SHARP INEQUALITIES
FOR POLYGAMMA FUNCTIONS

Bai-Ni Guo* — Feng Qi** — Jiao-Lian Zhao***
— Qiu-Ming Luo****

(Communicated by Stanisław Kanas)

ABSTRACT. In the paper, the authors review some inequalities and the (logarithmically) complete monotonicity concerning the gamma and polygamma functions and, more importantly, present a sharp double inequality for bounding the polygamma function by rational functions.

1. Introduction

We now prepare some notations, review some inequalities and the (logarithmically) complete monotonicity concerning the gamma and polygamma functions, and then state our main results.

1.1. Completely monotonic functions

Recall ([41: Chapter XIII] and [73: Chapter IV]) that a function $f(x)$ is said to be completely monotonic on an interval $I \subseteq \mathbb{R}$ if $f(x)$ has derivatives of all orders on $I$ and

$$0 \leq (-1)^k f^{(k)}(x) < \infty$$

(1.1)

holds for all $k \geq 0$ on $I$.

2010 Mathematics Subject Classification: Primary 33B15; Secondary 26A48, 26D07, 26D15.

Keywords: inequality, polygamma function, psi function, completely monotonic function, logarithmically completely monotonic function.

The second author was partially supported by the China Scholarship Council and the NNSF of China under Grant No. 11361038.
The third author was partially supported by the Natural Science Foundation of Shaanxi Province of China under Grant No. 2014JQ1006.
The fourth author was supported in part by the Natural Science Foundation Project of Chongqing in China under Grant No. CSTC2011JJA00024, the Research Project of Science and Technology of Chongqing Education Commission in China under Grant No. KJ120625, and the Fund of Chongqing Normal University in China under Grant Nos. 10XLR017 and 2011XLZ07.
The celebrated Bernstein-Widder Theorem [73, p. 161] states that a function \( f(x) \) is completely monotonic on \((0, \infty)\) if and only if
\[
f(x) = \int_{0}^{\infty} e^{-xs} d\mu(s),
\tag{1.2}
\]
where \( \mu \) is a nonnegative measure on \([0, \infty)\) such that the integral (1.2) converges for all \( x > 0 \). This means that a function \( f(x) \) is completely monotonic on \((0, \infty)\) if and only if it is a Laplace transform of the measure \( \mu \).

The completely monotonic functions have applications in different branches of mathematical sciences. For example, they play some role in combinatorics, numerical and asymptotic analysis, physics, potential theory, and probability theory.

The most important properties of completely monotonic functions can be found in [41, Chapter XIII], [73, Chapter IV] and closely related references therein.

1.2. Logarithmically completely monotonic functions

Recall also ([7,52]) that a function \( f \) is said to be logarithmically completely monotonic on an interval \( I \subseteq \mathbb{R} \) if it has derivatives of all orders on \( I \) and its logarithm \( \ln f \) satisfies
\[
0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty
\tag{1.3}
\]
for \( k \in \mathbb{N} \) on \( I \), where \( \mathbb{N} \) denotes the set of positive integers.

By looking through “logarithmically completely monotonic function” in the database MathSciNet, it is found that this phrase was first used in [7], but with no a word to explicitly define it. Thereafter, it seems to have been ignored by the mathematical community. In early 2004, this terminology was recovered in [52] and it was immediately referenced in [17,59]. A natural question that one may ask is: Whether is this notion trivial or not? In [52, Theorem 4], it was proved that all logarithmically completely monotonic functions are also completely monotonic, but not conversely. This result was formally published when revising [48]. Hereafter, this conclusion and its proofs were dug in [12,21,48] once and again. Furthermore, in the paper [12], the logarithmically completely monotonic functions on \((0, \infty)\) were characterized as the infinitely divisible completely monotonic functions studied in [34] and all Stieltjes transforms were proved to be logarithmically completely monotonic on \((0, \infty)\). For more information, please refer to [12].
1.3. The gamma and polygamma functions

It is well-known that the classical Euler gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt. \quad (1.4)$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and the $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. It is common knowledge that these functions are fundamental and important and that they have much extensive applications in mathematical sciences.

1.4. The first kind of inequalities for the psi and polygamma functions

In [35: Theorem 2.1] and [47: Theorem 1.3], the function $\psi(x) - \ln x + \frac{\alpha}{x}$ was proved to be completely monotonic on $(0, \infty)$ if and only if $\alpha \geq 1$, so is its negative if and only if $\alpha \leq \frac{1}{2}$. In [13: Theorem 2] and [43: Theorem 2.1], the function $e^x \Gamma(x)$ was proved to be logarithmically completely monotonic on $(0, \infty)$ if and only if $\alpha \geq 1$, so is its reciprocal if and only if $\alpha \leq \frac{1}{2}$. From these, the following double inequalities were derived and employed in [19: p. 131], [22: Lemma 2.2], [24: Lemma 1], [25: p. 223, Lemma 2.3], [26: p. 107, Lemma 3], [27: p. 853], [45: p. 55, Theorem 5.11], [50: p. 1625], [54: Lemma 1], [55: p. 79], and [60: p. 2155, Lemma 3]: for $x \in (0, \infty)$ and $k \in \mathbb{N}$, we have

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \quad (1.5)$$

and

$$\frac{(k - 1)!}{x^k} + \frac{k!}{2x^{k+1}} < \left| \psi^{(k)}(x) \right| < \frac{(k - 1)!}{x^k} + \frac{k!}{x^{k+1}}. \quad (1.6)$$

In [3: Theorem 9], it was proved that if $k \geq 1$ and $n \geq 0$ are integers, then

$$S_k(2n; x) < \left| \psi^{(k)}(x) \right| < S_k(2n + 1; x) \quad (1.7)$$

holds for $x > 0$, where

$$S_k(p; x) = \frac{(k - 1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p} B_{2i} \left[ \prod_{j=1}^{k-1} (2i + j) \right] \frac{1}{x^{2i+k}} \quad (1.8)$$

