Analyticity, rank one perturbations and the invariance of the left spectrum

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Abstract. We discuss the question of the analyticity of a rank one perturbation of an analytic operator. If $M_z$ is the bounded operator of multiplication by $z$ on a functional Hilbert space $H_κ$ and $f \in H$ with $f(0) = 0$, then $M_z + f \otimes 1$ is always analytic. If $f(0) \neq 0$, then the analyticity of $M_z + f \otimes 1$ is characterized in terms of the membership to $H_κ$ of the formal power series obtained by multiplying $f(z)$ by $\frac{i}{f(0) - z}$. As an application, we discuss the problem of the invariance of the left spectrum under rank one perturbation. In particular, we show that the left spectrum $σ_l(T + f \otimes g)$ of the rank one perturbation $T + f \otimes g$, with $g \in \ker(T^*)$, of a cyclic analytic left invertible bounded linear operator $T$ coincides with the left spectrum of $T$ except the point $\langle f, g \rangle$. In general, the point $\langle f, g \rangle$ may or may not belong to $σ_l(T + f \otimes g)$. However, if it belongs to $σ_l(T + f \otimes g) \{0\}$, then it is a simple eigenvalue of $T + f \otimes g$.

Mathematics Subject Classification. Primary 47B32; Secondary 47B13.

Keywords. Analytic, Rank one perturbation, Left spectrum.

1. Analyticity and the invariance of the left spectrum

Several examples of multiplication operators on reproducing kernel Hilbert spaces suggest that the essential spectrum and the left spectrum coincide. Since the essential spectrum of a bounded linear operator is always invariant under compact perturbations (see [3]), it is reasonable to ask whether the left spectrum is also invariant under a compact or a finite rank perturbation. We address the above problem for rank one perturbations on a testing ground. This note exploits Shimorin’s analytic model (see [10]) for analytic left-invertible operators to provide a solution to this problem when the operator in question is a cyclic analytic left-invertible operator, and the perturbation is of the form $f \otimes g$ with $g \in \ker T^*$. We see that solution to the

The work of the second author is supported by The Fields-Laval Post-doctoral Fellowship and Inspire Faculty Fellowship (DST/INSPIRE/04/2021/002555). The third author is supported by the postdoctoral fellowship of the National Board of Higher Mathematics.
above problem is closely related to the description of the hyper-range of the rank one perturbation of analytic operators. Before we state the main result, we fix some notations and collect the necessary preliminaries.

Let \( \mathbb{C} \) denote the set of complex numbers. For a positive real number \( r \), let \( D_r \) denote the open disc centred at 0 and of radius \( r \). For any holomorphic function \( \phi : D \to \mathbb{C} \) and an integer \( n \geq 0 \), let \( \hat{\phi}(n) \) denote the coefficient of \( z^n \) in the power series representation of \( \phi \). Let \( \partial \) denote the partial derivative with respect to \( w \). For a complex Hilbert space \( H \), let \( B(H) \) denote the unital \( C^* \)-algebra of bounded linear operators on \( H \). For \( T \in B(H) \), the kernel and range of \( T \) are denoted by \( \ker T \) and \( \text{ran} T \), respectively. We denote by \( \sigma_p(T), \sigma(T), \sigma_l(T) \) and \( \sigma_e(T) \) the point spectrum, spectrum, left spectrum and essential spectrum of \( T \), respectively. The spectral radius of \( T \) is denoted by \( r(T) \) (refer to [3] for definitions and basic spectral theory). For \( f, g \in H \), the bounded linear operator \( f \otimes g \) on \( H \) is given by

\[
f \otimes g(h) = \langle h, g \rangle f, \quad h \in H.
\]

A bounded linear operator \( T \) on \( H \) is left invertible if there exists a bounded linear operator \( L \) on \( H \) (a left-inverse) such that \( LT = I \). For a positive integer \( m \), an operator \( T \) on \( H \) is said to be \( m \)-cyclic if there is an \( m \)-dimensional vector subspace \( M \) of \( H \), called the cyclic space of \( T \), such that \( H \) is the closed linear span of \( \{ T^n h : n \geq 0, h \in M \} \). For simplicity, we refer \( 1 \)-cyclic operator as the cyclic operator. A bounded linear operator \( T \) on \( H \) is analytic if the hyper-range \( \bigcap_{n=0}^{\infty} T^n H \) of \( T \) is \( \{0\} \). We say that \( T \) on \( H \) has the wandering subspace property if

\[
H = \bigvee_{n \geq 0} T^n (\ker T^*) .
\]

Let \( H_\kappa \) be a reproducing kernel Hilbert space of complex-valued holomorphic functions defined on the unit disc and let \( \kappa : D \times D \to \mathbb{C} \) be the reproducing kernel for \( H_\kappa \), that is, \( \kappa(\cdot, w) \in H_\kappa \) and

\[
( f, \kappa(\cdot, w))_{H_\kappa} = f(w), \quad f \in H_\kappa, \ w \in D. \tag{1.1}
\]

We call \( H_\kappa \) a functional Hilbert space if \( z \) is a multiplier of \( H_\kappa \) (that is, \( zf \in H_\kappa \) for every \( f \in H_\kappa \)) and \( \kappa \) is normalized at the origin (that is, \( \kappa(z, 0) = 1 \) for every \( z \in D \)).

