On \((\varepsilon, \delta)\)-Freudenthal Kantor triple systems and anti-structurable algebras with certain conditions

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Abstract. In this paper we discuss a characterization of anti-structurable algebras in connection with their relation with \((-1, -1)\)-Freudenthal Kantor triple systems.

1. Introduction
Freudenthal, Tits \cite{52}, I.L. Kantor \cite{33, 34, 35} and Koecher \cite{38, 39} studied constructions of Lie algebras from nonassociative algebras and triple systems, in particular Jordan algebras, while B.N. Allison \cite{1, 2} defined the concept of structurable algebras, containing Jordan algebras. Recently, we have studied constructions of Lie (super)algebras from triple systems and anti-structurable algebras \cite{24, 26, 29, 30, 31}. As a continuation of \cite{29, 30} we are interested to characterize the structure properties of anti-structurable algebras. Especially, Jordan and Lie (super)algebras \cite{9, 12} play an important role in many mathematical and physical subjects \cite{5, 10, 11, 13, 15, 25, 28, 36, 46, 47, 51, 54, 55} and the construction and characterization of these algebras can be expressed in terms of triple systems \cite{19, 22, 23, 27, 37, 48} by the standard embedding method \cite{21, 40, 41, 49, 53}.

Summarizing the content of this paper we give an introduction in section \S1, definitions and preamble in section \S2, while in in section \S3 we give properties of anti-structurable algebras satisfying the second order condition, we discuss the notion left neutral pair for \((\varepsilon, \delta)\)-Freudenthal Kantor triple systems and give examples of anti-structurable algebras with left neutral pair.

2. Definitions and preamble
2.1. \((\varepsilon, \delta)\)-Freudenthal Kantor triple systems
In this paper triple systems are finite dimensional and defined over a field \(\Phi\) of characteristic \(\neq 2\) or \(3\). A vector space \(V\) over \(\Phi\) endowed with a trilinear operation \(V \times V \times V \rightarrow V, (x, y, z) \mapsto (xyz)\) is said to be a \textit{generalized Jordan triple system of second order} (for short GJTS of 2nd order) if the following conditions are fulfilled:

\[
(ab(xyz)) = ((abx)y)z - (x(by)z) + (xy(abz)), \tag{2.1}
\]

\[
K(K(a, b)x, y) - L(y, x)K(a, b) - K(a, b)L(x, y) = 0, \tag{2.2}
\]

for all \(a, b, x, y, z \in V\), where \(L(a, b)c := (abc)\) and \(K(a, b)c := (abc) - (bca), a, b, c \in V\).

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A Jordan triple system (for short JTS) satisfies (2.1) and \((abc) = (cba)\), for all \(a, b, c \in V\).

We can generalize the concept of GJTS of 2nd order as follows (see [13, 14, 17, 18, 19, 20, 21, 53] and the earlier references therein).

For \(\varepsilon = \pm 1\) and \(\delta = \pm 1\), a triple product that satisfies the identities

\[
(ab(xyz)) = ((ab)x)yz + \varepsilon (x(bay))z + (xy(ab)z),
\]

\[
K(K(a, b)x, y) - L(y, x)K(a, b) + \varepsilon K(a, b)L(x, y) = 0,
\]

for all \(a, b, x, y, z \in V\), where

\[
L(a, b)c := (abc), \quad K(a, b)c := (acb) - \delta(bca), \quad a, b, c \in V,
\]

is called an \((\varepsilon, \delta)\)-Freudenthal Kantor triple system (for short \((\varepsilon, \delta)\)-FKTS).

**Remark.** We note that \(K(b, a) = -\delta K(a, b)\).

**Remark.** The concept of GJTS of 2nd order coincides with that of \((-1, 1)\)-FKTS. Thus we can construct the corresponding simple Lie algebras by means of the standard embedding method ([6, 13, 14, 15, 16, 17, 21, 24, 26, 35, 53]).

An \((\varepsilon, \delta)\)-FKTS \(U\) is called unitary if the identity map \(Id\) is contained in \(K(U, U)\) i.e., if there exist \(a_i, b_i \in U\), such that \(\Sigma K(a_i, b_i) = Id\).

