Stress-energy tensor in the Bel-Szekeres space-time

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Abstract

In a recent work an approximation procedure was introduced to calculate the vacuum expectation value of the stress-energy tensor for a conformal massless scalar field in the classical background determined by a particular colliding plane wave space-time. This approximation procedure consists in appropriately modifying the space-time geometry throughout the causal past of the collision center. This modification in the geometry allows to simplify the boundary conditions involved in the calculation of the Hadamard function for the quantum state which represents the vacuum in the flat region before the arrival of the waves. In the present work this approximation procedure is applied to the non-singular Bel-Szekeres solution, which describes the head on collision of two electromagnetic plane waves. It is shown that the stress-energy tensor is unbounded as the killing-Cauchy horizon of the interaction is approached and its behavior coincides with a previous calculation in another example of non-singular colliding plane wave space-time.

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1 Introduction

It is known that exact gravitational plane waves are very simple time dependent plane symmetric solutions of Einstein’s equations [1]. Nevertheless, they show two main nontrivial global features, namely: i) the absence of a global Cauchy surface, which is a consequence of the focusing effect that the waves exert on null rays [2], ii) the presence of a Killing-Cauchy horizon which may be physically understood as the caustic produced by the focusing of null rays [3]. The inverse of the focusing time is a measure of the strength of the wave. For an Einstein-Maxwell plane wave such inverse time equals the electromagnetic energy per unit surface of the wave. This makes exact plane waves very different from their linearized counterparts, which have no focusing points and admit a globally hyperbolic space-time structure. One expects that exact plane waves may be relevant for the study of the strong time dependent gravitational fields that may be produced in the collision of black holes [4, 5] or to represent travelling waves on strongly gravitating cosmic strings [6]. In recent years these waves have been used in classical general relativity to test some conjectures on the stability of Cauchy horizons [7, 8], and in string theory to test classical and quantum string behaviour in strong gravitational fields [9, 10, 11]. Their interest also stems from the fact that plane waves are a subclass of exact classical solutions to string theory [12, 13, 14].

In Einstein-Maxwell theory the particular class of plane symmetric waves are seen to contain only a non-null component of the Ricci tensor and only a non-null component of the Weyl tensor. In particular, the single component of the Weyl tensor may be conveniently interpreted as the transverse wave component in the direction of propagation of the wave. In that sense, the modulus term of the Weyl component can be identified with the amplitude of the wave and the phase term with the polarization of the wave. Furthermore, depending on whether the Ricci component or the Weyl component is zero we will distinguish in between pure gravitational plane waves or pure electromagnetic plane waves respectively.

When we consider a plane wave collision, we should analyze separately the collision between pure gravitational waves, between pure electromagnetic waves or between mixed waves. Namely: i) when two pure gravitational plane waves interact, the focusing effect of each wave distorts the causal structure of the space-time near the null horizons that these waves contain and either a spacelike curvature singularity or a new regular Killing-Cauchy horizon is created, ii) when two pure electromagnetic plane waves interact, the situation is more subtle. In fact, in the full Einstein-Maxwell theory, Maxwell’s equations remain linear indicating non direct electromagnetic interaction between the waves. However, there is a non-linear interaction of the waves through the gravitational field generated by their electromagnetic energy, which is similar to the magnitude of the interaction between pure gravitational waves. In that sense, the collision of two electromagenic waves is seen to produce gravitational waves. iii) in the case of mixed collisions, the pure electromagnetic wave is partially reflected by the incident pure gravitational wave. The gravitational wave, however, is not necessarily reflected.

Note that, the presence of a Killing-Cauchy horizon in a colliding plane wave space-time implies a breakdown of predictability since the geometry beyond the horizon is not uniquely determined by the initial data posed by the incoming colliding waves. Also, the singularities derived from plane wave collisions are not the result of the collapse of matter but the result of the non-linear effects of pure gravity, i.e. the self-gravitation of the gravitation field.

When waves are coupled to quantum fields there is neither vacuum polarization nor the
spontaneous creation of particles. In that sense they behave very much as electromagnetic or Yang-Mills plane waves in flat space-time [15, 16]. However, as a result of the non-linear interaction, the creation of quantum particles is expected in a plane wave collision.

The interaction of quantum fields with colliding plane waves was first considered by Yurtsever [17] for the singular Khan-Penrose solution [18], which describes the collision of two plane impulsive gravitational waves. In that case, an unambiguous “out” vacuum state was possible to define in a relatively simple way. More recently, Dorca and Verdaguer [19, 20] noticed that the presence of a Killing-Cauchy horizon in a non-singular colliding plane wave space-time could be used to define an unambiguous “out” vacuum state related to the preferred Hadamard state introduced by Kay and Wald in more generic space-times with Killing-Cauchy horizons [21]. With this premise, Dorca and Verdaguer studied the interaction of quantum fields in a particular non-singular colliding plane wave space-time, the interaction region of which was isometric to a region inside the event horizon of a Schwarzschild black hole [22, 23]. Later on, the same premise was applied by Feinstein and Sebastián [24] to the Bel-Szekeres solution [25], which represents the head on collision of two electromagnetic plane waves with an interaction region isometric to the Bertotti-Robinson universe [26] filled with an uniform electric field. In all these examples it was found that the initial state, defined to be the vacuum state in the flat region before the arrival of the waves, contained a spectrum of “out” particles consistent, in the long wavelength limit, with a thermal spectrum with a temperature inversely proportional to the focusing time of the waves.

