ON THE GEOGRAPHY OF LINE ARRANGEMENTS

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Abstract. This is a short survey about geography of line arrangements in the projective plane. Our purpose is to explain various properties of line arrangements in terms of Chern invariants. We show some few new facts about their distribution, and we discuss two open problems in relation to geography over \( \mathbb{Q} \) and over \( \mathbb{C} \).

1. Definitions, examples, and combinatorial facts

Let \( k \) be an arbitrary field. The most relevant fields for us will be \( \mathbb{Q} \), \( \mathbb{R} \), \( \mathbb{C} \), and the algebraic closure \( \overline{\mathbb{F}_p} \) of the field of \( p \) elements \( \mathbb{F}_p \). The projective plane over \( k \) will be denoted by \( \mathbb{P}^2_k \).

Definition 1.1. A set of the form \( \{ [x, y, z] \in \mathbb{P}^2_k : ax + by + cz = 0 \} \) for some \( a, b, c \in k \) not all zero will be called line. A line arrangement is a finite collection of two or more lines.

Definition 1.2. The combinatorics of a line arrangement \( A \) is the data of its lines and their points of incidence. For \( m \geq 2 \), an \( m \)-point of \( A \) is a point which belongs to exactly \( m \) lines in \( A \). We denote the number of \( m \)-points by \( t_m \).

Our main reference on line arrangements is \( [\text{Hirz83}] \).

Example 1.3. An arrangement of \( d \) lines with \( t_d = 1 \) is called trivial. An arrangement is called quasi-trivial if \( t_{d-1} = 1 \).

Example 1.4. Line arrangements in \( \mathbb{P}^2_{\mathbb{R}} \) are the ones that can be drawn on paper. Each of them partitions \( \mathbb{P}^2_{\mathbb{R}} \) into polygons. If all polygons are triangles, then the arrangement is called simplicial. There is a vast literature on simplicial arrangements (cf. \( [\text{Gr05}] \)). They have not been classified yet. An example of such is the complete quadrilateral defined by the zeros of \( xyz(x - y)(x - z)(y - z) \). It is an arrangement of 6 lines with \( t_2 = 3 \), \( t_3 = 4 \), \( t_4 = 0 \) else. Regular polygons define families of simplicial arrangements: by taking the regular polygon of \( n \) lines and adding its \( n \) lines of symmetry, we get an arrangement of \( 2n \) lines. It has \( t_2 = n \), \( t_3 = n(n - 1)/2 \), \( t_n = 1 \), \( t_m = 0 \) else. (see e.g. \( [\text{Hirz83}, 1.1] \)).

Example 1.5. Let \( n \geq 4 \). Then the zeros of \( (x^n - y^n)(x^n - z^n)(y^n - z^n) \) in \( \mathbb{P}^2_{\mathbb{C}} \) define an arrangement of \( 3n \) lines with \( t_3 = n^2 \), \( t_n = 3 \), \( t_m = 0 \) else. They are called Ceva arrangements. For \( n = 3 \), the polynomial \( (x^3 - y^3)(x^3 - z^3)(y^3 - z^3) \) defines the dual Hesse arrangement which has 9 lines, \( t_3 = 12 \), \( t_m = 0 \) else. The Hesse arrangement is the arrangement of 12 lines joining the 9 inflection points of a given smooth projective cubic in \( \mathbb{P}^2_{\mathbb{C}} \). It turns...
out that they are all projectively equivalent, and they have \( t_2 = 12, \ t_4 = 9, \ t_m = 0 \) else. The dual lines defined by the nine 4-points are the 9 lines of the dual Hesse arrangement. We recall that points and lines are dual objects of each other, in the sense that an arrangement of lines corresponds to the collection of points in \( \mathbb{P}^2_k \) given by the 3 coefficients of each line, and vice-versa.

**Example 1.6.** Let \( k = \mathbb{F}_{p^n} \) for some prime \( p \) and \( n > 0 \). The set of \( p^{2n} + p^n + 1 \) lines in \( \mathbb{P}^2_k \) form an arrangement of lines with \( t_{p^n+1} = p^{2n} + p^n + 1, \ t_m = 0 \) else. We call it a \textit{finite projective plane arrangement}. For \( p = 2 \) and \( n = 1 \) we have the \textit{Fano arrangement} of seven lines with seven triple points.

