A note on Reed’s Conjecture about $\omega$, $\Delta$ and $\chi$ with respect to vertices of high degree

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Abstract
Reed conjectured that for every graph, $\chi \leq \left\lceil \frac{\Delta + \omega + 1}{2} \right\rceil$ holds, where $\chi$, $\omega$ and $\Delta$ denote the chromatic number, clique number and maximum degree of the graph, respectively. We develop an algorithm which takes a hypothetical counterexample as input. The output discloses some hidden structures closely related to high vertex degrees. Consequently, we deduce two graph classes where Reed’s Conjecture holds: One contains all graphs in which the vertices of degree at least 5 form a stable set. The other contains all graphs in which every induced cycle of odd length contains a vertex of at most degree 3.

Keywords: coloring, chromatic number, algorithm, Reed’s Conjecture, maximum degree, minimal counterexample

1. Introduction

One of the most prominent problems in combinatorial optimization is to decide whether the vertices of a graph can be feasibly colored by not more than a fixed number of different colors. If this fixed number is at least three, the aforementioned decision problem is known to be NP-complete. The associated optimization problem consists of computing the chromatic number $\chi$ of a graph. The determination of bounds for $\chi$ is a commonly used method to confine this optimization problem.

A lower bound for $\chi$ is the clique number $\omega$. A classical upper bound for $\chi$ in terms of the maximum degree $\Delta$ is provided by Brooks’ Theorem ([3]). It implies the bound $\chi \leq \Delta + 1$, which can be established due to an algorithmic approach (see [13], for example). Reed conjectured that, roughly speaking, the arithmetic medium of those bounds yields a new one for $\chi$.

Conjecture 1. ([16]) For every graph $G$, $\chi(G) \leq \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil$ holds.

Verifying this highly non-trivial conjecture would for instance imply that, essentially, the chromatic number never exceeds half the maximum degree in triangle-free graphs. One famous triangle-free graph is the Chvátal graph (cf. [4]). It provides one of the numerous graphs that require the rounding up.

For the following list of results on Reed’s Conjecture, we spare giving the definitions of the graph notions and refer the reader to the cited literature. The hereafter mentioned results point out some highlights of the ongoing research on Reed’s Conjecture. However, note that we do not claim that this list is exhaustive.

Conjecture [11] is introduced by Reed in [16]. There, he demonstrates the following result.

Theorem 1 (Reed [16]). There is a constant $\Delta_0$ such that for $\Delta \geq \Delta_0$, if $G$ is a graph of maximum degree $\Delta$ with no clique of size exceeding $k$ for some $k \geq \left\lfloor (1 - \frac{1}{100000000}) \Delta \right\rfloor$, then $\chi(G) \leq \frac{\Delta + 1 + k}{2}$.

Randerath and Schiermeyer [13] work with such constants, too.

Theorem 2 (Randerath et al. [13]). For every $k \geq 3$ there is a constant $c_k$ such that $\chi(G) \leq \frac{\Delta(G) + \omega(G) + 1}{2}$ for all graphs $G$ with $\Delta(G) \geq \frac{2V(G)}{k} + c_k \cdot (\omega(G))^{k-1}$.
Some authors presented results in terms of graph parameters such as the maximum degree, the clique number, the number of vertices, and the stability number. The latter indicates the size of a maximum stable set of a graph and is denoted by $\alpha$. For example, Reed proved that graphs with $\Delta(G) = |V(G)| - 1$ comply with the conjecture. Randerath and Schiermeyer improved this bound to $\Delta(G) \geq |V(G)| - 4$. Kohl and Schiermeyer showed that even less restrictive conditions to the maximum degree suffice since the conjecture holds for a graph $G$ with $\Delta(G) \geq |V(G)| - 7$ or $\Delta(G) \geq |V(G)| - \alpha(G) - 4$. The same authors verified the conjecture for graphs $G$ that obey both $\omega(G) \leq 2$ and $\Delta(G) \geq \frac{\alpha(|V(G)| - \alpha(G)) + 18}{2}$. If a graph $G$ fulfills $\chi(G) > \lceil \frac{\alpha(|V(G)|)}{2} \rceil$ or $\chi(G) > \frac{\alpha(|V(G)| - \alpha(G) + 3)}{2}$, then it obeys the conjecture, which was demonstrated by Rabern Gernert and Rabern observed that the conjecture is valid if $\omega(G) = \Delta(G)$ or if $\omega(G) = \Delta(G) + 1$ or if $\chi(G) \leq \omega(G) + 2$.