A further step in the study of the interaction of quantum fields with colliding plane waves is the computation of the expectation value of the stress-energy tensor. Again, this problem was first considered by Yurtsever [17] for the Khan-Penrose solution [18]. In that case it was possible to determine the behavior of the stress-energy tensor near the singularity of the interaction region. It was shown that for the conformal coupling case (i.e. \( \xi = 1/6 \)) the energy density and two of the principal pressures were positive and unbounded towards the singularity. This problem has been also considered by Dorca and Verdaguer in the mentioned above non-singular colliding plane wave spacetime with an interaction region isometric to an interior region of a Schwarzschild black hole. As in the case of Yurtsever for the Khan-Penrose solution, the expectation value of the stress-energy tensor was calculated in the state representing the Minkowski vacuum in the flat region before the arrival of the waves. This value was first computed in a region close to both the Killing-Cauchy horizon and the topological singularities, the folding singularities, that the colliding plane wave space-time contains [28]. In that particular region, the calculations were simplified due to the blueshift effect on the energy of the initial quantum modes as they reached the Killing-Cauchy horizon [29]. In a recent work [30], an approximation procedure was proposed by the author in order to calculate such an expectation value throughout the causal past region of the collision center. In both calculations, it was found that the stress-energy diverged as the Killing-Cauchy horizon was approached. The rest energy density was positive and unbounded towards the horizon. Two of the principal pressures were negative and of the same order of magnitude of the energy density. It was also pointed out that such a behavior suggested that the non singular Killing-Cauchy horizon is indeed unstable under quantum perturbations and a curvature singularity would be the general outcome of a generic plane wave space-time when backreaction is taking into account.
In the present paper, the approximation procedure introduced in [30] is applied to the non singular Bel-Szekeres space-time as a first attempt to generalize such an approximation to more generic colliding plane wave space-times. The plan of the paper is the following. In section 2 the geometry of the Bel-Szekeres solution is briefly reviewed. In section 3 an adequate approximation in the space-time geometry is introduced throughout the causal past of the collision center. Then, the mode solutions of a massless scalar field which represent the vacuum state before the arrival of the waves are calculated all over this particular region. In section 4 the Hadamard function, which is the key ingredient for the computation of the stress-energy tensor, is calculated and regularized by means of the point-splitting technique. In section 5 the vacuum expectation value of the stress-energy tensor is calculated. In section 6 a summary and some consequences of that result are given. In order to help to maintain the main body of the paper reasonably clear, some final results are stored in the Appendices.

2 Description of the geometry

The Bel-Szekeres solution [25] represents the collision of two electromagnetic shock waves followed by trailing radiation. The interaction region is isometric to the Bertotti-Robinson universe [26], which is the static conformally flat solution of Einstein-Maxwell equations with an uniform electric field. Such a geometry is similar to the throat of the Reissner-Nordstrom solution for the special case $M = Q$ [31]. The space-time contains four space-time regions, given by

\begin{align}
  ds^2_{IV} &= 4L_1L_2 dudv - dx^2 - dy^2, \\
  ds^2_{II} &= 4L_1L_2 dudv - \cos^2 v \left(dx^2 + dy^2\right), \\
  ds^2_{III} &= 4L_1L_2 dudv - \cos^2 u \left(dx^2 + dy^2\right), \\
  ds^2_I &= 4L_1L_2 dudv - \cos^2(u + v) dx^2 - \cos^2(u - v) dy^2,
\end{align}

where for convenience we have used $u$ and $v$ as dimensionless null coordinates, and where $L_1$ and $L_2$, are length parameters such that $L_1L_2$ is directly related to the focusing time of the collision, i.e. to the inverse of the strength of the waves, which is a measure of the amount of nonlinearity of the gravitational waves [28].

This colliding wave space-time, as shown in Fig. 1, consists of two approaching waves, regions II and III, in a flat background, region IV, and an interaction region, region I. The two waves move in the direction of two null coordinates $u$ and $v$, and since they have translational symmetry along the transversal $x$-$y$ planes, the interaction region retains a two-parameter symmetry group of motions generated by the Killing vectors $\partial_x$ and $\partial_y$. The four space-time regions are separated by the two null wave fronts $u = 0$ and $v = 0$. Namely, the boundary between regions I and II is $\{0 \leq u < \pi/2, \; v = 0\}$, the boundary between regions I and III is $\{u = 0, \; 0 \leq v < \pi/2\}$, and the boundary of regions II and III with region IV is $\{u \leq 0, \; v = 0\} \cup \{u = 0, \; v \leq 0\}$. Region I meets region IV only at the surface $u = v = 0$. The Killing-Cauchy horizon in the region I corresponds to the hypersurface $u + v = \pi/2$ and plane wave regions II and III meet such a Killing-Cauchy horizon only at $\mathcal{P} = \{u = \pi/2, \; v = 0\}$ and $\mathcal{P}' = \{u = 0, \; v = \pi/2\}$ respectively. Observe that plane wave regions II and III contain a singularity at $u = \pi/2$, for region II, and $v = \pi/2$, for region...
III. These singularities are not curvature singularities but a type of topological singularity commonly referred to as a folding singularity [28]. This terminology arises from the fact that the whole singularity \( u = \pi/2 \) in region II (or \( v = \pi/2 \) in region III) must be identified (i.e. “folded”) with \( \mathcal{P} \) (or \( \mathcal{P}' \)) (see [19] for more details and for a 3-dimensional plot of a space-time of this type).

3 Mode propagation

For simplicity we will consider in this section a massless scalar field, which satisfies the usual Klein-Gordon equation,

\[ \Box \phi = 0. \]  \hspace{1cm} (5)

Following the directions of the approximation procedure introduced in the previous work [30], we will be interested in the value of the quantum field \( \phi \) all over the causal past region of the collision center. The reason is essentially because the calculations can be greatly simplified in this region. We will start with the field solution in the flat region prior to the arrival of the waves, which is chosen to be the usual vacuum in Minkowski space-time. This vacuum solution will set a well posed initial value problem on the null boundary \( \Sigma = \{u = 0, \ v \leq 0\} \cup \{u \leq 0, \ v = 0\} \), by means of which a unique solution for the field equation can be found throughout the space-time, i.e., in the plane wave regions (regions II and III), and in the interaction region (region I).

We will consider the line element,

\[ ds^2 = 2e^{-M(u,v)}du dv - e^{-U(u,v)} \left( e^{V(u,v)} dx^2 + e^{-V(u,v)} dy^2 \right), \]  \hspace{1cm} (6)

which applies globally to the four space-time regions, and where the functions \( U, V \) and \( M \), can be directly read off (1)-(4). Then, the field equation can be separated in a plane-wave form solution for the transversal coordinates \( x \) and \( y \), with \( k_x \) and \( k_y \), respectively, as separation constants. This plane-wave separation is just a trivial consequence of the translational symmetry of the space-time on the planes spanned by the Killing vectors \( \partial_x \) and \( \partial_y \). The field solution is thus,

\[ \phi(u, v, x, y) = e^{U(u,v)/2} f(u, v) e^{ik_xx + ik_yy}, \]  \hspace{1cm} (7)

where the function \( f(u, v) \) satisfies the following second order differential equation,

\[ f_{,uv} + \Omega(u, v) f = 0; \quad \Omega(u, v) = -\left(\frac{e^{-U/2}}{e^{-U/2}}\right)_{,uv} + \frac{1}{2} e^{-M+U} \left( k^2_x e^{-V} + k^2_y e^V \right). \]  \hspace{1cm} (8)

