Magnetised relativistic accretion disc around a spinning charged accelerating black hole

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This paper studied the relativistic accretion thick disc model in the background of a spinning charged accelerating black hole described by the C-metric family. We generalise the construction of the thick disc model to the spinning charged C-metric. This work aims to study the effects of this background on the thick magnetised disc model via studying the properties of these equilibrium sequences of magnetised, non-self-gravitating discs. In particular, we analyse the influence of the strength of the magnetic field in this space-time. We show the properties of this relativistic accretion disc model and its dependence on the initial parameters. In principle, this space-time could be distinguished from a Kerr space-time by observing the features of accretion disc in its vicinity. Besides, this theoretical model can serve as the initial data for numerical simulations.

I. INTRODUCTION

Black holes are some of the most extreme objects as the strongest gravitational field sources known in our universe, especially with the nowadays outstanding series of available observations. In view of a general agreement, the observed properties of many astrophysical objects could be best explained in the framework of accretion disc models. In fact, the investigation of the proper disc models, by analytical or numerical setups, rely on the ability of constructing suitable representations based on physical assumptions. Among this successful theoretical models is thick accretion disc with a toroidal shape and no magnetic field which was first introduced in 1974 and presented in these seminal works [1, 8]. This model provides a general method to build equilibrium configurations of the perfect fluid matter orbiting around a stationary and axially symmetric black hole. After the confirmation of the significant role of magnetic field in astronomical phenomena [9], Komissarov proposed a magnetic version of this model [10]. This work involves an analytic solution for an axisymmetric, stationary torus with the constant specific angular momentum distribution and a toroidal magnetic field configuration, which can be served as criterion for numerical MHD.

These mentioned works considered accretion onto the Kerr black hole. In this work, we focus on the thick disc model in the space-time of a spinning charged accelerating black hole described by a generalized family of C-metric. The C-metric originally belongs to the large class of solutions discovered by Levi-Civita [11]. However, through a series of different transformations, from the Plebański-Demiański class of electrovacuum spacetimes [12], one could obtain the spinning charged C-metric, while it is not analytically pleasant. In 2004, Hong and Teo reexpressed the metric in a factorized version that is easier to work but also possible to be presented in Boyer-Lindquist-type coordinates [13, 14]. In fact, the maximal analytical extension of this metric describes two causally disconnected black holes accelerating in opposite directions [15].

In this paper we investigate this background with the aim of study the properties of magnetised tori and the morphology of the equipotential surfaces.

There are different motivations to consider this research. First of all, this is an exact solution to Einstein’s field equation even if it may sound to have nonphysical interpretation. However, the existence of singularities in solutions has also been interpreted as nonphysical for around a century, and people identify them with black holes with solid evidence. Besides, most of the works in the astrophysics area have been done by assuming that the Schwarzschild or Kerr metrics describe astrophysical compact objects in the relativistic astrophysical study. However, besides these setups, all astrophysical observations may not be fitted, in general, within the general theory of relativity by using the Schwarzschild or Kerr metric [16, 18].

In this perspective, the family of C-metric could be a hypothetical candidate for objects that exist in nature. In order to investigate this question, the study of its fingerprint in the observational data can be a proper first step. It may sound that the only source of information that we have in the strong gravity regime is coming from its environment, like the lensing or accretion discs, especially with the advent of horizon-scale observations of astrophysical black holes. There is almost a vast literature on the lensing in using this metric e.g. [19, 22]; however, to our knowledge, this work is one of the first (semi-)analytic works in studying accretion disc in this background.

