Quivers with relations for symmetrizable Cartan matrices II : Convolution algebras

C. Geiss, B. Leclerc, and J. Schröer

February 6, 2015

Abstract

We realize the enveloping algebra of the positive part of a symmetrizable Kac-Moody algebra as a convolution algebra of constructible functions on module varieties of some Iwanaga-Gorenstein algebras of dimension 1.

1 Introduction

Let $Q$ be a finite quiver without oriented cycles. Let $C$ be the symmetric generalized Cartan matrix corresponding to the undirected graph underlying $Q$, and let $g = g(C)$ be the Kac-Moody Lie algebra attached to $C$. Kac [K1] has shown that the dimension vectors of the indecomposable representations of $Q$ form the roots of the positive part $n$ of $g$. For quivers $Q$ of finite type, Ringel [Rin1, Rin2] found a direct construction of the Lie algebra $n$ itself, and of its enveloping algebra $U(n)$, in terms of the representation theory of $Q$. He used Hall polynomials counting extensions of representations over $\mathbb{F}_q$, and recovered $U(n)$ as the Hall algebra of the path algebra $\mathbb{F}_qQ$ specialized at $q = 1$. Later, Schofield [S] replaced counting points of varieties over $\mathbb{F}_q$ by taking Euler characteristic of complex varieties, and extended Ringel’s result to an arbitrary quiver (see also [Rie] in the finite-type case). Finally, Lusztig [L1] reformulated Schofield’s construction and obtained $U(n)$ as a convolution algebra of constructible functions over the affine spaces $\text{rep}(Q, d)$ of representations of $CQ$ with dimension vector $d$.

In this paper we prove a broad generalization of Schofield’s theorem. We take for $C$ an arbitrary symmetrizable generalized Cartan matrix $[K2]$ §1.1, §2.1. This means that there exists a diagonal matrix $D$ with positive integer diagonal entries such that $DC$ is symmetric. With this datum together with an orientation $\Omega$ of the graph naturally attached to $C$, we have associated in [GLS] a finite-dimensional algebra $H = H(C, D, \Omega)$ defined by a quiver with relations. This algebra makes sense over an arbitrary field $K$, but here we fix $K = \mathbb{C}$ so that varieties of $H$-modules are complex varieties. When $C$ is symmetric and $D$ is the unit matrix, $H$ is just the path algebra of the quiver $Q$ corresponding to $C$ and $\Omega$. In general, it is shown in [GLS] that $H$ is Iwanaga-Gorenstein of dimension 1, and that its category of locally free modules (i.e. modules of homological dimension $\leq 1$) carries an Euler form whose symmetrization is given by $DC$. The affine varieties $\text{rep}_{lf}(H, r)$ of locally free $H$-modules with rank vector $r$ are smooth and irreducible (Proposition [S1,L1]). By analogy with [S1,L1], we then introduce a convolution bialgebra $M = \mathcal{M}(H)$ of constructible functions on the varieties $\text{rep}_{lf}(H, r)$, We show that $M$ is a Hopf algebra isomorphic to the enveloping algebra of the Lie algebra of its primitive elements (Proposition [S3]). Let again $n$ denote the positive part of the symmetrizable Kac-Moody algebra $g$. Our main result is then:
Theorem 1.1. The Hopf algebra $\mathcal{M}(H)$ is isomorphic to $U(n)$.

This generalizes Schofield’s theorem in two directions. Firstly, to the best of our knowledge, Theorem 1.1 gives the first complex geometric construction of $U(n)$ for a symmetrizable generalized Cartan matrix. Secondly, note that if $D$ is a symmetrizer for $C$, then $kD$ is also a symmetrizer for every $k \in \mathbb{Z}_{>0}$. As $k$ increases, the categories of locally free modules over $H(C,kD,\Omega)$ become more and more rich and complicated, the dimension of the varieties $\text{rep}_{1,k}(H(C,kD,\Omega),r)$ increase and their orbit structure gets finer, but the convolution algebras $\mathcal{M}(H(C,kD,\Omega))$ remain the same. Thus for every symmetrizable Kac-Moody algebra $\mathfrak{g}$ we get an infinite series of exact categories $\text{rep}_{1,k}(H(C,kD,\Omega))$ ($k \geq 1$) whose convolution algebras $\mathcal{M}(H(C,kD,\Omega))$ are isomorphic to $U(n)$. This appears to be new, even when $C$ is symmetric.

Theorem 1.1 implies that every positive root of $\mathfrak{g}$ is the rank vector of an indecomposable object of $\text{rep}_{1,k}(H(C,kD,\Omega))$. However, the converse is not true, and already for a matrix $C$ of finite type $B_3$ with minimal symmetrizer $D$, one can find indecomposable locally free $H$-modules whose rank vector is not a root, see [GLS §13.7]. But by Theorem 1.1, any primitive element of $\mathcal{M}$ vanishes on such an indecomposable module.

The proof of Theorem 1.1 is largely inspired from [S], but with some non-trivial modifications. In fact when $C$ is symmetric and $D$ is the unit matrix the algebra $\mathcal{M}$ coincides with the algebra denoted by $R^+(Q)$ in [S]. However in all other cases, $\mathcal{M}$ is defined using filtrations which are not composition series, and so we have to adapt all the arguments of [S] to our setting (compare for instance [S] Lemma 3.6.2 with Proposition 4.13 below). We also note that, following [L1] and in contrast to [S], we systematically use the language of constructible functions and convolution products. This allows us to simplify some key steps (compare for instance the proof of [S] Theorem 4.3 with that of Proposition 5.2 below).

We hope that our complete and detailed exposition of the proof of Theorem 1.1 will also be useful for readers interested in Schofield’s original theorem. Indeed, although this important result has been cited by many authors (e.g. [L1], [J], [BT]), the manuscript [S] remains unpublished, and we do not know of any other proof available in the literature. The reader only interested in the case of path algebras may read our paper assuming everywhere that $C$ is symmetric and $D$ is the unit matrix.

Let us outline the structure of the paper and the main steps of the proof of Theorem 1.1. In Section 2 we review the definition of $H(C,D,\Omega)$ and the results of [GLS] which will be needed in the sequel. In Section 3 we introduce the algebra of constructible functions $\mathcal{M}(H)$ and show that it is a homomorphic image of $U(n)$ (Corollary 3.11). The rest of the paper is devoted to the proof that this homomorphism is an isomorphism. To do that we follow Schofield’s strategy and introduce in Section 4 a new algebra $\mathcal{D}$ constructed as a “limit” of certain algebras of constructible functions $\mathcal{D}_P$ indexed by the projective $H$-modules $P$. This algebra $\mathcal{D}$ contains two copies of $\mathcal{M}$ (Proposition 4.4 and Corollary 4.6). Various relations satisfied by the generators of $\mathcal{D}$ are obtained in §4.4. These relations are very close to the relations satisfied by the Chevalley generators of $U(g)$, but a few relations do not match. To overcome this problem, in Section 5 one considers a certain quotient $\mathcal{E}$ of a “Borel” subalgebra $\mathcal{D}^{>0}$ of $\mathcal{D}$, and one constructs a Lie algebra $\mathcal{L}$ of partially defined derivations on $\mathcal{E}$. One then shows that the generators of $\mathcal{L}$ satisfy all the defining relations of the Kac-Moody algebra $\mathfrak{g}$ (Theorem 5.8). Applying the Gabber-Kac theorem, one deduces that $\mathcal{L}$ is in fact isomorphic to $\mathfrak{g}$, and finally that $\mathcal{M}$ is isomorphic to $U(n)$ (Theorem 5.10). The paper closes with the description of two basic examples of algebras $\mathcal{M}$ (Section 6).
2 The algebra $H$

2.1 Definition of $H(C,D,\Omega)$

We retain the notation of [GLS]. Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a symmetrizable generalized Cartan matrix, and let $D = \text{diag}(c_1,\ldots,c_n)$ be a symmetrizer of $C$. This means that $c_i \in \mathbb{Z}_{\geq 0}$, and

$$c_i = 2, \quad c_{ij} \leq 0 \quad \text{for} \quad i \neq j, \quad c_ic_{ji} = c_jc_{ij}.$$  

When $c_{ij} < 0$ define

$$g_{ij} := |\gcd(c_{ij},c_{ji})|, \quad f_{ij} := |c_{ij}|/g_{ij}.$$

An orientation of $C$ is a subset $\Omega \subset \{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$ such that the following hold:

(i) $\{(i,j),(j,i)\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;

(ii) For each sequence of the form $((i_1,i_2),(i_2,i_3),\ldots,(i_t,i_{t+1}))$ with $t \geq 1$ and $(i_s,i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$ we have $i_1 \neq i_t$.

For an orientation $\Omega$ of $C$ let $Q := Q(C,\Omega) := (Q_0,Q_1)$ be the quiver with vertex set $Q_0 := \{1,\ldots,n\}$ and with arrow set

$$Q_1 := \{\alpha_{ij}^{(g)} : j \to i \mid (i,j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i : i \to i \mid 1 \leq i \leq n\}.$$  

Let

$$H := H(C,D,\Omega) := \mathbb{C}Q/I$$

where $\mathbb{C}Q$ is the path algebra of $Q$, and $I$ is the ideal of $\mathbb{C}Q$ defined by the following relations:

(H1) (nilpotency) for each $i$ we have

$$\varepsilon_i^{c_i} = 0;$$

(H2) (commutativity) for each $(i,j) \in \Omega$ and each $1 \leq g \leq g_{ij}$ we have

$$\varepsilon_i^{f_{ij}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}.$$  

This definition is illustrated by many examples in [GLS Section 13].

Clearly, $H$ is a finite-dimensional $\mathbb{C}$-algebra. It is known [GLS Theorem 1.1] that $H$ is Iwanaga-Gorenstein of dimension 1. This means that for an $H$-module $M$ we have

$$(\text{proj.dim}(M) < \infty) \iff (\text{inj.dim}(M) < \infty) \iff (\text{proj.dim}(M) \leq 1) \iff (\text{inj.dim}(M) \leq 1).$$

Moreover, if $M$ is a submodule of a projective $H$-module and if $\text{proj.dim}(M) \leq 1$, then $M$ is projective. Dually, if $M$ is a quotient module of an injective $H$-module and $\text{inj.dim}(M) \leq 1$ then $M$ is injective.

Note that if $C$ is symmetric and if $D = \text{diag}(1,\ldots,1)$, then $H$ is isomorphic to the path algebra $\mathbb{C}Q^0$, where $Q^0$ is the acyclic quiver obtained from $Q$ by deleting all loops $\varepsilon_i$. More generally, it is easy to see that if $C$ is symmetric and $D = \text{diag}(k,\ldots,k)$ for some $k > 0$, then $H$ is isomorphic to $RQ^0 := R \otimes_{\mathbb{C}} \mathbb{C}Q^0$, where $R$ is the truncated polynomial ring $\mathbb{C}[x]/(x^k)$. In that case, $H$-modules are nothing else than representations of $Q^0$ over the ring $R$. When $C$ is only symmetrizable, one has a similar picture by replacing the path algebra $RQ^0$ by a modulated graph over a family of truncated polynomial rings, as we shall now explain.
2.2 Modulated graphs

It was shown in [GLS, §5] that $H$ gives rise to a modulated graph, and that the category of $H$-modules is isomorphic to the category of representations of this modulated graph. This viewpoint, which is very close to Dlab and Ringel’s theory of species [DR], will be useful in several places below.

