Brane World Cosmology

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Abstract. Recent developments in the physics of extra dimensions have opened up new avenues to test such theories. We review cosmological aspects of brane world scenarios such as the Randall–Sundrum brane model and two–brane systems with a bulk scalar field. We start with the simplest brane world scenario leading to a consistent cosmology: a brane embedded in an Anti–de Sitter space–time. We generalise this setting to the case with a bulk scalar field and then to two–brane systems.

We discuss different ways of obtaining a low–energy effective theory for two–brane systems, such as the moduli space approximation and the low–energy expansion. A comparison between the different methods is given. Cosmological perturbations are briefly discussed as well as early universe scenarios such as the cyclic model and the born–again brane world model. Finally we also present some physical consequences of brane world scenarios on the cosmic microwave background and the variation of constants.
1. Introduction

Whilst theories formulated in extra dimensions have been around since the early twentieth century, recent developments have opened up new avenues which have enabled the cosmological consequences of such theories to be extracted. This, together with advances in high precision cosmological data, opens up the exciting possibility of testing and constraining extra–dimensional theories for the first time. In this review we explore these developments.

In the early twentieth century Kaluza and Klein \[1\], following Nordstrom, attempted to unify electromagnetism and gravity by proposing a theory in five space–time dimensions, with the extra spatial dimension compactified on a circle. More recently, attempts to construct a consistent theory incorporating both quantum mechanics and gravity have been constructed in more than four dimensions, with the extra, spatial dimensions compactified. In particular superstring theories, which include both quantum theory and gravity, are only consistent in ten dimensions \[2\], with the extra dimensions being spatial. The usual four dimensional physics is retrieved by compactifying on a compact manifold with dimensions typically of the order of the Planck scale. This is essentially a generalisation of Kaluza–Klein theory.

Another approach has emerged recently, motivated by M-theory. M-theory is an umbrella theory that, in certain limits, reduces to the five known string theories or to supergravity. Horava and Witten \[3\] showed that the strong coupling limit of the $E_8 \times E_8$ heterotic string theory at low energy is described by eleven dimensional supergravity with the eleventh dimension compactified on an orbifold with $Z_2$ symmetry, i.e. an interval. The two boundaries of space–time (i.e. the orbifold fixed points) are 10–dimensional planes, on which gauge theories (with $E_8$ gauge groups) are confined. Later Witten argued that 6 of the 11 dimensions can be consistently compactified on a Calabi–Yau threefold and that the size of the Calabi-Yau manifold can be substantially smaller than the space between the two boundary branes \[4\], i.e. Planckian. The branes are hypersurfaces embedded in the extra dimension. Thus, in this limit space–time looks five–dimensional with four dimensional boundary branes \[5\]. This has led to the concept of brane world models.

More generally speaking, in brane world models the standard model particles are confined on a hypersurface (called a brane) embedded in a higher dimensional space (called the bulk). Only gravity and other exotic matter such as the superstring dilaton field can propagate in the bulk. Our universe may be such a brane–like object. This idea was originally motivated phenomenologically (see \[3\]–\[10\]) and later revived by the recent developments in string theory. Within the brane world scenario, constraints on the size of extra dimensions become weaker because the standard model particles propagate only in three spatial dimensions. Newton’s law of gravity, however, is sensitive to the presence of extra–dimensions. Gravity is being tested only on scales larger than a tenth of a millimeter and possible deviations below that scale can be envisaged.

Arkani-Hamed, Dimopoulos and Dvali (ADD) \[11\] and \[12\], following an earlier
idea by Antoniadis [13], proposed an interesting, but simple model, considering a flat geometry in \((4 + d)\)-dimensions, in which \(d\) dimensions are compact with radius \(R\) (toroidal topology). All standard model particles are confined to the brane, but gravity can explore the extra \(d\) dimensions. This gives rise to a modification of the gravitational force law as seen by an observer on the brane. Two test masses, \(m_1\) and \(m_2\), at distances \(r\) apart will feel a gravitational potential of

\[
V(r) \approx \frac{m_1 m_2}{M_{\text{fund}}^{d+2} r^{d+1}}, \quad r \ll R
\]

and

\[
V(r) \approx \frac{m_1 m_2}{M_{\text{fund}}^{d+2} R^{d} r}, \quad r \gg R
\]

where \(M_{\text{fund}}\) is the fundamental Planck mass in the higher dimensional space–time. This allows one to identify the four–dimensional Planck mass \(M_{\text{Pl}}\),

\[
M_{\text{Pl}}^{2} = M_{\text{fund}}^{2+d} R^{d}.
\]

From above we see that on scales larger than \(R\) gravity behaves effectively as the usual four–dimensional gravity. However, on scales less than \(R\) there are deviations and gravity looks truly \(4 + d\) dimensional. Since gravity is tested only down to sizes of around a millimeter, \(R\) could be as large as a fraction of a millimeter.

The most important result of the ADD proposal is a possible resolution to the hierarchy problem, that is the large discrepancy between the Planck scale at \(10^{19}\) GeV and the electroweak scale at 100 GeV. The fundamental Planck mass could be comparable to the electroweak scale as long as the volume of the extra dimensional space is large enough. To realise this, their proposal requires more than one extra dimension.

However, progress was made by Randall and Sundrum, who considered non–flat, i.e. warped bulk geometries [14], [15]. Their models were formulated in one extra dimension with the bulk space–time being a slice of Anti–de Sitter space–time, i.e. a space–time with a negative cosmological constant. They proposed a two–brane model in which the hierarchy problem can be addressed. The large hierarchy is due to the highly curved AdS background which implies a large gravitational red–shift between the energy scales on the two branes. In this scenario, the standard model particles are confined on a brane with negative tension sitting at \(y = r_c\), whereas a positive tension brane is located at \(y = 0\), where \(y\) is the extra spatial dimension. The large hierarchy is generated by the appropriate inter–brane distance, i.e. the radion. It can be shown that the Planck mass \(M_{\text{Pl}}\) measured on the negative tension brane is given by \((k = \sqrt{-\Lambda_5 \kappa_5^2 / 6})\),

\[
M_{\text{Pl}}^{2} \approx e^{2k r_c} M_{5}^{3} / k,
\]

where \(M_{5}\) is the five–dimensional Planck mass and \(\Lambda_5\) the (negative) cosmological constant in the bulk. Thus, we see that, if \(M_{5}\) is not very far from the electroweak scale \(M_W \approx \text{TeV}\), we need \(k r_c \approx 50\), in order to generate a large Planck mass on our brane. Hence, by tuning the radius \(r_c\) of the extra dimension to a reasonable value, one
can obtain a very large hierarchy between the weak and the Planck scale. Of course, a complete realisation of this mechanism requires an explanation for such a value of the radion. In other words, the radion needs to be stabilised at a certain value. The stabilisation mechanism is not thoroughly understood, though models with a bulk scalar field have been proposed and have the required properties [13].

This two brane model was unrealistic because standard model particles are confined to the negative tension brane. However, Randall and Sundrum realised that due to the curvature of the bulk space time, Newton’s law of gravity could be obtained on a positive tension brane embedded in an infinite extra dimension with warped geometry. Small corrections to Newton’s law are generated and constrain the possible scales in the model to be smaller than a millimetre. This model does not solve the hierarchy problem, but has interesting cosmological implications.

A spectacular consequence of brane cosmology, which we discuss in detail in a later section, is the possible modification of the Friedmann equation at very high energy [17]. This effect was first recognised [18] in the context of inflationary solutions. As we will see, Friedmann’s equation has, for the Randall–Sundrum model, the form ([19] and [20])

$$H^2 = \frac{\kappa_5^4}{36} \rho^2 + \frac{8\pi G_N}{3} \rho + \Lambda,$$

relating the expansion rate of the brane $H$ to the (brane) matter density $\rho$ and the (effective) cosmological constant $\Lambda$. The cosmological constant can be tuned to zero by an appropriate choice of the brane tension and bulk cosmological constant, as in the Randall-Sundrum case. Notice that at high energies, for which

$$\rho \gg \frac{96\pi G_N}{\kappa_5^4},$$

where $\kappa_5^2$ is the five dimensional gravitational constant, the Hubble rate becomes

$$H \propto \rho,$$

while in ordinary cosmology $H \propto \sqrt{\rho}$. The latter case is retrieved at low energy, i.e.

$$\rho \ll \frac{96\pi G_N}{\kappa_5^4},$$

Of course modifications to the Hubble rate can only be significant before nucleosynthesis, though they may have drastic consequences on early universe phenomena such as inflation.

A natural extension to the Randall–Sundrum model is to consider a bulk scalar field. This occurs automatically in supersymmetric extensions and makes more contact to string theory, which has the dilaton field. The addition of a bulk scalar field allows other fundamental problems to be addressed, for instance that of the cosmological constant. Attempts have been made to invoke an extra–dimensional origin for the apparent (almost) vanishing of the cosmological constant. The self–tuning idea [21] advocates that the energy density on our brane does not lead to a large curvature of our universe. On the contrary, the extra dimension becomes highly curved, preserving a flat Minkowski brane.
with apparently vanishing cosmological constant. Unfortunately, the simplest realisation of this mechanism with a bulk scalar field fails due to the presence of a naked singularity in the bulk. This singularity can be shielded by a second brane whose tension has to be fine-tuned with the original brane tension \[22\]. In a sense, the fine-tuning problem of the cosmological constant reappears through the extra dimensional back-door. However, in supersymmetric extensions of the Randall–Sundrum model the bulk scalar field can give rise to a late-time acceleration of the universe once supersymmetry is broken on the physical brane world. The naked singularity mentioned above is automatically beyond the second brane. There is still a fine-tuning problem here, namely that of supersymmetry breaking. We discuss this in detail in later sections.

One might hope that brane world models could explain the origin of the universe itself. Attempts have been made in this direction by considering brane collisions. In this scenario the universe undergoes a series of big bangs and big crunches as the branes go through cycles of moving apart and colliding. This is an exciting development which we briefly discuss in the last section.

In this review we study the physics of one brane systems both with an empty bulk and with a bulk scalar field. Birkhoff’s theorem is discussed and examined in the case of a bulk scalar field. We then turn our attention to the case of two brane systems. The low-energy behaviour of two brane systems is emphasised. Throughout this review we discuss the cosmological implications and predictions of brane worlds. In particular, the CMB predictions of simple brane models are reviewed. Finally, we conclude by considering brane collisions both in the cyclic and born-again brane world scenarios. The review is an extension of an earlier review \[23\], with recent developments and new material included. There are other reviews on brane worlds and their cosmological implications (see e.g. \[24\] - \[32\]), each with a different emphasis.

2. A brane in an anti–de Sitter bulk space–time

We begin with a discussion of the simplest non-trivial (i.e. different from Minkowski space–time) bulk, the Anti–de Sitter (AdS) space–time. The AdS space–time is a maximally symmetric solution to Einstein’s equation with negative cosmological constant.

The first well motivated brane world model with an Anti–de Sitter bulk space–time was suggested by Randall and Sundrum. Initially, they proposed a two–brane scenario in five dimensions with a highly curved bulk geometry as an explanation for the large hierarchy between the Planck scale and the electroweak energy scale \[14\]. In their scenario the standard model particles live on a brane with (constant) negative tension, whereas the bulk is a slice of Anti–de Sitter (AdS) space–time. In the bulk there is another brane with positive tension. This is the so–called Randall–Sundrum I (RSI) model, in order to distinguish this setup from a one brane model, which they proposed immediately afterwards. Analysing the solution of Einstein’s equation on the positive tension brane and sending the negative tension brane to infinity, an observer
confined to the positive tension brane recovers Newton’s law if the curvature scale of the AdS space is smaller than a millimeter \[15\]. Note that the higher–dimensional space is non–compact, which must be contrasted with the Kaluza–Klein mechanism, where all extra–dimensional degrees of freedom are compact. This one–brane model, the so–called Randall–Sundrum II (RSII) model, will be our main focus in this section.

Before discussing the cosmological consequences of the second Randall–Sundrum model, let us first look at the static solution, introducing some valuable techniques. The (total) action consists of the Einstein-Hilbert action and the brane action, which have the form

\[ S_{\text{EH}} = - \int d^5x \sqrt{-g^{(5)}} \left( \frac{R}{2\kappa_5^2} + \Lambda_5 \right), \tag{9} \]
\[ S_{\text{brane}} = \int d^4x \sqrt{-g^{(4)}} (-\sigma). \tag{10} \]

Here, \( \Lambda_5 \) is the bulk cosmological constant and \( \sigma \) is the (constant) brane tension. \( \kappa_5 \) is the five–dimensional gravitational coupling constant. The brane is located at \( y = 0 \) and we assume a \( Z_2 \) symmetry, i.e. we identify \( y \) with \( -y \). The ansatz for the metric is

\[ ds^2 = e^{-2K(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2. \tag{11} \]

Einstein’s equations, derived from the action above, give two independent equations:

\[ 6K'' = -\kappa_5^2 \Lambda_5 \]
\[ 3K'' = \kappa_5^2 \sigma \delta(y). \]

The first equation can be easily solved:

\[ K = K(y) = \sqrt{-\frac{\kappa_5^2}{6} \Lambda_5} \ y \equiv ky, \tag{12} \]

which tells us that \( \Lambda_5 \) must be negative. If we integrate the second equation from \(-\epsilon\) to \(+\epsilon\), take the limit \( \epsilon \to 0 \) and make use of the \( Z_2 \) symmetry, we get

\[ 6K'|_0 = \kappa_5^2 \sigma \tag{13} \]

Together with eq. (12) this tells us that

\[ \Lambda_5 = -\frac{\kappa_5^2}{6} \sigma^2 \tag{14} \]

Thus, there must be a fine tuning between the brane tension and the bulk cosmological constant for static solutions to exist.

Randall and Sundrum have shown that there is a continuum of Kaluza–Klein modes for the gravitational field, which is different from the case of a periodic extra dimension where a discrete spectrum is predicted. The continuum of Kaluza–Klein modes lead to a correction to the force between two static masses on the brane. To be more specific, it was shown that the potential energy between two point masses confined on the brane is given by

\[ V(r) = \frac{G_N m_1 m_2}{r} \left( 1 + \frac{l^2}{r^2} + O(r^{-3}) \right). \tag{15} \]
As gravitational experiments show no deviation from Newton’s law of gravity on length scales larger than a millimeter [33], \( l \) has to be smaller than that length scale.

A remark is in order: we assumed a \( Z_2 \) symmetry, i.e. we are considering the orbifold \( S_1/Z_2 \). Most brane world scenarios assume this symmetry. The physical motivation for considering this orbifold compactification comes from heterotic M–theory (as mentioned in the introduction). Of course, assuming the \( Z_2 \) symmetry leads to a significant simplification of the equations and therefore simplicity is a good motivation, too.

In the following we derive Einstein’s equation for an observer on the brane.

2.1. Einstein’s equations on the brane

There are two ways of deriving the cosmological equations and we will describe both of them below. The first one is rather simple and makes use of the bulk equations only. The second method uses the geometrical relationship between four–dimensional and five–dimensional quantities. We begin with the simpler method.

