Strong Coupling Expansion of the Entanglement Entropy of Yang-Mills Gauge Theories

Jiunn-Wei Chen

Department of Physics and Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan and Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei 10617, Taiwan

Shou-Huang Dai

Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei 10617, Taiwan

Jin-Yi Pang

Department of Physics and Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan and Interdisciplinary Center for Theoretical Study and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China

Abstract

We calculate the entanglement entropy of the $SU(N)$ Yang-Mills gauge theories on the lattice under the strong coupling expansion in powers of $\beta = 2N/g^2$, where $g$ is the coupling constant. Using the replica method, our Lagrangian formalism maintains gauge invariance on the lattice. At $O(\beta^2)$ and $O(\beta^3)$, the entanglement entropy is solely contributed by the central plaquettes enclosing the conical singularity of the $n$-sheeted Riemann surface. The area law emerges naturally to the highest order $O(\beta^3)$ of our calculation. The leading $O(\beta)$ term is negative, which could in principle be canceled by taking into account the “cosmological constant” living in interface of the two entangled subregions. This unknown cosmological constant resembles the ambiguity of edge modes in the Hamiltonian formalism. We further speculate this unknown cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Furthermore, it could play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.
I. INTRODUCTION

Entanglement entropy is a measure of the level of entanglement between the degrees of freedom in two subregions of a physical system. In some systems, entanglement entropy plays the role of an order parameter to characterize quantum phase transitions [1, 3], while in others it demonstrates the scaling behavior [4, 5]. In field theory, a widely used method to calculate the entanglement entropy is the replica method [6]. This method calculates the trace of the reduced density matrix to the $n$-th power in the path integral formalism, which amounts to computing the free energy of the system on a $n$-sheeted Riemann surface, or equivalently on a cone with a conical angle of $2n\pi$. The entanglement entropy is then obtained as a response of the free energy to the change of the conical angle at $n = 1$.

In previous studies, while the computation of entanglement entropy for the scalar and spinor fields are considered straightforward, the gauge field case is more subtle. In the Hamiltonian approach [7–10], one might need to impose constraints such as the Gauss law or gauge fixing in order to get rid of the unphysical degrees of freedom. Then if one uses the gauge link language to discretize the theory, there is an ambiguity about which side the gauge links on the boundary belong to. Also, it was argued that the inability to decompose all gauge invariant states into direct products of gauge invariant states living in the two subregions might be a problem. It was proposed that these ambiguities might be compensated by edge modes which live in the interface of the two subregions [7].

This situation reminds us the difficulty of the canonical quantization of gauge fields. It is desirable to explore other approaches which might shed light on the problem from a different angle. Therefore, in this paper, we study the entanglement entropy on a lattice using the Lagrangian formalism [11, 12].

Our starting point is the replica method, which is assumed to be valid as long as the quantum field theory is local. Then we use the Wilson gauge action which has the advantage of being gauge invariant on a discrete lattice. This action is the sum of plaquettes which are Wilson loops living on the $n$-sheeted Riemann surface. There are two types of plaquettes on our squared lattice. If a plaquette encircles the tip of the cone, then it is formed by $4n$ links and called a central plaquette. Otherwise it is formed by 4 links, and is called a regular plaquette. When the conical angle or $n$ changes, only those plaquettes with $4n$ links response to this change. Therefore entanglement entropy necessarily involves those central
plaquettes. The fact that those central plaquettes all live in the interface between the two subregions naturally hints to the area law of entanglement entropy which states that the leading contribution to entanglement entropy scales as the area of the interface.

The connection to the area law can be further demonstrated order by order diagrammatically under the strong coupling expansion. Interestingly, we find the leading term in the strong coupling expansion to be negative. However, symmetries of the action allow a two-dimensional cosmological constant living in the interface [13] which could provide a positive contribution at an even lower order. We speculate this unknown two-dimensional cosmological constant is corresponding to the ambiguities encountered in the Hamiltonian approach.