The result mentioned last implies that Conjecture holds for perfect graphs. Many other hereditary graph classes have been analyzed by means of the conjecture. King, Reed and Vetta proved that the conjecture holds for the line graph of a graph as well as for the line graph of a multigraph. King and Reed showed that quasi-line graphs also comply with the conjecture. These results can be extended to the class of claw-free graphs, according to King. The class of claw-free graphs in particular contains the $K_4$-free graphs, that is, the complements of triangle-free graphs. Complements of triangle-free graphs form a subset of the class of almost-split graphs, which were introduced by Kohl and Schiermeyer. They verified the conjecture for this graph class. Recently, Aravind, Karthick and Subramanian verified the conjecture for some further graph classes which are defined by families of forbidden induced subgraphs, such as the {odd hole}-free graphs. Assuming that a considered graph $G$ contains an odd hole (otherwise, the previously mentioned result takes effect), they showed that the conjecture holds for $G$ if $G$ is $\{P_3, \text{house, dart, } P_2 \cup P_3\}$-free, or if $G$ is $\{P_3, \text{kite, bull, } (K_3 \cup K_1) \oplus K_1\}$-free, or if $G$ is $\{P_5, C_4\}$-free, or if $G$ is $\{\text{chair, bull, house, } W_4\}$-free, or if $G$ is $\{\text{chair, bull, house, dart}\}$-free. Finally, Reed’s Conjecture holds for planar and for toroidal graphs. This was deduced from some results of Thomassen as well as results of Albertson and Hutchinson by Gernert and Rabern.

Finally, Rabern observed that the join of two vertex-disjoint graphs obey the conjecture. In other words, Reed’s Conjecture holds for graphs with disconnected complement. Therefore, a connected complement is required for every - yet hypothetical - counterexample to Conjecture. Further properties of such a counterexample, mostly given in terms of graph parameters such as vertex and edge number, are listed by Gernert and Rabern. Their way to attack the conjecture led to our approach: we assume the existence of a counterexample which is then used as an input to an algorithm. The output of the algorithm discloses some immanent structures of the counterexample which are formed by high degree vertices. As a result, we deduce two new, non-trivial graph classes that obey Reed’s Conjecture.

2. Preparation

In order to present the main results correctly and compactly, we need some preparation. A graph class is called hereditary if it is closed under taking induced subgraphs. Thus, a hereditary graph class allows a description in terms of forbidden induced subgraphs. If $S$ is a set of forbidden induced subgraphs for a graph class $C$, then every graph in $C$ is $S$-free.

All graphs considered in this paper are finite, undirected and free of loops or parallel edges. Let $G$ denote such a graph. We say that a vertex coloring is feasible for $G$ if a color is assigned to every vertex of $G$ such that adjacent vertices get different colors. The least number of colors needed for such a feasible vertex coloring is called chromatic number and denoted by $\chi$. A clique is a complete graph. The size of a largest clique in $G$ is called clique number of $G$ and is denoted by $\omega$. A set of vertices in $G$ is called stable if the vertices in the set are pairwise not adjacent.

