I. APPROXIMATE GAUSSIAN DISTRIBUTION OF THE ATOMS

In order to show how the Gaussian distribution of the atomic positions arises, we consider here an atom of mass $m$ sitting in a one-dimensional optical trap of frequency $\omega$. The results will be straightforwardly generalizable to the three-dimensional case as the three Cartesian coordinates decouple and can be treated independently. We work in a regime of temperatures $T$ much lower compared to the trap depth, but larger than the trap frequency, i.e., $k_B T \gg \hbar \omega$. These assumptions allow us to treat the trap as a harmonic potential, yielding a Hamiltonian

$$\hat{H}_{\text{trap}} \approx \frac{\hat{p}^2}{2m} + m \frac{1}{2} \omega^2 \hat{x}^2,$$  \hspace{1cm} (S1)

where $\hat{x}$ and $\hat{p}$ are the quantum position and momentum operators, respectively. The thermal state of the system is described by the Gibbs form

$$\rho_{\text{th}} = \frac{1}{Z} e^{-\beta \hat{H}_{\text{trap}}},$$ \hspace{1cm} (S2)

where $\beta = 1/(k_B T)$ and $Z$ is the partition function

$$Z = \text{tr} \left\{ e^{-\beta \hat{H}_{\text{trap}}} \right\}.$$ \hspace{1cm} (S3)
Employing the standard mapping
\[ \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} (\hat{b}^\dagger - \hat{b}), \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{b}^\dagger + \hat{b}) \]  

(S4)
in terms of bosonic creation (\(\hat{b}^\dagger\)) and annihilation (\(\hat{b}\)) operators \([\hat{b}, \hat{b}^\dagger] = 1\), one readily obtains

\[ \hat{H}_{\text{trap}} = \hbar \omega \left( \hat{b}^\dagger \hat{b} + \frac{1}{2} \right) \]  

(S5a)
and

\[ Z = \sum_n e^{-\beta\hbar\omega(n+1/2)} = \frac{1}{2\sinh \left( \frac{\beta\hbar\omega}{2} \right)}. \]  

(S5b)

Calling \(|x\rangle\) the position eigenvector \(\hat{x}|x\rangle = x|x\rangle\), the probability density functions of the atomic position is defined as

\[ p_{\text{pos}}(x) = \langle x|\hat{p}_{\text{th}}|x\rangle. \]  

(S6)

Its analytical form can be extracted from the Feynman propagator for the harmonic oscillator \(K(x, y, t) = \langle x|e^{-i\hat{H}_{\text{trap}}/\hbar}|y\rangle\), which, in the time interval \(t \in (0, \pi/\omega)\), reads (see, e.g., [1, 2] for detailed derivations)

\[ K(x, y, t) = \sqrt{\frac{m\omega}{2\pi\hbar|\sin (\omega t)|}} \times \]  

\[ \times \exp \left\{ -\frac{m\omega}{2\hbar \sin (\omega t)} [(x^2 + y^2) \cos (\omega t) - 2xy] \right\}. \]  

(S7)
Substituting \(t \rightarrow -i\beta \hbar\) and \(y \rightarrow x\) one finds

\[ K(x, x, -i\beta) = \langle x|e^{-i\hat{H}_{\text{trap}}/\hbar}|x\rangle = \sqrt{\frac{m\omega}{2\pi\hbar \sinh (\omega\beta\hbar)}} \times \]  

\[ \times \exp \left\{ -\frac{m\omega}{\hbar \sinh (\omega\beta\hbar)} (\cosh (\omega\beta\hbar) - 1) x^2 \right\}. \]  

(S8)
Dividing by the partition function (S3) one finally finds the Gaussian

\[ p_{\text{pos}}(x) = \sqrt{\frac{m\omega (\cosh (\omega\beta\hbar) - 1)}{\pi\hbar \sinh (\omega\beta\hbar)}} \times \]  

\[ \times \exp \left\{ -\frac{m\omega}{\hbar \sinh (\omega\beta\hbar)} (\cosh (\omega\beta\hbar) - 1) x^2 \right\}. \]  

(S9)
The variance \(\sigma\) can be read off directly and amounts to

\[ \sigma^2 = \frac{\hbar \sinh (\omega\beta\hbar)}{2m\omega (\cosh (\omega\beta\hbar) - 1)}. \]  

(S10)
Since we assumed \(k_B T \gg \hbar \omega\), i.e., \(\omega\beta\hbar \ll 1\), we can expand this expression to lowest order, which yields

\[ \sigma^2 = \frac{1}{m\omega^2 \beta} = \frac{k_B T}{m\omega^2}, \]  

(S11)
as reported in the main text.

The distribution (S9) is straightforwardly generalized to three-dimensions and traps centered along a single linear chain at positions \(k\mathbf{R}_0 = (0, 0, k\mathbf{R}_0)\) with \(k\) an integer:

\[ p^{(k)}_{\text{pos}}(r) = \frac{1}{(2\pi)^{3/2} \sigma^3} \exp \left\{ -\frac{r_x^2}{2\sigma^2} - \frac{r_y^2}{2\sigma^2} - \frac{(r_z - (k-1)R_0)^2}{2\sigma^2} \right\}. \]  

(S12)
For clarity, we remark here that the indices in the expression above distinguish between Cartesian components only, e.g., \(r_1\) and \(r_2\) are the components of the same atom along the \(x\) and \(y\) directions. In the following, when necessary the trap index will always appear before the component one, e.g., \(r_{k,i}\) is the \(i\)-th component of the \(k\)-th atom’s position. For a ladder, a second set of position distributions \(p^{(k)}_{\text{pos}}(\mathbf{r})\) would be added with the same Gaussian form up to \(r_2 \rightarrow r_2 - R_0\).
II. DISTRIBUTION OF THE DISTANCES AND INTERACTIONS FOR A SINGLE CHAIN.

Here we focus on a single one-dimensional chain as most of the properties which affect the results in the main text are due to the presence of an extended longitudinal direction. Still, the considerations made for the marginal distributions for pairs of atoms directly apply to any regular lattice configuration as well. The distribution of the differences $d_k = r_{k+1} - r_k = (d_{k,1}, d_{k,2}, d_{k,3})$ can be found in the Supplemental Material of Ref. [3] and, for isotropic traps, reads

$$p_{\text{diff}}(d_1, \ldots, d_{L-1}) = \left[ \frac{\sigma^1 - L}{2\sqrt{L} (\sqrt{2\pi})^{L-1}} \right]^3 \exp \left( -\frac{1}{2\sigma^2} \sum_{k=1}^{L-1} \left[ d_{k,1}^2 + d_{k,2}^2 + d_{k,3}^2 + (d_{k,3} - R_0)^2 A_{kq} (d_{q,3} - R_0) \right] \right),$$  

where $A_{kq} = L - \max(k,q) - (L-k)(L-q)/L = (L - \max(k, q)) \min(k, q)/L$. The correlations between different components $d_{k,i}$ can be worked out via the inverse [4]

$$C = A^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

implying,

$$\langle d_{k,i} d_{q,j} \rangle - \langle d_{k,i} \rangle \langle d_{q,j} \rangle = \sigma^2 \delta_{ij} (2\delta_{k,q} - \delta_{k,q+1} - \delta_{k,q-1}).$$

Subsequent distances are therefore (anti-)correlated, and these correlations pass onto any (non-trivial) function of the distances, and in particular the energy displacements $\delta V_k = V(d_k) - V(R_0)$.

As a consistency check, we remark that $C(L)$ is a $(L-1) \times (L-1)$ matrix, whose determinant satisfies the recursion relation

$$\det C(L) = 2 \det C(L-1) - \det C(L-2)$$

with “seed” (or “initial conditions”) $\det C(2) = 2$ and $\det C(3) = 3$, which is solved by $\det C(L) = L$. Consequently, the factor $\left[ \sqrt{\det A} \right]^3$ produced by the Gaussian integration over all variables exactly cancels the $L^{-3/2}$ appearing in the normalization factor, as expected.

A. Marginal distribution for a single pair of atoms

The $d_k$s are identically distributed, so we can select any given one (and drop its index for brevity) and integrate over the other $L - 2$ variables from equation (S13). This yields

$$p_{\text{diff}}(d) = \frac{1}{(4\pi)^{3/2} \sigma^3} \exp \left( -\frac{1}{4\sigma^2} [d_1^2 + d_2^2 + (ds - R_0)^2] \right) = \frac{1}{(4\pi)^{3/2} \sigma^3} \exp \left( -\frac{1}{\sigma^2} [d_1^2 + d_2^2 + 2ds - 2R_0] \right),$$

where $d = |d|$ denotes the distance between a pair of neighboring atoms. The distribution for this new variable can be then obtained via a solid angle integration and reads

$$p_{\text{dist}}(d) = \frac{d^2}{4(\pi)^{1/2} \sigma^3} \exp \left( -\frac{1}{\sigma^2} (d^2 + R_0^2) \right) \int_0^{\pi/2} d\theta \sin \theta e^{-\frac{d R_0 \cos \theta}{2\sigma^2}} = \frac{d}{\sqrt{\pi \sigma R_0}} \exp \left( -\frac{1}{\sigma^2} (d^2 + R_0^2) \right) \sinh \left( \frac{d R_0}{2\sigma^2} \right).$$
The distribution of an energy shift $\delta V$ can be now found via a change of variables ($d \to d'(\delta V)$), according to

$$P(\delta V) = |d'(\delta V)| p_{\text{dist}}(d(\delta V)),$$

For the sake of generality, we keep $\alpha$ generic in

$$d(\delta V) = \left( \frac{C_\alpha}{V_0 + \delta V} \right)^{\frac{1}{\alpha}}, \quad (S19)$$

where $V_0 = C_\alpha / R_0^\alpha$, which implies

$$d'(\delta V) = -\frac{1}{\alpha} C_\alpha \frac{1}{(V_0 + \delta V)^{1+1/\alpha}}. \quad (S20)$$

