Explicit Construction of AG Codes from Generalized Hermitian Curves

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Abstract

We present multi-point algebraic geometric codes overstepping the Gilbert-Varshamov bound. The construction is based on the generalized Hermitian curve introduced by A. Bassa, P. Beelen, A. Garcia, and H. Stichtenoth. These codes are described in detail by constructing a generator matrix. It turns out that these codes have nice properties similar to those of Hermitian codes. It is shown that the duals are also such codes and an explicit formula is given.

Index Terms

Hermitian codes, algebraic geometric codes, asymptotically good tower, Gilbert-Varshamov bound.

I. INTRODUCTION

Let \( F \) be a function field over a finite field \( \mathbb{F}_l \). An algebraic geometric code is of the form \( C_{L}(D, G) \) with \( D = P_1 + \ldots + P_n \) where the \( P_j \)'s are pairwise-distinct places of degree one in \( F \), and \( G \) is a divisor of \( F \) such that \( \text{supp}(G) \cap \text{supp}(D) = \emptyset \).

And \( C_{L}(D, G) \) is defined by

\[
C_{L}(D, G) := \{(f(P_1), f(P_2), \ldots, f(P_n)) | f \in L(G)\},
\]

where \( L(G) \) denotes the Riemann-Roch space associated to \( G \), see [1], [2] as general references for all facts concerning algebraic geometric (AG) codes.

The Gilbert-Varshamov (GV) bound [1], [2] guarantees the existence of families of codes over the finite field with good asymptotic parameters; i.e., information rate and relative minimum distance. It is well known that the parameters of AG codes related to asymptotically good towers of function fields are better than the GV bound in a certain range of the rate [3], [4].

Denote by \( N(F) \) the number of rational places of \( F/\mathbb{F}_l \). Let \( N_l(g) := \max \{N(F)|F \text{ is a function field over } \mathbb{F}_l \text{ of genus } g\} \).

The real number

\[
A(l) := \limsup_{g \to \infty} \frac{N_l(g)}{g},
\]

is called Ihara’s quantity. The Drinfeld-Vladut bound [5] tells us that

\[
A(l) \leq \sqrt{l} - 1.
\]

If \( l \) is a square; then

\[
A(l) = \sqrt{l} - 1,
\]

which was first shown by Ihara [6]. Tsfasman, Vladut and Zink gave in [7] an independent proof of Equation (2). For \( l \) is a square, and \( l \geq 49 \), the GV bound was improved by the famous Tsfasman-Vladut-Zink theorem [7]. Also, for \( l = q^c \) with odd \( c > 1 \) and very large \( q \), there are improvements of the GV bound due to Niederreiter and Xing [2].

For applications to coding theory though, explicit construction of good towers are needed. In [8] A. Garcia, and H. Stichtenoth gave an explicit construction of a tower of Artin-Schreier extensions of function fields over \( \mathbb{F}_{q^2} \) attaining the Drinfeld-Vladut bound. Recently, in [9], A. Bassa, P. Beelen, A. Garcia, and H. Stichtenoth produced an explicit tower of function fields over finite fields \( \mathbb{F}_{q^{2b+1}} \) for any integer \( b \geq 1 \) and showed that this tower gives

\[
A(q^{2b+1}) \geq \frac{2(q^{b+1} - 1)}{q + 1 + \epsilon} \quad \text{with} \quad \epsilon = \frac{q - 1}{q^b - 1}.
\]

Using this tower they obtained an improvement of the GV bound for all non-prime fields \( \mathbb{F}_l \) with \( l \geq 49 \), except possibly \( l = 125 \) in [10].

Their construction can be restated as follows. Let \( \mathbb{F}_l \) be a non-prime field and write \( l = q^c \) with \( c \geq 2 \). Here the integer \( c \) can be even or odd. For every partition of \( c \) in relatively prime parts; i.e.,

\[
c = a + b \quad \text{with} \quad a \geq 1, b \geq 1,
\]

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where the greatest common factor of \( a \) and \( b \) is \( \gcd(a, b) = 1 \), we define
\[
H(x, y) := \frac{y^q}{x} + \frac{y^{q+1}}{x^q} + \ldots + \frac{y^{q-1}}{x^{q^{p-1}}}
\]
\[
+ \frac{y}{x^{q^{p}} + \frac{y^q}{x^{q^{p+1}}} + \ldots + \frac{y^{q^{a-1}}}{x^{q^{p^{a-1}}}}}.
\]
The asymptotically good tower \( \mathcal{F} = (F^{(1)} \subseteq F^{(2)} \subseteq F^{(3)} \subseteq \ldots) \) over \( \mathbb{F}_q \) is recursively given by the equation
\[
H(x, y) = 1. \tag{4}
\]
Precisely speaking,
1) \( F^{(1)} = \mathbb{F}_q(x_1) \) is the rational function field, and
2) \( F^{(i+1)} = F^{(i)}(x_{i+1}) \) with \( H(x_i, x_{i+1}) = 1 \), for all \( i \geq 1 \).

For the case \( a = b + 1 \), this tower implies Equation (3). For \( a = b = 1 \), this tower is identical with the one constructed in [8] and the second function field \( F^{(2)} \) is the Hermitian function field [1], [11]. To see this, we restate Equation (4) as follows
\[
\frac{y^q}{x} + \frac{y}{x^q} = 1.
\]
We replace \( xy \) by \( z \); then
\[
z + z^q = x^{q+1}, \tag{5}
\]
which is exactly the canonical definition of Hermitian curve. So the second function field \( F^{(2)} \) with general coefficients \( a, b \) can be regarded as the generalized Hermitian function field. And then the generalized Hermitian curve can be defined as follows.

**Definition 1.** The **generalized Hermitian curve** over \( \mathbb{F}_q \) with coprime integers \( a, b \in \mathbb{Z}^+ \) verifying \( a + b = c \), is defined by the affine equation
\[
\frac{y^q}{x} + \frac{y^{q+1}}{x^q} + \ldots + \frac{y^{q-1}}{x^{q^{p-1}}} + \frac{y}{x^{q^p}} + \frac{y^q}{x^{q^{p+1}}} + \ldots + \frac{y^{q^{a-1}}}{x^{q^{p^{a-1}}}} = 1.
\]

The AG codes arising from the Hermitian curve are widely investigated, which are called Hermitian codes. The advantage of these codes is that these codes are easy to describe and to encode and decode. Moreover, these codes often have excellent parameters.

