Finite dimensional simple modules
of deformed current Lie algebras

Kentaro Wada

Abstract. The deformed current Lie algebra was introduced in [W] to study the representation theory of cyclotomic $q$-Schur algebras at $q = 1$. In this paper, we classify finite dimensional simple modules of deformed current Lie algebras.

Contents

§ 0. Introduction

§ 1. Deformed current Lie algebras $\mathfrak{sl}^Q_m[x]$ and $\mathfrak{gl}^Q_m[x]

§ 2. Representations of $\mathfrak{sl}^Q_m[x]

§ 3. Representations of $\mathfrak{gl}^Q_m[x]

§ 4. Rank 1 case; some relations in $U(\mathfrak{sl}^Q_2[x])$

§ 5. Rank 1 case; finite dimensional simple modules of $U(\mathfrak{sl}^Q_2[x])$

§ 6. Finite dimensional simple $U(\mathfrak{sl}^Q_m[x])$-modules

§ 7. Finite dimensional simple $U(\mathfrak{gl}^Q_m[x])$-modules

Appendix A. Some combinatorics

References

§ 0. Introduction

0.1. The deformed current Lie algebra $\mathfrak{g}_Q(m)$ was introduced in [W] to study the representation theory of cyclotomic $q$-Schur algebras at $q = 1$. In this paper, we introduce the deformed current Lie algebra $\mathfrak{sl}^Q_m[x]$ and $\mathfrak{gl}^Q_m[x]$ over $\mathbb{C}$ associated with the special linear Lie algebra $\mathfrak{sl}_m$ and general linear Lie algebra $\mathfrak{gl}_m$ respectively. $\mathfrak{sl}^Q_m[x]$ (resp. $\mathfrak{gl}^Q_m[x]$) is a deformation of the current Lie algebra $\mathfrak{sl}_m[x] = \mathfrak{sl}_m \otimes_{\mathbb{C}} \mathbb{C}[x]$ (resp. $\mathfrak{gl}_m[x] = \mathfrak{gl}_m \otimes_{\mathbb{C}} \mathbb{C}[x]$) with deformation parameters $Q = (Q_1, Q_2, \ldots, Q_{m-1}) \in \mathbb{C}^{m-1}$. Note that $\mathfrak{sl}^Q_m[x]$ (resp. $\mathfrak{gl}^Q_m[x]$) is coincide with $\mathfrak{sl}_m[x]$ (resp. $\mathfrak{gl}_m[x]$) if $Q_i = 0$ for all $i = 1, 2, \ldots, m - 1$. The Lie algebra $\mathfrak{g}_Q(m)$ introduced in [W] is isomorphic to $\mathfrak{gl}^Q_m[x]$ under a suitable choice of deformation parameters $Q$ (Lemma 1.7).

0.2. The differences of the representation theory of $\mathfrak{sl}^Q_m[x]$ from one of $\mathfrak{sl}_m[x]$ appear in the following two points. The deformed current Lie algebra $\mathfrak{sl}^Q_m[x]$ has a family
of 1-dimensional representations \( \{ \mathcal{L}^\beta \mid \beta \in \prod_{i=1}^{m-1} \mathbb{B}(Q_i) \} \), where

\[
\mathbb{B}(Q_i) = \begin{cases} 
\{0\} & \text{if } Q_i = 0, \\
\mathbb{C} & \text{if } Q_i \neq 0,
\end{cases}
\]

although the 1-dimensional representation of \( \mathfrak{sl}_m[x] \) is only the trivial representation (Lemma \( \text{[3.2]} \)). (We remark that \( \mathcal{L}^{(0,\ldots,0)} \) is the trivial representation of \( \mathfrak{sl}_m(x) \).)

The second difference appears in the evaluation modules. For each \( \gamma \in \mathbb{C} \), we can consider the evaluation homomorphism \( \text{ev}_\gamma : U(\mathfrak{sl}_m^Q[x]) \to U(\mathfrak{sl}_m) \) which is a deformation of the evaluation homomorphism for \( \mathfrak{sl}_m[x] \) (see the paragraph \( \text{[1.5]} \) for the definition). Then we can consider the evaluation modules by regarding \( U(\mathfrak{sl}_m^Q[x]) \)-modules as \( U(\mathfrak{sl}_m^Q[x]) \)-modules through the evaluation homomorphism \( \text{ev}_\gamma \). The evaluation homomorphism \( \text{ev}_\gamma \) is surjective if \( \gamma \neq Q_i^{-1} \) for all \( i = 1, 2, \ldots, m-1 \) such that \( Q_i \neq 0 \). However, \( \text{ev}_\gamma \) is not surjective if \( \gamma = Q_i^{-1} \) for some \( i = 1, 2, \ldots, m-1 \). Moreover, in general, the evaluation module of a simple \( U(\mathfrak{sl}_m) \)-module at \( \gamma \in \mathbb{C} \) is not simple if \( \gamma = Q_i^{-1} \) for some \( i = 1, 2, \ldots, m-1 \) (see Remark \( \text{[5.10]} \)).

**0.3.** It is a purpose of this paper to classify the finite dimensional simple modules of \( \mathfrak{sl}_m^Q[x] \) and \( \mathfrak{gl}_m^Q[x] \). A classification of the finite dimensional simple modules for the original current Lie algebra is well-known (e.g. \( \text{[C], [CP]} \)). The classification for \( \mathfrak{sl}_m^Q[x] \) (resp. \( \mathfrak{gl}_m^Q[x] \)) is an analogue of the original case.

Since \( \mathfrak{sl}_m^Q[x] \) has the triangular decomposition (Proposition \( \text{[1.4]} \)), we can develop the usual highest weight theory (see \( \text{[2]} \)). In particular, any finite dimensional simple \( U(\mathfrak{sl}_m^Q[x]) \)-module is isomorphic to a highest weight module \( \mathcal{L}(u) \) of highest weight \( u \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \) (Proposition \( \text{[2.6]} \)). Then it is enough to determine the highest weights such that the corresponding simple highest weight modules are finite dimensional. We obtain a classification of such highest weights as follows. Let \( \mathbb{C}[x]_{\text{monic}} \) be the set of monic polynomials over \( \mathbb{C} \) with the indeterminate variable \( x \). For each \( Q \in \mathbb{C} \), put

\[
\mathbb{C}[x]_{\text{monic}}^Q = \begin{cases} 
\mathbb{C}[x]_{\text{monic}} & \text{if } Q = 0, \\
\{ \varphi \in \mathbb{C}[x]_{\text{monic}} \mid Q^{-1} \text{ is not a root of } \varphi \} & \text{if } Q \neq 0.
\end{cases}
\]

We define the map \( \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^Q) \times \mathbb{B}(Q_i) \to \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \),

\[
(\varphi, \beta) = ((\varphi_i, \beta_i))_{1 \leq i \leq m-1} \mapsto u(Q)(\varphi, \beta) = (u(Q)(\varphi, \beta)_{i,t})_{1 \leq i \leq m-1, t \geq 0},
\]

by

\[
u(Q)(\varphi, \beta)_{i,t} = \begin{cases} 
\gamma_{t,1}^i + \gamma_{t,2}^i + \cdots + \gamma_{t,n_i}^i & \text{if } Q_i = 0, \\
\gamma_{t,1}^i + \gamma_{t,2}^i + \cdots + \gamma_{t,n_i}^i + Q_i^{-1} t \beta_i & \text{if } Q_i \neq 0
\end{cases}
\]

(0.3.1)
where \( \varphi_i = (x - \gamma_{i,1})(x - \gamma_{i,2}) \ldots (x - \gamma_{i,m_i}) \) \( (1 \leq i \leq m - 1) \). Then we have the following classification of finite dimensional simple \( U(\mathfrak{sl}^Q_m[x]) \)-modules (Theorem 6.4).

Theorem: \( \{ \mathcal{L}(u^Q(\varphi,\beta)) \mid (\varphi,\beta) \in \prod_{i=1}^{m-1}(\mathbb{C}[x]_{\text{monic}} \times \mathbb{B}^{(Q_i)}) \} \) gives a complete set of isomorphism classes of finite dimensional simple \( U(\mathfrak{sl}^Q_m[x]) \)-modules.

We remark that \( \mathcal{L}(u^Q(\varphi,\beta)) \) is isomorphic to a subquotient of

\[
\big( \bigotimes_{j=1}^{m-1} \bigotimes_{k=1}^{n_j} L(\omega_j)^{\text{ev}_{y_j,k}} \big) \otimes \mathcal{L}^{\beta},
\]

where \( \{ \omega_j \mid 1 \leq j \leq m - 1 \} \) is the set of fundamental weights for \( \mathfrak{sl}_m \), \( L(\omega_j) \) \( (1 \leq j \leq m - 1) \) is the simple highest weight \( U(\mathfrak{sl}_m) \)-module of highest weight \( \omega_j \) and \( L(\omega_j)^{\text{ev}_{y_j,k}} \) is the evaluation module of \( L(\omega_j) \) at \( \gamma_{j,k} \).

We also see that any finite dimensional simple \( U(\mathfrak{gl}^Q_m[x]) \)-module is isomorphic to a highest weight module \( \mathcal{L}(\tilde{\mathfrak{u}}) \) of highest weight \( \tilde{\mathfrak{u}} \in \prod_{j=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \) (Proposition 3.3). Note that \( \mathfrak{sl}_m[x] \) is a Lie subalgebra of \( \mathfrak{gl}_m^Q[x] \) (Proposition 1.4 (iii)). The difference of representations of \( \mathfrak{gl}_m^Q[x] \) from one of \( \mathfrak{sl}_m[x] \) is given by the family of 1-dimensional \( U(\mathfrak{gl}_m^Q[x]) \)-modules \( \{ \tilde{\mathcal{L}}^h \mid h \in \prod_{t \geq 0} \mathbb{C} \} \). We remark that \( \tilde{\mathcal{L}}^h \) \( (h \in \prod_{t \geq 0} \mathbb{C}) \) is isomorphic to the trivial representation \( \mathcal{L}^{(0,...,0)} \) as a \( U(\mathfrak{sl}_m^Q[x]) \)-module when we restrict the action. We obtain the classification of finite dimensional simple \( U(\mathfrak{gl}_m^Q[x]) \)-modules as follows. We define the map \( \prod_{i=1}^{m-1}(\mathbb{C}[x]_{\text{monic}} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C} \to \prod_{j=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \),

\[
(\varphi, \beta, h) = ((\varphi_i, \beta_i)_{1 \leq i \leq m-1}, (h_t)_{t \geq 0}) \mapsto \tilde{\mathfrak{u}}^Q(\varphi, \beta, h) = (\tilde{\mathfrak{u}}^Q(\varphi, \beta, h)_{j,t})_{1 \leq j \leq m, t \geq 0}
\]

by

\[
\tilde{\mathfrak{u}}^Q(\varphi, \beta, h)_{j,t} = \begin{cases} 
\sum_{k=j}^{m-1} u^Q(\varphi, \beta)_{k,t} + h_t & \text{if } 1 \leq j \leq m - 1 \text{ and } t \geq 0, \\
h_t & \text{if } j = m \text{ and } t \geq 0,
\end{cases}
\]

where \( u^Q(\varphi, \beta)_{k,t} \) is determined by (0.3.1). Then we have the following classification of finite dimensional simple \( U(\mathfrak{gl}_m^Q[x]) \)-modules (Theorem 7.4).

Theorem: \( \{ \mathcal{L}(\tilde{\mathfrak{u}}^Q(\varphi, \beta, h)) \mid (\varphi, \beta, h) \in \prod_{i=1}^{m-1}(\mathbb{C}[x]_{\text{monic}} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C} \} \) gives a complete set of isomorphism classes of finite dimensional simple \( U(\mathfrak{gl}_m^Q[x]) \)-modules.

We remark that \( \mathcal{L}(\tilde{\mathfrak{u}}^Q(\varphi, \beta, h)) \) is isomorphic to a subquotient of

\[
\big( \bigotimes_{j=1}^{m-1} \bigotimes_{k=1}^{n_j} \tilde{L}(\tilde{\omega}_j)^{\tilde{\text{ev}}_{y_j,k}} \big) \otimes \tilde{\mathcal{L}}^{\beta} \otimes \tilde{\mathcal{L}}^h.
\]
(See §7 for definitions of $L(\tilde{\omega}_j)^{e\tilde{v}_{\gamma_j,k}}$, $\tilde{\mathcal{L}}^\beta$ and $\tilde{\mathcal{L}}^h$.) We also remark that

$$L(\tilde{\omega}^{(Q)}(\varphi, \beta, h)) \cong L(\mathbf{u}^{(Q)}(\varphi, \beta)),$$

$$L(\tilde{\omega}_j)^{e\tilde{v}_{\gamma_j,k}} \cong L(\omega_j)^{ev_{\gamma_j,k}}; \quad \tilde{\mathcal{L}}^\beta \cong \mathcal{L}^\beta \quad \text{and} \quad \tilde{\mathcal{L}}^h \cong \mathcal{L}^{(0,...,0)}$$

as $U(\mathfrak{s}(\mathfrak{m})[x])$-modules when we restrict the action.

**Acknowledgements:** This work was supported by JSPS KAKENHI Grant Number JP16K17565.

\section{Deformed current Lie algebras $\mathfrak{s}(\mathfrak{m})[x]$ and $\mathfrak{g}(\mathfrak{m})[x]$}

In this section, we give a definition of deformed current Lie algebras $\mathfrak{s}(\mathfrak{m})[x]$ and $\mathfrak{g}(\mathfrak{m})[x]$, and also give some basic facts. The definition of $\mathfrak{g}(\mathfrak{m})[x]$ in this section is different from one of $\mathfrak{g}(\mathfrak{m})[x]$ given in [W]. The relation between $\mathfrak{g}(\mathfrak{m})[x]$ and $\mathfrak{g}(\mathfrak{m})[x]$ is given in Lemma 1.7.

**Definition 1.1.** Put $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_{m-1}) \in \mathbb{C}^{m-1}$. We define the Lie algebra $\mathfrak{s}(\mathfrak{m})[x]$ over $\mathbb{C}$ by the following generators and defining relations:

**Generators:** $\mathcal{X}_{i,t}^\pm, \mathcal{J}_{i,t} (1 \leq i \leq m - 1, t \geq 0)$.

**Relations:**

(L1) $[\mathcal{J}_{i,s}, \mathcal{J}_{j,t}] = 0$,

(L2) $[\mathcal{J}_{j,s}, \mathcal{X}_{i,t}^\pm] = \pm a_{ji} \mathcal{X}_{i,s+t}^\pm$,

(L3) $[\mathcal{X}_{i,t}^+, \mathcal{X}_{j,s}^-] = \delta_{ij}(\mathcal{J}_{i,s+t} - Q_i \mathcal{J}_{i,s+t+1})$,

(L4) $[\mathcal{X}_{i,t}^+, \mathcal{X}_{j,s}^+] = 0 \quad \text{if} \ j \neq i \pm 1$,

(L5) $[\mathcal{X}_{i,t+1}^+, \mathcal{X}_{i\pm 1, s}^+] = [\mathcal{X}_{i,t}^+, \mathcal{X}_{i\pm 1, s+1}]$,

(L6) $[\mathcal{X}_{i,s}^+, [\mathcal{X}_{i,t}^+, \mathcal{X}_{i\pm 1, u}^\pm]] = [\mathcal{X}_{i,s}^- [\mathcal{X}_{i,t}^-, \mathcal{X}_{i\pm 1, u}^-]] = 0$,

where we put $a_{ji} = \begin{cases} 2 & \text{if} \ j = i, \\ -1 & \text{if} \ j = i \pm 1, \\ 0 & \text{otherwise}. \end{cases}$

We also define the Lie algebra $\mathfrak{g}(\mathfrak{m})[x]$ over $\mathbb{C}$ by the following generators and defining relations:

**Generators:** $\mathcal{X}_{i,t}^\pm (1 \leq i \leq m - 1, t \geq 0), \mathcal{J}_{j,t} (1 \leq j \leq m, t \geq 0)$.

**Relations:**

(L1) $[\mathcal{I}_{i,s}, \mathcal{I}_{j,t}] = 0$,

(L2) $[\mathcal{I}_{j,s}, \mathcal{X}_{i,t}^\pm] = \pm a'_{ji} \mathcal{X}_{i,s+t}^\pm$,

(L3) $[\mathcal{X}_{i,t}^+, \mathcal{X}_{j,s}^-] = \delta_{ij}(\mathcal{J}_{i,s+t} - Q_i \mathcal{J}_{i,s+t+1})$, where we put $\mathcal{J}_{i,t} = \mathcal{I}_{i,t} - \mathcal{I}_{i+1,t}$,
together with the relations (L4)-(L6) in the above. In the relation (L'2), we put \( a'_{ji} = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases} \)

1.2. We call \( \mathfrak{sl}_m^{(Q)}[x] \) (resp. \( \mathfrak{gl}_m^{(Q)}[x] \)) the deformed current Lie algebra associated with the special linear Lie algebra \( \mathfrak{sl}_m \) (resp. the general linear Lie algebra \( \mathfrak{gl}_m \)). If \( Q_i = 0 \) for all \( i = 1, 2, \ldots, m - 1 \), then \( \mathfrak{sl}_m^{(Q)}[x] \) (resp. \( \mathfrak{gl}_m^{(Q)}[x] \)) coincides with the current Lie algebra \( \mathfrak{sl}_m[x] = \mathfrak{sl}_m \otimes_{\mathbb{C}} \mathbb{C}[x] \) (resp. \( \mathfrak{gl}_m[x] = \mathfrak{gl}_m \otimes_{\mathbb{C}} \mathbb{C}[x] \)) associated with \( \mathfrak{sl}_m \) (resp. \( \mathfrak{gl}_m \)). We can also regard \( \mathfrak{sl}_m^{(Q)}[x] \) (resp. \( \mathfrak{gl}_m^{(Q)}[x] \)) as a filtered deformation of \( \mathfrak{sl}_m[x] \) (resp. \( \mathfrak{gl}_m[x] \)) in a similar way as in [W] Proposition 2.13.

1.3. For \( 1 \leq i \neq j \leq m \) and \( t \geq 0 \), we define an element \( \mathcal{E}_{i,j,t} \in \mathfrak{sl}_m^{(Q)}[x] \) (resp. \( \mathcal{E}_{i,j,t} \in \mathfrak{gl}_m^{(Q)}[x] \)) by

\[
\mathcal{E}_{i,j,t} = \begin{cases} \{ \mathcal{X}_{i,0}^+, \mathcal{X}_{i+1,0}^+, \ldots, \mathcal{X}_{i-2,0}^+, \mathcal{X}_{i-1,0}^+ \} & \text{if } j > i, \\ \mathcal{X}_{i,0}^-, \mathcal{X}_{i+1,0}^-, \ldots, \mathcal{X}_{i-2,0}^-, \mathcal{X}_{i-1,0}^- \} & \text{if } j < i. \end{cases}
\]

In particular, we have \( \mathcal{E}_{i,i+1,t} = \mathcal{X}_{i,t}^+ \) and \( \mathcal{E}_{i+1,i,t} = \mathcal{X}_{i,t}^- \).

Let \( \mathfrak{n}^+ \) and \( \mathfrak{n}^- \) be the Lie subalgebra of \( \mathfrak{sl}_m^{(Q)}[x] \) (also of \( \mathfrak{gl}_m^{(Q)}[x] \)) generated by

\[
\{ \mathcal{X}_{i,t}^+ \mid 1 \leq i \leq m - 1, \ t \geq 0 \} \quad \text{and} \quad \{ \mathcal{X}_{i,t}^- \mid 1 \leq i \leq m - 1, \ t \geq 0 \}
\]

respectively. Let \( \mathfrak{n}^0 \) (resp. \( \tilde{\mathfrak{n}}^0 \)) be the Lie subalgebra of \( \mathfrak{sl}_m^{(Q)}[x] \) (resp. \( \mathfrak{gl}_m^{(Q)}[x] \)) generated by

\[
\{ \mathcal{J}_{i,t} \mid 1 \leq i \leq m - 1, \ t \geq 0 \} \quad \text{(resp. } \{ \mathcal{I}_{i,t} \mid 1 \leq i \leq m, \ t \geq 0 \}).
\]

By the relation (L1) (resp. (L’1)), we see that \( \mathfrak{n}^0 \) (resp. \( \tilde{\mathfrak{n}}^0 \)) is a commutative Lie subalgebra of \( \mathfrak{sl}_m^{(Q)}[x] \) (resp \( \mathfrak{gl}_m^{(Q)}[x] \)).

