A STUDY ABOUT STABILITY OF TWO AND THREE SPECIES POPULATION MODELS

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ABSTRACT

In this article, we have discussed the stability of second order linear and non-linear systems by characteristic roots. In the case of non-linear system, we linearize the nonlinear system under certain specified conditions and study the stability of critical points of the linearized systems. Necessary theories have been presented, applied, and illustrated with examples. A self-contained theory for a homogeneous linear system of third order is built by using the basic concept of the differential equation.

Keywords: Stability, Two and three order species, Characteristic roots, Stability of linear and nonlinear system.

1. INTRODUCTION

In general, Stability means a situation in which something is not likely to move or change that is the state or quality of being stable.

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations have been the most important part of pure and applied mathematics. Only the simplest differential equations admit solutions given by explicit formulas. Many properties of solutions of differential equations may be determined without finding their exact form. The graphical solution is the special type of approximate solution. Here we will apply the graphical method to find the approximate solutions of second and third-order linear and nonlinear ordinary differential equations. The mathematical formulation of various special problems results in nonlinear differential equations. In many cases, it is possible to replace such a nonlinear equation by a suitable linear equation which approximates the actual nonlinear equation close enough to give valuable results. Such a “linearization” is not always feasible; and when it is not original nonlinear equation itself must be considered. While the general theory and methods of linear equations are must highly be developed, very little of the general character is known about nonlinear equations. The study of nonlinear equations is restricted to a variety of rather special cases and one must resort to various methods of approximation.

2. STABILITY OF SECOND ORDER SYSTEMS BY CHARACTERISTIC ROOTS

We consider the system

\[
\frac{dx}{dt} = P(x, y), \\
\frac{dy}{dt} = Q(x, y),
\]

Assume that (0, 0) is an isolated critical point of the system (1). Let C be a path of (1); let x=f(t), y=g(t) be a solution of (1) defining C parametrically. Let

\[
D(t) = \sqrt{[f(t)]^2 + [g(t)]^2}
\]

denote the distance between the critical point (0, 0) and the point R: [f(t), g(t)] on C. The critical point (0, 0) is called stable if for every number \( \epsilon > 0 \), there exists a number \( \delta > 0 \) such that the following is true: Every path C for which

\[
D(t_o) < \epsilon
\]

is defined for all \( t \geq t_o \) and such that...
\( D(t) < \epsilon \) for \( t_0 \leq t < \infty \)

There are three types of stability:

i) Stable

ii) Asymptotically stable

iii) Unstable

**2.1. Stability of Second Order Linear Systems**

We consider the linear system

\[
\begin{align*}
\frac{dx}{dt} &= ax + by, \\
\frac{dy}{dt} &= cx + dy,
\end{align*}
\]

Where \( a, b, c, \) and \( d \) are real constants. The origin \((0, 0)\) is clearly a critical point of (1).

We assume that

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0
\]

And hence \((0, 0)\) is the only critical point of (1).

Let

\[
\begin{align*}
x &= Ae^{\lambda t}, \\
y &= Be^{\lambda t},
\end{align*}
\]

be a solution of (1), then \( \lambda \) must satisfy quadratic equation

\[
\lambda^2 - (a + d)\lambda + (ad - bc) = 0
\]

Is called the characteristic equation of (1). Let \( \lambda_1, \lambda_2 \) be the roots of the characteristic equation (4). We shall prove that the critical point \((0, 0)\) of the system (1) depends upon the nature of the roots \( \lambda_1 \) and \( \lambda_2 \).

We would expect three possibilities, according as \( \lambda_1 \) and \( \lambda_2 \) are real and distinct, real and equal, or conjugate complex. But actually the situation here is not quite so simple and we must consider the following five cases:

- \( \lambda_1 \) and \( \lambda_2 \) are real, unequal, and of the same sign.
- \( \lambda_1 \) and \( \lambda_2 \) are real, unequal, and of opposite sign.
- \( \lambda_1 \) and \( \lambda_2 \) are real and equal.
- \( \lambda_1 \) and \( \lambda_2 \) are conjugate complex but not pure imaginary.
- \( \lambda_1 \) and \( \lambda_2 \) are pure imaginary.

