Specializing Koornwinder polynomials to Macdonald polynomials of type $B$, $C$, $D$ and $BC$

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Abstract
We study the specializations of parameters in Koornwinder polynomials to obtain Macdonald polynomials associated to the subsystems of the affine root system of type $(C_\infty \vee C_n)$ in the sense of Macdonald (Affine Hecke algebras and orthogonal polynomials, Cambridge tracts in mathematics, Cambridge Univ Press, 2003), and summarize them in what we call the specialization table. As a verification of our argument, we check the specializations to type $B$, $C$ and $D$ via Ram–Yip type formulas of non-symmetric Koornwinder and Macdonald polynomials.

Keywords Macdonald–Koornwinder polynomials · Askey–Wilson polynomials · Ram–Yip type formulas · Alcove walks · Affine Hecke algebras

1 Introduction
In [21], Macdonald introduced families of multivariate $q$-orthogonal polynomials associated to various root systems, which are today called the Macdonald polynomials. Each family has additional $t$ parameters corresponding to the Weyl group orbits in the root system. Following this work, in [17], Koornwinder introduced a multivariate analogue of Askey–Wilson polynomial, having additional five parameters aside from $q$, which is today called the Koornwinder polynomial. It was also shown in [17] that by specializing these five parameters, we can obtain the Macdonald polynomials of type $(BC_n, B_n)$ and $(BC_n, C_n)$ in the sense of [21]. Today, these families of multivariate $q$-orthogonal polynomials are called the Macdonald–Koornwinder polynomials [7, 11, 23, 35].
After the development of the representation theoretic approach [2–6, 24, 30, 31, 33, 36] using the (double) affine Hecke algebras, there appeared several versions of unified formulation of the Macdonald–Koornwinder polynomials [7, 11, 23, 35]. These studies are now called the Macdonald-Cherednik theory.

The specialization argument given by Koornwinder in [17] is now understood in a more general form. First, after the studies in [24, 30, 31, 33, 36], the Koornwinder polynomial can be formulated as the Macdonald polynomial associated to the affine root system of type \((C_n^\vee, C_n)\) in the sense of [23]. See also [34, 35] for the relevant explanation. Then, as mentioned in [23, p.12, (5.17)], the Macdonald polynomials associated to all the subsystems of type \((C_n^\vee, C_n)\) can be obtained by specializing the five parameters of the Koornwinder polynomial in the way respecting the orbits of the extended affine Weyl group acting on the affine root systems. See also the comment in [11, 6.19].

However, it seems that the detailed explanation of the specialization argument is not given in literature. The aim of this note is to clarify this point.

What troubled the authors in the early stages of the study is that there are tremendously many notations for the affine root systems and the parameters of Macdonald–Koornwinder polynomials, and that even for the work [21] and the book [23] both by Macdonald, there seems no explicit comparison in literature. To the authors’ best knowledge, in the present writing this note, the most general framework of the theory of Macdonald–Koornwinder polynomials is given by Stokman [35], which is based on the approach of Haiman [11]. It treats uniformly the four classes of Macdonald–Koornwinder polynomials: GL\(_n\), the untwisted case, the twisted case, and the Koornwinder case. The formulation by Macdonald in [23] treats the latter three cases along this classification.

Although it would be the best to work in the framework of [35], we gave up to do so due to the following reasons. First, since we are interested in the specialization of Koornwinder polynomials, we may ignore GL\(_n\) case, and the formulation of [23] will be enough. Second, we are also motivated by Ram–Yip type formulas of non-symmetric Macdonald–Koornwinder polynomials [26, 28], and will check our specialization argument in the level of those formulas. The calculations in the check are based on the paper [37] by the first named author, which mainly follows the notation in [23]. Let us mention that some specialization arguments are given in [35, Example 9.3.28, Remark 9.3.29].

After these considerations, we decided to use the notation in the following literature:

1. [23] for affine root systems.
2. [24] for the parameters of Koornwinder polynomials.

Let us explain (1) in detail. We use the word “affine root system” in the sense of [23, §1.2], which originates in [20]. The word “irreducible finite root system” means an irreducible root system in [23]. We also denote by \(\vee\) the dualizing of finite and affine root systems. Then, as explained in [23, §1.3], similarity classes of irreducible affine root systems are divided into three cases:

- Reduced and of the form \(S(R)\) with \(R\) an irreducible finite root system.
- Reduced and of the form \(S(R)^\vee\) with \(R\) an irreducible finite root system.
- Non-reduced and of the form \(S_1 \cup S_2\) with \(S_1\) and \(S_2\) reduced affine root systems.
The appearing $R$ is one of the types $A_n$, $B_n$, $C_n$, $D_n$, $BC_n$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$. According to the type of $R$, we say

- $S(R)$ is of type $X$ if $R$ is of type $X$,
- $S(R)^\vee$ is of type $X^\vee$ if $R$ is of type $X$,
- A non-reduced system $S_1 \cup S_2$ is of type $(X, Y)$ if $S_1$ and $S_2$ are of type $X$ and $Y$, respectively.

We refer to [23, (1.3.1)–(1.3.18)) for explicit descriptions of these irreducible affine root systems, although some of them will be displayed in the main text.

As explained in [23, §1.4], Macdonald developed a unified formulation to associate a family of $q$-orthogonal polynomials to each of the following pairs $(S, S')$ of irreducible affine root systems.

(a) $(S, S') = (S(R), S(R)^\vee)$ with $R$ an irreducible finite root system.
(b) $S = S' = S(R)^\vee$ with $R$ an irreducible finite root system.
(c) $S = S'$ is non-reduced of type $(X, Y)$.

For each pair $(S, S')$, we have the associated non-symmetric [23, §5.2] and symmetric [23, §5.3] Macdonald polynomials. For the reference in the main text, let us introduce:

**Definition 1.0.1** We call the non-symmetric and symmetric Macdonald polynomials associated to $(S, S')$ in the class (a), (b) and (c) the non-symmetric and symmetric Macdonald polynomials of type $X$, $X^\vee$ and $(X, Y)$, respectively.

In particular, the Koornwinder polynomial is the Macdonald polynomial of type $(C_n^\vee, C_n)$. The detail of the affine root system of type $(C_n^\vee, C_n)$ will be explained in Sect. 2.1.

As mentioned before, in [23, p.12], Macdonald gives a comment that the affine root system of type $(C_n^\vee, C_n)$ has as its subsystem all the non-reduced affine root systems and the classical affine root systems of type $B_n$, $B_n^\vee$, $C_n$, $C_n^\vee$, $BC_n$ and $D_n$. Also, at [23, (5.17)], he comments that an appropriate specialization of parameters in the Koornwinder polynomials yields the Macdonald polynomials associated to the corresponding subsystem. Let us state again that the aim of this note is to clarify this point.

Let us turn to the explanation of (2), the notation of the five parameters of the Koornwinder polynomial. We will use those introduced by Noumi in [24]:

\[ t, t_0, t_n, u_0, u_n. \]  

(1.0.1)

Let us call them the **Noumi parameters** for distinction. The details will be explained in Sect. 2.2.

Now we can explain the main result of this note.

**Theorem 1** (Propositions 2.3.1, 2.4.1–2.4.9) For each type $X$ listed in Table 1 and for each (not necessarily) dominant weight $\mu$ of type $C_n$, the specialization of the Noumi parameters in the (non-symmetric) Koornwinder polynomial with weight $\mu$ yields the (non-symmetric) Macdonald polynomial with $\mu$ of type $X$ in the sense of Definition 1.0.1.
Hereafter we refer Table 1 as the specialization table.

Let us explain how to read Theorem 1 and the specialization Table 1 in the case of type $C_n$. The associated Macdonald polynomial has the parameters $q$ and two kinds of $t$’s. The latter correspond to the two orbits of the extended affine Weyl group acting on the affine root system of type $C_n$, and we denote them by $t_s$ and $t_l$. Using them, we denote the symmetric Macdonald polynomial of type $C_n$ by $P^C_\mu (x; q, t_s, t_l)$ with dominant weight $\mu$. See Sect. 2.3 for the detail of these symbols for type $C_n$. We also have the Koornwinder polynomial $P_\mu (x; q, t, t_0, t_n, u_0, u_n)$ with the same dominant weight, whose detail will be explained in Sect. 2.2. Then, specializing the Noumi parameters as indicated in the type $C_n$ row in Table 1, we obtain $P^C_\mu (x; q, t_s, t_l)$. In other words, the following identity holds.

$$ P^C_\mu (x; q, t_s, t_l) = P_\mu (x; q, t, t_0, t_n, u_0, u_n) \quad (1.0.2) $$

See Proposition 2.3.1 for the detail of type $C_n$.

We derive each of the specializations in Sects. 2.3 and 2.4, as indicated in the specialization Table 1. Our argument is based on the fact that each family of Macdonald–Koornwinder polynomials is uniquely determined by the inner product. Thus, the desired specialization will be obtained by studying the degeneration of the weight function of the inner product, which is actually described in the formula [23, (5.1.7)]. See (2.2.10) for the precise statement. As commented at [23, (5.1.7)], all we have to do is to take care the correspondence of the orbits of the extended affine Weyl group.

In Sect. 3, as a verification of the specializing Table 1, we check the obtained specializations by using explicit formulas of Macdonald–Koornwinder polynomials. We focus on Ram–Yip type formulas [26, 28] which were mentioned before. These formulas give explicit description of the coefficients in the monomial expansion of non-symmetric Macdonald–Koornwinder polynomials as a summation of terms over the so-called alcove walks, the notion introduced by Ram [27]. We do this check for Ram–Yip formulas of type $B$, $C$ and $D$ in the sense of [28]. The check is done just in case-by-case calculation, but since the situation is rather complicated due to the notational problem of affine root systems and parameters, we believe that it has some importance. The result is as follows.
Theorem 2 (Propositions 3.3.4, 3.2.5 and 3.4.5) For each $\mu \in P_{C_n}$, we have
\[
E_\mu(x; q, t^R_m, 1, t^R_l, 1, t^R_l) = E^{B, RY}_\mu(x; q, t^R_m, t^R_l), \tag{1.0.3}
\]
\[
E_\mu(x; q, t^R_m, 1, t^R_s, 1, 1) = E^{C, RY}_\mu(x; q, t^R_s, t^R_m), \tag{1.0.4}
\]
\[
E_\mu(x; q, t, 1, 1, 1, 1) = E^{D, RY}_\mu(x; q, t). \tag{1.0.5}
\]

Here the left hand sides denote specializations of the non-symmetric Koornwinder polynomials $E_\mu(x)$, and the right hand side denotes the non-symmetric Macdonald polynomials of type $B, C$ and $D$ in the sense of [28]. For the detail, see the beginning of Sect. 3 for the explanation. Comparing these identities with the specialization Table 1, we find that $E^{B, RY}_\mu(x)$ is equivalent to the polynomial of type $B_n$, $E^{C, RY}_\mu(x)$ is to that of type $C_\vee^n$, and $E^{D, RY}_\mu(x)$ is to that of type $D_n$ in the sense of Definition 1.0.1.

Notation and terminology

Here are the notations and terminology used throughout in this paper.

- We denote by $\mathbb{Z}$ the ring of integers, by $\mathbb{N} = \mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$ the set of non-negative integers, by $\mathbb{Q}$ the field of rational numbers, and by $\mathbb{R}$ the field of real numbers.
- We denote by $e$ the unit of a group.

2 Specialization table of Koornwinder polynomials

The aim of this section is to give the detail of the specialization Table 1. As explained in Sect. 1, we use the affine root systems in the sense of Macdonald [20, 23]. Our main system is that of type $(C_\vee^n, C_n)$, which will be denoted by $S$. See (2.1.1) for the precise definition. According to the list of affine root systems in [23, §1.3], those in Table 1 are subsystems of $S$. Explicitly, the following types are the subsystems of type $(C_\vee^n, C_n)$.

\[
B_n, B_\vee^n, C_n, C_\vee^n, D_n, BC_n, (BC_n, C_n), (C_\vee^n, BC_n), (B_\vee^n, B_n). \tag{2.0.6}
\]

The details of these subsystems will be explained in Sects. 2.3 and 2.4.

2.1 Affine root system of type $(C_\vee^n, C_n)$

Here we introduce the notation for the affine root system of type $(C_\vee^n, C_n)$ in the sense of Macdonald [23], following Chapter 1 of loc. cit.

Let $n \in \mathbb{Z}_{\geq 2}$, and $E$ be the $n$-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$. We take and fix an orthonormal basis $\{\epsilon_i \mid i = 1, 2, \ldots, n\}$ of $E$. Thus, we may identify $E = (V, \langle \cdot, \cdot \rangle)$ with $V = \bigoplus_{i=1}^{n} \mathbb{R} \epsilon_i$. Let $F$ be the $\mathbb{R}$-linear space of affine linear functions $E \to \mathbb{R}$. The inner product $\langle \cdot, \cdot \rangle$ yields the isomorphism $F \cong V \oplus \mathbb{R}c$, where $c$ is the point at infinity.
where \( c \) is the constant function \( c(v) = 1 \) for any \( v \in V \). Hereafter we identify \( F \) and \( V \oplus \mathbb{R}c \) by this isomorphism.

We denote by \( S \) the affine root system of type \((C_n^\vee, C_n)\) in the sense of [23, §1.3, (1.3.18)]. Thus, \( S \) is a subset of \( F = V \oplus \mathbb{R}c \) given by

\[
S = O_1 \sqcup O_2 \sqcup \cdots \sqcup O_5,
\]

\[
O_1:=[\pm \epsilon_i + rc \mid 1 \leq i \leq n, \ r \in \mathbb{Z}], \quad O_2:=2O_1, \quad O_3:=O_1 + \frac{1}{2}c, \quad O_4:=2O_3 = O_2 + c, \quad O_5:=[\pm \epsilon_i \pm \epsilon_j + rc \mid 1 \leq i < j \leq n, \ r \in \mathbb{Z}]. \tag{2.1.1}
\]

An element of \( S \) is called an affine root, or just a root. Following the choice of [37], we consider the affine roots

\[
a_0:=-2\epsilon_1 + c, \quad a_j:=\epsilon_j - \epsilon_{j+1} \ (1 \leq j \leq n-1), \quad a_n:=2\epsilon_n. \tag{2.1.2}
\]

They form a basis of \( S \) in the sense of [23, §1.2]. Obviously we have

\[
\frac{1}{2}a_0 \in O_3, \quad a_0 \in O_4, \quad a_j \in O_5 \ (1 \leq j \leq n-1), \quad \frac{1}{2}a_n \in O_1, \quad a_n \in O_2.
\]

Below is the Dynkin diagram cited from [23, (1.3.18)]. The mark \( \ast \) above the index \( i \) implies that \( a_i, \frac{1}{2}a_i \in S \).

In fact, the description (2.1.1) gives the orbit decomposition of \( S \) by the action of the extended affine Weyl group. For the explanation, we need to introduce more symbols.

The inner product \( \langle \cdot, \cdot \rangle \) on \( V \) is extended to \( F = V \oplus \mathbb{R}c \) by

\[
\langle v + rc, w + sc \rangle := \langle v, w \rangle, \quad v, w \in V, \quad r, s \in \mathbb{R}.
\]

For a non-constant function \( f \in F \setminus \mathbb{R}c \), we define \( s_f \in \text{GL}_R(F) \) by

\[
F \ni g \mapsto s_f(g) := g - \langle g, f^\vee \rangle f, \quad f^\vee := \frac{2}{\langle f, f \rangle} f.
\]

It is the reflection with respect to the hyperplane \( H_f := f^{-1}(\{0\}) \subset V \). Now we consider the subset

\[
R := \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \pm 2\epsilon_i \mid 1 \leq i \leq n \} \subset S \cap V, \tag{2.1.3}
\]

which is in fact the finite root system of type \( C_n \). Among the affine roots \( a_i \) in (2.1.2), those except \( a_0 \) belong to \( R \), which are the simple roots of type \( C_n \). Then the finite Weyl group \( W_0 \) is the subgroup

\[
W_0 := \langle s_i \ (i = 1, 2, \ldots, n) \rangle \subset \text{GL}_R(V), \quad s_i := s_{a_i}. \tag{2.1.4}
\]
Note that each element in $W_0$ is an isometry for the inner product $\langle \cdot, \cdot \rangle$. 

Next, for $v \in V$, we define $t(v) \in \text{GL}_R(F)$ by

$$F \ni f \longmapsto t(v)(f) := f - \langle f, v \rangle c.$$  \hspace{1cm} (2.1.5)

Then, for $w \in W_0$, we have

$$w t(v) w^{-1} = t(wv).$$ \hspace{1cm} (2.1.6)

Let $P_{C_n} \subset F$ be given by

$$P_{C_n} := \mathbb{Z} \epsilon_1 \oplus \mathbb{Z} \epsilon_2 \oplus \cdots \oplus \mathbb{Z} \epsilon_n,$$  \hspace{1cm} (2.1.7)

which is in fact the weight lattice of the finite root system of type $C_n$. Then,

$$t(P_{C_n}) = \{ t(\mu) \mid \mu \in P_{C_n} \} \subset \text{GL}_R(F)$$  \hspace{1cm} (2.1.8)

is isomorphic to the additive group $P_{C_n}$. Viewing (2.1.6) as an action of $W_0$ on $t(P_{C_n})$, we can take the semi-direct product of (2.1.4) and (2.1.8) to obtain the extended affine Weyl group $W$ of type $(C_n^\vee, C_n)$:

$$W := t(P_{C_n}) \rtimes W_0 \subset \text{GL}_R(F).$$  \hspace{1cm} (2.1.9)

It acts on $S$ by permutation \cite[(1.4.6), (1.4.7)]{23}, and the orbits are given in (2.1.1) \cite[1.5]{23}.

Let us also give a description of $W$ as an abstract group. We set

$$s_0 := t(\epsilon_1) s_{2\epsilon_1} \in W.$$  \hspace{1cm} (2.1.10)

Then $W$ has a presentation with generators

$$W = \langle s_0, s_1, \ldots, s_n \rangle$$  \hspace{1cm} (2.1.11)

and the following relations.

$$s_i^2 = 1 \quad (0 \leq i \leq n),$$

$$s_i s_j = s_j s_i \quad (|i - j| > 1),$$

$$s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1} \quad (1 \leq j \leq n - 2),$$

$$s_i s_{i+1} s_i s_{i+1} = s_{i+1} s_i s_{i+1} s_i \quad (i = 0, n - 1).$$  \hspace{1cm} (2.1.12)

Hereafter the length $\ell(w)$ of $w \in W$ indicates that for a reduced expression in terms of the generators $\{s_i\}_{i=0}^n$. For later use, we write down a reduced expression of $t(\epsilon_i)$:

$$t(\epsilon_i) = s_{i-1} s_{i-2} \cdots s_1 s_0 s_1 s_2 \cdots s_{n-1} s_{n-2} \cdots s_{i+1} s_i \quad (1 \leq i \leq n).$$  \hspace{1cm} (2.1.13)
Let us also introduce \( F_{\mathbb{Z}} \subset F \) by

\[
F_{\mathbb{Z}} := \mathbb{P}{\cdot\mathfrak{c}} \oplus \frac{1}{2}\mathbb{Z}\mathfrak{c}.
\]

(2.1.14)

Then we have \( S \subset F_{\mathbb{Z}} \). We write down the action of \( W \) on \( F_{\mathbb{Z}} \):

\[
s_0(\epsilon_i) = \begin{cases} 
    c - \epsilon_1 & (i = 1) \\
    \epsilon_i & (i \neq 1)
\end{cases},
\]

\[s_j(\epsilon_i) = \begin{cases} 
    \epsilon_j & (i = j + 1) \\
    \epsilon_i & (i \neq j, j + 1)
\end{cases} \quad (1 \leq j \leq n - 1),
\]

\[s_n(\epsilon_i) = \begin{cases} 
    -\epsilon_n & (i = n) \\
    \epsilon_i & (i \neq n)
\end{cases},
\]

\[s_k(c) = c \quad (0 \leq k \leq n).
\]

By these formulas, we can check the orbit decomposition (2.1.1) directly.

