Zero sum cycles in complete digraphs

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Abstract

Given a non-trivial finite Abelian group \((A, +)\), let \(n(A) \geq 2\) be the smallest integer such that for every labelling of the arcs of the bidirected complete graph \(K^\pm_{n(A)}\) with elements from \(A\) there exists a directed cycle for which the sum of the arc-labels is zero. The problem of determining \(n(Z_q)\) for integers \(q \geq 2\) was recently considered by Alon and Krivelevich [2], who proved that \(n(Z_q) = O(q \log q)\). Here we improve their result and show that \(n(A)\) grows linearly. More generally we prove that for every finite Abelian group \(A\) we have \(n(A) \leq 8|A|\), while if \(|A|\) is prime then \(n(A) \leq 2^2|A|\).

As a corollary we also obtain that every \(K_{16q}\)-minor contains a cycle of length divisible by \(q\) for every integer \(q \geq 2\), which improves a result from [2].

1 Introduction

Zero-sum problems are a branch of Ramsey theory with an algebraic flavour. One of the earliest results of this form is the Erdős-Ginzburg-Ziv Theorem [5] which is considered to be the common ancestor of many zero-sum problems. It states that if \(k | m\) then among \(m + k - 1\) integers one can always find \(m\) whose sum is divisible by \(k\). Over time many problems of this and similar type have been considered, and zero-sum problems have established themselves as a well studied and substantial branch of contemporary combinatorics.

A much studied example of a zero-sum problem for graphs goes as follows. What is the smallest number \(n\) so that any complete graph with edges labelled by the elements of a finite group \(G\) contains a subgraph of a prescribed type in which the total weight of the edges is 0 in \(G\)? For examples and an overview of such results, the interested reader may consult e.g. [3, 1] and the survey article [4]. Here we consider this problem for complete directed graphs, where the desired subgraph is a directed cycle. This scenario was recently also considered by Alon and Krivelevich [2].

Given an integer \(n \geq 1\), we denote by \(K^\pm_n\) the complete digraph consisting of \(n\) vertices and whose arc-set consists of all ordered pairs of vertices. Given a set \(A\), for us an \(A\)-arc-labeling of \(K^\pm_n\) is simply a function \(w : A(K^\pm_n) \to A\). If \((A, +)\) is an Abelian group, we say that \(w\) is zero-sum-free if there is no directed cycle for which the sum of arc-labels is zero. In this paper, for a non-trivial finite Abelian group \((A, +)\), we are interested in determining the smallest integer \(n(A) \geq 2\) such that \(K^\pm_{n(A)}\) has no zero-sum-free \(A\)-arc-labeling, i.e. for every \(A\)-arc-labeling there is a directed cycle for which the sum of the arc-labels is zero.

In [2] Alon and Krivelevich proved that \(n(Z_q) \leq [2q \ln q]\) for every integer \(q \geq 2\) and \(n(Z_p) \leq 2p - 1\) for every prime number \(p\). Here we improve their results and show that \(n(A)\) grows at most linearly for every non-trivial finite Abelian group \(A\).

Theorem 1. For every non-trivial finite Abelian group \((A, +)\) we have \(n(A) \leq 8|A|\).

Our second main result improves the upper bound of [2] in the case of primes.

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Theorem 2. For every prime $p \geq 3$ we have $n(\mathbb{Z}_p) \leq \frac{3p-1}{2}$.

The motivation of Alon and Krivelevich for studying the function $n(\mathbb{Z}_q)$ came from a related problem about the containment of cycles with particular lengths in minors of complete graphs. Our improvement on the upper bound on $n(\mathbb{Z}_q)$ also directly improves the corresponding result from [2].

Corollary 1. Let $q \geq 2$ be an integer. Then every $K_{2n(\mathbb{Z}_q)}$-minor contains a cycle of length divisible by $q$. In particular, every $K_{16q}$-minor contains a cycle of length divisible by $q$, and if $p$ is a prime, then every $K_{3p-1}$-minor contains a cycle of length divisible by $p$.

2 Proofs

Let $n \geq 2$ be an integer and $(A, +)$ a non-trivial finite Abelian group. We say that an $A$-arc labeling $w$ of $\tilde{K}_n$ is $A$-complete at the vertices $u, v \in V(\tilde{K}_n)$ if for every $a \in A$ there is a directed path $P_a$ from $u$ to $v$ such that $\sum_{(x, y) \in P_a} w(x, y) = a$. Given a vertex $v$ of $\tilde{K}_n$ and an element $c \in A$ let $w'$ be the $A$-arc-labelling defined by

$$w'(x, y) = \begin{cases} w(x, y) & x, y \neq v \\ w(x, y) + c & x = v \\ w(x, y) - c & y = v \end{cases}$$

This operation on arc labellings is called a switching by $c$ at $v$, and is denoted by $S_{c,v}$. Two $A$-arc-labellings of $\tilde{K}_n$ are switching-equivalent if we can obtain one from the other by a sequence of switchings. The following properties of switchings follow directly from the definitions, and will be very important for our investigations.