Hence, the distribution of energy shifts for a pair is

$$P(\delta V | V_0, R_0, \sigma) = \frac{R_0}{\sigma} \left( \frac{1}{\alpha \sqrt{\pi} V_0} \right)^{1+\frac{\alpha}{2}} \times$$

$$\times e^{-\frac{R_0^2}{2\sigma^2} \left( 1 + \frac{\delta V}{V_0} \right)^{-\frac{1}{\alpha}}} \times$$

$$\times \sinh \left[ \frac{R_0^2}{2\sigma^2} \left( 1 + \frac{\delta V}{V_0} \right)^{-\frac{1}{\alpha}} \right]. \quad (S21)$$

It is relatively simple to see that, if we define the dimensionless quantities $\delta v = \delta V / V_0$ and $s = \sigma / R_0$, we can simplify this expression further:

$$P(\delta v | s) = \frac{1}{\alpha \sqrt{\pi} s} \left( 1 + \delta v \right)^{1+\frac{\alpha}{2}} \times$$

$$\times e^{-\frac{1}{4s^2} \left[ 1 + (1 + \delta v)^{-\frac{1}{\alpha}} \right]} \times$$

$$\times \sinh \left[ \frac{1}{2s^2} \left( 1 + \delta v \right)^{-\frac{1}{\alpha}} \right]. \quad (S22)$$

The probability distribution function in Eq. (S22) is defined in the domain $\delta v \in [-1, +\infty)$; for $\delta v = -1 + \varepsilon$, in the limit $\varepsilon \to 0^+$ it behaves as

$$P(\delta v | s) \propto \varepsilon^{-1-\frac{\alpha}{2}} e^{-\frac{1}{4s^2} \varepsilon^{-\frac{1}{\alpha}}} \sinh \left[ \frac{\varepsilon^{-\frac{1}{\alpha}}}{2s^2} \right] \to 0, \quad (S23)$$

as the (vanishing) exponential factor dominates. In the opposite limit $\delta v \to \infty$, instead, the distribution behaves asymptotically as

$$P(\delta v | s) \approx \frac{1}{2\alpha \sqrt{\pi} s^3} e^{-\frac{1}{4s^2} \delta v^{-1-3/\alpha}}. \quad (S24)$$

This shows that this distribution is fat-tailed. In particular, all the distribution moments $\langle \delta v^\beta \rangle$ with $\beta \geq 3/\alpha$ are not defined and, for both $\alpha = 3$ (dipole-dipole interactions) and $\alpha = 6$ (van der Waals), this includes all integer moments (e.g., the mean and variance). These fat tails are the consequence of the approximation of an atom’s position distribution as a Gaussian everywhere in space, i.e., including points much further away from the center of a trap than a few $\sigma$s. In other words, it appears to be an artifact of the description, rather than something occurring in a real experiment. The result of this approximation is to allow for an extremely small (but not vanishing) probability that two atoms can be arbitrarily close, which, due to the algebraic scaling of the interactions, produces considerable energy shifts. Moments like the mean and variance are therefore dominated by the very rare events in which two atoms lie very close to each other. The rarity of such events is encoded in the exponential suppression $e^{-(1/4s^2)}$ in Eq. (S24). In principle, these unphysical fat tails could affect our results, as it is known that, in the Anderson problem, the scaling of the localization length is modified when Cauchy-like distributions are chosen instead of more well-behaved ones. However, as mentioned above, the fat tails in our case are strongly suppressed and one needs to assess how likely it is to actually probe them in a simulation or an experiment. For that purpose, let us first notice
that the asymptotic behavior reported in (S24) emerges when the argument of the sinh function in Eq. (S22) is small, i.e., still assuming \( \delta v \gg 1 \), for

\[
\delta v \gg (2s^2)^{-\alpha}.
\]  

(S25)

The probability of generating an energy shift within the tails is

\[
P_s \equiv P\left(\delta v > (2s^2)^{-\alpha}\right) = \int_{(2s^2)^{-\alpha}}^{\infty} d\delta v P(\delta v | s).
\]  

(S26)

Employing now the asymptotic expression (S24) we obtain

\[
P_s = P\left(\delta v > (2s^2)^{-\alpha}\right) \approx \frac{4s^3}{3\sqrt{\pi}} e^{-\frac{1}{4s^2}}.
\]  

(S27)

This result apparently does not depend on \( \alpha \), but we need to remember that the derivation assumes \( \delta v \gg 1 \), and is therefore only consistent if \( (2s^2)^{-\alpha} \gg 1 \). Considering \( \alpha = 3 \) or 6, though, this is satisfied already for rather large disorder amplitudes, e.g., \( s = 0.3 \), which then yields \( P_{s,0,1} \approx 0.0013 \). Due to the exponential factor, these probabilities decrease very fast with \( s \). For \( s = 0.1 \), for instance, we get \( P_{0.1} \approx 10^{-14} \) and in the range spanned in the plots reported in the main text \( s \leq 5 \times 10^{-4} \) this becomes \( P_s < 10^{-200000} \), which is clearly impossible to observe in any reasonable experiment or numerical procedure. Hence, we can safely assume the unphysical fat tails to be completely irrelevant in the determination of our numerical results in the regime considered.

## III. HILBERT SPACE REDUCTIONS AND RESTRICTED HAMILTONIANS

The Hamiltonian introduced in the main text reads

\[
\hat{H} = \Omega \sum_{k=1}^{N} \hat{\sigma}_x^{(k)} + \Delta \sum_{k=1}^{N} \hat{n}_k + \sum_{k=1}^{N} \frac{1}{2} V(d_{km}) \hat{n}_m \hat{n}_k,
\]  

(S28)

where \( d_{km} \) denotes the distance between the \( k \)-th and \( m \)-th atoms. In order to exploit the large energy separations present in the system, we switch to the interaction picture

\[
\hat{H}_I(t) = e^{i\hat{H}_0 t} \hat{H}_I e^{-i\hat{H}_0 t} = \Omega \sum_k e^{i\hat{H}_0 t} \hat{\sigma}_x^{(k)} e^{-i\hat{H}_0 t}.
\]  

(S29)

Recalling that \( [\hat{\sigma}_x^{(k)}, \hat{n}_m] = 0 \) for every \( k \neq m \) and that \( \hat{\sigma}_x^{(k)} \hat{n}_k = (1 - \hat{n}_k) \hat{\sigma}_x^{(k)} \) we can simplify the \( k \)-th addend in Eq. (S29)

\[
e^{i\hat{H}_0 t} \hat{\sigma}_x^{(k)} e^{-i\hat{H}_0 t} = e^{i\hat{n}_k (\Delta + \sum_{m \neq k} V(d_{km}) \hat{n}_m)} \hat{\sigma}_x^{(k)}
\]

\[
\times e^{-i\hat{n}_k (\Delta + \sum_{m \neq k} V(d_{km}) \hat{n}_m)} = e^{i[2\hat{n}_k - 1] (\Delta + \sum_{m \neq k} V(d_{km}) \hat{n}_m) \hat{\sigma}_x^{(k)}},
\]  

(S30)

where in the first equality we singled out in the exponentials all the terms which depend upon \( \hat{n}_k \); all the remaining ones cancel out. The Hamiltonian \( \hat{H}_I \) can then be written as

\[
\hat{H}_I(t) = \Omega \sum_k e^{i[2\hat{n}_k - 1] (\Delta + \sum_{m \neq k} V(d_{km}) \hat{n}_m) \hat{\sigma}_x^{(k)}}.
\]  

(S31)

We apply a rotating-wave approximation to discard all terms which oscillate fast in time. This implies that the oscillation frequency \( \nu \) should be \( \gg \Omega \) for a term to be neglected. Note that the frequency \( \nu \) is however operator-valued:

\[
\omega = (2\hat{n}_k - 1) (\Delta + \sum_{m \neq k} V(d_{km}) \hat{n}_m).
\]  

(S32)
Since the prefactor $-1 \leq 2\hat{n}_k - 1 \leq 1$ is of order $O(1)$, it is the second factor which is decisive for the selection. We introduce now for every site $k$ a projector $\hat{P}_k$ over all states where there is a single excitation among the neighbors of $k$ and no additional one within a radius $2R_0$. Its specific structure depends clearly on the structure of the lattice, but if we define by $F_k$ the set of nearest-neighboring sites of $k$ and by $S_k$ the set of sites within a distance $2R_0$ from $k$ which are neither site $k$ itself nor one of the sites in $F_k$ (i.e., $k \notin S_k$, $F_k \cap S_k = \emptyset$), then we can give an implicit definition according to
\[
\hat{P}_k = \sum_{q \in F_k} \hat{n}_q \prod_{q' \in F_k, q' \neq q} (1 - \hat{n}_{q'}) \prod_{q'' \in S_k} (1 - \hat{n}_{q''}).
\] (S33)

