Tidal and centripetal stress in a model human falling deep into a Kerr black hole

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Abstract
I calculate the tidal and centripetal stress in the Misner–Thorne–Wheeler model human— a rigid prism— falling into a high-spin supermassive Kerr black hole at a constant, near-equatorial latitude with high energy, enough to surmount the radial potential barrier near the throat of the black hole. The stress in the model human would be nonlethal.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Misner, Thorne, and Wheeler [1] (hereinafter ‘MTW’) calculated the tidal stress in a model human— a rigid prism— falling toward, or into, an uncharged spherical static (i.e., Schwarzschild [2]) black hole. They calculated the radius from the center of the black hole at which the prism, falling radially inward, would be ripped apart. Depending on the mass of the black hole, this could occur outside the event horizon (for a collapsed star) or inside the event horizon (for a supermassive black hole).

This article extends that calculation, by example, to a rotating black hole: an idealized rotating black hole modeled by Kerr’s exact solution [3] of Einstein’s field equations for a
vacuum. Any fall into a Schwarzschild black hole would be fatal; a fall into a Kerr black hole may not be.

Specifically, the example simulates the constant-latitude, near-equatorial fall of the MTW prism into a 4 million solar mass black hole at a polar angle \(\theta = \tan^{-1}(1155/68)\) from the spin axis, with enough energy (seven rest masses) to surmount the radial potential barrier near the throat of the black hole and continue falling to arbitrarily negative values of the radial coordinate \(r\). The tidal torque on the prism is calculated, and the orientation of the prism is updated by integrating Euler’s equations of rigid-body motion. The normal tidal plus centripetal stress on the prism is also calculated; it would be well below the yield stress assumed by MTW.

2. Kerr spacetime

2.1. Metric

In ingoing Kerr coordinates \([u, r, \theta, \phi]\) and geometrized units in which \(G_N = c = 1\) and length is measured in centimeters, Kerr spacetime is described by the line element [4]

\[
ds^2 = -\left[ 1 - 2mr \rho^{-2} \right] du^2 + 2dr du + \rho^2 d\theta^2 \\
+ \rho^{-2} \left[ (r^2 + a^2) - \Delta a^2 \sin^2 \theta \right] \sin^2 \theta d\phi^2 \\
- 2a \sin^2 \theta d\phi dr - 4amr \sin^2 \theta \rho^{-2} d\phi du,
\]

where

\[\Delta = r^2 - 2mr + a^2,\]
\[\rho^2 = r^2 + a^2 \cos^2 \theta,\]

and the metric signature is \(-+++\). The covariant metric tensor components are:

\[g_{00} = -1 + 2mr / \rho^2,\]
\[g_{01} = g_{01} = 1,\]
\[g_{03} = g_{30} = -2amr \sin^2 \theta / \rho^2,\]
\[g_{13} = g_{31} = -a \sin^2 \theta,\]
\[g_{22} = \rho^2,\]
\[g_{33} = \left( (r^2 + a^2) - \Delta a^2 \sin^2 \theta \right) \sin^2 \theta / \rho^2,\]

and

\[g_{02} = g_{20} = g_{11} = g_{12} = g_{21} = g_{23} = g_{32} = 0.\]

2.2. Properties

The properties of Kerr spacetime have been reviewed by Wiltshire et al [5], Adamo and Newman [6], and Teukolsky [7], among others. Here I review only a few properties of Kerr spacetime and ingoing Kerr coordinates that may clarify the example presented here.
Each event \([u, r, \theta, \phi]\) is assumed to be identified with the event \([u, r, \theta, \phi - 2\pi]\). Hence the coordinate \(\phi\) need only range over the open interval \([0, 2\pi]\). The coordinate \(\theta\) ranges over \([0, \pi]\). The coordinates \(u\) and \(r\) range over the real numbers.

Let \(g_0, g_1, g_2,\) and \(g_3\) denote the coordinate basis vectors with covariant components

\[
\gamma_{\mu}^\nu = \begin{cases} 
1 & \text{if } \nu = \mu \\
0 & \text{if } \nu \neq \mu.
\end{cases}
\]

The metric tensor components do not depend on \(u\) or \(\phi\), so the coordinate basis vectors \(g_0\) and \(g_3\) are Killing vectors. The Killing vector \(g_0\) is timelike at infinity, i.e., at large positive or negative \(r\), so Kerr spacetime is stationary. The symmetry implied by the Killing vector \(g_3\) is axial symmetry.

The coordinate basis vector \(g_3\) is everywhere spacelike. The coordinate basis vector \(g_1\) is everywhere null; nevertheless, a displacement in the direction \(g_1\) is called a radial displacement and is measured in centimeters. The coordinate \(r\) is called the radius.

