BLOWING-UP SOLUTIONS OF A TIME-SPACE FRACTIONAL SEMI-LINEAR EQUATION WITH A STRUCTURAL DAMPING AND A NONLOCAL IN TIME NONLINEARITY

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Abstract

In this paper, we investigate the semilinear equation with a time-space fractional structural damping and a nonlocal in time nonlinearity

\[ D_0^{1+\alpha} u + (-\Delta)\alpha u + (-\Delta)^{\beta} D_0^{2\beta} u = I_0^{1-\gamma} |u|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \]

where \( p > 1, \alpha, \gamma \in (0, 1), \delta, \sigma \in (0, 1). D_0^{1+\alpha} \) is the Caputo fractional derivative and \( I_0^{1-\gamma} \) is the Riemann-Liouville fractional integral of order \( 1-\gamma \). We prove the non-existence of global solutions if

\[ 1 < p \leq \frac{2(2 + \alpha_1 - \gamma)}{(\alpha_1 - \gamma)} + 1, \]

for any space dimension \( N \geq 1 \). Then, we extend the result to the system

\[ D_0^{1+\alpha_1} u + (-\Delta)^{\alpha_1} u + (-\Delta)^{\beta_1} D_0^{2\beta_1} u = I_0^{1-\gamma_1} |v|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \]
\[ D_0^{1+\beta_2} v + (-\Delta)^{\alpha_2} v + (-\Delta)^{\beta_2} D_0^{2\beta_2} v = I_0^{1-\gamma_2} |u|^q, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \]

where \( p, q > 1, 0 < \delta_i, \alpha_i < 1 \) and \( \gamma_2 \in (0, 1) \). Also, we present the necessary conditions for the existence of local or global solutions.

Keywords and phrase: Cauchy problem, time-space fractional derivatives, structural damping, nonexistence

1 Introduction

We consider the following Cauchy problem

\[ D_0^{1+\alpha} u + (-\Delta)^{\alpha} u + (-\Delta)^{\beta} D_0^{2\beta} u = I_0^{1-\gamma} |u|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \quad (1.1) \]

with the initial conditions

\[ u(0, x) = u_0, \quad u_1(0, x) = u_1(x), \quad x \in \mathbb{R}^N, \quad (1.2) \]

where \( u = u(t, x), p > 1, 0 < \gamma < \alpha_2 < \alpha_1 < 1, 0 < \sigma < \delta < 1 \) and \( D_0^{\alpha} u = I_0^{1-\alpha} u_t, \) \( I_0^{1-\alpha} \) is the Riemann-Liouville fractional integral of order \( 1-\alpha \) which is defined for \( u \in C(0, t) \) as following

\[ I_0^{1-\alpha} u = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-\alpha} u(\tau) d\tau. \]

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The fractional Laplacian operator is defined as
\[ (-\Delta)^s u(t,x) = \frac{C(N,\delta)}{2} \int_{\mathbb{R}^N} \frac{2u(t,x) - u(t,x+y) - u(t,x-y)}{|y|^{N+2s}} dy, \]
where \( C(N,\delta) \) is a positive normalizing constant depending on \( N \) and \( \delta \). The term \( (-\Delta)^s \mathcal{D}_{0+}^{\alpha} u \) represents a generalized structural damping.

For the semilinear damped wave equation
\[ u_{tt} - \Delta u + u_t = |u(t)|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N. \]
(1.3)
Todorova and Yordanov [18] studied the global existence of mild solutions to (1.3), be it valid, if \( p > 1 + \frac{2}{N} \) with \( \|u_0\| \) and \( \|u_1\| \) sufficiently small. In addition, they proved that the mild solution cannot exist globally when \( 1 < p < 1 + \frac{2}{N} \) and \( \int u_i > 0, i = 0,1 \). Here, the critical exponent of the damped wave equation (1.3) is equal to the Fujita critical exponent for \( u_t - \Delta u = |u|^p \).

M. Kirane and N. Tatar [11] studied the particular case of (1.1) with \( \alpha_1 = \gamma = 1 \) and \( \delta = 0 \), they discussed the nonexistence of global solutions and established conditions for which the problem has no local weak solution.

D’Abbicco and Ebert [3] considered a more general case, they treated the following problem
\[ u_{tt} + (-\Delta)^p u + (-\Delta)^q u_t = |u(t)|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \]
where \( 2\delta < \alpha \). They proved that if \( 1 < p < 1 + \frac{2\alpha}{(N-2\delta)} \), then the solution blows-up in a finite time.

Motivated by the above results, the paper presents the results for the nonexistence of global solutions to problem (1.1) - (1.2). The analysis is based on the test function method.

The rest of the paper is organized as follows: In section 2, we recall some definitions about fractional calculus which will be used in the sequel. In Section 3, we study the absence of global weak solutions. In Section 4, we extend the previous results of Section 3 to a system of semilinear coupled equations. In Section 5, we establish the necessary conditions for local or global existence of problem (1.1) - (1.2).

2 Preliminary

In this section, we present some results and basic properties of fractional calculus.

