ON EQUIVARIANT TRIANGULATED CATEGORIES

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Abstract. Consider a finite group \( G \) acting on a triangulated category \( \mathcal{T} \). In this paper we try to understand when the category \( \mathcal{T}^G \) of \( G \)-equivariant objects in \( \mathcal{T} \) is triangulated. We prove that it is so in two cases: the action on the derived category \( D^b(A) \) induced by an action on an abelian category \( A \) and the action on the homotopy category \( H^0(A) \) of a pretriangulated DG-category \( A \) induced by an action on \( A \). Also, we show that the relation “to be an equivariant category with respect to a finite abelian group action” is symmetric on idempotent complete additive categories.

1. Introduction

Triangulated categories became very popular in algebra, geometry and topology in last decades. In algebraic geometry, they arise as derived categories of coherent sheaves on algebraic varieties or stacks. It turned out that some geometry of varieties can be captured well through their derived categories and homological algebra of these categories. Therefore it is always interesting and important to understand how different geometrical operations, constructions, relations look like on the derived category side.

In this paper we are interested in autoequivalences of derived categories or, more general, in group actions on triangulated categories. For \( X \) an algebraic variety, there are “expected” autoequivalences of \( D^b(\text{coh}(X)) \) which are induced by automorphisms of \( X \) or by tensoring into line bundles on \( X \). If \( X \) is a smooth Fano or if \( K_X \) is ample, essentially that is all: Bondal and Orlov have shown in [3] that for smooth irreducible projective variety \( X \) with \( K_X \) or \( -K_X \) ample all autoequivalences of \( D^b(\text{coh}(X)) \) are generated by automorphisms of \( X \), twists into line bundles on \( X \) and translations. On the contrary, varieties with zero \( K_X \) may have many non-trivial autoequivalences of \( D^b(\text{coh}(X)) \). For example, the autoequivalence group of derived category of abelian varieties was calculated by Orlov in [13].

Our goal is to study, for an action of a group \( G \) on a triangulated category \( \mathcal{T} \), the “quotient category” \( \mathcal{T}^G \), or the category of \( G \)-equivariant objects in \( \mathcal{T} \).

The motivation comes from the concept of a \( G \)-equivariant sheaf. If \( X \) is an algebraic variety and the group \( G \) (finite or reductive algebraic) acts freely on \( X \), then \( G \)-equivariant coherent sheaves on \( X \) correspond to coherent sheaves on the quotient variety \( X/G \). On the categorical level, the category \( \text{coh}^G(X) \) of \( G \)-equivariant coherent sheaves on \( X \) is equivalent to the category \( \text{coh}(X/G) \). For arbitrary \( G \)-actions, \( G \)-equivariant sheaves

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correspond to sheaves on the quotient stack $X//G$ which is different from the quotient variety in general.

Following Deligne [6], one can define action of a group on a category and consider equivariant objects in the category with respect to the action, see Section 3. For an action of a group $G$ on a category $\mathcal{C}$, we denote the category of $G$-equivariant objects in $\mathcal{C}$ by $\mathcal{C}^G$. For $\mathcal{C} = \text{coh}(X)$ and the action on $\mathcal{C}$ induced by an action of $G$ on a variety $X$, $G$-equivariant objects in $\mathcal{C}$ are $G$-equivariant sheaves on $X$. Another basic example of a group action on $\text{coh}(X)$ comes from twisting into line bundles. If $G \subset \text{Pic}(X)$ is a finite subgroup in the Picard group of $X$, then tensor product with bundles from $G$ defines an action of $G$ on $\text{coh}(X)$. In this case, $G$-equivariant objects in $\text{coh}(X)$ correspond to coherent sheaves on a non-ramified $|G|$-fold cover of $X$ which is given explicitly as the relative spectrum $\text{Spec}_X(\oplus_{L \in G} L^{-1})$.

Instead of abelian categories of sheaves, one could consider derived categories and group actions on them. What categories would equivariant objects form? For the two examples of actions mentioned above the result is not surprising. We have

**Theorem 1.1** (first proved in [7]). Suppose $G$ is a group, $X$ is an algebraic $G$-variety over $k$ and $\text{char}(k)$ does not divide $|G|$. Then $\mathcal{D}^b(\text{coh}(X))^G \cong \mathcal{D}^b(\text{coh}^G(X))$.

and

**Theorem 1.2** (see Theorem 5.4 below). Let $X$ be an algebraic variety over a field $k$ and $G \subset \text{Pic}(X)$ be a finite subgroup. Suppose $\text{char}(k)$ does not divide $|G|$. Let $G$ act on $\mathcal{D}^b(\text{coh}(X))$ by twisting into line bundles $L \in G$. Denote by $Y$ the relative spectrum $\text{Spec}_X(\oplus_{L \in G} L^{-1})$. Then $\mathcal{D}^b(\text{coh}(X))^G \cong \mathcal{D}^b(\text{coh}(Y))$.

Hence equivariant objects in triangulated categories are of some interest. Let $\mathcal{T}$ be a triangulated category, suppose one has an exact action of a group $G$ on $\mathcal{T}$. Consider the “quotient category” $\mathcal{T}^G$. It is natural to ask whether $\mathcal{T}^G$ is a triangulated category. Unfortunately, in general there are no reasons why this category should be triangulated.

There is a natural way to introduce a shift functor and a class of distinguished triangles on $\mathcal{T}^G$, see Definition 4.1. But it is not clear how to check that any morphism in $\mathcal{T}^G$ fits into a distinguished triangle without additional assumptions. That is related with non-functoriality of cones in triangulated categories, so no positive solution of this bug is expected in general setting.

In this paper we prove that $\mathcal{T}^G$ has a natural structure of a triangulated category in the two following cases: $\mathcal{T}$ is a derived category of an abelian category $\mathcal{A}$ and $\mathcal{T}$ has a DG-enhancement $\mathcal{A}$. In both cases we suppose that the $G$-action on $\mathcal{T}$ is induced by a $G$-action on $\mathcal{A}$. In the first case we simply demonstrate that $\mathcal{T}^G$ is equivalent to the derived category of $\mathcal{A}^G$, what can hardly be regarded as a new result. In the second case starting with a $G$-action on a DG-category $\mathcal{A}$, we construct DG-category $\mathcal{Q}_G(\mathcal{A})$ being a DG-enhancement of $H^0(\mathcal{A})^G$. Similar construction was supposed by P. Sosna in paper [15], which motivated our research of equivariant DG-categories. Compared to his
one, our construction has better functorial properties, in particular, quasi-equivalent $\mathcal{A}$ and $\mathcal{A}'$ produce quasi-equivalent $Q_G(A)$ and $Q_G(A')$.

As an application, we conclude that for smooth projective irreducible variety $X$ with ample $K_X$ or $-K_X$ and any action of a finite group $G$ on $\mathcal{T} = \mathcal{D}^b(\text{coh}(X))$, the equivariant category $\mathcal{T}^G$ is triangulated, see Theorem 5.5.

For an application where the use of DG-enhancements is essential, we consider the subcategory $\mathcal{D}^b_Z(X) \subset \mathcal{D}^b(\text{coh}(X))$ of objects supported in a closed subvariety $Z \subset X$ and a $G$-action on $\mathcal{D}^b_Z(X)$ induced by a $G$-action on $X$, see Theorem 6.11.

As for “non-abelian” group actions on derived categories of coherent sheaves, the situation with applications is not so good. To use our results, it suffices to lift a group action on a derived category onto DG-level. In particular, it requires to lift an autoequivalence of a derived category to a DG-autoequivalence of certain DG-enhancement. Ever for an autoequivalence $\mathcal{D}^b(\text{coh}(X)) \rightarrow \mathcal{D}^b(\text{coh}(X))$ given by a kernel on $X \times X$, it is not clear how to do it. It is known (see Lunts and Orlov’s paper [10]) that for a projective variety, DG-enhancement of its derived category of coherent sheaves is strongly unique. But this uniqueness is too flexible, it allows to lift action onto DG-level only in a very weak sense: a sense of quasi-functors, which is not suitable for our purposes. The problem of constructing a DG-enhancement of $\mathcal{D}^b(\text{coh}(X))$ with a compatible group action on it seems to be rather interesting to investigate.

Our treatment is based on descent theory. This point of view was developed by the author in [7] and involves the language of monads and modules. We consider equivariant categories like $\mathcal{T}^G$ or $\mathcal{D}^b(\text{coh}(X))^G$ as certain categories of descent data. Namely, with any action of a group $G$ on a category $\mathcal{C}$ we associate a comonad $T_G$ on $\mathcal{C}$ such that the corresponding category of comodules is equivalent to $\mathcal{C}^G$. Thus, key point in the proof of Theorems 5.1 and 6.7 is to show that a certain comparison functor is an equivalence. This is done using a rather specific but powerful special case of Beck theorem (see Proposition 2.8) which is valid for triangulated categories.

Consider a Galois covering $X \rightarrow Y$ of algebraic varieties with an abelian Galois group. Theorems 1.1 and 1.2 imply that either of the categories $\mathcal{D}^b(\text{coh}(X))$ and $\mathcal{D}^b(\text{coh}(Y))$ can be reconstructed from another one as a category of equivariant objects with respect to a certain group action. In Section 7, using the language of monads, we demonstrate that this situation is typical, proving the following reversion theorem:

**Theorem 1.3.** Let $\mathcal{B}$ and $\mathcal{C}$ be idempotent complete additive categories over an algebraically closed field $k$, suppose $\text{char}(k)$ does not divide $|G|$. Suppose $\mathcal{B} \cong \mathcal{C}^G$ for some action of a finite abelian group $G$ on $\mathcal{C}$. Then $\mathcal{C} \cong \mathcal{B}^{G^\vee}$ for some action of the dual group $G^{\vee}$ on $\mathcal{B}$.