Checking that the expression above satisfies $(\hat{P}_k)^2 = \hat{P}_k$ is straightforward if one recalls that $\hat{n}_q^2 = \hat{n}_q$ and $(1 - \hat{n}_q)^2 = 1 - \hat{n}_q \forall q$. The relevance of the projector $\hat{P}_k$ is that it precisely identifies the constraints – identified in the main text – under which a spin (or atom) is able to flip (or being excited/de-excited). Slightly more formally,
\[
(\Delta + \sum_{m \neq k} V(d_{km})\hat{n}_m)\hat{P}_k \approx (\Delta - V(R_0))\hat{P}_k = 0,
\] (S34)
where we have neglected all contributions from excitations beyond a distance of $2R_0$. Furthermore, note that according to definition (S33) $\hat{P}_k$ acts trivially on site $k$ and thus commutes with all local operators which instead exclusively act on that site; in particular, $\left[\hat{\sigma}^{(k)}_x, \hat{P}_k \right] = 0$. Defining for brevity $\hat{Q}_k = 1 - \hat{P}_k$ the projector onto the orthogonal subspace ($\hat{Q}_k^2 = \hat{Q}_k$, $\hat{Q}_k\hat{P}_k = 0$) we thus have
\[
\hat{\sigma}^{(k)}_x = \left(\hat{P}_k + \hat{Q}_k\right)\hat{\sigma}^{(k)}_x \left(\hat{P}_k + \hat{Q}_k\right) = \hat{P}_k\hat{\sigma}^{(k)}_x\hat{P}_k + \hat{Q}_k\hat{\sigma}^{(k)}_x\hat{P}_k + \hat{Q}_k\hat{\sigma}^{(k)}_x\hat{P}_k + \hat{Q}_k\hat{\sigma}^{(k)}_x\hat{Q}_k = \hat{P}_k\hat{\sigma}^{(k)}_x + \hat{Q}_k\hat{\sigma}^{(k)}_x.
\] (S35)

Hence, we can separate the interaction Hamiltonian $\hat{H}_I$ into two contributions:
\[
\hat{H}_I(t) \approx \Omega \sum_k \hat{P}_k\hat{\sigma}^{(k)}_x + e^{i(2\hat{n}_k-1)(\Delta + \sum_{m \neq k} V(d_{km})\hat{n}_m)} \hat{Q}_k\hat{\sigma}^{(k)}_x.
\] (S36)

The space of configurations onto which $\hat{Q}_k$ has support can be further split into three classes:

(A) States where site $k$ has two or more excited nearest neighbors;

(B) States where site $k$ has only one excited neighbor, but there is at least another excitation within a radius $2R_0$;

(C) States where no neighbors of $k$ are excited.

In case (A) the interaction potential on site $k$ is $\geq 2V(R_0)$; accounting for the facilitation condition $\Delta = -V(R_0)$ we find $\nu \gtrsim V(R_0) \gg \Omega$; these terms are thereby oscillating very fast and can be discarded. Terms of type (B) are facilitated by the single neighboring excitation, but the presence of an additional one within a distance $2R_0$ implies that
\[
\Delta + \sum_{m \neq k} V(d_{km})\hat{n}_m \geq V(2R_0)
\] (S37)
and therefore $\nu \gtrsim V(2R_0) \gg \Omega$, which allows us to neglect all type-(B) contributions as well. Terms belonging to class (C) are instead more delicate, since an appropriate combination of the interactions with many excitations at different distances could approximately cancel out the detuning $\Delta$. For instance, for dipole-dipole interactions ($\alpha = 3$) the potential obeys $V(\gamma R_0) = V(R_0)\gamma^{-3}$; considering a honeycomb lattice with 5 excited next-nearest neighbors at distance $R_3 = \sqrt{3}R_0$ and a single excited next-next-next-next-nearest (or fourth-nearest for brevity) neighbor at distance $R_4 = 3R_0$ one finds
\[
\Delta + \sum_{m \notin F_k} V(d_{km}) \rightarrow -V(R_0) + 5V(R_1) + V(R_4) = \approx -0.00071V(R_0).
\] (S38)
However, configurations such as this one always require a large local density of excitations, and hence can only affect Hilbert subspaces at higher energies than the ones considered in the main text, separated at least by some factors of $V(R_1) \gg \Omega$. As long as we consider the low-energy Hilbert subspaces, it is thus fine to neglect terms of type (C) as well. Overall, in the subspaces we are interested in we can approximate

$$\hat{H}_I(t) \approx \Omega \sum_k \hat{P}_k \hat{\sigma}_x^{(k)}.$$  \hfill (S39)

Going back to the original Schrödinger representation is now straightforward and yields

$$\hat{H} \approx \Omega \sum_k \hat{P}_k \hat{\sigma}_x^{(k)} + \Delta \sum_k \hat{n}_k + \sum_{m=1}^N \frac{1}{2} V(d_{km}) \hat{n}_m \hat{n}_k.$$  \hfill (S40)

Note that in the specific subspace (let us call it $H_1$) considered in the main text, the one including all possible one-excitation states plus all possible pairs of neighboring ones, the diagonal part $\hat{H}_0$ acts trivially as the null operator and can thus be discarded, implying

$$\hat{H}_{H_1} = \Omega \sum_k \hat{P}_k \hat{\sigma}_x^{(k)}.$$  \hfill (S41)

We remark that the same derivation can be followed in the presence of weak disorder by changing the definition of $\hat{H}_1$ in Eq. (S28) to

$$\hat{H}_1 = \Omega \sum_k \hat{\sigma}_x^{(k)} + \frac{1}{2} \sum_{k \neq q} \delta V(d_{kq}) \hat{n}_k \hat{n}_q.$$  \hfill (S42)

Since the second term is diagonal and commutes with $\hat{H}_0$, the calculation of the interaction picture is straightforward:

$$\hat{H}_I(t) = \Omega \sum_k e^{i t (2 \hat{n}_k - 1)} (\Delta + \sum_{m \neq k} V(d_{km}) \hat{n}_m) \hat{\sigma}_x^{(k)} + \frac{1}{2} \sum_{k \neq q} \delta V(d_{kq}) \hat{n}_k \hat{n}_q$$  \hfill (S43)

and one can follow the same steps outlined above.

### A. Hilbert space lattice structure

Having derived the restricted Hamiltonian (S41) we can now identify the geometric structure of the Hilbert space in the basis of eigenstates of $\hat{\sigma}_z^{(k)}$. To start with, we introduce the following definitions for the basis itself: we call $|M_k\rangle$ states with a single excitation present on site $k$, whereas we denote by $|N_{kq}\rangle$ states with a pair of excitations on sites $k$ and $q$. Fixing the number $N$ of tweezers, the Hilbert subspace we work in is therefore defined as

$$H_1 = \text{Span} \{ |M_k\rangle, |N_{kq}\rangle \mid k = 1, \ldots, N; q \in \mathcal{F}_k \},$$  \hfill (S44)

where we recall that $\mathcal{F}_k$ is the set of nearest neighbors of site $k$. Note that, since $|N_{kq}\rangle = |N_{qk}\rangle$ the pair states are doubly counted; however, this clearly still leads to the generation of the same vector space. Alternatively, one can also define an equivalence relation $|N_{kq}\rangle \sim |N_{ml}\rangle \iff (k = m \land q = l) \lor (k = l \land q = m)$ and take the quotient of the r.h.s. above. In the following, it is understood that the states $|N_{kq}\rangle$ are always taken from this space, i.e., we shall never consider states with two isolated excitations at distance $d > R_0$.

By construction, $\hat{H}_{H_1}, \hat{H}_1 \subseteq H_1$. Furthermore, we know that the action of $\hat{P}_k \hat{\sigma}_x^{(k)}$ is to flip the spin in site $k$ conditioned on the presence of a single excitation in $\mathcal{F}_k$ and no additional one in $\mathcal{S}_k$. This implies

$$\hat{P}_k \hat{\sigma}_x^{(k)} |M_l\rangle = \begin{cases} 0 & \text{if } l = k, \\ |N_{kl}\rangle & \text{if } l \in \mathcal{F}_k, \\ 0 & \text{otherwise}. \end{cases}$$  \hfill (S45)
Considering that \( l \in \mathcal{F}_k \iff k \in \mathcal{F}_l \), one can see that
\[
\hat{H}_{\mathcal{F}_k} |M_l\rangle = \Omega \sum_{k \in \mathcal{F}_l} |N_{kl}\rangle. \tag{S46}
\]
Similarly,
\[
\hat{P}_k \hat{\sigma}_z^{(k)} |N_{ql}\rangle = \begin{cases} |M_l\rangle & \text{if } q = k, \\ |M_q\rangle & \text{if } l = k, \\ 0 & \text{otherwise}, \end{cases} \tag{S47}
\]
since by construction the only facilitated spins are in sites \( q \) and \( l \). Hence,
\[
\hat{H}_{\mathcal{F}_k} |N_{ql}\rangle = \Omega (|M_q\rangle + |M_l\rangle). \tag{S48}
\]
Collecting these considerations, we can find the Hamiltonian matrix elements:
\[
\langle M_q | \hat{H}_{\mathcal{F}_l} | M_k \rangle = 0 \tag{S49a}
\]
\[
\langle N_{ml} | \hat{H}_{\mathcal{F}_l} | N_{kq} \rangle = 0 \tag{S49b}
\]
\[
\langle N_{ml} | \hat{H}_{\mathcal{F}_l} | M_k \rangle = \begin{cases} \Omega & \text{if } l = k, \\ \Omega & \text{if } m = k, \\ 0 & \text{otherwise}. \end{cases} \tag{S49c}
\]

Now, there are as many states \( |M_k\rangle \) as there are sites, so it is natural to make a connection: starting from the real-space geometry of the tweezer array, which defines the original lattice structure, we place for visual aid each state \( |M_k\rangle \) on the corresponding site \( k \). Crucially, each pair state \( |N_{kq}\rangle \) is exclusively connected (via the Hamiltonian) to the two one-excitation states \( |M_k\rangle \) and \( |M_q\rangle \), so it is placed as a mid-point between sites \( k \) and \( q \), changing the structure to a generalized Lieb lattice. Now, by drawing a link between any pair of sites every time the corresponding states yield a non-zero Hamiltonian matrix element one precisely reconstructs the kind of lattices we displayed in Fig. 1 in the main text.

