OCTONIONS, ALBERT VECTORS AND THE GROUP $E_6(F)$

JOHN N. BRAY, YEGOR STEPANOV, ROBERT A. WILSON

Abstract. We present a uniform approach to the construction of the groups of type $E_6$ over arbitrary fields without using Lie theory. This gives a simple description of the group generators and some of the subgroup structure. In the finite case our approach also permits relatively straightforward computation of the group order.

Contents

1. Introduction 1
2. Some properties of $\Omega_{2m}(F,Q)$ 2
3. Octonions 4
4. A basis for the split octonion algebra 6
5. Albert vectors 7
6. Some elements of $\text{SE}_6(F)$ 8
7. Action of $\text{SE}_6(F)$ on white points 11
8. White points in the case of a finite field 14
9. The stabiliser of a white point 15
10. Simplicity of $E_6(F)$ 22
11. Conclusions 24
References 24

1. Introduction

The construction of the groups of type $E_6$ goes back to the work of Dickson [7, 8]. He constructed the analog of the complex Lie group $E_6$ as a linear group in 27 variables which leaves a certain cubic form invariant. Jacobson, inspired by Dickson and by Chevalley’s Tôhoku paper [6], studied the automorphism group of an Albert algebra (this algebra consists of the $3 \times 3$ Hermitian matrices written over an octonion algebra), and the stabiliser of the determinant over the fields of characteristic not two or three in a series of papers [9, 10, 11]. For instance, he proved that if the Albert algebra contains nilpotent elements, then the group is simple. It must have been implicit that the determinant of the elements in the Albert algebra is essentially the same as Dickson’s cubic form, although Jacobson does not refer to Dickson. Moreover, although cases of characteristic 2 and 3 were of no problem to Dickson, they were still problematic in Jacobson’s construction. The series of papers by Aschbacher [1, 2, 3, 4, 5] also addresses the question of construction of the groups of type $E_6$ and $^2E_6$ without mentioning the Albert algebra or octonions at all. It is to be emphasised that Aschbacher’s construction is of a rather abstract nature and some of the structural aspects require further research. In a preprint by R. A. Wilson [14] the construction of finite simple groups $E_6(q)$, $F_4(q)$ and $^2E_6(q)$ is sketched.

In the late 1980s the problem of classifying the maximal subgroups came into prominence. Aschbacher’s study of the 27-dimensional module for $E_6$ reveals much
more structure rather than the standard 78-dimensional representation. However, Aschbacher does not give a complete list of maximal subgroups, which means there is still a need for a modern review of Dickson’s construction.

2. Some properties of $\Omega_{2m}(F, Q)$

Let $V$ be a vector space over a field $F$ of dimension $n$. We assume that there is a non-singular quadratic form $Q$ defined on $V$. Denote by $GO_n(F, Q)$ the group of non-singular linear transformations that preserve the form $Q$. In case of characteristic 2 we define the quasideterminant $qdet : GO_n(F, Q) \to F_2$ to be the map

$$1\text{mod 2}. $$

Further, the group $SO_n(F, Q)$ is the kernel of the (quasi)determinant map. Finally, the subgroup $\Omega_n(F, Q)$ of $SO_n(F, Q)$ is defined as the kernel of another invariant called the spinor norm. If the characteristic of the field is not 2, there exists a double cover of $\Omega_n(F, Q)$, denoted as $Spin_n(F, Q)$.

This section is devoted to some of the private life of the group $\Omega_{2m}(F, Q)$, which will be crucial in our further constructions. Consider the vector space $V$ of dimension $2m + 2$ over $F$ with a non-singular quadratic form $Q$ defined on it. Assuming that the Witt index of $Q$ is at least 1, we can pick the basis $B = \{v_1, w_1, ..., w_{2m}, v_2\}$ in $V$ such that $(v_1, v_2)$ is a hyperbolic pair. Consider the decomposition $V = \langle v_1 \rangle \oplus \langle w_1, ..., w_{2m} \rangle \oplus \langle v_2 \rangle$ and denote $W = \langle w_1, ..., w_{2m} \rangle$.

**Lemma 2.1.** The stabiliser in $\Omega_{2m+2}(F, Q)$ of the vector $v_1$ is a subgroup of shape $W : \Omega_{2m}(F, Q_W)$, and the stabiliser of the pair $(v_1, v_2)$ is a subgroup $\Omega_{2m}(F, Q_W)$.

Note that any element in $\Omega_{2m+2}(F, Q)$ which fixes $v_1$ also stabilises $(v_1)\perp$, so with respect to the basis $B$ it has the following matrix form:

$$\hat{A} \begin{pmatrix} 1 & 0 & 0 \\ u^T_2 & A & 0 \\ \mu & u_1 & \lambda \end{pmatrix},$$

where the matrix $A$ acts on $W$ as an element of $\Omega_{2m}(F, Q_W)$. Let $f$ be the polar form of $Q$. Then an element $\hat{A}$ in the stabiliser of $v_1$ acts on $v_2$ as

$$v_2 \mapsto (\mu \mid u_1 \mid \lambda).$$

The bilinear form $f$ is preserved, so we get

$$1 = f(v_1, v_2) = \mu f(v_1, v_1) + f(v_1, (0 \mid u_1 \mid 0)) + \lambda f(v_1, v_2) = \lambda,$

and hence $\lambda = 1$.

Since $(v_1, v_2)$ is a hyperbolic pair, the form $f$ on $V$ can be represented by the Gram matrix

$$[f]_B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & B & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
where $B$ is the Gram matrix of $f_W$, the restriction of $f$ on $W$. We explore the fact that an element in the stabiliser of $v_1$ preserves the form $f$:

\[
[f]_B = \hat{A} \cdot [f]_B \cdot \hat{A}^\top = \begin{pmatrix}
0 & 0 & 1 \\
0 & ABA^\top & ABu_1^\top + u_2^\top \\
1 & u_2 + u_1BA^\top & 2\mu + u_1Bu_1^\top
\end{pmatrix}.
\]

It follows that $u_2 = -u_1BA^\top$. We also have

\[
0 = Q(v_2) = Q(v_2\hat{A}) = Q((\mu \mid u_1 \mid 1)) = Q((\mu \mid u_1 \mid 0)) + Q(v_2) + f((\mu \mid u_1 \mid 0), v_2) = Q_W(u_1) + \mu,
\]

so $\mu = -Q_W(u_1)$. As a result, the general element $\hat{A}$ in the stabiliser of $v_1$ has the following form:

\[
\hat{A} = \begin{pmatrix}
1 & 0 & 0 \\
-AB^\top u_1^\top & A & 0 \\
-Q_W(u_1) & u_1 & 1
\end{pmatrix}.
\]

Witt’s lemma tells us that the group $GO_{2m}(F, Q_W)$ acts transitively on the non-zero vectors of each norm in $W$. In fact, the same is true for $\Omega_{2m}(F, Q_W)$ in case when $Q_W$ is of Witt index at least 1.

**Lemma 2.2.** The group $\Omega_{2m}(F, Q_W)$, where $Q_W$ is of Witt index at least 1, acts transitively on

\[
O_\lambda = \{ v \in W \mid Q_W(v) = \lambda, \ v \neq 0 \}
\]

for each value of $\lambda \in F$.

From now on we require that $Q_W$ has Witt index at least 2. The following technical results will be our main tool in the construction of certain orthogonal subgroups of $E_6(F)$.

**Theorem 2.3.** Let $Q_W$ be of Witt index at least 2. The subgroup $\Omega_{2m}(F, Q_W)$ is maximal in $W: \Omega_{2m}(F, Q_W)$.

**Proof.** Recall that $v_2 \in V$ is mapped under the action of $G = W: \Omega_{2m}(F, Q_W)$ to a vector of the form $(-Q_W(u) \mid u \mid 1)$, where $u$ is an element of $W$. Since the stabiliser of $v_2$ in $G$ is $\Omega_{2m}(F, Q_W)$, we conclude that the orbit of $v_2$ under the action of $G$ is the following set:

\[
O_G(v_2) = \left\{ (-Q_W(u) \mid u \mid 1) \mid u \in W \right\}.
\]

Since the elements of this orbit are in one-to-one correspondence with the cosets of $\Omega_{2m}(F, Q_W)$ in $G$, it is enough to show the primitive action on $O_G(v_2)$.

Consider the action of $G$ on $O_G(v_2)$. A general element in $G$ acts on the elements of $O_G(v_2)$ in the following way:

\[
(-Q_W(u) \mid u \mid 1) \mapsto (-Q_W(u) - uABv^\top - Q(v) \mid uA + v \mid 1) = (-Q_W(uA + v) \mid uA + v \mid 1).
\]

Note that $uABv^\top = f_W(uA, v)$. We see that this action is isomorphic to the action on $W$ defined by $u \mapsto uA + v$, where $u, v \in W$. In case when $A$ is the identity matrix, this map is a translation. On the other hand, taking $v = 0$, we obtain the action of $\Omega_{2m}(F, Q_W)$. Denote the group generated by the described action on $W$ as $A\Omega_{2m}(F, Q_W)$. 

\[
[\quad]
The action of $G = \Lambda\Omega_{2m}(F, Q_W)$ on $W$ is primitive if any $G$-congruence on the elements of $W$ is trivial, i.e. it is either equality or the universal relation. Now suppose $\sim$ is a non-trivial $G$-congruence. In particular, $\sim$ is not the equality relation, so we may assume that there are two different elements $v_1, v_2 \in W$ such that $v_1 \sim v_2$. Translating both $v_1$ and $v_2$ by $-v_2$ and using the fact that $G$ preserves $\sim$, we obtain $v_1 - v_2 \sim 0$. That is, $0 \sim v$ for some non-zero vector $v$. Denote by $O_\lambda$ the set

$$O_\lambda = \{ u \in W \mid u \neq 0, \ Q_W(u) = \lambda \}$$

and let $\lambda = Q_W(v)$. Note that the group $\Omega_{2m}(F, Q_W)$ acts transitively on $O_\lambda$ for all $\lambda \in F$, so from $0 \sim v$ we obtain $0 \sim vA$ for all $A \in \Omega_{2m}(F, Q_W)$, and hence $0 \sim v_\lambda$ for all $v_\lambda \in O_\lambda$.

Suppose $v \in W$ is such that $Q_W(v) = \lambda \neq 0$. Since the Witt index of $Q_W$ is at least 2, there exist two isotropic vectors $e_1, f_1 \in W$ spanning a totally isotropic subspace of dimension 2. Let $\lambda_1 = f_W(v, e_1)$, $\lambda_2 = f_W(v, f_1)$ and denote $u = \lambda_2 e_1 - \lambda_1 f_1$. Then $Q_W(u) = 0$ and so

$$Q_W(v + u) = Q_W(v) + f_W(v, u) = Q(v) + \lambda_2 f_W(v, e_1) - \lambda_1 f_W(v, f_1) = Q_W(v),$$

in other words, $f_W(v + u, v) = 0$. We readily see that $v + u \in O_\lambda$, and so $v \sim v + u$. Employing the translation by $-v$ we obtain $0 \sim u$. That is, 0 is congruent to some non-zero isotropic vector. Of course, this means that $0 \sim u_0$ for any $u_0 \in O_0$, since $\Omega_{2m}(F, Q_W)$ acts transitively on the isotropic vectors.

We say that two vectors $u, v \in W$ are in $\lambda$-relation if $u \sim v$ and $Q(u - v) = \lambda$. From the previous discussion we know that 0 is in 0-relation with some isotropic non-zero vector $u$. Choose $w \in W$ such that $(u, w)$ is a hyperbolic pair. Then for all non-zero $\xi \in F$ there is a chain $u \sim 0 \sim \xi w$. Indeed, since the action of $\Omega_{2m}(F, Q_W)$ is transitive on the elements of $O_0$, then $0 \sim u_0$ for all $u_0 \in O_0$ and so by transitivity $u_0 \sim u_0'$ for any two $u_0, u_0' \in O_0$. For any non-zero $\xi \in F$, $\xi w$ is a non-zero isotropic vector, hence the result. Since $u \sim \xi w$, we can translate both sides by $-\xi w$ to get $u - \xi w \sim 0$, where $Q_W(u - \xi w) = Q_W(u) + \xi^2 Q_W(w) - \xi f_W(u, w) = -\xi$. Therefore, having the 0-relation it is possible to obtain a $\xi$-relation for any $\xi \neq 0$ in $F$. Finally, using the transitivity of $\Omega_{2m}(F, Q_W)$ on every $O_\xi$, we obtain that $\sim$ is the universal congruence, and so the action of $\Lambda\Omega_{2m}(F, Q_W)$ on $W$ is primitive. \[\square\]

As we already know from Lemma 2.1, the stabiliser in $\Omega_{2m+2}(F, Q)$ of an isotropic vector $v_1 \in V$ is a subgroup of shape $W: \Omega(F, Q_W)$. We also note that every subgroup of $\Omega_{2m+2}(F, Q)$ containing $W: \Omega_{2m}(F, Q_W)$ as a subgroup, stabilises the 1-space spanned by $v_1$.

