KERNELS OF REPRESENTATIONS OF SEMISIMPLE DRINFELD DOUBLES

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Abstract. A description for kernels of representations of semisimple Drinfeld doubles $D(A)$ is given. Using this we also obtain a description of all normal Hopf subalgebras of $D(A)$. As an application, kernels of irreducible representations of $D(G)$ are computed and shown that they are all normal Hopf subalgebras of $D(G)$.

Introduction

Let $A$ be a semisimple Hopf algebra and $D(A)$ be its Drinfeld double. It is well known that the category $\text{Rep}(D(A))$ of representations $D(A)$ is a modular category [1] and is braided equivalent to the center $Z(\text{Rep}(A))$ of the tensor category $\text{Rep}(A)$. The Drinfeld doubles $D(A)$ play a very important role in the classification of semisimple Hopf algebras.

In this paper we will give a description of the kernels of the representations of $D(A)$. This will allow us to obtain a description of the normal Hopf subalgebras of $D(A)$ in terms of normal Hopf subalgebras of $A$ and $A^*$.

Recently the author introduced in [3] the notion of kernel of a representation of a semisimple Hopf algebra. It was proven that if the character of the representation is central in the dual Hopf algebra then the kernel is a normal Hopf subalgebra. It is not known whether the kernel is in general a normal Hopf subalgebra. In this paper we prove that in the case $D(G)$, of a Drinfeld double of a finite group $G$, all representation kernels are normal Hopf subalgebras of $D(G)$.

Fusion subcategories of the category $\text{Rep}(D(G))$ of a finite group $G$ were recently studied in [13]. They are parameterized in terms of a pair of commuting subgroups of $G$ and a $G$-invariant bicharacter defined on their product. The main ingredient used in [13] is the notion of centralizer for a fusion subcategory introduced in [10]. In this paper we will identify from the above mentioned parametrization all fusion

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subcategory of $\text{Rep}(D(G))$ that are of the form $\text{Rep}(D(G)//L)$ where $L$ is a normal Hopf subalgebra of $D(G)$. They correspond to those bicharacters from $[13]$ satisfying a stronger condition than that of $G$-invariance.

The paper is organized as follows. The first section recalls few basic results on Hopf algebras and the kernels of their characters that are needed in the paper.

The second section is concerned with fusion subcategories of $\text{Rep}(A)$ for a semisimple Hopf algebra $A$. Theorem 2.3 gives a necessary and sufficient condition for such a category to be normal (i.e of the form $\text{Rep}(A//L)$ for some normal Hopf subalgebra $L$ of $A$). A description of the commutator subcategory $\text{Rep}(A//L)^c$ is given in Proposition 2.4.

Next section studies kernels of representations of Drinfeld doubles $D(A)$. The key idea is to use a lemma from $[4]$ concerning fusion subcategories of a products of two fusion categories. This lemma can be regarded as a quantum analogue of Goursat’s lemma for groups. A description of normal Hopf subalgebras of $D(A)$ is also given in terms of normal Hopf subalgebras of $A$ and $A^*$. Two general examples are also considered in this section.

The last two sections are concerned with the special case $D(G)$, of Drinfeld doubles of finite groups. Some basic results from group representations that are needed are recalled in the first section. A basis for central characters in $D(G)$ is also given here. In the last section the parametrization from $[13]$ of fusion subcategories of $\text{Rep}(D(G))$ is recalled. This section gives a description of all normal Hopf subalgebras of $D(G)$ and shows that the kernel of any character of $D(G)$ is normal.

We work over the algebraic closed field $\mathbb{C}$. For a vector space $V$ by $|V|$ is denoted the dimension $\dim \mathbb{C}V$. We use Sweedler’s notation $\Delta(x) = \sum x_1 \otimes x_2$ for comultiplication. All the other Hopf notations are those used in $[11]$.

1. Preliminaries

1.1. Notations. Let $A$ be a finite dimensional semisimple Hopf algebra over $\mathbb{C}$. Then $A$ is also cosemisimple $[8]$. The character ring $C(A)$ of $A$ is a semisimple subalgebra of $A^*$ $[17]$ and it has a vector space basis given by the set $\text{Irr}(A)$ of irreducible characters of $A$. Moreover, $C(A) = \text{Cocom}(A^*)$, the space of cocommutative elements of $A^*$. By duality, the character ring of $A^*$ is a semisimple subalgebra of $A$ and $C(A^*) = \text{Cocom}(A)$. If $M$ is an $A$-module with character $\chi$ then $M^*$ is also an $A$-module with character $\chi^* = \chi \circ S$. This induces an involution “*” $: C(A) \rightarrow C(A)$ on $C(A)$. Let $m_A(\chi, \mu)$ be the multiplicity form
on $C(A)$. For $d \in \text{Irr}(A^*)$ denote by $C_d$ the simple subcoalgebra of $A$ whose character as $A^*$-module equals $d$ $[7]$. Denote by $t_A$ the integral in $A^*$ with $t_A(1) = |A|$. It is known that $t_A$ is also the regular character of $A$ $[11]$.

1.2. Kernels of characters for semisimple Hopf algebras. Let $M$ be a representation of $A$ which affords the character $\chi$. Define $\ker_A(\chi)$ as the set of all irreducible characters $d \in \text{Irr}(A^*)$ which act as the scalar $\epsilon(d)$ on $M$. Then Proposition 1.2 of $[3]$ implies that

$$\ker_A(\chi) = \{ d \in \text{Irr}(A^*) \mid \chi(d) = \epsilon(d)\chi(1) \}.$$ 

Similarly let $z_A(\chi)$ be the the set of all irreducible characters $d \in \text{Irr}(A^*)$ which act as a scalar $\alpha \epsilon(d)$ on $M$, where $\alpha$ is a root of unity. Then from the same proposition it follows

$$z_A(\chi) = \{ d \in \text{Irr}(A^*) \mid |\chi(d)| = \epsilon(d)\chi(1) \}.$$ 

Clearly $\ker_A(\chi) \subset z_A(\chi)$.

Since the sets $\ker_A(\chi)$ and $z_A(\chi)$ are closed under multiplication and “*” they generate Hopf subalgebras of $A$ denoted by $A_{\chi}$ and $Z_A(\chi)$, respectively (see $[3]$).

Remark 1.1.

Suppose that $K$ is a Hopf subalgebra of a semisimple Hopf algebra $A$ via $i : K \hookrightarrow A$. The restriction functor from $A$-modules to $K$-modules induces a map $\text{res} : C(A) \to C(K)$. It is easy to see that $\text{res} = i^*|_{C(A)}$, the restriction of the dual map $i^* : A^* \to K^*$ to the subalgebra of characters $C(A) \subset A^*$.

2. Fusion subcategories of $\text{Rep}(A)$

2.1. Fusion categories and their universal gradings. In this subsection we recall few facts on fusion categories from $[6]$ and $[4]$.

Let $\mathcal{C}$ be a fusion subcategory. Let $\mathcal{O}(\mathcal{C})$ be its set of simple objects considered up to isomorphism. Recall that $\mathcal{C}_{ad}$ is defined as the fusion subcategory of $\mathcal{C}$ containing $X \otimes X^*$ for each simple object $X$ of $\mathcal{C}$.

A grading of a fusion category $\mathcal{C}$ by a group $G$ is a map $\deg : \mathcal{O}(\mathcal{C}) \to G$ with the following property: for any simple objects $X, Y, Z \in \mathcal{C}$ such that $X \otimes Y$ contains $Z$ one has $\deg(Z) = \deg(X)\deg(Y)$. This corresponds to a decomposition $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, where $\mathcal{C}_g \subset \mathcal{C}$ is the full additive subcategory generated by simple objects of degree $g$. The subcategory $\mathcal{C}_1$ corresponding to $g = 1$ is a fusion subcategory; it is called the trivial component of the grading. A grading is said to be trivial if $\mathcal{C}_1 = \mathcal{C}$. It is said to be faithful if the map $\deg : \mathcal{O}(\mathcal{C}) \to G$ is surjective. For any fusion category $\mathcal{C}$, as explained in Sect. 3.2 of
there is a notion of universal grading whose group is called the universal grading group and it is denoted by $U_c$. Its trivial component is the fusion subcategory $C_{ad}$. The following Lemma appears in \[4\].

**Lemma 2.1.** Let $C$ be a fusion category and

$$C = \bigoplus_{g \in U_c} C_g$$

be its universal grading. There is a one-to-one correspondence between fusion subcategories $D \subset C$ containing $C_{ad}$ and subgroups $G \subset U_c$, namely

$$D \mapsto G_D := \{ g \in U_c | D \cap C_g \neq 0 \}$$

and

$$G \mapsto D_G := \bigoplus_{g \in G} C_g.$$

Let $A$ be a semisimple Hopf algebra. It is known that $\text{Rep}(A)$ is a fusion category. Moreover there is a maximal central Hopf subalgebra $K(A)$ of $A$ such that $\text{Rep}(A)_{ad} = \text{Rep}(A//K(A))$, see \[3\]. Since $K(A)$ is commutative it follows that $K(A) = k[U_A]$ where $U_A$ is the universal grading group of $\text{Rep}(A)$. For example if $A = kG$ then $K(A) = k\mathcal{Z}(G)$ and $U_A = \mathcal{Z}(G)$, the linear dual group of the center $\mathcal{Z}(G)$ of $G$.