with the usual convention that an empty sum is nil and $B_i$ for $i \geq 0$ are Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!} = 1 - \frac{x}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{x^{2j}}{(2j)!}, \quad |x| < 2\pi. \quad (1.9)$$
In [2], among other things, the following double inequalities were procured: for \( x > \frac{1}{2} \), we have
\[
\sum_{k=1}^{2N+1} \frac{B_{2k} \left( \frac{1}{2} \right)^k}{2k(x - \frac{1}{2})^{2k}} < \ln \left( x - \frac{1}{2} \right) - \psi(x) < \sum_{k=1}^{2N} \frac{B_{2k} \left( \frac{1}{2} \right)^k}{2k(x - \frac{1}{2})^{2k}} \tag{1.10}
\]
and
\[
\frac{(n - 1)!}{(x - \frac{1}{2})^n} + \sum_{k=1}^{2N+1} \frac{(n + 2k - 1)!B_{2k} \left( \frac{1}{2} \right)^k}{(2k)! (x - \frac{1}{2})^{n+2k}} < |\psi^{(n)}(x)| < \frac{(n - 1)!}{(x - \frac{1}{2})^n} + \sum_{k=1}^{2N} \frac{(n + 2k - 1)!B_{2k} \left( \frac{1}{2} \right)^k}{(2k)! (x - \frac{1}{2})^{n+2k}} \tag{1.11}
\]
where \( n \geq 1, N \geq 0 \), an empty sum is understood to be nil, and
\[
B_k \left( \frac{1}{2} \right) = \left( \frac{1}{2^{k-1}} - 1 \right) B_k, \quad k \geq 0. \tag{1.12}
\]
When replacing \( 2N \) by \( 2N - 1 \), inequalities (1.10) and (1.11) are reversed. In particular, for \( n = 1 \) and \( N = 0 \),
\[
\frac{1}{x - \frac{1}{2}} - \frac{1}{12(x - \frac{1}{2})^2} < \psi'(x) < \frac{1}{x - \frac{1}{2}}, \quad x > \frac{1}{2}. \tag{1.13}
\]

It is obvious that if taking \( x \to \left( \frac{1}{2} \right)^+ \) the lower and upper bounds in (1.11) tend to \(-\infty\) and \( \infty \) respectively, but the middle term tends to a limited constant. This implies that inequalities in (1.10) and (1.11), including (1.13), may be not ideal.

It is noted that the inequality (1.7) was deduced from [3] Theorem 8] which states that the functions
\[
F_n(x) = \ln \Gamma(x) - \left( x - \frac{1}{2} \right) \ln x + x - \frac{1}{2} \ln (2\pi) - \sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j - 1)x^{2j-1}} \tag{1.14}
\]
and
\[
G_n(x) = -\ln \Gamma(x) + \left( x - \frac{1}{2} \right) \ln x - x + \frac{1}{2} \ln (2\pi) + \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j - 1)x^{2j-1}} \tag{1.15}
\]
for \( n \geq 0 \) are completely monotonic on \((0, \infty)\).

In [40] Theorem 1], the convexity of the functions \( F_n(x) \) and \( G_n(x) \) were presented alternatively.

Stimulated by [49], the complete monotonicity of \( F_n(x) \) and \( G_n(x) \) were simply verified in [37] Theorem 2] again.

In [38], the complete monotonicity of the functions \( F_n(x) \) and \( G_n(x) \) was strengthened and generalized.

106
Stimulated by the double inequality (1.6) appeared in [19, p. 131], the complete monotonicity of the functions $F^{(k+1)}(x)$ and $G^{(k+1)}(x)$ for $k \in \mathbb{N}$ were recovered in [42, Theorem 3.1] by using similar method in [38, pp. 35–37, Section 2].

1.5. The second kind of inequalities for the psi and polygamma functions

In [31, Theorem 1], the function

$$g_{\alpha, \beta}(x) = \left[ e^{\psi(x+1)} \right]^\alpha \frac{(x+\beta)^{\alpha}}{(x+\beta)^{\beta}}$$

(1.16)

for real numbers $\alpha \neq 0$ and $\beta$ was shown to be logarithmically completely monotonic with respect to $x \in (\max\{0, -\beta\}, \infty)$ if and only if either $\alpha > 0$ and $\beta \geq 1$ or $\alpha < 0$ and $\beta \leq \frac{1}{2}$. As a result, the following double inequalities (1.17) and (1.18) were deduced in [51]: for $x \in (0, \infty)$ and $k \in \mathbb{N}$, we have

$$\ln\left(x + \frac{1}{2}\right) - \frac{1}{x} < \psi(x) < \ln(x + 1) - \frac{1}{x}$$

(1.17)

and

$$\frac{(k-1)!}{(x+1)^k} + \frac{k!}{x^{k+1}} < \left| \psi^{(k)}(x) \right| < \frac{(k-1)!}{(x+\frac{1}{2})^k} + \frac{k!}{x^{k+1}}.$$  

(1.18)

It is clear that the left-hand side inequality in (1.17) and the right-hand side inequality in (1.18) are better than the left-hand side inequality in (1.5) and the right-hand side inequality in (1.6). It is also easy to see that the right-hand side inequality in (1.17) and the left-hand side inequality in (1.18) are more exact than the right-hand side inequality in (1.5) and the left-hand side inequality in (1.6) when $x > 0$ is close enough to 0, but not when $x > 0$ is large enough.

For more information on further investigation of functions similar to (1.16), please refer to the research papers [30, 32, 33, 68], the expository article [45] and related references therein.

1.6. A sharp inequality for the psi function and related results

In [9, Lemma 1.7] and [57, Theorem 1], it was proved that the double inequality

$$\ln\left(x + \frac{1}{2}\right) - \frac{1}{x} < \psi(x) < \ln(x + e^{-\gamma}) - \frac{1}{x}$$

(1.19)

holds on $(0, \infty)$ and the scalars $\frac{1}{2}$ and $e^{-\gamma} = 0.56 \cdots$ in (1.19) are the best possible.

It is clear that the inequality (1.19) refines and sharpens (1.17). The inequality (1.19) has relations with (1.5) as (1.17) does.

More strongly, the function

$$Q(x) = e^{\psi(x+1)} - x$$

(1.20)
was proved in [57: Theorem 2] to be strictly decreasing and convex on \((-1, \infty)\) with \(\lim_{x \to \infty} Q(x) = \frac{1}{2}\). The basic tools of the proofs in [57] include
\[
\psi'(x)e^{\psi(x)} < 1, \quad x > 0
\] (1.21)
and
\[
[\psi'(x)]^2 + \psi''(x) > 0, \quad x > 0.
\] (1.22)

Among other things, the monotonicity and convexity of the function (1.20) were also derived in [16: Corollary 2, Corollary 3]: for all \(t > 0\), the function \(\exp\{\psi(x+t)\} - x\) is decreasing with respect to \(x \in [0, \infty)\); for all \(t > 0\), the digamma function can be written in a way:
\[
\psi(x+t) = \ln(x + \delta(x)), \quad x > 0,
\]
where \(\delta\) is decreasing convex function which maps \([0, \infty)\) onto \([e^{\psi(t)}, t - \frac{1}{2})\).