Assume that \( H_\kappa \) is a functional Hilbert space. By the closed graph theorem, the operator \( M_z \) of multiplication by the coordinate function \( z \) defines a bounded linear operator on \( H_\kappa \). If \( M_z^* \) denotes the Hilbert space adjoint of \( M_z \), then by (1.1),

\[
M_z^* \kappa(\cdot, w) = \overline{w} \kappa(\cdot, w), \quad w \in D. \tag{1.2}
\]

The reader is referred to [9] for the basics of reproducing kernel Hilbert spaces.

**Remark 1.1.** Assume that \( H_\kappa \) is a functional Hilbert space. Since \( \kappa \) is normalized at the origin, by (1.2), all constant functions belong to \( \ker M_z^* \). As \( z \)
is a multiplier, this shows that $\mathcal{H}_\kappa$ contains the linear space of polynomials. If, in addition, $\dim \ker \mathcal{M}_z^* = 1$, then $\ker \mathcal{M}_z^*$ is spanned by 1.

Here is the main result of this paper (see [8, Proposition 4.2] for a characterization of analytic bounded composition operators on directed graphs with one circuit in terms of divergence of a series).

**Theorem 1.2.** Let $\mathcal{H}_\kappa$ be a functional Hilbert space and let $f(z) = \sum_{j=0}^\infty \hat{f}(j)z^j$ belong to $\mathcal{H}_\kappa$. Assume that $\dim \ker \mathcal{M}_z^* = 1$. If the multiplication operator $\mathcal{M}_z$ on $\mathcal{H}_\kappa$ is left-invertible, then $\mathcal{M}_z + f \otimes 1$ is analytic if and only if exactly one of the following statements holds:

(i) $f(0) = 0$,

(ii) $f(0) \neq 0$ and $\sum_{j=0}^\infty \left( \sum_{i=0}^j \frac{\hat{f}(j-i)}{j!(0)^i} \right) z^j$ does not belong to $\mathcal{H}_\kappa$.

**Remark 1.3.** Note that the formal power series $\sum_{j=0}^\infty \left( \sum_{i=0}^j \frac{\hat{f}(j-i)}{j!(0)^i} \right) z^j$ is equal to the product of $f$ with the formal power series $\sum_{j=0}^\infty \left( \frac{z}{j!(0)} \right)^j$.

Theorem 1.2 is partly motivated by the wandering subspace problem (see Corollary 2.7). Moreover, it helps understand the problem of the invariance of the left spectra under compact perturbations. To see some general facts related to the latter problem, let $T, K \in \mathcal{B}(\mathcal{H})$. If $K$ is a compact operator, then

$$\sigma_l(T + K) \setminus \sigma_p(T + K) \subseteq \sigma_l(T), \quad \sigma_l(T) \setminus \sigma_p(T) \subseteq \sigma_l(T + K). \quad (1.3)$$

Indeed, if $\lambda \notin \sigma_l(T)$, then $T - \lambda$ is left-invertible $\Rightarrow T + K - \lambda$ is left semi-Fredholm

$\Rightarrow \text{ran}(T + K - \lambda)$ is closed and $\ker(T + K - \lambda)$ is finite dimensional, and hence either $\lambda \in \mathbb{C}\setminus\sigma_l(T + K)$ or $\lambda \in \sigma_p(T + K)$ completing the verification of the first inclusion in (1.3). The second inclusion in (1.3) can be obtained along similar lines. The obvious question that appears here is when equality occurs in these inclusions. This, plus a little more, yields the following proposition:

**Proposition 1.4.** Let $T, K \in \mathcal{B}(\mathcal{H})$. Assume that $K$ is a compact operator. Then the following statements are valid:

(i) if $\sigma_p(T) = \emptyset$, then $\sigma_l(T + K) = \sigma_l(T) \cup \sigma_p(T + K)$,

(ii) for every $\lambda \in \sigma_p(T + K) \setminus \sigma_l(T)$, $\ker(T + K - \lambda)$ is finite dimensional.

**Proof.** If $\sigma_p(T) = \emptyset$, then by (1.3),

$$\sigma_l(T) = \sigma_l(T) \setminus \sigma_p(T) \subseteq \sigma_l(T + K) \subseteq \sigma_l(T) \cup \sigma_p(T + K),$$

which yields (i). To see (ii), let $\lambda \in \mathbb{C}\setminus\sigma_l(T)$. For $h \in \mathcal{H}$, note that

$$(T + K - \lambda)h = 0 \text{ if and only if } (T - \lambda)h = -Kh. \quad (1.4)$$

Applying a left-inverse $L_\lambda$ of $T - \lambda$ on both sides, we get

$$h = -L_\lambda Kh. \quad (1.5)$$
Thus the dimension of \( \ker(T + K - \lambda) \) is the dimension of \( \ker(L_\lambda K + I) \). Since \( L_\lambda K \) is a compact operator, \( \ker(T + K - \lambda) \) is finite dimensional (see [3, Corollary 1.2.3 and Theorem 1.3.3]).

Thus the problem of invariance of the left spectra could be answered provided we determine the eigenvalues of the compact perturbation. Let us analyse this problem for \( K = f \otimes g \) with \( g \in \ker T^* \).

**Proposition 1.5.** Let \( T \in \mathcal{B}(\mathcal{H}) \) and \( f, g \in \mathcal{H} \). For \( \lambda \notin \sigma_l(T) \), the following statements are valid:

(i) the dimension of \( \ker(T + f \otimes g - \lambda) \) is at most 1,

(ii) if \( g \in \ker T^* \) and \( \lambda \neq \langle f, g \rangle \), then \( \lambda \notin \sigma_p(T + f \otimes g) \).

