ABELIAN QUOTIENTS OF THE CATEGORIES OF SHORT
EXACT SEQUENCES

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Abstract. We mainly investigate abelian quotients of the categories of short
exact sequences. The natural framework to consider the question is via iden-
tifying quotients of morphism categories as modules categories. These ideas
not only can be used to recover the abelian quotients produced by cluster-
tilting subcategories of both exact categories and triangulated categories,
but also can be used to reach our goal. Let $(C, E)$ be an exact category. We
denote by $E(C)$ the category of bounded complexes whose objects are given
by short exact sequences in $E$ and by $SE(E)$ the full subcategory formed by
split short exact sequences. In general, $E(C)$ is just an exact category, but
the quotient $E(C)/[SE(E)]$ turns out to be abelian. In particular, if $(C, E)$ is
Frobenius, we present three equivalent abelian quotients of $E(C)$ and point
out that the equivalences are actually given by left and right rotations. The
abelian quotient $E(C)/[SE(E)]$ admits some nice properties. We explicitly de-
scribe the abelian structure, projective objects, injective objects and simple
objects, which provide a new viewpoint to understanding Hilton-Rees Theo-
rem and Auslander-Reiten theory. Furthermore, we present some analogous
results both for $n$-exact versions and for triangulated versions.

1. Introduction

Cluster-tilting theory provides a way to construct abelian quotient categories.
Let $C$ be a triangulated category and $T$ be a cluster-tilting subcategory of $C$, then
the quotient $C/[T]$ is abelian; related works see [9, 24, 21] and [27]. The version of
exact categories see [12]. Different methods for understanding the abelian quotients
have been investigated further, for example, via localisations [7, 8], via cotorsion
pairs [33, 34, 30], via homotopical algebra [36] and so on.

Let $C$ be an abelian category. Denote by $E(C)$ the category of all short exact
sequences in $C$. It is well known that $E(C)$ is an exact category but it is not
abelian in general. Denote by $\text{Mor}(C)$ the morphism category of $C$, by $\text{Mono}(C)$
the monomorphism category of $C$, and by $\text{Epi}(C)$ the epimorphism category of $C$.
Then the three categories $E(C)$, $\text{Mono}(C)$ and $\text{Epi}(C)$ are equivalent. Note that
in the case when $C$ is the module category over a ring, then the monomorphism
category $\text{Mono}(C)$ is known as the submodule category. The structure of submodule
categories has been studied intensively by Ringel and Schmidmeier [39, 40]. Let
$S(A)$ be the submodule category of an artin $k$-algebra $A$. If $A = k[t]/(t^n)$, Ringel
and Zhang established two abelian quotients of $S(A)$ [41, Theorem 1]. Denote by

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\( \mathcal{U}_1 \) (resp. \( \mathcal{U}_2 \)) the full subcategory of \( \mathcal{S}(A) \) formed by objects of the form \( (X \xrightarrow{1} X) \oplus (0 \to Y) \) (resp. \( (X \xrightarrow{1} X) \oplus (Y \to P) \) with \( P \) projective-injective). They showed that the quotient categories \( \mathcal{S}(A) / [\mathcal{U}_1] \) and \( \mathcal{S}(A) / [\mathcal{U}_2] \) are equivalent to \( \text{mod-} \Pi_{n-1} \) where \( \Pi_{n-1} \) is the preprojective algebra of type \( A_{n-1} \). Recently due to Eiriksson \[13\] Theorem 1, the above result was generalized for any self-injective algebra of finite representation type by replacing \( \Pi_{n-1} \) with \( B \), the stable Auslander algebra of \( A \).

The present paper mainly studies the abelian quotients of the categories of short exact sequences. Our approach to understanding abelian quotients is via morphism categories. The following is a basic proposition.

**Proposition 1.1.** Let \( C \) be an additive category, then we have the following equivalences.

(a) \( \text{Mor}(C)[\mathcal{U}] \cong \text{mod-}C \), where \( \mathcal{U} \) is the full subcategory of \( \text{Mor}(C) \) consisting of \( (X \xrightarrow{1} X) \oplus (Y \to 0) \).

(b) \( \text{Mor}(C)[\mathcal{U}'] \cong (\text{mod-}C^{\text{op}})^{\text{op}} \), where \( \mathcal{U}' \) is the full subcategory of \( \text{Mor}(C) \) consisting of \( (X \xrightarrow{1} X) \oplus (0 \to Y) \).

Using Proposition 1.1, we realize some abelian quotient categories constructed by cluster-tilting subcategories. For example, we can reprove \[12\] Theorem 3.2, \[12\] Theorem 3.5 and \[27\] Corollary 4.4.

Let \( (C, \mathcal{E}) \) be an exact category. We denote by \( C^b(C) \) the category of bounded complexes over \( C \), by \( \mathcal{E}(C) \) the full subcategory of \( C^b(C) \) consisting of short exact sequences in \( \mathcal{E} \), by \( S\mathcal{E}(C) \) the full subcategory of \( \mathcal{E}(C) \) formed by split short exact sequences over \( C \). A short exact sequence \( 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0 \) is denoted by \( (X_1 \to X_2 \to X_3) \) for short. The following is our main theorem.

**Theorem 1.2.** Let \( (C, \mathcal{E}) \) be an exact category and \( X_\bullet : 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0 \) be a short exact sequence in \( \mathcal{E} \).

(a) If \( (C, \mathcal{E}) \) has enough projectives, denote by \( \mathcal{P} \) the full subcategory of \( C \) formed by all projectives, then we have the following equivalences:

\[
\alpha_1 : \mathcal{E}(C)/[S\mathcal{E}(C)] \cong (\text{mod-}C/\mathcal{P}), \quad X_\bullet \mapsto \text{Coker}(C/\mathcal{P})(-,-,f_2)
\]

\[
\alpha_2 : \mathcal{E}(C)/[PE(C)] \cong (\text{mod-}(C/\mathcal{P})^{\text{op}})^{\text{op}}, \quad X_\bullet \mapsto \text{Coker}(C/\mathcal{P})(f_2,-)
\]

where \( PE(C) \) is the full subcategory of \( \mathcal{E}(C) \) formed by \( (0 \to X \to X) \oplus (\Omega Y \to P \to Y) \).

(b) If \( (C, \mathcal{E}) \) has enough injectives, denote by \( \mathcal{I} \) the full subcategory of \( C \) formed by all injectives, then we have the following equivalences:

\[
\beta_1 : \mathcal{E}(C)/[SE(C)] \cong (\text{mod-}(C/\mathcal{I})^{\text{op}})^{\text{op}}, \quad X_\bullet \mapsto \text{Coker}(C/\mathcal{I})(f_1,-)
\]

\[
\beta_2 : \mathcal{E}(C)/[IE(C)] \cong \text{mod-}C/\mathcal{I}, \quad X_\bullet \mapsto \text{Coker}(C/\mathcal{I})(-,f_1)
\]

where \( IE(C) \) is the full subcategory of \( \mathcal{E}(C) \) formed by \( (X \to X \to 0) \oplus (Y \to I \to \Omega^{-1}Y) \).

In particular, if \( (C, \mathcal{E}) \) is a Frobenius category, then the quotient categories \( \mathcal{E}(C)/[S\mathcal{E}(C)], \mathcal{E}(C)/[PE(C)] \) and \( \mathcal{E}(C)/[IE(C)] \) are equivalent to abelian category \( \text{mod-}C/\mathcal{P} \). As we can see in Remark \[43\] the equivalences between the three quotient categories are given by left rotations and right rotations of short exact sequences.
If $(\mathcal{C}, \mathcal{E})$ is Frobenius, we can show that the three categories $\mathcal{E}(\mathcal{C})$, $\text{Mono}(\mathcal{C})$ and $\text{Epi}(\mathcal{C})$ are equivalent. Therefore, we have the following result, which generalizes [13, Theorem 1].

**Corollary 1.3.** Let $(\mathcal{C}, \mathcal{E})$ be a Frobenius category. Denote by $\mathcal{P}$ the full subcategory of projective-injective objects in $\mathcal{C}$, by $\mathcal{U}_1$ the full subcategory of $\text{Mono}(\mathcal{C})$ consisting of $(X \xrightarrow{1} X) \oplus (0 \rightarrow Y)$, by $\mathcal{U}_2$ the full subcategory of $\text{Mono}(\mathcal{C})$ consisting of $(X \xrightarrow{1} X) \oplus (Y \rightarrow P)$ with $P \in \mathcal{P}$ and by $\mathcal{U}_3$ the full subcategory of $\text{Mono}(\mathcal{C})$ consisting of $(0 \rightarrow X) \oplus (Y \rightarrow P)$ with $P \in \mathcal{P}$. Then each of the quotient categories $\mathcal{E}(\mathcal{C})/\mathcal{U}_1$, $\mathcal{E}(\mathcal{C})/\mathcal{U}_2$ and $\mathcal{E}(\mathcal{C})/\mathcal{U}_3$ is equivalent to $\text{mod-}\mathcal{C}/[\mathcal{P}].$

Our second part of the paper is to studying the properties of $\mathcal{E}(\mathcal{C})/[S\mathcal{E}(\mathcal{C})]$. We show that the abelian structure is given by pullback and pushout diagrams; see Theorem 1.2. We characterize the simple objects in $\mathcal{E}(\mathcal{C})/[S\mathcal{E}(\mathcal{C})]$ as the Auslander-Reiten sequences in $\mathcal{C}$; see Theorem 1.20. We describe the projective objects and injective objects in $\mathcal{E}(\mathcal{C})/[S\mathcal{E}(\mathcal{C})]$; see Proposition 4.11. In particular, if $(\mathcal{C}, \mathcal{E})$ has enough projectives, then each projective object in $\mathcal{E}(\mathcal{C})/[S\mathcal{E}(\mathcal{C})]$ is of the form $P_X : 0 \rightarrow \Omega X \rightarrow P \rightarrow X \rightarrow 0$ for some object $X$ in $\mathcal{C}$.

As applications, our results provide a new viewpoint to understanding Hilton-Reses Theorem and Auslander-Reiten theory. Now we assume that $(\mathcal{C}, \mathcal{E})$ is an exact category with enough projectives and injectives. By Theorem 1.2 we have a duality

$$\Phi : \text{mod-}\mathcal{C}/[\mathcal{P}] \rightarrow \text{mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \delta^* \mapsto \delta_*$$

where $\delta$ is a short exact sequence in $\mathcal{E}$, $\delta^*$ is the contravariant defect and $\delta_*$ is the covariant defect. Moreover, by restrictions and Proposition 4.11 we obtain the following two dualities

$$\Phi : \text{proj-}\mathcal{C}/[\mathcal{P}] \rightarrow \text{inj-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \mathcal{C}/[\mathcal{P}](\cdot, X) \mapsto \text{Ext}^1_{\mathcal{C}}(X, \cdot).$$

$$\Phi : \text{inj-}\mathcal{C}/[\mathcal{P}] \rightarrow \text{proj-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \text{Ext}^1_{\mathcal{C}}(\cdot, X) \mapsto \mathcal{C}/[\mathcal{I}](X, \cdot).$$

Hence, the following result seems natural.

**Theorem 1.4.** (Hilton-Reses Theorem, see [18, 32]) Let $(\mathcal{C}, \mathcal{E})$ be an exact category with enough projectives and injectives.

(a) There is an isomorphism between $\mathcal{C}/[\mathcal{P}](Y, X)$ and the group of natural transformations from $\text{Ext}^1_{\mathcal{C}}(X, \cdot)$ to $\text{Ext}^1_{\mathcal{C}}(Y, \cdot)$.

(b) There is an isomorphism between $\mathcal{C}/[\mathcal{I}](X, Y)$ and the group of natural transformations from $\text{Ext}^1_{\mathcal{C}}(\cdot, X)$ to $\text{Ext}^1_{\mathcal{C}}(\cdot, Y)$.

If furthermore, $\mathcal{C}$ is a dualizing $k$-variety, then $\mathcal{C}/[\mathcal{P}]$ and $\mathcal{C}/[\mathcal{I}]$ are also dualizing $k$-varieties. Thus we have two dualities $\Phi : \text{mod-}\mathcal{C}/[\mathcal{P}] \rightarrow \text{mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}$ and $D : \text{mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}} \rightarrow \text{mod-}\mathcal{C}/[\mathcal{I}]$. The composition of $\Phi$ and $D$ defines an equivalence

$$\Theta : \text{proj-}\mathcal{C}/[\mathcal{P}] \xrightarrow{\Phi} \text{inj-}(\mathcal{C}/[\mathcal{I}])^{\text{op}} \xrightarrow{D} \text{proj-}\mathcal{C}/[\mathcal{I}].$$

Now we have the following generalized Auslander-Reiten duality and defect formula.

**Theorem 1.5.** Let $(\mathcal{C}, \mathcal{E})$ be an $\text{Ext}$-finite exact category with enough projectives and injectives. Assume that $\mathcal{C}$ is a dualizing $k$-variety. Then there is an equivalence $\tau : \mathcal{C}/[\mathcal{P}] \cong \mathcal{C}/[\mathcal{I}]$ satisfying the following properties:

(a) $D\text{Ext}^1_{\mathcal{C}}(-, X) \cong \mathcal{C}/[\mathcal{P}](\tau^{-1}X, \cdot)$, $D\text{Ext}^1_{\mathcal{C}}(X, \cdot) \cong \mathcal{C}/[\mathcal{I}](\cdot, \tau X)$.

(b) $D\delta_* = \delta^*\tau^{-1}$, $D\delta^* = \delta_*\tau$ for each short exact sequence $\delta$ in $\mathcal{E}$.

Therefore, $\mathcal{C}$ has Auslander-Reiten sequences.
We point out that for $n$-exact categories and triangulated categories, by considering quotients of the categories of $n$-exact sequences and quotients of the categories of triangles, we obtain some analogous results.

This paper is organized as follows.

In Section 2, we make some preliminaries. We collect some definitions and facts on morphism categories, exact categories, quotient categories and functor categories.

In Section 3, we provide techniques to identify quotients of morphism categories as module categories. Subsection 3.1 is devoted to proving Proposition 1.1. In subsection 3.2, we apply Proposition 1.1 to exact categories. We show that some quotient categories of epimorphism categories are equivalent to module categories, see Theorem 3.2; see Corollary 3.1. We obtain certain recollements of abelian categories from the viewpoint of morphism categories, which implies Auslander’s formula; see Corollary 3.10. In subsection 3.3, we apply Proposition 1.1 to triangulated categories; see Proposition 3.19 and Corollary 3.20. In subsection 3.4, we give some examples.

In Section 4, we study the abelian quotients of the categories of short exact sequences. In subsection 4.1 we realize some quotients of these categories as module categories; see Theorem 1.2. In subsection 4.2 we describe the abelian structure of the quotients; see Theorem 4.8. In subsection 4.3 we study the projective objects and injective objects, which are applied to prove Hilton-Rees Theorem; see Proposition 4.11 and Theorem 1.4. In subsection 4.4, we will restrict our attention to the connection to Auslander-Reiten theory. We will prove Theorem 1.20 and Theorem 1.5. Subsection 4.5 is devoted to listing the higher versions on the abelian quotients of the categories of $n$-exact sequences.

In Section 5, we consider the abelian quotients of the categories of triangles. There are some parallel results.

2. Preliminaries

In this section, we make some preliminaries. Let $\mathcal{C}$ be an additive category. We denote by $\mathcal{C}(X,Y)$ the set of morphisms from $X$ to $Y$ in $\mathcal{C}$. The composition of $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$ is denoted by $gf$.

2.1. Morphism categories. Assume that $\mathcal{C}$ is an additive category. The morphism category of $\mathcal{C}$ is the category $\text{Mor}(\mathcal{C})$ defined by the following data. The objects of $\text{Mor}(\mathcal{C})$ are all the morphisms $f : X \to Y$ in $\mathcal{C}$. The morphisms from $f : X \to Y$ to $f' : X' \to Y'$ are pairs $(a,b)$ where $a : X \to X'$ and $b : Y \to Y'$ such that $bf = f'a$. The composition of morphisms is componentwise. We denote by $\text{Mono}(\mathcal{C})$ the full subcategory of $\text{Mor}(\mathcal{C})$ consisting of monomorphisms in $\mathcal{C}$, which is called the monomorphism category of $\mathcal{C}$. Dually, we define epimorphism category $\text{Epi}(\mathcal{C})$ of $\mathcal{C}$. In particular, if $\mathcal{C}$ is abelian, then $\text{Mor}(\mathcal{C})$ is an abelian category. In this case, $\text{Mono}(\mathcal{C})$ is an additive category of $\text{Mor}(\mathcal{C})$ which is closed under extensions, thus it becomes an exact category. Moreover, $\text{Mono}(\mathcal{C})$ is isomorphic to $\text{Epi}(\mathcal{C})$, where the isomorphism is given by cokernel functor.

2.2. Exact categories. We recall the notion of exact categories from [11]. Let $\mathcal{C}$ be an additive category. A kernel-cokernel pair $(i,p)$ in $\mathcal{C}$ is a pair of composable morphisms $X \xrightarrow{i} Y \xrightarrow{p} Z$ such that $i$ is a kernel of $p$ and $p$ is a cokernel of $i$. Assume
that \( \mathcal{E} \) is a class of kernel-cokernel pairs. A kernel-cokernel pair \((i, p)\) in \( \mathcal{E} \) is called a \textit{short exact sequence} in \( \mathcal{E} \), which is denoted by \( 0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0 \). A morphism \( p : Y \to Z \) is called \textit{admissible epimorphism} if there exists a morphism \( i : X \to Y \) such that \((i, p) \in \mathcal{E} \). \textit{Admissible monomorphisms} are defined dually.

