Quantization of Space-like States in Lorentz-Violating Theories

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Abstract. Lorentz violation frequently induces modified dispersion relations that can yield space-like states that impede the standard quantization procedures. In certain cases, an extended Hamiltonian formalism can be used to define observer-covariant normalization factors for field expansions and phase space integrals. These factors extend the theory to include non-concordant frames in which there are negative-energy states. This formalism provides a rigorous way to quantize certain theories containing space-like states and allows for the consistent computation of Cherenkov radiation rates in arbitrary frames and avoids singular expressions.

1. Introduction
Theories involving space-like solutions to the dispersion relation can yield negative-energy states in certain reference frames. The quantization of such a theory in these frames is known to be problematic. For example, the simple tachyonic neutrino model [1], proposed to accommodate anomalous neutrino mass measurement results, defined by the Lagrangian

\[ L = i\bar{\psi}\gamma_5\partial_5\psi - m_\nu\bar{\psi}\psi, \]  

leads to the dispersion relation \( p^2 = -m_\nu^2 \). Conventional attempts at quantization typically fail due to the impossibility of finding a reference frame-independent separation of particle and anti-particle states. This leads to inconsistencies in the usual re-interpretation of the negative-energy states in terms of anti-particles. The situation is demonstrated in figures 1 and 2 which shows the dispersion relation plotted in two different reference frames. Note that some of the anti-particle states in one frame are transformed into particle states in the other frame.

Similar issues can occur in Lorentz-violating theories, such as in the CPT-violating, massive photon model (with \( k_\mu^A \) a constant background vector field)

\[ \mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}k_\mu^A \epsilon_{\kappa\lambda\mu\nu}A^\lambda F^{\mu\nu} + \frac{1}{2}m_\gamma^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \]  

with momentum-space solution containing the perturbed dispersion relation

\[ R_T(p) = \frac{1}{4}(p^2 - m_\gamma^2)^2 - (p \cdot k_\gamma)^2 + p^2 k_\gamma^2 = 0, \]  

which has the observer-covariant factorization \( R_T(p) = R_+(p)R_-(p) \) with

\[ R_\pm(p) = \frac{1}{2}(p^2 - m_\gamma^2) \pm \sqrt{(p \cdot k_\gamma)^2 - p^2 k_\gamma^2}. \]
This factorization can be directly related to solutions to the modified Dirac equation “off-shell” spinor solutions

$$\frac{1}{2} (\not{p} - m - b_\mu \gamma_\mu) u_\pm(p) = R_\pm(p) u_\pm(p),$$

with $R_\pm(p) = \frac{1}{2} (p^2 - m^2 - b^2) \pm \sqrt{(b \cdot p)^2 - b^2 p^2}$ as above acting on spinors $u_\pm = (\not{p} + m - \gamma_5 b) w_\pm$
giving the condition

$$\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \sigma^{\mu\nu} p^\alpha b^\beta w_\pm = \pm \sqrt{(b \cdot p)^2 - b^2 p^2} w_\pm,$$

which demonstrates that they are eigenstates of the Pauli-Lubanski vector.

The factor $R_+(p) = 0$ has space-like solutions with $p^2 < 0$ at high-momenta. Figure 3 shows

$$\frac{1}{2} (\not{p} - m - b_\mu \gamma_\mu) u_\pm(p) = R_\pm(p) u_\pm(p),$$

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The factor $R_+(p) = 0$ has space-like solutions with $p^2 < 0$ at high-momenta. Figure 3 shows
the positive-energy particle states for the case of a pure timelike $b^\mu$ coefficient. Space-like states exist in the region where the dispersion relation plot dips below the light-cone. Figure 4 displays the energy and momentum in a highly boosted frame where the energy dips below zero in some region. A first attempt may be to re-interpret these negative-energy solutions as anti-particles and use conventional field normalization factors, but it turns out that this procedure leads to singular factors in the commutation relations that makes the quantization procedure suspect. Instead, an alternative procedure which makes use of an extended Hamiltonian formalism will be used that makes all of the states into particle states with an observer covariant interpretation of particle states that is valid in all frames. To identify the appropriate normalization factors, use is made of the classical mechanical limit of the theory.

