The Superposition of Two Identical States: The Empty State

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Abstract

In this paper we define and study a new class of states (the empty states). These states are the superposition of two identical states (self-superposition state). We defined three different representations of theses states, namely, the source, the operator, and the self-superposition representations. Then, we apply the empty state to an elementary example state and find its three different representations. We apply the empty state to Fock states and find an undetermined state. Then, we apply the empty state to the coherent state to produce the empty state of the coherent state (the EC state), and determine some of its mathematical, statistical and nonclassical properties.

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1 Introduction

Many states have been introduced in quantum optics to observe some nonclassical effects \(^1,2\), that are generally important for many applications in quantum optics, (for example see \(^3,4\)). A nonclassical state can be determined in many ways, such as finding quantum interferences in the photon number distribution or finding some negative regions in the Wigner function, etc \(^2,5,6\). An example of a nonclassical state is the Fock state or number state \(^7\).

One class of nonclassical states is the superposition of two states. One possible formula for these states is \(|\psi\rangle = C(|r\rangle + e^{i\phi}|re^{i\theta}\rangle)\), where the state \(|r\rangle\) can be any state with a continuous variable \(r\), phase parameters \(\phi\) and \(\theta\), and \(C\) is the normalization constant. If state \(|r\rangle\) is the Glauber coherent state \(^8\) and \(\theta\) equals \(\pi\), then the state \(|\psi\rangle\) becomes the Schrödinger cat states which perform some nonclassical properties \(^9–14\). Another example is where \(|r\rangle\) can be the squeezed coherent state, where here \(r\) refers to two independent parameters, we will have the superposition of two squeezed coherent states \(^15–17\).

Aside from state \(|\psi\rangle\), there are more general formulas to use to construct the superposition of two states (for example see \(^17–26\)). Although these general formulas are studied extensively, there are special cases in which no physics or at least no quantum optics has yet been attributed to them, to the best of our knowledge. These special cases are the superposition of two identical states or self-superposition states. When \(\theta\) equals zero in the above \(|\psi\rangle\), we can get a possible construction of a self-superposition state

\[|\psi\rangle = C(|r\rangle + e^{i\phi}|r\rangle).\] (1)

Another possible construction is

\[|\psi\rangle = \lim_{\theta \rightarrow 0} C(|r\rangle + e^{i\phi}|re^{i\theta}\rangle),\] (2)

and there are many other possibilities as well.

In general, these different constructions of a self-superposition state are not equivalent. On other words, their resultant states are not same. The possible outcomes of these different constructions can thus be classified as two different types. The first type gives the state \(|r\rangle\) multiplied by a coefficient. The second type gives a new state which cannot be described in terms of the state \(|r\rangle\). The first type is trivial since we start with a state and end up with the same state. Also, because we know that the superposition of two states produces a new state, the self-superposition state should also give a new state. Therefore, we cannot consider the first type to be a self-superposition state. The first type looks like a normal addition, and all observables will be invariant because we just altered the amplitude.

Now, the second type which gives a new state is truly what we can call the self-superposition. To further distinguish between the first and second types, we call the second type of state an empty state. The detailed discussion regarding self-
superposition states and the reason why they are named empty states, is discussed in the next section.

In the following sections, we introduce the general formulation of the empty states and define three different representations to describe them: the source, the operator and the self-superposition representations. Then, we apply the empty state to an example state and to the Fock states (empty-Fock states). We also apply the empty state to the coherent state (the EC state) and study its mathematical, statistical and nonclassical properties.

2 The Empty States

This section examines the different constructions of the self-superposition states, and the properties of empty states. Let us start by studying the possible constructions of self-superposition states, the first of which is the construction in Eq. (1)

\[ |\psi\rangle = C(|r\rangle + e^{i\phi}|r\rangle), \]

where the states \(|r\rangle\) and \(|\psi\rangle\) are assumed to be normalized. The normalization constant \(C\) reads

\[ C = \frac{1}{\sqrt{2(1 + \cos \phi)}} \] (3)

Then, the state \(|\psi\rangle\) yields

\[ |\psi\rangle = \frac{1 + e^{i\phi}}{\sqrt{2(1 + \cos \phi)}}|r\rangle = e^{i\phi'}|r\rangle, \] (4)

where \(\phi'\) is given by

\[ \tan \phi' = \frac{\sin \phi}{1 + \cos \phi}. \] (5)

The state \(|\psi\rangle\) is simply the state \(|r\rangle\) with a different amplitude, so this construction is trivial, and we cannot call it a self-superposition state.

Another possible construction is found in Eq. (2)

\[ |\psi\rangle = \lim_{\theta \to 0} C(|r\rangle + e^{i\phi}|re^{i\theta}\rangle). \]

Neglecting the limit, the normalization constant \(C\) is given as

\[ C = \frac{1}{\sqrt{2 + e^{i\phi}\langle r|re^{i\theta}\rangle + e^{-i\phi}\langle re^{i\theta}|r\rangle}}. \] (6)

Because we required \(\theta\) to go to zero, we can expand the inner product around \(\theta \approx 0\) as

\[ \langle r|re^{i\theta}\rangle \approx 1 + g_1(r)\theta + g_2(r)\theta^2 + \cdots, \quad \langle re^{i\theta}|r\rangle \approx 1 + g_1^*(r)\theta + g_2^*(r)\theta^2 + \cdots, \] (7)
where \( g_1(r) \) and \( g_2(r) \) are functions that depend on the state \( |r\rangle \). This expansion gives 1 when \( \theta \) equals 0, thus satisfying the normalization condition of \( |r\rangle \). The normalization constant can be rewritten as

\[
C = \frac{1}{\sqrt{2 + 2 \cos \phi + e^{i\phi}(g_1 \theta + g_2 \theta^2) + e^{-i\phi}(g_1^* \theta + g_2^* \theta^2)}}.
\]  

(8)

This normalization constant, \( C \), is valid as long as \( \theta \ll 1 \). Now, let us expand the state \( |re^{i\theta}\rangle \), which may be expanded as

\[
|re^{i\theta}\rangle \approx |r\rangle + ir \frac{d|r\rangle}{dr} \theta + \ldots,
\]

(9)

where we have treated the state \( |re^{i\theta}\rangle \) as a function. Then, the state \( |\psi\rangle \) in Eq.(2) becomes

\[
|\psi\rangle = \frac{|r\rangle + e^{i\phi} \left(|r\rangle + ir \frac{d|r\rangle}{dr} \theta\right)}{\sqrt{2 + 2 \cos \phi + e^{i\phi}(g_1 \theta + g_2 \theta^2) + e^{-i\phi}(g_1^* \theta + g_2^* \theta^2)}}.
\]

(10)

Consequently, if we set \( \theta \) to equal zero, we get exactly Eq.(4). Therefore, this state indicates that we obtained the same result as the previous construction, which thus yields to be a trivial case. Nevertheless, if and only if the phase \( \phi \) becomes equal to \( \pi \), and then \( \theta \) goes to zero, we will get zero. However, if we keep \( \phi = \pi \) but assuming that \( g_1 + g_1^* \) is zero, the state \( |\psi\rangle \) becomes finite and gives

\[
\lim_{\theta \to 0} |\psi_{\phi=\pi}\rangle = -\frac{ir \frac{d|r\rangle}{dr} \theta}{\sqrt{g_2 + g_2^* i\theta}} = -\frac{r}{\sqrt{g_2 + g_2^*}} \frac{d|r\rangle}{dr}.
\]

(11)

This result gives a new state other than \( |r\rangle \), since the derivative of a state generally gives a new state. It means that the construction in Eq.(2) is not trivial and constructs a self-superposition state. As discussed in the introduction, we give this type of states a special name which is the empty state. Before we progress, we want to point out certain important remarks. Making \( \theta \) go to zero is a requirement of constructing two identical states. Also, we found that phase \( \phi \) should be \( \pi \) for us to get a new state out of \( |r\rangle \). In other words, if \( \phi \) has other values rather than \( \pi \), we would not be able to find a new state. Also, note that \( g_1 + g_1^* \) has to be zero in order to get a nonvanishing state, so this requirement is a necessary condition for constructing the empty state.