In particular, if \( g \in \ker T^* \) and \( \sigma_p(T) = \emptyset \), then

\[
\sigma_p(T + f \otimes g) \subseteq \sigma_l(T) \cup \{ \langle f, g \rangle \}.
\]

**Proof.** By (1.5), any \( h \in \ker(T + f \otimes g - \lambda) \) is a multiple of \( L_\lambda f \), and hence (i) follows. If \( \lambda \) is an eigenvalue of \( T + f \otimes g \) with eigenvector \( h \), then by (1.5), \( \langle h, g \rangle \neq 0 \), and hence by (1.4), \( f = -\frac{(T-\lambda)h}{\langle h, g \rangle} \). Taking inner-product with \( g \), this implies that \( \lambda = \langle f, g \rangle \) (since \( g \in \ker T^* \)). This yields (ii). To see the remaining part, note that \( \sigma_p(T + f \otimes g) \setminus \sigma_l(T) \subseteq \{ \langle f, g \rangle \} \) (by (ii)) and apply Proposition 1.4. \( \square \)

Thus a solution to the problem of the invariance of the left spectra for the rank one perturbation \( f \otimes g \) with \( g \in \ker T^* \) boils down to deciding whether or not \( \langle f, g \rangle \) is an eigenvalue of \( T + f \otimes g \). This lies deeper! Nevertheless, as an application of the proof of Theorem 1.2, we obtain the following result (cf. [6, Theorem 1.1]):

**Theorem 1.6.** Let \( T \) be a cyclic analytic left invertible operator in \( \mathcal{B}(\mathcal{H}) \). Assume that \( f \in \mathcal{H} \) and \( g \in \ker T^* \). Then \( \sigma_l(T) \) is a subset of \( \sigma_l(T + f \otimes g) \) such that

\[
\sigma_l(T + f \otimes g) \setminus \{ \langle f, g \rangle \} = \sigma_l(T) \setminus \{ \langle f, g \rangle \},
\]

\[
r(T + f \otimes g) = \max \{ r(T), |\langle f, g \rangle| \}.
\]

Moreover, the following statements are valid:

(i) if \( \langle f, g \rangle \) belongs to \( \sigma_l(T + f \otimes g) \setminus \{0\} \), then it is a simple eigenvalue of \( T + f \otimes g \),

(ii) either \( \sigma_l(T + f \otimes g) = \sigma_l(T) \) or \( \sigma_l(T + f \otimes g) = \sigma_l(T) \cup \{ \langle f, g \rangle \} \).

**Remark 1.7.** In general, the inclusion \( \sigma_l(T) \subseteq \sigma_l(T + f \otimes g) \) and the inequality \( r(T + f \otimes g) \geq r(T) \) are strict. Further, any of following possibilities can occur:

\[
 r(T + f \otimes g) < |\langle f, g \rangle|, \quad r(T + f \otimes g) = |\langle f, g \rangle|, \quad r(T + f \otimes g) > |\langle f, g \rangle|.
\]

The proofs of Theorems 1.2 and 1.6 occupy a significant portion of Sect. 2. In the remaining part of this section, we explain the essential difference between the problems of the invariance of the essential spectra and that of left spectra with the help of one concrete family of multiplication operators. Recall that the Fredholm index is invariant under compact perturbations (see [3, Theorem 1.3.1]). So, if one can show that the dimension of
the cokernel is not preserved under perturbations, then so is the dimension of the kernel. Hence, the left spectrum may not be invariant under compact perturbations. The following proposition supports these speculations (cf. [11, Theorem 1]).

**Proposition 1.8.** Let $\mathcal{H}_\kappa$ be a functional Hilbert space and let $\mathcal{M}_z$ denote the operator of multiplication by $z$. Assume that the kernel $\kappa$ satisfies

$$\ker(\mathcal{M}_z^* - w) = \{\alpha \kappa(\cdot, w) : \alpha \in \mathbb{C}\}, \quad w \in \mathbb{D}. \tag{1.6}$$

If $f \in \mathcal{H}_\kappa$ and $S := \mathcal{M}_z + f \otimes 1$, then the following statements are valid:

(i) if $f(0) = 0$, then

$$\ker S^* = \begin{cases} \text{span}\{\kappa(\cdot, 0), \overline{\partial}\kappa(\cdot, w)|_{w=0}\} & \text{if } f'(0) = -1, \\
\text{span}\{\kappa(\cdot, 0)\} & \text{otherwise}, \end{cases}$$

(ii) if $f(0) \neq 0$, then

$$\ker S^* = \text{span}\{(1 + \overline{f'(0)})\kappa(\cdot, 0) - \overline{f(0)}\overline{\partial}\kappa(\cdot, w)|_{w=0}\}.$$

In particular, if $f(0) = 0$ and $f'(0) = -1$, then $S$ is not cyclic.

**Proof.** Recall from [5, Lemma 4.1] and [4, Lemma 1.22(ii)] that

$$e_n := \left. \frac{\partial^n \kappa(\cdot, w)}{n!} \right|_{w=0} \in \mathcal{H}_\kappa, \quad \langle f, e_n \rangle = \frac{(\partial^n f)(0)}{n!}, \quad f \in \mathcal{H}_\kappa, \quad n \geq 0,$$ \tag{1.7}

We claim that

$$\ker S^* = \{ae_0 + be_1 : b(1 + \overline{f'(0)}) + a\overline{f(0)} = 0\}. \tag{1.8}$$

To see this, note that since $\kappa(\cdot, 0) = 1$, by (1.7),

$$S^*(ae_0 + be_1) = \mathcal{M}_z^*(ae_0 + be_1) + 1 \otimes f(ae_0 + be_1) = b(1 + \overline{f'(0)}) + a\overline{f(0)}. \tag{1.9}$$

Thus, if $b(1 + \overline{f'(0)}) + a\overline{f(0)} = 0$, then $ae_0 + be_1 \in \ker S^*$. Conversely, let $h \in \ker S^*$. Note that $\mathcal{M}_z^* h = -\langle h, f \rangle$, and since $1 \in \ker \mathcal{M}_z^*$, $h$ belongs to ker $\mathcal{M}_z^2$. Since $\kappa$ satisfies (1.6), by [4, Lemma 1.22(ii)], $h \in \text{span}\{e_0, e_1\}$. This combined with (1.9) yields (1.8).

