Sonine Transform Associated to the Bessel-Struve Operator

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Abstract

In this paper we consider the Bessel-Struve operator \(l_\alpha\) and the Bessel-Struve intertwining operator \(\chi_\alpha\) and its dual, we define and study the Bessel-Struve Sonine transform \(S_{\alpha,\beta}\) on \(\mathcal{E}(\mathbb{R})\). We prove that \(S_{\alpha,\beta}\) is a transmutation operator from \(l_\alpha\) into \(l_\beta\) on \(\mathcal{E}(\mathbb{R})\) and we deduce similar result for its dual \(S^*_{\alpha,\beta}\) on \(\mathcal{E}'(\mathbb{R})\). Furthermore, invoking Weyl integral transform and the Dual Sonine transform \(^{t}S_{\alpha,\beta}\) on \(\mathcal{D}(\mathbb{R})\), we get a relation between the Bessel-Struve transforms \(\mathcal{F}^\alpha_{BS}\) and \(\mathcal{F}^\beta_{BS}\).

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1 Introduction

Among the formulae listed in Watson’s classical monograph [9] there is the following one due to Sonine [6]

\[ j_\alpha(\lambda x) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)} \int_0^1 (1 - t^2)^{\alpha - \beta - 1} j_\beta(\lambda x t) t^{2\beta + 1} dt \]

where \( j_\alpha \) is the normalized Bessel function

\[ j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1)z^{-\alpha} J_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n + \alpha + 1)} \]

K.Trimèche introduced the Sonine integral transform, in [8], by

\[ \forall x \geq 0, \quad S_{\alpha,\beta}(f)(x) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)} \int_0^1 (1 - r^2)^{\alpha - \beta - 1} f(rx) r^{2\beta + 1} dr \]

This transform is related to the Bessel operator defined by

\[ L_\alpha u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\alpha + 1}{x} \left( \frac{du}{dx}(x) - \frac{du}{dx}(0) \right), \quad x \in \mathbb{R} \]

where \( u \) designates an even function infinitely differentiable on \( \mathbb{R} \).

In this paper, we consider the differential operator \( l_\alpha \), \( \alpha > -\frac{1}{2} \), defined by

\[ l_\alpha u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\alpha + 1}{x} \left( \frac{du}{dx}(x) - \frac{du}{dx}(0) \right), \quad x \in \mathbb{R} \quad (1) \]

with \( u \) an infinitely differentiable function on \( \mathbb{R} \).

This operator is called Bessel-Struve operator. We remark that \( l_\alpha \) can be expressed in the following form

\[ \forall x \in \mathbb{R}^*, \quad l_\alpha(u)(x) = \frac{1}{|x|^{2\alpha + 1}} \frac{d}{dx} \left( |x|^{2\alpha + 1} (u'(x) - u'(0)) \right) \quad (2) \]

For \( \lambda \in \mathbb{C} \), the differential equation :

\[
\begin{align*}
\begin{cases}
    l_\alpha u = \lambda^2 u \\
    u(0) = 1, u'(0) = \frac{\lambda \Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{3}{2})}
\end{cases}
\end{align*}
\]
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possesses a unique solution denoted $\Phi_\alpha(\lambda)$ and called Bessel-Struve kernel. This kernel possesses the following integral representation:

$$\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{C}, \quad \Phi_\alpha(\lambda x) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_0^1 (1 - t^2)^{\alpha - \frac{1}{2}} e^{\lambda xt} dt \quad (3)$$

$\Phi_\alpha$ can be expressed using normalized Bessel function $j_\alpha$ and normalized Struve function $h_\alpha$ by:

$$\Phi_\alpha(x) = j_\alpha(ix) - i h_\alpha(ix), \quad x \in \mathbb{R} \quad (4)$$

where

$$h_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) z^{-\alpha} H_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n (z/2)^{2n+1}}{\Gamma(n + \frac{3}{2}) \Gamma(n + \alpha + \frac{3}{2})}$$

Therefore, the Bessel-Struve kernel can be extended to an analytic function on $\mathbb{C}$ and it has the form in a power series:

$$\Phi_\alpha(z) = \sum_{n=0}^{+\infty} \frac{z^n}{c_n(\alpha)}, \quad z \in \mathbb{C} \quad (5)$$

where

$$c_n(\alpha) = \frac{\sqrt{\pi n!} \Gamma\left(\frac{n}{2} + \alpha + 1\right)}{\Gamma(\alpha + 1) \Gamma(\frac{n+1}{2})}$$

The outline of the content of the paper is as follows:

In section 2, we give some results about harmonic analysis associated to Bessel-Struve operator which we will use later.

