GELFAND-KIRILLOV DIMENSIONS OF SIMPLE MODULES
OVER TWISTED GROUP ALGEBRAS \( k \ast A \)

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Abstract

When \( A \) is a free abelian group of finite rank \( n \) the twisted group algebra \( k \ast A \) of \( A \) over a field \( k \) is a quantum deformation of the coordinate ring of the torus \((k^\times)^n\). This algebra is also the localization of the quantum polynomial ring generated by the \( n \) variables \( X_1, \cdots, X_n \) subject only to the relations \( X_i X_j = q_{ij} X_j X_i \). The latter is an associated graded ring for many distinct quantum groups. It is known that the GK dimension of a finitely generated \( k \ast A \)-module is an integer. The Gelfand–Kirillov dimension is important and interesting invariant of noncommutative infinite dimensional algebras and their modules. In this article we attempt to characterize the simple \( k \ast A \)-modules by determining the set of integers that can be the Gelfand-Kirillov dimension of a simple \( k \ast A \)-module. In our answer to this question we invoke another important invariant, namely, the Krull dimension.

1. Introduction

Let \( k \) be a field and let \( k^\times \) denote the group \( k \backslash \{0\} \). The algebra of quantum polynomials \( \Lambda_{q,r} \) is the associative algebra generated over a field \( k \) by the variables \( X_1, X_1^{-1}, \cdots, X_r, X_r^{-1}, X_{r+1}, \cdots X_n \) subject to the relations

\[
X_i X_j = q_{ij} X_j X_i, \quad 1 \leq i, j \leq n \quad \text{and}
\]

\[
X_i^{-1} X_i = 1, \quad 1 \leq i \leq r,
\]

where \( q_{ij} \in k^\times \). Let \( q \) stand for the matrix of multi-parameters \( q_{ij} \), that is, \( q = (q_{ij}) \). The matrix \( q \) is assumed to be a multiplicatively antisymmetric matrix, that is, \( q_{ii} = 1 \) and \( q_{ji} = q_{ij}^{-1} \). These algebras which are quantum deformations of coordinate algebras of suitable affine varieties play an important role in noncommutative geometry \(^{[19]}\). The cases \( r = 0 \) and \( r = n \) are especially noteworthy and important: quantum affine spaces which are the algebras with \( r = 0 \) in the above definition occur as the associated graded rings of several important quantum groups \(^{[22]}\). The role of these algebras in quantum group theory is discussed in \(^{[7]}\). The case \( r = n \) is also interesting and the corresponding quantum polynomial rings known as \( n \)-dimensional...
quantum tori or, more simply quantum tori have found applications in the representation theory of torsion-free nilpotent groups [13]. For example, if $H$ is a finitely generated and torsion-free nilpotent group of class two with center $\zeta H$ then the central localization $kH(k\zeta H \setminus \{0\})^{-1}$ is an algebra of this type.

It is this latter type of algebras which we denote as $\Lambda_q$ that concerns us in this article. While many different aspects including irreducible and projective modules [3, 4, 5, 6, 8, 9, 10] automorphisms and derivations [11, 7, 11, 23, 24] Krull and global dimensions [2, 13, 21] etc. of these algebras have been studied in the recent past (a survey appears in [12]) our topic of focus is the Gelfand-Kirillov dimension of simple modules over these algebras. This question was first considered in [21]. We recall that the Gelfand-Kirillov dimension (GK dimension) is an important and well-behaved invariant of finitely generated algebras and their modules reflecting their asymptotic growth behaviour. Consequently the determination of the GK dimension of the simple modules may be regarded as a step towards characterizing the irreducible representations of an algebra.

The question of the GK dimension of the simple modules of the Weyl algebras $A_n(\mathbb{C})$ is discussed in [15] and in [21] it was shown that for the algebras $\Lambda_q$ having Krull dimension equal to one, each simple module has the same GK dimension, namely, $n - 1$. Some further results on this question were established in [16] and [17].

In general, however, for a given algebra there may be various distinct possibilities for the GK dimension of its simple modules. As an example of this the $n$-th complex Weyl algebra $A_n(\mathbb{C})$ has a simple module with GK dimension $j$ for each integer $j$ in the set \{n, n + 1, \ldots, 2n - 1\} (15).