The reason why we name these states as empty states is because in order to construct these states, the phase \( \phi \) has to equal \( \pi \). Now, in Eq.(2) when we take \( \theta \) to zero, and \( \phi \) equals to \( \pi \), we subtract two identical states as follows:

\[
|\psi\rangle = \lim_{\theta \to 0} C(|r\rangle - |re^{i\theta}\rangle).
\]

(12)

Intuitively, subtracting two identical states should give zero. Because of this first intuition, I named these states the empty states.
Note that Eq.(12) can be rewritten as

\[|\psi\rangle = \lim_{\theta \to 0} C(|r\rangle - |r + i r \theta\rangle) = \lim_{\theta \to 0} C(|r\rangle - |r + r \theta e^{i \pi/2}\rangle), \tag{13}\]

where we expanded \(e^{i \theta}\) around zero. This state reaches the empty state from a fixed phase which is \(\pi/2\). To generalize this result, we define this construction

\[|E_r\rangle = \lim_{|\Delta r| \to 0} C_{r}(|r + \Delta r\rangle - |r\rangle) = \lim_{|\Delta r| \to 0} C_{r}(|r + |\Delta r|e^{i \Delta \theta}\rangle - |r\rangle). \tag{14}\]

This construction is built on adding to \(r\) an arbitrary complex variable \(\Delta r\). Also, this state always gives a new state since it is just a generalization of Eq.(13). Therefore, this construction is the general formulation of the empty state, \(|E_r\rangle\), and it can be used as another definition of the empty state. The label \(E\) refers to the empty state, \(r\) is its variable, and \(\Delta r\) is a complex variable which we name the source of the empty state. This expression approaches the empty state from an arbitrary phase, since the source \(\Delta r\) can carry any phase. The formulation of Eq.(14) is named as the source representation of the empty state.

We note that any other construction of the empty state rather than Eq.(14) becomes either equivalent or a special case of it. For example, we may reach the empty state from this state

\[|\psi_1\rangle = \lim_{|\Delta r| \to 0} C_1(|r - \Delta r\rangle - |r\rangle), \]

or

\[|\psi_2\rangle = \lim_{|\Delta r| \to 0} C_2(|r + \Delta r\rangle - |r - \Delta r\rangle), \]

or may via other possible constructions. All these possibilities will be finally either an equivalent definition to our definition in Eq.(14) or a special case of our definition, as mentioned above.

Now let us study the empty state from Eq.(14). The normalization constant \(C_{r}\) is given as

\[C_{r} = \lim_{|\Delta r| \to 0} \frac{1}{\sqrt{2 - \langle r + \Delta r| r\rangle - \langle r| r + \Delta r\rangle}}. \tag{15}\]

Expanding the inner product \(\langle r| r + \Delta r\rangle\) around \(|\Delta r| \approx 0\). We find

\[\langle r| r + \Delta r\rangle \approx 1 + g_1(r)|\Delta r| - \frac{g_2(r)}{2}|\Delta r|^2 + \cdots, \tag{16}\]

and its complex conjugate

\[\langle r + \Delta r| r\rangle \approx 1 + g_1^*(r)|\Delta r| - \frac{g_2^*(r)}{2}|\Delta r|^2 - \cdots, \tag{17}\]

and their combination is

\[\langle r + \Delta r| r\rangle + c.c. \approx 2 + 2 \text{Re}(g_1(r))|\Delta r| - \text{Re}(g_2(r))|\Delta r|^2 + \cdots \tag{18}\]
If $g_1$ is a purely imaginary expression, Eq.(18) becomes

$$\langle r + \Delta r | r \rangle + \text{c.c.} \approx 2 - \text{Re}(g_2(r)) |\Delta r|^2 + \cdots \quad (19)$$

Then, substituting Eq.(18) into the normalization constant $C_r$ gives

$$C_r = \lim_{|\Delta r| \to 0} \frac{1}{\sqrt{2 - 2 \text{Re}(g_1(r)) |\Delta r| - \text{Re}(g_2(r)) |\Delta r|^2}}. \quad (20)$$

Now, because the expansion of this expression $|r + \Delta r\rangle - |r\rangle$ in Eq.(14) around $|\Delta r| \approx 0$ is in order of $|\Delta r|$, the expression of $\text{Re}(g_1(r))$ in Eq.(20) has to be zero to obtain a nonzero empty state. In other words, $\text{Re}(g_1(r)) = 0$ is the necessary condition for constructing a nonvanishing empty state. Applying this condition, the normalization constant $C_r$ becomes

$$C_r = \lim_{|\Delta r| \to 0} \sqrt{\frac{1}{\text{Re}(g_2(r))}} \frac{1}{|\Delta r|}. \quad (21)$$

Substituting this expression back into the empty state $|E_r\rangle$ yields

$$|E_r\rangle = \sqrt{\frac{1}{\text{Re}(g_2(r))}} \lim_{|\Delta r| \to 0} \frac{|r + \Delta r\rangle - |r\rangle}{|\Delta r|}. \quad (22)$$

Expanding the state $|r + \Delta r\rangle$ around $|\Delta r| \approx 0$ gives

$$|r + \Delta r\rangle \approx |r\rangle + \left( \frac{\partial |r + \Delta r\rangle}{\partial |\Delta r|} \right) \bigg|_{|\Delta r|=0} |\Delta r| + \cdots. \quad (23)$$

Substituting Eq.(23) into the empty state Eq.(22) yields

$$|E_r\rangle = \sqrt{\frac{1}{\text{Re}(g_2(r))}} \left( \frac{\partial |r + \Delta r\rangle}{\partial |\Delta r|} \right) \bigg|_{|\Delta r|=0}. \quad (24)$$

The expression of Eq.(24) is the final stage to determine the empty state described by one variable. This formula is equivalent to Eq.(14), and it provides another way to calculate the empty states. Note that Eq.(24) requires two inputs, the state $|r + \Delta r\rangle$ and $g_2(r)$ which is come from expansion of this inner product $\langle r | r + \Delta r \rangle$.

The formula of empty state in Eq.(24) cannot be simplified any further unless we assume new constrains. Such a constrain could be to assume that both $r$ and $\Delta r$ are real, so we can simplify Eq.(24) or Eq.(22) as

$$|E_r\rangle = \sqrt{\frac{1}{\text{Re}(g_2(r))}} \left( \frac{d|r\rangle}{d|r|} \right). \quad (25)$$

This state just emphasizes our previous statement that the empty state $|E_r\rangle$ always gives a new state, which differs from its original state $|r\rangle$. 
From Eq. (24) or Eq. (25), we can see that the empty state may be understood as an operator acting on the original state $|r\rangle$. Therefore, we can generally write any empty state as
\begin{equation}
|E_r\rangle = \hat{A}_r |r\rangle,
\end{equation}
where $\hat{A}_r$ is the generator operator of the empty state from the original state $|r\rangle$. Therefore, the empty state may be understood as an operator that depends on the variable $r$. The formulation of Eq. (26) is considered the operator representation of the empty states.