One-point codes from Hermitian curves were well-studied in the literature, and efficient methods to decode them were known [1], [12], [13], [14]. The minimum distance of Hermitian two-point codes had been first determined by M. Homma and S. J. Kim [15], [16], [17], [18]. The explicit formulas for the dual minimum distance of such codes were given by S. Park [19]. Recently, Hermitian codes from higher-degree places had been considered in [20]. The dual minimum distance of many three-point codes from Hermitian curves was computed in [21], by extending a recent and powerful approach by A. Couvreur [22]. H. Maharaj, G. L. Matthews and G. Pirsic determined explicit bases for large classes of Riemann-Roch spaces of the Hermitian function field [23]. These bases gave better estimates on the parameters of a large class of multi-point Hermitian codes.

In [24], the authors explicitly constructed multi-point codes from the generalized Hermitian curves with coefficients \( a = 1 \), and \( b = c - 1 \).

In this paper we investigate multi-point codes from the generalized Hermitian curves \( \mathcal{X} \) with \( a = b + 1 \). The advantage in this setting is that the related tower achieve the bound stated in Equation (5) as indicated in [9]. We introduce four important divisors as follows according to [9].

1) \( D := \sum_{\alpha, \beta} D_{\alpha, \beta} \), where \( D_{\alpha, \beta} := (x = \alpha, y = \beta) \) with \( \alpha, \beta \in \mathbb{F}_q \) satisfying \( H(\alpha, \beta) = 1 \);
2) \( P := (x = 0, y = 0) \), which would be split into two parts, namely \( P = P_1 + P_0 \) where \( P_1 := (x = 0, y = 0, x^{-q^p} y = a^{-1}) \) denotes a rational place, if \( a \) and \( q \) are coprime;
3) \( Q := (x = \infty, y = \infty) \);
4) \( V := (x = 0, y = \infty) \).

The divisors \( D, P, Q \) and \( V \) contain all the possible rational places on the curve. We define the algebraic geometric codes over \( \mathbb{F}_q \)
\[
C_{v, r, s, t} = C_{\mathcal{X}}(D, vP_1 + rP_0 + sQ + tV).
\]
For applications of such codes in practice one needs an explicit description, which means an explicit basis for the vector space \( \mathcal{L}(vP_1 + rP_0 + sQ + tV) \) or a generator matrix of the code \( C_{v, r, s, t} \). We discover that this problem is related to a point-counting problem. Pick’s theorem [25], [26] provides a simple formula for calculating for two-dimensional lattice point set. Let \( \Omega \) be a lattice polygon. Assume there are \( I \) lattice points in the interior of \( \Omega \), and \( M \) lattice points on its boundary. Let \( S \) denote the area of \( \Omega \). Then
\[
S = I + \frac{M}{2} - 1.
\]
The main technical part in this paper is to solve the related three-dimensional point-counting problem using Pick’s theorem. So we can describe the code $C_{v,r,s,t}$ by constructing a generator matrix. As in the Hermitian case, it turns out that the dual code of $C_{v,r,s,t}$ is of the same type. Finally, it is shown that the Goppa bound of $C_{v,r,s,t}$ improves the GV bound in a certain interval. For example, we find a $[496, 250, \geq 172]$-code over $\mathbb{F}_{32}$ overstepping the GV bound.

The paper is organized as follows. In Section 2, we introduce some arithmetic properties of the curve $X$ and describe all the rational places. In Section 3, we construct a basis for the Riemann-Roch space $\mathcal{L}(vP_1 + rP_0 + sQ + tV)$. Section 4 is devoted to investigating the parameters and the duality properties of $C_{v,r,s,t}$.

II. The Arithmetic Properties of the Curve

We start with some notations according to [9]. Let $q$ be a power of a prime $p$ and $\mathbb{F}_{q^e}$ be a finite field of cardinality $q^e$. For an integer $a \geq 1$, we define the function

$$\text{Tr}_a(x) := x + x^{q^1} + x^{q^2} + \ldots + x^{q^{a-1}}.$$ 

We assume $c$ is an odd number and fix a partition of $c$ into two consecutive integers; i.e., we write

$$c = a + b, \text{ with } a = b + 1, b \in \mathbb{Z}^+.$$ 

In this section we study the generalized Hermitian curve $X$ over $\mathbb{F}_{q^e}$

$$\text{Tr}_a \left( \frac{y^{q^a}}{x} \right) + \text{Tr}_a \left( \frac{x^{q^a}}{y} \right) = 1. \quad (6)$$

For abbreviation we set $N_k := (q^k - 1)/(q - 1)$ for every integer $k > 1$. For an element $f$ in the function field $\mathbb{F}_{q^e}(X)$ of $X$, define

$$\text{div}(f), \text{div}_0(f), \text{ and } \text{div}_{\infty}(f)$$

the principal divisor, zero divisor and the pole divisor of $f$ in $\mathbb{F}_{q^e}(X)$. Let $P := (x = 0, y = 0), Q := (x = \infty, y = \infty)$, and $V := (x = 0, y = \infty)$ be the divisors of $\mathbb{F}_{q^e}(X)$. For a divisor $D$ in $\mathbb{F}_{q^e}$, we denote by $\deg(D)$ the degree of $D$. Several results in [9] are restated by the following proposition.

Proposition 2 ([9]). 1) The curve $X$ has genus

$$g = \frac{1}{2} \left( (q^e - 2)(q^{a-1} + q^{b-1} - 2) + (q^e - q) \right).$$

2) $\text{div}(x) = P + q^{a-1}N_bV - q^bQ$, and $\text{div}(y) = q^bP - q^{b-1}N_aV - Q$.

3) $\deg(P) = q^{a-1} - 1$, $\deg(Q) = q^{b-1} - 1$, and $\deg(V) = q - 1$.

4) For each $\alpha \in \mathbb{F}_{q^e}^*$, there are $q^{a-1}$ elements $\beta \in \mathbb{F}_{q^e}$ such that $\text{Tr}_a \left( \frac{\beta}{\alpha^{q^a}} \right) = 1$, and for such pairs $(\alpha, \beta)$ there is a unique place $D_{\alpha,\beta}$ of degree one with $x \equiv \alpha \mod P_{\alpha,\beta}$ and $y \equiv \beta \mod P_{\alpha,\beta}$.

Let $D := \sum D_{\alpha,\beta}$. Then $\deg(D) = q^{e-1}(q^e - 1)$.

The following proposition describes all the rational places on the curve $X$.

Proposition 3. 1) There exists a unique rational place $P_1 := (x = 0, y = 0, x^{-b}y = a^{-1})$ in the divisor $P$ if and only if $p \nmid a$.

2) There exists a unique rational place $Q_1 := (x = \infty, y = \infty, x^{-1}y^b = b^{-1})$ in the divisor $Q$ if and only if $p \nmid b$.

