Highly incidental patterns on a quadratic hypersurface in \(\mathbb{R}^4\)

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Abstract

In [23], Sharir and Solomon showed that the number of incidences between \(m\) distinct points and \(n\) distinct lines in \(\mathbb{R}^4\) is
\[
O^* \left( m^{2/5} n^{4/5} + m^{1/2} n^{1/2} q^{1/4} + m^{2/3} n^{1/3} s^{1/3} + m + n \right),
\]
provided that no 2-flat contains more than \(s\) lines, and no hyperplane or quadric contains more than \(q\) lines, where the \(O^*\) hides a multiplicative factor of \(2^{\sqrt{\log m}}\) for some absolute constant \(c\).

In this paper we prove that, for integers \(m, n\) satisfying \(n^{9/8} < m < n^{3/2}\), there exist \(m\) points and \(n\) lines on the quadratic hypersurface in \(\mathbb{R}^4\),
\[
\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2^2 + x_3^2 - x_4^2\},
\]
such that (i) at most \(s = O(1)\) lines lie on any 2-flat, (ii) at most \(q = O(n/m^{1/3})\) lines lie on any hyperplane, and (iii) the number of incidences between the points and the lines is \(\Theta(m^{2/3} n^{1/2})\), which is asymptotically larger than the upper bound in (1), when \(n^{9/8} < m < n^{3/2}\). This shows that the assumption that no quadric contains more than \(q\) lines (in the above mentioned theorem of [23]) is necessary in this regime of \(m\) and \(n\).

By a suitable projection from this quadratic hypersurface onto \(\mathbb{R}^3\), we obtain \(m\) points and \(n\) lines in \(\mathbb{R}^3\), with at most \(s = O(1)\) lines on a common plane, such that the number of incidences between the \(m\) points and the \(n\) lines is \(\Theta(m^{2/3} n^{1/2})\). It remains an interesting question to determine if this bound is also tight in general.

Keywords. Combinatorial geometry, incidences.

1 Introduction

Let \(P\) be a set of \(m\) distinct points in \(\mathbb{R}^2\) and let \(L\) be a set of \(n\) distinct lines in \(\mathbb{R}^2\). Let \(I(P, L)\) denote the number of incidences between the points of \(P\) and the lines of \(L\); that is, the number of pairs \((p, \ell)\), such that \(p \in P\), \(\ell \in L\) and \(p \in \ell\). The classical Szemerédi–Trotter theorem [28] yields the worst-case tight bound
\[
I(P, L) = O \left( m^{2/3} n^{2/3} + m + n \right).
\]
Theorem 1.1 (Guth and Katz [8]). Let \( P \) be a set of \( m \) distinct points and \( L \) a set of \( n \) distinct lines in \( \mathbb{R}^3 \), and let \( s \leq n \) be a parameter, such that no plane contains more than \( s \) lines of \( L \). Then
\[
I(P, L) = O\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right).
\]

Remark. When \( s = \Theta(\sqrt{n}) \), this bound is known to be tight, by a generalization to three dimensions of Elekes’ planar construction of points and lines on an integer grid (see Guth and Katz [8] for the details). For smaller values of \( s \), it is an open problem to give lower bounds or improve the upper bound, and the case \( s = O(1) \) is of particular interest. In Theorem 1.5 we will give an improved upper bound, and it remains a question (see Question 4.1) whether it is tight.

In a recent paper of Sharir and Solomon [23], the following analogous and sharper result in four dimensions was established.

Theorem 1.2. Let \( P \) be a set of \( m \) distinct points and \( L \) a set of \( n \) distinct lines in \( \mathbb{R}^4 \), and let \( q, s \leq n \) be parameters, such that (i) each hyperplane or quadric contains at most \( q \) lines of \( L \), and (ii) each 2-flat contains at most \( s \) lines of \( L \). Then
\[
I(P, L) \leq 2^{c\sqrt{\log m}} \left(\frac{m^{2/5}n^{4/5}}{n^3/2} + m\right) + A\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right),
\]
where \( A \) and \( c \) are suitable absolute constants. When \( m \leq n^{6/7} \) or \( m \geq n^{5/3} \), there is the sharper bound
\[
I(P, L) \leq A\left(m^{2/5}n^{4/5} + m + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right).
\]

In general, except for the factor \( 2^{c\sqrt{\log m}} \), the bound is tight in the worst case, for any values of \( m, n, q, s \) and corresponding suitable ranges of \( q \) and \( s \).