If \(U\) is an \((\varepsilon, \delta)\)-FKTS and \(a, b \in U\) then \((a, b)\) is called a left neutral pair if \(L(a, b) = Id\).

For \(\delta = \pm 1\), a triple system \((a, b, c) \mapsto [abc], a, b, c \in V\) is called a \(\delta\)-Lie triple system (for short \(\delta\)-LTS) if the following three identities are fulfilled

\[
[abc] = -\delta [bac],
\]

\[
[abc] + [cba] + [cab] = 0,
\]

\[
[ab[xyz]] = [[ab]xyz] + [x[aby]z] + [xy[abz]],
\]

where \(a, b, x, y, z \in V\). An 1-LTS is a LTS while a \(-1\)-LTS is an anti-LTS, by [14].

**Proposition 2.1** ([14],[21]) Let \(U(\varepsilon, \delta)\) be an \((\varepsilon, \delta)\)-FKTS. If \(J\) is an endomorphism of \(U(\varepsilon, \delta)\) such that \(J < xyz > = < Jxjyjz >\) and \(J^2 = -\varepsilon Id\), then \((U(\varepsilon, \delta), [xyz])\) is a LTS (if \(\delta = 1\)) or an anti-LTS (if \(\delta = -1\)) with respect to the product

\[
[xyz] := < xJyz > - \delta < yJxz > + \delta < xJzy > - < yJzx >.
\]

**Corollary 2.1** Let \(U(\varepsilon, \delta)\) be an \((\varepsilon, \delta)\)-FKTS. Then the vector space \(T(\varepsilon, \delta) = U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)\) becomes a LTS (if \(\delta = 1\)) or an anti-LTS (if \(\delta = -1\)) with respect to the triple product defined by

\[
\begin{pmatrix}
L(a, d) - \delta L(c, b) - \varepsilon K(b, d) & \delta K(a, c) \\
-\varepsilon K(c, d) & (L(d, a) - \delta L(b, c))
\end{pmatrix}
\]

Thus we can obtain the standard embedding Lie algebra (if \(\delta = 1\)) or Lie superalgebra (if \(\delta = -1\)), \(L(\varepsilon, \delta) = D(T(\varepsilon, \delta), T(\varepsilon, \delta)) \oplus T(\varepsilon, \delta)\), associated to \(T(\varepsilon, \delta)\), where \(D(T(\varepsilon, \delta), T(\varepsilon, \delta))\) is the set of inner derivations of \(T(\varepsilon, \delta)\), i.e.

\[
D(T(\varepsilon, \delta), T(\varepsilon, \delta)) := \left\{ \begin{pmatrix} L(a, b) & \delta K(c, d) \\ -\varepsilon K(c, f) & \varepsilon L(b, a) \end{pmatrix} \right\} \text{span}, T(\varepsilon, \delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \right\} \text{span}.
\]

**Remark.** We note that \(L(\varepsilon, \delta) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_{1} \oplus L_{2}\) is the five graded Lie (super)algebra, such that \(L_{-1} \oplus L_1 = T(\varepsilon, \delta)\) and \(D(T(\varepsilon, \delta), T(\varepsilon, \delta)) = L_{-2} \oplus L_0 \oplus L_2\) with \([L_i, L_j] \subseteq L_{i+j}\).
2.2. δ-structurable algebras

The motivation for the study of such nonassociative algebras is as follows. The existence of the class of nonassociative algebras called structurable algebras is an important generalization of Jordan algebras giving a construction of Lie algebras. Hence from our concept, by means of triple products, we define a generalization to construct Lie superalgebras as well as Lie algebras.

Our start point briefly described in a historical setting is the construction of Lie (super) algebras starting from a class of nonassociative algebras. Hence within the general framework of $(\epsilon, \delta)$-FKTSs $(\epsilon, \delta = \pm 1)$ and the standard embedding Lie (super)algebra construction studied in [6, 7, 13, 14, 15, 26] (see also references therein) we define $\delta$-structurable algebras as a class of nonassociative algebras with involution which coincides with structurable algebras for $\delta = 1$ as introduced and studied in [1, 2]. Structurable algebras are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to GJTSs 2nd order, or $(-1,1]$-FKTSs, as introduced and studied in [33, 34] and further studied in [3, 4, 32, 42, 43, 44, 45, 50]. Their importance lies with constructions of 5-graded Lie algebras $L(U) := L(\epsilon, \delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, [L_i, L_j] \subseteq L_{i+j}$. For $\delta = -1$ the anti-structurable algebras defined here are a new class of nonassociative algebras that may similarly shed light on the notion of $(-1, -1)$-FKTSs hence (by [6, 7]) on the construction of Lie superalgebras and Jordan algebras as it will be shown.