**IV. BOUND ON THE NUMBER OF FLAT BANDS**

We provide an account of the lower bound of the number of flat bands \( n_{\text{flat}} \geq |n_1 - n_2| \) mentioned in the main text. We recall that \( n_{\text{flat}} \) denotes the number of flat bands in the model, \( n_1 \) the number of one-particle states per unit cell and \( n_2 \) the corresponding number of pair states per unit cell. Before doing that, however, we briefly comment on the fact that the spectrum of the hopping Hamiltonians (S41) is always symmetric with respect to \( \epsilon = 0 \). In fact, one can define the parity transformation
\[
\hat{U} = \hat{U}^\dagger = (-1)^{\sum_k \hat{n}_k} \tag{S50}
\]
which, in the subspace \( \mathcal{H}_1 \), acts according to \( \hat{U} |M_k\rangle = - |M_k\rangle \) on all one-excitation states and \( \hat{U} |N_{kq}\rangle = |N_{kq}\rangle \) on all pair states. Combined with Eqs. (S49a)-(S49c), this implies \( \hat{U}^\dagger \hat{H}_{\mathcal{F}_l} \hat{U} = -\hat{H}_{\mathcal{F}_l} \). Hence, if \( |\epsilon\rangle \) is an eigenvector of the Hamiltonian at energy \( \epsilon \), then \( \hat{U} |\epsilon\rangle \) is also an eigenvector, but at eigenvalue \( -\epsilon \), proving the symmetry of the spectrum under reflection \( \epsilon \to -\epsilon \).

We start directly from the synthetic lattice reconstructed in the Hilbert space according to the procedure described in the previous section. This structure is not in general a Bravais lattice and needs, as a first step, to be reduced to one by identifying an appropriate “basis”. This is a standard procedure in crystallography and solid state physics and we refer the reader to any good introductory textbook (see e.g., [5]). For the reader’s convenience, we however recall here just a few of the most basic concepts: a Bravais lattice is a lattice structure where the positions \( \vec{l} \) of the lattice sites can be written as discrete combinations
\[
\vec{l} = \sum_{i=1}^d z_i \vec{a}_i \quad \text{with} \quad z_i \in \mathbb{Z}. \tag{S51}
\]
of a set of \( d \) linearly-independent primitive lattice vectors \( \vec{a}_i \) \((i = 1 \ldots d)\), where \( d \) is the dimensionality of the system. If a site is located at the origin, all sites can be found this way and all points at positions \( \vec{l} \) are lattice sites. Any
lattice is, by definition, a periodically repeating pattern, and is therefore invariant under a certain set of translations by \( \vec{l} \) for some specific choice of the primitive lattice vectors. However, in many cases an additional set of \( B \) vectors \( \{ \vec{b}_1, \ldots, \vec{b}_B \} \), called “basis”, is required. In such cases, and fixing conventionally \( \vec{b}_1 = 0 \) which can be done without loss of generality, if one lattice point is located at the origin, every point at a position \( \vec{l} \) is also a lattice site, but not all lattice sites are at positions \( \vec{l} \). All of them are instead found at positions \( \vec{l} + \vec{b}_j \) with \( j = 1, \ldots, B \). We also remark that distances between sites in the synthetic lattice are not meaningful, being just a convenient way to visualize the structure of the Hilbert space. Hence, we are free to rescale the length of all (dimensionless) vectors \( \vec{a}_i, \vec{b}_j \) by a common factor. In all the examples discussed below the primitive lattice vectors have the same length and we shall choose to normalize them to unit length (\( |\vec{a}_i| = 1 \)). Also, for brevity in the following we refer to the \( \mathbb{R}^d \) space where these vectors live as the direct space.

We also introduce the reciprocal lattice vectors \( \vec{a}_i^* \) which satisfy the defining relations
\[
\vec{a}_i^* \cdot \vec{a}_j = 2\pi \delta_{ij}. \tag{S52}
\]
The reciprocal Bravais lattice is then reconstructed by taking integer combinations of these vectors, i.e.,
\[
\vec{G} = \sum_{i=1}^d z_i^* \vec{a}_i^* \quad \text{with} \quad z_i^* \in \mathbb{Z}. \tag{S53}
\]
We define a unit cell \( U^* \) which contains only one reciprocal lattice point. All possible translations \( \vec{G} \) of \( U^* \) cover the whole space \( \mathbb{R}^d \) without any overlaps. It can be visualized as a tessellation with \( U^* \) a tile. From a slightly different (but equivalent) perspective, one can define the equivalence relation between vectors \( \vec{k}, \vec{q} \in \mathbb{R}^d \) living in reciprocal space
\[
\vec{k} \sim \vec{q} \Leftrightarrow \exists \vec{G} | \vec{k} = \vec{q} + \vec{G} \tag{S54}
\]
with \( \vec{G} \) a reciprocal lattice vector. Hence, the unit cell may be defined as a quotient \( \mathbb{R}^d / \sim \). By defining quasi-momenta \( \vec{k} \) as reciprocal space vectors belonging to a unit cell \( U^* \), one can define a Fourier series in the usual way for any generic quantity \( A_{\vec{l}} \) living on the direct-space Bravais lattice
\[
\tilde{A}_{\vec{k}} = \sum_{\vec{l}} e^{-i\vec{k} \cdot \vec{l}} A_{\vec{l}}, \tag{S55}
\]
The corresponding inverse transform is also standard:
\[
A_{\vec{l}} = \int_{U^*} \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{l}} \tilde{A}_{\vec{k}}, \tag{S56}
\]
as can be shown remembering that
\[
\frac{\vec{G} \cdot \vec{l}}{2\pi} \in \mathbb{Z} \quad \tag{S57}
\]
and using the Poisson-summation-derived distributional identity
\[
\sum_{z \in \mathbb{Z}} e^{-i\alpha z} = \sum_{m \in \mathbb{Z}} 2\pi \delta(\alpha + 2\pi m), \tag{S58}
\]
with \( \alpha \in \mathbb{R} \) and \( \delta \) the Dirac delta. The choice of the unit cell is not unique; in the following we assume to be working in the first Brillouin zone \( B \) [5].

Clearly, the definitions above do not hinge upon working in a specific space and, indeed, one can analogously define a unit cell in direct space which contains a single Bravais lattice point. Hence, such a unit cell includes \( B \) synthetic lattice points. It is quite natural to subdivide them according to whether they are of the “one-excitation” or “pair” kind. As done in the main text, we define \( n_1 \) the number of one-excitation states in a unit cell and \( n_2 = B - n_1 \) the number of pair ones. For example,

- Synthetic square lattice (Lieb lattice): \( n_1 = 1, n_2 = 2, B = 3 \).
- Synthetic triangular lattice: \( n_1 = 1, n_2 = 3, B = 4 \).
• Synthetic honeycomb lattice: \( n_1 = 2, n_2 = 3, B = 5 \).

Since each synthetic lattice point can be uniquely associated to a given primitive lattice vector \( \vec{l} \) and basis vector \( \vec{b}_i \), we can unambiguously denote each state in the Hilbert subspace \( \mathcal{H}_1 \) as a tensor product \( |\vec{l}\rangle \otimes |\vec{b}_i\rangle \). For later convenience, we introduce now a new notation distinguishing between the basis vectors identifying one-excitation states \( (|\vec{b}_i\rangle \rightarrow |\mu_j\rangle, j = 1, \ldots, n_1) \) and pair states \( (|\vec{b}_i\rangle \rightarrow |\nu_j\rangle, j = 1, \ldots, n_2) \), so that the space of basis states is equivalently generated as

\[
\text{Span}\ \{ |\mu_1\rangle, \ldots, |\mu_{n_1}\rangle, |\nu_1\rangle, \ldots, |\nu_{n_2}\rangle \}.
\] (S59)

Consequently, there is a bijective correspondence between states \( |M_k\rangle \) and states \( |\vec{l}\rangle \otimes |\mu_i\rangle \) and between states \( |N_{kq}\rangle \) and states \( |\vec{l}\rangle \otimes |\nu_i\rangle \).

We also define the lattice translation operator \( T_{\vec{j}} \), where \( \vec{j} \) is a Bravais lattice vector, which acts on the positional degrees of freedom according to

\[
T_{\vec{j}} |\vec{l}\rangle = |\vec{l} + \vec{j}\rangle.
\] (S60)

By the straightforward quasi-momentum states definition

\[
|\vec{k}\rangle = \sum_{\vec{l}} \text{e}^{-i\vec{k} \cdot \vec{l}} |\vec{l}\rangle
\] (S61)

one also gets

\[
T_{\vec{j}} |k\rangle = \text{e}^{i\vec{k} \cdot \vec{j}} |\vec{k}\rangle.
\] (S62)