2.1.1. Friedmann’s equation from five–dimensional Einstein equations

In the following subsection we will set \( \kappa_5 \equiv 1 \) as this simplifies the equations a lot. We write the bulk metric as follows:

\[
 ds^2 = a^2 b^2 (dt^2 - dy^2) - a^2 \delta_{ij} dx^i dx^j .
\]  

(16)

This metric is consistent with homogeneity and isotropy on the brane located at \( y = 0 \). The functions \( a \) and \( b \) are functions of \( t \) and \( y \) only. Furthermore, we have assumed flat spatial sections, but it is straightforward to include a spatial curvature. For this metric, Einstein equations in the bulk read:

\[
a^2 b^2 G^0_0 \equiv 3 \left[ \frac{\dot{a}^2}{a^2} + \frac{\dot{b}}{ab} - \frac{a''}{a} + \frac{a'b'}{ab} + kb^2 \right] = a^2 b^2 \left[ \rho_B + \rho \delta(y - y_b) \right]
\]  

(17)

\[
a^2 b^2 G^5_5 \equiv 3 \left[ \frac{\dot{a}}{a} - \frac{\dot{b}}{ab} - 2 \frac{a''}{a^2} - \frac{a'b'}{ab} + kb^2 \right] = -a^2 b^2 T^5_5
\]  

(18)

\[
a^2 b^2 G^0_5 \equiv 3 \left[ -\frac{\dot{a}'}{a} + 2 \frac{\dot{a}a'}{a^2} + \frac{\dot{b}'}{ab} + \frac{a'b'}{ab} \right] = -a^2 b^2 T^0_5
\]  

(19)

\[
a^2 b^2 G^i_j \equiv 3 \left[ \frac{\dot{a}}{a} + \frac{b}{b} - \frac{\dot{b}}{b^2} - 3 \frac{a''}{a} - \frac{b''}{b} + \frac{b'^2}{b^2} \right] \delta^i_j = -a^2 b^2 \left[ \rho_B + \rho \delta(y - y_b) \right] \delta^i_j,
\]  

(20)

where the bulk energy–momentum tensor \( T^a_b \) has been kept general here. For the Randall–Sundrum model we will now take \( \rho_B = -p_B = \Lambda_5 \) and \( T^0_5 = 0 \). Later we will make use of these equations to derive Friedmann’s equation with a bulk scalar field. In the equations above, a dot represents the derivative with respect to \( t \) and a prime a derivative with respect to \( y \).
If one integrates the 00–component over $y$ from $-\epsilon$ to $\epsilon$ and use make use of the $Z_2$ symmetry (which implies here that $a(y) = a(-y)$, $b(y) = b(-y)$, $a'(y) = -a'(-y)$ and $b'(y) = -b'(-y)$), then, in the limit $\epsilon \to 0$, one gets

$$\left. \frac{a'}{a} \right|_{y=0} = \frac{1}{6} ab \rho. \quad (21)$$

Similarly, integrating the $ij$–component and using the last equation gives

$$\left. \frac{b'}{b} \right|_{y=0} = -\frac{1}{2} ab (\rho + p). \quad (22)$$

These two conditions are called the junction conditions and play an important role in cosmology, describing how the brane with some given energy–momentum tensor $T_{\mu\nu}$ can be embedded in a higher–dimensional space–time. The other components of the Einstein equations should be compatible with these conditions. It is not difficult to show that the restriction of the 05 component to $y = 0$ leads to

$$\dot{\rho} + 3 \dot{a} a (\rho + p) = 0, \quad (23)$$

where we have made use of the junction conditions (21) and (22). This is nothing but matter conservation on the brane, which must hold because we have assumed that matter is confined on the brane.

Restricting the 55–component of the bulk equations to the brane and using again the junction conditions gives

$$\ddot{a} - \frac{\dot{a} \dot{b}}{ab} + kb^2 = -\frac{a^2 b^2}{3} \left[ \frac{1}{12} \rho (\rho + 3p) + q_B \right]. \quad (24)$$

Changing to cosmic time $d\tau = ab dt$, writing $a = \exp(\alpha(t))$ and using the energy conservation gives (34, 35)

$$\frac{d(H^2 e^{4\alpha})}{d\alpha} = \frac{2}{3} \Lambda_5 e^{4\alpha} + \frac{d}{d\alpha} \left( \frac{e^{4\alpha} \rho^2}{36} \right). \quad (25)$$

In this equation $aH = da/d\tau$. This equation can easily be integrated to give

$$H^2 = \frac{\rho^2}{36} + \frac{\Lambda_5}{6} + \frac{\mu}{a^4}. \quad (26)$$

Let us split the total energy density and pressure into parts coming from matter and brane tension, i.e. we write $\rho = \rho_M + \sigma$ and $p = p_M - \sigma$. Then we obtain Friedmann’s equation

$$H^2 = \frac{8\pi G}{3} \rho_M \left[ 1 + \frac{\rho_M}{2\sigma} \right] + \frac{\Lambda_4}{3} + \frac{\mu}{a^4}, \quad (27)$$

where we have made the identification

$$\frac{8\pi G}{3} = \frac{\sigma}{18} \quad (28)$$

$$\frac{\Lambda_4}{3} = \frac{\sigma^2}{36} + \frac{\Lambda_5}{6}. \quad (29)$$

Imposing the fine–tuning relation (14) of the static Randall–Sundrum solution in the last equation, we see that $\Lambda_4 = 0$. If there is a small mismatch between the brane tension and
the five–dimensional cosmological constant, an effective four–dimensional cosmological constant is generated. Hence, the Randall–Sundrum setup does not provide a solution to the cosmological constant problem, as one has to impose a relation between brane tension and the cosmological constant in the bulk, i.e. equation (14).

Another result is that the four–dimensional Newton constant is directly related to the brane tension in this model.

The constant $\mu$ appears in the derivation above as an integration constant. The term including $\mu$ is sometimes called the dark radiation term (see e.g. [36]-[38]). The parameter $\mu$ can be obtained from a full analysis of the bulk equations [39]-[41] (we will discuss this in section 4). An extended version of Birkhoff’s theorem tells us that, if the bulk space–time is AdS, this constant is zero [42]. If the bulk is AdS–Schwarzschild instead, $\mu$ is non–zero but a measure of the mass of the bulk black hole. In the following we will assume that $\mu = 0$ and $\Lambda_4 = 0$.

The most significant change in Friedmann’s equation compared to the usual four–dimensional form is the appearance of a term proportional to $\rho^2$. This term implies that if the matter energy density is much larger than the brane tension, i.e. $\rho_M \gg \sigma$, the expansion rate $H$ is proportional to $\rho_M$, instead of $\sqrt{\rho_M}$. The expansion rate is, in this regime, larger in this brane world scenario. Only in the limit where the brane tension is much larger than the matter energy density is the usual behaviour $H \propto \sqrt{\rho_M}$ recovered. This is the most important change in brane world scenarios. As we will see later, it is quite generic and not restricted to the Randall–Sundrum brane world model.

At the time of nucleosynthesis the brane world corrections in Friedmann’s equation must be negligible, otherwise the expansion rate is modified and the results for the abundances of the light elements are unacceptably changed. This implies that $\sigma \geq (1\text{MeV})^4$. Note, however, that a much stronger constraint arises from current tests for deviation from Newton’s law [43] (assuming the Randall–Sundrum fine–tuning relation (14)): $\kappa_5^{-1} > 10^5 \text{TeV}$ and $\sigma \geq (100\text{GeV})^4$. Similarly, cosmology constrains the amount of dark radiation. It has been shown that the energy density in dark radiation can be at most be 10 percent of the energy density in photons [38].

Finally we can derive Raychaudhuri’s equation from Friedmann’s equation and the energy–conservation equation:

$$\frac{dH}{d\tau} = -4\pi G(\rho_M + p_M) \left[ 1 + \frac{p_M}{\sigma} \right].$$

2.1.2. Another derivation of Einstein’s equation In the following, we present a more geometrical and, hence, more powerful derivation of Einstein’s equation on the brane [44].

For this, consider an arbitrary (3+1) dimensional hypersurface $\mathcal{M}$ with unit normal vector $n_a$ embedded in a five–dimensional space–time. The induced metric and the extrinsic curvature of the hypersurface are defined as

$$h^a_{\ b} = \delta^a_{\ b} - n^a n_b,$$

$$K_{ab} = h^c_{\ a} h^d_{\ b} \nabla_c n_d.$$
For the derivation we need three equations; two of them relate four-dimensional quantities constructed from $h_{ab}$ to full five-dimensional quantities constructed from $g_{ab}$. We just state these equations here and refer to [45] for their derivation. The first equation is the Gauss equation, which reads

$$R^{(4)}_{abcd} = h^{j}_{a}h^{k}_{b}h^{l}_{c}h^{m}_{d}R_{jklm} - 2K_{a[c}K_{d]b}.$$  

This equation relates the four-dimensional curvature tensor $R^{(4)}_{abcd}$, constructed from $h_{ab}$, to the five-dimensional one and $K_{ab}$. The next equation is the Codazzi equation, which relates $K_{ab}$, $n_{a}$ and the five-dimensional Ricci tensor:

$$\nabla^{(4)}_{b}K^{c}_{a} - \nabla^{(4)}_{a}K = n^{c}h^{b}_{a}R_{bc}.$$  

One decomposes the five-dimensional curvature tensor $R_{abcd}$ into the Weyl tensor $C_{abcd}$ and the Ricci tensor:

$$R_{abcd} = \frac{2}{3} \left( G_{cd}h^{c}_{a}h^{d}_{b} + G_{cd}n^{c}n^{d} - \frac{1}{4}G \right) h_{ab} + KK_{ab} - K^{c}_{a}K_{bc} - \frac{1}{2} \left( K^{2} - K^{cd}K_{cd} \right) h_{ab} - E_{ab},$$  

where

$$E_{ab} = C_{abcd}n^{c}n^{d}.$$  

We would like to emphasise that this equation holds for any hypersurface. If one considers a hypersurface with energy momentum tensor $T_{ab}$, then there exists a relationship between $K_{ab}$ and $T_{ab}$ ($T$ is the trace of $T_{ab}$) [46]:

$$[K_{ab}] = -\kappa^{2}_{5} \left( T_{ab} - \frac{1}{3}h_{ab}T \right),$$  

where [...] denotes the jump:

$$[f](y) = \lim_{\epsilon \to 0} \left( f(y + \epsilon) - f(y - \epsilon) \right).$$  

These equations are called junction conditions and are equivalent, in the cosmological context, to the junction conditions (21) and (22). Splitting $T_{ab} = \tau_{ab} - \sigma h_{ab}$ and inserting the junction condition into equation (36), we obtain Einstein’s equation on the brane:

$$G^{(4)}_{ab} = 8\pi G \tau_{ab} - \Lambda_{4}h_{ab} + \kappa^{4}_{5}\pi_{ab} - E_{ab}.$$  

The tensor $\pi_{ab}$ is defined as

$$\pi_{ab} = \frac{1}{12} \tau_{ab} - \frac{1}{4}\tau_{ac}\tau^{c}_{b} + \frac{1}{8}h_{ab}\tau_{cd}\tau^{cd} - \frac{1}{24}\tau^{2}h_{ab},$$  

whereas

$$8\pi G = \frac{\kappa^{4}_{5}}{6},$$  

$$\Lambda_{4} = \frac{\kappa^{2}_{5}}{2} \left( \Lambda_{5} + \frac{\kappa^{2}_{5}}{6}\sigma^{2} \right).$$
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Note that in the Randall–Sundrum case we have $\Lambda_4 = 0$ due to the fine-tuning between the brane tension and the bulk cosmological constant. Moreover $E_{ab} = 0$ as the Weyl–tensor vanishes for an AdS space–time. In general, energy conservation and the Bianchi identities imply that

\[ \kappa_5^4 \nabla^a \pi_{ab} = \nabla^a E_{ab} \]  \hspace{1cm} (44)

on the brane.

Clearly, this method is very powerful, as it does not assume homogeneity and isotropy nor does it assume the bulk to be AdS. In the case of an AdS bulk and a Friedmann–Robertson–Walker brane, the previous equations reduce to the Friedmann equation and Raychaudhuri equation derived earlier.

To conclude this section, we have given two derivations of Einstein’s equation as perceived by an observer confined on a brane embedded in an Anti–de Sitter bulk. In the derivation the emphasis was given to the brane observer. This point of view, however, neglects the role of the bulk. In particular it does not say anything about the magnitude of $E_{ab}$ once the brane becomes inhomogeneous. In fact, the set of equations on the brane are not closed in general [47]. This is in particular a problem when discussing the evolution of cosmological perturbations: the evolution of the bulk gravitational field can not be solved using Einstein’s equation on the brane alone, needing instead a full five–dimensional analysis. We will come back to this point in a later section.

2.2. Slow–roll inflation on the brane

We have seen that the Friedmann equation on a brane is drastically modified at high energy where the $\rho^2$ term dominates. As a result the early universe cosmology in brane world models tends to be different from that of standard four–dimensional cosmology. Thus, it seems natural to look for brane effects on early universe phenomena such as inflation (see in particular [48] and [49]) and on phase–transitions [50].

Let us therefore consider a scalar field confined on the brane. The energy density and the pressure of this field are given by

\[ \rho_\phi = \frac{1}{2} \phi_{, \mu} \phi^{, \mu} + V(\phi), \]  \hspace{1cm} (45)

\[ p_\phi = \frac{1}{2} \phi_{, \mu} \phi^{, \mu} - V(\phi), \]  \hspace{1cm} (46)

where $V(\phi)$ is the potential energy of the scalar field. The full evolution of the scalar field is described by the (modified) Friedmann equation, the Klein–Gordon equation and the Raychaudhuri equation.

2.2.1. Single field slow–roll inflation  In the following we will discuss the case of slow–roll inflation. Therefore, the evolution of the fields is governed by (from now on a dot stands for the derivative with respect to cosmic time)

\[ 3H \dot{\phi} \approx - \frac{\partial V}{\partial \phi} \]  \hspace{1cm} (47)
It is easy to show that these equations imply that the slow–roll parameters are given by

\[ \epsilon \equiv - \frac{\ddot{H}}{H^2} = \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2 \left[ \frac{4\sigma(\sigma + V)}{(2\sigma + V)^2} \right] \]

\[ \eta \equiv \frac{V''}{3H^2} = \frac{1}{8\pi G} \left( \frac{V''}{V} \right) \left[ \frac{2\sigma}{2\sigma + V} \right]. \]

The modifications to General Relativity are contained in the square brackets of these expressions. They imply that for a given potential and given initial conditions for the scalar field the slow–roll parameters are suppressed compared to the predictions made in General Relativity. In other words, brane world effects ease slow–roll inflation. In the limit \( \sigma \ll V \) the parameters are heavily suppressed. It implies that steeper potentials can be used to drive slow–roll inflation. What does this mean for cosmological perturbations generated during inflation? In order to study cosmological perturbations, one needs to have information about the projected Weyl tensor, which encodes the dynamics of the bulk space–time. As already said, one needs a full five–dimensional analysis to study the evolution of \( \delta E_{\mu\nu} \). However, one can get some information by neglecting the backreaction of the bulk, i.e. by neglecting the contribution of \( E_{\mu\nu} \). Let us then consider scalar perturbations first. The perturbed line element on the brane has the usual four–dimensional form

\[ ds^2 = -(1 + 2A)dt^2 + 2\partial_i B dt dx^i + ((1 - 2\psi)\delta_{ij} + D_{ij} E) dx^i dx^j, \]

where the functions \( A, B, E \) and \( \psi \) depend on \( t \) and \( x^i \). We will make use of the gauge invariant quantity

\[ \zeta = \psi + H \frac{\delta \rho}{\rho}. \]

In General Relativity, the evolution equation for \( \zeta \) can be obtained from the energy conservation equation, which reads on large scales

\[ \dot{\zeta} = - \frac{H}{\rho + p} \delta p_{\text{nad}}. \]

In this equation \( \delta p_{\text{nad}} = \delta p_{\text{tot}} - c_s^2 \delta \rho \) is the non-adiabatic pressure perturbation. Thus, in the absence of non–adiabatic pressure perturbations the quantity \( \zeta \) is conserved on superhorizon scales. The energy conservation equation holds for the Randall–Sundrum model as well. Therefore, eq. (53) is still valid for the brane world model we consider. For inflation driven by a single scalar field, \( \delta p_{\text{nad}} \) vanishes and therefore \( \zeta \) is constant on superhorizon scales during inflation. Its amplitude is given in terms of the fluctuations in the scalar field on spatially flat hypersurfaces:

\[ \zeta = \frac{H \delta \phi}{\dot{\phi}}. \]

The quantum fluctuation in the (slow–rolling) scalar field obeys \( \langle (\delta \phi)^2 \rangle \approx (H/2\pi)^2 \), as the Klein–Gordon equation is not modified in the brane world model we consider. The
The amplitude of scalar perturbations is \[ A_S^2 = \frac{4\langle \zeta^2 \rangle}{25} \]. Using the slow–roll equations and eq. (54) one obtains \[ A_S^2 \approx \left( \frac{512\pi}{75M_{Pl}^4} \right) \frac{V^3}{V''} \left[ \frac{2\sigma + V}{2\sigma} \right]^3 \bigg|_{k=aH} \] \hspace{1cm} (55)

Again, the corrections are contained in the terms in the square brackets. For a given potential the amplitude of scalar perturbations is enhanced compared to the prediction of General Relativity.