In Section 2 we recall necessary facts about monads and comonads. In Section 3 we define group actions and equivariant objects and introduce adjoint functors, monads and comonads needed for descent theory. In Section 4, for a triangulated category $\mathcal{T}$ with a group action, we define shift functor and a class of distinguished triangles in $\mathcal{T}^G$. Under some conditions, in Sections 5 and 6 we prove that this definition does make $\mathcal{T}^G$
a triangulated category. In Section 5 we consider action of $G$ on a derived category $D^b(A)$ of an abelian category $A$, induced by a $G$-action on $A$. In Section 6 we consider action of $G$ on a DG-enhanced triangulated category $T$, induced by a $G$-action on a DG-enhancement $A$. In Section 7 we prove a “reversion theorem” for quotient categories modulo finite abelian group actions.

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2. Preliminaries on (co)monads

We recall some facts concerning (co)monads and (co)modules. More details can be found in books by Barr-Wells [2, chapter 3] and MacLane [11, chapter 6].

Let $\mathcal{C}$ be a category.

**Definition 2.1.** A comonad $T = (T, \varepsilon, \delta)$ on the category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and of natural transformations of functors $\varepsilon : T \to \text{Id}_\mathcal{C}$ and $\delta : T \to T^2 = TT$ such that the following diagrams are commutative:

\[
\begin{array}{ccc}
T & \xrightarrow{\delta} & T^2 \\
\downarrow{\delta} & & \downarrow{T\varepsilon} \\
T^2 & \xrightarrow{\varepsilon T} & T,
\end{array}
\quad
\begin{array}{ccc}
T & \xrightarrow{\delta} & T^2 \\
\downarrow{\delta} & & \downarrow{T\delta} \\
T^2 & \xrightarrow{\delta T} & T^3.
\end{array}
\]

**Definition 2.2.** Two comonads $T = (T, \varepsilon, \delta)$ and $T' = (T', \varepsilon', \delta')$ on the category $\mathcal{C}$ are isomorphic if there exists an isomorphism of functors $T \to T'$ compatible with $\varepsilon$-s and $\delta$-s.

**Example 2.3.** Consider a pair of adjoint functors: $P^* : \mathcal{B} \to \mathcal{C}$ (left) and $P_* : \mathcal{C} \to \mathcal{B}$ (right). Let $\eta : \text{Id}_\mathcal{B} \to P_*P^*$ and $\varepsilon : P^*P_* \to \text{Id}_\mathcal{C}$ be the natural adjunction morphisms. Define a triple $(T, \varepsilon, \delta)$ by taking $T = P^*P_*$ and $\delta = P^*\eta P_* : P^*P_* \to P^*P_*P^*P_*$. Then $T = (T, \varepsilon, \delta)$ is a comonad on $\mathcal{C}$.

**Definition 2.4.** The comonad introduced above will be denoted $T(P^*, P_*)$.

Essentially, any comonad can be obtained in this way from a pair of adjoint functors. This follows from the below construction due to Eilenberg-Moore.

**Definition 2.5.** Suppose $T = (T, \varepsilon, \delta)$ is a comonad on $\mathcal{C}$. A comodule over $T$ (it is sometimes called a $T$-coalgebra) is a pair $(F, h)$ where $F \in \text{Ob} \mathcal{C}$ and $h : F \to TF$ is a morphism satisfying the following two conditions:

1. the composition

   \[ F \xrightarrow{h} TF \xrightarrow{\varepsilon F} F \]

   is the identity;
(2) the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{h} & TF \\
\downarrow{h} & & \downarrow{Th} \\
T^2F & \xrightarrow{\delta F} & T^2F
\end{array}
\]

commutes.

A morphism between comodules is defined in the natural way.

All comodules over a given comonad \( T \) on \( \mathcal{C} \) form a category which is denoted \( \mathcal{C}_T \). Define a functor \( Q_*: \mathcal{C} \rightarrow \mathcal{C}_T \) by

\[ Q_*F = (TF, \delta F), \quad Q_*f = Tf, \]

define \( Q^*: \mathcal{C}_T \rightarrow \mathcal{C} \) to be the forgetful functor: \((F, h) \mapsto F \). Then the pair of functors \((Q^*, Q_*)\) is an adjoint pair and the comonad \( T(Q^*, Q_*) \) (see Definition 2.4) is \( T \).

**Proposition 2.6** (Comparison theorem, [2, 3.2.3], [11, 6.3]). Assume that a comonad \( T = (T, \varepsilon, \delta) \) on a category \( \mathcal{C} \) is defined by an adjoint pair of functors \( P^*: \mathcal{B} \rightarrow \mathcal{C}, P_*: \mathcal{C} \rightarrow \mathcal{B} \). Then there exist a unique (up to an isomorphism) functor (called comparison functor) \( \Phi: \mathcal{B} \rightarrow \mathcal{C}_T \) such that the diagram of categories

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{P_*} & \mathcal{C} \\
\downarrow{\Phi} & & \downarrow{Q_*} \\
\mathcal{C}_T & \xleftarrow{Q^*} & \mathcal{B}
\end{array}
\]

commutes, i.e. both triangles are commutative:

\[ \Phi P_* \cong Q_*, \quad Q^* \Phi \cong P^*. \]

We want to understand when comparison functor is an equivalence. Exact criterion is given by Beck theorem (see [2, 3.14] or [11, 6.7]) and is rather complicated. Below we present a simple sufficient condition on an adjoint pair providing comparison functor is an equivalence.

First we recall

**Definition 2.7.** A category \( \mathcal{C} \) is called idempotent complete (or Karoubian complete or Cauchy complete) if any projector in \( \mathcal{C} \) splits.

**Proposition 2.8** (see [12], Corollary 3.17 and Proposition 3.18, or [7], Corollaries 3.10 and 3.11). In the above notation suppose that the category \( \mathcal{B} \) is idempotent complete. If the natural morphism of functors \( \eta: \text{Id}_\mathcal{B} \rightarrow P_*P^* \) is a split monomorphism (i.e. has a left inverse morphism of functors) then the comparison functor \( \Phi: \mathcal{B} \rightarrow \mathcal{C}_T \) is an equivalence.

If the natural morphism \( \eta(F): F \rightarrow P_*(P^*(F)) \) splits for any object \( F \in \mathcal{B} \) then the comparison functor \( \Phi: \mathcal{B} \rightarrow \mathcal{C}_T \) is fully faithful.
The notion of a monad is dual to the notion of a comonad. We present below related definitions and facts.

**Definition 2.9.** A monad \( S = (S, \eta, \mu) \) on a category \( C \) consists of a functor \( S: C \to C \) and of natural transformations of functors \( \eta: \text{Id}_C \to S \) and \( \mu: T^2 = TT \to T \) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
S & \xrightarrow{\eta S} & S^2 \\
\downarrow{s} & & \downarrow{\mu} \\
S^2 & \xrightarrow{\mu} & S
\end{array}
\quad
\begin{array}{ccc}
S^3 & \xrightarrow{S \mu} & S^2 \\
\downarrow{\mu S} & & \downarrow{\mu} \\
S^2 & \xrightarrow{\mu} & S
\end{array}
\]

**Definition 2.10.** Consider a pair of adjoint functors: \( P_*: C \to B \) (left) and \( P_*: B \to C \) (right). The endofunctor \( S = P_\circ P*: C \to C \) together with natural adjunction morphisms forms a monad \( S = (S, \eta, \mu) \) on \( C \).

**Definition 2.11.** Suppose \( S = (S, \eta, \mu) \) is a monad on \( C \). A module over \( S \) is a pair \((F, h)\) where \( F \in \text{Ob}C \) and \( h: SF \to F \) is a morphism satisfying the following two conditions:

\[
1_F = h \circ \eta F: F \to F, \quad h \circ \mu F = h \circ Sh: S^2 F \to F.
\]

All modules over a given monad \( S \) on \( C \) form a category which is denoted \( C^S \). Define a functor \( Q^*: C \to C^S \) by

\[
Q^*F = (SF, \mu F), \quad Q^*f = Sf,
\]

let \( Q_*: C^S \to C \) be the forgetful functor. Then the pair of functors \( (Q^*, Q_*) \) is an adjoint pair and the monad \( S(Q^*, Q_*) \) is \( S \).

**Proposition 2.12** (Comparison theorem for monads). Assume that a monad \( S = S(P_*, P_*) \) on a category \( C \) is defined by an adjoint pair of functors \( P^*: C \to B, P_*: B \to C \). Then there exists a unique (up to an isomorphism) functor (called comparison functor) \( \Phi: B \to C^S \) such that the diagram of categories

\[
\begin{array}{ccc}
C & \xrightarrow{P_\circ P^*} & B \\
\downarrow{Q_\circ Q^*} & & \downarrow{\Phi} \\
C^S
\end{array}
\]

commutes, i.e. both triangles are commutative:

\[
\Phi P^* \cong Q^*, \quad Q_* \Phi \cong P_*.
\]
3. About group actions

Let $\mathcal{C}$ be a pre-additive category, linear over a field $k$. Let $G$ be a finite group, suppose that $|G| \neq 0$ in $k$.

**Definition 3.1.** A (right) action of $G$ on $\mathcal{C}$ is defined by the following data:
- family of autoequivalences $\phi_g : \mathcal{C} \rightarrow \mathcal{C}, g \in G$;
- family of isomorphisms $\varepsilon_{g,h} : \phi_g \phi_h \rightarrow \phi_{hg}$, for which all diagrams
  \[
  \begin{array}{ccc}
  \phi_f \phi_g \phi_h & \xrightarrow{\varepsilon_{g,h}} & \phi_f \phi_{hg} \\
  \downarrow{\varepsilon_{f,g}} & & \downarrow{\varepsilon_{f,gh}} \\
  \phi_g f \phi_h & \xrightarrow{\varepsilon_{g,f,h}} & \phi_{hgf}
  \end{array}
  \]
  are commutative.