Some elements of the covariant metric tensor blow up as \(\rho^2 \to 0\), e.g., as \(r \to 0\) in the equatorial plane \(\theta = \pi / 2\). Components of the Riemann curvature tensor \(R^\nu_{\mu\rho\lambda}\) also blow up at \(\rho^2 = 0\). So does the Kretschmann curvature scalar \(K = R^\nu_{\mu\rho\lambda}R^\rho_{\lambda\mu\nu}\) \([8]\), which does not depend on the choice of coordinate system. Hence the singularity at \(\rho^2 = 0\) is a physical singularity of Kerr spacetime curvature, not an artifact of the choice of ingoing Kerr coordinates.

Kerr spacetime depends on two parameters: \(m\) and \(a\). The parameter \(m\) is called the mass parameter, or mass, and the parameter \(a\) is called the specific angular momentum parameter, or the spin parameter. These names describe the effects of the spacetime on the motion of test particles at large radius \([3]\). However, Kerr spacetime contains no matter, spinning or otherwise, and no nongravitational fields. If \(a = 0\), the metric describes Schwarzschild spacetime in ingoing Kerr coordinates, which in this case are ingoing Eddington–Finkelstein coordinates.

Analysis of the equations of motion for outgoing uncharged massless particles (e.g., photons) in Kerr spacetime reveals that, if \(a^2 \leq m^2\), outgoing photons may remain, i.e. orbit, at radii \(r\) where \(\Delta = 0\) \([1]\). The larger of these radii, \(r = r_+ \equiv m + \sqrt{m^2 - a^2}\), is called the outer horizon, and the smaller radius, \(r = r_- \equiv m - \sqrt{m^2 - a^2}\), is called the inner horizon.

Photons at \(r < r_-\) may not propagate to any \(r > r_-\). Photons at \(r < r_+\) may not propagate to any \(r > r_+\), hence the outer horizon should appear black to any outside observer. Moreover, analysis of the equations of motion (see below) for massive particles shows that massive particles at \(r < r_+\) may not propagate to any \(r > r_+\). Hence the outer horizon has been called the event horizon \([9]\), beyond which observers on the outside may acquire no information about events on the inside. However, Hawking \([10]\) has conjectured that, when quantum effects are considered, information about the interior may leak out, so the outer horizon is an apparent horizon but not an event horizon.

If \(a^2 > m^2\), there are no horizons. The Kerr solution describes a spacetime with a ‘naked’ singularity, unclothed by a horizon, not a black hole \([11, 12]\).

Kerr’s solution may not be a good model of the interior of a real spinning black hole; see discussion in section 8.

2.3. Curvature

The Riemann tensor describes the acceleration of separation between nearby free particles, a consequence of spacetime curvature. It is defined in terms of the Christoffel symbols and their derivatives \([1]\):
\[ R^a_{\beta \gamma} \equiv \frac{\partial}{\partial x^\beta} \left( \Gamma^a_{\beta \lambda} \right) - \frac{\partial}{\partial x^\gamma} \left( \Gamma^a_{\beta \lambda} \right) + \Gamma^a_{\beta \lambda} \Gamma^\lambda_{\gamma \lambda} - \Gamma^a_{\beta \gamma} \Gamma^\lambda_{\lambda \lambda}. \] (12)

The Christoffel symbols in turn may be expressed in terms of derivatives of metric tensor components:
\[ \Gamma^a_{\beta \gamma} = \frac{1}{2} g^{a \lambda} \left( g_{\beta \lambda, \gamma} + g_{\gamma \lambda, \beta} - g_{\beta \gamma, \lambda} \right), \] (13)

where the comma denotes ordinary differentiation, for example
\[ g_{\beta \gamma, \lambda}. \]

Explicit expressions for the Christoffel symbol components and the Riemann tensor components of Kerr spacetime are lengthy and, for brevity, are not displayed here. However, it is easy to display them using a computer algebra system, such as Maxima \[13, 14], that supports the symbolic calculation of the tensors of general relativity.

2.4. Parameters

For the numerical example we assume the Kerr mass parameter is \( m = 4 \times 10^6 M_\odot \), where \( M_\odot \) is the Solar mass. This is roughly the mass of the supermassive black hole at the center of the Milky Way, near the radio source Sgr A* \[15–17\].

We assume the Kerr spin parameter is \( a = 0.998 m \), a plausible generic value for a supermassive black hole that grows by accretion from a thin disk \[18\]. The Sgr A* black hole may have a smaller spin \[19–21\].

In SI-units \( M_\odot = 1.984 \times 10^{30} \text{ kg} \) \[22\]. In \( G_N = c = 1 \) units \( M_\odot = 1.47662 \times 10^5 \text{ cm} \). Hence \( m \approx 5.91 \times 10^{11} \text{ cm} \), and \( a \approx 5.89 \times 10^{11} \text{ cm} \).