Let $(A, -)$ be a finite dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e. $\overline{x} = x, \overline{xy} = y x, x, y \in A$) over $\Phi$. The identity element of $A$ is denoted by 1. Since $\text{char} \Phi \neq 2$, by [1] we have $A = {\mathcal H} \oplus S$, where $\mathcal{H} = \{a \in A|\overline{a} = a\}$ and $S = \{a \in A|\overline{a} = -a\}$.

Suppose $x, y, z \in A$. Put

$$[x, y] := xy - yx, \quad [x, y, z] := (xy)z - x(yz).$$

(2.8)

We note that $[x, y, z] = -[z, y, x]$.

Let the operators $L_x$ and $R_x$ be defined by $L_x(y) := xy, R_x(y) := yx, x, y \in A$ and for $\delta = \pm 1$ define

$$\delta V_{x,y} := L_{\overline{y}} + \delta (R_x R_{\overline{y}} - R_y R_{\overline{x}}),$$

$$\delta B_A(x, y, z) := \delta V_{x,y}(z) = (xy)z + \delta[(z\overline{y})x - (z\overline{x})y], x, y, z \in A.$$  

(2.10)

$B_A(x, y, z)$ is called the triple system obtained from the algebra $(A, -)$. We will call $\overline{B_A(x, y, z)}$ the anti-triple system obtained from the algebra $(A, -)$. We write for short

$$V_{x,y} := \delta V_{x,y}, \quad B_A := (\delta B_A, A).$$

(2.11)

Remark. The upper left index notation is chosen in order not to be mixed with the upper right index notation of [1] which has a different meaning.

A unital non-associative algebra with involution $(A, -)$ is called a structurable algebra if the following identity is fulfilled

$$[V_{u,v}, V_{x,y}] = V_{u,v(x),y} - V_{x,v(u)y}.$$  

(2.12)

for $V_{u,v} = \overline{V_{v,u}}, V_{x,y} = \overline{V_{y,x}}, u, v, x, y \in A$, and we will call $(A, -)$ an anti-structurable algebra if the identity (2.12) is fulfilled for $V_{u,v} = \overline{V_{v,u}}, V_{x,y} = \overline{V_{y,x}}$.

If $(A, -)$ is structurable then, by [34], the triple system $B_A$ is called a generalized Jordan triple system (abbreviated GJTS) and by [8], $B_A$ is a GJTS of 2nd order, i.e. satisfies the identities (2.3) and (2.4). If $(A, -)$ is anti-structurable then we call $B_A$ an anti-GJTS.
3. Main results

3.1. Properties satisfying the second order condition

From now on we assume $\delta = -1$ (unless otherwise specified) and let $(A, -)$ be an anti-structurable algebra. Define $C(a, b, c) \in \text{End} A$ by

$$C(a, b, c) := \lfloor ab, d, c \rfloor - \lfloor a, b, d \rfloor c, \quad a, b, c, d \in A.$$  \hfill (3.13)

We say that $A$ satisfies condition $\mathcal{C}$ if

$$C(x, y, w) - C(w, y, x) = C(w, x, y) - C(y, x, w), \quad x, y, w \in A.$$  \hfill (3.14)

**Theorem 3.1.** Let $(A, -)$ be an anti-structurable algebra. Then the second order condition (2.4) and condition $\mathcal{C}$ are equivalent.

**Proof.** Suppose first that $A$ satisfies condition $\mathcal{C}$. We show then that the second order condition (2.4) is fulfilled for $\delta = -1 = \varepsilon$.