**Remark 3.2.** We do not require $\phi_e$ to be the identity functor, but the definition implies they are naturally isomorphic. Indeed, we have an isomorphism of functors $\varepsilon_{e,e} : \phi_e \phi_e \rightarrow \phi_e$, and since $\phi_e$ is fully faithful, we get an isomorphism
  \[
  \phi_e^{-1}(\varepsilon_{e,e}) : \phi_e \rightarrow \text{Id}.
  \]

Denote the inverse isomorphism $\text{Id} \rightarrow \phi_e$ by $u$.

**Example 3.3.** Suppose a group $G$ acts on a scheme $X$. Then $\phi_g = g^* : \text{coh}(X) \rightarrow \text{coh}(X)$ and canonical isomorphisms $g^* h^* \rightarrow (h g)^*$ define an action of $G$ on the category $\text{coh}(X)$.

Suppose $G$ acts on a category $\mathcal{C}$.

**Definition 3.4.** A $G$-equivariant object in $\mathcal{C}$ is a pair $(F, (\theta_g)_{g \in G})$ where $F \in \text{Ob} \mathcal{C}$ and $(\theta_g)_{g \in G}$ is a family of isomorphisms
  \[
  \theta_g : F \rightarrow \phi_g(F),
  \]
such that all diagrams
  \[
  \begin{array}{ccc}
  F & \xrightarrow{\theta_g} & \phi_g(F) \\
  \downarrow{\phi_h} & & \downarrow{\phi_g(\theta_h)} \\
  \phi_h(F) & \xrightarrow{\varepsilon_{g,h}} & \phi_g(\phi_h(F))
  \end{array}
  \]
are commutative. A morphism of $G$-equivariant objects from $(F_1, (\theta_1^g))$ to $(F_2, (\theta_2^g))$ is a morphism $f : F_1 \rightarrow F_2$ compatible with $\theta_g$, i.e. such $f$ that the below diagrams commute for all $g \in G$
  \[
  \begin{array}{ccc}
  F_1 & \xrightarrow{\theta_1^g} & \phi_g(F_1) \\
  \downarrow{f} & & \downarrow{\phi_g(f)} \\
  F_2 & \xrightarrow{\theta_2^g} & \phi_g(F_2)
  \end{array}
  \]
The category of $G$-equivariant objects in $\mathcal{C}$ is denoted $\mathcal{C}^G$.

**Remark 3.5.** It follows from the definition that “$\theta_e$ is identity”. More precisely, $\theta_e : F \to \phi_e(F)$ equals $u(F)$.

**Example 3.6.** In Example 3.3 $G$-equivariant objects are $G$-equivariant coherent sheaves on $X$.

Define the functor $p^* : \mathcal{C}^G \to \mathcal{C}$ as the forgetful functor: $p^*(F, (\theta_g)) = F$.

Define the functor $p_* : \mathcal{C} \to \mathcal{C}^G$ as follows:

$$p_*(F) = \left( \bigoplus_{h \in G} \phi_h(F), (\xi_g) \right),$$

where

$$\xi_g : \bigoplus_{h \in G} \phi_h(F) \to \bigoplus_{h \in G} \phi_g \phi_h(F)$$

is the collection of isomorphisms

$$\varepsilon_{g,h}^{-1} : \phi_h(F) \to \phi_g \phi_h(F).$$

**Lemma 3.7.** The functor $p^*$ is both left and right adjoint to the functor $p_*$.

*Proof.** First check that $p^*$ is left adjoint to $p_*$. Construct the unit morphism $\eta : \text{Id} \to p_* p^*$ of endofunctors on $\mathcal{C}^G$: for $\mathcal{F} = (F, (\theta_g)) \in \mathcal{C}^G$ take

$$\eta(\mathcal{F}) = \sum_h \theta_h : \mathcal{F} \to p_* p^* \mathcal{F} = (\bigoplus_h \phi_h(F), (\xi_g)).$$

Construct the counit morphism $\varepsilon : p_* p^* \to \text{Id}$ of endofunctors on $\mathcal{C}$: for $F \in \mathcal{C}$ take

$$\varepsilon(F) = u^{-1}(F) \text{pr}_e : \bigoplus \phi_h(F) \to \phi_e(F) \to F.$$  

One can check that these two morphisms satisfy necessary relations and hence the functors $p^*$ and $p_*$ are adjoint.

Likewise, to check that $p_*$ is left adjoint to $p^*$, we construct adjunction morphisms. Construct the unit $\eta' : \text{Id}_\mathcal{C} \to p^* p_*$: for $F \in \mathcal{C}$ take

$$\eta'(F) : F \to p^* p_* F = \bigoplus_h \phi_h(F)$$

to be the composition of $u(F) : F \to \phi_e(F)$ and the embedding of the summand $\phi_e(F)$.

Construct the counit $\varepsilon' : p_* p^* \to \text{Id}_{\mathcal{C}^G}$: for $\mathcal{F} = (F, (\theta_g)) \in \mathcal{C}^G$ take

$$\varepsilon'(\mathcal{F}) = \bigoplus_h \theta_h^{-1} : p_* p^* \mathcal{F} = (\bigoplus_h \phi_h(F), (\xi_g)) \to \mathcal{F}.$$  

These two morphisms of functors satisfy certain relations, therefore $p_*$ is left adjoint to $p^*$.

Following Definitions 2.4 and 2.10 one may consider

- the comonad $\mathcal{T}(p^*, p_*)$ on $\mathcal{C}$;
- the monad $\mathcal{S}(p^*, p_*)$ on $\mathcal{C}^G$;
- the comonad $\mathcal{T}(p_*, p^*)$ on $\mathcal{C}^G$.
Lemma 3.8. We have two equalities of natural transformations:

\[(3.1) \quad \varepsilon \circ \eta' = 1_{\text{Id}_C}\]

and

\[(3.2) \quad \varepsilon' \circ \eta = |G| \cdot 1_{\text{Id}_{C^G}}\]

Proof. It follows immediately from explicit formulas, see Proof of Lemma 3.7. \(\square\)

Definition 3.9. The monad \(S(p_*, p^*)\) and the comonad \(T(p^*, p_*)\) on \(C\) will be called associated with the action of \(G\) on \(C\).

Next proposition shows that modules/comodules over these monad/comonad are precisely \(G\)-equivariant objects in \(C\).

Proposition 3.10. The comparison functors

\[
(1) \quad C^G \to C_{T(p^*, p_*)}; \\
(2) \quad C^G \to C_{S(p^*, p_*)}
\]

are equivalences.

If, in addition, \(C\) is idempotent complete, then the comparison functors

\[
(3) \quad C \to (C^G)_{T(p^*, p_*)}; \\
(4) \quad C \to (C^G)_{S(p^*, p_*)}
\]

also are equivalences.

Proof. (1) First, we prove that the comparison functor \(\Phi: C^G \to C_{T(p^*, p_*)}\) is fully faithful. According to 2.8, we need to check that the unit of adjunction \(\eta: \text{Id} \to p_! p^*\) is a split embedding (for any object, but in fact the below splitting is functorial).

Indeed, for any object \(F \in C^G\) the morphism \(\eta(F): F \to p_! p^* F\) has a left inverse morphism \(\frac{1}{|G|} \varepsilon'(F)\), see Lemma 3.8.

Then we check that \(\Phi\) is essentially surjective. Indeed, take an object \((F, h)\) in \(C_{T(p^*, p_*)}\). Here \(h: F \to TF = \oplus_{h} \phi_h(F)\) is a morphism obeying

(a) \(\varepsilon(F) \circ h = \text{Id}_F\) and
(b) \(Th \circ h = \delta(F) \circ h\).

Components of \(h\) are some morphisms \(\theta_h: F \to \phi_h(F)\). Condition (a) imply that \(\theta_h\) is an isomorphism. Condition (b) imply that \(\varepsilon_{h,g} \phi_h(\theta_g) \theta_h = \theta_{gh}\), and therefore all \(\theta_h\) are invertible. We get that \((F, (\theta_g))\) is an equivariant object and \((F, h)\) is isomorphic to \(\Phi((F, (\theta_g)))\).

(2) is similar to (1).

(3) Since \(C\) is idempotent complete, one can use Proposition 2.8. It suffices to verify that the unit of adjunction \(\eta': \text{Id} \to p^* p_*\) has a left inverse morphism of functors. Indeed, \(\varepsilon\) is left inverse to \(\eta'\) by Lemma 3.8.

(4) is similar to (3). \(\square\)
Example 3.11. If $\mathcal{C}$ is not idempotent complete, then comparison functors (3) and (4) from Proposition 3.10 need not be equivalences. For example, take the category $k$–$\text{vect}$ of finite-dimensional vector spaces over a field $k$ as $\mathcal{C}$ and let $\mathcal{C}_0 \subset \mathcal{C}$ be its subcategory of even-dimensional spaces. Let the group $G = \mathbb{Z}/2\mathbb{Z}$ act trivially on $\mathcal{C}$. We claim that the comparison functor

$$\Phi_0: \mathcal{C}_0 \to (\mathcal{C}_0^G)_{(p^*, p_*)}$$

is not an equivalence. To see this, consider the commutative diagram of functors

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Phi} & (\mathcal{C}^G)_{(p^*, p_*)} \\
\sigma \downarrow & & \downarrow \tilde{\sigma} \\
\mathcal{C}_0 & \xrightarrow{\Phi_0} & (\mathcal{C}_0^G)_{(p^*, p_*)},
\end{array}$$

where $\Phi$ and $\Phi_0$ are comparison functors and $\sigma, \tilde{\sigma}$ denote fully faithful embeddings. Category $\mathcal{C}$ is idempotent complete, hence by Proposition 3.10 $\Phi$ is an equivalence. For a vector space, $\Phi(V)$ is an object $(V, h)$, where $V = (p_* V, (\xi_g)) = (V \oplus V, (\xi_g)) \in \mathcal{C}^G$. We see that $V \oplus V$ is even-dimensional and therefore $\Phi(V)$ belongs to the image of $\tilde{\sigma}$. It follows that $\tilde{\sigma}$ is an equivalence. Since $\sigma$ is not an equivalence, neither is $\Phi_0$.

