(1 − ϵ)-approximate fully dynamic densest subgraph: linear space and faster update time

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Abstract

We consider the problem of maintaining a (1 − ϵ)-approximation to the densest subgraph (DSG) in an undirected multigraph as it undergoes edge insertions and deletions (the fully dynamic setting). Sawlani and Wang [SW20] developed a data structure that, for any given ϵ > 0, maintains a (1 − ϵ)-approximation with $O(\log^4 n/\epsilon^6$) worst-case update time for edge operations, and $O(1)$ query time for reporting the density value. Their data structure was the first to achieve near-optimal approximation, and improved previous work that maintained a $(1/4 − \epsilon)$ approximation in amortized polylogarithmic update time [BHNT15]. In this paper we develop a data structure for (1 − ϵ)-approximate DSG that improves the one from [SW20] in two aspects. First, the data structure uses linear space improving the space bound in [SW20] by a logarithmic factor. Second, the data structure maintains a (1 − ϵ)-approximation in amortized $O(\log^2 n/\epsilon^4$) time per update while simultaneously guaranteeing that the worst case update time is $O(\log^3 n \log \log n/\epsilon^6)$. We believe that the space and update time improvements are valuable for current large scale graph data sets. The data structure extends in a natural fashion to hypergraphs and yields improvements in space and update times over recent work [BBCG22] that builds upon [SW20].

1 Introduction

The densest subgraph problem (DSG) is the following. Given an undirected (multi)graph $G = (V, E)$ find $\max_{S \subseteq V} \frac{|E(S)|}{|S|}$ where $E(S)$ is the set of edges with both endpoints in $S$. We use $\rho_G(S)$ to denote $\frac{|E(S)|}{|S|}$, the density of $S$ in the graph $G$. DSG is a very well-studied problem with both practical and theoretical appeal. On the practical side it is a core problem in network analysis and graph mining to find clusters and communities. In addition to being directly relevant, it is also a canonical problem in the field of dense subgraph discovery [GT15; LRJA10; TC21]. On the theoretical side DSG is polynomial-time solvable problem and has several important connections to network flow, arboricity, matchings, submodular, and supermodular optimization [BHNT15; CQT22; Fuj09; Fuj80; Gol84; PQ82; SW20]. There have been several recent works on algorithmic aspects of DSG and its variants. A particular emphasis has been on fast and scalable algorithms due to the large scale graph data driving many of the applications. For this reason, even though there are polynomial-time exact algorithms for DSG via network flow [Gol84; PQ82] and submodular function minimization, the recent focus has been on near-linear time constant factor and (1 − ϵ)-approximation algorithms.
and heuristics [BGM14; BGPSTWW20; BSW19; Cha00; CQT22; DCS17; KS09a].\(^1\) Another reason for the focus on approximation is that exact efficient algorithms are unlikely in other models of computing such as streaming [BHINT15; BKV12; MTVV15], parallel and distributed [BGM14; BKV12; GLM19], and dynamic [BHINT15; ELS15; SW20], which are of much importance. Constrained versions of DSG such as the densest \( k \)-subgraph problem [BCCFV10; FPK01; Kho06; Man17] are also of much interest and very well-studied, especially for their theoretical importance. In this paper we focus on the unconstrained DSG problem.

In this paper we are interested in the dynamic setting for DSG where the underlying graph \( G \) undergoes edge insertions and deletions. The goal is to maintain an approximation to the densest subgraph. This is the so-called fully dynamic setting.\(^2\) The two query operations to the data structure are: (i) report the (approximate) value of the densest subgraph and (ii) output an (approximately optimal) densest subgraph. Note that the densest subgraph can be very large, and hence the typical focus is to have a fast query time for the value, and to be able to output the densest subgraph in time proportional to its size. In many applications of interest the underlying graph is large and changes frequently [SMSLÖ17]. In addition to the practical interest, dynamic (graph) algorithms have seen a surge of interest and many exciting new results in recent years, and they have led to numerous breakthroughs in obtaining faster algorithms for a number of fundamental problems in a variety of models — we refer the reader to some surveys [Hen18; HHS21] rather than give pointers to the very large literature. For dynamic DSG, Sawlani and Wang [SW20], in an elegant recent work, developed a data structure that for any given \( \epsilon > 0 \), maintains a \((1 - \epsilon)\)-approximation to the value of the optimum density in worst-case update time of \( O((\log^4 n)/\epsilon^6) \) using \( O(m \log m) \) space, where \( n \) and \( m \) are the number of vertices and edges respectively. Their data structure reports the value of the density in \( O(1) \) time. Their work is the first to maintain an arbitrarily good approximation for the optimum density, improving previous works, and in particular the work of Bhattacharya et al. [BHINT15] that maintained a \((1/4 - \epsilon)\)-approximation with a polylogarithmic amortized update time.

1.1 Motivation and contribution

Our goal is to obtain improved data structures for dynamic DSG. Given the large scale graphs that are common today, a practical concern is the space usage. For instance, if a graph has a million edges then \( \log m \) (to base 2) is about 20. A data structure that uses \( O(m \log m) \) space instead of linear space, like the one in [SW20], may not be able to fit data in main memory even though the constant may appear small in theoretical analysis. Second, it is helpful to obtain improved update times since it will allow for more accurate estimates under some given budget on the update time. In this paper we develop fully dynamic data structures for \((1 - \epsilon)\)-approximate fully dynamic DSG that have the following features:

- The space used is linear in the size of the graph.
- The query time to answer the value of the densest subgraph is \( O(1) \).
- The data structure reports a value \( \lambda \) such that \( \lambda \geq (1 - \epsilon) \text{OPT} - O(\ln n/\epsilon) \) in amortized

\(^1\)Maximum flow has seen a spate of breakthroughs in the last decade or more and a near-linear time algorithm was announced just a few months ago [CKLPGS22]. Despite their theoretical importance, these new algorithms are far from practical at this point in time.

\(^2\)We focus on edge insertions and deletions assuming that the number of vertices is fixed and known. One can handle vertex additions and deletions via edge updates in standard ways. We believe that edge updates are more interesting from both a practical and theoretical point of view for DSG.
$O(\log n/\epsilon^2)$ time for edge updates. Here OPT is the optimum density. A $(1 - \epsilon)$ true approximation can be maintained in amortized $O(\log^2 n/\epsilon^4)$ time.

- The data structure can be extended to have a worst-case update time of $O(\log^3 n \log \log n/\epsilon^6)$ while maintaining the amortized bound of $O(\log^2 n/\epsilon^4)$ for edge operations.

Thus we are able to improve upon the data structure of [SW20] in terms of space, and also obtain faster worst-case and amortized update times. In addition to obtaining improved bounds, our data structure is quite simple and the analysis is self-contained. Like previous data structures for density, ours is also based on maintaining graph orientations to minimize arboricity. Along the way we obtain some new tradeoffs for maintaining arboricity that are of independent interest. We outline these in the next subsection.

**Extensions:** Our data structure for maintaining $(1 - \epsilon)$-approximation for DSG extends in a natural and relatively easy fashion to hypergraphs. We obtain a linear space data structure and the running time for a rank $r$ hypergraph is an $O(r^2)$ factor worse than it is for graphs. Recent work of Bera et al. [BBCG22] has extended the ideas in [SW20] to hypergraphs; our results improve upon theirs in a similar fashion as ours improve upon [SW20] for graphs in terms of space and update times. Bera et al. [BBCG22] showed that one can handle arbitrary edge-weights in graphs and hypergraphs via a sparsification technique [MPPTX15] and a black-box reduction to a data structure for the unweighted case. Their reduction is randomized and assumes an oblivious adversary. Moreover, the reduction requires guessing the optimum density and maintaining a logarithmic number of parallel copies of the unweighted data structure. This increases the space and update times by poly-logarithmic factors when compared to the unweighted case. We can employ their reduction and plug in our data structure for the unweighted case.

### 1.2 Technical overview in the context of related work

The optimum density of a given graph $G$ is closely related to the graph theoretic notion of **arboricity** that we now define. Given an undirected graph $G = (V,E)$, an orientation of $G$ is a directed graph $D = (V,A)$ that is obtained from $G$ by orienting each edge $\{u,v\} \in E$ either as the arc $(u,v)$ or as the arc $(v,u)$. Given an orientation $D$ of $G$, let $\text{arb}(G,D)$ be the maximum in-degree among vertices in $D$. The arboricity of $G$, denoted by $\text{arb}(G)$, is the minimum $\text{arb}(G,D)$ over all orientations $D$ of $G$. A well-known theorem of Nash-Williams [Nas64] implies that $\text{arb}(G)$ is equal to the minimum number of forests that are needed to cover the edge set $E$. Further, $\text{arb}(G)$ can be computed in polynomial time. One can show that $\rho(G)$, the optimum density of a graph $G$, corresponds to the **fractional** arboricity of $G$; by fractional one means that an edge $\{u,v\} \in E$ is now allowed to be fractionally oriented both as $(u,v)$ and $(v,u)$ (with the sum of the non-negative fractions summing to one) and the goal is to minimize the maximum fractional in-degree of the nodes. A fractional orientation can be viewed as a solution to an exact LP relaxation for DSG suggested by Charikar [Cha00] (this has been noted in several papers including [BGM14; BGPSTWW20; SW20]). It can be shown that the fractional and integral arboricity differ by at most 1.

Independent of the connection to DSG, dynamic maintainance of the arboricity of a graph has received attention in the data structures community since it has connections to the problem of maintaining fast adjacency queries in low arboricity graphs (such as planar graphs) [KNR92]. Several papers, starting with the work of Brodal and Fagerburg [BF99], developed dynamic algorithms (and analysis) for maintaining approximate arboricity. The initial papers had amortized complexity bounds assuming that the arboricity was guaranteed to be upper bounded by a given bound $\alpha$ [BF99; Kow07]. Kopeliwitz et al [KKPS14] were the first to obtain a data structure that had
polylogarithmic worst-case update time. Specifically, their algorithm maintained an orientation such that the maximum in-degree of the orientation is \((1 + \epsilon) \text{arb}(G) + O(\log n/\epsilon)\). However the update time depended on the arboricity (in the application of interest, arboricity was small and this was not a limitation). In these papers the focus was on maintaining a somewhat loose approximation to the arboricity (such as a constant factor with the precise constant left unspecified) as the approximation translated into running time rather than the quality of a solution. Subsequently, motivated by application to DSG, Bhattacharya et al. [BHNT15] developed a data structure that maintained constant factor approximation to the arboricity in poly-logarithmic amortized update time and linear space (see also [HNW20]). Their update time did not depend on the arboricity. Using this, [BHNT15] showed that a \((1/4 - \epsilon)\)-approximate densest subgraph can be maintained in amortized poly-logarithmic update time. We note that there is no data structure so far that can maintain a constant factor approximation to the arboricity with a worst-case poly-logarithmic update time.