The arguments presented so far suggest that, at least for scalar perturbations, perturbations in the bulk space–time are not important during inflation. This, however, might not be true for tensor perturbations, as gravitational waves can propagate into the bulk. For tensor perturbations, a wave equation for a single variable can be derived \[55\]. The wave equation can be separated into a four–dimensional and a five–dimensional part, so that the solution has the form \[ h_{ij} = A(y)h(x^\mu)e_{ij} \], where \( e_{ij} \) is a (constant) polarisation tensor. One finds that the amplitude for the zero mode tensor perturbation is given by \[55\]

\[ A_T^2 = \frac{4}{25\pi M_{Pl}^4} H^2 F^2(H/\mu)|_{k=aH}. \] \hspace{1cm} (56)

with

\[ F(x) = \left[ \sqrt{1 + x^2} - x^2 \sinh^{-1} \left( \frac{1}{x} \right) \right]^{-1/2}, \] \hspace{1cm} (57)

where we have defined

\[ \frac{H}{\mu} = \left( \frac{3}{4\pi\sigma} \right)^{1/2} H M_{Pl}. \] \hspace{1cm} (58)

It can be shown that modes with \( m > 3H/2 \) are generated but they decay during inflation. Thus, in this scenario one expects only the massless modes to survive until the end of inflation \[55, 56\]. From eqns. (55) and (55) one sees that the amplitudes of scalar and tensor perturbations are enhanced at high energies, though scalar perturbations are more enhanced than tensors. Thus, the relative contribution of tensor perturbations will be suppressed if inflation is driven at high energies.

Interestingly, to first order in the slow–roll parameters, the consistency relation between the amplitudes of scalar and tensor spectra and the spectral index of the tensor power spectrum

\[ \frac{A_T^2}{A_S^2} = \frac{n_T}{2} \] \hspace{1cm} (59)

found in General Relativity holds also for inflation driven on the brane in AdS \[57\]. However, it was shown in \[58\]–\[61\] that this degeneracy is broken at second order in the slow–roll parameters. Nevertheless, it will be difficult to distinguish observationally between models of inflation in General Relativity and the models of brane world inflation considered here. For a recent discussion on constraints on brane world inflation, see \[62\].
2.2.2. Two field slow–roll inflation  Let us consider now the case for two slowly rolling scalar fields, \( \phi \) and \( \chi \), driving a period of inflation in the brane world model we consider \[64\]. To ensure the slow roll behaviour of the fields, the slow–roll parameters

\[
\epsilon_\phi \equiv \frac{1}{16\pi G} \left( \frac{V_\phi}{V} \right)^2 \frac{4\lambda(\lambda + V)}{(2\lambda + V)^2},
\]

\[
\epsilon_\chi \equiv \frac{1}{16\pi G} \left( \frac{V_\chi}{V} \right)^2 \frac{4\lambda(\lambda + V)}{(2\lambda + V)^2},
\]

\[
\eta_{\phi\phi} \equiv \frac{1}{8\pi G} \left( \frac{V_{\phi\phi}}{V} \right) \left[ \frac{2\lambda}{2\lambda + V} \right],
\]

\[
\eta_{\phi\chi} \equiv \frac{1}{8\pi G} \left( \frac{V_{\phi\chi}}{V} \right) \left[ \frac{2\lambda}{2\lambda + V} \right],
\]

\[
\eta_{\chi\chi} \equiv \frac{1}{8\pi G} \left( \frac{V_{\chi\chi}}{V} \right) \left[ \frac{2\lambda}{2\lambda + V} \right],
\]

should be small. However, instead of working with the fields \( \phi \) and \( \chi \) directly, it is more convenient to consider the so–called adiabatic \( \sigma \) and entropy field \( s \). Fluctuations in \( s \) are related to entropy perturbations, whereas perturbations in \( \sigma \) describe purely adiabatic perturbations \[63\]. The field \( \sigma \) and \( s \) are defined through a field rotation:

\[
\delta \sigma = (\cos \theta) \delta \phi + (\sin \theta) \delta \chi
\]

\[
\delta s = - (\sin \theta) \delta \phi + (\cos \theta) \delta \chi,
\]

with

\[
\cos \theta = \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + \dot{\chi}^2}}, \quad \sin \theta = \frac{\dot{\chi}}{\sqrt{\dot{\phi}^2 + \dot{\chi}^2}}.
\]

The slow–roll parameters for \( \sigma \) and \( s \) are given by

\[
\epsilon = \frac{1}{16\pi G} \left( \frac{V_\sigma}{V} \right)^2 \approx \epsilon_\phi + \epsilon_\chi.
\]

and

\[
\eta_{\sigma\sigma} = \eta_{\phi\phi} \cos^2 \theta + 2\eta_{\phi\chi} \cos \theta \sin \theta + \eta_{\chi\chi} \sin^2 \theta,
\]

\[
\eta_{ss} = \eta_{\phi\phi} \sin^2 \theta - 2\eta_{\phi\chi} \cos \theta \sin \theta + \eta_{\chi\chi} \cos^2 \theta,
\]

\[
\eta_{ss} = (\eta_{\chi\chi} - \eta_{\phi\phi}) \sin \theta \cos \theta + \eta_{\phi\chi} (\cos^2 \theta - \sin^2 \theta).
\]

It can be shown that the total curvature perturbation \( \zeta \) and the entropy perturbation

\[
S = \frac{H \delta s}{\dot{\sigma}}
\]

obey the equations \[64\]

\[
\dot{\zeta} \simeq -2H \eta_{ss} S
\]

\[
\dot{S} \simeq [-2\epsilon - \eta_{ss} + \eta_{\sigma\sigma}] HS.
\]

These are the same expressions as found in General Relativity \[65\]. Note, however, that at very high energies, i.e. \( V \gg \lambda \), we have that the ratios \( \zeta/S \) and \( \dot{S}/S \) are
suppressed by a factor \( \approx \sqrt{\lambda/V} \), when compared to General Relativity. It implies that, for given potentials and initial conditions, entropy perturbations have less influence on adiabatic perturbations. Even if entropy perturbations are generated, the final correlation between the adiabatic and entropy modes would be suppressed.

### 2.3. Primordial black holes on the brane

Let us now turn our attention to another interesting consequence of brane worlds: the modification of the evaporation law of primordial black holes, which results in a higher lifetime and lower temperature at evaporation \([66],[67]\). It was shown that the mass–temperature relation for a black hole confined on the brane with radius \( r_0 \) much smaller than the AdS curvature length scale is modified. Consider such a small black hole on the brane. Near the horizon, the metric is (approximately) that of a 5D Schwarzschild black hole, which reads, using spherical coordinates (\( f = 1 - r_0^2/r^2 \))

\[
ds^2_5 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega_5^2.
\]

The near–horizon geometry induced on the brane is given by

\[
ds^2_4 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega_4^2,
\]

which is different from the four–dimensional Schwarzschild metric. However, as in General Relativity, it is the vicinity of the horizon which determines the thermodynamic quantities such as entropy and temperature. For the temperature one finds

\[
T_{BH} = \frac{\sqrt{3}}{32\pi} \left( \frac{M_5^3}{M} \right)^{1/2}.
\]

Note that \( T_{BH} \propto 1/\sqrt{M} \) instead of \( T_{BH} \propto 1/M \) as in General Relativity. Furthermore, it can be shown that Stefan’s law for a D–dimensional black hole reads \([67]\)

\[
\frac{dM}{dt} \approx g_D\sigma_D A_{\text{eff},D} T^D,
\]

in which \( g_D \) denotes the total number of effective bosonic and fermionic degrees of freedom, \( \sigma_D \) is the D–dimensional Stefan–Boltzmann constant defined per degree of freedom (\( \zeta(D) \) is the Riemann zeta function and \( \Omega_{D-2} \) the volume of a \( D-2 \) sphere)

\[
\sigma_D = \frac{\Omega_{D-2}}{4(2\pi)^{D-1}} \Gamma(D)\zeta(D)
\]

and \( A_{\text{eff},D} \) is the effective black hole area calculated from the induced metric.

This implies that, for a given mass, the lifetime \( t_{\text{evap}} \approx M/\dot{M} \) of such a small black hole on the brane can be much longer than black holes in General Relativity. One finds that, for a small five–dimensional black hole,

\[
t_{\text{evap},5D} \approx t_{\text{evap},4D} \left( \frac{l}{r_0} \right)^2,
\]

where \( t_{\text{evap},4D} \) is the lifetime of a black hole of the same mass as predicted with General Relativity. In fact, in \([67],[68]\) it was shown that such a small black hole can survive until the present epoch.
Once a small black hole has formed, it will accrete matter and radiation surrounding it. It is thought that in the standard cosmology this process does not lead to a significant enhancement of the black hole mass \[69\]. However, this is no longer true in brane world cosmology in the high energy regime, where Friedmann’s equation is modified. In this regime the radiation density obeys \(\rho_{\text{rad}} \propto t^{-1}\) instead of \(\rho_{\text{rad}} \propto t^{-2}\). As a result, the growth in the mass of the black hole is \(\dot{M} \propto t^{-1}\) (in General Relativity one finds \(\dot{M} \propto t^{-2}\)). As discussed in depth in \[68\], the consequence of this is that a small black hole can experience significant growth in the high energy regime, albeit with details depending on the accretion efficiency. Once the high energy regime is over, the accretion history is the same as in standard cosmology. In \[70\] one finds a detailed discussion on how these considerations modify some of the known constraints on primordial black holes.

For more work on black holes in brane cosmology, see \[71\]-\[73\]. For reviews, see \[31\] and \[74\].

### 2.4. Graviton Backreaction

We now discuss the production of gravitons by processes on the brane \[75\], \[76\]. As branes carry matter fields and gravity couples to matter, one can envisage the radiation of gravitons into the bulk. The effect of the graviton emission is to modify the background geometry. In turn, this leads to the expected dark radiation term on the brane. As a first step we will neglect the back–reaction of gravitons impinging on the brane. This is much more involved and will be briefly touched upon at the end of the subsection.

#### 2.4.1. Dark Radiation

Physically, matter particles on the brane can collide and lead to the presence of bulk gravitons. As such the brane gravitons carry an energy–momentum tensor and therefore lead to a modification of the bulk Einstein equations. Let us first assume that the gravitons escape radially from the brane. The energy–momentum tensor in the bulk is then

\[
T_{ab} = \sigma k_a k_b 
\]

where \(k_a\) is the null impulsion of the gravitons. The general situation when radiation is not radial will be dealt with later. One can choose the ansatz

\[
ds^2 = -f(r, v)dv^2 + 2drdv + r^2 dx^2
\]

where

\[
f(r, v) = k^2 r^2 - \frac{C(v)}{r^2}
\]

for the bulk metric. This is the so–called Vaidya metric. The function \(C(v)\) generalises the black hole mass as it occurs in the static solutions obeying Birkhoff’s theorem. The Weyl parameter \(C(v)\) satisfies

\[
\frac{dC}{dv} = \frac{2\kappa^2 \sigma}{3} r^3 \left( \dot{r} - \sqrt{f + \dot{r}^2} \right)^2
\]
in terms of the proper time on the brane. The brane junction conditions are now modified and lead to \((H = \dot{r}/r)\)

\[
H^2 = \frac{k_5^2}{18} \lambda \rho + \frac{k_2^2}{36} \rho^2 + \frac{C(v)}{a^4}
\]  

(79)

This is the Friedmann equation where the Weyl parameter \(C(v)\) plays the role of the dark radiation term. Similarly the total matter and pressure contents, \(\rho_m = \lambda + \rho\) and \(p_m = -\lambda + p\), satisfy a non–conservation equation

\[
\dot{\rho}_m + 3 \frac{\dot{r}}{r} (\rho_m + p_m) = -2\sigma 
\]  

(80)

One can see that the effect of the graviton emission is to induce a loss term in the conservation equation. The cosmology on the brane is now highly modified by the graviton radiation term. To go further we need to evaluate \(\sigma\).

Gravitons in the bulk are produced according to the tree level process \(\psi + \bar{\psi} \rightarrow \text{graviton}\) where \(\psi\) and \(\bar{\psi}\) are particles and antiparticles confined to the brane. As long as the temperature on the brane is much lower than both \(k_5^{-2/3}\) and \(k_4^{-1}\), then the temperature \(T\) is always bigger than the Hubble rate. Indeed at high energy \(H \sim k_5^2 T^4\) implies that \(H \ll T\) and similarly at low energy \(H \sim k_4 T^2\) implying that \(H \ll T\) in this regime as well. We have used Stefan’s law whereby \(\rho \sim T^4\) in the radiation epoch. Hence cosmological effects can be neglected and the scattering cross sections can be computed in Minkowski space on the brane. The interaction Lagrangian between matter fields and gravitons is given by

\[
S = \kappa_5 \int dm u_m(0) \int d^4 x \tau_{\mu \nu} h_{\mu \nu}^{(m)}
\]  

(81)

where \(m\) labels the continuous spectrum of Kaluza–Klein states. The wave function of the gravitons in the extra dimension is \(u_m(y)\) corresponding to the four dimensional field \(h_{\mu \nu}^{(m)}\). Here \(\tau_{\mu \nu}\) is the energy momentum of particles on the brane. The matrix element corresponding to this tree level interaction can be computed (averaged over spins)

\[
|M| = \kappa_5^2 |u_m(0)| A \frac{s^2}{8}
\]  

(82)

where \(s\) is the Mandelstam variable \(s = (p_1 + p_2)^2\) corresponding to the annihilation of particles with momenta \(p_{1,2}\). The factor \(A\) is given by \(A = 2/3\) for scalars, \(A = 1\) for fermions and \(A = 4\) for photons.

The Boltzmann equation in curved space is

\[
\dot{\rho} + 3H(\rho + p) = - \int dm \int \frac{d^3 p}{(2\pi)^3} C_m
\]  

(83)

where the collision term reads

\[
C_m = \frac{1}{2} \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} |M|^2 f_1 f_2 \delta^{(4)}(p_m - p_1 - p_2)
\]  

(84)

where \(f_i = 1/(e^{E_i/T} \pm 1)\) depending on the statistics of the particles. Here \(p_m\) is the momentum of the gravitons. Neglecting the contribution of the light gravitons \(m \ll k\), and using \(u_m(0) = 1/\sqrt{\pi}\) for \(m \gg k\) one finds that

\[
\dot{\rho} + 4H \rho = - \left[ \frac{315 \zeta(9/2) \zeta(7/2)}{512 \pi^2} \right] g(T) \kappa_5^2 T^8
\]  

(85)
where the effective number of species is given by

\[ g(T) = \frac{2g_s}{3} + 4g_v + (1 - 2^{-7/2})(1 - 2^{-5/2})g_f \]  \hspace{1cm} (86)

Here \( g_s \), \( g_f \) and \( g_v \) are the number of relativistic particles of spin 0, 1/2 and 1.

This allows one to identify the parameter \( \sigma \)

\[ \sigma = \frac{\alpha}{12} \kappa_5^2 \rho^2 \]  \hspace{1cm} (87)

where Stefan’s law reads \( \rho = \frac{g^2}{30} g_* T^4 \) where \( g_* = g_s + g_v + 7/8 g_f \) and we have defined the parameter

\[ \alpha = \frac{212625}{64\pi^7} \zeta(9/2)\zeta(7/2) \frac{g(T)}{g_*^2} \]  \hspace{1cm} (88)

The cosmological equations can now be solved in the high energy regime leading to the Weyl parameter on the brane

\[ C(t) \sim \alpha a^4 \]  \hspace{1cm} (89)

and

\[ a(t) \sim t^{1/(4+\alpha)} \]  \hspace{1cm} (90)

Numerically \( \alpha \approx 0.02 \) in the standard model of particle physics. The correction to the \( t^{1/4} \) behaviour in the high energy phase is tiny. Nevertheless, notice that the ratio

\[ \epsilon_W = \frac{\rho_W}{\rho_{rad}} \rightarrow \frac{\alpha}{4} \]  \hspace{1cm} (91)

at low energy. The presence of some extra radiation energy density during nucleosynthesis is highly constrained leading to

\[ \epsilon_W \leq 8.10^{-2} \]  \hspace{1cm} (92)

which is marginally higher than \( \alpha/4 \). Forthcoming constraints from CMB anisotropies will lead to a tightening of this bound. This might open up the possibility of confronting the prediction of brane models with experiment.

In the following we will present the case when radiation is not radial.