4. Triangulated structure on the quotient category of a triangulated category

Let $\mathcal{T}$ be a triangulated category with an action of a group $G$ by exact autoequivalences $\phi_g, g \in G$. The key question of this paper is to understand when the quotient category $\mathcal{T}^G$ is also triangulated.

Definition 4.1. Define a shift functor on $\mathcal{T}^G$: on objects $(F, (\theta_g))[1] = (F[1], (\theta_g[1]))$, on morphisms in $\mathcal{T}^G$ shift is the same as on morphisms in $\mathcal{T}$. Say that a triangle

$$(F_1, (\theta_g^1)) \xrightarrow{\alpha} (F_2, (\theta_g^2)) \xrightarrow{\beta} (F, (\theta_g)) \xrightarrow{\gamma} (F_1, (\theta_g^1))[1]$$

in $\mathcal{T}^G$ is distinguished if and only if the triangle

$$F_1 \xrightarrow{\alpha} F \xrightarrow{\beta} F_2 \xrightarrow{\gamma} F_1[1]$$

is distinguished in $\mathcal{T}$.

To check that this definition introduces a triangulated structure, one essentially have to demonstrate that every morphism fits into a distinguished triangle. Below we try to do it directly and see what goes wrong.

Take a morphism $f: (F_1, (\theta_g^1)) \to (F_2, (\theta_g^2))$ in $\mathcal{T}^G$. Consider a cone of $f: F_1 \to F_2$ in $\mathcal{T}$:

$$F_1 \xrightarrow{f} F_2 \to F \to F_1[1].$$
Let us introduce equivariant structure on $F$. For any $g \in G$ consider the diagram

$$
\begin{array}{cccc}
F_1 & \xrightarrow{f} & F_2 & \xrightarrow{\theta_1} & F & \xrightarrow{\theta_2} & F_1[1] \\
\phi_g(F_1) & \xrightarrow{\phi_g(f)} & \phi_g(F_2) & \xrightarrow{\phi_g} & \phi_g(F) & \xrightarrow{\phi_g[1]} & \phi_g(F_1)[1].
\end{array}
$$

By an axiom of a triangulated category, there exists a morphism $\theta_g : F \to \phi_g(F)$ completing the diagram. It suffices now to check that $\theta_g$-s are compatible for different $g$-s. But $\theta_g$ is not unique in general, so it is not clear how to do it.

5. Finite group quotients for derived categories

Suppose $\mathcal{T}$ is the bounded derived category of an abelian category $\mathcal{A}$. Consider the action of a group $G$ on $\mathcal{T}$ induced by a $G$-action on $\mathcal{A}$. For an abelian category $\mathcal{A}$ the category $\mathcal{A}^G$ is also abelian. In this section we prove that the category $\mathcal{T}^G$ is triangulated and that $\mathcal{T}^G \cong \mathcal{D}(\mathcal{A}^G)$.

**Theorem 5.1.** Let $\mathcal{A}$ be an abelian category with an action of a finite group $G$. Suppose $\mathcal{A}$ is linear over the ring $\mathbb{Z}[1/|G|]$. Let $\mathcal{D}(\mathcal{A})$ be its bounded derived category, it is equipped with an action of $G$ in the natural way. Then one has an equivalence $\mathcal{D}(\mathcal{A}^G) \to \mathcal{D}(\mathcal{A})^G$.

**Proof.** First of all, we note that $\mathcal{D}(\mathcal{A}^G)$ is idempotent complete by [1, Corollary 2.10].

Consider the functors $p^* : \mathcal{A}^G \to \mathcal{A}$ and $p_* : \mathcal{A} \to \mathcal{A}^G$ introduced in Section 3. Since they are exact, there exist derived functors $Rp^* : \mathcal{D}(\mathcal{A}^G) \to \mathcal{D}(\mathcal{A})$ and $Rp_* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}^G)$ which can be defined termwise. Also consider the adjoint functors $q^* : \mathcal{D}(\mathcal{A})^G \to \mathcal{D}(\mathcal{A})$ and $q_* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}^G)$; see Section 3.

Adjoint pairs $Rp^*, Rp_*$ and $q^*, q_*$ define two comonads on $\mathcal{D}(\mathcal{A})$, which are tautologically isomorphic.

Use Proposition 2.8 to check that the comparison functor

$$
\mathcal{D}(\mathcal{A}^G) \to \mathcal{D}(\mathcal{A})_{(Rp^*, Rp_*)}
$$

is an equivalence. We need to check that the canonical morphism of functors $\text{Id} \to Rp_* Rp^*$ on $\mathcal{D}(\mathcal{A}^G)$ is a split embedding. Indeed, for any

$$
\mathcal{F}^* = [\ldots \to (F^i, (\theta^i_g)) \to (F^{i+1}, (\theta^{i+1}_g)) \to \ldots] \in \mathcal{D}(\mathcal{A}^G)
$$

the morphism of complexes

$$
\mathcal{F}^* \to Rp_* Rp^* \mathcal{F}^*
$$

given by the family

$$
\oplus_h \theta^i_h : (F^i, (\theta^i_g)) \to (\oplus_{h \in G} \phi_h(F^i), (\xi^i_g)),
$$

has a left inverse morphism

$$
Rp_* Rp^* \mathcal{F}^* \to \mathcal{F}^*
$$
given by the family
\[ \frac{1}{|G|} \oplus_h (\vartheta_h^{-1}) : (\oplus_{h \in G} \phi_h(F^i), (\xi_g^i)) \to (F^i, (\vartheta_g^i)). \]

Clearly, this splitting is functorial.

We obtain a series of equivalences
\[ D(A^G) \to D(A)_{T(Rp^*, Rp^*)} = D(A)_{T(q^*, q^*)} \cong D(A)^G \]
where the latter equivalence is due to Proposition 3.10. □

As corollaries, we obtain theorems from the introduction.

**Theorem 5.2.** Let \( G \) be a finite group and \( X \) be a quasi-projective \( G \)-variety over a field \( k \). Suppose \( \text{char}(k) \) does not divide \( |G| \). Then
\[ D^b(\text{coh}(X))^G \cong D^b(\text{coh}^G(X)). \]
Informally, “passing to equivariant category commutes with passing to derived category”.

**Proof.** Take \( A = \text{coh}(X) \). Then \( A^G \cong \text{coh}^G(X) \). By Theorem 5.1 we get the result. □

**Corollary 5.3.** Suppose \( X \) is a Galois covering of a quasi-projective variety \( Y \) over a field \( k \) with a Galois group \( G \). Suppose \( \text{char}(k) \) does not divide \( |G| \). Then
\[ D^b(\text{coh}(X))^G \cong D^b(\text{coh}(Y)). \]

**Proof.** It follows from Theorem 5.2 and the well-known fact that \( \text{coh}^G(X) \cong \text{coh}(Y) \). □

**Theorem 5.4.** Let \( X \) be a quasi-projective algebraic variety over a field \( k \) and \( G \subset \text{Pic}(X) \) be a finite subgroup. Let \( G \) act on \( \text{coh}(X) \) by tensoring into line bundles of \( G \). Let
\[ Y = \text{Spec}_X \left( \bigoplus_{L \in G} L^{-1} \right) \]
be the relative spectrum. Suppose \( \text{char}(k) \) does not divide \( |G| \). Then
\[ D^b(\text{coh}(X))^G \cong D^b(\text{coh}(Y)). \]

**Proof.** Since \( \text{Pic}(X) \) is a not a set of line bundles, but a set of isomorphism classes of line bundles, certain care should be taken when defining \( G \)-action on \( \text{coh}(X) \). Let us do it in some details.

Clearly, \( G \cong \langle g_1 \rangle \times \ldots \times \langle g_m \rangle \) where \( g_i \in \text{Pic}(X) \) are elements of order \( n_i \). Choose a line bundle \( L_i \) on \( X \) representing each \( g_i \). Fix isomorphisms \( t_i : L_i^{n_i} \to O_X \) for each \( i \).

For \( g = \prod g_i^{d_i} \), \( 0 \leq d_i < n_i \) denote by \( L(g) \) the bundle \( \bigotimes_i L_i^{d_i} \). Define an action of \( G \) on \( \text{coh}(X) \). Let \( \phi_g : \text{coh}(X) \to \text{coh}(X) \) be \( L(g) \otimes - \). Isomorphisms \( \varepsilon_{g,h} : \phi_g \phi_h \to \phi_{gh} \) are defined through isomorphisms \( L(g) \otimes L(h) \cong L(hg) \) which are tautological or defined via \( t_i \).

Let
\[ R = \bigoplus_{g \in G} L(g)^{-1} \]
be a sheaf on $X$. With the use of $t_i$, one can introduce multiplication $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{R} \to \mathcal{R}$ making $\mathcal{R}$ a sheaf of $\mathcal{O}_X$-algebras. Let $Y = \text{Spec}_X \mathcal{R}$ be the relative spectrum of $\mathcal{R}$. Coherent sheaves on $Y$ are coherent sheaves of $\mathcal{R}$-modules on $X$. A coherent sheaf $\mathcal{F}$ on $X$ is a sheaf of $\mathcal{R}$-modules if it is equipped with a morphism $a: \mathcal{R} \otimes \mathcal{F} \to \mathcal{F}$ compatible with multiplication. One has

$$a \in \text{Hom}(\mathcal{R} \otimes \mathcal{F}, \mathcal{F}) = \prod_{g \in G} \text{Hom}(\mathcal{L}(g)^{-1} \otimes \mathcal{F}, \mathcal{F}) = \prod_{g \in G} \text{Hom}(\mathcal{F}, \mathcal{L}(g) \otimes \mathcal{F}) = \prod_{g \in G} \text{Hom}(\mathcal{F}, \phi_g(\mathcal{F})) \ni (\theta_g).$$

It can be checked that $a$ is compatible with multiplication in $\mathcal{R}$ iff $(\theta_g)$ is compatible with $\varepsilon_{g,h}$ in the sense of Definition 3.3. Thus coherent sheaves of $\mathcal{R}$-modules correspond to $G$-equivariant coherent sheaves on $X$ with respect to the action introduced above. Therefore

$$\text{coh}(X)^G \cong \text{coh}(Y).$$

Let $\mathcal{A} = \text{coh}(X)$, then $\mathcal{A}^G \cong \text{coh}(Y)$. By Theorem 5.1 we have

$$\mathcal{D}^b(\text{coh}(X))^G \cong \mathcal{D}^b(\text{coh}(Y)).$$

To complete this section, we demonstrate that its assumptions on the action are always fulfilled for some triangulated categories.

