A Simple and Efficient Estimation of the Average Treatment Effect in the Presence of Unmeasured Confounders

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Abstract

Wang and Tchetgen Tchetgen (2017) studied identification and estimation of the average treatment effect when some confounders are unmeasured. Under their identification condition, they showed that the semiparametric efficient influence function depends on five unknown functionals. They proposed to parameterize all functionals and estimate the average treatment effect from the efficient influence function by replacing the unknown functionals with estimated functionals. They established that their estimator is consistent when certain functionals are correctly specified and attains the semiparametric efficiency bound when all functionals are correctly specified. In applications, it is likely that those functionals could all be misspecified. Consequently their estimator could be inconsistent or consistent but not efficient. This paper presents an alternative estimator that does not require parameterization of any of the functionals. We establish that the proposed estimator is always consistent and always attains the semiparametric efficiency bound. A simple and intuitive estimator of the asymptotic variance is presented, and a small scale simulation study reveals that the proposed estimation outperforms the existing alternatives in finite samples.

Keywords: Average treatment effect; Unmeasured confounders; Semiparametric efficiency; Endogeneity.

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1 Introduction

A common approach to account for individual heterogeneity in the treatment effect literature on observational data is to assume that there exist confounders, and conditional on these confounders, there is no systematic selection into the treatment (i.e., the so-called Unconfounded Treatment Assignment condition suggested in Rosenbaum and Rubin (1983, 1984)). Under this assumption, several procedures for estimating the average treatment effect (hereafter ATE) have been proposed, including the weighting procedure (Rosenbaum (1987), Hirano, Imbens, and Ridder (2003), Tan (2010), Imai and Ratkovic (2014), Chan, Yam, and Zhang (2016), Yiu and Su (2018)); the matching procedure (Rosenbaum (2002), Rosenbaum et al. (2002), Dehejia and Wahba (1999)); and the regression procedure (Heckman, Ichimura, and Todd (1997), Heckman, Ichimura, and Todd (1998), Imbens, Newey, and Ridder (2006), Chen, Hong, and Tarozzi (2008)). For example survey, see Imbens and Wooldridge (2009) and Imbens and Rubin (2015). A critical requirement in this literature is that all confounders are observed and available to researchers. In applications, however, it is often the case that some confounders are either not observed or not available. In this case, the average treatment effect is only partially identified even with the aid of some instrumental variables (see Imbens and Angrist (1994), Angrist, Imbens, and Rubin (1996), Abadie (2003), Abadie, Angrist, and Imbens (2002), Tan (2006), Cheng, Small, Tan, and Have (2009), Ogburn, Rotnitzky, and Robins (2015) for examples).

Recently Wang and Tchetgen Tchetgen (2017) suggested a novel identification condition of ATE when some confounders are not available. Under their condition, they showed that the semiparametric efficient influence function of ATE depends on five unknown functionals. They proposed to parameterize all five functionals, estimate those functionals with appropriate parametric approaches, plug the estimated functionals into the influence function, and then estimate the ATE from the estimated influence function. They established that their estimator is consistent if certain functionals are correctly parameterized and attains the semiparametric efficiency bound if all functionals are correctly specified. In applications, it is quite possible that some or all of the five functionals are misspecified and consequently their estimator could be inefficient or worse, inconsistent. This paper proposes an alternative, intuitive and easy to compute estimation that does not require parameterization of any of the five unknown functionals. We establish that under some sufficient conditions the proposed estimator is consistent, asymptotically normally distributed and attains the semiparametric efficiency bound. Moreover, the proposed procedure provides a natural and convenient estimate of the asymptotic variance.

The paper is organized as follows. Section 2 describes the basic framework. Section 3 describes the proposed estimation and derives the large sample properties of the proposed estimator. Section 4 presents a consistent variance estimator. Since the proposed procedure depends on smoothing parameters, Section 5 presents a data driven method for selecting the smoothing parameters. Section 6 reports a small scale simulation study to evaluate the finite sample performance of the proposed estimator. Some concluding remarks are in Section 7. All technical proofs are relegated to the Appendix and the supplementary material.
2 Basic Framework

Let $D \in \{0, 1\}$ denote the binary treatment indicator, and let $Y(1)$ and $Y(0)$ denote the potential outcomes when an individual is assigned to the treatment and control group respectively. The parameter of interest is the population average treatment effect $\tau = \mathbb{E}[Y(1) - Y(0)]$. Estimation of $\tau$ is complicated by the presence of confounders and the fact that $Y(1)$ and $Y(0)$ cannot be observed simultaneously. To distinguish observed confounders from unobserved confounders, we shall use $X$ to denote the observed confounders and use $U$ to denote the unmeasured confounders. It is well established in the literature that, when all confounders are observed, the following Unconfounded Treatment Assignment condition is sufficient to identify $\tau$:

**Assumption 2.1.** $(Y(0), Y(1)) \perp (D, Z)|(X, U)$.

When $U$ is unmeasured, we have the classical omitted variable problem, causing the treatment indicator $D$ to be endogenous. To tackle the endogeneity problem, instrumental variable is often the preferred choice. Let $Z \in \{0, 1\}$ denote the variable satisfying the following classical instrumental variable conditions:

**Assumption 2.2 (Exclusion restriction).** $\forall z, d, Y(z, d) = Y(d)$, where $Y(z, d)$ is the response that would be observed if a unit were exposed to $d$ and the instrument had taken value $z$ to be well defined.

**Assumption 2.3 (Independence).** $Z \perp U|X$.

**Assumption 2.4 (IV relevance).** $Z \not\perp D|X$.

Wang and Tchetgen Tchetgen (2017) showed that Assumptions 2.1-2.4 alone do not identify $\tau$, but if in addition one of the following conditions holds:

1. there is no additive $U-Z$ interaction in $\mathbb{E}[D|Z, X, U]$:
   \[
   \mathbb{E}[D|Z = 1, X, U] - \mathbb{E}[D|Z = 0, X, U] = \mathbb{E}[D|Z = 1, X] - \mathbb{E}[D|Z = 0, X] .
   \]

2. there is no additive $U-d$ interaction in $\mathbb{E}[Y(d)|X, U]$:
   \[
   \mathbb{E}[Y(1) - Y(0)|X, U] = \mathbb{E}[Y(1) - Y(0)|X] ,
   \]

then ATE is identified and can be expressed as

\[
\tau = \mathbb{E}[\delta(X)] = \mathbb{E} \left[ \frac{\delta^Y(X)}{\delta^D(X)} \right] ,
\]

where

\[
\delta^Y(X) = \mathbb{E}[Y|Z = 1, X] - \mathbb{E}[Y|Z = 0, X] ,
\]

\[
\delta^D(X) = \mathbb{E}[D|Z = 1, X] - \mathbb{E}[D|Z = 0, X] ,
\]

\[
\delta(X) = \delta^Y(X)/\delta^D(X) .
\]
Furthermore, Wang and Tchetgen Tchetgen (2017) derived the efficient influence function for \( \tau \):

\[
\varphi_{eff}(D, Z, X, Y) = \frac{2Z - 1}{f_{Z|X}(Z|X)} \frac{1}{\delta^D(X)} \left\{ Y - D\delta(X) - E[Y|Z = 0, X] + E[D|Z = 0, X]\delta(X) \right\} + \delta(X) - \tau,
\]

where \( f_{Z|X}(Z|X) \) is the conditional probability mass function of \( Z \) given \( X \). Clearly, the efficient influence function depends on five unknown functionals: \( \delta(X), \delta^D(X), f_{Z|X}, p^Y_0(X) = E[Y|Z = 0, X] \) and \( p^D_0(X) = E[D|Z = 0, X] \). They proposed to parameterize all five functionals, estimate the functionals with appropriate parametric approaches, and plug the estimated functionals into the efficient influence function to estimate \( \tau \). They established that their estimator of \( \tau \) is consistent and asymptotically normally distributed if

- either \( \delta(X), \delta^D(X), p^Y_0(X) = E[Y|Z = 0, X] \) and \( p^D_0(X) = E[D|Z = 0, X] \) are correctly specified
- or \( \delta^D(X) \) and \( f_{Z|X} \) are correctly specified
- or \( \delta(X) \) and \( f_{Z|X} \) are correctly specified,

and their estimator attains the semiparametric efficiency bound only when all five functionals are correctly specified. The main goal of this paper is to present an alternative, intuitive and easy approach to compute estimator that does not require parameterization of any of the functionals and is always consistent and asymptotically normal and attains the semiparametric efficiency bound.