The term \( m^{2/3}n^{1/3}s^{1/3} \) comes from the planar Szemerédi–Trotter bound [2], and is unavoidable, as it can be attained if we densely pack points and lines into 2-flats, in patterns that realize the bound in [2].

Likewise, the term \( m^{1/2}n^{1/2}q^{1/4} \) comes from the bound of Guth and Katz [8] in three dimensions (as in Theorem 1.1), and is again unavoidable, as it can be attained if we densely “pack” points and lines into hyperplanes, in patterns that realize the bound in three dimensions.

In this paper we show that the condition in assumption (i) of Theorem 1.2 that quadrics also do not contain too many lines, cannot be dropped, by proving the following theorem.

Theorem 1.3. For each positive integer \( k \) and each \( \alpha > 0 \), there exists \( m = \Theta(k^{3+3\alpha}) \) points and \( n = \Theta(k^{2+4\alpha}) \) lines on the quadratic hypersurface
\[
S := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2^2 + x_3^2 - x_4^2\}
\]
in \( \mathbb{R}^4 \), such that there are at most \( O(1) \) lines lying on any 2-flat and \( O(k^{1+3\alpha}) \) lines lying on any hyperplane, and \( I(P, L) = \Theta(k^{3+4\alpha}) \).

\(^1\)The additional requirement in [8], that no regulus contains too many lines, is not needed for the bound given below.
Given integers $m$ and $n$, there are $k, \alpha$ such that $m = \Theta(k^{3+3\alpha})$ points and $n = \Theta(k^{2+4\alpha})$. Substituting these values in Theorem 1.3 we obtain the following corollary.

**Corollary 1.4.** For integers $m, n$, there is a configuration of $m$ points and $n$ lines in $\mathbb{R}^4$, such that all the points (resp., lines) are contained (resp., fully contained) in $S$, and (i) the number of lines in any common 2-flat is $O(1)$, (ii) the number of lines in a common hyperplane is $O(n/m^{1/3})$, and (iii) the number of incidences between the points and lines is $\Omega(m^{2/3}n^{1/2} + m + n)$.

**Remarks.** (1) For integers $m, n$, satisfying $n^{9/8} < m < n^{3/2}$, the number incidences $\Omega(m^{2/3}n^{1/2})$ in Corollary 1.4 is asymptotically larger than the bound of Theorem 3 for the number of incidences $O(m^{2/5}n^{4/5} + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3} + m + n) = O(m^{2/5}n^{4/5} + m^{5/12}n^{3/4} + m + n)$ (as $q = O(n/m^{1/3})$). This implies that the condition in assumption (i) of Theorem 1.2 cannot be dropped, in this regime of $m$ and $n$.

(2) We note that the number of 2-rich points determined by $n$ lines in $\mathbb{R}^4$ is $O(n^{3/2})$, provided that at most $O(\sqrt{n})$ of the lines lie on a common plane or regulus. To see this, project the lines onto some (generic) hyperplane $H$, such that no two lines are projected onto the same line, and similarly, no two 2-rich points are projected onto the same 2-rich point, and such that at most $O(\sqrt{n})$ lines lie on a common plane or regulus. Then, the number of 2-rich points in the configuration of $n$ lines in $\mathbb{R}^4$ is equal to the number of 2-rich points in the configuration of the projected lines onto $H$. By Guth and Katz [8], the number of 2-rich points determined by the projected lines is $O(n^{3/2})$, and therefore the same holds for the number of 2-rich points in the original configuration of lines in $\mathbb{R}^4$. We also notice that in a configuration of $m$ points and $n$ lines in $\mathbb{R}^4$, the 1-rich points (i.e., points that are incident to exactly one line) contribute at most $m$ incidences. Therefore, in Corollary 1.4 as $s = O(1)$, the assumption that $m \leq n^{3/2}$ causes no loss of generality.

**Proof Techniques.** It is a common practice to take geometric objects to be integer points on certain hypersurfaces (especially quadratic ones) and varieties passing through a lot of such points, in order to obtain lower bounds for their incidences. For some most recent applications of this method, see [24] [30] [33]. In this paper we obtain our incidence lower incidence bound by taking integer points and “low height” lines on the above hypersurface $S$.