By (3.14) we have

$$C(x, y, w) - C(w, y, x) = C(w, x, y) - C(y, x, w), \quad x, y, w \in A,$$

that is, by (3.13),

$$[x, y, z]w - [x, y, z]w - [w, y, z]x = [w, x, z, y] - [w, x, z]y - [y, x, z, w] + [y, x, z]w.$$  \hfill (3.15)

Then, by (2.8), the last identity is equivalent to

$$((xy)z)w - (xy)(zw) - ((xy)z)w = ((xy)z)w - ((xy)z)w,$$

and canceling the first with the third term and the fifth with the seventh term both in the left and in the right hand side of the last identity we obtain

$$\begin{align*}
(x(yz))w - (xy)(zw) - (w(yz))x + (wy)(zx) = \\
(w(xz))y - (w(xz))y = (w(xz))y - (w(xz))y.
\end{align*}$$

(3.15)

Set in (3.15) $x = d, y = c, z = a\bar{b} + b\bar{a}$, hence $z = \bar{c}$, and then, by using the involution properties, we obtain

$$\begin{align*}
(d(xz))w - (d(xz))w = (w(xz))d + (w(xz))d = \\
(w(xz))d - (w(xz))d = (c(zd))w + (c(zd))w.
\end{align*}$$

(3.16)

Now, by definitions (2.5) and (2.10) follows $L(x, y)z = (x\bar{y})z - (z\bar{y})x + (z\bar{y})y, x, y, z \in A$, so the identity (3.16) can be written

$$L(d, A(a, b)c) + L(c, A(a, b)d)w = A(c, d)(A(a, b)w).$$

(3.17)

for all $w \in A$, where

$$A(a, b) := L(a, b) + L(b, a), a, b \in A,$$

(3.18)

thus $A(a, b)c = (a\bar{b} + b\bar{a})c = zc$. Then, by (3.17),

$$L(d, A(a, b)c) + L(c, A(a, b)d) = A(c, d)A(a, b).$$

(3.19)

By [30] §2, we have $[A(a, b), L(c, d)] = L(A(a, b)c, d) - L(c, A(a, b)d)$, or equivalently,

$$A(a, b)L(c, d) = L(A(a, b)c, d) - L(c, A(a, b)d) + L(c, d)A(a, b).$$

(3.20)
Then, by (3.18), the identity (3.20) is equivalent to

\[ A(a,b)L(c,d) = L(A(a,b)c,d) - L(c,A(a,b)d) + A(c,d)A(a,b) - L(d,c)A(a,b). \]  

(3.21)

Now, by (3.19), \( A(c,d)A(a,b) - L(c,A(a,b)d) = L(d,A(a,b)c) \) so (3.21) is equivalent to

\[ A(a,b)L(c,d) = L(A(a,b)c,d) + L(d,A(a,b)c) - L(d,c)A(a,b), \]  

(3.22)

that is, by (3.18),

\[ A(a,b)L(c,d) = A(a,b)c,d) - L(d,c)A(a,b). \]  

(3.23)

Further, by (2.5) and (2.10), \( K(a,b)c = L(a,c)b + L(b,c)a = (a\overline{b} + b\overline{a})c = A(a,b)c, \) for all \( a, b, c \in A, \) so (3.23) is equivalent to \( K(a,b)L(c,d) = K(K(a,b)c,d) - L(d,c)K(a,b), \) that is the second order condition (2.4) is fulfilled.

Conversely, to show that (2.4) implies condition \( \mathcal{C} \) it is straightforward and we omit it here. □

Remark. An anti-structurable algebra satisfying the condition \( \mathcal{C} \) is a \((-1,-1)\)-FKTS.

3.2. Lie admissible structures

In this section we announce results demanding extensive proofs which will to be presented elsewhere.

Theorem 3.2 Let \((A,^-)\) be an anti-structurable algebra such that \(- = Id. \) Then \(A\) is a LTS with respect to the new product \([x,y,z] = B_A(x,y,z) - B_A(y,x,z), x,y,z \in A.\)

Theorem 3.3 Let \((A,^-)\) be an anti-structurable algebra satisfying the second order condition (2.4). Then

i) \(A\) is a Lie admissible, i.e. the Jacobi identity is fulfilled:

\[ [[x,y],z] + [[y,z],x] + [[z,x],y] = 0, x,y,z \in A, \]

ii) \([x,y,z] + [z,y,x]\) is totally symmetric in any exchanges of \(x, y, z \in A,\)

iii) \([h,x,y] = [x,h,y] = [x,y,h] = 0, \) for all \(h \in \mathcal{H}, x,y \in A.\)

