Semileptonic decays of heavy mesons

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Abstract

The semileptonic and leptonic decays of heavy mesons are studied as a phenomenological application and exploration of a heavy-quark limit of Dyson-Schwinger equations. The single form factor, $\xi(w)$, which characterises the semileptonic decay in this limit, is calculated and compares well with recent experimental extractions. We obtain a lower bound of $1/3$ on the slope-parameter $\rho^2 \equiv -\xi'(1)$, which, in calculations with realistic input, is exceeded by a great deal: agreement with experimental data requiring $\rho^2 \sim 1.2 - 1.6$. The flavour and momentum dependence of the light-quark propagators has observable consequences.

Key words: Electroweak interactions; Semileptonic decays $B \to D(D^*)\ell\nu$; Dyson-Schwinger equations; Confinement; Nonperturbative QCD; Quark models.
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1. Introduction.

Semileptonic decays of pseudoscalar mesons provide a means of measuring elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix; fundamental parameters of the standard model. For example: the $K_{e3}$ decay, $K \to \pi \nu\ell$, can be used to determine $|V_{us}|$; $D_{e3}$ to determine $|V_{cs}|$; and semileptonic decays of $B$ mesons can provide information about $|V_{cb}|$ and $|V_{ub}|$. With a single hadron in both the initial and final state, these processes are an ideal tool for studying the influence of nonperturbative, strong interaction dynamics on weak interaction processes. An accurate determination of the CKM matrix elements relies on developing a good understanding of these nonperturbative effects.
The simplest of these processes to study are those with a pseudoscalar meson in both the initial and final state. In this case the interaction is only sensitive to the vector part of the electroweak \((V-A)\) interaction and, of the two form factors, denoted \(f_\pm(t)\), \(f_+(t)\) is dominant for \(e\) and \(\mu\) final states, for which the leptonic current is approximately conserved. In theoretical studies of these decays of light pseudoscalar mesons, \(\pi\) and \(K\), nonperturbative dressing of the quark-W-boson vertex is important, with nonanalytic contributions associated with, for example, \(K-\pi\) loops, being significant for \(t \gtrsim 0\) \cite{1}. For heavy mesons, e.g. \(D\) and \(B\), such nonanalytic contributions are unimportant in the physical region: \(0 \lesssim t \leq (m_B - m_D)^2\). However, this does not automatically mean that one can use the bare vertex, \(\gamma_\mu\), because these are not the only dressing effects induced by the strong interaction.

The propagator for a dressed quark of flavour \(f\) can be written\(^1\)

\[
S_f(p) = -i\gamma \cdot p \sigma^f_V(p^2) + \sigma^f_S(p^2) = \frac{Z_f(p^2)}{i\gamma \cdot p + M_f(p^2)}.
\]  

The characteristic behaviour of \(S_f(p)\) in QCD can be estimated by studying the quark Dyson-Schwinger equation (DSE) \cite{2} and is illustrated in Fig. 1. As typical of these calculations, the different quark flavours are specified by the value of the current-quark mass at the renormalisation point, \(\mu \simeq 10\) GeV in this heuristic illustration using the simple model of Ref. \cite{3}, with (in MeV): \(m_u(\mu) \approx 1; m_s(\mu) \approx 25; m_c(\mu) \approx 300;\) and \(m_b(\mu) \approx 3000\). The intersection of the dotted, diagonal line with a given curve marks the solution of \(p = M_f(p)\) and defines the Euclidean constituent quark mass, \(M_E^f\) \cite{2}. The ratio \(M_E^f/m_f\) is a single, indicative and quantitative measure of the nonperturbative effects of gluon-dressing on the quark propagator.