**Theorem 5.5.** Suppose $X$ is a smooth irreducible projective variety over a field $k$ with ample canonical or anticanonical bundle. Let $G$ be a finite group acting on $\mathcal{T} = \mathcal{D}^b(\text{coh}(X))$, assume $\text{char}(k)$ does not divide $|G|$. Then the equivariant category $\mathcal{T}^G$ is triangulated.

**Proof.** By a result of Bondal and Orlov [5, Theorem 3.1], the group of isomorphism classes of exact autoequivalences $\mathcal{T} \to \mathcal{T}$ is

$$\mathbb{Z} \oplus (\text{Pic}(X) \rtimes \text{Aut}(X)),$$

where $\mathbb{Z}$ stands for translations. Since $G$ is finite, any $G$-action on $\mathcal{T}$ factorizes through $\text{Pic}(X) \rtimes \text{Aut}(X)$. Therefore the action is induced by a $G$-action on $\text{coh}(X)$, so Theorem 5.1 applies. □

### 6. Finite group quotients for enhanced triangulated categories

In this section we will use DG-enhancements to check that Definition 4.1 does introduce a triangulated structure on $\mathcal{T}^G$ in the following setting. Suppose we are given a pretriangulated DG-category $\mathcal{A}$ with an action of $G$ and a $G$-equivariant equivalence $H^0(\mathcal{A}) \to \mathcal{T}$. Then we produce another pretriangulated DG-category $Q_G(\mathcal{A})$ and an equivalence $H^0(Q_G(\mathcal{A})) \to \mathcal{T}^G$ which respects distinguished triangles. Therefore, we conclude that $\mathcal{T}^G$ is triangulated via Definition 4.1.

We refer to [4] or [9] for the definitions and basic facts concerning DG-categories.
A **DG-category** is a $k$-linear category such that all Hom spaces are differential complexes of $k$-vector spaces and composition of morphisms satisfies graded Leibniz rule. For a DG-category $\mathcal{A}$, by $H^0(\mathcal{A})$ the *homotopic* category of $\mathcal{A}$ is denoted. This is a category with the same objects as $\mathcal{A}$ and whose Hom spaces are zero homology of Hom spaces in $\mathcal{A}$. An additive functor $\Phi: \mathcal{A} \to \mathcal{B}$ between two DG-categories is a **DG-functor** if for any $X,Y \in \mathcal{A}$ the induced morphism $\text{Hom}_\mathcal{A}(X,Y) \to \text{Hom}_\mathcal{B}(\Phi(X),\Phi(Y))$ is a morphism of complexes. For any DG-functor $\Phi: \mathcal{A} \to \mathcal{B}$ one has an induced functor on homotopy categories $H_0(\Phi): H_0(\mathcal{A}) \to H_0(\mathcal{B})$. A DG-functor $\Phi$ is said to be **quasi-equivalence** if for any $X,Y \in \mathcal{A}$ the morphism $\text{Hom}_\mathcal{A}(X,Y) \to \text{Hom}_\mathcal{B}(\Phi(X),\Phi(Y))$ is a quasi-isomorphism of complexes and $H_0(\Phi)$ is essentially surjective.

Let $C_{DG}(k)$ denote the DG-category of complexes of $k$-vector spaces. For a DG-category $\mathcal{A}$, a (right) $\mathcal{A}$-module is a DG-functor from $\mathcal{A}^{op}$ to $C_{DG}(k)$. Denote by $\text{Mod}-\mathcal{A}$ the category of right $\mathcal{A}$-modules, it is a DG-category. It has shift and cones of closed morphisms of degree zero, its homotopy category $H_0(\text{Mod}-\mathcal{A})$ is triangulated. Yoneda embedding $h: \mathcal{A} \to \text{Mod}-\mathcal{A}$ is a fully faithful functor. Modules of the form $h^X, X \in \mathcal{A}$, are called free. The minimal full strict subcategory in $\text{Mod}-\mathcal{A}$, containing all free modules and closed under shifts and cones is called *pretriangulated hull* of $\mathcal{A}$, we denote it $\text{Pre-Tr}(\mathcal{A})$. Its homotopy category $H_0(\mathcal{A})$ is triangulated. An $\mathcal{A}$-module $M$ is said to be **semi-free** if there exists a filtration $0 = M_0 \subset M_1 \subset \ldots \subset M$ of submodules such that $\bigcup M_i = M$ and $M_i/M_{i-1}$ is isomorphic to a direct sum of shifts of some free modules. An $\mathcal{A}$-module is **perfect** if it is semi-free and isomorphic in $H_0(\text{Mod}-\mathcal{A})$ to a direct summand of some module in $\text{Pre-Tr}(\mathcal{A})$. DG-category of perfect $\mathcal{A}$-modules is denoted $\text{Perf}(\mathcal{A})$. Its homotopy category $H_0(\text{Perf}(\mathcal{A}))$ is also triangulated and it is an idempotent closure of $H_0(\text{Pre-Tr}(\mathcal{A}))$.

If Yoneda embedding $h: \mathcal{A} \to \text{Pre-Tr}(\mathcal{A})$ is a quasi-equivalence, then $\mathcal{A}$ is called *pre-triangulated*. If $h: \mathcal{A} \to \text{Pre-Tr}(\mathcal{A})$ is a DG-equivalence, then $\mathcal{A}$ is called *strongly pre-triangulated*. DG-category $\mathcal{A}$ is said to be *perfect* if the embedding $\mathcal{A} \to \text{Perf}\mathcal{A}$ is a quasi-equivalence. In all three cases the homotopy category $H_0(\mathcal{A})$ is triangulated.

Definitions of a group action and of an equivariant object are to be modified in the case of DG-categories, they are as follows:

**Definition 6.1.** A (right) action of a group $G$ on a DG-category $\mathcal{A}$ consists of the following data:

- family of DG-autoequivalences $\phi_g: \mathcal{C} \to \mathcal{C}, g \in G$;
- family of closed isomorphisms of degree 0: $\varepsilon_{g,h}: \phi_g \phi_h \to \phi_{hg}$, satisfying usual associativity conditions.

**Definition 6.2.** A $G$-equivariant object in a DG-category $\mathcal{A}$ is a pair $(F, (\theta_g)_{g \in G})$ where $F \in \text{Ob}\mathcal{A}$ and $(\theta_g)_{g \in G}$ is a family of closed isomorphisms of degree 0

$$\theta_g: F \to \phi_g(F),$$
satisfying usual associativity conditions. A morphism of $G$-equivariant objects from $(F_1, (\theta^1_g))$ to $(F_2, (\theta^2_g))$ is a morphism $f : F_1 \to F_2$ compatible with $\theta_g$.

**Proposition 6.3.** For an action of a group $G$ on a DG-category $\mathcal{A}$, the category of equivariant objects $\mathcal{A}^G$ is also a DG-category.

**Proof.** Indeed, it is clear that $$\text{Hom}_{\mathcal{A}^G}((F_1, (\theta^1_g)), (F_2, (\theta^2_g))) \subset \text{Hom}_\mathcal{A}(F_1, F_2)$$ is a subcomplex. □

For a pretriangulated DG-category $\mathcal{A}$ with a $G$-action the category $\mathcal{A}^G$ may not be pretriangulated, see an example below. But for a strongly pretriangulated category $\mathcal{A}$, the category $\mathcal{A}^G$ is also strongly pretriangulated, see [15, Prop. 3.7].

**Example 6.4.** We give an example of a pretriangulated DG-category $\mathcal{A}_0$ with a $G$-action such that the category $\mathcal{A}_0^G$ is not pretriangulated.

Let $\mathcal{C}$ be the category of $\mathbb{Z}/3\mathbb{Z}$-graded vector spaces. Denote by $V_i, (i = 0, 1, 2)$ the simple objects of $\mathcal{C}$. Let $\mathcal{A} = C^\bullet_{DG}(\mathcal{C})$ be the DG-category of complexes over $\mathcal{C}$. Let $M_i = V_0 \oplus [V_i \xrightarrow{1} V_i]$ be the complex located in degrees $-1$ and $0$.

Let $\mathcal{A}' \subset \mathcal{A}$ be the full subcategory whose objects are all objects of $\mathcal{A}$ except for those quasi-isomorphic to $V_0$, let $\mathcal{A}_0 \subset \mathcal{A}$ be the full subcategory such that $\text{Ob} \mathcal{A}_0 = \text{Ob} \mathcal{A}' \cup \{M_1, M_2\}$. Since $M_1 \cong V_0$ in $H^0(\mathcal{A})$, the category $\mathcal{A}_0$ is pretriangulated (but not strongly pretriangulated). Let the group $G = \mathbb{Z}/2\mathbb{Z} = \langle g \rangle$ act on $\mathcal{A}$ by permuting $V_1$ and $V_2$ and sending $V_0$ to itself. Then the subcategory $\mathcal{A}_0$ is invariant.

We claim that the category $\mathcal{A}_0^G$ is not pretriangulated. Indeed, there is an object $(V_0, (1)_g)[-1]$ in $\mathcal{A}_0^G$. But $\mathcal{A}_0^G$ contain no objects $(F, (\theta_g))$ quasi-isomorphic to $(V_0, (1)_g)$. Assume the contrary. Then $F$ is quasi-isomorphic to $V_0$. The definition of $\mathcal{A}_0$ implies that $F$ is either $M_1$ or $M_2$. In both cases $F$ is not DG-isomorphic to $\phi_g(F)$, we get the contradiction. Therefore $\mathcal{A}_0^G$ is not homotopically closed under shifts and hence is not triangulated.