**Projection to $\mathbb{R}^3$.** As remarked above, Guth and Katz [8] proved that the number of incidences between $m$ points and $n$ lines in $\mathbb{R}^3$ is $I(P, L) = O \left( m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3} + m + n \right)$, provided that no plane contains more than $s$ lines of $L$. When $s = \Theta(\sqrt{n})$, this bound is tight, by a generalization to three dimensions of Elekes’ construction of points and lines on an integer grid in the plane (see Guth and Katz [8] for the details). For smaller values of $s$, it is an open problem to give lower bounds or improve the upper bound, where the case $s = O(1)$, is of particular interest.

By choosing a generic projection from $\mathbb{R}^4$ to $\mathbb{R}^3$, we show that Corollary 1.4 directly implies the following Theorem.

**Theorem 1.5.** For integers $m, n$, there is a configuration of $m$ points and $n$ lines in $\mathbb{R}^3$, such that (i) the number of lines in any common plane is $s = O(1)$, and (ii) the number of incidences between the points and lines is $\Omega(m^{2/3}n^{1/2} + m + n)$.

**Remark.** When $n^{3/4} \ll m \ll n^{3/2}$, the term $m^{2/3}n^{1/2}$ dominates over $m$ and $n$, showing that in this regime of $m$ and $n$, the construction in Theorem 1.5 of $m$ points and $n$ lines with $O(1)$ lines in a common plane, yields a super-linear number of incidences. As observed above, the bound of

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2A regulus is a quadratic surface that is doubly ruled by lines. For more details about reguli, see e.g., Sharir and Solomon [22].
Guth and Katz [8] implies that the number of 2-rich points determined by the \( n \) lines is \( O(n^{3/2}) \), so the assumption that \( m \leq n^{3/2} \) causes no loss of generality.

**Background.** Incidence problems have been a major topic in combinatorial and computational geometry for the past thirty years, starting with the Szemerédi-Trotter bound [28] back in 1983. Several techniques, interesting in their own right, have been developed, or adapted, for the analysis of incidences, including the crossing-lemma technique of Székely [27], and the use of cuttings as a divide-and-conquer mechanism (e.g., see [2]). Connections with range searching and related problems in computational geometry have also been noted, and studies of the Kakeya problem (see, e.g., [29]) indicate the connection between this problem and incidence problems. See Pach and Sharir [14] for a comprehensive survey of the topic.

The simplest instances of incidence problems involve points and lines. Szemerédi and Trotter solved completely this special case in the plane [28]. Guth and Katz’s second paper [8] provides a worst-case tight bound in three dimensions, under the assumption that no plane contains too many lines; see Theorem 1.1. Under this assumption, the bound in three dimensions is significantly smaller than the planar bound (unless one of \( m, n \) is significantly smaller than the other), and the intuition is that this phenomenon should also show up as we move to higher dimensions. The first attempt in higher dimensions was made by Sharir and Solomon in [20]. In a recent work, Sharir and Solomon [23] gave a tight bound in four-dimensions provided that the number of lines fully contained in a common hyperplane or quadric is bounded by a parameter \( q \), and the number of lines fully contained in a common 2-flat is bounded by a parameter \( s \). Whereas the condition that no common hyperplane contains more than a bounded number of lines was known to be necessary, it remained an open question whether the condition that the number of lines in a common quadric is bounded is necessary. In this paper, we show that when \( n^{9/8} < m < n^{3/2} \), this condition is indeed necessary, by describing an explicit quadratic hypersurface in \( \mathbb{R}^4 \) containing more incidences than the bound prescribed by the main theorem of [23]. This is the content of Theorem 1.3 and Corollary 1.4.

We remark that in [30], another example of points on a quadratic hypersurface in \( \mathbb{F}^4 \) with highly incidental pattern was noticed. There \( \mathbb{F} \) is a finite field. Our current quadratic hypersurface and our counting techniques in \( \mathbb{R}^4 \) are slightly different. The reader may find it interesting to compare the results here to the results in [30].

