Large sample-to-sample fluctuations of the nonequilibrium critical current through mesoscopic Josephson junctions

P. Samuelsson$^a$ and H. Schomerus$^b$

$^a$ Department of Microelectronics and Nanoscience, Chalmers University of Technology and Göteborg University, S-41296 Göteborg, Sweden

$^b$ Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands

We present a theory for the nonequilibrium current in a mesoscopic Josephson junction which is coupled to a normal electron reservoir, and apply it to a chaotic junction. Large sample-to-sample fluctuations of the critical current $I_c$ are found, with rms $I_c \equiv \sqrt{\langle I_c^2 \rangle - \langle I_c \rangle^2} \approx \sqrt{N e \Delta / h}$, where the voltage difference $eV$ between the electron reservoir and the junction exceeds the superconducting gap $\Delta$ and the number of modes $N$ connecting the junction to the superconducting electrodes is large.

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FIG. 1. Three-terminal SNS junction, consisting of a mesoscopic scatterer (grey shaded) connected to two superconducting reservoirs via contacts 2 and 3 and a normal reservoir via contact 1. The black bar in contact 1 indicates a tunnel barrier, the arrows the direction of positive current flow.

Over the last years there has been an increased interest in the nonequilibrium Josephson current in mesoscopic multiterminal superconductor-normal metal-superconductor (SNS) junctions. Non-equilibrium in the junction is created by quasiparticle injection from one or several normal electron reservoirs, connected to the normal part of the SNS junction. By controlling the voltage applied between the normal reservoirs and the SNS junction, it has been shown in recent experiments that the Josephson current can be suppressed, reversed and in the case with injection from a superconducting reservoir, even enhanced.

The microscopic mechanism for these effects, nonequilibrium population of the current-carrying Andreev levels, was discussed by van Wees et al. in 1991. Thereafter, the nonequilibrium Josephson current in various multiterminal geometries has been studied in both diffusive and quantum-ballistic junctions. In Ref. it was pointed out that the nonequilibrium Josephson current in ballistic SNS junctions can not be described only in terms of the nonequilibrium population of Andreev levels. There is also a quantum-interference addition to the Josephson current, which results from the difference between the scattering-state wavefunctions for injected electrons and holes.

In this paper we develop a general theory of the nonequilibrium Josephson current in three-terminal SNS junctions (see Fig. 1), in terms of the scattering matrix for electrons and holes injected from the normal reservoir. The theory is then applied to a chaotic junction, in the limit of weak coupling to the normal reservoir and at zero temperature. We find that the quantum-interference contribution gives rise to sample-to-sample fluctuations of the critical current $I_c$ which are much larger than the equilibrium fluctuations. For a large voltage $V$ (with $eV \gtrsim \Delta$, the superconducting gap),

$$r m s \ I_c \equiv \sqrt{\langle I_c^2 \rangle - \langle I_c \rangle^2} \approx \sqrt{N e \Delta / h}, \quad (1)$$

hence the fluctuations are of the order of the ensemble-averaged critical current itself. (Here $N$ is the number of modes connecting the junction to each of the superconducting electrodes.) In this regime the current results from the quantum-interference contribution alone. For $eV \lesssim \Delta$ the critical current is of order $N e \Delta / h$, with fluctuations of order $e \Delta / h$.

A model of the junction is presented in Fig. 1. A mesoscopic scatterer is connected to two superconducting leads via ballistic contacts, each supporting $N$ transverse modes. The phase difference between the superconductors is $\phi$. The scatterer is also connected to a normal reservoir via a contact with $M$ modes, containing a tunnel barrier with transparency $\Gamma$. A voltage $V$ is applied between the SNS junction and the normal reservoir. We assume that the resistance of the injection contact is the dominating resistance of the junction, such that the potential drops completely over the injection point. In order to preserve nonequilibrium, the strength of the tunnel barrier $\Gamma$ is however limited by the requirement that the dwell time of the injected quasiparticles $\propto 1/\Gamma$ must be smaller than the inelastic scattering time in the junction.

Under these conditions, the distribution of the quasiparticles in the junction is determined by the distributions $n^{(b)} = n_F (E \mp e V) / N e$ of electrons (holes) in the reservoir, where $n_F = [1 + \exp(E/kT)]^{-1}$. The current in contact $j = 1, 2, 3$ can then be written as

$$I_j = \int dE n^{(b)} n^{(c)} \frac{d^2}{dE d\phi} T(j, \phi) \frac{dE}{h}.$$

Here $n^{(b)}$ is the Fermi function for normal electrons and $n^{(c)}$ for superconducting quasiparticles.

