A Note on Comparator-Overdrive-Delay Conditioning for Current-Mode Control

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Abstract

Comparator-overdrive-delay conditioning is a new control conditioning approach for high frequency current-mode control. No existing literature rigorously studies the effect of the comparator overdrive delay to the current-mode control. The results in this paper provide insights on the mechanism of comparator-overdrive-delay conditioning.

I. INTRODUCTION

High-frequency current-mode control is one of the most popular controller strategies for power converters [1], [2]. In high-frequency dc-dc converters, it is not accurate enough to just model the comparator as an ideal logic comparator. In a non-ideal comparator, there exists a propagation delay from the input to the output. This delay depends on the input overdrive, the difference between input voltages. This phenomenon is called comparator overdrive propagation delay [3]. We emphasize that this nonideality can be carefully designed and utilized to attenuate the interference in the current measurement.

II. THEORY

Theorem 1. Given a constant off-time current control loop with comparator overdrive delay, if the input is a ramp with slope $m_1$ and interference function $w(t)$, the condition to guarantee the continuous static current mapping is

$$V_{\text{trig}} \tau \geq m_1 K_3 \left( \frac{|W(\omega)|}{m_1} \right).$$

Proof. We start with defining the saturating integral operator

Definition 1. Given a function $f(x)$ that is continuous on the interval $[a,b]$, we divide the interval into $n$ subintervals of equal width $\Delta x$ and from each interval choose a point, $x_i^*$. The saturating partial sum of this sequence is

$$\hat{S}_{n+1} \triangleq u \left( \hat{S}_n + f(x_i^*[n+1]) \Delta x \right),$$

where $u(x)$ is a saturating function where $u(x) = 0$ if $x < 0$ and $u(x) = x$ if $x \geq 0$.

Definition 2. The saturating integral operator is defined as

$$\int_a^b f(x) \, dx \triangleq \lim_{n \to \infty} \hat{S}_n.$$  

We consider the following implicit mapping, which defines a function $M: b \to \theta$

$$\int_{-\infty}^{\theta} \left( x + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(w)| e^{j(\omega x + \varphi(\omega))} \, d\omega \right) \, dx = k,$$

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where \( k \) parametrizes the transconductance, effective capacitor, and threshold voltage as \( k \triangleq V_{\text{trig}}C_{\text{eff}}/G \). We define functions \( K_1 \) and \( K_2 \) as

\[
K_1(|W(\omega)|) \triangleq \sup_{b \in \mathbb{R}} \quad 0 \leq \varphi(\omega) \leq 2\pi \quad K_2(b, \varphi(\omega)),
\]

\[
K_2(b, \varphi(\omega)) \triangleq \int_{-\infty}^{\psi_h} \left( x + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)|e^{i(\omega x + \varphi(\omega))} \, d\omega - b \right) \, dx,
\]

where \( \psi_l \) and \( \psi_h \), as the function of \( b \) and \( \varphi(\omega) \), are the lowest and largest solution of the equation

\[
x + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)|e^{i(\omega x + \varphi(\omega))} \, d\omega - b = 0.
\]

It is not very clear if \( K_1 \) exists or not. We further introduce \( K_3 \) and \( K_4 \) to analyze. We define functions \( K_3 \) and \( K_4 \) as

\[
K_3(|W(\omega)|) \triangleq \max_{-A \leq b' \leq A} \quad 0 \leq \varphi'(\omega) \leq 2\pi \quad K_4(b', \varphi'(\omega)),
\]

\[
K_4(b', \varphi'(\omega)) \triangleq \int_{-\infty}^{\psi_h'} \left( x + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)|e^{i(\omega x + \varphi'(\omega))} \, d\omega - b' \right) \, dx,
\]

where \( \psi_l' \) and \( \psi_h' \), as the function of \( b' \) and \( \varphi'(\omega) \), are the lowest and largest solution of the equation

\[
x + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)|e^{i(\omega x + \varphi'(\omega))} \, d\omega - b' = 0.
\]

**Lemma 1.** We show that given \( b \) and \( \varphi(\omega) \), we can always find \( b' \) and \( \varphi'(\omega) \) such that \( K_4(b', \varphi'(\omega)) = K_2(b, \varphi(\omega)) \), and vice versa.

