Sending quantum information with Gaussian states

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1 Information characteristics of a quantum channel

During the last couple of years an impressive progress has been achieved in the theory of transmission of classical information through quantum communication channels (see [8] for a comprehensive survey). The problem of sending quantum information is much less understood; we refer in particular to the papers [10], [2], [1], initiating the study of this problem, where the reader can find further references. In this paper we make a contribution to this study by considering rather concrete situation: sending Gaussian (quasifree) states through linear Bosonic channels.

Consider quantum system in a Hilbert space $\mathcal{H}$, with a fixed density operator $\rho$. A channel is a transformation of quantum states as presented by density operators, given by the relation

$$ T[\rho] = \sum_j A_j \rho A_j^*; $$

where $A_j$ are bounded operators in $\mathcal{H}$ satisfying $\sum_j A_j^* A_j = I$. Let us denote $H(\rho) = -\text{Tr} \rho \log \rho$ the von Neumann entropy of a density operator $\rho$. We call $\rho$ the input state, and $T[\rho]$ the output state of the channel. There are three important entropy quantities related to the couple $(\rho, T)$:

1) The entropy of the input state $H(\rho)$;
2) The entropy of the output state $H(T[\rho])$;
3) The entropy exchange $H(\rho, T)$. 

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While the definition and the meaning of the first two entropies is obvious, the third quantity is somewhat more sophisticated. To define it one introduces the reference system, described by the Hilbert space $\mathcal{H}_R$, isomorphic to the Hilbert space $\mathcal{H}_Q = \mathcal{H}$ of the initial system. Then according to [10], [2], there exists purification of the state $\rho$, i.e. a unit vector $|\psi\rangle \in \mathcal{H}_Q \otimes \mathcal{H}_R$ such that

$$\rho = \text{Tr}_R |\psi\rangle \langle \psi|.$$  

The entropy exchange is then defined as

$$H(\rho, T) = H((T \otimes \text{Id})[|\psi\rangle \langle \psi|]),$$

that is as the entropy of the output state of the dilated channel $(T \otimes \text{Id})$ applied to the input which is purification of the input state $\rho$. One can then show that $H(\rho, T)$ is equal to the entropy increase in the channel environment $E$ provided the channel is represented by a unitary interaction with the environment system being initially in a pure state [10], [2].

From these three entropies one can construct three other quantities, bearing some analogy with the classical mutual information. In general, if $\rho_{12}$ is a density operator in a tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, and $\rho_1, \rho_2$ are the partial states of $\rho_{12}$, resp., in $\mathcal{H}_1, \mathcal{H}_2$, then one can introduce the quantity

$$C_{12} = H(\rho_1) + H(\rho_2) - H(\rho_{12}).$$

This quantity is nonnegative by subadditivity of quantum entropy, and to certain extent reflects information correlation between the two systems [10], although its operational meaning is still not completely clarified. Then from the above three entropies one can construct three quantum information quantities [10], [1]:

1) Quantum mutual information

$$I = H(\rho) + H(T[\rho]) - H(\rho, T),$$

reflecting quantum information transfer from the reference system $R$ to the output of the initial system $Q'$. The important component of it is the coherent information $H(T[\rho]) - H(\rho, T)$, supremum of which with respect to input states $\rho$ was conjectured as the quantum capacity of the channel $T$ [2].
2) **Loss**

\[ L = H(\rho) + H(\rho, T) - H(T[\rho]), \]

which reflects quantum information transfer from the reference system \( R \) to the output of the environment \( E' \).

3) **Noise**

\[ N = H(T[\rho]) + H(\rho, T) - H(\rho), \]

which reflects quantum information transfer from the output of the environment \( E' \) to the output of the initial system \( Q' \). In [1] quantum Wenn’s diagrams were introduced to visualize the relations between the entropy and the information quantities. However, in contrast to classical case, some areas in these diagrams representing conditional entropies may have negative measure. We use another graphic representation via the *information triangle*. In this representation the entropies \( H(\rho), H(T[\rho]), H(\rho, T) \) are associated with the sides of the triangle, and the information quantities \( I, L, N \) are attached to its vertices. The deficiency of this picture is that the representation of the information quantities is only qualitative: roughly, the bigger is the quantity - the bigger is distance from the corresponding vertex to the opposite side of the triangle, and vice versa.