### 3 Point Estimation

To motivate our estimation procedure, we rewrite the treatment effect coefficient. Applying the tower law of conditional expectation, we obtain:

\[
\tau = \mathbb{E} \left[ \frac{\delta^Y(X)}{\delta^D(X)} \right] = \mathbb{E} \left[ \frac{\mathbb{E}[Y|Z = 1, X]}{\delta^D(X)} - \frac{\mathbb{E}[Y|Z = 0, X]}{\delta^D(X)} \right]
\]

\[
= \mathbb{E} \left[ \frac{Z}{f_{Z|X}(1|X)} \cdot \frac{\mathbb{E}[Y|Z = 1, X]}{\delta^D(X)} - \frac{1 - Z}{f_{Z|X}(0|X)} \cdot \frac{\mathbb{E}[Y|Z = 0, X]}{\delta^D(X)} \right]
\]

\[
= \mathbb{E} \left[ \frac{Z}{f_{Z|X}(1|X)} \cdot \frac{\mathbb{E}[Y|Z, X]}{\delta^D(X)} - \frac{1 - Z}{f_{Z|X}(0|X)} \cdot \frac{\mathbb{E}[Y|Z, X]}{\delta^D(X)} \right]
\]

\[
= \mathbb{E} \left[ \left\{ \frac{2Z - 1}{f_{Z|X}(Z|X)} \right\} \frac{Y}{\delta^D(X)} \right].
\]

The above expression suggests a natural and intuitive plugin estimation, with \( f_{Z|X}(Z|X) \) and \( \delta^D(X) \) replaced by some consistent estimates. There are many approaches to estimate these functionals including parametric and nonparametric approaches, but as noted by Hirano, Imbens, and Ridder (2003), not all estimates can lead to efficient estimation of \( \tau \). In this paper, we present an intuitive and easy way to compute estimates of functionals that ensure efficiency of the plugin
estimation of $\tau$. To illustrate our procedure, we notice that the following conditions hold for any integrable functions $u_1(X)$ and $u_2(X)$:

$$E \left[ \frac{Z}{f_{Z|X}(1|X)} u_1(X) \right] = E[u_1(X)] = E \left[ \frac{1 - Z}{f_{Z|X}(0|X)} u_1(X) \right],$$  \quad (3.2)

$$E \left[ D \left\{ \frac{2Z - 1}{f_{Z|X}(Z|X)} \right\} u_2(X) \right] = E \left[ \delta^D(X) u_2(X) \right],$$  \quad (3.3)

and (3.2) and (3.3) uniquely determine $f_{Z|X}(Z|X)$ and $\delta^D(X)$. These conditions impose restrictions on the unknown functionals and they must be taken into account when estimating those functionals. One difficulty with these conditions is that they must be imposed in an infinite dimensional functional space. To overcome this difficulty, we propose to impose the conditions on a smaller sieve space. Specifically, let $u_K(X) = (u_{K,1}(X), \ldots, u_{K,K}(X))^\top$ denote a known basis functions that can approximate any suitable function $u(X)$ arbitrarily well (see Chen (2007) or Appendix A.1 for further discussion). Conditions (3.2) and (3.3) imply for any integers $K_1$ and $K_2$:

$$E \left[ \frac{Z}{f_{Z|X}(1|X)} u_{K_1}(X) \right] = E[u_{K_1}(X)] = E \left[ \frac{1 - Z}{f_{Z|X}(0|X)} u_{K_1}(X) \right],$$  \quad (3.4)

and

$$E \left[ D \left\{ \frac{Z}{f_{Z|X}(1|X)} - \frac{1 - Z}{f_{Z|X}(0|X)} \right\} u_{K_2}(X) \right] = E[\delta^D(X) u_{K_2}(X)].$$  \quad (3.5)

We shall construct estimates of the functionals by imposing the above conditions. To ensure consistency, we shall allow $K_1$ and $K_2$ to increase with sample size at appropriate rates.

### 3.1 Estimation of $f_{Z|X}(Z|X)^{-1}$

Consider estimation of $f_{Z|X}(Z|X)^{-1}$. An obvious approach is to solve the sample analogue of (3.4):

$$\frac{1}{N} \sum_{i=1}^N Z_i w_i u_{K_1}(X_i) = \frac{1}{N} \sum_{i=1}^N u_{K_1}(X_i);$$  \quad (3.6)

$$\frac{1}{N} \sum_{i=1}^N (1 - Z_i) w_i u_{K_1}(X_i) = \frac{1}{N} \sum_{i=1}^N u_{K_1}(X_i).$$  \quad (3.7)

But there are many solutions and all solutions are consistent estimates of $f_{Z|X}(Z|X)^{-1}$. The question is which solution is the best estimate of $f_{Z|X}(Z|X)^{-1}$ in the sense of ensuring efficient estimation of $\tau$. Let $\rho(v)$ denote a strictly increasing and concave function and let $\rho'(v)$ denote its first derivative. Denote

$$\hat{\rho}(X_i) \triangleq \frac{1}{N} \rho'\left(\hat{\lambda}_{K_1}^\top u_{K_1}(X_i)\right),$$

with $\hat{\lambda}_{K_1} \in \mathbb{R}^K$ maximizing the following objective function

$$\hat{G}(\lambda) \triangleq \frac{1}{N} \sum_{i=1}^N Z_i \rho(\lambda^\top u_{K_1}(X_i)) - \frac{1}{N} \sum_{i=1}^N \lambda^\top u_{K_1}(X_i).$$  \quad (3.8)
It is easy to show that $N\hat{p}(X)$ satisfies (3.6). Moreover, $N\hat{p}(X)$ can be interpreted as a generalized empirical likelihood estimator of $f_{Z|X}(1|X)^{-1}$ (see Appendix A.2) and hence is the best estimate. The fact that $\hat{G}(\lambda)$ is globally concave implies that its maximand is easy to compute.

Applying the same idea to (3.7), we have

$$\hat{q}(X_i) \triangleq \frac{1}{N} \rho'(\hat{\beta}_{K_1}^T u_{K_1}(X_i)),$$

with $\hat{\beta}_{K_1} \in \mathbb{R}^{K_1}$ maximizing the following globally concave objective function

$$\hat{H}(\beta) \triangleq \frac{1}{N} \sum_{i=1}^{N} (1 - Z_i) \rho(\beta^T u_{K_1}(X_i)) - \frac{1}{N} \sum_{i=1}^{N} \beta^T u_{K_1}(X_i).$$

(3.9)

Again, $N\hat{q}(X)$ satisfies (3.7) and can be interpreted as a generalized empirical likelihood estimator of $f_{Z|X}(0|X)^{-1}$.