**Proof.** We define

\[
\varphi'(\omega) = \varphi(\omega) + \omega \psi_l,
\]

\[
b' = b - \psi_l.
\]

We first show the corresponding \( \psi_h' \) and \( \psi_h \) as well as \( \psi_h \) and \( \psi_l \) satisfy the following relationships

\[
\psi_h' = \psi_h - \psi_l,
\]

\[
\psi_l' = 0.
\]

The relationship can be verified as

\[
\psi_h + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)|e^{i(\omega \psi_h + \varphi(\omega))} \, d\omega - b = 0,
\]

\[
\psi_h - \psi_l + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)|e^{i(\omega \psi_h + \varphi'(\omega) - \omega \psi_l)} \, d\omega - (b - \psi_l) = 0.
\]

We validate that \( \psi_h' \) is the solution

\[
\psi_h' + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)|e^{i(\omega \psi_h' + \varphi'(\omega))} \, d\omega - b' = 0.
\]
Similarly, we can validate that zero is the solution
\[
K_4(b, \varphi'(\omega)) = \int_0^{\psi_h} \left( x + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)| e^{j(\omega x + \varphi'(\omega))} d\omega - b \right) dx
\]
\[
= \int_0^{\psi_h} \left( x + \psi_l + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)| e^{j(\omega x + \varphi(\omega) + \omega\psi_l)} d\omega - b \right) dx
\]
\[
= \int_0^{\psi_l} \left( x + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |W(\omega)| e^{j(\omega x + \varphi(\omega))} d\omega - b \right) dx
\]
\[
= K_2(b, \varphi(\omega)).
\]  \hfill (18)

The opposite direction is similar so we omit the proof.

Therefore, \( K_3 \), defined as the maximum of \( K_4 \), should be equal to \( K_1 \), defined as the maximum of \( K_2 \),
\[
K_1\left(|W(\omega)|\right) = K_3\left(|W(\omega)|\right).
\]  \hfill (19)

\( K_3(|W(\omega)|) \) exists from the Weierstrass theorem, hence \( K_1(|W(\omega)|) \) exists.

The following Lemma[2] provides the condition that guarantees the static mapping to be continuous.

**Lemma 2.** For all \( k > K_1(|W(\omega)|) \), function \( \theta = \mathcal{M}(b) \) is continuous.

**Proof.** From the subadditivity,
\[
\left| \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \right| \leq \int_a^b |f(x) - g(x)| \, dx.
\]  \hfill (20)

Then we prove given \( b \) which approaches \( b_0 \), \( \theta \) approaches \( \theta_0 \). \( b \) and \( \theta \) satisfies the integral-threshold constraint
\[
\int_{-\infty}^{\theta} (x + w(x) - b) \, dx = k.
\]  \hfill (21)

\( b_0 \) and \( \theta_0 \) satisfies the integral-threshold constraint
\[
\int_{-\infty}^{\theta_0} (x + w(x) - b_0) \, dx = k.
\]  \hfill (22)

From (21) and (22),
\[
\int_{-\infty}^{\theta_0} (x + w(x) - b_0) \, dx - \int_{-\infty}^{\theta} (x + w(x) - b) \, dx
\]
\[
= \int_{-\infty}^{\theta} (x + w(x) - b) \, dx - \int_{-\infty}^{\theta_0} (x + w(x) - b) \, dx.
\]  \hfill (23)

Denote the right hand side of equation as RHS. Because the \( k > k_{\text{min}} \), \( \theta > \theta(b) \) and \( \theta_0 > \theta(b_0) \),
\[
|RHS| = \left| \int_{\theta}^{\theta_0} (x + w(x) - b) \, dx \right| \geq \mu_1 |\theta - \theta_0|,
\]  \hfill (24)

where
\[
\mu_1 = \min_{x \in [\theta_0, \theta]} (x + w(x) - b).
\]  \hfill (25)
Denote the right hand side of equation as LHS.

\[
|\text{LHS}| \leq \int_{-\infty}^{\theta_0} (x + w(x) - b_0) - (x + w(x) - b) \, dx \leq \mu_2 |b - b_0|,
\]

where

\[
\mu_2 = \max_{b \in [b_0, b]} \left( \theta_0 - \psi_l(b) \right).
\]

Therefore, we have

\[
|\theta - \theta_0| \leq \frac{\mu_2}{\mu_1} |b - b_0|.
\]

