Symmetric reduction of high-multiplicity one-loop integrals and maximal cuts

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Abstract: We derive useful reduction formulae which express one-loop Feynman integrals with a large number of external momenta in terms of lower-point integrals carrying easily derivable kinematic coefficients which are symmetric in the external momenta. These formulae apply for integrals with at least two more external legs than the dimension of the external momenta, and are presented in terms of two possible bases: one composed of a subset of descendant integrals with one fewer external legs, the other composed of the complete set of minimally-descendant integrals with just one more leg than the dimension of external momenta. In 3+1 dimensions, particularly compact representations of kinematic invariants can be computed, which easily lend themselves to spinor-helicity or trace representations. The reduction formulae have a close relationship with $D$-dimensional unitarity cuts, and thus provide a path towards computing full (all-$\epsilon$) expressions for scattering amplitudes at arbitrary multiplicity.

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1 Introduction

The computation of multi-loop helicity amplitudes in Yang-Mills theories has recently met
with great success, such as at two loops, five points for QCD [1–5], and, using myriad
mathematical tools, through to seven loops and six points for planar $\mathcal{N} = 4$ super Yang-
Mills [6]. Despite this, all multiplicity (all-$n$) analytic forms of one-loop amplitudes in Yang-
Mills theory are known only up to $\mathcal{O}(\epsilon)$, and only in cases with maximal supersymmetry
and MHV helicity structure [7], or all-plus and single-minus helicity configurations [8, 9].
Recently, an all-$n$ form for a two-loop partial all-plus amplitude in pure Yang-Mills has
been computed [10]. One of the principle barriers to computing these is a lack of a concise
analytic understanding of one-loop high-multiplicity integrals.

One way this is relevant to contemporary multi-loop computations is that all-$\epsilon$ analytic
expressions occur when applying one-loop results to multi-loop integrals [11, 12]; recent
computations of all-plus amplitudes at two loops reveal a tantalisingly one-loop like analytic
structure [4, 13–16]. Moreover, higher-in-epsilon terms of one-loop amplitudes contribute at
two-loops Such an amplitude $A_{n}^{2-\text{loop}}$ can be arranged in a way that manifests its universal
infra-red (IR) structure [17]:

$$A_{n}^{2-\text{loop}} = A_{n}^{\text{tree}} \gamma^2 + A_{n}^{1-\text{loop}} \gamma^1 + F_n,$$

(1.1)
where the universal infra-red terms $I^m$ are defined in [17]. In particular, the term $I^1$ contains infra-red poles in the dimensional regulator up to $O(\epsilon^{-2})$ which would cancel with the higher-in-$\epsilon$ terms in the one-loop amplitude $A^{1\text{-loop}}$ upon expansion, leaving sub-leading and extra finite terms in addition to $F_n$.

Beyond this, the all-$\epsilon$ one-loop structure of $N = 4$ super Yang-Mills is conjectured to be proportional to a dimension-shifted all-plus amplitude in pure Yang-Mills [18]; a conclusive proof of this is still outstanding, but will be addressed in a forthcoming paper [19]. Generalising results such as these which rely on particular contingencies requires a more complete understanding of the all-$n$ analytic structure of both tree amplitudes and integrals; this paper focuses on one-loop instances of the latter.

Techniques to reduce finite $n$-point one-loop Feynman integrals (from here on referred to as an $n$-gon) in integer dimension have long been well understood [20–24]. The use of dimensional regularisation to deal with both ultraviolet (UV) and infra-red (IR) divergencies led to more general formulae [25, 26] which both justify a small basis of scalar integrals for one-loop amplitudes and codify relations amongst the integrals as a particular case of dimensional recurrence identities [27]. These can in turn be used to solve the integrals themselves [27–31].

These formulae face issues, however, when $n > 2 + L$, for $L$ the dimension of the external momenta, in which case the derivation depends on the inversion of a singular matrix $S$. A formalism was in turn developed [33] in terms of defining a pseudo-inverse of $S$ to resolve high-$n$ integrals into lower-point ones. Although suitable for application in the form of computer algorithms, the approach breaks the cyclic symmetry of the external legs, obscuring the simplicity of the analytic structure and making it difficult to apply to possible conjectures of all-$n$ all-$\epsilon$ forms.

In this paper we present an alternative which preserves the symmetries when reduced to $(L + 1)$-gons, and allows simple basis choices to be made for the reduction to $(n - 1)$-gons. For the equal propagator-mass case where $L = 4$, the reduction formula from $n$-point integrals\footnote{We assume that the external legs are always kept in integer dimension, for example in the four-dimensional helicity scheme of dimensional regularisation [32].} $I_n$ to pentagon integrals $I_5$ is especially simple and takes the form

$$I_n = \frac{1}{2^{n-5}} \sum_{i_1, \ldots, i_{n-5}=1}^n \prod_{m} \xi_{[i_m]}^{P_i} I_5^{[P_i]} ,$$

where

$$\xi_{k_j}^{[k_1,k_2,k_3,k_4,k_5,k_6]} = (-1)^j \frac{2 \text{tr}_5 \left( q_{k_j+1} q_{k_j+2} q_{k_j+3} q_{k_j+4} q_{k_j+5} q_{k_j+6} \right)}{\text{tr}_5 \left( q_{k_1} q_{k_2} q_{k_3} q_{k_4} q_{k_5} q_{k_6} \right) q_{k_1}} ;$$

here $\text{tr}_5(p_1 p_2 \ldots) = \text{tr}(\gamma_5 p_1 p_2 \ldots)$, $q_{ij} = p_i + p_{i+1} + \ldots + p_{j-1}$ where $p_i$ is the (cyclically indexed) $i$th outgoing external momentum.