**2.2. Stability of Second Order Non-linear Systems**

Consider the non-linear real autonomous systems

\[
\begin{align*}
\frac{dx}{dt} &= P(x, y), \\
\frac{dy}{dt} &= Q(x, y),
\end{align*}
\]

Assume that the system (1) has an isolated critical point which will choose to be the origin \((0, 0)\). Further assume that the functions \( P \) and \( Q \) in the right members of (1) are such that \( P(x, y) \) and \( Q(x, y) \) can be written in the form

\[
\begin{align*}
P(x, y) &= ax + by + P_1(x, y), \\
Q(x, y) &= cx + dy + Q_1(x, y),
\end{align*}
\]

Where

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0
\]

And

\[
\begin{align*}
P_1(x, y) \text{ and } Q_1(x, y) \text{ have continuous first partial derivatives for all } (x, y), \text{ and are such that}
\end{align*}
\]

\[
\lim_{(x, y) \to (0, 0)} \frac{P_1(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \to (0, 0)} \frac{Q_1(x, y)}{\sqrt{x^2 + y^2}} = 0
\]

Thus the system under consideration may be written in the form

\[
\begin{align*}
\frac{dx}{dt} &= ax + by + P_1(x, y), \\
\frac{dy}{dt} &= cx + dy + Q_1(x, y),
\end{align*}
\]

Where \( a, b, c, d, P_1, \) and \( Q_1 \) satisfy the requirements (1) and (2).

If \( P(x, y) \) and \( Q(x, y) \) in (1) can be expanded in power series about \((0, 0)\), the system (1) takes the form

\[
\begin{align*}
\frac{dx}{dt} &= \frac{\partial P}{\partial x} x + \frac{\partial P}{\partial y} y + a_{12} x^2 + a_{22} xy + a_{21} y^2 + \ldots \\
\frac{dy}{dt} &= \frac{\partial Q}{\partial x} x + \frac{\partial Q}{\partial y} y + b_{12} x^2 + b_{22} xy + b_{21} y^2 + \ldots
\end{align*}
\]

This system is of the form (4), where \( P_1(x, y) \) and \( Q_1(x, y) \) are the terms of higher degree in the right members of the equations. The requirements 1 and 2 will be met,

\[
\frac{\partial(P, Q)}{\partial(x, y)} \neq 0
\]

Observe that the constant terms are missing in the expansions in the right members of (5), since \( P(0, 0) = Q(0, 0) = 0 \).

**2.2.1. Similar Stability of Non-linear Systems**

**Hypothesis:** We consider the non-linear system

\[
\begin{align*}
\dot{x} &= ax + by + P_1(x, y), \\
\dot{y} &= cx + dy + Q_1(x, y),
\end{align*}
\]

and the corresponding linear system

\[
\begin{align*}
\dot{x} &= ax + by, \\
\dot{y} &= cx + dy,
\end{align*}
\]

Where \( a, b, c, d, P_1, \) and \( Q_1 \) satisfies

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0
\]

and

\[
\lim_{(x, y) \to (0, 0)} \frac{P_1(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \to (0, 0)} \frac{Q_1(x, y)}{\sqrt{x^2 + y^2}} = 0
\]

Here \( P_1 \) and \( Q_1 \) are non-linear terms of \((x, y)\). Both the linear and the non-linear systems have an isolated critical point at \((0, 0)\).

Let \( \lambda_1 \) and \( \lambda_2 \) be the roots of the characteristic equation

\[
\lambda^2 - (a + d)\lambda + (ad - bc) = 0
\]
of the linear system (2).

- If both roots of the characteristic equation (3) of the linear system (2) are real and negative or conjugate complex with negative real parts, then not only (0, 0) is an asymptotically stable critical point of (2), but also (0, 0) is an asymptotically stable critical point of (1).

- If one of the two real roots of the characteristic equation (3) of the linear system (2) is positive or two conjugate complex with positive real parts, then not only (0, 0) is an unstable critical point of (2), but also (0, 0) is an unstable critical point of (1).

### 2.2.2. Dissimilar Stability of Non-linear Systems

#### Hypothesis:
We consider the non-linear systems

\[
\frac{dx}{dt} = ax + by + P_1(x, y), \\
\frac{dy}{dt} = cx + dy + Q_1(x, y),
\]

and the corresponding linear system is

\[
\frac{dx}{dt} = ax + by, \\
\frac{dy}{dt} = cx + dy,
\]

Where a, b, c, d, P_1, and Q_1 satisfies \( \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \)

and

\[
\lim_{(x,y) \to (0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y) \to (0,0)} \frac{Q_1(x,y)}{\sqrt{x^2+y^2}} = 0
\]

Here \( P_1 \) and \( Q_1 \) are non-linear terms of \( x, y \). Both the linear and the non-linear systems have an isolated critical point at \((0, 0)\). Let \( \lambda_1 \) and \( \lambda_2 \) be the roots of the characteristic equation

\[
\lambda^2 - (a + d)\lambda + (ad - bc) = 0
\]

of the linear system (2).

