Spectral flow for an integrable staggered superspin chain

Holger Frahm and Konstantin Hobuß

Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany

E-mail: frahm@itp.uni-hannover.de

Received 28 March 2017, revised 17 May 2017
Accepted for publication 7 June 2017
Published 28 June 2017

Abstract
The flow of the low energy eigenstates of a \( U_q[sl(2|1)] \) superspin chain with alternating fundamental (3) and dual (\( \bar{3} \)) representations is studied as function of a twist angle determining the boundary conditions. The finite size spectrum is characterized in terms of scaling dimensions and quasi momenta representing the two families of commuting transfer matrices for the model which are even and odd under the interchange \( 3 \leftrightarrow \bar{3} \), respectively. Based on the extrapolation of our finite size data we find that under a variation of the boundary conditions from periodic to antiperiodic for the fermionic degrees of freedom levels from the continuous part of the spectrum flow into discrete levels and vice versa.

Keywords: integrable superspin chains, non-compact conformal field theory, spectral flow

(Integrable two-dimensional vertex models and the corresponding \( (1+1) \)-dimensional quantum spin chains have provided many examples for lattice realizations of conformal field theories (CFTs) and thereby led to important insights into the critical properties of strongly correlated systems. For lattice models with finite dimensional representations for the local degrees of freedom and Hermitian Hamiltonian the unitary CFTs appearing in the continuum limit can be identified by relating Virasoro representations to the discrete finite size spectrum of the former [1, 2].

Recently, a growing number of lattice models related to order-disorder transitions in quantum Hall systems, the anti-ferromagnetic Potts model, intersecting loops or physical properties of two-dimensional polymers, has been found whose continuum limit—although having a finite-dimensional quantum space for the local degrees of freedom on the lattice—are CFTs with a non-compact target space [3–9]. This manifests itself in the emergence of a continuous component in the spectrum of scaling dimensions in the thermodynamic limit and,
possibly, a non-analytic dependence of the effective central charge of the lattice model on the twist of boundary conditions related to discrete levels appearing in the spectrum of the lattice model when the corresponding operator in the CFT becomes normalizable.

For some of these lattice models the spectral data obtained from their exact solution have allowed to identify the CFT describing their continuum limits:

- An integrable vertex model built from the three-dimensional fundamental and dual representations of the superalgebra \( sl(2|1) \) was shown to flow to a \( SU(2|1) \) Wess–Zumino–Novikov–Witten (WZNW) model at level \( k = 1 \) [3, 10]. For a physical modular invariant partition function one is forced to consider continuous values of the \( sl(2|1) \) charge quantum number leading to continua of scaling dimensions.

- A staggered six-vertex model related to the anti-ferromagnetic Potts model [4] (which also appears in the phase diagram of a staggered superspin chain built from four-dimensional representations of \( U_q[sl(2|1)] \), see [6]) has been proven to provide a realization of the \( SL(2, \mathbb{R})/U(1) \) Euclidean black hole sigma model [11, 12]. Here the identification was based on the density of states in the continuous spectrum [13–15].

- The \( a_{\chi}^2 \) model (equivalent to the 19-vertex Izergin–Korepin model [16]) in ‘regime III’ has a non-compact continuum limit again described by the \( SL(2, \mathbb{R})/U(1) \) black hole sigma model [7]. Here the spectrum of discrete levels appearing in the lattice model with twisted boundary conditions has been found to be consistent with the predicted appearance of levels related to the principal discrete representations of \( SL(2, \mathbb{R}) \) states [17, 18]. Similarly, the scaling limit of the general \( a_{\chi}^2 \) model has been shown to be a \( SL(N)/SO(N) \) gauged WZNW model [9].

In this paper we are studying the continuum limit of the mixed \( U_q[sl(2|1)] \) superspin chain based on the three dimensional atypical representation and its dual (‘quark’ and ‘antiquark’, labeled 3 and \( \bar{3} \) in the following) [19], an anisotropic deformation of the \( sl(2|1) \) superspin chain mentioned above. In previous work on this model, and similar as in the isotropic case, the existence of an exact zero energy state and of continua of scaling dimensions have been established [5]. Here we generalize the lattice model to include a twist allowing for an adiabatic change from periodic to anti-periodic boundary conditions for the fermionic degrees of freedom. Using the Bethe ansatz solution of this model we find the exact dependence of some low lying levels on the twist angle. Translating our results into the context of the field theory describing the continuum limit of the model we find that under the spectral flow states from the continuum part of the spectrum in the Neveu–Schwarz sector are mapped onto discrete levels in the Ramond sectors, and vice versa.

