A variational formulation of the wear contact problems for coated bodies

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Abstract. Plane wear contact problems for an elastic half-plane with thin coating and a rigid punch are considered. A local (differential) setting of the wear contact problem assumes that actual contact areas are unknown in advance, may be multiple (discrete), and may change through wear. The thin coating is modelled by a Winkler-type layer. An additional normal displacement of a substrate surface is proportional to a local pressure. A variational approach to the problem is used. The variational formulation in the form of a system of an evolutionary variational inequality and an ordinary differential equation of first order is obtained. An explicit Euler scheme is used for the time discretization. A spatial discretization of the problems is made using spaces of integrated fundamental solutions. The boundary element method is applied to construct such spaces, whose elements satisfy homogeneous equilibrium equations and compatibility conditions.

1. Introduction
Coatings are widely used in many mechanical systems to keep moving components from wearing excessively. Coatings may be fabricated in a variety of forms and structures using different kinds of materials. This leads to a set of mathematical models used to describe the coating systems for tribological applications. In this study a thin coating is simulated by an additional normal displacement of a substrate surface proportional to a local pressure [1]. It was shown in [2] that this model can be used when the coating compliance is greater than the substrate compliance.

Wear contact problems for elastic bodies with thin coatings have been systematically investigated by Goryacheva [3], Soldatenkov [4] and other researchers. These problems can be conditionally divided into three classes: the single contact problems, the periodic problems, and the multiple (discrete) contact problems. The latter class in particular includes problems with a bounded nominal contact region for regular wavy surfaces. Such problems cannot be reduced to periodic because of the nonuniform load distribution between individual contact spots. For each of the above classes of problems there are two subclasses: the problems with a constant contact area and the problems with a varying, usually growing contact area. In the latter, the parameters describing the shape and size of the actual contact areas are related to the unknowns. To derive them it is necessary to use additional conditions, which considerably complicate the methods of solving problems.
An alternative approach to wear contact problems with a changing contact area is to use their variational formulation. The variational method can be used both to study the existence and uniqueness of the solution of contact problems and to calculate the numerical solution [5, 6].

The purpose of this study is to obtain a stress variational formulation of the wear contact problem for an elastic half-plane with the thin coating and to develop a computational algorithm.

2. Problem statement

We consider a sliding contact of a rigid cylinder and an elastic homogeneous isotropic half-space with a thin coating. The cylinder moves along a generatrix with a constant speed \( V \), as shown in Figure 1. Tangential traction at the contact zone acts in the slip direction and causes wear of the coating. We assume that the wear-resistant properties of the coating do not change in the slip direction. In this case, the problem can be considered as a two-dimensional (plane strain) wear contact problem for a rigid punch and an elastic half-plane with a thin coating. The wear process is considered on a finite time interval \([0,T]\).

![Figure 1. A coated half-plane and a sliding punch](image)

Let the elastic half-plane occupy the semi-infinite domain \( \Omega = \{ x=(x_1,x_2,x_3): x_3 \leq 0, x_3 = 0 \} \) with the boundary \( \Gamma \) in the fixed rectangular coordinate system \( Ox_1x_2x_3 \). We denote by \( \vec{u}(x,t) \) a displacement vector, by \( \hat{\sigma}(x,t) \), \( \hat{\epsilon}(x,t) \) strain and stress tensors respectively at a point \( x \in \Omega \) at an instant of time \( t \in [0,T] \). We assume that the initial unstrained state of the elastic half-plane is unstressed. The displacements and deformations are considered small. The equilibrium state of the elastic half-plane is governed by the system of equations

\[
\text{Div} \hat{\sigma}(x,t) = 0, \quad \hat{\epsilon}(x,t) = \hat{\sigma}(x,t) = \hat{\epsilon}(x,t) = \hat{\sigma}(x,t) = \hat{\sigma}(x,t) \text{ in } \Omega \times (0,T),
\]

where \( L = L/2(\text{Grad} + \text{Grad}^T) \) and \( \hat{C} = \{C_{ijkl}\} \) is an elasticity tensor.

The boundary \( \Gamma \) is coated. The coating wear is characterized by a linear wear \( w(x,t) \). Hence the current coating thickness \( h(x,t) \) measured along the \( x_3 \)-axis is related to the wear \( w(x,t) \) by \( h(x,t) = h_0(x) - w(x,t) \), where \( h_0(x) \) is an initial thickness that is assumed to be small enough compared to the size of the contact area.