For a general multiplicatively antisymmetric matrix $q$ the nature of the simple modules of the algebra $\Lambda_q$ is not well-understood. But it has become evident that the matrix $q$ has an important role in determining the growth rates of the simple $\Lambda_q$ modules. So far in the results concerning the GK dimension of simple $\Lambda_q$-modules, this influence has manifested only indirectly via the ring-theoretic invariants of Krull and global dimensions. It was shown in [21] that these two dimensions coincide for the algebras $\Lambda_q$. This common value known as the dimension of the algebra $\Lambda_q$ is related to the transcendence degree of commutative subalgebras of $\Lambda_q$ – a point which will be more fully explained in Section 3. Our first theorem thus reads:

**Theorem 1.** Let $\Lambda_q$ be an $n$-dimensional quantum torus algebra with (Krull or global) dimension $n - 1$. For the GK dimension of a simple $\Lambda_q$-module $M$ the following dichotomy holds

$$\mathcal{G}(M) = 1, \quad \text{or} \quad \mathcal{G}(M) = \mathcal{G}(\Lambda_q) - \mathcal{G}(\mathcal{Z}(\Lambda_q)) - 1,$$

where $\mathcal{Z}(\Lambda_q)$ denotes the center of $\Lambda_q$. 

Following [21] we have the following

**Definition 2.** The $\lambda$-group $G(\Lambda_q)$ of a quantum torus algebra $\Lambda_q$ is defined as the subgroup of $k^\times$ generated by the multi-parameters $q_{ij}$.

For an arbitrary subgroup $H < k^\times$ we denote the torsion-free rank of $H$ by $\text{rk}(H)$. Note that

$$0 \leq \text{rk}(G(\Lambda_q)) \leq \frac{n(n-1)}{2}.$$ 

The algebra $\Lambda_q$ has so-called *scalar automorphisms*. These automorphisms are defined by $X_i \mapsto \beta_i X_i$, where $\beta_i \in k^\times$. The quantum torus algebras can be presented as an iterated skew-Laurent extension involving such automorphisms. It was observed in [21] that the GK dimensions of simple $\Lambda_q$-modules take non-negative integer values.

For a subset $U$ of non-negative integers and for an integer $k$, by $U + k$ we mean the set $\{u + k \mid u \in U\}$ Our second theorem concerns the GK dimensions of the simple modules of a skew-Laurent extension as in the preceding paragraph.

**Theorem 2.** Let $\Lambda_q$ be an $n$-dimensional quantum torus algebra and consider the skew-Laurent extension

$$\Gamma_{q,\sigma} = \Lambda_q[Y^{\pm 1}; \sigma],$$

where $\sigma \in \text{Aut}(\Lambda_q)$ is a scalar automorphism defined by $\sigma(X_i) = \beta_i X_i$. Let $\mathcal{GK}(\Lambda_q)$ be the (finite) set of GK dimensions of simple $\Lambda_q$-modules. Let $M$ be a simple $\Gamma_{q,\sigma}$-module. Denote by $H_\sigma$ the subgroup of $k^\times$ generated by the scalars $\beta_i$ for $1 \leq i \leq n$. If the subgroups $G(\Lambda_q)$ and $H_\sigma$ of $k^\times$ intersect trivially then

$$\mathcal{GK}(M) \in \{\text{rk}(H_\sigma), \cdots, n\} \cup (\mathcal{GK}(\Lambda_q) + 1).$$

**Remark 3.** Clearly, applying the last theorem iteratively allows us to compute the possible values of the GK dimension of simple $\Lambda_q$-modules provided the condition in the theorem is satisfied.

This article is organized as follows: In Section 2 we give an exposition of the GK dimension and also establish some facts as to how this dimension varies under a change of ground field. In Section 3 we describe the structure and fundamental properties of the quantum torus algebra. Section 4 is meant to be an exposition of the known facts concerning the GK dimensions of finitely generated $\Lambda_q$-modules, particularly the useful facts established in [14]. In Sections 5 and 6 we present the proofs of Theorems 1 and 2 respectively. All modules in this paper will be right modules.
2. The Gelfand-Kirillov Dimension

The Gelfand-Kirillov dimension is a measure of the growth of an algebraic structure in terms of a generating set but which turns out to be independent of such a choice. For an affine algebra $B$ over a field $k$ with the generating set

$$Y := \{(b_0 = 1), b_1, \ldots, b_m\}$$

we consider the (increasing) finite dimensional filtration $\{B_i\}_{i=0}^\infty$ of $B$ obtained as follows:

(i) $B_0 = k$,
(ii) $B_1 = \sum_{j=0}^{m} kb_j$,
(iii) $B_r = B_1^r$ for $r \geq 1$.

We note that each of the spaces $B_r$ has a finite spanning set and consequently a finite basis over $k$ (Henceforth we will refer to $B_i$ as the \textit{standard finite dimensional filtration} of $B$ with respect to the generating set $Y$). It is thus meaningful to consider the function $f_Y : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f_Y(i) := \dim_k(B_i)$. We think of $f_Y$ as the growth function of the algebra $B$ with respect to the generating set $Y$. For example, the growth function of the free algebra $k\{x, y\}$ with respect to the generating set $Y := \{1, x, y\}$ is $f_Y(i) = 2^i$.

