On the Geometry of Matrix Models for $N=1^*$

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Abstract

We investigate the geometry of the matrix model associated with an $\mathcal{N}=1$ super Yang-Mills theory with three adjoint fields, which is a massive deformation of $\mathcal{N}=4$. We study in particular the Riemann surface underlying solutions with arbitrary number of cuts. We show that an interesting geometrical structure emerges where the Riemann surface is related on-shell to the Donagi-Witten spectral curve. We explicitly identify the quantum field theory resolvents in terms of geometrical data on the surface.

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1 Introduction

Recently it has been proposed to use matrix models to extract information about holomorphic quantities of a certain class of $\mathcal{N} = 1$ gauge theories. More precisely, the exact superpotential and the condensates of chiral operators in the gauge theory can be computed from the free energy of an associated matrix model \cite{1}. This proposal has been extensively tested in the case of a pure $U(N)$ theory with an adjoint field with potential $W(\Phi)$ (and possibly matter in the fundamental representation), showing that the matrix model results and the quantum field theory ones agree. This model has an interesting geometrical structure in terms of a Riemann surface \cite{2,3,4,5}. In particular, the resolvents in the matrix model and in the field theory seem to be related to geometrical quantities on this surface. This geometrical structure is deeply related to the Seiberg-Witten curve of the $\mathcal{N} = 2$ theory of which the $\mathcal{N} = 1$ theory is a deformation. In this paper we will focus on the geometry of the $\mathcal{N} = 1$ theories that are obtained as deformations of the $\mathcal{N} = 4$ SYM. More precisely, we consider the model with three adjoint fields, a mass term for two of them and a generic potential for the third one. The associated matrix model was solved in the case of a single cut in \cite{1,6}, showing remarkable agreement with the quantum field theory results. In this paper, we are interested in the geometry of the matrix model for arbitrary number of cuts. Since the model can be considered as an $\mathcal{N} = 1$ deformation of an $\mathcal{N} = 2$ theory with a massive hypermultiplet, the corresponding geometry is expected to be related to the Donagi-Witten curve \cite{7}. The emergence of an elliptic curve in the case of a single cut \cite{1} is a first example of this connection. We will investigate the relation between the matrix model and the Donagi-Witten curve, and the geometrical structure that emerges in this way.

To be more concrete, we will study an $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $U(N)$ and three chiral fields $\Phi, X, Y$ in the adjoint representation. The theory is characterized by the superpotential

$$W(\Phi, X, Y) = i\Phi[X, Y] + mXY + W(\Phi),$$

where $W = \sum_{k=1}^{n+1} g_k \text{Tr} \Phi^k$ is a polynomial in $\Phi$ of degree $n + 1$. This model can be seen as a deformation of the $\mathcal{N} = 2$ theory with a massive adjoint hypermultiplet considered by Donagi and Witten \cite{7}. We are mainly interested in a classical vacuum where $X$ and $Y$ vanish and $W'(\Phi) = 0$, even though our results can be applied also to other vacua. The $N$ classical eigenvalues of $\Phi$ can be distributed over the $n$ points $\phi_i$ where $W'(x) = \prod_{i=1}^n (x - \phi_i) = 0$. The gauge group is correspondingly broken to $\prod_{i=1}^n U(N_i)$, where $N_i$ is the number of eigenvalues corresponding to the $\phi_i$ critical point of $W$. At low energy, we can write an effective superpotential

$$W_{\text{eff}}(S_i, \phi_i, m, \tau)$$

(1.2)

for the condensate superfields $S_i = \text{Tr}(W_a W^a)(i)$. In quantum field theory, the vacuum expectation value of $W_{\text{eff}}$ can be computed (in principle) from the knowledge of the underlying $\mathcal{N} = 2$ Seiberg-Witten curve. The
The addition of the superpotential $W(\Phi)$ indeed selects particular points in the moduli space of the $\mathcal{N} = 2$ theory, which can be found by explicitly solving the equations of motion. Once these points are found, $W_{\text{eff}}$ can be determined via

$$W_{\text{eff}} = \langle W(\Phi) \rangle_{\mathcal{N}=2} = \sum_{k=1}^{n+1} g_k \text{Tr} \langle \Phi^k \rangle_{\mathcal{N}=2}.$$  \hspace{1cm} (1.3)

In our case the $\mathcal{N} = 2$ theory is described by the Donagi-Witten curve for $U(N)$, which is an $N$-sheeted covering of a torus. The points selected by the equations of motion are typically points of partial or maximal degeneracy of the curve. In the generic vacuum $\prod_{i=1}^n U(N_i)$ we indeed expect that $N - n$ monopoles are massless.

In the corresponding matrix model, the vacuum $\prod_{i=1}^n U(N_i)$ is associated with an $n$-cut solution of the equations of motion. We will show that, as in [1], the matrix model can be associated with a Riemann surface of genus equal to the number of cuts. $W_{\text{eff}}$ is then computed from geometrical data on this surface. It is then interesting to investigate the relation of this geometrical structure with the Donagi-Witten curve. The single cut case was solved in [1,6]. It corresponds to a field theory massive vacuum where the $\mathcal{N} = 2$ spectral curve has maximal degeneracy and itself becomes a torus with modular parameter $\tau/N$. The emergence of such torus on the matrix model side was explicitly determined in [1,6]. The connection with the $\mathcal{N} = 2$ curve becomes more explicit when one considers the opposite case of maximal number of cuts. As in [3], we can introduce a degree $N + 1$ superpotential with $W'(x) = \epsilon \prod_{i=1}^N (x - \phi_i) = 0$. In the vacuum where all the $N$ eigenvalues of $\Phi$ are distinct the gauge group is broken to the maximal abelian subgroup $U(1)^N$. In the limit where the superpotential is turned off ($\epsilon \to 0$), we should recover the dynamic of the $\mathcal{N} = 2$ theory. We will show that the Riemann surface of the associated matrix model with $N$ cuts becomes in this limit the Donagi-Witten curve.