The construction of an integrable mixed superspin chain with alternating three-dimensional quark and anti-quark representations of \( U_q[sl(2|1)] \) is based on four \( \mathcal{R} \)-matrices acting on tensor products \( 3 \otimes 3, 3 \otimes \bar{3}, \bar{3} \otimes 3, \) and \( 3 \otimes \bar{3} \) satisfying Yang-Baxter equations

\[
\mathcal{R}_{12}^{(\omega_j, \omega_j)}(\lambda) \mathcal{R}_{13}^{(\omega_j, \omega_j)}(\lambda + \mu) \mathcal{R}_{23}^{(\omega_j, \omega_j)}(\mu) = \mathcal{R}_{23}^{(\omega_j, \omega_j)}(\mu) \mathcal{R}_{13}^{(\omega_j, \omega_j)}(\lambda + \mu) \mathcal{R}_{12}^{(\omega_j, \omega_j)}(\lambda),
\]

with \( \omega_j \in \{3, \bar{3}\} \) for \( j = 1, 2, 3 \) [19] (see also [20, 21]). Two families of row-to-row transfer matrices acting on a Hilbert space \( (3 \otimes 3)^{\otimes L} \) are constructed as the supertrace over auxiliary spaces \( A \cong \mathbb{C} \) of ordered products of these \( \mathcal{R} \)-matrices as [22]

\[
\tau^{(3)}(\lambda) = \text{str}_A \left( G^{(3)}(\alpha) \mathcal{R}_{A,2L}^{(3,3)}(\lambda) \mathcal{R}_{A,2L-1}^{(3,3)}(\lambda - i\gamma) \mathcal{R}_{A,2L-2}^{(3,3)}(\lambda) \ldots \mathcal{R}_{A,1}^{(3,3)}(\lambda - i\gamma) \right),
\]

\[
\tau^{(3)}(\lambda) = \text{str}_A \left( G^{(3)}(\alpha) \mathcal{R}_{A,2L}^{(3,3)}(\lambda + i\gamma) \mathcal{R}_{A,2L-1}^{(3,3)}(\lambda) \mathcal{R}_{A,2L-2}^{(3,3)}(\lambda + i\gamma) \ldots \mathcal{R}_{A,1}^{(3,3)}(\lambda) \right).
\]
Here $\gamma$ parametrizes the deformation parameter $q = \exp(-i\gamma)$, more details can be found in [5]. In equation (2) boundary conditions are controlled by the diagonal twist matrices $G^{(\omega)}(\alpha) = \exp \left( 2i\alpha J^{(\omega)}_3 \right)$. $J^{(\omega)}_i$ being the diagonal generator of the spin-subalgebra in the representation $\omega$. For $\alpha = 0$ the lattice model obeys periodic boundary conditions while for $\alpha = \pm \pi$ bosons are periodic while fermions obey antiperiodic boundary conditions. In the field theory describing the continuum limit of the lattice model these cases correspond to the Ramond (R) and Neveu–Schwarz (NS) sector, respectively. As a consequence of (1) the two transfer matrices commute, $[\tau^{(\omega_1)}(\lambda), \tau^{(\omega_2)}(\mu)] = 0$ for all $\lambda$, $\mu$ and $\omega_j \in \{3,3\}$. Local integrals of motion are generated by the double row transfer matrix

$$\tau(\lambda) = \tau^{(3)}(\lambda_3) \tau^{(3)}(\lambda_2),$$

(3)

For example, the Hamiltonian of the mixed $U_q[sl(2|1)]$ superspin chain is

$$H = i \frac{\partial}{\partial \lambda} \ln \left( \tau^{(3)}(\lambda) \left[ \tau^{(3)}(\lambda)^{-1} \right] \right) \bigg|_{\lambda = 0}.\tag{4}$$

For a staggered model as the one considered here we can introduce another combination of the single-row transfer matrices, i.e. $\tau^{(3)}(\lambda)[\tau^{(3)}(\lambda)]^{-1}$, whose logarithm, unlike (3), is odd under the exchange of the quark and antiquark representation. Among the conserved quantities generated by this object, and motivated by the analysis of a staggered six-vertex model by Ikhlef et al [13], see also [14, 15], we consider the ‘quasi momentum operator’

$$\mathcal{K} = \frac{\gamma}{2\pi(\pi - 2\gamma)} \ln \left( \tau^{(3)}(\lambda) \left[ \tau^{(3)}(\lambda)^{-1} \right] \right) \bigg|_{\lambda = 0}.\tag{5}$$