Closing this part, we recall the positive and negative parts of \( S \). Let us write \( S \) as

\[
S = \{ \pm \epsilon_i + \frac{1}{2}r\mathfrak{c}, \pm 2\epsilon_i + r\mathfrak{c} | 1 \leq i \leq n, r \in \mathbb{Z} \}
\]

\[
\cup \{ \pm \epsilon_i \pm \epsilon_j + r\mathfrak{c} | 1 \leq i < j \leq n, r \in \mathbb{Z} \}.
\]

It has the decomposition \( S = S^+ \cup S^- \) with the sets \( S^\pm \) of positive and negative roots, respectively. To describe \( S^\pm \), let us recall the decomposition of the finite root system \( R \) of type \( C_n \) (see (2.1.3)) into positive and negative roots:

\[
R = R_+ \cup R_-, \quad R_+ := \{ 2\epsilon_i | 1 \leq i \leq n \} \cup \{ \epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n \} \subset R,
\]

\[
R_- := - R_+.
\]

Then, the sets \( S_{\pm} \) are given by

\[
S^+ := \{ \alpha + r\mathfrak{c}, \alpha^+ + \frac{1}{2}r\mathfrak{c} | \alpha \in R_+, r \in \mathbb{N} \} \cup \{ \alpha + r\mathfrak{c}, \alpha^+ + \frac{1}{2}r\mathfrak{c} | \alpha \in R_-, r \in \mathbb{N} \},
\]

\[
S^- := - S^+.
\]

Using (2.1.2), we have \( a_i \in S^+ \) for each \( i = 0, 1, \ldots, n \). Moreover we have

\[
S^+ = \sum_{i=0}^{n} \mathbb{N}a_i \setminus \{0\}.
\]

(2.1.15)

We also define \( \tilde{R}, \tilde{R}_{\pm} \subset S \) by

\[
\tilde{R} := \tilde{R}_+ \cup \tilde{R}_-, \quad \tilde{R}_+ := \{ \epsilon_i, 2\epsilon_i | 1 \leq i \leq n \} \cup \{ \epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n \},
\]

\[
\tilde{R}_- := - \tilde{R}_+.
\]

(2.1.16)
Then, any \( a \in S \) can be presented as \( a = \alpha + rc \) with \( \alpha \in \tilde{R} \) and \( r \in \frac{1}{2}\mathbb{Z} \), and we denote
\[
\bar{a} := \alpha \in \tilde{R}.
\] (2.1.17)

### 2.2 Parameters, weight function and Koornwinder polynomials

In this subsection, we explain the parameters and the weight function for type \((C_n^\vee, C_n)\), and introduce the symmetric and non-symmetric Koornwinder polynomials. As for the parameters of Koornwinder polynomials, we mainly use the Noumi parameters in [24], as mentioned in Sect. 1. Due to the necessity in the specialization argument, we also give a summary of the comparison of the Noumi parameters with those given by Macdonald in [23], which we will refer as the Macdonald parameters.

We begin with explanation on the parameters in [23]. Let us write again the \( W \)-orbits (2.1.1) in \( S = O_1 \sqcup \cdots \sqcup O_5 \) and the affine roots \( a_i \) in (2.1.2):

\[
O_1 := \{ \pm \epsilon_i + rc \mid 1 \leq i \leq n, r \in \mathbb{Z} \}, \quad O_2 := 2O_1, \quad O_3 := O_1 + \frac{1}{2}c,
\]
\[
O_4 := 2O_3 = O_2 + c, \quad O_5 := \{ \pm \epsilon_i \pm \epsilon_j + rc \mid 1 \leq i < j \leq n, r \in \mathbb{Z} \}.
\]
\[
a_0 := -2\epsilon_1 + c, \quad a_j := \epsilon_j - \epsilon_{j+1} \quad (1 \leq j \leq n-1), \quad a_n := 2\epsilon_n.
\]
\[
\frac{1}{2}a_0 \in O_3, \quad a_0 \in O_4, \quad a_j \in O_5 \quad (1 \leq j \leq n-1), \quad \frac{1}{2}a_n \in O_1, \quad a_n \in O_2.
\]

We attach a parameter \( k_r \in \mathbb{R} \) to each \( W \)-orbit as
\[
k_r \longleftrightarrow O_r \quad (r = 1, 2, \ldots, 5),
\] (2.2.1)
and define the label \( k \) [23, §1.5] as a map on given by
\[
k : S \rightarrow \mathbb{R}, \quad k(a) := k_r \quad \text{for } a \in O_r.
\] (2.2.2)

Let us choose \( q \in \mathbb{R} \) satisfying \( 0 < q < 1 \), and define the set of parameters as
\[
\{ q^{k(a)} \mid a \in S \} = \{ q^{k_1}, q^{k_2}, \ldots, q^{k_5} \}.
\] (2.2.3)

We call \( q^{k} \)'s the Macdonald parameters. These are used in the formulation of Koornwinder polynomials in [23].

As mentioned at (1.0.1) and in the beginning of this Sect. 2.2, in the following argument, we will mainly use the Noumi parameters
\[
t, t_0, t_n, u_0, u_n
\]
introduced in [24]. As will be shown in Sect. 2.2.1 below, we have the following relation between the Macdonald parameters and the Noumi parameters.
\[
(q^{2k_1}, q^{2k_2}, q^{2k_3}, q^{2k_4}, q^{k_5}) = (t_n u_n, \frac{t_n}{u_n}, t_0 u_0, \frac{t_0}{u_0}, t).
\] (2.2.4)
Restating by (2.2.1), the Noumi parameters and the $W$-orbits correspond in the way

$$
\begin{align*}
 t_n u_n & \longleftrightarrow O_1, \quad t_n / u_n \longleftrightarrow O_2, \quad t_0 u_0 \longleftrightarrow O_3, \quad t_0 / u_0 \longleftrightarrow O_4, \quad t \longleftrightarrow O_5.
\end{align*}
$$

(2.2.5)

Now we introduce the base field for (non-symmetric) Koornwinder polynomials. Adding the square $t_1^2, t_1^{1/2}, u_1^{1/2}$ of the Noumi parameters and the new parameter $q^{1/2}$, we define the base field $\mathbb{K}$ to be the rational function field

$$
\mathbb{K} := \mathbb{Q}(t_1^2, t_1^{1/2}, u_1^{1/2}).
$$

(2.2.6)

Next, following [23, §5.1], we explain the weight function for (non-symmetric) Koornwinder polynomials. Using the exponent $e$ in the sense of [23, (1.4.5)], which is given by

$$
e = \frac{1}{i} \sum_{i=0}^{n} a_i = \frac{1}{2} c.
$$

We used the affine roots $a_i$ in (2.1.2). Also, using $L := P_{C^n}$, we set

$$
\Lambda := L \oplus \mathbb{Z} c_0 = P_{C^n} \oplus \frac{1}{2} \mathbb{Z} = \bigoplus_{i=1}^{n} \mathbb{Z} e_i \oplus \frac{1}{2} \mathbb{Z}.
$$

Note that we have $S \subset \Lambda$. For each $f = \mu + r c_0 \in \Lambda$, we define

$$
e^f := e^\mu q^r = e^\mu q^r.
$$

(2.2.7)

Then, for the label $k$ in (2.2.2), we define the weight function $\Delta_{S,k}$ [23, (5.1.7)] as

$$
\Delta_{S,k} := \prod_{a \in S^+} \Delta_a = \prod_{a \in S^+} \frac{1 - q^{k+2a} e^a}{1 - q^{k-a} e^a}.
$$

(2.2.8)

Here we used $S^+$ in (2.1.15) and set $k(2a) := 0$ if $2a \notin S$. As explained in [23, (5.1.14)], we can rewrite $\Delta_{S,k}$ as

$$
\Delta_{S,k} = \prod_{r=1}^{4} \prod_{a \in S^+ \cap O_r} \Delta_a \cdot \prod_{a \in S^+ \cap O_5} \Delta_a = \prod_{a \in R^+_e} \frac{(e^{2a}, q e^{-2a}; q)_{\infty}}{\prod_{r=1}^{4} (v_r e^a, v'_r e^{-a}; q)_{\infty}} \cdot \prod_{a \in R^+_e} \frac{(e^a, q e^{-a}; q)_{\infty}}{\prod_{r=5}^{k+1} (v_r e^a, q e^{-a}; q)_{\infty}},
$$

where we used GASPER and Rahman’s notation in [10] for $q$-shifted factorials

$$
(x; q)_{\infty} := \prod_{n=0}^{\infty} (1 - x q^n), \quad (x_1, \ldots, x_n; q)_{\infty} := \prod_{i=1}^{n} (x_i; q)_{\infty},
$$

(2.2.9)
and $R^+_s$ (resp. $R^+_l$) denotes the set of positive (resp. short) roots in the finite root system $R$ of type $C_n$. Explicitly, we have

$$R^+_s := \{ \epsilon_i | 1 \leq i \leq n \}, \quad R^+_l := \{ \epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n \}.$$ 

We also used the following $4 \times 2$ parameters $v_1, \ldots, v_4$ and $v'_1, \ldots, v'_4$.

$$(v_1, \ldots, v_4) := (q^{k_1}, -q^{k_2}, q^{k_3+\frac{1}{2}}, -q^{k_4+\frac{1}{2}}).$$

$$(v'_1, \ldots, v'_4) := (q^{k_1+1}, -q^{k_2+1}, q^{k_3+\frac{1}{2}}, -q^{k_4+\frac{1}{2}}).$$

Finally, as mentioned in the last part of [23, (5.1.7)], the following relation holds for each subsystem $S^0$ of the affine root system $S$.

$$\Delta_{S,k} \big|_{k(a) - k(2a) = 0 (a \notin S^0)} = \Delta_{S^0,k}. \quad (2.2.10)$$

For the complete set of the subsystems $S^0$ in $S$, see the comment in the beginning of this Sect. 2.

The weight function $\Delta_{S,k}$ defines an inner product on the space

$$\mathbb{K}[x_{\pm 1}] = \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}], \quad x_i := e^{\epsilon_i}$$

of the $n$-variable Laurent polynomials, where in the last part we used (2.2.7). Then, by [23, §5.2], we have the family of non-symmetric Koornwinder polynomials

$$E_{\mu}(x) = E_{\mu}(x; q, t, t_0, u_0, u_n) \in \mathbb{K}[x_{\pm 1}], \quad \mu \in PC_n, \quad (2.2.11)$$

as a unique orthogonal basis of the inner product on $\mathbb{K}[X_{\pm 1}]$ satisfying the triangular property. Moreover, by [23, §5.3], for a dominant weight $\mu$ in $PC_n$, we have the symmetric Koornwinder polynomial

$$P_{\mu}(x) = P_{\mu}(x; q, t, t_0, u_0, u_n) \in \mathbb{K}[x_{\pm 1}]^{W_0}. \quad (2.2.12)$$

2.2.1 Derivation of (2.2.4)

Let us derive the relation (2.2.4) between the Macdonald and Noumi parameters. We use the notation of the affine Hecke algebra given in [23, Chapter 4]. We make one modification: The base field $K$ is enlarged so that it contains $\tau_i$’s and $\tau_i'$’s defined blow, and $q^{1/2}$ (in the version of [23], it contains $q$ but doesn’t $q^{1/2}$).

Let $q$ be a real number such that $0 < q < 1$, and $K$ be a subfield of $\mathbb{R}$ containing $q^{1/2}$. We denote by $H$ the affine Hecke algebra associated to the extended affine Weyl group $W$ of (2.1.9) in the sense of [23, 4.1]. It is an associative $K$-algebra generated by

$$H = \langle T_0, T_1, \ldots, T_n \rangle \quad (2.2.13)$$
with certain defining relations, for which we refer [37, (2.2.3)–(2.2.5)].

**Remark 2.2.1** We give another description of the affine Hecke algebra $H$. As a $K$-linear space, it has the form

$$H = H_0 \otimes_K K[Y^{t(\epsilon_j)} \mid j = 1, 2, \ldots, n] \simeq H_0 \otimes_K K P_{C_n}, \quad (2.2.14)$$

where $H_0$ denotes the Hecke algebra associated to the finite Weyl group $W_0$ of type $C_n$ (see (2.1.4)), and $K P_{C_n}$ denotes the group algebra of the additive group $PC_n$. The commuting elements $Y^{t(\epsilon_j)}$'s are defined in [23, §3.2], and using the reduced expressions (2.1.13), we have the following relations between $Y^{t(\epsilon_j)}$'s and the generators $T_i$'s in (2.2.13).

$$Y^{t(\epsilon_j)} = T_{j-1}^{-1} \cdots T_1^{-1} T_0 \cdots T_{n-1} T_n T_{n-1} \cdots T_j. \quad (2.2.15)$$

Note that the ordering of $T_i$'s is opposite of those in some literature, for example [30, §2.2, p.399], [14, §3.1, p.312] and [8]. This discrepancy is reflected on the triangular property of the (non-symmetric) Koornwinder polynomials. Namely, the choices of the ordering on the space $K[x^\pm 1]$ and $K[x^\pm 1]W_0$, where the (non-symmetric) Koornwinder polynomials live, are in opposite between ours and those other literature.

Recall the Macdonald parameters $q^{k_1}, q^{k_2}, \ldots, q^{k_5}$ in (2.2.2). Following [23, (4.4.3)], we introduce the additional parameters $\kappa_i, \kappa'_i \in K$ for $i = 0, 1, \ldots, n$ as

$$k_1 = k(a_n) = \frac{1}{2}(\kappa_n + \kappa'_n), \quad k_2 = k(2a_n) = \frac{1}{2}(\kappa_n - \kappa'_n),$$
$$k_3 = k(a_0) = \frac{1}{2}(\kappa_0 + \kappa'_0), \quad k_4 = k(2a_0) = \frac{1}{2}(\kappa_0 - \kappa'_0),$$
$$k_5 = k(a_j) = \kappa_j = \kappa'_j \quad (1 \leq j \leq n - 1).$$

We also introduce $\tau_i, \tau'_i \in K$ for $i = 0, 1, \ldots, n$ by

$$\tau_i := q^{\kappa_i/2}, \quad \tau'_i := q^{\kappa'_i/2}.$$

By definition, we have

$$q^{k_1} = \tau_n \tau'_n, \quad q^{k_2} = \tau_n / \tau'_n, \quad q^{k_3} = \tau_0 \tau'_0, \quad q^{k_4} = \tau_0 / \tau'_0,$$
$$q^{k_5} = \tau_j \tau'_j \quad (1 \leq j \leq n - 1). \quad (2.2.16)$$

Using the parameters $\tau_i$ and $\tau'_i$, we explain the basic representation $\beta$ of $H$ [23, (4.3.10)], which actually goes back to Lusztig [19]. It is a faithful representation in the group algebra $A = KL$ of $L := Q_{C_n}^\lor = Q_{B_n} = PC_n$ given by

$$\beta : H \hookrightarrow \text{End}_K(K P_{C_n}), \quad \beta(T_i) := \tau_i s_i + b_i (1 - s_i) \quad (0 \leq i \leq n). \quad (2.2.17)$$

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where, expressing the element of $KP_{C_n}$ corresponding to $\alpha \in P_{C_n}$ as $e^{\alpha}$, the function $b_i$ is defined by

$$b_i = b_i(t_i, t'_i; x^{\alpha_i}) := \frac{\tau_i - \tau_{i-1} + (\tau'_i - \tau'_{i-1})x^{\alpha_i}/2}{1 - x^{\alpha_i}},$$

$$x^{\alpha_i} := \begin{cases} x_i/x_{i+1} = e^{\epsilon_i - \epsilon_{i+1}} & (1 \leq i \leq n - 1) \\ q\tau_1^{-2} = q e^{-2 \epsilon_1} & (i = 0) \\ x_n^2 = e^{2 \epsilon_n} & (i = n) \end{cases}. \tag{2.2.18}$$

Here the symbol $b_i(t, u; z)$ is borrowed from [23, (4.2.1)], and the symbol $x^{\alpha_i}$ is from [37]. Note that the representation $\beta$ is well defined although the function $b_i$ does not belong to the group algebra $KP_{C_n}$.

The relation of the Macdonald and Noumi parameters is obtained by the comparison between the realizations of the basic representation in $\beta$ and $\gamma$. Slightly extending the Noumi parameters as

$$t_0, t_n, u_0, u_n, t_j := t, u_j := 1 \ (j = 1, 2, \ldots, n - 1),$$

we define the function $d_i(z)$ for $i = 0, 1, \ldots, n$ as

$$d_i(z) = \frac{t_i^{1/2} - t_i^{-1/2} + (u_i^{1/2} - u_i^{-1/2})z^{1/2}}{1 - z}.$$

Then, comparing [24, p.52] and [23, §4.3] (see also [37, (2.2.8)–(2.2.11)]), we have the relation

$$b_i = d_i(x^{\alpha_i}). \tag{2.2.19}$$

This relation (2.2.19) yields the correspondence

$$(\tau_0, \tau_n, \tau'_0, \tau'_n, \tau_j = \tau'_j) = (t_0^{1/2}, t_n^{1/2}, u_0^{1/2}, u_n^{1/2}, t_j^{1/2}). \tag{2.2.20}$$

Combining it with (2.2.16), we obtain the relation (2.2.4).

2.3 Specialization to affine root system of type $C_n$

As an illustration of deriving the specialization Table 1, we explain how to find the parameter specialization for type $C_n$:

$$\begin{array}{c|ccc|ccc} t & t_0 & t_n & u_0 & u_n \\ \hline C_n & t_0 & t_n & u_0 & u_n & t_0^{1/2} & t_n^{1/2} \\ C_n' & t_0^{1/2} & t_n^{1/2} & 1 & 1 \end{array}$$

As mentioned in Sect. 1, we need to observe the correspondence of the orbits of extended affine Weyl groups of type $(C_n', C_n)$ and of type $C_n$. So we start with the
explanation on the description of the type $C_n$ as the affine root subsystem of the type $(C_n^\vee, C_n)$.

Using the description (2.1.1) of the affine root system $S$ of type $(C_n^\vee, C_n)$, let us consider the following subset $S_C$ of $S$.

$S_C := O_5^C \sqcup O_1^C$, $O_2^C := O_4 = \{ \pm 2\epsilon_i + r \mid 1 \leq i \leq n, r \in \mathbb{Z} \}$,

$O_5^C := O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, r \in \mathbb{Z} \}$. (2.3.1)

It is the affine root system of type $C_n$ in the sense of [23, §1.3, (1.3.4)]. The following gives a basis $\{ a_0^C, a_1^C, \ldots, a_n^C \}$ of $S_C$ in the sense of [23, §1.2].

$a_0^C := 2a_0 = -2\epsilon_1 + 1$, $a_j^C := a_j = \epsilon_j - \epsilon_{j+1}$ (1 $\leq j \leq n - 1$), $a_n^C := 2a_n = 2\epsilon_n$.

Here is the Dynkin diagram cited from [23, (1.3.4)]:

The description (2.3.1) gives the decomposition of the extended affine Weyl group $W_C$. To describe it, recall the finite Weyl group $W_0$ of type $C_n$ in (2.1.4), which can be rewritten as $W_0 = \langle s_1^C s_2^C \cdots s_n^C \rangle$. We also denote by

$L' = P_{C_n}^\vee = P_{B_n} := \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \mathbb{Z}\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$ (2.3.2)

the weight lattice of finite root system of type $B_n$. Then $W_C$ is given by

$W_C := W_0 \ltimes t(L') = W_0 \ltimes t(P_{B_n})$, (2.3.3)

and it acts on $S_C$ [23, §1.4, (1.4.6), (1.4.7)]. The corresponding $W_C$-orbits are given by the above $O_5^C$ and $O_1^C$ [23, §1.5]. By (2.3.1), we have $a_0^C, a_n^C \in O_2 \sqcup O_4 = O_1^C$ and $a_j^C \in O_5 = O_5^C$ (1 $\leq j \leq n - 1$).