Assume now that $f(0) = 0$. Thus by (1.8),

$$\ker S^* = \{ae_0 + be_1 : b(1 + \overline{f'(0)}) = 0\}.$$

In case $f'(0) = -1$, ker $S^* = \text{span}\{e_0, e_1\}$. Otherwise, ker $S^* = \text{span}\{e_0\}$. If $f(0) \neq 0$, then by (1.8), ker $S^*$ consists of elements of the form $ae_0 + be_1$ where $a = -\frac{b(1 + \overline{f'(0)})}{\overline{f(0)}}$ and we get the desired conclusion in (ii). Since $\mathcal{H}_\kappa$ contains all complex polynomials, it is easy to see using (1.7) that $\kappa(\cdot, 0)$ and $\overline{\partial}\kappa(\cdot, w)|_{w=0}$ are linearly independent. The remaining part now follows from (i) and [7, Proposition 1(i)]. \[\square\]
2. Proofs of Theorems 1.2 and 1.6

In the proof of Theorem 1.2, we employ reproducing kernel techniques to describe the hyper-range of a rank one perturbation of $\mathcal{M}_z$ by $f \otimes g$, where $g \in \ker \mathcal{M}_z^*$. This, combined with Shimorin’s analytic model yields the proof of Theorem 1.6. Recall from [10, Pg 154] that any analytic left-invertible operator $T$ is unitarily equivalent to the operator $\mathcal{M}_z$ of multiplication by $z$ on a reproducing kernel Hilbert space $\mathcal{H}_\kappa$ with kernel $\kappa$. Indeed, $\mathcal{H}_\kappa$ consists of $\ker T^*$-valued holomorphic functions on a disc centred at the origin in the complex plane and the reproducing kernel $\kappa$ of $\mathcal{H}_\kappa$ satisfies $\kappa(\cdot, 0) = 1$ (see [10, Eqn (2.6)]). If, in addition, $T$ is cyclic, then $\ker T^*$ is one-dimensional, and hence, in the proof of Theorem 1.6, we may assume that $\mathcal{H}_\kappa$ is a functional Hilbert space. We need several lemmas to prove Theorems 1.2 and 1.6. The first one is algebraic and holds for any bounded linear operator on a complex Hilbert space.

**Lemma 2.1.** Let $T \in \mathcal{B}(\mathcal{H})$ and let $f, g \in \mathcal{H}$. If $T^*g = 0$, then the following statements are valid:

(i) for any positive integer $n$,

$$
(T + f \otimes g)^n = T^n + \sum_{j=0}^{n-1} \langle f, g \rangle^{n-j-1} T^j f \otimes g 
$$

(ii) if $T$ is cyclic, then $T + f \otimes g$ is $2$-cyclic.

**Proof.** Suppose that $T^*g = 0$.

(i) Clearly, the formula (2.1) holds for $n = 1$. Assume the formula (2.1) for a positive integer $n \geq 2$. Since $T^*g = 0$, $(f \otimes g)T = f \otimes T^*g = 0$ and for any $h \in \mathcal{H}$,

$$(f \otimes g)(T^j f \otimes g)h = \langle h, g \rangle \langle T^j f, g \rangle f = 0, \quad j \geq 1.$$ 

This, combined with the induction hypothesis yields

$$
(T + f \otimes g)(T + f \otimes g)^n = T^{n+1} + \sum_{j=0}^{n-1} \langle f, g \rangle^{n-j-1} T^j + 1 f \otimes g
$$

$$
+ (f \otimes g)T^n + \sum_{j=0}^{n-1} \langle f, g \rangle^{n-j-1}(f \otimes g) T^j f \otimes g
$$

$$
= T^{n+1} + \sum_{j=0}^{n-1} \langle f, g \rangle^{n-j-1} T^j + 1 f \otimes g
$$

$$
+ \langle f, g \rangle^{n-1}(f \otimes g)^2
$$

$$
= T^{n+1} + \sum_{j=0}^{n} \langle f, g \rangle^{n-j} T^j f \otimes g.
$$

This completes the verification of (i).
(ii) Let $\xi \in \mathcal{H}$. A routine verification using (2.1) shows that for any integer $n \geq 1$,

$$(T + f \otimes g)^n \xi = T^n \xi + \langle \xi, g \rangle \sum_{j=0}^{n-1} (f, g)^{n-j-1} T^j f,$$

$$(T + f \otimes g)^{n-1} f = \sum_{j=0}^{n-1} (f, g)^{n-j-1} T^j f.$$

It is immediate that

$$T^n \xi = (T + f \otimes g)^n \xi - \langle \xi, g \rangle (T + f \otimes g)^{n-1} f.$$ 

Thus, if $\xi$ is a cyclic vector for $T$, then $T + f \otimes g$ is 2-cyclic with the set of cyclic vectors equal to $\{\xi, f\}$. □

We also need the following characterization of closed range multiplication operators $M_z$ on functional Hilbert spaces.

