Scaling Limit of Quantum Electrodynamics with Spatial Cutoffs

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Abstract. In this paper the Hamiltonian of quantum electrodynamics with spatial cutoffs is investigated. We define a scaled total Hamiltonian and consider its asymptotic behavior. In the main theorem, it is shown that the scaled total Hamiltonian converges to a self-adjoint operator in the strong resolvent sense, and effective potentials are derived.

1 Introduction

Quantum electrodynamics (QED) describes the system of Dirac fields coupled to quantized radiation fields. In this paper a scaled QED Hamiltonian is considered. In the main theorem, the effective potentials are obtained by taking a scaling limit of the scaled QED Hamiltonian. Let us define a QED Hamiltonian as an operator on a boson-fermion Fock space. The state space of QED is defined by the boson-fermion Fock space \( F_{\text{QED}} = F_{\text{Dirac}} \otimes F_{\text{rad}} \), where \( F_{\text{Dirac}} \) is the fermion Fock space on \( L^2(\mathbb{R}^3; \mathbb{C}^4) \) and \( F_{\text{rad}} \) is the boson Fock space on \( L^2(\mathbb{R}^3; \mathbb{C}^2) \). The field operators of the Dirac field and the radiation field are denoted by \( \psi(x) \) and \( A(x) \), respectively. Here we impose ultraviolet cutoffs on both \( \psi(x) \) and \( A(x) \). To define an interaction between the Dirac field and the radiation field, we introduce the electromagnetic current:

\[
J(x) = \begin{bmatrix} \rho(x) \\ J(x) \end{bmatrix},
\]

where \( \rho(x) = \psi^*(x)\psi(x) \) and \( J^j(x) = \psi^*(x)\alpha^j\psi(x), \ j = 1, 2, 3, \) with \( \alpha^j \in M_4(\mathbb{C}) \) satisfying the canonical anti-commutation relation \( \{\alpha^j, \alpha^l\} = 2\delta_{j,l} \). In this paper, instead of \( J(x) \), we consider the spatially localized electromagnetic current:

\[
J_\chi(x) = \begin{bmatrix} \rho_\chi(x) \\ J_\chi(x) \end{bmatrix},
\]

where \( \rho_\chi(x) = \chi(x)\rho(x) \) and \( J_\chi(x) = \chi(x)J(x) \) with a spatial cutoff \( \chi(x) \). Then the QED Hamiltonian with the spatial cutoff is defined by

\[
H = H_{\text{Dirac}} + H_{\text{rad}} + e \int_{\mathbb{R}^3} J_\chi(x) \cdot A(x) \, dx + \frac{e^2}{8\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_\chi(x)\rho_\chi(y)}{|x-y|} \, dx \, dy,
\]

(1)
where \( H_{\text{Dirac}} \) and \( H_{\text{rad}} \) are the free Hamiltonians of the Dirac field and the radiation field, respectively, \( e \in \mathbb{R} \) denotes the coupling constant, and \( \mathbf{J}_X(x) \cdot \mathbf{A}(x) = \sum_{j=1}^{3} J^j_X(x) A^j(x) \). \( H_{\text{Dirac}} \) and \( H_{\text{rad}} \) are denoted by formally

\[
H_{\text{Dirac}} = \sum_{s=\pm 1/2}^{1} \int_{\mathbb{R}^3} \sqrt{p^2 + M^2} \left( b^*_s(p) b_s(p) + d^*_s(p) d_s(p) \right) dp, \quad M > 0,
\]

\[
H_{\text{rad}} = \sum_{r=1,2} \int_{\mathbb{R}^3} |k| a^*_r(k) a_r(k) dk.
\]

It is seen that under some conditions on ultraviolet cutoffs and spatial cutoffs, \( H \) is a self-adjoint operator on \( \mathcal{F}_{\text{QED}} \) in [18], and the spectral properties of \( H \) also has been investigated in [3] [6] [18].

Now we consider the scaled QED Hamiltonian defined by

\[
H(\Lambda) = H_{\text{Dirac}} + \Lambda^2 H_{\text{rad}} + e\Lambda \int_{\mathbb{R}^3} \mathbf{J}_X(x) \cdot \mathbf{A}(x) dx + \frac{e^2}{8\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_\Lambda(x) \rho_\Lambda(y)}{|x-y|} dx dy, \quad \Lambda > 0,
\]  

(2)

and this is the main object in this paper. Historically Davies investigates a scaled Hamiltonian of the form \( H_0 + \Lambda^2 \phi(x) + \Lambda^2 H_b \) in [5] where \( H_0 = \frac{p^2}{2m} \) is a schrödinger operator, \( \phi(x) \) is the field operator of the scalar boson field, and \( H_b \) is the free Hamiltonian. Then an effective Hamiltonian \( H_0 + \kappa^2 \rho_\Lambda(x) \) is obtained by the scaling limit of the scaled Hamiltonian. This is the so called weak coupling limit. Regarding this scaling limit as \( \exp(-it\Lambda^2(\Lambda^{-2}H_0 + \Lambda^{-1} \Lambda^2 \phi(x) + H_b)) \), we may say that the weak coupling limit is to take \( t \to \infty, M \to \infty \) and \( \kappa \to 0 \) simultaneously. Roughly speaking it is a long time behavior of the time evolution of the Hamiltonian but with a simultaneous weak coupling limit between a particle and a scalar field. The scaled QED Hamiltonian \( H(\Lambda) \) in [2] is an extended model consided in Davies [5], and the unitary evolution of \( H(\Lambda) \) is given by

\[
e^{-itH(\Lambda)} = e^{-it\Lambda^2 \left( \int_{\mathbb{R}^3} \mathbf{J}_X(x) \cdot \mathbf{A}(x) dx + \frac{e^2}{8\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_\Lambda(x) \rho_\Lambda(y)}{|x-y|} dx dy \right)},
\]

(3)

where \( t\Lambda^2 \) is the scaled time and \( \xi \) is the scaled coupling constant. As a remark, \( H(\Lambda) \) is also derived from the transformation \( a_r(k) \mapsto \Lambda a_r(k) \), \( a^*_r(k) \mapsto \Lambda a^*_r(k) \). In this case, however, the ultraviolet cutoffs are independent of the scaling parameter \( \Lambda \).

In the main theorem, the asymptotic behavior of \( H(\Lambda) \) as \( \Lambda \to \infty \) is considered. To investigate it, we consider a dressing transformation, which is a unitary transformation, defined in [51]. Then by taking the scaling limit of \( H(\Lambda) \) as \( \Lambda \to \infty \), we have

\[
s - \lim_{\Lambda \to \infty} \left( H(\Lambda) - z \right)^{-1} = \left( H_{\text{Dirac}} + \frac{e^2}{8\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_\Lambda(x) \rho_\Lambda(y)}{|x-y|} dx dy - \frac{e^2}{4} \rho_{\text{eff}} - z \right)^{-1} P_{\text{rad}},
\]

(4)

where \( V_{\text{eff}} \) is a effective potential of the Dirac field given by

\[
V_{\text{eff}} = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbf{J}_X(x) \cdot \Delta(x-y) \mathbf{J}_X(y) \ dx dy,
\]

(5)

\( \Delta(z) = (\lambda^{i,j}(z) + \overline{\lambda}^{i,j}(-z))_{i,j} \) is \( 3 \times 3 \) matrix with \( \lambda^{i,j}(z) \) defined in \([42]\), and \( P_{\text{rad}} \) denotes the projection onto the linear subspace spanned by the Fock vacuum \( \Omega_{\text{rad}} \in \mathcal{F}_{\text{rad}} \). It is noted that \( V_{\text{eff}} \)
is an operator on $\mathcal{F}_{\text{Dirac}}$. Thus, by the scaling limit, we see formally that the density of the charge is changed as follows :

$$\rho(x)\rho(y) \quad \rightarrow \quad \rho(x)\rho(y) + \text{const.} |x-y| J(x) \cdot \nabla (x-y)J(y).$$

There are a lot of results on scaling limits of quantum field Hamiltonians, so far. As is mentioned above, the first rigorous result is obtained by Davies [5], and he derives $N$-body Schrödinger Hamiltonians with effective potentials from the Hamiltonians of the system of particles interacting with boson fields. Arai [11] considers an abstract scaling limit, and then apply it to a spin-boson model and the non-relativistic QED models in the dipole approximation. For further results on the non-relativistic QED models, refer to [8, 9, 12, 13]. Hiroshima considers the Hamiltonian of a system of particles interacting with bosons and fermions with effective potentials from the Hamiltonians of the system of particles interacting with bose fields. Arai [1] considers an abstract scaling limit, and then apply it to a spin-boson model and the Yukawa potential as an effective potential. On the recent research, Suzuki considers generalized spin-boson model [16] and generalized Nelson model [17], and Ohkubo investigates the so called Dereziński-Gérard model [15].