Equation (8) can be straightforwardly solved in the flat region (region IV). Then, this solution determines on the null boundary \( \Sigma \) a well posed set of initial conditions for the solutions of equation (8) in plane wave regions II and III. Finally, the field solution in regions II, III and IV is,
\[ \phi(u, v, x, y) = \frac{1}{\sqrt{2k_-(2\pi)^3}} e^{ik_xx + ik_yy} \begin{cases} \
\frac{1}{\cos u} e^{-i2k_+ \tan u - i2k_- v}, & \text{in region II,} \\
\frac{1}{\cos v} e^{-i2k_+ u - i2k_- \tan v}, & \text{in region III,} \\
\cos u \sin^2 v, & \text{in region IV,} 
\end{cases} \]

where we have used two new separation constants \( k_\pm \), which are related to the previous ones \( k_x \) and \( k_y \) by the relation \( 4k_+k_- = k_x^2 + k_y^2 \). For convenience, we define also the following set of dimensionless constants:

\[ \hat{k}_\pm = \sqrt{L_1L_2} k_\pm, \quad k_1 = \sqrt{L_1L_2} k_x, \quad k_2 = \sqrt{L_1L_2} k_y. \] (10)

Even though \( \Sigma = \{(u = 0, v < 0) \cup \{u < 0, v = 0\} \} \) is a null hypersurface, a well defined scalar product is given by (see [19] for details).

\[ (\phi_1, \phi_2) = -i \int dx dy \left[ \int_{-\infty}^0 \left( \phi_1 \partial_u \phi_2^* \right) \bigg|_{u=0} + \int_0^0 \left( \phi_1 \partial_v \phi_2^* \right) \bigg|_{u=0} \right] dv. \] (11)

Notice that the initial modes \((9)\) are well normalized on the boundary \( \Sigma \) between the flat region and the plane wave regions, and this means, from general grounds, that they remain well normalized on the boundary \( \Sigma_I = \{u = 0, 0 \leq v < \pi/2\} \cup \{0 \leq u < \pi/2, v = 0\} \) between the plane waves and the interaction region. Thus, the Cauchy problem for the interaction region is now well posed.

However, although it has been rather easy to find the solution of the field equation in regions II and III which smoothly matches with the Minkowski vacuum, it turns out to be a difficult problem for the interaction region. Observe that the Cauchy data for the interaction region is imposed by the field modes \((9a,b)\) on the lines \( \Sigma_I = \{u = 0, 0 \leq v < \pi/2\} \cup \{0 \leq u < \pi/2, v = 0\} \), which are characteristic lines for the differential equation \((8)\). Thus, the only independent Cauchy data are the values of the function \( f(u, v) \) on them. Furthermore, we are only interested in finding the field solution on the causal past of the collision center, which is determined by the simple condition \( u = v \). Then, the only relevant Cauchy data lie on the segments, \( \Sigma_I = \{u = 0, 0 \leq v < \pi/4\} \cup \{0 \leq u < \pi/4, v = 0\} \) (see [27] for details).

In order to solve this partial problem, we start with the following change of coordinates,

\[ t = u + v, \quad z = v - u, \] (12)

in equation \((8)\) and we obtain,

\[ f_{tt} - f_{zz} + \Omega(t, z)f = 0, \] (13)

where the term \( \Omega(t, z) \), using \((10)\), is given by

\[ \Omega(t, z) = \frac{k_1^2 + (\sin^2 t)/4}{\cos^2 t} + \frac{k_2^2 - (\sin^2 z)/4}{\cos^2 z}. \] (14)

From now on, we will denote the causal past region of the collision center by \( S = \{0 \leq t < \pi/2, -\pi/4 < z < \pi/4\} \). Observe that the behaviour of the variables \( t \) and \( z \) in equation \((14)\)
in region $S$ is very different. Since $t$ runs from 0 to $\pi/2$ and $z$ runs from $-\pi/4$ to $\pi/4$ in this region, the term (14) blows up as coordinate $t$ goes to $\pi/2$, but is perfectly smooth over the entire range of coordinate $z$. This fact suggests that in the whole region $S$ the physical results that we may expect are directly related to the coordinate $t$ and we may not expect any physically remarkable change if we take $z = 0$ in equation (13). However, if we want to be consistent with such an approximation, we must also modify the boundary conditions that lie on the line segments $\Sigma_i$. Since on the boundary $\Sigma_i$ we have that $t = \pm z$ and coordinate $t$ runs from $-\pi/4$ to $\pi/4$, we must also take $t = z = 0$. This means that the boundary conditions on $\Sigma_i$, given in (14a,b), reduce in such an approximation to the flat boundary conditions given in (14c). Therefore we change the mode propagation problem for the colliding wave space-time into a rather simpler Schrödinger-type problem, which is clear from Fig. 2, and which requires only that we find a solution to equation (13) with initial conditions given by the Minkowski flat modes below the hypersurface $\{t = 0, -\pi/4 < z < \pi/4\}$. Recall, however, that none of the discussion above is applicable when a solution of equation (13) in the neighborhood of the folding singularities $P$ and $P'$ is required. This is not only because in that case both coordinates $t$ and $z$ take values near $\pi/2$ and thus the potential term (14) is unbounded as $z \to \pi/2$, but also because the boundary conditions (14a,b) are also unbounded as the folding singularities at $t = \pm z = \pi/2$ are approached. In that case the mode propagation problem is much more complicated and a more detailed discussion is required (see [19, 20, 29, 24]) for details).

In fact, rather than relying on the discussed approximations in the exact field equation (13), it will be necessary to rewrite a new field equation using an adequate approximation to the space-time geometry throughout the causal past region of the collision center. This is essentially because the process of renormalization involves the subtraction of the infinite divergences that arise from the formal definition of the stress-energy tensor, and these divergences can be expressed as entirely geometric terms, which are independent of any possible approximations in the field equation.

Approximating the field equation (13) in the causal past of the collision center by taking $z = 0$ is essentially equivalent to changing the line element, in the causal past of the collision center, by a related line element obtained from (4) by setting $z = 0$, i.e.,

$$ds^2 = L_1 L_2 \left( dt^2 - dz^2 \right) - \cos^2 t \, dx^2 - \, dy^2. \quad (15)$$

We will suppose that the line element (15) applies all over the causal past of the collision center, not only in the interaction region but also through the plane wave regions II and III in the sense of Fig. 2. The plane wave collision starts at $t = 0$ but to avoid smoothness problems derived from such an approximation, we will suppose that (15) applies exactly on a range $\epsilon < t < \pi/2$, for a certain $\epsilon > 0$. In the range $0 \leq t \leq \epsilon$, as described below, we will interpolate a line element which smoothly matches with the flat space at $t = 0$. Nevertheless, the details of this matching will not affect the main physical features.