2.4.2. Gravitons bouncing off the brane

So far it has been assumed that radiation is radial. This is a drastic assumption. In particular this implies that gravitons cannot be emitted and then captured by the brane. An analysis which includes the effect of gravitons bouncing off the brane has been performed in [76]. The gravitons emitted after a particle–antiparticle annihilation on the brane are not all radial. In particular, the energy–momentum tensor is not well approximated by the form given in (75) but also contains non-radial terms proportional to \( \rho^2 \). Moreover, the gravitons may follow geodesics in the bulk which could lead to subsequent collisions with the brane. As a result, the non–conservation equation for the dark component depends on the amount of radiation being emitted and bouncing off the brane. This has been studied numerically.

In particular the ratio \( \epsilon_W \) has been evaluated and is enhanced in comparison with the value, \( \alpha/4 \), given in the previous section. Consequently, this tightens the bounds coming from nucleosynthesis and CMB. Finally, the dynamical consequences of a non–constant Weyl parameter in radiating branes have been investigated in [77].
2.5. Non-$Z_2$ Symmetric Branes

Most brane world scenarios assume a $Z_2$ symmetry about our brane, motivated by M-theory. However, a lot of recent models are not directly derived from M-theory and the motivation for maintaining the $Z_2$ symmetry is less clear. For example, multi-brane scenarios have been suggested which involve branes that, although lying between two bulk space times with the same cosmological constant, do not possess a $Z_2$ symmetry of the metric itself [78]. In this subsection we investigate the cosmological implications of relaxing the $Z_2$ symmetry about the brane world embedded in AdS–Schwarzschild [79], [80].

When one does not impose a $Z_2$ symmetry about our brane world the Friedmann equation on the brane acquires an extra term for the lack of $Z_2$ symmetry. The method of deriving the Friedmann equation is as outlined previously. However, the junction conditions are modified. Let us write [80]

$$a'(0^+) = -a'(0^-) + d(t).$$

(93)

where $d(t)$ is some function of time only and has yet to be determined. $d(t)$ represents the asymmetry of the metric across the brane and the fact that it is not identically zero for all time is what is referred to as the brane being ‘non-$Z_2$ symmetric’ in the rest of this review. Calculating the Israel junction conditions one finds that

$$d(t) = \frac{2F}{\rho_0 a^3},$$

(94)

where $F$ is an integration constant which, when non-zero, dictates to what extent the $Z_2$ symmetry is broken (in fact, it is related to the difference of the black hole masses on each sides of the branes, see [79]). This results in a modified Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^4 \rho_\lambda}{18} + \frac{\kappa^4}{36} \rho^2 - \frac{k}{a^2} + \frac{C}{a^4} + \frac{F^2}{(\rho + \rho_\lambda)^2 a^8}.$$  

(95)

where $\rho_\lambda$ is the energy density associated to the brane tension. So the absence of the $Z_2$ symmetry gives rise to an extra term in the Friedmann equation. For a radiation dominated Universe where $\rho = \gamma/a^4$, the extra term behaves as $F^2/\gamma^2$ as $\rho \to \infty$. This results in the Hubble constant being approximately constant at very early times, resulting in an inflation-like period. At late times the extra term behaves as $(F\rho/\gamma\rho_\lambda)^2$ as $\rho \to 0$, so the effects are no longer significant and our solution reverts to that analysed previously.

The effect of this extra term in the Friedmann equation has been analysed in [80], where it was shown that the universe undergoes a period of exponential expansion during a radiation dominated epoch. However, given that the universe must be described by the usual Friedmann equation by the time of nucleosynthesis the maximum expansion due to the F-term was found to be $10^4$, which is less than that needed to solve the standard cosmological problems.
2.6. Some final remarks

The Randall–Sundrum brane world is the simplest brane world model, which leads to a consistent cosmological framework. However, the model is related to deeper issues, such as the AdS–CFT correspondence (see e.g. [81]-[86]) and therefore to the holographic principle [87]. We cannot go into the details here. Similarly, we do not cover inflation driven by the trace anomaly of the conformal field theory living on the brane (see e.g. [88]-[91]). These are interesting and important developments which give insights into gravity and the consequences for early universe cosmology.

3. Including a Bulk Scalar Field

In this section we are going to generalise the results found in the previous section where the bulk was empty to a more realistic case where the effects of a bulk scalar field are included. This is motivated by supersymmetric extensions to the Randall–Sundrum model, and of course is a more realistic limit of string theory which automatically has the string dilaton field. The projective approach considered previously, where one focuses on the dynamics of the brane, can be extended to this case. Here one studies both the projected Einstein and the Klein-Gordon equation [92], [93]. Again, bulk effects do not decouple, so the dynamics are not closed. As we will see, there are now two objects representing the bulk back-reaction: the projected Weyl tensor $E_{\mu\nu}$ and the loss parameter $\Delta \Phi^2$. In the case of homogeneous and isotropic cosmology on the brane, the projected Weyl tensor is determined entirely up to a dark radiation term. Unfortunately, no information on the loss parameter is available. This prevents a rigorous treatment of brane cosmology in the projective approach. However, the inclusion of a bulk scalar field opens up other avenues. As we will see later in this section, the bulk scalar field can give rise to late-time acceleration of the universe, i.e. it could act as a quintessence field, and also leads to a variation in Newton’s constant.

Another route to analysing brane models is to study the motion of a brane in a bulk space–time. This approach is successful in the Randall-Sundrum case thanks to Birkhoff’s theorem which dictates a unique form for the metric in the bulk [42]. In the case of a bulk scalar field, no such theorem is available. One has to resort to various ansätze for particular classes of bulk and brane scalar potentials (see e.g. [94]-[101]). We discuss this in more detail in section 4.

3.1. BPS Backgrounds

3.1.1. Properties of BPS Backgrounds

As the physics of branes with bulk scalar fields is pretty complicated, we will start with a particular example where both the bulk and the brane dynamics are related and under control [103] (see also [104] and [105]). This is the situation in the BPS case and is automatic in the supersymmetric case. We specify
the bulk Lagrangian as
\[
S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g_5} \left( R - \frac{3}{4} \left( \left( \partial \phi \right)^2 + V(\phi) \right) \right)
\]  \hspace{1cm} (96)
where \( V(\phi) \) is the bulk potential. The boundary action depends on a brane potential \( U_B(\phi) \)
\[
S_B = -\frac{3}{2\kappa_5^2} \int d^4x \sqrt{-g_4} U_B(\phi_0)
\]  \hspace{1cm} (97)
where \( U_B(\phi_0) \) is evaluated on the brane. The BPS backgrounds arise as a special case of this general setting with a particular relationship between the bulk and brane potentials. This relation appears in the study of \( N = 2 \) supergravity with vector multiplets in the bulk. The bulk potential is given by
\[
V = \left( \frac{\partial W}{\partial \phi} \right)^2 - W^2
\]  \hspace{1cm} (98)
where \( W(\phi) \) is the superpotential. The brane potential is simply given by the superpotential
\[
U_B = W
\]  \hspace{1cm} (99)
We would like to mention that the last two relations have also been used to generate bulk solutions without necessarily imposing supersymmetry \([96,102]\). Notice that the Randall–Sundrum case can be retrieved by putting \( W = \text{cst} \). Supergravity puts further constraints on the superpotential which turns out to be of the exponential type \([103]\)
\[
W = 4ke^{\alpha \phi}
\]  \hspace{1cm} (100)
with \(\alpha = -1/\sqrt{12}, 1/\sqrt{3}\). In the following we will often choose this exponential potential with an arbitrary \(\alpha\) as an example. The actual value of \(\alpha\) does not play any role and will be considered generic.

The bulk equations of motion comprise the Einstein equations and the Klein-Gordon equation. In the BPS case and using the following ansatz for the metric
\[
ds^2 = a(y)^2 \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
\]  \hspace{1cm} (101)
these second order differential equations reduce to a system of two first–order differential equations
\[
\frac{a'}{a} = -\frac{W}{4},
\]  \hspace{1cm} (102)
\[
\phi' = \frac{\partial W}{\partial \phi}.
\]  
Notice that when \( W = \text{cst} \) one easily retrieves the exponential profile of the Randall-Sundrum model.

An interesting property of BPS systems can be deduced from the study of the boundary conditions. The Israel junction conditions reduce to
\[
\frac{a'}{a} \big|_B = -\frac{W}{4} \big|_B
\]  \hspace{1cm} (103)
and for the scalar field
\[ \phi'\big|_B = \frac{\partial W}{\partial \phi}\big|_B \] (104)

This is the main property of BPS systems: the boundary conditions coincide with the bulk equations, i.e. as soon as the bulk equations are solved one can locate the BPS branes anywhere in this background, there is no obstruction due to the boundary conditions. In particular two-brane systems with two boundary BPS branes admit moduli corresponding to massless deformations of the background. They are identified with the positions of the branes in the BPS background. We will come back to this later in section 5.

Let us treat the example of the exponential superpotential. The solution for the scale factor reads
\[ a = (1 - 4k\alpha^2 y)^{1/4\alpha^2}, \] (105)
and the scalar field is given by
\[ \phi = -\frac{1}{\alpha} \ln(1 - 4k\alpha^2 y). \] (106)

For \( \alpha \to 0 \), the bulk scalar field decouples and these expressions reduce to the Randall-Sundrum case. Notice a new feature here, namely the existence of singularities in the bulk, corresponding to
\[ a(y)|_{y_*} = 0 \] (107)

In order to analyse singularities it is convenient to use conformal coordinates
\[ du = \frac{dy}{a(y)}. \] (108)

In these coordinates light follows straight lines \( u = \pm t \). It is easy to see that the singularities fall into two categories depending on \( \alpha \). For \( \alpha^2 < 1/4 \) the singularity is at infinity \( u_* = \infty \). This singularity is null and absorbs incoming gravitons. For \( \alpha^2 > 1/4 \) the singularity is at finite distance. It is time-like and not wave-regular, i.e. the propagation of wave packets is not uniquely defined in the vicinity of the singularity. For all these reasons these naked singularities in the bulk are a major drawback of brane models with bulk scalar fields [106]. In the two-brane case the second brane has to sit in front of the naked singularity. This is automatic in the supersymmetric case, as required by conservation of global charges.

3.1.2. de Sitter and anti de Sitter Branes \ Let us modify slightly the BPS setting by detuning the tension of the BPS brane. This corresponds to adding or subtracting some tension compared to the BPS case
\[ U_B = TW \] (109)
where \( T \) is a real number. Notice that this modification only affects the boundary conditions; the bulk geometry and scalar field are still solutions of the BPS equations.
of motion. In this sort of situation, one can show that the brane does not stay static. For the detuned case, the result is either a boosted brane or a rotated brane. Defining by \( u(x^\mu) \) the position of the brane in conformal coordinates, one obtains
\[
(\partial u)^2 = \frac{1 - T^2}{T^2}.
\]
(110)
The brane velocity vector \( \partial_\mu u \) is of constant norm. For \( T > 1 \), the brane velocity vector is time-like and the brane moves at constant speed. For \( T < 1 \) the brane velocity vector is space-like and the brane is rotated\[107, 108\]. Going back to a static brane, we see that the bulk geometry and scalar field become \( x^\mu \) dependent. In the next section we will find many more cases where branes move in a static bulk or equivalently, a static brane borders a non-static bulk.

Let us now conclude this section by studying the brane geometry when \( T > 1 \). In particular one can study the Friedmann equation for the induced scale factor
\[
H^2 = \frac{T^2 - 1}{16} W^2,
\]
(111)
where \( W \) is evaluated on the brane. Of course we find that cosmological solutions are only valid for \( T > 1 \). Now in the Randall-Sundrum case \( W = 4k \) leading to
\[
H^2 = (T^2 - 1)k^2.
\]
(112)
In the case \( T > 1 \) the brane geometry is driven by a positive cosmological constant. This is a de Sitter brane. When \( T < 1 \) the cosmological constant is negative, corresponding to an AdS brane. We are going to generalise these results by considering the projective approach to the brane dynamics.

### 3.2. Bulk Scalar Fields and the Projective Approach

**3.2.1. The Friedmann Equation** We will first follow the traditional coordinate-dependent path. This will allow us to derive the matter conservation equation, the Klein-Gordon and the Friedmann equations on the brane. Then we will concentrate on the more geometric formulation where the role of the projected Weyl tensor will become transparent\[114, 115\]. Again, in this subsection we will put \( \kappa_5 \equiv 1 \).

We consider a static brane that we choose to put at the origin \( y = 0 \). and impose \( b(0, t) = 1 \). This guarantees that the brane and bulk expansion rates
\[
4H = \partial_\tau \sqrt{-g|_0}, \quad 3H_B = \partial_\tau \sqrt{-g_B|_0}
\]
(113)
coincide. We have identified the brane cosmic time \( d\tau = ab|_0 dt \). We will denote by prime the normal derivative \( \partial_n = \frac{1}{ab|_0} \partial_\tau \). Moreover we now allow for some matter to be present on the brane
\[
\tau^\mu_{\nu \text{ matter}} = (-\rho_m, p_m, p_m, p_m).
\]
(114)
The bulk energy-momentum tensor reads
\[
T_{ab} = \frac{3}{4} (\partial_a \phi \partial_b \phi) - \frac{3}{8} g_{ab} \left((\partial \phi)^2 + V\right).
\]
(115)
The total matter density and pressure on the brane are given by
\[ \rho = \rho_m + \frac{3}{2} U_B, \quad p = p_m - \frac{3}{2} U_B. \] (116)

The boundary condition for the scalar field is unchanged by the presence of matter on the brane.

The (05) Einstein equation leads to matter conservation
\[ \dot{\rho}_m = -3H(\rho_m + p_m). \] (117)

By restricting the (55) component of the Einstein equations we obtain
\[ H^2 = \frac{\rho_m^2}{36} - \frac{2}{3} Q - \frac{1}{9} E + \frac{\mu}{a^4} \] (118)
in units of \( \kappa^2 \). The last term is the dark radiation, whose origin is similar to the Randall-Sundrum case. The quantities \( Q \) and \( E \) satisfy the differential equations [35]
\[ \dot{Q} + 4HQ = HT_5^5, \]
\[ \dot{E} + 4HE = -\rho T_5^0. \]

These equations can be easily integrated to give
\[ H^2 = \frac{\rho_m^2}{36} + \frac{U_B \rho_m}{12} - \frac{1}{16a^4} \int d\tau da^4 \frac{d}{d\tau} (\dot{\phi}^2 - 2U) - \frac{1}{12a^4} \int d\tau a^4 \rho_m \frac{dU_B}{d\tau}, \] (119)
up to a dark radiation term and we have identified
\[ U = \frac{1}{2} \left( U_B^2 - \left( \frac{\partial U_B}{\partial \phi} \right)^2 + V \right). \] (120)

This is the Friedmann equation for a brane coupled to a bulk scalar field. Notice that retarded effects springing from the whole history of the brane and scalar field dynamics are present. In the following section we will see that these retarded effects come from the projected Weyl tensor. They arise from the exchange between the brane and the bulk. Notice that Newton’s constant depends on the value of the bulk scalar field evaluated on the brane \( (\phi_0 = \phi(t, y = 0)) \):
\[ \frac{8\pi G_N(\phi_0)}{3} = \frac{\kappa^2 U_B(\phi_0)}{12}. \] (121)

On cosmological scales, time variation of the scalar field induces a time variation of Newton’s constant. This is highly constrained experimentally [116], [117], leading to tight restrictions on the time dependence of the scalar field.

To get a feeling of the physics involved in the Friedmann equation, it is convenient to assume that the scalar field is evolving slowly on the scale of the variation of the scale factor. Neglecting the evolution of Newton’s constant, the Friedmann equation reduces to
\[ H^2 = \frac{8\pi G_N(\phi)}{3} \rho_m + \frac{U}{8} - \frac{\dot{\phi}^2}{16} \] (122)

Several comments are in order. First of all we have neglected the contribution due to the \( \rho_m^2 \) term as we are considering energy scales below the brane tension. Then the
main effect of the scalar field dynamics is to involve the potential energy $U$ and the kinetic energy $\dot{\phi}^2$. Although the potential energy appears with a positive sign we find that the kinetic energy has a negative sign. For an observer on the brane this looks like a violation of unitarity. We will reanalyse this issue in section 5, when considering the low energy effective action in four dimensions and we will see that there is no unitarity problem in this theory. The minus sign for the kinetic energy is due to the fact that one does not work in the Einstein frame where Newton’s constant does not vary, a similar minus sign appears also in the effective four–dimensional theory when working in the brane frame.