Suppose a triangulated category $\mathcal{T}$ has an enhancement: a pretriangulated DG-category $\mathcal{A}$ and an exact equivalence $H^0(\mathcal{A}) \to \mathcal{T}$. Suppose the finite group $G$ acts on both $\mathcal{A}$ and $\mathcal{T}$ compatibly. Then P. Sosna in [15] defines $\mathcal{T}^G$ as $H^0(\text{Pre-Tr}(\mathcal{A}^G))$. Below we demonstrate that this construction is, in general, dependent on the choice of enhancement.

**Example 6.5.** We give an example of two strongly pretriangulated categories $\mathcal{A}_1$ and $\mathcal{A}_2$ with actions of a finite $G$ and of $G$-equivariant quasi-equivalence $\mathcal{A}_1 \to \mathcal{A}_2$ such that the induced functor $$\text{Pre-Tr}(\mathcal{A}_1^G) \to \text{Pre-Tr}(\mathcal{A}_2^G)$$ is not a quasi-equivalence.
Let $\mathcal{C}$, $\mathcal{A}$, $V_i$ and $M_i$ be as in Example 6.3. Denote by $(\dim_0, \dim_1, \dim_2) \in \mathbb{Z}^3$ the dimension of objects in $\mathcal{C}$.

Consider the subcategory $\mathcal{A}_1 \subset \mathcal{A}$ generated by $M_1$ and $M_2$ by taking shifts and cones. Consider the subcategory $\mathcal{A}_2 \subset \mathcal{A}$ generated by $M_1$, $M_2$ and $V_0$. Clearly, $\mathcal{A}_i$ are strongly pretriangulated, the inclusion $\mathcal{A}_1 \to \mathcal{A}_2$ induces an equivalence $H^0(\mathcal{A}_1) \cong H^0(\mathcal{A}_2) \cong D^b(\text{vect})$ with $V_0 \cong M_1 \cong M_2$ in $H^0(\mathcal{A}_i)$ being the simple object. Hence, $\mathcal{A}_1$ and $\mathcal{A}_2$ are quasi-equivalent.

Let the group $G = \mathbb{Z}/2\mathbb{Z}$ act on $\mathcal{A}$ by permuting $V_1$ and $V_2$ and sending $V_0$ to itself. Then the subcategories $\mathcal{A}_1$ and $\mathcal{A}_2$ are invariant. Since they are strongly pretriangulated, the categories $\mathcal{A}_i^G$ are pretriangulated and Pre-Tr($\mathcal{A}_i^G$) is DG-equivalent to $\mathcal{A}_i^G$. Clearly, $H^0(\mathcal{A}_2^G) \cong D^b(\text{vect}^G)$, its simple objects are $(V_0, (1)_g)$ and $(V_0, \text{sign}(g))_g$. We claim that the subcategory $H^0(\mathcal{A}_1^G) \subset H^0(\mathcal{A}_2^G)$ does not contain objects isomorphic to $(V_0, (1)_g)$, and hence the inclusion $H^0(\mathcal{A}_1^G) \to H^0(\mathcal{A}_2^G)$ is not an equivalence.

Indeed, let $(N^\bullet, (\theta_g))$ be an object of $\mathcal{A}_1^G$. Note that
\[
\dim_1(N^i) + \dim_2(N^i) = \dim_0(N^i) + \dim_0(N^{i+1}).
\]
(This is true for $N^\bullet$ being any shift of $M_1$ and $M_2$ and therefore for any complex obtained from them by subsequent taking cones.) Since $N^\bullet$ is $G$-invariant, one has $\dim_1(N^i) = \dim_2(N^i)$. We deduce that
\[
\dim_0(N^i) = \dim_0(N^{i+1}) \pmod{2}.
\]
Since the complex $N^\bullet$ is finite, all $\dim_0(N^i) = 0 \pmod{2}$. Therefore
\[
\sum_i (-1)^i \dim_0 H^i(N^\bullet) = 0 \pmod{2}.
\]
Hence $N^\bullet$ is not homotopic to $V_0$.

This issue arises because the enhancement $\mathcal{A}$ may be “not enough symmetric”: objects $F$ and $\phi_\theta(F)$ that should be DG-isomorphic are only homotopic. Therefore the category $H^0(\mathcal{A}^G)$ lacks some desired objects. Fortunately, these missing objects can be recovered as certain direct summands of objects of $H^0(\mathcal{A}^G)$. More precisely, if $T$ is idempotent complete, then the idempotent completion of $H^0(\mathcal{A}^G)$ is the good candidate for $T^G$: it does not depend on the enhancement.

**Lemma 6.6.** Let $\mathcal{A}$ be an additive DG-category, acted by a finite group $G$. Then one has a natural equivalence
\[
H^0(\text{Perf}(\mathcal{A}^G)) \to H^0(\text{Perf}(\mathcal{A}))^G.
\]

**Proof.** Consider the DG-category $\mathcal{A}^G$ and adjoint functors $p^*: \mathcal{A}^G \to \mathcal{A}$ and $p_*: \mathcal{A} \to \mathcal{A}^G$ introduced in Section 5. They are both left and right adjoint to each other and we have natural transformations of adjunction: $\eta: \text{Id}_{\mathcal{A}^G} \to p_*p^*$ and $\epsilon': p_*p^* \to \text{Id}_{\mathcal{A}^G}$, such that $\epsilon' \eta = |G|$. These functors extend to adjoint functors Perf($\mathcal{A}^G$) $\to$ Perf($\mathcal{A}$) and Perf($\mathcal{A}$) $\to$ Perf($\mathcal{A}^G$), which we also denote $p^*$ and $p_*$ respectively. They also possess
natural transformations $\eta: \text{Id}_{\text{Perf}(\mathcal{A}^G)} \to p_*p^*$ and $\varepsilon': p_*p^* \to \text{Id}_{\text{Perf}(\mathcal{A}^G)}$ satisfying the same identity. The same is true for $H^0$: we have got adjoint functors

$$H^0(p^*): H^0(\text{Perf}(\mathcal{A}^G)) \to H^0(\text{Perf}(\mathcal{A}))$$

and $H^0(p_*): H^0(\text{Perf}(\mathcal{A})) \to H^0(\text{Perf}(\mathcal{A}^G))$ and functorial morphisms

$$\eta: \text{Id}_{H^0(\text{Perf}(\mathcal{A}^G))} \to H^0(p_*)H^0(p^*)$$

and

$$\varepsilon': H^0(p_*)H^0(p^*) \to \text{Id}_{H^0(\text{Perf}(\mathcal{A}^G))}$$

such that $\varepsilon'\eta = |G|$. This adjoint pair generates a comonad $T = T(H^0(p^*), H^0(p_*))$ on $H^0(\text{Perf}(\mathcal{A}))$. By Proposition 2.8, we have an equivalence

$$\Phi: H^0(\text{Perf}(\mathcal{A}^G)) \to H^0(\text{Perf}(\mathcal{A}))$$

Indeed, the natural morphism $\eta: \text{Id}_{H^0(\text{Perf}(\mathcal{A}^G))} \to H^0(p_*)H^0(p^*)$ has a left inverse morphism $\varepsilon'$. Each equivalence $\phi_g: \mathcal{A} \to \mathcal{A}$ extends to an equivalence $\mathcal{A}\text{–Mod} \to \mathcal{A}\text{–Mod}$ which restricts to an equivalence $\text{Perf}(\mathcal{A}) \to \text{Perf}(\mathcal{A})$. This defines an action of $G$ on $\text{Perf}(\mathcal{A})$. Clearly, the comonad $T$ is isomorphic to the comonad associated with the induced $G$-action on $H^0(\text{Perf}(\mathcal{A}))$, see Definition 3.9. Therefore by Proposition 3.10 one has an equivalence

$$H^0(\text{Perf}(\mathcal{A}))^G \to H^0(\text{Perf}(\mathcal{A}))_T,$$

this concludes the proof. □

As a corollary we get

**Theorem 6.7.** Let $\mathcal{T}$ be an idempotent complete triangulated category with an action of a finite group $G$. Suppose it has a $G$-equivariant enhancement: a pretriangulated DG-category $\mathcal{A}$ with a $G$-action and a $G$-equivariant exact equivalence $\varepsilon: H^0(\mathcal{A}) \to \mathcal{T}$. Then one has a natural equivalence

$$H^0(\text{Perf}(\mathcal{A}^G)) \to \mathcal{T}^G,$$

which is exact with respect to the standard triangulation of $H^0(\text{Perf}(\mathcal{A}^G))$ and the family of triangles in $\mathcal{T}^G$ introduced in 4.1.

Consequently, the category $H^0(\text{Perf}(\mathcal{A}^G))$ depends only on $\mathcal{T}$ and does not depend on the choice of $\mathcal{A}$ and of $G$-action on $\mathcal{A}$. Also, the category $\mathcal{T}^G$ is triangulated in the sense of Definition 4.1.