Observation 1. Let $w$ and $w'$ be switching-equivalent $A$-arc-labellings of $\tilde{K}_n$, then the following hold.

- $w$ is zero-sum-free if and only if $w'$ is so.
- $w$ is $A$-complete at $u, v$ if and only if $w'$ is so.

The following lemma is our main technical result on the way to the proof of Theorem 1.

Lemma 1. Let $n \geq 2$, $(A, +)$ a non-trivial finite Abelian group and $w$ an $A$-arc-labelling of $\tilde{K}_n$. Then there exists another $A$-arc-labelling $w'$ which is switching-equivalent to $w$ and at least one of the following holds.

- There exists a vertex set $V \subseteq V(\tilde{K}_n)$ of size $|V| \geq n - 4|A|$ and a proper subgroup $B < A$ such that $w'(x, y) \in B$ for every $(x, y) \in A(\tilde{K}_n[V])$.

- There exist vertices $u, v \in V(\tilde{K}_n)$ such that $w'$ is $A$-complete at $u, v$.

Proof. Suppose towards a contradiction that the claim was false, and let $A$ be a group of smallest size for which it fails. In particular, there exists some $n \in \mathbb{N}$ and an $A$-arc-labelling $w$ such that the following hold for every $A$-arc-labelling $w'$ that is switching-equivalent to $w$.

(i) For every $V \subseteq V(\tilde{K}_n)$ with $|V| \geq n - 4|A|$ and for every proper subgroup $B < A$ there exists $(x, y) \in A(\tilde{K}_n[V])$ such that $w'(x, y) \notin B$.

(ii) For every $u, v \in V(\tilde{K}_n)$ there exists some $a \in A$ such that there is no directed path $P_a$ in $\tilde{K}_n$ from $u$ to $v$ such that $\sum_{(x, y) \in P_a} w'(x, y) = a$.

In the following we will lead these assumptions towards a contradiction. To do so, we first prove the following auxiliary claim by induction on $i$.  

2
Claim. For every $0 \leq i \leq |A| - 1$ there exist pairwise distinct vertices $x_0, y_0, x_1, y_1, \ldots, x_{i-1}, y_{i-1}, x_i \in V(\tilde{K}_n)$ and an $A$-arc-labelling $w_i$ of $\tilde{K}_n$ that is switching-equivalent to $w$ such that the following hold.

- $w_i(x_{j-1}, x_j) = 0$ for every $1 \leq j \leq i$.
- Let $a_{i,j} := w_i(x_{j-1}, y_{j-1}) + w_i(y_{j-1}, x_j)$ for $1 \leq j \leq i$, and

\[ A_i := \left\{ \sum_{j \in J} a_{i,j} \mid J \subseteq \{1, \ldots, i\} \right\} \subseteq A. \]

Then $|A_i| \geq i + 1$.

Before moving on to the proof of the claim, note that the sequence $x_0, y_0, x_1, y_1, \ldots, x_{i-1}, y_{i-1}, x_i$ represents a chain of triangles connecting $x_0$ and $x_i$, see Figure 1. When going from $x_0$ to $x_i$, at every intermediate vertex $x_{j-1}$ we have the choice to either go directly to $x_j$ and pick up the label 0, or to take the detour $x_{j-1} \rightarrow y_{j-1} \rightarrow x_j$ and pick up labels summing up to $a_{i,j}$. Therefore, the set $A_i$ represents all possible arc-label sums along possible paths from $x_0$ to $x_i$ in this configuration.

Proof of the claim. For $i = 0$ note that, by definition, $A_i$ always includes 0 (the empty sum), so the claim holds trivially. Moving on to the inductive step, suppose $1 \leq i \leq |A| - 1$ and the claim holds for $i - 1$. Put $V := V(\tilde{K}_n) \setminus \{x_0, y_0, x_1, y_1, \ldots, x_{i-1}, y_{i-1}, x_i\}$. Then $|V| = n - (2i - 1) \geq n - 2|A| + 3$. Let $w_i$ be the $A$-arc-labelling of $\tilde{K}_n$ obtained from $w_{i-1}$ by switching by $w_{i-1}(x_{i-1}, y)$ at every vertex $y \in V$. Then $w_i$ clearly switching-equivalent to $w_{i-1}$ and hence to $w$, and by definition satisfies $w_i(x_{j-1}, x_j) = w_{i-1}(x_{j-1}, x_j) = 0$ for $1 \leq j \leq i - 1$ as well as $w_i(x_{i-1}, y) = 0$ for every $y \in V$.

