Particle dynamics near Kerr-MOG black hole

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Abstract This paper explores the dynamics of both neutral and charged particles orbiting near a rotating black hole in scalar–tensor–vector gravity. We study the conditions for the particle to escape at the innermost stable circular orbit. We investigate the stability of orbits through the effective potential and Lyapunov exponent in the presence of a magnetic field. The effective force acting on particle is also discussed. We also study the center of mass energy of particle collision near the horizon of this black hole. Finally, we compare our results with the particle motion around Schwarzschild, Kerr and Schwarzschild-MOG black holes. It is concluded that the external magnetic field, spin parameter and dimensionless parameter of the theory have strong effects on the particle dynamics in modified gravity.

1 Introduction

The mysteries of the universe have always been an interesting topic for physicists. Many cosmological observations indicate that the universe is facing accelerated expansion, which is believed to be due to the existence of a mysterious form of energy named dark energy. The matter which cannot be seen directly but can be observed by its gravitational effects on visible matter is known as dark matter (DM). This does not interact with the electromagnetic force and light. Modified gravity theories help to uncover the enigmatic nature of dark energy and DM. These theories are constructed by modifying the matter or gravitational part of the Einstein–Hilbert action.

Modified theories with additional fields (scalar or vector) like scalar–tensor theories are a generalization of the tensor theory (general relativity). In Brans–Dicke theory (an example of scalar–tensor theories), the gravitational field is obtained by a tensor field $R$ and a massless scalar field $\phi$. The vector–tensor theories are formulated by adding a dynamical vector field coupled to gravity in the Einstein–Hilbert action. Moffat \cite{1} formulated a modified theory of gravity (MOG) termed scalar–tensor–vector gravity (STVG), which acts as an alternative of DM. This theory introduces new fields in general relativity which makes the gravitational field stronger. Its action consists of the usual Einstein–Hilbert term associated with the metric tensor $g_{\mu\nu}$, a massive vector field $\phi_{\mu}$ and three scalar fields which represent the running values of the gravitational constant $G$, coupling constant $\omega$ (determining the coupling strength between matter and vector field) and the vector field’s mass $\mu$ (adjusting the coupling range). The scalar field $G = G_N(1 + \alpha)$ is the strength of the gravitational attraction, where $G_N$ is Newton’s gravitational constant and $\alpha$ is a dimensionless parameter of the theory. The vector field produces a repulsive gravitational force which is related to a fifth force charge proportional to mass–energy. This theory helps to explain the solar system, rotational curves of galaxies, motion of galaxy clusters, gravitational lensing of galaxy and cluster of galaxies without DM \cite{2}.

Moffat and Toth \cite{3,4} studied static spherically symmetric vacuum solutions for a flat FRW model and also discussed the origin of inertia in STVG theory. Deng et al. \cite{5} discussed the modifications of STVG and constraints on its parameters. Mishra and Singh \cite{6} studied the galaxy rotational curves and compared the form of the acceleration law in fourth order gravity with STVG as well as modified Newtonian dynamics. Moffat and Rahvar \cite{7} used the weak-field approximation to test the dynamics of a cluster of galaxies and found that this theory is consistent with the observational data from the solar system to megaparsec scales. Roshan \cite{8} discussed some cosmological solutions for a flat FRW model using the Noether symmetry approach. Sharif and Yousaf \cite{9} used this approach to find anisotropic exact solutions of locally rotationally symmetric Bianchi type-I model. Mureika et al. \cite{10} analyzed the thermodynamics of Schwarzschild-MOG
as well as Kerr-MOG black hole (BH) and found a change in the entropy area law with the increase of the parameter $\alpha$.

The dynamics of particles (neutral or charged, massive or massless) around a BH is one of the most interesting problems in BH astrophysics. This plays a key role in understanding the geometrical structure of spacetime. New observational evidence for BHs provides new motivations for the investigation of general relativistic dynamics of particles and electromagnetic fields in the vicinity of BHs. Astronomical observations over the last decade indicate the existence of stellar-mass and supermassive BHs in some X-ray binary systems and in galactic centers.