In section 3, we deal with Sonine transform defined by

$$\forall x \in \mathbb{R}, \quad S_{\alpha,\beta}(f)(x) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1) \Gamma(\alpha - \beta)} \int_0^1 (1 - r^2)^{\alpha - \beta - 1} f(rx) r^{2\beta + 1} dr$$

Firstly, we find that $S_{\alpha,\beta}$ verifies the following relation

$$\forall f \in \mathcal{E}(\mathbb{R}), \quad l_\alpha(S_{\alpha,\beta}(f)) = S_{\alpha,\beta}(l_\beta(f))$$

Next, we prove that $S_{\alpha,\beta}$ is an isomorphism from $\mathcal{E}(\mathbb{R})$ into itself. These two statements allows us to say that $S_{\alpha,\beta}$ is a transmutation operator from $l_\alpha$ into $l_\beta$ on $\mathcal{E}(\mathbb{R})$.

In section 4, we define the dual of $S_{\alpha,\beta}$ on $\mathcal{E}'(\mathbb{R})$ denoted $S_{\alpha,\beta}^*$. We prove that
$S_{*,\alpha,\beta}$ is a transmutation operator from $l_\beta$ into $l_\alpha$ on $\mathcal{E}'(\mathbb{R})$. Furthermore, we consider the dual Sonine integral transform $tS_{\alpha,\beta}$ in the following sense

$$\int_{\mathbb{R}} S_{\alpha,\beta}(g(x)) f(x) |x|^{2\alpha+1} dx = \int_{\mathbb{R}} tS_{\alpha,\beta}(f)(g(x)) |x|^{2\beta+1} dx$$

We express $tS_{\alpha,\beta}$ using Weyl integral transform associated to Bessel-Struve operator introduced by the authors in [1] and we find a relation between the Bessel-Struve transforms $\mathcal{F}_{B,S}^\alpha$ and $\mathcal{F}_{B,S}^\beta$ on $\mathcal{D}(\mathbb{R})$.

Similar results about Sonine transform have been obtained by K.Trimèche in [8] for the Bessel operator. Recently, Mourou in [3], [4] and Soltani in [7] obtain analogous results in the framework of Dunkl operator.

## 2 Bessel-Struve intertwining operator and its dual

$\mathcal{E}(\mathbb{R})$ designates the space of infinitely differentiable functions $f$ on $\mathbb{R}$, provided with the topology defined by the semi norms

$$p_{n,a}(f) = \sup_{0 \leq k \leq n} \left| f^{(k)}(x) \right|_{x \in [-a,a]}$$

where $a > 0$ and $n \in \mathbb{N}$.

The Bessel-Struve intertwining operator on $\mathbb{R}$ denoted $\chi_\alpha$, introduced by L. Kamoun and M. Sifi in [2], is defined by:

$$\chi_\alpha(f)(x) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})} \int_0^1 (1 - t^2)^{\alpha - \frac{1}{2}} f(\sqrt{t}x) dt \quad , f \in \mathcal{E}(\mathbb{R}), \; x \in \mathbb{R} \quad (6)$$

It verifies the following properties

Let $f$ be in $\mathcal{E}(\mathbb{R})$

$$\chi_\alpha(f)(0) = f(0) \quad (7)$$

$$[\chi_\alpha(f)]'(0) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{3}{2})} f'(0) \quad (8)$$

$$\forall \lambda \in \mathbb{R}, \quad \Phi_\alpha(\lambda \cdot) = \chi_\alpha(e^{\lambda \cdot}) \quad (9)$$

We consider the differential operator $\frac{d}{dx^2} = \frac{1}{2x} \frac{d}{dx}$. 
Theorem 2.1 [2, Theorem 1] The operator $\chi_\alpha$, $\alpha > -\frac{1}{2}$, is a topological isomorphism from $\mathcal{E}(\mathbb{R})$ onto itself. The inverse operator $\chi^{-1}_\alpha$ is given for all $f$ in $\mathcal{E}(\mathbb{R})$ by

(i) if $\alpha = r + k$, $k \in \mathbb{N}$, $-\frac{1}{2} < r < \frac{1}{2}$

$$\chi^{-1}_\alpha f(x) = \frac{2\sqrt{\pi}}{\Gamma(\alpha + 1)\Gamma(\frac{1}{2} - r)}x(\frac{d}{dx^2})^{k+1}\left[\int_0^x (x^2 - t^2)^{-r-\frac{1}{2}}|t|^{2\alpha+1}f(t)\,dt\right]$$

(ii) if $\alpha = \frac{1}{2} + k$, $k \in \mathbb{N}$

$$\chi^{-1}_\alpha f(x) = \frac{2^{2k+1}k!}{(2k+1)!}x(\frac{d}{dx^2})^{k+1}(x^{2k+1}f(x)), \ x \in \mathbb{R}$$

Proposition 2.1 [2, Proposition 1] $\chi_\alpha$ is a transmutation operator from $l_\alpha$ into $D^2$ on $\mathcal{E}(\mathbb{R})$. Namely $\chi_\alpha$ is an isomorphism from $\mathcal{E}(\mathbb{R})$ into itself and verifies

$$l_\alpha \circ \chi_\alpha = \chi_\alpha \circ D^2$$

Definition 2.1 We define the dual transform of $\chi_\alpha$, denoted $\chi^*_\alpha$, on $\mathcal{E}'(\mathbb{R})$ by

$$\langle \chi^*_\alpha(T), f \rangle = \langle T, \chi_\alpha f \rangle, \quad T \in \mathcal{E}'(\mathbb{R}), \ f \in \mathcal{E}(\mathbb{R})$$

Theorem 2.2 [1, Corollary 3.1] $\chi^*_\alpha$ is an isomorphism from $\mathcal{E}'(\mathbb{R})$ into itself.