3) The divisor $V$ can be written as $V = \sum_{\mu^a = 1} V_{\mu}$, where $V_{\mu} := (x = 0, y = \infty, x^{b-1}N_a y^{a-1}N_b = \mu)$ denotes a rational place in $V$ just in case $p = 2$.

Therefore, all the possible rational places on the curve $X$ are the following: $D_{\alpha,\beta}$, $P_1$, $Q_1$, and $V_{\mu}$.

Proof. 1) Suppose that $P_1$ is a rational place in $P$. It can be deduced by the divisors of $x$ and $y$ that $v_{P_1}(x^{-b}y) = 0$ and $v_{P_1}(x^{-1}y^b) > 0$. We can assume that $x^{-b}y \equiv \gamma \mod P_{\gamma}$. Taking evaluation in Equation (6), we have $\text{Tr}_a \gamma = 1$. Actually, it is shown in [9] that the conorm of $P$ with respect to $\mathbb{F}_{q}/\mathbb{F}_{q^e}$ is

$$\text{Con}_{\mathbb{F}_{q}/\mathbb{F}_{q^e}}(P) := \sum_{\text{Tr}_a \gamma = 1, \gamma \in \mathbb{F}_{q}} \overline{\mathcal{P}}_{\gamma},$$

where $\overline{\mathcal{P}}_{\gamma} := (x = 0, y = 0, x^{-b}y = \gamma)$ denotes a rational place in the function field $\mathbb{F}_{q}(X)$. Hence, each rational place in the function field $\mathbb{F}_{q}(X)$ can be written as $P_\gamma := (x = 0, y = 0, x^{-b}y = \gamma)$ with $\text{Tr}_a \gamma = \gamma + \gamma^q + \gamma^{q^2} + \ldots + \gamma^{q^{a-1}} = 1$ and $\gamma \in \mathbb{F}_{q^e}$. 

$$\quad (7)$$
Now we only need to show that there exist a unique solution in Equation (7) if and only if \( p \nmid a \). If \( \gamma \) is a solution in Equation (7), then
\[
\gamma^q + \gamma^q + \gamma^q + \ldots + \gamma^q = 1. 
\]
(8)

Combining Equations (7) and (8) we find that \( \gamma^a = \gamma \) and therefore \( \gamma \in \mathbb{F}_q \cap \mathbb{F}_q = \mathbb{F}_q \). Now Equation (7) becomes \( a \gamma = 1 \), and then \( \gamma = a^{-1} \) just in case \( p \nmid a \). In other words, \( P_1 := (x = 0, y = 0, x - q^y y = a^{-1}) \) represent a rational place in \( \mathbb{F}_q(X) \) when \( p \nmid a \).

2) Similar to the proof of assertion 1).

3) Suppose that \( V_{\mu} \) is a rational place in \( V \). It is easy to show that \( v_{\nu}, (x^{q-1}y^{q-1}N_\nu) = 0 \). Let us assume that \( x^{q-1}y^{q-1}N_\nu \equiv \mu \mod V_{\mu} \). Multiplying both sides of Equation (6) with \( y q x \), we obtain
\[
\left( x^{q-1}y^{q-1}N_\nu \right)^{q-1} + 1 = \kappa,
\]
where \( v_{\nu}(\kappa) > 0 \). Then we have \( \mu^{q-1} = -1 \). Note that the equation \( \mu^{q-1} = -1 \) has \( q - 1 \) solutions in \( \mathbb{F}_q \) if and only if \( 2 | q \). Now the assertion 3) can be deduced similarly.

\[ \square \]

III. Explicit bases for Riemann-Roch spaces

In the rest of this paper, we shall always assume that \( p \nmid a \). Now by Proposition 3, the rational place \( P_1 \) exists. The divisor \( P \) can be decomposed by \( P = P_1 + P_0 \), with \( \text{deg}(P_1) = 1 \), and \( \text{deg}(P_0) = q^{a-1} - 1 \). We remark that the assumption \( p \nmid a \) is not essential. If \( p \nmid a \), then \( p \nmid b \). And we obtain a rational place \( Q_1 \) in \( Q \). All the results in both cases are similar.

Our next aim is to determine a basis for a space \( \mathcal{L}(vP_1 + rP_0 + sQ + tV) \) and then we can construct a generator matrix for our AG codes. Let \( u := a - y \frac{q^x - 1}{x} \), \( w := \frac{x^y - 1}{x u} \), and \( z := \frac{y^x}{x^q} \) be three elements in \( \mathbb{F}_q(X) \). We want to construct a basis of \( \mathcal{L}(vP_1 + rP_0 + sQ + tV) \) where all elements are of the form
\[
x^i z^j w^k \text{ with } (i, j, k) \in \mathbb{Z}^3.
\]

**Proposition 4.** The divisors of \( u, w, \) and \( z \) are given by
\[
\begin{align*}
\text{div}(u) &= (q^c - 1)P_1 - N_c V, \\
\text{div}(z) &= -q^{b-1}N_c V + (q^c - 1)Q, \\
\text{div}(w) &= (q^c - 1)P_0 - (q^{a-1} - 1)N_c V.
\end{align*}
\]

**Proof.** The divisor of \( u \) is computed in (9), the others can be calculated directly. \[ \square \]

We denote by \( \lfloor x \rfloor \) the largest integer not greater than \( x \) and by \( \lceil x \rceil \) the smallest integer not less than \( x \). It is easy to show that \( j = \left\lfloor \frac{\alpha}{\beta} \right\rfloor \) is equivalent to
\[
\begin{align*}
j \in \mathbb{Z} & \quad \text{and} \quad \alpha \leq \beta j < \alpha + \beta.
\end{align*}
\]
Let us denote the lattice point set
\[
\Omega_{v, r, s, t} := \{(i, j, k) \mid -v \leq i, \quad -r \leq i + (q^c - 1)k < -r + (q^c - 1), \\
- s \leq -q^a i + (q^c - 1)j < (q^c - 1) - s, \\
-t \leq (q^{a-1}N_b)i - (q^{b-1}N_c)j - (q^{a-1} - 1)N_c k \},
\]
or equivalently,
\[
\begin{align*}
\Omega_{v, r, s, t} := \{(i, j, k) \mid i \geq -v, j &= \left\lfloor \frac{q^a i - s}{q^c - 1} \right\rfloor, k = \left\lfloor \frac{-i - r}{q^a - 1} \right\rfloor, \\
(q^{a-1}N_b)i - (q^{b-1}N_c)j + t \geq (q^{a-1} - 1)N_c k \}.
\end{align*}
\]

With these notations we have the following proposition.