Hussain and Jamil [11] studied timelike geodesics around Schwarzschild-MOG BH and found that the stability of orbits increases due to the presence of a vector field in STVG theory. Pradhan [12] explored circular geodesics near a Kerr–Newman–Taub–NUT spacetime and found that the energy gain is maximum for zero NUT parameter and also for a maximum spin value. Babar et al. [13] studied the motion of charged particles in the vicinity of a weakly magnetized naked singularity and explored the escape velocity of particles orbiting in the innermost stable circular orbits (ISCOs). Soroushfar et al. [14] discussed the geodesics around a charged rotating BH in $f(R)$ gravity and found that the shape of an orbit depends on the value of energy, angular momentum, charge as well as the cosmological constant. Sharif and Iftikhar [15] studied this phenomenon around a higher dimensional BH and found that higher dimensions have strong effects on particle motion.

Bardeen [16] investigated the characteristics of a Kerr BH and its circular orbits. Aliev and Ozdemir [17] discussed charged particle motion around a rotating BH and found that a magnetic field has a strong effect, enlarging the region of stability close to the event horizon. Frolov and Stejkovic [18] explored particle motion around a five-dimensional rotating BH and found that there do not exist SCOs in equatorial planes. Aliev and Gumrukcuoglu [19] studied charged rotating BHs on a 3-brane and found that negative tidal charge increases the horizon radius as well as the radii of the photosphere.

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where $\xi_{(i)} = (1, 0, 0, 0)$ and $\xi_{(0)} = (0, 0, 0, 1)$. When $a = 0$, the metric (1) reduces to the Kerr metric, further $a = \alpha = 0$ leads to the Schwarzschild metric and for $a = 0$, we obtain Schwarzschild-MOG metric. The Kerr-MOG metric (1) is singular if $\rho$ or $\Delta$ vanishes. The curvature and coordinate singularities correspond to $\rho = 0$ and $\Delta = 0$, respectively. The horizons of (1) can be obtained by

$$r_{\pm} = GM \pm \sqrt{G^2 M^2 - a^2 - \alpha G_N G M^2},$$

where the ± sign corresponds to the event and Cauchy horizons, respectively. The ergosphere can be obtained by solving $g_{tt} = 0$

$$r_{es} = GM \pm \sqrt{G^2 M^2 - a^2 - \alpha G_N G M^2}.$$  

For $\theta = 0, \pi$, both ergosphere and event horizon coincide. The extremal condition is located at $G^2 M^2 = a^2 + \alpha G_N G M^2$. The corresponding angular velocity is

$$\Omega_H = \frac{\alpha}{2G^2 M^2 - \alpha G N G M^2 + 2GM \sqrt{G^2 M^2 - a^2 - \alpha G_N G M^2}}.$$  

We consider the equatorial plane to find the motion of test particle, i.e., $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$.

The motion of a neutral particle can be illustrated by the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\sigma\eta} \dot{x}^\sigma \dot{x}^\eta,$$

where $\dot{x}^\sigma = u^\sigma = \frac{dx^\sigma}{d\tau}$ is the four velocity of particle and $\tau$ is the proper time. For a Kerr-MOG BH, the Lagrangian becomes

$$2\mathcal{L} = -\left(\frac{r^2 - 2GMr + \alpha G_N G M^2}{r^2}\right) \dot{t}^2$$

$$-2a \left(\frac{2GMr - \alpha G_N G M^2}{r^2}\right) \dot{t} \dot{\phi}$$

$$+ \frac{r^2}{\Delta} \dot{r}^2 + \left[r^2 (r^2 + a^2) + a^2 \left(2GMr - \alpha G_N G M^2\right)\right] \dot{\phi}^2.$$  

It is clear from Eq. (2) that $t$ and $\phi$ are cyclic coordinates. Corresponding to these cyclic coordinates there are two constants of motion, i.e., the total energy $E$ and azimuthal angular momentum $L_z$, which are conserved along geodesics. The generalized momenta are

$$- \dot{t} = - \left(\frac{r^2 - 2GMr + \alpha G_N G M^2}{r^2}\right) t$$

$$- a \left(\frac{2GMr - \alpha G_N G M^2}{r^2}\right) \phi = E,$$

$$\dot{r} = \frac{r^2 - 2GMr + \alpha G_N G M^2}{r^2},$$

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The total angular momentum is given as

$$L^2 = \left(\frac{r^2 \dot{\phi}}{r^2 \dot{\phi}}\right)^2 + \left(\frac{r^2 \dot{\phi} \sin \theta}{r^2 \dot{\phi} \sin \theta}\right)^2,$$

where $v^2 = -r \dot{\theta}, \dot{\theta}$ is the initial angular velocity of the particle. Using the value of $\dot{\phi}$ from Eq. (6), we have

$$L^2 = \left[\left(r^2 - 2GMr + \alpha G_N G M^2\right) L_z + a \left(2GMr - \alpha G_N G M^2\right) E\right]^2$$