**Theorem 2.4.** Let $Q_W$ be of Witt index at least 2. Any subgroup $H$ such that

$$(9) \quad W: \Omega_{2m}(F, Q_W) \leq H < \Omega_{2m+2}(F, Q),$$

stabilises the 1-space $\langle v_1 \rangle$.

3. **Octonions**

In this section we discuss the definition and some properties of octonion algebras. Our usual point of reference will be [13].

**Definition 3.1.** Let $F$ be any field. An octonion algebra $\mathbb{O} = \mathbb{O}_F$ is an 8-dimensional composition algebra, i.e. it admits a norm defined as a quadratic form $N: \mathbb{O} \to F$ such that the polar form of $N$ is non-degenerate and $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{O}$.

The multiplicative identity in $\mathbb{O}$ is denoted as $1_\mathbb{O}$, and throughout the paper we sometimes omit the subscript. Denote the polar form of $N$ by $\langle \cdot, \cdot \rangle$ and define the **trace** of an octonion via

$$(10) \quad T(x) = \langle x, 1_\mathbb{O} \rangle.$$
where $1_{\mathcal{O}}$ is the multiplicative identity in $\mathcal{O}$. Octonion algebras are quadratic: an arbitrary element $x \in \mathcal{O}$ satisfies the equation
\begin{equation}
 x^2 - T(x) \cdot x + N(x) \cdot 1_{\mathcal{O}} = 0.
\end{equation}

The conjugation in $\mathcal{O}$ is the mapping $\overline{\cdot} : \mathcal{O} \to \mathcal{O}$ defined by
\begin{equation}
 \overline{x} = T(x) \cdot 1_{\mathcal{O}} - x.
\end{equation}

We call $\overline{x}$ the conjugate of $x$. The following lemma summarises the properties of $\mathcal{O}$ related to octonion conjugation.

**Lemma 3.2.** For all $x, y \in \mathcal{O}$ the following identities hold:
\begin{enumerate}
  \item $(i)$ $xx = x \overline{x} = N(x) \cdot 1_{\mathcal{O}},$
  \item $(ii)$ $\overline{xy} = y \overline{x},$
  \item $(iii)$ $\overline{x} = x,$
  \item $(iv)$ $x + y = \overline{x} + \overline{y},$
  \item $(v)$ $N(x) = N(\overline{x}),$
  \item $(vi)$ $\langle x, y \rangle = \langle \overline{x}, y \rangle.$
\end{enumerate}

Furthermore, we have the following important properties.

**Lemma 3.3.** For all $x, y, z \in \mathcal{O}$ the following identities hold:
\begin{enumerate}
  \item $(i)$ $x(\overline{yz}) = N(x)y,$
  \item $(ii)$ $(\overline{xy})y = N(y)x,$
  \item $(iii)$ $x(\overline{yz}) + y(\overline{xz}) = \langle x, y \rangle \cdot z,$
  \item $(iv)$ $(\overline{xy})z + (\overline{xz})y = \langle y, z \rangle \cdot x.$
\end{enumerate}

Octonion algebras are alternative, which means that the octonion multiplication satisfies the following laws.

**Lemma 3.4.** For all $x, y \in \mathcal{O}$ the following are true:
\begin{enumerate}
  \item $(i)$ $(xx)y = x(xy),$
  \item $(ii)$ $(yx)x = y(xx),$
  \item $(iii)$ $(xy)x = x(yx).$
\end{enumerate}

Finally, we must emphasise the most important property of the octonionic multiplication. The following theorem is the special case of Theorem 1.6.2 in [13].

**Theorem 3.5.** Any 8-dimensional composition algebra is neither associative nor commutative.

While the last theorem suggests that the calculations involving the elements of $\mathcal{O}$ can be quite tedious and uncomfortable, in some cases the following lemmata can make our life easier.

**Lemma 3.6.** If $x, y, z \in \mathcal{O}$, then $T(xy) = T(yx)$ and $T(x(yz)) = T((xy)z)$.

Note that although we have 3-associativity for the trace in general, we cannot derive the generalised associativity in this case. Although there is no associativity in general, in some cases the so-called Moufang law can help us with the bracketing.

**Lemma 3.7.** For all $x, y, z \in \mathcal{O}$, the following identities hold:
\begin{align}
 x(yz)x &= (xy)(xz), \\
 x(yz)y &= ((xy)z)y, \\
 (xy)z &= x(y(xz)).
\end{align}

We call an octonion $x \in \mathcal{O}$ invertible if $N(x) \neq 0$. In this case $x^{-1} = N(x)^{-1} \overline{x}$. If there exists an isotropic octonion in $\mathcal{O}$, i.e. an element $x \in \mathcal{O}$ such that $N(x) = 0$, then we call $\mathcal{O}$ a split octonion algebra. As a special case of Theorem 1.8.1 in [13], we get the following.
Theorem 3.8. Over any given field $F$ there is a unique, up to isomorphism, split composition algebra.

It turns out that any isotropic octonion left- and right-annihilates a 4-dimensional subspace of split octonion algebra $O$.

Lemma 3.9. Let $O$ be a split octonion algebra. Then for any isotropic $x \in O$, the following is true:

\[(14) \dim_F(Ox) = \dim_F(xO) = 4.\]

Moreover, $Ox$ is the set of octonions that are right-annihilated by $x$, and $xO$ is the set of octonions that are left-annihilated by $x$.

Proof. We prove the statement for the right multiplication by $x$. The proof for the left multiplication is essentially the same. The map $R_x: O \to O \quad y \mapsto xy$ is an $F$-linear map with $\text{Im}(R_x) = Ox$, which is a totally isotropic subspace of $O$.

Indeed, $(y(xy))(x) = y(xf)g = 0$ for any $y \in O$. Since $N$ is non-singular and its polar form is non-degenerate, we conclude that $\dim_F(Ox) \leq 4$.

If $x \neq 0$ and $yx = 0$, then $y$ is isotropic for if that were not the case, we would get $x = y^{-1}(yx) = y^{-1} \cdot 0 = 0$, a contradiction. It follows that $\dim_F(\ker(R_x)) \leq 4$.

The Rank–Nullity theorem implies that $\dim_F(Ox) = \dim_F(\ker(R_x)) = 4$. $\square$

Finally, we determine the centre of $O$ and also the elements in an octonion algebra that ‘associate’ with all other elements. By the centre of an octonio n algebra $O$ we understand $\{c \in O \mid cx = xc \text{ for all } x \in O\}$. In the literature, for example, in [16], it is sometimes required that central elements also “associate” with all other elements. We do not require this in our definition, however, it will become obvious that we have this property free of charge.

Lemma 3.10. The centre of an octonion algebra $O = O_F$ is $F \cdot 1_O$.

Lemma 3.11. If $u \in O$ satisfies

\[(15) (xu)y = x(uy)\]

for all $x, y \in O$, then $u \in F \cdot 1_O$.

Corollary 3.12. Suppose that $u \in O$ is an invertible octonion. Then

\[(16) (AB)(uB) = N(u)AB\]

holds for all $A, B \in O$ if and only if $u \in F \cdot 1_O$.

Remark. The statement of this corollary holds even if $u$ is not invertible, but this requires a different, more hands-on proof.

In the subsequent constructions we consider certain subalgebras of $O$. We say that an $F$-subalgebra $S$ of $O$ is sociable if $S$ contains $F \cdot 1_O$ and for all $x, y \in S$ and for all $z \in O$ we have $(xy)z = x(yz)$.

4. A BASIS FOR THE SPLIT OCTONION ALGEBRA

Theorem 3.8 allows us to choose a basis for $O$ and use it in our further constructions. Otherwise speaking, we redefine the split octonion algebra $O$ over $F$ in the following way.

Definition 4.1. If $F$ is any field, then the split octonion algebra over $F$ is defined as an 8-dimensional vector space $O = O_F$ with basis $\{e_i \mid i \in \pm I\}$, where $I = \{0, 1, \omega, \overline{\omega}\}$, $\pm I = \{\pm 0, \pm 1, \pm \omega, \pm \overline{\omega}\}$ and bilinear multiplication given by the following table.
In other words, we get

(i) \( e_1 e_\omega = -e_\omega e_1 = e_\omega \);

(ii) \( e_1 e_0 = -e_0 e_1 = e_1 \);

(iii) \( e_{-1} e_1 = -e_0 \) and \( e_0 e_0 = e_0 \);

and images under negating all subscripts (including 0), and multiplying all subscripts by \( \omega \), where \( \omega^2 = \overline{\omega} = 1 \). All other products of basis vectors are 0. Essentially, this is the same basis as given in section 4.3.4 of [15]. Thus, \( e_0 \) and \( e_{-0} \) are orthogonal idempotents with \( e_0 + e_{-0} = 1_\mathcal{O} \). Now, if \( x = \sum_{i \in \mathcal{I}} \lambda_i e_i \), then the trace and the norm can be defined in the following way:

\[
\begin{align*}
T(x) &= \lambda_0 + \lambda_{-0}, \\
N(x) &= \lambda_{-1} \lambda_1 + \lambda_{-\omega} \lambda_{-\overline{\omega}} + \lambda_\omega \lambda_{-\overline{\omega}} + \lambda_{-\omega} \lambda_{-\overline{\omega}} + \lambda_0 \lambda_{-0}.
\end{align*}
\]

Note that whenever we obtain an octonion which is a scalar multiple of \( 1_\mathcal{O} \), we understand it as a field element. The fact that our newly defined algebra \( \mathcal{O} \) is indeed a composition algebra can be verified by a tedious but straightforward computation. Note that \( N(e_i) = 0 \) for \( i \neq \pm 0 \), so \( \mathcal{O} \) is indeed a split octonion algebra.

The involution \( x \mapsto \mathcal{I} \) is the extension by linearity of

\[
e_i \mapsto -e_i \ (i \neq \pm 0), \ e_0 \leftrightarrow e_{-0}.
\]

5. Albert vectors

For further discussion we consider \( \mathcal{O} = \mathcal{O}_F \) to be an octonion algebra over the field \( F \). The Albert space \( \mathcal{J} = \mathcal{J}_F \) is the 27-dimensional vector space spanned by the elements of the form

\[
(a, b, c | A, B, C) = \begin{pmatrix}
a & C & B \\
\overline{C} & b & A \\
\overline{B} & \overline{A} & c
\end{pmatrix},
\]

where \( a, b, c, A, B, C \in \mathcal{O} \) and furthermore \( a, b, c \in \langle 1_\mathcal{O} \rangle \). To denote certain subspaces of \( \mathcal{J} \) we use the following intuitive notation. The 10-dimensional subspace spanned by the Albert vectors of the form \((a, b, 0 | 0, 0, C)\) is denoted \( \mathcal{J}_{10}^{BC} \), while the 8-space spanned by the vectors \((0, 0, 0 | A, 0, 0)\) is denoted \( \mathcal{J}_{8}^{A} \) and so on. That is, the subscript determines the dimension and the superscript shows which of the six ‘coordinates’ we use to span the corresponding subspace. Of course, this notation is by no means complete as it does not allow us to denote any possible subspace of \( \mathcal{J} \). If this is the case, we specify the spanning vectors and denote the corresponding space in some other manner.