For \( 0 < \alpha < 1 \) and \( T > 0 \), the Riemann-Liouville derivatives of order \( \alpha \) for a continuous function \( f \) are defined as
\[ D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau, \quad D_{0T}^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (\tau-t)^{-\alpha} f(\tau) d\tau. \]

For \( 0 < \alpha < 1 \) and \( f \in AC[0,T] \), the Caputo derivatives of fractional order \( \alpha \) are defined as
\[ D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau, \quad D_{0T}^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T (\tau-t)^{-\alpha} f'(\tau) d\tau. \]

Assume \( D_{0+}^\alpha f \in L^1(0,T) \), \( g \in C^1([0,T]) \) and \( g(T) = 0 \), then we have the following formula of integration by parts
\[ \int_0^T g(t) D_{0+}^\alpha f(t) dt = \int_0^T (f(t) - f(0)) D_{0+}^\alpha g(t) dt. \]

**Lemma 1.** 1. Suppose \( f \in C(0,\infty) \), \( p > q > 0 \), and \( D_{0+}^p f(t) \) exists, then for \( t > 0 \)
\[ D_{0+}^p \left( f_{0+}^{p-q} \right)(t) = D_{0+}^{p-q} f(t). \]
In particular, when \( p = n \) we have
\[
D_{\alpha T}^n (I_{\alpha T}^q f)(t) = (-1)^n D_{\alpha T}^{n-q} f(t).
\]

2. Let \( n - 1 \leq q < n \), then for every \( t > 0 \),
\[
\begin{align*}
I_{\alpha T}^{m-p} D_{\alpha T}^q f(t) &= D_{\alpha T}^{p+q-m} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{m-p-k} D_{\alpha T}^{q-k} f(T)}{\Gamma(q-k)\Gamma(m-k-p+1)}, \\
\end{align*}
\]
\[
\tag{2.2}
\]

**Proof.**

1. Let \( m-1 \leq p < m \) and \( n-1 \leq p-q < n \), we have
\[
D_{\alpha T}^p (I_{\alpha T}^q f)(t) = (-1)^p D_{\alpha T}^{m-n}(I_{\alpha T}^{m-p+q} f)(t) \\
= (-1)^p D_{\alpha T}^{m-p} f(t) \\
= D_{\alpha T}^{p-q} f(t).
\]

2. Using (2.1), we can write
\[
\begin{align*}
I_{\alpha T}^{m-p} D_{\alpha T}^q f(t) &= D_{\alpha T}^{p+q-m} [I_{\alpha T}^p D_{\alpha T}^q f(t)] \\
&= D_{\alpha T}^{p+q-m} \left\{ f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{n-k} D_{\alpha T}^{q-k} f(T)}{\Gamma(q-k+1)} \right\} \\
&= D_{\alpha T}^{p+q-m} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{m-p-k} D_{\alpha T}^{q-k} f(T)}{\Gamma(q-k)\Gamma(m-k-p+1)}
\end{align*}
\]
due to the following equality [19]
\[
I_{\alpha T}^q D_{\alpha T}^q f(t) = f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{q-k} D_{\alpha T}^{q-k} f(T)}{\Gamma(q-k+1)}.
\]

\(\square\)

**Lemma 2.** Let \( p \) and \( q \) be real numbers, if \( m-1 \leq p < m \) and \( n-1 \leq q < n \), then
\[
D_{\alpha T}^p D_{\alpha T}^q f(t) = D_{\alpha T}^{p+q} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{n-k} D_{\alpha T}^{q-k} f(T)}{\Gamma(q-k)\Gamma(m-k-p+1)}.
\]
\[
\tag{2.3}
\]

**Proof.** By the semigroup property of fractional integrals and (2.2), we can write
\[
\begin{align*}
D_{\alpha T}^p D_{\alpha T}^q f(t) &= D_{\alpha T}^m \left[ (-1)^m I_{\alpha T}^{m-p} D_{\alpha T}^q f(t) \right] \\
&= D_{\alpha T}^m \left[ D_{\alpha T}^{p+q-m} f(t) - \sum_{k=1}^{n} C_k (T-t)^{m-p-k} D_{\alpha T}^{q-k} f(T) \right] \\
&= D_{\alpha T}^{p+q} f(t) - \sum_{k=1}^{n} (-1)^{n-k} \frac{(T-t)^{p-k} D_{\alpha T}^{q-k} f(T)}{\Gamma(q-k)\Gamma(1-k-p)(m-k-p+1)}.
\end{align*}
\]
\(\square\)
For \( T > 0 \) and \( \eta \gg 1 \), if we set
\[
\varphi_1(t) = \begin{cases} 
(1 - \frac{t}{T})^\eta, & 0 < t \leq T, \\
0, & t > T,
\end{cases}
\]
then
\[
D_t^\theta \varphi_1(t) = \frac{(\eta + 1)}{\Gamma(\eta + 1 - \alpha)} T^{-\alpha} \left(1 - \frac{t}{T}\right)^{\eta-\alpha}.
\]