Let $D$ be a fusion subcategory of $\text{Rep}(A)$ and $O(D)$ be its set of objects. Then $I_D := \cap_{V \in O(D)} \text{Ann}_A(V)$ is a Hopf ideal in $A$ \[4\] and $D = \text{Rep}(A/I_D)$. For a fusion category $D \subset \text{Rep}(A)$ define its regular character as $r_D := \sum_{X \in \text{Irr}(D)} \dim_C(X) \chi_X$ where $\text{Irr}(D)$ is the set of irreducible objects of $D$ and $\chi_X$ is the character of $X$ as $A$-module. Thus $r_D \in C(A)$.\[5\]

**Remark 2.2.** Let $B$ be a normal Hopf subalgebra of a semisimple Hopf $A$ and $\chi$ a character of $A$ affording the representation $M_\chi$. Then $M_\chi$ is a representation of $A//B$ if and only if $A_\chi \supset B$.

**Theorem 2.3.** Let $A$ be a finite dimensional semisimple Hopf algebra and $D$ be a fusion subcategory of $\text{Rep}(A)$. Then $D = \text{Rep}(A//L)$ for some normal Hopf subalgebra $L$ of $A$ if and only if the regular character $r_D$ of $D$ is central in $A^*$. In this case $L = A_{r_D}^\ast$.

**Proof.** If $D = \text{Rep}(A//L)$ then by Theorem 2.4 of \[3\] it follows that $r_D$ is an integral of $(A//L)^\ast$. Since this is a normal Hopf subalgebra of $A^\ast$, it follows from Lemma 1 of \[3\] that $r_D$ is central in $A^\ast$. Conversely, if $r_D$ is central in $A^\ast$ then by Proposition 3.3 of \[3\] it follows that $A_{r_D}^\ast$ is a normal Hopf subalgebra of $A$. Since $r_D^2 = |D|r_D$ the same Proposition implies that $D = \text{Rep}(A//A_{r_D}^\ast)$.
2.2. The commutator subcategory. Recall that the notion of com-
mutator subcategory from [6]. If \( \mathcal{D} \) is a fusion subcategory of \( \mathcal{C} \) then
\( \mathcal{D}^{co} \) is the full abelian subcategory of \( \mathcal{C} \) generated by those objects \( X \)
such that \( X \otimes X^* \in \mathcal{O}(\mathcal{D}) \).

In this subsection we will describe the category \( \text{Rep}(A//L)^{co} \) for \( L \)
a normal Hopf subalgebra of \( A \). For \( a, b \in A \) define \([a, b] = ab - ba\) the usual commutator. Then for \( L \) a Hopf subalgebra of \( A \) let \([A, L] \) be the ideal generated by \([a, l] \) with \( a \in A \) and \( l \in L \).

Proposition 2.4. Let \( L \) be a normal Hopf subalgebra of \( A \). Then \([A, L] \)
is a Hopf ideal of \( A \) and

\[
\text{Rep}(A//L)^{co} = \text{Rep}(A/[A, L])
\]

Proof. To see that \([A, L] \) is a Hopf ideal note that

\[
\Delta([a, l]) = \sum a_1l_1 \otimes a_2l_2 - \sum l_1a_1 \otimes l_2a_2 \\
= \sum (a_1a_1 - l_1a_1) \otimes a_2l_2 + \sum l_1a_1 \otimes (a_2l_2 - l_2a_2)
\]

Now consider \( M \) an irreducible \( A/[A, L] \)-module affording the character \( \chi \in C(A) \). Since \((la)_m = (al)_m\) for all \( a \in A \), \( l \in L \) and \( m \in M \) it follows that left multiplication by \( l \) on \( M \) is a morphism of \( A \)
module. Schur's Lemma implies that each \( l \in L \) acts by a scalar on \( M \). Thus \( \chi \downarrow^A_L = \chi(1)\psi \) for some linear character \( \psi \) of \( L \). Then

\[
(\chi\chi^*) \downarrow^A_L = \chi(1)^2\psi\psi^{-1} = \chi(1)^2\epsilon_L.
\]

Conversely suppose that \( M \otimes M^* \) is a trivial \( L \)-module for an irre-
ducible \( A \)-module. Let \( A = I_M \oplus \text{Ann}_A(M) \) be the decomposition of \( A \)
in two-sided ideals where \( I_M \) is the minimal ideal in \( A \) corresponding of \( M \). It is well known (see [16] for example) that the minimal ideal \( I_M \) in \( A \) corresponding of \( M \) satisfies \( I_M \cong M \otimes M^* \) where \( I_M \) is regarded as \( A \)-module by the adjoint action. Therefore \( l_1xSl_2 = \epsilon(l)x \) for all \( x \in I_M \) and \( l \in L \). Then \( lx = (l_1xS(l_2))l_3 = xl \) for all \( x \in I_M \) and \( l \in L \). Then

\((la - al)m = 0\) for all \( a \in A \). Indeed if \( a \in \text{Ann}_A(M) \) this is clear since
\((la).m = (al).m = 0\). On the other hand if \( a \in I_M \) then \( al - la = 0 \) and thus \((al - la).m = 0\). This shows that \( M \in \text{Rep}(A/[A, L]) \).

Definition 2.6. Let \( L \) be a normal Hopf subalgebra of \( A \). An irre-
ducible character \( \alpha \) of \( L \) is called \( A \)-stable if there is a character
\( \chi \in \text{Rep}(A) \) such that \( \chi \downarrow^A_L = \frac{\chi(1)}{\epsilon(1)}\alpha \). Such a character \( \chi \) is said to seat
over \( \alpha \).

Denote by \( G_\alpha(L) \) the set of all \( A \)-stable linear characters of \( L \).

Lemma 2.7. Let \( M \) be an irreducible module of \( A \) affording a character
\( \chi \). Then \( M \in \mathcal{O}(\text{Rep}(A//L)^{co}) \) if and only if \( L \) acts trivially on some
tensor power $M^\otimes n$ of $M$. In these conditions $\chi \downarrow^A_L = \chi(1)\psi$ for some $A$-stable linear character $\psi$ of $L$.

**Proof.** Suppose that $L$ acts trivially on some tensor power $M^\otimes n$ of $M$. This means that $\chi^n \downarrow^A_L = \chi(1)^n \epsilon_L$. Let

$$\chi \downarrow^A_L = \sum_{\alpha \in \text{Irr}(L)} m_\alpha \alpha$$

for some nonnegative integers $m_\alpha$ and

$$\chi^{-1} \downarrow^A_L = \sum_{\alpha \in \text{Irr}(L)} n_\alpha \alpha$$

for some nonnegative integers $n_\alpha$. Recall that $m_L(\epsilon_L, \alpha \beta) > 0$ if and only if $\alpha = \beta^*$. Since $\chi^n \downarrow^A_L = \chi(1)^n \epsilon_L$ this implies that $\chi \downarrow^A_L = \frac{\chi(1)}{\alpha(1)} \alpha$ and $\chi^{-1} \downarrow^A_L = \frac{\chi(1)^{-1}}{\alpha(1)} \alpha^*$ for a fixed character $\alpha$. It follows by counting the multiplicity of $\epsilon_L$ in $\chi^n \downarrow^A_L$ that $\alpha(1) = 1$. Thus $\alpha$ is an $A$-stable linear character of $L$.

Conversely suppose that $M \in \mathcal{O}(\text{Rep}(A//L)^{\text{co}})$ and let

$$\chi \downarrow^A_L = \sum_{\alpha \in \text{Irr}(K)} m_\alpha \alpha.$$ 

Since $m_L(\epsilon_L, \alpha \beta^*) = \delta_\alpha, \beta$, counting the multiplicity of $\epsilon_L$ in $\chi \downarrow^A_L \chi^* \downarrow^A_L$ implies that $\chi \downarrow^A_L = \chi(1)\alpha$ for a $L$-linear character $\alpha$. If $n$ is the order of $\alpha$ in $G(L^*)$ then clearly $\chi^n \downarrow^A_L = \chi(1)^n \epsilon_L$ which shows that $L$ acts trivially on $M^\otimes n$.

By duality it follows that the subcategory $\text{Rep}(L^*)^{\text{co}}$ of $\text{Rep}(A^*)$ has a similar description as in Proposition 2.4.

3. **Kernels of characters of representations of Drinfeld doubles**

The Drinfeld double $D(A)$ of $A$ is defined as follows: $D(A) \cong A^{\text{cop}} \otimes A$ as coalgebras and multiplication is given by

$$(g \triangleright h)(f \trianglerighthook l) = \sum g(h_1 \rightarrow f \leftarrow S^{-1} h_3) \trianglerighthook h_2 l.$$ 

Its antipode is given by $S(f \triangleright h) = S^{-1}(h)S(f)$.

It is known that $\text{Rep}(D(A^*)) = \text{Rep}(A^*)^{\text{rev}} \boxtimes \text{Rep}(A)$ since $D(A)$ is a cocycle twist of $A^{\text{cop}} \otimes A$. It is also known that $D(A)$ is a semisimple Hopf algebra if and only if $A$ is a semisimple Hopf algebra [11].