Lacking the associativity in \( \mathcal{O} \) we also need to be slightly careful when we calculate the determinant of \( X \). For these purposes we define the Dickson–Freudenthal determinant as

\[
\Delta(X) = abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + T(ABC).
\]

This is a cubic form on \( \mathcal{J} \) and it can be shown that it is equivalent to the original Dickson’s cubic form used to construct the group of type \( \mathrm{E}_6 \).

| \( e_{-1} \) | \( e_{\overline{\omega}} \) | \( e_{\omega} \) | \( e_0 \) | \( e_{-0} \) | \( e_{-\omega} \) | \( e_{-\overline{\omega}} \) | \( e_1 \) |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | \( e_{-1} \) | \( e_{\overline{\omega}} \) | \( -e_\omega \) | \( -e_0 \) |
| 0 | 0 | -\( e_{-1} \) | \( e_{\overline{\omega}} \) | 0 | 0 | -\( e_\omega \) | \( e_{-\overline{\omega}} \) |
| 0 | -\( e_{-1} \) | 0 | \( e_\omega \) | 0 | -\( e_{-0} \) | 0 | -\( e_{-\overline{\omega}} \) |
| \( e_0 \) | -\( e_{-1} \) | 0 | 0 | \( e_\omega \) | \( e_{-\overline{\omega}} \) | 0 | 0 |
| 0 | \( e_{-1} \) | \( e_{\overline{\omega}} \) | \( e_\omega \) | 0 | \( e_{-0} \) | 0 | 0 |
| \( e_{-\omega} \) | -\( e_{-\overline{\omega}} \) | 0 | \( e_0 \) | 0 | \( e_{-\overline{\omega}} \) | \( -e_\omega \) | 0 |
| \( e_{-\overline{\omega}} \) | \( e_{\overline{\omega}} \) | 0 | -\( e_0 \) | 0 | \( e_{-\overline{\omega}} \) | \( -e_\omega \) | 0 |
| \( e_1 \) | -\( e_{-0} \) | -\( e_{-\overline{\omega}} \) | \( e_{-\overline{\omega}} \) | \( e_1 \) | 0 | 0 | 0 |
We define the group $\text{SE}_6(F)$ or $\text{SE}_6(F, \mathbb{O})$ if we want to specify the octonion algebra, to be the group of all $F$-linear maps on $\mathbb{J}$ preserving the Dickson–Freudenthal determinant. If $F = \mathbb{F}_q$, then we denote this by $\text{SE}_6(q)$. The group $\text{E}_6(F)$ is defined as the quotient of $\text{SE}_6(F)$ by its centre. Suppose $M$ is a $3 \times 3$ matrix written over $\mathbb{O}$. If $M$ is written over any sociable subalgebra of $\mathbb{O}$, then for an element $X \in \mathbb{J}$ the mapping $X \mapsto M^\top XM$ makes sense. Indeed, every entry in the matrix $M^\top XM$ is a sum of the terms of the form $m_1xm_2$, where $m_1$ and $m_2$ belong to the same sociable subalgebra, and so $(m_1x)m_2 = m_1(xm_2)$.

Suppose $X = (a, b, c | A, B, C)$ and $Y = (d, e, f | D, E, F)$. Define the mixed form $M(Y, X)$ as

\begin{equation}
M(Y, X) = \frac{1}{\alpha(\alpha - 1)} \Delta(X + \alpha Y) - \frac{1}{\alpha - 1} \Delta(X + Y) + \frac{1}{\alpha} \Delta(X) - (\alpha + 1)\Delta(Y),
\end{equation}

for any $\alpha \notin \{0, 1\}$.

We colour the non-zero Albert vectors in $\mathbb{J}$ according to the following rules.

**Definition 5.1.** A non-zero Albert vector $X \in \mathbb{J}$ is called

(i) white if $M(Y, X) = 0$ for all $Y \in \mathbb{J}$;
(ii) grey if $\Delta(X) = 0$ and there exists $Y \in \mathbb{J}$ such that $M(Y, X) \neq 0$;
(iii) black if $\Delta(X) \neq 0$ and $X$ is not white.

A white/grey/black point is a 1-dimensional subspace of $\mathbb{J}$ spanned by a white/grey/black vector.

For example, the vector $(0, 0, 1 | 0, 0, 0)$ is white, because if $Y$ is an arbitrary Albert vector, then $M(Y, X) = 0$. Similarly, $(\lambda, 1, 1 | 0, 0, 0)$, where $\lambda \neq 0$, is black, since in this case $\Delta(X) = \lambda \neq 0$, and it is certainly not white as there exists $Y = (a, b, c | A, B, C)$ such that $M(Y, X) \neq 0$:

\begin{equation}
M(Y, X) = \lambda(bc - A\bar{A}) + (ac - B\bar{B}) + (ab - C\bar{C}).
\end{equation}

Taking, for instance, $Y = (0, 1, 1 | 0, 0, 0)$, we get $M(Y, X) = \lambda \neq 0$. Finally, $(0, 1, 1 | 0, 0, 0)$ is grey as $\Delta(X) = 0$ and for $Y = (a, b, c | A, B, C)$ the value of $M$ is given by

\begin{equation}
M(Y, X) = (ac - B\bar{B}) + (ab - C\bar{C}),
\end{equation}

so we may take $Y = (1, 1, 0 | 0, 0, 0)$ to get $M(Y, X) = 1 \neq 0$. Later we will also show that if $X$ is white, then $\Delta(X) = 0$. The terms white, grey and black were introduced by Cohen and Cooperstein [12]. In the paper by Aschbacher [11] they are called 'singular', 'brilliant non-singular' and 'dark' respectively. Jacobson [10] uses the terms 'rank 1', 'rank 2' and 'rank 3'.

It is clear that the action of $\text{SE}_6(F)$ preserves the colour, except possibly in case $F = \mathbb{F}_2$, when white and grey vectors might conceivably be intermixed. However, as we will see now, this does not happen.

Consider $X = (a, b, c | A, B, C)$ and $Y = (0, 0, 1 | 0, 0, 0)$. Then we find $\Delta(X + Y) - \Delta(X) = ab - C\bar{C}$, which is a quadratic form with 17-dimensional radical in $\mathbb{J}$. In the case when $Y = (0, 1, 1 | 0, 0, 0)$ we get $\Delta(X + Y) - \Delta(X) = a + ab + ac - B\bar{B} - C\bar{C}$. If $F = \mathbb{F}_2$, we have $a^2 = a$, so the latter form is quadratic.
with 9-dimensional radical. This shows that \((0, 0, 1 \mid 0, 0, 0)\) and \((0, 1, 1 \mid 0, 0, 0)\) are in different orbits of the isometry group for any field.

6. Some elements of \(\text{SE}_6(F)\)

Throughout this section, let \(X = (a, b, c \mid A, B, C)\) to be an arbitrary element of \(\mathbb{J} = \mathbb{J}_F\). We encode some of the elements of \(\text{SE}_6(F)\) by the \(3 \times 3\) matrices written over social subalgebras of \(\mathbb{O} = \mathbb{O}_F\). As we mentioned before, if such a matrix \(M\) is written over any sociable subalgebra of \(\mathbb{O}\), then the expression \(\mathbb{M}^\top X M\) makes sense. Furthermore, the action of the form \(X \mapsto \mathbb{M}^\top X M\) is obviously \(F\)-linear. If two matrices \(M\) and \(N\) are written over the same sociable subalgebra, then we have enough associativity to see that the action by the product \(MN\) is the same as the product of the actions, that is

\[
(\mathbb{N}\mathbb{M})^\top X (MN) = \mathbb{N}^\top (\mathbb{M}^\top X M)N.
\]

In general, the action by the product of two matrices is not defined whereas the product of the actions still is. Note that also \(-I_3\) acts trivially on \(\mathbb{J}\).

We first notice that the elements

\[
\delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]

preserve the Dickson–Freudenthal determinant. Their actions are given by

\[
\delta : (a, b, c \mid A, B, C) \mapsto (b, a, c \mid \mathbb{B}, \mathbb{A}, \mathbb{C}), \\
\tau : (a, b, c \mid A, B, C) \mapsto (c, a, b \mid C, A, B).
\]

Now let \(x\) be any octonion and consider the matrices

\[
M_x = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M'_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \quad M''_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}.
\]

Note that the elements \(M'_x, M''_x\) can be obtained from \(M_x\) by applying the triality element \(\tau\), so to show that all three families described above preserve the Dickson–Freudenthal determinant, we only need to consider one of them.

**Lemma 6.1.** The elements \(M_x\), where \(x \in \mathbb{O}\) is any octonion, preserve the Dickson–Freudenthal determinant, and hence they encode the elements of \(\text{SE}_6(F)\).

**Proof.** The action of \(M_x\) on \(\mathbb{J}\) is given by

\[
M_x : (a, b, c \mid A, B, C) \mapsto (a, b + aN(x) + T(\mathbb{C}C), c \mid A + \mathbb{B}B, B, C + ax).
\]

The individual terms in the Dickson–Freudenthal determinant are being mapped in the following way:

\[
\begin{align*}
abc & \mapsto abc + a^2cN(x) + ac T(\mathbb{C}C), \\
-aA & \mathbb{A} & \mapsto -aA\mathbb{A} - aT(ABx) - aN(x)N(B), \\
-bB & \mathbb{B} & \mapsto -bB\mathbb{B} - aN(x)N(B) - T(\mathbb{C}C)B\mathbb{B}, \\
-cC & \mathbb{C} & \mapsto -cC\mathbb{C} - ac T(\mathbb{C}C) - a^2cN(x), \\
T(ABC) & \mapsto T(ABC) + B\mathbb{B}T(\mathbb{C}C) + 2aN(x)N(B) + a T(ABx).
\end{align*}
\]

It is visibly obvious now that all the necessary terms on the right-hand side cancel out, so the result follows. \(\square\)

It is obvious enough that we can also consider

\[
L_x = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L'_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}, \quad L''_x = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
for an arbitrary \( x \in \mathcal{O} \). A similar straightforward calculation as in Lemma 6.1 can be performed to show that these are also the elements of \( \text{SE}_0(F) \). Further in this paper we will be able to show that the actions of the elements \( M_x, M'_x, M''_x, L_x, L'_x \) and \( L''_x \) generate the whole group \( \text{SE}_0(F) \).

Finally, we consider the elements of the form

\[
P_u = \begin{pmatrix} u & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P'_u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \pi \end{pmatrix}, \quad P''_u = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u \end{pmatrix},
\]

where \( u \) is an octonion of norm one. The action of the element \( P_u \) on \( \mathcal{J} \) is given by

\[
P_u : (a, b, c \mid A, B, C) \mapsto (a, b, c \mid uA, Bu, \overline{uC\overline{u}}).
\]

It is a matter of straightforward computation to show that the elements \( P_u \) preserve the Dickson–Freudenthal determinant. Indeed, we have

\[
\begin{align*}
\text{abc} & \mapsto \text{abc}, \\
A\overline{A} & \mapsto a(uA)\overline{(\pi\overline{u})} = aN(uA) = aN(A)N(u) = a\overline{A}, \\
B\overline{B} & \mapsto b(Bu)\overline{(\pi\overline{u})} = bN(Bu) = b\overline{B}, \\
C\overline{C} & \mapsto c(\pi\overline{c}\overline{C})u = cN(\pi\overline{C})N(u) = c\overline{C},
\end{align*}
\]

and for the last term we get

\[
\begin{align*}
T((uA)(Bu)(\pi\overline{C})) & = T(((\pi\overline{C})(uA))(Bu)) = T((\pi(C(\pi(uA))))(Bu)) \\
& = T((\pi(CA))(Bu)) = T((Bu)(\pi(CA))) = T(B(u(\pi(CA)))) = T(B(\pi(C))) = T(BCA) = T(ABC).
\end{align*}
\]

On the other hand, it is not difficult to see that \( P_u = M_{u^{-1}L_1M_{u^{-1}L_1} \cdots L_{u^{-1}}}, \) so the fact that the matrices \( P_u \) preserve the determinant follows from the calculations already done for the elements \( M_x \) and \( L_x \). We also notice that the elements \( P_u \) preserve the quadratic form \( Q_8^C \) defined on \( \mathcal{J}_8^C \) via

\[
Q_8^C((0, 0, 0 \mid 0, 0, C)) = C\overline{C}.
\]

We finish this section by showing that the action of the elements \( P_u \) on \( \mathcal{J}_{10}^{ab\overline{C}} \), as \( u \) ranges through all the octonions of norm one, is that of \( \Omega_8(F, Q_8^C) \) when \( \mathcal{O} \) is split.

**Lemma 6.2.** If \( \mathcal{O} \) is split, the actions of the elements \( P_u \) on \( \mathcal{J}_8^C \), as \( u \) ranges through all the octonions of norm one, generate a group of type \( \Omega_8^+(F) \). The action on \( \mathcal{J}_{10}^{ab\overline{C}} \) is also that of \( \Omega_8^+(F) \).