2. Classical Mechanics Lagrangians in the SME

The traditional method of computing Lagrangians in the SME [2] uses a Legendre transformation in conjunction with the group velocity to obtain the appropriate Lagrange functions from the dispersion relation. The process can be made to appear covariant by introducing an arbitrary path parametrization and a four-velocity $u^\mu = dx^\mu/d\lambda$. For example, the fermion dispersion relation with an external $b^\mu$-parameter considered in Ref. [2] leads to the dispersion relation of the CPT-violating massive photon model (see Eqs. (2) and (3)) by making the replacements $b^\mu \rightarrow k_A^\mu$, and $m^2 \rightarrow m_\gamma^2 - k_A^2$. The relevant dispersion relation is

$$ \mathcal{R}(p) = \frac{1}{4}(p^2 - m^2 + b^2)^2 - (b \cdot p)^2 + m^2b^2 = 0. \tag{7} $$

The group velocity $\vec{v} = dx/dt$ is computed using implicit differentiation as

$$ v_i = \frac{\partial p_0}{\partial p^i}, \tag{8} $$

which is well-defined away from any singular points where the energy surfaces become degenerate. The relevant Lagrangians are computed using the Legendre transformation $L = \vec{u} \cdot \vec{p} - p^0(\vec{p})$. The expression for $\vec{v}(\vec{p})$ defined in Eq. (8) is inverted for $\vec{p}(\vec{v})$ to give

$$ L_{\pm} = -m\sqrt{1 - \vec{v}^2} \mp \sqrt{(b \cdot \vec{v})^2 - b^2(1 - \vec{v}^2)}, \tag{9} $$

where the signs represent two valid solutions that reduce to the standard case when $b^\mu \rightarrow 0$.

The above procedure can be made explicitly covariant by introducing a zero-component of the four-velocity, $u^0(\lambda)$, and an arbitrary parametrization $\lambda(t)$ so that the action appears covariant in terms of the four-velocity $u^\mu = dx^\mu/d\lambda$, as

$$ L_{\pm}[u^\mu, x] = -m\sqrt{u^2} \mp \sqrt{(b \cdot u)^2 - b^2u^2}. \tag{10} $$

The above approach has some issues which make a relativistic Hamiltonian description difficult. For one, the function $u^\mu$ has a “gauge” degree of freedom due to the re-parametrization invariance and therefore it is not fully determined by the equation of motion. Computing the momentum

$$ p^\mu = -\frac{\partial L}{\partial \dot{u}^\mu} \tag{11} $$

which is insensitive to a scaling of the four-velocity. This means that Eq. (11) is not invertible to find $u(p)$ unless some condition on $u^\mu$ is enforced (such as $u^2 = 1$ when $\lambda = \tau$ is taken as the proper time). Calculation of the relativistic Hamiltonian gives

$$ \mathcal{H} = p^0u^0 - \vec{p} \cdot \vec{u} - L = 0, \tag{12} $$
since \( L = -u_\mu p^\mu \), yielding no useful Hamiltonian formalism. It is also unclear how the Legendre transformation specifically maps the Lagrange functions \( L_+ \) and \( L_- \) to the hypersurfaces defined by \( R_+(p) = 0 \) and \( R_-(p) = 0 \).

### 3. Extended Hamiltonian Formalism

The relativistic Hamiltonian can be extended to “off-shell” values using an extended Hamiltonian formalism originally due to Dirac. The procedure is to introduce a new variable \( e(\lambda) \) as a Lagrange multiplier [3]

\[
S^*_\pm = -\int \left[ me^{-1}u^2 \pm \sqrt{(b \cdot u)^2 - b^2 u^2} - \frac{e}{m} R_\pm(p, x) \right] d\lambda,
\]

with the observer covariant factorization

\[
R(p) = R_+(p)R_-(p),
\]

and

\[
R_\pm = \frac{1}{2} \left( p^2 - m^2 - b^2 \right) \pm \sqrt{(b \cdot p)^2 - b^2 p^2},
\]

which agrees with the previous action “on-shell” where \( R_\pm = 0 \), but modified action applies for unconstrained variations of \( u_\mu \) and \( p_\mu \) yielding an extended relativistic Hamiltonian

\[
H^*_{\pm} = -\frac{e}{m} R_\pm(p, x) = -\frac{e}{2m} \left( p^2 - m^2 - b^2 \pm 2\sqrt{(b \cdot p)^2 - b^2 p^2} \right),
\]