The organization of the paper is as follows: the space-time is briefly explained in [14]. The relativistic magnetised tori presented in section [11]. The results and discussion are presented in Section [14] and the conclusions are summarised in Section [15]. In this paper, the geometrized units where \( c = 1 \), and \( G = 1 \), also the signature \((- + + +)\) are used.
II. THE SPINNING CHARGED C-METRIC

The family of C-metric has accelerating nature and is considered as describing an accelerating black hole [15]. The spinning charged C-metric in Boyer-Lindquist-type coordinates [14] reads as

$$ds^2 = \frac{1}{\Omega^2} \left( -\frac{f}{\Sigma} \left[ dt - a \sin^2 \theta \frac{d \varphi}{K} \right]^2 + \frac{\Sigma}{f} dr^2 + \Sigma r^2 \frac{d \theta^2}{g} + \frac{g \sin^2 \theta}{\Sigma r^2} \left[adt - (r^2 + a^2) \frac{d \varphi}{K} \right]^2 \right),$$

where

$$\Omega = 1 + \alpha r \cos \theta,$$

$$f(r) = (1 - \alpha^2 r^2) \left( 1 - \frac{2m}{r} + \frac{e^2 + a^2}{r^2} \right),$$

$$g(\theta) = (e^2 + a^2) \alpha^2 \cos^2 \theta + 2ma \cos \theta + 1,$$

$$\Sigma(r, \theta) = \frac{a^2}{r^2} \cos^2 \theta + 1,$$

$$\xi = \alpha^2 (e^2 + a^2) + 1,$$

$$K = \xi + 2ma,$$

where $$t \in (-\infty, +\infty)$$, $$\theta \in (0, \pi)$$, $$r \in (0, +\infty)$$. The metric has four independent parameters: the mass $$m$$, the electric charge $$e$$, the rotation $$a$$, and the so-called acceleration parameter $$\alpha$$.

In this metric $$r = 0$$ is the curvature singularity, and there is a conical singularity on the $$\theta$$ axis. However, the parameter $$K$$ affects the distribution of conical defects in the space-time and allows $$\varphi$$ to be a 2$$\pi$$-periodic. In fact, the conical deficit is associated to the presence of a cosmic string. The deficit along both axis $$\theta = 0$$, and $$\theta = 2\pi$$ are not the same, and this imbalance tension is the origin of the driven acceleration. The parameter $$K$$ regulates the distribution of tensions along either axis. It is also worth mentioning that a negative deficit is also possible; however, this would be sourced by a negative energy object.

Almost all studies with this metric are revolved around the coordinate ranges, which are dictated by the metric functions and their root configurations. First of all the conformal factor $$\Omega$$ determines the location of the boundary

$$r_b = -\frac{1}{\alpha \cos \theta}.$$

In addition, the roots of metric function $$f(r)$$ correspond to horizons. Thus, $$f(r)$$ should have at least one root for $$r \in (0, \frac{b}{a})$$ to have a black hole in the space-time. However, mostly with charge, generic configurations have different distinct horizons in a pair of inner and outer horizons. Like the regular Reissner-Nordstrom solution, they typically approach one another and vanish in the case of a larger charge. Furthermore, when the acceleration horizon is present, there is a second outer acceleration horizon, and both intersect with the boundary. In general, the pairs of horizons separated the space-time into different regions which share the same signature. For astrophysics point of view, we are interested in studying the accretion disc in the outer communication region between the outer horizon and the acceleration horizon. In Figures 1 and 2, the place of inner and outer horizon have presented for the chosen parameters. Therefore, by increasing $$e$$ the place of horizons become closer to each other, and to the black hole. However, since the place of accelerating horizon is just depend on $$\alpha$$, the valid region becomes wider. The same behaviour expected for increasing $$a$$, as the metric function $$f(r)$$ is symmetric in parameters $$e$$ and $$a$$.

Finally, from the analyzing the $$\theta$$-coordinate, the metric function $$g(\theta)$$ should have solution and be positive on $$[0, \pi]$$. Therefore it requires to have

$$e^2 + a^2 \leq m^2,$$  \hspace{1cm} (9)

also the following condition should be fulfilled
FIG. 3: Allowed parametric regions for the spinning charged C-metric as a function of e. The regions marked out with hatching corresponds to the forbidden regions. First one represent the result for $a = 0$, and the second one for $a = 0.5$.

$$2ma \leq \begin{cases} 2\sqrt{\xi - 1} & \xi > 2, \\ \xi & 0 < \xi \leq 2. \end{cases}$$ (10)

In Figures 3 and 4 show different parametric setups. In this Figure the hatched part is the excluded region by equation (10) for chosen parameters. We see that this condition acts as an upper bound on the rotation or on the acceleration for smaller parameter values. In Figure 5 the metric function $g(\theta)$ is plotted for chosen parameters. As it has shown, the forbidden region is the hatched part, which shrinks as $a$ and $e$ increase.