3. The geodesic

For the purpose of calculating the trajectory of the prism through the spacetime, I model the prism as a test particle. It follows a timelike geodesic, a solution of the geodesic equation of motion, for which Carter found a first integral \[1, 4\]:
\[ \frac{du}{d\tau} = \left( -a \left( aE \sin^2 \theta - \Phi \right) + \left( r^2 + a^2 \right) \left( \sqrt{R} + P \right) / \mu \rho^2 \right), \] (15)
\[ \frac{dr}{d\tau} = \sqrt{R} / \mu \rho^2, \] (16)
\[ \frac{d\theta}{d\tau} = \sqrt{\Theta} / \mu \rho^2, \] (17)
and
\[ \frac{d\phi}{d\tau} = \left( -(aE - \Phi / \sin^2 \theta) + a \left( \sqrt{R} + P \right) / \mu \rho^2 \right), \] (18)

where
\[ \Theta = Q - \cos^2 \theta \left( a^2 - E^2 \right) + \Phi^2 / \sin^2 \theta, \] (19)
\[ P = E \left( r^2 + a^2 \right) - a \Phi, \] (20)
and
\[ R = P^2 - \Delta(\mu^2r^2 + K). \]  

(21)

where \( \mu, E, \Phi, \) and \( K \) are constants of motion described in the following section, and
\[ Q = K - (\Phi - aE)^2. \]  

(22)

The signs of \( \sqrt{R} \) and \( \sqrt{Q} \) may be positive or negative. The sign of \( \sqrt{R} \) should be
negative for a future-directed, radially inbound \( (r \text{ decreasing}) \) portion of a geodesic. The sign
of \( \sqrt{Q} \) should be positive for a descending \( (\theta \text{ increasing}) \) portion of a geodesic and negative
for an ascending \( (\theta \text{ decreasing}) \) portion of a geodesic.

These equations may be integrated numerically to find the geodesic followed by the
MTW prism in this example.

3.1. Constants of motion

The constants of motion appearing in the four-velocity are \( \mu \), the rest mass; \( E \), the energy; \( \Phi \),
the angular momentum about the spacetime axis of symmetry; and \( K \), Carter’s fourth constant
of motion [4], which is a generalized total angular momentum.

Test particles of different rest masses would fall in the same manner, so it is convenient
to assume \( \mu = 1 \), instead of using the actual rest mass of the prism.

A constant-latitude fall requires \( d\theta/d\tau = 0 \), which, from (17), requires \( \Theta = 0 \). Requiring
the right-hand side of (19) to vanish implies
\[ Q = \cos^2 \theta \left( a^2 \left( \mu^2 - E^2 \right) + \Phi^2 \sin^2 \theta \right), \]  

(23)

and, using (22),
\[ K = (\Phi - aE)^2 + \cos^2 \theta \left( a^2 \left( \mu^2 - E^2 \right) + \Phi^2 \sin^2 \theta \right). \]  

(24)

After Hamilton [23], I choose \( \Phi \) to stationarize \( \Theta \) with respect to variations in \( \theta \):
\[ 0 = \frac{\partial \Theta}{\partial \theta}. \]  

(25)

which implies
\[ \Phi^2 = a^2 \sin^4 \theta(E - \mu)(E + \mu). \]  

(26)

I choose the positive root
\[ \Phi = a \sin^2 \theta \sqrt{(E - \mu)(E + \mu)} \]  

(27)

for the example geodesic.

I choose \( E/\mu = 7 \), which is large enough for the infalling prism to fall monotonically to
arbitrarily large negative values of \( r \).

3.2. Initial conditions

I will impose initial conditions at \( r = 0 \) and integrate the four-velocity forward and backward
in proper time \( \tau \) from there. At \( r = 0 \), I assume:
\[ \tau = 0 \]  

(28)
\[ [u, r, \theta, \phi] = \begin{bmatrix} 0, 0, \tan^{-1}(1155/68), 0 \end{bmatrix} \]  

3.3. Integration

Figure 1 shows the coordinates \( u, r, \) and \( \phi \) of the MTW prism as a function of proper time along the geodesic, obtained by numeric integration using the function \( \text{rk} \) of the dynamics package of Maxima release 5.29.1, built using CLISP version 2.49. A proper time step of \( 3 \times 10^6 \) cm and 64 significant digits floating-point precision were used.

4. The tetrad

4.1. Constructing a tetrad at \( r = 0 \)

At \( r = 0 \) a tetrad is constructed using the Gram–Schmidt orthonormalization algorithm adapted to signature \(-+++\) spacetime.