We distinguish two cases.

Case 1. There exists a non-trivial subgroup $\{0\} \neq B \leq A$ such that $B \subseteq A_{i-1}$.

Let $H$ be the quotient group $A/B$ and let us define an $H$-arc-labelling $\tilde{w}$ of the complete subdigraph $\tilde{K}_n[V]$ by putting $\tilde{w}_i(x, y) = w_i(x, y) + B$ for all $x, y \in V$. We call $\tilde{w}_i$ the $B$-factor of $w_i$ restricted to $\tilde{K}_n[V]$. By our initial minimality assumption on $|A|$, the claim of the lemma has to hold for $H$, the complete digraph $\tilde{K}_n[V]$ and the $H$-arc-labelling $\tilde{w}_i$. It follows that there exists an $H$-arc-labelling $\tilde{w}_i'$ of $\tilde{K}_n[V]$ switching-equivalent to $\tilde{w}_i$ such that at least one of the following holds.

(i') There exists a vertex set $W \subseteq V$ of size $|W| \geq |V| - 4|H|$ and a proper subgroup $\tilde{G} < H$ such that $\tilde{w}_i'(x, y) \in \tilde{G}$ for every $(x, y) \in A(\tilde{K}_n[W])$.

(ii') There exist vertices $u', v' \in V(\tilde{K}_n)$ such that $\tilde{w}_i'$ is $H$-complete at $u', v'$.

Since $\tilde{w}_i$ and $\tilde{w}_i'$ are switching-equivalent $H$-arc-labellings on $\tilde{K}_n[V]$, there exists a sequence $S_{c_i + B, v_i}$, $i \in I$ of switchings that transform $\tilde{w}_i$ to $\tilde{w}_i'$. Applying the corresponding sequence $S_{c_i, v_i}$, $i \in I$ of switchings to $w_i$ one obtains an $A$-arc-labelling $w_i'$ of $\tilde{K}_n$ which is switching-equivalent to $w_i$, and hence to $w$, and whose $B$-factor restricted to $\tilde{K}_n[V]$ is $\tilde{w}_i'$. We next consider two cases depending on whether (i') or (ii') occurs.
Case 1 + (i’) holds. As $B$ is a non-trivial subgroup of $A$, we have $|H| \leq |A|/2$ and hence $|W| \geq |V| - 4|H| \geq |V| - 2|A| \geq n - 2|A| + 3 - 2|A| \geq n - 4|A|$. Let $G$ be the pre-image of $G$ under the canonical group homomorphism $A \to H = A/B$. Then $G$ is a proper subgroup of $A$, and by the assumption on $\bar{w}_i'$ and the fact that $\bar{w}_i'$ is the $B$-factor of $w_i'$ restricted to $\tilde{K}_n[V]$, we find that $w_i'(x, y) \in G$ for every $(x, y) \in A(\tilde{K}_n[W])$. The existence of $W$ and $w_i'$ yields a contradiction to property (i).

Case 2. There is no non-trivial subgroup $\{0\} \neq B \subseteq A$ such that $B \subseteq A_{i-1}$.

By property (i) applied to the set $V$ (which satisfies $|V| \geq n - 4|A|$) and to the trivial subgroup $\{0\} < A$ there exist distinct vertices $y_{i-1}, x_i \in V$ such that $a := w_i(y_{i-1}, x_i) \neq 0$. Let us add $y_{i-1}, x_i$ to the already constructed sequence $x_0, y_0, x_1, y_1, \ldots, y_{i-2}, x_{i-1}$. By the definition of $w_i$, the resulting sequence clearly satisfies the first property we need. For the second property first note that we have $a_{i,j} = w_i(x_{j-1}, y_{j-1}) + w_i(y_{j-1}, x_j) = w_i-1(x_{j-1}, y_{j-1}) + w_i-1(y_{j-1}, x_j) = a_{i-1,j}$ for all $1 \leq j \leq i - 1$ and $a_{i,i} = w_i(x_{i-1}, y_{i-1}) + w_i(y_{i-1}, x_i) = w_i(y_{i-1}, x_i) = a$ which is by assumption non-zero. Recall that $A_i = \{\sum_{j \leq i} w_i, j \mid \{1, \ldots, i\}\} = A_{i-1} + \{0, a\}$. By the inductive assumption on $A_{i-1}$, to obtain $|A_i| \geq i + 1$ it suffices to show that $A_i$ is a proper superset of $A_{i-1}$.

