Reconstructing three-dimensional densities from two-dimensional observations of molecular gas

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ABSTRACT
Star formation has long been known to be an inefficient process, in the sense that only a small fraction $\epsilon_{ff}$ of the mass of any given gas cloud is converted to stars per cloud free-fall time. However, developing a successful theory of star formation will require measurements of both the mean value of $\epsilon_{ff}$ and its scatter from one molecular cloud to another. Because $\epsilon_{ff}$ is measured relative to the free-fall time, such measurements require accurate determinations of cloud volume densities. Efforts to measure the volume density from two-dimensional projected data, however, have thus far relied on treating molecular clouds as simple uniform spheres, while their real shapes are likely filamentary and their density distributions far from uniform. The resulting uncertainty in the true volume density is likely one of the major sources of error in observational estimates of $\epsilon_{ff}$. In this paper, we use a suite of simulations of turbulent, magnetized, radiative, self-gravitating star-forming clouds in order to examine whether it is possible to obtain more accurate volume density estimates and thereby reduce this error. We create mock observations from the simulations, and show that current analysis methods relying on the spherical assumption likely yield $\sim 0.5$ dex errors in volume density estimates, corresponding to a $0.25$ dex scatter in $\epsilon_{ff}$, comparable to the scatter in observed cloud samples. We build a predictive model that uses information accessible in two-dimensional measurements – most significantly the Gini coefficient of the surface density distribution – to produce estimates of the volume density with $\sim 0.2$–$0.3$ dex less scatter. Application of this model to observational estimates of $\epsilon_{ff}$ should greatly reduce one of the largest sources of error, and allow significantly more accurate determinations of its true mean value and scatter of $\epsilon_{ff}$.

Key words: stars: formation – ISM: structure.

1 INTRODUCTION

Because of the wide range of physical processes involved, star formation is one of the least understood phenomena in the universe. However, it is also one of the most important, because star formation plays a key role in the evolution of galaxies and sets the initial conditions for planet formation. One major unsolved problem in this field is why star formation is such an inefficient process. For a star-forming region, the depletion time $t_{\text{dep}} = M_{\text{gas}}/M_\star$ is the ratio of the gas mass and the star formation rate (SFR). It is a characteristic timescale of star formation. By contrast, the natural timescale for a cloud collapsing under its own gravity is the free-fall time,

$$t_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho}},$$

where $G$ is the gravitational constant and $\rho$ is the volume density. The star formation efficiency, defined as (Krumholz & McKee 2005)

$$\epsilon_{ff} = \frac{t_{\text{ff}}}{t_{\text{dep}}} = \sqrt{\frac{3\pi}{32G\rho M_{\text{gas}}}},$$

characterises the efficiency of the star formation process. A value of $\epsilon_{ff} \sim 1$ for a given star-forming region indicates that the region is giving birth to stars with little resistance to self-gravity, i.e., all the gas collapses into stars in a single free-fall time. On the contrary, if $\epsilon_{ff}$ is low, this implies that some other process, for example magnetic or turbulent pressure, is obstructing free-fall collapse and impeding star formation (Federrath & Banerjee 2015; Federrath 2018b).

Zuckerman & Evans (1974) were the first to point out that comparing the Milky Way’s star formation rate ($\sim 1 M_\odot$ yr$^{-1}$), total mass of molecular clouds ($\sim 10^9 M_\odot$), and typical molecular cloud free-fall time ($\sim 10$ Myr) implies that molecular clouds have $\epsilon_{ff} \ll 1$, and Krumholz & Tan (2007) extended this conclusion to the denser parts of molecular clouds traced by molecules such as HCN. Krumholz et al. (2019) summarise more recent observations on both sub-galactic and whole-galaxy scales, and show that these yield $\epsilon_{ff}$ estimates consistent with a near universal value $\epsilon_{ff} \sim 0.01$ (e.g. Heyer et al. 2016; Ochsendorf et al. 2017; Onus et al. 2018; Utomo et al. 2018). These results have a study-to-study dispersion of...
\( \approx 0.3 \) dex, and a dispersion of about \( 0.3 - 0.5 \) dex within any single study.

The origin of the low observed value of \( \epsilon_{\text{eff}} \) is one of the major puzzles in star formation theory. To explain it, different groups have built models that can be classified into two main types. One group of theorists explain this phenomenon by focusing on galactic scale physical processes (e.g. Kim et al. 2011; Ostriker & Shetty 2011; Faucher-Giguère et al. 2013), while others construct their models by summing up star formation in individual molecular clouds, each of which has a small value of \( \epsilon_{\text{eff}} \) due to some internal regulation process (e.g. Elmegreen & Parravano 1994; Krumholz et al. 2011; Federrath & Klessen 2012). Both classes of models predict similarly-low \( \epsilon_{\text{eff}} \) values on average, but they differ substantially in their predictions for the dispersion of \( \epsilon_{\text{eff}} \) on sub-galactic scales – models where star formation is regulated only on galactic scales generally predict much larger dispersions than those where it is regulated on the cloud scale (Lee et al. 2016; Krumholz & McKee 2020). This provides a strong motivation for measuring the cloud-scale distribution of \( \epsilon_{\text{eff}} \) values with enough fidelity that we can determine not just its mean value, but also its dispersion. Such measurements also offer an invaluable opportunity to test prescriptions for star formation and feedback in large-scale galaxy and cosmological simulations, since different prescriptions for these processes yield differing distributions of \( \epsilon_{\text{eff}} \) (e.g., Semenov et al. 2016; Grudić et al. 2019; Grisdale et al. 2019; Fujimoto et al. 2019). In order to take advantage of this opportunity, however, we must be able to separate true dispersion from observational errors; the more we can decrease observational errors in measurements of \( \epsilon_{\text{eff}} \), the more we can constrain theoretical models.

Examining equation 2, we can see that the value of \( \epsilon_{\text{eff}} \) is related to the SFR, gas mass and volume density. All three parameters carry observational uncertainties, but the volume density is the dominant one. While the other two parameters can be obtained from two-dimensional surface density maps, the volume density is an inherently three-dimensional property, estimates of which are inevitably subject to projection effects. The scale of volume density uncertainties depends on the measurement method. One method is to estimate \( \rho \) with density-sensitive multiline spectroscopy (Gao & Solomon 2004; Ginsburg et al. 2013; Leroy et al. 2017; Onus et al. 2018), but this is observationally expensive, and requires significant calibration with uncertain theoretical models. A more direct approach is to derive \( \rho \) from the column density \( \Sigma \) of observed star-forming gas, relying on assumptions about the line-of-sight depth. For extragalactic observations on scales \( \gtrsim 100 \) pc, one can estimate the depth from consideration of hydrostatic balance within a galactic disc (e.g., Utomo et al. 2018), but this approach is not available for surveys focusing on nearby molecular clouds on smaller scales, which are the measurements that are most valuable for testing theoretical models.

Instead, the most common approach in the literature is to assume the cloud being observed is approximately spherical, so its depth along the line sight is comparable to its size in the plane of the sky. Heyer et al. (2016), for example, identify dense clumps in ATLASGAL dust maps, and for each clump they measure the total area \( A \) and total mass \( M_{\text{cloud}} \). From these two they compute the mean surface density \( \Sigma = M_{\text{cloud}}/A \) and assign a mean radius \( R_{\text{eff}} = \sqrt{A/\pi} \). Therefore, under the spherical assumption the spherical volume density \( \rho_{\text{sph}} \) is simply

\[
\rho_{\text{sph}} = \frac{3M_{\text{cloud}}}{4\pi R_{\text{eff}}^3} = \frac{3\Sigma}{4\sqrt{\pi}A}.
\]

A number of other authors have used the same basic approach in the Milky Way (e.g. Krumholz et al. 2012; Lada et al. 2013; Evans et al. 2014; Pokhrel et al. 2020) and in the Large Magellanic Cloud (Ochsendorf et al. 2017).