Although the entropy and information quantities described above were studied in some detail from the general point of view, they are far from being completely understood, and concrete examples in which they can be explicitly evaluated are certainly welcome. In quantum statistics there is one large class of states for which many explicit calculations are possible – the so called quasifree states of canonical commutation relations, in many respect analogous to the classical Gaussian probability distributions. They are the states of the maximal entropy among all states with fixed second moments, for example, mean energy for a quadratic Hamiltonian. The aim of the present paper is the study the behavior of the information triangle for Gaussian input state and the most common attenuation/amplification channel.

## 2 Quantum Gaussian states

In this Section we repeat some results of [3], [7], [9] and give a new variant of the expression for the entropy of a general quantum Gaussian state. Let
\( q_j, p_j \) be the canonical observables satisfying the Heisenberg CCR

\[
[q_j, p_k] = i \delta_{jk} \hbar I, \quad [q_j, q_k] = 0, \quad [p_j, p_k] = 0.
\]

Let us introduce the column vector

\[
R = [q_1, \ldots, q_s; p_1, \ldots, p_s]^T.
\]

We also introduce real column 2\( s \)-vector \( z = [x_1, \ldots, x_s; y_1, \ldots, y_s]^T \), and the unitary operators in \( \mathcal{H} \)

\[
V(z) = \exp i \sum_{j=1}^s (x_j q_j + y_j p_j) = \exp i R^T z.
\]

The operators \( V(z) \) satisfy the Weyl-Segal CCR

\[
V(z) V(z') = \exp [i/2\Delta(z, z')] V(z + z'),
\]

where

\[
\Delta(z, z') = \hbar \sum_{j=1}^s (x'_j y_j - x_j y'_j)
\]

is the canonical symplectic form. The Weyl-Segal CCR is the rigorous counterpart of the Heisenberg CCR, involving only bounded operators. We denote by

\[
\Delta = \begin{bmatrix}
0 & \hbar & 0 \\
0 & \hbar & \ddots \\
\ddots & \ddots & \ddots \\
\hbar & 0 & 0 \\
-h & 0 & 0 \\
0 & -h & \ddots \\
0 & 0 & -h \\
0 & \ddots & \ddots \\
0 & 0 & 0
\end{bmatrix}
\]

the \( (2s) \times (2s) \)-skew-symmetric commutation matrix of components of the vector \( R \). Most of the results below are valid for the case where the commutation matrix is arbitrary skew-symmetric matrix, not necessarily of the canonical form \((2.2)\).
The density operator $\rho$ is called Gaussian, if its quantum characteristic function has the form

$$\text{Tr}\rho V(z) = \exp(i m^T z - \frac{1}{2} z^T \alpha z),$$

where $m$ is column $(2s)$-vector and $\alpha$ is real symmetric $(2s) \times (2s)$-matrix.

One can show that

$$m = \text{Tr}\rho R ; \quad \alpha - \frac{i}{2} \Delta = \text{Tr}\rho R R^T$$

(cf. [6], [7]). The mean $m$ can be arbitrary vector; in what follows we will be interested in the case $m = 0$. The necessary and sufficient condition on the correlation matrix $\alpha$ is the matrix uncertainty relation

$$\alpha - \frac{i}{2} \Delta \geq 0.$$  

(2.3)

This condition is equivalent to its transpose $\alpha + \frac{i}{2} \Delta \geq 0$, and to the following matrix generalization of the Heisenberg uncertainty relation

$$\Delta^{-1} \alpha \Delta^{-1} + \frac{1}{4} \alpha^{-1} \geq 0,$$  

(2.4)

which is obtained by combining together (2.3) and its transpose. The state $\rho$ is pure if and only if the equality holds in this equation, or

$$(\Delta^{-1} \alpha)^2 = - \frac{1}{4} I.$$  

(2.5)

