Finite temperature gauge theory from the transverse lattice

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Numerical computations are performed and analytic bounds are obtained on the excited spectrum of glueballs in $SU(3)$ gauge theory, by transverse lattice Hamiltonian methods. We find an exponential growth of the density of states, implying a finite critical temperature. It is argued that the Nambu-Goto string model lies in a different universality class.

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INTRODUCTION

The behavior of QCD at finite temperature is of great theoretical and experimental interest. Traditionally, finite temperature calculations of a field theory are performed in Euclidean space with periodic boundary conditions imposed in the time direction. However, the same information can be extracted from a Minkowski space calculation by examining the density of states in the spectrum as a function of energy. In particular, an exponentially increasing density of states implies a finite critical temperature $T_c$. There is some experimental evidence of this exponential increase in the hadron spectrum \cite{9}. Recently, there has been renewed interest in the lightcone formulation of thermal field theory in Minkowski space \cite{2}. In this letter we investigate the aspects of a lightcone formulation of QCD relevant for finite temperature and, in particular, its implications for the string picture of QCD.

To leading order of the $1/N_c$ expansion of QCD \cite{4}, we compute the density of glueball states per unit mass via the transverse lattice approach to non-abelian gauge theory \cite{5}. This is a Hamiltonian method that combines light-front quantization in the two “longitudinal” space-time directions, $x^\pm = (x^0 \pm x^3)/\sqrt{2}$, with a lattice in the remaining “transverse” directions, $\{x^1, x^2\}$. It has been successfully used to compute the properties of the lightest glueballs and mesons (for a review, see Ref. \cite{6}). In these calculations, effective Hamiltonians on coarse lattices were tuned to optimize Lorentz covariance of the lightest boundstates. The method also yields the spectrum of heavier boundstates. A rapid exponential rise in the density of states suggests that there may be enough states at relatively low masses, where the coarse lattice calculation is still accurate for aggregate quantities, to observe this behaviour unambiguously.

Using the detailed methods described in Refs. \cite{7}, we demonstrate that in an intermediate mass range, $\sim 1–3$ times the lightest glueball mass, the density of states does indeed show a degree of universality with respect to cutoffs. We find an exponentially increasing density of states, implying a finite-temperature Hagedorn transition. Although the method is not well-suited to producing an accurate prediction of the critical temperature, we make a numerical estimate and produce analytic bounds on it. One application of interest is to the string picture of large-$N_c$ gauge theory. Results from previous Euclidean lattice computations \cite{8} indicate a $T_c$ lower than that of the Nambu-Goto string model \cite{9}, suggesting that a QCD string in the large-$N_c$ limit has more worldsheet degrees of freedom than this free bosonic string. Our calculations suggest that this is due at least in part to longitudinal oscillations. This is also consistent with the Nambu-Goto string description appearing in a distinct, unstable phase of transverse lattice gauge theory discovered by Klebanov and Susskind \cite{10}.

TRANSVERSE LATTICE GAUGE THEORY

The transverse lattice is formulated in the following manner. Two coordinates $x^\alpha, \alpha \in \{0,3\}$, are continuous while two directions $x = \{x^1, x^2\}$ are discretized as a lattice, spacing $a$. The longitudinal continuum gauge potentials $A^\alpha(x)$ lie at sites $x$. The transverse flux link fields $M_r(x), r \in \{1,2\}$, lie on the link between $x$ and $x + a\hat{r}$, representing the gauge fields polarized in the $x$ directions. In general there are also fermi fields $\Psi(x)$, but we will not need these for the glueball spectrum at large-$N_c$. For a coarse transverse lattice, the strategy is to perform a color-dieletric expansion \cite{11} of the most general lightcone gauge Hamiltonian, renormalisable with respect to the continuum coordinates $x^\alpha$, in powers of $M_r$.

Provided these link fields are chosen sufficiently heavy, one can truncate this expansion to study the low lying hadron boundstates dominated by just a few particles of these fields. The remaining couplings in the effective Hamiltonian can then be accurately constrained by optimizing symmetries broken by the cutoffs in low energy observables. Typically this method is viable in a window between small lattice spacings, where the fields become
too light to justify the color-dieletric expansion, and large lattice spacings where Lorentz covariance breaks down uncontrollably.