We denote by $L^1_\alpha(\mathbb{R})$ the space of measurable functions $f$ verifying

$$\int_{\mathbb{R}} |f(t)| |t|^{2\alpha+1} \, dt < +\infty$$

Definition 2.2 For $f \in L^1_\alpha(\mathbb{R})$ with bounded support, we define the integral transform $W_\alpha$ by

$$W_\alpha f(y) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_{|y|}^{+\infty} (x^2 - y^2)^{\alpha-\frac{1}{2}} x f(sgn(y)x) \, dx, \ y \in \mathbb{R}^*$$

$W_\alpha$ is called Weyl integral transform associated to Bessel-Struve operator.
Remark 2.1 By a change of variable, \( W_\alpha f \) can be written

\[
W_\alpha f(y) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} |y|^{2\alpha + 1} \int_{1}^{+\infty} (t^2 - 1)^{\alpha - \frac{1}{2}} t f(ty) \, dt, \quad y \in \mathbb{R}^* \quad (12)
\]

We designate by \( K_0 \) the space of functions \( f \) infinitely differentiable on \( \mathbb{R}^* \) with bounded support verifying for all \( n \in \mathbb{N} \),

\[
\lim_{y \to 0^+} y^n f^{(n)}(y) \quad \text{and} \quad \lim_{y \to 0^-} y^n f^{(n)}(y)
\]

exist.

Remark 2.2 From Lemma 3.1 of [1], one can see that, if \( f \) belongs to \( D(\mathbb{R}) \) then \( W_\alpha (f) \) belongs to \( K_0 \). On the other hand, Proposition 3.2 of [1] says that \( W_\alpha \) is a bounded operator from \( L^1_\alpha(\mathbb{R}) \) into \( L^1(\mathbb{R}) \).

Proposition 2.2 Let \( f \) be a function in \( \mathcal{E}(\mathbb{R}) \) and \( g \) a function in \( L^1_\alpha(\mathbb{R}) \) with bounded support, the operators \( \chi_\alpha \) and \( W_\alpha \) are related by the following relation

\[
\int_{\mathbb{R}} \chi_\alpha f(x) g(x) |x|^{2\alpha + 1} \, dx = \int_{\mathbb{R}} f(x) W_\alpha g(x) \, dx \quad (13)
\]

Proof. By a change of variable, we have

\[
\int_{\mathbb{R}} \chi_\alpha f(x) g(x) |x|^{2\alpha + 1} \, dx = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{\mathbb{R}} \left( \int_{0}^{x} (x^2 - y^2)^{\alpha - \frac{1}{2}} f(y) \, dy \right) x g(x) \, dx
\]

Using Chasles relation and applying Fubini’s theorem, we obtain

\[
\int_{\mathbb{R}} \chi_\alpha f(x) g(x) |x|^{2\alpha + 1} \, dx = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{0}^{+\infty} \left( \int_{y}^{+\infty} (x^2 - y^2)^{\alpha - \frac{1}{2}} x g(x) \, dx \right) f(y) \, dy
\]

\[- \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{-\infty}^{0} \left( \int_{y}^{-\infty} (x^2 - y^2)^{\alpha - \frac{1}{2}} x g(-x) \, dx \right) f(y) \, dy
\]

Finally, by a change of variable, we get

\[
\int_{\mathbb{R}} \chi_\alpha f(x) g(x) |x|^{2\alpha + 1} \, dx = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{\mathbb{R}} \left( \int_{|y|}^{+\infty} (x^2 - y^2)^{\alpha - \frac{1}{2}} x g((\text{sign}(y))x) \, dx \right) f(y) \, dy
\]

\[
= \int_{\mathbb{R}} f(y) W_\alpha g(y) \, dy \quad \square
\]
Proposition 3.4 of \[1\] allows us to give the following definition

**Definition 2.3** We define the operator $V_\alpha$ on $K_0$ as follows

(i) If $\alpha = k + \frac{1}{2}$, $k \in \mathbb{N}$ and $f \in K_0$

\[ V_\alpha f(x) = (-1)^{k+1} \frac{2^{2k+1}k!}{(2k+1)!} \left( \frac{d}{dx^2} \right)^{k+1} (f(x)), \quad x \in \mathbb{R}^* \]