- If the roots of the characteristic equation (3) are pure imaginary, then although \((0, 0)\) is a stable critical point of (2), the critical point \((0, 0)\) of (1) may be asymptotically stable, stable or unstable.

Here we give some Illustrative Examples of similar and dissimilar stability of Non Linear System

#### Example-2.2.1.
We consider the system

\[
\dot{x} = x + x^2 - 3xy, \\
\dot{y} = -2x + y + 3y^2,
\]

The critical points are given by

\[
x (1+x-3y) = 0, \\
-2x + y + 3y^2 = 0.
\]

Solving (2) the critical points are \((0, 0)\), \((0, -\frac{1}{3})\), \((1, \frac{2}{3})\), \((2, 1)\).

(i) For \((0, 0)\) the corresponding linear system is

\[
\dot{x} = x, \\
\dot{y} = -2x + y,
\]

The characteristic equation of (3) is

\[
\lambda^2 - 2\lambda + 1 = 0 \\
\Rightarrow (\lambda - 1)^2 = 0 \\
\therefore \lambda = 1, 1
\]

Here the roots are real, equal and both positive and the system (1) is not such that \(a= d \neq 0, b= c= 0\).

Hence the critical point \((0, 0)\) is unstable.

(ii) For \((0, -\frac{1}{3})\) we use

\[
x = \xi + 1, \quad y = \eta + \frac{2}{3}
\]

in (1) and obtain

\[
\dot{\xi} = 2\xi + \xi^2 - 3\xi\eta, \\
\dot{\eta} = -2\xi - \eta + 3\eta^2,
\]

The characteristic equation of the corresponding linear part of (4) is

\[
\lambda^2 - \lambda - 2 = 0 \\
\Rightarrow (\lambda - 1)(\lambda + 2) = 0 \\
\therefore \lambda = -1, 2
\]

Here the roots are real, unequal and opposite signs.

Hence the critical point \((0, -\frac{1}{3})\) is unstable.

(iii) For \((1, \frac{2}{3})\) we use

\[
x = \xi + 1, \quad y = \eta + \frac{2}{3}
\]

in (1) and obtain

\[
\dot{\xi} = \xi - 3\eta + \xi^2 - 3\xi\eta, \\
\dot{\eta} = -2\xi + 5\eta + 3\eta^2,
\]

The characteristic equation of the corresponding linear part of (5) is

\[
\lambda^2 - 6\lambda - 1 = 0 \\
\Rightarrow \lambda = 3 + \sqrt{10}, 3 - \sqrt{10}
\]

Here the roots are real, unequal and opposite signs.

Hence the critical point \((1, \frac{2}{3})\) is unstable.

(iv) For \((2, 1)\) we use

\[
x = \xi + 2, \quad y = \eta + 1
\]

in (1) and we obtain

\[
\dot{\xi} = 2\xi - 6\eta + \eta^2 - 3\xi\eta, \\
\dot{\eta} = -2\xi + 7\eta + 3\eta^2,
\]

The characteristic equation of the corresponding linear part of (6) is

\[
\lambda^2 - 9\lambda + 2 = 0
\]
Here the roots are real, unequal and both positive.
Hence (2, 1) is unstable.

From (1)
\[
\frac{dy}{dx} = \frac{-2x^2 + 3y^2}{x(x^2 - 3y)} = c \text{ (say)}
\]
\[c = 0, \quad \left(1 + \frac{1}{2}\right)^2 = \frac{2}{3}\left(x + \frac{1}{2}\right)\]
c → ∞, \quad x(1 + x - 3y) = 0 \implies x = 0, x - 3y + 1 = 0.
The required figure is shown below:

\[\text{Figure 1.} \quad \text{3-Dimensional figure for example-2.2.1, using MATHEMATICA software.}\]

**Example-2.2.2.** We consider
\[
\begin{align*}
\dot{x} &= 8x - y^2, \\
\dot{y} &= -6y + 6x^2,
\end{align*}
\]
The critical points are given by
\[
\begin{align*}
8x - y^2 &= 0, \\
-6y + 6x^2 &= 0,
\end{align*}
\]
Solving (2) the critical points are (0, 0), (2, 4).