$$+ \frac{r^2}{\Delta} \dot{r}^2.$$  

Using the normalization condition, $g_{\sigma \eta} u^\sigma u^\eta = -1$, it follows that

$$\dot{r}^2 = \dot{E}^2 + \frac{1}{r^2} \left(a^2 \dot{E}^2 - L_z^2\right) + \frac{1}{r^2} \left(2GMr - \alpha G_N G M^2\right)$$

$$\left(aE - L_z\right)^2 - \frac{\Delta}{r^2}.$$  

Equations (5)–(8) are useful for discussing various features of particle motion in the case of (1). From Eq. (8), we obtain

$$\frac{1}{2} \left(E^2 - 1\right) = \frac{1}{2} \dot{r}^2 + U_{eff}(r, E, L_z),$$

where

$$U_{eff}(r, E, L_z) = \frac{GM}{r^3} \left(\dot{E}^2 - 1\right) + \frac{\alpha G_N G M^2}{2r^2} \left(aE - L_z\right)^2.$$  

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The maximum and minimum values of the effective potential $U_{\text{eff}}$ determine the unstable and stable circular orbits, respectively. The radius of the innermost stable circular orbit ($r_0$) can be found by solving $\frac{dU_{\text{eff}}}{dr} = 0$. We have solved $\frac{dU_{\text{eff}}}{dr} = 0$ using Mathematica 8.0 and found three roots of $r$. We have ignored the two imaginary values of $r$ and the remaining root is for the radius of the ISCO ($r_0$). The energy and azimuthal angular momentum corresponding to $r_0$ are

$$E_0 = \frac{1}{r_0} \left[ \frac{r_0^2 - 2GMr_0 + aG_NM^2 \mp a\sqrt{GM(r_0 - M\alpha G_N)}}{\sqrt{r_0^2 - 3GMr_0 + 2aG_NM^2 \mp 2a\sqrt{GM(r_0 - M\alpha G_N)}}} \right],$$

(9)

$$L_{z0} = \frac{1}{r_0} \left[ \frac{\mp \sqrt{GM(r_0 - M\alpha G_N)}(a^2}{r_0^2 + 2a\sqrt{GM(r_0 - M\alpha G_N)} } - aG_NM^2 \alpha \left[ \frac{r_0^2 - 3GMr_0 + 2aG_NM^2}{2a\sqrt{GM(r_0 - M\alpha G_N)}} \right]^{\frac{1}{2}} \right],$$

(10)

where $r_0$ is the radius of the ISCO and the ± signs correspond to the counter-rotating and co-rotating orbits, respectively. For $\alpha = 0$, Eqs. (9) and (10) reduce to case of the Kerr BH [28,29].

Consider a particle orbiting in ISCO colliding with another particle which is at rest. After collision, there are three cases (depending upon the collision process) for the particle motion, i.e., either it is captured by BH, or bounded near a BH or escaping to infinity. If there is a small change in energy and angular momentum, then the particle’s orbit will slightly be perturbed and the particle remains bounded. But for a large change, it may be captured by BH or escaping to infinity. After collision, particle will be in a new plane with respect to the original equatorial plane. Thus the particle would have a new energy and azimuthal angular momentum. For simplicity, we assume that, after collision, the initial radial velocity and azimuthal angular momentum do not change and the particle attains an escape velocity $v_{\text{esc}}$ orthogonal to the equatorial plane. The new energy and azimuthal angular momentum of particle are

$$L_{\text{new}}^2 = r_0^2 v_{\perp}^2 + \left( r_0^2 - 2GMr_0 + aG_NM^2 \right) L_{z0} + a(2GMr_0 - aG_NM^2) E_0^2,$$

$$E_{\text{new}}^2 = 1 - \frac{2GM}{r_0} + \frac{\left[ L_{z0}^2 - a^2 (E_0^2 - 1) + aG_NM^2 \right]}{r_0^2} - \frac{2GM}{r_0^3} (aE_0 - L_{z0}) - \frac{aG_NM^2}{r_0^4} (aE_0 - L_{z0})^2.$$

(11)

After collision, the particle attains greater energy and angular momentum as compared to before collision. We observe from the above equation that $E_{\text{new}} \to 1$ as $r_0 \to \infty$. Thus for unbounded motion, the particle requires $E_{\text{new}} \geq 1$ to escape, whereas the particle cannot escape for $E_{\text{new}} < 1$.