A class of kernel-cokernel pairs \( \mathcal{E} \) is called an \textit{exact structure} of \( \mathcal{C} \) if \( \mathcal{E} \) is closed under isomorphisms and satisfies the following axioms:

\begin{itemize}
  \item[(E0)] Identity morphisms are admissible epimorphisms.
  \item[(E0)\^{op}] Identity morphisms are admissible monomorphisms.
  \item[(E1)] The composition of two admissible epimorphisms is an admissible epimorphism.
  \item[(E1)\^{op}] The composition of two admissible monomorphisms is an admissible monomorphism.
  \item[(E2)] Given a short exact sequence \( 0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0 \) in \( \mathcal{E} \) and a morphism \( \varphi : X \to X' \) in \( \mathcal{C} \), there exists a commutative diagram
    \[
    \begin{array}{ccc}
    0 & \to & X & \xrightarrow{i} & Y & \xrightarrow{p} & Z & \to & 0 \\
    & \downarrow{\varphi} & \quad & \downarrow{\varphi'} & \quad & \downarrow{\phi} & \quad & \downarrow{\phi'} & \quad & \downarrow{0} \\
    0 & \to & X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z' & \to & 0
    \end{array}
    \]
    such that the second row belongs to \( \mathcal{E} \). In this case, \((0 \to X \xrightarrow{(i, \varphi)} Y \oplus X' \xrightarrow{(\varphi', -i')} Y' \to 0) \in \mathcal{E} \).
  \item[(E2)\^{op}] Given a short exact sequence \( 0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0 \) in \( \mathcal{E} \) and a morphism \( \phi : Z' \to Z \) in \( \mathcal{C} \), there exists a commutative diagram
    \[
    \begin{array}{ccc}
    0 & \to & X & \xrightarrow{i} & Y & \xrightarrow{p} & Z & \to & 0 \\
    & \downarrow{\phi} & \quad & \downarrow{\phi'} & \quad & \downarrow{\phi} & \quad & \downarrow{\phi} & \quad & \downarrow{0} \\
    0 & \to & X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z' & \to & 0
    \end{array}
    \]
    such that the first row belongs to \( \mathcal{E} \). In this case, \((0 \to Y' \xrightarrow{(p', \phi)} Z' \oplus Y \xrightarrow{(\phi, -p)} Z \to 0) \in \mathcal{E} \).
\end{itemize}

An \textit{exact category} is an additive category \( \mathcal{C} \) admits an exact structure \( \mathcal{E} \), which is denoted by \((\mathcal{C}, \mathcal{E})\).

For example, an additive category is an exact category with respect to the class of split short exact sequences, which are isomorphic to \( 0 \to X \xrightarrow{(1)} X \oplus Y \xrightarrow{(0, 1)} Y \to 0 \) for some \( X, Y \in \mathcal{C} \). An abelian category \( \mathcal{C} \) is an exact category where the exact structure is given by all the kernel-cokernel pairs in \( \mathcal{C} \).

An object \( P \) of an exact category \((\mathcal{C}, \mathcal{E})\) is called \textit{projective} if for each admissible epimorphism \( p : Y \to Z \) and each morphism \( f : P \to Z \), there exists a morphism \( g : P \to Y \) such that \( f = pg \). The full subcategory of projectives is denoted by \( \mathcal{P} \). We say an exact category \((\mathcal{C}, \mathcal{E})\) has \textit{enough projective objects} if for each object \( X \in \mathcal{C} \) there is an admissible epimorphism \( p : P \to X \) with \( P \in \mathcal{P} \). Dually, we can define injective objects. The full subcategory of injectives is denoted by \( \mathcal{I} \). An exact category is \textit{Frobenius} provided that it has enough projectives and injectives and, moreover, the classes of projectives and injectives coincide. If an exact category \((\mathcal{C}, \mathcal{E})\) has enough projectives, then we can consider the projective resolutions and
Ext functors as right derived functors of Hom as in abelian categories. Hence, 
$\text{Ext}_1^C(Z,X)$ parameterizes the short exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$ up to equivalence.

### 2.3. Quotient categories.

Let $\mathcal{C}$ be an additive category. An *ideal* $\mathcal{I}$ of $\mathcal{C}$ is a class of additive subgroups $\mathcal{I}(X,Y)$ of $\mathcal{C}(X,Y)$ such that $hg \in \mathcal{I}(X,W)$ for each $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{I}(Y,Z)$ and $h \in \mathcal{C}(Z,W)$. Assume that $\mathcal{I}$ is an ideal of $\mathcal{C}$, then by definition, the *quotient category* $\mathcal{C}/\mathcal{I}$ has the same objects as $\mathcal{C}$ and has morphisms $\mathcal{C}/\mathcal{I}(X,Y) = \mathcal{C}(X,Y)/\mathcal{I}(X,Y)$. For example, the Jacobson radical $J_\mathcal{C}$ of $\mathcal{C}$ is an ideal of $\mathcal{C}$. Suppose that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$. We denote by $[\mathcal{D}](X,Y)$ the subset of morphisms of $\mathcal{C}(X,Y)$ which factor through an object in $\mathcal{D}$. It is easy to see that $[\mathcal{D}]$ is an ideal of $\mathcal{C}$, thus we have a quotient category $\mathcal{C}/[\mathcal{D}]$ and a quotient functor $Q : \mathcal{C} \to \mathcal{C}/[\mathcal{D}]$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. The image of $f$ under $Q$ is denoted by $\bar{f}$. It is well known that for each additive functor $F : \mathcal{C} \to \mathcal{E}$, if $F(\mathcal{D}) = 0$, then there is a unique functor $F' : \mathcal{C}/[\mathcal{D}] \to \mathcal{E}$ such that $F'Q = F$.

Let $F : \mathcal{C} \to \mathcal{D}$ be a full and dense functor. If each morphism $f \in \mathcal{C}(X,Y)$ with $F(f) = 0$ factors through an object $Z$ with $F(Z) = 0$, then the functor $F$ is called *objective* (see [11]). In this case, there is an equivalence $\mathcal{C}/[\text{Ker}F] \cong \mathcal{D}$, where $\text{Ker}F$ is the full subcategory of $\mathcal{C}$ formed by $X$ with $F(X) = 0$.

### 2.4. Functor categories.

Let $\mathcal{C}$ be an additive category. A right $\mathcal{C}$-*module* is a contravariantly additive functor $F : \mathcal{C} \to \mathcal{Ab}$ where $\mathcal{Ab}$ is the category of abelian groups. Denote by $\text{Mod}\mathcal{C}$ the category of right $\mathcal{C}$-modules. It is well known that $\text{Mod}\mathcal{C}$ is an abelian category. The $\mathcal{C}$-module $\mathcal{C}(-,X)$ is a projective object of $\text{Mod}\mathcal{C}$ for each object $X \in \mathcal{C}$. Moreover, each projective object is a direct summand of $\mathcal{C}(-,X)$ for some $X \in \mathcal{C}$. By definition, a $\mathcal{C}$-module $F$ is called *finitely presented* (or *coherent*) if there exists an exact sequence $\mathcal{C}(-,X) \to \mathcal{C}(-,Y) \to F \to 0$. We denote by $\text{mod}\mathcal{C}$ the full subcategory of $\text{Mod}\mathcal{C}$ formed by finitely presented $\mathcal{C}$-modules, by $\text{proj}\mathcal{C}$ (resp. $\text{inj}\mathcal{C}$) the full subcategory of $\text{mod}\mathcal{C}$ consisting of projective (resp. injective) objects. It is known that $\text{mod}\mathcal{C}$ is closed under cokernels and extensions.

Moreover, we have the following result.

#### Proposition 2.1.

([3], [25] Lemma 4.1) Let $\mathcal{C}$ be an additive category. Then $\text{mod}\mathcal{C}$ is abelian if and only if $\mathcal{C}$ admits weak kernels.

Recall that a morphism $f : X \to Y$ in $\mathcal{C}$ is a *weak kernel* of $g : Y \to Z$ if $gf = 0$ and for each morphism $h : W \to Y$ such that $gh = 0$, there exists a morphism $p : W \to X$ such that $fp = h$.

#### Remark 2.2.

Assume that $\mathcal{C}$ admits weak kernels. For later use, we recall the abelian structure of $\text{mod}\mathcal{C}$. Let $\alpha : F_1 \to F_2$ be a morphism in $\text{mod}\mathcal{C}$ with the following presentation:

$$
\begin{array}{c}
\mathcal{C}(-,X_1) \ar[d] \\
\mathcal{C}(-,X_2) \ar[d] \ar[r]^-{\alpha} & \mathcal{C}(-,Y_2) \ar[d] \\
F_1 \\
\mathcal{C}(-,Y_1) \ar[r]^-{F_2} & 0
\end{array}
$$

Then $\text{Coker}(\mathcal{C}(-,Y_1 \oplus X_2) \to \mathcal{C}(-,Y_2))$ is a cokernel of $\alpha$. Suppose that $Z_1 \to Y_1 \oplus X_2$ is a weak kernel of $Y_1 \oplus X_2 \to Y_2$ and $Z_2 \to Z_1 \oplus X_1$ is a weak kernel of $Z_1 \oplus X_1 \to Y_1$, then $\text{Coker}(\mathcal{C}(-,Z_2) \to \mathcal{C}(-,Z_1))$ is a kernel of $\alpha$. 


Let $D$ be a full subcategory of $C$. A morphism $f : D \to X$ is called a right $D$-approximation of $X$ if $D \in D$ and each morphism $g : D' \to X$ with $D' \in D$ factors through $f$. The category $D$ is called contravariantly finite if each object in $C$ admits a right $D$-approximation. A contravariantly finite and covariantly finite subcategory is called functorially finite.

Example 2.3. (a) Let $C$ be an abelian category, then $\text{mod-}C$ is abelian.

(b) Let $C$ be an exact category with enough projectives. Denote by $\mathcal{P}$ the subcategory of projectives. If $\mathcal{M}$ is a contravariantly finite subcategory of $C$, then $\text{mod-}\mathcal{M}$ is abelian. Moreover, if $\mathcal{M}$ contains $\mathcal{P}$, then $\text{mod-}\mathcal{M}/[\mathcal{P}]$ is still abelian (see [12 Lemma 2.3]). In particular, $\text{mod-}C/\mathcal{P}$ is abelian.

(c) Let $C$ be a triangulated category, then $\text{mod-}C$ is abelian.

The following result generalizes [2 Proposition 4.1] slightly.

Proposition 2.4. Let $C$ be an additive category and $D$ be a contravariantly finite subcategory. Then

(a) $\text{Mod-}C/[\mathcal{D}] \cong \{F \in \text{Mod-}C | F(\mathcal{D}) = 0\}$.

(b) $\text{mod-C}/[\mathcal{D}] \cong \text{mod}\_\text{q}C = \{F \in \text{mod-C} | F(\mathcal{D}) = 0\}$.

Proof. (a) It follows from the universal property of quotient functors. For convenience, we identify $\text{Mod-}C/[\mathcal{D}]$ and $\{F \in \text{Mod-}C | F(\mathcal{D}) = 0\}$.

(b) For each object $X \in C$, we assume that $f : D \to X$ is a right $D$-approximation of $X$. Since $\text{Im}\text{C}(-, f) = [\mathcal{D}]/(-, X)$, we have the following exact sequence

$$C(-, D) \xrightarrow{C(-, f)} C(-, X) \to C(-, X)/[\mathcal{D}]/(-, X) \to 0.$$  

It follows that $C/[\mathcal{D}](-, X) \in \text{mod-C}$, since $C/[\mathcal{D}](Y, X) = C(Y, X)/[\mathcal{D}](Y, X)$ for each $Y \in C$. Thus $C/[\mathcal{D}](-, X) \in \text{mod}\_\text{q}C$ since $f$ is a right $D$-approximation. Consequently, $\text{mod-C}/[\mathcal{D}] \subseteq \text{mod}\_\text{q}C$. On the other hand, for each $F \in \text{mod}\_\text{q}C$, there is an exact sequence $C(-, X_1) \to C(-, X_2) \to F \to 0$ with $F(\mathcal{D}) = 0$. The following exact sequence

$$C(-, X_1)/[\mathcal{D}]/(-, X_1) \to C(-, X_2)/[\mathcal{D}]/(-, X_2) \to F \to 0$$

shows that $F \in \text{mod-C}/[\mathcal{D}]$. \hfill $\Box$

Let $k$ be a commutative artinian ring and $E$ be the injective envelope of $k$. Set $D = \text{Hom}_k(-, E)$. A $k$-linear additive category $C$ is called dualizing $k$-variety if the functor $D : \text{Mod-}C \to \text{Mod-}C^{\text{op}}$ given by $D(F)(X) := D(F(X))$, induces a duality $D : \text{mod-}C \to \text{mod-}C^{\text{op}}$.

Example 2.5. (a) Let $A$ be an artin $k$-algebra. Denote by $\text{mod-}A$ the category of finitely presented right $A$-modules, and by $\text{proj-}A$ the full subcategory of $\text{mod-}A$ formed by projective $A$-modules. Then both $\text{mod-}A$ and $\text{proj-}A$ are dualizing $k$-varieties.

(b) Let $C$ be a dualizing $k$-variety, then $\text{mod-}C$ is a dualizing $k$-variety. Moreover, $\text{mod-}C$ is an abelian category with enough projectives and enough injectives.

(c) Any functorially finite subcategory of a dualizing $k$-variety is also a dualizing $k$-variety.

(d) Let $C$ be a dualizing $k$-variety and $D$ be a contravariantly finite subcategory, then $C/[\mathcal{D}]$ is a dualizing $k$-variety.
Proof. Since one can find (a) and (b) in \([4]\) and find (c) in \([5]\), we only prove (d). Let 
\(F \in \text{mod-\(\mathcal{C}/[\mathcal{D}]\)}\), then by Proposition 2.1, \(DF\) can be viewed as a finitely presented 
\(\mathcal{C}^\text{op}\)-module which vanishes on \(\mathcal{D}\). Thus \(DF \in \text{mod-}(\mathcal{C}/[\mathcal{D}])^\text{op}\). Conversely, we can 
show that if \(F \in \text{mod-}(\mathcal{C}/[\mathcal{D}])^\text{op}\), then \(DF \in \text{mod-\(\mathcal{C}/[\mathcal{D}]\)}\]. \(\square\)

3. Identifying Quotients of Morphism Categories as Module Categories

Our approach to understanding the categories of short exact sequences will be 
based on viewing them as morphism categories, which we are able to identify with 
certain module categories. In this section we provide techniques needed for such 
identifications.

3.1. Basic case: additive categories. Let \(\mathcal{C}\) be an additive category. For two 
objects \(f : X \to Y\) and \(f' : X' \to Y'\) in \(\text{Mor}(\mathcal{C})\), we define \(\mathcal{R}(f, f')\) (resp. \(\mathcal{R}'(f, f')\)) 
to be the set of morphisms \((a, b)\) such that there is some morphism \(p : Y \to X'\) 
such that \(f'p = b\) (resp. \(pf = a\)). Then \(\mathcal{R}\) and \(\mathcal{R}'\) are ideals of \(\text{Mor}(\mathcal{C})\).

**Lemma 3.1.** Let \(\mathcal{C}\) be an additive category, then we have the following equivalences.

(a) \(\text{Mor}(\mathcal{C})/\mathcal{R} \cong \text{mod-\(\mathcal{C}\)}\).

(b) \(\text{Mor}(\mathcal{C})/\mathcal{R}' \cong (\text{mod-\(\mathcal{C}^\text{op}\)})^\text{op}\).

**Proof.** (a) We define a functor \(\alpha : \text{Mor}(\mathcal{C}) \to \text{mod-\(\mathcal{C}\)}\) by mapping \(f : X \to Y\) to \(F = \text{Coker}(\mathcal{C}(-, f) : \mathcal{C}(-, X) \to \mathcal{C}(-, Y))\). The functor \(\alpha\) is dense and full by Yoneda’s 
lemma. Suppose that \((a, b)\) is a morphism from \(f : X \to Y\) to \(f' : X' \to Y'\). If 
\(\alpha(a, b) = 0\), then the following diagram

\[
\begin{array}{ccc}
\mathcal{C}(-, X) & \xrightarrow{c(-, f)} & \mathcal{C}(-, Y) \\
\downarrow{c(-, a)} & & \downarrow{c(-, b)} \\
\mathcal{C}(-, X') & \xrightarrow{c(-, f')} & \mathcal{C}(-, Y')
\end{array}
\]

is commutative and each row is exact. There exists a morphism \(\mathcal{C}(-, p) : \mathcal{C}(-, Y) \to \mathcal{C}(-, X')\) 
such that \(\mathcal{C}(-, f')\mathcal{C}(-, p) = \mathcal{C}(-, b)\), that is, \(f'p = b\). Therefore, the 
functor \(\alpha\) induces an equivalence \(\text{Mor}(\mathcal{C})/\mathcal{R} \cong \text{mod-\(\mathcal{C}\)}\). We can show (b) similarly. \(\square\)

We denote by \(\mathcal{U}\) the full subcategory of \(\text{Mor}(\mathcal{C})\) consisting of \((X \xrightarrow{1} X) \oplus (Y \to 0)\) 
and by \(\mathcal{U}'\) the full subcategory of \(\text{Mor}(\mathcal{C})\) consisting of \((X \xrightarrow{1} X) \oplus (0 \to Y)\).

**Lemma 3.2.** Let \((a, b)\) be a morphism from \(f : X \to Y\) to \(f' : X' \to Y'\). Then the 
following holds.

(a) The morphism \(b\) factors through \(f'\) if and only if \((a, b)\) factors through some 
object in \(\mathcal{U}\).

