STANDARD AND GENERALIZED NEWTONIAN GRAVITIES AS “GAUGE” THEORIES OF THE EXTENDED GALILEI GROUP - I: THE STANDARD THEORY

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Abstract

Newton’s standard theory of gravitation is reformulated as a gauge theory of the extended Galilei Group. The Action principle is obtained by matching the gauge technique and a suitable limiting procedure from the ADM-De Witt action of general relativity coupled to a relativistic mass-point.

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1 Introduction

The present work is the first of a series in which a suitable reformulation of the gauge procedure is exploited for dealing with classical non-relativistic systems. In particular, this is the first of two papers in which the gauge technique is applied to extended Galilei group. The general scope of this treatment is to reformulate firstly standard Newton’s theory as a general manifestly-covariant Galilean gauge theory, and, secondly, to seek possible generalizations of it.

As is well-known, a geometrical four-dimensional formulation of Newton’s gravitational theory has been developed already in the thirties by Elie Cartan [1]. More recent formulations of the classical theory of gravitation in geometrical terms have been proposed by Havas [2], Anderson [3], Trautman [4], Künzle [5] and Kuchař [6]. Analyses of the classical theory as a non-relativistic limit of general relativity has been made by Dautcourt [7], Künzle [5], Ehlers [8], Malament [9] and others. In all these papers, the Newtonian theory is reobtained by describing its inertial-gravitational structure in terms of an affine connection compatible with the temporal flow $t_\mu$ and a rank-three spatial metric $h^{\mu\nu}$. While the curvature of the four-dimensional affine connection is different from zero because of the presence of matter, the Newtonian flatness of the absolute three-space is guaranteed by the further requirement that Poisson’s equation be satisfied, in the covariant form $R_{\mu\nu} = 4\pi G \rho(z) t_\mu t_\nu$, where $R_{\mu\nu}$ is the Ricci tensor of the affine connection and $\rho(z)$ is the matter density. In this way the four-dimensional description is dynamical, while the three-dimensional one is not. A further typical feature of this geometrical formulation is the fact that, unlike the case of general relativity in which there is a unique compatible affine connection, the curved four-dimensional affine structure can be separated out in a flat affine (inertial) structure and a gravitational (force) potential; this splitting, however, cannot be done in a unique way, unless special boundary conditions are externally provided.

While the four-dimensional point of view about Newton’s theory shows remarkable geometrical insights and even provides a better foundation for Newtonian kinematics than does the traditional point of view (see, in this connection Earman and Friedman [10]), it
does not lend itself to any easy generalization. And since what we want to obtain in the end is precisely a true generalization of Newton’s theory allowing for a dynamical three-space, we will adopt here a completely independent procedure based on a three-dimensional level of description from the beginning. With this in view, the present paper should be read also as a first necessary step towards the searched generalized formulation. Our approach develops through the following steps:

(1) First, we exploit the gauge methodology originally applied by Utiyama \[11\] to the Lorentz group within the field theoretic framework, in order to find all the inertial-gravitational fields which can be coupled to a non-relativistic mass-point. To achieve this result, we apply a suitably adapted Utiyama procedure to the Galilei group. Specifically, we consider a Galilei invariant Action corresponding to the projective canonical realization which describes a free particle of mass \(m\). The requirement of invariance (properly quasi-invariance, according to what usually obtains in the case of groups with non trivial cohomology structure) of the Action under localized Galilei transformations, leads directly to the following results: (a) the introduction of eleven compensating gauge fields (one more than the order of the standard Galilei group due to the central extension of it); (b) the characterization of their Galilean transformation properties, and ; (c) the explicit form of the Action describing the dynamics of the mass-point interacting with the gauge fields playing the role of external fields. A geometrical interpretation of these latter fields is then exhibited by evidentiating their relation to the so-called Galilei and Newtonian Structures studied by Künzle \[5\] and Kuchař \[6\].

(2) Second, we look for a possible field Action capable of describing the dynamics of these fields. In realizing this program we are guided by the following facts: (a) The non-relativistic limit of the relativistic Lagrangian for a mass-point in a pseudo-Riemannian space is precisely the Galilei matter Lagrangian we have already obtained through the gauge procedure; (b) the non-relativistic limit of Einstein equations leads to the geometric Cartan structure with Newton’s equations; on the other hand: (c) none among the existing four-dimensional formulations of Newtonian gravitation is cast in a variational form.
The explicit construction of the fundamental Galilei Action is performed by matching the results obtained through the above gauge technique and a suitable non-relativistic limiting procedure (for $c^2 \to \infty$) from the four-dimensional level. Precisely, the limiting procedure is applied to the Einstein-Hilbert-De Witt action for the gravitational field plus a matter action corresponding to a single mass point, under the assumption of the existence of a global 3+1 splitting of the total Action, and of a suitable parametrization of the 4-metric tensor in terms of powers of $c^2$. Once the expansion in powers of $1/c^2$ is explicitly calculated, we make the Ansatz of identifying the wanted Galilean Action $A$ with the zeroth order term of the expansion itself.

The resulting Action contains 27 fields, i.e., 16 fields over and above the gauge fields obtained through the gauge technique. These fields are not coupled to matter and lack a-priori any definite transformation property. Their role is nonetheless essential (in the spirit of the Einstein-Kretschmann debate[12], one would say) as auxiliary fields, to the effect that they guarantee a Galilean general-covariance of the three-dimensional theory (where of course the absolute nature of time is preserved). In fact, once appropriate transformation properties for the auxiliary fields are postulated, the fundamental Galilei Action turns out to be quasi-invariant under the local Galilei transformations.

As expected, a constraint analysis shows that the theory has no physical field degrees of freedom so that it is essentially Newton’s theory expressed as a gauge invariant theory or, stated in other words, as a theory cast in a form valid in arbitrary (absolute time respecting) Galilean reference frames. This formulation implies, of course, flatness of the three-space metric $g_{ij}$ (expressed through the vanishing of the three-dimensional Ricci tensor: $R_{ij} = 0$) and validity of the Poisson equation in a suitable Galilei-covariant form (see Section 7.1). As far as we know, a manifestly-covariant formulation of Newton’s theory of gravitation has never been proposed until now.

In Section 2, the free mass-point realization of the extended Galilei group is expounded together some preliminaries and notations. Section 3 is devoted to the gaugeization of the group: gauge compensating fields and their group transformation properties are derived.
by imposing \textit{quasi-invariance} of the action. The equations of motion of the mass-point in presence of the \textit{gauge "external"} fields are discussed in Section 4 together with a proper characterization of the relations between various kinds of "observers" and "forms" of the fields. Section 5 is dedicated to a summary of the known facts about Galilean and Newtonian geometrical structures and to the correspondence between these latter and our \textit{gauge} fields. In section 6 the main \textit{Ansatz} for the selection of the fundamental Galilean Action is discussed. Section 7 is devoted to a discussion of the constraint analysis of the covariant form of Newton’s theory in \textit{arbitrary - absolute time respecting} reference systems (Section 7.1) and in \textit{Galilean} reference systems (Section 7.2), respectively.

2 Preliminaries on the Galilei free mass-point realization

The Lagrangian and the action for a free non relativistic mass-point can be written,

\[ L_M = \frac{1}{2} m \delta_{ij} \dddot{x}_i \dddot{x}_j, \quad A_M = \int_{t_1}^{t_2} dt \ L_M, \quad \left( \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \right), \quad (2.1) \]

respectively.

The variational principle \( \delta A_M = 0 \), with variations which vanish at the end points, gives the Euler-Lagrange equations:

\[ \delta L = \frac{\partial L}{\partial x^i} \dot{x}^i \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = -m \delta_{ij} \dddot{x}_j \overset{=}{=} 0, \quad (2.2) \]

where \( \overset{=}{=} \) means that the equality is satisfied on the extremals.

The infinitesimal Galilei transformations of the configuration variables will be written as:

\[ \left\{ \begin{array}{l}
\delta t = -\varepsilon \\
\delta x^i = \varepsilon^i + c_{jk}^i \omega^j x^k - v^i t \\
\delta \dot{x}^i = c_{jk}^i \omega^j \dot{x}^k - v^i 
\end{array} \right. \quad (2.3) \]

where \( \varepsilon, \varepsilon^i, \omega^i, v^i \) are the infinitesimal parameters of time translation, space translations, space rotations and pure Galilei transformations (Galilei boosts), respectively, and the \( c_{jk}^i \)'s are the standard structure-constants of the SO(3) group.
We remark that, for a given infinitesimal time transformation \( t \to t^\ast = t + \delta t, \ f(t) \to f^\ast(t^\ast) \), the symbol \( \delta \) means \( \delta f(t) \equiv f^\ast(t^\ast) - f(t) \). On the other hand, the \textit{equal time} infinitesimal transformation will be denoted by \( \delta_0 f(t) \equiv f^\ast(t) - f(t) = \delta f - \delta t \cdot \frac{df}{dt} \).

The \textit{equal-time} configuration variables transformations corresponding to the transformation \((2.3)\) are:

\[
\begin{align*}
\delta_0 t &= 0 \\
\delta_0 x^i &= \varepsilon^i + c_{jk}^i \omega^j x^k - tv^i + e\ddot{x}^i \\
\delta_0 \dot{x}^i &= c_{jk}^i \omega^j \dot{x}^k - v^i + e \dddot{x}^i.
\end{align*}
\]

Note that, while \( \delta_0 \) commutes with time derivative, so that

\[
\delta_0 \frac{df(t)}{dt} = \frac{d}{dt} \delta_0 f(t),
\]

we have instead

\[
\delta \frac{df(t)}{dt} = \delta_0 \frac{df(t)}{dt} + \frac{d^2 f(t)}{dt^2} \delta t = \frac{d}{dt} \delta f(t) - \frac{df(t)}{dt} \frac{d(\delta t)}{dt}.
\]

Finally, notice that, since the Galilei transformations are not, in general, fixed-time transformations, the variation of the action must be explicitly written in the form:

\[
\Delta S_M = \int_{t_1}^{t_2} dt \left[ \delta L_M + L_M \frac{d}{dt} \delta t \right] = \int_{t_1}^{t_2} dt \left[ \delta_0 L_M + \frac{d}{dt} (L_M \delta t) \right].
\]

Under the transformations \((2.3)\), we have:

\[
\Delta S_M = \int_{t_1}^{t_2} dt \left[ \frac{d}{dt} \left( -m \delta_{ij} x^i v^j \right) \right],
\]

so that the action \((2.1)\) results \textit{quasi-invariant} under them. As a consequence of this \textit{quasi-invariance} of the action, we have the Noether identity:

\[
\dot{G} \equiv -\delta_0 x^i \mathcal{E}_i \equiv 0,
\]

where the constant of the motion \( G \) is given by:

\[
G = \frac{\partial L_M}{\partial \dot{x}^i} \delta_0 x^i - \varepsilon L_M + m \delta_{ij} x^i v^j.
\]

Before proceeding, let us fix some notation in connection with more general classes of functions that we will have to consider; precisely: 1) \( f(z,t) \), with \( z \) and \( t \) independent variables; and 2) \( f(x(t),t) \).
The variations in case 1) when \( t \to t^* = t + \delta t, \ z \to z^* = z + \delta z \), will be denoted by

\[
\delta f(z, t) = f^*(z^*, t^*) - f(z, t) \\
\delta_0 f(z, t) = f^*(z, t) - f(z, t) = \\
= \delta f(z, t) - \frac{\partial f(z, t)}{\partial z^k} \delta z^k - \frac{\partial f(z, t)}{\partial t} \delta t .
\] (2.11)

On the other hand, in case 2), it is convenient to distinguish three kinds of variations: if

\[
\begin{align*}
& t \to t^* = t + \delta t \\
& x(t) \to x^*(t^*) = x(t) + \delta x(t) = x(t) + \delta_0 x(t) + \frac{dx(t)}{dt} \delta t ,
\end{align*}
\] (2.12)

we shall define:

\[
\begin{align*}
\delta f(x(t), t) &= f^*(x^*(t^*), t^*) - f(x(t), t) \\
\delta_0 f(x(t), t) &= f^*(x(t), t) - f(x(t), t) = \\
&= \delta f(x(t), t) - \frac{\partial f(x(t), t)}{\partial x^k} \delta x^k(t) - \frac{\partial f(x(t), t)}{\partial t} \delta t \\
\delta_0 \eta f(x(t), t) &= f^*(x^*(t), t) - f(x(t), t) = \\
&= \delta f(x(t), t) - \left[ \frac{\partial f(x(t), t)}{\partial x^k} \frac{dx^k(t)}{dt} + \frac{\partial f(x(t), t)}{\partial t} \right] \delta t .
\end{align*}
\] (2.13)

Let us now turn to the Hamiltonian formalism. The canonical momenta and the Hamiltonian function are (\( \bar{f}(p, q) \) denotes a function in phase-space),

\[
\begin{align*}
p_i &= \frac{\partial L_M}{\partial \dot{x}^i} = m \delta_{ij} \dot{x}^j \\
\bar{H} &= p_i \dot{x}^i - L_M = \frac{1}{2m} \delta_{ij} p_i p_j ,
\end{align*}
\] (2.14)

respectively. Then, the conserved quantity \( G \) becomes:

\[
\begin{align*}
\bar{G} &= \varepsilon \bar{H} + \varepsilon^i p_i + \omega^i \bar{J}_i + v^i \bar{K}_i \\
&= \varepsilon \bar{H} + (\eta^i - tv^i)p_i + m\delta_{ij} v^i x^j \geq 0 ,
\end{align*}
\] (2.15)

where, for future convenience, we have introduced the infinitesimal transformation descriptors

\[
\eta^i(x^j) = \varepsilon^i + c_{jk}^i \omega^j x^k .
\] (2.16)

From Eq.(2.15), we obtain the following independent constants of the motion:

\[
\bar{H} , \ p_i , \ \bar{K}_i = m\delta_{ij} x^j - t p_i , \ \bar{J}_i = c_{ijk}^i x^j p_k .
\] (2.17)
The constants of motion (2.17) provide a projective canonical realization of the Lie algebra of the Galilei group in terms of Poisson brackets:

\[
\{K_i, H\} = \mathcal{P}_i \\
\{K_i, \mathcal{P}_j\} = \delta_{ij}m \\
\{\mathcal{P}_i, \mathcal{J}_j\} = c_{ij}^k \mathcal{P}_k \\
\{K_i, \mathcal{J}_j\} = c_{ij}^k \mathcal{K}_k \\
\{\mathcal{J}_i, \mathcal{J}_j\} = c_{ij}^k \mathcal{J}_k ,
\]

(2.18)

where:

\[
H = \bar{H} , \mathcal{P}_i = \bar{p}_i , K_i = \bar{K}_i , J_i = \bar{J}_i .
\]

(2.19)

Alternatively, the realization can be considered as a true realization [13] of the centrally-extended Galilei group via the central charge \( M = m \).

The generator of the equal-time Galilei transformation in phase-space, say \( \bar{\delta}_0 \), is given by the expression \( \bar{G} \) defined in eq. (2.15). We have

\[
\begin{cases}
\delta_0 t = 0 \\
\delta_0 x^i = \{x^i, G\} = \epsilon \frac{1}{m} \delta^{ij} p_j + \eta^i \\
\Rightarrow \quad \bar{\delta}_0 x^i\bigg|_{p = \partial L_M / \partial \dot{x}} = \delta_0 x^i = \delta x^i + \varepsilon \dot{x}^i 
\end{cases}
\]

(2.20)

Notice that, within the Hamiltonian formalism, the variation \( \delta_0 \dot{x}^i \) can be obtained only by using the equations of motion. In fact, we have:

\[
\delta_0 \dot{x}^i = \frac{\partial \eta^i(x)}{\partial x^k} \dot{x}^k - v^i - \varepsilon \ddot{x}^i = \frac{\partial \eta^i(x)}{\partial x^k} \dot{x}^k - v^i - \varepsilon \frac{1}{m} \delta^{ij} \mathcal{E} L_j
\]

(2.21)

Under the transformations (2.4), which are the configuration-space analogues of the transformations (2.20), we have:

\[
\Delta S_M = \int_{t_1}^{t_2} dt \left[ \frac{d}{dt} \left( \varepsilon L_M - m \delta_{ij} x^i v^j \right) \right].
\]

(2.22)

The resulting Noether’s constants of the motion are clearly the same as those associated with the transformations (2.3) and (2.20).

