NONLINEAR STABILITY OF THIN, RADIALLY STRATIFIED DISKS

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ABSTRACT

We perform local numerical experiments to investigate the nonlinear stability of thin, radially stratified disks. We demonstrate the presence of radial convective instability when the disk is nearly in uniform rotation and show that the net angular momentum transport is strictly inward, consistent with previous investigations of vertical convection. We then show that a convectively unstable equilibrium is stabilized by differential rotation. Convective instability is determined by the Richardson number $R_i \equiv N_r^2/(q\Omega)^2$, where $N_r$ is the radial Brunt-Väisälä frequency and $q\Omega$ is the shear rate. Classical convective instability in a nonshearing medium ($R_i \to -\infty$) is suppressed when $R_i \gtrsim -1$; i.e., when the shear rate becomes greater than the growth rate. Disks with a nearly Keplerian rotation profile and radial gradients on the order of the disk radius have $R_i \gtrsim -0.01$ and are therefore stable to local nonaxisymmetric disturbances. One implication of our results is that the “baroclinic” instability recently claimed by Klahr & Bodenheimer is either global or nonexistent. We estimate that our simulations would detect any genuine growth rate $\gtrsim 0.0025\Omega$.

Subject headings: accretion, accretion disks — galaxies: nuclei — solar system: formation

Online material: color figures

1. INTRODUCTION

In order for astrophysical disks to accrete, angular momentum must be removed from the disk material and transported outward. In many disks, this outward angular momentum transport is likely mediated internally by magneto-hydrodynamic (MHD) turbulence driven by the magnetorotational instability (MRI; see Balbus & Hawley 1998). A key feature of this transport mechanism is that it arises from a local shear instability and is therefore very robust. In addition, MHD turbulence transports angular momentum outward; some other forms of turbulence, such as convective turbulence, appear to transport angular momentum inward (Stone & Balbus 1996; Cabot 1996). The mechanism is only effective, however, if the plasma in the disk is sufficiently ionized to be well coupled to the magnetic field (Blaes & Balbus 1994; Hawley et al. 1995; Gammie 1996; Jin 1996; Wardle 1999; Fleming et al. 2000; Balbus & Terquem 2001; Sano & Stone 2002a, 2002b; Salmeron & Wardle 2003; Salmeron 2004; Kunz & Balbus 2004; Desch 2004). In portions of disks around young, low-mass stars, in cataclysmic variable disks in quiescence, and in X-ray transients in quiescence (Stone et al. 2000; Gammie & Menou 1998; Menou 2000), the plasma may be too neutral for the MRI to operate. This presents some difficulties for understanding the evolution of these systems, since no robust transport mechanism akin to MRI-induced turbulence has been established for purely hydrodynamic Keplerian shear flows.

Such a mechanism has been claimed recently by Klahr & Bodenheimer (2003), who find vortices and an outward transport of angular momentum in the nonlinear outcome of their global simulations. The claim is that this nonlinear outcome is due to a local instability (the “global baroclinic instability”) resulting from the presence of an equilibrium entropy gradient in the radial direction. The instability mechanism invoked (Klahr 2004) is an interplay between transient amplification of linear disturbances and nonlinear effects. The existence of such a mechanism would have profound implications for understanding the evolution of weakly ionized disks.

In a companion paper (Johnson & Gammie 2005, hereafter Paper I), we have performed a linear stability analysis for local nonaxisymmetric disturbances in the shearing-wave formalism. While the linear theory uncovers no exponentially growing instability (except for convective instability in the absence of shear), interpretation of the results is somewhat difficult due to the non-normal nature of the linear differential operators: one has a coupled set of differential equations in time rather than a dispersion relation, which results in a nontrivial time dependence for the perturbation amplitudes $\delta(t)$. In addition, transient amplification does occur for a subset of initial perturbations, and linear theory cannot tell us what effect this will have on the nonlinear outcome. For these reasons, and in order to test for the presence of local nonlinear instabilities, we here supplement our linear analysis with local numerical experiments.

We begin in \S 2 by outlining the basic equations for a local model of a thin disk. In \S 3 we summarize the linear theory results from Paper I. We describe our numerical model and nonlinear results in \S\S 4 and 5 and discuss the implications of our findings in \S 6.