This paper is organized as follows. In Section 2, we introduce the Dirac field and the quantized radiation field with ultraviolet cutoffs, and by introducing spatial cutoffs, we define the QED Hamiltonian on the boson-fermion Fock space $\mathcal{F}_{\text{QED}} = \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{rad}}$, and state the main theorem. In Section 3, we give the proof of the main theorem.

## 2 Definitions and Main Results

### 2.1 Dirac Fields

Let us first define the Dirac field [19]. The state space of the Dirac field is defined by

$$\mathcal{F}_{\text{Dirac}} = \bigoplus_{n=0}^{\infty} (\otimes_{a}^{n} L^{2}(R^{3};C^{4})),
$$

where $\otimes_{a}^{n}$ denotes the $n$-fold anti-symmetric tensor product with $\otimes_{a}^{0} L^{2}(R^{3};C^{4}) := C$. For $\xi = \xi^{i}(\xi_{1}, \ldots, \xi_{4}) \in L^{2}(R^{3};C^{4})$, we denote the annihilation operator by $B(\xi)$, and for $\eta = \eta(\eta_{1}, \ldots, \eta_{4}) \in L^{2}(R^{3};C^{4})$ the creation operator $B(\eta)$. The creation operators and annihilation operators satisfy the canonical anti-commutation relations :

$$\{B(\xi), B^{\ast}(\eta)\} = (\xi, \eta)_{L^{2}(R^{3};C^{4})}, \quad \{B(\xi), B(\eta)\} = 0,$$

where $\{X,Y\} = XY + YX$. In this paper the inner product $(y,x)_{\mathcal{H}}$ on a Hilbert space $\mathcal{H}$ is linear in $x$ and antilinear in $y$. Let $\Omega_{\text{Dir}} = \{1,0,0, \cdots \} \in \mathcal{F}_{\text{Dirac}}$ be the Fock vacuum. The finite particle subspace over $N \subset L^{2}(R^{3};C^{4})$ is defined by

$$\mathcal{F}^{\text{fin}}_{\text{Dir}}(N) = \text{L.h}\{B^{\ast}(\xi_{1}) \cdots B^{\ast}(\xi_{n})\Omega_{\text{Dir}} \mid \xi_{j} \in N, j = 1, \cdots, n, n \in N\}. \quad (6)$$

In particular we simply call $\mathcal{F}^{\text{fin}}_{\text{Dir}}(L^{2}(R^{3};C^{4}))$ the finite particle subspace. For $f \in L^{2}(R^{3})$ let us set

$$b_{1/2}^{\ast}(f) = B^{\ast}(\xi^{i}(f,0,0,0)), \quad b_{-1/2}^{\ast}(f) = B^{\ast}(\xi^{i}(0,f,0,0)),
$$

$$d_{1/2}^{\ast}(f) = B^{\ast}(\xi^{i}(0,0,f,0)), \quad d_{-1/2}^{\ast}(f) = B^{\ast}(\xi^{i}(0,0,0,f)).$$
Then it is seen that
\[ \{ b_3(f), b^*_3(g) \} = \{ d_3(f), d^*_3(g) \} = \delta_{s, \tau}(f, g)_{L^2(\mathbb{R}^3)}, \]
\[ \{ b_s(f), b^*_s(g) \} = \{ d_s(f), d^*_s(g) \} = \{ b_s(f), d_s(g) \} = \{ b_s(f), d^*_s(g) \} = 0. \]

In this paper we denote the domain of operator \( X \) by \( \mathcal{D}(X) \). Let \( A \) be a self-adjoint operator on \( L^2(\mathbb{R}^3; \mathbb{C}^4) \). The second quantization of \( A \) is defined by
\[
d\Gamma_l(A) = \bigoplus_{n=0}^{\infty} \left( \sum_{j=1}^{n} (I \otimes \cdots \otimes \overline{A}_{jth} \otimes I \cdots \otimes I) \right).
\]

Let \( f \in \mathcal{D}(A) \). Then it follows that
\[
[d\Gamma_l(A), b_s(f)] = -b_s(Af), \quad [d\Gamma_l(A), b^*_s(f)] = b^*_s(Af),
\]
\[
[d\Gamma_l(A), d_s(f)] = -d_s(Af), \quad [d\Gamma_l(A), d^*_s(f)] = d^*_s(Af),
\]

on the finite particle subspace. Now let us define the Dirac field. The energy of an electron with momentum \( p \) is given by
\[
E_M(p) = \sqrt{M^2 + p^2}, \quad M > 0,
\]
where the constant \( M > 0 \) denotes the mass of an electron. The free Hamiltonian of the Dirac field is given by
\[
H_{\text{Dirac}} = d\Gamma_l(E_M).
\]

Let
\[
h_D(p) = \alpha \cdot p + \beta M, \quad s(p) = s \cdot p,
\]
where \( \alpha^j, j = 1, 2, 3, \) and \( \beta \) are \( 4 \times 4 \) matrices satisfying the canonical anti-commutation relations
\[
\{ \alpha^j, \alpha^k \} = 2 \delta_{jk}, \quad \{ \alpha_j, \beta \} = 0, \quad \beta^2 = I,
\]
and \( s = (s_j)_{j=1}^3 \) is the angular momentum of the spin. Let
\[
f_s^j(p) = \frac{\chi_\text{Dirac}(p) u^j_s(p)}{\sqrt{(2\pi)^3 E_M(p)}}, \quad g_s^j(p) = \frac{\chi_\text{Dirac}(p) v^j_s(p)}{\sqrt{(2\pi)^3 E_M(p)}},
\]
where \( \chi_\text{Dirac} \) is a cutoff function and, \( u_s = (u^j_s)_{j=1}^3 \) and \( v_s = (v^j_s)_{j=1}^3, \) denote the positive and negative energy part with spin \( s, \) respectively, satisfying
\[
h_D(p) u_s(p) = E_M(p) u_s(p), \quad s(p) u_s(p) = s|p| u_s(p),
\]
\[
h_D(p) v_s(p) = -E_M(p) v_s(p), \quad s(p) v_s(p) = s|p| v_s(p),
\]
and we set \( v^j_s(p) = v^j_s(-p) \). The field operator \( \psi(x) = \{ \psi_1(x), \cdots, \psi_4(x) \} \) is defined by
\[
\psi_l(x) = \sum_{s = \pm 1/2} (b_s(f^l_{s,x}) + d^*_s(g^l_{s,x})),
\]
where \( f^l_{s,x}(p) = f^l_s(p)e^{-ip \cdot x} \) and \( g^l_{s,x}(p) = g^l_s(p)e^{-ip \cdot x} \). We suppose the assumption (A.1) below.
(A.1) (Ultraviolet cutoff for the Dirac field)
\[ \chi_{\text{Dirac}} \text{ satisfies that} \]
\[ \int_{\mathbb{R}^3} \left| \frac{\chi_{\text{Dirac}}(p)\hat{d}^s(p)}{\sqrt{E_M(p)}} \right|^2 dp < \infty, \quad \int_{\mathbb{R}^3} \left| \frac{\chi_{\text{Dirac}}(p)\hat{v}^s(p)}{\sqrt{E_M(p)}} \right|^2 dp < \infty. \]

