FURTHERANCE OF NUMERICAL RADIUS INEQUALITIES OF HILBERT SPACE OPERATORS

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ABSTRACT. If $A, B$ are bounded linear operators on a complex Hilbert space, then

$$w(A) \leq \frac{1}{2} \left( \|A\| + \sqrt{r(|A|\|A^*\|)} \right),$$

$$w(AB \pm BA) \leq 2\sqrt{2}\|B\| \sqrt{w^2(A) - c^2(\Re(A)) + c^2(\Im(A))},$$

where $w(\cdot)$, $\|\cdot\|$, $c(\cdot)$ and $r(\cdot)$ are the numerical radius, the operator norm, the Crawford number and the spectral radius respectively, and $\Re(A)$, $\Im(A)$ are the real part, the imaginary part of $A$ respectively. The inequalities obtained here generalize and improve on the existing well known inequalities.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B}(\mathcal{H})$ be the collection of all bounded linear operators on $\mathcal{H}$. As usual the norm induced by the inner product $\langle \cdot, \cdot \rangle$ is denoted by $\|\cdot\|$. For $A \in \mathcal{B}(\mathcal{H})$, let $\|A\|$ be the operator norm of $A$, i.e., $\|A\| = \sup_{\|x\|=1}\|Ax\|$. For $A \in \mathcal{B}(\mathcal{H})$, $A^*$ denotes the adjoint of $A$ and $|A|$, $|A^*|$ respectively denote the positive part of $A$, $A^*$, i.e., $|A| = (A^*A)^{\frac{1}{2}}$, $|A^*| = (AA^*)^{\frac{1}{2}}$. Let $S_\mathcal{H}$ denote the unit sphere of the Hilbert space $\mathcal{H}$. The numerical range of $A$, denoted by $W(A)$, is defined as $W(A) := \{\langle Ax, x \rangle : x \in S_\mathcal{H}\}$. Considering the continuous mapping $x \mapsto \langle Ax, x \rangle$ from $S_\mathcal{H}$ to the scalar field $\mathbb{C}$, it is easy to see that $W(A)$ is a compact subset of $\mathbb{C}$ if $\mathcal{H}$ is finite dimensional. The famous Toeplitz-Hausdorff theorem states that the numerical range is a convex set. The numerical radius and the Crawford number of $A$, denoted as $w(A)$ and $c(A)$, respectively, are defined as

$$w(A) := \sup_{x \in S_\mathcal{H}} |\langle Ax, x \rangle|$$

and

$$c(A) := \inf_{x \in S_\mathcal{H}} |\langle Ax, x \rangle|.$$
The numerical radius is a norm on $B(H)$ satisfying the following inequality
\[
\frac{1}{2} \|A\| \leq w(A) \leq \|A\|. \tag{1.1}
\]
Clearly, (1.1) implies that the numerical radius norm is equivalent to the operator norm. The inequality (1.1) is sharp, $w(A) = \|A\|$ if $AA^* = A^*A$ and $w(A) = \frac{\|A\|}{2}$ if $A^2 = 0$. For further readings on the numerical range and the numerical radius of bounded linear operators, we refer to the book [12]. The spectral radius of $A$, denoted as $r(A)$, is defined as
\[
r(A) := \sup_{\lambda \in \sigma(A)} |\lambda|,
\]
where $\sigma(A)$ is the spectrum of $A$. Since $\sigma(A) \subseteq \overline{W(A)}$, $r(A) \leq w(A)$. Also, $r(A) = w(A)$ if $A^*A = AA^*$. Kittaneh [16, Th. 1] and [17, Th. 1] improved on the inequality (1.1), to prove that
\[
\frac{1}{4} \|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2} \|A^*A + AA^*\| \tag{1.2}
\]
and
\[
w(A) \leq \frac{1}{2} \left( \|A\| + \sqrt{\|A^2\|} \right), \tag{1.3}
\]
respectively. Bhunia and Paul [10, Cor. 2.5] improved on the right hand inequalities of both (1.1) and (1.2) to prove that
\[
w^2(A) \leq \min_{0 \leq \alpha \leq 1} \|\alpha|A|^2 + (1 - \alpha)|A^*|^2\|. \tag{1.4}
\]
In [9, Th. 2.1], Bhunia and Paul also improved on the left hand inequalities of both (1.1) and (1.2) to prove that
\[
\frac{1}{4} \|A^*A + AA^*\| \leq \frac{1}{8} \left( \|A + A^*\|^2 + \|A - A^*\|^2 \right)
\leq \frac{1}{8} \left( \|A + A^*\|^2 + \|A - A^*\|^2 \right) + \frac{1}{8} c^2(A + A^*) + \frac{1}{8} c^2(A - A^*)
\leq w^2(A).
\]
Fong and Holbrook [11] obtained the remarkable numerical radius inequality that
\[
w(AB + BA) \leq 2\sqrt{2}\|B\|w(A). \tag{1.5}
\]
Hirzallah and Kittaneh [14] improved on the inequality (1.5) in the following form:
\[
w(AB \pm BA) \leq 2\sqrt{2} \|B\| \sqrt{w^2(A) - \frac{|\|\Re(A)\| - \|\Im(A)\| |}{2}}. \tag{1.6}
\]
Over the years many mathematicians have developed various inequalities improving (1.1), we refer to [1, 3, 4, 5, 6, 7, 8] and references therein. In this paper, we obtain an improvement and generalization of the inequality (1.3). Some inequalities for the numerical radius of the commutators of bounded linear operators are also obtained, which improve on (1.5).
2. Improvement of inequality (1.3)