By counting pairs of lines in two different ways, we obtain that any arrangement of \( d \) lines satisfies

\[
\binom{d}{2} = \sum_{m \geq 2} \binom{m}{2} t_m,
\]

which is a purely combinatorial fact. Another general statement is the following, which is almost purely combinatorial. It is essentially \cite{deBrEr48}, which is purely combinatorial, up to the point when we prove that a non quasi-trivial \( A \subset \mathbb{P}^2_k \) with \( \sum_{m \geq 2} t_m = d \) is in fact a finite projective plane arrangement.

**Theorem 1.7.** A nontrivial arrangement of \( d \) lines \( A \) satisfies

\[
\sum_{m \geq 2} t_m \geq d.
\]

Equality holds if and only if \( A \) is either quasi-trivial or a finite projective plane arrangement.

**Proof.** This first part is taken from \cite{U11} Remark 7.4]. Let us label the \( m \)-points of the arrangement from 1 to \( r = \sum_{m \geq 2} t_m \), and the lines from 1 to \( d \). We define

\[
a_{i,j} = \begin{cases} 
1 & \text{if the line } j \text{ contains point } i \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( L_j \) be the vector \((a_{i,j})_{1 \leq i \leq r}\). We want to prove that the \( L_j \) are linearly independent in \( \mathbb{Q}^r \). Suppose not, say that \( L_1 = \sum_{j=2}^{d} x_j L_j \) for some \( x_j \in \mathbb{Q} \). Then, by taking the usual inner product in \( \mathbb{Q}^r \), we have

\[
x_j = \frac{L_1 \cdot L_j - 1}{1 - L_j \cdot L_j} < 0
\]

for all \( j > 1 \). But the coordinates of \( L_1 \) are either 1 or 0, and so it is impossible that every \( x_j \) is negative. This proves the inequality part of the statement.

For the second part of the statement, observe first that a quasi-trivial arrangement and a finite projective plane arrangement have \( r = d \). Conversely, assume that \( A \) satisfies \( r = d \) and that it is not quasi-trivial. Define the matrix \( A = (L_j)_{1 \leq j \leq d} \), and let \( n_j \) be the number of points in the line \( L_j \), and let \( q_i \) be the number of lines passing through the \( i \)-th point. We now use \cite{HP79} Section 1].
We first prove the following lemma:

**Lemma 1.8.** There is a permutation matrix \( P \) such that \((PA)_{ii} = 0\) for every \( 1 \leq i \leq d \).

**Proof.** Define the matrix \( T; T_{ij} = 1, 1 \leq i \leq d, 1 \leq j \leq d \), so

\[
A'(T - A) = A'T - A' = \begin{bmatrix}
0 & n_1 - 1 & n_1 - 1 & \ldots & n_1 - 1 \\
n_2 - 1 & 0 & n_2 - 1 & \ldots & n_2 - 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n_d - 1 & n_d - 1 & n_d - 1 & \ldots & 0
\end{bmatrix}.
\]

Thus we have

\[
det(A'(T - A)) = \prod_{j=1}^{d} (n_j - 1)det \begin{bmatrix}
0 & 1 & \ldots & 1 \\
1 & 0 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 0
\end{bmatrix} \neq 0.
\]

Then \( T - A \) is a matrix with entries in \( \{0, 1\} \) and non zero determinant, hence there is an addend in the determinant formula which is non zero, meaning in our case that it is the multiplication of only ones. Therefore there must be a row permutation \( P \) such that \( P(T - A) \) has only 1 in its diagonal, and so \( PA \) has only 0 on its diagonal. \( \square \)

Hence by reordering the \( m \)-points, we can assume that \( A \) has \( a_{kk} = 0 \), that is, the line \( L_k \) does not contain the \( k \)-th point. In this way we have that \( n_k \geq g_k \) for every \( k \). But then \( \sum_j n_j = \sum_j \sum_j a_{ij} = \sum_i g_i \) implies \( n_k = g_k \) for every \( k \). This in turn implies that there is a line passing through any pair of points. Indeed this gives the second equality in the count

\[
\#\{(h, k) : \text{there is a line containing point } h \text{ and point } k\} = \sum_{i=1}^{d} \binom{n_i}{2} = \sum_{i=1}^{d} \binom{g_i}{2} = \sum_{m \geq 2} \binom{m}{2} t_m = \binom{d}{2}.
\]