Let us introduce the function

$$g(x) = (x + 1) \log(x + 1) - x \log x, \quad x > 0.$$

We shall also use the matrix function $\text{abs}(\cdot)$, which is defined as follows: for a diagonalizable matrix $M = T \text{diag}(m_j) T^{-1}$, we put $\text{abs}M = T \text{diag}(|m_j|) T^{-1}$. In [9] it was shown that the entropy of the Gaussian state is equal to

$$H(\rho) = \frac{1}{2} \text{Sp}G((-\Delta^{-1} \alpha)^2),$$

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where
\[ G(a^2) = (a + \frac{1}{2}) \log(a + \frac{1}{2}) - (a - \frac{1}{2}) \log(a - \frac{1}{2}), \]
and \( Sp \) denotes trace of a matrix, as distinct from trace of operator. The matrix \( \Delta^{-1} \alpha \) has purely imaginary eigenvalues \( \pm ia_j \) and is diagonalizable.

Since \( G(a^2) = g(|a| - \frac{1}{2}) \), we obtain another expression
\[ H(\rho) = \frac{1}{2} Spg(\text{abs}(\Delta^{-1} \alpha) - \frac{I}{2}), \quad (2.6) \]
which will be used in the sequel.

### 3 Purification of Gaussian states

Let us denote \( \mathcal{H}_Q = \mathcal{H}_1 \) the Hilbert space of irreducible representation \( z \rightarrow V_1(z) \) of the CCR (2.1), \( \mathcal{H}_R = \mathcal{H}_2 \) the Hilbert space of irreducible representation \( z \rightarrow V_2(z) \) of the CCR
\[ V_2(z)V_2(z') = \exp[-i/2\Delta(z, z')]V_2(z + z'). \]

For example, \( V_2(z) = \exp i \sum_{j=1}^{s} (x_j p_j^{(2)} + y_j q_j^{(2)}) \), where \( q_j^{(2)}, p_j^{(2)} \) satisfy the Heisenberg CCR in \( \mathcal{H}_2 \). In \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) the operators \( V(z_1, z_2) = V_1(z_1) \otimes V_2(z_2) \) satisfy the CCR
\[ V(z_1, z_2)V(z'_1, z'_2) = \exp[i/2\Delta(z_1, z_2; z'_1, z'_2)]V(z_1 + z'_1, z_2 + z'_2), \]

where
\[ \Delta(z_1, z_2; z'_1, z'_2) = \Delta(z_1, z'_1) - \Delta(z_2, z'_2). \]

Following [4] we introduce Gaussian state \( \rho_{12} \) in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) with the correlation matrix
\[ \alpha_{12} = \begin{bmatrix} \alpha & \Delta \sqrt{-(\Delta^{-1} \alpha)^2 - I/4} \\ -\Delta \sqrt{-(\Delta^{-1} \alpha)^2 - I/4} & \alpha \end{bmatrix}, \]
that is
\[ \text{Tr}_{12} V(z_1, z_2) = \exp[-\frac{1}{2}(z_1^T \alpha z_1 + z_2^T \alpha z_2 + z_1^T \Delta \sqrt{-(\Delta^{-1} \alpha)^2 - I/4} z_2 - z_2^T \Delta \sqrt{-(\Delta^{-1} \alpha)^2 - I/4} z_1)]. \]
Obviously, $\rho_1 = \text{Tr}_2 \rho_{12}$.

Let us show that $\rho_{12}$ is pure. With

$$\Delta_{12} = \begin{bmatrix} \Delta & 0 \\ 0 & -\Delta \end{bmatrix},$$

we have

$$\Delta_{12}^{-1} \alpha_{12} = \begin{bmatrix} \Delta^{-1} \alpha & \sqrt{-(\Delta^{-1} \alpha)^2 - 1/4} \\ \sqrt{-(\Delta^{-1} \alpha)^2 - 1/4} \cdot \alpha^{-1} \end{bmatrix},$$

and it is easy to check that $(\Delta_{12}^{-1} \alpha_{12})^2 = -1/4$. By the criterium (2.3) $\rho_{12}$ is pure.

