THEOREMS OF BARTH-LEFSCHETZ TYPE ON KÄHLER
MANIFOLDS OF NON-NEGATIVE BISECTIONAL CURVATURE

MEEYOUNG KIM AND JON WOLFSON

0. Introduction

The general philosophy of Barth-Lefschetz type theorems is that a subvariety $Y$ of "small" codimension in a fixed variety $X$ must be subject to stringent topological restrictions. For example, in the case of a complete intersection $Y$ in the $n$-dimensional complex projective space $\mathbb{P}^n$, the Lefschetz hyperplane theorem gives cohomological restrictions on $Y$.

In 1970 W. Barth [B] discovered that even when $Y$ is not a complete intersection, one can compare the cohomology of $\mathbb{P}^n$ to that of $Y$ and prove theorems similar to the Lefschetz theorem. Since Barth’s foundational work, there have been many studies in this direction, for example, those by Sommese [S] and Fulton-Lazarsfeld [F-L].

In [S-W], R. Schoen and J. Wolfson showed that a variant of an argument of Frankel together with Morse theory on a space of paths leads to a proof of homotopic Barth-Lefschetz type theorems for complex submanifolds of compact Kähler manifolds of non-negative holomorphic bisectional curvature. To use the technique of [S-W] in manifolds other than $\mathbb{P}^n$ it is necessary to compute a numerical invariant, called the complex positivity, which measures the positivity of the holomorphic bisectional curvature. In this note, we compute the complex positivity of all the compact hermitian symmetric spaces. In fact this number plays an important role in much work centered on hermitian symmetric spaces, for example the metric rigidity theorems of Mok [M1]. Combining the computations of complex positivity with the results of [S-W] we conclude:

Theorem 0.1. Let $V$ be a Kähler manifold. Suppose that $M, N \subset V$ are compact complex submanifolds of complex dimensions $m, n$, respectively. Let

$$i_* : \pi_j(N, N \cap M) \to \pi_j(V, M),$$

be the homomorphism induced by the inclusion.

(i) If $V = U(p + q)/S(U(p) \times U(q))$, the complex grassmannian, then $i_*$ is an isomorphism for $j \leq n + m - 2pq + p + q - 1$, and is surjective for $j = n + m - 2pq + p + q$.

(ii) If $V = SO(2+p)/SO(2) \times SO(p)$, the complex quadric, then $i_*$ is an isomorphism for $j \leq n + m - p - 1$, and is surjective for $j = n + m - p$.

(iii) If $V = Sp(r)/U(r)$, then $i_*$ is an isomorphism for $j \leq n + m - r^2$, and is surjective for $j = n + m - r^2 + 1$.

(iv) If $V = SO(2r)/U(r)$, then $i_*$ is an isomorphism for $j \leq n + m - r^2 + 3r - 3$, and is surjective for $j = n + m - r^2 + 3r - 2$.

Date: February 24, 2022.
The second author was partially supported by NSF grant DMS-9802487.
(v) If \( V = E_6/\text{Spin}(10) \times T^1 \), then \( i_* \) is an isomorphism for \( j \leq n + m - 21 \), and is surjective for \( j = n + m - 20 \).

(vi) If \( V = E_7/E_6 \times T^1 \), then \( i_* \) is an isomorphism for \( j \leq n + m - 37 \), and is surjective for \( j = n + m - 36 \).

Case (i) is stated in [S-W] and case (ii) is stated incorrectly there, both without proof. We remark that this result also follows from computations of Goldstein [G] and Sommese’s approach to Barth-Lefschetz type theorems. On the other hand using the result of [S-W] the proof of the theorem is reduced to what are essentially standard computations in hermitian symmetric spaces. Further, the method of proof of the theorem yields better results when some additional information is available about the ambient curvature along either \( M \) or \( N \). This should prove especially useful if either \( M \) or \( N \) is a hypersurface. This is discussed in more detail in Section 1. Finally the method described in [S-W] worked in hermitian symmetric spaces. Consequently it was necessary to understand how well the method of [S-W] worked in hermitian symmetric spaces. This is accomplished here.