The thin coating is modelled by a Winkler-type layer. The additional normal displacement \( s(x,t) \) due to transverse compression of the coating is related to the normal stress \( \sigma_{22} \) on \( \Gamma \) by

\[
s(x,t) = B(x,w)\sigma_{22}(x,t),
\]
where $B(x,w)$ is the coating compliance depending upon the coating structure and the wear $w(x,t)$. The compliance of a heterogeneous coating with Young’s modulus $E^c = E^c(x)$ and Poisson’s ratio $\nu^c = \nu^c(x)$ is given by

$$B(x,w) = \int_0^1 \frac{(1 - 2\nu^c)(1 + \nu^c)}{(E^c \cdot (1 - \nu^c))} dx_2, \quad h_0(x) - w > 0.$$  

(3)

Let $\Gamma^c_p$ denote the part of the coating surface $\Gamma^c$ on which contact with the punch is possible. The position and size limits of $\Gamma^c_p$ are determined based on geometric considerations. The actual contact areas are unknown in advance, may be multiple (discrete), and may change through wear. The punch shape is described by the function $\Phi(x)$ whose value at a point $x \in \Gamma^c_p$ is equal to the distance from this point to the punch surface measured along the $x_2$-axis. The distance $\Phi(x)$ is measured with respect to the undeformed state of the coating and the half-plane at the initial instant of time $t = 0$. The position of the punch is specified by the displacement $\delta_j(t)$ and the angle of rotation $\varphi_j(t)$.

The boundary conditions are applied at the undeformed surface $\Gamma$ of the elastic half-plane under above assumption about the coating. The contact between the punch and the elastic half-plane with the Winkler-type coating is described by linearized boundary conditions of unilateral contact with wear

$$u_2(x,t) + s(x,t) \leq w(x,t) + \Phi(x) + \delta_2(t) + \varphi_j(t)x_j, \quad \sigma_{22}(x,t) \leq 0, \quad \sigma_{12}(x,t) = 0,$$

$$\sigma_{22}(x,t)[u_2(x,t) + s(x,t) - w(x,t) - \Phi(x) - \delta_2(t) - \varphi_j(t)x_j] = 0 \quad \text{on} \quad \Gamma^c_p \times (0,T),$$

(4)

where $\Gamma^c_p$ is the projection of $\Gamma^c_p$ onto the boundary $\Gamma$ along the $x_2$-axis. The rest of the boundary $\Gamma \setminus \Gamma^c_p$ is free from external loads

$$\sigma_{22}(x,t) = \sigma_{12}(x,t) = 0 \quad \text{on} \quad (\Gamma \setminus \Gamma^c_p) \times (0,T).$$

(5)

The external force $R_j(t)$ and moment $M_j(t)$ are applied to the punch. The equilibrium conditions for the punch can be written as

$$\int_{\Gamma^c_p} \sigma_{22}(x,t) dx_2 = R_j(t), \quad \int_{\Gamma^c_p} \sigma_{22}(x,t)x_j dx_2 = M_j(t), \quad t \in [0,T].$$

(6)

The coating wear rate $\dot{w}$ has a power law dependence on the contact pressure

$$\dot{w}(x,t) = K_w \left| \sigma_{22}(x,t) \right|^{\alpha} \quad \text{on} \quad \Gamma^c_p \times (0,T),$$

(7)

where $K_w$ is a wear rate coefficient and $\alpha$ is an exponent. In the general case for heterogeneous coating, the coefficient $K_w$ and exponent $\alpha$ are functions of the coordinate $x$ and the linear wear $w$. The initial wear is absent

$$w(x,0) = 0.$$  

(8)

We denote by $R^2$ the 2-dimensional Euclidean space and by $S^2$ the space of second-order symmetric tensors on $R^2$. The local (differential) setting of the problem can now be stated as follows.

Problem A. Find a displacement field $\hat{u} : \Omega \times [0,T] \to R^2$, a strain field $\hat{\sigma} : \Omega \times [0,T] \to S^2$, a stress field $\hat{\dot{u}} : \Omega \times [0,T] \to R$, an additional displacement $s : \Gamma^c_p \times [0,T] \to R$, a linear wear $w : \Gamma^c_p \times [0,T] \to R$, a punch displacement $\delta_j : [0,T] \to R$, and an angle of the punch rotation $\varphi_j : [0,T] \to R$ such that (1)-(8) is fulfilled.
The conditions for the existence and uniqueness of a solution to unilateral contact problems can be found in [5, 6]. We assume that a unique solution to the problem (1)-(8) exists.