We further speculate that the two-dimensional cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. They can play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

This paper is organized as follows. In Sec. II, we briefly review the notion of entanglement entropy and the replica method. We carry out the calculation of the entanglement entropy of Yang-Mills fields on the lattice under the strong coupling expansion in Sec. III. In Sec. IV, we discuss the cancellation of the negative term by taking into account the cosmological constant, and the ambiguity arising from it and conclude.

II. ENTANGLEMENT ENTROPY AND THE REPLICA METHOD

Suppose our system occupies an infinitely large and flat $D + 1$ dimensional spacetime and is divided into two semi-infinite subregions $A$ and $B$. They are divided by an infinite and flat $D - 1$ dimensional space-like boundary. The entanglement entropy (EE) of a quantum theory between the two subregions is defined by the von Neumann entropy. With some simple algebra, it can be re-expressed as:

$$S_{EE} = - \text{Tr}[\rho_V \ln \rho_V] = - \left. \frac{\partial}{\partial n} \right|_{n \to 1} \ln \text{Tr}[\rho^n_V]$$

(1)

where $\rho_A = \text{Tr}_B[\rho]$ is the reduced density matrix by tracing out the degrees of freedom in region $B$. This expression is called the replica method because it involves $n$ copies of $\rho_A$.

An elegant path integral formulation to compute the entanglement entropy using the
replica method was first introduced in [6] (see also [14]). In this set up, one recalls that \( \rho_{ij} = \langle i | e^{-H/T} | j \rangle \) and \( \text{Tr}[\rho] \) is the partition function calculated in finite temperature field theory with appropriate boundary conditions (periodic and anti-periodic boundary conditions for bosons and fermions, respective) imposed for fields at Euclidean time \( \tau = 0 \) and \( 1/T \), where \( T \) is the temperature. Then \( \text{Tr}[\rho^2] \) can be computed by doubling the period (by imposing appropriate boundary conditions at \( \tau = 0 \) to \( 2/T \)). Similarly, \( \text{Tr}[\rho^2_A] \) is computed by doubling the period (from 0 to \( 2/T \)) for region \( A \) while region \( B \) still has the single period (from 0 to \( 1/T \)) as shown in the left plot of Fig. 1(A), which is equivalent to performing the path integral on a 2-sheeted Riemann surface in the right plot. One can generalize this set up to \( \text{Tr}[\rho^n_A] \) for an arbitrary \( n \). There is no restriction on the space partition between \( A \) and \( B \). The sizes of \( A, B, \) and \( T \) can be either finite or infinite.

In this paper, we will just concentrate on the simplest case with the sizes of space and (Euclidean) time to be both infinite (i.e. \( T = 0 \)) and the interface between \( A \) and \( B \) to be a flat infinite plane. In this limit, the \( n \)-sheeted Riemann surface has a conical structure as shown in Fig. 1(B) with the time and longitudinal spacial direction (the direction that is perpendicular to the interface) lying on the cone while the space on the interface transverse to the cone.

As a result, \( \text{Tr}[\rho^n_A] \) becomes a partition function \( Z_n \) on the \( n \)-sheeted Riemann surface, or, in our case, a cone with \( 2n\pi \) conical angle, normalized by \( n \)-copies of the partition function on the ordinary Euclidean space \( Z_1^n \):

\[
\text{Tr}[\rho^n_A] = \frac{Z_n}{Z_1^n},
\]

which ensures that as \( n = 1 \), \( \text{Tr}[\rho^n_A] = 1 \). The entanglement entropy is then given by

\[
S_{EE} = -\frac{\partial}{\partial n} (\ln Z_n - n \ln Z_1) \bigg|_{n \to 1}^{n=1+\epsilon} - \frac{1}{\epsilon} [\ln Z_{1+\epsilon} - (1 + \epsilon) \ln Z_1]_{\epsilon \to 0}.
\]

Note that \( n \) is taken as an integer in the integral of \( Z_n \). After one obtains the analytic expression for \( T'r[\rho^n] \), then \( n \) can be analytically extended to non-integers to carry out the differentiation at \( n = 1 \).

