COHOMOLOGY OF $SL_3(\mathbb{Z})$ WITH COEFFICIENTS IN THE STANDARD REPRESENTATION

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Abstract. This paper is a natural continuation of a joint paper with Bajpai, Harder and Moya Giusti [BHHM], even though it began as an answer to Goncharov’s question. It that paper, we had complete description for all representations except for odd symmetric powers and their dual ones. For those representations we were left with two options: certain one dimensional module is a ghost space or not. Here we find the $H^2(SL_3(\mathbb{Z}), V_3)$ has ghost classes. It means that it is generated by a class from the cohomology of the Borel subgroup.

With the techniques developed here, we show that the $d_2$ map of the spectral sequence for the boundary cohomology of $GL_4(\mathbb{Z})$ is non-trivial if and only if there is a ghost class in $GL_3(\mathbb{Z})$ (see Propositions 11 and 12.) We use [EVGS] to show that a spectral sequence related to $GL_4(\mathbb{Z})$ does not degenerate at $E_2$-level. Then $d_2$ is non-trivial. Therefore, we obtain that $H^2(SL_3(\mathbb{Z}), V_3))$ is a ghost space, where $V_3$ is the standard representation.

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1. INTRODUCTION

1.1. Summary of results and methods. This project began by a series of questions by Goncharov on cohomology of $GL_4(\mathbb{Z})$ and a congruence subgroup of it $\Gamma_1(4, p)$. He needs those computations in relation to multiple
zeta values and multiple polylogarithms at the roots of unity. By $\Gamma_1(n,P)$ we mean the stabilizer modulo $p$ of the vector $[1,0,\ldots,0]$ in $GL_n(\mathbb{Z})$.

One of the questions the Goncharov asked was to compute carefully $H^3(GL_4(\mathbb{Z}), \det)$. Solving this problem has a consequence for the cohomology on $GL_3(\mathbb{Z})$.

The current paper can be considered as a continuation of the joint paper with Bajpai, Harder and Moya Giusti [BHHM] on cohomology of $SL_3(\mathbb{Z})$. In that paper we showed that $H^2(SL_3(\mathbb{Z}), V_3)$ is one dimensional; however, we could not describe the Hecke module. There are exactly two possibilities. Each of them describes uniquely the corresponding Hecke module. The first one is that the cohomology has a ghost class and the other option is that it doesn’t. Here we show that it has a ghost class. Therefore,

$$H^2(SL_3(\mathbb{Z}), V_3) = H^1(B, V_3).$$

The approach is the following. Both the ghost classes and the $d_2$ maps are directly related to the boundary cohomology of $GL_3(\mathbb{Z})$ and of $GL_4(\mathbb{Z})$. A ghost class in $GL_3$ means that we have a nontrivial connecting homomorphism $\delta : H^1(B, V_3^+) \to H^2_\partial(GL_3(\mathbb{Z}), V_3^+)$. On the other hand, by construction of a $d_2$ map, we invert an isomorphism $\text{im}(\delta) = \ker(\ldots)$ and compose that with an inclusion. If $\text{im}(\delta) \neq 0$ then $d_2 \neq 0$.

On the other hand, if $d_2 = 0$, then $H^3(GL_4(\mathbb{Z}), \det)$ has to be two-dimensional. However, from a result of Elbaz-Vincent, Gangl and Soulé [EVGS] we have that the third cohomology is one-dimensional. Therefore $d_2 \neq 0$.

Examining carefully the cohomology of $GL_4(\mathbb{Z})$, led us to proving the existence of a ghost class in $GL_3(\mathbb{Z})$, which in turn corrects the cohomology of $GL_4(\mathbb{Z})$. The corrected version has the same dimensions as the previously published version [H3]. The only difference is that at one instance the Hecke module should be changed. This was the initial question by Goncharov - to compute $H^3(GL_4(\mathbb{Z}), \det)$, see Theorem 19.

1.2. Technical background. Now, let us recall what is boundary cohomology. Let $\Gamma$ be $GL_3(\mathbb{Z})$ or $GL_4(\mathbb{Z})$; let $G$ be $GL_3(\mathbb{R})$ or $GL_4(\mathbb{R})$; and let $K$ be a maximal compact group in $G$. We can take $K$ to be $O_3(\mathbb{R})$ or $O_4(\mathbb{R})$, respectively. Let $S$ be the locally symmetric space defined by the double quotient

$$S = \Gamma \backslash G / (K \times \mathbb{R}_{>0}).$$

Then the group cohomology of $\Gamma$ with coefficients in a representation $V$ is isomorphic to the cohomology of $S$ with coefficient in the corresponding local system $\tilde{V}$. That is

$$H^q(\Gamma, V) = H^q(S, \tilde{V}).$$

The Borel-Serre compactification of $S$ (see [BoSe]), which we will denote by $\overline{S}$, is a compactification obtain by glueing strata $S_P$ corresponding to the parabolic subgroups (of $GL_3$ or $GL_4$). The glueing is done in the following way. Strata corresponding to maximal parabolic subgroup (rank 1) are glued along strata of higher rank. For instance for $GL_3$, we have two maximal parabolic subgroups $Q_{12}$ and $Q_{23}$ and a minimal parabolic subgroup $Q_0$. 


Then the strata $S_{Q_{12}}$ and $S_{Q_{23}}$ are glued together along $S_{Q_0}$. For $Q_{12}$, $Q_{12}$ and $Q_0$, we take the following representatives

**Maximal parabolic subgroups (rank 1):**

\[
Q_{12} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad Q_{23} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.
\]

**Minimal parabolic subgroup (rank 2):**

\[
Q_0 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.
\]

The compactification $\overline{S}$ has the same homotopy type as the locally symmetric space $S$. We have $H^q(\overline{S}, i_\ast \tilde{V}) = H^q(S, V)$, where $i : S \to \overline{S}$ is the inclusion. Let $\partial \overline{S}$ be the boundary of $\overline{S}$. We define the boundary cohomology of $\Gamma$ to be

\[
H^q_{\partial}(\Gamma, V) := H^q(\partial \overline{S}, j_\ast i_\ast \tilde{V}),
\]

where $j : \partial \overline{S} \to \overline{S}$ is the inclusion of the boundary into the compactification.

If $\Gamma = GL_3(\mathbb{Z})$, then the boundary cohomology $H^q_{\partial}(\Gamma, V)$ can be computed via Mayer-Vietoris exact sequence

\[
\to H^{q-1}(Q_0, V) \to \to H^q_{\partial}(GL_3(\mathbb{Z}), V) \to H^q(Q_{12}, V) + H^q(Q_{23}, V) \to H^q(Q_0, V)
\]

For $GL_4(\mathbb{Z})$, we consider the following representatives for each class of parabolic subgroups.

**Maximal parabolic subgroups (rank 1):**

\[
P_{13} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad P_{12,34} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad P_{24} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.
\]

**Intermediate parabolic subgroups (rank 2):**

\[
P_{12} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad P_{23} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \quad P_{34} = \begin{pmatrix} * & * & * \\ 0 & 0 & * \end{pmatrix}.
\]

**Minimal parabolic subgroup (rank 3):**

\[
B = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.
\]

For $\Gamma = GL_4(\mathbb{Z})$, one can compute the boundary cohomology in terms of a spectral sequence. Let $E_{1}^{p,q} = \bigoplus H^q(P, V)$, where the direct sum is taken over parabolic subgroups $P$ of rank $p + 1$. From the glueing of the strata we naturally have the map

\[
d_1 : E_{1}^{p,q} \to E_{1}^{p+1,q}.
\]
For any spectral sequence there is a standard way of constructing the \(d_2\)-map

\[
d_2 : E^{p,q}_2 \to E^{p+2,q-1}_2.
\]

One of the \(d_2\)’s in this paper is the one the spectral sequence for the boundary cohomology of \(GL_4(\mathbb{Z})\) with coefficient in the determinant representation, \(\text{det}\). There are two more \(d_2\) maps corresponding to two other spectral sequences both related to \(GL_3(\mathbb{Z})\). One of them is related to potentially ghost classes in \(GL_3(\mathbb{Z})\) and the other to ghost classes in \(GL_3(\mathbb{Z})\). (see Section 4, more specifically, Propositions 11 and 12.)

What are ghost classes and what are potentially ghost classes? Potentially ghost classes are defined as those cohomological classes inside the boundary cohomology that come from higher rank parabolic subgroups (not from the maximal parabolic subgroups). They can be defined via a filtration of the spectral sequence; however, we do not need it in such generality. For our purposes, for \(\Gamma = GL_3(\mathbb{Z})\), the potentially ghost classes are the ones in the image of the connecting homomorphism

\[
p_{\text{Gh}} q(GL_3(\mathbb{Z}), V) := \text{im} \left[ H^{q-1}_q(Q_0, V) \to H^q_0(GL_3(\mathbb{Z}), V) \right].
\]

Now, we are going to define Eisenstein cohomology. It is the image of the compositions \(H^i(\Gamma, V) = H^i(S, V) = H^i(\mathcal{S}, i_* V)\) and restriction homomorphism \(j^* : H^q(\mathcal{S}, i_* V) \to H^q(\partial \mathcal{S}, j^* i_* V) = H^q_0(GL_3(\mathbb{Z}), V)\). Then the ghost classes are defined as the intersection of potentially ghost classes and the Eisenstein cohomology

\[
\text{Gh}^q(GL_3(\mathbb{Z}), V) := p_{\text{Gh}} q(GL_3(\mathbb{Z}), V) \cap H^q_{\text{Eis}}(GL_3(\mathbb{Z}), V)
\]

We compute cohomology of the parabolic subgroups by using Kostant’s formula and the Leray-Serre spectral sequence. Let \(P\) be a parabolic subgroup. Let \(N_P\) be the nilpotent radical of \(P\) and let \(M_P = P/N_P\) be the Levi quotient. Then we have the Leray-Serre spectral sequence

\[
H^j(M_P, H^i(N_P, V)) \Rightarrow H^{i+j}(P, V).
\]

This gives a sort of induction from lower rank reductive groups to ones with higher rank, (since each Levi quotient \(M_P\) is a product of groups of lower rank.) To start all that we need to compute the cohomology of the nilpotent radical \(N_P\). It is done using the Kostant formula.