The transfer matrices (2) (and therefore $H$ and $\mathcal{K}$) can be diagonalized using the nested algebraic Bethe ansatz [23, 24]. The resulting expressions obtained within this framework depend on the choice of grading for the underlying superalgebra [22, 25–28]. As in [3, 5] we use $[p_0, p_1, p_2] = [0, 1, 0]$. In the sector of the Hilbert space with fixed quantum numbers related to the $U(1)$ subalgebras of $U_q[sl(2|1)]$, i.e. charge $b = (N_1 - N_2)/2$ and $z$-component of the spin $j_z = L - (N_1 + N_2)/2$ with non-negative integers $N_{1,2}$ the spectrum of the transfer matrices can be parametrized $N_1 + N_2$ complex rapidities $\{\lambda^{(1)}_j\}_{j=1}^{N_1}$ and $\{\lambda^{(2)}_j\}_{j=1}^{N_2}$ solving the Bethe equations,

$$\frac{\sinh(\lambda^{(1)}_j + i\gamma)}{\sinh(\lambda^{(1)}_j - i\gamma)} = e^{i\alpha} \prod_{k=1}^{N_1} \frac{\sinh(\lambda^{(1)}_j - \lambda^{(2)}_k + i\gamma)}{\sinh(\lambda^{(1)}_j - \lambda^{(2)}_k - i\gamma)}, \quad j = 1, \ldots, N_1,$n

$$\frac{\sinh(\lambda^{(2)}_j + i\gamma)}{\sinh(\lambda^{(2)}_j - i\gamma)} = e^{i\alpha} \prod_{k=1}^{N_2} \frac{\sinh(\lambda^{(2)}_j - \lambda^{(1)}_k + i\gamma)}{\sinh(\lambda^{(2)}_j - \lambda^{(1)}_k - i\gamma)}, \quad j = 1, \ldots, N_2.\tag{6}$$

The corresponding eigenvalues of the Hamiltonian (4) and the quasi momentum (5) are

$$E(\{\lambda^{(1)}_j\}, \{\lambda^{(2)}_j\}) = 4L \cot(\gamma) + 2 \left\{ \sum_{k=1}^{N_1} \frac{\sin(2\gamma)}{\cos(2\gamma) - \cosh(2\lambda^{(1)}_k)} + \sum_{k=1}^{N_2} \frac{\sin(2\gamma)}{\cos(2\gamma) - \cosh(2\lambda^{(2)}_k)} \right\},$$

$$\mathcal{K}(\{\lambda^{(1)}_j\}, \{\lambda^{(2)}_j\}) = \frac{\gamma}{2\pi(\pi - 2\gamma)} \left( \sum_{k=1}^{N_1} \ln \left( \frac{\sinh(\lambda^{(1)}_j + i\gamma)}{\sinh(\lambda^{(1)}_j - i\gamma)} \right) - \sum_{k=1}^{N_2} \ln \left( \frac{\sinh(\lambda^{(2)}_j + i\gamma)}{\sinh(\lambda^{(2)}_j - i\gamma)} \right) \right).\tag{7}$$

Note, the definition of $\gamma$ in [5] differs from the one used here by a factor of 2.
Previous studies of this model [3, 5, 6] have shown that the spectrum of the transfer matrix (3) in the sector with charge \( b = 0 \) (i.e. \( N_1 = N_2 = N \)) contains (after scaling by a factor of two) that of the antiferromagnetic spin-1 XXZ Heisenberg model with twist \( \varphi = \pi + \alpha \) [29]; for a subset of eigenstates of the mixed superspin chain the Bethe roots degenerate as \( \lambda^{(1)} = \hat{\lambda}^{(2)} = \lambda_j \). These corresponding eigenvalues can be identified with eigenenergies of the spin-1 chain in the sector with \( j_3 = L - N \), see [5, 6]. At low energies the operator content of the effective field theory for the spin-1 XXZ model is given in terms of composites of a \( U(1) \) Kac-Moody field and Ising operators [30–32]. This provides a subset of the finite size spectrum of the superspin chain [3, 5]. After rescaling of the finite size energy gaps, \( \Delta E \to (L\Delta E/2\pi V_F) \equiv X \) (\( V_F = \pi/\gamma \) is the Fermi velocity of the gapless excitations), the corresponding effective scaling dimensions of the superspin chain (4) are