The one-side inequality (1.21) was deduced in [16: Corollary 2] and recovered in [11: Lemma 1.1].

The sharp double inequality (1.19) is the special case \(t = 1\) of the following double inequality obtained in [16: Corollary 3]: for all \(x > 0\) and \(t > 0\), it holds that
\[
\ln\left(x + \frac{2t - 1}{2}\right) < \psi(x + t) < \ln(x + \exp(\psi(t))).
\] (1.23)

It is worthwhile to remark that the left-hand side inequality in (1.23) for \(x + t \leq \frac{1}{2}\) is meaningless.

Replacing \(x\) by \(x + t\) in (1.19) yields
\[
\ln\left(x + t + \frac{1}{2}\right) - \frac{1}{x + t} < \psi(x + t) < \ln(x + t + e^{-\gamma}) - \frac{1}{x + t}
\] (1.24)
for all \(x > 0\) and \(t > 0\). The left-hand side inequality in (1.21) extends and refines the corresponding one in (1.23) and their right-hand side inequalities do not contain each other.

For information about the history and background of the function (1.20) and inequalities (1.21) and (1.23), please refer to the expository papers [45,46] and lots of references therein.

The inequality (1.22) was first obtained in the proof of [4] p. 208, Theorem 4.8] and recovered in [8: Theorem 2.1] and [11: Lemma 1.1].

In [10: Remark 1.3], it was pointed out that the inequality (1.22) is the special case \(n = 1\) of [10: Lemma 1.2] which reads
\[
(-1)^n \psi^{(n+1)}(x) < \frac{n}{\sqrt{(n-1)!}} \left[(-1)^{n-1} \psi^{(n)}(x) \right]^{1+1/n}
\] (1.25)
for \(x > 0\) and \(n \in \mathbb{N}\). This inequality can be restated more meaningfully as
\[
\sqrt[n+1]{\frac{\psi^{(n+1)}(x)}{n!}} < \sqrt[n]{\frac{\psi^{(n)}(x)}{(n-1)!}}.
\] (1.26)
SHARP INEQUALITIES FOR POLYGAMMA FUNCTIONS

In [6], Lemma 4.6, the inequality (1.22) was generalized to the $q$-analogue. In [18,53], the divided difference

$$
\Delta_{s,t}(x) = \begin{cases} 
\left[ \frac{\psi(x + t) - \psi(x + s)}{t - s} \right]^2 + \frac{\psi''(x + t) - \psi''(x + s)}{t - s}, & s \neq t \\
\left[ \psi'(x + s) \right]^2 + \psi'''(x + s), & s = t
\end{cases}
$$

for $|t - s| < 1$ and $-\Delta_{s,t}(x)$ for $|t - s| > 1$ were proved to be completely monotonic with respect to $x \in (-\min\{s,t\},\infty)$. Consequently, the function $[\psi'(x)]^2 + \psi''(x)$ appearing in (1.22) is completely monotonic on $(0,\infty)$.

For $m, n \in \mathbb{N}$, let

$$
f_{m,n}(x) = \psi^{(n)}(x) + [\psi^{(m)}(x)]^2, \quad x > 0.
$$

In [27], it was revealed that the functions $f_{1,2}(x)$ and $f_{m,2n-1}(x)$ are completely monotonic on $(0,\infty)$, but the functions $f_{m,2n}(x)$ for $(m, n) \neq (1, 1)$ are not monotonic and does not keep the same sign on $(0,\infty)$. This means that $f_{1,2}(x)$ is the only nontrivial completely monotonic function on $(0,\infty)$ among all functions $f_{m,n}(x)$ for $m, n \in \mathbb{N}$.

In [55, Theorem 3], the function

$$
\Delta_\lambda(x) = [\psi'(x)]^2 + \lambda \psi'''(x)
$$

was shown to be completely monotonic on $(0,\infty)$ if and only if $\lambda \leq 1$.

For real numbers $s, t, \alpha = \min\{s, t\}$ and $\lambda$, define

$$
\Delta_{s,t;\lambda}(x) = \begin{cases} 
\left[ \frac{\psi(x + t) - \psi(x + s)}{t - s} \right]^2 + \lambda \frac{\psi''(x + t) - \psi''(x + s)}{t - s}, & s \neq t \\
\left[ \psi'(x + s) \right]^2 + \lambda \psi'''(x + s), & s = t
\end{cases}
$$

with respect to $x \in (-\alpha, \infty)$. In [54], the following complete monotonicity were established:

1. For $0 < |t - s| < 1$,
   (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq 1$,
   (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda \geq \frac{1}{|t-s|}$;

2. For $|t - s| > 1$,
   (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq \frac{1}{|t-s|}$,
   (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda \geq 1$;

3. For $s = t$, the function $\Delta_{s,s;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq 1$;

4. For $|t - s| = 1$,
   (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda < 1$,
   (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda > 1$,
   (c) and $\Delta_{s,t;1}(x) \equiv 0$. 

109
These results generalize the claim in the proof of [36]. For detailed information, see related texts remarked in the expository article [61].

In [10: Remark 2.3], it was pointed out that the inequality

$$\psi''(x) + \left[ \psi'(x + \frac{1}{2}) \right]^2 < 0 \quad (1.31)$$

for $x > 0$ is a direct consequence of [10: Theorem 2.2]: for $x > 0$, $1 \leq k \leq n - 1$ and $n \in \mathbb{N}$, we have

$$(n - 1) \left[ \frac{\psi(k)(x + \frac{1}{2})}{(-1)^{k-1}(k-1)!} \right]^{n/k} < (-1)^{n+1} \psi^{(n)}(x) < (n - 1) \left[ \frac{\psi(k)(x)}{(-1)^{k-1}(k-1)!} \right]^{n/k} \quad (1.32)$$

which can be rewritten as

$$\sqrt{k \left| \frac{\psi(k)(x + \frac{1}{2})}{(k-1)!} \right|} < \sqrt{n \left| \frac{\psi(n)(x)}{(n-1)!} \right|} < \sqrt{k \left| \frac{\psi(k)(x)}{(k-1)!} \right|}. \quad (1.33)$$

The lower bound in the inequality (1.33) was refined in [24].