Another interesting remark is that in three dimensions, there are certain quadratic surfaces, called reguli, such that if one allows too many lines to lie on such a regulus, the number of 2-rich points determined by them can be larger than the Guth-Katz bound [8] of \( O(n^{3/2}) \). The quadratic hypersurface in \( \mathbb{R}^4 \) presented in this paper can be thought of as a higher degree analog of regulus. However, if one only cares about incidences between points and lines (instead of the number of 2-rich points determined by the lines), the existence of many lines on a regulus (or any quadratic surface in \( \mathbb{R}^3 \)) do not yield more than a linear number of incidences.

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2 Proof of Theorem 1.3

Proof. We start by recalling the quadric

\[ S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2^2 + x_3^2 - x_4^2\}, \]

on which the construction takes place, and define the set of points by

\[ P = \{(x_1, x_2, x_3, x_4) \in S \mid x_i \in \mathbb{Z}, i = 1, \ldots, 4, \ |x_1| \leq 200k^{2+2\alpha}, |x_2|, |x_3|, |x_4| \leq 100k^{1+\alpha}\}, \]

and the set of lines

\[ L = \{(x + tv \mid t \in \mathbb{R}) \subseteq S \mid x = (x_1, \ldots, x_4), \ v = (v_1, \ldots, v_4), \ x_i, v_i \in \mathbb{Z}, i = 1, \ldots, 4, \]
\[ |x_1| \leq k^{2+2\alpha}, |x_2|, |x_3|, |x_4| \leq k^{1+\alpha}, \]
\[ \frac{k^{1+2\alpha}}{2} \leq |v_1| \leq 8k^{1+2\alpha}, |v_2|, |v_3| \leq k^{\alpha}, \ v_4^2 = v_2^2 + v_3^2, \]
\[ v_1 = 2x_2v_2 + 2x_3v_3 - 2x_4v_4, \]
\[ \gcd(v_2, v_3, v_4) = 1, \text{ and } |v_4| \geq \frac{k^{\alpha}}{2}, \]

for any positive integer \( k \) and any \( \alpha > 0 \).

Since a point on \( S \) is uniquely determined by its last three coordinates, we have

\[ |P| = |\{(x_2, x_3, x_4) \in \mathbb{Z}^3 \mid |x_2|, |x_3|, |x_4| \leq 100k^{1+\alpha}\}| = \Theta(k^{3+3\alpha}). \]

The analysis of (an asymptotically tight bound on) the number of lines of \( L \) is a bit more involved. A line \{(x + tv \mid t \in \mathbb{R}) \subseteq S \in (5)\} (assuming \( x \in S, |x_1| \leq k^{2+2\alpha}, |x_2|, |x_3|, |x_4| \leq k^{1+\alpha}\) is fully contained in \( S \) if and only if

\[ v_1 = 2x_2v_2 + 2x_3v_3 - 2x_4v_4 \text{ and } v_4^2 = v_2^2 + v_3^2. \]

It follows by Benito and Varona [1] Theorem 1] that the number of primitive integer triples \((v_2, v_3, v_4)\) (i.e., without a common divisor) satisfying \(v_4^2 = v_2^2 + v_3^2, \ |v_2|, |v_3| \leq k^{\alpha}\), and \( |v_4| \geq \frac{k^{\alpha}}{2}\) is \( \Theta(k^{\alpha}) \). For each such \((v_2, v_3, v_4)\), we claim that there are \( \Omega(k^{3+3\alpha}) \) (and trivially also \( O(|P|) = O(k^{3+3\alpha}) \)) points \( x \in P \), such that \( v_1 = 2x_2v_2 + 2x_3v_3 - 2x_4v_4 \) satisfying \( \frac{k^{1+2\alpha}}{4} \leq |v_1| \leq 8k^{1+2\alpha} \). Indeed, note that \( |v_2|, |v_3| \leq |v_4| \). Choosing \( |x_2|, |x_3| \leq \frac{|v_4|}{4}, \frac{k^{1+\alpha}}{2} \leq |x_4| \leq k^{1+\alpha} \) (there are at least \( \frac{k^{3+3\alpha}}{32} \) choices of such triples \((x_2, x_3, x_4)\)) implies that

\[ 2|x_2v_2 + 2x_3v_3| \leq 2|x_2||v_2| + 2|x_3||v_3| \leq 2\frac{|x_4|}{4}(|v_2| + |v_3|) \leq |x_4||v_4|, \]

Here \( |v_1| \geq |x_4||v_4| \geq \frac{k^{1+2\alpha}}{4} \). The inequality \( |v_1| \leq 8k^{1+2\alpha} \) is immediate.