We will also analyze the geometry of the matrix model for an arbitrary number of cuts. We will show that, upon minimization, the Riemann surface always becomes a covering of a torus and we will discuss the relation of this surface with the Donagi-Witten construction. We will also identify the field theory resolvents with geometrical quantities on the Riemann surface. All the results have a direct analogue in the case of a pure $U(N)$ theory [1,3,6,9]. We will mainly consider the on-shell theory, where a minimization with respect to the moduli has been performed. The organization of the paper is as follows. In Section 2, we discuss the Riemann surface associated with the off-shell theory. In Section 3, we will discuss the conditions following from the minimization and the relation of the on-shell theory with the Donagi-Witten curve. In Section 4, we discuss the identification of the field theory resolvents with geometrical quantities on the Riemann surface. Finally, in the Appendix we briefly discuss the loop equations and some identities for the off-shell theory.
2 The Riemann surface

Following [1], we can compute $W_{\text{eff}}$ from the large $\hat{N}$ expansion of a matrix model:

$$
\int D\hat{\Phi} D\hat{X} D\hat{Y} e^{\frac{1}{g_s} \text{Tr}\{i\hat{\Phi}[\hat{X},\hat{Y}]+m\hat{X}\hat{Y}+W(\hat{\Phi})\}},
$$

(2.1)

where $\hat{\Phi}, \hat{X}, \hat{Y}$ are $\hat{N} \times \hat{N}$ hermitian matrices. The model can be solved integrating out the matrices $\hat{X}, \hat{Y}$ and diagonalizing the remaining matrix $\hat{\Phi}$. The saddle point equation of motion reads

$$
W'(\lambda_I) = g_s \sum_{J \neq I} \left[ \frac{1}{\lambda_I - \lambda_J} - \frac{1}{\lambda_I - \lambda_J + im} - \frac{1}{\lambda_I - \lambda_J - im} \right],
$$

(2.2)

where $\lambda_I$ are the eigenvalues of $\hat{\Phi}$. As usual, in the large $\hat{N}$ limit, the eigenvalues will be spread over $n$ cuts around each solution of $W'(\hat{\Phi})$. We will denote the fraction of eigenvalues for each cut as $\hat{N}_i$. According to the Dijkgraaf–Vafa prescription the filling fractions $S_i = g_s \hat{N}_i$ are identified with the field theory condensates and the effective superpotential corresponding to the $\prod_{i=1}^{n} U(N_i)$ vacuum is [1]

$$
W_{\text{eff}} = \sum_i \left( N_i \frac{\partial F}{\partial S_i} - 2\pi i \tau S_i \right),
$$

(2.3)

where $F(S_i)$ is the matrix model free energy. Unless explicitly stated, we will consider the case with the maximal allowed number of cuts.

Information about the model is encoded in the resolvent $\omega(x) \equiv \frac{1}{N} \text{Tr} \frac{1}{x - \Phi}$. As usual, $\omega(x)$ has cuts corresponding to the distribution of the matrix model eigenvalues. For this particular model, it is useful to define [8, 1, 6] the function

$$
G(x) = U(x) + iS \left[ \omega(x + \frac{i}{2} m) - \omega(x - \frac{i}{2} m) \right],
$$

(2.4)

where $S \equiv g_s \hat{N}$ is the 't Hooft coupling and $U(x)$ is defined by the property

$$
U\left(x + \frac{i}{2} m\right) - U\left(x - \frac{i}{2} m\right) = W'(x).
$$

(2.5)

Then as a consequence of (2.2) [8, 1, 6]

$$
G\left(x + \frac{i}{2} m \pm i\epsilon\right) = G\left(x - \frac{i}{2} m \mp i\epsilon\right) \quad \text{for } x \text{ on a cut.}
$$

(2.6)

Equation (2.6) means that $G$ is well-defined on a surface obtained from the $x$ plane after $n$ identifications, as shown in figure 1. If one adds the point $x = \infty$, this space is topologically a Riemann surface $\Sigma$ of genus $n$. Notice that this way of describing a Riemann surface is somewhat reminiscent of the light cone parameterization of moduli space of punctured Riemann surfaces [9].
Figure 1: Cuts in the $x$ plane for the function $G$. Lower cuts are identified with upper cuts as shown using dashed lines. $A_i$ and $B_i$ form a basis of cycles for $\Sigma$.

The function $G$ and the Riemann surface $\Sigma$ allow to write the effective superpotential (2.3) in a more convenient form. First of all, integrating equation (2.4) around the cuts we obtain the relation

$$S_i = \frac{1}{2\pi} \int_{A_i} G \, dx ,$$

(2.7)

where $A_i$ are the cycles encircling the cuts in the $x$ plane. Moreover, as shown in [1], $\partial F/\partial S_i$ can be written in terms of integrals over the cycles $B_i$ going from a lower cut to an upper one. The superpotential then reads

$$W_{\text{eff}} = i \sum_i \left( N_i \int_{B_i} G \, dx - \tau \int_{A_i} G \, dx \right).$$

(2.8)

