Fractional Diffusion Equation for a Power-Law-Truncated Lévy Process

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Truncated Lévy flights are stochastic processes which display a crossover from a heavy-tailed Lévy behavior to a faster decaying probability distribution function (pdf). Putting less weight on long flights overcomes the divergence of the Lévy distribution second moment. We introduce a fractional generalization of the diffusion equation, whose solution defines a process in which a Lévy flight of exponent \( \alpha \) is truncated by a power-law of exponent \( 5 - \alpha \). A closed form for the characteristic function of the process is derived. The pdf of the displacement slowly converges to a Gaussian in its central part showing however a power law far tail. Possible applications are discussed.

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Since Lévy flights have been introduced into statistical physics, it has become clear that special attention must be given to the fact that due to their heavy tails they are characterized by diverging moments. A few approaches have been suggested to overcome this divergence. These include the introduction of the concept of Lévy walks \(^1\), confining Lévy flights by external potentials \(^2\), and introduction of truncation procedures \(^3\) \(^4\). Each of the approaches represents a different physical situation, but they all made it possible for Lévy processes to be applicable in a variety of areas ranging from Hamiltonian dynamics \(^5\) to spectral diffusion in single molecule spectroscopy \(^6\), from bacterial motion \(^7\) to the albatross flights \(^8\) and also in analysis of economical data \(^9\) \(^10\). Here we concentrate on the truncation of the flights.

In many cases the Lévy-flight behavior corresponds to intermediate asymptotics. At very large values of the variable some cutoff enters, so that the moments exist. Truncated Lévy flights, a process showing a slow convergence to a Gaussian, were introduced by Mantegna and Stanley \(^3\) and have been used especially in econophysics, see Refs. \(^4\) \(^10\). The truncated Lévy flight is a Markovian jump process, with the length of jumps showing a power-law behavior up to some large scale. At larger scales the power-law behavior crosses over to a faster decay, so that the second moment of the jump lengths exists. In this case the central limit theorem applies, so that at very long times the distribution of displacements converges to a Gaussian: this convergence however may be extremely slow. The original work concentrated on numerical simulations of the process which assumed a \( \theta \)-function cutoff. Koponen \(^4\) slightly changed the model by replacing the \( \theta \)-function cutoff by an exponential one and obtained a useful analytical representation for the model. However, further investigations have shown that the models with sharp (\( \theta \)-function or exponential) cutoffs predicting a Gaussian or an exponential tail of the pdf are not always appropriate \(^10\).

In mathematical physics, it is often convenient to have a deterministic equation for the pdf of a process, an analogue of the diffusion or Fokker-Planck equation (FPE), to be solved under given initial and boundary conditions. For the case of anomalous transport, fractional generalizations of such equations may be relevant \(^11\). However, these classes of equations are valid only for processes showing exact scaling in the force-free limit. The truncated Lévy flights are not such a process. As mentioned, in the course of time the process displays a crossover between the two regimes: at shorter times the characteristics of this random process behave as those of a Lévy flight, while at long times they are close to ones for normal diffusion. Processes showing a crossover can often be described by equations containing derivatives of different order in the same variable. A known example here is the telegrapher’s equation with the second and the first-order temporal derivatives, describing the crossover from ballistic transport to the diffusion behavior. However, fractional generalizations of the telegrapher’s equation describe Lévy-walk-like processes \(^12\) and not truncated Lévy flights.

The equation we propose for truncated Lévy flights with the power-law cutoff has the following form:

\[
\left( 1 - C_\alpha \frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}} \right) \frac{\partial p(x,t)}{\partial t} = D \frac{\partial^2 p(x,t)}{\partial x^2},
\]

where \( D \) is the diffusion coefficient governing the long-time asymptotic behavior, and the scale factor \( C_\alpha = D/K_\alpha \) is a coefficient governing the intermediate-time Lévy-like one. The dimension of \( C_\alpha \) is \([C_\alpha] = [L^{2-\alpha}]\). In Eq. \( \frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}} \) denotes the symmetric Riesz-Weyl operator...
The Hilbert transform corresponding Riemann-Liouville operators, and \( \hat{f}(x) \) for the Green’s function of this equation, corresponding to a weight function \(- |\alpha| \phi(x)\) which describes a crossover between the characteristic function of some probability distribution and \(\phi(k)\), for which \(\beta = 2\) gives us a known form \(-k^2 \phi(k)\). This is the reason to include the “minus” sign in Eq. (2). Note however that \(\phi(k)\) corresponds to \(\phi(k)\).