(ii) If $\alpha = k + r$, $k \in \mathbb{N}$, $-\frac{1}{2} < r < \frac{1}{2}$ and $f \in K_0$

\[ V_\alpha f(x) = c_1 \int_{|x|}^{+\infty} (y^2 - x^2)^{-r-\frac{1}{2}} \left( \frac{d}{dy^2} \right)^{k+1} (f)(sgn(x)y) y \, dy, \quad x \in \mathbb{R}^* \]

where \[ c_1 = \frac{(-1)^{k+1}2^{2k+1}\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)} \]

**Proposition 2.3** For $f$ in $K_0$, $V_\alpha(f)$ belongs to $L^1_\alpha(\mathbb{R})$ and has a bounded support.

**Proof.** If supp$(f)$ is included in $[-a,a]$ then it’s clear that supp$(V_\alpha)$ is included in $[-a,a]$.

For $\alpha = k + \frac{1}{2}$, we obtain from \[1\] Lemma 3.2

\[ V_\alpha f(x) = (-1)^{k+1} \frac{2^{2k+1}k!}{(2k+1)!} \sum_{i=0}^{k+1} \beta_i^{k+1} x^{-2k-2+i} f(i)(x) \]

Since $f$ be in $K_0$, we get $|x|^{2k+2}V_\alpha(f) \in L^1(\mathbb{R})$ which proves that $V_\alpha f$ belongs to $L^1_\alpha(\mathbb{R})$.

For $\alpha = k + r$, if $x > 0$, from \[1\] Lemma 3.2

\[ V_\alpha f(x) = c_1 \sum_{i=0}^{k+1} \beta_i^{k+1} \int_{x}^{+\infty} (y^2 - x^2)^{-r-\frac{1}{2}} y^{-2k-1+i} f(i)(y) \, dy \]

By a change of variables,

\[ x^{2\alpha+1}V_\alpha f(x) = c_1 \sum_{i=0}^{k+1} \beta_i^{k+1} \int_{1}^{+\infty} (t^2 - 1)^{-r-\frac{1}{2}} t^{-2k-1}(tx)^i f(i)(tx) \, dt \]
If $x < 0$, by a change of variables and using [1, Lemma 3.2], we can write

$$V^{\alpha}f(x) = c_1 \sum_{i=0}^{k+1} \beta_i^{k+1} \int_{-\infty}^{x} (y^2 - x^2)^{-r-\frac{1}{2}} y^{-2k-1+i} f^{(i)}(y) dy$$

and

$$|x|^{2\alpha+1} V^{\alpha}f(x) = c_1 \sum_{i=0}^{k+1} \beta_i^{k+1} (\text{sign}(x))^{2k+1} \int_{1}^{+\infty} (t^2-1)^{-r-\frac{1}{2}} t^{-2k-1}(tx)^{i} f^{(i)}(tx) dt$$

Therefore, we see that $\lim_{x \to 0^+} |x|^{2\alpha+1} V^{\alpha}f(x)$ exists from dominated convergence theorem. Since $V^{\alpha}f$ is with bounded support then $V^{\alpha}f$ belongs to $L^1_\alpha(\mathbb{R})$ \Box

Proposition 2.4 [1, Remark 3.3] The operators $V^{\alpha}$ and $\chi^{\alpha-1}$ are related by the following relation

$$\int_{\mathbb{R}} V^{\alpha}f(x) g(x) |x|^{2\alpha+1} dx = \int_{\mathbb{R}} f(x) \chi^{\alpha-1} g(x) dx \quad (14)$$

for all $f \in K_0$ and $g \in \mathcal{E}(\mathbb{R})$

Lemma 2.1 Let $f$ be a function in $K_0$ then $W^{\alpha}(V^{\alpha}(f)) = f$ on $\mathbb{R}^*$

Proof. Using Proposition 2.3 and relation (13), we get, for all $g \in \mathcal{E}(\mathbb{R})$

$$\int_{\mathbb{R}} W^{\alpha}(V^{\alpha}(f))(x) g(x) dx = \int_{\mathbb{R}} V^{\alpha}(f)(x) \chi^{\alpha}(g(x)) |x|^{2\alpha+1} dx$$

By relation (14), we deduce that

$$\int_{\mathbb{R}} W^{\alpha}(V^{\alpha}(f))(x) g(x) dx = \int_{\mathbb{R}} f(x) g(x) dx$$

Therefore

$$W^{\alpha}(V^{\alpha}(f))(x) = f(x), \quad a.e. \ x \in \mathbb{R}$$

Since $W^{\alpha}(V^{\alpha}(f))$ and $f$ are both continuous functions on $\mathbb{R}^*$, we get for all $x \in \mathbb{R}^*$, $W^{\alpha}(V^{\alpha}(f))(x) = f(x)$.
Sonine integral transform

Throughout this section $\alpha$ and $\beta$ are two real numbers verifying $\alpha > \beta > \frac{-1}{2}$.