**Proposition 1.4.**

(i) \( \{ \mathcal{E}_{i,j,t} \mid 1 \leq i \neq j \leq m, \ t \geq 0 \} \cup \{ \mathcal{J}_{i,t} \mid 1 \leq i \leq m - 1, \ t \geq 0 \} \) gives a basis of \( \mathfrak{sl}_m^{(Q)}[x] \).

(ii) \( \{ \mathcal{E}_{i,j,t} \mid 1 \leq i \neq j \leq m, \ t \geq 0 \} \cup \{ \mathcal{I}_{j,t} \mid 1 \leq j \leq m, \ t \geq 0 \} \) gives a basis of \( \mathfrak{gl}_m^{(Q)}[x] \).

(iii) There exists an injective homomorphism of Lie algebras

\[
\Upsilon : \mathfrak{sl}_m^{(Q)}[x] \to \mathfrak{gl}_m^{(Q)}[x] \text{ such that } \mathcal{X}_{i,t}^+ \mapsto \mathcal{X}_{i,t}^+, \text{ and } \mathcal{J}_{i,t} \mapsto \mathcal{I}_{i,t} - \mathcal{I}_{i+1,t}. \]

(iv) We have the triangular decomposition

\[
\mathfrak{sl}_m^{(Q)}[x] = \mathfrak{n}^- \oplus \mathfrak{n}^0 \oplus \mathfrak{n}^+ \quad \text{and} \quad \mathfrak{gl}_m^{(Q)}[x] = \mathfrak{n}^- \oplus \tilde{\mathfrak{n}}^0 \oplus \mathfrak{n}^- \quad \text{(as vector spaces).}
\]
In particular,
\[
\{\mathcal{E}_{i,j,t} \mid 1 \leq i < j \leq m, \, t \geq 0\} \quad (\text{resp.} \quad \{\mathcal{E}_{i,j,t} \mid 1 \leq j < i \leq m, \, t \geq 0\})
\]
gives a basis of \(n^+\) (resp. \(n^-\)), and
\[
\{\mathcal{J}_{i,t} \mid 1 \leq i \leq m-1, \, t \geq 0\} \quad (\text{resp.} \quad \{\mathcal{I}_{j,t} \mid 1 \leq j \leq m, \, t \geq 0\})
\]
gives a basis of \(n^0\) (resp. \(\tilde{n}^0\)).

Proof. (i) and (ii) are proven in a similar way as in the proof of [W, Proposition 2.6]. By checking the defining relations, we see that \(\Upsilon\) is well-defined. We also see that \(\Upsilon\) is injective by investigating the basis given in (i) and (ii) under the homomorphism \(\Upsilon\). Then we have (iii). (iv) follows from (i) and (ii).

1.5. Evaluation homomorphisms and evaluation modules. The general linear Lie algebra \(\mathfrak{gl}_m\) is a Lie algebra over \(\mathbb{C}\) generated by \(e_i, f_i \quad (1 \leq i \leq m-1)\) and \(K_j \quad (1 \leq j \leq m)\) together with the following defining relations:
\[
\begin{align*}
[K_i, K_j] &= 0, \quad [K_j, e_i] = a_{ij}^{} e_i, \quad [K_j, f_i] = -a_{ji}^{} f_i, \\
[e_i, f_j] &= \delta_{ij} H_i, \quad \text{where} \quad H_i = K_i - K_{i+1}, \\
[e_i, e_j] &= [f_i, f_j] = 0 \quad \text{if} \quad j \neq i \pm 1, \quad [e_i, [e_i, e_{i+1}]] = [f_i, [f_i, f_{i+1}]] = 0.
\end{align*}
\]

The special linear Lie algebra \(\mathfrak{sl}_m\) is a Lie subalgebra of \(\mathfrak{gl}_m\) generated by \(e_i, f_i, H_i \quad (1 \leq i \leq m-1)\).

For each \(\gamma \in \mathbb{C}\), by checking the defining relations, we have the homomorphisms of algebras (evaluation homomorphism)
\[
ev_{\gamma} : U(\mathfrak{gl}_m^{(Q)}[x]) \rightarrow U(\mathfrak{sl}_m) \quad \text{by} \quad X_{i,t}^+ \mapsto (1 - Q_i \gamma) \gamma^{t} e_i, \quad X_{i,t}^- \mapsto \gamma^{t} f_i, \quad \mathcal{J}_{i,t} \mapsto \gamma^{t} H_i
\]
and
\[
\tilde{\ev}_{\gamma} : U(\mathfrak{gl}_m^{(Q)}[x]) \rightarrow U(\mathfrak{gl}_m) \quad \text{by} \quad X_{i,t}^+ \mapsto (1 - Q_i \gamma) \gamma^t e_i, \quad X_{i,t}^- \mapsto \gamma^t f_i, \quad \mathcal{I}_{j,t} \mapsto \gamma^t K_j.
\]
Clearly, the homomorphism \(\ev_{\gamma}\) (resp. \(\tilde{\ev}_{\gamma}\)) is surjective if \(\gamma \neq Q_i^{-1}\) for all \(i = 1, \ldots, m-1\) such that \(Q_i \neq 0\).

For a \(U(\mathfrak{sl}_m)\)-module \(M\) (resp. a \(U(\mathfrak{gl}_m)\)-module \(M\)), we can regard \(M\) as a \(U(\mathfrak{gl}_m^{(Q)}[x])\)-module (resp. a \(U(\mathfrak{gl}_m^{(Q)}[x])\)-module) through the evaluation homomorphism \(\ev_{\gamma}\) (resp. \(\tilde{\ev}_{\gamma}\)). We call it the evaluation module, and denote it by \(M^{\ev_{\gamma}}\) (resp. \(M^{\tilde{\ev}_{\gamma}}\)).

1.6. In the rest of this section, we give a relation with the Lie algebra \(\mathfrak{g}_{\mathbb{Q}}(\mathfrak{m})\) introduced in [W, Definition 2.2].

Let \(\mathfrak{m} = (m_1, \ldots, m_r)\) be an \(r\)-tuple of positive integers such that \(\sum_{k=1}^{r} m_k = m\). Put \(\Gamma(\mathfrak{m}) = \{(i, k) \mid 1 \leq i \leq m_k, \, 1 \leq k \leq r\}\) and \(\Gamma'(\mathfrak{m}) = \Gamma(\mathfrak{m}) \setminus \{(m_r, r)\}\). Then
we have the bijective map

\[ \zeta : \Gamma(m) \to \{1, 2, \ldots, m\} \] such that \((i, k) \mapsto \sum_{j=1}^{k-1} m_j + i.\]

For \((i, k) \in \Gamma(m)\) and \(j \in \mathbb{Z}\) such that \(1 \leq \zeta((i, k)) + j \leq m\), put \((i + j, k) = \zeta^{-1}(\zeta((i, k)) + j)\). For \((i, k) \in \Gamma'(m)\) and \((j, l) \in \Gamma(m)\), put \(a'_{(j,l)(i,k)} = a'_{(i,l)(j,k)}\).

Take \(\hat{Q} = (\hat{Q}_1, \ldots, \hat{Q}_{r-1}) \in \mathbb{C}^{r-1}\). Then the Lie algebra \(\mathfrak{g}_{\hat{Q}}(m)\) in [W, Definition 2.2] is defined by the generators \(X^\pm_{(i,k),t}\), \(I_{(j,l),t}\) \((i, k) \in \Gamma'(m), (j, l) \in \Gamma(m), t \geq 0\) together with the following defining relations:

\[
[X_{(i,k),s}, I_{(j,l),t}] = 0, \quad [I_{(j,l),s}, X^\pm_{(i,k),t}] = \pm a'_{(j,l)(i,k)} X^\pm_{(i,k),s+t},
\]

\[
[X^+_{(i,k),t}, X^-_{(j,l),s}] = \delta_{(i,k),(j,l)} \begin{cases} J(i,k)_{s+t} & \text{if } i \neq m_k, \\ -\hat{Q}_k J(m_k,k)_{s+t} + J(m_k,k)_{s+t+1} & \text{if } i = m_k, \end{cases}
\]

\[
[X^\pm_{(i,k),t}, X^\pm_{(j,l),s}] = 0 \quad \text{if } (j, l) \neq (i \pm 1, k),
\]

\[
[X^+_{(i,k),t+1}, X^+_{(i\pm 1,k),s}] = [X_{(i,k),t}, X^+_{(i\pm 1,k),s+1}], \quad [X^-_{(i,k),t+1}, X^-_{(i\pm 1,k),s}] = [X^-_{(i,k),t}, X^-_{(i\pm 1,k),s+1}],
\]

\[
[X^\pm_{(i,k),s}, [X^+_{(i,k),t}, X^+_{(i\pm 1,k),u}]] = [X^-_{(i,k),s}, [X^-_{(i,k),t}, X^-_{(i\pm 1,k),u}]] = 0,
\]

where we put \(J(i,k), = I_{(i,k),t} - I_{(i+1,k),t}\). Then we have the following isomorphism between \(\mathfrak{gl}_m^{(Q)}[x]\) and \(\mathfrak{g}_{\hat{Q}}(m)\) under the suitable choice of the deformation parameters \(Q\).

**Lemma 1.7.** Assume that \(\hat{Q}_i \neq 0\) for all \(i = 1, 2, \ldots, r - 1\). We take \(Q = (Q_1, Q_2, \ldots, Q_{m-1}) \in \mathbb{C}^{m-1}\) as

\[
Q_i = \begin{cases} \hat{Q}_i^{-1} & \text{if } \zeta^{-1}(i) = (m_k, k) \text{ for some } k = 1, 2, \ldots, r - 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Then we have the isomorphism of Lie algebras \(\Phi : \mathfrak{gl}_m^{(Q)}[x] \to \mathfrak{g}_{\hat{Q}}(m)\) such that

\[
X^+_{i,t} \mapsto \begin{cases} X^+_{\zeta^{-1}(i),t} & \text{if } \zeta^{-1}(i) \neq (m_k, k) \text{ for all } k = 1, \ldots, r - 1, \\ -\hat{Q}_k X^+_{\zeta^{-1}(i),t} & \text{if } \zeta^{-1}(i) = (m_k, k) \text{ for some } k = 1, \ldots, r - 1, \end{cases}
\]

\[
X^-_{i,t} \mapsto X^-_{\zeta^{-1}(i),t}, \quad I_{j,t} \mapsto I_{\zeta^{-1}(j),t}.
\]

**Proof.** We see the well-definedness of \(\Phi\) by checking the defining relations. The inverse homomorphism of \(\Phi\) is given by

\[
X^+_{(i,k),t} \mapsto \begin{cases} X^+_{\zeta((i,k)),t} & \text{if } i \neq m_k, \\ -\hat{Q}_k X^+_{\zeta((i,k)),t} & \text{if } i = m_k, \end{cases} \quad X^-_{(i,k),t} \mapsto X^-_{\zeta((i,k)),t}; \quad I_{(j,l),t} \mapsto I_{\zeta((j,l)),t}.
\]

\(\square\)
§ 2. Representations of \( \mathfrak{sl}_m^{(Q)}[x] \)

In this section, we give some fundamental results for finite dimensional \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-modules by using the standard argument.

2.1. Put \( \mathfrak{h} = \bigoplus_{i=1}^{m-1} C J_{i,0} \subset \mathfrak{sl}_m^{(Q)}[x] \), then \( \mathfrak{h} \) is a commutative Lie subalgebra of \( \mathfrak{sl}_m^{(Q)}[x] \). (Note that, if \( Q_i = 0 \) for all \( i = 1, \ldots, m - 1 \), \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{sl}_m(x) \).) Let \( \mathfrak{h}^* \) be the dual space of \( \mathfrak{h} \). Thus, if \( \tilde{\alpha} \) is a highest weight \( \mathfrak{sl}_m^{(Q)} \)-module, \( \tilde{\mathfrak{h}} \) is a partial order on \( \mathfrak{h}^\ast \). For each \( i = 1, 2, \ldots, m - 1 \), we take \( \alpha_i \in \mathfrak{h}^\ast \) as \( \alpha_i(J_{j,0}) = a_{ji} \) for \( j = 1, \ldots, m - 1 \). Put \( Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathfrak{h}^\ast \). We define the partial order on \( \mathfrak{h}^\ast \) by \( \lambda \geq \mu \) if \( \lambda - \mu \in Q^+ \) for \( \lambda, \mu \in \mathfrak{h}^\ast \).

2.2. For \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module \( M \), we consider the decomposition \( M = \bigoplus_{\lambda \in \mathfrak{h}^\ast} \tilde{M}_\lambda \), where \( \tilde{M}_\lambda = \{ x \in M \mid (h - \lambda(h))^N \cdot x = 0 \text{ for } h \in \mathfrak{h} \text{ and } N \gg 0 \} \), namely \( M = \bigoplus_{\lambda \in \mathfrak{h}^\ast} \tilde{M}_\lambda \) is the decomposition to the generalized simultaneous eigenspaces for the action of \( \mathfrak{h} \). By the relation (L2), we have

\[
\mathfrak{X}^{\pm}_{i,t} \cdot \tilde{M}_\lambda \subset \tilde{M}_{\lambda \pm \alpha_i} \quad (1 \leq i \leq m - 1, t \geq 0).
\]

Thus, if \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module \( M \neq 0 \) is finite dimensional, there exists \( \lambda \in \mathfrak{h}^\ast \) such that \( \tilde{M}_\lambda \neq 0 \) and \( \mathfrak{X}^{\pm}_{i,t} \cdot \tilde{M}_\lambda = 0 \) for all \( i = 1, 2, \ldots, m - 1 \) and \( t \geq 0 \). On the other hand, \( \tilde{M}_\lambda (\lambda \in \mathfrak{h}^\ast) \) is closed under the action of \( \mathfrak{n}^0 \) by the relation (L1). Thus, we can take a simultaneous eigenvector \( v \in \tilde{M}_\lambda \) for the action of \( \mathfrak{n}^0 \). Then we have the following lemma.

Lemma 2.3. For a finite dimensional \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module \( M \neq 0 \), there exists \( v_0 \in M \) (\( v_0 \neq 0 \)) satisfying the following conditions:

(i) \( \mathfrak{X}^{\pm}_{i,t} \cdot v_0 = 0 \) for all \( i = 1, \ldots, m - 1 \) and \( t \geq 0 \),

(ii) \( J_{i,t} \cdot v_0 = u_{i,t} v_0 \) (\( u_{i,t} \in \mathbb{C} \)) for each \( i = 1, \ldots, m - 1 \) and \( t \geq 0 \).

Moreover, if \( M \) is simple, we have \( M = U(\mathfrak{sl}_m^{(Q)}[x]) \cdot v_0 \).

2.4. Highest weight modules. For \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module \( M \), we say that \( M \) is a highest weight module if there exists \( v_0 \in M \) satisfying the following conditions:

(i) \( M \) is generated by \( v_0 \) as a \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module.

(ii) \( \mathfrak{X}^{\pm}_{i,t} \cdot v_0 = 0 \) for all \( i = 1, \ldots, m - 1 \) and \( t \geq 0 \).

(iii) \( J_{i,t} \cdot v_0 = u_{i,t} v_0 \) (\( u_{i,t} \in \mathbb{C} \)) for each \( i = 1, \ldots, m - 1 \) and \( t \geq 0 \).

In this case, we say that \( (u_{i,t})_{1 \leq i \leq m - 1, t \geq 0} \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \) is the highest weight of \( M \), and that \( v_0 \) is a highest weight vector of \( M \).

Let \( M \) be a highest weight \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module with a highest weight \( u = (u_{i,t})_{1 \leq i \leq m - 1, t \geq 0} \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \) and a highest weight vector \( v_0 \in M \). Thanks to the triangular decomposition (Proposition \([\text{L}4]\) (iv)) together with the above conditions, we have \( M = U(\mathfrak{n}^-) \cdot v_0 \). Let \( \lambda_u \in \mathfrak{h}^\ast \) be as \( \lambda_u(J_{i,0}) = u_{i,0} \) for \( i = 1, \ldots, m - 1 \). By \( M = U(\mathfrak{n}^-) \cdot v_0 \) and the relation (L2), we have the weight space decomposition

\[
(2.4.1) \quad M = \bigoplus_{\mu \in \mathfrak{h}^\ast} M_\mu, \quad \text{where } M_\mu = \{ x \in M \mid h \cdot x = \mu(h) \cdot x \text{ for } h \in \mathfrak{h} \}.
\]
and we also have \( \dim \mathbb{C} M_{x_0} = 1 \).

### 2.5. Verma modules.

For \( u = (u_{i,t}) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \), let \( \mathcal{J}(u) \) be the left ideal of \( U(\mathfrak{sl}_m^{(Q)}[x]) \) generated by \( X_{i,t}^+ (1 \leq i \leq m - 1, t \geq 0) \) and \( J_{i,t} - u_{i,t} (1 \leq i \leq m - 1, t \geq 0) \). We define the Verma module \( \mathcal{M}(u) = U(\mathfrak{sl}_m^{(Q)}[x]) / \mathcal{J}(u) \). Then \( \mathcal{M}(u) \) is a highest weight module of highest weight \( u \), and any highest weight module of highest weight \( u \) is realized as a quotient of the Verma module \( \mathcal{M}(u) \). By the weight space decomposition \((2.4.1)\), we see that \( \mathcal{M}(u) \) has the unique maximal proper submodule \( \text{rad} \mathcal{M}(u) \). Put \( \mathcal{L}(u) = \mathcal{M}(u) / \text{rad} \mathcal{M}(u) \), then we have the following proposition.

**Proposition 2.6.** For \( u = (u_{i,t}) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \), a highest weight simple \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module of highest weight \( u \) is isomorphic to \( \mathcal{L}(u) \). Moreover, any finite dimensional simple \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module is isomorphic to \( \mathcal{L}(u) \) for some \( u = (u_{i,t}) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C} \).

**Proof.** By Lemma 2.3 a finite dimensional simple \( U(\mathfrak{sl}_m^{(Q)}[x]) \)-module is a highest weight module. Then we have the proposition by the above arguments. \( \square \)

---

### § 3. Representations of \( \mathfrak{gl}_m^{(Q)}[x] \)

For finite dimensional \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-modules, we can develop a similar argument as in the case of \( U(\mathfrak{sl}_m^{(Q)}[x]) \) discussed in the previous section. In this section, we give only some notation for \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-modules.

### 3.1. Highest weight modules.

For \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-module \( M \), we say that \( M \) is a highest weight module if there exists \( v_0 \in M \) satisfying the following conditions:

(i) \( M \) is generated by \( v_0 \) as a \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-module.

(ii) \( X_{i,t}^+ \cdot v_0 = 0 \) for all \( i = 1, \ldots, m - 1 \) and \( t \geq 0 \).

(iii) \( J_{j,t} \cdot v_0 = \tilde{u}_{j,t} v_0 \) (\( \tilde{u}_{j,t} \in \mathbb{C} \)) for each \( j = 1, \ldots, m \) and \( t \geq 0 \).

In this case, we say that \((\tilde{u}_{j,t})_{1 \leq j \leq m, t \geq 0} \in \prod_{j=1}^{m} \prod_{t \geq 0} \mathbb{C} \) is the highest weight of \( M \), and that \( v_0 \) is a highest weight vector of \( M \).