**Lemma 3 (8).** Let \( \varphi_1 \) as in (2.4), for \( \eta > \frac{p}{\alpha - \theta} - 1 \),
\[
\int_0^T D_t^\theta \varphi_1 = \frac{C_1}{\eta - \theta} T^{1-\theta},
\]
and
\[
\int_0^T \varphi_1^{p'/p} |D_t^\theta \varphi_1|^{p'} = CT^{1-p'\theta},
\]
where
\[
C = \frac{\eta^{p'}}{\eta + 1 - p'\theta} \left[ \frac{\Gamma(\eta - \theta)}{\Gamma(\eta + 1 - 2\theta)} \right]^{p'/p}.
\]

**Lemma 4 (11).** Let \( \psi \in C^2(\mathbb{R}^N) \) be a real function decreasing in \(|\xi| > 1 \). Assume that

i. \( \psi > 0 \) is compactly supported,

ii. \( \psi \leq |\xi|^{-\alpha} \) for \( 0 < \alpha < N + 2p\sigma \) and \(|\xi| \) large enough.

Then there exist \( C_1 > 0 \) and \( C_2 > 0 \) such that
\[
|(-\Delta)^\sigma \psi| \leq C_1 |\xi|^{-(N+2\sigma)} \quad \text{and} \quad \left| \frac{(-\Delta)^\sigma \psi}{\psi^{1/p}} \right|^{\frac{1}{p-1}} \leq C_2.
\]

## 3 Blow-up of solutions

In this section, we first give the definition of weak solution of (1.1) - (1.2). After we prove the blow-up of solutions.

**Definition 1.** Let \( 0 < \alpha_i < 1, \ p > 1 \). For \( u_0, u_1 \in L^p_{\text{loc}}(\mathbb{R}^N) \), the function \( u \in L^p_{\text{loc}}(Q_T) \) is a weak solution of problem (1.1) - (1.2) if
\[
\int_{Q_T} u D_t^{\alpha_1} \phi + \int_{Q_T} u(-\Delta)u \phi + \int_{Q_T} u(-\Delta)^\gamma \phi = \int_{Q_T} u_0 D_t^{\alpha_1} \phi(0) + \int_{Q_T} u_1 D_t^{\alpha_2} \phi + \int_{Q_T} u_0(-\Delta)u \phi + \int_{Q_T} u_1(-\Delta)^\gamma \phi,
\]
for \( Q_T := [0, T] \times \mathbb{R}^N, \ \phi > 0, \phi \in C^0_c((0, T] \times \mathbb{R}^N) \) with \( \text{supp} \phi \subset \mathbb{R}^N \) and \( \phi(T, \cdot) = 0 \).

**Theorem 1.** Assume that \( u_0 = 0, u_1 \in L^1(\mathbb{R}^N) \) and \( u_1 \geq 0 \). If
\[
1 < p \leq p_* := \frac{2(2 + \alpha_1 - \gamma)}{(\frac{2\alpha_1 + 1}{\alpha} N + 2\gamma - 2\alpha_1 - 2)_+ + 1},
\]
then any solution to (1.1) - (1.2) blows up in a finite time.
Proof. We assume the contrary. Let
\[ \phi(t, x) = D_\eta^{1-\gamma} \varphi(t, x) = D_\eta^{1-\gamma} (\varphi_1(t)\varphi_2(x)), \]
where \( \varphi_1(t) \) and \( \varphi_2(x) = \psi(T^{-0}/2\chi) \) are defined as in (2.4) and (2.5).
According to (3.1) and Lemma 2 we have
\[
\int_{Q_T} |u|^p \varphi + \int_{Q_T} u_1 \varphi_2 D_\eta^{1+\alpha_1-\gamma} \varphi_1 = \int_{Q_T} u \varphi_2 D_\eta^{2+\alpha_1-\gamma} \varphi_1 + \int_{Q_T} (\varphi_1 D_\eta^2 + \varphi_2 D_\eta^{1+\alpha_2-\gamma}) \varphi_1 + \int_{Q_T} u(-\Delta)^\sigma \varphi_2 D_\eta^{1-\gamma} \varphi_1. \tag{3.2}
\]
Therefore, by Lemma 3 we get
\[
\int_{Q_T} |u|^p \varphi + CT^{-(\alpha_1-\gamma)} \int_{\mathbb{R}^N} u_1(x) \varphi_2(x) dx \leq \int_{Q_T} |u| \left( \varphi_2 D_\eta^{2+\alpha_1-\gamma} \varphi_1 + D_\eta^{1+\alpha_2-\gamma} \varphi_1 \right) \left( -\Delta \varphi_2 + D_\eta^{1-\gamma} \varphi_1 \right) \right).
\]
Using the Young inequality with parameters \( p \) and \( p' = \frac{p}{p-1} \), we obtain
\[
\int_{Q_T} |u|^p \varphi + CT^{-(\alpha_1-\gamma)} \int_{\mathbb{R}^N} u_1(x) \varphi_2(x) dx \leq \frac{1}{3p} \int_{Q_T} |u|^p \varphi + \frac{3p'-1}{p'} \int_{\mathbb{R}^N} \varphi_2 \left( \varphi_1 D_\eta^{2+\alpha_1-\gamma} \varphi_1 \right)^{p'} \left( \varphi_1 D_\eta^{1+\alpha_2-\gamma} \varphi_1 \right)^{p'} + \frac{1}{3p} \int_{Q_T} |u|^p \varphi + \frac{3p'-1}{p'} \int_{\mathbb{R}^N} \varphi_2 \left( \varphi_1 D_\eta^{1+\alpha_2-\gamma} \varphi_1 \right)^{p'} \left( \varphi_1 D_\eta^{1-\gamma} \varphi_1 \right)^{p'}, \tag{3.3}
\]
where \( \Sigma = [0, T] \times supp \varphi_2 \). Using Lemma 4 it holds
\[
\left| \frac{\varphi_2(x)}{\varphi_1^{1/p}} \right|^{p'} \left| \frac{-\Delta \varphi_1}{\varphi_1^{1/p}} \right|^{p'} \leq C_2.
\]
Passing to the scaled variables
\[
\tau = \frac{t}{T}, \quad \xi = \frac{x}{T^{\alpha_2/2}}, \quad \theta = \frac{\alpha_1 + 1}{\sigma} \quad \text{and} \quad T \gg 1.
\]
Hence
\[
\left( 1 - \frac{1}{p} \right) \int_{Q_T} |u|^p \varphi + CT^{-(\alpha_1-\gamma)} \int_{\mathbb{R}^N} u_1(x) \varphi_2(x) dx \leq C \left( T^{-p'(2+\alpha_1-\gamma)+\frac{\alpha_2}{2}+1} + T^{-p'(\theta_2+1+\alpha_2-\gamma)+\frac{\alpha_2}{2}+1} + T^{-p'(\theta_2+1-\gamma)+\frac{\alpha_2}{2}+1} \right)
\leq C \left( T^{-p'(2+\alpha_1-\gamma)+\frac{\alpha_2}{2}+1} + 2T^{-p'(\theta_2+1-\gamma)+\frac{\alpha_2}{2}+1} \right)
\leq CT^{-p'(2+\alpha_1-\gamma)+\frac{\alpha_2}{2}+1}. \tag{3.4}
\]
Under the condition \( u_1(x) \geq 0 \), we obtain
\[
\frac{1}{p'} \int_{Q_T} |u|^p \varphi \leq CT^{-p'(2+\alpha_1-\gamma)+\frac{\alpha_2}{2}+1}. \tag{3.5}
\]
Since 