For $i = 1, \ldots, n$, let $H_i$ be the subalgebra of $H$ generated by $\varepsilon_i$. Thus $H_i$ is isomorphic to $\mathbb{C}[x]/(x^r)$. For $(i, j) \in \Omega$ we define

$$iH_j := H_i \text{Span}_C(\alpha_{ij}^{(g)} | 1 \leq g \leq g_{ij})H_j.$$ 

It is shown in [GLS] that $iH_j$ is an $H_i$-$H_j$-bimodule, which is free as a left $H_i$-module and free as a right $H_j$-module. An $H_i$-basis of $iH_j$ is given by

$$\{\alpha_{ij}^{(g)}, \alpha_{ij}^{(g)} \varepsilon_j, \ldots, \alpha_{ij}^{(g)} \varepsilon_j^{g_j-1} | 1 \leq g \leq g_{ij}\}.$$ 

In particular, we have an isomorphism of left $H_j$-modules: $iH_j \cong H_j^{\Omega_j}$.

The tuple $(H_i, iH_j | (i, j) \in \Omega)$ is called the oriented modulation associated with $H(C, D, \Omega)$. A representation $(M_i, M_{ij})$ of this modulation consists of finite-dimensional $H_i$-modules $M_i$ for each $i = 1, \ldots, n$, and of $H_i$-linear maps

$$M_{ij}: iH_j \otimes_{H_i} M_j \to M_i$$

for each $(i, j) \in \Omega$. Representations of this modulation form an abelian category $\text{rep}(C, D, \Omega)$ isomorphic to the category of $H$-modules [GLS, Proposition 5.1]. (Here we identify the category of $H$-modules with the category of representations of the quiver $Q(C, \Omega)$ satisfying the relations (H1) and (H2).) Given a representation $(M_i, M_{ij})$ in $\text{rep}(C, D, \Omega)$ the corresponding $H$-module $(M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ is obtained as follows: the $\mathbb{C}$-linear map $M(\varepsilon_i): M_i \to M_i$ is given by

$$M(\varepsilon_i)(m) := \varepsilon_i m.$$ 

(here we use that $M_i$ is an $H_i$-module), and for $(i, j) \in \Omega$, the $\mathbb{C}$-linear map $M(\alpha_{ij}^{(g)}): M_j \to M_i$ is defined by

$$M(\alpha_{ij}^{(g)})(m) := M_{ij}(\alpha_{ij}^{(g)} \otimes m).$$

The maps $M(\alpha_{ij}^{(g)})$ and $M(\varepsilon_i)$ satisfy the defining relations (H1) and (H2) of $H$ because the maps $M_{ij}$ are $H_i$-linear.

2.3 Locally free $H$-modules

We say that an $H$-module $M = (M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ is locally free if for every $i$ the $H_i$-module $M_i$ is free. By [GLS, Theorem 1.1], $M$ is locally free if and only if $\text{proj.dim}(M) \leq 1$. The full subcategory $\text{rep}_{lf}(H)$ whose objects are the finite-dimensional locally free modules is closed under extensions, kernels of epimorphisms and cokernel of monomorphisms, and it has Auslander-Reiten sequences [GLS, Lemma 3.8, Theorem 3.9].

The rank vector of $M \in \text{rep}_{lf}(H)$ is the $n$-tuple of integers $\text{rank}(M) = (\text{rk}(M_i))$. For $j = 1, \ldots, n$ we denote by $E_j$ the unique locally free $H$-module with rank vector $(\delta_{ij} | i = 1, \ldots, n)$. In other words, $E_j$ is nothing else than $H_j$ regarded as an $H$-module in the obvious way.

For $M, N \in \text{rep}_{lf}(H)$ the integer

$$\langle M, N \rangle_H := \dim H(M, N) - \dim \text{Ext}_H^1(M, N)$$
depends only on the rank vectors \( \text{rank}(M) \) and \( \text{rank}(N) \), see [GLS, Proposition 4.1]. The map \( (M,N) \mapsto (M,N)_H \) thus descends to a bilinear form on the Grothendieck group \( \mathbb{Z}^n \) of \( \text{rep}_{\text{L1}}(H) \), given on the basis \( \alpha_i = \text{rank}(E_i) \) by

\[
\langle \alpha_i, \alpha_j \rangle_H = \begin{cases} 
   c_{ij} & \text{if } (j,i) \in \Omega, \\
   c_i & \text{if } i = j, \\
   0 & \text{otherwise}.
\end{cases}
\]

Let \((-, -)_H\) be the symmetrization of \((-, -)_H\) defined by \((a, b)_H := (a, b)_H + (b, a)_H\), and let \(q_H\) be the quadratic form defined by \(q_H(a) := \langle a, a \rangle_H\). The forms \(q_H\) and \((-, -)_H\) are called the homological bilinear forms of \(H\).

Note that \((-, -)_H\) is nothing else than the symmetric bilinear form

\[
\langle \alpha_i, \alpha_j \rangle = c_{ij}, \quad (1 \leq i, j \leq n)
\]

associated with the symmetric matrix \(DC\).

### 3 The convolution algebra \(M\)

#### 3.1 Definition of the algebra \(M\)

Let \(d \in \mathbb{N}^n\) be a dimension vector. Let \(\text{rep}(H, d)\) be the affine complex variety of \(H\)-modules with dimension vector \(d = (d_1, \ldots, d_n)\). By definition the closed points in \(\text{rep}(H, d)\) are tuples

\[
M = (M(a))_{a \in Q_1} \in \prod_{a \in Q_1} \text{Hom}_C(\mathbb{C}^{d(a)}, \mathbb{C}^{d(a)})
\]

of \(\mathbb{C}\)-linear maps such that

\[
M(e_i)^{c_i} = 0
\]

and for each \((i, j) \in \Omega\) and \(1 \leq g \leq g_{ij}\) we have

\[
M(e_i)^{f_i}M(\alpha_j^{(g)}) = M(\alpha_j^{(g)})M(e_i)^{f_{ij}}.
\]

The group \(G_d := \text{GL}_{d_1} \times \cdots \times \text{GL}_{d_n}\) acts on \(\text{rep}(H, d)\) by conjugation. The \(G_d\)-orbit of \(M \in \text{rep}(H, d)\) is denoted by \(O_M\). The \(G_d\)-orbits of \(\text{rep}(H, d)\) are in one-to-one correspondence with isomorphism classes of \(H\)-modules with dimension vector \(d\).

Recall that a constructible function on a complex algebraic variety \(V\) is a map \(\varphi : V \to \mathbb{C}\) such that the image of \(\varphi\) is finite, and for each \(a \in \mathbb{C}\) the preimage \(\varphi^{-1}(a)\) is a constructible subset of \(V\). Let \(\mathcal{F}_d\) be the complex vector space of constructible functions \(f : \text{rep}(H, d) \to \mathbb{C}\) which are constant on \(G_d\)-orbits, and let

\[
\mathcal{F} = \mathcal{F}(H) = \bigoplus_{d \in \mathbb{N}^n} \mathcal{F}_d.
\]

We endow \(\mathcal{F}\) with a convolution product \(*\) defined as in [L1, §10.12] or [L2, §2.1], using Euler characteristics of constructible subsets. Namely, we put

\[
(f * g)(X) = \int_{Y \subseteq X} f(Y)g(X/Y) d\chi_X, \quad (f, g \in \mathcal{F}, \ X \in \text{rep}(H)).
\]

Here, the integral is taken on the variety of all \(H\)-submodules \(Y\) of \(X\), and for a constructible function \(\varphi\) on a variety \(V\), we set

\[
\int_{Y \subseteq V} \varphi(Y) d\chi_X = \sum_{a \in \mathbb{C}} a \cdot \chi(\varphi^{-1}(a)).
\]
It is well-known that \((\mathcal{F}, \star)\) has the structure of an \(\mathbb{N}^n\)-graded associative \(\mathbb{C}\)-algebra, see e.g. [BT §4.2].

Let \(e_i = (0, \ldots, c_i, \ldots, 0) \in \mathbb{N}^n\) be the dimension vector of \(E_i\). Let \(\theta_i \in \mathcal{F}\) denote the characteristic function of the \(G_{e_i}\)-orbit of \(\text{rep}(H, e_i)\) corresponding to \(E_i\).

**Definition 3.1.** We denote by \(\mathcal{M} = \mathcal{M}(H)\) the subalgebra of \((\mathcal{F}, \star)\) generated by \(\theta_i\) \((1 \leq i \leq n)\), and we set \(\mathcal{M}_d = \mathcal{M} \cap \mathcal{F}_d\) \((d \in \mathbb{N}^n)\).

Note that \(\mathcal{M}_d\) is a finite-dimensional vector space. The unit element \(1_{\mathcal{M}}\) of \(\mathcal{M}\) is the characteristic function of the zero \(H\)-module.

**Lemma 3.2.** Let \(f \in \mathcal{M}_d\) and \(X \in \text{rep}(H, d)\). If \(X\) is not locally free then \(f(X) = 0\).

**Proof.** For an \(H\)-module \(X\) and a sequence \(i = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k\), we have, by definition of the convolution product \(\star\),

\[
(\theta_{i_1} \star \cdots \star \theta_{i_k})(X) = \chi(F_{X,i}),
\]

where \(F_{X,i}\) is the constructible set of flags of \(H\)-modules

\[
0 = X_0 \subset X_1 \subset \cdots \subset X_k = X \quad \text{with} \quad X_j/X_{j-1} \cong E_{i_j} \quad (1 \leq j \leq k).
\]

By [GLS Lemma 3.8] the category of locally free \(H\)-modules is stable under extensions, hence if \(X\) is not locally free we have \(F_{X,i} = \emptyset\) for every sequence \(i\). This shows that \((\theta_{i_1} \star \cdots \star \theta_{i_k})(X) = 0\) for every sequence \(i\), and thus, by definition of \(\mathcal{M}\), that \(f(X) = 0\) for every \(f \in \mathcal{M}\).

**Remark 3.3.** When the Cartan matrix \(C\) is symmetric and \(D\) is the unit matrix, the algebra \(H\) is the path algebra \(\mathbb{C}Q^+\) (see §2.1). In that case \(\mathcal{M}(H)\) coincides with the algebra \(R^+(\mathbb{C}Q^n)\) of [S §2.3], and with the algebra \(\mathcal{M}_0(\Omega)\) of [L1 §10.19].