Theorem 3.4 Let \((A,^-)\) be an anti-structurable algebra satisfying the second order condition (2.4) and let \(F(x,y,z) \in EndA\) be defined by

\[ F(x,y,z)w := [x\overline{y},w,z] + [z,x\overline{y},w] + ([x,y,w] - [y,x,w])z, \quad x,y,z,w \in A. \]

Then the following identities are fulfilled:

i) \(F(x,y,z) = -F(y,x,z), \quad x,y,z \in A,\)

ii) \(F(x,y,z) + F(y,z,x) + F(z,x,y) = 0, \quad x,y,z \in A.\)

Remark. We have also \(K(u,v)K(x,y) + K(x,y)K(u,v) = K(K(u,v)x,y) + K(x,K(u,v)y), \) for \(x,y,u,v \in A\) so the set of \(K(x,y), x,y \in A,\) form a Jordan algebra (see [30] for details).

3.3. Left neutral pair and invertible elements

In this section we discuss the notion of left neutral pair for \((\varepsilon, \delta)\)-FKTSs.

Lemma 3.1 Let \(U\) be an \((\varepsilon, -1)\)-FKTS, \(\varepsilon = \pm 1, \) with product \((abc), a,b,c \in U, \) and a left neutral pair \((u,v), u,v \in U.\) Then \(R(v,u)^2 = Id, \) where \(R(v,u)w := (wvu), u,v,w \in U.\)
Proposition 3.1
Let $\Box$

The proof is a direct consequence of Lemma 3.3 for we have

$$2K(u,v)y - 2L(y,v)u + 2\varepsilon K(u,v)L(v,y)v = 0$$

or equivalent $(uvy) + (yvu) - (yvu) + \varepsilon(u(vvy)u) = 0$ so $y = -\varepsilon(u(vvy)u)$, for all $y \in U$.

On the other hand, by (2.3), replacing $b = y = v, x = z = u$ follows

$$(av(uvu)) = ((av)vu) + \varepsilon(u(vav)u) + (uvav),$$

hence $(av) = ((av)vu) + \varepsilon(u(vav)u) + (uvav)$, that is $(av)vu) = -\varepsilon(u(vav)u)$, for all $a \in U$. Since we have shown above that $a = -\varepsilon(u(vav)u)$ follows from the last identity that $a = ((av)vu)$, for all $a \in U$, hence $R(v,u)^2 = Id$. □

Lemma 3.2
Let $U$ be an $(\varepsilon,\delta)$-FKTS, $\varepsilon,\delta = \pm 1$, with product $(abc), a,b,c \in U$, and a left neutral pair $(u,v), u,v \in U$. Then

$$R(v,u)^2 = 2\delta + 1)R(v,u) + (2\delta + 1)Id = 0, \quad \text{where } R(v,u)w := (wvu), u,v,w \in U. \quad (3.24)$$

Proof. We remark first that (3.24) is equivalent to $(R(v,u) - Id)(R(v,u) - (2\delta + 1)Id) = 0$. Then the proof is clear from Lemma 3.1 for the case $\delta = -1$ while the case corresponding $\delta = 1$ follows from Lemma 1 ([15]). □

Remark. For $\delta = -1$ we note that the following decomposition is valid $U := U(\varepsilon, -1) = U_1(u,v) \oplus U_{-1}(u,v)$, while, for $\delta = 1$ we have $U := U(\varepsilon, 1) = U_1(u,v) \oplus U_3(u,v)$, by [19], where

$${U_i}(\varepsilon, \delta) = \{x \in U|R(v,u)x = ix\}. \quad (3.25)$$

Lemma 3.3
Let $U$ be an $(\varepsilon, -1)$-FKTS, $\varepsilon = \pm 1$, with product $(abc), a,b,c \in U$, and a left neutral pair $(u,v), u,v \in U$. Then we have $U_1(u,v) = \{x \in U|K(x,u)v = 2x\}$ and $U_{-1}(u,v) = \{x \in U|K(x,u) = 0\}$, where $U_i(\varepsilon, -1)$ are defined by (3.25).