However, the errors and biases that result from the spherical assumption are at present poorly understood. For decades, filamentary structures have been observed to be a common feature of the interstellar medium (ISM) (e.g. Schneider & Elmegreen 1979; Dobashi et al. 2005; Arzoumanian et al. 2011; André et al. 2014; Kainulainen et al. 2016). Contours identified on the surface density map of these structures are elongated. Thus, the volume density derived under the spherical assumption may be quite different from the true mean density. Moreover, even for molecular clouds with perfectly spherical shapes, the mean volume density may still not reflect the mean free-fall time of the whole region. As shown in equation 1, \( t_{\text{ff}} \propto \rho^{-0.5} \), which is a non-linear correlation. Thus, if the molecular cloud has a non-uniform mass distribution (which is very likely), the value of \( t_{\text{ff}} \) determined by integrating sub-regions would not be equal to the value calculated with the mean density of the whole region (e.g. Hennebelle & Chabrier 2011; Federrath & Klessen 2012; Federrath 2013; Salim et al. 2015).

Given the importance of volume density measurements and the potential problems of the commonly-used spherical assumption, our goal is to find an improved method to estimate the three dimensional (3D) volume density from two dimensional (2D) observations. Since the true value of volume density can only be determined with 3D data, we turn to numerical simulations, from which we can obtain all 3D properties of the simulated molecular clouds. Using these simulations, we generate mock observations and place surface density contours over them. For each contour, we calculate the true volume density, together with a number of other parameters (mean surface density, velocity dispersion, mass of enclosed stars, etc.) that would be accessible in realistic 2D observations. We use these data to both calibrate the expected error in estimates of \( \epsilon_{\text{eff}} \) that rely on the spherical assumption, and to develop a predictive model for the volume density that can be used to reduce this error.

This paper is structured as follows. Section 2 summarises the simulation data and the data analysis methods. Section 3 presents the results of the analysis and the predictive model. Section 4 discusses the physical meaning behind the proposed model, telescope beam effects, and possible future work in this area. Section 5 concludes the work done in the paper.

2 SIMULATIONS AND ANALYSIS METHODS

The simulations we use in our study are from the work of Cunningham et al. (2018, hereafter C18). We choose these simulations because they include detailed treatments of many physical processes: gravity, magnetic fields, turbulence, mechanical jets/outflows, and radiation feedback. Moreover, these simulations produce star formation efficiencies and initial mass function (IMF) peaks that are both stable in time and are close matches to recent observations, and they span a wide range of turbulent and magnetic field characteristics, allowing us to check for systematic variations with these properties. We start this section with a brief introduction to the main features of the C18 simulations, and then describe the data analysis methods we apply in the remainder of this section.

2.1 Summary of simulations

C18 uses the ORION2 adaptive mesh refinement (AMR) code (Li et al. 2012). It solves the equations of ideal magnetohydrodynamics (MHD) using the scheme of Mignone et al. (2012), together with
coupled self-gravity (Truelove et al. 1998; Klein et al. 1999) and radiation transfer (Krumholz et al. 2007). The C18 simulations include driven turbulence, produced following the driving recipe of Mac Low (1999). They include protostellar outflows following the procedure described in Cunningham et al. (2011); star formation follows the sink particle algorithm of Krumholz et al. (2004), while protostar evolution and radiative feedback use the model developed by Offner et al. (2009). We refer readers to C18 for full details on how each of these physical processes are implemented.

C18 includes nine individual simulations with slightly different initial conditions, whose properties are summarised in Table 1. For all simulations, the AMR hierarchy is initialized on a 256^3 base grid denoted as \( L = 0 \). The highest refinement level \( L_{\text{max}} = 4 \) (so the highest resolution is \( 1/2^4 \) times of that of the base grid) for three simulations and \( L_{\text{max}} = 3 \) for the other six. The values of \( L_{\text{max}} \) for each simulation are listed in Table 1. The initial state consists of molecular gas with solar metallicity, a mean molecular weight 2.33 \( m_p \) and an initial temperature \( T_g = 10 \) K. Thus, the initial sound speed is \( c_s = 0.19 \) km s\(^{-1}\). The simulation domain is a periodic box with size \( L = 0.65 \) pc and mean density \( \bar{\rho} = 4.46 \times 10^{-20} \) g cm\(^{-3}\), which corresponds to a total mass of \( M = 185 \) M\(_\odot\).

In all cases the simulations begin with a uniform medium, and are run for two box crossing times with gravity disabled and turbulent driving turned on, so that the turbulence reaches a statistical steady state. After that point, gravity is turned on; in half the simulations driving continues, while in the other half it is disabled at this point, so that turbulence decays freely. In addition to this variation in driving, the simulations vary in their degree of magnetisation. All simulations begin with a uniform magnetic field, whose strength is parameterised in terms of the mass-to-flux ratio normalised to the critical value, \( \mu_\phi = M/M_\phi \), where \( \Phi \) is the magnetic flux through the simulation domain and \( M_\phi = \Phi/2\pi \sqrt{G} \) is the magnetic critical mass (Mouschovias & Spitzer 1976). The C18 simulations include cases with \( \mu_\phi = 1.56, 2.17, 23.1 \) and \( \infty \) (i.e., no magnetic fields), each run with driving turned on and off, for a total of eight models. In addition, one of the non-driven models is run with protostellar outflows disabled, yielding a total of nine cases.

### 2.2 Data analysis methods

For the analysis in this paper, we use only the last snapshots, which are taken at the times listed in Table 1; here \( t = 0 \) corresponds to the time at which gravity is switched on. The analysis procedure consists of three steps: creating and selecting contours, measurement of the true 3D volume density, and measurement of 2D contour properties. The details of each step will be illustrated below.

#### 2.2.1 Creating and selecting contours

The first step of the simulation data analysis is to generate and select surface density contours. We start by making projection maps for every snapshot at the native resolution of the simulations along each of the three cardinal axes, yielding 27 gas column density maps. On each map, we define 30 levels of surface density \( \Sigma \), uniformly spaced in logarithm between the mean value of the map, \( \Sigma_1 \) and the maximum value \( \Sigma_{\text{max}} \). We start from \( \Sigma \) rather than from a lower surface density contour because we want to focus on the high-density regions where star formation occurs. From the smallest to the largest of the determined column density levels, we draw contours on the \( \Sigma \) map for each level. The set of closed contours generated by this procedure forms the basic data set we will analyse in the remainder of this work.

To select contours suitable for further analysis, we discard those that fail to meet three conditions. First, we project each contour onto the two axes of the \( \Sigma \) map, and measure the lengths \( L_1 \) and \( L_2 \) of the one-dimensional (1D) projections on both axes. We only retain contours with \( L_1, L_2 < L/2 \). The reason is that the C18 simulations use periodic boundary conditions, which makes it hard to define the shape and the centre of mass of the contours that cover a significant fraction of the computational box. Second, we discard contours with a mean radius \( R_{\text{eff}} = \sqrt{A/\pi} < L/100 \), where \( A \) is the area enclosed by the contour. As shown in Federrath et al. (2011), one needs about 30 pixels across a structure to adequately resolve its internal turbulent motions. Since our simulation maps are either 2048\(^2\) or 4096\(^2\) in size, this condition ensures that contours are resolved by a minimum radius of \( \approx 20–40 \) pixels, depending on the maximum resolution of the simulation. Thus this criterion guarantees that the internal structures of the selected contours are well resolved. Third, we retain only contours enclosing at least one sink particle. This selection rule is intended to mimic observations, which usually focus on regions selected around observed protostars. With these criteria above, our 27 maps yield 761 contours suitable for further analysis.