To obtain a \((1 - \epsilon)\)-approximation to density with worst-case update time, Sawlani and Wang [SW20] use two important ideas. First, they exploited the fact that density corresponds to fractional arboricity. This additional flexibility allowed them to make copies of edges and assume that the arboricity is sufficiently large. They are then able to use the above mentioned algorithm of Kopelwitz et al [KKPS14] which maintain \((1 + \epsilon)\)-approximation to arboricity but has an additive error which can be absorbed when arboricity is large. To overcome the issue that the update time of the data structure in [KKPS14] has a dependence on the arboricity, [SW20] uses a simple form of scaling by “guessing/estimating” the optimum density. However, the price to make this idea work is that [SW20] need to maintain \(\Omega(\log m)\) copies of the data structure from [KKPS14], one for each potential value of the density (within a factor of 2). They need to dynamically adjust all these data structures simultaneously, and need additional modifications to overcome the running time dependency in [KKPS14] on the arboricity, which could translate to bad running times for the copies of the data structure corresponding to scales smaller than the optimum density. This is the reason for an additional logarithmic factor in the space and some complexity in the implementation of the data structure.

In this paper we build on previous ideas but take a somewhat different approach. The algorithms for maintaining a low arboricity orientation (here and in prior work) use certain invariants that try to balance the in-degrees between adjacent vertices — the high-level goal is to orient each edge towards the lower degree vertex. They flip (i.e., reorient) edges to maintain the invariants as edges are inserted and deleted. However, flipping one edge can lead to flipping an adjacent edge and eventually cause a cascading sequence of flips, so we require a careful analysis to argue about the update time and the quality of the orientation. Our first idea is that one can maintain an orientation of the graph such that maximum in-degree is at most \((1 + \epsilon) \text{arb}(G) + O(\log n/\epsilon)\) in amortized \(O(\log n/\epsilon)\) update time. To obtain a worst-case guarantee we alter the update algorithm in two important, but relatively simple (in retrospect) ways. Note that, unlike the work in [KKPS14], the update time does not depend on the arboricity while providing the same guarantee. As far as we are aware, there was no previous dynamic algorithms that maintains an orientation with maximum in-degree arbitrarily close to arboricity in worst-case (or even amortized) polylogarithmic update time in the large arboricity regime. Once we have the above guarantee, we can use the idea of duplicating edges (which can be done implicitly and does not add to the space) to maintain a \((1 - \epsilon)\)-approximation for fractional orientations which corresponds to density.

Our data structures maintain a local optimality invariant on the orientation that differs in a simple but crucial way from that of [KKPS14]. In a certain sense, we take a first-principles approach to maintaining orientations with both additive and multiplicative slack, given that density maintenance can absorb the additive slack via edge duplication. We believe that this leads to a
clean and improved data structure. The transparency of the analysis also allowed us to improve the worst-case update by exploiting the different behavior of the basic data structure when the arboricity is large and when it is small.

1.3 Other related work

The first polynomial-time algorithm for DSG was via a reduction to network flow [Gol84; PQ82]; the decision problem of whether \( \rho(G) \geq \lambda \) can be solved via \( s-t \) maxflow in an auxiliary graph which has \( |E| \) edges and \( |E| \) vertices. Combining this binary search over \( \lambda \) yields an algorithm to find the optimum density. This leads to a near-linear time algorithm via the current fastest algorithm for \( s-t \) maximum flow when the edge and vertex weights are polynomially bounded [CKLPGS22]. One can also derive a polynomial-time algorithm via reduction to submodular function minimization via the following observation: for any graph \( G = (V, E) \), the set function \( f : 2^V \rightarrow \mathbb{R}_+ \) defined by \( f(S) = |E(S)| \) is supermodular.\(^3\) The supermodularity perspective allows one to handle generalizations of DSG and other problems — we refer the reader to [CQT22; Fuj09; VBK21]. Charikar, in an influential work [Cha00], showed that a simple greedy algorithm [AITT00] yields a \( \frac{1}{2} \)-approximation for DSG, and he also described an LP relaxation that is exact for DSG. The dual of this LP can be viewed as finding the minimum degree fractional orientation of the given graph. We discuss more details in section 2. The LP relaxation has led to several fast \((1-\epsilon)\)-approximation algorithms via mathematical programming and flow techniques [BGM14; BGPSTWW20; BSW19; CQT22].

As we remarked, algorithms for DSG have been explored in the last few years in streaming, mapreduce, parallel and distributed, and dynamic settings, starting with the work of [BKV12]. In terms of dynamic data structures, around the same time as [BHNT15], Epasto et al [ELS15] described a data structure that maintained a \((1/2-\epsilon)\)-approximation with insertions only in amortized \( O\left(\frac{1}{\epsilon^2} \log^2 n\right) \) time; they also generalized their result to handle random deletions with slightly worse update time. Hu, Wu and Chan [HWC17] were the first to consider dynamic densest subhypergraph. Their results are parametrized by the rank \( r \) of the hypergraph. They maintained a \( \frac{1-\epsilon}{2} \)-approximation in the insertions only case, and a \( \frac{1-\epsilon}{r^2} \)-approximation in the fully dynamic case. Their update time is amortized \( \text{poly}\left(\frac{r}{\epsilon} \log n\right) \). Recently Bera et al. [BBCG22], building upon [SW20], showed that one can maintain a \((1-\epsilon)\)-approximation for densest subhypergraph in the fully dynamic setting with a worst-case update time of \( \text{poly}\left(\frac{r}{\epsilon} \log n\right) \). As we mentioned, they also extended the algorithm to the weighted case, via randomization, against oblivious adversaries.

Kannan and Vinay introduced a directed graph version of DSG [KV99]. Charikar [Cha00] showed that it can be solved exactly via a reduction to polynomial number of DSG instances with vertex weights. Improvements in the running time were made by Saha and Khuller [KS09b]. Sawlani and Wang [SW20] showed that one can obtain a \((1-\epsilon)\)-approximation for directed DSG via \( O\left(\log n/\epsilon\right) \) instances of DSG with vertex weights. [SW20] claimed that their dynamic data structure for DSG extended to vertex-weighted graphs, and via their reduction, claimed a fully dynamic \((1-\epsilon)\)-approximate algorithm for directed DSG. We encountered some technical difficulties while trying to extend our data structure to the vertex-weighted setting; we were also unable to verify the correctness of the data structure in [SW20] due to an important missing technical detail. Bhattacharya et al. [BHNT15] describe a dynamic data structure for directed DSG that maintains a \((1/8-\epsilon)\)-approximation in amortized polylogarithmic update time. They also rely on a reduction to undirected graphs, but their reduction is based on the one in [KS09b] and loses a factor of 2 in the approximation unlike the one in [SW20]. In future work we plan to address dynamic DSG for vertex-weighted undirected graphs and directed DSG.

\(^3\)A real-value set function \( f : 2^V \rightarrow \mathbb{R} \) is submodular iff \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \) for all \( A, B \subseteq V \). A set function is supermodular iff \(-f\) is submodular.
2 Preliminaries

Let $G = (V, E)$ be an undirected multigraph. Let $A$ denote the set arcs obtained by bi-directing each edge. A fractional orientation of $G$ is a function $y : A \rightarrow [0, 1]$ such that for each undirected edge $e$ we have $y(a) + y(a') = 1$, where $a$ and $a'$ are the bi-directed arcs corresponding to $e$. $y$ is an integral orientation if $y(a) \in \{0, 1\}$ for each arc. We call an integral orientation simply an orientation, and use $D$ to denote the directed graph induced by the orientation. All quantities discussed here will refer to the current state of the grant and orientation as it evolves with edge insertions and deletions. For a directed graph $D$ and a vertex $v$ we use $\delta^-(v)$ and $\delta^+(v)$ to denote the set of incoming arcs into $v$, and the set of outgoing arcs out of $v$ respectively. We let $N^-(v)$ to denote the in-neighborhood of $v$ with respect to $D$; i.e., the endpoints $w$ of arcs $(w, v) \in \delta^-(v)$ directed into $v$. Similarly, we let $N^+(v)$ to denote the out-neighborhood of $v$ with respect to $D$; i.e., the endpoints $w$ of arcs $(v, w) \in \delta^+(v)$ directed out of $v$.

2.1 Min-max orientations and an LP relaxation for DSG

We consider the LP formulation for DSG from [Cha00]. Recall that the objective is to find a set $S \subseteq V$ to maximize the quantity $|E(S)|/|S|$. A natural way to write this is via indicator variables $x_v, v \in V$ for inclusion in the optimum set $S$. An edge $e$ can be taken only if both $u$ and $v$ are in the set $S$. Hence one can express the objective as max $\sum_{e = (u, v) \in E} \min\{x_u, x_v\}$ with $x \in \{0, 1\}^V$. To express this via an LP relaxation, one can normalize the denominator with the constraint $\sum_v x_v = 1$, and rewrite the convex objective $\sum_{e = (u, v) \in E} \min\{x_u, x_v\}$ as a linear objective via additional variables (which we omit). The LP and its dual are described in fig. 1. The dual LP describes a fractional orientation of $E$ to minimize the maximum in-degree of any vertex. Here, for an edge $e$ and endpoint $v$, $y(e, v) \geq 0$ represents the fractional amount of $e$ directed towards $v$. We let $\text{OPT}_{LP}$ denote the common optimum value of the above linear programs. Charikar showed that the LP is an exact relaxation for DSG and hence $\text{OPT}_{LP} = \rho(G)$.

Given a fractional orientation $y$, and a vertex $v$, we let $\bar{y}(v) \overset{\text{def}}{=} \sum_{e \in \partial(v)} y(e, v)$ denote the weighted in-degree of $v$. The dual LP wants to minimize max$_v \bar{y}(v)$.

2.2 Locally optimal and approximate locally optimal orientations

A useful and important idea for arboricity and density maintenance is the idea of an (approximately) locally optimal orientation that has been explored in previous work. We discuss some basics for the sake of completeness before stating a specific approximate variant that we work with. Consider the following local constraint for $y$.

\[ \text{For each edge } e = \{u, v\}, \text{ if } y(e, v) > 0 \text{ then } \bar{y}(v) \leq \bar{y}(u). \]

\[ \overset{\text{def}}{=} \text{Here a pair } \{u, v\} \text{ will repeat in the sum according to the multiplicity of the edge.} \]
The idea is that if $y$ wants to minimize the maximum in-degree, then it should never fractionally direct an edge towards an endpoint with (strictly) larger in-degree. We say that $y$ is locally optimal when it satisfies the condition above for all arcs. The following lemma can be shown.