The paper is structured as follows: in section 2 the notation is introduced, and we review derivations of some previously known results. In section 3 we explicitly study
Figure 1. Diagrammatic representation of the general one-loop integral function $I^D_n$ in eq. (2.1). The external momenta $p_i$ should always be considered outgoing.

the case where the external dimensions are in four-dimensional Minkowski space, giving explicit analytic expressions for the various kinematic determinants and coefficients for the pentagon and hexagon cases. In section 4 we derive new formulae for reducing loop integrals in the singular cases where kinematic determinants vanish. In section 5, the connection to a type of maximal cut on the pentagon is discussed.

2 Review and definitions

Basing our notation and analysis on [25, 26], a general $D$-dimensional one-loop scalar integral corresponding to the momentum routing in figure 1 is defined as

$$I^D_n = \int \frac{d^D \ell}{(2\pi)^D (\ell^2 - M_1^2)((\ell - q_2)^2 - M_2^2) \cdots ((\ell - q_n)^2 - M_n^2)}$$

(2.1)

where the momenta

$$q_i = p_1 + p_2 + \cdots + p_{i-1}.$$  

(2.2)

Upon integration, the denominator can be expressed as a homogeneous second-order polynomial of Feynman parameters $a_i$

$$i(-1)^{n+1}(4\pi)^{D/2} I^D_n = \Gamma(n-D/2) \int_0^1 d^n a_i \delta (1 - \Sigma_i a_i) \frac{1}{\left[\Sigma_{i,j=1}^n a_i S_{ij} a_j\right]^{n-D/2}}$$

$$\equiv I_n[1] ;$$

(2.3)

where the $n$-gon matrix

$$S_{ij} = \frac{1}{2} \left( M_i^2 + M_j^2 - q_{ij}^2 \right) ,$$

(2.4)

In terms of graph polynomials, this is just a representation the $F$ (second Symanzik) polynomial as a bilinear map of the Feynman parameters.
with $q_{ij}$ defined as

$$q_{ij} = q_j - q_i = \sum_{i}^{j-1} p_i ;$$

(2.5)

these fully characterise the integral together with the dimension $D$ and the numerator which we express in the argument of $I_n$ and is in general a monomial of Feynman parameters.

To avoid repeated explicit denotations of determinants, we introduce

$$\Upsilon_n \equiv \det S ,$$

(2.6)

and the Gram determinant, $\Delta_n \equiv \det[2p_i \cdot p_j]$, can be expressed in terms of the $n$-gon matrix as a Cayley-Menger determinant:

$$\Delta_n = 2^{n-1} \begin{vmatrix} 0 & -1^T \\ 1 & S \end{vmatrix} ,$$

(2.7)

where $1$ here represents a column of $n$ entries of value 1.

We can derive linear relations between integrals that are useful in practical calculations by considering alternative ways of deriving the same quantities [26]. Consider the cancellation of a propagator in eq. (2.1):

$$J_1 = i (-1)^{n+1}(4\pi)^{D/2} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^2 - M^2_1}{(\ell^2 - M^2_1)((\ell - q_1)^2 - M^2_2)\cdots((\ell - q_{n-1})^2 - M^2_n)} ;$$

(2.8)

whence we can derive a relation by considering the evaluation of this integral in two different ways. Firstly, the obvious

$$(2.9)$$

comes from a cancellation with the propagator; the superscript (1) here means pinching the propagator between $n$ and 1. Secondly, leaving the numerator in place and Feynman parametrising leads to the identity

$$- I^{(1)}_{n-1}[1] = (n - 1 - D) I^{D+2}_{n}[1] - 2 \sum_{j=1}^{n} S_{ij} I_{n}[a_j] .$$

(2.10)

Generalising to any inverse propagator in the numerator and, where $\Upsilon_n \neq 0$, inverting gives

$$I_n[a_i] = \sum_{j=1}^{n} S^{-1}_{ij} I^{(j)}_{n-1}[1] + (n - 1 - D) \sum_{j=1}^{n} S^{-1}_{ij} I^{D+2}_{n}[1] .$$

(2.11)

Summing the left-hand side using $\sum_{i=1}^{n} a_i = 1$ gives

$$I_n[1] = \sum_{j=1}^{n} c_j I^{(j)}_{n-1}[1] + (n - 1 - D) c_0 I^{D+2}_{n}[1] ,$$

(2.12)

where we have now defined

$$c_i = \sum_{j=1}^{n} S^{-1}_{ij}$$

(2.13)
and

\[ c_0 \equiv \sum_{j=1}^{n} c_j \]

\[ = (2)^{1-n} \frac{\Delta_n}{\Upsilon_n}. \tag{2.14} \]

Equation (2.12) clearly diverges if \( \Upsilon_n = 0 \). Introducing \( L \) for the dimension of the external momenta, then the condition on the vanishing of \( \Upsilon_n \) is

\[ n > L + 2. \tag{2.15} \]

We explore the case where \( L = 4 \) in section 3, and derive the formula for the general-\( L \) cases defined by equation (2.15) in section 4.