(i) For (0, 0) the corresponding linear system is
\[
\begin{align*}
\dot{x} &= 8x, \\
\dot{y} &= -6y,
\end{align*}
\]
The characteristic equation of (3) is
\[
\lambda^2 - 2\lambda - 48 = 0 \\
\implies (\lambda + 6)(\lambda - 8) = 0
\]
\[\therefore \lambda = -6, 8\]
Here the roots are real, unequal and of opposite signs.
Hence the critical point (0, 0) is unstable.

(ii) For (2, 4) we use
\[
\lambda = \frac{9 + \sqrt{73}}{2}, \quad \frac{9 - \sqrt{73}}{2}
\]
x = \xi + 2, \quad y = \eta + 4 \text{ in (1) and obtain}
\[
\begin{align*}
\dot{\xi} &= 8\xi - 8\eta - \eta^2, \\
\dot{\eta} &= 24\xi - 6\eta + 6\xi^2,
\end{align*}
\]
The characteristic equation of the linear part of (4) is
\[
\lambda^2 - 2\lambda + 144 = 0
\]
\[\lambda = 1 \pm \sqrt{143} i\]
Here the roots are conjugate complex with real part positive.
Hence (2, 4) is unstable.

From (1)
\[
\frac{dy}{dx} = \frac{-6y + 6x^2}{8x - y^2} = c \text{ (say)}
\]
c = 0, \quad y = x^2
\[c \rightarrow \infty, \quad 8x - y^2 = 0 \implies y^2 = 8x\]
The required figure is shown below:

\[\text{Figure 2.} \quad \text{3-Dimensional figure for example-2.2.2, using MATHEMATICA software.}\]

**Example-2.2.3.** We consider the system
\[
\begin{align*}
\dot{x} &= x - y + x^2, \\
\dot{y} &= 12x - 6y + xy,
\end{align*}
\]
The critical points are given by
\[
\begin{align*}
x - y + x^2 &= 0, \\
12x - 6y + xy &= 0,
\end{align*}
\]
Solving (2) the critical points are (0, 0), (2, 6), (3, 12).

(i) For (0, 0) the corresponding linear system is
\[
\begin{align*}
\dot{x} &= x - y, \\
\dot{y} &= 12x - 6y,
\end{align*}
\]
The characteristic equation of (3) is
\[
\lambda^2 + 5\lambda + 6 = 0
\]
\[\therefore \lambda = -2, -3\]
Here the roots are real, unequal and both negative.

Hence the critical point (0, 0) is asymptotically stable.

(ii) For (2, 6) we use

\[ x = \xi + 2, \quad y = \eta + 6 \]  \text{in (1) and obtain}

\[ \dot{\xi} = 5\xi - \eta + \xi^2, \]
\[ \dot{\eta} = 18\xi - 4\eta + \eta\xi. \]

The characteristic equation of linear part of (4) is

\[ \lambda^2 - \lambda - 2 = 0 \]
\[ \Rightarrow (\lambda - 2)(\lambda + 1) = 0 \]
\[ \therefore \lambda = 2, -1 \]

Here the roots are real, unequal and of opposite signs.

Hence the critical point (2, 6) is unstable.

(iii) For (3, 12) we use

\[ x = \xi + 3; \quad y = \eta + 12 \]  \text{in (1) and obtain}

\[ \dot{\xi} = 7\xi - \eta + \xi^2, \]
\[ \dot{\eta} = 24\xi - 3\eta + \eta\xi. \]

The characteristic equation of linear part of (5) is

\[ \lambda^2 - 4\lambda + 3 = 0 \]
\[ \therefore \lambda = 1, 3 \]

Here the roots are real, unequal and both positive.

Hence the critical point (3, 12) is unstable.

From (1)

\[ \frac{dy}{dx} = \frac{12x - 6y + xy}{x - y + x^2} = c \text{ (say)} \]
\[ c = 0, \quad y = \frac{12x}{6-x} \]
\[ c \to \infty, \quad \left( y + \frac{1}{2} \right) = \left( x + \frac{1}{2} \right)^2 \]

The required figure is given in the following figure:

**Figure 3.** 3-Dimensional figure for example 2.2.3, using MATHEMATICA software.

### 3. **Stability of Third Order Linear Systems by Characteristic Roots**

We consider the homogeneous linear system

\[ \frac{dx}{dt} = a_1x + b_1y + c_1z, \]  \text{(1)}

\[ \frac{dy}{dt} = a_2x + b_2y + c_2z, \]
\[ \frac{dz}{dt} = a_3x + b_3y + c_3z, \]

Where the coefficients \( a_1, \ b_1, \ c_1, \ a_2, \ b_2, \ c_2 \) are real constants.