Next, we explain the parameters for $S_C$. Similarly as in Sect. 2.2, we attach parameters $k_s^C, k_l^C \in \mathbb{R}$ to the $W_C$-orbits as

$k_s^C \longleftrightarrow O_5^C, \quad k_l^C \longleftrightarrow O_1^C$, (2.3.4)

and define the label $k^C : S_C \rightarrow \mathbb{R}$ in the same way as $k : S \rightarrow \mathbb{R}$ in (2.2.2). We also denote

$t_l^C := q^{k_l^C}$, $t_s^C := q^{k_s^C}$ (1 $\leq j \leq n - 1$), (2.3.5)

We now argue that under the specialization

$(t, t_0, t_n, u_0, u_n) \mapsto (t_s^C, (t_l^C)^2, (t_l^C)^2, 1, 1)$,
the non-symmetric Koornwinder polynomials degenerate into the non-symmetric Macdonald polynomials of type \( C_n \). Recalling that both polynomials are determined uniquely by the inner products, or by the weight functions, we see that it is enough to check that the weight function \( \Delta_{S,k} \) in (2.2.8) of type \( (C_n, C_n) \) degenerates to that of type \( C_n \). The latter weight function is given by [23, (5.1.7)]:

\[
\Delta^C = \Delta_{S^C,k}^C := \prod_{a \in (S^C)^+} \frac{1 - q^{kC(2a)}e^{a}}{1 - q^{kC(a)}e^{a}}.
\]

Here \((S^C)^+ \subset S^C\) is the set of positive roots with respect to the basis \( \{a_0^C, a_1^C, \ldots, a_n^C\} \), i.e., \((S^C)^+ := \sum_{l=0}^{n} \mathbb{N}a_l^C \setminus \{0\}\), and \(k^C : S^C \rightarrow \mathbb{R}\) is the extension of the label \( k^C \) (see (2.3.4)) by \( k^C(2a) := 0 \) \( (a \notin S) \). Recalling (2.2.10), we have

\[
\Delta_{S,k} \Big|_{k(a) - k(2a) = 0 \, (a \in S \setminus S^C)} = \Delta_{S^C,k}.
\]

Thus, the desired specialization is given by

\[
k(a) - k(2a) \mapsto 0 \, \ (a \in S \setminus S^C), \quad k(a) - k(2a) \mapsto k^C(a) \, \ (a \in S^C). \quad (2.3.6)
\]

Since (2.3.1) yields \( S \setminus S^C = O_1 \sqcup O_3, S^C = O_5^C \sqcup O_1^C, O_5^C = O_5 \) and \( O_1^C = O_2 \sqcup O_4 \), the map (2.3.6) can be rewritten in terms of \( k_1, k_2, \ldots, k_5 \) and \( k_s^C, k_l^C \) as

\[
k_1 - k_2, k_3 - k_4 \mapsto 0, \quad k_2, k_4 \mapsto k_s^C, \quad k_5 \mapsto k_l^C.
\]

Using (2.2.4) and (2.3.5), and assuming \( u_0, u_n > 0 \), we can further rewrite it as

\[
(t_nu_n)/t_n, \quad (t_0u_0)/t_0 \mapsto 1, \quad (t_0u_0)/t_0, \quad t_nu_n \mapsto (t_s^C)^2,
\]

\[
t \mapsto t_s^C \iff (t, t_0, t_n, u_0, u_n) \mapsto \left( (t_s^C)^2, (t_l^C)^2, 1, 1 \right). \quad (2.3.7)
\]

Now we suppress the superscript \( C \) in \( t_s^C \) and \( t_l^C \), and denote by

\[
E^C_\mu(x; q, t_s, t_l), \quad \mu \in P_{C_n}
\]

the non-symmetric Macdonald polynomial of type \( C_n \) (Definition 1.0.1). Similarly, for a dominant \( \mu \in P_{C_n} \), we denote by \( P^C_\mu(x; q, t_s, t_l) \) the symmetric Macdonald polynomials of type \( C_n \). Then the conclusion of this Sect. 2.3 is:

**Proposition 2.3.1** For any \( \mu \in P_{C_n} \), we have

\[
E^C_\mu(x; q, t_s, t_l) = E_\mu(x; q, t_s, t_l^2, t_l^2, 1, 1).
\]

Also, for a dominant weight \( \mu \), we have

\[
P^C_\mu(x; q, t_s, t_l) = P_\mu(x; q, t_s, t_l^2, t_l^2, 1, 1).
\]
The following table shows the comparison of the correspondence (2.2.5) between the Noumi parameters and the $W$-orbits with that (2.3.4) between the parameters of type $C_n$ and the $W^C$-orbits.

| Type$(C_n^\vee, C_n)$ | Type$C_n$ |
|------------------------|-----------|
| $t_n u_n$ ←→ $O_1$    | $t_0 u_0$ ←→ $O_3$ |
| $t_n / u_n$ ←→ $O_2^C$ | $t_l^C$ ←→ $O_l = O_2 \sqcup O_4$ |
| $t_0 / u_0$ ←→ $O_4$   | $t_s^C$ ←→ $O_s = O_5$ |

**Remark 2.3.2** One may wonder whether it is possible to see the specialization (2.3.7) on the level of affine Hecke algebras. To clarify the point, let us denote by $H^C$ the affine Hecke algebra associated to the group $W^C$ (2.3.3) in the sense of [23, 4.1]. It is an associative algebra over $K \subset \mathbb{R}$ (see Sect. 2.2), and as a $K$-linear space, it has the form $H^C = H_0 \otimes_K K[Y^\lambda \mid \lambda' \in P_{B_n}] \simeq H_0 \otimes_K K P_{B_n}$ by [23, (4.2.7), (4.3.1)]. Here we used similar notation as in (2.2.14). In particular, $H_0$ is the Hecke algebra associated to the finite Weyl group $W_0$ of type $C_n$, and the part $K[Y^\lambda \mid \lambda' \in P_{B_n}]$ is a commutative subalgebra. Also, following [23, (4.4.2)], we define $\tau_{C,i} = \tau_{C,i}^t := q^{rC}/2$, where $r := s (a_i \in O_s^C)$ and $r := l (a_i \in O_l^C)$. Then, using the function (2.2.18) with the parameters $\tau_{C,i}$ and $\tau_{C,i}^t$ instead of $\tau_i$ and $\tau_i^t$, we have a faithful $H^C$-module

$$\beta^C : H^C \longrightarrow \text{End}_K (K P_{C_n})$$

which is the basic representation of type $C_n$. The basic representations $\beta$ (2.2.17) and $\beta^C$ sit in the following diagram.

$$\begin{array}{c}
\begin{array}{c}
H \\
\downarrow \beta \\
\text{End}_K (K P_{C_n})
\end{array} & \text{H}^C & \begin{array}{c}
\downarrow \beta^C \\
\text{End}_K (K P_{C_n})
\end{array}
\end{array}$$

One can see that the specialization (2.3.7) maps $\beta(T_j) \mapsto \beta^C(T_j)$ $(1 \leq j \leq n - 1)$, but the images of $\beta(T_i)$ is not equal to $\beta^C(T_i)$ for $i = 0, n$. Thus, it is unclear whether we can see the specialization on the level of affine Hecke algebras $H$ and $H^C$.

### 2.4 Specialization to other subsystems

For all the subsystems of the affine root system $S$ of type $(C_n^\vee, C_n)$, we can make similar arguments as in Sect. 2.3, which will yield the specialization Table 1. In this subsection, we list all the arguments except type $C_n$ which is already done. Let us write again the specialization table:
A remark is in order on the treatment of the type $BC_n$ and the non-reduced systems. As we have seen in Sect. 2.2.1, the argument on the specialization to type $C_n$ used the extended affine Weyl group of type $C_n$. In contrast, as commented at the beginning of Sect. 5.1 and (5.1.7) in [23], we don’t have the extended affine Weyl groups (or the affine Hecke algebras) associated to the type $BC_n$ and the non-reduced systems, so we cannot follow the argument in Sect. 2.2.1. However, the (non-symmetric) Macdonald polynomials for non-reduced systems are defined as the specialization of Koornwinder polynomials in [23], and thus the situations are easier than reduced systems.

### 2.4.1 Type $B_n$

For $n \in \mathbb{Z}_{\geq 3}$, the following subset $S^B \subset S$ forms the affine root system of type $B_n$ in the sense of [23, §1.3, (1.3.2)].

$$S^B := O^B_s \sqcup O^B_l,$$

where

$$O^B_s := O_1 = \{ \pm \epsilon_i + r \mid 1 \leq i \leq n, \ r \in \mathbb{Z} \},$$

$$O^B_l := O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, \ r \in \mathbb{Z} \}.$$  (2.4.1)

Using the symbol $L' = P^\vee_{B_n} = P_{C_n} = \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i$ in [23, 1.4], the extended affine Weyl group is given by

$$W^B := W^B_0 \times t(L') = W^B_0 \times t(P_{C_n}) \cong W.$$  (2.4.2)

Here $W^B_0$ denotes the Weyl group of the finite root lattice $B_n$. The group $W^B$ acts on $S^B$ by permutation, and the $W^B$-orbits are given by $O^B_s$ and $O^B_l$. We attach parameters $k^B_s$ and $k^B_l$ to the $W^B$-orbits as

$$O^B_s \longleftrightarrow k^B_s, \quad O^B_l \longleftrightarrow k^B_l,$$

and define the label $k^B$ by

$$k^B : S^B \longrightarrow \mathbb{R}, \quad k^B(a) := k^B_s (a \in O^B_s), \quad k^B(a) := k^B_l (a \in O^B_l).$$
Mimicking the relation (2.2.4), we introduce the parameters of type $B_n$ by

$$\begin{align*}
t_s^B &:= q^{k_s^B}, \quad t_l^B := q^{k_l^B}.
\end{align*}$$

(2.4.3)

They correspond to the $W^B$-orbits as $t_s^B \leftrightarrow O_s^B$ and $t_l^B \leftrightarrow O_l^B$.

The weight function of type $B_n$ is given by

$$\Delta^B = \Delta^B_{\mu^B := \prod_{a \in (S^B)^+} \frac{1 - q^{k^B(2a)} e^a}{1 - q^{k^B(a)} e^a}.$$  

Then, (2.2.10) yields

$$\Delta_{\mu^B} \bigg|_{k(\mu) - k(2\mu) = 0 \, (a \in S \setminus S^B)} = \Delta^B_{\mu^B}.$$  

Thus the desired specialization is given by

$$k(2a) \rightarrow 0 \quad (a \in S \setminus S^B), \quad k(a) - k(2a) \rightarrow k^B(a) \quad (a \in S^B).$$

By (2.4.1), we have $S \setminus S^B = O_2 \sqcup O_3 \sqcup O_4$, $S^B = O_s^B \sqcup O_l^B$, $O_s^B = O_1$ and $O_l^B = O_5$. Then, we can rewrite the specialization in terms of $k_1, \ldots, k_5$ and $k_s^B, k_l^B$ as

$$k_2 - 0, \quad k_3 - k_4, \quad k_4 - 0 \rightarrow 0, \quad k_1 \rightarrow k_s^B, \quad k_5 \rightarrow k_l^B.$$  

Using (2.2.4) and (2.4.3), and assuming $t_0, u_0 > 0$, we have

$$\begin{align*}
\frac{t_n}{u_n} \cdot \frac{t_0 u_0}{u_0} \rightarrow 1, \quad t_n u_n \rightarrow (t_s^B)^2, \\
t \rightarrow t_l^B \iff (t, t_0, t_n, u_0, u_n) \rightarrow (t_l^B, 1, t_s^B, 1, t_s^B).
\end{align*}$$

(2.4.4)

Now we suppress the superscript $B$ in the parameters, and denote by

$$E^B_{\mu}(x; q, t_s, t_l), \quad \mu \in P_{B_n} := \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i \oplus \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$$

the non-symmetric Macdonald polynomial of type $B_n$ (Definition 1.0.1). Having that $P_{C_n} \subset P_{B_n}$, we conclude:

**Proposition 2.4.1** For any $\mu \in P_{C_n}$, we have

$$E^B_{\mu}(x; q, t_s, t_l) = E_{\mu}(x; q, t_l, 1, t_s, 1, t_s).$$

Also, for a dominant weight $\mu$, we have

$$P^B_{\mu}(x; q, t_s, t_l) = P_{\mu}(x; q, t_l, 1, t_s, 1, t_s)$$

for the symmetric Macdonald polynomials of type $B_n$. 

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Remark 2.4.2 We can make a similar observation as in Remark 2.3.2. Let us denote by $H^B$ the affine Hecke algebra for the extended Weyl group $W^B$ in the sense of [23, Chap. 4]. As a linear space over the base field $K$, we have $H^B \simeq H_0 \otimes_K K P_{C_n} \simeq H$. Denoting by $\beta^B$ the basic representation of $H^B$, we have the following diagram.

$$
\begin{array}{ccc}
H & \xrightarrow{\beta} & H^B \\
\downarrow & & \downarrow \\
\text{End}_K(K P_{C_n}) & \longrightarrow & \text{End}_K(K P_{B_n})
\end{array}
$$

As in Remark 2.3.2, we can that the specialization (2.4.4) maps $\beta(T_j) \mapsto \beta^B(T_j)$ for $j = 1, 2, \ldots, n - 1$, but the images of $\beta(T_i)$ is not equal to $\beta^B(T_i)$ for $i = 0, n$.

2.4.2 Type $B_n^\vee$

For $n \in \mathbb{Z}_{\geq 3}$, the following subset $S^{B_n^\vee} \subset S$ forms the affine root system of type $B_n^\vee$ in the sense of [23, §1.3, (1.3.3)].

$$
S^{B_n^\vee} := O^{B_n^\vee}_I \cup O^{B_n^\vee}_S, \quad O^{B_n^\vee}_I := O_2 = \{ \pm 2\epsilon_i + 2r \mid 1 \leq i \leq n, \ r \in \mathbb{Z} \}, \\
O^{B_n^\vee}_S := O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, \ r \in \mathbb{Z} \}.
$$

Using the symbol $L = L' = P_{B_n} = P_{C_n} = \oplus_{i=1}^n \mathbb{Z} \epsilon_i$ in [23, 1.4], the extended affine Weyl group is given by

$$
W^{B_n^\vee} := W^B_0 \ltimes t(L') = W^B_0 \ltimes t(P_{C_n}) \simeq W.
$$

It acts on $S^{B_n^\vee}$, and the $W^{B_n^\vee}$-orbits are $O^{B_n^\vee}_S$ and $O^{B_n^\vee}_I$. We attach parameters to these orbits as

$$
k^{B_n^\vee}_S \longleftrightarrow O^{B_n^\vee}_S, \quad k^{B_n^\vee}_I \longleftrightarrow O^{B_n^\vee}_I,
$$

and define the label $k^{B_n^\vee} : S^{B_n^\vee} \rightarrow \mathbb{R}$ as before. We also introduce another set of parameters as

$$
l^{B_n^\vee}_I := \tau^{B_n^\vee}_{2,n} = q^{B_n^\vee}_n, \quad l^{B_n^\vee}_I := \tau^{B_n^\vee}_{2,j} = q^{B_n^\vee}_{j} \quad (0 \leq i \leq n - 1).
$$

They correspond to the $W^{B_n^\vee}$-orbits as $l^{B_n^\vee}_S \leftrightarrow O^{B_n^\vee}_S$ and $l^{B_n^\vee}_I \leftrightarrow O^{B_n^\vee}_I$. 

\[ \text{Springer} \]
The weight function of type $B_n^\vee$ is given by
\[
\Delta^{B_n^\vee} = \Delta_{SB_n^\vee,k^{B_n^\vee}} := \prod_{a \in (SB_n^\vee)^+} \frac{1 - q^{k^{B_n^\vee}(2a)}e_a}{1 - q^{k^{B_n^\vee}(a)}e_a}.
\]

Then, (2.2.10) yields
\[
\Delta_{S,k} \big|_{k(a)-k(2a)=0, (a \in S \setminus SB_n^\vee)} = \Delta_{SB_n^\vee,k}.
\]

Thus the specialization from type $(C_n^\vee, C_n)$ to type $B_n^\vee$ is given by
\[
k(a) - k(2a) \mapsto 0 \quad (a \in S \setminus SB_n^\vee), \quad k(a) - k(2a) \mapsto k^B(a) \quad (a \in SB_n^\vee).
\]

By (2.4.5), we have $S \setminus SB_n^\vee = O_1 \sqcup O_3 \sqcup O_4, S_n^\vee = O_s^{B_n^\vee} \sqcup O_l^{B_n^\vee}, O_s^{B_n^\vee} = O_5$ and $O_l^{B_n^\vee} = O_2$. Then the above specialization can be written as
\[
k_1 - k_2, k_3 - k_4, k_4 - 0 \mapsto 0, \quad k_2 \mapsto k_l^{B_n^\vee}, \quad k_5 \mapsto k_s^{B_n^\vee}.
\]

Using (2.2.4) and (2.4.7), and assuming $t_0, u_n, u_0 > 0$, we can further rewrite it as
\[
(t_n/u_n, t_0/u_0) \mapsto (1, t_n/u_n) \mapsto (t_l^{B_n^\vee})^2, \\
t \mapsto t_s^{B_n^\vee} \iff (t, t_0, t_n, u_0, u_n) \mapsto (t_s^{B_n^\vee}, 1, (t_l^{B_n^\vee})^2, 1, 1).
\]

Now we suppress the superscript $B_n^\vee$ in the parameters, and denote by
\[
E_{B_n^\vee}(x; q, t, t_l), \quad \mu \in P_{B_n^\vee} = P_{C_n}
\]
the non-symmetric Macdonald polynomial of type $B_n^\vee$ (Definition 1.0.1). The conclusion of this Sect. 2.4.2 is:

**Proposition 2.4.3** For any $\mu \in P_{C_n}$, we have
\[
E_{B_n^\vee}(x; q, t, t_l) = E_{\mu}(x; q, t, t_l^2, 1, 1).
\]

**2.4.3 Type $C_n^\vee$**

For $n \in \mathbb{Z}_{\geq 2}$, the following subset $S_{C_n^\vee} \subset S$ forms the affine root system of type $C_n^\vee$ in the sense of [23, §1.3, (1.3.5)].

\[
\begin{align*}
S_{C_n^\vee} := & \, O_{s_{C_n^\vee}} \cup O_{l_{C_n^\vee}}, \quad O_{s_{C_n^\vee}} := O_1 \sqcup O_3 = \{ \pm \epsilon_i + \frac{1}{2} r \mid 1 \leq i \leq n, \ r \in \mathbb{Z} \}, \\
O_{l_{C_n^\vee}} := & \, O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, \ r \in \mathbb{Z} \}.
\end{align*}
\]

(2.4.8)
Using $L = L' = P^\vee_{C_n} = P_{B_n} = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$, the extended affine Weyl group is given by

$$W^\vee_{C_n} := W_0 \ltimes t(L') = W_0 \ltimes t(P_{B_n}) = W^\vee.$$  

The $W^\vee_{C_n}$-orbits on $S^\vee_{C_n}$ are $O^\vee_{S}$ and $O^\vee_{I}$. We define the label $k^\vee_{S} : S^\vee_{C_n} \to \mathbb{R}$ using the correspondence

$$k^\vee_{S} \leftrightarrow O^\vee_{S}, \quad k^\vee_{I} \leftrightarrow O^\vee_{I}.$$  

Mimicking the relation (2.2.4), we define another set of parameters as

$$t^\vee_{S} := q^{k^\vee_{S}}, \quad t^\vee_{I} := q^{k^\vee_{I}}.$$  

They correspond to the $W^\vee_{C_n}$-orbits as $t^\vee_{S} \leftrightarrow O^\vee_{S}$ and $t^\vee_{I} \leftrightarrow O^\vee_{I}$. The weight function is given by

$$\Delta^\vee_{C_n} = \Delta^\vee_{S^\vee_{C_n}, k^\vee_{S}} := \prod_{a \in (S^\vee_{C_n})^+} \frac{1 - q^{k^\vee_{S}(2a)}e^a}{1 - q^{k^\vee_{S}(a)}e^a}.$$  

Then (2.2.10) yields

$$\Delta_{S,k} \bigg|_{k(a) - k(2a) = 0 (a \in S \setminus S^\vee_{C_n})} = \Delta^\vee_{S^\vee_{C_n}, k}.$$  

Thus the specialization to type $C_n^\vee$ is given by

$$k(a) - k(2a) \mapsto 0 \quad (a \in S \setminus S^\vee_{C_n}), \quad k(a) - k(2a) \mapsto k_B(a) \quad (a \in S^B).$$  

By (2.4.8), we have $S \setminus S^\vee_{C_n} = O_2 \sqcup O_4$, $S^\vee_{C_n} = O^\vee_{S} \sqcup O^\vee_{I}$, $O^\vee_{S} = O_1 \sqcup O_3$ and $O^\vee_{I} = O_5$. Then we can rewrite the above specialization as

$$k_2 \to 0, \quad k_4 \to 0 \mapsto 0, \quad k_1, k_3 \mapsto k^\vee_{S}, \quad k_5 \mapsto k^\vee_{I}.$$  

