Sum-free Sets of Integers with a Forbidden Sum

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Abstract

A set of integers is sum-free if it contains no solution to the equation \( x + y = z \). We study sum-free subsets of the set of integers \( [n] = \{1, \ldots, n\} \) for which the integer \( 2n + 1 \) cannot be represented as a sum of their elements. We prove a bound of \( O(2^{n/3}) \) on the number of these sets, which matches, up to a multiplicative constant, the lower bound obtained by considering all subsets of \( B_n = \{\lceil \frac{2}{3}(n+1) \rceil, \ldots, n\} \). A main ingredient in the proof is a stability theorem saying that if a subset of \( [n] \) of size close to \( |B_n| \) contains only a few subsets that contradict the sum-freeness or the forbidden sum, then it is almost contained in \( B_n \). Our results are motivated by the question of counting symmetric complete sum-free subsets of cyclic groups of prime order. The proofs involve Freiman’s 3k − 4 theorem, Green’s arithmetic removal lemma, and structural results on independent sets in hypergraphs.

1 Introduction

For an abelian additive group \( G \), a set \( A \subseteq G \) is sum-free if there are no \( x, y, z \in A \) such that \( x + y = z \). The study of sum-free sets was initiated in 1916 by Schur [30], who proved that the set of nonzero integers cannot be partitioned into a finite number of sum-free sets. His work was originally motivated by an attempt to prove the famous Fermat’s Last Theorem, which states that the set of all \( k \)th powers of nonzero integers is sum-free for every \( k \geq 3 \). To date, over a century later, sum-free sets play a fundamental role in the area of additive combinatorics and enjoy an intensive and fruitful line of research.

Sum-free subsets of the set of integers \( [n] = \{1, \ldots, n\} \) have attracted a significant attention in the literature over the years. It is easy to show that the largest size of a sum-free subset of \( [n] \) is \( \lceil n/2 \rceil \), attained by the set of odd integers in \( [n] \) and by the integer interval \( \lfloor (n/2) + 1, n \rfloor \). Cameron and Erdős [9] raised the question of counting the sum-free subsets of \( [n] \) and conjectured that there are \( O(2^{n/2}) \) such sets. Their conjecture was confirmed more than a decade later by Green [16] and by Sapozhenko [27] independently. More recently, Alon, Balogh, Morris, and Samotij [3] proved a refined version of the conjecture, providing a bound of \( 2^{O(n/m)} \cdot \binom{|n/2|}{m} \) on the number of sum-free subsets of \( [n] \) of size \( m \) for every \( 1 \leq m \leq \lfloor n/2 \rfloor \). The study of structural characterization of sum-free sets of integers was initiated by Freiman [15] who showed, roughly speaking, that every sum-free subset of \( [n] \) of density greater than 5/12 either consists entirely of odd integers or is close to an interval. Freiman’s result was extended to all sum-free subsets of \( [n] \) of density greater than 2/5 in an unpublished work by Deshouillers, Freiman, and Sós. However, their characterization does not hold for smaller subsets, and the 2/5 barrier was handled by a

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more complicated characterization due to Deshouillers, Freiman, Sós, and Temkin [11], that was recently further extended by Tran [31].

Another setting of great interest in the study of sum-free sets is the cyclic group $\mathbb{Z}_p$ of prime order $p$. The largest size of a sum-free subset of $\mathbb{Z}_p$ is known to be $\lfloor (p + 1)/3 \rfloor$. An explicit characterization of the sum-free sets that attain the largest size was provided in the late sixties by Yap [32, 33], Diananda and Yap [13], and Rhemtulla and Street [25]. In 2004, Green [16] proved an essentially tight upper bound of $2 \cdot 2^{\frac{1}{3} \cdot o(\frac{1}{3})} \cdot p$ on the number of sum-free subsets of $\mathbb{Z}_p$. As for their structure, Deshouillers and Lev [12] proved, improving on Lev [24] and Deshouillers and Freiman [10], that every sum-free subset of $\mathbb{Z}_p$ whose density is at least 0.318 is contained, up to an automorphism, in a bounded-size central interval of $\mathbb{Z}_p$.

For a prime $p$, a set $S \subseteq \mathbb{Z}_p$ is said to be symmetric if $x \in S$ implies that $-x \in S$, and complete if every element of $\mathbb{Z}_p \setminus S$ can be represented as a sum of two (not necessarily distinct) elements of $S$. The family of symmetric complete sum-free subsets of $\mathbb{Z}_p$ has been considered in the literature, motivated by several applications, such as the study of regular triangle-free graphs with diameter 2 [19], random sum-free sets of positive integers [7, 6], and dioid partitions of the group $\mathbb{Z}_p$ [20] (see [8] for a survey). In a recent work [21], a full characterization was provided for the symmetric complete sum-free subsets of $\mathbb{Z}_p$ of size at least $\left\lfloor \frac{1}{3} (p + 1) \right\rfloor$, where $c > 0$ is a universal constant. Somewhat surprisingly, this characterization reduces the challenge of counting the symmetric complete sum-free subsets of $\mathbb{Z}_p$ of a given sufficiently large size to counting certain sets of integers. As will be shortly explained, this question motivates the study of sum-free sets of integers with a forbidden sum, considered in the current work and described below.

### 1.1 Sum-free Sets of Integers with a Forbidden Sum

Let $n \geq 1$ be an integer. For a set $A \subseteq [n]$ and an integer $k \geq 0$, denote

$$kA = \left\{ \sum_{i=1}^{k} a_i \mid a_1, \ldots, a_k \in A \right\},$$

and let $\sum A = \cup_{k \geq 0} (kA)$. In this work we study sum-free subsets of $[n]$ for which the integer $2n + 1$ is a forbidden sum, that is, sets $A \subseteq [n]$ satisfying $A \cap 2A = \emptyset$ and $2n + 1 \notin \sum A$. For example, it is easy to see that the interval $B_n = \left[ \left\lfloor \frac{2}{3} (n + 1) \right\rfloor, n \right]$ satisfies these properties and has size $|B_n| = \left\lfloor \frac{1}{3} (n + 1) \right\rfloor$. We start with the extremal question of how large can such a set be, and prove the following tight upper bound.

**Theorem 1.1.** For every integer $n$, every sum-free set $A \subseteq [n]$ such that $2n + 1 \notin \sum A$ satisfies

$$|A| \leq \left\lfloor \frac{1}{3} (n + 1) \right\rfloor.$$

In fact, we prove a stronger statement than that of Theorem 1.1, providing the same bound under the weaker assumption $2n + 1 \notin (3A) \cup (4A)$ rather than $2n + 1 \notin \sum A$. This is tight in the sense that the bound is no longer true if we only require $2n + 1 \notin 3A$ (see Section 3.4).

Equipped with the tight answer to the extremal question, we turn to provide a corresponding robust stability theorem, which roughly speaking says the following: If a subset of $[n]$ of size close to $|B_n|$ contains only a few subsets that contradict the sum-freeness or the forbidden sum, then
it is almost contained in \( B_n \). To state the result precisely, let us introduce the following notation. For an integer \( n \), let \( \mathcal{F}_n^{(3)} \) denote the collection of all sets \( \{x, y, z\} \subseteq [n] \) of distinct \( x, y, z \) satisfying \( x + y = z \) or \( x + y + z = 2n + 1 \). For \( k \geq 4 \), let \( \mathcal{F}_n^{(k)} \) denote the collection of all sets \( \{x_1, \ldots, x_k\} \subseteq [n] \) of distinct \( x_1, \ldots, x_k \) satisfying \( \sum_{i=1}^{k} x_i = 2n + 1 \). Our stability theorem is the following.

**Theorem 1.2.** For every \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon) > 0 \) such that for every sufficiently large integer \( n \) the following holds. Every set \( A \subseteq [n] \) of size \( |A| \geq \left( \frac{1}{3} - \delta \right) \cdot n \) contains at least \( \delta \cdot n^{k-1} \) sets from \( \mathcal{F}_n^{(k)} \) for some \( k \in \{3, 4, 5\} \) or satisfies \( |A \setminus B_n| \leq \epsilon n \).

The proof of Theorem 1.2 employs the celebrated Freiman’s 3k – 4 theorem [14] as well as Green’s arithmetic removal lemma [17]. We remark that, in contrast to Theorem 1.1, it is essentially unavoidable to involve the sets of \( \mathcal{F}_n^{(5)} \) in the statement of Theorem 1.2. To see this, consider the set of odd integers in the interval \([1, \lfloor n/2 \rfloor]\) and observe that it is disjoint from \( B_n \), contains no set of \( \mathcal{F}_n^{(3)} \cup \mathcal{F}_n^{(4)} \), and yet has size \( |B_n| \).

We finally turn to the question of counting the sum-free sets \( A \subseteq [n] \) with \( 2n + 1 \notin \sum A \). Considering all subsets of \( B_n \) easily yields a lower bound of \( 2^{\left\lfloor \frac{n}{3} \right\rfloor + 1} \). We prove that this is tight up to a multiplicative constant.

**Theorem 1.3.** There are \( \Theta(2^{n/3}) \) sum-free sets \( A \subseteq [n] \) satisfying \( 2n + 1 \notin \sum A \).

The proof of the upper bound in Theorem 1.3 involves two main components. The first is the study of structural characterization of independent sets in hypergraphs developed by Saxton and Thomason [28] and by Balogh, Morris, and Samotij [5], building on a technique of Kleitman and Winston [22] (see also [26]). We employ a general theorem of [5] transferring stability results to bounds on the number of independent sets in hypergraphs. This allows us to derive from Theorem 1.2 that most sum-free subsets of \( [n] \) with \( 2n + 1 \) as a forbidden sum are almost contained in the set \( B_n \). The second component of the proof, used to count these sets, is an \( O(2^{n/3}) \) bound on the number of sets \( A \subseteq [n] \) for which \( n + 1 \notin 3A \) (see Theorem 3.12). Note that the sum-freeness constraint is not considered here). This bound, which might be of independent interest, is tight up to a multiplicative constant, as follows by considering all subsets of the interval \([\left\lfloor \frac{1}{3}n + 2 \right\rfloor, n]\). Its proof is inspired by a counting technique due to Alon et al. [3], and uses Janson’s inequality and a bound of Green and Morris [18] on the number of sets of integers with small sumset.

Although the current work focuses on the forbidden sum \( 2n + 1 \), we remark that Theorem 1.3 can be extended to other forbidden sums around \( 2n \) as well (see Section 3.5). However, for even forbidden sums in this regime, the situation is somewhat different in the sense that a similar bound of \( 2^{\left\lfloor \frac{1}{3}n + o(1) \right\rfloor} \) holds even without assuming the sum-freeness of the sets (see Proposition 3.22).

### 1.2 Symmetric Complete Sum-free Sets in Cyclic Groups

For a prime \( p \), we consider the family of symmetric complete sum-free subsets of the cyclic group \( \mathbb{Z}_p \). It is easy to deduce from the classification results of [32,33,13,25] that the largest possible size of such a set is \( \left\lfloor (p + 1)/3 \right\rfloor \), attained uniquely, up to an automorphism, by the set \( \{k + 1, 2k + 1\} \) for \( p = 3k + 2 \) and by the set \( \{k\} \cup [k + 2, 2k - 1] \cup \{2k + 1\} \) for \( p = 3k + 1 \) where \( k \geq 4 \).

In a recent work [21], the characterization of symmetric complete sum-free subsets of \( \mathbb{Z}_p \) of largest size was extended to a linear range of sizes. It was shown there, using a structural result
of [12], that for all sufficiently large primes \( p \), every symmetric complete sum-free subset of \( \mathbb{Z}_p \) of (even) size \( s \in [0.318p, \frac{p-1}{2}] \) is, up to an automorphism, of the form

\[
S_T = [p - 2s + 1, 2s - 1] \cup \pm(s + T),
\]

where \( T \subseteq [0, 2t - 1] \) is a set of \( t \) integers for \( t = \frac{p - 3s + 1}{2} \). While \( S_T \) is symmetric for every set of integers \( T \), sufficient and necessary conditions on \( T \) for which \( S_T \) is complete and sum-free were provided in [21]. These conditions reduce the challenge of counting the symmetric complete sum-free subsets of \( \mathbb{Z}_p \) of a given sufficiently large size to a question of counting certain sets of integers. As an application of our Theorem 1.3, we make a step towards this challenge and provide a tight estimation for the number of symmetric complete sum-free sets \( S_T \) of size \( s \) that satisfy \( s \in S_T \) (equivalently, \( 0 \in T \)).