Lastly, we may introduce a third representation of the empty state. Before doing so, we recall the definition of the empty state Eq. (14)
\begin{equation}
|E_r\rangle = \lim_{|\Delta r| \to 0} C_r (|r + \Delta r\rangle - |r\rangle)
\end{equation}
When $|\Delta r|$ goes to zero, both states will be identical. One can ask why this state does not give a zero? The reason is that the normalization constant $C_r$ as found in Eq. (21) is balanced off the zero of subtraction of the two identical states. Therefore, we should be able to write the empty state as
\begin{equation}
|E_r\rangle = \mathbb{N}(|r\rangle - |r\rangle).
\end{equation}
Because this state explicitly represents the state $|r\rangle$, and it cannot be written as the formulas of either Eq. (24) or Eq. (25), the coefficient $\mathbb{N}$ has to be undetermined. We name the formulation of Eq. (27) the self-superposition representation of empty state.

### 2.1 Many Variables

The above discussion of the empty state is for a single variable case. Here we are going to generalize it to many variables. Suppose that we have a state with $N$-variables described as $|r_1, r_2, \cdots, r_i, \cdots, r_N\rangle$. Then, The empty state of the variable $r_i$ can be written as
\begin{equation}
|E_{r_i}\rangle = \lim_{|\Delta r_i| \to 0} C_{r_i} (|r_1, r_2, \cdots, r_i + \Delta r_i, \cdots, r_N\rangle - |r_1, r_2, \cdots, r_i, \cdots, r_N\rangle),
\end{equation}
where $C_{r_i}$ is the normalization constant, function of all variables. Following a similar procedure as in our previous discussion of the empty state of one variable, we can write this empty state as
\begin{equation}
|E_{r_i}\rangle = \sqrt{\frac{1}{\text{Re}(g_{r_i})}} \lim_{|\Delta r_i| \to 0} \frac{\partial |r_1, r_2, \cdots, r_i + \Delta r_i, \cdots, r_N\rangle}{\partial |\Delta r_i|},
\end{equation}
where $g_{r_i}$ came from this inner product expansion
\begin{equation}
\langle r_1, r_2, \cdots, r_i + \Delta r_i, \cdots, r_N | r_1, r_2, \cdots, r_i, \cdots, r_N \rangle + \text{c.c.} \approx 2 - \text{Re}(g_{r_i}) |\Delta r_i|^2 + \cdots.
\end{equation}
The expression in Eq. (29) is the general case of the empty states when the necessary condition is satisfied. To conclude this section, the self-superposition state is named the empty state, and it can be represented by three different representations, namely, the source, the operator, and the self-superposition representations. In the next section, we give an elementary example of empty states and introduce the empty state of the Fock states.

3 An Elementary Example And The Empty State of The Fock States

In the previous section, we defined the empty state as Eq. (14). In this section, we provide an elementary example state of the empty state and find its representations. We also introduce the empty state of the Fock state (empty-Fock state).

3.1 The Elementary Example

Let us start with an elementary example by defining the state $|R\rangle$ as

$$
|R\rangle = F(|n\rangle + R|m\rangle),
$$

(31)

where $R$ is assumed to be real for simplicity, $F$ is the normalization constant, and $|n\rangle$ and $|m\rangle$ are some number states, where $n \neq m$. The normalization constant $F$ is found to be

$$
F = \frac{1}{\sqrt{1 + R^2}}.
$$

(32)

The probabilities of finding the state $|n\rangle$ or $|m\rangle$ are given as

$$
P_n = |\langle n|R \rangle|^2 = \frac{1}{1 + R^2}, \quad \text{and} \quad P_m = |\langle m|R \rangle|^2 = \frac{R^2}{1 + R^2}.
$$

(33)

Next, the state $|R + \Delta R\rangle$ can be written as

$$
|R + \Delta R\rangle = \frac{|n\rangle + (R + \Delta R)|m\rangle}{\sqrt{1 + (R + \Delta R)^2}}.
$$

(34)

Note that $R$ is real and $\Delta R$ also is real. Then, we can find the expression of this inner product and its complex conjugate as

$$
\langle R + \Delta R|R \rangle + c.c. \approx 2 - \frac{(\Delta R)^2}{(1 + R^2)^2} + \cdots
$$

(35)

This expression is similar to Eq. (19), where $\text{Re}(g_2(r))$ here is $1/(1 + R^2)^2$, so we can write the empty state of the state $|R\rangle$ as

$$
|E_R\rangle = \lim_{\Delta R \to 0} C_R(|R + \Delta R\rangle - |R\rangle) = (1 + R^2)\frac{d|R\rangle}{dR}.
$$

(36)
Expanding this state by using both definitions $|R\rangle$ and $F$ gives
\begin{equation}
|E_R\rangle = \frac{-R|n\rangle + |m\rangle}{\sqrt{1 + R^2}}.
\end{equation}

This state is the empty state of the $|R\rangle$ state. This state is indeed a new state, since it is a different state than $|R\rangle$. To see that we calculated the probabilities of finding the state $|n\rangle$ or $|m\rangle$
\begin{equation}
P_n = |\langle n|E_R\rangle|^2 = \frac{R^2}{1 + R^2}, \quad \text{and} \quad P_m = |\langle m|E_R\rangle|^2 = \frac{1}{1 + R^2}.
\end{equation}
These two probabilities in Eq.(38) are indeed different than the probabilities of Eq.(33). In fact, the two probabilities are switched. To emphasize that the state $|R\rangle$ and its empty state are not the same, we calculate this inner product
\begin{equation}
\langle R|E_R\rangle = \langle E_R|R\rangle = 0.
\end{equation}
The two states are orthogonal to each other, so they are different.

Then, the empty state of $|R\rangle$ state can be represented by using the operator representation (see Eq.(26)) as
\begin{equation}
|E_R\rangle = \hat{A}_R|R\rangle = (1 + R^2) \frac{d|R\rangle}{dR} = \left[-R|n\rangle\langle n| + \frac{1}{R}|m\rangle\langle m|\right]|R\rangle,
\end{equation}
where $\hat{A}_R$ is the generator operator of the empty state $|E_R\rangle$, and it can be written explicitly as
\begin{equation}
\hat{A}_R = (1 + R^2) \frac{d}{dR} = \left[-R|n\rangle\langle n| + \frac{1}{R}|m\rangle\langle m|\right].
\end{equation}
From the above expression we can see that this operator has two expressions. Finally, the self-superposition representation of $|E_R\rangle$ state can be written as
\begin{equation}
|E_R\rangle = \aleph(|R\rangle - |R\rangle).
\end{equation}

### 3.2 The Empty State of The Fock State (Empty-Fock State)

In this section, we discuss the empty state of the Fock state (Empty-Fock state). The Fock states or the number states are represented as $|n\rangle$ state, where $n$ can be any positive integer number. Because the number $n$ is an integer, we cannot apply Eq.(24) to find an expression to describe the empty state of the Fock states. However, we can solve this problem by using an auxiliary state. The auxiliary state is a state that has a continuous variable, say $r$, so when taking this variable to zero or infinite, we are left with a number state. The state $|R\rangle$ can be an auxiliary state, since when $R$ goes to zero, we left with $|n\rangle$ state, and when $R$ goes to $\infty$, we left with $|m\rangle$ state (see Eq.(31)).
Using the state $|R\rangle$ as an auxiliary state, the empty-Fock state can be represented using the self-superposition representation as

$$|E_n\rangle = |E_{R=0}\rangle = \lim_{R \to 0} \mathfrak{N}(|R\rangle - |R\rangle) = \mathfrak{N}(|n\rangle - |n\rangle) = |m\rangle,$$  \hspace{1cm} (43)

where we replace the state $|R=0\rangle$ to its corresponding state $|n\rangle$ from Eq.(31), and the resultant state of $|E_{R=0}\rangle$ is $|m\rangle$ state. The state in Eq.(43) tells us that there is strong dependence on the source state, the auxiliary state $|R\rangle$. If we use a different state of $|R\rangle$ having a different value of $m$, and the same value of $n$, we will get another result for the empty-Fock state, because we have two different sources states. Note that $m$ has to be not equal to $n$, as if they are equal we cannot reach this conclusion. Therefore, using the self-superposition representation for empty-Fock states gives us no clear state. Then, using the source representation, however, we can write Eq.(43) as

$$|E_n\rangle = \lim_{R \to 0} \lim_{|\Delta R| \to 0} C_R(|R + \Delta R\rangle - |R\rangle).$$  \hspace{1cm} (44)

This representation clarifies the source state $|R\rangle$ and the source $\Delta R$.