\[ p \leq p^* , \]

we have to distinguish two cases.

In case \( p < p^* \): if a solution of \((1.1) \rightarrow (1.2)\) exists globally, then taking \( T \to +\infty \), we get

\[ \lim_{T \to +\infty} \int_{Q_T} |u|^p \varphi < 0. \]

Contradiction the fact that \( \int_0^\infty \int_{\mathbb{R}^N} |u|^p \varphi > 0 \).

In case \( p = p^* \): we repeat the same calculation as above by taking \( \varphi_2(x) = \psi\left(\frac{|x|}{B_{\epsilon T}}\right) \), where \( 1 < B < T \) and when \( T \) goes to infinity we don’t have \( B \) goes to infinity at the same time, employing the Hölder’s inequality instead of Young’s, we obtain

\[ \int_{\Sigma_B} |u|\varphi_2 D^{2+\alpha_1 - \gamma}_t \varphi_1 \leq B^{p - \frac{\alpha}{\gamma}} \left( \int_{\Sigma_B} |u|^p \varphi \right)^{1/p} . \]  

(3.6)

Using Lemma 4 and the Hölder inequality, we have

\[ \int_{\Sigma_B} |u|-\Delta \varphi_2 D^{1+\alpha_2 - \gamma}_t \varphi_1 \leq CT^{-\theta(\delta - \sigma) + \alpha_2} B^{\frac{\delta - N}{\rho}} \left( \int_{\Sigma_B} |u|^p \varphi \right)^{1/p} , \]  

(3.7)

and

\[ \int_{\Sigma_B} |u|-\Delta \varphi_2 D^{1+\gamma}_t \varphi_1 \leq B^{2\rho - \frac{\alpha}{\rho}} \left( \int_{\Sigma_B} |u|^p \varphi \right)^{1/p} , \]  

(3.8)

where \( \Sigma_B = [0, T] \times \text{supp} \varphi_2 \). Combining (3.6), (3.7) with (3.8), we get

\[ \int_{Q_T} |u|^p \varphi + CT^{-\alpha_1 - \gamma} \int_{\mathbb{R}^N} u_1(x) \varphi_2(x) dx \leq CB^{\rho - \frac{\alpha}{\rho}} \left( \int_{\Sigma_B} |u|^p \varphi \right)^{1/p} + CT^{-\theta(\delta - \sigma) + \alpha_2} B^{2\delta - \frac{N}{\rho}} \left( \int_{\Sigma_B} |u|^p \varphi \right)^{1/p} + C B^{2\rho - \frac{\alpha}{\rho}} \left( \int_{\Sigma_B} |u|^p \varphi \right)^{1/p} . \]

Thus, passing the limit as \( T \to +\infty \) and then where \( B \to +\infty \), with \( p > \frac{N}{N - 2\sigma} \) we have

\[ \lim_{T \to +\infty} \int_{Q_T} |u|^p \varphi < 0. \]