### 3.2. Verma modules.

For \( \tilde{u} = (\tilde{u}_{j,t}) \in \prod_{j=1}^{m} \prod_{t \geq 0} \mathbb{C} \), let \( \mathcal{J}(\tilde{u}) \) be the left ideal of \( U(\mathfrak{gl}_m^{(Q)}[x]) \) generated by \( X_{i,t}^+ (1 \leq i \leq m - 1, t \geq 0) \) and \( J_{j,t} - \tilde{u}_{j,t} (1 \leq j \leq m, t \geq 0) \). We define the Verma module \( \mathcal{M}(\tilde{u}) = U(\mathfrak{gl}_m^{(Q)}[x]) / \mathcal{J}(\tilde{u}) \). Then \( \mathcal{M}(\tilde{u}) \) is a highest weight module of highest weight \( \tilde{u} \), and any highest weight module of highest weight \( \tilde{u} \) is realized as a quotient of the Verma module \( \mathcal{M}(\tilde{u}) \). \( \mathcal{M}(\tilde{u}) \) has the unique maximal proper submodule \( \text{rad} \mathcal{M}(\tilde{u}) \). Put \( \mathcal{L}(\tilde{u}) = \mathcal{M}(\tilde{u}) / \text{rad} \mathcal{M}(\tilde{u}) \), then we have the following proposition.

**Proposition 3.3.** For \( \tilde{u} = (\tilde{u}_{j,t}) \in \prod_{j=1}^{m} \prod_{t \geq 0} \mathbb{C} \), a highest weight simple \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-module of highest weight \( \tilde{u} \) is isomorphic to \( \mathcal{L}(\tilde{u}) \). Moreover, any finite dimensional simple \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-module is isomorphic to \( \mathcal{L}(\tilde{u}) \) for some \( \tilde{u} = (u_{j,t}) \in \prod_{j=1}^{m} \prod_{t \geq 0} \mathbb{C} \).
\section*{4. Rank 1 Case: Some Relations in $U(\mathfrak{sl}_2^{(Q)}[x])$}

\subsection*{4.1.}
Take $Q \in \mathbb{C}$, then $\mathfrak{sl}_2^{(Q)}[x]$ is a Lie algebra over $\mathbb{C}$ generated by $\mathcal{X}_t^\pm$ and $\mathcal{J}_t$ ($t \in \mathbb{Z}_{\geq 0}$) together with the following defining relations:

\begin{align*}
\text{(L1)} & \quad [\mathcal{J}_s, \mathcal{J}_t] = 0, \\
\text{(L2)} & \quad [\mathcal{J}_s, \mathcal{X}_t^\pm] = \pm 2\mathcal{X}_{s+t}^\pm, \\
\text{(L3)} & \quad [\mathcal{X}_t^+, \mathcal{X}_s^-] = \mathcal{J}_{s+t} - Q\mathcal{J}_{s+t+1}, \\
\text{(L4)} & \quad [\mathcal{X}_t^\pm, \mathcal{X}_s^\pm] = 0.
\end{align*}

(In the rank 1 case, we omit the first index of the generators since it is trivial.) By checking the defining relations, we see that there exists the algebra anti-automorphism $\dagger : U(\mathfrak{sl}_2^{(Q)}[x]) \to U(\mathfrak{sl}_2^{(Q)}[x])$ such that

$$\dagger(\mathcal{X}_t^+) = \mathcal{X}_t^-, \quad \dagger(\mathcal{X}_t^-) = \mathcal{X}_t^+, \quad \dagger(\mathcal{J}_t) = \mathcal{J}_t.$$ 

Clearly, $\dagger^2$ is the identity on $U(\mathfrak{sl}_2^{(Q)}[x])$.

\subsection*{4.2.}
For $t, b \in \mathbb{Z}_{\geq 0}$, we define an element $\mathcal{X}_t^{+(b)}$ (resp. $\mathcal{X}_t^{-(b)}$) of $U(\mathfrak{sl}_2^{(Q)}[x])$ by

$$\mathcal{X}_t^{\pm(b)} = \frac{(\mathcal{X}_t^\pm)^b}{b!}.$$ 

For convenience, we put $\mathcal{X}_t^{\pm(b)} = 0$ for $b \in \mathbb{Z}_{<0}$.

For $t, p, h \in \mathbb{Z}_{\geq 0}$, we define an element $\mathcal{X}_t^{+(p);h}$ (resp. $\mathcal{X}_t^{-(p);h}$) of $U(\mathfrak{sl}_2^{(Q)}[x])$ by

$$\mathcal{X}_t^{\pm((p);h)} = 1, \quad \mathcal{X}_t^{\pm((p);h)} = \sum_{w=0}^{p} \binom{p}{w} (-Q)^w \mathcal{X}_{t+h+w}^{\pm} \text{ for } p > 0.$$

Clearly, we have $\dagger(\mathcal{X}_t^{+(p);h}) = \mathcal{X}_t^{-(p);h}$. For examples, we have

\begin{align*}
\mathcal{X}_t^{+(0);h} & = 1, \quad \mathcal{X}_t^{\pm((1);h)} = \mathcal{X}_t^\pm + (-Q)\mathcal{X}_{t+h}^\pm, \\
\mathcal{X}_t^{\pm((2);h)} & = \mathcal{X}_{t+2h}^\pm + 2(-Q)\mathcal{X}_{t+2h+1}^\pm + (-Q)^2\mathcal{X}_{t+2h+2}^\pm, \\
\mathcal{X}_t^{\pm((3);h)} & = \mathcal{X}_{t+3h}^\pm + 3(-Q)\mathcal{X}_{t+3h+1}^\pm + 3(-Q)^2\mathcal{X}_{t+3h+2}^\pm + (-Q)^3\mathcal{X}_{t+3h+3}^\pm.
\end{align*}

For $s, p \in \mathbb{Z}_{\geq 0}$, we define an element $\mathcal{J}_s^{(p)}$ of $U(\mathfrak{sl}_2^{(Q)}[x])$ inductively on $p$ by

$$\mathcal{J}_s^{(0)} = 1, \quad \mathcal{J}_s^{(p)} = \frac{1}{p} \sum_{z=1}^{p} (-1)^{z-1} \left( \sum_{w=0}^{z} \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z)} \right) \text{ for } p > 0.$$
For examples, we have

\[ \mathcal{J}_s^{(0)} = 1, \quad \mathcal{J}_s^{(1)} = \mathcal{J}_s + (-Q)\mathcal{J}_{s+1}, \]
\[ \mathcal{J}_s^{(2)} = \frac{1}{2} \left( (\mathcal{J}_s^2 - \mathcal{J}_2s) + 2(-Q)(\mathcal{J}_s\mathcal{J}_{s+1} - \mathcal{J}_{2s+1}) + (-Q)^2(\mathcal{J}_{s+1}^2 - \mathcal{J}_{2s+2}) \right), \]
\[ \mathcal{J}_s^{(3)} = \frac{1}{3} \left( (\mathcal{J}_s^3 - 2\mathcal{J}_s\mathcal{J}_{2s} + \mathcal{J}_{3s}) + 3(-Q)(\mathcal{J}_s^2\mathcal{J}_{s+1} - \mathcal{J}_s\mathcal{J}_{2s+1} - \mathcal{J}_{s+1}\mathcal{J}_{2s} + \mathcal{J}_{3s+1}) 
+ (-Q)^2(3\mathcal{J}_s\mathcal{J}_{s+1}^2 - 2\mathcal{J}_s\mathcal{J}_{2s+2} - 4\mathcal{J}_{s+1}\mathcal{J}_{2s+1} + 3\mathcal{J}_{3s+2}) 
+ (-Q)^3(\mathcal{J}_{s+1}^3 - 2\mathcal{J}_{s+1}\mathcal{J}_{2s+2} + \mathcal{J}_{3s+3}) \right). \]

**Lemma 4.3.** For \( s, t, p \in \mathbb{Z}_{\geq 0} \), we have the following relations in \( U(\mathfrak{sl}_2^{(Q)}[x]) \).

(i) \[ [\mathcal{J}_s^{(p)}_\ast, \mathcal{X}_t^+] = \sum_{z=1}^{p} (-1)^{z+1}(z + 1)\mathcal{J}_s^{(p-z)}\mathcal{X}_t^{+(z);s}. \]

(ii) \[ [\mathcal{J}_s^{(p)}_\ast, \mathcal{X}_t^-] = -\sum_{z=1}^{p} (-1)^{z+1}(z + 1)\mathcal{X}_t^{-+(z);s}\mathcal{J}_s^{(p-z)}. \]

**Proof.** (ii) follows from (i) by applying the algebra anti-automorphism \( \dagger \) defined in (4.1.1). Then, we prove only (i) by the induction on \( p \).

If \( p = 0 \), (i) is clear. If \( p > 0 \), by the definition (4.2.2), we have

\[ \mathcal{J}_s^{(p)}\mathcal{X}_t^+ = \frac{1}{p} \sum_{z=1}^{p} (-1)^{z+1} \left( \sum_{w=0}^{z} \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \right) \mathcal{J}_s^{(p-z)}\mathcal{X}_t^+. \]

By the assumption of the induction, we have

\[ \mathcal{J}_s^{(p)}\mathcal{X}_t^+ = \frac{1}{p} \sum_{z=1}^{p} (-1)^{z+1} \left( \sum_{w=0}^{z} \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \right) \times \left( \mathcal{X}_t^+ \mathcal{J}_s^{(p-z)} + \sum_{k=1}^{p-z} (-1)^{k+1}(k+1)\mathcal{J}_s^{(p-z-k)}\mathcal{X}_t^{+(k);s} \right) \]
\[ = \frac{1}{p} \sum_{z=1}^{p} (-1)^{z+1} \sum_{w=0}^{z} \binom{z}{w} (-Q)^w (\mathcal{X}_t^+ \mathcal{J}_{zs+w} + 2\mathcal{X}_t^{+(k);s}) \mathcal{J}_s^{(p-z)} \]
\[ + \frac{1}{p} \sum_{z=1}^{p} \sum_{w=0}^{z} \sum_{k=1}^{p-z} (-1)^{z+k} \binom{z}{w} (k+1)(-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z-k)}\mathcal{X}_t^{+(k);s}. \]
Applying the assumption of the induction again, we have

\[(4.3.1)\]
\[
\mathcal{J}_s^{(p)} \mathcal{X}_t^+ = \mathcal{X}_t^+ \frac{1}{p} \sum_{z=1}^{p} (-1)^{z-1} \sum_{w=0}^{z} \left( \frac{z}{w} \right) (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z)}
\]
\[
+ 2 \frac{1}{p} \sum_{z=1}^{p} (-1)^{z-1} \sum_{w=0}^{z} \left( \frac{z}{w} \right) (-Q)^w \mathcal{J}_s^{(p-z)} \mathcal{X}_t^+ \mathcal{J}_{z+zs+w}
\]
\[
- 2 \frac{1}{p} \sum_{z=1}^{p} (-1)^{z-1} \sum_{w=0}^{z} \left( \frac{z}{w} \right) (-Q)^w \sum_{k=1}^{p-z} (-1)^{k+1}(k+1) \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+(k);s}
\]
\[
+ \frac{1}{p} \sum_{z=1}^{p} \sum_{w=0}^{p-z} \sum_{k=1}^{p-z} (-1)^{z+k} \left( \frac{z}{w} \right) (k+1)(-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+(k);s}.
\]

By the definition \[(4.2.2)\], we have

\[(4.3.2)\]
\[
\mathcal{X}_t^+ \frac{1}{p} \sum_{z=1}^{p} (-1)^{z-1} \sum_{w=0}^{z} \left( \frac{z}{w} \right) (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z)} = \mathcal{X}_t^+ \mathcal{J}_s^{(p)}.
\]

By the definition \[(4.2.1)\], we have

\[(4.3.3)\]
\[
\sum_{z=1}^{p} (-1)^{z-1} \sum_{w=0}^{z} \left( \frac{z}{w} \right) (-Q)^w \sum_{k=1}^{p-z} (-1)^{k+1}(k+1) \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+(k);s}
\]
\[
= \sum_{z=1}^{p} (-1)^{z-1} \sum_{w=0}^{z} \left( \frac{z}{w} \right) (-Q)^w \sum_{k=1}^{p-z} (-1)^{k+1}(k+1) \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+(k);s}.
\]

Put

\[(*)\]
\[
= \sum_{z=1}^{p} (-1)^{z-1} \sum_{w=0}^{z} \left( \frac{z}{w} \right) (-Q)^w \sum_{k=1}^{p-z} (-1)^{k+1}(k+1) \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+(k);s}.
\]

By the definition \[(4.2.1)\], we also have

\[(*)\]
\[
= \sum_{z=1}^{p} (-1)^{z-1} \sum_{w=0}^{z} \left( \frac{z}{w} \right) (-Q)^w \sum_{k=1}^{p-z} (-1)^{k+1}(k+1) \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+(k);s}
\]
\[
\times \left( \sum_{l=0}^{k} \binom{k}{l} (-Q)^l \mathcal{X}_t^{+(l);s+zs+w+k}ight)
\]
\[
= \sum_{z=1}^{p} \sum_{k=1}^{p-z} (-1)^{z+k}(k+1) \mathcal{J}_s^{(p-z+k)} \sum_{w=0}^{z} \sum_{l=0}^{k} \binom{k}{l} (-Q)^{w+l} \mathcal{X}_t^{+(l);s+zs+w+l}.
\]
Put \( z' = z + k \) and \( w' = w + l \), we have

\[
(\ast) = \sum_{z' = 2}^{p} \sum_{k=1}^{z' - 1} (-1)^{z'} (k + 1) \mathcal{J}_s^{(p - z')} \sum_{w' = 0}^{\min\{k, w'\}} (-Q)^{w'} \mathcal{X}_t^{+z' + w' + s}.
\]

By the induction on \( k \), we can show that

\[
(4.3.4) \quad \sum_{l = \max\{0, w' - (z' - k)\}}^{\min\{k, w'\}} \binom{z' - k}{w' - l} \binom{k}{l} = \binom{z'}{w'}.
\]

Then, we have

\[
(\ast) = \sum_{z' = 2}^{p} (-1)^{z'} \left( \sum_{k=1}^{z' - 1} (k + 1) \right) \mathcal{J}_s^{(p - z')} \sum_{w' = 0}^{\min\{k, w'\}} (-Q)^{w'} \mathcal{X}_t^{+z' + w' + s},
\]

and by the definition of \((4.2.4)\), we have

\[
(4.3.5) \quad (\ast) = \sum_{z' = 2}^{p} (-1)^{z'} \frac{(z' - 1)(z' + 2)}{2} \mathcal{J}_s^{(p - z')} \mathcal{X}_t^{+((z') + s)}.
\]

By the definition \((4.2.2)\), we have

\[
(4.3.6) \quad \sum_{z=1}^{p} \sum_{w=0}^{p-z} \binom{z}{w} (k + 1)(-Q)^{w} \mathcal{J}_{zs+w} \mathcal{J}_s^{(p - z - k)} \mathcal{X}_t^{+(k+s)}
\]

\[
= \sum_{k=1}^{p-1} \sum_{z=1}^{p-k} \binom{z}{w} (k + 1)(-Q)^{w} \mathcal{J}_{zs+w} \mathcal{J}_s^{(p - z - k)} \mathcal{X}_t^{+(k+s)}
\]

\[
= \sum_{k=1}^{p-1} (-1)^{k+1}(k + 1)(p - k) \left( \frac{1}{p - k} \sum_{z=1}^{p-k} (-1)^{z-1} \sum_{w=0}^{z} \binom{z}{w} (z - Q)^{w} \mathcal{J}_{zs+w} \mathcal{J}_s^{((p-k)-z)} \mathcal{X}_t^{+(k+s)}
\]

\[
= \sum_{k=1}^{p-1} (-1)^{k+1}(k + 1)(p - k) \mathcal{J}_s^{(p-k)} \mathcal{X}_t^{+(k+s)}.
\]

Combining \((4.3.1)\) with \((4.3.6)\), we have

\[
\mathcal{J}_s^{(p)} \mathcal{X}_t^+ = \mathcal{X}_t^+ \mathcal{J}_s^{(p)} + \frac{1}{p} \sum_{z=1}^{p} (-1)^{z+1}(2 + (z - 1)(z + 2) + (z + 1)(p - z)) \mathcal{J}_s^{(p - z)} \mathcal{X}_t^{+(z+s)}
\]

\[
= \mathcal{X}_t^+ \mathcal{J}_s^{(p)} + \sum_{z=1}^{p} (-1)^{z+1}(z + 1) \mathcal{J}_s^{(p - z)} \mathcal{X}_t^{+(z+s)}.
\]
Lemma 4.4. For $s, t, h \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$, we have

$$[\mathcal{X}_t^+, \mathcal{X}_s^{-(p); h}] = \sum_{w=0}^{p} \binom{p}{w} (-Q)^w \mathcal{J}_{s+t+ph+w}^{(1)}.$$

Proof. By the definitions (4.2.1), (4.2.2) and the defining relation (L3), we have

$$\mathcal{X}_t^+ \mathcal{X}_s^{-(p); h} = \sum_{w=0}^{p} \binom{p}{w} (-Q)^w \mathcal{X}_s^{-(p); h} \mathcal{X}_t^+ + \mathcal{J}_{s+t+ph+w}^{(1)} + (-Q)\mathcal{J}_{s+t+ph+w+1}^{(1)}$$

Lemma 4.5. For $s, t, h \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$, we have the following relations.

(i) $[\mathcal{J}_s^{(1)}, \mathcal{X}_t^{+(p); h}] = 2\mathcal{X}_t^{+(p+1); h}$.

(ii) $[\mathcal{J}_s^{(1)}, \mathcal{X}_t^{-(p); h}] = -2\mathcal{X}_t^{-(p+1); h}$.

Proof. (ii) follows from (i) by applying the algebra anti-automorphism $\dagger$ defined in (4.1.1). Then, we prove (i).