Since \( b_s(f) \) and \( d_s(f) \) are bounded with
\[ \|b_s(f)\| = \|d_s(f)\| = \|f\|, \]
we see that \( \psi(x) = \{\psi_1(x), \ldots, \psi_4(x)\} \) is bounded with
\[ \|\psi_l(x)\| \leq M'_D, \quad l = 1, \ldots, 4, \]
where \( M'_D = \sum_{s=\pm 1/2} \left( \left\| \frac{\chi_{\text{Dirac}}(p)\hat{d}^s(p)}{\sqrt{2\pi}E_M} \right\| + \left\| \frac{\chi_{\text{Dirac}}(p)\hat{v}^s(p)}{\sqrt{2\pi}E_M} \right\| \right). \]

\[ 2.2 \text{ Quantized Radiation Fields} \]

Next let us introduce the radiation field quantized in the Coulomb gauge. The Hilbert space for the quantized radiation field is given by
\[ \mathcal{F}_{\text{rad}} = \bigotimes_{n=0}^{\infty} \bigotimes_s^{n} L^2(\mathbb{R}^3; \mathbb{C}^2), \]
where \( \bigotimes^n_s \) denotes the n-fold symmetric tensor product with \( \bigotimes^0 L^2(\mathbb{R}^3; \mathbb{C}^2) := \mathbb{C} \). We denote the creation operator on \( \mathcal{F}_{\text{rad}} \) by \( A^*(\xi) \), \( \xi = (\xi_1, \xi_2) \in L^2(\mathbb{R}^3; \mathbb{C}^2) \), and the annihilation operator by \( A(\eta) \), \( \eta = (\eta_1, \eta_2) \in L^2(\mathbb{R}^3; \mathbb{C}^2) \). Let \( \Omega_{\text{rad}} = \{1, 0, 0, \ldots\} \in \mathcal{F}_{\text{rad}} \) be the Fock vacuum. The finite particle subspace on \( \mathcal{D} \subset \mathcal{F}_{\text{rad}} \) is defined by
\[ \mathcal{F}^\text{fin}_{\text{rad}}(\mathcal{D}) = \text{L.h.} \{A^*(\xi_1)\cdots A^*(\xi_n)\Omega_{\text{rad}} \mid \xi_j \in \mathcal{D}, j = 1, \ldots, n, \ n \in \mathbb{N}\}. \]

For simplicity we call \( \mathcal{F}^\text{fin}_{\text{rad}}(L^2(\mathbb{R}^3; \mathbb{C}^2)) \) the finite particle subspace. The creation operator and the annihilation operator satisfy the canonical commutation relation on the finite particle subspace :
\[ [A(\xi), A^*(\eta)] = (\xi, \eta)_{L^2(\mathbb{R}^3; \mathbb{C}^2)}, \quad [A(\xi), A^*(\eta)] = [A^*(\xi), A^*(\eta)] = 0, \]
where \( [X, Y] = XY - YX \). For \( f \in L^2(\mathbb{R}^3) \) let us set
\[ a^+_1(f) = A^*((f, 0)), \quad a^+_2(f) = A^*((0, f)). \]

Then it follows that on the finite particle subspace
\[ [a_r(f), a_r^+(g)] = \delta_{r'r}(f, g), \quad [a_r(f), a_r(g)] = [a_r^+(f), a_r^+(g)] = 0. \]
Let \( S \) be a self-adjoint operator on \( L^2(\mathbb{R}^3; \mathbb{C}^2) \). The second quantization of \( S \) is defined by
\[ d\Gamma_\Sigma(S) = \bigoplus_{n=0}^{\infty} \left( \sum_{j=1}^{n} (I \otimes \cdots I \otimes \sum_{jth} S \otimes I \cdots \otimes I) \right). \]
Let \( f \in \mathcal{D}(S^{-1/2}) \). It is seen that \( a_r(f) \) and \( a_r^*(f) \) are relatively bounded with respect to \( d\Gamma_b(S) \) with

\[
\|a_r(f)\Psi\| \leq \frac{|f|}{\sqrt{\omega}} \|d\Gamma_b(S)^{1/2}\Psi\|, \quad (14)
\]

\[
\|a_r^*(f)\Psi\| \leq \frac{|f|}{\sqrt{\omega}} \|d\Gamma_b(S)^{1/2}\Psi\| + |f|\|\Psi\|, \quad (15)
\]

for \( \Psi \in \mathcal{D}(d\Gamma_b(S)^{1/2}) \). We also see that

\[
[d\Gamma_b(S), a_r(f)] = -a_r(Sf), \quad [d\Gamma_b(S), a_r^*(f)] = a_r^*(Sf), \quad f \in \mathcal{D}(S), \quad (16)
\]

on the finite particle subspace.

Let us define the quantized radiation field. The one particle energy of photon with momentum \( k \) is given by

\[
\omega(k) = |k|. \quad (17)
\]

Then the free Hamiltonian of the radiation field is given by

\[
H_{\text{rad}} = d\Gamma_b(\omega). \quad (18)
\]

Let

\[
h_j^k(\mathbf{k}) = \frac{\chi_{\text{rad}}(\mathbf{k})e_j^k(\mathbf{k})}{\sqrt{2(2\pi)^3\omega(\mathbf{k})}}, \quad (19)
\]

where \( \chi_{\text{rad}} \) is a ultraviolet cutoff function and \( e_r(\mathbf{k}) = (e_j^k(\mathbf{k}))_{j=1}^3 \), \( r = 1, 2 \), denotes the polarization vectors satisfying

\[
e_r(\mathbf{k}) \cdot e_{r'}(\mathbf{k}) = \delta_{r,r'}, \quad \mathbf{k} \cdot e_r(\mathbf{k}) = 0, \quad \text{a.e.} \ \mathbf{k} \in \mathbb{R}^3.
\]

It is seen in \cite{4} that any polarization vectors satisfy that

\[
\sum_{r=1,2} e_j^r(\mathbf{k})e_j^r(\mathbf{k}) = \delta_{j,j} - \frac{k^j k^j}{|k|^2}. \quad (20)
\]

We introduce the following conditions.

**A.2 (Ultraviolet cutoff for the radiation field)**

\( \chi_{\text{rad}} \) satisfies that \( \chi_{\text{rad}}(-\mathbf{k}) = \chi_{\text{rad}}(\mathbf{k}) \) and

\[
\int_{\mathbb{R}^3} \left| \frac{\chi_{\text{rad}}(\mathbf{k})}{\sqrt{\omega(\mathbf{k})}} \right|^2 d\mathbf{k} < \infty, \quad \int_{\mathbb{R}^3} \left| \frac{\chi_{\text{rad}}(\mathbf{k})}{\omega(\mathbf{k})} \right|^2 d\mathbf{k} < \infty.
\]

The quantized radiation field \( A(x) = (A_j^j(x))_{j=1}^3 \) is defined by

\[
A_j^j(x) = \sum_{r=1,2} (a_r(h_j^r(x)) + a_r^*(h_j^r(x))), \quad x \in \mathbb{R}^3, \quad (21)
\]

where \( h_j^r(x) = h_j^r(\mathbf{k})e^{-ik \cdot x} \). It is seen that \( A(x) \) is relatively bounded with respect to \( H_{\text{rad}}^{1/2} \)

\[
\|A_j^j(x)\Psi\| \leq \sum_{r=1,2} (2M_{R}^{2,j,r}H_{\text{rad}}^{1/2}\Psi\| + M_{R}^{1,j,r}\|\Psi\|), \quad (22)
\]

where \( M_{R}^{1,j,r} = \frac{1}{\sqrt{2(2\pi)^3}} \left| \frac{\chi_{\text{rad}}e_j^r}{\sqrt{\omega}} \right|, \) \( k = 1, 2, \) \( r = 1, 2, \) \( j = 1, 2, 3. \)
2.3 Total Hamiltonian

The total Hilbert space of quantum electrodynamics is defined by

$$F_{\text{QED}} = F_{\text{Dirac}} \otimes F_{\text{rad}},$$

and the free Hamiltonian on $F_{\text{QED}}$ by

$$H_0 = H_{\text{Dirac}} \otimes I + I \otimes H_{\text{rad}}.$$  (23)

To define the interaction, we introduce an assumption on the spatial cutoff functions $\chi$:

(A.3) **(Spatial cutoffs)**

$\chi$ satisfies that

1. $\int_{\mathbb{R}^3} |\chi(x)| \, dx < \infty$ and
2. $\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\chi(x) \cdot \chi(y)|}{|x - y|} \, dx \, dy < \infty$.