The time dependence of the scalar field is determined by the Klein-Gordon equation. The dynamics is completely specified by

$$\ddot{\phi} + 4H\dot{\phi} + \frac{1}{2}(3 - \omega_m) \rho_m \frac{\partial U_B}{\partial \phi} = - \frac{\partial U}{\partial \phi} + \Delta \Phi_2,$$  \tag{123}$$

where $p_m = \omega_m \rho_m$. We have identified

$$\Delta \Phi_2 = \phi''|_0 - \frac{\partial U_B}{\partial \phi} \frac{\partial^2 U_B}{\partial \phi^2}|_0.$$  \tag{124}$$

This cannot be set to zero and requires the knowledge of the scalar field in the vicinity of the brane. When we discuss cosmological solutions below, we will assume that this term is negligible.

The evolution of the scalar field is driven by two effects. First of all, the scalar field couples to the trace of the energy–momentum tensor via the gradient of $U_B$. Secondly, the field is driven by the gradient of the potential $U$, which might not necessarily vanish.

### 3.2.2. The Friedmann equation vs the projected Weyl tensor

We are now coming back to the origin of the non-trivial Friedmann equation. Using the Gauss-Codazzi equation one can obtain the Einstein equation on the brane \cite{22, 23}

$$\bar{G}_{ab} = -\frac{3}{8} U h_{ab} + \frac{U_B}{4} \tau_{ab} + \pi_{ab} + \frac{1}{2} \partial_a \phi \partial_b \phi - \frac{5}{16} (\partial \phi)^2 h_{ab} - E_{ab}.$$  \tag{125}$$

Now the projected Weyl tensor can be determined in the homogeneous and isotropic cosmology case. Indeed only the $E_{00}$ component is independent. Using the Bianchi identity $\bar{D}^a \bar{G}_{ab} = 0$ where $\bar{D}_a$ is the brane covariant derivative, one obtains that

$$\dot{E}_{00} + 4HE_{00} = \partial_\tau \left( \frac{3}{16} \dot{\phi}^2 + \frac{3}{8} U \right) + \frac{3}{2} H \dot{\phi}^2 + \frac{\dot{U}_B}{4} \rho_m.$$  \tag{126}$$

leading to

$$E_{00} = \frac{1}{a^4} \int d\tau a^4 \left( \partial_\tau \left( \frac{3}{16} \dot{\phi}^2 + \frac{3}{8} U \right) + \frac{3}{2} H \dot{\phi}^2 + \frac{\dot{U}_B}{4} \rho_m \right)$$  \tag{127}$$

Upon using

$$\bar{G}_{00} = 3H^2$$  \tag{128}$$

one obtains the Friedmann equation. It is remarkable that the retarded effects in the Friedmann equation all spring from the projected Weyl tensor. Hence the projected Weyl tensor proves to be much richer in the case of a bulk scalar field than in the empty bulk case.
3.2.3. Self-Tuning and Accelerated Cosmology  

The dynamics of the brane is not closed, it is an open system continuously exchanging energy with the bulk. This exchange is characterised by the dark radiation term and the loss parameter. Both require a detailed knowledge of the bulk dynamics. This is of course beyond the projective approach where only quantities on the brane are evaluated. In the following we will assume that the dark radiation term is absent and that the loss parameter is negligible. Furthermore, we will be interested in the effects of a bulk scalar field for late-time cosmology (i.e. well after nucleosynthesis) and not in the case for inflation driven by a bulk scalar field (see e.g. [109]-[113]).

Let us consider the self-tuned scenario as a solution to the cosmological constant problem. It corresponds to the BPS superpotential with $\alpha = 1$. In that case the potential $U = 0$ for any value of the brane tension. The potential $U = 0$ can be interpreted as a vanishing of the brane cosmological constant. The physical interpretation of the vanishing of the cosmological constant is that the brane tension curves the fifth dimensional space-time leaving a flat brane intact. Unfortunately, the description of the bulk geometry in that case has shown that there was a bulk singularity which needs to be hidden by a second brane whose tension is fine tuned with the first brane tension. This reintroduces a fine tuning in the putative solution to the cosmological constant problem [22].

Let us generalise the self-tuned case to $\alpha \neq 1$, i.e. $U_B = TW, T > 1$ and $W$ is the exponential superpotential. The resulting induced metric on the brane is of the FRW type with a scale factor

$$a(t) = a_0 \left( \frac{t}{t_0} \right)^{1/3+1/6\alpha^2}$$

leading to an acceleration parameter

$$q_0 = \frac{6\alpha^2}{1 + 2\alpha^2} - 1$$

For the supergravity value $\alpha = -\frac{1}{\sqrt{12}}$ this leads to $q_0 = -4/7$ and equation of state $\omega = -5/7$. This is in coincidental agreement with the supernovae results. This model can serve as a brane quintessence model [103],[114]. We will comment on the drawbacks of this model later. See also [119] and [120] for similar ideas.

3.2.4. The brane cosmological eras  

Let us now consider the possible cosmological scenarios with a bulk scalar field [113],[115]. We assume that the potential energy of the scalar field $U$ is negligible throughout the radiation and matter eras before serving as quintessence in the recent past.

At very high energy, i.e. energies above the tension of the brane, the non-conventional cosmology driven by the $\rho_m^2$ term in the Friedmann equation is obtained. Assuming radiation domination, the scale factor behaves like

$$a = a_0 \left( \frac{t}{t_0} \right)^{1/4}$$
and the scalar field
\[ \phi = \phi_i + \beta \ln \left( \frac{t}{t_0} \right) \]  
(132)

In the radiation-dominated era no modification is present, provided
\[ \phi = \phi_i \]  
(133)

which is a solution of the Klein-Gordon equation as the trace of the energy–momentum tensor for radiation vanishes (together with a decaying solution, which we have neglected). In the matter-dominated era the scalar field evolves due to the coupling to the trace of the energy–momentum tensor. This has two consequences. Firstly, the kinetic energy of the scalar field starts contributing in the Friedmann equation. Secondly, the effective Newton constant does not remain constant. The cosmological evolution of Newton’s constant is severely constrained since nucleosynthesis \[116,117\]. This restricts the possible time variation of \( \phi \).

In order to be more quantitative let us come back to the exponential superpotential case with a detuning parameter \( T \). The time dependence of the scalar field and scale factor become
\[ \phi = \phi_1 - \frac{8}{15} \alpha \ln \left( \frac{t}{t_e} \right) \]
\[ a = a_e \left( \frac{t}{t_e} \right)^{5 - \frac{8}{15} \alpha^2} \]

where \( t_e \) and \( a_e \) are the time and scale factor at matter-radiation equality. Notice the slight discrepancy of the scale factor exponent with the standard model value of \( 2/3 \). The redshift dependence of the Newton constant is
\[ \frac{G_N(z)}{G_N(z_e)} = \left( \frac{z + 1}{z_e + 1} \right)^{4\alpha^2/5} \]  
(134)

For the supergravity model with \( \alpha = -\frac{1}{\sqrt{12}} \) and \( z_e \approx 10^3 \) this leads to a decrease by (roughly) 37% since nucleosynthesis. This is marginally compatible with experiments \[116,117\].

Finally let us analyse the possibility of using the brane potential energy of the scalar field \( U \) as the source of acceleration now. We have seen that when matter is negligible on the brane, one can build brane quintessence models. We now require that this occurs only in the recent past. As can be expected, this leads to a fine-tuning problem as
\[ M^4 \sim \rho_c \]  
(135)

where \( M^4 = (T - 1)^{3W}/2\alpha \) is the amount of detuned tension on the brane. Of course this is nothing but a reformulation of the usual cosmological constant problem. Provided one accepts this fine tuning, as in most quintessence models, the exponential model with \( \alpha = -\frac{1}{\sqrt{12}} \) is a cosmologically consistent quintessence model with a five-dimensional origin.
3.3. Brief summary

In this section we have explored the cosmology of a brane world model with a bulk scalar field, as motivated by supersymmetry. This is a complicated model, with more consequences than the Randall–Sundrum model. The main difference is that the gravitational constant becomes time dependent. As such it has much in common with scalar–tensor theories \[118\], but there are important differences due to the projected Weyl tensor \( E_{\mu\nu} \) and its time evolution. The bulk scalar field can play the role of the quintessence field, as discussed above, but it could also play a role in an inflationary era in the very early universe (see e.g. \[109\]-\[113\]). In any case, the cosmology of such a system is much richer and, because of the variation of the gravitational constant, more constrained. Similarly, it can leave a trace in the CMB anisotropies, which will also constrain the parameters of the theory, and result in the variation of fundamental constants. These results are discussed in sections 5 and 6 of this review.

4. Birkhoff Theorem and its Violations

We have already mentioned the extended Birkhoff theorem in section 2. It states that for the case of a vacuum bulk space–time, the bulk is necessarily static, in certain coordinates. A cosmologically evolving brane is then moving in that space–time, whereas for an observer confined to the brane the motion of the brane will be seen as an expanding (or contracting) universe. In the case of a scalar field in the bulk, a similar theorem is unfortunately not available, which makes the study of such systems much more complicated. We will now discuss these issues in some detail, following in particular \[99\] and \[100\]. As we will see later in this section, the violation of Birkhoff theorem in the presence of a bulk scalar field is due to the projected Weyl tensor.

4.1. Motion in AdS-Schwarzschild Bulk

We have already discussed the static background associated with BPS configurations (including the Randall–Sundrum case) in the last section. Here we will focus on other backgrounds for which one can integrate the bulk equations of motion. Let us write the following ansatz for the metric

\[
 ds^2 = -A^2(r)dt^2 + B^2(r)dr^2 + R^2(r)d\Sigma^2 \tag{136}
\]

where \( d\Sigma^2 \) is the metric on the three–dimensional symmetric space of curvature \( q = 0, \pm 1 \). In general, the function \( A, B \) and \( R \) depend on the type of scalar field potential. This is to be contrasted with the case of a negative bulk cosmological constant where Birkhoff’s theorem states that the most general solution of the (bulk) Einstein equations is given by \( A^2 = f, B^2 = 1/f \) and \( R = r \) where

\[
 f(r) = q + \frac{r^2}{l^2} - \frac{\mu}{r^2}. \tag{137}
\]
We have denoted by \( l = 1/k = \sqrt{-6/(\Lambda_5 \kappa_5^2)} \) the AdS scale and \( \mu \) the black hole mass (see section 2). This solution is the so-called AdS-Schwarzschild solution.

Let us now study the motion of a brane of tension \( T/l \) in such a background. The equation of motion is determined by the junction conditions. The resulting equation of motion for a boundary brane with a \( Z_2 \) symmetry is

\[
\left( \dot{r}^2 + f(r) \right)^{1/2} = \frac{T}{l} r
\]

for a brane located at \( r = 40 \). Here \( \dot{r} \) is the velocity of the brane measured with respect to the proper time on the brane. This leads to the following Friedmann equation

\[
H^2 \equiv \left( \frac{\dot{r}}{r} \right)^2 = \frac{T^2 - 1}{l^2} - \frac{q}{r^2} + \frac{\mu}{r^4}.
\]

(139)

So the brane tension leads to an effective cosmological constant \( (T^2 - 1)/l^2 \). The curvature gives the usual term familiar from standard cosmology while the last term is the dark radiation term whose origin springs from the presence of a black-hole in the bulk. At late time the dark radiation term is negligible for an expanding universe; we retrieve the cosmology of a FRW universe with a non-vanishing cosmological constant. The case \( T = 1 \) corresponds of course to the Randall–Sundrum case.

Notice that all the bulk physics in the absence of a bulk scalar field is captured by the unique choice of the static background solution. This solution is only parametrised by the \( AdS \) curvature and the black hole mass. These two numbers are all that is needed to describe the motion of any brane in a static bulk. We will now contrast this situation with the case of a bulk scalar field.

### 4.2. Violating Birkhoff Theorem

We now turn to a general analysis of the brane motion in a static bulk with a bulk scalar field. To do that it is convenient to parametrise the bulk metric slightly differently

\[
ds^2 = -f^2(r)h(r)dt^2 + \frac{dr^2}{h(r)} + r^2d\Sigma_q^2.
\]

(140)

where \( q = 0, \pm 1 \) is the spatial curvature. Now, the Einstein equations lead to (redefining \( \phi \rightarrow \frac{\kappa_5^2}{2s_5}\phi \) and \( V \rightarrow \frac{3}{8s_5^2} V \))

\[
\frac{3}{r^2} \left( h + \frac{rh'}{2} - q \right) = -\kappa_5^2 \left( \frac{h\phi'^2}{2} + V \right)
\]

(141)

\[
\frac{3}{r^2} \left( h + \frac{rh'}{2} - q + \frac{hrf'}{f} \right) = \kappa_5^2 \left( \frac{h\phi'^2}{2} - V \right)
\]

(142)

and the Klein Gordon equation

\[
h\phi'' + \left( \frac{3h}{r} + \frac{hf'}{f} + h' \right) \phi' = \frac{dV}{d\phi}.
\]

(143)

Subtracting eq. (141) from (142) and solving the resulting differential equation, we obtain

\[
f = \exp \left( \frac{\kappa_5^2}{3} \int dr \phi'^2 \right).
\]

(144)
This gives the link between the bulk scalar field and the bulk metric.

Another piece of information is obtained by evaluating the spatial trace of the projected Weyl tensor. This is obtained by computing both the bulk Weyl tensor and the vector normal to the moving brane

$$\frac{\mu}{r^3} \equiv -\frac{E^i_i}{3} = \frac{r}{4f^2} \left( \frac{hf^2}{r^2} \right)' + \frac{q}{2r^2}. \tag{145}$$

This is the analogue of the dark radiation term for a general background. The equations of motion can be cast in the form

$$\mu' = -\frac{\kappa_5^2}{3}(\mu - \frac{kr^2}{2})r\phi'^2, \tag{146}$$
$$\mathcal{H}' + 4\frac{\mu}{r^3} = -\frac{2\kappa_5^2}{3}(\mathcal{H} - \frac{q}{r^2})r\phi'^2, \tag{147}$$
$$\kappa_5^2 V = 6\mathcal{H} + \frac{3}{4}r\mathcal{H}' - 3\frac{\mu}{r^4}, \tag{148}$$

where we have defined

$$\mathcal{H} = \frac{q - h}{r^2}. \tag{149}$$

This allows us to retrieve easily some of the previous solutions.

Choosing $\phi$ to be constant leads to $f = 1$, $\mu$ is constant and

$$\mathcal{H} = -\frac{1}{l^2} + \frac{\mu}{r^4} \tag{150}$$

This is the AdS-Schwarzschild solution as specified by Birkhoff theorem.

Another interesting case is obtained for spatially flat sections $q = 0$

$$\frac{\kappa_5^2}{3}r\phi' = -\frac{d\ln \mu}{d\phi}, \tag{151}$$
$$d \left( \frac{\mathcal{H}}{\mu^2} \right) = \frac{1}{\mu} d \left( r^{-4} \right), \tag{152}$$
$$\frac{\kappa_5^2}{6} V = -\frac{3}{4\kappa_5^2} \frac{d\mu}{d\phi} \frac{d}{d\phi} \left( \frac{\mathcal{H}}{\mu} \right) + \mathcal{H} \tag{153}$$

In this form it is easy to see that the dynamics of the bulk are completely integrable. First of all the solutions depend on an arbitrary function $\mu(\phi)$ which determines the dynamics. Notice that

$$f = \frac{\mu_0}{\mu} \tag{154}$$

where $\mu_0$ is an arbitrary constant. The radial coordinate $r$ is obtained by simple integration of eq. (151)

$$r = r_0 e^{-\frac{\kappa_5^2}{3} \int \frac{d\phi}{\mu} d\phi}. \tag{155}$$

Finally the rest of the metric follows from

$$h = -\frac{4\kappa_5^2}{3} r^2 \mu^2 \int d\phi \frac{d\phi}{d\mu} e^{\frac{\kappa_5^2}{3} \int \frac{d\phi}{\mu} d\phi} \tag{156}$$
The potential $V$ then follows from (153). This is remarkable and shows why Birkhoff’s theorem is not valid in the presence of a bulk scalar field. It is due to the infinite number of choices for the dark radiation function $\mu(\phi)$. Any choice of this function leads to a new static background in the bulk.