**Theorem 1.4.** For every sufficiently large prime \( p \) and every even integer \( s \in [0.318p, \frac{p-1}{2}] \), there are \( \Theta(2^{(\frac{1}{3} + o(1))n}) \) symmetric complete sum-free sets \( S_T \subseteq \mathbb{Z}_p \) of size \( s \) that satisfy \( s \in S_T \).

It will be interesting to figure out if a similar upper bound holds for all symmetric complete sum-free sets \( S_T \) of a sufficiently large size \( s \) as well.

**Outline.** The rest of the paper is organized as follows. In Section 2 we gather several definitions and results used throughout the paper. In Section 3 we study sum-free subsets of \([n]\) avoiding the forbidden sum \( 2n + 1 \). Theorem 1.1 is proved in Section 3.1. For the counting question, we first prove in Section 3.2 a weaker upper bound of \( 2^{(\frac{1}{3} + o(1))n} \), and then in Section 3.3 prove our stability result Theorem 1.2 and use it to prove the tight answer given in Theorem 1.3. In Section 3.4 we discuss the tightness of Theorem 1.1 and in Section 3.5 we discuss extensions of Theorem 1.3 to additional forbidden sums. Finally, in Section 4 we present our application to counting symmetric complete sum-free subsets of cyclic groups of prime order and prove Theorem 1.4.

### 2 Preliminaries

#### 2.1 Additive Combinatorics

For an abelian additive group \( G \) and two sets \( A, B \subseteq G \) define \( A + B = \{a + b \mid a \in A, b \in B\} \). We use the notations \( kA = \{\sum_{i=1}^{k} a_i \mid a_1, \ldots, a_k \in A\} \) for \( k \geq 0 \), and \( \sum A = \cup_{k \geq 0}(kA) \). The set \( A \) is **sum-free** if \( A \cap 2A = \emptyset \), **complete** if \( G \setminus A \subseteq 2A \), and **symmetric** if \( A = -A \). The following simple claim is well known.

**Claim 2.1.** For every nonempty sets \( A, B \subseteq \mathbb{Z} \), \( |A + B| \geq |A| + |B| - 1 \). In particular, for every \( k \geq 0 \), \( |kA| \geq k \cdot |A| - (k - 1) \).

The following classical theorem, proved in 1959 by Freiman [14], shows that sets of integers with small sumset are highly structured.

**Theorem 2.2 (Freiman’s 3k − 4 Theorem [14]).** Every finite set \( A \subseteq \mathbb{Z} \) satisfying \( |A + A| \leq 3|A| - 4 \) is contained in an arithmetic progression of length at most \( |A + A| - |A| + 1 \).
We also need a recent result due to Tran [31] on the structure of large sum-free subsets of \([n]\) (see also [11]).

**Theorem 2.3 ([31]).** There exists a constant \(c > 0\) such that for every integer \(n\) and real \(\eta \in \left[\frac{2}{3}, c\right]\) the following holds. Every sum-free set \(A \subseteq [n]\) of size \(|A| \geq \left(\frac{2}{3} - \eta\right) \cdot n\) satisfies one of the following alternatives.

1. All the elements of \(A\) are congruent to 1 or 4 modulo 5.
2. All the elements of \(A\) are congruent to 2 or 3 modulo 5.
3. All the elements of \(A\) are odd.
4. \(\min(A) \geq |A|\).
5. \(A \subseteq \left[\left(\frac{4}{3} - 200\eta^{1/2}\right)n, (\frac{4}{3} + 200\eta^{1/2})n\right] \cup \left[\left(\frac{4}{3} - 200\eta^{1/2}\right)n, n\right]\).

**2.2 Green’s Arithmetic Removal Lemma**

In 2005, Green [17] proved an arithmetic removal lemma for abelian groups, motivated by well-known removal lemmas in graph theory (see [23] for an alternative proof and an extension). Among other applications, he used it to prove that every ‘almost’ sum-free subset of \([n]\) can be made sum-free by removing relatively few elements. We state below the arithmetic removal lemma of [17] and a variant of its application to sum-freeness, a proof of which is included for completeness.

**Theorem 2.4 ([17]).** For every \(\varepsilon > 0\) and an integer \(k \geq 3\) there exists \(\delta = \delta_k(\varepsilon) > 0\), such that for every integer \(N\) the following holds. Let \(A_1, \ldots, A_k\) be subsets of an abelian additive group \(G\) of size \(|G| = N\) such that the number of zero-sum \(k\)-tuples in \(\prod_{i=1}^{k} A_i\) (that is, \(k\)-tuples \((x_1, \ldots, x_k) \in \prod_{i=1}^{k} A_i\) satisfying \(\sum_{i=1}^{k} x_i = 0\)) is at most \(\delta \cdot N^{k-1}\). Then, there exist subsets \(A_i' \subseteq A_i\), \(i \in [k]\), with \(|A_i'| \leq \varepsilon \cdot N\), for which there are no zero-sum \(k\)-tuples in \(\prod_{i=1}^{k} (A_i \setminus A_i')\).

**Corollary 2.5.** For every \(\varepsilon > 0\) there exists \(\delta' = \delta'(\varepsilon) > 0\), such that for every sufficiently large integer \(n\) the following holds. Let \(\mathcal{F}\) denote the collection of all sets \(\{x, y, z\} \subseteq [n]\) of distinct \(x, y, z\) satisfying \(x + y = z\). If a set \(A \subseteq [n]\) contains at most \(\delta' \cdot n^2\) sets from \(\mathcal{F}\) then there exists a subset \(A' \subseteq A\) of size \(|A'| \leq \varepsilon \cdot n\) for which \(A \setminus A'\) is sum-free.

**Proof:** For \(\varepsilon > 0\) let \(\delta = \delta_3(\varepsilon)\), where \(\delta_3\) is as in Theorem 2.4 and define \(\delta' = \frac{\varepsilon}{6}\). We apply Theorem 2.4 with the group \(G = \mathbb{Z}_{2n}\) and \(k = 3\). Identify a set \(A \subseteq [n]\) as a subset of \(G\) in the natural way, and consider the sets \(A_1 = A_2 = A\) and \(A_3 = -A\). Observe that for \(x, y, z \in [n]\), the equality \(x + y = z\) over \(G\) is equivalent to the same equality over the integers. Assuming that \(A\) contains at most \(\delta' \cdot n^2\) sets from \(\mathcal{F}\), the number of ordered triples \((x, y, z) \in A^3\) such that \(x + y = z\) is at most \(3! \cdot \delta' n^2 + 3n \leq 8\delta' n^2 = \delta \cdot |G|^2\), where the inequality holds assuming that \(n\) is sufficiently large. By Theorem 2.4 there exists a set \(A' \subseteq A\) of size \(|A'| \leq 3 \cdot \frac{\varepsilon}{6} \cdot |G| = \varepsilon \cdot n\) such that \(A \setminus A'\) is sum-free over the group \(G\), thus over the integers as well.

As another application of Theorem 2.4 we show that for every fixed \(k\), if a subset of \([n]\) includes a relatively few \(k\)-subsets with a given sum then it has a large subset including no \(k\)-tuples with this sum at all.
Corollary 2.6. For every $\epsilon > 0$ and an integer $k \geq 3$ there exists $\delta'' = \delta_k'(\epsilon) > 0$, such that for every sufficiently large integer $n$ the following holds. For an integer $\ell$, let $F$ denote the collection of all sets $\{x_1, \ldots, x_k\} \subseteq [n]$ of distinct $x_1, \ldots, x_k$ satisfying $\sum_{i=1}^k x_i = \ell$. If a set $A \subseteq [n]$ contains at most $\delta'' \cdot n^{k-1}$ sets from $F$ then there exists a subset $A' \subseteq A$ of size $|A'| \leq \epsilon \cdot n$ for which $\ell \notin k(A \setminus A')$.

Proof: For $\epsilon > 0$ and $k \geq 3$ let $\delta'' = \delta_k'(\epsilon)$, where $\delta_k$ is as in Theorem 2.4. It can be assumed that $\ell \in [k, kn]$, as otherwise $\ell \notin kA$ for every $A \subseteq [n]$. We apply Theorem 2.4 with the group $G = \mathbb{Z}_{kn}$ and the integer $k$. Identify a set $A \subseteq [n]$ as a subset of $G$ in the natural way, and consider the sets $A_1 = \cdots = A_{k-1} = A$ and $A_k = A - \ell$. Observe that for $x_1, \ldots, x_k \in [n]$, the equality $\sum_{i=1}^k x_i = k \cdot \ell$ can be written as $\sum_{i=1}^k x_i = \ell$, and that it holds over $G$ if and only if it holds over the integers. Assuming that $A$ contains at most $\delta'' \cdot n^{k-1}$ sets from $F$, the number of ordered $k$-tuples $(x_1, \ldots, x_k) \in A^k$ such that $\sum_{i=1}^k x_i = \ell$ is at most

$$k! \cdot \delta'' \cdot n^{k-1} + \binom{k}{2} n^{k-2} \leq 3^k |G| \leq 3^{k!} \cdot \delta'' \cdot n^{k-1} = \frac{3k!}{2^{k-1}} \cdot \delta'' \cdot |G| \leq \delta'' \cdot |G|^k,$$

where the first inequality holds assuming that $n$ is sufficiently large. By Theorem 2.4 there exists a set $A' \subseteq A$ of size $|A'| \leq k \cdot \frac{1}{k} \cdot |G| = \epsilon \cdot n$ such that $\ell \notin k(A \setminus A')$ over the group $G$, thus over the integers as well. 

### 2.3 Independent Sets in Hypergraphs

Structural results on independent sets in hypergraphs were found in recent years as a strong tool in proving extremal, structural, and counting results in combinatorics (see, e.g., [28, 5, 26]). We state below a theorem of Balogh, Morris, and Samotij [5] that provides a general framework to derive counting statements from supersaturation and stability results.

We start with a few notations. For a hypergraph $\mathcal{H}$, denote by $V(\mathcal{H})$ the set of its vertices and by $E(\mathcal{H}) \subseteq P(V(\mathcal{H}))$ the set of its hyperedges. Let $v(\mathcal{H}) = |V(\mathcal{H})|$ and $e(\mathcal{H}) = |E(\mathcal{H})|$. The hypergraph $\mathcal{H}$ is $k$-uniform if $|e| = k$ for every $e \in E(\mathcal{H})$. For a set $A \subseteq V(\mathcal{H})$ let $\mathcal{H}[A]$ be the subhypergraph of $\mathcal{H}$ induced by $A$. An independent set in $\mathcal{H}$ is a subset of $V(\mathcal{H})$ containing no hyperedge of $\mathcal{H}$. Let $\mathcal{I}(\mathcal{H})$ denote the family of independent sets in $\mathcal{H}$, and for an integer $m$, let $\mathcal{I}(\mathcal{H}, m)$ denote the family of independent sets in $\mathcal{H}$ of size $m$. For a set $T \subseteq V(\mathcal{H})$ define $\deg_{\mathcal{H}}(T) = |\{e \in E(\mathcal{H}) \mid T \subseteq e\}|$, and for an integer $\ell$, let

$$\Delta_{\ell}(\mathcal{H}) = \max\{\deg_{\mathcal{H}}(T) \mid T \subseteq V(\mathcal{H}) \text{ and } |T| = \ell\}.$$

We also need the following definitions of density and stability of hypergraphs used in [5] (see also [2, 29]).

**Definition 2.7.** Let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of hypergraphs, and let $\alpha \in (0, 1)$ be a real number.

1. We say that $\mathcal{H}$ is $\alpha$-dense if for every $\epsilon > 0$ there exist $\delta > 0$ and $n_0$ such that the following holds. For every $n \geq n_0$ and a set $A \subseteq V(\mathcal{H}_n)$ with $|A| \geq (\alpha + \epsilon) \cdot v(\mathcal{H}_n)$, $e(\mathcal{H}_n[A]) \geq \delta \cdot e(\mathcal{H}_n)$.

2. For a sequence $\mathcal{B}$ of sets $\mathcal{B}_n \subseteq P(V(\mathcal{H}_n))$, we say that $\mathcal{H}$ is $(\alpha, \mathcal{B})$-stable if for every $\epsilon > 0$ there exist $\delta > 0$ and $n_0$ such that the following holds. For every $n \geq n_0$ and a set $A \subseteq V(\mathcal{H}_n)$ with $|A| \geq (\alpha - \delta) \cdot v(\mathcal{H}_n)$, it holds that $e(\mathcal{H}_n[A]) \geq \delta \cdot e(\mathcal{H}_n)$ or $|A \setminus B| \leq \epsilon \cdot v(\mathcal{H}_n)$ for some $B \in \mathcal{B}_n$. 