Proof. Consider $x \in U_{-1}(u,v)$. Then, by (3.25), $(xvu) = -x$. Since $(uvx) = x$ then $(xvu) + (uxv) = 0$, hence $K(u,v)xv = 0$. Since, by (2.4),

$$K(u,x) = K((wvu),x) = -K(u,(uxv)) - \delta K(u,v)K(u,v)xv = -K(u,x)$$

then it follows $K(u,x) = 0$.

Conversely, let $K(u,x) = 0$ hence $K(u,x)v = 0$. Thus we get $(xvu) + (uxv) = 0$ hence $(xvu) = -x$, that is $x \in U_{-1}(u,v)$. Hence $U_{-1}(u,v) = \{x \in U|K(x,u) = 0\}$.

On the other hand, let $x \in U_1(u,v)$. Then, by (3.25), $(xvu) = x$. Since $(uvx) = x$ then $(xvu) + (uxv) = 2x$, hence $K(u,v)xv = 2x$.

Conversely, if $K(x,u)v = 2x$ then $(xvu) + (uxv) = 2x$, hence $(xvu) = x$, that is $x \in U_1(u,v)$. Hence $U_1(u,v) = \{x \in U|K(x,u) = 0\}$. □

Theorem 3.5
Let $U$ be an $(\varepsilon,\delta)$-FKTS, $\varepsilon,\delta = \pm 1$, with product $(abc), a,b,c \in U$, and a left neutral pair $(u,v), u,v \in U$. Then, $U := U(\varepsilon, \delta) = U_1(u,v) \oplus U_{-1}(u,v)$, where for $\delta = -1$ we have $U_1(u,v) = \{x \in U|K(x,u)v = 2x\}$ and $U_{-1}(u,v) = \{x \in U|K(x,u) = 0\}$, while for $\delta = 1$ we have $U_1(u,v) = \{x \in U|K(x,u) = 0\}$ and $U_3(u,v) = \{x \in U|K(x,u) = 2x\}$.

Proof. The proof is a direct consequence of Lemma 3.3 for $\delta = -1$, while for $\delta = 1$ the proof follows from [15]. □

Proposition 3.1
Let $U := U(\varepsilon, -1)$ be an $(\varepsilon, -1)$-FKTS, $\varepsilon = \pm 1$, with product $(abc), a,b,c \in U$, and a left neutral pair $(u,v), u,v \in U$. Then $U_1(u,v) \simeq \tilde{K} := \{K(a,b)\}_{a,b \in U}^{\text{span}}$ as a JTS with respect to a map $\eta : \tilde{K} \rightarrow U_1(u,v), \eta(K(a,b)) = K(a,b)v$. 
Proof. We show that $\eta$ is a bijection and

$$
\eta \{ K(a,b)K(c,d)K(e,f) \} = (\eta (K(a,b)) \eta (K(c,d)) \eta (K(e,f))).
$$

(3.26)

Indeed, from (2.4) follows the identity $K((uxz), y) + K(x, (uvy)) - K(u, K(x, y)v) = 0$, hence $K(K(x, y)v, u) = 2K(x, y)$, since $(u, v)$ is a left neutral pair. Then, by Lemma 3.3, the last identity implies $K(x, y)v \in U_1(u, v)$.

Now, if $\theta := \frac{1}{2} K(., u)$, then $(\theta \circ \eta)(K(x, y)) = \frac{1}{2} K(K(x, y)v, u) = K(x, y)$.

Similarly, for $x \in U_1(u, v)$, $(\eta \circ \theta)(x) = \frac{1}{2} \eta (K(x, u)) = \frac{1}{2} K(x, u)v = x$. Hence $\eta$ is a bijection with $\eta^{-1} = \theta = \frac{1}{2} K(., u)$. Furthermore, the left hand side of (3.26) is equal to

$$
K(K(a,b)c, K(e,f)d) v + K(K(e,f)c, K(a,b)d) v,
$$

by Proposition 2.3 ([30]), and equals the right hand side (3.26), by the definition of $\eta$. □

Remark. We note that $U_3(u, v) \simeq K := \{ K(a,b) | a, b \in U \}$ span as a JTS under the assumption of existence of a left neutral pair $(u, v), u, v \in U$ and $\delta = 1$, by [15].