**Lemma 2.2.** Let $\mathcal{H}_K$ be a functional Hilbert space and let $M_z$ denote the operator of multiplication by $z$. Assume that $\dim \ker M_z^* = 1$. Then $\text{ran} \, M_z$ is closed if and only if

$$\text{ran} \, M_z = \{ f \in \mathcal{H}_K : f(0) = 0 \}.$$ 

In particular, if $M_z$ is left-invertible, then for any holomorphic function $h : \mathbb{D} \to \mathbb{C}$,

$$h \in \mathcal{H}_K \text{ if and only if } zh \in \mathcal{H}_K.$$ 

**Proof.** By Remark 1.1, $\ker M_z^*$ is spanned by the constant function 1. It follows that

$$\text{ran} \, M_z \subseteq \overline{\text{ran} \, M_z} = (\ker M_z^*)^\perp = \{1\}^\perp = \{ f \in \mathcal{H}_K : f(0) = 0 \}. \quad (2.2)$$

This yields the desired equivalence. To see the remaining part, assume that $M_z$ is left-invertible. Note that if $zh \in \mathcal{H}_K$, then $zh \in \{ f \in \mathcal{H}_K : f(0) = 0 \}$, and hence $zh$ belongs to the range of $M_z$. Thus there exists $g \in \mathcal{H}_K$ such that $zh = zg$. It is now easy to see that $h = g \in \mathcal{H}_K$. □

The following lemma required in the proof of Theorem 1.6 describes the hyper-range of rank one perturbations of left invertible analytic operators (cf. [2, Lemma 2.4]).

**Lemma 2.3.** For $r > 0$, let $\mathcal{H}$ be a Hilbert space of complex-valued holomorphic functions on $\mathbb{D}_r$ such that $\mathcal{H}$ contains all complex polynomials in $z$. Assume that

$$h \in \mathcal{H} \text{ if and only if } zh \in \mathcal{H}. \quad (2.3)$$

Let $M_z$ denote the operator of multiplication by $z$ and let $f, g \in \mathcal{H}$ be such that $g \in \ker M_z^*$. Then the following statements are valid:

(i) if $\langle f, g \rangle = 0$, then $M_z + f \otimes g$ is analytic,
(ii) If \( f(0) \neq 0 \) and \( \langle f, g \rangle \neq 0 \), then the hyper-range \( \mathcal{R}_\infty \) of \( \mathcal{M}_z + f \otimes g \) is given by
\[
\mathcal{R}_\infty = \begin{cases} 
\text{span}\{h_0\} & \text{if } h_0 \text{ belongs to } \mathcal{H}, \\
\{0\} & \text{otherwise},
\end{cases}
\]
where \( h_0 := \sum_{j=0}^\infty \left( \sum_{i=0}^j \hat{f}(j-i) \right) z^i \).

(iii) If \( \langle f, g \rangle \neq 0 \) and \( h_0 \) converges in \( \mathcal{H} \), then \( h_0 \) is an eigenfunction of \( \mathcal{M}_z + f \otimes g \) with respect to the simple eigenvalue \( \langle f, g \rangle \).

**Proof.** Let \( \mathcal{M}_{f,g} := \mathcal{M}_z + f \otimes g \). Note that \( h \in \mathcal{R}_\infty \) if and only if for every integer \( n \geq 1 \), there exists \( h_n \in \mathcal{H} \) such that \( h = \mathcal{M}_{f,g} h_n \). Thus, by Lemma 2.1(i), \( h \in \mathcal{R}_\infty \) if and only if there exists \( \{h_n\}_{n \geq 1} \subseteq \mathcal{H} \) such that
\[
h = z^n h_n + \langle h_n, g \rangle \sum_{j=0}^{n-1} \langle f, g \rangle^{n-j-1} z^j f, \quad n \geq 1. \tag{2.4}
\]
(i): In case \( \langle f, g \rangle = 0 \), \( h = z^n h_n + \langle h_n, g \rangle z^{n-1} f \) for every positive integer \( n \), and hence \( h \) has zero at 0 of arbitrary order. Thus \( h = 0 \) in this case.

(ii): Assume that \( \hat{f}(0) \neq 0 \) and \( \langle f, g \rangle \neq 0 \). Fix an integer \( n \geq 1 \). Letting \( f(z) = \sum_{k=0}^\infty \hat{f}(k) z^k \) in (2.4), we get
\[
h = z^n h_n + \langle h_n, g \rangle \sum_{k=0}^\infty \sum_{i=0}^{n-1} \hat{f}(k) \langle f, g \rangle^{n-i-1} z^{i+k}, \quad n \geq 1.
\]
Comparing the coefficients on both sides, we obtain
\[
\hat{h}(j) = \langle h_n, g \rangle \sum_{i=0}^j \langle f, g \rangle^{n-i-1} \hat{f}(j-i), \quad j = 0, \ldots, n-1, \quad n \geq 1. \tag{2.5}
\]

Letting \( j = 0 \) in the above identity, we get \( \hat{h}(0) = \langle h_n, g \rangle \langle f, g \rangle^{n-1} \hat{f}(0) \). Thus, (2.5) simplifies to
\[
\hat{h}(j) = \frac{\hat{h}(0)}{\hat{f}(0)} \sum_{i=0}^j \frac{\hat{f}(j-i)}{\langle f, g \rangle^i}, \quad j = 0, \ldots, n-1, \quad n \geq 1,
\]
where we used the assumption that \( \hat{f}(0) = \hat{f}(0) \) and \( \langle f, g \rangle \) are nonzero. Since \( n \) was arbitrary, we obtain that \( h = \frac{\hat{h}(0)}{\hat{f}(0)} h_0 \). Thus, if \( h_0 \) does not belong to \( \mathcal{H} \), then \( \mathcal{R}_\infty = \{0\} \).