**Lemma 2.1.** There is an optimum solution to the dual LP satisfying the local optimality condition. Conversely, if $y$ satisfies the local optimality condition above, then $y$ is optimal.

We now consider approximations of the local optimality condition. For $\alpha, \beta \geq 0$, we say that $y$ is $(\alpha, \beta)$-locally optimal, or a local $(\alpha, \beta)$-approximation, if for all edges $e = \{u, v\}$,

$$
\text{If } y(e, v) > 0, \text{ then } \bar{y}(v) < (1 + \alpha)\bar{y}(u) + \beta.
$$

Intuitively, the condition states that $y$ should never fractionally direct an edge towards an endpoint with substantially larger in-degree.

This local optimality condition extends ideas from [KKPS14; SW20]. In terms of the definition above, [SW20] considered $(0, \epsilon \lambda)$ where $\lambda$ is a constant factor estimate of the optimum density, which is inspired by [KKPS14] who consider $(0, c)$ for some fixed constant $c$. A simple but key idea in our work is to introduce the multiplicative dimension given by $\alpha > 0$. Following similar ideas in previous work we show that approximate local optimality implies approximate global optimality.

**Lemma 2.2.** Let $\alpha, \beta > 0$ with $\alpha < c / \log n$ for a sufficiently small constant $c > 0$. Let $\mu = \max_v \bar{y}(v)$. Let $k = \lceil \log_{1+\epsilon}(n) \rceil$, and suppose that every arc $(u, v)$ is $(\alpha, \beta)$-locally optimal. Then

$$
\mu \leq e^{O\left(\sqrt{\alpha \log(n)}\right)} \left( \text{OPT}_{\text{LP}} + O\left(\sqrt{\frac{\log(n)}{\alpha}}\right) \beta \right).
$$

In particular, given $\epsilon \in (0, 1)$, for $\beta = O(1)$ and $\alpha = ce^{2}/\log(n)$ for a sufficiently small constant $c$, then the claim is as follows: if every arc $(u, v)$ is $(ce^{2}/\log n, O(1))$-locally optimal, then $\mu \leq (1 + \epsilon) \text{OPT}_{\text{LP}} + O(\ln(n)/\epsilon)$.

*Proof.* We define an increasing sequence $\mu_0 < \mu_1 < \cdots$ where $\mu_0 = 0$ and $\mu_i = (1 + \alpha)\mu_{i-1} + \beta$ for $i \geq 1$. Let $k \in \mathbb{Z}_{\geq 0}$ be the unique index such that $\mu_{k-1} \leq \mu < \mu_k$. Observe that

$$
\mu_{k-i} \geq \frac{\mu}{(1 + \alpha)^i} - i\beta \geq e^{-\alpha i} \mu - i\beta
$$

for each $i \in \{1, \ldots, k\}$.

For each index $i \in \{1, \ldots, k\}$, let $S_i = \{v : d^-(v) \geq \mu_{k-i}\}$. Note that $S_i$ is nonempty for all $i \leq k$. Additionally, by $(\alpha, \beta)$-local optimality, we have $N^-(S_i) \subseteq S_{i+1}$ for each $i$. 

![Figure 1: LP for DSG and its dual.](image-url)
Now, let $\epsilon > 0$ be a sufficiently small parameter to be chosen later. Since $S_1$ is non-empty and $|S_i| \leq n$ for all $i$, there must be an index $i \leq O(\log(n)/\epsilon)$ such that $|S_{i+1}| \leq (1 + \epsilon)|S_i|$. Consider the subgraph induced by $S_{i+1}$. Every vertex in $S_i \subseteq S_{i+1}$ is the head of at least \[ e^{-\alpha i} \mu - i \beta \geq e^{-O(\alpha \log(n)/\epsilon)} \mu - O\left(\frac{\log(n)}{\epsilon}\right) \beta \] fractional edges in the orientation, and the underlying undirected edges are all contained in the subgraph induced by $S_{i+1}$. Therefore we have \[ \text{OPT}_{LP} \geq \frac{|E(S_{i+1})|}{|S_{i+1}|} \geq \left( e^{-O(\alpha \log(n)/\epsilon)} \mu - O\left(\frac{\log(n)}{\epsilon}\right) \beta \right) \frac{|S_i|}{|S_{i+1}|} \geq \frac{1}{1 + \epsilon} \left( e^{-O(\alpha \log(n)/\epsilon)} \mu - O\left(\frac{\log(n)}{\epsilon}\right) \beta \right). \] Rearranging, \[ \mu \leq e^{O(\alpha \log(n)/\epsilon)} \left( (1 + \epsilon) \text{OPT}_{LP} + O\left(\frac{\log(n)}{\epsilon}\right) \beta \right) \leq e^{O(\alpha \log(n)/\epsilon) + \epsilon} \left( \text{OPT}_{LP} + O\left(\frac{\log(n)}{\epsilon}\right) \beta \right). \] For $\epsilon = O\left(\sqrt{\alpha \log(n)}\right)$, we have \[ \mu \leq e^{O\left(\sqrt{\alpha \log(n)}\right)} \left( \text{OPT}_{LP} + O\left(\sqrt{\frac{\log(n)}{\alpha}}\right) \beta \right), \] as desired. ■

The proof also shows that it is easy to extract an approximate densest subgraph from $y$; it is always a prefix of the list of vertices in descending order of in-degree. Moreover, identifying a prefix reduces to keeping track of the cardinalities of the sets $S_i$ as defined in the proof, and identifying an index $i$ such that $|S_{i+1}| \leq (1 + \epsilon)|S_i|$. The data structures described in the rest of this article maintain integral orientations of undirected and unweighted graphs. It will be clear from their description that it is easy to maintain a list of the vertices in decreasing order of in-degrees, as well as the cardinalities of the sets $S_i$ so that we always know which prefix of the list induces an approximate densest subgraph. We refer to this as an implicit representation of the approximate densest subgraph; in particular, we can list off the vertices in an approximate densest subgraph in $O(1)$ time per vertex. Later on, to simplify the presentation of these data structures, we will focus on the aspects maintaining the orientation is nearly optimal maximum in-degree rather than on aspects of maintaining an approximate densest subgraph. We assume that an implicit list representation of an approximate densest subgraph, as described above, is maintained in the background with no significant overhead.

3 Data structure with amortized update time guarantee

In this section we describe a simple fully dynamic data structure for maintaining $(1 - \epsilon)$-approximate densest subgraph that has polylogarithmic amortized update time. It is based on maintaining an approximate orientation of the graph. Here we recall that $\text{OPT}_{LP}$ refers the optimum fractional orientation and $\text{arb}(G)$ refers the optimum integral orientation. We have $\text{OPT}_{LP} \leq \text{arb}(G) \leq \text{OPT}_{LP} + 1$. Our data structures work with integral orientations, and they have additive error when comparing with $\text{OPT}_{LP}$ or $\text{arb}(G)$. We address the additive error after the following theorem.
**Theorem 3.1.** Consider the task of approximating the densest subgraph in an unweighted graph dynamically updated by edge insertions and deletions. Let $\epsilon > 0$ be given. Then one can maintain an orientation (explicitly) with maximum in-degree at most $(1 + \epsilon)\text{OPT}_{LP} + O(\log(n)/\epsilon)$, and a subgraph (implicitly) with density at least $(1 - \epsilon)\text{OPT}_{LP} - O(\log(n)/\epsilon)$, in $O(\log(n)/\epsilon^2)$ amortized time per update. The data structure uses $O(m + n)$ space.

Theorem 3.1 gives a $(1 - \epsilon)$-approximation in $O(\log(n)/\epsilon^2)$ amortized time when the density is at least $\Omega(\log(n)/\epsilon^2)$. This regime may already be of interest for many applications of densest subgraph. To obtain an unconditional $(1 - \epsilon)$-approximation, one may simply duplicate each edge $O(\log(n)/\epsilon^2)$ times (as done by [SW20]), which ensures the density is sufficiently large, while increasing the time for each edge insertion and deletion multiplicatively by the same factor. We note that to maintain linear space usage, one needs to make minor modifications so that the copies of an edge use the same auxiliary data. We address these changes at the end of this section (in section 3.4).

**Corollary 3.2.** Under the same conditions of theorem 3.1, one can maintain a subgraph with density at least $(1 - \epsilon)\text{OPT}_{LP}$ in $O\left(\log^2(n)/\epsilon^4\right)$ amortized time per update and $O(m + n)$ space.

### 3.1 High-level overview

Let $\alpha = c\epsilon^2/\ln(n)$, for a sufficiently small constant $c$. We present a data structure that tries to maintain a $(1 + O(\alpha), O(1))$-locally optimal orientation as edges are inserted and deleted. At a high-level, it automatically flips an arc whenever it detects that local optimality for that arc is no longer satisfied. In designing such a data structure there are two high-level concerns. The first is to develop an organizing system to efficiently detect arcs that violate the inequality. The second is to control or account for the running time spent on “cascades”, where flipping one arc leads to violating local optimality for other adjacent arcs, hence further arc flips.

**Arc labels.** The main ingredient in the data structure, and the only auxiliary data stored in the data structure beyond the orientation itself, is a set of integer endpoint labels $\varphi(u \mid a)$ and $\varphi(v \mid a)$ for each arc $a = (u, v)$. The labels play a key role both in maintaining the local optimality conditions, and as a foothold for an amortized analysis that can account for cascades of flips.

When an arc $a = (u, v)$ is added to the orientation, we record the values of $d^{-}(u)$ and $d^{-}(v)$ (just after adding the arc $a$) as $\varphi(u \mid a)$ and $\varphi(v \mid a)$, respectively. Periodically the data structure resets $\varphi(u \mid a)$ and $\varphi(v \mid a)$ to the current values of $d^{-}(u)$ and $d^{-}(v)$.

As mentioned above, the first role of the labels is to help maintain and certify local optimality. As we will show in section 3.2, the data structure maintains $\varphi(u \mid a)$ and $\varphi(v \mid a)$ such that

$$\varphi(u \mid a) \leq (1 + \alpha)d^{-}(u) + O(1), \quad d^{-}(v) \leq (1 + \alpha)\varphi(v \mid a) + O(1), \quad \text{and} \quad \varphi(u \mid a) \leq \varphi(v \mid a) + 1. \quad (1)$$

Combining these inequalities implies $(1 + O(\alpha), O(1))$-local optimality.