Theorem 2.8 (Theorems 5.4 and 6.3 in [5]). Let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of $k$-uniform hypergraphs for an integer $k$, and let $\alpha \in (0, 1)$ and $c > 0$. Let $p \in [0, 1]^\mathbb{N}$ be a sequence of real numbers satisfying that for every sufficiently large integer $n$ and every $\ell \in [k]$,

$$\Delta_\ell(\mathcal{H}_n) \leq c \cdot p_n^{\ell-1} \cdot \frac{e(\mathcal{H}_n)}{\nu(\mathcal{H}_n)}.$$

Let $H$ be a $k$-uniform hypergraph and let $\alpha \in (0, 1)$ and $c > 0$. Let $p \in [0, 1]$. Let $\ell \in [k]$. Let $\mathcal{B}_n \subseteq \mathcal{P}(V(\mathcal{H}_n))$. If $\mathcal{H}$ is $(\alpha, \mathcal{B})$-stable then for every $\gamma > 0$ there exist $\beta > 0$ and $C > 0$ such that for every sufficiently large $n$ and every $m \geq C p_n \nu(\mathcal{H}_n)$ there are at most

$$\left(1 - \beta\right)^m \cdot \left(\frac{\alpha \cdot \nu(\mathcal{H}_n)}{m}\right)$$



independent sets $I \in \mathcal{I}(\mathcal{H}_n, m)$ such that $|I \setminus B| \geq \gamma m$ for every $B \in \mathcal{B}_n$.

2.4 Janson’s Inequality

Janson’s inequality is a useful tool to bound the probability that no event of a collection of ‘mostly’ independent events occurs. See, e.g., [4, Chapter 8].

Lemma 2.9 (Janson’s Inequality). Let $\{B_i\}_{i \in J}$ be a family of subsets of a finite set $X$ and let $p \in [0, 1]$. Denote

$$\mu = \sum_{i \in J} p^{|B_i|} \quad \text{and} \quad \Delta = \sum_{i \sim j} p^{|B_i \cup B_j|},$$

where $i \sim j$ means that $i$ and $j$ are distinct indices in $J$ satisfying $B_i \cap B_j \neq \emptyset$. Let $R$ be a random subset of $X$, where every element of $X$ is chosen to be in $R$ independently with probability $p$. Then, the probability that no $B_i$ for $i \in J$ is contained in $R$ is at most $\max(e^{-\mu/2}, e^{-\mu^2/(2\Delta)})$.

2.5 Sets of Integers with Small Sumset

We need the following bound due to Green and Morris [18] on the number of sets of integers with a bounded-size sumset.

Theorem 2.10 (Theorem 1.1 in [18]). For every $\delta > 0$ and $\lambda > 0$, for a sufficiently large integer $\ell$ the following holds. For every $k \in \mathbb{N}$ there are at most

$$2^{\delta \ell} \cdot \binom{\lambda \ell/2}{\ell} \cdot k^{\lfloor \Lambda + \delta \rfloor}$$

sets $S \subseteq [k]$ with $|S| = \ell$ and $|S + S| \leq \lambda \cdot \ell$.

We also use, in this context, the following simple bound given in [3] on the number of integer partitions of an integer $k$ into $\ell$ distinct parts, i.e., the number of sets of $\ell$ positive integers whose sum is $k$.

Lemma 2.11 (Lemma 5.1 in [3]). For every two positive integers $k$ and $\ell$, the number of partitions of $k$ into $\ell$ distinct parts is at most

$$\left(\frac{e^2 k}{\ell^2}\right)^\ell.$$
3 Sum-free Sets of Integers with a Forbidden Sum

We study the sum-free sets $A \subseteq [n]$ that satisfy $2n + 1 \notin \sum A$, where the integer $2n + 1$ is referred to as a forbidden sum. We start with the extremal question of how large can such a set be, and then turn to study the number of these sets.

3.1 The Maximum Size

The following theorem confirms Theorem 1.1.

**Theorem 3.1.** For every integer $n$, every sum-free set $A \subseteq [n]$ such that $2n + 1 \notin (3A) \cup (4A)$ satisfies

$$|A| \leq \left\lfloor \frac{1}{3}(n + 1) \right\rfloor.$$ 

**Proof:** Let $A \subseteq [n]$ be a sum-free set such that $2n + 1 \notin (3A) \cup (4A)$. Denote $C = A \cup (A + A) \subseteq [2n]$.

By the sum-freeness of $A$ and Claim 2.1,

$$|C| = |A| + |A + A| \geq |A| + (2|A| - 1) = 3|A| - 1.$$ 

We claim that $|C| \leq n$. Otherwise, by the pigeonhole principle, there exists $1 \leq i \leq n$ for which $\{i, 2n + 1 - i\} \subseteq C$. Notice that $i$ belongs to either $A$ or $A + A$ and that $2n + 1 - i$, which is larger than $n$, belongs to $A + A$. This implies that $2n + 1 \in (3A) \cup (4A)$, in contradiction to our assumption. It follows that $3|A| - 1 \leq |C| \leq n$, hence $|A| \leq \left\lfloor \frac{1}{3}(n + 1) \right\rfloor$ as required. □

**Remark 3.2.** We note that the assumption $2n + 1 \notin (3A) \cup (4A)$ in Theorem 3.1 cannot be relaxed to the assumption $2n + 1 \notin 3A$. See Section 3.4 for a detailed discussion.

To obtain a matching lower bound, consider the interval $B_n = \left[\left\lceil \frac{2}{3}(n + 1) \right\rceil, n\right]$ whose size is

$$|B_n| = n - \left(\left\lceil \frac{2}{3}(n + 1) \right\rceil - 1\right) = \left\lfloor \frac{1}{3}(n + 1) \right\rfloor.$$ 

Clearly, the sum of every two element of $B_n$ is smaller than $2n + 1$ and the sum of every three is larger than $2n + 1$. This implies that $2n + 1 \notin \sum B_n$ and, in particular, that $2n + 1 \notin (3B_n) \cup (4B_n)$. Combining it with Theorem 3.1, we derive the following corollary.

**Corollary 3.3.** For every integer $n$, the following holds.

1. The maximum size of a sum-free set $A \subseteq [n]$ such that $2n + 1 \notin \sum A$ is $\left\lfloor \frac{1}{3}(n + 1) \right\rfloor$.

2. The maximum size of a sum-free set $A \subseteq [n]$ such that $2n + 1 \notin (3A) \cup (4A)$ is $\left\lfloor \frac{1}{3}(n + 1) \right\rfloor$. 


3.2 Supersaturation

We turn to prove a supersaturation result for sum-free subsets of \([n]\) with \(2n + 1\) as a forbidden sum. Namely, we show that every set \(A \subseteq [n]\) of size linearly larger than the bound given in Theorem 3.1 contains many subsets that contradict the sum-freeness or the forbidden sum. To state it formally, we recall the following notation.

**Definition 3.4.** For an integer \(n\), let \(\mathcal{F}_n^{(3)}\) denote the collection of all sets \(\{x, y, z\} \subseteq [n]\) of distinct \(x, y, z\) satisfying \(x + y = z\) or \(x + y + z = 2n + 1\). For \(k \geq 4\), let \(\mathcal{F}_n^{(k)}\) denote the collection of all sets \(\{x_1, \ldots, x_k\} \subseteq [n]\) of distinct \(x_1, \ldots, x_k\) satisfying \(\sum_{i=1}^{k} x_i = 2n + 1\).

The proof of the supersaturation result, stated below, uses the corollaries of Green’s arithmetic removal lemma given in Section 2.2.

**Theorem 3.5.** For every \(\epsilon > 0\) there exists \(\delta = \delta(\epsilon) > 0\) such that for every sufficiently large integer \(n\) the following holds. Every set \(A \subseteq [n]\) of size \(|A| \geq \left(\frac{1}{3} + \epsilon\right) \cdot n\) contains at least \(\delta \cdot n^{k-1}\) sets from \(\mathcal{F}_n^{(k)}\) for some \(k \in \{3, 4\}\).

**Proof:** For a given \(\epsilon > 0\), define \(\delta = \min(\delta'(\xi), \delta''(\xi), \delta'''(\xi))\), where \(\delta', \delta'', \text{ and } \delta'''\) are as in Corollaries 2.5 and 2.6. Assume by contradiction that for a sufficiently large integer \(n\), a set \(A \subseteq [n]\) of size \(|A| \geq \left(\frac{1}{3} + \epsilon\right) \cdot n\) contains fewer than \(\delta \cdot n^{k-1}\) sets from \(\mathcal{F}_n^{(k)}\) for every \(k \in \{3, 4\}\). Applying Corollary 2.5 we obtain a set \(D_1 \subseteq A\) of size \(|D_1| \leq \frac{\xi}{6} \cdot n\) for which \(A \setminus D_1\) is sum-free. Applying Corollary 2.6 twice with \(\ell = 2n + 1\) and \(k \in \{3, 4\}\), we obtain two sets \(D_2, D_3 \subseteq A\), each of which is of size at most \(\frac{\xi}{6} \cdot n\), such that \(2\log + 1 \notin (A \setminus D_2)\) and \(2\log + 1 \notin (A \setminus D_3)\). Consider the set \(B = A \setminus (D_1 \cup D_2 \cup D_3)\), and notice that \(B\) is sum-free and satisfies \(2\log + 1 \notin (3B) \cup (4B)\). The size of \(B\) satisfies

\[
|B| \geq |A| - |D_1| - |D_2| - |D_3| \geq \left(\frac{1}{3} + \epsilon\right) \cdot n - 3 \cdot \frac{\xi}{6} \cdot n \geq \left(\frac{1}{3} + \frac{\xi}{2}\right) \cdot n.
\]

We get a contradiction to Theorem 3.1, so we are done. \(\square\)

**Theorem 3.5** combined with a result of [5] on counting independent sets in hypergraphs (see Section 2.3) already allows us to derive a bound of \(2^{\Theta(\log^{1+\epsilon})\cdot n}\) on the number of sum-free sets \(A \subseteq [n]\) with \(2\log + 1 \notin (3A) \cup (4A)\), and, in particular, on those that satisfy \(2\log + 1 \notin \sum A\). We include a proof of this bound for didactical reasons and then turn to prove the tighter bound of \(O(2^{n/3})\) (see Section 3.3).

**Corollary 3.6.** There are \(2^{\Theta(\log^{1+\epsilon})\cdot n}\) sum-free sets \(A \subseteq [n]\) satisfying \(2\log + 1 \notin (3A) \cup (4A)\).

**Proof:** Let \(\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}\) be a sequence of 4-uniform hypergraphs defined as follows. For every \(n\), \(\mathcal{H}_n\) is the hypergraph on the vertex set \([n]\) whose hyperedges are all 4-subsets of \([n]\) that contain at least one of the sets in \(\mathcal{F}_n^{(3)} \cup \mathcal{F}_n^{(4)}\). Observe that \(e(\mathcal{H}_n) = \Theta(n^3)\), and that \(\Delta_1(\mathcal{H}_n) = \Theta(n^2)\), \(\Delta_2(\mathcal{H}_n) = \Theta(n)\), \(\Delta_3(\mathcal{H}_n) = \Theta(n)\), and \(\Delta_4(\mathcal{H}_n) = 1\). Hence, for every sufficiently large \(n\), every \(\ell \in [4]\), some constant \(c > 0\), and \(p_n = 1/\sqrt{n}\), we have

\[
\Delta_\ell(\mathcal{H}_n) \leq c \cdot p_n^{\ell-1} \cdot \frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)}.
\]
Since every sum-free set $A \subseteq [n]$ satisfying $2n + 1 \notin (3A) \cup (4A)$ forms an independent set in $\mathcal{H}_n$, it suffices to bound from above the size of $\mathcal{I}(\mathcal{H}_n)$.

To this end, let us show that the sequence of hypergraphs $\mathcal{H}$ is $\frac{1}{3}$-dense (recall Definition 2.7). Item 1. Let $\varepsilon > 0$ be a constant. By Theorem 3.5 for some $\delta > 0$ and every sufficiently large integer $n$, every set $A \subseteq [n]$ of size $|A| \geq (\frac{1}{3} + \varepsilon) \cdot n$ contains at least $\delta \cdot n^{k-1}$ sets from $F_n^{(k)}$ for some $k \in \{3, 4\}$. Observe that this implies, using $e(\mathcal{H}_n) = \Theta(n^3)$, that such an $A$ satisfies $e(\mathcal{H}_n[A]) \geq \delta' \cdot e(\mathcal{H}_n)$ for some $\delta' > 0$, as required.