**Proposition 5.** There exists a constant \( C \) depend on the \( r, s, t \); such that, for \( v \geq C \), the number of \( \Omega_{v, r, s, t} \) verifies
\[
\# \Omega_{v, r, s, t} = 1 - g + v + (q^{a-1} - 1)r + q^{b-1} s + (q - 1)t.
\]

We shall give a proof of Proposition 5 later. The following proposition is the main result of this paper which can be applied to encoding multi-point codes.

**Proposition 6.** The elements \( x^i z^j w^k \text{ with } (i, j, k) \in \Omega_{v, r, s, t} \) form a basis of \( \mathcal{L}(vP_1 + rP_0 + sQ + tV) \).
Proof. Proposition 5 and Proposition 6 imply
\[
\text{div}(x^i z^j w^k) = i \text{div}(x) + j \text{div}(z) + k \text{div}(w)
= i P_0 + i P_1 + q^{a-1}N_b i V - q^a i Q
- q^{b-1}N_c j V + (q^c - 1) j Q
+ (q^c - 1) k P_0 - (q^{a-1} - 1) N_c k V
= i P_1 + (i + (q^c - 1) k) P_0 + (-q^a i + (q^c - 1) j) Q
+ (q^{a-1}N_b i - q^{b-1}N_c j - (q^{a-1} - 1) N_c k) V.
\]
Thus, \(x^i z^j w^k \in \mathcal{L}(v P_1 + r P_0 + s Q + t V)\) if and only if the following conditions hold
\[
P_1 : -v \leq i,
P_0 : -r \leq i + (q^c - 1) k,
Q : -s \leq -q^a i + (q^c - 1) j,
V : -t \leq (q^{a-1}N_b i) - (q^{b-1}N_c j) - (q^{a-1} - 1) N_c k.
\]
Hence, all the elements in \(\{x^i z^j w^k|(i, j, k) \in \Omega_{v,r,s,t}\}\) are contained in \(\mathcal{L}(v P_1 + r P_0 + s Q + t V)\).

Note that for \((i, j, k) \in \Omega_{v,r,s,t}\) both \(j\) and \(k\) are determined by \(i\) which is exactly the valuation of \(x^i z^j w^k\) with respect to \(P_1\). In other words, all elements in \(\{x^i z^j w^k|(i, j, k) \in \Omega_{v,r,s,t}\}\) have different valuations with respect to \(P_1\), so they are linearly independent.

Denote by \(d\) the dimension of \(\mathcal{L}(v P_1 + r P_0 + s Q + t V)\). To complete the proof, we only need to show that \(d = \# \Omega_{v,r,s,t}\) for large \(v\). By Riemann-Roch theorem, for large \(v\), we have
\[
d = 1 - g + v + (q^{a-1} - 1) r + q^{b-1} s + (q - 1) t. \tag{9}
\]
The proposition now follows from combining Equation (9) and Propositions 5.

\[\square\]

Corollary 7. The elements \(x^i y^j u^k\) with \((i, j, k) \in \Omega'_{v,r,s,t}\) form a basis of the Riemann-Roch space \(\mathcal{L}(v P_1 + r P_0 + s Q + t V)\), where \(\Omega'_{v,r,s,t}\) denotes the lattice point set
\[
\Omega'_{v,r,s,t} := \{(i, j, k) | -v \leq i + q^b j + (q^c - 1) k
- r \leq i + q^b j < -r + (q^c - 1)
- s \leq -q^a i - j < -s + (q^c - 1)
- t \leq q^{a-1}N_b i - q^{b-1}N_a j - N_c k \}.
\]

Proof. By definition,
\[
x^i z^j w^k = x^{i-q^b j} z^{j+q^a k} u^{-k}.
\]
Let \(i' := i - q^b j - k, j' := j + q^a k, k' := -k\). Then \(\Omega'_{v,r,s,t}\) becomes
\[
\{(i', j', k') | -v \leq i' + q^b j' + (q^c - 1) k'
- r \leq i' + q^b j' < -r + (q^c - 1)
- s \leq -q^a i' - j' < -s + (q^c - 1)
- t \leq q^{a-1}N_b i' - q^{b-1}N_a j' - N_c k' \},
\]
which is equal to \(\Omega_{v,r,s,t}\). So we have \(\{x^i y^j u^k|(i, j, k) \in \Omega_{v,r,s,t}\} = \{x^i z^j w^k|(i, j, k) \in \Omega_{v,r,s,t}\}\).

In order to count the elements of \(\Omega_{v,r,s,t}\) we need some preparations.

Lemma 8. Assume that \(\alpha\) is an integer.

1) Let \(L_{\alpha}^{(1)} := \{(i, j)|0 \leq i < q^c - 1, \text{ and } q^{a-1}N_b i - q^{b-1}N_c j = -\alpha\}\). Then
\[
\# L_{\alpha}^{(1)} = \begin{cases} q-1 & \text{for } q^{b-1}|\alpha, \\ 0 & \text{otherwise.} \end{cases}
\]

2) Let \(L_{\alpha}^{(2)} := \{(i, j)|0 \leq i < q^c - 1, \text{ and } -q^a i + (q^c - 1) j = -(q^c - 1)q - \alpha\}\). Then \#\(L_{\alpha}^{(1)} = 1\).

3) Let \(L_{\alpha}^{(3)} := \{(m, l)|0 \leq m < q^b - 1, \text{ and } q^{b-1}l - N_c = \alpha\}\). Then \#\(L_{\alpha}^{(3)} = 1\).

Proof. We prove only the first assertion of this lemma. The other conclusions can be deduced similarly. Note that the greatest common factor of \(q^{a-1}N_b\) and \(q^{b-1}N_c\) is \(\text{gcd}(q^{a-1}N_b, q^{b-1}N_c) = q^{b-1}\). The equation \(q^{a-1}N_b i - q^{b-1}N_c j = -\alpha\) has integer solutions if and only if \(k^{b-1}\) is an integer. If \(k^{b-1} \nmid \alpha\), then \#\(L_{\alpha}^{(1)} = 0\). If \(k^{b-1} \mid \alpha\), we can assume that \((i_0, j_0)\) is an
integer solution. We claim that all the other solutions are given by \((i_0 + N_c \lambda, j_0 + q N_b \lambda)\) with \(\lambda \in \mathbb{Z}\). It is easy to check that \((i_0 + N_c \lambda, j_0 + q N_b \lambda)\) is an integer solution. Conversely, let \((i, j)\) be an integer solution different from \((i_0, j_0)\). Substituting both solutions into the equation we obtain

\[ q N_b (i - i_0) - N_c (j - j_0) = 0. \]

So \(N_c (i - i_0) \) and \(q N_b (j - j_0)\). Hence, \(L^{(1)}_\alpha\) can be written as

\[ L^{(1)}_\alpha = \{(i_0 + N_c \lambda, j_0 + q N_b \lambda) \mid 0 \leq i_0 + N_c \lambda < q^{c} - 1\}. \]

Hence, \(#L^{(1)}_\alpha = q - 1\).