The last task we need to perform here is when $R$ being generally any complex value, will Eq.(43) be invariant? Note, we cannot use the derivative to $|R\rangle$ as we did in Eq.(36). We need to use the general definition of the empty state Eq.(14) or Eq.(24). From the LHS of Eq.(36), the normalization constant $C_R$ is found to be

$$C_R = \frac{1}{\sqrt{\left[\frac{1}{\sqrt{1+|R+\Delta R|^2}} - \frac{1}{\sqrt{1+|R|^2}}\right]^2 + \left[\frac{R + \Delta R}{\sqrt{1+|R+\Delta R|^2}} - \frac{R}{\sqrt{1+|R|^2}}\right]^2}}.$$  \hspace{1cm} (45)

Because we are interested in the Fock states, we need to use two limits $|\Delta R| \to 0$ and $R \to 0$. After we carry on the calculations of the two limits, we obtain

$$|E_n\rangle = \lim_{R \to 0} \lim_{|\Delta R| \to 0} C_R(|R + \Delta R\rangle - |R\rangle) = \mathfrak{N}(|n\rangle - |n\rangle) = e^{i\Delta \theta}|m\rangle,$$  \hspace{1cm} (46)

where $\Delta \theta$ is the phase of $\Delta R$, means $\Delta R = |\Delta R|e^{i\Delta \theta}$. We got the same state $|m\rangle$ of the real $R$ treatment in Eq.(43). However, the additional thing here is the external phase, so the state is undermined and the complex amplitude as well.

The last thing we want to say in this section is that although we found that the empty-Fock state is a pure state of a Fock state, we cannot guarantee that this will be always the case. Maybe if we used another auxiliary state to find the empty-Fock state, we could obtain a mixed state, so the general statement that can be said now is that the empty-Fock state highly depends on its source state, its auxiliary state. In the next section, we talk about the empty state of the coherent state.
4 The Empty State of The Coherent State (EC State)

It is known that the coherent state is a quantum state, and it is the closest state to classical physics [8,28,29]. This state is used to build some nonclassical states and is applied to an enormous number of applications (for example, see [27,30–34]). In this section, we employ the empty state to the coherent state to produce what we call the empty state of the coherent state, or shortly the (EC state). In this section we study this state and find its mathematical and physical properties.

4.1 The EC State

The coherent state is defined as the solution of this equation $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where $|\alpha\rangle$ is the coherent state, which can be written as

$$|\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$  \hspace{1cm} (47)

Now, the definition of the empty state of the coherent state or the EC state can be found from Eq.(14)

$$|E_\alpha\rangle = \lim_{|\Delta\alpha| \to 0} C_\alpha(|\alpha + \Delta\alpha\rangle - |\alpha\rangle)$$  \hspace{1cm} (48)

The inner product of two general coherent states $|\alpha\rangle$ and $|\beta\rangle$ is given as

$$\langle \beta |\alpha\rangle + \text{c.c.} = e^{-|\bf{d}|^2/2} |\beta|^2 + (\beta^* \alpha + \alpha^* \beta)$$  \hspace{1cm} (49)

Using this relation, we find the inner product of $|\alpha + \Delta\alpha\rangle|\alpha\rangle$ and its complex conjugate as

$$\langle \alpha + \Delta\alpha |\alpha\rangle + \text{c.c.} = 2e^{-|\Delta\alpha|^2/4} \cos(|\Delta\alpha||\alpha| \sin(\Delta\theta - \theta)).$$  \hspace{1cm} (50)

This expression can be expanded around $|\Delta\alpha| \approx 0$ to give

$$\langle \alpha + \Delta\alpha |\alpha\rangle + \text{c.c.} \approx 2 - \left[1 + |\alpha|^2 \sin(\Delta\theta - \theta)^2\right] |\Delta\alpha|^2 + \cdots,$$  \hspace{1cm} (51)

where $\Delta\alpha$ is expressed as $|\Delta\alpha|e^{i\Delta\theta}$ and $\alpha$ is written as $|\alpha|e^{i\theta}$. Since the linear term of $|\Delta\alpha|$ is absence, that means the necessary condition is satisfied, so the EC state has a nonvanishing empty state. This equation is similar to Eq.(19) with $g_2$ equal to $[1 + |\alpha|^2 \sin(\Delta\theta - \theta)^2]$, so we can write the EC state as

$$|E_\alpha\rangle = \frac{1}{\sqrt{1 + |\alpha|^2 \sin(\Delta\theta - \theta)^2}} \lim_{|\Delta\alpha| \to 0} \frac{|\alpha + \Delta\alpha\rangle - |\alpha\rangle}{|\Delta\alpha|}$$  \hspace{1cm} (52)
Now, let us substitute the expression of $|\alpha\rangle$ from Eq. (47) into Eq. (52). We then find that

$$
|E_\alpha\rangle = \frac{1}{\sqrt{1 + |\alpha|^2\sin(\Delta\theta - \theta)^2}} \lim_{|\Delta\alpha|\to 0} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}|\Delta\alpha|} \left[ e^{-\frac{1}{2}|\alpha + \Delta\alpha|^2} (\alpha + \Delta\alpha)^n - e^{-\frac{1}{2}|\alpha|^2} \alpha^n \right] |n\rangle.
$$

(53)

The expression inside the bracket inside the summation can be further expanded as

$$
e^{-\frac{1}{2}|\alpha|^2} \left[ e^{-\frac{1}{2}|\Delta\alpha|^2 - |\alpha||\Delta\alpha| \cos(\Delta\theta - \theta)} \left( |\alpha| e^{i\theta} + |\Delta\alpha| e^{i\Delta\theta} \right)^n - |\alpha|^n e^{i\theta} \right].
$$

(54)

This expression can be expanded around $|\Delta\alpha| \approx 0$. It gives

$$
e^{-\frac{1}{2}|\alpha|^2} \alpha^n \left[ \frac{e^{i(\Delta\theta - \theta)n}}{|\alpha|} - |\alpha| \cos(\Delta\theta - \theta) \right] |\Delta\alpha| + \cdots = \left( \frac{\partial |\alpha + \Delta\alpha\rangle}{\partial |\Delta\alpha|} \right)_{|\Delta\alpha|=0} |\Delta\alpha| + \cdots.
$$

(55)

By substituting this expression back into Eq. (53), the source amplitude $|\Delta\alpha|$ will cancel out from the denominator and numerator. We are left with

$$
|E_\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left[ \frac{e^{i(\Delta\theta - \theta)n}}{|\alpha|} - |\alpha| \cos(\Delta\theta - \theta) \right] |\Delta\alpha| |n\rangle.
$$

(56)

This expression can be further simplified, if we use this derivation

$$
\frac{\partial |\alpha\rangle}{\partial |\alpha|} = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left[ \frac{n}{|\alpha|} - |\alpha| \right] |n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{n\alpha^n}{\sqrt{n!}} |n\rangle - |\alpha||\alpha\rangle,
$$

(57)

which can be rewritten as

$$
e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{n\alpha^n}{\sqrt{n!}} |n\rangle = |\alpha| \frac{\partial |\alpha\rangle}{\partial |\alpha|} + |\alpha|^2 |\alpha\rangle.
$$