(b) The morphism \(a\) factors through \(f\) if and only if \((a, b)\) factors through some 
object in \(\mathcal{U}'\).

**Proof.** We only prove (a). Suppose that there is a morphism \(p : Y \to X'\) 
such that \(f'p = b\), then \((a, b)\) factors through \(X \oplus X'\xrightarrow{(0, 1)} X'\) as \((a, b) = ((a - 
pf, 1), f'(\frac{1}{\pf}) \cdot p)\). Conversely, if \((a, b)\) factors through some object \(A \oplus B\xrightarrow{(0, 1)} B\) 
in \(\mathcal{U}\). Assume that \((a, b) = ((a_2, a_2'), b_2)((\frac{a_1}{a_1'}, b_1),\) then the morphism 
\(p = a_2'b_1 : Y \to X'\) satisfies \(f'p = b_2b_1 = b\). \(\square\)
Lemma 3.1 and Lemma 3.2 imply the following proposition.

**Proposition 3.3.** Let $C$ be an additive category, then the following holds.
(a) $\text{Mor}(C)/[U] \cong \text{mod-}C$.
(b) $\text{Mor}(C)/[U'] \cong (\text{mod-}C^{\text{op}})^{\text{op}}$.

### 3.2 Second case: exact categories

In this subsection we assume that $(C,E)$ is an exact category with enough projectives. We denote by $\mathcal{P}$ the full subcategory of $\mathcal{C}$ consisting of projectives. Assume that $\mathcal{M}$ is a full subcategory of $\mathcal{C}$. We denote by $\Omega\mathcal{M}$ the full subcategory of $\mathcal{C}$ formed by objects $\Omega M$ such that there is a short exact sequence $0 \to \Omega M \to P \to M \to 0$ in $\mathcal{E}$ with $M \in \mathcal{M}$ and $P \in \mathcal{P}$, by $\mathcal{M}_L$ the full subcategory of $\mathcal{C}$ consisting of objects $X$ such that there is a short exact sequence $0 \to X \to M_1 \to M_2 \to 0$ in $\mathcal{E}$ with $M_i \in \mathcal{M}$.

For convenience, we fix some notations. We denote by $\mathcal{U}$ the full subcategory of $\text{Epi}(\mathcal{M})$ consisting of $(M \xrightarrow{1} M) \oplus (M' \to 0)$, by $\mathcal{V}$ the full subcategory of $\text{Epi}(\mathcal{M})$ consisting of $(M \xrightarrow{k} M) \oplus (P \to M')$ with $P \in \mathcal{P}$, by $U'$ the full subcategory of $\text{Mono}(\mathcal{M})$ consisting of $(0 \to M) \oplus (M' \xrightarrow{1} M')$ and by $V'$ the subcategory of $\text{Mono}(\mathcal{M})$ consisting of $(0 \to M) \oplus (\Omega M' \to P)$ with $P \in \mathcal{P}$. We denote by $\text{Ad-Epi}(\mathcal{M})$ the full subcategory of $\text{Epi}(\mathcal{M})$ consisting of admissible epimorphisms $f : M_1 \to M_2$ with $M_i \in \mathcal{M}$.

**Definition 3.4.** A full subcategory $\mathcal{M}$ of $\mathcal{C}$ is called rigid if $\text{Ext}_C^1(M,M') = 0$ for each objects $M, M' \in \mathcal{M}$.

**Remark 3.5.** Let $\mathcal{M}$ be a rigid subcategory of $\mathcal{C}$. If $0 \to X \xrightarrow{k} M_1 \xrightarrow{f} M_2 \to 0$ is a short exact sequence with $M_i \in \mathcal{M}$, then $k$ is a left $\mathcal{M}$-approximation of $X$.

**Proof.** For each $M \in \mathcal{M}$, applying $\text{C}(-,M)$ to the exact sequence $0 \to X \xrightarrow{k} M_1 \xrightarrow{f} M_2 \to 0$, we have the following exact sequence

$$0 \to \text{C}(M_2, M) \to \text{C}(M_1, M) \to \text{C}(X, M) \to \text{Ext}_C^1(M_2, M) = 0.$$ 

Hence, $k$ is a left $\mathcal{M}$-approximation of $X$. \hfill $\Box$

**Lemma 3.6.** Let $\mathcal{M}$ be a full subcategory of $\mathcal{C}$ containing $\mathcal{P}$. Assume that the following diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & X \\
\downarrow^g & & \downarrow^a \\
0 & \longrightarrow & X' \\
\end{array}
\quad
\begin{array}{ccc}
& & 0 \\
\downarrow^b & & \\
& & \\
\end{array}
\begin{array}{ccc}
M_1 & \longrightarrow & M_2 \\
\downarrow^f & & \downarrow^b \\
M_1' & \longrightarrow & M_2' \\
\end{array}
$$

is commutative with rows in $\mathcal{E}$ and $M_i, M_i' \in \mathcal{M}$. Consider the following statements:
(a) The morphism $b$ factors through $f'$.
(b) The morphism $b$ factors through $f$.
(c) The morphism $a$ factors through some object in $\mathcal{U}$.
(d) The morphism $g$ factors through $k$.
(e) The morphism $g$ factors through some object in $\mathcal{M}$.

Then (a) $\iff$ (b) $\iff$ (c) $\iff$ (d) $\iff$ (e). Moreover, if $\mathcal{M}$ is rigid, then all the statements are equivalent.

**Proof.** We note that (b) $\iff$ (e) follows from Lemma 3.2, (b) $\iff$ (d) is easy, (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (e) are trivial. We prove (a) $\Rightarrow$ (b). Suppose that there is a morphism
Proof. Let \( M = M_1 \) such that \( f'M = b \). There exist two morphisms \( u : M_2 \to P \) and \( v : P \to M_2' \) such that \( P \in \mathcal{P} \) and \( b - f'p = vu \). Since \( f' \) is an admissible epimorphism and \( P \) is projective, there is a morphism \( w : P \to M_1' \) such that \( f'w = v \). Thus, \( f'(p + wu) = f'p + vu = b \).

Now assume that \( M \) is rigid. It remains to prove (e) \( \Rightarrow \) (d). Suppose that there exist two morphisms \( g_1 : X \to M \) and \( g_2 : M \to X' \) with \( M \in \mathcal{M} \) such that \( g = g_2g_1 \). Since \( k : X \to M_1 \) is a left \( \mathcal{M} \)-approximation of \( X \) by Remark \( \ref{rem:approximation} \), \( g_1 \) factors through \( k \), thus \( g \) factors through \( k \).

**Lemma 3.7.** Let \( \mathcal{M} \) be a full subcategory of \( \mathcal{C} \) containing \( \mathcal{P} \). Assume that the following diagram

\[
\begin{array}{c}
0 \rightarrow X \xrightarrow{k} M_1 \xrightarrow{f} M_2 \xrightarrow{0} \\
\downarrow g \quad \downarrow a \quad \downarrow b \\
0 \rightarrow X' \xrightarrow{k'} M_1' \xrightarrow{f'} M_2' \xrightarrow{0}
\end{array}
\]

is commutative with rows in \( \mathcal{E} \) and \( M_1, M_1' \in \mathcal{M} \). Consider the following statements:

(a) The morphism \( a \) in \( \mathcal{M}/[\mathcal{P}] \) factors through \( f' \).
(b) The morphism \( (a, b) \) factors through some object in \( \mathcal{V} \).
(c) The morphism \( g \) factors through some object in \( \Omega M \).

Then (a) \( \Leftrightarrow \) (b) \( \Rightarrow \) (c). Moreover, if \( \mathcal{M} \) is rigid, then all the statements are equivalent.

**Proof.** (a) \( \Rightarrow \) (b). Suppose that there is a morphism \( p : M_2 \to M_1' \) such that \( pf = a \). Since \( \mathcal{C} \) has enough projectives, there is an admissible epimorphism \( a_1 : P \to M_1' \) with \( P \in \mathcal{P} \). Since \( a - pf \) factors through \( a_1 \), we assume that \( a - pf = a_1a_2 \) where \( a_2 : M_1 \to P \). Now we have the following commutative diagram

\[
\begin{array}{c}
0 \rightarrow X \xrightarrow{k} M_1 \xrightarrow{f} M_2 \xrightarrow{0} \\
\downarrow a_2 \quad \downarrow f' \quad \downarrow b' \\
P \xrightarrow{f'a_1} M_2' \xrightarrow{0} \\
\downarrow a_1 \quad \downarrow \quad \downarrow \\
0 \rightarrow X' \xrightarrow{k'} M_1' \xrightarrow{f'} M_2' \xrightarrow{0}
\end{array}
\]

with exact rows. Since \( f'a_1a_2k = f'(a - pf)k = bfk - f'pfk = 0 \), there exists a morphism \( b' : M_2 \to M_2' \) such that \( b'f = f'a_1a_2 \). Since \( bf = f'a = f'(a - a_1a_2) + f'a_1a_2 = (f'p + b')f \) and \( f \) is an epimorphism, we have \( b = f'p + b' \). Thus the following diagram

\[
\begin{array}{c}
M_1 \xrightarrow{f} M_2 \\
\downarrow a \quad \downarrow \quad \downarrow (f', p) \\
M_2 \oplus P \xrightarrow{(0, f'a_1)} M_2 \oplus M_2' \\
\downarrow \quad \downarrow \quad \downarrow \\
M_1' \xrightarrow{f'} M_2'
\end{array}
\]
is commutative. In other words, \((a, b)\) factors through \((M_2 \oplus P \xrightarrow{(1 \ 0 ; a_1)} M_2 \oplus M'_2) \in V\).

(b) \Rightarrow (a). Assume that the morphism \((a, b)\) factors through \((M \oplus P \xrightarrow{(1 \ 0 ; \pi)} M \oplus M') \in V\). Suppose that the following diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{f} & M_2 \\
\downarrow{a_1} & & \downarrow{b_1} \\
M \oplus P & \xrightarrow{(1 \ 0 ; \pi)} & M \oplus M'
\end{array}
\]

is commutative. Let \(p = a_2 b_1\), then \(pf = a_2 b_1 f = a_2 a_1\), thus \(a = a_2 a_1 = pf\).

(b) \Rightarrow (c) is trivial.

Now assume that \(\mathcal{M}\) is rigid. It remains to prove (c) \Rightarrow (a). Suppose that \(g\) has a factorization \(X \xrightarrow{\Omega \ M} X'\). Then by Remark 3.8 we complete the following commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{k} & X & \xrightarrow{\ k} & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{1} & 0 \\
\downarrow{g_1} & & \downarrow{a_1} & & \downarrow{b_1} & & \ & & \\
0 & \xrightarrow{i} & \Omega M & \xrightarrow{\pi} & P & \xrightarrow{1} & M & \xrightarrow{1} & 0 \\
\downarrow{g_2} & & \downarrow{a_2} & & \downarrow{b_2} & & \ & & \\
0 & \xrightarrow{\ k'} & X' & \xrightarrow{\ f'} & M'_1 & \xrightarrow{\ f''} & M'_2 & \xrightarrow{1} & 0
\end{array}
\]

with exact rows and \(P \in \mathcal{P}\). Since \((a - a_2 a_1)k = k'(g - g_2 g_1) = 0\), there exists a morphism \(p : M_2 \to M'_1\) such that \(a - a_2 a_1 = pf\). Therefore, \(a = pf\). \(\square\)

**Lemma 3.8.** Let \(\mathcal{M}\) be a full subcategory of \(\mathcal{C}\) containing \(\mathcal{P}\), then

(a) \(\text{Ad-Epi}(\mathcal{M})/[\mathcal{U}] \cong \text{Mor}(\mathcal{M}/[\mathcal{P}])/R\).

(b) \(\text{Ad-Epi}(\mathcal{M})/[\mathcal{V}] \cong \text{Mor}(\mathcal{M}/[\mathcal{P}])/R'\).

**Proof.** Define a functor

\[\alpha : \text{Ad-Epi}(\mathcal{M}) \to \text{Mor}(\mathcal{M}/[\mathcal{P}]), \quad (M_1 \xrightarrow{f} M_2) \mapsto (M_1 \xrightarrow{f} M_2).\]

For each object \(f : M_1 \to M_2\) in \(\text{Mor}(\mathcal{M}/[\mathcal{P}])\), there is an admissible epimorphism \(\pi : P \to M_2\) with \(P \in \mathcal{P}\) since \(\mathcal{C}\) has enough projectives. Thus \((f, \pi) : M_1 \oplus P \to M_2\) is an object in \(\text{Ad-Epi}(\mathcal{M})\) such that \(\alpha(f, \pi) = f\). Therefore, \(\alpha\) is dense.

Assume that \(f : M_1 \to M_2\) and \(f' : M'_1 \to M'_2\) are objects in \(\text{Ad-Epi}(\mathcal{M})\) and \((a, b)\) is a morphism in \(\text{Mor}(\mathcal{M}/[\mathcal{P}])\) from \(f\) to \(f'\). Then \(b_1 = b_2\), thus \(bf = f' a\) factors through some object \(P \in \mathcal{P}\). Assume that \(bf - f' a = vu\) where \(u : M_1 \to P\) and \(v : P \to M_2\). Since \(f'\) is an admissible epimorphism and \(P\) is projective, there exists a morphism \(w : P \to M'_1\) such that \(f'w = v\). Now \((a + wu, b)\) is a morphism in \(\text{Ad-Epi}(\mathcal{M})\) from \(f\) to \(f'\) since \(bf = f'(a + wu)\). Thus, \(\alpha(a + wu, b) = (\ a, \ b)\) and the functor \(\alpha\) is full.
(a) The functor \( \alpha \) induces a full and dense functor \( \tilde{\alpha} : \text{Ad-Epi}(\mathcal{M}) \to \text{Mor}(\mathcal{M}/[\mathcal{P}])/\mathcal{R} \).

By the equivalence of (a) and (c) in Lemma 3.6, we have \( \text{Ad-Epi}(\mathcal{M})/\mathcal{U} \cong \text{Mor}(\mathcal{M}/[\mathcal{P}])/\mathcal{R} \).

(b) The functor \( \alpha \) induces a full and dense functor \( \tilde{\alpha} : \text{Ad-Epi}(\mathcal{M}) \to \text{Mor}(\mathcal{M}/[\mathcal{P}])/\mathcal{R}' \).

By the equivalence of (a) and (b) in Lemma 3.7, we have \( \text{Ad-Epi}(\mathcal{M})/\mathcal{V} \cong \text{Mor}(\mathcal{M}/[\mathcal{P}])/\mathcal{R}' \). \( \square \)

Lemma 3.8 and Lemma 3.1 imply the following theorem, which will be crucially used in section 4 to describe the categories of short exact sequences.

**Theorem 3.9.** Let \((\mathcal{C}, \mathcal{E})\) be an exact category with enough projectives. If \( \mathcal{M} \) is a full subcategory of \( \mathcal{C} \) containing \( \mathcal{P} \), then

(a) \( \text{Ad-Epi}(\mathcal{M})/\mathcal{U} \cong \text{mod-}(\mathcal{M}/[\mathcal{P}]) \).

(b) \( \text{Ad-Epi}(\mathcal{M})/\mathcal{V} \cong (\text{mod-}(\mathcal{M}/[\mathcal{P}])^{\text{op}})^{\text{op}} \).

**Lemma 3.10.** Let \( \mathcal{M} \) be a full subcategory of \( \mathcal{C} \) containing \( \mathcal{P} \), then \( \text{Ad-Epi}(\mathcal{M}) \cong \text{Epi}(\mathcal{M}) \).

**Proof.** We claim that the inclusion \( \text{Ad-Epi}(\mathcal{M}) \to \text{Epi}(\mathcal{M}) \) is dense. Indeed, assume that \( f : M_1 \to M_2 \) is an epimorphism, then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \Omega M_2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Omega M_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \longrightarrow & M_1 \\
\downarrow f & & \downarrow f \\
P & \longrightarrow & M_2 \\
\end{array}
\]

with exact rows in \( \mathcal{E} \) and \( P \in \mathcal{P} \). Thus the exact sequence \( 0 \to X \to M_1 \oplus P \xrightarrow{(f, \pi)} M_2 \to 0 \) belongs to \( \mathcal{E} \). Consequently, \( (M_1 \oplus P \xrightarrow{(f, \pi)} M_2) \in \text{Ad-Epi}(\mathcal{M}) \). A direct computation shows that \((M_1 \xrightarrow{\pi} M_2)\) is isomorphic to \((M_1 \oplus P \xrightarrow{(f, \pi)} M_2)\) in \( \text{Epi}(\mathcal{M}) \). \( \square \)

**Remark 3.11.** Let \((\mathcal{C}, \mathcal{E})\) be an exact category with enough projectives, then \( \mathcal{C}(\mathcal{E}) \), Ad-Epi(\(\mathcal{C}\)) and Epi(\(\mathcal{C}\)) are equivalent.

**Corollary 3.12.** Let \( \mathcal{M} \) be a full subcategory of \( \mathcal{C} \) containing \( \mathcal{P} \). If \( \mathcal{M} \) is closed under kernel of epimorphisms, denote by \( \text{Mono}_C(\mathcal{M}) \) the full subcategory of \( \text{Mono}(\mathcal{M}) \) consisting of monomorphisms \( f \) such that \( \text{Coker}(f) \in \mathcal{M} \), then

(a) \( \text{Mono}_C(\mathcal{M})/[\mathcal{U}'] \cong \text{mod-}(\mathcal{M}/[\mathcal{P}]) \).

(b) \( \text{Mono}_C(\mathcal{M})/[\mathcal{V}'] \cong (\text{mod-}(\mathcal{M}/[\mathcal{P}])^{\text{op}})^{\text{op}} \).