By the definition (4.2.1), we have

$$\mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p); h} = \sum_{w=0}^{p} \binom{p}{w} (-Q)^w \mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p+1); h} + 2\mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p+1); h}$$

Applying Lemma 4.3 (i), we have

$$\mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p); h} = \sum_{w=0}^{p} \binom{p}{w} (-Q)^w \mathcal{X}_t^{+(p+1); h} \mathcal{J}_s^{(1)} + 2\mathcal{X}_t^{+(p+1); h} \mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p+1); h}$$

Then, by the definition (4.2.1) again, we have

$$\mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p); h} = \mathcal{X}_t^{+(p); h} \mathcal{J}_s^{(1)} + 2\sum_{w=0}^{p} \binom{p}{w} (-Q)^w \mathcal{X}_t^{+(p+1); h} \mathcal{J}_s^{(1)} + (-Q)\mathcal{X}_t^{+(p+1); h} \mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p+1); h}.$$
\[
\begin{align*}
&= \mathcal{X}_{s+t+ph}^+ + \sum_{w=1}^p \left\{ \binom{p}{w} + \binom{p}{w-1} \right\} (-Q)^w \mathcal{X}_{s+t+ph+w}^+ + (-Q)^{p+1} \mathcal{X}_{s+t+ph+p+1}^+ \\
&= \sum_{w=0}^{p+1} \binom{p+1}{w} (-Q)^w \mathcal{X}_{s+t-h+(p+1)h+w}^+ \\
&= \mathcal{X}_{s+t-h}^{+(p+1)h}.
\end{align*}
\]

Thus, we have (i). □

**Lemma 4.6.** For \( s, t, c \in \mathbb{Z}_{\geq 0} \), we have
\[
[\mathcal{X}_t^+, \mathcal{X}_s^{-(c)}] = \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} - \mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-(1);s+t)}.
\]

**Proof.** We prove the lemma by the induction on \( c \). If \( c = 0 \), it is clear. If \( c = 1 \), it is the defining relation (L3). If \( c > 1 \), by the assumption of the induction, we have
\[
\begin{align*}
\mathcal{X}_t^+ \mathcal{X}_s^{-(c)} &= \frac{1}{c} \mathcal{X}_t^+ \mathcal{X}_s^{-(c-1)} \mathcal{X}_s^-
\end{align*}
\]
Then, by the defining relations (L3), (L4) and Lemma 4.3 (ii), we have
\[
\begin{align*}
\mathcal{X}_t^+ \mathcal{X}_s^{-(c)} &= \frac{1}{c} \left\{ \mathcal{X}_s^{-(c-1)} \left( \mathcal{X}_s^+ \mathcal{X}_t^+ + \mathcal{J}_{s+t}^{(1)} \right) + \mathcal{X}_s^{-(c-2)} \left( \mathcal{X}_s^- \mathcal{J}_{s+t}^{(1)} - 2 \mathcal{X}_s^{-(1);s+t)} \right) \\
&\quad - \mathcal{X}_s^{-(c-3)} \mathcal{X}_s^- \mathcal{X}_s^{-(1);s+t)} \right\} \\
&= \frac{1}{c} \left\{ c \mathcal{X}_s^{-(c)} \mathcal{X}_t^+ + \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} + (c - 1) \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} \\
&\quad - 2 \mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-(1);s+t)} - (c - 2) \mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-(1);s+t)} \right\} \\
&= \mathcal{X}_s^{-(c)} \mathcal{X}_t^+ + \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} - \mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-(1);s+t)}.
\end{align*}
\]

□

**4.7.** A partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a non-increasing sequence of non-negative integers which has only finitely many non-zero terms. The size of a partition \( \lambda \) is the sum of all terms of \( \lambda \), and we denote it by \( |\lambda| \). Namely, we have \( |\lambda| = \sum_{i \geq 1} \lambda_i \). If \( |\lambda| = n \), we say that \( \lambda \) is a partition of \( n \), and we denote it by \( \lambda \vdash n \). The length of \( \lambda \) is the maximal \( i \) such that \( \lambda_i \neq 0 \), and we denote the length of \( \lambda \) by \( \ell(\lambda) \). For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), let \( m_j(\lambda) \ (j \in \mathbb{Z}_{\geq 0}) \) be the multiplicity of \( j \) in \( \lambda \). Then, for a partition \( \lambda \) and \( t, h \in \mathbb{Z}_{\geq 0} \), we define an element \( \mathcal{X}_t^{+(\lambda,h)} \) (resp. \( \mathcal{X}_t^{-(\lambda,h)} \)) of \( U(\mathfrak{sl}_2^{(Q)}[x]) \) by
\[
(4.7.1) \quad \mathcal{X}_t^{\pm(\lambda,h)} = \prod_{j \geq 1} \frac{(\mathcal{X}_t^{\pm((j);h)})^{m_j(\lambda)}}{m_j(\lambda)!},
\]
where we note the defining relation (L4). Clearly, we have \( \hat{\tau}(\mathcal{X}_t^{(\lambda,h)}) = \mathcal{X}_t^{-(\lambda,h)} \). For examples, we have

\[
\begin{align*}
\mathcal{X}_t^{\pm(0);h} &= 1, \quad \mathcal{X}_t^{\pm((1);h)} = \mathcal{X}_t^{\pm(1);h}, \\
\mathcal{X}_t^{\pm((2);h)} &= \mathcal{X}_t^{\pm(2);h}, \quad \mathcal{X}_t^{\pm((1,1);h)} = \left(\mathcal{X}_t^{\pm((1);h)}\right)^2 / 2!, \\
\mathcal{X}_t^{\pm((3);h)} &= \mathcal{X}_t^{\pm((2);h)} \mathcal{X}_t^{\pm((1);h)} \mathcal{X}_t^{\pm((1,1);h)} = \left(\mathcal{X}_t^{\pm((1);h)}\right)^3 / 3!, \\
\mathcal{X}_t^{\pm((3,2,2,1,1);h)} &= \left(\mathcal{X}_t^{\pm((3);h)}\right)^2 \left(\mathcal{X}_t^{\pm((2);h)}\right)^3 \left(\mathcal{X}_t^{\pm((1);h)}\right)^2 / 2! 3! 2!. 
\end{align*}
\]

For \( t, h, k, b, p \in \mathbb{Z}_{\geq 0} \), we define an element \( \mathcal{X}_t^{+(b;p|k;h)} \) (resp. \( \mathcal{X}_t^{-(b;p|k;h)} \)) of \( U(\mathfrak{sl}_2^{(Q)}[x]) \) by

\[
(4.7.2) \quad \mathcal{X}_t^{\pm(b;p|k;h)} = \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))}.
\]

Note the defining relation (L4), we see that \( \hat{\tau}(\mathcal{X}_t^{+(b;p|k;h)}) = \mathcal{X}_t^{-(b;p|k;h)} \). For examples, we have

\[
\begin{align*}
\mathcal{X}_t^{\pm((k;p|0);h)} &= \mathcal{X}_t^{\pm(b-p)}, \quad \mathcal{X}_t^{\pm((k;p|1);h)} = \mathcal{X}_t^{\pm((1);h)} \mathcal{X}_t^{\pm(b-p-1)}, \\
\mathcal{X}_t^{\pm((k;p|2);h)} &= \mathcal{X}_t^{\pm((2);h)} \mathcal{X}_t^{\pm(b-p-1)} + \mathcal{X}_t^{\pm((1,1);h)} \mathcal{X}_t^{\pm(b-p-2)}, \\
\mathcal{X}_t^{\pm((k;p|3);h)} &= \mathcal{X}_t^{\pm((3);h)} \mathcal{X}_t^{\pm(b-p-1)} + \mathcal{X}_t^{\pm((2,1);h)} \mathcal{X}_t^{\pm(b-p-2)} + \mathcal{X}_t^{\pm((1,1,1);h)} \mathcal{X}_t^{\pm(b-p-3)}.
\end{align*}
\]

For the element \( \mathcal{X}_t^{\pm((b;p|k);h)} \in U(\mathfrak{sl}_2^{(Q)}[x]) \), we prepare the following technical formulas.

**Lemma 4.8.** For \( t, h, k, b, p \in \mathbb{Z}_{\geq 0} \), we have the following equations for the element \( \mathcal{X}_t^{\pm((b;p|k);h)} \) of \( U(\mathfrak{sl}_2^{(Q)}[x]) \).

(i) If \( b-p < 0 \), we have \( \mathcal{X}_t^{\pm((b;p|k);h)} = 0 \).

(ii) If \( k = 0 \), we have \( \mathcal{X}_t^{\pm((b;p|0);h)} = \mathcal{X}_t^{\pm(b-p)} \).

If \( k = 1 \), we have \( \mathcal{X}_t^{\pm((b;p|1);h)} = \mathcal{X}_t^{\pm((1);h)} \mathcal{X}_t^{\pm(b-p-1)} \).

(iii) If \( p = b \), we have \( \mathcal{X}_t^{\pm((b;b|k);h)} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases} \)

(iv) If \( b, p > 0 \), we have \( \mathcal{X}_t^{\pm((b;p|k);h)} = \mathcal{X}_t^{\pm(b-1;p-1|k;h)} \).

(v) If \( b, k > 0 \), we have

\[
\mathcal{X}_t^{\pm((b;p|k);h)} = \frac{1}{k} \sum_{z=1}^{k} z \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-1;p|k-z;h)}. 
\]
(vi) If $b > 0$, we have

$$(b - p + k)\mathcal{X}_t^{\pm(b;p;k:h)} = \mathcal{X}_t^{\pm(b-1;p;k:h)} + \sum_{z=1}^{k} (z + 1) \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-1;p;k-z;h)}.$$  

**Proof.** (i), (ii), (iii) and (iv) are clear from definitions.

We prove (v). Note that $\sum_{z \geq 1} z m_z(\lambda) = k$ for a partition $\lambda$ of $k$. Then, by the definition (4.7.2), we have

$$\mathcal{X}_t^{\pm(b;p;k;h)} = \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))} = \frac{1}{k} \sum_{\lambda \vdash k} \left( \sum_{z \geq 1} z m_z(\lambda) \right) \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))}.$$  

On the other hand, by the definition (4.7.1), we have

$$\mathcal{X}_t^{\pm(\lambda;h)} = \prod_{j \geq 1} \left( \frac{\mathcal{X}_t^{\pm((j);h)} m_j(\lambda)}{m_j(\lambda)!} \right) = \frac{1}{m_z(\lambda)!} \mathcal{X}_t^{\pm((z);h)} \frac{(\mathcal{X}_t^{\pm((z);h)} m_z(\lambda) - 1)!}{m_z(\lambda)} \prod_{j \geq 1 \atop j \neq z} \mathcal{X}_t^{\pm((j);h)} m_j(\lambda)!$$

for each $z$ such that $m_z(\lambda) \neq 0$. Thus, we have

$$(b - p + k)\mathcal{X}_t^{\pm(b;p;k;h)} = \frac{1}{k} \sum_{\lambda \vdash k} \sum_{z \geq 1 \atop m_z(\lambda) 
eq 0} z \mathcal{X}_t^{\pm((z);h)} \frac{(\mathcal{X}_t^{\pm((z);h)} m_z(\lambda) - 1)!}{m_z(\lambda)} \prod_{j \geq 1 \atop j \neq z} \mathcal{X}_t^{\pm((j);h)} m_j(\lambda)! \mathcal{X}_t^{\pm(b-p-\ell(\lambda))}$$

$$= \frac{1}{k} \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm((z);h)} \sum_{z \geq 1 \atop m_z(\lambda) 
eq 0} \left( \frac{(\mathcal{X}_t^{\pm((z);h)} m_z(\lambda) - 1)!}{m_z(\lambda)} \prod_{j \geq 1 \atop j \neq z} \mathcal{X}_t^{\pm((j);h)} m_j(\lambda)! \right) \mathcal{X}_t^{\pm(b-p-\ell(\lambda))}$$

$$= \frac{1}{k} \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm((z);h)} \prod_{\mu \vdash k-z \atop j \geq 1} \left( \frac{(\mathcal{X}_t^{\pm((z);h)} m_j(\mu))}{m_j(\mu)!} \right) \mathcal{X}_t^{\pm((b-1)p-\ell(\mu)+1)}$$

$$= \frac{1}{k} \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm((z);h)} \prod_{\mu \vdash k-z \atop j \geq 1} \left( \frac{(\mathcal{X}_t^{\pm((z);h)} m_j(\mu))}{m_j(\mu)!} \right) \mathcal{X}_t^{\pm((b-1)p-\ell(\mu))}$$

$$= \frac{1}{k} \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm((b-1)p;\hbar-k-z;h)}.$$

We prove (vi). By the definition (4.7.2), we have

$$(b - p + k)\mathcal{X}_t^{\pm(b;p;k;h)} = k \mathcal{X}_t^{\pm(b;p;k;h)} + \sum_{\lambda \vdash k} \ell(\lambda) \mathcal{X}_t^{\pm((\lambda);h)} \mathcal{X}_t^{\pm((b-p-\ell(\lambda))} + \sum_{\lambda \vdash k} (b - p - \ell(\lambda)) \mathcal{X}_t^{\pm((\lambda);h)} \mathcal{X}_t^{\pm((b-p-\ell(\lambda))}.$$
Note that \( \ell(\lambda) = \sum_{z \geq 1} m_z(\lambda) \), \((b - p - \ell(\lambda)) \mathcal{A}_t^{\pm(b-p-\ell(\lambda))} = \mathcal{A}_t^{\pm(b-p-\ell(\lambda)-1)} \) and the defining relation (L4), we have

\[
(b - p + k) \mathcal{A}_t^{\pm(b,p,k)} = k \mathcal{A}_t^{\pm(b,p,k)} + \sum_{\lambda \leq k} \left( \sum_{z \geq 1} m_z(\lambda) \right) \mathcal{A}_t^{\pm(\lambda,k)} \mathcal{A}_t^{\pm(b-\ell(\lambda))} + \mathcal{A}_t^{\pm \lambda \mathcal{A}_t^{\pm(\lambda,k)}} + \mathcal{A}_t^{\pm \lambda \mathcal{A}_t^{\pm(b-1-p-\ell(\lambda))}}.
\]

Then, by (v) and the definition (4.7.2), we have

\[
(b - p + k) \mathcal{A}_t^{\pm(b,p,k)} = k \mathcal{A}_t^{\pm(b,p,k)} + \sum_{z = 1}^{k} \mathcal{A}_t^{\pm((z,k)-p)} \mathcal{A}_t^{\pm(b-1;p,k-z;k)} + \mathcal{A}_t^{\pm \lambda \mathcal{A}_t^{\pm(b-1-p-\ell(\lambda))}}.
\]

Then, by (v) and the definition (4.7.2), we have

\[
(b - p + k) \mathcal{A}_t^{\pm(b,p,k)} = \mathcal{A}_t^{\pm(b-1;p,k)} + \sum_{z = 1}^{k} (z+1) \mathcal{A}_t^{\pm((z,k)-p)} \mathcal{A}_t^{\pm(b-1;k-z;k)}.
\]

\[\square\]

**Lemma 4.9.** For \( s, t, c, p, k \in \mathbb{Z}_{\geq 0} \), we have

\[
(4.9.1)\quad [\mathcal{A}_t^{b,c} \mathcal{A}_s^{-(c,p,k,s+t)}] = \sum_{z=0}^{k} \sum_{w=0}^{k-z} \binom{k-z}{w} (-Q)^w \mathcal{A}_s^{-(c,p+1|z,s+t)} \mathcal{A}_s^{(1)}(k-z+1)(s+t)+w - (k+1) \mathcal{A}_s^{-(c,p+1|k+1,s+t)}.
\]

**Proof.** If \( c = 0 \), the equation (4.9.1) follows from Lemma 4.8 (i) and (iii). Then, we prove (4.9.1) by the induction on \( k \) in the case where \( c > 0 \).

If \( k = 0 \), we see that (4.9.1) is just the formula in Lemma 4.8 by Lemma 4.8 (ii).

If \( k > 0 \), by Lemma 4.8 (v) and the defining relation (L4), we have

\[
\mathcal{A}_t^{c,p,k} \mathcal{A}_s^{-(c-p,k,s+t)} = \frac{1}{k} \sum_{z=1}^{k} z \mathcal{A}_t^{c,p,k} \mathcal{A}_s^{(c-1,p,k-z,s+t)} \mathcal{A}_s^{-(z,s+t)}.
\]

Applying the assumption of the induction, we have

\[
\mathcal{A}_t^{c,p,k} \mathcal{A}_s^{-(c,p,k,s+t)}
\]
Applying Lemma 4.4 and Lemma 4.5 (ii), we have

\[
\begin{align*}
\mathcal{X}_s^{-(c-1;p(k-1+z);s+t)} & = 1 \\
\sum_{y=0}^{k-z} \mathcal{X}_s^{-(c-1;p+1|y;s+t)} & \left( \sum_{w=0}^{k-z-y} (-Q)^w \mathcal{J}^{(1)}_{(k-z-y+1)(s+t) + w} \right) \\
-(k - z + 1) \mathcal{X}_s^{-(c-1;p+1|k-z+1;s+t)} & \mathcal{X}_s^{-(z);s+t}.
\end{align*}
\]

Applying Lemma 4.3 and Lemma 4.5 (ii), we have

\[
\begin{align*}
\mathcal{X}_s^{-(c;p|k;s+t)} + \mathcal{X}_s^{-(c-1;p|k-z;z+t)} & = \frac{1}{k} \sum_{z=1}^{k} \mathcal{X}_s^{-(c-1;p|k-z;z+t)} \mathcal{X}_s^{-(z);s+t} + \sum_{w=0}^{z} \left( \sum_{w=0}^{k-z-y} (-Q)^w \mathcal{J}^{(1)}_{(z+1)(s+t) + w} \right) \\
+ \frac{1}{k} \sum_{z=1}^{k} \sum_{y=0}^{k-z} \mathcal{X}_s^{-(c-1;p+1|y;s+t)} & \left( \sum_{w=0}^{k-z-y} (-Q)^w \mathcal{J}^{(1)}_{(k-z-y+1)(s+t) + w} \right) \\
& \times \left( \mathcal{X}_s^{-(z);s+t} \mathcal{J}^{(1)}_{(k-z-y+1)(s+t) + w} - 2 \mathcal{X}_s^{-(z+1);s+t} \mathcal{J}^{(1)}_{s+(k-z-y)(s+t) + w} \right) \\
& - \frac{1}{k} \sum_{z=1}^{k} (k - z + 1) \mathcal{X}_s^{-(c-1;p+1|k-z+1;s+t)} \mathcal{X}_s^{-(z);s+t}.
\end{align*}
\]

Put

\[
\begin{align*}
(*1) & = \frac{1}{k} \sum_{z=1}^{k} \mathcal{X}_s^{-(c-1;p|k-z;z+t)} \mathcal{X}_s^{-(z);s+t} \mathcal{X}_t^{+} \\
(*2) & = \frac{1}{k} \sum_{z=1}^{k} \mathcal{X}_s^{-(c-1;p|k-z;z+t)} \sum_{w=0}^{z} \left( \sum_{w=0}^{k-z-y} (-Q)^w \mathcal{J}^{(1)}_{(z+1)(s+t) + w} \right) \\
(*3) & = \frac{1}{k} \sum_{z=1}^{k} \sum_{y=0}^{k-z} \mathcal{X}_s^{-(c-1;p+1|y;s+t)} \sum_{w=0}^{k-z-y} \left( \sum_{w=0}^{k-z-y} (-Q)^w \mathcal{J}^{(1)}_{(k-z-y+1)(s+t) + w} \right) \\
(*4) & = \frac{1}{k} \sum_{z=1}^{k} \sum_{y=0}^{k-z} \mathcal{X}_s^{-(c-1;p+1|y;s+t)} \sum_{w=0}^{k-z-y} \left( \sum_{w=0}^{k-z-y} (-Q)^w \mathcal{J}^{(1)}_{s+(k-z-y)(s+t) + w} \right) \\
(*5) & = \frac{1}{k} \sum_{z=1}^{k} (k - z + 1) \mathcal{X}_s^{-(c-1;p+1|k-z+1;s+t)} \mathcal{X}_s^{-(z);s+t},
\end{align*}
\]

then we have

\[
(4.9.2) \quad \mathcal{X}_t^{+} \mathcal{X}_s^{-(c;p|k;s+t)} = (*1) + (*2) + (*3) - 2(*4) - (*5).
\]
By Lemma 4.8 (v) together with (L4), we have

\[(4.9.3)\]
\[(*1) = \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{X}_t^+.
\]

Put \(z' = k - z\) in \((*)2\) and apply Lemma 4.8 (iv), we have

\[(4.9.4)\]
\[(*2) = \frac{1}{k} \sum_{z'=0}^{k-1} (k - z') \mathcal{X}_s^{-(c;p+1|z';s+t)} \sum_{w=0}^{k-z'} \binom{k - z'}{w} (-Q)^w \mathcal{J}^{(1)}_{(k-z'+1)(s+t)+w}.
\]

Put \(h = z + y\) in \((*)3\), we have

\[(4.9.5)\]
\[(*3) = \frac{1}{k} \sum_{h=1}^{k} \sum_{z=1}^{h} \mathcal{X}_s^{-(c-1;p+1|h-z;s+t)} \sum_{w=0}^{k-h} \binom{k - h}{w} (-Q)^w \mathcal{J}^{(1)}_{(k-h+1)(s+t)+w}.
\]

Applying Lemma 4.8 (v) together with (L4), we have

\[(4.9.6)\]
\[(*3) = \sum_{z=0}^{k} \sum_{w=0}^{k-z} \binom{k - z}{w} (-Q)^w \mathcal{X}_s^{-(c;p+1|z;s+t)} \mathcal{J}^{(1)}_{(k-z+1)(s+t)+w}.
\]

By \((4.9.4)\) and \((4.9.5)\), we have

\[(4.9.6)\]
\[(*2) + (*3) = \sum_{z=0}^{k} \sum_{w=0}^{k-z} \binom{k - z}{w} (-Q)^w \mathcal{X}_s^{-(c;p+1|z;s+t)} \mathcal{J}^{(1)}_{(k-z+1)(s+t)+w}.
\]

We also have

\[(4.9.7)\]
\[(*4) = \frac{1}{k} \sum_{y=0}^{k-y} \sum_{z=1}^{k-y} \mathcal{X}_s^{-(c-1;p+1|y;s+t)} \sum_{w=0}^{k-z-y} \binom{k - z - y}{w} (-Q)^w \mathcal{X}_s^{-(z+1;s+t)} \mathcal{J}^{(1)}_{(k-z-y)(s+t)+w}.
\]

Put \(h = k - y + 1\), we have

\[(4.9.7)\]
\[(*4) = \frac{1}{k} \sum_{h=2}^{k+1} \sum_{z=1}^{h-1} \mathcal{X}_s^{-(c-1;p+1|h+1;s+t)} \sum_{w=0}^{h-z-1} \binom{h - z - 1}{w} (-Q)^w \mathcal{X}_s^{-(z+1;s+t)} \mathcal{J}^{(1)}_{(h-z-1)(s+t)+w}.
\]

Put

\[(\sharp) = \sum_{w=0}^{h-z-1} \binom{h - z - 1}{w} (-Q)^w \mathcal{X}_s^{-(z+1;s+t)} \mathcal{J}^{(1)}_{(h-z-1)(s+t)+w}.
\]
By (4.2.1), we have
\[
\# = \sum_{w=0}^{h-z-1} \left( \frac{h-z-1}{w} \right) (-Q)^w \sum_{y=0}^{z+1} \left( \frac{z+1}{y} \right) (-Q)^y \mathcal{X}_{s+h(s+t)+w+y}^{-}.
\]

Put \( y' = w + y \), we have
\[
\# = \sum_{y'=0}^{h} \left( \frac{h}{y'} \right) \left( \frac{h-(z+1)}{w} \right) \left( \frac{z+1}{y' - w} \right) (-Q)^{y'} \mathcal{X}_{s+h(s+t)+y'}^{-}.
\]

Note that \( \sum_{w=\max\{0,y'-(z+1)\}}^{\min\{h-(z+1),y'\}} \left( \frac{h-(z+1)}{w} \right) \left( \frac{z+1}{y' - w} \right) = \left( \frac{h}{y'} \right) \) by (4.3.4), we have
\[
\# = \sum_{y'=0}^{h} \left( \frac{h}{y'} \right) (-Q)^{y'} \mathcal{X}_{s+h(s+t)+y'}^{-} = \mathcal{X}_{s}^{-}(h(s+t)).
\]

(Use (4.2.1) again.) Then, we have
\[
(*) = \frac{1}{k} \sum_{h=2}^{k+1} \left( \sum_{z=1}^{h-1} \mathcal{X}_{s}^{-}(c-1;p+1|k-h+1;s+t) \mathcal{X}_{s}^{-}(h(s+t)) \right)\mathcal{X}_{s}^{-}(s+1)\mathcal{X}_{s}^{-}(h(s+t)).
\]

Then we have
\[
2(*) + (*) = \sum_{z=1}^{k+1} z \mathcal{X}_{s}^{-}(c-1;p+1|k-z+1;s+t) \mathcal{X}_{s}^{-}(z(s+t)) = (k+1) \mathcal{X}_{s}^{-}(c;p+1|k+1;s+t),
\]

where the last equation follows from Lemma 4.8 (v).