### 3.2 Varieties of locally free \(H\)-modules

Let \(d \in \mathbb{N}^n\) be a dimension vector. If \(M \in \text{rep}(H, d)\) is locally free, its rank vector is \(r = (r_1, \ldots, r_n)\) where \(r_i := d_i/c_i\). Hence locally free modules can only exist if \(d_i\) is divisible by \(c_i\) for every \(i\). In this case we say that \(d\) is \(c\)-divisible. Let \(\text{rep}_{1.f.}(H, r)\) be the union of all orbits \(O_M\) of locally free modules \(M\) of rank vector \(r\). By Lemma 3.2 the support of every constructible function \(f \in \mathcal{M}_d\) is contained in \(\text{rep}_{1.f.}(H, r)\). Consider the natural projection

\[
\pi : \text{rep}_{1.f.}(H, r) \to \text{rep}(H_1, d_1) \times \cdots \times \text{rep}(H_n, d_n), \quad (M(\alpha))_{\alpha \in Q_i} \mapsto (M(\varepsilon_1), \ldots, M(\varepsilon_n)).
\]

The image of \(\pi\) is \(O_{E_{i_1}^{r_{i_1}}} \times \cdots \times O_{E_{i_n}^{r_{i_n}}}\), where \(O_{E_{i_j}^{r_{i_j}}}\) is the \(G_{d_i}\)-orbit of the free \(H_i\)-module \(E_{i_j}^{r_{i_j}}\) of rank \(r_i\). (Note that \(\text{rep}(H_i, d_i)\) is just a point if \(c_i = 1\).) We identify \(\text{Im}(\pi)\) with the \(G_d\)-orbit \(O_{E^r}\) of the locally free \(H\)-module \(E^r := \bigoplus_{i=1}^n E_{i_j}^{r_{i_j}}\). In particular, \(O_{E^r}\) is smooth and irreducible of dimension

\[
\sum_{i=1}^n c_i^2 r_i^2 - \sum_{i=1}^n c_i r_i^2.
\]

Here, the summands of the first sum are the dimensions of the groups \(G_{d_i}\), while the summands of the second sum are the dimensions of the endomorphism rings \(\text{End}_{H_i}(E_{i_j}^{r_{i_j}})\).

**Proposition 3.4.** Let \(d\) be \(c\)-divisible as above. Set \(r_i := d_i/c_i\) and \(r = (r_1, \ldots, r_n)\). Then \(\text{rep}_{1.f.}(H, r)\) is a non-empty open subset of \(\text{rep}(H, d)\). Moreover we have:
(i) The restriction \( \tilde{\pi} : \text{rep}_{l.f.}(H, r) \to \mathcal{O}_{E^r} \) of \( \pi \) to its image defines a vector bundle of rank \( \sum_{(i,j) \in \Omega} c_i |c_{ij}|r_i r_j \). In particular, \( \text{rep}_{l.f.}(H, r) \) is smooth and irreducible of dimension

\[
\sum_{i=1}^n c_i (c_i - 1) r_i^2 + \sum_{(i,j) \in \Omega} c_i |c_{ij}|r_i r_j = \dim(G_d) - q_H(r).
\]

(ii) If \( q_H(r) \leq 0 \) then \( \text{rep}_{l.f.}(H, r) \) has infinitely many \( G_d \)-orbits.

Proof. Since the locally free modules are exactly the modules with projective dimension at most 1 (see [GLS Proposition 3.5]), upper semicontinuity yields that \( \text{rep}_{l.f.}(H, r) \) is open in \( \text{rep}(H, d) \).

By \$2.2\$ the fibre \( \pi^{-1}(E_1^\oplus r_1, \ldots, E_n^\oplus r_n) \) can be identified with

\[
\bigoplus_{(i,j) \in \Omega} \text{Hom}_H \left( H_j \otimes_{H_i} E_j^\oplus r_j, E_i^\oplus r_i \right) = \bigoplus_{(i,j) \in \Omega} \text{Hom}_H \left( E_i^\oplus |c_{ij}|r_j, E_i^\oplus r_i \right).
\]

Therefore we easily calculate that

\[
\dim \left( \pi^{-1}(E_1^\oplus r_1, \ldots, E_n^\oplus r_n) \right) = \sum_{(i,j) \in \Omega} c_i |c_{ij}|r_i r_j.
\]

Next, notice that \( \tilde{\pi} \) is by construction \( G_d \)-equivariant. Since \( \mathcal{O}_{E^r} \) is a single \( G_d \)-orbit, all fibers of \( \tilde{\pi} \) are isomorphic, and, in particular, are vector spaces of the same dimension.

Consider the trivial vector bundle

\[
X := \left( \bigoplus_{a \in Q_1^c} \text{Hom}_C(\mathbb{C}^{d(a)}, \mathbb{C}^{d_i(a)}) \right) \times \mathcal{O}_{E^r}.
\]

over \( \mathcal{O}_{E^r} \). A point of \( X \) is given by a tuple \( M = \left( \left( M(\alpha_{ij}^{(g)}) \right)_{(i,j) \in \Omega; 1 \leq g \leq r_j}, (M(\epsilon_i))_{i=1,\ldots,n} \right) \) of \( \mathbb{C} \)-linear maps. Obviously, the map \( \mu : X \to X \) defined by

\[
\mu(M) := \left( (M(\epsilon_i))^{(g)}, M(\alpha_{ij}^{(g)}) - M(\alpha_{ij}^{(g)}) M(\epsilon_j)^{(g_j)}, (M(\epsilon_i)) \right)
\]

is an endomorphism of the vector bundle \( X \), and by construction \( \text{Ker}(\mu) = \text{rep}_{l.f.}(H, d) \). Since by the above consideration, the fibre

\[
\text{Ker}(\mu)_{(M(\epsilon_i))} = \tilde{\pi}^{-1}(M(\epsilon_i))
\]

is of constant dimension for all \( (M(\epsilon_i)) \in \mathcal{O}_{E^r} \), we have that \( \text{Ker}(\mu) \) is a vector bundle of the claimed rank over \( \mathcal{O}_{E^r} \). This proves (i).

The one-dimensional torus \( \{ (\lambda \text{id}_{d_1}, \ldots, \lambda \text{id}_{d_n}) \mid \lambda \in \mathbb{C}^* \} \subset G_d \) acts trivially on \( \text{rep}_{l.f.}(H, r) \). So the maximal dimension of a \( G_d \)-orbit is \( \dim(G_d) - 1 \). Hence, by (i), if \( q_H(r) \leq 0 \), every \( G_d \)-orbit has dimension at most \( \dim(\text{rep}_{l.f.}(H, r)) - 1 \). This proves (ii).

Remark 3.5. The vector bundle structure of Proposition 3.4 is inspired by [Bo Sec. 2]. The same statement remains true for any algebraically closed field, with basically the same proof. Notice that we have a natural action of \( \text{Aut}_H(E^r) = \prod_{i=1}^n \text{GL}_{r_i}(H_i) \) on \( \tilde{\pi}^{-1}(E^r) \) by conjugation, such that the \( G_d \)-orbits on \( \text{rep}_{l.f.}(H, d) \) are in bijection with the \( \text{Aut}_H(E^r) \)-orbits on \( \tilde{\pi}^{-1}(E^r) \).
3.3 Bialgebra structure of $\mathcal{M}$

Consider the direct product of algebras $H \times H$. Modules for $H \times H$ are pairs $(X_1, X_2)$ of modules for $H$. An $H \times H$-submodule of $(X_1, X_2)$ is a pair $(Y_1, Y_2)$ where $Y_1$ is an $H$-submodule of $X_1$ and $Y_2$ is an $H$-submodule of $X_2$. Note that we can regard $H \times H$ as the algebra $H(C \oplus C, D \oplus D, \Omega \oplus \Omega)$, where $C \oplus C$ (resp. $D \oplus D$) means the block diagonal matrix with two diagonal blocks equal to $C$ (resp. $D$), and $\Omega \oplus \Omega$ is the obvious orientation of $C \oplus C$ induced by the orientation $\Omega$ of $C$. Therefore we can define as above a convolution algebra $\mathcal{F}(H \times H)$.

We have an algebra embedding of $\mathcal{F}(H) \otimes \mathcal{F}(H)$ into $\mathcal{F}(H \times H)$ by setting

$$(f \otimes g)(X,Y) = f(X)g(Y).$$

Following Ringel [Rin2] (see also [BT] §4.3), one introduces a map $c : \mathcal{F}(H) \to \mathcal{F}(H \times H)$ by

$$c(f)(X,Y) = f(X \oplus Y).$$

**Proposition 3.6.** The map $c$ restricts to a homomorphism $c : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ for the convolution product such that

$$c(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i.$$

This makes $\mathcal{M}$ into a cocommutative bialgebra.

**Proof.** We first show that $c$ is a homomorphism from $\mathcal{F}(H)$ to $\mathcal{F}(H \times H)$ for the convolution product. Indeed, on the one hand we have for $f, g \in \mathcal{F}(H)$

$$(c(f \ast g))(X,Y) = \int_{Z \subseteq X \oplus Y} f(Z)g((X \oplus Y)/Z) d\chi,$$

and on the other hand

$$(c(f) \ast c(g))(X,Y) = \int_{Z_1 \subseteq X, Z_2 \subseteq Y} f(Z_1 \oplus Z_2)g((X \oplus Y)/(Z_1 \oplus Z_2)) d\chi.\quad (1)$$

To show that the two integrals are the same, we consider the $\mathbb{C}^*$-action on $X \oplus Y$ given by

$$\lambda \cdot (x,y) = (\lambda x, y), \quad (x \in X, y \in Y, \lambda \in \mathbb{C}^*).$$

This induces a $\mathbb{C}^*$-action on the variety of submodules $Z$ of $X \oplus Y$, whose fixed points are exactly the submodules of the form $Z = Z_1 \oplus Z_2$ with $Z_1 \subseteq X$ and $Z_2 \subseteq Y$. Moreover, for a submodule $Z$ of $X \oplus Y$ and $\lambda \in \mathbb{C}^*$, the $H$-module $\lambda \cdot Z$ is isomorphic to $Z$, so for every $f \in \mathcal{F}(H)$ we have $f(\lambda \cdot Z) = f(Z)$, and therefore (1) and (2) are equal. It follows that $c$ restricts to a homomorphism from $\mathcal{M}(H)$ to $\mathcal{M}(H \times H)$. Since $E_i$ is indecomposable, we have

$$c(\theta_i)(X,Y) = \theta_i(X \oplus Y) = \begin{cases} 1 & \text{if } X \cong E_i \text{ and } Y = \{0\}, \text{ or } X = \{0\} \text{ and } Y \cong E_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $c(\theta_i)$ can be identified with $\theta_i \otimes 1 + 1 \otimes \theta_i \in \mathcal{M}(H) \otimes \mathcal{M}(H) \subset \mathcal{M}(H \times H)$. Finally, since $\mathcal{M}(H)$ is generated by the $\theta_i$'s and $c$ is multiplicative, this implies that the image $c(\mathcal{M}(H))$ is indeed contained in $\mathcal{M}(H) \otimes \mathcal{M}(H)$. \qed

An element $f$ of $\mathcal{M}$ is called primitive if $c(f) = f \otimes 1 + 1 \otimes f$.

**Lemma 3.7.** An element $f$ of $\mathcal{M}$ is primitive if and only if $f$ is supported only on indecomposable modules.
Proof. This follows immediately from the equality \( f(X \otimes Y) = c(f)(X, Y) \).

It is easy to see that if \( f \) and \( g \) are primitive then the Lie bracket

\[
[f, g] := f \ast g - g \ast f
\]
is also primitive. Hence the subspace \( \mathcal{P}_M \subset \mathcal{M} \) of primitive elements has the natural structure of a Lie algebra.

**Proposition 3.8.** \((\mathcal{M}, \ast, c)\) is a Hopf algebra isomorphic to the universal enveloping algebra \( U(\mathcal{P}_M) \).