We now check that if \( h_0 \in \mathcal{H} \), then \( \mathcal{R}_\infty \) is spanned by \( h_0 \). To see this, assume that \( h_0 \in \mathcal{H} \). Note that \( h_0 \) belongs to the hyper-range of \( \mathcal{M}_{f,g} \) provided there exists a sequence \( \{h_n\}_{n \geq 1} \in \mathcal{H} \) such that (2.4) holds for \( h = h_0 \). It follows from the definition of \( \hat{h}_0(\cdot) \) and (2.3) that \( h_0 - \sum_{j=0}^{n-1} \frac{z^j f}{\langle f, g \rangle^j} \) is divisible by \( z^n \) in \( \mathcal{H} \). Thus there exists a sequence \( \{h_n\}_{n \geq 1} \in \mathcal{H} \) such that
\[
h_0 - \sum_{j=0}^{n-1} \frac{z^j f}{\langle f, g \rangle^j} = z^n h_n, \quad n \geq 1. \tag{2.6}
\]
Fix \( n \geq 1 \). Comparing the coefficient of \( z^n \) on both sides, we obtain
\[
\hat{h}_0(n) - \sum_{j=0}^{n-1} \frac{\hat{f}(n-j)}{(f, g)^j} = \hat{h}_n(0).
\]
However, by the definition of \( \hat{h}_0 \), this simplifies to
\[
\frac{\hat{f}(0)}{(f, g)^n} = \hat{h}_n(0). \tag{2.7}
\]
Since any \( \phi \in \mathcal{H} \) can be written as \( \phi = \hat{\phi}(0) + z\psi \) for \( \psi \in \mathcal{H} \) (see (2.3)) and \( g \in \ker \mathcal{M}_z^* \), we have \( \langle \phi, g \rangle = \langle \hat{\phi}(0), g \rangle \), and hence
\[
\langle h_n, g \rangle = \langle \hat{h}_n(0), g \rangle \overset{(2.7)}{=} \frac{\langle \hat{f}(0), g \rangle}{(f, g)^n} = \frac{1}{(f, g)^{n-1}}.
\]
Combining this with (2.6), we see that \( \{h_n\}_{n \geq 1} \) satisfies
\[
h_0 = z^n h_n + \langle h_n, g \rangle \sum_{j=0}^{n-1} (f, g)^{n-j-1} z^j f, \quad n \geq 1.
\]
This completes the proof of (ii).

(iii) Assume that \( \langle f, g \rangle \neq 0 \) and \( h_0 \in \mathcal{H} \). By (ii), \( \mathcal{R}_\infty \) is one-dimensional invariant subspace of \( \mathcal{M}_z + f \otimes g \), and hence there exists \( \lambda \in \mathbb{C} \) such that \( (\mathcal{M}_z + f \otimes g) h_0 = \lambda h_0 \). Since \( g \in \ker \mathcal{M}_z^* \),
\[
(\mathcal{M}_z + f \otimes g) h_0 = \sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} \frac{\hat{f}(j-i)}{(f, g)^i} \right) z^{j+1} + \sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} \frac{\hat{f}(j-i)}{(f, g)^i} \right) (z^j, g) f
\]
\[
= \sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} \frac{\hat{f}(j-i)}{(f, g)^i} \right) z^{j+1} + (f, g) \sum_{k=0}^{\infty} \hat{f}(k) z^k
\]
\[
= \lambda \sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} \frac{\hat{f}(j-i)}{(f, g)^i} \right) z^{j}.
\]
Comparing the constant terms on both sides, we get \( \lambda = \langle f, g \rangle \). Since \( \mathcal{R}_\infty \) contains eigenfunction corresponding to any nonzero eigenvalue of \( \mathcal{M}_z + f \otimes g \) and since \( \mathcal{R}_\infty \) is one dimensional, \( \langle f, g \rangle \) is a simple eigenvalue of \( \mathcal{M}_z + f \otimes g \), completing the proof.

**Proof of Theorem 1.2.** Since \( \mathcal{H}_\kappa \) is a functional Hilbert space, \( \langle f, 1 \rangle = f(0) \). By Lemma 2.2, (2.3) holds for \( \mathcal{H} = \mathcal{H}_\kappa \). The desired conclusion is now immediate from Lemma 2.3 (with \( g = 1 \)).

Here is a particular case of Theorem 1.2.

**Corollary 2.4.** Assume the hypotheses of Theorem 1.2. If \( \mathcal{M}_z \) is left-invertible, and \( f \) is a polynomial of degree \( n \), then the operator \( \mathcal{M}_z + f \otimes 1 \) is analytic if and only if exactly one of the following holds:

(i) \( f(0) = 0 \),
(ii) \( f(0) \neq 0, f(f(0)) \neq 0 \) and \( \frac{1}{f(0) - z} \notin \mathcal{K} \).