This complication can be easily avoided by turning to a re-parameterization invariant formulation of the free mass-point system. Using coordinates \( t(\lambda), x^i(\lambda) \), the Lagrangian
and the Action become
\[
\hat{L}_M(\lambda) = \frac{1}{2} m \frac{\delta_{ij} x^{\prime i}(\lambda) x^{\prime j}(\lambda)}{t'(\lambda)} ,
\]
\[
\hat{S}_M = \int_{\lambda_1}^{\lambda_2} d\lambda \, \hat{L}_M(\lambda) ,
\tag{2.23}
\]
respectively, where \( f'(\lambda) \equiv \frac{d}{d\lambda} f(\lambda) \). As before, in this enlarged space, we can define again an infinitesimal transformation \( \hat{\delta} \) and an “equal-\( \lambda \)” one, say \( \hat{\delta}_0 \). The associated Euler-Lagrange equations and canonical momenta are
\[
\begin{align*}
\hat{\mathcal{E}} \mathcal{L}_i = \frac{d}{d\lambda} \left[ \frac{m \delta_{ij} x^{\prime i}(\lambda) x^{\prime j}(\lambda)}{t'^2(\lambda)} \right]_0 = 0 \\
\hat{\mathcal{E}} \mathcal{L}_i = -\frac{d}{d\lambda} \left[ \frac{m \delta_{ij} x^{\prime i}(\lambda)}{t'(\lambda)} \right]_0 = 0 ,
\end{align*}
\tag{2.24}
\]
\[
\begin{align*}
\hat{E} &= -\frac{\partial \hat{L}}{\partial t'} = \frac{m \delta_{ij} x^{\prime i}(\lambda) x^{\prime j}(\lambda)}{2 t'^2(\lambda)} = \frac{1}{2m} \delta^{ij} \hat{p}_i \hat{p}_j \\
\hat{\dot{p}}_i &= \frac{\partial \hat{L}}{\partial x^{\prime i}} = \frac{m \delta_{ij} x^{\prime j}(\lambda)}{t'(\lambda)} = m \delta_{ij} \dot{x}^j (t) = p_i ,
\end{align*}
\tag{2.25}
\]
respectively, where we have defined the Poisson brackets so that:
\[
\{t(\lambda), \hat{E}(\lambda)\}' = -1 \\
\{x^i(\lambda), \hat{p}_j(\lambda)\}' = \delta^j_i .
\tag{2.26}
\]
In the enlarged phase-space, coordinatized by \( (t, \ x^i, \ \hat{E}, \ \hat{p}_i) \), we obtain a vanishing canonical Hamiltonian and the first-class constraint
\[
\hat{\chi} \equiv \hat{E} - \frac{1}{2m} \delta^{ij} \hat{p}_i \hat{p}_j \approx 0 .
\tag{2.27}
\]
The constraint \( \hat{\chi} \) generates the following transformation of the configurational variables:
\[
\hat{\delta}_0 t(\lambda) = -\alpha(\lambda) , \quad \hat{\delta}_0 x^i(\lambda) = -\alpha(\lambda) \frac{1}{m} \delta^{ij} \hat{p}_j .
\]
This is the re-parameterization gauge transformation \( \lambda \to \lambda - \alpha(\lambda)/t' \). The Lagrangian is obviously \textit{quasi-invariant} under this operation since:
\[
\hat{\delta}_0 \hat{L}_M = \frac{d}{d\lambda} \left[ \frac{-\alpha(\lambda)}{t'} \hat{L}_M \right] .
\tag{2.28}
\]
The canonical generators of the extended Galilei algebra are now:
\[
\hat{\mathcal{H}} = \hat{E} , \quad \hat{\mathcal{P}}_i = \hat{p}_i , \quad \hat{\mathcal{K}}_i = m \delta_{ij} x^j - t \hat{p}_i , \quad \hat{\mathcal{J}}_i = c_{ij}^k x^j \hat{p}_k , \quad \hat{\mathcal{M}} = m .
\tag{2.29}
\]
and satisfy the Lie-algebra (2.18) with the primed Poisson-brackets (2.26). Consequently, the generator of the complete phase-space Galilei transformation \( \delta_0 \), which is now given by:

\[
\hat{G} = \varepsilon \hat{E} + \varepsilon^i \hat{p}_i + \omega^i \hat{J}_i + v^i \hat{K}_i
\]

\[
= \varepsilon \hat{E} + (\eta^i - tv^i)\hat{p}_i + m\delta_{ij}v^i x^j\ ,
\]

yields the following “equal-\( \lambda \)” infinitesimal transformations:

\[
\begin{align*}
\delta_0 \lambda &= 0 \\
\delta_0 t &= \{ t \ , \ \hat{G} \}' = -\varepsilon = \delta_0 t = \delta t \\
\delta_0 x^i &= \{ x^i \ , \ \hat{G} \}' = \varepsilon^i + c_{jk}^i \omega^j x^k - tv^i = \delta_0 x^i = \delta x^i,
\end{align*}
\]

which coincide with the transformations (2.3).

We have now \( \hat{\delta_0} \hat{p}_i \bigg|_{\hat{p}=\partial \hat{L}_M/\partial x'_{\lambda}} = \hat{\delta_0} \left[ m \delta_{ij} x^j \right] \), without any use of Euler-Lagrange equations. On the other hand, under the transformations (2.31), it follows:

\[
\hat{\delta_0} \hat{L}_M = \frac{d}{d\lambda} \left[ -m\delta_{ij} v^i x^j \right],
\]

and

\[
\hat{\delta_0} \hat{\chi} = 0 \ ,
\]

so that the canonical generators (2.29) are again constants of the motion. Furthermore, the first-class constraint is Galilei invariant, and the quasi-invariance of the Lagrangian is an effect of the central-charge term alone.

3 “Gauging” the extended Galilei algebra for the free mass-point

We proceed now to gauging the Galilei transformations along the standard line of Utiyama [11]. Since the Newtonian time is absolute, the most general finite transformation allowed for the time coordinate is of the form \( t \rightarrow t' = t + f(t) \). Therefore a first guess of how to gauge the group amounts to replace the complete infinitesimal generators (2.15) and (2.30) with:

\[
\bar{G} = \varepsilon(t) \bar{H} + \varepsilon^i(x,t) p_i + \omega^i(x,t) \bar{J}_i + v^i(x,t) \bar{K}_i
\]

\[
= \varepsilon(t) \bar{H} + \left[ \eta^i(x,t) - tv^i(x,t) \right] p_i + m \delta_{ij}v^i(x,t) x^j\ ,
\]

10
and

\[ \hat{G} = \varepsilon(t) \hat{E} + \varepsilon^i(x, t) \hat{p}_i + \omega^i(x, t) \hat{J}_i + v^i(x, t) \hat{K}_i \]

\[ = \varepsilon(t) \hat{E} + [\eta^i(x, t) - tv^i(x, t)] \hat{p}_i + m \delta_{ij} v^j(x, t) x^j, \]

respectively, with \( \varepsilon \) independent of \( x \), and

\[ \eta^i(x, t) = \varepsilon^i(x, t) + c_{jk}^i \omega^j(x, t) x^k. \] (3.3)

Notice that we have not absorbed the term \( tv^i(x, t) \) into \( \eta^i(x, t) \), as it would be natural from the point of view of the configuration space. Actually, this would not be as much natural in phase-space and, in addition, from the group-theoretical point of view, it would be confusing: it would mix the role of the central charge with that of the three-dimensional Euclidean subalgebra.

We see from Eq.(3.1) and Eq.(3.2) that gauging the extended Galilei algebra in the case of the mass-point realization is equivalent to gauging the algebra generated by energy (respectively time-translation, within the re-parameterization invariant picture), linear momentum, and the central-charge \( \mathcal{M} = m \), with parameters \( \varepsilon(t), \eta^i(x, t) - tv^i(x, t), \delta_{ij} v^j(x, t) x^j \), respectively.

Corresponding to the complete generator (3.1), we obtain:

\[
\begin{aligned}
\delta_0 t &= 0 \\
\delta_0 x^i &= \varepsilon(t) \frac{\delta_{ij} p_j}{m} + \eta^i(x, t) - t v^i(x, t) \\
\end{aligned}
\] (3.4)

and

\[
\begin{aligned}
\delta_0 p_i &= p_k \frac{\partial}{\partial x^i} [\eta^k(x, t) - t v^k(x, t)] + m \frac{\partial}{\partial x^i} [\delta_{ik} x^j v^j(x, t)] \\
\delta_0 H &= \frac{1}{m} \delta_{ij} p_i \delta p_j \\
\delta_0 J_i &= c_{ij}^k p_k [\eta^j(x, t) - t v^j(x, t)] - c_{ij}^k x^j p_r \frac{\partial}{\partial x^k} [\eta^r(x, t) - t v^r(x, t)] \\
&- mc_{ij}^k x^j \frac{\partial}{\partial x^k} [\delta_{im} x^l v^m(x, t)] \\
\delta_0 K_i &= \varepsilon(t) p_i + tp_k \frac{\partial}{\partial x^i} [\eta^k(x, t) - t v^k(x, t)] \\
&+ m \delta_{ij} [\eta^j(x, t) - t v^j(x, t)] - m t \frac{\partial}{\partial x^i} [\delta_{im} x^l v^m(x, t)]. \\
\end{aligned}
\] (3.5)
Finally, using \( p_i = m \delta_{ij} \dot{x}^j \), we obtain
\[
\begin{align*}
\delta_0 t &= 0 \\
\delta_0 \dot{x}^i &= \varepsilon(t) \dot{x}^i + \eta^i(x, t) - t \nu^i(x, t) \\
\delta_0 \ddot{x}^i &= \frac{d}{dt} [\varepsilon(t) \dot{x}^i] + x^k \frac{\partial}{\partial x^k} [\eta^i(x, t) - t \nu^i(x, t)] \\
&\quad + \frac{\partial}{\partial t} [\eta^i(x, t) - t \nu^i(x, t)].
\end{align*}
\tag{3.6}
\]

Eqs. (3.6) reduce to Eqs. (2.4) in the limit of global flat symmetry and they can be taken as a definition of the Galilei gauge transformations in configuration space. Notice that the quasi-invariance of the Lagrangian under the global flat transformations is now broken and that \( \delta_0 p_i \big|_{p_i = m \delta_{ij} \dot{x}^j} \) is not equal to \( \delta_0 [m \delta_{ij} \dot{x}^j] \).

On the other hand, within the re-parameterization invariant picture, corresponding to the complete generator (3.2), we get:
\[
\begin{align*}
\hat{\delta}_0 \lambda &= 0 \\
\hat{\delta}_0 t(\lambda) &= -\varepsilon(t(\lambda)) \\
\hat{\delta}_0 x^i(\lambda) &= \eta^i(x, t) - t \nu^i(x, t) \\
\hat{\delta}_0 t'(\lambda) &= -t' \frac{d\varepsilon(t(\lambda))}{dt} \\
\hat{\delta}_0 x''(\lambda) &= x^k \frac{\partial}{\partial x^k} [\eta^i(x, t) - t \nu^i(x, t)] + t' \frac{\partial}{\partial t} [\eta^i(x, t) - t \nu^i(x, t)],
\end{align*}
\tag{3.7}
\]

and
\[
\begin{align*}
\hat{\delta}_0 \hat{p}_i &= -\hat{p}_k \frac{\partial}{\partial x^k} [\eta^j(x, t) - t \nu^j(x, t)] - m \frac{\partial}{\partial x^k} [\delta_{ik} x^j v^k(x, t)] \\
\hat{\delta}_0 \hat{E} &= \hat{E} \frac{d\varepsilon(t)}{dt} + \hat{p}_i \frac{\partial}{\partial t} [\eta^j(x, t) - t \nu^j(x, t)] + m \frac{\partial}{\partial t} [\delta_{ik} x^j v^j(x, t)] \\
\hat{\delta}_0 \hat{J}_i &= c_{ij} k \hat{p}_k [\eta^j(x, t) - t \nu^j(x, t)] - c_{ij} k x^j \hat{p}_r \frac{\partial}{\partial x^k} [\eta^r(x, t) - t \nu^r(x, t)] \\
&\quad - m c_{ij} k x^j \frac{\partial}{\partial x^k} [\delta_{lm} x^l v^m(x, t)] \\
\hat{\delta}_0 \hat{K}_i &= \varepsilon(t) \hat{p}_i + t \hat{p}_k \frac{\partial}{\partial x^k} [\eta^j(x, t) - t \nu^j(x, t)] \\
&\quad + m \delta_{ij} [\eta^j(x, t) - t \nu^j(x, t)] - m t \frac{\partial}{\partial x^i} [\delta_{lm} x^l v^m(x, t)].
\end{align*}
\tag{3.8}
\]

In conclusion, from Eq. (3.6), we see that \( x(t) \) undergoes a time-dependent general Euclidean coordinate transformation (although deformed by an effect from Galilei boosts) plus a velocity-dependent transformation induced by the condition \( \delta_0 t = 0 \). On the other hand,
from Eq.(3.7), we see that \( t(\lambda) \) undergoes a transformation which does not involve the Euclidean coordinates and, simultaneously, \( x(\lambda) \) undergoes a time-dependent general Euclidean coordinate transformation (still deformed by an effect from Galilei boosts). Being a consequence of the absolute nature of the Newtonian time, these features do not appear in the gauging of the Poincaré realization corresponding to the free relativistic mass-point with Lagrangian \( L_M = -m\sqrt{\eta_{\mu\nu}x'^{\mu}(\lambda)x'^{\nu}(\lambda)} \) (see, for instance, [14]). Actually, the gauging of the Poincaré transformations induces a general coordinate transformation of \( x^\mu(\lambda) \) which is indistinguishable from the simple gauging of the space-time translations alone. As well-known, the gauging of the Lorentz transformations is made evident by means of the “soldering procedure” which amounts to say that the vectors belonging to the tangent bundle of the curved space-time must transform as four vectors under “local” Lorentz transformation. The “soldering” is done by means of a set of vierbeins \( E_A^\mu(x) \) so that the flat transformation \( x'^\mu \to x'^\nu + \varepsilon^\mu_\nu x'^\mu \) (where \( \varepsilon^\mu_\nu \) are the parameters of the Lorentz transformations) is replaced by \( E_A^\mu(x)x'^\mu \to E_A^\mu(x)x'^\mu + \varepsilon^S_B E_A^\mu(x)x'^\mu. \)

We want to discuss now the problem of the invariance (possibly quasi-invariance) of the Lagrangian with respect to the local Galilei transformations just defined. In the Newtonian case, unlike the relativistic one, the absolute nature of time prevents us from using the standard “soldering” procedure, if not for the Euclidean subalgebra generated by \( \hat{P}_k \) and \( \hat{J}_i \). Actually, while the effect of the space rotations in the term \( \eta^i(x,t) \) is indistinguishable from the effect generated in it by space translations, the effect of “local” space rotations and translations can be distinguished for three-vectors by introducing the “soldering” with dreibeins \( E_i^a(x,t) \). Notice, however, that further complications arise here from the fact that the general Euclidean transformations are time-dependent and that time translations and Galilei boosts introduce new terms into the transformation of the velocity. Accordingly, the rule for the transition from the global flat transformation of three-velocity to its general transformation is obtained by defining \( \omega^a(x,t) = E_i^a(x,t)\omega^i(x,t) \), and by imposing, within the two alternative pictures introduced above, the following transformation properties

\[
\begin{align*}
\delta_0 \dot{x}^i &= \frac{d}{dt}[\varepsilon \dot{x}^i] + c_{ij}^k \omega^j \dot{x}^k - v^i \\
\Rightarrow \delta_{[t]}[E_i^a \dot{x}^i] &= \frac{d}{dt}[\varepsilon E_i^a \dot{x}^i] + c_{bc}^a \omega^b E_i^c \dot{x}^i + [?] 
\end{align*}
\]
where \( \eta \) is the transformation. Then, by imposing
\[
\Rightarrow \delta_{0[\lambda]}[E^a_i x^i] = c_{bc}^a \omega^b x^i + [?] ,
\]
where the question marks in Eqs. (3.9) and (3.10) stand for possible terms connected to time translation and Galilei boosts: this point will be settled later on.

By means of the invertible \( dreibeins E^a_i \), a Euclidean metric in the three-space is naturally introduced in the form
\[
\delta_{ij} \Rightarrow g_{ij} \equiv \delta_{ab} E^a_i E^b_j ,
\]
while the inverse metric can be likewise defined in term of the inverse \( dreibeins H^i_a (H^i_a E^a_b = \delta^i_b , H^i_a E^a_j = \delta^i_j) \) as:
\[
\delta^{ij} \rightarrow g^{ij} \equiv \delta^{ab} H^i_a H^j_b , \quad g^{ij} g_{jk} = \delta^i_k .
\]

Let us now discuss, within the standard picture, the issue of the invariance of the Lagrangian under the newly introduced local transformations, by considering first the local time-independent Euclidean transformations \( \delta^{[\varepsilon]}_0 (\varepsilon = 0, v^i = 0, \omega^i = \omega^i(x), \varepsilon^i = \varepsilon^i(x)) \); then pure time transformations \( \delta^{[\varepsilon]}_0 (\varepsilon = \varepsilon(t), v^i = 0, \omega^i = 0, \varepsilon^i = 0) \), and the combination of both \( \delta^{[\varepsilon]}_0 \) and \( \delta^{[\varepsilon]}_0 \), say \( \delta^{[\varepsilon]}_0 (\varepsilon = \varepsilon(t), v^i = 0, \omega^i = \omega^i(x), \varepsilon^i = \varepsilon^i(x)) \); finally the most general transformations including local time-dependent Euclidean transformations and Galilei boosts.

i) In the case of local time-independent Euclidean transformations \( \delta^{[\varepsilon]}_0 \), defined by \( \varepsilon = 0, v^i = 0, \omega^i = \omega^i(x), \varepsilon^i = \varepsilon^i(x) \), we have:
\[
\delta^{[\varepsilon]}_0 \dot{x}^i = x^k \frac{\partial}{\partial x^k} \left[ \varepsilon^i(x) + c_{ji} \omega^j(x) x^j \right]
\]
\[
= x^k \eta^{[\varepsilon]}_{c[i]}(x) ,
\]
where \( \eta^{[\varepsilon]}_{c[i]}(x) = \varepsilon^i(x) + c_{ji} \omega^j(x) x^j \) is the restriction of \( \eta^i(x,t) \) to time independent Euclidean transformation. Then, by imposing
\[
\delta^{[\varepsilon]}_0 [E^a_i \dot{x}^i] = c_{bc}^a \omega^b(x) [E^c_i \dot{x}^i] ,
\]
it follows:
\[
\delta^{[\varepsilon]}_0 [E^a_i (x,t)] = \delta^{[\varepsilon]}_0 E^a_i (x,t) + \frac{\partial E^a_i (x,t)}{\partial x^k} \delta^{[\varepsilon]}_0 x^k
\]
\[
= - \frac{\partial \eta^{[\varepsilon]}_{c[i]}(x)}{\partial x^k} E^a_i + c_{bc}^a \omega^b(x) E^c_i ,
\]
\[
\delta_0^{[\epsilon]} E^a_i = -\frac{\partial \eta^i_{[\epsilon]}(x)}{\partial x^k} E^a_i + c_{bc} \omega^b(x) E^c_i - E^a_{i,k} \eta^k_{[\epsilon]}(x). \tag{3.16}
\]

In agreement to the conventions established in Eqs. (2.13). It is then immediately seen that the invariance of the Lagrangian under time-independent local Euclidean transformation can be recovered by means of the substitution:

\[
\dot{x}^i \Rightarrow E^a_{i} \dot{x}^i, \tag{3.17}
\]

which leads to

\[
L_M = \frac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j \Rightarrow L_M^{[\epsilon]} \equiv \frac{1}{2} m \delta_{ab} E^a_i \dot{E}^b_j \dot{E}^j \dot{x}^i = \frac{1}{2} mg_{ij} \dot{x}^i \dot{x}^j. \tag{3.18}
\]

In fact, since the transformation rules for the three-dimensional metric field (3.11) are given by:

\[
\delta_0^{[\epsilon]} g_{ij} = \delta_0^0 g_{ij} + g_{ij,k} \eta^k_{[\epsilon]}(x) = -\frac{\partial \eta^i_{[\epsilon]}(x)}{\partial x^j} g_{kj} - \frac{\partial \eta^k_{[\epsilon]}(x)}{\partial x^j} g_{ik}, \tag{3.19}
\]

(which are the correct infinitesimal transformation properties of a covariant second-rank three-dimensional tensor), it follows:

\[
\delta_0^{[\epsilon]} L_M^{[\epsilon]} = 0. \tag{3.20}
\]

Therefore, the preliminary guess for the gauging of the Galilei transformation (see Eqs. 3.1, 3.4-3.6, or 3.2, 3.7 and 3.8) is made geometrically consistent by replacing the flat metric \(\delta_{ij}\) and \(\delta^{ij}\) in those formulae by \(g_{ij}\) and \(g^{ij}\), respectively.