For completeness, we give the Lagrangian, used in Ref. [2], that we shall employ here. It contains all allowed terms up to order \( (M) \) for the large-\( N_c \) limit:

\[
L = \sum_x \int dx^- \sum_{a,\beta = 1}^{N_c} \sum_{r=1,2} - \frac{1}{2G^2} \text{Tr} \left\{ F^{\alpha\beta}(x) \right\} - \mu^2 \text{Tr} \left\{ M_r M_r^\dagger \right\} \\
+ \frac{\lambda_1}{N_c} \text{Tr} \left\{ M_1 M_2 M_1^\dagger M_2^\dagger \right\} + \text{H. C.}
\]

where \( F^{\alpha\beta}(x) \) is the continuum field strength in the \((x^0, x^3)\) planes at each site \( x \). We have defined \( M_r = M_r^\dagger \) and hold \( G^2 N_c \) finite as \( N_c \to \infty \). In the \( N_c \to \infty \) limit, glueball states consists of connected loops of transverse flux links and are absolutely stable to decay.

The reader is referred to Ref. [7] for details of the construction of the lightcone Hamiltonian and Fock space, the renormalisation, and the determination of various couplings appearing in Eqn. (1). These couplings are accurately constrained by optimizing covariance of low-lying glueball eigenfunctions and rotational invariance of the heavy-source potential. The relatively large number of couplings means the Hamiltonian is highly ‘improved’ and can give cut-off independent results on quite coarse lattices. Taking the fundamental scale to be the string tension \( \sigma \), Refs. [7] investigated individual glueball masses up to about 1.5 times the lightest glueball mass and the method was shown to produce accurate results, even for lattice spacings of order \( 1/\sqrt{\sigma} \). We investigate here whether any part of the higher excited spectrum, probed at finite temperature, is similarly under control.

**Density of States**

We employ a DLCQ cutoff \( K \) in the \( x^- \) direction [12] to obtain a finite Fock space basis at finite transverse lattice spacing \( a \). Therefore, we must study the stability of results under variation of both \( K \) and \( a \). Figs. [1] and [2] display the distribution of glueball masses \( M_i \), in units of the string tension \( \sqrt{\sigma} \), for the \( i^{th} \) state in the spectrum.

The results in Fig.[1] are shown for fixed \( a = 1.44/\sqrt{\sigma} \), the lattice spacing at which the best covariance of the low-lying spectrum was obtained for Lagrangian [11], and increasing values of the DLCQ cutoff \( K = 8, 10, 12 \), where \( K \to \infty \) is the limit of infinite dimensional Fock space. The largest basis at \( K = 12 \) contains 654,948 states in total, though only the lowest 20,000 are plotted for file storage reasons. At the largest masses displayed, the DLCQ cutoff is introducing non-universal artifacts because it cuts off the maximum number of \( M \)-particles in a state. At lower masses however, there is universal behavior of the slope of the distribution. We repeated the calculation for \( K = 12 \) at four different lattice spacings, shown in Fig[2] the couplings of [11] re-optimized for covariance of the low-lying spectrum at each. Again, for the range of masses plotted, the slope is relatively universal.

The approximately universal linear growth observed in this region implies that the density of states per unit mass, \( \rho(M) = dt/dM \), grows exponentially with mass \( M \)

\[
\rho(M) \sim e^{M/T_c}.
\]

This exponential behaviour establishes itself already in the first 100 or so states, occurring at less than twice the lightest glueball mass, which is within the expected range of validity of the method [7]. It is perhaps surprising from the figures that the universal growth appears to extend to much higher masses. One would expect the individual highly excited glueball masses to shift around relative to their true values due to cutoff dependence. However, an aggregate quantity like the density of states appears to be less sensitive to cutoff dependence.

**Critical Temperature and Strings**

It is well known [1] that exponential growth of the density of states leads to a diverging canonical partition function for temperature \( T > T_c \). To determine an accurate value for \( T_c \), one needs to know the density for asymptotically high masses \( M \), when power corrections to the expression [2] at lower masses can be neglected. In principle, knowledge of the functional form of power corrections could enable one to extract \( T_c \) accurately from lower masses, but in practice the form is unknown for QCD. Therefore, although exponential growth is clearly occurring, we can only extract a rough estimate for \( T_c \) from the numerical data, based on linear fits to Figs. [1] and [2].