Now, we can apply Item 1 of Theorem 2.8 to obtain that for every $\gamma > 0$ there exists $C > 0$ such that for every sufficiently large $n$ and every $m \geq C p_n^*(\mathcal{H}_n) = C \sqrt{n}$,

$$|\mathcal{I}(\mathcal{H}_n, m)| \leq \left(\frac{\frac{1}{3} + \gamma}{m}\right)^n.$$

It follows that

$$|\mathcal{I}(\mathcal{H}_n)| \leq \sum_{m=0}^{\sqrt{n}} \binom{n}{m} + \sum_{m=\sqrt{n}}^{\frac{\sqrt{n}}{3}} \left(\frac{\frac{1}{3} + \gamma}{m}\right)^n \leq n^{O(\sqrt{n})} + 2^{\left(\frac{1}{3} + \gamma\right) \cdot n} \leq 2^{\left(\frac{1}{3} + 2\gamma\right) \cdot n}.$$

Since the bound holds for every $\gamma > 0$, the result follows.

3.3 The Tight Bound – Proof of Theorem 1.3

In this section we estimate the number of sum-free sets $A \subseteq [n]$ with $2n + 1 \notin \Sigma A$ and confirm Theorem 1.3. Recall that a lower bound of $2^{\frac{2}{3}(n+1)}$ follows by considering all subsets of the set $B_n = \lfloor \frac{2}{3}(n+1) \rfloor, [n]$. For the upper bound, we prove the following stronger statement.

**Theorem 3.7.** There are $O(2^{n/3})$ sum-free sets $A \subseteq [n]$ satisfying $2n + 1 \notin (3A) \cup (4A) \cup (5A)$.

**A roadmap for the proof of Theorem 3.7.** In the proof, we count separately the sets that include ‘many’ elements which do not belong to $B_n$ and the sets that are almost contained in $B_n$. The former sets are considered in Section 3.3.1 where we prove our stability result Theorem 3.2. Combine it with a result of [5] on counting independent sets in hypergraphs, and obtain the required bound (see Corollary 3.11). To count the sets that are almost contained in $B_n$, we consider two subcases: The sets whose elements are all greater than $n/2$ are considered in Section 3.3.2 (see Corollary 3.13), and those that include at least one smaller element are considered in Section 3.3.3 (see Lemma 3.16). We finally put everything together and derive Theorem 3.7 in Section 3.3.4.

**Remark 3.8.** For simplicity of presentation, we omit throughout this section floor and ceiling signs whenever the implicit assumption that a certain quantity is integer makes no essential difference in the argument.

3.3.1 Stability

We restate and prove our stability result (recall Definition 3.4).

**Theorem 3.9.** For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every sufficiently large integer $n$ the following holds. Every set $A \subseteq [n]$ of size $|A| \geq (\frac{1}{3} - \delta) \cdot n$ contains at least $\delta \cdot n^{k-1}$ sets from $F_n^{(k)}$ for some $k \in \{3, 4, 5\}$ or satisfies $|A \setminus \left[\frac{2n}{3} + 1, n\right]| \leq \epsilon n$. 

We need the following simple lemma.

**Lemma 3.10.** For $\varepsilon > 0$ and a sufficiently large integer $n$, let $A \subseteq [n]$ be a set of size $|A| \geq \left(\frac{2}{3} - \varepsilon\right) \cdot n$ satisfying either $2n + 2 \notin 3A$ or $2n + 1 \notin 3A$. Then, $|A \cap \left[\frac{2n}{3} + 1, n\right]| \leq 3\varepsilon n$.

**Proof:** We prove the lemma under the assumption $2n + 2 \notin 3A$. The case of $2n + 1 \notin 3A$ is similar. Assume by contradiction that $|A \cap \left[\frac{2n}{3} + 1, n\right]| > 3\varepsilon n$. Denote $\max(A) = \frac{2n}{3} + a$, and notice that $a \in (3\varepsilon n, \frac{2n}{3}]$. Consider the interval $[\frac{2n}{3} - 2n + 2, \frac{2n}{3} + a]$ of length $3a - 1$, and observe that it contains $\left\lceil\frac{3a - 2}{2}\right\rceil$ pairwise disjoint sets $\{x, y\}$ of distinct $x, y$ satisfying $x + y = \frac{4n}{3} - a + 2$. Since $\frac{2n}{3} + a \in A$ and $2n + 2 \notin 3A$, it follows that at least one element from every such set does not belong to $A$, hence

$$|A| \leq \max(A) - \left\lceil\frac{3a - 2}{2}\right\rceil = \frac{2n}{3} - \left\lfloor\frac{a - 1}{2}\right\rfloor < \left(\frac{2}{3} - \varepsilon\right) \cdot n,$$

in contradiction.

**Proof of Theorem 3.9** For a given $\varepsilon > 0$ define

$$\delta = \min \left(\delta'(\frac{\varepsilon}{96}), \delta''(\frac{\varepsilon}{96}), \delta'(\frac{\varepsilon}{96}), \delta''(\frac{\varepsilon}{24})\right),$$

where $\delta'$, $\delta''$, $\delta'''$, and $\delta'''$ are as in Corollaries 2.5 and 2.6. Notice that it can be assumed, whenever needed, that $\varepsilon$ is sufficiently small (because the statement of the theorem is stronger for smaller values of $\varepsilon$). For a sufficiently large integer $n$, let $A \subseteq [n]$ be a set of size $|A| \geq (\frac{1}{3} - \delta) \cdot n$. Assume that for each $k \in \{3, 4, 5\}$, fewer than $\delta \cdot n^{k-1}$ of the sets in $F_n^{(k)}$ are contained in $A$. Our goal is to prove that $|A \setminus \left[\frac{2n}{3} + 1, n\right]| \leq 3\varepsilon n$.

We first apply Corollaries 2.5 and 2.6 once and three times respectively, to obtain a set $A' \subseteq A$ such that $A'$ is sum-free and satisfies $2n + 1 \notin (3A') \cup (4A') \cup (5A')$ and $|A \setminus A'| \leq 4 \cdot \frac{\varepsilon}{96} \cdot n = \frac{\varepsilon n}{24}$. Observe that

$$|A'| = |A| - |A \setminus A'| \geq \left(\frac{1}{3} - \delta - \frac{\varepsilon}{24}\right) \cdot n \geq \left(\frac{1}{3} - \frac{\varepsilon}{12}\right) \cdot n,$$

and that, by Theorem 3.1, we have $|A'| \leq \frac{4}{3}$ (recall Remark 3.8).

Consider the set $C = A' \cup (A' + A') \subseteq [2n]$. Observe that $|C| \leq n$, as otherwise, by the pigeonhole principal, there exists $1 \leq i \leq n$ for which $\{i, 2n + 1 - i\} \subseteq C$, in contradiction to $2n + 1 \notin (3A') \cup (4A')$. By the sum-freeness of $A'$ and (1), it follows that

$$|A'| + |A' + A'| = |C| \leq n \leq \frac{|A'|}{\frac{1}{3} - \frac{\varepsilon}{12}} = \frac{3|A'|}{1 - \frac{\varepsilon}{4}} \leq 3|A'| \cdot \left(1 + \frac{\varepsilon}{3}\right).$$

Rearranging, we obtain that

$$|A' + A'| \leq |A'| \cdot (2 + \varepsilon) \leq 3|A'| - 4.$$

Hence we can apply Theorem 2.2 to obtain that $A'$ is contained in an arithmetic progression of length at most

$$|A' + A'| - |A'| + 1 \leq |A'| \cdot (1 + \varepsilon) + 1 \leq \frac{n}{3} \cdot (1 + \varepsilon) + 1.$$
It follows that one can exclude at most \( \frac{\epsilon n}{3} + 1 \) elements from \( A' \) to get a subset \( A'' \subseteq A' \) contained in an arithmetic progression of length precisely \( d \).

\[
|A \setminus A''| \leq |A \setminus A'| + \frac{en}{3} + 1 \leq \frac{en}{24} + \frac{en}{3} + 1 = \frac{3en}{8} + 1 \tag{2}
\]

and, using (1),

\[
|A''| = |A| - |A \setminus A''| \geq \left( \frac{1}{3} - \frac{\epsilon}{24} \right) \cdot n - \left( \frac{3en}{8} + 1 \right) \geq \left( \frac{1}{3} - \frac{\epsilon}{2} \right) \cdot n. \tag{3}
\]

Since \( A'' \subseteq A' \), we have \( 2n + 1 \notin (3A'') \cup (4A'') \cup (5A'') \).

Let \( B \) denote an arithmetic progression of length \( \frac{\epsilon}{3} \) containing the set \( A'' \), and let \( d \) denote its difference. Observe that \( d \in \{1, 2, 3\} \), as otherwise \( |B| \leq \frac{n}{2} \), in contradiction to the bound given in (3) on the size of \( A'' \subseteq B \). We consider below each of the three possible values of \( d \) separately.

1. \( d = 1 \): In this case \( B = [a, a + \frac{n}{3} - 1] \) for some \( a \in [1, \frac{2n}{3} + 1] \). It suffices to show that

\[
a \geq \left( \frac{2}{3} - \frac{\epsilon}{2} \right) \cdot n,
\]

as desired.

By \( A'' \subseteq B \), we have that \( A'' + A'' \subseteq B + B = [2a, 2a + \frac{2n}{3} - 2] \). Recall that \( |A''| \geq \left( \frac{1}{3} - \frac{\epsilon}{2} \right) \cdot n \), hence by Claim 2.1 it follows that \( |A'' + A''| \geq 2|A''| - 1 \geq \left( \frac{2}{3} - \epsilon \right) \cdot n - 1 \). We conclude that \( A'' \) includes all but at most \( \frac{\epsilon n}{3} \) of the elements of \( B \), and that \( A'' + A'' \) includes all but at most \( \epsilon n \) of the elements of \( B + B \). Since \( A'' \) is sum-free, it follows that the intervals \( B \) and \( B + B \) intersect at no more than \( \frac{\epsilon n}{2} + \epsilon n = \frac{3\epsilon n}{2} \) elements, implying that \( a + \frac{\epsilon n}{3} \leq 2a + \frac{3\epsilon n}{2} \), thus

\[
a \geq \left( \frac{2}{3} - \frac{\epsilon}{2} \right) \cdot n.
\]

We next use the fact that \( 2n + 1 \notin 4A'' \), which implies that \( A'' \) is disjoint from

\[
(2n + 1) - 3A'' \subseteq (2n + 1) - 3B = [n - 3a + 4, 2n - 3a + 1].
\]

By Claim 2.1 and (3), \( |3A''| \geq 3|A''| - 2 \geq \left( 1 - \frac{\epsilon}{3} \right) \cdot n - 2 \). Therefore, the set \( (2n + 1) - 3A'' \) includes all but at most \( \frac{\epsilon n}{2} \) of the elements of \( (2n + 1) - 3B \). This implies that the intervals \( B \) and \( (2n + 1) - 3B \) intersect at no more than \( \frac{\epsilon n}{2} + \frac{\epsilon n}{2} = \epsilon n \) elements, hence either

\[
a + \frac{n}{3} \leq n - 3a + 4 + 2\epsilon n \quad \text{or} \quad 2n - 3a + 2 \leq a + 2\epsilon n.
\]

The first possibility is ruled out using \( a \geq \left( \frac{1}{3} - \frac{\epsilon}{2} \right) \cdot n \), so we derive that \( a \geq \left( \frac{2}{3} - \frac{\epsilon}{2} \right) \cdot n \).

