Informatic error-disturbance relation in the qubit case

Li-Yi Hsu

1Department of Physics, Chung Yuan Christian University, Chungli, 320, Taiwan, Republic of China

In 1927, Heisenberg heuristically disclosed the tradeoff between the error in the measurement and the caused disturbance on another complementary observable. In quantum theory, most uncertainty relations are proposed to describe the level of unavoidable uncertainty in the measurement process. In this paper, we study the error-disturbance relation from an information perspective. We ask how much information, rather than how much uncertainty, can be gained during two sequential measurements. To achieve the optimal information gain, we argue that the strategy for an "intelligent" prior apparatus is to clone the unknown state and, for the posterior apparatus, the swapping operation should be performed in the posterior apparatus. We propose a coarse-grain random access code, and therein information causality as a physical principle can be exploited to derive the upper-bound of information gain. Finally, we conjecture the information gain of the position and momentum using coarse-grain measurements.

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As a fundamental principle in quantum physics, there is always tradeoff in measuring two non-commuting observables. To access some observable of an quantum object, one must specify definite experiments as the measurement processes. In Heisenberg’s original thought, determining an electron’s position with the attainable accuracy must reduce the measurement precision of another complementary observable, such as momentum. In detail, when an electron is scattered by photons, the position is instantly determined. However, the momentum undergoes a “discontinuous change” due to the scattering. General speaking, in the sequential quantum measurements, the prior measurement may include error, and the caused disturbance generates a discontinuous change for the latter measurement outcome. Hence the term “error-disturbance relation” is designated as a suitable name of uncertainty principle.

Researchers attempt to tackle quantum measurement uncertainty using alternative approaches. Considering the thought experiment of observing an electron using an imaginary gamma-ray microscope, Heisenberg put down the relation \( \Delta x \Delta p \sim h \) in 1927. Robertson’s well-known formula has been introduced in many standard quantum physics textbooks, which states that the product of two standard deviations for two non-commuting observables is upper-bounded [4], therein the error-and-disturbance scenario is poorly represented. On the other hand, entropic uncertainty has been also investigated in [28]. As proposed by Busch, Lahti, and Werner (BLW), the Wasserstein distance between probability distributions is exploited [2] and, potential experimental schemes have been also proposed [9, 11]. Recently, based on the error-and-disturbance scenario, Ozawa’s reformulation has attracted much more attention [12, 14]; therein, the error and disturbance is recalled and operationally defined [15]. In particular, the qubit case with discrete measurement outcomes [17, 18] has been claimed to be experimentally verified [17, 18].

However, the Ozawa and BLW’s approaches have been debated in the field [19–21], partially due to lack of well specification on the usage “uncertainty”. For Robertson’s relation, uncertainty implicitly indicates the product of two standard deviations; however, the indication is not useful in the recent studies. On the other hand, measurements are exploited to know the information of an object. A more meaningful question is to ask how much information, rather than its uncertainty, can be obtained in sequential measurements. Thanks to the quantum information science, physicists can quantify information using mutual information. Inevitable uncertainty never be removed, which implicitly indicates the impossibility of accessing full information. In this paper, we propose the upper bound for information that one can obtain from the measurements of the two non-commuting observables.

Throughout we employ the von Neumann quantum measurement model, which can be briefly stated as follows. The goal is to learn the information of an unknown state of the quantum object \( O \) through sequential measurements. In detail, a prior measurement on the observable \( A \) is performed using apparatus \( A \), and a later measurement on the observable \( B \) is performed using apparatus \( B \). Before the readouts, the quantum system \( P_A \) and \( P_B \) each as the probe or pointer in the apparatus \( A \) and \( B \) interacts with the object \( O \), respectively. The process is depicted in Figure 1. Hereafter the object \( O \) can be regarded as a qubit with binary measurement outcomes. In addition, let the observables \( A \) and \( M_A \) \((B \) and \( M_B \)) be the spin observables in the same direction.

Without loss of generality, the initial state of the composite system \( O + P \) is \(|\psi\rangle \otimes |0\rangle \) before turning on the interaction at time \( t \). Notably, \(|\psi\rangle \) is an unknown two-level state, and \(|0\rangle \) is a fixed probe state; and these two states are completely uncorrelated. At time \( t + \Delta t \), the interaction is turned off. \( U \) denotes the unitary time evolution.
evolution of $O + P$ during the interval $(t, t + \Delta t)$. After the turn-off of the interaction the sharp measurement on the observable $M$ is performed. In Ozawa’s formulation \cite{22,23} the noise value $\epsilon(\psi)$ is defined as