The exact field equation for this approximate space-time is,

$$(\Box + \xi R) \phi = 0, \quad (16)$$

where it is necessary to consider a coupling curvature term in the field equation because, although the exact space-time is a vacuum solution, we have a bounded nonzero value for $R$
in the approximated space-time. In order to solve this new field equation, we start rewriting the line element \((\mathbb{I})\) in the following general way,

\[
ds^2 = (f_1 f_2 f_3) \, dt^* \, dt - \left( \frac{f_1 f_2}{f_3} \right) \, dz^2 - \left( \frac{f_2 f_3}{f_1} \right) \, dx^2 - \left( \frac{f_1 f_3}{f_2} \right) \, dy^2,
\]

(17)

where the \(f_i\) are functions of coordinate \(t\) alone, which for values of \(0 < \epsilon < t < \pi/2\), can be straightforwardly determined by direct comparison with \((\mathbb{I})\) as \(f_1(t) = \sqrt{L_1 L_2}\), \(f_2(t) = \sqrt{L_1 L_2} \cos t\), \(f_3(t) = \cos t\). For values \(t \leq 0\) we take \(f_1(t) = f_2(t) = \sqrt{L_1 L_2}\), \(f_3(t) = 1\), which correspond to their values in flat space. Finally, in the interval \(0 \leq t \leq \epsilon\), we smoothly interpolate each \(f_i(t)\) \((i = 1, 2, 3)\) between these values. Also, in order to prevent singularities in the field equation, we conveniently reparametrize coordinate \(t\), by \(t^*(t)\), as follows,

\[
\frac{dt^*}{dt} = \frac{1}{f_3(t)}.
\]

(18)

Now, we use the following ansatz for the field solutions,

\[
\phi_k = h(t^*) \, e^{i k_3 z + i k_y y + i k_x x},
\]

(19)

where the plane wave factor in coordinates \(x, y\) is related to the translational symmetry of the space-time along the transversal directions \(x, y\), and the plane wave factor in coordinate \(z\) is just a consequence of our approximation. Then equation \((\mathbb{I})\) directly leads to the following Schrödinger-like differential equation for the function \(h(t^*)\),

\[
h_{t^* t^*} + \omega^2(t) \, h = 0, \quad V(t) \equiv \omega^2(t) = f_0'^2(t) + f_1'^2(t) \, k_x^2 + f_2'^2(t) \, k_y^2 + f_3'^2(t) \, k_z^2,
\]

(20)

where the function \(f_0(t)\) stands for,

\[
f_0'^2(t) = [f_1(t) f_2(t) f_3(t)] \xi R.
\]

(21)

Such differential equation can be WKB solved, essentially because the short wavelength condition holds, i.e. \(\omega^{-1} \partial h / \partial t^* \ln \omega \ll 1\). Observe that this condition reduces to \((\partial t / \partial t^*) \, dV / dt \ll 2 \omega^3\), which becomes particularly accurate when the Killing-Cauchy horizon is approached since in that case \(dt / dt^* = f_3(t) \rightarrow 0\). Therefore, the mode solutions \(\phi_k\) which reduce to the flat mode solutions in the region prior to the arrival of the waves, are

\[
\phi_k = \frac{\hat{\omega}^{1/2}}{\sqrt{(2\pi)^3 2 k_3 W(t)}} \, e^{ik_3 z + i k_y y + i k_x x - i \int^{t^*} W(\zeta) d\zeta^*},
\]

(22)

where we denote \(\hat{\omega}^2 = k_1^2 + k_2^2 + k_3^2\) with \(k_1 = \sqrt{L_1 L_2} \, k_x\), \(k_2 = \sqrt{L_1 L_2} \, k_y\), \(k_3 = k_z\) and where \(W(t)\) stands for an adiabatic series in powers of the time-dependent frequency \(\omega(t)\) of the modes and its derivatives. Up to adiabatic order four (i.e. up to terms involving four derivatives of \(\omega(t)\)) \(W(t)\) it is given by,

\[
W(t) = \omega + \frac{A_2}{\omega^3} + \frac{B_2}{\omega^5} + \frac{A_4}{\omega^7} + \frac{B_4}{\omega^9} + \frac{C_4}{\omega^{11}},
\]

(23)

where, using the notation \(\dot{V} \equiv dV / dt^*\),
\[ A_2 = -\frac{\dot{V}}{8}, \quad B_2 = \frac{5}{32} \dot{V}^2, \tag{24} \]

\[ A_4 = -\frac{\ddot{V}}{32}, \quad B_4 = -\frac{28 \dot{V} \dddot{V} + 19 \dddot{V}^2}{128}, \quad C_4 = \frac{221}{258} \dot{V}^2 \dddot{V}, \quad D_4 = -\frac{1105}{2048} \dddot{V}^4, \]

and \( A_n, B_n, \ldots \) denote the \( n \) adiabatic terms in \( W(t) \). Up to adiabatic order zero it is simply \( W(t) = \omega(t) \). Observe the two following facts:

(i) Near the horizon \( t = \pi/2 \) we have \( W(t) \approx \omega(t) \). This is because the higher adiabatic corrections vanish at the horizon.

(ii) In the flat region prior to the arrival of the waves we have \( W(t) = \hat{\omega} = (k_1^2 + k_2^3 + k_3^3)^{1/2} \).

In that case, since \( f_3 = 1 \), we can use (18) to set \( t^* = t \), where without loss of generality we choose the value \( t^* = 0 \) at \( t = 0 \). Therefore, the mode solutions (22) in the flat region reduce to,

\[ \phi_{k_1}^{IV} = \frac{1}{\sqrt{(2\pi)^3 2k_1}} e^{i k_1 x + i k_2 y + i k_3 z - i \hat{\omega} t}, \]

which indeed are the flat mode solutions defined in (4c), recalling that the new separation constant \( k_z = k_3 \) is related to the original \( k_\pm \) by the ordinary null momentum relations, i.e.,

\[ \hat{\omega} = \hat{k}_+ + \hat{k}_-, \quad k_z = \hat{k}_+ - \hat{k}_-. \tag{25} \]

It is important to understand that we are constructing a set of mode solutions as an adiabatic series in terms of derivatives of the frequency \( \omega(t) \) in the differential equation (20). This procedure is similar but not equivalent to the construction of an adiabatic vacuum state where the field modes are expanded as an adiabatic series in terms of the derivatives of the metric coefficients (see for example [32] for details). In fact, observe for instance that the term \( f_3^2(t) \) in (20) involves two derivatives of the metric since it is directly related to the curvature scalar. Thus, it would be a second order term for an eventual adiabatic vacuum, but it is simply a zeroth order term in our adiabatic series in derivatives of \( \omega(t) \).