Assuming generic parameters \( \{M_i, q_i\} \) to avoid IR divergences allows us to set \( D \) to be an integer, and leads to the integer version of the \( n = L + 1 \) reduction formula

\[ I_{L+1}[1] = \frac{1}{2} \sum_{j=1}^{n} c_j I_L^{(j)}[1]. \tag{2.16} \]

This case was proved long ago [20–24], and indeed the generalisation to higher \( n \) was also proved, first by Melrose [22] and very shortly afterwards by Petersson [23]. Continuation of this to the case where \( D \) is not an integer is the main result of this paper.

It is worthwhile highlighting that Petersson [23] defines the \( c_i \) in a different manner:

\[ c_i = \frac{1}{(l^+_i - q_{i-1})^2 - M_i} + \frac{1}{(l^+_i - q_{i-1})^2 - M_i}, \tag{2.17} \]

where \( l^+_i \) are the two solutions to the \( L \) equations

\[ (\ell - q_{j-1})^2 - M_j^2 = 0 \quad j \neq i. \tag{2.18} \]

These can be recognised as maximal cut constraints [34] for the \( L \)-point integral \( I_L^{(i)} \), although they were not considered as such at the time. This is expanded upon for the \( L = 4 \) case and more generally to non-integer \( D \) in section 5.

3 Four-dimensional Minkowski space

The various kinematic coefficients (\( c_i, \Upsilon_n, \Delta_n \) etc...) can of course be enumerated using computer algebra systems, however we show in this section that it is in fact quite easy to derive conveniently compact expressions where \( L = 4 \) and the internal masses are degenerate.

3.1 Pentagon

Our first example is the version of the shift relation (2.12) which reduces the pentagon, \( I_5 \), to boxes, \( I_4^{(j)} \), and a dimension shifted pentagon

\[ I_5 = \frac{1}{2} \left[ \sum_{j=1}^{5} c_j I_4^{(j)} + (4 - D)c_0 I_5^{D+2} \right]. \tag{3.1} \]
The Gram determinant is given by

\[
\Delta_5 = \det_{i,j \neq 5} [2p_i \cdot p_j] \\
= 16 \det_p \det[p_i^2] \\
= -16 \epsilon_{p_1p_2p_3p_4} \epsilon_{p_1p_2p_3p_4} \\
= \text{tr}^2(1234) \equiv (\text{tr}(\gamma_5 p_1 p_2 p_3 p_4))^2 .\tag{3.2}
\]

For simplicity considering the all-massless (i.e. propagators as well as external momenta) case, then the pentagon determinant is, written in Mandlestam \((s_{ij} = (p_i + p_j)^2)\) notation,

\[
\Upsilon_5 = \begin{vmatrix}
0 & 0 & s_{12} & s_{45} & 0 \\
0 & 0 & 0 & s_{23} & s_{51} \\
s_{12} & 0 & 0 & 0 & s_{34} \\
s_{45} & s_{23} & 0 & 0 & 0 \\
0 & s_{51} & s_{34} & 0 & 0 \\
\end{vmatrix} = -\frac{1}{16} s_{12}s_{23}s_{34}s_{45}s_{51} .\tag{3.3}
\]

The \(c_i\)’s can readily be determined by noting that, from conservation of momentum

\[
-\frac{1}{2} \begin{pmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\text{tr}(4512) \\
\text{tr}(5123) \\
\text{tr}(1234) \\
\text{tr}(2345) \\
\text{tr}(3451) \\
\end{pmatrix} = \begin{pmatrix}
s_{23}s_{34} \\
s_{34}s_{45} \\
s_{45}s_{51} \\
s_{51}s_{12} \\
s_{12}s_{23} \\
\end{pmatrix}, \tag{3.4}
\]

and as

\[S_{ij}c_j = 1 , \tag{3.5}\]

we can identify

\[c_i = -\frac{1}{16 \Upsilon_5} s_{(i+1)(i+2)} s_{(i+2)(i+3)} \text{tr} ((i - 2)(i - 1)i(i + 1)) .\tag{3.6}\]

The indices take on a cyclic definition (i.e. \(i + 1 = 1\) where \(i = 5\)).

