Dynamic multipolar polarizabilities and hyperpolarizabilities of the Sr lattice clock

Fang-Fei Wu,1,4 Yong-Bo Tang,2,3,* Ting-Yun Shi,1 and Li-Yan Tang1,†
1 State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, People’s Republic of China
2 College of Engineering Physics, Shenzhen technology University, Shenzhen 518118, China
3 College of Physics and Materials Science, Henan Normal University, Xinxiang 453007, Peoples Republic of China and
4 University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China

I. INTRODUCTION

The last few years have witnessed significant advances in optical clocks, which enable a wide range of applications, such as redefine the unit of time [1, 2], test the local Lorentz invariance [3, 4], probe dark matter and dark energy [5, 6], search variations of the fundamental constants [7, 8], and detect gravitational wave [14]. At present, the highest fractional accuracy of optical clocks has reached the level of $10^{-19}$ based on the Al$^+$ [11], while the uncertainty for the Sr [12, 13] and Yb [14] optical lattice clocks has achieved an accuracy of $10^{-18}$ level. Aiming to develop optical clock with uncertainty and stability below $10^{-18}$, a better understanding and meticulous control of the atom-field interactions would benefit for the realization of a new generation of higher-precision optical clocks.

Stark shift as one significant sources of systematic uncertainty for most clocks [12–14], it is closely related to the polarizabilities and hyperpolarizabilities of clock states. Employing a magic wavelength optical lattice [13–17] can eliminate the leading-order of Stark shift, but can not cancel the residual multipolar and higher-order Stark shifts for optical lattice clocks. At the level of $10^{-19}$ accuracy, the effects on the systematic uncertainty of optical clock from the multipolar and higher-order atom-field interaction need to be quantitatively evaluated [18–21].

For the Sr clock, the differential dynamic multipolar polarizability of $\Delta\alpha_{QM}^\omega(\omega)$ at the magic wavelength has contradictions among available theoretical and experimental results. The latest measurement is $-8.01(33) \times 10^{-5}$ a.u. [21], which disagrees with the recent theoretical result of $2.80(36) \times 10^{-5}$ a.u. [20] and previous experimental value of $0.0(2.6) \times 10^{-5}$ a.u. [22]. Especially the sign in $\Delta\alpha_{QM}^\omega(\omega)$ between the measurement [21] and theory [20] is opposite each other. In addition, the differential dynamic hyperpolarizability at the magic wavelength also has discrepancy in theory and experiment. The recent RIKEN experimental result of $-2.10(7) \times 10^7$ a.u. [21] agrees well with the SYRTE measurement of $-2.01(45) \times 10^7$ a.u. [22, 23] and the theoretical calculation of $-1.5(4) \times 10^7$ a.u. [20], but it is inconsistent with the zero value of $-1.3(1.3) \times 10^7$ a.u. measured by JILA [12]. Especially, the single-electron approximated result of $-3.74 \times 10^7$ a.u. [19] is not within the error bar of any other existing theoretical and experimental results. Therefore, carrying out an independent theoretical calculation is expected to solve these discrepancies.

In this paper, we develop an effective method by combining the Dirac-Fock plus core polarization (DFCP) and relativistic configuration interaction (RCI) approaches for the relativistic calculation of the divalent atoms, and apply it to calculate the dynamic multipolar polarizabilities and hyperpolarizabilities at the magic wavelength for the Sr clock states by employing the sum-over-states method. The detailed comparisons for the energies, reduced matrix elements and static dipole polarizabilities between our results and other published literatures are also made. Our work not only resents an independent test for the previous calculations of Ref. [20], but also would stimulate further investigations on the differential multipolar polarizability of the Sr clock. The atomic units (a.u.) are used throughout of this work except specifically mentioned.

*Email Address: ybtang@htu.edu.cn
†Email Address: lytang@wipm.ac.cn
II. THEORETICAL FRAMEWORK

A. The combination method of DFCP and RCI

The basic strategy of present theoretical method is that a
divalent electron atom is simplified as a frozen core part
and valence electron part. The calculation process can
be divided into three steps. The first step is the Dirac-
Fock (DF) calculation of frozen core part to obtain the
core orbital functions \( \psi(r) \) [24].