Obvious in this figure is a signal difference between light- and heavy-quarks: for light-quarks, \(M_E^f/m_f \gtrsim 20\), while, for heavy-quarks, \(M_E^f/m_f \lesssim 5\). This is a manifestation of the fact that the heavy-quark mass functions are well represented by a constant whereas those of light-quarks are not; the accuracy of this approximation improving with increasing current-quark mass. The same is true of \(Z_f(p^2)\); i.e., \(Z_f(p^2) \approx 1\). This feature provides the basis for a simplification of our study of semileptonic decays of heavy mesons: \textit{in the kinematic region explored by the decays} one can approximate the dressed heavy-quark

\[1\] We employ a Euclidean space formulation with \(\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \gamma^\dagger_\mu = \gamma_\mu\) and \(a \cdot b = \sum_{i=1}^4 a_i b_i\). A timelike vector, \(Q_\mu\), has \(Q^2 < 0\).

\[2\] Confinement entails that there is no real solution of \(p^2 + M_f(p^2) = 0\); i.e., no “pole-mass” for the quarks.
Fig. 1. The behaviour of $M_f(p^2)$: $u/d$-quark - solid line; $s$-quark - long-dashed line; $c$-quark - dashed line; and $b$-quark - dot-dash line, obtained using the model of Ref. [3]. See also Refs. [4,5]. The diagonal dotted line is $p = M(p)$.

propagator as

$$S_{f=c,b} \approx \frac{1}{i \gamma \cdot p + \hat{M}_f},$$  \tag{2}$$

where $\hat{M}_f \approx M_f^E$; i.e., in the first instance, one can ignore the momentum dependence of gluon dressing of the heavy-quark propagator. In the DSE framework, “heavy-quark symmetry” [6] finds its foundation in this.

Returning to the dressed quark-W-boson vertex, which describes the coupling of dressed-quarks to the W-boson, the vector piece satisfies the identity

$$Q_{\mu} i V_{\mu}^{f_1 f_2}(p; Q) = S_{f_1}^{-1}(p_+) - S_{f_2}^{-1}(p_-) - (m_{f_1} - m_{f_2}) \Gamma_i^{f_1 f_2}(p; Q),$$  \tag{3}$$

where $p_+ = p + \hat{\eta} Q$, $p_- = p - \hat{\eta} Q$, $\hat{\eta} = (1 - \eta)$, and $Q$ is the total momentum. Here $\eta$ is the momentum partitioning parameter that arises because of the arbitrariness in the definition of the relative momentum in a covariant formalism, and $\Gamma_i^{f_1 f_2}(p; Q)$ is the flavour-dependent scalar vertex. (In the absence of interactions $\Gamma_i^{f_1 f_2}(p; Q) = I_D$.) In studying the semileptonic decays of light mesons, the dressing of the vertex implied by this Ward-Takahashi identity was itself important, even neglecting the nonanalytic contributions mentioned in the introduction [1]. However, in the case of heavy-quarks, the
ability to neglect the momentum dependence of gluon dressing entails Eq. (2) and \((m_{f_1} - m_{f_2})I_{f_1 f_2}(p; Q) \approx (\tilde{M}_{f_1} - \tilde{M}_{f_2})I_D;\) and hence Eq. (3) is satisfied approximately by the bare vertex. This justifies the approximation

\[
V_{f_1 f_2}^f(p; Q) = \gamma_\mu ,
\]

amplifying the heavy-quark simplification in the study of these decays.

2. Semileptonic and leptonic decays.
Herein we study the impulse approximation to the semileptonic \(B^0 \to D^- \ell^+ \nu\) decay amplitude, defined by

\[
\langle D^-(k)|\bar{b}\gamma_\mu c|B^0(P)\rangle \equiv f_+(t)(K + P)_\mu - f_-(t)(K - P)_\mu
\]

\[
= N_c \int \frac{d^4\ell}{(2\pi)^4} tr_D \left[ \tilde{\Gamma}_{D^-}(\ell + \hat{n}P; -K)S_d(\ell + \hat{n}(P + K))\Gamma_{B^0}(\ell + \hat{n}K; P) \times S_b(\ell - \eta P + \hat{n}K)iV^{bc}_\mu (\ell + (1 - 2\eta)K; K - P)S(c(\ell + \hat{n}P - \eta K)) \right],
\]