The $\rho(v)$ function can be any increasing and strictly concave function. Some examples include

$\rho(v) = -\exp(-v)$ for the exponential tilting (Kitamura and Stutzer, 1997, Imbens, Spady, and Johnson, 1998), $\rho(v) = \log(1 + v)$ for the empirical likelihood (Owen, 1988, Qin and Lawless, 1994), $\rho(v) = -(1 - v)^2/2$ for the continuous updating of the generalized method of moments (Hansen, 1982, Hansen, Heaton, and Yaron, 1996) and $\rho(v) = v - \exp(-v)$ for the inverse logistic.

### 3.2 Estimation of $\delta^D(X)$ and $\tau$

Having estimated $f_{Z|X}(Z|X)^{-1}$, we now apply the same principle to estimate $\delta^D(X)$. But there is one difference. Here $\delta^D(X) \in [-1, 1]$ and the $\rho(v)$ function is not suitable. We shall use the following strictly convex function

$$f(x) = \log(e^x + e^{-x})$$

whose derivative is the tanh function $f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ with range $[-1, 1]$. We estimate $\delta^D(X)$ by

$$\hat{\delta}^D(X) = f'(\hat{\gamma}_{K_2}^T u_{K_2}(X)),$$

with $\hat{\gamma}_{K_2} \in \mathbb{R}^{K_2}$ maximizing the following globally concave function

$$\hat{F}(\gamma) = \frac{1}{N} \sum_{i=1}^{N} D_i \{Z_i N\hat{p}(X_i) - (1 - Z_i) N\hat{q}(X_i)\} \cdot \gamma^T u_{K_2}(X_i) - \frac{1}{N} \sum_{i=1}^{N} f(\gamma^T u_{K_2}(X_i)).$$

Again, $\hat{\delta}^D(X)$ can be interpreted as a generalized empirical likelihood estimator and hence is the best estimate.

Finally, the plugin estimator of $\tau$ is given by

$$\hat{\tau} = \sum_{i=1}^{N} \{Z_i \hat{p}(X_i) - (1 - Z_i) \hat{q}(X_i)\} Y_i / \hat{\delta}^D(X_i).$$
3.3 Large Sample Properties

To establish the large sample properties of $\hat{\tau}$, we shall impose the following assumptions:

**Assumption 3.1.** \( \mathbb{E} \left[ \frac{1}{\delta^2(\hat{X})^2} \right] < \infty \) and \( \mathbb{E} \left[ \frac{Y^2}{\delta^2(\hat{X})^2} \right] < \infty. \)

**Assumption 3.2.** The support \( \mathcal{X} \) of \( r \)-dimensional covariate \( X \) is a Cartesian product of \( r \) compact intervals.

**Assumption 3.3.** We assume that there exist three positive constants \( \eta_1 > \eta_2 > 1 > \eta_3 > 0 \) such that
\[
\eta_2 \leq \frac{1}{f_{Z|X}(z|x)} \leq \eta_1 \quad \text{and} \quad -\eta_3 \leq \delta^D(x) \leq \eta_3, \quad \forall (z, x) \in \{0, 1\} \times \mathcal{X}.
\]

**Assumption 3.4.** There are \( \lambda_K, \beta_K, \gamma_K, \psi_{1K}, \psi_{0K}, \phi_{1K} \) and \( \phi_{0K} \) in \( \mathbb{R}^K \) and \( \alpha > 0 \) such that
\[
\sup_{x \in \mathcal{X}} \left| (\rho')^{-1} \left( \frac{1}{f_{Z|X}(1|x)} \right) - \lambda_K^T u_K(x) \right| = O(K^{-\alpha}), \quad \sup_{x \in \mathcal{X}} \left| (\rho')^{-1} \left( \frac{1}{f_{Z|X}(0|x)} \right) - \beta_K^T u_K(x) \right| = O(K^{-\alpha}),
\]
\[
\sup_{x \in \mathcal{X}} \left| (f')^{-1} \left( \delta^D(x) \right) - \gamma_K^T u_K(x) \right| = O(K^{-\alpha}),
\]
\[
\sup_{x \in \mathcal{X}} \left| \frac{p_1^Y(x)}{\delta^D(x)} - \psi_{1K}^T u_K(x) \right| = O(K^{-\alpha}), \quad \sup_{x \in \mathcal{X}} \left| \frac{p_0^Y(x)}{\delta^D(x)} - \psi_{0K}^T u_K(x) \right| = O(K^{-\alpha}),
\]
\[
\sup_{x \in \mathcal{X}} \left| \frac{p_1^Y(x)}{\delta^D(x)^2} - \phi_{1K}^T u_K(x) \right| = O(K^{-\alpha}), \quad \sup_{x \in \mathcal{X}} \left| \frac{p_0^Y(x)}{\delta^D(x)^2} - \phi_{0K}^T u_K(x) \right| = O(K^{-\alpha}),
\]
as \( K \to \infty \), where \( p_z^Y(x) = \mathbb{E}[Y|Z = z, X = x] \) for \( z \in \{0, 1\} \).

**Assumption 3.5.** \( K_1 \asymp K_2 \asymp K \in \mathbb{N}, \ \zeta(K)^4 K^3/N \to 0 \) and \( \sqrt{N} K^{-\alpha} \to 0 \), where \( \zeta(K) = \sup_{x \in \mathcal{X}} \|u_K(x)\| \) and \( \|\cdot\| \) is the usual Frobenius norm defined by \( \|A\| = \sqrt{\text{tr}(AA^T)} \) for any matrix \( A \).

**Assumption 3.6.** \( \rho \) is a strictly concave function defined on \( \mathbb{R} \), i.e. \( \rho''(\gamma) < 0 \), \( \forall \gamma \in \mathbb{R} \), and the range of \( \rho' \) contains \([\eta_2, \eta_1]\).

Assumption 3.1 ensures the asymptotic variance to be bounded. Assumption 3.2 restricts the covariates to be bounded. This condition, though restrictive, is commonly imposed in the nonparametric regression literature. Assumption 3.3 requires the probability function to be bounded away from 0 and 1. Condition of this sort is familiar in the literature. Assumption 3.4 is needed to control for the approximation bias, and they are commonly imposed in the nonparametric literature. Assumption 3.5 imposes restrictions on the smoothing parameter so that the proposed estimator of ATE is root-N consistent. This condition, however, is practically unhelpful. We shall present a data driven approach to determine \( K_1 \) and \( K_2 \). Assumption 3.6 is a mild restriction on \( \rho \) and is satisfied by all important special cases considered in the literature.

Under the above assumptions, the following theorem establishes the consistency, asymptotic normality and the semiparametric efficiency of \( \hat{\tau} \).

**Theorem 3.7.** Suppose that the average treatment effects is identified in (2.1), under Assumptions 3.1-3.6, we have
1. \( \hat{\tau} \xrightarrow{p} \tau \);
2. \( \sqrt{N}(\hat{\tau} - \tau) \xrightarrow{d} N(0, V_{\text{eff}}) \),

where \( V_{\text{eff}} = \mathbb{E} \left[ \varphi_{\text{eff}}(D, Z, X, Y)^2 \right] \) is the efficient variance bound developed in Wang and Tchetgen Tchetgen (2017).

Sketched proof can be found in Appendix A.4 and detailed proofs are provided in the supplementary material.

## 4 Variance Estimation

To conduct the statistical inference on \( \tau \), we need a consistent estimator of the asymptotic variance of \( \hat{\tau} \). Note that the asymptotic variance of \( \hat{\tau} \),

\[
\mathbb{E} \left[ \left( \frac{2Z - 1}{f_{Z|X}(Z|X)} \frac{1}{\delta^2(X)} \left\{ Y - D\delta(X) - \mathbb{E}[Y|Z = 0, X] + \mathbb{E}[D|Z = 0, X]\delta(X) \right\} + \delta(X) - \tau \right)^2 \right],
\]

depends on five unknown functionals. Direct estimation of the variance requires replacing the five unknown functionals with consistent estimators. In this section, we present an alternative estimation that does not require estimation of those functionals.

