Local heterotic geometry in holomorphic coordinates

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Abstract
In the same spirit as done for N=2 and N=4 supersymmetric non-linear σ models in 2 space-time dimensions by Zumino and Alvarez- Gaumé and Freedman, we analyse the (2,0) and (4,0) heterotic geometry in holomorphic coordinates. We study the properties of the torsion tensor and give the conditions under which (2,0) geometry is conformally equivalent to a (2,2) one. Using additional isometries, we show that it is difficult to equip a manifold with a closed torsion tensor, but for the real 4 dimensional case where we exhibit new examples. We show that, contrarily to Callan, Harvey and Strominger ’s claim for real 4 dimensional manifolds, (4,0) heterotic geometry is not necessarily conformally equivalent to a (4,4) Kähler Ricci flat geometry. We rather prove that, whatever the real dimension be, they are special quasi Ricci flat spaces, and we exemplify our results on Eguchi-Hanson and Taub-NUT metrics with torsion.
1 Introduction

In 1979, B. Zumino [1] proved the importance of complex geometry for the study of supersymmetric $\sigma$-models. In particular, using holomorphic coordinates, he showed that $N = 2$ supersymmetry in two space-time dimensions requires the bosonic part of the Lagrangian density to be built on a Kähler manifold. The work of Alvarez-Gaumé and Freedman has generalised this approach to $N = 4$ supersymmetry which was shown to require hyperkähler manifolds [2]. The heterotic supersymmetries lead to generalisations of the aforementioned geometries: in the $(2,0)$ [3, 4] and in the $(4,0)$ cases [5, 6, 7, 8], it was shown that one needs target manifolds with torsion. It is the aim of the present work to analyse the properties of such manifolds equipped with torsion using holomorphic coordinates.

In section 2, we recall the necessary and sufficient conditions on the target space metric and torsion tensors for $(2,0)$ heterotic supersymmetry and express them in holomorphic coordinates. Various geometrical objects are then given in such coordinates, in the same spirit as done by Alvarez-Gaumé and Freedman [9] and Hull [4]. A torsion potential $b_{\mu\nu}$ and a vector $V_\mu$ (dual of the torsion tensor in the special case of 4 real dimensions) are then introduced and used to express the relevant geometrical objects.

We then analyse the necessary and sufficient conditions under which, given a complex Riemannian manifold of real dimension $2N$, a $(2,0)$ heterotic geometry (metric $g_{\mu\nu} +$ torsion) and a $(2,2)$ one (Kähler metric $\hat{g}_{\mu\nu}$, no torsion) have conformally related metrics:

$$\hat{g}_{\mu\nu} = e^{-2f} g_{\mu\nu}.$$ 

This appears to be a very restrictive condition as the conformal factor has to be a real function, harmonic with respect to the Kähler metric laplacian, whatever the dimension of the manifold be:

$$\hat{\Delta} e^{2f} = 0.$$ 

In order to describe geometries leading to one-loop finite non-linear $\sigma$ models in two space time dimensions, following Friedan [10] in the torsionless case, Friedling and Van de Ven [11] introduced “generalised quasi Ricci flat” metrics through [1]:

$$\exists \ W_\mu \text{ and } \chi_\mu \text{ such that } Ric_{(\mu\nu)} = \nabla_{(\mu} W_{\nu)}$$

$$Ric_{[\mu\nu]} = \frac{1}{2} T^\rho_{\mu\nu} W_\rho + \nabla_{[\mu} \chi_{\nu]}.$$ (1)

We then express these conditions in holomorphic coordinates and also discuss the conditions that restrict the holonomy group from $U(N)$ to $SU(N)$: indeed, it was argued by Hull [4] that such a requirement is necessary for off-shell one-loop finiteness of $(2,0)$ non-linear $\sigma$ models in two space time dimensions. We show that the metric has to be a special “generalized quasi Ricci flat” one [1].

Several examples of metrics with special isometries are then constructed in Section 3. Despite the large isometry groups considered, which enforce conformal equivalence with

$^1 A_{[\lambda\mu\nu]\rho} \overset{\text{def}}{=} \frac{1}{2}(A_{\lambda\mu\nu\rho} - A_{\lambda\nu\mu\rho})$ and $A_{(\lambda\mu\nu)\rho} \overset{\text{def}}{=} \frac{1}{2}(A_{\lambda\mu\nu\rho} + A_{\lambda\nu\mu\rho})$. Moreover $\nabla_\mu$ is the ordinary covariant derivative with the symmetric Christoffel connection.

$^2 \chi_\mu = -W_\mu$ where $W_\mu$ is related to the trace of the torsion tensor.
the Kähler torsionless case, (2,0) supersymmetry is not sufficient to define uniquely the metrics and several examples of supplementary properties are analysed. In particular, we generalise the (torsionless) metrics of LeBrun \cite{12} which have a vanishing scalar curvature, to target spaces with torsion and obtain as a special case an easy derivation of Eguchi-Hanson metric with torsion \cite{3} directly in holomorphic coordinates. Unfortunately, despite their conformal equivalence with regular Kähler ones, all the metrics we obtained are singular.

Section 4 is devoted to (4,0) heterotic geometry. The necessary and sufficient conditions for (4,0) supersymmetry are given (as shown in \cite{7}, they are slightly less restrictive than often asserted ( see for example \cite{8})), and we prove that in any 4N dimensional case, the metric has to be a special “generalised quasi Ricci flat” one where the vector $V_\mu$ depends on a single complex function $F$. We also show that scalar flatness is obtained on general grounds only for 4 real dimensions.

We then analyse the possible conformal equivalence of such metrics with torsionless hyperkähler ones involved in (4,4) supersymmetry and show that, even in 4 real dimensions, contrarily to Callan et al.’s claim \cite{14}, (4,0) world sheet supersymmetry does not imply that the corresponding metric is conformally equivalent to a Ricci flat Kähler one.

Section 5 is then devoted to a detailed study of Eguchi-Hanson and Taub-NUT metrics with torsion \cite{13} in holomorphic coordinates. While Eguchi-Hanson with torsion is indeed conformally equivalent to its torsionless counterpart, this fails to be true for Taub-NUT with torsion. To conclude some remarks are offered in Section 6.

2 (2,0) heterotic geometry in complex coordinates

2.1 Generalities and notations

As explained in the introduction, we consider 2N real dimensional complex Riemannian manifolds with torsion and assume that the metric is hermitian with respect to the covariantly constant complex structure $J$ and that the torsion tensor three form is closed. These conditions are necessary and sufficient to build a (2,0) supersymmetric $\sigma$ model with a Wess-Zumino term \cite{3}. All these hypothesis write :

- distance :
  \[ d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad g_{\mu\nu} = g_{\nu\mu} \]  
  \[ g = g^{\mu\nu}(2) \]

- torsion : the torsion tensor $T_{\mu\nu\rho} = g_{\mu\sigma}T^\sigma_{\nu\rho}$ is fully skew-symmetric and its associated three form is closed
  \[ T = {\frac{1}{3!}}T_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho, \quad dT = 0 \]

- $J_\mu^\nu$ is an almost complex structure :
  \[ J_\mu^\nu J_\nu^\rho = - \delta_\mu^\rho \]

- integrability condition on $J_\mu^\nu$ to be a complex structure :
  \[ N_{\mu\nu}^\rho \equiv J_\lambda^\mu(\partial_\lambda J_\nu^\rho - \partial_\nu J_\lambda^\rho) - (\mu \leftrightarrow \nu) = 0 \]

\[ ^{3} \text{first obtained by Delduc and Valent} \cite{13}. \]
- Hermiticity: the metric is hermitian with respect to the complex structure

\[ g_{\mu \lambda} J^{\lambda}_\nu + J^{\lambda}_\mu g_{\lambda \nu} = 0 \]  

(6)

which is equivalent to the statement that \( J_{\mu \nu} \triangleq g_{\mu \sigma} J^{\sigma}_{\rho \nu} \) is skew symmetric and therefore locally defines a two-form:

\[ \omega = \frac{1}{2} J_{\mu \nu} dx^\mu \wedge dx^\nu \]  

(7)

- Covariant constancy of \( J^{\mu}_{\nu} \):

\[ D_{\mu} J^{\rho}_{\nu} \equiv \partial_{\mu} J^{\rho}_{\nu} + \Gamma_{\mu \lambda \nu}^{\rho} J^{\lambda}_{\nu} - \Gamma_{\nu \mu \lambda}^{\lambda} J^{\rho}_{\lambda} = 0 \]  

(8)

(The connection with torsion is taken as \( \Gamma_{\mu \rho \nu}^{\rho} = \gamma_{\mu \rho \nu} - \frac{1}{2} T_{\mu \rho \nu} \) where \( \gamma_{\mu \rho \nu} \) is the usual symmetric Christoffel connection). As a consequence of equ. (8), the integrability condition (5) reduces to an algebraic constraint on the torsion:

\[ N^{\rho}_{\mu \nu} \equiv T^{\rho}_{\mu \nu} - J^{\lambda}_{\mu \nu} T^{\rho}_{\lambda \sigma} + (J^{\lambda}_{\mu \sigma} T^{\rho}_{\lambda \nu} - J^{\lambda}_{\nu \sigma} T^{\rho}_{\lambda \mu}) J^{\rho}_{\lambda} = 0 \]  

(9)

(compare with equ.(21) of \[8\]) and then one obtains:

\[ d\omega = -\frac{1}{2} T_{\nu \rho \mu} J^{\rho}_{\mu} dx^\mu \wedge dx^\nu \wedge dx^\rho = \frac{1}{2} T_{\rho \mu \nu} J^{\mu}_{\rho} J^{\nu}_{\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \]  

(10)

In the absence of torsion, equations (2,4,6,8) define a Kähler manifold.