Let \( \vec{e}_0, \vec{e}_1, \vec{e}_2, \) and \( \vec{e}_3 \) denote the basis vectors of the tetrad. Basis vector \( \vec{e}_0 \) has contravariant components \( e^{\mu}_0 \). The timelike basis vector is \( \vec{e}_0 \), the four-velocity of the test particle to which the tetrad frame is attached:

\[ e_0 = \begin{bmatrix} 7 - 4\sqrt{3}, -4\sqrt{3}, 0, 0 \end{bmatrix} \]  

The remaining basis vectors are obtained from the four-vectors \( \vec{g}_1, \vec{g}_2, \) and \( \vec{g}_3 \). One obtains

\[ e_1 = \begin{bmatrix} \sqrt{97 - 56\sqrt{3}}, \frac{49 - 28\sqrt{3}}{\sqrt{97 - 56\sqrt{3}}}, 0, 0 \end{bmatrix} \]  

\[ e_2 = \begin{bmatrix} 0, 0, \frac{2599649364962450237}{900635231936800000000000}, 0 \end{bmatrix} \]
and
\[ e_3 = \begin{bmatrix} 1155 & 1155 & 0 & 4296849004503079274887 \\ 68 & 68 & 148604813269387200000000000000000 & .333 \\ \end{bmatrix}. \] (33)

It is straightforward to verify that
\[ g_{\mu\nu} e^\mu_i e^\nu_j = \begin{cases} -1 & \text{if } i = j = 0 \\ 1 & \text{if } i = j \neq 0 \\ 0 & \text{if } i \neq j. \end{cases} \] (34)

### 4.2. Parallel-propagating the tetrad

If a basis vector \( e_i \) at event \( x^\nu \) is parallel-propagated along a small displacement \( dx^\nu \), its components change by small amounts \( de_i = -\Gamma^\mu_{\nu\rho} e^\mu_i dx^\nu \). If the displacement occurs along a geodesic during a small lapsed \( \tau \) of proper time, one may write
\[ \frac{de_i}{d\tau} = -\Gamma^\mu_{\nu\rho} e^\mu_i \frac{dx^\nu}{d\tau} = -\Gamma^\mu_{\nu\rho} e^\mu_i e^\nu_0, \quad \text{where } i = 0, 1, 2, 3. \] (35)

To parallel-propagate the tetrad basis vectors along the geodesic, these equations, for \( i = 1, 2, 3 \), are integrated numerically simultaneously with Carter’s first integrals, (15)–(18), of the geodesic equations. For \( i = 0 \), (35) is just the geodesic equation, which may be integrated numerically, or one may use Carter’s first integrals.

### 5. The tidal tensor

The Riemann tensor describes the acceleration of separation between nearby free particles. In Kerr spacetime, a vacuum spacetime, this deviation is solely a tidal effect: the Ricci curvature tensor, a trace of the Riemann tensor, vanishes, so the Weyl conformal tensor, the traceless part of the Riemann tensor, equals the Riemann tensor. The Weyl tensor describes tidal acceleration.

If parts of the MTW prism were not bound together by cohesive or, if compressed, repulsive electromagnetic forces, they would fall along separate geodesics, changing their distance from one another. But the parts of the prism are bound together by electromagnetic forces. The forces that are exerted to keep the distances between parts from changing may be calculated using Newton’s second law of motion in an inertial frame, such as the tetrad frame.

Let primed (‘) indices denote tetrad-frame coordinates. Thus \( R^\ell_{\ell'\ell'} \) denotes Riemann tensor components in tetrad coordinates \( x^\ell \), which I will name \([\tau, x', y', z']\), and \( R^\alpha_{\alpha\beta\gamma} \) denotes Riemann tensor components in the Kerr coordinates \( x^\nu \) named \([u, r, \theta, \phi]\).

To transform the contravariant index from Kerr to tetrad coordinates, one contracts it on the partial derivative of contravariant tetrad vector components with respect to Kerr vector components:
\[ R^{\alpha'}_{\alpha'\beta'\gamma'} = R^\alpha_{\alpha\beta\gamma} x^{\alpha'}_{\alpha}. \] (36)

Then, to transform the covariant indices from Kerr to tetrad coordinates, one contracts them on partial derivatives of contravariant Kerr vector components with respect to tetrad vector components:
The partial derivatives of contravariant Kerr vector components with respect to tetrad vector components are just the tetrad basis vectors, so

$$R_{\alpha \beta \gamma \delta} = R_{\alpha \beta \gamma \delta}^0 x_\alpha x_\beta x_\gamma x_\delta.$$

At each step of propagating the tetrad basis vectors along the geodesic, we have the tetrad basis vectors at the current event $x^\nu$. The derivatives $x^\nu_\alpha$ (in general, $x^\nu_\alpha$) may be calculated by matrix inversion: using the chain rule for derivatives, the identity tensor $g^{\lambda \nu}$ may be expressed as the total derivative

$$g^{\lambda \nu} = \frac{\partial x^\mu_\nu}{\partial x^\lambda_\mu} x^\lambda_\mu,$$

which indicates $x^\nu_\alpha$ is the inverse of $x^\lambda_\mu$. This identity may be represented as the matrix equation

$$A X = I,$$

where $I$ is the identity matrix, with $g^{\lambda \nu}$ the element at row $\lambda$ and column $\nu$; $X$ is the matrix to be determined, with $x^\nu_\alpha$ the element at row $\mu$ and column $\nu$, and $A$ is the matrix with $x^\lambda_\mu$ the element at row $\lambda$ and column $\mu$. Hence $X = A^{-1}$, and $x^\nu_\alpha$ is the row $\mu$, column $\nu$ element of $A^{-1}$.