This equation is solved numerically for a range of parameters, and the results are presented in Fig. 2. The current $I_j$ is found to be weakly dependent on $\phi$, consistent with the observation in Ref. 4.

Finally, we consider the effect of nonequilibrium on the Josephson phase. The phase difference $\phi$ is given by the sum of the phases of the individual currents $I_j$, i.e.,

$$\phi = \sum_j I_j / e V.$$
\[ I_j = \int_{-\infty}^{\infty} dE (i_j^e n^e + i_j^h n^h + i_j^s n^s) , \]  
(2)

with \( i_j^{\pm(h)} \) the current density of the scattering states resulting from injected electron (hole) quasiparticles from the normal reservoir and \( i_j^s \) the total current density for quasiparticles injected from the superconductors (\( i^s = 0 \) for subgap energies \(|E| < \Delta \)).

The current \( I = (I_2 + I_3)/2 \) flowing between the superconductors can be rewritten by using the current conservation for each energy \( i_j^{e,h} = i_j^{\pm h} \) and the fact that no current is flowing in the injection lead in equilibrium, \( i_j^e + i_j^h = 0 \). It takes then the form \( I = I^{\text{eq}} + I^{\text{sq}} \), where the equilibrium current at \( eV = 0 \) is given by

\[ I^{\text{eq}} = \frac{1}{2} \int_{-\infty}^{\infty} dE \left[ \frac{e}{2} (n^e + n^h - 2n_F) + i^e \frac{e}{2} (n^e - n^h) \right] . \]  
(3)

Here the current densities \( i^e = i_2^e + i_3^e = i_5^e + i_6^e \) and \( i^h = (i_2^h - i_3^h + i_5^h - i_6^h)/2 \) are the sum and the difference of the current densities of the scattering states for injected electrons and holes. The contribution \( \propto i^+ \) to \( I^{\text{eq}} \) results from the nonequilibrium population of the Andreev levels, while the current \( \propto i^- \) accounts for the quantum-interference contribution as well as for an asymmetric splitting of the injected current \( I_1 = \frac{1}{2} \int dE (i_1^e - i_1^h) (n^e - n^h)/2 \).

We will now express the current densities in terms of the scattering matrix \( S \) of injected quasiparticles from the reservoir. The current densities are calculated most conveniently in the contacts \( j = 1, 2, 3 \), where the wavefunctions are plane-wave solutions to the Bogoliubov-de Gennes equation. A wave incident on the scatterer from leads 2 and 3 is described by the \( 4N \)-vector of wavefunction coefficients \( c_{n} = (c_{1}^{e}, c_{2}^{e}, c_{1}^{h}, c_{2}^{h}) \). The superscript \( +(-) \) denotes a positive (negative) sign of the wave vector. Correspondingly the outgoing wave is given by \( c_{\text{out}} = (c_{2}^{e}, c_{2}^{e}, c_{1}^{h}, c_{1}^{h}) \). At the NS interfaces, Andreev reflection is described by the scattering matrix

\[ S_A = \alpha \left( \begin{array}{cc} 0 & r_A \\ r_A^* & 0 \end{array} \right) , \quad r_A = \left( \begin{array}{cc} e^{i\varphi}/2 & 0 \\ 0 & e^{-i\varphi}/2 \end{array} \right) , \]  
(4)

such that \( c_{\text{out}} = S_A c_{\text{in}} \), with \( \alpha = \exp[-i \arccos(E/\Delta)] \). The wavefunctions in the three contacts are then matched with help of the \((2N+M) \times (2N+M)\) scattering matrix \( S' \) of the normal region (including the tunnel barrier),

\[ S' = \begin{pmatrix} r_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & r_{33} \end{pmatrix} . \]  
(5)

We introduce a non-unitary matrix \( S_N \), describing only the scattering between the contacts \( j = 2 \) and 3,

\[ S_N = \begin{pmatrix} S_0(E) & 0 \\ 0 & S_0(-E) \end{pmatrix} , \quad S_0 = \begin{pmatrix} t_{22} & t_{23} \\ t_{32} & r_{33} \end{pmatrix} , \]  
(6)

such that \( c_{\text{out}} = S_N c_{\text{in}} \), and matrices which involve also contact 1,

\[ T = \begin{pmatrix} t_{12}(E) & t_{13}(E) & 0 \\ 0 & 0 & t_{12}(-E) & t_{13}(-E) \end{pmatrix} , \]  
(7)

\[ T' = \begin{pmatrix} t_{21}(E) & 0 \\ t_{31}(E) & 0 \\ 0 & t_{21}(-E) & 0 \\ 0 & 0 & t_{31}(-E) \end{pmatrix} , \quad R = \begin{pmatrix} r_{11}(E) & 0 \\ 0 & r_{11}^*(-E) \end{pmatrix} . \]  
(8)