By definition the function $g_2(n) = 2^n$ represents exponential growth whereas the function $g_3(n) = n^2 + n$ represents polynomial (quadratic growth). If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are eventually monotone functions then $f$ and $g$ are said to represent the same growth if $f(i) \leq cg(mi)$ and $g(j) \leq c_1f(m_1j)$ for natural numbers $c, m, c_1$ and $m_1$ and almost all $i$ and $j$. This relation induces a partitioning of all such functions into equivalence classes known as \textit{growth classes}. The various different choices of a generating set of an algebra $B$ give rise to equivalent growth functions [18]. The formula

$$\limsup \frac{\log_n \dim_k(B_n)}{n}$$

serves to define an \textit{asymptotic growth rate} of the algebra $B$ with respect to a fixed generating set. The useful numerical invariant arising from this expression is known as the Gelfand-Kirillov dimension (GK dimension) of the algebra $B$ with respect to the ground field $k$. It is independent of the choice of a generating set $Y$ employed to obtain the filtration $B_i$.

For example, when $B$ is finite dimensional the sequence $\{B_r\}_{r=1}^\infty$ must stop growing after a finite number of steps and therefore the GK dimension of $B$ is zero. For many important examples of infinite-dimensional associative affine algebras the GK dimension is a positive integer. In general, however, the GK dimension of an affine algebra can be any real number $r$ such that

$$r \in \{0\} \cup \{1\} \cup (2, \infty).$$
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We note that by the famous Bergman Gap Theorem the GK dimension of a finitely generated algebra cannot lie strictly between one and two. The definition of the GK dimension can be readily extended to the finitely generated modules of an affine algebra $\mathcal{B}$. To this end, let $\{\mathcal{B}_i\}_{i=1}^{\infty}$ be the standard finite dimensional filtration of $\mathcal{B}$ with respect to a generating set $Y$ as defined above and let $M_0$ be a finite dimensional generating subspace of $M$, that is, $M = kX$ where $X$ is a set of generators for $M$ as a $\mathcal{B}$-module. The growth function associated to $M$ with respect to the given choices of generating sets ($Y$ and $X$) is $f_{Y,X}(i) = \dim_k(M_0\mathcal{B}_i)$ and the GK dimension of $M$ ($\mathcal{GK}(M)$) is defined as the asymptotic growth rate:

$$\mathcal{GK}(M) = \limsup_n \log_n \dim_k(M_0\mathcal{B}_n).$$

As we would expect $\mathcal{GK}(M)$ is independent of the choices of generating sets for $M$ and $\mathcal{B}$. The filtration $M_0\mathcal{B}_i$ will be referred to as the standard finite dimensional filtration of $M$ with respect to (the generating sets) $Y$ and $X$. For further details on this dimension for algebras and modules we refer the reader to the references [18], [20] and [28]. We conclude this discussion with the following fact that we will be needing in the proof of Theorem 1 in Section 5. Essentially, it asserts that the GK dimension of algebras and modules remains invariant in passing to a finite extension of the ground field.

**Proposition 4.** Let $k \subset K$ be a finite extension of fields. Let $\mathcal{B}$ be an affine $K$-algebra with a finitely generated module $M$. The following assertions hold:

(i) The GK dimensions of $\mathcal{B}$ regarded as an affine $k$-algebra and as an affine $K$-algebra coincide.

(ii) Similarly, the GK dimension of $M$ does not depend on the choice of the ground field ($k$ or $K$).

**Proof.** Let $Z := \{z_1, \ldots, z_t\}$ be a basis for $K$ over $k$ and as before we fix a generating set

$$Y := \{(b_0 = 1), b_1, \ldots, b_m\}$$

of $\mathcal{B}$ as a $K$-algebra. Then $W := Y \cup Z$ generates $\mathcal{B}$ over $k$. We denote the standard finite dimensional filtration of $\mathcal{B}$ regarded as a $k$-algebra with respect to $W$ as $\{\tilde{\mathcal{B}}_j\}_{j=1}^{\infty}$. As before, we let $\{\mathcal{B}_i\}_{i=0}^{\infty}$ stand for standard filtration of $\mathcal{B}$ (as $K$-algebra) with respect to $Y$. Let $\tilde{f}$ and $f$ respectively denote the corresponding growth functions. Thus $f(i) = \dim_K(\mathcal{B}_i)$ and $\tilde{f}(j) = \dim_k(\tilde{\mathcal{B}}_j)$ for all $i$ and $j$. We observe that

$$\tilde{\mathcal{B}}_1 \leq \mathcal{B}_1 \leq \tilde{\mathcal{B}}_2$$

(3)
as $k$-spaces. By definition $\tilde{B}_i = \tilde{B}_1^i$ and similarly $B_j = B_1^j$ for all $i$ and $j$ and so (3) clearly implies that

$$\tilde{B}_i \leq B_i \leq \tilde{B}_2^i$$

as $k$-spaces. Now

$$f(i) = \dim_K(B_i) = \frac{1}{t^i} \dim_k(B_i).$$

which gives $\tilde{f}(i) \leq tf(i)$. But (4) also yields

$$f(i) \leq tf(i) \leq \tilde{f}(2i).$$

By definition, the last two facts mean that $f$ and $\tilde{f}$ belong to the same growth class and this establishes part (i) of the proposition.