To illustrate the idea, we denote:

\[
g_1(Z, X; \lambda) \equiv Z \rho \left( \lambda^T u_{K_1}(X) \right) u_{K_1}(X) - u_{K_1}(X),
\]

\[
g_2(Z, X; \beta) \equiv (1 - Z) \rho \left( \beta^T u_{K_1}(X) \right) u_{K_1}(X) - u_{K_1}(X),
\]

\[
g_3(Z, D, X; \lambda, \beta, \gamma) \equiv D \left\{ Z \cdot \rho \left( \lambda^T u_{K_1}(X) \right) - (1 - Z) \cdot \rho \left( \beta^T u_{K_1}(X) \right) \right\} u_{K_2}(X) - f' \left( \gamma^T u_{K_2}(X) \right) u_{K_2}(X),
\]

\[
g_4(Z, D, X, Y; \lambda, \beta, \gamma, \tau) \equiv \left\{ Z \cdot \rho \left( \lambda^T u_{K_1}(X) \right) - (1 - Z) \cdot \rho \left( \beta^T u_{K_1}(X) \right) \right\} Y / f' \left( \gamma^T u_{K_2}(X) \right) - \tau,
\]

and

\[
g(Z, D, X, Y; \theta) \equiv \begin{pmatrix} g_1(Z, X; \lambda) \\ g_2(Z, X; \beta) \\ g_3(Z, D, X; \lambda, \beta, \gamma) \\ g_4(Z, D, X, Y; \lambda, \beta, \gamma, \tau) \end{pmatrix}
\]

with \( \theta \equiv (\lambda, \beta, \gamma, \tau)^T \). Let \( \hat{\theta} \equiv (\hat{\lambda}_{K_1}, \hat{\beta}_{K_1}, \hat{\gamma}_{K_2}, \hat{\tau})^T \) and \( \theta^* \equiv (\lambda^*_K, \beta^*_K, \gamma^*_K)^T \). Then \( \hat{\theta} \) is the moment estimator solving the following moment condition:

\[
\frac{1}{N} \sum_{i=1}^{N} g(Z_i, D_i, X_i, Y_i; \hat{\theta}) = 0.
\] (4.1)

Applying Mean Value Theorem, we obtain

\[
0 = \frac{1}{N} \sum_{i=1}^{N} g(Z_i, D_i, X_i, Y_i; \theta^*) + \frac{1}{N} \sum_{i=1}^{N} \frac{\partial g(Z_i, D_i, X_i, Y_i; \hat{\theta})}{\partial \theta} (\hat{\theta} - \theta^*)
\] (4.2)
where \( \hat{\theta} = (\hat{\lambda}_{K_1}, \hat{\beta}_{K_1}, \hat{\gamma}_{K_2}, \hat{\tau})^\top \) lies on the line joining \( \hat{\theta} \) and \( \theta^* \). We show in the supplemental material that

\[
\frac{1}{N} \sum_{i=1}^{N} \frac{\partial g(Z_i, D_i, X_i, Y_i; \hat{\theta})}{\partial \theta} = \mathbb{E} \left[ \frac{\partial g(Z, D, X, Y; \theta^*)}{\partial \theta} \right] + o_p(1) \tag{4.3}
\]

Note that

\[
\hat{\tau} - \tau = \mathbf{e}_{2K_1+K_2+1}^\top (\hat{\theta} - \theta^*) \tag{4.4}
\]

where \( \mathbf{e}_{2K_1+K_2+1} \) is a \((2K_1 + K_2 + 1)\)-dimensional column vector whose last element is 1 and other components are all of 0's.

Combining (4.2), (4.3) and (4.4), we obtain

\[
\sqrt{N}(\hat{\tau} - \tau) = -\mathbf{e}_{2K_1+K_2+1}^\top \left\{ \mathbb{E} \left[ \frac{\partial g(Z, D, X, Y; \theta^*)}{\partial \theta} \right] + o_p(1) \right\}^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} g(Z_i, D_i, X_i, Y_i; \theta^*),
\]

which in turn implies

\[
V_{eff} = \lim_{N \to \infty} Var(\sqrt{N}(\hat{\tau} - \tau)) = \lim_{N \to \infty} \mathbf{e}_{2K_1+K_2+1}^\top \left\{ L \cdot \Omega \cdot (\hat{L}^{-1})^\top \right\} \mathbf{e}_{2K_1+K_2+1},
\]

where

\[
L = \mathbb{E} \left[ \frac{\partial g(Z, D, X, Y; \theta^*)}{\partial \theta} \right], \\
\Omega = \mathbb{E} \left[ g(Z, D, X, Y; \theta^*) g(Z, D, X, Y; \theta^*)^\top \right].
\]

Therefore, we can define the sandwich estimator for the efficient variance \( V_{eff} \) by

\[
\hat{V} = \mathbf{e}_{2K_1+K_2+1}^\top \left\{ \hat{L}^{-1} \cdot \hat{\Omega} \cdot (\hat{L}^{-1})^\top \right\} \mathbf{e}_{2K_1+K_2+1},
\]

where

\[
\hat{L} = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial g(Z_i, D_i, X_i, Y_i; \hat{\theta})}{\partial \theta}; \\
\hat{\Omega} = \frac{1}{N} \sum_{i=1}^{N} g(Z_i, D_i, X_i, Y_i; \hat{\theta}) g(Z_i, D_i, X_i, Y_i; \hat{\theta})^\top.
\]

**Theorem 4.1.** Under Assumptions 3.1-3.6, \( \hat{V} \) is a consistent estimator for the asymptotic variance \( V_{eff} \).
5 Selection of Tuning Parameters

The large sample properties of the proposed estimator permit a wide range of values of \( K_1 \) and \( K_2 \). This presents a dilemma for applied researchers who have only one finite sample and would like to have some guidance on the selection of smoothing parameters. In this section, we present a data-driven approach to select \( K_1 \) and \( K_2 \). Notice that \( f_{Z|X}(1|X)^{-1}, f_{Z|X}(0|X)^{-1} \) and \( \delta^D(X) \) satisfy the following regression equations:

\[
\mathbb{E} \left[ Zf_{Z|X}(1|X)^{-1} | X \right] = 1 , \]
\[
\mathbb{E} \left[ (1 - Z)f_{Z|X}(0|X)^{-1} | X \right] = 1 , \]
\[
\mathbb{E} \left[ D \{ Zf_{Z|X}(1|X)^{-1} - (1 - Z)f_{Z|X}(0|X)^{-1} \} | X \right] = \delta^D(X) .
\]

Since \( N\hat{p}(X), N\hat{q}(X) \) and \( \hat{\delta}^D(X) \) are consistent estimators of \( f_{Z|X}(1|X)^{-1}, f_{Z|X}(0|X)^{-1} \) and \( \delta^D(X) \) respectively, the mean-squared-error (MSE) of the nuisance parameters (\( \hat{\lambda}_{K_1}, \hat{\beta}_{K_1} \)) and \( \hat{\gamma}_{K_2} \) are defined by

\[
MSE_1(K_1) = \sum_{i=1}^{N} \{ Z_i N\hat{p}(X_i) - 1 \}^2 + \sum_{i=1}^{N} \{ (1 - Z_i) N\hat{q}(X_i) - 1 \}^2 ,
\]
\[
MSE_2(K_1, K_2) = \sum_{i=1}^{N} \left\{ D_i \{ Z_i N\hat{p}(X_i) - (1 - Z_i) N\hat{q}(X_i) \} - \hat{\delta}^D(X_i) \right\}^2 .
\]