(58)

Then, the EC state can be written as

$$
|E_\alpha\rangle = \frac{e^{i\delta\theta \frac{\partial |\alpha\rangle}{\partial |\alpha|}} + i|\alpha| |\alpha\rangle \sin (\delta\theta)}{\sqrt{1 + |\alpha|^2\sin(\delta\theta)^2}},
$$

(59)

where $\delta\theta$ is the phase difference and equals $\Delta\theta - \theta$. This equation is the complete expression of the EC state. As we can see, the EC state is a new state and differs from the coherent state. Also, the effect of the phase difference $\delta\theta$ can change the physical quantities, as we will see. That means the source’s phase $\Delta\theta$ is matter in order to reach the EC state. To illustrate this fact, suppose we reach the EC state from $\Delta\alpha$ having the same phase of $\alpha$, means $\theta = \Delta\theta$, or $\delta\theta = 0$. The expression in Eq. (59) simplifies to be just the first derivation. Therefore, maybe we can say that the EC state is a degenerate state.
Note that if we have more than one EC state, we need to write $\delta \theta$ in the label of the EC state as $|E_{\alpha, \delta \theta}\rangle$. However, in this article, we do not have two different EC states at the same time, so we do not need to write the angle $\delta \theta$ in the label of the EC state.

We can define a parameter that measures the ratio of the new state to the original state (here the derivation is the new state). We name this parameter the **emptiness** $E$. The emptiness is the ratio of the absolute coefficient of the new state (derivation state) to the absolute coefficient of the original state (coherent state):

$$E = \frac{|e^{i\delta \theta}|}{|i| \sin(\delta \theta)} = \frac{1}{|\alpha| \sin(\delta \theta)}. \quad (60)$$

If this parameter is infinity, that means it is purely a new state, and if it gives 0, that means it is a purely coherent state. It becomes infinite when $\delta \theta$ becomes 0, so this case is the maximum emptiness. Also, the emptiness becomes almost zero when $|\alpha| \gg 1$, and that case is the minimum emptiness. For a given value of $|\alpha|$, the minimum emptiness occurs when $\delta \theta$ equals $\pi/2$.

Also, note that when $\alpha$ goes to zero in the coherent state, we get the vacuum state $|0\rangle$. However, when $\alpha$ goes to zero in the EC state, we get one photon state $|1\rangle$.

Thus, by using the self-superposition representation, we can write

$$|E_0\rangle = \aleph(|0\rangle - |0\rangle) = \lim_{\alpha \to 0} \lim_{\Delta \alpha \to 0} C_\alpha(|\alpha + \Delta \alpha\rangle - |\alpha\rangle) = |1\rangle. \quad (61)$$

This result agrees with our previous discussion about Eq. (43), which is that the self-superposition of Fock-states gives us a new state. In the next section, we talk about some of the mathematical properties of EC state.

### 4.2 Identities and mathematical properties

In this section, we find some of the mathematical properties of the EC state. First, let us find the operator representation of the EC state, which means we want to find the generator operator $\hat{A}_\alpha$ of the EC state. If we add a photon to the coherent state by applying the operator $\hat{a}$, then the resulting state is given as

$$\hat{a}^\dagger |\alpha\rangle = \left( \frac{\partial}{\partial \alpha} + \frac{\alpha^*}{2} \right) |\alpha\rangle = \frac{e^{-\frac{1}{2} |\alpha|^2}}{\alpha} \sum_{n=0}^{\infty} \frac{\alpha^n n}{\sqrt{n!}} |n\rangle. \quad (62)$$

Note here the derivation is with respect to $\alpha$ and not to $|\alpha|$ as we had in the EC state. The relationship between the two derivatives is given as

$$\frac{\partial |\alpha\rangle}{\partial \alpha} = \frac{|\alpha|}{\alpha} \frac{\partial |\alpha\rangle}{\partial |\alpha|} + \frac{|\alpha|^2}{2\alpha} |\alpha\rangle, \quad (63)$$
where we can see the difference between the two derivatives. Then, by taking advantage of Eq.(58), we can simplify the derivation in Eq.(57) as

\[ \frac{\partial |\alpha\rangle}{\partial |\alpha|} = \left( \frac{\alpha}{|\alpha|} \hat{a}^\dagger - |\alpha| \right) |\alpha\rangle = \left( \frac{\alpha}{|\alpha|} \hat{a}^\dagger \right) |\alpha\rangle \]

(64)

Then, using this equation allows us to write Eq.(59) as

\[ |E_\alpha\rangle = \left[ e^{i\delta\theta/\alpha} \hat{a}^\dagger - \left[ 1 - i \sin(\delta\theta) \right] |\alpha\rangle \right] |\alpha\rangle = \hat{A}_\alpha |\alpha\rangle. \]

(65)

The expression inside the bracket is the generator operator \( \hat{A}_\alpha \). In case \( \delta\theta \) equals 0, this operator becomes

\[ \hat{A}_\alpha = \left( e^{i\theta} \hat{a}^\dagger - e^{-i\theta} \hat{a} \right), \]

(66)

where again \( \theta \) is the complex phase of \( \alpha \). This operator in this particular case is a skew-hermitian operator, and commutes with itself. A similar operator to \( \hat{A}_\alpha \) is introduced by Yuen [37]. His operator is \( \hat{b} = \mu \hat{a} + \nu \hat{a}^\dagger \), where \( |\mu|^2 - |\nu|^2 = 1 \). This operator is satisfying this relation \([ \hat{b}, \hat{b}^\dagger ] = 1 \). Our operator is different than this operator because we have \( \nu' = e^{i\theta} \mu' = e^{-i\theta} \), which satisfies \( |\mu'|^2 - |\nu'|^2 = 0 \) and \([ \hat{A}_\alpha, \hat{A}_\alpha^\dagger ] = 0 \). So it is similar but not the same. Also, if we calculate the commutation relation of the general \( \hat{A}_\alpha \), we get

\[ [ \hat{A}_\alpha, \hat{A}_\alpha^\dagger ] = \frac{\sin(\delta\theta)^2}{1 + |\alpha|^2 \sin(\delta\theta)^2} [\hat{a}, \hat{a}^\dagger]. \]

(67)

We also may connect the EC state to other known states. For example, we can connect the EC state to the Agarwal state or photon-added coherent state [36] throughout the annihilation operator as

\[ a^\dagger |\alpha, 1\rangle = \sqrt{1 + |\alpha|^2} |\alpha, 1\rangle, \]

(68)

where \( |\alpha, 1\rangle \) is the Agarwal state.

Secondly, we can find some identities of the EC state Eq.(59). Let us start by the annihilation operator \( \hat{a} \) acting on \( |E_\alpha\rangle \). First, the annihilation operator acting on the derivation of Eq.(64) gives

\[ \hat{a} \left( \frac{\partial |\alpha\rangle}{\partial |\alpha|} \right) = \frac{\alpha}{|\alpha|} \hat{a}^\dagger |\alpha\rangle - |\alpha| \hat{a} |\alpha\rangle, \]

(69)

\[ = \frac{\alpha}{|\alpha|} (\hat{a}^\dagger \hat{a} + 1) |\alpha\rangle - |\alpha| |\alpha\rangle, \]

\[ = \alpha \left( \frac{\alpha}{|\alpha|} \hat{a}^\dagger - |\alpha| |\alpha\rangle \right), \]

\[ = \alpha \left( \frac{\partial |\alpha\rangle}{\partial |\alpha|} \right) + \frac{\alpha}{|\alpha|} |\alpha\rangle. \]