Let us take a solution of the system (1) of the form

\[ x = Ae^{\lambda t}, \]
\[ y = Be^{\lambda t}, \]
\[ z = Ce^{\lambda t}, \]

Where \( A, B, C \) and \( \lambda \) are definite constants.

If we substitute (2) into (1) we obtain

\[ A\lambda e^{\lambda t} = a_1Ae^{\lambda t} + b_1Be^{\lambda t} + c_1Ce^{\lambda t}, \]
\[ B\lambda e^{\lambda t} = a_2Ae^{\lambda t} + b_2Be^{\lambda t} + c_2Ce^{\lambda t}, \]
\[ C\lambda e^{\lambda t} = a_3Ae^{\lambda t} + b_3Be^{\lambda t} + c_3Ce^{\lambda t}. \]

Which implies that

\[ (a_1 - \lambda)A + b_1B + c_1C = 0, \]
\[ a_2A + (b_2 - \lambda)B + c_2C = 0, \]
\[ a_3A + b_3B + (c_3 - \lambda)C = 0. \]

This system obviously has the trivial solution \( A=B=C=0 \). But this would only lead to the trivial solution \( x=0, y=0 \) and \( z=0 \) of the system. A necessary and sufficient condition that this system have a nontrivial solution is that the determinant

\[ \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{vmatrix} = 0. \]  \text{(4)}

Or

\[ (a_1 - \lambda)((b_2 - \lambda)(c_3 - \lambda) - b_3c_2) - b_1(a_2(c_3 - \lambda) - c_2a_3) + c_1(a_2b_3 - a_3(b_2 - \lambda)) = 0, \]

Or, \[ (a_1 - \lambda)(b_2c_3 - b_3\lambda - c_2\lambda + \lambda^2 - b_3c_2 - b_1(a_2c_3 - \lambda a_2 - a_3c_2) + c_1(a_2b_3 - a_3b_2 + a_3\lambda)) = 0, \]

Or, \[ \lambda - (a_1 + b_2 + c_3)^2 + (a_1b_2 + a_1c_3 + b_2c_3 - b_3c_2 - a_2b_1 - a_3c_1)\lambda - (a_1b_3c_2 - a_1b_2c_3 + a_2b_1c_3 - a_2b_3c_1 + a_2b_2c_1) = 0. \]  \text{(5)}

in the unknown \( \lambda \). This equation is called the characteristic equation associated with the system (1).

Its roots \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are called the characteristic roots of the given system.
Five cases must be considered:

- The roots $\lambda_1, \lambda_2$ and $\lambda_3$ are real and distinct.
- The roots $\lambda_1, \lambda_2$ and $\lambda_3$ are real and any two roots of them are equal and other is distinct.
- The roots $\lambda_1, \lambda_2$ and $\lambda_3$ are real and equal.
- The roots $\lambda_1$ is real and $\lambda_2$ and $\lambda_3$ are conjugate complex.
- The roots $\lambda_1$ is real and $\lambda_2$ and $\lambda_3$ are purely imaginary.

Case- 1
The roots of the characteristic equation (5) are real and distinct

If the roots $\lambda_1, \lambda_2$ and $\lambda_3$ of the characteristic equation (5) are real and distinct, it appears that we should expect three distinct solutions of the form (2), one corresponding to each of the three distinct roots. This is indeed the case. Furthermore, three distinct solutions are linearly independent. We summarize this case in the following theorem.

Theorem-4.1.1:

Hypothesis: The roots $\lambda_1, \lambda_2$ and $\lambda_3$ of the characteristic equation (4) associated with the system (1) are real and distinct. Assume that all the characteristic roots are negative.

- The critical point $(0, 0, 0)$ of the linear system (1) is asymptotically stable. The system (1) has three nontrivial linearly independent solutions of the form.

$$x = A_1 e^{\lambda_1 t},$$
$$y = B_1 e^{\lambda_2 t},$$
$$z = C_1 e^{\lambda_3 t},$$

Where $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3$ and $C_3$ are constants.

The general solution of the system (1) may thus be written as

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t},$$
$$y = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t},$$
$$z = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t},$$

Where $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ are arbitrary constants.