2.2 Charged case

The theoretical and experimental evidence indicates that a magnetic field must be present in the vicinity of BHs. It arises due to plasma in the surroundings of a BH [30] and plays an important role in the formation, structure and evolution of planets, stars, galaxies and possibly the entire universe [31]. The magnetic field has strong effects around the event horizon but it does not change the geometry of BH; rather the motion of charged particles is affected [32,33]. We assume that a particle has an electric charge and its motion is affected by the magnetic field in the BH exterior. Since the Kerr-MOG BH is non-vacuum, we follow [34] to calculate the four-vector potential given as

$$A^\alpha = \left[ \frac{-Qr}{r^2 - 2f} + \frac{\tilde{B}}{2} \left( 1 + \frac{2f_2}{r^2} \right) \right] \delta^\alpha_{(t)} + \left( \frac{\tilde{B}}{2} \left( 1 + \frac{2f_2}{r^2} \right) \right) \delta^\alpha_{(\phi)},$$

where $f = f_1 r + f_2$ with $f_1, f_2$ are constants and $\tilde{B}$ is the magnetic field. For simplicity, we take $Q = 0$ and also for the equatorial plane $\theta = \pi/2$. Thus the above equation becomes

$$A^\alpha = \frac{\tilde{B}}{2} \left( 1 + \frac{f_2}{r^2} \right) \delta^\alpha_{(t)} + \frac{\tilde{B}}{2} \left( 1 + \frac{2f_2}{r^2} \right) \delta^\alpha_{(\phi)}.$$

For (1), the above equation takes the form

$$A^\alpha = \frac{\tilde{B}}{2} \left( 1 + \frac{f_2}{r^2} \right) \delta^\alpha_{(t)} + \frac{\tilde{B}}{2} \left( 1 + \frac{2f_2}{r^2} \right) \delta^\alpha_{(\phi)}.$$

The magnetic field for an observer having four velocity $u_\eta$ is defined as

$$\tilde{B}^\alpha = -\frac{1}{2} e^{\eta \gamma \delta} F_{\gamma \delta} u_\eta,$$

where $e^{\eta \gamma \delta} = e^{\eta \gamma \delta} = \sqrt{-g}$ and $e^{\eta \gamma \delta}$ is the Levi-Civita symbol, and $g = det(g_{\sigma \eta})$ and $\varepsilon_{0123} = 1$. The Maxwell field tensor is $F_{\sigma \eta} = A_{\eta \sigma} - A_{\sigma \eta}$.

The Lagrangian of a particle having electric charge $q$ and mass $m$ is

$$\mathcal{L} = \frac{1}{2} g_{\sigma \eta} u^\sigma u^\eta + \frac{q}{m} A_\sigma u^\sigma.$$
The generalized four momentum is given by

\[ p_\sigma = m u_\sigma + q A_\sigma. \]

In the presence of a magnetic field, the constants of motion are

\[
\dot{r} = \frac{1}{r^3} [r(8a^3 B G^2 M^2 - 2a G(2a^2 B - a E + L)M r)
+ a^2 (-2a B + E) 
\times r^2 + (2a B + E)r^4 + a a G_N G M^2
\times (-a E r + a^2 B (-4 G M + 3 r)
+ r(L + Br(2 G M + r) - Br a G_N G M^2)], \tag{12}
\]

\[
\dot{\phi} = \frac{1}{r^3} [r(2a E G M r + a^2 B (8 G^2 M^2 - 8 G M r + r^2)
- (2 G M r - r)r \times (L + Br^2))
+ a G_N G M^2 (-a E r + a^2 B (-4 G M + 3 r)
+ r(L + 2 B \times G M r - Br a G_N G M^2)], \tag{13}
\]

where \( B = \frac{q B}{2m} \). Using the normalization condition, we obtain

\[
\dot{r}^2 = [r(a^2 B (4 G M - 3 r)(8 G^2 M^2 - 6 G M r - r^2)
- 4a^2 B E r (-4 G^2 M^2 + 2 G M r - r^2)
+ 2 G M r + r^2 - 4a E r^2 (G M + Br^3)
+ a^2 r (-2 BL (8 G^2 M^2 - 8 G r) \times M r + r^3) + 2 G M r (-6 G^2 M^2 + 4 G M r + r^2)
+ r(-r + E^2 (2 G M
+ r)) + r^2 (L^2 + B^2 r^5) - 2 G M (r^2 + (L + Br^2) r^3))
- a G_N G M^2 (16 a^4 B^2 G^2 M^2
- 8 a^2 B G (3 a^2 B - a E + L)
\times M r + ((3 a^2 B - a E + L)^2 - 16 a^2 B^2 G^2 M^2) r^2)
+ 4 B G (2 a^2 B - a \times E + L) M r^3
+(1 + 2 a B (2 a B - E) r^4 + 4 B G M r^5
- B^2 r^6 + Br \times a G_N G M^2 (2 a B (8 G M - 6 r)
+ 2a E r - r(2 L + B r (2 G M + r))
+ Br a G_N G M^2)], \tag{14}
\]

