Buckling and Postbuckling Analyses of Structure using Absolute Nodal Coordinate Formulation

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Abstract. In this paper, the three dimensional higher order beam element based on the absolute nodal coordinate formulation (ANCF) is used to study both the classical buckling and the nonlinear postbuckling problem. The analyses are performed using the Newton-Raphson method and the arc length method. The Newton-Raphson method is used for the Euler buckling whereas the Crisfield’s arc length method is applied to track the equilibrium path of the William’s Toggle. The solutions agree well with the analytical one or that given by the commercial finite element software ANSYS. Hence, the validity of the analyses is demonstrated.

1. Introduction
The absolute nodal coordinate formulation (ANCF) was originally proposed by Shabana in 1996 [1] for the dynamic analysis of flexible multibody system, especially for problems in which the components exhibit large deformation and rotation. The kinematic description based on the global coordinate system endows ANCF with the advantages of constant mass matrix, zero Coriolis and centrifugal force in the system equation of motion. Such valuable properties promoted fast development of the ANCF finite element (FE). Correspondingly, extensive applications of ANCF in different fields can be seen in the last two decades [2]. In the literature, most of the implementations are limited to the dynamic analysis. Few papers focus on the buckling and postbuckling analyses. As a nonlinear formulation, the ANCF is suited for the large deformation analysis. Additionally, it allows for the development of the non-incremental solution procedure. The implementation of nonlinear general continuum mechanics theory [3] has made it a general formulation for the structural analysis. All the illustrated advantages indicate that the nonlinear structural analysis including the buckling and postbuckling problem can be a potential application direction of the ANCF. It is the objective of this paper to show the validity of the ANCF application in the buckling and postbuckling analyses. In this paper, the application of ANCF element in the study of classical buckling by the means of Newton-Raphson (NR) method and nonlinear postbuckling response using the arc length method (ALM) are demonstrated. The remaining paper is organized as follows: In section 2, the adopted three dimensional ANCF higher order beam element is introduced briefly. In section 3, the Euler buckling of a column is shown. In section 4, the postbuckling of the William’s Toggle is presented. At last, the obtained conclusions are provided in section 5.

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2. Three dimensional ANCF higher order beam element

It is reported that the initially proposed fully parameterized ANCF beam elements [2] suffer from locking. Since the unbalanced order of displacement interpolation in axial and lateral directions is the main reason behind the locking phenomenon. An efficient way to alleviate the locking problem is to increase the interpolation order in the lateral directions. Correspondingly, a higher order beam element with 42 nodal coordinates (HOBE42) was proposed by Shen et al. [4]. Alleviating the locking effect, the HOBE42 is expected to correctly implement the nonlinear structural analysis. The element is introduced briefly herein for the sake of completeness.

The HOBE42 adopts quadratic interpolation in transverse directions whereas cubic polynomial in longitudinal direction. Besides all the gradient vectors, it also uses the curvature vectors as the nodal coordinates [5]. The position vector is represented as below:

\[ \mathbf{r} = \mathbf{S} \mathbf{e} \]  
(1)

Where, the vector of the element coordinate is \( \mathbf{e} = [(e' x)^T (e' z)^T]^T \) and the nodal coordinate vector is:

\[ \mathbf{e'} = \begin{bmatrix} r' x & r'^2 x & r'^3 x & r'' x & r''' x & r'''' x \\ r' y & r'^2 y & r'^3 y & r'' y & r''' y & r'''' y \\ r' z & r'^2 z & r'^3 z & r'' z & r''' z & r'''' z \end{bmatrix}^T \]  
(2)

Where, \( r'_x = \partial \mathbf{e'} / \partial x, \ r'_y = \partial \mathbf{e'} / \partial y, \ r'_z = \partial \mathbf{e'} / \partial z, \ r'''_z = \partial^2 \mathbf{e'} / \partial z^2 \) are gradient and curvature vectors at node \( i \). The shape function matrix is:

\[ \mathbf{S} = \begin{bmatrix} s_1 \mathbf{I} & s_2 \mathbf{I} & \ldots & s_{14} \mathbf{I} \end{bmatrix} \]  
(3)

Where, \( \mathbf{I} \) is the \( 3 \times 3 \) identity matrix, \( s_i, \ i = 1, 2, ..., 14 \) are shape functions shown as below:

\[
\begin{align*}
s_1 &= l - 3 \xi^2 + 2 \eta^3, \\
s_2 &= l (\xi - 2 \xi^2 + \eta^3), \\
s_3 &= l (\eta - 3 \eta^2 + \xi^3), \\
s_4 &= l (\eta^2 - \xi^2 - 3 \xi \eta), \\
s_5 &= l^2 (\eta^2 - \xi^2), \\
s_6 &= l (\eta^2 - 3 \eta^3 + \xi^3), \\
s_7 &= \frac{l^2}{2} (\xi^2 - \eta^2 - 3 \xi \eta), \\
s_8 &= 3 \xi^2 - 2 \xi^3, \\
s_9 &= l (5 \eta^2 - 3 \eta^3), \\
s_{10} &= l \xi \eta, \\
s_{11} &= l \xi \eta^2, \\
s_{12} &= l^2 \eta^2, \\
s_{13} &= \frac{l^2}{2} (\xi \eta^2), \\
s_{14} &= \frac{l^2}{2} (\xi \eta^2) \end{align*}
\]  
(4)

Where, the dimensionless coordinates are defined as \( \xi = x/l, \eta = y/l \) and \( \zeta = z/l \), \( l \) is the length of the element.

3. Euler Buckling of a column

Under the buckling phenomenon of a column, a very slight increment in the axial load leads to significant change in the axial deflection. In other words, the small increase in the axial force will cause the loss of the lateral stiffness of the column making the structure unstable. Based on this property, one can predict the buckling load numerically. The classical Euler buckling problem is used in this section to show the validity of the method.