Case-2
The roots of the characteristic equation (5) are one distinct and two equal

If the roots $\lambda_1$ of the characteristic equation is real and distinct, it appear that we should expect one distinct solution of the form

$$x = A_1 e^{\lambda_1 t},$$
$$y = B_1 e^{\lambda_2 t},$$
$$z = C_1 e^{\lambda_3 t},$$

Further more this solution are linearly independent. And the roots $\lambda_2$ and $\lambda_3$ of the characteristic equation (5) are real and equal, it would appear that, we could find only one solution of the form (2). Except in that subcase $a_1 = b_2 = c_3 \neq 0, a_2 = b_3 = c_1 = 0$. This would lead us to expect a second and third solution of the form

$$x = A e^{\lambda t},$$
$$y = B e^{\lambda t},$$
$$z = C e^{\lambda t},$$

And

$$x = (A_2 t + A_3) e^{\lambda t},$$
$$y = (B_2 t + B_3) e^{\lambda t},$$
$$z = (C_2 t + C_3) e^{\lambda t},$$

We can summarize this by the following theorem.

Theorem-4.1.2:

Hypothesis: If the roots of the characteristic equation (5) one of which $\lambda_1$ is real, distinct and $\lambda_2, \lambda_3$ are real, equal and the system (1) is not such that $a_1 = b_2 = c_3 \neq 0, a_2 = b_3 = c_1 = 0$. Also let $\lambda$ be their common value.

- The system (1) has one non-trivial and two linearly independent solutions of the form

$$x = A_1 e^{\lambda_1 t},$$
$$y = B_1 e^{\lambda_2 t},$$
$$z = C_1 e^{\lambda_3 t},$$

Where $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ and $A$, $B$, $C$ are definite constants and $A_2, B_2, C_2$ not all zero such that $\frac{b_2}{a_2} = \frac{b_3}{a_2}$ and $\frac{c_2}{a_2} = \frac{c_3}{a_2}$.

Hence the general solution of the system (1) is

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t},$$
$$y = B_1 e^{\lambda_2 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t},$$
$$z = C_1 e^{\lambda_3 t} + C_2 e^{\lambda_3 t} + C_3 e^{\lambda_3 t},$$

Where $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ are arbitrary constants.

Case-3
The roots of the characteristic equation (5) are real and equal:

If the roots of the characteristic equation (5) are real and equal, it should appear that we could find only one solution of the form (2) except in that subcase.

This would help us to except a second and third solution of the form

$$x = A e^{\lambda t},$$
$$y = B e^{\lambda t},$$
$$z = C e^{\lambda t},$$

And

$$x = (A_2 t + A_3) e^{\lambda t},$$
$$y = (B_2 t + B_3) e^{\lambda t},$$
$$z = (C_2 t + C_3) e^{\lambda t},$$

Hence the general solution of the system (1) is

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t},$$
$$y = B_1 e^{\lambda_2 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t},$$
$$z = C_1 e^{\lambda_3 t} + C_2 e^{\lambda_3 t} + C_3 e^{\lambda_3 t},$$

Where $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ are arbitrary constants.
And
\[\begin{align*}
x &= (A_1 t + A_2)e^{\lambda t}, \\
y &= (B_1 t + B_2)e^{\lambda t}, \\
z &= (C_1 t + C_2)e^{\lambda t},
\end{align*}\]

Theorem 4.1.3:

Hypothesis: If the roots of the characteristic equation (5) associated with the system (1) are real and equal and the system (1) is not such that \(a_1 = b_2 = c_2 \neq 0\), \(a_1 = b_3 = c_4 = 0\), we could find only one solution of the form (2) except in that subcase.

Case 4

The roots of the characteristic equation (5) are real and conjugate complex.

If the root \(\lambda_1\) is real, we would expect one solution of the form
\[\begin{align*}
x &= A e^{\lambda t}, \\
y &= B e^{\lambda t}, \\
z &= C e^{\lambda t},
\end{align*}\]

And express the first solution of (7) and (8) in the form
\[\begin{align*}
x &= (A_1 + iA_2)e^{\alpha t}, \\
y &= (B_1 + iB_2)e^{\alpha t}, \\
z &= (C_1 + iC_2)e^{\alpha t},
\end{align*}\]

Where \(A_1, A_2, B_1, B_2, C_1, C_2\) are arbitrary constants.

Case 4

The roots of the characteristic equation (5) are real and conjugate complex.