For a vertex $v$ in a graph $G$, a vertex adjacent to $v$ is called a neighbor. The number of neighbors of $v$ in $G$ is the $G$-degree of $v$, sometimes denoted by $\deg_G(v)$. If it is clear from the context which graph is considered, we simply write degree instead of $G$-degree. The minimum of all vertex degrees in $G$ is called the minimum degree of $G$ and is denoted by $\delta(G)$; the maximum of all vertex degrees in $G$ is called the maximum degree of $G$ and is denoted by $\Delta(G)$. If the $G$-degree of both endvertices of an edge is $\Delta(G)$, then the edge is a heavy edge. By $C_n$, $n \in \mathbb{N}$, $n \geq 3$, we denote a cycle on $n$ vertices. A cycle $C_n$ is called odd if $n$ is odd. Moreover, a cycle $C$ is a heavy cycle of $G$ if for all $v \in V(C)$, the $G$-degree of $v$ is at least $\Delta(G) - 1$.
A graph $G$ is a \textit{minimal counterexample to Reed’s Conjecture} if
\[
\left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil < \chi(G)
\]
holds and all proper induced subgraphs of $G$ obey the conjecture. That is, a minimal counterexample is an inclusionwise vertex-minimal counterexample to Reed’s Conjecture. Since there is no risk of confusion, in the proofs of our claims we abbreviate \textit{counterexample to Reed’s Conjecture} by \textit{counterexample}.

A vertex $v \in V(G)$ is called \textit{color-critical} if $\chi(G - v) < \chi(G)$. A graph $G$ is called \textit{color-critical} if every vertex in $G$ is color-critical. If, in addition, $\chi(G) = k$, then $G$ is sometimes called \textit{k-color-critical}.

\textbf{Proposition 1} (Toft \cite{18}, Diestel \cite{5}). A graph $G$ contains at least one induced $k$-color-critical subgraph $H_k$ for $1 \leq k \leq \chi(G)$ with $\chi(H_k) = k$.

This result follows directly from Theorem 2.1 by Toft \cite{18} for the case $k < \chi(G)$. The main idea of the proof for $k = \chi(G)$ is to successively remove vertices from a $k$-chromatic graph until the graph becomes critically $k$-chromatic, as suggested by Diestel \cite{5}. We mimic this idea of successively deleting vertices in the algorithm \textsc{HEAVY STABLE SETS} presented below.

In order to prove our claims, we need some results on color-critical graphs. In \cite{18}, Toft stated that the only 1-color-critical graph is $K_1$ and that the only 2-color-critical graph is $K_2$. In addition, Toft recapitulated a result of König (referring to \cite{12}): the only 3-color-critical graphs are the odd cycles. To facilitate citation, we wrap them into the following proposition.

\textbf{Proposition 2} (Toft \cite{18}, König \cite{12}). Let $k \in \mathbb{N}$ and let $G$ be a $k$-color-critical graph. Then
\begin{itemize}
  \item $G \cong K_1$ if and only if $k = 1$,
  \item $G \cong K_2$ if and only if $k = 2$,
  \item $G \cong C_{2n+1}$, $n \in \mathbb{N}$, if and only if $k = 3$.
\end{itemize}

In color-critical graphs, every vertex has at least $\chi(G) - 1$ neighbors.

\textbf{Proposition 3.} Let $G$ be a color-critical graph. Then $\delta(G) \geq \chi(G) - 1$.

Observe that Reed’s Conjecture holds for graphs having maximum degree at most 4.

\textbf{Proposition 4} (Gernert et al. \cite{8}). Let $G$ be a countereexample to Reed’s Conjecture. Then $\Delta(G) \geq 5$.

3. Main results

In the following, we present the algorithm \textsc{HEAVY STABLE SETS} (cf. Figure \ref{fig:algorithm}). Applied to a given graph $G$, this algorithm removes one by one vertices. First, it chooses a vertex of maximum degree, and removes it from $G$, hence constructing $G'$. In this new graph, the second vertex is chosen such that it is of maximum degree in $G'$. This idea is iterated in every step. In other words, in the graph that results by removing all vertices that were already chosen, the algorithm chooses a vertex which is of maximum degree in the actually considered graph.