### 3.2 Hexagon

At \(n = 6\) the reduction formula (2.12) is simplified by the fact that \(\Delta_6 = 0\) for \(L = 4\), and thus a hexagon can be written entirely in terms of pentagons [26]

\[I_6[1] = \frac{1}{2} \sum_{j=1}^6 c_j I_5^{(j)}[1] .\tag{3.7}\]

It is useful to have a compact value for \(\Upsilon_6\). It was shown explicitly [35] that for the all-massless case

\[\Upsilon_6 = -\frac{1}{64} \text{tr}^2(123456) .\tag{3.8}\]
This is in fact also the case for the generic external mass \((p_i^2 = m_i^2)\) and equal internal mass \((M_i^2 = M^2)\) case; this can be shown by considering the Cayley-Menger matrix \(C\)

\[
C = \begin{pmatrix}
0 & -1 \\
1 & S
\end{pmatrix}.
\]

The minors of \(C\) can be identified for the diagonal elements, by using the fact that they correspond pentagon Cayley-Menger determinants which, from equation (2.7), are proportional to Gram determinants. Explicitly the adjugate to the matrix \(C\) can be partially filled in

\[
C_{\text{adj}} = \begin{pmatrix}
Y_6 & * & * & * & * & * & * \\
* & -32\Delta_5^{(1)} & * & * & * & * & * \\
* & * & -32\Delta_5^{(2)} & * & * & * & * \\
* & * & * & -32\Delta_5^{(3)} & * & * & * \\
* & * & * & * & -32\Delta_5^{(4)} & * & * \\
* & * & * & * & * & -32\Delta_5^{(5)} & * \\
* & * & * & * & * & * & -32\Delta_5^{(6)}
\end{pmatrix},
\]

where \(\Delta_5^{(j)}\) is the Gram determinant of the pentagon with the \(j\)th propagator pinched, and \(*\) represents entries yet to be determined.

From the fact that \(C\) is a symmetric 7x7 matrix of rank 6, \(C_{\text{adj}}\) must be a symmetric rank 1 matrix, thus expressible as the outer product of two instances of the same vector \(b = (b_0, b)^T\)

\[
C_{ij} = b_i b_j \quad i, j \in \{0, \ldots, 6\},
\]

where \(b\) satisfies the conditions

\[
b_1^2 = Y_6, \quad (3.12)
\]

\[
b_i^2 = 32\Delta_5^{(i-1)} \quad i \neq 0.
\]

As

\[
CC_{\text{adj}} = \text{Diag}(\Delta_6) = 0,
\]

we can deduce from equation (2.14) that

\[
1 \cdot b = \sum_{j=1}^{6} b_j = 0,
\]

\[
\sum_{j=1}^{6} S_{ij} \cdot b_{j+1} = -b_0.
\]

The latter two conditions imply that

\[
c_i = -\frac{b_i}{b_0}.
\]

For equal mass propagators the identity

\[
\sum_{j=1}^{6} S_{ij} (-1)^j \text{tr}_5 ((j + 1)(j + 2)(j + 3)(j + 4)) = \frac{1}{2} \text{tr}_5(123456)
\]

\[
(3.18)
\]
can be used to deduce that

\[ c_i = (-1)^{i} 2 \text{tr}_5 \left( (i + 1)(i + 2)(i + 3)(i + 4) \right) \]

\[ \Rightarrow c_i^2 = 4 \frac{\Delta_5^{(i)}}{\text{tr}_5(123456)} \] (3.19)

and matching with equations (3.17), (3.12) and (3.13) gives

\[ \Upsilon_6 = -\frac{1}{64} \frac{\Delta_5}{\text{tr}_5(123456)}. \] (3.20)

This remarkably compact form of the hexagon determinant is valid for all cases where the external momenta \( p_i \) are massive or null, but the propagators have equal mass. These expressions are practically useful for computing loop amplitudes or form factors with six or more external legs and equal internal masses, especially when paired with the analysis in the following sections. A Petersson style analysis in terms of the unparametrised Feynman integral is carried out in section 5.

4 Reduction formulae

We begin this section by introducing a specific notation for the kinematic coefficients of the reduction formula (2.12) in the \( L + 2 \rightarrow L + 1 \) case

\[ \xi_i \equiv c_i \quad \text{for } L + 2 \rightarrow L + 1 \text{- point reduction}. \] (4.1)

In this section we show that these particular variables are all that are needed for the \( n \geq L + 2 \) reduction. In particular we show that the coefficients of reduction consist simply of a product of the \( \xi \)s corresponding to all possible descendant \( (L + 2) \)-gons of the given \( n \)-gon. Conversely, a reduction to \( (n - 1) \)-gons requires a choice of \( (L + 2) \)-dimensional basis with coefficients corresponding to the \( \xi \)s from a reduction of the \( (L + 2) \)-gon complementary to said basis; this is illustrated in figure 2 below. We also explain how to deal with Feynman parameters in the numerator.

4.1 Generalised reduction of scalar integrals

The \( L \)-dimensional version of equation (3.7) is

\[ I_{L+2} = \frac{1}{2} \sum_{j=1}^{L+2} \xi_j I_{L+1}^{(j)}, \] (4.2)

and with this notation, the reduction of a general descendant \( L + 2 \)-gon is

\[ I_{L+2}^{(i_1,i_2,\ldots,i_{n-L-2})} = \sum_{j \in \mathcal{E}_{L+2}^{(i_1,i_2,\ldots,i_{n-L-2})}} \xi_j^{(i_1,i_2,\ldots,i_{n-L-2})} I_{L+1}^{(i_1,i_2,\ldots,i_{n-L-2},j)}; \] (4.3)

where \( \mathcal{E}_{L+2}^{(i_1,i_2,\ldots,i_{n-L-2})} \) is the set \( \{1, \ldots, n\}/\{i_1, i_2, \ldots i_{n-L-2}\} \) which indexes the descendant \( (L + 2) \)-gon.
It should be emphasised that there are two notation schemes for integral reduction: descendant notation indexed by the propagators shrunk from the ancestor $n$-gon, and the ascendant notation indexed by the propagators expanded. To translate between the two

$$I^{(i_1,i_2,...,i_{n-L-2})}_{(i_1,i_2,...,i_{L+2})} \equiv I^{(i_1,i_2,...,i_{L+2})}_{(i_1,i_2,...,i_{n-L-2})}$$

for $i_k \in \mathcal{E}_{L+2}$, where the descendant notation is on the left and the ascendant notation is on the right.