In the following, due to (4,5), one can choose a coordinate system where \( J \) is diagonal and constant. The real coordinates \( x^\mu \) being split into the complex ones \( z^i \) and \( \bar{z}^i \), one has:

\[ \{ J^i_{\bar{j}} = J^i_{\bar{j}} = 0 \ , \ J_j^i = -J^{\bar{i}}_{\bar{j}} = i\delta^{\bar{i}}_j \} \Leftrightarrow \{ J_{ij} = J_{i\bar{j}} = 0 \ , \ J_{i\bar{j}} = J_{j\bar{i}} = ig_{ji} \} \]  

(11)

2.2 The geometrical objects in complex coordinates

In these complex coordinates, equation (6) gives:

\[ g_{ij} = g_{i\bar{j}} = 0 \]

The distance becomes:

\[ d\tau^2 = 2g_{ij} dz^i d\bar{z}^j \]  

(12)

and the complex structure two-form

\[ \omega = ig_{ij} dz^i \wedge d\bar{z}^j \]  

(13)

The covariant constancy of \( J \) then implies:

\[ (8) \Rightarrow \Gamma^i_{jk} = \Gamma^i_{j\bar{k}} = \Gamma_{ij\bar{k}} = \Gamma_{j\bar{k}i} = 0 \]  

(14)
Using the well known expression of the symmetric connection $\gamma^i_{\mu\nu}$ corresponding to the hermitian metric $g_{\mu\nu}$:

$$\gamma^i_{jk} = \frac{1}{2} g^{i\bar{l}} [g_{\bar{k},j} + g_{\bar{j},k}]$$

$$\gamma^i_{j\bar{k}} = \gamma^i_{k\bar{j}} = \frac{1}{2} g^{i\bar{l}} [g_{\bar{k},j} - g_{\bar{j},k}]$$

$$\gamma^i_{j\bar{k}} = 0$$  (15)

and conjugate expressions for $\gamma^i_{\mu\nu}$, equations (14) give:

$$T^i_{ijk} = T^i_{i\bar{j}k} = 0$$

$$T^i_{ij\bar{k}} = g^{i\bar{k}}_{\bar{l},j} - g^{i\bar{j}}_{\bar{l},k}$$

$$\Gamma^i_{jk} = \frac{1}{2} T^i_{jk} = g^{i\bar{l}} [g_{\bar{j},k} - g_{\bar{k},j}] = 2 \gamma^i_{j\bar{k}}$$  (16)

Of course, the algebraic constraint (9) is identically satisfied.

The closedness of the torsion, equ. (3), then writes:

$$[T^i_{ijk}, T^i_{i\bar{j}k}] + [T^i_{ij\bar{k}}, T^i_{i\bar{j}k}] = 0$$  (17)

or equivalently:

$$g_{i[k,\bar{l}],j} = g_{j[k,\bar{l}],i}$$

Equation (17) can also be written:

$$D_{i}T^i_{k|\ell} + D_{j}T^i_{ij\bar{k}} = T^\nu_{ij} T^i_{k\bar{\ell}} - 2 T^\nu_{i[k} T^i_{\ell]j}$$  where $\nu \equiv (n, \bar{n})$  (18)

The closedness of the torsion may also be solved through the introduction of a skew-symmetric torsion potential $b_{\mu\nu}$ (13):

$$T_{\mu\nu\rho} = b_{\nu\rho,\mu} + b_{\rho\mu,\nu} + b_{\mu\nu,\rho}$$

or, in complex coordinates

$$T_{ijk} = b_{jk,i} - b_{ik,j} + b_{ij,k}$$

Equation (16) then implies the existence of a vector potential $(\chi_i, \chi_{\bar{i}})$ and of a “gauge” freedom $(v_i, v_{\bar{i}})$ such that (3, 8):

$$b_{ij} = v_{j,i} - v_{i,j}$$

$$b_{i\bar{j}} + g_{i\bar{j}} = \chi_{j,i} - v_{i,j}$$

Due to equation (14), the Riemann tensor defined by

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - (\rho \leftrightarrow \sigma)$$  (19)

As usual, a comma indicates a derivative with respect to the coordinate.
has many vanishing components in holomorphic coordinates

\[ R_{jk\lambda}^i = R_{j\kappa\lambda}^i = 0. \]  

(20)

The surviving ones are

\[ R_{jk}^i = -g^{ij} [\partial_j T_{k\bar{n}} - 2T_{\bar{n}|k}^m g_l |_{m,j}] \]
\[ R_{jk}^i = -R_{j\bar{k}}^i = -\partial_i \Gamma^i_{jk} - D_k T_{j\bar{l}}^i + T_{l|k}^m T_{j\bar{n}}^i \]
\[ R_{j\bar{k}}^i = g^{ij} [\partial_{\bar{i}} T_{\bar{k}j} - 2T_{\bar{n}|k}^m g_l |_{m,j}] \]  

(21)

and similar expressions for \( R_{j\kappa\lambda}^i \).

The Ricci tensor

\[ Ric_{\mu\nu} \equiv R_{\mu\nu}^\sigma \]

then writes

\[ Ric_{ij} = D_i T_{kj}^k + T_{ij}^k T_{lk} \]
\[ Ric_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \| g \| + D_k T_{ij}^k - T_{il}^k T_{j\bar{k}}^l \]  

(23)

(for vanishing torsion, one recovers the usual results \( [1] \)).

We introduce the vector:

\[ V_\mu = \frac{1}{2} j^\nu j^\lambda T^\rho_{\nu\lambda} \Leftrightarrow \left( V_i = T_{ki}^k = -T_{ki}^k \right) \]

(24)

and obtain, by contraction of the closedness relation \( [18] \),

\[ (D_i V_j + D_j V_i) + (D_k T_{ij}^k + D_k T_{ji}^k) - 2(T_{ij}^k V_k + T_{ji}^k V_k + T_{il}^k T_{j\bar{k}}^l) = 0 \]

(25)

and

\[ g^{ij} (D_i V_j + D_j V_i) = g^{ij} (T_{ki}^k T_{kj}^l + 2V_i V_j) \]

(26)

Moreover, as a consequence of equation \( [18] \), we find:

\[ D_{[i} T_{j]}^{k \bar{i}} = D_{[i} T_{j]}^{k \bar{i}} \]

which gives by contraction

\[ D_k T_{ij}^k - D_k T_{ji}^k + D_i V_j - D_j V_i = 0. \]

(27)

Using these equations, the Ricci tensor may be expressed in holomorphic coordinates as:

\[ Ric_{ij} = D_j V_i + 2\partial_i (D_j V_j) \]
\[ Ric_{i\bar{j}} = D_j V_i + \frac{1}{2} \left[ \partial_{\bar{i}} (\partial_j \log \| g \| - 2V_j) + \partial_j (\partial_{\bar{i}} \log \| g \| - 2V_i) \right]. \]

(28)

Finally, the scalar curvature

\[ R \equiv g^{\mu\nu} Ric_{\mu\nu} \]

writes:

\[ R = 2g^{ij} \left[ \partial_i \partial_{\bar{j}} \log \| g \| + 2V_i V_j - \frac{1}{2} (D_i V_j + D_j V_i) \right] \]

(29)
2.3 Generalised Quasi Ricci flat metrics

For further use, we recall that in [10], Friedan defines “quasi Ricci flat metrics” through (no torsion)

\[ \exists \ V_\mu \ \text{such that} \ \text{Ric}_{\nu\mu} = D_\mu V_\nu + D_\nu V_\mu \]

Such geometries lead to one-loop finiteness for the D=2 non-linear σ models built on such target space metrics. They have been studied by Bonneau and Delduc and some explicit expressions have been obtained [15]. Here, in the presence of torsion, the Ricci tensor is no longer symmetric and, following Friedling and van de Ven [11], we define “generalised quasi Ricci flatness” by the same one loop finiteness requirement. This leads to the definition:

\[ \exists \ W_\mu \text{ and } \chi_\mu \ \text{such that} \ \text{Ric}_{(\mu\nu)} = \nabla_{(\mu} W_{\nu)} \]
\[ \text{Ric}_{[\mu\nu]} = \frac{1}{2} T^\rho_{\mu\nu} W_\rho + \nabla_{[\mu} \chi_{\nu]} \]  

(30)

In this (2,0) heterotic geometry, conditions (30) simplify to:

\[ \text{Ric}_{ij} = D_j W_i + \partial_{[i} (W + \chi)_{j]} \]
\[ \text{Ric}_{i\bar{j}} = D_{\bar{j}} W_i + \partial_{[i} (W + \chi)_{\bar{j}]} . \]  

(31)

2.4 SU(N) holonomy and generalised quasi Ricci flatness

Let us recall that the holonomy group of 2N dimensional (with or without torsion) manifolds with a covariantly constant complex structure is a subgroup of U(N). Moreover, particular cases where the holonomy is SU(N) play a special role: in the absence of torsion, this means Ricci flatness; in the present case, the vanishing of the U(1) part of the Riemann curvature

\[ C_{\mu\nu} = J^\rho_{\mu} R^\rho_{\sigma\mu\nu} \]

writes:

\[ C_{\mu\nu} = 2 \partial_{[\mu} \Gamma_{\nu]} = 0 \]  

(32)

where \( \Gamma_\mu = J^\rho_\mu \Gamma^\rho_{\nu\mu} \), (a priori not a vector), is found in holomorphic coordinates as [4] :

\[ \Gamma_i = i [\partial_i \log \|g\| - 2 V_i] , \ \Gamma_i = -i [\partial_i \log \|g\| - 2 V_i] . \]  

(33)

Using the tensor \( C_{\mu\nu} \), equation (28) writes

\[ \text{Ric}_{ij} = D_j W_i + \frac{i}{2} C_{ij} , \ \text{Ric}_{i\bar{j}} = D_{\bar{j}} W_i + \frac{i}{2} C_{i\bar{j}} \]  

(34)

Then SU(N) holonomy \( \Leftrightarrow C_{\mu\nu} = 0 \), leads to a special case of “generalised quasi Ricci flatness” (31) with

\[ W_i = V_i , \ \chi_i = -V_i . \]

As a consequence, and as will be exemplified in subsection (3.3.2), SU(N) holonomy is a more restrictive requirement that the sole “generalised quasi Ricci flatness” of Friedling and van de Ven.