Before we describe the construction of the relativistic thick disc model, we shortly discuss the modified von Zeipel radius or radius of gyration which is useful in the concept of thick disc model. This radius is the constant surfaces $R$, which for an axisymmetric and stationary metric is defined by

$$R = \frac{g_{\phi\phi}^2}{g_{\theta\theta} - g_{\theta\phi}g_{\phi\phi}},$$ (11)

with respect to the stationary observers, and known as von Zeipel cylinders [23]. For different sets of parameters in this space-time, we see them in Figure 6. This radius helps to analyze circular particle motion and have an intuitive image of them in this space-time, also they relate to the inertial forces. In the concept of the thick disc model by von Zeipel theorem, we can conclude that for a constant angular momentum distribution, the surface of constant $R$ and constant $\Omega$ coincide. This model is explained briefly in the next section.

### III. RELATIVISTIC THICK DISC MODEL

The thick disc model presents a general method of constructing perfect fluid equilibria, which is the simplest analytical model of not self-gravitating discs with no accretion flow based on Boyer’s condition [24]. Also, this models radiatively inefficient discs. In fact, this model is proper framework to describe properties of the astrophysical object, where the radial pressure gradients can not negligible and leads to a significant vertically growth in the size of the disc.

In this model, the equation of state is taken to be barotropic, and self-gravity is negligible. Also, rotation of perfect fluid is assumed to be in the azimuthal direction, then the four-velocity and stress-energy tensor simplifies to

$$u^\mu = (u^t, 0, 0, u^\phi),$$ (12)

$$T^\mu_\nu = wu^\mu u^\nu - \delta^\mu_\nu p,$$ (13)

here $w$ is the enthalpy, and $p$ is pressure. In the energy-momentum tensor, the dissipation due to the viscosity and the heat conduction are neglected. The relativistic Euler equation by considering the projection of conservation of stress-energy tensor into the plane normal to four-velocity is then written as [25],
FIG. 5: Plots of metric function $g(\theta)$. The red line corresponds to $g(\theta) = 0$, and the hatched region shows $g(\theta) < 0$.

FIG. 6: Von Zeipel cylinders with respect to the stationary observers. The plots show constant $R$ surfaces where the circular time-like motion is possible.
\[
\int_0^p \frac{dp}{w} = -\ln |u_t| + \ln |(u_t)_{in}| + \int_{\ell_{in}}^\ell \frac{\Omega d\ell}{1 - \Omega^2},
\]
where \(\ell = -\frac{u_w}{\Omega},\) and \(\Omega = \frac{u^\theta}{\ell},\) also the subscript \(\text{in}\) refers to the inner edge of the disc. The integrability condition implies \(\Omega = \Omega(\ell),\) and this relation satisfies the general relativistic version of the von Zeipel theorem for a toroidal magnetic field which states the surfaces of constant \(p\) coincide with constant \(w\) if and only if constant \(\Omega\) and constant \(\ell\) coincide, in another word the surfaces \(R = \text{constant}\) \((11),\) coincide with the surfaces \(\Omega = \text{constant}.\)

However, for the non magnetised version, the surface of equal \(\Omega, \ell, p\) and \(w\) all coincide.

The generalization of the thick disc model by considering magnetic field has been developed by \([10],\) which is explained briefly in the following subsection.