The operator defined by Eqs. (2) is a fractional generalization of the second derivative: in Fourier-representation \(\hat{f}(x)\) simply corresponds to \(- |\alpha| \phi(x)\) which describes a crossover between \(\alpha = 2\) (normal diffusion) and to \(\alpha < 2\) (Lévy-like superdiffusion). It can be shown, that contrary to the equations used in [14, 15] which describe processes getting more and more anomalous in the course of time (retarding subdiffusion or accelerating superdiffusion), equation (6) describes processes getting less anomalous, such as in our case, tending to normal diffusion. General properties of this equation will be considered elsewhere.

Using the Fourier-representation of the Riesz-Weyl derivative in Eq. (1) we get for the characteristic function of the distribution:

\[
\left( 1 + C_\alpha |k|^{2-\alpha} \right) \frac{\partial f(x)}{\partial t} = - Dk^2 f(k, t). \tag{6}
\]

The Green’s function of this equation, corresponding to the initial condition \(f(k, 0) = 1\) (i.e. \(p(x, 0) = \delta(x)\)) then reads:

\[
f(k, t) = \exp \left( \frac{-Dk^2}{1 + C_\alpha |k|^{2-\alpha}} \right). \tag{7}
\]

We postpone the proof of the fact that this is indeed a characteristic function of some probability distribution \(p(x, t)\) until later on and discuss first the main properties of such a solution.

The function \(f(k, t)\) is differentiable twice for each \(\alpha\); its second derivative \(f''(k, t)\) for \(\alpha = 0\) is \(-2Dt\), so that the second moment of the distribution evolves in a diffusive manner: \(\langle x^2(t) \rangle = 2Dt\), as in normal diffusion. However, in the intermediate domain of \(x\) the distribution shows the behavior typical for Lévy flights; namely for \(k\) large enough, i.e. for \(C_\alpha |k|^{2-\alpha} \gg 1\), the characteristic function has the form

\[
f(k, t) = \exp \left( - \frac{D}{C_\alpha} |k|^\alpha t \right), \tag{8}
\]

i.e. corresponds to the characteristic function of the Lévy distribution. Assuming \(\alpha < 2\) we get the following expansion for \(f(k, t)\) near \(k = 0\):

\[
f(k, t) \simeq 1 - Dtk^2 + DC_\alpha t |k|^{4-\alpha} + \ldots \tag{9}
\]

From this expression it is evident that \(f(k, t)\) always lacks the fourth derivative at \(k = 0\) (for \(1 < \alpha < 2\) it even lacks the third derivative), which means that the fourth moment of the corresponding distribution diverges. The absence of higher moments of the distribution explains the particular nature of the truncation implied by our model: The Lévy distribution is truncated by a power law with a power between 3 and 5. Thus, for all \(0 < \alpha < 2\) the corresponding distributions have a finite second moment and, according to the central limit theorem (slowly!) converge to a Gaussian. For the case \(0 < \alpha < 1\) (when \(\langle |x|^\beta \rangle < \infty\)) the speed of this convergence is given by the Berry-Esseen theorem, as noted in Ref. [17]. The convergence criteria for \(1 < \alpha < 2\) can be obtained using theorems of Ch. XVI of Ref. [17].

This transition from the initial Lévy-like distribution to a Gaussian is illustrated in Fig.1, obtained by a numerical inverse Fourier-transform of the characteristic function, Eq. (6). Here the case \(\alpha = 1, D = C = 1\) is shown. To put the functions for \(t = 0.001\) and for \(t = 1000\) on the same plot we rescale them in such a way that the characteristic width of the distribution \(W(t)\) (defined by \(\int_0^\infty p(x, t)dx = \frac{1}{2}\)) is the same. The behavior of the pdf to be at the origin \(p(0, t)\) as a function of \(t\) is shown on the double logarithmic scales in Fig.2. Note the crossover from the initially fast decay \(p(0, t) \propto t^{-1/\alpha}\) (Lévy superdiffusion) to the final form \(p(0, t) \propto t^{-1/2}\) typical for diffusion.

The asymptotics of the pdf at large \(x\) is determined by the first non-analytical term in the expansion, Eq. (7), i.e. \(\propto DC_\alpha t |k|^{4-\alpha}\). By making the inverse Fourier transformation of this term and using the Abel method of summation of improper integral, we get

\[
p(x, t) \simeq \frac{\Gamma(5-\alpha) \sin(\pi\alpha/2)}{\pi} \frac{DC_\alpha t}{x^{5-\alpha}}, \quad x \to \infty, \tag{10}
\]

where we use that \(\int_0^\infty d\xi \xi^{4-\alpha} \cos \xi = \sin(\pi\alpha/2)\Gamma(5-\alpha)\). Thus, in our case the Lévy-distribution is truncated not by a \(\theta\)- or exponential function, but by a steeper powerlaw, with a power \(\beta = 5 - \alpha\).
For example a Lévy flight truncated by another, faster decaying power-law, is a much better model for the behavior of commodity prices. Thus, the discussion in Ref. [10] shows that the cumulative distribution function of cotton prices may correspond to a power-law behavior of $1 - F(x) = \int_x^\infty p(x)dx \propto x^{-\alpha}$ with the power $\alpha = 1.7$ in its middle part and with the far tail decaying as a power-law $1 - F(x) \propto x^{-\beta}$ with $\beta \approx 3$. Thus, our equation (which is definitely the simplest form of the equation for truncated Lévy flights) adequately describes this very interesting case giving $\beta = 3.3$. It is highly probable that fractional equations of the type considered here might be a valuable tool in economic research.