A beam with one end clamped and the other end free is adopted. The dimension of the square cross section is \( A = 0.01 \times 0.01 \text{m}^2 \), the length \( L = \text{lm} \) and the corresponding area moment of inertia is \( I = 8.333 \times 10^{-10} \text{m}^4 \). The Young’s modulus is \( E = 2 \times 10^{11} \text{N/m}^2 \) and the Poisson’s ratio is \( \nu = 0.3 \). The beam is meshed using 10 HOBE42 elements.

![Figure 1. Configuration of a column.](image)

According to the Euler’s buckling method, the analytical solution in the fixed-free case is:
The corresponding axial load can be identified as the critical buckling load $P_c$ predicted numerically. 

\[ P_c = \frac{\pi^2EI}{(2L)^2} = 411.3 \text{N} \]  

(5)

Based on the above discussion, the Euler’s buckling load can be obtained numerically. The axial compressive force $P$ is applied at the free end and is gradually increased in discrete load steps. The small constant lateral perturbation force $P_\delta = 0.01 \text{N}$ is applied at the free end to model the imperfection in the system. The nonlinear static analysis using full Newton-Raphson iteration is performed at each load step to obtain the corresponding lateral deflection. Then, the full loading equilibrium path, which is the change of the dimensionless axial load $P / P_c$ with respect to the ratio of lateral deflection $\delta / \delta_c$, is plotted as shown in figure 2, where $\delta_c$ is the lateral deflection at $P = 0$. Due to the $P \cdot \delta$ effect [6], the lateral deflection at the free end keeps on increasing with the increment of the axial load instead of remaining constant. At the beginning, as the axial compressive force increases gradually, the lateral deflection increases slowly. On the contrary, as the axial load approaches the Euler’s buckling load, the change in lateral deflection becomes asymptotic which symbolizes loss of the lateral stiffness. The corresponding axial load can be identified as the critical buckling load $P_c$, predicted numerically.

![Figure 2. Buckling of a column.](image)

It can be seen in the figure 2 that the result obtained is in good agreement with the analytical result, i.e. $P_c / P_c = 1$. It can be concluded that the ANCF can successfully predict the Euler’s buckling load.

4. **Postbuckling of a William’s Toggle**
In the field of nonlinear finite element method, the Newton-Raphson method is widely used to follow the equilibrium path as shown in section 3. But the NR method is not capable to follow the equilibrium path after the buckling phenomenon occurs. To circumvent this problem, certain load control and displacement control techniques were proposed but they are not very general [7]. Wempner [8] introduced the idea of using the arc-length control to track the nonlinear equilibrium path. Later, Riks [9] formulated the general equations which can be used to predict the complete equilibrium path. However, the solution techniques adopted by those equations destroy the banded nature of the tangential stiffness matrix. Hence, its implementation in the nonlinear finite element code is not straightforward. To solve the aforementioned problems, Crisfield [10] proposed a modification of the arc length method which utilizes the tangential stiffness matrix directly from the nonlinear FEM theory and exhibits better convergence properties as well. This made the arc length method compatible with the existing nonlinear FE code.

The Crisfield’s ALM can be readily used in combination with the ANCF theory to study the nonlinear postbuckling response. In this section, the William’s toggle problem is studied and the dimensions adopted in [11] are used. As shown in figure 3(a), two bars of length $L = 2.5 \text{m}$ are connected by pin joint at the middle. The height of the middle pin joint is $d = 0.025 \text{m}$ from the base line and both the other ends are pinned. The dimensions of the bar cross section are $b = h = 0.01 \text{m}$, Young’s modulus
$E = 5 \times 10^{11} \text{N/m}^2$ and Poisson’s ratio is $\nu = 0.3$. Both bars have the same material and geometric properties. Hence, the principle of symmetry is applied. As shown in figure 3(b), only the left bar is analyzed and meshed using 20 HOBE42 elements. The vertical downward force is gradually applied.

Figure 3. William’s Toggle. (a) The whole structure. (b) The simplified half structure.

The equilibrium path obtained by the arc length method, the Newton-Raphson method based on the ANCF HOBE42 and the three dimensional Timoshenko beam element BEAM188 using the commercial FE software ANSYS are shown in figure 4. The converged result obtained by using 100 BEAM188 elements with nonlinear geometric option on in ANSYS is used as the reference. It can be found that the three solutions agree well. While the ALM based on ANCF is able to follow the full path, the Newton-Raphson method fails to follow after the limit point, instead it jumps past the postbuckle snap through region of the path.

Figure 4. Postbuckling of a William’s Toggle (Pin-Pin).

In the second case, the William’s Toggle is considered with both ends clamped. The equilibrium path obtained is different from that of the previous case but still follows the snap through pattern. Both results are compared with the plot obtained from the commercial finite element software ANSYS. The comparison is shown in figure 5.

Figure 5. Postbuckling of a William’s Toggle (Clamped-Clamped).
Same conclusion can be drawn in the clamped-clamped case. Thus, the validity of the ANCF application in the postbuckling analysis is verified.

5. Conclusion
The absolute nodal coordinate formulation based on the nonlinear continuum mechanics can not only be used in the dynamic analysis but can also perform well in the structural analysis. In this paper, it is utilized to predict the Euler buckling load of a column. The natural characteristic of ANCF makes it suitable for the nonlinear structural analysis. Therefore, combined with the Crisfield’s arc length method, the ANCF is applied to obtain the nonlinear postbuckling equilibrium path of the William’s Toggle. Hence, it is shown that the ANCF can successfully be applied in the study of buckling and nonlinear postbuckling analyses.

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