We now let $M$ be as in the theorem and define $X = \{m_1, \ldots, m_l\}$ to be finite set of generators for $M$ as $\mathcal{B}$-module. We set $M_0 = KX$ and $\tilde{M}_0 = kX$. Clearly, $\{M_0\mathcal{B}_i\}_{i=0}^\infty$ and $\{\tilde{M}_0\tilde{\mathcal{B}}_j\}_{j=0}^\infty$ are the standard filtrations of $M$ appearing in the computation of the GK dimension of $M$ with respect to the fields $K$ and $k$ respectively. Clearly $\tilde{M}_0$ is a $k$-subspace of $M_0$ such that $M_0 \leq \tilde{M}_0\tilde{\mathcal{B}}_1$ and noting (4) we have

$$\tilde{M}_0\tilde{\mathcal{B}}_i \leq M_0\mathcal{B}_i \leq \tilde{M}_0\tilde{\mathcal{B}}_{2i+1}$$

as $k$-subspaces of $M$. Now an argument analogous to the above for part (i) shows that the growth functions of two filtrations of $M$ with respect to $k$ and $K$ belong to the same growth class and consequently the GK dimension of $M$ is independent of the choice of the ground field ($k$ or $K$).

□

3. THE $n$-DIMENSIONAL QUANTUM TORUS

The definition of the quantum torus given in the introduction in terms of generators and relations is a rather simple one. As the name suggests, there is a second definition based on the idea of quantum deformation of the Laurent polynomial ring

$$L := k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$$

which is the coordinate algebra of the $n$-torus. This deformation can be achieved by passing to the Laurent polynomial ring $L[t, t^{-1}]$ and defining a new multiplication by $X_i X_j = t X_j X_i$ and finally specializing to the case $t = q$ ([28 Problem 16.A.3]).

As already noted the algebras $\Lambda_q$ have the structure of a twisted group algebra $k * A$ of a free abelian group of rank $n$ over $k$. We briefly recall this kind of a structure and refer the keen reader to [25] as an excellent reference on this subject. For a given group $G$ a $k$-algebra $\tilde{R}$ is said to be twisted group
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Let $G$ be a group and consider a map $\gamma : G \times G \to k^\times$ that satisfies:

$$\gamma(g_1, g_2) = \gamma(g_2, g_1)$$

which is the 2-co-cycle condition in terms of group cohomology (each twisted group algebra $k\ast G$ arises from an element in $H^2(G, k^\times)$). For a subgroup $H$ of $G$ the $k$-linear span of $\overline{H} := \{\overline{h} \mid h \in H\}$ is a sub-algebra of $R$ which is a twisted group algebra $k\ast H$ with the defining cocycle being the restriction of $\gamma$ to $H \times H$. This rule defines a one-one correspondence between subgroups of $G$ and certain subalgebras of $k\ast G$. For example, in the case $G$ abelian it can be shown that the center of $k\ast G$ is of the form $k\ast Z$ for a suitable subgroup $Z \leq G$ (e.g. [24, Lemma 1.1]). In the case of the quantum tori $\Lambda_q = k\ast A$ the subgroups $B$ for which the subalgebra $k\ast B$ is commutative play an important role. For example, it was conjectured in [21] and shown in [14] that the supremum of the ranks of such subgroups coincides with both the Krull and global dimensions of the algebra $\Lambda_q$.

Using the fact that $A \cong \mathbb{Z}^n$ is an ordered group (with the lexicographic order) it is not difficult to see that $k\ast A$ is a domain if and only if each product $a \cdot a$ with $a \in A$ is of the form $\mu \tilde{a}$ for $\mu \in k^\times$ and $a \in A$. Let $M$ be a finitely generated $k\ast A$-module. If $A_0$ is a subgroup of $A$ with finite index then it is not difficult to see that $M$ is a finitely generated $k\ast A_0$-module. Moreover the algebra $k\ast A$ is a finite normalizing extension of its subalgebra $k\ast A_0$, that is, it is generated over the latter by a finite number of normal elements namely the images in $k\ast A$ of a transversal $T$ of $A_0$. We recall that in a ring $T$ an element $t \in T$ normalizes a subring $S$ if $tS = St$.