Finally, we use the fact that \( 2n + 1 \notin 3A'' \), which implies that \( A'' \) is disjoint from

\[
(2n + 1) - 2A'' \subseteq (2n + 1) - 2B = \left[ \frac{4n}{3} - 2a + 3, 2n - 2a + 1 \right].
\]

Recalling that \( |A'' + A''| \geq \left( \frac{2}{3} - \epsilon \right) \cdot n - 1 \), the set \( (2n + 1) - 2A'' \) includes all but at most \( \epsilon n \) of the elements of \( (2n + 1) - 2B \). It follows that the intervals \( B \) and \( (2n + 1) - 2B \) intersect at no more than \( \frac{\epsilon n}{2} + \epsilon n = \frac{3\epsilon n}{2} \) elements, hence either

\[
a + \frac{n}{3} \leq \frac{4n}{3} - 2a + 3 + \frac{3\epsilon n}{2} \quad \text{or} \quad 2n - 2a + 2 \leq a + \frac{3\epsilon n}{2}.
\]

The first possibility is ruled out using \( a \geq \left( \frac{2}{3} - \frac{\epsilon}{2} \right) \cdot n \), so we derive that \( a \geq \left( \frac{2}{3} - \frac{\epsilon}{2} \right) \cdot n \), and we are done.
2. $d = 2$: In this case the elements of $A''$ are all even or all odd. We show that this is impossible. In the former alternative, the set $D_0 = \{k \mid 2k \in A''\}$ is a sum-free subset of $[\frac{n}{2}]$. However, the largest size of a sum-free subset of $[n]$ is $\lceil \frac{n}{3}\rceil$, implying $|A''| = |D_0| \leq \frac{n}{3}$, in contradiction to (3). For the latter alternative, where the elements of $A''$ are all odd, consider the set $D_1 = \{k \mid 2k - 1 \in A''\} \subseteq [n']$ where $n' = \frac{n}{2}$. Hence there are at least $\frac{n}{3}$ even integers in $D_0$, and thus all but at most $\frac{3en'}{2}$ of the elements of $A''$ are in $[\frac{2n}{3}]$. Let $E$ denote the set of odd integers in $[\frac{2n}{3}]$ and note that $|E| = \frac{n}{3}$. Using (3), it follows that there exists a set $F \subseteq E$ satisfying $E \setminus F \subseteq A''$ and $|F| \leq \frac{en'}{2} + \frac{3en'}{2} = 2en$. To obtain a contradiction we show that $2n + 1 \in 5A''$. Indeed, for every odd integers $x_1, x_2, x_3, x_4 \in \left[\frac{n}{2}, \frac{3n}{2}\right]$ there exists $x_5 \in E$ such that $\sum_{i=1}^{5} x_i = 2n + 1$, hence there are at least $(\frac{n}{2})^4$ 5-tuples of elements of $E$ with sum $2n + 1$. However, at most $5|F| \cdot \left(\frac{n}{3}\right)^3 < en^4$ of them involve elements of $F$, so assuming that $\varepsilon$ is sufficiently small, there must exist a 5-tuple of elements of $E \setminus F \subseteq A''$ whose sum is $2n + 1$, as desired.

3. $d = 3$: In this case all the elements of $A''$ are congruent to $r$ modulo 3 for some $r \in \{0, 1, 2\}$. We show that this is impossible. If $r = 0$ then the set $\{k \mid 3k \in A''\}$ is a sum-free subset of $[\frac{n}{3}]$, hence $|A''| \leq \frac{n}{6}$, in contradiction to (3). So assume that $r \in \{1, 2\}$. Denote by $[n]_r$ the set of integers in $[n]$ which are congruent to $r$ modulo 3, and notice, using (3), that there exists a set $F \subseteq [n]_r$ satisfying $[n]_r \setminus F \subseteq A''$ and $|F| \leq \frac{en}{3}$. For the given $n$ and $r \in \{1, 2\}$, let $t$ be the unique integer in $\{3, 4, 5\}$ satisfying $t \cdot r = 2n + 1 \pmod{3}$. To obtain a contradiction we show that $2n + 1 \in tA''$. By the definition of $t$, for every $x_1, \ldots, x_{t-1} \in [n]_r \cap \left[\frac{n}{3} + 1, \frac{2n}{3}\right]$ there exists $x_t \in [n]_r$ such that $\sum_{i=1}^{t} x_i = 2n + 1$, hence there are at least $(\frac{n}{3(t-1)})^{t-1}$ $t$-tuples of elements of $[n]_r$ with sum $2n + 1$. However, at most $t|F| \cdot \left(\frac{n}{3}\right)^{t-2} < en^{t-1}$ of them involve elements of $F$, so assuming that $\varepsilon$ is sufficiently small, there must exist a $t$-tuple of elements of $[n]_r \setminus F \subseteq A''$ whose sum is $2n + 1$, as desired.

The proof is completed.

Theorem 3.9 combined with a result of [5] on counting independent sets in hypergraphs (see Section 2.3) gives us the following corollary.

**Corollary 3.11.** For every $\gamma > 0$, there are $o(2^{n/3})$ sum-free sets $A \subseteq [n]$ satisfying

$$2n + 1 \notin (3A) \cup (4A) \cup (5A) \text{ and } |A \setminus \left[\frac{2n}{3} + 1, n\right]| \geq \gamma n.$$  

**Proof:** Let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of $5$-uniform hypergraphs defined as follows. For every $n$, $\mathcal{H}_n$ is the hypergraph on the vertex set $[n]$ whose hyperedges are all $5$-subsets of $[n]$ that contain at least one of the sets in $\mathcal{F}_n^{(3)} \cup \mathcal{F}_n^{(4)} \cup \mathcal{F}_n^{(5)}$. Observe that $e(\mathcal{H}_n) = \Theta(n^4)$ and that $\Delta_1(\mathcal{H}_n) = \Theta(n^3)$, $\Delta_2(\mathcal{H}_n) = \Theta(n^2)$, $\Delta_3(\mathcal{H}_n) = \Theta(n^2)$, $\Delta_4(\mathcal{H}_n) = \Theta(n)$, and $\Delta_5(\mathcal{H}_n) = 1$. Hence, for every sufficiently large $n$, every $\ell \in [5]$, some constant $c > 0$, and $p_n = 1/\sqrt{n}$, we have

$$\Delta_\ell(\mathcal{H}_n) \leq c \cdot p_n^{\ell-1} \cdot \frac{e(\mathcal{H}_n)}{\nu(\mathcal{H}_n)}.$$  

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Since every sum-free set \( A \subseteq [n] \) satisfying \( 2n + 1 \notin (3A) \cup (4A) \cup (5A) \) forms an independent set in \( \mathcal{H}_n \), it suffices to bound from above, for a given \( \gamma > 0 \), the number of sets \( I \in \mathcal{I}(\mathcal{H}_n) \) satisfying \( |I \setminus \frac{2n}{3} + 1, n| \geq \gamma n \).

To this end, let us show that the sequence of hypergraphs \( \mathcal{H} \) is \((\frac{1}{3}, B)\)-stable, where \( B = (B_n)_{n \in \mathbb{N}} \) is defined by \( B_n = \{ \left\lfloor \frac{2n}{3} + 1, n \right\rfloor \} \) (recall Definition 2.7, Item 2). Let \( \varepsilon > 0 \) be a constant. By Theorem 3.9 for some \( \delta > 0 \) and every sufficiently large integer \( n \), every set \( A \subseteq [n] \) of size \( |A| \geq (\frac{1}{3} - \delta) \cdot n \) contains at least \( \delta \cdot n^{k-1} \) sets from \( \mathcal{F}_n^{(k)} \) for some \( k \in \{3, 4, 5\} \) or satisfies \( |A \setminus \left\lfloor \frac{2n}{3} + 1, n \right\rfloor| \leq \varepsilon \cdot n \). In the former case we have, using \( e(\mathcal{H}_n) = \Theta(n^4) \), that \( e(\mathcal{H}_n[A]) \geq \delta' \cdot e(\mathcal{H}_n) \) for some \( \delta' > 0 \), as required.

Now, we can apply Item 2 of Theorem 2.8 to obtain that for every \( \gamma > 0 \) there exist \( \beta > 0 \) and \( C > 0 \) such that for every sufficiently large \( n \) and every \( m \geq C \sqrt{n} \), there are at most \( (1 - \beta)^m \cdot \binom{n/3}{m} \) sets \( I \in \mathcal{I}(\mathcal{H}_n, m) \) satisfying \( |I \setminus \left\lfloor \frac{2n}{3} + 1, n \right\rfloor| \geq \gamma m \). In particular, this bound holds on the number of sets \( I \in \mathcal{I}(\mathcal{H}_n, m) \) satisfying \( |I \setminus \left\lfloor \frac{2n}{3} + 1, n \right\rfloor| \geq \gamma n \), hence the total number of sets \( I \in \mathcal{I}(\mathcal{H}_n) \) satisfying \( |I \setminus \left\lfloor \frac{2n}{3} + 1, n \right\rfloor| \geq \gamma n \) is at most

\[
\sum_{m=0}^{\frac{C}{\sqrt{n}}} \binom{n}{m}^3 (1 - \beta)^m \cdot \binom{n/3}{m} \leq n^{O(\sqrt{n})} + (1 - \beta)^C \sqrt{n} \cdot \sum_{m=0}^{n/3} \binom{n/3}{m} \leq o(2^{n/3}),
\]

so we are done.

3.3.2 Sets with No Small Elements

Our goal in this section is to prove an upper bound on the number of sets \( A \subseteq [n] \), all of whose elements are greater than \( \frac{n}{2} \), that satisfy \( 2n + 1 \notin 3A \). To do so, we prove the following theorem, which yields the required bound as an easy corollary. (Recall Remark 3.8)

**Theorem 3.12.** There are \( O(2^{n/3}) \) sets \( A \subseteq [n] \) satisfying \( n + 1 \notin 3A \).

**Corollary 3.13.** There are \( O(2^{n/3}) \) sets \( A \subseteq \left\lfloor \frac{n}{2} + 1, n \right\rfloor \) satisfying \( 2n + 1 \notin 3A \).

**Proof of Corollary 3.13.** Map every set \( A \subseteq \left\lfloor \frac{n}{2} + 1, n \right\rfloor \) satisfying \( 2n + 1 \notin 3A \) to the set \( A' = \{ x - \frac{n}{2} \mid x \in A \} \subseteq \left\lfloor \frac{n}{2} \right\rfloor \), which satisfies \( \frac{n}{2} + 1 \notin 3A' \). By Theorem 3.12, the number of possible distinct sets \( A' \) is \( O(2^{n/3}) \). Since the mapping is injective, the corollary follows.

The proof of Theorem 3.12 employs a counting technique due to \([3]\). For every set \( S \subseteq \left\lfloor \frac{n}{2} \right\rfloor \), we consider all sets \( A \subseteq [n] \) with \( n + 1 \notin 3A \) such that \( S = A \cap \left\lfloor \frac{n}{2} \right\rfloor \). The following two claims provide upper bounds on the number of sets \( A \) associated with a given \( S \). The first is particularly useful for sets \( S \) with a large sumset \( S + S \), and the second, whose proof uses Janson’s inequality, is useful for sets \( S \) whose elements are, in average, significantly smaller than \( \frac{n}{2} \).

**Claim 3.14.** For every integer \( n \) and a set \( S \subseteq \left\lfloor \frac{n}{3} \right\rfloor \), the number of sets \( A \subseteq [n] \) such that \( n + 1 \notin 3A \) and \( S = A \cap \left\lfloor \frac{n}{3} \right\rfloor \) is at most \( 2^{2n/3 - |S + S|} \).

**Proof:** Fix a set \( S \subseteq \left\lfloor \frac{n}{3} \right\rfloor \). In order to specify a set \( A \subseteq [n] \) satisfying \( S = A \cap \left\lfloor \frac{n}{3} \right\rfloor \) one has to specify the elements of \( A \cap \left\lfloor \frac{n}{3} + 1, n \right\rfloor \). However, the assumption \( n + 1 \notin 3A \) implies that the \( |S + S| \) elements of the set \((n + 1) - (S + S) \subseteq \left\lfloor \frac{n}{3} + 1, n \right\rfloor \) do not belong to \( A \). This implies that the number of sets \( A \subseteq [n] \) such that \( n + 1 \notin 3A \) and \( S = A \cap \left\lfloor \frac{n}{3} \right\rfloor \) is at most \( 2^{2n/3 - |S + S|} \), as required.
Lemma 2.11 implies that to bound the above, observe that the function $g$ defined by $g(x) = (\frac{x^2}{\ell^2})^\ell \cdot e^{-x/(32\ell)}$ satisfies for every $x \geq \ell^2 / \delta$,

$$\frac{g(x + 32\ell)}{g(x)} = \left(1 + \frac{32\ell}{x}\right)^\ell \cdot e^{-1} \leq e^{32\ell^2/x} \cdot e^{-1} \leq e^{32\ell - 1} \leq \frac{1}{2},$$

assuming that $\delta$ is sufficiently small. By $g(\ell^2 / \delta) = (\frac{\ell^2}{\delta})^\ell \cdot e^{-\ell^2/(32\delta)}$ and the fact that $g$ is decreasing on $[0, \infty)$, it follows that (5) is bounded from above by

$$64\ell \cdot \left(\frac{\ell^2}{\delta}\right)^\ell \cdot e^{-\ell^2/(32\delta)} \cdot 2^{2n/3} \leq e^{-\ell} \cdot 2^{2n/3},$$
where we again use the fact that $\delta$ is sufficiently small. Summing over all integers $\ell$ we obtain a bound of $O(2^{2n/3})$.