4 Hadamard function in the interaction region

The key ingredient to calculate the vacuum expectation value of the stress-energy tensor is the Hadamard function \( G^{(1)}(x, x') \), which is defined as the vacuum expectation value of the anticommutator of the field, i.e.,

\[ G^{(1)}(x, x') = \langle \{ \phi(x), \phi(x') \} \rangle = \sum_k \left\{ u_k(x) u^*_k(x') + u_k(x') u^*_k(x) \right\}. \tag{26} \]

This Hadamard function contains non-physical divergence terms which can be subtracted by the following point-splitting prescription,

\[ G^{(1)}_{\text{H}}(x, x') = G^{(1)}(x, x') - S(x, x'), \tag{27} \]

where \( S(x, x') \) is the midpoint expansion of a locally constructed quantity commonly referred as a Hadamard elementary solution (see for example [33]) and given by
\[
S(x, x') = \frac{1}{8\pi^2} \left\{ -\frac{2}{\sigma} - 2\Delta^{(2)}_{\bar{\mu}\bar{\nu}} \frac{\sigma_{\bar{\mu}} \sigma_{\bar{\nu}}}{\sigma} - 2\Delta^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \frac{\sigma_{\bar{\mu}} \sigma_{\bar{\nu}} \sigma_{\bar{\rho}} \sigma_{\bar{\sigma}}}{\sigma} - \frac{a_1(0)}{2} \ln(\mu^{-2}\sigma) \\
- \left[ (a_1(0) \Delta^{(2)}_{\bar{\mu}\bar{\nu}} + a_1(2)_{\bar{\mu}\bar{\nu}}) \sigma^{\bar{\mu}} \sigma^{\bar{\nu}} - \frac{1}{2} a_2(0) \sigma \right] \ln(\mu^{-2}\sigma) - \frac{3}{4} a_2(0) \sigma \right\}, \tag{28}
\]

where the coefficients \(\Delta^{(2)}_{\bar{\mu}\bar{\nu}}, \Delta^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}}\) and \(a_1(0)\) are written in Appendix A. We use the standard definition for the geodetic biscalalar \(\sigma(x, x') = (1/2)s^2(x, x')\), being \(s(x, x')\) the proper distance between the points \(x\) and \(x'\) on a non-null geodesic connecting them. Also, \(\sigma_{\bar{\mu}}(x, x') = (\partial/\partial x^\bar{\mu})\sigma(x, x')\) is a geodesic tangent vector at the point \(\bar{x}\) with modulus \(s(x, x')\), being \(\bar{x}\) the midpoint between \(x\) and \(x'\) on the geodesic. The parameter \(\mu\) in the logarithmic term of \(\sigma\) is an arbitrary length parameter, which is related to the two-parameter ambiguity of the point-splitting regularization scheme \[33\]. Then we can compute \(\langle T_{\mu\nu} \rangle\) by means of the following differential operation,

\[
\langle T_{\mu\nu}(x) \rangle = \lim_{x \to x'} \mathcal{D}_{\mu\nu} G^{(1)}(x, x'), \tag{29}
\]

where \(\mathcal{D}_{\mu\nu}\) is a nonlocal differential operator, which in the conformal coupling case \((\xi = 1/6)\) is given by,

\[
\mathcal{D}_{\mu\nu} = \frac{1}{6} \left( \nabla_{\mu'} \nabla_{\nu'} + \nabla_{\nu'} \nabla_{\mu} \right) - \frac{1}{24} g_{\mu\nu} \left( \nabla_{\alpha'} \nabla^\alpha + \nabla_{\alpha} \nabla^\alpha \right) - \frac{1}{12} (\nabla_{\mu} \nabla_{\nu} + \nabla_{\nu} \nabla_{\mu}) + \frac{1}{48} g_{\mu\nu} \left( \nabla_{\alpha} \nabla^\alpha + \nabla_{\alpha} \nabla^\alpha \right) - \frac{1}{12} \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right). \tag{30}
\]

However, the above differential operation and its limit have no immediate covariant meaning because \(G^{(1)}(x, x')\) is not an ordinary function but a biscalalar and the differential operator \(\mathcal{D}_{\mu\nu}\) is nonlocal, thus we need to deal with the nonlocal formalism of bitensors (see, for example \[38, 39\] or the Appendix B of reference \[29\] for a review on this subject).

The regularization procedure \[27\], however, fails to give a covariantly conserved stress-energy tensor essentially because the locally constructed Hadamard function \(\sigma\) is not in general symmetric on the endpoints \(x\) and \(x'\) (i.e., it satisfies the field equation at the point \(x\) but fails to satisfy it at \(x'\) (see \[34\] for details). Thus, to ensure covariant conservation, we must introduce an additional prescription:

\[
\langle T_{\mu\nu}(x) \rangle = \langle T_{\mu\nu}^B(x) \rangle - \frac{a_2(0)}{64\pi^2} g_{\mu\nu}. \tag{31}
\]

Note that this last term is responsible for the trace anomaly in the conformal coupling case, because even though \(\langle T_{\mu\nu}^B(x) \rangle\) has null trace when \(\xi = 1/6\), the trace of \(\langle T_{\mu\nu}(x) \rangle\) is given by \(\langle T^\mu_\mu \rangle = -a_2(0)/(16\pi^2)\). The regularization prescription just given in \[31\] satisfies the well-known four Wald’s axioms \[33, 35, 36, 37\], a set of properties that any physically reasonable expectation value of the stress-energy tensor of a quantum field should satisfy. There is still an ambiguity in this prescription since two independent conserved local
curvature terms, which are quadratic in the curvature, can be added to this stress-energy tensor. In particular, the $\mu$-parameter ambiguity in (28) is a consequence of this (see [33] for details). Such a two-parameter ambiguity, however, cannot be resolved within the limits of the semiclassical theory, it may be resolved in a complete quantum theory of gravity [33]. Note, however, that in some sense this ambiguity does not affect the knowledge of the matter distribution because a tensor of this kind belongs properly to the left hand side of Einstein equations, i.e. to the geometry rather than to the matter distribution.