Within the re-parameterization invariant picture, the invariance is recovered by the corresponding substitution,

\[
x'^i \Rightarrow E^a_i x'^i \tag{3.21}
\]

which leads to

\[
\hat{L}_M(\lambda) \Rightarrow \hat{L}_M^{[\epsilon]}(\lambda) = \frac{1}{2} m g_{ij} x'^i(\lambda) x'^j(\lambda) \frac{1}{t'(\lambda)}. \tag{3.22}
\]

In fact, from Eq. (3.19) with \(\delta_0^{[\epsilon]}\) formally replaced by \(\delta_0^{[\epsilon]}\), it follows:

\[
\delta_0^{[\epsilon]} \hat{L}_M^{[\epsilon]} = 0. \tag{3.23}
\]
ii) Consider now the pure time-translations $\hat{\delta}_0^{[t]}$. For the sake of simplicity, we shall confine to the re-parameterization invariant picture. We have:

$$\begin{cases}
\hat{\delta}_0^{[t]} t &= -\epsilon(t) \\
\hat{\delta}_0^{[t]} x^i &= 0
\end{cases} \quad (3.24)$$

It is immediate to see that, by means of a *einbein* substitution of the form

$$t' \Rightarrow \Theta(t) t' , \quad (3.25)$$

the Lagrangian:

$$\hat{L}_M^{[e]}(\lambda) \equiv \frac{1}{2} m \delta_{ij} x'^i(\lambda) x'^j(\lambda) \Theta(t) t'(\lambda) , \quad (3.26)$$

is invariant under the transformations $\hat{\delta}_0^{[t]}$, provided we impose:

$$\hat{\delta}_0^{[t]}[\Theta(t)t'] = 0 \quad , \quad (3.27)$$

which, in turn, gives:

$$\hat{\delta}_0^{[t]} \Theta(t) = \hat{\delta}_0^{[t]}(t) - \varepsilon(t) \frac{d\Theta(t)}{dt} = \dot{\varepsilon}(t) \Theta(t) \quad . \quad (3.28)$$

iii) We can summarize the results found up to now in the re-parameterization invariant picture, by saying that the modified matter Lagrangian

$$\hat{L}_M^{[e]}(\lambda) \equiv \frac{1}{2} m g_{ij} x'^i(\lambda) x'^j(\lambda) \Theta(t) t'(\lambda) , \quad (3.29)$$

is strictly invariant under the local transformations:

$$\begin{cases}
\hat{\delta}_0^{[e]} t &= \hat{\delta}_0^{[e]} t = -\epsilon(t) \\
\hat{\delta}_0^{[e]} x^i &= \hat{\delta}_0^{[e]} x^i = \eta_{[e]}^i(x) \\
\hat{\delta}_0^{[e]} t' &= \hat{\delta}_0^{[e]} t' = -\varepsilon(t) t' \\
\hat{\delta}_0^{[e]} x'^i &= \hat{\delta}_0^{[e]} x'^i = x'^k \frac{\partial \eta_{[e]}^i(x)}{\partial x^k}
\end{cases} \quad (3.30)$$

if we adopt the following transformation rules for the fields:

$$\begin{cases}
\hat{\delta}_0^{[e]} \Theta(t) &= \hat{\delta}_0^{[e]} \Theta(t) = \dot{\varepsilon}(t) \Theta(t) \\
\hat{\delta}_0^{[e]} E^a_i &= \hat{\delta}_0^{[e]} E^a_i = - \frac{\partial \eta_{[e]}^i(x)}{\partial x^k} E^a_i + c_{be}^a \omega^b(x) E^c_i \\
\hat{\delta}_0^{[e]} g_{ij} &= \hat{\delta}_0^{[e]} g_{ij} = - \frac{\partial \eta_{[e]}^i(x)}{\partial x^j} g_{kj} - \frac{\partial \eta_{[e]}^k(x)}{\partial x^j} g_{ik}
\end{cases} \quad (3.31)$$
iv) Finally, we have to show that the invariance (possibly quasi-invariance) of the matter Lagrangian under the most general transformations, including local time-dependent Euclidean transformations and Galilei boosts, can be recovered by introducing four additional external fields, say $A_0(x, t), A_i(x, t)$. Within the re-parameterization invariant picture, our scope will be achieved if we succeed in defining a new matter Lagrangian $\hat{L}_M^g$ such that:

a) under the gauge transformations (3.7) and the corresponding transformations induced on all the additional fields, be quasi-invariant in the form:

$$\hat{\delta}_{0[\lambda]} \hat{L}_M^g = \frac{dF}{d\lambda},$$  \hspace{1cm} (3.32)

b) its global flat limit, together with that of its transformation properties, coincide with the expression (2.23) and (2.32), respectively.

First of all, in order to reproduce the cocycle term of Eq.(2.32) in the global flat limit, $F$ will be chosen as $F = -mg_{ij}v^i x^j$. Then, the matter Lagrangian will be defined as

$$\hat{L}_M^g(\lambda) = \frac{1}{\Theta t'} \left[ g_{ij} x'^i x'^j + 2A_i x'^i t' + 2A_0 t' t' \right].$$  \hspace{1cm} (3.33)

Second, since we have to preserve the transformation properties of $E_i^a, g_{ij}$ and $\Theta$ already established for the case $v^i(x, t) = 0$ in Eqs.(3.31), we shall assume

$$\begin{align*}
\hat{\delta}_{0[\lambda]} \Theta &= \hat{\delta}_0 \Theta - \epsilon(t) \frac{d\Theta(t)}{dt} = \dot{\epsilon}(t) \Theta(t), \\
\hat{\delta}_{0[\lambda]} E_i^a &= \hat{\delta}_0 E_i^a + E_i^a \tilde{\eta}^k(x, t) - \frac{\partial E_i^a}{\partial t} \epsilon(t) \\
&= c \epsilon^a_{bc} \omega^{bc}(x, t) E_i^a - \frac{\partial \tilde{\eta}^k(x, t)}{\partial x^i} E_k^a, \\
\hat{\delta}_{0[\lambda]} g_{ij} &= \hat{\delta}_0 g_{ij} + g_{ij,k} \tilde{\eta}^k(x, t) - \frac{\partial g_{ij}}{\partial t} \epsilon(t) \\
&= -\frac{\partial \tilde{\eta}^k(x, t)}{\partial x^i} g_{kj} - \frac{\partial \tilde{\eta}^k(x, t)}{\partial x^i} g_{ij},
\end{align*}$$  \hspace{1cm} (3.34)

where we have introduced the notation $\tilde{\eta}^k(x, t) \equiv \eta^k(x, t) - tv^k(x, t)$. Then, the quasi-invariance (3.32) of the matter Lagrangian (3.33) is guaranteed if the additional fields
\( A_0(\mathbf{x}, t) \), \( A_i(\mathbf{x}, t) \) transform as follows:

\[
\begin{align*}
\hat{\delta}_{0(x)} A_0 &= \hat{\delta}_0 A_0 - \varepsilon \frac{\partial A_0}{\partial t} + A_{i,j} \tilde{\eta}^j \\
&= 2\varepsilon A_0 - A_i \frac{\partial \tilde{\eta}^i}{\partial t} + \Theta \frac{\partial \mathcal{F}}{\partial t} \\
\hat{\delta}_{0(x)} A_i &= \hat{\delta}_0 A_i - \varepsilon \frac{\partial A_i}{\partial t} + A_{i,j} \tilde{\eta}^j \\
&= \dot{\varepsilon} A_i - A_j \frac{\partial \tilde{\eta}^j}{\partial x^i} - g_{ij} \frac{\partial \tilde{\eta}^j}{\partial t} + \Theta \frac{\partial \mathcal{F}}{\partial x^i}.
\end{align*}
\]

(3.35)

Notice that the fields \( E^0_i \), \( g_{ij} \), \( \Theta \), \( A_0 \) and \( A_i \) have the global flat Galilean limits \( \delta^0_i \), \( \delta_{ij} \), 1, 0 and 0, respectively. Here, the \( A_i \)'s play the role of the components \( E^0_i(\mathbf{x}) \) of the relativistic vierbeins, while the two fields \( \Theta \) and \( A_0 \) correspond to a splitting of the vierbein component \( E^0_0(\mathbf{x}) \), as it will be clear later on.

In a similar way, it can be seen that, within the standard picture, the Lagrangian

\[
L^2_M(t) = \frac{1}{\Theta} \left( \frac{m}{2} \left[ g_{ij} \dot{x}^i \dot{x}^j + 2A_i \dot{x}^i + 2A_0 \right] \right),
\]

(3.36)

is quasi-invariant under the field transformation rules (3.34), (3.35), with \( \delta_0 = \hat{\delta}_0 \), and the mass-point coordinate transformations given by (3.6), in the sense that we have

\[
\delta_{0[t]} L^q_M = \frac{d\mathcal{F}}{dt} + \varepsilon \frac{d[L^q_M]}{dt},
\]

(3.37)

so that

\[
\Delta S^q_M = \int_{t_1}^{t_2} dt \left[ \frac{d\mathcal{F}}{dt} + \frac{d[\varepsilon L^q_M]}{dt} \right] = 0.
\]

(3.38)

The condition for the invariance of the theory under the gauged Galilei transformations can now be easily formulated also in the Hamiltonian formalism. From Eq. (3.33), we get the following expressions for energy and linear momentum

\[
\begin{align*}
\tilde{E} &= \frac{\partial \hat{L}^q_M}{\partial t'} = \frac{1}{\Theta t'^2} \left( \frac{m}{2} g_{ij} x^i x'^j - \frac{m}{\Theta} A_0 \right) \\
\hat{p}_i &= \frac{\partial \hat{L}^q_M}{\partial x'^i} = \frac{m}{\Theta t'} \left( g_{ij} x'^j + A_i t' \right).
\end{align*}
\]

(3.39)

while the first-class constraint (2.27) becomes

\[
\dot{x}_g = \frac{1}{\Theta} \left[ \hat{E} + \frac{m}{\Theta} A_0 \right] - \frac{g_{ij}}{2m} \left[ \hat{p}_i - \frac{m}{\Theta} A_i \right] \left[ \hat{p}_j - \frac{m}{\Theta} A_j \right].
\]

(3.40)
Since, under the transformation (3.8) (with $\delta_{ij} \Rightarrow g_{ij}$), we have:

\[
\hat{\delta}_{0[\lambda]} \hat{\chi} = \frac{1}{\Theta} \hat{\delta}_{0[\lambda]} \hat{E} - \frac{1}{\Theta^2} \left[ \hat{E} + \frac{m}{\Theta} \hat{A}_0 \right] \hat{\delta}_{0[\lambda]} \Theta + \frac{1}{\Theta^2} m \hat{\delta}_{0[\lambda]} A_0
\]

\[
- \frac{1}{2m} \hat{\delta}_{0[\lambda]} g^{ij} \left[ \hat{\dot{p}}_i - \frac{m}{\Theta} A_i \right] \left[ \hat{\dot{p}}_j - \frac{m}{\Theta} A_j \right]
\]

\[
- \frac{g^{ij}}{m} \left[ \hat{\dot{p}}_i - \frac{m}{\Theta} A_i \right] \left[ \hat{\dot{p}}_j - \frac{m}{\Theta} A_j \right] \left[ \hat{\delta}_{0[\lambda]} \hat{\delta}_{0[\lambda]} A_j + \frac{m}{\Theta^2} A_j \hat{\delta}_{0[\lambda]} \Theta \right] ,
\]

the invariance of the constraint

\[
\hat{\delta}_{0[\lambda]} \hat{\chi}_g = 0
\]

which is the Hamiltonian analogue to the invariance of the Lagrangian, is guaranteed if the fields transform according to

\[
\begin{align*}
\hat{\delta}_{0[\lambda]} \Theta &= \hat{\dot{\chi}}(t) \Theta(t) \\
\hat{\delta}_{0[\lambda]} g_{ij} &= - \frac{\partial \hat{\eta}^k(x, t)}{\partial x^i} g_{kj} - \frac{\partial \hat{\eta}^k(x, t)}{\partial x^i} g_{kj} \\
\hat{\delta}_{0[\lambda]} A_0 &= 2 \hat{\dot{\varepsilon}} A_0 - A_i \frac{\partial \hat{\eta}^i}{\partial t} - \Theta \frac{\partial}{\partial t} \left[ g_{ij} v^i x^j \right] \\
\hat{\delta}_{0[\lambda]} A_i &= \hat{\dot{\varepsilon}} A_i - A_j \frac{\partial \hat{\eta}^j}{\partial x^i} - g_{ij} \frac{\partial \hat{\eta}^j}{\partial t} - \Theta \frac{\partial}{\partial x^i} \left[ g_{ij} v^i x^j \right].
\end{align*}
\]

Note that since

\[
\delta f(x(t), t) = \hat{\delta}_{0[\lambda]} f(x(t(\lambda)), t(\lambda)) ,
\]

Eqs.(3.43) can be easily adapted to the standard picture.

Let us remark in addition that were it not for the presence of the *einbein* $\Theta$, the modified Hamiltonian constraints (3.40) could have been made invariant only in the *week* sense $\hat{\delta}_{0[\lambda]} \hat{\chi}_g = \varepsilon \hat{\chi}_g \simeq 0$.

Finally, the *cocycle* term which appeared in a generic form in the transformations (3.35) of the fields $A_0$ and $A_i$, is now *explicitly determined* (up to a constant) in a form which reproduces the standard expression in the global flat Galilean limit. This is due to the fact that the Hamiltonian formalism of the reparametrization invariant scheme requires a first class constraint which, in absence of external fields, says that the Galilei Casimir invariant representing the internal energy ($E - \frac{1}{2m} \delta_{ij} p_i p_j$) is equal to zero.

In presence of external fields, the constraints given by Eqs.(3.40), says again that the Casimir invariant vanishes. This is clearly a consequence of the fact that time is absolute
and of the fact that preservation of the constraints (i.e. of the vanishing of the Casimir invariant) is possible only provided the projective realization is taken into account through the cocycle terms in Eq.(3.43). The analogue of this phenomenon in the standard Lagrangian picture is expressed by the fact that the identification of the total energy in different coordinate systems connected by general Galilean coordinate transformation requires explicitly the appearance of the cocycle term within the transformation rules of the relevant quantities. On the other hand it is obvious that the cocycle term is not associated to actual forces since it appears in the variation of the Lagrangian as a total derivative irrelevant for the equations of motion.

4 Dynamics of a mass-point in the external ”gauge” fields

We want to discuss now the equations of motion of the mass-point acted upon, as a test particle, by the newly introduced compensating external fields. For the sake of simplicity, we will use the Lagrangian in the standard picture:

$$L_M^g(t) = \frac{m}{\Theta} \left[ \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + A_i \dot{x}^i + A_0 \right]. \quad (4.1)$$

Upon variation, we get the Euler-Lagrange equations:

$$\ddot{x}^i + \Gamma^i_{ki} \dot{x}^k \dot{x}^l = -\frac{\dot{\Theta}}{\Theta} \left[ \dot{x}^i + g^{ij} A_j \right] - g^{ij} \frac{\partial g_{ji}}{\partial t} \dot{x}^l + g^{ij} \left[ \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right] \dot{x}^l, \quad (4.2)$$

where:

$$\Gamma^l_{ij} = \binom{l}{i,j} = \frac{1}{2} H^l_a \left[ E^a_{i,j} + E^a_{j,i} \right], \quad (4.3)$$

is the three-dimensional metric affinity of $g_{ij}$.