In the next Proposition, we establish an analogue of Sonine formula

**Proposition 3.1** We have the following relation

$$\Phi_\alpha(\lambda x) = a_{\alpha, \beta} \int_0^1 (1 - t^2)^{\alpha - \beta - 1} \Phi_\beta(\lambda x t) t^{2\beta + 1} dt$$  \hspace{1cm} (15)

where

$$a_{\alpha, \beta} = \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)}$$

**Proof.** From relation (5), we have

$$\int_0^1 (1 - t^2)^{\alpha - \beta - 1} \Phi_\beta(\lambda x t) t^{2\beta + 1} dt = \sum_{n=0}^{+\infty} \frac{(\lambda x)^n}{c_n(\beta)} I_n(\alpha, \beta)$$

where

$$I_n(\alpha, \beta) = \int_0^1 (1 - y)^{\alpha - \beta - 1} y^{\beta + \frac{n}{2} + 1} dy$$

Then

$$\int_0^1 (1 - t^2)^{\alpha - \beta - 1} \Phi_\beta(\lambda x t) t^{2\beta + 1} dt = \frac{\Gamma(\alpha - \beta)\Gamma(\beta + 1)}{2\Gamma(\alpha + 1)} \Phi_\alpha(\lambda x)$$

\(\square\)

**Definition 3.1** Let $f$ be a continuous function on $\mathbb{R}$. We define the Sonine integral transform as in [3] by, for all $x \in \mathbb{R}$

$$S_{\alpha, \beta}(f)(x) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)} \int_0^1 (1 - r^2)^{\alpha - \beta - 1} f(r x) r^{2\beta + 1} dr$$  \hspace{1cm} (16)

**Remark 3.1** The following relation yields from relation (15)

$$S_{\alpha, \beta}(\Phi_\beta(\lambda \cdot))(x) = \Phi_\alpha(\lambda x), \quad x \in \mathbb{R}$$
Proposition 3.2 For $f$ bounded continuous function on $\mathbb{R}$, the function $S_{\alpha,\beta}(f)$ is continuous on $\mathbb{R}$ and we have

\[ \|S_{\alpha,\beta}(f)\|_\infty \leq \|f\|_\infty \]  

where \( \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| \)

**Proof.** The result follows from continuity’s theorem and the fact that

\[ \int_0^1 (1 - r^2)^{\alpha-\beta-1} r^{2\beta+1} dr = \frac{\Gamma(\beta+1)\Gamma(\alpha-\beta)}{2\Gamma(\alpha+1)} \]

□

Theorem 3.1 For $f$ a function of class $C^2$ on $\mathbb{R}$, $S_{\alpha,\beta}(f)$ is a function of class $C^2$ on $\mathbb{R}$ and we have

\[ \forall x \in \mathbb{R}, \ l_\alpha(S_{\alpha,\beta}(f))(x) = S_{\alpha,\beta}(l_\beta(f))(x) \]  

(18)

**Proof.** Using the theorem of derivation under the integral sign, we can prove that $S_{\alpha,\beta}(f)$ is of class $C^2$ on $\mathbb{R}$ and

\[ \forall x \in \mathbb{R}, \ [S_{\alpha,\beta}(f)]'(x) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)} \int_0^1 (1 - r^2)^{\alpha-\beta-1} f'(r x) r^{2\beta+2} dr \]

By making the change of variable $t = rx$, we get for all $x \in \mathbb{R}^*$

\[ [S_{\alpha,\beta}(f)]'(x) - [S_{\alpha,\beta}(f)]'(0) = \frac{2\Gamma(\alpha + 1)\text{sign}(x)|x|^{-2\alpha-1}}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)} \int_0^x (x^2 - t^2)^{\alpha-\beta-1}[f'(t) - f'(0)] |t|^{2\beta+2} dt \]

Next, invoking relation (2), we obtain by integration by parts

\[ [S_{\alpha,\beta}(f)]''(x) - [S_{\alpha,\beta}(f)]''(0) = \frac{\Gamma(\alpha + 1)|x|^{-2\alpha-1}}{\Gamma(\beta + 1)\Gamma(\alpha - \beta + 1)} \int_0^x (x^2 - t^2)^{\alpha-\beta-1} f'(t) |t|^{2\beta+1} dt \]

we derive the two sides of the equation above, we obtain by virtue of the theorem of derivation under the integral sign

\[ [S_{\alpha,\beta}(f)]'''(x) = \frac{(2\alpha + 1)\Gamma(\alpha + 1)(-\text{sign}(x))|x|^{-2\alpha-2}}{\Gamma(\beta + 1)\Gamma(\alpha - \beta + 1)} \int_0^x (x^2 - t^2)^{\alpha-\beta-1} f'(t) |t|^{2\beta+1} dt \]
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\[ + \frac{\Gamma(\alpha + 1)2x|x|^{-2\alpha - 1}}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)} \int_0^x (x^2 - t^2)^{\alpha-\beta-1}l_\beta(f)(t) |t|^{2\beta + 1}dt \]