\footnote{We use equations [21][22][24].}
2.5 (2,0) versus (2,2) geometries

Given a complex Riemannian manifold \(- J^\nu_{\mu} \) being the complex structure - equipped with a metric \( g_{\mu\nu} \) and a torsion tensor \( T_{\mu\nu\rho} \) satisfying equations (34, 35, 36), we address the following question:

"does there exist a conformal transformation that transforms this (2,0) geometry into a (2,2) one", i.e. that relates the metric \( g_{\mu\nu} \) and the torsion tensor \( T_{\mu\nu\rho} \) to a Kähler metric \( \hat{g}_{\mu\nu} \) (no torsion) through:

\[
\hat{g}_{\mu\nu} = e^{-2f} g_{\mu\nu}
\]

(the positivity of the distance requires the reality of \( f \)).

The Kähler condition writes in complex coordinates:

\[
\hat{g}_{ij, k} = \hat{g}_{kj, i} \quad \hat{g}_{ij, k} = \hat{g}_{ik, j}
\]

which leads, using equation (16), to

\[
T_{ijk} = g_{ik, j} - g_{jk, i} = 2(g_{ik} \partial_j f - g_{jk} \partial_i f)
\]

\[
T_{i,jk} = g_{ik, j} - g_{jk, i} = 2(g_{ik} \partial_j f - g_{jk} \partial_i f)
\]

(36)

The vector \( V_\mu \) is then a gradient:

\[
V_i = T^k_{ki} = 2(N - 1) \partial_i f \quad V_i = T^k_{ki} = 2(N - 1) \partial_i f .
\]

This is a very restrictive condition as it means that the whole torsion tensor depends on a single real function \( f \). Notice that equation (36) may be rewritten as:

\[
T_{ijk} = \frac{1}{(N - 1)} [g_{ik} V_j - g_{jk} V_i] \quad T_{i,jk} = \frac{1}{(N - 1)} [g_{ik} V_j - g_{jk} V_i]
\]

(38)

The closedness relation (27) writes

\[
D_i V^i + D_j V^j = 2 \frac{N - 2}{N - 1} V_i V^i \quad \text{where} \quad V^i = g^{ij} V_j \quad V^j = g^{ij} V_i .
\]

(39)

Equation (27) gives \( D_i V^i = D_i V^j \) and, for \( N \neq 2 \), \( D_i V^i = D_j V_j \). As a consequence, the conformal factor \( e^{2f} \) satisfies:

\[
\hat{e}^{2f} \overset{\text{def}}{=} 2 \hat{g}^{ij} \hat{\nabla}_i \partial_j e^{2f} = 2 \hat{g}^{ij} \partial_i \partial_j e^{2f} = 4 \hat{g}^{ij} e^{2f} \left[ \partial_i \partial_j f + 2 \partial_i f \partial_j f \right]
\]

\[
= 4 e^{2f} g^{ij} \left[ D_i \partial_j f - V_i \partial_j f + \frac{1}{(N - 1)} V_i \partial_j f \right]
\]

\[
= \frac{2 e^{4f}}{N - 1} \left[ D_i V^i - \frac{N - 2}{N - 1} V_i V^i \right] = 0 .
\]

(40)

and we have also

\[
\Delta f \overset{\text{def}}{=} 2 \hat{g}^{ij} D_i \partial_j f = \frac{8(N - 2)}{(N - 1)^2} \| V \|^2 .
\]

(41)

Notice that using (13, 30) one has

\[
d\omega = 2 \omega \wedge df .
\]

(42)

Finally, the scalar curvatures are related by:

\[
R[g, T] = e^{-2f} \left[ \hat{R}[\hat{g}, \hat{T}] \equiv 0 + \frac{2N(N - 2)}{(N - 1)^2} \| V \|^2 \right]
\]

(43)
3 (2,0) heterotic geometry in 2 complex dimensions

3.1 The torsion tensor

In that special case $T_{ijk}$ has as many components as the vector $V_i$ and from definition (24) one obtains:

$$T_{ijk} = g_{ik}V_j - g_{jk}V_i$$

(44)

This relation is a duality one, first written in real coordinates in [13]

$$T_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}V^\sigma$$

The closedness relation (26), when compared to equation (27) gives

$$D_i V^i = D_j V^j = 0$$

(45)

and the scalar curvature reduces to:

$$R = 2g^{ij}(\partial_i \partial_j \log \det \|g\| + 2V_i V_j)$$

(46)

3.2 From (2,2) to (2,0) supersymmetry through a conformal rescaling of the metric

When one compares equations (13) and (37,38), one sees that in 2 complex dimensions one gets conformal equivalence between (2,0) and (2,2) geometries, if and only if the vector $V$ is the gradient of a real function:

$$V_j = 2\partial_j f, \quad V^j = 2\partial^j f$$

(47)

with the constraints

$$\hat{\Delta} e^{2f} = \hat{\Delta} f = 0$$

(48)

and, in that case, the scalar curvatures are proportional:

$$R[g, T] = e^{-2f} \hat{R}[\hat{g}, \hat{T} \equiv 0]$$

(49)

We now construct some examples of (2,0) heterotic metrics with particular isometries.

3.3 Special cases with linear U(N) isometry

3.3.1 Generalities

Let us consider the special family of hermitian metrics $g_{ij}$ ($i, j = 1,..,N$) with linear U(N) symmetry

$$g_{ij} = A(s)\delta_{ij} + B(s)\bar{z}^i z^j, \quad s = \sum_{i=1}^{N} \bar{z}^i z^i$$

(50)
For $N \geq 3$, this implies $\dot{A}(s) - B(s) = 0$ which means that the metric is Kähler and that no torsion can be put on a manifold with such isometries. For $N = 1$, the torsion tensor identically vanishes whereas for $N = 2$ (2 complex dimensions), equation (53) gives
\[ \frac{d}{ds}[s^2(\dot{A}(s) - B(s))] = 0 \Rightarrow \dot{A}(s) - B(s) = \frac{L}{2s^2} \] (54)

In such a case, the vector $V_i$ writes
\[ V_i = T^k_{ki} = \left[ \frac{\dot{A}(s) - B(s)}{A(s)} \right] \bar{z}^k \text{def} \frac{\partial}{\partial s} \log \gamma(L, s) \] (55)

Due to the reality of the functions $A(s)$ and $B(s)$ and consequently of $\gamma(L, s)$, the looked-for metric is conformally equivalent to a Kähler one through the conformal transformation $\hat{g} = \gamma(L, s)g$ (compare equations (47) and (55)). We then rescale the looked-for functions $A(s)$ and $C(s)$ according to :
\[ A(s) = \gamma(L, s)\eta(s) , \quad C(s) = \gamma(L, s)\mu(s) \] (56)

where, as a consequence of equations (54,55) we have :
\[ \mu(s) = \frac{d(s\eta(s))}{ds} , \quad \frac{d\gamma}{ds} = \frac{L}{2s\eta} \] (57)

In this $U(2)$ case, it is convenient to use the coordinates
\[ z^1 = \sqrt{s} \cos \frac{\theta}{2} e^{i(\phi+\psi)/2} , \quad z^2 = \sqrt{s} \sin \frac{\theta}{2} e^{i(\phi-\psi)/2} \]

to write the distance (12) :
\[ d\tau^2 = \frac{C(s)}{2s}(ds)^2 + 2sA(s)(\eta_1^2 + \eta_2^2) + 2sC(s)\eta_3^2 \] (58)

where the expressions for the $\eta_i$ are given in subsection (5.1).
The conformal rescaling then gives:

\[ d\tau^2 = \gamma(L, s) \left\{ \frac{1}{2s} \frac{d\sigma(s)}{ds} (ds)^2 + 2\sigma(s)(\eta_1^2 + \eta_2^2) + 2s \frac{d\sigma(s)}{ds} \eta_3^2 \right\} \]  

(59)

where \( \sigma(s) = s\eta(s) \).

As mentioned in the Introduction, despite the large isometry group considered, (2,0) supersymmetry is not sufficient to fix the geometry of the manifold: other constraints are needed. Three of them will be considered in this work:

- more supersymmetries: see next section,
- "generalised quasi Ricci flatness" as defined in the previous section,
- scalar-curvature flatness.