From these ingredients, the coefficients \( c \) can be calculated and the current densities in Eq. (2) are obtained from the quantum mechanical expression for current. The current densities \( i^+ \) and \( i^- \) follow after some matrix algebra,

\[ i^+(E) = \frac{e}{h} \text{tr} \left( T' \left( 1 - S_N S_A^0 \right)^{-1} \sigma_3' (1 - S_NS_A)^{-1} T' \right) , \]  
(9)

\[ i^-(E) = \frac{e}{h} \text{tr} \left( T' \left( 1 - S_A S_N^* \right)^{-1} \sigma_3' (1 - S_NS_A)^{-1} T' \sigma_3 \right) . \]  
(10)

After some further matrix algebra we obtain from Eqs. (6) and (8) the expressions (for subgap energies \(|E| < \Delta \))

\[ i^+(E) = \frac{2e}{i\hbar} \text{tr} \left( S' \frac{d}{d\phi} S \right) , \quad i^-(E) = \frac{2e}{i\hbar} \text{tr} \left( S' \frac{d}{d\phi} S \tau_z \right) , \]  
(11)

with \( \tau_z = \text{diag} (1,-1,-1) \). The expression for \( i^+ \) is well known [3,14]. Eqs. (3) and (4) are our general results for the nonequilibrium Josephson current.

In general, the current flowing between the superconductors contains also the part of the injected current which is asymmetrically split between contacts 2 and 3 [13]. This is not the case when the SNS junction is weakly coupled to the reservoir (\( \Gamma \ll 1 \)), because the injected current is then negligible. In this limit the matrix \( S_N = S_{N0} + \Gamma \delta S_N \) can be expanded to first order in \( \Gamma \), where \( S_{N0} \) is unitary. The two current densities \( i^+ \) and \( i^- \) have the same discrete spectrum of Andreev levels, given by the solutions \( E_n \) of \( \det(1 - S_A S_{N0}) = 0 \), but different spectral weights. The current density \( i^+ \) reduces to the well-known expression for the closed junction,

\[ i^+(E) = \sum_n I_n^+ \delta(E - E_n) , \quad I_n^+ = \frac{2e}{\hbar} \frac{dE_n}{d\phi} . \]  
(12)
can be written as a sum over the currents \( I_n^+ \) and \( I_n^- \) carried by the individual Andreev levels with positive energies \( E_n \). Eq. (12) provides a simple picture where in equilibrium all Andreev levels carry the currents \( I_n^+ \).

Increasing the voltage, the Andreev levels one by one switch from \( I_n^+ \) to \( I_n^- \) when the voltage is passing through \( eV = E_n \). At \( eV \geq \Delta \), all levels carry the current \( I_n^- \).

In terms of the transmission eigenvalues \( T_n \) of the matrix \( S_0 \), the Andreev bound-state energies are given by

\[
E_n = \Delta (1 - T_n \sin^2 \phi/2)^{1/2},
\]

hence the relation

\[
I_n^+ = - (\epsilon \Delta / 2h) T_n \sin \phi (1 - T_n \sin^2 \phi/2)^{-1/2}.
\]

(13)

The statistical properties of the equilibrium current \( I^{eq} \) are known \([1]\), with \( \langle I^{eq} \rangle \approx N e \Delta / h \) and rms \( I^{eq} \approx e \Delta / h \).

For \( eV \geq \Delta \) the current is

\[
I = \sum_n I_n^- = \sum_n R_n I_n^+.
\]

(11b)

where the unitary matrix \( U \) diagonalizes the unitary matrix product \( S_{N0} = U \text{diag}(\lambda) U^\dagger \). One can show with help of the corresponding eigenvalue equation that the ratios \( |R_n| \leq 1 \). It should be pointed out that the matrix \( \delta S_n \) can be expressed in terms of the closed junction scattering matrix \( S_{N0} \), i.e., the current density \( \langle \delta J \rangle \) depends manifestly on the properties of the contact between the normal reservoir and the SNS junction.