By (4.9.2), (4.9.3), (4.9.6) and (4.9.7), we have
\[
\mathcal{X}_{s}^{+} \mathcal{X}_{s}^{-}(c;p|k;s+t) = \mathcal{X}_{s}^{-}(c;p|k+1;s+t) \mathcal{X}_{s}^{+} + \sum_{k=1}^{k-1} \sum_{z=0}^{k-z} \left( \frac{k-z}{w} \right) (-Q)^w \mathcal{X}_{s}^{-}(c;p+1|z;s+t) \mathcal{J}_{(k-z+1)(s+t)+w}^{(1)}
- (k+1) \mathcal{X}_{s}^{-}(c;p+1|k+1;s+t).
\]
\[
\square
\]
Proposition 4.10. For \( s, t, b, c \in \mathbb{Z}_{\geq 0} \), we have

\[
(4.10.1) \quad [\mathcal{X}_t^{+(b)} \mathcal{X}_s^{-(c)}] = \sum_{p=1}^{\min\{b, c\}} \sum_{k=0}^{p-k} \sum_{l=0}^{p-k} (-1)^{k+l} \mathcal{X}_s^{-(c-p-k; s+t)} \mathcal{J}_{s+t}^{(p-(k+l))} \mathcal{X}_t^{+(b-p; l;s+t)}.
\]

Proof. We prove (4.10.1) by the induction on \( b \). If \( b = 1 \), (4.10.1) follows from Lemma 4.6 together with Lemma 4.8.

If \( b > 1 \), we have

\[
\mathcal{X}_t^{+(b)} \mathcal{X}_s^{-(c)}
= \frac{1}{b} \mathcal{X}_t^{+(b-1)} \mathcal{X}_s^{-(c)}
= \frac{1}{b} \mathcal{X}_t^{+(b-1)} \left( \mathcal{X}_s^{-(c)} \mathcal{J}_{s+t}^{(1)} - \mathcal{X}_s^{-(c-2)} \mathcal{J}_{s+t}^{(1)} \right) \mathcal{X}_t^{+(b-1)}
\]

by the assumption of the induction. Applying Lemma 4.6 and Lemma 4.9 we have

\[
\mathcal{X}_t^{+(b)} \mathcal{X}_s^{-(c)}
= \frac{1}{b} \left\{ \left( \mathcal{X}_s^{-(c)} \mathcal{X}_t^{+(b-1)} - \mathcal{X}_s^{-(c-2)} \mathcal{X}_t^{+(b-1)} \right) \mathcal{J}_{s+t}^{(1)} \right\}
\]

On the other hand, by Lemma 4.3 we have

\[
\mathcal{X}_s^{-(c-p; k; s+t)} \mathcal{X}_t^{+(b-1; p; l; s+t)}
= \mathcal{X}_s^{-(c-p; k; s+t)} \left( \mathcal{J}_{s+t}^{(p-(k+l))} \mathcal{X}_t^{+(b-1; p; l; s+t)} \right)
\]

Put

\[
(*1) = b \mathcal{X}_s^{-(c)} \mathcal{X}_t^{+(b)} + \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} \mathcal{X}_t^{+(b-1)} - \mathcal{X}_s^{-(c-2)} \mathcal{J}_{s+t}^{(1)} \mathcal{X}_t^{+(b-1)},
\]

\[
(*2) = \sum_{p=1}^{\min\{b, c\}} \sum_{k=0}^{p-k} \sum_{l=0}^{p-k} (-1)^{k+l} \mathcal{X}_s^{-(c-p-k; s+t)} \mathcal{J}_{s+t}^{(p-(k+l))} \mathcal{X}_t^{+(b-1; p; l; s+t)}.
\]
\[ (*)_3 = \sum_{p=1}^{\min\{b,c\}} \sum_{p-k=0}^{p-(k+l)} \sum_{l=0}^{z=1} (-1)^{k+l+z}(z+1) \]
\[ \times \mathcal{X}_s^{-(c; p; k; s+t)} \mathcal{J}_{s+t}^{(p-(k+l)-z)} \mathcal{X}_t^{+((z); s+t)} \mathcal{X}_t^{+(b-1; p; l; s+t)}, \]
\[ (*)_4 = \sum_{p=1}^{\min\{b,c\}} \sum_{p-k=0}^{p-(k+l)} \sum_{k-z=0}^{z=1} \sum_{l-w=0}^{w=0} (-1)^{k+l} \left( \frac{k-z}{w} \right)(-Q)^w \]
\[ \times \mathcal{X}_s^{-(c+1; p+1; s+t)} \mathcal{J}_{s+t}^{(p-(k+l))} \mathcal{X}_t^{+(b-1; p; l; s+t)}, \]
\[ (*)_5 = \sum_{p=1}^{\min\{b,c\}} \sum_{p-k=0}^{p-(k+l)} \sum_{l=0}^{z+1} (-1)^{k+l+1} (k+1) \mathcal{X}_s^{-(c; p+1; k+1; s+t)} \mathcal{J}_{s+t}^{(p-(k+l))} \mathcal{X}_t^{+(b-1; p; l; s+t)}, \]

then we have
\[ (4.10.2) \quad \mathcal{X}_t^{+(b)} \mathcal{X}_s^{-(c)} = \frac{1}{b} \{ (*)_1 + (*)_2 + (*)_3 + (*)_4 + (*)_5 \}, \]

where we note that \( \mathcal{X}_t^{+(b-1; p; l; s+t)} = 0 \) if \( p = b \) by Lemma 4.8 (i).

By Lemma 4.8 (ii) together with (L4), we have
\[ (4.10.3) \quad (*)_1 = b \mathcal{X}_s^{-(c)} \mathcal{X}_t^{+(b)} + \mathcal{X}_s^{-(c; 1; l; s+t)} \mathcal{J}_{s+t}^{(p)} \mathcal{X}_t^{+(b; 1; l; s+t)} = \mathcal{X}_s^{-(c; 1; l; s+t)} \mathcal{J}_{s+t}^{(p)} \mathcal{X}_t^{+(b; 1; l; s+t)}. \]

Put \( l' = l + z \) in \( (*)_3 \), we have
\[ (*)_3 = \sum_{p=1}^{\min\{b,c\}} \sum_{k-0}^{p-k} \sum_{l'=0}^{l'} \sum_{z=1} (-1)^{k+l'}(z+1) \mathcal{X}_s^{-(c; p; k; s+t)} \mathcal{J}_{s+t}^{(p-k-l')} \mathcal{X}_t^{+((z); s+t)} \mathcal{X}_t^{+(b-1; p; l'-z; s+t)}. \]

Then we have
\[ (*)_2 + (*)_3 = \sum_{p=1}^{\min\{b,c\}} \sum_{k-0}^{p-k} \sum_{l=0}^{z=1} (-1)^{k+l} \mathcal{X}_s^{-(c; p; k; s+t)} \mathcal{J}_{s+t}^{(p-(k+l))} \]
\[ \times \left( \mathcal{X}_t^{+(b-1; p; l; s+t)} + \sum_{l'=1}^{l} (z+1) \mathcal{X}_t^{+((z); s+t)} \mathcal{X}_t^{+(b-1; p; l'-z; s+t)} \right), \]
where we note that \( \sum_{z=1}^{0} \mathcal{X}_t^{+(z)} \mathcal{X}_t^{+(b-1:p[l-z];s+t)} = 0 \). Applying Lemma \[4.8\] (vi), we have

\[
(4.10.4)
\]

\[
(\ast 2) + (\ast 3) = \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^{p-k} \sum_{l=0}^{p-k} (-1)^{k+l} (b - p + l) \mathcal{X}_s^{-(c;p[k];s+t)} \mathcal{J}_{s+t}^{(p-(k+l))} \mathcal{X}_t^{+(b;p[l];s+t)} \\
= (b - 1) \mathcal{X}_s^{-(c;1[l];s+t)} \mathcal{J}_{s+t}^{(1)} \mathcal{X}_t^{+(b;1[l];s+t)} - b \mathcal{X}_s^{-(c;1[l];s+t)} \mathcal{J}_{s+t}^{(0)} \mathcal{X}_t^{+(b;1[l];s+t)} \\
- (b - 1) \mathcal{X}_s^{-(c;1[l];s+t)} \mathcal{J}_{s+t}^{(0)} \mathcal{X}_t^{+(b;1[l];s+t)} \\
+ \sum_{p=2}^{\min\{b,c\}} \sum_{k=2}^{p-k} \sum_{l=0}^{p-k} (-1)^{k+l} (b - p + l) \mathcal{X}_s^{-(c;p[k];s+t)} \mathcal{J}_{s+t}^{(p-(k+l))} \mathcal{X}_t^{+(b;p[l];s+t)}.
\]

Put \( p' = p + 1 \) in \((\ast 4)\), we have

\[
(\ast 4) = \sum_{p'=2}^{\min\{b,c\}} \sum_{k=2}^{p'-k-1} \sum_{l=0}^{p'-k-1} \sum_{z=0}^{k-z} (-1)^{k+l} \left( \binom{k-z}{w} \right) (-Q)^w \\
\times \mathcal{X}_s^{-(c;p'[z];s+t)} \mathcal{J}_{(k-z+1)[s+t]+w}^{(p'-(k+l)-1)} \mathcal{X}_t^{+(b-1;p'[-1];s+t)},
\]

where we note that \( \mathcal{X}_t^{+(b-1;p'[-1];s+t)} = 0 \) if \( p' = b + 1 \), and \( \mathcal{X}_s^{-(c;p'[z];s+t)} = 0 \) if \( p' = c + 1 \) by Lemma \[4.8\] (i). Note that

\[
\sum_{k=0}^{p-1} \sum_{l=0}^{p-k-1} \sum_{z=0}^{k-z} = \sum_{k=0}^{p-1} \sum_{z=0}^{k-z} \sum_{l=0}^{p-k-1} = \sum_{z=0}^{p-1} \sum_{l=0}^{p-k-1} \sum_{k=0}^{z},
\]

we have

\[
(\ast 4) = \sum_{p=2}^{\min\{b,c\}} \sum_{k=0}^{p-1} \sum_{z=0}^{p-k-1} \mathcal{X}_s^{-(c;p[z];s+t)} \\
\times \left( \sum_{k=0}^{p-1} \sum_{w=0}^{k-z} (-1)^{k+l} \left( \binom{k-z}{w} \right) (-Q)^w \mathcal{J}_{(k-z+1)[s+t]+w}^{(p-(k+l)-1)} \right) \mathcal{X}_t^{+(b-1;p[-1];s+t)}.
\]

Put \( k' = k - z + 1 \), we have

\[
\sum_{k=0}^{p-1} (-1)^{k+l} \left( \binom{k}{w} \right) (-Q)^w \mathcal{J}_{(k-z+1)[s+t]+w}^{(p-(k+l)-1)} \\
\sum_{k=0}^{p-1} (-1)^{k'+z+l-1} \left( \binom{k'-1}{w} \right) (-Q)^w \mathcal{J}_{k'[s+t]+w}^{(p-k'-z-1)}.
\]
Applying Lemma 4.8 (iv), we have

\[ \sum_{w=0}^{k'-1} \binom{k'-1}{w} (-Q)^w \mathcal{J}^{(1)}_{k'(s+t)+w} = \sum_{w=0}^{k'} \binom{k'}{w} (-Q)^w \mathcal{J}_{k'(s+t)+w}. \]

Thus we have

\[
\begin{aligned}
&\sum_{k'=	ext{fin}} \sum_{w=0}^{p-1} (-1)^{k'+l} \binom{k' - z}{w} (-Q)^w \mathcal{J}^{(1)}_{(k'-z+1)(s+t)+w} \mathcal{J}_{s+t}^{(p-(k+l)-1)} \\
= (-1)^{z+l} \sum_{k'=1}^{p-1} \sum_{w=0}^{k'-1} (-1)^{k'-1} \binom{k'}{w} (-Q)^w \mathcal{J}_{k'(s+t)+w} \mathcal{J}_{s+t}^{(p-k'-z-l)} \\
= (-1)^{z+l}(p - l - z) \mathcal{J}_{s+t}^{(p-l-z)},
\end{aligned}
\]

where the last equation follows from (4.2.2). Then we have

\[
\sum_{p=2}^{\min\{b,c\}} \sum_{z=0}^{p-2} \sum_{l=0}^{p-z} (p - l - z)(-1)^{z+l} \mathcal{X}_{s}^{-(c;p|z;s+t)} \mathcal{J}_{s+t}^{(p-l-z)} \mathcal{X}_{t}^{+(b-1;p-1|l;s+t)}.
\]

Applying Lemma 4.8 (iv), we have

\[
(4.10.5) \quad (\ast4) = \sum_{p=2}^{\min\{b,c\}} \sum_{z=0}^{p-1} \sum_{l=0}^{p-z} (p - l - z)(-1)^{z+l} \mathcal{X}_{s}^{-(c;p|z;s+t)} \mathcal{J}_{s+t}^{(p-l-z)} \mathcal{X}_{t}^{+(b;p|l;s+t)}.
\]

Put \( p' = p + 1 \) in (\ast5), we have

\[
(\ast5) = \sum_{p'=2}^{\min\{b,c\}} \sum_{k=0}^{p'-1} \sum_{l=0}^{p'-k-1} (1)_{k+l-1} \mathcal{X}_{s}^{+(b-1;p'-1|l;s+t)} \mathcal{J}_{s+t}^{(p'-k-1)} \mathcal{X}_{t}^{+(b-1;p'-1|l;s+t)},
\]

where we note that \( \mathcal{X}_{t}^{+(b-1;p'-1|l;s+t)} = 0 \) if \( p' = b + 1 \), and \( \mathcal{X}_{s}^{-(c;p'|k+1;s+t)} = 0 \) if \( p' = c + 1 \) by Lemma 4.8 (i). Put \( k' = k + 1 \), we have

\[
(\ast5) = \sum_{p'=2}^{\min\{b,c\}} \sum_{k'=1}^{p'-k'} \sum_{l=0}^{k'-1} (1)_{k+l} \mathcal{X}_{s}^{-(c;p'k';s+t)} \mathcal{J}_{s+t}^{(p'-k'-l)} \mathcal{X}_{t}^{+(b-1;p'-1|l;s+t)}
\]

\[
= \sum_{p=2}^{\min\{b,c\}} \sum_{k=0}^{p-1} \sum_{l=0}^{p-k} \sum_{k'=1}^{l} (1)_{k+l} \mathcal{X}_{s}^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{(p-k-l)} \mathcal{X}_{t}^{+(b-1;p-1|l;s+t)}.
\]
Applying Lemma 4.8 (iv), we have

\[(4.10.6) \quad \star 5 \quad \langle s \rangle = \sum_{p=2}^{\min \{b, c\}} \sum_{k=0}^{p-1} \sum_{l=0}^{t-1} k(1)^{k+l} \alpha^{-c;b(pk; s;t)} \mathcal{J}_{s+t}^{(p-k-1)} \alpha^{+b(pk; s;t)}.
\]

By (4.10.2), (4.10.3), (4.10.4), (4.10.5) and (4.10.6), we have

\[
\alpha^+(b) \alpha^-(c) = \alpha^-(c) \alpha^+(b) + \sum_{p=1}^{\min \{b, c\}} \sum_{k=0}^{p-1} \sum_{l=0}^{t-1} (-1)^{k+l} \alpha^{-c;b(pk; s;t)} \mathcal{J}_{s+t}^{(p-k+l)} \alpha^{+b(pk; s;t)}.
\]

\[\square\]

§ 5. Rank 1 case; finite dimensional simple modules of \(U(\mathfrak{sl}_2^Q[x])\)

In this section, we classify the finite dimensional simple \(U(\mathfrak{sl}_2^Q[x])\)-modules.

5.1. 1-dimensional representations. First, we consider 1-dimensional representations of \(\mathfrak{sl}_2^Q[x]\). Let \(L = \mathbb{C}v\) be a 1-dimensional \(U(\mathfrak{sl}_2^Q[x])\)-module with a basis \(\{v\}\), then \(\mathcal{J}_t (t \geq 0)\) acts on \(v\) as a scalar multiplication. If \(\alpha^+_t \cdot v \neq 0\) (resp. \(\alpha^-_t \cdot v \neq 0\)), then \(\alpha^+_t \cdot v\) (resp. \(\alpha^-_t \cdot v\)) is an eigenvector for the action of \(\mathcal{J}_t\) whose eigenvalue is different from one of \(v\) by the defining relation (L2). This is a contradiction since \(L\) is 1-dimensional. Thus, we have \(\alpha^\pm_t \cdot v = 0\) for \(t \geq 0\). Moreover, by the defining relation (L3), we have \((\mathcal{J}_t - Q \mathcal{J}_{t+1}) \cdot v = (\alpha^+_t \alpha^-_t - \alpha^-_t \alpha^+_t) \cdot v = 0\).

This implies that \(\mathcal{J}_t \cdot v = 0\) for \(t \geq 0\) if \(Q = 0\), and that \(\mathcal{J}_t \cdot v = Q^{-t} \mathcal{J}_0 \cdot v\) for \(t > 0\) if \(Q \neq 0\).

We define the set \(\mathbb{B}^Q\) by

\[
\mathbb{B}^Q = \begin{cases} 
\{0\} & \text{if } Q = 0, \\
\mathbb{C} & \text{if } Q \neq 0.
\end{cases}
\]

For each \(\beta \in \mathbb{B}^Q\), we can define the 1-dimensional \(U(\mathfrak{sl}_2^Q[x])\)-module \(L^\beta = \mathbb{C}v_0\) such that

\[
\alpha^\pm_t \cdot v_0 = 0, \quad \mathcal{J}_t \cdot v_0 = \begin{cases} 
0 & \text{if } Q = 0, \\
Q^{-t} \beta v_0 & \text{if } Q \neq 0 \quad (t \in \mathbb{Z}_{\geq 0})
\end{cases}
\]

by checking the defining relations of \(\mathfrak{sl}_2^Q[x]\). Note that \(L^0\) is the trivial representation. Now we obtain the following lemma.