Then, using this expression into \( \hat{a} \) acting on the EC state yields

\[ \hat{a} |E_\alpha\rangle = \alpha |E_\alpha\rangle + \frac{e^{i\delta\theta} \alpha}{|\alpha| \sqrt{1 + |\alpha|^2 \sin(\delta\theta)^2}} |\alpha\rangle. \]

(70)
It gives the same state plus the coherent state. For the creation operator, we have

\[ \langle E_\alpha | \hat{a}^\dagger \rangle = \langle E_\alpha | \alpha^* + \alpha \frac{\alpha^* e^{-i\delta \theta}}{|\alpha|\sqrt{1 + |\alpha|^2 \sin(\delta \theta)^2}}. \]  

(71)

If \( \delta \theta \) equals zero, this identity Eq.(70) is simplified to

\[ \hat{a}|E_\alpha \rangle = \alpha |E_\alpha \rangle + \frac{\alpha}{|\alpha|}|\alpha\rangle. \]  

(72)

Also, we can derive

\[ \hat{a}^2|E_\alpha \rangle = \alpha^2 |E_\alpha \rangle + \frac{2 e^{i\delta \theta} \alpha^K}{|\alpha|\sqrt{1 + |\alpha|^2 \sin(\delta \theta)^2}}|\alpha\rangle. \]  

(73)

Third is the eigen-equation of the EC state. From Eq.(70) and Eq.(73), we noticed that if we multiply Eq.(70) by 2 and divide Eq.(73) by \( \alpha \), we get exactly same second term for both equations, meaning

\[ 2\hat{a}|E_\alpha \rangle = 2\alpha |E_\alpha \rangle + \frac{2 e^{i\delta \theta} \alpha K}{|\alpha|\sqrt{1 + |\alpha|^2 \sin(\delta \theta)^2}}|\alpha\rangle. \]  

(74)

\[ \frac{\hat{a}^2}{\alpha}|E_\alpha \rangle = \alpha |E_\alpha \rangle + \frac{2 e^{i\delta \theta} \alpha K}{|\alpha|\sqrt{1 + |\alpha|^2 \sin(\delta \theta)^2}}|\alpha\rangle. \]  

(75)

where \( K = 1/\sqrt{1 + |\alpha|^2 \sin(\delta \theta)^2} \). Then, by subtracting the two equations, we can observe the eigen-equation of EC state

\[ \hat{A}_\alpha |E_\alpha \rangle \equiv \left(2\hat{a} - \frac{\hat{a}^2}{\alpha}\right)|E_\alpha \rangle = \alpha |E_\alpha \rangle \]  

(76)

where \( \hat{A}_\alpha \equiv \left(2\hat{a} - \frac{\hat{a}^2}{\alpha}\right) \). The EC state is not the only possible eigenfunction of this operator, the coherent state \( |\alpha\rangle \) itself is a possible solution as well. Also, the operator \( \hat{A}_\alpha \) is just a special case of the C.Brif operator \[38\].

Fourth, let us turn to calculate the inner products. Starting from the coherent with the derivation state \( \langle \alpha | \frac{\partial}{\partial \beta} | \beta \rangle \). By using Eqs.(49, 62, 64), we can find this product as

\[ \langle \alpha | \left( \frac{\partial}{\partial \beta} | \beta \right) \rangle = \langle \alpha | \left( \left( \frac{\beta}{|\beta|} \hat{a}^\dagger | \beta \right) - | \beta \rangle \langle \beta | \right) \rangle \]

\[ = \frac{\beta}{|\beta|} \langle \alpha | \hat{a}^\dagger | \beta \rangle - | \beta \rangle \langle \alpha | \beta \rangle \]

\[ = \langle \alpha | \beta \rangle \left[ \frac{\beta \alpha^*}{|\beta|} - | \beta \rangle \right] \]

\[ = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \beta \alpha^*} \left[ \frac{\beta \alpha^*}{|\beta|} - | \beta \rangle \right]. \]  

(77)

Note that if the derivation is for \( \alpha \) instead of \( \beta \), this inner product becomes zero, which means \( \langle \alpha | \left( \frac{\partial}{\partial \alpha} | \alpha \right) \rangle = 0 \). Using the relation in Eq.(77) and Eq.(58), we can derive the inner product of the EC state as

\[ \langle \alpha | E_\beta \rangle = \frac{e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \beta \alpha^*}}{\sqrt{1 + |\beta|^2 \sin(\delta \theta)^2}} \left[ e^{i\delta \theta} \left( \frac{\beta \alpha^*}{|\beta|} - | \beta \rangle \right) + i | \beta \rangle \sin(\delta \theta) \right]. \]  

(78)
In case both of them are $\alpha$, we get
\[ \langle \alpha | E_{\alpha} \rangle = \frac{i|\alpha| \sin(\delta \theta)}{\sqrt{1 + |\alpha|^2 \sin(\delta \theta)^2}} \] (79)

Next, the inner product of two derivation states can be found as
\[
\left( \frac{\partial}{\partial |\alpha|} \right) \left( \frac{\partial}{\partial |\beta|} \right) = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \beta \alpha^*} \left[ \frac{\alpha^* \beta}{|\alpha||\beta|} + \frac{\alpha^2 \beta^2}{|\alpha||\beta|} - \frac{|\beta| \alpha^* \beta}{|\alpha|} - \frac{|\alpha| \alpha^* \beta}{|\beta|} + |\alpha||\beta| \right].
\] (80)

This relation can be used to calculate the inner product of the EC states, if we wish. Also, note that when $\beta$ equals $\alpha$, we get 1. The next section discusses the statistical properties of the EC state.

### 4.3 Statistical properties

The statistical properties of a photonic state is an excellent way to know the photon distribution, the average number of photons, photons fluctuations, etc. In this section, we find some of the statistical properties of the EC state. Let us start by studying the photon probability distribution, which can be found by multiplying Eq.(56) by $\langle n \rangle$ and then square it. We found
\[
P_n = |\langle n | E_{\alpha} \rangle|^2 = \frac{e^{-|\alpha|^2} |\alpha|^{2n} \left[ n^2 + \cos(\delta \theta)^2 (|\alpha|^2 - 2n) \right]}{n! (1 + |\alpha|^2 \sin(\delta \theta)^2)}. \] (81)

This distribution is illustrated in Fig.(1). In the case when $\alpha$ becomes zero, one photon always will survive, which agrees with Eq.(61).

From Fig.(1) we can see one point with a zero probability at the angle $\delta \theta = 0$. This zero point occurs for $n = |\alpha|^2$. It indicates the maximum quantum interference. This result agrees with the definition of the emptiness in Eq.(60) $E$, where when the emptiness is maximized, the quantum interference is maximized as well. Therefore, the emptiness can be used to measure the strength of the quantum interference of the EC state.

Then, we calculate the average photon number of the EC state. Using Eqs.(70,71), we can calculate the average photon number $\langle \hat{n} \rangle$, which gives
\[ \langle \hat{n} \rangle = \langle E_{\alpha} | \hat{a}^\dagger \hat{a} | E_{\alpha} \rangle = |\alpha|^2 + \frac{1 + 2|\alpha|^2 \sin(\delta \theta)^2}{1 + |\alpha|^2 \sin(\delta \theta)^2} = |\alpha|^2 + M, \] (82)

where $M$ is defined as
\[ M = \frac{1 + 2|\alpha|^2 \sin(\delta \theta)^2}{1 + |\alpha|^2 \sin(\delta \theta)^2}. \] (83)

The maximum value of Eq.(82) is $2 + |\alpha|^2$, and that happens when $\delta \theta$ equals $\pi/2$ and $|\alpha|^2 \gg 1$. In contrast, the minimum value is $|\alpha|^2 + 1$, which happens when $\delta \theta$
equals 0. Therefore, this average generally is bounded by an upper and lower bound described as

\[ 1 \leq M \leq 2 \]
\[ 1 + |\alpha|^2 \leq \langle \hat{n} \rangle \leq 2 + |\alpha|^2. \] (84)

This bound indicates that we always have at least one photon on average.