2. BASIC EQUATIONS

The most relevant results from Klahr & Bodenheimer (2003) are those that came out of their two-dimensional (radial-azimuthal) simulations, since the salient feature supposedly giving rise to the instability is a radial entropy gradient. The simplest model to use for a local verification of their global results is the two-dimensional shearing sheet (see, e.g., Goldreich & Tremaine 1978). This local approximation is made by expanding the equations of motion in the ratio of the disk scale height $H$ to the local radius $R$, and is therefore only valid for thin disks ($H/R \ll 1$). The vertical
structure is removed by using vertically integrated quantities for the fluid variables.\(^3\) The basic equations that one obtains are
\[
\frac{d\Sigma}{dt} + \Sigma \nabla \cdot \mathbf{v} = 0, 
\]
\[
\frac{d\mathbf{v}}{dt} + \nabla P - \frac{\nabla P}{\Sigma} + 2\Omega \times \mathbf{v} - 2q\Omega^2 x \hat{x} = 0, 
\]
\[
\frac{d \ln S}{dt} = 0,
\]
where \(\Sigma\) and \(P\) are the two-dimensional density and pressure, \(S = P\Sigma^{-1}\) is the fluid entropy,\(^4\) \(\mathbf{v}\) is the fluid velocity, and \(d/dt\) is the Lagrangian derivative. The third and fourth terms in equation (2) represent the Coriolis and centrifugal forces in the local model expansion, where \(\Omega\) is the local rotation frequency, \(x\) is the radial Cartesian coordinate, and \(q\) is the shear parameter (equal to 1.5 for a disk with a Keplerian rotation profile). The gravitational potential of the central object is included as part of the centrifugal force term in the local-model expansion, and we ignore the self-gravity of the disk.

3. SUMMARY OF LINEAR THEORY RESULTS

An equilibrium solution to equations (1)–(3) is
\[
P = P_0(x), 
\]
\[
\Sigma = \Sigma_0(x), 
\]
\[
\mathbf{v} \equiv \mathbf{v}_0 = \left(-q\Omega x + \frac{P_0'}{2\Omega \Sigma_0}\right) \hat{y}, 
\]
where a prime denotes an \(x\)-derivative. One can regard the background flow as providing an effective shear rate
\[
\tilde{q}\Omega \equiv -v'_0
\]
that varies with \(x\), in which case \(v_0 = -\int_{r_0}^{r} \tilde{q}(s) \, ds\). Due to this background shear, localized disturbances can be decomposed in terms of “shwaves,” which are Fourier modes in a frame comoving with the shear. These have a time-dependent radial wave-number given by
\[
\tilde{k}_r(t, x) \equiv k_{x0} + \tilde{q}(x)\Omega k_y t, 
\]
where \(k_{x0}\) and \(k_y\) are constants. Here \(k_y\) is the azimuthal wave number of the shwave.

In the limit of zero stratification,
\[
P_0(x) \to \text{constant}, 
\]
\[
\Sigma_0(x) \to \text{constant}, 
\]
\[
\mathbf{v}_0 \to -q\Omega x \hat{y}, 
\]
\[
\tilde{q} \to q, 
\]
\[
\tilde{k}_r \to k_r \equiv k_{x0} + q\Omega k_y t. 
\]
In Paper I, we analyze the time dependence of the shwave amplitudes for both an unstratified equilibrium and a radially stratified equilibrium. As discussed in more detail in Paper I, applying the shwave formalism to a radially stratified shearing sheet effectively uses a short-wavelength WKB approximation and is therefore only valid in the limit \(k_r L \gg 1\), where the background varies on a scale of \(L \sim H \ll R\). The disk scale height \(H \equiv c_s \Omega\), where \(c_s = (\gamma P_0/\Sigma_0)^{1/2}\).

There are three nontrivial shwave solutions in the unstratified shearing sheet, two nonvortical and one vortical. The radial stratification gives rise to an additional vortical shwave. In the limit of tightly wound shwaves (\(|k_r| \gg k_y\)), the nonvortical and vortical shwaves are compressive and incompressive, respectively. The former are the extension of acoustic modes to nonaxisymmetric, and, to leading order in \((k_r L)^{-1}\), they are the same both with and without stratification. Since the focus of our investigation is on convective instability and the generation of vorticity, we repeat here only the solutions for the incompressive vortical shwaves and refer the reader to Paper I for further details on the nonvortical shwaves.