**Theorem 1. (Kostant)** Let \(V\) be a representation of highest weight \(\lambda\). Let \(N_P\) be nilpotent radical, and let \(\rho\) be half of the sum of the positive roots. Then

\[
H^i(N_P, V) = \bigoplus_{\omega} L_{\omega(\lambda+\rho)-\rho},
\]

where the sum is taken over the representatives of the quotient \(W_P/\mathcal{W}\) with minimal length such that their length is exactly \(i\). In the above notation \(L_{\lambda}\) means representation of \(N_P\) with highest weight \(\lambda\); and \(\mathcal{W}\) and \(W_P\) are the Weyl groups of \(G\) and of \(P\), respectively.

Another technique that we employ is the use of homological Euler characteristics. Intuitively it is the sum of alternating dimensions of cohomology groups; however it is computed via considering torsion elements of the arithmetic group. It is useful because it gives an extra invariant to work with.
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2. Euler characteristic

The homological Euler characteristic $\chi_h$ of a group $\Gamma$ with coefficients in a representation is defined as

$$\chi_h(\Gamma, V) = \sum_{i=0}^{\infty} (-1)^i \dim H^i(\Gamma, V).$$

For details on the above formula see [Br], [Se]. We recall the definition of orbifold Euler characteristic. If $\Gamma$ is torsion free, then the orbifold Euler characteristic is defined as $\chi_{orb}(\Gamma) = \chi_h(\Gamma)$. If $\Gamma$ has torsion elements and admits a finite index torsion free subgroup $\Gamma'$, then the orbifold Euler characteristic of $\Gamma$ is given by

$$\chi_{orb}(\Gamma) = \frac{1}{[\Gamma : \Gamma']} \chi_h(\Gamma').$$

One important fact is that, following Minkowski, every arithmetic group of rank greater than one contains a torsion free finite index subgroup and therefore the concept of orbifold Euler characteristic is well defined in this setting. If $\Gamma$ has torsion elements then we make use of the following formula discovered by Chiswell in [Ch].

$$\chi_h(\Gamma, V) = \sum_{(T)} \chi_{orb}(C(T)) \text{tr}(T^{-1}|V).$$

Otherwise, we use the formula described in equation (1). The sum runs over all the conjugacy classes in $\Gamma$ of its torsion elements $T$, denoted by $(T)$, and $C(T)$ denotes the centralizer of $T$ in $\Gamma$. From now on, orbifold Euler characteristic $\chi_{orb}$ will be simply denoted by $\chi$. Orbifold Euler characteristic has the following properties.

1. If $\Gamma$ is finitely generated torsion free group then $\chi(\Gamma)$ is defined as $\chi_h(\Gamma, \mathbb{Q})$.
2. If $\Gamma$ is finite of order $|\Gamma|$ then $\chi(\Gamma) = \frac{1}{|\Gamma|}$.
3. Let $\Gamma$, $\Gamma_1$ and $\Gamma_2$ be groups such that $1 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 1$ is exact then $\chi(\Gamma) = \chi(\Gamma_1)\chi(\Gamma_2)$.

Let us denote

$$T_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, T_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } T_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$  

A key result that we are going to use is a computationally effective way of computing the homological Euler characteristic, developed in [H1], [H2]. We know that when $\Gamma$ is $GL_n(\mathbb{Z})$ one has an expression of the form

$$\chi_h(\Gamma, V) = \sum_A \text{Res}(f_A) \chi(C(A)) \text{tr}(A^{-1}|V),$$

where $f_A$ denotes the characteristic polynomial of the matrix $A$.

**Proposition 2.** If $V$ is a representation of $GL_n(\mathbb{Z})$, then for all $n$, we have

$$H^i(SL_n(\mathbb{Z}), V) = H^i(GL_n(\mathbb{Z}), V) + H^i(GL_n(\mathbb{Z}), V \otimes \det).$$
Proof. If $V$ is a finite dimensional representation of $GL_n(\mathbb{Z})$ then we can consider it as a representation of $SL_n(\mathbb{Z})$. As a representation of $SL_n(\mathbb{Z})$ the induced representation to $GL_n(\mathbb{Z})$ is $\text{Ind}(V) = V + V \otimes \text{det}$. Therefore, $H^i(SL_n(\mathbb{Z}), V) = H^i(GL_n(\mathbb{Z}), \text{Ind}(V)) = H^i(GL_n(\mathbb{Z}), V) + H^i(GL_n(\mathbb{Z}), V \otimes \text{det})$.

Corollary 3. If $n$ even then

$$\chi_h(SL_n(\mathbb{Z}), V) = \chi_h(GL_n(\mathbb{Z}), V) + \chi_h(GL_n(\mathbb{Z}), V \otimes \text{det}).$$

It follows directly from the previous proposition.

Now we will explain (4) in detail. The summation is over all possible block diagonal matrices $A \in \Gamma$ satisfying the following conditions:

- The blocks in the diagonal belong to the set \{1, -1, T_3, T_4, T_6\}.
- The blocks $T_3, T_4$ and $T_6$ appear at most once and 1, -1 appear at most twice.
- A change in the order of the blocks in the diagonal does not count as a different element.

So, for example, if $n > 10$, the sum is empty and $\chi_h(\Gamma, V) = 0$.

In this case, one can see that every $A$ satisfying these properties has the same eigenvalues as $A^{-1}$. Even more, every such $A$ is conjugate, over $\mathbb{C}$, to $A^{-1}$ and therefore $Tr(A^{-1}|V) = Tr(A|V)$. We will use these facts in what follows.

Let us explain briefly the notation $\text{Res}(f)$. Let $f_1 = \prod_i(x-\alpha_i)$ and $f_2 = \prod_j(x-\beta_j)$ be two polynomials. Then by the resultant of $f_1$ and $f_2$, we mean $\text{Res}(f_1, f_2) = \prod_{i,j}(\alpha_i - \beta_j)$. If the characteristic polynomial $f$ is a power of an irreducible polynomial then we define $\text{Res}(f) = 1$. Let $f = f_1f_2\ldots f_d$, where each $f_i$ is a power of an irreducible polynomial over $\mathbb{Q}$ and they are relatively prime pairwise. Then, we define $\text{Res}(f) = \prod_{i<j}\text{Res}(f_i, f_j)$.

For any torsion free arithmetic subgroup $\Gamma \subset SL_n(\mathbb{R})$ we have the Gauss-Bonnet formula

$$\chi_h(\Gamma \backslash X) = \int_{\Gamma \backslash X} \omega_{GB}$$

where $\omega_{GB}$ is the Gauss-Bonnet-Chern differential form and $X = SL_n(\mathbb{R})/SO(n, \mathbb{R})$, see [Harder]. This differential form is zero if $n > 2$ and therefore for any torsion free congruence subgroup $\Gamma \subset SL_n(\mathbb{Z})$, $\chi_h(\Gamma \backslash X) = 0$. We will make use of this fact in the calculation of the homological Euler characteristic of $GL_4(\mathbb{Z})$.

2.1. Torsion elements. At this note, let $\Phi_n$ be the $n$-th cyclotomic polynomial then we list all the characteristic polynomials of torsion elements in $GL_4(\mathbb{Z})$ in the following table.

If a torsion element contains in its characteristic polynomial a factor of $\Phi_3^3$ or $\Phi_5^3$, then its centralizer will contain a subgroup commensurable to $GL_3(\mathbb{Z})$. However, the orbifold Euler characteristic of $GL_3(\mathbb{Z})$ is zero. Thus, it will not contribute to the sum of orbifold Euler characteristics. Similarly, if a torsion element contains in its characteristic polynomial a factor of $\Phi_4^4$ or $\Phi_2^4$, then its centralizer will be $GL_4(\mathbb{Z})$. However, the orbifold Euler characteristic of $GL_4(\mathbb{Z})$ is zero. For all $\Phi_n$, when $n = 5, 8, 10, 12$, we are going to show that again the Euler characteristics of their centralizers is zero. For such a torsion elements $A$, we have that the set $M(A)$ of all matrices
X, that commute with such a torsion element \( A \) contain polynomials in \( A \). Since \( \Phi_n(A) = 0 \), we obtain that \( M(A) \subset \mathbb{Z}[x]/\langle \Phi_n \rangle \) has finite index in ring of integers of the cyclotomic field obtained by adjoining a primitive \( n \)-root of 1. By the Dirichlet unit theorem we have that \( \mathbb{Z}[x]/\langle \Phi_n \rangle \) contains a free group \( \mathbb{Z} \) inside the group of units. Moreover the centralizer \( C(A) \) is exactly the invertible elements of \( M(A) \), which is the group of units in \( M(A) \). Since \( M(A) \) has a finite index in \( \mathbb{Z}[x]/\langle \Phi_n \rangle \), we obtain that the centralizer \( C(A) \) is the groups of units \( M(A) \) which is a finite index in the group of units in the cyclotomic field \( \mathbb{Q}[x]/\langle \Phi_n \rangle \). Note that \( \chi(\mathbb{Z}) \) it the same as the Euler characteristic of a circle, which is zero. Thus, \( \chi(C(A)) = 0 \).