Lemma contains the theorem. Hence the proof is done.

\[\blacksquare\]

**Theorem 2.** Given a constant off-time current control loop with comparator overdrive delay, if the input is a ramp with slope \(m_1\) and interference function \(w(t)\), the maximum comparator overdrive delay \(t_{\text{max}}\) is

\[
t_{\text{max}} = \frac{A_{ub}}{m_1} + \sqrt{\left( \frac{A_{ub}}{m_1} \right)^2 + \frac{2}{m_1} (V_{\text{th}} \tau + B)},
\]

where

\[\tau = C_{\text{eff}} G, \quad B = \int_{-\infty}^{+\infty} \left| \frac{W(\omega)}{\omega} \right| d\omega.\]

**Proof.** We omitted the discrete-time notation \([n]\) for the ease of derivation. The current error \(i_e \triangleq i_c - i_p\) is defined as the difference between peak current command \(i_c\) and actual peak current \(i_p\).

Given that the integrator does not saturate after \(t > t_c\), the crossing time \(t_c\) and on-time \(t_{\text{on}}\) satisfy

\[
\frac{1}{C_{\text{eff}}} \int_{t_c}^{t_{\text{on}}} g \left( i_e + m_1 (t - t_{\text{on}}) + w(t) \right) \, dt = V_{\text{th}}.\]

The transconductance \(g\) is bounded from above by \(G\). \(G\) provides the shortest overdrive delay. Therefore we do the worst-case analysis by substituting \(g\) by \(G\). Then the worst-case comparator time constant is defined as \(\tau_c \triangleq C_{\text{eff}} / G\)

\[
\int_{t_c}^{t_{\text{on}}} \left( i_e + m_1 (t - t_{\text{on}}) + w(t) \right) \, dt = V_{\text{th}} \tau_c.
\]

**A. Sector Boundedness of Nonlinearity \(\psi\)**

Although \(\psi\) is not differentiable everywhere, we can still prove that \(\psi\) is sector bounded by \([-2A_{ub}/t_{\text{od}}^{\text{min}}, +\infty)\) or equivalently in math as

\[
\left( \psi(\hat{t}_{\text{on}}) + \frac{A_{ub}}{t_{\text{od}}^{\text{min}}} \hat{t}_{\text{on}} \right) \hat{t}_{\text{on}} > 0,
\]

where \(t_{\text{od}}^{\text{min}}\) is the minimum comparator overdrive delay.

(1) We perturb \(i_e\) by \(d_i^e\), which is a positive infinitesimal. Because of the continuity property of the static mapping, the resulting increase on the on-time \(dt_{\text{on}}^+\) is also a positive infinitesimal. However, the variation on crossing time \(\Delta t_c^+\) might be a positive large number

\[
\int_{t_c + \Delta t_c^+}^{t_{\text{on}} + dt_{\text{on}}^+} \left( i_e + d_i^e \right) + m_1 (t - t_{\text{on}} - dt_{\text{on}}^+) + w(t) \, dt = V_{\text{th}} \tau_c.
\]
\[
\int_{t_c + \Delta t_c^+}^{t_{on}} -i_e + m_1(t - t_{on}) + w(t) \, dt + \int_{t_c}^{t_{on} + \Delta t_c^+} -i_c + m_1(t - t_{on}) + w(t) \, dt = V_{th} \tau_c. \tag{34}
\]

Because \(-i_e + m_1(t - t_{on}) + w(t)\) is positive for all \(t \geq t_c\), we have
\[
\int_{t_c + \Delta t_c^+}^{t_{on} + \Delta t_c^+} -i_e + m_1(t - t_{on} - dt_{on}^+) + w(t) \, dt \geq \int_{t_c + \Delta t_c^+}^{t_{on}} -i_e + m_1(t - t_{on}) + w(t) \, dt. \tag{35}
\]

Equation (35) can be algebraically transformed as
\[
-(i_e + di_e^+)(t_{on} + dt_{on}^+ - t_c - \Delta t_c^+) + \int_{t_c + \Delta t_c^+}^{t_{on} + \Delta t_c^+} w(t) \, dt - \frac{m_1}{2}(t_c + \Delta t_c^+ - t_{on} - dt_{on}^+)^2 \\
\geq -i_e(t_{on} - t_c - \Delta t_c^+) - \frac{m_1}{2}(t_c + \Delta t_c^+ - t_{on})^2 + \int_{t_c + \Delta t_c^+}^{t_{on}} w(t) \, dt. \tag{36}
\]