**Proof.** By assumption, the kernel functor \( \text{Ker} : \text{Epi}(\mathcal{M}) \to \text{Mono}_C(\mathcal{M}) \) induces two equivalences

\[
\text{Epi}(\mathcal{M})/\mathcal{U} \cong \text{Mono}_C(\mathcal{M})/\mathcal{U}', \quad \text{Epi}(\mathcal{M})/\mathcal{V} \cong \text{Mono}_C(\mathcal{M})/\mathcal{V}'.
\]

The corollary follows from Theorem 3.9 and Lemma 3.10. We can compare (a) with \( \text{[17]} \) Theorem 3.3]. \( \square \)

**Lemma 3.13.** Let \( \mathcal{M} \) be a full and rigid subcategory of \( \mathcal{C} \) containing \( \mathcal{P} \). Then

(a) \( \mathcal{M}_L/[\mathcal{M}] \cong \text{Ad-Epi}(\mathcal{M})/\mathcal{U} \).

(b) \( \mathcal{M}_L/[\Omega \mathcal{M}] \cong \text{Ad-Epi}(\mathcal{M})/\mathcal{V} \).
**Proof.** Define a functor by

\[ \beta : \text{Ad-} \text{Epi}(\mathcal{M}) \to \mathcal{M}_L, \quad (M_1 \xrightarrow{f} M_2) \mapsto \text{Ker}(f). \]

Then \( \beta \) is dense. For each morphism \( g : X \to X' \) in \( \mathcal{M}_L \), there exists the following diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{k} & X \\
\downarrow{g} & & \downarrow{a} \\
0 & \xrightarrow{k'} & X'
\end{array}
\quad \begin{array}{ccc}
M_1 & \xrightarrow{f} & M_2 \\
\downarrow{b} & & \downarrow{b} \\
M'_1 & \xrightarrow{f'} & M'_2
\end{array}
\]

with rows in \( \mathcal{E} \) and \( M_1, M'_1 \in \mathcal{M} \). Since \( \mathcal{M} \) is rigid, by Remark\(^\text{5.5}\) there exists a morphism \( a : M_1 \to M'_1 \) such that \( ak = k'g \). Then there is a morphism \( b : M_2 \to M'_2 \) such that \( bf = f'a \). Hence \( \beta(a, b) = g \) and the functor \( \beta \) is full.

(a) The functor \( \beta \) induces a full and dense functor \( \tilde{\beta} : \text{Ad-}\text{Epi}(\mathcal{M}) \to \mathcal{M}_L/\lceil \mathcal{M} \rceil \).

We note that \( \tilde{\beta}(\mathcal{U}) = 0 \). The equivalence \( \text{Ad-}\text{Epi}(\mathcal{M})/\lceil \mathcal{U} \rceil \cong \mathcal{M}_L/\lceil \mathcal{M} \rceil \) follows from the equivalent statements of (c) and (e) in Lemma\(^\text{3.6}\).

(b) The functor \( \beta \) induces a full and dense functor \( \tilde{\beta} : \text{Ad-}\text{Epi}(\mathcal{M}) \to \mathcal{M}_L/\lceil \Omega \mathcal{M} \rceil \).

Since \( \tilde{\beta}(\mathcal{V}) = 0 \), the equivalence \( \text{Ad-}\text{Epi}(\mathcal{M})/\lceil \mathcal{V} \rceil \cong \mathcal{M}_L/\lceil \Omega \mathcal{M} \rceil \) follows from the equivalent statements of (b) and (c) in Lemma\(^\text{3.7}\). \( \square \)

By Lemma\(^\text{3.13}\) and Theorem\(^\text{3.9}\) we have the following corollary, where (a) was appeared in \([12, \text{Theorem 3.2}]\).

**Corollary 3.14.** Let \( (\mathcal{C}, \mathcal{E}) \) be an exact category with enough projectives. If \( \mathcal{M} \) is a rigid and full subcategory of \( \mathcal{C} \) containing \( \mathcal{P} \), then

(a) \( \mathcal{M}_L/\lceil \mathcal{M} \rceil \cong \text{mod-}(\mathcal{M}/\lceil \mathcal{P} \rceil) \).

(b) \( \mathcal{M}_L/\lceil \Omega \mathcal{M} \rceil \cong (\text{mod-}(\mathcal{M}/\lceil \mathcal{P} \rceil))^\text{op} \).

Suppose that \( \mathcal{M} \) is a contravariantly finite subcategory of \( \mathcal{C} \) containing \( \mathcal{P} \), then by Proposition\(^\text{5.3}\) \( \text{Mor}(\mathcal{M})/\lceil \mathcal{U} \rceil \cong \text{mod-} \mathcal{M} \) is abelian. Moreover, \( \text{Epi}(\mathcal{M})/\lceil \mathcal{U} \rceil \cong \text{mod-} \mathcal{M}/\lceil \mathcal{P} \rceil \) is abelian by Theorem\(^\text{3.9}\). The following result is a variant of \([2, \text{Theorem 3.7}]\).

**Proposition 3.15.** Let \( \mathcal{C} \) be an abelian category with enough projectives. If \( \mathcal{M} \) is a contravariantly finite subcategory containing all projectives, then there exists a recollement

\[
\begin{array}{ccc}
\text{Epi}(\mathcal{M})/\lceil \mathcal{U} \rceil & \xrightarrow{i^*} & \text{Mor}(\mathcal{M})/\lceil \mathcal{U} \rceil \\
\downarrow{i} & & \downarrow{j^*} \\
\mathcal{C} & \xleftarrow{j_*} & \mathcal{C}
\end{array}
\]

of abelian categories.

**Proof.** Consider the cokernel functor

\[ \text{Cok} : \text{Mor}(\mathcal{M}) \to \mathcal{C}, \quad (M_1 \xrightarrow{f} M_2) \mapsto \text{Coker}(f). \]

Since \( \text{Cok}(X \xrightarrow{a} X) = 0 \) and \( \text{Cok}(X \to 0) = 0 \) for each \( X \in \mathcal{M} \), the functor \( \text{Cok} \) induces a functor \( j^* : \text{Mor}(\mathcal{M})/\lceil \mathcal{U} \rceil \to \mathcal{C} \). For each \( X \in \mathcal{C} \), there exists an exact sequence \( P_1 \xrightarrow{\theta} P_0 \to X \to 0 \) with \( P_i \in \mathcal{P} \) since \( \mathcal{C} \) has enough projectives. It is easy to check that the functor

\[ j_1 : \mathcal{C} \to \text{Mor}(\mathcal{M})/\lceil \mathcal{U} \rceil, \quad X \mapsto (P_1 \xrightarrow{\theta} P_0) \]
epimorphisms. Thus we have an exact sequence
\[ M \rightarrow \ker(a) \rightarrow \ker(b) \rightarrow X \rightarrow 0 \]
where \( h \) is the composition of \( b \) and the natural inclusion \( \ker(a) \hookrightarrow M_1 \). Define a functor by
\[ j_* : C \rightarrow \text{Mor}(M)/[U], \quad X \mapsto (M_2 \xrightarrow{h} M_1). \]

It is routine to prove that \((j_*, j^*)\) and \((j^*, j_1)\) are adjoint pairs. Moreover, \( j_* \) and \( j_1 \) are fully-faithful. We note that \( \ker(j^*) = \text{Epi}(M)/[U] \), so by [27] Remark 3.17 we complete the proof. Actually, the functors \( i^*, i_* \) and \( i^! \) are described as follows:

\[ i^* : \text{Mor}(M)/[U] \rightarrow \text{Epi}(M)/[U], \quad (M_1 \xrightarrow{f} M_2) \mapsto (M_1 \oplus P \xrightarrow{(f,\pi)} M_2) \]
\[ i_* : \text{Epi}(M)/[U] \rightarrow \text{Mor}(M)/[U], \quad (M_1 \xrightarrow{f} M_2) \mapsto (M_1 \xrightarrow{f} M_2) \]
\[ i^! : \text{Mor}(M)/[U] \rightarrow \text{Epi}(M)/[U], \quad (M_1 \xrightarrow{f} M_2) \mapsto (M_1 \xrightarrow{f} \text{Im}(f)) \]

where \( \pi : P \rightarrow M_2 \) is an epimorphism with \( P \in \mathcal{P} \).

**Corollary 3.16.** Let \( C \) be an abelian category with enough projectives. Then there exists a recollement

\[
\begin{array}{ccc}
\text{mod}(C)/[P]) & \xrightarrow{i_*} & \text{mod-}C \\
\text{mod-}C & \xrightarrow{j_*} & C \\
\end{array}
\]

of abelian categories. Therefore, we have an equivalence \( \text{mod-}C/[\text{mod-}C/[P]] \cong C \).

**Remark 3.17.** Following Lenzing [28], the equivalence \( \text{mod-}C/[\text{mod-}C/[P]] \cong C \) is called Auslander’s formula; see [3].

### 3.3. Third case: triangulated categories

Let \( C \) be a right triangulated category with suspension functor \( \Sigma \) and \( M \) be a full subcategory of \( C \). We denote by \( M + \Sigma M \) the full subcategory of \( C \) consisting of objects \( X \) such that there is a right triangle \( M_1 \rightarrow M_2 \rightarrow X \rightarrow \Sigma M_1 \) with \( M_1 \in M \). A full subcategory \( M \) is called rigid if \( C(M, M') = 0 \) for each \( M, M' \in M \).

**Lemma 3.18.** ([1] Lemma 1.3) Let \( X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1 \) be a right triangle, then the following holds.

(a) \( f_{i+1} \) is a weak cokernel of \( f_i \) for \( i = 1, 2 \).

(b) If \( \Sigma \) is fully-faithful, then \( f_i \) is a weak kernel of \( f_{i+1} \) for \( i = 1, 2 \).

The following result generalizes [21] Proposition 6.2 from triangulated categories to right triangulated categories.

**Proposition 3.19.** Let \( C \) be a right triangulated category and \( M \) be a rigid subcategory. If \( \Sigma \) is fully-faithful, then \( (M + \Sigma M)/[\Sigma M] \cong \text{mod-}M \).

**Proof.** By Lemma 3.11 we only need to show that \( (M + \Sigma M)/[\Sigma M] \cong \text{Mor}(M)/\mathbb{R} \).

For each morphism \( f_1 : M_1 \rightarrow M_2 \) in \( M \), we assume that \( M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} X \xrightarrow{f_3} \Sigma M_1 \) is a right triangle. The assignment \((M_1 \xrightarrow{f_1} M_2) \mapsto X\) defines a dense functor \( F : \text{Mor}(M) \rightarrow M + \Sigma M \). Assume that \( g : X \rightarrow X' \) is a morphism in \( M + \Sigma M \). Since \( M \) is rigid, \( f_3 gf_2 = 0 \). Thus by Lemma 3.18 there exists a morphism \( b : M_2 \rightarrow M_1 \)

such that $gf_2 = f_2'b$. Since $f_1'b = 0$, there is a morphism $a : M_1 \to M_2$ such that $bf_2 = f_2'a$ by Lemma 3.18. Hence, $F(a, b) = g$ and the functor $F$ is full.

\[
\begin{array}{c|c|c|c|c|c}
M_1 & f_1 & M_2 & f_2 & X & f_3 \\
\downarrow a & \downarrow b & \downarrow 1 & \downarrow g & \downarrow \Sigma a & \downarrow \Sigma a \\
M_1' & f_1' & M_2' & f_2' & X' & f_3' \\
\end{array}
\]

We note that $F$ induces a full and dense functor $\tilde{F} : \text{Mor}(\mathcal{M}) \to \mathcal{M} \cdot \Sigma \mathcal{M}/[\Sigma \mathcal{M}]$. As in the above diagram, we assume that $\tilde{F}(a, b) = g = 0$. Since $\mathcal{M}$ is rigid, $f_3$ is a left $\Sigma \mathcal{M}$-approximation, thus $g$ factors through $f_3$. Hence, $f_2'b = gf_2 = 0$ and $b$ factors through $f_3'$, which implies that $\text{Mor}(\mathcal{M})/R \cong (\mathcal{M} \cdot \Sigma \mathcal{M})/[\Sigma \mathcal{M}]$. We complete the proof. \hfill $\square$

We recall that a full subcategory $\mathcal{M}$ of a triangulated category $\mathcal{C}$ is \textit{cluster-tilting} if $\mathcal{M}$ is rigid and $\mathcal{C} = \mathcal{M} \cdot \Sigma \mathcal{M}$.

**Corollary 3.20.** ([19] [21] [27]) Let $\mathcal{C}$ be a triangulated category with suspension functor $\Sigma$ and $\mathcal{M}$ be a cluster-tilting subcategory, then there is an equivalence of categories $\mathcal{C}/[\Sigma \mathcal{M}] \cong \text{proj-}\mathcal{M}$.

**3.4. Examples.** Let $A$ be an Artin $k$-algebra and $\mathcal{C}$ be a full subcategory of $\text{mod-}A$ containing $A$. Denote by $\mathcal{U}$ the full subcategory of $\text{Epi}(\mathcal{C})$ consisting of $(\mathcal{C} \xrightarrow{f} \mathcal{C}) \oplus (Y \rightarrow 0)$. Assume that all the indecomposable objects in $\mathcal{C}$ are $M_1, M_2, \cdots, M_n$. Set $M = \bigoplus_{i=1}^n M_i$. Then $B = \text{End}_A(M)$ is called \textit{Auslander algebra} of $\mathcal{C}$ and $\overline{B} = \text{End}_A(M)$ is called \textit{stable Auslander algebra} of $\mathcal{C}$. It is easy to see that $\overline{B} = B/BeB$, where $e$ is the idempotent given by $\text{Hom}_A(M, A)$.

**Proposition 3.21.** With notations as above. Then

(a) $\text{Mor}(\mathcal{C})/\mathcal{U} \cong \text{proj-}B$.

(b) $\text{Epi}(\mathcal{C})/\mathcal{U} \cong \text{mod-}\overline{B}$.

**Proof.** (a) Since $\mathcal{C} = \text{add}M$ and the functor $\text{Hom}_A(M, -)$ induces an equivalence $\text{add}M \cong \text{proj-}B$, we have the following equivalence

$\alpha : \text{Mor}(\mathcal{C})/\mathcal{U} = \text{Mor}(\mathcal{C})/R \cong \text{Mor}(\text{proj-}B)/R \cong \text{mod-}B$

which mapping a morphism $f : X \to Y$ to $\text{CokerHom}_A(M, f)$.

(b) Assume that $f : X \to Y$ is a morphism in $\mathcal{C}$, then

$f$ is an epimorphism

$\iff \text{Hom}_B(\text{Hom}_A(M, A), \text{CokerHom}_A(M, f)) = 0$

$\iff \text{Hom}_B(eB, \alpha(f)) = 0$

$\iff \alpha(f)e = 0$

$\iff \alpha(f) \in \text{mod-}B/BeB = \text{mod-}\overline{B}$

where the first if and only if condition follows from [11] Section 6. Thus, the functor $\alpha$ induces an equivalence $\text{Epi}(\mathcal{C})/\mathcal{U} \cong \text{mod-}\overline{B}$. \hfill $\square$

**Example 3.22.** Let $A$ be a representation-finite Artin $k$-algebra. Then

$\text{Epi}(\text{mod-}A)/\mathcal{U} \cong \text{mod-}\overline{B}$

where $\overline{B}$ is the stable Auslander algebra of $A$. In the case when $A$ is self-injective, this equivalence was proved in [13] Theorem 1] using the language of submodule category.
Example 3.23. Let $A$ be an Artin algebra of CM-finite type. By definition, an algebra $A$ is called CM-finite type if the number of indecomposable Gorenstein projective modules up to isomorphisms is finite. Denote by $\text{Gproj}_A$ the full subcategory of $\text{mod-A}$ formed by Gorenstein projective modules. Then

$$\text{Epi}(\text{Gproj}_A)/[\mathcal{U}] \cong \text{mod-}\bar{B}$$

where $\bar{B}$ is the sable Auslander Cohen-Macaulay algebra of $A$.

4. Abelian quotients of the categories of short exact sequences

In this section, we assume that $(\mathcal{C}, \mathcal{E})$ is an exact category. We always view a short exact sequence $0 \to X_1 \overset{f_1}{\to} X_2 \overset{f_2}{\to} X_3 \to 0$, sometimes $(X_1 \to X_2 \to X_3)$ for short, as a complex $X_\bullet$ concentrated on degree 1, 2 and 3. When we say $\varphi_\bullet : X_\bullet \to Y_\bullet$ is a morphism between two short exact sequences $X_\bullet$ and $Y_\bullet$, we means that the following diagram

$$
\begin{array}{ccccccccc}
0 & \to & X_1 & \overset{f_1}{\to} & X_2 & \overset{f_2}{\to} & X_3 & \to & 0 \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \\
0 & \to & Y_1 & \overset{g_1}{\to} & Y_2 & \overset{g_2}{\to} & Y_3 & \to & 0
\end{array}
$$

is commutative. We denote by $C^b(\mathcal{C})$ the category of bounded complexes over $\mathcal{C}$, by $\mathcal{E}(\mathcal{C})$ the full subcategory of $C^b(\mathcal{C})$ consisting of short exact sequences in $\mathcal{E}$, and by $SE(\mathcal{C})$ the full subcategory of $\mathcal{E}(\mathcal{C})$ formed by split short exact sequences over $\mathcal{C}$.

Throughout this section, if $(\mathcal{C}, \mathcal{E})$ has enough projectives, we always denote by $\mathcal{P}$ the full subcategory of $\mathcal{C}$ formed by all projectives. Similarly, if $(\mathcal{C}, \mathcal{E})$ has enough injectives, we always denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ formed by all injectives.

