Generators of Nonassociative Simple Moufang Loops over Finite Prime Fields

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We present an elementary proof that the nonassociative simple Moufang loops over finite prime fields are generated by three elements. In the last section, we conclude that integral Cayley numbers of unit norm are generated multiplicatively by three elements.

Key Words: simple Moufang loops, economical generators, integral Cayley numbers, octonions.

1. SIMPLE MOUFTANG LOOPS

The first class of nonassociative simple Moufang loops was discovered by L. Paige in 1956 [9], who investigated Zorn’s and Albert’s construction of simple alternative rings. M. Liebeck proved in 1987 [7] that there are no other finite nonassociative simple Moufang loops. We can briefly describe the class as follows:

For every finite field $\mathbb{F}$, there is exactly one simple Moufang loop. Recall Zorn’s multiplication

$$\begin{pmatrix} a & c \\ \beta & d \end{pmatrix} \begin{pmatrix} c & \alpha \\ \delta & d \end{pmatrix} = \begin{pmatrix} ac + \alpha \cdot \delta & a\gamma + \alpha d - \beta \times \delta \\ \beta c + b\delta + \alpha \times \gamma & \beta \cdot \gamma + bd \end{pmatrix},$$

where $a, b, c, d \in \mathbb{F}$, $\alpha, \beta, \gamma, \delta \in \mathbb{F}^3$, and where $\alpha \cdot \beta$ (resp. $\alpha \times \beta$) denotes the dot product (resp. vector product) of $\alpha$ and $\beta$. Probably the easiest way to think of this multiplication is to consider the usual matrix multiplication with the added antidiagonal matrix

$$\begin{pmatrix} 0 & -\beta \times \delta \\ \alpha \times \gamma & 0 \end{pmatrix}.$$
We define the determinant of

\[ M = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \]

by \( \det M = ab - \alpha \cdot \beta \). Then \( \mathcal{L} = \{ M; \det M \neq 0 \} \) turns out to be a nonassociative Moufang loop, and so does \( \mathcal{M} = \{ M; \det M = 1 \} \). One can show that \( Z(\mathcal{L}) \), the center of \( \mathcal{L} \), consists of all elements

\[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \]

for \( 0 \neq a \in F \). Thus \( Z(\mathcal{M}) \) is at most a two-element subloop of \( \mathcal{M} \). More precisely, \( Z(\mathcal{M}) \) is trivial if and only if \( \text{char} F \), the characteristic of \( F \), equals 2 (cf. Lemma 3.2 [9]). Finally, \( \mathcal{M}/Z(\mathcal{M}) \) was found to be simple (and nonassociative) in [9].

Obviously, a finite simple Moufang loop is either associative (whence a finite simple group), or nonassociative—an element of the class introduced above. Each finite simple group is known to be generated by just two elements. (See [11] for more details.) The proof of this fact depends heavily on the classification of finite simple groups. As we have seen, nonassociative simple Moufang loops admit a much simpler classification. They cannot be generated by two elements only, since every Moufang loop is diassociative (cf. [8], [2], or [10]).

In this paper, we show that \( \mathcal{M}/Z(\mathcal{M}) \) is generated by three elements when \( F \) is a (finite) prime field. The approach taken here is elementary. Possible generalizations of this result would probably require more detailed methods than those used here, or a completely different approach (cf. L. E. Dickson’s proof that \( SL_2(q) \) is two-generated for odd \( q \neq 9 \) [4], or [5]).

2. GENERATORS

It is inconvenient to work with the quotient \( \mathcal{M}/Z(\mathcal{M}) \). We shall readily identify the elements \( M \) and \(-M\) in all of our computations, and multiply by \(-1\) freely. For \( M \in \mathcal{M} \), a matrix, let \( M' \) denote the transpose of \( M \).