This leads to a contradiction.

\[ \Box \]

### 4 Case of a 2 × 2 system

This part is concerned with the study of the following system

\[
\begin{cases}
D^{1+\alpha_1}_0 u + (-\Delta)^\gamma u + (-\Delta)^\beta D^{\alpha_2}_0 u = f^{1+\gamma}_0 |v|^p, & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\
D^{1+\beta_1}_0 v + (-\Delta)^\gamma v + (-\Delta)^\beta D^{\alpha_2}_0 v = f^{1+\gamma}_0 |u|^p, & (t, x) \in (0, \infty) \times \mathbb{R}^N,
\end{cases}
\]

(4.1)

supplemented with the initial conditions

\[
\begin{cases}
u(0, x) = u_0(x), u_0(0, x) = u_1(x), & x \in \mathbb{R}^N, \\
v(0, x) = v_0(x), v_0(0, x) = v_1(x), & x \in \mathbb{R}^N,
\end{cases}
\]

(4.2)

where \( p > 1, q > 1, 0 < \gamma_1 < \alpha_2 < \alpha_1 < 1, 0 < \gamma_2 < \beta_2 < \beta_1 < 1 \) and \( 0 < \sigma_1 < \delta_1 < 1 \).
Theorem 2. Assume that $u_0, v_0 = 0$, whereas $u_1, v_1 \in L^1(\mathbb{R}^N)$ and $u_1, v_1 \geq 0$. If

$$N < \max \left\{ \left( \frac{2 + \beta_1 - \gamma_2}{q} + q(1 + \alpha_1 - \gamma_1) \left( \frac{2 + \alpha_1 - \gamma_1}{q} + \frac{1}{q} + p(1 + \beta_1 - \gamma_2) \right) \right) \right\},$$

then any solution $(u, v)$ to $(4.1) - (4.2)$ blows-up in a finite time.

Proof. The proof proceeds by contradiction. Let

$$\phi(t, x) = D^\frac{1}{p} \bar{\phi}(t, x) = D^\frac{1}{p} \phi_i(t) \phi_2(x), \quad i = 1, 2,$$

where $\phi_1$ is defined as in (2.4) however with condition $\eta > \left\{ \frac{p}{\gamma - (2 + \alpha_1 - \gamma_1)}, \frac{p}{\gamma - (2 + \beta_1 - \gamma_2)} \right\}$ and $\phi_2(x) = \psi \left( \frac{|t|}{\gamma - p} \right)$ is defined above.

The weak solutions to system $(4.1) - (4.2)$ reads as

$$\int_{Q_T} |v|^{\frac{p}{q}} + \int_{Q_T} u \varphi_2 D^{1 + \alpha_1 - \gamma_1} \varphi_1$$

$$= \int_{Q_T} u \varphi_2 D^{2 + \alpha_1 - \gamma_1} \varphi_1 + \int_{Q_T} u (-\Delta)^{\frac{1}{2}} \varphi_2 D^{1 + \alpha_2 - \gamma_1} \varphi_1 + \int_{Q_T} u (-\Delta)^{\frac{1}{2}} \varphi_2 D^{1 + \gamma_1} \varphi_1, \quad (4.3)$$

and

$$\int_{Q_T} |v|^{\frac{p}{q}} + \int_{Q_T} v \varphi_2 D^{1 + \beta_1 - \gamma_2} \varphi_1$$

$$= \int_{Q_T} v \varphi_2 D^{2 + \beta_1 - \gamma_2} \varphi_1 + \int_{Q_T} v (-\Delta)^{\frac{1}{2}} \varphi_2 D^{1 + \beta_2 - \gamma_2} \varphi_1 + \int_{Q_T} v (-\Delta)^{\frac{1}{2}} \varphi_2 D^{1 + \gamma_2} \varphi_1. \quad (4.4)$$

Using the Hölder inequality, we obtain

$$\int_{Q_T} u \varphi_2 D^{2 + \alpha_1 - \gamma_1} \varphi_1 \leq \left( \int_{Q_T} |u|^{q} \bar{\phi} \right)^{1/q} \left( \int_{Q_T} \varphi_2 \varphi_1^{\frac{q}{q'}} |D^{2 + \alpha_1 - \gamma_1} \varphi_1|^{q'} \right)^{1/q'}, \quad (4.5)$$

$$\int_{Q_T} u (-\Delta)^{\frac{1}{2}} \varphi_2 D^{1 + \alpha_2 - \gamma_1} \varphi_1$$

$$\leq \left( \int_{Q_T} |u|^{q} \bar{\phi} \right)^{1/q} \left( \int_{Q_T} \varphi_2^{\frac{q}{q'}} |(-\Delta)^{\frac{1}{2}} \varphi_2|^{q'} \varphi_1^{\frac{q}{q'}} |D^{1 + \alpha_2 - \gamma_1} \varphi_1|^{q'} \right)^{1/q'}, \quad (4.6)$$

and

$$\int_{Q_T} u (-\Delta)^{\frac{1}{2}} \varphi_2 D^{1 + \gamma_1} \varphi_1$$

$$\leq \left( \int_{Q_T} |u|^{q} \bar{\phi} \right)^{1/q} \left( \int_{Q_T} \varphi_2^{\frac{q}{q'}} |(-\Delta)^{\frac{1}{2}} \varphi_2|^{q'} \varphi_1^{\frac{q}{q'}} |D^{1 + \gamma_1} \varphi_1|^{q'} \right)^{1/q'}, \quad (4.7)$$