A more general structure than a twisted group algebra is a crossed product $R\sharp G$. Here the ground ring $R$ need not be a field and the scalars in $R$ need not be central in $R\sharp G$. As in the case of twisted group algebras a copy $G$ of $G$ is an $R$-module basis for $R\sharp G$ and multiplication of the basis elements is defined exactly as in (5) above. However, as already remarked the scalars in $R$ need not commute with the basis elements $\tilde{g}$ for $g \in G$ but we have $\tilde{g}r = \sigma_g(r)\tilde{g}$ for $r \in R$ and $\sigma_g \in \text{Aut}(R)$. We refer the interested reader to the text [26] for further details concerning crossed products. Besides being generalizations of twisted group algebras, crossed products also arise as suitable localizations of the former rings. It is in this form that a crossed product will arise in our proof of Theorem 2 in Section 6. As we wish to study simple modules over the
algebra $\Lambda_q$ the following proposition already noted in [22] is of key importance for us.

**Proposition 5** (Nullstellensatz). Let $M$ be a simple $\Lambda_q$-module. Then the endomorphism algebra of $M$ is algebraic over the underlying field $k$.

**Proof.** Consider the quantum affine space $\Lambda_{q,0}$ in $m$ variables with multi-parameter matrix $q$. It is sufficient to prove the proposition for this ring as each quantum torus algebra $\Lambda_q$ can be obtained as a homomorphic image of a ring of this type.

Let $\Lambda := \Lambda_{q,0}$. By Theorem 3.11 of [20] it suffices to show that $\Lambda[Y]$ is generically flat over $k[Y]$. Now $k[Y]$ is generically flat as an algebra over the integral domain $k[Y]$ ([20] Corollary 9.4.7). As $\Lambda[Y]$ is built from the polynomial ring $k[Y]$ by a succession of normalizing extensions it follows from Theorem 9.4.10 of [20] that $\Lambda[Y]$ is generically flat over $k[Y]$. □

4. The GK dimension of finitely generated $\Lambda_q$-modules

The GK dimension is a particularly well-behaved dimension for the finitely generated modules over the algebras we are studying. One reason for this is that the Hilbert-Samuel machinery which works for almost commutative algebras can be adapted for our class of algebras as well (Section 5 of [21]). For example, we have the following:

**Proposition 6.** Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence of $\Lambda_q$-modules. Then

$$\mathcal{G}(M) = \max\{\mathcal{G}(L), \mathcal{G}(N)\}$$

**Proof.** See Lemma 5.5 of [21]. □

In [14] the finitely generated modules over crossed products of a free abelian group of finite rank over a division ring $D$ were studied with group-theoretic applications in mind. A dimension for finitely generated modules which was shown to coincide with the Gelfand–Kirillov dimension (measured relative to $D$) was introduced and studied. We aim to employ this dimension in our investigation of simple modules over the algebras we are considering here which are special cases of the aforementioned crossed products. We state below its definition and some key properties which were established in [14]. This dimension is used in conjunction with an appropriate notion of a critical module to be discussed below.
Definition 7 ([14]). Let $M$ be a finitely generated $\Lambda_q$-module. The dimension $\dim M$ of $M$ is the maximum $r$, where $0 \leq r \leq n$ so that for some subset $\mathcal{I} := \{i_1, i_2, \ldots, i_r\}$ of the indexing set $\{1, \ldots, n\}$ the module $M$ is not torsion as $\Lambda_{q,\mathcal{I}}$-module where $\Lambda_{q,\mathcal{I}}$ denotes the subalgebra of $\Lambda_q$ generated by the variables $X_{i}$ for $i \in \mathcal{I}$ and their inverses.

Remark 8. In [14] it was shown that the dimension $\dim M$ of $M$ in the sense of the last definition coincides with the GK dimension of $M$.

The following facts from [14] which we state for the algebras $\Lambda_q$ were shown for more general crossed products.

Lemma 9. [14, Lemma 2.3] Let $M$ be a finitely generated $\Lambda_q$-module with GK dimension $d$ and let $\Lambda_1$ be the subalgebra of $\Lambda_q$ generated by the variables $\{X_{i_1}, X_{i_2}, \ldots, X_{i_d}\}$ and their inverses. Then $M$ cannot not embed a $\Lambda_1$-module which is free of infinite rank.

Proof. Noting Remark 8 this follows from Lemma 2.3 of [14]. □

Definition 10. A nonzero $\Lambda_q$-module $N$ is said to be critical if for each non-zero submodule $N$ of $M$

$$\mathcal{G}(M/N) < \mathcal{G}(M).$$

Proposition 11. [14, Proposition 2.5] Every non-zero $\Lambda_q$-module contains a finitely generated critical submodule.

5. The dichotomy result

In this section we will establish the dichotomy result (Theorem B).