**Proof.** Consider the action on the last octonionic ‘co-ordinate’, i.e. \( C \mapsto \pi\overline{C}\pi \). We will show now that this map can be represented as a product of two reflexions. To avoid any predicaments in characteristic 2, we notice that since \( \langle x, y \rangle = T(x\overline{y}) \), we get

\[
\frac{2\langle x, y \rangle}{\langle y, y \rangle} = \frac{x}{N(y)}.
\]

Now, the reflexion in the hyperplane orthogonal to an arbitrary element \( v \in \mathcal{O} \) is the map

\[
r_v : x \mapsto x - \frac{T(x\overline{v})}{N(u)} \cdot u = x - \frac{x\pi + u\pi}{N(u)} \cdot u = x - \frac{(x\pi)u - u\pi u}{N(u)} = -\frac{u\pi u}{N(u)},
\]

It is easy to see now that the given action of \( P_u \) on \( \mathcal{J}_8^C \) is the composition \( r_v \circ r_1 \). As \( u \) ranges through all octonions of norm one, we get the action of \( \Omega_8(F, Q_8^C) \) on \( \mathcal{J}_8^C \). Since we assume that \( \mathcal{O} \) is split, the form \( Q_8 \) is of plus type, so we may denote this group as \( \Omega_8^+(F) \). When acting on \( \mathcal{J}_{10}^{ab\overline{C}} \), the space \( \mathcal{J}_2^{ab} \) is fixed pointwise and the form \( ab - C\overline{C} \) is preserved, so we again get the action of \( \Omega_8^+(F) \).

\[\square\]
7. Action of SE$_6(F)$ on white points

In this paper we will be mostly interested in the action of SE$_6(F)$ on the white points. Let $X = (a, b, c | A, B, C)$ be an arbitrary white vector. A white vector $W$ determines the quadratic form $\Delta(X + W) - \Delta(X) = M(W, X)$ on $\mathcal{J}$. Its radical is 17-dimensional and for any non-zero $\lambda \in F$ we have $\Delta(X + \lambda W) - \Delta(X) = \lambda(\Delta(X + W) - \Delta(X))$, so the form determined by $\lambda W$ has the same radical. Thus, the 17-dimensional space is determined by the white point $\langle W \rangle$.

For example, for the white vector $(0, 0, 1 | 0, 0, 0)$ the quadratic form is $ab - C\overline{C}$, whose radical is $J^A_B$. For the vector $(0, 0, 0 | 0, 0, D)$ with $D \neq 0 \neq D\overline{D}$ the form is $\tilde{Q}(X) = T(D(AB - c\overline{C}))$ with $\tilde{B}(X, Y) = T(D(AB' + A'B - c\overline{C'} - c\overline{C}))$ being its polar form, where $Y = (a', b', c' | A', B', C')$. Now $X$ is in the radical of $\tilde{Q}$ if and only if $\tilde{Q}(X) = 0$ and $\tilde{B}(X, Y) = 0$ for all $Y$. Taking $Y = (a', b', 1 | 0, 0, 0)$ gives us $T(D\overline{C}) = 0$ and taking $Y = (a', b', 0 | 0, B', 0)$ gives us $T(DA'B') = T((DA)B') = 0$ for all $B'$, so $DA = 0$. If $Y = (a', b', 0 | A', 0, 0)$ then $T(D(A'B)) = T((BD)A') = 0$ for all $A'$, so we get $BD = 0$. Finally, setting $Y = (a', b', 0 | 0, 0, C')$ gives us $T(cD\overline{C'}) = 0$ for all $C'$, so $cD = 0$, and thus $c = 0$. Therefore the radical is

\begin{equation}
\{(a, b, 0 | A, B, C) \mid DA = BD = T(D\overline{C}) = 0\}.
\end{equation}

To obtain 17-spaces determined by other “co-ordinate” white vectors we apply a suitable power of $\tau$ to these two.

Next, we derive a system of conditions for an arbitrary vector $X \in \mathcal{J}$ to be white.

**Lemma 7.1.** An Albert vector $X = (a, b, c | A, B, C)$ is white if and only if the following conditions hold:

\begin{equation}
\begin{aligned}
A\overline{A} &= bc, \\
B\overline{B} &= ca, \\
C\overline{C} &= ab, \\
AB &= c\overline{C}, \\
BC &= a\overline{A}, \\
CA &= b\overline{B}.
\end{aligned}
\end{equation}

If $X$ is white, then $\Delta(X) = 0$.

**Proof.** Let $Y = (d, e, f | D, E, F)$. We rewrite $M(Y, X)$ in the form

\[ M(Y, X) = (bc - A\overline{A})d + (ac - B\overline{B})e + (ab - C\overline{C})f + T(D(BC - a\overline{A}) + Q(CA - b\overline{B}) + R(AB - c\overline{C})). \]

It is visibly clear now that if all the conditions in the statement are satisfied, then $M(Y, X) = 0$. Now, taking $Y = (1, 0, 0 | 0, 0, 0)$ forces $bc - A\overline{A} = 0$. Similarly, we may take $Y = (0, 1, 0 | 0, 0, 0)$ to get $ac - B\overline{B} = 0$ and, say, $Y = (0, 0, 0 | D, 0, 0)$ to obtain $T(D(BC - a\overline{A})) = 0$ which forces $BC - a\overline{A} = 0$ as $D \in \mathcal{O}$ can be arbitrary. The other conditions are proved similarly.

Finally, if $X$ is white, then we get $T(ABC) = T(aA\overline{A}) = T(abc) = 2abc$. Also $bB\overline{B} = bca$, and so on. Overall we get

\[ \Delta(X) = abc - abc - bca - cab + 2abc = 0 \]

as required. This completes the proof. \hfill \Box

Finally, we investigate the orbits of SE$_6(F)$ on Albert vectors. One of our main goals is to show that SE$_6(F)$ acts transitively on white points.

**Lemma 7.2.** Suppose $X$ is an arbitrary Albert vector. Then $X$ can be mapped under the action of SE$_6(F)$ to a vector of the form $(a, b, c | 0, 0, 0)$ with $(a, b, c) \neq (0, 0, 0)$. In case when $\mathcal{O}$ is split, $X$ can be mapped to precisely one of the following:
(i) \((0,0,1 \mid 0,0,0)\), a white vector;
(ii) \((0,1,1 \mid 0,0,0)\), a grey vector; or
(iii) \((\lambda,1,1 \mid 0,0,0)\) where \(\lambda \neq 0\), a black vector.

In the last case there is one orbit for each non-zero value of \(\lambda\).

Proof. These vectors are indeed in the different orbits, except possibly for the white and grey vectors, since they have different values of \(\Delta\). We have already shown that these particular white and grey vectors are in different orbits in case of any field.

First, we show that each orbit of \(\text{SE}_6(F)\) contains an Albert vector of the form \((a,b,c \mid 0,0,0)\). Suppose that \(X = (a,b,c \mid A,B,C)\) is non-zero. If \((a,b,c) = (0,0,0)\), then after applying the triality element \(\tau\) a suitable number of times we may assume \(C \neq 0\). Consider the action of the element \(L_x\) on the Albert vector \((0,0,0 \mid A,B,C)\):

\[
L_x : (0,0,0 \mid A,B,C) \mapsto (\text{T}(Cx),0,0 \mid A,B + \overline{A}x,C),
\]

so we are allowed to choose orbit representatives with \((a,b,c) \neq (0,0,0)\).

As before, using a suitable power of \(\tau\), we may assume \(c \neq 0\). Now we apply the element \(M_x\) with \(x = -c^{-1}B\) to \(X\), which gives us the vector of the form \((a,b,c \mid A,0,C)\), where the 'coördinate' \(c\) stays the same, while \(a, b, A, C\) are possibly different. Next, the vector \((a,b,c \mid A,0,C)\) is being mapped to the vector of the form \((a,b,c \mid 0,0,C)\) under the action of \(L_x\) with \(x = -c^{-1}A\), where the value of \(c\) stays the same while the values of \(a, b, C\) may be adjusted.

If \(a = b = 0, C \neq 0\), then we apply the element \(L_x\) with \(x\) such that \(\text{T}(Cx) \neq 0\) to get the vector of the form \((\text{T}(Cx),0,0 \mid 0,0,C)\), i.e. we may assume that \(a \neq 0\). With the latter assumption we apply the element \(M_x\) with \(x = -a^{-1}C\) to \((a,b,c \mid 0,0,C)\) to get the vector of the form \((a,b,c \mid 0,0,0)\) with the value of \(b\) being adjusted.

Finally, we use the elements \(\tau\), \(P_\alpha\) and \(P'''\) to standardise the vector of the form \((a,b,c \mid 0,0,0)\) to one the forms in the statement. \(\square\)

Note that the last part of the proof of this lemma used the fact that the map \(N : \mathbb{O} \to F\) is onto, which is the case when \(\mathbb{O}\) is split. However, this is not true in any octonion algebra, which possibly leads to a bigger number of orbits. A vector of the form \((a,b,c \mid 0,0,0)\) is white if and only if precisely one of the \(a, b, c\) is non-zero, so we get the transitive action of \(\text{SE}_6(F)\) on white points regardless of the chosen octonion algebra.

Furthermore, we used the fact that \(N\) is a non-singular quadratic form on \(\mathbb{O}\), i.e. provided \(C \neq 0\), the map \(x \mapsto \text{T}(Cx)\) is surjective. This follows from the fact that the norm is a non-singular quadratic form on \(\mathbb{O}\), something that should hold for any octonion algebra.

Later we will use the transitivity on white points to calculate the group order in case \(F = \mathbb{F}_q\) by finding the stabiliser of a white point and calculating the number of white points in case of a finite field.

Lemma 7.3. Let \(\mathbb{O}\) be an arbitrary octonion algebra over \(F\). Let \(X \in \mathbb{J}\) be white and let \(\mathbb{J}_{17}\) be the 17-dimensional subspace of \(\mathbb{J}\) determined by \(X\). The stabiliser in \(\text{SE}_6(F)\) of \((X)\), and even of \(X\), is transitive on the white points spanned by the vectors in \(\mathbb{J}_{17} \setminus \langle X \rangle\) (there are no such white points when \(\mathbb{O}\) is non-split). It is also transitive on the white points spanned by the vectors in \(\mathbb{J} \setminus \mathbb{J}_{17}\).

Proof. Without loss of generality assume \(X = (0,0,1 \mid 0,0,0)\). As we know, the white point \(\langle X \rangle\) determines the 17-space \(\mathbb{J}_{17}^{AB}\). We also note that \(X\) is stabilised by the actions of the elements \(M_x, L_x, M'_x\) and \(L''_x\). Those act on the elements in
$J_{17}^{aB}$ in the following way:

\[
\begin{align*}
M_x &: (0, 0, c \mid A, B, 0) \mapsto (0, 0, c \mid A + xB, B, 0), \\
L_x &: (0, 0, c \mid A, B, 0) \mapsto (0, 0, c \mid A, B + x, 0), \\
M'_x &: (0, 0, c \mid A, B, 0) \mapsto (0, 0, c + T(\mathbf{A}) \mid A, B, 0), \\
L''_x &: (0, 0, c \mid A, B, 0) \mapsto (0, 0, c + T(Bx) \mid A, B, 0).
\end{align*}
\]

It follows that a general white vector $(0, 0, c \mid A, B, 0) \in J_{17}^{aB} \setminus \langle X \rangle$ can easily be mapped to $(0, 0, 0 \mid A, B, 0)$ using the action of $M'_x$ or $L''_x$ for some suitable $x \in \mathbb{O}$. A vector $(0, 0, 0 \mid A, B, 0)$ is white if $(A, B) \neq (0, 0)$ and $\mathbf{A}B = AB = 0$. It is obvious enough that $\mathcal{J}_{17}^{AB} \setminus \langle X \rangle$ is empty if $\mathbb{O}$ is not split, so we only need to show transitivity on the corresponding white points in case when $\mathbb{O}$ is split.

If $B = 0$ then evidently $A \neq 0$ and so we can apply the duality element $\delta$ to obtain a white vector of the form $(0, 0, 0 \mid A, B, 0)$ with $B \neq 0$. If now $A \neq 0$, we act by $M_x$ to obtain $(0, 0, 0 \mid A + xB, B, 0)$. Our aim is to show that there exists such $x \in \mathbb{O}$ that $A + xB = 0$. Denote $U = \{y \in \mathbb{O} \mid \mathcal{O}B = y\}$. Since for all $x \in \mathbb{O}$ we have $(xB)B = x(BB) = 0$, we conclude that $\mathcal{O}B \subseteq U$. Furthermore, we know that both subspaces are four-dimensional, so $\mathcal{O}B = U$. As $AB = 0$, we have $A \in U$, and therefore there exists $y = xB \in U$ such that $A + y = 0$.