The Hamiltonian can now be generically characterized as a sum of terms

\[
\hat{H}_{\mathcal{H}_1} = \Omega \sum_{\vec{l}} \sum_{\vec{j} \neq 0} \sum_{m=1}^{n_1} \sum_{n=1}^{n_2} \left( C_{\vec{j},m,n} |\mu_m\rangle \langle \nu_n| + D_{\vec{j},m,n} |\nu_n\rangle \langle \mu_m| \right) |\vec{l} + \vec{j}\rangle \langle \vec{l}|,
\] (S63)

where \( C_{\vec{j}} \) and \( D_{\vec{j}} \) are collections of connectivity matrices with elements 1 (if two states are linked) and 0 (if the two states are not). For instance, if the Hamiltonian can cause a hop from \( \vec{l} \) to \( \vec{l} + \vec{a}_1 \) accompanied by a change \( |\mu_1\rangle \rightarrow |\nu_1\rangle \), then \( D_{\vec{a}_1,1,1} = 1 \). Note that these are, in general, rectangular matrices of size \( n_1 \times n_2 \). Furthermore, to ensure that \( H \) is Hermitian they must satisfy

\[
C_{-\vec{j},m,n} = D_{\vec{j},m,n}^* = D_{\vec{j},m,n},
\] (S64)

where the last equality comes from the fact that they are defined to be real (their elements being either 0 or 1). Note that no terms \( \propto |\mu_m\rangle \langle \mu_m| \) or \( |\nu_n\rangle \langle \nu_n| \) appear, as one-excitation states are exclusively connected to pair ones and vice versa (see Eqs. (S49a)-(S49c)). Neither \( C \) nor \( D \) depends explicitly on \( \vec{l} \), as the form of the Hamiltonian is independent of the choice of the origin. In this form, it is not difficult to exploit this symmetry of the Hamiltonian under discrete lattice translations to partially diagonalize it in terms of Fourier modes:
\[ \hat{H}_\eta = \Omega \sum_l \sum_j \sum_{m=1}^{n_1} \sum_{n=1}^{n_2} \left( C_{j,m,n} |\mu_m \rangle \langle \nu_n| + C_{-j,m,n} |\nu_n \rangle \langle \mu_m| \right) T_j \hat{T}_l^{\dagger} = \]
\[ = \Omega \sum_j \sum_{m=1}^{n_1} \sum_{n=1}^{n_2} \left( C_{j,m,n} |\mu_m \rangle \langle \nu_n| + C_{-j,m,n} |\nu_n \rangle \langle \mu_m| \right) T_j \int \frac{d^d k}{(2\pi)^d} |\tilde{k}\rangle \langle \tilde{k}| = \]
\[ = \Omega \int_B \frac{d^d k}{(2\pi)^d} \sum_j \sum_{m=1}^{n_1} \sum_{n=1}^{n_2} \left( C_{j,m,n} |\mu_m \rangle \langle \nu_n| + C_{-j,m,n} |\nu_n \rangle \langle \mu_m| \right) e^{i\tilde{k} \cdot \tilde{j}} |\tilde{k}\rangle \langle \tilde{k}| = \]
\[ = \Omega \int_B \frac{d^d k}{(2\pi)^d} \sum_{m=1}^{n_1} \sum_{n=1}^{n_2} \left( \sum_{j} C_{j,m,n} e^{i\tilde{k} \cdot \tilde{j}} \right) |\mu_m \rangle \langle \nu_n| + \left( \sum_{j} C_{-j,m,n} e^{i\tilde{k} \cdot \tilde{j}} \right) |\nu_n \rangle \langle \mu_m| \right) |\tilde{k}\rangle \langle \tilde{k}| = \]
\[ = \Omega \int_B \frac{d^d k}{(2\pi)^d} \sum_{m=1}^{n_1} \sum_{n=1}^{n_2} \left( \tilde{C}_{-k,m,n} |\nu_n \rangle \langle \nu_n| + \tilde{C}_{-k,m,n}^* \right) |\tilde{k}\rangle \langle \tilde{k}|, \]
where again
\[ \tilde{C}_{-k,m,n} = \left( \sum_j C_{j,m,n} e^{i\tilde{k} \cdot \tilde{j}} \right) \]
is, for every \( \tilde{k} \in B \), a rectangular \( n_1 \times n_2 \) matrix. Calling now
\[ \tilde{M}_k = \sum_{m=1}^{n_1} \sum_{n=1}^{n_2} \left[ \tilde{C}_{-k,m,n} |\mu_m \rangle \langle \nu_n| + \text{h.c.} \right], \]
we can represent it as a matrix in the basis \( \{ |\mu_1 \rangle, \ldots, |\mu_{n_1} \rangle; |\nu_1 \rangle, \ldots, |\nu_{n_2} \rangle \} \), which yields
\[ M_{\tilde{k}} = \begin{pmatrix} 0 & \tilde{C}_{-\tilde{k}} \\ \tilde{C}_{-\tilde{k}}^\dagger & 0 \end{pmatrix}. \]
Due to this particular block structure,
\[ \text{Rank } \{ M_{\tilde{k}} \} = \text{Rank } \left\{ \tilde{C}_{-\tilde{k}} \right\} + \text{Rank } \left\{ \tilde{C}_{-\tilde{k}}^\dagger \right\}. \]
Furthermore, the rank of a rectangular matrix is never greater than its shortest side. In this case,
\[ \text{Rank } \left\{ \tilde{C}_{-\tilde{k}} \right\} \leq \min \{ n_1, n_2 \}, \]
which in turn implies that the rank of the square matrix \( M_{\tilde{k}} \) is \( \leq 2 \min \{ n_1, n_2 \} \). This means that the size of the kernel of \( M_{\tilde{k}} \) has a lower bound
\[ \dim (\text{Ker } M_{\tilde{k}}) = B - \text{Rank } \{ M_{\tilde{k}} \} \geq \]
\[ \geq \left( n_1 + n_2 \right) - 2 \min \{ n_1, n_2 \} = \]
\[ = \max \{ n_1, n_2 \} - \min \{ n_1, n_2 \} = \]
\[ = |n_1 - n_2|. \]
Hence, if \( |n_1 - n_2| \geq 1 \) then for every \( \tilde{k} \) one can find a kernel vector \( |v_{\tilde{k}} \rangle \) in the basis such that \( \tilde{M}_{\tilde{k}} |v_{\tilde{k}} \rangle = 0 \). Correspondingly, \( \hat{H}_H |\tilde{k}\rangle \otimes |v_{\tilde{k}} \rangle = 0 \forall \tilde{k} \) and the set of all these states forms a zero-energy flat band. Clearly, if
\(|n_1 - n_2| > 1\) then more than one choice of \(|v\rangle_{\langle k|}\) can be made per each quasi-momentum \(\vec{k}\), each identifying an independent flat band. Hence, calling the number of flat bands in the model \(n_{\text{flat}}\), consistently with the main text notation,

\[
n_{\text{flat}} = \dim \left( \text{Ker} M_{\vec{k}} \right) \geq |n_1 - n_2|, \tag{S72}
\]

which proves the bound.

The general rules for filling the matrix elements of \(\tilde{C}_{\vec{k}}\) are the following:

- Choose \(n\)-th column \(1 \leq n \leq n_2\).
- Consider the two possible ways in which a particle can hop from the intermediate state \(|\nu_n\rangle\) within the basis to its neighbors \(|\mu_m\rangle\) and \(|\mu_p\rangle\).
- Add \(e^{i\vec{k} \cdot \vec{j}_m}\) to \(C_{-\vec{k},m,n}\) and \(e^{i\vec{k} \cdot \vec{j}_p}\) to \(C_{-\vec{k},p,n}\), where \(\vec{j}_m/p\) are the lattice vectors pointing to the arrival lattice sites.

In the next sections we work out some examples among the ones displayed in the main text. For simplicity, we set \(\Omega = 1\).

**A. Example: the triangular lattice**

The triangular lattice is a two-dimensional Bravais lattice with primitive lattice vectors

\[
\vec{a}_1 = a (1, 0)^T \quad \text{and} \quad \vec{a}_2 = a \left( \cos \frac{\pi}{3}, \sin \frac{\pi}{3} \right)^T, \tag{S73}
\]

with \(a\) the real-space lattice spacing. In the Hilbert space, we have again a triangular structure where a new site is added on each link.

It is not difficult to see that this reduces to a pure triangular lattice by choosing a basis of 4 sites, a single one-excitation one \((n_1 = 1)\) and 3 pair ones \((n_2 = 3)\). The primitive lattice vectors will be the same as above, where we fix for simplicity \(a = 1\). The basis states can be chosen according to:

- \(|\mu_1\rangle\) : a one-excitation site at \(\vec{b} = 0\).
- \(|\nu_1\rangle\) : a pair site at \(\vec{b} = \vec{a}_1/2\).
- \(|\nu_2\rangle\) : a pair site at \(\vec{b} = \vec{a}_2/2\).
- \(|\nu_3\rangle\) : a pair site at \(\vec{b} = (\vec{a}_1 - \vec{a}_2)/2\).

The matrix \(\tilde{C}_{-\vec{k}}\) is now a \(1 \times 3\) matrix whose elements can be computed via the procedure outlined above:

\[
\tilde{C}_{-\vec{k},1,1} : \text{from basis state } |\nu_1\rangle \text{ one can reach state } |\mu_1\rangle \text{ within the same Bravais lattice site } (\Rightarrow +1) \text{ or state } |\mu_1\rangle \text{ at the neighboring site } \vec{j} = \vec{a}_1 \left( \Rightarrow +e^{i\vec{k} \cdot \vec{a}_1} \right). \tag{S74}
\]

\[
\tilde{C}_{-\vec{k},1,2} : \text{from basis state } |\nu_2\rangle \text{ one can reach state } |\mu_1\rangle \text{ within the same site } (\Rightarrow +1) \text{ or state } |\mu_1\rangle \text{ at the neighboring site } \vec{j} = \vec{a}_2 \left( \Rightarrow +e^{i\vec{k} \cdot \vec{a}_2} \right). \tag{S75}
\]

\[
\tilde{C}_{-\vec{k},1,3} : \text{from state } |\nu_3\rangle \text{ one can reach state } |\mu_1\rangle \text{ within the same site } (\Rightarrow +1) \text{ or state } |\mu_1\rangle \text{ at the neighboring site } \vec{j} = \vec{a}_1 - \vec{a}_2 \left( \Rightarrow +e^{i\vec{k} \cdot (\vec{a}_1 - \vec{a}_2)} \right). \tag{S75}
\]

Collecting all terms, the matrix \(\tilde{C}_{-\vec{k}}\) reads

\[
\tilde{C}_{-\vec{k}} = \left( \begin{array}{ccc} 1 + e^{i\vec{k} \cdot \vec{a}_1} & 1 + e^{i\vec{k} \cdot \vec{a}_2} & 1 + e^{i\vec{k} \cdot (\vec{a}_1 - \vec{a}_2)} \end{array} \right) \equiv \vec{w}_{\vec{k}}^\dagger \tag{S74}
\]

and is equivalent to a three-dimensional vector \(\vec{w}_{\vec{k}}\). Thus, the total matrix \(M_{\vec{k}}\) can be expressed as

\[
M_{\vec{k}} = \left( \begin{array}{c} \vec{w}_{\vec{k}}^\dagger \\ \vec{w}_{\vec{k}} \\ 0 \end{array} \right). \tag{S75}
\]
There are two kernel states corresponding to four-dimensional vectors \((0, v_{k,1})\) and \((0, v_{k,2})\) with
\[
\vec{w}^\dagger_k \cdot \vec{v}_{k,1/2} = 0. \tag{S76}
\]
These states thus reconstruct two flat bands, in line with the bound \(n_{flat} \geq 2\) of this case.