To do so, first note that there has to exist some integer $c \geq 0$ such that $ca \notin A_{i-1}$, as otherwise $A_{i-1}$ would contain the cyclic subgroup of $A$ generated by $a$, and hence we would be in Case 1. So let us fix the smallest such integer $c$ (then $c \geq 1$ since $0a = 0 \in A_{i-1}$). By the minimality of $c$ we have $(c - 1)a + a \in A_{i-1}$ and hence $ca = (c - 1)a + a \in A_{i-1} + \{0, a\} = A_i$, which in turn shows $A_i \setminus A_{i-1} \neq \emptyset$, as required. This proves the assertion of the inductive claim, and concludes the proof of our claim.

To finish the proof of the lemma, we apply the claim with $i = |A| - 1$, and obtain a chain of triangles connecting some vertices $x_0$ and $x_{|A|-1}$ and an $A$-arc-labelling $w_{|A|-1}$ switching equivalent to $w$. Note that in this case we necessarily have $A_{|A|-1} = A$, and hence it follows, that inside this chain of triangles there exists for every $a \in A$ a directed path $P_a$ from $x_0$ to $x_{|A|-1}$ whose arc-labels sum up to $a$, i.e. $w_{|A|-1}$ is $A$-complete at $x_0, x_{|A|-1}$. As $w_{|A|-1}$ is switching-equivalent to $w$, this contradicts property (ii). This final contradiction finishes the proof of the Lemma
Using Lemma 1 we can now give the proof of Theorem 1.

**Proof of Theorem 1.** We want to show that $n(A) \leq 8|A|$ for every non-trivial Abelian group $(A, +)$.

Suppose towards a contradiction that there exists a finite Abelian group $(A, +)$ such that $n(A) > 8|A|$ and suppose that the size of $A$ is the smallest possible. By the definition of $n(A)$ there has to exist a zero-sum free $A$-arc-labelling $w$ of $\tilde{K}_{n(A)}$. We apply Lemma 1 with $n = n(A)$ and the $A$-arc-labelling $w$ to find an $A$-arc-labelling $w'$ of $\tilde{K}_{n(A)}$ that is switching-equivalent to $w$ and one of the following holds.

(i) There exists a vertex set $V \subseteq V(\tilde{K}_n)$ such that $|V| \geq n(A) - 4|A| > 8|A| - 4|A| = 4|A|$ and a proper subgroup $B < A$ such that $w'(x, y) \in B$ for every $(x, y) \in A(\tilde{K}_n[V])$.

(ii) There exist vertices $u, v \in V(\tilde{K}_n)$ such that $w'$ is $A$-complete at $u, v$.

Since $w'$ is switching-equivalent to $w$, by Observation 1 $w'$ is also a zero-sum free $A$-arc-labelling of $\tilde{K}_n$.

**Case 1:** (i) holds. If $B = \{0\}$ then any digon in $\tilde{K}_n[V]$ would contradict our assumption that $w'$ is a zero-sum free $A$-arc-labelling of $\tilde{K}_n$, and hence we may assume $B \neq \{0\}$. By the minimality of $|A|$, we must then have $n(B) \leq 8|B| \leq 4|A| < |V|$, where we used that $B$ is a proper subgroup of $A$, and hence $|B| \leq |A|/2$.

Now, by the definition of $n(B)$, $w'$ cannot be a zero-sum-free $B$-labelling of $\tilde{K}_n[V]$, i.e. there has to exist a directed cycle $C$ in $\tilde{K}_n[V]$ whose arc-labels sum up to zero in $B$, and hence in $A$. This contradicts the assumption that $w'$ is a zero-sum free $A$-arc-labelling of $\tilde{K}_n$.

**Case 2:** (ii) holds. Let $a := -w'(v, u) \in A$. As $w'$ is $A$-complete at $u, v$ there exists a directed path $P_a$ in $\tilde{K}_n$ from $u$ to $v$ such that $\sum_{(x, y) \in A(P_a)} w'(x, y) = a$. Let $C$ be the directed cycle obtained from $P_a$ by adding the arc $(v, u)$. Then we have

$$\sum_{(x, y) \in A(C)} w'(x, y) = w'(v, u) + \sum_{(x, y) \in A(P_a)} w'(x, y) = -a + a = 0,$$

which again contradicts the assumption that $w'$ is a zero-sum free $A$-arc-labelling of $\tilde{K}_n$.