Now, the elements $P''_u$ with $N(u) = 1$ act on the Albert vectors of the form $(0, 0, 0 \mid 0, B, 0)$ as

\[
(0, 0, 0 \mid 0, B, 0) \mapsto (0, 0, 0 \mid 0, \mathbf{BB}, 0),
\]

and as $u$ ranges through all the octonions of norm $1$ the action generated is that of $\Omega_{17}^{aB} (F)$ which in case when $\mathbb{O}$ is split is transitive on isotropic vectors, i.e. those with $BB = 0$. It follows that $SE_6(F)$ is indeed transitive on the white points spanned by the vectors in $J_{17}^{aB} \setminus \langle X \rangle$.

To show the transitivity on white points spanned by the vectors in $J_{17}^{aB} \setminus \mathcal{J}_{17}^{aB}$ we prove that every white point spanned by a white vector $(a, b, c \mid A, B, C) \in J_{17}^{aB} \setminus \mathcal{J}_{17}^{aB}$ can be mapped to the white point spanned by $(1, 0, 0 \mid 0, 0, 0)$. Note that we require $(a, b, C) \neq (0, 0, 0)$.

In case $(a, b) = (0, 0)$ we choose $x \in \mathbb{O}$ such that $T(Cx) \neq 0$ and apply the element $L_x$, which maps our vector $(0, 0, c \mid A, B, C)$ to $(T(Cx), 0, c \mid A, B + x, C)$. If, on the other hand, $a = 0$ and $b \neq 0$, we apply $\delta$. Hence, we may assume that we deal with a vector $(a, b, c \mid A, B, C)$ with $a \neq 0$. Take $x = -a^{-1}C$ and act by the element $M_x$:

\[
M_x : (a, b, c \mid A, B, C) \mapsto (a, b + aa^{-2}CC - T(a^{-1}CC), c \mid A - a^{-1}CCB, B, 0).
\]

The whiteness conditions imply $CC = ab$ and $BC = a\mathbf{A}$, so additionally we get $b + aa^{-2}CC - T(a^{-1}CC) = b + b - T(b) = 0$ and $A - a^{-1}CCB = A - A = 0$. This means that the given $M_x$ acts on the elements of $\mathbb{O} \setminus \mathcal{J}_{17}^{aB}$ in the following way:

\[
M_x : (a, b, c \mid A, B, C) \mapsto (a, 0, c \mid 0, B, 0),
\]

where $a \neq 0$. It is still white, so $BB = ca$. Finally, we act by $L''_x$ with $y = -a^{-1}B$:

\[
L''_x : (a, 0, c \mid 0, B, 0) \mapsto (a, 0, 0 \mid 0, 0, 0),
\]

where $a \neq 0$. In other words, any white point spanned by an element in $\mathbb{O} \setminus \mathcal{J}_{17}^{aB}$ can be mapped by the action of the stabiliser of $\langle X \rangle$ to the white point spanned by $(1, 0, 0 \mid 0, 0, 0)$.

\[\square\]

**Lemma 7.4.** The action of $SE_6(F)$ on white points is primitive.

**Proof.** From the previous Lemma it follows that if $\mathbb{O}$ is non-split, then the action of $SE_6(F)$ on white points in $2$-transitive and hence primitive. It remains to prove the statement in the case when $\mathbb{O}$ is split.
Suppose $X, Y \in \mathcal{J}$ are white vectors such that $\langle X \rangle \neq \langle Y \rangle$. Define $\sim$ to be an $\text{SE}_q(F)$-congruence on white points and let $\langle X \rangle \sim \langle Y \rangle$. Our aim is to show that this generates the universal congruence. Since for $\mathcal{O}$ split the action on the white vectors is transitive, we may assume $X = (0, 0, 1 \mid 0, 0, 0)$. As mentioned in the beginning of this section, $\langle X \rangle$ determines the 17-dimensional space $\mathbb{J}_{17}^{cAB}$. We now distinguish two cases.

If $Y \in \mathbb{J}_{17}^{cAB}$, then acting by the stabiliser of $\langle X \rangle$ we get $\langle X \rangle \sim \langle Y \rangle$ for all white $Y \in \mathbb{J}_{17}^{cAB} \setminus \langle X \rangle$. Take $\hat{Y} = (0, 0, 0 \mid e_0, 0, 0) \in \mathbb{J}_{17}^{cAB}$ and $\hat{X} = (0, 1, 0 \mid 0, 0, 0) \notin \mathbb{J}_{17}^{cAB}$. As we see from the earlier calculations, both $\langle \hat{Y} \rangle$ and $\langle \hat{X} \rangle$ are white vectors in this subspace.

On the other hand, if $Y$ lies outside of $\mathbb{J}_{17}^{cAB}$, then we get $\langle X \rangle \sim \langle \hat{Y} \rangle$ for all white $\hat{Y} \in \mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ since the stabiliser of $\langle X \rangle$ is transitive on the white points spanned by those. In particular, we may take $\hat{Y} = (1, 0, 0 \mid 0, 0, 0)$. Acting by the stabiliser of $\langle \hat{Y} \rangle$ on both sides in $\langle X \rangle \sim \langle \hat{Y} \rangle$, we map $\langle X \rangle$ to $\langle \hat{X} \rangle$ with $\hat{X} = (0, 0, 0 \mid e_0, 0, 0)$. Note that both $X$ and $\hat{X}$ are not in $\mathbb{J}_{17}^{ABC}$ which is the 17-space determined by $\hat{Y}$, but $\hat{X} \in \mathbb{J}_{17}^{cAB}$ and by transitivity we get $\langle X \rangle \sim \langle \hat{X} \rangle$. Again, we act by the stabiliser of $\langle X \rangle$ to ensure $\langle X \rangle \sim \langle \hat{X} \rangle$ for all white points $\langle \hat{X} \rangle$ spanned by $\hat{X} \in \mathbb{J}_{17}^{cAB}$, i.e. our $\text{SE}_q(F)$-congruence is trivial in this case as well.

8. White points in the case of a finite field

In this section $F$ is a finite field of $q$ elements, that is, $F = \mathbb{F}_q$. Our aim is to count the white points in this case.

**Theorem 8.1.** If $F = \mathbb{F}_q$, then there are precisely

$$\frac{q^{12} - 1}{q^4 - 1}$$

white vectors in $\mathcal{J}$.

**Proof.** In the proof we study the series of subspaces

$$0 < J_{10}^{abC} < J_{26}^{AB} < \mathcal{J}.$$

First, $(a, b, 0 \mid 0, 0, 0) \in \mathbb{J}_{10}^{abC}$ is white if and only if $ab - C \overline{C} = 0$. We notice that $ab - C \overline{C}$ is a quadratic form of plus type defined on $\mathbb{J}_{10}^{abC}$, so there are $(q^5 - 1)(q^4 + 1)$ white vectors in this subspace.

Next, suppose $(a, b, c \mid A, B, C) \in \mathbb{J} \setminus \mathbb{J}_{26}^{ABC}$ is white. Then $C = B \overline{A}c^{-1}$, $b = A \overline{A}c^{-1}$ and $\overline{A} = B \overline{B}c^{-1}$. We may choose $A, B, c$ to be arbitrary (with $c \neq 0$), so there are $q^{16}(q - 1)$ white vectors in $\mathbb{J} \setminus \mathbb{J}_{26}^{ABC}$.

Finally, we investigate the white vectors $(a, b, 0 \mid A, B, C) \in \mathbb{J}_{26}^{ABC} \setminus \mathbb{J}_{10}^{abC}$. The conditions for such a vector to be white take the following form:

$$\begin{align*}
A \overline{A} &= B \overline{B} = AB = 0, \\
C \overline{C} &= ab, \\
BC &= a \overline{A}, \\
CA &= b \overline{B}.
\end{align*}$$

(40)

Note that we also require $(A, B) \neq (0, 0)$. In case $A = 0, B \neq 0$ we apply $\delta$ followed by $\gamma$ to $(a, b, 0 \mid A, B, C)$ in order to obtain a vector of the form $(a, b, 0 \mid A, B, C)$ with $A \neq 0$. Note that the values of $a, b, A, B, C$ are not the same as in the initial Albert vector. So, assuming $A \neq 0$, we have $AB = 0$ exactly when $B$ is in a particular
4-dimensional subspace of \( \mathbb{O} \) and any such \( B \) satisfies \( B\overline{B} = 0 \). For any octonion \( x \), the action by the element \( L_x \) establishes a bijection between the white vectors of the form \((*,*,0 \mid A,B,*)\) and those of the form \((*,0 \mid A,B + Ax,*)\). Left-multiplication by \( \overline{A} \) annihilates a 4-dimensional subspace of \( \mathbb{O} \) (see Lemma 3.19), so by the rank-nullity theorem we conclude that the image \( \mathcal{A} = \{\overline{Ax} \mid x \in \mathbb{O}\} \) is also 4-dimensional. Note that \( \mathcal{A}(\overline{Ax}) = (\overline{A\overline{x}})x = 0 \), for any \( x \in \mathbb{O} \), so \( \mathcal{A} \) is the 4-space of all octonions left-annihilated by \( A \), and therefore it contains \( -B \). Now we pick an octonion \( x \) such that \( \overline{Ax} = -B \) to obtain the bijection between the white vectors of the form \((*,*,0 \mid A,B,*)\) with \( A \neq 0 \) and those of the form \((*,*,0 \mid A,0,*)\). An Albert vector \((a,b,0 \mid A,0,C)\) is white if and only if \( A\overline{A} = C\overline{C} = CA = 0 \) and \( a = 0 \), with no dependence on \( b \). As before, \( C \) lies in a particular 4-dimensional subspace of \( \mathbb{O} \), hence \((0,b,0 \mid 0,0,C)\) lies in a particular 5-dimensional subspace of \( \mathbb{J}_{10}^{abc} \), so for any choice of the pair \((A,B)\) there are \( q^5 \) white vectors. If \( A \neq 0 \), then there are \((q^4 - 1)(q^3 + 1)\) choices for \( A \), and for each of these \( q^4 \) choices for \( B \). If \( A = 0 \), we have \((q^4 - 1)(q^3 + 1)\) choices for \( B \). It follows that in total there are

\[
q^5(q^4(q^4 - 1)(q^3 + 1) + (q^4 - 1)(q^3 + 1)) = q^5(q^8 - 1)(q^3 + 1)
\]

white vectors in \( \mathbb{J}_{26}^{abABC} \). The calculations above give the numbers of white vectors in certain subsets of \( \mathbb{J} \) as shown in the following table.

| subset | \( \mathbb{J}_{10}^{abc} \) | \( \mathbb{J}_{26}^{abABC} \setminus \mathbb{J}_{10}^{abc} \) | \( \mathbb{J} \setminus \mathbb{J}_{26}^{abABC} \) |
|--------|-----------------|-----------------|-----------------|
| number of white vectors | \((q^5 - 1)(q^4 + 1)\) | \(q^5(q^8 - 1)(q^3 + 1)\) | \(q^{16}(q - 1)\) |

Adding these numbers gives the result. \( \square \)

**Corollary 8.2.** There are precisely

\[
(41) \quad \frac{(q^{12} - 1)(q^8 - 1)}{(q^4 - 1)(q - 1)}
\]

white points in the case \( F = \mathbb{F}_q \).

### 9. The Stabiliser of a White Point

In this section we assume that \( \mathbb{O} \) is a split octonion algebra. It is our aim now to obtain the stabiliser in \( \text{SE}_6(F) \) of a white point. In particular, we prove the following result.

**Theorem 9.1.** If \( \mathbb{O} \) is split, then the stabiliser of a white vector in \( \text{SE}_6(F) \) is isomorphic to the group generated by the actions of the elements \( M_x, L_x, M'_x \) and \( L'_x \) on \( \mathbb{J} \) as \( x \) ranges over \( \mathbb{O} \) and this is a group of shape

\[
(42) \quad F^{16}:\text{Spin}^+_10(F).
\]

The stabiliser of a white point is isomorphic to

\[
(43) \quad F^{16}:\text{Spin}^+_10(F).F^x,
\]

where \( F^x \) is the multiplicative group of the field \( F \).

This whole section is devoted to proving this result. Some of this proof is in the running text, and some of it is contained in a series of technical lemmata. First, we prove that no invertible \( F \)-linear maps on \( \mathbb{O} \) can change the order of the octonion product.