Let us stress that the function $\Theta(t)$ which appears in the Lagrangian (4.1) and in the equations of motion (4.2) has no real dynamical content. In fact, by redefining the evolution
parameter $t$ and the field $A_0, A_i$ according to

\[
\begin{aligned}
T(t) & \equiv \int_0^t d\tau \Theta(\tau) \\
\tilde{A}_0 & \equiv A_0 / \Theta^2 \\
\tilde{A}_i & \equiv A_i / \Theta 
\end{aligned}
\]  

(4.4)

it follows:

\[
\frac{d^2x^i}{dT^2} + \Gamma^i_{kl} \frac{dx^k}{dT} \frac{dx^l}{dT} = -g^{ij} \frac{\partial g_{jl}}{\partial T} \frac{dx^l}{dT} + g^{ij} \left[ \frac{\partial \tilde{A}_l}{\partial x^j} - \frac{\partial \tilde{A}_i}{\partial x^l} \right] \frac{dx^l}{dT} + g^{ij} \left[ \frac{\partial A_0}{\partial x^j} - \frac{\partial A_i}{\partial T} \right].
\]  

(4.5)

In order to gain a better physical insight, we will discuss the standard Newtonian gravitational problem in general Galilean coordinates. Before doing that, let us reproduce in our notations the Kuchař’s classification of the relevant reference frames obtained by passive coordinate transformations:

1) **non rotating observers**: $\Theta = 1, A_i = 0$;
2) **rigid observers**, $\Theta = 1, g_{ij} = \delta_{ij}$ (an example belonging to this class is provided by eqs.(4.19) below; Wheeler’s definition of *Galilean observer* is a subcase of *rigid observers*, see eq.(4.22);
3) **freely-falling observers**: $\Theta = 1, A_0 = 0$
4) **Gaussian** (freely falling, non rotating) observer: $\Theta = 1, A_i = 0, A_0 = 0$ (these are the analogues of the *Fermi-Walker observers* in general relativity);
5) **Galilean observers** (rigid, non rotating): $\Theta = 1, g_{ij} = \delta_{ij}, A_i = 0, A_0 = -\varphi$: they are called *absolute Galilean observers* by Wheeler [15] and *inertial* by Levi-Civita [16];
6) **inertial observers** (rigid, freely falling, non rotating): $\Theta = 1, g_{ij} = \delta_{ij}, A_i = 0, A_0 = 0$;

Let $[y^a, t]$ be the coordinates used by a *Galilean observer* (rigid and non-rotating). Newton’s equations can be written:

\[
m \frac{d^2y^a}{dt^2} = -m \delta^{at} \frac{\partial \varphi}{\partial y^t},
\]  

(4.6)

where $\varphi(y, t)$ is the gravitational potential satisfying the Poisson equation ($\Delta = \delta^{ij} \partial_i \partial_j$)

\[
\Delta \varphi(z, t) = 4\pi Gm \delta[z - y(t)]
\]  

(4.7)
This situation corresponds to:

\[ \Theta = 1, \quad g_{ij} = \delta_{ij}, \quad A_0 = -\varphi, \quad A_i = 0. \quad (4.8) \]

Eqs. (4.4) and (4.7) are form-invariant under the group of Galilei transformations (2.3) that connect Galilean observers. In particular, the gravitational potential is a scalar under these transformations (see for example Wheeler [15] \S 12.17).

Let us perform now a passive transformation to a coordinate system \((T, \xi^a)\):

\[ \begin{align*}
  t & \rightarrow T(t) \\
y^a & \rightarrow \xi^a(x(t), t).
\end{align*} \quad (4.9)\]

We have:

\[ \frac{d^2 \xi^a}{dT^2} = \frac{1}{T^2} \left[ \frac{\partial \xi^a}{\partial x^i} \frac{\partial x^i}{T} + \frac{\partial \xi^a}{\partial t} \right] + \frac{\dot{T}}{T^2} \left[ \frac{\partial^2 \xi^a}{\partial x^i \partial x^j} \dot{x}^i \dot{x}^j + 2 \frac{\partial \xi^a}{\partial x^i} \dot{x}^i + \frac{\partial^2 \xi^a}{\partial t \partial t} \right] \quad (4.10)\]

and, after some algebraic manipulations, using Eq.(4.3),

\[ \dot{x}^i + \frac{\partial x^i}{\partial y^a} \frac{\partial^2 \xi^a}{\partial x^b \partial x^j} \dot{x}^b \dot{x}^j = -\frac{\partial x^i}{\partial y^a} \left[ 2 \frac{\partial^2 \xi^a}{\partial x^i \partial t} \dot{x}^i + \frac{\partial^2 \xi^a}{\partial t \partial t} \right] \]

\[ = -\frac{\partial x^i}{\partial y^a} \frac{\dot{T}}{T} \left[ \frac{\partial \xi^a}{\partial x^i} \dot{x}^i + \frac{\partial \xi^a}{\partial t} \right] \]

\[ = -\frac{\partial x^i}{\partial y^a} \dot{T}^2 \delta_{ab} \frac{\partial \varphi}{\partial y^b}, \quad (4.11)\]

where \(\dot{x}\) means derivation respect the evolution parameter \(t\). The identification of the equations (4.11) and (4.12) expresses the non-relativistic equivalence principle which is implicit in the application of the gauge technique to a space-time symmetry group. We have:

\[ \begin{align*}
  \Theta & = \dot{T} \\
g_{ij} & = \delta_{ab} \frac{\partial \xi^a}{\partial x^i} \frac{\partial \xi^b}{\partial x^j} \quad ; \quad E_i^a = \frac{\partial \xi^a}{\partial x^i} \\
g^{ij} & = \delta_{ab} \frac{\partial x^i}{\partial \xi^a} \frac{\partial x^j}{\partial \xi^b} \quad ; \quad H_i^a = \frac{\partial x^i}{\partial y^a} \\
A_0 & = -\varphi \dot{T}^2 + \frac{1}{2} \delta_{ab} \frac{\partial \xi^a}{\partial t} \frac{\partial \xi^b}{\partial t} \\
A_i & = \delta_{ab} \frac{\partial \xi^a}{\partial x^i} \frac{\partial \xi^b}{\partial t}. \quad (4.12)\]

Note that
\[ \tilde{A} = \frac{1}{\Theta^2} (A_0 - g^{ij} A_i A_j) = -\varphi \] (4.13)
Then this last combination of the fields \( A_0 \) and \( A_i \) has to be identified with Newton’s potential, while the \( A_i \)'s are inertial fields.

Let us remark that, while the Poisson equation (4.7) can be rewritten in the new coordinates \( z^b \to \xi^b(z',t) \) in the form
\[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial z'^i} \left[ \sqrt{gg}^{ij} \frac{\partial \varphi}{\partial z'^j} \right] = 4\pi Gm\delta[z' - x(t)] , \] (4.14)
this equation does not determine the gravitational potential since this latter is not a geometrical object and its functional form must depend on other fields. Therefore, the transformed Poisson equation (4.14) cannot be considered the equation for the field \( A_0 \) in an arbitrary reference frame and its integration cannot give rise to a term like \( \frac{1}{2} \epsilon_{ab} \frac{\partial \xi^a}{\partial t} \frac{\partial \xi^b}{\partial t} \), as it should be according to Eq.(4.12). Instead, it is the quantity \( \tilde{A} \) of Eqs.(4.13) that plays now the role of the Newton’s potential \( \varphi \). Then Eqs.(4.14) can be rewritten in the form
\[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial z'^i} \left[ \sqrt{gg}^{ij} \frac{\partial \tilde{A}}{\partial z'^j} \right] = -4\pi Gm\delta[z' - x(t)] . \] (4.15)

The existence of the cocycle terms in the transformation rules of the potentials (see Eqs.(3.43)) does not invalidate the non-relativistic equivalence principle since, as shown at the end of the previous section, it affects only the definition of the total energy in different coordinate systems. The dynamics of a mass-point in the above external fields can be made Galilean-generally-covariant only if the second Galilean Casimir invariant \( (E - P^2/2m) \) vanishes, as in the flat case.

Let us consider now various cases of passive coordinate transformations. For instance the kinematical group of passive coordinate transformations for the rigid observers (i.e., an arbitrary rigid but rotating frame). First, there are the Galilean coordinate transformations (according to Wheeler’s terminology), which preserve the rigid and non-rotating character of the coordinates:
\[
\begin{aligned}
t & \to t - \varepsilon \\
y^a & \to \xi^a(x, t) = R_i^a x^i + \varepsilon^a(t) \\
R_i^a & = \text{cost. .}
\end{aligned}
\] (4.16)
We have now $\Theta = 1$, $g_{ij} = \delta_{ij}$, $A_0 = -\varphi + \frac{1}{2} \delta_{ab} \dot{\varepsilon}^a \varepsilon^b$, $A_i = \delta_{ab} R_i^a \varepsilon^b$, and Eq. (4.6) is correspondingly modified.

These results can be obtained as a particular case ($\dot{R}_i^a = 0$) of the passive coordinate transformation corresponding to an arbitrary rigid and rotating motion

$$
\begin{align*}
\{ t & \rightarrow t - \varepsilon \\
q^a & \rightarrow \xi^a(x, t) = R_i^a(t) x^i + \varepsilon^a(t) 
\}.
\end{align*}
$$

(4.17)

In this case we have:

$$
\begin{align*}
\frac{\partial \xi^a}{\partial x^j} & = E_j^a(x, t) = R_i^a(t) \\
\frac{\partial \xi^a}{\partial t} & = \dot{R}_i^a(t) x^i + \varepsilon^a(t) \\
\frac{\partial^2 \xi^a}{\partial x^j \partial t} & = \ddot{R}_i^a(t) x^i + \dddot{\varepsilon}^a(t),
\end{align*}
$$

(4.18)

and Eqs. (4.12) become:

$$
\begin{align*}
\Theta & = 1 \\
g_{ij} & = \delta_{ij} ; \frac{\partial g_{ij}}{\partial t} = 0 ; \Gamma^k_{ij} = 0 \\
A_0 & = -\varphi + \frac{1}{2} \delta_{ab} [\dot{R}_j^a x^j + \varepsilon^a] [\ddot{R}_j^b x^j + \dddot{\varepsilon}^b] \\
A_i & = \delta_{ab} R_i^a [\dot{R}_j^b x^j + \varepsilon^b].
\end{align*}
$$

(4.19)

Then, the equations of motions take the form:

$$
\begin{align*}
\dot{x}^i & = \delta^{ik} \left\{ \frac{\partial A_k}{\partial t} - \frac{\partial A_0}{\partial x^k} + \left[ \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} \right] \dot{x}^j \right\} \\
& = \delta^{ik} \left\{ -\frac{\partial \varphi}{\partial x^k} - \delta_{ab} R_i^a \varepsilon^b - 2\delta_{ab} R_i^a \ddot{R}_j^b x^j \\
& \quad - \frac{1}{2} \delta_{ab} [R_i^a \dddot{R}_j^b x^j + \ddot{R}_i^a \dot{R}_j^b x^j] - \frac{1}{2} \delta_{ab} \left[ R_k^a \dddot{R}_j^b + \ddot{R}_k^a \dot{R}_j^b \right] x^j \right\}.
\end{align*}
$$

(4.20)

Since the angular velocity vector can be expressed as

$$
\omega^k(t) = \frac{1}{2} \epsilon^{krs} \delta_{ab} R_r^a(t) \dot{R}_s^b(t),
$$

(4.21)

the physical meaning of the various terms can be immediately identified as follows:

$$
\begin{align*}
m \left[ \frac{\partial A_k}{\partial x^k} - \frac{\partial A_{ij}}{\partial x^j} \right] \dot{x}^j & = 2m R_j^a(t) \dot{R}_k^a(t) \dot{x}^j = 2m [\ddot{x} \wedge \dot{x}]_k \quad \text{(Coriolis force)} \\
m \left[ \frac{\partial A_k}{\partial t} - \frac{\partial A_0}{\partial x^k} \right] & = -m \frac{\partial \varphi}{\partial x^k} \quad \text{(gravitational force)} \\
& -m \delta_{ab} R_k^a \varepsilon^b \quad \text{(translational inertial force)} \\
& + m [\dddot{x} \wedge (\dddot{x} \wedge \dot{x})]_k \quad \text{(centrifugal force)} \\
& + m [\dddot{x} \wedge \dot{x}]_k \quad \text{(Jacobi force)}.
\end{align*}
$$

(4.22)
Finally, the equations of motion, which are the transformed of Newton’s equations (4.6) in this generalized reference frame, are:

$$m \ddot{x}^k = -m \delta^{kt} \frac{\partial \mathcal{L}}{\partial \dot{x}^t} - m \delta^{kt} R^a_l \dot{\varepsilon}^a + m \dot{\omega} \wedge (\mathcal{L} \wedge \dot{x})^k + 2m \dot{\omega} \wedge \dot{x}^k + m \dot{\omega} \wedge x^k.$$ \hspace{1cm} (4.23)

On the other hand the Kinematical group of the subclass of rigid observers, called Galilean by Wheeler, is defined by Eqs.(4.16). Finally, the Kinematical group of Galilean observers (Wheeler’s absolute Galilean) is defined by $R^a_i = \text{const}$ and $\varepsilon^a = \text{const}$.

When $\dot{R}^a_i = 0$, the modification of Eq.(4.6) is due only to the translational-inertial force. Of course, corresponding to a transformation of the form (4.9), metric, affinity, and dreibeins are equivalent to the global flat ones and therefore do not represent true additional dynamical variables. It will be interesting to see whether it is possible to build up a Galilean theory in which the metric and the fields $A_i$ assume an intrinsic dynamical content. This question will be dealt with in the following Sections.

5 Galilean limit of the relativistic mass-point theory

In order to understand the generality of the results so far obtained, it is profitable to make recourse to the axiomatic formulation of the so-called Newtonian space-time structures and to reconsider our formulation as a suitable non-relativistic limit of a Poincaré invariant theory. Axiomatic formulations of the possible geometries of Newtonian space-times has been introduced by Havas, Trautman, Künzle, and Kuchař.

Following Künzle, we define:

- A Galilei structure over a four-dimensional manifold $\mathcal{V}$ is a pair $(h^{\mu\nu}, t_\nu)$, where $h^{\mu\nu}$ is a symmetric covariant tensor of rank 3 and $t_\mu$ is a 1-form having the property that is the generator of the kernel of $h^{\mu\nu}$, $\forall x \in \mathcal{V}$.

The triple $(\mathcal{V}; h^{\mu\nu}, t_\nu)$ is called a Galilei Manifold. A vector $u^\mu$ is called a unit time-like vector if $u^\mu t_\mu = 1$, and a contravariant tensor is called space-like if it vanish when contracted with $t_\mu$ on any index.
A linear symmetric connection is called a Galilei connection $\nabla$ if it is defined on a Galilei manifold $(V; h^{\mu\nu}, t_\nu)$ and satisfies
\[ \nabla_\rho h^{\mu\nu} = \nabla_\rho t_\mu = 0. \] (5.1)

It can be shown that such a connection exist if and only if $t_\mu$ is a closed 1-form and it is uniquely defined up to an arbitrary two-form $\chi$ according to:
\[ \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + t_\beta h^{\alpha\rho} \chi_{\rho\gamma} + t_\gamma h^{\alpha\rho} \chi_{\rho\beta}, \] (5.2)
where $\Gamma^\alpha_{\beta\gamma}$ are the connection coefficients and
\[ \Gamma^\alpha_{\beta\gamma} \equiv h^{\alpha\rho} \left[ \frac{1}{2} h_{\beta\mu} h_{\gamma\nu} h^{\mu\nu} + u^\sigma h_{\rho(\beta t_\gamma)} - h_{\rho(\beta t_\gamma)} h^{\alpha\rho} - t_\beta u^\alpha \right]; \] (5.3)
here $u^\alpha$ ($u^\alpha t_\alpha = 1$) is an arbitrary given time-like unit vector field, and $\gamma_{\alpha\beta}$ is the associated covariant space metric defined by
\[ \gamma_{\alpha\beta} u^\beta = 0 \quad \text{and} \quad \gamma_{\alpha\beta} \gamma^{\rho\beta} = \delta^\beta_\alpha - t_\alpha u^\beta. \] (5.4)

A symmetric Galilei connection $\nabla$ is called Newtonian, and the quadruple $(V; h^{\mu\nu}, t_\nu; \Gamma)$ is called correspondingly a Newtonian Manifold, if the two-form $\chi$ in Eq.(5.2) is closed.

Newtonian manifolds in which $t_\mu$ is also exact will be called Special Newtonian Manifolds: the hypothesis that $t_\mu$ be an exact one form implies the existence of a global absolute time. The standard Newtonian connection is indeed obtained by choosing $\chi_{\alpha\beta} = \varphi_{,\alpha} t_\beta - \varphi_{,\beta} t_\alpha$ where $\varphi$ is the Newton’s potential.

Finally, the covariant space metric can be introduced by means of the relations:
\[ h_{\mu\nu} h^{\nu\rho} = \delta^\rho_\mu - t_\mu u^\rho, \quad h_{\mu\nu} u^\nu = 0. \] (5.5)

It is clear that the contravariant metric allows to assign lengths to space-like vectors but no lengths whatsoever to time-like vectors. These being the premises, we have:

1) the field equations, i.e. Newton’s law of gravitation $\Delta \varphi = 4\pi G \rho$ ($\rho$ mass density), can be rewritten as $R_{\alpha\beta} = 4\pi G pt_\alpha t_\beta$;
2) the matter Lagrangian for a mass-point, in the given Newtonian space-time, is (see Ref. [6]):

\[
A_M = \int d\lambda L_M(\lambda) ,
\]

\[
L_M = \frac{m}{2} \left( \frac{1}{t_\mu x^\mu} \right) h_{\mu\nu} x^\mu x^\nu - m \varphi (t_\rho x^\rho) .
\]

(5.6)

Note that only within Special Newtonian Manifolds the first of Eqs. (5.6) can be rewritten in the form \( A_M = \int dT \tilde{L}_M(T) \).

At this point, if our fields \( g_{ij}, \Theta, A_i, A_0 \) are supposed to be resident within a Special Newtonian space-time, it is easy to make all the identifications which are relevant to our formulation. In particular, in a convenient coordinate chart \( x^\mu = [t; x^k] \) of the Special Newtonian space time, we have:

\[
\begin{align*}
\varphi &= -\frac{1}{\Theta^2} \left( A_0 - 1 \frac{g^{ij}}{2} A_i A_j \right) \\
t_\mu &= [\Theta(t); \vec{0}] = [\tilde{T}; \vec{0}] = t_\mu \\
h^{\mu\nu} &= \begin{vmatrix} 0 & 0 \\ 0 & g^{ij} \end{vmatrix} \\
[\nabla^0]_0^i &= g^{ij} \left[ \frac{\partial A_j}{\partial t} - 1 \frac{\partial A_0}{\Theta} A_j - \frac{\partial A_0}{\partial x^j} \right] \\
[\nabla^0]_0^i &= \frac{1}{\Theta} \left[ 1 - g^{ij} A_j \right] \\
[\nabla^i]_k &= \Gamma^i_{jk} \\
[\nabla^i]_k &= 0
\end{align*}
\]

(5.7)

so that the Lagrangian (5.6) becomes:

\[
L_M^q = \frac{m}{t_\mu x^\mu} \frac{1}{2} h_{\mu\nu} x^\mu x^\nu - m \varphi (t_\rho x^\rho) \\
= \frac{m}{\Theta t^\nu} \left[ 1 \frac{1}{2} g^{ij} x^i x^j + A_i x^i t^i + A_0 t^\nu t^\nu \right] ,
\]

(5.8)

which is precisely our Lagrangian (3.33) (see also Eqs. (4.13)).