This finishes the proof of Theorem 1.

We now continue with groups of prime order and prove Theorem 2. It will be convenient to first prove the following auxiliary result.

**Lemma 2.** Let $p \geq 3$ be a prime number and $w$ a zero-sum free $\mathbb{Z}_p$-arc-labelling of $\tilde{K}_3$. Then there exists a vertex $v \in V(\tilde{K}_n)$ and non-trivial directed paths $P_1, P_2$ in $\tilde{K}_3$ ending at $v$ such that the three values 0, $\sum_{(x, y) \in A(P_1)} w(x, y)$, $\sum_{(x, y) \in A(P_2)} w(x, y)$ are pairwise distinct.

**Proof.** Since $w$ is a zero-sum free arc-labelling, there exists, in particular, no directed cycle with all arc labels zero. It follows that we can order the vertices of $\tilde{K}_3$ as $v_1, v_2, v_3$ such that $w(v_i, v_j) \neq 0$ for $1 \leq i < j \leq 3$. Let $a := w(v_2, v_3)$. If $w(v_1, v_2) + a \notin \{0, a\}$, then the claim of the lemma is satisfied with $v = v_3$ and the paths $P_1 = (v_2, v_3), P_2 = (v_1, v_2), (v_2, v_3)$. Hence, moving on, we may assume that $w(v_1, v_2) + a \in \{0, a\}$. As $w(v_1, v_2) \neq 0$, it necessarily follows that $w(v_1, v_2) + a = 0$, i.e. $w(v_1, v_2) = -a$. If $w(v_1, v_3) \neq a$ then the claim of the lemma is satisfied with $v = v_3$ and the paths $P_1 = (v_1, v_3), P_2 = (v_2, v_3)$. Thus, in what follows we may assume that $w(v_1, v_3) = a$. Next, note that $w(v_3, v_2) \neq -a$, as otherwise the digon spanned by the vertices $v_2, v_3$ would form a directed cycle whose arc labels sum up to zero. Therefore, if $w(v_3, v_2) \neq 0$ then the statement of the lemma holds with $v = v_2$ and the paths $P_1 = (v_1, v_2), P_2 = (v_3, v_2)$. As a result, we may also assume that $w(v_3, v_2) = 0$. Now, however, it is the vertex $v = v_2$ and the paths $P_1 = (v_1, v_3), (v_3, v_2), P_2 = (v_1, v_2)$ that fulfill the desired property.

□

5
We are now prepared for the proof of Theorem 2

Proof of Theorem 2: Suppose towards a contradiction that there exists a prime number $p \geq 3$ such that $n(Z_p) > \frac{3p-1}{2}$. Then for $n = \frac{3p-1}{2}$ there must exist a zero-sum free $Z_p$-arc-labelling $w$ of $\bar{K}_n$. In order to lead this assumption towards a contradiction, let us first prove the following claim by induction on $i$.

Claim. For $i = 0, 1, 2, \ldots, \frac{p-1}{2}$ there are distinct vertices $x_0, y_0, z_0, x_1, y_1, z_1, \ldots, x_{i-1}, y_{i-1}, z_{i-1}, x_i$ in $\bar{K}_n$ and a $Z_p$-arc-labelling $w_i$ of $\bar{K}_n$ switching-equivalent to $w$ such that the following hold for every $1 \leq j \leq i$.

- $w_i(x_{j-1}, x_j) = 0$.
- There exist two directed paths $P_{1,j}, P_{2,j}$ in $\bar{K}_n[V_j]$, $V_j := \{x_{j-1}, y_{j-1}, z_{j-1}, x_j\}$ starting at $x_{j-1}$ and ending in $x_j$ such that the three values $a_{i,j}^{(1)} = \sum_{(x,y) \in A(P_{1,j})} w_i(x,y)$ and $a_{i,j}^{(2)} = \sum_{(x,y) \in A(P_{2,j})} w_i(x,y)$ are pairwise different.

Before moving on to the proof of the claim, note that the setting is analogous to the one in the proof of Theorem 1. However, instead of a chain of triangles we have a chain of $\bar{K}_4$'s connecting $x_0$ and $x_i$. When going from $x_0$ to $x_i$, at every intermediate vertex $x_{j-1}$ we have the choice to either go directly to $x_j$ and pick up labels summing up to $a_{i,j}^{(1)}$ or $a_{i,j}^{(2)}$, respectively. If we put

$$A_i = \{0, a_{i,1}^{(1)}, a_{i,1}^{(2)} \} + \{0, a_{i,2}^{(1)}, a_{i,2}^{(2)} \} + \cdots + \{0, a_{i,i}^{(1)}, a_{i,i}^{(2)} \} \subseteq Z_p,$$

then $A_i$ represents all possible arc-label sums along possible paths from $x_0$ to $x_i$ in this configuration.