**Lemma 9.2.** There are no invertible \( F \)-linear maps \( \phi, \psi : \mathbb{O} \to \mathbb{O} \) such that for all \( A, B \in \mathbb{O} \) it is true that \( AB = (B\psi)(A\phi) \).
Proof. For the sake of finding a contradiction, suppose that \( \phi, \psi : \mathbb{O} \to \mathbb{O} \) are invertible \( F \)-linear maps such that the identity \( AB = (B\psi)(A\phi) \) holds for all \( A, B \in \mathbb{O} \). In particular, substituting \( A = 1_\mathbb{O} \), we get \( B = (B\psi)u \) for all \( B \in \mathbb{O} \), where \( u = 1_\mathbb{O} \), so \( B\psi = Bu^{-1} \) for all \( B \in \mathbb{O} \), which means that the map \( \psi \) is right multiplication by \( u^{-1} \). Note that the existence of \( u^{-1} \) follows from the invertibility of the map \( \psi \). Thus, our identity has the form \( AB = (Bu^{-1})(A\phi) \) for all \( A, B \in \mathbb{O} \). We can substitute \( B = u \) which immediately gives us \( A\phi = Au \) for all \( A \in \mathbb{O} \), so the map \( \phi \) is right multiplication by \( u \). Finally, we get \( AB = (Bu^{-1})(Au) \) for all \( A, B \in \mathbb{O} \) and specifically for \( B = 1_\mathbb{O} \) we get \( A = u^{-1}(Au) \), or likewise \( uA = Au \) for all \( A \in \mathbb{O} \). Therefore \( u \) is a scalar multiple of \( 1_\mathbb{O} \), i.e. \( u = \mu \cdot 1_\mathbb{O} \) for some \( \mu \in F \). Since the linear maps \( \phi \) and \( \psi \) are invertible, \( \mu \) is non-zero, and we get \( AB = (Bu^{-1})(Au) = (\mu^{-1}\mu \cdot 1_\mathbb{O})BA = BA \) for all \( A, B \in \mathbb{O} \), which is definitely not true as \( \mathbb{O} \) is not commutative. \( \square \)

Second, we show that if two invertible linear maps commute with the octonion product, then these are mutually invertible scalar multiplication maps.

**Lemma 9.3.** Suppose \( \phi, \psi : \mathbb{O} \to \mathbb{O} \) are two invertible \( F \)-linear maps such that \( AB = (A\phi)(B\psi) \) for all \( A, B \in \mathbb{O} \). Then \( \psi : x \mapsto \mu x \) for some non-zero \( \mu \in F \) and \( \phi = \psi^{-1} \), i.e. \( \phi : x \mapsto \mu^{-1}x \).

**Proof.** Suppose \( \phi, \psi : \mathbb{O} \to \mathbb{O} \) are \( F \)-linear maps such that \( AB = (A\phi)(B\psi) \) for all \( A, B \in \mathbb{O} \). When \( A = 1_\mathbb{O} \) we get \( B\psi = uB \) for all \( B \in \mathbb{O} \) where \( u = (1_\mathbb{O}\phi)^{-1} \), so the map \( \psi \) is left multiplication by \( u \). Substituting \( B = 1_\mathbb{O} \) on the other hand gives us \( A = (A\phi)(1_\mathbb{O}\psi) \) for all \( A \) and so \( A\phi = Av \) where \( v = (1_\mathbb{O}\psi)^{-1} \), so \( \phi \) is the right multiplication by \( v \). Therefore the condition in this case becomes \( AB = (Av)(uB) \) for all \( A, B \in \mathbb{O} \). Substituting \( B = u^{-1} \), we get \( Au^{-1} = Av \) for all \( A \in \mathbb{O} \), and therefore \( v = u^{-1} \), and our identity turns out to be \( AB = (Au^{-1})(uB) \) for all \( A, B \in \mathbb{O} \). Now since \( u \) is invertible, we can write \( u^{-1} = N(u)^{-1} \). Finally, by Corollary 3.12, \( u \) must be a scalar multiple of \( 1_\mathbb{O} \), i.e. \( u = \mu \cdot 1_\mathbb{O} \). \( \square \)

The statements in Lemmas 9.2 and 9.3 are true even when \( \mathbb{O} \) is not split. Everything is ready now for the investigation of the white vector stabiliser. Since it was shown that the group \( SE_8(F) \) acts transitively on the set of white points, it is sufficient to study the stabiliser of a specific white vector. For instance, it is convenient to take \( v = (0, 0, 1 | 0, 0, 0) \). First thing to notice is that \( v \) is invariant under the action of the elements of the form

\[
L'_x = \begin{pmatrix}
1 & 0 & x \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad M'_y = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix},
\]

where \( x, y \in \mathbb{O} \).

**Lemma 9.4.**

(a) Let \( Q \) be any of the \( \{L, L', L'', M, M', M''\} \). Then the actions on \( \mathfrak{J} \) of the elements \( Q_x \) where \( x \) ranges over \( \mathbb{O} \) generate an elementary abelian group isomorphic to \( F^8 \).

(b) Let \( (R, S) \) be any pair from the set \( \{(L, M''), (L', M), (L'', M')\} \) or any of the \( \{(L, M'), (L', M''), (L'', M)\} \). Then the actions of \( R_x \) and \( S_x \) as \( x \) ranges through \( \mathbb{O} \), generate an elementary abelian group isomorphic to \( F^{16} \).

**Proof.** To show part (a) for the elements \( L_x, L'_x, L''_x \) it is enough to consider just, say, \( L''_x \) as to obtain the result for the rest of them we can apply the action of the triality element

\[
\tau = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]
Similarly, out of $M_x, M'_x, M''_x$ we only need to consider, for instance, $M'_x$. The actions of $L''_x$ and $M'_y$ on $\mathcal{J}$ are given by

\begin{align}
L''_x &: (a, b, c | A, B, C) \mapsto (a, b, c + ax\bar{x} + T(Bx) | A + \bar{C}x, B + \bar{A}\bar{x}, C), \\
M'_y &: (a, b, c | A, B, C) \mapsto (a, b, c + by\bar{y} + T(\bar{y}A) | A + By, B + \bar{y}\bar{C}, C).
\end{align}

We notice that the action is nontrivial whenever $x$ and $y$ are non-zero. The element $M'_y$ sends $(a, b, c + ax\bar{x} + T(Bx) | A + \bar{C}x, B + a\bar{x}, C)$ to

$$(a, b, c + ax\bar{x} + T(Bx) + by\bar{y} + T(\bar{y}A) | A + \bar{C}x + by, B + a\bar{x} + \bar{y}\bar{C}, C),$$

and the element $L''_x$ sends $(a, b, c + by\bar{y} + T(\bar{y}A) + ax\bar{x} + T(Bx) | A + by + \bar{C}x, B + \bar{y}\bar{C} + a\bar{x}, C)$ to

$$(a, b, c + by\bar{y} + T(\bar{y}A) + ax\bar{x} + T(Bx) | A + by + \bar{C}x, B + \bar{y}\bar{C} + a\bar{x}, C).$$

Hence, the actions of these elements commute. Similarly, it is straightforward to verify that the actions of $L''_x$ and $L''_y$ commute as well as the actions of $M'_x$ and $M'_y$. Moreover, the element $L''_y$ sends $(a, b, c + ax\bar{x} + T(Bx) | A + \bar{C}x, B + ax, C)$ to

$$(a, b, c + ax\bar{x} + T(Bx) + ay\bar{y} + T(By) + aT(\bar{y}y) | A + \bar{C}x + \bar{C}y, B + a\bar{x} + ay, C),$$

so the action of $L''_{x+y}$ is the same as the product of the actions of $L''_x$ and $L''_y$. A similar calculation shows that the action of $M'_{x+y}$ is the same as the product of the actions of $M'_x$ and $M'_y$. It follows that the action of $L''_x$ on $\mathcal{J}$, $x \in \mathbb{O}$ generates an abelian group $(F^8, +)$ as well as the action of the element $M'_y$, $y \in \mathbb{O}$. We simply denote the abelian group $(F^8, +)$ as $F^n$ in our further discussion.

To prove part (b) we need to verify that the intersection of the corresponding abelian groups, isomorphic to $F^8$ and generated by the actions of $L''_x$ and $M'_y$ is trivial. Suppose that the actions of $L''_x$ and $M'_y$ are equal. Then, according to (15), in the fourth “coordinate” we have

$$A + \bar{C}x = A + By$$

for arbitrary $A, B, C \in \mathbb{O}$. In other words, we get $\bar{C}x = By$ for arbitrary octonions $B$ and $C$. In particular, if $B = 1_0$ and $C = 0$, we get $y = 0$ and if $B = 0$ and $C = 1_0$ we obtain $x = 0$. So, the intersection of two copies of $F^8$ consists of the identity element, as needed, and the result follows. Again, to get (b) for the rest of the pairs in the first set we apply the triality element. The calculations for the second set of pairs are of the same nature. 

The next observation is that our white vector $v$ is also invariant under the action of the elements

\begin{align}
M_x &= \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
L_y &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix}.
\end{align}

First, we show that the actions of these on $\mathbb{J}^{abc}_{10}$ generate a group of type $\Omega^{10}_0(F)$. As we will see further, instead of arbitrary octonions it is enough for $x$ to range through the scalar multiples of the basis elements $e_1$. Define the quadratic form $Q_{10}$ on $\mathcal{J}$ via

$$Q_{10}((a, b, c | A, B, C)) = ab - C\bar{C}.$$  

We notice that $Q_{10}$ is of plus type, so for convenience we denote the group $\Omega_{10}(F, Q_{10})$ as $\Omega^{\pm}_{10}(F)$.

To construct $\Omega^{\pm}_{10}(F)$ we follow the series of steps. First, we consider the 4-space $V_4$ spanned by the Albert vectors of the form $(a, b, 0 | 0, 0, C_{-1}e_{-1} + C_1e_1)$. 

\[1\]
Lemma 9.5. The actions of the elements $M_{\lambda e_{\pm 1}}$ and $L_{\lambda e_{\pm 1}}$ on $V_4$, where $\lambda \in F$, generate a group of type $\Omega^+_4(F)$.

Proof. Consider the vectors $v_1, v_2, v_3$ and $v_4$ defined as

\[
v_1 = (1,0,0 | 0,0,0), \quad v_2 = (0,1,0 | 0,0,0), \quad v_3 = (0,0,0 | 0,0,e_{-1}), \quad v_4 = (0,0,0 | 0,0,e_1).
\]

It is clear that these span $V_4$, so define $B = \{v_1, v_4, v_2\}$ to be the basis of our 4-space. The element $M_{\lambda e_{-1}}$ acts on the basis elements in the following way:

\[
\begin{align*}
v_1 &\mapsto (1,0,0 | 0,0,\lambda e_{-1}) = v_1 + \lambda v_3, \\
v_4 &\mapsto (0,\lambda,0 | 0,0,e_1) = v_4 + \lambda v_2, \\
v_3 &\mapsto (0,0,0 | 0,0,e_{-1}) = v_3, \\
v_2 &\mapsto (0,1,0 | 0,0,0) = v_2.
\end{align*}
\]

As we can see, with respect to the basis $B$ the action can be written as a $4 \times 4$ matrix

\[
\begin{pmatrix}
1 & 0 & \lambda & 0 \\
0 & 1 & 0 & \lambda \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & \lambda \\
0 & 1 \\
\lambda & 0 \\
0 & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]

where $\otimes$ is the Kronecker product. Similarly, the action of $M_{\lambda e_1}$ on $B$ is given by

\[
\begin{align*}
v_1 &\mapsto (1,0,0 | 0,0,\lambda e_1) = v_1 + \lambda v_3, \\
v_4 &\mapsto (0,\lambda,0 | 0,0,e_1) = v_4, \\
v_3 &\mapsto (0,\lambda,0 | 0,0,e_{-1}) = v_3 + \lambda v_2, \\
v_2 &\mapsto (0,1,0 | 0,0,0) = v_2,
\end{align*}
\]

so the corresponding $4 \times 4$ matrix has the form

\[
\begin{pmatrix}
1 & \lambda & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \lambda \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & \lambda \\
0 & 1
\end{pmatrix}.
\]

Now, for convenience, we do the same calculations for $L_{-\lambda e_{-1}}$: it acts on the elements of $B$ as

\[
\begin{align*}
v_1 &\mapsto (1,0,0 | 0,0,0) = v_1, \\
v_4 &\mapsto (\lambda,0,0 | 0,0,e_{-1}) = \lambda v_1 + v_4, \\
v_3 &\mapsto (0,\lambda,0 | 0,0,e_{-1}) = v_3, \\
v_2 &\mapsto (0,1,0 | 0,0,\lambda e_{-1}) = \lambda v_3 + v_2,
\end{align*}
\]

and it can be written in the matrix form as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\lambda & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \lambda & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
\lambda & 1
\end{pmatrix}.
\]