\[
X_{\{m,w\}}(\varphi = \alpha + \pi) = -\frac{1}{4} \delta_{m+w \in 2\mathbb{Z}} + \frac{m^2}{2k} + \frac{k}{2} \left( w + \frac{\varphi \chi}{\pi} \right)^2, \quad k = \frac{\pi}{\pi - 2\gamma},
\]

where the integer \( m \) is the quantum number \( j_3 \) of the corresponding state in the superspin chain while \( w \) is related to the vorticity of that state. Note that the energy of the level \( (m, w) = (0, -1) \) in the Ramond sector \( (\varphi = \pi) \) vanishes identically. The Bethe root configuration corresponding to this exact zero mode in the \( (b, j_3) = (0, 0) \) sector of the periodic superspin chain is highly degenerate, i.e. \( \hat{\lambda}^{(0)} \equiv 0 \) for all \( j = 1, \ldots, L \) and \( a = 1, 2 \). This is the ground state of the periodic superspin chain (and the related spin-1 XXZ model with antiperiodic boundary conditions where the zero-mode has been shown to be a singlet under an exact dynamical lattice supersymmetry [33]) for anisotropies \( 0 \leq \gamma \leq \pi/4 \).

In the large \( L \) limit the possible root configurations solving the Bethe equation (6) can be classified using the string hypothesis. A class of low energy excitations in the zero charge sector \( b = 0 \) has been identified with collections of \( O(L) \) ‘strange 2-strings’ consisting of complex conjugate pairs of rapidities \( \lambda^{(1)} = (\hat{\lambda}^{(2)})^* \). These strange strings come in two types, i.e. \( (\pm) \) with \( \text{Im}(\hat{\lambda}^{(1)}) = \pm \gamma/2 \). Solutions to the Bethe equations consisting of \( N_\pm \) type-(\pm) strange strings with \( N_+ + N_- = L - j_3 \) but \( \Delta N = N_+ - N_- \neq 0 \) have been found to form a continuous component of the finite size spectrum starting at levels (8) with \( m + w \in 2\mathbb{Z} \) [3, 5].

Based on these insights from the lattice model it has been argued that the continuum limit of the isotropic \( \gamma = 0 \) superspin chain flows to a \( SU(2|1) \) WZNW model at level \( k = 1 \) [3, 10]. The generic class I irreducible representations of the affine superalgebra \( sl(2|1) \) are built over the typical representations \( [b, j = \frac{1}{2}] \) with charge \( b \in \mathbb{C} \) and spin \( j = \frac{1}{2} \). To obtain a physical modular invariant partition function from the characters for these representations which can be compared with the values of the lattice model one is forced to consider continuous values of the charge \( b \) which, after analytical continuation \( b \to i\beta \) does yield continua starting at scaling dimensions \( X^{\text{eff}, R}(m, w) \equiv X^{\text{eff}}(\varphi = \pi) \in \mathbb{N} + \frac{1}{4} \) in the Ramond sector [10]. Note, however, that this proposal does not capture the primary fields with integer scaling dimension \( X^{\text{eff}, R} \) (i.e. \( m + w \) odd in (8)), in particular the \( sl(2|1) \) singlet state with \( X^{\text{eff}, R} = 0 \). Since modular invariance appears to preclude the appearance of this singlet on its own it has been argued in [10] that this state is an artifact of the lattice model which disappears in the continuum limit once the partition function is properly normalized.

\(^2\)The eigenvalues \( K \) of the quasi momentum operator, equation (7), are odd under the interchange \( \{ \lambda^{(1)} \} \leftrightarrow \{ \hat{\lambda}^{(2)} \} \) changing the sign of the charge \( b \). As an immediate consequence \( K = 0 \) for the levels from the spin-1 XXZ chain subset of the spectrum.
Here we will not try to extend this picture to describe the continuum spectrum of critical exponents of the anisotropic superspin chain. Instead we focus on another characteristic feature of certain conformal field theories with non-compact target space, i.e. the appearance of a discrete component of the spectrum. To be specific we study the spectrum of the $U_q[sl(2|1)]$ superspin chain as a function of the twist $\varphi_1 = \alpha + \pi$ which, in the continuum limit, corresponds to the spectral flow between the Neveu–Schwarz sector and the Ramond sector.