Below will list all torsion elements together with their centralizers \( C(A) \), orbifold Euler characteristic of their centralizers \( \chi(C(A)) \), and their resultants \( R(f) \) and finally the product \( \chi(C(A))R(f) \).

Also if \( \Phi_n^2 \) is the characteristic polynomial of a torsion element then its centralizer is commensurable to \( \text{GL}_2(\mathbb{Z}[\xi_n]) \) where \( \xi_n \) is a primitive 3-rd, 4-th or 6-th root of 1. However, \( \chi(C(A)) = \chi(\text{GL}_2(\mathbb{Z}[\xi_n])) = 0 \).

| S.No. | Polynomial | Centralizer \( C(A) \) | \( \det(A) \) | \( \chi(C(A)) \) | \( R(f) \) | \( \chi(C(A))R(f) \) |
|-------|------------|------------------------|--------|----------------|--------|----------------|
| 3     | \( \Phi_1^2 \Phi_2 \) | \( \text{GL}_2(\mathbb{Z}) \times \text{GL}_2(\mathbb{Z}) \) | 1      | \( \frac{1}{24} \) | 24     | \( \frac{36}{24} \) |
| 4     | \( \Phi_1^2 \Phi_3 \) | \( \text{GL}_2(\mathbb{Z}) \times C_6 \) | 1      | \( -\frac{1}{24} \cdot \frac{1}{3} \) | 32     | \( -\frac{32}{24} \) |
| 5     | \( \Phi_1^2 \Phi_4 \) | \( \text{GL}_2(\mathbb{Z}) \times C_4 \) | 1      | \( -\frac{1}{24} \cdot \frac{1}{4} \) | 22     | \( -\frac{44}{24} \) |
| 6     | \( \Phi_1^2 \Phi_6 \) | \( \text{GL}_2(\mathbb{Z}) \times C_6 \) | 1      | \( -\frac{1}{24} \cdot \frac{1}{6} \) | 12     | \( -\frac{24}{24} \) |
| 8     | \( \Phi_1 \Phi_3 \Phi_3 \) | \( C_2 \times C_2 \times C_6 \) | \( -1 \) | \( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} \) | 2 \times 3 | \( \frac{1}{3} \) |
| 9     | \( \Phi_1 \Phi_3 \Phi_4 \) | \( C_2 \times C_2 \times C_4 \) | \( -1 \) | \( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} \) | 2 \times 2 | \( \frac{1}{4} \) |
| 10    | \( \Phi_1 \Phi_3 \Phi_6 \) | \( C_2 \times C_2 \times C_6 \) | \( -1 \) | \( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} \) | 2 \times 3 | \( \frac{1}{3} \) |
| 12    | \( \Phi_2^2 \Phi_3 \) | \( \text{GL}_2(\mathbb{Z}) \times C_6 \) | 1      | \( -\frac{1}{24} \cdot \frac{1}{6} \) | 12     | \( -\frac{12}{24} \) |
| 13    | \( \Phi_2^2 \Phi_4 \) | \( \text{GL}_2(\mathbb{Z}) \times C_4 \) | 1      | \( -\frac{1}{24} \cdot \frac{1}{4} \) | 23     | \( -\frac{23}{24} \) |
| 14    | \( \Phi_2^2 \Phi_6 \) | \( \text{GL}_2(\mathbb{Z}) \times C_6 \) | 1      | \( -\frac{1}{24} \cdot \frac{1}{6} \) | 32     | \( -\frac{72}{24} \) |
| 16    | \( \Phi_3 \Phi_4 \) | \( C_4 \times C_4 \) | 1      | \( \frac{1}{12} \cdot \frac{1}{4} \) | 12     | \( \frac{24}{24} \) |
| 17    | \( \Phi_3 \Phi_6 \) | \( C_4 \times C_6 \) | 1      | \( \frac{1}{12} \cdot \frac{1}{6} \) | 22     | \( \frac{64}{24} \) |
| 19    | \( \Phi_4 \Phi_4 \) | \( C_4 \times C_6 \) | 1      | \( \frac{1}{12} \cdot \frac{1}{6} \) | 12     | \( \frac{24}{24} \) |

Table 1: Torsion elements in \( \text{GL}_4(\mathbb{Z}) \).
Then the Euler characteristic \( \chi_h(GL_4(\mathbb{Z}), \mathbb{Q}) \) is just the sum of the elements

\[
\chi_h(GL_4(\mathbb{Z}), \mathbb{Q}) = \sum_A \chi(C(A))R(f),
\]

which is the sum of the entries in the last column. Note that all fractions with denominators \( 24^2 \) add up to zero. The remaining one are \( \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \). Therefore, we obtain the following.

**Lemma 4.**

\[
\chi_h(GL_4(\mathbb{Z}), \mathbb{Q}) = 1.
\]

For the representation \( \det \), we have that \( Tr(A|\det) = \det(A) \). Therefore,

\[
\chi_h(GL_4(\mathbb{Z}), \det) = \sum_A \det(A)\chi(C(A))R(f).
\]

Note that all the entries with denominators \( 24^2 \) come from torsion element with determinant 1, and all entries with denominators 2 or 4 come from torsion elements with determinant \(-1\). Again, the sum of \( \det(A)\chi(C(A))R(f) = \chi(C(A))R(f) \) over torsion elements of determinant = 1 add up to zero. The remaining torsion elements have determinant \(-1\). Therefore

\[
\chi_h(GL_4(\mathbb{Z}), \det) = \sum_A \det(A)\chi(C(A))R(f) = \sum_{A:\det(A)=-1} \det(A)\chi(C(A))R(f) = -\sum_{A:\det(A)=-1} \chi(C(A))R(f) = -\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right) = -1.
\]

Therefore, we obtain the following.

**Lemma 5.**

\[
\chi_h(GL_4(\mathbb{Z}), \det) = -1.
\]

From Lemmas 4 and 4, and Corollary 3, we have that

**Corollary 6.**

\[
\chi_h(SL_4(\mathbb{Z}), \mathbb{Q}) = 0.
\]

3. **Cohomology of \( GL_4(\mathbb{Z}) \) with coefficients in \( \det \) as vector spaces**

We would like to compute the cohomology of \( GL_4(\mathbb{Z}) \) with coefficients in \( \det \).

In order to do that, first we need the following result by Elbas-Vincent, Gangle and Soule [?]

**Theorem 7.**

\[
H^i(SL_4(\mathbb{Z}), \mathbb{Q}) = \begin{cases} 
\mathbb{Q} & \text{for } i = 0 \ or \ 3 \\
0 & \text{otherwise}
\end{cases}
\]
From Corollary 3, we know that
\[ \dim[H^i(GL_4(\mathbb{Z}), \det)] = \dim[H^i(SL_4(\mathbb{Z}), \mathbb{Q})] - \dim[H^i(GL_4(\mathbb{Z}), \mathbb{Q})]. \]
From the Theorem 7 \( \dim[H^i(SL_4(\mathbb{Z}), \mathbb{Q})] \) completely. We need to examine \( \dim[H^i(GL_4(\mathbb{Z}), \mathbb{Q})] \).

**Proposition 8.**
\[ H^i(GL_4(\mathbb{Z}), \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases} \]

**Proof.** Obviously, \( H^0(GL_4(\mathbb{Z}), \mathbb{Q}) = \mathbb{Q} \). Also, by Theorem 7, we have that \( H^i(GL_4(\mathbb{Z}), \mathbb{Q}) = 0 \) for \( i \neq 0, 3 \) and \( H^3(GL_4(\mathbb{Z}), \mathbb{Q}) = 0 \) or \( \mathbb{Q} \), since \( H^i(GL_4(\mathbb{Z}), \mathbb{Q}) \) is a direct summand of \( H^i(SL_4(\mathbb{Z}), \mathbb{Q}) = 0 \). There are only two possibilities \( H^3(GL_4(\mathbb{Z}), \mathbb{Q}) = 0 \) or \( \mathbb{Q} \). If \( H^3(GL_4(\mathbb{Z}), \mathbb{Q}) = \mathbb{Q} \), then \( \chi_h(GL_4(\mathbb{Z}), \mathbb{Q}) = 0; \) however, \( \chi_h(GL_4(\mathbb{Z}), \mathbb{Q}) = 1. \) Therefore, \( H^3(GL_4(\mathbb{Z}), \mathbb{Q}) \) cannot be \( \mathbb{Q} \). The only other possibility is \( H^3(GL_4(\mathbb{Z}), \mathbb{Q}) = 0 \), which is the statement of the Proposition.