The second role is to help amortize the time spent processing the arcs, particularly in the presence of cascades. At a high-level, the data structure only does work on an arc $a = (u, v)$ if $d^{-}(u)$ is much smaller than $\varphi(u \mid a)$ or $d^{-}(v)$ is much larger than $\varphi(v \mid a)$. Meanwhile $\varphi(u \mid a)$ and $\varphi(v \mid a)$ reflect the values of $d^{-}(u)$ and $d^{-}(v)$ at an earlier amount of time. Thus we only process an arc $a$ after $d^{-}(u)$ or $d^{-}(v)$ has deviated substantially from the point when the labels for $a$ were set. These observation translates to an amortized running time via a charging scheme described in section 3.3.
\begin{figure}[h]
\begin{lstlisting}
insert(e = \{u, v\})
1. We assume \(d^-(v) \leq d^-(u)\). (Otherwise swap \(u\) and \(v\).)
2. Add \(a = (u, v)\) to the orientation, set \(\varphi(v|a) = d^-(v)\), set \(\varphi(u|a) = d^-(u)\) and call check-inc(\(v\)).

delete(e = \{u, v\})
1. We assume \(e\) is oriented as \(a = (u, v)\).
2. Delete \(a\) from the orientation and call check-dec(\(v\)).

check-inc(\(v\))

/* We call this routine whenever \(d^-(v)\) has increased (always by 1). */
1. While there are arcs \(a = (u, v) \in \delta^-(v)\) s.t. \(d^-(v) > (1 + \alpha)\varphi(v|a) + 1\)
   A. If \(d^-(u) < d^-(v)\):
      1. Flip \(a\) to \((v, u)\), and set \(\varphi(u|a) = d^-(u)\) and \(\varphi(v|a) = d^-(v)\).
      /* This also restores \(d^-(v)\) to its previous value and fixes the invariant for all arcs in \(\delta^-(v)\). */
      2. Recurse by calling check-inc(\(u\)), and return.
   B. Otherwise set \(\varphi(u|a) = d^-(u)\) and \(\varphi(v|a) = d^-(v)\).

check-dec(\(u\))

/* We call this routine whenever \(d^-(u)\) has decreased (always by 1). */
1. While there is an arc \(a = (u, v) \in \delta^+(u)\) s.t. \(\varphi(u|a) > (1 + \alpha)d^-(u) + 1\)
   A. If \(d^-(u) < d^-(v)\):
      1. Flip \(a\) to \((v, u)\). Set \(\varphi(u|a) = d^-(u)\) and \(\varphi(v|a) = d^-(v)\).
      /* This also restores \(d^-(u)\) to its previous value. */
      2. Recurse by calling check-dec(\(v\)), and return.
   B. Otherwise set \(\varphi(u|a) = d^-(u)\) and \(\varphi(v|a) = d^-(v)\).
\end{lstlisting}
\caption{Dynamically approximating the min-max orientation in an unweighted graph with fast amortized update times.}
\end{figure}

The data structure: Pseudocode for the data structure is presented in fig. 2. Clearly it is very simple. At a high-level, the data structure adds and deletes arcs as requested and then makes local flips and resets arc labels to repair the inequalities in (1) whenever they are violated. When inserting an edge \(e = \{u, v\}\), it orients \(e\) towards the vertex with smaller in-degree. When deleting an edge \(e\), it removes the corresponding oriented arc. These operations increase or decrease the in-degree \(d^-(v)\) of an endpoint \(v\), and may violate the inequalities in (1) above, which relate \(d^-(v)\) to the labels \(\varphi(v|a)\) for arcs \(a\) in \(\delta^-(v)\) or \(\delta^+(v)\). To check and repair these inequalities we introduce two subroutines check-inc(\(v\)) and check-dec(\(v\)).

We call check-inc(\(v\)) whenever the in-degree of a vertex \(v\) is increased. The subroutine checks for any arcs \(a = (u, v) \in \delta^-(v)\) where \(d^-(v)\) has become too large relative to \(\varphi(v|a)\). For each such arc \(a\), depending on whether or not \(d^-(u) < d^-(v)\), it either flips \(a\) (restoring \(d^-(v)\) to its previous value) and resets \(\varphi(v|a)\), or relabels \(\varphi(v|a) = d^-(v)\). A flip would increase \(d^-(u)\), so in this case we recurse on \(u\).
The other routine, \texttt{check-dec}(u), is similar to \texttt{check-inc} except it is for the case where the in-degree of a vertex \( u \) is decreased. \texttt{check-dec}(u) makes sure that \( d^-(u) \) is not too much smaller than \( \varphi(u | a) \) for any arc \( a \in \delta^+(u) \). When violations are found, we either flip the violating arc or reset its label. A flip leads to a recursive call to \texttt{check-dec} on the opposite endpoint, hence possibly more flips.

The point of the calls to \texttt{check-inc} and \texttt{check-dec} is to ensure that the label inequalities are met for all arcs in the orientation. We call these subroutines appropriately whenever an in-degree change and a violation might be created. The conditional loops in these subroutines ensure the subroutines do not terminate until all violating labels are addressed.

### 3.2 Maintaining a \((1 + O(\alpha), O(1))\)-locally optimal orientation

We now prove formally that the data structure maintains a \((1 + O(\alpha), O(1))\)-locally optimal orientation. The local optimality is certified via the arc labels as described above.

**Lemma 3.3.** For all arcs \( a = (u, v) \), we have \( \varphi(u | a) \geq \varphi(v | a) - 1 \).

*Proof.* The labels \( \varphi(u | a) \) and \( \varphi(v | a) \) are set only in two situations. The first setting is when we add \( a \) to the orientation, either upon inserting \( \{u, v\} \), or from flipping \( (v, u) \) to \( (u, v) \). When inserting \( \{u, v\} \), the choice of orientation implies that \( d^-(v) \leq d^-(u) \) before adding \( a \), so we have \( \varphi(v | a) \leq \varphi(u | a) + 1 \) as desired. When flipping \( (v, u) \) to \( (u, v) \), we have \( d^-(v) \leq d^-(u) - 1 \) before flipping, hence \( \varphi(v | a) \leq \varphi(u | a) + 1 \).

The second setting where we reset \( \varphi(u | a) \) and \( \varphi(v | a) \) is after we choose not to flip \( (u, v) \) in either \texttt{check-inc}(v) or \texttt{check-inc}(u). In either case we would have just verified that \( d^-(u) \geq d^-(v) \), hence \( \varphi(u | a) \geq \varphi(v | a) \) as well.

**Lemma 3.4.** For all arcs \( a = (u, v) \), we have \( d^-(v) \leq (1 + \alpha)\varphi(v | a) + 1 \).

*Proof.* Fix \( a \). If \( d^-(v) \) momentarily increases as to violate the desired inequality, then in the subsequent call to \texttt{check-inc}(v), the data structure will continue to process edges in \( \delta^-(v) \) until either (a) it flips some arc (possibly \( a \)) in \( \delta^-(v) \) or (b) resets \( \varphi(v | a) \). In event (a), \( d^-(v) \) is decreased to its previous value before the inequality was violated. In event (b), we set \( \varphi(v | a) = d^-(v) \) which satisfies the inequality.

**Lemma 3.5.** For all arcs \( a = (u, v) \), \( \varphi(u | a) \leq (1 + \alpha)(d^-(u) + 1) \).

*Proof.* The proof is similar to the proof of lemma 3.4. Fix \( a \). If \( d^-(u) \) momentarily decreases as to violate the desired inequality, then in the subsequent call to \texttt{check-dec}(u), the data structure will continue to pull process edges in \( \delta^+(u) \) until either (a) it flips some arc (possibly \( a \)) in \( \delta^+(u) \) or (b) resets \( \varphi(u | a) \). In event (a), \( d^-(u) \) is increased to its previous value before the inequality was violated. In event (b), we set \( \varphi(u | a) = d^-(u) \) which satisfies the inequality.

**Lemma 3.6.** The orientation is always \((1 + O(\alpha), O(1))\)-locally optimal.

*Proof.* Fix \( a = (u, v) \). By lemmas 3.3–3.5 we have
\[
d^-(v) \leq (1 + \alpha)\varphi(v | a) + 1 \leq (1 + \alpha)\varphi(u | a) + 2 + \alpha \leq (1 + \alpha)^2 d^-(u | a) + 3 + 3\alpha,
\]
as desired.
3.3 Running time analysis

The description above establishes that the data structure maintains a \((1 + O(\alpha), O(1))\)-locally optimal orientation, which implies global optimality. In this section we address the remaining issue of (amortized) running time. We mention that in addition to the work described in the pseudocode, the data structure also maintains a list representation of the vertices of an approximate densest subgraph in the background. As discussed at the end of section 2.2, this is fairly simple to do with negligible overhead as it largely consists of maintaining a list of the vertices in decreasing order of in-degree. We have omitted these details from the pseudocode as we feel they distract from the main points of the analysis.

We now focus on analyzing the algorithm pertaining to the pseudocode. To simplify the discussion we first explain how, with some simple auxiliary data structures, each step in the pseudocode takes constant time.

In particular, we explain how to organize the arcs so that in the loops of \texttt{check-inc} and \texttt{check-dec}, each arc \(a\) can be generated in \(O(1)\) time. For each vertex \(v\), we maintain the arcs in \(\delta^-(v)\), and the arcs in \(\delta^+(v)\), in order of \(\varphi(v|a)\), in two nested doubly linked lists. We first describe the construction for \(\delta^-(v)\). We place each arc in \(\delta^-(v)\) in a doubly linked list consisting of all arcs \(a\) in \(\delta^-(v)\) with the same label \(\varphi(v|a)\). We then place these lists in an outer doubly linked list, in order of label. We also maintain maintain a pointer to the location of the first list of arcs \(a\) with label \(\varphi(v|a)\) greater than equal to \(d^-(v)\). This allows for the following constant time operations. First, we can retrieve the minimum or maximum label in constant time. Second, we can insert a new arc \(a \in \delta^-(v)\) with label \(a = d^-(v)\) in constant time.

For \(\delta^+(u)\), we construct the same data structure as described above except with respect to the labels \(\varphi(u|a)\).

With the arcs in \(\delta^+(v)\) and \(\delta^-(v)\) sorted as described above, we can make each iteration of the loops in \texttt{check-inc} and \texttt{check-dec} run in \(O(1)\) time by querying these data structures for the minimum or maximum label arc. When updating \(\varphi(u|a)\) or \(\varphi(v|a)\) for an arc \(a = (u,v)\), since these labels are set to \(d^-(u)\) and \(d^-(v)\), we can use our additional pointers to insert them into the appropriate lists in \(O(1)\) time.