Taking into account the above relation $(4.3), (4.5), (4.6)$ and $(4.7)$, we find

$$\int_{Q_T} |v|^{\frac{p}{q}} + \int_{Q_T} u \varphi_2 \leq \left( \int_{Q_T} |u|^{q} \bar{\phi} \right)^{1/q} A, \quad (4.8)$$
we have set
\[
\mathcal{A} = \left( \int_{Q_T} \varphi_2 \varphi_1 |D_{\eta T}^{2+\gamma_1-\gamma_2} \varphi_1|^{\rho} \right)^{1/q'} + \left( \int_{Q_T} \varphi_2^\prime \varphi_1^\prime |(-\Delta)^{\eta_1} \varphi_2^\prime \varphi_1^\prime |D_{\eta T}^{1+\gamma_2-\gamma_1} \varphi_1|^{\rho} \right)^{1/q'}
\]
\[
+ \left( \int_{Q_T} \varphi_2^\prime \varphi_1^\prime |(-\Delta)^{\eta_1} \varphi_2^\prime \varphi_1^\prime |D_{\eta T}^{1+\gamma_2-\gamma_1} \varphi_1|^{\rho} \right)^{1/q'}.
\]
Similarly, we get
\[
\int_{Q_T} |u|^q \tilde{\varphi} + CT^{-(\beta_1-\gamma_2)} \int_{\mathbb{R}^N} v_1 \varphi_2 \leq \left( \int_{Q_T} |v|^p \tilde{\varphi} \right)^{1/p} \mathcal{B},
\] (4.9)
with
\[
\mathcal{B} = \left( \int_{Q_T} \varphi_2 \varphi_1^{-\rho} |D_{\eta T}^{2+\gamma_1-\gamma_2} \varphi_1|^{\rho} \right)^{1/p'} + \left( \int_{Q_T} \varphi_2^\prime \varphi_1^\prime^{-\rho} |(-\Delta)^{\eta_1} \varphi_2^\prime \varphi_1^\prime |D_{\eta T}^{1+\gamma_2-\gamma_1} \varphi_1|^{\rho} \right)^{1/p'}
\]
\[
+ \left( \int_{Q_T} \varphi_2^\prime \varphi_1^\prime^{-\rho} |(-\Delta)^{\eta_1} \varphi_2^\prime \varphi_1^\prime |D_{\eta T}^{1+\gamma_2-\gamma_1} \varphi_1|^{\rho} \right)^{1/p'}.
\]
Therefore, as \(u_1, v_1 \geq 0\), we obtain
\[
\int_{Q_T} |v|^p \tilde{\varphi} \leq \left( \int_{Q_T} |u|^q \tilde{\varphi} \right)^{1/q} \mathcal{A},
\] (4.10)
and
\[
\int_{Q_T} |u|^q \tilde{\varphi} \leq \left( \int_{Q_T} |v|^p \tilde{\varphi} \right)^{1/p} \mathcal{B}.
\] (4.11)
Now, combining (4.10) and (4.11), we write
\[
\left\{ \begin{aligned}
\left( \int_{Q_T} |v|^p \tilde{\varphi} \right)^{1/q} &\leq \mathcal{B}^{1/p} \mathcal{A}, \\
\left( \int_{Q_T} |u|^q \tilde{\varphi} \right)^{1/p} &\leq \mathcal{A}^{1/q} \mathcal{B}.
\end{aligned} \right.
\] (4.12)
Using Lemma\(^3\) Lemma\(^3\) and making the change of variables
\[
x = \xi T^{\theta_1} \quad \text{with} \quad \theta_1 = \frac{\alpha_1 + 1}{\sigma_1} \quad \text{in} \quad \mathcal{A},
\]
\[
x = \xi T^{\theta_2} \quad \text{with} \quad \theta_2 = \frac{\beta_1 + 1}{\sigma_2} \quad \text{in} \quad \mathcal{B}.
\]
We obtain the estimates
\[
\left\{ \begin{aligned}
\left( \int_{Q_T} |v|^p \tilde{\varphi} \right)^{1/q} &\leq T^{\theta_1}, \\
\left( \int_{Q_T} |u|^q \tilde{\varphi} \right)^{1/p} &\leq T^{\theta_2}.
\end{aligned} \right.
\] (4.13)
Choose $\phi$ where

$$l_1 = \left( -2 + \beta_1 - \gamma_2 + \frac{1}{p'} \left( \frac{\theta_1 N}{2} - 1 \right) \right) \frac{1}{q} - \left( 2 + \alpha_1 - \gamma_1 \right) + \frac{1}{q'} \left( \frac{\theta_1 N}{2} - 1 \right),$$

and

$$l_2 = \left( -2 + \alpha_1 - \gamma_1 + \frac{1}{q'} \left( \frac{\theta_1 N}{2} - 1 \right) \right) \frac{1}{p} - \left( 2 + \beta_1 - \gamma_2 \right) + \frac{1}{p'} \left( \frac{\theta_2 N}{2} - 1 \right).$$