In June 2001 Meeyoung Kim died in a tragic drowning accident. She was 35 years old and had a promising mathematical future ahead of her. Her mathematics was centered around the relation between topology and algebraic geometry. She had wanted the work reported on in this paper to be a beginning, not an end. Her passing leaves all who knew her saddened with a profound sense of loss.

1. The index of a critical point

Let \( V \) be a complete Kähler manifold of complex dimension \( v \), with complex structure \( J \) and Levi-Civita connection \( \nabla \). Let \( M \) and \( N \) be complex submanifolds of complex dimensions \( m \) and \( n \), respectively. We denote, by \( \Omega(V;M,N) = \Omega \), the space of piecewise smooth paths \( \gamma : [0,1] \to V \) constrained by the requirements that \( \gamma(0) \in M \) and \( \gamma(1) \in N \). Consider the energy of a path \( E(\gamma) = \int_0^1 |\dot{\gamma}|^2 dt \) as a function on \( \Omega \). It is shown in [S-W] that \( \gamma \) is a critical point of \( E \) if:

(i) \( \gamma \) is a smooth geodesic
(ii) \( \gamma \) is normal to \( M \) and \( N \) at \( \gamma(0) \) and \( \gamma(1) \), respectively.

Let \( W_1, W_2 \in T_{\gamma(0)}M \). If \( \gamma \) is a critical point of \( E \) then the second variation of \( E \) along \( \gamma \) is:

\[
\frac{1}{2} E_{\gamma\gamma}(W_1, W_2) = \left( \langle \nabla_{\dot{\gamma}} W_1, \dot{\gamma} \rangle \right)^1_0 + \int_0^1 \langle \nabla_{\dot{\gamma}} W_1, \nabla_{\dot{\gamma}} W_2 \rangle dt
- \int_0^1 \langle R(\dot{\gamma}, W_1)\dot{\gamma}, W_2 \rangle dt.
\] (1.1)

where \( R \) denotes the curvature tensor of \( V \).

Suppose that \( \gamma \) is a nontrivial critical point and that \( W(0) \) is a vector in \( T_{\gamma(0)}M \). Parallel translate \( W(0) \) along \( \gamma \) to construct a vector field \( W = W(t) \) along \( \gamma \). Of
course, \( W(1) \) need not be tangent to \( N \) at \( \gamma(1) \) so \( W \) is not necessarily an element of \( T_N \). However formally we have:

\[
\frac{1}{2} E_{**}(W, W) = \langle \nabla_W W, \dot{\gamma} \rangle |_{0}^{1} - \int_{0}^{1} \langle R(\dot{\gamma}, W) \dot{\gamma}, W \rangle dt. \tag{1.2}
\]

\( V \) is Kähler so \( JW \) is also parallel along \( \gamma \). \( M \) is complex so \( JW(0) \in T_{\gamma(0)}M \). Thus we also have:

\[
\frac{1}{2} E_{**}(JW, JW) = \langle \nabla_{JW} JW, \dot{\gamma} \rangle |_{0}^{1} - \int_{0}^{1} \langle R(\dot{\gamma}, JW) \dot{\gamma}, JW \rangle dt. \tag{1.3}
\]

Adding (1.2) and (1.3) and using \( \nabla_{JW} JW = -\nabla_W W \) we have:

\[
\frac{1}{2} E_{**}(JW, JW) + \frac{1}{2} E_{**}(W, W) = -\int_{0}^{1} \left( \langle R(\dot{\gamma}, W) \dot{\gamma}, W \rangle + \langle R(\dot{\gamma}, JW) \dot{\gamma}, JW \rangle \right) dt. \tag{1.4}
\]

Using the symmetries of the curvature tensor we have:

\[
\langle R(\dot{\gamma}, W) \dot{\gamma}, W \rangle + \langle R(\dot{\gamma}, JW) \dot{\gamma}, JW \rangle = \langle R(\dot{\gamma}, J\dot{\gamma}) W, JW \rangle. \tag{1.5}
\]

This expression is the holomorphic bisectional curvature of the complex lines \( \dot{\gamma} \wedge J\dot{\gamma} \) and \( W \wedge JW \).