Then

\[ l_\alpha(S_{\alpha,\beta}(f))(x) = [S_{\alpha,\beta}(f)]''(x) + \frac{2\alpha + 1}{x}([S_{\alpha,\beta}(f)]'(x) - [S_{\alpha,\beta}(f)]'(0)) \]

\[ = \frac{2\Gamma(\alpha + 1)\text{sign}(x)|x|^{-2\alpha}}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)} \int_0^x (x^2 - t^2)^{\alpha-\beta-1}l_\beta(f)(t) |t|^{2\beta + 1}dt \]

\[ \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)} \int_0^1 (1 - u^2)^{\alpha-\beta-1}l_\beta(f)(xu) u^{2\beta + 1}du \]

Proposition 3.3 For all \( f \) in \( \mathcal{E}(\mathbb{R}) \) the function \( S_{\alpha,\beta}(f) \) belongs to \( \mathcal{E}(\mathbb{R}) \). The operator \( S_{\alpha,\beta} \) is continuous from \( \mathcal{E}(\mathbb{R}) \) into itself.

Proof. We deduce the wanted result using the theorem of derivation and proposition 3.2.

Remark 3.2 Let \( f \) be a function of class \( C^1 \) on \( \mathbb{R} \), we have

\[ S_{\alpha,\beta}(f)(0) = f(0) \quad \text{(19)} \]

\[ [S_{\alpha,\beta}(f)]'(0) = \frac{\Gamma(\alpha + 1)\Gamma(\beta + \frac{3}{2})}{\Gamma(\beta + 1)\Gamma(\alpha + \frac{3}{2})}f'(0) \quad \text{(20)} \]

Theorem 3.2 The Sonine transform is a topological isomorphism from \( \mathcal{E}(\mathbb{R}) \) into itself. Furthermore, it verifies

\[ S_{\alpha,\beta} = \chi_\alpha \circ \chi_\beta^{-1} \quad \text{(21)} \]

The inverse operator is

\[ S_{\alpha,\beta}^{-1} = \chi_\beta \circ \chi_\alpha^{-1} \quad \text{(22)} \]

Proof. We denote

\[ P_{\alpha,\beta} = S_{\alpha,\beta}(f) - \chi_\alpha \circ \chi_\beta^{-1} \]

Using theorem 3.1 relation (9) and proposition 2.1 we have

\[ l_\alpha P_{\alpha,\beta}(\Phi_\beta(\lambda.)) = S_{\alpha,\beta} \circ l_\beta \circ \chi_\beta(e^{\lambda.}) - l_\alpha \circ \chi_\alpha(e^{\lambda.}) \]

\[ = \lambda^2 S_{\alpha,\beta} \circ \chi_\beta(e^{\lambda.}) - \lambda^2 \chi_\alpha(e^{\lambda.}) \]

\[ = \lambda^2 P_{\alpha,\beta}(\Phi_\beta(\lambda.)) \]
Furthermore, from relations (7) and (19), we deduce that

\[ P_{\alpha,\beta}(\Phi_{\beta}(\lambda.))(0) = 0 \]

and, from relations (8) and (20), we get

\[ [P_{\alpha,\beta}(\Phi_{\beta}(\lambda.))]'(0) = 0 \]

From the uniqueness of the solution of the differential equation \( l_{\alpha}u = \lambda^2u \) with the initial condition \( u(0) = u'(0) = 0 \), we obtain \( P_{\alpha,\beta}(\Phi_{\beta}(\lambda.)) = 0 \). Thus the density of the family \( \{\Phi_{\alpha}(\lambda.)\}_{\lambda \in \mathbb{R}} \) in \( \mathcal{E}(\mathbb{R}) \) implies that for all \( f \in \mathcal{E}(\mathbb{R}) \), \( P_{\alpha,\beta}(f) = 0 \) which proves the relation (21). \( \Box \)

**Remark 3.3** By theorem 3.1 and theorem 3.2, we conclude that \( S_{\alpha,\beta} \) is a transmutation operator from \( l_{\alpha} \) into \( l_{\beta} \) on \( \mathcal{E}(\mathbb{R}) \).