### 3.3.2 Generalised Quasi Ricci flat metrics

Due to the linear \( U(N) \) isometry, the Ricci tensor is symmetric and, when compared to equations (28), conditions (31) write:

\[ D_j[W_i - V_i] = 0 \]  

(60)

and:

\[ D_j[W_i - V_i] = \partial_i \partial_j \log \frac{\det \|g\|}{\gamma^2(L, s)} \]  

(61)

Using equations (54,55), (60) gives:

\[ W_i = V_i + \kappa C(s)z^i \]  

Equation (61) then leads to:

\[ \kappa \frac{d(sA(s))}{ds} = \frac{d}{ds} \log(A(s)C(s)) - \frac{L}{s^2 A(s)} \]  

(62)

Under the conformal rescaling (56), this differential equation gives, after a first integration [7]

\[ \frac{d\sigma(s)}{ds} = \frac{s}{\sigma(s)} \exp(\kappa \gamma(L, s)\sigma(s)) \]  

(63)

We were not able to solve explicitly the system (54,55), but the distance (59) writes:

\[ d\tau^2 = \gamma(L, s) \left\{ \frac{\sigma}{2s^2(\sigma)} \exp(-\kappa \gamma \sigma)(d\sigma)^2 + 2\sigma(\eta_1^2 + \eta_2^2) + \frac{2s^2(\sigma)}{\sigma} \exp(\kappa \gamma \sigma)\eta_3^2 \right\} \]  

(64)

and the geometrical objects are:

\[ V_i = \partial_i \log \gamma(L, s) \quad, \quad \det \|g\| = \gamma^2(L, s) \exp(\kappa \gamma(L, s)\sigma(s))/4 \]

\[ Ric_{ij} = D_i V_j = D_j V_i \quad, \quad Ric_{ij} = D_j V_i + \frac{\kappa^2}{2} \partial_i \partial_j [\gamma(L, s)\sigma(s)] \]  

(65)

- An inessential integration constant has been suitably chosen.
In the special case $\kappa = 0$, equations (60,61) give with (33,55):

\[ W_i = V_i \quad \Gamma_i = \Gamma_i = 0 \]

and one gets $SU(2) \equiv Sp(1)$ holonomy. The isometries then enforce the solution to be Eguchi-Hanson metric with torsion $8$:

\[
A(s) = \frac{\gamma(L,s)}{2s} \sqrt{s^2 + \lambda^2} \quad C(s) = \frac{\gamma(L,s)}{2s} \frac{s^2}{\sqrt{s^2 + \lambda^2}} \\
\frac{d}{ds} \gamma(L,s) = \frac{L}{\sqrt{s^2 + \lambda^2}}
\]

$L$ and $\lambda^2$ being constants and

\[
V_i = \partial_i \log \gamma(L,s) \quad \det \| g \| = \frac{\gamma^2(L,s)}{4}, \quad R = 0.
\]

This illustrates how one-loop finiteness, i.e. generalised quasi Ricci flatness is a less restrictive requirement than $SU(N)$ holonomy.

For vanishing torsion ($L=0$), $\gamma$ is a constant which may be taken to be 1, and we recover the metric first derived by Bonneau and Delduc $[15]$. Equation (63) integrates to:

\[
s^2(\sigma) = \frac{2}{\kappa^2} \left[ 1 - (1 + \kappa \sigma)e^{-\kappa \sigma} \right] - d
\]

where $d$ is a real integration constant. This metric has a non-vanishing scalar curvature

\[
R = 4\kappa \left( 1 + \frac{\kappa s^2(\sigma)e^{\kappa \sigma}}{2\sigma} \right).
\]

The distance (64) specifies to:

\[
d\tau^2 = \frac{\kappa e^{-\kappa \sigma}}{2s^2(\sigma)} (d\sigma)^2 + 2\sigma(\eta_1^2 + \eta_2^2) + \frac{2s^2(\sigma)}{\kappa e^{-\kappa \sigma}} \eta_3^2.
\]

Using this form of the distance, one can check that it is indeed regular for $\kappa \leq 0$ and $d > 0$. We take for variable $\sigma \in [\sigma_0, +\infty[$ where $\sigma_0$ is defined by $s(\sigma_0) = 0$. For $\sigma \to +\infty$ this distance is asymptotically flat (similarly to the torsionless Taub-NUT) and it exhibits a bolt $n=2$ for $\sigma \to \sigma_0$ (similarly to the torsionless Eguchi-Hanson). This means that, starting for $\kappa = 0$ from an asymptotically locally euclidean metric we get, for $\kappa < 0$, an asymptotically locally flat metric.

\[8\] which is discussed in subsect. (5.1).
3.3.3 Metrics with a vanishing scalar curvature

When the isometry group is U(2), equation (46) gives

\[
R = 2 \left[ \frac{D + s(D)}{C} + \frac{D}{A} + \frac{L^2}{2s^3A^2C} \right],
\]

where \( D = \frac{d}{ds} \log \det ||g|| \). \( \text{(69)} \)

After the conformal rescaling \( \gamma(L,s) \), the condition \( R = 0 \) becomes a \( L \) independent third order differential equation:

\[
s \frac{d}{ds} \left( \frac{\ddot{\sigma}}{\dot{\sigma}} \right) + \frac{\ddot{\sigma}}{\dot{\sigma}} + 2s \frac{\dddot{\sigma}}{\dot{\sigma}} = 0
\]

which integrates in a first step to

\[
\sigma^2 \left( \frac{s \ddot{\sigma}}{\dot{\sigma}} \right) = c
\]

and then the looked-for metric is given by:

\[
A(s) = \frac{\gamma(L,s)}{s} \sigma(s), \quad C(s) = \frac{\gamma(L,s)}{s} \frac{[\sigma^2(s) + 2c\sigma(s) + d]}{\sigma(s)}
\]

\[
s \frac{d}{ds} \gamma(L,s) = \frac{L}{2\sigma(s)}, \quad s \frac{d}{ds} \sigma(s) = \frac{[\sigma^2(s) + 2c\sigma(s) + d]}{\sigma(s)}
\]

\( \text{(70)} \)

\( L, c \) and \( d \) being constants. The geometrical objects are:

\[
V_i = \partial_i \log \gamma(L,s), \quad \det ||g|| = \frac{\gamma^2(L,s)[\sigma^2 + 2c\sigma + d]}{s^2}
\]

\[
Ric_{ij} = D_i \dot{V}_j = D_j \dot{V}_i, \quad Ric_{ij} = Ric_{ji} = D_j \dot{V}_i + \partial_i \partial_j \log \frac{[\sigma^2 + 2c\sigma + d]}{s^2}.
\]

\( \text{(71)} \)

As announced in subsection (3.3.1), these metrics are obtained through a conformal rescaling \( \gamma(L,s) \) of the Kähler, scalar flat, torsionless ones which are known as LeBrun metrics \( \text{[12]} \). These last ones are asymptotically euclidean and regular for suitable choices of the parameters \( c \) and \( d \). They give counterexamples to the Generalised Positive Action conjecture of Hawking and Pope \( \text{[16]} \). On the contrary, for \( L \neq 0 \), our metrics \( \text{(70)} \), although still asymptotically euclidean, are always singular: we have checked this on the expression of the curvature tensor.

In the special case \( c = 0 \), one recovers Eguchi-Hanson metric with torsion \( (d = -\lambda^2) \)

\[
A(s) = \frac{\gamma(L,s)}{2s} \sqrt{s^2 + \lambda^2}, \quad C(s) = \frac{\gamma(L,s)}{2s} \frac{s^2}{\sqrt{s^2 + \lambda^2}}
\]

\[
s \frac{d}{ds} \gamma(L,s) = \frac{L}{\sqrt{s^2 + \lambda^2}}.
\]

\( \text{(72)} \)

Let us emphasize that we have obtained a new and easy derivation of Eguchi-Hanson metric with torsion directly in complex coordinates.

---

\( ^9 \) scalar flatness being conserved through a conformal transformation in 2 complex dimensions \( \text{[49]} \).
3.4 More examples : Calabi metrics with torsion

3.4.1 Generalities

In [17] Calabi exhibits Kähler torsionless metrics in 2N real dimensions. He considers the special family of metrics with a linear O(N) symmetry and depending only on N coordinates:

\[
x_i = z^i + \bar{z}^i \quad i = 1, ..N \quad , \quad s = \sum_{i=1}^{N} x_i^2
\]

\[
g_{ij} = A(s)\delta_{ij} + B(s)x_ix_j
\]

Then, with \(C(s) = A(s) + sB(s)\) we have

\[
g^{ij} = \frac{1}{A(s)}\delta_{ij} - \left[ \frac{B(s)}{A(s)C(s)} \right] x_ix_j
\]

and \(\det g = A(s)C(s)\). From (16), the torsion tensor is found to be

\[
T_{ijk} = [2\dot{A}(s) - B(s)](x_j\delta_{ik} - x_i\delta_{jk})
\]

and its closedness gives

\[
2\frac{d}{ds}[2\dot{A}(s) - B(s)]\delta_{ij}x_kx_l + [2\dot{A}(s) - B(s)]\delta_{ij}\delta_{kl} - \ (i \leftrightarrow l) = 0
\]

For \(N \neq 2\), this implies a vanishing torsion tensor. On the contrary, for \(N = 2\) (2 complex dimensions), equation (76) gives :

\[
\frac{d}{ds} \left[ s(2\dot{A}(s) - B(s)) \right] = 0 \ \Rightarrow \ 2\dot{A}(s) - B(s) = \frac{L}{s}
\]

In such a case, the vector \(V_i\) writes

\[
V_i = T^k_{ki} = \left[ \frac{2\dot{A}(s) - B(s)}{A(s)} \right] x_i \ \overset{\text{def}}{=} \partial_i \log \gamma(L, s)
\]

Here again, due to the reality of the functions \(A(s)\) and \(B(s)\) and consequently of \(\gamma(L, s)\), the looked-for metric is conformally equivalent to a Kähler one through the conformal transformation \(\hat{g} = \gamma(L, s)g\) (compare equations (47) and (78)). We then rescale the looked-for functions \(A(s)\) and \(C(s)\) according to :

\[
A(s) = \gamma(L, s)\eta(s) \quad , \quad C(s) = \gamma(L, s)\mu(s)
\]

where, as a consequence of equations (77,78) :

\[
\mu(s) = \eta(s)s\frac{d}{ds}\log[s\eta^2(s)] \quad , \quad s\frac{d\gamma}{ds} = \frac{L}{2\eta}
\]

We now add further geometrical constraints.