In order to linearize our notation, we introduce two mappings \( u, l : \mathbb{F}_3 \longrightarrow \mathcal{M} \) defined by

\[ u(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad l(\alpha) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \]
Next consider $t : \mathbb{F}^3 \setminus \{0\} \rightarrow \mathbb{F}^3$ given by

$$t(\alpha_1, \alpha_2, \alpha_3) = \begin{cases} (-\alpha_1^{-1}, 0, 0) & \text{if } \alpha_1 \neq 0, \\ (0, -\alpha_2^{-1}, 0) & \text{if } \alpha_1 = 0, \alpha_2 \neq 0, \\ (0, 0, -\alpha_3^{-1}) & \text{otherwise}. \end{cases}$$

Note that $\alpha \cdot t(\alpha) = -1$. Finally, let $s(\alpha)$ stand for the matrix

$$\begin{pmatrix} 0 & \alpha \\ t(\alpha) & 0 \end{pmatrix}.$$

Recall that, for matrices in $\mathcal{M}$,

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}^{-1} = \begin{pmatrix} b & -\alpha \\ -\beta & a \end{pmatrix},$$

and also the two special cases of (4.1), (4.2) [9]:

$$l(\alpha) = s(\alpha)'u(t(\alpha))(-s(\alpha))',$$

$$u(\alpha) = s(\alpha)l(t(\alpha))(-s(\alpha)).$$

Observe that $l(\alpha)^{-1} = l(-\alpha)$, and $u(\alpha)^{-1} = u(-\alpha)$.

We start our search for generators with the following result due to Paige:

**Proposition 2.1.** Every simple Moufang loop $\mathcal{M}/Z(\mathcal{M})$ is generated by

$$\{u(\alpha), l(\alpha); 0 \neq \alpha \in \mathbb{F}^3\}.$$

**Proof.** Combine Lemmas 4.2 and 4.3 [9].

Let us identify all nonzero elements of $\mathbb{P}^3 = \mathbb{F}^3/\mathbb{F}$ with those vectors from $\mathbb{F}^3$ whose first nonzero coordinate equals 1. Also, let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$, as usual.

**Proposition 2.2.** Assume that $\mathbb{F}$ is a prime field. Then $\mathcal{M}/Z(\mathcal{M})$ is generated by

$$\{u(\alpha), l(\alpha); \alpha \in \mathbb{P}^3\}.$$

**Proof.** First check that $u(aa)u(ba) = u((a + b)\alpha)$, and $l(aa)l(ba) = l((a + b)\alpha)$ for all $\alpha \in \mathbb{F}^3, a, b \in \mathbb{F}$. Given $0 \neq \beta \in \mathbb{F}^3$, there is $a \in \mathbb{F}$ and $\alpha \in \mathbb{P}^3$ such that $\beta = a\alpha$. Since $\mathbb{F}$ is prime, we can use $\alpha$ as an exponent, and write $u(\beta) = u(\alpha)^a$, $l(\beta) = l(\alpha)^a$. We are finished, by Proposition 2.1.
Proposition 2.3. Assume that $\mathbb{F}$ is prime. Then $\mathcal{M}/\mathbb{Z}(\mathcal{M})$ is generated by

$$
S = \{u(e_1), u(e_2), u(e_3)\} \cup \{s(\alpha), s(\alpha)^{\prime}; \alpha \in \mathbb{F}^3\}.
$$

Proof. Observe that, for $\alpha \in \mathbb{F}^3$, we have $t(\alpha) = \{e_1, e_2, e_3\}$. Given $t(\alpha)$ with $\alpha \in \mathbb{F}^3$, (1) yields $l(\alpha) = s(\alpha)^{\prime}u(-e_i)(-s(\alpha)^{\prime})$ for some $e_i, 1 \leq i \leq 3$. In particular, the elements $l(e_i), 1 \leq i \leq 3$, are generated by $S$.

Symmetrically, given $u(\alpha)$ with $\alpha \in \mathbb{F}^3$, equation (2) yields $u(\alpha) = s(\alpha)(-e_i)(-s(\alpha))$ for some $e_i, 1 \leq i \leq 3$; and we are done by Proposition 2.2. ■

Lemma 2.1. Assume $\alpha, \beta \in \mathbb{F}^3 \setminus \{0\}$ are such that $t(\alpha) = t(\beta)$. Then $s(\beta) = s(\alpha)(-\alpha \times \beta)$, and $s(\beta)^{\prime} = s(\alpha)^{\prime}u(\alpha \times \beta)$.

Proof. By definition, we have

$$
s(\alpha)s(\beta) = \begin{pmatrix} \alpha \cdot t(\beta) & t(\alpha) \times t(\beta) \\ \alpha \times \beta & t(\alpha) \cdot t(\beta) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ \alpha \times \beta & -1 \end{pmatrix}.
$$

Thus $s(\beta) = -s(\alpha)^{-1}(-\alpha \times \beta) = s(\alpha)(-\alpha \times \beta)$. As for the remaining equation, start with $s(\alpha)^{\prime}s(\beta)^{\prime} = -u(\alpha \times \beta)$.