We are now in a position to perform (see for example [14]) in a general way the non-relativistic limit of the mass-point Lagrangian that is invariant with respect to the Poincaré group gauged à la Utiyama. We have:

\[
L_M^{Rg} = -mc \sqrt{-g_{\mu\nu} x^\mu x^\nu} ,
\]

(5.9)

where we have assumed the \((-1, +1, +1, +1)\) convention for the signature of the metric \( g_{\mu\nu} \).
In order to perform the limit, we must write the explicit dependence of the metric tensor on \( c^2 \) (velocity of light). As shown by Dautcourt [7], Ehlers [8] and Künzle [17], the correct parameterization to start with is:

\[
g_{\mu\nu} = -c^2 t_{\mu} t_{\nu} + \overset{\circ}{g}_{\mu\nu},
\]

and, for the inverse metric,

\[
g^{\mu\nu} = h^{\mu\nu} - \frac{1}{c^2} \kappa^{\mu\nu}.
\]

From the relation

\[
g_{\mu\nu} g^{\nu\rho} = \delta^{\mu}_\rho,
\]

we see that, at first order in \( 1/c^2 \), the following identities are fulfilled:

\[
\begin{cases}
    h^{\mu\nu} t_{\nu} &= 0 \\
    h^{\mu\nu} \overset{\circ}{g}_{\nu\rho} &= \delta^{\mu}_\rho - t_{\rho} \kappa^{\mu\nu} t_{\nu}.
\end{cases}
\]

Therefore, in the limit \( c \to +\infty \), we must identify \( h^{\mu\nu} \) with the Newtonian space metric and \( t_{\mu} t_{\nu} \) with the Newtonian time metric.

All the objects of the standard (Special Newtonian Manifold \( t_{\mu} = t_{\mu} \)) Newtonian theory can then be reconstructed by the relations:

\[
\begin{cases}
    u^{\mu} t_{\mu} &= 1 \\
    u^{\nu} \overset{\circ}{g}_{\nu\rho} &= 2 \varphi t_{\rho} \\
    h^{\mu\nu} \overset{\circ}{g}_{\nu\rho} &= \delta^{\mu}_\rho - t_{\rho} \kappa^{\mu\nu} t_{\nu} \\
    h_{\mu\nu} &= \overset{\circ}{g}_{\mu\nu} - 2 \varphi t_{\mu} t_{\nu}
\end{cases}
\]

that give the expressions \( \varphi, u^{\mu}, h^{\mu\nu}, h_{\mu\nu} \) as functions of \( t_{\mu} \) and \( \overset{\circ}{g}_{\mu\nu} \). The fields so defined automatically fulfill all the conditions required for the underlying space-time to be a Newtonian manifold. Within the coordinate chart used in Eqs.(5.7), it results

\[
\overset{\circ}{g}_{\mu\nu} = h_{\mu\nu} + \begin{vmatrix} 2 \varphi \Theta^2 & 0 \\ 0 & 0 \end{vmatrix}.
\]

By inserting the metric tensor (5.10) into the relativistic Lagrangian (5.9) and taking into account Eqs.(5.14), the expansion in terms of \( c^2 \) becomes:

\[
L = c^2 \left[ -m t_{\nu} x^{\nu} \right] + \left[ \frac{1}{t_{\rho} x^{\rho}} \frac{m}{2} h_{\mu\nu} x^{\mu} x^{\nu} - m \varphi t_{\rho} x^{\rho} \right] + O(1/c^2).
\]
We see that the zero\textsuperscript{th} order part in $c^2$, reproduces the matter Lagrangian (5.6), while the coefficient of $c^2$ (which in the adapted frame (5.7) becomes $\left[-m \frac{dT}{dt} \frac{d\lambda}{dx} = -m\Theta \frac{d\lambda}{dx}\right]$) is, within the standard picture, the central charge $-m\Theta$ of the extended Galilei Group ($-m$ if one chooses $t = T$).

6 Searching for field equations

The expressions of the fields appearing in Eq.(4.12) are the generic-frame expressions of external fields resident in a spatially flat Newtonian space. On the other hand, having in mind the relations among the three-dimensional gauge fields that we have introduced and the four-dimensional metric, as contained in Eqs.(5.7,5.10,5.14), it seems natural to look for dynamical field equations in three dimensions by exploiting some limiting procedure over a four-dimensional theory. Now, let us observe that the contraction $c \to \infty$ from the Poincaré algebra to the extended Galilei algebra is a well defined procedure in the case of the single mass-point that we have studied in the previous section, due to the fact that the action is a Poincaré invariant: it amounts indeed to a uniquely defined contraction on the scalar representation. On the other hand, the field equations of motion do not transform like a scalar representation and it is well known that contracting a non-trivial representation is a delicate matter: a-priori, different contractions of the same equations could result. Therefore taking into account the fact that the Galilean matter Lagrangian (5.8) is nothing but the zero\textsuperscript{th} order term of the $1/c^2$ expansion (5.16) of the general relativistic mass-point Lagrangian, we will build up the wanted Galilean variational problem by means of the zero\textsuperscript{th}-order term of the $1/c^2$ expansion of the full four-dimensional Einstein-Hilbert action for the gravitational field plus a single mass-point [18]:

$$S = S_F + S_M = \frac{c^3}{16\pi G} \int d^4z \sqrt{-g} \ 4R - mc \int d\lambda \sqrt{-g_{\mu\nu} x^\mu x^\nu} \quad (6.1)$$

First of all, we shall restrict ourselves to globally hyperbolic space-time manifolds for which a global 3+1 splitting exists. The associated action principle will be formulated by re-expressing the Einstein-Hilbert Action (6.1) in the form given by Arnowitt-Deser-Misner
This is tantamount to assume the existence of an absolute-time foliation of space-time \((t = t(z'))\) and of a global coordinate system in which \(t_\mu = (\Theta(t), \vec{0})\) as in Eq.\((5.6)\). In such a system, \(g_{\mu\nu}\) must be of the form of Eq.\((5.10)\) with \(t_\mu = t_\mu\) and the integration measure \(d^4z\) can be rewritten as \(dt d^3z\).

Then, owing to Eqs.\((5.7, 5.10, 5.14)\), we can put

\[
g_{\mu\nu} = -c^2 t_\mu t_\nu + \tilde{g}_{\mu\nu}
\]

\[
= -c^2 \begin{vmatrix} \Theta^2 & 0 & 2A_0 & A_j \\ 0 & 0 & g_{ij} & \frac{1}{c^2} \end{vmatrix} + \frac{1}{c^2} \begin{vmatrix} 2\alpha_0 & \alpha_j \\ \alpha_i & \gamma_{ij} \end{vmatrix} \cdots \beta_{ij} + O\left(\frac{1}{c^6}\right)
\]

(6.2)

and

\[
g^{\mu\nu} = \begin{vmatrix} 0 & 0 & 1 & -g^{ik} A_k \\ -g^{ik} A_k & g^{ij} & g^{kl}(\gamma_{kl} + A_k A_i) & \frac{1}{c^2} \end{vmatrix} \cdots + O\left(\frac{1}{c^6}\right).
\]

(6.3)

While some terms of order \(c^{-4}\) do contribute to the zero\(^{th}\) order term we are interested in, no term of order \(c^{-6}\) can survive the contraction, so that they will be ignored from now on. Note that the parameterization \((5.2)\) of \(g_{\mu\nu}\) is the most general one which can be locally connected to the flat Minkowski metric \(\eta_{\mu\nu}\) by means of a general coordinate transformation. By relaxing this last restriction, i.e. allowing for \(\Theta = \Theta(t, z')\), one obtains a parameterization of \(g_{\mu\nu}\), which needs also a Weyl transformation to be locally connected with \(\eta_{\mu\nu}\). This corresponds in some way to allow for a classical analogue of the dilaton degree of freedom dynamically interacting with the other fields by paying the price of abandoning the Galilean interpretation of \(\Theta(t)\) given in the previous section. Yet, this additional liberty allows for interesting results and will be exploited explicitly in the second paper of the present series.

Following Arnowitt, Deser and Misner [19], we write the covariant and the contravariant four-dimensional metric in the form:

\[
^4g_{\mu\nu} = \begin{vmatrix} -N^2 + 3g^{ij} N_i N_j & N_j \\ N_i & 3g_{ij} \end{vmatrix}
\]

\[
^4g^{\mu\nu} = \begin{vmatrix} \frac{1}{N^2} \frac{3g^{ik} N_k}{N^2} & \frac{3g^{ij}}{N^2} \frac{N_j N_k}{N^2} \\ \frac{3g^{il} N_l}{N^2} & g^{ij} - \frac{3g^{il} N_i N_j}{N^2} \end{vmatrix}
\]

(6.4)
where $g_{ij}^3 g^{jk} = \delta_i^k$. Then, neglecting surface terms, the action (5.1) can be written:

\[
S = \frac{c^3}{16\pi G} \int dt d^3z \sqrt{3g} N \left[ 3R + 3g^{ik} 3g^{jl} (K_{ij} K_{kl} - K_{ik} K_{jl}) \right] + mc \int d\lambda \left( N^2 - 3g^{ij} N_i N_j \right) t' t' - N_i x'^i x'^j + mc \int d\lambda \sqrt{\left( N^2 - 3g^{ij} N_i N_j \right) t' t' - 3g^{ij} x'^i x'^j},
\]

where:

\[
\begin{align*}
3g &= \det 3g_{ij} \\
\Gamma_{ij}^k &= \frac{3g_{kl}}{2} (3g_{li,j} + 3g_{lj,i} - 3g_{ij,l}) \\
3R_{ij} &= \Gamma_{ik,j} - \Gamma_{ij,k} + \Gamma_{ik,l} \Gamma_{jl}^l - \Gamma_{ij,l} \Gamma_{lk}^l \\
3R &= \frac{3g^{ij}}{2} 3R_{ij} \\
K_{ij} &= \frac{1}{2N} \left( \nabla_i N_j + \nabla_j N_i - \frac{\partial g_{ij}}{\partial t} \right) \quad \text{(extrinsic curvature)}
\end{align*}
\]

having denoted by $\nabla$ the covariant three-space derivative with respect to the Christoffel connection of $3g_{ij}$. In terms of the notation we have previously introduced for the expansion of the covariant four-dimensional metric, we have

\[
\begin{align*}
3g_{ij} &\equiv g_{ij} + \frac{1}{c^2} \gamma_{ij} + \frac{1}{c^4} \beta_{ij} + O(\frac{1}{c^6}) \\
3R &= R + \frac{1}{c^2} R_1 (g_{ij}, \gamma_{ij}) + \frac{1}{c^4} R_2 (g_{ij}, \gamma_{ij}, \beta_{ij}) + O(\frac{1}{c^6}) \\
N_i &\equiv A_i + \frac{1}{c^2} \alpha_i + \frac{1}{c^4} \beta_i + O(\frac{1}{c^6}) \\
N^2 &\equiv c^2 \Theta^2 - 2A + \frac{2}{c^2} \left[ \alpha_0 - g^{ij} \alpha_i A_j - \frac{1}{2} \gamma_{rs} g^{ji} g^{sj} A_i A_j \right] + O(\frac{1}{c^4}) \\
NK_{ij} &\equiv 3B_{ij} = B_{ij} + \frac{1}{c^2} B_{ij}^{(1)} + O(\frac{1}{c^4})
\end{align*}
\]

where we have defined:

\[
\begin{align*}
A &= A_0 - \frac{1}{2} g^{ij} A_i A_j \\
B_{ij} &= \frac{1}{2} [\nabla_i A_j + \nabla_j A_i - \frac{\partial g_{ij}}{\partial t}] \\
B_{ij}^{(1)} &= \frac{1}{2} [\nabla_i \alpha_j + \nabla_j \alpha_i - \frac{\partial g_{ij}}{\partial t} - A_k g^{kl} (3\nabla_j \gamma_{il} + 3\nabla_l \gamma_{ij} - 3\nabla_i \gamma_{lj})].
\end{align*}
\]
Then, inserting eqs. (5.6-5.8) into eq. (5.5), it follows:

\[
S = S_F + S_M =
\]

\[
+ c^2 \left\{ \frac{1}{16\pi G} \int dt d^3 z \sqrt{g} \left[ \Theta R_1 + \frac{\Theta}{2} g^{ij} \gamma_{ij} R - \frac{A}{\Theta} R + \Theta g^{ij} g^{jl} (B_{ij} B_{kl} - B_{ik} B_{jl}) \right] \right\}
\]

\[
+ m \int d\lambda \Theta t' \left\{ \frac{1}{2} g_{ij} (x^i + g^{ik} A_k t') (x^j + g^{jk} A_j t') + A_0 t'^3 \right\}
\]

\[
+ O(1/c^2) ,
\]

so that the zero-th-order term, identified as the total Action \( \bar{S} \), is:

\[
\bar{S} = \frac{1}{16\pi G} \int dt d^3 z \sqrt{g} \left[ \Theta R_2 + \frac{1}{2} g^{ij} \gamma_{ij} \Theta R_1 + \frac{1}{2} g^{ij} \beta_{ij} \Theta R \right. \\
- \frac{1}{2} \Theta \left( g^{ik} g^{jl} - \frac{1}{2} g^{ij} g^{kl} \right) \gamma_{ij} \gamma_{kl} R - \frac{A}{\Theta} R_1 \\
- \frac{1}{\Theta} \left( \alpha_0 - g^{ij} A_i \alpha_j + \frac{1}{2} A g^{ij} \gamma_{ij} + \frac{1}{2} \gamma_{ij} g^{im} A_i A_m \right) \right] R - \frac{A^2}{2\Theta^3} R \\
\frac{2}{\Theta} \Theta g^{ik} g^{jl} (B_{ij} B_{kl} - B_{ik} B_{jl}) - \frac{2}{\Theta} g^{ik} g^{jr} \gamma_{rs} g^{sl} (B_{ij} B_{kl} - B_{ik} B_{jl}) \\
+ \frac{A}{\Theta} g^{ik} g^{jl} (B_{ij} B_{kl} - B_{ik} B_{jl}) \\
+ m \int d\lambda \frac{m}{\Theta t'} \left[ \frac{1}{2} g_{ij} (x^i + g^{ik} A_k t') (x^j + g^{jk} A_j t') + A_0 t'^3 \right].
\]

(6.9)

We see that 27 fields survive the contraction, namely \( \Theta, A, A_i, g_{ij}, \alpha_0, \alpha_i, \gamma_{ij}, \beta_{ij} \), where \( A \) has been used as independent variable instead of \( A_0 \); on the other hand \( \beta_0 \) and \( \beta_i \) disappear at this order.

Let us now look for the invariance of the Action. In section 3 we have shown that,
provided we assume the transformation properties (3.43, the matter part of the Action (6.10) is invariant under the *gauged* Galilei transformation). Now, in order to formulate the *local Galilei invariance* for the action (6.10), appropriate *gauge* transformations for the fields $\gamma_{ij}$, $\beta_{ij}$, $\alpha_i$ and $\alpha_0$ must be postulated. The correct choice is

$$\begin{align*}
\delta \gamma_{ij} &= -\frac{\partial \eta^k(x,t)}{\partial x^i} \gamma_{kj} - \frac{\partial \eta^i(x,t)}{\partial x^i} \gamma_{kj} \\
\delta \alpha_0 &= 2\dot{\epsilon} \alpha_0 - \alpha_i \frac{\partial \tilde{\eta}^i}{\partial t} \\
\delta \alpha_i &= \dot{\epsilon} \alpha_i - \alpha_j \frac{\partial \tilde{\eta}^j}{\partial x^i} - \gamma_{ij} \frac{\partial \tilde{\eta}^j}{\partial t},
\end{align*}
$$

(6.11)

and, indeed, by direct calculation it turns out the the total variation of the Action (6.10) under the transformation $\delta$ defined by (3.7), (3.43) and (6.11) is given by:

$$\delta \tilde{S} = \int dtd^3z \left\{ \dot{\epsilon} \tilde{\mathcal{L}} + \Theta \mathcal{E}_{\mathcal{L}}_A \left( \frac{\partial F}{\partial t} - A_r g^r s \frac{\partial F}{\partial s} \right) + \Theta \mathcal{E}_{\mathcal{L} A_i} \frac{\partial F}{\partial z^i} + \frac{1}{8\pi G} \frac{\partial}{\partial z^i} \left( \frac{\sqrt{g}}{\Theta^2} [B^{ij} - (\text{Tr} B) g^{ij}] \frac{\partial F}{\partial z^j} \right) \right\}.
$$

(6.12)

As a matter of fact, this result means that a *quasi-invariance* of the Galilean total Action (6.10) holds, *modulo* the Euler-Lagrange equations for the fields $A_i$ ($\mathcal{E}_{\mathcal{L} A_i}$) and $A$ ($\mathcal{E}_{\mathcal{L} A}$) that are given by:

$$\begin{align*}
\mathcal{E}_{\mathcal{L} A} &= \frac{\delta \tilde{S}}{\delta A} = \frac{\partial \tilde{S}}{\partial z^k} \frac{\delta \tilde{S}}{\delta \partial_k A_i} \\
\mathcal{E}_{\mathcal{L} A_i} &= \frac{\delta \tilde{S}}{\delta A_i} - \frac{\partial}{\partial z^k} \frac{\delta \tilde{S}}{\delta \partial_k A_i}.
\end{align*}
$$

(6.13)

Let us remark that this peculiarity is precisely what it should be expected in the case of a variational principle corresponding to a *singular* Lagrangian.