We now move onto the amortized analysis of the data structure with the understanding that each line of the pseudocode takes constant time. The main issue is that the recursive calls in \texttt{check-inc} and \texttt{check-dec} can potentially lead to many flips for a single insertion and edge deletion, and the challenge is to amortize these flips. At a high-level, the analysis observes that an arc \(a = (u,v)\) is processed only when \(d^-(u)\) or \(d^-(v)\) have deviated substantially from the labels \(\varphi(u|a)\) and \(\varphi(v|a)\), respectively. We devise a charging scheme that allows us to amortize the time processing \(a\) against the change to \(d^-(u)\) or \(d^-(v)\). Meanwhile each edge insertion or edge deletion (after accounting for the full chain of local flips) ultimately changes the in-degree of a single vertex by 1, which is reflected in the amortized cost.

**Lemma 3.7.** \texttt{insert} and \texttt{delete} take \(O(1/\alpha) = O(\log(n)/\epsilon^2)\) amortized time.

**Proof.** Observe that the net effect of \texttt{insert} on the vertex in-degrees is to increase \(d^-(x)\) of a single vertex \(x\) by 1. Similarly, \texttt{delete} decreases \(d^-(x)\) of a single vertex \(x\) by 1.

Now, the running time in \texttt{insert} is proportional to the number of arcs considered in the while loop (step (1)) of \texttt{check-inc}, over all recursive calls to \texttt{check-inc} (plus a constant amount of work). The running time in \texttt{delete} is proportional to the number of arcs considered in the while loops of \texttt{check-dec}.

Our amortized analysis is a fractional charging scheme, where we distribute fractional credits to each arc that accumulate and pay for processing the arc later. The credits are generated as
follows. Whenever either insert or delete results in changing the in-degree of a vertex $x$ (by $1$), we spread $O(1/\alpha)$ credits uniformly over the arcs in $\delta^-(x)$; thus each arc $a \in \delta^-(x)$ receives a credit of $\Omega(1/\alpha \delta^-(x))$. One unit of credit will pay for a constant amount of work, so this adds an amortized cost of $O(1/\alpha)$ to both insert and delete.

Claim: Any arc $a$ processed by check-inc or check-dec has acquired at least 1 unit of credit since $\varphi(a)$ was last set.

Consider first check-inc; suppose $a = (u, v)$ is processed in the loop. Since $d^-(v) > (1 + \alpha)\varphi(v \mid a) + 1$, and $\varphi(v \mid a)$ had been set to $d^-(v)$ earlier, $d^-(v)$ must have gained at least $\Omega(\alpha d^-(v))$ edges since $a$ was last labeled. Each edge gained by $\delta^-(v)$ contributes $\Omega(\frac{1}{\alpha d^-(v)})$ credits to $a$ and thus $a$ has one unit of credit to pay by the time $d^-(v)$ increases to greater than $(1 + \alpha)\varphi(v \mid a) + 1$. This proves the part of the claim concerning check-inc.

Now consider check-dec($u$); suppose $a = (u, v)$ is processed in the loop. Recall that $\varphi(u \mid a)$ had been set to $d^-(u)$ earlier; now (when $a$ is processed) we have $\varphi(u \mid a) > (1 + \alpha)d^-(u) + 1$. Therefore $\delta^-(u)$ must have lost at least $\Omega(\max\{1, \alpha d^-(u)\})$ arcs since $\varphi(u \mid a)$ was set. Each edge lost contributes $\Omega(1/\alpha \max\{1, d^-(u)\})$ credits to $a$. Multiplying these quantities together shows that $a$ has acquired at least one credit before it is processed, as claimed.

Now, the claim implies that the constant work in every iteration of the loop in check-inc (except the last, over all recursive calls), as well as for every recursive call in check-dec, is paid for by existing credit. All put together, the overall amortized running time of each operation is bounded above by the initial amortized cost, $O(1/\alpha)$.

Remark 3.1. Of course one could have the insertion of an edge $e$ pay for the amortized cost of deleting $e$ later, and claim that deletion takes $O(1)$ amortized time. We do not emphasize this distinction.

3.4 Extending to fractional orientations.

As discussed above, to obtain a proper $(1 + \epsilon)$-approximation to the fractional arboricity, one can duplicate each edge $k = O(\log(n)/\epsilon^2)$ times for sufficiently large $C > 0$, and interpret each “duplicate” as a fractional edge of weight $1/k$. This increases the running time of all operations by $O(\log(n)/\epsilon^2)$. Additionally it would increase the space by a $O(\log(n)/\epsilon^2)$-factor. We would like to avoid this additional space overhead and here we will explain how to simulate the duplication approach in linear space.

Let $e = \{u, v\}$ be a fixed edge. In an orientation, $e$ is directed as either $a_1 = (u, v)$ or $a_2 = (v, u)$. If we duplicate $e$ $k$ times, then some duplicates will be of the form $a_1$ and the rest will be of the form $a_2$.

Now, rather than record each copy separately, we can instead record numerically how many copies of $e$ are oriented in each direction. Additionally, for each orientation $a_i$ of $e$, we will maintain one pair of labels $\varphi(u \mid a_i)$ and $\varphi(v \mid a_i)$ that serve all copies of that arc, rather than having each copy of $a_i$ have its own set of labels. Thus when the data structure resets the labels for one copy of the arc $a_i$, this automatically resets the labels for all copies of the arc $a_i$ simultaneously.

We argue that resetting the labels of all the copies of an arc $a_i$, rather than a particular copy, still preserves correctness. In general, we reset the labels $\varphi(u \mid a_i)$ and $\varphi(v \mid a_i)$ when doing so would preserve the inequalities in lemmas 3.3-3.5. In particular, if it is valid to update $\varphi(u \mid a_i)$ and $\varphi(v \mid a_i)$ for one particular copy of $a_i$, then it is valid to update the labels for all copies of $a_i$. Thus no error is introduced; if anything, updating the labels of the copies of $a_i$ can be understood as “free” updates that only help the data structure.
The final point to address is for the nested lists that maintain the arcs in $\delta^-(v)$ and $\delta^+(v)$ in order of $a$. Here we take advantage of the fact that all copies of an arc have the same label. Consider an arc $a \in \delta^-(v)$ (say). Rather than store each copy of $a$ separately, we have a node representing $a$ along with the number of copies of $a$ that are in $\delta^-(v)$. Since all the copies of the same arc have the same label, they would occupy the same place in the list anyway. Now removing a copy of an arc from the list corresponds to decrementing the corresponding counter, unless it was the last copy in which case the node for that arc is removed. Likewise inserting an arc corresponds to incrementing a counter unless it is the first copy in which case a new node is created.

4 (1 + $\epsilon$, log($n$))-approximate orientation with worst-case updates

The previous section gives a data structure that dynamically maintains a (1 + $\epsilon$, log($n$)/$\epsilon$)-approximate orientation in $O(\log(n)/\epsilon^2)$ amortized time per update. This section extends that data structure to obtain $O\left(\log^2(n)\log(\text{OPT}_{LP} + \log(n)/\epsilon^4)\right)$ worst-case time per update while retaining the same amortized running time. Formally, we will prove the following.

**Theorem 4.1.** Let $G$ be an unweighted and undirected graph over $n$ vertices, dynamically updated by edge insertions and deletions. Then one can maintain an orientation of $G$ with maximum in-degree at most $(1 + \epsilon)\text{OPT}_{LP} + O(\log(n)/\epsilon)$, and (implicitly) a subgraph with density at least $(1 - \epsilon)\text{OPT}_{LP} - O(\log(n)/\epsilon)$, in $O(\log(n)/\epsilon^2)$ amortized time and $O\left(\log^2(n)\log(\text{OPT}_{LP} + \log(n)/\epsilon^4)\right)$ worst-case time per edge insertion or deletion. The data structure uses $O(m + n)$ space.

As before, we can also obtain a fractional orientation of maximum load at most a (1 + $\epsilon$)-factor of the fractional arboricity by implicitly duplicating each edge $O(\log(n)/\epsilon^2)$ times. Again, to maintain linear space storage, some minor adjustments are required so that the “copies” of an edge use the same data. We briefly comment on the adjustments at the end of this section (section 4.4). Altogether we obtain the following bounds that increase the running times in theorem 4.1 by roughly a $O(\log(n)/\epsilon^2)$-factor.

**Corollary 4.2.** Let $G$ be an unweighted and undirected graph over $n$ vertices, dynamically updated by edge insertions and deletions. Then one can maintain a fractional orientation of $G$ with maximum in-degree at most $(1 + \epsilon)\text{OPT}_{LP}$, and (implicitly) a subgraph with density at least $(1 - \epsilon)\text{OPT}_{LP}$, in $O\left(\log^2(n)/\epsilon^4\right)$ amortized time and $O\left(\log^3(n)\log(\text{OPT}_{LP} + \log(n) + \log(1/\epsilon))/\epsilon^6\right)$ worst-case time per edge update. The data structure uses $O(m + n)$ space.

4.1 High-level overview

As mentioned above, the new data structure is based on the amortized data structure form the previous section. To motivate the changes we first explain where the previous (purely) amortized approach can have bad worst-case performance. There are two factors that are unbounded: the depth of recursive calls to check-inc or check-dec, and the length of the loop within a single call to check-inc or check-dec. We discuss them separately and consider the recursive aspect first. Consider the subroutine check-inc in fig. 2. Each time we flip an arc in step (1.A.1) we also make a recursive call to the opposite endpoint, which may trigger further flips and recursive calls. The total number of flips can potentially be very large. Similarly check-dec can have many flips via recursive calls.
### check-inc\(v\)

```c
/* We call this routine whenever \(d^-(v)\) has increased (always by 1). */

1. For up to \(C/\alpha\) arcs \(a = (u,v) \in \delta^-(v)\) s.t. \(d^-(v) \geq (1 + \alpha/2)\varphi(v|a)\), in increasing order of \(\varphi(v|a)\), for a sufficiently large constant \(C\):

   A. If \(d^-(v) \geq (1 + \alpha)(d^-(u) + 1)\):
      1. Flip \(a\) to \((v,u)\) and set \(\varphi(u|a) = d^-(u)\) and \(\varphi(v|a) = d^-(v)\).
      /* This restores \(d^-(v)\) to its previous value. */
      2. Call check-inc\(u\) and return.
   B. Otherwise set \(\varphi(v|a) = d^-(v)\) and \(\varphi(u|a) = d^-(u)\).

### check-dec\(u\)

/* We call this routine whenever \(d^-(u)\) has decreased (always by 1). */

1. For up to \(C/\alpha\) arcs \(a = (u,v) \in \delta^+(u)\) s.t. \(\varphi(u|a) \geq (1 + \alpha/2)d^-(u)\), in decreasing order of \(\varphi(u|a)\), for a sufficiently large constant \(C\):

   A. If \(d^+(u) \geq (1 + \alpha)(d^+(u) + 1)\):
      1. Flip \(a\) to \((v,u)\). Set \(\varphi(u|a) = d^-(u)\) and \(\varphi(v|a) = d^-(v)\).
      2. Call check-dec\(u\) and return.
   B. Otherwise set \(\varphi(v|a) = d^-(v)\) and \(\varphi(u|a) = d^-(u)\).
```

Figure 3: Revised implementations of `check-inc` and `check-dec` (cf. fig. 2) for fast worst-case update times.