Moreover, each line \( \ell \) satisfying the above conditions is incident to \( O(k) \) different points of \( P \) (and can thus be expressed in \( O(k) \) different ways as \{(x + tv \mid t \in \mathbb{R}) \subseteq S \}, for \( |x_1| \leq k^{2+2\alpha}, |x_2|, |x_3|, |x_4| \leq k^{1+\alpha}\)). Indeed, parameterize \( \ell \) as \{(x + tv \mid t \in \mathbb{R}) \subseteq S \}, where \( x, v \) satisfy

\[ |v_1| \geq \frac{k^{1+2\alpha}}{4}, \ |x_1| \leq k^{2+2\alpha}, \]

and \( v = (v_1, v_2, v_3, v_4) \) is primitive (i.e., its coordinates do not have a common factor). Notice that if \( |t| > 8k \), then the first coordinate of \( x + tv \) has absolute value greater than \( k^{2+2\alpha} \), and that if
If \( t \notin \mathbb{Z} \), then \( x + tv \notin \mathbb{Z}^4 \) (since \( v \) is primitive and \( x \in \mathbb{Z}^4 \)). In either case, \( x + tv \notin P \). This implies that
\[
\ell \cap P \subseteq \{ x + tv \mid t \in \mathbb{Z}, \ |t| \leq 8k \},
\]
and thus \( |\ell \cap P| \leq 16k = O(k) \) as claimed. Therefore, the total number of lines is \( \Omega(\frac{k^{3+3\alpha}k^\alpha}{k}) = \Omega(k^{2+4\alpha}) \).

It is easy to see that each line in \( L \) is incident to \( \Omega(k) \) points in \( P \). It follows that \( |L| = O(k^{2+4\alpha}) \). Hence \( |L| = \Theta(k^{2+4\alpha}) \).

Since each line has \( \Theta(k) \) integer points in \( P \) on it, we have
\[
I(P, L) = \Theta(k^{3+4\alpha}).
\]

We now bound the number of lines fully contained in any 2-flat, and then bound the number of lines on any hyperplane. The bounds will be uniform (i.e., independent of the specific 2-flat or hyperplane).

Let \( \pi \) denote any 2-flat, and we analyze the number of lines that are fully contained in \( \pi \cap S \). We claim that \( \pi \) contains no planes, so \( \pi \not\subset S \). Assume the contrary, then we parameterize
\[
\pi = \{(u_1s + r_1t + w_1, u_2s + r_2t + w_2, u_3s + r_3t + w_3, u_4s + r_4t + w_4) \mid s, t \in \mathbb{R} \},
\]
for constants \( u_i, r_i, w_i \in \mathbb{R}, \ i = 1, 2, 3, 4 \) where \( (u_1, u_2, u_3, u_4) \) and \( (r_1, r_2, r_3, r_4) \) are both nonzero and not proportional to each other. Comparing the coefficients of quadratic terms in the identity
\[
u_1s + r_1t + w_1 \equiv (u_2s + r_2t + w_2)^2 + (u_3s + r_3t + w_3)^2 - (u_4s + r_4t + w_4)^2,
\]
we deduce that \( (u_2, u_3, u_4) \) and \( (r_2, r_3, r_4) \) are proportional to each other. Hence we may assume \( u_2 = u_3 = u_4 = 0 \). But this forces \( u_1 = 0 \), a contradiction. Therefore \( \pi \) is not contained in \( S \). Thus the intersection \( \pi \cap S \) is a curve of degree at most two, so there are at most two lines fully contained in \( \pi \cap S \).

Next, we take any hyperplane \( H \), and analyze the number of lines fully contained in \( S \cap H \). The surface \( S \cap H \) is a quadratic 2-surface contained in \( H \). We will use the classification of (real) quadratic surfaces in \( \mathbb{R}^3 \) (see, e.g., Sylvester’s original paper [20]), and distinguish between two cases.