Proof of the claim. The claim holds for $i = 0$ by simply selecting an arbitrary vertex $x_0 \in V(\bar{K}_n)$ and putting $w_0 := w$. Now let $1 \leq i \leq \frac{p-1}{2}$ and suppose the claim holds for $i - 1$.

Put $W = \{x_0, y_0, z_0, x_1, y_1, z_1, \ldots, x_{i-1}, y_{i-1}, z_{i-1}, x_i\}$, $V = V(\bar{K}_n) \setminus W$ and let $w_i$ be the $Z_p$-arc-labelling of $\bar{K}_n$ defined by switching by $w_{i-1}(x_{i-1}, y)$ at every $y \in V$. Then $w_i$ is clearly switching-equivalent to $w_{i-1}$, and hence to $w$, by definition agrees with $w_{i-1}$ on all arcs spanned by $W$, and $w_i(x_{i-1}, y) = 0$ for every $y \in V$. In particular, for $1 \leq j \leq i - 1$ we have $w_i(x_{j-1}, x_j) = 0$ and the values $0, a_{i,j}^{(1)} = a_{i-1,j}^{(1)}$ and $a_{i,j}^{(2)} = a_{i-1,j}^{(2)}$ are pairwise different. Let us now pick three distinct vertices $V_3 = \{v_1, v_2, v_3\} \subseteq V$ arbitrarily. Since $w$ is a zero-sum free arc-labelling and $w_i$ is switching-equivalent to $w$, by Observation 1 $w_i$ is also zero-sum free. In particular, the restriction of $w_i$ to $A(\bar{K}_n[V_3])$ is also a zero-sum free $Z_p$-arc-labelling. By the application of Lemma 2 to $\bar{K}_n[V_3]$ we find that there is a vertex $x_1 \in V_3$ and two distinct directed paths $P_1$ and $P_2$ in $\bar{K}_n[V_3]$ of positive length ending at $x_1$, such that $0, b_{1,j}^{(1)} = \sum_{(x,y) \in A(P_1)} w_i(x,y)$ and $b_{1,j}^{(2)} = \sum_{(x,y) \in A(P_2)} w_i(x,y)$ are pairwise distinct. Let us denote by $u_1$ and $u_2$ the starting vertices of $P_1$ and $P_2$, respectively, and put $\{y_1, z_1\} = V_3 \setminus \{x_1\}$. Furthermore define $P_1$ and $P_2$ as the directed $x_{i-1}, x_i$-paths in $\bar{K}_n[V_i]$, $V_i = \{x_{i-1}, y_{i-1}, z_{i-1}, x_i\}$ obtained by extending $P_1$ and $P_2$ with the arcs $(x_{i-1}, u_1)$ and $(x_{i-1}, u_2)$, respectively. Then we have $w_i(x_{i-1}, x_i) = 0$, and the three values $0, a_{i,j}^{(1)} = w_i(x_{i-1}, u_1) + b_{1,j}^{(2)} = 0 + b_{1,j}^{(2)} = b_{1,j}^{(1)}$ and $a_{i,j}^{(2)} = w_i(x_{i-1}, u_2) + b_{1,j}^{(2)} = 0 + b_{1,j}^{(2)} = b_{1,j}^{(2)}$ are pairwise different, as required.

To conclude the proof of Theorem 2 let $x_0, y_0, z_0, x_1, y_1, z_1, \ldots, x_{\frac{p-1}{2}}, y_{\frac{p-1}{2}}, z_{\frac{p-1}{2}}, x_{\frac{p-1}{2}}$ and $w_{\frac{p-1}{2}}$ be as given by the above claim when applied with $i = \frac{p-1}{2}$. Let $A_{\frac{p-1}{2}}$ be the set of corresponding path sums, and put $a := -w_{\frac{p-1}{2}}(x_{\frac{p-1}{2}}, x_0)$. Since $p$ is a prime number, the iterative application of the Cauchy-Davenport Theorem directly yields that $A_{\frac{p-1}{2}} = Z_p$ must hold, and hence our chain of $\bar{K}_4$'s, in particular, contains a path $P_a$ from $x_0$ to $x_{\frac{p-1}{2}}$ such that $\sum_{(x,y) \in A(P_a)} w_{\frac{p-1}{2}}(x,y) = a$. Let $C$ be the directed cycle obtained from $P_a$ by adding the arc $(x_{\frac{p-1}{2}}, x_0)$. Then we have

$$\sum_{(x,y) \in A(C)} w_{\frac{p-1}{2}}(x,y) = w_{\frac{p-1}{2}}(x_{\frac{p-1}{2}}, x_0) + \sum_{(x,y) \in A(P_a)} w_{\frac{p-1}{2}}(x,y) = -a + a = 0.$$
However, this is impossible as \( w \) is switching equivalent to a zero-sum-free arc-labeling, and as such is zero-sum-free as well. This final contradiction shows that our very initial assumption that \( n(\mathbb{Z}_p) > \frac{3p-1}{2} \) was wrong and hence concludes the proof.