If $\chi \in L^{6/5}(\mathbb{R}^3)$, the Hardy-Littlewood-Sobolev inequality (e.g. [14]; 4.3 Theorem) shows that $\chi$ satisfies the condition (ii) in (A.3).

Now let us define the interaction. The electromagnetic current is denoted by

$$J(x) = \begin{bmatrix} \rho(x) \\ J(x) \end{bmatrix},$$

where $\rho(x) = \psi^*(x) \psi(x)$ and $J^j(x) = \psi^*(x) \alpha^j \psi(x)$, $j = 1, 2, 3$, with $\alpha^j \in M_4(\mathbb{C})$ satisfying $\{\alpha^j, \alpha^{j'}\} = 2\delta_{j,l}$. Let us define the functional on $\mathcal{D}(I \otimes H_{\text{rad}}^{1/2}) \times F_{\text{QED}}$ by

$$\ell_1(\Psi, \Phi) = \sum_{j=1}^{3} \int_{\mathbb{R}^3} \chi(x)(J^j(x) \otimes A^j(x) \Psi, \Phi)_{F_{\text{QED}}} \, dx$$

for $\Psi \in \mathcal{D}(I \otimes H_{\text{rad}}^{1/2})$ and $\Phi \in F_{\text{QED}}$. By (22), (13) and (A.3), we see that

$$|\ell_1(\Psi, \Phi)| \leq \left( L_1 \| (I \otimes H_{\text{rad}}^{1/2}) \Psi \| + R_1\|\Psi\| \right) \|\Phi\|,$$  (24)

where

$$L_1 = 2\|\chi\|_{L^1} \sum_{j,l,r} |\alpha_{j,l}^r| \alpha_{j,l}^r M_{\alpha}^l M_{\alpha}^l M_{\alpha}^{2,j,r}, \quad R_1 = \|\chi\|_{L^1} \sum_{j,l,r} |\alpha_{j,l}^r| \alpha_{j,l}^r M_{\alpha}^l M_{\alpha}^l M_{\alpha}^{1,j,r}.$$  (25)

By the Riesz representation theorem, there exists a unique vector $\Xi_\Psi \in F_{\text{QED}}$ such that

$$\ell_1(\Psi, \Phi) = (\Xi_\Psi, \Phi) \quad \text{for all} \quad \Phi \in F_{\text{QED}}.$$

Let us define the linear operator $H'_i : F_{\text{QED}} \to F_{\text{QED}}$ by

$$H'_i : \Psi \mapsto \Xi_\Psi.$$  (26)
It is seen from (24) that
\[ \|H'_I\Psi\| \leq L_4\|(I \otimes H_{rad}^{1/2})\Psi\| + R_1\|\Psi\|. \tag{27} \]

We may express \(H'_I\) formally by
\[ H'_I = \sum_{j=1}^{3} \int_{\mathbb{R}^3} \chi(x) J^j(x) \otimes A^j(x) \, dx. \]

In a similar way to \(H'_I\), let us define the functional \(\ell_{II} : \mathcal{F}_{\text{Dirac}} \times \mathcal{F}_{\text{Dirac}} \rightarrow \mathbb{C}\) by
\[ \ell_{II}(\Psi, \Phi) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\chi(x) \chi(y)}{|x - y|} \rho(x) \rho(y) \Psi(\Phi) \, dx \, dy. \]

By (13) and (A.3), we see that
\[ |\ell_{II}(\Psi, \Phi)| \leq \left( M_{II} \sum_{l,v} (M^l_D M^v_D)^2 \right) \|\Psi\| \|\Phi\|, \tag{28} \]

where \(M_{II} := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\chi(x) \chi(y)|}{|x - y|} \, dx \, dy\). Then by using the Riesz representation theorem again, it is seen that there exists a unique vector \(\Upsilon_{\Psi} \in \mathcal{F}_{\text{Dirac}}\) such that for all \(\Phi \in \mathcal{F}_{\text{Dirac}}\),
\[ \ell_{II}(\Psi, \Phi) = (\Upsilon_{\Psi}, \Phi). \]

Then we can define the linear operator \(H_{\text{long}} : \mathcal{F}_{\text{Dirac}} \rightarrow \mathcal{F}_{\text{Dirac}}\) by
\[ H_{\text{long}} : \Psi \mapsto \Upsilon_{\Psi}. \tag{29} \]

\(H_{\text{long}}\) is expressed by formally
\[ H_{\text{long}} = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\chi(x) \chi(y)}{|x - y|} \rho(x) \rho(y) \, dx \, dy. \]

Physically \(H_{\text{long}}\) derives from the longitudinal photons \([2]\). By (28), it is seen that \(H_{\text{long}}\) is bounded with
\[ \|H_{\text{long}}\| \leq M_{II} \sum_{l,v} (M^l_D M^v_D)^2. \tag{30} \]

Let
\[ H'_{II} = H_{\text{long}} \otimes I. \tag{31} \]

Then the total Hamiltonian is given by
\[ H = H_0 + eH'_I + e^2 \frac{8}{8\pi} H'_{II}, \tag{32} \]

with the coupling constant \(e \in \mathbb{R}\). On the self-adjointness of \(H\), the following Lemma follows.

**Lemma** ([13], Lemma 1.1 ) Assume that (A.1)-(A.3) hold. Then \(H\) is self-adjoint on \(\mathcal{D}(H_0)\), and essentially self-adjoint on any core of \(H_0\) and bounded from below.
In particular $H$ is essentially self-adjoint on

$$D_0 = \mathcal{F}_{\text{Dirac}}^{\text{fin}}(\mathcal{D}(E_M)) \hat{\otimes} \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D}(\omega)),$$

where $\hat{\otimes}$ denotes the algebraic tensor product.

The scaled QED-Hamiltonian is defined by

$$H(\Lambda) = H_{\text{Dirac}} \otimes I + \Lambda^2 I \otimes H_{\text{rad}} + e\Lambda H'_I + \frac{e^2}{8\pi} H''_I,$$

(34)

where $\Lambda > 0$ denotes the scaling parameter. We are concerned with the asymptotic behavior of $H(\Lambda)$ as $\Lambda \to \infty$. The strategy is that we use the dressing transformation defined in (51), and take the scaling limit of the unitary transformed Hamiltonian. The following theorem is the main result in this paper.

**Theorem 2.1** Assume (A.1)-(A.3). Then for $z \in \mathbb{C} \setminus \mathbb{R}$

$$s - \lim_{\Lambda \to \infty} \left( H(\Lambda) - z \right)^{-1} = \left( H_{\text{Dirac}} + \frac{e^2}{8\pi} H_{\text{long}} - \frac{e^2}{4} V_{\text{eff}} - z \right)^{-1} \otimes P_{\Omega_{\text{rad}}},$$

(35)

where

$$V_{\text{eff}} = \sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} J_j(x) \cdot \triangle(x-y) J_l(y) \, dx \, dy,$$

(36)

where $\triangle(z) = (\lambda^{(i)}(z) + \lambda^{(i)}(-z))^3_{i=1}$ is the $3 \times 3$ matrix, and $\lambda^{(i)}(z)$ is a function defined in (42).

By the general theorem ([17], Lemma 2.7) on resolvent convergence, we obtain the following corollary.

**Corollary 2.2** Assume (A.1)-(A.3). Then

$$s - \lim_{\Lambda \to \infty} e^{-itH(\Lambda)} (I \otimes P_{\Omega_{\text{rad}}}) = e^{-it \left( H_{\text{Dirac}} + \frac{e^2}{8\pi} H_{\text{long}} - \frac{e^2}{4} V_{\text{eff}} \right)} \otimes P_{\Omega_{\text{rad}}}.$$

(37)

### 3 Proof of Theorem 2.1

To prove the Theorem 2.1, we apply the abstract scaling limit considered in [1].