We shall be interested in the particular subclass of Gaussian states most familiar in quantum optics, which we call gauge-invariant. These are the states having the P-representation

$$\rho = \pi^{-s} |\det N|^{-1} \int \exp(-\zeta^\dagger N^{-1} \zeta) \langle \zeta | \zeta \rangle d^2 \zeta$$

(see e.g. [3], Sec. V, 5. II). Here $\zeta \in \mathbb{C}^s$, $|\zeta\rangle$ are the coherent vectors in $\mathcal{H}$, $a|\zeta\rangle = \zeta |\zeta\rangle$, $N$ is positive Hermitian matrix such that

$$N = \text{Tr} \rho a^\dagger a$$

(we use here vector notations, where $a = [a_1, \ldots, a_s]^T$ is a column vector and $a^\dagger = [a_1^\dagger, \ldots, a_s^\dagger]$ is a row vector) and $a_j = \frac{1}{\sqrt{2}} (q_j + ip_j).$ As shown in [3], the correlation matrix of such states is

$$\alpha = \hbar \begin{bmatrix} \text{Re}N + 1/2 & -\text{Im}N \\ \text{Im}N & \text{Re}N + 1/2 \end{bmatrix},$$

The real $2s \times 2s$– matrices of such form can be rewritten as complex $s \times s$– matrices, by using the correspondence

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \leftrightarrow A + iB,$$

which is in fact algebraic isomorphism, provided $A^T = A, B^T = -B$. Apparently,

$$\frac{1}{2} \text{Sp} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \text{Sp}(A + iB).$$
By using this correspondence, we have
\[ \alpha \leftrightarrow \hbar(N + I/2), \quad \Delta \leftrightarrow -i\hbar I, \]
and
\[ \Delta^{-1} \alpha \leftrightarrow i(N + I/2). \]
In particular, the formula (2.6) becomes
\[ H(\rho) = \text{Spg}(N), \]
which is well known (see e.g. [9]) and confirms (2.6).

For future use we also need the correspondence
\[ \Delta_{12}^{-1} \alpha_{12} \leftrightarrow \left[ i(N + I/2) \frac{\sqrt{N^2 + N}}{\sqrt{N^2 + N}} -i(N + I/2) \right]. \quad (3.8) \]
For the case of one degree of freedom we shall be interested in the following Section, \( N \) is just nonnegative number and \( \rho \) is elementary Gaussian state with the characteristic function
\[ \exp \left[ -\frac{\hbar}{2} \left( N + \frac{1}{2} \right) |z|^2 \right], \quad (3.9) \]
where we put \(|z|^2 = (x^2 + y^2)|.\]

## 4 Attenuation/amplification channel

Let us consider CCR with one degree of freedom described by one mode annihilation operator \( a = \frac{1}{2\hbar}(q + ip) \), and let \( a_0 \) be another mode in the Hilbert space \( \mathcal{H}_0 = \mathcal{H}_E \) describing environment. Let the environment be initially in the vacuum state, which is described by the characteristic function (3.9) with \( N = 0 \) i.e. \( \exp[-\frac{\hbar}{2}|z|^2]. \)

The linear attenuator with coefficient \( k < 1 \) is described by the transformation
\[ a \rightarrow ka + \sqrt{1 - k^2}a_0 \]
in the Heisenberg picture. Similarly, the linear amplifier with coefficient \( k > 1 \) is described by the transformation
\[ a \rightarrow ka + \sqrt{k^2 - 1}a_0. \]
It follows that the corresponding transformations $T_k[\rho]$ of states in the Schrödinger picture have, correspondingly, the characteristic functions

$$\text{Tr} T_k[\rho] V(z) = \text{Tr} \rho V(kz) \exp\left[-\frac{\hbar}{4}(1 - k^2)|z|^2\right], \quad k < 1, \quad (4.10)$$

$$\text{Tr} T_k[\rho] V(z) = \text{Tr} \rho V(kz) \exp\left[-\frac{\hbar}{4}(k^2 - 1)|z|^2\right], \quad k > 1, \quad (4.11)$$

see [3]. Let the input state $\rho$ of the system have the characteristic function (3.9), i.e. $\exp\left[-\frac{\hbar}{4}\left(N + \frac{1}{2}\right)|z|^2\right]$. The entropy of $\rho$ is

$$H(\rho) = g(N).$$

From (4.10), (4.11) we find that the output state $T_k[\rho]$ is again elementary Gaussian with $\bar{N}$ replaced by

$$\bar{N} = k^2 N, \quad k < 1; \quad \bar{N} = k^2 N + (k^2 - 1), \quad k > 1.$$