The smoothing parameters \( K_1 \) and \( K_2 \) shall be chosen to minimize \( MSE_1 \) and \( MSE_2 \). Specifically, denote the upper bounds of \( K_1 \) and \( K_2 \) by \( \bar{K}_1 \) and \( \bar{K}_2 \) (e.g. \( \bar{K}_1 = \bar{K}_2 = 5 \) in our simulation studies). The data-driven \( K_1 \) and \( K_2 \) are given by

\[
\hat{K}_1 = \arg \min_{K_1 \in \{1, \ldots, \bar{K}_1 \}} MSE_1(K_1) ,
\]
\[
\hat{K}_2 = \arg \min_{K_2 \in \{1, \ldots, \bar{K}_2 \}} MSE_2(\hat{K}_1, K_2) .
\]

6 Simulation Studies

In this section, we conduct a small scale simulation study to evaluate the finite sample performance of the proposed estimator. To evaluate the performance of our estimator against the existing alternatives, particularly the estimators proposed by Wang and Tchetgen Tchetgen (2017), we adopt the exact same design (i.e., the same data generating processes (DGP)). In each Monte Carlo run, we generate sample of data from DGP for two sizes: \( N = 500 \) and \( N = 1000 \) respectively, and from each sample we compute our estimator and other existing estimators. We then repeat the Monte Carlo runs for 500 times.

The observed baseline covariates are \( X = (1, X_2) \), where \( X \) include an intercept term and a continuous random variable \( X_2 \) uniformly distributed on the interval \((-1, -0.5) \cup (0.5, 1)\). The unmeasured confounder \( U \) is a Bernoulli random variable with mean 0.5. The instrumental variable \( Z \),
treatment variable $D$ and outcomes variable $Y \in \{0, 1\}$ are generated according to the simulation design of Wang and Tchetgen Tchetgen (2017). The true value of the average treatment effect is $\tau = 0.087$.

We compute the proposed estimator (cbe), the naive estimator, the multiply robust estimator (mr) and the bounded multiply robust estimator (b-mr) proposed by Wang and Tchetgen Tchetgen (2017). Details of calculations are given below.

1. the proposed estimator (cbe) is computed with $\rho(v) = \log(1 + v)$;
2. the naive estimator is computed by the difference of group means between treatment and control groups;
3. the multiply robust estimator (mr) and the bounded multiply robust estimator (b-mr) are computed by the procedures proposed by Wang and Tchetgen Tchetgen (2017).

The multiply robust estimator (mr) and the bounded multiply robust estimator (b-mr) proposed by Wang and Tchetgen Tchetgen (2017) depend on parameterization of five unknown functionals. In their paper they considered several models, denoted by $M_1$, $M_2$ and $M_3$ (see Wang and Tchetgen Tchetgen (2017) for a detailed discussion of the model specification). Following Wang and Tchetgen Tchetgen (2017), we consider scenarios where some or all functionals are misspecified.

Table 1 reports the bias, standard deviation (Stdev), and the root mean square error (RMSE) of $\hat{\tau}$ from the 500 Monte Carlo runs. In each Monte Carlo run, we use the data driven approach to select $K_1$ and $K_2$, and their histograms are depicted in Figure 1. The estimated asymptotic variances are reported in Table 2.

Glancing at these tables, we have the following observations:

1. The naive estimator has large bias. This is not surprising since it ignores the confounding effect.
2. The multiple robust estimators (mr) of Wang and Tchetgen Tchetgen (2017) has huge bias when some functionals are misspecified.
3. The bounded multiple robust estimator (b-mr) of Wang and Tchetgen Tchetgen (2017) is more robust than mr-estimator, but it still has a significant bias if some functionals are misspecified. And the bias does not vanish as the sample size increases. Moreover, if all functionals are misspecified, the bias of b-mr estimator is substantially large.
4. The proposed estimator (cbe) is unbiased for both $N = 500$ and $N = 1000$. Its performance (Bias, Stdev, RMSE) is comparable to Wang and Tchetgen Tchetgen (2017) ‘s estimator when all functionals are correctly parameterized.
Table 1: Simulation results of estimated average treatment effects

| Estimators    | N = 500 |        |        |        |
|---------------|---------|--------|--------|--------|
|               |         | Bias   | Stdev  | RMSE   |
| Naive         |         | -0.057 | 0.045  | 0.073  |
| mr(All)       |         | 0.003  | 0.139  | 0.139  |
| mr(M₁)        |         | 0.004  | 0.139  | 0.139  |
| mr(M₂)        |         | -0.004 | 0.163  | 0.163  |
| mr(M₃)        |         | -30.973| 883.036| 884.579|
| mr(None)      |         | -13.887| 419.412| 419.648|
| b-mr(All)     |         | 0.006  | 0.145  | 0.145  |
| b-mr(M₁)      |         | -0.015 | 0.163  | 0.164  |
| b-mr(M₂)      |         | -0.010 | 0.207  | 0.207  |
| b-mr(M₃)      |         | 0.008  | 0.142  | 0.143  |
| mr(None)      |         | -0.137 | 0.648  | 0.663  |
| cbe           |         | 0.003  | 0.152  | 0.152  |

| Estimators    | N = 1000|        |        |        |
|---------------|---------|--------|--------|--------|
|               |         | Bias   | Stdev  | RMSE   |
| Naive         |         | -0.056 | 0.031  | 0.064  |
| mr(All)       |         | -0.002 | 0.102  | 0.102  |
| mr(M₁)        |         | -0.0005| 0.102  | 0.102  |
| mr(M₂)        |         | -0.011 | 0.121  | 0.121  |
| mr(M₃)        |         | -94.930| 1737.95| 1740.541|
| mr(None)      |         | 9.708  | 240.259| 240.455|
| b-mr(All)     |         | 0.003  | 0.104  | 0.104  |
| b-mr(M₁)      |         | -0.021 | 0.134  | 0.136  |
| b-mr(M₂)      |         | -0.008 | 0.141  | 0.141  |
| b-mr(M₃)      |         | 0.002  | 0.103  | 0.103  |
| b-mr(None)    |         | 0.224  | 0.638  | 0.676  |
| cbe           |         | 0.004  | 0.110  | 0.110  |

The true value for of the average treatment effects is 0.087. Bias, standard deviation (Stdev), root mean squared error (RMSE) of each estimator after $J = 500$ Monte Carlo trials are reported. All: all of the three models $M₁, M₂, M₃$ are correctly specified; $M₁$: only the model $M₁$ is correctly specified; $M₂$: only the model $M₂$ is correctly specified; $M₃$: only the model $M₃$ is correctly specified; None: all of the models are misspecified.
Table 2: Simulation results of estimated efficient deviation

| Methods | Situation | Deviation Estimate |
|---------|-----------|--------------------|
| All     | 3.04      |                    |
| \(M_1\) | 3.19      |                    |
| mr      | \(M_2\)  | 3.22               |
|         | \(M_3\)  | 2260.0             |
| None    | 3596.7    |                    |
| All     | 3.04      |                    |
| \(M_1\) | 3.19      |                    |
| b-mr    | \(M_2\)  | 3.22               |
|         | \(M_3\)  | 2078.0             |
| None    | 3572.2    |                    |
| cbe     | —         | 3.41               |

The true value of efficient deviation is 3.04. All: all of the three models \(M_1, M_1, M_3\) are correctly specified; \(M_1\): only the model \(M_1\) is correctly specified; \(M_2\): only the model \(M_2\) is correctly specified; \(M_3\): only the model \(M_3\) is correctly specified; None: all of the models are misspecified.
Figure 1: Histogram of $K_1$ & $K_2$

(a) $n=500$

(b) $n=1000$
5. In variance estimation, both the multiple robust estimator (mr) and the bounded multiple robust estimator (b-mr) have large biases when some functionals are misspecified. In contrast, the proposed variance estimator is consistent.