10 A dot indicates a derivative with respect to the variable \(s\).
3.4.2 Generalised Quasi Ricci flat metrics

Due to the linear O(N) isometry, the Ricci tensor is symmetric and then equations (60,61) hold. Using equations (77,78), we obtain:

\[ W_i = V_i \]

and:

\[ \log \det \| g \| = \text{constant} \iff \frac{d}{ds} (s \eta^2) = \text{constant} \quad (81) \]

Then, the looked-for metric here also has SU(2) \( \equiv \text{Sp}(1) \) holonomy and writes:

\[
A(s) = \frac{\gamma(L,s)}{2\sqrt{s}} \sqrt{s + \lambda} \quad , \quad C(s) = \frac{\gamma(L,s)}{2\sqrt{s}} \frac{s}{\sqrt{s + \lambda}}
\]

\[ s \frac{d}{ds} \gamma(L,s) = L \sqrt{\frac{s}{s + \lambda}} \quad (82) \]

L and \( \lambda \) being constants and

\[
V_i = \partial_i \log \gamma(L,s) \quad , \quad \det \| g \| = \frac{\gamma^2(L,s)}{4}
\]

\[
Ric_{ij} = D_i V_j = D_j V_i \quad , \quad Ric_{ij} = Ric_{ji} = D_j V_i = D_i V_j \quad , \quad R = 0 \quad . \quad (83)
\]

3.4.3 Metrics with a vanishing scalar curvature

In this case, equation (80) gives

\[
R = 4 \left[ \frac{D + 2s \dot{D}}{C} + \frac{D}{A} + \frac{L^2}{sA^2C} \right] \quad \text{where} \quad D \overset{\text{def}}{=} \frac{d}{ds} \log \det \| g \| \quad . \quad (84)
\]

After the conformal rescaling (79), the condition \( R = 0 \) becomes a \( L \) independent differential equation \(^{11}\):

\[
\left( \frac{s \ddot{\eta}^2}{\dot{\eta}^2} \right) \left[ 1 + s \frac{d}{ds} \log \eta \right] + s \frac{d}{ds} \left( \frac{\ddot{\eta}^2}{\dot{\eta}^2} \right) = 0
\]

which integrates in a first step to

\[
\frac{s \dot{\eta} (\dot{\eta}^2)}{\dot{\eta}^2} = \kappa
\]

Taking \( u = (s \eta^2(s)) \) as a new variable, one is led to a Riccati equation:

\[
2\kappa \frac{d \eta}{du} + \frac{1}{u} \eta^2 = 1
\]

We then find two families of solutions which hereagain are obtained from the torsionless case \( (L=0) \) through a conformal rescaling of the metric:

\(^{11}\) \( (\ddot{s} \dot{\eta}^2) = \frac{d}{ds}(s \dot{\eta}^2) = \frac{d}{ds} \left[ \frac{d}{ds}(s \eta^2) \right] \)
- the quasi Ricci flat one \( (82) \) for \( \kappa = 0 \),

- a new one, given through the following equations:

\[
A[v(s)] = v[g_1 + L \log v] \frac{I_1(v) - dK_1(v)}{I_0(v) + dK_0(v)}
\]

\[
C[v(s)] = v[g_1 + L \log v] \frac{I_0(v) + dK_0(v)}{I_1(v) - dK_1(v)}
\]

(85)

and where \( s = c^2[I_0(v) + dK_0(v)] \) defines the function \( v(s) \); \( L, g_1, c \) and \( d \) are constants and \( I(v) \) and \( K(v) \) the usual Bessel functions. Moreover,

\[
V_i = \partial_i \log[g_1 + L \log v], \quad \det\|g\| = v^2 [g_1 + L \log v]^2
\]

\[
Ric_{ij} = D_i V_j = D_j V_i, \quad Ric_{ij} = Ric_{ji} = D_j V_i + \partial_i \partial_j \log v^2(s).
\]

(86)

4 (4,0) heterotic geometry

We add to conditions (38) for (2,0) supersymmetry, the following ones:

- apart from the first complex structure, labelled as \( J_{3\mu}^\nu \), there does exist another one, \( J_{\mu}^\nu \), integrable, covariantly constant and anticommuting with \( J_{3\mu}^\nu \):

\[
J_{\mu}^\nu J_{1\nu}^\rho = -\delta_\mu^\rho \quad (87)
\]

\[
N_{\nu\mu}^\rho \equiv J_{1\mu}^\lambda (\partial_\nu J_{1\nu}^\rho - \partial_\nu J_{1\lambda}^\rho) - (\mu \leftrightarrow \nu) = 0 \quad (88)
\]

\[
D_\mu J_{1\nu}^\rho = 0 \quad (89)
\]

\[
J_{1\mu}^\nu J_{3\nu}^\rho = -J_{3\mu}^\nu J_{1\nu}^\rho \quad (90)
\]

- hermiticity: the metric is also hermitian with respect to the complex structure \( J_{1\mu}^\nu \),

\[
g_{\mu\lambda} J_{1\nu}^\lambda + J_{1\mu}^\nu g_{\lambda\nu} = 0 \quad (91)
\]

As a consequence (see for example ref.[7]),

\[
J_{2\nu}^\mu \overset{\text{def}}{=} J_{3\mu}^\rho J_{4\nu}^\rho \quad (92)
\]

is a third complex structure, integrable and covariantly constant; moreover the metric is also hermitian with respect to \( J_{2\mu}^\nu \) and the triplet of complex structures \( J_{a\mu}^\nu \) \( (a = 1,2,3) \) satisfies a quaternionic multiplication law:

\[
J_{a\mu}^\nu J_{b\nu}^\rho = -\delta_\mu^\rho + \epsilon_{abc} J_{c\mu}^\rho \quad \text{with} \quad \epsilon_{123} = +1 \quad (93)
\]
Moreover, the generalized Nijenhuis tensors
\[ N_{ab}^\rho \equiv [J_a^\lambda (\partial_\lambda J_b^\rho - \partial_\nu J_b^\rho) - (\mu \leftrightarrow \nu)] + (a \leftrightarrow b) \] (94)
vanish, which ensures the N=4 supersymmetry algebra [7]. The sufficient character of equations (87-91) is often missed in the literature where (93,94) are added as independent conditions although they are direct consequences of the others.

The dimension of the manifold has to be a multiple of 4, and we shall now translate these new conditions in complex coordinates adapted to the complex structure \( J_3 \) (see (11)). As a consequence of equation (93), all components of the tensors:
\[ K_{\nu}^\mu = \frac{1}{2}(J_1 - iJ_2)^\nu_{\mu}, \quad \bar{K}_{\nu}^\mu = \frac{1}{2}(J_1 + iJ_2)^\nu_{\mu} \] (95)
vanish, but for
\[ K_{\bar{i}}^j = J_{\bar{i}}^j = -iJ_{\bar{i}}^j, \quad \bar{K}_{\bar{i}}^j = J_{\bar{i}}^j = iJ_{\bar{i}}^j \] (96)
with \( i, j, \bar{i}, \bar{j} = 1,2,...2N \). The hermiticity of the metric then implies the skew-symmetry of \( K_{\mu\nu} \) and \( \bar{K}_{\mu\nu} \) which therefore locally define (2,0) and (0,2) forms:
\[ \omega_1 - i\omega_2 = (K_{\bar{i}}^j{g}_{jk})dz^i \wedge dz^k = J_{\bar{i}k}dz^i \wedge dz^k \]
\[ \omega_1 + i\omega_2 = (\bar{K}_{\bar{i}}^j{g}_{jk})dz^\bar{i} \wedge dz^\bar{k} = J_{\bar{i}k}dz^\bar{i} \wedge dz^\bar{k} \] (97)
In this coordinate system, the covariant constancy of the complex structures \( J_1 \) and \( J_2 \) implies:
\[ \partial_\bar{i}K_{jk} = T_{\bar{i}j}^lK_{lk} - T_{\bar{i}k}^lK_{lj}, \quad \partial_\bar{i}K_{jk} = \Gamma_{\bar{i}j}^lK_{lk} - \Gamma_{\bar{i}k}^lK_{lj} \] (98)
and the complex conjugate equations. Multiplying equations (98) by \( (K^{-1})^{jk} = (\bar{K})^{jk} \) gives:
\[ \frac{1}{2}(K^{-1})^{jk}\partial_\bar{i}K_{jk} = -V_i \]
\[ \frac{1}{2}(K^{-1})^{jk}\partial_\bar{i}K_{jk} = -\Gamma_{\bar{i}j}^l = V_i - \partial_\bar{i}\log \det \|g\| \] (99)
Then, with
\[ F = \det \|K_{ij}\| \] (100)
(K being a 2Nx2N skew-symmetric matrix), we find
\[ V_i = \frac{1}{2}\partial_\bar{i}\log \left(\frac{\det \|g\|^2}{F}\right) \quad V_i = \frac{1}{2}\partial_\bar{i}\log F \] (101)
As a consequence \( FF^* \propto (\det \|g\|)^2 \). We emphasize that \textit{a priori} \( F \) is a complex function. This will be important in the following (subsection 5.2). Notice also that \( \Gamma_\mu \) of equation (83) is a true gradient vector:
\[ \Gamma_\mu = -i \partial_\mu \log \frac{\det \|g\|}{F} \]
and then that $C_{\mu\nu}$ vanishes. This is not a surprise as the holonomy is now $\text{Sp}(N) \subset \text{SU}(2N)$.