If the root \(\lambda_1\) is real, we would expect one solution of the form
\[\begin{align*}
x &= A e^{\lambda t}, \\
y &= B e^{\lambda t}, \\
z &= C e^{\lambda t},
\end{align*}\]

Also this distinct solution are linearly independent and if the roots \(\lambda_2\) and \(\lambda_3\) are conjugate complex as \(\alpha \pm i \beta\), then we will obtain two distinct solutions
\[\begin{align*}
x &= A_1^* e^{\alpha + i\beta}, \\
y &= A_2^* e^{\alpha - i\beta}, \\
z &= B_1^* e^{\alpha + i\beta}, \\
z &= B_2^* e^{\alpha - i\beta}.
\end{align*}\]

Thus the real part is
\[\begin{align*}
x &= e^{\alpha t}(A_1 \cos \beta - A_2 \sin \beta), \\
y &= e^{\alpha t}(B_1 \cos \beta - B_2 \sin \beta), \\
z &= e^{\alpha t}(C_1 \cos \beta - C_2 \sin \beta),
\end{align*}\]

And the imaginary part is
\[\begin{align*}
x &= e^{\alpha t}(A_2 \cos \beta - A_1 \sin \beta), \\
y &= e^{\alpha t}(B_2 \cos \beta - B_1 \sin \beta), \\
z &= e^{\alpha t}(C_2 \cos \beta - C_1 \sin \beta).
\end{align*}\]
Also this solutions (10) and (11) are linearly independent.

Now we can verify this by the Wronskian determinant for these solutions we find

\[
W(t) = \begin{vmatrix} A e^{\lambda_1 t} & e^{\alpha t}(A_2 \cos \beta t - A_2 \sin \beta t) & e^{\alpha t}(A_2 \cos \beta t + A_2 \sin \beta t) \\ B e^{\lambda_2 t} & e^{\alpha t}(B_2 \cos \beta t - B_2 \sin \beta t) & e^{\alpha t}(B_2 \cos \beta t + B_2 \sin \beta t) \\ C e^{\lambda_3 t} & e^{\alpha t}(C_2 \cos \beta t - C_2 \sin \beta t) & e^{\alpha t}(C_2 \cos \beta t + C_2 \sin \beta t) \end{vmatrix} \neq 0
\]  

Since the Wronskian determinant \( W(t) \) is not equal to zero. Thus the solutions (10) and (11) are needed linearly independent. Hence a linear combination of these two real solutions the general solution of the system (1). We can summarize these by the following theorem.

**Theorem-4.1.4:**

**Hypothesis:** If the roots of the characteristic equation (5) one of which \( \lambda_1 \) is real and \( \lambda_2, \lambda_3 \) are conjugate complex numbers \( \alpha \pm i \beta \).

The system (1) has one non-trivial linearly independent solution two real linearly dependent solutions of the form

\[
x = A_1 e^{\lambda_1 t}, \quad y = B_1 e^{\lambda_1 t}, \quad z = C_1 e^{\lambda_1 t},
\]

Where \( A_1, B_1, C_1 \) are definite constants.

The general solution may thus be written by

\[
x = K_1 A e^{\lambda_1 t} + K_2 e^{\alpha t}(A_2 \cos \beta t - A_2 \sin \beta t), \quad y = K_2 B e^{\alpha t}(B_2 \cos \beta t - B_2 \sin \beta t) + K_2 e^{\alpha t}(B_2 \cos \beta t - B_2 \sin \beta t),
\]

Where \( K_1, K_2, K_3 \) are arbitrary constant

Case-5

The roots of the characteristic equation (5) are real and purely imaginary:

If the root \( \lambda_1 \) is real, we would accept one distinct solution of the form

\[
x = A e^{\lambda_1 t}, \quad y = B e^{\lambda_1 t},
\]

and if the roots \( \lambda_2 \) and \( \lambda_3 \) are the purely imaginary numbers \( \pm bi \), then we will obtain two distinct solutions

\[
x = A_1^* e^{iat}, \quad x = A_2^* e^{-iat}, \quad y = B_1^* e^{iat}, \quad y = B_2^* e^{-iat},
\]

of the form (2), one corresponding to each of the complex roots. However the solutions (14) are complex solutions. In order to obtain real solutions in this case we first express the complex constant \( A_1^* \), \( B_1^* \), \( C_1^* \) in the form

\[
A_1^* = A_1 + iA_2,
\]

\[
B_1^* = B_1 + iB_2,
\]

\[
C_1^* = C_1 + iC_2,
\]

Where \( A_1, B_1, C_1, A_2, B_2, C_2 \) are real.

Now we apply Euler’s formula

\[e^{i\theta} = \cos \theta + i \sin \theta\]

and express the first solution of (14) in the form

\[
x = (A_1 + iA_2)(\cos \beta t + i \sin \beta t),
\]

\[
y = (B_1 + iB_2)(\cos \beta t + i \sin \beta t),
\]

\[
z = (C_1 + iC_2)(\cos \beta t + i \sin \beta t),
\]

Where \( A_1, A_2, B_1, B_2, C_1, C_2 \) are constants.