**Proof.** A nonzero element \( f \) of \( \mathcal{M} \) is group-like if \( c(f) = f \otimes f \). Arguing as in [BT] §4.5, we see that the only group-like element is the unit element \( 1_M \). Indeed, if \( f \) is group-like for any \( H \)-module \( X \) and \( k \in \mathbb{N} \) we have \( f(X^\otimes k) = f(X)^k \). If \( f(X) \neq 0 \) for a module \( X \), then the decomposition of \( f \) with respect to the direct sum \( \bigoplus_d \mathcal{M}_d \) has infinitely many nonzero components, a contradiction. Hence \( f = \lambda 1_M \) where \( \lambda = \lambda^2 \) and \( \lambda \neq 0 \), so \( f = 1_M \).

Therefore, we can repeat the last part of the proof of [Rin2 Theorem]: by [Sw] Lemma 8.0.1, \( \mathcal{M} \) is an irreducible cocommutative coalgebra, hence a Hopf algebra [Sw Theorem 9.2.2]. It then follows from [Sw Theorem 13.0.1] that \( \mathcal{M} \) is isomorphic as a Hopf algebra to the universal enveloping algebra \( U(\mathcal{P}_M) \) of the Lie algebra \( \mathcal{P}_M \).

**Remark 3.9.** When the Cartan matrix \( C \) is symmetric and \( D \) is the unit matrix, the Lie algebra \( \mathcal{P}_M \) coincides with the Lie algebra \( L^+(\mathbb{C}Q^\circ) = L^+(Q^\circ) \) of [S] §2.6.

### 3.4 Relations in \( \mathcal{M} \)

For \( f \in \mathcal{M} \) we denote by \( \text{ad} \ f \) the endomorphism of \( \mathcal{M} \) defined by

\[
\text{ad} \ f(g) := [f, g], \quad (g \in \mathcal{M}).
\]

**Proposition 3.10.** The generators \( \theta_i \) of \( \mathcal{M} \) satisfy the relations:

\[
(\text{ad} \ \theta_i)^{1-c_{ij}}(\theta_j) = 0, \quad (1 \leq i \neq j \leq n).
\]

**Proof.** Since \( \theta_i \in \mathcal{P}_M \), we have

\[
\Theta_{ij} := (\text{ad} \ \theta_i)^{1-c_{ij}}(\theta_j) \in \mathcal{P}_M
\]

for all \( j \neq i \). By Lemma 3.2 and Lemma 3.7 to check that \( \Theta_{ij} = 0 \) it is therefore sufficient to check that there is no indecomposable locally free \( H \)-module with dimension vector \( (1-c_{ij})e_i + e_j \).

Let us assume that \( (i, j) \in \Omega \), and consider a locally free module \( M \) with this dimension vector. Then, by §2.2, \( M \) is given by an \( H_i \)-linear map

\[
M_j : iH_j \otimes H_i M_j \to M_i,
\]

where \( M_j = H_j \) and \( M_i = H_i^{\otimes (1-c_{ij})} \). Now, \( iH_j \otimes H_i M_j = iH_j \) is a free \( H_i \)-module of rank \( f_{ij}g_{ij} = -c_{ij} \), so \( M_i \) contains a direct summand \( N_i \) isomorphic to \( H_i \) such that \( N_i \cap \text{Im}(M_{ij}) = \{0\} \). It follows that \( M \) has a direct summand isomorphic to \( E_i \), and therefore \( M \) is not indecomposable.

The case \( (j, i) \in \Omega \) is dual, and one can argue similarly.

Let \( g \) be the symmetrizable Kac-Moody Lie algebra over \( \mathbb{C} \) with Cartan matrix \( C \). It is defined by the following presentation. There are \( 3n \) generators \( e_i, f_i, h_i \) \( (1 \leq i \leq n) \) subject to the relations:
\[\begin{align*}
(i) \quad [e_i, f_j] &= \delta_{ij} h_i; \\
(ii) \quad [h_i, h_j] &= 0; \\
(iii) \quad [h_i, e_j] &= c_{ij} e_j, \quad [h_i, f_j] = -c_{ij} f_j; \\
(iv) \quad (\text{ad} e_i)^{1-c_{ij}}(e_j) &= 0, \quad (\text{ad} f_i)^{1-c_{ij}}(f_j) = 0 \quad (i \neq j).
\end{align*}\]

Let \( n \) be the Lie subalgebra generated by \( e_i \) \((1 \leq i \leq n)\). Then \( U(n) \) is the associative \( \mathbb{C} \)-algebra with generators \( e_i \) \((1 \leq i \leq n)\) subject to the relations

\[(\text{ad} e_i)^{1-c_{ij}}(e_j) = 0, \quad (i \neq j).\]

**Corollary 3.11.** The assignment \( e_i \mapsto \theta_i \) extends to a surjective algebra homomorphism

\[F : U(n) \to \mathcal{M}.\]

**Proof.** This follows from Proposition 3.10. \( \square \)

### 4 The algebra \( \mathcal{D} \)

#### 4.1 Definition of \( \mathcal{D}_P \)

Let \( P \) be a finite-dimensional projective \( H \)-module. Let \( \text{Gr}(P) \) denote the variety of all \( H \)-submodules of \( P \). Let \( \text{Gr}_{\text{lf}}(P) \) denote the constructible subset of \( \text{Gr}(P) \) consisting of all locally free submodules. The group \( \text{Aut}_H(P) \) of automorphisms of \( P \) acts on \( \text{Gr}(P) \), hence it acts diago- 

\[\begin{align*}
(f \ast g)(X,Y) &= \int_{Z \in \text{Gr}_{\text{lf}}(P)} f(X,Z)g(Z,Y)d\chi, \quad (f,g \in \mathcal{C}_P, \ X,Y \in \text{Gr}_{\text{lf}}(P)).
\end{align*}\]

This makes \((\mathcal{C}_P, \ast)\) into an associative \( \mathbb{C} \)-algebra, with unit element the function \( 1_P \) defined by

\[1_P(X,Y) = \begin{cases} 1 & \text{when } X = Y, \\ 0 & \text{otherwise.} \end{cases}\]

For \( i = 1, \ldots, n \), define elements of \( \mathcal{C}_P \) by

\[\begin{align*}
x_{i,P}(X,Y) &= \begin{cases} 1 & \text{when } X \subset Y \text{ and } Y/X \cong E_i, \\ 0 & \text{otherwise}, \end{cases} \\
y_{i,P}(X,Y) &= \begin{cases} 1 & \text{when } Y \subset X \text{ and } X/Y \cong E_i, \\ 0 & \text{otherwise}. \end{cases}
\end{align*}\]

**Definition 4.1.** Let \( \mathcal{D}_P \) denote the subalgebra of \((\mathcal{C}_P, \ast)\) generated by \( x_{i,P}, y_{i,P}, \ (1 \leq i \leq n)\).
4.2 Definition of $\mathcal{D}$

Let $A := \mathbb{C}(\overline{x}_i, \overline{y}_i; 1 \leq i \leq n)$ be the free associative algebra generated by non-commuting variables $\overline{x}_i$, $\overline{y}_i$, $(1 \leq i \leq n)$. We have a surjective algebra homomorphism $\overline{\pi}_P : A \to \mathcal{D}_P$ given by

$$\overline{\pi}_P(\overline{x}_i) = x_i, \quad \overline{\pi}_P(\overline{y}_i) = y_i,$$

(1 \leq i \leq n).

Define $K$ as the intersection of Ker($\overline{\pi}_P$) when $P$ runs over all isomorphism classes of projective $H$-modules.

**Definition 4.2.** Let $\mathcal{D} = \mathcal{D}(H) := A/K$. Then $\mathcal{D}$ is generated by

$$x_i := \overline{x}_i \mod K, \quad y_i := \overline{y}_i \mod K, \quad (1 \leq i \leq n),$$

and we have natural surjective algebra homomorphisms $\pi_p : \mathcal{D} \to \mathcal{D}_P$ such that

$$\pi_p(x_i) = x_i, \quad \pi_p(y_i) = y_i, \quad \pi_p(1) = 1_p.$$

**Remark 4.3.** When the Cartan matrix $C$ is symmetric and $D$ is the unit matrix, the algebra $\mathcal{D}(H)$ is nothing else than the algebra $R'(H)$ of [S] §3.

The following is the extension to our setting of [S] Theorem 3.5.

**Proposition 4.4.** We have an injective algebra homomorphism $\Phi : \mathcal{M} \to \mathcal{D}$ such that

$$\Phi(\theta_i) = x_i, \quad (1 \leq i \leq n).$$

*Proof.* If $X$ is not a submodule of $Y$, we have $(x_{i_1, P} \cdots x_{i_k, P})(X, Y) = 0$. Otherwise, if $X \subset Y$ are locally free submodules of $P$ then $(x_{i_1, P} \cdots x_{i_k, P})(X, Y)$ is equal to the Euler characteristic of the constructible set of flags:

$$\{(X = X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = Y) \mid X_j/X_{j-1} \cong E_{ij}, \quad (1 \leq j \leq k)\}.$$

Quotienting each step of the flags by $X$, we see that this set has the same Euler characteristic as

$$\{(0 = Z_0 \subset Z_1 \subset \cdots \subset Z_{k-1} \subset Z_k = Y/X) \mid Z_j/Z_{j-1} \cong E_{ij}, \quad (1 \leq j \leq k)\}.$$

So we get

$$(x_{i_1, P} \cdots x_{i_k, P})(X, Y) = \left\{ \begin{array}{ll} (\theta_{i_1} \cdots \theta_{i_k})(Y/X) & \text{if } X \subset Y, \\ 0 & \text{otherwise.} \end{array} \right. \quad (3)$$

For a sequence $\mathbf{i} = (i_1, \ldots, i_k)$ let us write for short

$$x_{i_1, P} := x_{i_1} \cdots x_{i_k}, \quad \theta_i := \theta_{i_1} \cdots \theta_{i_k}, \quad x_i := x_{i_1} \cdots x_{i_k}.$$

By (3), every relation $\sum a_i \theta_i = 0$ in $\mathcal{M}$ implies a relation $\sum a_i x_{i_1, P} = 0$ in $\mathcal{D}_P$. Hence we have a well-defined homomorphism $\Phi_P$ from $\mathcal{M}$ to $\mathcal{D}_P$ mapping $\theta_i$ to $x_{i, P}$. Since this can be done for every projective module $P$, the map $\Phi_P$ can be lifted to a homomorphism $\Phi$ from $\mathcal{M}$ to $\mathcal{D}$ mapping $\theta_i$ to $x_i$.

Suppose now that $\sum a_i x_i = 0$ is a relation in $\mathcal{D}$, and let $X$ be a locally free $H$-module. Take a projective cover $p_X : \mathcal{P} \to X$ and set $Y = \text{Ker}(p_X)$. Then $Y \subset P$ and $P/Y \cong X$. We have $\sum a_i x_i = 0$ in $\mathcal{D}_P$, so by (3) again

$$\sum a_i x_i(Y, P) = \sum a_i \theta_i(X) = 0.$$

Since this holds for any $X$, it follows that $\sum a_i \theta_i = 0$ is a relation in $\mathcal{M}$. Therefore $\Phi$ is injective. \[\square\]
Note that the definitions of \( x_{i,P} \) and \( y_{i,P} \) immediately imply:

**Lemma 4.5.** The assignments \( x_i \mapsto y_i, \ y_i \mapsto x_i \ (1 \leq i \leq n) \) extend to an anti-automorphism \( \tau \) of \( D \).