6. The histograms in Figure 1 reveal that for both \( N = 500 \) and \( N = 1000 \), \( K_1 = 2 \) and \( K_2 = 2 \) are most preferred, suggesting that the growing rate of \( K_1 \) and \( K_2 \) is slow, an observation consistent with Assumption 3.5. Overall, the simulation results show that the proposed estimator out-performs the existing estimators.

7 Concluding Remarks

Most of the existing treatment effect literature on observational data assume that all confounders are observed and available to researchers. In applications, it is often the case that some confounders are not observed or not available. Wang and Tchetgen Tchetgen (2017) studied identification and estimation of the average treatment effect when some confounders are not observed. They propose to parameterize five unknown functionals and show that their estimation is consistent when certain functionals are correctly specified and is efficient when all functionals are correctly specified. This paper proposes an alternative estimation. Unlike Wang and Tchetgen Tchetgen (2017), the proposed estimation does not parameterize any of the functionals and is always consistent. Moreover, the proposed estimator attains the semiparametric efficiency bound. A simple asymptotic variance estimator is presented, and a small scale simulation study suggests the practicality of the proposed procedure.

Our procedure only applies to the binary treatment with unmeasured confounders. However, other forms of treatment, such as multiple valued or continuous treatment, may arise in applications. Extension of the proposed methodology to those forms of treatment with unmeasured confounders is certainly of great interest. This extension shall be pursued in a future project.

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A Appendix

A.1 Discussion on $u_K$

To construct our estimator, we need to specify the sieve basis $u_K(X)$. Although the approximation theory is derived for general sequences of sieve basis, the most common class of functions are power series and splines. In particular, we can approximate any function $f : \mathbb{R}^r \to \mathbb{R}$ by $\tilde{\gamma}^\top K \tilde{u}_K(x)$, where $\tilde{u}_K(x)$ is a prespecified sieve basis. Because $\tilde{\gamma}^\top K \tilde{u}_K(x) = \tilde{\gamma}^\top K A_{K}^{-1} K A_{K} \tilde{u}_K(x)$, we can also use $u_K(x) = A_{K} \times K \tilde{u}_K(x)$ as the new basis for approximation. By choosing $A_{K} \times K$ appropriately we obtain a system of orthonormal basis (with respect to some weights). In particular, we choose $A_{K} \times K$ so that

$$\mathbb{E} \left[ u_K(X) u_K^\top (X) \right] = I_{K \times K} \ .$$

(A.1)

We define the usual Frobenius norm $\|A\| \triangleq \sqrt{\text{tr}(AA^\top)}$ for any matrix $A$. Define

$$\zeta(K) \triangleq \sup_{x \in \mathcal{X}} \|u_K(x)\| .$$

(A.2)

In general, this bound depends on the array of basis that is used. Newey (1994, 1997) showed that

1. for power series: there exists a universal constant $C_0 > 0$ such that $\zeta(K) \leq C_0K$;

2. for regression splines: there exists a universal constant $C_0 > 0$ such that $\zeta(K) \leq C_0\sqrt{K}$.

A.2 Duality of Constrained Optimization

Let $L(v, v_0)$ be a distance measure that is continuously differentiable in $v \in \mathbb{R}$, non-negative, strictly convex in $v$ and $L(v_0, v_0) = 0$. The general idea of calibration is to minimize the aggregate distance between the final weights to a given vector of design weights subject to moment constraints. Being motivated by (3.4), we consider to construct the calibration weights $\{w_i\}_{i=1}^N$ by solving the following constrained optimization problem:

$$\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^N L(w_i, 1) , \\
\text{subject to} & \quad \frac{1}{N} \sum_{i=1}^N Z_i w_i u_{K_1}(X_i) = \frac{1}{N} \sum_{i=1}^N u_{K_1}(X_i) = \frac{1}{N} \sum_{i=1}^N (1 - Z_i) w_i u_{K_1}(X_i) ,
\end{align*}$$

(A.3)

where $K_1 \to \infty$ as the sample size $N \to \infty$, yet with $K_1/N \to 0$. The constrained optimization problem stated above is equivalent to two separate constrained optimization problems.

$$\text{Minimize} \quad \sum_{i=1}^N Z_i L(Np_i, 1) \quad \text{subject to} \quad \sum_{i=1}^N Z_i p_i u_{K_1}(X_i) = \frac{1}{N} \sum_{i=1}^N u_{K_1}(X_i) ,$$

(A.4)
Minimize \( \sum_{i=1}^{N} (1 - Z_i) L(Nq_i, 1) \) subject to \( \sum_{i=1}^{N} (1 - Z_i) q_i u_{K_1}(X_i) = \frac{1}{N} \sum_{i=1}^{N} u_{K_1}(X_i) \). 

(A.5)

Because the primal problems (A.4) and (A.5) are convex separable programs with linear constraints, Tseng and Bertsekas (1987) showed that the dual problems are unconstrained convex maximization problems that can be solved by numerical efficient and stable algorithms.

We show the dual of (A.4) is the unconstrained optimization (3.8) by using the methodology introduced in Tseng and Bertsekas (1987). Let \( g(v) = L(1 - v, 1), g'(v) = \frac{\partial g(v)}{\partial v}, E_{K_1 \times N} \triangleq (u_{K_1}(X_1), \ldots, u_{K_1}(X_N)), s_i \triangleq 1 - Z_i N p_i, i = 1, \ldots, N, \) and \( s \triangleq (s_1, \ldots, s_N)^\top \), then we can rewrite the problem (A.4) as

\[
\min_{s} \sum_{i=1}^{N} Z_i g(s_i) \quad \text{subject to} \quad E_{K_1 \times N} \cdot s = 0.
\]

For every \( j \in \{1, \ldots, N\} \), we define the conjugate convex function (Tseng and Bertsekas, 1987) of \( Z_j g(\cdot) \) to be

\[
l_j(u_j) = \sup_{s_j} \{ u_j s_j - Z_j g(s_j) \} = \sup_{p_j} \{-Z_j N p_j u_j + u_j - Z_j g(1 - Z_j N p_j) \} \\
= \sup_{p_j} \{-Z_j N p_j u_j + u_j - Z_j g(1 - N p_j) \} \\
= -Z_j N p_j^* u_j + u_j - Z_j g(1 - N p_j^*),
\]

where the third equality follows by \( Z g(1 - Z N p_j) = Z g(1 - N p_j) \), and \( p_j^* \) satisfies the first order condition:

\[-Z_j u_j = -Z_j g'(1 - N p_j^*) \Leftrightarrow p_j^* = \frac{1}{N} \left( 1 - (g')^{-1}(u_j) \right); \]

then we can have

\[
l_j(u_j) = -Z_j u_j \left( 1 - (g')^{-1}(u_j) \right) + u_j - Z_j g \left( (g')^{-1}(u_j) \right) \\
= -Z_j \left( g \left( (g')^{-1}(u_j) \right) + u_j - u_j (g')^{-1}(u_j) \right) + u_j \\
= -Z_j \rho(u_j) + u_j,
\]

where

\[
\rho(u) \triangleq g \left( (g')^{-1}(u) \right) + u - u (g')^{-1}(u).
\]

By Tseng and Bertsekas (1987), the dual problem of (A.4) is

\[
\min_{\lambda} \sum_{j=1}^{N} l_j(\lambda^\top E_j) = \min_{\lambda} \sum_{j=1}^{N} l_j(\lambda^\top u_{K_1}(X_j))
\]
\[ = \min_{\lambda} \sum_{j=1}^{N} \left\{ -Z_j \rho \left( \lambda^T u_{K_1}(X_j) \right) + \lambda^T u_{K_1}(X_j) \right\} \]

\[ = - \max_{\lambda} \sum_{j=1}^{N} \left\{ Z_j \rho \left( \lambda^T u_{K_1}(X_j) \right) - \lambda^T u_{K_1}(X_j) \right\} \]

\[ = - \max_{\lambda} \hat{G}(\lambda) , \]

where \( E_j \) is the \( j \)-th column of \( E_{K_1 \times N} \), i.e., \( E_j = u_{K_1}(X_j) \), which is our formulation (3.8).