Finally we give the proof of Corollary \( \dag \). Even though the argument is identical to the one in \[2\], we include it for the reader’s convenience.

**Proof of Corollary \( \dag \)** Let \( q \geq 2 \) and \( G \) be a \( K_{2n(\mathbb{Z}_q)} \)-minor. Label the \( 2n(\mathbb{Z}_q) \) super nodes of this minor as \( X_i^+, X_i^- \) for \( 1 \leq i \leq n(\mathbb{Z}_q) \). By definition, for every \( i \) there is a unique edge \( x_i^+ x_i^- \) in \( G \) connecting a vertex in \( X_i^+ \) to a vertex in \( X_i^- \), and for every \( i \neq j \) the induced subgraph \( G[X_i^+ \cup X_j^-] \) is a tree. Let us define \( w(i, j) \in \mathbb{Z}_q \) to be one plus the length of the unique path connecting \( x_i^+ \) and \( x_j^- \) in this tree taken modulo \( q \). This results in a \( \mathbb{Z}_q \)-arc-labelling of the complete digraph \( \tilde{K}_{n(\mathbb{Z}_q)} \) on the vertex-set \( \{1, \ldots, n(\mathbb{Z}_q)\} \). Then, by the definition of the function \( n(\mathbb{Z}_q) \) there has to exist a directed cycle in this auxiliary complete digraph whose arc-labels sum up to zero. Expanding this cycle into a cycle in \( G \) by replacing arcs in \( \tilde{K}_{n(\mathbb{Z}_q)} \) with the connecting corresponding paths in \( G \) yields a cycle of length divisible by \( q \) in \( G \), as desired. \[\Box\]

### 3 Concluding remarks

In Theorem \( \dag \) we proved an upper bound on the function \( n(A) \) for general finite Abelian groups \( A \), which shows that this function grows at most linearly with \( |A| \). As demonstrated by Theorem \( \mathbf{2} \) this upper bound can be improved at least when \( A = \mathbb{Z}_p \) for some prime \( p \geq 3 \), but we believe that improvement should be possible in general. To support this belief we first remark that, by slightly adjusting the proofs of Lemma \( \dag \) and Theorem \( \dag \) one can obtain the following result.

**Theorem 3.** Let \( (A, +) \) be a non-trivial finite Abelian group and let \( p \) be the smallest prime divisor of \( |A| \). Then we have \( n(A) \leq \frac{2p^2}{(p-1)^2} |A| \).

The statement of Theorem \( \dag \) can be recovered by noting that we always have \( p \geq 2 \).

Next, let us consider Theorem \( \mathbf{2} \). Its proof is based on Lemma \( \dag \) which allowed us to build a long chain of \( \tilde{K}_t \)'s and hence to prove the \( \mathbb{Z}_p \)-completeness of the given arc-labelling. An improvement on this auxiliary result would directly improve the upper bound on the function \( n(A) \) for groups of prime order. In this direction we propose the following conjecture.

**Conjecture 1.** Let \( t \geq 3 \) an integer, \( p \geq 3 \) a prime number, and \( w \) a zero-sum-free \( \mathbb{Z}_p \)-arc-labelling of \( \tilde{K}_t \). Then there exist a vertex \( v \in V(\tilde{K}_t) \) and non-trivial directed paths \( P_1, P_2, \ldots, P_{t-1} \) in \( \tilde{K}_t \) ending at \( v \) such that the \( t-1 \) values \( \sum_{(x,y) \in A(P_i)} w(x,y) \), \( 1 \leq i \leq t-1 \) are all non-zero and pairwise distinct.

Note that Lemma \( \dag \) resolves the case \( t = 3 \), and if the conjecture is true for a given \( t \), then, following the arguments from the proof of Theorem \( \mathbf{2} \) one can obtain \( n(\mathbb{Z}_q) \leq t[\frac{p-1}{p-1}] + 1 \) whenever \( p \geq 3 \) is a prime.