To generalise equation (4.2) to $n > L + 2$, we start by looking at the $n = L + 3$ case. Equation (2.10) gives

$$-I^{(i)}_{L+2}[1] = (L + 1 - D) I^{D+2}_{L+3}[1] - 2 \sum_{j=1}^{n} S_{ij} I_{L+3}[a_j] ,$$

which with some linear algebra (shown in appendix A.1) leads to

$$I_{L+3} = \frac{1}{2} \sum_{j \in \mathcal{E}_{L+2}} \xi^{(k)}_j I^{(i)}_{L+2} .$$

The manifest symmetry is broken by an explicit choice of descendant $(L + 2)$-gon (the choice of pinched propagator $k$), however it is restored at the level of $(L + 1)$-gons

$$I_{L+3} = \frac{1}{4} \sum_{j,k=1}^{n} \xi^{(j)}_k \xi^{(k)}_j I^{(j,k)}_{L+1} .$$

The derivation generalises without any new algebraic obstruction aside from the opacity of heavily indexed notation: the reduction formula for $I_n$ to $I_{n-1}$ with a choice of basis corresponding to a given choice of complementary $L + 2$-gon is

$$I_n = \frac{1}{2} \sum_{j \in \mathcal{E}_{L+2}} \xi^{[k_1,...,k_{L-2}]}_j I^{(j)}_{n-1} , \quad \text{for } n \geq L + 2 ,$$

where $\{k_1,...,k_n\} = \{1,...,n\}/\{i_1,...,i_{n-L-2}\}$, the $i_n$s being the pinched propagators from the $n$-gon to form the $(L + 2)$-gon.\footnote{This is a more compact labelling of each $(L + 1)$-gon in the large-$n$ case, but less in the small-$n$ case as, for example, $I^{(1)}_{L+1} \equiv I^{[1,2,3,4,5]}_{L+1}$ for a hexagon descendant.}

For the $L = 4$ equal-internal-mass case:\footnote{It is a remarkable feature of $\text{tr}_5$ that the expression in equation (4.10) is so broadly applicable.}

$$\xi^{[k_1,k_2,k_3,k_4,k_5,k_6]}_{k_j} = (-1)^j \frac{2 \text{tr}_5 (q_{k_j+1} k_{j+2} q_{k_{j+2}} k_{j+3} q_{k_{j+3}} k_{j+4} q_{k_{j+4}} k_{j+5})}{\text{tr}_5 (q_{k_1} k_2 q_{k_3} k_4 q_{k_4} k_5 q_{k_5} q_{k_6} k_1)} ,$$

with the $j$s defined cyclically over $\{1,...,6\}$. 

\[\text{(4.10)}\]
Figure 2. Left: a choice of hexagon is determined by a choice of six external legs of $I_n$. The coefficients are the $\xi_i$s corresponding to the reduction of this hexagon to pentagons. Right: the basis of the reduction it corresponds to are the six $(n-1)$-gons which are not ancestors of this hexagon.

4.2 Feynman parameters in the numerator

It is sometimes necessary to reduce $n$-gons with Feynman parameters in the numerator. In [26], a formula is derived for the reduction of an $n$-point integral with a pair of Feynman parameters in the numerator. We repeat the derivation here for completeness.

We begin with a method analogous to used in section 2

$$ J_{n;i} \equiv \Gamma(n-1-D/2) \int_0^1 \cdots \int_0^{1-a_1-a_2-\cdots-a_{n-1}} \frac{a_k}{(a^T S a)^{n+1-D/2}} \bigg|_{a_i=1-a_1-\cdots-a_{n-1}} ; $$

the $\hat{a}_i$ signifies the omission of $a_i$; applying the derivative gives

$$ J_{n;i,k} = (\delta_{nk} - \delta_{ik}) I_{n-1}^{D+2} - 2 \sum_{j=1}^n (S_{nj} - S_{ij}) I_n[a_j a_k] . $$

Alternatively, integration gives

$$ J_{n;i} = I_{n-1}^{(i)}[a_k] - I_{n-1}^{(n)}[a_k] , $$

where $I^{(k)}[a_k] = 0$.

Now combining (4.12) and (4.13) allows us to define

$$ \Phi_k \equiv \delta_{ik} I_{n-1}^{D+2} - 2 \sum_{j=1}^n S_{ij} I_n[a_j a_k] + I_{n-1}^{(i)}[a_k] $$

which is independent of the choice of index $i$ on the right-hand side.