Thus we have all the factors needed on the right-hand side of (38) to transform the Riemann tensor to the tetrad frame. The only components needed are $R_{\alpha \beta \gamma \delta}$ for $\alpha = 1, 2, 3$ and $\beta = 1, 2, 3$, which appear in the equation of geodesic deviation of two nearby geodesics, $x^\nu(\tau)$ and $x^\nu(\tau + n^\nu(\tau)$, initially separated by a small three-vector $n^\nu$ and at rest in the tetrad frame. The equation of geodesic deviation is

$$\frac{d^2 x^\nu}{d\tau^2} = R_{\alpha \beta \gamma \delta}^\nu x^\alpha_\beta x^\gamma_\delta,$$

and $x^\nu_0 = 1$. The rank-1 tidal tensor is

$$C^\nu_\beta = R_{\alpha \beta \gamma \delta}^\nu x^\gamma_\delta,$$

and we may write

$$\frac{d^2 n^\nu}{d\tau^2} = C^\nu_\beta n^\beta.$$

Figure 2 shows the diagonal (in a matrix representation) components of the tidal tensor, which cause longitudinal tidal stress (tension if positive; compression if negative) in the MTW prism, to a degree that depends on the orientation of the prism in the tetrad frame. The diagonal components may also apply torques, accelerating rotation about the center of mass.

Figure 3 shows the off-diagonal components of the tidal tensor, which cause shear stress in the MTW prism, causing acceleration of rotation about the center of mass.

6. The model human

MTW modeled the falling human as a large brick: a rigid right rectangular prism of depth $D = 20$ cm, width $W = 20$ cm, height $H = 180$ cm, and mass $M = 7.5 \times 10^3$ g, uniform in composition, that yields—crushes or fractures—if internal pressure or tension reaches 100 atmospheres ($1.01 \times 10^5$ dyn cm$^{-2}$).
Figure 4 shows the prism and the body-fixed axes used to describe positions with respect to the body. The origin of the Euclidean body-fixed coordinate system is the center of mass of the body. The $z$ axis extends through the center of the top of the head. We may imagine that the body faces in the $x$ direction and the $y$-axis points in the left-hand direction.

The $x$, $y$, and $z$ axes are principal axes of inertia with moments of inertia

$$I_1 = 2.05 \times 10^8 \text{ g cm}^2,$$

$$I_2 = 2.05 \times 10^8 \text{ g cm}^2,$$

$$I_3 = 5.0 \times 10^6 \text{ g cm}^2,$$

respectively [25].

7. Rotation, tidal stress, and centripetal stress

An arbitrary orientation of the prism within the tetrad frame can be described in terms of Euler (or Eulerian) angles $\phi$, $\theta$, and $\psi$ [26]. We suppose that the prism’s body-fixed axes $x$, $y$, and $z$ initially coincide with the tetrad-frame axes $x'$, $y'$, and $z'$. The prism and its axes are then rotated right-handedly first through an angle $\phi$ about the $z$ axis, then through an angle $\theta$ about the $x$ axis, then finally through an angle $\psi$ about the $z$ axis, which now points in a different direction within the tetrad frame.

A particle of the prism at position $r \equiv [x, y, z]^T$ in body-frame coordinates is, before Euler rotations, at position $r' \equiv [x', y', z']^T = r$ in tetrad-frame coordinates. After right-
Figure 3. $C_{yx}$ is the acceleration in the $x'$ direction that a free particle would experience per cm of displacement in the $y'$ direction from the origin of the tetrad frame. It equals $C_{yx}'$, the acceleration in the $y'$ direction that a free particle would experience per cm of displacement in the $x'$ direction. $C_{zx}$ is the acceleration in the $x'$ direction per cm of displacement in the $z'$ direction. It equals $C_{zx}'$, $C_{zy}'$ is the acceleration in the $y'$ direction per cm of displacement in the $z'$ direction. It equals $C_{zy}$.

Figure 4. The Misner–Thorne–Wheeler model of a human: a right rectangular prism 180 cm tall, 20 cm wide, and 20 cm deep. The prism has a mass of $7.5 \times 10^4$ g and is uniform in composition. The $x$, $y$, and $z$ axes shown are fixed to the body, which rotates in the tetrad frame. The tetrad-frame axes $x'$, $y'$, and $z'$ are also shown; in this illustration the body-frame axes coincide with the tetrad-frame axes.
handed rotation through $\phi$ about the $z$ axis, then $\theta$ about the $x$ axis, then $\psi$ about the $z$ axis, the tetrad-frame coordinates of the particle are

$$x' = (\cos \phi \cos \psi - \sin \phi \sin \psi \cos \theta)x + (\cos \phi \sin \psi \cos \theta + \sin \phi \cos \psi)y + (\sin \psi \sin \theta)z,$$

$$y' = - (\sin \phi \cos \psi \cos \theta + \cos \phi \sin \psi)x + (\cos \phi \cos \psi \cos \theta - \sin \phi \sin \psi)y + (\cos \psi \sin \theta)z,$$

$$z' = \sin \phi \sin \theta x - \cos \phi \sin \theta y + \cos \theta z.$$  

This may be written as a matrix equation

$$r' = E(\phi, \theta, \psi)r,$$

where $E(\phi, \theta, \psi)$ is the Euler transformation matrix, which depends on $\phi$, $\theta$, and $\psi$.