**Lemma 9.** Suppose that \(0 \leq m < q^{b-1},\) and \(s, t \geq 0\). Let \(\Psi_m\) be a lattice point set

\[ \Psi_m := \{(i, j) \mid 0 \leq i < q^{c} - 1, \]

\[-t \leq q^{b-1} N_b i - q^{b-1} N_c j + N_c m, \]

\[-s - (q^c - 1) q \leq -q^a i + (q^c - 1) j \}. \]

Then

\[ \#\Psi_m = \left( q^c + 1 \right) q / 2 + \left\lfloor \frac{t + N_c m}{q^{b-1}} \right\rfloor (q - 1) + s, \]

(10)

and

\[ \sum_{m=0}^{q^{b-1}-1} \#\Psi_m = \frac{1}{2} \left( q^{c+b-1} + q^{c+b} - q^c + q \right) + q^{b-1} s + (q - 1) t. \]

(11)

![Fig. 1. The lattice point set \(\Psi_m\)](image)

**Proof.** We split \(\Psi_m\) into three parts \(\Psi^{(0)}, \Psi^{(1)}\) and \(\Psi^{(2)}\); namely

\[ \Psi^{(0)} := \{(i, j) \mid 0 \leq i < q^{c} - 1, \]

\[0 \leq q^{a-1} N_b i - q^{b-1} N_c j, \]

\[-(q^c - 1) q \leq -q^a i + (q^c - 1) j \}, \]

\[ \Psi^{(1)} := \{(i, j) \mid 0 \leq i < q^{c} - 1, \]

\[-t - N_c m \leq q^{a-1} N_b i - q^{b-1} N_c j < 0, \]

\[-(q^c - 1) q \leq -q^a i + (q^c - 1) j \}, \]

(12)

and

\[ \Psi^{(2)} := \{(i, j) \mid 0 \leq i < q^{c} - 1, \]

\[-t - N_c m \leq q^{a-1} N_b i - q^{b-1} N_c j, \]

\[-s - (q^c - 1) q \leq -q^a i + (q^c - 1) j < -(q^c - 1) q \}. \]

Then we have \(#\Psi_m = #\Psi^{(0)} + #\Psi^{(1)} + #\Psi^{(2)}\). Equation (10) now follows from the following assertions

1) \(#\Psi^{(0)} = q (q^c + 1) / 2\).
2) \( \#\Psi^{(1)} = (q - 1) \left\lfloor \frac{t + N_e m}{q^{b-1}} \right\rfloor \).

3) \( \#\Psi^{(2)} = s \).

Let \( O = (0, 0) \), \( A = (0, -q) \), and \( B = (q^c - 1, q^{b+1} - q) \). Denote by \( S \) the area of the triangle \( \triangle OAB \), and by \( M \) the number of lattice points in the boundary of \( \triangle OAB \). Applying Pick’s Theorem, the number \( I \) of lattice points within \( \triangle OAB \) verifies

\[ I = S - M/2 + 1. \]

It is easy to see that \( S = q(q^c - 1)/2 \). In the same notations of Lemma 8 we have \( M = \#L^{(1)}_0 + \#L^{(2)}_0 + q - 1 + 1 = 2q \). Since \( \Psi^{(0)} \) contains exactly the lattice points in the triangle \( \triangle OAB \) except the vertex \( B \), we see that

\[ \#\Psi^{(0)} = \#\triangle OAB - 1 = I + M - 1 \]

\[ = S + M/2 \]

\[ = q(q^c + 1)/2. \]

This proves the first assertion. Put \( \alpha := q^{b-1} N_c j - q^{a-1} N_b i \). For \( 0 \leq i < q^c - 1 \) and \( \alpha > 0 \), we have

\[ -q^a i + (q^c - 1) j = -q^a i + \frac{(\alpha + q^{a-1} N_b i)(q - 1)}{q^{b-1}} \]

\[ = \frac{(q - 1)\alpha - q^{a-1} i}{q^{b-1}} \]

\[ > -q(q^c - 1). \]

So Condition (12) is invalid, and therefore

\[ \Psi^{(1)} = \bigcup_{\alpha=1}^{t+N_e m} L^{(1)}_\alpha. \]

Applying Lemma 8 we get

\[ \#\Psi^{(1)} = \sum_{\alpha=1}^{t+N_e m} \#L^{(1)}_\alpha \]

\[ = (q - 1) \left\lfloor \frac{t + N_e m}{q^{b-1}} \right\rfloor , \]

which completes the proof of the second assertion. Similar to the proof above, we find that \( \#\Psi^{(2)} = s \).

In order to complete the proof, we introduce a lattice point set

\[ \Phi_t := \{(m, l) | 0 \leq m < q^{b-1}, 0 < l \leq \frac{t + N_e m}{q^{b-1}} \}. \]

Then

\[ \#\Phi_t = \sum_{m=0}^{q^{b-1}-1} \left\lfloor \frac{t + N_e m}{q^{b-1}} \right\rfloor . \]  \hfill (13)

Note that

\[ \Phi_t = \Phi_0 \bigcup \left( \bigcup_{\alpha=1}^{t} \#L^{(3)}_\alpha \right) . \]  \hfill (14)

Fig. 2. The lattice point set \( \Phi_t \)
Using Pick’s Theorem, it is easy to show that

$$\#\Phi_0 = (N_c - 1)(q^{b-1} - 1)/2.$$  \hfill (15)

By Lemma 8 it follows that

$$\sum_{\alpha=1}^{t} \#L_{\alpha}^{(3)} = t.$$ \hfill (16)

Hence, we deduce that

$$\#\Phi_t = \#\Phi_0 + \sum_{\alpha=1}^{t} \#L_{\alpha}^{(3)} = (N_c - 1)(q^{b-1} - 1)/2 + t.$$ \hfill (17)

If we combine Equations (13) and (17), then we get

$$\sum_{m=0}^{q^{b-1} - 1} \left[ \frac{t + N_c m}{q^{b-1}} \right] = (N_c - 1)(q^{b-1} - 1)/2 + t.$$ \hfill (18)