The function $F$ being rescaled such that

$$FF^* = (\det \|g\|)^2,$$

(102)

which has no consequence on the vector $V_\mu$, equations (28) give:

$$\text{Ric}_{ij} = D_jV_i, \quad \text{Ric}_{i\bar{j}} = D_{\bar{j}}V_i$$

(103)

We obtain the special case of “generalised quasi Ricci flatness” (see (31) for $W = V$ and $\chi = -V$) that leads to $\text{SU}(2N)$ holonomy (see subsection (2.4)).

To summarize: $(4,0)$ heterotic geometry implies, whatever the dimension of the manifold be, generalised quasi Ricci flatness

$$\text{Ric}_{\mu\nu} = D_{\nu}V_\mu$$

(104)

where the vector $V_\mu$, related to the torsion tensor, is given by

$$V_i = T^k_{ki} = \frac{1}{2} \partial_i \log F^*, \quad V_{\bar{i}} = T^k_{\bar{k}i} = \frac{1}{2} \partial_{\bar{i}} \log F, \quad FF^* = (\det \|g\|)^2.$$  

(105)

Notice that, in the absence of torsion, the Ricci tensor vanishes [2], [9]. Moreover, equation (104) generalises in $4N$ dimensions the corresponding one found in 4 dimensions [13] where $V_\mu$ is the dual of the torsion tensor

$$V_\mu^{4\text{dim.}} = \frac{1}{3!} \epsilon_{\mu\rho\sigma} T^{\rho\sigma}$$

(106)

Finally, the scalar curvature reads

$$R = g^{ij}(D_iV_j + D_jV_i) = \frac{1}{2}g^{ij}(D_iD_j \log F + D_jD_i \log F^*)$$

(107)

On the contrary of 4 dimensional case (as a consequence of equ. (45)), it no longer vanishes on general grounds.

If the function $F$ is real ($F = \text{det} \|g\|$), equation (105) shows that $V_\mu$ is a gradient, which reminds us the conformal equivalence relation (37). The conformal factor would then be

$$e^{-2f} = (F)\frac{1}{2(N-1)}.$$  

However, equation (38), the other condition for conformal invariance, does not hold on general grounds for $N \neq 2$. We then specify to 4 real dimensions manifolds where, due to (44), it holds true. We are then in position to compare our results to previous ones obtained by Callan, Harvey and Strominger (in the appendix of [14]). They claimed that there exists a conformal rescaling of the metric,

$$\hat{g} = (\exp 2\phi)g$$

(108)
where $\phi$ is defined through equations similar to (100) and the first of (99) ($\phi$ of ref. [14] is equal to $-\frac{1}{4}\log F$), such that an equation similar to (48) holds:
\[
\hat{\Delta} \Omega \equiv \hat{\Delta} e^{-2\phi} = \hat{\Delta} e^{2f} = 0, \quad \Delta \phi = \Delta f = 0.
\quad (109)
\]

Unfortunately, they implicitly suppose that their function $\phi$ is real, which, as will be shown on an explicit example (equ. (146) in subsect. 5.2), is generally wrong. As a consequence and as explained in subsection (3.2), the “metric” $\hat{g}$ introduced by Callan et al., does not lead to a real distance, which is unacceptable. Then the assertion that “$N=4$ world sheet supersymmetry imply that the [corresponding] sigma model metric is conformal to a Ricci flat Kähler metric” is generally wrong. Rather, we have shown that the metric is a generalised quasi Ricci flat one with a vanishing scalar curvature. Notice also that they forget about the second of (99), then missing the relationship between $\phi$ and $\det \|g\|$. Finally, the correct version of equation (109) is not the Laplace condition on function $\phi$ when $\phi$ is a true complex function, but, using equation (45), valid in two complex dimensions, and result (105):
\[
\begin{align*}
g^{ij} D_i \partial_j \phi &= g^{ij} D_j \partial_i \phi^* = 0 \\
\Rightarrow \Delta \phi &\equiv g^{ij} (D_i \partial_j + D_j \partial_i) \phi = g^{ij} D_j \partial_i (\phi - \phi^*) \neq 0.
\end{align*}
\quad (110)
\]

We now construct some examples of (4,0) heterotic geometries in complex coordinates.

5 Examples of (4,4) and (4,0) geometries in complex coordinates

In the construction of four dimensional hyperkähler metrics, the most useful tool was certainly the curvature self-duality requirement. This led to Eguchi-Hanson [18] and Taub-NUT metrics which are particular cases of the multicentre metrics [19]. More recently, but following the same technique, Atiyah and Hitchin have obtained a genuinely new hyperkähler metric [20] which is deeply related with the dynamics of a system of two magnetic monopoles [21].

For all these metrics, at least one choice of holomorphic coordinates is known. They are given in [22] for Taub-NUT, in [23] for Eguchi-Hanson, in [22] for the multicentre and in [24] for Atiyah-Hitchin. Furthermore, there are important examples, mainly due to Calabi, for which the use of holomorphic coordinates was essential [17],[25].

Similarly, (4,0) geometries have been recently obtained by Delduc and Valent [13] as extensions of Taub-NUT and Eguchi-Hanson metrics with torsion, using harmonic superspace and curvature self-duality. It is the aim of this work to explore the advantages of holomorphic coordinates: we shall give examples of these ones and compute for every case the vector $V_i$ and the function $F$ of the preceding section.
5.1 Eguchi-Hanson (with and without torsion)

We extract from [13] the vierbeins and the complex structures of Eguchi-Hanson metric with torsion:

\[
e_0 = \frac{1}{2} \gamma_0(s) \sqrt{\frac{s}{s^2 - a^2}} ds + \frac{2\rho}{\sqrt{s}} \eta_3 \quad ; \quad e_3 = \gamma_0(s) \sqrt{\frac{s^2 - a^2}{s^2}} \eta_3
\]

\[
e_{1,2} = \gamma_0(s) \sqrt{s} \eta_{1,2} \quad \text{with} \quad \frac{d}{ds} \gamma_0^2(s) = \frac{L}{s^2 - a^2 + \rho^2}
\]

(111)

The one forms \( \eta_i \) (i=1,2,3) satisfy

\[
d\eta_i = -\epsilon_{ijk} \eta_j \wedge \eta_k
\]

(112)

and can be parametrised in spherical coordinates as:

\[
\eta_1 = \frac{1}{2} (\cos \phi \, d\theta + \sin \phi \sin \theta \, d\psi)
\]

\[
\eta_2 = \frac{1}{2} (-\sin \phi \, d\theta + \cos \phi \sin \theta \, d\psi)
\]

\[
\eta_3 = \frac{1}{2} (d\phi + \cos \theta \, d\psi)
\]

(113)

The euclidean distance is \((x^\mu \equiv [s, \theta, \phi, \psi])\)

\[
d\tau^2 = e_0^2 + \sum_{i=1}^{3} e_i^2 = g_{\mu\nu} dx^\mu dx^\nu
\]

(114)

and the complex structures two forms are

\[
\omega_i = e_0 \wedge e_i - \frac{1}{2} \epsilon_{ijk} e_j \wedge e_k
\]

(115)

One can check that in the following change of coordinates:

\[
x^\mu [s, \theta, \phi, \psi] \rightarrow y^\mu [s' = s, \theta' = \theta, \phi' = \phi + \beta - \phi_0(s), \psi' = \psi]
\]

with \( \tan \phi_0(s) = \frac{\rho}{\sqrt{s^2 - a^2}} \), \( \beta \) constant,

(116)

\( g_{\mu\nu}(\rho, a^2, \gamma_0(s)) \) transforms to \( g'_{\mu\nu} \equiv g_{\mu\nu}(\rho = 0, a'^2 = a^2 - \rho^2, \gamma_0(s)) \). This means that under a conformal rescaling \( \gamma_0^2(s) \), Eguchi-Hanson metric with torsion transforms to Eguchi-Hanson without torsion, in accordance with Callan et al.’s claim [14]. With regard to the complex structures

\[
\Omega = \cos \alpha \, \omega_3 + \sin \alpha \, (\cos \beta \, \omega_1 + \sin \beta \, \omega_2) \equiv \frac{1}{2} \Omega_{\mu\nu} dx^\mu \wedge dx^\nu
\]

(117)

one obtains:

\[
\Omega_{s\theta} = \frac{\gamma_0^2}{4} \sqrt{s^2 - a^2} \sin \alpha \cos(\beta + \phi)
\]
complex heterotic geometry

\[ \Omega_{s\phi} = \frac{\gamma_0^2}{4} \cos \alpha \]
\[ \Omega_{s\psi} = \frac{\gamma_0^2}{4} \left[ \cos \alpha \cos \theta + \frac{s}{\sqrt{s^2 - a^2}} \sin \alpha \sin \theta \sin(\beta + \phi) \right] \]
\[ \Omega_{\theta\phi} = \frac{\gamma_0^2}{4} \sqrt{s^2 - a^2 + \rho^2} \sin \alpha \sin(\beta + \phi - \phi_0(s)) \]
\[ \Omega_{\theta\psi} = \frac{\gamma_0^2}{4} \left[ -s \cos \alpha \sin \theta + \sqrt{s^2 - a^2 + \rho^2} \sin \alpha \sin(\beta + \phi - \phi_0(s)) \right] \]
\[ \Omega_{\phi\psi} = \frac{\gamma_0^2}{4} \sqrt{s^2 - a^2 + \rho^2} \sin \alpha \cos(\beta + \phi - \phi_0(s)) \] (118)

One can also check that in coordinates \( y^\mu \)

\[ \Omega'_\mu = \Omega_{\mu\nu}(\rho = 0, a'^2 = a^2 - \rho^2, \gamma_0(s)) \]

As a consequence, the search for an holomorphic system of coordinates associated to the complex structure

\[ J = \cos \alpha \ J_3 + \sin \alpha \ (\cos \beta \ J_1 + \sin \beta \ J_2) \] (119)

is the same as in the torsionless case.