**Proof.** Assume that \( \mathcal{M} \) is left-invertible. Let \( f(z) = \sum_{k=0}^{n} \hat{f}(k)z^k, \) \( z \in \mathbb{D}, \) and assume that \( f(0) \neq 0. \) Since \( \mathcal{K} \) contains all polynomials (see Remark 1.1),

\[
h_0 \in \mathcal{K} \iff h_1 := \sum_{j=n}^{\infty} \left( \sum_{i=j-n}^{j} \frac{\hat{f}(j-i)}{f(0)^j} \right) z^j \in \mathcal{K}, \tag{2.8}
\]

where \( h_0 \) is as given in Lemma 2.3(ii). Also, since \( \hat{f}(j) = 0 \) for \( j > n, \)

\[
h_1 = \sum_{j=n}^{\infty} \left( \sum_{i=j-n}^{j} \frac{\hat{f}(j-i)}{f(0)^j} \right) z^j
= \left( \sum_{k=0}^{n} \hat{f}(k)f(0)^k \right) \sum_{j=n}^{\infty} \frac{z^j}{f(0)^j}
= f(f(0)) \sum_{j=n}^{\infty} \frac{z^j}{f(0)^j}. \tag{2.9}
\]

This together with (2.8) shows that \( h_0 \in \mathcal{K} \) if and only if either \( f(f(0)) = 0 \) or \( \frac{1}{f(0) - z} \in \mathcal{K}. \) One may now apply Theorem 1.2. \( \square \)

**Remark 2.5.** Assume that \( f \) is a polynomial such that \( f(0) \neq 0 \) and \( f(f(0)) \neq 0 \). By (2.8) and (2.9), \( h_0 \in \mathcal{K} \) if and only if \( \frac{1}{f(0) - z} \in \mathcal{K}. \)

We now complete the proof of Theorem 1.6.

**Proof of Theorem 1.6.** In view of the discussion prior to Lemma 2.1, we may assume that \( T = \mathcal{M} \) is acting on a functional Hilbert space \( \mathcal{K} \). Since \( \mathcal{M} \) is an analytic injection, \( \sigma_p(\mathcal{M}) = \emptyset. \) Hence, by Proposition 1.4,

\[
\sigma_l(\mathcal{M} + f \otimes g) = \sigma_l(\mathcal{M}) \cup \sigma_p(\mathcal{M} + f \otimes g).
\]

Consider the following cases:

- If \( \langle f, g \rangle = 0, \) then by Lemma 2.3(i), \( \mathcal{M} + f \otimes g \) is analytic, and hence \( \sigma_p(\mathcal{M} + f \otimes g) \subseteq \{0\} = \{\langle f, g \rangle\}. \)
- If \( \langle f, g \rangle \neq 0, \) then by Proposition 1.5(ii) and Lemma 2.3, \( \sigma_p(\mathcal{M} + f \otimes g) \subseteq \{\langle f, g \rangle\} \) (since \( 0 \notin \sigma_l(\mathcal{M})). \)

All conclusions now follow from (i) and (ii) of Proposition 1.4. \( \square \)

The following is a consequence of Theorem 1.6.

**Corollary 2.6.** Under the hypotheses of Theorem 1.2, if \( \mathcal{M} \) is left-invertible, \( \mathcal{M} + f \otimes 1 \) is analytic and \( f(0) \neq 0, \) then

\[
\sigma_l(\mathcal{M} + f \otimes 1) = \sigma_l(\mathcal{M}), \quad r(\mathcal{M} + f \otimes 1) = r(\mathcal{M}).
\]

Following \[10\], we call the operator \( T' := T(T^*T)^{-1} \) the *Cauchy dual* of \( T \) whenever \( T \) is left-invertible. For operators satisfying the so-called kernel condition (see \[1, \text{Sect. 2}\]), the wandering subspace property is preserved under the rank one perturbation \( 1 \otimes 1. \)
Corollary 2.7. Assume the hypotheses of Theorem 1.2. Assume that $\mathcal{M}_z$ is left-invertible and satisfies the kernel condition:

$$\mathcal{M}_z^* \mathcal{M}_z (\ker \mathcal{M}_z^*) \subseteq \ker \mathcal{M}_z^*.$$ 

If $\mathcal{M}_z$ has the wandering subspace property, then the operator $\mathcal{M}_z + 1 \otimes 1$ also has the wandering subspace property.

Proof. Recall the fact that for any left-invertible operator $S$ on $\mathcal{H}$, $S$ is analytic if and only if the Cauchy dual $S'$ of $S$ has the wandering subspace property (see [10, Corollary 2.8]). Because of this fact, it suffices to check that if $\mathcal{M}_z'$ is analytic, then so is $(\mathcal{M}_z + 1 \otimes 1)'$.

Let $T$ be a left invertible operator and $S = T + f \otimes g$ with $f \in \ker T^*$ of unit norm and $g \in \mathcal{H}$. Note that $S^* S = (T^* + g \otimes f)(T + f \otimes g) = T^* T + g \otimes g$.

It is easy to see using $T^* T' = (T^* T)^{-1}$ that

$$(S^* S)^{-1} = (T^* T)^{-1} - (1 + \|T' g\|^2)^{-1} (T^* T)^{-1} g \otimes (T^* T)^{-1} g.$$ 

One may now verify that $S'$ is given by

$$(T + f \otimes g)' = T' + (1 + \|T' g\|^2)^{-1} (f - T' g) \otimes (T^* T)^{-1} g.$$ (2.10)

Since $\mathcal{M}_z$ satisfies the kernel condition, by [1, Proposition 2.1],

$$(\mathcal{M}_z^* \mathcal{M}_z)^{-1} \ker \mathcal{M}_z^* = \ker \mathcal{M}_z^*.$$ 

Thus $\mathcal{M}_z' 1 = \beta z$ for some scalar $\beta$. Apply now the formula (2.10) to $f = g = 1$ to conclude that $(\mathcal{M}_z + 1 \otimes 1)'$ is of the form $\mathcal{M}_z' + (\alpha' - \beta' z) \otimes 1$ for some constants $\alpha', \beta'$ with $0 \leq |\alpha'| \leq 1$. The desired conclusion now follows from Theorem 1.2. \qed

3. An example

We conclude this note with one motivating example illustrating the general picture of the invariance of the left spectra.