Using Equations (10) and (18) we obtain

$$\sum_{m=0}^{q^{b-1} - 1} \#\Psi_{s,t}^{m} = \frac{1}{2} \left( q^{c+b-1} + q^{c+b} - q^c + q \right) + q^{b-1}s + (q - 1)t,$$

which completes the proof. \hfill \Box

**Lemma 10.** Let $\Omega_{v,0,s,t}$ be the lattice point set $\Omega_{v,r,s,t}$ with $r = 0$. For $v \geq v_0 := (q^c - 1)(q^b + q^{b-1} - 1)$, and $s,t \geq 0$, we have

$$\#\Omega_{v,0,s,t} = 1 - g + v + q^{b-1}s + (q - 1)t.$$ \hfill

**Proof.** In order to calculate the number of the set $\Omega_{v,0,s,t}$, we fix the index $k$, and define

$$\Theta_k := \{(i,j)|(i,j,k) \in \Omega_{v,0,s,t}\}.$$ \hfill

Precisely speaking,

$$\Theta_k := \{(i,j)|-v \leq i \leq v, \quad 0 \leq i + (q^c - 1)k < q^c - 1,$$

$$-s \leq -q^a i + (q^c - 1)j < (q^c - 1) - s,$$

$$-t \leq (q^{a-1}N_b)i - (q^{b-1}N_c)j - (q^{a-1} - 1)N_ck \}.$$ \hfill (22)

Then we have

$$\Omega_{v,0,s,t} = \left( \bigcup_{k=-\infty}^{\infty} \Theta_k \right) \bigcup \left( \bigcup_{k=q^a+q^{b-1}}^{\infty} \Theta_k \right).$$ \hfill

We count it by two steps.

1) $\# \left( \bigcup_{k=q^a+q^{b-1}}^{\infty} \Theta_k \right) = v - v_0$. Let $(i,j) \in \Theta_k$, and $M := -q^a i + (q^c - 1)j$. Inequality (22) tells us $M < (q^c - 1)(k - 1)$. For $k \geq q^b + q^{b-1}$, we have

$$\begin{align*}
(q^{a-1}N_b)i - (q^{b-1}N_c)j - (q^{a-1} - 1)N_ck \\
= (q^{a-1}N_b)i - q^{b-1}M + q^a i - (q^{a-1} - 1)N_ck \\
= -q^{a-1}i - q^{b-1}M - (q^{a-1} - 1)N_ck \\
> \frac{q^{a-1}(q^c - 1)(k - 1)}{q - 1} - \frac{q^{b-1}(q^c - 1)}{q - 1} - (q^{a-1} - 1)N_ck \\
= N_c (k - q^b - q^{b-1}) \\
\geq -t,
\end{align*}$$

\hfill (23)
which means that Condition (22) is invalid. We claim that
\[ \bigcup_{k = q^b + q^{b - 1}}^{\infty} \Theta_k = \left\{ (i, j) \mid -v \leq i < -v_0, \; j = \left\lfloor \frac{q^a i - s}{q^c - 1} \right\rfloor \right\}. \]  

(23)

To see this, we shall write down Condition (20) for various \( k \). We put \( v_\mu := (q^c - 1)(q^b + q^{b - 1} - 1 + \mu) \) for \( \mu \in \mathbb{N} \). Then
\[
\begin{align*}
-v_1 & \leq i < -v_0 \quad \text{for } k = q^b + q^{b - 1}, \\
-v_2 & \leq i < -v_1 \quad \text{for } k = q^b + q^{b - 1} + 1, \\
-v_3 & \leq i < -v_2 \quad \text{for } k = q^b + q^{b - 1} + 2, \\
\ldots \\
-v_{\mu + 1} & \leq i < -v_\mu \quad \text{for } k = q^b + q^{b - 1} + \mu.
\end{align*}
\]

Combining Condition (20) for \( k \geq q^b + q^{b - 1} \), we get \( i < -v_0 \). We note that Conditions (19) and (21) are independent of \( k \). Then \( \bigcup_{k = q^b + q^{b - 1}}^{\infty} \Theta_k \) becomes
\[
\{ (i, j) \mid -v \leq i, \\
\quad i < v_0, \\
\quad -s \leq -q^a i + (q^c - 1)j < (q^c - 1) - s \}
\]

which implies Equation (23). By Equation (23) one can easily verify that \( \# \left( \bigcup_{k = q^b + q^{b - 1}}^{\infty} \Theta_k \right) = v - v_0 \).

2) \( \# \left( \bigcup_{k = -\infty}^{q^b + q^{b - 1}} \Theta_k \right) = 1 - g + v_0 + q^b - 1 + q - 1 \).

It is important to write \( k = q^b l + m \) with \( 0 \leq m < q^b - 1 \) and \( l \leq q \). Let \( \tilde{i} := i + (q^c - 1)k \), and \( \tilde{j} = j + q^a k - l \). Then \( \Theta_k \) becomes
\[
\tilde{\Theta}_{l,m} := \{ (\tilde{i}, \tilde{j}) \mid \tilde{i} \geq -v + (q^c - 1)k, \\
\quad 0 \leq \tilde{i} < q^c - 1, \\
\quad -s - (q^c - 1)l \leq -q^a \tilde{i} + (q^c - 1)\tilde{j} < -s - (q^c - 1)(l - 1), \\
\quad -t \leq q^a - 1 N b \tilde{i} - q^b - 1 N_c \tilde{j} + N_c m \}
\]

and then
\[
\begin{align*}
\bigcup_{k = -\infty}^{q^b + q^{b - 1}} \Theta_k &= \bigcup_{m = 0}^{q^b - 1} \bigcup_{l = -\infty}^{q - 1} \tilde{\Theta}_{l,m}. \\
\end{align*}
\]

Set \( \Psi_m := \bigcup_{l = -\infty}^{q - 1} \tilde{\Theta}_{l,m} \), we have
\[
\# \left( \bigcup_{k = -\infty}^{q^b + q^{b - 1}} \Theta_k \right) = \sum_{m = 0}^{q^b - 1} \# \Psi_m.
\]

Note that Inequality (24) is invalid since \( -v + (q^c - 1)k \leq -v + v_0 \leq 0 \). We see that Conditions (25) and (27) are independent of \( l \), so we can combine Condition (26) for \( l \leq q \). Let \( s_\mu := s + (q^c - 1)(q - \mu) \) for \( \mu \in \mathbb{N} \). Then Condition (26) can be expressed as
\[
\begin{align*}
-s_0 & \leq -q^a \tilde{i} + (q^c - 1)\tilde{j} < -s_1 \quad \text{for } l = q, \\
-s_1 & \leq -q^a \tilde{i} + (q^c - 1)\tilde{j} < -s_2 \quad \text{for } l = q - 1, \\
-s_2 & \leq -q^a \tilde{i} + (q^c - 1)\tilde{j} < -s_3 \quad \text{for } l = q - 2, \\
\ldots \\
-s_\mu & \leq -q^a \tilde{i} + (q^c - 1)\tilde{j} < -s_{\mu + 1} \quad \text{for } l = q - \mu.
\end{align*}
\]