In the looked-for coordinate system \( z^\mu (z^i, \bar{z}^i) \), the complex structure tensor \( \Omega'_\mu \) must take the form \( \Omega'_\mu \): \n
\[ \Omega'_{\mu} = \frac{\partial x^\rho}{\partial z^\mu} \Omega^\sigma_{\rho} \frac{\partial z^\nu}{\partial x^\sigma} \]

From

\[ (\Omega'_\mu - i\delta'_\mu) \partial_\nu z^i = 0 \quad i = 1, 2. \] (121)

This homogeneous system of four partial derivatives equations can be shown to reduce itself to two independent equations:

\[ -i \partial_{\phi'} z + \frac{s \cos \alpha}{s^2 - a'^2} \partial_{\phi'} z + \frac{\sin \alpha}{\sqrt{s^2 - a'^2}} \left[ \cos \phi' \partial_{\theta'} z + \frac{\sin \phi'}{\sin \theta'} (\partial_{\psi'} z - \cos \theta' \partial_{\phi'} z) \right] = 0 \] (122)

\[ i \partial_{\psi'} z - \frac{\cos \alpha}{\sin \theta'} (\partial_{\psi'} z - \cos \theta' \partial_{\phi'} z) + \sin \alpha \left[ \sqrt{s^2 - a'^2} \cos \phi' \partial_{\phi'} z - \frac{s}{\sqrt{s^2 - a'^2}} \sin \phi' \partial_{\phi'} z \right] = 0 \] (123)

Special solutions are obtained for \( \alpha = 0 \) \( (J_3 \text{ diagonal}) \) \( \exists \):

\[ z^1 = (s^2 - a'^2)^{1/4} \cos(\theta'/2) \exp i(\phi' + \psi')/2 \]
\[ z^2 = (s^2 - a'^2)^{1/4} \sin(\theta'/2) \exp i(\phi' - \psi')/2 \] (124)
and for $\alpha = \pi/2$ ($J_1$ diagonal)

\[
\begin{align*}
z^1 &= \sqrt{s^2 - a'^2} \sin \theta' \cos \phi' - is \cos \theta' \\
z^2 &= i\psi' + \log \frac{\sqrt{s^2 - a'^2} (\cos \theta' \cos \phi' + i \sin \phi')}{\sqrt{(z^1)^2 + a'^2}}
\end{align*}
\] (125)

In the coordinate system (124), one obtains

\[
\begin{align*}
\omega_1 - i\omega_2 &= (\gamma_0(s))^2 dz^1 \wedge dz^2 \\
\omega_3 &= ig_{ij} dz^i \wedge d\bar{z}^j
\end{align*}
\] (126)

with

\[
\begin{align*}
g_{1\bar{1}} &= \frac{(\gamma_0(s))^2}{2} \left[ \frac{s}{s^2 - a'^2} |z^2|^2 + \frac{1}{s} |z^1|^2 \right] \\
g_{22} &= \frac{(\gamma_0(s))^2}{2} \left[ \frac{s}{s^2 - a'^2} |z^1|^2 + \frac{1}{s} |z^2|^2 \right] \\
g_{12} &= -\frac{(\gamma_0(s))^2}{2} \frac{a'^2}{s(s^2 - a'^2)} z^z \bar{z}^1 \\
|z^1|^2 + |z^2|^2 &\equiv t = \sqrt{s^2 - a'^2}
\end{align*}
\] (127)

and

\[
\det \|g\| = \frac{(\gamma_0(s))^4}{4}.
\]

The torsion tensor is then obtained as

\[
T = \frac{L}{2(s^2 - a'^2)} [(\bar{z}^2 d\bar{z}^1 - \bar{z}^1 d\bar{z}^2) dz^1 \wedge dz^2 + c.c.]
\] (128)

and its closedness is readily verified.

Notice that with $\gamma_0^2$ changed to $\gamma$, $t = \sqrt{s^2 - a'^2}$ changed to $s$ and $a'^2$ to $-\lambda^2$, one reproduces equations (66,72).

The vector $V_i$ and function $F$ are then obtained from equations (97,100, 101 and 126):

\[
V_i = \frac{1}{2} \partial_i \log \left( \frac{(\gamma_0(s))^2}{2} \right)^2 ; \quad F = F^* = \left( \frac{(\gamma_0(s))^2}{2} \right)^2
\] (129)

and

\[
\begin{align*}
d\omega_3 &= \frac{1}{2} \omega_3 \wedge d(\log \det \|g\|) \\
\omega_1 - i\omega_2 &= \frac{1}{2} (\omega_1 - i\omega_2) \wedge d(F).
\end{align*}
\] (130)

(131)

Here it is obvious that Callan et al’ s assertion works : indeed Eguchi-Hanson with torsion is conformally equivalent to its torsionless counterpart, with the conformal factor

\[
e^{2f} = \frac{(\gamma_0(s))^2}{2}.
\]
The usual hyperkähler Eguchi-Hanson metric is obtained for $L = 0$ ($\gamma_0$ constant which may be taken to be 1) and, still in coordinates (124) the Kähler potential is $K(t)$ with:

$$
\frac{dK}{dt} = \frac{\sqrt{t^2 + a^2}}{2t}
$$

In coordinates (123), corresponding to $J_1$ diagonal, we found in the same limit $L = \rho = 0$

$$
K = -\frac{s}{2}
$$

### 5.2 Taub-NUT (with and without torsion)

Here too we extract from [13] the vierbeins and the complex structures:

$$
e_0 = \frac{1}{2} \gamma_0(s) a(s) \left[ \frac{1 + \lambda s}{\sqrt{s}} ds + 2 \rho s^{3/2} \sigma_3 \right] ; \quad e_3 = \gamma_0(s) a(s) \sqrt{s} \sigma_3
$$

$$
e_{1,2} = \gamma_0(s) a(s) (1 + \lambda s) \sqrt{s} \sigma_{1,2} \quad \text{with} \quad a(s) = \frac{1}{\sqrt{(1 + \lambda s)(1 + \rho^2 s^2)}}
$$

and

$$
\frac{d}{ds} \gamma_0^2(s) = \frac{L}{s^2}
$$

The one forms $\sigma_i$ ($i=1,2,3$) satisfy

$$
d\sigma_i = \epsilon_{ijk} \sigma_j \wedge \sigma_k
$$

and can be parametrised in spherical coordinates as:

$$
\sigma_1 = \frac{1}{2} \left[ - \sin \psi \, d\theta - \cos \psi \sin \theta \, d\phi \right]
$$

$$
\sigma_2 = \frac{1}{2} \left[ \cos \psi \, d\theta - \sin \psi \sin \theta \, d\phi \right]
$$

$$
\sigma_3 = \frac{1}{2} \left[ d\psi - \cos \theta \, d\phi \right]
$$

The euclidean distance is ($x^\mu \equiv [s, \theta, \phi, \psi]$)

$$
d\tau^2 = e_0^2 + \sum_{i=1}^{3} e_i^2 = g_{\mu\nu} dx^\mu dx^\nu
$$

Let us define

$$
\mathcal{K}_i = e_0 \wedge e_i - \frac{1}{2} \epsilon_{ijk} e_j \wedge e_k \quad ; \quad i, j, k = 1, 2, 3
$$

from which we deduce the complex structures two forms

$$
\omega_1 \pm i \omega_2 = \left[ (\pm i \sin \psi - \cos \theta \cos \psi) \mathcal{K}_1 - (\pm i \cos \psi + \cos \theta \sin \psi) \mathcal{K}_2 \right.
$$

$$
\left. + \sin \theta \mathcal{K}_3 \right] \exp \pm i \phi
$$

$$
\omega_3 = \sin \theta [\cos \psi \mathcal{K}_1 + \sin \psi \mathcal{K}_2] + \cos \theta \mathcal{K}_3
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$$

$$
e_{1,2} = \gamma_0(s) a(s) (1 + \lambda s) \sqrt{s} \sigma_{1,2} \quad \text{with} \quad a(s) = \frac{1}{\sqrt{(1 + \lambda s)(1 + \rho^2 s^2)}}
$$

and

$$
\frac{d}{ds} \gamma_0^2(s) = \frac{L}{s^2}
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The one forms $\sigma_i$ ($i=1,2,3$) satisfy

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$$
\sigma_1 = \frac{1}{2} \left[ - \sin \psi \, d\theta - \cos \psi \sin \theta \, d\phi \right]
$$

$$
\sigma_2 = \frac{1}{2} \left[ \cos \psi \, d\theta - \sin \psi \sin \theta \, d\phi \right]
$$

$$
\sigma_3 = \frac{1}{2} \left[ d\psi - \cos \theta \, d\phi \right]
$$

The euclidean distance is ($x^\mu \equiv [s, \theta, \phi, \psi]$)

$$
d\tau^2 = e_0^2 + \sum_{i=1}^{3} e_i^2 = g_{\mu\nu} dx^\mu dx^\nu
$$

Let us define

$$
\mathcal{K}_i = e_0 \wedge e_i - \frac{1}{2} \epsilon_{ijk} e_j \wedge e_k \quad ; \quad i, j, k = 1, 2, 3
$$

from which we deduce the complex structures two forms

$$
\omega_1 \pm i \omega_2 = \left[ (\pm i \sin \psi - \cos \theta \cos \psi) \mathcal{K}_1 - (\pm i \cos \psi + \cos \theta \sin \psi) \mathcal{K}_2 \right.
$$

$$
\left. + \sin \theta \mathcal{K}_3 \right] \exp \pm i \phi
$$

$$
\omega_3 = \sin \theta [\cos \psi \mathcal{K}_1 + \sin \psi \mathcal{K}_2] + \cos \theta \mathcal{K}_3
$$