We show an example surface density map and contours, in this case for simulation lo projected along the \( \hat{x} \)-axis, in Figure 1. The white circles are the projected positions of sink particles, the blue contours are from level 6/30 (\( \Sigma = 0.29 \) g cm\(^{-2}\)) and the yellow contours are from level 11/30 (\( \Sigma = 0.80 \) g cm\(^{-2}\)). For reference, the mean surface density of this map is \( \bar{\Sigma} = 0.084 \) g cm\(^{-2}\).

#### 2.2.2 Measuring the effective volume density

As mentioned in the introduction, for a molecular cloud with a non-uniform mass distribution, a simple mean volume density \( \bar{\rho} \) does not reflect the mean free-fall time of the whole cloud, and using it may

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1 Note that \( \bar{\Sigma} \) is identical for each projection of a single simulation, but differs between the simulations, because at the snapshots we use, different simulations have converted different fractions of their gas to stars.


lead to significant uncertainties when inferring the value of $\epsilon_{\text{ff}}$. We therefore define the effective volume density $\rho_{\text{eff}}$ to be a free-fall-time-weighted mean density that is more suitable for calculating $\epsilon_{\text{ff}}$. For a molecular cloud with non-uniform density and a fixed value of $\epsilon_{\text{ff}}$, the SFR is given by (Hennebelle & Chabrier 2011; Federrath & Klessen 2012),

$$M_* = \int \rho_{\text{eff}} \frac{\rho}{\epsilon_{\text{ff}}(\rho)} \, dV = \epsilon_{\text{ff}} \int \frac{32G}{3\pi} \rho^{3/2} \, dV,$$

where the integral is over the cloud volume. We therefore define the effective free-fall time for the whole mass to be

$$t_{\text{ff,eff}} = \sqrt{\frac{3\pi}{32G\rho_{\text{eff}}}},$$

where $\rho_{\text{eff}}$ is our effective volume density, defined implicitly by demanding

$$M_* = \epsilon_{\text{ff}} \frac{M_{\text{gas}}}{t_{\text{ff,eff}}} = \epsilon_{\text{ff}} \left( \frac{32G}{3\pi} \rho_{\text{eff}}^{1/2} \right) \int \rho \, dV.$$

Equating equation 4 and equation 6, we therefore define $\rho_{\text{eff}}$ as

$$\rho_{\text{eff}} = \left( \frac{\int \rho^{3/2} \, dV}{\int \rho \, dV} \right)^2,$$

which is more suitable for calculating $\epsilon_{\text{ff}}$ as in equation 2. For each selected contour, we measure $\rho_{\text{eff}}$ by evaluating the integrals in equation 7 over a volume defined by the projection of the contour along the line of sight through the full volume of the simulation.

### 2.2.3 Measurement of 2D contour properties

To build a model that can predict $\rho_{\text{eff}}$ from 2D observations, we need to determine contour properties that may be related to $\rho_{\text{eff}}$. To illustrate our procedure we will use the contour located on the mid-right side of Figure 1 as an example. We zoom in on this contour in Figure 2, where we show the contour, its centre of mass (CoM) position, major axis direction, and plane-of-sky magnetic field ($B_{\text{pos}}$) direction.

From each selected contour we determine 10 parameters. The 1st parameter is the spherical density $\rho_{\text{sph}}$ as defined in equation 3, which we will compare with the value of $\rho_{\text{eff}}$ defined in equation 7. The 2nd is the mean radius of the contour, $R_{\text{eff}}$. The 3rd is the ratio between the mean column density of the contour and that of the whole column density map $\Sigma_{\text{contour}}/\Sigma$; we choose the ratio instead of the absolute value in order to minimise the effect of the difference of gaseous mass between simulations. As stellar feedback may change the mass distribution of a molecular cloud, we select as the 4th parameter the ratio between the total mass of the sink particles inside the contour and the total gas mass of the contour, $M_{\text{s}}/M_{\text{contour}}$.

The 5th parameter is the line-of-sight (los) velocity dispersion $\sigma_{v,\text{los}}^2$. We define it as follows, roughly mimicking the way it might be measured from a position-position-velocity data cube using an optically thin tracer: for each pixel $i$ in the projected map that lies inside the contour of interest, we first compute the first moment of the los velocity, $v_{i,\text{los}} = \sum_j M_j v_{ij,\text{los}}/\sum_j M_j$, where $M_j$ and $v_{ij,\text{los}}$ are the mass and los velocity of each cell $j$ along a particular line of sight $i$ through the projected map. We further define the mean los velocity $\bar{v}_{\text{los}}$ as the mean of the $v_{i,\text{los}}$ values, and the los velocity dispersion by $\sigma_{v,\text{los}}^2 = \sum_i (v_{i,\text{los}} - \bar{v}_{\text{los}})^2/N_p$, where $N_p$ is the total number of pixels included in the contour. Thus, the los velocity dispersion is the root mean square velocity of all computational cells within the contour, measured in the frame where the CoM velocity is zero.

To describe the shapes of our contours, we introduce the ellipticity

| Name       | $\mu_\phi$ | Outflows | Driving | $L_{\text{max}}$ | Time (Myr) | $M_{\text{s}}/M_{\text{contour}}$ |
|------------|------------|----------|---------|------------------|------------|----------------------------------|
| lo         | 1.56       | √        | ×       | 3                | 1.911      | 0.10                             |
| loDrive    | 1.56       | √        | ×       | 3                | 1.843      | 0.070                            |
| loNW       | 1.56       | ×        | ×       | 3                | 1.640      | 0.13                             |
| lo2        | 2.17       | √        | ×       | 4                | 1.547      | 0.057                            |
| lo2Drive   | 2.17       | √        | √       | 3                | 1.824      | 0.080                            |
| hi         | 23.1       | √        | ×       | 4                | 1.390      | 0.060                            |
| hiDrive    | 23.1       | √        | √       | 3                | 1.535      | 0.034                            |
| hydro      | ∞          | √        | ×       | 4                | 1.319      | 0.052                            |
| hydroDrive | ∞          | √        | √       | 3                | 1.505      | 0.052                            |

Table 1. The short names and main differences of all 9 simulations in C18. The 1st column is the name of each simulation. The 2nd column is the mass-to-flux ratio normalised to the critical value ($\mu_\phi$). The 3rd and 4th columns indicate whether protostellar outflows and turbulent driving are included in the simulation. The 5th column shows the highest refinement level $L_{\text{max}}$, which is related to the maximum linear resolution by $\Delta x = (524 \text{ AU})/2^{L_{\text{max}}}$, where the integral is over the cloud volume. The 6th column is the ratio between the total mass of sink particles and total mass inside the simulation box.
where \(a\) is the semi-major axis length of the contour and \(b\) is the semi-minor axis length; \(e \sim 0\) corresponds to an extremely elongated contour and \(e \sim 1\) describes a nearly circular contour. To determine \(a\) and \(b\), we first calculate the CoM of the contour. Then for each pixel in the contour with a mass \(M_p\) and a displacement from the CoM, \(\Delta \mathbf{x} = (\Delta x_{p,1}, \Delta x_{p,2})\), we define the inertia tensor \(I\) as

\[
I_{ij} = (-1)^{i+j} \sum_p m_p \Delta x_{p,i} \Delta x_{p,j},
\]

where the sum runs over all pixels interior to the contour. The eigenvalues of \(I\) are \(a\) and \(b\) (where \(a \geq b\) by convention), and the corresponding eigenvectors define the directions of the major and minor axes.