Since \( \dim[H^i(GL_4(\mathbb{Z}), \det)] = \dim[H^i(SL_4(\mathbb{Z}), \mathbb{Q})] - \dim[H^i(GL_4(\mathbb{Z}), \mathbb{Q})] \), we obtain the following.

**Corollary 9.**
\[ H^i(GL_4(\mathbb{Z}), \det) = \begin{cases} \mathbb{Q} & \text{for } i = 3 \\ 0 & \text{otherwise} \end{cases} \]

4. **Spectral sequences**

Our approach is the following. First, we consider a couple of different spectral sequences related to \( GL_3 \) that relate the corresponding \( d_2 \) maps to ghost classes in \( GL_3 \) or to something that we call **potential ghost classes**. Then we consider the spectral sequence for the boundary cohomology of \( GL_4 \). When we restrict the last spectral sequence to a maximal parabolic subgroup containing \( GL_3 \), we reduce it to one of the spectral sequences related to \( GL_3 \). In this way, we can say whether we have a ghost class in \( GL_3 \) after examining cohomology of \( GL_4 \).

4.1. **A couple of spectral sequences related to** \( GL_3(\mathbb{Z}) \). Let \( Q_{12} \) and \( Q_{23} \) be representatives of the two classes of maximal parabolic groups of \( GL_3 \). Let \( Q_0 \) be the Borel subgroup of \( GL_3 \). Let \( H^q_3(GL_3(\mathbb{Z}), V) \) and \( H^q_{Eis}(GL_3(\mathbb{Z}), V) \) be the boundary and the Eisenstein cohomology of \( GL_3(\mathbb{Z}) \) with coefficients in \( V \). For the boundary cohomology of \( GL_3(\mathbb{Z}) \), the spectral sequence gives a long exact sequence
\[ \cdots \rightarrow H^{q-1}(Q_0, V) \rightarrow H^q_{\partial}(GL_3(\mathbb{Z}), V) \rightarrow H^q(Q_{12}, V) + H^q(Q_{23}, V) \rightarrow H^q(Q_0, V) \rightarrow \cdots \]

**Definition 10.** We call an element \( g \in H^q_{\partial}(GL_3(\mathbb{Z}), V) \) a potentially ghost class in \( g = \delta(g_0) \), where \( \delta : H^{q-1}(Q_0, V) \rightarrow H^q_{\partial}(GL_3(\mathbb{Z}), V) \) is the connecting homomorphism. We call \( g \in H^q_{\partial}(GL_3(\mathbb{Z}), V) \) a ghost class if \( g \) is a potentially ghost class and \( g \in H^q_{Eis}(GL_3(\mathbb{Z}), V) \).
Now we define the first spectral sequence related to $GL_3(\mathbb{Z})$. Let
\[
\partial E_0^{0,q} = H_0^q(GL_3(\mathbb{Z}), V) \\
\partial E_1^{1,q} = H^q(Q_{12}, V) + H^q(Q_{23}, V) \\
\partial E_2^{2,q} = H^q(Q_0, V)
\]

**Proposition 11.** A class $g$ from the boundary cohomology $H_0^q(GL_3(\mathbb{Z}), V)$ is a potentially ghost class if and only if $d_2(g) \neq 0$.

**Proof.** By definition
\[
\partial E_2^{0,q} = \ker \left[ d_1^{0,q} : H_0^q(GL_3(\mathbb{Z}), V) \to H^q(Q_{12}, V) + H^q(Q_{23}, V) \right]
\]
and
\[
\partial E_2^{2,q-1} = \coker \left[ d_1^{1,q-1} : H_0^{q-1}(Q_{12}, V) + H^{q-1}(Q_{23}, V) \to H^{q-1}(Q_0, V) \right].
\]

Using the exact sequence (5), we can express $E_2^{0,q}$ in the following way
\[
\partial E_2^{0,q} = \ker [d_1^{0,q}] = \text{im}[\delta].
\]
Again using the exact sequence (5), we can express $E_2^{2,q-1}$ as
\[
\partial E_2^{2,q-1} = \coker [d_1^{1,q-1}] = \partial E_2^{2,q-1}/\text{im}[d_1^{1,q-1}] = \partial E_1^{2,q-1}/\ker[\delta].
\]
The $d_2$ map sends $\partial E_2^{0,q}$ to $\partial E_2^{2,q-1}$ in the following way. Let $x \in \partial E_2^{0,q} = \text{im}[\delta]$. Then, there exists an element $\tilde{x} \in \partial E_2^{2,q-1} = H^{q-1}(Q_0, V)$ such that $\delta(\tilde{x}) = x$. The choice of $\tilde{x}$ is unique up addition with an element of $\text{ker}[\delta]$. Therefore, $\tilde{x}$ is uniquely defined modulo $\ker[\delta]$. Then the map sending $x$ to $\tilde{x}$ modulo $\ker[\delta]$ is well defined. Call this map $d_2$. Then $d_2$ sends $x$ from $\partial E_2^{0,q} = \text{im}[\delta]$ to $\tilde{x}$ modulo $\text{im}[\delta]$ in $\partial E_2^{2,q-1}/\ker[\delta] = E_2^{2,q-1}$.

Now we can prove the Proposition. If $x$ is a potentially ghost class, then $x$ belongs to $\partial E_1^{0,q} = H_0^q(GL_3(\mathbb{Z}), V)$ and $x$ is an image of an element from $H^{q-1}(Q_0, V)$ under the connecting homomorphism $\delta$. Therefore, it is in the kernel of the map
\[
d_1^{0,q} : \left( \partial E_1^{0,q} = H_0^q(GL_3(\mathbb{Z}), V) \right) \to \left( \partial E_1^{1,q} = H^q(Q_{12}, V) + H^q(Q_{23}, V) \right)
\]
Recall that $\text{ker}[d_1^{0,q}]$ is exactly $\partial E_2^{0,q}$. Therefore, $x$ is a nonzero element of $\partial E_2^{0,q}$. Now we have to show that $d_2(x) \neq 0$.

Since $x$ is potentially ghost, we obtain that $x$ is in the image of $\delta$. Let $\tilde{x} \in H^{q-1}(Q_0, V) = \partial E_2^{2,q-1}$ be a pre-image of $x$, that is $\delta(\tilde{x}) = x$. The map $x \mapsto \tilde{x}$ is defined uniquely modulo $\text{ker}[\delta]$. By definition of the map $d_2$, we have that $d_2(x) = \tilde{x}$ modulo $\ker[\delta]$.

Assume that $d_2(x) = 0$. We show that this assumption leads to a contradiction. Therefore, $x$ being a potentially ghost class implies that $d_2(x) \neq 0$. If $d_2(x) = 0$ then $\tilde{x} \in \ker[\delta]$. Therefore, $\tilde{x} + \text{ker}[\delta] = \ker[\delta]$. In particular, $x = \delta(\tilde{x}) = \delta(\tilde{x} + \text{ker}[\delta]) = \delta(\text{ker}[\delta]) = 0$. Then $x = 0$. However, $x$ is a potentially ghost. In particular $x \neq 0$. We arrived at a contradiction. That proves one of the implications of the Proposition.

Conversely: Suppose $d_2$ is a non-zero map. We have to show that there is a potentially ghost class in $H_0^q(GL_3(\mathbb{Z}), V)$. If $d_2 \neq 0$ then there is an element $x \neq 0$ such that $d_2(x) \neq 0$. Then $x$ belongs to $\partial E_2^{0,q} = \ker[d_1^{0,q}] : \partial E_1^{0,q} \to$
Since \( \partial E^{1,q}_1 = H^q_\delta(GL_3(\mathbb{Z}), V) \) and \( \partial E^{1,q}_1 = H^q_\delta(Q_{12}, V) + H^q_\delta(Q_{23}, V) \), we obtain that
\[
\ker[\partial_{E^{1,q}_1} : \partial E^{1,q}_1 \to \partial E^{1,q}_1] = \ker[H^q_\delta(GL_3(\mathbb{Z}), V) \to H^q_\delta(Q_{12}, V) + H^q_\delta(Q_{23}, V)]
\]
Since, we have the long exact sequence (5), we obtain that the kernel above is the image of the connecting homomorphism \( \delta \). Therefore, \( x \in \ker[\partial_{E^{1,q}_1}] = \im[\delta] \). Since \( x \neq 0 \) is in the image of \( \delta \), we have that \( x \) is a potentially ghost class.

Now let us consider the second exact sequence. Let
\[
\begin{align*}
Eis E^{0,q}_1 &= H^q_{Eis}(GL_3(\mathbb{Z}), V) \\
Eis E^{1,q}_1 &= H^q(Q_{12}, V) + H^q(Q_{23}, V) \\
Eis E^{2,q}_1 &= H^q(Q_0, V)
\end{align*}
\]
Denote by
\[
\partial d_2 : \partial E^{0,q}_2 \to \partial E^{2,q-1}_2
\]
the \( d_2 \) map for the first spectral sequence \( E_2 \) that we considered. And let
\[
Eis d_2 : Eis E^{0,q}_2 \to Eis E^{2,q-1}_2
\]
the \( d_2 \) map for the second spectral sequence \( Eis_2 \) that we considered.