Let \( \{W_1(0), \ldots, W_m(0), JW_1(0), \ldots, JW_m(0)\} \) be an orthonormal framing of \( T_{\gamma(0)}M \). For each \( i = 1, \ldots, m \), parallel translate \( W_i(0) \) along \( \gamma \) to construct parallel vector fields \( \{W_1, \ldots, W_m, JW_1, \ldots, JW_m\} \) along \( \gamma \). Note that the vectors \( W_i(1), JW_i(1) \) are perpendicular to both \( \dot{\gamma}(1) \) and \( J\dot{\gamma}(1) \). Thus the vector space

\[
S = \text{span}\{W_1(1), \ldots, W_m(1), JW_1(1), \ldots, JW_m(1)\}
\]

is a complex \( m \)-dimensional space lying in a complex \( (v-1) \)-dimensional subspace of \( T_{\gamma(1)}V \). It follows that the subspace \( S \cap T_{\gamma(1)}N \) has complex dimension at least equal to

\[
m + n - (v-1).
\]

Moreover, the vector fields \( \{W, JW\} \) with \( W(1), JW(1) \in S \cap T_{\gamma(1)}N \) are parallel and lie in \( T_{\gamma} \).

**Theorem 1.1.** Suppose that \( V \) is a Kähler manifold of positive holomorphic bisectional curvature, that \( M \) and \( N \) are complex submanifolds and that \( \gamma \) is a nontrivial critical point of the energy on \( \Omega(V; M, N) \). Then,

\[
\text{index}(\gamma) \geq m + n - (v-1).
\]

**Proof.** There are at least \( m + n - (v-1) \) pairs \( \{W, JW\} \) that are parallel along \( \gamma \) and lie in \( T_{\gamma} \). For each such pair, using the curvature assumption (1.4) and (1.5) we have:

\[
E_{**}(W, W) + E_{**}(JW, JW) = -2 \int_{0}^{1} \langle R(\dot{\gamma}, J\dot{\gamma}) W, JW \rangle dt < 0.
\]

The result follows.
Let $V$ be a Kähler manifold. Fix $x \in V$ and let $X \wedge JX$ be a complex line in $T_x V$. Let $\mathcal{C}(x, X \wedge JX)$ be the cone:

$$\mathcal{C}(x, X \wedge JX) = \{y \in T_x V : \langle R(X, JX)y, Jy \rangle > 0\}.$$ 

Note that $\mathcal{C}$ is a complex cone; if $Y \in \mathcal{C}$ then $JY \in \mathcal{C}$.

**Definition 1.1.** Denote the set of complex subspaces of $\mathcal{C}(x, X \wedge JX)$ by $\mathcal{L} = L(x, X \wedge JX)$ and define

$$\ell(x, X \wedge JX) = \max_{L \in \mathcal{L}} \dim_{\mathbb{C}}(L).$$ 

Then define,

(i) $\ell(x) = \inf_{X \wedge JX} \ell(x, X \wedge JX) \quad \text{and} \quad \ell = \inf_{x \in V} \ell(x).$

We say that $\ell$ is the complex positivity of $V$.

If $V$ is a hermitian symmetric space then clearly $\mathcal{C}(x, X \wedge JX)$ and hence $\ell(x, X \wedge JX)$ are independent of $x \in V$.

**Remark:** If $V$ is a compact hermitian symmetric space and $e$ is the identity define the symmetric bilinear form $H_X(W, Z) = \langle R(X, JX)W, JZ \rangle$, where $X, W, Z \in T_x V$. Then for any $X \neq 0$, $H_X$ is positive semi-definite. Denote the null-space by $N_X$. Then it follows easily that $\ell(X \wedge JX)$ is the complementary dimension of $N_X$. For the four classical families of hermitian symmetric spaces it is easy to compute $\dim_{\mathbb{C}} N_X$ directly (see [M2]). For example, if $V = \text{Gr}(p, p + q; \mathbb{C})$, the complex Grassmann manifold and $X \in T_x V$ then $X \neq 0$ is a matrix with $1 \leq \text{rank}(X) \leq \min(p, q)$. Then,

$$\dim_{\mathbb{C}} N_X = (p - \text{rank}(X))(q - \text{rank}(X)).$$ 

Therefore, $\dim_{\mathbb{C}} N_X$ is maximal and $\ell(X \wedge JX)$ is minimal when $\text{rank}(X) = 1$.