In [55: Theorem 1], the inequality (1.31) was generalized to the complete monotonicity: for real number $\alpha \in \mathbb{R}$ and $x > -\min\{0, \alpha\}$,

1. the function $\psi''(x) + [\psi'(x + \alpha)]^2$ is completely monotonic if and only if $\alpha \leq 0$;
2. the function $-\{\psi''(x) + [\psi'(x + \alpha)]^2\}$ is completely monotonic if $\alpha \geq \sup_{x \in (0, \infty)} \frac{x}{\phi^{-1}(2(x + 1)^2 - 1)|e^{2x}|}$, \quad (1.34)

where $\phi^{-1}$ denotes the inverse function of $\phi(x) = x \coth x$ on $(0, \infty)$.

In passing, it is noted that the results demonstrated in [44] have very close relations with the above mentioned conclusions.

1.7. Some more results

In recent years, inequalities and (logarithmically) complete monotonicity relating to the polygamma functions have been investigated in other directions.

In [63][66], the function

$$\left[ \frac{\Gamma(b)}{\Gamma(a)} \right]^{1/(b-a)} \quad (1.35)$$

was bounded by means and the digamma function of means. In [64][67], the difference $\psi^{(n)}(b) - \psi^{(n)}(a)$ was bounded by a linear combination of $\psi^{(n+1)}(a)$ and $\psi^{(n+1)}(b)$, whose coefficients are means. In [14][65], several inequalities involving the digamma function $\psi(x)$, the trigamma function $\psi'(x)$, and means
SHARP INEQUALITIES FOR POLYGAMMA FUNCTIONS

were discovered. There are more details on the history and new results of this field in the expository articles [45,46,61].

In [69], the monotonicity and logarithmic convexity of the function
\[
\frac{\Gamma(x + y + 1)/\Gamma(y + 1)}{(x + y + 1)^\alpha}^{1/x}
\]
were discussed. In [15], the geometric convexity of the function
\[
[\Gamma(x)]^{1/(x-1)}
\]
on \((1, \infty)\) was established. From this some inequalities for
\[
\frac{\Gamma(x + 1)^{1/x}}{\Gamma(y + 1)^{1/y}}
\]
were derived. For more information on the origin and background of this topic, please see [22,50,62] and related references therein.

In [39,70], necessary and sufficient conditions on \(\alpha\) and \(\beta\) such that the function
\[
x^\alpha[\Gamma(x)]^\beta/\Gamma(\beta x)
\]
is logarithmically completely monotonic on \((0, \infty)\) were established.

In the newly published papers [20,23,29,58,71,72], there exist some new ideas and results related to complete monotonicity, special functions such as the gamma and polygamma functions, and the like.

1.8. Main results of this paper

The main aim of this paper is to sharpen the double inequality (1.18) and to generalize the sharp inequality (1.19) to the cases for polygamma functions.

The main result of this paper may be stated as the following theorem.

**Theorem 1.** For \(x > 0\) and \(k \in \mathbb{N}\), the double inequality
\[
\frac{(k-1)!}{x + \left[\frac{(k-1)!}{|\psi^{(1)}(x)|}\right]^{1/k} k} + \frac{k!}{x^{k+1}} < |\psi^{(k)}(x)| < \frac{(k-1)!}{(x + \frac{1}{2})^k} + \frac{k!}{x^{k+1}}
\]
holds and the constants \(\left[\frac{(k-1)!}{|\psi^{(1)}(x)|}\right]^{1/k}\) and \(\frac{1}{2}\) in (1.40) are the best possible.

As direct consequences of Theorem 1, the following corollaries may be derived.

**Corollary 1.** For \(x > 0\) and \(k \in \mathbb{N}\), the double inequality
\[
\frac{(k-1)!}{x + \left[\frac{(k-1)!}{|\psi^{(1)}(x)|}\right]^{1/k} k} < |\psi^{(k)}(x + 1)| < \frac{(k-1)!}{(x + \frac{1}{2})^k}
\]
is valid and the scalars \(\left[\frac{(k-1)!}{|\psi^{(1)}(x)|}\right]^{1/k}\) and \(\frac{1}{2}\) in (1.41) are the best possible.
Corollary 2. Under the usual convention that an empty sum is understood to be nil, the double inequalities

\[
\sum_{i=1}^{m} \frac{1}{(x+i-1)^{k+1}} + \frac{(k-1)!}{\left\{ x + m - 1 + \left[ \frac{(k-1)!}{\psi^{(k)}(1)} \right]^{1/k} \right\}^k} < \left| \psi^{(k)}(x) \right| < k! \sum_{i=1}^{m} \frac{1}{(x+i-1)^{k+1}} + \frac{(k-1)!}{(x+m-\frac{1}{2})^k}
\]  

(1.42)

and

\[
\sum_{i=1}^{m-1} \frac{1}{(x+i)^{k+1}} \left[ \frac{(k-1)!}{\psi^{(k)}(1)} \right]^{1/k} - k! \sum_{i=1}^{m-1} \frac{1}{(x+i)^{k+1}} < \left| \psi^{(k)}(x+m) \right| < \frac{(k-1)!}{(x+\frac{1}{2})^k} - k! \sum_{i=1}^{m-1} \frac{1}{(x+i)^{k+1}}
\]  

(1.43)

hold for \( x > 0 \) and \( k, m \in \mathbb{N} \). Meanwhile, the quantities \( \left[ \frac{(k-1)!}{\psi^{(k)}(1)} \right]^{1/k} \) and \( \frac{1}{2} \) in inequalities (1.42) and (1.43) are the best possible.

Remark 1. When approximating the psi function \( \psi(x) \) and polygamma functions \( \psi^{(k)}(x) \) for \( k \in \mathbb{N} \), the double inequalities (1.19) and (1.40) are more accurate than (1.5) and (1.6) as long as \( x \) is enough close to 0 from the righthand side. For example, the right-hand side inequality in (1.18) and (1.40) has been applied in the proof of [51: Theorem 1] to prove that the inequality

\[
\frac{1 + 2t}{2t^2} \left[ \ln \Gamma \left( \frac{t}{1 + 2t} \right) - \ln \Gamma(t) \right] < 1 - \psi(t)
\]  

(1.44)

is valid for \( t > 0 \).