Let us now discuss the cosmological consequences of this theorem. The Friedmann equation for a cosmological brane in the static background specified by $\mu(\phi)$ is

$$H^2 = \mathcal{H} + \frac{k_5^4}{36} \mu^2 \rho^2$$

where $H$ is the Hubble parameter on the brane in cosmic time. In general the dynamics are extremely intricate. One can simplify and retrieve standard cosmology by studying the vicinity of a critical point $\frac{d\mu}{d\phi} = 0$. Parameterising

$$\mu = \frac{6A}{k_5^2} + B\phi^2$$

leads to the Friedmann equation

$$H^2 = \frac{k_5^4}{36} (\rho^2 - \theta) \mu^2 + \frac{\mu}{a^4} + o(a^{-4})$$

Here $\theta$ is an arbitrary integration constant. Notice that this is a small deviation from the Randall-Sundrum case as the scalar field behaves as

$$\phi = r^{-B/A}$$

and goes to zero at large distances. Hence, standard cosmology is retrieved at low energy and long distance. Indeed, $\mu(\phi)$ becomes constant and putting $\theta = 0$ one obtains the brane Friedmann equation with its characteristic $\rho^2$ term.

5. Cosmology of a Two–Brane System

In this section we will once more include an ingredient suggested by particle physics theories, in particular M–theory. So far we have assumed that there is only one brane in the whole space–time. According to string theory, there should be at least another brane in the bulk. Indeed, in heterotic M–theory these branes are the boundaries of the bulk space–time \cite{3}. Another motivation is the hierarchy problem. Randall and Sundrum proposed a two–brane model (one with positive and one with negative tension), embedded in a five–dimensional AdS space–time. In their scenario the standard model particles would be confined on the negative tension brane. As they have shown, in this case gravity is weak due to the warping of the bulk space–time. However, as will become clear from the results in this section, in order for this model to be consistent with gravitational experiments, the interbrane distance has to be fixed \cite{122}. This can be achieved, for example, with a bulk scalar field. As shown in \cite{122} and \cite{123}, gravity in the two–brane model of Randall-Sundrum is described by a scalar–tensor theory, in which the interbrane distance, the radion, plays the role of a scalar field. Introduction of a bulk scalar field modifies the Brans–Dicke parameter (see \cite{124} and \cite{125}) and introduces a second scalar field in the low–energy effective theory. Hence, in the case of two branes
and a bulk scalar field, the resulting theory at low energy is a bi–scalar–tensor theory \[126\], \[127\].

In the following we will investigate the cosmological consequences when the distance between the branes is not fixed (for some aspects not covered here see e.g. \[128\]-\[141\]). Motivation for this comes, for example, from a recent claim that the fine–structure constant might slowly evolve with time \[142\].

5.1. The low–energy effective action

In order to understand the cosmology of the two–brane system, we derive the low-energy effective action using the moduli space approximation. The moduli space approximation gives the low–energy limit effective action for the two–brane system, i.e. for energies much smaller than the brane tensions. Later in this section we will compare the moduli space approximation to another method.

In the static BPS solutions described in the section 3, the brane positions can be chosen arbitrarily. In other words, they are moduli fields. It is expected that by putting some matter on the branes, these moduli fields become time-dependent, or, if the matter is inhomogeneously distributed, space–time dependent. Thus, the first approximation is to replace the brane positions with space–time dependent functions. Furthermore, in order to allow for the gravitational zero mode, we will replace the flat space–time metric \(\eta_{\mu\nu}\) with \(g_{\mu\nu}(x^\alpha)\). We assume that the evolution of these fields is slow, which means that we neglect terms like \((\partial \phi)^3\) when constructing the low–energy effective action.

As already mentioned, the moduli space approximation is only a good approximation at energies much less than the brane tension. Thus, in the moduli space approximation we do not recover the quadratic term in the Friedmann equation. (See \[143\] for these issues.) We are interested in the late time effects, where the corrections have to be small.

Replacing \(\eta_{\mu\nu}\) with \(g_{\mu\nu}(x^\alpha)\) in \[101\] and collecting all the terms one finds from the five–dimensional action after an integration over \(y\): \[161\]

\[
S_{\text{MSA}} = \int d^4x \sqrt{-g_4} \left[ f(\phi, \sigma) R(4) + \frac{3}{4} a^2(\phi) \frac{U_B(\phi)}{\kappa_5^2} (\partial \phi)^2 ight. \\
\left. - \frac{3}{4} a^2(\sigma) \frac{U_B(\sigma)}{\kappa_5^2} (\partial \sigma)^2 \right].
\]

with

\[ f(\phi, \sigma) = \frac{1}{\kappa_5^2} \int_\phi^\sigma dy a^2(y), \]

(162)

with \(a(y)\) given by \[105\]. The moduli \(\phi\) and \(\sigma\) represent the location of the two branes. Note that the kinetic term of the field \(\phi\) has the wrong sign. This is an artifact of the frame we use here. As we will see below, it is possible to go to the Einstein frame with a simple conformal transformation, in which the sign in front of the kinetic term is correct for both fields.
In the following we will concentrate on the BPS system with exponential superpotential from section 3. Let us redefine the fields according to

\[
\tilde{\phi}^2 = \left(1 - 4k\alpha^2 \phi \right)^{2\beta}, \quad \tilde{\sigma}^2 = \left(1 - 4k\alpha^2 \sigma \right)^{2\beta},
\]

with \(\beta = \frac{2\alpha^2 + 1}{4\alpha^2}\); and then

\[
\tilde{\phi} = Q \cosh R, \quad \tilde{\sigma} = Q \sinh R.
\]

A conformal transformation \(\tilde{g}_{\mu\nu} = Q^2 g_{\mu\nu}\) leads to the Einstein frame action:

\[
S_{\text{EF}} = \frac{1}{2k\kappa^2(2\alpha^2 + 1)} \int d^4x \sqrt{-g} \left[ \mathcal{R} - \frac{12\alpha^2}{1 + 2\alpha^2} \frac{(\partial Q)^2}{Q^2} - \frac{6}{2\alpha^2 + 1} (\partial R)^2 \right].
\]

Note that in this frame both fields have the correct sign in front of the kinetic terms. For \(\alpha \to 0\) (i.e. the Randall–Sundrum case) the \(Q\)–field decouples. This reflects the fact that the bulk scalar field decouples, and the only scalar degree of freedom is the distance between the branes. One can read off the gravitational constant to be

\[
16\pi G = 2k\kappa^2(1 + 2\alpha^2).
\]

The matter sector of the action can be found easily: if matter lives on the branes, it “feels” the induced metric. That is, the action has the form

\[
S_{(1)}^m = S_{(1)}^m(\Psi_1, g_{\mu\nu}^{(1)}) \quad \text{and} \quad S_{(2)}^m = S_{(2)}^m(\Psi_2, g_{\mu\nu}^{(2)}),
\]

where \(g_{\mu\nu}^{(i)}\) denotes the induced metric on each branes. In going to the Einstein frame one gets

\[
S_{(1)}^m = S_{(1)}^m(\Psi_1, A^2(Q, R)g_{\mu\nu}) \quad \text{and} \quad S_{(2)}^m = S_{(2)}^m(\Psi_2, B^2(Q, R)g_{\mu\nu}),
\]

where matter now couples explicitly to the fields via the functions \(A\) and \(B\), which we will give below (neglecting derivative interactions). Note that we have assumed in (167) that the matter fields on the branes do not directly couple to the bulk scalar field.

The theory derived with the help of the moduli space approximation has the form of a multi–scalar–tensor theory, in which matter on both branes couples differently to the moduli fields. We note that methods different from the moduli–space approximation have been used in the literature in order to obtain the low–energy effective action or the resulting field equations for a two–brane system (see in particular [144]–[149]). Qualitatively, the features of the resulting theories agree with the moduli–space approximation discussed above. We will come back to this point in section 5.4.

In the following we will discuss observational constraints imposed on the parameters of the theory.

### 5.2. Observational constraints

In order to constrain the theory, it is convenient to write the moduli Lagrangian in the form of a non-linear sigma model with kinetic terms

\[
\gamma_{ij} \partial \phi^i \partial \phi^j,
\]

(169)
where \( i = 1, 2 \) labels the moduli \( \phi^1 = Q \) and \( \phi^2 = R \). The sigma model couplings are here

\[
\gamma_{QQ} = \frac{12\alpha^2}{1 + 2\alpha^2 Q^2}, \quad \gamma_{RR} = \frac{6}{1 + 2\alpha^2}
\]

(170)

Notice the potential danger of the \( \alpha \to 0 \) limit, the RS model, where the coupling to \( Q \) becomes very small. In an ordinary Brans-Dicke theory with a single field, this would correspond to a vanishing Brans-Dicke parameter which is ruled out experimentally. Here we will see that the coupling to matter is such that this is not the case. Indeed we can write the action expressing the coupling to ordinary matter on our brane as

\[
A = a(\phi) f^{-1/2}(\phi, \sigma), \quad B = a(\sigma) f^{-1/2}(\phi, \sigma),
\]

(171)

where we have neglected the derivative interaction.

Let us introduce the parameters

\[
\alpha_Q = \partial_Q \ln A, \quad \alpha_R = \partial_R \ln A.
\]

(172)

We find that \( (\lambda = 4/(1 + 2\alpha^2)) \)

\[
A = Q^{-\frac{\alpha^2}{2}} (\cosh R)\hat{4},
\]

(173)

leading to

\[
\alpha_Q = -\frac{\alpha^2\lambda}{2} \frac{1}{Q}, \quad \alpha_R = \frac{\lambda \tanh R}{4}.
\]

(174)

Observations constrain the parameter

\[
\theta = \gamma^{ij}\alpha_i\alpha_j
\]

(175)

to be less than \( 10^{-5} \) today. We obtain therefore a bound on

\[
\theta = \frac{1}{3} \frac{\alpha^2}{1 + 2\alpha^2} + \frac{\tanh^2 R}{6(1 + 2\alpha^2)}.
\]

(176)

The bound implies that

\[
\alpha \leq 10^{-2}, \quad R \leq 0.2
\]

(177)

today. The smallness of \( \alpha \) indicates a strongly warped bulk geometry such as an Anti–de Sitter space–time. In the case \( \alpha = 0 \), we can easily interpret the bound on \( R \). Indeed in that case

\[
\tanh R = e^{-k(\sigma - \phi)},
\]

(178)

i.e. this is nothing but the exponential of the radion field measuring the distance between the branes. Thus we find that gravity experiments require the branes to be sufficiently far apart. When \( \alpha \neq 0 \) but small, one way of obtaining a small value of \( R \) is for the hidden brane to become close to the would-be singularity where \( a(\sigma) = 0 \).

Constraints on the parameters also arise from nucleosynthesis. Nucleosynthesis constrains the effective number of relativistic degrees of freedom at this epoch. In brane worlds the energy conservation equation implies

\[
\rho a^3 \neq \text{const.},
\]

(179)
resulting in a different expansion rate than that given by general relativity, giving rise to constraints on the parameters. One finds $\alpha \leq 0.1$ and $R \leq 0.4$ at the time of nucleosynthesis.

We would like to mention that the parameter $\theta$ can be calculated also for matter on the negative tension brane. Then, following the same calculations as above, it can be seen that the observational constraint for $\theta$ cannot be satisfied. Thus, if the standard model particles are confined on the negative tension brane, the moduli have necessarily to be stabilised. In the following we will assume that the standard model particles are confined on the positive tension brane and study the cosmological evolution of the moduli fields.

5.3. Cosmological implications

The discussion in the last subsection raises an important question: the parameter $\alpha$ has to be chosen rather small for the theory to be consistent with observations. Similarly the field $R$ has to be small too. The field $R$ is dynamical and one would like to know if the cosmological evolution drives the field $R$ to small values such that it is consistent with the observations today. Otherwise are there natural initial conditions for the field $R$? In the following we study the cosmological evolution of the system in order to answer these questions.

5.3.1. Cosmological attractor solutions

The field equations for a homogeneous and isotropic universe can be obtained from the action. The Friedmann equation reads

$$H^2 = \frac{8\pi G}{3} \left( \rho_1 + \rho_2 + V_{\text{eff}} + W_{\text{eff}} \right) + \frac{2\alpha^2}{1 + 2\alpha^2} \dot{\phi}^2 + \frac{1}{1 + 2\alpha^2} \dot{R}^2.$$  (180)

where we have defined $Q = \exp \phi$. The field equations for $R$ and $\phi$ read

$$\ddot{R} + 3H\dot{R} = -8\pi G \frac{1 + 2\alpha^2}{6} \left[ \frac{\partial V_{\text{eff}}}{\partial R} + \frac{\partial W_{\text{eff}}}{\partial R} \right] + \alpha_R^{(1)} (\rho_1 - 3p_1) + \alpha_R^{(2)} (\rho_2 - 3p_2).$$  (181)

$$\ddot{\phi} + 3H\dot{\phi} = -8\pi G \frac{1 + 2\alpha^2}{12\alpha^2} \left[ \frac{\partial V_{\text{eff}}}{\partial \phi} + \frac{\partial W_{\text{eff}}}{\partial \phi} \right] + \alpha_\phi^{(1)} (\rho_1 - 3p_1) + \alpha_\phi^{(2)} (\rho_2 - 3p_2).$$  (182)

The coupling parameters are given by

$$\alpha_\phi^{(1)} = -\frac{2\alpha^2}{1 + 2\alpha^2}, \quad \alpha_\phi^{(2)} = -\frac{2\alpha^2}{1 + 2\alpha^2},$$  (183)

$$\alpha_R^{(1)} = \tanh R \frac{1}{1 + 2\alpha^2}, \quad \alpha_R^{(2)} = \left( \tanh R \right)^{-1} \frac{1}{1 + 2\alpha^2}. $$  (184)

We have included matter on both branes as well as potentials $V_{\text{eff}}$ and $W_{\text{eff}}$. We now concentrate on the case where matter is only on our brane.
We will discuss the inflationary epoch in more detail below. In the radiation-dominated epoch the trace of the energy-momentum tensor vanishes, so that $R$ and $\phi$ quickly become constant. The scale factor scales like $a(t) \propto t^{1/2}$.

In the matter-dominated era the solution to these equations is given by

$$\rho_1 = \rho_e \left( \frac{a}{a_e} \right)^{-3 - 2\alpha^2/3}, a = a_e \left( \frac{t}{t_e} \right)^{2/3 - 4\alpha^2/27},$$

(185)

together with

$$\phi = \phi_e + \frac{1}{3} \ln \frac{a}{a_e}, R = R_0 \left( \frac{t}{t_e} \right)^{-1/3} + R_1 \left( \frac{t}{t_e} \right)^{-2/3},$$

(186)
as soon as $t \gg t_e$. Note that $R$ indeed decays. This implies that small values of $R$ compatible with gravitational experiments are favoured by the cosmological evolution. Note, however, that the size of $R$ in the early universe is constrained by nucleosynthesis as well as by the CMB anisotropies. A large discrepancy between the values of $R$ during nucleosynthesis and now induces a variation of the particle masses, or equivalently Newton’s constant, which is excluded experimentally. One can show that by putting matter on the negative tension brane as well, the field $R$ evolves even faster to zero [126]. This behaviour is reminiscent of the attractor solution in scalar-tensor theories [151].

In the five-dimensional picture the fact that $R$ is driven to small values means that the negative tension brane is driven towards the bulk singularity. In fact, solving the equations numerically for more general cases suggest that $R$ can even be negative, which is, in the five-dimensional description, meaningless as the negative tension brane would move through the bulk singularity. Thus, in order to make any further progress, one has to understand the bulk singularity better. Of course, one could simply assume that the negative tension brane is destroyed when it hits the singularity. A more interesting alternative would be if the brane is repelled instead. It was speculated that this could be described by some effective potential in the low-energy effective action [126].