Rewriting this, we have

\[
x = [(A_1 \cos \beta t - A_2 \sin \beta t)] + i[(A_2 \cos \beta t + A_1 \sin \beta t)],
\]

\[
y = [(B_1 \cos \beta t - B_2 \sin \beta t)] + i[(B_2 \cos \beta t + B_1 \sin \beta t)],
\]

\[
z = [(C_1 \cos \beta t - C_2 \sin \beta t)] + i[(C_2 \cos \beta t + C_1 \sin \beta t)].
\]

It can be shown that a pair \([f_1(t), f_2(t)]\) and \([h_1(t), h_2(t)]\) of complex function is a solution of the system (1) if and only if the pair \([f_1(t), g_1(t)]\) consisting of their real parts and the pair \([f_2(t), g_2(t), h_2(t)]\) consisting of their imaginary parts are solutions of (15). Thus both the real part

\[
x = (A_1 \cos \beta t - A_2 \sin \beta t),
\]

\[
y = (B_1 \cos \beta t - B_2 \sin \beta t),
\]

\[
z = (C_1 \cos \beta t - C_2 \sin \beta t),
\]

and the imaginary part

\[
x = (A_2 \cos \beta t + A_1 \sin \beta t),
\]

\[
y = (B_2 \cos \beta t + B_1 \sin \beta t),
\]

\[
z = (C_2 \cos \beta t + C_1 \sin \beta t),
\]

of the solution (15) of the system (1) are also solution of the system (14).

Also the solutions (16) and (17) are linearly independent.

We can verify this by evaluating the Wronskian determinant for these solutions. We find

\[
W(t) = \begin{vmatrix} A e^{\lambda_1 t} & A_1 \cos \beta t - A_2 \sin \beta t & (A_2 \cos \beta t + A_1 \sin \beta t) \\ B e^{\lambda_2 t} & B_1 \cos \beta t - B_2 \sin \beta t & (B_2 \cos \beta t + B_1 \sin \beta t) \\ C e^{\lambda_3 t} & C_1 \cos \beta t - C_2 \sin \beta t & (C_2 \cos \beta t + C_1 \sin \beta t) \end{vmatrix} \neq 0
\]

Since the Wronskian determinant \( W(t) \) in (18) is not equal to zero. Thus the solutions (16) and (17) are linearly independent. Hence a linear combination of these two real solutions provides the general solution of the system (1). We can summarize these results by the following theorem.

**Theorem-4.1.5:**
Hypothesis: If the roots of the characteristic equation (5) one of which \( \lambda_1 \) is real and distinct and \( \lambda_2, \lambda_3 \) are the purely imaginary numbers \( \pm \beta t \).

The system (1) has one non-trivial and two real linearly independent solutions of the form

\[
x = A e^{\lambda_1 t}, \quad y = B e^{\lambda_1 t}, \quad z = C e^{\lambda_1 t},
\]

and

\[
x = (A_1 \cos \beta t - A_2 \sin \beta t), \quad x = (A_2 \cos \beta t - A_1 \sin \beta t),
\]

\[
y = (B_1 \cos \beta t - B_2 \sin \beta t), \quad y = (B_2 \cos \beta t - B_1 \sin \beta t),
\]

\[
z = (C_1 \cos \beta t - C_2 \sin \beta t), \quad z = (C_2 \cos \beta t - C_1 \sin \beta t),
\]

Where \( A_1, A_2, B_1, B_2, C_1, C_2 \) and \( A, B, C \) are definite real constants.

The general solution of the system (1) may thus be written by

\[
x = K_1 e^{\lambda_1 t} + [K_2(A_1 \cos \beta t - A_2 \sin \beta t) + K_3(A_2 \cos \beta t - A_1 \sin \beta t)],
\]

\[
y = K_1 e^{\lambda_1 t} + [K_2(B_1 \cos \beta t - B_2 \sin \beta t) + K_3(B_2 \cos \beta t - B_1 \sin \beta t)],
\]

\[
z = K_1 e^{\lambda_1 t} + [K_2(C_1 \cos \beta t - C_2 \sin \beta t) + K_3(C_2 \cos \beta t - C_1 \sin \beta t)],
\]

Where, \( K_1, K_2, K_3 \) are arbitrary constants.

6. CONCLUSIONS

In this article, we have discussed about the stability of second and third order linear and non-linear differential equations by characteristic roots. We discussed these methods with a hypothesis and illustrative examples which will be helpful for the further investigation.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest related to the publication of this article.

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