which can be written as

\[
\frac{1}{2} \left( E^2 - 1 \right) = \frac{1}{2} \dot{r}^2 + U_{\text{eff}}(r, E, L, \phi, B),
\]

where

\[
U_{\text{eff}}(r, E, L, \phi, B) = -\frac{1}{2a^6}[r(a^4 B^2 (4 G M - 3 r)
\times (8 G^2 M^2 - 6 G M r - r^2) - 4a^3
\times B E r (-4 G^2 M^2 + 2 G M r + r^2)
- 4a E r^2 (G M + Br^3)
+ a^2 r (-2 B L (8 G^2 M^2 - 8 G M r + r^2)
+ 2 B^2 r^3 (-6 G^2 M^2 + 4 G M r + r^2)
+ r(-r + E^2 (2 G M + r))
+ r^2 (-L^2 r - 2 \times B L r^3)
- B^2 r^5 + 2 G M (r^2 + (L + Br^2))]
- a G_N G M^2 
\times (16 a^4 B^2 G^2 M^2 - 8 a^2 B G (3 a^2 B - a E + L)
\times M r + ((3 a^2 B - a E + L)^2 - 16 a^2 B^2 G^2 M^2) r^2)
+ 4 B G (2 a^2 B - a E + L)
\times M r^3 + (1 + 2 a B (2 a B - E) r^4
+ 4 B G M r^5 - B^2 r^6 + B x a G_N G M^2 (2 a B (8 G M - 6 r)
+ 2a E r - r(2 L + B r (2 G M + r))
+ Br a G_N G M^2)], \tag{15}
\]

Equations (12)–(15) are invariant under the transformations \( \phi \rightarrow -\phi \), \( L_z \rightarrow -L_z \), \( B \rightarrow -B \).

3 Escape velocity and effective potential

In this section, we discuss the properties of the escape velocity and the effective potential of a particle when \( G = 1 \), \( M = 1 \), \( G_N = 1 \), \( E = 1 \). Figure 1 shows the escape velocity for a particle moving around the Kerr-MOG BH in the presence of a magnetic field. In the upper panel, the left graph indicates that particles with large angular momentum have less possibilities to escape as compared to those having small angular momentum. The right graph shows that the escape velocity increases with the increase of a magnetic field. Particles attain more energy due to large values of a magnetic field and can easily escape. In the lower panel, the left graph provides a comparison for escape velocity of the Kerr-MOG BH with Schwarzschild and Kerr BHs. This shows that the Kerr-MOG BH has higher \( v_{\text{esc}} \) as compared to the Schwarzschild and Schwarzschild-MOG BH; also \( v_{\text{esc}} \) increases with increasing value of the parameter \( \alpha \).

The effect of spin on the escape velocity is shown in the right graph. This also provides a comparison of the escape velocity of the particle around the Schwarzschild-MOG BH with Kerr-MOG BH. We see that with the increase of the spin of the BH, particles have more possibilities to escape as \( v_{\text{esc}} \) is high for large values of the spin parameter \( a \). A rotating BH (\( a \neq 0 \)) may provide a sufficient amount of energy to the particle due to which it can escape to infinity as compared to a non-rotating BH. It is also noted that particles in the vicinity of the Schwarzschild-MOG BH can escape easily as compared to the Kerr-MOG BH. We conclude that \( v_{\text{esc}} \)
becomes almost constant as the particle moves away from the BH.

Figure 2 shows the behavior of $U_{\text{eff}}$ against $r$. The stable and unstable circular orbits correspond to minimum and maximum values of the effective potential, respectively. In the upper panel, the left graph shows that initially the orbits are unstable and then become stable, stability increases with the increase of $L$. The right graph is plotted for different values of $B$. We observe that the orbits are initially stable, then become unstable and stability decreases for increasing value of $B$. In the lower panel, the left graph shows that the particle motion becomes more unstable for high values of $\alpha$. The right graph indicates that circular orbits for the Kerr-MOG BH are more unstable as compared to Kerr and Schwarzschild BH. The last graph is plotted for different values of spin parameter $a$, which indicate that the stability of circular orbits decreases with the increase of $a$. Thus the motion of the particle will be more unstable for large value of $a$. We note that circular orbits around the Schwarzschild-MOG BH are more stable as compared to the Kerr-MOG BH.