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. Let us set

$$\mathcal{Z} = \mathcal{X} \otimes \mathcal{Y}.$$

Let $A$ and $B$ be non-negative self-adjoint operators on $\mathcal{X}$ and $\mathcal{Y}$, respectively, and we assume $\ker B \neq \{0\}$. Let $P_B : \mathcal{Y} \to \ker B$ the orthogonal projection. We consider a family of symmetric operators $\{C(\Lambda)\}_{\Lambda > 0}$ satisfying the conditions:

(C.1) For all $\varepsilon > 0$ there exists a constant $\Lambda(\varepsilon) > 0$ such that for all $\Lambda > \Lambda(\varepsilon)$,

$$\mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B) \subset \mathcal{D}(C(\Lambda)),$$

and there exists $b(\varepsilon) \geq 0$ such that

$$\|C(\Lambda) \Xi\| \leq \varepsilon \|(A \otimes I + \Lambda I \otimes B) \Xi\| + b(\varepsilon) \|\Xi\|.$$
(C.2) There exists a symmetric operator \( C \) on \( \mathbb{Z} \) such that \( D \otimes \ker B \subset D(C) \) and for all \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[
\lim_{\Lambda \to \infty} C(\Lambda)(A \otimes I + \Lambda I \otimes B - z) = C(A - z)^{-1} \otimes P_B.
\]

**Theorem A** ([1], Theorem 2.1)
Assume (C.1) and (C.2). Then (i)-(iii) follows.
(i) there exists \( \Lambda_0 \geq 0 \) such that for all \( \Lambda > \Lambda_0 \),
\[
X(\Lambda) = A \otimes I + \Lambda I \otimes B + C(\Lambda)
\]
is self-adjoint on \( \mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B) \) and uniformly bounded from below for \( \Lambda \), furthermore \( X(\Lambda) \) is essentially self-adjoint on any core of \( A \otimes I + I \otimes B \).
(ii) Let \( X = A \otimes I + (I \otimes P_B)C(I \otimes P_B) \). Then \( X \) is self-adjoint on \( \mathcal{D}(A \otimes I) \) and bounded from below, and essentially self-adjoint on any core of \( A \otimes I \).
(iii) Let \( z \in \bigcap_{\Lambda \geq \Lambda_0} \rho(X(\Lambda)) \cap \rho(X) \) where \( \rho(O) \) denotes the resolvent set of an operator \( O \). Then, it follows that
\[
\lim_{\Lambda \to \infty} (X(\Lambda) - z)^{-1} = (X - z)^{-1}(I \otimes P_B).
\]

Now let us consider \( H_{\text{QED}} \). Let
\[
\Pi^j(x) = \sum_{r=1,2} \left( -a_r \left( \frac{h^j_r x}{\omega} \right) + a_r \left( \frac{h^j_r x}{\omega} \right) \right).
\] (38)
In a similar way to \( H^I \), we can define the operator
\[
T = \sum_{j=1}^3 \int_{\mathbb{R}^3} \chi(x) \left( J^j(x) \otimes \Pi^j(x) \right) dx.
\] (39)
By the canonical commutation relations of \( a_r(f) \) and \( a^*_r(g) \), we have
\[
\left[ \Pi^j(x), \Pi^j(y) \right] = 0,
\] (40)
follows. By (20), we also see that
\[
\left[ A^j(x), \Pi^j(y) \right] = \lambda^{j,j}(x - y),
\] (41)
where
\[
\lambda^{j,j}(x) = \int_{\mathbb{R}^3} \frac{\left| \chi_{\text{rad}}(k) \right|^2}{(2\pi)^3 k^3} \left( \delta_{j,j} - \frac{k^j k^j}{|k|^2} \right) e^{-ik \cdot x} d^3k.
\] (42)
By (14) and (15), it is seen that
\[
\| \Pi^j(x) \Psi \| \leq \sum_{r=1,2} (M_R^{1,j,r} \| H_{\text{rad}}^{1/2} \Psi \| + M_R^{2,j,r} \| \Psi \|).\]
(43)
By (16), it is seen that
\[
\left[ \Pi^j(x), H_{\text{rad}} \right] = -iA^j(x),
\] (44)
Lemma 3.1 Assume (A.1) - (A.3). Then there exists \( \theta_j(t) \in [-|t|, |t|], j = 1, \cdots, 4 \), such that on \( \mathcal{D}_0 \).

(i) \[
U(t)^{-1}(H_{\text{Dirac}} \otimes I)U(t) = H_{\text{Dirac}} \otimes I + (-it)U(\theta_1(t))^{-1}[T, H_{\text{Dirac}} \otimes I]U(\theta_1(t)),
\]

(ii) \[
U(t)^{-1}(I \otimes H_{\text{rad}})U(t) = I \otimes H_{\text{rad}} - tH_1' + \frac{it^2}{2}U(\theta_2(t))^{-1}[T, H_1']U(\theta_2(t)),
\]

(iii) \[
U(t)^{-1}H_1'U(t) = H_1' + (-it)U(\theta_3(t))^{-1}[T, H_1']U(\theta_3(t)),
\]

(iv) \[
U(t)^{-1}H_2'U(t) = H_2' + (-it)U(\theta_4(t))^{-1}[T, H_2']U(\theta_4(t)).
\]

(Proof) Let us only prove (ii). Other cases can be proven in a similar manner to (ii). Let \( \Psi \in \mathcal{D}_0 \) and \( \Phi \in \mathcal{F}_{\text{QED}} \). We set

\[
F_{\Phi, \Psi}(t) = (\Phi, U(t)^{-1}I \otimes H_{\text{rad}})U(t)\Psi.
\]

By the strong differentiability of \( U(t)\Psi \) with respect to \( t \), Taylor’s theorem shows that there exists \( \theta_2(t) \in [-|t|, |t|] \) such that

\[
F_{\Phi, \Psi}(t) = F_{\Phi, \Psi}(0) + \frac{t}{1!}F'_{\Phi, \Psi}(0) + \frac{t^2}{2!}F''_{\Phi, \Psi}(\theta_2(t)),
\]

where \( F' = \frac{dF}{dt} \) and \( F'' = \frac{d^2F}{dt^2} \). By (45), we obtain (48), since \( \Phi \in \mathcal{F}_{\text{QED}} \) is arbitrary.

By Lemma 3.1 we obtain the following corollary.

Corollary 3.2 Assume (A.1) - (A.3). Then it follows that

\[
U \left( \frac{e}{\Lambda} \right)^{-1}H(\Lambda)U \left( \frac{e}{\Lambda} \right) = \tilde{H}_0(\Lambda) + K(\Lambda),
\]

where

\[
\tilde{H}_0(\Lambda) = \left( H_{\text{Dirac}} + \frac{e^2}{8\pi}H_{\text{long}} \right) \otimes I + \Lambda^2 H_{\text{rad}},
\]

and

\[
K(\Lambda) = -i \frac{e}{\Lambda} \left( \theta_1 \left( \frac{e}{\Lambda} \right) \right)^{-1}[T, H_{\text{Dirac}} \otimes I]U \left( \theta_1 \left( \frac{e}{\Lambda} \right) \right) + \frac{ie^2}{2} \left( \theta_2 \left( \frac{e}{\Lambda} \right) \right)^{-1}[T, H_1']U \left( \theta_2 \left( \frac{e}{\Lambda} \right) \right) - ie^2 \left( \theta_3 \left( \frac{e}{\Lambda} \right) \right)^{-1}[T, H_1']U \left( \theta_3 \left( \frac{e}{\Lambda} \right) \right) - i \frac{e^3}{8\pi \Lambda} \left( \theta_4 \left( \frac{e}{\Lambda} \right) \right)^{-1}[T, H_2']U \left( \theta_4 \left( \frac{e}{\Lambda} \right) \right).
\]
By Corollary 3.2, it follows that for \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[
(H(\Lambda) - z)^{-1} = U \left( \frac{e}{\Lambda} \right) \left( \hat{H}_0(\Lambda) + K(\Lambda) - z \right)^{-1} U \left( \frac{e}{\Lambda} \right)^{-1}.
\] (54)

In the following proposition, we will prove that \( \hat{H}_0(\Lambda) \) and \( K(\Lambda) \) satisfy the condition (C.1) and (C.2) with applying \( \hat{H}_0(\Lambda) \) to \( X_0(\Lambda) \) and \( K(\Lambda) \) to \( C(\Lambda) \).