### III. CALCULATION ON AN N-SHEETED LATTICE MANIFOLD

We now discrete the spacetime on the \( n \)-sheeted manifold by a squared lattice. Firstly, the spacetime is decomposed into the direct product of \( 1+1 \) dimensional \( n \)-sheeted lattice
Figure 1: The $n$-sheeted Riemann surface in the replica trick: (A) illustrates the geometric structure of the $n$-sheeted manifold in the case of $n = 2$ arising from the replica trick, on which the partition function $Z_n$ is computed. The unshaded and shaded parts label the subregion A and B, respectively. The B subregion is traced out in the reduced density matrix $\rho_A$. The 2 sheets in the right figure are on top of each other and attached at the branch point, meaning that they have the same $x_\perp$ coordinates. The right figure also shows how the edges of the cuts on each sheet are identified. If one discretize such an $n$-sheeted manifold, then a central plaquette which encloses the conical point has $4n$ links, compared to 4 links in a regular plaquette. This geometry is equivalent to a cone with $2n\pi$ conical angle depicted in (B).

with discrete parallel coordinates $(x_\parallel, \tau)$ and transverse $D-1$ dimensional discrete transverse coordinates $(x_\perp)$. The spacetime is discretized by a squared lattice with the end point of the cut on each sheet (or the conical singularity corresponding to the tip of the cone) not sitting on a site. Later we will show that our result is independent of how the discretization on the cone is performed as long as the conical singularity is encircled by a Wilson loop in the action.

A parallel plaquette lying on the $(x_\parallel, \tau)$ plane is labelled by the parallel index $p_\parallel$, the sheet number $k$ and the transverse coordinate $(x_\perp)$. The transverse plaquettes lying in the
co-dimensions are labelled by the transverse index \( p_\perp \) and the sheet index \( k \). Especially, each of the central plaquettes is made of \( 4n \) links to form a closed plaquette enclosing the conical point. We mark these parallel plaquettes with index \( p_\parallel = 0 \).

Recall that the partition function of the lattice gauge theory on a one-sheet manifold is given by

\[
Z = \int \mathcal{D}U \exp \left\{ -\beta \sum_p \left[ 1 - \frac{1}{N} \text{Re tr} U_p \right] \right\}
\]

\[
\xrightarrow{a \to 0} \int \mathcal{D}A \exp \left\{ - \int d^4x \left[ \frac{1}{4g^2} \text{tr} F^2 \right] \right\}.
\]

where \( \beta = 2N/g^2 \) and \( p \) is denoted as the index of plaquette. The plaquette \( U_p \) is the Wilson loop at location \( p \) composed of the ordered product of four gauge links, \( U_p = \prod_{l \in \partial p} U_l \). The action recovers the Yang-Mills action in the continuum limit by setting the lattice spacing \( a \to 0 \).

To construct the lattice system in the general \( D + 1 \) dimensions whose \( 1 + 1 \) dimensions is an \( n \)-sheeted manifold, we will rewrite the partition function in (4) in terms of the transverse plaquettes \( U_{p_\perp}^{(k)} \), the parallel but non-central plaquettes \( U_{p_\parallel \neq 0}(x_\perp) \) and the central plaquettes \( U(x_\perp) = \prod_{l \in \partial (p_\parallel = 0)} U_l \) which enclose the conical singularity (as shown in Fig. (1A)). We also introduce an extra \( 1/n \) factor to the central plaquette terms. This is because a central plaquette is composed of \( 4n \) gauge links. It encircles the \( F_{01} \) flux over an area \( na^2 \) and it will contribute a factor \( \propto n^2a^4F_{01}^2 \) to the action which is inconsistent with the contribution from the transverse plaquettes which scales as \( na^4 \). Therefore the central plaquette contribution is multiplied by a factor \( 1/n \) to compensate this effect. The partition function on the \( n \)-sheeted
Riemann surface now reads
\[
Z_n = \int \left[ \prod_{k=1}^{n} D U^{(k)} \right] \exp \left\{ \frac{\beta}{N} \sum_{k=1}^{n} \sum_{p_{\perp}} \text{Re} \text{ tr} \left[ U^{(k)}_{p_{\perp}} - 1 \right] + \frac{\beta}{N} \sum_{x_{\perp}} \sum_{p_{\parallel} \neq 0} \text{Re} \text{ tr} \left[ U^{(k)}_{p_{\parallel}}(x_{\perp}) - 1 \right] \right. \\
+ \frac{\beta}{nN} \sum_{x_{\perp}} \text{Re} \text{ tr} \left[ U(x_{\perp}) - 1 \right] \left. \right\} 
\]