Remark 2. Integrating on both sides of (1.19) arrives at

\[
\frac{1}{2} \ln 2 - x - \ln x + \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) < \ln \Gamma(x)
\]  

\[
< \gamma e^{-\gamma} - x - \ln x + (x + e^{-\gamma}) \ln(x + e^{-\gamma}), \quad x > 0.
\]  

(1.45)

It may be verified that the functions

\[
\ln \Gamma(x) - x - \ln x + \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right)
\]  

(1.46)

and

\[
-x - \ln x + (x + e^{-\gamma}) \ln(x + e^{-\gamma}) - \ln \Gamma(x)
\]  

(1.47)

are strictly increasing on \((0, \infty)\). Consequently, we procure

\[
\ln \Gamma(x) < \frac{1}{2} \ln(2\pi) - 1 - x - \ln x + \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right), \quad x > 0.
\]  

(1.48)
The right-hand side inequality in (1.45) and the inequality (1.48) are not included each other.

Similar argument on (1.40) reveals that the positive functions
\[ \left( \frac{(k-1)!}{x^k} + \frac{(k-2)!}{(x + \frac{1}{2})^{k-1}} \right) \left| \psi^{(k-1)}(x) \right| \]
and
\[ \left| \psi^{(k-1)}(x) \right| - \frac{(k-1)!}{x^k} - \frac{(k-2)!}{\left\{ x + \left[ \frac{(k-1)!}{\psi^{(k)}(1)} \right]^{1/k} \right\}^{k-1}} \]
for \( k \in \mathbb{N} \) are strictly decreasing on \((0, \infty)\). Basing on this, we guess that these two functions are completely monotonic on \((0, \infty)\).

**Remark 3.** The double inequality (1.45) and the inequality (1.48) may be rewritten and combined as
\[ \frac{\sqrt{2}}{xe^x} \left( x + \frac{1}{2} \right)^{x+1/2} < \Gamma(x) < \min \left\{ \frac{e^{\gamma/e^\gamma}}{xe^x} (x + e^{-\gamma})e^{x-e^{-\gamma}}, \sqrt{\frac{2\pi}{e}} \cdot \frac{1}{xe^x} \left( x + \frac{1}{2} \right)^{x+1/2} \right\}, \quad x > 0. \] (1.51)

When \( x \) is smaller, this double inequality is better than
\[ \exp\left[ -x - \frac{1}{2} \psi\left( x + \frac{1}{3} \right) \right] < \frac{\Gamma(x)}{\sqrt{2\pi x^x}} < \exp\left[ -x - \frac{1}{2} \psi(x) \right], \quad x > 0, \] (1.52)
\[ \frac{x^{x[1-\ln x+\psi(x)]}}{e^x} < \Gamma(x) \leq \frac{x^{x[1-\ln x+\psi(x)]}}{e^{x-1}}, \quad x \in (0, 1] \] (1.53)
and
\[ \frac{\sqrt{2\pi} x^{x[1-\ln x+\psi(x)]}}{e^x} < \Gamma(x) \leq \frac{x^{x[1-\ln x+\psi(x)]}}{e^{x-1}}, \quad x \in [1, \infty), \] (1.54)
which was established in [5] and [28, Theorem 5] respectively.

For more information on bounding the gamma function \( \Gamma(x) \), please refer to the expository texts in [28] and closely related references therein.

## 2. Proofs of Theorem 1 and corollaries

Now we are in a position to prove Theorem 1 and the above corollaries.

**Proof of Theorem 1** For \( x > 0 \) and \( k \in \mathbb{N} \), let
\[ h_k(x) = \left[ \frac{(k-1)!}{\psi^{(k)}(x)} - \frac{k!}{x^{k+1}} \right]^{1/k} - x. \] (2.1)
Using the right-hand side inequality in (1.7) for $n \geq 0$ yields

$$h_k(x) > \left[ \frac{(k-1)!}{S_k(2n+1; x) - \frac{k!}{x^{k+1}}} \right]^{1/k} - x$$

$$= \left\{ \frac{(k-1)!}{x^k} - \frac{k!}{2x^{k+1}} + \sum_{i=1}^{2n+1} B_{2i} \left[ \prod_{j=1}^{k-1} (2i+j) \right] \frac{1}{x^{2i}} \right\}^{1/k} - x$$

$$= x \left\{ \left[ \frac{1}{1 - \frac{k}{2x} + \sum_{i=1}^{2n+1} \frac{B_{2i}}{(k-1)!} \left[ \prod_{j=1}^{k-1} (2i+j) \right] \frac{1}{x^{2i}}} \right]^{1/k} - 1 \right\}$$

$$= \frac{1}{u} \left\{ \left[ \frac{1}{1 - \frac{k}{2u} + \sum_{i=1}^{2n+1} \frac{B_{2i}}{(k-1)!} \left[ \prod_{j=1}^{k-1} (2i+j) \right] u^{2i}} \right]^{1/k} - 1 \right\},$$

$$\rightarrow \frac{1}{2}$$

as $u \rightarrow 0^+$, or say, $x \rightarrow \infty$. Similarly, making use of the left-hand side inequality in (1.7) for $n \geq 0$ results in

$$h_k(x) < \left[ \frac{(k-1)!}{S_k(2n; x) - \frac{k!}{x^{k+1}}} \right]^{1/k} - x$$

$$= \left\{ \frac{(k-1)!}{x^k} - \frac{k!}{2x^{k+1}} + \sum_{i=1}^{2n} B_{2i} \left[ \prod_{j=1}^{k-1} (2i+j) \right] \frac{1}{x^{2i}} \right\}^{1/k} - x$$

$$= x \left\{ \left[ \frac{1}{1 - \frac{k}{2x} + \sum_{i=1}^{2n} \frac{B_{2i}}{(k-1)!} \left[ \prod_{j=1}^{k-1} (2i+j) \right] \frac{1}{x^{2i}}} \right]^{1/k} - 1 \right\}$$

$$= \frac{1}{u} \left\{ \left[ \frac{1}{1 - \frac{k}{2u} + \sum_{i=1}^{2n} \frac{B_{2i}}{(k-1)!} \left[ \prod_{j=1}^{k-1} (2i+j) \right] u^{2i}} \right]^{1/k} - 1 \right\},$$

$$\rightarrow \frac{1}{2}$$

as $u \rightarrow 0^+$, or say, $x \rightarrow \infty$. In a word, it follows that

$$\lim_{x \to \infty} h_k(x) = \frac{1}{2}$$

(2.2)
SHARP INEQUALITIES FOR POLYGAMMA FUNCTIONS

By the well-known recurrence formula \[ p. 260, 6.4.6 \]
\[
\psi^{(n-1)}(x + 1) = \psi^{(n-1)}(x) + \frac{(-1)^{n-1}(n-1)!}{x^n}
\] (2.3)
and the integral representation \[ p. 260, 6.4.1 \]
\[
\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}e^{-xt}} dt
\] (2.4)
for \( x > 0 \) and \( n \in \mathbb{N} \), we have
\[
\lim_{x \to 0^+} h_k(x) = \lim_{x \to 0^+} \left[ \frac{(k-1)!}{(1)^{k+1}\psi^{(k)}(x + 1)} \right]^{1/k} = \left[ \frac{(k-1)!}{\psi^{(k)}(1)} \right]^{1/k}.
\] (2.5)