**Proposition 3.3** Assume (A.1) - (A.3).

1. For \( \varepsilon > 0 \), there exists \( \Lambda(\varepsilon) \geq 0 \) such that for all \( \Lambda > \Lambda(\varepsilon) \),
\[
\|K(\Lambda)\Psi\| \leq \varepsilon \|\hat{H}_0(\Lambda)\Psi\| + \nu(\varepsilon)\|\Psi\|,
\] \( \Psi \in D_0 \),
\] (55)
holds, where \( \nu(\varepsilon) \) is a constant independent of \( \Lambda \geq \Lambda(\varepsilon) \).

2. For all \( z \in \mathbb{C} \setminus \mathbb{R} \), it follows that
\[
s - \lim_{\Lambda \to \infty} K(\Lambda) \left( \hat{H}_0(\Lambda) - z \right)^{-1} = K(H_{\text{Dirac}} + \frac{e^2}{8\pi}H_{\text{long}} - z)^{-1} \otimes P_{\Omega_{\text{end}}},
\] (56)
where
\[
K = -\frac{ie^2}{2}[T, H^I_f].
\] (57)

To prove Proposition 3.3 let us prove Lemma 3.4 - Lemma 3.7.

**Lemma 3.4** Assume (A.1)-(A.3). Then it follows that
\[
\left\| [H_{\text{Dirac}}, J^j(x)] \right\| \leq c^j,
\] (58)
where \( c^j = \frac{2}{\sqrt{2\pi}} \sum_{l', l} \sum_y |\alpha_{l', l}^j| \left( \| \sqrt{EM} \chi_{D} u_s^l \| + \| \sqrt{EM} \chi_{D} v_s^l \| \right) \), and
\[
\left\| [p(x)p(y), J^j(x)] \right\| \leq d^j,
\] (59)
where \( d^j = 4 \sum_{l', l, y, y'} |\alpha_{l', l}^j| \left( M_{l}^I M_{l'}^I \right)^2 M_{l}^I M_{l'}^I \).

**Proof**

By using \([X, YZ] = [X, Y]Z + Y[X, Z]\), we see that
\[
[H_{\text{Dirac}}, J^j(x)] = \sum_{l, l'} \alpha_{l, l'}^j \left( [H_{\text{Dirac}}, \psi_l^r(x)] \psi_l^r(x) + \psi_l^r(x)[H_{\text{Dirac}}, \psi_l^r(x)] \right).
\]

By the commutation relations (7) and (8), we have
\[
[H_{\text{Dirac}}, \psi_l^r(x)] = \sum_s \left( b_s^r(E_{M} f_{s,x}^l) - d_s(E_{M} g_{s,x}^l) \right),
\]
\[
[H_{\text{Dirac}}, \psi_l^r(x)] = \sum_s \left( -b_s(E_{M} f_{s,x}^l) + d_s(E_{M} g_{s,x}^l) \right).
\]
Since

\[ [\rho(x)\rho(y), J^j(x)] = \sum_{i,l} \alpha^i_{j,l} \left( [\rho(x)\rho(y), \psi_i^j(x)] \psi_l^l(x) + \psi_i^j(x)[\rho(x)\rho(y), \psi_l^l(x)] \right). \]

Then by (12) and (13), we obtain (58). We also see that

\[ \| [\rho(x)\rho(y), \psi_i^j(x)] \| \leq 2\| \rho(x) \| \| \rho(y) \| \| \psi_i^j(x) \|, \]

(60)

where \( \psi_i^j(x) = \psi_l^l(x) \) or \( \psi_i^j(x) \), it follows from (13), that

\[ \| [\rho(x)\rho(y), \psi_i^j(x)] \| \leq 2 \sum_{\nu,\nu'} \left( M^\nu_0 M^{\nu'}_D \right)^2 M^\nu_D. \]

(61)

Then, by (13), (3) and (61), we have (59). □

**Lemma 3.5** There exist \( a_j > 0, b_j > 0, j = 1, 2, \) independent of \( s \), such that for \( \Psi \in D_0 \)

\begin{align*}
(i) \quad & \| (U(s)^{-1}[T, H_{\text{Dirac}} \otimes I]U(s)\Psi \| \leq a_1 \| I \otimes H^{1/2}_{\text{rad}} \Psi \| + b_1 \| \Psi \|, \\
(ii) \quad & \| (U(s)^{-1}[T, H^3_{\text{Dirac}}]U(s)\Psi \| \leq a_2 \| I \otimes H^{1/2}_{\text{rad}} \Psi \| + b_2 \| \Psi \|. \end{align*}

(62) (63)

**Proof**

(i) Let \( \Psi \in D_0 \) and \( \Phi \in F_{\text{QED}} \). We see that

\[ (\Phi, U(s)^{-1}[T, H_{\text{Dirac}} \otimes I]U(s)\Psi) = \int_{\mathbb{R}^3} \chi(x) \left( (H_{\text{Dirac}}, J^j(x)) \otimes I \right) U(s)\Phi, (I \otimes \Pi^j(x))U(s)\Psi \right) \, dx. \]

Note that \( [H_{\text{Dirac}}, J^j(x)] \) is a bounded operator by (58). Then, by the Schwarz inequality, we have

\begin{align*}
(\Phi, U(s)^{-1}[T, H_{\text{Dirac}} \otimes I]U(s)\Psi) & \leq \left( \int_{\mathbb{R}^3} |\chi(x)| \| (H_{\text{Dirac}}, J^j(x)) \otimes I \right) U(s)\Phi \|^2 \, dx \right)^{1/2} \times \left( \int_{\mathbb{R}^3} |\chi(x)| \| (I \otimes \Pi^j(x))U(s)\Psi \|^2 \, dx \right)^{1/2}.
\end{align*}

(64)

By (58), we have

\[ \left\| (H_{\text{Dirac}}, J^j(x)) \otimes I \right) U(s)\Phi \| \leq c_j \| \Psi \|. \]

(65)

Since \( I \otimes \Pi^j(x) \) and \( U(s) \) commute on \( D_0 \) for each \( x \in \mathbb{R}^3 \), we have by (43) that

\[ \| I \otimes \Pi^j(x)U(s)\Psi \| = \| I \otimes \Pi^j(x)\Psi \| \leq \sum_{r=1,2} \left( M^r_0 M^r_3 \right)^2 \| I \otimes H_{\text{rad}}^{1/2} \Psi \| + M^3_0 \| \Psi \|. \]

(66)

Then by (65), (66) and (64), we see that there exist \( a_1 \geq 0 \) and \( b_1 \geq 0 \) such that

\[ (\Phi, U(s)^{-1}[T, H_{\text{Dirac}} \otimes I]U(s)\Psi) \leq \left( a_1 \| I \otimes H_{\text{rad}}^{1/2} \Psi \| + b_1 \| \Psi \| \right) \| \Phi \|
\]

for \( \Phi \in F_{\text{QED}} \). Hence we have

\[ \| U(s)^{-1}[T, H_{\text{Dirac}} \otimes I]U(s)\Psi \| \leq a_1 \| I \otimes H_{\text{rad}}^{1/2} \Psi \| + b_1 \| \Psi \|. \]

(67)
(ii) Let $\Psi \in \mathcal{D}_0$ and $\Phi \in \mathcal{F}_{\text{QED}}$. Then we have

\[
(\Phi, U(s)^{-1}[T, H_{\text{long}} \otimes I]U(s)\Psi) = \int_{\mathbb{R}^3} \chi(x) \frac{\chi(y) \chi(z)}{|y - z|} (U(s)\Phi, ([J^j(x), \rho(y)\rho(z)] \otimes \Pi^j(x))U(s)\Psi) dxdydz
\]

\[
= \int_{\mathbb{R}^3} \chi(x) \frac{\chi(y) \chi(z)}{|y - z|} (\{[\rho(z)\rho(y), J^j(x)] \otimes I\}U(s)\Phi, (I \otimes \Pi^j(x))U(s)\Psi) dxdydz.
\]