## 4 The Dual Sonine transform

**Definition 4.1** Since \( l_{\alpha} \) is a bounded linear operator from \( \mathcal{E}(\mathbb{R}) \) into itself, we define, for \( T \in \mathcal{E}'(\mathbb{R}) \), \( l_{\alpha}T \) the compactly supported distribution on \( \mathbb{R} \) by

\[ \langle l_{\alpha}T, f \rangle = \langle T, l_{\alpha}f \rangle, \quad f \in \mathcal{E}(\mathbb{R}) \]

**Theorem 4.1** The dual transform \( S_{\alpha,\beta}^* \) of \( S_{\alpha,\beta} \) defined on \( \mathcal{E}'(\mathbb{R}) \) by

\[ \langle S_{\alpha,\beta}^*T, f \rangle = \langle T, S_{\alpha,\beta}f \rangle, \quad f \in \mathcal{E}(\mathbb{R}) \]

is an isomorphism of \( \mathcal{E}'(\mathbb{R}) \) into itself, satisfying the intertwining relation

\[ l_{\beta}(S_{\alpha,\beta}^*T) = S_{\alpha,\beta}^*(l_{\alpha}T), \quad T \in \mathcal{E}'(\mathbb{R}) \]  \( (23) \)

**Proof.** From theorem 3.2, we deduce by duality that \( S_{\alpha,\beta}^* \) is an isomorphism from \( \mathcal{E}'(\mathbb{R}) \) into itself. Using theorem 3.1, we obtain

\[ \langle l_{\beta}S_{\alpha,\beta}^*T, f \rangle = \langle S_{\alpha,\beta}^*l_{\alpha}T, f \rangle \]

which gives the wanted result. \( \Box \)
Remark 4.1 From theorem [3,2] we deduce that
\[ S_{\alpha,\beta}^* = (\chi_\beta^{-1})^* \circ \chi_\alpha^* \] (24)

Definition 4.2 We define the Bessel-Struve transform on \( \mathcal{E}'(\mathbb{R}) \) by
\[ \forall T \in \mathcal{E}'(\mathbb{R}), \forall \lambda \in \mathbb{R}, \mathcal{F}_{B,S}^\alpha(T)(\lambda) = < T, \Phi_\alpha(-i\lambda) > \] (25)

Proposition 4.1 [1, Proposition 4.4] For all \( T \in \mathcal{E}'(\mathbb{R}) \),
\[ \mathcal{F}_{B,S}^\alpha(T) = \mathcal{F} \circ \chi_\alpha^*(T) \] (26)
where \( \mathcal{F} \) is the classical Fourier transform on \( \mathcal{E}'(\mathbb{R}) \).

The Bessel-Struve translation operator is given by
\[ \tau_a f(x) = \chi_{\alpha,a} \chi_{\alpha,x} \left[ \chi_\alpha^{-1}(f)(a + x) \right] \]
It is shown in [2, Proposition 4] that this translation operator is the unique solution \( C^\infty \) on \( \mathbb{R} \times \mathbb{R} \) of the Cauchy problem
\[
\begin{dcimaary}
  l_{\alpha,a} u(a, x) = l_{\alpha,x} u(a, x) \\
  u(0, x) = f(x) \\
  D_a u(0, x) = D f(x)
\end{dcimaary}
\]
where \( f \in \mathcal{E}(\mathbb{R}) \)

This translation operator has the following properties

(i) The operator \( \tau_a \) is a linear continuous operator from \( \mathcal{E}(\mathbb{R}) \) into itself.

(ii) \( \forall u \in \mathcal{E}(\mathbb{R}), \forall a, x \in \mathbb{R}, \) we have
\[ \tau_a u(x) = \tau_x u(a), \quad \tau_0 u(x) = u(x) \]
\[ \tau_a \circ \tau_x = \tau_x \circ \tau_a, \quad l_\alpha \circ \tau_a = \tau_a \circ l_\alpha \]

(iii) The operator \( \tau_a \) verifies the following type product formula
\[ \forall a, x \in \mathbb{R}, \quad \tau_a(\Phi_\alpha(\lambda))(x) = \Phi_\alpha(\lambda x) \Phi_\alpha(\lambda a) \] (27)

Definition 4.3 The convolution product of two elements \( T \) and \( K \) in \( \mathcal{E}'(\mathbb{R}) \) is defined by
\[ \langle T \ast_\alpha K, f \rangle = \langle T_a, K_x, \tau_a f(x) \rangle, \quad f \in \mathcal{E}(\mathbb{R}) \] (28)
The convolution product $\star_\alpha$ satisfies the following property

**Proposition 4.2** Let $T, K \in \mathcal{E}'(\mathbb{R})$ then

$$\mathcal{F}_{BS}^\alpha(T \star_\alpha K) = \mathcal{F}_{BS}^\alpha(T) \mathcal{F}_{BS}^\alpha(K)$$

(29)

**Proof.** From relations (25) and (28), we have

$$\mathcal{F}_{BS}^\alpha(T \star_\alpha K)(\lambda) = \langle T_x, < K_y, \tau_x \Phi_\alpha(\lambda y) \rangle$$

(30)