The following technical lemma confirms that \textsc{HEAVY STABLE SETS} (Figure \ref{fig:algorithm}) terminates. Moreover, Lemma \ref{lem:termination} lists some properties of the vertex sets which were removed from the input graph.

\textbf{Lemma 1.} Let $G$ be a graph, and let $k \in \{2, \ldots, \chi(G) - 1\}$. If $G$ and $k$ are taken as input for \textsc{heavy stable sets}, then the algorithm terminates. Moreover, for all $r \in \{0, \ldots, 2k - 1\}$,
\begin{enumerate}
  \item $\Delta(G_{r+1}) \leq \Delta(G) - (r + 1)$, with equality if and only if $S_r \neq \emptyset$,
  \item $G_{r+1} = G - \bigcup_{i=0}^{r} S_i$,
  \item $S_r$ is a (possibly empty) stable set,
  \item for all $v \in S_r$, $v$ has $\Delta(G) - r$ neighbors in $G - \bigcup_{i=0}^{r} S_i$.
\end{enumerate}
Input: Graph $G$, $k \in \{1, \ldots, \chi(G) - 1\}$
Output: For all $i \in \{0, \ldots, 2k\}$: Stable Sets $S_i$, Graphs $G_{i+1}$ with $G_{i+1} = G_i - S_i$

$G_0 \leftarrow G$
$r \leftarrow 0$
for all $i = 0$ to $2k - 1$ do
    $S_i \leftarrow \emptyset$
    end for
    while $\Delta(G_r - S_r) = \Delta(G) - r$ do
        choose $v \in V(G_r - S_r)$ such that the $(G_r - S_r)$-degree of $v$ is $\Delta(G) - r$, break ties by selecting a vertex with higher $G$-degree
        $S_r \leftarrow S_r \cup \{v\}$
        $G_{r+1} \leftarrow G_r - S_r$
        while $\Delta(G_r - S_r) < \Delta(G) - r$ do
            if $r = 2k - 1$ then
                return
            else
                $r \leftarrow r + 1$
                $G_r \leftarrow G_{r-1} - S_{r-1}$
            end if
        end while
    end while
end for

Figure 1: The procedure HEAVY STABLE SETS

Proof. The algorithm starts with $r = 0$. Since $G = G_0 - S_0$, $\Delta(G_0 - S_0) \geq \Delta(G) - 0$ is trivially true. Thus, the algorithm enters the outer while-loop. As long as

$$\Delta(G_r - S_r) = \Delta(G) - r$$

holds for any fixed $r \in \{0, \ldots, 2k - 1\}$, the outer while-loop is repeated. After every enlargement of $S_r$, the procedure tests if (1) is still true and enters the inner while-loop if and only if it is not. The inner while-loop is repeated until $r$ is large enough such that (1) is true again or until $r$ reaches the value of $2k - 1$, forcing the algorithm to stop. In particular, every time the inner while-loop is entered or iterated, either the algorithm stops or $r$ is increased by one. So, in order to show that the procedure terminates, it suffices to prove that for every $r \in \{0, \ldots, 2k - 1\}$, the inner while-loop is entered or iterated. To attain this result, let $S^i_r$ be the set $S_r$ after the $i$th vertex was added, for some $i \in \mathbb{N}_0$. Since $\|S^i_r\| = i$,

$$G_r - S_r^{i+1} \subseteq G_r - S_r^i$$

holds. That is, every time the outer while-loop is repeated for some fixed $r$, the considered graph is a proper subgraph of the graph considered in the previous iteration. Therefore, at some point of the iteration,

$$\Delta(G_r - S_r) < \Delta(G) - r$$

holds, contradicting (1). Hence, the inner while-loop is entered and the algorithm terminates.

Point 1 of Lemma 1 Observe that (2) implies $\Delta(G_{r+1}) = \Delta(G_r - S_r) \leq \Delta(G) - r - 1$ and that $S_r = \emptyset$ if and only if (1) was false at any time of the procedure.