Lemma 5.2. Any 1-dimensional \(U(\mathfrak{sl}_2^Q[x])\)-module is isomorphic to \(L^\beta\) for some \(\beta \in \mathbb{B}^Q\).

5.3. Recall from §2 a finite dimensional simple \(U(\mathfrak{sl}_2^Q[x])\)-module is isomorphic to a simple highest weight module \(L(u)\) for some highest weight \(u = (u_t) \in \prod_{t \geq 0} \mathbb{C}\) (Proposition 2.6), where we omit the first index for the highest weight. Then, in
order to classify the finite dimensional simple \( U(\mathfrak{sl}_2^{(Q)}[x]) \)-module, it is enough to classify the highest weight \( u \) such that \( \mathcal{L}(u) \) is finite dimensional.

In order to obtain a necessary condition for \( u \) such that \( \mathcal{L}(u) \) is finite dimensional, we prepare the following lemma.

**Lemma 5.4.** Let \( M \) be a finite dimensional \( U(\mathfrak{sl}_2^{(Q)}[x]) \)-module. Take an element \( v \in M \) satisfying

\[
X_t^+ \cdot v = 0, \quad \mathcal{J}_t \cdot v = u_t v \quad (t \in \mathbb{Z}_{\geq 0}), \quad X_0^{-}(n) \cdot v \neq 0 \text{ and } X_0^{-(n+1)} \cdot v = 0
\]

for some \( u_t \in \mathbb{C} \) (\( t \in \mathbb{Z}_{\geq 0} \)) and \( n \in \mathbb{Z}_{\geq 0} \). (In fact, a such element exists by Lemma 2.3.) Then, for \( s, t \in \mathbb{Z}_{\geq 0} \), we have

\[
\sum_{w=0}^{n} \binom{n}{w} (-Q)^w \mathcal{J}_{t+ns+w}^{(1)} \cdot v = \sum_{k=0}^{n-1} (-1)^{n-k+1} \left( \sum_{w=0}^{k} \binom{k}{w} (-Q)^w \mathcal{J}_{t+kw+w}^{(1)} \right) \mathcal{J}_s^{(n-k)} \cdot v.
\]

**Proof.** By the assumption \( X_0^{-(n+1)} \cdot v = 0 \) and Proposition 4.10 we have

\[
0 = X_s^{+(n)} \mathcal{J}_0^{-(n+1)} \cdot v = \left( X_0^{-(n+1)} X_s^{+(n)} + \sum_{p=1}^{n} \sum_{k=0}^{p} \sum_{l=0}^{p-k} (-1)^{k+l} X_0^{-(n+1;p(k);s)} \mathcal{J}_s^{(p-k)} \mathcal{J}_s^{-(n-k)} \right) \cdot v.
\]

By the definition, we have \( X_s^{+(n;p;l;s)} = \sum_{\lambda \vdash l} \mathcal{J}_s^{+(\lambda;r)} \mathcal{J}_s^{-(n-p-\ell(\lambda))} \). Thus, by the definition of \( X_s^{+(\lambda;r)} \) and the assumption \( X_t^+ \cdot v = 0 \ (t \geq 0) \), we have

\[
X_s^{+(n;p;l;s)} \cdot v = \begin{cases} v & \text{if } l = 0 \text{ and } p = n, \\ 0 & \text{otherwise.} \end{cases}
\]

Then (5.4.1) implies that

\[
0 = \sum_{k=0}^{n} (-1)^k X_0^{-(n+1;n;k;s)} \mathcal{J}_s^{(n-k)} \cdot v.
\]

By the definition, we have

\[
X_0^{-(n+1;n;k;s)} = \sum_{\lambda \vdash k} \mathcal{J}_s^{-(\lambda;r)} \mathcal{J}_s^{-(1-\ell(\lambda))} = \begin{cases} \mathcal{J}_s^{-} & \text{if } k = 0, \\ \mathcal{J}_s^{-(\lambda;r)} & \text{if } k \neq 0. \end{cases}
\]
Thus, we have

\[ 0 = X_0^{-} J_s^{(n)} \cdot v + \sum_{k=1}^{n} (-1)^k X_0^{-((k):s)} J_s^{(n-k)} \cdot v. \]

By multiplying \( X_0^+ \) from left to this equation, we have

\[ 0 = X_0^+ X_0^{-} J_s^{(n)} \cdot v + \sum_{k=1}^{n} (-1)^k X_0^+ X_0^{-((k):s)} J_s^{(n-k)} \cdot v \]

\[ = \sum_{k=0}^{n} (-1)^k \left( \sum_{w=0}^{k} \binom{k}{w} (-Q)^w J_{t+ks+w} \right) J_s^{(n-k)} \cdot v, \]

where we use Lemma 4.4 and the fact \( X_0^+ J_s^{(n-k)} \cdot v = 0 \). This implies the Lemma. \( \square \)

This Lemma implies the following proposition which gives a necessary condition for \( u \) such that \( \mathcal{L}(u) \) is finite dimensional.

**Proposition 5.5.** Let \( M \) be a finite dimensional \( U(\mathfrak{sl}_2^{(Q)}[x]) \)-module. Take an element \( v \in M \) satisfying

\[ X_0^+ \cdot v = 0, \quad J_t \cdot v = u_t v \quad (t \in \mathbb{Z}_{\geq 0}), \quad X_0^{-} J_s^{(n)} \cdot v \neq 0 \text{ and } X_0^{-} J_s^{(n+1)} \cdot v = 0 \]

for some \( u_t \in \mathbb{C} \) \( (t \in \mathbb{Z}_{\geq 0}) \) and \( n \in \mathbb{Z}_{\geq 0} \).

(i) If \( Q = 0 \), we have \( u_0 = n \), and there exist \( \gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{C} \) such that

\[ u_t = p_t(\gamma_1, \gamma_2, \ldots, \gamma_n) \quad (t > 0), \]

where \( p_t(\gamma_1, \ldots, \gamma_n) = \gamma_1^t + \gamma_2^t + \cdots + \gamma_n^t \).

(ii) If \( Q \neq 0 \), there exist \( \beta, \gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{C} \) such that

\[ u_0 = n + \beta \text{ and } u_t = p_t(\gamma_1, \gamma_2, \ldots, \gamma_n) + Q^{-t} \beta \quad (t > 0), \]

where \( p_t(\gamma_1, \ldots, \gamma_n) = \gamma_1^t + \gamma_2^t + \cdots + \gamma_n^t \).

**Proof.** (i). Assume that \( Q = 0 \). Then, \( \mathfrak{sl}_2^{(0)}[x] \) coincides with the current Lie algebra \( \mathfrak{sl}_2[x] \) of \( \mathfrak{sl}_2 \). Moreover, the Lie subalgebra of \( \mathfrak{sl}_2[x] \) generated by \( X_0^+ \) and \( J_0 \) is isomorphic to \( \mathfrak{sl}_2 \). Thus, by the representation theory of \( \mathfrak{sl}_2 \), we have \( u_0 = n \).

For \( u_1, \ldots, u_n \), there exist \( \gamma_1, \ldots, \gamma_n \in \mathbb{C} \) such that

\[ u_k = p_k(\gamma_1, \ldots, \gamma_n) \quad \text{for } k = 1, \ldots, n \]

by Lemma A.2.
By the definition, we have
\[ J^{(k)}_1 = \frac{1}{k} \sum_{z=1}^{k} (-1)^{z-1} J_z J^{(k-z)}_1 \tag{5.5.2} \]
since we assume \( Q = 0 \). By the induction on \( k \) together with (5.5.1), (5.5.2) and (A.1.1), we can show that
\[ J^{(k)}_1 v = e_k(\gamma_1, \ldots, \gamma_n) v \text{ for } k = 1, \ldots, n, \tag{5.5.3} \]
where \( e_k(\gamma_1, \ldots, \gamma_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_k} \).

By the induction on \( t \), we prove that
\[ u_t = p_t(\gamma_1, \ldots, \gamma_n) \quad (t > 0). \tag{5.5.4} \]

If \( t \leq n \), (5.5.4) follows from (5.5.1). If \( t > n \), by Lemma 5.4 in the case where \( s = 1 \), we have
\[ u_t v = \sum_{k=0}^{n-1} (-1)^{n-k+1} J_{(t-n)+k} J^{(n-k)}_1 v. \tag{5.5.5} \]

By the assumption of the induction together with (5.5.3) and (A.1.2), we have
\[ u_t v = \sum_{k=0}^{n-1} (-1)^{n-k+1} p_{t-n+k}(\gamma_1, \ldots, \gamma_n) e_{n-k}(\gamma_1, \ldots, \gamma_n) v = p_t(\gamma_1, \ldots, \gamma_n) v. \]

(ii). Assume that \( Q \neq 0 \). For \( u_0, u_1, \ldots, u_n \), there exist \( \beta, \gamma_1, \ldots, \gamma_n \in \mathbb{C} \) such that
\[ u_0 = n + \beta \text{ and } u_k = p_k(\gamma_1, \ldots, \gamma_n) + Q^{-k} \beta \text{ for } k = 1, \ldots, n \tag{5.5.6} \]
by Lemma A.2.

By the induction on \( k \), we prove that
\[ J^{(k)}_0 v = e_k(\theta_1, \theta_2, \ldots, \theta_n) v \text{ for } k = 1, \ldots, n, \]
where \( \theta_i = 1 - Q \gamma_i \) (1 \leq i \leq n) and \( e_k(\theta_1, \ldots, \theta_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \theta_{i_1} \theta_{i_2} \cdots \theta_{i_k} \).

In the case where \( k = 1 \), we have \( J^{(1)}_0 v = (J_0 + (-Q)J_1) v \). Then we have
\[ J^{(1)}_0 v = e_1(\theta_1, \ldots, \theta_n) v \] by (5.5.5).

In the case where 1 < \( k \leq n \), by the definition, we have
\[ J^{(k)}_0 v = \frac{1}{k} \sum_{z=1}^{k} (-1)^{z-1} \left( \sum_{w=0}^{z} \binom{z}{w} (-Q)^w J_w \right) J^{(k-z)}_0 v. \]
Applying the assumption of the induction to the right-hand side, we have

\[(5.5.7) \quad \mathcal{J}_0^{(k)} \cdot v = \frac{1}{k} \sum_{z=1}^{k} (-1)^{z-1} e_{k-z}(\theta_1, \ldots, \theta_n) \left( \sum_{w=0}^{z} \binom{z}{w} (-Q)^w J \right) \cdot v, \]

where we note that \( \mathcal{J}_0^{(0)} = e_0(\theta_1, \ldots, \theta_n) = 1. \) On the other hand, by (5.5.5), we have

\[(5.5.8) \quad \sum_{z=0}^{k} \binom{z}{w} (-1)^w \mathcal{J}_w \cdot v = p_z(\theta_1, \ldots, \theta_n) v \quad (1 \leq z \leq k \leq n), \]

where we note that \( \sum_{z=0}^{n} \binom{n}{w} (-1)^w = 0. \) By (5.5.7) and (5.5.8) together with (A.1.1), we have (5.5.6).

By the induction on \( t \), we prove that

\[(5.5.9) \quad u_t = p_t(\gamma_1, \ldots, \gamma_n) + Q^{-t} \beta \quad (t > 0). \]

If \( t \leq n \), (5.5.9) follows from (5.5.5). If \( t > n \), by Lemma 5.4 in the case where \( s = 0 \), we have

\[
\sum_{w=0}^{n} \binom{n}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w} \cdot v = \sum_{k=0}^{n-1} (-1)^{n-k+1} \left( \sum_{w=0}^{k} \binom{k}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w} \right) \mathcal{J}_0^{(n-k)} \cdot v.
\]

By (5.5.6), we have

\[(5.5.10) \quad \sum_{w=0}^{n} \binom{n}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w} \cdot v = \sum_{k=0}^{n-1} (-1)^{n-k+1} e_{n-k}(\theta_1, \ldots, \theta_n) \left( \sum_{w=0}^{k} \binom{k}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w} \right) \cdot v.
\]

On the other hand, for \( k \geq 0 \), we have

\[(5.5.11) \quad \sum_{w=0}^{k} \binom{k}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w} = \sum_{w=0}^{k+1} \binom{k+1}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w} \]
since $\mathcal{J}_{(t-n-1)+w}^{(1)} = \mathcal{J}_{(t-n-1)+w} + (-Q)\mathcal{J}_{(t-n-1)+w+1}$. Then, by (5.5.11) and the assumption of the induction, we have

\begin{equation}
\sum_{w=0}^{n} \binom{n}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} \cdot v
\end{equation}

\begin{equation}
= (-Q)^{n+1} \mathcal{J}_t \cdot v + \sum_{w=0}^{n} \binom{n+1}{w} (-Q)^w (p_{(t-n-1)+w}(\gamma_1, \ldots, \gamma_n) + Q^{-(t-n-1)+w}) v
\end{equation}

and

\begin{equation}
\sum_{w=0}^{k} \binom{k}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} \cdot v
\end{equation}

\begin{equation}
= \sum_{w=0}^{k+1} \binom{k+1}{w} (-Q)^w (p_{(t-n-1)+w}(\gamma_1, \ldots, \gamma_n) + Q^{-(t-n-1)+w}) v
\end{equation}

for $k = 0, 1, \ldots, n - 1$. Moreover, by the direct calculations, we have

\begin{equation}
\sum_{w=0}^{k+1} \binom{k+1}{w} (-Q)^w (p_{(t-n-1)+w}(\gamma_1, \ldots, \gamma_n) + Q^{-(t-n-1)+w}) v = p_{k+1}^{(\gamma)}(\theta_1, \ldots, \theta_n)
\end{equation}

for $k \geq 0$, where $p_{k+1}^{(\gamma)}(\theta_1, \ldots, \theta_n) = \gamma_1^{t-n-1}\theta_{k+1} + \gamma_2^{t-n-1}\theta_{k+1} + \cdots + \gamma_n^{t-n-1}\theta_{k+1}$.

Then, by (5.5.10), (5.5.12), (5.5.13) and (5.5.14), we have

\begin{equation}
(-Q)^{n+1} \mathcal{J}_t \cdot v - (-Q)^{n+1} \mathcal{J}_t \cdot v = \sum_{k=0}^{n-1} (-1)^{n-k+1} e_{n-k}(\theta_1, \ldots, \theta_n)p_{k+1}^{(\gamma)}(\theta_1, \ldots, \theta_n).
\end{equation}

Applying (A.3.2) to the right-hand side, we have

\begin{equation}
\mathcal{J}_t \cdot v = (p_t(\gamma_1, \ldots, \gamma_n) + Q^{-t} \beta) v.
\end{equation}

\[\square\]

**5.6.** By Lemma 5.2 and Proposition 5.5 we see that the highest weight $u = (u_t)_{t \geq 0}$ of a simple highest weight $U(\mathfrak{sl}_2^{(Q)}[x])$-module $\mathcal{L}(u)$ has the form

\begin{equation}
\begin{cases}
  n & \text{if } Q = 0, \\
  n + \beta & \text{if } Q \neq 0,
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
  p_t(\gamma_1, \gamma_2, \ldots, \gamma_n) & \text{if } Q = 0, \\
  p_t(\gamma_1, \gamma_2, \ldots, \gamma_n) + Q^{-t} \beta & \text{if } Q \neq 0.
\end{cases}
\end{equation}
for some \( n \in \mathbb{Z}_{\geq 0} \) and \( \beta, \gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{C} \) if \( \mathcal{L}(u) \) is finite dimensional.

Let \( \mathbb{C}[x] \) be the polynomial ring over \( \mathbb{C} \) with the indeterminate variable \( x \), and let \( \mathbb{C}[x]_{\text{monic}} \) be the subset of \( \mathbb{C}[x] \) consisting of monic polynomials. We define the set \( \mathbb{C}[x]_{\text{monic}}^{(Q)} \) by

\[
\mathbb{C}[x]_{\text{monic}}^{(Q)} = \begin{cases} 
\mathbb{C}[x]_{\text{monic}} & \text{if } Q = 0, \\
\{ \varphi \in \mathbb{C}[x]_{\text{monic}} \mid Q^{-1} \text{ is not a root of } \varphi \} & \text{if } Q \neq 0.
\end{cases}
\]

Recall that

\[
\mathbb{B}(Q) = \begin{cases} 
\{0\} & \text{if } Q = 0, \\
\mathbb{C} & \text{if } Q \neq 0.
\end{cases}
\]

We define the map

\[
(5.6.2) \quad \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}(Q) \to \prod_{t \geq 0} \mathbb{C}, \quad (\varphi, \beta) \mapsto u^{(Q)}(\varphi, \beta) = (u^{(Q)}(\varphi, \beta)_t)_{t \geq 0}
\]

by

\[
u^{(Q)}(\varphi, \beta)_t = \begin{cases} 
\deg \varphi + \beta & \text{if } t = 0, \\
p_t(\gamma_1, \gamma_2, \ldots, \gamma_n) & \text{if } t > 0 \text{ and } Q = 0, \\
p_t(\gamma_1, \gamma_2, \ldots, \gamma_n) + Q^{-t} \beta & \text{if } t > 0 \text{ and } Q \neq 0,
\end{cases}
\]

when \( \varphi = (x - \gamma_1)(x - \gamma_2) \cdots (x - \gamma_n) \). We see that the map \((5.6.2)\) is injective, and it gives a bijection between \( \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}(Q) \) and the set of highest weight \( u = (u_t)_{t \geq 0} \) satisfying \((5.6.1)\), where we note that

\[
p_t(\gamma_1, \ldots, \gamma_n; Q^{-1}, \ldots, Q^{-1}) + Q^{-t}(\beta + k) = p_t(\gamma_1, \ldots, \gamma_n) + Q^{-t}(\beta + k).
\]

Then we have the following corollary of Lemma \( \ref{lem:5.2} \) and Proposition \( \ref{prop:5.5} \).

**Corollary 5.7.** Any finite dimensional simple \( U(\mathfrak{sl}_2^{(Q)}[x]) \)-module is isomorphic to \( \mathcal{L}(u^{(Q)}(\varphi, \beta)) \) for some \( (\varphi, \beta) \in \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}(Q) \). Moreover, \( \mathcal{L}(u^{(Q)}(\varphi, \beta)) \not\cong \mathcal{L}(u^{(Q)}(\varphi', \beta')) \) if \( (\varphi, \beta) \neq (\varphi', \beta') \).