Our improvement of the inequality (1.3), is stated as the following theorem:

**Theorem 2.1.** Let $A \in \mathcal{B}(\mathcal{H})$. Then, $w(A) \leq \frac{1}{2} \left( \|A\| + \sqrt{r \left( \|A\| A^* \right)} \right)$.

**Remark 2.2.** If $A \in \mathcal{B}(\mathcal{H})$, then $r \left( \|A\| A^* \right) \leq w \left( \|A\| A^* \right) \leq \|A\| A^* \| = \|A^2\|$. Hence, Theorem 2.1 improves (1.3). To show proper improvement we consider $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$. Then $|A| = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$ and $A^* = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$. It is easy to see that $r \left( \|A\| A^* \right) = 9 < \| (\|A\| A^*) \| = \|A^2\| = \sqrt{59 + 10\sqrt{34}} \approx 10.83$.

In order to prove Theorem 2.1 we need the following sequence of lemmas. First lemma can be found in [18].

**Lemma 2.3.** ([18, Cor. 2]) Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \|A^{1/2}B^{1/2}\|.$$ 

The second lemma which contains a mixed schwarz inequality, can be found in [13, pp. 75-76].

**Lemma 2.4.** ([13, pp. 75-76]) Let $A \in \mathcal{B}(\mathcal{H})$. Then

$$|\langle Ax, x \rangle| \leq \langle \|A\| x, x \rangle^{1/2} \langle A^* x, x \rangle^{1/2}, \forall \ x \in \mathcal{H}.$$ 

The third lemma is as follows.

**Lemma 2.5.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then

$$r(AB) = \|A^{1/2}B^{1/2}\|^2.$$ 

**Proof.** By commutativity property of the spectral radius we have that

$$r(AB) = r \left( A^{1/2}B^{1/2}A^{1/2}B^{1/2} \right) = r \left( A^{1/2}B^{1/2}B^{1/2}A^{1/2} \right) = r \left( A^{1/2}B^{1/2} \right) \left( A^{1/2}B^{1/2} \right)^* = \|A^{1/2}B^{1/2} \left( A^{1/2}B^{1/2} \right)^* \|^2 = \|A^{1/2}B^{1/2}\|^2.$$ 

\[ \square \]

Now we prove Theorem 2.1.

**Proof of Theorem 2.1.** Let $x \in S_\mathcal{H}$. Then by Lemma 2.4 we get,

$$|\langle Ax, x \rangle| \leq \langle \|A\| x, x \rangle^{1/2} \langle A^* x, x \rangle^{1/2}$$

$$\leq \frac{1}{2} \left( \langle \|A\| x, x \rangle + \langle A^* x, x \rangle \right)$$

$$\leq \frac{1}{2} \|A| + |A^*| \|$$

$$\leq \frac{1}{2} \left( \|A\| + \|A^{1/2}A^*|^{1/2}\| \right), \text{ by Lemma 2.3}$$

$$= \frac{1}{2} \left( \|A\| + \sqrt{r \left( \|A\| A^* \right)} \right), \text{ by Lemma 2.5}.$$
Hence, by taking supremum over $x \in S_\mathcal{H}$ we get,

$$
\|A\| \leq \frac{1}{2} \left( \|A\| + \sqrt{r(|A|A^*)} \right),
$$

This completes the proof.

As an application of Theorem 2.1, we prove the following corollary.