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$$

$$
e_{1,2} = \gamma_0(s) a(s) (1 + \lambda s) \sqrt{s} \sigma_{1,2} \quad \text{with} \quad a(s) = \frac{1}{\sqrt{(1 + \lambda s)(1 + \rho^2 s^2)}}
$$

and

$$
\frac{d}{ds} \gamma_0^2(s) = \frac{L}{s^2}
$$

The one forms $\sigma_i$ ($i=1,2,3$) satisfy

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d\sigma_i = \epsilon_{ijk} \sigma_j \wedge \sigma_k
$$

and can be parametrised in spherical coordinates as:

$$
\sigma_1 = \frac{1}{2} \left[ - \sin \psi \, d\theta - \cos \psi \sin \theta \, d\phi \right]
$$

$$
\sigma_2 = \frac{1}{2} \left[ \cos \psi \, d\theta - \sin \psi \sin \theta \, d\phi \right]
$$

$$
\sigma_3 = \frac{1}{2} \left[ d\psi - \cos \theta \, d\phi \right]
$$

The euclidean distance is ($x^\mu \equiv [s, \theta, \phi, \psi]$)

$$
d\tau^2 = e_0^2 + \sum_{i=1}^{3} e_i^2 = g_{\mu\nu} dx^\mu dx^\nu
$$

Let us define

$$
\mathcal{K}_i = e_0 \wedge e_i - \frac{1}{2} \epsilon_{ijk} e_j \wedge e_k \quad ; \quad i, j, k = 1, 2, 3
$$

from which we deduce the complex structures two forms

$$
\omega_1 \pm i \omega_2 = \left[ (\pm i \sin \psi - \cos \theta \cos \psi) \mathcal{K}_1 - (\pm i \cos \psi + \cos \theta \sin \psi) \mathcal{K}_2 \right.
$$

$$
\left. + \sin \theta \mathcal{K}_3 \right] \exp \pm i \phi
$$

$$
\omega_3 = \sin \theta [\cos \psi \mathcal{K}_1 + \sin \psi \mathcal{K}_2] + \cos \theta \mathcal{K}_3
$$
We now look for a new system of coordinates \( z^1, z^2, \bar{z}^1, \bar{z}^2 \) which diagonalize the complex structures according to (11). With \( \Omega \) defined through (11 7), one obtains

\[
\begin{align*}
\Omega_{s\theta} &= \frac{\gamma_0^2}{4(1 + \rho^2 s^2)} (1 + \lambda s) \sin \alpha \sin(\phi - \beta) \\
\Omega_{s\phi} &= \frac{\gamma_0^2}{4(1 + \rho^2 s^2)} \left[ \lambda s \sin \alpha \sin \theta \cos(\theta) \cos(\phi - \beta) - \cos \alpha (1 + \lambda s \sin^2 \theta) \right] \\
\Omega_{s\psi} &= \frac{\gamma_0^2}{4(1 + \rho^2 s^2)} \left[ \sin \alpha \sin \phi \csc \theta + \cos \alpha \cos \theta \right] \\
\Omega_{\theta\phi} &= -\frac{s \gamma_0^2}{4(1 + \rho^2 s^2)} \left[ \sin \alpha \cos \theta \cos(\phi - \beta) - \rho s \cos \phi \sin(\phi - \beta) \right] \\
\Omega_{\theta\psi} &= \frac{\gamma_0^2}{4(1 + \rho^2 s^2)} \left[ \sin \alpha \sin(\phi - \beta) - \rho s \cos \phi \cos(\phi - \beta) \right] - \rho s \cos \alpha \sin \theta \\
\Omega_{\phi\psi} &= -\frac{s \gamma_0^2}{4(1 + \rho^2 s^2)} \left[ \sin \alpha \sin(\phi - \beta) - \rho s \cos \phi \cos(\phi - \beta) \right] - \rho s \cos \alpha \sin \theta
\end{align*}
\]
and
\[
\det ||g|| = \left[ \frac{s(\gamma_0(s))^2}{8} \right]^2 \frac{(1 + \rho^2 s^2 \cos^2 \theta) \sin^2 \theta}{(1 + \rho^2 s^2)^2}.
\]

The torsion tensor is then obtained but, as its expression is lengthy, we do not give it here. The vector \( V_i \) and function \( F \) then result from equations (97,100,101 and 144):
\[
V_i = \frac{1}{2} \partial_i \log F^* ; \quad F = \left[ \frac{s(\gamma_0(s))^2}{8} \right]^2 \frac{(1 - i \rho s \cos \theta)^2 \sin^2 \theta}{(1 + \rho^2 s^2)^2} \exp -2i\phi
\]
and we have
\[
d\omega_3 = \frac{1}{2} \omega_3 \wedge \left( (d(\log \det ||g||) + \frac{i}{2} d^c \log(\frac{F}{F^*})) \right)
\]
\[
d(\omega_1 - i\omega_2) = \frac{1}{2} (\omega_1 - i\omega_2) \wedge d \log F
\]
\[
d(\omega_1 + i\omega_2) = \frac{1}{2} (\omega_1 + i\omega_2) \wedge d \log F^*,
\]
with the notations
\[
d = d' + d'', \quad d^c = i(d' - d''), \quad d'f = z^i \frac{\partial}{\partial z^i} f, \quad d''f = \bar{z}^i \frac{\partial}{\partial \bar{z}^i} f.
\]

Notice the important difference with Eguchi-Hanson: here the function \( F \) is a complex one. One should wonder whether, in an analytic change of coordinates, the function \( F \) might become real. \( F \) being a determinant, \( \log F \rightarrow \log F + h(z^i) \). Then
\[
\log \frac{F}{F^*} \rightarrow \log \frac{F'}{F'^*} = \log \frac{F}{F^*} + h(z^i) - \bar{h}(\bar{z}^i).
\]
A real \( F' \) then needs
\[
\partial_i \partial_j \left[ \log \frac{F}{F^*} \right] = 0.
\]
One computes
\[
\log \frac{F}{F^*} = -4i\phi + 2 \log \frac{(1 - i \rho s \cos \theta)}{(1 + i \rho s \cos \theta)} = -2(z^1 - \bar{z}^1) - 8i\phi
\]
and, for example
\[
\partial_2 \partial_2 \left[ \log \frac{F}{F^*} \right] = -8i\partial_2 \partial_2 \phi = -8i\partial_2 \left[ \frac{\rho s}{2(1 + \lambda s)} \right] = -2i\rho s \cos \theta \frac{1 + \rho^2 s^2}{(1 + \lambda s)^2} \neq 0.
\]
As a consequence, there is no conformal equivalence between this (4,0) geometry and an hyperkähler one and this gives a counterexample to Callan et al.’s assertion.

Let us observe that, contrarily to Eguchi-Hanson’s case, the new parameter \( \rho \) plays a distinguished role as it signals a new geometry. The second parameter in the torsion, \( L \) (\( \Leftrightarrow \) the function \( \gamma_0^2(s) \)), can be re-absorbed by a conformal transformation which, by the way, is missing in the present status of the Harmonic superspace approach [13].

The usual hyperkähler Taub-NUT metric is obtained for \( \rho = L = 0 \) (\( \gamma_0(s) \) constant which may be taken to be 1) and, in coordinates (143) the Kähler potential is:
\[
K = \frac{s}{2} \left[ 1 + \frac{1}{4} \lambda s(1 + \cos^2 \theta) \right]. \quad (147)
\]
6 Concluding remarks

We think to have clearly exemplified the advantages of using holomorphic coordinates to describe heterotic geometry. This enables us to obtain new and easy derivations of Eguchi-Hanson metric with torsion \([13]\) and generalisations of the Kähler quasi Ricci flat metrics of Bonneau and Delduc \([13]\) and of the scalar flat metrics of LeBrun \([12]\).

We have also proven that, contrarily to Callan et al.’s claim \([14]\), even for a 4 dimensional manifold, (4,0) heterotic geometry is not simply conformally equivalent to a (4,4) HyperKähler one.

It may be interesting to summarise the results in the language of the holonomy group:

| PROPERTIES | no torsion | with torsion |
|------------|------------|--------------|
| real dimension \(N\) \(\text{holonomy } O(N)\) | susy (1, 1) | (1, 0) heterotic geometry |
| real dimension \(2N\) \(\text{holonomy } U(N)\) | susy (2, 2) Kähler manifold | (2, 0) heterotic geometry |
| real dimension \(2N\) \(\text{holonomy } SU(N)\) | susy (2, 2) Kähler + Ricci flat | (2, 0) heterotic geometry special quasi Ricci flat space \((W=-\chi=V \text{ where } V \text{ is a vector related to the torsion tensor})\) |
| real dimension \(4N\) \(\text{holonomy } Sp(N)\) | susy (4, 4) hyperkähler \(\rightarrow \text{ Ricci flat}\) | (4, 0) heterotic geometry special quasi Ricci flat space \((W=-\chi=V \text{ where } V \text{ depends on a single complex function } F)\) |

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