**Theorem 1.2.** Suppose that $V$ is a complete Kähler manifold of non-negative holomorphic bisectional curvature. Let $M$ and $N$ be complex submanifolds of complex dimensions $m$ and $n$, respectively, and $\gamma$ be a nontrivial critical point of energy on $\Omega(V; M, N)$. Then

$$\text{index}(\gamma) \geq m + n - (v - 1) - (v - \ell).$$

**Proof.** The argument in the proof of Theorem 1.2 shows that if $W, JW \in S \cap T_\gamma(1) N$ then

$$E_{\ast\ast}(W, W) + E_{\ast\ast}(JW, JW) = -2 \int_0^1 \langle R(\dot{\gamma}, J\dot{\gamma})W, JW \rangle \leq 0.$$ 

To get strict inequality we want

$$\langle R(\dot{\gamma}, J\dot{\gamma})W, JW \rangle > 0$$

at $\gamma(0)$. This is insured by requiring that:

$$W(0), JW(0) \in L \cap T_\gamma(0) M$$

where $L \in L(\gamma(0), \dot{\gamma} \wedge J\dot{\gamma})$. Let $L \in L(\gamma(0), \dot{\gamma} \wedge J\dot{\gamma})$ be of maximal dimension. Then the complex dimension of $L \cap T_\gamma(0) M$ is at least $\dim_{\mathbb{C}} L + m - (v - 1)$. Parallel transport $L \cap T_\gamma(0) M$ along $\gamma$ to $\gamma(1)$ and denote the resulting subspace by $T$. Then $T \cap T_\gamma(1) N$ has complex dimension at least $n + \dim_{\mathbb{C}} L + m - (v - 1) - (v - 1)$. The result follows.
Corollary 1.3. Under the same hypotheses as the theorem assume that
\[ \ell(x, X \wedge JX) \geq \ell_0, \]
for every \( x \in M \) and every complex line \( X \wedge JX \) normal to \( T_x M \) then
\[ \text{index}(\gamma) \geq m + n - (v - 1) - (v - \ell_0). \]

2. Applications

Let \( V \) be a complete Kähler manifold of non-negative holomorphic bisectional curvature, of complex dimension \( v \) and with complex positivity \( \ell \). Let \( M, N \subset V \) be complex submanifolds of complex dimensions \( m, n \), respectively and suppose that \( M \) is compact and \( N \) is a closed subset of \( V \). The Morse theory of the energy functional on the path space \( \Omega \) is described in [S-W]. Combining this Morse theory with Theorem 1.2 it follows that:

Theorem 2.1. Suppose that,
\[ \lambda_0 = n + m - v - (v - \ell) \geq 0. \]
Then relative homotopy groups \( \pi_j(\Omega, N \cap M) \) are zero for \( 0 \leq j \leq \lambda_0 \).

Theorem 2.1 and the long exact homotopy sequence of the pair \( (\Omega, N \cap M) \) imply that the homomorphism induced by the inclusion:
\[ i_* : \pi_j(N \cap M) \rightarrow \pi_j(\Omega) \quad (2.1) \]
is an isomorphism when \( j < n + m - v - (v - \ell) \) and is a surjection when \( j = n + m - v - (v - \ell) \).

Consider the fibration:
\[ \Omega(V; M, x) \longrightarrow \Omega(V; M, N) \]
\[ \downarrow e \quad (2.2) \]
\[ N \]
where \( e \) is the evaluation map \( e : \gamma \mapsto \gamma(1) \) and \( x \in N \). It is well-known that the homotopy groups of the fiber \( \Omega(V; M, x) \) satisfy:
\[ \pi_j(\Omega(V; M, x)) \simeq \pi_{j+1}(V, M), \quad (2.3) \]
for all \( j \). The long exact homotopy sequence of the fibration is:
\[ \cdots \longrightarrow \pi_{j+1}(N) \longrightarrow \pi_j(\Omega(V; M, x)) \longrightarrow \pi_j(\Omega) \]
\[ \longrightarrow \pi_j(N) \longrightarrow \pi_{j-1}(\Omega(V; M, x)) \longrightarrow \cdots \quad (2.4) \]
Thus, using (2.3), the long exact sequence (2.4) becomes:
\[ \cdots \longrightarrow \pi_{j+1}(N) \rightarrow \pi_{j+1}(V, M) \rightarrow \pi_j(\Omega) \rightarrow \pi_j(N) \rightarrow \pi_j(V, M) \rightarrow \cdots \quad (2.5) \]
We have:
**Theorem 2.2.** Let $V$ be a complete Kähler manifold of non-negative holomorphic bisectional curvature, of complex dimension $v$ and with complex positivity $\ell$. Let $M, N \subset V$ be complex submanifolds of complex dimensions $m, n$, respectively, such that $M$ is compact and $N$ is a closed subset of $V$. Then the homomorphism induced by the inclusion $\iota^*: \pi_j(N, N \cap M) \to \pi_j(V, M)$ is an isomorphism for $j \leq n + m - v - (v - \ell)$ and is a surjection for $j = n + m - v - (v - \ell) + 1$.