4.1. Realizing quotients of the categories of short exact sequences as module categories.

Theorem 4.1. Let $(\mathcal{C}, \mathcal{E})$ be an exact category and $X_\bullet : 0 \to X_1 \overset{f_1}{\to} X_2 \overset{f_2}{\to} X_3 \to 0$ be a short exact sequence in $\mathcal{E}$.

(a) If $(\mathcal{C}, \mathcal{E})$ has enough projectives, then we have the following equivalences:

$$
\alpha_1 : \mathcal{E}(\mathcal{C})/[SE(\mathcal{C})] \cong \text{mod-}\mathcal{C}/[\mathcal{P}], \quad X_\bullet \mapsto \text{Coker}(\mathcal{C}/[\mathcal{P}](f_1, f_2))
$$

$$
\alpha_2 : \mathcal{E}(\mathcal{C})/[PE(\mathcal{C})] \cong (\text{mod-}\mathcal{C}/[\mathcal{P}])^{\mathcal{op}}^{\mathcal{op}} \quad X_\bullet \mapsto \text{Coker}(\mathcal{C}/[\mathcal{P}](f_2, -))
$$

where $PE(\mathcal{C})$ is the full subcategory of $\mathcal{E}(\mathcal{C})$ formed by $(0 \to X \to X) \oplus (\Omega Y \to P \to Y)$.

(b) If $(\mathcal{C}, \mathcal{E})$ has enough injectives, then we have the following equivalences:

$$
\beta_1 : \mathcal{E}(\mathcal{C})/[IE(\mathcal{C})] \cong (\text{mod-}\mathcal{C}/[\mathcal{I}])^{\mathcal{op}} \quad X_\bullet \mapsto \text{Coker}(\mathcal{C}/[\mathcal{I}](f_1, -))
$$

$$
\beta_2 : \mathcal{E}(\mathcal{C})/[IE(\mathcal{C})] \cong \text{mod-}\mathcal{C}/[\mathcal{I}] \quad X_\bullet \mapsto \text{Coker}(\mathcal{C}/[\mathcal{I}](f_1, -))
$$

where $IE(\mathcal{C})$ is the full subcategory of $\mathcal{E}(\mathcal{C})$ formed by $(X \to X \to 0) \oplus (Y \to I \to \Omega^{-1}Y)$.

Proof. We only prove (a). We have two equivalences $\mathcal{E}(\mathcal{C})/[SE(\mathcal{C})] \cong \text{Ad-Epi}(\mathcal{C})/[\mathcal{U}]$ and $\mathcal{E}(\mathcal{C})/[PE(\mathcal{C})] \cong \text{Ad-Epi}(\mathcal{C})/[\mathcal{V}]$, where $\mathcal{U}$ (resp. $\mathcal{V}$) is the full subcategory of $\text{Ad-Epi}(\mathcal{C})$ consisting of $(X \overset{1}{\to} X) \oplus (Y \to 0)$ (resp. $(X \overset{1}{\to} X) \oplus (P \to Y)$ with $P \in \mathcal{P}$). Then (a) follows from Theorem 3.9. \qed
Corollary 4.2. Let \((\mathcal{C}, \mathcal{E})\) be a Frobenius category, then all the quotient categories \(\mathcal{E}(\mathcal{C})/\lbrack SE(\mathcal{C})\rbrack, \mathcal{E}(\mathcal{C})/\lbrack PE(\mathcal{C})\rbrack\) and \(\mathcal{E}(\mathcal{C})/\lbrack IE(\mathcal{C})\rbrack\) are equivalent to abelian category \(\text{mod-}\mathcal{C}/[\mathcal{P}]\).

Remark 4.3. Let \((\mathcal{C}, \mathcal{E})\) be a Frobenius category. Suppose that \(X_\bullet : 0 \to X_1 \to X_2 \to X_3 \to 0\) is a short exact sequence in \(\mathcal{E}\), then we have the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \Omega X_3 & \to & P & \to & X_3 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & X_1 & \to & X_2 & \to & X_3 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & X_1 & \to & I & \to & \Omega^{-1}X_1 & \to & 0
\end{array}
\]

with rows in \(\mathcal{E}\). Hence, the equivalences between the quotient categories in Corollary 4.2 can be described as the following rotations:

\[
\alpha = \alpha_2^{-1}\beta_1 : \mathcal{E}(\mathcal{C})/\lbrack SE(\mathcal{C})\rbrack \cong \mathcal{E}(\mathcal{C})/\lbrack PE(\mathcal{C})\rbrack, \\
X_\bullet \mapsto (\Omega X_3 \to P \oplus X_1 \to X_2)
\]

\[
\beta = \beta_2^{-1}\alpha_1 : \mathcal{E}(\mathcal{C})/\lbrack SE(\mathcal{C})\rbrack \cong \mathcal{E}(\mathcal{C})/\lbrack IE(\mathcal{C})\rbrack, \\
X_\bullet \mapsto (X_2 \to X_3 \oplus I \to \Omega^{-1}X_1)
\]

By the dual of Lemma 3.10, the following is an equivalent statement of Corollary 4.2. We can compare it with [13, Theorem 1].

Corollary 4.4. Let \((\mathcal{C}, \mathcal{E})\) be a Frobenius category. Denote by \(\mathcal{P}\) the full subcategory of projective-injective objects in \(\mathcal{C}\), by \(\mathcal{U}_1\) the full subcategory of \(\text{Mono}(\mathcal{C})\) consisting of \((X \xrightarrow{1} X) \oplus (0 \to Y)\), by \(\mathcal{U}_2\) the full subcategory of \(\text{Mono}(\mathcal{C})\) consisting of \((X \xrightarrow{1} X) \oplus (Y \to P)\) with \(P \in \mathcal{P}\) and by \(\mathcal{U}_3\) the full subcategory of \(\text{Mono}(\mathcal{C})\) consisting of \((0 \to X) \oplus (Y \to P)\) with \(P \in \mathcal{P}\). Then all the quotient categories \(\text{Mono}(\mathcal{C})/\lbrack \mathcal{U}_1\rbrack, \text{Mono}(\mathcal{C})/\lbrack \mathcal{U}_2\rbrack\) and \(\text{Mono}(\mathcal{C})/\lbrack \mathcal{U}_3\rbrack\) are equivalent to \(\text{mod-}\mathcal{C}/[\mathcal{P}]\).

4.2. Abelian structure. Let \((\mathcal{C}, \mathcal{E})\) be an exact category with enough projectives. Then by Theorem 4.1, \(\mathcal{E}(\mathcal{C})/\lbrack SE(\mathcal{C})\rbrack\) is equivalent to \(\text{mod-}\mathcal{C}/[\mathcal{P}]\) thus has an abelian structure. In this subsection, we will prove that for general exact category \((\mathcal{C}, \mathcal{E})\), the quotient category \(\mathcal{E}(\mathcal{C})/\lbrack SE(\mathcal{C})\rbrack\) always has an abelian structure given by pushout and pullback diagrams.

Lemma 4.5. Assume that the following diagram

\[
\begin{array}{cccccc}
X_\bullet & 0 & \to & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \to & 0 \\
\downarrow \phi_\bullet & & \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 \\
Y_\bullet & 0 & \to & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \to & 0
\end{array}
\]
is commutative with rows in $\mathcal{E}$. Then the following diagram

$$
\begin{array}{c}
\begin{array}{c}
K(\varphi_{*}) \rightarrow 0 \rightarrow X_{1} \rightarrow X_{2} \oplus Y_{1} \rightarrow Z \rightarrow 0 \\
\downarrow \kappa_{*} \downarrow \downarrow \downarrow \downarrow \downarrow \\
X_{\bullet} \rightarrow 0 \rightarrow X_{1} \rightarrow f_{1} \rightarrow f_{2} \rightarrow X_{3} \rightarrow 0 \\
\downarrow \pi_{*} \downarrow \varphi_{1} (I) \downarrow a_{1} \downarrow \\
I(\varphi_{*}) \rightarrow 0 \rightarrow Y_{1} \rightarrow Z \rightarrow X_{3} \rightarrow 0 \quad (4.1) \\
\downarrow \iota_{*} \downarrow h_{1} \downarrow \downarrow \downarrow \downarrow \downarrow \\
Y_{\bullet} \rightarrow 0 \rightarrow Y_{1} \rightarrow g_{1} \rightarrow g_{2} \rightarrow Y_{3} \rightarrow 0 \\
\downarrow c_{*} \downarrow h_{1} \downarrow (\frac{h_{2}}{a_{2}}) \downarrow (\frac{1}{g_{3}g_{2}}) \downarrow \\
C(\varphi_{*}) \rightarrow 0 \rightarrow Z \rightarrow X_{3} \oplus Y_{2} \rightarrow Y_{3} \rightarrow 0 \\
\end{array}
\end{array}
$$

is commutative with rows in $\mathcal{E}$, moreover, $\varphi_{*} = i_{*} \pi_{*}$.

**Proof.** By [11 Proposition 3.1], the morphism $\varphi_{*}$ factors through some short exact sequence $I(\varphi_{*})$ in $\mathcal{E}$ in such a way that $\varphi_{*} = i_{*} \pi_{*}$ and the squares (I) and (II) are both pushout and pullback diagrams. The sequences $K(\varphi_{*})$ and $C(\varphi_{*})$ belong to $\mathcal{E}$ since the squares (I) and (II) are both pushout and pullback diagrams. $\square$

**Lemma 4.6. ([15 Proposition 1.1])** Let $\varphi_{*} : X_{\bullet} \rightarrow Y_{\bullet}$ be a morphism in $\mathcal{E}(\mathcal{C})$. Then the following statements are equivalent.

(a) There is a morphism $p_{1} : X_{2} \rightarrow Y_{1}$ such that $\varphi_{1} = p_{1}f_{1}$.
(b) There is a morphism $p_{2} : X_{3} \rightarrow Y_{2}$ such that $\varphi_{3} = g_{2}p_{2}$.
(c) The morphism $\varphi_{*}$ is homotopic to zero.
(d) The morphism $\varphi_{*}$ factors through a split short exact sequence.
(e) The morphism $\varphi_{*} = 0$ in $\mathcal{E}(\mathcal{C})/[SE(\mathcal{C})]$.

**Lemma 4.7.** Let $\varphi_{*} : X_{\bullet} \rightarrow Y_{\bullet}$ be a morphism in $\mathcal{E}(\mathcal{C})$. Then $\varphi_{*}$ is a monomorphism in $\mathcal{E}(\mathcal{C})/[SE(\mathcal{C})]$ if and only if $(\frac{f_{1}}{\varphi_{1}})$ is a section.

**Proof.** For the “if” part, assume that there exists a morphism $(f_{1}', \varphi_{1}') : X_{2} \oplus Y_{1} \rightarrow X_{1}$ such that $(f_{1}', \varphi_{1}') (\frac{f_{1}}{\varphi_{1}}) = 1$. Suppose that $\psi_{*} : Z_{\bullet} \rightarrow X_{\bullet}$ is a morphism such that $\varphi_{*} \psi_{*} = 0$.

$$
\begin{array}{c}
\begin{array}{c}
Z_{\bullet} \rightarrow 0 \rightarrow Z_{1} \rightarrow Z_{2} \rightarrow Z_{3} \rightarrow 0 \\
\downarrow \psi_{*} \downarrow \downarrow \downarrow \downarrow \downarrow \\
X_{\bullet} \rightarrow 0 \rightarrow X_{1} \rightarrow f_{1} \rightarrow f_{2} \rightarrow X_{3} \rightarrow 0 \\
\downarrow \varphi_{*} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
Y_{\bullet} \rightarrow 0 \rightarrow Y_{1} \rightarrow g_{1} \rightarrow g_{2} \rightarrow Y_{3} \rightarrow 0 \\
\end{array}
\end{array}
$$

By Lemma 4.6, there is a morphism $p_{1} : Z_{2} \rightarrow Y_{1}$ such that $\varphi_{1} \psi_{1} = p_{1}h_{1}$. Thus there exists a morphism $q_{1} = (f_{1}', \varphi_{1}') (\frac{\psi_{1}}{p_{1}}) : Z_{2} \rightarrow X_{1}$ such that $q_{1}h_{1} = (f_{1}' \psi_{2} + \varphi_{1}' p_{1}) h_{1} = (f_{1}' f_{1} + \varphi_{1}' \varphi_{1}) \psi_{1} = \psi_{1}$. We infer that $\psi_{*} = 0$ by Lemma 4.6 again.
For the “only if” part, there is a morphism $p_1 = (0, 1) : X_2 \oplus Y_1 \to Y_1$ such that $\varphi_1 = p_1 (f_1')$, so we have $\varphi_* h_* = i_* \pi_* k_* = 0$ by Lemma 4.6 and Lemma 4.4. Since $\varphi_*$ is a monomorphism, we have $h_* = 0$, thus $f_1'$ is a section by Lemma 4.3. \hfill \Box

When $C$ is a certain abelian category, the following theorem was appeared in [14, Theorem 2.5].

**Theorem 4.8.** Let $(C, \mathcal{E})$ be an exact category. Then the quotient $\mathcal{E}(C)/[S\mathcal{E}(C)]$ is an abelian category whose kernels and cokernels are given by pullback and pushout diagrams.

**Proof.** Suppose that $\varphi_* : X_* \to Y_*$ is a morphism in $\mathcal{E}(C)$. As notations in diagram (4.1), we claim that $h : K(\varphi_*) \to X_*$ is a kernel of $\varphi_*$. By Lemma 4.7, $h$ is a monomorphism. Since $\varphi_1 = (0, 1) (f_1')$, it follows from Lemma 4.4 that $\varphi_* h_* = 0$. Assume that there is a morphism $\psi_* : Z_* \to X_*$ such that $\varphi_* \psi_* = 0$, then by Lemma 4.6 there is a morphism $p_1 : Z_2 \to Y_1$ such that $\varphi_1 \psi_1 = p_1 h_1$. Since $(f_1') h_1 = \phi_p \psi_1$, we obtain the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Z_1 & \stackrel{h_1}{\longrightarrow} & Z_2 & \stackrel{h_2}{\longrightarrow} & Z_3 & \longrightarrow & 0 \\
& \downarrow & & & \downarrow & & & \downarrow & \\
0 & \longrightarrow & X_1 & \stackrel{(f_1')}{\longrightarrow} & X_2 \oplus Y_1 & \stackrel{(p_1 + h_1)}{\longrightarrow} & Z & \longrightarrow & 0 \\
\end{array}
$$

By a direct checking, we have $\psi_* = h \theta_*$. Dually we can show that $\psi_* : Y_* \to C(\varphi_*)$ is a cokernel of $\varphi_*$. It remains to show that $\text{Coker}(\text{Ker}(\varphi_*)) \cong \text{Ker}(\text{Coker}(\varphi_*))$, that is, $\text{Coker}(h_*) \cong \text{Ker}(h_*)$. Indeed, the following commutative diagram

$$
\begin{array}{ccccccc}
\text{Coker}(h_*) & 0 & \longrightarrow & X_2 \oplus Y_1 \stackrel{\left(\begin{array}{cc} a_1 & 0 \\ 1 & -h_1 \end{array}\right)}{\longrightarrow} X_2 \oplus Z \stackrel{\left(\begin{array}{cc} f_1 & -h_2 \\ a_1 & 0 -1 \end{array}\right)}{\longrightarrow} X_3 \longrightarrow 0 \\
& \downarrow & & & \downarrow & & & \downarrow & \\
\text{I}(\varphi_*) & 0 & \longrightarrow & X_2 \oplus Y_1 \stackrel{\left(\begin{array}{cc} 1 & 0 \\ 0 & h_1 \end{array}\right)}{\longrightarrow} X_2 \oplus Z \stackrel{\left(\begin{array}{cc} 0 & h_2 \\ 0 & -1 \end{array}\right)}{\longrightarrow} X_3 \longrightarrow 0 \\
\end{array}
$$

shows that $\text{Coker}(h_*) \cong \text{I}(\varphi_*)$. The following commutative diagram

$$
\begin{array}{ccccccc}
\text{I}(\varphi_*) & 0 & \longrightarrow & Y_1 \stackrel{\left(\begin{array}{cc} h_1 \\ 0 \end{array}\right)}{\longrightarrow} Z \oplus Y_2 \stackrel{\left(\begin{array}{cc} h_2 & 0 \\ 0 & 1 \end{array}\right)}{\longrightarrow} X_3 \oplus Y_2 \longrightarrow 0 \\
& \downarrow & & & \downarrow & & & \downarrow & \\
\text{Ker}(\varphi_*) & 0 & \longrightarrow & Y_1 \stackrel{\left(\begin{array}{cc} h_1 \\ 0 \end{array}\right)}{\longrightarrow} Z \oplus Y_2 \stackrel{\left(\begin{array}{cc} h_2 & 0 \\ 0 & -1 \end{array}\right)}{\longrightarrow} X_3 \oplus Y_2 \longrightarrow 0 \\
\end{array}
$$

implies that $\text{I}(\varphi_*) \cong \text{Ker}(\varphi_*)$. We are done. \hfill \Box

**Remark 4.9.** Let $(C, \mathcal{E})$ be an exact category with enough projectives. Then $\mathcal{E}(C)/[S\mathcal{E}(C)] \cong \text{mod-}C/[\mathcal{P}]$ thus has an abelian structure. Theorem 4.8 tells us that the quotient $\mathcal{E}(C)/[S\mathcal{E}(C)]$ has an abelian structure given by pullout and pullback diagrams. In fact, the two abelian structures are the same by Remark 2.2.