Using (2.2.4) and (2.4.10), we can rewrite it as

$$t_n \mapsto 1, \quad t_n u_n, t_0 u_0 \mapsto (t^\vee_{S})^2, \quad \left(\begin{array}{c} \frac{t_n}{u_n} \\frac{t_0}{u_0} \end{array}\right) \mapsto \left(\begin{array}{c} t^\vee_{S} \quad t^\vee_{I} \quad t^\vee_{S} \quad t^\vee_{I} \end{array}\right).$$  

We suppress the superscript $C^\vee$ in the parameters, and denote by

$$E^\vee_{\mu}(x; q, t_5, t_I), \quad \mu \in P^\vee_{C_n} = P_{B_n}.$$  

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the non-symmetric Macdonald polynomial of type \( C_n^\vee \) (Definition 1.0.1). Noting that \( P_{C_n} \subset P_{B_n} \), we have the conclusion:

**Proposition 2.4.4** For any \( \mu \in P_{C_n} \), we have

\[
E_{\mu}^{C^\vee}(x; q, t_s, t_l) = E_{\mu}(x; q, t_l, t_s, t_s, t_s, t_s)
\]

### 2.4.4 Type \( BC_n \)

For \( n \in \mathbb{Z}_{\geq 1} \), the following subset \( S_{BC} \subset S \) forms the affine root system of type \( BC_n \) in the sense of [23, §1.3, (1.3.6)].

\[
S_{BC} := O_s^{BC} \sqcup O_m^{BC} \sqcup O_l^{BC}, \quad O_s^{BC} := O_1 = \{ \pm \epsilon_i + r \mid 1 \leq i \leq n, \ r \in \mathbb{Z} \}, \quad O_l^{BC} := O_4 = \{ \pm 2\epsilon_i + 2r + 1 \mid 1 \leq i \leq n, \ r \in \mathbb{Z} \}, \quad O_m^{BC} := O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, \ r \in \mathbb{Z} \}.
\]

(2.4.11)

Hereafter we assume \( n \geq 2 \) to make the argument compatible with that so far. The Dynkin diagram is then given by

\[
0 \quad 1 \quad 2 \quad \ldots \quad n-1 \quad n
\]

(2.4.12)

Recall the comment in the beginning of this Sect. 2.4. We will not introduce a new extended affine Weyl group, but consider the group \( W \) of type \( (C_n^\vee, C_n) \) (see (2.1.9)). It acts on \( S_{BC} \), and the \( W \)-orbits are given by \( O_s^{BC} \), \( O_m^{BC} \) and \( O_l^{BC} \). Hence, we already have the correspondence between the Macdonald parameters of type \( (C_n^\vee, C_n) \) and the \( W \)-orbits on \( S_{BC} \). Let us denote

\[
t_s^{BC} := q^{k_1}, \quad t_m^{BC} := q^{k_5}, \quad t_l^{BC} := q^{k_4},
\]

(2.4.13)

which correspond to the \( W \)-orbits \( O_s^{BC}, O_m^{BC} \) and \( O_l^{BC} \), respectively.

Following [23, (5.1.77)], we define the weight function \( \Delta_{S_{BC,k}} \) of type \( BC_n \) to be the specialization of \( \Delta_{S,k} \) of type \( (C_n^\vee, C_n) \). In other words, we take the right hand side of (2.2.10) as the definition:

\[
\Delta^{BC} = \Delta_{S_{BC,k}} := \Delta_{S,k} \big|_{k(a) - k(2a) = 0 \ (a \in S \backslash S_{BC})}.
\]

By (2.4.11), we have \( S \backslash S_{BC} = O_2 \sqcup O_3 \) and \( S_{BC} = O_s^{BC} \sqcup O_m^{BC} \sqcup O_l^{BC} = O_1 \sqcup O_5 \sqcup O_4 \). Then, we can see that the specialization to type \( BC_n \) is given by

\[
k_2 - 0, \ k_3 - k_4 \mapsto 0.
\]

Using (2.2.4) and (2.4.13), we can rewrite it as

\[
t_{n}, \ (t_0u_0)/t_0 \mapsto 1, \quad t_n u_n \mapsto (t_s^{BC})^2, \quad t_0 \mapsto (t_l^{BC})^2, \quad t \mapsto t_m^{BC}
\]
\( \iff (t, t_0, t_n, u_0, u_n) \mapsto \left( t_m^{BC}, (t_l^{BC})^2, t_s^{BC}, 1, t_s^{BC} \right) \).

Now we suppress the superscript \( BC \) in the parameters, and denote by

\[ E_{\mu}^{BC}(x; q, t_s, t_m, t_l), \quad \mu \in P_{Cn} \]

the non-symmetric Macdonald polynomial of type \( BC_n \) (Definition 1.0.1). Then the conclusion is:

**Proposition 2.4.5** For any \( \mu \in P_{Cn} \), we have

\[ E_{\mu}^{BC}(x; q, t_s, t_m, t_l) = E_{\mu}(x; q, t_m, t_l^2, t_s, 1, t_s). \]

### 2.4.5 Type \( D_n \)

For \( n \in \mathbb{Z}_{\geq 4} \), the following subset \( S^D \subset S \) forms the affine root system of type \( D_n \) in the sense of [23, §1.3, (1.3.7)].

\[ S^D := O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r | 1 \leq i < j \leq n, r \in \mathbb{Z} \}. \tag{2.4.14} \]

![Diagram of affine root system of type Dn](image)

Using the Weyl group \( W_0^D \) and the weight lattice

\[ L' = P_{D_n} := \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n \oplus \mathbb{Z}\left( \epsilon_1 + \cdots + \epsilon_n \right) \tag{2.4.15} \]

of the finite root system of type \( D_n \), the extended affine Weyl group is given by

\[ W^D := W_0^D \rtimes t(L') = W_0^D \rtimes t(P_{D_n}). \tag{2.4.16} \]

It acts on \( S^D \) by permutation, and there is a unique orbit. Attaching \( k^D \in \mathbb{R} \) to this unique orbit, we define the label by \( k^D(a) := k^D(a \in S^D) \), and introduce

\[ t^D := q^{k^D}. \tag{2.4.17} \]

The weight function is given by

\[ \Delta^D = \Delta_{S^D, k^D} := \prod_{a \in (S^D)^+} \frac{1 - q^{k^B(2a)}e^a}{1 - q^{k^B(a)}e^a}. \]

The relation (2.2.10) yields

\[ \Delta_{S, k} \mid _k(a) - k(2a) = 0 (a \in S \setminus S^D) = \Delta_{S^D, k}. \]
Thus, the specialization to type $D_n$ is given by

\[ k(a) - k(2a) \rightarrow 0 \quad (a \in S \setminus S^D), \quad k(a) - k(2a) \rightarrow k^D(a) \quad (a \in S^D). \]

By (2.4.14), we have $S \setminus S^D = O_1 \sqcup \cdots \sqcup O_4$ and $S^B = O_5$. Then, we can rewrite the specialization in terms of $k_1, \ldots, k_5$ and $k^D$ as

\[ k_2 - 0, \quad k_3 - k_4, \quad k_4 - 0 \rightarrow 0, \quad k_1 \rightarrow k^B_s, \quad k_5 \rightarrow k^B_l. \]

Using (2.2.4) and (2.4.17), we have

\[ t_n u_n, \quad t_n/u_n, \quad t_0 u_0, \quad t_0/u_0 \rightarrow 1, \]
\[ t \leftrightarrow t^D \iff (t, t_0, t_n, u_0, u_n) \rightarrow (t^D, 1, 1, 1, 1). \]

We suppress the superscript $D$ in the parameters, and denote by

\[ E^D_{\mu}(x; q, t), \quad \mu \in P_{D_n} \]

the non-symmetric Macdonald polynomial of type $D_n$ (Definition 1.0.1). Since $P_{C_n} \subset P_{D_n}$, we have:

**Proposition 2.4.6** For any $\mu \in P_{C_n}$, we have

\[ E^D_{\mu}(x; q, t) = E_{\mu}(x; q, t, 1, 1, 1, 1). \]

### 2.4.6 Type $(BC_n, C_n)$

For $n \in \mathbb{Z} \geq 1$, the following subset $S^{BC,C} \subset S$ forms the affine root system of type $(BC_n, C_n)$ in the sense of [23, §1.3, (1.3.15)].

\[
S^{BC,C} := O^{BC,C}_s \sqcup O^{BC,C}_m \sqcup O^{BC,C}_l,
\]
\[
O^{BC,C}_s := O_1 = \{ \pm \epsilon_i + r \mid 1 \leq i \leq n, \quad r \in \mathbb{Z} \},
\]
\[
O^{BC,C}_l := O_2 \sqcup O_4 = \{ \pm 2 \epsilon_i + r \mid 1 \leq i \leq n, \quad r \in \mathbb{Z} \},
\]
\[
O^{BC,C}_m := O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, \quad r \in \mathbb{Z} \}.
\]

\[ (2.4.18) \]

The diagram is for $n \geq 2$, and hereafter we assume this condition. The mark * above the index $n$ implies that there is a basis $\{a^{BC,C}_i\}_{i=0}^n$ such that $a^{BC,C}_n, 2a^{BC,C}_n \in S^{BC,C}$. There are three $W$-orbits $O^{BC,C}_s, O^{BC,C}_m$ and $O^{BC,C}_l$. We introduce the parameters

\[ t^{BC,C}_s := q^{k_1}, \quad t^{BC,C}_m := q^{k_5}, \quad t^{BC,C}_l := q^{k_2}. \]

\[ (2.4.19) \]
which correspond to the $W$-orbit $O_{BC}^{S,C}_s$, $O_{BC}^{m,C}_m$ and $O_{BC}^{l,C}_l$, respectively.

Similarly as in the previous Sect. 2.4.4, the weight function $\Delta_{S,C,k}^{BC}$ of type $(BC_n, C_n)$ is defined by the specialization of $\Delta_{S,k}$ as

$$\Delta_{S,C,k}^{BC} = \Delta_{S,C,k}^{S,C} \big|_{k(a)-k(2a)=0 \ (a \in S)}.$$

By (2.4.18), we have $S \setminus S_{BC}^{C} = O_3$, $O_{l}^{BC,C} = O_2 \sqcup O_4$, which implies that the specialization to type $(BC_n, C_n)$ is given by

$$k_3 - k_4 \mapsto 0, \quad k_2 \mapsto k_4.$$

Using (2.2.4) and (2.4.20), and assuming $u_0 > 0$, we can rewrite it as

$$(t_0 u_0)/t_0 u_0 \mapsto 1, \quad t_n u_n \mapsto (t_s^{BC,C})^2, \quad t_{n/u_0} \mapsto (t_l^{BC,C})^2, \quad t \mapsto t_m^{BC,C}$$

$$\iff (t, t_0, t_n, u_0, u_n) \mapsto (t_s^{BC,C}, (t_s^{BC,C})^2, t_s^{BC,C}, 1, t_s^{BC,C}/t_l^{BC,C}).$$

We suppress the superscript $BC, C$ in the parameters, and denote by

$$E_{\mu}^{BC,C}(x; q, t_s, t_m, t_l), \quad \mu \in P_{C_n}$$

the non-symmetric Macdonald polynomial of type $(BC_n, B_n)$ (Definition 1.0.1). The conclusion of this Sect. 2.4.6 is:

**Proposition 2.4.7** For any $\mu \in P_{C_n}$, we have

$$E_{\mu}^{BC,C}(x; q, t_s, t_m, t_l) = E_{\mu}(x; q, t_m, t_l^2, t_s t_l, 1, t_s/t_l).$$

2.4.7 Type $(C_n^\vee, BC)$

For $n \in \mathbb{Z}_{\geq 1}$, the following subset $S_{C_n^\vee, BC} \subset S$ forms the affine root system of type $(BC_n, C_n)$ in the sense of [23, §1.3, (1.3.16)].

$$S_{C_n^\vee, BC} := O_{s}^{C_n^\vee, BC} \sqcup O_{m}^{C_n^\vee, BC} \sqcup O_{l}^{C_n^\vee, BC},$$

$$O_{s}^{C_n^\vee, BC} := O_1 \sqcup O_3 = \{ \pm \epsilon_i + \frac{1}{2} r \ | \ 1 \leq i \leq n, \ r \in \mathbb{Z} \}, \quad (2.4.21)$$

$$O_{l}^{C_n^\vee, BC} := O_2 = \{ \pm 2 \epsilon_i + 2 r \ | \ 1 \leq i \leq n, \ r \in \mathbb{Z} \},$$

$$O_{m}^{C_n^\vee, BC} := O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \ | \ 1 \leq i < j \leq n, \ r \in \mathbb{Z} \}.$$

Hereafter we assume $n \geq 2$. Then the Dynkin diagram is given by

$$0 \quad 1 \quad 2 \quad \ldots \quad n-1 \quad n$$

(2.4.22)
There are three \( W \)-orbits \( O_s^\vee,BC \), \( O_m^\vee,BC \) and \( O_l^\vee,BC \), and the parameters are defined to be
\[
\begin{align*}
t_s^\vee,BC &= q^{k_1} , \\
t_m^\vee,BC &= q^{k_5} , \\
t_l^\vee,BC &= q^{k_2}.
\end{align*}
\] (2.4.23)

The weight function of type \((C_n^\vee, BC_n)\) is defined by
\[
\Delta^\vee,BC = \Delta^S_{C_v,BC}, k := \Delta^S, k |_{k(a) - k(2a) = 0, (a \in S \setminus S^C_v,BC)}.
\]

By (2.4.21), we have \( S \setminus S^C_v,BC = O_4 \) and \( O^C_v,BC = O_1 \sqcup O_3 \), which implies that
\[
k_4 \mapsto 0, \quad k_1 \mapsto k_3
\]
give the desired specialization. Using (2.2.4) and (2.4.23), we can rewrite it as
\[
t_0/u_0 \mapsto 1, \quad t_0u_0, t_nu_n \mapsto (t_s^\vee,BC)^2, \quad t_n/u_n \mapsto (t_l^\vee,BC)^2, \quad t \mapsto t_m^\vee,BC
\]
\[
\iff (t, t_0, t_n, u_0, u_n) \mapsto (t_m^\vee,BC, t_s^\vee,BC, t_s^\vee,BC, t_s^\vee,BC, t_l^\vee,BC, t_l^\vee,BC, t_l^\vee,BC)/t_l^\vee,BC).
\]

We suppress the superscript \( C_v, BC \) in the parameters, and denote by
\[
E^C_v,BC \mu(x; q, t_s, t_m, t_l), \quad \mu \in PC_n
\]
the non-symmetric Macdonald polynomial of type \((C_n^\vee, BC_n)\) (Definition 1.0.1). The conclusion of this Sect. 2.4.6 is:

**Proposition 2.4.8** For any \( \mu \in PC_n \), we have
\[
E^C_v,BC \mu(x; q, t_s, t_m, t_l) = E_\mu(x; q, t_m, t_s, t_s, t_l).
\]

### 2.4.8 Types \((C_2, C_2^\vee)\) and \((B_n^\vee, B_n)\)

The affine root systems of type \((C_2, C_2^\vee)\) and of type \((B_n^\vee, B_n)\) with \( n \in \mathbb{Z}_{\geq 3} \) in the sense of [23, §1.3, (1.3.17)] are given by the following subset \( S^{B^\vee, B} \subset S \).
\[
\begin{align*}
S^{B^\vee, B} &:= O_s^{B^\vee, B} \sqcup O_m^{B^\vee, B} \sqcup O_l^{B^\vee, B}, \\
O_s^{B^\vee, B} &:= O_1 = \{ \pm \epsilon_i + r \mid 1 \leq i \leq n, r \in \mathbb{Z} \}, \\
O_l^{B^\vee, B} &:= O_2 = \{ \pm 2 \epsilon_i + 2r \mid 1 \leq i \leq n, r \in \mathbb{Z} \}, \\
O_m^{B^\vee, B} &:= O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, r \in \mathbb{Z} \}.
\end{align*}
\] (2.4.24)

In the case \( n \geq 3 \), the Dynkin diagram is given by

![Dynkin diagram](image)
The $W$-orbits are $O_{s}^{B_{\vee},B}$, $O_{m}^{B_{\vee},B}$ and $O_{l}^{B_{\vee},B}$. The corresponding parameters are defined to be
\[ t_{s}^{B_{\vee},B} := q^{k_{1}} , \quad t_{m}^{B_{\vee},B} := q^{k_{5}} , \quad t_{l}^{B_{\vee},B} := q^{k_{2}} . \]

(2.4.25)

The weight function $\Delta_{S^{B_{\vee},B},k}$ is defined by
\[ \Delta^{B_{\vee},B} = \Delta_{S^{B_{\vee},B},k} := \Delta_{S,k} \big|_{k(a) - k(2a) = 0 \text{ for } a \in S^{B_{\vee},B}} . \]

By (2.4.24), we have $S \setminus S^{B_{\vee},B} = O_{3} \sqcup O_{4}$, which implies that
\[ k_{3} - k_{4}, \quad k_{4} \rightarrow 0 \]
gives the specialization to type $(C_{2}, C_{2}')$ and $(B_{n}', B_{n})$. Using (2.2.4) and (2.4.25), we can rewrite it as
\[ t_{0}u_{0}, \quad t_{0}/u_{0} \rightarrow 1 , \quad t_{n}u_{n} \leftrightarrow (t_{s}^{B_{\vee},B})^{2} , \quad t_{n}/u_{n} \leftrightarrow (t_{l}^{B_{\vee},B})^{2} , \quad t \leftrightarrow t_{m}^{B_{\vee},B} \leftrightarrow (t_{0}^{B_{\vee},B}, 1, t_{s}^{B_{\vee},B}t_{l}^{B_{\vee},B}, 1, t_{s}^{B_{\vee},B}/t_{l}^{B_{\vee},B}) . \]

We suppress the superscript $B_{\vee}, B$ in the parameters, and denote by
\[ E_{\mu}^{B_{\vee},B}(x; q, t_{s}, t_{m}, t_{l}) , \quad \mu \in P_{C_{n}} \]
the non-symmetric Macdonald polynomial of types $(C_{2}, C_{2}')$ and $(B_{n}', B_{n})$ (Definition 1.0.1). The conclusion of this Sect. 2.4.6 is:

**Proposition 2.4.9** For any $\mu \in P_{C_{n}}$, we have
\[ E_{\mu}^{B_{\vee},B}(x; q, t_{s}, t_{m}, t_{l}) = E_{\mu}(x; q, t_{m}, 1, t_{s}t_{l}, 1, t_{s}/t_{l}) . \]

### 2.5 Relation to Koornwinder’s specializations in admissible pairs

As mentioned in Sect. 1, in the original theory [21], Macdonald used admissible pairs to formulate his family of multivariate orthogonal polynomials for general root systems. Here, an admissible pair means a pair $(R, S)$ of root systems satisfying the following conditions.

- Both $R$ and $S$ span the common finite-dimensional Euclidean space $V$.
- $S$ is reduced.
- The Weyl groups are identical, i.e., $W_{R} = W_{S}$.

In [17, §6.1], Koornwinder obtained Macdonald polynomials of the admissible pairs
\[ (R, S) = (R_{BC_{n}}, S_{B_{n}}), \quad (R_{BC_{n}}, S_{C_{n}}) \]
by specializing the parameters in his polynomials. The parameters in [17] are denoted as
\[ a, b, c, d, t, q, \]
and we call them the Koornwinder parameters. The root systems \( R_{BCn}, S_{Bn} \) and \( S_{Cn} \) are given by
\[
R_{BCn} := \{ \pm \epsilon_i \mid 1 \leq i \leq n \} \cup \{ \pm 2 \epsilon_i \mid 1 \leq i \leq n \} \cup \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n \}, \\
S_{Bn} := \{ \pm \epsilon_i \mid 1 \leq i \leq n \} \cup \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n \}, \\
S_{Cn} := \{ \pm \epsilon_i \mid 1 \leq i \leq n \} \cup \{ \frac{1}{2}(\pm \epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq n \}. 
\] (2.5.1)

Using them, the specializations in [17, §6.1] are described as
\[
(R_{BCn}, S_{Bn}) : (a, b, c, d, t, q) \mapsto (q^{1/2}, -q^{1/2}, ab b_B^{1/2}, -b_B^{1/2}, t_B, q), \quad (2.5.2) \\
(R_{BCn}, S_{Cn}) : (a, b, c, d, t, q) \mapsto (ac b_C^{1/2}, q ac b_C^{1/2}, -b_C^{1/2}, -qb_C^{1/2}, t_C, q^2). \quad (2.5.3)
\]

In loc. cit., these results are given without explaining how to derive them. We guess that they are derived by the comparison of the weight functions of inner products, as we did in the previous Sects. 2.3 and 2.4.