We assume $\Upsilon_n \neq 0$ and $\Delta_n \neq 0$; using the trick

$$ \sum_{j} S_{ij}^{-1} \Phi_k = S_{ik}^{-1} I_{n}^{D+2} - 2 I_n[a_i a_k] + S_{ij}^{-1} I_{n-1}^{(j)}[a_k] , $$

where, also

$$ \sum_{j} S_{ij}^{-1} \Phi_k = c_l \Phi_k , $$

which is independent of the choice of index $i$ on the right-hand side.
and using the fact that $\sum_j S_{ij} c_j = 1$, then
\[ c_0 \Phi_k = c_0 I_n^{D+2} - 2I_n[a_k] + c_j I_{n-1}^{(j)}[a_k]. \tag{4.17} \]
Combining with equations \eqref{eq:4.15} and \eqref{eq:2.12} gives
\[ I_n[a_i a_k] = \frac{1}{2} \left[ \frac{1}{2} \left[ S_{ik}^{-1} + (n - 2 - D) \frac{c_i c_k}{c_0} I_n^{D+2} \right] + \frac{1}{4} (n - 2 - D) \left[ c_k S_{ij}^{-1} + c_i S_{kj}^{-1} - \frac{c_i c_k c_j}{c_0} \right] I_{n-1}^{(j); D+2} + \frac{1}{4} S_{ij}^{-1} S_{kl}^{-1} I_{n-2}^{(j,l)} \right], \tag{4.18} \]
where we implicitly sum over $j$ consistent with summation convention. Note that equation \eqref{eq:4.18} applies for all cases where $n \leq L + 1$; a good consistency check can be carried out by confirming that summing equation \eqref{eq:4.18}, $\sum_{j=1}^n I_n[a_j a_i] = I_n[a_i]$, reduces to equation \eqref{eq:2.11}.

For $S_{ik} \neq 0$ we identify
\[ I_n[a_i a_k] = -\frac{1}{2} \frac{\partial}{\partial S_{ik}} I_n^{D+2}[1], \tag{4.19} \]
while for a $S_{ik} = 0$, these can be related to a non-zero entry through $\Phi_k$ equivalence.

The $n = L + 2$ ($\Delta_n = c_0 = 0$) case can also be derived,
\[ I_{L+2}[a_i a_k] = \frac{1}{2} \frac{\partial}{\partial S_{ik}} \left( I_{L+2}^{D+2} \right) = \frac{1}{2} \left[ \sum_{j=1}^{L+2} \xi_j I_{L+1}^{(j)}[a_i a_k] - \frac{1}{2} \sum_{j=1}^{L+2} \frac{\partial \xi_j}{\partial S_{ik}} I_{L+1}^{(j)}[1] \right]; \tag{4.20} \]
and making the observation that
\[ (S\xi)_l = 1 \Rightarrow \left( \frac{\partial S\xi}{\partial S_{ik}} \right)_l = 0 \Rightarrow \frac{\partial \xi_j}{\partial S_{ik}} = - \left( S_{ik}^{-1} \xi_k + S_{kl}^{-1} \xi_l \right) \tag{4.21} \gives the generic formula for a monomial of Feynman parameters in terms of lower-point amplitude.

The reduction of any $n$-point integral for $n > L + 2$ follows by replacing the reduction in equation \eqref{eq:4.20} by the relevant choice of reduction formulæ; the matrix $S$ used in the simplification of the derivative in equation \eqref{eq:4.22} needs to be replaced by the appropriate $L + 2$-gon $S_{ij}$ matrix for each $\xi$, as described in the previous section.

Note that combining equation \eqref{eq:4.18} with equation \eqref{eq:4.19} generates the partial differential equations for $I_n^{D+2}$ in terms of the variables $S_{ik}$.

We also note here that the analysis generalises to terms with a greater number of Feynman parameters in the numerator, $N$, by generalising $\Phi_k \rightarrow \Phi_N \equiv \Phi_{k_1, k_2, \ldots, k_N}$
\[ \Phi_N \equiv \sum_{j=1}^N \delta_{ik_1} I_n^{D+2}(a_{k_1} a_{k_2} \cdots a_{k_N}) - 2S_{ij} I_n \left[ a_j \prod_{l=1}^N a_{k_l} \right] + I_{n-1}^{(i)} \prod_{l=1}^N a_{k_l}. \tag{4.23} \]
We leave such derivations to the reader.
Figure 3. The quadruple cut can be solved and the solutions plugged into the pinched propagator to give $c_4$.

5 Maximal cuts

The Petersson formula (2.17) implies there is another way of finding $c$ by solving the on-shell conditions of the one-mass boxes of the kind depicted in figure 3.