**Proof.** For $\lambda_0 = n + m - v - (v - \ell)$ consider the diagram:

$$
\begin{array}{ccccccccc}
\pi_{\lambda_0+1}(N) & \to & \pi_{\lambda_0+1}(V, M) & \to & \pi_{\lambda_0}(\Omega) & \to & \pi_{\lambda_0}(N) & \to & \pi_{\lambda_0}(V, M) \\
\uparrow_{\simeq} & & \uparrow & & \uparrow_{\text{onto}} & & \uparrow_{\simeq} & & \uparrow \\
\pi_{\lambda_0+1}(N) & \to & \pi_{\lambda_0+1}(N, N \cap M) & \to & \pi_{\lambda_0}(N \cap M) & \to & \pi_{\lambda_0}(N) & \to & \pi_{\lambda_0}(N, N \cap M)
\end{array}
$$

The vertical arrows are induced by inclusion. The top row is the long exact sequence (2.5). The bottom row is the long exact sequence of the pair $(N, N \cap M)$. The result follows using Theorem 2.1 and the commutativity of the diagram.

**Corollary 2.3.** Under the same hypotheses as in Theorem 2.2, if $j \leq 2m - v - (v - \ell) + 1$

then

$$\pi_j(V, M) = 0.$$ 

**Proof.** Apply Theorem 2.2 to the case $N = M$. 

**Corollary 2.4.** Under the same hypothesis as in Theorem 2.2, if $j \leq \min(2m - v - (v - \ell) + 1, n + m - v - (v - \ell))$

then

$$\pi_j(N, N \cap M) = 0.$$ 

**Proof.** Follows from Corollary 2.3 and Theorem 2.2.

**Remark:** Under the same hypotheses as Theorem 2.2 assume, in addition, that

$$\ell(x, X \wedge JX) \geq \ell_0,$$

for every $x \in M$ and every complex line $X \wedge JX$ normal to $T_x M$. Then by Corollary 1.3 the ranges of validity of the results of this section are improved by replacing $\ell$ by $\ell_0$. For example suppose $V = \text{Gr}(p, p + q; \mathbb{C})$ and $M$ is a complex hypersurface. If every normal $(1,0)$-vector $X$ along $M$ has rank$(X) \geq r > 1$ then for any complex submanifold $N$ the index range in Theorem 2.2 is increased by $r - 1$. 


3. The Compact Hermitian Symmetric Spaces

We exploit the curvature computation in [Bo], to compute the complex positivity of the hermitian symmetric spaces. Recall the notation of [Bo]. Let $G$ be a compact connected simple Lie group and $K$ be the identity component of the fixed point set of an involutive automorphism of $G$. We assume that $K$ has infinite center. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$ and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Denote the complexifications of these Lie algebras by $\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$. Let $\Sigma$ be the system of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Let $\mathfrak{b}_{\alpha}$ be the one-dimensional eigenspace of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ corresponding to $\alpha \in \Sigma$. We have,

$$[\mathfrak{b}_\alpha, \mathfrak{b}_\beta] = \begin{cases} \mathfrak{b}_{\alpha+\beta}, & \text{if } \alpha + \beta \text{ is a root } (\alpha + \beta \neq 0), \\ 0, & \text{if } \alpha + \beta \text{ is not a root } (\alpha + \beta \neq 0), \end{cases} \quad (3.1)$$