\[
\rightarrow \int \left[ \prod_{k=1}^{n} D A^{(k)} \right] \exp \left\{ -\frac{1}{4g^2} \sum_{k=1}^{n} \int d^2 x_{\perp} \int_{\mathbb{R}^2 - \{0\}} d^2 x_{\parallel} \text{tr} F^{(k)2} - \frac{1}{4g^2} \sum_{k=1}^{n} \int d^2 x_{\perp} \int_{\{0\}} d^2 x_{\parallel} \text{tr} F^{(k)2} \right\}
\]

\[
= \int \left[ \prod_{k=1}^{n} D A^{(k)} \right] \exp \left\{ -\frac{1}{4g^2} \sum_{k=1}^{n} \int d^2 x_{\perp} \int_{\mathbb{R}^2 - \{0\}} d^2 x_{\parallel} \text{tr} F^{(k)2} \right\}
\]

where we have used the boundary condition
\[
F^{(k)}(x_{\perp}, x_{\parallel} = 0) = F^{(l)}(x_{\perp}, x_{\parallel} = 0), \quad (k \neq l), \quad (8)
\]
i.e. the field strength on each sheet should be equal. This condition is natural if we impose
\[2\pi\] rotation symmetry on the \[n\]-sheeted surface.

Expanding Eq. (6) to the second order of \[\beta\], we have
\[
Z_n = e^{-\beta n N - \frac{\beta}{2} n N_{\perp}} \int \left[ \prod_{k=1}^{n} D U^{(k)} \right] \left\{ 1 + \frac{\beta}{N} \sum_{k=1}^{n} \sum_{p_{\perp}} \text{Re} \text{ tr} \left[ U^{(k)}_{p_{\perp}} \right] \\
+ \frac{\beta}{N} \sum_{x_{\perp}} \sum_{p_{\parallel} \neq 0} \text{Re} \text{ tr} \left[ U^{(k)}_{p_{\parallel}}(x_{\perp}) \right] + \frac{\beta}{nN} \sum_{x_{\perp}} \text{Re} \text{ tr} \left[ U(x_{\perp}) \right] \right. \\
+ \frac{\beta^2}{2! N^2} \sum_{k,l=1}^{n} \sum_{p_{\parallel} \neq 0} \left[ \text{Re} \text{ tr} U^{(k)}_{p_{\parallel}} \right] \left[ \text{Re} \text{ tr} U^{(l)}_{p_{\parallel}} \right] \\
+ \frac{\beta^2}{2! N^2} \sum_{x_{\perp}, y_{\perp}} \sum_{k,l=1}^{n} \sum_{p_{\parallel} \neq 0} \left[ \text{Re} \text{ tr} U^{(k)}_{p_{\parallel}}(x_{\perp}) \right] \left[ \text{Re} \text{ tr} U^{(l)}_{p_{\parallel}}(y_{\perp}) \right] \\
+ \frac{\beta^2}{2! N^2} \sum_{x_{\perp}, y_{\perp}} \sum_{k,l=1}^{n} \sum_{p_{\parallel} \neq 0} \left[ \text{Re} \text{ tr} U^{(k)}_{p_{\parallel}}(x_{\perp}) \right] \left[ \text{Re} \text{ tr} U^{(l)}_{p_{\parallel}}(y_{\perp}) \right] \\
+ \text{cross-term} + O(\beta^3) \right\} \cdot (9)
\]