### A. Magnetised version

The evolution of an ideal magnetised fluid is described by the following conservation laws \([26, 27],\)

\[
\begin{align*}
\nabla_\nu (\rho u^\nu) &= 0, \\
\n\nabla_\nu T^{\nu\mu} &= 0, \\
\n\nabla_\nu (b^\nu u^\mu - b^\mu u^\nu) &= 0,
\end{align*}
\]

The baryon conservation, energy-momentum conservation and induction equation, respectively. Here, \(b^\mu = (0, B)\) and is related to the magnetic pressure in the fluid frame as \(|b|^2 = 2\rho_m\) \([26, 27].\) The total energy-momentum tensor of the fluid together with the electromagnetic field, considering the variation in pressure and density are adiabatic, reads as \([26],\)

\[
T^{\nu\mu} = (w + |b|^2) u^\nu u^\mu + \left(\rho_{\text{gas}} + \frac{1}{2} |b|^2\right) g^{\mu\nu} - b^\nu b^\mu,
\]

where \(w\) is enthalpy, \(\rho_{\text{gas}}\) is the gas pressure. We proceed here following \([10],\) by assuming purely rotational fluid motion and purely toroidal magnetic field,

\[
u^r = u^\theta = b^r = b^\theta = 0.
\]

Thus, considering the assumptions, the only task is to solve the following equation,

\[
\int_0^p \frac{dp}{w} + \int_0^{\tilde{p}_m} \frac{d\tilde{p}_m}{\tilde{w}} = -\ln |u_t| - \ln |(u_t)_{in}| + \int_{\ell_{in}}^{\ell} \frac{\Omega d\ell}{1 - \Omega^2},
\]

where \(\tilde{p}_m = L\rho_m,\) and \(\tilde{w} = Lw.\) Also, \(L = g_{\phi\phi}^2 - g_{tt} g_{\phi\phi}.\) Adopting \([10],\) to be able to solve equations, we assume the following relations

\[
p = Kw^\kappa, \quad \tilde{p}_m = K_m \tilde{w}^\eta
\]

where \(K, \kappa, K_m\) and \(\eta\) are constants. \(\tilde{p}_m\) can be rewritten in terms of the magnetic pressure as \(p_m = K_m L^{-1} w^\eta.\) In \([20]\) the constant of integration was chosen in a way that on the surface of the disc and its inner edge, i.e. \(u_t = (u_t)_{in},\) and \(\ell = \ell_{in},\) we have vanishing pressures. Thus, the equation \((20)\) can fully integrate

\[
W - W_{\text{in}} + \frac{\kappa}{\kappa - 1} \frac{p}{w} + \frac{\eta}{\eta - 1} \frac{p_m}{w} = \int_{\ell_{in}}^{\ell} \frac{\Omega d\ell}{1 - \Omega^2},
\]

where \(W = \ln |u_t|\). This equation implies \(\Omega = \Omega(\ell)\) and the surface of equal \(\Omega, \ell, p,\) and \(\rho\) coincide for vanishing \(p_m\) \([25].\) So, by specifying \(\Omega = \Omega(\ell),\) one can construct the model and then \(W\) and \(p\) easily are followed. Also, \(\ell\) needs to be specified to fix the geometry of the equipotential surfaces. In fact, by considering a constant distribution profile \(\ell = \ell_0,\) the right-hand side of the equation vanishes. However, \(\ell_0\) should be chosen in the interval between the marginally stable orbit \(l_{ms}\) and the marginally bound orbit \(l_{mb},\) to constructing a finite-size disc. Therefore, the disc surface is fully determined by the choice of \(W_{\text{in}}\) independently of the magnetic field \([28]\) and the value of \(\ell_0\) determines the total potential. In this case the total potential obtains as

\[
W(r, \theta) = \frac{1}{2} \ln \frac{L}{A},
\]

where \(A = g_{\phi\phi}^2 + 2\ell_0^2 g_{\theta\theta} + \ell_0^2 g_{tt}\). Thus \(W\) satisfies this relation \([25],\)

\[
\begin{cases}
W_{\text{in}} \leq W_{\text{cusp}} & \text{if } |l_{ms}| < |\ell_0| < |l_{mb}|, \\
W_{\text{in}} < 0 & \text{if } |\ell_0| \geq |l_{mb}|.
\end{cases}
\]