Finally, the action on $B$ of $L_{-\lambda e_1}$ is given by

\[
\begin{align*}
v_1 &\mapsto (1,0,0 | 0,0,0) = v_1, \\
v_4 &\mapsto (0,\lambda,0 | 0,0,e_1) = v_4, \\
v_3 &\mapsto (\lambda,0,0 | 0,0,e_{-1}) = \lambda v_1 + v_3, \\
v_2 &\mapsto (0,1,0 | 0,0,\lambda e_1) = \lambda v_3 + v_2,
\end{align*}
\]
and in the matrix form we get
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\lambda & 0 & 1 & 0 \\
0 & \lambda & 0 & 1
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

As we know,
\[
\langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \mid \lambda \in F \rangle \cong \mathrm{SL}_2(F).
\]

It follows that
\[
\cong \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \lambda \in F \rangle \cong \mathrm{SL}_2(F),
\]

and since
\[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

we finally get
\[
\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \rangle \cong \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F).
\]

Now, \( \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F) \cong \Omega_4^+(F) \) (see, for example [17]), and this finishes the proof.

\[\square\]

In our construction we use the results of section 2. Consider the 6-space \( V_6 \) spanned by the Albert vectors \((a, b, 0 \mid 0, 0, C)\), where \( C \in \langle e_{-1}, e_{+e}, e_{-e}, e_1 \rangle \). Our copy of \( \Omega_4^+(F) \) preserves two isotropic Albert vectors in \( V_6 \):

\[
(u_\omega) = (0, 0, 0 \mid 0, 0, e_\omega), \\
u_{-\omega} = (0, 0, 0 \mid 0, 0, e_{-\omega}).
\]

The element \( M_{u_\omega} \) preserves \( u_\omega \) but not \( u_{-\omega} \). Therefore, adjoining \( M_{u_\omega} \) to \( \Omega_4^+(F) \), we obtain a subgroup of \( V_4: \Omega_4^+(F) \) (Lemma 2.1), and since \( \Omega_4^+(F) \) is maximal in the latter (Theorem 2.3), we conclude that the action of \( M_{l_{e_{+e}}, l_{e_{-e}}} \) and \( M_{u_\omega} \) on \( V_6 \) is that of \( V_4: \Omega_4^+(F) \). That is, we have constructed the group \( V_4: \Omega_4^+(F) \) as the stabiliser of \( u_\omega \) in \( \Omega_6^+(F) \). Now we use the result of Theorem 2.4. The element \( M_{u_{-\omega}} \) preserves \( V_6 \) but it does not preserve \( u_\omega \), and as a consequence it does not preserve the 1-space \( \langle u_{\omega} \rangle \). Therefore, if we adjoin \( M_{u_{-\omega}} \) to our copy of \( V_4: \Omega_4^+(F) \), we get the action of the group \( \Omega_6^+(F) \) on \( V_6 \).

Similarly, we consider the 8-space \( V_8 \) spanned by the vectors \((a, b, 0 \mid 0, 0, C)\) with \( C \in \langle e_{-1}, e_{+e}, e_\omega, e_{-e}, e_{-e}, e_1 \rangle \). Consider two isotropic Albert vectors

\[
(u_\omega) = (0, 0, 0 \mid 0, 0, e_\omega), \\
u_{-\omega} = (0, 0, 0 \mid 0, 0, e_{-\omega}),
\]

which are fixed by our copy of \( \Omega_6^+(F) \). The action of the element \( M_{u_\omega} \) on \( V_8 \) preserves \( u_\omega \) but not \( u_{-\omega} \) and therefore adjoining this element to \( \Omega_6^+(F) \) we get the action of the group \( V_6: \Omega_6^+(F) \). Next, the element \( M_{u_{-\omega}} \) does not preserve the 1-space \( \langle u_\omega \rangle \), so appending it to \( V_6: \Omega_6^+(F) \) we get the action of the group \( \Omega_8^+(F) \) on \( V_8 \).
Finally, we consider the 10-space $\mathbb{J}^{abc}_{10}$ with two isotropic Albert vectors
\begin{align}
  u_0 &= (0,0,0 \mid 0,0,e_0), \\
  u_{-0} &= (0,0,0 \mid 0,0,e_{-0}).
\end{align}

Following the same procedure, we adjoin the element $M_{e_0}$ which fixes $u_0$ but not $u_{-0}$ to get the action of the group of shape $V_8: \Omega^+_8(F)$. Appending the action of $M_{e_{-0}}$, which does not preserve $\langle u_0 \rangle$, to this yields the action of $\Omega^+_10(F)$ on $\mathbb{J}^{abc}_{10}$.

Lemma 9.4 allows us to conclude that we have shown the following result.

**Lemma 9.6.** The actions of $M_x$ and $L_x$ on $\mathbb{J}^{abc}_{10}$ generate the group $\Omega^+_10(F)$ as $x$ ranges through $\mathbb{O}$.

Now we need to understand the action of the elements $M_x$ and $L_x$ on the whole 27-space $\mathbb{J}$.

**Lemma 9.7.** Suppose an element $g$ of the stabiliser in $\text{SE}_6(F)$ of $v$ preserves the decomposition of the Albert space into the direct sum of the form $\mathbb{J} = \mathbb{J}^a_{10} \oplus \mathbb{J}^{ab}_{16} \oplus \mathbb{J}^{abc}_{10}$.

(a) If the action of $g$ on the 10-space $\mathbb{J}^{abc}_{10}$ is given by
\begin{align}
  (1,0,0 \mid 0,0,0) &\mapsto (\lambda,0,0 \mid 0,0,0), \\
  (0,1,0 \mid 0,0,0) &\mapsto (0,\lambda^{-1},0 \mid 0,0,0), \\
  (0,0,0 \mid 0,0,C) &\mapsto (0,0,0 \mid 0,0,C),
\end{align}
then $\lambda$ is a square in $F$.

(b) On the other hand, an action of the type
\begin{align}
  (1,0,0 \mid 0,0,0) &\mapsto (0,\lambda,0 \mid 0,0,0), \\
  (0,1,0 \mid 0,0,0) &\mapsto (\lambda^{-1},0,0 \mid 0,0,0), \\
  (0,0,0 \mid 0,0,C) &\mapsto (0,0,0 \mid 0,0,C)
\end{align}
is impossible.

(c) Finally, if the action on the 10-space is trivial, then the action on the corresponding 16-space is that of $\pm 1_{16}$ (hence, the action on $\mathbb{J}$ is that of $P_{3,1}$).

**Proof.** We are considering the elements that fix $\mathbb{J}^C_{8}$ pointwise and either fix or swap the 1-dimensional spaces $\mathbb{J}^a_{10}$ and $\mathbb{J}^b_{10}$. So we may assume that these elements respectively fix or swap the corresponding 17-spaces $\mathbb{J}^{abc}_{17}$ and $\mathbb{J}^{bac}_{17}$. In particular, their intersection, i.e. the space $\mathbb{J}^C_{17}$ is fixed. If the action of the stabiliser swaps $\mathbb{J}^a_{17}$ and $\mathbb{J}^b_{17}$ while leaving the 1-space $\mathbb{J}^C_{17}$ in its place, then it also swaps the 8-spaces $\mathbb{J}^a_{8}$ and $\mathbb{J}^b_{8}$ as these subspaces are the intersections of the 17-space $\mathbb{J}^{ab}_{17}$ with $\mathbb{J}^{bac}_{17}$ and $\mathbb{J}^{abc}_{17}$ respectively.

Suppose now that an element $g$ in the stabiliser acts in the following manner:
\begin{align}
g : (a,b,c \mid A,B,C) &\mapsto (\lambda a, \lambda^{-1}b,c \mid A\phi,B\psi,C),
\end{align}
where $\phi,\psi : \mathbb{O} \to \mathbb{O}$ are invertible $F$-linear maps. As this action is supposed to preserve the determinant, it has to preserve the cubic term $T(ABC)$ in particular, i.e. we must have $T(ABC) = T((A\phi)(B\psi)C)$ for all $A,B,C \in \mathbb{O}$. This is equivalent to the condition $AB = (A\phi)(B\psi)$ for all $A,B \in \mathbb{O}$, since the original identity is equivalent to $(A\phi,\overline{C}) = ((A\phi)(B\psi),\overline{C})$. By Lemma 9.3 we find that $A\phi = \mu^{-1}A$ and $B\psi = \mu B$ for all $A,B \in \mathbb{O}$ and some non-zero $\mu \in F$. The individual terms in the determinant are being changed in the following way:
\begin{align}
abc &\mapsto abc, \\
aAa &\mapsto \lambda \mu^{-2}aAa, \\
bBb &\mapsto \lambda^{-1}\mu^2bBb, \\
cCc &\mapsto cCc, \\
T(ABC) &\mapsto T(ABC).
\end{align}

It follows that in order to preserve the determinant we must have $\lambda^{-1}\mu^2 = 1$, i.e. $\lambda = \mu^2$. 
In case when $g$ acts as
\[ g : (a, b, c \mid A, B, C) \mapsto (\lambda^{-1}b, \lambda a, c \mid B\psi, A\phi, C), \]
we get $T(ABC) = T((B\psi)(A\phi)C)$ for all $A, B, C \in \mathcal{O}$. This holds if and only if $AB = (B\psi)(A\phi)$ for all $A, B \in \mathcal{O}$. Lemma 9.2 asserts that there are no such maps $\phi$ and $\psi$, and so this rules out the latter case.

Finally, if we assume the trivial action on $\mathbb{J}^{\text{bc}}_{10}$, then we get $\lambda = 1$, i.e. $\mu^2 = 1$, so the action on $\mathbb{J}$ is indeed that of $F_{\pm 1}$.

Now let $X = (a, b, c \mid A, B, C)$ and let $Y = (a', b', c' \mid A', B', C')$. An isometry which maps $X$ to $Y$ and $v$ to $\lambda v$ must send $\Delta(X + v) - \Delta(X) = ab - CC'$ to $\Delta(Y + \lambda v) - \Delta(Y) = \lambda(a'b' - C'C')$. The 17-dimensional radical of both of these forms is fixed, and the quadratic form $ab - CC'$ is being scaled by a factor of $\lambda$. In particular, when $\lambda = 1$, the quadratic form is being preserved. So, the action of the vector stabiliser on the 10-dimensional quotient is that of a subgroup of $GO_{10}^+(F)$.

Consider the white vectors of the form $(a, 0, c \mid A, B, 0)$ and $(0, b, c \mid A, B, 0)$ with $a, b \neq 0$. In the first case the conditions for being white are
\[
\begin{aligned}
A\overline{A} &= 0, \\
B\overline{B} &= ac, \\
a\overline{A} &= 0, \\
AB &= 0.
\end{aligned}
\]

In other words, we have a white vector of the form $(a, 0, B\overline{B}/a \mid 0, B, 0)$. For the second vector we get
\[
\begin{aligned}
bc &= A\overline{A}, \\
B\overline{B} &= 0, \\
b\overline{B} &= 0,
\end{aligned}
\]
so the vector has the form $(0, b, A\overline{A}/b \mid A, 0, 0)$. The elements $M'_x$ and $L''_x$ transform these in the following way:
\[
\begin{aligned}
M'_x : (a, 0, B\overline{B}/a \mid 0, B, 0) &\mapsto (a, 0, B\overline{B}/a \mid 0, B, 0), \\
M'_x : (0, b, A\overline{A}/b \mid A, 0, 0) &\mapsto (0, b, A\overline{A}/b + bx + T(xA) \mid A + bx, 0, 0), \\
L''_x : (a, 0, B\overline{B}/a \mid 0, B, 0) &\mapsto (a, 0, B\overline{B}/a + ax + T(Bx) \mid 0, B + ax, 0), \\
L''_x : (0, b, A\overline{A}/b \mid A, 0, 0) &\mapsto (0, b, A\overline{A}/b \mid A, 0, 0).
\end{aligned}
\]

Note that we already have an elementary abelian group $F^{16}$ acting on the 17-space $\mathbb{J}^{\text{ab}}_{17}$. We can now invoke Lemma 9.7 to conclude that the action of the stabiliser on the remaining 10-space $\mathbb{J}^{\text{bc}}_{10}$ is that of $\Omega_{10}^+(F)$ and the kernel of the action on $\mathbb{J}$ has order no more than two.

**Theorem 9.8.** The actions of the elements $M_x$ and $L_x$ on $\mathbb{J}$ where $x$ ranges through a split octonion algebra $\mathcal{O}$ generate a group of type $\text{Spin}^+_1(F)$ understood as $\Omega_{10}^+(F)$ in case of characteristic 2.