Here \[N\] is the number of plaquettes on a single sheet for plaquettes not encircling the conical singularity, including both the parallel and transverse plaquettes. \[N_{\perp}\] is the number
of plaquettes encircling the conical singularity. For the SU(\(N > 2\)) gauge theory,

\[
Z_n = e^{-\beta n N - \frac{\beta}{2} N_\perp} \left\{ 1 + \frac{\beta^2}{2! N^2} \frac{1}{4} \sum_{k,l=1}^{n} \sum_{p} \int \left[ \prod_{k=1}^{n} DU^{(k)} \right] \left[ \text{tr} U_p^{(k)} \right] \left[ \text{tr} U_q^{(\perp)} \right] \\
+ \frac{\beta^2}{2! N^2} \frac{1}{4} \sum_{x, y} \sum_{x, y, k, l=1}^{n} \sum_{p, q} \int \left[ \prod_{k=1}^{n} DU^{(k)} \right] \left[ \text{tr} U_p^{(x)} \right] \left[ \text{tr} U_q^{(y)} \right] \right\}
\]

(10)

In the last line we have used

\[
\int dU U_i U^\dagger_{kl} = \frac{1}{N} \delta_{il} \delta_{jk}. \tag{11}
\]

As a result,

\[
\int DU \text{tr} U_p \text{tr} U_p^\dagger = \int dU_1 dU_2 dU_3 dU_4 \sum_{a,b,c,d=1}^{N} \sum_{i,j,k,l=1}^{N}
\times U_{1,ab} U_{2,be} U_{3,cd} U_{4,da}
\times U_{4,ij} U_{3,jk} U_{2,kl} U_{1,li}
\]

\[
= \frac{1}{N^2} \sum_{a,b,c,d=1}^{N} \sum_{i,j,k,l=1}^{N} \delta_{ai}^2 \delta_{bl}^2 \delta_{ck}^2 \delta_{dj}^2
\]

\[
= 1. \tag{12}
\]

Similarly,

\[
\int DU \text{tr} U(x_\perp) \text{tr} U^\dagger(x_\perp) = 1. \tag{13}
\]

Note that this result is independent of \(n\). This means no matter how we discretize the lattice, as long as the conical singularity is encircled by the same number of plaquettes in the action, the result will be the same. But if one chooses to put the conical singularity on a cite with no plaquette encircling the conical singularity, then those effects will all vanish in that calculation.

Taking the combination

\[
\ln Z_n - n \ln Z_1 = -\beta N_\perp \left[ \frac{1}{n} - n \right] + \frac{\beta^2 N_\perp}{2! N^2} \left( \frac{1}{n^2} - n \right) + O(\beta^3) \tag{14}
\]
then the Renyi entropy is

\[ S_n = -\beta N \frac{1 + n}{n} + \frac{\beta^2 N}{2! N^2} \left( 1 + \frac{n + n^2}{n^2} \right) + O(\beta^3). \]  

(15)

Using the identities

\[ \int dUU_{i_1 j_1} U_{i_2 j_2} \ldots U_{i_N j_N} = \frac{1}{N!} \epsilon_{i_1 i_2 \ldots i_N} \epsilon_{j_1 j_2 \ldots j_N} \]  

(16)

\[ \int dUU_{i_1 j_1}^\dagger U_{i_2 j_2}^\dagger \ldots U_{i_N j_N}^\dagger = \frac{1}{N!} \epsilon_{i_1 i_2 \ldots i_N} \epsilon_{j_1 j_2 \ldots j_N} \]  