In order to investigate the mesoscopic fluctuations of the nonequilibrium current in more detail we now apply our theory to a chaotic SNS junction, in the limit of weak coupling to the normal reservoir \([4]\). Here we only consider the simplest case, in which the dwell time in the superconducting scatterer (with the superconducting leads replaced by normal ones) \( t_{dwell} < \hbar / \Delta \). (Our main conclusions should apply also for the opposite case.) For such a junction we can neglect the energy dependence of \( S' \), which is then distributed with the so-called Poisson kernel \( P(S') \propto |\text{det}(1 - (S'^\dagger)S')|^{-2N+M+1} \), where \( (S') \) is the ensemble-averaged scattering matrix \([4]\). (The magnetic field \( B = 0 \), which gives a symmetric scattering matrix \( S' = S'\dagger \).) Furthermore, the current density for energies outside the gap vanishes \([3]\). Using the energy symmetries \( i^+(E) = -i^-(E) \) and \( i^-(E) = i^+(E) \), the total current at zero temperature,

\[
I = - \sum_{E_n < eV} I_n^+ + \sum_{E_n < eV} I_n^- \equiv I^+ + I^-,
\]

(12)

can be written as a sum over the currents \( I_n^+ \) and \( I_n^- \) carried by the individual Andreev levels with positive energies \( E_n \). Eq. (12) provides a simple picture where...
as a function of junction modes $N$. The mean critical current is about $eV_{\Delta}$ (in general the energies lie in the interval $[\Delta \cos \phi/2, \Delta]$). In the range $0.54\Delta \leq eV \leq 0.98\Delta$ the critical phase is determined by the condition $\cos \phi_e/2 = eV/\Delta$ that the first Andreev bound state drops below $eV$, with only small fluctuations due to the high density of transmission eigenvalues $T_n \approx 1$. Hence the critical current is $I_c = I^{eq}(\phi_e)$. In this regime the quantum-interference contribution $I^-$ in Eq. (12) does not play any role because $\langle I^+ \rangle \gg \text{rms} I^-$. For a voltage $eV \approx 0.98\Delta$ very close to the gap, $I^+$ and $I^-$ are both of order $\sqrt{Ne\Delta}/h$, and the critical current starts to deviate from what one would expect from a pure nonequilibrium population of the Andreev levels. (For increasing $N$ the cross-over voltage $eV \rightarrow \Delta$.) In parallel the fluctuations of the critical current increase. The critical current remains constant for $eV \geq \Delta$, where it is given solely by $I^-$. The critical current for $eV \geq \Delta$ and its fluctuations as a function of junction modes $N$ are shown in the upper panel of Fig. 3. The mean critical current is $\langle I_c \rangle \approx 0.16\sqrt{N/3}e\Delta/h$. The fluctuations are of the same order, $\text{rms} I_c \approx 0.1\sqrt{N/3}e\Delta/h$, which is by a factor of about $\sqrt{N}/3$ larger than the equilibrium fluctuations. Hence the $N$ dependence in Eq. (13) carries over to the average critical current and its fluctuations.

Finally let us consider the dependence of the critical current on the number of injection modes $M$. This number is significant because the current $I^-$ depends manifestly on the coupling of the reservoir to the junction [see Eq. (11)], in contrast to the current $I^+$ which only depends on properties of the decoupled junction. The lower panel of Fig. 3 shows that the critical current and its fluctuations at $eV \geq \Delta$ are suppressed when $M$ is increased. The functional dependence is approximately $M^{-1/3}$. The curves flatten out when $M$ becomes larger than the total number $2N$ of modes connected to the superconductors. Thus, for an experimental observation of the large fluctuations predicted above, an injection contact with few modes is favorable.

In conclusion, we have studied the nonequilibrium Josephson current in a mesoscopic SNS junction connected to a normal electron reservoir. It is found that the current can be expressed in terms of the scattering matrix for the quasiparticles injected from the normal reservoir, Eqs. (6) and (7). As an application we considered the nonequilibrium current in a chaotic Josephson junction at zero temperature, weakly coupled to the normal reservoir. It is found that the fluctuations of the critical current for a voltage $eV \geq \Delta$ are of order $\text{rms} I_c \approx \sqrt{N\Delta e}/h$, which is of the same order as the mean critical current itself, and much larger than the equilibrium fluctuations (of order $\Delta e/h$).

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