Hence, by taking the limit as $T \to \infty$ in (4.13), we obtain

$$\begin{cases}
\int_0^\infty \int_{\mathbb{R}^N} |\varphi|^{p'} \varphi < 0, \\
\int_0^\infty \int_{\mathbb{R}^N} |u|^{q'} \varphi < 0,
\end{cases}$$

which is a contradiction. Then $(u, v)$ cannot be a global solution. \qed

5 Necessary conditions for local and global existence

In this part, we establish the necessary conditions for the local and global existence of solutions to the problem (1.1) - (1.2).

**Theorem 3.** Let $u_0 = 0$ and let $u$ be a local solution to problem (1.1) - (1.2), then exists positive constant $C$, such that

$$\inf_{|x| \to \infty} u_1(x) \leq CT^{-\frac{\rho}{p'}(2+\alpha_1-\gamma)+(1+\alpha_1-\gamma)}.$$

**Proof.** Using the Young inequality in the right hand side of (3.2), we obtain

$$\int_{Q_T} u_1 \varphi_2 D_{\tilde{\sigma}}^{1+\alpha_1-\gamma} \varphi_1 \leq \frac{3^{p'-1}}{p'} \int_{Q_T} \varphi_1^{-p'/p} \varphi_2^{-p'/p} \left( \varphi_2' D_{\tilde{\sigma}}^{2+\alpha_1-\gamma} \varphi_1 |p' | + |(-\Delta)^{\tilde{\sigma}} \varphi_2 D_{\tilde{\sigma}}^{1+\alpha_1-\gamma} \varphi_1 |p' | + |(-\Delta)^{\tilde{\sigma}} \varphi_2 D_{\tilde{\sigma}}^{1+\alpha_1-\gamma} \varphi_1 |p' | \right).$$

Let $\psi$ the first eigenfunction of $(-\Delta)^{\tilde{\sigma}}$ with $\lambda_0$ the first eigenvalue on $\Omega$ (17)

$$\begin{cases}
(-\Delta)^{\tilde{\sigma}} \psi(x) = \lambda_0 \psi(x), & x \in \Omega, \\
\psi(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \\
\psi(x) \geq 0, & x \in \mathbb{R}^N,
\end{cases}$$

(5.1)

where $\Omega := \{ x \in \mathbb{R}^N : 1 < |x| < 2 \}$ and $\theta = (\alpha, \sigma)$.

Choose $\varphi_2(x) = \psi(x/R)$ and $\xi = x/R$, so that $(-\Delta)^{\tilde{\sigma}} \varphi_2(x) = (-\Delta)^{\tilde{\sigma}} \psi(x/R) = R^{-2\theta}(-\Delta)^{\tilde{\sigma}} \psi(\xi)$, we obtain

$$T^{-1-(1+\alpha_1-\gamma)} \int_{\Omega} u_1(R\xi) \psi(\xi) \leq C \left( T^{1-p'(2+\alpha_1-\gamma)} + \lambda_0 R^{-2p'\tilde{\sigma}} T^{1-p'(1+\alpha_2-\gamma)} + \lambda_0 R^{-2p'\tilde{\sigma}} T^{1-p'(1-\gamma)} \right) \int_{\Omega} \psi(\xi).$$

Thus

$$T^{-1-(1+\alpha_1-\gamma)} \inf_{|\xi| > 1} u_1(R\xi) \int_{\Omega} \psi(\xi) \leq C \left( T^{-p'(2+\alpha_1-\gamma)} + R^{-2p'\tilde{\sigma}} T^{-p'(1+\alpha_2-\gamma)} + R^{-2p'\tilde{\sigma}} T^{-p'(1-\gamma)} \right) \int_{\Omega} \psi(\xi).$$

(5.2)
By passing to the limit in (5.2), as \( R \to \infty \), we get
\[
T^{-(1+\alpha_1-\gamma)} \inf_{|x| \to \infty} u_1(x) \leq CT^{-p'(2+\alpha_1-\gamma)}. \tag{5.3}
\]

\[\square\]

**Corollary 1.** Assume that \( \lim_{|x| \to \infty} \inf u_1(x) = +\infty \), then problem \((1.1) - (1.2)\) does not admit any local solution.

**Corollary 2.** Suppose that problem \((1.1) - (1.2)\) has a global solution, then \( \lim_{|x| \to \infty} \inf u_1(x) = 0. \)

**Proof.** The proof proceeds by contradiction. Suppose
\[
C = \lim_{|x| \to \infty} \inf u_1 > 0.
\]
Using \((5.3)\) for \( T > 1 \), we get
\[
T^{-(1+\alpha_1-\gamma)+p'(2+\alpha_1-\gamma)} \leq C.
\]
We get a contradiction when \( T \to \infty \).