Proposition 3.2 Let $U$ be an $(\varepsilon, -1)$-FKTS, $\varepsilon = \pm 1$, with product $(abc), a, b, c \in U$, and a left neutral pair $(u, v), u, v \in U$. Then the maps $U_u, U_v$ are invertible, where $U_x : U \to U, U_x(y) := (xyx)$.

Proof. Since $L(u, v) = Id$ then $[L(x, y), L(u, v)] = 0, x, y \in U$.

By (2.3), $L((xyu), v) = -\varepsilon L((uyx), v)$. Putting $y = v$ in the last identity and applying both sides to $u$ follows $R^2(v, u) = -\varepsilon U_u U_v$. Thus the map $U_u$ is onto and $U_v$ is one to one. Since $U$ is finite dimensional, $U_u$ is one to one, thus invertible so $U_v$ is invertible. □

Remark. An analogous result was obtained for the case of a $(\varepsilon, 1)$-FKTS in [15].

Remark. These results can be applied to the $(-1, -1)$-FKTS $U := M_{k,k}(\Phi)$ of square matrices of order $k$ over $\Phi$ with product $(xyz) = xy^\top z - zy^\top x + zx^\top y$ ([29]), where $x^\top$ denotes the transposed matrix of $x$. Also, we emphasize that this triple system is an anti-structurable algebra satisfying the second order condition.

Proposition 3.3 Let $U$ be an unitary $(-1, -1)$-FKTS, with product $(abc), a, b, c \in U$, and a left neutral pair $(u, v), u, v \in U$. Then $U_3(\Phi) = \{ E, F, H \}$ span as a subalgebra of the corresponding Lie superalgebra $L(U)$ of $U$ and $L(v, u) = Id$.

Proof. From the unitary property follows $I \in \tilde{K} := \{ K(a,b) | a, b \in U \}$ span. Let $L(U)$ be the corresponding Lie superalgebra of $U$ ([15, 29]). Then $E, F, H \in L(U)$ where

$$
E := \begin{pmatrix} 0 & Id \\ 0 & 0 \end{pmatrix}, \quad F := \begin{pmatrix} 0 & 0 \\ -Id & 0 \end{pmatrix}, \quad H := [E, F] = \begin{pmatrix} -Id & 0 \\ 0 & Id \end{pmatrix}.
$$

On the other hand, since $\begin{pmatrix} L(u, v) & 0 \\ 0 & -L(v, u) \end{pmatrix} \in L(U)$ by [15], and $L(u, v) = Id$ it follows $L(v, u) = Id$. □

3.4. Examples of $(-1, -1)$-FKTS with left neutral pairs

We give examples of $(-1, -1)$-FKTS with left neutral pairs and invertible elements. Let

$$
GL_k(\Phi) := \{ A \in M_{k,k}(\Phi) | \det A \neq 0 \}.
$$

If $u \in GL_k(\Phi)$ then set $v = (u^\top)^{-1}$, where the involution is transposition and so $L(u, v)z = uu^{-1}z - zu^{-1}u + u^\top (u^\top)^{-1} = z$. Thus there exists a left neutral pair $(u, (u^\top)^{-1})$. Also we have

$$
U_u z = u^\top zu - u^\top zu + u^\top uz, \quad U_{(u^\top)^{-1}} = (u^\top)^{-1}((u^\top)^{-1})(u^\top)^{-1} z = (u^\top)^{-1} u^{-1} z,
$$
thus by straightforward calculation follows $U_u U_{(u^T)^{-1}} z = z$. Then the map $U_u$ is invertible. This implies that with any $u \in GL_k(\Phi)$ there can be constructed a left neutral pair $(u, (u^T)^{-1})$.

Set $O(\Phi) := \{ A \in M_{k,k}(\Phi) \mid AA^T = Id \}$. Then in the example above, if any element $u \in O(\Phi)$ it follows that $(u, u)$ is a left neutral pair, i.e. $u$ is a left unit element.

**Theorem 3.6** Let $U$ be a $(-1, -1)$-FKTS. Then $(u, v)$ is a left neutral pair if and only if $(v, u)$ is a left neutral pair.

**Proof.** We shall prove that $L(u, v) = Id$ if and only if $L(v, u) = Id$.

If $L(u, v) = Id$ then $[L(u, v), L(v, x)] = 0$ so $L((uvx), x) = L((uvx), (vxu)) = 0$, by (2.3), hence $L(v, x - (vxu))v = 0$, since $L(u, v) = Id$. Now, since $U_v$ is invertible follows from the last identity that $(vxu) = x$, hence $L(v, u) = Id$.