Then, after using the identity in Eq. (73), we can find the average of the square photon number operator as

\[ \langle \hat{n}^2 \rangle = \langle \hat{a}^{\dagger} \hat{a} \hat{a}^{\dagger} \hat{a} \rangle = |\alpha|^4 + 5|\alpha|^2 + \frac{1+2|\alpha|^2 \sin(\delta \theta)^2}{1 + |\alpha|^2 \sin(\delta \theta)^2} = |\alpha|^4 + 5|\alpha|^2 + M. \] (85)

The photon number fluctuations \( \langle \Delta n \rangle \) are given as

\[ \langle \Delta n \rangle = \sqrt{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2} = \sqrt{|\alpha|^2(5 - 2M) + M(1 - M)}, \] (86)

This expression is always real and positive, and it becomes equal to \( \sqrt{3} |\alpha| \) when the phase difference \( \delta \theta \) equals 0. And the minimum fluctuations happens when the phase difference equals to \( \pi/2 \) (see Fig.(2)).

The next statistical property is the classification of the photon number distributions. If we want to know whether the EC state is a sub- or super- Poissonian distribution, we can use the Mandel Q-parameter [41]. The Mandel Q-parameter is one of the parameters that enables us to know the category of the distribution. It is defined as

\[ Q = \frac{\langle \Delta \hat{n} \rangle^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle}. \] (87)
If $Q$ is $-1 \leq Q < 0$, the statistics become a sup-Poissonian distribution. Also, if $Q$ is $Q > 0$, the statistics become a super-Poissonian distribution, and if $Q$ is $-1$, the statistics become the number state distribution. By substituting these Eqs.\(82, 86\) into the Mandel Q-parameter Eq.\(87\), we find

$$Q = \frac{2|\alpha|^2(2 - M) - M^2}{|\alpha|^2 + M}$$ \hspace{1cm} (88)$$

This expression exhibits both sub- and super- Poissonian distribution (see Fig.(4)), which means it can be used to produce photon antibunching.

Next, we calculate the quantum phase distribution $P(\varphi)$ of EC state. We selected the Pegg-Barnett formulation of the phase distribution $[39, 40]$. This distribution is defined as

$$P(\varphi) = \frac{1}{2\pi}|\langle \varphi | r \rangle|^2,$$ \hspace{1cm} (89)$$

where the state $|\varphi \rangle$ is defined as

$$|\varphi \rangle = \sum_{n=0}^{\infty} e^{in\varphi}|n\rangle.$$ \hspace{1cm} (90)$$

Then, using Eq.\(56\) it is allows us to find this distribution for the EC state

$$P(\varphi) = \frac{1}{2\pi}|\langle \varphi | E_\alpha \rangle|^2 = \frac{e^{-|\alpha|^2}}{2\pi[1 + |\alpha|^2 \sin(\delta \theta)^2]} \left| \sum_{n=0}^{\infty} \frac{e^{-i(n\varphi - \theta)}|\alpha|^n}{\sqrt{n!}} \left( \frac{ne^{i\delta \theta}}{|\alpha|} - |\alpha| \cos(\delta \theta) \right) \right|^2$$ \hspace{1cm} (91)$$

This distribution becomes close to the phase distribution of the coherent state when $\delta \theta$ becomes $\pi/2$. In this case, there is one peak. However, when $\delta \theta$ becomes 0, this
distribution gives two peaks. In the next section, we talk about the nonclassicality properties of the EC state.

4.4 Nonclassical Properites

In this section we find some nonclassical properties of the EC state. We started first the squeezing properties and then the quasi-probability distributions.

4.4.1 Squeezing Properties

Some of the squeezing properties can be found by finding the quadrature operators. These quadrature operators $\hat{X}_1$ and $\hat{X}_2$ are the operators of the dimensionless position and momentum. They are defined as

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2}, \quad \hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}.\quad (92)$$

Whenever the fluctuations of one of these two operators of a given state are less than 1/4, that state is squeezed. Namely, the condition reads as

$$\langle (\Delta \hat{X}_i)^2 \rangle < \frac{1}{4}.\quad (93)$$

Now let us find the expectation values of these operators for the EC state. Before doing so, let us write certain helpful mathematical relations by using these Eqs.(70,73,79). The helpful relations are

$$\langle E_\alpha |\hat{a}| E_\alpha \rangle = \alpha \left( 1 - iK^2 \sin(\delta \theta)e^{i\delta \theta} \right),$$
\[ \langle E_\alpha | \hat{a}^2 | E_\alpha \rangle = \alpha^2 \left( 1 - 2iK^2 \sin(\delta \theta)e^{i\delta \theta} \right), \]

\[ \langle E_\alpha | \hat{a} \hat{a}^\dagger | E_\alpha \rangle = 1 + |\alpha|^2 + K^2(1 + 2|\alpha|^2 \sin(\delta \theta)^2), \]  

(94)

where \( K \) is the expression \( 1/\sqrt{1 + |\alpha|^2 \sin(\delta \theta)^2} \). Note that \( K \) is always bounded between 0 and 1. Then, the expectation values of the operators \( \hat{X}_1 \) and \( \hat{X}_2 \) for the EC state are given by

\[ \langle E_\alpha | \hat{X}_1 | E_\alpha \rangle = \frac{\alpha + \alpha^*}{2} + |\alpha|K^2 \sin(\delta \theta) \sin(\Delta \theta), \]  

(95)

\[ \langle E_\alpha | \hat{X}_2 | E_\alpha \rangle = \frac{\alpha - \alpha^*}{2i} - |\alpha|K^2 \sin(\delta \theta) \cos(\Delta \theta). \]  

(96)

Recall that \( \delta \theta \) is defined as \( \Delta \theta - \theta \). Therefore, in addition to the phase difference \( \delta \theta \), we need to know the source phase \( \Delta \theta \). We note that if \( \Delta \theta \) equals \( \theta \), the above expressions tend to give exactly the same results as the coherent states, where the second term becomes zero.

Next, the expectation values for the square of \( \hat{X}_1 \) and \( \hat{X}_2 \) operators are given as

\[ \langle E_\alpha | \hat{X}_1^2 | E_\alpha \rangle = \frac{1}{4} + \frac{\alpha^2 + \alpha^*}{4} + \frac{|\alpha|^2}{2} + \frac{K^2}{2} \left[ 1 + 2|\alpha|^2 S_1 \right], \]  

(97)

where \( S_1 = \sin(\delta \theta) [\sin(\delta \theta) + \sin(2\Delta \theta - \delta \theta)] \) and

\[ \langle E_\alpha | \hat{X}_2^2 | E_\alpha \rangle = \frac{1}{4} - \frac{\alpha^2 + \alpha^*}{4} + \frac{|\alpha|^2}{2} + \frac{K^2}{2} \left[ 1 + 2|\alpha|^2 S_2 \right], \]  

(98)

where \( S_2 = \sin(\delta \theta) [\sin(\delta \theta) - \sin(2\Delta \theta - \delta \theta)] \). The fluctuations of these operators are given as

\[ \langle (\Delta \hat{X}_1)^2 \rangle = \langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2 = \frac{1}{4} + K^2 \left[ \frac{1}{2} - |\alpha|^2 K^2 \sin(\delta \theta)^2 \sin(\Delta \theta)^2 \right]. \]  

(99)

\[ \langle (\Delta \hat{X}_2)^2 \rangle = \langle \hat{X}_2^2 \rangle - \langle \hat{X}_2 \rangle^2 = \frac{1}{4} + K^2 \left[ \frac{1}{2} - |\alpha|^2 K^2 \sin(\delta \theta)^2 \cos(\Delta \theta)^2 \right]. \]  