Since \( L(\cdot) \) is strictly convex, i.e., \( L''(v) > 0 \), and \( g''(v) = L''(1 - v) \), then \( g(\cdot) \) is also strictly convex and \( g'(\cdot) \) is strictly increasing. Note that

\[ \rho(v) = g((g^{-1}(v)) + v - v(g^{-1}(v)) \Leftrightarrow \rho(g'(v)) = g(v) + g'(v) - vg'(v) . \]

Differentiating \( v \) on both sides in above equation yields:

\[ \rho'(g'(v)) g''(v) = g'(v) + g''(v) - g'(v) - vg''(v) = (1 - v)g''(v) . \]

Since \( g''(v) > 0 \), we can have

\[ \rho'(g'(v)) = 1 - v , \]

then we differentiate \( v \) on both sides to get \( \rho''(g'(v)) g''(v) = -1 \), which implies

\[ \rho''(v) = -\frac{1}{g''((g^{-1}(v))} < 0 . \]

Therefore, the convexity of \( L(\cdot) \) is equivalent to the concavity of \( \rho(\cdot) \).

### A.3 Convergence Rates of Estimated Weights

The following result ensures the consistency of \( N \hat{p}(X) \), \( N \hat{q}(X) \) and \( \hat{\delta}^D(X) \) as well as their convergence rates. The proof is presented in Section 2 of the supplemental material.

**Proposition A.1.** Under Assumptions 3.2-3.6, we have

\[ \sup_{x \in \mathcal{X}} |N \hat{p}(x) - f_Z|_{X} (1|x)^{-1}| = O_p \left( \zeta(K)K^{-\alpha} + \zeta(K)\sqrt{\frac{K}{N}} \right) , \]

\[ \int_{\mathcal{X}} |N \hat{p}(x) - f_Z|_{X} (1|x)^{-1}|^2 dF_X(x) = O_p \left( K^{-2\alpha} + \frac{K}{N} \right) , \]

\[ \frac{1}{N} \sum_{i=1}^{N} |N \hat{p}(X_i) - f_Z|_{X} (1|X_i)^{-1}|^2 = O_p \left( K^{-2\alpha} + \frac{K}{N} \right) , \]
and

\[
\sup_{x \in \mathcal{X}} |N \hat{q}(x) - f_{Z|X}(0|x)^{-1}| = O_p \left( \zeta(K) K^{-\alpha} + \zeta(K) \sqrt{\frac{K}{N}} \right),
\]
\[
\int_{\mathcal{X}} |N \hat{q}(x) - f_{Z|X}(0|x)^{-1}|^2 dF_X(x) = O_p \left( K^{-2\alpha} + \frac{K}{N} \right),
\]
\[
\frac{1}{N} \sum_{i=1}^{N} |N \hat{q}(X_i) - f_{Z|X}(0|X_i)^{-1}|^2 = O_p \left( K^{-2\alpha} + \frac{K}{N} \right),
\]

and

\[
\sup_{x \in \mathcal{X}} |\hat{\delta}^D(x) - \delta^D(x)| = O_p \left( \zeta(K) K^{-\alpha} + \zeta(K) \sqrt{\frac{K}{N}} \right),
\]
\[
\int_{\mathcal{X}} |\hat{\delta}^D(x) - \delta^D(x)|^2 dF_X(x) = O_p \left( K^{-2\alpha} + \frac{K}{N} \right),
\]
\[
\frac{1}{N} \sum_{i=1}^{N} |\hat{\delta}^D(X_i) - \delta^D(X_i)|^2 = O_p \left( K^{-2\alpha} + \frac{K}{N} \right).
\]

A.4 Sketched Proof of Theorem 3.7

The detailed proof of Theorem 3.7 is given in the supplementary material. Here we present the outline of whole the proof. By Assumption 3.5, \( K_1 \asymp K_2 \asymp K \), without loss of generality, we assume that \( K_1 = K_2 = K \). We introduce the following notation: let \( G^*(\lambda), \lambda_K^* \) and \( p^*(X) \) be the theoretical counterparts of \( \hat{G}(\lambda), \hat{\lambda}_K \) and \( \hat{p}(X) \) defined by

\[
G^*(\lambda) = \mathbb{E}[\hat{G}_K(\lambda)] = \mathbb{E}\left[ Z' \left( \lambda^\top u_K(X) \right) \right] - \lambda^\top u_K(X),
\]
\[
\lambda_K^* = \arg \max G^*(\lambda), \quad p^*(X) = \frac{1}{N} \rho'((\lambda_K^*)^\top u_K(X)).
\]

We also introduce the following notation:

\[
p_Y^Y(X) = \mathbb{E}[Y|Z = 1, X], \quad p_Y^0(X) = \mathbb{E}[Y|Z = 0, X], \quad \delta^Y(X) = p_Y^Y(X) - p_Y^0(X),
\]
\[
\hat{\Psi}_K = - \int_{\mathcal{X}} \frac{p_Y^Y(x)}{\delta^D(x)} f_{Z|X}(1|x) \rho''(\lambda_K^* u_K(x)) u_K(x) dF_X(x),
\]
\[
\Psi_K = - \int_{\mathcal{X}} \frac{p_Y^Y(x)}{\delta^D(x)} f_{Z|X}(1|x) \rho''((\lambda_K^*)^\top u_K(x)) u_K(x) dF_X(x),
\]
\[
\hat{\Sigma}_K = \frac{1}{N} \sum_{i=1}^{N} Z_i \rho''(\lambda_K^* u_K(X_i)) u_K(X_i) u_K(X_i)^\top,
\]
\[
\Sigma_K = - \mathbb{E}\left[ f_{Z|X}(1|X) \rho''((\lambda_K^*)^\top u_K(X)) u_K(X) u_K(X)^\top \right],
\]
\[
\hat{Q}_K(X) = \hat{\Psi}_K \hat{\Sigma}_K^{-1} u_K(X), \quad Q_K(X) = \Psi_K \Sigma_K^{-1} u_K(X).
\]
where $\hat{\lambda}_K$ lies on the line joining $\lambda_K$ and $\lambda_K'$. Note that $Q_K(X)$ is the weighted $L^2$ projection of $-p_1^X(X)/\delta^D(X)$ on the space linearly spanned by $u_K(X)$. Note that

$$\sqrt{N}(\hat{\tau} - \tau) = \sqrt{N} \sum_{i=1}^{N} Z_i \hat{p}(X_i)Y_i/\hat{\delta}^D(X_i) - \sqrt{N} \sum_{i=1}^{N} (1 - Z_i)\hat{q}(X_i)Y_i/\hat{\delta}^D(X_i).$$