Proposition 2.4. Assume $\mathbb{F}$ is prime. Then $\mathcal{M}/\mathbb{Z}(\mathcal{M})$ is generated by

$$
\{u(e_i), s(e_i); 1 \leq i \leq 3\}.
$$

Proof. Thanks to Proposition 2.3, we only need to generate elements $s(\alpha)$ and $s(\alpha)^{\prime}$, for $\alpha \in \mathbb{F}^3$. First of all, note that $s(e_i)^{\prime} = -s(e_i)$, and, by (1), $l(e_i) = s(e_i)^{\prime}u(e_i)^{-1}s(e_i)$, for $1 \leq i \leq 3$. Also, $s(0, 1, a) = s(e_2)(e_1)^{-a}$ and $s(0, 1, a)^{\prime} = s(e_2)^{\prime}u(e_1)^{a}$ for all $a \in \mathbb{F}$. Similarly, $s(1, a, 0) = s(e_1)^{-a}l(e_3)$ and $s(1, a, 0)^{\prime} = s(e_1)^{\prime}u(e_3)^{a}$ for all $a \in \mathbb{F}$. Next, by (1) and (2),

$$
l(1, a, 0) = s(1, a, 0)^{\prime}u(-e_1)(-s(1, a, 0)^{\prime}),
$$

$$
u(1, a, 0) = s(1, a, 0)l(-e_1)(-s(1, a, 0)).
$$

For $0 \neq a \in \mathbb{F}$, we have $l(a^{-1}, 1, 0) = l(1, a, 0)^{a^{-1}}, u(a^{-1}, 1, 0) = u(1, a, 0)^{a^{-1}}$. Hence we obtained $l(-a, 1, 0)$ and $u(-a, 1, 0)$ for all $a \in \mathbb{F}$.\[\]
Finally, Lemma 2.1 yields

\[ s(1, a, b) = s(1, a, 0)l(b(-a, 1, 0)) = s(1, a, 0)b(-a, 1, 0)^b, \]
\[ s(1, a, b)' = s(1, a, 0)'u(b(a, -1, 0)) = s(1, a, 0)'u(-a, 1, 0)^{-b}, \]

for all \( b \in \mathbb{F}. \)

We may further reduce the number of generators one by one down to three. The equations (3) below have been carefully chosen. They are satisfied independently of \( \text{char} \mathbb{F}, \) and are as simple as the author was able to find. We leave the somewhat lengthy verification to the reader. Let

\[ x = \begin{pmatrix} 0 & e_3 \\ -e_3 & 1 \end{pmatrix}. \]

Then

\[ s(e_1) = s(e_3)s(e_2), \]
\[ s(e_2) = [u(e_1)u(e_3) \cdot u(e_2)u(e_1)][u(e_3)^{-1} \cdot u(e_2)s(e_3)], \]
\[ u(e_3) = x^{-1}s(e_3), \]
\[ s(e_3) = [u(e_2)u(e_1) \cdot x][xu(e_1)] \cdot [x^2u(e_2) \cdot u(e_1)x^2u(e_1)]. \]

**Theorem 2.1.** Assume that \( \mathbb{F} \) is a prime field. Then the simple Moufang loop \( \mathcal{M}/Z(\mathcal{M}) \) is generated by the three elements

\[ \begin{pmatrix} 1 & (1, 0, 0) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & (0, 1, 0) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & (0, 0, 1) \\ (0, 0, -1) & 1 \end{pmatrix}. \]

3. INTEGRAL CAYLEY NUMBERS OF UNIT NORM

The smallest nonassociative simple Moufang loop \( \mathcal{M}_{120} \) of 120 elements is constructed over the binary field. As Paige showed in [9], it is isomorphic to the integral Cayley numbers of unit norm modulo their center. The aim of this short section is to offer one explicit isomorphism, and to conclude that the loop of integral Cayley numbers of unit norm is generated multiplicatively by three elements. The isomorphism will allow us to perform calculations inside integral Cayley numbers more efficiently than by the conventional rules.