where \(t = -(P - K)^2, \tilde{\Gamma}_{B^0,D^-}(k; P)^T = C^\dagger \Gamma_{B^0,D^-}(-k; P)C, C = \gamma_2\gamma_4\) is the charge conjugation matrix, and \(\Gamma_{B^0,D^-}\) are the meson Bethe-Salpeter amplitudes \[\]

\[
N_c \int \frac{d^4k}{(2\pi)^4} tr_D \left[ \tilde{\Gamma}_{B^0,D^-}(k; -P)\partial_\mu S_d(k + \hat{n}P)\Gamma_{B^0,D^-}(k; P)S_b/c(k - \eta P) \right. \\
+ \left. \tilde{\Gamma}_{B^0,D^-}(k; -P)S_d(k + \hat{n}P)\Gamma_{B^0,D^-}(k; P)\partial_\mu S_b/c(k - \eta P) \right] = 2P_\mu ,
\]

which is the consistent, canonical normalisation in impulse approximation [7].

Simultaneously we study the leptonic decay of a heavy, pseudoscalar meson, \(M\), which is described by one, dimensioned coupling, \(f_M\):

\[
\langle 0|\bar{f}_2\gamma_\mu \gamma_5 f_1|\Phi_M(P)\rangle \equiv f_M P_\mu = N_c \int \frac{d^4k}{(2\pi)^4} tr_D \left[ \gamma_5 \gamma_\mu S_{f_1}(k + \hat{n}P)\Gamma_{M}(k; P)S_{f_2}(k - \eta P) \right] .
\]

With this normalisation, \(f_\pi \simeq 131\text{ MeV}\). Equation (7) is exact but approximation enters in the calculation of the Schwinger functions that appear in it; i.e., the quark propagators and Bethe-Salpeter amplitude.

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2.1 Heavy-quark limit. At this point, no assumption has been made about the form of the Schwinger functions in Eqs. (5) and (7). In a study of light-meson decays, these functions were determined from model DSEs, and the integrals then evaluated [1]. For heavy mesons, where the application of DSEs has not hitherto been extensive, we will explore the consequences of Eqs. (2) and (4), in an estimation of the behaviour of the form factors and decay constants: an exploration of a heavy-quark limit in our framework.

To be explicit, we set \( \eta = 1 \) in Eqs. (5) and (7) (recall \( \hat{\eta} = 1 - \eta \)), so that the heavy-quark carries all the momentum of the heavy meson, in which case the Bethe-Salpeter amplitude constrains the momentum of the light-quark in the meson. We write the heavy-meson momentum as \( P = (\hat{M}_f + E)v \), where \( E \equiv M_H - \hat{M}_f \), with \( M_H \) the mass of the heavy meson, and \( v \) is a timelike unit-vector, \( v^2 = -1 \). Equation (2) then yields

\[
S_{fQ}(p - P) = \frac{-1}{2} \frac{1 + i\gamma \cdot v}{p \cdot v + E} + O \left( \frac{|p|}{M_f} \frac{E}{M_f} \right).
\]

(8)

The bound state amplitudes, present in the integrands that arise in the calculation of physical processes, ensure that \( |p|/\hat{M}_f \ll 1 \).

Using Eq. (2), or (8), the dressed quark-W-boson vertex is given by Eq. (4). For the heavy-meson Bethe-Salpeter amplitude we employ the Ansatz

\[
\Gamma_{B^0,D^-}(k; P) = \gamma_5 \left( 1 - \frac{1}{2} i\gamma \cdot v \right) \frac{\varphi(k^2)}{N_{B^0,D^-}},
\]

(9)

which is motivated by the Bethe-Salpeter equation studies of Ref. [8]. This function describes the momentum distribution of the light-quark, which we assume to be the same in both \( B \) and \( D \) mesons. Our estimate of the leptonic decay constants and form factors will be robust if the results are insensitive to the form of \( \varphi(k^2) \).