The second step is to solve the followed DFCP equation
to obtain the single-electron wave functions \( \phi(r) \),
\[
h_{\text{DFCP}}(r)\phi(r) = \varepsilon\phi(r),
\]
(1)
where \( h_{\text{DFCP}}(r) \) represents the DFCP Hamiltonian,
\[
h_{\text{DFCP}}(r) = c\alpha \cdot p + (\beta - 1)c^2 + V_N(r) + V_{\text{core}}(r),
\]
(2)
where \( \alpha \) and \( \beta \) are the \( 4 \times 4 \) Dirac matrices, \( p \) is the
momentum operator for the valence electron, \( V_N(r) \)
is the Coulomb potential between a valence electron and
nucleus, \( V_{\text{core}}(r) \) represents the interaction potential be-
tween core electrons and a valence electron, which is ap-
proximated as a DF potential and a semi-empirical one.

\[V_{\text{core}}(r) = V_{\text{DF}}(r) + V_1(r),\]
(3)
with
\[V_1(r) = \frac{\alpha_{\text{core}}}{2r^4}(1 - \exp\left(\frac{r^6}{\rho_{\kappa}^6}\right)),\]
(4)
where \( \alpha_{\text{core}} = 5.812 \) a.u. [26] is the static dipole polariz-
ability of the Sr\(^{2+}\) core. \( \rho_{\kappa} \) is the radial cutoff parameter
which is tuned to reproduce the experimental binding
energy of the lowest state of each \( \kappa \) angular quantum
number. The values of our cutoff parameters for different \( \kappa \) are listed in Table I. The core wave functions \( \psi(r) \)
are constructed by the single-electron wave functions
\[\phi(r) = \text{obtained in the second step [29, 30].}\]

The third step is configuration interaction calculation of
a divalent electron atom. The eigen equation can be expressed as
\[
(\sum_i h_{\text{DFCP}}(r_i) + V_{ij})\Psi(\pi J M) = E\Psi(\pi J M).
\]
(5)
The two-particle interaction potential is
\[V_{ij} = \frac{1}{r_{ij}} + V_2(r_{ij}),\]
(6)
the first term is the Coulomb interaction between two
valence electrons, the second term is two-body core-
polarization interaction with the functional form [27, 28],
\[
V_2(r_{ij}) = -\frac{\alpha_{\text{core}}^2 r_i \cdot r_j}{r_i^3 r_j^3}\sqrt{\frac{1}{\left[1 - \exp\left(\frac{r_i^6}{\rho_{\kappa}^6}\right)\right]\left[1 - \exp\left(\frac{r_j^6}{\rho_{\kappa}^6}\right)\right]}}.
\]
(7)
where \( \rho_{\kappa} \) is fine-tuned on the \( \rho_{\kappa} \) that optimized in the
second step to get accurate energy for the divalent atoms.

The wave function \( \Psi(\pi J M) \) with parity \( \pi \), angular
momentum \( J \), and magnetic quantum number \( M \)
of the system is expanded as a linear combination of the
configuration-state wave functions \( \Phi_I(\pi J M) \), which
are constructed by the single-electron wave functions
\( \phi(r) \) obtained in the second step [29, 30].
\[
\Psi(\pi J M) = \sum_I C_I |\Phi_I(\pi J M)|,
\]
(8)
where \( C_I \) and \( \sigma \) are the expansion coefficients and the
additional quantum number to define each configuration
state uniquely, respectively. Throughout the present cal-
lculations, the basis functions are constructed by using the
Notre Dame basis sets [31].

| \( \kappa \) | \( \rho_{\kappa} \) (in a.u.) for different quantum state |
|--------|----------------------------------|
| \(-1\)  | 2.02950                          |
| \(1\)   | 1.94995                          |
| \(-2\)  | 1.95360                          |
| \(2\)   | 2.35035                          |
| \(-3\)  | 2.36185                          |

TABLE I: The radial cutoff parameter \( \rho_{\kappa} \) (in a.u.) for
different quantum state.

B. Dynamic multipolar polarizability and
hyperpolarizability

For an atom exposed under a linear polarized laser
field with the laser frequency \( \omega \), the dynamic magneto-
dipole and electric-quadrupole polarizabilities for the ini-
tial state \( |0\rangle \equiv |n_0, J_0 = 0\rangle \) (where \( n_0 \) represents all other
quantum numbers) are written as [32]
\[
\alpha^{M1}(\omega) = \frac{2}{3} \sum_n \frac{\Delta E_{n0}|(|0\rangle M1|n_J_n\rangle|^2}{\Delta E_{n0} - \omega^2},
\]
(9)
\[
\alpha^{E2}(\omega) = \frac{1}{30}(\alpha\omega)^2 \sum_n \frac{\Delta E_{n0}|(|0\rangle Q|n_J_n\rangle|^2}{\Delta E_{n0} - \omega^2},
\]
(10)
where \( \alpha \) is the fine structure constant, \( M1 \) and \( Q \) are,
respectively, the magnetic-dipole and electric-quadrupole
transition operators. \( \Delta E_{n0} \) represents the transition
energy between the initial state \( |0\rangle \) and the intermediate
state \( |n_J_n\rangle \).