3.1. **Fusion subcategories of direct products of categories.** Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two fusion categories. Identify them with the corresponding fusion subcategories of $\mathcal{C}^1 \boxtimes \mathcal{C}^2$ given by $\mathcal{C}^1 \boxtimes 1$ and $1 \boxtimes \mathcal{C}^2$ respectively. Then every simple object of $\mathcal{C}^1 \boxtimes \mathcal{C}^2$ is of the form $X_1 \boxtimes X_2$ where $X_i$ is a simple object of $\mathcal{C}^i$. 
Let $\mathcal{D} \subset \mathcal{C}^1 \boxtimes \mathcal{C}^2$ be a fusion subcategory. Define $\mathcal{L}^i(\mathcal{D}) := \mathcal{D} \cap \mathcal{C}^i$, $i = 1, 2$. Let also $\mathcal{K}^i(\mathcal{D})$ be the fusion subcategory generated by all simple objects $X_1$ of $\mathcal{C}^1$ such that $X_1 \boxtimes X_2 \in \mathcal{D}$ for some simple object $X_2$ of $\mathcal{C}^2$. Similarly define the fusion subcategory $\mathcal{K}^2(\mathcal{D})$. Clearly $\mathcal{L}^1(\mathcal{D}) \boxtimes \mathcal{L}^2(\mathcal{D}) \subset \mathcal{D} \subset \mathcal{K}^1(\mathcal{D}) \boxtimes \mathcal{K}^1(\mathcal{D})$.

The following Lemma is taken from [4].

**Lemma 3.1.** Let $\mathcal{D} \subset \mathcal{C}^1 \boxtimes \mathcal{C}^2$ be a fusion subcategory. Then there is a group $\mathcal{X}$ and faithful $\mathcal{X}$-gradings $\mathcal{K}_i(\mathcal{D}) = \oplus_{x \in \mathcal{X}} \mathcal{K}_i(\mathcal{D})_x$ with trivial components $\mathcal{L}^i(\mathcal{D})$, $i = 1, 2$ such that

$$ (3.2) \quad \mathcal{D} = \oplus_{x \in \mathcal{X}} \mathcal{K}_1(\mathcal{D})_x \boxtimes \mathcal{K}_2(\mathcal{D})_x. $$

**Proof.** First, let us show that

$$ (3.3) \quad \mathcal{D} \supset \mathcal{K}_i(\mathcal{D})_{ad} \boxtimes \mathcal{K}_2(\mathcal{D})_{ad} = (\mathcal{K}_1(\mathcal{D}) \boxtimes \mathcal{K}_2(\mathcal{D}))_{ad}, $$

where $\mathcal{K}_i(\mathcal{D})_{ad}$ denotes the adjoint subcategory of $\mathcal{K}_i(\mathcal{D})$. To prove this, note that if $X_1 \boxtimes X_2$ is an object of $\mathcal{C}^1 \boxtimes \mathcal{C}^2$ that is also in $\mathcal{O}(\mathcal{D})$ then $(X_1 \otimes X_2^*) \boxtimes (X_2 \otimes X_2^*) \in \mathcal{D}$. Since $X_i \otimes X_i^*$ contains the unit object it follows that $(X_1 \otimes X_1^*) \boxtimes 1 \in \mathcal{D}$ and $1 \boxtimes (X_2 \otimes X_2^*) \in \mathcal{D}$. Therefore

$$ (3.4) \quad \mathcal{K}_i(\mathcal{D}) \subset \mathcal{L}_i(\mathcal{D}) \quad \text{for } i = 1, 2. $$

But $\mathcal{L}_1(\mathcal{D}) \boxtimes \mathcal{L}_2(\mathcal{D}) \subset \mathcal{D}$, so Equation 3.4 implies equation 3.3. Now let $U_{\mathcal{K}_i(\mathcal{D})}$ be the universal grading group of $\mathcal{K}_i(\mathcal{D})$. By Equation 3.3 and Lemma 2.1,

$$ \mathcal{D} = \oplus_{\gamma \in \Gamma} (\mathcal{K}^1(\mathcal{D}) \boxtimes \mathcal{K}^2(\mathcal{D}))_{\gamma}, $$

for some subgroup $\Gamma \in U_{\mathcal{K}_1(\mathcal{D}) \boxtimes \mathcal{K}_2(\mathcal{D})} = U_{\mathcal{K}_1(\mathcal{D})} \times U_{\mathcal{K}_2(\mathcal{D})}$.

By the definition of $\mathcal{K}^i(\mathcal{D})$, the subgroup $\Gamma \in U_{\mathcal{K}_i(\mathcal{D})}$ has the following property: the maps $\Gamma \twoheadrightarrow U_{\mathcal{K}_1(\mathcal{D})}$ and $\Gamma \twoheadrightarrow U_{\mathcal{K}_2(\mathcal{D})}$ are surjective. Goursat’s lemma for groups (see [15]) implies that $\Gamma$ equals the fiber product $U_{\mathcal{K}_1(\mathcal{D})} \times_{\mathcal{X}} U_{\mathcal{K}_2(\mathcal{D})}$ for some group $\mathcal{X}$ equipped with epimorphisms $U_{\mathcal{K}_i(\mathcal{D})} \twoheadrightarrow \mathcal{X}$, $i = 1, 2$. These epimorphisms define faithful $\mathcal{X}$-gradings of $\mathcal{K}_i(\mathcal{D})$ such that Equation 3.2 holds. Formula 3.2 implies that $\mathcal{L}_i(\mathcal{D}) := \mathcal{D} \cap \mathcal{C}_i$ equals the trivial component of the $\mathcal{X}$-grading of $\mathcal{K}_i(\mathcal{D})$.

The following remark is straightforward.

**Remark 3.5.** Let $\mathcal{C}$ be a fusion category and $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ be a faithful grading of $\mathcal{C}$. For any object $X_g \in \mathcal{C}_g$ one has $X_g \otimes n \in \mathcal{C}_1$ where $n \geq 1$ is the order of $g \in G$. Indeed $\mathcal{C}_g \otimes n \subset \mathcal{C}_g^n = \mathcal{C}_1$.

**Remark 3.6.**
If $\alpha$ is an irreducible $A$-stable character of $L$ then formulae from [2] shows that in this situation

$$\alpha \uparrow^A_L = \frac{\sum_{\chi \in A_\alpha} \chi(1)^2}{\sum_{\chi \in A_\alpha} \chi(1)} \chi$$

where $A_\alpha$ is the set of all irreducible characters $\chi$ of $A$ that seat over $\alpha$. Also if $t_{A//L} \in (A//L)^*$ is the integral on $A//L$ with $t_{A//L}(1) = \frac{|A|}{|L|}$ it follows from Theorem 4.3 of [2] that

$$(3.7) \quad \frac{\alpha \uparrow^A_L}{\alpha(1)} = \frac{\chi t_{A//L}}{\chi(1)}$$

for any $\chi \in A_\alpha$.

Let $T \subset \text{Irr}(A)$ be a set of irreducible characters of of $A$ and $\chi$ another character of $A$. Denote by $T_\chi$ the set all irreducible constituents of $\mu \chi$ where $\mu \in T$. Also let $\text{Irr}(\chi)$ be the set of all irreducible constituents of $\chi$.

**Lemma 3.8.** Suppose $A$ is a semisimple Hopf algebra and $L$ is a normal Hopf subalgebra. Suppose that there is a finite group $G$ and faithful grading on $\text{Rep}(A) = \oplus_{g \in G} \text{Rep}(A)_g$ with the trivial component $\text{Rep}(A)_1 = \text{Rep}(A//L)$. Then for any $\chi_g \in \text{Rep}(A)_g$ it follows that $\chi_g \downarrow^A_L = \chi(1)\psi_g$ where $\psi_g$ is a $A$-stable linear character of $L$. Moreover

$$\text{Rep}(A)_g = \text{Rep}(A//L)\chi_g = \text{Irr}(\psi_g \uparrow^A_L)$$

**Proof.** By Remark 3.5 it follows that $\chi^\circ_g \in \text{Rep}(A)_1 = \text{Rep}(A//L)$. Then Lemma 2.7 implies that $\chi_g \downarrow^A_L = \chi(1)\psi_g$ where $\psi_g$ is a $A$-stable linear character of $L$. The last equalities follows from Remark 3.6. \[\Box\]

### 3.2. Hopf subalgebras of Drinfeld doubles $D(A)$

If $H$ is a Hopf subalgebra of $D(A)$ then

$$\text{Rep}(H^*) \subset \text{Rep}(D(A)^*) = \text{Rep}(A) \boxtimes \text{Rep}(A^*)^\text{rev}$$

Then Lemma 3.1 implies that any Hopf subalgebra $H$ of $D(A)$ is completely determined by four fusion subcategories satisfying the following properties:

$$(3.9) \quad (\mathcal{K}^1)_{\text{ad}} \subset \mathcal{L}^1 \subset \mathcal{K}^1 \subset \text{Rep}(A)$$

$$(3.10) \quad (\mathcal{K}^2)_{\text{ad}} \subset \mathcal{L}^2 \subset \mathcal{K}^2 \subset \text{Rep}(A^*)$$

with faithful gradings
(3.11) \[ \mathcal{K}^i = \bigoplus_{x \in \mathcal{X}} (\mathcal{K}^i)_x \]

by a given group \( \mathcal{X} \) such that \( \mathcal{K}^i_1 = \mathcal{L}_i \) for \( i = 1, 2 \). In this situation

(3.12) \[ \text{Rep}(H^*) = \bigoplus_{x \in \mathcal{X}} (\mathcal{K}^1)_x \boxtimes (\mathcal{K}^2)_x. \]

3.3. Kernels of representations of Drinfeld doubles \( D(A) \).

**Theorem 3.13.** Let \( M \) be a \( D(A) \)-module with character \( \chi \). Then there is a finite group \( \mathcal{X} \) and irreducible characters \( \eta_x \) and \( d_x \) of \( A \) and \( A^* \) respectively with \( \eta_1 = \epsilon_A \) and \( d_1 = 1 \) such that

\[ \ker_{D(A)}(\chi) = \bigcup_{x \in \mathcal{X}} (\ker_{A^*}(\chi \downarrow_{A^*})\eta_x \times \ker_A(\chi \downarrow_A)d_x). \]

Moreover \( \eta^n_x \in \ker_{A^*}(\chi \downarrow_{A^*}) \) and \( d^n_x \in \ker_A(\chi \downarrow_A) \) for some \( n \geq 1 \) and all \( x \in \mathcal{X} \).