**Proposition 12.** The map \( Eis d_2(x) \) is non-zero if and only if \( x \) is a ghost class.

**Proof.** Let \( x \) be a ghost class. Then \( x \) is in \( H^q_{Eis}(GL_3(\mathbb{Z}), V) \) and \( x \) is in the image of \( \delta \). If \( x \) is in the image of \( \delta \) from the long exact sequence it follows that \( x \) is in the kernel of \( H^q_\delta \to H^q(Q_{12}, V) + H^q(Q_{23}, V) \). Also \( x \) belongs to \( H^q_{Eis}(GL_3(\mathbb{Z}), V) = Eis E^{0,q}_1 \). Therefore, \( x \) belongs \( \ker[\partial Eis E^{0,q}_1 \to Eis E^{1,q}_1] \).

The last module is by definition \( Eis E^{2,q}_2 \). So far we obtained that if \( x \) is a ghost class then \( x \) is in \( Eis E^{0,q}_2 \). We are going to show that \( Eis d_2(x) \neq 0 \).

Now, we have to define the map \( Eis d_2 \). Let \( y \) be an element of \( Eis E^{0,q}_2 = \ker[\partial Eis E^{1,q}_1 \to Eis E^{1,q}_1] \). Then \( y \) is an element of \( \ker[\partial Eis E^{0,q}_1 \to Eis E^{1,q}_1] \), since \( Eis E^{0,q}_1 \subset Eis E^{1,q}_1 \) and \( Eis E^{1,q}_1 = Eis E^{1,q}_1 \). Then \( Eis d_2(y) = \partial d_2(y) \) by construction of \( d_2 \) maps in spectral sequences.

In particular, for \( y = x \), where \( x \) is a ghost class, we have that \( x \) is potentially ghost. Therefore \( \partial d_2(x) \neq 0 \). Since \( Eis d_2(x) = \partial d_2(x) \neq 0 \).

Conversely: Let \( Eis d_2(x) \neq 0 \). Then \( \partial d_2(x) \neq 0 \). Therefore \( x \) is a potentially ghost class. Since \( x \) is in \( Eis E^{0,q}_2 = \ker[\partial Eis E^{0,q}_1 \to Eis E^{1,q}_1] \), in particular \( x \) is in \( Eis E^{0,q}_2 \) which is exactly \( H^q_{Eis}(GL_3(\mathbb{Z}), V) \). If \( x \) is a potentially ghost class and \( x \) is in the Eisenstein cohomology \( H^q_{Eis}(GL_3(\mathbb{Z}), V) \), then \( x \) is a ghost class.

**Lemma 13.** We have that if \( d_2 \) is a non-zero map, then the connecting homomorphism \( \delta \) has nonzero image. That is, \( d_2 \neq 0 \) iff \( \im[\delta] \neq 0 \).

**Proof.** We have that in the case of boundary cohomology of \( GL_3(\mathbb{Z}) \), we have that \( d_2 \) is an isomorphism. Then it is nonzero iff the domain is nonzero. Finally, the domain is
\[
\im[\delta] = \ker[H^2_\delta(GL_3(\mathbb{Z}), V_\lambda) \to H^2(Q_1, V_\lambda) + H^2(Q_2, V_\lambda)]
\]
From the Mayer-Vietoris exact sequence. Therefore \( d_2 \neq 0 \) iff \( \im[\delta] \neq 0 \). □
We will use the following notation. The notation \((a, b)\) means the first cohomology of \(GL_2(\mathbb{Z})\) with coefficients in the highest weight representation \(V_{a,b}\) that reduces to the character \((\lambda_1, \lambda_2) \mapsto \lambda_1^a \lambda_2^b\). That is, \((a, b) = H^1(GL_2(\mathbb{Z}), V_{a,b})\). It is quite suitable for computation since we immediately see whether two representations are isomorphic or not, because we see explicitly their characters. We also use \((a|b)\), which stands for \(H^0(GL_1(\mathbb{Z}), V_a) \otimes H^0(GL_1(\mathbb{Z}), V_b)\), where \(V_a\) is the one dimensional representation with a character \(\lambda \mapsto \lambda^a\). From Kostant’s formula we have that \(H^1(GL_2(\mathbb{Z}), V_{a,b})\) is mapped to \(H^0(GL_1(\mathbb{Z}), V_{b-a+1}) \otimes H^0(GL_1(\mathbb{Z}), V_{a+1})\). With our notation, we have \((a, b) \to (b-a|a+1)\). The kernel of this map is the interior cohomology \((a, b) = H^1_{int}(GL_2(\mathbb{Z}), V_{a,b})\). We denote this interior cohomology by \(\overline{(a, b)}\).

Also \((a, b|c)\) stands for \((a, b) \otimes (c) = H^1(GL_2(\mathbb{Z}), V_{a,b}) \otimes H^0(GL_1(\mathbb{Z}), V_c)\). Similarly,

\[
\overline{(a, b|c)} = (a, b) \otimes (c) = H^1_{int}(GL_2(\mathbb{Z}), V_{a,b}) \otimes H^0(GL_1(\mathbb{Z}), V_c)
\]

\[
(a|b, c) = (a) \otimes (b, c) = H^0(GL_1(\mathbb{Z}), V_a) \otimes H^1(GL_2(\mathbb{Z}), V_{b,c})
\]

and

\[
(a|b, c) = (a, b) \otimes (c) = H^0(GL_1(\mathbb{Z}), V_a) \otimes H^1_{int}(GL_2(\mathbb{Z}), V_{b,c}).
\]

If we have to write \(H^0(GL_2(\mathbb{Z}), V_{a,b})\) then we simply write \((a|b)\), since this is naturally isomorphic to \(H^0(GL_1(\mathbb{Z}), V_a) \otimes H^0(GL_1(\mathbb{Z}), V_b)\).

We use the notation \((a, b, c)\) for the induced character \((\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda_1^a \lambda_2^b \lambda_3^c\) on the diagonal.

The notation we use for permutation is the following: 123 is the trivial permutation 231 is simply how the ordered set 123 is permuted; 123 is sent to the ordered set 231 element by elements: 1 goes to the first position, 2 goes to the first position, and 3 goes to the second position.

| \(w\) | \(w(\rho) - \rho\) |
|-----|-----------------|
| 123 | \((a, b, c)\) |
| 132 | \((a, c - 1, b + 1)\) |
| 213 | \((b - 1, a + 1, c)\) |
| 231 | \((b - 1, c - 1, a + 2)\) |
| 312 | \((c - 2, a + 1, b + 1)\) |
| 321 | \((c - 2, b, a + 2)\) |

With our notation, \((1, 0, 0)\) is the character of the standard representation, \((1, 1, 0)\) is the character of the dual representation of the standard representation and \((1, 1, 1)\) is the determinant representation. We need to twist the representation \(V_{1,0,0}\) by the determinant \(V_{1,1,1}\) in order to have a nontrivial cohomology. This twist makes no difference if we restrict to \(SL_3(\mathbb{Z})\). We have \(H^i(GL_3(\mathbb{Z}), V_{1,0,0}) = 0\) for all \(i\), because \(-I\) acts nontrivially on \(V_{1,0,0}\); however, if we twist by the determinant, we have nontrivial and actually a very interesting cohomology groups. Note that \((1, 0, 0) \otimes (1, 1, 1) = (2, 1, 1)\).
5.1. $H^q(GL_3(\mathbb{Z}), V_{1,1,0})$.

| $l$  | $w$ | $w(p) - p$  | $Q_{12}$ | $Q_{23}$ | $Q_0$ |
|-----|-----|--------------|---------|---------|-------|
| 0   | 123 | (1, 1, 0)    | -       | -       | -     |
| 1   | 132 | (1, −1, 2)   | (1, −1|2) | -       | -     |
| 1   | 213 | (0, 2, 0)    | (0|2, 0) | (0|2, 0) | (0|2, 0) |
| 2   | 231 | (0, −1, 3)   | -       | -       | -     |
| 2   | 312 | (−2, 2, 2)   | (−2|2|2) | (−2|2|2) |
| 3   | 321 | (−2, 1, 3)   | -       | -       | -     |

Therefore,

$$H^q(Q_{12}) = \begin{cases} 
(1, −1|2) & q = 2 \\
0 & \text{otherwise,}
\end{cases}$$

$$H^q(Q_{23}) = \begin{cases} 
(0|2, 0) & q = 1 \\
(0|2, 0) + (−2|2|2) & q = 2 \\
0 & \text{otherwise}
\end{cases}$$

and

$$H^q(Q_0) = \begin{cases} 
(0|2, 0) & q = 1 \\
(−2|2|2) & q = 2 \\
0 & \text{otherwise,}
\end{cases}$$

We have that $\dim(2, 0) = \dim H^1(GL_2(\mathbb{Z}), V_{2,0}) = \dim S_4 = 0$ where $S_4$ is the space of cusp forms of weight 4. Therefore $(0|2, 0) = 0$. Also $(1, −1) = H^1(GL_2(\mathbb{Z}), V_{1, −1})$ has boundary part $−2|2|2$ and interior part isomorphic to $(2, 0)$ which vanishes. Therefore $(1, −1|2) = (−2|2|2)$.