5.3.2. Boundary inflation driven by an inflaton field on the brane Let us consider now an inflationary epoch, driven by an inflaton field embedded on the positive tension brane [18], [152]. The action for the inflaton field $\chi$ is, in the Einstein frame,

$$S_{\text{inflaton}} = \int \sqrt{-g} d^4x \left[ -\frac{1}{2} A^2(\phi, R) (\partial \chi)^2 - A^4(\phi, R) V_0 \chi^n \right],$$

(187)

where we have assumed a typical chaotic inflationary potential for the inflaton field. Explicitly, the potential reads

$$V(\chi, \phi, R) = V_0 \exp \left( -\frac{8\alpha^2}{1 + 2\alpha^2} \phi \right) (\cosh R)^{4} \chi^n.$$

(188)

It can be shown that the field $R$ is driven towards zero during the inflationary epoch. The reason is the same as in the matter era: the trace of the energy-momentum tensor

‡ For $\alpha = 0$ the theory is equivalent to the Randall–Sundrum model. In this case the bulk singularity is shifted towards the Anti-de Sitter boundary.
of the inflaton field is not zero. In fact, it can be shown that $R$ decays quickly to zero according to

$$R(t) \propto \ln \left[ \frac{1 + \exp(-4ct)}{1 - \exp(-4ct)} \right], \quad (189)$$

where $c$ is a constant depending on the initial values of $\chi$ and $\phi$ and the energy scale $V_0$. In any case, $R$ decays quickly to zero and its energy density becomes negligible. We assume in the following that $R$ does not play a role in the last 60 e–folds of the inflationary era. We are then left with an inflationary scenario with two scalar fields. It is clear from eq. (188) that the effective energy scale

$$V_{\text{eff}} = V_0 \exp \left( -\frac{8\alpha^2}{1 + 2\alpha^2 \phi} \right) \quad (190)$$

depends on the evolution of $\phi$. For example, for $n = 2$, the mass of the inflaton is not constant, but rather depends on the evolution of $\phi$, since we consider the problem in the Einstein frame in which the Planck mass is constant, but the masses of particles vary. This has the effect that the background solutions are slightly modified relative to the case in General Relativity.

The perturbations in this scenario were discussed in [152] for the cases of $n = 2$ and $n = 4$. In the case of $n = 2$ and $\alpha = 0.01$, entropy perturbations do not play a role at all. The amplitude of the spectrum of entropy perturbations is around one percent that of the curvature perturbation. Similar conclusions hold for the case of $n = 4$. In figure 1 we plot the spectral index of the curvature perturbation $n_R$ versus $r_T = P_T/16P_R$ ($P_T$ is the tensor power spectrum and $P_R$ is the curvature perturbation power spectrum) for different cases of $\alpha$. As can be seen, increasing $\alpha$ results in an increase of the tensor perturbation relative to the curvature perturbation and in a smaller value of $n_R$. The potential with $n = 4$ is already under pressure observationally, and increasing $\alpha$ makes this worse. At least for the potentials considered here, a large coupling parameter $\alpha$ is not desirable.

### 5.3.3. Variation of constants

In a realistic brane world model, i.e. in a model which is based on a fundamental theory such as superstring theory, it is expected that the moduli fields couple to all forms of matter embedded on the brane [153]. At the classical level, this automatically leads to the conclusion that gauge coupling parameters, such as the fine structure constant, are not fundamental parameters but depend on the vacuum expectation values of the moduli fields themselves. Quantum mechanically, an additional dependence of the couplings to the moduli fields is generated by a conformal anomaly when switching between the Jordan and the Einstein frame [118]. It can, however, be shown that one–loop corrections cancel and the variation of constants is generated only by directly coupling the gauge fields to the bulk scalar and having moduli–dependent Yukawa couplings. In the following we assume that the Yukawa couplings do not depend on the moduli fields. The expression for $\delta \alpha_{\text{EM}}/\alpha_{\text{EM}}$ is, in general, a complicated function of both $R$ and $\phi$. However, since $R$ rapidly decays to zero, we can neglect the variation
Figure 1. As can be seen, increasing $\alpha$ results in a decrease of $n_R$, the power law index of the curvature perturbation power spectrum and in an increase of $r_T = P_T/16P_R$. Taken from [152].

The analysis was carried out in [154] and the result is shown in figure 2 under different assumptions for the value of $\theta$.

It is interesting to note that these observations indicate that $\beta \approx \alpha$, which are a priori independent parameters.

§ We are considering the chiral limit here.
5.3.4. Open questions In this subsection, we have discussed some of the cosmological consequences of brane world moduli. It is clearly a problem for the theory that the negative tension brane collapses during the cosmological evolution, although this attractor solution makes the theory more compatible with observations. Nevertheless, theories of this kind suffer from the existence of singularities and it is conceivable that stringy effects will “smooth out” this singularity. The cosmological implications of this might be interesting in their own right.

Another potentially interesting question is how Kaluza–Klein corrections affect the above results (see [155] for an approach to include Kaluza–Klein corrections). Although it is unlikely that these corrections prevent the collapse of the negative tension brane, one can imagine that they influence the time taken for this to happen, influencing the dependence of \( R(t) \) on \( t \).

5.4. Moduli versus Projective Approaches

So far we have considered three different approaches to the brane dynamics. The first one consists in solving the five dimensional equations exactly (see Section 4). This is
only possible in a few cases, especially when no matter is present on the branes. The second one is the projective approach (see Section 2 and Section 3). In that case one writes down the four–dimensional equations on the branes. In general one cannot find a closed system of equations. Indeed the projected Weyl tensor is calculable only in an isotropic FRW universe and reduces to the dark radiation term. In the case of a bulk scalar field we have seen that the second derivative of the scalar field with respect to the bulk coordinate is unconstrained from the point of view of the brane. Thus, the full dynamics of the bulk scalar field is not understood. Finally, we have presented the moduli space approximation in the case of a two–brane system, whereby the whole dynamics have been reduced to a four–dimensional description involving the low–energy degrees of freedom.

In the following we will show how the projective approach and the moduli space approximation are linked. This is only the case in the two–brane system. In the one–brane system where the bulk is infinite, the calculation of the projected Weyl tensor is an open problem.

We will restrict the analysis to the Randall–Sundrum case, following [145]. In that case the low energy degrees of freedom are the graviton and the radion. It is convenient to choose a gauge where the radion appears explicitly in the five dimensional metric

\[ ds^2 = e^{2\phi} dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu \]  

(192)

and the branes are placed at \( y = 0 \) and \( y = \rho \). In following we will consider that \( \phi(x^\mu) \) is a function in four dimensions whereas \( g_{\mu\nu}(y, x^\mu) \) depends on all five–dimensional coordinates. Decomposing the extrinsic curvature

\[ e^{-\phi} K_{\mu\nu} = \Sigma_{\mu\nu} + \frac{Q}{4} g_{\mu\nu} \]  

(193)

the junction conditions become

\[ \Sigma_{\nu} - \frac{3}{4} \delta_{\nu} Q |_0 = \frac{\kappa^2}{2} \left( -\lambda_1 \delta_{\nu} + T^{(1)\mu} \right) \]  

(194)

on the first brane of tension \( \lambda_1 \) and carrying the energy momentum tensor \( T^{(1)\mu}_{\nu} \). Together with the Einstein equations, these junction conditions specify the dynamics of the system.

As we have already mentioned several times, we will be interested in the low–energy limit of the brane system. This is the regime where effects relevant for the CMB physics take place. High energy effects would imply a modification of the early Universe features which might leave an imprint on the primordial fluctuations. At low energy on the branes, it is legitimate to use a derivative expansion as long as typical length scales \( L \) on the branes are much larger than the curvature of the bulk \( l = 1/k \). Using dimensional analysis the Hubble rate on the brane is \( H = O(L^{-1}) \). The Friedmann equation when \( \rho \ll \lambda \) leads to \( \rho = O(L^{-2}\kappa_4^{-2}) \). Using \( \lambda = O(l^{-2}\kappa_4^{-2}) \), this implies that

\[ \frac{\rho}{\lambda} = O \left( \frac{L}{l} \right)^2 \]  

(195)
which is much smaller than one. Hence energy densities on the brane are much lower than the brane tension. Within this approximation, the bulk metric can be expanded

\[ g_{\mu\nu}(y, x^\mu) = a^2(y) [g_{\mu\nu}(x^\mu) + g^{(1)}_{\mu\nu} + \ldots] \]  

(196)

where at each order the tensor \( g^{(n)}_{\mu\nu} \) involves only \((2n)\) derivatives on the brane, i.e. corresponds to terms of order \( O((l/L)^{2n}) \). At each order in this approximation scheme, the function \( g^{(n)}_{\mu\nu}(y, x^\mu) \) of \( y \) is valid throughout the bulk. Hence it encapsulates non-local effects due to the presence of both branes.

In the following we will analyse the zeroth and first order terms of the derivative expansion. At zeroth order, we obtain the equations governing the Randall-Sundrum model in the absence of matter on the branes. At first order in the derivative expansion, the components of the extrinsic curvature tensor are given by

\[ \Sigma^{(1)}_{\mu\nu} = \frac{l}{a^2} \left[ \frac{1}{2} (R^\mu_\nu - \frac{1}{4} \delta^\mu_\nu R) + \frac{ye^\phi}{l} (D^\mu D_\nu \phi - \frac{1}{4} \delta^\mu_\nu D^2 \phi) \right. \]

\[ + \left. \frac{y^2 e^{2\phi}}{l^2} + \frac{ye^\phi}{l} \right] (D^\mu \phi D_\nu \phi - \frac{1}{4} \delta^\mu_\nu D^2 \phi) + \frac{l}{2a^4} E^\mu_\nu \]  

(197)

where covariant derivatives are taken with respect to the zeroth order metric \( g_{\mu\nu}(x^\mu) \).

Similarly the trace part reads

\[ Q^{(1)} = \frac{l}{a^2} \left[ \frac{R}{6} + \frac{ye^\phi}{l} (2D^2 \phi + D^\alpha \phi D_\alpha \phi) - \frac{y^2 e^{2\phi}}{l^2} D^\alpha \phi D_\alpha \phi \right] \]  

(198)

The Einstein equations lead to the first order metric being a function of the zeroth order terms

\[ g^{(1)}_{\mu\nu} = -\frac{l^2}{2} \left( \frac{1}{a^2} - 1 \right) (R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R) \]

\[ + \frac{l^2}{2} \left( \frac{1}{a^2} - 1 - \frac{2ye^\phi}{l a^2} \right) (D_\mu D_\nu \phi + \frac{1}{2} g_{\mu\nu} D^\alpha \phi D_\alpha \phi - \frac{y^2 e^{2\phi}}{a^2} (D_\mu \phi D_\nu \phi \right. \]

\[ - \frac{1}{2} g_{\mu\nu} D^\alpha \phi D_\alpha \phi) - \left( \frac{1}{a^4} - 1 \right) E_{\mu\nu} \]  

(199)

Notice that the metric \( g_{\mu\nu} \) and \( \phi \) are not fixed yet. The equations of motion for gravity and \( \phi \) follow from the junction conditions. On the first brane, they can be cast in the form of Einstein equations for the induced metric \( g_{\mu\nu} \)

\[ G^\mu_\nu = \frac{\kappa_5^2}{l} T^{(1)\mu}_\nu + E^\mu_\nu \]  

(200)

where \( G^\mu_\nu \) is the Einstein tensor on the first brane. Of course this is nothing but the expected Einstein equation on the first brane. On the second brane, the induced metric is conformally related to the induced metric on the first one \( f_{\mu\nu} = \Omega^2 g_{\mu\nu} \), where \( \Omega = \exp(-e^\phi) \) and

\[ G^\mu_\nu = -\frac{\kappa_5^2}{l} T^{(2)\mu}_\nu + \frac{E^\mu_\nu}{\Omega^2} \]  

(201)

Notice that the projected Weyl tensor appears in both Einstein equations on the branes. Using the explicit relation between \( f_{\mu\nu} \) and \( g_{\mu\nu} \) one can express the Einstein tensor on
the second brane in terms of the Einstein tensor on the first brane and tensors depending on the conformal factor $\Omega$. This allows one to obtain $E^\mu_\nu$ from both equations

\[
E^\mu_\nu = -\frac{\kappa^2}{l} \frac{1 - \Psi}{\Psi} (T^{(1)\mu}_\nu + (1 - \Psi) T^{(2)\mu}_\nu) - \frac{1}{\psi} [(D^\mu D_\nu \Psi - \delta^\mu_\nu D^2 \Psi)

+ \frac{w(\Psi)}{\Psi} (D^\mu \Psi D_\nu \Psi - \frac{1}{2} \delta^\mu_\nu D^\alpha \Psi D_\alpha \Psi)]
\]

(202)

where $\Psi = 1 - \Omega^2$ and

\[
w(\Psi) = \frac{3\Psi}{2(1 - \Psi)}.
\]

(203)

Using the tracelessness of $E^\mu_\nu$, one obtains the field equation governing the dynamics of $\Psi$. Moreover, now that we have determined $E^\mu_\nu$, the Einstein equations on the first brane lead to a closed set of equations. These equations of motion are equivalent to the ones obtained with the moduli approximation provided one identifies

\[
\Psi = \frac{1}{\cosh^2(R)}
\]

(204)

This shows the equivalence between the moduli space approach and the projective approach. The solutions of the moduli space equations of motion allow one to determine both $E^\mu_\nu$ and the first order metric $g^{(1)}_{\mu\nu}$. Of course, this result implies that the equations of motion of the moduli space approach give an accurate description of the brane dynamics (at lowest order in the projective approach). For other work in this direction, we refer to [144]-[149] and [156]-[161].

Having shown the relationship between the projective approach and moduli space approximation we now discuss cosmological perturbations and the effect the brane world moduli have on the CMB anisotropies.

6. Cosmological Perturbations

In this section we turn our attention to cosmological perturbations in brane world scenarios. It might be that extra dimensions have an important impact on the evolution (and maybe on the generation) of cosmological perturbations. Therefore, cosmological observations might be able to constrain extra–dimensional models, particularly with the advent of high precision data.

There has been a huge number of publications related to cosmological perturbations in brane world scenarios, see [162]-[189]. In this review we just touch the topic briefly and describe some important results, since the details are very technical. Instead we refer for more details to the reviews of brane world perturbations in [190] and [191].

6.1. Cosmological Perturbations: Methods

Generally speaking, there are two points of view one can take. The first is the point of view from a brane observer. In this approach one starts with the Einstein equations on the brane and considers perturbations of them. As we have seen, the projected Weyl
tensor $E_{\mu\nu}$ describes the influence of the bulk. We will see below that the field equations on the brane do not close and hence this approach is incomplete.

The second approach takes the point of view from the bulk. Here, one can make two choices of coordinates: (a) the first choice of coordinates assumes the brane to be at rest and straight, even in the presence of perturbations (Gaussian normal coordinate system). It turns out that in this coordinate system the junction conditions are easy to implement. However, the bulk is non–static and, more annoyingly, a coordinate singularity appears at finite distance from the brane [41]. One way to regulate this singularity is to introduce a “regulator brane”, on which boundary conditions have to be chosen. This method was used in [192] and we will come back to this later. In addition to the singularity, apart from the case of a de–Sitter or Minkowski brane, the field equations are non–separable. (b) Because of the difficulties in the Gaussian normal coordinate system, one may choose coordinates in which the bulk metric take a simple form. Choosing a proper gauge in the bulk, one can then find the general solution to the bulk equations [174]. The price to be paid when choosing such a coordinate system is that the brane moves during the cosmological evolution and the junction conditions are difficult to implement.

In two–brane systems, the situation is not easier: whatever coordinates one chooses, there are two junction conditions to be fulfilled.

We discuss now the brane point of view in more detail.