**Proof.** Since \( \ker_{D(A)}(\chi) \) is the set of simple objects of a fusion subcategory \( \text{Rep}(A^{(A)})^{\text{rev}} \boxtimes \text{Rep}(A) \) one can apply Lemma 3.11. Note that \( \mathcal{L}_1(\ker_{D(A)}(\chi)) \) has as set of simple objects the set \( \ker_{A^*}(\chi \downarrow_{A^*}) \) and \( \mathcal{L}_2(\ker_{D(A)}(\chi)) \) has as set of simple objects the set \( \ker_A(\chi \downarrow_A) \).

Let \( L_{A^*} := A^*_{(\chi^D_A(A))} \subset A^* \) and \( L_A := A_{(\chi^D_A(A))} \subset A \) be the Hopf subalgebras determined by \( \ker_{A^*}(\chi \downarrow_{A^*}) \) and \( \ker_A(\chi \downarrow_A) \) respectively. Similarly define \( K_{A^*} \) as the Hopf subalgebra of \( A^* \) determined by \( \mathcal{K}_1(\ker_{D(A)}(\chi)) \) and \( K_A \subset A \) as the Hopf subalgebra of \( A \) generated by \( \mathcal{K}_2(\ker_{D(A)}(\chi)) \). Then by Lemma 3.11 one has that \( (K_{A^*})_{ad} \subset L_A \) and \( (K_{A^*}) {ad} \subset L_A. \) Applying Lemma 3.8 it follows that there are irreducible characters \( \chi_x \) and \( d_x \) of \( A \) respectively \( A^* \) such that

\[ \ker_{D(A)}(\chi) = \bigcup_{x \in \mathcal{X}} \ker_{A^*}(\chi \downarrow_{A^*})\eta_x \times \ker_A(\chi \downarrow_A)d_x. \]

Remark 3.14 implies that \( \eta^n_x \in \ker_{A^*}(\chi \downarrow_{A^*}) \) and \( d^n_x \in \ker_A(\chi \downarrow_A) \) for some \( n \geq 1 \). \( \square \)

**Remark 3.14.**

If \( \eta \bowtie d \in \ker_{D(A)}(\chi) \) one can prove something a little stronger than \( \eta^n \in \ker_{A^*}(\chi \downarrow_{A^*}) \) and \( d^n \in \ker_A(\chi \downarrow_A) \).

Clearly \( C_{\eta} \bowtie C_d \subset D(A)_\chi \). Then from 3.10 above it follows that \( C_{d^*}C_d \subset L_A \wedge \). Thus for all \( m \in M \) and \( x \in C_d \) one has \( d^*xm = \epsilon(d)\epsilon(x)m \). Multiplying this equality by \( \eta^* \) and noticing that \( \eta^*d^* \in \ker_{D(A)}(\chi) \) one obtains: \( \epsilon(d)\epsilon(x)(\chi^*m) = (\chi^*d^*)xm = \chi(1)\epsilon(d)xm. \)
A basis of the simple coalgebra $C_d$ is given by the comatrix entries $x_{ij}^d$ with $1 \leq i, j \leq \ell(d)$. Therefore if $x = x_{ij}^d$ with $i \neq j$ then $x_{ij}^d m = 0$ for all $m \in M$. If $i = j$ then the above formula implies that $x_{ii}^d m = \frac{1}{\chi(1)} \eta^* m$. This shows that $x_{ii}^d m$ does not depend on $i$.

Recall that the exponent of $A$ is the smallest positive number $n > 0$ such that $h^n = \epsilon(h) I$ for all $h \in A$. The generalized power $h^n$ is defined by $h^n = \sum (h) h_1 h_2 \ldots h_n$. The exponent of a finite dimensional semisimple Hopf algebra is always finite and divides the third power of the dimension of $A$ \[5\]. If $h = d$ one has

$$d^n = \sum_{i,j_1, j_2, \ldots, j_{n-1}} x_{i,j_1,j_2,\ldots,j_{n-1}} x_{j_1,j_2,\ldots,j_{n-1}}$$

and therefore $d^n m = \sum_{i=1}^{\ell(d)} x_{ii}^d m = \epsilon(d) x_{11}^n m$. If $n$ is divisible by the exponent of $A$ then $(x_{ii}^d)^n m = m$ for all $m \in M$. Thus one obtains that $(x_{ii}^d)^n \in \ker_A (\chi^{D(A)})$.

3.4. Normal Hopf subalgebras of Drinfeld doubles $D(A)$. In this subsection we give a description of normal Hopf subalgebras of Drinfeld doubles $D(A)$.

**Remark 3.15.**

Suppose that $K$ is a normal Hopf subalgebra of $A$ and let $H = A//K$ with the natural projection $\pi_K : A \to H$. Then $H^* \subset A^*$ via $\pi_K^*$. Thus Remark 1.1 shows that $\pi_K |_{C(A^*)}$ is the restriction map of $A^*$-characters to $H^*$. This implies that an irreducible character $d$ of $A^*$ seats over an irreducible character $x$ of $H^*$ if and only if $m_{\pi_K(d), x} > 0$. Suppose now that $x$ is an $A^*$-stable character of $H^*$. In this situation we denote by $C_{\pi_K^{-1}(x)}$ the subcoalgebra of $A$ generated by the set $A_x$ of all characters $d \in \text{Irr}(A^*)$ that seat over $x$, i.e with the property $\pi_K(d) = \epsilon(d)x$. Also Remark 3.6 shows that the set $A_x$ is precisely the set of irreducible constituents of $\Lambda_{A^*} d$ for any $A^*$-character $d$ seating over $x$. Here $\Lambda_{A^*} \in K$ is the idempotent integral of $K$.

**Theorem 3.16.** Any normal Hopf subalgebra $H$ of $D(A)$ is of the following form:

$$H = \bigoplus_{x \in \mathcal{X}} C_{x}^{A} \rtimes C_{\pi_{L_2}^{-1}(\psi(x))}$$

where $L_1$ and $L_2$ are normal Hopf subalgebras of $A$, $\mathcal{X} \subset G_{st}(L_1)$ is a subgroup of $A$-stable linear characters of $L_1$ and $\psi : \mathcal{X} \to G_{st}((A//L_2)^*)$ is a group monomorphism into the group of $A^*$-stable linear characters of $(A//L_2)^*$. 
Proof. By Theorem 2.4 of [3] any normal Hopf subalgebra \( H \) of \( D(A) \) is the kernel of a character \( \chi \) of \( D(A) \), central in \( D(A)^* \). It follows that its restrictions to \( A \) and \( A^* \) are also central and therefore \( A_\chi \) and \( A^*_\chi \) are normal Hopf subalgebras of \( A \) respectively \( A^* \). Put \( L_2 = A^{(\chi^*_\chi^*_D(A))} \) and let \( L_1 \) be such that \( A^{(\chi^*_\chi^*_D(A))} = (A//L_1)^* \). With the notations from previous Theorem suppose that \( \eta \trianglerighteeq \eta \cap \Lambda_1 \). By Equation 3.11 this is equivalent to \( \ker_{D(A)}(\chi) \). For a given \( x \in X \).

Since \( \eta \subseteq \ker_{D(A)}(\chi) \) for some \( A \)-stable linear character \( f_x \) of \( L_1 \). This shows that \( \mathcal{X} \) can be regarded as a subgroup of \( G_{st}(L_1) \). By duality, the same argument applied for \( (K^2)_x \) gives that \( d^{(A//L_2)^*} = \epsilon(d)g_x \) for some \( A^* \)-stable linear character of \( (A//L_2)^* \). Therefore \( \mathcal{X} \) can also be identified with a subgroup of \( G_{st}((A//L_2)^*) \). Using these identifications one can define now the map \( \psi(f_x) = g_x \) for all \( x \in X \). By Equation 3.11 this is a group monomorphism from \( \mathcal{X} \) to \( G_{st}((A//L_2)^*) \) and the proof is finished.

Corollary 3.17. With the notations from Theorem 3.16 it follows that
\[
\text{Rep}(H^*) \subset \text{Rep}(A//L_1)^\psi \otimes \text{Rep}(L_1)^\psi
\]

Remark 3.18.