We curtail this scenario by increasing the requirements for a flip. Before, we always flipped an arc from \((u,v)\) to \((v,u)\) if \(u\) is the smaller degree vertex. Now we only flip if \(u\) is substantially smaller than \(v\): namely, only if \(d^-(v) \geq (1 + \alpha)(d^-(u) + 1)\). For `check-inc\(v\)` this has the following effect. When recursing to \(u\), we know that \(d^-(u)\) is smaller than \(d^-(v)\) was (at the time of calling `check-inc\(v\)`) by a \((1 + \alpha)\)-factor. Meanwhile the in-degrees are integral, bounded above by \(O(\text{arb}(G) + \log(n)/\epsilon)\) (pending proof of correctness) and bounded below by 0. Therefore there are at most \(O(\log_{1+\alpha}(\text{arb}(G) + \log(n)/\epsilon)) = O(\log(\text{arb}(G) + \log(n)/\epsilon)/\alpha)\) recursive calls. The recursion depth for `check-dec` is bounded similarly; here the degrees increase a \((1 + \alpha)\)-multiplicative factor with each recursive call.

We note that a similar argument as described above, except in purely additive terms, limits the recursive depth in previous work [KKPS14; SW20].

The second issue is that the loops may be very long when there are many labels that require updating, but are not actually flipped. (Flipping restores the in-degree to the previous value and terminates the loop.) To try to limit the number of such label updates, we adjust the data structure to process extra arcs to “get ahead” of the expiring labels. More precisely, we adjust the loop to try to process \(O(1/\alpha)\) arcs even if not all of them are critically outdated. This helps the data structure stay ahead of a glut of labels about to expire. That said, we only process arc labels that are at least a little outdated, and this allows us to retain the amortized running time. The formal description of the algorithms are in fig. 3.
4.2 Maintaining a \((1 + O(\alpha), O(1))\)-locally optimal orientation

In this section we prove that the data structure maintains a \((1 + O(\alpha), O(1))\)-locally optimal orientation. By lemma 2.2 this implies the global optimality conditions described in theorem 4.1. The overall analysis is structured similarly to that of the previous data structure in section 3.2; in particular, lemmas 4.3–4.6 are in one-to-one correspondence with lemmas 3.3–3.6.

**Lemma 4.3.** For every arc \(a = (u, v)\), we have \(\varphi(v \mid a) \leq (1 + \alpha)(\varphi(u \mid a) + 1)\).

**Proof.** For any arc \(a = (u, v)\), the algorithm sets \(\varphi(u \mid a)\) and \(\varphi(v \mid a)\) to \(d^-(u)\) and \(d^-(v)\) only after verifying that \(d^-(v) \leq (1 + \alpha)(d^-(u) + 1)\).

**Lemma 4.4.** For every arc \(a = (u, v)\), we have \(d^-(v) \leq (1 + \alpha)(\varphi(v \mid a) + 1)\).

**Proof.** Fix \(a\). Let us call \(a\) **bad** if it violates the claimed inequality, and **dangerous** if

\[d^-(v) \geq (1 + \alpha/2)\varphi(v \mid a).\]

We want to show that \(a\) is never bad. Now, for \(a\) to be bad it must first be dangerous, which makes it eligible to be processed in the loop for check-inc\((v)\). After \(a\) becomes dangerous, \(d^-(v)\) must still increase by \(\Omega(\alpha d^-(v))\) before \(a\) becomes bad, and each increment presents an opportunity to process \(a\). We want to argue that \(a\) will be processed before it comes bad.

Recall that check-inc\((v)\) processes the arc \(b \in \delta^-(v)\) with minimum label \(\varphi(v \mid b)\). Let us say an arc \(\varphi(v \mid b)\) has higher priority than \(a\) if \(\varphi(v \mid b) \leq \varphi(v \mid a)\). When \(a\) becomes dangerous, there are at most \(d^-(v)\) other arcs with higher priority. Additionally, as long as \(a\) remains dangerous, there will be no new higher priority arcs because each new label is set to \(d^-(v)\), and \(d^-(v) > \varphi(v \mid a)\).

Each time \(d^-(v)\) increases, if we do not process \(a\), then we instead process \(C/\alpha\) higher-priority arcs for a suitably large constant \(C\). Each higher-priority arc \(b\) is either flipped or has \(\varphi(v \mid b)\) reset to be greater than \(\varphi(v \mid a)\). Either way, the arc \(b\) will no longer be higher-priority, and the number of higher-priority arcs has decreased by 1.

To recap, each unit increase to \(d^-(v)\) processes \(C/\alpha\) higher-priority arcs, so we would process all higher-priority arcs before \(d^-(v)\) changes by enough to make \(a\) bad. This forces us to process \(a\) before \(a\) becomes bad. Processing \(a\) will either reset \(\varphi(v \mid a)\) or flip \(a\); in either case, \(a\) will no longer be dangerous. \(\blacksquare\)

**Lemma 4.5.** For every arc \(a = (u, v)\), we have \(\varphi(u \mid a) \leq (1 + \alpha)(d^-(u) + 1)\).

**Proof.** The proof is very similar to the proof of lemma 4.4. Fix \(a\). Let us call \(a\) **bad** if it violates the claimed inequality, and **dangerous** if

\[d^-(u) \geq (1 + \alpha/2)\varphi(u \mid a).\]

Whenever \(\varphi(u \mid a)\) is set, it is always set to \(d^-(u)\), in which case it is not dangerous. For \(a\) to become bad, it must first be dangerous. If \(a\) is dangerous, then it is eligible to be processed for the loop in check-dec\((u)\) (which resets \(\varphi(u \mid a)\)). Additionally, after \(a\) becomes dangerous, \(d^-(u)\) must still decrease by \(\Omega(\alpha d^-(u))\) before \(a\) becomes bad, and each increment is an opportunity to process \(a\). We want to argue that \(a\) must be processed before it comes bad.

check-dec\((u)\) repeatedly processes the arc \(b \in \delta^-(u)\) with maximum label \(\varphi(u \mid b)\). Let us say an arc \(\varphi(u \mid b)\) has higher priority than \(a\) if \(\varphi(u \mid b) \geq \varphi(u \mid a)\). There are at most \(d^-(u)\) other arcs with higher priority when \(a\) becomes dangerous. As long as \(a\) remains dangerous, there are no new higher priority arcs because each new label is set to \(d^-(u)\), and \(d^-(u) < \varphi(u \mid a)\).
Each decrease in $d^-(u)$ that does not process $a$ must process $C/\alpha$ higher-priority arcs for a suitably large constant $C$. Each processed arc is no longer higher-priority after processing, so the number of higher-priority arcs decreases by 1 with each iteration.

To recap, each unit decrease to $d^-(u)$ processes and removes $C/\alpha$ higher-priority arcs. There are only $d^-(u)$ higher-priority arcs total and by the time $d^-(u)$ decreases by $\Omega(ad^-(u))$, all higher-priority arcs will be processed. Thus the data structure processes $a$ before $a$ becomes bad. Processing $a$ resets $\varphi(u \mid a)$ or flips $a$, and $a$ is no longer dangerous.

Lemma 4.6. The data structure maintains $(1 + O(\alpha), O(1))$-local optimality.

Proof. For each arc $a = (u, v)$, by lemmas 4.3–4.5, we have

$$d^-(v) \leq (1 + \alpha)\varphi(v \mid a) + 1 + \alpha \leq (1 + \alpha)^2\varphi(u \mid a) + 2 + O(\alpha) \leq (1 + \alpha)^3d^-(u) + 3 + O(\alpha),$$

as desired.

4.3 Update times

It remains to establish the running times claimed in theorem 4.1. By employing the same auxiliary data structures described in section 3.3, we may assume that each step in the pseudocode takes constant time. The analysis is then largely reduced to counting the total number of arcs processed by check-inc and check-dec.

We first re-establish the amortized bounds.

Lemma 4.7. Each edge insertion and edge deletion takes $O(\log(n)/\epsilon^2)$ amortized time.

Proof. The argument is similar to that of the amortized data structure (lemma 3.7, section 3), so we restrict ourselves to a sketch.

We first discuss edge insertion. Here the running time is proportional to the number of arcs processed in the loop of check-inc. The key points to amortizing the number of arcs is as follows. First, each edge insertion results in the in-degree of exactly one vertex increasing by 1. Second, in order for an arc $a = (u, v)$ to be processed in the loop check-inc($u$), $d^-(v)$ must have increased from $\varphi(v \mid a)$ by a $(1 + \Omega(\alpha))$-factor. These factors allow us to apply the same charging scheme as in lemma 3.7 to obtain the $O(\log(n)/\epsilon^2)$ amortized running time.

Edge deletion is similar. The running time is proportional to the total number of arcs flipped over all recursive calls. An arc is flipped only if $d^-(u)$ has decreased by a $(1 + \Omega(\alpha))$-factor since $\varphi(u \mid a)$ was last set. Additionally each edge deletion results in decreasing the in-degree of exactly one vertex, by 1. These factors allow us to apply the same charging scheme as in lemma 3.7 to obtain the $O(\log(n)/\epsilon^2)$ amortized running time.

Now we analyze worst-case bounds.

Lemma 4.8. Each edge insertion and edge deletion takes $O(\log(\text{arb}(G) + \log(n)/\epsilon)/\alpha^2) = O\left(\log(\text{OPT}_{LP} + \log(n)/\epsilon)\log^2(n)/\epsilon^4\right)$ worst-case time.