$$
\epsilon(\psi) = \langle \psi \otimes \xi | (M_A^{\text{out}} - A^{\text{in}})^2 | \psi \otimes \xi \rangle^{1/2},
$$

where $M_A^{\text{out}} = U(t) \otimes M_A U(t)$ and $A^{\text{in}} = A \otimes I$. The disturbance value is $\eta(\psi)$ defined as

$$
\eta(\psi) = \langle \psi \otimes \xi | (B^{\text{out}} - B^{\text{in}})^2 | \psi \otimes \xi \rangle^{1/2},
$$

where $B^{\text{out}} = U(t) (B \otimes I) U(t)$ and $B^{\text{in}} = B \otimes I$. For more discussions on the limitation of operator $U$, readers can refer to \cite{24}. For simplicity, we weaken the condition such that all kinds of unitarity are feasible for the object-probe interactions.

Case (a) Let $U$ be the identity operator. Obviously, $B^{\text{out}} = B^{\text{in}}$, hence $\eta = 0$. The object state $\psi$ is undisturbed, and no information can be gained.

Case (b) Let $U$ be the SWAP operation,

$$
U_{\text{SWAP}} |\psi\rangle |0\rangle = |0\rangle |\psi\rangle.
$$

(1)

As a result, we have

$$
\langle M^{\text{out}} \rangle = \langle A^{\text{in}} \rangle,
$$

(2)

and the error $\epsilon(\psi) = 0$. The state of the object after the unitary evolution is always fixed, which is completely uncorrelated with the initial state. In this case, the state is regarded as mostly disturbed. As a result, it is possible that $\epsilon(\psi) \eta(\psi) = 0$.

The following remarks refer to Ozawa’s scenario. First, Heisenberg’s picture is usually exploited in Ozawa’s formulation. Here we exploit Schrödinger’s picture, which aids in elucidating the substantial role of quantum cloning. We can describe the sequential measurements as the quantum circuit, as shown in Fig. 1(b). Second, as previously mentioned, we focus on information gain. How the state is disturbed is not our main concern. We will show that error comes from imperfect quantum cloning. Third, the concept of joint measurement is exploited in Ozawa’s study \cite{13}. Nevertheless, the role of respective unitary evolution in either apparatus $A$ or $B$ is not clear described.

The following interesting question naturally arises. Can $\epsilon = 0$ and $\eta = 0$ simultaneously? The intuitive answer is positive with a $1 \leftrightarrow 2$ cloning machine (PCM), that is,

$$
U_{\text{PCM}} |\psi\rangle |0\rangle = |0\rangle |\psi\rangle.
$$

(3)

If one were to perfect $1 \leftrightarrow 2$ cloning ($U_A = U_{\text{PCM}}$) followed by the swapping operation between the object and probe $B$ ($U_B = U_{\text{SWAP}}$), the states of both probes $P_A$ and $P_B$ would be the same state, $|\psi\rangle$; hence it were to access full information of the measured observables $A$ and $B$ both. Therefore, the quantum no-cloning prevents the simultaneous vanishing of noise and disturbance. For physical realization, we propose that the optimal information gain where $U_A$ as the optimal $1 \rightarrow 2$ cloner and $U_B$ as the swapping operation are used, respectively. Notably, we do not know of other non-trivial or accidental means for getting optimal information gain, which is outside the scope of this paper.

To tackle the error-disturbance relation from an informational perspective, we propose the coarse-grained random access code (CRAC) with the following scenario. Initially, Alice has a two-bit database $x_A x_B \in \{00, 01, 10, 11\}$, which the distant Bob would like to access. Alice and Bob share an ensemble of the Bell state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle |1\rangle - |1\rangle |0\rangle)$ as their physical resource. To help Bob, Alice is permitted to perform classical one-bit communication with Bob. Unlike the random access code in \cite{25}, where Bob wants to access only $x_A$ or $x_B$, he attempts to access both $x_A$ and $x_B$ simultaneously. In this case, Bob sequentially measures the observables

$$
A = \hat{a} \cdot \vec{\sigma} \quad \text{and} \quad B = \hat{b} \cdot \vec{\sigma},
$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\sigma_x$, $\sigma_y$, and $\sigma_z$ are Pauli matrices. Without loss of generality, the unit vectors $\hat{a}$ and $\hat{b}$ lie in the $(x-y)$ equator of the Bloch sphere and divided the equator into four quadrants $Q_{x \pm x \pm}$, as depicted in Fig. 2. Alice random chooses a unit Bloch vector $\hat{\varphi}_{x_A x_B} \in Q_{x \pm x \pm}$ with the angle $\varphi_{x_A x_B}$ in the equator. The corresponding state of the phase $\varphi_{x_A x_B}$ is