Given a datum \( L_1, L_2 \mathcal{X} \) and \( \psi \) as in the above Theorem it does not necessarily follows that \( H \) is a normal Hopf subalgebra of \( D(A) \). Compatibility conditions between \( L_1, L_2 \) and \( \psi \) should be imposed in order to get a normal Hopf subalgebra. We will see in Theorem 3.20 that such a necessary condition is that \( L_1 \) and \( L_2 \) commute elementwise. In the case of the Drinfeld double of a group \( G \) we will give in Theorem 5.3 the necessary and sufficient conditions that has to be satisfied by this datum in order to get a normal Hopf subalgebra of \( D(G) \).

Remark 3.19. Let \( K \) be a Hopf subalgebra of \( A \) and \( \Lambda_K \) be its idempotent integral. Then it is well known (see [3] for example) that the induced module \( A \otimes_K k \) is isomorphic to \( A\Lambda_K \) via the map \( a \otimes_K 1 \mapsto a\Lambda_K \).

3.5. Commutativity between \( L_1 \) and \( L_2 \).

Theorem 3.20. Suppose that \( K := (A//L_1)^* \triangleright L_2 \) is a normal Hopf subalgebra of \( D(A) \). Then \( L_1 \) and \( L_2 \) commute elementwise.

Proof. One has by [3] that \( K \) is a normal Hopf subalgebra of \( D(A) \) if and only if \( K = \ker_{D(A)}(\epsilon_{K}^{1\uparrow_{K}^{D(A)}}) \). This is equivalent to \( L_2 = K \cap A \subset \ker_{D(A)}(\epsilon_{K}^{1\uparrow_{K}^{D(A)}}) \) and \( (A//L_1)^* = K \cap A^* \subset \ker_{D(A)}(\epsilon_{K}^{1\uparrow_{K}^{D(A)}}) \).
Note that $k \uparrow_{K}^{D(A)} = D(A) \otimes (A//L_1)_{\epsilon} \otimes L_2$. Thus if $K$ is a normal Hopf subalgebra of $D(A)$ then $m((f \triangleright b) \otimes_K 1) = \epsilon(m)((f \triangleright b) \otimes_K 1)$ for all $m \in L_2$ and any $f \in A^*$, $b \in A$. But if $m \in L_2$ then one has

$$m((f \triangleright b) \otimes_K 1) = \left((f_1(Sm_2)f_3(m_1)f_2 \triangleright m_2b) \otimes_K 1\right)$$

This implies that $(f_1(Sm_2)f_3(m_1)f_2 \triangleright b) \otimes_K 1) = \epsilon(m)((f \triangleright b) \otimes_K 1)$ and previous Remark gives that

$$\text{(3.21)}$$

$$(f_1(Sm_2)f_3(m_1)f_2 \triangleright b)(t_{A//L_1} \triangleright \Lambda_{L_2}) = \epsilon(m)(f \triangleright b)(t_{A//L_1} \triangleright \Lambda_{L_2})$$

for all $f \in A^*$ and $b \in A$. Put $b = 1$ and apply $\text{id} \otimes \epsilon$ in the previous equality. Then one obtains that $(f_1(Sm_2)f_3(m_1)f_2 t_{A//L_1} = \epsilon(m) t_{A//L_1}$. Evaluating both sides at any $l \in L_1$ it follows that $Sm_2lm_1 = \epsilon(l)m$ which shows that $tm = ml$. Thus $L_1$ and $L_2$ commute elementwise. 

**Remark 3.22.**

The proof of the previous Theorem shows that the Equation [3.21] is necessary in order for $K$ to be a normal Hopf subalgebra of $D(A)$. Writing a similar condition for $(A//L_1)^* = K \cap A^* \subset \ker_{D(A)}(\epsilon_{K} \uparrow_{K}^{D(A)})$ it follows that both conditions together are necessary and sufficient for $K$ to be a normal Hopf subalgebra of $D(A)$.

At the end of this section we give two general examples of normal Hopf subalgebras of $D(A)$ of the type described in the previous Theorem.

**Proposition 3.23.** Let $A$ be a semisimple Hopf algebra and $K(A)$ its maximal central Hopf subalgebra. Then $K(A)$ is a normal Hopf subalgebra of $D(A)$.

**Proof.** If $x \in K(A)$ then $a_1 x S(a_2) = \epsilon(a)x$ for all $a \in A$ since $x$ is central in $A$. On the other hand all $h \in A$ and $f \in A^*$ one has

$$<(h \triangleright \otimes \text{id})(f_2 x S(f_1)) = (h \triangleright \otimes \text{id})(f_2 x_1 \rightarrow Sf_1 \leftarrow Sx_3 \triangleright x_2)$$

$$= [f_2 x_1 \rightarrow Sf_1 \leftarrow Sx_3] (h) x_2$$

$$= f_2(h_1) f_1(S(x_1) S(h_2) x_3) x_2$$

$$= f(S(x_1) S(h_2) x_3) x_1 x_2$$

$$= \epsilon(h) f(S(x_1) x_3) x_2$$
since $x \in K(A)$. This shows that
\begin{equation}
(3.24) \quad f_x S(f_1) = \epsilon \otimes f(S(x_1)x_2) x_2 \in K(A).
\end{equation}

Thus $K(A)$ is closed under the adjoint action of $D(A). \square$

Note that this Hopf subalgebra corresponds to $L_1 = A$, $L_2 = K(A)$ with $\mathcal{X}$ and $\psi$ trivial.

**Proposition 3.25.** Let $A$ be a semisimple Hopf algebra. Then $(A//K(A))^* \bowtie A$ is a normal Hopf subalgebra of $D(A)$.

**Proof.** Note that
\[
(A//K(A))^* = \{ f \in A^* \mid f(ax) = \epsilon(x)f(a) \}
\]
for all $a \in A$ and $x \in K(A)$.

Since $K(A) \subset Z(A)$ it follows that $(b \rightarrow f \leftarrow c) \in (A//K(A))^*$ for all $f \in (A//K(A))^*$. Indeed $(b \rightarrow f \leftarrow c)(ax) = f(caxb) = f(cabx) = \epsilon(x)f(cab) = (b \rightarrow f \leftarrow c)(a)\epsilon(x)$. Thus $a.(f \bowtie b) = a_1(f \bowtie b)S_{a_2} = a_1 \rightarrow f \leftarrow S_a \bowtie b \bowtie S_{a_4} \in (A//K)^* \bowtie A$ if $f \bowtie b \in (A//K)^* \bowtie A$. This shows that $(A//K)^* \bowtie A$ is closed under the adjoint action of $D(A)$ by elements of $A$.

In order to show that $(A//K)^* \bowtie A$ is also closed under the adjoint action of $D(A)$ by elements of $A^*$ note that
\[
g.(f \bowtie b) = g_2(f \bowtie b)Sg_1 = g_2f(Sb_1 \rightarrow S(g_1) \leftarrow b_3) \bowtie b_2.
\]

Then it is enough to check that if $f \in (A//K)^*$ then $g_2f(b \rightarrow S(g_1) \leftarrow c) \in (A//K)^*$ for all $g \in A^*$. Indeed, for all $a \in A$ and $x \in K(A)$ one has
\[
[g_2f(b \rightarrow Sg_1 \leftarrow c)](ax) = g_2(a_1x_1)f(a_2x_2)Sg_1(ca_3x_3b) = g_2(a_1x_1)f(a_2)Sg_1(ca_3x_2b) = g(Sbsx_2Sca_3Sca_1x_1)f(a_2) = g(Sbsa_3Sca_1)f(a_2)\epsilon(x) = [g_2f(b \rightarrow Sg_1 \leftarrow c)](a)\epsilon(x)
\]

Note that this Hopf subalgebra corresponds to $L_1 = K(A)$, $L_2 = A$ with $\mathcal{X}$ and $\psi$ trivial.

4. **Representations of $D(G)$ and Central Characters**

4.1. **The Drinfeld double $D(G)$**. Let $D(G)$ be the Drinfeld double of $G$. As a coalgebra $D(G) = kG^{*\text{cop}} \otimes kG$ and the multiplication is given by
\[
(p_x \bowtie g)(p_y \bowtie h) = p_x p_{g^{y^{-1}}} \bowtie gh = \delta_{x,g^{y^{-1}}} p_x \bowtie gh.
\]
The antipode is given by the formula \( S(p_x \triangleright g) = g^{-1}p_x^{-1} = p_{g^{-1}x^{-1}g}^{-1} \).

A vector space basis for \( D(G) \) is given by \( \{ p_x \triangleright y \}_{x \in G} \) where \( \{ p_x \}_{x \in G} \) is the dual basis of the basis of \( kG \) given by the group elements.

### 4.2. Irreducible representations of \( D(G) \)

Let \( \mathcal{R} \) be a set of representatives of conjugacy classes of \( G \). The irreducible representations of \( D(G) \) are parameterized by pairs \((a, \gamma)\) where \( a \in \mathcal{R} \) and \( \gamma \in \text{Irr}(C_G(a)) \) is an irreducible character of the centralizer \( C_G(a) \) of \( a \) in \( G \). Their characters are denoted by \( \overline{(a, \gamma)} \) respectively.

### 4.3. Few results on group representations

In this subsection we give some results on group representations that are needed in the next sections. Let \( H \) be a subgroup of \( G \). Define by \( \text{core}_G(H) \) the largest normal subgroup of \( G \) contained in \( H \). Then \( \text{core}_G(H) = \cap_{g \in G} gHg^{-1} \).