We are interested in the geometrical structure of the problem. To this purpose, we can re-formulate the matrix model data in the following way. The coordinate $x$ of the matrix model plane is not a well-defined function on the Riemann surface $\Sigma$. Its differential $dx$, though, is well-defined. It has a double pole in the point $x = \infty$ and is regular otherwise; its $A$ periods are zero, and its $B$ periods are $im$. Moreover we can see from eq. (2.4) that the function $G$ has a single pole of order $n + 1$ at the point $x = \infty$. By Riemann-Roch this property singles out $G$ up to an additive constant. Thus our geometrical data are

- a Riemann surface $\Sigma$ of genus $n$
- a differential $dx$ with a double pole and periods

$$\int_{A_i} dx = 0 , \quad \int_{B_i} dx = im .$$

(2.9)

Let us count the number of moduli of our data. We have $3n - 3 + 1$ moduli from the moduli space of Riemann surfaces of genus $n$ with one puncture. Again from Riemann-Roch one gets $n + 1$ for the number of differentials with a double pole; integrating this
to obtain \( x \) involves an extra additive constant. Taking away \( 2n \) from (2.9) one gets finally
\[
(3n - 2) + (n + 1) + 1 - (2n) = 2n.
\]

From the matrix model point of view, these \( 2n \) moduli are easily interpreted as the classical vacua (points \( \phi_i \) in which \( W'(\phi_i) = 0 \), and the filling fractions \( S_i \). Alternatively, we can also interpret the \( 2n \) moduli as the zeros of \( dx \). A meromorphic differential with a double pole must indeed have \( 2n \) zeros; in the \( x \) plane they are the end-points of the cuts, denoted by small circles in figure 1. We see then that \( dx \) has precisely the role played by \( dw \) in the light-cone parameterization of the moduli space of Riemann surfaces with punctures in [9].

It is interesting to compare our data with the case of a pure \( U(N) \) gauge theory. In that case, the Riemann surface has genus \( n \) and can be explicitly written as
\[
y^2 = W'(x)^2 + f_{n-1}(x), \tag{2.10}
\]
where \( f_{n-1} \) is a polynomial of degree \( n - 1 \), whose coefficients are related to the moduli \( S_i \). The role of the meromorphic function \( G \) is played by \( y \): all the relevant formulae are obtained with the substitution \( G\, dx \to y\, dx \). An important difference with respect to our case, is that the matrix model plane for pure \( U(N) \) corresponds to just one sheet of the surface (2.10). On this sheet we have the relation \( y = W'(x) - 2\omega(x) \), instead of equation (2.4). All quantities are then continued to the second sheet. In our case, the matrix model plane is topologically identified with the entire Riemann surface.

### 3 Minimizing the superpotential

The results obtained so far from the matrix model apply to an off-shell theory. In order to obtain the vacuum value of the superpotential we further have to minimize equation (2.8) with respect to \( S_i \). We refer to the theory after minimization as the on-shell theory.

Define the differentials
\[
\omega_i = \frac{1}{2\pi} \frac{\partial}{\partial S_i} G\, dx. \tag{3.11}
\]

The \( \omega_i \) have no poles and are therefore holomorphic differentials. Indeed, when we differentiate equation (2.4), the \( \omega_i \) get contributions only from the difference of the resolvents; the simple pole in the resolvent, which behaves at infinity as \( \omega(x) \sim 1/x \), cancels in the difference. From equation (2.7) it also follows that the \( \omega_i \) form a

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1. The condition of having specified periods has an analogue in that case, where the differential was required to have purely imaginary periods. A differential with a double pole and purely imaginary periods is uniquely determined on any Riemann surface by a procedure similar to that in [9]; then imposing the actual value of these periods as in (2.9) constrains to a sub-variety of the moduli space. The only real difference with [9] is that \( dw \) has two single poles while our \( dx \) has one double pole.

2. Notice the difference with the case of pure gauge [8,14] where one of the derivatives of \( ydx \) with respect to the \( S_i \) is a meromorphic differential with a simple pole. This difference is consistent with the fact that \( \Sigma \) has genus \( N \), while the hyperelliptic curve considered in [8,14] has only genus \( N - 1 \).

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basis of canonically normalized holomorphic differentials: $\int_{A_i} \omega_j = \delta_{ij}$. Minimizing equation (2.8) with respect to $S_i$ we obtain

$$\sum_i N_i \int_{B_i} \omega_j = \int_{B_j} \sum_i N_i \omega_i = \tau ,$$

(3.12)

where we used the symmetry of the period matrix $\int_{B_i} \omega_i$.

The effects of the minimization translate into the behavior of the holomorphic differential $\Omega = \sum_i N_i \omega_i$. Its $A$-periods are $\int_{A_i} \Omega = N_i$. The minimization tells us that, on-shell, the $B$-periods of $\Omega$ are all equal, $\int_{B_i} \Omega = \tau$. To see what this means, consider the map

$$P \mapsto z(P) : = \int_{P_0}^P \Omega$$

(3.13)

from the Riemann surface to $\mathbb{C}$. The map (3.13) could be considered an “incomplete Jacobi map”, in the sense that we only consider the integral of one of the holomorphic differentials. For a generic Riemann surface, one would not be able to do better and identify points to make the image compact. However, in our case, since all the $B$-periods are equal and all the $A$-periods are integers, (3.13) is a well-defined map to a torus defined by the identifications

$$z \sim z + \tau , \quad z \sim z + \tilde{N} ,$$

(3.14)

where $\tilde{N}$ is the highest common factor of the $N_i$. Thus, we have shown that using the equations of motion for $S_i$, $\Sigma$ becomes a covering of a torus of modular parameter $\tau/\tilde{N}$.