**Corollary 2.6.** Let $A \in \mathcal{B}(\mathcal{H})$. If $r(|A|A^*) = 0$, then $w(A) = \frac{\|A\|}{2}$.

**Proof.** It follows from (1.1) and Theorem 2.1 that

$$
\frac{\|A\|}{2} \leq w(A) \leq \frac{1}{2} \left( \|A\| + \sqrt{r(|A|A^*)} \right).
$$

This implies that if $r(|A|A^*) = 0$, then $w(A) = \frac{\|A\|}{2}$. □

**Remark 2.7.** It should be mentioned here that the converse of Corollary 2.6 does not hold if dim($\mathcal{H}$) $\geq$ 3. As for example, we consider $A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then we see that $w(A) = \frac{3}{2} = \frac{\|A\|}{2}$, but $r(|A|A^*) \neq 0$.

The following corollary is an immediate consequence of Theorem 2.1.

**Corollary 2.8.** Let $A \in \mathcal{B}(\mathcal{H})$. If $w(A) = \frac{1}{2} \left( \|A\| + \sqrt{\|A^2\|} \right)$, then $r(|A|A^*) = \|A^2\|$.

**Proof.** Using Remark 2.2, it follows from Theorem 2.1 that

$$
w(A) \leq \frac{1}{2} \left( \|A\| + \sqrt{r(|A|A^*)} \right) \leq \frac{1}{2} \left( \|A\| + \sqrt{\|A^2\|} \right).
$$

This implies that if $w(A) = \frac{1}{2} \left( \|A\| + \sqrt{\|A^2\|} \right)$, then $r(|A|A^*) = \|A^2\|$. □

**Remark 2.9.** It should be mentioned that the converse of Corollary 2.8 is not true. Considering the same example as in Remark 2.7, i.e., $A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we see that $r(|A|A^*) = \|A^2\| = 1$, but $w(A) = \frac{3}{2} < 2 = \frac{1}{2} \left( \|A\| + \sqrt{\|A^2\|} \right)$.

We give a sufficient condition for $w(A) = \frac{1}{2} \left( \|A\| + \sqrt{r(|A|A^*)} \right)$, when $A$ is a complex $n \times n$ matrix.

**Proposition 2.10.** Let $A$ be a complex $n \times n$ matrix. Suppose $A$ satisfies either one of the following conditions.

(i) $A$ is unitarily similar to $[\alpha] \oplus B$, where $B$ is an $(n-1) \times (n-1)$ matrix with $\|B\| \leq |\alpha|$.

(ii) $r(|A|A^*) = 0$.

Then, $w(A) = \frac{1}{2} \left( \|A\| + \sqrt{r(|A|A^*)} \right)$. 

Proof. Let \((i)\) holds. Then \(w(A) = |\alpha|\) and \(\|A\| = |\alpha|\). Also it is not difficult to verify that \(r(|A||A^*|) = |\alpha|^2\). Hence, \(\frac{1}{2} \left(\|A\| + \sqrt{r(|A||A^*|)}\right) = |\alpha|\). Now let \((ii)\) holds. Then from Corollary \(2.6\) we get, \(w(A) = \frac{1}{2} \left(\|A\| + \sqrt{r(|A||A^*|)}\right) = \frac{\|A\|^2}{2}\). Thus, we complete the proof.

Next we give a generalized result of Theorem \(2.1\). For this purpose we need the following lemma, which is the generalization of Lemma \(2.4\).

**Lemma 2.11.** ([19, Th. 5]). Let \(A, B \in \mathcal{B}(\mathcal{H})\) be such that \(|A|B = B^*|A|\) and let \(f, g\) be non-negative continuous functions on \([0, \infty]\) satisfy \(f(t)g(t) = t, \forall t \geq 0\). Then, \(|\langle ABx, y\rangle| \leq r(B)\|f(|A|)x\|\|g(|A^*|)y\|, \forall x, y \in \mathcal{H}\).

Using Lemma \(2.11\) and proceeding similarly as in Theorem \(2.1\), we can prove the following theorem.

**Theorem 2.12.** Let \(A, B \in \mathcal{B}(\mathcal{H})\) be such that \(|A|B = B^*|A|\) and let \(f, g\) be as in Lemma \(2.11\). Then
\[
w(AB) \leq \frac{r(B)}{2} \left(\max \{\|f(|A|)\|, \|g(|A^*|)\|\} + \|f(|A|)\| \|g(|A^*|)\|\right).
\]

Considering \(f(t) = g(t) = \sqrt{t}\) in Theorem \(2.12\) we get the following corollary.