Finally, unless \( A \) is a quasi-trivial arrangement, there are 4 points in the arrangement, with no 3 in a line, then we get that for every pair of points there is a line not passing through either of them. Let \( P_1 \) and \( P_2 \) be two points, and let \( L_k \) be a line not containing them. Then \( g_i = n_k \) and \( g_j = n_k \). Therefore \( g_1 = \ldots = g_d \), and so \( n_j = g_i \) for every \( i \neq j \).

Assume now that \( n_i = g_j = q + 1 \) for all \( i, j \), for some \( q \). So we have that \( d = q^2 + q + 1 = t_{q+1}, t_m = 0 \) else. We recall that our line arrangement \( A \) is in \( \mathbb{P}_k^2 \) for some field \( k \).

By a change of coordinates, we can assume that four of the \((q + 1)\)-points of \( A \) are \([1, 0, 1], [0, 1, 1], [1, 0, 0], \) and \([0, 1, 0] \). Hence we also have the \((q + 1)\)-point \([0, 0, 1]\), and the lines \( x = 0, z = 0, x = z, \) and \( y = z \).

Let \( k' = \{ x \in k \text{ such that } [1, x, 0] \text{ is a } (q + 1)\text{-point}\} \). We will show that this set is a subfield of \( k \). Clearly \( 0, 1 \in k' \). Since we know it has exactly \( q \) elements, we only need to show that \( k' \) is a ring. From now on, we will use the notation \([a, b, c] - [c, d, e]\) for the line through the points \([a, b, c] \) and \([c, d, e] \).
We have \([1, c, 0] - [1, 0, 1] \cap \{x = 0\} = [0, -c, 1], [0, -c, 1] - [1, 0, 0]\) and \(\{x = z\} = [1, -c, 1]\), and \([1, -c, 1] - [0, 0, 1] \cap \{z = 0\} = [1, -c, 0]\). We also have \([1, c, 0] - [0, 1, 1] \cap \{x = z\} = [1, c + 1, 1]\), and \([1, c + 1, 1] - [0, 0, 1] \cap \{z = 0\} = [1, c + 1, 0]\). Thus if \(c \in k\), then \(c + 1 \in k\) and \(c \not\in k\). By interchanging the roles of \(x\) and \(y\), we get that if \([c, 1, 0]\) is a \((q + 1)\)-point, then so is \([c + 1, 1, 1]\).

Let \(a, b \in k\)' with \(a \neq b, b \neq 0\). Then \([1, a, 0]\) and \([1, b, 0]\) are \((q + 1)\)-points, and by the previous paragraph we know that this implies that \([1, a + 1, 1]\) and \([b^{-1} + 1, 1, 1]\) are \((q + 1)\)-points. Observe that \([1, a + 1, 1] - [b^{-1} + 1, 1, 1] \cap \{z = 0\} = [1, -a, 0]\). Thus \(ab \not\in k\).

We now want to show that \(c \in k\) implies \(c^2 \in k\). Say \(c \neq 0, 1, -1\). If \(c + 1 = 1\) for \(m < 5\), then \(c^2\) is \(1, -c - 1\) or \(-1\), already in \(k\). Note that \(c(c + 1) = c^2 + c \in k\)', and so \(c^2 + c + 1 \in k\). Also \(c^2 + c + 1 \neq c - 1\) (or we would already have \(c^2 = -2 \in k\)'). So \((c - 1)(c^2 + c + 1) = c^3 - 1 \in k\). Thus \(c^3 \in k\). As \(c^3 \neq c\), then \(c^4 \in k\). As \(k\)' is finite there must be an \(n\) such that \(c^m = 1\). Multiplying \(c^4\) by \(c\) enough times, we get \(c^{m-1} \in k\), and then \(c^{m-1} c^3 = c^2 \in k\).