### 3.1 The $\mathcal{N} = 2$ case

To understand our result better, it is convenient to consider first the case $n = N$, maximal number of cuts $N$ and $N_i = 1$. The gauge group is completely broken to the maximal abelian subgroup $U(1)^N$. This is the situation where we expect to recover information about the underlying $\mathcal{N} = 2$ theory. Following [3], we take a potential $W(\Phi)$ of degree $N + 1$ with $W'(x) = \epsilon \Pi_i (x - \phi_i)$. Since all $N_i = 1$, the eigenvalues of $\Phi$ are all distinct and classically coincide with the $N$ number $\phi_i$. The $\mathcal{N} = 1$ theory we are considering differs from an $\mathcal{N} = 2$ theory only for the presence of the potential $W(\Phi)$. If we turn off the deformation by sending $\epsilon \to 0$, we recover the $\mathcal{N} = 2$ theory in the point of the moduli space specified by the VEVs $\phi_i$ [3].

The $\mathcal{N} = 2$ theory we obtain in this way has gauge group $U(N)$, a massive adjoint hypermultiplet [7] and coupling constant $\tau$. The corresponding Seiberg-Witten curve was determined in [7]. The curve can be written as an $N$-sheeted covering of a base torus (of modular parameter $\tau$) expressed by the equation

$$F_N(v, z) = \det(v - \Phi(z)) = 0 ,$$

(3.15)

where $\Phi$ is a section of a $U(N)$-Higgs-bundle on the torus. The theory is a massive deformation of $\mathcal{N} = 4$ SYM and the presence of the torus reflects the S-duality of the
original $\mathcal{N} = 4$ theory. Equation (3.15) gives the Donagi-Witten curve as a degree $N$ polynomial in $v$ with coefficients that are elliptic functions on the base torus. The curve (3.15) is uniquely characterized by the existence of a meromorphic function, $v$, with $N$ single poles at the $N$ counter-images $p_i$ of the point $z = 0$ on the base torus. In one of the points, $p_0$, the residue has to be $-(N-1)m$, while the other $N-1$ points have residue $m$. The existence of a meromorphic function with these properties uniquely determines the curve (3.15) [7][10].

We will now show that, after minimization, the matrix model Riemann surface $\Sigma$ for $n = N$ and $N_i = 1$ becomes the Donagi-Witten curve describing the $\mathcal{N} = 2$ theory. We have already seen that the surface $\Sigma$ becomes a covering of a torus of modular parameter $\tau$ ($\tilde{N} = 1$ in this case). We will shortly see that the order $\tilde{n}$ of this covering is actually $N$. We can construct, on shell, the meromorphic function $v$ that characterizes a Donagi-Witten curve. It is easier to find first the differential $dv$, which should have poles of order 2 with coefficient equal to minus the residue of $v$. We have at our disposal at least two differentials with double poles in some or all the points $p_i$. One is the pull-back of the differential $\mathcal{P}(z)dz$ on the base torus. Once pulled back from $T^2$ to $\Sigma$, this indeed provides double poles with coefficient one on every point $p_i$ over $z = 0$. Another one is $dx$ which has a double pole at $x = \infty$. It is natural to suppose that $x = \infty$ corresponds to $p_0$ [3]. We can construct a differential with the desired properties by adding $dx$ and $\mathcal{P}(z)dz$. This easily provides us with a differential with double poles in $p_i$ with coefficients $-m(-\tilde{n} + 1, 1, \ldots, 1)$. We should also require $dv$ to have zero periods, since we want to integrate it to give a well defined meromorphic function $v$ on $\Sigma$. We can still add a piece with $dz$ without adding any pole. This way we end up with a differential $adx - m(\mathcal{P}(z) + b)dz$, where the constants $a$ and $b$ have to be determined imposing the vanishing of all periods. Only two of the $2N$ conditions on periods are non trivial. Indeed, the pull-back of any $f(z)dz$ has possibly non–vanishing periods only around the two cycles of $T^2$. The same statement holds for $dx$; since the periods of $dx$ are all equal (2.9), the only independent non–vanishing period is around the cycle $B$, corresponding to one of the cycle of the base torus. Thus, we reduce to only two equations, which determine uniquely $a$ and $b$. The result is

$$dv = 2\pi dx - m\left(\mathcal{P}(z) + \frac{\pi^2}{3} E_2(\tau)\right)dz,$$

where $E_2(\tau)$ is a standard Eisenstein series. This can be integrated up to a constant to give $^4$

$$v = 2\pi x + m\frac{\theta_1'(\pi z|\tau)}{\theta_1(\pi z|\tau)} \equiv 2\pi x + mh_1(z).$$

This expression is well-defined since $x$ and $h_1(z)$ are both periodic along the $A$ cycle and the coefficients are chosen in order to cancel their jump along $B$. To determine

$^3$Notice that $dz$ is not vanishing in $x = \infty$. This is consistent with the fact that the point at infinity is not a branch point in the Donagi-Witten curve.

$^4$Here $h_1(z) = \zeta(z) - \frac{\theta_1(\omega_1)}{\theta_1(\omega_2)}z$ for a torus with periods $2\omega_1$ and $2\omega_2$. $\zeta(z)$ is the Weierstrass zeta function: it is defined by having a simple pole in $z = 0$ and quasi-periodicity properties $\zeta(z+2\omega_i) = \zeta(z) + 2\zeta(\omega_i)$. $h_1(z)$ is periodic along $A$ and $h_1(z + 2\omega_2) = h_1(z) - \frac{\theta_1'}{\theta_1}$.