After this simplification, we have

$$H^q(Q_{12}) = \begin{cases} 
(1, −1|2) & q = 2 \\
0 & \text{otherwise,}
\end{cases}$$

$$H^q(Q_{23}) = \begin{cases} 
(0|2, 0) & q = 2 \\
0 & \text{otherwise}
\end{cases}$$

For the boundary cohomology we have the long exact sequence (5). The boundary map $H^2(Q_{12}, V_{1,1,0}) \to H^2(Q_0, V_{1,1,0})$ is nontrivial. Since both spaces are one-dimensional, we have that this map is an isomorphism. Therefore $H^3_b(GL_3(\mathbb{Z}), V_{1,1,0})$ is the following extension:

$$0 \to H^1(Q_0, V_{1,1,0}) \to H^3_b(GL_3(\mathbb{Z}), V_{1,1,0}) \to H^3(Q_{12}, V_{1,1,0}) \to 0.$$

Explicitly, the short exact sequence is

$$0 \to (0|2|0) \to H^3_b(GL_3(\mathbb{Z}), V_{1,1,0}) \to (−2|2|2) \to 0.$$

Therefore $(0|2|0)$ is the potentially ghost class for the coefficient system $V_{1,1,0}$, which we will denote by $pG^3_b(GL_3(\mathbb{Z}), V_{1,1,0}) = (0|2|0)$, which lives
inside $H^2_\rho(GL_3(\mathbb{Z}), V_{1,1,0})$. The ghost space $G\!h^2(GL_3(\mathbb{Z}), V_{1,1,0})$ is the intersection of the potentially ghost space with the Eisenstein cohomology. From [BHLM], we know that the Eisenstein cohomology in that case is one dimensional. Suppose $G\!h^2(GL_3(\mathbb{Z}), V_{1,1,0}) = 0$. Then we have that $pG\!h^2(GL_3(\mathbb{Z}), V_{1,1,0}) \cap H^2_{Eis}(GL_3(\mathbb{Z}), V_{1,1,0}) = G\!h^2(GL_3(\mathbb{Z}), V_{1,1,0}) = 0$. Therefore, the composition of the inclusion $H^2_{Eis}(GL_3(\mathbb{Z}), V_{1,1,0}) \to H^2_\rho(GL_3(\mathbb{Z}), V_{1,1,0})$ and the projection $H^2_\rho(GL_3(\mathbb{Z}), V_{1,1,0}) \to (-2|2|2)$ is nontrivial, where $(-2|2|2)$ is a one dimensional summand of $\ker[H^2(Q_{12}, V_{1,1,0}) + H^2(Q_{23}, V_{1,1,0}) \to H^2(Q_0, V_{1,1,0})]$.

Therefore $H^2_{Eis}(GL_3(\mathbb{Z}), V_{1,1,0}) = (-2|2|2)$.

We obtain the following.

**Proposition 14.** (a) If $G\!h^2(GL_3(\mathbb{Z}), V_{1,1,0}) = 0$ then $H^2_{Eis}(GL_3(\mathbb{Z}), V_{1,1,0}) = (-2|2|2)$;
(b) If $G\!h^2(GL_3(\mathbb{Z}), V_{1,1,0}) \neq 0$ then $H^2_{Eis}(GL_3(\mathbb{Z}), V_{1,1,0}) = G\!h^2(GL_3(\mathbb{Z}), V_{1,1,0}) = (0|2|0)$.

**Remark 15.** The two spaces $(-2|2|2)$ and $(0|2|0)$ are non-isomorphic Hecke modules. Therefore, the two modules cannot be interchanged in the above Proposition.

5.2. $H^q(GL_3(\mathbb{Z}), V_{2,1,1})$. For the representation $V_{2,1,1}$, which is the standard representation twisted by the determinant, we have the following.

| $l$ | $w$ | $w(\rho) - \rho$ | $Q_{12}$ | $Q_{23}$ | $Q_0$ |
|-----|-----|-----------------|----------|----------|------|
| 0   | 123 | (2,1,1)         | -        | -        | -    |
| 1   | 132 | (2,0,2)         | (2,0|2)   | (2|0|2)   |      |
| 1   | 213 | (0,3,1)         | (0—3,1) | -        | -    |
| 2   | 231 | (0,0,4)         | (0|0|4)  | (0|0|4)  |      |
| 2   | 312 | (−1,3,2)        | -        | -        | -    |
| 3   | 321 | (−1,1,4)        | -        | -        | -    |

Therefore,

$$H^q(Q_{12}) = \begin{cases} (2|0|2) + (0|0|4) & q = 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^q(Q_{23}) = \begin{cases} (0|3,1) & q = 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^q(Q_0) = \begin{cases} (2|0|2) & q = 1 \\ (0|0|4) & q = 2 \\ 0 & \text{otherwise} \end{cases}$$

We have that $\dim(2,0) = \dim H^1(GL_2(\mathbb{Z}), V_{2,0}) = \dim S_4 = 0$ where $S_4$ is the space of cusp forms of weight 4. Therefore $(2,0|2) = 0$. Also $(3,1) = H^4(GL_2(\mathbb{Z}), V_{3,1})$ has boundary part $(0|4)$ and interior part isomorphic to $(2,0)$ which vanishes. Therefore $(0|3,1) = (0|0|4)$.
After this simplification, we have

\[ H^0(Q_{12}) = \begin{cases} 
0 & q = 0 \\
(0|0|4) & q = 2 \\
0 & \text{otherwise},
\end{cases} \]

\[ H^0(Q_{23}) = \begin{cases} 
(0|2|0) & q = 2 \\
0 & \text{otherwise}
\end{cases} \]

For the boundary cohomology we have the long exact sequence 5. The boundary map \( H^2(Q_{23}, V_{2,1,1}) \to H^2(Q_0, V_{2,1,1}) \) is nontrivial. Since both spaces are one-dimensional, we have that this map is an isomorphism. Therefore \( H^2_{\partial}(GL_3(\mathbb{Z}), V_{2,1,1}) \) is the following extension:

\[ 0 \to H^1(Q_0, V_{2,1,1}) \to H^2_{\partial}(GL_3(\mathbb{Z}), V_{2,1,1}) \to H^2_{\partial}(Q_{12}, V_{2,1,1}) \to 0. \]

Explicitly, the short exact sequence is

\[ 0 \to (2|0|0) \to H^2_{\partial}(GL_3(\mathbb{Z}), V_{2,1,1}) \to (0|0|4) \to 0. \]

Therefore \((2|0|2)\) is the potentially ghost class for the coefficient system \( V_{2,1,1} \), which we will denote by \( pGh^2(GL_3(\mathbb{Z}), V_{2,1,1}) = (2|0|2) \), which lives inside \( H^2_{\partial}(GL_3(\mathbb{Z}), V_{2,1,1}) \). The ghost space \( Gh^2(GL_3(\mathbb{Z}), V_{2,1,1}) \) is the intersection of the potentially ghost space with the Eisenstein cohomology. From \( [BHHM] \), we know that the Eisenstein cohomology in that case is one dimensional. Suppose \( Gh^2(GL_3(\mathbb{Z}), V_{2,1,1}) = 0 \). Then we have that \( pGh^2(GL_3(\mathbb{Z}), V_{2,1,1}) \cap H^2_{\text{Eis}}(GL_3(\mathbb{Z}), V_{2,1,1}) = Gh^2(GL_3(\mathbb{Z}), V_{2,1,1}) = 0 \). Therefore, the composition of the inclusion \( H^2_{\text{Eis}}(GL_3(\mathbb{Z}), V_{2,1,1}) \to H^2_{\partial}(GL_3(\mathbb{Z}), V_{2,1,1}) \) and the projection \( H^2_{\partial}(GL_3(\mathbb{Z}), V_{2,1,1}) \to (0|0|4) \) is nontrivial, where \((0|0|4)\) is a one dimensional summand of \( \ker[H^2(Q_{12}, V_{2,1,1}) + H^2(Q_{23}, V_{2,1,1}) \to H^2(Q_0, V_{2,1,1})] \).

Therefore \( H^2_{\text{Eis}}(GL_3(\mathbb{Z}), V_{2,1,1}) = (0|0|4) \)

We obtain the following.

**Proposition 16.** (a) If \( Gh^2(GL_3(\mathbb{Z}), V_{2,1,1}) = 0 \) then \( H^2_{\text{Eis}}(GL_3(\mathbb{Z}), V_{2,1,1}) = (0|0|4) \);

(b) If \( Gh^2(GL_3(\mathbb{Z}), V_{2,1,1}) \neq 0 \) then \( H^2_{\text{Eis}}(GL_3(\mathbb{Z}), V_{2,1,1}) = Gh^2(GL_3(\mathbb{Z}), V_{2,1,1}) = (2|0|2) \).

**Remark 17.** The two spaces \((0|0|4)\) and \((2|0|2)\) are non-isomorphic Hecke modules. Therefore, the two modules cannot be interchanged in the above Proposition.