We next count the sets $A \in C$ for which $S_A \in S(k, \ell)$ for $k$ and $\ell$ satisfying $k < \ell^2 / \delta$. To do so, we consider the following two cases defined by the size of the sumset $S_A + S_A$. Note that one can assume here, whenever needed, that $\ell$ is sufficiently large, as for a constant $\ell$ and for $k < \ell^2 / \delta$ there is only a constant number of sets $S_A \in S(k, \ell)$ and they correspond to $O(2^{2n/3})$ sets $A$.

1. Consider the sets $A \in C$ with $S_A \in S(k, \ell)$ satisfying $|S_A + S_A| \geq \ell / \delta$. Combining (4) with Claim 3.14, the number of these sets for a given $\ell$ is at most

$$\sum_{k=1}^{\ell^2 / \delta} \left( \frac{2^2 k}{\ell^2} \right)^{\ell / \delta} 2^{2n/3 - \ell / \delta} \leq \ell^2 \left( \frac{2^2}{\delta} \right)^{\ell / \delta} 2^{2n/3 - \ell / \delta} \leq e^{-\ell} 2^{2n/3},$$

where we have used $k < \ell^2 / \delta$ and that $\delta$ is sufficiently small. Summing over all integers $\ell$ we obtain a bound of $O(2^{2n/3})$.

2. Consider the sets $A \in C$ with $S_A \in S(k, \ell)$ satisfying $|S_A + S_A| \leq \ell / \delta$, and recall that by Claim 2.14 we have $|S_A + S_A| \geq 2\ell - 1$. For such a set $S_A$ let $\lambda \in [2 - \delta, 1 / \delta]$ be a real number satisfying $\lambda \ell \leq |S_A + S_A| \leq (1 + \delta) \cdot \lambda \ell$. By Theorem 2.10 for a sufficiently large $\ell$, the number of such sets $S_A$ is at most

$$2^{\delta \ell} \cdot \frac{(1 + \delta) \lambda \ell / 2}{\ell} \cdot k^{(1 + \delta) \lambda + \delta} \leq 2^{O(\delta \ell)} \cdot \frac{(1 + \delta) \lambda \ell / 2}{\ell},$$

where for the inequality we have used $k < \ell^2 / \delta$, $\lambda \leq 1 / \delta$, and the assumption that $\delta$ is sufficiently small. By Claim 3.14 we obtain that, for given $\ell$ and $\lambda$, the number of sets $A$ as above is at most

$$2^{O(\delta \ell)} \cdot \frac{(1 + \delta) \lambda \ell / 2}{\ell} \cdot 2^{2n/3 - \lambda \ell} \leq 2^{O(\delta \ell)} \cdot 2^{(1 + \delta) \lambda \ell / 2} \cdot 2^{2n/3 - \lambda \ell} \leq 2^{-\ell / 2} \cdot 2^{2n/3}.$$  

Summing over $O(1)$ values of $\lambda$, say $\lambda = (2 - \delta)(1 + \delta)^j$ for $0 \leq j \leq \frac{2 \ln(1 / \delta)}{\delta}$, and over all integers $\ell$, we obtain a bound of $O(2^{2n/3})$.

Summing all the obtained bounds, we get the required bound of $O(2^{2n/3})$, thus the proof is completed.

3.3.3 Sets that Include a Small Element and are Almost Contained in $B_n$  

Here we show an easy bound on the number of sum-free sets $A \subseteq [n]$ with $2n + 1 \not\in 3A$ that intersect $[n]$ and are almost contained in $[\frac{2n}{3} + 1, n]$. (Recall Remark 3.8)

**Lemma 3.16.** For every sufficiently small $\gamma > 0$, there are $o(2^{n/3})$ sum-free sets $A \subseteq [n]$ satisfying $A \cap [\frac{n}{2}] \neq \emptyset$, $2n + 1 \not\in 3A$, and $|A \setminus [\frac{2n}{3} + 1, n]| \leq \gamma n$.

**Proof:** Fix a set $S \subseteq [\frac{2n}{3}]$ for which there exists $z \in S \cap [\frac{n}{2}]$. Observe that if $z \in [\frac{n}{2}]$ then there is a collection of at least $n / 12$ pairwise disjoint sets $\{x, y\} \subseteq [\frac{2n}{3} + 1, n]$ such that $x - y = z$. In addition, if $z \in [\frac{n}{2}, \frac{n}{2}]$ then there is a collection of at least $n / 12$ pairwise disjoint sets $\{x, y\} \subseteq [\frac{2n}{3} + 1, n]$ such
Lemma 3.18. that \( x + y + z = 2n + 1 \). In both cases, every sum-free set \( A \subseteq [n] \) with \( 2n + 1 \notin 3A \) such that \( S = A \setminus \left[ \frac{2n}{3} + 1, n \right] \) does not contain any of these sets, hence the number of such sets \( A \) is at most \( 2^{n/3-n/6} \cdot 3^{n/12} = 2^{n/6} \cdot 3^{n/12} \). Summing over all choices of sets \( S \subseteq \left[ \frac{2n}{3} \right] \) of size at most \( \gamma n \) such that \( S \cap \left[ \frac{2n}{3} \right] \neq \emptyset \), we get that the total number of sets \( A \subseteq [n] \) satisfying \( A \cap \left[ \frac{2n}{3} \right] \neq \emptyset, 2n + 1 \notin 3A \), and \( |A \setminus \left[ \frac{2n}{3} + 1, n \right]| \leq \gamma n \), is at most

\[
\sum_{m=1}^{\gamma n} \binom{2n/3}{m} \cdot 2^{n/6} \cdot 3^{n/12} \leq 2^{H(3/2) \cdot 2n/3} \cdot 2^{n/6} \cdot 3^{n/12} \leq o(2^{n/3}),
\]

where \( H \) stands for the binary entropy function and \( \gamma \) is assumed to be sufficiently small.

3.3.4 Putting Everything Together

We are finally ready to prove Theorem 3.7 which confirms Theorem 1.3.

Proof of Theorem 3.7 Let \( \gamma > 0 \) be a sufficiently small constant. By Corollary 3.11 there are \( o(2^{n/3}) \) sum-free sets \( A \subseteq [n] \) satisfying \( 2n + 1 \notin (3A) \cup (4A) \cup (5A) \) and \( |A \setminus \left[ \frac{2n}{3} + 1, n \right]| \geq \gamma n \). Hence, it suffices to bound the number of sum-free sets \( A \subseteq [n] \) satisfying \( 2n + 1 \notin 3A \) and \( |A \setminus \left[ \frac{2n}{3} + 1, n \right]| < \gamma n \). By Lemma 3.16 at most \( o(2^{n/3}) \) of them intersect \( \left[ \frac{2n}{3} \right] \), and by Corollary 3.13 at most \( O(2^{n/3}) \) of them do not. This completes the proof.

3.4 On the Tightness of Theorem 3.1

Theorem 1.1 provides a tight upper bound of \( \left[ \frac{1}{3}(n + 1) \right] \) on the size of every sum-free set \( A \subseteq [n] \) satisfying \( 2n + 1 \notin \sum A \). In fact, the bound is shown in Theorem 3.1 even under the weaker assumption \( 2n + 1 \notin (3A) \cup (4A) \). We claim that Theorem 3.1 is tight not only because of the bound that it provides, but also in the sense that the assumption \( 2n + 1 \notin (3A) \cup (4A) \) cannot be relaxed to \( 2n + 1 \notin 3A \). To see this, let \( n \) be an integer satisfying \( n = 2 \pmod{5} \), and consider the set

\[
A = \{ x \in [n] \mid x = 1 \text{ or } 4 \pmod{5} \}.
\]  

(6)

Notice that over \( \mathbb{Z}_5 \) we have \( 2 \{1, 4\} = \{0, 2, 3\} \), implying that \( A \) is sum-free. We also have, over \( \mathbb{Z}_5 \), that \( 3 \{1, 4\} = \{0, 2, 3\} + \{1, 4\} = \{1, 2, 3, 4\} \), implying that \( 2n + 1 \), which is divisible by 5, is not in \( 3A \). It follows that for every integer \( n \) such that \( n = 2 \pmod{5} \) there exists a sum-free set \( A \subseteq [n] \) satisfying \( 2n + 1 \notin 3A \) whose size is \( |A| = \left[ \frac{2n}{5} \right] \). Another example of such a set is given by the set of all integers of \( [n] \) that are congruent to 2 or 3 modulo 5. The following theorem shows that these constructions achieve the largest possible size of a set with these properties.

Theorem 3.17. For every sufficiently large integer \( n \), every sum-free set \( A \subseteq [n] \) such that \( 2n + 1 \notin 3A \) satisfies

\[
|A| \leq \left[ \frac{2n}{5} \right].
\]

The proof of Theorem 3.17 uses a recent characterization of large sum-free subsets of \( [n] \) due to Tran [31] (see Theorem 2.3). We start with the following lemma.

Lemma 3.18. For every integer \( n \), every nonempty sum-free set \( A \subseteq [n] \) such that \( \min(A) > \frac{n}{3} \) and \( 2n + 1 \notin 3A \) satisfies \( |A| \leq \left[ \frac{1}{5}(n + 1) \right] \).

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Proof: Let \( A \subseteq [n] \) be a sum-free set with \( 2n + 1 \not\in 3A \), and denote \( k = \min(A) > \frac{n}{3} \). Assume first that \( k \leq \left\lfloor \frac{n}{2} \right\rfloor \). Consider the collection of pairwise disjoint pairs

\[
B_1(k) = \{(x, x + k) \mid x \in \left[k, n-k\right]\},
\]

that involves all the elements of \([k, n-k] \cup [2k, n]\) (Note that \( k \leq n-k < 2k \leq n \)). Since \( A \) is sum-free and includes \( k \), for every pair \((x, y) \in B_1(k)\) we have \( x \notin A \) or \( y \notin A \). Additionally, consider the collection of pairwise disjoint pairs

\[
B_2(k) = \{(x, 2n + 1 - 2x) \mid x \in \left[n-k+1, \left\lfloor \frac{1}{3}(2n+1)\right\rfloor \right]\},
\]

that involves elements from \([n-k+1, 2k-1]\) (Note that \( n-k+1 \leq 2k \)). Notice that for every pair \((x, y) \in B_2(k)\) we have \( 2x + y = 2n + 1 \), so by \( 2n + 1 \not\in 3A \) we have \( x \notin A \) or \( y \notin A \). Since the pairs of \( B_1(k) \cup B_2(k) \) are pairwise disjoint, it follows that the size of the set \( A \subseteq [k, n] \) satisfies

\[
|A| \leq (n-k+1) - |B_1(k)| - |B_2(k)| = (n-k+1) - (n-2k+1) - \left(\left\lfloor \frac{1}{3}(2n+1)\right\rfloor - (n-k)\right) = n - \left\lfloor \frac{1}{3}(2n+1)\right\rfloor = \left\lfloor \frac{1}{3}(n+1)\right\rfloor,
\]

as required. Assume next that \( k > \left\lfloor \frac{n}{2} \right\rfloor \). Here, \( A \subseteq \left\lfloor \frac{n}{2} \right\rfloor + 1, n \) and a similar argument implies that

\[
|A| \leq \left\lfloor \frac{n}{2} \right\rfloor - B_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = \left\lfloor \frac{n}{2} \right\rfloor - \left(\left\lfloor \frac{1}{3}(2n+1)\right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor\right) = n - \left\lfloor \frac{1}{3}(2n+1)\right\rfloor = \left\lfloor \frac{1}{3}(n+1)\right\rfloor,
\]

so we are done. ❄️

Proof of Theorem 3.17 For a sufficiently large integer \( n \), let \( A \subseteq [n] \) be a sum-free set such that \( 2n + 1 \notin 3A \). Apply Theorem 2.3 with \( \eta = \frac{4}{n} \). Assume by contradiction that \( |A| > \left\lfloor \frac{2n}{3} \right\rfloor \geq \left(\frac{5}{3} - \eta\right) \cdot n \). It suffices to prove that \( A \) does not satisfy any of the five alternatives in Theorem 2.3.

Alternatives (1) and (2) are not satisfied because at most \( \left\lfloor \frac{2n}{3} \right\rfloor \) of the elements of \([n]\) are congruent to 1 or 4 modulo 5, and at most \( \left\lfloor \frac{2n}{3} \right\rfloor \) of them are congruent to 2 or 3 modulo 5.