In [24, p.54], Noumi gave the correspondence between the Noumi parameters \( q, t, t_0, t_n, u_0, u_n \) and the Koornwinder parameters \( a, b, c, d, t, q \). The correspondence is that \( q \) and \( t \) are common, and
\[
(t_0, t_n, u_0, u_n) = (-cd/q, -ab, -c/d, -a/b). 
\]

We can then rewrite the specialization (2.5.2) to the admissible pair \( (R_{BCn}, S_{Bn}) \) as
\[
(t, t_0, t_n, u_0, u_n) \mapsto (t_B, 1, a_B b_B, 1, a_B). 
\]

Thus, setting \( t_B = t_m^{B, B}, a_B = t_s^{B, B} / t_l^{B, B} \) and \( b_B = (t_l^{B, B})^2 \), we see that it coincides with the specialization to type \( (B_n^{\vee}, B_n) \) in Sect. 2.4.8.

Let us remark that a similar rewriting of the specialization (2.5.3) to the admissible pair \( (R_{BCn}, S_{Cn}) \) does not have a corresponding one in Table 1. This impossibility seems to be due to the fact that the root system \( S_{Cn} \) in (2.5.1) cannot be treated in the formulation of [23].

### 2.6 The rank one case

This subsection is added after the referee comments. We would like to appreciate the referees’ suggestions.

As explained in the beginning of Sect. 2.1, the argument so far assumes the rank \( n \geq 2 \). The purpose of this Sect. 2.6 is to study the excluded case \( n = 1 \). As mentioned
in the beginning of Sect. 1, the Koornwinder polynomial is designed to give a multi-
variable analogue of the Askey–Wilson polynomial. So it is natural to study what our
specialization argument yields in the rank one case. The argument is similar to the
previous one, so we only give an outline.

Let $E = (\mathbb{R} \epsilon, \langle \cdot, \cdot \rangle)$ be the 1-dimensional Euclidean space with basis $\epsilon$, and $F$ be
the $\mathbb{R}$-linear space of affine linear functions on $E$. We identify $F \xrightarrow{\sim} \mathbb{R} \epsilon \oplus \mathbb{R} c$ by the
inner product $\langle \cdot, \cdot \rangle$ as in the rank $n \geq 2$ case (Sect. 2.1). The affine root system of type
$(C_1^\vee, C_1)$ is the subset $S = S_{C_1^\vee, C_1} \subset E$ given by

$$
S = O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4, \quad O_1 := \pm \epsilon + \mathbb{Z} c, \quad O_2 := 2O_1, \quad O_3 := O_1 + \frac{1}{2} c, \\
O_4 := 2O_3 = O_2 + c.
$$

We take the basis $\{\frac{1}{2} a_0, a_0, \frac{1}{2} a_1, a_1\}$ of $S$ with $a_0 := -2 \epsilon + c$ and $a_1 := 2 \epsilon$. The Dynkin
diagram of $S$ is shown in the next line, where the mark * has the same meaning as in
the rank $n \geq 2$ case.

\begin{center}
\begin{tabular}{c|c|c}
\hline
0 & * & 1 \\
\hline
\end{tabular}
\end{center}

Next, as in (2.1.4), we denote the simple reflections by $s_0 := s_{a_0}$ and $s_1 := s_{a_1}$, and
define the finite Weyl group by $W_0 := \langle s_0 \rangle \subset \text{GL}_\mathbb{R}(F)$. The extended affine Weyl
group is defined by $W := t(P_L) \rtimes W_0 \subset \text{GL}_\mathbb{R}(F)$, where $P := \mathbb{Z} \epsilon \subset F$ is the weight
lattice “of type $C_1$”. Then we have $W = \langle s_0, s_1 \rangle$ as in (2.1.11), and the subsets
$O_1, O_2, O_3, O_4 \subset S$ are $W$-orbits of $\frac{1}{2} a_1, a_1, \frac{1}{2} a_0, a_0$, respectively.

We attach parameters $k_1, k_2, k_3, k_4$ to these $W$-orbits under the correspondence
$k_i \leftrightarrow O_i$ as in (2.2.1). Choosing $q \in \mathbb{R}$ with $0 < q < 1$, we set $k : S \to \mathbb{R}$ by
$k(a) := k_i$ for $a \in O_i$ as in (2.2.2). We call $\{q^{k(a)} \mid a \in S\} = \{q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}\}$
the set of Macdonald parameters as in (2.2.3). We also have the Noumi parameters
$t_0, t_1, u_0, u_1$, which correspond to the Macdonald parameters by the relation

$$
(q^{2k_1}, q^{2k_2}, q^{2k_3}, q^{2k_4}) = (t_1 u_1, t_1, t_0 u_0, t_0).
$$

We define the base field to be $\mathbb{K} := \bigotimes(q^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, u_1^{\frac{1}{2}})$ as in (2.2.6).

By the general theory [23, §§5.2–5.3], we have the one-variable symmetric Laurent polynomial

$$
P_l(x) = P_l(x; q, t_0, u_0, u_1) \in \mathbb{K}[x^{\pm 1}]^{W_0}
$$

for each dominant weight $\lambda = l \epsilon \in P_+ := \mathbb{N} \epsilon$, where $W_0 = \langle s_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts on
$\mathbb{K}[x^{\pm 1}]$ by $s_1(x) = x^{-1}$. The Laurent polynomial $P_l(x)$ ($l \in \mathbb{N}$) is equal to the Askey–
Wilson polynomial [1]. Let us briefly explain the correspondence, referring to [24, §3],
[33], [25] and [23, §§6.4–6.6] for the detail. We use Gasper and Rahman’s notation.
for $q$-shifted factorials \((2.2.9)\) and $q$-hypergeometric series

$$s+1\phi_s\left[ a_1, \ldots, a_{s+1}; b_1, \ldots, b_s; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_{s+1}; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \frac{z^k}{(q; q)_k}.$$ 

The Askey–Wilson polynomial is now defined to be

$$p_l\left( \frac{1}{2}(x + x^{-1}); q, a, b, c, d \right) := \frac{a^{-l} (ab, ac, ad; q)_l}{(abcd; q)_l} \cdot 4\phi_3\left[ q^{-l}, q^{l-1}abcd, ax, a/x; ab, ac, ad; q, q \right]. \quad (2.6.2)$$

Although the form \((2.6.2)\) is asymmetric with respect to the parameters $a, b, c, d$, the polynomial actually has the parameter symmetry, which can be seen from the recurrence relation [1, (1.24)–(1.27)]. See also [37, §4, Remark 4.1.2] for the relation between the recurrence relation and the Yip-type formula of Littlewood–Richardson coefficients of Koornwinder polynomials and the reduction to the rank one Askey–Wilson case. Using $(x; q)_l := (x; q)_\infty / (q^l x; q)_\infty$ for $l \in \mathbb{N}$, we have the correspondence [1, p.5]

$$p_l\left( \frac{1}{2}(x + x^{-1}); q, a, b, c, d \right) = 2^l (abcdq^{l-1}; q)_l \cdot P_l(x; q, t_0, t_1, u_0, u_1),$$

where the Askey–Wilson parameters $a, b, c, d$ correspond to the Macdonald parameters by

$$(a, b, c, d) = (\frac{1}{2}t_1 u_0^1, -\frac{1}{2}t_0^1 u_0^{-1}, t_1^{-1} u_1^{-\frac{1}{2}}, -t_0^{-1} u_1^{\frac{1}{2}}).$$

Combining it with \((2.6.1)\), we see that the Askey–Wilson parameters correspond to the $W$-orbits in $S$ by $(a, b, c, d) \leftrightarrow (O_3, O_4, O_1, O_2)$.

We now turn to the specialization argument. We list up the subsystems of $S = S_{C_1}^{\gamma}, C_1$ and the corresponding specialization rules in Table 2. The Dynkin diagrams are borrowed from [23, §1.3]. The “Noumi” column shows the specialization of the Noumi parameters $t_0, t_1, u_0, u_1$ in the same way as the specialization Table 1. The “Askey–Wilson” column shows the specialization of the Askey–Wilson parameters.

The specialization rules for the types $(C_1^\gamma, BC_1, (BC_1, C_1)$ and $BC_1$ are obtained by making $n = 1$ and deleting the $t$ column in the specialization Table 1. We can obtain the type $A_1$ by a similar argument as the reduced subsystems of $(C_n^\gamma, C_n)$, noting that we have four embeddings $S^{A_1} \leftrightarrow S_{C_1}^{\gamma}, C_1$ as indicated in the “orbits” column in Table 2.

Table 2 yields the degeneration scheme (Fig. 1) of $q$-hypergeometric orthogonal polynomials which respects the embeddings of affine root systems into $(C_1^\gamma, C_1)$. Our degeneration scheme seems to be new.

For comparison, let us recall the Askey scheme of $q$-hypergeometric orthogonal polynomials (also called the $q$-Askey scheme, see [16, p.413] for example). It shows the classification and the behavior under parameter specializations of $q$-hypergeometric
orthogonal polynomials. Among those polynomials, we could only find the continuous $q$-Jacobi polynomial and the Rogers polynomial in our Fig. 1 at this moment. As we will explain below, the former appears naturally, but the appearance of the Rogers polynomial is tricky. It might be possible that all the polynomials in our Fig. 1 can be expressed as those in the $q$-Askey scheme. However, according to the quite different forms of our scheme and the $q$-Askey scheme, we can say that the parameter specializations taken in the $q$-Askey scheme do not necessarily respect the affine root system structures.

**Remark 2.6.1** Recently, Koornwinder [18] proposed new degeneration schemes of $q$-hypergeometric orthogonal polynomials, called $q$-Verde-Star and $q$-Zhedanov schemes. These schemes look quite different from ours, and the relation is unclear at this moment,
Among the specialized polynomials appearing in Table 2 and Fig. 1, the type \((BC_1, C_1)\) is essentially the same with the \emph{continuous q-Jacobi polynomial} \(P_l^{(\alpha, \beta)}(x; q)\) [16, §14.10]. The relation with the Askey–Wilson polynomial is given by

\[
P_l^{(\alpha, \beta)}(x; q) = (\text{const.}) \cdot p_n(x; q, q^{\frac{1}{2}}(\alpha+\frac{1}{4}), q^{\frac{1}{2}}(\alpha+\frac{3}{4}), -q^{\frac{1}{2}}(\beta+\frac{1}{4}), -q^{\frac{1}{2}}(\beta+\frac{3}{4})).
\]

The appearance of \(P_l^{(\alpha, \beta)}(x; q)\) is natural in view of the fact discovered by Koornwinder [17, p.195] that the polynomial \(P_l^{(\alpha, \beta)}(x; q)\) is the Macdonald symmetric polynomial of the admissible pair \(R = S = BC_1\) (see Sect. 2.5), which corresponds to the non-reduced affine root system \((BC_1, C_1)\).

Let us also recall that the Macdonald symmetric polynomial of type \(A_1\) is essentially equal to the \emph{Rogers} or the \emph{continuous q-ultraspherical polynomial} \(C_n(x; a|q)\). See [23, §6.3], [10, §7.4] and [16, §14.10.1] for the detail. The generating function is given by

\[
\sum_{l=0}^\infty C_l(x; a|q)y^l = \frac{(ayz, ay/z; q)_\infty}{(tz, t/z; q)_\infty}
\]

with \(x = (z+z^{-1})/2\). The Rogers polynomials are obtained by specializing parameters of the Askey–Wilson polynomials in several different ways. One of them is shown in [16, p.420, (14.1.20)]:

\[
C_l(x; a|q) = (\text{const.}) \cdot p_l(x; q, a, -a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}}),
\]

which seems to be the most famous one, but does not appear in our Table 2. However, there is another one which we learned from [29, (6.5b)]:

\[
C_l(x; a^2|q^2) := (\text{const.}) \cdot p_l(x; q, a, -a, q^{\frac{1}{2}}, -q^{\frac{1}{2}}).
\]

This relation appears in the third embedding \(S^{A_1} \sim O_2 \subset SC_1^{C_1}\) of type \(A_1\) in Table 2. Indeed, the embedded \(S^{A_1}\) is identified with the orbit \(O_2\) of \emph{long roots}, so the shift parameter \(q_A\) for the embedded system should be the square of the parameter \(q\) for the ambient system \(SC_1^{C_1}\), and we have the parameter \(q^2\) in the Rogers polynomial and the parameter \(q\) in Askey–Wilson polynomial as in (2.6.3).

### 3 Specialization in Ram–Yip type formula

In this section, we give a partial check of the specialization Table 1 in the level of \emph{Ram–Yip type formulas}. Precisely speaking, we show that the non-symmetric Koornwinder polynomial degenerates to the non-symmetric Macdonald polynomials of types.
Let us explain what we mean by the word *Ram–Yip type formulas*. In [28], Ram and Yip derived explicit formulas of non-symmetric Macdonald polynomials of reduced affine root systems using alcove walks. Their argument is designed to work in general setting, and the details are later given by Orr and Shimozono in [26], which derives among many results an explicit formula of the non-symmetric Koornwinder polynomial. We call all of these formulas Ram–Yip type formulas of non-symmetric Macdonald polynomials.

A caution is now in order. The realization of affine root systems in [28] is different from our default one in [23]. For distinction, we denote by $S^X_{RY}$ the affine root system of type $X$ used in [28], and call the non-symmetric Macdonald polynomials of type $X$ treated in loc. cit. the polynomial of *Ram–Yip type $X$*.

Let us summarize the results given in this Sect. 3 in the following Table 3.

As mentioned above, we treat the types $B_n$, $C_n$ and $D_n$ in the sense of [28], each in Sects. 3.3, 3.2 and 3.4, respectively. Since the types $B_n$ and $C_n$ have discrepancy from those in our default [23], we use the symbols $B^{RY}_n$ and $C^{RY}_n$ in Table 3. The type $D_n$ has no discrepancy, and we use the symbol $D_n$. The $B^{RY}_n$ row in Table 3 indicates the specialization of the Noumi parameters to obtain the non-symmetric polynomial of Ram–Yip type $B_n$. More explicitly, denoting the latter by $E^{B,RY}_\mu(x)$, we have

$$E_\mu(x; q, i_m^{RY}, 1, i_l^{RY}, 1, i_l^{RY}) = E^{B,RY}_\mu(x; q, i_m^{RY}, i_l^{RY}).$$

This equality will be shown in Proposition 3.3.4. The $B_n$ row in Table 3 is a copy from the specialization Table 1, which we give in the intention of checking the specialization argument in Sects. 2.3 and 2.4. As for the other types, Table 3 claims that the type $D_n$ is clean, but that the type $C^{RY}_n$ (Ram–Yip type $C_n$) is a little confusing, which turns out to correspond to the type $B_n^\vee$ in the sense of [23].

### 3.1 Ram–Yip type formula of type $(C_n^\vee, C_n)$

In this subsection, we recall Ram–Yip type formula of type $(C_n^\vee, C_n)$, i.e., the Ram–Yip type formula of non-symmetric Koornwinder polynomial, derived in [26]. For the notation, we follow [37].

| Table 3 | Specialization table for Ram–Yip formulas |
|---------|------------------------------------------|
| $B^{RY}_n$ | Sect. 3.3 | $\tau_m^{RY}$ | $\tau_l^{RY}$ | $\tau_l^{RY}$ | $\tau_l^{RY}$ |
| $B_n$ | $\tau_l$ | $\tau_s$ | $\tau_s$ |
| $C^{RY}_n$ | Sect. 3.2 | $\tau_m^{RY}$ | $\tau_s^{RY}$ | $\tau_s^{RY}$ | $\tau_s^{RY}$ |
| $B^\vee_n$ | $\tau_s$ | $\tau_l^2$ | $\tau_l^2$ | $\tau_l^2$ |
| $D_n$ | Sect. 3.4 | $\tau$ | $\tau$ | $\tau$ | $\tau$ |
3.1.1 Alcove walks of type \((C_n^\vee, C_n)\)

Here we recall the alcove walks of type \((C_n^\vee, C_n)\) introduced in [26]. See [37, §2.1.3] for more information and illustrated examples.

We keep the notation for the affine root system \(S\) of type \((C_n^\vee, C_n)\) introduced in Sect. 2.1. Thus, the system \(S\) is realized in \(F = V \oplus \mathbb{R} c, V = \mathbb{R}^{n} = \mathbb{R}^{n} \) with the \(W\)-orbit decomposition \(S = O_1 \sqcup \cdots \sqcup O_5\). We write again the decomposition (2.1.1) and the affine roots \(a_i\) in (2.1.2):

\[
O_1:=\{\pm \varepsilon_i + rc \mid 1 \leq i \leq n, r \in \mathbb{Z}\}, \quad O_2:=2O_1, \quad O_3:=O_1 + \frac{1}{2}c, \\
O_4:=2O_3 = O_2 + c, \quad O_5:=\{\pm \varepsilon_i \pm \varepsilon_j + rc \mid 1 \leq i < j \leq n, r \in \mathbb{Z}\}.
\]

\[a_0:=-2\varepsilon_1 + c, \quad a_j:=\varepsilon_j - \varepsilon_{j+1} \quad (1 \leq j \leq n - 1), \quad a_n:=2\varepsilon_n. \tag{3.1.1}\]

\[
\frac{1}{2}a_0 \in O_3, \quad a_0 \in O_4, \quad a_j \in O_5 \quad (1 \leq j \leq n - 1), \quad \frac{1}{2}a_n \in O_1, \quad a_n \in O_2.
\]

For each affine root \(a \in S\), we denote by \(H_a:=\{x \in V \mid a(x) = 0\}\) the associated hyperplanes in \(V\). An alcove is a connected component of \(V \setminus \bigcup_{a \in S} H_a\). The distinguished alcove

\[A:=\{x \in V \mid a_i(x) > 0 \ (i = 0, 1, \ldots, n)\} \subseteq V \tag{3.1.2}\]

is called the fundamental alcove. We have a bijection

\[W \ni w \longmapsto wA \in \pi_0(V \setminus \bigcup_{a \in S} H_a).\]

The boundary of an alcove \(wA\) consists of \(n + 1\) hyperplanes, each of which is called an edge of \(wA\). For an edge \(H\) of an alcove \(wA\), there exists \(i \in \{0, 1, \ldots, n\}\) such that \(H\) separates \(wA\) and \(ws_iA\). The edge \(H\) has two sides facing \(wA\) and \(ws_iA\), and we call them \(wA\)-side and \(ws_iA\)-side, respectively.

We assign + or – to each side of an edge of an alcove \(wA\) as follows. Let \(\{H_{\gamma_i} \mid i = 0, 1, \ldots, n\}\) be the hyperplanes surrounding the alcove \(wA\). Reordering \(\gamma_i\)’s if necessary, we can assume that \(H_{\gamma_i}\) separates \(wA\) and \(ws_iA\) for each \(i\). Then, using the notation (2.1.17), the assignment is given by:

\[
\text{if } \gamma_i \in \widetilde{R}_+, \text{ then assign } + \text{ to } wA\text{-side, and } - \text{ to } ws_iA\text{-side,} \\
\text{if } \gamma_i \in \widetilde{R}_-, \text{ then assign } - \text{ to } wA\text{-side, and } + \text{ to } ws_iA\text{-side.} \tag{3.1.3}\]

In Fig. 2, we give an illustration for the case \(n = 2\).

Next, we introduce alcove walks for the system \(S\). Let \(w, z \in W\) be arbitrary, and take a reduced expression \(w = s_{i_1}s_{i_2} \cdots s_{i_r}\) with \(r = \ell(w)\). For a bit sequence \(b = (b_1, b_2, \ldots, b_r) \in \{0, 1\}^r\), consider the sequence of alcoves

\[p = (p_0:=ZA, \ p_1:=zs_{i_1}A, \ p_2:=zs_{i_1}s_{i_2}A, \ \ldots, \ p_r:=zs_{i_1} \cdots s_{i_r}A),\]
which is called an alcove walk with start $z$ of type $\vec{w}:=(i_1, i_2, \ldots, i_r)$. Note that it depends on the choice of the reduced expression of $w$, which is indicated in the symbol $\vec{w}$. We denote by $\Gamma(\vec{w}, z)$ the set of all alcove walks with start $z$ of type $\vec{w}$.