After this preliminary introduction on the point-splitting regularization technique, we may proceed to calculate the Hadamard function $G^{(1)}(x,x')$ in the interaction region for the initial vacuum state defined by the modes $\phi_k$, (22). The Hadamard function can be written as,

$$G^{(1)}(x,x') = \sum_k \phi_k(x) \phi_k^*(x') + c.c. \quad (32)$$

Note that solutions $\phi_k$ contain the function $h(t^*)$, which cannot be calculated analytically but may be approximated up to any adiabatic order as described in (23)-(24). Thus, we have the inherent ambiguity of where to cut the adiabatic series. In fact, this is an asymptotic expansion, which has a well established ultraviolet limit but it may have convergence problems in the low-energy limit. However, observe from (18) and (20) that since $dt/dt^* \rightarrow 0$ and $V(t) = \omega^2(t) \rightarrow k_1^2$ towards the horizon, the adiabatic series (23) reduces to $W \simeq \omega$ near the horizon. This means that we could cut the adiabatic series (23) at order zero if we were interested in a calculation near the horizon. However, this is only partially true. In fact, it would be true if we were only interested in the particle production problem [24] but it is not sufficient for the calculation of the vacuum expectation value of the stress-energy tensor. This is because $G^{(1)}$ calculated with $h(t^*)$ at order zero does not reproduce the short-distance singular structure of a Hadamard elementary solution (28) in the coincidence limit $x \rightarrow x'$. The smallest adiabatic order for the function $h(t^*)$ which we need to recover the singular structure of $G^{(1)}$ is order four, basically because our adiabatic construction of the mode solutions is similar (but not equivalent) to an adiabatic vacuum state (see [32] for details).

Although expanding the function $h(t^*)$ in (32) up to adiabatic order four will give an accurate value for the stress-energy tensor near the horizon, it will also give a suitable approximate value for this tensor all over the causal past of the collision center (region $S$ in Fig. 2). The reason is that even though the short-wavelength condition, i.e. $\omega^{-1}d/dt^* \ln \omega \ll 1$, is particularly accurate near the horizon it also holds throughout region $S$.

In the mode sum (32) we use the shortened notation $\sum_k \equiv \int_0^{\infty} dk_-/k_- \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y$ or equivalently $\sum_k \equiv (L_1L_2)^{-1} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3/\hat{\omega}$, where the change of variables (25) and the usual notation (10) have been used. Therefore we have,

$$G^{(1)}(x,x') = \frac{1}{2(2\pi)^3 L_1L_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_1 dk_2 dk_3}{\sqrt{W(t)W(t')}} \times \quad (33)$$

$$e^{-i \int_{\tau'}^{\tau^*} W(\zeta) d\zeta^* + ik_x(x-x') + ik_y(y-y') + ik_z(z-z')} + c.c.$$

We assume that the points $x$ and $x'$ are connected by a non-null geodesic in such a way that they are at the same proper distance $\epsilon$ from a third midpoint $\bar{x}$. We parametrize the
geodesic by its proper distance $\tau$ and with abuse of notation we denote the end points by $x$ and $x'$, which should not be confused with the third component of $(t, z, x, y)$. Then we expand the integrand function in powers of $\epsilon$ and we finally integrate term by term to get an expression up to $\epsilon^2$. The details of such a tedious calculation can be found in ref. [29].

The result is,

\begin{equation}
G^{(1)}(x, x') = \hat{A} + \sigma \hat{b} + C_{\alpha\beta} \sigma^\alpha \sigma^\beta + D_{\alpha\beta\gamma\delta} \sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta + \frac{1}{8\pi^2} \left\{ -\frac{2}{\sigma} - 2\Delta^{(2)}_{\mu\nu} \frac{\sigma^\mu \sigma^\nu}{\sigma} + 2\Delta^{(4)}_{\mu\nu\rho\sigma} \frac{\sigma^\mu \sigma^\nu \sigma^\rho \sigma^\sigma}{\sigma} - a_1^{(0)} L - \left( (a_1^{(0)} \Delta^{(2)}_{\mu\nu} + a_1^{(2)} \mu\nu) \sigma^\mu \sigma^\nu - \frac{1}{2} a_2^{(0)} \sigma \right) \hat{L} \right\},
\end{equation}

where $L$ is a logarithmic term defined as $L = 2\gamma + \ln(\sigma \xi R/2)$, being $\gamma$ Euler’s constant and where all the involved coefficients $\hat{A}, \hat{b}, C_{\alpha\beta}$... are given in Appendix A. According to (27), the Hadamard function can be regularized using the elementary Hadamard solution (28) and finally the regularized expression for $G^{(1)}(x, x')$ up to order $\epsilon^2$ is,

\begin{equation}
G^{(1)}_B(x, x') = \hat{A} + \sigma \hat{B} + C_{\alpha\beta} \sigma^\alpha \sigma^\beta + D_{\alpha\beta\gamma\delta} \sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta + \frac{1}{8\pi^2} \left\{ -a_1^{(0)} \hat{L} - \left( (a_1^{(0)} \Delta^{(2)}_{\mu\nu} + a_1^{(2)} \mu\nu) \sigma^\mu \sigma^\nu - \frac{1}{2} a_2^{(0)} \sigma \right) \hat{L} \right\},
\end{equation}

where $\hat{L}$ is a bounded logarithmic term given by, $\hat{L} = 2\gamma + \ln(\mu^2 \xi R/2)$, $\mu$ being the arbitrary length parameter introduced in (28), and where the coefficient $\hat{B}$ is $\hat{B} = \hat{b} + 3 a_2^{(0)}/(32\pi^2)$, which is given also in Appendix A. From (35) we can directly read off the regularized mean square field in the “in” vacuum state as \( \langle \phi^2 \rangle = \hat{A}/2 - a_1^{(0)} \hat{L}/(16\pi^2) \). It is important to remark, however, that the term $D_{\alpha\beta\gamma\delta}$ in (35) appears only as a consequence of our approximate procedure of calculating the Hadamard function, i.e. using an adiabatic order four expansion for the initial modes in powers of the mode frequency $\omega(t)$ and its derivatives. Had we used an exact expression for the initial modes (or an adiabatic vacuum state [32]), such a term would not appear.

### 5 Expectation value of the stress-energy tensor

To calculate the vacuum expectation value of the stress-energy tensor we have to apply the differential operator (30) to (35). As we have already pointed out, this is not straightforward because we work with nonlocal quantities. Note first that the operator (30) acts on bitensors which depend on the end points $x$ and $x'$, but the expression (35) for $G^{(1)}_B$ depends on the midpoint $\bar{x}$. This means that we need to covariantly expand (35) in terms of the endpoints $x$ and $x'$. Also, the presence of quartic $\sigma^\mu$ terms in (35) gives, after differentiation, path dependent terms which must be conveniently averaged. The details of such a calculation may be found for instance in [29]. Then, in the orthonormal basis $\theta_1 = g_{tt}^{1/2} \ dt, \theta_2 = g_{zz}^{1/2} \ dz, \theta_3 = g_{xx}^{1/2} \ dx, \theta_4 = g_{yy}^{1/2} \ dy$, using the trace anomaly prescription (31), we obtain the following expectation values $\langle T_{\mu\nu} \rangle$ in the conformal coupling case and for values $0 < \epsilon < t < \pi/2$ of coordinate $t$.
\[ \langle T_{\mu\nu} \rangle = \frac{2\gamma + \ln(\mu^2 R/12)}{2880 (L_1 L_2)^2 \pi^2} \text{diag}(-1, -1, 1, -1) + \frac{1}{4} \langle T_\tau^\tau \rangle g_{\mu\nu} + \text{diag}(\rho_1(t), -\rho_2(t), \rho_1(t) + 2\rho_2(t), -\rho_2(t)), \]