Let \( N \) be a normal subgroup of \( G \). Then \( G \) acts on the irreducible characters of \( N \).

Let \( \alpha \) be an irreducible character of \( N \) and \( \chi \) be an irreducible character of \( G \). The set of characters lying over a given \( \alpha \) is denoted by \( \text{Irr}(G) |_{\alpha} \).

**Lemma 4.1.** Let \( N \) be a normal subgroup of \( G \) and \( \alpha \in \text{Irr}(N) \). Then the induced character \( \alpha \uparrow_G^N \) vanishes outside \( N \). If \( \alpha \) is a \( G \)-stable character of \( N \) then \( \alpha \uparrow_G^N(g) = \frac{G}{|N|} \text{Res}(g) \) for all \( g \in N \).

**Proof.** Denote by \( M_{\alpha} \) the representation afforded by \( \alpha \). Let \( G = \bigcup_{i=1}^{s} x_iN \) a coset decomposition of \( G \). Then \( g(x_i \otimes_N m) = x_j \otimes_N (x_j^{-1}gx_i).m \) where \( j \) is chosen such that \( gx_iN = x_jN \). Thus if \( g \notin N \) then \( i \neq j \) and \( \alpha \uparrow_G^N(g) = 0 \). On the other hand if \( g \in N \) then \( \alpha \uparrow_G^N(g) = \sum_{i=1}^{s} \alpha(x_i^{-1}gx_i) \). But if \( \alpha \) is \( G \)-stable then \( \alpha(x_i^{-1}gx_i) = \alpha(g) \) for all \( i \) and the formula follows. \( \square \)

Let \( a \in G \) and denote \( \tilde{C}_G(a) := \text{core}_G(C_G(a)) \) the core of the centralizer \( C_G(a) \).

For any \( a \in G \) let \( N(a) \) be the smallest subgroup of \( G \) containing \( a \). It is easy to see that \( N(a) \) is generated by the conjugacy class of \( a \).

**Lemma 4.2.** If \( a \in G \) then \( N(a) \) and \( \tilde{C}_G(a) \) form a pair of commuting groups.

**Proof.** One has \( \tilde{C}_G(a) = \cap_{g \in G} C_G(a)g^{-1} = \cap_{g \in G} C_G(gag^{-1}) \). Since \( N(a) \) is generated by the conjugacy class of \( a \) it is enough to verify that any element of this conjugacy class commutes with each element of \( \tilde{C}_G(a) \). Let \( z \in \tilde{C}_G(a) \) and \( a \in G \) be fixed. Then there is \( z' \in C_G(a) \) such that \( z = g\z'g^{-1} \) and therefore \( z(gag^{-1}) = (g\z'g^{-1})(gag^{-1}) = g\z'ag^{-1} = gaz'g^{-1} = (gag^{-1})z \). \( \square \)
4.4. **Kernels in** $kG^*$. In this subsection we apply the previous constructions of $\ker_A \chi$ and $z_A \chi$ for $A = kG^*$. Note that in this case $\text{Irr}(A) \cong G$.

Let $\chi \in \text{Irr}(G)$. It follows that $h \in \ker_{kG} \chi$ if and only if $N(h) \subset \ker_{kG} \chi$.

**Lemma 4.3.** For any $h \in G$ one has

$$kG^*_h = k[G/N(h)]^*.$$ 

**Proof.** By its definition $kG^*_h$ is determined by all the characters $\chi \in \text{Irr}(G)$ such that $\chi(h) = \chi(1)$. These are precisely the characters of $G/N(h)$. □

**Lemma 4.4.** For any $h \in G$ one has

$$Z_{kG^*} h = k[G/[G, N(h)]]^*.$$ 

**Proof.** By its definition $Z_h$ is generated by the set of irreducible characters $\chi$ of $G$ with the property that $\chi(h) = \omega \chi(1)$ for some root of unity $\omega$. Since $N(h)$ is generated by the conjugacy class of $h$ it follows that every element of $N(h)$ acts as a scalar on the representation $M_\chi$ afforded by $\chi$. Then if $n \in N(h)$ and $g \in G$ it follows that $gng^{-1}$ acts as identity on $M_\chi$. Thus $\chi \in \text{Irr}(G/[G, N(h)])$. Conversely for any $\chi \in \text{Irr}(G/[G, N(h)])$ one has that left multiplication by any $n \in N(h)$ is a morphism of $kG$ modules and Schur’s lemma implies the conclusion. □

**Lemma 4.5.** Let $h \in G$ and $\chi \in Z_{kG^*} h$ with $\chi(h) = \omega \chi(1)$ with $\omega \in k^*$. Then all irreducible characters of $G$ which satisfy $\mu(h) = \omega \mu(1)$ are constituents of $\chi \in \text{Irr}(G)$ where $\epsilon \in \text{Irr}(G)$. This follows from Theorem 4.3 of [2] that $\mu \chi^*$ has all the constituents inside $\epsilon \uparrow_{G(h)}^G$. Thus $m_G(\mu, \chi \epsilon \uparrow_{G(h)}) = m_G(\mu \chi^*, \epsilon \uparrow_{G(h)}) > 0$. □

4.5. **Central characters in** $D(G)$.

**Lemma 4.6.** Let $c = \sum_{h \in G} \chi_h \triangleright h$ be a character of $D(G)^*$ with $\chi_h \in C(G)$. Then $c$ is central in $D(G)$ if and only if $\chi_{ghg^{-1}} = \chi_h$ and $\chi_h$ vanishes on $G \setminus C_G(h)$.

**Proof.** The character $c$ is central if and only it is invariant under the adjoint action of $D(G)$ on itself. Thus $c$ is central if and only if $gcg^{-1} = g$ for all $g \in G$ and $\delta_{x,1}c = \delta_{x,1}c$ for all $x \in G$. The first condition is
equivalent to $\chi_{ghg^{-1}} = \chi_h$ for all $g \in G$. For the second condition one has

$$p_{x,c} = \sum_{uv = x} p_v p_u^{-1} \chi_{h} \triangleright h \triangleright h,$$

$$= \sum_{h \in G} \sum_{uv = x} p_v (\chi_h \triangleright h) p_u^{-1} \chi_{h} \triangleright h,$$

$$= \sum_{h \in G} \sum_{uv = x} (p_v \chi_{h} p_{hu^{-1}h^{-1}} \triangleright h),$$

$$= \sum_{h \in G} \sum_{\{u \mid uh_{h^{-1}} = x\}} p_{hu^{-1}h^{-1}} \chi_h \triangleright h.$$

Suppose that $c$ is central. Then $p_{x,c} = \delta_{x,1} c$ if and only if

$$\sum_{\{u \mid uh_{h^{-1}} = x\}} p_{hu^{-1}h^{-1}} \chi_h = \delta_{x,1} \chi_h$$

for all $h \in G$. If $x = 1$ this means precisely that $\chi_h$ is zero outside $C_G(h)$. The converse is immediate.  

Remark 4.7. Let $H$ be a subgroup of $G$. A character $\chi \in C(G)$ vanishes outside $H$ if and only if it vanishes outside $\text{core}_G(H)$.

Theorem 4.8. A basis for $Z(D(G)) \cap C(D(G)^*)$ is given by the elements $p_{\mathcal{D}} \triangleright z_{\mathcal{C}}$ where $\mathcal{D}$ and $\mathcal{C}$ run through all conjugacy classes of $G$ that centralize each other element-wise.

Proof. Let $a \in \mathcal{C}$. Then $\bar{C}_G(a) = \cap_{g \in G} g C_G(a) g^{-1} = \cap_{x \in \mathcal{C}} C_G(x)$. Thus a conjugacy class $\mathcal{D}$ centralizes each element of another conjugacy class $\mathcal{C}$ if and only if $\mathcal{D}$ is contained in $\bar{C}_G(a)$. In this case the above remark implies $p_{\mathcal{D}}$ vanishes outside the centralizer of each element of $\mathcal{C}$. The previous lemma implies that $p_{\mathcal{D}} \triangleright z_{\mathcal{C}}$ is central in $D(G)$. The same lemma also implies that any central character is a linear combination of such characters. \hfill $\Box$

5. Normal Hopf subalgebras of Drinfeld doubles $D(G)$

For a subgroup $N$ of $G$ define $C_G(N) := \cap_{n \in N} C_G(n)$ to be the subgroup of the elements of $G$ that commute with each element of $N$. If $N$
is a normal subgroup of $G$ let $G_{st}(N)$ be the group of linear characters of $N$ that are stable under the action of $G$ induced by the conjugation on $N$.

It is not difficult to check the following result:

**Lemma 5.1.** Let $N$ be a normal subgroup of $G$ and $x, x' \in G_{st}(N)$. Then

$$x \uparrow_N^G x' \uparrow_N^G = \frac{|G|}{|N|} (xx') \uparrow_N^G.$$  

If $x$ is a character of $N$ denote by $C_x \uparrow_N^G$ the subcoalgebra of $kG^*$ generated by $x \uparrow_N^G$. Also if $N$ is a normal subgroup of $G$ with $\pi_N : G \to G/N$ the natural group projection and $g \in G$ denote by $C_{\pi_N^{-1}(\bar{g})}$ the vector space with a basis given by the elements $gn$ with $n \in N$.