Other useful identities are: $\zeta'(z) = -\mathcal{P}(z)$, $\omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \pi i/2$, $\zeta(\omega_1)\omega_1 = \pi^2 E_2(\tau)/12$. 
the order \( \tilde{n} \) of the covering, it is now enough to compute the value of the residue of \( x \) at \( x = \infty \). From equation (2.4) it follows

\[
\frac{\partial}{\partial \Sigma_i} G dx = m \frac{dx}{x^2},
\]

so that \( dz \equiv \Omega = \left( \frac{mN}{2\pi} \right) \frac{dx}{x^2} \). It follows that, in local coordinates, \( x = -mN/(2\pi z) \). This fixes the residue of \( v \) near \( x = \infty \) to be \( -(N - 1)m \). Thus \( \tilde{n} = N \), and our proof is completed.

Once the algebraic curve is given, eq. (3.17) determines the map to the matrix model plane \( x \). The function \( G \) is also uniquely determined by the requirement of having a single pole of order \( N + 1 \) at \( x = \infty \), even though its explicit form can be difficult to find. Using the results of [7], we can determine the expression of the curve and \( G \) for small values of \( N \). The spectral curve (3.15) can be written as a pair of equations [7]

\[
y^2 = (t - e_1)(t - e_2)(t - e_3),
\]

\[
F_N(v, t, y) = 0,
\]

where the first equation is the standard representation of the torus as a cubic, while the second is a polynomial of degree \( N \) in \( v \), giving the \( N \)-sheeted covering of the torus.

As shown in [7], \( F \) is also a polynomial in \( x \) and \( y \).

For example, for \( N = 2 \) and \( W'(\Phi) = \Phi^2 - A_2 \), we have

\[
F_2(v, x, y) = v^2 - t - A_2,
\]

\[
G = y + v^3 - \frac{3A_2}{2}v.
\]

Analogously, for \( N = 3 \) and \( W'(\Phi) = \Phi^3 - A_2\Phi - A_3 \) we have

\[
F_3(v, x, y) = v^3 - v(3t + A_2) + 2y - A_3,
\]

\[
G = v^4 - v^2(6t - \frac{2}{3}A_2) + 8vy - 3t^2 - \frac{2}{3}A_2 t.
\]

The degree \( N \) polynomial in \( F \) can be related to \( W' \) by analyzing the large \( x \) behavior of \( G \) (see equation 2.4). In both cases, \( x \) is determined by equation (3.17). The structure of these two examples suggests that in general \( G \) is a linear combination of the polynomials \( P_k \) defined in [7].

For generic \( N \) it is probably more convenient to use an alternative expression for the Donagi-Witten curve [10],

\[
\sum_{n=-\infty}^{\infty} (-)^n q^{\frac{1}{2}n(n-1)} e^{nzh(x - nm)} = 0,
\]

where \( H \) is a degree \( N \) polynomial, which we can roughly identify with \( W' \). Another advantage of this expression is that it is naturally written in terms of the matrix model variable \( x \). Indeed, as shown in [10], to go from the polynomial equation (3.19) to the expression above, a change of variables \( v \rightarrow v - mh_1(z) \) is required, which, by equation (3.17) exactly defines the function \( x \).
3.2 The general case

We can also study the general case with arbitrary \( n \) and \( N \). In this case, the group is broken to \( \prod_{i=1}^{n} U(N_i) \) and there are some non-abelian gauge factors at low energy. For these vacua, we could also introduce, as in [12], integer numbers \( b_i \) labeling the type of confinement in each factor. The numbers \( b_i \) appear in the matrix model expression for \( W_{\text{eff}} \) as

\[
W_{\text{eff}} = \sum_i \left( N_i \frac{\partial F}{\partial S_i} - 2\pi i \tau S_i \right) - \sum_{i=2}^{n} 2\pi i b_i S_i .
\]

(3.23)

The minimization procedure then fixes the periods of \( \Omega \) to be \( N_i \) and \( \tau + b_i \). As before, we conclude that the map \( z : \Sigma \rightarrow \mathbb{C} \) is well defined if we make the identifications \( z \sim z + \tau \) and \( z \sim z + t \), where \( t \) is the highest common factor of the integers \( N_i \) and \( b_i \). It was shown in [12] that \( t \) defines the index of confinement of the vacuum.

We see that \( \Sigma \) becomes a covering of a torus of modular parameter \( \tau/t \). We can further show that \( \Sigma \) is a \( N/t \)-sheeted covering of the base torus, and express \( \Sigma \) as an algebraic equation. Indeed, the argument we used to identify the function \( v \) can be repeated almost verbatim in the general case. The only difference is now the ratio of the residues in the points \( p_i \): equation (3.17) is replaced by

\[
v = \frac{2\pi}{t} dx + mh_1(z) .
\]

(3.24)

Since the behavior of \( x \) at infinity is still given by \( x = -mN/(2\pi z) \), the residue at infinity is now \( N/t \). It follows that the number of sheets is \( \tilde{n} = N/t \). The genus \( n \) curve \( \Sigma \) is then expressed as an element of the spectral family \( F_{\tilde{n}}(v, z) = 0 \). Since the arithmetic genus of a curve of the family \( F_{\tilde{n}}(v, z) = 0 \) is \( \tilde{n} > n \), the algebraic curve will be singular and \( \Sigma \) will correspond to its normalization.