Unlike the case of complex numbers \( \mathbb{C} \) and quaternions \( \mathbb{H} \), there are still many names for the eight-dimensional real algebra \( \mathbb{O} = \mathbb{H} \times \mathbb{H} \) obtained from \( \mathbb{H} \) by the Cayley-Dickson process: Cayley numbers, algebra of octaves,
octonions. We prefer to use the name octonions. Recall that for $(q, Q), (r, R) \in \mathbb{H} \times \mathbb{H}$ the multiplication in $\mathbb{O}$ is defined by
\[
(q, Q) \cdot (r, R) = (qr - \overline{RQ}, Rq + Q\overline{r}),
\]
where the bar indicates conjugation in $\mathbb{H}$. An alternative way to describe the multiplication is to introduce a new unit $e$, regard $\mathbb{O}$ as $\mathbb{H} + \mathbb{H}e$, and write $(q + Qe) \cdot (r + Re) = (qr - \overline{RQ}) + (Rq + Q\overline{r})e$. (For an excellent discussion concerning the notation in $\mathbb{O}$, see [3].)

When $\mathbb{O}$ is viewed as an eight-dimensional real vector space, we will use Dickson’s notation for its basis, namely $1, i, j, k, e, ie, je, ke$. This is not the best choice when one wishes to describe the multiplicative relations between basis elements in a compact way (see [1], [3]), but it seems to be the best choice for what follows.

The conjugate of the octonion $a = q + Qe$ is defined as $\overline{a} = \overline{q} - Qe$. Its norm $N(a)$ is then the non-negative real number $a\overline{a}$. For any $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, a set of integral elements of $\mathbb{F}$ is defined as a maximal subset of $\mathbb{F}$ containing 1, closed under multiplication and subtraction, and such that both $N(a)$ and $a + \overline{a}$ are integers for each element $a$ of the set. Such a set is unique for $\mathbb{F} = \mathbb{R}, \mathbb{C},$ and $\mathbb{H}$. In the case of the octonions there are seven such sets, all isomorphic. For the rest of this paper, we select the one which Coxeter denotes by $J$, and calls integral Cayley numbers.

By $J'$ we mean the 240 elements of $J$ with norm one. See [3], p.29, for the list of all elements of $J'$. No three of the basis elements generate $J'$. Following common practice, let $h = 1/2 \cdot (i + j + k + e) \in J'$. Coxeter knew that $i, j,$ and $h$ generated $J$ by multiplication and subtraction, but he did not notice that $J'$ is generated by the above elements just under multiplication. From 2.1, we know that there must be three elements generating $J'/\{1, -1\}$. Indeed, $i, j,$ and $h$ do the job. One of the possible isomorphisms $\varphi: J'/\{1, -1\} \to M_{120}$ is determined by
\[
\begin{align*}
    i &\mapsto \begin{pmatrix} 0 & e_3 \\
    e_3 & 0 \end{pmatrix}, \\
    j &\mapsto \begin{pmatrix} 0 & e_2 \\
    e_2 & 0 \end{pmatrix}, \\
    h &\mapsto \begin{pmatrix} 1 & (0, 1, 0) \\
    (1, 0, 1) & 1 \end{pmatrix},
\end{align*}
\]
which, however, is rather tedious to check by hand. The author acknowledges that he used his own set of GAP 4 libraries [6] to confirm the computation. Since $i^2 = -1$, we see that $i, j,$ and $h$ generate $J'$.

In order to be able to carry out calculations in $J'$, we would like to know $\varphi(e)$.

**Lemma 3.1.** (i) $e = -(jh \cdot hi) \cdot kh$ in $\mathbb{O}$.
(ii) The element corresponding to $e$ under $\varphi$ is
\[
\begin{pmatrix} 0 & (1, 1, 1) \\
    (1, 1, 1) & 0 \end{pmatrix}.
\]
Proof. Consider the multiplicative relations in [3], p.567 (or p.28). In particular, we have $hi = -1 - ih$. Hence

\[-(jh \cdot hi) \cdot kh = (jh + jh \cdot ih) \cdot kh = (jh + k - h - ih) \cdot kh = jh \cdot kh + k \cdot kh - h \cdot kh - ih \cdot kh = (-i + h - kh) + (-h) - (k - h) - (j - h - kh) = -i - j - k + 2h,
\]

which equals $e$, since $h = 1/2 \cdot (i + j + k + e)$. 

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