(17)

we can find the non-vanishing contribution coming from the \( N \)-th order of \( \beta \). It is obtained via

\[
\int \mathcal{D}U \left( \text{tr}U_p \right)^N = \int dU_1 dU_2 dU_3 dU_4 \sum_{a_1, b_1, c_1, d_1 = 1}^N \sum_{a_2, b_2, c_2, d_2 = 1}^N \ldots \sum_{a_N, b_N, c_N, d_N = 1}^N \times U_{1, a_1 b_1} U_{2, b_1 c_1} U_{3, c_1 d_1} U_{4, d_1 a_1} \times U_{1, a_2 b_2} U_{2, b_2 c_2} U_{3, c_2 d_2} U_{4, d_2 a_2} \times \ldots \times U_{1, a_N b_N} U_{2, b_N c_N} U_{3, c_N d_N} U_{4, d_N a_N} 
\]

\[
= \frac{1}{(N!)^4} \sum_{a_1, b_1, c_1, d_1 = 1}^N \sum_{a_2, b_2, c_2, d_2 = 1}^N \ldots \sum_{a_N, b_N, c_N, d_N = 1}^N \epsilon_{a_1 a_2 \ldots a_N}^2 \epsilon_{b_1 b_2 \ldots b_N}^2 \epsilon_{c_1 c_2 \ldots c_N}^2 \epsilon_{d_1 d_2 \ldots d_N}^2 
\]

\[
= 1 \quad \text{(18)}
\]

and also

\[
\int \mathcal{D}U \left[ \text{tr}U(x_L) \right]^N = 1. \quad \text{(19)}
\]

This type of contribution appears at the \( N \)-th order of \( \beta \),

\[
Z_{(N)}^n = \int \left[ \prod_{k=1}^n \mathcal{D}U^{(k)} \right] \frac{\beta N}{N!N} \left\{ \sum_{k=1}^n \sum_{p \neq 0} \text{Re} \left[ U_p^{(k)} \right] + \frac{1}{n} \sum_{x_L} \text{Re} \left[ \mathcal{U}(x_L) \right] \right\}^N 
\]

\[
= \frac{\beta N}{N!N} \left\{ \sum_{k=1}^n \sum_{p \neq 0} \frac{1}{2^N} \int \prod_{k=1}^n \mathcal{D}U^{(k)} \right\} \left[ 2 \left( \text{tr}U_p^{(k)} \right)^N \right] 
\]

\[
+ \frac{1}{n^N} \sum_{x_L} \frac{1}{2^N} \int \prod_{k=1}^n \mathcal{D}U^{(k)} \right\} \left[ 2 \left( \text{tr}U(x_L^+) \right)^N \right] 
\]

\[
+ \text{cross-term} \}
\]

\[
= \frac{\beta N}{N!N2^N-1} \left\{ nN + \frac{N_\perp}{nN} + \text{cross-term} \right\} \quad \text{(20)}
\]
where $U_{p\neq 0}^{(k)}$ includes the transverse plaquettes $U_{p\bot}^{(k)}$ and the parallel plaquettes $U_{p\parallel \neq 0}(x_\bot)$.

For SU(2) theory, we have an additional $O(\beta^2)$ contribution,

$$
\ln Z_n^{(N=2)} = \left( -\beta n N - \frac{\beta}{n} N_\bot \right)
+ \ln \left\{ 1 + \frac{\beta^2}{2! N^2} \frac{1}{2} n N + \frac{\beta^2}{2! N^2} \frac{1}{2} N_\bot + \frac{\beta^2}{2! N^2} \frac{1}{2} n N + \frac{\beta^2}{2! N^2} \frac{1}{2} n N_\bot + O(\beta^4) \right\}
= -\beta n N - \frac{\beta}{n} N_\bot
+ \frac{\beta^2}{2! N^2} n N + \frac{\beta^2}{2! N^2} \frac{1}{n^2} N_\bot + O(\beta^4).
$$

(21)