**Remark 4.10.** Let $\varphi_* : X_* \to Y_*$ be a monomorphism in $\mathcal{E}(C)/[S\mathcal{E}(C)]$. Then by Lemma 4.5, we have $\varphi_* = i_* \pi_*$. Note that $\pi_* : X_* \to I(\varphi_*)$ is both a monomorphism and an epimorphism, thus it is an isomorphism. Therefore, for convenience
when we mention a monomorphism \( \varphi_* : X_* \to Y_* \) in \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \), we can assume that \( \varphi_1 = 1 \).

### 4.3. Projective objects, injective objects and Hilton-Rees Theorem

Let \((\mathcal{C}, \mathcal{E})\) be an exact category. We first provide the projective objects and injective objects in \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \).

**Proposition 4.11.** Let \((\mathcal{C}, \mathcal{E})\) be an exact category.

(a) Each short exact sequence \( P_X : 0 \to \Omega X \xrightarrow{f_1} P \xrightarrow{f_2} X \to 0 \) in \( \mathcal{E} \) with \( P \) projective is a projective object in \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \).

(b) If \((\mathcal{C}, \mathcal{E})\) has enough projectives, then each projective object in \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \) is of the form \( P_X \) for some object \( X \) in \( \mathcal{C} \). In this case, \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \) has enough projectives.

**Proof.** (a) Assume that \( \varphi_* : Y_* \to Z_* \) is an epimorphism and \( \psi_* : P_X \to Z_* \) is a morphism in \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \). By the dual version of Remark 4.10, we assume that \( \varphi_3 = 1 \). Since \( P \) is projective, we obtain a morphism \( \phi_* : P_X \to Y_* \) such that \( \phi_3 = \psi_3 \). Since \( \psi_3 = \varphi_3 \phi_3 \), it follows that \( \psi_* = \varphi_* \phi_* \) by Lemma 4.6. Therefore, \( P_X \) is projective.

(b) Suppose that \( X_* : 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0 \) is an object in \( \mathcal{E}(\mathcal{C}) \). Since \( \mathcal{C} \) has enough projectives, there exists a short exact sequence \( P_{X_3} : 0 \to \Omega X_3 \xrightarrow{g_1} P \xrightarrow{g_2} X_3 \to 0 \) in \( \mathcal{E} \) with \( P \) projective. Thus we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
P_{X_3} & 0 & \longrightarrow & \Omega X_3 & \xrightarrow{g_1} & P & \xrightarrow{g_2} & X_3 & \longrightarrow & 0 \\
\downarrow{\varphi_*} & & \downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi} & & \downarrow{\varphi} & \\
X_* & 0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \longrightarrow & 0
\end{array}
\]

Consequently, \( \varphi_* : P_{X_3} \to X_* \) is an epimorphism where \( P_{X_3} \) is projective by (a). In particular, assume that \( X_* \) is a projective object in \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \), then there is an epimorphism \( \varphi_* : P_{X_3} \to X_* \). Since \( X_3 \) is projective, \( \varphi_* \) is split. Thus each projective object of \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \) is of the form \( P_X \) for some object \( X \) in \( \mathcal{C} \). \( \square \)

**Corollary 4.12.** Let \((\mathcal{C}, \mathcal{E})\) be a Frobenius category, then \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \) is a Frobenius abelian category.

**Remark 4.13.** Assume that \((\mathcal{C}, \mathcal{E})\) is an exact category with enough projectives. If \( \mathcal{C} \) admits an additive generator \( M \), then \( P_M \) is a projective generator for \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \). Therefore, \( \mathcal{E}(\mathcal{C})/[{SE}(\mathcal{C})] \cong \text{mod-End}P_M \cong \text{mod-}B \), where \( B \) is the stable Auslander algebra of \( \mathcal{C} \). See subsection 3.4 for more details.

Recall that given a short exact sequence \( \delta : 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0 \) in \( \mathcal{E} \), we define the contravariant defect \( \delta^* \) and the covariant defect \( \delta_* \) by the following exact sequence of functors

\[
0 \to \mathcal{C}(\mathcal{C}, X_1) \xrightarrow{\mathcal{C}(\cdot, f_1)} \mathcal{C}(\mathcal{C}, X_2) \xrightarrow{\mathcal{C}(\cdot, f_2)} \mathcal{C}(\mathcal{C}, X_3) \to \delta^* \to 0,
\]

\[
0 \to \mathcal{C}(X_3, \mathcal{C}) \xrightarrow{\mathcal{C}(f_2, \cdot)} \mathcal{C}(X_2, \mathcal{C}) \xrightarrow{\mathcal{C}(f_1, \cdot)} \mathcal{C}(X_1, \mathcal{C}) \to \delta_* \to 0.
\]

**Example 4.14.** (a) Let \( \delta = P_X : 0 \to \Omega X \to P \to X \to 0 \) with \( P \in \mathcal{P} \). Then \( \delta^* = \mathcal{C}/[\mathcal{P}](\mathcal{C}, X) \) and \( \delta_* = \mathcal{E}_{\mathcal{C}}^1(X, \mathcal{C}) \).
Let $\delta = I_X : 0 \to X \to I \to \Omega^{-1}X \to 0$ with $I \in \mathcal{I}$. Then $\delta^* = \text{Ext}_C^1(-, X)$ and $\delta_* = \mathcal{C}/\mathcal{I}(X, -)$.

**Remark 4.15.** Let $(\mathcal{C}, \mathcal{E})$ be an exact category with enough projectives and injectives.

(a) In Theorem 4.11 the equivalence $\alpha_1 : \mathcal{E}(\mathcal{C})/[S\mathcal{E}(\mathcal{C})] \cong \text{mod-}\mathcal{C}/[\mathcal{P}]$ is given by $\delta \mapsto \delta^*$, and the equivalence $\beta_1 : \mathcal{E}(\mathcal{C})/[S\mathcal{E}(\mathcal{C})] \cong (\text{mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}})^{\text{op}}$ is given by $\delta \mapsto \delta_*$.

(b) In $\text{mod-}\mathcal{C}/[\mathcal{P}]$, each projective object is of the form $\mathcal{C}/[\mathcal{P}](-, X)$, and each injective object is of the form $\text{Ext}_\mathcal{C}^1(-, X)$.

**Proof.** (a) Assume that $\delta : 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0$ is a short exact sequence in $\mathcal{E}$. Recall that $\alpha_1(\delta) = \text{Coker}(\mathcal{C}/[\mathcal{P}](-, f_2))$ and $\delta^* = \text{Coker}(\mathcal{C}(-, f_2))$. Since $\delta^*(\mathcal{P}) = 0$, we can view $\delta^*$ as a finitely presented $\mathcal{C}/[\mathcal{P}]$-module by Proposition 2.4

(b) It follows from Proposition 4.11 and Example 4.14 since $\mathcal{E}(\mathcal{C})/[S\mathcal{E}(\mathcal{C})] \cong \text{mod-}\mathcal{C}/[\mathcal{P}]$. □

**Proposition 4.16.** Let $(\mathcal{C}, \mathcal{E})$ be an exact category with enough projectives and injectives. Then there is a duality

$$\Phi : \text{mod-}\mathcal{C}/[\mathcal{P}] \to \text{mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \delta^* \mapsto \delta_*.$$ 

Moreover, by restrictions, we obtain the following two dualities

$$\Phi : \text{proj-}\mathcal{C}/[\mathcal{P}] \to \text{inj-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \mathcal{C}/[\mathcal{P}](-, X) \mapsto \text{Ext}_\mathcal{C}^1(X, -).$$

$$\Phi : \text{inj-}\mathcal{C}/[\mathcal{P}] \to \text{proj-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \text{Ext}_\mathcal{C}^1(-, X) \mapsto \mathcal{C}/[\mathcal{I}](X, -).$$

**Proof.** It is a direct consequence of Remark 4.15(a) and Example 4.14 □

The following result is implied in Proposition 4.16.

**Theorem 4.17.** (Hilton-Rees Theorem, see [13, 32]) Let $(\mathcal{C}, \mathcal{E})$ be an exact category with enough projectives and injectives.

(a) There is an isomorphism between $\mathcal{C}/[\mathcal{P}](Y, X)$ and the group of natural transformations from $\text{Ext}_\mathcal{C}^1(X, -)$ to $\text{Ext}_\mathcal{C}^1(Y, -)$.

(b) There is an isomorphism between $\mathcal{C}/[\mathcal{I}](X, Y)$ and the group of natural transformations from $\text{Ext}_\mathcal{C}^1(-, X)$ to $\text{Ext}_\mathcal{C}^1(-, Y)$.

The following is a variant of [41 Section 7].

**Proposition 4.18.** Let $(\mathcal{C}, \mathcal{E})$ be an exact category with enough projectives and $F$ be an object in $\text{mod-}\mathcal{C}/[\mathcal{P}]$. Then there exists a short exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\mathcal{E}$, such that the following sequence

$$\cdots \to \mathcal{C}/[\mathcal{P}](-, \Omega^2 X_3) \to \mathcal{C}/[\mathcal{P}](-, \Omega X_1) \to \mathcal{C}/[\mathcal{P}](-, \Omega X_2) \to \mathcal{C}/[\mathcal{P}](-, \Omega X_3) \to \mathcal{C}/[\mathcal{P}](-, X_1) \to \mathcal{C}/[\mathcal{P}](-, X_2) \to \mathcal{C}/[\mathcal{P}](-, X_3) \to F \to 0$$

is a projective resolution of $F$. Moreover, if $(\mathcal{C}, \mathcal{E})$ has enough injectives, then the following sequence

$$0 \to F \to \text{Ext}_\mathcal{C}^1(-, X_1) \to \text{Ext}_\mathcal{C}^1(-, X_2) \to \text{Ext}_\mathcal{C}^1(-, X_3) \to \text{Ext}_\mathcal{C}^2(-, X_1) \to \cdots$$

is an injective resolution of $F$. 

Proof. The existence of short exact sequence \(0 \to X_1 \to X_2 \to X_3 \to 0\) follows from the equivalence \(E(C)/[SE(C)] \cong \text{mod-}C/[P]\). Thus we have an exact sequence \(0 \to C(-, X_1) \to C(-, X_2) \to C(-, X_3) \to F \to 0\).

A direct checking proves that the sequence
\[
C/[P](-, X_1) \to C/[P](-, X_2) \to C/[P](-, X_3) \to F \to 0
\]
is exact. The following commutative diagram
\[
\begin{array}{c}
0 \to \Omega X_3 \to P \to X_3 \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to X_1 \to X_2 \to X_3 \to 0
\end{array}
\]
implies that \(\Omega X_3 \to X_1 \to X_2 \to X_3\) is a left triangle in \(C/[P]\). Since \(C/[P](X, -)\) is a homological functor, the sequence
\[
\cdots \to C/[P](-, \Omega X_2) \to C/[P](-, \Omega X_3) \to C/[P](-, X_1) \to \\
C/[P](-, X_2) \to C/[P](-, X_3)
\]
is exact. The sequences (4.4) and (4.5) together show that sequence (4.2) is a projective resolution of \(F\).

Since the sequence (4.3) is exact, it remains to show that \(\text{Ext}^{i}_C(-, X)\) is injective for \(i \geq 2\). Indeed, since \((C, \mathcal{E})\) has enough injectives, by choosing injective envelopes, we have
\[
\text{Ext}^{i}_C(-, X) \cong \text{Ext}^{i-1}_C(-, \Omega^{-1}X) \cong \cdots \cong \text{Ext}^{1}_C(-, \Omega^{-1+1}X).
\]

4.4. Simple objects and Auslander-Reiten theory. In this subsection, we always assume that \((C, \mathcal{E})\) is an Ext-finite \(k\)-linear exact category, where Ext-finite means that all morphism and extension modules \(C(X, Y)\) and \(\text{Ext}^{i}_C(X, Y)\) have finite length over \(k\).

Recall that a non-split exact sequence \(0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0\) is called an Auslander-Reiten sequence if the following two conditions are satisfied:

(a) If \(g : X_1 \to Y\) is not a section, then \(g\) factors through \(f_1\).
(b) If \(h : Z \to X_3\) is not a retraction, then \(h\) factors through \(f_2\).

We say \(C\) has right (resp. left) Auslander-Reiten sequences if each non-projective (resp. non-injective) object is the ending (resp. starting) term of an Auslander-Reiten sequence. We say \(C\) has Auslander-Reiten sequences if it has both right and left Auslander-Reiten sequences.

Lemma 4.19. Let \(X_\bullet\) be a simple object in \(E(C)/[SE(C)]\), then \(X_\bullet\) is isomorphic to \(X'_\bullet: 0 \to X'_1 \xrightarrow{f'_1} X'_2 \xrightarrow{f'_2} X'_3 \to 0\), where \(X'_1\) and \(X'_3\) are indecomposable.

Proof. Assume that \(X_\bullet\) is of the form \(0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0\) with \(f_1, f_2 \in J_C\). Suppose that \(X_1 = X'_1 \oplus X'_2\), where \(X'_1\) is indecomposable. Then there exist two canonical morphisms \(i : X'_1 \to X_1\) and \(\pi : X_1 \to X'_1\) such that \(\pi i = 1\). Considering the pushout of \(f_1\) and \(\pi\), we have the following commutative diagram
\[
\begin{array}{c}
X_\bullet \\
\downarrow \phi_\bullet \\
X'_\bullet
\end{array}
\begin{array}{c}
0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0 \\
\downarrow \pi \downarrow \phi_2 \\
0 \to X'_1 \xrightarrow{g_1} X'_2 \xrightarrow{g_2} X'_3 \to 0
\end{array}
\]
whose second row belongs to \( \mathcal{E} \). Since \( X_\bullet \) is a simple object, \( \varphi_\bullet \) is either zero or a monomorphism. Noting \( \varphi_\bullet \) is an epimorphism, we claim that \( \varphi_\bullet \) is a monomorphism; thus it is an isomorphism. Otherwise, \( \varphi_\bullet = 0 \), thus there exists a morphism \( p : X_2 \to X'_1 \) such that \( \pi = pf_1 \). The following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X'_1 & \xrightarrow{i} & X' & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X'_1 & \xrightarrow{\pi} & X'_1 & \longrightarrow & 0 & \longrightarrow & 0 \\
\end{array}
\]

implies that \( f_1 \notin J_C \). It is a contradiction. Therefore, \( X_\bullet \) is isomorphic to \( X'_\bullet \). \( \square \)

We compare the following result with [42, Propostion 14] and [6, Proposition 4.1].

**Theorem 4.20.** Let \((\mathcal{C}, \mathcal{E})\) be an Ext-finite \( k \)-linear exact category.

(a) Assume that \( X_\bullet : 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0 \) is a non-split short exact sequence in \( \mathcal{E} \) where \( X_1 \) and \( X_3 \) are indecomposable. Then \( X_\bullet \) is a simple object in \( \mathcal{E}(\mathcal{C})/([\mathcal{E}(\mathcal{C})] \] if and only if \( X_\bullet \) is an Auslander-Reiten sequence in \( \mathcal{C} \).

(b) There is a bijection between the set of isoclasses of simple objects in \( \mathcal{E}(\mathcal{C})/([\mathcal{E}(\mathcal{C})] \) and the set of isoclasses of Auslander-Reiten sequences in \( \mathcal{C} \).

**Proof.** (a) For the “only if” part, suppose that \( \varphi_1 : X_1 \to Y_1 \) is not a section, then we have the following commutative diagram

\[
\begin{array}{ccccccccc}
X_\bullet & \longrightarrow & 0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y_\bullet & \longrightarrow & 0 & \longrightarrow & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & X_3 & \longrightarrow & 0 \\
\end{array}
\]

which is induced by the pushout of \( f_1 \) and \( \varphi_1 \). Since \( f_1 \) and \( \varphi_1 \) are not sections and \( X_1 \) is indecomposable, the morphism \( (f_1, \varphi_1) \) is not a section. Thus \( \varphi_\bullet = 0 \) is not a monomorphism by Lemma 4.7. We infer that \( \varphi_\bullet = 0 \) since \( X_\bullet \) is a simple object. It follow from Lemma 4.6 that \( \varphi_1 \) factors through \( f_1 \). Dually, we can prove that if \( \varphi_3 : Z_3 \to X_3 \) is not a retraction, then \( \varphi_3 \) factors through \( f_2 \). Thus \( X_\bullet \) is an Auslander-Reiten sequence.

For the “ if” part, assume that \( \varphi_\bullet : X_\bullet \to Y_\bullet \) is a morphism in \( \mathcal{E} \). If \( \varphi_1 \) is a section, then \( (\varphi_1, \varphi_\bullet) \) is a section, thus \( \varphi_\bullet \) is a monomorphism. If \( \varphi_1 \) is not a section, then \( \varphi_1 \) factors through \( f_1 \) since \( X_\bullet \) is an Auslander-Reiten sequence, thus \( \varphi_\bullet = 0 \). Therefore, each morphism \( \varphi_\bullet : X_\bullet \to Y_\bullet \) is either a monomorphism or a zero morphism. It means that \( X_\bullet \) is a simple object in \( \mathcal{E}(\mathcal{C})/([\mathcal{E}(\mathcal{C})] \).