The remaining two bands can be calculated instead by squaring \(M_k\):
\[
M_k^2 = \begin{pmatrix}
\vec{w}^\dagger_k \cdot \vec{w}_k & 0 \\
0 & \vec{w}_k \otimes \vec{w}_k^\dagger
\end{pmatrix}. \tag{S77}
\]

From the symmetric structure of the spectrum and the presence of two flat bands, we can simply infer the non-zero ones as (see Fig. 1 in the main text)
\[
\pm \sqrt{\vec{w}^\dagger_k \cdot \vec{w}_k} = \pm \sqrt{3 + \cos (k \cdot \vec{a}_1) + \cos (k \cdot \vec{a}_2) + \cos (k \cdot (\vec{a}_1 - \vec{a}_2))}. \tag{S78}
\]

Choosing the reciprocal lattice vectors as
\[
\vec{a}_1^* = \frac{4\pi}{\sqrt{3}} \begin{pmatrix} \cos \frac{\pi}{6}, -\sin \frac{\pi}{6} \end{pmatrix}^\top \quad \text{and} \quad \vec{a}_2^* = \frac{4\pi}{\sqrt{3}} (0, 1)^\top \tag{S79}
\]
the first Brillouin zone \(\mathcal{B}\) is an hexagon in \(\vec{k}\) space identified by the conditions
\[
\begin{align*}
\left| k \cdot \vec{a}_1^* \right| & \leq \frac{1}{2} |\vec{a}_1^*|^2 \right) \cap \\
\left| k \cdot \vec{a}_2^* \right| & \leq \frac{1}{2} |\vec{a}_2^*|^2 \right) \cap \\
\left| k \cdot (\vec{a}_1^* - \vec{a}_2^*) \right| & \leq \frac{1}{2} |(\vec{a}_1^* - \vec{a}_2^*)|^2.
\end{align*} \tag{S80}
\]

**B. Example: the honeycomb lattice**

The honeycomb lattice is a triangular Bravais lattice with primitive lattice vectors
\[
\vec{a}_1 = a (1, 0)^T \quad \text{and} \quad \vec{a}_2 = a \begin{pmatrix} \cos \frac{\pi}{3}, \sin \frac{\pi}{3} \end{pmatrix}^T, \tag{S81}
\]
where the lattice spacing \(a\) is \(\sqrt{3}\) times the edge of the hexagons, plus a basis of two vectors
\[
\vec{b}_1 = 0 \quad \text{and} \quad \vec{b}_2 = \frac{2\vec{a}_2 - \vec{a}_1}{3}. \tag{S82}
\]

In the synthetic lattice, this gives rise to a structure with a basis of 5 elements: 2 one-excitation sites \((n_1 = 2)\) and 3 pair ones \((n_2 = 3)\), which we choose as follows:

- \(|\mu_1\rangle\) : a one-excitation site at \(\vec{b} = 0\).
- \(|\mu_2\rangle\) : a one-excitation site at \(\vec{b} = \frac{2\vec{a}_2 - \vec{a}_1}{3}\).
- \(|\nu_1\rangle\) : a pair site at \(\vec{b} = \frac{2\vec{a}_2 - \vec{a}_1}{6}\).
- \(|\nu_2\rangle\) : a pair site at \(\vec{b} = \frac{2\vec{a}_1 - \vec{a}_2}{6}\).
- \(|\nu_3\rangle\) : a pair site at \(\vec{b} = -\frac{\vec{a}_1 + \vec{a}_2}{6}\).

We thus see that the \(\vec{C}_{k}\) are \(2 \times 3\) matrices and that there is at least one flat zero-energy band. The matrix elements can be identified column by column as follows:
\(|\nu_1\rangle\) : From \(|\nu_1\rangle\) one can jump to \(|\mu_1\rangle\) or to \(|\mu_2\rangle\) remaining in the same Bravais lattice site.

\(|\nu_2\rangle\) : From \(|\nu_2\rangle\) one can jump to \(|\mu_1\rangle\) in the same site or to \(|\mu_2\rangle\) changing site by \(\vec{j} = \vec{a}_1 - \vec{a}_2\) (\(\Rightarrow e^{i\vec{k} \cdot (\vec{a}_1 - \vec{a}_2)}\)).

\(|\nu_3\rangle\) : From \(|\nu_3\rangle\) one can jump to \(|\mu_1\rangle\) in the same site or to \(|\mu_2\rangle\) changing site by \(\vec{j} = -\vec{a}_2\) (\(\Rightarrow e^{-i\vec{k} \cdot \vec{a}_2}\)).

Hence,

\[
\vec{C}_{-\vec{k}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\vec{k} \cdot (\vec{a}_1 - \vec{a}_2)} & e^{-i\vec{k} \cdot \vec{a}_2} \\ \end{pmatrix} \equiv \begin{pmatrix} \vec{w}_{k,1}^\dagger \\ \vec{w}_{k,2}^\dagger \end{pmatrix},
\]

with \(\vec{w}_{k,1/2}\) three-dimensional vectors. The matrix \(M_{-\vec{k}}\) is thus

\[
M_{-\vec{k}} = \begin{pmatrix} 0 & 0 & \vec{w}_{k,1}^\dagger \\ 0 & 0 & \vec{w}_{k,2}^\dagger \end{pmatrix}.
\]

The kernel state is a five-dimensional vector \((0, 0, \vec{v}_{-\vec{k}})\) which satisfies \(\vec{w}_{k,1/2}^\dagger \cdot \vec{v}_{-\vec{k}} = 0\).

To identify the remaining non-zero bands, we again take the square of the total matrix \(M_{-\vec{k}}\):

\[
M_{-\vec{k}}^2 = \begin{pmatrix} \vec{w}_{k,1}^\dagger \cdot \vec{w}_{k,1} & \vec{w}_{k,1}^\dagger \cdot \vec{w}_{k,2} & 0 \\ \vec{w}_{k,2}^\dagger \cdot \vec{w}_{k,1} & \vec{w}_{k,2}^\dagger \cdot \vec{w}_{k,2} & 0 \\ 0 & 0 & \vec{w}_{k,1} \otimes \vec{w}_{k,1}^\dagger & \vec{w}_{k,2} \otimes \vec{w}_{k,2}^\dagger \end{pmatrix},
\]

where the first block is \(2 \times 2\) and the second one \(3 \times 3\). We can now diagonalize the first block to find (see Fig. 1 in the main text)

\[
\lambda_{\vec{k},\pm} = \frac{1}{2} \left( |\vec{w}_{k,1}|^2 + |\vec{w}_{k,2}|^2 \right) \pm \sqrt{\left( |\vec{w}_{k,1}|^2 - |\vec{w}_{k,2}|^2 \right)^2 + 4 |\vec{w}_{k,1}^\dagger \cdot \vec{w}_{k,2}|^2},
\]

with \(\lambda_{\vec{k},\pm} \geq 0\). The four non-trivial bands will thus correspond to \(\pm \sqrt{\lambda_{\vec{k},+}}\) and \(\pm \sqrt{\lambda_{\vec{k},-}}\). Working out the scalar products

\[
|\vec{w}_{k,1}|^2 = |\vec{w}_{k,2}|^2 = 3
\]

and

\[
|\vec{w}_{k,1}^\dagger \cdot \vec{w}_{k,2}| = \left| 1 + e^{i\vec{k} \cdot (\vec{a}_1 - \vec{a}_2)} + e^{-i\vec{k} \cdot \vec{a}_2} \right|
\]

we obtain by substitution

\[
\lambda_{\vec{k},\pm} = 3 \pm \left| 1 + e^{i\vec{k} \cdot (\vec{a}_1 - \vec{a}_2)} + e^{-i\vec{k} \cdot \vec{a}_2} \right| = 3 \pm \sqrt{3 + 2 \cos \left( \vec{k} \cdot (\vec{a}_1 - \vec{a}_2) \right) + 2 \cos \left( \vec{k} \cdot \vec{a}_2 \right) + 2 \cos \left( \vec{k} \cdot \vec{a}_1 \right)}.
\]

From the first equality we see that the second addend is always \(\leq 3\). It is \(3\) only when \(\vec{k} = 0\) (up to reciprocal lattice translations \(\vec{G}\), see (S33)). Hence, \(\lambda_- (\vec{k} = 0) = 0\) is a minimum and \(\lambda_+ (\vec{k} = 0) = 6\) is a maximum. The bands \(\pm \sqrt{\lambda_{\vec{k},-}}\) touch at \(\vec{k} = 0\) with linear dispersion. Second, the argument of the absolute value will vanish when

\[
\vec{k} \cdot (\vec{a}_1 - \vec{a}_2) = \pm \frac{2\pi}{3} + 2\pi n, \quad -\vec{k} \cdot \vec{a}_2 = \pm \frac{4\pi}{3} + 2\pi m
\]

where the signs must be chosen consistently. Up to reciprocal lattice translations, one can choose

\[
\vec{k} = \pm \frac{1}{3} (\vec{a}_2 - \vec{a}_1),
\]

identifying the points at the vertices of the hexagonal first Brillouin zone (one can verify these points lie at the boundary of two of the conditions in (S80)). Therefore, the two upper bands \(\sqrt{\lambda_{\vec{k},+}}\) and \(-\sqrt{\lambda_{\vec{k},-}}\) touch at the vertices of the first Brillouin zone with linear dispersion and similarly do the lower bands \(-\sqrt{\lambda_{\vec{k},+}}\) and \(-\sqrt{\lambda_{\vec{k},-}}\).
V. TRANSFER MATRIX APPROACH