In the unstratified shearing sheet, the solution for the incompressive shwave is given by
\[
\delta v_x = \delta v_{x0} \frac{k_r^2}{k_y^2}, 
\]
\[
\delta v_y = -\frac{k_y}{k_x} \delta v_{x0}, 
\]
\[
\frac{\delta \Sigma}{\Sigma_0} = \frac{\delta P}{\gamma P_0} = \frac{1}{\epsilon_x k_x} \left[ k_r \frac{\delta v_{x0} k_r^2}{k_y^2} + 2(q-1)\Omega k_y \frac{\delta v_{x0} k_r^2}{k_y^2} \right], 
\]
where \(k^2 = k_x^2 + k_y^2\), \(k_{x0}\) and \(\delta v_{x0}\) are the values of \(k\) and \(\delta v_x\) at \(t = 0\), and an overdot denotes a time derivative.\(^5\)

The kinetic energy for a single incompressive shwave can be defined as
\[
E_k \equiv \frac{1}{2} \Sigma_0 (\delta v_x^2 + \delta v_y^2) = \frac{1}{2} \Sigma_0 \delta v_{x0}^2 \frac{k_{x0}^2}{k_y^2}, 
\]

an expression that varies with time and peaks at \(k_r = 0\). If one defines an amplification factor for an individual shwave,
\[
\mathcal{A} \equiv \frac{E_k(k_r = 0)}{E_k(t = 0)} = 1 + \frac{k_{x0}^2}{k_y^2}, 
\]
it is apparent that an arbitrary amount of transient amplification in the kinetic energy of an individual shwave can be obtained as one increases the amount of swing for a leading shwave \((k_r \ll -k_y)\).

This transient amplification of local nonaxisymmetric disturbances is reminiscent of the “swing amplification” mechanism that occurs in disks that are marginally stable to the axisymmetric gravitational instability (Goldreich & Lynden-Bell 1965; Julian & Toomre 1966; Goldreich & Tremaine 1978). In that context, nonaxisymmetric shwaves experience a short period of exponential growth near \(k_y = 0\) as they swing from leading to trailing. In order for this mechanism to be effective in destabilizing a disk, however, a feedback mechanism is required to convert trailing shwaves into leading shwaves (see, e.g., Binney & Tremaine 1987 in the context of shearing waves in galactic disks). The arbitrarily large amplification implied by equation (18) has led some authors to argue for a bypass transition to turbulence in hydrodynamic Keplerian shear flows (Chagelishvili et al. 2003; Umurhan & Regev 2004).

\(^3\) This vertical integration is not rigorous.

\(^4\) For a non–self-gravitating disk, the two-dimensional adiabatic index \(\gamma = (3\gamma_{3D} - 1)/(\gamma_{3D} + 1)\) (e.g., Goldreich et al. 1986).

\(^5\) As discussed in Paper I, this solution is valid for all time only in the short-wavelength limit \((k_r H \gg 1)\); for \(H_k \lesssim 0(1)\), an initially leading incompressive shwave turns into a compressive shwave near \(k_r = 0\).
Afshordi et al. 2005). The reasoning is that nonlinear effects somehow provide the necessary feedback. We show in Paper I that an ensemble of incompressive shwaves drawn from an isotropic, Gaussian random field has a kinetic energy that is a constant, independent of time. This indicates that a random set of vortical perturbations will not extract energy from the mean shear. It is clear, however, that the validity of this mechanism as a transition to turbulence can only be fully explored with a nonlinear analysis or numerical experiment. To date, no such study has demonstrated a transition to turbulence in Keplerian shear flows.

In the presence of radial stratification, there are two linearly independent incompressive shwaves. The radial velocity perturbation satisfies the following equation (we use a subscript \(s\) to distinguish the stratified from the unstratified case):

\[
(1 + \tilde{\tau}^2) \frac{d^2 \delta v_{r}}{d \tilde{\tau}^2} + 4\tilde{\tau} \frac{d \delta v_{r}}{d \tilde{\tau}} + (\text{Ri} + 2) \delta v_{r} = 0, \tag{19}
\]

where

\[
\tilde{\tau} \equiv \tilde{k}_r / k_y = \frac{\tilde{q}}{\Omega} \Omega t + k_{s} / k_y \quad \tag{20}
\]

is the time variable and

\[
\text{Ri} \equiv \frac{N^2}{(\tilde{q} \Omega)^2} \tag{21}
\]

is the Richardson number, a measure of the relative importance of buoyancy and shear (Miles 1961; Howard 1961; Chimonas 1970). Here

\[
N^2 \equiv - \frac{c^2}{L_3 L_P} \tag{22}
\]

is the square of the local Brunt-Väisälä frequency in the radial direction, where \(L_P \equiv \gamma P_0 / P_0\) and \(L_S \equiv \gamma S_0 / S_0\) are the equilibrium pressure and entropy length scales in the radial direction.