For SU(3) theory,

$$
\ln Z_n^{(N=3)} = \left( -\beta n N - \frac{\beta}{n} N_\bot \right)
+ \ln \left\{ 1 + \frac{\beta^2}{2! N^2} \frac{1}{2} n N + \frac{\beta^2}{2! N^2} \frac{1}{2} n N_\bot + \frac{\beta^3}{3! N^3} \frac{1}{2} n N + \frac{\beta^3}{3! N^3} \frac{1}{n^3} N_\bot + O(\beta^4) \right\}
= -\beta n N - \frac{\beta}{n} N_\bot
+ \frac{\beta^2}{2! N^2} \frac{1}{2} n N + \frac{\beta^2}{2! N^2} \frac{1}{n^2} N_\bot + \frac{\beta^3}{3! N^3} \frac{1}{2} \frac{1}{n^2} N_\bot + O(\beta^4).
$$

(22)

So Renyi entropy for SU(N) theory to the order of $\beta^3$ is

$$
S_n^{(N=2)} = -\beta N_\bot \frac{1+n}{n} + \frac{\beta^2 N_\bot}{2! N^2} \left( \frac{1+n+n^2}{n^2} \right) + O(\beta^4);
$$

(23)

$$
S_n^{(N=3)} = -\beta N_\bot \frac{1+n}{n} + \frac{\beta^2 N_\bot}{2! N^2} \left( \frac{1+n+n^2}{n^2} \right) + \frac{\beta^3 N_\bot}{3! N^3 2^2} \frac{1+n+n^2}{n^3} + O(\beta^4);
$$

(24)

$$
S_n^{(N>3)} = -\beta N_\bot \frac{1+n}{n} + \frac{\beta^2 N_\bot}{2! N^2} \left( \frac{1+n+n^2}{n^2} \right) + O(\beta^4).
$$

(25)

As a result, entanglement entropy of SU(N) gauge theory in the strong coupling expansion is

$$
S_{EE}^{(N=2)} = \frac{A_\bot}{a_\bot^2} \left[ -\frac{4 N^2}{\lambda} + 6 \frac{N^2}{\lambda^2} + O(\frac{N^4}{\lambda^4}) \right] + \delta S_{EE}^{(N=2)},
$$

(26)

$$
S_{EE}^{(N=3)} = \frac{A_\bot}{a_\bot^2} \left[ -\frac{4 N^2}{\lambda} + \frac{3 N^2}{\lambda^2} + 4 \frac{N^3}{3 \lambda^3} + O(\frac{N^4}{\lambda^4}) \right] + \delta S_{EE}^{(N=3)},
$$

(27)

$$
S_{EE}^{(N>3)} = \frac{A_\bot}{a_\bot^2} \left[ -\frac{4 N^2}{\lambda} + \frac{3 N^2}{\lambda^2} + O(\frac{N^4}{\lambda^4}) \right] + \delta S_{EE}^{(N>3)}.
$$

(28)

where $A_\bot$ is the area of interface, $a_\bot$ is lattice spacing in the transverse space and where $\lambda = g^2 N$ is the t’Hooft coupling. As we argued after Eq. (13), this result is independent
of how the lattice is discretize, as long as the conical singularity is encircled by the same number of plaquettes in the action. The additional term $\delta S_{EE}$ will be explained in Sec. IV.

Likewise, the $U(1)$ result can also be obtained using the same method:

$$S_{EE}^{(U(1))} = \frac{A_\perp}{a_\perp^2} \left[ \frac{4}{g^2} + \frac{3}{g^4} + O\left(\frac{1}{g^8}\right) \right] + \delta S_{EE}^{(U(1))} \quad (29)$$

**IV. DISCUSSION AND CONCLUSION**

In the result Eqs. (26-28) and (29), one finds that the first order term in $S_{EE}$ has a negative contribution while all the rest orders contribute to entanglement entropy positively. It turns out with the conical structure of space time, we are allowed to introduce more local operators in the continuum action

$$S = \int d^2x_\perp d^2x_\parallel \left[ -\frac{1}{4} F^2 + c_4 + c_2 \delta^{(2)}(x_\parallel, \tau) \right]. \quad (30)$$