By the Schwarz inequality, we see that

\[
| (\Phi, U(s)^{-1}[T, H_{\text{H}}]U(s)\Psi) | \leq \left( \int_{\mathbb{R}^3} |\chi(x)| \frac{|\chi(y)\chi(z)|}{|y - z|} \|([\rho(z)\rho(y), J^j(x)] \otimes I)U(s)\Phi\|^2 dxdydz \right)^{1/2}
\]

\[
\times \left( \int_{\mathbb{R}^3} |\chi(x)| \frac{|\chi(y)\chi(z)|}{|y - z|} \|(I \otimes \Pi^j(x))U(s)\Psi\|^2 dxdydz \right)^{1/2}.
\]

(67)

By (59), (66) and (67), there exists $a_2 \geq 0$ and $b_2 > 0$ such that

\[
| (\Phi, U(s)^{-1}[T, H_{\text{H}}]U(s)\Psi) | \leq \left( a_2 \|I \otimes H_{\text{rad}}^{1/2}\Psi\| + b_2 \|\Psi\| \right) \|\Phi\|,
\]

for $\Phi \in \mathcal{F}_{\text{QED}}$. Then we have

\[
\|U(s)^{-1}[T, H_{\text{H}}]U(s)\Psi\| \leq a_2 \|I \otimes H_{\text{rad}}^{1/2}\Psi\| + b_2 \|\Psi\|.
\]

Thus the proof is completed. $\blacksquare$

It is seen that $|\lambda^{j,l}(z)|$ is uniformly bounded with respect to $z$, namely

\[
|\lambda^{j,l}(z)| \leq \gamma^{j,l} := \int_{\mathbb{R}^3} \left| \frac{\chi_{\text{rad}}(k)}{(2\pi)^3 |k|^2} \left( \delta_{j,l} - \frac{k^j k^l}{|k|^2} \right) \right| dk.
\]

(68)

**Lemma 3.6** Assume (A.1) - (A.3). Then there exist constants $a_3 > 0$ and $b_3 > 0$, independent of sufficiently small $s$, such that for $\Psi \in \mathcal{D}_0$,

\[
\|U(s)^{-1}[T, H_{\text{H}}]U(s)\Psi\| \leq a_3 \|I \otimes H_{\text{rad}}\Psi\| + b_3 \|\Psi\|.
\]

(69)

**Proof** Let $\Psi \in \mathcal{D}_0$ and $\Phi \in \mathcal{F}$. By the equality $[A \otimes B, C \otimes D] = [A, B] \otimes CD + CA \otimes [B, D]$ and the commutation relations (44), we have

\[
(\Phi, U(s)^{-1}[T, H_{\text{H}}]U(s)\Psi)
\]

\[
= \sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi(x)\chi(y) \left( U(s)\Phi, [J^j(x) \otimes \Pi^j(x), J^l(y) \otimes A^l(y)]U(s)\Psi \right) dxdy
\]

\[
= \sum_{j,l} \left( X_{j,l}(\Phi, \Psi) + Y_{j,l}(\Phi, \Psi) \right),
\]

(70)

where

\[
X_{j,l}(\Phi, \Psi) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi(x)\chi(y) \left( ([J^j(y), J^l(x)] \otimes I)U(s)\Phi, (I \otimes \Pi^j(x)A^l(y))U(s)\Psi \right) dxdy,
\]

(71)

\[
Y_{j,l}(\Phi, \Psi) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi(x)\chi(y) \lambda^{j,l}(x - y) \left( U(s)\Phi, (J^j(y)J^l(x) \otimes I)U(s)\Psi \right) dxdy.
\]

(72)
By the Schwarz inequality,

\[
|X_{j,l}(\Phi, \Psi)| \leq \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\chi(x)\chi(y)||((J^i(y), J^j(x)) \otimes I)U(s)\Phi|^2 \, dx \, dy \right\}^{1/2} \\
\times \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\chi(x)\chi(y)||((I \otimes \Pi^j(x)A^i(y))U(s)\Psi|^2 \, dx \, dy \right\}^{1/2}.
\]

(73)

We see that \(|||J^i(y), J^j(x)||| \leq 2|||J^i(y)||| |||J^j(x)|||\), and hence, we have from (13) that

\[
|||J^i(y), J^j(x)||| \leq 2 \sum_{\mu, \nu, \mu', \nu'} |\alpha_{\mu, \mu'}| |\alpha_{\nu, \nu'}| M_{D}^{\mu} M_{D}^{\nu} M_{D}^{\mu'} M_{D}^{\nu'}.
\]

(74)

In a similar way to Lemma 3.1, it is seen that there exists \(\tau(s) \in [-|s|, |s|]\) such that for \(\Psi \in \mathcal{D}_0\)

\[
U(s)^{-1} \left( I \otimes \Pi^j(x)A^i(y) \right) U(s)\Psi = \left( I \otimes \Pi^j(x)A^i(y) \right) \Psi + \frac{-is}{1!} U(\tau(s))^{-1} [T, I \otimes \Pi^j(x)A^i(y)] U(\tau(s))\Psi.
\]

(75)

Let \(\Xi \in \mathcal{G}_{\text{QED}}\). Then by the commutativity of \(I \otimes \Pi^j(x)\) and \(T\), we have

\[
(\Xi, [T, I \otimes \Pi^j(x)A^i(y)]U(\tau(s))\Psi) = (\Xi, I \otimes \Pi^j(x)[T, I \otimes A^i(y)]U(\tau(s))\Psi) = \int_{\mathbb{R}^3} \chi(z) \lambda^{k,l}(z-y)(\Xi, [J^k(z) \otimes \Pi^j(x), I \otimes A^i(y)]U(\tau(s))\Psi) \, dz
\]

\[
= \int_{\mathbb{R}^3} \chi(z) \lambda^{k,l}(z-y)((J^k(z) \otimes I)\Xi, U(\tau(s))(I \otimes \Pi^j(x))\Psi) \, dz.
\]

(76)

Then we obtain by (43) that

\[
\left| \langle (J^k(z) \otimes I)\Xi, [T, I \otimes \Pi^j(x)A^i(y)]U(\tau(s))\Psi \rangle \right| \leq \int_{\mathbb{R}^3} \chi(z) \left\| \lambda^{k,l}(z-y) \right\| \left\| (J^k(z) \otimes I)\Xi \right\| \left\| (I \otimes \Pi^j(x))\Psi \right\| \, dz
\]

\[
\leq \|\chi\|_{L^1} \sum_{\nu, \nu'} |\alpha_{\nu, \nu'}|^2 M_{D}^{\nu} M_{D}^{\nu'} \left( \sum_{r=1,2} (M_{R}^{k,j,r})^2 \|\Pi^j(x)\Psi\|_2 + M_{R}^{3,j,r} \|\Psi\| \right) \|\Xi\|,
\]

(77)

where \(\gamma^{k,l}\) is defined in (63). Hence we obtain that

\[
\|U(\tau(s))^{-1}[T, I \otimes \Pi^j(x)A^i(y)]U(\tau(s))\Psi\| \leq q_{j,l} \|\Pi^j(x)\Psi\| + \tilde{q}_{j,l} \|\Psi\|,
\]

(78)

where

\[
q_{j,l} = \|\chi\|_{L^1} \sum_{k,r,\nu, \nu'} |\alpha_{\nu, \nu'}|^2 M_{D}^{\nu} M_{D}^{\nu'} M_{R}^{k,j,r},
\]

\[
\tilde{q}_{j,l} = \|\chi\|_{L^1} \sum_{k,r,\nu, \nu'} \gamma^{k,l} |\alpha_{\nu, \nu'}|^2 M_{D}^{\nu} M_{D}^{\nu'} M_{R}^{3,j,r}.
\]
It is seen that there exist \( c_{j,l} \geq 0 \) and \( d_{j,l} \geq 0 \) such that

\[
\|I \otimes \Pi^j(x)A^j(y)\| \leq c_{j,l} \|I \otimes H_{rad}\| + d_{j,l} \|\Psi\|.
\]  

(79)

By (78), (79) and (75), we have

\[
\|U(s)^{-1}(I \otimes \Pi^j(x)A^j(y))U(s)\| \leq (c_{j,l} + s\|q\|)\|I \otimes H_{rad}\| + (d_{j,l} + s\|q\|)\|\Psi\|.
\]  

(80)