By virtue of (2.3) and (2.4), straightforward computation yields
\[
h'_k(x) = \frac{d}{dx} \left\{ \left[ \frac{(k-1)!}{(-1)^{k+1}\psi^{(k)}(x + 1)} \right]^{1/k} - x \right\}
\]
\[
= -\frac{\psi^{(k+1)}(x + 1)}{k\psi^{(k)}(x + 1)} \left[ \frac{(k-1)!}{(-1)^{k+1}\psi^{(k)}(x + 1)} \right]^{1/k} - 1
\]
\[
= \left[ k^{1/k} \psi^{(k+1)}(x + 1) \left[ \frac{(k-1)!}{\psi^{(k)}(x + 1)} \right]^{1+1/k} \right]^{1/k} - 1
\]
\[
= \left[ k^{1/k} \psi^{(k+1)}(x + 1) \sqrt[k]{\frac{(k-1)!}{\psi^{(k)}(x + 1)}} \right]^{k+1/k} - 1.
\]
By virtue of the inequality (1.26), it follows that \( h'_k(x) < 0 \) on \((0, \infty)\), which means that the functions \( h_k(x) \) for \( k \in \mathbb{N} \) are strictly decreasing on \((0, \infty)\).

In conclusion, from a combination of the decreasing monotonicity of \( h_k(x) \) with the limits (2.2) and (2.5), Theorem 1 follows immediately. \( \square \)

Remark 4. Here we provide a simple proof for the limit (2.2) as follows. In \[ p. 260, 6.4.11 \], it is listed that
\[
\psi^{(n)}(z) \sim (-1)^{n-1} \left[ \frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^\infty B_{2k} \frac{(2k + n - 1)!}{(2k)!z^{2k+n}} \right]
\] (2.6)
for \( z \to \infty \) in \( |\arg z| < \pi \). If we substitute (2.6) into the definition (2.1) of \( h_k \), then we obtain (2.2) readily.

Proof of Corollary 1. This follows from
\[
\left| \psi^{(k)}(x) \right| - \frac{k!}{x^{k+1}} = \left| \psi^{(k)}(x + 1) \right|,
\]
an equivalence of (2.3), for \( k \in \mathbb{N} \) and \( x > 0 \). \( \square \)
Proof of Corollary 2. Utilizing the identity (2.3) and the integral expression (2.4) shows
\[ |\psi^{(k)}(x + m)| = |\psi^{(k)}(x)| - \sum_{i=1}^{m} \frac{k!}{(x + i - 1)^{k+1}} \]
for \( k, m \in \mathbb{N} \) and \( x > 0 \). Combining this with (1.40) and the case \( x + m \) of (1.40) respectively leads to the inequalities (1.42) and (1.43). Corollary 2 is proved. □

Remark 5. This article is a simplified and updated version of the preprint [56].

REFERENCES

[1] ABRAMOWITZ, M.—STEGUN, I. A.: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards, Applied Mathematics Series 55, U.S. Government Printing Office, Washington, 1965.

[2] ALLASIA, G.—GIORDANO, C.—PEČARIĆ, J.: *Inequalities for the gamma function relating to asymptotic expansions*, Math. Inequal. Appl. 5 (2002), 543–555.

[3] ALZER, H.: *On some inequalities for the gamma and psi functions*, Math. Comp. 66 (1997), 373–389.

[4] ALZER, H.: *Sharp inequalities for the digamma and polygamma functions*, Forum Math. 16 (2004), 181–221.

[5] ALZER, H.—BATIR, N.: *Monotonicity properties of the gamma function*, Appl. Math. Lett. 20 (2007), 778–781; [http://dx.doi.org/10.1016/j.aml.2006.08.026](http://dx.doi.org/10.1016/j.aml.2006.08.026).

[6] ALZER, H.—GRINSHPAN, A. Z.: *Inequalities for the gamma and q-gamma functions*, J. Approx. Theory 144 (2007), 67–83.

[7] ATANASSOV, R. D.—TSOUKROVSKI, U. V.: *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. 41 (1988), No. 2, 21–23.

[8] BATIR, N.: *An interesting double inequality for Euler’s gamma function*, J. Inequal. Pure Appl. Math. 5 (2004), No. 4, Art. 97; [http://www.emis.de/journals/JIPAM/article452.html](http://www.emis.de/journals/JIPAM/article452.html).

[9] BATIR, N.: *Inequalities for the gamma function*, Arch. Math. (Basel) 91 (2008), 554–563.

[10] BATIR, N.: *On some properties of digamma and polygamma functions*, J. Math. Anal. Appl. 328 (2007), 452–465; [http://dx.doi.org/10.1016/j.jmaa.2006.05.065](http://dx.doi.org/10.1016/j.jmaa.2006.05.065).

[11] BATIR, N.: *Some new inequalities for gamma and polygamma functions*, J. Inequal. Pure Appl. Math. 6 (2005), No. 4, Art. 103; [http://www.emis.de/journals/JIPAM/article577.html](http://www.emis.de/journals/JIPAM/article577.html).

[12] BERG, C.: *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. 1 (2004), 433–439.

[13] CHEN, C.-P.—QI, F.: *Logarithmically completely monotonic functions relating to the gamma function*, J. Math. Anal. Appl. 321 (2006), no. 1, 405–411; [http://dx.doi.org/10.1016/j.jmaa.2005.08.056](http://dx.doi.org/10.1016/j.jmaa.2005.08.056).

[14] CHU, Y.-M.—ZHANG, X.-M.—TANG, X.-M.: *An elementary inequality for psi function*, Bull. Inst. Math. Acad. Sin. (N.S.) 3 (2008), 373–380.

[15] CHU, Y.-M.—ZHANG, X.-M.—ZHANG, Z.-H.: *The geometric convexity of a function involving gamma function with applications*, Commun. Korean Math. Soc. 25 (2010), 373–383; [http://dx.doi.org/10.4134/CKMS.2010.25.3.373](http://dx.doi.org/10.4134/CKMS.2010.25.3.373).
SHARP INEQUALITIES FOR POLYGAMMA FUNCTIONS

[16] ELEZOVIC, N.—GIORDANO, C.—PECARIĆ, J.: The best bounds in Gautschi’s inequality, Math. Inequal. Appl. 3 (2000), 239–252.