**Example 3.1.1** (Alcove walks of type $(C_2\vee, C_2)$) We cite from [37, Example 2.1.1] some illustrated examples of alcove walks in the case $n=2$. Let us take $w, z \in W$ as $w = s_1 s_2 s_1 s_0$ and $z = e$. Then we have the following two alcove walks which belong to $\Gamma(\vec{w}, z)$.

$$p_1 := (A, s_2 A, s_2 s_1 A, s_2 s_1 s_0 A), \quad p_2 := (A, s_1 A, s_1 s_2 A, s_1 s_2 s_1 A, s_1 s_2 s_1 s_0 A).$$

We depict them in Fig. 3, where the grayed alcove is the fundamental alcove $A$, and the number $i=0, 1, 2$ on each hyperplane $H$ indicates that $H$ belongs to the $W$-orbit of $H_{\alpha_i}$.

Let us explain how we depicted alcove walks in Fig. 3. Hereafter, for an alcove walk $p \in \Gamma(\vec{w}, z)$ and $k = 1, 2, \ldots, r := \ell(w)$, the $k$-th step of $p$ means the transition $p_{k-1} \rightarrow p_k$. Then the $k$-th bit $b_k \in \{0, 1\}$ and the $k$-th step drawing correspond as in Table 4, where we denote by $v_{k-1} \in W$ the element giving the $(k-1)$-th alcove $p_{k-1}$, i.e., we have $p_{k-1} = v_{k-1} A$. We call the $k$-th step with $b_k = 1$ a crossing, and the step with $b_k = 0$ a folding, corresponding to the drawing.

Take $z, w \in W$ and a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$. For each $p = (z A, \ldots, z s_{i_1}^{b_1} \cdots s_{i_r}^{b_r} A) \in \Gamma(\vec{w}, z)$, we define $e(p) \in W$ to be the element corresponding to the endpoint of $p$, i.e.,

$$e(p):= z s_{i_1}^{b_1} s_{i_2}^{b_2} \cdots s_{i_r}^{b_r}. \quad (3.1.4)$$
Fig. 3 Alcove walks $p_1$ and $p_2$ of type $(C_2^\vee, C_2)$

| Table 4 | The correspondence of bit and alcove walk |
|---------|------------------------------------------|
| $b_k$   | 1 | 0 |
| Crossing | $p_{k-1}$ | $p_k$ | $p_{k-1} = p_k$ | $v_{k-1}s_k A$ |
| Folding  | |

| Table 5 | Four classes of steps of alcove walks |
|---------|---------------------------------------|
| Positive crossing | Negative crossing | Positive folding | Negative folding |
| $-$ | $+$ | $+$ | $-$ | $+$ | $+$ | $-$ | $+$ |
| $p_{k-1}$ | $p_k$ | $p_{k-1} = p_k$ | $v_{k-1}s_k A$ | $p_{k-1} = p_k$ | $v_{k-1}s_k A$ |

Also, for $k = 1, 2, \ldots, r$, we define $h_k(p) \in S$ to be the affine root associated to the hyperplane $H$ separating $p_{k-1} = v A$ and $v s_k A$. We also divide steps of $p \in \Gamma(\overrightarrow{w}, z)$ into the four classes in Table 5 according to the signs on the sides of edges in (3.1.3).

Along the classification in Table 5, for each $p \in \Gamma(\overrightarrow{w}, z)$, we define $\varphi_{\pm}(p) \subseteq \{1, \ldots, r\}$ by

$$\varphi_+(p) := \{k \mid \text{the } k\text{-th step of } p \text{ is a positive folding}\},$$

$$\varphi_-(p) := \{k \mid \text{the } k\text{-th step of } p \text{ is a negative folding}\}.$$
3.1.2 Ram–Yip type formula

Recall that we introduced in (2.2.11) the non-symmetric Koornwinder polynomial

$$E_\mu(x) = E_\mu(x; q, t, t_0, u_0) \in \mathbb{K}[x^{\pm 1}].$$

for each $\mu \in P_{C_n}$. We now explain Ram–Yip type formula of type $(C_\vee, C_n)$ in [26, 28], which expresses the monomial expansion of $E_\mu(x)$ with the coefficients given by summation over alcove walks.

For any $a = a + rc \in S$, $\alpha \in \tilde{R}$ (see (2.1.16) and (2.1.17)), we define

\begin{equation}
\begin{aligned}
q_{sh}(a + rc) := q^{-r}, & \quad t^{ht(a + rc)} := t^{(\rho_s^{C_\vee}C, a)}(t_0 t_n)^{(\rho_0^{C_\vee}C, a)}, \\
rho_s^{C_\vee}C := \sum_{i=1}^n (n - i)\epsilon_i, & \quad \rho_0^{C_\vee}C := \frac{1}{2} \sum_{i=1}^n \epsilon_i.
\end{aligned}
\end{equation}

(3.1.5)

Let us given $v, w \in W$ and a reduced expression of $w$. Then, for an alcove walk $p \in \Gamma(\vec{w}, z)$, we define $d(p)$ and $wt(p)$ by the decomposition of the element $e(p) \in W$ (see (3.1.4)) along $W = t(P_{C_n}) \rtimes W_0$ as

\begin{equation}
e(p) = t(wt(p))d(p), \quad d(p) \in W_0, \quad wt(p) \in P_{C_n}.
\end{equation}

(3.1.6)

Also, for $\mu \in P_{C_n}$, we denote the shortest element in the coset $t(\mu)W_0$ by

\begin{equation}
w(\mu) \in W.
\end{equation}

(3.1.7)

In the case $\mu = \epsilon_i, i = 1, \ldots, n$, the element $w(\epsilon_i)$ is given by

\begin{equation}
w(\epsilon_i) = s_{i-1} \cdots s_0.
\end{equation}

(3.1.8)

**Fact 3.1.2** ([28, Theorem 3.1], [26, Theorem 3.13]) For any $\mu \in P_{C_n}$, take a reduced expression $w(\mu) = s_{i_1} \cdots s_{i_r}$ of $w(\mu) \in W$. Then

$$E_\mu(x) = \sum_{p \in \Gamma(w(\mu), e)} f_p t_{d(p)}^{\frac{1}{2}} x^{wt(p)}, \quad f_p := \prod_{k \in \varphi_+(p)} \psi_{i_k}^+(q^{sh(-\beta_k)} t^{ht(-\beta_k)}) \prod_{k \in \varphi_-(p)} \psi_{i_k}^-(q^{sh(-\beta_k)} t^{ht(-\beta_k)}).$$
Here we set $\beta_k := s_i s_{i-1} \cdots s_{i+k-1} (a_i)$ for $k = 1, \ldots, r$, and $\psi_i(z)$ for $i = 0, 1, \ldots, n$ are given by

$$\psi_j^\pm(z) := \pm \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - z^{\pm 1}} \quad (1 \leq j \leq n - 1),$$

$$\psi_0^\pm(z) := \pm \frac{u_n^{-\frac{1}{2}} - u_n^{\frac{1}{2}}}{1 - z^{\pm 1}} + z^{\pm 1}(u_0^{-\frac{1}{2}} - u_0^{\frac{1}{2}}),$$

$$\psi_n^\pm(z) := \pm \frac{(t_n^{-\frac{1}{2}} - t_n^{\frac{1}{2}}) + z^{\pm 1}(t_0^{-\frac{1}{2}} - t_0^{\frac{1}{2}})}{1 - z^{\pm 1}}.$$  \hspace{1cm} (3.1.9)

### 3.2 Ram–Yip type $C_n$

In this subsection, we show that the Ram–Yip formula of the non-symmetric Macdonald polynomial of type $C_n$ in the sense of [28] can be obtained from the Ram–Yip type formula of type $(C_\vee, C_n)$ (Fact 3.1.2) by the corresponding specialization in Table 3:

$$t_0 = u_0 = u_n = 1.$$  

See Proposition 3.2.5 for the precise statement.

A caution on the notation is in order. In [28], the Ram–Yip formula for what they call type $C_n$ is derived using the affine root system of type $C_\vee$ in the sense of loc. cit. As mentioned before, it turns out that both the polynomial and the root system are different from those in [23]. For distinction, we denote by $E^{C,RY}_\mu(x)$ and $S^{C_\vee,RY}$ the polynomial and the system treated in [28], and call them the Macdonald polynomial of Ram–Yip type $C_n$ and the affine root system of Ram–Yip type $C_\vee$, respectively.

#### 3.2.1 Affine root system of Ram–Yip type $C_\vee$

We start with the explanation on the system $S^{C_\vee,RY}$. Let $S$ be the affine root system of type $(C_n, C_n)$ in (2.1.1). The affine root system $S^{C_\vee,RY}$ of Ram–Yip type $C_n$ is the subset of $S$ given by

$$S^{C_\vee,RY} := O_1 \cup O_5 \{\pm \epsilon_i + rc \mid 1 \leq i \leq n, r \in \mathbb{Z}\}$$

$$= \cup\{\pm \epsilon_i \pm \epsilon_j + rc \mid 1 \leq i < j \leq n, r \in \mathbb{Z}\},$$  \hspace{1cm} (3.2.1)

where we used the $W$-orbits in (2.1.1). The basis of $S^{C_\vee,RY}$ in [28] is given by

$$a_0^{C_\vee,RY} := -(\epsilon_1 + \epsilon_2) + c, \quad a_j^{C_\vee,RY} := a_j = \epsilon_j - \epsilon_{j+1} \quad (j = 1, \ldots, n - 1),$$

$$a_n^{C_\vee,RY} := \epsilon_n.$$  

Note that we have $a_j^{C_\vee,RY} = a_j$ in (2.1.2), but the other two roots are different from those in (2.1.2).
Next, we turn to the extended affine Weyl group. The reflections associated to the above basis are denoted by

\[ s_{C_0} := s_{a_0^{C,RY}}, \quad s_i = s_{a_i^{C,RY}} \quad (i = 1, \ldots, n), \]  

(3.2.2)

where we used \( s_i \in W_0 \) in (2.1.4). Note that we have the common \( s_n \) although \( a_n^{C,RY} \neq a_n \). We also consider the automorphism group \( \Omega^{C^{\vee}, RY} \) of the extended Dynkin diagram of type \( B_n \):

![Diagram]

Explicitly, using the weight lattice \( P_{C_n} = \bigoplus_{i=1}^{n} \mathbb{Z} \epsilon_i \oplus \mathbb{Z}_{12} (\epsilon_1 + \cdots + \epsilon_n) \) of type \( B_n \) in (2.3.2), we have

\[ \Omega^{C^{\vee}, RY} := P_{C_n}^{\vee} / Q_{C_n}^{\vee} = P_{B_n} / Q_{B_n} = \{ \pi^{C^{\vee}} | (\pi^{C^{\vee}})^2 = e \}. \]

The generator \( \pi^{C^{\vee}} \) flips the diagram by transposing the vertices 0 ↔ 1. Then, the extended affine Weyl group \( W^{C^{\vee}, RY} \) is defined to be the subgroup of \( \text{GL}(V) \) generated by the reflections in (3.2.2) and \( \pi^{C^{\vee}} \). In other words, we have

\[ W^{C^{\vee}, RY} := \langle s_0^{C^{\vee}}, s_1, \ldots, s_n, \pi^{C^{\vee}} \rangle. \]

As an abstract group, \( W^{C^{\vee}, RY} \) is presented by these generators with the following relations.

\[
\begin{align*}
\pi^{C^{\vee}} s_0^{C^{\vee}} &= s_1 \pi^{C^{\vee}}, \\
s_0^{C^{\vee}} s_1 &= s_1 s_0^{C^{\vee}}, \\
s_0^{C^{\vee}} s_2 &= s_2 s_0^{C^{\vee}}, \\
s_0^{C^{\vee}} s_1 s_0^{C^{\vee}} &= s_2 s_0^{C^{\vee}}, \\
s_1 s_i s_1 &= s_i s_1 s_i + 1 (1 \leq i \leq n - 2), \\
s_n s_{n-1} s_n s_{n-1} &= s_{n-1} s_n s_{n-1} s_n.
\end{align*}
\]

(3.2.3)

In the second line, we abusively denoted \( s_0 := s_0^{C^{\vee}} \). Let us write down the action of \( W^{C^{\vee}, RY} \) on \( F_{\mathbb{Z}} = P_{C_n} \oplus \frac{1}{2} \mathbb{Z} \) in (2.1.14).

\[
\begin{align*}
s_0^{C^{\vee}} (\epsilon_i) &= \begin{cases} 
  c - \epsilon_2 & (i = 1) \\
  c - \epsilon_1 & (i = 2) \\
  \epsilon_i & (i \neq 1, 2)
\end{cases}, \\
s_j (\epsilon_i) &= \begin{cases} 
  \epsilon_j & (i = j + 1) \\
  \epsilon_{j+1} & (i = j) \\
  \epsilon_i & (i \neq j, j + 1)
\end{cases} \\
s_n (\epsilon_i) &= \begin{cases} 
  -\epsilon_n & (i = n) \\
  \epsilon_i & (i \neq n)
\end{cases}, \\
\pi^{C^{\vee}} (\epsilon_i) &= \begin{cases} 
  c - \epsilon_1 & (i = 1) \\
  \epsilon_i & (i \neq 1)
\end{cases}.
\end{align*}
\]
We can see from this action that $W^{C\vee,RY}$ preserves $S^{C\vee,RY} \subset S$, and the description $S^{C\vee,RY} = O_1 \sqcup O_5$ in (3.2.1) is actually the decomposition into $W^{C\vee,RY}$-orbits.

In fact, as the following lemma shows, the group $W^{C\vee,RY}$ is identical to $W$ in (2.1.9).

**Lemma 3.2.1** The following gives a group isomorphism $\varphi^C : W \xrightarrow{\sim} W^{C\vee,RY}$.

$$\varphi^C(s_i) := s_i \quad (1 \leq i \leq n), \quad \varphi^C(s_0) := \pi^C.$$

In particular, we have the following relations of subgroups in $GL_{\mathbb{R}}(F_Z, F_Z) = V \oplus \mathbb{R}c$.

$$W \cong W^{C\vee,RY} = t(P_{C_n}) \ltimes W_0.$$  

**Proof** We regard $W$ as the group with the presentation $\langle s_0, s_1, \ldots, s_n \rangle$ in (2.1.11). Since $\varphi^C(s_0s_1s_0) = \pi^C s_1\pi^C s_0^C$, we have the surjectivity of the homomorphism $\varphi^C$ up to well-definedness. Thus, it is enough to show that the defining relations (2.1.12) of $W$ are mapped by $\varphi^C$ to those (3.2.3) of $W^{C\vee,RY}$. The non-trivial parts are those containing $s_0 \in W$. As for the fourth relation $s_0s_1s_0s_1 = s_1s_0s_1s_0$ in (2.1.12), the application of $\varphi^C$ yields

$$\varphi^C(s_0s_1s_0s_1) = \varphi^C(s_1s_0s_1s_0) \iff \pi^C s_1\pi^C s_1 = s_1\pi^C s_1\pi^C s_1$$

$$\iff s_0^C s_1 = s_1s_0^C,$$

which is in the third line of (3.2.3). The other relations are similarly checked. \qed

For later use, we write down the reduced expression of $t(\epsilon_i) \in W^{C\vee,RY}$ for $i = 1, 2, \ldots, n$.

$$t(\epsilon_1) = \pi^C s_1 \cdots s_n s_{n-1} \cdots s_1$$

$$t(\epsilon_2) = \pi^C s_0^C s_1 \cdots s_n s_{n-1} \cdots s_2$$

$$t(\epsilon_i) = \pi^C s_{i-1} \cdots s_2 s_0^C s_1 \cdots s_n s_{n-1} \cdots s_i \quad (3 \leq i \leq n).$$  

(3.2.4)

### 3.2.2 Ram–Yip formula of non-symmetric Macdonald polynomials of Ram–Yip type $C_n$

Recalling the $W^{C\vee,RY}$-orbit decomposition $S^{C\vee,RY} = O_1 \sqcup O_5$ in (3.2.1), we take parameters in the correspondence

$$t^R_Y \longleftrightarrow O_1, \quad t^R_Y \longleftrightarrow O_5.$$

For each $\mu \in P_{C_n}$, we have the non-symmetric Macdonald polynomial of Ram–Yip type $C_n$, which is then denoted by

$$E_{\mu}^{C,RY}(x) = E_{\mu}^{C,RY}(x; q, t^R_Y, t^R_Y) \in \mathbb{K}_{C,RY}[x^{\pm 1}].$$
Fig. 4 The fundamental alcove $A^{C, \text{RY}}$ of Ram–Yip type $C_2$

Below we explain the explicit formula of $EC^{C, \text{RY}}_{\mu}(x)$ given in [28].

For each $a = \alpha + rc \in SC^{C, \text{RY}} \subset P_{C_n} \oplus \mathbb{R}c$, we define $q^{shC}(a)$ and $t^{htC}(a)$ by

\[
q^{shC}(\alpha + rc) := q^{-r}, \quad t^{htC}(\alpha + rc) := (t_s^{\text{RY}})^{(\rho^{C,s}_r \cdot \alpha)}(t_m^{\text{RY}})^{(\rho^{C,m}_s \cdot \alpha)},
\]

\[
\rho^{C,s}_r := \sum_{i=1}^{n} \epsilon_i, \quad \rho^{C,m}_s := \sum_{i=1}^{n} (n - i) \epsilon_i.
\] (3.2.5)

We also denote the fundamental alcove of $SC^{C, \text{RY}}$ by

\[
A^{C, \text{RY}} := \left\{ x \in V \mid \alpha^{C, \text{RY}}(x) \geq 0, \ i = 0, 1, \ldots, n \right\}.
\]

Then we have $A^{C, \text{RY}} = A \cup s_0A$, where $A$ is the fundamental alcove (3.1.2) and $s_0$ is the 0-th reflection associated to $a_0 = -2\epsilon_1 + c \in S$ (3.1.1), both of type $(C_\vee, C_n)$. Note that $a_0 \neq a_0^{C, \text{RY}}$, so that the corresponding hyperplanes and reflections are different. See Fig. 4 for the case $n = 2$.

Finally, for each $\mu \in P_{C_n}$, we denote the shortest element in the coset $t(\mu)W_0$ by

\[
w^{C}(\mu) \in W^{C, \text{RY}}.
\] (3.2.6)

Fact 3.2.2 [28, Theorem 3.1] Let $\mu \in P_{C_n}$ be arbitrary, and take a reduced expression $w^{C}(\mu) = (\pi^{C})^k s_{i_1} \cdots s_{i_t}$ with $k \in \{0, 1\}$, using the abbreviated symbols in (3.2.3).
Then, we have

$$E_{\mu}^{C, \text{RY}}(x) = \sum_{p \in \Gamma_{C}(\mu, \text{e})} f_{p}^{C} \frac{1}{d_{\text{d}(p)}} x^{\text{wt}(p)},$$

$$f_{p}^{C} := \prod_{k \in \varphi^{+}(p)} (\psi_{ik}^{C})^{+} (q^{sh(-\beta_{k})} t^{ht(-\beta_{k})}) \prod_{k \in \varphi^{-}(p)} (\psi_{ik}^{C})^{-} (q^{sh(-\beta_{k})} t^{ht(-\beta_{k})}),$$

where $\beta_{k} := s_{i_{k}} s_{i_{k-1}} \cdots s_{i_{1}} (s_{i_{r}}^{C, \text{RY}})$ for $k = 1, 2, \ldots, r$, and $(\psi_{i}^{C})^{\pm}(z)$ for $i = 0, 1, \ldots, n$ is given by

$$(\psi_{i}^{C})^{\pm}(z) := \pm \frac{(t_{m}^{\text{RY}})^{\frac{1}{2}} - (t_{m}^{\text{RY}})^{\frac{1}{2}}}{1 - z^{\pm 1}} (0 \leq i \leq n - 1),$$

$$(\psi_{n}^{C})^{\pm}(z) := \pm \frac{(t_{s}^{\text{RY}})^{\frac{1}{2}} - (t_{s}^{\text{RY}})^{\frac{1}{2}}}{1 - z^{\pm 1}}. \quad (3.2.7)$$

### 3.2.3 Specialization to type $C_{n}$

In this part, we check that the specialization $t_{0} = u_{0} = u_{n} = 1$ of the Ram–Yip type formula for the non-symmetric Koornwinder polynomial $E_{\mu}(x)$ (Fact 3.1.2) is equal to the Ram–Yip formula for the non-symmetric Macdonald polynomial $E_{\mu}^{C, \text{RY}}(x)$ of Ram–Yip type $C_{n}$ (Fact 3.2.2). Using (2.2.11), we denote the specialized non-symmetric Koornwinder polynomial by

$$E_{\mu}^{\text{sp}, C}(x) = E_{\mu}^{\text{sp}, C}(x; q, t, t_{n}) := E_{\mu}(x; q, t, 1, t_{n}, 1, 1). \quad (3.2.8)$$

We denote by

$$\Gamma_{0}(\mu, \text{e}) \subset \Gamma(\mu, \text{e})$$

the subset consisting of alcove walks without folding by $s_{0}$. We first show that under the specialization $t_{0} = u_{0} = u_{n} = 1$, the summation over $\Gamma(\mu, \text{e})$ in Fact 3.1.2 reduces to that over $\Gamma_{0}(\mu, \text{e})$.