This give a total condition \( -s_0 \leq -q^a \tilde{i} + (q^c - 1)\tilde{j} \). So \( \Psi_m \) can be rewritten as
\[
\begin{align*}
\Psi_m &= \{ (\tilde{i}, \tilde{j}) \mid 0 \leq \tilde{i} < q^c - 1, \\
\quad -t \leq q^a - 1 N b \tilde{i} - q^b - 1 N_c \tilde{j} + N_c m, \\
\quad -s - (q^c - 1)q \leq -q^a \tilde{i} + (q^c - 1)\tilde{j} \}
\end{align*}
\]
Then we have by Lemma \ref{lem9} that
\[
\# \left( q^b + q^{b-1} - 1 \sum_{k=-\infty}^{\Omega_k} \right) = \sum_{m=0}^{q^{b-1}-1} \# \Psi_m = \frac{1}{2} (q^{c+b-1} + q^{c+b} - q^c + q) + q^{b-1}s + (q-1)t = 1 - g + v_0 + q^{b-1}s + (q-1)t.
\]

In summary,
\[
\# \Omega_{v,0,s,t} = 1 - g + v + q^{b-1}s + (q-1)t.
\]

This completes the proof. \hfill \Box

**Lemma 11.** If \( s + q^a r = \hat{s} + \sigma(q^c - 1) \), \( t - q^a^{-1} N_b - (q^c - 1) \sigma = \hat{t} + N_c \lambda \), then
\[
v P_1 + r P_0 + s Q + t V \sim (v + (q^c - 1) \lambda - r) P_1 + \hat{\sigma} Q + \hat{\lambda} V.
\]

**Proof.** By direct computation, we have
\[
v P_1 + r P_0 + s Q + t V + \text{div}(x(q^c - 1) \lambda - r) (x(q^c - 1) x - \lambda) \text{div}(x) + (q^a \lambda - \sigma) \text{div}(z) - \lambda \text{div}(w)
\]
\[
= v P_1 + r P_0 + s Q + t V + \text{div}(x(q^c - 1) \lambda - r) (P + q^a^{-1} N_b V - q^a Q) + (q^a \lambda - \sigma)(-q^a^{-1} N_c V + (q^c - 1)Q)
\]
\[
- \lambda (q^c - 1) P_0 - (q^a - 1) N_b V = (v + (q^c - 1) \lambda - r) P_1 + \hat{\sigma} Q + \hat{\lambda} V.
\]

\hfill \Box

**Lemma 12.** Suppose that \( s + q^a r = \hat{s} + \sigma(q^c - 1) \) with \( 0 \leq \hat{s} < q^c - 1 \), and \( t - q^a^{-1} N_b - (q^c - 1) \sigma = \hat{t} + N_c \lambda \) with \( 0 \leq t < N_c \). Let \( \hat{v} := v + (q^c - 1) \lambda - r \). Then
\[
\# \Omega_{v,r,s,t} = \# \Omega_{\hat{v},0,\hat{s},\hat{t}}.
\]

**Proof.** The proof of Lemma \ref{lem11} leads us to make a transformation \( i = \hat{i} + (q^c - 1) \lambda - r, j = \hat{j} + q^a \lambda - \sigma \), and \( k = \hat{k} - \lambda \). So we obtain
\[
\hat{\Omega}_{v,r,s,t} := \{ (i, j, k) | -\hat{v} \leq i \}
\]
\[
0 \leq \hat{i} + (q^c - 1) \hat{k} < 0 + (q^c - 1),
\]
\[
-\hat{s} \leq -q^a \hat{i} + (q^c - 1) \hat{j} < (q^c - 1) - \hat{s},
\]
\[
-\hat{t} \leq (q^a - 1) N_b \hat{i} - (q^b - 1) N_c \hat{j} - (q^a - 1) N_c \hat{k} \}
\]
which implies the lemma. \hfill \Box

Now Proposition \ref{prop6} follows easily from Lemmas \ref{lem10} and \ref{lem12}.

**IV. THE PROPERTIES OF THE CODES**

In this section, we study the linear code
\[
C_{v,r,s,t} = C_{\mathcal{L}}(D, v P_1 + r P_0 + s Q + t V).
\]
The length of \( C_{v,r,s,t} \) is \( n := \deg(D) = (q^c - 1) q^{c-1} \). For convenience we set \( G := v P_1 + r P_0 + s Q + t V \) with
\[
\deg(G) = v + (q^a - 1) r + q^{b-1} s + (q-1) t.
\]

It is well known that the dimension of an AG code \( C_{\mathcal{L}}(D, G) \) is given by
\[
\dim C_{\mathcal{L}}(D, G) = \dim \mathcal{L}(G) - \dim \mathcal{L}(G-D).
\] \hfill (28)

Set \( R := n + 2g - 2 \). If \( \deg(G) > R \), then the Riemann-Roch Theorem and Equation \hfill (28) \ yield
\[
\dim C_{v,r,s,t} = (1 - g + \deg(G)) - (1 - g + \deg(G-D)) = \deg D = n,
\]
which is trivial. So we should only consider the case $0 < \deg(G) < R$.

**Definition 13.** Two codes $C_1, C_2 \subseteq \mathbb{F}_q^n$ are said to be **equivalent** if there is a vector $a = (a_1, a_2, \ldots, a_n) \in (\mathbb{F}_q^n)^n$ such that $C_2 = a \cdot C_1$; i.e.,

$$C_2 = \{(a_1c_1, a_2c_2, \ldots, a_nc_n)| (c_1, c_2, \ldots, c_n) \in C_1\}.$$ 

Denote by $C^\perp$ the dual of $C$. The code $C$ is called **self-dual** (resp. **self-orthogonal**) if $C = C^\perp$ (resp. $C \subseteq C^\perp$).

**Proposition 14 ([11]).** Suppose $G_1$ and $G_2$ are divisors with $G_1 \sim G_2$ and $\supp G_1 \cap \supp D = \supp G_2 \cap \supp D = \emptyset$, then $C_{X}(D, G_1)$ and $C_{X}(D, G_2)$ are equivalent.

**Proposition 15.** Suppose that $s + q^a r = \hat{s} + \sigma(q^c - 1)$ with $0 < \hat{s} < q' - 1$, and $t - q^a - 1 N_b - (q^c - 1) \sigma = \hat{t} + N_c \lambda$ with $0 < t < N_c$. Let $\tilde{v} := v + (q^c - 1) \lambda - r$. Then the code $C_{v,r,s,t}$ is equivalent to $C_{\tilde{v},0, \tilde{s}, \tilde{t}}$.