The ground state of the former is found in the sector with quantum numbers $(b, j_3) = (0, 0)$. For even $L$ it is realized in the XXZ subspace of this sector and the corresponding Bethe root configuration can be mapped to that of the singlet $(j_3 = 0)$ ground state of the periodic spin-1 chain. Its effective scaling dimension is $X_{\text{eff,NS}}^{(0,0)}(0,0) \equiv X_{\text{eff}}^{(0,0)}(\varphi = 0) = -\frac{1}{4}$ (leading to an effective central charge $c^{\text{eff}} = 3$). For odd $L$ the lowest state in the NS sector is doubly degenerate. One of the Bethe root configurations for this state consists of $N_L = (L \pm 1)/2$ type-$(\pm)$ strange strings, i.e. $\Delta N = 1$. Its scaling dimension exhibits a logarithmic dependence on the system size, $X_0 = -\frac{1}{4} + O((\Delta N/\log L)^2)$ [3, 5], consistent with the emergence of a continuum of levels starting at $X_{\text{eff,NS}}^{(0,0)}(0,0) = -\frac{1}{4}$ in the thermodynamic limit.

Adiabatically changing the twist angle we can follow this state under the spectral flow. For $|\varphi| < \varphi_c = \pi/k$ its scaling dimension $X_{(m,w)}^{(0,0)}(\varphi)$ stays within the continuum above $X_{\text{eff}}^{(0,0)}(\varphi)$. As the twist approaches $\pm \varphi_c$, the strange string with largest roots goes to $\infty$. Beyond $\pm \varphi_c$, the root configuration changes and the finite size scaling dimension of the state deviates from (8). Unlike the higher excitations ($|\Delta N| > 1$) within this class of Bethe states it splits off from the emerging continuum above $X_{\text{eff}}^{(0,0)}(\varphi)$, see figure 1. Based on our finite size data we conjecture the following $\varphi$-dependence of the scaling dimension of the scaling dimension:

$$X_{(0,0)}^{*}(\varphi) = \begin{cases} 
-\frac{1}{4} + \frac{k}{2} \left(\frac{\varphi}{\pi}\right)^2 - \frac{2k-1}{(k-1)^2} \left(\frac{\varphi}{\pi}\right)^2 & \text{for } \varphi \leq |\varphi| \leq \varphi_{c,2} \\
\frac{1}{4} \left(1 - \frac{\varphi}{\pi}\right)^2 + \frac{1}{4} \frac{1}{k-1} & \text{for } |\varphi - \pi| \leq |\varphi_{c,2} - \pi|
\end{cases}$$

(9)

where $\varphi_{c,2} = \pi - \frac{\pi}{k(k-1)}$. Hence we find that one state from the continuum above $X_{\text{eff,NS}}^{(0,0)}(0,0)$ evolves, under the spectral flow $\varphi = 0 \ldots \pi$, to a discrete level with dimension

$$X_{(0,0)}^{*}(\pi) = \frac{1}{4} \frac{\pi - 2\gamma}{\pi + 2\gamma} = \frac{1}{4} \frac{1}{2k - 1}$$

(10)

in the Ramond sector. The Bethe root configuration for this state for $\varphi = \pi$ consists of $(L-1)/2$ of the usual 2-strings (consisting of two complex conjugate rapidities from the same level [34]) and, in addition, one single root at $\infty$ on either level, i.e. 3

$$\lambda^{(1)} = -\lambda^{(2)} = \left\{ \mu^{\pm}_k : \text{Im}(\mu^{\pm}_k) \simeq \pm \frac{\gamma}{2}, k = 1 \ldots \frac{L-1}{2}, \infty \right\}.$$  

(11)

Note that there are strong logarithmic finite size corrections to scaling to (10), similar as for the states in the continuum, see figure 2. Increasing the twist further we find that $X_{(0,0)}^{*}(\varphi) = X_{(0,0)}^{*}(2\pi - \varphi)$. For $\varphi > 2\pi - \varphi_{c,2}$ this level coincides with

$$X_{(0,-2)}^{*} = X_{(0,0)}^{*}(\varphi) - \frac{2k - 1}{(k-1)^2} \left(\frac{1}{2} - \frac{k}{2} \frac{\varphi}{\pi} - 2\right)^2,$$

(12)

and disappears in the continuum above $X_{\text{eff}}^{(0,-2)}(\varphi)$ at $\varphi = 2\pi - \varphi_c$.