For the \( J_0 = 0 \) state, the dynamic hyperpolarizability
\( \gamma_0(\omega) \) is expressed as,
\[
\gamma_0(\omega) = \frac{1}{9} T(1, 0, 1, \omega, -\omega, \omega) + \frac{2}{45} T(1, 2, 1, \omega, -\omega, \omega),
\]
(11)
with \( T(J_{\alpha}, J_{\beta}, J_{\gamma}; \omega_1, \omega_2, \omega_3) \) expressed as the followed
general formula [33],
where $D_{\mu i}$ is the dipole transition operator, and $\omega_i$ are the frequencies of the external electric field in the three directions with $\omega_1 = \omega_1 + \omega_2 + \omega_3$. $\sum$ implies a summation over the 24 terms generated by permuting the pairs $(-\omega_1/D_{\mu i})$, $(\omega_1/D_{\mu i})$, $(\omega_2/D_{\mu i})$, $(\omega_3/D_{\mu i})$, the superscripts $\mu_i$ are introduced for the purpose of labeling the permutations $33, 34$, and the prime over the summation means that the intermediate state of $|m_a, J_a \rangle \equiv \langle m_a, J_a |$ are included in Eq. (12).

It’s noted that the relationship between our hyperpolarizability $\gamma_0(\omega)$ and the $\beta(\omega)$ of Porsev et al. $[20]$ is $\gamma_0(\omega) = 4\beta(\omega)^{[2]}$, which indicates both of $T(1,0,1,\omega, -\omega, \omega)$ and $T(1,2,1,\omega, -\omega, \omega)$ terms in Eq. (11) are four times of $Y_{101}(\omega)$ and $Y_{121}(\omega)$ of Ref. [20], respectively. Compared with the dynamic multipolar polarizabilities, the calculation of the dynamic hyperpolarizabilities using the sum-over-states method is much more challenging, since the $T(J_a, J_b, J_c, \omega_1, \omega_2, \omega_3)$ term involves three summation over a large number of intermediate states. This makes it more difficult to calculate the dynamic hyperpolarizability of the clock atoms with high accuracy.

In present work, we perform a large-scale configuration-interaction calculations by constructing sufficient configurations in an appropriate cavity to make sure the completeness of intermediate states, which guarantees the accuracy of our calculations for the dynamic multipolar polarizabilities and hyperpolarizabilities.

III. RESULTS AND DISCUSSIONS

A. Comparisons of energies, reduced matrix elements and static dipole polarizabilities

In order to test the correctness and reliability of our method, we make detailed comparisons of the energies, reduced matrix elements and static dipole polarizabilities in Tables IV and V between present results and other available values. From the comparison of the energies in Table IV, it is seen that the difference for all the reduced matrix elements between our results and the values of Ref. [37] are within 2% except the $5s^21S_0 \rightarrow 5s5p^3P^0$ and $5s^21S_0 \rightarrow 5s6p^3P^0$ transitions. And from the static electric dipole polarizability in Table IV we can see that our values are 202.02 a.u. and 465.81 a.u. for the $5s^21S_0$ and $5s5p^3P^0$ clock states respectively, which agree with the results of 198.9 a.u. and 453.4 a.u. of Safronova et al. [37] within 3%.

Since in the later calculations for the dynamic multipolar polarizabilities and hyperpolarizabilities, we will replace our energies with the NIST energies [38], the error bar of our values mainly comes from reduced matrix elements. From the comparison of the reduced matrix elements in Table IIII we can see that the difference for all the parameters and results of Ref. [37] within 3%.