We first derive the influence function of $\sqrt{N} \sum_{i=1}^{N} Z_i \hat{p}(X_i)Y_i/\hat{\delta}^D(X_i)$, and similarly obtain that of $\sqrt{N} \sum_{i=1}^{N} (1 - Z_i)\hat{q}(X_i)Y_i/\hat{\delta}^D(X_i)$. We can decompose $\sqrt{N} \sum_{i=1}^{N} Z_i \hat{p}(X_i)Y_i/\hat{\delta}^D(X_i)$ as follows:

$$\sqrt{N} \sum_{i=1}^{N} Z_i \hat{p}(X_i)Y_i/\hat{\delta}^D(X_i)$$

$$= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{Z_i}{\hat{\delta}^D(X_i)} \{N\hat{p}(X_i) - Np^*(X_i)\} Y_i - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{Z_i}{\hat{\delta}^D(X_i)} \{N\hat{p}(X_i) - Np^*(X_i)\} Y_i \quad (A.6)$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{Z_i}{\hat{\delta}^D(X_i)} Np^*(X_i)Y_i - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{Z_i}{\hat{\delta}^D(X_i)} Np^*(X_i)Y_i \quad (A.7)$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{Z_i}{\hat{\delta}^D(X_i)} (N\hat{p}(X_i) - Np^*(X_i)) Y_i - \int_{X} p_1^Y(x) f_{Z|X}(1|x) \frac{\delta^D(X_i)}{\hat{\delta}^D(x)} (N\hat{p}(X) - Np^*(X))dF_X(x) \right\} \quad (A.8)$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \left( Np^*(X_i) - \frac{1}{f_{Z|X}(1|X_i)} \right) Z_i Y_i - E \left[ p_1^Y(X) \frac{\delta^D(X_i)}{\hat{\delta}^D(X)} f_{Z|X}(1|X) \left( Np^*(X) - \frac{1}{f_{Z|X}(1|X)} \right) \right] \right\} \quad (A.9)$$

$$+ \sqrt{N} E \left[ p_1^Y(X) \frac{\delta^D(X)}{\hat{\delta}^D(X)} f_{Z|X}(1|X) \left( Np^*(X) - \frac{1}{f_{Z|X}(1|X)} \right) \right] \quad (A.10)$$

$$+ \sqrt{N} \int_{X} p_1^Y(x) f_{Z|X}(1|x) (N\hat{p}(X) - Np^*(X))dF_X(x) - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} [Z_i \rho'((\lambda_K)^T u_K(X_i)) - 1]Q_K(X_i) \quad (A.11)$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} [Z_i \rho'((\lambda_K)^T u_K(X_i)) - 1]Q_K(X_i) - Q_K(X_i) \quad (A.12)$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ [Z_i \rho'((\lambda_K)^T u_K(X_i)) - 1]Q_K(X_i) + \frac{p_1^Y(X_i)}{\delta^D(X_i)} \left( \frac{Z_i}{f_{Z|X}(1|X_i)} - 1 \right) \right\} \quad (A.13)$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{Z_i Y_i}{f_{Z|X}(1|X_i)\delta^D(X_i)} - \frac{p_1^Y(X_i)}{\delta^D(X_i)} \left( \frac{Z_i}{f_{Z|X}(1|X_i)} - 1 \right) \right\} \quad (A.14)$$

The following lemmas are proved in the supplemental material.

**Lemma A.2.** Under Assumptions 3.1-3.6, the terms (A.6) (A.8), (A.9), (A.10), (A.11), (A.12) and (A.13) are of $o_p(1)$.
Lemma A.3. Under Assumptions 3.1-3.6, (A.7) has the following equivalent linear expression:

\[
(A.7) = -\frac{1}{\sqrt{N}} \sum_{i=1}^{N} D_i \cdot \frac{2Z_i - 1}{f_{Z|X}(Z_i|X_i)} \cdot \frac{p_Y(X_i)}{\delta^D(X_i)^2} + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{2Z_i - 1}{\delta^D(X_i)^2} \cdot \frac{\mathbb{E}[D_i|Z_i, X_i]}{f_{Z|X}(Z_i|X_i)} p_Y(X_i) + o_p(1) .
\]

By Lemmas A.2 and A.3, we can obtain that

\[
\sqrt{N} \sum_{i=1}^{N} Z_i \hat{p}(X_i) Y_i / \hat{\delta}^D(X_i)
\]

\[
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{Z_i Y_i}{f_{Z|X}(1|X_i) \delta^D(X_i)} - \frac{p_Y(X_i)}{\delta^D(X_i)} \left( \frac{Z_i}{f_{Z|X}(1|X_i)} - 1 \right) \right\}
\]

\[- \frac{1}{\sqrt{N}} \sum_{i=1}^{N} D_i \cdot \frac{2Z_i - 1}{f_{Z|X}(Z_i|X_i)} \cdot \frac{p_Y(X_i)}{\delta^D(X_i)^2} + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{2Z_i - 1}{\delta^D(X_i)^2} \cdot \frac{\mathbb{E}[D_i|Z_i, X_i]}{f_{Z|X}(Z_i|X_i)} p_Y(X_i) + o_p(1) .
\]

Symmetrically, we have

\[
\sqrt{N} \sum_{i=1}^{N} (1 - Z_i) \hat{q}(X_i) Y_i / \hat{\delta}^D(X_i)
\]

\[
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \frac{(1 - Z_i) Y_i}{f_{Z|X}(0|X_i) \delta^D(X_i)} - \frac{p_Y(X_i)}{\delta^D(X_i)} \left( \frac{1 - Z_i}{f_{Z|X}(0|X_i)} - 1 \right) \right\}
\]

\[- \frac{1}{\sqrt{N}} \sum_{i=1}^{N} D_i \cdot \frac{2Z_i - 1}{f_{Z|X}(Z_i|X_i)} \cdot \frac{p_Y(X_i)}{\delta^D(X_i)^2} + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{2Z_i - 1}{\delta^D(X_i)^2} \cdot \frac{\mathbb{E}[D_i|Z_i, X_i]}{f_{Z|X}(Z_i|X_i)} p_Y(X_i) + o_p(1) .
\]

Therefore,

\[
\sqrt{N}(\hat{\tau} - \tau) = \sqrt{N} \sum_{i=1}^{N} \left\{ Z_i \hat{p}(X_i) Y_i - (1 - Z_i) \hat{q}(X_i) Y_i - \tau \right\}
\]

\[
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[ \frac{2Z_i - 1}{\delta^D(X_i)f_{Z|X}(Z_i|X_i)} Y_i - \tau \right]
\]

\[- \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{p_Y(X_i)}{\delta^D(X)} \left\{ \frac{Z_i}{f_{Z|X}(1|X_i)} - 1 \right\}
\]

\[+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{p_Y(X_i)}{\delta^D(X)} \left\{ \frac{1 - Z_i}{f_{Z|X}(0|X_i)} - 1 \right\}
\]

\[- \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \delta(X_i) \left\{ \frac{2Z_i - 1}{f_{Z|X}(Z_i|X_i)} \frac{D_i}{\delta^D(X_i)} - \frac{2Z_i - 1}{f_{Z|X}(Z_i|X_i)} \frac{\mathbb{E}[D_i|Z_i, X_i]}{\delta^D(X_i)} \right\} + o_p(1) \quad \text{[since } \delta(X) = \frac{\delta^Y(X)}{\delta^D(X)} \text{]}
\]

\[
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varphi_{eff}(D_i, Z_i, X_i, Y_i) + o_p(1)
\]
where
\[
\varphi_{eff}(D_i, Z_i, X_i, Y_i) = \frac{2Z_i - 1}{f_{Z|X}(Z_i|X_i)} \frac{1}{\delta^D(X_i)} \left\{ Y_i - D_i \delta(X_i) - \mathbb{E}[Y_i|Z_i = 0, X_i] + \mathbb{E}[D_i|Z_i = 0, X_i] \delta(X_i) \right\} + \delta(X_i) - \tau ,
\]
is the efficient influence function given in Wang and Tchetgen Tchetgen (2017).