**Proof.** It follows easily from Lemma 11 and Proposition 14.

We use the following lemma to calculate the dual of $C_{v,r,s,t}$.

**Lemma 16 ([11]).** Let $\tau$ be an element of the function field of $X$ such that $v_{P_i}(\tau) = 1$ for all rational places $P_i$ contained in the divisor $D$. Then the dual of $C_{X}(D, G)$ is

$$C_{X}(D, G)^\perp = C_{X}(D, D - G + \div(d\tau) - \div \tau).$$

**Proposition 17.** The dual of $C_{v,r,s,t}$ is

$$C_{v,r,s,t}^\perp = C_{-1-v, -1-r, -s, -t},$$

where $A = q^{c+a} + q^c - q^a - 2$, and $B = (q^a - 1)N_b - 1$.

**Proof.** Consider the element

$$\tau := \prod_{\alpha \in \mathbb{F}_q} (x - \alpha) = x^{q^c} - x.$$ 

Then $\tau$ is a prime element for all places $D_{\alpha, \beta}$, and its divisor is

$$\div(\tau) = \div_0(x) + D - q^c \div_{\infty}(x) = P + q^a - 1 N_b V + D - q^{c+a} Q.$$ 

It follows from [9] that

$$\Diff(F/K(x)) = (q^c + q^a - 2)Q + ((q^a - 1)N_b + (q^a - 1)N_c - 1) V.$$ 

So the divisor of $d\tau$ is

$$\div(d\tau) = \div(-dx) = -2 \div_{\infty}(x) + \Diff(F/K(x)) = -2q^a Q + (q^c + q^a - 2)Q + ((q^a - 1)N_b + (q^a - 1)N_c - 1) V = (q^c - q^a - 2)Q + ((q^a - 1)N_c + (q^a - 1)N_c - 1) V.$$ 

Let $\eta := d\tau/\tau$ be a Weil differential. Set $A := q^{c+a} + q^c - q^a - 2$, and $B := (q^a - 1)N_c - 1$. The divisor of $\eta$ is

$$\div(\eta) = \div(d\tau) - \div(\tau) = (q^c - q^a - 2)Q + ((q^a - 1)N_c + (q^a - 1)N_c - 1) V$$

By Lemma 16 the dual of $C_{v,r,s,t}$ is

$$C_{v,r,s,t}^\perp = C_{X}(D, D - vP_1 - rP_0 - sQ - tV + \div(\eta)) = C_{X}(D, (-1-v)P_1 + (-1-r)P_0 + (A - s)Q + (B - t)V) = C_{-1-v, -1-r, -A - s, -B - t}.$$ 

**Proposition 18.** Suppose that $0 < \deg(G) < R$. Then the following holds:
1) The dimension of \( C_r \) is given by

\[
\dim C_{v,r,s,t} = \begin{cases} 
#\Omega_{v,r,s,t} & \text{for } 0 \leq \deg(G) < n, \\
n - #\Omega_{v,r,s,t}^\perp & \text{for } n \leq \deg(G) \leq R.
\end{cases}
\]

where \( \Omega_{v,r,s,t}^\perp := \Omega - 1 - v, 1 - r, A - s, B - t \).

2) The minimum distance \( d \) of \( C_r \) satisfies

\[
d \geq n - v - (q^a - 1)r - q^{b-1}s - (q - 1)t.
\]

Proof. 1) For \( 0 \leq \deg(G) < n \), we have by Proposition 6 and Equation (28) that

\[
\dim C_{v,r,s,t} = \dim L(G) = #\Omega_{v,r,s,t}.
\]

For \( n \leq \deg(G) \leq R \), Proposition 17 yields

\[
\dim C_{v,r,s,t} = n - \dim C_{v,r,s,t}^\perp = n - #\Omega_{v,r,s,t}^\perp.
\]

2) The inequality follows from Goppa bound.

By Proposition 6 or Corollary 7, one can easily specify a generator matrix for the code \( C_{v,r,s,t} \). We fix an ordering of the set

\[
T := \left\{ (\alpha, \beta) \in \mathbb{F}_q^* \times \mathbb{F}_q^* \ \middle| \ \operatorname{Tr}_b\left(\frac{\beta^a}{\alpha}\right) + \operatorname{Tr}_a\left(\frac{\beta}{\alpha^q}\right) = 1 \right\}.
\]

For \( (i, j, k) \in \mathbb{Z}^3 \) we define the vector

\[
E_{i,j,k} := \left\{ \alpha^i \beta^j \left( a^{-1} - \frac{\beta^a}{\alpha} - \frac{\beta^q}{\alpha^q} \right)^k \ (\alpha, \beta) \in T \right\} \in \mathbb{F}_q^n.
\]

Proposition 19. Suppose that \( 0 \leq \deg(G) < n \). Let \( m := \dim C_{v,r,s,t} \) and \( (i_\lambda, j_\lambda, k_\lambda) \) with \( 1 \leq \lambda \leq m \) be all elements in \( \Omega'_{v,r,s,t} \). Then the \( m \times n \) matrix whose rows are \( E_{i_1,j_1,k_1}, \ldots, E_{i_m,j_m,k_m} \) is the generator matrix of \( C_{v,r,s,t} \).

Proof. Corollary 7

Example 20. Let us consider the case \( q = 2 \), \( c = 5 \), \( a = 3 \), and \( b = 2 \), then \( n = 496 \), and \( g = 75 \). By Proposition 15, we should only consider the codes \( C_{v,r,s,t} \) with \( r = 0, 0 \leq s < 31, 0 \leq t < 31 \). Applying Proposition 17, we find that the dual code of \( C_{v,r,s,t} \) is \( C_{-1-v,1-r,278-s,92-t} \). Using Proposition 18, we can determine the dimension and the Goppa bound for \( C_{v,r,s,t} \). The GV bound is the best lower bound which is known from elementary coding theory. However, its proof is not constructive. It does not provide a simple algebraic algorithm for the construction of good long codes. While our codes \( C_{v,r,s,t} \) can be constructed explicitly by applying Proposition 19, and it turns out that the Goppa bound of \( C_{v,r,s,t} \) improves the GV bound in a certain interval, see Figure 3. For instance, we find that the code \( C_{324,0,0,0} \) is a \([496, 250, \geq 172]\)-code over \( \mathbb{F}_{32} \), which oversteps the GV bound.

![Figure 3: Bound for \( \mathbb{F}_{q^c} = \mathbb{F}_{32} \)](image)
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