By analogy to the free mass-point case, the terms $c^4...+c^2...$ of eqs.(6.9) could be rewritten as $c^2(\mathcal{M} + c^2 \mathcal{N})$ where $\mathcal{M} + c^2 \mathcal{N}$ ought to be interpreted as the central charge of the *asymptotic* Galilei group. To avoid an infinite central-charge, the theory should, in some sense, provide the condition $\mathcal{N} = 0$ automatically. The discussion of the *asymptotic* Galilei group will be dealt with in a separate paper. This analysis will require taking into account the $1/c^2$ expansion of the neglected surface terms, as they are needed in the evaluation of the asymptotic Poincaré group in the case of asymptotically-flat space-times (see for example ref.[23]).
7 Galilean Covariant Formulations of Newtonian Gravity

7.1 The Newtonian Theory in Arbitrary Reference Frames

In Section 4 it was shown that the field $\Theta(t)$ has no real dynamical content since its effect amounts only to a redefinition of the evolution parameter $t$ in the expression $T(t) = \int_0^t d\tau \Theta(\tau)$. It is easy to show that this fact is still true for its role within the total Action \[ (6.10) \]. Indeed, if we redefine the fields $A_0, A_i, \alpha_0$ and $\alpha_i$ as follows (see eq.(4.4); from now on $\nabla$ will be replace by $\nabla$),

\[
\begin{align*}
\tilde{A}_0 & \equiv \frac{A_0}{\Theta^2} ; \quad \tilde{A} \equiv \frac{A}{\Theta^2} \\
\tilde{A}_i & \equiv \frac{A_i}{\Theta} \\
\tilde{\alpha}_0 & \equiv \frac{\alpha_0}{\Theta^2} ; \quad \tilde{\alpha}_i \equiv \frac{\alpha_i}{\Theta} \\
\tilde{B}_{ij} & \equiv \frac{B_{ij}}{\Theta} = \frac{1}{2} \left[ \nabla_i \tilde{A}_j + \nabla_j \tilde{A}_i - \frac{\partial g_{ij}}{\partial T} \right] \\
\tilde{B}_{(ij)} & \equiv \frac{B_{(ij)}}{\Theta} = \frac{1}{2} \left[ \nabla_i \tilde{\alpha}_j + \nabla_j \tilde{\alpha}_i - \frac{\partial \gamma_{ij}}{\partial T} - \tilde{A}_k g^{kl} (\nabla_l \gamma_{ji} + \nabla_j \gamma_{il} - \nabla_l \gamma_{ij}) \right],
\end{align*}
\]

the Action \[ (6.10) \] becomes:

\[
\tilde{S} = \int dT d^3z \tilde{L} = \frac{1}{16\pi G} \int dT d^3z \left[ \tilde{R}_2 - \tilde{A} \tilde{R}_1 \right. \\
- \sqrt{g} R \left( \frac{\tilde{A}}{2} + \tilde{\alpha}_0 - g^{ij} \tilde{A}_i \tilde{\alpha}_j + \frac{1}{2} \tilde{A} g^{ij} \gamma_{ij} + \frac{1}{2} g_{ij} g^{im} \tilde{A}_i \tilde{A}_m \right) \\
+ 2 g^{ik} g^{jl} (\tilde{B}_{ij} \tilde{B}_{kl} - \tilde{B}_{ik} \tilde{B}_{jl}) - 2 g^{ik} g^{jl} (\tilde{B}_{ij} \tilde{B}_{kl} - \tilde{B}_{ik} \tilde{B}_{jl}) \left. \\
+ A g^{ik} g^{jl} (\tilde{B}_{ij} \tilde{B}_{kl} - \tilde{B}_{ik} \tilde{B}_{jl}) \right] \\
+ m \int dT dz \left[ \frac{1}{2} g_{ij} \left( \frac{dx^i}{dT} + g^{ik} \tilde{A}_k \right) \left( \frac{dx^j}{dT} + g^{jk} \tilde{A}_k \right) + \tilde{A} \right] \delta^3[z - x(T)],
\]

where, for future convenience, we have introduced the notations

\[
\begin{align*}
\tilde{R}_1(g, \gamma) & = \sqrt{g} R_1(g, \gamma) + \frac{1}{2} g^{ij} \gamma_{ij} \sqrt{g} R \\
\tilde{R}_2(g, \gamma, \beta) & = \sqrt{g} R_2(g, \gamma, \beta) + \frac{1}{2} g^{ij} \gamma_{ij} R_1 \\
& + \frac{1}{2} \sqrt{g} R \left[ g^{ij} \beta_{ij} - (g^{ik} g^{jl} - \frac{1}{2} g^{ij} g^{kl}) \gamma_{ij} \gamma_{kl} \right].
\end{align*}
\]

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It is then seen that the Action (7.2) is independent of \( \Theta(t) \).

For future reference we give here the explicit expressions of the quantities \( \tilde{R}_1 \) and \( \tilde{R}_2 \). They are:

\[
\begin{align*}
\tilde{R}_1(g, \gamma) &= \sqrt{g} \left[ -\left( R^{ij} - 1 \frac{g^{ij} R}{2} \right) \gamma_{ij} + \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right) \nabla_k \nabla_l \gamma_{ij} \right] \\
\tilde{R}_2(g, \gamma, \beta) &= \sqrt{g} \left[ -\left( R^{ij} - 1 \frac{g^{ij} R}{2} \right) \beta_{ij} + \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right) \nabla_k \nabla_l \beta_{ij} \right] \\
&+ \frac{\sqrt{g}}{2} g^{ab} \gamma_{ab} \left[ -\left( R^{ij} - 1 \frac{g^{ij} R}{2} \right) \gamma_{ij} + \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right) \nabla_k \nabla_l \gamma_{ij} \right] \\
&+ \sqrt{g} g^{ab} \gamma_{ij} \beta_{ab} \left( R^{ij} - 1 \frac{g^{ij} R}{2} \right) \\
&+ \sqrt{g} g^{ij} g^{js} \gamma_{rs} \left[ \nabla_a \nabla_b \gamma_{ij} + \nabla_i \nabla_j \gamma_{ab} - \nabla_i \nabla_j \beta_{ab} - \nabla_i \nabla \beta_{ab} \right] \\
&+ \frac{1}{2} \frac{\sqrt{g}}{2} g^{ij} g^{js} \gamma_{rs} \left[ \nabla_r \gamma_{ij} \nabla_a \gamma_{ab} - \frac{1}{4} \nabla_r \gamma_{ij} \nabla_s \gamma_{ab} + \frac{3}{4} \nabla_r \gamma_{ia} \nabla_s \gamma_{jb} \right. \\
&\left. - \frac{1}{2} \nabla_r \gamma_{ia} \nabla_j \gamma_{sb} - \nabla_r \gamma_{is} \nabla_a \gamma_{jb} \right] .
\end{align*}
\] **(7.4)**

We shall deal now with the problem of investigating the true dynamical degrees of freedom of the theory by means of a constraint analysis within the Hamiltonian formalism.

The canonical momenta \( \dot{f} = \frac{\partial f}{\partial T} \) are defined by:

\[
\begin{align*}
\pi^{ij} &= \frac{\delta S}{\delta \dot{g}_{ij}} = \frac{-\sqrt{g}}{16\pi G} \left[ \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right) - \left( g^{im} \gamma_{mn} g^{nk} g^{il} + g^{ik} g^{jm} \gamma_{mn} g^{nl} \right) - g^{im} \gamma_{mn} g^{nj} g^{kl} - g^{ij} g^{km} \gamma_{mn} g^{nl} \right] \tilde{B}^{(ij)}_{kl} \\
\pi^i_{\gamma} &= \frac{\delta S}{\delta \dot{\gamma}_{ij}} = \frac{-\sqrt{g}}{16\pi G} \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right) \tilde{B}_{kl} \\
p_i &= \frac{\delta S}{\delta \dot{x}^i} = g_{ij} \left( \frac{dx^i}{dT} + g^{ik} \tilde{A}_k \right) \delta^3 \left[ x - x(T) \right],
\end{align*}
\] **(7.5)**

and

\[
\begin{align*}
\pi_A &= \frac{\delta S}{\delta \dot{A}} = 0 \\
\pi_{\alpha a} &= \frac{\delta S}{\delta \dot{\alpha}_a} = 0 \\
\pi^i_{\beta} &= \frac{\delta S}{\delta \dot{\beta}_{ij}} = 0 \\
\pi_i &= \frac{\delta S}{\delta \dot{\tilde{A}}_i} = 0 \\
\pi^i_{\alpha} &= \frac{\delta S}{\delta \dot{\alpha}_i} = 0
\end{align*}
\] **(7.6)**

Since the Lagrangian \( \tilde{\mathcal{L}} \) is independent of the corresponding velocities, the latter momenta define in fact 14 primary constraints.
The Dirac Hamiltonian density is given by:

\[
\hat{H}_d = \frac{1}{16\pi G} \left\{ \tilde{A} \hat{R}_1 - \hat{R}_2 + \sqrt{g} R \left( \frac{\tilde{A}^2}{2} + \tilde{\alpha}_0 - g^{ij} \tilde{A}_i \tilde{\alpha}_j + \frac{1}{2} \gamma_{ij} g^{iv} g^{js} \tilde{A}_v \tilde{A}_s \right) \right\} \\
+ \frac{32\pi G}{\sqrt{g}} \left( g_{ik} g_{jl} - \frac{1}{2} g_{ij} g_{kl} \right) \pi^i_\gamma \pi^j_\kappa - 16 \frac{\pi^i_\gamma}{\sqrt{g}} \left( g_{ik} g_{jl} - \frac{1}{2} g_{ij} g_{kl} \right) \pi^k_\gamma \pi^l_\gamma \\
+ \frac{32\pi G}{\sqrt{g}} \left( g_{ik} \gamma_{jl} - \frac{1}{2} g_{ij} \gamma_{kl} \right) \pi^i_\gamma \pi^j_\kappa \\
+ \left( \frac{1}{2m} g^{ij} p_i p_j - m \tilde{A} \right) \delta^3 [z - x(T)] \\
- [\tilde{\alpha}_k - \tilde{A}_i \gamma_{jk} \tilde{\phi}_k - \tilde{A}_i \gamma^{ij} \tilde{\phi}_k] + \lambda^A \pi_A + \lambda^{\alpha} \pi_{\alpha} + \lambda^i \pi^i + \lambda^i \gamma^{ij} \pi^i \pi^j 
\]

where we have introduced \textit{ad hoc} notations for the following important quantities

\[
\left\{ \begin{array}{l}
\tilde{\phi}_k = 2 g_{ij} \nabla_k \pi^j + p_k \delta^3 [z - x(T)] + 2 \nabla_i [\pi^r \gamma_{sk} - \pi^s \gamma_{rk}] \\
\tilde{\phi}^i = \nabla_l \pi^l_\gamma 
\end{array} \right. 
\]

(7.8)

We will apply now the Dirac-Bergmann procedure. By imposing time-conservation of the \textit{primary} constraints, we obtain the 14 \textit{secondary} (not all independent) ones:

\[
\left\{ \begin{array}{l}
\dot{\pi}_{\alpha} = - \frac{1}{16\pi G} \sqrt{g} R \approx 0 \\
\dot{\pi}^{ij} = - \frac{1}{16\pi G} \left[ R^{ij} - \frac{1}{2} g^{ij} R \right] \approx 0 \\
\dot{\pi}_A = - \frac{\tilde{R}_1}{16\pi G} + m \delta^3 [z - x(T)] - \sqrt{g} \tilde{A} \approx 0 \\
\dot{\pi}^i = g^{ik} \tilde{\phi}_k + \frac{\sqrt{g}}{16\pi G} R \left( g^{ik} \tilde{\alpha}_k - g^{ik} \gamma_{kl} g^{lm} \tilde{A}_m \right) \approx 0 \\
\dot{\pi}^i_\alpha = \tilde{\phi}^i + \frac{\sqrt{g}}{16\pi G} R g^{kl} \tilde{A}_l \approx 0 
\end{array} \right. 
\]

(7.9)

An equivalent, more expressive, set of 10 \textit{secondary} constraints is:

\[
\left\{ \begin{array}{l}
\chi_{\alpha} \equiv \sqrt{g} R \approx 0 \\
\chi^{ij}_R \equiv \sqrt{g} R^{ij} - \frac{1}{3} \sqrt{g} R \approx 0 \\
\chi_1 \equiv - \frac{1}{16\pi G} \tilde{R}_1 + m \delta^3 [z - x(T)] + \frac{16\pi G}{\sqrt{g}} \left( g_{ik} g_{jl} - \frac{1}{2} g_{ij} g_{kl} \right) \pi^i_\gamma \pi^j_\gamma \approx 0 \\
\tilde{\phi}_k \approx 0 \\
\tilde{\phi}^i \approx 0 
\end{array} \right. 
\]

(7.10)

By imposing time-conservation of the \textit{secondary} constraints, we obtain the \textit{tertiary} con-
constraints in the form:

\[
\begin{align*}
\dot{\chi}_{00} & \simeq 0 \\
\dot{\chi}^{ij}_R & \simeq -16\pi G \sqrt{g} \left[ \frac{1}{2} \nabla^i \nabla^j \left( \frac{\pi_\gamma}{\sqrt{g}} \right) + \Delta \left( \frac{\pi^{ij}_\gamma}{\sqrt{g}} \right) - \frac{1}{2} g^{ij} \Delta \left( \frac{\pi_\gamma}{\sqrt{g}} \right) \right] \simeq 0 \\
\dot{\chi}_1 & \simeq \sqrt{g} g^{kl} g^{rs} \nabla_k \nabla_l \nabla^r \nabla^s - \frac{(16\pi G)^2}{2\sqrt{g}} g_{rs} \pi^{ks}_\gamma g_{jt} - \frac{1}{2} g_{ij} g_{kl} \pi^{ij}_\gamma \pi^{kl}_\gamma \simeq 0 \\
\dot{\phi}_k & \simeq 0 \\
\dot{\Phi} & \simeq 0 ,
\end{align*}
\] (7.11)

where $\Delta$ is the three-dimensional Laplace operator ($\Delta = \nabla^2$). In this way we get seven (not independent) tertiary constraints. Note in this connection that, because of the constraint $R^{ij} \simeq 0$, the covariant derivatives commute on the constraint’s surface. Also, recall finally that all the fields’ momenta are tensor densities of weight $+1$.

At this point, it is expedient to introduce the well-known transverse-traceless-decomposition of symmetric tensors, which, due to the secondary constraint $\dot{\chi}^{ij}_R \simeq 0$, can now be referred to the globally flat metric (on the constraints surface) $g_{ij}$. Specifically:

\[
\begin{align*}
\pi^{ij}_T & = \pi^{ij}_T + \frac{1}{2} [g^{ij} \pi_T - \Delta^{-1} \nabla^i \nabla^j \pi_T] + \nabla^i \pi^j_L + \nabla^j \pi^i_L \\
\gamma^{ij} & = \gamma^{ij}_T + \frac{1}{2} [g^{ij} \pi_T - \Delta^{-1} \nabla^i \nabla^j \pi_T] + \nabla^i \gamma^j_L + \nabla^j \gamma^i_L \\
\pi^{ij}_\gamma & = \pi^{ij}_T + \frac{1}{2} [g^{ij} \pi_\gamma T - \Delta^{-1} \nabla^i \nabla^j \pi_\gamma T] + \nabla^i \pi^j_\gamma L + \nabla^j \pi^i_\gamma L \\
\beta^{ij} & = \beta^{ij}_T + \frac{1}{2} [g^{ij} \pi_T - \Delta^{-1} \nabla^i \nabla^j \pi_T] + \nabla^i \beta^j_L + \nabla^j \beta^i_L \\
\pi^{ij}_\beta & = \pi^{ij}_T + \frac{1}{2} [g^{ij} \pi_\beta T - \Delta^{-1} \nabla^i \nabla^j \pi_\beta T] + \nabla^i \pi^j_\beta L + \nabla^j \pi^i_\beta L \\
\lambda^{ij}_\beta & = \lambda^{ij}_T + \frac{1}{2} [g^{ij} \pi_\beta T - \Delta^{-1} \nabla^i \nabla^j \pi_\beta T] + \nabla^i \lambda^j_\beta L + \nabla^j \lambda^i_\beta L ,
\end{align*}
\] (7.12)

where $g^{ij} \gamma^{ij}_T = \nabla^i \gamma^{TT}_a = 0$, $g^{ij} \beta^{ij}_T = \nabla^i \beta^{TT}_a = 0$ and $g_{ij} \pi^{ij}_T = \nabla_i \pi^{ai}_T = 0$, $g_{ij} \pi^{ij}_\gamma T = \nabla_i \pi^{ai}_\gamma T = 0$, $g_{ij} \pi^{ij}_\beta T = \nabla_i \pi^{ai}_\beta T = 0$. In terms of these quantities, it can be seen that the chain that starts from the primary constraint $\pi^{ij}_\beta \simeq 0$ gets contributions only from the TT part, $\pi^{ij}_T$. Consistently, the longitudinal and trace parts, $\pi^{ij}_\beta L$ and $\pi^{ij}_{\beta T}$, do not generate any chain.

At this stage of the procedure, the constraints $\dot{\Phi}^k \simeq 0$, $\dot{\chi}^{ij}_R \simeq 0$ can be rewritten as:

\[
\begin{align*}
\dot{\Phi}^k & = \nabla^k \nabla_i \pi^{iL}_\gamma + \Delta \pi^{kL}_\gamma \\
\dot{\chi}^{ij}_R & = 16\pi G \left[ \Delta \pi^{ij}_T + \Delta \cdot (\nabla^i \pi^{jL}_\gamma + \nabla^j \pi^{iL}_\gamma) + g^{ij} \Delta \cdot \nabla_k \pi^{kL}_\gamma + \nabla^i \nabla^j \nabla_k \pi^{kL}_\gamma \right] .
\end{align*}
\] (7.13)
It is then apparent from these expressions that, provided that the asymptotic boundary conditions are such as to allow the inference $\Delta f = 0 \implies f = 0$, a condition that is also necessary to the inversion of the transverse-traceless decomposition, the constraints $\tilde{\chi}_{ij}$ and $\tilde{\Phi}^k$ are equivalent to:

$$\begin{cases}
\pi_{ij}^T \approx 0 , \\
\pi_i^L \approx 0 .
\end{cases} \quad (7.14)$$

Therefore we have in fact only three independent tertiary constraints, namely $\tilde{\chi}_1 \approx 0$ and the two independent components of the first line of eqs.$(7.14)$.