Invoking relations (27) and (25), we obtain the desired result. □

**Theorem 4.2** We have

1. For all $T \in \mathcal{E}'(\mathbb{R})$

$$\mathcal{F}_{BS}^\alpha(T) = \mathcal{F}_{BS}^\beta(S^*_{\alpha,\beta} T)$$

2. For all $T, K \in \mathcal{E}'(\mathbb{R}),$

$$S^*_{\alpha,\beta}(T \star_\alpha K) = S^*_{\alpha,\beta}(T) \star_\beta S^*_{\alpha,\beta}(K)$$

**Proof.** The first statement can be deduced from relations (26) and (24). The second statement follows by applying $\mathcal{F}_{BS}^\beta$ on both sides and using statement 1 and relation (29). □

**Definition 4.4** For $f$ continuous function on $\mathbb{R}$, with compact support, we define the Dual Sonine transform denoted $^tS_{\alpha,\beta}$ by

$$^tS_{\alpha,\beta}(f)(x) = a_{\alpha,\beta} \int_{|x|}^{+\infty} (y^2 - x^2)^{\alpha - \beta - 1} y f(sgn(x)y)dy, \quad x \in \mathbb{R}^*$$

(30)

where $a_{\alpha,\beta}$ is given in Proposition 3.1.

**Remark 4.2** Invoking relations (30) and (11), we get

$$^tS_{\alpha,\beta}(f)(x) = \frac{\sqrt{\pi}\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta + \frac{1}{2})} W_{\alpha - \beta - \frac{1}{2}}(f)(x), \quad x \in \mathbb{R}$$

(31)

**Proposition 4.3** the Dual Sonine transform verifies the following relation for all $f \in \mathcal{D}(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R}),$

$$\int_{\mathbb{R}} S_{\alpha,\beta}(g)(x) f(x) |x|^{2\alpha + 1} dx = \int_{\mathbb{R}} ^tS_{\alpha,\beta}(f)(x) g(x) |x|^{2\beta + 1} dx$$

(32)
**Proof.** We obtain the result using relations (16) and (30) and Fubini’s theorem.

**Theorem 4.3** For all \( f \) in \( \mathcal{D}(\mathbb{R}) \), we have

\[
^{t}S_{\alpha,\beta}(f) = V_{\beta}(W_{\alpha}(f))
\]  
(33)

**Proof.** From relations (32), (21), (13) and (14) we obtain for \( f \) in \( \mathcal{D}(\mathbb{R}) \) and \( g \in \mathcal{E}(\mathbb{R}) \)

\[
\int_{\mathbb{R}} ^{t}S_{\alpha,\beta}(f)(y)g(y)|y|^{2\beta+1}dy = \int_{\mathbb{R}} V_{\beta}(W_{\alpha}(f))(y)g(y)|y|^{2\beta+1}dy
\]

As the functions \( ^{t}S_{\alpha,\beta}(f) \) and \( V_{\beta}(W_{\alpha}(f)) \) are with compact support , then

\[
^{t}S_{\alpha,\beta}(f) = V_{\beta}(W_{\alpha}(f)) \quad \text{a.e}
\]

Since both functions are continuous on \( \mathbb{R}^* \), we get
\[
\forall x \in \mathbb{R}^*, \quad ^{t}S_{\alpha,\beta}(f)(x) = V_{\beta}(W_{\alpha}(f))(x)
\]  

**Definition 4.5** The Bessel-Struve transform is defined on \( L^{1}_{\alpha}(\mathbb{R}) \) by

\[
\forall \lambda \in \mathbb{R}, \quad F_{B,S}^{\alpha}(f)(\lambda) = \int_{\mathbb{R}} f(x) \Phi_{\alpha}(-i\lambda x) |x|^{2\alpha+1}dx
\]  
(34)

**Proposition 4.4** Let \( f \) be a function in \( L^{1}_{\alpha}(\mathbb{R}) \) with bounded support, then we have

\[
F_{B,S}^{\alpha}(f) = F \circ W_{\alpha}(f)
\]  
(35)

where \( F \) is the classical Fourier transform defined on \( L^{1}(\mathbb{R}) \) by

\[
F(g)(\lambda) = \int_{\mathbb{R}} g(x)e^{-i\lambda x}dx
\]

**Proof.** We proceed in similar way as proposition 3.2 in [1]  

**Corollary 4.1** For all \( f \in \mathcal{D}(\mathbb{R}) \), we have the following decomposition:

\[
F_{BS}^{\alpha}(f) = F_{BS}^{\beta} \circ ^{t}S_{\alpha,\beta}(f)
\]

**Proof.** Let \( f \in \mathcal{D}(\mathbb{R}) \) with support included in \([-a, a]\). Invoking relation (31), we can see that \( ^{t}S_{\alpha,\beta}(f) \) is continuous function on \( \mathbb{R}^* \) with support included in \([-a, a]\) and verifying \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^+} f(x) \) exist. Then we deduce the result from relation (35), theorem 4.3 and Lemma 2.1  

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