### 6. Ghost classes in \( GL_3(\mathbb{Z}) \)

We are going to consider three subsections. The first one will be a preliminary on boundary cohomology of \( GL_4(\mathbb{Z}) \) with coefficients in \( \text{det} = (1,1,1,1) \). The second one will assume that there are no ghost classes in \( H^2(GL_3(\mathbb{Z}), V_{1,1,0}) \), which will lead to a contradiction. And the third one where we consider non-triviality of ghost classes in \( H^2(GL_3(\mathbb{Z}), V_{1,1,0}) \).
6.1. Preliminaries. In the table below we consider the action of all permutations of four elements on the weight $(1,1,1,1)$. The second column gives the length of the permutation. The third column gives $w(p) - p$. And the rest of the columns give the corresponding cohomology of the parabolic subgroups. We use the notation $(1,1,0|2)$ to denote $H^2(GL_3(Z), V_{1,1,0}) \otimes H^0(GL_1(Z), V_2)$.

From table 2, we deduce the following. For the maximal parabolic groups $P_{13}, P_{12,34}$ and $P_{24}$, we have

$$H^q(P_{13}, det) = \begin{cases} (1,1,0|2) + (0|0|0|4) & \text{for } q = 3 \\ 0 & \text{otherwise,} \end{cases}$$

$$H^q(P_{12,34}, det) = \begin{cases} (1,-1|2|2) + (0|0|3,1) & \text{for } q = 3 \\ 0 & \text{otherwise,} \end{cases}$$

$$H^q(P_{24}, det) = \begin{cases} (0|2,1,1) + (-2|2|2|2) & \text{for } q = 3 \\ 0 & \text{otherwise,} \end{cases}$$

For the intermediate parabolic subgroups, we have

$$H^q(P_{12}, det) = \begin{cases} (1,-1|2|2) + (0|0|0|4) & \text{for } q = 3 \\ 0 & \text{otherwise,} \end{cases}$$

$$H^q(P_{23}, det) = \begin{cases} (-2|2|2|2) + (0|0|0|4) & \text{for } q = 3 \\ 0 & \text{otherwise,} \end{cases}$$

$$H^q(P_{34}, det) = \begin{cases} (0|0|3,1) + (-2|2|2|2) & \text{for } q = 3 \\ 0 & \text{otherwise,} \end{cases}$$

For the minimal parabolic subgroup, we have

$$H^q(B, det) = \begin{cases} (0|2|0|2) & \text{for } q = 2 \\ (0|0|0|4) + (-2|2|2|2) & \text{for } q = 3 \\ (-2|0|2|4) & \text{for } q = 8 \\ 0 & \text{otherwise,} \end{cases}$$

6.2. Assume there are no ghost classes in $H^2(GL_3(Z), V_{1,1,0})$. Denote by $Gh^2(GL_3(Z), V_{1,1,0})$ the ghost classes in $H^2(GL_3(Z), V_{1,1,0})$. Then $Gh^2(GL_3(Z), V_{1,1,0}) = 0$ if and only if $Gh^2(GL_3(Z), V_{2,1,1}) = 0$, since $(2,1,1)$ is dual to $(1,1,0)$. This is the case by assumption.

Suppose $H^4_{\text{Eis}}(SL_4(Z), \Q) \neq 0$ then by Poincare duality $H^4_{\text{Eis}}(SL_4(Z), \Q)$ will have half of the dimension of $H^4_{\text{Eis}}(SL_4(Z), \Q)$. In particular $H^4_{\text{Eis}}(SL_4(Z), \Q) \neq 0$. However, from a Theorem 7, we have that $H^4(SL_4(Z), \Q) = 0$. In particular, $H^4_{\text{Eis}}(SL_4(Z), \Q) = 0$. This is is a contradiction due to the assumption that $H^4_{\text{Eis}}(SL_4(Z), \Q) \neq 0$. Therefore $H^4_{\text{Eis}}(SL_4(Z), \Q) = 0$. The induced representation from $SL_4(Z)$ to $GL_4(Z)$ of the trivial representation is $Ind(\Q) = \Q + det$. Therefore, we obtain that $H^4_{\text{Eis}}(SL_4(Z), \Q) = H^4_{\text{Eis}}(GL_4(Z), \Q) + H^4_{\text{Eis}}(GL_4(Z), det) = 0$. We deduce that $H^4_{\text{Eis}}(GL_4(Z), det) = 0$.

We are going to use a spectral sequence that converges to the boundary cohomology. Let

$$E^{0,q}_1 = H^q(P_{13}, det) + H^q(P_{12,34}, det) + H^q(P_{24}, det)$$

$$E^{1,q}_1 = H^q(P_{12}, det) + H^q(P_{23}, det) + H^q(P_{34}, det)$$

$$E^{2,q}_1 = H^q(B, det)$$
COHOMOLOGY OF $SL_3(\mathbb{Z})$ WITH COEFFICIENTS IN $V_{17}^3$

$w \quad l \quad w(\rho) - \rho \quad P_{13} \quad P_{12,34} \quad P_{24} \quad P_{12} \quad P_{23} \quad P_{34} \quad B$

| 1234 | 0 | (1,1,1,1) | - | - | - | - | - | - | - |
| 1243 | 1 | (1,1,0,2) | (1,1,0|2) | - | - | - | - | - | - |
| 1324 | 1 | (1,0,2,1) | - | - | - | - | - | - | - |
| 1342 | 2 | (1,0,0,3) | - | - | - | - | - | - | - |
| 1423 | 2 | (1,-1,2,2) | (1,-1|2|2) | - | - | - | - | - | - |
| 1432 | 3 | (1,-1,1,3) | - | - | - | - | - | - | - |
| 2134 | 1 | (0,2,1,1) | - | - | (0|2,1,1) | - | - | - | - |
| 2143 | 2 | (0,2,0,2) | - | - | - | - | - | - | - |
| 2314 | 2 | (0,0,3,1) | (0|0,3,1) | - | - | - | (0|0|3,1) | - |
| 2341 | 3 | (0,0,0,4) | (0|0,0|4) | - | - | (0|0|0|4) | - | - | - |
| 2413 | 3 | (0,-1,3,2) | - | - | - | - | - | - | - |
| 2431 | 4 | (0,-1,1,4) | - | - | - | - | - | - | - |
| 3124 | 2 | (-1,2,2,1) | - | - | - | - | - | - | - |
| 3142 | 3 | (-1,2,0,3) | - | - | - | - | - | - | - |
| 3214 | 3 | (-1,1,3,1) | - | - | - | - | - | - | - |
| 3241 | 4 | (-1,1,0,4) | - | - | - | - | - | - | - |
| 3412 | 4 | (-1,-1,3,3) | - | - | - | - | - | - | - |
| 3421 | 5 | (-1,-1,2,4) | - | - | - | - | - | - | - |
| 4123 | 3 | (-2,2,2,2) | - | - | (2|2,2|2) | - | (2|2|2|2) | (2|2|2|2) | - |
| 4132 | 4 | (-2,2,1,3) | - | - | - | - | - | - | - |
| 4213 | 4 | (-2,1,3,2) | - | - | - | - | - | - | - |
| 4231 | 5 | (-2,1,1,4) | - | - | - | - | - | - | - |
| 4312 | 5 | (-2,0,3,3) | - | - | - | - | - | - | - |
| 4321 | 6 | (-2,0,2,4) | - | - | - | - | - | - | (2|0|2|4) |

Table 2. Cohomology of the parabolic subgroups
We have that $E^{1,2}_1 = 0$ and $E^{2,2}_1 = H^2(B, \det) = (0|2|0|2)$. Therefore, $E^{3,2}_2 = E^{2,2}_1/im[E^{1,2}_1 \to E^{2,2}_1] = E^{3,2}_1 = (0|2|0|2)$. Since $H^3_3(GL_4(\mathbb{Z}), \det) = 0$, we have that $E^{3,2}_3 = 0$. It implies that $E^{3,2}_2 = (0|2|0|2)$ is in the image of the $d_2$ map of the spectral sequence. That is

$$E^{2,2}_2 = \text{im}[d_2].$$

Due to the assumption that there are no ghost classes with coefficients in $V_{1,1,0}$, from Proposition 14 we have that $H^2_{Eis}(GL_3(\mathbb{Z}), V_{1,1,0}) = (-2|2|2)$. That assumption implies that there are no ghost classes with coefficients in $V_{2,1,1}$. Then from Proposition 16 we have that $H^2_{Eis}(GL_3(\mathbb{Z}), V_{2,1,1}) = (0|0|4)$. Therefore, $(1, 1, 0|2) = (-2|2|2|2)$ and $(0|2, 1, 1) = (0|0|0|4)$ Therefore,

$$H^3(P_{13}, \det) = (-2|2|2|2)$$

and

$$H^3(P_{24}, \det) = (0|0|0|4).$$

For $P_{12,34}$, we have the following simplification.

$$(1, -1|2|2) = H^1(GL_2(\mathbb{Z}), V_{1,-1}) \otimes H^1(GL_1(\mathbb{Z}), V_2) \otimes H^1(GL_1(\mathbb{Z}), V_2)$$

It is isomorphic to the boundary cohomology

$$H^1(GL_1(\mathbb{Z}), V_{-2}) \otimes H^1(GL_1(\mathbb{Z}), V_2) \otimes H^1(GL_1(\mathbb{Z}), V_2) \otimes H^1(GL_1(\mathbb{Z}), V_2) = (-2|2|2|2)$$

Similarly, $(0|0|3, 1) = (0|0|0|4)$ Therefore

$$H^3(P_{12,34}, \det) = (1, -1|2|2) + (0|0|3, 1) = (-2|2|2|2) + (0|0|0|4).$$

Since $H^3$ of maximal parabolic subgroups contribute to $E^{p,q}_1$ for $p = 0$ and $q = 3$, we have that

$$E^{0,3}_1 = (-2|2|2|2) + (-2|2|2|2) + (0|0|0|4) + (0|0|0|4).$$

Then $E^{0,3}_2$ is a subspace of $E^{0,3}_1$, therefore it involves only the Hecke modules $(-2|2|2|2)$ and $(0|0|0|4)$. However, none of them is isomorphic to $E^{2,2}_2 = (0|2|0|2)$ as one dimensional Hecke modules. Therefore, the $d_2$ map

$$d_2 : E^{0,3}_2 \to E^{2,2}_2$$

is the zero map; that is $d_2 = 0$. This contradicts the isomorphism (6).