There are elements $e_\alpha \in \mathfrak{b}_\alpha$ such that

$$[h, e_\alpha] = 2\pi i (h) e_\alpha, \quad [e_\alpha, e_{-\alpha}] = (2\pi i)^{-1} h_\alpha, \quad (h \in \mathfrak{t}).$$

$\mathfrak{g}$ is spanned by $\mathfrak{t}$ and $e_\alpha + e_{-\alpha}, i(e_\alpha - e_{-\alpha})$, for $\alpha \in \Sigma$. Define $N_{\alpha, \beta}$ by:

$$[e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha+\beta}, \quad \alpha, \beta \in \Sigma, \alpha + \beta \neq 0.$$

Then ([Bo]),

$$N_{\alpha, \beta} = -N_{-\alpha, -\beta}. \quad (3.2)$$

The set of positive complementary roots of $G/K$, i.e. the set of positive roots of $\mathfrak{g}$ which are not the roots of $\mathfrak{t}$, will be denoted by $\Psi$. We put

$$n^+ = \sum_{\alpha \in \Psi} \mathfrak{b}_\alpha, \quad n^- = \sum_{-\alpha \in \Psi} \mathfrak{b}_\alpha.$$

We identify $\mathfrak{p}$ with the tangent space of $G/K$ at $K$ (or at the identity $e$). Then $T_e G/K \otimes \mathbb{C}$ is identified with $n^+ \oplus n^-$. Complex conjugation of $\mathfrak{p} \otimes \mathbb{C}$ becomes $e_\alpha \mapsto e_{-\alpha}$ and the assignment $e_\alpha \mapsto e_\alpha + e_{-\alpha}$ defines an isomorphism of complex vector spaces between $n^+$ and $\mathfrak{p}$. Here we endow $\mathfrak{p}$ with the complex structure induced from that of $G/K$.

For $\alpha \in \Sigma$, let $\omega_\alpha$ be the Maurer-Cartan form on $G_{\mathbb{C}}$ (the complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$) defined by,

$$\omega_\alpha(e_\alpha) = 1, \quad \omega_\alpha(e_\beta) = 0, \quad \alpha \neq \beta, \quad \omega_\alpha(t) = 0, \quad t \in \mathfrak{t}_{\mathbb{C}}.$$

This induces a Kähler metric on $G/K$ with orthonormal basis $\{e_\alpha : \alpha \in \Psi\}$.

Note that $e_\alpha$ and $e_{-\alpha}$ are $(1,0)$ and $(0,1)$ vectors, respectively. The curvature tensor with respect to the basis $\{e_\alpha : \alpha \in \Psi\}$ is

$$R(e_{-\gamma}, e_{\delta})e_\alpha = \text{ad}[e_\delta, e_{-\gamma}](e_\alpha) = \left[ [e_\delta, e_{-\gamma}], e_\alpha \right] = \sum_{\beta} R_{\alpha \gamma \delta}^\beta e_\beta.$$

Writing,

$$e_\alpha = \frac{1}{\sqrt{2}} (X_\alpha - iJX_\alpha), \quad e_{-\alpha} = \frac{1}{\sqrt{2}} (X_\alpha + iJX_\alpha),$$

the real tangent vectors $\{X_\alpha : \alpha \in \Psi\}$ to $G/K$ at $K$ form a basis for $\mathfrak{p}$ and using the Kähler symmetries of curvature,

$$\langle R(X_\alpha, JX_{\beta})X_{\gamma}, JX_{\delta} \rangle = - \langle R(e_\alpha, e_{-\beta})e_{\gamma}, e_{-\delta} \rangle.$$
For $\alpha, \beta \in \Sigma$, we will denote the Killing form $\kappa(h_\alpha, h_\beta) = \text{Tr}(\text{ad} h_\alpha \text{ ad} h_\beta)$ by $(\alpha, \beta)$.

**Lemma 3.1.** ([Bo] 1.4) If $\alpha, \beta \in \Psi$, then $\alpha + \beta$ is not a root and therefore $(\alpha, \beta) \geq 0$.