The cusp point is obtained at the intersection of the specific angular momentum and the Keplerian one. Also, the gas pressure at the center \(p_c,\) reads as

\[
p_c = w_c (W_{\text{in}} - W_c) \left(\frac{\kappa}{\kappa - 1} + \frac{\eta}{\beta_c (\eta - 1)}\right)^{-1},
\]

where the subscript \(c\) refers to the mentioned quantity at the center. Also, \(\beta_c = p_c / p_{nc}\) is the magnetization parameter at the center. The variables of model are then \(W, w, p, p_m, u^t, u^\phi, b^t\) and \(u^\phi.\) So by using equation of state, one can find \(K\) and \(K_m,\) then the solution is obtained utilizing \([22]\) and \([23].\)

In what follows we build the thick disc model in this space-time; however, because of the conical deficit the disk does not lie in the equatorial plane and finding conditions of existence of equipotential surfaces are more challenging as in the Kerr metric.
FIG. 7: Possible region for having the thick disc model. The dashed curves shows when $\partial_r W = 0$ and $\partial_\theta W = 0$. The dark blue regions shows area where the condition for a maximum of $W(r, \theta)$ are fulfilled. The white area depicts where the condition for a minimum of $W(r, \theta)$ are fulfilled.

IV. RESULTS AND DISCUSSION

In this section, we analyzed the impact of the different parameters of the model on the morphology of the equipotential surfaces. In Figures 7 and 8, we examine the possibility of having solutions for this disc’s model depending on the variation of the parameters. Figure 7 shows the regions in the non-spinning charged C-metric. In the panel, the intersection of the dashed curve with the white area is where we can choose the centre of the disc, and in the dark-blue region, we can have the cusp points. In general, as the acceleration parameter $\alpha$ increases, almost the possibility of having solutions decreases dramatically. On the contrary, the charge parameter $e$ has an imperceptible impact; however, it positively contributes to having solutions, especially its effect manifests more for a relatively large $\alpha$. For example, as seen in the second row, for vanishing $e$, we do not have a solution (first plot); however, by increasing $e$, we obtain solutions (second plot). Since, by increasing $e$, the centre’s location in the white area, and the cusp’s location in the dark-blue area move away from each other leading to increase the possibility of having solutions. In addition, from the panel, the possibility of existence of two cusps is also predicted.

In Figure 8, by using the information in Figure 7, we choose a model with an inner cusp, a centre and an outer cusp to see the largest possible model. The cusp point is the location where accretion can start. In the columns 2 and 3 of the Figure 8 we consider also rotation parameter $a$. In general, the effect of rotation parameter $a$ on having solutions is not strong compared to $\alpha$ but stronger than the charge parameter $e$. In fact, parameter $a$, like $\alpha$, has a negative effect on having solutions. Contrary to $e$, by increasing $a$, the centre and the cusp’s location approach one another, and gradually we do not have any solutions. As Figure 8 shows, for vanishing rotation in the first column, there is a possibility of having an inner cusp and an outer cusp specified by the red curves. Moreover, by increasing charge, the closed equipotential surfaces also become larger, as predicted by Figure 7. However, by considering rotation in the second and third column, the possibility of having the inner cusp disappear. Besides, by increasing $a$ we obtain smaller closed equipotential surfaces. However, the possibility of having the outer cusp leads matter to flow outwards. In addition, by increasing $a$ we obtain smaller closed equipotential surfaces. In addition, by increasing $a$ a disc becomes more oriented from the horizontal axis.

In general, as $e$ increases, we expect the matter is concentrated closer to the inner edge of the disc since the slope to reach the cusp is steeper. On the contrary, the higher values of $\alpha$ spread the matter more through the disc since the value of the equipotential surface at the centre and the cusp become closer as $\alpha$ increases.

In Figures 9, we examine the effect of the magnetization parameter $\beta_c$ and the high dependency of the disc
structure and its orientation on the parameter $e$ in the vicinity of the compact object for a fixed value of acceleration parameter $\alpha$ and vanishing rotation. In fact, comparing columns shows that the magnetization parameter does not influence the geometry of the disc; however, it changes the distribution of matter inside the disc and shifts the location of the rest-mass density maximum as the dashed lines figure. In addition, comparing rows show that we have a larger oriented disc for larger values of $e$. Moreover, matter is more concentrated in the inner part of the disc as was predicted in the previous Figures.