5.1. **Hopf subalgebras of $D(G)$**. Let $N$ and $M$ be normal subgroups of $G$, $\mathcal{X} \subset G_{st}(N)$ and $\psi : \mathcal{X} \to G/M$ a monomorphism of groups.

Let $D(N, M, \mathcal{X}, \psi)$ be the subcoalgebra of $D(G)$ given by:

$$D(N, M, \mathcal{X}, \psi) := \oplus_{x \in \mathcal{X}} C_x \uparrow_N^G \bowtie C_{\psi_0(x)}.$$  

**Theorem 5.2.** Any Hopf subalgebra of $D(G)$ is of the type $D(N, M, \mathcal{X}, \psi)$ for a given datum as above.

**Proof.** Since $x$ is $G$-stable by Frobenius reciprocity it follows that $x \uparrow_N^G = \sum_{\chi \in \Lambda_N} \chi(1)\chi$ and therefore $\dim_k C_x \uparrow_N^G = \frac{|G|}{|N|}$. Then $\dim_k D(N, M, \mathcal{X}, \psi) = \frac{|\mathcal{X}| |G||M|}{|N|}$. Let $\psi_0(x) \in G$ such that $\psi(x) = \psi_0(x)M$ for all $x \in M$. Since $\psi$ is a group morphism it follows that $(\psi_0(x)\Lambda_M)(\psi_0(x')\Lambda_M) = \psi_0(xx')\Lambda_M$ for all $x, x' \in M$. Here $\Lambda_M = \frac{1}{|M|} \sum_{m \in M} m$. Let

$$\Lambda_{D(N, M, \mathcal{X}, \psi)} = \frac{|N|}{|\mathcal{X}| |G||M|} \sum_{x \in \mathcal{X}} x \uparrow_N^G \bowtie \psi_0(x)\Lambda_M.$$  

One has

$$\Lambda^2_{D(N, M, \mathcal{X}, \psi)} = \left(\frac{|N|}{|\mathcal{X}| |G||M|}\right)^2 \sum_{x, x' \in \mathcal{X}} x \uparrow_N^G x' \uparrow_N^G \bowtie \psi_0(x)\Lambda_M \psi_0(x')\Lambda_M$$

$$= \frac{|N|}{|\mathcal{X}| |G||M|} \sum_{x, x' \in \mathcal{X}} (xx') \uparrow_N^G \bowtie \psi_0(xx')\Lambda_M$$

by the above lemma. This shows that $D(N, M, \mathcal{X}, \psi)$ is a Hopf subalgebra of $D(G)$.

In order to see that any Hopf subalgebra $H$ of $D(G)$ is of this type one has to apply the results from subsection 3.2 to the case $A = kG$. Note
that in this case one can write $\mathcal{L}^1 = \text{Rep}(G/L_1)$ and $\mathcal{K}^1 = \text{Rep}(G/K_1)$ for some normal subgroups $L_1$ and $K_1$ of $G$. Condition 3.9 translates by $K_1 \subset L_1 \subset Z_G(K_1)$ where $Z_G(K_1)$ is defined by $Z_G(K_1)/K_1 = Z(G/K_1)$. On the other hand one can write $\mathcal{L}^2 = \text{Rep}(kL_2^*)$ and $\mathcal{K}^2 = \text{Rep}(kK_2^*)$ for some subgroups $L_2$ and $K_2$ of $G$. Then condition 3.10 is satisfied if $L_2$ is a normal subgroup of $K_2$. Clearly the second grading $(i = 2)$ of 3.11 implies that $L_2$ is a normal subgroup of $K_2$ and $\mathcal{X} \cong K_2/L_2$. In this case the grading becomes $K_2 = \bigcup_{x \in K_2/L_2} xL_2$. Applying Lemma 3.8 to the first grading of 3.11 $(i = 1)$ it follows that $\mathcal{K}^1_x$ is given by all the characters of $G$ that seat over a $G$-stable linear character $\psi_x$ of $L_1$. Thus $\mathcal{X}$ can also be identified with a subgroup of $G_u(L_1)$ by $x \mapsto \psi_x$. Thus formula 3.12 implies that $H = D(L_1, L_2, \mathcal{X}, \psi)$ where $\psi$ sends $\psi_x$ to the class of $x$ modulo $K_2$ based on the two identifications of $\mathcal{X}$.

\section{Normal Hopf subalgebras of $D(G)$}

\textbf{Theorem 5.3.} A Hopf subalgebra $D(N,M,\mathcal{X},\psi)$ of $D(G)$ is normal if and only $\psi(\mathcal{X}) \subset C_G(N)/M \cap Z(G/M)$ and $[N,M] = 1$.

\textbf{Proof.} In order to see when $D(N,M,\mathcal{X},\psi)$ is a normal Hopf subalgebra of $D(G)$ it is enough [9] to see when its integral $\Lambda_{D(N,M,\mathcal{X},\psi)}$ is central in $D(G)$. In order to do this one needs to verify the conditions from Lemma 4.6.

First note that $\psi_0(x)m \neq \psi_0(y)m'$ for $x \neq y$ and for any $m,m' \in M$ since $\psi$ is a monomorphism. The equality of characters $\chi_{ghg^{-1}} = \chi_h$ from Lemma 4.6 is satisfied if and only if $g\psi_0(x)Mg^{-1} = \psi_0(x)M$ for all $g \in G$. This is equivalent to the fact that $\psi(x) \in Z(G/M)$. On the other hand for the second condition note by that Lemma 4.1 $x_1^{G} \not\subset N$ is zero outside $N$ and does not vanishes on any element of $N$. Thus the second condition of Lemma 4.6 is equivalent to $N \subset C_G(\psi_0(x)m)$ or $\psi(x) \in C_G(N)/M$.

\section{Kernels of irreducible characters of $D(G)$}

Since $D(G) = kG^\ast \otimes kG$ the dual Hopf algebra $D(G)^\ast$ can be identified with $kG^\ast \otimes kG^\text{op}$ via $< f \otimes l, p_x \otimes g >= < f, g > p_x$, $l >$ for any $g,x,l \in G$ and $f \in kG^\ast$. For a subgroup $H$ of $G$ denote by $(G/H)_l$ a set of representatives for the left cosets of $H$ in $G$.

\textbf{Lemma 5.4.} For a representation $(\hat{a}, \hat{\gamma})$ of $D(G)$ one has

$$(\hat{a}, \hat{\gamma}) = \sum_{g \in (G/C_G(a))_l} \sum_{z \in C_G(a)} \gamma(z)p_{gag^{-1}} \otimes gzg^{-1}.$$
Proof. It is enough to show that

\[ (a, \gamma)(p_x \bowtie l) = \gamma(g^{-1}lg) \]

if there is \( g \in G \) such that \( x = gag^{-1} \) and \( g^{-1}lg \in C_G(a) \) and zero otherwise.

The representation corresponding to \((a, \gamma)\) is given by \( kG \otimes kC_G(a) M_\gamma \) where \( M_\gamma \) is the module affording the character \( \gamma \). The action of \( kG^* \) is given by \( p_x(g \otimes kC_G(a) m) = \delta_{x,gag^{-1}}(g \otimes kC_G(a) m) \) and the action of \( kG \) is the action of induced module. Using Lemma 4.1 a straightforward computation implies formula 5.5. □

Proposition 5.6. Let \( M \) be subgroups of \( G \), \( \gamma \) a character of \( M \) and \( a \in G \). Then the set \( S \) of pairs \((\chi, l) \in Z_{kG^*}a \times \text{core}_G M \) such that

\[ \frac{\chi(a)}{\chi(1)} = \left( \frac{\gamma(g\phi g^{-1})}{\gamma(1)} \right)^{-1} \]

for all \( g \in G \) is of the form

\[ S = \bigcup_{i=0}^{s-1} (\text{Irr}(G)|f_0^i \times l_0^i \text{core}_G(\ker_M(\gamma))) \]

for some \( G \)-stable character \( f_0 \) of \( N(a) \) of order \( s \) and some \( l_0 \in Z_{C_G(a)} \gamma \) such that \( l_0^i \in \text{core}_G(\ker_M(\gamma)) \).

Proof. Let

\[ H := \{ \omega \mid \text{there is } (\chi, l) \in S \text{ with } \omega = \frac{\chi(a)}{\chi(1)} \text{ for some } (\chi, l) \in S \} \]

It can be checked that \( H \) is a subgroup of \( k^* \). Since \( k \) is algebraically closed and \( H \) is finite it follows that \( H \) is a cyclic group. Therefore \( H = \{1, \omega, \cdots, \omega^{s-1}\} \) for some root of unity \( \omega \) of order \( s \). Let \( \chi_0 \) and \( l_0 \) such that \((\chi_0, l_0) \in S \) and \( \frac{\chi_0(a)}{\chi_0(1)} = \omega = \left( \frac{\gamma(g\phi g^{-1})}{\gamma(1)} \right)^{-1} \) for all \( g \in G \).