Without loss of generality, we consider $c_4$; the on-shell conditions read

$$\ell^2 = 0,$$
$$\left(\ell - p_1\right)^2 = 0,$$
$$\left(\ell - p_1 - p_2\right)^2 = 0,$$
$$\left(\ell + p_5\right)^2 = 0;$$

their two solutions lie in complex momentum space, expressed in the spinor-helicity formalism as

$$\ell_{ab} = t^+_a = \begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix} \lambda_1 \bar{\lambda}_5,$$
$$l^-_{ab} = \left\langle 1 2 \right| \begin{bmatrix} 2 & 34 & 5 \\ 5 & 2 \end{bmatrix} \lambda_5 \bar{\lambda}_1.$$

Applying these solutions into the pinched propagator:

$$\left(\ell^+ - q_3\right)^2 = \frac{[2345]}{[52]},$$
$$\left(\ell^- - q_3\right)^2 = \frac{\left\langle 2345\right|}{\left\langle 52\right\rangle}.$$ 

so that, using the definition (2.17),

$$c_4 = \frac{[2345]}{[52]} + \frac{[52]}{\left\langle 2345\right|},$$
$$= \frac{\text{tr}(2345)}{s_{23}s_{34}s_{45}}.$$
Figure 4. The pentagon $D$-dimensional pole corresponds to the unique solution $l_0$ to the cut constraints (5.10).

which matches equation (3.6), showing consistency between the “Feynman-parametric”, $S_{ij}$-matrix-based Melrose picture [22] and the unparametrised cut-constraint based Petersson analysis [23].

5.1 The $D$-dimensional pole

Here we look at the singular behaviour missed by the four-dimensional cut constraints. Again specifying to the $L - 4$ case, the $4 - 2\epsilon$ pentagon dimension-shift identity (2.12) is

$$I_5 = \frac{1}{2} \left[ \sum_{j=1}^{n} c_j I_{4}^{(j)} + \epsilon c_0 I_5^{0-2\epsilon} \right]. \quad (5.9)$$

This equation has interesting properties when one considers the maximal cut of the pentagon. Cut conditions for the equal internal mass pentagon depicted in figure 4 are

$$\ell^2 - M^2 = \ell^{[4]}^2 + \ell^{[-2\epsilon]}^2 - M^2 = \ell^2 - \mu^2 = 0,$$
$$\ell - p_1)^2 - \mu^2 = 0,$$
$$\ell - p_1 - p_2)^2 - \mu^2 = 0,$$
$$\ell + p_5 + p_4)^2 - \mu^2 = 0,$$
$$\ell + p_5)^2 - \mu^2 = 0; \quad (5.10)$$

where we have defined $\mu^2 = M^2 - \ell^{[-2\epsilon]}^2$. The unique solution to the conditions (5.10) is

$$l_0^2 = \frac{\text{tr}_5 (\gamma^\mu 12345)}{2 \text{tr}_5 (2345)}$$
$$\mu_0^2 = l^2 = \frac{16\Upsilon}{\Delta_5} = \frac{1}{c_0}. \quad (5.11)$$

where in the spinor helicity formalism, if e.g. the external momenta 5 and 1 are massless ($m_1^2 = m_2^2 = 0$),

$$\text{tr}_5 (\sigma^{ab} 12345) = \lambda^a_1[1234|5] \lambda^b_5 - \lambda^a_5[5432|1] \lambda^b_1$$

which can comfortably be generalised to the massive case over a given basis using the various schemes available [36–39].
Considering the integral explicitly and re-introducing the normalisation of equation (2.1)

\[ (2\pi)^{4-2\epsilon} I_5 = \int \frac{d^{4-2\epsilon} \ell}{((\ell^2 - M^2)((\ell - q_1)^2 - M^2)((\ell - q_2)^2 - M^2)((\ell - q_3)^2 - M^2)((\ell - q_4)^2 - M^2)} , \]

we follow the standard procedure of separating the integral measure:

\[ d^{4-2\epsilon} \ell = -\frac{1}{2} \Omega_{S-2\epsilon} \frac{d^4 l d\mu^2}{(\mu^2)^{1+\epsilon}} \]

where \( \Omega_{S-2\epsilon} \) is the unit surface area of the sphere, \( S_{-2\epsilon} \), as is normally extracted from the loop integration step. Upon shifting the \( \mu^2 \) variable such that it matches the definition from the conditions (5.10), this gives the integral

\[ = -\frac{1}{2} \Omega_{S-2\epsilon} \int_{M^2}^{\infty} \frac{d\mu^2}{(\mu^2 - M^2)^{1+\epsilon}} \int \frac{d^4 l}{((l^2 - \mu^2)((l - q_1)^2 - \mu^2)((l - q_2)^2 - \mu^2)((l - q_3)^2 - \mu^2)((l - q_4)^2 - \mu^2) .} \]

The residues of the integrand can be captured changing the \( \mu^2 \) integration region to a large enough contour on the complex plane, and taking the limit \( \epsilon \to 0 \); we can see that where the “four-dimensional” residue is simply manifestly captured by the pole at

\[ \mu^2 = M^2 \]

and need only be combined with four other conditions (for example the box cut conditions (5.4)).

The \( D \)-dimensional pole, on the other hand, is buried in the solution to the on-shell conditions (5.10); it lies at

\[ \mu^2 = \frac{1}{c_0} . \]

It is not a coincidence that this is the inverse of the coefficient of the shifted integral in equation (2.12). as indicated by Schnetz for the Petersson formula in the integer dimension case [40], one can view the cut version of equation (5.9) as being analogous to a sum over residues; it can be seen as a “freezing out” of propagators into rational coefficients.