With the result of Lemma 9.3 we conclude that the stabiliser of a white vector is indeed a group of shape $F^{16}:\text{Spin}^+_1(F)$ as usual understood as $F^{16}:\Omega_{10}^+(F)$ in case of characteristic 2.

Now we have enough ingredients to produce the vector stabiliser. As before, we consider the stabiliser of the white vector $v = (0, 0, 1 \mid 0, 0, 0)$. As we know from Theorem 9.8 and Lemma 9.7 the actions of the elements $M_x$ and $L_x$ on $\mathbb{J}$ generate a group of type $\text{Spin}^+_1(F)$. It is easy to check that this copy of $\text{Spin}^+_1(F)$ normalises the elementary abelian group $F^{16}$ from Lemma 9.3. A straightforward computation illustrates the following result:
(51)

\begin{align*}
(M_x')^{L_y} & \text{ acts as } M_x', \\
(M_x')^{M_y} & \text{ acts as } L''_{-xy} \cdot M_x', \\
(L_x')^{L_y} & \text{ acts as } M'_{-xy} \cdot L''_x, \\
(L_x')^{M_y} & \text{ acts as } L''_x,
\end{align*}

where the products in the right-hand side are understood as the products of the actions rather than as the matrix products. Furthermore, the intersection of the groups Spin\(^+\)\(_{10}(F)\) and \(F^{16}\) is trivial: the action of Spin\(^+\)\(_{10}(F)\) preserves the decomposition \(\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_{16} \oplus \mathcal{J}_{16}^{A} \oplus \mathcal{J}_{16}^{B} \oplus \mathcal{J}_{16}^{C}\), while any non-trivial action of the elementary abelian group \(F^{16}\) fails to do so. Indeed, a general element in \(F^{16}\) has the form \(M_x' \cdot L''_y\) for some \(x, y \in \mathcal{O}\), and it sends an Albert vector \((0,0,1)_q\) to \((0,0,0)_q\).

Next, we consider the white point \((v)\) spanned by our white vector. The stabiliser in \(\text{SE}_6(F)\) of \((v)\), where \(v = (0,0,1 | 0,0,0)\), maps \(v\) to \(\lambda v\) for some non-zero \(\lambda \in F\). For instance, this can be achieved by the elements

\[P_{u^{-1}} = \text{diag}(1, u^{-1}, u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & u \end{pmatrix}\]

with \(u\) being an invertible octonion of arbitrary norm. Indeed, any such element \(P_{u^{-1}}\) sends \((0,0,1 | 0,0,0)\) to \((0,0,N(u) | 0,0,0)\) and since \(N(u)\) can be any non-zero field element, we get an abelian group \(F^{\times}\) on top of the vector stabiliser. This finishes the proof of the main theorem in this section.

Now, since the vector stabiliser is generated by the actions of \(M_x, L_x, M_x', L''_x\) on \(\mathcal{J}\), and the subgroup of \(\text{SE}_6(F)\) generated by \(M_x, M_x', L_x, L''_x, L''_x\) acts transitively on the white points, we make the following conclusion.

**Theorem 9.9.** The group \(\text{SE}_6(F)\) is generated by the actions of \(M_x, M_x', M''_x\) and \(L_x, L''_x, L''''_x\) on \(\mathcal{J}\) as \(x\) ranges through \(\mathcal{O}\).

If \(F = \mathbb{F}_q\), the the stabiliser of a white vector is a group of shape \(q^{16} : \text{Spin}^+_{10}(q)\), while the stabiliser of a white point is a group of shape \(q^{16} : \text{Spin}^+_{10}(q).C_q - 1\). As a consequence, we now have:

\[|\text{SE}_6(q)| = q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^2 - 1).
\]

We obtain \(\text{E}_6(q)\) as the quotient of \(\text{SE}_6(q)\) by any scalars it contains. Note that \(\text{SE}_6(q)\) contains non-trivial scalars if and only if \(q \equiv 1 \pmod{3}\), so

\[|\text{E}_6(q)| = \frac{1}{\gcd(3, q - 1)} q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^2 - 1).
\]

**10. Simplicity of \(\text{E}_6(F)\)**

The classical way of showing the simplicity of certain groups is the following lemma.

**Lemma 10.1** (Iwasawa). If \(G\) is a perfect group acting faithfully and primitively on a set \(\Omega\), and the point stabilizer \(H\) has a normal abelian subgroup \(A\) whose conjugates generate \(G\), then \(G\) is simple.

First, we show that the subgroup of \(\text{SE}_6(F)\) stabilising all the white points simultaneously acts on \(\mathcal{J}\) by scalar multiplications, and hence the action of \(\text{E}_6(F)\) on the set of white points is faithful.
Lemma 10.2. The subgroup in SE₆(F) stabilising simultaneously all white points is the group of scalars.

Proof. Consider the action of this stabiliser on $\mathbb{J}^{abc}_{10}$ and pick the basis

$$
\begin{align*}
  v_1 &= (1, 0, 0 | 0, 0, 0), \\
  v_2 &= (0, 1, 0 | 0, 0, 0), \\
  v_{i+2} &= (a_i, b_i, 0 | 0, 0, C_i),
\end{align*}
$$

where $1 \leq i \leq 8$ and $C_i \in \mathbb{C}$. Since in particular we stabilise $\langle v_1, ..., v_{10} \rangle$, the action on $\mathbb{J}_{10}^{ABC}$ is that of a $10 \times 10$ diagonal matrix $\text{diag}(\lambda_1, ..., \lambda_{10})$ with respect to the basis $\{v_1, ..., v_{10}\}$. Consider an Albert vector $v = (a, b, 0 | 0, 0, C)$, where $C = C_1 + \cdots + C_8$ and $a, b$ are such that $v$ is white, i.e. $C^2 = ab$. Now, if $F \neq \mathbb{F}_2$, we can choose $a, b \in F$ in such a way that $v$ can be written as a linear combination $v = \alpha v_1 + \beta v_2 + v_3 + \cdots + v_{10}$ with $\alpha \neq 0$. The stabiliser of all white point maps $v$ to $\lambda v$ for some non-zero $\lambda \in F$, so this ensures that $\lambda = \lambda_1 = \lambda_3 = \cdots = \lambda_{10}$. We now adjust the chosen values of $a$ and $b$ to obtain a linear combination with $\beta \neq 0$, and so $\lambda = \lambda_2 = \lambda_3 = \cdots \lambda_{10}$. It follows that the action on $\mathbb{J}_{10}^{abc}$ is just the multiplication by $\lambda$.

When $F = \mathbb{F}_2$, we take $\mathbb{C}$ to be the split octonion algebra with our favourite basis $\{e_i \mid i \in \pm\{0, 1, \omega, \overline{\omega}\}\}$. For the 10-space $\mathbb{J}_{10}^{abc}$ we choose the basis

$$
\begin{align*}
  v_1 &= (1, 0, 0 | 0, 0, 0), \\
  v_2 &= (0, 1, 0 | 0, 0, 0), \\
  v_{i+2} &= (0, 0, 0 | 0, 0, e_i),
\end{align*}
$$

and then proceed in the same manner. The vector $v = v_1 + \cdots + v_{10}$ is white and since there is a single choice for a non-zero scalar in $\mathbb{F}_2$, it is being fixed and the action on the whole 10-space in this case is that of $\text{diag}(1, ..., 1)$.

Now, by using the triality element, we map $\mathbb{J}_{10}^{abc}$ to $\mathbb{J}_{10}^{beA}$ and further to $\mathbb{J}_{10}^{caB}$ and so we obtain that the stabiliser of all white points acts on $\mathbb{J}$ by scalar multiplications. That is, the stabiliser of all the white points is trivial in $E_6(F)$.

From Lemma 7.4, we know that the action of $E_6(F)$ on the white points is primitive. We need to show that the group is perfect.

Lemma 10.3. The group $SE_6(F)$ is perfect.

Proof. This does not present great difficulties. A very straightforward computation shows that

\begin{align*}
  (L'_{-1})^{-1} \cdot L'_{z} \cdot L''_{-1} \cdot (L''_{z})^{-1} &\text{ acts as } M_x, \\
  (L_{-1})^{-1} \cdot L''_{z} \cdot L_{-1} \cdot (L''_{z})^{-1} &\text{ acts as } M'_x, \\
  (L'_{-1})^{-1} \cdot L_{z} \cdot L'_{-1} \cdot (L_{z})^{-1} &\text{ acts as } M''_x, \\
  (M'_{-1})^{-1} \cdot M''_{z} \cdot M'_{-1} \cdot (M''_{z})^{-1} &\text{ acts as } L_x, \\
  (M''_{-1})^{-1} \cdot M_x \cdot M'_z \cdot (M_z)^{-1} &\text{ acts as } L'_{z}, \\
  (M_{-1})^{-1} \cdot M'_z \cdot M'_{-1} \cdot (M'_z)^{-1} &\text{ acts as } L''_{z},
\end{align*}

where as before $A \cdot B$ is understood as the product of the actions by the matrices $A$ and $B$. Hence, every generator is in fact a commutator.

Finally, using the Iwasawa’s Lemma we obtain the following theorem.

Theorem 10.4. The group $E_6(F)$ is simple.
11. Conclusions

We have managed to obtain a self-contained construction of a group of type $E_6$ over an arbitrary field $F$. However, we want to point out that there is space for future research. For example, the main result in Section 9 depends on the fact that the underlying octonion algebra $\mathcal{O}$ is split. This completely covers the possibilities in case $F = \mathbb{F}_q$, but it seems to be quite interesting to understand to what extent it is possible to generalise our result in the case when a non-split octonion algebra exists. The main problem here is to be able to tell whether the actions of the matrices $M_x$ and $L_x$ on $\mathbb{J}_{10}$ generate $\Omega(\mathbb{J}_{10}, \mathbb{Q}_{10})$. At this stage it is possible to prove the following proposition.

**Proposition 11.1.** The actions of the elements $M_x$ and $L_x$ on $\mathbb{J}_{10}$ where $x$ ranges through a non-split octonion algebra $\mathcal{O}$, generate at most a group of type $\Omega(\mathbb{J}_{10}, \mathbb{Q}_{10})$.

**Proof.** It is straightforward to verify that the image of $(a, b, h | 0, 0, C)$ under the action of $M_x$ coincides with the image under the reflexion in $(0, N(x), 0 | 0, 0, x)$ followed by a reflexion in $(0, 0, 0 | 0, 0, x)$. The norm of both vectors is $N(x)$, so we conclude that $M_x$ acts as an element of $\Omega(\mathbb{J}_{10}, \mathbb{Q}_{10})$.

For the element $L_x$ we take the vectors $(N(x), 0, 0 | 0, 0, x)$ and $(0, 0, 0 | 0, 0, x)$ to obtain the same conclusion. □

Now suppose that $V$ is a vector space over $F$ with a quadratic form $Q$, such that $V = \langle e, f \rangle \oplus W$, where $\langle e, f \rangle$ is a hyperbolic pair and $W = \langle e, f \rangle^\perp$. Consider an element $g$ in $\text{CGO}(V, Q)$ which scales $Q$ by some $\lambda \neq 0$. Then $V = \langle e^g, f^g \rangle \oplus W^g$ and $\langle e^g, f^g \rangle$ is isometric to $\langle e, f \rangle$. Therefore, $W^g$ is isometric to $W$, and so there exists an isometry $h$ in $\text{GO}(V, Q)$ such that $\langle e^g, f^g \rangle^h = \langle e, f \rangle$. It follows that $(W^g)^h = W$, and $gh$ is a $\lambda$-scaling of $Q$ which fixes $\langle e, f \rangle$ and $W$. Hence, $gh$ is a $\lambda$-scaling of $Q_W$.

Consider a $\lambda$-similarity on $\mathcal{O} = \mathcal{O}_F$ that sends $1_{\mathcal{O}}$ to some $u \in \mathcal{O}$. Then it gives rise to an element in the stabiliser of a white point which scales $Q_8$ by $N(u)$. In other words, we have shown the following.

**Proposition 11.2.** If $\mathcal{O}$ is an arbitrary octonion algebra over $F$, then the elements in the stabiliser of a white point can only scale a white vector by $\lambda$, where $\lambda 
 F$ is such that there exists $u \in \mathcal{O}$ with $N(u) = \lambda$.

It is easy to check that all such scalings are possible. For example, the elements $P_{u^{-1}}$, defined in [52], do the job.

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School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, UK
E-mail address: j.m.bray@qmul.ac.uk, e.stepanov@qmul.ac.uk, r.a.wilson@qmul.ac.uk