Theorem 1. Let $\Lambda_q$ be an $n$-dimensional quantum torus algebra with (Krull or global) dimension $n - 1$. For the GK dimension of a simple $\Lambda_q$-module $M$ the following dichotomy holds

$$\mathcal{G}(M) = 1, \quad \text{or} \quad \mathcal{G}(M) = \mathcal{G}(\Lambda_q) - \mathcal{G}(Z(\Lambda_q)) - 1,$$

where $Z(\Lambda_q)$ denotes the center of $\Lambda_q$.

Proof. As in Section 3 we may write $\Lambda_q$ as a twisted group algebra $\Lambda_q := k \ast A$ for a free abelian group $A$ with rank $n$. Moreover, we let $\bar{a}$ stand for the image of $a \in A$ in $\Lambda_q$. It is not difficult to see that $Z(\Lambda_q)$ has the form $k \ast Z$ for a suitable subgroup $Z$ of $A$ (e.g., Lemma 1.1 of [24]). As

$$\mathcal{G}(k \ast Z) = l$$
(e.g., [21]), the latter alternative in the assertion of the theorem then reads
\[ G_K(M) = \text{rk}(A) - \text{rk}(Z) - 1. \]

Let \( P \) be the annihilator of \( M \) in \( k * Z \). Clearly \( P \) is a prime ideal of \( k * Z \). The action of \( k * Z \) on \( M \) gives an embedding
\[ (k * Z)/P \longrightarrow \text{End}_{\Lambda_q}(M). \]

As \( k * A \) satisfies Nullstellensatz (Proposition 5), \( \text{End}_{\Lambda_q}(M) \) is algebraic over \( k \) and so \( (k * Z)/P \) is also algebraic. A commutative affine algebraic domain is a field [27] and therefore \( P \) is a maximal ideal of \( k * Z \).

Set \( K = (k * Z)/P \) and \( Q = P\Lambda_q \). Clearly, \( M \) is a simple \( \Lambda_q/Q \)-module. By [26, Chapter 1, Lemmas 1.3 and 1.4] the \( k \)-algebra \( \Lambda_q/Q \) is a twisted group algebra \( K * A/Z \) of \( A/Z \) over \( K \) with a transversal \( T \) for \( Z \) in \( A \) yielding a \( K \)-basis as the set \( \{ \bar{t} + Q \mid t \in T \} \). Moreover, the elements \( \zeta + Q \), where \( \zeta \in k * Z \) constitute a copy of \( K \) in \( \Lambda_q/Q \). We note that the group-theoretic commutator \([\bar{t}_1 + Q, \bar{t}_2 + Q]\) with values in the unit group of \( \Lambda_q/Q \) for any \( t_1, t_2 \in T \) satisfies
\[ [\bar{t}_1 + Q, \bar{t}_2 + Q] = [\bar{t}_1, \bar{t}_2] + Q \in k^\times + Q. \]

Now, in view of Proposition 5.1(c) of [18] we have
\[ G_K - \text{dim}_{k*A}(M) = G_K - \text{dim}_{K*A/Z}(M). \]

In the last equation in both LHS and RHS the GK dimension is being measured relative to \( k \). Since \( K \) is finitely generated and algebraic over \( k \) therefore \([K:k] < \infty\) and in view of Proposition [14](ii) it suffices to determine the possible values of the GK dimension of the simple \( K * A/Z \)-module \( M \) measured relative to \( K \).

Our main point in passing to the algebra \( K * A/Z \) is that as a \( K \)-algebra it is central, that is, has center \( K \). Indeed, as we already saw in Section 3 the center of \( K * A/Z \) is of the form \( K * Y \) for a subgroup \( Y \) of \( A/Z \). If the image \( \bar{t} + Q \) of some coset \( t + Z \in Y \) centralized all elements of \( K * A/Z \) then using (6) we have,
\[ [\bar{t}, \bar{t}_1] + Q = 1 + Q. \quad \forall t_1 \in T. \]

As \([\bar{t}, \bar{t}_1] \in k^\times \), it follows that \([\bar{t}, \bar{t}_1] = 1 \). Thus, if \( \bar{t} + Q \) lies in the center of \( K * A/Z \) then \( \bar{t} \) must be in the center of \( k * Z \) of \( k * A \). Since \( t \in T \), this is possible only if \( t = 1 \).