Figure 10 presents the profound impact of $\alpha$ on the geometry of the disc for a fixed value of $e$. In fact, according to the last row of Figure 7 the possibility of having solutions for larger $\alpha$ depends on having large values for $e$, so the effect of higher $\alpha$ on the disc could be neutralized partially with the higher charge.

Figure 11 shows dependency of the disc structure on the parameter $a$ for the fixed parameters $\alpha$, $e$, and $\beta_e$. As we expected from Figure 8, increasing $a$ decreases the disc size and change the distribution of matter inside the disc. Furthermore, we do not have an inner cusp for any rotation parameters. In addition, increasing $a$ shifts the disc farther from the compact object, contrary to an increase in $e$, which shift the disc closer to the central object. In general, $a$ and $e$ play the opposite role in any aspects regarding the disc properties.

V. SUMMARY AND CONCLUSION

In this paper, we analysed equilibrium sequences of magnetised, non-self-gravitating discs around a spinning charged accelerating black hole. This solution is described via the C-metric which is briefly explained in the Section 1. In this procedure we considered the approach of Komissarov [10] to attach a dynamically toroidal magnetic field to the model.

More precisely, we have analysed the influence of the magnetisation parameter $\beta_e$, charge $e$, rotation $a$, and accelerating parameter $\alpha$ on the structure of the magnetised thick disc model. We have shown that changing parameter $\beta_e$ has a noticeable effect on the location and amplitude of the rest-mass density maximum, also distributing the matter inside the disc. The effect of magnetisation parameter is in complete agreement with previous studies using this model [11]. Furthermore, in this case, the range of isodensity contours is increasing, which is compatible with the increase of rest-mass density in the inner part of the disc. Indeed, this result remains valid for any chosen value of other parameters.

On the other hand, we have seen the effect of varying the metric parameters: charge $e$, acceleration $\alpha$, and rotation $a$ on the disc’s geometry and its overall shape. We have shown that we can have the thick disc model for relatively small values of $\alpha$, and by increasing $\alpha$ the disc structure becomes smaller and gradually vanished. Additionally, $a$ has a similar effect on the structure: by increasing $a$, the disc becomes thinner and smaller and more oriented until it vanishes completely. On the contrary to these two parameters, an increase in $e$, increases the disc size and possibility of having a solution. In addition, we have seen that $e$ and $a$ change the distribution of matter inside the disc in opposite way. Besides, increasing $a$ shifts the disc farther from the compact object, contrary to an increase in $e$. However, we should mention that the strength of the parameters are not the same, as $\alpha$ has the strongest and $e$ has the weaker effect on the disc structure, in comparison.

As a further step of this work can be a study on the oscillation of the disc in this setup which is in progress. In addition, this is interesting to examine different angular momentum profiles in this space-time. In addition study the time-like circular motion. It is also of some interest to apply these models as the initial conditions in the numerical simulations and test their ability to account for observable constraints of astrophysical systems.

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FIG. 8: Contour map of the equipotential surfaces. The red lines show the equipotential corresponding to the inner and outer cusps.

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FIG. 9: Contour map of the rest-mass density of magnetised disc. The dashed lines point the center of the disc located at $r_c = 7.5$. The column 1 shows highly magnetised disc and the column 2 depicts low magnetised one.

FIG. 10: Contour map of the rest-mass density of magnetised highly disc. The dashed lines point the center of the disc located at $r_c = 6.5$.

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$\alpha = 0.001, L^2 = 12.6, W_{in} = -0.041615, e = 0.7, a = 0.1 \quad \alpha = 0.001, L^2 = 12.6, W_{in} = -0.041421, e = 0.7, a = 0.3$

$\alpha = 0.001, L^2 = 12.6, W_{in} = -0.041218, e = 0.7, a = 0.5 \quad \alpha = 0.001, L^2 = 12.6, W_{in} = -0.041006, e = 0.7, a = 0.7$

FIG. 11: Contour map of the rest-mass density of highly magnetised disc for various spin values. The dashed lines point the center of the disc. Those solution have the same parameters ($\alpha$, $L$ and $e$) of the non-rotating solution given at the bottom left of the Figure 9.