For alternative (3), notice that if all the elements of \( A \) are odd then \( 2n + 1 \notin 4A \). By Theorem 3.1 using \( 2n + 1 \notin 3A \), we get that \( |A| \leq \left\lfloor \frac{1}{3}(n+1)\right\rfloor \), in contradiction.

For alternative (4), use Lemma 3.18 to obtain that if \( \min(A) \geq \frac{4}{7} \) then \( |A| \leq \left\lfloor \frac{1}{3}(n+1)\right\rfloor \), again in contradiction.

Finally, notice that if alternative (5) holds, with \( \eta = \frac{2}{n} \), then there exists a constant \( d > 0 \) for which

\[
A \subseteq \left\lfloor \frac{n}{5} - d \cdot \sqrt{n} \right\rfloor + \left\lfloor \frac{2n}{5} + d \cdot \sqrt{n} \right\rfloor \cup \left\lfloor \frac{4n}{5} - d \cdot \sqrt{n}, n \right\rfloor.
\]

Denote \( B = \left[\frac{4}{5}, \frac{2n}{5}\right] \cup \left[\frac{4n}{5}, n\right] \) and notice that the assumption \( |A| > \left\lfloor \frac{2n}{3} \right\rfloor \) implies that there exists \( D \subseteq B \) of size \( |D| \leq 3d \cdot \sqrt{n} \) such that \( B \setminus D \subseteq A \). To obtain a contradiction, we show that \( 2n + 1 \in D \). Indeed, for every \( x, y \in \left[\frac{4n}{5} + 1, \frac{9n}{10}\right] \subseteq B \) there exists \( z \in \left[\frac{9}{10}, \frac{2n}{3}\right] \subseteq B \) such that \( x + y + z = 2n + 1 \), hence there are at least \( \left(\frac{n}{5}\right)^2 \) triples of elements of \( B \) with sum \( 2n + 1 \). However, at most \( 3|D| \cdot |B| < 4d \cdot n^{1.5} \) of them involve elements of \( D \), so for a sufficiently large \( n \) there must exist a triple of elements of \( B \setminus D \subseteq A \) whose sum is \( 2n + 1 \), and we are done. ❄️
While the bound given in Theorem 3.17 is tight for integers \( n \) satisfying \( n = 2 \) (mod 5) (see (6)), it turns out that this is not the case in general, as is shown in the following theorem.

**Theorem 3.19.** There exists a constant \( \eta > 0 \) such that for every sufficiently large integer \( n \) such that \( n \neq 2 \) (mod 5), every sum-free set \( A \subseteq [n] \) such that \( 2n + 1 \not\in 3A \) satisfies \( |A| \leq (\frac{2}{3} - \eta) \cdot n \).

**Proof:** For a sufficiently large integer \( n \) such that \( n \neq 2 \) (mod 5), let \( A \subseteq [n] \) be a sum-free set satisfying \( 2n + 1 \not\in 3A \). Apply Theorem 2.3 with a sufficiently small constant \( \eta > 0 \). Assume by contradiction that \( |A| > (\frac{2}{3} - \eta) \cdot n \). It suffices to prove that \( A \) does not satisfy any of the five alternatives in Theorem 2.3.

Assume that alternative (1) holds, that is, all the elements of \( A \) are congruent to 1 or 4 modulo 5. Denote by \([n]_r\) the set of integers in \([n]\) which are congruent to \( r \) modulo 5. Letting \( A = A_1 \cup A_4 \) where \( A_r \subseteq [n] \), for \( r \in \{1,4\} \), the assumption \( |A| > (\frac{2}{3} - \eta) \cdot n \) implies that \( |[n]_r \setminus A_r| < \eta \cdot n \) for \( r \in \{1,4\} \). To obtain a contradiction, we show that \( 2n + 1 \in 3A \). Observe that by \( n \neq 2 \) (mod 5) there exist \( r_1, r_2, r_3 \in \{1,4\} \) such that \( r_1 + r_2 + r_3 = 2n + 1 \) (mod 5). Further, for every \( x \in [n]_{r_1} \cap [\frac{n}{2} + 1, n] \) and \( y \in [n]_{r_2} \cap [\frac{n}{2} + 1, n] \) there exists \( z \in [n]_{r_3} \) such that \( x + y + z = 2n + 1 \), hence there are at least \((\frac{n}{5})^2\) triples in \([n]_{r_1} \times [n]_{r_2} \times [n]_{r_3} \) with sum \( 2n + 1 \). However, at most \( 3 \cdot \eta \cdot n = 3\eta n^2 \) of them involve elements not in \( A \), so for a sufficiently small \( \eta \) there must exist a triple of elements of \( A \) whose sum is \( 2n + 1 \), as required.

Alternative (2) is handled similarly to the way alternative (1) is, so we omit the details.

For alternative (3), notice that if all the elements of \( A \) are odd then \( 2n + 1 \not\in 4A \). By Theorem 3.1 using \( 2n + 1 \not\in 3A \), we get that \( |A| \leq (\frac{1}{2}(n + 1)) \), in contradiction.

For alternative (4), use Lemma 3.18 to obtain that if \( \min(A) \geq |A| > \frac{\eta}{5} \) then \( |A| \leq (\frac{1}{3}(n + 1)) \), again in contradiction.

Finally, notice that if alternative (5) holds then for \( d = 200\eta^{1/2} \) we have

\[
A \subseteq \left[ \left( \frac{1}{5} - d \right) \cdot n, \left( \frac{2}{5} + d \right) \cdot n \right] \cup \left[ \left( \frac{4}{5} - d \right) \cdot n, n \right].
\]

Denote \( B = \left[ \frac{n}{5}, \frac{2n}{5} \right] \cup \left[ \frac{4n}{5}, n \right] \) and notice that there exists \( D \subseteq B \) of size \( |D| \leq 3dn \) such that \( B \setminus D \subseteq A \). To obtain a contradiction, we show that \( 2n + 1 \in 3A \). Recall that there are at least \((\frac{n}{5})^2\) triples of elements of \( B \) with sum \( 2n + 1 \) (see the proof of Theorem 3.17). However, at most \( 3|D| \cdot |B| < 4dn^2 \) of them involve elements of \( D \), so for a sufficiently small \( \eta \) there must exist a triple of elements of \( B \setminus D \subseteq A \) whose sum is \( 2n + 1 \), and we are done.

\[\blacksquare\]

### 3.5 Other Forbidden Sums

Although the current work focuses on sum-free sets \( A \subseteq [n] \) that satisfy \( 2n + 1 \not\in \sum A \), it is natural to consider other forbidden sums, different from \( 2n + 1 \), as well. We first observe that Theorem 1.3 can be used to obtain a tight estimation for the number of sum-free subsets of \([n]\) with an odd forbidden sum sum around \( 2n \).

**Corollary 3.20.** For every fixed integer \( k \in \mathbb{Z} \), there are \( \Theta(2^{n/3}) \) sum-free sets \( A \subseteq [n] \) satisfying \( 2(n + k) + 1 \not\in \sum A \).

**Proof:** Apply Theorem 1.3 to get that there are \( \Theta(2^{(n+k)/3}) = \Theta(2^{n/3}) \) sum-free sets \( A \subseteq [n + k] \) satisfying \( 2(n + k) + 1 \not\in \sum A \). Considering the subsets of \([n]\) with this property, the estimation is affected by no more than a multiplicative constant factor of \( 2^k \), so we are done.

\[\blacksquare\]
For the forbidden sum $2n$, we prove the following extremal result, whose proof resembles that of Theorem 1.1.

**Theorem 3.21.** For every integer $n$, the maximum size of a sum-free set $A \subseteq [n]$ such that $2n \notin \sum A$ is $\lfloor \frac{1}{3} (n-1) \rfloor$.

**Proof:** For the upper bound, let $A \subseteq [n]$ be a sum-free set such that $2n \notin \sum A$. Consider the set $C = A \cup (A + A)$. By the sum-freeness of $A$ and Claim 2.11 $|C| = |A| + |A + A| \geq 3|A| - 1$. We claim that $|C| \leq n - 2$. To see this, notice that $1 \notin A$ and that $n \notin C$, hence $C \subseteq [2, 2n-2] \setminus \{n\}$. If $|C| > n - 2$ then, by the pigeonhole principle, there exists $2 \leq i \leq n - 1$ for which $\{i, 2n-i\} \subseteq C$, implying that $2n \in \sum A$, in contradiction. It follows that $3|A| - 1 \leq |C| \leq n - 2$, hence $|A| \leq \lfloor \frac{1}{3} (n-1) \rfloor$ as required. The set $\lfloor \frac{1}{3} (2n+1) \rfloor, n-1$ implies a matching lower bound. ■

We remark that it can be verified that the proof technique of Theorem 3.7 can be used to obtain a tight bound of $O(2^{n/3})$ on the number of sum-free sets $A \subseteq [n]$ satisfying $2n \notin (3A) \cup (4A) \cup (5A)$. Combining it with the lower bound that follows from Theorem 3.21 we get that there are $\Theta(2^{n/3})$ sum-free sets $A \subseteq [n]$ such that $2n \notin \sum A$ (and, as in Corollary 3.20 one can derive such a bound for every even forbidden sum around $2n$). However, we point out a significant difference between the forbidden sums $2n$ and $2n+1$. We prove below a bound of $2^{(\frac{1}{3}+o(1))n}$ on the number of sets $A \subseteq [n]$ satisfying $2n \notin \sum A$, which holds even without assuming the sum-freeness of the sets. This is in contrast to the forbidden sum $2n+1$ that is avoided by the $2^{(n/2)}$ subsets of $[n]$ that consist of only even integers.

**Proposition 3.22.** There are $2^{(\frac{1}{3}+o(1))n}$ sets $A \subseteq [n]$ satisfying $2n \notin \sum A$.

The proof uses the following special case of a result of Alon [11].

**Theorem 3.23 (11).** For every $\varepsilon > 0$ and every sufficiently large integer $n$, every set $A \subseteq \mathbb{Z}_n$ of size $|A| \geq (\frac{1}{3} + \varepsilon) \cdot n$ contains a set $B \subseteq A$ satisfying $0 < |B| \leq 3$ and $\sum_{b \in B} b = 0$ (in $\mathbb{Z}_n$).

**Proof of Proposition 3.22.** We first claim that for every $\varepsilon > 0$ and every sufficiently large integer $n$, every set $A \subseteq [n]$ of size $|A| \geq (\frac{1}{3} + \varepsilon) \cdot n$ contains at least $\delta \cdot n^{k-1}$-k-subsets of $[n]$ with sum $2n$ for some $k \in \{3, 4, 6\}$. To see it, for a given $\varepsilon > 0$ define $\delta = \min(\delta'_3(\frac{\varepsilon}{6}), \delta'_4(\frac{\varepsilon}{6}), \delta_6''(\frac{\varepsilon}{6}))$ where $\delta'_3$, $\delta'_4$, and $\delta_6''$ are as in Corollary 2.6. Assume by contradiction that for a sufficiently large $n$, a set $A \subseteq [n]$ of size $|A| \geq (\frac{1}{3} + \varepsilon) \cdot n$ contains fewer than $\delta \cdot n^{k-1}$ k-subsets of $[n]$ with sum $2n$ for every $k \in \{3, 4, 6\}$. Applying Corollary 2.6 three times with $\ell = 2n$ and $k \in \{3, 4, 6\}$ we obtain that there exist sets $D_1, D_2, D_3 \subseteq A$, each of which is of size at most $\frac{\varepsilon}{6} \cdot n$, such that $2n \notin 3(A \setminus D_1) \cup 4(A \setminus D_2) \cup 6(A \setminus D_3)$. Hence, the set $B = A \setminus (D_1 \cup D_2 \cup D_3)$ has size $|B| \geq (\frac{1}{3} + \frac{\varepsilon}{6}) \cdot n$, and yet satisfies $2n \notin (3B) \cup (4B) \cup (6B)$, in contradiction.