[17] GRINSHSPAN, A. Z.—ISMAIL, M. E. H.: Completely monotonic functions involving the gamma and q-gamma functions, Proc. Amer. Math. Soc. 134 (2006), 1153–1160.

[18] GUO, B.-N.—QI, F.: A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications, J. Korean Math. Soc. 48 (2011), 655–667; http://dx.doi.org/10.4134/JKMS.2011.48.3.655

[19] GUO, B.-N.—CHEN, R.-J.—QI, F.: A class of completely monotonic functions involving the polygamma functions, J. Math. Anal. Approx. Theory 1 (2006), 124–134.

[20] GUO, B.-N.—QI, F.: A completely monotonic function involving the tri-gamma function and with degree one, Appl. Math. Comput. 218 (2012), 9890–9897; http://dx.doi.org/10.1016/j.amc.2012.03.075

[21] GUO, B.-N.—QI, F.: A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 72 (2010), 21–30.

[22] GUO, B.-N.—QI, F.: An extension of an inequality for ratios of gamma functions, J. Approx. Theory 163 (2011), 1208–1216; http://dx.doi.org/10.1016/j.jat.2011.04.003

[23] GUO, B.-N.—QI, F.: Monotonicity of functions connected with the gamma function and the volume of the unit ball, Integral Transforms Spec. Funct. 23 (2012), 701–708; http://dx.doi.org/10.1080/10652469.2011.627511

[24] GUO, B.-N.—QI, F.: Refinements of lower bounds for polygamma functions, Proc. Amer. Math. Soc. 141 (2013), 1007–1015; http://dx.doi.org/10.1090/S0002-9939-2012-11387-5.

[25] GUO, B.-N.—QI, F.: Some properties of the psi and polygamma functions, Hacet. J. Math. Stat. 39 (2010), 219–231.

[26] GUO, B.-N.—QI, F.: Two new proofs of the complete monotonicity of a function involving the psi function, Bull. Korean Math. Soc. 47 (2010), 103–111; http://dx.doi.org/10.4134/bkms.2010.47.1.103

[27] GUO, B.-N.—QI, F.—SRIVASTAVA, H. M.: Some uniqueness results for the nontrivially complete monotonicity of a class of functions involving the polygamma and related functions, Integral Transforms Spec. Funct. 21 (2010), 849–858; http://dx.doi.org/10.1080/10652461003748112

[28] GUO, B.-N.—ZHANG, Y.-J.—QI, F.: Refinements and sharpenings of some double inequalities for bounding the gamma function, J. Inequal. Pure Appl. Math. 9 (2008), No. 1, Art. 17; http://www.emis.de/journals/JIPAM/article953.html

[29] GUO, B.-N.—ZHAO, J.-L.—QI, F.: A completely monotonic function involving the tri- and tetra-gamma functions, Math. Slovaca 63 (2013), 469–478; http://dx.doi.org/10.2478/s12175-013-0109-2

[30] GUO, S.—QI, F.—SRIVASTAVA, H. M.: A class of logarithmically completely monotonic functions related to the gamma function with applications, Integral Transforms Spec. Funct. 23 (2012), 557–566; http://dx.doi.org/10.1080/10652469.2011.611331

[31] GUO, S.—QI, F.—SRIVASTAVA, H. M.: Necessary and sufficient conditions for two classes of functions to be logarithmically completely monotonic, Integral Transforms Spec. Funct. 18 (2007), 819–826; http://dx.doi.org/10.1080/10652460701528933

[32] GUO, S.—QI, F.—SRIVASTAVA, H. M.: Supplements to a class of logarithmically completely monotonic functions associated with the gamma function, Appl. Math. Comput. 197 (2008), 768–774; http://dx.doi.org/10.1016/j.amc.2007.08.011

[33] GUO, S.—SRIVASTAVA, H. M.: A class of logarithmically completely monotonic functions, Appl. Math. Lett. 21 (2008), 1134–1141.
HORN, R. A.: *On infinitely divisible matrices, kernels and functions*, Z. Wahrscheinlichkeitstheorie Verw. Geb. **8** (1967), 219–230.

ISMAIL, M. E. H.—MULDOON, M. E.: *Inequalities and monotonicity properties for gamma and \(q\)-gamma functions*. In: Approximation and Computation: A Festschrift in Honour of Walter Gautschi (R. V. M. Zahar, ed.), Internat. Ser. Numer. Math. **119**, Birkhäuser, Basel, 1994, pp. 309–323.

KAZARINOFF, D. K.: *On Wallis’ formula*, Edinburgh Math. Notes **40** (1956), 19–21.

KOUMANDOS, S.: *Remarks on some completely monotonic functions*, J. Math. Anal. Appl. **324** (2006), 1458–1461.

KOUMANDOS, S.—PEDERSEN, H. L.: *Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler’s gamma function*, J. Math. Anal. Appl. **355** (2009), 33–40.

LÜ, Y.-P.—SUN, T.-C.—CHU, T.-C.: *Necessary and sufficient conditions for a class of functions and their reciprocals to be logarithmically completely monotonic*, J. Inequal. Appl. **2011** (2011), Article 36; [http://dx.doi.org/10.1186/1029-242X-2011-36](http://dx.doi.org/10.1186/1029-242X-2011-36)

MERKLE, M.: *Logarithmic convexity and inequalities for the gamma function*, J. Math. Anal. Appl. **203** (1996), 369–380.

MITRINOVIĆ, D. S.—PEČARIĆ, J. E.—FINK, A. M.: *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

MORTICI, C.: *Very accurate estimates of the polygamma functions*, Asymptot. Anal. **68** (2010), 125–134; [http://dx.doi.org/10.3233/ASY-2010-0983](http://dx.doi.org/10.3233/ASY-2010-0983)

MULDOON, M. E.: *Some monotonicity properties and characterizations of the gamma function*, Aequationes Math. **18** (1978), 54–63.