Using Eqs. (4), (8) and (9), the normalisation condition, Eq. (6), can be written

\[
N^2_H = \frac{1}{M_H} \frac{N_c}{32\pi^2} \int_0^\infty du \varphi(z)^2 \left( \sigma^f_S(z) + \sqrt{u} \sigma^f_V(z) \right) \equiv \frac{1}{M_H \kappa^2_f},
\]

(10)

where \( z = u - 2E\sqrt{u} \) and \( H = B^0, D^-, \ldots \), etc., with the light-quark flavour, \( f \), chosen appropriately. Clearly \( N^2_{H^+_f} = \text{constant} = N^2_{H^-_f} = N^2_{H^+_z} M_{H^+_z} \). Similarly,
we find for a heavy-meson leptonic decay constant:

\[ f_H = \frac{\kappa_f N_c}{\sqrt{M_H}} \frac{1}{8\pi^2} \int_0^\infty du \left( \sqrt{u} - E \right) \varphi(z) \left( \sigma^f_S(z) + \frac{1}{2} \sqrt{u} \sigma^f_V(z) \right), \]  

(11)

which entails that

\[ f_H \sqrt{M_H} = \text{constant}, \]  

(12)

except for deviations due to differences between \( u/d \)- and \( s \)-quark propagation characteristics.

Considering Eq. (5), we find, at leading order in \( 1/M_H \),

\[ f_\pm(t) = \frac{1}{2} \frac{M_D \pm M_B}{\sqrt{M_D M_B}} \xi(w), \]  

(13)

\[ \xi(w) = \kappa_d^2 \frac{N_c}{32\pi^2} \int_0^1 d\tau \frac{1}{W} \int_0^\infty du \varphi(z_W)^2 \left( \sigma^d_S(z_W) + \sqrt{u/W} \sigma^d_V(z_W) \right), \]  

(14)

with \( W = 1 + 2\tau(1 - \tau)(w - 1), z_W = u - 2E\sqrt{u/W} \) and \( \xi(w) \)

\[ w = \frac{M_B^2 + M_D^2 - t}{2M_BM_D} = v_B \cdot v_D. \]  

(15)

The canonical normalisation of the Bethe-Salpeter amplitude automatically ensures that \( \xi(w = 1) = 1 \). Equation (13) is an example of a general result that, in the “heavy-quark limit” the semileptonic decays of heavy mesons are described by a single, universal function [6].

Equation (14) yields the result: \( \rho^2 \geq \rho^2_{\text{min}}, \) where\(^4\)

\[ \rho^2_{\text{min}} \equiv -\frac{d\xi(w)}{dw} \bigg|_{w=1}^{E=0} = \frac{1}{3} \left( 1 + \frac{1}{2} \int_0^\infty du \varphi(u)^2 \sqrt{u} \sigma^f_V(u) \right), \]  

(16)

\(^4\) The minimum physical value of \( w \) is \( w_{\text{min}} = 1 \), which corresponds to maximum momentum transfer with the final state meson at rest; the maximum value is \( w_{\text{max}} \approx (M_B^2 + M_D^2)/(2M_BM_D) = 1.6 \), which corresponds to maximum recoil of the final state meson with the charged lepton at rest.

\(^5\) For \( \varphi(z) \) and \( \sigma^f_{V/S}(z) \) non-negative, non-increasing, convex-up functions of their argument, which includes \( \varphi = \text{constant} \) and a free-particle propagator, \( -\xi'_E(w) \) takes its minimum value at \( E = 0 \).
which entails an upper bound on the slope of $\xi (w): \rho^2 \gtrsim \frac{1}{3}$, independent of the details of our model and consistent with Ref. [9].

2.1 Light-quark propagator. The leptonic decay constant and $\xi (w)$ depend on the light-quark propagator and the momentum distribution of the light-quark in the heavy meson, described by $\varphi (k^2)$. Many observables involving hadrons composed of light-quarks have been studied using the $u/d$- and $s$-quark propagators specified by

$$
\bar{\sigma}_S^f (x) = 2\bar{m}_f \mathcal{F}(2(x + \bar{m}_f^2)) + \mathcal{F}(b_1 x) \mathcal{F}(b_3 x) \left( b_0^f + b_2^f \varphi (\varepsilon x) \right),
$$