Conversely, if $L(v, u) = Id$ follows then that $L(u, v) = Id$, by an analogous proof. □

3.5. Almost complex structure and complex structure

Let $U$ be an $(\epsilon, \delta)$-FKTS and $T(\delta)$ be the $\delta$-Lie triple systems associated with $U$ ([21]). Let us set $E := \begin{pmatrix} 0 & Id \\ 0 & 0 \end{pmatrix}$, $F := \begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix}$ and $J = \delta E - \epsilon F$. Then it follows $J^2 = -\epsilon \delta Id$.

We shall call it an $(\epsilon, \delta)$-almost complex structure. Thus we can define the following operation

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y], \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad Y = \begin{pmatrix} z \\ w \end{pmatrix} \in T(\delta).$$

Then from Proposition 6.1 ([21]) we have that the identity $N(X, Y) = 0$, for $X, Y \in T(\delta)$, is equivalent to $K(x, y) = L(y, x) - \delta L(x, y)$.

**Remark.** For anti-structurable algebras with product (2.10), $\epsilon = \delta = -1$. Since then $K(x, y) = L(y, x) + L(x, y)$ we can easily show that there exists a complex structure on the associated anti-LTS $T(-1)$ if $Id \in \mathcal{K} := \{ K(a, b) \mid a, b \in U \}_{span}$ (i.e. unitary).

A structurable algebra has no property of complex structure.

**Remark.** For the $(-1, -1)$-FKTS $U := M_{k,k}(\Phi)$ of square matrices of order $k$ over $\Phi$ with product $(xyz) = xy^Tz - zy^Tx + zx^Ty$ ([29]), since $K(x, y) = L(y, x) + L(x, y)$ we can easily show that there exists a complex structure on the associated anti-LTS $T(-1)$. Moreover, by [29], the standard embedding Lie superalgebra $L(U)$ corresponding to the $(-1, -1)$-FKTS above is $\mathfrak{osp}(2m|4m)$ or $\mathfrak{osp}(2m + 1|4m + 2)$ if $k = 2m$ or $k = 2m + 1$, respectively.

3.6. Examples of anti-structurable algebras with left neutral pairs

We give examples of anti-structurable algebras with left neutral pairs and invertible elements.

From the fact that $U := M_{k,k}(\Phi)$ with the product $(xyz) = xy^Tz - zy^Tx + zx^Ty$ is an anti-structurable algebra satisfying the second order condition (2.4) we have the following.

Let $(u, v), u, v \in U$ be a left neutral pair and $GL_k(\Phi) := \{ A \in M_{k,k}(\Phi) \mid \det A \neq 0 \}$. If $u \in GL_k(\Phi)$ then set $v = (u^T)^{-1}$, where the involution is transposition and so $L(u, v)z = uu^Tz - zu^{-1}u + uz^Tu^{-1} = z$. Thus there exists a left neutral pair $(u, (u^T)^{-1})$. Also we have

$$U_u z = u^Tzu - u^Tzu + u^T uz, \quad U_{(u^T)^{-1}} = (u^T)^{-1}((u^T)^{-1})(u^T)^{-1}z = (u^T)^{-1}u^{-1}z,$$

thus by straightforward calculation follows $U_u U_{(u^T)^{-1}} z = z$. Then the map $U_u$ is invertible. This implies that with any $u \in GL_k(\Phi)$ there can be constructed a left neutral pair $(u, (u^T)^{-1})$.

Set $O(\Phi) := \{ A \in M_{k,k}(\Phi) \mid AA^T = Id \}$. Then in the example above, if any element $u \in O(\Phi)$ it follows that $(u, u)$ is a left neutral pair, i.e. $u$ is a left unit element.

**Theorem 3.7** Let $(\mathcal{A}, -)$ be an anti-structurable algebra satisfying the second order condition (2.4). Then $(u, v)$ is a left neutral pair if and only if $(v, u)$ is a left neutral pair.

**Proof.** It is a direct consequence of Theorem 3.6 and Theorem 3.1 and its remark. □
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