Figure 5 shows the orientation of the prism and its axes within the tetrad frame after rotations through $\phi = 0$ rad, then $\theta = \pi/2$ rad, and finally $\psi = 0$ rad. The $x$ axis still coincides with the $x'$ axis, but the $y$ axis now coincides with the $z'$ axis, and the $z$ axis points in the $-y'$ direction. I assume this is the orientation of the prism at $\tau = 0$, and I assume rotation rates are zero at $\tau = 0$. Prior and subsequent orientations are obtained by integrating Euler’s equations \cite{26} for the rotation of a rigid body:

$$N_1 = I_1 \omega_1 - (I_2 - I_3) \omega_2 \omega_3,$$

$$N_2 = I_2 \omega_2 - (I_3 - I_1) \omega_3 \omega_1,$$

$$N_3 = I_3 \omega_3 - (I_1 - I_2) \omega_1 \omega_2,$$

where $N_1$, $N_2$, and $N_3$ are the components of the torque $\mathbf{N} = [N_1, N_2, N_3]^T$ about the $x$, $y$, and $z$ axes (in this case applied by tidal stresses), an overdot denotes a derivative with respect to proper time $\tau$, and

$$\omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi,$$

$$\omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi,$$

$$\omega_3 = \dot{\phi} \cos \theta + \dot{\psi}$$

are the components of the instantaneous rotation rate $\omega$ in the $x$, $y$, and $z$ directions.

The torque $\mathbf{N}$ depends on the tidal tensor and the orientation of the prism in the tetrad frame---i.e., the Euler angles:
\[ N = \int_{-H/2}^{H/2} \int_{-W/2}^{W/2} \int_{-D/2}^{D/2} \mathbf{r} \times \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C}\mathbf{E}(\phi, \theta, \psi) \mathbf{r} \ \rho \ \text{d}x \ \text{d}y \ \text{d}z, \]  
\tag{57}

where \( \mathbf{C} \) is the matrix representation of the rank-1 tidal tensor, with \( C_{\beta \gamma}^\alpha \) the element at column \( \alpha' \) and row \( \beta' \), and \( \rho = M/(D \ W \ H) \), not to be confused with the conformal radius \( \rho \) of (3), is the mass density of the prism.

Equation (57) follows from the equation \( \mathbf{N} = \mathbf{r} \times \mathbf{F} \) for the torque \( \mathbf{N} \) exerted by a force \( \mathbf{F} \) acting at a displacement \( \mathbf{r} \) from the origin. In this case, we consider a vanishingly small volume of the prism, of measure \( dV = dx \, dy \, dz \), with mass \( dM = \rho \, dV \), at position \( \mathbf{r} \) in the body frame. The position in the tetrad frame is \( \mathbf{E}(\phi, \theta, \psi) \mathbf{r} \). A free-falling mass at this position would exhibit a tidal acceleration \( \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \mathbf{r} \), implying a force

\[ d\mathbf{F}' = \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \mathbf{r} \ dM. \]  
\tag{58}

In the body frame, the force is

\[ d\mathbf{F} = \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \mathbf{r} \ dM, \]  
\tag{59}

and the torque is

\[ d\mathbf{N} = \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \mathbf{r} \ dM. \]  
\tag{60}

The integral of this vanishingly small contribution to torque over the whole mass, i.e., over the whole volume, of the prism yields (57). The integrand is quadratic in \( x, y, \) and \( z \), hence the integral can be expressed in closed form. The torque components are

\[
N_1 = \frac{M}{12} \left( \left( \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \right)_{3,2} W^2 - \left( \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \right)_{2,3} H^2 \right); \tag{61}
\]

\[
N_2 = \frac{M}{12} \left( \left( \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \right)_{1,3} H^2 - \left( \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \right)_{3,1} D^2 \right); \tag{62}
\]

\[
N_3 = \frac{M}{12} \left( \left( \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \right)_{2,1} D^2 - \left( \mathbf{E}(\phi, \theta, \psi)^{-1} \mathbf{C} \mathbf{E}(\phi, \theta, \psi) \right)_{1,2} W^2 \right). \tag{63}
\]