$$
\bar{\sigma}_V^f (x) = \frac{2(x + \bar{m}_f^2) - 1 + e^{-2(x+\bar{m}_f^2)}}{2(x + \bar{m}_f^2)^2},
$$

where $\mathcal{F}(y) \equiv (1 - e^{-y})/y$, $x = p^2/(2D)$ and: $\bar{\sigma}_V^f (x) = 2D \sigma_V^f (p^2)$; $\bar{\sigma}_S^f (x) = \sqrt{2D} \sigma_S^f (p^2)$; $\bar{m}_f = m_f / \sqrt{2D}$, with $D$ a mass scale. This form is motivated by extensive studies of the DSE for the dressed-quark propagator [2] and combines the effects of confinement and dynamical chiral symmetry breaking with free-particle behaviour at large spacelike-$p^2$; i.e., asymptotic freedom.

The parameters $\bar{m}_f, b_{1,3}^f$ in Eqs. (17), (18) were determined in a $\chi^2$-fit to a range of light-hadron observables, which is described in Ref. [10] and leads to the values in Table 1. In the fit, the difference between the $u$- and $d$-quarks was neglected and only that minimal difference between $u$- and $s$-quarks allowed that was necessary to ensure: $\langle \bar{s}s \rangle < \langle \bar{u}u \rangle$; and $m_s/m_u \gg 1$. ($\epsilon = 10^{-4}$ is included in Eqs. (17), (18) only for the purpose of separating the small- and intermediate-$p^2$ behaviour of the algebraic form, characterised by $b_0$ and $b_2$; a separation in magnitude observed in numerical studies.) These light-quark propagators have since been used to successfully explore and predict a range of other hadronic observables [1,11]. We use them herein.

Table 1

|       | $\bar{m}_f$ | $b_0^f$ | $b_1^f$ | $b_2^f$ | $b_3^f$ |
|-------|-------------|---------|---------|---------|---------|
| $u$   | 0.00897     | 0.131   | 2.90    | 0.603   | 0.185   |
| $s$   | 0.224       | 0.105   | 2.90    | 0.740   | 0.185   |

3. Calculated heavy-meson observables.

In this study we have introduced one free parameter, $E \equiv M_H - \hat{M}_E^f$, which is a measure of the binding energy in a heavy meson, and an Ansatz for the Bethe-Salpeter amplitude, Eq. (9), which involves a single function, $\varphi (k^2)$. Our calculated form of $\xi (w)$, and the heavy-meson leptonic decay constants,
will depend on these quantities. Our predictions will be robust if they are independent of the details of our Ansatz for $\varphi(k^2)$.

To explore this we considered the following four one-parameter forms:

$$
\begin{align*}
\varphi_A(k^2) &= \exp\left(-\frac{k^2}{\Lambda^2}\right), \\
\varphi_B(k^2) &= \frac{\Lambda^2}{k^2 + \Lambda^2} \left(\frac{\Lambda^2}{k^2 + \Lambda^2}\right)^2 \theta \left(1 - \frac{k^2}{\Lambda^2}\right),
\end{align*}
$$

(19)

For a particular form, to fit the parameters $(E, \Lambda)$, we performed a $\chi^2$ fit to the following parametrisation of the experimental data [12]

$$
\xi(w) = \frac{2}{w + 1} \exp\left((1 - 2\rho^2)\frac{w - 1}{w + 1}\right), \quad \rho^2 = 1.53 \pm 0.36 \pm 0.14,
$$

(20)

to $f_D = 216 \pm 15$ MeV and $f_B = 206 \pm 30$ MeV, which, in the absence of experimental data, is our weighted average of the lattice-QCD results collected in Ref. [13]. Using $M_D = 1.87$ GeV, $M_{Ds} = 1.97$ GeV and $M_B = 5.27$ GeV, we obtain the results presented in Fig. 2 and Eq. (21), energies in GeV and $\rho^2$ dimensionless.