Hence by applying (74) and (80), to (73), we see that there exist constant \( a \geq 0 \) and \( b \geq 0 \) independent of sufficiently small \( s \), such that

\[
|X_{i,l}^j(\Phi, \Psi)| \leq \left(a\|I \otimes H_{rad}\| + b\|\Psi\|\right)\|\Phi\|.
\]  

(81)

Furthermore we can see by (13) under (A.3) that

\[
|Y_{i,l}^j(\Phi, \Psi)| \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left|\chi(x)\chi(y)\lambda^j(x-y)\right||(U(s)\Phi, (J^j(y)J^j(x) \otimes I)U(s)\Psi)||dxdy
\]

\[
\leq \|\chi\|^2_\mathbb{L}^2 \gamma^j \left(\sum_{\nu,\nu',\mu,\mu'} M^\nu_D M^\nu_{D'} M^\mu_D M^\mu_{D'}\right)\|\Phi\| \|\Psi\|.
\]  

(82)

By (81), (82) and (70), it can be seen that there exist \( a_3 \geq 0 \) and \( b_3 \geq 0 \) independent of sufficiently small \( s \) such that

\[
|\Phi, U(s)^{-1}[T, H^j]\|U(s)\Psi| \leq (a_3\|I \otimes H_{rad}\| + b_3\|\Psi\|)\|\Phi\|.
\]

and hence we obtain that

\[
|\Phi, U(s)^{-1}[T, H^j]\|U(s)\Psi| \leq a_3\|I \otimes H_{rad}\| + b_3\|\Psi\|.
\]

Thus the proof is completed. \( \blacksquare \)

**Lemma 3.7** Assume (A.1)-(A.3). Then

\[
s - \lim_{\Lambda \to \infty} K(\Lambda)\Psi = K\Psi, \quad \Psi \in \mathcal{D}_0.
\]  

(83)

where \( K \) is an operator defined in (57).

**(Proof)**

By Lemma 3.5 it is sufficient to prove that

\[
\lim_{t \to 0} U(t)^{-1}[T, H^j]\|U(t)\Psi = [T, H^j]\|\Psi,
\]  

(84)

for \( \Psi \in \mathcal{D}_0 \). In a similar way to Lemma 3.1 there exists \( \theta(t) \in [-t, t] \) such that

\[
U(t)^{-1}[T, H^j]\|U(t)\Psi = [T, H^j]\|\Psi - itU(\theta(t))^{-1}[T, H^j]U(\theta(t))\Psi.
\]

Then by Lemma 3.6 we obtain that

\[
\|U(t)^{-1}[T, H^j]\|U(t)\Psi - [T, H^j]\|\Psi| \leq t \left(a_3\|I \otimes H_{rad}\| + b_3\|\Psi\|\right),
\]
and hence (84) follows. ■

(Proof of Proposition 3.3)

(1) By Lemma 3.5 and Lemma 3.6, (55) follows.

(2) It is seen that

\[ K(\Lambda) \left( \tilde{H}_0(\Lambda) - z \right)^{-1} = K(\Lambda)(H_{\text{Dirac}} + \frac{e^2}{8\pi} H_{\text{long}} - z)^{-1} \otimes P_{\Omega_{\text{rad}}} + K(\Lambda) \left( \tilde{H}_0(\Lambda) - z \right)^{-1} \left( I \otimes P_{\Omega_{\text{rad}}}^{\perp} \right), \]

where \( P_{\Omega_{\text{rad}}}^{\perp} = 1 - P_{\Omega_{\text{rad}}} \). By Lemma 3.7, we have

\[ s - \lim_{\Lambda \to \infty} K(\Lambda) \left( (H_{\text{Dirac}} + \frac{e^2}{8\pi} H_{\text{long}} - z)^{-1} \otimes P_{\Omega_{\text{rad}}} \right) \Psi = K \left( (H_{\text{Dirac}} + \frac{e^2}{8\pi} H_{\text{long}} - z)^{-1} \otimes P_{\Omega_{\text{rad}}} \right) \Psi. \]

By (55), we see that for \( \varepsilon > 0 \) there exists \( \Lambda(\varepsilon) \geq 0 \) such that for all \( \Lambda > \Lambda(\varepsilon) \)

\[ \|K(\Lambda) \left( \tilde{H}_0(\Lambda) - z \right)^{-1} \Xi\| \leq \varepsilon \Xi \| + (\varepsilon |z| + v(\varepsilon)) \left( \tilde{H}_0(\Lambda) - z \right)^{-1} \Xi \|,
\]

follows for \( \Xi \in \mathcal{F}_{\text{QED}} \). Furthermore we see that \( \lim_{\Lambda \to \infty} \| (\tilde{H}_0(\Lambda) - z)^{-1} (I \otimes P_{\Omega_{\text{rad}}}^{\perp}) \Psi \| = 0 \), and hence we obtain that

\[ \lim_{\Lambda \to \infty} \left\| K(\Lambda) \left( \tilde{H}_0(\Lambda) - z \right)^{-1} \left( I \otimes P_{\Omega_{\text{rad}}}^{\perp} \right) \Psi \right\| = 0. \]

By (86) and (87), we obtain (56). ■

(Proof of Theorem 2.1)

We see that for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[ \left( H(\Lambda) - z \right)^{-1} = U \left( \frac{e}{\Lambda} \right) \left( \tilde{H}_0(\Lambda) + K(\Lambda) - z \right)^{-1} U \left( \frac{e}{\Lambda} \right)^{-1}. \]

It is seen from (55) and (56) that \( \tilde{H}_0(\Lambda) \) and \( K(\Lambda) \) satisfy (C.1) and (C.2) with applying \( \tilde{H}_0(\Lambda) \) to \( X_0(\Lambda) \) and \( K(\Lambda) \) to \( C(\Lambda) \). Then by Proposition 3.3 we obtain that

\[ s - \lim_{\Lambda \to \infty} \left( H(\Lambda) - z \right)^{-1} = \left( (H_{\text{Dirac}} + \frac{e^2}{8\pi} H'_\text{II}) \otimes I + K_{\text{rad}} - z \right)^{-1} \left( I \otimes P_{\Omega_{\text{rad}}} \right), \]

where

\[ K_{\text{rad}} = -\frac{ie^2}{2}(I \otimes P_{\Omega_{\text{rad}}})[T,H'_\text{II}](I \otimes P_{\Omega_{\text{rad}}}). \]

Let us compute \( K_{\text{rad}} \). It is seen that

\[ (\Omega_{\text{rad}}, \Pi^I(x)A^I(y)\Omega_{\text{rad}}) = \frac{-i}{2} \lambda^{I,I}(x-y). \]
By (41), we see that for \( \Psi = \Psi_{\text{Dir}} \otimes \Omega_{\text{rad}} \) and \( \Psi_{\text{Dir}} \in \mathcal{F}_{\text{Dirac}} \),

\[
(\Psi, [T, H'_I]\Psi)
= \sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi(x) \chi(y) \left( \Psi, [J^j(x), J^l(y)] \otimes \Pi^j(x) A^l(y) \Psi \right) dxdy
+ \sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi(x) \chi(y) \left( \Psi, J^l(y) J^j(x) \otimes [\Pi^j(x), A^l(y)] \Psi \right) dxdy
= -\frac{i}{2} \sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi(x) \chi(y) \lambda^{j,l}(x-y) \left( \Psi_{\text{Dir}}, [J^j(x), J^l(y)] \Psi_{\text{Dir}} \right) dxdy
- \frac{i}{2} \sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi(x) \chi(y) \lambda^{j,l}(x-y) \left( \Psi_{\text{Dir}}, J^l(y) J^j(x) \Psi_{\text{Dir}} \right) dxdy. \tag{89}
\]

Thus we have

\[
(I \otimes P_{\Omega_{\text{rad}}}) [T, H'_I] (I \otimes P_{\Omega_{\text{rad}}}) = -\frac{i}{2} \sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi(x) \chi(y) \lambda^{j,l}(x-y) \left( J^j(x) J^l(y) + J^l(y) J^j(x) \right) dxdy.
\]

Then the theorem follows. \( \blacksquare \)

Acknowledgments
It is pleasure to thank Professor F. Hiroshima for his advice and comments.

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