On Fine-Grained Exact Computation in Regular Graphs

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We show that there is no subexponential time algorithm for computing the exact solution of the maximum independent set problem in $d$-regular graphs, for any constant $d > 2$, unless ETH fails. We also discuss the extensions of our construction to other problems and other classes of graphs, including 5-regular planar graphs.

1 Introduction

The independent set problem is a fundamental graph covering problem that asks for a set of pairwise nonadjacent vertices; we are interested to maximize the size of such a set and in particular find a maximum size such set. This is known as maximum independent set (MIS) problem.

MIS is hard to approximate within any constant factor in general graphs[13], however, on the bounded degree or regular graphs, a simple greedy algorithm provides a constant-factor approximation of the optimum solution. Given its hardness, researchers have studied the time complexity of the exact problem, i.e. they seek for fastest possible exponential exact algorithm for the problem.

Many NP-hard problems are solvable exactly in time $\tilde{O}(2^n)^1$. In particular for the independent set problem the trivial algorithm of testing all possible solutions yields such a running time. There are several improvements over the trivial upper bound both in general graphs and graphs of bounded degree [1, 4, 11, 12]; however all of them have a running time of the form $\tilde{O}(c^n)$ for a certain constant $c < 2$. Hence, all of them are exponential to $n$. On the other hand, there are NP-hard problems that are subexponential time solvable; e.g. exploiting bidimensionality theory, it is possible to solve several NP-hard problems in time $2^{O(\sqrt{n})}$ in excluded minor graphs [2]. Such differences raise the question of what NP-hard problems have subexponential time algorithms?

One of the main tools designed to better understand the exact complexity of hard computational problems is the Exponential Time Hypothesis (ETH) [7]. Assuming ETH,

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1We hide a $\text{Poly}(n)$ factor in $\tilde{O}$ notation.
there is no algorithm with running time $2^{o(n)}$ to solve the 3-SAT problem, where $n$ stands for the number of variables in the formula. It has been proven that under the same assumption several other known problems, such as $k$-SAT \cite{10} MIS in bounded degree graphs \cite{8} are ETH-hard.

Both of the above results, for $k$-SAT and MIS on bounded degree graphs, are showing the hardness of problem in a sparse instances of the input. The main challenge to prove such a hardness is to transfer an arbitrary instance to an instance that is sparse or it has certain structural property. If the transformation takes subexponential time then we can link them to the existing known ETH-hard problems to show they are also ETH-hard. These reductions are best known as sparsification lemmas.

In this work, we continue a similar spirit by providing a transformation from generic bounded degree graphs to $d$-regular graphs, a quite restricted well-structured graphs. A graph $G$ is $d$-regular if degree of every vertex of $G$ is exactly $d$. We show that for every integer $d > 2$ the maximum independent set problem is ETH-hard.

One of the related work is the result of Mohar \cite{9}; he showed the independent set and vertex cover are NP-complete in 3-regular planar graphs. His reduction in a sense is similar to the one of Johnson and Szegedy \cite{8} in general graphs. Following, by now the standard, technique of replacing vertices of high degree by paths/cycles and then analyzing the connections.

Similarly, Fleischner et al. \cite{3} showed that MIS is NP-hard even in 3 and 4-regular Hamiltonian planar graphs\footnote{They used a claim in the book \cite{5} that states MIS is NP-complete in cubic planar graphs. In the book, the authors cite the paper of Garey et al. \cite{6}. To our understanding, this is an incorrect referencing. However later, Mohar \cite{9} showed the hardness of the problem in the claimed class. Therefore the result of Fleischner et al. \cite{3} is valid.}. Their reduction is a bit more involved than the two others as they had to support the Hamiltonicity of the underlying graph. All of the above constructions are specialized for their specific purpose and we do not see a direct extension of mentioned papers to general graphs for every constant $d > 2$.

We do not limit ourselves to general regular graphs, similar to our predecessors, we also discuss hardness of MIS in regular planar graphs. We show that our simple construction extends to 5-regular planar graphs. Since there is no 6-regular planar graph, this together with previous results shows that in any non-trivial $d$-regular planar graph ($d > 2$), the problem is NP-hard. Note that $d = 1$ is a matching and $d = 2$ is a disjoint union of cycles. In both of them a linear time greedy algorithm gives an optimal solution: take a vertex of least degree into MIS and then together with its neighbor(s) remove them from the graph, repeat this process until reaching an empty graph.