Considering \(di_e^+\) and \(dt_{on}^+\) are infinitesimals, (36) can be rearranged as
\[
-i_e dt_{on}^+ - di_e^+(t_{on} - t_c - \Delta t_c^+) + dt_{on}^+ w(t_{on}) + m_1(t_c + \Delta t_c^+ - t_{on}) \geq 0. \tag{37}
\]

We substitute (38) into (37). In this way, (37) is simplified into (39)
\[
-(i_e + di_e^+) + m_1(t_c + \Delta t_c^+ - t_{on}) + w(t_c + \Delta t_c^+) = 0, \tag{38}
\]

\[
(t_{on} - t_c - \Delta t_c^+) di_e^+ \leq (w(t_{on}) - w(t_c + \Delta t_c^+)) dt_{on}^+. \tag{39}
\]

We observe that \((t_{on} - t_c - \Delta t_c^+)\), and \(di_e^+, dt_{on}^+\) are all positive. Therefore, the right derivative of the function from \(t_{on}\) to \(i_e\) is bounded by
\[
0 \leq \frac{di_e^+}{dt_{on}^+} \leq \frac{w(t_{on}) - w(t_c + \Delta t_c^+)}{t_{on} - (t_c + \Delta t_c^+)} \leq \frac{2A_{ub}}{t_{on}^m}, \tag{40}
\]

where \(t_{on}^m\) is the minimum comparator overdrive delay.

(2) We perturb \(i_e\) by \(di_e^-\) which is a negative infinitesimal. A similar inequality is obtained as
\[
(t_{on} - t_c - \Delta t_c^-) di_e^- \leq (w(t_{on}) - w(t_c + \Delta t_c^-)) dt_{on}^- . \tag{41}
\]

The left derivative of the function from \(t_{on}\) to \(i_e\) is bounded by
\[
\frac{di_e^-}{dt_{on}^-} \geq \frac{w(t_{on}) - w(t_c + \Delta t_c^-)}{t_{on} - (t_c + \Delta t_c^-)} \geq -\frac{2A_{ub}}{t_{on}^m} . \tag{42}
\]

(3) Therefore, \(\psi\) is sector bounded by
\[
\left(\psi(t_{on}) + \frac{A_{ub}}{t_{on}^m} \right) t_{on} \geq 0. \tag{43}
\]
B. Bounds of the Comparator Overdrive Delay

From the circle criterion, a sufficient and necessary condition for the stability of current control loop is

$$\min \left\{ \frac{di^+}{dt^+_on}, \frac{di^-}{dt^-_on} \right\} > -\frac{m_1}{2}. \quad (44)$$

From (42)

$$\frac{di^+}{dt^+_on} \geq 0. \quad (45)$$

From (40)

$$\frac{di^-}{dt^-_on} \geq \frac{w(t_{on}) - w(t_c)}{t_{on} - t_c}. \quad (46)$$

We can prove (44) if we can show the following:

$$\frac{w(t_{on}) - w(t_c)}{t_{on} - t_c} > -\frac{m_1}{2}. \quad (47)$$

Given that integrator does not saturate after $t > t_c$, the crossing time $t_c$ and on-time $t_{on}$ satisfy

$$\frac{1}{C_{eff}} \int_{t_c}^{t_{on}} g \left( i_e + m_1 (t - t_{on}) + w(t) \right) dt = V_{th}. \quad (48)$$

The transconductance $g$ is bounded from above by $G$. $G$ provides the shortest overdrive delay. Therefore we do the worst-case analysis by substituting $g$ by $G$. Then the worst-case comparator time constant is defined as $\tau_c \triangleq C_{eff}/G$

$$\int_{t_c}^{t_{on}} \left( i_e + m_1 (t - t_{on}) + w(t) \right) dt = V_{th} \tau_c. \quad (49)$$

Equation (49) can be equivalently written as

$$\frac{1}{2} m_1 (t_{on} - t_c)^2 - w(t_c)(t_{on} - t_c) + \int_{t_c}^{t_{on}} w(t) \, dt = V_{th} \tau_c. \quad (50)$$