$$
|\varphi_{x_A x_B}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi_{x_A x_B}} |1\rangle),
$$

where the orthogonal state in the equator is $|\varphi_{x_A x_B}^\perp\rangle = \frac{1}{\sqrt{2}}(|0\rangle - e^{i\varphi_{x_A x_B}} |1\rangle)$ \cite{26}. Up to a global phase, the Bell state can be revised as

$$
|\Phi\rangle = \frac{1}{\sqrt{2}}(|\varphi_{x_A x_B}\rangle |\varphi_{x_A x_B}^\perp\rangle - |\varphi_{x_A x_B}^\perp\rangle |\varphi_{x_A x_B}\rangle).
$$

The bits $x_w = (1 - \Theta(\hat{\varphi}_{x_A x_B} \cdot \hat{\varphi})) \in \{0, 1\}$, where $w \in \{a, b\}$ and the Heaviside step function

$$
\Theta(z) = \begin{cases} 
1, & z > 0 \\
0, & z \leq 0.
\end{cases}
$$

The word “coarse-grained” indicates Alice can choose any $|\varphi_{x_A x_B}\rangle$ to encode her two-bit database and, she only concerns which quadrant includes $\varphi_{x_A x_B}$. We state the proposed protocol as follows.

(1). Encoding phase: Alice perform the projective measurement with the random-chosen orthonormal basis states \{|$\varphi_{x_A x_B}\rangle$, $|\varphi_{x_A x_B}^\perp\rangle$\}. 

(2). Communication phase: Alice announces the classical bit $\beta = 1\ (0)$ if the post-selected state is $|\varphi_{x,\hat{A}}\rangle$ ($|\varphi_{x,\hat{B}}^\perp\rangle$).

(3). Decoding phase: As shown in Fig. 1, Bob sequentially performs the sharp measurement $A$ and $B$ with the outcomes $O_A$ and $O_B \in \{1,-1\}$, respectively. Finally Bob’s guessing answers on $x_A$ and $x_B$ are $g_A = O_A + \beta \pmod{2}$ and $g_B = O_B + \beta \pmod{2}$, where $O_A = \frac{1}{2} - \frac{1}{2}2^i$ and $O_B = -\frac{1}{2}2^i \in \{0,1\}$, respectively.

According to information causality [23, 27], we have
\[
\sum_{W \in \{A,B\}} I(x_W : g_W) \leq 1, \tag{4}
\]
where $I(x : g)$ denotes the mutual information between the random variables $x$ and $g$. Let the state $|\varphi_{x,\hat{A}}\rangle$ be Bob’s qubit state after Alice’s local measurement. Ineq. (4) can be revised as follows
\[
\sum_{W \in \{A,B\}} I(x_W : o_W) \leq 1. \tag{5}
\]

To evaluate the upper bound of the mutual information, we exploit the following lemma.

**Evan-Schulman Lemma:** Consider a cascade of two communication channels: $X \rightarrow Y \rightarrow Z$, with $X$, $Y$, $Z$ being random variables. Let $Y$ and $Z$ be the input and output of the symmetric channel $C$, respectively, with the successful transmission probability $\frac{1 + \xi}{2}$. We have
\[
\frac{I(X;Z)}{I(X;Y)} \leq \xi^2. \tag{6}
\]

Interested readers can refer to [28, 29] for the detailed rigorous proof.

Without loss of generality, assume the bias parameter $0 \leq \xi \leq 1$, and hereafter let the $X$ and $Y$ be the input and output of an error-free channel and hence we have $I(X;Y) = 1$. Let variables $Y$ and $Z$ be $x'_W$ and $o'_W$, respectively. Finally we have
\[
I(x'_W : o'_W) \leq \xi^2. \tag{7}
\]

To achieve the optimal encoding, the unitary operator $U_A$ corresponds to the phase covariant $1 \rightarrow 2$ cloner (PCC), which reads [30, 33]
\[
U_{PCC} |0\rangle |0\rangle = |0\rangle |0\rangle , \tag{8}
\]
and
\[
U_{PCC} |1\rangle |0\rangle = \cos \eta |1\rangle |0\rangle + \sin \eta |0\rangle |1\rangle. \tag{9}
\]

Specifically, no ancilla qubit is required in the cloning process.