Proof. Consider first edge insertion. The running time is proportional to the number of arcs processed by check-inc. Each call to check-inc processes at most $O(1/\alpha)$ arcs and makes at most one recursive call. Each recursive call check-inc($v$) increases $d^-(v)$ by a $(1 + \alpha)$-approximate factor over the previous call, and $d^-(v)$ is bounded above by $O(\text{arb}(G) + \log(n)/\epsilon)$ by lemma 4.6. Thus there are at most $O(\log_{1+\alpha}(\text{arb}(G) + \log(n)/\epsilon)) = O(\log(\text{arb}(G) + \log(n)/\epsilon)/\alpha)$ recursive calls.

The running time for edge deletion follows by analyzing check-dec similarly.
4.4 Fractional orientations

As with the data structure in section 3, the data structure here can be modified to implicitly simulate duplicate arcs without increasing the space usage. The adjustments are the same as in section 3 so we limit ourselves to a sketch. The main idea is to use the same labels $\phi(u \mid a)$ and $\phi(v \mid a)$ for all copies of the same arc $a$. That is, when we update the labels of one copy of $a$ it automatically propagates to all copies of $a$. As before, this does not create any issues because labels are only made when they are safe with respect to the inequalities in lemmas 4.3–4.5, and if it is safe to relabel one copy of an arc, it is safe to relabel all copies of the arc. The other adjustments discussed in section 3.4 extend here in a straightforward fashion.

5 Improved worst-case updates for small arboricity

For densest subgraph, we now have the following worst case update times:

1. $O\left(\log (n) \log (\text{OPT}_{\text{LP}})/\epsilon^4\right)$ time for a $(1 + \epsilon, \log (n)/\epsilon)$-bicriteria approximation.

2. $O\left(\log^3(n)(\log(\text{OPT}_{\text{LP}}) + \log \log (n))/\epsilon^6\right)$ time for a $(1 + \epsilon)$-factor approximation.

Note that the first data structure has a faster update time but the additive error implies that it is only good when density is $\Omega\left(\log(n)/\epsilon^2\right)$. The second running time is slower because we implicitly duplicate edges to artificially increase the arboricity to be at least $\Omega\left(\log(n)/\epsilon^2\right)$. Now, in the small-arboricity regime that necessitates the slower running time, the $\log(\text{OPT}_{\text{LP}})$-factor is also negligible and the second running time is closer to $O\left(\log^3(n)/\epsilon^6\right)$. So we have faster running times in the “high-arboricity” and “low-arboricity” regimes, taken separately. The goal in this section is to unify these ideas and obtain a faster running time overall.

To obtain a faster worst-case running time for densest subgraph, we will build on ideas in the previous section to develop a data structure whose output is only valid in the low-arboricity regime. This data structure will then be run in parallel with the faster data structure mentioned above that does not duplicate edges, which is both (a) accurate in the high-arboricity regime and (b) correctly signals if we are in a high- or low-arboricity setting. Overall, the data structure will still retain linear space and the same amortized running times. The bounds we obtain for the low-arboricity setting is as follows.

**Theorem 5.1.** Let $\epsilon, T > 0$ be given with $\epsilon$ sufficiently small and $T > \Omega(\log(n)/\epsilon^2)$. In $O(\log(n)/\epsilon^2)$ amortized time and $O\left(\log^2(n) \log(T)/\epsilon^4\right)$ worst-case time per edge insertion or deletion, one can maintain an orientation of $G$, and (implicitly) the vertices of a subgraph of $G$, such that if $\text{OPT}_{\text{LP}} \leq T$, then the maximum in-degree is at most $(1 + \epsilon) \text{OPT}_{\text{LP}} + O(\log(n)/\epsilon)$, and the density of the subgraph is at least $(1 - \epsilon) \text{OPT}_{\text{LP}} - O(\log(n)/\epsilon)$.

Before proving theorem 5.1, we complete the discussion of how to apply it to obtain faster worst-case updates for dynamic densest subgraph in general. We first observe that, like the data structures in the previous section, the additive $O(\log(n)/\epsilon)$-factor can be removed by implicitly duplicating each edge $O(\log(n)/\epsilon^2)$ times. This increases the running times by a $O(\log(n)/\epsilon^2)$ factor but keeps everything else (and in particular the space) the same. (To maintain linear space usage, one makes the exact same adjustments as in section 4.4.) Second, we set $T = O(\log(n)/\epsilon^2)$, since above this threshold we already faster update times. Putting these ideas together we obtain the following.
Corollary 5.2. Let $G$ be an undirected graph over $n$ vertices dynamically updated by edge insertions and deletions. Let $\epsilon > 0$ be sufficiently small. Let $T = O(\log(n)/\epsilon^2)$. In $O\left(\frac{\log(n)}{\epsilon^4}\right)$ amortized time and $O\left(\frac{\log^3(n)(\log\log(n) + \log(1/\epsilon))}{\epsilon^6}\right)$ worst-case time per edge insertion or deletion, one can maintain a fractional orientation with the following property.

1. If the maximum in-degree is at least $(1 + \epsilon)T$, then $\text{OPT}_{LP} \geq T$.

2. If the maximum in-degree is at most $(1 + \epsilon)T$, then the maximum in-degree is at most an $(1 + \epsilon)\text{OPT}_{LP}$. In this case, the data structure also implicitly provides a list-representation of the vertices of a $(1 - \epsilon)$-approximate densest subgraph.

By running the data structure in Corollary 5.2 in parallel with the data structure from theorem 4.1, we obtain the following bounds for dynamically approximating the densest subgraph.

Corollary 5.3. Let $G$ be an undirected graph over $n$ vertices dynamically updated by edge insertions and deletions. Let $\epsilon > 0$ be sufficiently small. Then one can maintain an $(1 - \epsilon)$-approximation of the density, and an implicit list-representation of the vertices of an $(1 - \epsilon)$-approximate densest subgraph, with linear space and within the following time bounds.

(i) $O\left(\frac{\log(n)}{\epsilon^2}\right)$ amortized time per edge insertion or deletion.

(ii) $O\left(\frac{\log(n)\log(\text{OPT}_{LP})}{\epsilon^4} + \frac{\log^3(n)(\log\log(n) + \log(1/\epsilon))}{\epsilon^6}\right)$ worst-case time per edge insertion or deletion.

We now focus on proving theorem 5.1 for the remainder of this section.

5.1 High-level overview

The new data structure takes the data structure from the previous section and incorporates one simple idea: a threshold $T$. For a vertex $v$, let

$$d_T^-(v) \overset{\text{def}}{=} \min\{d^-(v), T\}.$$  

$d_T^-(v)$ truncates the in-degree of $v$ to be at most $T$.

Rather than trying to minimize the maximum in-degree $d^-(v)$, the new data structure tries to minimize the maximum truncated in-degree $d_T^-(v)$. Local optimality conditions are redefined in terms of $d_T^-(v)$, and we now only adjust arcs when we find errors with respect to $d_T^-(v)$. An important consequence is that recursive calls effectively end at vertices with in-degree greater than $T$. This effectively replaces “$\log(\text{OPT}_{LP})$”-factors in the running times from section 4 with “$\log(T)$”-factors, even when $\text{arb}(G)$ is much greater than $T$.

We now describe the changes more precisely. Let $T > 0$ be a fixed value; $T$ should be at least $c_T \log(n)/\epsilon$ for a sufficiently large constant $c_T > 0$. As before, let $\alpha = \epsilon^2/\log(n)$. The labels $\varphi(u \mid a)$ and $\varphi(v \mid a)$ of an arc $a$ are now set to $d_T^-(u)$ and $d_T^-(v)$. In check-inc($v$), we only process an arc $a = (u, v)$ if $d_T^-(v)$ significantly exceeds $\varphi(v \mid a)$, and then we only flip $a$ if $d_T^-(v)$ significantly exceeds $d_T^-(u)$. (This is as opposed to acting on $d^-(v)$ and $d^-(u)$ in the previous section.) Likewise, in check-dec($u$), we only process an arc $a = (u, v)$ if $d_T^-(u)$ is significantly less than $\varphi(u \mid a)$ and then we flip $a$ if $d_T^-(u)$ is significantly less than $d_T^-(v)$. Note that the labels are always at most $T$, and the data structure does not flip arcs when the concerned vertex has in-degree much greater than $T$. In particular we permit violations of the local optimality criteria when the in-degrees exceed $v$.  

19
except

(Note that a priori we first address the issue of whether maintaining local optimality for arcs below the threshold \( T \) when the in-degrees start to exceed \( T \) correct itself when the in-degrees of its endpoints fall below \( T \). For example, it is not clear that a “rogue” arc violating local optimality in the original sense will suffice to obtain global optimality, at least in the restricted setting where \( \text{OPT} \leq (1 - O(\epsilon))T \). (Note that a priori the data structure can have in-degrees greater than \( T \) even if \( \text{OPT} \leq T \), as \( \text{OPT} \) fluctuates above and below \( T \).) The following lemma is similar to lemma 2.2 except \( d_T(\cdots) \) takes the role of \( d^-(\cdots) \).

Figure 4: Revised versions of check-inc and check-dec (cf. fig. 3) that only maintains a locally optimal orientation whenever the arboricity is less than \( T \).

Incorporating the threshold \( T \) has the running time advantage of ending the recursion calls when the in-degrees start to exceed \( T \). Consequently the recursive depth is reduced from \( O(\log(\text{arb}(G))/\alpha) \) to \( O(\log(T)/\alpha) \). However it raises technical issues as well. As mentioned above, the data structure will certainly not maintain local optimality for arcs when the in-degrees are larger than \( T \). We need to redefine a new notion of local optimality for the truncated in-degrees, and show that they are sufficient for global optimality when \( \text{OPT}_{LP} \leq T \). It is also no longer clear how violations to large in-degree might corrupt arcs where the in-degrees and labels are below \( T \). For example, it is not clear that a “rogue” arc violating local optimality in the original sense will correct itself when the in-degrees of its endpoints fall below \( T \).

The pseudocode is presented in fig. 4. It is obtained by taking the pseudocode in the previous section and replacing \( d^-(\cdots) \) with \( d_T(\cdots) \) everywhere.