There are three systematic sources of error in our estimate. The first two come from the variation of the data in Figs. [1] and [2] which we take as indicative of the error due to residual dependence on the cutoffs \( K \) and \( a \). The third comes from the linear fits themselves, since power corrections to Eqn. [2] in the actual data will make the fit result for \( T_c \) depend upon the range of masses chosen for the fit. Those intervals are naturally limited above and
below: for all the data, smooth linear behaviour does not start until the block of points beginning at $M \approx 6\sqrt{\sigma}$; DLCQ artifacts limit the maximum useful value of $t$, roughly $\ln t = 6, 7, 8$ for the $K = 8, 10, 12$ data respectively. Within these limits, we performed linear fits to the data sets in the figures for a variety of different mass intervals. The maximum variation of $T_c$ with interval chosen was of order $\pm 15\%$, which we take as indicative of the error made by using simplified fitting function (2).

As a result, we estimate $T_c = 0.63(1)(9)(9)\sqrt{\sigma}$, where the first error is due to $K$, second due to $a$, and third from the fits.

We can also deduce analytic bounds on $T_c$. In Ref. [11], we derived in transverse lattice gauge theory an approximate analytic formula for the number $d_n$ of highly excited glueball states of mass squared $M^2 = 2\pi n\sigma$, $n$ integer.
The derivation was based on the assumption that the glueball masses were governed only by longitudinal dynamics of the transverse link fields and a fixed number of such links in a state. Assuming a transverse degeneracy factor $g^p$ for the number of distinct states with $p$ links, the formula of Ref. [11] trivially generalises to

$$d_n = \exp\{2\sqrt{-n\text{Li}_2(-g^2)}\},$$

where $\text{Li}_2$ is the dilogarithm function. This leads to a Hagedorn temperature

$$T_c = \sqrt{\frac{\pi\sigma}{-2\text{Li}_2(-g^2)}},$$

The maximum degeneracy one could expect is from a long free chain of $p$ links, which on a square transverse lattice gives $g = 4$. This provides an upper bound on the density of states and hence a lower bound $T_c = 0.54\sqrt{\sigma}$. Conversely, setting $g = 1$ corresponds to neglecting transverse structure altogether. This is a lower bound on the density of states, implying $T_c = 1.375\sqrt{\sigma}$ is an upper bound. The numerical solution investigated above is somewhere between these extremes; in reality there are non-trivial longitudinal and transverse dynamics.

Recent large-$N_c$ extrapolations of Euclidean lattice computations [8] are more suited to providing accurate determinations of the critical temperature, obtaining $T_c = 0.596(4)\sqrt{\sigma}$. This is less than the result, $T_c/\sqrt{\sigma} = \sqrt{3/2\pi} \approx 0.69$, for the non-interacting Nambu-Goto string [10]. This would suggest that the large-$N_c$ gauge theory flux string has more worldsheet degrees of freedom than a simple bosonic one, which has only transverse oscillations. In making this comparison based on $T_c$ with a free string, it is important to use results from the large-$N_c$ limit, since only in this limit do lightcone flux strings not split and join. At finite $N_c$, by examining only $T_c$ one cannot separate physically distinct world-sheet effects from string interaction effects proportional to $1/N_c^2$. Also, the Hagedorn determination of $T_c$ directly from the density of states is strictly correct only for a gas of free hadrons/strings.

Although Euclidean lattice results provide an accurate value for $T_c$, they do not explain why it differs from the bosonic string value. Transverse lattice gauge theory can provide an explanation with the help of an older analysis by Klebanov and Susskind [10]. They noted that choosing a sufficiently tachyonic link field mass $\mu_h$ (see Eqs. [10]) would drive the theory into a new phase, where glueball states were unstable to becoming infinitely long chains of transverse links, each link with frozen longitudinal dynamics. In this phase, there are only transverse dynamics (this is the complete opposite of the scenario used above to derive the bounds). The spectrum of this phase is exactly that of the Nambu-Goto string [10]. Thus, at least some of the extra degrees of freedom present in the QCD phase of transverse lattice gauge theory, which can lower the critical temperature, appear to be longitudinal oscillations of the flux string.

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