Using these conditions for reexpressing the constraints $\chi_1 \approx 0$ and $\tilde{\chi}_1 \approx 0$ in terms of the transverse traceless variables, we obtain:

$$\begin{align}
\chi_1 &= \frac{\sqrt{g}}{16\pi G} \Delta \gamma^T + m\delta^3[Z - x(t)] \\
&+ \frac{16\pi G}{\sqrt{g}} \frac{1}{4} \left[ (\pi_{ij})^T \right]^2 - (\Delta^{(-1)} \nabla^i \nabla^j \pi_{ij}) \cdot (\Delta^{(-1)} \nabla^i \nabla^j \pi_{ij}) \right] \approx 0 , \\
\tilde{\chi}_1 &= -\pi_{ij} \cdot (\Delta \cdot \nabla \gamma^T) + \left[ \frac{1}{2} \nabla^i \nabla^j \gamma^T - (\nabla^i \nabla^j \nabla \gamma^T) \right] \cdot (\Delta^{(-1)} \nabla^i \nabla^j \pi_{ij}) \\
&- \frac{16\pi G}{\sqrt{g}} \frac{\pi_{ij}}{8} \left[ (\pi_{ij})^T \right]^2 - (\Delta^{(-1)} \nabla^i \nabla^j \pi_{ij}) \cdot (\Delta^{(-1)} \nabla^i \nabla^j \pi_{ij}) \right] \approx 0 . \quad (7.15)
\end{align}$$

At this stage we have the following situation: 

(i) $\pi_{\beta T} \approx 0$, $\pi_{\beta L} \approx 0$ do not generate secondary constraints; 
(ii) $\pi_{\alpha 0} \approx 0$ generate the secondary $\chi_{\alpha 0} \approx 0$ and no tertiary; 
(iii) $\pi_i^0 \approx 0$ generate the secondaries $\tilde{\Phi}_i \approx 0$, i.e. $\pi_i^L \approx 0$ and no tertiary; 
(iv) $\pi_i^0 \approx 0$ generate the secondaries $\phi_i \approx 0$ and no tertiary; 
(v) $\pi_A \approx 0$ generates the secondary $\chi_1 \approx 0$ and then the tertiary $\tilde{\chi}_1 \approx 0$; 
(vi) $\pi_{ij}^T \approx 0$ (only two independent constraints) generate $\chi_{ij}^T \approx 0$ (only two independent constraints due to the Bianchi identities) and then the two tertiaries $\chi_{ij}^R \approx 0$.

We have only to find the quaternary constraints generated by the time derivatives of the tertiary constraints $\tilde{\chi}_1 \approx 0$ and $\chi_{ij}^T \approx 0$. While $\tilde{\chi}_1 \approx 0$ is given in appendix A, we have:

$$\begin{align}
\chi_{ij}^T &= \frac{\sqrt{g}}{2 \cdot 16\pi G} \Delta \Delta \gamma_{ij}^T + \frac{16\pi G}{2 \sqrt{g}} \left[ \nabla^r \pi_{ij} \Delta \nabla^s \pi_{ij} \right. \\
&- \nabla^r \nabla^s \nabla^i \nabla^j \Delta^{(-1)} \pi_{ij} \nabla^r \nabla^s \nabla^i \nabla^j \Delta^{(-1)} \pi_{ij} \right. \\
&+ \frac{1}{2} \nabla^i \nabla^r \Delta^{(-1)} \pi_{ij} \nabla^s \pi_{ij} + \frac{1}{2} \nabla^i \nabla^r \Delta^{(-1)} \pi_{ij} \nabla^s \pi_{ij} + \frac{1}{2} \nabla^r \nabla^s \Delta^{(-1)} \pi_{ij} \nabla^i \nabla^j \pi_{ij} \\
&- \frac{1}{2} \nabla^r \nabla^s \Delta^{(-1)} \pi_{ij} \Delta \pi_{ij} + \frac{1}{2} \nabla^r \nabla^s \pi_{ij} \right] \approx 0 . \quad (7.16)
\end{align}$$

Before ending the discussion of these chains of constraints, let us remark that the relevant
sector of solutions of eqs. (7.15) is
\[
\begin{align*}
\pi_{\gamma_T} & \simeq 0 \\
\sqrt{g} \Delta \gamma_T & \simeq -16\pi G m \delta^3 [z - \mathbf{x}(t)].
\end{align*}
\] (7.17)

Using eqs. (7.17) inside eqs. (7.16), we get
\[
\ddot{\chi}_{ij} R \simeq \sqrt{g} 2 \cdot 16 \pi G g^{ir} g^{js} \Delta \Delta \gamma_T \simeq 0,
\]
which implies \( \gamma_{rs}^{TT} \simeq 0 \). By using eqs. (7.17) and \( \gamma_{rs}^{TT} \simeq 0 \) in \( \ddot{\chi}_1 \) (see Appendix A), we obtain
\[
\ddot{\chi}_1 \simeq \frac{\sqrt{g}}{16\pi G} \left[ \left( \Delta \gamma_T - 2 \Delta \cdot \nabla^k \gamma^L_k \right) g^{ij} + 2 \nabla^i \nabla^j \nabla^k \gamma^L_k \right] \cdot \nabla_i \nabla_j \left( \tilde{A} - \frac{1}{4} \gamma_T \right),
\] (7.18)

which implies \( \tilde{A} - \frac{1}{4} \gamma_T \simeq 0 \) as a relevant solution. Therefore we get in the end
\[
\Delta \tilde{A} \simeq -4\pi G m \delta^3 [z - \mathbf{x}(t)],
\] (7.19)
i.e. the Poisson equation in a *three-dimensional general covariant* form. This means that \( \ddot{\chi}_1 \simeq 0 \) is the equation which replaces the Poisson equation in an *arbitrary-absolute time respecting* frame; the important result just obtained is that, provided that the Newton potential \( A_0 = -\varphi \), seen by the *Galilean* observers, is replaced by the effective potential \( \tilde{A} = \frac{A_0}{\Theta^2} - \frac{1}{2} g^{ij} A_i A_j \), then we get the Poisson equation for \( \tilde{A} \) as the most relevant solution (see Eq. 4.15) in every *allowed* reference frame.

Some words should be spent about the “invariance” of Eq. (7.19). Since we have shown in Eq. (6.12) that the Action is *quasi-invariant* modulo the equation of motion, one could expect that Eq. (7.19) be invariant under all the *local* Galilei transformations, just as all other equations are. Yet, this is not true because Eq. (6.12) is not invariant under *local* Galilei boosts because it gets contributions from the *cocycle* term. This does not invalidated the invariance of the theory, however. Indeed, since the Action is *quasi-invariant modulo* equations of motion, there is anyway a *conserved charge* associated to the boosts [22]. Therefore, the full invariance of Poisson equation should be accounted for by the transformations generated by these *conserved charge*.

Finally, time conservation of the *quaternary* constraints gives the *quinquenary* constraints. One of these latter, precisely that following from the \( \chi_1 \) chain, fixes the *multiplier*
\( \lambda_A \). On the other hand, the chain originated by \( \chi_R^{rs} \) continues along three more time derivations. To avoid cumbersome expression, we give the simplified forms of the leading terms for the previous relevant sector; using all the constraints already worked out, it follows:

\[
\begin{align*}
(3) \chi_1 &= \frac{d^3}{dT^3} \chi_1 \simeq \frac{\sqrt{g}}{16\pi G} \left[ (\Delta \gamma_T - 2\Delta \cdot \nabla \gamma^k_{\gamma^k_L}) \right] g^{ij} + 2\nabla^i \nabla^j \nabla \gamma^k_{\gamma^k_L} \cdot \nabla \nabla \lambda_A + \ldots \simeq 0 \\
(3) \chi_R^{rs} &= \frac{d^3}{dT^3} \chi_R^{rs} \simeq \Delta \Delta \pi_{TT}^{rs} \simeq 0 \\
(4) \chi_R^{rs} &= \frac{d^4}{dT^4} \chi_R^{rs} \simeq \frac{\sqrt{g}g^{ri}g^{sj}}{16\pi G} \Delta \Delta \beta_{TT}^{ij} + \ldots \simeq 0 \\
(5) \chi_R^{rs} &= \frac{d^5}{dT^5} \chi_R^{rs} \simeq \frac{\sqrt{g}g^{ri}g^{sj}}{16\pi G} \Delta \Delta \lambda_{TT}^{ij} + \ldots \simeq 0 ,
\end{align*}
\]

(7.20)

The last one ends the chain and fixes the transverse-traceless part \( \lambda_{ij}^{\beta TT} \) of the multipliers \( \lambda_{ij}^{\beta} \).

Therefore, since \( \lambda_A \) is determined from eq.(7.20, the chain of \( \pi_A \simeq 0 \) contains two pairs of second class constraints \( (\pi_A, \chi_1), (\chi_1, \chi_1) \). On the other hand, each of the two independents chains of \( \pi_{ij}^{TT} \simeq 0 \) contains three pairs of second class constraints \( (\pi_{ij}^{TT}, \chi_R^{rs}), (\chi_R^{rs}, \chi_R^{rs}), (\chi_R^{rs}, \chi_R^{rs}) \), since the sixth time derivative of these primary constraints determine the two independent Dirac multipliers \( \lambda_{ij}^{\beta TT} \).

In conclusion there are 18 first-class constraints and 16 second-class constraints. While the variables \( \tilde{A}, \gamma_T, \beta_{ij}^{TT}, \gamma_{ij}^{TT} \) and two components of \( g_{ij} \) are determined by half of the second-class constraints (the other half determines their canonical momenta), the variables \( \tilde{A}_I, 3 \) of the \( g_{ij}, \alpha_i, \gamma_i^L, \alpha_0, \pi_T \) (conjugated to one of the \( g_{ij} \)), \( \beta_i^L, \beta^T \) are gauge variables (their conjugated variables are determined by the first class constraints); correspondingly, the eleven Dirac multipliers \( \lambda_i, \lambda^{\alpha}, \lambda_i^\alpha, \lambda^{\beta TT}, \lambda_i^{\beta L} \) remain arbitrary.

Thus, apart from the particle degrees of freedom, no physical field degrees of freedom survive, as indeed it should be, and the role of the Newton potential is taken by \( \tilde{A} \), which satisfies a Poisson equation in the most relevant sector of solutions. It would be interesting to see whether unconventional sectors are allowed corresponding to more general solutions.
for the gravitational potential

The logical connections of the various constraints involved is described in Fig.1, which summarizes what is being fixed by each chains.

We can conclude this section by noting that, in force of eqs. (7.10) and (7.11), the condition for the finiteness of the central-charge term is indeed satisfied, and that $N = 0$ holds. It remains an open task that of performing the $1/c^2$ expansion of the neglected surface term. It is likely that clarifying this issue will be relevant also to the understanding of the role of the cocycle contribution to the local Galilei transformations of the Poisson equation.

### 7.2 The Newtonian Theory in Galilean Reference Frames

Starting from the general scheme of the 27-fields theory it is now interesting to see that, by confining to a post-Newtonian like parameterization for the four-dimensional covariant metric tensor, defined by $\Theta = 1$, $g_{ij} = \delta_{ij}$, $A_i = 0$ and $A = -\varphi$ (i.e. the fields as seen by the Galilean observers: see Eq. (4.8)), one obtains the maximum of similarity to Newton’s theory, i.e. a non-generally covariant formulation which is valid only in Galilean reference frame connected by global Galilei transformations. It should be clear, however, that in this way we are dealing in fact with a different variational problem with respect to the previous one. Putting

$$4g_{\mu\nu} = \left| -c^2 - 2\varphi + \frac{2a_0}{c^2} \right| \left| \delta_{ij} + \frac{\alpha_i}{c} + \frac{\alpha_j}{c} \right|,$$

the explicit expressions of the quantities $R_1$ and $R_2$ defined in Eq. (5.7) become:

$$\begin{align*}
R_1 &= \delta_{ij}\delta^{rs}[\gamma_{ir,sj} - \gamma_{ij,rs}] \\
R_2 &= \gamma_{lm}\delta^{tr}\delta^{ms}\delta^{ij}[\gamma_{ij,rs} + \gamma_{rs,ij} - 2\gamma_{ir,sj}] \\
&\quad + \delta^{lm}\delta^{rs}\delta^{ij}[\gamma_{rs,lm} - \frac{1}{4}\gamma_{ij,rs,m} - \frac{3}{4}\gamma_{ir,lm} - \frac{1}{2}\gamma_{ir,ts,m} - \frac{1}{2}\gamma_{ir,lm} - \frac{1}{2}\gamma_{ir,lm}]
\end{align*}$$

This could possibly be of some interest in connection with the debate about the so-called fifth force.

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2 This could possibly be of some interest in connection with the debate about the so-called fifth force.
Figure 1: What is being fixed by the constraints' chains for the 27-fields theory.
and the total action $\tilde{S}$ (6.10) results:

$$
\tilde{S} \equiv \frac{1}{16\pi G} \int dt d^3 z L_f + m \int dt d^3 z L_m \delta^3[z - x(t)]
$$

$$
= \frac{1}{16\pi G} \int dt d^3 z \left[ (\varphi + \frac{1}{2} \delta^{ij} \gamma_{ij}) R_1 + R_2 \right] + m \int dt d^3 z \left[ \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j - \varphi \right] \delta^3[z - x(t)].
$$

(7.23)

It is seen that the matter Lagrangian $L_m$ has precisely the form which is to be expected for a Galilean observer if Eq.(4.8) are inserted in Eq.(4.1). Therefore $(t, z)$ define a system of coordinates for a Galilean reference frame.

Note that:

(1) $\alpha_0$ and $\alpha_i$ do not appear in the lagrangian $L_f$;

(2) $L_f$ depends on $\beta_{ij}$ in a pure additive way through the term $\delta^{ij} \delta^{rs} [\beta_{ir,sj} - \beta_{ij,rs}]$ (see Eqs.(7.22)), which is again a surface term; moreover $\beta_{ij}$ is not coupled to the other fields.

We can put accordingly $\alpha_0 = \alpha_i = \beta_{ij} = 0$ without altering the dynamics of this theory which indeed depends now on $\varphi$ and $\gamma_{ij}$ only.

Let us note, moreover, that the $c^4$-order term in the expansion (6.9) is automatically zero in this case, while the $c^2$-order term becomes:

$$
\frac{1}{16\pi G} \int dt d^3 z \left[ R_1 - 16\pi G m \delta^3[z - x(t)] \right].
$$

(7.24)

The resulting Lagrangian is:

$$
\mathcal{L} = \frac{1}{16\pi G} \left[ (\varphi + \frac{1}{2} \delta^{ij} \gamma_{ij}) R_1 + R_2 \right] + m \left[ \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j - \varphi \right] \delta^3[z - x(t)] ,
$$

(7.25)

where $R'_2 = R_2|_{\beta_{ij}=0}$, does not depend on the velocities $\dot{\varphi}$ and $\dot{\gamma}_{ij}$, so that all the field-momenta play the role of primary constraints.

The Hamiltonian formulation is defined by:

$$
\begin{cases}
\pi_{\varphi} = 0 \\
\pi_{\gamma} = 0 \\
p_k = m \delta_{ki} \dot{x}^l \\
H_c = \frac{1}{16\pi G} \int d^3 z \left[ -(\varphi + \frac{1}{2} \delta^{ij} \gamma_{ij}) R_1 - R'_2 \right] + \int d^3 z \left[ \frac{1}{2m} \delta^{ij} p_i p_j + m \varphi \right] \delta^3[z - x(t)],
\end{cases}
$$

(7.26)
so that we have the 7 primary constraints:

\[
\begin{align*}
\pi_{\phi} & \simeq 0 \\
\pi_{rs}^\gamma & \simeq 0 .
\end{align*}
\] (7.27)

The Dirac Hamiltonian is:

\[
H_d = H_c + \int d^3z \left[ \lambda^\varphi(z,t)\pi_{\phi} + \lambda_\gamma^\gamma(z,t)\pi_{rs}^\gamma \right],
\] (7.28)

where the \(\lambda\)'s are the Dirac multipliers. Time-conservation of these constraints generate the 7 secondary constraints:

\[
\begin{align*}
\chi_{\phi}(z,t) & \equiv \dot{\pi}_{\phi}(z,t) = \{\pi_{\phi}(z,t), H_d\} = \frac{1}{16\pi G} R_1 - m\delta^3 z - x(t) \simeq 0 \\
\chi_{ij}^\gamma(z,t) & \equiv \dot{\pi}_{ij}^\gamma(z,t) = \{\pi_{ij}^\gamma(z,t), H_d\} \\ &= \frac{1}{16\pi G} \left\{ \frac{1}{2} \left[ \delta^{ir}\delta^{js} - \frac{1}{2} \delta^{ij}\delta_{rs} \right] \delta^{ab} \left[ \gamma_{ab,rs} + \gamma_{rs,ab} - \gamma_{ar,sb} - \gamma_{as,rb} \right] \\
& \quad + \left[ \delta^{ir}\delta^{js} - \delta^{ij}\delta^{rs} \right] \partial_r \partial_s \phi \right\} \simeq 0 ,
\end{align*}
\] (7.29)

while their time conservation gives the following condition on \(\lambda_{ij}^\gamma\) and \(\lambda^\varphi\):

\[
\begin{align*}
\psi_{\phi}(z,t) & \equiv \dot{\chi}_{\phi}(z,t) = \{\chi_{\phi}(z,t), H_d\} \\
& = \frac{1}{16\pi G} \delta_{ij}\delta_{rs} \left[ \lambda_{ir,sj}^\gamma - \lambda_{ij,rs}^\gamma \right] + p_k \delta^{kl} \partial_l \delta^3 z - x(t) \simeq 0 \\
\psi_{ij}^\gamma(z,t) & \equiv \dot{\chi}_{ij}^\gamma(z,t) = \{\chi_{ij}^\gamma(z,t), H_d\} \\ &= \frac{1}{16\pi G} \left\{ \frac{1}{2} \left[ \delta^{ir}\delta^{js} - \frac{1}{2} \delta^{ij}\delta_{rs} \right] \delta^{ab} \left[ \lambda_{ab,rs}^\gamma + \lambda_{rs,ab}^\gamma - \lambda_{ar,sb}^\gamma - \lambda_{as,rb}^\gamma \right] \\
& \quad + \left[ \delta^{ir}\delta^{js} - \delta^{ij}\delta^{rs} \right] \partial_r \partial_s \phi \right\} \simeq 0 .
\end{align*}
\] (7.30)

The secondary constraints are just the Euler-Lagrange field equations. Let us remark that the constraint \(\chi_{\phi} \simeq 0\) is just the expected condition for the vanishing of the \(c^2\) term (7.24) (in the Lagrangian description this term vanishes because of the Euler-Lagrange equation for \(\varphi\)). On the other hand, the mass-point equations are

\[
\dot{p}_k = \{p_k, H_d[T]\} = -m\partial_k \varphi ,
\] (7.31)

i.e. the standard Newton’s equations with potential \(\varphi\). Finally, evaluating \(R_1\) from the contraction \(\delta_{ij}\chi_{ij}^\gamma \simeq 0\) and by substituting it in the constraint \(\chi_{\phi} \simeq 0\), we obtain the classical Poisson equation for the potential \(\varphi(z,t)\), i.e.:

\[
\delta^{ij}\partial_i\partial_j \varphi(z,t) = 4\pi Gm\delta^3 z - x(t) .
\] (7.32)
Let us remark that, if we put an inhomogeneous solution of eq. (7.32) into eq. (7.31), one should get the motion of the particle under the usual field reaction; this gives rise to problems of self-energy similar to those of, e.g., the special relativistic electromagnetic case.