To obtain the Euler angles, I integrate nonsingular Hamiltonian equations [27–29] formulated in terms of Euler parameters \( \epsilon = [\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3] \), which are normalized to unity \( (\epsilon \cdot \epsilon = 1) \) and related to the Euler angles by

\[
\phi = \tan^{-1} \left( \frac{\epsilon_3}{\epsilon_0} \right) + \tan^{-1} \left( \frac{\epsilon_2}{\epsilon_1} \right), \tag{64}
\]

\[
\theta = 2 \sin^{-1} \left( \sqrt{\epsilon_1^2 + \epsilon_2^2} \right), \tag{65}
\]

and

\[
\psi = \tan^{-1} \left( \frac{\epsilon_3}{\epsilon_0} \right) - \tan^{-1} \left( \frac{\epsilon_2}{\epsilon_1} \right). \tag{66}
\]
After the constraint $\epsilon \cdot \epsilon = 1$ is incorporated, the resulting unconstrained Hamiltonian equations for the rotational dynamics are

\[
\dot{\epsilon}_0 = \left(-\epsilon_1 \omega_1 - \epsilon_2 \omega_2 - \epsilon_3 \omega_3\right) / 2,
\]

\[
\dot{\epsilon}_1 = \left(\epsilon_0 \omega_1 - \epsilon_3 \omega_2 - \epsilon_2 \omega_3\right) / 2,
\]

\[
\dot{\epsilon}_2 = \left(\epsilon_3 \omega_1 + \epsilon_0 \omega_2 - \epsilon_1 \omega_3\right) / 2,
\]

\[
\dot{\epsilon}_3 = \left(-\epsilon_2 \omega_1 + \epsilon_1 \omega_2 + \epsilon_0 \omega_3\right) / 2
\]

and Euler’s equations (51)–(53).

Equations (51)–(53), (67)–(70) are integrated forward and backward in proper time from $\tau = 0$, where the initial conditions $\epsilon = [1/\sqrt{2}, 1/\sqrt{2}, 0, 0]$ and $\omega_1 = \omega_2 = \omega_3 = 0$ are imposed. The Euler angles obtained from the resulting history of $\epsilon$ using (64)–(66), are plotted in figure 6.

Figure 7 shows the first (left) and last (right) frames of a supplementary animation that shows the rotation of the prism in the tetrad frame, starting about three seconds before reaching the radius of the singularity, and ending about three seconds after reaching the radius of the singularity. (Figure 5 showed the orientation of the prism at the radius of the singularity and is the middle frame of the animation.)
In a manner similar to the calculation of torques in (57), we calculate the aggregate tidal force exerted on the top half of the prism in the body frame: 

\[
F \equiv \begin{bmatrix} F_x, F_y, F_z \end{bmatrix}^T = \int_0^{H/2} \int_{-W/2}^{W/2} \int_{-D/2}^{D/2} E(\phi, \theta, \psi)^{-1} \mathbf{CE}(\phi, \theta, \psi) \rho \ dx \ dy \ dz. 
\]

(71)

The integrand is linear in \(x, y, \) and \(z\), so \(F\) may be calculated using a lumped-mass model, i.e., as the tidal force on a point of mass \(M/2\) located at the mean position of the mass, namely \(x = 0, y = 0, \) and \(z = H/4\).

The aggregate tidal force exerted on the bottom half of the prism in the body frame is equal but opposite. If the top and bottom halves of the prism were not attached by cohesion, they would accelerate apart, if \(F_z\) is positive, away from \(z = 0\). The tension that holds the top and bottom halves together must be \(F_z\). If \(F_z\) is negative, then the force is repulsive; the prism is compressed.

The average normal tidal stress over the square where the \(z = 0\) plane bisects the MTW prism is \(\sigma_z = F_z/(WD)\). The variation with proper time is plotted in figure 8. Positive stress is tensile; negative stress is compressive. The greatest tensile and compressive stresses are in the \(z\) direction, because of the orientation assumed at \(\tau = 0\), and because the Euler angles do not change much near \(\tau = 0\) (see figure 6).

The stress \(\sigma_z\) is small until \(\tau \approx -10^{10} \text{ cm}\), a fraction of a second before the prism reaches \(r = 0\). Then \(\sigma_z\) begins to increase, qualitatively as one would expect [1] in the Newtonian approximation or in the Schwarzschild limit \(\alpha = 0\). Then \(\sigma_z\) decreases suddenly and becomes negative, reaching \(\sigma_z \approx -0.22 \text{ atm}\); the prism is compressed along the \(z\) axis. The compression \(-\sigma_z\) plotted is that required by the lower half of the prism to bear the weight of the upper half of the prism.

Figure 8. Average normal tidal stress (atmospheres) in each of three directions (\(x\): forward, \(y\): left, \(z\): up) through a plane bisecting the prism. Positive stress is tensile; negative stress is compressive.
The compression decreases rapidly and becomes tension after $\tau = 0$, reaching a peak of $\sigma_z \approx 0.21$ atm before decreasing toward zero for large negative $r$. Hence the greatest normal tidal stress encountered in this scenario is the compression $-\sigma_z \approx 0.22$ atm just before $\tau = 0$ and just outside $r = 0$. This is much less than the yield stress of 100 atm assumed by MTW. Thus we see that the prism would withstand the tidal stress in this scenario.