In this way, because \(k\)' has finite order, given a nonzero \(c \in k\)', we have \(c^{-1} \in k\). Therefore given \(a, b \in k\)', we have that \(a + b = b(ab^{-1} + 1) \in k\)'. This completes the proof that \(k' = \mathbb{F}_q\), and so \(\mathcal{A}\) is projectively equivalent to a copy of the finite projective plane arrangement \(\mathbb{P}^2_{\mathbb{F}_q}\) in \(\mathbb{P}^2_{\mathbb{F}_k}\).

**Remark 1.9.** In [215], the first author explores configurations of \(r\) \(m\)-points and \(d\) curves in Hirzebruch surfaces where \(r = d\), realizing certain Ryser designs over \(\mathbb{F}_p^r\).

### 2. Chern invariants

We now define key combinatorial invariants for line arrangements. Surprisingly, we will see that various general properties of line arrangements can be expressed with these invariants.

**Definition 2.1.** Let \(\mathcal{A}\) be an arrangement of \(d\) lines. We define the integers

\[
\bar{c}_1(\mathcal{A}) = 9 - 5d + \sum_{m \geq 2} (3m - 4)t_m \quad \text{and} \quad \bar{c}_2(\mathcal{A}) = 3 - 2d + \sum_{m \geq 2} (m - 1)t_m.
\]

They are called the *Chern numbers* of \(\mathcal{A}\).

**Remark 2.2.** The Chern numbers of line arrangements come from the following. Let \(\sigma: X \to \mathbb{P}^2\) be the blow-up of all the \(m\)-points with \(m > 2\). Let \(D\) be the reduced total transform of the arrangement under \(\sigma\), and so it contains all strict transforms of the lines and all exceptional divisors of \(\sigma\). Let \(\Omega^1_X(\log D)\) be the rank two vector bundle on \(X\) of log differentials with poles in \(D\). Let \(c_1(\Omega^1_X(\log D)\)) be the Chern classes of the dual of \(\Omega^1_X(\log D)\). Then the Chern numbers of the line arrangement are \(\bar{c}_1 = c_1 \cdot c_1\) and \(\bar{c}_2 = c_2\). For details see e.g. [110a] Sections 2 and 4, where it is done in more generality for arrangements of curves in algebraic surfaces. See [178] for a specific reference to invariants of line arrangements.

**Proposition 2.3.** If \(\mathcal{A}\) has \(t_d = t_{d-1} = 0\), then its Chern numbers are positive.
Proof. We note that a quasi-trivial arrangement has \( \bar{c}_2 = 0 \). Suppose \( d = 4 \). Then \( \mathcal{A} \) only has nodes (under the conditions of the proposition) and therefore \( \bar{c}_2(\mathcal{A}) = 1 \). We now argue by induction on \( d \).

Assume that \( \mathcal{A} \) has \( d+1 \geq 5 \) lines, and let \( L \in \mathcal{A} \) be a line passing by \( t \geq 3 \) points (it must exist by the assumptions). The arrangement \( \mathcal{A} \setminus L \) is not trivial, and so

\[
\bar{c}_2(\mathcal{A} \setminus L) - 5 + 2t \geq \bar{c}_2(\mathcal{A} \setminus L) + 1 \geq 1
\]

and

\[
\bar{c}_2(\mathcal{A} \setminus L) - 2 + t \geq \bar{c}_2(\mathcal{A} \setminus L) + 1 \geq 1.
\]

\( \square \)

Proposition 2.4. (see [So84, Theorem (5.1)]) Let \( \mathcal{A} \) be an arrangement of \( d \) lines such that \( t_2 = 0 \). Then,

\[
\frac{2d - 6}{d - 2} \leq \frac{\bar{c}_2}{\bar{c}_1} \leq 3.
\]

Left equality holds if and only if \( t_2 = \binom{d}{2} \) (i.e. the arrangement has only nodes), and right equality holds if and only if \( \sum_{m \geq 2} t_m = d \) (and so \( \mathcal{A} \) is a finite projective plane arrangement).