From Proposition 4.4 and Lemma 4.5 it follows that

**Corollary 4.6.** We have an injective algebra anti-homomorphism \( \Psi : M \to D \) such that

\[
\Psi(\theta_i) = y_i, \quad (1 \leq i \leq n).
\]

### 4.3 Bialgebra structure of \( D \)

Consider the algebras \( C_{(P,P')}((H \times H), (P,P')) \) is a finite-dimensional projective \( H \times H \)-module, that is, a pair of finite-dimensional projective \( H \)-modules. We have an algebra embedding \( C_p(H) \otimes C_{p'}(H) \to C_{(P,P')}((H \times H)) \) given by

\[
(f \otimes g)((x,y),(x',y')) = f(x,y)g(x',y'), \quad (f \in C_p(H), \ g \in C_{p'}(H), \ x,y \in P, \ x',y' \in P').
\]

We introduce \( \mathbb{C} \)-linear maps \( d_{(P,P')} : C_{P \oplus P'}(H) \to C_{(P,P')}((H \times H)) \) given by

\[
(d_{(P,P')})(f)((x,y),(x',y')) = f(x \oplus X', Y \oplus Y').
\]

**Proposition 4.7.** The map \( d_{(P,P')} \) restricts to an algebra homomorphism \( D_{P \oplus P'} \to D_P \otimes D_{P'} \) such that

\[
d_{(P,P')}\left(x_{i,P} \oplus 1_p \right) = x_{i,P} \oplus 1_{p}, \quad d_{(P,P')}\left(y_{i,P} \oplus 1_p \right) = y_{i,P} \oplus 1_{p}, \quad (1 \leq i \leq n).
\]

**Proof.** The proof is similar to the proof of Proposition 4.6. We show that \( d_{(P,P')} \) is a homomorphism from \( C_{P \oplus P'}(H) \) to \( C_{(P,P')}((H \times H)) \) for the convolution product. Indeed, on the one hand we have for \( f,g \in C_{P \oplus P'}(H) \)

\[
(d_{(P,P')})(f \ast g)((x,y),(x',y')) = \int_{Z \in P \oplus P'} f(X \oplus X',Z)g(Z,Y \oplus Y')d\chi,
\]

and on the other hand

\[
(d_{(P,P')})(f) \ast (d_{(P,P')})(g))((x,y),(x',y')) = \int_{Z \in P \oplus P',Z_1 \in P',Z_2 \in P'} f(X \oplus X',Z_1 \oplus Z_2)g(Z_1 \oplus Z_2,Y \oplus Y')d\chi.
\]

To show that the two integrals are the same, we consider the \( \mathbb{C}^* \)-action on \( P \oplus P' \) given by

\[
\lambda \cdot (p,p') = (\lambda p,p'), \quad (p \in P, \ p' \in P', \ \lambda \in \mathbb{C}^*).
\]

This induces a \( \mathbb{C}^* \)-action on the variety of submodules \( Z \) of \( P \oplus P' \), whose fixed points are exactly the submodules of the form \( Z = Z_1 \oplus Z_2 \) with \( \lambda \in P \) and \( Z_2 \leq P' \). In particular this action fixes \( X \oplus X' \) and \( Y \oplus Y' \). Since by definition \( f \) and \( g \) are constant on the orbits of \( \mathbb{C}^* \subset \text{Aut}(P \oplus P') \) for the diagonal action, we see that (4) and (5) are equal.

We have:

\[
(d_{(P,P')})(x_{i,P \oplus P'})(X,Y,Y') = x_{i,P \oplus P'}(X \oplus X',Y \oplus Y')
\]

\[
= \begin{cases} 
1 & \text{if } X \oplus X' \subset Y \oplus Y' \text{ and } (Y \oplus Y')/(X \oplus X') \cong E_i, \\
0 & \text{otherwise}.
\end{cases}
\]
Since $X, Y \subseteq P$ and $X', Y' \subseteq P'$, and since $P \cap P' = \{0\}$, we have $X \oplus X' \subseteq Y \oplus Y'$ if and only if $X \subseteq Y$ and $X' \subseteq Y'$. Moreover in this case, $(Y \oplus Y')/(X \oplus X') = (Y/X) \oplus (Y'/X')$. So since $E_i$ is indecomposable we have that $(Y \oplus Y')/(X \oplus X') \cong E_i$ if and only if $Y/X \cong E_i$ and $Y'/X' \cong E_i$. Thus $d_{(p,p')}(x_{i,P} \oplus 1_{p'} + 1_p \otimes x_{i,p'})$ can be identified with $x_{i,p} \otimes 1_{p'} + 1_p \otimes x_{i,p'}$. Similarly, $d_{(p,p')}(y_{i,P} \oplus 1_{p'} + 1_p \otimes y_{i,p'})$ can be identified with $y_{i,p} \otimes 1_{p'} + 1_p \otimes y_{i,p'}$. It follows that $d_{(p,p')}$ restricts to an algebra homomorphism $D_{p \oplus p'} \to D_p \otimes D_{p'}$. 

We can now lift the maps $d_{(p,p')}: D_{p \oplus p'} \to D_p \otimes D_{p'}$ to an algebra homomorphism $d: D \to D \otimes D$,

making $(D, \ast, d)$ into a cocommutative bialgebra. It follows from Proposition 4.7 that $d(x_i) = x_i \otimes 1 + 1 \otimes x_i$, $d(y_i) = y_i \otimes 1 + 1 \otimes y_i$, $(1 \leq i \leq n)$, that is, $x_i$ and $y_i$ are primitive. Let $P_D$ be the subspace of primitive elements of $D$, a complex Lie algebra. In particular, $P_D$ contains the following distinguished elements:

$$h_i := [x_i, y_i], \quad (1 \leq i \leq n).$$

The following is an adaptation to our setting of [S, Theorem 3.3].

**Proposition 4.8.** $(D, \ast, d)$ is a cocommutative Hopf algebra isomorphic to the universal enveloping algebra $U(P_D)$. It is $\mathbb{Z}^n$-graded via

$$\deg(x_i) = \alpha_i, \quad \deg(y_i) = -\alpha_i, \quad (1 \leq i \leq n).$$

**Proof.** Analogous to the proof of Proposition 3.8. 

**Corollary 4.9.** The sub-Lie algebra of $P_D$ generated by the elements $x_i$ is isomorphic to $P_{\mathbb{M}}$. 

**Proof.** This follows from Proposition 4.4. 

### 4.4 Relations in $D$

The following is the extension to our setting of [S, §3.6.1].

**Proposition 4.10.** If $i \neq j$ we have $[x_i, y_j] = 0$. On the other hand $h_i = [x_i, y_i] \neq 0$ for every $i = 1, \ldots, n$.

**Proof.** Let $i \neq j$. It is enough to check that for every $P$ we have $x_{i,P} \ast y_{j,P} - y_{j,P} \ast x_{i,P} = 0$. Let $X, Y \in \text{Gr}_1(P)$, and denote by $X_k$ (resp. $Y_k$) the subspace of $X$ (resp. $Y$) sitting on vertex $k$ of the quiver of $H$. We have

$$x_{i,P} \ast y_{j,P}(X,Y) = \chi \left( \{ Z \in \text{Gr}_1(P) \mid X \subseteq Z, Y \subseteq Z, Z|X \cong E_i, Z|Y \cong E_j \} \right),$$

$$y_{j,P} \ast x_{i,P}(X,Y) = \chi \left( \{ Z \in \text{Gr}_1(P) \mid Z \subseteq X, Z \subseteq Y, Z|X \cong E_j, Z|Y \cong E_i \} \right).$$

Since $i \neq j$, these two products are zero unless $X_k = Y_k$ for $k \neq i$ and $k \neq j$. Moreover for these products to be nonzero we should also have $X_j \subseteq Y_i, Y_j \subseteq X_i, Y_i/X_i \cong E_i$ and $X_j/Y_j \cong E_j$. In this case, the two constructible sets above are both reduced to a point, and have Euler characteristic 1. So, for any pair $(X,Y)$ the difference $[x_{i,P}, y_{j,P}](X,Y)$ vanishes.

On the other hand, let $P_i$ be the projective cover of $E_i$. Then

$$[x_{i,P_i}, y_{i,P_i}](P_i, P_i) = -y_{i,P} \ast x_{i,P}(P_i, P_i) = -1,$$

since the constructible set $\{ Z \subseteq P_i \mid P_i|Z \cong E_i \}$ is reduced to a point. Therefore $[x_{i,P_i}, y_{i,P} \neq 0$, hence $[x_i, y_i] \neq 0$. 

\[\square\]
Given a projective $H$-module $P$ we put

$$h_{i,p} := [x_{i,p}, y_{i,p}] = \pi_p(h_i) \in D_p.$$  

Recall from [GLS, §9] the following definition.

**Definition 4.11.** Let $X$ be an $H$-module. For $i = 1, \ldots, n$ we denote by $\text{sub}_i(X)$ the largest submodule $U$ of $X$ which is supported on the vertex $i$. Dually we denote by $\text{fac}_i(X)$ the largest quotient module $X/U$ which is supported on the vertex $i$.

Thus $\text{sub}_i(X)$ and $\text{fac}_i(X)$ are $H$-modules. Note that, even if $X$ is locally free, $\text{sub}_i(X)$ and $\text{fac}_i(X)$ are not necessarily free $H$-modules. We denote by $s_i(X)$ the multiplicity of $E_i$ as an indecomposable summand of $\text{sub}_i(X)$. Similarly we denote by $t_i(X)$ the multiplicity of $E_i$ as an indecomposable summand of $\text{fac}_i(X)$.

**Lemma 4.12.** Suppose that $s_i(X) = k \geq r \geq 0$. Then

$$\chi(\{Z \subset X \mid Z \cong E_i^\oplus r\}) = \binom{k}{r}.$$  

Similarly, if $t_i(X) = l \geq r \geq 0$, then

$$\chi(\{Z \subset X \mid X/Z \cong E_i^\oplus r\}) = \binom{l}{r}.$$  

**Proof.** Clearly, $Z$ is a submodule of $X$ isomorphic to $E_i^\oplus r$ if and only if $Z$ is a submodule of $\text{sub}_i(X)$ isomorphic to $E_i^\oplus r$. Thus we want to calculate the Euler characteristic of

$$S = \{Z \subset \text{sub}_i(X) \mid Z \cong E_i^\oplus r\}.$$  

We can write $\text{sub}_i(X) \cong E_i^\oplus k \oplus V$, where $V$ is an $H$-module contained in $\text{Ker}(\epsilon_i^{c_i-1})$. The map $Z \mapsto \epsilon_i^{c_i-1}(Z)$ is a fibration from $S$ to the Grassmannian $\text{Gr}(r, \mathbb{C}^k)$ with fibers isomorphic to $\mathbb{C}^d$, where

$$d = r \dim \left(\text{Ker} \left(\epsilon_i^{c_i-1} \big|_{E_i^\oplus (k-r) \oplus V} \right)\right) = r((k-r)(c_i-1) + \dim V).$$