Example 3.1. Let $\mathcal{H}_K$ be a functional Hilbert space such that all complex polynomials are dense in $\mathcal{H}_K$. Let $\mathcal{M}_z$ denote the operator of multiplication by $z$ on $\mathcal{H}_K$. For scalars $a, b \in \mathbb{C}$, let $f(z) = az + b$, $z \in \mathbb{D}$. By Corollary 2.4, $\mathcal{M}_z + f \otimes 1$ is analytic if and only if $b = 0$ or

$$b \neq 0, a \neq -1 \text{ and } \frac{1}{b - z} \notin \mathcal{H}_K.$$ 

Assume that $\mathcal{M}_z$ is left invertible. By Theorem 1.6,

$$\begin{align*}
\sigma_l(\mathcal{M}_z + f \otimes 1) \setminus \{b\} &= \sigma_l(\mathcal{M}_z) \setminus \{b\}, \\
r(\mathcal{M}_z + f \otimes 1) &= \max\{r(\mathcal{M}_z), |b|\}. 
\end{align*}$$ (3.1)

Furthermore, we have the following:
• Assume that $b = 0$ and $a \neq 0$. Then $f = az$ and
\[(\mathcal{M}_z + f \otimes 1)(1) = (1 + a)z.\]
Thus $1 \in \ker(\mathcal{M}_z + f \otimes 1)$ if and only if $a = -1$. Hence, by Proposition 1.5, $\ker(\mathcal{M}_z - z \otimes 1)$ is spanned by \{1\}. In this case, $\mathcal{M}_z - z \otimes 1$ is analytic and
\[\sigma_1(\mathcal{M}_z - z \otimes 1) = \sigma_1(\mathcal{M}_z) \cup \{0\}, \quad r(\mathcal{M}_z - z \otimes 1) = r(\mathcal{M}_z).\]
• Assume that $b \neq 0$, $a \neq -1$ and $\frac{1}{b - z} \in \mathcal{H}_\kappa$ (so that $|b| \geq 1$). Then by Lemma 2.3(iii) and Remark 2.5, $b \in \sigma_p(\mathcal{M}_z + f \otimes 1)$, and hence by (3.1), we obtain
\[\sigma_1(\mathcal{M}_z + f \otimes 1) = \sigma_1(\mathcal{M}_z) \cup \{b\}, \quad r(\mathcal{M}_z + f \otimes 1) = \max\{r(\mathcal{M}_z), |b|\}.\]
• Assume that $b \neq 0$, $a \neq -1$ and $\frac{1}{b - z} \notin \mathcal{H}_\kappa$. Then, by Lemma 2.3(ii) and Remark 2.5, $\mathcal{M}_z + f \otimes 1$ is analytic. Also, by Proposition 1.5(ii), $0$ does not belong to $\sigma_p(\mathcal{M}_z + f \otimes 1)$. Since the point spectrum of an analytic operator is contained in \{0\}, $\sigma_p(\mathcal{M}_z + f \otimes 1) = \emptyset$. It now follows from (3.1) that
\[\sigma_1(\mathcal{M}_z + f \otimes 1) = \sigma_1(\mathcal{M}_z), \quad r(\mathcal{M}_z + f \otimes 1) = r(\mathcal{M}_z).\]

It is evident that the above discussion extends, with suitable modifications, to the case when $f$ is a polynomial.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

[1] Anand, A., Chavan, S., Jabłoński, Z.J., Stochel, J.: A solution to the Cauchy dual subnormality problem for 2-isometries. J. Funct. Anal. 277, 108292 (2019)
[2] Anand, A., Chavan, S., Trivedi, S.: Analytic m-isometries without the wandering subspace property. Proc. Am. Math. Soc. 148, 2129–2142 (2020)
[3] Chavan, S., Misra, G.: Notes on the Brown-Douglas-Fillmore theorem, Cambridge-IISc Series, p. xi+246. Cambridge University Press, Cambridge (2021)
[4] Cowen, M., Douglas, R.: Complex geometry and operator theory. Acta Math. 141, 187–261 (1978)
[5] Curto, R., Salinas, N.: Generalized Bergman kernels and the Cowen-Douglas theory. Am. J. Math. 106, 447–488 (1984)
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[6] Das, S., Sarkar, J.: Left-invertibility of rank-one perturbations. Complex Anal. Oper. Theory. 16(109), 22 (2022)

[7] Herrero, D.: On multicyclic operators. Integr. Equ. Oper. Theory. 1, 57–102 (1978)

[8] Jabłoński, Z.J., Kośmider, J.: $m$-isometric composition operators on directed graphs with one circuit. Integr. Equ. Oper. Theory. 93, 26 (2021)

[9] Paulsen, V., Raghupathi, M.: An introduction to the theory of reproducing kernel Hilbert spaces. Cambridge Studies in Advanced Mathematics, vol. 152, p. 182. Cambridge University Press, Cambridge (2016)

[10] Shimorin, S.: Wold-type decompositions and wandering subspaces for operators close to isometries. J. Reine Angew. Math. 531, 147–189 (2001)

[11] Stampfli, J.G.: Perturbations of the shift. J. Lond. Math. Soc. 40, 345–347 (1965)

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Received: December 27, 2022.
Accepted: March 13, 2023.