As for lower bounds, it is easy to see that \( n(\mathbb{Z}_q) \geq q + 1 \) for every integer \( q \geq 2 \). For this, fix some linear ordering \( v_1, \ldots, v_q \) of the vertices of \( \tilde{K}_q \) and define a \( \mathbb{Z}_q \)-arc-labelling of \( \tilde{K}_q \) by setting the label of an arc \( (v_i, v_j) \) to be equal to 1 if \( i < j \) and 0 otherwise. Then every directed cycle in \( \tilde{K}_q \) uses at least one and at most \( q-1 \) arcs of the form \( (v_i, v_j) \) with \( i < j \), and hence the sum of its arc-labels is non-zero in \( \mathbb{Z}_q \). Hence, this arc-labelling is zero sum-free, and shows that \( n(\mathbb{Z}_q) > q \), as required.

Surprisingly, for non-cyclic Abelian groups we did not manage to obtain a general linear lower bound in terms of the group order.

**Question 1.** Do we have \( n(A) > |A| \) for every non-trivial finite Abelian group \( (A, +) \)?

A natural way to answer Question \( \dag \) would be via the following product inequality.
Question 2. Is it true that if \((A_1,+)\) and \((A_2,+)\) are non-trivial finite Abelian groups, then
\[ n(A_1 \times A_2) - 1 \geq (n(A_1) - 1)(n(A_2) - 1). \]

Let now \((A,+)\) be a non-trivial finite Abelian group and suppose that the answer to Question 2 is positive. Then, by the Fundamental Theorem on Finite Abelian Groups, \(A\) is isomorphic to the direct product \(\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k}\), for some \(q_1, \ldots, q_k\). Therefore, it follows that \(n(A) - 1 \geq (n(\mathbb{Z}_{q_1}) - 1) \cdots (n(\mathbb{Z}_{q_k}) - 1) \geq q_1 \cdots q_k = |A|\). However, a positive answer to Question 2 would actually prove much more. Let \(\ell \in \mathbb{N}\) be arbitrary and let \(A^\ell\) denote the \(\ell\)-fold direct product of \(A\) with itself. Then by Theorem 1 we have
\[ (n(A) - 1)^\ell \leq n(A^\ell) - 1 \leq 8|A^\ell| - 1 < 8|A|^\ell, \]
\[ i.e. \quad n(A) \leq 8^{1/\ell}|A| + 1. \]
Taking the limit as \(\ell \to \infty\) we would obtain \(n(A) \leq |A| + 1\), which in turn would result in the exact answer \(n(A) = |A| + 1\).

For small groups we managed to check by computer [6] that \(n(A) = |A| + 1\) holds for all Abelian groups \((A,+)\) of order at most 6, and we believe that this equality might actually be true in general.

As a final comment, let us mention that the proof of Lemma 1 would still work (without any noteworthy changes), if in the definition of \(A\)-completeness we would require paths of length at least two. Using this stronger statement in the proof of Theorem 1 one can show that in every labelling of the complete digraph on \(n \geq 8|A|\) vertices with labels from an Abelian group \((A,+)\) there exists a directed cycle of length at least three whose arc-labels sum up to zero. This slightly stronger statement has the following immediate consequence concerning the undirected version of the zero-sum cycle problem.

Corollary 2. Let \((A,+)\) be a finite Abelian group, \(n \geq 8|A|\), and let \(w : E(K_n) \to A\) be an edge-labelling of the complete undirected graph of order \(n\). Then there exists a cycle \(C\) in \(K_n\) such that \(\sum_{e \in E(C)} w(e) = 0\).

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References
[1] N. Alon and Y. Caro, On three zero-sum Ramsey-type problems. Journal of Graph Theory, 17(22), 177–192, 1993.
[2] N. Alon and M. Krivelevich, Divisible subdivisions, arXiv preprint, arXiv:2012.05112 (2020)
[3] N. Alon and N. Linial, Cycles of length 0 modulo \(k\) in directed graphs. Journal of Combinatorial Theory, Series B, 47(1), 114–119, 1989.
[4] Y. Caro, Zero-sum problems — a survey, Discrete Mathematics 152:93–113, (1996).
[5] P. Erdős, A. Ginzburg, A. Ziv, Theorem in additive number theory, Bulletin of the Research Council of Israel 10F:41–43 (1961).
[6] M. Scheucher, A SAT-model for computing \(n(A)\),
  [http://page.math.tu-berlin.de/~scheuch/suppl/SAT/zerosumcycles/zerosumcycles_sat.py](http://page.math.tu-berlin.de/~scheuch/suppl/SAT/zerosumcycles/zerosumcycles_sat.py).