As already noted in Section 3 by the theorem of Brookes [13] the dimension of \( k * A \) equals the supremum of the ranks of the subgroups \( B \leq A \) such that the subalgebra \( k * B \) is commutative. In the present situation this means the existence of a subgroup \( B \) of \( A \) with rank \( n - 1 \) such that \( k * B \) is commutative.
In passing to $K \ast A/Z$ although the center becomes equal to the base field a small difficulty appears, namely, that $A/Z$ need not be torsion-free. To overcome this we may replace $A$ by a subgroup $A_0$ of finite index such that $A_0/Z$ is torsion-free. Then $B_0 := A_0 \cap B$ is a subgroup of $A_0$ with rank $n - 1$ and clearly $k \ast B_0$ is a commutative sub algebra of $k \ast A_0$. Evidently, $k \ast B_0Z$ is commutative and therefore by the preceding paragraph $\text{rk}(B_0Z) = \text{rk}(B_0)$. Replacing $B_0$ by $B_0Z$ if necessary we may assume that $B_0 \geq Z$. In view of (6) the subalgebra $K \ast B_0/Z$ of $K \ast A_0/Z$ is commutative. We obviously have $\text{rk}(B_0/Z) = n - 1 - \text{rk}(Z) = \text{rk}(A_0/Z) - 1 = \text{rk}(A/Z) - 1$.

The last equation means that $K \ast A_0/Z$ is a $n - \text{rk}(Z)$-dimensional quantum torus over $K$ with (Krull or global) dimension $n - \text{rk}(Z) - 1$. As a module over the sub-algebra $K \ast A_0/Z$ the simple $K \ast A/Z$-module $M$ need not remain simple. However as $K \ast A/Z$ is a finite normalizing extension of $K \ast A_0/Z$ (Section 3) the $K \ast A/Z$-module $M$ decomposes as a finite direct sum of simple $K \ast A_0/Z$-modules (e.g., Exercise 15A.3 [28]). We thus have

$$M = N_1 \oplus N_2 \oplus \cdots \oplus N_s$$

as $K \ast A_0/Z$-modules. By Lemma 2.7 of [14],

$$\mathcal{GK} - \dim_{K \ast A/Z}(M) = \mathcal{GK} - \dim_{K \ast A_0/Z}(M).$$

Moreover the GK dimension of a finite direct sum of modules is the maximum of the GK dimensions of the summands (Proposition 5.1 of [18]). In view of these remarks, to establish the theorem it suffices to show that if $F$ is field and $F \ast \mathbb{Z}^r$ is a twisted group algebra with center $F$ and dimension equal to $r - 1$ then for any simple $F \ast \mathbb{Z}^r$-module $N$ the following dichotomy holds

$$\mathcal{GK}(N) = 1 \text{ or } \mathcal{GK}(N) = r - 1.$$  

But this is precisely the content of [16, Theorem 2.1].

6. Proof of Theorem 2

We recall (Definition 2) that the $\lambda$-group $G(\Lambda_q)$ of $\Lambda_q$ is the subgroup of $k^\times$ generated by the entries of the matrix $q$. We also recall our notation that for a subset $U$ of non-negative integers $U + 1$ is the set $U + 1 = \{u + 1 \mid u \in U\}$. We then have

**Theorem 2.** Let $\Lambda_q$ be an $n$-dimensional quantum torus algebra and consider the skew-Laurent extension

$$\Gamma_{q,\sigma} = \Lambda_q[Y^\pm; \sigma],$$
where \( \sigma \in \text{Aut}(\Lambda_q) \) is a scalar automorphism defined by \( \sigma(X_i) = \beta_i X_i \). Let \( \mathfrak{G}K(\Lambda_q) \) be the (finite) set of GK dimensions of simple \( \Lambda_q \)-modules. Let \( M \) be a simple \( \Gamma_{q,\sigma} \)-module. Denote by \( H_\sigma \) the subgroup of \( k^\times \) generated by the scalars \( \beta_i \) for \( 1 \leq i \leq n \). If the subgroups \( G(\Lambda_q) \) and \( H_\sigma \) of \( k^\times \) intersect trivially then

\[
\mathcal{Gk}(M) \in \{ \text{rk}(H_\sigma), \cdots, n \} \cup (\mathfrak{G}K(\Lambda_q) + 1).
\]

**Proof.** Writing \( \Gamma \) for \( \Gamma_{q,\sigma} \) and \( \Lambda \) for \( \Lambda_q \). Noting Proposition 11, we let \( N \) be a finitely generated critical \( \Lambda \)-submodule of \( M \). Consider the \( \Gamma \)-submodule \( N' \) of \( M \) generated by \( N \):

\[
N' := N\Gamma = \sum_{i \in \mathbb{Z}} NY^i.
\]

Since \( N \) is assumed to be critical, therefore \( N \neq 0 \) and \( N' = M \). If the sum in (8) is direct, then \( N\Gamma \cong N \otimes_\Lambda \Gamma \) and Lemma 2.4 of [14] gives

\[
\mathcal{Gk} - \text{dim}(M) = \mathcal{Gk} - \text{dim}(N) + 1.
\]

Moreover, since the (left) \( \Lambda \)-module \( \Gamma \) is free, it is faithfully flat, and it follows from this that \( N \) must be a simple \( \Lambda \)-module. Noting equation (9) we thus obtain

\[
\mathcal{Gk}(M) \in \mathfrak{G}k(\Lambda_q) + 1.
\]

Clearly the assertion of the theorem holds true in this case. We are thus left with the possibility where the sum \( \sum_{i \in \mathbb{Z}} NY^i \) fails to be direct. We know from Lemma 2.4 of [14] that in this case

\[
\mathcal{Gk}(N) = \mathcal{Gk}(M)
\]

recalling that the dimension being referred to in this same lemma coincides with the GK dimension measured relative to the ground field \( k \). Let \( d := \mathcal{Gk}(M) \).