If the equation of \( H \) can be expressed as \( x_1 = \varphi(x_2, x_3, x_4) \), where \( \varphi \) is a linear form, then each point \( x \in H \cap S \) satisfies the equations
\[
\begin{aligned}
x_2^2 + x_3^2 - x_4^2 &= \varphi(x_2, x_3, x_4), \\
x &\in H.
\end{aligned}
\]
This is either a cone, i.e., is linearly equivalent to \( x_2^2 + x_3^2 - x_4^2 = 0 \), or a hyperboloid of one or two sheets, i.e., is linearly equivalent to \( x_2^2 + x_3^2 - x_4^2 = 1 \) or \( x_2^2 + x_3^2 - x_4^2 = -1 \), respectively. It is easy to verify (and well known) that there are no lines on the hyperboloid of two sheets. We therefore assume that \( S \cap H \) is either a cone or a hyperboloid of one sheet. In these cases, there are at most two lines of \( L \) with any given direction that are fully contained in \( S \cap H \). Note that if a line \( \{ x + tv \mid t \in \mathbb{R} \} \in L \) is fully contained in \( S \cap H \), then \( v_1 = \tilde{\varphi}(v_2, v_3, v_4) \) (where we let \( \tilde{\varphi} \) denote the linear homogeneous part of \( \varphi \)), and \( v_2^2 = v_3^2 + v_4^2 \) (being the homogeneous part of degree two in \( t \)), \( |v_2|, |v_3| \leq k^{\alpha} \) and \( |v_4| \geq k^{\alpha} \). As observed above, there are \( O(k^{\alpha}) \) such triples \( (v_2, v_3, v_4) \). Therefore, the number of lines in \( L \) that lie in \( S \cap H \) is \( O(k^{\alpha}) \).

In the remaining case, the equation of \( H \) is of the form \( \varphi(x_2, x_3, x_4) = 0 \), where \( \varphi \) is a linear form. We can assume, without loss of generality, that the equation of \( H \) is \( x_2 = \psi(x_3, x_4) \), where
ψ is a linear form (the remaining case $x_4 = 0$ is simpler to handle). In this case, for every point $x \in S \cap H$, we have
\[
\begin{cases}
x_1 = \psi(x_3, x_4)^2 + x_3^2 - x_4^2, \\
x \in H.
\end{cases}
\]

The classification of (real) quadratic surfaces implies that this can be an elliptic paraboloid, a parabolic cylinder or a hyperbolic paraboloid. An elliptic paraboloid contains no lines and the corresponding case is trivial. If $S \cap H$ is a parabolic cylinder, then all lines on it are parallel. It is straightforward that there are $O(k^{2+2\alpha})$ points in $P$ that lie on it (by counting possible pairs $(x_3, x_4)$). Hence there are $O(k^{1+2\alpha})$ lines in $L$ that are fully contained in $S \cap H$. In the rest of the discussion we assume $S \cap H$ is a hyperbolic paraboloid. In this case, similarly to the case of the one-sheeted hyperboloid, there are at most two lines with the same direction. Moreover, the direction $(v_1, v_2, v_3, v_4)$ of any line on $S \cap H$ satisfies $v_2 = \tilde{\psi}(v_3, v_4)$ and $v_3^2 = v_1^2 + v_2^2$ (where $\tilde{\psi}$ denote the linear homogeneous part of $\psi$). Thus once we fix $v_1$ and “$v_3$ or $v_4$” (depending on $\tilde{\psi}$), we have limited the possible direction $(v_1, v_2, v_3, v_4)$ in a set with $\leq 2$ elements. Hence there are $O(k^{1+3\alpha})$ lines that are fully contained in $S \cap H$.

Finally, we show that for $\alpha < \frac{1}{2}$, the number of incidences is (asymptotically) larger than
\[
\Theta \left( m^{2/5}n^{4/5} + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3} + m + n \right),
\]
which is the bound of Theorem 3 with $m = \Theta(k^{3+3\alpha}), n = \Theta(k^{2+4\alpha}), q = O(k^{1+3\alpha})$, and $s = O(1)$. We have
\[
m^{2/5}n^{4/5} = O(k^{\frac{6+16\alpha + 8+16\alpha}{5}}) = O(k^{\frac{6+6\alpha + 4+4\alpha + 1+3\alpha}{5}}),
\]
and the exponent is smaller than $3 + 4\alpha$, as $\alpha < \frac{1}{2}$. Similarly,
\[
m^{1/2}n^{1/2}q^{1/4} = O(k^{\frac{6+16\alpha + 4+4\alpha + 1+3\alpha}{4}}) = O(k^{\frac{11+17\alpha}{4}}),
\]
and the exponent is smaller than $3 + 4\alpha$, as $\alpha < \frac{1}{2} < 1$. Similarly,
\[
m^{2/3}n^{1/3} = O(k^{\frac{6+16\alpha + 2+4\alpha}{3}}) = O(k^{\frac{8+10\alpha}{3}}),
\]
and the exponent is smaller than $3 + 4\alpha$ for every $\alpha$. Since both $m$ and $n$ are $O(k^{3+4\alpha})$, the claim is proved. \hfill \Box