This result deserves some comments. We know that the relevant geometry for arbitrary \( n \) and \( N \) can be determined as a particular point in the moduli space of the underlying \( U(N) \) \( \mathcal{N} = 2 \) theory. In a vacuum with \( \prod_{i=1}^{n} U(N_i) \) only \( n \) photons remain massless and this requires that \( N - n \) monopoles are massless and condensate in order to give mass to all the other degrees of freedom. The associated curve has then \( N - n \) nodes. The sub-variety of the moduli space where \( N - n \) monopoles are massless has dimension \( n \). The point in this sub-variety associated to the \( \mathcal{N} = 1 \) vacuum can be determined by explicitly solving the equations of motion for the potential \( W(\Phi) \). This minimization was explicitly done in the case of a pure gauge theory in [2]. In that case, one can explicitly check that a potential \( W(\Phi) \) of degree \( n + 1 \) selects a point in the \( \mathcal{N} = 2 \) moduli space of the form

\[
y^2 = P_N(x)^2 - 1 = \prod_{k=1}^{N-n} (x - u_k)^2(W'(x)^2 + f_{n-1}) ,
\]

(3.25)

where the \( N - n \) double zeros correspond to a degeneracy of the curve associated with \( N - n \) massless monopoles. The reduced genus \( n - 1 \) curve \( y^2 = (W'(x)^2 + f_{n-1}) \) is exactly that describing the matrix model geometry. It would be interesting to repeat this
analysis for the case with a massive hypermultiplet. The $\mathcal{N} = 1$ vacuum $\prod_{i=1}^{n} U(N_i)$ should be associated with an element of the spectral family $F_N(v, z) = 0$ with $N - n$ nodes, thus degenerating to a genus $n$ surface. The matrix model computation suggests that the normalization of such curve can be also find by normalizing a curve $F_n(v, z)$ of Donagi-Witten type but with a different base torus of modular parameter $\tau/t$.

Finally we would like to comment about the cases where some of the cuts coincide. In such cases, the genus $n$ curve $\Sigma$ degenerates. The normalization of the resulting curve has genus equal to the number of cuts. We can for example make contact with the examples discussed in [1,6]. These correspond to completely massive vacua and are associated with a single cut. The previous discussion can be repeated with $n$ replaced by the number of cuts ($n \to 1$) and the only non trivial $N_i = N$. We exhibited $\Sigma$ as a 1-sheeted covering (i.e. an isomorphism) of a torus of modular parameter $\tau/N$. For $n = 1$, the Donagi-Witten curve becomes quite trivial, $F_1(v, x, y) = v - A$, and indeed describes a torus. Equation (3.17) then gives $\frac{2\pi}{m} x = A - \pi \frac{q_1(z|\tau)}{q_1(z|\tau)}$. This is exactly the map from the torus to the matrix model plane given in [16]. The field theory interpretation of this result is simple. Indeed, as it is well known [7], the massive vacua of the $U(N)$ theory with bare coupling $\tau$ correspond to points in the moduli space of the $\mathcal{N} = 2$ theory where the genus $N$ curve $F_N(v, x, y) = 0$ maximally degenerates and becomes a torus of modular parameter $\tau/N$.

4 Resolvents

The interesting quantities to compute in quantum field theory are the resolvents [4]

$$R(x) = \text{Tr}(\mathcal{W}_\alpha \mathcal{W}_\alpha x - \Phi) ,$$

$$T(x) = \text{Tr}(\frac{1}{x - \Phi}) .$$

The knowledge of $R(x)$ and $T(x)$ allows to compute all the vacuum expectation values of operators in the chiral ring [4]. In this section, we will discuss a possible identification of $R$ and $T$ with geometrical quantities. In the case of pure gauge, $R$ can be related to the meromorphic function $y$ defining the curve (see equation (2.10)) while $T(x)dx$ becomes a meromorphic differential on the curve [3][4][11][15].

4.1 Identification of the resolvents

We expect that, as in [4], $R(x)$ is identified with the matrix model resolvent

$$R(x) = S \omega(x) .$$

In the case of pure gauge, this identification descends from the comparison of the matrix model loop equations with the Ward identities in field theory [4]. As shown in [4], the equations for $R(x)$, which can be deduced in the quantum field theory using the Konishi anomaly, are formally identical to the matrix model Ward identities for
ω(\(x\)). In our case, the Ward identities are more complicated to write (some explicit relations are discussed in the Appendix), but we expect that the general philosophy still applies.

Quantum mechanically, we may expect that small cuts are opened around the classical eigenvalues of \(\Phi\), analogously to what happens for the matrix model. Integrals of \(R(\(x\))\) around a critical point of \(W\) define the condensates \(\text{Tr}(W_\alpha W^\alpha)_{(i)}\). The contour integrals of \(R\) around a cut are mapped to integrals of \(\omega(\(x\))\) around its cut in the matrix model plane (these cuts are indicated with dotted lines in figure 2), which define the quantities \(S_i\)

\[
\text{Tr}(W_\alpha W^\alpha)_{(i)} = \frac{1}{2\pi i} \oint R(\(x\))d\(x\) = \frac{1}{2\pi i} \oint S\omega(\(x\))d\(x\) = S_i . \tag{4.3}
\]

![Figure 2: The \(C_i\) contours encircle the cuts of \(\omega(\(x\))\), denoted by dotted lines. The contours \(A, A^*\), \(B, B^*\) delimit a simply connected region in the \(x\) plane.](image)

From quantum field theory it is also obvious that the integrals of \(T(\(x\))\) around a critical point of \(W\) are integers

\[
\frac{1}{2\pi i} \oint T(\(x\))d\(x\) = N_i . \tag{4.4}
\]