(100)

In the above expressions, when \( \delta \theta \) equals zero, they gives 3/4. This is the minimum fluctuations of one photon. In Fig.(4) we representes the fluctuations of \( \hat{X}_1 \) operator for some angles of \( \delta \theta \) and \( \Delta \theta \). We can see that the squeezing can occur for some values of \( \Delta \theta \). Applying Eq.(93) to Eq.(99) gives us the squeezing condition

\[ \frac{1}{4} < |\alpha|K \sin(\delta \theta) \sin(\Delta \theta). \]  

(101)

The maximum squeezing occurs when \( \delta \theta \) equals \( \pi/2 \) and \( \Delta \theta \) equals \( \pi/2 \) or 0 for \( \hat{X}_2 \) operaator, and \( |\alpha| \) equals \( \sqrt{3} \). When that happens, Eq.(99) gives 3/16, and Eq.(100) gives 3/8.
The fluctuation of $\hat{X}_1$ operator of the EC state for some values of $\Delta \theta$ and $\delta \theta$. Each plot has 4 values of $\delta \theta$ which are form the top to down 0, $\pi/8$, $\pi/4$, $\pi/2$. The straight line in all figures is for $\delta \theta = 0$.

The multiplication of Eq.(99) and Eq.(100) always satisfies these upper and lower bounds

$$\frac{1}{16} \leq \langle \Delta \hat{X}_1 \rangle^2 \langle \Delta \hat{X}_2 \rangle^2 \leq \frac{9}{16} \quad (102)$$

For example, in case both $\delta \theta$ and $\Delta \theta$ equal $\pi/2$, the multiplication yields

$$\langle \Delta \hat{X}_1 \rangle^2 \langle \Delta \hat{X}_2 \rangle^2 = \frac{1}{16} \frac{(3 + |\alpha|^2)(3 + |\alpha|^4)}{(1 + |\alpha|^2)^3}. \quad (103)$$

This relation starts from 9/16 when $|\alpha|$ is zero, and finishes at 1/16 when $|\alpha| \gg 1$.

Next, we study the quasi-probability distributions for the EC state.

### 4.4.2 Quasi-Probability Distributions

In quantum optics, there are some quasi-probability distributions that give an indication of whether a light state represents some nonclassical features or does not. The first distribution is the $Q(\alpha)$, or Husimi function [42]. This function gives us an idea of the behaviour of the interested state in the phase space. It is defined as

$$Q = \frac{\langle \alpha | \hat{\rho} | \alpha \rangle}{\pi}, \quad (104)$$

where $\hat{\rho}$ is the density operator. The density operator of the EC state is $|E_\alpha\rangle\langle E_\alpha|$. Applying this density operator to $Q(\alpha)$ yields $Q(\alpha) = |\langle \alpha | E_\beta \rangle|^2 / \pi$. Using the identity in Eq.(78) allows us to calculate the $Q(\alpha)$ distribution

$$Q = \frac{K^2}{\pi} e^{-|\alpha - \beta|^2} \left[ |\alpha - \beta|^2 \cos(\delta \theta)^2 + |\alpha|^2 \sin(\delta \theta)^2 + \frac{\alpha^* \beta - \beta^* \alpha}{2i} \sin(2\delta \theta) \right], \quad (105)$$

where $K$ is $1/\sqrt{1 + |\beta|^2 \sin(\delta \theta)^2}$. This function is always positive. In Fig.(5) we plot Eq.(105) for some values of $\delta \theta$. We plotted $Q(\alpha)$ in Fig.(5) versus the real and.
imaginary parts of $\alpha = \alpha_r + i\alpha_i$. From the same figure, we can see how $Q(\alpha)$ gives a semi-crescent shape which indicates there is a quantum interference. This nonclassical

![Figure 5: The $Q(\alpha)$ function of the EC state for different phase differences $\delta \theta$. The axis are the real and imaginary parts of $\alpha$. The value of $\beta$ is real and equals 1.](image)

behaviour is maximized when $\delta \theta$ equals 0. In this case, the Hsuimi function looks like a ring, and its equation can be reduced to the following

$$Q(\alpha) = \frac{|\alpha - \beta|^2}{\pi} e^{-|\alpha - \beta|^2}.$$  \hspace{1cm} (106)

Also, we note that as the angle $\delta \theta$ increases, the ring deforms. That deformation is maximized at the angle $\pi/2$.

The quantum antinormally ordered characteristic function is defined as

$$C_A(\lambda) = \int d^2 \alpha Q(\alpha)e^{\lambda^* \alpha - \lambda \alpha^*}.$$  \hspace{1cm} (107)

Substituting the expression of $Q(\alpha)$ into this formula yields $C_A(\lambda)$ to be

$$C_A(\lambda) = \frac{K^2}{2} e^{\beta^* \lambda - \beta \lambda^* - |\lambda|^2} \left[2 + |\beta|^2 + \beta^* \lambda - \beta \lambda^* - 2|\lambda|^2 - (\beta^* \lambda - \beta \lambda^* + |\beta|^2) \cos(2\delta \theta) + i(\beta^* \lambda + \beta \lambda^*) \sin(2\delta \theta) \right].$$  \hspace{1cm} (108)

Using this equation allows us to find the Wigner characteristic function, where it is defined as

$$C_W(\lambda) = C_A(\lambda)e^{\frac{1}{2}|\lambda|^2}.$$  \hspace{1cm} (109)
This characteristic function enables us to find the Wigner function $W(\alpha)$. The Wigner function is a quasi-probablity distribution and gives a clear sign for whether a state has nonclassical properties or does not. If there are some negative regions in the Wigner function, then the state is a nonclassical state.

The Wigner function can be defined as

$$W(\alpha) = \frac{1}{\pi^2} \int d^2 \lambda C_W(\lambda)e^{\lambda^* \alpha - \lambda \alpha^*}. \quad (110)$$

Using this definition, we find the Wigner function as

$$W(\alpha) = -\frac{K^2}{\pi}e^{-2|\alpha - \beta|^2}\left[2 - 8|\alpha|^2 - |\beta|^2(5 + 3 \cos(2\delta \theta)) + (\alpha^* \beta + \alpha \beta^*)(6 + 2 \cos(2\delta \theta)) + 2i(\alpha^* \beta - \alpha \beta^*) \sin(2\delta \theta)\right]. \quad (111)$$

When $\delta \theta$ equals zero, this function reduces to

$$W(\alpha) = \frac{2}{\pi}(4|\alpha - \beta|^2 - 1)e^{-2|\alpha - \beta|^2}. \quad (112)$$

The Wigner function is illustrated in Fig.(6). From Eq. (111) and Fig.(6), we can say that the EC state is always a nonclassical state. In other words, the Wigner function always has some negative regions. Also, the maximum negative value of $W(\alpha)$ occurs when $\delta \theta$ equals 0.

Final remarks about the nonclassical properties of the EC state. We found that the extreme nonclassical behaviour or higher negativity of the Wigner function is...
occurs when $\delta \theta$ equals 0. Also, the squeezing properties and $Q(\alpha)$ function maximize the nonclassical behaviour of the EC state when $\delta \theta$ equals 0. That angle is that same angle which maximize the emptiness in Eq.(60). This means the emptiness is a good way to represent the nonclassical behaviour of EC state. Also, it means the derivation state in Eq.(57) is by itself a nonclassical state.

5 Conclusion

The empty states are a new class of states, and we believe that these states are promising. We think that because of their unexpected behaviour starting from the initial thought of them to give nothing to their nonclassical behaviour as for the EC state. These unexpected properties of the empty states let us believe that these states will find their way in the applications of physics.

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