and one of Hamilton’s equations gives the velocity as

\[
u^\mu = -\frac{\partial H^*_\pm}{\partial p_\mu} = \frac{e}{m} \left( p^\mu \mp \frac{(b \cdot p)b^\mu - b^2 p^\mu}{\sqrt{(b \cdot p)^2 - b^2 p^2}} \right),
\]

which can be inverted to give

\[
p^\mu = \frac{m u^\mu}{e} \pm \frac{(u \cdot b)b^\mu - b^2 u^\mu}{\sqrt{(b \cdot u)^2 - b^2 u^2}},
\]

providing a well-defined Legendre transformation between the Lagrangian and extended Hamiltonian functions, at least away from singular points. Behavior near singular points is discussed further in Refs. [3, 4].

### 4. Quantization

A typical field expansion contains the observer-covariant phase space factor

\[
\int \frac{d^3 \vec{p}}{2 p^0(p)} = \int d^4 p \delta(p^2 - m^2)\Theta(p^0),
\]

which is problematic in frames where space-like states satisfy \( p^0 \leq 0 \) and the naive commutators

\[
[a(p), a^\dagger(p)] = (2\pi)^3 2p^0(p)\delta(\vec{p} - \vec{p}'),
\]

vanish or go negative, leading to interpretational difficulties. To circumvent this problem, the correct relativistic Hamiltonian for the theory in Eq. (16) can be used in the delta function to
put the particles properly on-shell according to their classical mechanical limit. In this way the phase-space integral becomes

\[ \int d^4p \left( \frac{-\epsilon}{2m} \right) \delta(\mathcal{H}^\pm_+(p)) = \int \frac{d^3\vec{p}}{\Lambda^\mu_\pm(p)}, \]  

(21)

with

\[ \Lambda^\mu_\pm(p) = -\left( \frac{2m}{\epsilon} \right) \left( \frac{\partial \mathcal{H}^\pm_+}{\partial \rho^0} \right) = 2 \left( p^0 \pm \frac{(b \cdot p)b^0 - b^2p^0}{\sqrt{(b \cdot p)^2 - b^2p^2}} \right), \]  

(22)

which enforces the correct dynamics on surface \( \mathcal{H}^\pm_+ = 0 \). Since \( u^2 = u_0^2 - \vec{u}^2 = 1 \) (in proper time parameterization), the factor \( u^0 = -\partial \mathcal{H}^\pm_+ / \partial \rho^0 > 0 \) in all frames, so that the denominator in the phase-space factor in Eq. (21) is positive definite. Using this factor in the field expansions (with the appropriate photon extended Hamiltonian [5]) yields

\[ A^\mu(x) = \int \sum_{\pm} \frac{d^3\vec{p}}{\Lambda^\mu_\pm(p)} \epsilon^\mu a^\dagger(p) e^{-ip \cdot x} + c.c., \]  

(23)

and the corresponding phase space factors are well-defined in all frames, not just concordant [6] ones in which the Lorentz-violating parameters are small. Defining the canonical momenta \( \pi^\mu = \partial \mathcal{L} / \partial \dot{A}_\mu \) gives

\[ \pi^\mu(x) = F^{\mu 0}(x) + \epsilon^{\mu \alpha \beta}(k_{AF})_\alpha A_\beta(x) - \eta^{\mu 0} \frac{1}{2} \partial_\nu A^\nu(x). \]  

(24)

Imposing the canonical quantization rules

\[ [A_\mu(t, \vec{x}), \pi^\nu(t, \vec{y})] = i \delta^\nu_\mu \delta^3(\vec{x} - \vec{y}), \quad [A_\mu(t, \vec{x}), A_\nu(t, \vec{y})] = 0, \]  

(25)

yields the momentum-space state algebra

\[ [a(p), a^\dagger(\vec{p}')] = (2\pi)^3 \Lambda^\mu_\pm(p) \delta(p - \vec{p}'), \]  

(26)

which has a positive-definite \( \Lambda^\mu_\pm(p) > 0 \) in all observer frames.

5. Summary

Lorentz-violating field theories that lead to space-like states are not automatically ruled out at the quantum level, even though there may exist frames in which the energy runs negative. By using an appropriate modified Hamiltonian formalism, it is possible (at least in certain specific cases) to define positive-definite phase space factors that can be used to expand the fields and impose a consistent separation between particle and anti-particle states. The corresponding phase-space factors are important in consistent calculations like Cherenkov processes [7, 8].

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