**Theorem 4.** Suppose \((1.1) - (1.2)\) admit a global weak solution, then exists positive constant \( C \) such that
\[
\lim_{|x| \to \infty} \left[ \inf_{|x| \to \infty} u_1(x) \right] \leq C.
\]

**Proof.** Using the relation \((5.2)\), we have
\[
\inf_{|\xi|>1} u_1(\xi) \int_T^\Omega \psi(\xi) \leq C \left[ T^{(1+\alpha_1-\gamma)-p'(2+\alpha_1-\gamma)} + R^{-2\beta p'-\gamma} T^{(1+\alpha_1-\gamma)-p'(1+\alpha_2-\gamma)} \right] \int_T^\Omega \psi(\xi),
\]
which implies that
\[
\inf_{|x|>R} u_1(x) \int_T^\Omega \psi(\xi) \leq C T^{(1+\alpha_1-\gamma)-p'(2+\alpha_1-\gamma)} + T^{(1+\alpha_1-\gamma)-p'(1+\alpha_2-\gamma)} R^{-2\beta p'} \int_T^\Omega \psi(\xi).
\]
The right-hand side have a minimum at
\[
T^3 = \left[ \frac{p'(2+\alpha_1-\gamma) - (1+\alpha_1-\gamma)}{(1+\alpha_1-\gamma) - p'(1+\alpha_2-\gamma)} \right]^{\frac{1}{\gamma'}} R^{\frac{2\gamma}{1+\alpha_1-\gamma}},
\]
whereupon
\[
\inf_{|\xi|>R} \left[ u_1(x) \right] \left[ \frac{2\gamma}{1+\alpha_1-\gamma} \right] \left[ \frac{\alpha_1-\gamma}{(1+\alpha_1-\gamma) - p'(1+\alpha_2-\gamma)} \right] \int_T^\Omega \psi(\xi)
\leq C 2^{\frac{2\gamma}{1+\alpha_1-\gamma}} (\alpha_1-\gamma) \left( 1+\alpha_1-\gamma \right) \int_T^\Omega \psi(\xi).
\]
Finally, dividing by
\[
\int_T^\Omega \psi(\xi) \leq C
\]
we obtain
\[
\lim_{|x| \to \infty} \left[ \inf_{|x| \to \infty} u_1(x) \right] \leq C.
\]
\[\square\]
References

[1] M. Bonforte, J.L. Vázquez, Quantitative local and global a priori estimates for fractional nonlinear diffusion equations, Advances in Mathematics, (2014), 242–284.

[2] M. D’Abbicco, The influence of a nonlinear memory on the damped wave equation, Nonlinear Analysis: Theory, Methods and Applications, (2014), 130–145.

[3] M. D’Abbicco, M.R. Ebert, T. Picon, Global existence of small data solutions to the semilinear fractional wave equation, New Trends in Analysis and Interdisciplinary Applications, (2017), 465–471.

[4] M. D’Abbicco, S. Lucente, M. Reissig, Semi-linear wave equations with effective damping, Chinese Annals of Mathematics, series B, (2013), 345–380.

[5] M.F. de Almeida, L.C. Ferreira, Self-similarity, symmetries and asymptotic behavior in Morrey spaces for a fractional wave equation, Differential and Integral Equations, (2012), 957–976.

[6] P.T. Duong, Some results on the global solvability for structurally damped models with special nonlinearity, Ukrainian Mathematical Journal, (2019), 1395–1418.

[7] Y. Fujita, Integrodifferential equation which interpolates the heat equation and the wave equation, Osaka Journal of Mathematics, (1990), 309–321.

[8] K.M. Furati, M. Kirane, Necessary conditions for the existence of global solutions to systems of fractional differential equations, Fractional Calculus and Applied Analysis, (2008), 281–298.

[9] K.M. Furati, N-e. Tatar, An existence result for a nonlocal fractional differential problem, Journal of Fractional Calculus, (2004), 43–51.

[10] M. Kirane, Y. Laskri, Nonexistence of global solutions to a hyperbolic equation with a space–time fractional damping, Applied Mathematics and Computation, (2005), 1304–1310.

[11] M. Kirane, N-e. Tatar, Nonexistence of solutions to a hyperbolic equation with a time fractional damping, Zeitschrift für Analysis und ihre Anwendungen, (2006), 131–142.

[12] A.K. Mezadek, M. Reissig, Semi-linear fractional σ-evolution equations with mass or power non-linearity, Nonlinear Differential Equations and Applications NoDEA, (2018), 42.

[13] V. Pata, M. Squassina, On the strongly damped wave equation, Communications in Mathematical Physics, (2005), 511–533.

[14] V. Pata, S. Zelik, Smooth attractors for strongly damped wave equations, Nonlinearity, (2006), 1495.

[15] D.T. Pham, M.K. Mezadek, M. Reissig, Global existence for semi-linear structurally damped σ-evolution models, Journal of Mathematical Analysis and Applications, (2015), 569–596.

[16] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives: Theory and Applications, (1987).

[17] R. Servadei, E. Valdinoci, On the spectrum of two different fractional operators, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, (2014), 831–855.

[18] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, Journal of Differential Equations, (2001), 464–489.

[19] Z. Yong, W. Jinrong, Z. Lu, Basic theory of fractional differential equations, (2016).