### 3 Proof of Theorem 1.5

The proof of Theorem 1.5 follows easily by Corollary 1.3 together with the following lemma.

**Lemma 3.1.** Let $L$ be a set of $n$ lines in $\mathbb{R}^4$ such that at most $s$ lines lie on a common 2-flat. There exists a projection from $\mathbb{R}^4$ onto a hyperplane $H \subset \mathbb{R}^4$, such that at most $s$ lines lie on any common plane in $H$.

**Proof of Lemma 3.1.** Let $\pi_1, \ldots, \pi_k$ denote the set of 2-flats containing at least two lines in $L$, then $k \leq \binom{n}{2}$. For a generic hyperplane $H \subset \mathbb{R}^4$, the projection $p : \mathbb{R}^4 \to H$ maps $\pi_i$ onto a plane $\pi'_i$ contained in $H$. We pick, as we may, a hyperplane $H$, so that $p$ is bijective on $\pi_1, \ldots, \pi_k$. Denote by $L'$ the set of projected lines in $\mathbb{R}^3$. It is easy to verify that the set of planes in $H$ containing at least two lines in $L'$ consists precisely of $\pi'_1, \ldots, \pi'_k$. Moreover, the number of lines in $L'$ that are contained in $\pi'_i$ is equal to the number of lines in $L$ that are contained in $\pi_i$, thus completing the proof. \hfill \Box
4 Discussion and open questions

In Corollary 1.4, we show a concrete irreducible quadratic hypersurface $S$ in $\mathbb{R}^4$, together with a set of $m$ points and $n$ lines that lie on $S$, for $n^{9/8} < m < n^{3/2}$, such that (i) the number of lines in any common 2-flat is $O(1)$, (ii) the number of lines in any common hyperplane is $O(n/m^{1/3})$, and (iii) the number of incidences between the points and lines is $\Omega(m^{2/3}n^{1/2})$, which is asymptotically larger than $\Theta(m^{2/3}n^{1/5} + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3} + m + n)$ in this regime of $m$ and $n$. A natural question is to extend this result to other regimes by a similar construction. The condition (i) is natural and should not be hard to achieve, since if a plane is not contained in a quadratic hypersurface, then by the generalized version of Bézout’s theorem $5$ it can contain at most two lines. Here are a few natural questions that arise

1. Can we generalize our construction, such that in (ii) we are allowed to have a more general $q$, not necessarily $\sim n/m^{1/3}$, s.t. the number of lines in any common hyperplane is $O(q)$, and we still get a lower bound of incidences asymptotically larger than $\Theta(m^{2/5}n^{4/5} + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3} + m + n)$?

2. Can we find a similar construction when $m < n^{9/8}$?

3. How powerful is the natural generalization of this construction for $\mathbb{R}^d$, when $d > 4$? Notice that for $d > 4$, finding the precise bound for the number of incidences between a set $P$ of $m$ points and a set $L$ of $n$ lines in $\mathbb{R}^d$ is already an interesting open question. It is probably too early for us to answer this question before we find the correct bound.

4. In three dimensions, it remains a question to determine if Theorem 1.5 is tight.

**Question 4.1.** Let $P$ be a set of $m$ distinct points and $L$ a set of $n$ distinct lines in $\mathbb{R}^3$, and assume that no plane contains more than $s = O(1)$ lines of $L$. Then what is a good or tight upper bound of $I(P,L)$? Would $O(m^{2/3}n^{1/2} + m + n)$ suffice?

We do not know the answer to this question yet. It seems to require new techniques.

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