6.1.1. The brane point of view  We have seen that Einstein’s equations on the brane have the form

$$G_{\mu\nu} = 8\pi G \tau_{\mu\nu} - \Lambda_{4} h_{\mu\nu} + \kappa_{5}^{4} \pi_{\mu\nu} - E_{\mu\nu}$$

The projected Weyl tensor is tracefree: $E_{\mu}^{\mu} = 0$. Let us therefore take the point of view that $E_{\mu\nu}$ represents a new energy–momentum tensor, representing the influence of the bulk gravitational field. The fluid corresponding to $E_{\mu\nu}$ has been named “Weyl–fluid”. The most significant contribution from this fluid is its anisotropic stress $\delta \pi_{\text{Weyl}}$. Consider the off–diagonal component of the perturbed Einstein equation. Choosing the longitudinal gauge, we can write the perturbed metric on the brane as

$$ds^{2} = -(1 - 2\psi)dt^{2} + (1 + 2\phi)dx^{2}.$$  

Assuming that the anisotropic stress of the normal matter is zero, the off–diagonal component of Einstein’s equation leads to [51]

$$\phi + \psi = -8\pi G \delta \pi_{\text{Weyl}}.$$  

Therefore, even in the absence of anisotropic stress of matter, the Weyl fluid induces an anisotropic stress, so that $\phi + \psi \neq 0$. In addition, there is an entropy perturbation induced by the Weyl fluid. These are the two effects the bulk gravitational field has on cosmological perturbations. In particular, it might leave a measurable signal in the microwave background radiation.
The Sachs–Wolfe effect, relating temperature fluctuations at the last scattering surface to metric/matter perturbations, reads now in the brane world theory we consider \[ \frac{\delta T}{T} = \left( \frac{\delta T}{T} \right)_{\text{GR}} + \frac{8\rho_{\text{Weyl}}}{3\rho_{\text{cdm}}} S^* - 8\pi G \delta \pi_{\text{Weyl}} + \frac{16\pi G}{a^{5/2}} \int da \, a^{5/2} \delta \pi_{\text{Weyl}} \] (208)

The first term is the usual expression for the temperature fluctuation obtained in General Relativity. The second term includes an entropy perturbation $S^*$, induced by the Weyl fluid and is related to $\delta \rho_{\text{Weyl}}$ and $\rho = \rho_{\text{rad}} + \rho_{\text{cdm}}$ by $S^* = \delta \rho_{\text{Weyl}}/\rho_{\text{Weyl}} - \delta \rho/(3(\rho + p))$. The last term, however, represents an integral over the history of the anisotropic stress of the Weyl fluid. Because there is no equation for $\pi_{\text{Weyl}}$ when considering the brane alone, one cannot make a prediction for the temperature anisotropy of the cosmic microwave background radiation. For this, the full five–dimensional problem has to be solved.

6.2. Scalar Perturbations: CMB Anisotropies

However, some interesting progress has been made by using low–energy approximations discussed in the last section. As we have seen, one considers scales much below the brane tension and scales much larger than the AdS curvature scale. The equation for the Weyl tensor reads then

$$\nabla^\mu E_{\mu\nu} = 0.$$

(209)

The idea is to introduce a second brane (so–called regulator brane) in the bulk, on which proper boundary conditions can be chosen. The Weyl tensor is then specified by the interbrane distance and the energy–momentum tensor of matter on both branes. The evolution equation for the interbrane distance is then obtained from eq. (209).

To be more concrete, consider again the five–dimensional line element

$$ds^2 = e^{2\phi} dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu.$$  

(210)

The physical distance $D$ between the branes is given by $D = \exp(\phi) \Delta y$, where $\Delta y$ is the coordinate distance. Remember that we have defined $\Psi = 1 - \exp(-2e^\phi)$. Hence, $\Psi$ is completely specified by the interbrane distance. As established in the last section on the effective action, both the projective approach and the moduli space approximation lead to the conclusion that the radion couples to the trace of the energy–momentum tensor and therefore that the theory has much in common with a scalar–tensor theory (see in particular eq. (202)). Thus, fluctuations of the radion satisfy a wave equation which is similar to that of the scalar field in a scalar–tensor theory. This then specifies the full evolution of the Weyl tensor and, in particular, one is able to calculate the effective anisotropic stress $\pi_{\text{Weyl}}$ induced by the bulk gravitational field, so that one can calculate the temperature fluctuations (see eq. (208)) completely.

So far, two cases have been studied in the literature. Let us describe them briefly.

In the first case, the second brane was regarded as a regulator brane in order to deal with boundary conditions in the bulk \[192\]. The interbrane distance was kept fixed, which can be obtained by choosing $\rho_{\text{reg.brane}} = -\rho_{\text{our.brane}}$. Then, assuming adiabatic
perturbations, there is only one remaining free parameter specifying the amplitude of fluctuations in the dark radiation: \( \delta C_{\text{Weyl}} = \delta \rho_{\text{Weyl}}/\rho_{\text{rad}} \). The results are shown in the figure 3.

\[
\frac{\delta \rho_c}{\rho} = \delta C,
\]

\[\delta C = -2 \zeta, \quad \delta C = -\zeta, \quad \delta C = 0, \quad \delta C = \zeta, \quad \delta C = 2 \zeta, \quad \text{WMAP}\]

Figure 3. Effects of the bulk gravitational field in brane worlds for constant radion and adiabatic perturbations. The quantity \( \delta C_{\text{Weyl}} \) quantifies the amplitude of the perturbations in the Weyl fluid. \( \zeta \) denotes here the curvature perturbation due to dark radiation. Taken from [192].

It was found in this case that perturbations on large angular scales are affected most (i.e. multipole numbers smaller than 200), whereas the effect of the Weyl fluid is small on smaller scales.

Let us now discuss the second case studied in the literature [193]. Here, the branes are free to move and there is no potential energy for the radion. The negative tension brane contains no matter. The background evolution was studied in the last section. Hence, this setup reflects a two–brane system with specific matter content on the two branes. For the perturbations, it was assumed that initially, fluctuations in the radion do not contribute to the total curvature perturbation and that they are subsequently sourced by fluctuations in matter. The other perturbations between matter and radiation are assumed to be adiabatic. The results are shown in the figure 4.

The case of \( R_{\text{ini}} = 0 \) corresponds to the ΛCDM case. It can be seen from the normalised spectra that all scales are influenced and that the spectra for \( R_{\text{ini}} > 0 \) are damped, compared to the ΛCDM case.

It is not surprising that the resulting CMB power spectra look different for the two cases taken from the literature. In both cases, a different behaviour of the radion is assumed and also the nature of fluctuations in both cases is different. Thus, the behaviour of the projected Weyl tensor in both cases is different.

To summarise, although some progress has been made, it is clear that brane worlds will not make a definite prediction for CMB perturbations. A lot will depend on the dynamics of the two branes, if the radion has a potential or not, if the perturbations in
the radion initially play a role and whether or not there is matter on the second brane. However, we will be able to constrain different models for the initial conditions in the very early universe and, maybe, different models for inflation and the cyclic universe.

6.3. Tensor Modes

We have seen in section 2 that, during inflation, heavy modes decay and the zero mode is the only one which survives. However, are heavy modes generated during the radiation or matter dominated epochs? As is the case in General Relativity, the zero mode remains frozen outside the horizon, but new effects might appear when the massive modes re-enter the horizon. The only way to study this effect is to study the five–dimensional problem. To lowest order, one can employ a “near–brane and late–time approximation” and study the solution of the bulk equation near the brane [194, 195]. In this case, the bulk equation is separable and can be solved explicitly. It was shown that the massive modes decay even outside the horizon [195]. However, nothing can be said about the creation of heavy modes, since this is a higher–order effect. The problem was attacked analytically in [196] and numerically in [197]. It was shown that massive modes are sourced by the initial zero mode, which re-enters the horizon and is subsequently damped. This effect has been studied numerically in [197] and the result is shown in figure 5.

7. Brane Collisions

The brane dynamics may lead to a collision between branes, either boundary branes in the Randall-Sundrum model or a floating five–brane with one of the two boundary
Figure 5. Damping of tensor modes in the Randall–Sundrum brane world due to the creation of massive modes. Here, \( \epsilon_* = \rho_0 / \lambda \) at the start of integration and \( \gamma \) is a parameter specifying the brane position. Taken from [197].

branes for the heterotic M-theory. This possibility has been in particular investigated in the context of early universe cosmology [198]–[213]. In all these cases, the brane collision is not well described at the level of a five–dimensional effective field theory. In particular, one may envisage that the collision may be modified by the exchange of open strings or open membranes when the branes get closer than a distance of the order of the string scale. In that case, one may have a correct description of the collision. Several scenarios have been considered. One appealing possibility is that the singular behaviour at the brane collision is resolved and a finite bounce happens, i.e. the two branes approach within a minimal distance and rebound before flying away in opposite directions. Another possibility, which has been considered in heterotic M-theory under the name of a “small instanton”, is that the impinging brane remains stuck to the other brane, implying a change in the matter content. Finally the two branes may go past each other, exchanging some energy in the process. The resulting configuration going from a big crunch to a big bang situation. This is the scenario used in the cyclic model.

We will first discuss the cyclic model, a brane–inspired four dimensional model advocated to be an alternative to inflation. These models are controversial and subject to fierce debates. In order to illustrate the possible problems occurring during a brane
collision, we will consider the born again scenario. Again, we will be brief and refer to the reviews [214]-[216] for more details.

The subject of brane collisions is still in its infancy, deserving further study in view of its cosmological relevance. In particular, the question about how perturbations evolve during the brane collision is an important one and has sparked a lot of activity [217]-[235].

7.1. The cyclic model

The cyclic model [205] is an off–shoot of the ekpyrotic model [198]-[200] whereby the big bang originated from the collision of two nearly BPS branes. The primordial fluctuations have been argued to follow from the small ripples on the colliding branes. This scenario has led to heated exchanges within the cosmology community. The cyclic scenario picks up most of the ingredients of the ekpyrotic universe within a purely four–dimensional description. At its most basic level, it is described by a scalar–tensor theory involving one scalar degree of freedom which is inspired by the radion of the Randall–Sundrum model. The potential can be seen in figure 6 and can be separated into four different regions. There are four phases. Currently the field evolves slowly along a quintessence–like potential leading to the acceleration of the expansion of the universe. Later the field will roll down the negative part of the potential, where the universe undergoes a slow contracting epoch. Quantum fluctuations in the field $\phi$ are created during this phase. Then the field becomes kinetically dominated and rushes towards a singularity at infinity, where it bounces back to the region of acceleration. The acceleration era is useful in diluting the entropy created in each cycle, overcoming the problems of the cyclic universe envisaged by Tolman in the thirties. Now this scenario relies crucially on the behaviour of the universe at the singular bounce.

The bounce occurs when the field $\phi$ is infinite corresponding to the collision of the branes. The description of the physics at the bounce is an open question. In the following section, we will describe an attempt to understand the behaviour of the brane model at a singular bounce. It relies on the techniques, like the moduli space approximation, that we have already presented. The model was dubbed the born–again brane world [206].

7.2. Born–again brane world

The born–again brane world is a model of the early universe, in which two boundary branes collide. In doing so, the brane tensions change signs. In this model, there are no repeated cycles and as such it has to be distinguished from the cyclic model.

Let us focus on the Randall–Sundrum model when the boundary branes have detuned tensions. We have already shown that this leads to the motion of the boundary branes when the tensions are increased compared to the critical value corresponding to the BPS static configuration. In particular this leads to the collision of the two boundary
branes. More precisely the Einstein frame potential reads
\[
V(\eta) = (T_1 - 1) \cosh^4 \left( \frac{\eta}{2} \right) + (T_2 - 1) \sinh^4 \left( \frac{\eta}{2} \right)
\]
where \(\eta\) is a normalised field related to
\[
\eta = -\ln \left| \frac{\sqrt{1 - \Psi} - 1}{\sqrt{1 - \Psi} + 1} \right|
\]
The two branes are infinitely separated when \(\eta = 0\) and collide when \(\eta = \infty\). If \(T_1 > 1\) and \(T_1 + T_2 < 2\) the potential has a maximum. For \(\eta\) large enough, the branes are attracted towards each other leading to a brane collision. The branes collide when \(\Psi = 0\). Now in the brane frame of the positive tension brane the symmetry \(\Psi \rightarrow -\Psi\) exchanges positive branes and negative branes.

In a realistic setting, our universe cannot be on the negative tension brane. Hence before the collision our brane was the negative tension brane which became the positive tension brane after the collision. Before the collision the negative tension brane frame can be obtained from the Einstein frame putting \(f_{\mu\nu} = g_{\mu\nu}/\Phi\). The field \(\Phi\) vanishes linearly at the collision leading to a divergence of the Hubble rate in the Einstein metric. This divergence is non–existent in the brane frame of the negative tension brane. In

**Figure 6.** Examples of cyclic potentials. In region (a), the potential is such that the field leads to the present acceleration of the expansion of the universe. In region (b), the universe is contracting and in region (c) the universe is driven towards the brane collision when the field value is \(-\infty\). Taken from \[234\].
that frame, the metric goes smoothly from the pre–collision regime (a contraction in the Einstein frame) to the post–collision regime (an expansion in the Einstein frame). So it seems that by going to the brane frame where matter couples directly to gravity, one can extrapolate from the pre–big bang contracting phase to the big bang epoch.

This would justify the smooth passage used in the cyclic universe but for a hitch. The curvature perturbation $\zeta$ possesses a logarithmic divergence at the collision. This divergence is present both in the Einstein and brane frames, indicating its fundamental nature. This indicates that the brane model is unstable at the collision. This divergence is at the origin of the fierce debate on cosmological perturbation in colliding brane models 217–235.

In conclusion, there is yet no known prescription to go through a cosmological singularity such as the one springing from the collision of two branes (for some recent progress, however, see 229 and 232). Even in the simplest setting such as the born–again brane world scenario, divergences show an intrinsic instability of the system. One may hope that by going to string theory new results on the resolution of cosmological singularities could be obtained.

8. Summary

In this review we have discussed aspects of brane world cosmology, covering both the simplest Randall–Sundrum model and models with a bulk scalar field, the latter being motivated by supersymmetry. We have seen that these models are formulated in such a way that their predictions can now be tested against precision data. We have discussed the early time cosmology when the Friedmann equation is modified and considered its effect on inflation, considered modifications to general relativity and how this constrains brane world models and discussed also the CMB predictions. More theoretical questions as well as speculative ideas concerning the very origin of the universe have also been considered. However, our review is certainly not complete. For example, we have been unable to cover some recent developments, such as brane cosmology in Gauss–Bonnet theory 236–239 and the cosmological consequences of induced gravity on the brane 240–243. These topics give rise to interesting cosmological consequences; we refer to the original literature for more details. Similarly, several authors have studied brane models in dimensions higher than five (see e.g. 249–254). In this case the compactification is different as is the recovery of Einstein gravity. Covering this is beyond the scope of this review. However, we noted that in six–dimensional models with a bulk scalar field the problem with the self–tuning scenario is alleviated 255.

Brane world cosmology is in its infancy and there are many open questions. Here we summarise some of them.

- In the case of the Randall–Sundrum model, the homogeneous cosmological evolution is well understood. However, the evolution of cosmological perturbations is still not fully understood. This is because the effects of the bulk gravitational field, encoded in the projected Weyl–tensor, on CMB physics and Large Scale Structures are
unknown. Whilst progress has been made in this direction, as discussed in section 6, there is still much to do.

- For models with bulk scalar fields, we have presented some results on the cosmological evolution of a homogeneous brane where we assumed that the bulk scalar field does not vary strongly around the brane; this needs investigating. Furthermore, for models with two branes, the cosmology has only been explored in the low energy regime, for instance with the moduli space approximation. As yet the cosmology of the high energy regime is an open question and can only be explored using techniques which go beyond the low energy effective action.

- Both the bulk scalar field as well as the interbrane distance in two brane models could play an important role at least during some part of the cosmological history. For example one of the fields could play the role of dark energy. In that case, it is only natural that masses of particles, as well as other parameters, will vary. Preliminary work in this direction was discussed in section 5. However, more work needs to be done in this regard.

- The bulk singularity seems to play an important role in a cosmological setting. We have seen in section 5 that the negative tension brane moves towards the bulk singularity. Consequently cosmology requires a detailed understanding of the singularity. For example, the brane might be repelled by the singularity; if this were the case it may have cosmological consequences. More understanding of this is needed.

- Brane collisions provide an exciting development into the possible origin of the universe. However, much work is needed on how cosmological perturbations arise in this scenario and how such perturbations evolve before and after the bounce \[217-235\]. The problem is again one of the singularity. Whilst progress has been made there are still unanswered questions and one may well have to go beyond the current brane description and into the string theory to understand this singularity.

- To date brane worlds are phenomenological models and have yet to make contact with an underlying theory. For example, whilst they are motivated by string theory the current models have not arisen as an approximation to string theory. Similarly, they have yet to be embedded in a realistic theory of particle physics. Much work is needed in this direction.

Brane cosmology has opened up new horizons, enabling theories in extra dimensions to be explored and tested. Many interesting questions have arisen. With new techniques and high precision data cosmology will play an increasingly important role in testing ideas beyond the standard model of particle physics.

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