The contradiction is due to the assumption that there are no ghost classes with coefficients in $V_{1,1,0}$.

6.3. **Existence of ghost classes in $GL_3(\mathbb{Z})$.** Due to the contradiction in the previous section we have that there are ghost classes in $GL_3(\mathbb{Z})$. More precisely,

**Theorem 18.** (a) There is are non-trivial ghost classes in $H^2(GL_3(\mathbb{Z}), V_{1,1,0})$ and in $H^2(GL_3(\mathbb{Z}), V_{2,1,1})$. Equivalently,

$$Gh^2(GL_3(\mathbb{Z}), V_{1,1,0}) \neq 0$$

and in

$$Gh^2(GL_3(\mathbb{Z}), V_{2,1,1}) \neq 0.$$
(b) The group cohomology of $H^2(GL(\mathbb{Z}), V_{1,1,0})$ and $H^2(GL(\mathbb{Z}), V_{2,1,1})$ are concentrated in degree 2 and consist of ghost classes and the zero vector. Namely

\[ H^2(GL(\mathbb{Z}), V_{1,1,0}) = Gh^2(GL(\mathbb{Z}), V_{1,1,0}) = (0|2|0). \]

and

\[ H^2(GL(\mathbb{Z}), V_{2,1,1}) = Gh^2(GL(\mathbb{Z}), V_{2,1,1}) = (2|0|2). \]

Proof. Part (a) follows from the last Subsection. Part (b) follows from Propositions 14 and 16.

7. Application of Ghost Classes in $GL(\mathbb{Z})$ to the Cohomology $H^q(GL(\mathbb{Z}), det)$

The author published a paper on cohomology of $GL(\mathbb{Z})$ with non-trivial coefficients. The coefficients were $\text{Sym}^nV \otimes \text{det}$ is the $n$-th symmetric powers of the standard representation twisted by the determinant. The statements there are correct with one modification: The cohomology with coefficients in $\text{det}$ have the same dimension as stated in the paper; however, the corresponding Hecke modules are different.

In this Subsection we will find exactly which Hecke module occurs in the cohomology groups.

Since $Gh^2(GL(\mathbb{Z}), V_{1,1,0}) = (0|2|0)$ and $Gh^2(GL(\mathbb{Z}), V_{2,1,1}) = (2|0|2)$, we have that

\[ H^3(P_{13}, det) = (1, 1, 0|2) = (-0|2|0|2) \]

and

\[ H^3(P_{24}, det) = (0|2, 1, 1) = (-0|2|0|2). \]

The module $(0|0|0|4)$ appears once in $E_1^{0,3}$ twice in $E_1^{1,3}$ and once in $E_1^{2,3}$. It does not appear anywhere else in the spectral sequence. Therefore, it does not appear at the $E_2$-level. Similarly, $(-2|2|2|2)$ appears once in $E_1^{0,3}$ twice in $E_1^{1,3}$ and once in $E_1^{2,3}$. It does not appear anywhere else in the spectral sequence. Therefore, it does not appear at the $E_2$-level. Therefore, the only non zero terms at the $E_2$-level are

\[ E_2^{0,3} = (0|2|0|2) + (0|2|0|2), \]

\[ E_2^{2,2} = (0|2|0|2) \]

\[ E_2^{2,6} = (-2|0|2|4) \]

The $d_2$-map is non-trivial $d_2 : E_2^{0,3} \rightarrow E_2^{2,2}$. (Otherwise we will have non-trivial 4-th cohomology of $GL(\mathbb{Z})$.) Therefore, the $E_3$ level is different from the $E_2$-level.

\[ E_3^{0,3} = (0|2|0|2) \]

and

\[ E_3^{2,6} = (-2|0|2|4) \]

The spectral sequence stabilizes at the $E_3$-level. Therefore,

\[ H_3^q(GL(\mathbb{Z}), det) = \begin{cases} 
(0|2|0|2) & q = 3 \\
(-2|0|2|4) & q = 8 \\
0 & \text{otherwise}
\end{cases} \]
We have that $H^q(\text{GL}_4(\mathbb{Z}), \det)$ is concentrated in degree 3. In general, the interior cohomology of $\text{GL}_4(\mathbb{Z})$ is concentrated in degree 4 and 5. Think of the cuspidal cohomology as the analytic version of the interior cohomology. Therefore, the groups cohomology and the Eisenstein cohomology coincide in this case; that is, $H^q(\text{GL}_4(\mathbb{Z}), \det) = H^q_{\text{Eis}}(\text{GL}_4(\mathbb{Z}), \det)$. Then, we obtain a minor correction of the paper [H3], stated in the following theorem.

**Theorem 19.**

$$H^q(\text{GL}_4(\mathbb{Z}), \det) = \begin{cases} (0|2|0|2) & q = 3 \\ 0 & \text{otherwise} \end{cases}$$

8. **Final remarks**

**Remark 20.** (On cohomology of $\text{GL}_4(\mathbb{Z})$) In the paper [H3], we considered cohomology of $\text{GL}_4(\mathbb{Z})$ with coefficients a family of representations $\text{Sym}^{n-4}V_4 \otimes \det$. The output was a family of Hecke modules. For $n = 4$, that is, for the representation, $\det$, we took the Hecke module corresponding to $n = 0$. However, the case $n = 0$, should have been considered independently from the case $n > 0$. This is what we did at the end of this paper. Both in the current paper and in [H3], we have that $H^q(\text{GL}_4(\mathbb{Z}), \det)$ is concentrated in degree $q = 3$ and it is one dimensional. The difference is the corresponding Hecke module. In [H3], we obtained $H^3(\text{GL}_4(\mathbb{Z}), \det) = (0|0|0|4)$, while the corrected version, presented here is

$$H^3(\text{GL}_4(\mathbb{Z}), \det) = (0|2|0|2),$$

which stands for $H^0(B, V_{0,2,0,2})$.

We also believe that there are other coefficient systems in $\text{GL}_4(\mathbb{Z})$ that give ghost classes.

**Conjecture 21.** (On ghost classes in $\text{GL}_4(\mathbb{Z})$)

(a) If we denote by $\text{Gh}^q(\Gamma, V)$ the ghost space of $H^q(\Gamma, V)$, then the only nontrivial ghost spaces of $\text{GL}_4(\mathbb{Z})$ with coefficients in any finite dimensional highest weight representation are $\text{Gh}^2(\text{GL}_3(\mathbb{Z}), S_{2n+1}V_3 \otimes \det)$ and $\text{Gh}^2(\text{GL}_4(\mathbb{Z}), S_{2n+1}V_3^*)$, where $S_{2n+1}V_3$ is any odd symmetric power of the standard representation $V_3$ and $S_{2n+1}V_3^*$ is its dual. Each of those spaces is one dimensional.

(b) From (Harder, Jitendra, Matias and me) it follows that the potentially ghost classes in those cases are one-dimensional. A reformulation of this conjecture is

$$\text{Gh}^2(\text{GL}_3(\mathbb{Z}), S_{2n+1}V_3 \otimes \det) = p\text{Gh}^2(\text{GL}_3(\mathbb{Z}), S_{2n+1}V_3 \otimes \det)$$

and

$$\text{Gh}^2(\text{GL}_3(\mathbb{Z}), S_{2n+1}V_3^*) = p\text{Gh}^2(\text{GL}_3(\mathbb{Z}), S_{2n+1}V_3^*).$$

(c) For all other representations $V$ of $\text{GL}_3(\mathbb{Z})$ we have that $\text{Gh}^2(\text{GL}_3(\mathbb{Z}), V) = p\text{Gh}^2(\text{GL}_3(\mathbb{Z}), V) = 0$. A reformulation of the conjectures in parts (a) and (b) is

$$\text{Gh}^2(\text{GL}_3(\mathbb{Z}), V) = p\text{Gh}^2(\text{GL}_3(\mathbb{Z}), V),$$

for all finite dimensional highest weight representations $V$ of $\text{GL}_3(\mathbb{Z})$. 
Recall that $\Gamma_1(n,p)$ is the stabilizer mod $p$ of the vector $[1,0,\ldots, 0]$ in $GL_n(\mathbb{Z})$

**Conjecture 22.** *(On ghost classes in $\Gamma_1(3,p)$)*

From considerations of $\Gamma_1(4,p)$, we expect that

$$Gh^2(\Gamma_1(3,p), V) = pGh^2(\Gamma_1(3,p), V)$$

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