Point 2 of Lemma 1 Let $r \in \{0, \ldots, 2k - 1\}$. Since $G_{r+1} = G_r - S_r$ and $G_0 = G$,

$$G_{r+1} = G_r - S_r = G_{r-1} - S_{r-1} - S_r = \ldots = G_0 - \bigcup_{i=0}^{r} S_i = G - \bigcup_{i=0}^{r} S_i.$$

Point 3 and 4 of Lemma 1 Let $S_r \neq \emptyset$. Note that a vertex $v$ is put into $S_r$ after the conditions $\Delta(G_r - S_r) = \Delta(G) - r$ and $\deg_{G_r-S_r}(v) = \Delta(G) - r$ are verified. In particular, $v$ has $\Delta(G) - r$ neighbors in

$$G - \bigcup_{i=0}^{r-1} S_i,$$

and Point 3 follows if $v$ has no neighbors in $S_r$. After $v$ is put into $S_r$, the former neighbors of $v$ in the updated graph $G_r - S_r$ have $(G_r - S_r)$-degree at most $\Delta(G_r - S_r) - 1$. Hence, they will not be put into $S_r$. Hence, $S_r$ is stable and thus Point 3 and Point 4 follow.
We apply heavy stable sets to a minimal counterexample to Reed’s Conjecture. Roughly speaking, the union of the constructed stable sets induces a color-critical subgraph in the counterexample. Every vertex in this union has relatively high degree.

**Theorem 3.** Let $G$ be a minimal counterexample to Reed’s Conjecture. Then $G$ contains a 2-color-critical subgraph consisting of vertices with $G$-degree $\Delta(G)$. Moreover, for every $k \in \mathbb{N}$, $2 \leq k \leq \chi(G) - 1$, $G$ contains a $(k+1)$-color-critical subgraph $H$ such that $\text{deg}_G(v) \geq \max\{\delta(G), \Delta(G) - 2k + 3\}$ holds for all $v \in V(H)$.

**Proof.** Let $k \in \{1, \ldots, \chi(G) - 1\}$ and let $G$ be a minimal counterexample. Use $G$ and $k$ as input for heavy stable sets. Let $S$ be the graph induced by

$$G_{2k} = G - \bigcup_{i=0}^{2k-1} S_i,$$

and assume that $S$ is $k$-colorable. By Lemma[1] Point[2]

$$\chi'(G_{2k}) = \chi(G) - k$$

holds, thus $\chi(G) \leq \chi(G_{2k}) + \chi(S)$. That is, $\chi(G_{2k}) \geq \chi(G) - k$. By Lemma[1] Point[1] $\Delta(G) - 2k \geq \Delta(G_{2k})$. Since $G_{2k}$ is an induced subgraph of $G$, $\omega(G) \geq \omega(G_{2k})$ holds. Given that $G$ is a counterexample,

$$\chi(G_{2k}) \geq \chi(G) - k \geq \left\lfloor \frac{\Delta(G) + \omega(G) + 1}{2} \right\rfloor - k$$

holds. Note that $k > 0$ and $S_0 \neq \emptyset$. Hence, $G_{2k}$ is both a proper induced subgraph of $G$ and a counterexample, contradicting the choice of $G$. Thus, $S$ is at least $(k+1)$-chromatic. Due to Proposition[1] $S$ contains an induced $(k+1)$-color-critical subgraph.

Let $H$ be such a $(k+1)$-color-critical subgraph and let $v \in V(H)$. If $k = 1$, then $v \in S_0$ or $v \in S_1$. If $v \in S_0$, then $\text{deg}_G(v) = \Delta(G)$. If $v \in S_1$, then $v$ has at least one neighbor in $H \subseteq (S_0 \cup S_1)$, by Proposition[3]. Due to Lemma[1] Point[3] $v$ has $\Delta(G) - 1$ neighbors in $G - (S_0 \cup S_1)$, hence $\text{deg}_G(v) = \Delta(G)$.