The closely related problems of finding and counting cliques and finding a minimum vertex cover have also been discussed in this paper. The simple gadget construction facilitates further customization to obtain lowerbounds on such problems.

Before delving into the technical details, let us introduce the notations that are used throughout the paper. $\mathbb{N}$ denotes the set of natural numbers and for a set of integers $\{1, \ldots, k\}$ we write $[k]$. Degree of a vertex $v$ in a graph $G = (V, E)$ is written as $d_v$. We write $\Delta$ for the maximum degree of a graph. For a vertex $v$, the closed neighborhood of $v$ is denoted by $N[v]$ (it contains $v$ and all of its neighbors).
2 MIS has no Subexponential Algorithm on $d$-regular Graphs

In this section, we show the ETH-hardness of finding MIS in $d$-regular graphs.

2.1 Graph Construction

Let $G$ be a graph of maximum degree $\Delta$. We construct a gadget $G_\Delta$ where all of its vertices except one of them ($v_\Delta$) has degree $\Delta$. Then for every vertex $v \in V(G)$, if $\Delta - d_v = d > 0$ we connect the vertex $v$ to $d$ copies $G^1_v, \ldots, G^d_v$ of the $G_\Delta$ gadgets by adding edges from $v$ to all of these gadgets. This way the resulted graph $G'$ is a $\Delta$-regular graph. The construction of gadgets is such that from a maximum independent set in $G'$ we can derive a maximum independent set of $G$.

We may further assume that the maximum degree $\Delta$ is an odd integer. Otherwise, we add a complete graph on $\Delta + 2$ vertices to the original graph. The resulting graph has an odd maximum degree $\Delta + 1$ and it is clear that in polynomial time we can construct MIS of the original graph from MIS of the new graph and vice versa. Hence, we have the following assumption in the rest of paper.

Assumption: $\Delta$ is an odd number bigger than 1.

2.2 Gadget Construction

For a vertex $v$ of degree $d_v < \Delta$ we construct $\delta_v = \Delta - d_v$ distinct gadgets $H^1_v, \ldots, H^{\delta_v}_v$ as follows (all of them have the same structure). In the following we explain the construction of a single gadget, let say $H_v$.

First create $(\Delta - 1)/2$ complete bipartite graphs $K_1, \ldots, K_{(\Delta-1)/2}$ with partitions of size $\Delta - 1$. We name the partitions of the $i$'th bipartite graph $A_i, B_i$, for $i \in [(\Delta-1)/2]$. Add $\Delta - 1$ vertices $a_1, \ldots, a_{(\Delta-1)/2}, b_1, \ldots, b_{(\Delta-1)/2}$ to the gadget $H$. Connect all vertices of partition $A_i$ (resp. $B_i$) to $a_i$ (resp. $b_i$). Then connect all $a_i, b_i$'s ($i \in [(\Delta-1)/2]$) to a new vertex $h$. The construction of $H$ is completed. By construction, every vertex except $h$, has degree $\Delta$. The degree of $h$ is $2(\Delta - 1)/2 = \Delta - 1$. $h$ is the vertex that connects our gadget to the graph $G$.

Whenever it is necessary, if a gadget $H$ is the $j$'th gadget of a vertex $v$, to distinguish different gadgets, we add indices $v, j$ to $H$ and all of the aforementioned vertices and partitions. E.g. instead of a vertex $h$ we may write $h^j_v$.

The construction of the auxiliary graph $G'$ is pretty simple: take $G$ as a base, then for every $v \in G$ connect all of its gadgets, i.e. $H^j_v$'s, to $v$ by adding edges $\{h^j_v, v\}$ for $j \in [\delta_v]$. Let us make some observation on $G'$. First observe that every vertex of $G'$ has degree exactly $\Delta$.

We formalize the second observation for bounding order of $G'$ in the following.

Observation 1. Order of an attached gadget to any vertex is $O(\Delta^2)$. Since there are at most $\Delta$ such gadgets attached to a vertex $v$, $G'$ has $O(\Delta^3|V(G)|)$ vertices. As the number of edges of each gadget is at most $\Delta$ times more than its vertices, $G'$ has $O(\Delta^4|V(G)| + |E(G)|)$ edges.
2.3 From an MIS in $G'$ to an MIS in $G$

The main observation on each individual gadget is the following (we ignore the indices of the gadget for simplicity). In any MIS of $G'$, for a gadget $H$, from each bipartite graph $K_i$ in $H$ we have to take one of its partitions: $A_i$ or $B_i$, entirely into the MIS. The design of $H$ is such that, after the previous selection we can take either of the sets $a_i$'s or $b_i$'s in the solution. But then we are not able to take the vertex $h$ in the MIS. Consequently, vertex $v$ (a vertex of $G$ that is connected to the gadget $H$ in $G'$) is freely available to join MIS later. Hence, the existence of $v$ in MIS merely depends on the structure of $G$, not its connected gadgets. We prove these claims formally in the following.