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
& E & \Lambda & f_D & f_{D*} & f_B & \rho^2 \\
A & 0.640 & 1.03 & 0.227 & 0.245 & 0.135 & 1.55 \\
B & 0.567 & 0.843 & 0.227 & 0.239 & 0.135 & 1.56 \\
C & 0.612 & 1.32 & 0.227 & 0.242 & 0.135 & 1.55 \\
D & 0.643 & 1.02 & 0.272 & 0.296 & 0.162 & 1.21 \\
\hline
\end{array}
$$

(21)

4. Observations and Conclusions.

One observes from Eq. (21) that, independent of $\varphi(k^2)$, a description of the data requires a “binding energy” in $B, D$ mesons of $E \simeq 0.6$ GeV, which is consistent with the values obtained in Bethe-Salpeter equation studies of these systems: $E \sim 0.4 - 0.6$ [5,14]. Further, the size-parameter $\Lambda \sim 1$ GeV; and comparing this with an analogous parametrisation of the pion Bethe-Salpeter amplitude shows $\Lambda \sim (1.5-2.0) \Lambda_\pi$ [15]. Hence, our Bethe-Salpeter amplitude represents the heavy meson as an object of small spacetime extent; occupying an intrinsic volume of only 5-20% that of the pion.

Globally, considering Eq. (21) and Fig. 2, the Ansätze: $A, B, C$, for the Bethe-Salpeter amplitude provide equivalent descriptions of all observables. The step-function, Ansatz $D$, which is qualitatively different in form and corresponds simply to the introduction of a cut-off in the integrals that yield observables,
Fig. 2. Our calculated form of $\xi(w)$ compared with experimental fits [12]. Experiment: dashed line, Eq. (20); short-dashed line, the linear fit $\xi(w) = 1 - \rho^2(w - 1)$, $\rho^2 = 0.91 \pm 0.15 \pm 0.06$. Calculation: solid line, $\xi(w)$ calculated with forms $A$, $B$ or $C$ in Eq. (19) using the parameters in Eq. (21), there is no discernible difference on the scale of this figure; dot-dash line, $\xi(w)$ calculated with form $D$. not surprisingly leads to a poorer description. The fact that even this form is not completely unreasonable is due to the presence of the light-quark propagators in our study, which describe the confinement of light-quarks in the heavy meson and restrict their propagation range. It is the combination of the light-quark propagators, determined in studies of light-hadron observables, and the heavy-meson Bethe-Salpeter amplitudes that prescribe the values of the observables; i.e., as observed in Ref. [16], these observables are strongly influenced by the nonperturbative characteristics of confinement, dynamical chiral symmetry breaking and bound state formation.

In our simple analysis, the calculated values of the leptonic decay constants are in quantitative agreement with the results in Ref. [5]. In particular, with the realistic Ansätze: $A, B, C$, we note that $f_{D_s}/f_D \approx 1.07$, whereas Eq. (12) suggests the result $f_{D_s}/f_D = \sqrt{M_D/M_{D_s}} = 0.97$. This difference, $\sim 10\%$, is an indication of the influence and magnitude of light-quark effects in heavy-meson observables. Further, we note that while our value of $f_D = 227$ MeV, agrees with that found in numerical simulations of lattice-QCD, our value of $f_B = 135$ MeV is $\sim 35\%$ lower, and there appears no way to reduce this discrepancy. Our quantitative agreement with Ref. [5] suggests that the value inferred from lattice simulations may be erroneous.

We obtain significant curvature in $\xi(w)$ on the physically accessible domain
in $B \rightarrow Dℓν$ decays: $1 < w \lesssim 1.6$; i.e., as illustrated in Fig. 2, a comparison of our calculated result with the linear fit of Ref. [12] is unfavourable. In the present study, no choice of our two parameters yields a straight line, and trying to fit the short-dashed line in Fig. 2 leads to a value of $\xi(w_{\text{max}}) \sim 20\%$ higher than that inferred from experiment. We note that using any of the three, realistic, one-parameter Ansätze for the Bethe-Salpeter amplitude, our calculated result is in exact agreement with the experimental fit to the data, Eq. (20). The agreement between our calculation and the other nonlinear fits described in Ref. [12]: \(\exp(-\rho^2(w - 1))\), \(\rho^2 = 1.27 \pm 0.29 \pm 0.12\); and \((2/(w + 1))^{2\rho^2}\), \(\rho^2 = 1.42 \pm 0.32 \pm 0.13\), is not quite as good, with the deviation reaching $\sim 5\%$ at $w_{\text{max}}$. From this we conclude that $\rho^2$ in the range $\sim 1.2 - 1.6$ is admissible, with $\rho^2 \simeq 1.55$ most likely.