3. Variational formulation

A suitable approach for solving unilateral contact problems with contact areas that are unknown in advance and may be multiple (discrete) is to use the variational method [5].

We introduce a real Hilbert space $U(\Omega)=\{\tilde{\tau}=(\tau_{ij})\in[L^2(\Omega)]^{2}\times 2 : \tilde{\tau}_{ij}/\partial x_{j} \in L^2(\Omega), i, j = 1, 2\}$ with the inner product $(\tilde{\sigma}, \tilde{\tau}_{ij}) = \int_{\Omega} \sigma_{ij} \tau_{ij} + (\partial \sigma_{ij}/\partial x_{j})(\partial \tau_{ij}/\partial x_{j}) d\Omega$. We say that a stress field $\tilde{\tau}=(\tau_{ij})\in U(\Omega)$ satisfies the compatibility conditions if there exists a displacement field $\tilde{v}\in[W^1_0(\Omega)]^2$ such that $\tilde{\tau} = \hat{C} \cdot \tilde{v}$. We select a subspace $U_{0}(\Omega) \subset U(\Omega)$ of the stress fields satisfying the homogeneous equilibrium equations and the compatibility conditions

$$U_{0}(\Omega) = \{\tilde{\tau} \in U(\Omega) : \text{Div} \tilde{\tau} = 0, \tilde{\tau} = \hat{C} \cdot \tilde{v}, \tilde{v} \in [W^1_0(\Omega)]^2\}.$$ 

There exist trace operators, which map an element $\tilde{\tau} \in U_{0}$ onto displacement and stress vectors on the boundary $\Gamma_p$. We create a set of statically admissible stress fields satisfying the compatibility conditions

$$V(t) = \{\tilde{\tau} \in U_{0}(\Omega) : \tau_{ij}(x) \leq 0, x \in \Gamma_p; \tau_{ij}(x) = 0, x \in \Gamma \setminus \Gamma_p; \tau_{ij}(x) = 0, x \in \Gamma; \int_{\Gamma_p} \tau_{ij} d\Gamma_p = R(t), \int_{\Gamma_p} \tau_{ij} v_{ij} d\Gamma_p = M_{ij}(t)\}, \quad t \in \{0, T\}.$$ 

The set $V(t), t \in \{0, T\}$, is assumed to be nonempty. Here it is easy to check that the set $V(t)$ is closed and convex in $U_{0}$.

We define the boundary linear $b(\cdot) : U_{0}(\Omega) \to R$ and bilinear $a(\cdot) : U_{0}(\Omega) \otimes U_{0}(\Omega) \to R$, $d(\cdot) : U_{0}(\Omega) \otimes H^{1/2}(\Gamma_p) \to R$ forms as follows

$$b(\tilde{\tau}) = <\tau_{ij}, \Phi>_{\Gamma_p}, \quad a(\tilde{\sigma}, \tilde{\tau}_{ij}) = <\tau_{ij}, \tilde{v}_{ij}(\tilde{\sigma})>_{\Gamma_p}, \quad d(\tilde{w}, \tilde{\tau}_{ij}) = <\tau_{ij}, \tilde{w}>_{\Gamma_p},$$

where $<\cdot, \cdot>_{\Gamma_p}$ is a canonical bilinear form on $[H^{1/2}(\Gamma_p)] \otimes [H^{1/2}(\Gamma_p)]$ and $v_{ij}(\tilde{\sigma})$ is a component of the displacement vector $\tilde{v}$, which corresponds to the stress field $\tilde{\sigma}$. We also define the scalar function of three arguments $e(\cdot, \cdot, \cdot) : U_{0}(\Omega) \otimes U_{0}(\Omega) \otimes H^{1/2}(\Gamma_p) \to R$ as follows

$$e(\tilde{\sigma}, \tilde{\tau}_{ij}, \tilde{w}) = <\tau_{ij}, s(\tilde{\sigma})>_{\Gamma_p},$$

where $s(\tilde{\sigma})$ is the additional displacement given by (2). In (2) the coating compliance $B(x, w)$ is calculated according to (3) and consequently depends on the third argument $w$ of the function $e(\cdot, \cdot, \cdot)$.

Using the results of [5, 6], a stress variational formulation of the wear contact problem (1)-(8) is obtained as follows.