\textsuperscript{3}This configuration has already been observed for $L = 3$ in [5] but not considered further.
Concluding our analysis of the scaling dimensions under the spectral flow we note that starting from the lowest state in the continuum above $X_{\text{eff}}(1, -1)$ ($\varphi = \pi$) for even $L$ we have observed another discrete level in the spectrum of the superspin chain $X^{\ast}_{(0,0)}(\varphi)$ = $X_{\text{eff}}^{\ast}(m, w)(\varphi)$ = $X_{\text{eff}}^{\ast}(m, w)(\varphi) = \frac{2k - 1}{(k - 1)^2} \left( 1 - k \frac{|\varphi - \pi|}{\pi} \right)^2$, $|\varphi - \pi| > \frac{2\pi}{k}$. (13)

Let us now analyze the quasi momentum of the state with dimension $X^{\ast}_{(0,0)}(\varphi)$ (9) identified above. An operator similar to (5) has first been introduced for a $Z_2$-staggered six-vertex model where its (real) eigenvalues have been identified with the quantum number parametrizing the spin $j$ of the $SL(2, \mathbb{R})$ affine primaries from the continuous series, $j = -1/2 + iK$ [13]. Based on this identification the staggered six-vertex model has been shown to be described by the $SL(2, \mathbb{R})/U(1)$ Euclidean black hole CFT in the continuum limit.

For the superspin chains considered here the quasi momentum allows to label the continua of scaling dimensions in a similar way [35]: in fact starting with the lowest state in the NS sector but outside of the XXZ subspace the corresponding eigenvalue $K$ is found to take real values for $|\varphi| < \varphi_c$, see figure 3. The amplitude vanishes as $1/\log L$ in the thermodynamic limit. This agrees with the density of states observed in the continua [3, 5].

Figure 1. Evolution of the lowest states in the NS sector of the superspin chain of odd length $L$ with the twist angle $\alpha = -\pi + \varphi$: bullets are the scaling dimensions obtained from the solution of the Bethe equation (6) for $L = 27$ evolving from a root configuration consisting of strange strings with $|\Delta N| = 1$ for $\varphi = 0$, i.e. the NS ground state, for anisotropy $\gamma = 17\pi/40$. Open circles show the flow of scaling dimensions with $|\Delta N| = 3$ and 5 for the same parameters. Black lines indicate the lowest effective scaling dimensions $X_{\text{eff}}^{\ast}(m, w)(\varphi)$ of the superspin chain, i.e. $(m, w) = (0,0), (1, -1), (2,0)$, and $(0,-1)$, for this anisotropy. The shaded areas indicate the observed continua of scaling dimensions starting at $X_{\text{eff}}^{\ast}(m, w)$ with even $m + w$. The conjectured $\varphi$-dependence of the discrete level $X_{(0,0)}^{\ast}$, equation (9), is shown in red (dash-dotted lines indicate continuations of the functions appearing in the piecewise definition of $X_{(0,0)}^{\ast}$ beyond their domain of definition).

Concluding our analysis of the scaling dimensions under the spectral flow we note that starting from the lowest state in the continuum above $X_{(1, -1)}^{\ast}(\varphi = \pi)$ (for even $L$) we have observed another discrete level in the spectrum of the superspin chain $X_{(1, -1)}^{\ast}(\varphi) = X_{(1, -1)}^{\ast}(\varphi) = \frac{2k - 1}{(k - 1)^2} \left( 1 - k \frac{|\varphi - \pi|}{\pi} \right)^2$, $|\varphi - \pi| > \frac{2\pi}{k}$.
Figure 2. Scaling dimension of the discrete level in the Ramond sector, $\varphi = \pi$, as a function of the anisotropy parameter $\gamma$: bullets are the numerical data for system size $L = 501$, dash-dotted line is the conjectured value (10) in the thermodynamic limit. The inset shows the $L$ dependence of the finite size data (●) for $\gamma = 17\pi/40$ together with the extrapolation based on an assumed rational dependence of the data on $1/\log L$ (dashed line).