3.1 Effective force

The effective force acting on a particle provides information as regards the motion, i.e., whether it is attracted towards the BH or moving away from it [35]. We study the particle motion in the background of the Kerr-MOG BH where attractive as well as repulsive gravitational forces can be produced by STVG. Here, we find the effective force acting on particle using Eq. (15) [36]:

$$
F = \frac{-1}{2} \frac{dU_{\text{eff}}}{dr} 
$$

$$
= - \frac{GM}{2r^4} [L(3L + Br^2 - 6aE + 24a^2B) + 3a^2E^2 - 12a^3BE \\
+ 24a^4B^2 - B^2r^4 + r^2 + 4a^2B^2r^2] \\
- \frac{(aG \sqrt{GM})^2}{2r^5} [4BL - 4aBE + B^2 \\
+ 12a^2B^2] - \frac{(aG \sqrt{GM})^2(GM)}{r^6} \\
\times [BL(10a^2 - 3r^2) - B^2r^4 - 3aBEr^2 \\
+ 6a^2B^2r^2 + 10] \\
\times a^3BE - 30a^4B^2] - \frac{(aG \sqrt{GM})^3}{40a^4B^2} \\
+ \frac{(GM)^2}{r^5} [16a^2BL + 48a^4B^2 \\
- 16a^3BE + 6a^2B^2r^2] + \frac{G \sqrt{GM}^2}{2r^5} [2L(L - 2aE + 6a^2B) \\
+ r^2 + 4a^2B^2r^2 - 2aBEr^2 + 2a^2E^2 \\
- 12a^3B + 18a^4B^2] + \frac{1}{2r^3} [L(1 + 2a^2B)]
$$
We see that the first, second and third terms are attractive if
\[ L(3L + Br^2 - 6aE + 24a^2B) > 3a^2E^2 - 12a^3BE + 21a^4B^2 - B^2r^4 + r^2 + 4a^2B^2r^2, 4BL > -4aBE + B^2 + 12a^2B^2 \] and
\[ BL(10a^2 - 3r^2) > -B^2r^4 - 3aBEr^2 + 6a^2B^2r^2 + 10a^2BE - 30a^4B^2, \]
respectively. The fourth term is also attractive. The fifth, sixth and seventh terms are repulsive if
\[ 16a^2BL > 48a^4B^2 - 16a^3BE + 6a^2B^2r^2, \]
\[ 2L(L - 2aE + 6a^2B) > r^2 + 4a^2B^2r^2 - 2aBEr^2 + 2a^2E^2 - 12a^3B + 18a^4B^2 \] and
\[ L(1 + 2a^2B) > a^2 - 3a^4B^2 + 4a^3BE - a^2E^2 - B^2r^4, \]
respectively. The last three terms are also repulsive.

Figure 3 describes the behavior of the effective force as a function of \( r \). In the upper panel, the left graph shows that the effective force acting on particles is more attractive for large values of a magnetic field. The right graph shows the comparison of the effective force acting on a particle around the
Kerr-MOG BH with the Kerr and Schwarzschild BHs. We see that the effective force on a particle for the Kerr-MOG BH attains higher values than the Kerr and Schwarzschild BHs and increases for increasing values of $\alpha$. This means that repulsion in reaching the singularity for the Kerr-MOG BH is higher than that in Schwarzschild and Kerr BHs. The behavior of the effective force for different values of the spin parameter is represented in lower graph showing that the effective force acting on particles increases with the increase of the spin parameter and becomes almost constant as the particle moves away from the BH. It is also observed that the effective force on particles in the vicinity of Schwarzschild-MOG BH is small as compared to the Kerr-MOG BH.