In spite of what could appear from Eqs. (7.30), not all the 7 secondary constraints are independent: as a matter of fact only four of them are independent and, correspondingly, only four multipliers are determined by eqs. (7.30). In order to see this explicitly, let us complete the constraint analysis of the theory. First of all, we note that three combination of the primary constraints, given by

\[ \Pi^k = \partial_t \pi^{kl}_\gamma , \]  

are first class. This can be easily checked thanks to the fact that the following six relations

\[ \partial_t \chi^{kl}_\gamma \equiv 0 \]
\[ \partial_t \psi^{kl}_\gamma \equiv 0 . \]  

hold identically. Consequently, one has to expect that three of the \( \gamma_{ij} \) and three of the \( \lambda_{ij}^\gamma \) are free quantities. In order to evidentiate explicitly the multipliers and the fields that are determined by the constraints, it is profitable again to parameterize \( \gamma_{ij} \) and \( \lambda_{ij}^\gamma \) in terms of the transverse-traceless decomposition of symmetric tensors, as:

\[ \gamma_{ij} = \zeta_{ij}^{TT} + \frac{1}{2} \left[ \delta_{ij} \zeta^T - \Delta^{-1} \zeta_{ij}^T \right] + \zeta_{i,j} + \zeta_{j,i} \]
\[ \lambda_{ij}^\gamma = \lambda_{ij}^{\gamma TT} + \frac{1}{2} \left[ \delta_{ij} \lambda_{i,j}^{\gamma T} - \Delta^{-1} \lambda_{ij}^{\gamma T} \right] + \lambda_{i,j}^\gamma + \lambda_{j,i}^\gamma , \]  

where \( \delta_{ij} \zeta_{ij}^{TT} = \delta_{ij} \zeta_{ai,j}^{TT} = 0 \) and \( \delta_{ij} \lambda_{ij}^{\gamma TT} = \delta_{ij} \lambda_{ai,j}^{\gamma TT} = 0 \). In terms of these quantities, the secondary and tertiary constraints (Eqs. (7.29) and (7.30)) become:

\[ \left\{ \begin{array}{lcl} \chi_{i,j} & \approx & 0 \Rightarrow \Delta \zeta^T + 16\pi Gm\delta^3 [z - x(t)] \approx 0 \\ \lambda_{i,j}^{\gamma} & \approx & 0 \Rightarrow \left[ \delta_{ir} \delta_{js} - \delta_{ij} \delta_{rs} \right] \partial_r \partial_s [\phi - \frac{1}{2} \zeta^T] + \delta_{ir} \delta_{js} \delta_{lm} \zeta_{i,j,lm}^{TT} \approx 0 , \end{array} \right. \]  

\[ \left\{ \begin{array}{lcl} \psi_{i,j} & \approx & 0 \Rightarrow \Delta \lambda^{\gamma TT} - 16\pi Gp_k \delta^{kl} \partial_l \delta^3 [z - x(t)] \approx 0 \\ \psi_{i,j}^{\gamma} & \approx & 0 \Rightarrow \left[ \delta_{ir} \delta_{js} - \delta_{ij} \delta_{rs} \right] \partial_r \partial_s [\lambda_{\phi} - \frac{1}{3} \lambda^{\gamma TT}] + \delta_{ir} \delta_{js} \delta_{lm} \lambda_{i,j,lm}^{\gamma TT} \approx 0 , \end{array} \right. \]  

respectively. The transverse-traceless decomposition shows that the equations \( \chi_{i,j}^\gamma \approx 0 \) cannot be solved for the fields \( \zeta_i \) and the multipliers \( \lambda_i^\gamma \). In particular, as to the multipliers,
we can solve only for:

\[
\begin{align*}
\lambda^\varphi &= \lambda^\varphi[z; p_k, x^k] \\
\lambda^T &= \lambda^T[z; p_k, x^k] \\
\lambda^T_{ij} &= \lambda^T_{ij}[z; p_k, x^k],
\end{align*}
\]

(7.38)

where asymptotic boundary conditions for the \( \lambda \)'s allowing for the inference \( \Delta f = 0 \implies f = 0 \) have been assumed. Substituting these expressions for the multipliers, the Dirac Hamiltonian becomes:

\[
H_d = H_c + \int d^3z \left[ \lambda^\varphi[z; p_k, x^k] \pi_\varphi + \lambda^T_{ij}[z; p_k, x^k] \pi_{ij} + \frac{1}{2} \left( \delta_{ij} \lambda^T[z; p_k, x^k] - \frac{1}{\sqrt{2}} \lambda^T_{ij} \pi_{ij} \right) - 2 \lambda^T_{ij} \partial_\gamma \pi_{ij} \right],
\]

(7.39)

an expression which shows that the undetermined multipliers \( \lambda^T_i \) are associated to the first class constraints \( \Pi^k \), as it must be.

As a consequence, the variational problem must be independent of the quantities \( \zeta_i \) of Eqs. (7.35). In fact, in terms of the transverse-traceless quantities, we have:

\[
\begin{align*}
R_1 &= -\Delta \zeta^T \\
R'_2 &= \frac{3}{8} \delta^{kl} \zeta^T \zeta^T_k = \frac{\delta^{kl} \delta^{ij}}{10} \delta^{im} \zeta^{TT}_{il} \zeta^{TT}_{js,lm} + \delta^{ij} \zeta_i \Delta \zeta^T + \frac{\partial F^k [\zeta^{TT}_{ij}, \zeta^T, \zeta_i]}{\partial z^k}
\end{align*}
\]

(7.40)

Thus, neglecting the total divergence \( F^k \), and thanks to suitable cancellations, the variational problem for a Galilean observer, can be reformulated, as an effective theory, only in terms of \( \zeta^T \) and \( \zeta^{TT}_{ij} \), in the form:

\[
S = \frac{1}{16 \pi G} \int dt d^3z \left[ -\varphi \Delta \zeta^T - \frac{1}{8} \zeta^T \Delta \zeta^T - \frac{1}{2} \delta^{ij} \delta^{rs} \delta^{lm} \zeta^{TT}_{il} \zeta^{TT}_{js,lm} \right] + m \int dt d^3z \left[ \frac{1}{2} \delta^{ij} \dot{x}^i \dot{x}^j - \varphi \right] \delta^3 \left[ \mathbf{z} - \mathbf{x}(t) \right].
\]

(7.41)
Let us remark that in eq. (7.41) the term depending on $\zeta_{ij}^{TT}$ is decoupled from the other degrees of freedom. Therefore, in order to get a variational principle for the Poisson equation, only the auxiliary, non propagating, variable $\zeta^T$ is needed.

This theory turns out to be quasi-invariant under the global infinitesimal transformations, which constitute the kinematical group of the Galilean reference frames (2.3), as it should be, provided that $\varphi$ is a scalar field and $\gamma_{ij}$ is a covariant space 2-tensor, i.e.

$$
\begin{aligned}
\delta \varphi &= 0 \\
\delta \gamma_{ij} &= -\omega^j [c_{li}^k \gamma_{kj} + c_{lj}^k \gamma_{ik}]
\end{aligned}
$$

(7.42)

As a consequence of these transformation properties, the transverse traceless components transform according to

$$
\begin{aligned}
\delta \zeta^T &= 0 \\
\delta \zeta_i &= -\omega^j [c_{li}^k \zeta_k - \frac{1}{2} (\Delta^{-1} \zeta^T)_k] \\
\delta \zeta_{ij}^{TT} &= -\omega^k [c_{li}^k \zeta_{kj}^{TT} + c_{lj}^k \zeta_{ik}^{TT}]
\end{aligned}
$$

(7.43)

so that, finally,

$$
\delta S = m \int dt \frac{d}{dt} \left[ \delta_{ij} v^i x^j \right] .
$$

(7.44)

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Appendix A: Explicit expression for the constraints $\bar{\chi}_1$ of the 27-fields theory.

Using the notations: $f_i = \nabla_i f$, $f^i = \nabla^i f$ and $(K_\eta) = (16\pi G/\sqrt{g})$, the explicit expression of the constraints $\bar{\chi}_1$ takes the form:
\[ \ddot{x}_i = -\frac{1}{(K_\eta)} \tilde{A}_{ij} \gamma^{T} ; j + \frac{1}{4 (K_\eta)} \gamma^{T} ; i \gamma^{T} ; j + \]

\[ \frac{1}{(K_\eta)} \tilde{A}_{ij} \gamma^{T} ; ij - \frac{1}{4 (K_\eta)} \gamma^{T} ; ij \]

\[ \frac{1}{2 (K_\eta)} \gamma^{T} ; ij \gamma^{TT} ; ij k - \frac{2}{(K_\eta)} \tilde{A}_{ij} \gamma^{L} ; jk + \]

\[ \frac{1}{2 (K_\eta)} \gamma^{T} ; i \gamma^{L} ; jk + \frac{2}{(K_\eta)} \tilde{A}_{ij} \gamma^{L} ; ij k - \]

\[ \frac{1}{2 (K_\eta)} \gamma^{T} ; ij \gamma^{L} ; kij - \frac{1}{(K_\eta)} \gamma^{TT} ; ij k \gamma^{L} ; ij l - \]

\[ \frac{2}{(K_\eta)} \gamma^{L} ; i j k \gamma^{L} ; ij l + \frac{2}{(K_\eta)} \gamma^{L} ; i j k \gamma^{L} ; ik l + \]

\[ \frac{1}{2 (K_\eta)} \gamma^{T} ; ij \gamma^{L} ; jk - \frac{1}{2 (K_\eta)} \gamma^{T} ; ij \gamma^{L} ; ik + \]

\[ \frac{(K_\eta)}{4} \tilde{A}_{ij} \Delta^{-1} \pi \gamma T ; ij k \Delta^{-1} \pi \gamma T ; jk - \frac{(K_\eta)}{16} \gamma^{T} ; i \Delta^{-1} \pi \gamma T ; j k \Delta^{-1} \pi \gamma T ; i j k + \]

\[ \frac{(K_\eta)}{2} \gamma^{T} ; ij \Delta^{-1} \pi \gamma T ; i k \Delta^{-1} \pi \gamma T ; i j k + \frac{(K_\eta)}{4} \gamma^{TT} ; ij k \Delta^{-1} \pi \gamma T ; i j k \Delta^{-1} \pi \gamma T ; i j k + \]

\[ \frac{(K_\eta)}{2} \gamma^{L} ; i j k \Delta^{-1} \pi \gamma T ; i k \Delta^{-1} \pi \gamma T ; i j k + \frac{(K_\eta)}{8} \Delta^{-1} \pi \gamma T ; ij k \Delta^{-1} \pi \gamma T ; i j k + \]

\[ \frac{(K_\eta)}{2} \gamma^{TT} ; ij k \Delta^{-1} \pi \gamma T ; i k \Delta^{-1} \pi \gamma T ; i j k + \]

\[ \frac{(K_\eta)}{4} \gamma^{TT} ; ij k \Delta^{-1} \pi \gamma T ; i k \Delta^{-1} \pi \gamma T ; ij k + \frac{(K_\eta)}{2} \gamma^{L} ; i j k \Delta^{-1} \pi \gamma T ; i j k \Delta^{-1} \pi \gamma T ; j k l + \]

\[ \frac{(K_\eta)}{2} \gamma^{TT} ; ij j k \Delta^{-1} \pi \gamma T ; j k l \Delta^{-1} \pi \gamma T ; i j k + \frac{(K_\eta)}{2} \gamma^{L} ; i j k \Delta^{-1} \pi \gamma T ; j k l \Delta^{-1} \pi \gamma T ; i j k + \]

\[ \frac{(K_\eta)}{2} \gamma^{TT} ; ij j k \Delta^{-1} \pi \gamma T ; j k l \Delta^{-1} \pi \gamma T ; i j k - \frac{(K_\eta)}{2} \Delta^{-1} \pi \gamma T ; i j k \Delta^{-1} \pi \gamma T ; i j k + \]

\[ (K_\eta) \tilde{A}_{ij} \Delta^{-1} \pi \gamma T ; ij k \gamma^{T} ; i j - \frac{5 (K_\eta)}{8} \gamma^{T} ; ij \Delta^{-1} \pi \gamma T ; i j + \]

\[ \frac{(K_\eta)}{2} \gamma^{TT} ; ij k \Delta^{-1} \pi \gamma T ; i j k - (K_\eta) \gamma^{L} ; i j k \Delta^{-1} \pi \gamma T ; i j k + \]

\[ \frac{(K_\eta)}{2} \gamma^{TT} ; ij j k \Delta^{-1} \pi \gamma T ; j k l \gamma^{L} ; i j k - \frac{3 (K_\eta)}{2} \gamma^{TT} ; ij j k \Delta^{-1} \pi \gamma T ; j k l \gamma^{L} ; i j k - \]

\[ (K_\eta) \gamma^{TT} ; ij k \Delta^{-1} \pi \gamma T ; i j k - (K_\eta) \gamma^{L} ; i j k \Delta^{-1} \pi \gamma T ; i j k + \]

\[ (K_\eta) \gamma^{TT} ; ij j k \Delta^{-1} \pi \gamma T ; j k l \gamma^{L} ; i j k - \frac{3 (K_\eta)}{2} \gamma^{TT} ; ij j k \Delta^{-1} \pi \gamma T ; j k l \gamma^{L} ; i j k - \]

\[ (K_\eta) \gamma^{TT} ; ij j k \Delta^{-1} \pi \gamma T ; j k l \gamma^{L} ; i j k - (K_\eta) \gamma^{L} ; i j k \Delta^{-1} \pi \gamma T ; i j k + \]

\[ (K_\eta) \gamma^{TT} ; ij k \Delta^{-1} \pi \gamma T ; i j k - (K_\eta) \gamma^{L} ; i j k \Delta^{-1} \pi \gamma T ; i j k + \]
\[
\frac{(K_\eta)}{4} \Delta^{-1} \gamma^T_{;i} \Delta^{-1} \pi_{T;ij} \pi_{T;ij} + \frac{(K_\eta)}{2} \pi_{i; j} \pi_{T;ij} - \frac{5 (K_\eta)}{4} \bar{A}_{i} \pi_{T;ij}^2 + \\
\frac{5 (K_\eta)}{16} \gamma^T_{;i} \pi_{T;ij} \pi_{T;ij}^2 + \frac{(K_\eta)}{2} \gamma^L_{i; j} \pi_{T;ij}^2 + \frac{(K_\eta)^3}{8} \Delta^{-1} \pi_{T;ij} \Delta^{-1} \pi_{T;ij}^2 - \\
\frac{(K_\eta)}{8} \pi_{T;ij}^4 - (K_\eta) \bar{A}_{i} \pi_{T;ij} \pi_{T;ij}^2 + (K_\eta) \bar{A}_{i} \Delta^{-1} \pi_{T;ij} \pi_{T;ij}^2 - \\
\frac{(K_\eta)}{4} \gamma^T_{;i} \Delta^{-1} \pi_{T;ij} \pi_{T;ij} + \frac{\gamma^L_{i; j}}{2} \pi_{T;ij}^2 - \\
\frac{(K_\eta)}{8} \gamma^T_{;i} \pi_{T;ij} \pi_{T;ij} + \frac{(K_\eta)}{2} \gamma^L_{i; j} \pi_{T;ij} + \\
\frac{(K_\eta)}{2} \bar{A} \pi_{T;ij} \pi_{T;ij} - \frac{(K_\eta)}{8} \gamma^T \Delta^{-1} \pi_{T;ij} \pi_{T;ij} - \\
\frac{(K_\eta)}{4} \gamma^T_{i; j} \pi_{T;ij} \pi_{T;ij} - \frac{(K_\eta)}{2} \gamma^L_{i; j} \pi_{T;ij} + \\
\frac{(K_\eta)}{8} \Delta^{-1} \gamma^T_{;ij} \pi_{T;ij} + \frac{(K_\eta)}{2} \gamma^L_{i; j} \Delta^{-1} \pi_{T;ij} \pi_{T;ij}^{jk}
\]

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