As discussed in Paper I, eq. (19) is the same equation that one obtains for the incompressive shwaves in a shearing, stratified atmosphere.

| Run | Description | \(L_x (L_y = 12)\) | \(N_y\) | \(N_x\) | Figure(s) |
|-----|-------------|------------------|-------|-------|-----------|
| 1   | Linear theory test | 0.375 | 16, 32, 64 | 64\(N_x\) | 1         |
| 2   | External potential, \(\Omega = 0\) | 12 | 512 | \(N_y\) | 2–4       |
| 3   | External potential, \(\Omega = 1\) | 12 | 512 | \(N_y\) | 5–6       |
| 4   | Uniform rotation (\(q = 0\)) | 12, 6, 3, 1.5 | 512 | \(N_y\) | 7–9       |
| 5   | External potential, \(\Omega = 1\), boost | 12 | 128, 256, 512 | \(N_y\) | 10        |
| 6   | Small shear (\(-\infty \leq \text{Ri} \leq -1\)) | 12 | 128, 256, 512 | \(N_y\) | 11        |
| 7   | Small shear (\(q = 0.2, \text{Ri} \lesssim -1\)) | 12 | 128, 256, 512 | \(N_y\) | 12        |
| 8   | Aliasing (\(q = 0.2, \text{Ri} \lesssim -1\)) | 12 | 128, 256, 512 | \(N_y\) | 13        |
| 9   | Parameter survey | 12 | 512 | \(N_y\) | 14        |
| 10  | Keplerian disk (\(q = 1.5, \text{Ri} \approx -0.004\)) | 12 | 512 | \(N_y\) | 15        |
The solutions for the other perturbation variables are related to $\delta v_{ys}$ by

$$
\frac{\delta \Sigma_s}{\Sigma_0} = \frac{1}{L_s} \int \delta v_{ys} \, dt,
$$

$$
\delta P_s = \frac{\gamma \Omega}{k_y c_s} \left[ \tilde{\alpha} \frac{d}{d\tilde{\tau}} \left( \frac{\delta v_{ys}}{c_s} \right) + 2(\bar{q} - 1) \frac{\delta v_{ys}}{c_s} \right].
$$

Since the solutions to equation (19) are hypergeometric functions, which have a power-law time dependence, it cannot in
general be accurately treated with a WKB analysis; there is no asymptotic region in time where equation (19) can be reduced to
a dispersion relation. If, however, there is a region of the disk where the effective shear is zero, $\tilde{\alpha} \rightarrow$ constant and equation (19)
can be expressed as a WKB dispersion relation:

$$
\omega^2 = \frac{k_y^2}{k_y^2 + k_{y,0}^2} N_x^2,
$$

with $\delta(t) \propto \exp(-i\omega t)$. For $\bar{q} \approx 0$ and $N_x^2 < 0$, then, there is
convective instability. For disks with nearly Keplerian rotation profiles and modest radial gradients, $\bar{q} \approx 1.5$, and one would
expect that the instability is suppressed by the strong shear. Due
to the lack of a dispersion relation, however, there is no clear
cutoff between exponential and oscillatory time dependence, and
establishing a rigorous analytic stability criterion is difficult.

For $\bar{q} \neq 0$, the asymptotic time dependence for each pertur-
bation variable at late times is

$$
\delta P_s \propto \delta v_{ys} \propto t^{(\alpha-1)/2},
$$

$$
\delta \Sigma_s \propto \delta v_{ys} \propto t^{(\alpha-1)/2},
$$

where

$$
\alpha \equiv \sqrt{1 - 4Ri}.
$$

The density and $y$-velocity perturbations therefore grow asympto-
tically for $\alpha > 1$; i.e., $Ri < 0$. For small Richardson number,
as is expected for a nearly Keplerian disk with modest radial
gradients, $\alpha \approx 1 - 2Ri$ and the asymptotic growth is extremely
slow:

$$
\delta \Sigma_s \sim \delta v_{ys} \sim \tau^{-Ri}.
$$

The energy of an ensemble of shwaves grows asymptotically
as $t^{2\alpha-1}$, or $t^{1-4Ri}$ for small $Ri$. The ensemble energy growth is

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**Fig. 2.**—Evolution of $v_x$ as a function of time for run 2 (external potential, nonrotating frame).