The 7th and 8th parameters are the projected, mass-weighted mean magnetic field strengths in the plane of sky \(B_{\text{pos}}\) and in the line of sight \(B_{\text{los}}\); the former is approximately measurable using Zeeman splitting, and the latter using dust polarisation. We define the 9th parameter \(\theta\) as the angle between the major axis and \(B_{\text{pos}}\). For consistency we always choose the smaller angle between the two directions, thus \(\theta \in [0, \pi/2] \text{ radian}\).

The 10th and last parameter is the Gini coefficient \(g\) (Gini 1936) of the column densities of the pixels \(\Sigma\) enclosed by the contour; \(g = 0\) describes a contour with uniform surface density while \(g \sim 1\) corresponds to a contour with a highly concentrated mass distribution.

### 3 RESULTS

Having created our sample with the selection of 761 contours and measured the interesting contour properties, we now investigate whether it is possible to build a model that can predict \(\rho_{\text{eff}}\) from the contour properties. We start by examining the difference between \(\rho_{\text{eff}}\) and \(\rho_{\text{sph}}\) in Section 3.1. Then we utilize the method of multiple linear fitting to build our model. In the remainder of this section, we describe the effectiveness of our model under different conditions.

#### 3.1 Comparing \(\rho_{\text{eff}}\) and \(\rho_{\text{sph}}\)

For each selected contour, we define

\[
Q_{\text{sph}} \equiv \frac{\rho_{\text{sph}}}{\rho_{\text{eff}}}
\]

as the ratio of the spherical approximation density to the effective density; values of \(Q_{\text{sph}} \sim 1\) indicate that the spherical density overestimates the effective density, while values \(Q_{\text{sph}} < 1\) indicate underestimates. This will be our figure of merit for the remainder of the paper, i.e., this quantity characterises how well we can approximate the true, 3D density given the projected information to which we have access. A perfect model would yield a distribution of \(Q\) values that is a \(\delta\) function at \(Q = 1\). For the 761 selected contours, the mean value of \(Q_{\text{sph}}\) is \(Q_{\text{sph}} = 1.32\), and the median value is \(Q_{\text{sph,med}} = 0.89\). We show the full histogram of log \(Q_{\text{sph}}\) in Figure 3 with the contours’ simulation sources labeled. From Figure 3 we can see that although the distribution of log \(Q_{\text{sph}}\) is centered reasonably well around log \(Q_{\text{sph}} = 0\), it is slightly more weighted to log \(Q_{\text{sph}} < 0\), with 421 > 761/2 samples lying at log \(Q_{\text{sph}} < 0\). The distribution of log \(Q_{\text{sph}}\) values varies between individual simulations. Most log \(Q_{\text{sph}}\) values for the hydro simulation, for example, are less than 0. To quantify the dispersion of log \(Q_{\text{sph}}\), we determine the 16th and 84th percentiles of log \(Q_{\text{sph}}\), which we show as black vertical dashed lines in Figure 3. We define the dispersion

\[
\sigma_{\text{sph}} \equiv \frac{1}{2} \left( Q_{\text{sph,84}} - Q_{\text{sph,16}} \right),
\]

where \(Q_{\text{sph,16}}\) and \(Q_{\text{sph,84}}\) are the 16th and 84th percentile values, respectively. Thus, for a Gaussian distribution of log \(Q_{\text{sph}}\) values, \(\sigma_{\text{sph}}\) is just the usual Gaussian dispersion. For the data shown in Figure 3, \(\sigma_{\text{sph}} = 0.45\) dex. Therefore, the volume density determined under the spherical cloud assumption carries an uncertainty of \(\Delta \rho_{\text{sph}} \approx 0.45\) dex.

#### 3.2 Building the predictive model

To reduce the uncertainty carried by \(\rho_{\text{sph}}\), we next build a model to predict the value of \(\rho_{\text{eff}}\) from 2D contour properties by multiple linear fitting (MLF). As some parameters introduced in Section 2.2.3 have wide ranges, we carry out our fits using log-scaled variables. The dependent variable is \(Y = \log(\rho_{\text{eff}}/\rho_{\text{sph}}) = -\log Q_{\text{sph}}\), while the six independent variables are

\[
X = \left[ \log R_{\text{eff}}, \log \frac{\Sigma_{\text{contour}}}{\sum}, \log \frac{M_s}{M_{\text{contour}}}, \log \sigma_{v_{\text{los}}}, \varepsilon, g \right].
\]

We omit the magnetic variables for now, because they are not available for the simulations that do not include magnetic fields; we revisit these variables in Section 3.3. After fitting we obtain the coefficient vector \(\mathbf{k}\) and the intercept \(b\). Thus, the predicted effective volume density \(\rho_p\) is

\[
\rho_p = C \rho_{\text{sph}} \left( 10^k X + b \right) \rho_{\text{sph}},
\]

where \(C \equiv 10^k X + b\) is the correction factor. By analogy with \(Q_{\text{sph}}\) and \(\sigma_{\text{sph}}\) as defined in Section 3.1, we now define \(Q_p = \rho_p/\rho_{\text{eff}}\) and \(\sigma_p\) as the ratio of the predicted and effective densities and half of the distance between the 16th and 84th percentiles of log \(Q_p\), respectively.
The histogram of log $Q_{\text{sph}}$ values is the same as that shown in Figure 3. The two blue dashed lines show the 16th and 84 percentiles of log $Q_{\text{sph}}$, and the two orange dashed lines show the 16th and 84 percentiles of log $Q_p$. The predictive model substantially reduces the error in estimates of the effective density.

We report the best-fitting values of $k$ and $b$ as Fit 1 in Table 2. The coefficient of determination for this fit is $R^2 = 0.76$, indicating a strong correlation and justifying our choice of MLF. With the fitted relation, we predict the effective volume density $\rho_p$ for every contour in the sample. We compare the normalized histograms of log $Q_{\text{sph}}$ and log $Q_p$ in Figure 4. It is obvious that log $Q_p$ is much more narrowly distributed around zero than log $Q_{\text{sph}}$, and the resulting dispersion, $\sigma_p = 0.19$ dex, is substantially smaller. Thus, this fitted relation reduces the uncertainty in the effective volume density by

$$\Delta \sigma = \sigma_{\text{sph}} - \sigma_p = 0.26 \text{ dex}. \quad (14)$$

We use $\Delta \sigma$, the amount by which a given model reduces the scatter in log($\rho_p/\rho_{\text{eff}}$) compared to log($\rho_{\text{sph}}/\rho_{\text{eff}}$), as our figure of merit for evaluating our predictive model from this point forward.

### Table 2

| Quantity                | Fit 1 | Fit 2 |
|-------------------------|-------|-------|
| log($R_{\text{eff}}$/pc) | 0.39  | 0.35  |
| log($\Sigma_{\text{contour}}$/\Sigma) | 0.12  | -0.11 |
| log($M_\star$/M$_{\text{contour}}$) | 0.027 | 0.017 |
| log($\sigma_{v,los}$/cm/s) | -0.094 | -0.041 |
| log($B_{v,\text{pos}}$/G) | —     | 0.072 |
| log($B_{v,\text{los}}$/G) | —     | 0.16  |
| $\theta$                | —     | -0.011|
| $e$                     | 0.24  | 0.34  |
| $g$                     | 3.8   | 3.5   |
| $b$                     | 0.073 | 0.96  |
| $R^2$                   | 0.76  | 0.78  |
| $\Delta \sigma$ (dex)  | 0.26  | 0.22  |

Table 2. Results of MLF for the correction factor $C$ between $\rho_{\text{eff}}$ and $\rho_{\text{sph}}$ (see equation 13). The top block of rows show the fit coefficients $k$, and the last three rows provide the intercept $b$, the coefficients of determination $R^2$, and the amount $\Delta \sigma$ by which the fit reduces the dispersion of log $Q$.