**5.8.** Recall the evaluation modules from the paragraph \( \ref{5.5} \). Let \( L(2) \) be the two-dimensional simple \( U(\mathfrak{sl}_2) \)-module, and \( v_0 \in L(2) \) be a highest weight vector. We consider the evaluation module \( L(2)^{ev, \gamma} \) at \( \gamma \in \mathbb{C} \), then we see that

\[
X_t^+ \cdot v_0 = 0, \quad J_t \cdot v_0 = \gamma^t v_0 \quad (t \geq 0)
\]

in \( L(2)^{ev, \gamma} \).
For \((\varphi = (x - \gamma_1)(x - \gamma_2)\ldots(x - \gamma_n), \beta)\in \mathbb{C}[x]_{\text{monic}}^Q \times \mathbb{B}^Q\), we consider the \(U(\mathfrak{sl}_2^Q[x])\)-module
\[
\mathcal{N}_{(\varphi, \beta)} = L(2)^{ev_{\gamma_1}} \otimes L(2)^{ev_{\gamma_2}} \otimes \cdots \otimes L(2)^{ev_{\gamma_n}} \otimes \mathcal{L}^\beta,
\]
where \(\mathcal{L}^\beta\) is the 1-dimensional \(U(\mathfrak{sl}_2^Q[x])\)-module given in the paragraph 5.8.1. Let \(v_0^{(k)}\in L(2)^{ev_{\gamma_k}} (1 \leq k \leq n)\) be a highest weight vector, and \(\mathcal{L}^\beta = \mathbb{C}w_0\). Put \(v_{(\varphi, \beta)} = v_0^{(1)} \otimes v_0^{(2)} \otimes \cdots \otimes v_0^{(n)} \otimes w_0\). Then, for \(t \geq 0\), we have
\[
(5.8.1) \quad \mathcal{X}_t^+ \cdot v_{(\varphi, \beta)} = 0
\]
and
\[
(5.8.2) \quad \mathcal{J}_t \cdot v_{(\varphi, \beta)} = \begin{cases} 
(n + \beta)v_{(\varphi, \beta)} & \text{if } t = 0, \\
pt(\gamma_1, \gamma_2, \ldots, \gamma_n)v_{(\varphi, \beta)} & \text{if } t > 0 \text{ and } Q = 0, \\
pt(\gamma_1, \gamma_2, \ldots, \gamma_n) + Q^{-1}\beta)v_{(\varphi, \beta)} & \text{if } t > 0 \text{ and } Q \neq 0.
\end{cases}
\]

Let \(\mathcal{N}'_{(\varphi, \beta)}\) be the \(U(\mathfrak{sl}_2^Q[x])\)-submodule of \(\mathcal{N}_{(\varphi, \beta)}\) generated by \(v_{(\varphi, \beta)}\). Then (5.8.1) and (5.8.2) imply that \(\mathcal{N}'_{(\varphi, \beta)}\) is a highest weight module of highest weight \(u^Q(\varphi, \beta)\), and \(\mathcal{N}'_{(\varphi, \beta)}/\text{rad}\mathcal{N}'_{(\varphi, \beta)}\) is isomorphic to the simple highest weight module \(\mathcal{L}(u^Q(\varphi, \beta))\). From the construction, \(\mathcal{L}(u^Q(\varphi, \beta)) \cong \mathcal{N}'_{(\varphi, \beta)}/\text{rad}\mathcal{N}'_{(\varphi, \beta)}\) is finite dimensional for each \((\varphi, \beta)\in \mathbb{C}[x]_{\text{monic}}^Q \times \mathbb{B}^Q\). Combining with Corollary 5.7, we have the following classification of finite dimensional simple \(U(\mathfrak{sl}_2^Q[x])\)-modules.

**Theorem 5.9.** For \((\varphi, \beta)\in \mathbb{C}[x]_{\text{monic}}^Q \times \mathbb{B}^Q\), the highest weight simple \(U(\mathfrak{sl}_2^Q[x])\)-module \(\mathcal{L}(u^Q(\varphi, \beta))\) of highest weight \(u^Q(\varphi, \beta)\) is finite dimensional, and we have that
\[
\mathcal{L}(u^Q(\varphi, \beta)) \cong \mathcal{L}(u^Q(\varphi', \beta')) \iff (\varphi, \beta) = (\varphi', \beta')
\]
for \((\varphi, \beta), (\varphi', \beta')\in \mathbb{C}[x]_{\text{monic}}^Q \times \mathbb{B}^Q\). Moreover,
\[
\{\mathcal{L}(u^Q(\varphi, \beta)) \mid (\varphi, \beta)\in \mathbb{C}[x]_{\text{monic}}^Q \times \mathbb{B}^Q\}
\]
gives a complete set of isomorphism classes of finite dimensional simple \(U(\mathfrak{sl}_2^Q[x])\)-modules.

**Remark 5.10.** If \(Q \neq 0\), the evaluation module \(L(2)^{ev_{Q^{-1}}}\) at \(Q^{-1}\) is not simple. Recall that \(L(2)\) is the two dimensional simple \(U(\mathfrak{sl}_2)\)-module with a highest weight vector \(v_0\). Put \(v_1 = f \cdot v_0\). Then we see that \(U(\mathfrak{sl}_2^Q[x]) \cdot v_1 = \mathbb{C}v_1\) is a proper \(U(\mathfrak{sl}_2^Q[x])\)-submodule of \(L(2)^{ev_{Q^{-1}}}\). Moreover, we have \(L(2)^{ev_{Q^{-1}}}/\mathbb{C}v_1 \cong \mathcal{L}^1\) and \(\mathbb{C}v_1 \cong \mathcal{L}^{-1}\) as \(U(\mathfrak{sl}_2^Q[x])\)-modules.
§ 6. Finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$-modules

In this section, we classify the finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$-modules. By Proposition 2.6, any finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$-module is isomorphic to the simple highest weight module $\mathcal{L}(u)$ of highest weight $u = (u_i, t) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$. Thus, it is enough to classify the highest weight $u$ such that $\mathcal{L}(u)$ is finite dimensional.

6.1. 1-dimensional representations. First, we consider the 1-dimensional representations of $\mathfrak{sl}_m^{(Q)}[x]$. For each $i = 1, 2, \ldots, m-1$, by checking the defining relations, we have the homomorphism of algebras

$$\iota_i : U(\mathfrak{sl}_2^{(Q)}[x]) \to U(\mathfrak{sl}_m^{(Q)}[x]) \text{ by } X_i^\pm \mapsto X_i^\pm, \ J_i \mapsto J_i, \ (t \geq 0). \tag{6.1.1}$$

Let $L = \mathbb{C}v$ be a 1-dimensional $U(\mathfrak{sl}_m^{(Q)}[x])$-module. For each $i = 1, 2, \ldots, m-1$, when we regard $L$ as a $U(\mathfrak{sl}_2^{(Q)}[x])$-module through the homomorphism $\iota_i$, we see that $L$ is isomorphic to $\mathcal{L}^{\beta_i}$ for some $\beta_i \in \mathbb{B}^{(Q_i)}$ by Lemma 5.2. Thus, we have

$$\mathcal{X}_{i,t} \cdot \nu = 0, \quad J_{i,t} \cdot \nu = \begin{cases} 0 & \text{if } Q_i = 0, \\ Q_i^{-t} \beta_i \nu & \text{if } Q_i \neq 0 \end{cases} \quad (1 \leq i \leq m-1, \ t \geq 0) \tag{6.1.2}$$

for some $\beta = (\beta_i)_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$.

On the other hand, by checking the defining relations, we can define the 1-dimensional $U(\mathfrak{sl}_m^{(Q)}[x])$-module $\mathcal{L}^\beta = \mathbb{C}v$ by (6.1.2) for each $\beta = (\beta_i) \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$. Now we proved the following lemma.

Lemma 6.2. Any 1-dimensional $U(\mathfrak{sl}_m^{(Q)})$-module is isomorphic to $\mathcal{L}^\beta$ for some $\beta \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$.

6.3. For $u = (u_i, t) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$, let $v_0$ be a highest weight vector of the simple highest weight $U(\mathfrak{sl}_m^{(Q)}[x])$-module $\mathcal{L}(u)$. When we regard $\mathcal{L}(u)$ as a $U(\mathfrak{sl}_2^{(Q)}[x])$-module through the homomorphism $\iota_i$ in (6.1.1) for each $i = 1, \ldots, m-1$, we see that the $U(\mathfrak{sl}_2^{(Q)}[x])$-submodule of $\mathcal{L}(u)$ generated by $v_0$ is a highest weight $U(\mathfrak{sl}_2^{(Q)}[x])$-module of highest weight $u_i = (u_i, t) \in \prod_{t \geq 0} \mathbb{C}$ with the highest weight vector $v_0$. Then, if $\mathcal{L}(u)$ is finite dimensional, we see that $u_i = u_i^{(Q)}(\varphi_i, \beta_i)$ for some $(\varphi_i, \beta_i) \in \mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}^{(Q_i)}$ by Theorem 5.9 (or Corollary 5.7).

For $(\varphi, \beta) = ((\varphi_i, \beta_i))_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} \left( \mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}^{(Q_i)} \right)$, we define

$$u^{(Q)}(\varphi, \beta) = (u^{(Q)}(\varphi, \beta))_{1 \leq i \leq m-1, t \geq 0} \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$$
by

(6.3.1) \[ u^{(Q)}(\varphi, \beta)_{i,t} = \begin{cases} \deg \varphi_i + \beta_i & \text{if } t = 0, \\ p_t(\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,m_i}) & \text{if } t > 0 \text{ and } Q_i = 0, \\ p_t(\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,m_i}) + Q_i^{-t} \beta_i & \text{if } t > 0 \text{ and } Q_i \neq 0, \end{cases} \]

when \( \varphi_i = (x - \gamma_{i,1})(x - \gamma_{i,2}) \ldots (x - \gamma_{i,m_i}) \) \((1 \leq i \leq m - 1)\). Then we have that

\[ (u^{(Q)}(\varphi, \beta)_{i,t})_{t \geq 0} = u^{(Q_i)}(\varphi_i, \beta_i) \]

for each \( i = 1, 2, \ldots, m - 1 \). From the definition, we see that

\[ u^{(Q)}(\varphi, \beta) = u^{(Q)}(\varphi', \beta') \iff (\varphi, \beta) = (\varphi', \beta') \]

for \((\varphi, \beta), (\varphi', \beta') \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}^{(Q_i)})\). By the above argument, any finite dimensional simple \( U(\mathfrak{sl}_n^{(Q)})\)-module is isomorphic to \( L(u^{(Q)}(\varphi, \beta)) \) for some \((\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})\).

On the other hand, for each \((\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}^{(Q_i)})\), we can construct a finite dimensional highest weight \( U(\mathfrak{sl}_m^{(Q)}[x])\)-module of highest weight \( u^{(Q)}(\varphi, \beta) \) as follows.

Let \( \omega_j \) \((1 \leq j \leq m - 1)\) be the fundamental weight of \( \mathfrak{sl}_m \), and \( L(\omega_j) \) be the simple highest weight \( U(\mathfrak{sl}_m)\)-module of highest weight \( \omega_j \). Let \( v_0 \in L(\omega_j) \) be a highest weight vector, then we have \( e_i \cdot v_0 = 0 \) and \( H_i \cdot v_0 = \delta_{ij} v_0 \) \((1 \leq i \leq m - 1)\) by the definition. Recall that \( L(\omega_j)^{ev_\gamma} \) is the evaluation module of \( L(\omega_j) \) at \( \gamma \in \mathbb{C} \).

From the definition, we see that

(6.3.2) \[ X_{i,t}^+ \cdot v_0 = 0, \quad J_{i,t} \cdot v_0 = \delta_{ij} \gamma_i^t v_0 \quad (1 \leq i \leq m - 1, t \geq 0) \]

in \( L(\omega_j)^{ev_\gamma} \).

For \((\varphi, \beta) = ((\varphi_i, \beta_i))_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}^{(Q_i)})\), we consider the \( U(\mathfrak{sl}_m^{(Q)}[x])\)-module

\[ \mathcal{N}_{(\varphi, \beta)} = \left( \bigotimes_{j=1}^{m-1} \bigotimes_{k=1}^{n_j} L(\omega_j)^{ev_{\gamma_{j,k}}} \right) \otimes \mathcal{L}^\beta, \]

where \( n_j \) and \( \gamma_{j,k} \) \((1 \leq k \leq n_j)\) are determined by \( \varphi_j = (x - \gamma_{j,1})(x - \gamma_{j,2}) \ldots (x - \gamma_{j,n_j}) \) for each \( j = 1, 2, \ldots, m - 1 \), and \( \beta = (\beta_i)_{1 \leq i \leq m-1} \). Let \( v_0^{(j,k)} \in L(\omega_j)^{ev_{\gamma_{j,k}}} \)

\((1 \leq j \leq m - 1, 1 \leq k \leq n_j)\) be a highest weight vector, and \( \mathcal{L}^\beta = \mathbb{C} v_0 \). Put

\[ v_{(\varphi, \beta)} = \left( \bigotimes_{j=1}^{m-1} \bigotimes_{k=1}^{n_j} v_0^{(j,k)} \right) \otimes w_0 \in \mathcal{N}_{(\varphi, \beta)}, \]

then we have

(6.3.3) \[ X_{i,t}^+ \cdot v_{(\varphi, \beta)} = 0, \quad J_{i,t} \cdot v_{(\varphi, \beta)} = u^{(Q)}(\varphi, \beta)_{i,t} v_{(\varphi, \beta)} \quad (1 \leq i \leq m - 1, t \geq 0) \]
by (6.3.3). Let $N'_{(\varphi, \beta)}$ be the $U(\mathfrak{sl}_m^{(Q)}[x])$-submodule of $N_{(\varphi, \beta)}$ generated by $v_{(\varphi, \beta)}$. Then (6.3.3) implies that $N'_{(\varphi, \beta)}$ is a finite dimensional highest weight module of highest weight $u^{(Q)}(\varphi, \beta)$. Then we obtain the following classification of finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$-modules.

**Theorem 6.4.** For $(\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}^{(Q_i)})$, the highest weight simple $U(\mathfrak{sl}_m^{(Q)}[x])$-module $L(u^{(Q)}(\varphi, \beta))$ of highest weight $u^{(Q)}(\varphi, \beta)$ is finite dimensional, and we have that

$$L(u^{(Q)}(\varphi, \beta)) \cong L(u^{(Q)}(\varphi', \beta')) \iff (\varphi, \beta) = (\varphi', \beta')$$

for $(\varphi, \beta), (\varphi', \beta') \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}^{(Q_i)})$. Moreover,

$$\{ L(u^{(Q)}(\varphi, \beta)) \mid (\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}^{(Q_i)}) \}$$

gives a complete set of isomorphism classes of finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$-modules.

§ 7. Finite dimensional simple $U(\mathfrak{gl}_m^{(Q)}[x])$-modules

In this section, we classify the finite dimensional simple $U(\mathfrak{gl}_m^{(Q)}[x])$-modules. By Proposition 1.4 (iii), $\mathfrak{sl}_m^{(Q)}[x]$ is a Lie subalgebra of $\mathfrak{gl}_m^{(Q)}[x]$. The difference of representations of $\mathfrak{gl}_m^{(Q)}[x]$ from one of $\mathfrak{sl}_m^{(Q)}[x]$ is given by the family of 1-dimensional $U(\mathfrak{gl}_m^{(Q)}[x])$-modules $\{ \mathcal{L}^h \mid h \in \prod_{i=0}^{m-1} \mathbb{C} \}$. We remark that $\tilde{\mathcal{L}}^h$ (for $h \in \prod_{i=0}^{m-1} \mathbb{C}$) is isomorphic to the trivial representation as a $U(\mathfrak{sl}_m^{(Q)}[x])$-module when we restrict the action.

### 7.1. 1-dimensional representations.

For $\beta = (\beta_i)_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$, by checking the defining relations, we can define the 1-dimensional $U(\mathfrak{gl}_m^{(Q)}[x])$-module $L^\beta = \mathbb{C} v$ by

$$X_{i, t}^+ \cdot v = 0, \quad J_{i, t} \cdot v = \begin{cases} 0 & \text{if } Q_i = 0, \\
Q_i^{-1} \beta_i v & \text{if } Q_i \neq 0 \quad (1 \leq i \leq m-1, t \geq 0), \end{cases}$$

$$I_{j, t} \cdot v = \left( \sum_{k=j}^{m-1} J_{k, t} \right) \cdot v \quad (1 \leq j \leq m-1, t \geq 0), \quad I_{m, t} \cdot v = 0 \quad (t \geq 0).$$

Note that $J_{j, t} = I_{j, t} - I_{j+1, t}$ in $U(\mathfrak{gl}_m^{(Q)}[x])$, we see that $\tilde{\mathcal{L}}^\beta \cong \mathcal{L}^\beta$ as $U(\mathfrak{sl}_m^{(Q)}[x])$-modules when we restrict the action on $\tilde{\mathcal{L}}^\beta$ to $U(\mathfrak{sl}_m^{(Q)}[x])$ through the injective homomorphism $\Upsilon$ in the proposition 1.4 (iii).
For \( h = (h_t)_{t \geq 0} \in \prod_{t \geq 0} \mathbb{C} \), we can also define the 1-dimensional \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-module \( \tilde{L}^h = \mathbb{C} v \) by
\[
\mathcal{X}^\pm_{i,t} \cdot v = 0, \quad \mathcal{I}_{j,t} \cdot v = h_t v \quad (1 \leq i \leq m - 1, 1 \leq j \leq m, t \geq 0).
\]

We see that \( \tilde{L}^h \cong \mathcal{L}^0 \) as \( U(\mathfrak{sl}^Q_m[x]) \)-modules when we restrict the action on \( \tilde{L}^h \) to \( U(\mathfrak{sl}^Q_m[x]) \) where \( \mathcal{L}^0 = (0)_1 \leq i \leq m-1 \in \prod_{i=1}^{m-1} \mathcal{B}^{(Q_i)} \) (i.e. \( \mathcal{L}^0 \) is the trivial representation). Then we have the following classification of 1-dimensional \( U(\mathfrak{gl}_m^{(Q)}) \)-modules:

**Lemma 7.2.** Any 1-dimensional \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-module is isomorphic to \( \tilde{L}^\beta \otimes \tilde{L}^h \) for some \( \beta \in \prod_{i=1}^{m-1} \mathcal{B}^{(Q_i)} \) and \( h \in \prod_{t \geq 0} \mathbb{C} \). We have that
\[
\tilde{L}^\beta \otimes \tilde{L}^h \cong \tilde{L}^{\beta'} \otimes \tilde{L}^{h'} \iff (\beta, h) = (\beta', h').
\]

Moreover, we see that \( \tilde{L}^\beta \otimes \tilde{L}^h \cong \mathcal{L}^0 \) as \( U(\mathfrak{sl}^Q_m[x]) \)-modules when we restrict the action on \( \tilde{L}^\beta \otimes \tilde{L}^h \) to \( U(\mathfrak{sl}^Q_m[x]) \).

**Proof.** Let \( \mathcal{L} = \mathbb{C} v \) be a 1-dimensional \( U(\mathfrak{gl}_m^{(Q)}) \)-module. By restricting the action on \( \mathcal{L} \) to \( U(\mathfrak{sl}^Q_m[x]) \) through the injective homomorphism \( \Upsilon \) in the proposition 7.4 (iii), we have
\[
(7.2.1) \quad \mathcal{X}^\pm_{i,t} \cdot v = 0,
\]
\[
\mathcal{I}_{i,t} \cdot v = (\mathcal{I}_{i,t} - \mathcal{I}_{i+1,t}) \cdot v = \begin{cases} 0 & \text{if } Q_i = 0, \\ Q_i^{-1} \beta_i v & \text{if } Q_i \neq 0 \end{cases} \quad (1 \leq i \leq m - 1, t \geq 0)
\]
for some \( \beta = (\beta_i) \in \prod_{i=1}^{m-1} \mathcal{B}^{(Q_i)} \) by Lemma 6.2.

On the other hand, for \( t \in \mathbb{Z}_{\geq 0} \), there exists \( h_t \in \mathbb{C} \) such that
\[
(7.2.2) \quad \mathcal{I}_{m,t} \cdot v = h_t v
\]
since \( \dim \mathcal{L} = 1 \). Then \( (7.2.1) \) and \( (7.2.2) \) imply that
\[
\mathcal{I}_{j,t} \cdot v = (\sum_{k=j}^{m-1} \mathcal{J}_{k,t} + h_t) \cdot v \quad (1 \leq j \leq m - 1, t \geq 0), \quad \mathcal{I}_{m,t} \cdot v = h_t v \quad (t \geq 0).
\]

Then we see that \( \mathcal{L} \cong \tilde{L}^\beta \otimes \tilde{L}^h \). The remaining statements are clear. \( \square \)

**7.3.** For \( \tilde{u} = (\tilde{u}_{j,t}) \in \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C} \), let \( u_0 \) be a highest weight vector of the simple highest weight \( U(\mathfrak{gl}_m^{(Q)}[x]) \)-module \( \mathcal{L}(\tilde{u}) \). By restricting the action on \( \mathcal{L}(\tilde{u}) \) to \( U(\mathfrak{sl}^Q_m[x]) \), Theorem 6.4 implies that
\[
(7.3.1) \quad \tilde{u}_{i,t} - \tilde{u}_{i+1,t} = u^{(Q)}(\varphi, \beta)_{i,t} \quad (1 \leq i \leq m - 1, t \geq 0)
\]
For some \((\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})\) if \(L(\tilde{u})\) is finite dimensional.