Due to the tight-binding structure of $\hat{H}_{H_1}$, the Schrödinger equation

$$\hat{H}_{H_1} |\psi\rangle = \epsilon |\psi\rangle \quad (S92)$$

can be conveniently rewritten in terms of the state components over the “synthetic lattice states” $|A_i\rangle$, $|B_i\rangle$, $|C_i\rangle$, $|D_i\rangle$ and $|E_i\rangle$: defining in fact

$$|\psi\rangle = \sum_i [A_i |A_i\rangle + B_i |B_i\rangle + C_i |C_i\rangle + D_i |D_i\rangle + E_i |E_i\rangle] \quad (S93)$$
equation (S92) can be recast as the system

$$\begin{align*}
\epsilon A_i &= \delta A_i, A_i + C_i + C_{i+1} \\
\epsilon B_i &= \delta B_i, B_i + D_i + D_{i+1} \\
\epsilon C_i &= A_i + A_{i-1} + E_i \\
\epsilon D_i &= B_i + B_{i-1} + E_i \\
\epsilon E_i &= \delta E_i, E_i + C_i + D_i.
\end{align*} \quad (S94)$$

Applying the “detangling” transformation

$$\begin{align*}
X_+^i &= \frac{A_i \pm B_i}{\sqrt{2}} \quad Y_+^i = \frac{C_i \pm D_i}{\sqrt{2}} \\
\delta X_+^i &= \frac{\delta A_i \pm \delta B_i}{2},
\end{align*} \quad (S95)$$
one can then reformulate the problem in the new picture, where it reads

$$\begin{align*}
\epsilon X_+^i &= \delta X_+^i, X_+^i + \delta X_-^i, X_-^i + X_+^{i+1} + Y_+^{i+1} \\
\epsilon X_-^i &= \delta X_-^i, X_-^i + \delta X_+^i, X_+^i + Y_-^i + Y_-^{i+1} \\
\epsilon Y_+^i &= X_+^i + X_-^{i+1} + \frac{2}{\epsilon - \delta E_i}, Y_+^i \\
\epsilon Y_-^i &= X_-^i + X_+^{i-1} \\
\epsilon E_i &= \frac{\sqrt{2}}{\epsilon - \delta E_i}, Y_+^{i+1}.
\end{align*} \quad (S96)$$

As sketched in Fig. 2 in the main text, this can be interpreted as a different structure where “+” variables leave on the one-dimensional “stub” lattice and “−” ones on the remaining simple chain. It is also clear that the random numbers $\delta X_-^i$ effectively act as random hopping amplitudes between $X$ sites from the chain to the stub lattice and vice versa.

The last equation in (S96) has been substituted into the third one to replace $E_i$, so that we can now close the equations without the variables $E_i$, which drop out. By combining the first two equations with the third and fourth one step to the right ($i \rightarrow i+1$) and singling out to the l.h.s. the “rightmost” variables $Y_+^{i+1}, X_-^{i+1}$ we find

$$\begin{align*}
Y_+^{i+1} &= (\epsilon - \delta X_+^i), X_+^i, X_-^i - \delta X_-^i, X_+^{i+1} + Y_+^{i+1} \\
Y_-^{i+1} &= (\epsilon - \delta X_+^i), X_-^i, X_+^i - \delta X_-^i, X_-^{i+1} - Y_-^{i+1} \\
X_-^{i+1} &= -X_+^i + \left(\epsilon - \frac{2}{\epsilon - \delta E_{i+1}}\right), Y_+^{i+1} \\
X_+^{i+1} &= -X_-^i + \epsilon Y_-^{i+1}.
\end{align*} \quad (S97)$$

By introducing the shorthand

$$\begin{align*}
\alpha_+^i &= \epsilon - \delta X_+^i \quad \text{and} \quad \gamma_i = \epsilon - \frac{2}{\epsilon - \delta E_{i+1}}, \quad (S98)
\end{align*}$$
substituting the first two equations in the third and fourth ones and rearranging the order one gets

\begin{align*}
X_{i+1}^+ &= -X_i^+ + \gamma_i \alpha_i^+ X_i^- - \gamma_i \delta X_i^+ X_i^- - \gamma_i Y_i^+ \\
X_{i+1}^- &= -X_i^- + \epsilon \alpha_i^+ X_i^- - \epsilon \delta X_i^- X_i^- - \epsilon Y_i^- \\
Y_{i+1}^+ &= \alpha_i^+ X_i^+ + \delta X_i^- X_i^- Y_i^+ \\
Y_{i+1}^- &= \alpha_i^+ X_i^- + \delta X_i^- X_i^- Y_i^- .
\end{align*}
(S99)

This can be now interpreted as a matrix-vector multiplication of the form

\begin{equation}
\begin{pmatrix}
X^- \\
Y^+
\end{pmatrix}_{i+1} = T_i \begin{pmatrix}
X^+ \\
Y^-
\end{pmatrix}_i ,
\end{equation}
(S100)

where the transfer matrix \( \hat{T}_i \) reads

\[ T_i = \begin{pmatrix}
\gamma_i \alpha_i^+ - 1 & -\gamma_i \delta X_i^- & -\gamma_i & 0 \\
-\epsilon \delta X_i^- & -1 + \epsilon \alpha_i^+ & 0 & -\epsilon \\
\alpha_i^+ & -\delta X_i^- & -1 & 0 \\
-\delta X_i^- & \alpha_i^+ & 0 & -1
\end{pmatrix} .
\]

At any fixed value of the energy \( \epsilon \), these transfer matrices can be used in a sequence to iteratively reconstruct the components of the vector \( |\psi\rangle \):

\begin{equation}
\begin{pmatrix}
X^+ \\
X^- \\
Y^+ \\
Y^-
\end{pmatrix}_n = \left( \prod_{i=n}^1 T_i \right) \begin{pmatrix}
X^+ \\
X^- \\
Y^+ \\
Y^-
\end{pmatrix}_1 .
\end{equation}
(S101)

The boundary conditions then distinguish which values of \( \epsilon \) are actually part of the spectrum of \( \hat{H}_{H_i} \). Product of random matrices such as \( \prod_{i=1}^n T_i \) are of course dependent on the specific realization of the disorder. However, their asymptotic properties (for \( n \gg 1 \)) are often more general and only depend instead on the probability distribution function the disorder satisfies. In particular, taking a randomly-selected “seed” vector \( \vec{v}_0 \), the norm

\[ \| \vec{v}_n \| = \left\| \prod_{i=n}^1 \hat{T}_i \vec{v}_0 \right\| ,
\]
(S102)

tends to behave exponentially as \( e^{-n/\xi} \). The scale \( \xi \) is the localization length at energy \( \epsilon \).

Each matrix \( \hat{T}_i \) is \( 4 \times 4 \) and symplectic, i.e., if we define

\[ J = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} ,
\]
(S103)

then \( (T_i)^T J T_i = J \), which implies in particular that its determinant is 1 and that its spectrum comes in pairs of inverse eigenvalues \( \lambda_{1/2}, 1/\lambda_{1/2} \). This is a particular case of a more general class of systems [6]. It is intuitive to understand that, at every matrix multiplication, (at most) two directions in the four-dimensional space it acts upon get stretched, suggesting a long-distance exponential scaling of the form \( e^{-n/\xi_1} e^{-n/\xi_2} \) with \( \xi_1 < \xi_2 \), where therefore \( \xi_1 \) is the dominant scale and \( \xi_2 \) the subleading one. The specific numerical method employed for the extraction of \( \xi_{1/2} \) can be found in [7].

VI. LOCALIZATION AND SCALING EXponents

In Table I we list the scaling exponents \( \nu_i \) extracted from the localization lengths \( \xi_i \sim s^{\nu_i} \), \( i = 1, 2 \) at a given disorder strength \( s \) as described in Fig. 3 of the main text. For comparison, and further to the discussion in the main
of Rabi frequency $\Omega_{i,i}$ an excitation at site 1, the second pulse then exploits the blockade mechanism to create a superposition of spin-up the state-preparation time to a minimum (implying higher values of $\Omega_{Q}$) factors in the second, and fourth lines of (S104). In practice the choice of $\Omega_{Q}$ affects the sites of the (synthetic) Lieb ladder are drawn from a uniform interval $[\frac{-W}{2}, \frac{W}{2}]$. The second (third) column accounts for a situation where only pair sites (all sites of the Lieb ladder) are affected. On the other hand, the values we obtain for the disorder distribution drawn from the flat disorder energies $\epsilon$ at energies $\epsilon = \{1, 1.1, 1.1\}$ does not seem to be close to either the anomalous or the edge scaling exponents and, based solely on the present analysis, cannot be simply attributed to the disorder acting on only the pair state sites. On the other hand, the values we obtain for the disorder distribution drawn from (S12), i.e. $\nu(\epsilon = \sqrt{2}) \approx \{1, 1.1, 1.1\}$ does not seem to be close to either the anomalous or the edge scaling exponents and, based solely on the present analysis, cannot be simply attributed to the disorder acting on only the pair state sites.