If $k > 2$, it suffices to prove that $\text{deg}_G(v) \geq \Delta(G) - 2k + 3$. To obtain this result, let $v \in V(H)$. Then $v \in S_j$ for some $j \in \{0, \ldots, 2k - 1\}$. Note that if $j \leq 2k - 3$, then, by Lemma[4] Point[3] $v$ has $\Delta(G) - j$ neighbors in $G - \bigcup_{i=0}^{2k-1} S_i$. In this case, the claim follows, since

$$\text{deg}_G(v) \geq \Delta(G) - j \geq \Delta(G) - 2k + 3.$$

Further note that if $j = 2k - 1$, then, again by Lemma[1] Point[3] $v$ has $\Delta(G) - 2k + 1$ neighbors in $G_{2k} = G - S$. Moreover, $H \subseteq S$, and by Proposition[3] $v$ has at least $k$ neighbors in $H$. It follows that

$$\text{deg}_G(v) \geq \Delta(G) - 2k + 1 + k = \Delta(G) - k + 1 \geq \Delta(G) - 2k + 3.$$  (3)

Hence in order to complete the proof, we have to consider the case $j = 2k - 2$. Recall that by Lemma[1] Point[2]

$$G_{2k-1} = G - \bigcup_{i=0}^{2k-2} S_i,$$

holds. By Lemma[1] Point[3] $v$ has $\Delta(G) - 2k + 2$ neighbors in $G_{2k-1}$. If $v$ has at least one neighbor in

$$\bigcup_{i=0}^{2k-2} S_i,$$

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the proof is completed. Assume the contrary. Since \( H \) is a \((k+1)\)-color-critical graph, \( v \) has at least 
\( k \) neighbors in \( H \). By our assumption, every neighbor of \( v \) in \( H \) was added to the vertex set that 
induces \( H \) after \( v \) was added. Hence,
\[
deg_G(v) = \Delta(G) - 2k + 2
\]
and \( N_H(v) \subseteq S_{2k+1} \). Let \( x \in N_H(v) \). According to (3),
\[
deg_G(x) \geq \Delta(G) - 2k + 3
\]
holds. Recall that by Lemma 1 Point 2 \( G_{2k-2} = G - \bigcup_{i=0}^{2k-3} S_i \) holds. Since \( v \in S_{2k-2} \), by 
Lemma 1 Point 1 we have \( \Delta(G_{2k-2}) = \Delta(G) - 2k + 2 \). Therefore, the \( G_{2k-2}\)-degree of \( x \) is not only 
at most \( \Delta(G) - 2k + 2 \) but is exactly of that value. Otherwise, since \( v \) and \( x \) are adjacent, the \( G_{2k-1}\)-
degree of \( x \) is at most \( \Delta(G) - 2k \), contradicting the fact that \( x \in S_{2k-1} \). Thus, the \( G_{2k-1}\)-degree of 
\( x \) is identical with the \( G_{2k-1}\)-degree of \( v \). Choosing \( v \) and not \( x \) to be put into \( S_{2k-2} \) contradicts the 
tie-break rule of the algorithm, since the \( G\)-degree of \( x \) is higher than the \( G\)-degree than \( v \), which can be 
be seen by (4) and (5). This completes the proof.

Proposition 2 provides the sets of 2- and 3-critical graphs. Hence, we adapt the result of Theorem 3.

**Corollary 1.** Let \( G \) be a minimal counterexample to Reed’s Conjecture. Then \( G \) contains a heavy 
edge and a heavy odd cycle.

**Proof.** Let \( G \) be a vertex-minimal counterexample to Conjecture 1. By Theorem 3, \( G \) contains a 2-
respectively 3-color-critical subgraph \( H \) such that for all \( v \in V(H) \), \( \deg_G(v) = \Delta(G) \) respectively 
\( \deg_G(v) \geq \Delta(G) - 1 \). By Proposition 2 \( H \) is an edge respectively \( H \) is an odd cycle.

Corollary 1 allows us to find two classes of graphs for which Reed’s Conjecture holds.