Then \( \chi_0 \bigg|_{N(a)} = \chi_0(1) f_0 \) for some \( f_0 \in G_{st}(N(a)) \) of order \( s \). Also note that \( g_0^s \in \text{core}_G(\ker_M \gamma) \). Lemma 4.5 implies the conclusion of the Proposition. □

Theorem 5.7. Let \( a \in \mathcal{R} \) and \( \gamma \in \text{Irr}(C_G(a)) \). Then the Hopf subalgebra \( D_{(a, \gamma)} := D(G)_{(a, \gamma)} \) is a normal Hopf subalgebra of \( D(G) \). Moreover with the above notations one has

\[ D_{(a, \gamma)} = D(N(a), \text{core}_G(\ker_{C_G(a)}(\gamma))), < f_0 >, \psi \]

for some \( G \)-stable linear character \( f_0 \) of \( N(a) \) and some monomorphism \( \psi : < f > \rightarrow Z(G/\text{core}_G(\ker_{C_G(a)}(\gamma))) \cap C_G(N(a))/(\ker_{C_G(a)}(\gamma)) \).
Proof. First we will describe ker($\hat{a, \gamma}$). An irreducible character of $D(G)^*$ is given by $\chi \bowtie l$ with $\chi \in \text{Irr}(G)$ and $l \in G$. It follows that $\chi \bowtie l \in \ker_{D(G)}(a, \gamma)$ if and only if $\hat{(a, \gamma)}(\chi \bowtie l) = \chi(1)$ $\gamma(1) \frac{|G|}{|C_G(a)|}$.

It follows from Lemma 5.4 that $\hat{(a, \gamma)}(\chi \bowtie l) = \chi(a) \gamma\left(\sum_{\{g \in (G/C_G(a))l \mid l \in gC_G(a)g^{-1}\}} g^{-1}lg\right)$.

Since the above the sum has $\frac{|G|}{|C_G(a)|}$ terms and one has $|\chi(h)| \leq \chi(1)$ and $|\gamma(g^{-1}lg)| \leq \gamma(1)$ we deduce that the above equality is satisfied if and only if there is a root of unity $\omega \in k^*$ such that $\chi(a) = \omega \chi(1)$ and $l \in \text{core}_G(Z_{C_G(a)} \gamma)$ with the property that $\gamma(g^{-1}lg) = \omega^{-1} \gamma(1)$ for all $g \in G$. By Proposition 5.6 it follows that there is $f_0$ a $G$-stable linear character of $N(a)$ and $l_0 \in G$ such that

$$\ker_{D(G)}(a, \gamma) = \bigcup_{i=0}^{s} (\text{Irr}(G)|f_i^n \times l_i^n \text{core}_G(\ker_{C_G(a)} \gamma))$$

Thus one can take $\mathcal{X}$ to be the group generated by $f_0$ and define $\psi$ by sending $f_i^n$ to the class of $g_0^n$ modulo $\text{core}_G(\ker_{C_G(a)} \gamma)$, for all $1 \leq i \leq s - 1$. Note that $[N(a), C_G(a)] = 1$ by Lemma 1.2. It can be easily checked that the map $\psi$ satisfies the additional hypothesis from Theorem 5.3. □

The description of the kernels from the previous theorem implies the following corollary:

**Corollary 5.8.** With the above notations one has

$$D_{(1, \gamma)} = kG^* \bowtie \ker_G \gamma$$

**Proposition 5.9.** If $(a, \gamma)$ is a representation of $D(G)$ then

$$Z_{D(G)}(a, \gamma) = k[G/[G, N(a)]]^* \bowtie \text{core}_G(Z_{C_G(a)} \gamma)$$

Proof. The proof is similar to the proof of Theorem 5.7. Lemma 4.4 is needed in order to compute $Z_{kG^* \cdot a}$. □

5.4. **Fusion subcategories of** $\text{Rep}(D(G))$. Let $\mathcal{D}$ be a fusion subcategory of $C$. Following [12] $\mathcal{D}$ is completely determined by two canonical normal subgroups $K_\mathcal{D}$ and $H_\mathcal{D}$ of $G$ and a $G$-invariant bicharacter $B_\mathcal{D} : K_\mathcal{D} \times H_\mathcal{D} \to k^*$. The subgroups $K_\mathcal{D}$ and $H_\mathcal{D}$ are defined as follows:

$$K_\mathcal{D} := \{gag^{-1} \mid g \in G \text{ and } (a, \gamma) \in \mathcal{D} \text{ for some } \gamma\}$$
and $H_p$ is the normal subgroup of $G$ such that $\mathcal{D} \cap \text{Rep}(G) = \text{Rep}(G/H_p)$.

Note that $K_p$ is the fusion subcategory of $kG^*$ determined by restricting all simple objects of $\mathcal{D}$ to $kG^*$.

The bicharacter

$$B_\mathcal{D} : K_\mathcal{D} \times H_\mathcal{D} \to k^*$$

is defined by

$$B_\mathcal{D}(g^{-1}ag, h) := \frac{\gamma(ghg^{-1})}{\gamma(1)}$$

if $(a, \gamma) \in \mathcal{D}$. This is well defined and does not depend on $\gamma$ by [13].

Recall that a bicharacter is called $G$-invariant if and only if $B(xkx^{-1}, h) = B(k, h)$ and $B(k, xhx^{-1}) = B(k, h)$ for all $x \in G$, $k \in K$ and $h \in H$.

Conversely, any two normal subgroups $K$ and $H$ of $G$ that centralize each other element-wise together with a $G$-invariant bicharacter $B : K \times H \to k^*$ give rise to a fusion category denoted by $S(K, H, B)$ in [13]. It is defined as the full abelian subcategory of $\text{Rep}(D(G))$ generated by the objects $(a, \gamma)$ such that $a \in K \cap \mathcal{R}$ and $\gamma \in \text{Irr}(C_G(a))$ such that $\gamma(h) = B(a, h)\gamma(1)$ for all $h \in H$.

5.5. Normal fusion subcategories of $\text{Rep}(D(G))$. In this section we will identify all the normal fusion subcategories $S(K, H, B)$ of $\text{Rep}(D(G))$.

Remark 5.10. Let $B$ be a normal Hopf subalgebra of a semisimple Hopf algebra $A$ and $a \in A$. Then $a(A//B) \neq 0$ if and only if $a\Lambda_B \neq 0$.

Theorem 5.11. Using the notations from Theorem 5.3, the fusion subcategory $\text{Rep}(D(G)//D(N, M, X, \psi))$ can be identified with $S(K, H, B)$ where $K := N$, $H := < \psi_0(x), M | x \in X >$ and $B : K \times H \to k^*$ is given by $B(n, \psi_0(x)m) = x(n)^{-1}$.

Proof. Let $\mathcal{D} := \text{Rep}(D(G)//D(N, M, X, \psi))$. For the subgroup $H_{\mathcal{D}}$, one has to look at $\text{Rep}(D(G)//D(N, M, X, \psi)) \cap \text{Rep}(kG)$. Thus

$$H_{\mathcal{D}} = \cap_{\chi \in \text{Irr}(G)} \{ (i, \chi) \in D \} \ker G \chi.$$

Since $D_{(i, \chi)} = kG^* \cong \ker G \chi$ it follows that $D_{(i, \chi)} \supset D(N, M, X, \psi)$ if and only if $\ker G \chi \supset \langle \psi(x), m | x \in X, m \in M >$. Thus $H_{\mathcal{D}} = \langle \psi(x), m | x \in X, m \in M >$. The subgroup $K_{\mathcal{D}}$ is generated by all $x \in G$ such that $p_x(D(G)//D(N, M, X, \psi)) \neq 0$. The above remark implies that $K_{\mathcal{D}}$ is given by those $x \in G$ such that $p_x\Lambda_{D(N, M, X, \psi)} \neq 0$. Formula for $\Lambda_{D(N, M, X, \psi)}$ from Theorem 5.2 shows that $x \in K_{\mathcal{D}}$ if and only if $(f^i \uparrow^G_N)(x) \neq 0$ for some $i$. Lemma 4.1 shows that $K_{\mathcal{D}} \subset N$. Since $f$ is linear it follows that $K_{\mathcal{D}} = N$.

In order to describe the bicharacter $B$ suppose that $(a, \gamma) \in \text{Rep}(D(G)//D(N, M, X, \psi))$. Then $D_{(a, \gamma)} \supset D(N, M, X, \psi)$. 


Thus using again Lemma 4.1 and Remark 3.6 one has \( B(ga^{-1}, h) = \gamma(g^{-1} h g) = \frac{\gamma \left( g^{-1} h g \right)}{\gamma(1)} = x(a)^{-1} \).

\[ \begin{array}{c}
\text{Theorem 5.12.} \\
A \text{ category } S(K, H, B) \text{ is normal if and only if } \\
B(ga^{-1}, h) = B(a, h) = B(a, yhy^{-1}) \\
\text{for all } a, x, y, h \in G. \text{ In these conditions} \\
S(K, H, B) = \text{Rep} \left( \frac{D(G)}{D(K, K^\perp, X, \psi)} \right) \\
\text{where } X = \{ B(-, h) \mid h \in H \} \text{ and } \psi : X \to Z(G/K^\perp) \cap C_G(K)/K^\perp \text{ is given by } \psi(x) = \bar{h} \text{ for any } h \in H \text{ with } x = B(-, h). \\
\text{Proof.} \text{ It can easily be checked that } X \text{ is a group of } G\text{-stable linear characters of } K \text{ and that } \psi \text{ takes values in } Z(G/K^\perp) \cap C_G(K)/K^\perp. \\
The rest of the theorem follows from Theorem 5.11. \square
\end{array} \]

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