In analogy with [5], we may expect \(T(\(x\))\) to be associated with a differential on the Riemann surface. Consider the differential

\[
t(\(x\))d\(x\) = \left[T(\(x - \frac{i}{2}m\)) - T(\(x + \frac{i}{2}m\))\right]d\(x\) ; \tag{4.5}
\]

in a similar way as for the matrix model resolvent, the residue cancels from the two \(\pm im/2\) pieces, and the differential is holomorphic around \(x = \infty\). We now conjecture that \(t(\(x\))d\(x\)) can be extended to a holomorphic differential defined on the entire Riemann surface. Its periods around the \(A_i\) cycles are \(N_i\), by equation (4.4). These are the same \(A_i\) periods as \(dz\). Since an holomorphic differential is completely specified by its \(A_i\) periods we conclude that

\[
\Omega \equiv dz = \frac{1}{2\pi i} t(\(x\))d\(x\) = \frac{1}{2\pi i} \left[T(\(x - \frac{i}{2}m\)) - T(\(x + \frac{i}{2}m\))\right]d\(x\) . \tag{4.6}
\]
It then follows that also the $B_i$ periods are completely specified (this argument was also used in a similar way in [5])
\[
\frac{1}{2\pi i} \int_{B_i} t(x) dx = \tau . \tag{4.7}
\]

The validity of equation (4.6) is strengthened by the following formal argument. Using the definition of $dz$ and the identification (4.2), we have
\[
dz = \frac{1}{2\pi} \sum_i N_i \frac{d}{dS_i} G dx = -\frac{1}{2\pi i} \sum_i N_i \frac{d}{dS_i} \left[ R \left( x + \frac{im}{2} \right) - R \left( x - \frac{im}{2} \right) \right] , \tag{4.8}
\]
which, by comparison with (4.6), leads to the suggestive equation
\[
\frac{1}{N} \sum_i N_i \frac{\partial}{\partial S_i} \text{Tr} \left( W \alpha W^* \right) = \text{Tr} \left( \frac{1}{x - \Phi} \right) . \tag{4.9}
\]

### 4.2 Field theory expectation values

We can use $t(x) dx$ to compute the field theory expectation values of $\langle \text{Tr} \Phi^k \rangle$. To this purpose, we can integrate $x^k t(x) dx$ on a small contour around $x = \infty$,
\[
\langle \text{Tr} \left[ (\Phi + \frac{im}{2})^k - (\Phi - \frac{im}{2})^k \right] \rangle = \int_{\infty} x^k dz . \tag{4.10}
\]

We can deform the previous contour integral in the $x$ plane until it encircles the cycles $A_i$ and $A_i^*$ (see figure 2), obtaining
\[
\int_{\infty} x^k dz = \sum_i \left( \int_{A_i} x^k dz + \int_{A_i^*} x^k dz \right) = \sum_i \int \left[ (x_0^{(i)} + \frac{im}{2})^k - (x_0^{(i)} - \frac{im}{2})^k \right] dz , \tag{4.11}
\]
where $x_0^{(i)}(z)$ are maps from the $A$ cycle of the base torus to contours around the cuts of the resolvents on the real axis (indicated by dots in figure 2). We call $C_i$ these contours. Comparing equations (4.10) and (4.11) we obtain the useful formula
\[
\langle \text{Tr} \Phi^k \rangle = \sum_i \int_{C_i} x^k dz = \sum_i \int x_0^{(i)}(z)^k dz . \tag{4.12}
\]

Formula (4.12) was derived in [6] in the case of a single cut.

We can also compute the vacuum value of the effective potential in terms of the function $x_0$. This will also strengthen our identification of $t(x) dx$ with $dz$. On shell, we can write $W_{\text{eff}}$ as
\[
\sum_i \left( \int_{A_i} dz \int_{B_i} G dx - \int_{B_i} dz \int_{A_i} G dx \right) .
\]

By Riemann bilinear relations [13], this expression is also equal to $\text{Res}_{\infty}(z(x)G(x)dx)$. The $U$ piece in the definition (2.4) of $G$ is the only one which can contribute to this
residue. We can at this point try to invert the proof of the Riemann bilinear relations. To this purpose we choose a base point, say on the real axis of $x$, and modify all the $A_i$ and $B_i$ periods in such a way that they bound a simply connected region (see figure 2). This way we build a polygon whose sides are $A_i$, $B_i$ and their opposites $A_i^*$, $B_i^*$, identified in such a way as to reconstruct the Riemann surface $\Sigma$, similarly to [13]. In the interior of this polygon (a simply connected region) the function $z$ is well-defined. The Riemann bilinear relations are then demonstrated by deforming a contour integral of $zGdx$ around $x = \infty$ to the perimeter of the polygon and by exploiting the periodicities of $z$. If we now substitute $G(x)$ with $U(x)$, we obtain extra contributions coming from the fact that $U(x)$ is a well defined function on the plane $x$ but not on the Riemann surface. But we can now exploit that to our advantage:

$$W_{\text{eff}} = \text{Res}_\infty(zUdx) = - \sum_i \left( \int_{A_i + A_i^*} zUdx + \int_{B_i + B_i^*} zUdx \right)$$

$$= - \sum_i \left( \int_{A_i} (z + \tau)U(x_0 + im) + \int_{A_i^*} zU(x_0 - im)dx - \int_{B_i} Udx \right)$$

$$= - \sum_i \left( \int_{C_i} zW'(x)dx - \int_{B_i} U(x)dx \right)$$

$$= \sum_i \left( \int_{C_i} W(x)dz \right) + N \int_{im/2}^{im/2} U(x)dx . \quad (4.13)$$

In second line of 4.13 we first used equation (2.5) and the fact that $U(x)$ is not well-defined on $\Sigma$ to evaluate the integrals over $A_i$; the integrals over $B_i$ instead just behave as for the usual Riemann argument. We have also used that $\int A_i U = 0$. We then integrated by parts the integrals over $C_i$, and then used appropriate integrals of (2.5) again to put the second piece in the final form.