First we explain how to construct an MIS in a single gadget $H$.

**Lemma 2.** Let $H$ be a gadget constructed as above, then it has an MIS $I$, such that the vertex $h \not\in I$ and the size of $I$ is $(\Delta - 1)^2/2 + \Delta - 1$.

**Proof.** We first constructively show that an independent set of the claim size and structure exists; then we prove it is a maximum independent set. To construct $I$, take all vertices in partitions $A_i$ ($i \in [(\Delta - 1)/2]$) into $I$, then add all vertices with labels $b_i$ to $I$. The size of $I$ is as claimed, it does not contain a vertex $h$, and it is an independent set of $H$. It lefts to show there is no independent set $I'$ of larger size in $H$.

Since $K_i$, the $i$'th complete bipartite graph of the gadget, is a complete bipartite graph, we can take at most $\Delta - 1$ vertex of it in the MIS. We first show that we should take exactly $\Delta - 1$ vertices from each of such bipartite graphs into any MIS.

For the sake of contradiction, suppose in one of these $K_i$'s, let call it $K$, an MIS $I'$ of $H$ has at most $t \leq \Delta - 2$ vertices of $K$. If $t > 0$, then w.l.o.g. suppose the selected vertices of $K$ are in its $B$ part\(^3\). Since $t > 0$ and all of these vertices are connected to the same vertex $b$ in $H - K$ it means $b$ is not in the independent set, hence if we take the entire $B$ part of $K$ into the independent set, the resulting set is still an independent set and larger than $I'$, a contradiction.

It remains to show the claim holds for the case of $t = 0$. $t = 0$ means no vertex of $K$ is in $I'$, then we should have both $a_i, b_i \in I$ (otherwise we add one side of $K$ to $I'$ and make a larger independent set). If this is the case, we remove $a_i$ from $I$ and add all vertices of the $A$ partition of $K$ to the independent set to make it larger, a contradiction.

Therefore, in any maximum independent set $I'$, for every bipartite graph $K_i$, entirety of one of its partitions is in $I'$. For the remaining undecided vertices, observe that we may take at most $(\Delta - 1)/2$ other vertices in the maximum independent set, this is forced by the choice of the corresponding partitions of bipartite graphs. \hfill \Box

Now we are ready to establish a connection between MIS of $G$ and $G'$ by the following lemma.

**Lemma 3.** Given an integer $k$, there is an independent set $I'$ of $G'$ of size at least $k + \sum_{v \in V(G)}(\Delta - d_v) \cdot ((\Delta - 1)^2/2 + \Delta - 1)$ if and only if there is an independent set $I$ of size at least $k$ in $G$. Moreover, we can construct $I$ from $I'$ and vice versa in linear time.

\(^3\)Clearly if a vertex from the $B$ part of $K$ is in any independent set of $H$ then no vertex from its $A$ part can contribute to that independent set, as $K$ is a complete bipartite graph.
Proof. The only if direction is straightforward: initialize $I' = I$ then add all maximum independent sets of all gadgets, computed by the approach explained in the proof of Lemma 2, to $I'$. The size of $I'$ is as claimed. On the other hand, none of the vertices of gadgets that are connected to the vertices of $G$ are in $I'$. It means there is no conflict between choices in gadgets and vertices in $I$, hence $I'$ is an independent set of the claimed size.

For the if part, by Lemma 2 there are at most $\sum_{v \in V(G)} (\Delta - d_v) \cdot (\Delta - 1)^2 / 2 + \Delta - 1$ vertices in $I'$ that are in $G' - G$. Hence, at least $k$ vertices $I = \{u_1, \ldots, u_k\}$ of $I'$ are belonging to both $G$ and $G'$, thus $I$ is an independent set of size $k$ in $G$.

The main theorem is the consequence of the previous lemmas and the sparsification lemma for the independent set problem.

Theorem 4. There is no algorithm with running time $2^{o(|E|)}$ to solve the maximum independent set problem in $\Delta$-regular graphs unless ETH fails.