(b) It follows from Lemma 4.19 and (a). \( \square \)

From now on to the end of this subsection, we assume that \((\mathcal{C}, \mathcal{E})\) is an exact category with enough projectives and injectives. If \( \mathcal{C} \) is also a dualizing \( k \)-variety, then \( \mathcal{C}/[\mathcal{P}] \) and \( \mathcal{C}/[\mathcal{I}] \) are also dualizing \( k \)-varieties (see Example 2.5(d)). We have
two dualities $\Phi : \text{mod-}C/|P| \to \text{mod-}(C/|I|)\text{op}$ and $D : \text{mod-}(C/|I|)\text{op} \to \text{mod-}C/|I|$. The composition of $\Phi$ and $D$ defines an equivalence

$$\Theta : \text{mod-}C/|P| \xrightarrow{\Phi} \text{mod-}(C/|I|)\text{op} \xrightarrow{D} \text{mod-}C/|I|.$$  

We consider the following restriction

$$\Theta : \text{proj-}C/|P| \xrightarrow{\Phi} \text{inj-}(C/|I|)\text{op} \xrightarrow{D} \text{proj-}C/|I|.$$  

Since the projective object in $\text{mod-}C/|I|$ is of the form $C/|I|(-, Y)$, we have

$$\Theta(C/|P|(-, X)) = D\text{Ext}^1_C(X, -) \cong C/|I|(-, Y)$$  

for some $Y \in C$. Therefore, there is an equivalence $\tau : C/|P| \cong C/|I|$ mapping $X$ to $Y$. The functor $\tau$ induces an equivalence $\tau^{-1} \circ \text{mod-}C/|P| \cong \text{mod-}C/|I|$, $F \mapsto F\tau^{-1}$, such that $D\Phi = \tau^{-1}$. Assume that $\delta$ is a short exact sequence in $\mathcal{E}$, then $D\Phi(\delta^* ) = D\delta_\tau$. On the other hand, $\tau^{-1}(\delta^*) = \delta^* \tau^{-1}$. Hence, we have $D\delta_\tau = \delta^* \tau^{-1}$.

To summarize, we have the following generalized Auslander-Reiten duality and defect formula.

**Theorem 4.21.** Let $(C, \mathcal{E})$ be an Ext-finite exact category with enough projectives and injectives. Assume that $C$ is a dualizing $k$-variety. Then there is an equivalence $\tau : C/|P| \cong C/|I|$ satisfying the following properties:

(a) $D\text{Ext}^1_C(X, -) = C/|P|(-, \tau X)$ and $D\delta_\tau = \delta^* \tau^{-1}$. Then $D\delta^* = \delta_\tau \tau^{-1}$ follows from $D\delta_\tau = \delta^* \tau^{-1}$. If $\delta = I_X$, then $\delta^* = \text{Ext}^1_C(X, -)$ and $\delta_\tau = C/|I|(X, \tau -) \cong C/|P|(-, \tau X)$. Since $D\delta^* = \delta_\tau$, we have $D\text{Ext}^1_C(X, -) \cong C/|P|(-, \tau X)$. The last assertion follows from (a) and [29, Theorem 1.1]; see [31, Theorem 3.6] for the exact version.

**Corollary 4.22.** ([38, Theorem 7.1.3]) Let $C$ be a dualizing $k$-variety, then $\text{mod-}C$ has Auslander-Reiten sequences.

**Proof.** By Example 2.5(b), $\text{mod-}C$ is a dualizing $k$-variety, moreover, it is an abelian category with enough projectives and enough injectives. Thus the consequence follows from Theorem 4.21.

**Remark 4.23.** Let $A$ be an Artin $k$-algebra and $C = \text{mod-}A$. Then $C$ satisfies all the conditions in Theorem 4.21 and the functor $\tau : C/|P| \cong C/|I|$ is given by $DTr$.  

4.5. **Higher version.** In this subsection, we assume that $n$ is an integer greater than or equal to 1 and $(C, \mathcal{E}_n)$ is an $n$-exact category in the sense of Jasso (see [22]). Let $X_\bullet : 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n+1}} X_{n+2} \to 0$ be an $n$-exact sequence in $\mathcal{E}_n$, we say $f_1$ is an admissible monomorphism and $f_{n+1}$ is an admissible epimorphism. We always view $X_\bullet$ as a complex over $C$ concentrated on degree $1, 2, \ldots, n + 2$. We denote by $\mathcal{E}_n(C)$ the category of all $n$-exact sequences in $\mathcal{E}_n$ where the morphisms between two $n$-exact sequences are given by morphisms of complexes, and by $C\mathcal{E}_n(C)$ the full subcategory of $\mathcal{E}_n(C)$ formed by contractible $n$-exact sequences.
Assume that \((\mathcal{C}, \mathcal{E}_n)\) is an \(n\)-exact category. An object \(P \in \mathcal{C}\) is projective if for each admissible epimorphism \(f : X \to Y\) the sequence \(\mathcal{C}(P, X) \to \mathcal{C}(P, Y) \to 0\) is exact. The full subcategory of \(\mathcal{C}\) formed by projectives is denoted by \(\mathcal{P}\). We say \((\mathcal{C}, \mathcal{E}_n)\) has enough projectives if for each object \(X \in \mathcal{C}\) there is an \(n\)-exact sequence \(P_X : 0 \to Y \to P_n \to \cdots \to P_1 \to X \to 0\) in \(\mathcal{E}_n\) with \(P_i \in \mathcal{P}\). We denote by \(P\mathcal{E}_n(\mathcal{C})\) the full subcategory of \(\mathcal{E}_n(\mathcal{C})\) formed by \(P_X\) and contractible \(n\)-exact sequences \(X_\bullet\) with \(f_{n+1} = 1\). Dually, we define injective objects and \((\mathcal{C}, \mathcal{E}_n)\) has enough injectives. The full subcategory of \(\mathcal{C}\) formed by injectives is denoted by \(\mathcal{I}\). We denote by \(I\mathcal{E}_n(\mathcal{C})\) the full subcategory of \(\mathcal{E}_n(\mathcal{C})\) formed by \(I_X : 0 \to X \to I_1 \to \cdots \to I_n \to Y \to 0\) with \(I_i \in \mathcal{I}\) and contractible \(n\)-exact sequences \(X_\bullet\) with \(f_1 = 1\).

With notations as above, we have the following higher version of Theorem 4.11.

**Theorem 4.24.** Let \((\mathcal{C}, \mathcal{E}_n)\) be an \(n\)-exact category.

(a) If \((\mathcal{C}, \mathcal{E}_n)\) has enough projectives, then we have the following equivalences:

\[
\mathcal{E}_n(\mathcal{C})/[C\mathcal{E}_n(\mathcal{C})] \cong \text{mod-}\mathcal{C}/[\mathcal{P}], \mathcal{E}_n(\mathcal{C})/[P\mathcal{E}_n(\mathcal{C})] \cong (\text{mod-}(\mathcal{C}/[\mathcal{P}])^{\text{op}})^{\text{op}}.
\]

(b) If \((\mathcal{C}, \mathcal{E}_n)\) has enough injectives, then we have the following equivalences:

\[
\mathcal{E}_n(\mathcal{C})/[C\mathcal{E}_n(\mathcal{C})] \cong (\text{mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}})^{\text{op}}, \mathcal{E}_n(\mathcal{C})/[I\mathcal{E}_n(\mathcal{C})] \cong \text{mod-}\mathcal{C}/[\mathcal{I}].
\]

**Lemma 4.25.** ([22] Proposition 4.8) Let \((\mathcal{C}, \mathcal{E}_n)\) be an \(n\)-exact category. If the sequence

\[
0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to 0
\]

is an \(n\)-exact sequence in \(\mathcal{E}_n\), then the following statements are equivalent.

(a) The diagram

\[
\begin{array}{ccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_{n+1} \\
\downarrow{\phi_1} & & \downarrow{\phi_2} & & \cdots & & \downarrow{\phi_n} & & \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & Y_{n+1}
\end{array}
\]

is an \(n\)-pushout and \(n\)-pullback diagram.

(b) The sequence

\[
\cdots \to X_{n+1} \oplus Y_n \xrightarrow{(\phi_{n+1}, g_n)} Y_{n+1} \to 0
\]

is an \(n\)-exact sequence in \(\mathcal{E}_n\).

(c) There exists a commutative diagram

\[
\begin{array}{ccccccccccc}
0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_{n+1} & \xrightarrow{f_{n+1}} & X_{n+2} & 0 \\
\downarrow{\phi_1} & & \downarrow{\phi_2} & & \cdots & & \downarrow{\phi_n} & & \downarrow{1} & & \downarrow{1} & & \\
0 & \xrightarrow{g_1} & Y_1 & \xrightarrow{g_2} & Y_2 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & Y_{n+1} & \xrightarrow{g_{n+1}} & X_{n+2} & 0
\end{array}
\]

such that the second row is an \(n\)-exact sequence in \(\mathcal{E}_n\).

The following lemma is a higher version of Lemma 4.5.
Lemma 4.26. Assume that the following diagram

\[
\begin{array}{c}
X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \xrightarrow{0} \\
Y_0 \xrightarrow{\psi_0} Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots \xrightarrow{g_n} Y_{n+1} \xrightarrow{g_{n+1}} Y_{n+2} \xrightarrow{0}
\end{array}
\]

is commutative with rows in \(E_n\). Then we get the following commutative diagram

\[
\begin{array}{c}
K(\varphi_0) \xrightarrow{k_0} K(\varphi_0) \xrightarrow{\phi_0} X_1 \oplus Y_1 \xrightarrow{(f_1, 0)} X_2 \oplus Y_2 \xrightarrow{(f_2, 0)} \cdots \xrightarrow{(f_n, 0)} X_{n+1} \oplus Y_{n+1} \xrightarrow{(f_{n+1}, 0)} X_{n+2} \oplus Y_{n+2} \xrightarrow{0} \\
I(\varphi_0) \xrightarrow{i_0} I(\varphi_0) \xrightarrow{\psi_0} X_1 \xrightarrow{(0, 0)} X_2 \xrightarrow{(0, 0)} \cdots \xrightarrow{(0, 0)} X_{n+1} \xrightarrow{(0, 0)} X_{n+2} \xrightarrow{0} \\
C(\varphi_0) \xrightarrow{c_0} C(\varphi_0) \xrightarrow{\chi_0} Z_1 \xrightarrow{(h_1, 0)} Z_2 \xrightarrow{(h_2, 0)} \cdots \xrightarrow{(h_n, 0)} Z_{n+1} \xrightarrow{(h_{n+1}, 0)} Z_{n+2} \xrightarrow{0}
\end{array}
\]

with rows in \(E_n\), moreover, \(\varphi_0 = i_0 \pi_0 \) in \(E_n(\mathcal{C})/([E_n(\mathcal{C})])\).

Proof. By [22 Proposition 4.9], there exist two morphisms \(\pi : X_0 \to I(\varphi_0)\) and \(i : I(\varphi_0) \to Y_0\) such that \(I(\varphi_0) \in E_n\) and \(\varphi_0 = i_0 \pi_0 \) in \(E_n(\mathcal{C})/([E_n(\mathcal{C})])\). The sequences \(K(\varphi_0)\) and \(C(\varphi_0)\) are \(n\)-exact sequences in \(E_n\) by Lemma 4.25.

\[\square\]

Theorem 4.27. Let \((\mathcal{C}, E_n)\) be an \(n\)-exact category. Then the category \(E_n(\mathcal{C})/([E_n(\mathcal{C})])\) is an abelian category whose kernels and cokernels are given by \(n\)-pullback and \(n\)-pushout diagrams.

Proof. The proof is similar to that of Theorem 4.28. \[\square\]

Definition 4.28. (see [20]) An \(n\)-exact sequence \(0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n+1}} X_{n+2} \to 0\) is called an \(n\)-Auslander-Reiten sequence if the following conditions are satisfied:

(a) All \(f_i \in J_\mathcal{C}\).
(b) If \(g : X_1 \to Y\) is not a section, then \(g\) factors through \(f_1\).
(c) If \(h : Z \to X_{n+2}\) is not a retraction, then \(h\) factors through \(f_{n+1}\).

The following is a higher analogue of Theorem 4.29.

Theorem 4.29. Let \((\mathcal{C}, E_n)\) be an Ext-finite \(k\)-linear \(n\)-exact category.

(a) Assume that \(X_\bullet : 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n+1}} X_{n+2} \to 0\) is an \(n\)-exact sequence in \(E_n\), where \(f_i \in J_\mathcal{C}, X_1\) and \(X_{n+2}\) are indecomposable. Then \(X_\bullet\) is simple in \(E_n(\mathcal{C})/([E_n(\mathcal{C})])\) if and only if \(X_\bullet\) is an \(n\)-Auslander-Reiten sequence in \(\mathcal{C}\).

(b) There is a bijection between the set of isoclasses of simple objects in \(E_n(\mathcal{C})/([E_n(\mathcal{C})])\) and the set of isoclasses of \(n\)-Auslander-Reiten sequences in \(\mathcal{C}\).
**Proposition 4.30.** Let $(C, E_n)$ be an $n$-exact category.

(a) Each $n$-exact sequence $P_X : 0 \to Y \xrightarrow{f_1} P_n \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} P_2 \xrightarrow{f_n} P_1 \xrightarrow{f_{n+1}} X \to 0$ in $E_n$ with $P_i$ projective is a projective object in $E_n(C)/[C E_n(C)]$.

(b) If $(C, E_n)$ has enough projectives, then each projective object in $E_n(C)/[C E_n(C)]$ is of the form $P_X$ for some object $X$ in $C$. In this case, $E_n(C)/[C E_n(C)]$ has enough projectives.

The following is a combination of [26 Theorem 1.3] and [19 Lemma 3.5].

**Lemma 4.31.** Let $C$ be an $n$-abelian category with enough projectives.

(a) There exists an abelian category $\mathcal{A}$ with enough projectives such that $C$ is an $n$-cluster-tilting subcategory of $\mathcal{A}$, moreover, the class of projectives in $C$ and the class of projectives in $\mathcal{A}$ coincide.

(b) For each $n$-exact sequence

$$0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to 0,$$

there exist two long exact sequences

1. $0 \to \text{Ext}_A^n(-, X_1) \to \text{Ext}_A^n(-, X_2) \to \cdots \to \text{Ext}_A^n(-, X_{n+2}) \to \text{Ext}_A^n(X, -) \to \cdots$

2. $0 \to \text{Ext}_A^n(-, X_1) \to \text{Ext}_A^n(-, X_2) \to \cdots \to \text{Ext}_A^n(-, X_{n+2}) \to \text{Ext}_A^n(X, -) \to \cdots$

From now on to the end of this section, we assume that $C$ is an $n$-abelian category with enough projectives and injectives. The full subcategory of $C$ consisting of projectives (resp. injectives) is denoted by $P$ (resp. $I$). By Lemma 4.31 we always view $C$ as an $n$-cluster tilting subcategory of an abelian category $\mathcal{A}$. For convenience, we denote $\text{Ext}^n_A(-, X)|_C$ by $\text{Ext}^n_C(-, X)$ and denote $\text{Ext}_A^n(X, -)|_C$ by $\text{Ext}_C^n(X, -)$ for short.

**Definition 4.32.** ([23 Definition 3.1]) Given an $n$-exact sequence

$$\delta : 0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to 0,$$

we define the **contravariant defect** $\delta^*$ and the **covariant defect** $\delta_*$ by the following exact sequence of functors

$$0 \to C(-, X_1) \xrightarrow{c(-, f_1)} C(-, X_2) \xrightarrow{c(-, f_2)} \cdots \xrightarrow{c(-, f_{n+1})} C(-, X_{n+2}) \to \delta^* \to 0,$$

$$0 \to C(X_{n+2}, -) \xrightarrow{c(f_{n+1}, -)} \cdots \xrightarrow{c(f_2, -)} C(X_2, -) \xrightarrow{c(f_1, -)} C(X_1, -) \to \delta_* \to 0.$$

**Example 4.33.** (a) Let $\delta = P_X : 0 \to Y \xrightarrow{f_1} P_n \xrightarrow{f_2} \cdots \xrightarrow{f_n} P_1 \xrightarrow{f_{n+1}} X \to 0$ with $P_i \in P$, then $\delta^* = C/[P](-, X)$ and $\delta_* = \text{Ext}^n_C(X, -)$.

(b) Let $\delta = I_X : 0 \to X \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} I_n \xrightarrow{f_{n+1}} Y \to 0$ with $I_i \in I$, then $\delta^* = \text{Ext}^n_C(-, X)$ and $\delta_* = C/[I](X, -)$.

**Remark 4.34.** (a) In Theorem 4.24 the equivalence $\alpha : E_n(C)/[C E_n(C)] \cong \text{mod-}C/[P]$ is given by $\delta \mapsto \delta^*$, and the equivalence $\beta : E_n(C)/[C E_n(C)] \cong (\text{mod-}(C/[I]))^{op}$ is given by $\delta \mapsto \delta_*$. (b) In $\text{mod-}C/[P]$, each projective object is of the form $C/[P](-, X)$, and each injective object is of the form $\text{Ext}^n_C(-, X)$. 
Proposition 4.35. Let $\mathcal{C}$ be an $n$-abelian category with enough projectives and injectives. Then there is a duality
\[ \Phi : \text{mod-}\mathcal{C}/[\mathcal{P}] \rightarrow \text{mod-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \delta^* \mapsto \delta_*. \]
Moreover, by restrictions, we obtain the following two dualities
\[ \Phi : \text{proj-}\mathcal{C}/[\mathcal{P}] \rightarrow \text{inj-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \mathcal{C}/[\mathcal{P}](\cdot, X) \rightarrow \text{Ext}^n(X, \cdot)_\mathcal{C}, \]
\[ \Phi : \text{inj-}\mathcal{C}/[\mathcal{P}] \rightarrow \text{proj-}(\mathcal{C}/[\mathcal{I}])^{\text{op}}, \quad \text{Ext}^n(\cdot, X)_\mathcal{C} \rightarrow \mathcal{C}/[\mathcal{I}](\cdot, \cdot). \]

Theorem 4.36. (Higher Hilton-Rees Theorem) Let $\mathcal{C}$ be an $n$-abelian category with enough projectives and injectives.