Hence

$$\chi(S) = \chi(\text{Gr}(r, \mathbb{C}^k))\chi(\mathbb{C}^d) = \binom{k}{r}.$$  

For the second statement, let $Y$ be the (unique) submodule of $X$ such that $X/Y \cong \text{fac}_i(X)$. Using the map $Z \mapsto U:= Z/Y$ we see that the set $\{Z \subset X \mid X/Z \cong E_i^\oplus r\}$ has the same Euler characteristic as

$$T = \{U \subset \text{fac}_i(X) \mid \text{fac}_i(X)/U \cong E_i^\oplus r\}.$$  

As above the map $U \mapsto \epsilon_i^{c_i-1}(U)$ is a fibration from $T$ to $\text{Gr}(l-r, \mathbb{C}^l)$ with fibers isomorphic to affine spaces, so

$$\chi(T) = \chi(\text{Gr}(l-r, \mathbb{C}^l)) = \binom{l}{r}. \quad \square$$

The following is the extension to our setting of [S, Lemma 3.6.2].
Proposition 4.13. For $X,Y \in \text{Gr}_{1,1}(P)$ we have

$$h_{i,p}(X,Y) = \begin{cases} 
    s_i(P/X) - t_i(X), & \text{if } X = Y, \\
    0, & \text{otherwise.}
\end{cases}$$

More generally, for $f \in D_p$ we have

$$(h_{i,p} \ast f)(X,Y) = (s_i(P/X) - t_i(X))f(X,Y),$$

$$(f \ast h_{i,p})(X,Y) = (s_i(P/Y) - t_i(Y))f(X,Y).$$

Proof. We have $(x_{i,p} \ast y_{i,p})(X,Y) = \chi(S)$ and $(y_{i,p} \ast x_{i,p})(X,Y) = \chi(I)$ where

$$S = \{ Z \in \text{Gr}_{1,1}(P) \mid X \subset Z, Y \subset Z, Z/X \cong E_i \cong Z/Y \},$$

$$I = \{ Z \in \text{Gr}_{1,1}(P) \mid Z \subset X, Z \subset Y, X/Z \cong E_i \cong Y/Z \}.$$

By the first isomorphism theorem, we have $(X + Y)/Y \cong X/(X \cap Y)$. So since $X$ and $Y$ are locally free, $X + Y$ and $X \cap Y$ are either both locally free or both non locally free.

(a) Suppose $X + Y$ and $X \cap Y$ are locally free. We have two subcases.

(a1) If $X = Y$ then $S = \{ Z \in \text{Gr}_{1,1}(P) \mid X \subset Z, Z/X \cong E_i \}$. The map $Z \mapsto U := Z/X$ is an isomorphism from $S$ to $S' = \{ U \mid U \subset P/X, U \cong E_i \}$. Hence by Lemma 4.12, $\chi(S) = \chi(S') = s_i(P/X)$. On the other hand, $I = \{ Z \in \text{Gr}_{1,1}(P) \mid Z \subset X, X/Z \cong E_i \}$. Hence by Lemma 4.12, $\chi(I) = t_i(X)$. So $\chi(S) - \chi(I) = s_i(P/X) - t_i(X)$, as required.

(a2) If $X \neq Y$ then, $S \neq \emptyset$ if and only if $I \neq \emptyset$, and in that case $S$ and $I$ are reduced to single points, namely $S = \{X + Y\}$ and $I = \{X \cap Y\}$. Hence we always have $\chi(S) - \chi(I) = 0$.

(b) Suppose $X + Y$ and $X \cap Y$ are not locally free. Then $X \neq Y$. If $S \neq \emptyset$ there exists $Z$ such that $X + Y \subset Z \subset P$ and $Z/Y \cong E_i$. Hence $M = (X + Y)/Y$ is isomorphic to a non trivial submodule of $E_i$. The map $Z \mapsto U := Z/Y$ maps $S$ isomorphically to $\{ U \subset P/Y \mid M \subset U, U \cong E_i \}$. This is an affine space (of dimension $(s_i(P/Y) - 1)(\dim(M) - 1)$), hence $\chi(S) = 1$. Since $N := X/(X \cap Y)$ we see that $I = \{ Z \in X \mid X/Z \cong N, X/Z \cong E_i \}$. This is an affine space (of dimension $(t_i(X) - 1)(\dim(N) - 1)$), hence $\chi(I) = 1$. So we have again $\chi(S) - \chi(I) = 0$.

In conclusion we have $\chi(S) - \chi(I) = 0$, unless $X = Y$ where $\chi(S) - \chi(I) = s_i(P/X) - t_i(X)$. This proves the first part of the proposition.

For the second part we have

$$(h_{i,p} \ast f)(X,Y) = \int_{Z \in \text{Gr}_{1,1}(P)} h_{i,p}(X,Z)f(Z,Y)d\chi.$$

Since $h_{i,p}(X,Z) = 0$ unless $Z = X$, we see that $(h_{i,p} \ast f)(X,Y) = h_{i,p}(X,X)f(X,Y)$. Similarly, $(f \ast h_{i,p})(X,Y) = f(X,Y)h_{i,p}(Y,Y)$. Thus the second part follows immediately from the first one. 

The following is the extension to our setting of [S] Theorem 3.6.2.

Proposition 4.14. We have $[h_i, h_j] = 0$ in $\mathcal{D}$.

Proof. We have to show that for every projective $H$-module $P$ there holds $\pi_p([h_i, h_j]) = 0$. By Proposition 4.13 we see that, for $X, Y \in \text{Gr}_{1,1}(P)$,

$$\pi_p([h_i, h_j])(X,Y) = [h_{i,p}, h_{j,p}](X,Y) = (s_i(P/X) - t_i(X) - s_i(P/Y) + t_i(Y))h_{j,p}(X,Y) = 0,$$

because $h_{j,p}(X,Y) = 0$ unless $X = Y$. Hence $[h_{i,p}, h_{j,p}] = 0$, as required.
The following is the extension to our setting of [S Corollary 3.7.1] and [S Theorem 3.9].

**Proposition 4.15.** Assume that \( j = i \), or that \((j,i) \in \Omega\) (so that all arrows between \( i \) and \( j \) in the quiver of \( H \) are of the form \( \alpha_{ji}(x) \colon i \to j \)). Then in \( D \) we have

\[
[h_{j,x_i}] = c_{ji}x_i, \quad \quad \quad [h_{j,y_i}] = -c_{ji}y_i.
\]

**Proof.** Let \( P \) be a projective \( H \)-module. Let \( X, Y \) be locally free submodules of \( P \). Since \( H \) is Iwanaga-Gorenstein of dimension 1, \( X \) and \( Y \) are also projective (see §2.1 and §2.3). By Proposition 4.13 we have

\[
[h_{j,P,x_i,P}](X,Y) = (s_j(P/X) - t_j(X) - s_j(Y) + t_j(Y)x_i,p(X,Y).
\]

By definition of \( x_i,p \) we have \( x_i,p(X,Y) = 0 \) unless \( X \subset Y \) and \( Y/X = E_i \). Since \( X \) and \( Y \) are projective, this implies that

\[
Y = P_i \oplus P', \quad X = \text{Ker}(P_i \to E_i) \oplus P', \quad P = P_i \oplus P' \oplus P'',
\]

where \( P_i \) is the projective cover of \( E_i \) and \( P', P'' \) are projective. Hence \( P/Y \cong P'' \) and \( P/X \cong E_i \oplus P'' \). By [GLS] Proposition 3.1,

\[
\text{Ker}(P_i \to E_i) = \bigoplus_{k \in \Omega(-,i)} P^{[k]}_k
\]

where \( \Omega(-,i) = \{ k = 1, \ldots, n \mid (k,i) \in \Omega \} \). Therefore if \( j = i \) then \( t_j(Y) - t_j(X) = 1 \) and \( s_j(P/X) - s_j(Y) = s_j(E_i) = 1 \), thus

\[
[h_{j,P,x_i,P}](X,Y) = 2x_i,p(X,Y).
\]

On the other hand, if \((j,i) \in \Omega\) then \( t_j(Y) - t_j(X) = -|c_{ji}| = c_{ji} \), and \( s_j(P/X) - s_j(Y) = 0 \), thus

\[
[h_{j,P,x_i,P}](X,Y) = c_{ji}x_i,p(X,Y).
\]

So in both cases we conclude that \([h_{j,x_i}] = c_{ji}x_i \). The identity \([h_{j,y_i}] = -c_{ji}y_i \) is proved similarly.

\( \square \)

Note that the relation of Proposition 4.15 also holds when \( c_{ji} = 0 \), that is, when there is no arrow between \( i \) and \( j \). In summary, in the Lie algebra \( P_D \) of primitive elements of \( D \), the following identities are satisfied:

(i) \([x_i,y_j] = 0\), if \( i \neq j\);

(ii) \([x_i,y_i] = h_i\), \([h_i,x_i] = 2x_i\), \([h_i,y_i] = -2y_i\);

(iii) \((\text{ad}x_i)^{1-c_{ij}}(x_j) = 0 = (\text{ad}y_i)^{1-c_{ij}}(y_j)\), if \( i \neq j\);

(iv) If \((j,i) \in \Omega\) or \( c_{ij} = 0 \) then \([h_{j,x_i}] = c_{ji}x_i\) and \([h_{j,y_i}] = -c_{ji}y_i\).

Relation (iii) for \( y_i \)'s follows from Corollary 4.6, taking into account that the Serre relations are left-right symmetric.

These are almost all the defining relations of the Kac-Moody algebra \( g \). The only relations of the complete presentation of \( g \) which have not been proved are relations (iv) when \((i,j) \in \Omega\). To check that these relations are **not** satisfied, one can calculate easily that if \((i,j) \in \Omega\),

\[
[h_{j,P,x_i,P}](\text{Ker}(P_i \to E_i), P_i) = 0, \quad x_i,P(\text{Ker}(P_i \to E_i), P_i) = 1,
\]

hence \([h_{j,x_i}] \neq c_{ji}x_i\).
5 The Lie algebra $\mathcal{L}$ of partially defined derivations

5.1 The ideal $J$

**Definition 5.1.** Let $J$ be the linear subspace of $\mathcal{D}$ consisting of all elements $x$ such that for every projective $H$-module $P$ and every locally free submodules $X \subseteq Y$ of $P$ such that $P/X$ is injective, we have $\pi_p(x)(X,Y) = 0$.

We denote by $\mathcal{D}^{\geq 0}$ (resp. $\mathcal{D}^{>0}$) the subalgebra of $\mathcal{D}$ generated by the elements $x_i$, $h_i$ (1 $\leq i \leq n$) (resp. $x_i$ (1 $\leq i \leq n$)). By Proposition 4.4 $\mathcal{D}^{\geq 0}$ is isomorphic to $\mathcal{M}$. The following is the extension to our setting of $\text{[S]}$ Theorem 4.3.