Figure 3. Quasi momentum of the discrete level for $\gamma = 17\pi/40$ versus twist angle $\varphi$: filled (open) symbols are the numerical data for the real (imaginary) part of $\mathcal{K}$ for system sizes $L = 3$ and 27. Dash-dotted line (in red) is the conjecture for $\text{Im}(\mathcal{K})$, equation (14), saturating at $\varphi = \varphi_c$.2
For $|\varphi| > \varphi_c$, however, i.e. as the discrete level (9) leaves the continuum of scaling dimensions, the quasi momentum becomes imaginary with a linear dependence on the twist angle $\varphi$

$$K^*(\varphi) = \frac{i}{2(\pi - 2\gamma)}(|\varphi| - \varphi_c) = i\left(\frac{k|\varphi|}{2\pi} - \frac{1}{2}\right) \quad \text{for } \varphi_c \leq \varphi \leq \varphi_{c,2}$$  \hspace{1cm} (14)

(we have chosen the branch of the logarithm in (5) such that $\text{Im}(K) \in [0,(k-1)/2]$.) Corrections to scaling in this expression are small which allows to observe this behaviour already for $L = 3$, see figure 3. As for the scaling dimension (9) the $\varphi$-dependence of the quasi momentum changes when the twist is increased beyond $|\varphi| = \varphi_{c,2}$: while $K$ remains purely imaginary its value saturates at

$$\text{Im}(K^*(\varphi)) = \frac{4\gamma^2}{\pi^2 - 4\gamma^2} = \frac{(k - 1)^2}{2k - 1} \quad \text{for } \varphi \geq \varphi_{c,2}.$$ \hspace{1cm} (15)

Note that $\text{Im}(K^*(\pi))$ is smaller than the maximum $\gamma/(\pi - 2\gamma) = (k - 1)/2$ which is possible according to the definition (5).

The emergence of a discrete level out of the continuum at finite twist is strongly reminiscent to the situation in the $a_2^{(2)}$ spin chain [7]: in the black hole CFT describing the low energy behaviour of this lattice model it is understood as a consequence of the inclusion of principal discrete representations of $SL(2, \mathbb{R})$ as Kac-Moody primaries in the coset theory. For operators corresponding to normalizable states in the parent WZNW theory of the model the spin of these representations (related to the momentum along the non-compact direction of the target space, an infinite cigar for the $SL(2, \mathbb{R})/U(1)$ sigma model) is restricted to values $j \leq -1/2$. To satisfy this bound a finite, non-zero twist has to be applied [17, 18]. For the bosonic $SL(2, \mathbb{R})/U(1)$ coset at level $k$ the spectrum of discrete representations also needs to be truncated by the ‘unitarity bound’ $j \geq (k - 1)/2$ to guarantee non-negative conformal weights [12, 17, 36].

The appearance of the discrete level $X^*$ (9) in the spectrum of the superspin chain at twist $\varphi_c$, accompanied with the change of quasi momentum from real to imaginary can be interpreted in a similar way: states in the continuum of scaling dimensions have real $K$, the discrete levels are characterized by imaginary quasi momentum. The observed bound $\text{Im}(K) \geq 0$ for states in the lattice model can be attributed to the normalizability of the corresponding primaries. Continuing (14) to $|\varphi| < \varphi_c$ would yield imaginary quasi momenta $-1/2 \leq \text{Im}(K) < 0$ which is not realized in the spectrum of the superspin chain. As for a restriction of $\text{Im}(K)$ from above (similar to the unitarity bound in the string theory) our data for the lattice model do not provide a conclusive answer. We would need to see a level being ‘absorbed’ by the continuum at such a bound under the spectral flow (as is the case for the $a_2^{(2)}$ spin chain [7]). It would be tempting to associate this bound with $\text{Im}(K) \leq (k - 1)/2$ as implied by our choice of the branch for the logarithm in (5). Since the quasi momentum for the discrete level studied above saturates at the value (15) below $(k - 1)/2$, however, our data cannot support this conjecture.

To summarize, we have studied the spectral flow for a staggered superspin chain with continuous and discrete components in the spectrum of scaling dimensions under a twist in the boundary conditions. Based on the exact solution of this model we identified a state from the continuous part of the spectrum in the Neveu–Schwarz sector (i.e. with anti-periodic boundary conditions for the fermionic degrees of freedom) which under variation of the twist flows to a discrete level in the Ramond sector which has not been identified previously. Levels can be attributed to the continuous or discrete part of the spectrum based on their quasi momentum. The appearance of discrete levels in the spectrum of the lattice model has been argued to
be related to the normalizability of primary fields in the non-rational conformal field theory describing the continuum limit.

Acknowledgments

This work has been carried out within the research unit *Correlations in Integrable Quantum Many-Body Systems* (FOR2316). Funding by the Deutsche Forschungsgemeinschaft under grant No. Fr 7379-1 is gratefully acknowledged.