(a) There is an isomorphism between $\mathcal{C}/[\mathcal{P}](Y, X)$ and the group of natural transformations from $\text{Ext}^n(\cdot, X)$ to $\text{Ext}^n_\mathcal{C}(\cdot, Y)$.

(b) There is an isomorphism between $\mathcal{C}/[\mathcal{I}](X, Y)$ and the group of natural transformations from $\text{Ext}^n(\cdot, X)$ to $\text{Ext}^n_\mathcal{C}(\cdot, Y)$.

Theorem 4.37. Let $\mathcal{C}$ be an $n$-abelian category with enough projectives and injectives. Assume that $\mathcal{C}$ is a dualizing $k$-variety. Then there is an equivalence $\tau_n : \mathcal{C}/[\mathcal{P}] \cong \mathcal{C}/[\mathcal{I}]$ satisfying the following properties:

(a) $\text{DExt}^n_\mathcal{C}(\cdot, X) = \mathcal{C}/[\mathcal{P}](\tau_{n-1}^{-1} X, \cdot)$, $\text{DExt}^n_\mathcal{C}(X, \cdot) = \mathcal{C}/[\mathcal{I}](\cdot, \tau_n X)$.

(b) $D\delta_1 = \delta_2^* \tau_{n-1}$, $D\delta_2 = \delta_3^* \tau_n$ for each $n$-exact sequence $\delta$.

Remark 4.38. Let $A$ be an Artin $k$-algebra and $\mathcal{C}$ be an $n$-cluster-tilting subcategory of $\text{mod-}A$. Then $\mathcal{C}$ contains proj-$A$ and inj-$A$, moreover, $\mathcal{C}$ is a dualizing $k$-variety since it is functorially finite. Indeed, the functor $\tau_n : \mathcal{C}/[\mathcal{P}] \cong \mathcal{C}/[\mathcal{I}]$ in Theorem 4.37 is given by $\text{DT}_r \Omega^{n-1}$.

5. Abeian Quotients of the Categories of Triangles

Let $\mathcal{C}$ be a triangulated category with the suspension functor $\Sigma$. We denote by $\Delta(\mathcal{C})$ the category of triangles in $\mathcal{C}$, where the objects are the triangles $X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1)$ and the morphisms from $X_\bullet$ to $Y_\bullet$ are the triples $\varphi_\bullet = (\varphi_1, \varphi_2, \varphi_3)$ such that the following diagram is commutative:

\[
\begin{array}{cccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} \Sigma X_1 \\
| & \varphi_1 & | & \varphi_2 & | & \varphi_3 \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} \Sigma Y_1 \\
\end{array}
\]

Let $X_\bullet$ and $Y_\bullet$ be two triangles, we denote $\mathcal{R}_2(X_\bullet, Y_\bullet)$ (resp. $\mathcal{R}_2'(X_\bullet, Y_\bullet)$) the class of morphisms $\varphi_\bullet : X_\bullet \rightarrow Y_\bullet$ such that there is a morphism $p : X_3 \rightarrow Y_2$ such that $g_2 p = \varphi_3$ (resp. $p f_2 = \varphi_2$). It is easy to see that $\mathcal{R}_2$ and $\mathcal{R}_2'$ are ideals of $\Delta(\mathcal{C})$.

The first part of the following result is implied in [35].

Theorem 5.1. Let $\mathcal{C}$ be a triangulated category, then we have the following two equivalences.

(a) $\Delta(\mathcal{C})/\mathcal{R}_2 \cong \text{mod-}\mathcal{C}$.

(b) $\Delta(\mathcal{C})/\mathcal{R}_2' \cong (\text{mod-}\mathcal{C})^{\text{op}}$.

Proof. Define a functor $\alpha : \Delta(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$ by taking a triangle $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1$ to $f_2 : X_2 \rightarrow X_3$. It is routine to check that $\Delta(\mathcal{C})/\mathcal{R}_2 \cong \text{Mor}(\mathcal{C})/\mathcal{R}$ and $\Delta(\mathcal{C})/\mathcal{R}_2' \cong \text{Mor}(\mathcal{C})/\mathcal{R}'$. Then the result follows from Lemma 3.4. \qed
Let \((C, \mathcal{E})\) be a Frobenius category. We denote by \(\mathcal{P}\) the full subcategory of \(C\) formed by projectives. It is well known that the quotient category \(C/|\mathcal{P}|\) is a triangulated category. The following corollary follows from Theorem 4.1 and Theorem 5.1.

**Corollary 5.2.** Let \((C, \mathcal{E})\) be a Frobenius category, then the categories \(\mathcal{E}(C)/|SE(C)|\), \(\Delta(C)/|\mathcal{P}|\)/\(\mathcal{R}_2\) and \(\text{mod-}C/|\mathcal{P}|\) are equivalent.

**Remark 5.3.** Let \(C\) be a triangulated category. Assume that \(X_1\) and \(Y_1\) are two triangles. We denote by \(\mathcal{R}_1(X_1, Y_1)\) (resp. \(\mathcal{R}_3(X_1, Y_1)\)) the class of morphisms \(\varphi_1 : X_1 \to Y_1\) such that there is a morphism \(p : X_2 \to Y_1\) (resp. \(p : \Sigma X_1 \to Y_3\)) such that \(g_1p = \varphi_2\) (resp. \(g_3p = \Sigma \varphi_1\)). Then \(\mathcal{R}_1\) and \(\mathcal{R}_3\) are ideals of \(\Delta(C)\). Moreover, we have equivalences \(\Delta(C)/\mathcal{R}_1 \cong \Delta(C)/\mathcal{R}_2 \cong \Delta(C)/\mathcal{R}_3\), which are given by rotations. We can see Remark 4.3 for comparison.

Similarly, we have equivalences \(\Delta(C)/\mathcal{R}_1' \cong \Delta(C)/\mathcal{R}_2' \cong \Delta(C)/\mathcal{R}_3'\), where \(\mathcal{R}_1'(X_1, Y_1)\) (resp. \(\mathcal{R}_3'(X_1, Y_1)\)) is the class of morphisms \(\varphi_1' : X_1 \to Y_1\) such that there is a morphism \(p : X_2 \to Y_1\) (resp. \(p : \Sigma X_1 \to Y_3\)) such that \(pf_1 = \varphi_1\) (resp. \(pf_3 = \varphi_3\)).

From now on, we assume that \(C\) is a triangulated category. We will give some basic properties on the abelian category \(\Delta(C)/\mathcal{R}_2\).

**Proposition 5.4.** Let \(\varphi_1 : X_1 \to Y_1\) be a morphism in \(\Delta(C)\). Then we have

(a) The following statements are equivalence:
   (i) \(\varphi_1 = 0\) in \(\Delta(C)/\mathcal{R}_2\).
   (ii) \(\varphi_3\) factors through \(g_2\).
   (iii) \(g_3\varphi_3 = 0\).
   (iv) \((\Sigma \varphi_1)f_3 = 0\).
   (v) \(\varphi_1\) factors through \(f_1\).
   (vi) \(\varphi_1 = 0\) in \(\Delta(C)/\mathcal{R}_1\).

(b) The zero objects in \(\Delta(C)/\mathcal{R}_2\) are of the form \((X \xrightarrow{f_1} X \to 0 \to \Sigma X) \oplus (0 \to Y \xrightarrow{\varphi_1} Y \to 0)\).

(c) \(\varphi_1\) is a monomorphism in \(\Delta(C)/\mathcal{R}_2\) if and only if \((f_1, \varphi_1)\) is a section.

**Proof.** (a) It is clear.

(b) Assume that \(X_1 : X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1\) is a zero object in \(\Delta(C)/\mathcal{R}_2\). Then \(\text{Id}_{X_1} = 0\), thus \(f_3 = 0\) by (a). Therefore, \(X_1\) is isomorphic to \((X_1 \xrightarrow{f_1} X_1 \to 0 \to \Sigma X_1) \oplus (0 \to X_3 \xrightarrow{\varphi_1} X_3 \to 0)\).

(c) The proof is similar to that of Lemma 4.7.

**Remark 5.5.** (a) Denote by \(\mathcal{U}\) the full subcategory of \(\Delta(C)\) formed by \((X \xrightarrow{f_1} X \to 0 \to \Sigma X) \oplus (0 \to Y \xrightarrow{\varphi_1} Y \to 0)\). By Proposition 5.4(b) there is a dense functor \(\beta : \Delta(C)/|\mathcal{U}| \to \Delta(C)/\mathcal{R}_2\). But we point out that \(\beta\) is not an equivalence in general, because a morphism \(\varphi_1 : X_1 \to Y_1\) in \(\Delta(C)\) such that \(\varphi_1 = 0\) in \(\Delta(C)/\mathcal{R}_2\) does not imply that \(\varphi_1\) factors through some object in \(\mathcal{U}\).

(b) Assume that there is a commutative diagram

\[
\begin{array}{c}
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1 \\
\downarrow \varphi_1 \downarrow \varphi_2 \downarrow \Sigma \varphi_1 \\
Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \xrightarrow{g_3} \Sigma Y_1
\end{array}
\]
whose rows are triangles. It is well known that there is a morphism $\varphi_3 : X_3 \to Y_3$ such that the above diagram is commutative. But the morphism $\varphi_3$ is not unique in general. Assume that $\varphi'_3 : X_3 \to Y_3$ is another morphism satisfying required condition. Set $\varphi_\bullet = (\varphi_1, \varphi_2, \varphi_3)$ and $\varphi'_\bullet = (\varphi_1, \varphi_2, \varphi'_3)$. Then we have $\varphi_\bullet = \varphi'_\bullet$ in $\Delta(\mathcal{C})/\mathcal{R}_2$ by Proposition 5.4(a).

Recall that a commutative diagram
\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X_2 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} \\
Y_1 & \xrightarrow{g_1} & Y_2
\end{array}
\]
is called a homotopy cartesian if
\[
\begin{array}{ccc}
X_1 & \xrightarrow{(f_1, \varphi_1)} & X_2 \oplus Y_1 \\
\downarrow{\varphi_2} & & \downarrow{\varphi_3} \\
Y_2 & \xrightarrow{\delta} & \Sigma X_1
\end{array}
\]
is a triangle, where $\delta$ is called a differential.

The following result is well known, for example, see [25, Appendix A].

**Lemma 5.6.** Let $(\mathcal{C}, \Delta, \Sigma)$ be a pre-triangulated category. Then $\Delta$ satisfies axiom (TR4) if and only if for each commutative diagram
\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \Sigma X_1 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi_3} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \Sigma X_1
\end{array}
\]
with rows in $\Delta$, there exists a morphism $\varphi_3 : X_3 \to Y_3$ such that the whole diagram is commutative and the following diagram
\[
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & X_3 \\
\downarrow{\varphi_2} & & \downarrow{\varphi_3} \\
Y_2 & \xrightarrow{g_2} & Y_3
\end{array}
\]
is a homotopy cartesian.

**Lemma 5.7.** Assume that the following
\[
\begin{array}{ccc}
X_\bullet & \xrightarrow{X_\bullet} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \Sigma X_1 \\
\downarrow{\varphi_\bullet} & & \downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi_3} & & \downarrow{\Sigma \varphi_1} \\
Y_\bullet & \xrightarrow{Y_\bullet} & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \Sigma Y_1
\end{array}
\]
is a morphism of triangles. Then we have the following commutative diagram

\[
\begin{array}{c}
K(\varphi^*) & X_1 \xrightarrow{(f_1, \varphi_1)} X_2 \oplus Y_1 \xrightarrow{f_2 - h_1} Z \xrightarrow{f_3} \Sigma X_1 \\
& h_* & & h_2 & & h_2 \\
I(\varphi^*) & Y_1 \xrightarrow{a_1} Z \xrightarrow{a_2} X_3 \xrightarrow{h_3} \Sigma Y_1 \\
& \phi & & \phi & & \phi \\
C(\varphi^*) & Z \xrightarrow{(a_2, h_3)} X_3 \oplus Y_2 \xrightarrow{(\varphi_3, g_2)} Y_3 \xrightarrow{g_3} \Sigma Z
\end{array}
\]

such that each row is a triangle and \( \varphi^* = i_\ast \pi_* \) in \( \Delta(C)/R_2 \).

**Proof.** We extend the morphism \( h_3 = \Sigma \varphi_1 \cdot f_3 : X_3 \to \Sigma Y_1 \) to a triangle \( I(\varphi^*) \).

By Lemma 5.7 and its dual, we choose two morphisms \( a_1 : X_2 \to Z \) and \( a_2 : Z \to Y_2 \) such that the associated squares are commutative, and \( K(\varphi^*) \) and \( C(\varphi^*) \) are triangles. We have \( \varphi^* = i_\ast \pi_* \) by Proposition 5.4(a). \( \square \)

**Theorem 5.8.** Let \( C \) be a triangulated category. Then the category \( \Delta(C)/R_2 \) is an abelian category where the kernels and cokernels are given by homotopy cartesian diagrams.

**Proof.** Given a morphism \( \varphi^* : X^* \to Y^* \) in \( \Delta(C) \), as notations in Lemma 5.7 we can show that \( K(\varphi^*) \) is a kernel of \( \varphi^* \), \( C(\varphi^*) \) is a cokernel of \( \varphi^* \) and \( I(\varphi^*) \) is the image of \( \varphi^* \). \( \square \)

The following result is a triangulated analogue of Theorem 4.8.

**Theorem 5.9.** Let \( C \) be a Hom-finite Krull-Smidt \( k \)-linear triangulated category.

(a) Assume that \( X^* : X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1 \) is a triangle such that \( f_3 \neq 0 \). Then \( X^* \) is a simple object in \( \Delta(C)/R_2 \) if and only if \( X^* \) is an Auslander-Reiten triangle in \( C \).

(b) There is a bijection between the set of isoclasses of simple objects in \( \Delta(C)/R_2 \) and the set of isoclasses of Auslander-Reiten triangles in \( C \).

Let \( \delta : X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1 \) be a triangle. The contravariant defect \( \delta^* \) and the covariant defect \( \delta_* \) are defined by the following exact sequence of functors

\[
C(-, X_1) \xrightarrow{C(-, f_1)} C(-, X_2) \xrightarrow{C(-, f_2)} C(-, X_3) \to \delta^* \to 0,
\]

\[
C(X_3, -) \xrightarrow{C(f_2, -)} C(X_2, -) \xrightarrow{C(f_1, -)} C(X_1, -) \to \delta_* \to 0.
\]

**Example 5.10.** Let \( \delta = P_X : \Sigma^{-1}X \to 0 \to X \xrightarrow{1} X \), then \( \delta^* = C(-, X) \) and \( \delta_* = C(\Sigma^{-1}X, -) \).
Remark 5.11. Let $C$ be a triangulated category. Then the equivalence $\Delta(C)/R_2 \cong \text{mod-}C$ is given by $\delta \mapsto \delta^*$ and the equivalence $\Delta(C)/R_1 \cong (\text{mod-}C)^{\text{op}}$ is given by $\delta \mapsto \delta_*$. Since $\Delta(C)/R_2 = \Delta(C)/R_1$ by Proposition 5.4(a), we have a duality
\[
\phi : \text{mod-}C \rightarrow (\text{mod-}C)^{\text{op}}, \delta^* \mapsto \delta_*.
\]
By restriction, we have two dualities
\[
\phi : \text{proj-}C \rightarrow \text{inj-}C^{\text{op}}, C(-,X) \mapsto C(\Sigma^{-1}X,-).
\]
\[
\phi : \text{inj-}C \rightarrow \text{proj-}C^{\text{op}}, C(-,\Sigma X) \mapsto C(X,-).
\]
Therefore, $\text{mod-}C$ is a Frobenius abelian category. So is $\Delta(C)/R_2$. Moreover, each projective-injective object in $\Delta(C)/R_2$ is of the form $X \rightarrow 0 \rightarrow \Sigma X \xrightarrow{1} \Sigma X$.

Theorem 5.12. Let $C$ be a Hom-finite Krull-Smidt $k$-linear triangulated category. Assume that $C$ is a dualizing $k$-variety. Then there is an equivalence $\tau : C \cong C$ such that $D\Sigma(-,X,\tau \cdot X)$ for each $X \in C$ and $D\delta^* = \delta_* \tau$ for each triangle $\delta$.

Proof. The composition of $\phi : \text{mod-}C \rightarrow (\text{mod-}C)^{\text{op}}$ and $D : (\text{mod-}C)^{\text{op}} \rightarrow \text{mod-}C$ is an equivalence $\theta = D\phi : \text{mod-}C \cong \text{mod-}C$. By restriction, we have an equivalence $\theta : \text{proj-}C \cong \text{proj-}C$. Since $\theta(C(-,X)) = D\Sigma^{-1}X, \Sigma$ for some $Y \in C$, there is an equivalence $\tau : C \cong C$ mapping $X$ to $Y$. In this case, $D\Sigma^{-1}X, \Sigma$ for each triangle $\delta$. Since $\tau$ induces an equivalence $\tau^{-1} : \text{mod-}C \cong \text{mod-}C$ such that $\theta = \tau^{-1}$, we have $D\delta = D\phi(\delta^*) = \tau^{-1}(\delta^*) = \delta^*\tau^{-1}$ for each triangle $\delta$. Thus $D\delta^* = \delta_* \tau$. □

Remark 5.13. In Theorem 5.12 if we set $F = \tau \Sigma : C \cong C$, then $D\Sigma(-,\tau \cdot X)$ for each $X \in C$. Thus the functor $F$ is known as a Serre functor.

Remark 5.14. Given an $n$-angled category $C$ in the sense of Geiss-Keller-Oppermann (see [16]), one can consider the quotient of the category of $n$-angles and obtain some similar results. We leave them to the readers.

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