**Proposition 5.2.** $J \cap \mathcal{D}^{\geq 0}$ is a two-sided ideal of $\mathcal{D}^{\geq 0}$.

**Proof.** Let $x \in J \cap \mathcal{D}^{\geq 0}$ and let $f = \pi_p(x)$ for some projective $H$-module $P$. Let $X \subseteq Y$ be locally free submodules of $P$ such that $P/X$ is injective. Then, by definition of $J$ we have $f(X,Y) = 0$, and we want to show that for every $i = 1, \ldots, n$,

$$(h_i \cdot f)(X,Y) = (f \cdot h_i,P)(X,Y) = (x_i, P \cdot f)(X,Y) = (f \cdot x_i, P)(X,Y) = 0.$$

By Proposition 4.13 we have

$$(h_i \cdot f)(X,Y) = h_i(X,Y) = 0, \quad (f \cdot h_i,P)(X,Y) = h_i(P,Y) f(X,Y) = 0.$$

Next, combining Proposition 4.13 and the definition of $x_i, P$ we easily see that every $(U,V)$ in the support of an element $g \in \mathcal{D}^{\geq 0}$ satisfies $U \subseteq V$. We therefore have

$$(x_i \cdot f)(X,Y) = \int_{X \subseteq Z \subseteq Y} f(Z,Y)dZ.$$

Now, since $P/X$ is injective and $H$ is Iwanaga-Gorenstein of dimension 1, for every locally free $H$-module $Z$ such that $X \subseteq Z \subseteq P$, we have that $P/Z$ is injective. Hence the function $Z \mapsto f(Z,Y)$ vanishes in the range $X \subseteq Z \subseteq Y$, and $(x_i, P \cdot f)(X,Y) = 0$. The proof that $(f \cdot x_i, P)(X,Y) = 0$ is similar.

The following is the extension to our setting of $\text{[S]}$ Lemma 4.1.

**Proposition 5.3.** We have $J \cap \mathcal{D}^{>0} = \{0\}$.

**Proof.** Let $x$ be a nonzero element of $\mathcal{D}^{>0}$. Then $f : = \Phi^{-1}(x)$ is a non zero element of $\mathcal{M}$. Let $X$ be an $H$-module in the support of $f$. We denote by $I(X)$ an injective envelope of $X$, and by $\psi : P(I(X)) \rightarrow I(X)$ a projective cover of $I(X)$. Then we have

$$(\pi_p(I(X))(x))(\psi^{-1}(0), \psi^{-1}(X)) = f(\psi^{-1}(X)/\psi^{-1}(0)) = f(X) \neq 0.$$

On the other hand we have $\psi^{-1}(0) \subseteq \psi^{-1}(X) \subseteq P(I(X))$ and $P(I(X))/\psi^{-1}(0) \cong I(X)$ is injective. Thus $x \notin J$.

Recall from 4.4 the definition of $s_i(X)$ and $t_i(X)$.

**Lemma 5.4.** If $X$ is projective and $Y$ is injective we have

$$c_i t_i(X) = (\text{rank}(X), \alpha_i)_H, \quad c_i s_i(Y) = (\alpha_i, \text{rank}(Y))_H.$$
Proof. Since \( X \) is projective \( \langle \text{rank}(X), \alpha_i \rangle_H = \langle X, E_i \rangle_H = \dim \text{Hom}_H(X, E_i) = c_i t_j(X) \). Similarly, since \( Y \) is injective \( \langle \alpha_i, \text{rank}(Y) \rangle_H = \langle E_i, Y \rangle_H = \dim \text{Hom}_H(E_i, Y) = c_i s_i(Y) \). \( \square \)

The following is the extension to our setting of [S, Theorem 4.4].

**Proposition 5.5.** Let \( x \in D^{\geq 0} \) be of degree \( \beta \). Then

\[
[x, h_i] + \frac{1}{c_i} (\beta, \alpha_i) x
\]

belongs to the ideal \( J \).

Proof. Let \( f = \pi_P(x) \) for some projective \( H \)-module \( P \). Let \( X \subseteq Y \) be locally free submodules of \( P \) such that \( P/X \) is injective. If \( \text{rank}(Y) - \text{rank}(X) \neq \beta \), we have \( f(X,Y) = [h_i, f](X,Y) = 0 \). So we can assume \( \text{rank}(Y) - \text{rank}(X) = \beta \). By Proposition 4.13 and Lemma 5.4 we then have

\[
[f, h_i, P](X,Y) = -(s_i(P/X) - s_i(P/Y)) + (t_j(Y) - t_i(X)) f(X,Y)
\]

\[
= -\frac{1}{c_i} (\alpha_i, \beta)_H + (\beta, \alpha_i)_H f(X,Y)
\]

\[
= -\frac{1}{c_i} (\beta, \alpha_i) f(X,Y).
\]

Since this holds for any projective \( H \)-module \( P \), it follows that \( [x, h_i] + \frac{1}{c_i} (\beta, \alpha_i) x \in J \). \( \square \)

5.2 The algebra \( \mathcal{E} \)

The next definition is based on Proposition 5.2.

**Definition 5.6.** Denote by \((\mathcal{E}, \ast)\) the associative \( \mathbb{C} \)-algebra

\[
\mathcal{E} := D^{\geq 0}/(J \cap D^{\geq 0}).
\]

By Proposition 5.3 the assignment

\[
x_i \mapsto x_i \mod J, \quad (1 \leq i \leq n)
\]

extends to an injective algebra homomorphism \( D^{\geq 0} \to \mathcal{E} \). We can therefore write for short \( x_i \in \mathcal{E} \) instead of \( x_i \mod J \). We will also allow ourselves to write \( h_i \in \mathcal{E} \) for the class of \( h_i \) modulo \( J \). The algebra \( \mathcal{E} \) is \( \mathbb{N}^n \)-graded via

\[
\deg(x_i) = \alpha_i, \quad \deg(h_i) = 0, \quad (1 \leq i \leq n).
\]

By Proposition 5.5 the following identity holds in \( \mathcal{E} \):

\[
[h_i, f] = \frac{1}{c_i} (\beta, \alpha_i) f, \quad (1 \leq i \leq n),
\]

for every \( f \in \mathcal{E} \) with \( \deg(f) = \beta \).
5.3 The Lie algebra $\mathcal{L}$

5.3.1 Derivations

A right derivation on $E$ is a $C$-linear map $\mathbf{x} : E \to E$, $f \mapsto (f)\mathbf{x}$
such that

$$(f \ast g)\mathbf{x} = (f)\mathbf{x} \ast g + f \ast (g)\mathbf{x}, \quad (f, g \in E).$$

It is easy to check that right derivations on $E$ form a $C$-vector space, and a Lie algebra for the
bracket:

$$(f)[x, y] := ((f)x)y - ((f)y)x, \quad (f \in E). \quad (7)$$

For example, for every $i = 1, \ldots, n$, the maps $x_i, h_i$ defined by

$$(f)x_i := [f, x_i], \quad (f)h_i := [f, h_i], \quad (f \in E),$$

are right derivations on $E$. Note that, by (6), we have

$$(f)h_i = -\frac{1}{c_i}(\beta, \alpha_i)f$$

for $f \in E$ of degree $\beta$.

5.3.2 Partially defined derivations

We say that a $C$-linear map $\mathbf{z} : V \to E$ is a partially defined right derivation on $E$ if $V$ is a subspace
of $E$ of finite codimension and for every $f, g \in V$ such that $f \ast g \in V$ we have

$$(f \ast g)\mathbf{z} = (f)\mathbf{z} \ast g + f \ast (g)\mathbf{z}.$$  

Given two partially defined derivations $\mathbf{z}'$ and $\mathbf{z}''$, we write $\mathbf{z}' = \mathbf{z}''$ if $(f)\mathbf{z}' = (f)\mathbf{z}''$ for every $f$ in
a subspace of $E$ of finite codimension contained in the intersection of the domains of $\mathbf{z}'$ and $\mathbf{z}''$. This is an equivalence relation (because the intersection of three subspaces of finite codimension
has finite codimension). We denote by $\mathcal{R}$ the set of congruence classes of partially defined right derivations on $E$ for the congruence relation $\equiv$. For simplicity, we shall use the same notation for
a partially defined derivation $\mathbf{z}$ and its congruence class. Then $\mathcal{R}$ becomes a $C$-vector space if we define $\mathbf{z}' + \mathbf{z}''$ by

$$(f)(\mathbf{z}' + \mathbf{z}'') := (f)\mathbf{z}' + (f)\mathbf{z}''$$

for $f$ in the intersection of the domains of $\mathbf{z}'$ and $\mathbf{z}''$. It is also easy to check that the same formula as (7) endows $\mathcal{R}$ with the structure of a Lie algebra.

For instance, we can define elements $y_i$ of $\mathcal{R}$ by

$$(f)y_i := [f, y_i] \mod J, \quad (f \in D^{\geq 0}, \ 1 \leq i \leq n).$$

The fact that for $f \in D^{\geq 0}$ we have $[f, y_i] \in D^{\geq 0}$ follows from the identities

$$[x_j, y_i] = \delta_{ij}h_i, \quad [f \ast g, y_i] = [f, y_i] \ast g + f \ast [g, y_i], \quad (1 \leq i, j \leq n, \ f, g \in D^{\geq 0}).$$

Note that the derivations $x_i$ and $h_i$ also give rise to well-defined elements of $\mathcal{R}$, which we continue
to denote by $x_i$ and $h_i$ according to our convention.
5.3.3 Definition of $\mathcal{L}$ and relations in $\mathcal{L}$

**Definition 5.7.** Let $\mathcal{L}$ denote the Lie subalgebra of $\mathcal{R}$ generated by $x_i, h_i, y_i$, $(1 \leq i \leq n)$.

The following theorem is an analogue of [S] Theorem 4.6.

**Theorem 5.8.** The following identities hold in $\mathcal{L}$:

(i) $[x_i, y_j] = \delta_{ij} h_i$;

(ii) $[h_i, h_j] = 0$;

(iii) $[h_i, x_j] = c_{ij} x_i, [h_j, y_i] = -c_{ji} y_i$;

(iv) $(\text{ad} x_i)^{1-c_{ij}}(x_j) = 0, (\text{ad} y_i)^{1-c_{ij}}(y_j) = 0$ ($i \neq j$).

**Proof.** Let us denote by $\pi$ the class in $\mathcal{E}$ of an element $u$ of $D^{>0}$. By definition, for $f \in D^{>0}$, we have

$$(f)[x_i, y_j] = [[[f, x_j], y_j] - [[f, y_j], x_j] - [f, [x_j, y_j]] = \delta_{ij} [f, h_i] = \delta_{ij} \cdot (f) h_i.$$  

This proves (i). The proof of (ii) follows immediately from (ii). Next, assume that $f \in \mathcal{E}$ is homogeneous of degree $\beta$. Then

$$((f)[h_j, x_i] = [[[f, h_j], x_i] - [[f, x_i], h_j] - [f, [h_j, x_i]] = -\frac{1}{c_j} (\beta, \alpha_j) [f, x_i] + \frac{1}{c_j} (\beta + \alpha_i, \alpha_j) [f, x_i] = c_{ji} \cdot (f) x_i.$$  

Similarly, one has $(f)[h_j, y_i] = -c_{ji} \cdot (f) y_i$ for $f \in D^{>0}$. This proves (iii). Finally, (iv) follows from Proposition 3.10, Proposition 4.4 and Corollary 4.6. 

Note that $\mathcal{L}$ is a $\mathbb{Z}^n$-graded Lie algebra via

$$\text{deg}(x_i) = \alpha_i, \quad \text{deg}(y_i) = -\alpha_i, \quad \text{deg}(h_i) = 0.$$  

**Lemma 5.9.** The dimension of the subspace of $\mathcal{L}$ spanned by the $h_i$ $(1 \leq i \leq n)$ is equal to the rank of the Cartan matrix $C$.