ORCID

Holger Frahm  
https://orcid.org/0000-0003-4629-6612

References

[1] Blöte H W J, Cardy J L and Nightingale M P 1986 Conformal invariance, the central charge and universal finite-size amplitudes at criticality  *Phys. Rev. Lett.* 56 742–5
[2] Affleck I 1986 Universal term in the free energy at a critical point and the conformal anomaly  *Phys. Rev. Lett.* 56 746–8
[3] Essler F H L, Frahm H and Saleur H 2005 Continuum limit of the integrable $s(2/1)$ 3-3 superspin chain  *Nucl. Phys.* B 712 513–72
[4] Ikhlef Y, Jacobsen J L and Saleur H 2008 A staggered six-vertex model with non-compact continuum limit  *Nucl. Phys.* B 789 483–524
[5] Frahm H and Martins J M 2011 Finite size properties of staggered $U_q[sl(2|1)]$ superspin chains  *Nucl. Phys.* B 847 220–46
[6] Frahm H and Martins J M 2012 Phase diagram of an integrable alternating $U_q[sl(2|1)]$ superspin chain  *Nucl. Phys.* B 862 504–52
[7] Vernier E, Jacobsen J L and Saleur H 2014 Non compact conformal field theory and the $a_2^{(2)}$ (Izergin–Korepin) model in regime III  *J. Phys. A: Math. Theor.* 47 285202
[8] Frahm H and Martins J M 2015 Finite-size effects in the spectrum of the $OSp(3|2)$ superspin chain  *Nucl. Phys.* B 894 665–84
[9] Vernier E, Jacobsen J L and Saleur H 2016 The continuum limit of $a^{(2)}_{N-1}$ spin chains  *Nucl. Phys.* B 911 52–93
[10] Saleur H and Schomerus V 2007 On the $SU(2|1)$ WZW model and its statistical mechanics applications  *Nucl. Phys.* B 775 312–40
[11] Witten E 1991 String theory and black holes  *Phys. Rev.* D 44 314–24
[12] Dijkgraaf R, Verlinde H and Verlinde E 1992 String propagation in black hole geometry  *Nucl. Phys.* B 371 269–314
[13] Ikhlef Y, Jacobsen J L and Saleur H 2012 An integrable spin chain for the $SL(2,R)/U(1)$ black hole sigma model  *Phys. Rev. Lett.* 108 081601
[14] Candu C and Ikhlef Y 2013 Non-linear integral equations for the $SL(2,R)/U(1)$ black hole sigma model  *J. Phys. A: Math. Theor.* 46 415401
[15] Frahm H and Seel A 2014 The staggered six-vertex model: conformal invariance and corrections to scaling  *Nucl. Phys.* B 879 382–406
[16] Izergin A G and Korepin V E 1981 The inverse scattering method approach to the quantum Shabat–Mikhailov model  *Commun. Math. Phys.* 79 303–16
[17] Hamany A, Prezas N and Troost J 2002 The partition function of the two-dimensional black hole conformal field theory  *J. High Energy Phys.* JHEP04/2002014
[18] Ribault S and Schomerus V 2004 Branes in the 2D black hole  *J. High Energy Phys.* JHEP02(2004)019
[19] Gade R M 1999 An integrable $s(2|1)$ vertex model for the spin quantum Hall critical point  *J. Phys. A: Math. Gen.* 32 7071–82
[20] Perk J H H and Schultz C L 1981 New families of commuting transfer matrices in \( q \)-state vertex models Phys. Lett. A \textbf{84} 407–10

[21] Kulish P P and Sklyanin E K 1980 Zapiski Nauchn. Seminarov LOMI \textbf{95} 129–60

[22] Kulish P P and Reshetikhin N Y 1983 Diagonalisation of \( GL(N) \) invariant transfer matrices and quantum \( N \)-wave system (Lee model) J. Phys. A: Math. Gen. \textbf{16} L591–6

[23] Essler F H L and Korepin V E 1992 Higher conservation laws and algebraic Bethe ansätze for the supersymmetric \( t-J \) model Phys. Rev. B \textbf{46} 9147–62

[24] Alcaraz F C and Martins J M 1989 Conformal invariance and the operator content of the XXZ model J. Phys. A: Math. Gen. \textbf{22} 1829–58

[25] Frahm H and Hobuß K 2017 unpublished

[26] Maldacena J and Ooguri H 2001 Strings in \( AdS_3 \) and the \( SL(2, R) \) WZW model. Part 1: the spectrum J. Math. Phys. \textbf{42} 2929–60