Proof. Johnson and Szegedy [8] showed that there is no subexponential time algorithm to compute an MIS in graphs of degree at most 3, unless ETH fails. As explained earlier, in the description for Assumption 2.1, w.l.o.g. we can assume the input graph has an odd maximum degree. Thus, in Lemma 3 we provided a reduction from an independent set problem in graphs of maximum degree $\Delta$ to $\Delta$-regular graphs. By Observation 1 the size of each gadget is $\text{Poly}(\Delta)$ (independent of the order of $G$), thus our reductions are fine grain, therefore the theorem follows.

2.4 Extensions

The gadget construction simply extends to vertex cover and clique problems. On the other hand, another extension is to set up a similar lower bound in planar graphs. Our gadgets are not planar but it is easy to modify the most interior part of the gadgets (the bipartite graphs) to obtain planar gadgets. We explain the case of 5-regular planar graphs then we talk about the extension to the maximum clique problem.

Regular Planar Graphs

As discussed earlier, it is well known that the MIS problem is hard in 3, 4-regular planar graphs. We do not know if there is any result to show the hardness for 5-regular planar graphs. Here we present a simple construction to show the hardness of MIS (and consequently minimum vertex cover) in these graphs. The construction is similar as before, we keep vertices $a_i, b_i$ as we had, however, instead of bipartite graphs in the gadget, we insert a modified Icosahedron as drawn in the Fig 1), we call this graph $\mathcal{X}$.

Lemma 5. $\mathcal{X}$ has a maximum independent set of size 4 and both vertices $a, b$ will be in any MIS.

Proof. One can observe that vertices $a, b, f, k$ all together form an independent set of the claimed attributes, We prove that this is the only MIS of $\mathcal{X}$ by showing that in
any MIS, both vertices $a, b$ are present. Except $a, b$, every other vertex has degree 5 and every two non-adjacent vertices share at most 2 neighbors. Hence, if there are two vertices $x, y \in V(X) - \{a, b\}$ in an MIS $I$, then $|N[x] \cup N[y]| \geq 6 + 6 - 2 = 10$. Latter means all vertices of $X$ except at most two of them, let call them $u, v$, are in the closed neighborhood of $x, y$. Clearly both $u, v$ are in $I$ otherwise size of $I$ is less than 4. If $\{u, v\} = \{a, b\}$ we are done; otherwise, w.l.o.g. let suppose $u \notin \{a, b\}$. Since $|N[u]| = 6$ and $u$ is not neighbor of $x, y$, $u$ covers at least $6 - 2 - 2$ vertices that are not in $N[x] \cup N[y]$. It means $x, y, u$ together are neighbor of all vertices of $X$, hence, $v$ cannot be in $I$, a contradiction.

The rest of the proof is straightforward from above lemma and our general construction. Construct a gadget $H$ by taking 2 copies $X_1, X_2$ of $X$ and adding a vertex $h$. Then connect $a_i, b_i \in X_i$ to $h$ (we added indices to vertices of $X$ to distinguish the disjoint copies of them). Now we can attach copies of a gadget $H$ to every vertex that has a degree less than 5 in a given planar graph.

**Theorem 6.** MIS problem is NP-hard in 5-regular planar graphs.

**Proof.** By result of Mohar [9] we know that MIS problem is NP-hard in cubic planar graphs. Given a cubic planar graph $G$ construct a 5-regular planar graph $G'$ as explained above. $G$ has an independent set of size $k$ if and only if $G'$ has an independent set of size $k + 4\Sigma_{v\in V(G)}(5 - d_v)$. Since $G'$ is a 5-regular graph the claim follows in a same line as for general graphs. 

**Triangles and Cliques**

The gadgets do not have a triangle as a subgraph, on the other hand, the original connections in the graph $G$ are untouched, hence there is a clique on at least $k \geq 3$ vertices in $G'$ if and only if there is a clique of order $k$ in $G$. Since the transformation from...
$G$ to $G'$ happens in linear time on graphs of bounded degree, essentially every hardness result in graphs of bounded degree, for finding triangles or small cliques extends to the regular graphs.

3 Conclusion and Future Directions

In this work, we showed the maximum independent set problem has no subexponential algorithm in $d$-regular graphs. Our construction with simple modifications extends to other covering problems and also to other classes of graphs. We believe this work could ease the way to obtain fine-grain reductions for other problems.

We considered the independent set problem, one of the most basic problems were its sparsification lemma is known. Another interesting direction is to consider the $k$-SAT problem when the corresponding graph has the same degree for all variables and clauses.

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