We use the following theorem to compute the holomorphic bisectional curvature:

**Theorem 3.2.** ([Bo] 2.1, 2.2)

$$R^\gamma_{\beta \delta \alpha} = \begin{cases} 0, & \text{if } \beta + \alpha \neq \gamma + \delta \\ (\beta, \delta), & \text{if } \beta = \gamma, \alpha = \delta \\ N_{\alpha, -\delta} N_{\beta, -\gamma}, & \text{if } \beta + \alpha = \gamma + \delta, \beta \neq \gamma. \end{cases}$$

Hence,

**Lemma 3.3.**

$$\langle R(\sum_\alpha a_\alpha X_\alpha, J \sum_\alpha a_\alpha X_\alpha) \sum_\beta b_\beta X_\beta, J \sum_\beta b_\beta X_\beta \rangle = \sum_{\alpha, \gamma} a_\alpha^2 b_\gamma^2 \langle \alpha, \gamma \rangle + \sum_{\alpha \neq \beta, \gamma \neq \delta, \alpha + \gamma = \beta + \delta} a_\alpha a_\beta b_\gamma b_\delta N_{\alpha, -\delta} N_{\beta, -\gamma}.$$  \hspace{1cm} (*3.3*)

**Proof.** For $\alpha = \beta$ and $\gamma \neq \delta$, by Theorem 3.2

$$\langle R(X_\alpha, JX_\alpha)X_\gamma, JX_\delta \rangle = -\langle R(e_\alpha, e_\alpha)e_\gamma, e_\delta \rangle = R^\delta_{\gamma \alpha} = 0.$$

Similarly, for $\alpha \neq \beta$ and $\gamma = \delta$,

$$\langle R(X_\alpha, JX_\beta)X_\gamma, JX_\gamma \rangle = 0.$$

Therefore,

$$\langle R(\sum_\alpha a_\alpha X_\alpha, J \sum_\alpha a_\alpha X_\alpha) \sum_\beta b_\beta X_\beta, J \sum_\beta b_\beta X_\beta \rangle = \sum_{\alpha, \gamma} a_\alpha^2 b_\gamma^2 \langle R(X_\alpha, JX_\alpha)X_\gamma, JX_\gamma \rangle + \sum_{\alpha \neq \beta, \gamma \neq \delta} a_\alpha a_\beta b_\gamma b_\delta \langle R(X_\alpha, JX_\beta)X_\gamma, JX_\delta \rangle.$$  \hspace{1cm} (*3.3*)

Using the orthonormality of $\{e_\alpha : \alpha \in \Psi\}$ and Theorem 3.2, we have

$$\langle R(X_\alpha, JX_\alpha)X_\gamma, JX_\gamma \rangle = R^\gamma_{\alpha \alpha} (e_\gamma, e_\gamma) = (\alpha, \gamma).$$

For $\alpha \neq \beta, \gamma \neq \delta$, by Theorem 3.2, we have

$$\langle R(X_\alpha, JX_\beta)X_\gamma, JX_\delta \rangle = R^\delta_{\gamma \beta} = \begin{cases} N_{\alpha, -\delta} N_{\gamma, -\gamma}, & \text{if } \alpha + \gamma = \beta + \delta, \\ 0, & \text{otherwise}. \end{cases}$$

The lemma follows. \hfill \Box

Suppose that $e_{\alpha_j}, e_{-\alpha_j}$ for $\alpha_j \in \Psi, j = 1, \ldots, k$, lie in an abelian subspace of $\mathfrak{p}$. Then

$$N_{\alpha_i, -\alpha_j} = 0, \text{ if } i \neq j, \ i, j = 1, \ldots, k. \hspace{1cm} (*3.3)$$

Let $X_\alpha$ be the vector associated to the root $\alpha_i$. From Lemma 3.3 it follows that:

$$\langle R(X_\alpha, JX_\alpha)Y, JY \rangle = 0, \ i \neq j,$$

for any vector $Y$. This implies:
Lemma 3.4. Suppose that $X = \sum_{i=1}^{k} a_i X_{\alpha_i}$, where $e_{\alpha_i}, e_{-\alpha_i}$, $i = 1, \ldots, k$, lie in an abelian subspace of $p$. Then for any $i = 1, \ldots, k$,

$$\ell(X_{\alpha_i} \wedge JX_{\alpha_i}) \leq \ell(X \wedge JX).$$

Proof. From (3.3) and Lemma 3.3:

$$\langle R(\sum_{i=1}^{k} a_i X_{\alpha_i}, J \sum_{i=1}^{k} a_i X_{\alpha_i}) \sum_{j} b_{\beta_j} X_{\beta_j}, J \sum_{j} b_{\beta_j} X_{\beta_j} \rangle = \sum_{i=1}^{k} \sum_{\beta} a_i^2 b_{\beta_i}^2 (\alpha_i, \beta).$$

Since $(\gamma, \delta) \geq 0$ for $\gamma, \delta \in \Psi$, the result follows.

