := \sum_{m=0}^{\infty} \tilde{a}_m |k_+(m), k_-(m), k_+(m), k_-(m)\rangle,
\]

\[ \tag{A27} \]

with

\[
\tilde{a}_m = a_m \left( \sum_{l=0}^{\infty} |a_l|^2 \right)^{-1}.
\]

\[ \tag{A28} \]

The states belonging to the infrarred spectrum of \( \hat{H} \) solve the three constraints \( \hat{H}_\pm \) and \( \hat{D} \) when quantum corrections of the Planck order are neglected.

2. Algebraic Quantization and physical states:

Given the solutions (A4) to the Master Constraint, one can ask which is the relation between the states annihilated by \( \hat{M} \) and the solutions (19) found within the Algebraic Quantization approach.

They can be easily computed by means of (A4) just solving the equation \( \hat{M}\Psi = 0 \). In this case, we set \( t = 1 \) and \( k = 0 \) in (A4). After applying the map (A3), the resulting solutions
are

\[ f(t = 1; k = 0, j = m, j' = \epsilon m) = \sum_{l=0}^{\infty} (-1)^l |k_+(l), k_-(l), k'_+(l), k'_- (l)\rangle, \quad (A29) \]

where \( f(t = 1; k = 0, j = m, j' = \epsilon m) = \Psi_{m, \epsilon} \). These states solve simultaneously the three constraints \( \hat{H}_+ \) and \( \hat{D} \). Nevertheless, they do not belong to \( \mathcal{H}_{\text{kin}} \). Hence additional considerations are necessary in order to endow them with Hilbert space structure.

**Appendix B: Constraint observable algebra**

In this Appendix we will study several properties of the quantum constraints \( \hat{H}_+ \) and \( \hat{D} \) defined in Eq. \((38)\). In particular, we are interested in the determination of their action on the eigenfunctions of \( \hat{H} \). This space of states \( \{|q_3, p_3\}_{tk}\) or \( \{|x, k, q_3, p_3\}\) is characterized by the eigenvalue of the Casimir \( \hat{C} \) which are labeled by \( t \) or \( x \), depending if it corresponds to the discrete or the continuous spectrum; \( k \), which is the eigenvalue of the constraint \( \hat{H}_- \); and \( \lambda_H \), the eigenvalue of the Hamiltonian \( \hat{H} \). For simplicity, we will restrict the study to \( \{|q_3, p_3\}_{tk}\), but the very same conclusions are also valid for \( \{|x, k, q_3, p_3\}\).

Since the quantum constraints fulfill the commutation relations \((30)\), it seems natural to introduce the ladder operators

\[ \hat{K}_\pm = \hat{H}_+ \pm i\hat{D}. \quad (B1) \]

Their commutators with the constraint \( \hat{H}_- \) can be straightforwardly deduced by means of the commutation relations \((30)\), yielding

\[ [\hat{H}_-, \hat{K}_\pm] = \pm 2\hat{K}_\pm. \quad (B2) \]

Now, using these commutators, one can easily conclude that the operators \( \hat{K}_\pm \) acting on states of the form \( |q_3, p_3\rangle_{tk} \) shift the label \( k \) in two units, that is,

\[ H_-(\hat{K}_\pm |q_3, p_3\rangle_{tk}) = (k \pm 2)\hat{K}_\pm |q_3, p_3\rangle_{tk}. \quad (B3) \]

Moreover, the square of the norms of \( \hat{K}_\pm |q_3, p_3\rangle_{tk} \) fulfill

\[ ||\hat{K}_\pm |q_3, p_3\rangle_{tk}||^2 = (2\lambda_h - k^2 \pm 2k)|| |q_3, p_3\rangle_{tk}||^2. \quad (B4) \]

This result, together with the relations \((38)\) and the reality of the coefficients \( \tilde{a}_m \) in Eq. \((A27)\) for the normalized eigenfunctions \( |q_3, p_3\rangle_{tk} \), allow us to conclude that the action of the
operators $\hat{K}_\pm$ is given by
\[ \hat{K}_\pm |q_3, p_3\rangle_{tk} = (-1)^{r_\pm(q_3, p_3, k, t)} \sqrt{2\lambda h - k^2 \pm 2k} |q_3, p_3\rangle_{t(k\pm 2)}, \tag{B5} \]
with $r_\pm(q_3, p_3, k, t)$ some integers that can depend on the corresponding quantum numbers, such that $r_\pm(q_3, p_3, k, t) + r_\mp(q_3, p_3, k \pm 2, t)$ are even integers. It is worth commenting that the action of $\hat{K}_\pm$ on the eigenstates $\{|q_3, p_3\rangle_{tk}\}$ with $k = \pm 2t$, respectively, is by annihilation. For the generalized eigenfunctions $\{|x, k, q_3, p_3\rangle\}$, it only happens for $k = \pm 1$ and $x = 0$.

Now, a straightforward calculation allows us to conclude that
\[ \hat{H}_+ = \frac{1}{2}(\hat{K}_+ + \hat{K}_-), \quad \hat{D} = \frac{i}{2}(\hat{K}_- - \hat{K}_+). \tag{B6} \]
Therefore, the action of the constraints on the solution space provided by condition (59) mixes states with labels $k \pm 2$.

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