Table V lists the dynamic magnetic-dipole, electric-quadrupole, and hyperpolarizabilities of the $5s^21S_0$ and $5s5p^3P^0$ clock states at the 813.4280(5) nm magic wavelength. A direct comparison between our work and the calculations of Porsev et al. [20] are also given in this table. The differential $M1$ polarizability $\Delta \alpha_M^{(1)}(\omega)$ is determined thoroughly by $\alpha_M^{(1)}(\omega)$, since the $\alpha_M^{(1)}(\omega)$ polarizability is more than 3 orders of magnitude larger than $\alpha^{(0)}(\omega)$ polarizability. The differential $E2$ polarizability $\Delta \alpha_E^{(2)}(\omega)$ is an order of magnitude larger than $\Delta \alpha_M^{(1)}(\omega)$. The final value of the differential dynamic multipolar polarizability $\Delta \alpha_M^{(2)}(\omega)$ is 2.689(4) $\times 10^{-5}$ a.u., which agrees well with the CI+All-order result of 2.80(36) $\times 10^{-5}$ a.u. [20].

The detailed comparison for the differential multipolar polarizability between theory and experiment are summarized in the Fig. I. It is seen that for $\Delta \alpha_M^{(2)}(\omega)$, there are obvious differences among values of CI+All-order method [20] and the single-electron Fues’ model.
TABLE II: Comparison of energy (in cm$^{-1}$) for some selective low-lying states.

| State     | Present   | NIST [36] | Diff.   |
|-----------|-----------|-----------|---------|
| $5s^2 P_0$ | -134991.48 | -134891.36 | -0.01%  |
| $5p^5 S_0$ | -104184.28 | -104305.54 | -0.116% |
| $5p^2 P_0$ | -99511.11  | -99703.93  | -0.193% |
| $5p^2 P_0$ | -97619.69  | -97737.14  | -0.120% |
| $5p^2 S_0$ | -96437.02  | -96453.36  | -0.110% |
| $5p^2 P_0$ | -93816.62  | -93845.05  | -0.030% |
| $5p^2 S_0$ | -92285.67  | -92300.80  | -0.016% |
| $5p^2 P_0$ | -91375.90  | -91385.20  | -0.010% |
| $5p^5 P_0$ | -120241.53 | -120579.86 | -0.281% |
| $5p^4 P_0$ | -109953.10 | -110143.08 | -0.090% |
| $4d^5 P_0$ | -97533.06  | -97605.30  | -0.074% |
| $5d^7 P_0$ | -95471.60  | -95487.70  | -0.015% |
| $5p^4 P_0$ | -93174.39  | -93185.32  | -0.012% |
| $5p^5 P_0$ | -91004.58  | -91911.51  | -0.008% |
| $5p^5 P_0$ | -120066.55 | -120393.03 | -0.271% |
| $5p^5 P_0$ | -113209.78 | -113198.92 | -0.010% |
| $5p^5 P_0$ | -100941.97 | -101029.05 | -0.086% |
| $5p^5 P_0$ | -100754.48 | -100798.97 | -0.044% |
| $5p^5 P_0$ | -98662.01  | -98663.32  | -0.029% |
| $5p^5 P_0$ | -97520.99  | -97594.64  | -0.075% |
| $5p^5 P_0$ | -95944.97  | -95990.51  | -0.047% |
| $5p^5 P_0$ | -119998.82 | -119998.82 | -0.252% |
| $5p^5 P_0$ | -101482.42 | -101630.52 | -0.146% |
| $5p^5 P_0$ | -100106.28 | -101070.47 | -0.064% |
| $5p^5 P_0$ | -100843.28 | -100924.30 | -0.080% |
| $5p^5 P_0$ | -98545.47  | -98515.62  | -0.030% |
| $5p^5 P_0$ | -97484.48  | -97560.78  | -0.078% |
| $5p^5 P_0$ | -96142.22  | -96146.95  | -0.005% |
| $5s^4 D_1$ | -116417.74 | -116738.33 | -0.275% |
| $5s^4 S_1$ | -105819.31 | -105858.60 | -0.037% |
| $5s^4 D_1$ | -99792.66  | -99890.46  | -0.098% |
| $5s^4 P_0$ | -99310.15  | -99497.26  | -0.188% |
| $5s^4 S_1$ | -97457.80  | -97472.69  | -0.015% |
| $5s^4 D_1$ | -95168.98  | -95211.54  | -0.045% |
| $5s^4 D_2$ | -116359.19 | -116678.58 | -0.274% |
| $5s^4 D_2$ | -114362.94 | -114747.68 | -0.335% |
| $5s^4 S_1$ | -100399.00 | -100169.92 | -0.131% |
| $5s^4 D_2$ | -99778.03  | -99875.38  | -0.097% |
| $5s^4 P_0$ | -99040.99  | -99222.73  | -0.183% |
| $5s^4 P_0$ | -97769.91  | -97936.53  | -0.170% |