The prism, rotating in the inertial tetrad frame, would be subject to centripetal stress in addition to tidal stress. We now calculate the centripetal stress and the combined centripetal stress and tidal stress.

The components $\omega_i$ of the angular velocity of the MTW prism, obtained by integrating Euler’s equations, are plotted in figure 9. The angular velocity component $\omega_3$ about the long ($z$) axis of the prism is much smaller than the other components, hence the rotation is about an axis perpendicular to the $z$ axis, which simplifies the estimation of centripetal stress.

The components $\omega_i$ of the angular velocity of the MTW prism, obtained by integrating Euler’s equations, are plotted in figure 9. The angular velocity component $\omega_3$ about the long ($z$) axis of the prism is much smaller than the other components, hence the rotation is about an axis perpendicular to the $z$ axis, and the centripetal stress is in the $z$ direction.

I estimate the maximum (over position) centripetal stress using the approximation

$$\sigma_z \approx \frac{1}{2} \rho \omega^2 \left( \frac{H}{2} \right)^2$$

for the maximum centripetal stress in a thin rod of uniform composition and cross-section, mass density $\rho$, and length $H$, rotating at angular velocity $\omega$ about an axis perpendicular to the rod and intersecting the rod at the center. The maximum stress occurs at the center of the rod, i.e., the waist of the MTW prism. The angular velocity history for this scenario is plotted in figure 10. The resulting centripetal stress plus tidal stress is shown in figure 11, along with tidal stress alone. The peak centripetal stress occurs at positive $\tau$ and overlaps the peak of tensile tidal stress, producing a peak centripetal stress plus tidal stress of $\sigma_z \approx 0.26$ atm. This, too, is much less than the yield stress assumed by MTW. For a human, the stress would be momentarily uncomfortable, based on the discomfort criterion of Taylor and Wheeler [30].
The calculations above show that an MTW prism falling deep into a Kerr black hole along a particular geodesic would withstand the theoretical tidal and centripetal stress. A prism falling into a real spinning black hole would be subjected to additional stresses and to radiation (e.g., [31]).

8. Discussion

The calculations above show that an MTW prism falling deep into a Kerr black hole along a particular geodesic would withstand the theoretical tidal and centripetal stress. A prism falling into a real spinning black hole would be subjected to additional stresses and to radiation (e.g., [31]).
At the outer horizon, a falling object might [32], or might not [10], encounter a non-classical firewall that destroys it. The debate seems unsettled; see, e.g., [33].

Inside the outer horizon, a prism that approaches the segment of the inner horizon at which the coordinate $u$ blows up would encounter infinitely blueshifted radiation from the external Universe [9, 34]. The prism in this example does not approach that segment of the inner horizon, hence it avoids such radiation as well as the weak, oscillatory, null curvature singularity that is predicted to form there [35–37] as a result of superposition of infalling metric perturbations and outflowing metric perturbations generated by matter that collapsed to form the black hole.

The prism is predicted to approach and cross the other segment of the inner horizon, the segment called the outgoing inner horizon by Marolf and Ori [38]. They predict that an infalling observer approaching the Kerr outgoing inner horizon would experience, to leading order in a linear expansion, a finite change in metric perturbation during a proper time interval that decreases exponentially as the coordinate time $u$ (their $v$) at which the observer falls through the event horizon blows up. In this late-infall time limit, the proper time during which an observer would experience this finite metric change vanishes; the change becomes a step discontinuity in the perturbed metric, which Marolf and Ori call an effective gravitational shock wave. The prism avoids this shock by avoiding late infall. It would nevertheless be stressed to some extent that would depend on the details of collapse and accretion.

The prism would probably encounter infalling matter that collapsed to form the black hole or fell in later. Inward of this matter, the spacetime would not be Kerr, and the validity of this simulation would end, even if the prism does not collide with the matter (for example, if the matter consists of undisrupted stars). This end of validity would probably occur at some $r > 0$ and $\tau < 0$ before some and perhaps before any of the interesting simulated dynamics and stresses occur.

9. Conclusion

I have simulated numerically the fall of the MTW prism into a high-spin ($a/m = 0.998$) supermassive ($m = 4 \times 10^{6}M_{\odot}$) Kerr black hole at a constant, near-equatorial latitude ($\theta = \tan^{-1}(1155/68)$) with enough energy ($E = 7 \mu$) to surmount the radial potential barrier near the throat of the black hole. I have also simulated the rotation of the prism within a co-moving inertial frame of reference, driven by tidal torque, and calculated the tidal and centripetal stresses in the prism assuming a near-worst-case orientation of the prism at zero radius, and zero angular momentum there. The calculated maximum stress is about 0.26 atmospheres, much less than the yield stress assumed by MTW.

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