Now we can state the main result for this paper as follows. According to (5) and (7), the information gain
\[
I = \sum_{W \in \{A,B\}} I(x'_W : o'_W) \leq \xi_A^2 + \xi_B^2 \leq 1, \tag{10}
\]

where
\[
\xi_A = |\hat{a} \cdot \hat{\varphi}_{x,\hat{A}}| \sin \eta, \quad \xi_B = |\hat{b} \cdot \hat{\varphi}_{x,\hat{B}}| \cos \eta. \tag{11}
\]

The details for calculating $\xi_A$ and $\xi_B$ are given in the Supplemental material. Notably, the bias parameters each are the products of two parts. One part is from the cloning coefficients ($\sin \eta, \cos \eta$), and the other is from the measurement process ($|\hat{a} \cdot \hat{\varphi}_{x,\hat{A}}|, |\hat{b} \cdot \hat{\varphi}_{x,\hat{B}}|$). We consider the following specific cases.

Case (a) Consider either $\xi_A = 1$ or $\xi_B = 1$. To fulfill $\xi_A = 1\ (\xi_B = 1)$, the equation $|\sin \eta| = |\hat{a} \cdot \hat{\varphi}_{x,\hat{A}}| = 1\ ((|\cos \eta| = |\hat{b} \cdot \hat{\varphi}_{x,\hat{B}}| = 1)$ must hold; hence $\xi_B = \cos \eta = 0\ (\xi_A = \sin \eta = 0)$. Here, $U_A$ is only a swapping operator (an identity operator). Furthermore, $\hat{a}$ ($\hat{b}$) and the unknown $\hat{\varphi}_{x,\hat{A}}$ are accidentally in either the same or opposite direction. In CRAC, when $|\hat{a} \cdot \hat{\varphi}_{x,\hat{A}}| = 1\ (|\hat{b} \cdot \hat{\varphi}_{x,\hat{B}}| = 1)$, Alice and Bob’s outcomes must either perfect or anti-perfect correlated, which leads to 1-bit of information gain.

Case (b) Consider the symmetric cloning with $\eta = \frac{\pi}{4}$. The condition $|\hat{a} \cdot \hat{\varphi}_{x,\hat{A}}| = |\hat{b} \cdot \hat{\varphi}_{x,\hat{B}}| = 1$ yields 1-bit of information gain. However, the latter measurement is meaningless because $\hat{b}$ is either parallel or anti-parallel to $\hat{a}$.

Case (c) Let the observables of these two measurements be most incompatible (e.g., $\hat{a} \cdot \hat{b} = 0$). In this case, we set $\hat{a} \cdot \hat{\varphi}_{x,\hat{A}} = \cos \delta$ and $\hat{b} \cdot \hat{\varphi}_{x,\hat{B}} = \sin \delta$. The optimal value $I$ in (10) can be achieved using the symmetric cloning with the condition $\delta = \frac{\pi}{4}$. As a result,
\[
I \leq \frac{1}{2}. \tag{12}
\]

In the general CRAC, Alice has an $N$-bit local database $\vec{\varphi} = x_1, \ldots, x_2$, where $x_i = (1 - \Theta(\hat{\varphi}_2 \cdot \hat{m}_i)) \forall i$, and the measurement basis states $\{|\varphi_{x,\hat{A}}\rangle, |\varphi_{x,\hat{B}}^\perp\rangle\}$ are exploited during the encoding phase followed by classical one-bit communication. Similarly, Bob wants to access the $i$-th bit $x_i$ by measuring $\hat{m}_i \cdot \hat{\sigma}$ with $\forall i = 1, \ldots, N$. The proposed optimal strategy for Bob is to performs $1 \rightarrow N$ phase covariant cloning and measures the observable $\hat{m}_i \cdot \hat{\sigma}$ on the object with the $i$-th copied state. Thus the information gain from measuring $\hat{m}_i \cdot \hat{\sigma}$ is $I_i \leq \xi_i^2$. Based on (11), we conjecture that the $i$-th bias parameter
\[
\xi_i = (\hat{e}_i \cdot \hat{\vec{n}})(\hat{m}_i \cdot \hat{\varphi}_2),
\]
where $\{\hat{e}_1, \ldots, \hat{e}_N\}$ is a set of the orthonormal basis vectors in the $N$ dimensional space $\mathbb{R}^N$. The vector $\hat{\vec{n}} \in \mathbb{R}^N$ and $|\hat{\vec{n}}| \leq 1$. Here the vectors $\hat{e}_1, \ldots, \hat{e}_N$ and $\hat{\vec{n}}$ should be parameterized according to the $1 \rightarrow N$ cloning process. Such conjecture ensures that preservation of information causality,
\[
\sum_{i=1}^{N} I_i \leq 1.
\]
FIG. 1: (color online). (a) The whole unitary operator $U$ can be decomposed as $U = U_B U_A$. Firstly, during the time interval $[t, t + \Delta t_A]$, the evolutions of the pair-wise interactions between the object and the probe $P_A$ ($P_B$) are represented by unitary operator $U_A$ ($U_B$). The initial states of the probes $P_A$ and $P_B$ are $|0\rangle_A$ and $|0\rangle_B$, respectively. To gain optimal information, $U_A$ should be the optimal $1\rightarrow2$ cloner, and $U_B$ the swapping operator.