Problem B. Find a stress field $\tilde{\sigma} : \{0, T\} \to V$ and a linear wear $w : \{0, T\} \to H^{1/2}(\Gamma_p)$ such that $w(x, 0) = 0$ and

$$a(\tilde{\sigma}(t), \tilde{\tau} - \tilde{\sigma}(t)) + e(\tilde{\sigma}(t), \tilde{\tau} - \tilde{\sigma}(t), w) - d(w(t), \tilde{\tau} - \tilde{\sigma}(t)) - b(\tilde{\tau} - \tilde{\sigma}(t)) \geq 0 \quad \forall \tilde{\tau} \in V(t), \quad (9)$$

$$\dot{w} = K_w |\tilde{\tau}|^{p}. \quad (10)$$
Theorem 1. Let the functions that determine the elastic properties of the half-space, the coating compliance, the external loads and the contact conditions be bounded with the appropriate norm and have the following smoothness properties
\[ C_{ijkl} \in L_p(\Omega), \quad B \in L_p(\Gamma_p), \quad R_2 \in L_2(0,T), \quad M_1 \in L_2(0,T), \quad \Phi \in H^{1/2}(\Gamma_p), \]
\[ K_w \in L_2(\Gamma_p), \quad \alpha \in L_2(\Gamma_p). \]  
(11) 

Assume that the elasticity tensor satisfies the usual properties of symmetry
\[ C_{ijkl} = C_{ijlk} \leq C_1, \quad C_1 = \text{const} > 0 \]  
(12) 

and ellipticity
\[ C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq C_2\varepsilon_{ij}\varepsilon_{ij}, \quad C_2 = \text{const} > 0, \quad \forall\{\varepsilon_{ij}\} \in L_2(\Omega). \]  
(13) 

In addition, assume that the coating compliance is positive
\[ B > 0. \]  
(14) 

Then a solution to the problem A is a solution to the problem B.

Theorem 2. Let \{\hat{\sigma},w\} be a solution of the problem B and the conditions (11)-(14) are satisfied. Then there exist displacement \( \hat{u}(\hat{\sigma}) \) and strain \( \hat{\varepsilon}(\hat{\sigma}) \) fields of the elastic half-space, an additional displacement \( \hat{s}(\hat{\sigma}) \), punch displacement \( \delta_z \), and angle of the punch rotation \( \varphi_z \) satisfying (at least in the generalized sense) all the conditions of the problem A.

4. Computational algorithm

The time discretization of the problem is based on an explicit Euler scheme. We divide the time axis into \( K \) equal intervals of length \( \Delta t \) and refer \( t^k = k\Delta t, k = 0, K \). We denote the value of \( \hat{\sigma}(x,t) \), \( w(x,t) \), \( R_2(t) \) and \( M_1(t) \) at the instant of time \( t^k \) by \( \hat{\sigma}^k(x) \), \( w^k(x) \), \( R_2^k \) and \( M_1^k \) respectively. Similarly we define \( V^k = V(t) \big|_{t^k} \). As a result, we obtain
\[ w^k(x) = w^{k-1}(x) + \Delta t \cdot K_w(x,w^{k-1}) \left[ \hat{\sigma}_{ij}^{k-1}(x) \right]_{ij}, \]  
(15) 
\[ a(\hat{\sigma}^k,\hat{\varepsilon}^k - \hat{\sigma}^k) + e(\hat{\sigma}^k,\hat{\varepsilon}^k - \hat{\sigma}^k,\hat{w}^k) - d(\hat{\varepsilon}^k,\hat{\sigma}^k) - b(\hat{\varepsilon}^k,\hat{\sigma}^k) \geq 0 \quad \forall \hat{\varepsilon} \in V^k, \]  
(16) 
\[ w^0(x) = 0. \]  
(17) 

The variational inequality (16) must be solved at each step of the integration process (15)-(17). This inequality corresponds to a contact problem for an elastic half-plane with a Winkler-type coating which is indented by a rigid punch which shape is described by the function \( \Phi(x) + w^k(x) \), i.e., the current coating wear is taken into account.

It follows from (12)-(13) that the bilinear form \( a(\cdot,\cdot) \) is continuous symmetric and positive definite on \( U(\Omega) \otimes U(\Omega) \), and it follows from (14) that the function \( e(\cdot,\cdot,w^k) \) is a continuous symmetric and positive definite on \( U(\Omega) \otimes U(\Omega) \) bilinear form with respect to the first two arguments for a fixed third one. Consequently, it can be proved the following theorem.