### Figure 5

Same as Figure 4, but now showing the results for effective densities predicted using Fit 2, which includes magnetic field information. Note that the comparison set of $Q_{\text{sph}}$ values shown here (blue histogram) is slightly different than that in Figure 4, because in this figure we omit the purely hydrodynamic simulations, whereas in the previous figure we included all simulations.

#### 3.3 The effect of including magnetic field data

To check the effect of including magnetic field data on our density predictions, we perform another MLF on the seven MHD simulations in our data set (hydro and hydroDrive excluded), to which we refer as Fit 2. This fit includes log $B_{\text{pos}}$, log $B_{\text{los}}$, and $\theta$ (the angle between the plane of sky magnetic field direction and the contour major axis) in the vector of independent variables $X$. We report the results of this fit in Table 2. We define log $Q_{p,2}$ as the logarithm of the ratio of predicted $\rho_p$ and effective densities, in analogy with log $Q_p$, and we plot the normalized histograms of log $Q_{p,2}$ and log $Q_{\text{sph},2}$ in Figure 5; note that the distribution of log $Q_{\text{sph}}$ shown here is slightly different than that shown in Figure 4, since the former includes the contours from hydro and hydroDrive, while this figure excludes them. The dispersion of $Q_{\text{sph}}$ for this sample is $\sigma_{\text{sph},2} = 0.41$ dex, and the dispersion of $Q_{p,2}$ is $\sigma_{p,2} = 0.19$ dex. Thus, $\Delta \sigma_2 = \sigma_{\text{sph},2} - \sigma_{p,2} = 0.22$ dex. Comparing the results from this and the previous fit, we find fairly minor differences in the fit coefficients and intercepts. The $R^2$ value only increases by about 0.02 from Fit 1 to Fit 2, and $\Delta \sigma_2$ is nearly the same as $\Delta \sigma$. This indicates that a model including magnetic field information does not significantly reduce the uncertainty on $\rho_{\text{eff}}$ in comparison to one omitting it. Moreover, as summarized in the review by Crutcher (2012), magnetic field measurements are observationally expensive: determination of $B_{\text{pos}}$ requires measurements of polarised dust continuum emission or absorption, while $B_{\text{los}}$ requires Zeeman effect measurements. Due to the long observation times required, these are difficult to obtain for a large sample. Considering the small gains that we have found from including magnetic field information and the difficulty in obtaining it, we generally suggest using Fit 1 to predict the effective density, unless there is magnetic field information available, in which case Fit 2 can be used.
3.4 Dependence on physical conditions: turbulence, magnetic fields, and outflows

We obtain the relation in Section 3.2 by performing MLF on all nine C18 simulations. However, the ambient conditions (mean magnetic field strength, presence or absence of turbulence driving) vary between individual simulations. If the coefficients of the model fit depend on ambient conditions, this may reduce the reliability of our model under specific circumstances. To check whether this is a concern, we use the two linear models developed in Section 3.2 and Section 3.3 to determine the values of $\sigma_p$ and $\sigma_{p,2}$, the dispersions in $\log Q_p$ and $\log Q_{p,2}$, for different subsets of the simulations. We divide the simulations into those with driven versus decaying turbulence, into simulations with different mass-to-flux ratios, and into simulations that do or do not include protostellar outflows. We plot the results in the top panel of Figure 6; for comparison we also show $\sigma_{\text{sph}}$, the dispersion in $Q_{\text{sph}}$ for the same set of simulations. Note that the model obtained via Fit 2 is only applicable for simulation sets excluding hydro and hydroDrive. We provide a full tabulation of the results in Table 3.

From this plot we can see that for subsets including magnetic fields, there is no significant difference in $\sigma_p$ for different models. After applying both prediction models, the dispersion of $Q_p$ values is decreased for each subset of the simulations to $\sigma_p \in [0.14, 0.30]$ dex; the improvement compared to the simple spherical assumption is in the range $\Delta \sigma \in [0.04, 0.32]$ dex. Therefore, we find relatively little variation in the performance of our prediction model in different simulation subsets; $\sigma_{\text{sph}}$ and $\sigma_p$ values vary between different sets of simulations, but relatively modestly, so that the errors in the predicted models lie in the range $\approx 0.14 - 0.30$ dex for each subset of the simulations. Including B-field information only brings slight improvement from Fit 1 to Fit 2, and the difference is negligible for the weak B-field subset ($\mu_B = 23.1$). We do find a rough negative correlation between $\sigma_p$ and the number of contours (plotted in the bottom panel of Figure 6) available for a particular simulation subset, but the physical mechanism behind this trend is unclear. More theoretical work is needed to understand how variations in the ambient conditions affect the relationship between the sky-projected and volumetric quantities, and how they might affect our model. Nonetheless, we can state at this point that the relationship between $\rho_{\text{eff}}$ and $\rho_{\text{sph}}$ does not seem to depend strongly on the physical conditions present in the star-forming region.

4 DISCUSSION

Although our model has proven effective in reducing the uncertainty in observational inferences of $\rho_{\text{eff}}$, the physical mechanisms leading to this model are still unclear. In this section, we begin to investigate this question by examining the predictive power of each individual parameter in Section 4.1. We then extend our model to account for finite resolution effects in Section 4.2. Finally, we discuss the implications of our findings for observational efforts to measure $\rho_{\text{eff}}$ and its variation in Section 4.3. We end this section by discussing the limitations of our model, and areas for future improvement, in Section 4.4.

4.1 Predictive power of individual parameters

An obvious question that follows from the success of our MLF model in reducing uncertainties in $\rho_{\text{eff}}$ is, which parameters have the most predictive power? We have already seen that magnetic field information adds little accuracy, and we now seek to extend this analysis to the remaining parameters. To investigate this issue, we carry out simple linear fits on the whole sample using only one independent variable each time, and measure the $R^2$ and $\Delta \sigma$ (equation 14) values for the fit; the latter characterises the amount by which a model including only that parameter is able to improve estimates of $\rho_{\text{eff}}$ relative to the naive spherical assumption. We tabulate the results in Table 4. The larger $\Delta \sigma$ is, the more the corresponding parameter can reduce the uncertainty in the effective volume density. The table reveals that the parameters vary widely in their importance. The Gini coefficient $g$ is the most important factor in our model, and by itself it accounts for most of the improvement: $\Delta \sigma = 0.22$ dex for $g$ alone, versus $\Delta \sigma = 0.26$ dex for Fit 1, using all the variables. Next, log $R_{\text{eff}}$ and log($\Sigma_{\text{contour}}/\Sigma$) have medium predictive power, while the other parameters have limited influence on the fitted relation.

To explain this difference, we need to reexamine equation 13. Our model is to multiply $\rho_{\text{sph}}$ by a correction factor $C$. Thus, if one parameter can reveal how far the object is away from a spherical, uniform-density cloud, then we would expect it to have strong predictive power, or large $\Delta \sigma$. To start with, $g$ describes how concentrated the mass distribution is on the 2D projected map, which is strongly related to the volume-density profile. A larger $g$ corresponds to a larger $\int \rho^{3/2} \, dV$ term and hence a larger $\rho_{\text{eff}}$, which is consistent with the positive coefficient of $g$. At the same time, contours with larger $\Sigma_{\text{contour}}/\Sigma$ and larger $R_{\text{eff}}$ might on average be more collapsed along the line of sight, which would suggest a reason for their predictive power: they can flag deviations from the simple spherical assumption. However, the low $R^2$ values of these two individual parameter fits indicate that this is not a strict relation.
constraint on the 3D shape of the gas. Finally, closotoisothermal. Similarly, ellipticity may describe how close the is important only close to protostars, and thus most of the gas is though our simulations include stellar radiation feedback, this effect and velocity fluctuations in the case of (near-)isothermal turbulence; (2010) and Federrath & Banerjee (2015) explain this phenomenon et al. 2008). Passot & Vázquez-Semadeni (1998), Federrath et al. found in several observations (e.g. Goodman et al. 2009; Pineda the estimated density and the true effective density correlation between density and velocity dispersion to their
Also, ellipticity may describe how close the 2D contour shape is to a circle, but this apparently provides limited constraint on the 3D shape of the gas. Finally, log(M_{s}/M_{contour}) has the smallest Δσ and R^2 values. The reason may be that, once sink particles form in the C18 simulations, the local density profile evolves very little; it likely remains close to the usual ρ ∝ r^{-3/2} form expected for free-fall collapse. As a result, the fraction of the available mass that has already accreted, as parameterised by M_{s}/M_{contour}, has very limited predictive power.