Under the action of the linear isotropy group any tangent vector of a hermitian symmetric space can be written as the sum of vectors all of which lie in a maximal abelian subspace $a \subset p$. Therefore the infimum

$$\inf_{X \wedge JX} \ell(X \wedge JX)$$

is achieved when $X$ is spanned by one base element, i.e. $X = a_{\alpha} X_{\alpha}$. Fix $\alpha \in \Psi$ and set $\Psi'_\alpha := \{ \gamma \in \Psi : (\alpha, \gamma) \neq 0 \}$.

Proposition 3.5. For fixed $X_{\alpha}$:

(i) $\langle R(X_{\alpha}, JX_{\alpha}) \sum_{\beta} b_{\beta} X_{\beta}, J \sum_{\beta} b_{\beta} X_{\beta} \rangle = \sum_{\beta} a_{\alpha}^2 b_{\beta}^2 (\alpha, \beta)$.

(ii) The span of $\{X_{\gamma}, JX_{\gamma} : \gamma \in \Psi'_\alpha\}$ is a maximal subspace of the cone $C(X_{\alpha} \wedge JX_{\alpha})$.

Proof. (i) immediately follows from Lemma 3.3. (ii) then follows from (i).

Corollary 3.6. For a compact hermitian symmetric space the complex positivity is equal to

$$\ell = |\Psi'_\alpha|,$$

for any $\alpha \in \Psi$.

Proof. By Proposition 3.3, it follows that $\ell = \inf_{\alpha \in \Psi} |\Psi'_\alpha|$. However, the elements of $\Psi$ are all highest weight vectors and hence they are equivalent under the action of the isotropy group. Therefore $|\Psi'_\alpha|$ is independent of $\alpha \in \Psi$.

By Corollary 3.6 the complex positivities of the compact hermitian symmetric spaces are:

- $\text{Gr}(p, p+q; \mathbb{C})$: $\ell = p+q-1$.
- $\text{Gr}(2, p+2; \mathbb{R})$: $\ell = p-1$.
- $Sp(r)/U(r)$: $\ell = r$.
- $SO(2r)/U(r)$: $\ell = (r-1) + (r-2)$.
- $E_6/(\text{Spin}(10) \times T^1)$: $\ell = 11$.
- $E_7/(E_6 \times T^1)$: $\ell = 17$.

Combining this computation with Theorem 2.2 gives Theorem 0.1.
REFERENCES

[B] Barth, W., Transplanting cohomology classes in complex projective space, Amer. J. Math. 92 (1970), 951–967.

[Bo] Borel, A., On the curvature tensor of the hermitian symmetric manifolds, Ann. Math. 71 (1960), 508–521.

[F-L] Fulton, W., and Lazarsfeld, R., Connectivity and its applications in algebraic geometry, Algebraic Geometry, Springer Lecture Notes in Math. 862 (1981), 26–92.

[G] Goldstein, N., Ampleness and connectedness in complex $G/P$, Trans. AMS, 274 (1982), 361–373.

[M1] Mok, N., Uniqueness theorems of Kähler metrics of semipositive bisectional curvature on compact hermitian symmetric spaces, Math. Ann. 276 (1987) 177-204.

[M2] Mok, N., Metric rigidity theorems on hermitian locally symmetric manifolds, World Scientific, Singapore, 1989.

[S-W] Schoen, R., and Wolfson, J., Theorems of Barth-Lefschetz type and Morse theory on the space of paths, Math Z. 229 (1998), 77-87.

[S] Sommese, A., Complex subspaces of homogeneous complex manifolds II-Homotopy Results, Nagoya Math. J. 86 (1982), 101–129.