QI, F.: *A completely monotonic function involving the divided difference of the psi function and an equivalent inequality involving sums*, ANZIAM J. **48** (2007), 523–532; [http://dx.doi.org/10.1017/S1446181100003199](http://dx.doi.org/10.1017/S1446181100003199)

QI, F.: *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Article ID 493058, 84 pp.; [http://dx.doi.org/10.1155/2010/493058](http://dx.doi.org/10.1155/2010/493058)

FENG Qi—QIU-MING LUO: *Bounds for the ratio of two gamma functions: from Wendel’s asymptotic relation to Elezović-Giordano-Pečarić’s theorem*, J. Inequal. Appl. **2013** (2013), 542, 20 pages; [http://dx.doi.org/10.1186/1029-242X-2013-542](http://dx.doi.org/10.1186/1029-242X-2013-542)

QI, F.: *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct. **18** (2007), 503–509; [http://dx.doi.org/10.1080/10652460701358976](http://dx.doi.org/10.1080/10652460701358976)

QI, F.—CHEN, C.-P.: *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), 603–607; [http://dx.doi.org/10.1016/j.jmaa.2004.04.026](http://dx.doi.org/10.1016/j.jmaa.2004.04.026)

QI, F.—CUI, R.-Q.—CHEN, C.-P.—GUO, B.-N.: *Some completely monotonic functions involving polygamma functions and an application*, J. Math. Anal. Appl. **310** (2005), 303–308; [http://dx.doi.org/10.1016/j.jmaa.2005.02.016](http://dx.doi.org/10.1016/j.jmaa.2005.02.016)

QI, F.—GUO, B.-N.: *A logarithmically completely monotonic function involving the gamma function*, Taiwanese J. Math. **14** (2010), 1623–1628.

QI, F.—GUO, B.-N.: *An inequality involving the gamma and digamma functions*, [http://arxiv.org/abs/1101.4698](http://arxiv.org/abs/1101.4698)

QI, F.—GUO, B.-N.: *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), No. 1, Art. 8, 63–72; [http://rgmia.org/v7n1.php](http://rgmia.org/v7n1.php)

QI, F.—GUO, B.-N.: *Completely monotonic functions involving divided differences of the di- and tri-gamma functions and some applications*, Commun. Pure Appl. Anal. **8** (2009), 1975–1989; [http://dx.doi.org/10.3934/cpaa.2009.8.1975](http://dx.doi.org/10.3934/cpaa.2009.8.1975)
SHARP INEQUALITIES FOR POLYGAMMA FUNCTIONS

[54] QI, F.—GUO, B.-N.: Necessary and sufficient conditions for a function involving divided differences of the di- and tri-gamma functions to be completely monotonic, http://arxiv.org/abs/0903.3071

[55] QI, F.—GUO, B.-N.: Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic, Adv. Appl. Math. 44 (2010), 71–83; http://dx.doi.org/10.1016/j.aam.2009.03.003

[56] QI, F.—GUO, B.-N.: Sharp inequalities for polygamma functions, http://arxiv.org/abs/0903.1984

[57] GUO, B.-N.—QI, F.: Sharp inequalities for the psi function and harmonic numbers, Analysis (Munich) 34 (2014), 201–208; http://dx.doi.org/10.1515/anly-2014-0001

[58] QI, F.—GUO, B.-N.: Some completely monotonic functions involving the gamma and polygamma functions, J. Aust. Math. Soc. 80 (2006), 81–88; http://dx.doi.org/10.1017/S1446788700011393

[59] QI, F.—GUO, B.-N.—CHEN, C.-P.: Some completely monotonic functions involving the gamma and polygamma functions, J. Comput. Appl. Math. 233 (2010), 2149–2160; http://dx.doi.org/10.1016/j.cam.2009.09.044

[60] QI, F.—GUO, B.-N.—LUO, Q.-M.: Bounds for the ratio of two gamma functions—from Wendel’s and related inequalities to logarithmically completely monotonic functions, Banach J. Math. Anal. 6 (2012), 132–158.

[61] QI, F.—WEI, C.-F.—GUO, B.-N.: Complete monotonicity of a function involving the ratio of gamma functions and applications, Banach J. Math. Anal. 6 (2012), 35–44.

[62] SONG, Y.-Q.—CHU, Y.-M.—WU, L.-L.: An elementary double inequality for gamma function, Int. J. Pure Appl. Math. 38 (2007), 549–554.

[63] WU, L.-L.—CHU, Y.-M.: An inequality for the psi functions, Appl. Math. Sci. (Ruse) 2 (2008), 545–550.

[64] WU, L.-L.—CHU, Y.-M.: Inequalities for the generalized logarithmic mean and psi functions, Int. J. Pure Appl. Math. 48 (2008), 117–122.

[65] ZHANG, X.-M.—CHU, Y.-M.: A double inequality for the gamma and psi functions, Int. J. Mod. Math. 3 (2008), 67–73.

[66] ZHAO, T.-H.—CHU, Y.-M.: A class of logarithmically completely monotonic functions associated with a gamma function, J. Inequal. Appl. 2010 (2010), Article ID 392431, 11 pp.; http://dx.doi.org/10.1155/2010/392431

[67] ZHAO, T.-H.—CHU, Y.-M.—JIANG, Y.-P.: Monotonic and logarithmically convex properties of a function involving gamma functions, J. Inequal. Appl. 2009 (2009), Article ID 728612, 13 pp.; http://dx.doi.org/10.1155/2009/728612

[68] ZHAO, T.-H.—CHU, Y.-M.—WANG, H.: Logarithmically complete monotonicity properties relating to the gamma function, Abstr. Appl. Anal. 2011 (2011), Article ID 896483, 13 pp.; http://dx.doi.org/10.1155/2011/896483

[69] ZHAO, J.-L.—GUO, B.-N.—QI, F.: A refinement of a double inequality for the gamma function, Publ. Math. Debrecen 80 (2012), 333–342; http://dx.doi.org/10.5486/PMD.2012.5010

[70] ZHAO, J.-L.—GUO, B.-N.—QI, F.: Complete monotonicity of two functions involving the tri- and tetra-gamma functions, Period. Math. Hungar. 65 (2012), 147–155; http://dx.doi.org/10.1007/s10998-012-9562-x
[73] WIDDER, D. V.: The Laplace Transform, Princeton University Press, Princeton, NJ, 1946.

Received 21.11.2011
Accepted 4.8.2012

*School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City
Henan Province
454000
CHINA
E-mail: bai.ni.guo@hotmail.com

**College of Mathematics
Inner Mongolia University for Nationalities
Tongliao City
Inner Mongolia Autonomous Region
028043
CHINA
E-mail: qifeng618@gmail.com

***Department of Mathematics and Informatics
Weinan Teachers University
Weinan City
Shaanxi Province
714000
CHINA
E-mail: zhaojl2004@gmail.com

****Department of Mathematics
Chongqing Normal University
Chongqing City
401331
CHINA
E-mail: luomath@126.com