**Fig. 3.**—Plot of $N_x^2$ (left) and $S$ (right) as a function of $x$ for run 2. Both are averaged over $y$. The dotted line shows the equilibrium profile, and the solid lines show snapshots during the nonlinear regime. Growth initially occurs at the minimum in $N_x^2$, and the entropy eventually settles to a nearly constant value.
thus nearly linear in time for small $Ri$, independent of the sign of $Ri$.7

The velocity perturbations are changed very little by a weak radial gradient. One would therefore expect that, at least in the linear regime, transient amplification of the kinetic energy for an individual shwave would be relatively unaffected by the presence of stratification. There is, however, an associated density perturbation in the stratified shearing sheet that is not present in the unstratified sheet.8 This results in transient amplification of the potential energy of an individual shwave. We do not derive in Paper I a general closed-form expression for the energy of an ensemble of incompressive shwaves in the stratified shearing sheet, so it is not entirely clear what effect this qualitatively new piece of the energy will have on an ensemble of shwaves in the linear regime.

Due to the subtleties involved in interpreting the transient amplification of linear disturbances, and since the linear theory

7 We show in Paper I that there is also linear growth in the energy of an ensemble of compressive shwaves.

8 The amplitude of the density perturbation in the unstratified sheet is an order of magnitude lower than the velocity perturbations in the short-wavelength limit; see eq. (16).
indicates weak power-law rather than exponential asymptotic growth in radially stratified disks, we expect that stability will be decided by higher order (nonlinear) interactions. This suggests a nonlinear, numerical study, which we describe here in the context of local numerical experiments in a radially stratified shearing sheet.

4. NUMERICAL MODEL

To investigate local nonlinear effects in a radially stratified thin disk, we integrate the governing equations (1)–(3) with a hydrodynamics code based on ZEUS (Stone & Norman 1992). This is a time-explicit, operator-split, finite-difference method on a staggered mesh. It uses an artificial viscosity to capture shocks. The computational grid is $L_x \times L_y$ in physical size with $N_x \times N_y$ grid cells, where $x$ is the radial coordinate and $y$ is the azimuthal coordinate. The boundary conditions are periodic in the $y$-direction and shearing-periodic in the $x$-direction. The shearing-box boundary conditions are described in detail in Hawley et al. (1995). As described in Masset (2000) and Gammie (2001), advection by the linear shear flow can be done by interpolation. Rather than using a linear interpolation scheme as in Gammie (2001), we now do the shear transport with the same upwind advection algorithm used in the rest of the code. This is less diffusive than linear interpolation, and the separation of the shear from the bulk fluid velocity means that one is not Courant-limited by large shear velocities at the edges of the computational domain.

We use the following equilibrium profile, which in general gives rise to an entropy that varies with radius:

$$h_0(x) = h_a \left[ 1 - \epsilon \cos \left( \frac{2\pi x}{L_x} \right) \right], \quad \Sigma_0(x) = \frac{h_0(\Gamma - 1)}{\Gamma K},$$

$$P_0(x) = K \Sigma_0^{\Gamma - 1},$$

where $h_a$, $\epsilon$, $\Gamma$, and $K$ are model parameters. The flow is maintained in equilibrium by setting the initial velocity according to equation (6). This equilibrium yields a Brunt-Väisälä frequency

$$N_c^2(x) = \frac{(\gamma - \Gamma)(h_a)^2}{\gamma(\Gamma - 1)h_0},$$

which can be made positive, negative, or zero by varying $\gamma - \Gamma$.

We fix some of the model parameters to yield an equilibrium profile that is appropriate for a thin disk. In particular, we want $H/L_p \sim H/R \ll 1$ in order to be consistent with our use of a razor-thin (two-dimensional) disk model. In addition, we want the equilibrium values for each fluid variable to be of the same order to ensure the applicability of our linear analysis. These requirements can be met by choosing $K = 1$, $\epsilon = 0.1$, $L_x = 12$, and $h_a = \tilde{c}_s^2 \Gamma/(\Gamma - 1)$, where $\tilde{c}_s \equiv (\langle P_0/\Sigma_0 \rangle)^{1/2} \equiv 1$ is (to within a factor of $\sqrt{\gamma}$) the $x$-average of the sound speed. Since the equilibrium profile changes with $\Gamma$, we choose a fixed value of $\Omega = 4/3$, which for $\gamma = \Gamma$ corresponds to a three-dimensional adiabatic index of $7/5$. These numbers yield $|H/L_p| \leq 0.2$. Our unfixed model parameters are thus $L_y$, $q$, and $\gamma$.