Thus

$$H(T_k[\rho]) = g(k^2 N), \quad k < 1; \quad H(T_k[\rho]) = g(k^2 N + (k^2 - 1)), \quad k > 1.$$

Now we calculate the entropy exchange $H(\rho, T_k)$. The (pure) input state $\rho_{12}$ of the extended system $H_1 \otimes H_2$ is characterized by the $2 \times 2$–matrix (3.8). The action of the extended channel $(T \otimes \text{Id})$ transforms this matrix into

$$\Delta_{12}^{-1} \sigma_{12} \leftrightarrow \begin{bmatrix} i(\bar{N} + \frac{1}{2}) & k\sqrt{N^2 + N} \\ k\sqrt{N^2 + N} & -i(N + \frac{1}{2}) \end{bmatrix}.$$ 

From formula (2.6) we deduce $H(\rho, T_k) = g(|\lambda_1| - \frac{1}{2}) + g(|\lambda_2| - \frac{1}{2})$, where $\lambda_1, \lambda_2$ are the eigenvalues of the complex matrix in the right-hand side. The eigenvalues are: $\lambda_1 = \frac{i}{2}$,

$$\lambda_2 = -i[(1 - k^2)N + \frac{1}{2}], \quad k < 1; \quad i[(k^2 - 1)(N + 1) + \frac{1}{2}], \quad k > 1.$$

Therefore we obtain

$$H(\rho, T_k) = g((1 - k^2)N) \quad k < 1; \quad g((k^2 - 1)(N + 1)), \quad k > 1.$$
The behavior of the entropies $H(T_k[\rho]), H(\rho, T_k)$ as functions of $k$ is clear from Fig.1. In particular, for all $N$ the coherent information $H(T_k[\rho]) - H(\rho, T_k)$ turns out to be positive for $k > 1/\sqrt{2}$ and negative otherwise. It tends to $-H(\rho)$ for $k \to 0$, is equal to $H(\rho)$ for $k = 1$, and quickly tends to zero as $k \to \infty$ (Fig.2; on both plots $N = 1$). The behavior of the information triangle shows that loss dominates for $k \to 0$, mutual information for $k \approx 1$, while the noise - as $k \to \infty$. This agrees with what one should expect on physical grounds from quantities presenting quantum mutual information, loss and noise and gives further support for their use in quantum information theory. However, negativity of the coherent information for $k < 1/\sqrt{2}$ looks somewhat mysterious and waits for a physical explanation.

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References

[1] C. Adami and N. J. Cerf, “Capacity of noisy quantum channels”, Phys. Rev. A, vol. A56, pp. 3470-3485, 1972.

[2] H. Barnum, M. A. Nielsen, B. Schumacher, “Information transmission through noisy quantum channels”, LANL Report no. quant-ph/9702049, Feb. 1997.

[3] C. W. Helstrom, Quantum detection and estimation theory, chapter 5, Academic press, 1976.

[4] A. S. Holevo, “Generalized free states of the C*-algebra of the CCR. ”, Theor. Math. Phys., vol. 6, no.1, pp. 3-20, 1971.

[5] A. S. Holevo, “Towards the mathematical theory of quantum communication channels”, Problems of Information Transm., vol. 8, no.1, pp. 63-71, 1972.
[6] A. S. Holevo, “Some statistical problems for quantum Gaussian states”, *IEEE Transactions on Information Theory*, vol. IT-21, no.5, pp. 533-543, 1975.

[7] A. S. Holevo, *Probabilistic and statistical aspects of quantum theory*, chapter 5, North-Holland, 1982.

[8] A. S. Holevo, “Coding theorems for Quantum Channels”, *Tamagawa University Research Review*, No.4, 1998.

[9] A. S. Holevo, M. Sohma and O. Hirota, “The capacity of quantum Gaussian channels ”, Preprint 1998.

[10] G. Lindblad, “Quantum entropy and quantum measurements”, *Lect. Notes Phys.*, vol. 378, Quantum Aspects of Optical Communication, Ed. by C. Benjaballah, O. Hirota, S. Reynaud, pp.71-80, 1991.