### VII. Initial State Preparation and Evolution

In this section we consider the preparation of the state $|\psi_{\text{loc}}\rangle = \frac{1}{\sqrt{4}}(|A_{i}| + |B_{i}| - |E_{i}| - |E_{i+1}|)$ localized at rungs $i, i+1$ of the ladder. We assume that each atom in the ladder can be addressed individually with a laser pulse of Rabi frequency $\Omega_{R}$ and duration $\tau$ so that the atomic spin evolves according to

$$U(\theta \equiv \Omega_{R}\tau) = e^{-i\frac{\theta}{2}\sigma_{z}} = \left(\begin{array}{cc} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{array}\right)$$

if the laser detuning $\Delta$ is set such that it is resonant with the transition of the addressed atom. If $\Delta \gg \Omega_{R}$, instead, it acts trivially like an identity operator. Specifically, we will distinguish two special cases, namely $\Delta = 0$ in addition to $\Delta = -V(R_{0})$ corresponding to the blockade and facilitation condition respectively. The state $|\psi_{\text{loc}}\rangle$ can be obtained by application of six pulses on initially all atoms in the spin-down state as $|\psi_{\text{loc}}\rangle = F_{\text{b}}(2\pi)F_{\text{f}}(2\pi)F_{\text{b}}(\pi)F_{\text{f}}(\pi)B_{4}(\pi)B_{1}(\frac{\pi}{2})|\psi_{\text{loc}}\rangle$, where now $B_{j}(\theta), F_{j}(\theta)$ stand for the laser pulse of area $\theta = \Omega_{R}\tau$ in the blocked (B) and facilitated (F) regime applied at site $j = 1, \ldots, 4$ labeling the effective plaquette formed by the four sites corresponding to the $i$-th and $(i + 1)$-th rung of the ladder, see Eq. (S104). Here, the first pulse creates an excitation at site 1, the second pulse then exploits the blockade mechanism to create a superposition of spin-up states at sites 1 and 4. Next, the pulse in the facilitated regime applied at site 2 creates a superposition of the form $-i|\uparrow\rangle + |\downarrow\rangle$ if and only if a single nearest-neighbor is already excited, and so forth. We have omitted the global $-i$ factors in the second, and fourth lines of (S104). In practice the choice of $\Omega_{R}$ is a trade-off between the need to keep the state-preparation time to a minimum (implying higher values of $\Omega_{R}$) and the upper bounds imposed for keeping the blockade and facilitation conditions preserved, see [9] for details of these issues.
Once the state $|\psi_{\text{loc}}\rangle$ has been prepared, it evolves according to $H_{\text{eff}}$, Eq. (S108). The probability of excitation at rung $i$ is obtained as $P_\alpha^i = n_\alpha^i / \sum_{i=1}^L n_\alpha^i$. Here $n_\alpha^i = \langle\psi(t)|\hat{n}_\alpha^i|\psi(t)\rangle$, $\alpha = u, l$ for the upper and lower leg of the ladder respectively and $L$ denotes the length of the ladder. These probabilities are then averaged over the disorder $P_\alpha^i \to \langle P_\alpha^i \rangle_{\delta V} = p_\alpha^i$ and used to define the average position and the standard deviation of the excitations according to

$$\bar{x}_\alpha = \sum_i p_\alpha^i i \quad \text{(S105)}$$

$$\langle \Delta x^\alpha \rangle^2 = \sum_i p_\alpha^i i^2 - (\bar{x}_\alpha)^2 = \sum_i p_\alpha^i (i - \bar{x}_\alpha)^2. \quad \text{(S106)}$$

**VIII. NUMERICAL SIMULATION OF THE SPIN DYNAMICS**

We consider the full Hamiltonian Eq. (S28) which we express in units of the Rabi frequency as

$$\Omega^{-1}\hat{H} = \sum_k \hat{\sigma}_x^{(j)} + (-\tilde{V}(R_0)) n_k + \tilde{V}(R_0) \sum_{k>j} \frac{\hat{n}_k \hat{n}_j}{|k-j|^{\alpha}}, \quad \text{(S107)}$$

where $\tilde{V}(R_0) = V(R_0)/\Omega$ (in what follows we label all dimensionless quantities by tilde). This leads to the following effective Hamiltonian on the Lieb lattice of length $L$

$$\Omega^{-1}\hat{H}_{\text{eff}} = \hat{\mathcal{H}}_0 \otimes \mathbb{1}_L + \hat{\mathcal{H}}_0^{\text{dis}} + \left[ \hat{\mathcal{H}}_1 \otimes \hat{G}_L + \text{H.c.} \right], \quad \text{(S108)}$$

expressed in the basis

$$\{|A_1\rangle, \ldots, |A_L\rangle, |B_1\rangle, \ldots, |B_L\rangle, \ldots, |E_L\rangle\}, \quad \text{(S109)}$$

where

$$\hat{\mathcal{H}}_0 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0
\end{pmatrix}, \quad \text{(S110)}$$
The two red solid lines correspond to a cut for fixed values of \( \Omega/V(R_0) \), \( \Omega/V(R_0) = 1/20 \) (upper line) and \( \Omega/V(R_0) = 1/200 \) (lower line).

(b) Comparison between the evolution of \( |\psi_{\text{loc}}\rangle \) generated by \( H \), Eq. (S28), dashed line and \( H_{\text{eff}} \), solid line, in a ladder of \( L = 4 \) and for \( \Omega/V(R_0) = 1/200 \). (c) Same as (b) with \( \Omega/V(R_0) = 1/200 \). Here we have used \( \Omega/t = 4.3 \) for each respective \( \Omega \) and averaged over 100 disorder realizations.

\[ \hat{H}_1 \] is a 5 \( \times \) 5 matrix with only non-zero entries \((\hat{H}_1)_{1,3} = (\hat{H}_1)_{2,4} = 1\), \((G_L)_{ij} = \delta_{i,j-1}\) is a \( L \times L \) matrix with ones on the first upper diagonal and

\[
\hat{H}_0 = \text{diag} \left( \delta_{A_1}, ..., \delta_{A_{L-1}}, \delta_{A_L} = 0, 
\delta_{B_1}, ..., \delta_{B_{L-1}}, \delta_{B_L} = 0, 
\delta_{C_1} = 0, ..., \delta_{C_L} = 0, 
\delta_{D_1} = 0, ..., \delta_{D_L} = 0, 
\delta_{E_1}, ..., \delta_{E_L} \right) 
\]

is a 5\( \times \)5 diagonal disorder matrix, where we impose open boundary conditions (OBC) by requiring that \( \delta_{A_L} = \delta_{B_L} = 0 \), since spin configurations corresponding to the \( A_L, B_L \) basis elements are missing; in fact, these would be pair states occupying a site on the \( L \)-th rung and one on the (non-existent) \( (L+1) \)-th. Analogously we enforce the OBC in the coupling matrix by setting all Hamiltonian elements corresponding to the \( A_L, B_L \) basis elements to zero. Here, \( \delta_{\Xi_j} = \Omega^{-1} (V(d_{\Xi_j}) - V(R_0)) \), where \( \Xi_j = A_j, ..., E_j \) and \( d_{\Xi_j} \) is a shorthand for the spin separation in the given configuration \( \Xi_j \). We note that since configurations \( C, D \) correspond to single spin excitation, the associated disorder is vanishing by definition, \( \delta_{C_j} = \delta_{D_j} = 0, \forall j \). The disorder energies \( \delta_{\Xi_j} \) are generated from first drawing a specific realization of atomic positions at each site of the lattice in all three spatial directions with isotropic Gaussian distribution of width \( s \).

We then exactly evolve an initial state

\[
|\psi_0\rangle = \sum_{j=1}^{5L} c_j |b_j\rangle ,
\]

as \( |\psi(t)\rangle = \exp \left[ -i t \hat{H}_{\text{eff}} \right] |\psi_0\rangle \), where \( b_j \) are the elements of the basis (S109) [strictly speaking there are only 5\( L - 2 \) non-trivial elements due to the OBC].

We note that the result of the evolution depends on two independent parameters, \( s \) and the ratio \( V(R_0)/\Omega \), where the Rabi frequency should further satisfy \( \Omega \ll V(2R_0) \) for the effective Hamiltonian (S108) to be valid. In Fig. S1(a) we present the results of the simulation analogous to that performed in Fig. 4, showing \( \Delta x \) in the \( s - \Omega/V(R_0) \) plane. We observe that the maximum of \( \Delta x \) as a function of the disorder gets shifted towards higher disorder strength as \( \Omega \) is increased.

The dependence of \( \Delta x \) in Fig. S1(a) can be intuitively understood as follows. Smaller values of \( \Omega/V(R_0) \) correspond to larger diagonal disorder elements \( \delta_{ij} \). Since it is the disorder which couples the flat and dispersive bands, the smaller the \( s \), the smaller the \( \Omega \) that is sufficient to cause the excitation hopping and thus the increase in \( \Delta x \). As \( s \) is increased, Anderson localization becomes more and more relevant and, correspondingly, the localization length at \( \epsilon = 0 \) shrinks. Eventually, the state becomes capable of propagating over distances comparable to the localization length. Further increasing \( s \) then reduces this scale, corresponding to the decrease in \( \Delta x \). Clearly, by increasing \( \Omega/V(R_0) \) the hopping
amplitude becomes more relevant with respect to the typical energy shifts and the localization length is thus increased. Higher values of $s$ are then required to localize the state again. In Fig. 5b,c we show a comparison between the exact evolution according to the full Hamiltonian (S28), dashed line, and $H_{\text{eff}}$, solid line. As expected, the predictions of the two models show an agreement in the regime where $\Omega \ll V(2R_0)$, Fig. S1(c) ($V(R_0)/\Omega = 200$). On the other hand for larger $\Omega$, the two models start to differ as shown in Fig. S1(b) ($V(R_0)/\Omega = 20$).

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