The sinusoidal equilibrium profile we are using generates radial oscillations in the shearing sheet, since the analytic equilibrium differs from the numerical equilibrium because of truncation error. We apply an exponential damping term to the governing equations in order to reduce the spurious oscillations and therefore get cleaner growth rate measurements. We damp the oscillations until their amplitude is equal to that of machine-level noise and subsequently apply low-level random perturbations to trigger any instabilities that may be present.

As a test for our code, we evolve a particular solution for the incompressive shwaves in the radially stratified shearing sheet (eqs. [19] and [23]–[25]). The initial conditions are $\delta v_r/\tilde{c}_s = 6\Sigma_0/\Sigma_0 = 1 \times 10^{-4}$ and $k_y = -128\pi/L_y$. We set $L_y = 0.375$ and $k_y = 2\pi/L_y$ in order to operate in the short-wavelength regime, and the other model parameters are $q = 1.5$ and $\gamma - \Gamma = -0.3102$. The latter value yields a minimum value for $N_c^2(x) \approx -0.01$. The results of the linear theory test are shown in Figure 1.

5. NONLINEAR RESULTS

Table 1 gives a summary of the runs that we have performed. A detailed description of the setup and results for each is given in the following subsections. Our primary diagnostic is a measurement of growth rates, and the probe that we use for these measurements is an average over azimuth of the absolute value of $\langle |v_r|/h_0 \rangle$, at the minimum in $N_c^2$. Measuring $v_r$ allows us to demonstrate the damping of the initial radial oscillations, and averaging over azimuth masks the interactions between multiple Fourier components with different growth rates in our measurements. We reference this probe with the following definition:

$$v_r \equiv \langle |v_r(x_{\text{min}}, y)| \rangle,$$

in which angle brackets denote an average over $y$, and $x_{\text{min}}$ is the $x$-value at which $N_c^2(x)$ is a minimum.

5.1. External Potential in a Nonrotating Frame

As a starting problem, we investigate a stratified flow with $v_0 = 0$. Such a flow can be maintained in equilibrium by replacing the tidal force in equation (2) with an external potential $\Phi = -h_0$. This can be done in either a rotating or nonrotating frame. It is a particularly simple way of validating our study of convective instability in the shearing sheet. The condition $v_0 = 0$ implies
\( \tilde{q} = 0 \), and therefore equation (26) should apply in the WKB limit, with the expected growth rate obtained by evaluating equation (32) locally.\(^9\) We have performed a fiducial run with an imposed external potential in a nonrotating frame (\( \Omega = 0 \) in eq. [2]) to compare with the outcome expected from the Schwarzschild stability criterion implied by equation (26). We set \( \gamma - \Gamma = -0.3102 \), corresponding to \( N_{x, \text{min}} = -0.01 \), and \( L_x = L_x \). The expected growth rate for this Schwarzschild-unstable equilibrium is 0.1 \( \text{in units of the average radial sound-crossing time} \). The numerical resolution for the fiducial run is 512 \( \times \) 512, and all the variables are randomly perturbed at an amplitude of \( \delta \Sigma/\Sigma_0 = \delta P/P_0 = \delta \omega/c_s = 1.0 \times 10^{-12} \).

A plot of \( \nu \) as a function of time is given in Figure 2, showing the initial damping followed by exponential growth in the linear regime. The analytic growth rate is shown on the plot for comparison. A least-squares fit of the data in the range \( 100 \leq \tau \leq 250 \) yields a measured growth rate of 0.0978.\(^{10}\) Figure 3 shows a cross-section of \( N_x^2 \) as a function of \( x \) after the instability has begun to set in, along with cross-sections of the entropy early and late in the nonlinear regime. The growth is initially concentrated

\(^9\) The fastest growing WKB modes are the ones with a growth rate corresponding to the minimum in \( N_x^2 \).

\(^{10}\) Measurements of the growth rate earlier in the linear regime or over a larger range of data yield results that differ from this value by at most 5%. 
near the minimum points in $N^2(x)$. Eventually the entropy turns over completely and settles to a nearly constant value. Figure 4 shows two-dimensional snapshots of the entropy in the nonlinear regime. Runs with the same equilibrium profile except $\gamma - \Gamma \geq 0$ are stable. There is also a long-wavelength axisymmetric instability that is present for $\gamma - \Gamma < 0$ even in the absence of the small-scale nonaxisymmetric modes. We measure its growth rate to be 0.07. Due to the long-wavelength nature of these modes, they are not treatable by a local linear analysis.