where we have used for simplicity the notation \( z = \sin t \). The trace anomaly term in that case is given by \((31)\) as,

\[ \frac{1}{4} \langle T_\tau^\tau \rangle g_{\mu\nu} = -\frac{a_2}{64 \pi^2} g_{\mu\nu} = -\frac{\text{diag}(1, -1, -1, -1)}{2880 (L_1 L_2)^2 \pi^2}, \]

and the functions \( \rho_1(t) \) and \( \rho_2(t) \) are given by,

\[ \rho_1(t) = \frac{-247247 + 1456234 z^2 + 792789 z^4}{69189120 (L_1 L_2)^2 \pi^2 (1-z)^2 (1+z)^2}, \]

\[ \rho_2(t) = \frac{502931 + 179686 z^2 + 923887 z^4}{69189120 (L_1 L_2)^2 \pi^2 (1-z)^2 (1+z)^2}. \]

Recall that \( \rho_1(t) \) is a positive definite function in an interval \( 0 < \epsilon < t < \pi/2 \). Both functions are unbounded at the horizon \( t = \pi/2 \), and the expectation value of the stress-energy tensor near the horizon is approximately given by,

\[ \langle T_{\mu\nu} \rangle|_{t=\pi/2} = \text{diag}(\rho(t), -\rho(t), 3\rho(t), -\rho(t)) + \frac{1}{4} \langle T_\tau^\tau \rangle g_{\mu\nu}, \]

where \( \rho(t) = \Lambda (L_1 L_2)^{-2} \cos t^{-4} \) and \( \Lambda \approx 0.0029 \). The behavior of \( \langle T_{\mu\nu} \rangle \) entirely agrees with the previous result \((30)\). For values of \( t \leq 0 \), \( \langle T_{\mu\nu} \rangle = 0 \), and to be consistent with the approximation we have used for the space-time geometry, we should require that the value of \( \langle T_{\mu\nu} \rangle \) \((30)\), which is valid for \( 0 < \epsilon < t < \pi/2 \), goes smoothly to zero as \( t \to 0 \). In fact, this can be achieved using an adequate matching of the line element \((13)\) with the flat line element through the interval \( 0 < t \leq \epsilon \).

Observe that the logarithmic term in the stress-energy tensor \((30)\) appears as a consequence of a similar term in the Hadamard function \((33)\). The argument of this logarithm depends on the curvature scalar and thus it will grow unbounded as the flat region is approached. However, the coefficient that will appear in front of such a logarithm in the Hadamard function \((33)\), depends only on locally constructed curvature terms. Therefore, with an adequate matching of the space-time geometry, this coefficient will also smoothly vanish towards the flat space region, below \( t = 0 \), and it will not give a contribution to the stress-energy tensor. The details of such a matching, however, will not affect the main features of the stress-energy tensor \((30)\), particularly when the Killing-Cauchy horizon is approached.

We must recall, however, that although the value \((30)\) for \( \langle T_{\mu\nu} \rangle \) satisfies asymptotically the conservation equation near the Killing-Cauchy horizon, it does not satisfy exactly the conservation equation throughout region \( S \), essentially because it is obtained by means of an approximation in the field modes. Nevertheless, we could obtain a truly conserved \( \langle T_{\mu\nu} \rangle \), in the context of the present approximation, by solving the conservation equation considering a \( \langle T_{\mu\nu} \rangle \) given by the following set of non-null components \( \{ \langle T_t^t(t) \rangle, \langle T_z^z(t) \rangle, \langle T_x^x(t) \rangle, \langle T_y^y(t) \rangle \} \).
with the conditions: i) $\langle T_y(t) \rangle = \langle T_z(t) \rangle$, which is compatible with (36) and it is a physical consequence of the isotropy of the metric (15) along the $y$-$z$ directions, ii) trace anomaly condition, i.e. $\langle T_x(t) \rangle = \langle T_{\mu\mu} \rangle - \langle T_t(t) \rangle - 2 \langle T_z(t) \rangle$, iii) the ansatz $\langle T_t(t) \rangle = \rho_1(t)$, which is the approximate value of $\langle T_t(t) \rangle$ obtained in our calculation. Finally, the conservation equation gives straightforwardly values for $\langle T_z(t) \rangle$ and $\langle T_x(t) \rangle$ which are compatible with the values $\langle T_z(t) \rangle = \rho_2(t)$ and $\langle T_x(t) \rangle = -\rho_1(t) - 2\rho_2(t)$ obtained in our approximation. In particular, they have the same behaviour near the Killing-Cauchy horizon.

Inspection of (36) shows that not only is the weak energy condition satisfied [40], which means that the energy density is nonnegative for any observer, but also the strong energy condition is satisfied.

6 Conclusions

We have calculated the expectation value of the stress-energy tensor of a massless scalar field in a space-time representing the head on collision of two electromagnetic plane waves throughout the causal past of the collision center and in the field state which corresponds to the physical vacuum state before the collision takes place. We have performed the calculations in this particular region essentially because, following the directions of a previous work [30], we could introduce a suitable approximation to the space-time metric (see Fig. 2). This approximation not only has allowed us to dramatically simplify the calculations but also to keep unchanged the main physical features, in particular the behavior of the stress-energy tensor near the Killing-Cauchy horizon of the interaction region. In fact, such an approximation is also valid for more generic plane wave space-times, and this will be the subject of a forthcoming paper.

The results we have obtained are entirely compatible with the previous result [30] and they may be briefly described as follows: before the collision of the waves $\langle T_{\mu\nu} \rangle = 0$, which correspond to the lower edge Fig. 2. Then, after the collision the value of $\langle T_{\mu\nu} \rangle$ starts to increase until it grows unbounded towards the Killing-Cauchy horizon of the interaction region. The weak energy condition is satisfied, the rest energy density is positive and diverges as $\cos^{-4} t$. Two of the principal pressures are negative and of the same order of magnitude of the energy density. The strong energy condition is also satisfied, $\langle T_{\mu} \rangle$ is finite but $\langle T_{\mu\nu} \rangle \langle T_{\mu\nu} \rangle$ diverges at the horizon and we may use ref [41] on the stability of Cauchy horizons to argue that the horizon will acquire by backreaction a curvature singularity. Thus, contrary to simple plane waves, which do not polarize the vacuum [13, 16], the nonlinear collision of these waves polarize the vacuum and the focusing effect that the waves exert seems to produce, in general, an unbounded positive energy density at the focusing points. Therefore, when the colliding waves produce a Killing-Cauchy horizon, that horizon may be, in general, unstable by vacuum polarization.