Our calculation is not alone in indicating significant curvature in $\xi(w)$; there are qualitative similarities with the results of Refs. [16–18], for example. However, our calculation is unique in being able to encompass a value of $\xi(w_{\text{max}}) \lesssim 0.5$ with physically reasonable and internally consistent values of our two parameters, $E$ and $Λ$. The detailed, constrained description of the light-quark component of the heavy meson, via its propagator and its momentum distribution function (Bethe-Salpeter amplitude), is responsible for this.

A quantitative step in extending this study is a thorough analysis of the DSE for a heavy-quark, such as being explored in Ref. [19], which could provide the basis for an estimation of corrections to the “heavy-quark” limit expressed by Eqs. (2) and (8). One could then employ the calculated heavy-quark propagator in Bethe-Salpeter equation studies of heavy-meson bound states, perhaps in a manner analogous to Ref. [20]. In this way $E$ and $Λ$, which are fitting-parameters herein, would be correlated and their values fixed by the mass scale in the gluon propagator. This would lead to predictions of leptonic decay constants and semileptonic form factors that are less dependent on existing data.

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**References**

[1] Yu. L. Kalinovsky, K. L. Mitchell and C. D. Roberts, “$K_{e3}$ and $π_{e3}$ transition form factors”, nucl-th/9610047, to appear in Phys. Lett. B.

[2] C. D. Roberts and A. G. Williams, Prog. Part. Nucl. Phys. **33** (1994) 477.

[3] M. R. Frank and C. D. Roberts, Phys. Rev. C **53** (1996) 390.
[4] A. G. Williams, G. Krein and C. D. Roberts, Ann. Phys. 210 (1991) 464.
[5] H. J. Munczek and P. Jain, Phys. Rev. D 46 (1992) 438.
[6] N. Isgur and M. B. Wise, Phys. Lett. B 237 (1990) 527.
[7] C. D. Roberts, Nucl. Phys. A 605 (1996) 475.
[8] C. J. Burden, et al., “Ground-state spectrum of light-quark mesons”, nucl-th/9605027, to appear in Phys. Rev. C.
[9] C. G. Boyd, et al., Phys. Rev. D 55 (1997) 3027.
[10] C. J. Burden, C. D. Roberts and M. J. Thomson, Phys. Lett. B 371 (1996) 163.
[11] M. A. Pichowsky and T.-S. H. Lee, Phys. Lett. B 379 (1996) 1; “Exclusive diffractive processes and the quark substructure of mesons”, nucl-th/9612049, to appear in Phys. Rev. D.
[12] J.E. Duboscq et al., Phys. Rev. Lett. 76 (1996) 3899.
[13] J. D. Richman and P. R. Burchat, Rev. Mod. Phys. 67 (1995) 893.
[14] A. J. Sommerer, J. R. Spence and J. P. Vary, Phys. Rev. C 49 (1994) 513; A. A. El-Hady, et al., “$B \to D(D^*)$ form-factors in a Bethe-Salpeter model”, hep-ph/9605397.
[15] R. Alkofer, A. Bender and C. D. Roberts, Int. J. Mod. Phys. A 10 (1995) 3319.
[16] S. Simula, Phys. Lett. B 373 (1996) 193.
[17] M. A. Ivanov and T. Mitzutani, Few-Body Systems 20 (1996) 49.
[18] A. A. El-Hady, et al., Phys. Rev. D 51 (1995) 5245.
[19] C. J. Burden, “The analytic structure of heavy quark propagators”, hep-ph/9702411.
[20] C. J. Burden and D.-S. Liu, Phys. Rev. D 55 (1997) 367.