5.2. External Potential in a Rotating Frame

We have performed the same test as described in § 5.1 in a rotating frame ($\Omega = 1$ in eq. [2]). Figure 5 shows the exponential growth in the linear regime for this run, with a measured growth rate of 0.0977. Figure 6 shows snapshots of the entropy in the nonlinear regime. The results are similar to the nonrotating case, except that (1) rotation suppresses the long-wavelength axisymmetric instability, (2) the nonlinear outcome exhibits more coherent structures in the rotating case, including transient vortices, and (3) these coherent structures eventually become unstable to a Kelvin-Helmholtz-type instability.

5.3. Uniform Rotation

Having demonstrated the viability of simulating convective instability in the local model, we now turn to the physically realistic equilibrium described in § 4. We begin by setting the shear parameter $q$ to zero in order to make contact with the results of §§ 5.1 and 5.2. This is analogous to a disk in uniform rotation. The other model parameters are the same as for the previous runs. While there is still an effective shear $-0.05 \leq \tilde{q} \leq 0.05$, near $\tilde{q} = 0$ one expects the modes to obey equation (26) in the WKB limit. Figures 7 and 8 give the linear and nonlinear results for this run. The measured growth rate in the linear regime is 0.0809.

Consistent with results from numerical simulations of vertical convection (Stone & Balbus 1996; Cabot 1996), the angular momentum transport associated with radial convection is slightly inward. Figure 9 shows the evolution of the dimensionless angular momentum flux

$$\alpha \equiv \frac{1}{L_x L_y \langle P_0 \rangle} \int \Sigma \delta v_r \delta v_r \, dx \, dy,$$

(34)

where $\langle P_0 \rangle$ is the radial average of the equilibrium pressure, for an extended version of run 4. The thin line shows a version of the data that has been boxcar-smoothed to show the bias toward a negative flux.

There are two reasons for the larger error in the measured growth rate for this run: (1) the equilibrium velocity gives rise to numerical diffusion due to the motion of the fluid variables with respect to the grid, and (2) since the growing modes are being advected in the azimuthal direction, the maximum growth does not occur at the grid scale. The latter effect can be seen in Figure 8; several grid cells are required for a well-resolved wavelength. In order to resolve smaller wavelengths, we have repeated this run with $L_y = 6, 3,$ and 1.5. The results are plotted in Figure 7, along with the results from the $L_y = 12$ run. The measured growth rate for the $L_y = 1.5$ run is 0.0924.

To quantify the effects of numerical diffusion, we have performed a series of tests similar to run 2 (external potential in a rotating frame) but with an overall boost in the azimuthal direction. Figure 10 shows measured growth rates as a function of boost at three different numerical resolutions. The largest boost magnitude in this plot corresponds to the velocity at the minimum in $N^2$ for a run with $q = 1.5$. This highlights the importance of the fluid velocity with respect to the grid in determining numerical damping in ZEUS.

5.4. Shearing Sheet

To investigate the effect of differential rotation upon the growth of this instability, we have performed a series of simulations with nonzero values of $q$. Intuitively, one expects the instability to be suppressed when the shear rate is greater than the growth rate; i.e., for $\text{Re} \gtrsim -1$. Figure 11 shows growth rates from a series of runs with $N^2_{\text{min}} = -0.01$ and small, nonzero values of $q$ at three numerical resolutions. This figure clearly demonstrates our main result: convective instability is suppressed by differential rotation. The expected growth rate from linear theory $\langle (|N^2|)^{1/2} \rangle$ at $\tilde{q} = 0$ is shown in Figure 11 as a dotted line. If there is a radial position where $\tilde{q}(x) = 0$ (i.e., $\text{Re} = -\infty$), $v_r$ at that position looks similar to those in the previous runs (very little deviation from a straight line); these measurements are indicated on the plot with filled symbols. For $q \gtrsim 0.055$ there is no longer any point where $\tilde{q}(x) = 0$; in that case $v_r$ was measured at the radial average between the minimum in $N^2(x)$ and the minimum in $\tilde{q}(x)$, since this is where the maximum growth occurred. The data for these measurements, which are indicated in Figure 11 with open symbols, are not as clean as they are for the runs with $\text{Re} = -\infty$ (see Fig. 12). All of the growth rate measurements in Figure 11 were obtained by a least-squares fit of the data in the range $1 \times 10^{-9} < \nu_c \lesssim 1 \times 10^{-5}$. The dashed line in Figure 11 indicates the value of $q$ for which $\text{Re}_{\text{min}} = -1$.