In the more generic case when the wave collision produces a spacelike singularity it seems clear that the vacuum expectation value of the stress-energy tensor will also grow unbounded near the singularity. In fact, in a forthcoming paper we will extend the approximation introduced in the present work to a more generic plane wave spacetime with the objective of more generally proving that the negative pressures associated to the quantum fields could not prevent the formation of the singularity.
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A Useful adiabatic expansions

The coefficients for the Hadamard function in (35), using for simplicity the notation $z = \sin t$, are:

$$\bar{A} = \frac{1}{60 L_1 L_2 \pi^2},$$

$$\bar{B} = \frac{525 - 426 z^2 + 485 z^4}{60480 (L_1 L_2)^2 \pi^2 (1 - z)^2 (1 + z)^2},$$

$$C_{zz} = \frac{3597 + 2002 z^2 + 3107 z^4}{332640 L_1 L_2 \pi^2 (1 - z)^2 (1 + z)^2},$$

$$C_{xx} = \frac{561 + 2222 z^2 + 1524 z^4}{166320 (L_1 L_2)^2 \pi^2 (1 - z)(1 + z)},$$

$$C_{yy} = \frac{1}{L_1 L_2} C_{zz},$$

$$D_{zzzz} = \frac{4433 + 12974 z^2 + 3513 z^4}{823680 \pi^2 (1 - z)^2 (1 + z)^2},$$

$$D_{xxxx} = \frac{5005 + 31200 z^2 + 45867 z^4}{4324320 (L_1 L_2)^2 \pi^2},$$

$$D_{gggg} = \frac{1}{(L_1 L_2)^2} D_{zzzz},$$

$$D_{xxxx} = \frac{1001 + 4030 z^2 + 2848 z^4}{864864 L_1 L_2 \pi^2 (1 - z)(1 + z)},$$

$$D_{zzgg} = \frac{143 + 260 z^2 + 120 z^4}{20592 L_1 L_2 \pi^2 (1 - z)^2 (1 + z)^2},$$

$$D_{xxgg} = \frac{1}{L_1 L_2} D_{zzzz}.$$

The coefficients for the midpoint expansion of the locally constructed Hadamard function are:
\[ a_{1}^{(0)} = -R \left( \xi - \frac{1}{6} \right), \quad \Delta^{(2)}_{\mu\nu} = \frac{1}{12} R_{\mu\nu}, \]
\[ a_{2}^{(0)} = \frac{1}{2} \left( \frac{1}{6} - \xi \right)^{2} R^{2} + \frac{1}{6} \left( \frac{1}{5} - \xi \right) R_{,\alpha}^{\alpha} - \frac{1}{180} R^{\alpha\beta} R_{,\alpha\beta} + \frac{1}{180} R^{\alpha\beta,\gamma\delta} R_{,\alpha\beta\gamma\delta}, \]
\[ a_{1}^{(2)}_{\mu\nu} = \frac{1}{24} \left( \frac{1}{10} - \xi \right) R_{,\mu\nu}^{\alpha} + \frac{1}{120} R_{\mu\nu,\alpha}^{\alpha} - \frac{1}{90} R_{,\mu}^{\alpha} R_{,\alpha\nu} + \]
\[ \frac{1}{180} R_{,\mu}^{\alpha\beta} R_{,\alpha\beta\nu} + \frac{1}{180} R_{,\mu}^{\alpha\beta\gamma} R_{,\alpha\beta\gamma\nu}, \]
\[ \Delta^{(4)}_{\mu\nu\rho\tau} = \frac{3}{160} R_{,\mu\nu\rho\tau} + \frac{1}{288} R_{,\mu\nu}^{\rho\tau} + \frac{1}{360} R_{,\mu}^{\alpha\beta} R_{,\alpha\beta\rho\tau}. \]

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**Figure captions**

**Fig. 1** The colliding wave space-time consists of two approaching waves, regions II and III, in a flat background, region IV, and an interaction region, region I. The two waves move in the direction of two null coordinates \( u \) and \( v \). The four space-time regions are separated by the two null wave fronts \( u = 0 \) and \( v = 0 \). The boundary between regions I and II is \( \{0 \leq u < \pi/2, \ v = 0\} \), the boundary between regions I and III is \( \{u = 0, \ 0 \leq v < \pi/2\} \), and the boundary of regions II and III with region IV is \( \Sigma = \{u \leq 0, \ v = 0\} \cup \{u = 0, \ v \leq 0\} \). Region I meets region IV only at the surface \( u = v = 0 \). The Killing-Cauchy horizon in the region I corresponds to the hypersurface \( u + v = \pi/2 \) and plane wave regions II and III meet such a Killing-Cauchy horizon only at \( \mathcal{P} = \{u = \pi/2, \ v = 0\} \) and \( \mathcal{P}' = \{u = 0, \ v = \pi/2\} \) respectively. The hypersurfaces \( u = \pi/2 \) in region II and \( v = \pi/2 \) in region III are a type of topological singularities commonly referred as folding singularities and they must be identified with \( \mathcal{P} \) and \( \mathcal{P}' \) respectively.

**Fig. 2** The subset of Cauchy data which affects the evolution of the quantum field along the center \( u = v \) of the plane wave collision lies on the segments \( \Sigma_I = \{0 \leq u < \pi/4, \ v = 0\} \cup \{u = 0, \ 0 \leq v < \pi/4\} \). Region \( \mathcal{S} \) is the causal future of this Cauchy data (or equivalently, the causal past of the collision center).

We change the *mode propagation problem* for the plane wave collision, in the causal past of the collision center, region \( \mathcal{S} \), by a much simpler Schrödinger-type problem which consists in eliminating the dependence of the field equation on coordinate \( z \) by taking \( z = 0 \), and substituting the Cauchy data which come from the single plane wave regions on segments \( \Sigma_I \) by much simpler Minkowski Cauchy data. This procedure is essentially equivalent to modifying the space-time geometry in the causal past of the collision center by eliminating the dependence on coordinate \( z \) in the line element, setting \( z = 0 \), and smoothly matching this line element, through plane wave regions II and III, with the flat spacetime below the segment \( \{t = 0, \ -\pi/4 < z < \pi/4\} \).
\[ P(u=\pi/2,v=0) \quad u+v=\pi/2 \quad P'(u=0,v=\pi/2) \]