Since g is the dominant factor here, we provide a simplified model to predict ρ_{eff} using it alone:

\[ ρ_p = 10^{k_g g + b_g} ρ_{sph} = 10^{0.4 g - 0.94} ρ_{sph}, \]

where k_g is the slope and b_g is the intercept from the linear regression. This simplified model can reduce the uncertainty in ρ_{eff} by Δσ_g = 0.22 dex. As a consistency check, we note that a spherical cloud with uniform density has surface density Gini coefficient g_{sph} = 0.2. Inserting this value into equation 15 yields ρ_p = 10^{-0.06} ρ_{sph}, so we would correctly recover ρ_p ≈ ρ_{sph}.

### 4.2 Finite resolution effects

Both the numerical model in Section 3.2 and the simplified model in Section 4.1 are derived from projection maps created at the native resolution of the simulations, so we are effectively considering only cases where the internal structures of the selected contours are very well-resolved. In real observations the resolution may be limited, and may vary between observations depending on the instrument and the distance to the target. This might have non-trivial effects: a larger beam size will smear details of the contours, and the inferred value of g, for example, is very likely to decrease when high-Σ peaks are smeared out by low resolution. To explore this effect, we apply a series of Gaussian filters to our projection maps; we consider kernels with standard deviation (not full width at half maximum, FWHM) \( w = L/1600, L/800, L/400, 3L/800, L/200, L/160, 3L/400, 7L/800 \) and \( L/100 \), where L is the size of the simulation box. We do not consider larger beam sizes because this leaves too small a dynamic range between the size of contours we can resolve and the size scale at which the periodic nature of our simulation box begins to create problems. For each of the smoothed maps, we repeat the analysis presented in Section 2.2. Note that the 30 contour levels are separately calculated for each Gaussian-filtered map, and thus are not the same for maps with different levels of beam-smearing, since the contour levels depend on the maximum surface density Σ_{max}. Similar

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### Table 3.

| Quantity | All | drive | noDrive | 1.56(NW) | 1.56 | 2.17 | 23.1 | inf |
|----------|-----|-------|---------|---------|------|------|------|-----|
| log Q_{sph,16} | -0.54 | -0.31 | -0.77 | -0.26 | -0.32 | -0.50 | -0.38 | -0.72 |
| log Q_{sph,50} | -0.048 | 0.0057 | -0.27 | -0.022 | 0.0035 | -0.087 | 0.072 | -0.22 |
| log Q_{sph,84} | 0.37 | 0.39 | 0.33 | 0.55 | 0.35 | 0.39 | 0.44 | 0.17 |
| σ_p | 0.45 | 0.35 | 0.55 | 0.41 | 0.34 | 0.45 | 0.41 | 0.44 |

### Table 4.

| Quantity | Intercept | Coefficient | R^2 | Δσ (dex) |
|----------|-----------|-------------|-----|---------|
| log(Σ_{contour}/Σ) | -0.94 | 4.4 | 0.69 | 0.22 |
| log(R_{contour}/pc) | -0.23 | 0.35 | 0.081 | 0.042 |
| e | 1.2 | 0.66 | 0.21 | 0.028 |
| log(σ_{v,los}/(cm/s)) | -0.044 | 0.22 | 0.010 | 0.015 |
| log(M_{s}/M_{contour}) | 1.9 | -0.39 | 0.027 | 0.0072 |
| -0.076 | -0.0088 | 0.0011 | 0.0020 |

Table 3. Values of log Q_{sph}, log Q_{p} and log Q_{p,2} for different sets of simulations. The first row lists the name of different simulation subsets, where ‘All’ means all simulations and the remaining seven columns correspond to the same subsets of the simulations used in Figure 6. In the 1st column, Q is the ratio between the estimated density and the true effective density ρ_{eff}, and σ is the dispersion of log Q (Eq 11). The subscripts ‘sph’, ‘p’ and ‘p,2’ in the 1st column indicate the value of log Q obtained using the spherical assumption, and the predictive models from Fit 1 and Fit 2, respectively. The subscripts ‘16’, ‘50’, ‘84’ indicate the 16th, 50th and 84th percentile values. Note that Fit 2 is not applicable to simulation sets including hydro or hydroDrive, because those simulations did not include magnetic fields.

Table 4. Results of MLF performed on the whole sample with only one independent variable each time. The variables are ranked from top to bottom according to their Δσ values. For comparison, Δσ = 0.26 dex for Fit 1, which uses all the non-magnetic variables.
to Figure 1, we show a Gaussian-filtered, $x$-axis projected column density map of simulation lo in Figure 7. The Gaussian kernel applied on this map is $w = L/100$, which is shown as the pink circle in the right-upper corner. The contours shown are also from level 6 ($\Sigma = 0.17 \text{ g/cm}^2$) and level 11 ($\Sigma = 0.32 \text{ g/cm}^2$).

Since $g$ is the dominant factor in our model and is also likely to be the parameter that is most sensitive to resolution effects, we only study the effect of beam size on the simplified model shown in equation 15, which has $g$ as its sole parameter. We begin by investigating the effect of beam size on the values of $g$. For each contour selected from a Gaussian-filtered map, we first determine the Gini coefficient $g$ from the convolved $\Sigma$ map, then calculate the intrinsic Gini coefficient $g_0$ from the same area, but with $\Sigma$ values from the original (un-smoothed) map. We show the distribution of $g$ and $g_0$ values as a function of beam size in Figure 8; note that $g_0$ depends on beam size, because, although the map used to compute $g_0$ does not depend on resolution, the shapes of the contours do, since we are matching the contour shapes determined from the smoothed map. For $w < L/1000$, we see that $g \approx g_0$, and we recover the highly-resolved limit that we have considered thus far. Larger $w/L$ ratios lead to a divergence between $g$ and $g_0$. Therefore, the values of $k_g$ and $b_g$ in equation 15 need to be corrected for the beam size.

To study how $k_g$ and $b_g$ change with $w/L$, we collect contour properties from the maps with the same beam size and then perform linear regressions with only $g$ for each value of $w/L$. We show our best fits for $k_g$ and $b_g$ as a function of beam size in the top and bottom panels of Figure 9, respectively. We also show polynomial fits (3rd order for $k_g$, 2nd order for $b_g$) to the results, which capture the variation with high accuracy:

$$k_{g,p} = 2.6 \times 10^6 \left( \frac{w}{L} \right)^3 - 3.7 \times 10^4 \left( \frac{w}{L} \right)^2 - 47 \left( \frac{w}{L} \right) + 4.6,$$  

(16)

$$b_{g,p} = -4.9 \times 10^3 \left( \frac{w}{L} \right)^2 + 1.4 \times 10^2 \left( \frac{w}{L} \right) - 1.1.$$  

(17)

These fits allow us to predict the effective volume density accounting for beam size effects:

$$\rho_p = 10^{k_{g,p} + b_{g,p}} \rho_{\text{sph}},$$  

(18)

where $k_{g,p}$ and $b_{g,p}$ are determined by equation 16 and equation 17. The distributions of $Q_{\text{sph}}$ and $Q_p$ resulting from this procedure are shown in Figure 10. This plot reveals several interesting conclusions. First, $Q_{\text{sph}}$ is centred around 0 for highly-resolved observations ($w/L \lesssim 0.001$, i.e., ~1000 resolution elements across the molecular cloud), but drops to negative values for lower resolution. This means that $\rho_{\text{sph}}$ calculated from poorly-resolved observations will tend to underestimate $\rho_{\text{eff}}$, which leads to an overestimate of $\epsilon_{\text{eff}}$. This offset in $\rho_{\text{eff}}$ can be as large as $-0.18$ dex when $w/L = 0.01$. 