**Proof.** Since $H^0(\mathcal{A}) \cong \mathcal{T}$, the category $H^0(\mathcal{A})$ is idempotent complete, so the natural embedding $i: H^0(\mathcal{A}) \to H^0(\text{Perf}(\mathcal{A}))$ is an equivariant equivalence. Using lemma 6.6, we get a sequence of equivalences

$$H^0(\text{Perf}(\mathcal{A}^G)) \xrightarrow{\Psi} H^0(\text{Perf}(\mathcal{A}))^G \xrightarrow{i^{-1}} H^0(\mathcal{A})^G \xrightarrow{\varepsilon^G} \mathcal{T}^G.$$

It remains to check that a triangle in $H^0(\text{Perf}(\mathcal{A}^G))$ is distinguished if its image under the above equivalence in $\mathcal{T}^G$ is distinguished.
Consider the commutative diagram of functors

\[
\begin{array}{ccc}
H^0(\text{Perf}(A)) & \xrightarrow{i^{-1}} & H^0(A) \\
\downarrow{\Psi} & \downarrow{p^*} & \downarrow{p^*} \\
H^0(\text{Perf}(A^G)) & \xrightarrow{i^{-1}} & H^0(A)^G \\
H^0(\text{Perf}(A^G)) & \xrightarrow{\epsilon} & \mathcal{T} \\
\end{array}
\]

where \(p^*\) denote certain forgetful functors. Let \(\Delta\) be a triangle in \(H^0(\text{Perf}(A^G))\). By the definition of a distinguished triangle in \(\mathcal{T}^G\), the triangle \(\epsilon^G i^{-1} \Psi(\Delta)\) is distinguished \(\Longleftrightarrow \epsilon^G i^{-1} p^* \Psi(\Delta)\) is distinguished \(\Longleftrightarrow \epsilon^G i^{-1} p^* \Psi(\Delta)\) is distinguished (because \(\epsilon^G\) and \(i\) are equivalences) \(\Longleftrightarrow H^0(p^*)(\Delta)\) is distinguished. So we need to demonstrate that \(\Delta\) is distinguished \(\Longleftrightarrow H^0(p^*)(\Delta)\) is distinguished. Implication \(\Longrightarrow\) is clear. To check the opposite, suppose that \(H^0(p^*)(\Delta)\) is distinguished, then \(H^0(p_*) H^0(p^*)(\Delta)\) is also distinguished. By the proof of Lemma 6.6 the morphism \(\text{Id} \to H^0(p_*) H^0(p^*)\) is a split embedding of functors. Hence \(\Delta\) is distinguished as a direct summand of a distinguished triangle \(H^0(p_*) H^0(p^*)(\Delta)\) (see [13, Proposition 1.2.3]). \(\Box\)

A case when \(\mathcal{T}\) is not idempotent complete can be reduced to the one considered above.

The idea is straightforward: we extend \(\mathcal{T}\) to its idempotent completion \(\bar{\mathcal{T}}\). On the level of DG-enhancements, this is done by passing from a DG-category \(\mathcal{A}\) to the category of perfect complexes \(\text{Perf}(\mathcal{A})\). Then we restrict equivalence \(H^0(\text{Perf}(\mathcal{A}^G)) \to H^0(\text{Perf}(\mathcal{A}))^G \to \bar{\mathcal{T}}^G\) to certain smaller subcategories. Below we do it in some details.

Suppose \(\mathcal{A}\) is a pretriangulated DG-category with a \(G\)-action. Let \(\Psi: H^0(\text{Perf}(\mathcal{A}^G)) \to H^0(\text{Perf}(\mathcal{A}))^G\) be the equivalence from Lemma 6.6 and \(p^*: \text{Perf}(\mathcal{A}^G) \to \text{Perf}(\mathcal{A})\) be the forgetful functor.

**Definition 6.8.** Denote by \(Q_G(\mathcal{A})\) the full subcategory of \(\text{Perf}(\mathcal{A}^G)\), whose objects are such \(M\) that \(\Psi(M)\) is isomorphic in \(H^0(\text{Perf}(\mathcal{A}))^G\) to an object of \(H^0(\mathcal{A})^G\). Or, equivalently, such \(M\) that \(p^* M \in \text{Perf}(\mathcal{A})\) is quasi-isomorphic to an object of \(\mathcal{A}\).

**Theorem 6.9.** Suppose \(\mathcal{A}\) is a pretriangulated DG-category with a \(G\)-action. Then

1. \(Q_G(\mathcal{A})\) is a strongly pretriangulated DG-category.
2. There exists an equivalence \(\Gamma: H^0(Q_G(\mathcal{A})) \to H^0(\mathcal{A})^G\).
3. Suppose \(\mathcal{A}_1, \mathcal{A}_2\) are two pretriangulated DG-categories equipped with a \(G\)-action. Then for any \(G\)-equivariant DG-functor \(\phi: \mathcal{A}_1 \to \mathcal{A}_2\) one has a DG-functor \(Q_G(\phi): Q_G(\mathcal{A}_1) \to Q_G(\mathcal{A}_2)\) such that the diagram

\[
\begin{array}{ccc}
H^0(Q_G(\mathcal{A}_1)) & \xrightarrow{\Gamma_1} & H^0(\mathcal{A}_1)^G \\
H^0(Q_G(\mathcal{A}_2)) & \xrightarrow{\Gamma_2} & H^0(\mathcal{A}_2)^G \\
\end{array}
\]

commutes (up to an isomorphism). Moreover, if \(\phi\) is a quasi-equivalence then \(Q_G(\phi)\) is also a quasi-equivalence.
(4) Functors $Q_G(\phi)Q_G(\psi)$ and $Q_G(\phi\psi)$ are isomorphic, if exist.

Proof. (1) To show that $Q_G(A)$ is a strongly pretriangulated DG-category, it suffices to check that $Q_G(A)$ is closed under shifts and cones in $\text{Perf}(A^G)$. This is clear: consider the case of cones. Suppose $f: M \to N$ is a morphism in $\text{Perf}(A^G)$ and $K = C(f)$ is its cone in $\text{Perf}(A^G)$. Then $p^*(K)$ is a cone of the morphism $p^*(f): p^*(M) \to p^*(N)$ in $\text{Perf}(A)$. Since $p^*(M), p^*(N) \in \text{Ob Perf}(A)$ are quasi-isomorphic to objects of $A$ and $A$ is pretriangulated, $p^*(K)$ is also quasi-isomorphic to an object of $A$. Therefore, $K$ lies in $Q_G(A)$.

(2) Denote by $\overline{H^0(A)}^G$ the closure of $H^0(A)^G$ in $H^0(\text{Perf}(A))^G$ under isomorphisms. Then one has a commutative diagram of functors where vertical arrows denote embeddings of fully faithful subcategories.

\[
\begin{array}{ccc}
H^0(\text{Perf}(A^G)) & \xrightarrow{\Psi} & H^0(\text{Perf}(A))^G \\
\downarrow & & \downarrow \\
H^0(Q_G(A)) & \xrightarrow{\Psi} & \overline{H^0(A)}^G \\
\downarrow \Gamma & & \downarrow \sigma \\
H^0(A)^G & & H^0(A)^G.
\end{array}
\]

By the definition of $Q_G(A)$, the functor $\Psi: H^0(Q_G(A)) \to \overline{H^0(A)}^G$ is a well-defined equivalence. By the definition of $\overline{H^0(A)}^G$, the embedding $\sigma$ is an equivalence. Define $\Gamma$ as a composition of $\Psi$ and an inverse functor to $\sigma$. Clearly, $\Gamma$ is an equivalence.

(3) To prove this part, suppose $\phi: A_1 \to A_2$ is a DG-functor compatible with $G$-actions. Consider the commutative diagram:

\[
\begin{array}{ccc}
H^0(\text{Perf}(A_1^G)) & \xrightarrow{\Psi_1} & H^0(\text{Perf}(A_1))^G \\
\downarrow & & \downarrow \\
H^0(Q_G(A_1)) & \xrightarrow{\Gamma_1} & H^0(A_1)^G \\
\downarrow & & \downarrow \\
H^0(\text{Perf}(A_2^G)) & \xrightarrow{\Psi_2} & H^0(\text{Perf}(A_2))^G \\
\downarrow & & \downarrow \\
H^0(Q_G(A_2)) & \xrightarrow{\Gamma_2} & H^0(A_2)^G.
\end{array}
\]

Definition of $Q_G(A_i)$ and diagram chase show that the functor $H^0(\phi^G)$ sends objects of subcategory $H^0(Q_G(A_1))$ to the objects of $H^0(Q_G(A_2))$. Therefore
Clearly, \( G^2 \): \( \text{Perf}(A_1^G) \to \text{Perf}(A_2^G) \) restricts to a functor \( Q_G(\phi): Q_G(A_1) \to Q_G(A_2) \) such that \( H^0(Q_G(\phi)) \) completes the diagram.

Finally, if \( \phi \) is a quasi-equivalence, then \( H^0(\phi)^G: H^0(A_1)^G \to H^0(A_2)^G \) is an equivalence. Since \( \Gamma_i \) are equivalences, \( H^0(Q_G(\phi)) \) is an equivalence.

(4) Functors \( Q_G(\phi), Q_G(\psi) \) and \( Q_G(\phi\psi) \) are restrictions of \( \phi, \psi \) and \( \phi\psi \) respectively. This implies the result immediately. \( \square \)

**Corollary 6.10.** Let \( T \) be a triangulated category with an action of a finite group \( G \). Suppose it has a \( G \)-equivariant enhancement: a pretriangulated DG-category \( A \) with a \( G \)-action and a \( G \)-equivariant exact equivalence \( \epsilon: H^0(A) \to T \). Then there exists an exact equivalence \( H^0(Q_G(A)) \to T^G \), hence \( T^G \) is a triangulated category.

**Proof.** By Theorem 6.9 there is an equivalence

\[
H^0(Q_G(A)) \xrightarrow{\Gamma} H^0(A)^G \xrightarrow{\epsilon^G} T^G.
\]

By the proof of Theorem 6.9, \( \Gamma \) is a restriction of the equivalence \( H^0(\text{Perf}(A^G)) \to H^0(\text{Perf}(A))^G \), which is exact by Theorem 6.7 Therefore \( \Gamma \) (and \( \epsilon^G \Gamma \)) is also exact. \( \square \)

**Theorem 6.11.** Let \( X \) be a quasi-projective variety over \( k \) with an action of a finite group \( G \). Let \( Z \subset X \) be its closed \( G \)-invariant subvariety. Suppose \( \text{char} k \) does not divide \( |G| \). Denote by \( \mathcal{D}^b_Z(\text{coh}(X)) \) the full subcategory in \( \mathcal{D}^b(\text{coh}(X)) \) of objects supported in \( Z \). Clearly, \( G \) acts on both \( \mathcal{D}^b(\text{coh}(X)) \) and \( \mathcal{D}^b_Z(\text{coh}(X)) \) by pull-back functors. Then the category \( \mathcal{D}^b_Z(\text{coh}(X))^G \) is a triangulated category and has a DG-enhancement.