**Proof.** Let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be such that for every $i$ we have $\sum_{j=1}^n \lambda_j c_{ji} = 0$. Define $h := \sum_{j=1}^n \lambda_j h_j$. Then, for every $\beta = \sum_{i=1}^n b_i \alpha_i \in \mathbb{N}^n$ we have

$$\sum_{j=1}^n \frac{\lambda_j}{c_j} (\alpha_j, \beta) = \sum_{i=1}^n b_i \sum_{j=1}^n \lambda_j c_{ji} = 0.$$  

It then follows from (ii) that $h = 0$.

Conversely, let $h := \sum_{j=1}^n \lambda_j h_j$ and suppose that $h = 0$. We know that for every $k > 0$ the homogeneous component of $\mathcal{M}$ of degree $k \alpha_i$ is nonzero. Indeed the constructible function $\theta_i^{<k}$ takes the value $k$! on $E_i^{\leq k}$. Since $\mathcal{M} \cong D^{>0}$ embeds in $\mathcal{E}$, this shows that the homogeneous component $\mathcal{E}_{k\alpha_i}$ of $\mathcal{E}$ of degree $k \alpha_i$ is nonzero. By our assumption, we have $(f) h = 0$ for every $f$ in a subspace $V$ of $\mathcal{E}$ of finite codimension. The intersection

$$V \cap \left( \bigoplus_{k>0} \mathcal{E}_{k\alpha_i} \right)$$

must be of finite codimension in $\bigoplus_{k>0} \mathcal{E}_{k\alpha_i}$. Therefore, for every $i = 1, \ldots, n$ we can find $k > 0$ and a nonzero element $f \in \mathcal{E}_{k\alpha_i}$ such that

$$0 = (f) h = \left( \sum_{j=1}^n \frac{\lambda_j}{c_j} (\alpha_j, k \alpha_i) \right) = k \left( \sum_{i=1}^n \lambda_j c_{ji} \right) f = 0.$$  

Thus for every $i$ we have $\sum_{j=1}^n \lambda_j c_{ji} = 0$. 


5.4 Main result

Following [K2, §1.2] define a Lie algebra $\tilde{g}$ by the following presentation. The generators are $\tilde{x}_i, \tilde{y}_i, \tilde{h}_i$ $(1 \leq i \leq n)$, and the relations:

$$[\tilde{x}_i, \tilde{y}_j] = \delta_{ij}\tilde{h}_i, \quad [\tilde{h}_i, \tilde{h}_j] = 0, \quad [\tilde{h}_i, \tilde{x}_j] = c_{ij}\tilde{x}_j, \quad [\tilde{h}_i, \tilde{y}_j] = -c_{ij}\tilde{y}_j.$$ 

This is a $\mathbb{Z}^n$-graded Lie algebra via:

$$\deg(\tilde{x}_i) = \alpha_i, \quad \deg(\tilde{y}_i) = -\alpha_i, \quad \deg(\tilde{h}_i) = 0.$$ 

By [K2 Theorem 1.2, Theorem 9.11], $\tilde{g}$ has a unique $\mathbb{Z}^n$-graded maximal ideal $r$ not intersecting the subspace $\tilde{h}$ spanned by the $\tilde{h}_i$ $(1 \leq i \leq n)$. Moreover $r$ is generated by the elements

$$(\text{ad} \tilde{x}_i)^{1-c_{ij}}(\tilde{x}_j), \quad (\text{ad} \tilde{y}_i)^{1-c_{ij}}(\tilde{y}_j), \quad (i \neq j).$$

Hence, the assignment $\tilde{x}_i \mapsto e_i, \tilde{y}_i \mapsto f_i, \tilde{h}_i \mapsto h_i$ $(1 \leq i \leq n)$ induces a Lie algebra isomorphism $\tilde{g}/r \cong g$ (the Gabber-Kac theorem). To be more precise, we deal here with the Lie algebras denoted by $\tilde{g}'$ and $g'$ in [K2]. This version of the Gabber-Kac theorem for $\tilde{g}'$ and $g'$ is explained in [K2 Remark 1.5].

The following theorem is our main result. It is an analogue of [S Theorem 4.7] for our algebra $H(C, D, \Omega)$.

**Theorem 5.10.**

(i) The Lie algebra $\mathcal{L}$ is isomorphic to $g$.

(ii) The Lie algebra $\mathcal{P}_M$ is isomorphic to the positive part $n$ of $g$.

(iii) The homomorphism $F : U(n) \rightarrow M$ is an isomorphism of Hopf algebras.

**Proof.**

By Theorem 5.8 we have a surjective Lie algebra homomorphism from $\tilde{g}$ to $\mathcal{L}$ mapping $\tilde{x}_i$ to $x_i$, $\tilde{h}_i$ to $h_i$, and $\tilde{y}_i$ to $y_i$. Its kernel is a $\mathbb{Z}^n$-graded ideal $s$ of $\tilde{g}$ not intersecting $h$, because by Lemma 5.9 the space spanned by the $h_i$’s is isomorphic to the space spanned by the $\tilde{h}_i$’s. Hence by the Gabber-Kac theorem, $s \subseteq r$ and $g$ is a homomorphic image of $\mathcal{L}$. Since all the defining relations of $g$ are already satisfied on $\mathcal{L}$, this proves (i).

Therefore the Lie subalgebra of $\mathcal{L}$ generated by the $x_i$’s is presented by the first relations of Theorem 5.8 (iv). But, by construction, this Lie algebra is a homomorphic image of the Lie algebra $\mathcal{P}_{D^{>0}}$ of primitive elements of $D^{>0}$ by mapping $x_i$ to $x_i$. This proves that the $x_i$’s cannot satisfy more relations than the $e_i$’s. Since $\mathcal{P}_M \cong \mathcal{P}_{D^{>0}}$, this shows (ii).

Finally, (ii) implies that $F$ is an isomorphism of algebras. Since both families of generators $e_i$ and $\theta_i$ are primitive, this is in fact an isomorphism of Hopf algebras. This proves (iii). 

6 Examples

We give a description of the algebra $M$ in two basic cases corresponding to [GLS] §13.5, §13.6.

6.1 Dynkin type $A_2$

Let

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
with symmetrizer $D = \text{diag}(2, 2)$ and $\Omega = \{(1, 2)\}$. Thus $C$ is a Cartan matrix of Dynkin type $A_2$ with a non-minimal symmetrizer. We have $f_{12} = f_{21} = 1$. Thus $H = H(C, D, \Omega)$ is given by the quiver

$$
\begin{array}{c}
\circlearrowright_{e_1} \\
1 \\
\circlearrowright_{e_2}
\end{array}
$$

with relations $e_1^2 = e_2^2 = 0$ and $e_1 \alpha_{12} = \alpha_{12} e_2$. There are 4 isomorphism classes of indecomposable locally free $H$-modules, displayed as follows in [GLS] §13.5:

$$
E_1 = P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
P_2 = I_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \\
E_2 = I_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \\
X = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.
$$

Here, the numbers 1, 2, in the pictures of the modules correspond to composition factors. Note that $P_2$ and $X$ have the same rank vector.

Denote by $1_M$ the characteristic function of the $G_4$-orbit of a locally free $H$-module $M$ of dimension vector $d$. Thus, $\theta_1 = 1_{E_1}$ and $\theta_2 = 1_{E_2}$. We have

$$
\theta_2 \ast \theta_1 = 1_{E_1 \oplus E_1}, \\
\theta_1 \ast \theta_2 = 1_{P_2} + 1_X + 1_{E_1 \oplus E_2}, \\
[\theta_1, \theta_2] = 1_{P_2} + 1_X.
$$

The enveloping algebra $\mathcal{M} \cong U(n)$ has a Poincaré-Birkhoff-Witt basis given by

$$
\theta_2^a \ast [\theta_1, \theta_2]^b \ast \theta_1^c = a!b!c!d! \sum_{k=0}^{b} 1_{E_1 \oplus P_2 \oplus X^{b-1} \oplus P_1^k}, \\
(a, b, c, d \in \mathbb{Z}_{\geq 0}).
$$

### 6.2 Dynkin type $B_2$

Let

$$
C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}
$$

with symmetrizer $D = \text{diag}(2, 1)$ and $\Omega = \{(1, 2)\}$. Thus $C$ is a Cartan matrix of Dynkin type $B_2$. We have $f_{12} = 1$ and $f_{21} = 2$. Then $H = H(C, D, \Omega)$ is given by the quiver

$$
\begin{array}{c}
\circlearrowright_{e_1} \\
1 \\
\circlearrowright_{e_2}
\end{array}
$$

with relation $e_1^2 = 0$. There are 5 isomorphism classes of indecomposable locally free $H$-modules, displayed as follows in [GLS] §13.6:

$$
E_1 = P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
P_2 = I_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \\
I_2 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \\
E_2 = I_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \\
X = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.
$$

Note that $P_2$ and $X$ have the same rank vector. We have

$$
\theta_1 = 1_{E_1}, \\
\theta_2 = 1_{E_2}, \\
[\theta_1, \theta_2] = 1_{P_2} + 1_X, \\
[[\theta_1, \theta_2], \theta_2] = 2 \cdot 1_{I_1}.
$$

The enveloping algebra $\mathcal{M} \cong U(n)$ has a Poincaré-Birkhoff-Witt basis given by

$$
\theta_2^a \ast [[\theta_1, \theta_2], \theta_2]^b \ast [\theta_1, \theta_2]^c \ast \theta_1^d = 2^a b! c! d! \sum_{k=0}^{c} 1_{E_1 \oplus P_2 \oplus X^{c-1} \oplus P_1^k}, \\
(a, b, c, d \in \mathbb{Z}_{\geq 0}).
$$

22
References

[Bo] K. Bongartz, *A Geometric version of the Morita equivalence*, J. Algebra **139** (1991), 159–171.

[BT] T. Bridgeland, V. Toledano Laredo, *Stability conditions and Stokes factors*, Invent. Math. **187** (2012), 61–98.

[DR] V. Dlab, C.M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. 6 (1976), no. 173, v+57 pp.

[GLS] C. Geiss, B. Leclerc, J. Schröer, *Quivers with relations for symmetrizable Cartan matrices I : Foundations*, Preprint (2014), 67 pp., arXiv:1410.1403.

[J] D. Joyce, *Configurations in abelian categories. II. Ringel-Hall algebras*, Adv. Math. **210** (2007), 635–706.

[K1] V. Kac, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. **56** (1980), 57–92.

[K2] V. Kac, *Infinite dimensional Lie algebras*, (3rd ed.) Cambridge 1990.

[L1] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421.

[L2] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), 129–139.

[Rie] C. Riedtmann, *Lie algebras generated by indecomposables*, J. Algebra **170** (1994), 526–546.

[Rin1] C. M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583–591.

[Rin2] C. M. Ringel, *Lie algebras arising in representation theory*, in: Representations of algebras and related topics, Kyoto, 1990. London Math. Soc. Lecture Note Ser., **168**, 284–291, Cambridge University Press 1992.

[S] A. Schofield, *Quivers and Kac-Moody Lie algebras*, unpublished manuscript.

[Sw] M. E. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

---

Christof Geiss : Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, 04510 México D.F., México.
email: christof@math.unam.mx

Bernard Leclerc : Normandie Univ, France; UNICAEN, LMNO F-14032 Caen, France; CNRS UMR 6139, F-14032 Caen, France; Institut Universitaire de France.
email: bernard.leclerc@unicaen.fr

Jan Schröer : Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
email: schroer@math.uni-bonn.de