Let us now prove that the solution $p(x,t)$ is a pdf, i.e. a non-negative normalized function of $x$ for any $t$. The normalization is trivial and follows from the fact that $\int_0^\infty p(x,t)dx = 1$ for all $t$. We start from defining a function $G(u,t)$, $u > 0$, such that its Laplace transform in variable $u$ is

$$\tilde{G}(s,t) = \int_0^\infty e^{-su}G(u,t) = \exp \left(-\frac{s}{1 + A_\alpha s^{1-\alpha/2}}\right)$$

with $A_\alpha = C_\alpha/D^{1-\alpha/2}$. By comparing Eqs. (11) and (12), one sees that the characteristic function $\tilde{\phi}(k,t)$ can be rewritten in the following form:

$$\tilde{\phi}(k,t) = \int_0^\infty e^{-uDk^2}G(u,t)du.$$  

Indeed, the transition from Eq. (12) to Eqs. (11) is nothing else but the change of variable $s \rightarrow Dk^2$ in Eq. (11). It is clear that $\tilde{G}(0,t) = 1$ and, moreover, as we proceed to show, $\tilde{G}(s,t)$ is completely monotonic, i.e. it is non-negative, and the signs of its derivatives alternate. Then, according to Bernstein’s theorem, $\tilde{G}(s,t)$ is a Laplace-transform of some probability density. Now, we can perform the inverse Fourier-transform in the Eq. (12) and get

$$p(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi Du}} \exp \left(-\frac{x^2}{4Du}\right) G(u,t)du$$

which is a non-negative function (since the integrand is a product of two non-negative functions). Eq. (13) provides a subordination transformation: the truncated Lévy flights can be considered as a process subordinated to a Wiener process under the operational time given by the function $G(u,t)$ [13]. The small-$u$ behavior of this function (corresponding to the large-s one of its Laplace-transform, $G(s,t) = \exp (-A_\alpha^{-1}s^{1+\alpha/2})$) is approximately a one-sided (extreme) Lévy law of index $\alpha/2$. However, at large $u$ this Lévy law is truncated.

We now give a proof that the function $\tilde{G}(s,t) = \exp \left(-\frac{s}{1 + A_\alpha s^{1-\alpha/2}}\right)$ is a completely monotonic function in variable $s$. This function has a form $\exp(-\psi(s))$ and...
therefore is completely monotonic if the function $\psi(s)$ is positive and possesses a completely monotonic derivative. In our case $\psi(s) = s/(1 + As^b)$ with $b = 1 - \alpha/2$, $0 < b < 1$, $A > 0$. Its derivative is given by

$$
\psi'(s) = \frac{1}{1 + As^b} \left[ 1 - b - \frac{As^b}{1 + As^b} \right]. \tag{14}
$$

This function is a product of two functions. The first one is completely monotonic since it has a form $g(h(s))$ with $g(y) = 1/(1 + y)$ being completely monotonic and with $h(s) = As^b$ being a positive function with a completely monotonic derivative. The second function has the same form, now with $g(y) = 1 - by/(1 + y)$. This function $g(y)$ is positive for all $y > 0$, and its derivatives read

$$
g^{(n)}(y) = (-1)^n b^n n! (1 + y)^{-n-1}.
$$

The subordination property also sheds light on the possible nature of truncated Lévy distributions in economic processes. The truncated Lévy process can be interpreted as a simple random walk with a finite variance. However, the number of steps of the random walk (the number of transactions) per unit time is not fixed, but fluctuates strongly. The implications to economics of such models were considered in [19]. In our case the distribution function of the number of steps has itself a form of a truncated one-sided Lévy law.

Let us summarize our findings. We proposed a fractional generalization of a diffusion equation which describes power-law truncated Lévy flights, a random process showing a slow convergence to a Gaussian. We show that the solution of this equation is a pdf and give numerical results for the case $\alpha = 1$. Moreover, we argue that the truncated Lévy flights can be represented as a random process subordinated to a Wiener process, which might be helpful in econophysical applications. We end by noting that the equation discussed is a special case of distributed-order fractional diffusion equations. Modifications of our equation should be able to describe other types of truncation; however, the equation discussed here is definitely the simplest one.

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