**Proof.** Denote by \( C_{DG}(\mathcal{O}_X-\text{mod}) \) the DG-category of complexes of sheaves of \( \mathcal{O}_X \)-modules. Let \( \mathcal{I} \) be the full subcategory in \( C_{DG}(\mathcal{O}_X-\text{mod}) \) whose objects are left bounded complexes of injective \( \mathcal{O}_X \)-modules with finite coherent cohomology. It is well-known (see, for example, [3, Paragraph 3, Ex. 3] or [4, Section 5]) that \( \mathcal{I} \) is a DG-enhancement of \( \mathcal{D}^b(\text{coh}(X)) \). Clearly, \( G \) acts on \( C_{DG}(\mathcal{O}_X-\text{mod}) \) by pullbacks and \( \mathcal{I} \) is an invariant subcategory. Denote by \( \mathcal{I}_Z \) the full subcategory in \( \mathcal{I} \) of complexes whose cohomology sheaves are supported in \( Z \). Clearly, \( \mathcal{I}_Z \) is a \( G \)-invariant pretriangulated DG-subcategory in \( C_{DG}(\mathcal{O}_X-\text{mod}) \) and \( H^0(\mathcal{I}_Z) \cong \mathcal{D}^b_Z(\text{coh}(X)) \). Hence, Corollary 6.10 can be applied. We obtain that \( \mathcal{D}^b_Z(\text{coh}(X))^G \) is triangulated and has an enhancement \( Q_G(\mathcal{I}_Z) \). \( \square \)

7. **Reversion for descent categories**

Let \( B \) and \( C \) be idempotent complete additive categories. According to Proposition 3.10, if \( B \) is equivalent to a comodule category \( C_T \) for some comonad \( T \) on \( C \), then \( C \) is equivalent to a comodule category \( B_T \) for some comonad \( T' \) on \( B \). That is, the relation “to be a comodule category of” on the class of idempotent complete additive categories is symmetric.

In this section we go a bit further and demonstrate that the relation “to be a quotient category modulo finite abelian group action” is also symmetric.
Assume that $C$ is a $k$-linear additive category and $G$ is a finite group acting on $C$. Define a comonad on the category $C^G$. Let $k[G]$ be the regular representation of $G$ with the basis $e_g, g \in G$. Take $R: C^G \to C^G$ to be the tensoring by the regular representation: 

$$R((F, (\theta_g))) = k[G] \otimes (F, (\theta_g)).$$

Define morphism of functors $\varepsilon_R: R \to \text{Id}$ via the morphism of representations $k[G] \to k$ such that $e_g \mapsto 1$. Define morphism of functors $\delta_R: R \to RR$ via the morphism of representations $k[G] \to k[G] \otimes k[G]$ such that $e_g \mapsto e_g \otimes e_g$. Clearly, we obtain a comonad $(R, \varepsilon_R, \delta_R)$, denote it by $R$. 

**Proposition 7.1.** Comonad $T(p^*_*. p^*)$ is isomorphic to $R$.

**Proof.** Define an isomorphism of endofunctors $\beta: p^*_*p^* \to k[G] \otimes -$ for an object $F = (F, (\theta_g))$ we take an isomorphism

$$\beta(F): p^*_*p^*F = (\oplus_h \phi_h(F), (\xi_g)) \to k[G] \otimes F$$

which on the summand $\phi_h(F)$ is

$$\theta_h^{-1}: \phi_h(F) \to e_{h^{-1}} \otimes F.$$ 

Check that $\beta(F)$ is compatible with the equivariant structures. It follows from the diagram

$$\begin{array}{ccc}
\phi_h(F) & \xrightarrow{\theta_h^{-1}} & e_{h^{-1}} \otimes F \\
\downarrow{\xi_g} & & \downarrow{\theta_g} \\
\phi_g\phi_{h^{-1}}(F) & \xrightarrow{\phi_g(\theta_h^{-1})} & e_{gh^{-1}} \otimes \phi_g(F),
\end{array}$$

which is commutative by the definition of an equivariant object.

It remains to check that $\beta$ is compatible with $\varepsilon$-s and $\delta$-s, we skip this. 

From now on we suppose that the group $G$ is abelian, the field $k$ is algebraically closed and $\text{char}(k)$ does not divide $|G|$. Let $G^\vee = \text{Hom}(G, k^*)$ be the dual group to $G$, that is, the group of characters of $G$. Define an action of $G^\vee$ on the category $C^G$ by twisting: for $\chi \in G^\vee$ let

$$\phi_{\chi}((F, (\theta_h))) = (F, (\theta_h)) \otimes \chi = (F, (\theta_h \cdot \chi(h))).$$

For $\chi, \psi \in G^\vee$ the equivariant objects $\phi_{\chi}(\phi_{\psi}((F, (\theta_h))))$ and $\phi_{\psi \chi}((F, (\theta_h)))$ are the same, let isomorphisms

$$\varepsilon_{\chi, \psi}: \phi_{\chi} \circ \phi_{\psi} \to \phi_{\psi \chi}$$

be identities.

**Theorem 7.2.** Let $k$ be an algebraically closed field, $G$ be a finite abelian group such that $\text{char}(k)$ does not divide $|G|$. Suppose $C$ is a $k$-linear additive idempotent complete category and $G$ acts on $C$.

Then

$$(C^G)^{G^\vee} \cong C.$$
Proof. Consider the adjoint functors $q^* : (C^G)^{G^\vee} \to C^G$ and $q_* : C^G \to (C^G)^{G^\vee}$ (see Section 3 for the definition). We claim that the comonad $T(q^*, q_*)$ on $C^G$ is isomorphic to the comonad $\mathcal{R}$.

Indeed, the endofunctor $q^* q_* : C^G \to C^G$ is isomorphic to $R = k[G] \otimes -$:

$$q^* q_*(\mathcal{F}) = \oplus_{\chi \in G^\vee} (\chi \otimes \mathcal{F}) \cong k[G] \otimes \mathcal{F}$$

since $\oplus_{\chi \in G^\vee} \chi \cong k[G]$ as a representation. Fix an isomorphism $\gamma : \oplus_{\chi \in G^\vee} \chi \to k[G]$ such that $\gamma(\oplus 1) = e_e$, we also denote by $\gamma$ the corresponding isomorphism $q^* q_* \to R$.

To check that $\gamma$ is compatible with counit and comultiplication in $T(q^*, q_*)$ and $\mathcal{R}$, consider the diagrams of representations

$$
\begin{array}{ccc}
\oplus_{\chi \in G^\vee} \chi & \xrightarrow{\gamma} & k[G] \\
\downarrow^{p_{X \alpha}} & & \downarrow^{\varepsilon_R} \\
k & \xrightarrow{\eta} & k,
\end{array}
\begin{array}{ccc}
\oplus_{\chi \in G^\vee} \chi & \xrightarrow{\gamma} & k[G] \\
\downarrow^{\delta_R} & & \\
(\oplus_{\chi \in G^\vee} \chi) \otimes (\oplus_{\chi \in G^\vee} \chi) & \xrightarrow{\gamma \otimes \gamma} & k[G] \otimes k[G]
\end{array}
$$

(here $\eta$ on the summand $\chi$ is a direct sum of identity maps to $\oplus_{\chi_1 \chi_2 = \chi} \chi_1 \otimes \chi_2$). It suffices to prove that both diagrams are commutative on the element $\oplus 1 \in \oplus \chi$, which is true.

Therefore, one has:

$$(C^G)^{G^\vee} \cong (C^G)_{T(q^*, q_*)} \cong (C^G)_{\mathcal{R}} \cong (C^G)_{T(p^*, p^*)} \cong C,$$

where the first and the fourth equivalences are due to Proposition 3.10 and the third one is by Proposition 7.1.

As an immediate corollary we get

**Theorem 7.3.** Let $\mathcal{B}$ and $\mathcal{C}$ be idempotent complete additive categories over an algebraically closed field $k$, suppose $\text{char}(k)$ does not divide $|G|$. Suppose $\mathcal{B} \cong C^G$ for some action of a finite abelian group $G$ on $\mathcal{C}$. Then $\mathcal{C} \cong \mathcal{B}^{G^\vee}$ for some action of $G^\vee$ on $\mathcal{B}$.

**Example 7.4.** Suppose $G$ is a finite abelian group. Let $X$ be an algebraic $G$-variety over $k$, let $\mathcal{C} = D^b(\text{coh}(X))$. Then $C^G \cong D^b(\text{coh}^G(X))$ (see Theorem 5.2), the group $G^\vee$ acts on $D^b(\text{coh}^G(X))$ by twisting into characters of $G$. By Theorem 7.2 we have

$$D^b(\text{coh}^G(X))^{G^\vee} \cong D^b(\text{coh}(X)).$$

**Example 7.5.** Suppose $X$ is a Galois covering of an algebraic variety $Y$ with the abelian Galois group $G$. Let $\mathcal{C} = D^b(\text{coh}(X))$, by Corollary 5.3 we have $C^G \cong D^b(\text{coh}^G(X)) \cong D^b(\text{coh}(Y))$. Under the equivalence $\text{coh}^G(X) \cong \text{coh}(Y)$ equivariant line bundles $\mathcal{O}_X \otimes \chi, \chi \in G^\vee$, correspond to some line bundles on $Y$. Therefore the group $G^\vee$ acts on $D^b(\text{coh}(Y))$ by tensoring in line bundles. This is actually the action introduced in Theorem 5.4. So in this case Theorem 7.2 is covered by Theorem 5.4. We have

$$D^b(\text{coh}(Y))^{G^\vee} \cong D^b(\text{coh}(X)).$$
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