Proof. The left inequality is equivalent to

\[
0 \leq (d - 2)\bar{c}_2^2 - (2d - 6)\bar{c}_2 = \sum_{m \geq 2} t_m(-m^2 + m(1 + d) + (2 - 2d))
\]

but \( -m^2 + m(1 + d) + (2 - 2d) \geq 0 \) for all \( 2 \leq m \leq d - 1 \). Moreover we have \( -m^2 + m(1 + d) + (2 - 2d) > 0 \) for all \( 3 \leq m \leq d - 2 \). But recall that by hypothesis \( t_2 = t_{d - 1} = 0 \). This proves the first inequality.

The second inequality is equivalent to showing that

\[
\bar{c}_2^2 - 3\bar{c}_2 = d - \sum_{m \geq 2} t_m \leq 0,
\]

but this follows from Theorem 1.7.

\( \square \)

All statements about Chern numbers so far have been proved combinatorially, independent of the ground field \( k \). The next theorem shows constrains on these numbers for \( k = \mathbb{R} \) and \( k = \mathbb{C} \).

Theorem 2.5. Let \( \mathcal{A} \) be an arrangement of \( d \) lines with \( t_d = t_{d-1} = 0 \).

1) If \( k = \mathbb{R} \), then \( \bar{c}_2^2 \leq \frac{5}{3}\bar{c}_2 \). Equality is achieved if and only if \( \mathcal{A} \) is simplicial (see Example 1.4).

2) If \( k = \mathbb{C} \), then \( \bar{c}_2^2 \leq \frac{8}{3}\bar{c}_2 \). Equality is achieved if and only if \( \mathcal{A} \) is the dual Hesse arrangement (see Example 1.5).

Proof. We follow [Hirz83, p.115] for the proof of 1). As we noted in Example 1.4 a real arrangement partitions \( \mathbb{P}^2 \) in polygons. This can be used to compute the topological Euler characteristic of \( \mathbb{P}^2 \), which is equal to 1. With that one obtains

\[
\sum_{m \geq 3} (m - 3)p_m = -3 - \sum_{m \geq 2} (m - 3)t_m.
\]

5
where \( p_m \) is the number of \( m \)-gons. On the other hand, one can check that
\[
5 \bar{c}_2 - 2 \bar{c}_1^2 = -3 - \sum_{m \geq 2} (m - 3) t_m,
\]
and so we get what we want for 1).

The claim in 2) is essentially the Hirzebruch-Sakai inequality \[\text{Hirz83}\], which comes form the Bogomolov-Miyaoka-Yau inequality for algebraic surfaces. See \[\text{Hirz83}, \text{So84, Theorem 5.3}, \text{U08, Proposition II.8}\] for details.

\[\square\]

3. Density of Chern slopes

Analogous to the geography problem for surfaces of general type (cf. \[\text{PS71}\]), we can talk about the geography problem for line arrangements over a fixed field \( k \): Given \((a, b) \in \mathbb{Z}^2\), is there a line arrangement over \( k \) with \( \bar{c}_1^2 = a \) and \( \bar{c}_2 = b \)? From now on, we will restrict all line arrangements of \( d \) lines to satisfy \( t_d = t_{d-1} = 0 \). We recall that for every \( k \)
\[
2 - \frac{2}{d - 2} \leq \frac{\bar{c}_1^2}{\bar{c}_2} \leq 3
\]
by Proposition 2.4. The geography problem for line arrangements could be hard to solve in general. A slightly easier variant of the geography problem is to ask: What positive rational numbers can appear as the quotient \( \frac{\bar{c}_1^2}{\bar{c}_2} \) of a line arrangement? Our focus in this section is to obtain constraints for the possible values of the Chern slope \( \frac{\bar{c}_1^2}{\bar{c}_2} \) for a fixed \( k \). For example, we have already seen that \( \frac{\bar{c}_1^2}{\bar{c}_2} = 3 \) can only be realised by a finite projective plane arrangement, or that over \( \mathbb{C} \) the only line arrangement satisfying \( \frac{\bar{c}_1^2}{\bar{c}_2} = \frac{8}{3} \) is the dual Hesse arrangement. Aside from the results we already have about specific values of the Chern slope, our goal now is to find all accumulation points of Chern slopes for a given field \( k \). We start with a simple corollary of Proposition 2.4.