3.2 Lyapunov exponent

The Lyapunov exponent measures the average rate of expansion or contraction of trajectories in a phase space. The positive and negative Lyapunov exponents indicate divergence and convergence between neighboring orbits [37]. Using Eq. (15), we can find the Lyapunov exponent as [38]
Figure 4 shows the graph of the Lyapunov exponent as a function of $B$. In the upper panel, the left graph indicates that the Lyapunov exponent has decreasing behavior for higher values of angular momentum representing that orbits are more unstable for small values of the angular momentum compared to large values. The right graph gives a comparison for the Kerr-MOG BH with the Kerr, Schwarzschild and Schwarzschild-MOG BHs. This shows that, for the Kerr-MOG BH, instability of circular orbits is higher than for the Kerr, Schwarzschild and Schwarzschild-MOG BHs and instability increases with the increase of $\alpha$. The behavior of the Lyapunov exponent for different values of the spin parameter is shown in the lower graph. It is noted that orbits are more unstable with large values of $a$ as compared to small values.

4 Center of mass energy

The center of mass energy of two colliding particles can be obtained by adding their masses and kinetic energies depending upon the interacting particles and gravitational field around the astrophysical object. It is interesting to discuss the particle collision as it is a naturally occurring process in the universe. In the following, we discuss the CME for neutral as well as charged particles.

4.1 Neutral case

Let us consider two neutral particles with the same rest-mass $m_0$ but different four velocities $u^\sigma_1$ and $u^\eta_2$ colliding with each other. The conserved energy and angular momentum of colliding particles are $E_1$, $E_2$, $L_1$ and $L_2$. The CME of colliding particles is defined as

$$\left(\frac{E_{cm}}{\sqrt{2m_0}}\right)^2 = 1 - g_{\sigma\eta}u^\sigma_1 u^\eta_2. \quad (16)$$

Inserting the values of $g_{\sigma\eta}, u^\sigma_1$ and $u^\eta_2$ from Eqs. (5)–(8), the CME becomes

$$\left(\frac{E_{cm}}{\sqrt{2m_0}}\right)^2 = \frac{1}{r^2\Delta} \left[ (r(r(a^2 + r(-2GM + r))) 
+ (r^3 + a^2(2GM + r))E_1E_2 
- 2aGME_1L_2 - L_1(2aGME_2) 
+ (r - 2MG)L_2 + \alpha G_N GM^2 (r^2 
- a^2E_1L_2 + L_1(aE_2 - L_2) 
+ aE_1L_2)) - \sqrt{S_1S_2}, \right) \quad (17)$$
where

\[ S_i = r^4 E_i^2 + r^2 (\alpha^2 E_i^2 - L_i^2) + (2GMr - \alpha G_N GM^2) \]

\[ \times (aE_i - L_i)^2 - \Delta \frac{r^2}{r^2}, \quad i = 1, 2. \]

For \( a = \alpha = 0 \) and \( \alpha = 0 \), Eq. (17) reduces to Schwarzschild and Kerr BHs, respectively [22].

4.2 Charged case

Here we consider particle collision in the vicinity of a magnetic field. In this case, the CME takes the following form:

\[
\left( \frac{E_{cm}}{\sqrt{2m_0}} \right)^2 = \frac{1}{r^4 \Delta} \left[ -B^2 G^3 M^6 r^2 \alpha^2 G_N^2 (-8\alpha^2 BGM + 6a^2 B) \right.
\]

\[ \times r + 2BGMr^2 + Br^3 - a(E_1 + E_2) \]

\[ + r(L_1 + L_2) + \alpha G_N GM^2 \]

\[ \times (16a^3 B^2 G^2 M^2 + 24\alpha^3 B^2 GMr) \]

\[ - 9\alpha^2 B^2 r^2 + 16a^2 B^2 G^2 M^2 r^2 \]

\[ - 8a^2 B^2 GMr^2 + r^4 - 4a^2 B^2 r^4 \]

\[ - 4B^2 GMr^5 + B^2 r^6 + (E_1 + E_2) \]

\[ \times (-4\alpha^3 BGMr + 3a^3 Br^2) + 2a BGMr^3 + a Br^4 + r(a^2 B(4GM - 3r) - 2BGMr^2 \]

\[ + aBr_2 - rL_2) - r(-32a^4 B^2 G^3 M^3 + 48a^4 B^2 G^2 M^2 r - 14a^4 B^2 G) \]

\[ \times Mr^2 - a^2 r^3 - 3a^4 B^2 r^3 \]

\[ + 12a^2 B^2 G^2 M^2 r^3 \]

\[ + 2GMr^4 - 8a^2 B^2 G^2 \]

\[ \times r^4 - r^5 - 2a^2 B^2 r^5 - 2B^2 GMr^6 \]

\[ + B^2 r^7 + (E_1 + E_2)(-8a^3 B^2 G^2 M^2) \]