**Corollary 2.13.** Let \(A, B \in \mathcal{B}(\mathcal{H})\) be such that \(|A|B = B^*|A|\). Then
\[
w(AB) \leq \frac{r(B)}{2} \left(\|A\| + \sqrt{r(|A||A^*|)}\right)
\leq \frac{1}{4} \left(\|B\| + \sqrt{r(|B||B^*|)}\right) \left(\|A\| + \sqrt{r(|A||A^*|)}\right).
\]

**Remark 2.14.** If \(A, B \in \mathcal{B}(\mathcal{H})\) be such that \(|A|B = B^*|A|\), then Alomari [2, Cor. 3.2] proved that
\[
w(AB) \leq \frac{1}{4} \left(\|B\| + \sqrt{\|B^2\|}\right) \left(\|A\| + \sqrt{\|A^2\|}\right).
\]

Clearly our inequalities in Corollary \(2.13\) improve on the inequality \((2.1)\).

3. Improvement of inequality \((1.5)\)

In order to obtain an improvement of the inequality \((1.5)\) we need the following lemma \([9]\). First, we note the Cartesian decomposition of \(A \in \mathcal{B}(\mathcal{H})\), i.e., \(A = \Re(A) + i\Im(A)\), where \(\Re(A) = \frac{A + A^*}{2}\) and \(\Im(A) = \frac{A - A^*}{2i}\).

**Lemma 3.1.** ([9, Cor. 2.3]) Let \(A \in \mathcal{B}(\mathcal{H})\). Then
\[
\|AA^* + A^*A\| \leq 4 \left[w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}\right].
\]

Now we prove the desired result.

**Theorem 3.2.** Let \(A, B, X, Y \in \mathcal{B}(\mathcal{H})\). Then
\[
w(AXB \pm BYA) \leq 2\sqrt{2}\|B\| \max \{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}.
\]
Proof. First we assume that \( \|X\| \leq 1 \) and \( \|Y\| \leq 1 \). Let \( x \in S_\mathcal{H} \). Then we have
\[
\left| \langle (AX + YA)x, x \rangle \right| \leq |\langle AXx, x \rangle| + |\langle YAx, x \rangle| = |\langle Xx, A^*x \rangle| + |\langle A x, Y^*x \rangle| \leq \|A^*x\| + \|Ax\|, \text{ by Cauchy Schwarz inequality}
\]
\[
\leq \sqrt{2(\|A^*x\|^2 + \|Ax\|^2)}, \text{ by convexity of } f(x) = x^2
\]
\[
\leq \sqrt{2\|AA^* + A^*A\|} \leq 2\sqrt{\|A\| \|A^*\|} \leq 2\sqrt{\|A\| \|A^*\|} \leq 2\sqrt{2w^2(A) - c^2(\Re(A)) + c^2(\Im(A))}, \text{ by Lemma } 3.1.
\]
Hence, by taking supremum over \( \|x\| = 1 \) we get,
\[
w(AX + YA) \leq 2\sqrt{2} \max \{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}. \tag{3.1}
\]
Now we consider the general case, i.e., \( X, Y \in \mathcal{B}(\mathcal{H}) \) be arbitrary operators. If \( X = Y = 0 \) then Theorem 3.2 holds trivially. Let \( \max \{\|X\|, \|Y\|\} \neq 0 \). Then clearly \( \frac{X}{\|X\|} \|X\| \leq 1 \) and \( \frac{Y}{\|Y\|} \|Y\| \leq 1 \). So, replacing \( X \) and \( Y \) by \( X \) and \( Y \) respectively, in (3.1) we get,
\[
w(AX + YA) \leq 2\sqrt{2} \max \{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}} \tag{3.2}
\]
Now replacing \( X \) by \( XB \) and \( Y \) by \( BY \) in (3.2) we get,
\[
w(AXB + BYA) \leq 2\sqrt{2} \max \{\|XB\|, \|BY\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}},
\]
which implies that
\[
w(AXB + BYA) \leq 2\sqrt{2}\|B\| \max \{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}. \tag{3.3}
\]
On the basis of Theorem 3.2 we prove the following corollary.

Corollary 3.3. Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then
\[
w(AB \pm BA) \leq 2\sqrt{2}\|B\| \sqrt{w^2(A) - \frac{c^2(\Re(B)) + c^2(\Im(B))}{2}}. \tag{3.3}
\]
and
\[
w(AB \pm BA) \leq 2\sqrt{2}\|A\| \sqrt{w^2(B) - \frac{c^2(\Re(B)) + c^2(\Im(B))}{2}} \tag{3.4}
\]
Proof. By considering \( X = Y = I \) in Theorem 3.2 we get, (3.3). Interchanging \( A \) and \( B \) in (3.3) we get, (3.4).

\[\square\]

Remark 3.4. Clearly, the inequality (3.3) is stronger than the inequality (1.5).