**Lemma 3.2.3** Let $\mu \in P_{C_{n}}$ be arbitrary, and take a reduced expression $w(\mu) = s_{i_{1}} \cdots s_{i_{r}}$ for the element $w(\mu) \in W$ given in (3.1.7). Then

$$E_{\mu}^{\text{sp}, C}(x) = \sum_{p \in \Gamma_{0}(\mu, \text{e})} f_{p}^{\frac{1}{d_{\text{d}(p)}}} x^{\text{wt}(p)},$$

$$f_{p} := \prod_{k \in \varphi^{+}(p)} (\psi_{ik}^{\text{sp}, C})^{+} (q^{sh(-\beta_{k})} t^{ht(-\beta_{k})}) \prod_{k \in \varphi^{-}(p)} (\psi_{ik}^{\text{sp}, C})^{-} (q^{sh(-\beta_{k})} t^{ht(-\beta_{k})}),$$

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where we used

\[
(\psi^{\text{sp}, C}_i)^\pm(z) := \pm \frac{t_i - \frac{1}{2}}{1 - z^{\pm 1}} \quad (i = 1, \ldots, n - 1), \quad (\psi^{\text{sp}, C}_n)^\pm(z) := \pm \frac{t_n - \frac{1}{2}}{1 - z^{\pm 1}}.
\]

(3.2.9)

**Proof** The specialization \( u_0 = u_n = 1 \) yields \( \psi_0^{\pm}(z) = 0 \) by (3.1.9). Thus, no folding step by \( s_0 \) appear in the summation in Fact 3.1.2. Also, a direct calculation shows that under \( t_0 = 1 \), \( \psi_i^{\pm}(z) \) is equal to \( (\psi_i^{\text{sp}, C})^{\pm}(z) \) for \( i = 1, \ldots, n \).

\( \square \)

Comparing (3.2.7) and (3.2.9), we have

\[
(\psi^{\text{sp}, C}_i)^\pm(z) \bigg|_{t = t_0^{\text{RY}}} = (\psi^C_i)^\pm(z), \quad (\psi^{\text{sp}, C}_n)^\pm(z) \bigg|_{t_n = t_0^{\text{RY}}} = (\psi^C_n)^\pm(z).
\]

(3.2.10)

Hence, to check the identification of \( E^{C, \text{RY}}_{\mu}(x) \) with \( E^{\text{sp}, C}_{\mu}(x) \), it is enough to construct a bijection

\[
\Gamma_0(\overrightarrow{w(\mu), e}) \longrightarrow \Gamma_C(\overrightarrow{w_C(\mu), e}).
\]

between the sets of alcove walks.

**Lemma 3.2.4** For any \( \mu \in P_{C_n} \), take a reduced expression \( w(\mu) = s_{i_1} \cdots s_{i_\ell} \) of the element \( w(\mu) \in W \) in (3.1.7), and set

\[
I := \{ r \in \{1, 2, \ldots, \ell \} \mid i_r \neq 0 \} = \{ k_1 < k_2 < \cdots < k_s \} \quad (s \leq \ell),
\]

\[
J := \{ (b_1, b_2, \ldots, b_\ell) \in \{0, 1\}^\ell \mid b_i = 1 \ (i \notin I) \}.
\]

Also, define \( \theta^C : J \rightarrow \{0, 1\}^s \) by

\[
J \ni (b_1, b_2, \ldots, b_\ell) \mapsto (b_{k_1}, b_{k_2}, \ldots, b_{k_s}) \in \{0, 1\}^s.
\]

Then the following statements hold.

1. The length of \( w_C(\mu) \in W(C^\vee, \text{RY}) \) is \(|I| = s\), and we can write \( w_C(\mu) \) by

\[
w_C(\mu) = \begin{cases} s_{j_1} s_{j_2} \cdots s_{j_s} & (s \in 2\mathbb{N}) \\ \pi^{C^\vee} s_{j_1} s_{j_2} \cdots s_{j_s} & (s \notin 2\mathbb{N}) \end{cases}
\]

with some \( j_r \)'s, where we used the abbreviation in (3.2.3).

2. The map \( \tilde{\theta}^C : J \rightarrow \{0, 1\}^s \) induces a bijection

\[
\tilde{\theta}^C : \Gamma_0(\overrightarrow{w(\mu), e}) \longrightarrow \Gamma_C(\overrightarrow{w_C(\mu), e}),
\]

\[
p = (A, s_{l_1}^{b_1} A, \ldots, s_{l_1}^{b_1} \cdots s_{l_\ell}^{b_\ell} A)
\]
\[ \begin{aligned}
\overset{\sim}{\longrightarrow} & \begin{cases}
(A_C, s_{j_1}^{b_{k_1}} A_C, \ldots, s_{j_s}^{b_{k_s}} A_C) & (s \in 2\mathbb{N}) \\
(\pi^C \cdot A_C, \pi^C \cdot s_{j_1}^{b_{k_1}} A_C, \ldots, \pi^C \cdot s_{j_s}^{b_{k_s}} A_C) & (s \not\in 2\mathbb{N})
\end{cases} \\
\end{aligned} \]

3. For any \( p \in \Gamma_0(w(\mu), e) \), we have

\[ \begin{aligned}
\text{wt}(p) &= \text{wt}(\tilde{\vartheta}^C(p)), \\
\text{d}(p) &= \text{d}(\tilde{\vartheta}^C(p)).
\end{aligned} \]

**Proof** 1. It is enough to show \( \varphi^C(w(\mu)) = w_C(\mu) \) for any \( \mu \in P_{C_n} \). First, we can see \( \varphi^C(w(\epsilon_i)) = w_C(\epsilon_i) \) by the comparison between the reduced expressions (2.1.13) and (3.2.4). Since \( \varphi^C \) is a group isomorphism by Lemma 3.2.1, we see that \( \varphi^C(w(\mu)) = w_C(\mu) \) for any \( \mu \in P_{C_n} \).

2. It is an immediate consequence of the item (1) and the bijectivity of \( \vartheta \).

3. We want to show that for any \( p \in \Gamma_0(w(\mu), e) \), expressing \( e(p) = t(\text{wt}(p)) \text{d}(p) \), we have

\[ \begin{aligned}
\text{wt}(p) &= \text{wt}(\tilde{\vartheta}^C(p)), \\
\text{d}(p) &= \text{d}(\tilde{\vartheta}^C(p)).
\end{aligned} \]

For any \( i = 1, 2, \ldots, n \), we have \( \varphi^C(t(\epsilon_i)) = t(\epsilon_i) \) by the comparison of (2.1.13) with (3.2.4). Thus we have \( t(\text{wt}(p)) = t(\text{wt}(\tilde{\vartheta}^C(p))) \) for any \( p \), which means \( \text{wt}(p) = \text{wt}(\tilde{\vartheta}^C(p)) \). On the other hand, since \( \varphi^C|_{W_0} = \text{id}_{W_0} \), we have \( \text{d}(p) = \text{d}(\tilde{\vartheta}^C(p)) \) for any \( p \). Thus the statement is proved.

Combining this lemma with (3.2.10), we obtain the desired identification

\[ E_{\mu}^{\text{sp}, (x; q, t = t^R_Y m, t_n = t^R_Y s)} = E_{\mu}^{C, (x; q, t^R_Y s, t^R_Y m)}. \]

The definition (3.2.8) of \( E_{\mu}^{\text{sp}, C}(x) \) yields:

**Proposition 3.2.5** For any \( \mu \in P_{C_n} \), we have

\[ E_{\mu}(x; q, t^R_Y m, 1, t^R_Y s, 1, 1) = E_{\mu}^{C, (x; q, t^R_Y s, t^R_Y m)}. \]

Comparing this result with the specialization Table 1, we see that it corresponds to type \( B_n^\vee \). Thus, the Macdonald polynomial of Ram–Yip type \( C_n \) is the Macdonald polynomial of type \( B_n^\vee \) in the sense of Definition 1.0.1.

### 3.3 Ram–Yip type \( B_n \)

The Ram–Yip formula of non-symmetric Macdonald polynomial of type \( B_n \) is derived in [28] using the affine root system of type \( B_n^\vee \) in the sense of loc. cit. In this subsection, we give a similar argument as in the previous Sect. 3.2 to type \( B_n \), and show that under the specialization

\[ t_n = u_n, \quad t_0 = u_0 = 1 \]
we can recover the non-symmetric Macdonald polynomial of type $B_n$ in the sense of [28] from the non-symmetric Koornwinder polynomial.

We will use similar terminologies on the affine root system and the non-symmetric Macdonald polynomials as in Sect. 3.2. We denote by $S^{B_n,\text{RY}}$ and $E^{B_n,\text{RY}}(x)$ those considered in [28] for type $B$, and call them the affine root system of Ram–Yip type $B_n$ and the Macdonald polynomial of Ram–Yip type $B_n$, respectively.

3.3.1 Affine root system of Ram–Yip type $B_n^\vee$

Using the symbols in (2.1.1), the affine root system $S^{B_n,\text{RY}}$ of Ram–Yip type is given by

$$S^{B_n,\text{RY}} := (O_2 \sqcup O_4) \sqcup O_5 = \{ \pm 2\epsilon_i + r \mid 1 \leq i \leq n, \, r \in \mathbb{Z} \} \sqcup \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, \, r \in \mathbb{Z} \}. \quad (3.3.1)$$

The choice of the basis in [28] is given by

$$a^{B_n,\text{RY}}_0 := a_0 = -2\epsilon_1 + c, \quad a^{B_n,\text{RY}}_j := a_j = \epsilon_j - \epsilon_{j+1} \quad (j = 1, \ldots, n-1),$$

$$a^{B_n,\text{RY}}_n := a_n = 2\epsilon_n,$$

where $a_i$’s are in (2.1.2). Thus, the associated reflections are $s_{a^{B_n,\text{RY}}_i} = s_i$ in (2.1.4) and (2.1.10).

We turn to the explanation of the extended affine Weyl group. Let $\Omega_{B_n^\vee}$ be the automorphism group of the extended Dynkin diagram of type $C_n$:

```
  1  2  2  . . .  2  2  <  1
```

Explicitly, we have

$$\Omega_{B_n^\vee} := P_{B_n^\vee} / Q_{B_n^\vee} = P_{C_n} / Q_{C_n} = \langle \pi^{B_n^\vee} \mid (\pi^{B_n^\vee})^2 = e \rangle.$$ 

Then, the extended affine Weyl group $W^{B_n,\text{RY}}$ is the subgroup of $\text{GL}_\mathbb{R}(V)$, $V = \bigoplus_{i=1}^n \mathbb{R}\epsilon_i$ given by

$$W^{B_n,\text{RY}} := \{ s_0, s_1, \ldots, s_n, \pi_{B_n^\vee} \}.$$
As an abstract group, \( W^{B^\vee,RY} \) has a presentation with these generators and the following relations.

\[
\begin{align*}
    s_i^2 &= 1 \quad (i = 0, \ldots, n), \\
    s_i s_j &= s_j s_i \quad (|i - j| > 1), \\
    s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad (i = 1, \ldots, n - 2), \\
    s_i s_{i+1} s_i s_{i+1} &= s_{i+1} s_i s_{i+1} s_i \quad (i = 0, n - 1), \\
    \pi^{B^\vee} s_i &= s_{n-i+1} \pi^{B^\vee} \quad (i = 0, 1, \ldots, n).
\end{align*}
\]

Let us write down the action of \( W^{B^\vee,RY} \) on \( F_Z = P_{C_n} \oplus \frac{1}{2} \mathbb{Z}c \) (2.1.14).

\[
\begin{align*}
    s_0(\epsilon_i) &= \begin{cases} 
        c - \epsilon_1 & (i = 1) \\
        \epsilon_i & (i \neq 1)
    \end{cases}, \\
    s_j(\epsilon_i) &= \begin{cases} 
        \epsilon_j & (i = j + 1) \\
        \epsilon_j + 1 & (i = j) \\
        \epsilon_i & (i \neq j, j + 1)
    \end{cases}, \\
    s_n(\epsilon_i) &= \begin{cases} 
        -\epsilon_n & (i = n) \\
        \epsilon_i & (i \neq n)
    \end{cases}, \\
    \pi^{B^\vee}(\epsilon_i) &= \frac{1}{2}(c - \epsilon_{n-i+1}).
\end{align*}
\]

We can see from this action that \( W^{B^\vee,RY} \) acts on \( S^{B^\vee,RY} \), and the description \( S^{B^\vee,RY} = O_1 \sqcup O_5 \) in (3.3.1) is actually the decomposition into \( W^{B^\vee,RY} \)-orbits.

The group \( W^{B^\vee,RY} \) also has the following descriptions.

\[
\begin{align*}
    W^{B^\vee,RY} &= \Omega_{B^\vee} \rtimes W = t(P_{B_n}) \rtimes W_0, \\
    P_{B_n} &= \mathbb{Z}c_1 \oplus \cdots \oplus \mathbb{Z}c_n \oplus \mathbb{Z} \frac{1}{2}(c_1 + \cdots + c_n),
\end{align*}
\]

where we used \( t \) in (2.1.5). For later use, we write down reduced expressions of \( t(\epsilon_i) \)'s.

\[
\begin{align*}
    t(\epsilon_i) &= s_{i-1} \cdots s_1 s_0 s_1 \cdots s_n s_{n-1} \cdots s_i \quad (i = 1, 2, \ldots, n), \\
    t\left(\frac{1}{2}(c_1 + \cdots + c_n)\right) &= \pi^{B^\vee}(s_n \cdots s_1) \cdots (s_n s_{n-1}) s_n.
\end{align*}
\]

### 3.3.2 Ram–Yip formula of non-symmetric Macdonald polynomial of type \( B_n \)

Next we consider the parameters for Macdonald polynomials. Recalling the \( W^{B^\vee,RY} \)-orbit decomposition \( S^{B^\vee,RY} = O_5 \sqcup (O_2 \sqcup O_4) \) in (3.3.1), we take parameters \( t^{RY}_m \) and \( t^{RY}_l \) in the correspondence

\[
\begin{align*}
    t^{RY}_m &\leftrightarrow O_5, \\
    t^{RY}_l &\leftrightarrow O_2 \sqcup O_4.
\end{align*}
\]

We have the non-symmetric Macdonald polynomial of Ram–Yip type \( B_n \) for \( \mu \in P_{B_n} \) in (2.3.2), which is then denoted by

\[
E^{B^\vee,RY}_\mu(x) = E^{B^\vee,RY}_\mu(x; q, t^{RY}_m, t^{RY}_l) \in \mathbb{K}_{B,RY}[x^{\pm 1}],
\]

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Fig. 5 The fundamental alcove of Ram–Yip type $B_3^\vee$

For each $a = \alpha + rc \in S^B,RY$, we define $q^{shB}(a)$ and $t^{htB}(a)$ by

\[ q^{shB}(\alpha + rc) = q^{-r}, \quad t^{htB}(\alpha + rc) = \prod_{k \in \varphi^+} (\psi_{ik}^B)^{q^{shB}(\beta_k) \cdot t^{htB}(\beta_k)}, \quad \rho^B_m := \sum_{i=1}^{n} (n-i)\epsilon_i, \quad \rho^B_l := \frac{1}{2} \sum_{i=1}^{n} \epsilon_i. \]

We denote the fundamental alcove of Ram–Yip type $B_n^\vee$ by

\[ A^{B^\vee,RY} := \{ x \in V \mid a_i^{B^\vee}(x) \geq 0, \ i = 0, 1, \ldots, n \}. \]

See Fig. 5 for the case $n = 3$. We have $A^{B^\vee,RY} = A$ in (3.1.2).

We also denote by $w_B(\mu) \in W_{\rho^B,RY}$ the set of alcove walks with start $z \in W_{\rho^B,RY}$ of type $\rightarrow w$.

**Fact 3.3.1** [28, Theorem 3.1] Let $\mu \in P_{B_n}$ be arbitrary, and take a reduced expression $w_B(\mu) = (\pi^B)^k s_{i_1} s_{i_2} \cdots s_{i_r}, \ k \in \{0, 1\}$. Then we have

\[ E^{B,RY}_\mu(x) = \sum_{\mu \in \Gamma_{\rho^B}(\overrightarrow{w(\mu), \epsilon})} f^B_p \frac{1}{d(p)} x^{\text{wt}(p)}, \]

where $\beta_k := s_{i_r} \cdots s_{i_{k+1}} (a^B_i)$ for $k = 1, 2, \ldots, r$, and $(\psi^B_i)^{\pm}(z)$ for $i = 0, 1, \ldots, n$ is given by

\[ (\psi^B_j)^{\pm}(z) := \pm \frac{(t^R_m)^{-\frac{1}{2}} - (t^R_l)^{\frac{1}{2}}}{1 - z^{\pm1}} \quad (1 \leq j \leq n - 1), \]

\[ (\psi^B_i)^{\pm}(z) := \pm \frac{(t^R_l)^{-\frac{1}{2}} - (t^R_i)^{\frac{1}{2}}}{1 - z^{\pm1}} \quad (i = 0, n). \]
3.3.3 Specialization to type $B_n$

In this part, we check that the specialization $t_n = u_n$, $t_0 = u_0 = 1$ of $E_\mu(x)$ in Fact 3.1.2 is equal to $E_{\mu,R_\gamma}^B(x)$ in Fact 3.3.1. Using (2.2.11), we denote the specialized non-symmetric Koornwinder polynomial by

$$E_\mu^{sp,B}(x) = E_\mu^{sp,B}(x; q, t, t_n) := E_\mu(x; q, t, t_1, t_n, 1, t_n).$$  \hspace{1cm} (3.3.6)

**Lemma 3.3.2** The map $s_i \mapsto s_i$ ($i = 0, \ldots, n$) defines an injective group homomorphism $W \rightarrow W^{B_\gamma}$. 