**Corollary 3.1.** If \( r \) is an accumulation point of Chern slopes, then \( r \in [2, 3] \).

**Proof.** By Proposition 2.4 we know that \( 1 \leq \frac{\bar{c}_1^2}{\bar{c}_2} \leq 3 \). We note that after fixing the number of lines there are only finitely many different combinatorial arrangements, and so finitely many possible Chern slopes. Let \( s \) denote the Chern slope of some line arrangement, or that over \( \mathbb{C} \) the only line arrangement satisfying \( \frac{\bar{c}_1^2}{\bar{c}_2} = \frac{8}{3} \) is the dual Hesse arrangement. Aside from the results we already have about specific values of the Chern slope, our goal now is to find all accumulation points of Chern slopes for a given field \( k \). We start with a simple corollary of Proposition 2.4.

The following is inspired by the density lemma in \[\text{E15, Lemma 11.1}\].

**Lemma 3.2.** Let \( k \) be an infinite field. Let \( \mathcal{A}_n \) be a collection of arrangements of \( l(n) \) lines over a field \( k \) with \( \lim_{n \to \infty} \frac{\bar{c}_1^2}{\bar{c}_2} = c > 2 \) and \( \lim_{n \to \infty} l(n) = \infty \). Assume there is \( h \in [1, 2] \) such that \( \lim_{n \to \infty} \frac{\bar{c}_1^2}{l(n)^h} = a > 0 \). Then Chern slopes of line arrangements over \( k \) are dense in \( [2, c] \).
Proof. Let $x \in \mathbb{R}_{>0}$. We choose $n_0 > 0$ such that $d(n) = \lfloor xl(n)^{h-1} \rfloor$ are positive integers for all $n > n_0$, where $[y]$ is the integral part of $y$.

For $n > n_0$, we consider the arrangements of $d(n) + l(n)$ lines $\mathcal{A}_n^r$ over $k$ defined as $\mathcal{A}_n^r$ together with $d(n)$ general lines, this is, $d(n)$ lines which add only nodes and no other $m$-points to $\mathcal{A}_n^r$. Then

$$\frac{c_2^2(\mathcal{A}_n^r)}{c_2(\mathcal{A}_n^r)} = \frac{c_2^2(\mathcal{A}_n) + 2l(n)d(n) + d(n)^2 - 6d(n)}{c_2(\mathcal{A}_n) + l(n)d(n) + \frac{d(n)^2}{2} - \frac{3d(n)}{2}},$$

and so

$$\frac{c_2^2(\mathcal{A}_n^r)}{c_2(\mathcal{A}_n^r)} = \frac{c_2^2(\mathcal{A}_n) + 2\frac{d(n)}{l(n)^{h-1}} + \frac{d(n)^2}{l(n)^{h-1}} - \frac{6d(n)}{l(n)^h}}{c_2(\mathcal{A}_n) + \frac{d(n)}{l(n)^{h-1}} + \frac{d(n)^2}{2l(n)^h} - \frac{3d(n)}{2l(n)^h}}.$$

Then, if $h < 2$, we have $\lim_{n \to \infty} \frac{c_2^2(\mathcal{A}_n^r)}{c_2(\mathcal{A}_n^r)} = \frac{a+2x+x^2}{\frac{a}{x} + \frac{a}{x} + \frac{a}{x}} = f(x)$, and if $h = 2$, we get $\lim_{n \to \infty} \frac{c_2^2(\mathcal{A}_n^r)}{c_2(\mathcal{A}_n^r)} = \frac{a+2x+x^2}{\frac{a}{x} + \frac{a}{x} + \frac{a}{x}} = g(x)$. We note that both real functions $f(x)$ and $g(x)$ are continuous in $\mathbb{R}_{>0}$, and their range is $]2, c[.$

Proposition 3.3. Any $r \in [2, 3]$ is an accumulation point of Chern slopes of arrangements over $\mathbb{P}_p$.

Proof. We apply Lemma 3.2 for the collection $\mathcal{A}_n$ of finite projective plane arrangements (Example 1.6) given by $\mathbb{P}_2^{\mathbb{Z}}$, where $l(n) = p^{2n} + p^n + 1$. Here $c = 3$ and we use $h = \frac{3}{2}$.

Proposition 3.4. Any $r \in [2, \frac{5}{2}]$ is an accumulation point of Chern slopes of arrangements over $\mathbb{R}$.