5.2 A truncated local optimality condition

We first address the issue of whether maintaining local optimality for arcs below the threshold \( T \) suffices to obtain global optimality, at least in the restricted setting where \( \text{OPT}_{LP} \leq (1 - O(\epsilon))T \). (Note that a priori the data structure can have in-degrees greater than \( T \) even if \( \text{OPT}_{LP} \leq T \), as \( \text{OPT}_{LP} \) fluctuates above and below \( T \).) The following lemma is similar to lemma 2.2 except \( d_T(\cdots) \) takes the role of \( d^-(\cdots) \).
Lemma 5.4. Let $\alpha, \beta > 0$ with $\alpha = o(\log(n))$. Let $T > 0$ Suppose that for every arc $a = (u, v)$, we have $d_T^{-}(v) \leq (1 + \alpha)d_T^{-}(u) + O(1)$. Let $\mu = \max_v d_T^{-}(v)$. Then

$$\mu \leq e^{O\left(\sqrt{\alpha \log(n)}\right)} \left(\text{OPT}_{\text{LP}} + O\left(\sqrt{\frac{\log(n)}{\alpha}}\right)\beta\right).$$

In particular, for $\beta = O(1)$, and $\alpha \leq c\epsilon^2/\log(n)$ for a sufficiently small constant $c > 0$, we have

$$\mu \leq (1 + \epsilon)\text{OPT}_{\text{LP}} + O(\log(n)/\epsilon).$$

Proof sketch. The claim follows from the exact same proof as lemma 2.2, except with $\mu$ now equal to $\max_v d_T^{-}(v)$ instead of $\max_v d^{-}(v)$.

5.3 Maintaining truncated local optimality

We now show that the data structure maintains the truncated local optimality conditions described in section 5.2, for $\alpha = O(\epsilon^2/\log(n))$ and $\beta = O(1)$. The high-level structure is similar to the analysis for the previous data structure in section 4.2, and the details of the proofs are similar as well.

Lemma 5.5. For every arc $a = (u, v)$, we have

$$\varphi(v | a) \leq (1 + \alpha)(\varphi(u | a) + 1).$$

Proof sketch. The proof ideas are essentially the same as lemma 4.3. In short, $\varphi(u | a)$ and $\varphi(v | a)$ are set to $d_T^{-}(u)$ and $d_T^{-}(v)$ only in situations where the claimed inequality is satisfied.

Lemma 5.6. For every arc $a = (u, v)$, we have

$$d_T^{-}(v) \leq (1 + \alpha)(\varphi(v | a) + 1)$$

Proof sketch. The proof ideas are essentially the same as for lemma 4.4, and we restrict ourselves to a sketch. We can define notions of $a$ being dangerous and bad based on when $d_T^{-}(v)$ is large enough to being to threaten the desired inequality, and when $a$ actually violates the inequality, respectively. Similar to the proof lemma 4.4, we have the fact once $a$ becomes dangerous, $d_T^{-}(v)$ still has to increase by a $(1 + \Omega(\alpha))$-factor for $a$ to be bad. We then argue that one of the calls to check-inc($v$) from these increments would have to process $a$ before $d_T^{-}(v)$ was large enough to make $a$ bad. Processing $a$ either resets the labels for $a$ or flips it; either way $a$ is no longer dangerous.

Lemma 5.7. For every arc $a = (u, v)$, we have $\varphi(u | a) \leq (1 + \alpha)\left(d_T^{-}(u) + 1\right)$.

We omit the proof of lemma 5.7 as it is essentially the same as the proof lemma 4.5, in the same way that the proof of lemma 5.6 matches the proof of lemma 4.4.

Lemma 5.8. The data structure maintains $(1 + O(\alpha), O(1))$-local optimality.

Proof sketch. The claim follows from lemmas 5.5–5.7 in the exact same way that lemma 4.6 follows from lemmas 4.3–4.5.
5.4 Running time analysis

Lemma 5.9. Each edge insertion and deletion takes $O(1/\alpha) = O(\log(n)/\epsilon^2)$ amortized time.

We refer the reader to the proof of lemma 4.7, which can be applied here with essentially no changes except the argument is now based on the truncated in-degrees.

Lemma 5.10. Each edge insertion and deletion takes $O(\log(n) \log(T)/\epsilon^4)$ worst-case time.

Proof sketch. The proof is similar to that of lemma 4.8 in section 4. Consider first edge insertion. The two key ideas from that proof are as follows. First, each call to check-inc($v$) takes $O(1/\alpha)$ time excluding recursive calls. Second, each recursive call to check-inc($v$), $d^-(v)$ increases by a $(1 + \alpha)$-factor. The difference now is that we recursion stops when $d^-(v)$ exceeds $T$. The total running time is thus $O(\log_{1+\alpha}(T)/\alpha) = O(\log(T)/\alpha^2) = O\left(\log(T) \log^2(n)/\epsilon^4\right)$, as desired.

Likewise the key ideas used to bounding edge deletion in lemma 4.8 carry over here; the difference now is that the depth of the recursion is $O(\log(T)/\alpha^2)$. The running time follows.

6 Extending to hypergraphs

Recall that a hypergraph generalizes undirected graphs by allowing each edge to have any number of endpoints. Let $G = (V, E)$ be a hypergraph. The maximum number of endpoints in any edge is called the rank of the hypergraph; we let $r$ denote the rank of $G$. The size of the hypergraph is defined as the sum, over all edges, of the number of endpoints in that edge. We let $p$ denote the size of $G$. For a set of vertices $S \subseteq V$, let $E(S)$ denote the set of hyperedges with all endpoints in $S$. The density of a set $S$ is defined as $|E(S)|/|S|$. The densest subhypergraph problem is to find the set $S \subseteq V$ that maximizes the density. This can be solved optimally via a reduction to network flow or via submodular function minimization.

To generalize our data structures to hypergraphs we need to generalize the notion of orientations to hypergraphs in a natural fashion. An orientation of a hypergraph $G = (V, E)$ consists of selecting, for each edge $e \in E$, an endpoint $v \in e$ called the head. In this case we say that $e$ is directed to $v$. Given an orientation of $G$, the in-degree of a vertex $v$, denoted $d^-(v)$, is defined as the number of edges for which $v$ is the head. One can define fractional orientations of hypergraphs analogously; here each edge $e$ is associated with a convex combination of endpoints that fractionally act as the head of $e$.

In hypergraphs, as in graphs, it is easy to see that the density of any subgraph is bounded above by the maximum in-degree of any orientation.

Recall the dual LPs for densest subgraph and fractional orientations in fig. 1. One can easily generalize these LPs to hypergraphs, as noted in prior work. To extend the densest subgraph LP, for each summand corresponding to an edge $e$ in the objective, we take the minimum over all endpoints in $e$. In the dual LP for fractional orientations, we now have a variable $y(e,v)$ for every edge $e$ and every endpoint $v \in e$, and these must sum to at least one for every edge $e$. For the remainder of this section, we let OPT$_{LP}$ denote the common optimum values of these LPs for the hypergraph $G$.

Next, we define a notion of approximate local optimality of orientations of hypergraphs that leads to approximate densest subgraphs. Let $G = (V, E)$ by a hypergraph and fix an orientation of $G$. Let $\alpha, \beta \geq 0$. We say that the orientation is $(\alpha, \beta)$-locally optimal, or a local $(\alpha, \beta)$-approximation, if for all edges $e \in E$ with head $v \in e$, and all other endpoints $u \in e$, we have

$$d^-(u) \leq (1 + \alpha)d^-(v) + O(1).$$
Lemma 6.1. Let $e \in (0, 1)$. Let $G = (V, E)$ be an oriented hypergraph and let $\mu = \max_v d^-(v)$. Suppose the orientations is $(ce^2/\log(n), O(1))$-locally optimal for a sufficiently small constant $c$. Then $\mu \leq (1 + \epsilon) \text{OPT}_{LP} + O(\ln(n)/\epsilon)$.

We omit the proof as it is essentially the same as lemma 2.2. Here we point out that in oriented hypergraphs, for a set of vertices $S, N^+(S)$ is defined as the set of vertices that are an endpoint to an edge directed towards a vertex in $S$.

The data structures for graphs generalize in a straightforward fashion. The main difference is that we have a label $\varphi(v | e)$ for every edge $e$ and every endpoint $v$ of $e$. check-inc($v$) is generalized as follows. Recall that in graphs, check-inc($v$) processes an edge $e$ oriented to $v$ when $d^-(v)$ has grown significantly larger than $\varphi(v | e)$. In this case it either makes a favorable flip to the smaller in-degree endpoint, or relabels the endpoints of $e$. Both of these ideas generalize to hyperedges. The difference is that we now have to check all the endpoints of $e$ to identify the endpoint of minimum in-degree. If the minimum in-degree is smaller (or substantially smaller) than that of the head, then we make the minimum in-degree endpoint the head and recurse on that vertex; otherwise we relabel all the endpoints of $e$. In particular, processing an edge $e$ now takes time proportional to the number of endpoints of $e$, which is at most the rank $r$ of the hypergraph. Likewise check-dec($u$) generalizes to hypergraphs in a straightforward manner with an additional running time overhead of $r$. The proof of correctness follows by the exact same arguments as for graphs and is therefore omitted.

The following theorem extends theorem 4.1 to hypergraphs.

Theorem 6.2. Let $G$ be an unweighted and undirected hypergraph over $n$ vertices and rank $r$, dynamically updated by edge insertions and deletions. The one can maintain an orientation of $G$ with maximum in-degree $(1 + \epsilon) \text{OPT}_{LP} + O(\log(n)/\epsilon)$, and (implicitly) a subgraph of density $(1 - \epsilon) \text{OPT}_{LP} - O(\log(n)/\epsilon)$, in $O(r \log(n)/\epsilon^2)$ amortized time and $O \left( r \log^3(n) \log(\text{OPT}_{LP} + \log(n)/\epsilon)/\epsilon^4 \right)$ worst-case time per edge insertion or deletion. The data structure uses $O(n)$ space.

We can also extend the truncated data structure from section 5 to hypergraphs as described above, and run it parallel with theorem 6.2 similar to the combination presented in section 5. Here, however, a nonzero value of $\text{OPT}_{LP}$ may be as small as $1/r$. Thus to diminish the additive error in theorem 6.2 we have to duplicate each edge $O(r \ln(n)/\epsilon^2)$ time, rather than $O(\ln(n)/\epsilon^2)$ as in graphs. Altogether we obtain the following theorem generalizing Corollary 5.3 to hypergraphs of rank $r$.

Theorem 6.3. Let $G$ be an undirected hypergraph of rank $r$ over $n$ vertices dynamically updated by edge insertions and deletions. Let $\epsilon > 0$ be sufficiently small. Then one can maintain a $(1 - \epsilon)$-approximation of the density, and an implicit list-representation of the vertices of an $(1 - \epsilon)$-approximate densest subgraph, with linear space and within the following time bounds.

(i) $O \left( r^2 \log^2(n)/\epsilon^4 \right)$ amortized time per edge insertion or deletion.

(ii) $O \left( \frac{r \log(n) \log(\text{OPT}_{LP})}{\epsilon^4} + \frac{r^2 \log^3(n)(\log \log(n) + \log(1/\epsilon))}{\epsilon^6} \right)$ worst-case time per edge insertion or deletion.
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