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**Figure 7.** The Gaussian-filtered column density map of simulation lo, projected along the $x$-axis. The size of the Gaussian kernel applied on this map is shown as the pink circle in the right-upper corner. Its radius is $w = L/100$ (note that this is the Gaussian sigma, not the FWHM). The white circles are the projected positions of sink particles. The contours shown represent level 6 ($\Sigma = 0.17 \text{ g/cm}^2$) and level 11 ($\Sigma = 0.32 \text{ g/cm}^2$) of the 30 column density levels determined from the map.

**Figure 8.** The distributions of Gini coefficients computed on the beam-smoothed maps, $g$, and using the same contours applied to the original (un-smoothed) map, $g_0$, as a function of smoothing kernel dispersion $w/L$. The orange band is for $g$ and the blue band is for $g_0$. The upper and lower limits of each band are the 84th and 16th percentiles, while the middle dot points indicate the 50th percentiles. The dashed line is the value of $k_{\text{sph}} = 0.2$ for a surface density map projected by a uniform spherical cloud.

**Figure 9.** Top panel: best-fit coefficient $k_g$ as a function of beam size $w/L$. Bottom panel: same as top panel, but for the intercept $b_g$. In both panels, blue points indicate the numerical results, and orange lines indicate the polynomial fits given by equation 16 and equation 17, respectively.

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**Figure 10.** This plot reveals several interesting conclusions. First, $Q_{\text{sph}}$ is centred around 0 for highly-resolved observations ($w/L \lesssim 0.001$, i.e., ~1000 resolution elements across the molecular cloud), but drops to negative values for lower resolution. This means that $\rho_{\text{sph}}$ calculated from poorly-resolved observations will tend to underestimate $\rho_{\text{eff}}$, which leads to an overestimate of $\epsilon_{\text{eff}}$. This offset in $\rho_{\text{eff}}$ can be as large as $-0.18$ dex when $w/L = 0.01$. 

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**Figure 11.** This plot reveals several interesting conclusions. First, $Q_{\text{sph}}$ is centred around 0 for highly-resolved observations ($w/L \lesssim 0.001$, i.e., ~1000 resolution elements across the molecular cloud), but drops to negative values for lower resolution. This means that $\rho_{\text{sph}}$ calculated from poorly-resolved observations will tend to underestimate $\rho_{\text{eff}}$, which leads to an overestimate of $\epsilon_{\text{eff}}$. This offset in $\rho_{\text{eff}}$ can be as large as $-0.18$ dex when $w/L = 0.01$. 

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**Figure 12.** This plot reveals several interesting conclusions. First, $Q_{\text{sph}}$ is centred around 0 for highly-resolved observations ($w/L \lesssim 0.001$, i.e., ~1000 resolution elements across the molecular cloud), but drops to negative values for lower resolution. This means that $\rho_{\text{sph}}$ calculated from poorly-resolved observations will tend to underestimate $\rho_{\text{eff}}$, which leads to an overestimate of $\epsilon_{\text{eff}}$. This offset in $\rho_{\text{eff}}$ can be as large as $-0.18$ dex when $w/L = 0.01$. 

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**Figure 13.** This plot reveals several interesting conclusions. First, $Q_{\text{sph}}$ is centred around 0 for highly-resolved observations ($w/L \lesssim 0.001$, i.e., ~1000 resolution elements across the molecular cloud), but drops to negative values for lower resolution. This means that $\rho_{\text{sph}}$ calculated from poorly-resolved observations will tend to underestimate $\rho_{\text{eff}}$, which leads to an overestimate of $\epsilon_{\text{eff}}$. This offset in $\rho_{\text{eff}}$ can be as large as $-0.18$ dex when $w/L = 0.01$. 

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**Figure 14.** This plot reveals several interesting conclusions. First, $Q_{\text{sph}}$ is centred around 0 for highly-resolved observations ($w/L \lesssim 0.001$, i.e., ~1000 resolution elements across the molecular cloud), but drops to negative values for lower resolution. This means that $\rho_{\text{sph}}$ calculated from poorly-resolved observations will tend to underestimate $\rho_{\text{eff}}$, which leads to an overestimate of $\epsilon_{\text{eff}}$. This offset in $\rho_{\text{eff}}$ can be as large as $-0.18$ dex when $w/L = 0.01$. 

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**Figure 15.** This plot reveals several interesting conclusions. First, $Q_{\text{sph}}$ is centred around 0 for highly-resolved observations ($w/L \lesssim 0.001$, i.e., ~1000 resolution elements across the molecular cloud), but drops to negative values for lower resolution. This means that $\rho_{\text{sph}}$ calculated from poorly-resolved observations will tend to underestimate $\rho_{\text{eff}}$, which leads to an overestimate of $\epsilon_{\text{eff}}$. This offset in $\rho_{\text{eff}}$ can be as large as $-0.18$ dex when $w/L = 0.01$.
corresponding to a systematic overestimate of $\epsilon_{\text{ff}}$ by $\approx 0.1$ dex. Our predictive model corrects this systematic error, so $\log Q_{\text{sph}}$ is centred around 0, independent of beam size, with a maximum offset of only 0.04 dex. The predictive model also continues to reduce the dispersion in $\rho_{\text{eff}}$ estimates, though the improvement $\Delta \sigma$ decreases from 0.21 dex at high resolution to 0.12 dex at the coarsest resolution we consider. This degradation in performance is not surprising, since we have access to less information about the internal density structure of objects in the coarser observations. In summary, our correction model, equation 18, can both eliminate the resolution-dependent offset between $\rho_{\text{sph}}$ and $\rho_{\text{eff}}$ and reduce the uncertainty of $\rho_{\text{sph}}$, which can greatly enhance the accuracy of $\epsilon_{\text{ff}}$ measurements.

### 4.3 Implications for previous measurements of $\epsilon_{\text{ff}}$, and for star formation theories

As shown in Figure 10, for $w/L \geq 0.001$, $\rho_{\text{sph}}$ underestimates $\rho_{\text{eff}}$, which leads to a systematic overestimate of $\epsilon_{\text{ff}}$: a simple linear fit to our results gives

$$\Delta \epsilon_{\text{ff}} = -0.5 \log Q_{\text{sph},50} = 9.0 \frac{w}{L} \text{ dex},$$

where $w$ is the resolution and $L$ is the map size. To examine the possible $\Delta \epsilon_{\text{ff}}$ caused by beam size effects in observations, we take the example of the $\epsilon_{\text{ff}}$ study by Ochsendorf et al. (2017). They use the Magellanic Mopra Assessment (MAGMA) DR3 (Wong et al. 2011) CO intensity map to determine molecular cloud mass in the Large Magellanic Cloud (LMC), which has a beam size of 45'' FWHM and a map size of 3.6 deg$^2$. Inserting these factors into equation 19 predicts $\Delta \epsilon_{\text{ff}} = 0.03$ dex, which is a relatively small offset, and significantly smaller than the scatter determined by Ochsendorf et al. (2017) as $\sigma_{\epsilon_{\text{ff}}} \approx 0.4$ dex. This result suggests that the possible overestimation of $\epsilon_{\text{ff}}$ may not be significant in observations. This result, however, needs further investigation since equation 19 is fitted with the fixed simulation domain size $L$, which is not the exact equivalent of the observed map size in a real galaxy. We discuss this issue further in Section 4.4. Nonetheless, this result suggests that the bias in $\epsilon_{\text{ff}}$ measurements due to finite resolution is not a severe effect.