Theorem 3. The variational inequality (16) is equivalent to the following minimization problem: Find an element \( \hat{\sigma}^k \in V^k \) such that
\[ J(\hat{\sigma}^k) = \inf_{\hat{\varepsilon} \in V^k} \left\{ J(\hat{\varepsilon}) = \frac{1}{2} a(\hat{\varepsilon},\hat{\varepsilon}) + \frac{1}{2} e(\hat{\varepsilon},\hat{\varepsilon},\hat{w}^k) - d(\hat{\varepsilon},\hat{\sigma}^k) - b(\hat{\varepsilon}) \right\}. \]  
(18) 

Further we discretize the problem (18) with respect to the spatial coordinates. The main difficulty of using stress variational formulations is to approximate the space \( U_0 \), whose elements satisfy, at least in the generalized sense, the equilibrium equations and the compatibility conditions.
The spaces of integrated fundamental solutions [7] can be effectively used to build the approximations.

The solution to the problem (18) can be represented in the form

\[ \sigma_{ij}^n(x) = \int_{\Gamma_y} g_{ij}^n(x, \xi) q(\xi) d\Gamma_y(\xi), \]  

(19)

where \( \hat{G}^n(x, \xi) = \{ g_{ij}^n \} \) is the stress tensor of Flamant’s solution to the problem of a half-plane loaded by a normal force and \( \sigma_{zz}(\xi) \) is the required normal stress on \( \Gamma_y \).

The boundary-element approach is used to construct a finite-dimensional space of integrated fundamental solutions based on the integral representation (19). Let us select boundary elements and perform a triangulation \( T_{\Gamma_y}(\Gamma_y) \). We construct a finite-dimensional space of the boundary elements \( X^h(\Gamma_y) \). Using the integral representation (19), we define a linear integral operator \( \hat{c} = H(\sigma_{zz}) \) on \( X^h(\Gamma_y) \). The range of the operator \( H \) forms a finite-dimensional space of integrated fundamental solutions.

It follows from the properties of elastic potentials [7] that elements of the space \( Z^h(\Omega) \) satisfy the homogeneous equilibrium equations and the compatibility conditions in the domain \( \Omega \). The space of stresses \( U_0 \) is approximated by the finite-dimensional space of integrated fundamental solutions \( Z^h(\Omega) \) and the space of wear functions \( H^{\text{div}}(\Gamma_y) \) is approximated by the space of boundary elements \( X^h(\Gamma_y) \).

We use the Ritz method to approximate the minimization problem (18). As a result, it is necessary to solve a square programming problem with restrictions in the form of equalities and inequalities on each temporary layer. To simplify the restrictions, the linear transformation of variables is offered. A modification of the conjugate gradient method considering specifics of the restrictions is used for the numerical solution of the problems.

5. Conclusion
A variational approach to wear contact problems for coated bodies is presented. The local (differential) setting of the wear contact problem assumes that the actual contact areas are unknown in advance, may be multiple (discrete), and may change through wear. The thin coating is modelled by a Winkler-type layer. The additional normal displacement of a substrate surface is proportional to a local pressure. The variational formulation of the problem in the form of a system of an evolutionary variational inequality and an ordinary differential equation of first order is obtained. An explicit Euler scheme is used for the time discretization. A spatial discretization of the problems is made using the spaces of the integrated fundamental solutions. Elements of these spaces satisfy, at least in the generalized sense, the equilibrium equations and compatibility conditions. The boundary element method is applied to construct the spaces of integrated fundamental solutions.

The numerical algorithm developed in present study is implemented in a software written in FORTFAN. As an example, the wear problem for the elastic coated half-plane and the rigid punch with a regular wavy relief is considered, taking into account the change in the sizes of the actual contact areas. The obtained numerical results are used to verify robustness and computational efficiency of the variational approach to be used in this study.

References
[1] Shtaerman I Ya 1949 Contact Problems in Theory of Elasticity (Moscow-Leningrad: Gostekhizdat) 270
[2] Aleksandrov V M and Mhitaryan S M 1983 Contact Problems for Bodies with Thin Coatings and Stringers (Moscow: Nauka) 488
[3] Goryacheva I G 1998 Contact Mechanics in Tribology (Dordrecht etc.: Kluwer) 344
[4] Soldatenkov I A 2010 Wear Contact Problem with Applications to Engineering Wear Calculations (Moscow: Fizmatkniga) 160
[5] Kravchuk A S and Neittaanmäki P J 2007 Variational and quasi-variational inequalities in mechanics (Springer) 329
[6] Shillor M, Sofonea M and Telega J J 2004 Models and Analysis of Quasistatic Contact, Lect. Notes Phys. 655 (Berlin Heidelberg: Springer) 262
[7] Aleksidze M A 1991 Fundamental Functions in Approximate Solutions of Boundary Value Problems (Moscow: Nauka) 488