Some of the growth in Figure 11 appears to be due to aliasing. This is a numerical effect in finite-difference codes that results in an artificial transfer of power from trailing shwaves into leading shwaves as the shwave is lost at the grid scale. One expects aliasing to occur approximately at intervals of

$$\Delta \tau = \frac{N_x}{n_y \, L_x},$$

(35)
where $n_y$ is the azimuthal wave number. This interval corresponds to $\Delta k_y(t) = 2\pi/dx$, where $dx = L_y/N_y$ is the radial grid scale. On the basis of expression (35), aliasing effects should be more pronounced at lower numerical resolution because the code has less time to evolve a wave before the wavelength of the wave becomes smaller than the grid scale. It can be seen from the far right data point in Figure 11 (run 7 in Table 1) that the measured growth rate decreases with increasing resolution. The evolution of $n_y$ for this run is shown in Figure 12.

The effects of aliasing can be seen explicitly by evolving a single wave, as was done for our linear theory test (Fig. 1). Figure 13 shows the evolution of the density perturbation for a single wave using the same parameters that were used for run 7: $L_x = L_y = 12$, $N_{x,\text{min}}^2 = -0.01$, and $q = 0.2$. The initial wave vector used was $(k_{x0}, k_y) = (-8\pi/L_x, 8\pi/L_y)$. This corresponds to $n_y = 4$, and the expected aliasing interval (eq. [35]) is therefore $\Delta \tilde{\tau} = N_x/4$. Runs at three numerical resolutions are plotted in Figure 13, and the aliasing interval at each resolution is consistent with expression (35). It is clear from Figure 13 that a lower resolution results in a larger overall growth at the end of the run. It also appears that the growth seen in Figure 13 requires a negative entropy gradient. We have performed this same test with $N_{x,\text{min}}^2 > 0$, and while aliasing occurs at the same interval, there is no overall growth in the perturbations. This may be due to the

Fig. 8.—Snapshots of the entropy in the nonlinear regime for run 4. Dark shades indicate values above (red in electronic edition) and below (blue in electronic edition) the average value. Note that the maximum growth does not occur for modes at the grid scale. [See the electronic edition of the Journal for a color version of this figure.]
fact that the perturbations decay asymptotically for $N_2^2 > 0$ (see eq. [30]).

Figure 14 summarizes the parameter space we have surveyed, indicating that there is instability only for $\tilde{\eta} \approx 0$ and $N_2^2 < 0$. The numerical resolution in all of these runs is $512 \times 512$. Figure 15 shows the evolution of the radial velocity in run 10, a run with realistic parameters for a disk with a nearly Keplerian rotation profile and radial gradients on the order of the disk radius: $q = 1.5$ and $N_2^2_{\text{min}} = -0.01$ (corresponding to $\text{Ri} \approx -0.004$). Clearly, no instability is occurring on a dynamical timescale. This plot is typical of all runs for which the evolution was stable. To give a sense for the minimum growth rate that we are able to measure, we have also plotted in Figure 15 the results from several unstable runs with $q = 0$ and a boost equivalent to the velocity at the minimum in $N_2^2$ for run 10. It is difficult to measure a growth rate for the smallest value of $N_2^2_{\text{min}}$, but it is clear that there is activity present in this run that does not occur in the stable run. On the
basis of Figure 15, a conservative estimate for the minimum growth rate that should be detectable in our simulations is $0.0025 \Omega$.

6. IMPLICATIONS

Our results are consistent with the idea that nearly Keplerian disks are stable, or at most very weakly unstable, to local nonaxisymmetric disturbances. Because our numerical resolution is finite, we cannot exclude the possibility of instability appearing at even higher resolution. But Figure 11 demonstrates that convective instabilities that are present when the shear is nearly zero are stabilized by differential rotation. Perturbations simply do not have time to grow before they are pulled apart by the shear.

An important implication of our results is that the instability claimed by Klahr & Bodenheimer (2003) is not a linear or nonlinear local nonaxisymmetric instability. Figure 13 suggests that the results of Klahr & Bodenheimer (2003) may be due, at least in part, to aliasing. They use a finite difference code at fairly low numerical resolution ($N_x = 128^2$), and growth on long timescales requires $N_x^2 < 0$. [See the electronic edition of the Journal for a color version of this figure.]

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