If $W(\omega)/\omega$ is absolute integrable

$$\left| \int_{-\infty}^{\infty} f(\tau) \, d\tau \right| = \left| \int_{-\infty}^{\infty} \left( \frac{W(\omega)}{j\omega} + \pi W(0) \delta(\omega) \right) e^{j\omega t} \, d\omega \right| \leq \int_{-\infty}^{\infty} \left| \frac{W(\omega)}{\omega} \right| \, d\omega. \quad (51)$$

We can bound the integral term in (50) as

$$\left| \int_{t_c}^{t_{on}} w(t) \, dt \right| \leq \int_{-\infty}^{\infty} \left| \frac{W(\omega)}{\omega} \right| \, d\omega. \quad (52)$$

We define variable delay $t_{od} \triangleq t_{on} - t_c$. From (50),

$$t_{od} = \frac{w(t_c)}{m_1} \pm \sqrt{\left( \frac{w(t_c)}{m_1} \right)^2 + \frac{2}{m_1} \left( k - \int_{t_c}^{t_{on}} w(t) \, dt \right)}. \quad (53)$$

Because $V_{th} \tau_c > B$, we narrow down two solutions in (53) to one feasible solution

$$t_{od} = \frac{w(t_c)}{m_1} + \sqrt{\left( \frac{w(t_c)}{m_1} \right)^2 + \frac{2}{m_1} \left( V_{th} \tau_c - \int_{t_c}^{t_{on}} w(t) \, dt \right)}. \quad (54)$$
From (52), the variable delay is bounded from above by $t_u$ and from below by $t_l$

$$t_u \triangleq \frac{w(t_c)}{m_1} + \sqrt{\left(\frac{w(t_c)}{m_1}\right)^2 + \frac{2}{m_1}(V_{th}\tau_c + B)},$$

(55)

$$t_l \triangleq \frac{w(t_c)}{m_1} + \sqrt{\left(\frac{w(t_c)}{m_1}\right)^2 + \frac{2}{m_1}(V_{th}\tau_c - B)}.$$

(56)

We prove that $V_{th}\tau_c$ can guarantee the stability by using (44). The lower sector bound of the nonlinear function in the feedback path is

$$\frac{w(t_{on}) - w(t_c)}{t_{on} - t_c} \geq -A - w(t_c) \geq -A - \frac{w(t_c)}{t_l} =$$

(57)

$$\frac{-A - w(t_c)}{m_1^2 + \sqrt{\left(\frac{w(t_c)}{m_1}\right)^2 + \frac{2}{m_1}(V_{th}\tau_c - B)}} =$$

(58)

$$\frac{w(t_c)}{m_1} + \sqrt{\left(\frac{w(t_c)}{m_1}\right)^2 + \frac{2}{m_1}(V_{th}\tau_c - B)} - 1 + \frac{w(t_c)}{A} m_1 \geq$$

(59)

$$= \frac{w(t_c)}{A} + \sqrt{\left(\frac{w(t_c)}{A}\right)^2 + \frac{2m_1}{A^2}(V_{th}\tau_c - B)} - 1 + \frac{w(t_c)}{A} m_1 \geq$$

(60)

$$= \frac{w(t_c)}{A} + \sqrt{\left(\frac{w(t_c)}{A}\right)^2 + 8} - \frac{m_1}{2}.$$ 

(61)

The derivation from (60) to (61) is because the following auxiliary function is monotonically decreasing in the domain $x \in [-1, 1]$ if $\mu \geq 8$:

$$y \triangleq \frac{-1 - x}{x + \sqrt{x^2 + \mu}}.$$

(62)

For $\mu = 8$ and $y_{min} = y|_{x=1} = -0.5$, which guarantees the stability of the system.

The longest delay can be obtained from (55) as

$$t_{od}^{max} = \frac{A}{m_1} + \sqrt{\left(\frac{A}{m_1}\right)^2 + \frac{2}{m_1}(V_{th}\tau_c + B)}.$$

(63)

III. CONCLUSION

This paper develops the three theoretical results for high-frequency current-mode control using comparator-overdrive-delay conditioning: (1) the continuity condition of the static current mapping; (2) the closed-form sector bounds of the nonlinearity in the dynamical current mapping; (3) the closed-form upper and lower bounds of the comparator overdrive delay time.
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