Proof. We apply Lemma 3.2 for the collection $\mathcal{A}_n$ of arrangements of $2n$ lines given by regular polygons of $n$ sides (see Example 1.4). Here $c = \frac{5}{2}$ and we take $h = 2$.

We note that the simplicial arrangements given by regular polygons are not defined over $\mathbb{Q}$ in general. This is because all realizations are projectively equivalent strictly over $\mathbb{R}$, and for $n > 6$ the regular $n$-gon is not defined by lines over $\mathbb{Q}$. See [Cun11] Theorem 3.6 for details.

With respect to Chern slopes, the highest family for line arrangements defined over $\mathbb{Q}$, we can produce the following:

Example 3.5. For any $n \geq 3$, consider the lines $\{y = \alpha z/2\} , \{x = \alpha z/2\}, \{y = x + (\beta - n + 1)z\}, \{y = -x + (\beta + 1)z\}$, with $\alpha$ and $\beta$ sweeping all non negative integers up to $2n$ and $2n - 2$, respectively. This is an $n$ by $n$ array of “right triangle arrangements” of $8n$ lines. We note that when “$n = \infty$” we get an infinite simplicial arrangement in $\mathbb{R}^2$ with only right isosceles triangles. For a fixed $n$, it has $t_2 = 6n^2 + 6n - 8$, $t_3 = 2n^2 - 6n + 8$, $t_4 = 2n^2 + 2n - 3$, $t_{2n-1} = 2$, $t_{2n+1} = 2$, $t_m = 0$ else. Hence its Chern Slope is equal to

$$\frac{38n^2 - 18n - 7}{16n^2 - 8n - 2},$$

which converges to 2.375 as $n$ tends to infinity.

By Lemma 3.2, Example 3.5 implies that over $\mathbb{Q}$, any $r \in [2, 2.375]$ is an accumulation point of Chern Slopes.
Conjecture 3.6. The set of accumulation points of Chern slopes of arrangements over $\mathbb{Q}$ is $[2, \frac{5}{2}]$.

As explained in [Hirz83, Sect.(1.2)], one way of obtaining line arrangements in $\mathbb{P}_2^C$ is through finite reflection groups. In [ShTo54], the authors classified all finite unitary reflection groups. These are divided into three families and 34 exceptional groups. Hirzebruch uses this classification to show some line arrangements obtained from these groups which cannot be realised over $\mathbb{R}$. A complete study can be found in [OrSo83]. Using the notation of [ShTo54], the finite unitary groups which define line arrangements non-realisable over $\mathbb{R}$ are found in the family $G(m, p, n)$ (which contains the Ceva family) and in the exceptional groups $G_{24}, G_{25}, G_{26}$ and $G_{27}$. One thing to observe out of all this is that none of the examples coming from finite unitary reflection groups give us a family of line arrangements with Chern slope converging to something bigger that $\frac{5}{2}$. In fact, we have not found any family with that property. This should be seen as evidence for the following conjecture.

Conjecture 3.7. The set of accumulation points of Chern slopes of arrangements over $\mathbb{C}$ is $[2, \frac{5}{2}]$.

We point out that in order to understand the behaviour of the Chern slope of complex line arrangements, it suffices to understand those arrangements defined over $\mathbb{Q}$. This is because for every complex line arrangement, there exists a line arrangement defined over $\mathbb{Q}$ which has the same combinatorial data. Indeed, given a complex line arrangement, we can write down an affine system of polynomial equations defined over $\mathbb{Q}$ describing the way in which the lines intersect. For this, regard the coefficients of the lines as variables, and 3 by 3 determinants equal to zero as equations declaring concurrence of 3 lines. But we also need to say that some lines do not concur. For that we introduce extra variables to multiply these determinants, so that we impose that these multiplications are equal to 1. To avoid homogeneous issues, we declare from the beginning that the lines do not contain $[1, 0, 0]$. Thus we obtain a finite set of polynomials defined over $\mathbb{Q}$ describing the combinatorial data of the arrangement. If this system has a solution over $\mathbb{C}$, then it has a solution in $\mathbb{Q}$ by Hilbert’s Nullstellensatz.

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