This analysis can be carried forward to the \( n \)-gon reduction formula (4.9); this is straightforward as we observe that in the hexagon case

\[ I_6 = \frac{1}{2} \sum_{j=1}^{6} \xi_j I_5^{(j)} \]

taking the solution (5.11) for e.g. \( I_5^{(6)} \) gives

\[ (\ell + p_6) = 2l_0 \cdot p_6 + p_6^2 = \frac{\text{tr}_5(612345)}{\text{tr}_5(1234)} = \frac{2}{\xi_6} . \]
Consistent with the Petersson formula for a hexagon, the coefficient of the descendant box solutions satisfy the equations
\[ \frac{1}{2} \left( \xi_i c_j^{(i)} + \xi_j c_i^{(j)} \right) = \frac{1}{Q_i(l^+)Q_j(l^+)} + \frac{1}{Q_i(l^-)Q_j(l^-)}, \] (5.21)

where \( Q_i(l^\pm) \) is the inverse “frozen out” \( i \)th propagator with the null cut solutions of the other propagators \( l^\pm \) inserted. Note that the Petersson form on the right-hand side of equation (5.21) does not factor in such a simple manner, but is averaged over the all positive and all negative products. In this sense the unique solution to the \( D \)-dimensional constraint is simpler as it avoids the need to average:
\[ \prod_{m}^{n-L-1} \xi_{[P,lm]}^{[P,lm]} = \prod_{m}^{n-L-1} \frac{2^{n-L-2}}{Q(\ell_{lm})}. \] (5.22)

6 Concluding remarks

The formulae presented in this paper for the reduction of integrals should greatly simplify high-multiplicity amplitude computations. As well as being useful for reducing high-multiplicity integrals, the reduction formulae (4.9) also presents a very simple interpretation in terms of “freezing out” of propagators.

This integration with unitarity cuts leaves open the prospect of computing all-epsilon, all-multiplicity closed forms for scattering amplitudes, which as well as being a step forwards methodically, could also further more general analytic understanding of scattering amplitudes. The remarkable simplicity of the solution to the pentagon equal-mass conditions (5.10) opens up the prospect of a “generalised unitarity” approach to \( D \)-dimensional unitarity, avoiding the need for integrand reconstruction.

One could pose the question as to whether the “freezing out” can be extended to simplify multi-loop relations where reductions and relations between integrals are more involved.

We leave these questions to future work.

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A Explicit derivations

A.1 \( n = L + 3 \) reduction

Beginning with equation (4.5), we have

\[- I_{L+2}^{(i)}[1] = (L + 1 - D) I_{L+3}^{D+2}[1] - 2 \sum_{j=1}^{n} S_{ij} I_{L+3}[a_j] , \tag{A.1}\]

and then multiplying by \( \xi^{(k)}_i \) and summing

\[ \frac{1}{2} \sum_{i \in E_{L+2}} \xi^{(k)}_i I_{L+2}^{(i)}[1] = \sum_{i \in E_{L+2}} \xi^{(k)}_i \sum_{j=1}^{n} S_{ij} I_{L+3}[a_j] \]

\[ = \sum_{i \in E_{L+2}} \xi^{(k)}_i \sum_{j \in E_{L+2}} S_{ij} I_{L+3}[a_j] + \sum_{i \in E_{L+2}} \xi^{(k)}_i S_{ik} I_{L+3}[a_k] , \tag{A.2}\]

which, using the identity

\[ \sum_{i \in \mathcal{E}} S_{ij} \xi^{(k)}_i = 1 , \tag{A.3}\]

and \( \sum_{j=1}^{n} a_j = 1 \) gives

\[ \frac{1}{2} \sum_{j \in E_{L+2}} \xi^{(k)}_i I_{L+2}^{(i)}[1] = I_{L+3}[1] + \left( \sum_{i \in E_{L+2}} \xi^{(k)}_i S_{ik} - 1 \right) I_{L+3}[a_k] . \tag{A.4}\]

Recalling equation (2.14) (and defining \( \xi^{(k)}_k = 0 \))

\[ \sum_{i=1}^{L+3} \xi^{(k)}_i = 2^{L+1} \frac{\Delta^{(k)}_{L+2}}{\Upsilon^{(k)}_{L+2}} = 0 , \tag{A.5}\]

and, as \( \Upsilon_{L+3} = 0 \),

\[ S_{ik} = S_{i(k+1)} + \text{column operations} , \tag{A.6}\]

then we deduce that

\[ \sum_{i \in E_{L+2}} \xi^{(k)}_i S_{ik} - 1 = 0 . \tag{A.7}\]

Thus

\[ I_{L+3} = \frac{1}{2} \sum_{j \in E_{L+2}} \xi^{(k)}_i I_{L+2}^{(i)} , \tag{A.8}\]

which is equation (4.6).

The reduction formula (4.2) can now be applied to each \((L + 2)\)-gon to give

\[ I_{L+3} = \frac{1}{4} \sum_{j,k=1}^{n} \xi^{(j)}_k \xi^{(k)}_j I_{L+1}^{(j,k)} , \tag{A.9}\]

which is equation (4.7).
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