\[ \times r + 4a^3 BGM^2 + 2a^3 Br^3 + 2a Br^5) \]

\[ + E_1 E_2(-a^2 r^2 - a^2 r^3 - r^5 \]

\[ - 2a^2 GMr^2 + r(Br^2(-2GM + r) + a^2 B(8G^2 M^2 - 8GMr + r^2) \]

\[ + 2a BGMr E_1 L_2 + r L_1 (Br^3(-2GM + r) + a^2 B(8G^2 M^2 - 8GMr \]

\[ \text{Fig. 5 The CME with respect to } r \text{ for } L_1 = 1 \text{ and } L_2 = 1.5. \text{ Here, the vertical lines are event horizons} \]
The CME as a function of $r$ for finite CME only finite values of the angular momentum is allowed. The behavior of the CME as a function of $r$ for different values of the spin parameter is depicted in Fig. 5. In the upper panel, the left graph gives a comparison of the CME for colliding particle near a Kerr-MOG with Schwarzschild and Kerr BHs. The collision occurring near the horizon of the Kerr-MOG BH can produce high energy as compared to Schwarzschild and Kerr BHs and increases with the increase of the parameter $\alpha$. The right graph is plotted for different values of the spin parameter. We see that the CME strongly depends on the rotation of the BH. A high energy can be achieved with the maximum spin. The CME for different values of the parameter $\alpha$ (left) and spin parameter (right) in the presence of a magnetic field is shown in the lower panel. It is noted that a maximum CME can be produced in the presence of a magnetic field as compared to its absence. The CME has decreasing behavior with the increase of radial distance $r$ and becomes almost constant away from the event horizon of BH. The CME for colliding particles does not diverge in the absence/presence of a magnetic field.

Figure 6 is depicted for different values of $L_1$ in the presence as well absence of a magnetic field. We see that the CME decreases with the increase of the angular momentum. The particle colliding with small angular momentum can produce high energy as compared to a particle with large angular momentum. Initially, the CME decreases and then becomes constant with the increase of radial distance $r$. It is also observed that the CME is finite for finite values of the angular momentum and the system attains more energy in the presence of a magnetic field than in its absence.

5 Concluding remarks

In this paper, we have studied the dynamics of particles around the Kerr-MOG BH in the absence/presence of a magnetic field. We have explored the geodesics for both neutral and charged particles. We have graphically discussed conditions for a particle to escape to infinity after its collision with another particle. The effect of magnetic field, angular momentum, parameter $\alpha$ as well as spin parameter $a$ on the motion of neutral and charged particles is also analyzed graphically. It is seen that the escape velocity increases with the increase of $a$ and magnetic field but decreases with the increase of $L$. The particles can attain more energy in the presence of a magnetic field and can escape easily. The escape
velocity also depends upon the spin of BH. There will be more possibilities to escape to infinity for large values of the spin parameter. It is found that the escape velocity of particle around a Schwarzschild BH is smaller than Kerr and Kerr-MOG BH. Particles cannot escape easily in the vicinity of the Kerr-MOG BH as compared to Kerr, Schwarzschild and Schwarzschild-MOG BHs.

We have explored the stability of circular orbits by the effective potential. It is observed that the effective potential increases for large values of a magnetic field, which indicates that the presence of a magnetic field increases the stability of particle orbits. We have compared the stability of circular orbits around the Kerr-MOG BH with the Kerr and Schwarzschild BHs, finding indications that circular orbits around the Kerr-MOG BH are more unstable than Kerr, Schwarzschild and Schwarzschild-MOG BHs. We note that a large magnetic field leads to unstable motion. The rotation of BH has strong effects on the stability of orbits, which decreases with the increase of $a$. We have also discussed the instability of circular orbits through the Lyapunov exponent as a function of the magnetic field $B$.

Finally, we have calculated the CME for two interacting particles around the Kerr-MOG BH. It is found that particle collision can produce a high energy near Kerr-MOG BH as compared to Kerr, Schwarzschild and Schwarzschild-MOG BHs. We observe that the CME increases with the increase of the spin parameter as well as parameter $\alpha$ but decreases with the increase of the angular momentum. The CME is finite for finite values of the angular momentum and can be infinite for diverging angular momentum. We conclude that the external magnetic field, parameter $\alpha$, and spin parameter affect the motion of particles in STVG. It is worth mentioning here that our work is the generalization of [11], reducing to the case of a Schwarzschild-MOG BH [11] when $a = 0$ and to a Schwarzschild BH [39] for $a = \alpha = 0$.

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