The $c_4$ term is a four dimensional cosmological constant which does not contributes to the entanglement entropy. However, the $c_2$ term, which breaks the translational symmetry while moving on the cone, can contribute to the entanglement entropy. Assuming $c_2$ is a smooth function of $n$, then

$$c_2 = c'_2 (n - 1) + O\left((n - 1)^2\right), \quad (31)$$

where we have used the fact that $c_2$ should vanish at $n = 1$ where translational symmetry is recovered. Therefore there is an extra unknown contribution to the entanglement entropy which also obey the area law:

$$\delta S_{EE}^{(N)} = A_\perp c'_2 (N), \quad (32)$$

where we have shown the $N$ dependence explicitly since different theories would have different $c_2$ counterterms.

The negative term in our result could in principle be compensated by the two dimensional cosmological constant. This uncertainty resembles the ambiguity of edge modes in the Hamiltonian method of the lattice gauge theory’s entanglement entropy.

We further speculate that the two-dimensional cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Also, they could play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.
In summary, we have calculated the entanglement entropy of the $SU(N)$ Yang-Mills gauge theories on the lattice under the strong coupling expansion in powers of $\beta = 2N/g^2$. Using the replica method, our Lagrangian formalism maintains gauge invariance on the lattice. At $O(\beta^2)$ and $O(\beta^3)$, the entanglement entropy is solely contributed by the central plaquettes enclosing the conical singularity of the $n$-sheeted Riemann surface. The area law emerges naturally to the highest order $O(\beta^3)$ of our calculation. The leading $O(\beta)$ term is negative, which could in principle be canceled by taking into account the cosmological constant living in interface of the two entangled subregions. This unknown cosmological constant resembles the ambiguity of edge modes in the Hamiltonian formalism. We have further speculated that this unknown cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Furthermore, it could play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

Acknowledgments

We would like to thank Michael Endres, Feng-Li Lin, Masahiro Nozaki, Chen-Te Ma, Jackson Wu and Yun-Long Zhang for helpful discussions. This work is supported by the MOST, NTU-CTS and the NTU-CASTS of Taiwan. JYP is supported in part by NSFC under grant No. 11125524 and 1221504. SHD is supported by Grant No. NSC103-2811-M-002-134.

[1] T. J. Osborne and M. A. Nielsen, Phys.Rev. A66, 032110 (2002).
[2] C.-Y. Huang and F.-L. Lin, Phys. Rev. A 81, 032304 (2010), 0911.4670.
[3] J.-W. Chen, S.-H. Dai, and J.-Y. Pang (2014), 1411.2916.
[4] G. Vidal, J. Latorre, E. Rico, and A. Kitaev, Phys.Rev.Lett. 90, 227902 (2003), quant-ph/0211074.
[5] J. Latorre, E. Rico, and G. Vidal, Quant.Inf.Comput. 4, 48 (2004), quant-ph/0304098.
[6] J. Callan, Curtis G. and F. Wilczek, Phys.Lett. B333, 55 (1994), hep-th/9401072.
[7] W. Donnelly and A. C. Wall (2014), 1412.1895.
[8] W. Donnelly, Phys.Rev. D85, 085004 (2012), 1109.0036.
[9] H. Casini, M. Huerta, and J. A. Rosabal, Phys.Rev. D89, 085012 (2014), 1312.1183.
[10] P. Buividovich and M. Polikarpov, Phys.Lett. B670, 141 (2008), 0806.3376.
[11] D. N. Kabat, Nucl.Phys. B453, 281 (1995), hep-th/9503016.
[12] A. Velytsky, Phys.Rev. D77, 085021 (2008), 0801.4111.
[13] J. H. Cooperman and M. A. Luty, JHEP 1412, 045 (2014), 1302.1878.
[14] M. P. Hertzberg and F. Wilczek, Phys.Rev.Lett. 106, 050404 (2011), 1007.0993.