Now, let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of 6-uniform hypergraphs defined as follows. For every $n$, $\mathcal{H}_n$ is the hypergraph on the vertex set $[n]$ whose hyperedges are all 6-subsets of $[n]$ that contain 3, 4, or 6 distinct elements of $[n]$ whose sum is $2n$. It is straightforward to verify that
Since the bound holds for every \( \gamma \) \( \geq \) 1 symmetric complete sum-free subset of \( \mathbb{Z} \) from above the size of \( I \) such that for every sufficiently large \( n \), every \( \ell \in [6] \), some constant \( c > 0 \), and \( p_n = 1/\sqrt{n} \), we have

\[
\Delta_\ell(H_n) \leq c \cdot p_n^{\ell-1} \cdot e(H_n) / v(H_n).
\]

Since every set \( A \subseteq [n] \) satisfying \( 2n \not\in A \) forms an independent set in \( \mathcal{H}_n \), it suffices to bound from above the size of \( I(H_n) \). By the above supersaturation result, the sequence of hypergraphs \( \mathcal{H} \) is \( \frac{1}{2^t} \)-dense (recall Definition 2.7, Item 1). Applying Item 1 of Theorem 2.8 we obtain that for every \( \gamma > 0 \) there exists \( C > 0 \) such that for every sufficiently large \( n \) and every \( m \geq C p_n v(H_n) = C \sqrt{n} \),

\[
|I(H_n,m)| \leq \left( \frac{1 + \gamma}{m} \right)^n.
\]

It follows that

\[
|I(H_n)| \leq \sum_{m=0}^{C \sqrt{n}} \binom{n}{m} + \sum_{m=C \sqrt{n}}^{(1+\gamma)n} \binom{(1+\gamma)n}{m} \leq n^{O(\sqrt{n})} + 2^{(1+\gamma)n} \leq 2^{(1+2\gamma)n}.
\]

Since the bound holds for every \( \gamma > 0 \), the result follows.

\[\Box\]

## 4 Symmetric Complete Sum-free Sets in Cyclic Groups

In this section we relate our study of sum-free subsets of \([n]\) with \(2n + 1\) as a forbidden sum to counting symmetric complete sum-free sets in cyclic groups of prime order (see Section 2.1 for the definitions).

For a prime \( p \), an even integer \( s \in [\frac{p-3}{4}, \frac{p-1}{3}] \), and a set of integers \( T \subseteq [0,2t-1] \) where \( t = (p - 3s + 1)/2 \geq 1 \), consider the set \( S_T \subseteq \mathbb{Z}_p \) defined by

\[
S_T = [p - 2s + 1, 2s - 1] \cup \pm(s + T).
\]

Observe that \( |T| = t \) if and only if \( |S_T| = (4s - p - 1) + 2t = s \), and that \( 0 \in T \) if and only if \( s \in S_T \). As mentioned earlier, it was shown in [21] that for every sufficiently large prime \( p \), every symmetric complete sum-free subset of \( \mathbb{Z}_p \) of even size \( s \geq 0.318p \) is, up to an automorphism, of the form \( S_T \) for some set of integers \( T \subseteq [0,2t-1] \) of size \( t \). Moreover, those sets \( T \) for which \( S_T \) is complete and sum-free were fully characterized in [21]. To state the characterization result, we need the following definition.

**Definition 4.1.** For an integer \( t \geq 1 \) we say that a set \( T \subseteq [0,2t-1] \) is \( t \)-special if it satisfies

1. \( |T| = t \),
2. \( 2t - 1 \not\in 3T \), and
3. \( [0,2t - 1 + \min(T)] \setminus (2t - 1 - T) \subseteq T + T \).

Note that the addition is over the integers.
Theorem 4.2 (21). For every sufficiently large prime $p$ and every even integer $s \in [0.318p, \frac{p-1}{3}]$, the following holds. The symmetric complete sum-free subsets of $\mathbb{Z}_p$ of size $s$ are precisely all the dilations of the sets $S_T$ for $t$-special sets $T \subseteq [0, 2t - 1]$ where $t = \frac{(p - 3s + 1)}{2}$.

The above theorem reduces the challenge of counting the symmetric complete sum-free subsets of $\mathbb{Z}_p$ of a sufficiently large size to counting the $t$-special sets for an appropriate $t$ (There is an exact multiplicative gap of $(p - 2)/2$ between the two; see [21, Theorem 1.2]). As an application of Theorem 1.3, we prove below the following tight estimation for the number of $t$-special sets $T \subseteq [0, 2t - 1]$ that satisfy $0 \in T$.

Theorem 4.3. There are $\Theta(2^{t/3})$ $t$-special sets $T \subseteq [0, 2t - 1]$ satisfying $0 \in T$.

This easily implies, for a sufficiently large $s$, a tight estimation for the number of symmetric complete sum-free sets $S_T \subseteq \mathbb{Z}_p$ of size $s$ that satisfy $s \in S_T$, confirming Theorem 1.4.

Proof of Theorem 1.4. Let $p$ be a sufficiently large prime and let $s \in [0.318p, \frac{p-1}{3}]$ be an even integer. Denote $t = \frac{(p - 3s + 1)}{2}$. By Theorem 4.2, the number of symmetric complete sum-free sets $S_T \subseteq \mathbb{Z}_p$ of size $s$ that satisfy $s \in S_T$ is equal to the number of $t$-special sets that satisfy $0 \in T$. By Theorem 1.3, the latter is equal to $\Theta(2^{t/3}) = \Theta((p-3s)/6)$, so we are done.

4.1 Counting Special Sets – Proof of Theorem 4.3

We start with the following simple claim that shows that for a set that includes 0, the first and second conditions in Definition 4.1 imply the third.

Claim 4.4. For every integer $t \geq 1$ and a set $T \subseteq [0, 2t - 1]$ satisfying $0 \in T$, $|T| = t$, and $2t - 1 \notin 3T$, the following holds.

1. For every $\ell \in [0, t - 1]$ exactly one of the elements $\ell$ and $2t - 1 - \ell$ belongs to $T$.

2. For every $\ell_1, \ell_2 \in T$, if $\ell_1 + \ell_2 \in [0, 2t - 1]$ then $\ell_1 + \ell_2 \in T$.

3. $T$ is $t$-special.

Proof: Assume that a set $T \subseteq [0, 2t - 1]$ satisfies $0 \in T$, $|T| = t$, and $2t - 1 \notin 3T$.

1. Let $\ell \in [0, t - 1]$. The elements $\ell$ and $2t - 1 - \ell$ cannot both belong to $T$ as their sum, together with 0, is $2t - 1$, which does not belong to $3T$. In addition, if $T$ does not include one of $\ell$ and $2t - 1 - \ell$ for some $\ell \in [0, t - 1]$ then its size cannot reach $t$. This completes the proof of the first item.

2. For $\ell_1, \ell_2 \in T$ assume that $\ell_1 + \ell_2 \in [0, 2t - 1]$. By $2t - 1 \notin 3T$, it follows that $(2t - 1) - (\ell_1 + \ell_2) \notin T$, hence by Item 1, $\ell_1 + \ell_2 \in T$.

3. To prove that $T$ is $t$-special it suffices to show that $[0, 2t - 1] \setminus (2t - 1 - T) \subseteq T + T$. Indeed, every $\ell \in [0, 2t - 1]$ for which $2t - 1 - \ell \notin T$ satisfies, by Item 1, $\ell = \ell + 0 \in T + T$, and we are done.
Consider the following notion of sets that are closed under addition.

**Definition 4.5.** For an integer \( n \), we say that a set \( A \subseteq [n] \) is closed under addition if for every \( x, y \in A \) such that \( x + y \in [n] \) we have \( x + y \in A \). Let \( \mathcal{A}_n \) denote the collection of all sets \( A \subseteq [n] \) that are closed under addition and satisfy \( 2n + 1 \notin 3A \).

In the following lemma, we relate counting special sets that include 0 to counting sets that are closed under addition.

**Lemma 4.6.** For an integer \( t \geq 1 \), let \( T_t \) be the collection of all \( t \)-special sets \( T \subseteq [0, 2t - 1] \) satisfying \( 0 \in T \). Then, \( |T_t| = |\mathcal{A}_{t-1}| \).

**Proof:** For an integer \( t \geq 1 \), consider the function \( g \) that maps every set \( T \in T_t \) to the set

\[
g(T) = T \cap [t-1].
\]

To prove the lemma, it suffices to show that \( g \) is a bijection from \( T_t \) to \( \mathcal{A}_{t-1} \). We first observe that for every \( T \in T_t \) we have \( g(T) \in \mathcal{A}_{t-1} \). Indeed, for \( T \in T_t \) we have \( 2t - 1 \notin 3T \) and thus, by \( g(T) \subseteq T \), we also have \( 2t - 1 \notin 3g(T) \). Further, using Item 2 of Claim 4.4, it follows that \( g(T) \) is closed under addition as a subset of \([t-1]\), hence \( g(T) \in \mathcal{A}_{t-1} \).

The function \( g \) is injective by Item 1 of Claim 4.4 and the fact that \( 0 \in T \) for every \( T \in T_t \). To prove that \( g \) is surjective, take a set \( A \in \mathcal{A}_{t-1} \) and define

\[
T = \{0\} \cup A \cup \{(2t-1) - \ell \mid \ell \in [t-1] \setminus A\}.
\]

We clearly have \( g(T) = A \), \( 0 \in T \), and \( |T| = t \). By Item 3 of Claim 4.4, to prove that \( T \in T_t \) it suffices to show that \( 2t - 1 \notin 3T \). Assume by contradiction that there are \( x_1, x_2, x_3 \in T \) such that \( x_1 + x_2 + x_3 = 2t - 1 \). Observe, using \( 2t - 1 \notin 3A \), that exactly one of \( x_1, x_2, x_3 \), say \( x_3 \), is in \([t, 2t-1]\). Since \( 2t - 1 \notin T \), it follows that \( x_3 \in [t, 2t-2] \) and thus \( x_1 + x_2 \in [t-1] \). Using the fact that \( A \) is closed under addition, we get that \( x_1 + x_2 \in T \). However, \( x_3 = (2t-1) - (x_1 + x_2) \in T \), in contradiction to the fact that \( T \), by definition, contains no two elements whose sum is \( 2t - 1 \), and we are done.

We turn to estimate the size of \( \mathcal{A}_n \). To do so, we need the following lemma.

**Lemma 4.7.** For an integer \( n \geq 1 \), let \( D_n \) be the collection of all sum-free sets \( D \subseteq [n] \) that satisfy \( 2n + 1 \notin \sum D \). Then, \( |\mathcal{A}_n| \leq |D_n| \).

**Proof:** For an integer \( n \geq 1 \), consider the function \( f \) that maps every set \( A \in \mathcal{A}_n \) to the set \( f(A) \) defined by the following process: Go over the elements of \( A \) from the largest one to the smallest one, and exclude every element of \( A \) that forms a sum of two smaller elements in the set. To prove the lemma, it suffices to show that \( f \) is an injective function from \( \mathcal{A}_n \) to \( D_n \).

We first show that for every \( A \in \mathcal{A}_n \) we have \( f(A) \in D_n \). Let \( A \in \mathcal{A}_n \). It follows from the definition that \( f(A) \) is a sum-free subset of \([n]\). Assume by contradiction that \( 2n + 1 \in \sum f(A) \), that is, there exist \( k \geq 3 \) and \( x_1, \ldots, x_k \in f(A) \) such that \( \sum_{i=1}^{k} x_i = 2n + 1 \). We claim that the integers \( x_1, \ldots, x_k \) can be partitioned into 3 sets, each of which is of sum at most \( n \). To see this, start with the singleton partition and repeatedly combine pairs of sets whose joint sum is at most
This process must terminate with exactly 3 sets, since there are no 4 integers with total sum at most $2n + 1$ such that the sum of every two of them is larger than $n$. Since $A \subseteq [n]$ is closed under addition and contains $f(A)$, it follows that $2n + 1 \in 3A$, in contradiction.

To complete the proof, we show that $f$ is injective. For distinct sets $A_1, A_2 \in A_n$, let $x$ be the smallest integer that belongs to exactly one of them, and assume without loss of generality that $x \in A_1$ and $x \notin A_2$. Since $A_2$ is closed under addition, $x$ is not a sum of two elements of $A_2$, and by the minimality of $x$, it is not a sum of two elements of $A_1$ as well. Therefore, $x \in f(A_1)$ and $x \notin f(A_2)$, hence $f(A_1) \neq f(A_2)$.

**Corollary 4.8.** $|A_n| = \Theta(2^{n/3})$.

**Proof:** For the upper bound, combine Lemma 4.7 with Theorem 1.3. For the lower bound, consider all subsets of the set $[[\frac{2}{3}(n + 1)], n]$ to obtain that $|A_n| \geq 2^{\lfloor \frac{1}{3}(n + 1) \rfloor}$.

Finally, to derive Theorem 4.3 combine Lemma 4.6 with Corollary 4.8.

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