TABLE III: Comparison of energy (in cm$^{-1}$) for some selective low-lying states.

| Transition  | Present   | Ref. [37] | Diff.   |
|-------------|-----------|-----------|---------|
| $5s^2 P_0$  | 5.307     | 5.272     | 0.066%  |
| $5s^2 P_0$  | 5.152     | 5.158     | -0.004% |
| $5s^2 P_0$  | 0.235     | 0.228     | -0.03%  |
| $5s^2 P_0$  | 2.760     | 2.712     | 0.048%  |
| $5s^2 P_0$  | 2.002     | 1.970     | 0.015%  |
| $5s^2 P_0$  | 2.457     | 2.460     | -0.015% |
| $5s^2 P_0$  | 2.655     | 2.619     | 0.026%  |
| $5s^2 P_0$  | 0.523     | 0.516     | 0.015%  |
| $5s^2 P_0$  | 1.167     | 1.161     | 0.005%  |

*Ref. [38]*, *Ref. [39]*, *Ref. [40]*, *Ref. [41]*, *Ref. [42]*, *Ref. [43]*, *Ref. [44]*

TABLE IV: Contributions to the static dipole polarizability (in a.u.) of the $5s^2 P_0$ and $5s^2 P_0$ states.

| Core       | Present   | Ref. [37] | Diff.   |
|------------|-----------|-----------|---------|
| $5s^2 P_0$ | 189.947   | 187.4     | 0.012%  |
| $5s^2 P_0$ | 0.234     | 0.25      | -0.07%  |
| $5s^2 P_0$ | 0.236     | 0.34      | -0.004% |
| $5s^2 P_0$ | 0.976     | 0.95      | 0.02%   |
| $5s^2 P_0$ | 48.932    | 47.6      | 0.02%   |
| $5s^2 P_0$ | 1.734     | 1.69      | 0.02%   |
| $5s^2 P_0$ | 7.855     | 7.8       | 0.02%   |
| Tail       | 4.813     | 4.60      | 0.02%   |
| Valance    | 196.206   | 193.54    | 0.02%   |
| Core       | 5.812     | 5.29      | 0.02%   |
| Total      | 202.02    | 198.9     | 0.02%   |

FIG. 1: (Color online) Comparison of the $\Delta \alpha^{QM}(\omega)$ (in a.u.). The green line represents measurement results. The blue line represents our present value, and the magenta line denotes other theoretical results.
TABLE V: The dynamic magnetic-dipole and electric-quadrupole polarizabilities (in a.u.) for the $5s^2 1S_0$ and $5s5p^3 P_0^o$ clock states at the 813.4280(5) nm magic wavelength. $\Delta \alpha^{M1}(\omega) = \alpha^{M1}_{5s^2}(\omega) - \alpha^{M1}_{1S_0}(\omega)$ and $\Delta \alpha^{E2}(\omega) = \alpha^{E2}_{5s^2}(\omega) - \alpha^{E2}_{1S_0}(\omega)$ represent the difference for the clock states of the dynamic magnetic-dipole and electric-quadrupole polarizabilities, respectively. And $\Delta \alpha^{QM}(\omega) = \Delta \alpha^{M1}(\omega) + \Delta \alpha^{E2}(\omega)$. The numbers in parentheses are computational uncertainties. The numbers in the square brackets denote powers of ten.

| Polarizability | Present | DFCP+RCI | CI+PT | CI+All-order |
|----------------|---------|----------|-------|--------------|
| $\alpha^{M1}_{5s^2}(\omega)$ | 2.12(13)[-9] | 2.19[-9] | 2.37[-9] |
| $\alpha^{M1}_{1S_0}(\omega)$ | -5.05(31)[-6] | -5.09[-6] | -5.08[-6] |
| $\Delta \alpha^{M1}(\omega)$ | -5.05(31)[-6] | -5.09[-6] | -5.08[-6] |
| $\alpha^{E2}_{5s^2}(\omega)$ | 9.26(56)[-5] | 8.61[-5] | 8.87(26)[-5] |
| $\alpha^{E2}_{1S_0}(\omega)$ | 12.44(76)[-5] | 12.11[-5] | 12.22(25)[-5] |
| $\Delta \alpha^{E2}(\omega)$ | 3.18(94)[-5] | 3.50[-5] | 3.31(36)[-5] |
| $\Delta \alpha^{QM}(\omega)$ | 2.68(94)[-5] | 2.90[-5] | 2.80(36)[-5] |