As an application of the inequality (3.3) we prove the following result.
Corollary 3.5. Let \( A, B \in \mathcal{B}(\mathcal{H}) \) and let \( B \neq 0 \). If \( w(AB \pm BA) = 2\sqrt{2}\|B\|w(A) \), then \( 0 \in \overline{W(\Re(A)) \cap W(\Im(A))} \).

Proof. Let \( w(AB \pm BA) = 2\sqrt{2}\|B\|w(A) \). Then it follows from (3.3) that

\[
W(A) = \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}.
\]

Hence, \( c^2(\Re(A)) + c^2(\Im(A)) = 0 \), i.e., \( c(\Re(A)) = c(\Im(A)) = 0 \). Therefore, there exist norm one sequences \( \{x_n\} \) and \( \{y_n\} \) in \( \mathcal{H} \) such that \( |\langle \Re(A)x_n, x_n \rangle| \to 0 \) and \( |\langle \Im(A)y_n, y_n \rangle| \to 0 \) as \( n \to \infty \). So, \( 0 \in \overline{W(\Re(A)) \cap W(\Im(A))} \).

For our next result we need the following three lemmas, the first two of which can be found in [1] and [15], respectively.

Lemma 3.6. ([1, Remark 2.2]) Let \( A, B, X, Y \in \mathcal{B}(\mathcal{H}) \). Then

\[
w^2(AX \pm BY) \leq \|AA^* + Y^*Y\|\|X^*X + BB^*\|.
\]

Lemma 3.7. ([15, Th. 1.1]) Let \( A, B, X, Y \in \mathcal{B}(\mathcal{H}) \). Then

\[
\left\| \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} A & X \\ \|X\| & \|B\| \end{pmatrix} \right\|.
\]

The next lemma is as follows.

Lemma 3.8. Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then \( \|AA^* + B^*B\| \leq \mu(A, B) \), where

\[
\mu(A, B) = \frac{1}{2} \left[ \|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|BA\|^2} \right].
\]

Proof. \( AA^* + B^*B \) being a self-adjoint operator, we have

\[
\|AA^* + B^*B\| = r(AA^* + B^*B)
\]

\[
= r \begin{pmatrix} AA^* + B^*B & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= r \begin{pmatrix} |A^*| & |B| \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |A^*| & 0 \\ 0 & |B| \end{pmatrix}
\]

\[
= r \begin{pmatrix} |A^*|^2 & |A^*||B| \\ |B||A^*| & |B|^2 \end{pmatrix}, \quad r(XY) = r(YX)
\]

\[
= \left\| \begin{pmatrix} \|A\|^2 & \|A^*||B| \\ \|B||A^*| & \|B\|^2 \end{pmatrix} \right\|
\]

\[
\leq \left\| \begin{pmatrix} \|A\|^2 & \|A^*||B| \\ \|B||A^*| & \|B\|^2 \end{pmatrix} \right\|, \quad \text{by Lemma 3.7}
\]

\[
= \left\| \begin{pmatrix} \|A\|^2 & \|BA\| \\ \|BA\| & \|B\|^2 \end{pmatrix} \right\|
\]

\[
= \frac{1}{2} \left[ \|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|BA\|^2} \right].
\]
Hence,
\[ \|AA^* + B^*B\| \leq \mu(A, B). \]

**Remark 3.9.** Notice that \( \mu(A, B) \leq \max\{\|A\|^2, \|B\|^2\} + \|AB\| \). In particular, if \( A = B \) then \( \mu(A, A) = \|A\|^2 + \|A^2\| \). Hence, we have \( \|AA^* + A^*A\| \leq \|A\|^2 + \|A^2\| \).

Now we are in a position to prove the following result.

**Theorem 3.10.** Let \( A, B, X, Y \in B(\mathcal{H}) \). Then
\[ w(AX \pm BY) \leq \sqrt{\mu(A, Y) \mu(B, X)}. \]

**Proof.** The proof follows from Lemma 3.6 and Lemma 3.8.

An application of Theorem 3.10 we get the following corollary.

**Corollary 3.11.** Let \( A, B \in B(\mathcal{H}) \). Then
\[ w(AB \pm BA) \leq \sqrt{\|A\|^2 + \|A^2\|} (\|B\|^2 + \|B^2\|). \]

**Remark 3.12.** Let \( A, B \in B(\mathcal{H}) \) with \( A^2 = B^2 = 0 \). Then it follows from Corollary 3.11 that \( w(AB \pm BA) \leq \|A\|\|B\| < 2\sqrt{\|A\|\|B\|} = \sqrt{2}\|A\|\|B\|. \)

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