**Proof** Obvious from the structure $W^{B_\gamma} = \Omega_{B_\gamma} \rtimes W$ in (3.3.2). \hfill \square

**Lemma 3.3.3** For any $\mu \in P_{C_n}$, take a reduced expression $w(\mu) = s_i_1 s_i_2 \cdots s_i_r$ of the element $w(\mu) \in W$ in (3.1.7). Then, we have

$$E_\mu^{sp,B}(x) = \sum_{p \in \Gamma(w(\mu), e)} f_p \prod_{k \in \varphi_+(p)} \psi_{ik}^{sp,B}(z)^{i_k} \prod_{k \in \varphi_-(p)} \psi_{ik}^{sp,B}(z)^{\beta_k},$$

where $\beta_k := s_i s_{i_1} \cdots s_{i_k}$ for $k = 1, 2, \ldots, r$, and $(\psi_{ik}^{sp,B})^\pm(z)$ for $i = 0, 1, \ldots, n$ is given by

$$(\psi_{ik}^{sp,B})^\pm(z) := \pm \frac{t^{-\frac{1}{n}} - t^\frac{1}{n}}{1 - z^{\pm1}} (1 \leq j \leq n - 1),$$

$$(\psi_{ik}^{sp,B})^\pm(z) := \pm \frac{t^{-\frac{1}{n}} - t^\frac{1}{n}}{1 - z^{\pm1}} (i = 0, n).$$  \hspace{1cm} (3.3.7)

**Proof** A direct calculation with $t_n = u_n$ and $t_0 = u_0 = 1$ yields the result. \hfill \square

Since $P_{C_n} \subset P_{B_n}$, we have:

**Proposition 3.3.4** For any $\mu \in P_{C_n}$, the following equality holds.

$$E_\mu^B(x; q, t_m^B, 1, t_l^B, 1, t_l^B) = E_\mu^{B,R_\gamma}(x; q, t_m^B, 1, t_l^B).$$  \hspace{1cm} (3.3.8)

**Proof** By (3.3.6), it is enough to show $E_\mu^{sp,B}(x; q, t_m^B, t_l^B) = E_\mu^B(x; q, t_m^B, t_l^B)$. Comparing (3.3.5) and (3.3.7), we have

$$(\psi_{ik}^{sp,B})^\pm(z) \bigg|_{t_m = t_m^B} = (\psi_{ik}^{B})^\pm(z), \quad (\psi_{ik}^{sp,B})^\pm(z) \bigg|_{t_l = t_l^B} = (\psi_{ik}^{B})^\pm(z).$$

The embedding $W \rightarrow W^{B_\gamma}$ in Lemma 3.3.2 implies that we have $\Gamma(\mu, e) = \Gamma_B(\mu, e)$ for any $\mu \in P_{B_n} \cap P_{C_n} = P_{C_n}$. Then, the result follows from Lemma 3.3.3. \hfill \square

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Comparing this result with the specialization Table 1, we see that the specialization (3.3.8) corresponds to type $B_n$. Thus, the Macdonald polynomial of Ram–Yip type $B_n$ is the Macdonald polynomial of type $B_n$ in the sense of Definition 1.0.1.

### 3.4 Type $D_n$

By Proposition 2.4.6, we know that the specialization

$$t_n = u_n = t_0 = u_0 = 1$$

yields the non-symmetric Macdonald polynomial $E_D^\mu(x)$ of type $D_n$. In this subsection, we reprove it by using the Ram–Yip formula of type $D_n$, in which case there is no discrepancy between [23, 28], so we use our default notation for the affine root system and the non-symmetric Macdonald polynomials based on [23].

#### 3.4.1 Ram–Yip affine root system of type $D$

Recall the affine root system $S_D$ of type $D_n$ given in (2.4.14):

$$S_D := O_5 = \{ \pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, \ r \in \mathbb{Z} \}.$$  

A basis given by

$$a_0^D := -\epsilon_1 - \epsilon_2 + c, \quad a_j^D := a_j = \epsilon_j - \epsilon_{j+1} \quad (1 \leq j \leq n - 1), \quad a_n^D := \epsilon_{n-1} + \epsilon_n.$$  

Denoting $s_n^D := s_{a_n^D}$, the finite Weyl group is given by $W_0^D := \langle s_1, \ldots, s_{n-1}, s_n^D \rangle \simeq \{ \pm 1 \}^{n-1} \rtimes S_n$. Also, recall the weight lattice $P_D$ in (2.4.15):

$$P_D := \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n \oplus \mathbb{Z}\frac{1}{2} (\epsilon_1 + \cdots + \epsilon_n)$$

and the extended affine Weyl group $W^D = W_0^D \ltimes t(P_D)$ in (2.4.16). The group $W^D$ has another description:

$$W^D = \left\{ s_0^D, s_1, \ldots, s_{n-1}, s_n^D, \pi_1^D, \pi_{n-1}^D, \pi_n^D \right\}. \quad (3.4.1)$$

Here $\pi_1^D$, $\pi_{n-1}^D$ and $\pi_n^D$ denotes the generators of the automorphic group

$$\Omega_D := P_D / Q_D = \left\{ \pi_0^D = e, \pi_1^D, \pi_{n-1}^D, \pi_n^D \right\}$$

of the extended Dynkin diagram of type $D_n$:
As an abstract group, $W^D$ is presented by the generators \((3.4.1)\) and the following relations.

\[
s_i^2 = (s_i^D)^2 = e, \\
s_0^D s_1 = s_1 s_0^D, \quad s_{n-1} s_n^D = s_n^D s_{n-1}, \quad s_i s_j = s_j s_i \quad (|i - j| > 1), \\
s_0^D s_2 s_0^D = s_2 s_0^D s_2, \quad s_n^D s_{n-2} s_n^D = s_{n-2} s_n^D s_{n-2}, \quad s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i (i = 1, \ldots, n-2), \\
\pi_1^D s_0 = s_1 \pi_1^D, \quad \pi_{n-1}^D s_0 = s_n^D \pi_{n-1}^D, \quad \pi_n^D s_0 = s_n^D \pi_n^D, \\
\pi_n^D s_1 = s_n \pi_n^D, \quad \pi_n^D s_1 = s_n \pi_n^D, \quad \pi_{n-1}^D s_1 = s_{n-1} \pi_{n-1}^D (i = 2, \ldots, n-2), \\
\pi_n^D s_i = s_{n-1} \pi_n^D \quad (i = 2, \ldots, n-2), \quad (\pi_1^D)^2 = (\pi_1^D)^2 = e \quad (i = 1, \ldots, n).
\]

\[(3.4.2)\]

Although it will not be used explicitly, let us write down the action of $W^D$ on $F_Z$ \((2.1.14)\).

\[
s_0^D (\epsilon_i) = \begin{cases} 
  c - \epsilon_2 & (i = 1) \\
  c - \epsilon_1 & (i = 2) \\
  \epsilon_i & (i \neq 1, 2) 
\end{cases}, \quad s_j (\epsilon_i) = \begin{cases} 
  \epsilon_j & (i = j + 1) \\
  \epsilon_j + 1 & (i = j) \\
  \epsilon_i & (i \neq j, j + 1) 
\end{cases}, \quad \epsilon_i \quad (i \neq 1, 2).
\]

\[
s_n^D (\epsilon_i) = \begin{cases} 
  -\epsilon_n & (i = n - 1) \\
  -\epsilon_{n-1} & (i = n) \\
  \epsilon_i & (i \neq n - 1, n) 
\end{cases}, \quad \pi_n^D (\epsilon_i) = \frac{1}{2} c - \epsilon_{n-1+i} \quad (i = 0, \ldots, n),
\]

\[
\pi_1^D (\epsilon_i) = \begin{cases} 
  c - \epsilon_1 & (i = 1) \\
  \epsilon_i & (i \neq 1) 
\end{cases}, \quad \pi_{n-1}^D (\epsilon_i) = \begin{cases} 
  \frac{1}{2} c + \epsilon_n & (i = 1) \\
  \frac{1}{2} c - \epsilon_{n-1+i} & (i \neq 1) 
\end{cases}.
\]

We also write down reduced expressions of $t(\epsilon_i) \in W^D$:

\[
t(\epsilon_1) = \pi_1^D, \quad t(\epsilon_2) = \pi_1^D s_0^D s_1, \quad t(\epsilon_i) = \pi_1^D s_{i-1} \cdots s_0^D s_1 \cdots s_{i-1} \quad (i = 3, \ldots, n).
\]

\[(3.4.3)\]

### 3.4.2 Ram–Yip formula of non-symmetric Macdonald polynomial of type $D$

There is a unique $W^D$-orbit on the affine root system $S^D$, i.e., $O_5$, and correspondingly we set the parameter

\[
t \longleftrightarrow O_5.
\]

See also equation \((2.4.17)\). For $\mu \in P_{D_n}$, the non-symmetric Macdonald polynomial of type $D_n$ is denoted by

\[
E_\mu^D (x) = E_\mu^D (x; q, t).
\]
For each \( a = \alpha + rc \in S^D \), we define \( sh^D(a) \) and \( ht^D(a) \) by

\[
q^{sh^D(\alpha + rc)} = q^{-r}, \quad t^{ht^D(\alpha + rc)} = t^{(\rho^D, \alpha)}, \quad \rho^D = \sum_{i=1}^{n} (n - i) \epsilon_i.
\]

(3.4.4)

We also denote by \( w_D(\mu) \in W^D \) the shortest element in the coset \( t(\mu)W_0^D \). For \( \mu = \epsilon_i, i = 1, 2, \ldots, n \), they are given by

\[
w_D(\epsilon_1) = \pi_1^D, \quad w_D(\epsilon_2) = \pi_1^D s_0^D, \quad w_D(\epsilon_i) = \pi_1^D s_{i-1} \cdots s_2 s_0^D \quad (3 \leq i \leq n).
\]

(3.4.5)

The fundamental alcove of type \( D_n \) is denoted by

\[ A^D := \{ x \in V | a_i^D(x) \geq 0, \ i = 0, 1, \ldots, n \} . \]

Finally, we denote by \( \Gamma_D(\vec{w}, z) \) the set of all alcove walks with start \( z \in W^D \) of type \( \vec{w} \).

**Fact 3.4.1** [28, Theorem 3.1] For \( \mu \in PD_n \), take a reduced expression \( w_D(\mu) = \pi_j^D s_i \cdots s_r \) of the element \( w_D(\mu) \in W^D \) with some \( j \in \{0, 1, n-1, n\} \). Then we have

\[
E^D_\mu(x) = \sum_{p \in \Gamma_D(w(\mu), e)} f_p^D \frac{1}{d(p)} x^{wt(p)},
\]

\[
f_p^D := \prod_{k \in \varphi_+ (p)} (\psi^D_{i_k} + (q^{sh^D(-\beta_k)} t^{ht^D(-\beta_k)}) \prod_{k \in \varphi_- (p)} (\psi^D_{i_k} - (q^{sh^D(-\beta_k)} t^{ht^D(-\beta_k)})
\]

(3.4.6)

where \( \beta_k := s_i \cdots s_{i_k+1}(a^D_{i_r}) \) for \( k = 1, 2, \ldots, r \), and \( (\psi^D_{i})^\pm(z) \) for \( i = 0, 1, \ldots, n \) is given by

\[
(\psi^D_{i})^\pm(z) := \pm t^{-\frac{1}{2}} - t^{\frac{1}{2}} \frac{1}{1 - z^\pm 1}.
\]

For distinction, we denote by \( E^{D,RY}_\mu(x; q, t) \) the right hand side of (3.4.6).

### 3.4.3 Specialization to type \( D_n \)

In this part, we specialize \( t_n = u_n = t_0 = u_0 = 1 \) in \( E^D_\mu(x) \) in Fact 3.1.2, and show that it is equal to \( E^{D,RY}_\mu(x) \) in Fact 3.4.1. We denote the specialized Koornwinder polynomial by

\[
E^{sp,D}_\mu(x; q, t):=E_\mu(x; q, t, 1, 1, 1, 1).
\]

(3.4.7)
Let $\Gamma_{0,n}(\mu, e) \subset \Gamma(\mu, e)$ be the subset consisting of alcove walks without folding by $s_0$ or $s_n$.

**Lemma 3.4.2** For any $\mu \in P_{C_n}$, take a reduced expression $w(\mu) = s_1s_2 \cdots s_n$ of the element $w(\mu) \in W$ in (3.1.7). Then we have

\[
E_{\mu,D}^{sp}(x) = \sum_{p \in \Gamma_{0,n}(w(\mu), e)} f_p^{1/2} x^{wt(p)},
\]

\[
f_p := \prod_{k \in \varphi_+ (p)} (\psi_{t_k}^{sp,D})^+ (q^{sh(-\beta_k)} t^{ht(-\beta_k)}) \prod_{k \in \varphi_- (p)} (\psi_{t_k}^{sp,D})^- (q^{sh(-\beta_k)} t^{ht(-\beta_k)}),
\]

\[
(\psi_i^{sp,D})^\pm (z) := \pm \frac{t^\frac{1}{\ell} \mp t^\frac{1}{\ell}}{1 - z^{\pm 1}} \quad (i = 1, 2, \ldots, n - 1).
\]

**Proof** The specialization $t_0 = t_n = u_0 = u_n = 1$ in (3.1.9) yields $\psi_0^\pm (z) = \psi_n^\pm (z) = 0$. Thus the folding steps by $s_0$ or $s_n$ do not appear in the summation of Fact 3.1.2. $\square$

Thus, it is enough to construct a bijection $\Gamma_{0,n}(\mu, e) \rightarrow \Gamma_D(wD(\mu), e)$.

**Lemma 3.4.3** The following gives an injective group homomorphism $\varphi^D : W \rightarrow W^D$.

\[
\varphi^D(s_0) = \pi_1^D, \quad \varphi^D(s_i) = s_i \quad (1 \leq i \leq n - 1), \quad \varphi^D(s_n) = e.
\]

**Proof** We can check that the relations (2.1.12) of $W$ are mapped by $\varphi^D$ to those (3.4.2) of $W^D$. Indeed, as for the final relation $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$ in (2.1.12), we have

\[
\varphi^D(s_0 s_1 s_0 s_1) = \varphi^D(s_1 s_0 s_1 s_0) \iff \pi_1^D s_1 \pi_1^D s_1 = s_1 \pi_1^D s_1 \pi_1^D \iff s_0^D s_1 = s_1^D s_0,
\]

which is in the second line of (3.4.2). The other relations can be checked similarly. $\square$

**Lemma 3.4.4** For any $\mu \in P_{C_n}$, take a reduced expression $w(\mu) = s_1s_2 \cdots s_{\ell}$ of the element $w(\mu) \in W$ in (3.1.7), and set

- $I_0 := \{ r \in \{1, 2, \ldots, \ell \} \mid i_r \neq 0 \}$, \quad $I_n := \{ r \in \{1, 2, \ldots, \ell \} \mid i_r \neq n \}$,
- $I := I_0 \cup I_n = \{ k_1 < k_2 < \cdots < k_s \} \quad (s \leq \ell)$,
- $J := \{(b_1, b_2, \ldots, b_\ell) \in [0, 1]^\ell \mid b_i = 1 (i \notin I)\}$.

Using them, define $\vartheta^D : J \rightarrow \{0, 1\}^s$ by

\[
J \ni (b_1, b_2, \ldots, b_\ell) \longmapsto (b_{k_1}, b_{k_2}, \ldots, b_{k_s}) \in \{0, 1\}^s.
\]

1. The length of $w_D(\mu) \in W^D$ is equal to $|I| = s$, and

\[
w_D(\mu) = \begin{cases} s_{j_1} s_{j_2} \cdots s_{j_s} & (|I_0| \in 2\mathbb{Z}) \\
\pi_1^D s_{j_1} s_{j_2} \cdots s_{j_s} & (|I_0| \notin 2\mathbb{Z}) \end{cases}.
\]

\[\end{document}\]
2. The map $\tilde{\varphi}^D : J \to \{0, 1\}^s$ induces a bijection

$$\tilde{\varphi}^D : \Gamma_{0,n}(\overrightarrow{w(\mu)}, e) \to \Gamma_D(\overrightarrow{w_D(\mu)}, e),$$

$$(A, s_{j_1}^{b_{j_1}} A, \ldots, s_{j_l}^{b_{j_l}} \cdots s_{j_k}^{b_{j_k}} A) \mapsto \left( (A_D, s_{j_1}^{b_{j_1}} A_D, \ldots, s_{j_l}^{b_{j_l}} A_D) \right) (|I_0| \in 2\mathbb{Z})$$

$$(\pi_1^D A_D, \pi_1^D s_{j_1}^{b_{j_1}} A_D, \ldots, \pi_1^D s_{j_l}^{b_{j_l}} \cdots s_{j_k}^{b_{j_k}} A_D) (|I_0| \notin 2\mathbb{Z}).$$

3. For any $p \in \Gamma_{0,n}(\overrightarrow{w(\mu)}, e)$, we have $\text{wt}(p) = \text{wt}(\tilde{\varphi}^D(p))$, $d(p) = d(\tilde{\varphi}^D(p))$.

**Proof** 1. It is enough to show $\varphi^D(w(\mu)) = w_D(\mu)$ for any $\mu \in P_{C_n}$. By the reduced expressions (3.1.8) and (3.4.5), we have $\varphi^D(w(\epsilon_i)) = w_D(\epsilon_i)$ for each $i = 1, 2, \ldots, n$. Then, since $\varphi^D$ is a group homomorphism by Lemma 3.4.3, we find the desired equality.

2. It is an immediate consequence of (1) and the bijectivity of $\tilde{\varphi}^D$.

3. Similarly as (1), we have $\varphi^D(t(\epsilon_i)) = t(\epsilon_i)$ for each $i = 1, 2, \ldots, n$, and thus $t(\text{wt}(p)) = t(\text{wt}(\tilde{\varphi}^D(p)))$ for each $p \in \Gamma_{0,n}(\overrightarrow{w(\mu)}, e)$, which implies $\text{wt}(p) = \text{wt}(\tilde{\varphi}^D(p))$. As for the remaining $\varphi^D(d(p)) = d(\tilde{\varphi}^D(p))$, since $\varphi^D(s_n) = e$ and $\varphi^D$ preserves $s_1, s_2, \ldots, s_{n-1}$, the specialization $t_n = 1$ yields $d(p) = t_d(\tilde{\varphi}^D(p))$, which gives the consequence.

Thus we have $E_{\mu}^{sp,D}(x; q, t) = E_{\mu}^{D,RY}(x; q, t)$ for any $\mu \in P_{C_n} \subset P_{D_n}$. Using (3.4.7), we have the conclusion:

**Proposition 3.4.5** For any $\mu \in P_{C_n}$, the following equality holds.

$$E_{\mu}(x; q, t, 1, 1, 1, 1) = E_{\mu}^{D,RY}(x; q, t).$$

4 Concluding remarks

The original motivation of our study on specialization is to find some explicit formula of symmetric Macdonald–Koornwinder polynomials, bearing in mind the Macdonald tableau formula [22, Chap. VI, (7.13), (7.13′)] for type GL. Certain progress has been developed for such tableau formulas of type $B, C, D$ and $(C_n^\vee, C_n)$ by the recent papers [9, 12, 13], although the connection to Ram–Yip type formulas seems to be still unclear.

Another interesting theme is the $t = \infty$ limit. By Sanderson [32] and Ion [14], it is known that the graded character of the level one (thin) Demazure module of an affine Lie algebra of type $X_l^{(r)}$, $X = A, D, E$, is equal to the non-symmetric Macdonald polynomial of the corresponding type specialized at $t = \infty$ if $X_l^{(r)} \neq A_{2j}^{(2)}$, and equal to non-symmetric Koornwinder polynomial specialized at $t = \infty$ in $A_{2j}^{(2)}$. There are vast amount of literature on this topic from representation-theoretic, combinatoric, and geometric points of view. For example, Orr and Shimozono [26] studied the relation of the limits and quantum Bruhat graphs. Let us also mention the article [8] by Chihara,
where the Demazure specialization for type $A^{(2)}_{2l}$ is identified with the graded character of a Demazure slice of the same type $A^{(2)}_{2l}$.

Returning to our study, it would be interesting to find a concrete connection between our argument and the argument given in [8]. Let us close this note by a naive explanation on why the non-symmetric Koornwinder polynomial is related to the representation theory of the affine Lie algebra of type $A^{(2)}_{2l}$. According to [14, §3.2] and [8, §1.5], one considers the specialization of the Noumi parameters

\[(t, t_0, t_n, u_0, u_n) = (t, t, t, 1, t) \quad (4.0.8)\]

Here we exchanged the specialized value of $(t_0, u_0)$ and $(t_n, u_n)$ in loc. cit., due to the numbering of roots explained below. Comparing (4.0.8) and the specialization Table 1, we find that (4.0.8) is included as the case $t_m = t_f^2 = t_s = t$ in the $BC_n$ specialization of Sect. 2.4.4:

\[
\begin{array}{c|cccc}
B & t & t_0 & t_n & u_0 & u_n \\
C & t_m & t_f^2 & t_s & 1 & t_s \\
\end{array}
\]

Let us write again the Dynkin diagram (2.4.12) of the affine root system of type $BC_n$:

```
0 1 2 . . . n-1 n
```

This is in fact the Dynkin diagram for the affine Lie algebra of type $A^{(2)}_n$ for even $n$ [15, p.55, §4.8, Table Aff 2], with the numbering of the roots 0, 1, . . . , $n$ reversed. Thus, very naively speaking, we can read the result of Ion on the Koornwinder specialization [14, §3] from our specialization Table 1.

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