Notably, only $1\rightarrow2$ phase covariant cloning is “simple enough” that can be done using the single pair-wise interaction between the object and the probe. As for $1\rightarrow N$ cloning process with $N > 2$ should involve more quantum objects. In this case, the quantum cloning as the pre-process should be performed before any measured object enters into any apparatus.

At the end of the paper, we proposed a method of employing the CRAC in the continuous variables, such as position and momentum of a one-dimensional quantum system. Instead of precisely measuring the position and momentum, we suppose there are two coarse-grained measurements. Where one is performed to answer “Is the object at left or at right?”, and the other is performed to answer “Does the object go left or goes right?” As a famous example, the physical realization of answering the prior question can be regarded as the “which way” measurement in the double-slit experiment [34]. On this issue, Bohr argued that a measurement capable of definitively discerning two positions (one-bit information) must produce an “uncontrollable change in the momentum” (zero information) [35]. Therefore, at most one-bit information can be gained. On the other hand, the corresponding observables for the coarse-grained position and momentum should be less incompatible. According to (12), the information gain can be greater than 0.5 bit. Let $I_{pm}$ be the information gain from the coarse-grained position-and-momentum experiment. Based on the above argument, we conjecture that

$$\frac{1}{2} \leq I_{pm} \leq 1.$$ 

In summary, we propose an alternative way of studying sequential quantum measurement from the infomation perspective. The imperfect quantum cloning and information causality are key to determining the upper bound of the information gain. We also show that the fidelity of quantum cloning is limited by information causality. Anyway, we only propose a toy model. The constraint on time evolution unitary between the object and either probe should be seriously considered for further study.

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[1] Bohr gave a full account of measuring of an electron using a γ-ray microscope, which is published in [2]. Heisenberg finally referred to Bohr’s scenario instead of his own. Interesting readers can refer to [8].

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SUPPLEMENTAL MATERIAL

Without loss of generality, the spin observables along the directions of the unit vectors $\hat{a} = (a_x, a_y, 0)$ and $\hat{b} = (a_x, a_y, 0)$ are sequentially measured using the apparatus $A$ and $B$, respectively. The sharp projectors $\Pi^\pm_A = \frac{1}{2}(I \pm \hat{a} \cdot \vec{\sigma})$ ($\Pi^\pm_B = \frac{1}{2}(I \pm \hat{b} \cdot \vec{\sigma})$) are exploited to measure the observable $A$ ($B$). Let the probe state of apparatus $A$ be $|0_A\rangle$. After the unitary operation $U_A$ as the phase covariant $1 \rightarrow 2$ cloning, the density matrix of the probe $A$ is

$$\rho_A = tr_S\{|\psi\rangle \langle \psi|\},$$

where

$$|\psi\rangle = U_A|\varphi_{x_{A}x_{B}}\rangle_S|0_A\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi_{x_{A}x_{B}}}(\cos \eta |1\rangle |0\rangle + \sin \eta |0\rangle |1\rangle)).$$

The measurement outcome $o_A \in \{-1, 1\}$ can be obtained with the probability

$$p_A = tr(\rho_A \Pi^\pm_A) = \frac{1}{2}(1 + o_A(\hat{a} \cdot \varphi_{x_{A}x_{B}}) \cos \eta). \quad (S1)$$

Virtually, if $sgn(o_A) = sgn(\varphi_{x_{A}x_{B}} \cdot \hat{a})$, the decoding is successful under the mapping on $o_A : 1 \rightarrow 0$ and $-1 \rightarrow 1$. According to (S1), the successful decoding probability $p_A = \frac{1}{2}(1 + \hat{a} \cdot \varphi_{x_{A}x_{B}} \cos \eta) = \frac{1}{2}(1 + \xi_A)$. Therefore,

$$\xi_A = |\hat{a} \cdot \varphi_{x_{A}x_{B}}| \cos \eta.$$

Next, after the unitary evolution $U_B$, the state of the probe $P_B$ is swapped with that of the object. Hence the density matrix of the probe is

$$\rho_B = tr_A\{|\psi\rangle \langle \psi|\}.$$ 

Using the above similar calculation and argument, we have the biased parameter

$$\xi_B = |\hat{b} \cdot \varphi_{x_{A}x_{B}}| \sin \eta.$$