The final expression for $W_{\text{eff}}$ in (4.13) is consistent with the identification (4.12). Indeed, if $W \equiv \sum_p g_p \text{Tr} \Phi^p$, we know $W_{\text{eff}} = \sum g_p \langle \text{Tr} \Phi^p \rangle$. So equation (4.13) can be written as

$$\sum_i \int_{C_i} W(x)dz \sim \langle \sum_k g_k \text{Tr} \Phi^k \rangle , \quad (4.14)$$

modulo pieces which, for a given $W$, depend on $g_k$, but not on $\tau$ and on the choice of a particular vacuum. Formula (4.13) was derived in a different way in the one-cut case in [14].

Formula (4.12) gives a prescription for computing the quantities $\langle \text{Tr} \Phi^k \rangle$ purely in terms of matrix model data; the function $x_0(z)$ can be interpreted as the quantum distribution of field theory eigenvalues. We should note, however, that there is an ambiguity in the definition of $\langle \text{Tr} \Phi^k \rangle$ in field theory. The condensates can be computed as the order parameters $u_k = \langle \text{Tr} \Phi^k \rangle_{N=2}$ of the $N = 2$ vacuum that is selected by the potential $W(\Phi)$. In presence of a mass $m$, $u_k$ can mix with all the other order parameters $u_j, j < k$ [14]. The results for the condensates obtained with different methods could be related by a change of basis in the $u_i$; consistency
requires the coefficients of such redefinition to be vacuum independent. In the case of a single cut, it was explicitly checked in [6] that the condensates computed with the matrix model prescription are indeed related by a vacuum-independent redefinition to the condensate computed using the Donagi-Witten curve.

Acknowledgments

We would like to thank Annamaria Sinkovics and Stefan Theisen for interesting discussions. This work is partially supported by the EU contract HPRN–CT–2000–00122. M.P. is supported by the European Commission Marie Curie Postdoctoral Fellowship under contract number HPRN–CT–2001–01277. A. Z. are partially supported by INFN and MURST, and by the European Commission TMR program HPRN-CT-2000-00131, wherein he is associated to the University of Padova.

5 Appendix: Loop equations

The identifications of Section 4 could be strengthened by comparing the matrix model loop equations with the Konishi anomaly equations in field theory. A complete set of loop equations uniquely determining the resolvents would also give a convenient description of the off-shell theory. In the case of pure gauge, the loop equations for the matrix model give relation (2.10) for $y = W'(x) - 2\omega(x)$, which uniquely determines the Riemann surface associated with the matrix model; this result is also valid off-shell. In the case of $\mathcal{N} = 1^*$, the loop equations are more complicated. Here we will make a first step in the study of the loop equations and we derived a suggestive relation satisfied by the resolvent. In the following, we will refer to the Ward identity for the matrix model, which are identities for $\omega(x)$. As shown in [4], identical equation for $R(x)$ can be deduced from the quantum field theory Ward identities. Identities for $T(x)$ are then obtained by differentiating those for $R(x)$, as a consequence of the superfield formalism introduced in [4].

The best way to find the loop equations is to consider the matrix model before integrating out $\hat{X}$ and $\hat{Y}$, and making appropriate changes of coordinates. The ones we consider here are

$$\delta \hat{\Phi} = \frac{1}{x - \Phi}, \quad \delta \hat{X} = \frac{1}{x - \Phi + im/2} \hat{X} \frac{1}{x - \Phi - im/2}.$$  

The equations that we get from these are

$$\omega^2(x) = -\text{Tr} \left[ (W'(-\hat{\Phi}) + [\hat{X}, \hat{Y}]) \frac{1}{x - \Phi} \right],$$

$$\omega(x + im/2)\omega(x - im/2) = \text{Tr} \left[ \hat{Y} \hat{X} \frac{1}{x - \Phi - im/2} - \hat{X} \hat{Y} \frac{1}{x - \Phi + im/2} \right]. \quad \text{(5.15)}$$

By considering repeated translations of (5.15) by $\pm im/2$, we can find a combination in which $\hat{X}$ and $\hat{Y}$ disappear:

$$\sum_{n=-\infty}^{\infty} \left[ \omega(x + in + \frac{m}{2}) - \omega(x + i(n + 2) \frac{m}{2}) \right]^2 = 2 \sum_{n=-\infty}^{\infty} \text{Tr} \left[ \frac{W'(-\hat{\Phi})}{x - \Phi + in \frac{m}{2}} \right]. \quad \text{(5.16)}$$
One can actually write this equation in terms of $G$ by completing a square. Defining $X_n(x) \equiv X(x + in\frac{m}{2})$ for any function $X(x)$, we have
\[
\sum_{n=-\infty}^{\infty} G_n^2 = \sum_{n=-\infty}^{\infty} U_n^2 - 2 \sum_{n=-\infty}^{\infty} f_n,
\] (5.17)
where we have defined $f(x) = \text{Tr}[W'(\hat{\Phi}) - W'(x))/(x - \hat{\Phi})]$. The formal analogy with the pure gauge case is evident. In that case, the loop equations are $y^2 = W'^2 + f$. In our case, $G$ plays the role of $y$ and $U$ the role of $W'$. This would become of course more than an analogy if one considers the limit $m \to \infty$.

It is not clear whether equation (5.17), involving an infinite series of shifts, is well-defined. If so, it could be interpreted as an equation determining $G$, and implicitly $\Sigma$, in dependence of the polynomials $U$ and $f$. It would be interesting to investigate whether $G$ is uniquely determined by such equation. Similar issues were studied in [17]. It would be also interesting to study equation (5.17) after minimization, where it should be related to the Donagi-Witten curve. The form of equation (5.17) is very suggestive in comparison to the Lax matrix form of the Donagi-Witten curve given in equation (3.22). We leave the investigation of all these issues to future work.

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