For \(t \in \mathbb{Z}_{\geq 0}\), let \(h_t \in \mathbb{C}\) be such that

\[
\tilde{u}_{m,t} = h_t.
\]

By (7.3.1) and (7.3.2), we have

\[
\tilde{u}_{j,t} = \sum_{k=j}^{m-1} u^{(Q)}(\varphi, \beta)_{k,t} + h_t \quad (1 \leq j \leq m-1, \, t \geq 0), \quad \tilde{u}_{m,t} = h_t \quad (t \geq 0)
\]

for some \((\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})\) and \(h = (h_t) \in \prod_{t \geq 0} \mathbb{C}\) if \(L(\tilde{u})\) is finite dimensional.

For \((\varphi, \beta, h) = ((\varphi_i, \beta_i)_{1 \leq i \leq m-1}, (h_t)_{t \geq 0}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}\), we define

\[
\tilde{u}^{(Q)}(\varphi, \beta, h) = (\tilde{u}^{(Q)}(\varphi, \beta, h)_{j,t})_{j=1, t \geq 0} \in \prod_{j=1}^{m} \prod_{t \geq 0} \mathbb{C}
\]

by

\[
\tilde{u}^{(Q)}(\varphi, \beta, h)_{j,t} = \begin{cases} 
\sum_{k=j}^{m-1} u^{(Q)}(\varphi, \beta)_{k,t} + h_t & \text{if } 1 \leq j \leq m-1 \text{ and } t \geq 0, \\
h_t & \text{if } j = m \text{ and } t \geq 0.
\end{cases}
\]

From the definition, we see that

\[
\tilde{u}^{(Q)}(\varphi, \beta, h) = \tilde{u}^{(Q)}(\varphi', \beta', h') \iff (\varphi, \beta, h) = (\varphi', \beta', h')
\]

for \((\varphi, \beta, h), (\varphi', \beta', h') \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}\). By the above argument, any finite dimensional simple \(U(\mathfrak{gl}_m^{(Q)}[x])\)-module is isomorphic to \(L(\tilde{u}^{(Q)}(\varphi, \beta, h))\) for some \((\varphi, \beta, h) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}\).

On the other hand, for each \((\varphi, \beta, h) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}\), we can construct a finite dimensional highest weight \(U(\mathfrak{gl}_m^{(Q)}[x])\)-module of highest weight \(\tilde{u}^{(Q)}(\varphi, \beta, h)\) as follows.

Let \(P = \bigoplus_{i=1}^{m-1} \mathbb{Z} \varepsilon_i\) be the weight lattice of \(\mathfrak{gl}_m\). Put \(\tilde{\omega}_l = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_l\) for \(l = 1, 2, \ldots, m-1\). Let \(L(\tilde{\omega}_l)\) be the simple highest weight \(U(\mathfrak{gl}_m^{(Q)})\)-module of highest weight \(\tilde{\omega}_l\), and \(v_0 \in L(\tilde{\omega}_l)\) be a highest weight vector. Then, we have

\[
e_i \cdot v_0 = 0 \quad (1 \leq i \leq m-1) \quad \text{and} \quad K_j \cdot v_0 = \begin{cases} 
v_0 & \text{if } 1 \leq j \leq l, \\
0 & \text{if } l < j \leq m.
\end{cases}
\]
Recall that $L(\tilde{\omega})^{ev\gamma}$ is the evaluation module of $L(\tilde{\omega})$ at $\gamma \in \mathbb{C}$. From the definition, we see that

\[(7.3.3)\]

$$X_{i,t}^+ \cdot v_0 = 0 \quad (1 \leq i \leq m - 1, \ t \geq 0), \quad \mathcal{I}_{j,t} \cdot v_0 = \begin{cases} \gamma^j v_0 & \text{if } 1 \leq j \leq l, \\ 0 & \text{if } l < j \leq m \ (t \geq 0) \end{cases}$$

in $L(\tilde{\omega})^{ev\gamma}$. (We remark that $L(\tilde{\omega})^{ev\gamma} \cong L(\omega)^{ev\gamma}$ as $U(\mathfrak{sl}_m^{(Q)}[x])$-modules when we restrict the action on $L(\tilde{\omega})^{ev\gamma}$ to $U(\mathfrak{sl}_m^{(Q)}[x])$.)

For $(\varphi, \beta, h) = ((\varphi_i, \beta_i)_{1 \leq i \leq m-1}, (h_t)_{t\geq 0}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}(Q_i)) \times \prod_{t\geq 0} \mathbb{C}$, we consider the $U(\mathfrak{gl}_m^{(Q)}[x])$-module

$$\tilde{N}_{(\varphi, \beta, h)} = \left( \bigotimes_{l=1}^{m-1} \bigotimes_{k=1}^{n_l} L(\tilde{\omega})^{ev_{l,k}} \right) \otimes \tilde{L}^\beta \otimes \tilde{L}^h,$$

where $n_l$ and $\gamma_{l,k}$ ($1 \leq k \leq n_l$) are determined by $\varphi_l = (x-\gamma_{l,1})(x-\gamma_{l,2}) \cdots (x-\gamma_{l,n_l})$ for each $l = 1, 2, \ldots, m - 1$, and we put $\beta = (\beta_i)_{1 \leq i \leq m-1}$ and $h = (h_t)_{t\geq 0}$. Let $v_{0}^{(l,k)} \in L(\tilde{\omega})^{ev_{l,k}}$ ($1 \leq l \leq m - 1, 1 \leq k \leq n_l$) be a highest weight vector, $\tilde{L}^\beta = C w_0$ and $\tilde{L}^h = C z_0$. Put $v_{(\varphi, \beta, h)} = (\otimes_{l=1}^{m-1} \otimes_{k=1}^{n_l} v_{0}^{(l,k)}) \otimes w_0 \otimes z_0 \in \tilde{N}_{(\varphi, \beta, h)}$, then we have

\[(7.3.4)\]

$$X_{i,t}^+ \cdot v_{(\varphi, \beta, h)} = 0, \quad \mathcal{I}_{j,t} \cdot v_{(\varphi, \beta, h)} = \tilde{u}^{(Q)}(\varphi, \beta, h)_{j,t} v_{(\varphi, \beta, h)}$$

for $1 \leq i \leq m - 1, 1 \leq j \leq m$ and $t \geq 0$ by (7.3.3). Let $\tilde{N}'_{(\varphi, \beta, h)}$ be the $U(\mathfrak{gl}_m^{(Q)}[x])$-submodule of $\tilde{N}_{(\varphi, \beta, h)}$ generated by $v_{(\varphi, \beta, h)}$. Then (7.3.4) implies that $\tilde{N}'_{(\varphi, \beta, h)}$ is a finite dimensional highest weight module of highest weight $\tilde{u}^{(Q)}(\varphi, \beta, h)$. Now we obtain the following classification of finite dimensional simple $U(\mathfrak{gl}_m^{(Q)}[x])$-modules.

**Theorem 7.4.** For $(\varphi, \beta, h) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}(Q_i)) \times \prod_{t\geq 0} \mathbb{C}$, the highest weight simple $U(\mathfrak{gl}_m^{(Q)}[x])$-module $\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h))$ of highest weight $\tilde{u}^{(Q)}(\varphi, \beta, h)$ is finite dimensional, and we have that

$$\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h)) \cong \mathcal{L}(\tilde{u}^{(Q)}(\varphi', \beta', h')) \iff (\varphi, \beta, h) = (\varphi', \beta', h')$$

for $(\varphi, \beta, h), (\varphi', \beta', h') \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}(Q_i)) \times \prod_{t\geq 0} \mathbb{C}$. Moreover,

$$\{ \mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h)) \mid (\varphi, \beta, h) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]^{(Q_i)}_{\text{monic}} \times \mathbb{B}(Q_i)) \times \prod_{t\geq 0} \mathbb{C} \}$$

gives a complete set of isomorphism classes of finite dimensional simple $U(\mathfrak{gl}_m^{(Q)}[x])$-modules.

We also have the following corollary.
Corollary 7.5. For \((\varphi, \beta, h) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}} \times \mathbb{B}(Q_i)) \times \prod_{i \geq 0} \mathbb{C}\), we have

\[
\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h)) \cong \mathcal{L}(u^{(Q)}(\varphi, \beta))
\]

as \(U(\mathfrak{sl}_m^{(Q)}[x])\)-modules when we restrict the action on \(\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h))\) to \(U(\mathfrak{sl}_m^{(Q)}[x])\).

Proof. We prove that \(\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h))\) is also simple when we restrict the action to \(U(\mathfrak{sl}_m^{(Q)}[x])\). Then the isomorphism follows from the definitions of \(\tilde{u}^{(Q)}(\varphi, \beta, h)\) and \(u^{(Q)}(\varphi, \beta)\).

Let \(v_0 \in \mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h))\) be a highest weight vector as the \(U(\mathfrak{gl}_m^{(Q)}[x])\)-module. Then we have

\[
\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h)) = U(n^-) \cdot v_0
\]

by the triangular decomposition in Proposition 4.4 (iv). This implies that

\[
(7.5.1) \quad \mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h)) = U(\mathfrak{sl}_m^{(Q)}[x]) \cdot v_0.
\]

Assume that \(\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h))\) is not simple as a \(U(\mathfrak{sl}_m^{(Q)}[x])\)-module by the restriction, then \(\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h))\) contains a non-zero proper simple \(U(\mathfrak{sl}_m^{(Q)}[x])\)-submodule which is a highest weight \(U(\mathfrak{sl}_m^{(Q)}[x])\)-module. This implies that there exist an element \(w_0 \in \mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h))\) such that \(X_{i,t}^+ \cdot w_0 = 0\) \((1 \leq i \leq m-1, t \geq 0)\) and \(w_0 \notin \mathbb{C}v_0\). Then \(U(\mathfrak{gl}_m^{(Q)}[x]) \cdot w_0\) turns out to be a non-zero proper \(U(\mathfrak{gl}_m^{(Q)}[x])\)-submodule of \(\mathcal{L}(\tilde{u}^{(Q)}(\varphi, \beta, h))\). This is a contradiction. \(\square\)

Appendix A. Some Combinatorics

A.1. Let \(\mathbb{Z}[x_1, \ldots, x_n]\) be the ring of polynomials in independent variables \(x_1, \ldots, x_n\) over \(\mathbb{Z}\). For \(k \in \mathbb{Z}_{>0}\), put

\[
p_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k \in \mathbb{Z}[x_1, \ldots, x_n],
\]

\[
e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2} \cdots x_{i_k} \in \mathbb{Z}[x_1, \ldots, x_n].
\]

Namely, \(p_k(x_1, \ldots, x_n)\) is the power sum symmetric polynomial of degree \(k\), and \(e_k(x_1, \ldots, x_n)\) is the elementary symmetric polynomial of degree \(k\). We also put \(e_0(x_1, \ldots, x_n) = 1\). Then, for \(k > 0\), we have

\[
(A.1.1) \quad ke_k(x_1, \ldots, x_n) = \sum_{z=1}^{k} (-1)^{z-1} p_z(x_1, \ldots, x_n)e_{k-z}(x_1, \ldots, x_n)
\]

by [M §1 (2.11')]. For \(s > n\), we have

\[
0 = \sum_{z=1}^{s} (-1)^{z-1} p_z(x_1, \ldots, x_n)e_{s-z}(x_1, \ldots, x_n)
\]
= \sum_{z=1}^{s-1} (-1)^{z-1} p_z(x_1, \ldots, x_n) e_{s-z}(x_1, \ldots, x_n) + (-1)^{s-1} p_s(x_1, \ldots, x_n)

= \sum_{z=s-n}^{s-1} (-1)^{z-1} p_z(x_1, \ldots, x_n) e_{s-z}(x_1, \ldots, x_n) + (-1)^{s-1} p_s(x_1, \ldots, x_n),

where we note that $e_{s-z}(x_1, \ldots, x_n) = 0$ if $z < s - n$. Put $w = z - s + n$, we have

$$\sum_{w=0}^{n-1} (-1)^{n-w+1} p_{s-n+w}(x_1, \ldots, x_n) e_{n-w}(x_1, \ldots, x_n) = p_s(x_1, \ldots, x_n)$$

for $s > n$.

**Lemma A.2.** For $n \in \mathbb{Z}_{>0}$ and $u_1, u_2, \ldots, u_n \in \mathbb{C}$, the simultaneous equations

$$\begin{align*}
p_1(x_1, x_2, \ldots, x_n) &= u_1, \\
p_2(x_1, x_2, \ldots, x_n) &= u_2, \\
&\vdots \\
p_n(x_1, x_2, \ldots, x_n) &= u_n
\end{align*}$$

(A.2.1)

has a solution in $\mathbb{C}$.

**Proof.** We prove the lemma by the induction on $n$. In the case where $n = 1$, it is clear. If $n > 1$, the equations (A.2.1) are equivalent to the equations

$$\begin{align*}
p_1(x_1, x_2, \ldots, x_{n-1}) &= u_1 - x_n, \\
p_2(x_1, x_2, \ldots, x_{n-1}) &= u_2 - x_n^2, \\
&\vdots \\
p_{n-1}(x_1, x_2, \ldots, x_{n-1}) &= u_{n-1} - x_n^{n-1}, \\
p_n(x_1, x_2, \ldots, x_{n-1}) &= u_n - x_n^n.
\end{align*}$$

(A.2.2)

By (A.2.1), we have

$$p_n(x_1, x_2, \ldots, x_{n-1})$$

$$= \sum_{i=1}^{n-1} (-1)^{i+n-1} p_i(x_1, x_2, \ldots, x_{n-1}) e_{n-i}(x_1, x_2, \ldots, x_{n-1}),$$

where we note that $e_n(x_1, x_2, \ldots, x_{n-1}) = 0$. On the other hand, we can write

$$e_{n-i}(x_1, x_2, \ldots, x_{n-1}) = \sum_{\lambda \vdash n-i} \alpha_{\lambda} p_{\lambda}(x_1, x_2, \ldots, x_{n-1})$$
for some $\alpha_\lambda \in \mathbb{C}$, where $p_\lambda(x_1, x_2, \ldots, x_{n-1}) = \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(x_1, x_2, \ldots, x_{n-1})$ for $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n - i$. Thus we have

$$p_n(x_1, x_2, \ldots, x_{n-1}) = \sum_{i=1}^{n-1} \sum_{\lambda \vdash n-i} (-1)^{i+n-1} \alpha_\lambda p_i(x_1, x_2, \ldots, x_{n-1}) p_\lambda(x_1, x_2, \ldots, x_{n-1}).$$

(Note that $\{p_\mu(x_1, x_2, \ldots, x_{n-1}) \mid \mu \vdash k\}$ is not linearly independent if $k \geq n$. For an example, we have $p_{(3)}(x_1, x_2) = \frac{3}{2} p_{(2,1)}(x_1, x_2) - \frac{1}{2} p_{(1,1,1)}(x_1, x_2).$

Then the equations (A.2.2) are equivalent to the equations

\[
\begin{aligned}
\begin{cases}
p_1(x_1, x_2, \ldots, x_{n-1}) = u_1 - x_n, \\
p_2(x_1, x_2, \ldots, x_{n-1}) = u_2 - x_n^2, \\
\vdots \\
p_{n-1}(x_1, x_2, \ldots, x_{n-1}) = u_{n-1} - x_n^{n-1}, \\
\sum_{i=1}^{n-1} \sum_{\lambda \vdash n-i} (-1)^{i+n-1} \alpha_\lambda (u_i - x_n^i) \prod_{j=1}^{\ell(\lambda)} (u_{\lambda_j} - x_n^{\lambda_j}) = u_n - x_n^n \quad \cdots (\ast 1).
\end{cases}
\end{aligned}
\]

Let $\beta_n$ be a solution of the equation (\ast 1) for the variable $x_n$. By the assumption of the induction, the simultaneous equations

\[
\begin{aligned}
\begin{cases}
p_1(x_1, x_2, \ldots, x_{n-1}) = u_1 - \beta_n, \\
p_2(x_1, x_2, \ldots, x_{n-1}) = u_2 - \beta_n^2, \\
\vdots \\
p_{n-1}(x_1, x_2, \ldots, x_{n-1}) = u_{n-1} - \beta_n^{n-1}
\end{cases}
\end{aligned}
\]

for variables $x_1, x_2, \ldots, x_{n-1}$ has a solution. We denote it by $(x_1, x_2, \ldots, x_{n-1}) = (\beta_1, \beta_2, \ldots, \beta_{n-1})$. Then $(x_1, x_2, \ldots, x_n) = (\beta_1, \beta_2, \ldots, \beta_n)$ gives a solution of (A.2.3). \hfill \square

A.3. We consider some modifications of the formulas (A.1.1) and (A.1.2) as follows. Let $b = (b_1, \ldots, b_n)$ be $n$ independent variables, and we consider the ring of polynomials $\mathbb{Z}[x_1, \ldots, x_n][b_1, \ldots, b_n]$. For $k \in \mathbb{Z}_{>0}$, put

$$e_k^{(b)}(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (b_{i_1} + b_{i_2} + \cdots + b_{i_k}) x_{i_1} x_{i_2} \cdots x_{i_k} \in \mathbb{Z}[x_1, \ldots, x_n][b_1, \ldots, b_n]$$

and

$$p_k^{(b)}(x_1, \ldots, x_n) = b_1 x_1^k + b_2 x_2^k + \cdots + b_n x_n^k \in \mathbb{Z}[x_1, \ldots, x_n][b_1, \ldots, b_n].$$
We also put $e_0 = 1$. Note that $e_k(x_1, \ldots, x_n) = 0$ if $k > n$. Put $1 = (1, 1, \ldots, 1)$, then we have $e_k(x_1, \ldots, x_n) = ke_k(x_1, \ldots, x_n)$ and $p_k(x_1, \ldots, x_n) = p_k(x_1, \ldots, x_n)$.

We consider the generating functions $E(t)$, $E^{(b)}(t)$ and $P^{(b)}(t)$ by

$$E(t) = \sum_{k \geq 0} e_k(x_1, \ldots, x_n) t^k \in \mathbb{Z}[x_1, \ldots, x_n][b_1, \ldots, b_n][[t]],$$

$$E^{(b)}(t) = \sum_{k \geq 0} e_{k+1}^{(b)}(x_1, \ldots, x_n) t^k \in \mathbb{Z}[x_1, \ldots, x_n][b_1, \ldots, b_n][[t]],$$

$$P^{(b)}(t) = \sum_{k \geq 0} (-1)^k p_{k+1}^{(b)}(x_1, \ldots, x_n) t^k \in \mathbb{Z}[x_1, \ldots, x_n][b_1, \ldots, b_n][[t]].$$

Then, we have

$$E(t) = \prod_{i=1}^n \left( 1 + x_i t \right), \quad P^{(b)}(t) = \sum_{i=1}^n \frac{b_i x_i}{1 + x_i t}$$

and

$$P^{(b)}(t)E(t) = \sum_{i=1}^n b_i x_i \left( \prod_{j=1, j \neq i}^n (1 + x_i t) \right) = E^{(b)}(t).$$

This implies that, for $k \geq 0$,

$$e_{k+1}^{(b)}(x_1, \ldots, x_n) = \sum_{z=0}^k (-1)^z p_{z+1}^{(b)}(x_1, \ldots, x_n) e_{k-z}(x_1, \ldots, x_n).$$

In the case where $k = n$, we have

$$\sum_{z=0}^n (-1)^z p_{z+1}^{(b)}(x_1, \ldots, x_n) e_{n-z}(x_1, \ldots, x_n) = 0$$

since $e_{n+1}(x_1, \ldots, x_n) = 0$. This implies that

$$\sum_{z=0}^{n-1} (-1)^{n-z+1} p_{z+1}^{(b)}(x_1, \ldots, x_n) e_{n-z}(x_1, \ldots, x_n) = p_{n+1}^{(b)}(x_1, \ldots, x_n).$$

**References**

[C] V. Chari, *Integrable representations of affine Lie algebras*, Invent. Math. **85** (1986), 317-335.

[CP] V. Chari and A. Pressley, *New unitary representations of loop groups*, Math. Ann. **275** (1986), 87-104.

[M] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd edition, Oxford Univ. Press, 1995.
K. Wada, *New Realization of Cyclotomic q-Schur Algebras*, Publ. RIMS Kyoto Univ. **52** (2016), 497-555.

**Department of Mathematics, Faculty of Science, Shinshu University, Asahi 3-1-1, Matsumoto 390-8621, Japan**

*E-mail address: wada@math.shinshu-u.ac.jp*