However, it is not only the mean value of $\epsilon_{\text{ff}}$ that is crucial for theories of star formation. Its spread, $\sigma_{\epsilon_{\text{ff}}}$, is also important, because theoretical models predict widely differing values of $\sigma_{\epsilon_{\text{ff}}}$. For example, Lee et al. (2016) calculate $\sigma_{\epsilon_{\text{ff}}}$ for different theoretical models, predicting values of 0.24 dex for the turbulence-regulated model of Krumholz & McKee (2005) and 0.12 or 0.13 dex for the multi-free-fall model of Hennebelle & Chabrier (2011), depending on the choice of parameters. Models in which $\epsilon_{\text{ff}}$ increases with time as a cloud evolves give larger dispersions: $\sigma_{\epsilon_{\text{ff}}} = 0.54$ dex for $\epsilon_{\text{ff}} \propto t$ (Murray & Chang 2015; Lee et al. 2015), and 0.9 dex for $\epsilon_{\text{ff}} \propto t^2$ (Feldmann & Gnedin 2011). In observations of Milky Way molecular clouds that use the spherical approximation to determine $\epsilon_{\text{ff}}$, for example, Lada et al. 2013; Evans et al. 2014; Heyer et al. 2016, $\sigma_{\epsilon_{\text{ff}}}$ is estimated to be about 0.35 dex, which is significantly larger than the spread predicted by the first two models, and much smaller than the value expected from the time-dependent models.

Section 3.1 suggests a somewhat different interpretation: there we show that $\rho_{\text{sph}}$ typically differs from $\rho_{\text{eff}}$ by $\sigma_{\text{sph}} \approx 0.45$ dex, so even if $\epsilon_{\text{ff}}$ were perfectly constant in reality, a measurement of it that relies on the spherical assumption would be expected to show a dispersion $\sigma_{\epsilon_{\text{ff}},\text{sph}} \approx 0.23$ dex. Conversely, the intrinsic scatter in $\epsilon_{\text{ff}}$ suggested by an observed dispersion of 0.35 dex is $\sigma_{\epsilon_{\text{ff}},\text{intrinsic}} \approx \sqrt{0.35^2 - 0.23^2} = 0.26$ dex. This result directly casts doubt on the star formation models predicting larger $\epsilon_{\text{ff}}$ scatter. It suggests that a significant part of the observed scatter is not reflective of true scatter in $\epsilon_{\text{ff}}$, but instead represents observational error induced by reliance on the spherical assumption. This conclusion is consistent with the analysis of Krumholz & McKee (2020), who argue based on statistical modelling of star clusters and pre-cluster gas clumps that the intrinsic spread in $\epsilon_{\text{ff}}$ must be substantially smaller than the observed spread.

### 4.4 Future work

Although our predictive model has proven its ability to reduce the uncertainty of effective volume density estimates, there is still much room for future improvement. The first step would be to enlarge the sample with data from different simulations. Although the C18 simulations capture many of the physical processes and conditions in dense, star-forming molecular clouds, and span a very wide range of physical parameters (magnetic field strength, turbulent driving), they still have several limitations. For example, they apply purely solenoidal turbulent driving, whereas in reality both solenoidal modes from galactic differential rotation and compressive modes from stellar feedback may be present (Federrath 2018a,b). Another limitation is from their radiative transfer methods. They assume the gas and the dust share the same temperature. This assumption of strong coupling is valid at densities above $\sim 10^4 - 10^5$ cm$^{-3}$ (Goldsmith 2001), but may fail for lower density, non-self-gravitating regions, which leads the simulations to overstate the dust cooling rate for the gas. If we were to extend our analysis to other simulations without these limitations, we might extend the range of our contour sample and obtain better fits.

Another potential area of improvement is the fitting method. Our current MLF method is justified by its high $R^2$ results, but the resulting model is highly dominated by $g$. Moreover, the variables used in the MLF may not be completely independent of each other. A contour with small $R_{\text{eff}}$, for example, is more likely to have large $\Sigma$ because we are focusing on the centre of a molecular cloud. A linear relation, in this case, may not be the ideal form, and we should explore the possibility of other forms of correlations. If we were able to enlarge
the sample size with more simulations, one possible approach would be to utilize machine learning to discover the underlying relations.

In Section 4.2 we use the ratio between the beam size and map size $w/L$ for analysing the effects of beam-smearing. Expressing the results in terms of $w/L$ has the advantage that it makes the results dimensionless. However, the simulated cloud size is actually infinite because of the periodic boundary condition applied in the C18 simulations, while $L$ is only the simulation domain size and should neither be seen as the equivalent of a molecular cloud size nor as a projection map size in observations. Real molecular clouds have edges, and our simulations do not. Since this problem originates from the simulations themselves, we probably cannot overcome it using the C18 data. Instead, a better approach would be to start from galactic-scale simulations, form molecular clouds self-consistent within them, and continue zooming in until we reach the dense clump scale often used in $\sigma_{\text{eff}}$ estimates. This would provide a sample of simulated molecular clouds with well-defined physical sizes, from which we could derive relations for beam-size effects more comparable to observations.

5 CONCLUSION

This work aims at obtaining precise measurements of the star formation efficiency of molecular clouds. Making these measurements requires that we estimate the volume densities of gas clouds seen only in projection; these estimates are a major source of error, and reducing them is the primary goal of this work. We use a suite of simulations of star formation from Cunningham et al. (2018) to investigate the nature of this error. We first evaluate the effect of assuming that the clouds we see are spherical and uniform density, the most common approach in the current literature. Then we develop a numerical model that can predict the effective volume density of a projected 2D contour from its observable properties substantially more accurately than the simple spherical assumption. We build this model with multiple linear fitting, and the high coefficient of determination we obtain ($R^2 \sim 0.76$) demonstrates that this produces reliable results.

We find that the volume density determined from the spherical assumption has a significant scatter, $\sigma_{\text{sph}} = 0.45$ dex, compared to the true, free-fall time weighted mean density, which is the quantity of interest for measurements of the star formation efficiency. Considering this effect, the scatter in the star formation efficiency typically found in recent studies relying on the spherical assumption, $\sigma_{\text{eff}} \sim 0.35$ dex, likely represents a true, intrinsic scatter in the star formation efficiency of no more than 0.26 dex, imposing strong constraints on theoretical models.

By comparison, when we apply our linear model, using all the observable parameters we tested, we reduce the uncertainty of the mean density by as much as $\Delta \sigma = 0.26$ dex. We also evaluate the influence of individual parameters in our predictive model, and suggest physical explanations of their significance and relative predictive power. In cases where we observe only the mass, area, column density, and the Gini coefficient of a target cloud, a simplified model can still decrease the uncertainty by $\Delta \sigma = 0.22$ dex. This improvement is sufficient to roughly halve the uncertainties of recent star formation efficiency measurements, and thus is very substantial. In addition, we investigate the effect of the telescope beam size on our simplified model and provide a corrected version to minimize this effect.

Despite its good performance, this model still has much room for future development. We can extend its applicable range by including more simulations spanning a larger variety of physical conditions.

Rebuilding the model with machine learning may also enhance its capabilities.

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DATA AVAILABILITY

The simulation data underlying this article will be shared upon reasonable request to the corresponding author.

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