The course of reasoning that we will now follow is based on the arguments in the proof of Theorem 3.9 in [21] and is as follows: we begin by passing to a suitable (right) Ore localization of \( \Gamma \) such that the corresponding localization of \( M \) is a non-zero finite dimensional vector space over some division ring \( \mathcal{D} \) embedded in this localization. The generators in the set \( \{ X_i^{\pm 1}, X_2^{\pm 1}, \cdots, X_n^{\pm 1}, Y^{\pm 1} \} \) act \( \mathcal{D} \)-semi-linearly on this vector space and can thus be represented by invertible matrices with entries in \( \mathcal{D} \). Using the tools of Dieudonne determinant and Malcev-Neumann completion, it was shown in [21] that the commutation relations (1) imply certain relations among the multiparameters \( q_{ij} \). We will be employing this device of [21].

In view of Definition 7 and the succeeding remark, we let \( \mathcal{I} := \{ X_{i_1}, \cdots, X_{i_d} \} \) be a maximal subset of the set \( \{ X_1, \cdots, X_n \} \) such that \( N \) (and therefore \( M \)) is
not $\mathcal{S}$-torsion where $\mathcal{S}$ stands for the set of non-zero elements of the subalgebra $\Lambda_{q,I}$ generated by the variables $X_i$ for $i \in \mathcal{I}$ together with their inverses. Thus $\mathcal{S} = \Lambda_{q,I} \setminus \{0\}$. It is a known fact that the subset $\mathcal{S}$ is an Ore subset of $\Gamma$ (e.g., [21]).

We may thus pass to the (right) Ore localization $\Gamma\mathcal{S}^{-1}$ of $\Gamma$ at $\mathcal{S}$. As $M$ is not $\mathcal{S}$-torsion the corresponding module of fractions $M\mathcal{S}^{-1}$ is nonzero. Let $\mathcal{D} \subseteq \Gamma\mathcal{S}^{-1}$ be the quotient division ring $\Lambda_{q,I}\mathcal{S}^{-1}$. Note that $\Gamma\mathcal{S}^{-1}$ is a crossed product $\mathcal{D}^\# Z^{n-d}$. We claim that

$$s := \dim_{\mathcal{D}} M\mathcal{S}^{-1} < \infty.$$  

Indeed, if this was not true, we would obtain a contradiction to Lemma 9 as is easy to see. Let $\mathcal{J}$ be the complement of $\mathcal{I}$ in $\{1, \ldots, n\}$. Let $G_\mathcal{I}$ denote the subgroup of $G(\Lambda_q)$ defined as

$$G_\mathcal{I} = \langle q_{kl} \mid k, l \in \mathcal{I} \rangle.$$  

Next, for each $j \in \mathcal{J}$, we define

$$G_{\mathcal{I},j} = \langle q_{kj} \mid k \in \mathcal{I} \rangle.$$  

We will also need to refer to the subgroup $H_\mathcal{I}$ of $H_\sigma$ defined as follows:

$$H_\mathcal{I} := \langle \beta_i \mid i \in \mathcal{I} \rangle.$$  

By the hypothesis in the theorem, we have

$$YX_j = \beta_j X_j Y, \quad \forall j \in \mathcal{J}.$$  

This is precisely the situation of Section 3.9 of [21] and exactly as in that section the following dependence relations must hold:

$$\beta^*_j \in \langle G_\mathcal{I}, G_{\mathcal{I},j}, H_\mathcal{I} \rangle.$$  

By the assumption in the theorem, $G(\Lambda_q) \cap H_\sigma = 1$ and therefore $\beta_j^* \in H_\mathcal{I}$. But this means that

$$\text{rk}(H_\sigma) \leq |\mathcal{I}| = d = \mathcal{G}\mathcal{K}(M).$$  

It remains to show that $\mathcal{G}\mathcal{K}(M) < n + 1$. To this end we suppose that $\mathcal{G}\mathcal{K}(M) = n + 1$. Since $\Gamma$ is a twisted group algebra it follows by Definition 7 and Remark 8 that $M$ embeds a copy of the right regular module $\Gamma$ and as it is simple, coincides with the regular $\Gamma$-module. But this means that $\Gamma$ is a division ring which is clearly not true. Our proof is now complete. □

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