**Theorem 4.** Let \( \Delta_0 \in \mathbb{N} \). If Reed’s Conjecture holds for graphs with maximum degree at most \( \Delta_0 \), 
then the conjecture holds

1. for graphs in which the vertices of degree at least \( \Delta_0 + 1 \) form a stable set.
2. for graphs in which every induced odd cycle contains a vertex of degree at most \( \Delta_0 - 1 \).

In particular, if for a graph \( G \) the vertices of degree at least 5 form a stable set or if all induced cycles 
in \( G \) that are of odd length contain a vertex of degree at most 3, then Reed’s Conjecture holds for \( G \).

**Proof.** Let the conjecture hold for graphs with maximum degree at most \( \Delta_0 \). Then every 
counterexample contains a minimal counterexample that has maximum degree at least \( \Delta_0 + 1 \) as induced 
subgraph. According to Corollary 1 every minimal counterexample contains a heavy edge (respectively 
a heavy cycle). Hence, every counterexample contains an edge where the endvertices are of degree at least \( \Delta_0 + 1 \) (respectively an odd cycle where all vertices in the cycle are of degree at least \( \Delta_0 \)). Thus, Claim 1 (respectively Claim 2) follows.

In particular, by Proposition 1 Reed’s Conjecture holds for all graphs with \( \Delta = 4 \).

If all induced odd cycles contain a vertex of degree at most 3, then there is another, straightforward 
way to validate Reed’s Conjecture. A counterexample \( G \) is at least 5-colorable, since the conjecture 
holds for graphs with \( \chi \leq 4 \). Thus, \( G \) contains a 5-color critical graph whose minimum degree is at least 4.
But every subgraph of \( G \) which is induced by vertices of degree at least 4 is bipartite, a contradiction.
However, color-critical graphs are not excluded by means of the minimum degree if the difference 
between the maximum degree and the clique number is large enough.

**Observation 1.** Let \( \Delta_0 \in \mathbb{N} \). Graphs in which the vertices of degree at least \( \Delta_0 + 1 \) form a stable set 
and graphs in which every induced odd cycle contains a vertex of degree at most \( \Delta_0 - 1 \) can be 
recognized in polynomial time.

The first class mentioned in the observation is recognized by simply checking if an edge is formed 
in the set of vertices with degree at least \( \Delta_0 + 1 \). The recognition of the second class requires, 
especially, testing if the graph induced by vertices of degree at least \( \Delta_0 + 1 \) is bipartite.

**Theorem 5.** Let \( G \) be a graph in which the vertices of degree at least 5 form a stable set or in 
which all induced odd cycles contain a vertex of degree at most 3. Then a coloring of \( G \) that obeys 
Reed’s Conjecture can be found in polynomial time.
Proof. Let $G$ be a graph in which all induced odd cycles contain a vertex of degree at most 3. Let $B'$ be the graph induced by all vertices of degree at least 4. Observe that $B'$ is bipartite. Let $B$ be an inclusionwise maximal bipartite subgraph of $G$ that contains $B'$. We use breadth-first search in order to color $B'$ with the colors 1 and 2. Let $R$ be the graph induced by $V \setminus V(B)$. Assume $R$ contains a vertex of degree at least 2, say $v$. Then the $R$-degree of $v$ is at least 4, since otherwise, $v$ can be added to $B$. This contradicts the choice of $B$. Hence, the maximum degree of $R$ is 1. Thus, $R$ is bipartite. We therefore again use breadth-first search to color $R$ with colors 3 and 4. This way, we provide a 4-coloring.

Let $G$ be a graph in which all vertices of degree at least 5 form a stable set. Let $S$ be an inclusionwise maximal stable set which contains all vertices of degree at least 5. Observe that $R$, which is the graph induced by $V(G) \setminus S$, has $\Delta(R) \leq 3$: Every hypothetical vertex with $R$-degree 4 has $G$-degree at least 5 and is therefore in $S$. We color $S$ with one color. A 3-coloring of $R$ is provided by Lovász [13].

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