C. Comparison of hyperpolarizabilities

The dynamic hyperpolarizabilities of the $5s^2 1S_0$ and $5s5p^3 P_0^o$ clock states at 813.4280(5) nm magic wavelength for the Sr atom are presented in Table V. Since there is a factor of four difference in the definition of the hyperpolarizability between our $\gamma_0(\omega)$ and $\beta(\omega)$ of Porsey [20], we use $\frac{1}{\hbar \omega} T(1,0,1,\omega,\omega,\omega) = \frac{2}{\hbar \omega} Y_{101}(\omega)$, and $\frac{1}{\hbar \omega} T(1,2,1,\omega,\omega,\omega) = \frac{2}{\hbar \omega} Y_{211}(\omega)$ to make a direct comparison with the calculations of Porsey et al. [21]. Our values for both terms are closer to the CI+All-order values than the CI+perturbation theory (PT) results of Ref. [22]. The difference of $\frac{1}{\hbar \omega} T(1,0,1,\omega,\omega,\omega)$ term for the $5s^2 1S_0$ state is within the error bar of any theoretical and experimental results. Two independent theoretical results between our DFCP+RCI value $-2.09(43) \times 10^7$ a.u. and the CI+All-order result $-1.5(4) \times 10^7$ a.u. of Ref. [20] are both in good agreement with the recent high-accuracy measurement of $-2.10(7) \times 10^7$ a.u. in RIKEN [21] and $-2.01(45) \times 10^7$ a.u. in SYRTE [22].

IV. SUMMARY

We have developed the DFCP+CI method for the calculations of the atomic structure properties for divalent atoms. We carried out the calculations of the dynamic magnetic-dipole, electric-quadrupole polarizabilities and hyperpolarizabilities at the magic wavelength for the $5s^2 1S_0$ and $5s5p^3 P_0^o$ clock states of the Sr atom. For the differential hyperpolarizability, our result of $-2.09(43) \times 10^7$ a.u. is in good agreement with the theoretical value of S. G. Porsey et al. [20] and the measurement results of Refs. [21] [22], but disagrees with the zero value measured by JILA [12]. For the differential multipolar polarizability of $\Delta \alpha^{QM}(\omega)$, two independent theoretical calculations from our DFCP+RCI method and the CI+All-order approach of Porsey et al. [20] are consistent with each other, but both have obvious difference from the recent experimental measurement [21], even the sign of the values is opposite. So the difference about $\Delta \alpha^{QM}(\omega)$ in the Sr clock is still pending, which calls for further experimental investigation to resolve this discrepancy.
TABLE VI: The dynamic hyperpolarizabilities (in a.u.) for the 5s2 1S0 and 5s5p3 3P6 clock states at the 813.4280(5) nm magic wavelength. \( \gamma T(1, 0, 1, \omega, -\omega, \omega) = \gamma Y_{101}(\omega) \) and \( \gamma T(1, 2, 1, \omega, -\omega, \omega) = \gamma Y_{121}(\omega) \), where the definition of \( Y_{101}(\omega) \) and \( Y_{121}(\omega) \) can refer to the Ref. [20]. The numbers in parentheses are computational uncertainties. The numbers in the square brackets denote powers of ten.

| Hyperpolarizability | \( 5s2 1S0 \) | \( 5s5p3 3P6 \) |
|---------------------|----------------|------------------|
|                     | Present | Ref. [20] | Present | Ref. [20] |
| \( \frac{1}{36} \gamma T(1, 0, 1, \omega, -\omega, \omega) \) | -6.70[5] | -6.18[5] | -6.06[5] | -7.88[6] | -7.57[6] | -7.50[6] |
| \( \frac{1}{18} \gamma T(1, 2, 1, \omega, -\omega, \omega) \) | 1.49[6] | 1.41[6] | 1.33[6] | -1.22[7] | -6.58[6] | -3.03[6] |
| Total for \( \gamma T_{\gamma 0}(\omega) \) | 8.20[5] | 7.90[5] | 7.25[5] | -2.01[7] | -1.42[7] | -1.05[7] |
| Recommended \( \gamma T_{\gamma 0}(\omega) \) | 8.20(2.0)[5] | 7.90(65)[5] | -2.01(43)[7] | -1.42(37)[7] |
| Recommended \( \gamma \Delta T_{\gamma 0}(\omega) \) | -2.09(43)[7] | 1.5(4)[7] |

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