Hilbert Schemes of Points on Surfaces

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Abstract

The Hilbert scheme $S^{[n]}$ of points on an algebraic surface $S$ is a simple example of a moduli space and also a nice (crepant) resolution of singularities of the symmetric power $S^{(n)}$. For many phenomena expected for moduli spaces and nice resolutions of singular varieties it is a model case. Hilbert schemes of points have connections to several fields of mathematics, including moduli spaces of sheaves, Donaldson invariants, enumerative geometry of curves, infinite dimensional Lie algebras and vertex algebras and also to theoretical physics. This talk will try to give an overview over these connections.

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0. Introduction

The Hilbert scheme $S^{[n]}$ of points on a complex projective algebraic surface $S$ is a parameter variety for finite subschemes of length $n$ on $S$. It is a nice (crepant) resolution of singularities of the $n$-fold symmetric power $S^{(n)}$ of $S$. If $S$ is a K3 surface or an abelian surface, then $S^{[n]}$ is a compact, holomorphic symplectic (thus hyperkähler) manifold. Thus $S^{[n]}$ is at the same time a basic example of a moduli space and an example of a nice resolution of singularities of a singular variety. There are a number of conjectures and general phenomena, many of which originating from theoretical physics, both about moduli spaces for objects on surfaces and about nice resolutions of singularities. In all of these the Hilbert scheme of points can be viewed as a model case and sometimes as the main motivating example. Hilbert schemes of points on a surface have connections to many topics in mathematics, including moduli spaces of sheaves and vector bundles, Donaldson invariants, Gromov-Witten invariants and enumerative geometry of curves, infinite dimensional Lie algebras and vertex algebras, noncommutative geometry and also theoretical physics.

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It is usually best to look at the Hilbert schemes $S^{[n]}$ for all $n$ at the same time, and to study their invariants in terms of generating functions, because new structures emerge this way. For Euler numbers, Betti numbers and conjecturally for the elliptic genus these generating functions will be modular forms and Jacobi forms. This fits into general conjectures from physics about invariants of moduli spaces. Also the cohomology rings of the $S^{[n]}$ for different $n$ are closely tied together.  

The direct sum over $n$ of all the cohomologies is a representation for the Heisenberg algebra modeled on the cohomology of $S$, and the cohomology rings of the $S^{[n]}$ can be described in terms of vertex operators. In the case that the canonical divisor of the surface $S$ is trivial, this leads to an elementary description of the cohomology rings of the $S^{[n]}$, which coincides with the orbifold cohomology ring of the symmetric power, giving a nontrivial check of a conjecture relating the cohomology ring of a nice resolution of an orbifold to the recently defined orbifold cohomology ring.

The Hilbert schemes $S^{[n]}$ are closely related to other moduli spaces of objects on $S$, including moduli of vector bundles and moduli of curves e.g. via the Serre correspondence and the Mukai Fourier transform. This leads to applications to the geometry and topology of these moduli spaces, to Donaldson invariants, and also to formulas in the enumerative geometry of curves on surfaces and Gromov-Witten invariants. We want to explain some of these results and connections. We will not attempt to give a complete overview, but rather give a glimpse of some of the more striking results.

1. The Hilbert scheme of points

In this article $S$ will usually be a smooth projective surface over the complex numbers. We will study the Hilbert scheme $S^{[n]} = \text{Hilb}^n(S)$ of subschemes of length $n$ on $S$. The points of $S^{[n]}$ correspond to finite subschemes $W \subset S$ of length $n$, in particular a general point corresponds just to a set of $n$ distinct points on $S$. $S^{[n]}$ is projective and comes with a universal family $Z_n(S) \subset S^{[n]} \times S$, consisting of the $(W, x)$ with $x \in W$. An important role in applications of $S^{[n]}$ is played by the tautological vector bundles $L^{[n]} := \pi_∗q^∗(L)$ of rank $n$ on $S^{[n]}$. Here $\pi : Z_n(S) \to S^{[n]}$ and $q : Z_n(S) \to S$ are the projections and $L$ is a line bundle on $S$.

Closely related to $S^{[n]}$ is the symmetric power $S^{(n)} = S^n/G_n$, the quotient of $S^n$ by the action of the symmetric group $G_n$. The points of $S^{(n)}$ correspond to effective 0-cycles $\sum n_i[x_i]$, where the $x_i$ are distinct points of $S$ and the sum of the $n_i$ is $n$. The forgetful map

$$ \rho : S^{[n]} \to S^{(n)}, \quad W \mapsto \sum_{x \in S} \text{len}(O_{W,x})[x] $$

is a morphism. The symmetric power $S^{(n)}$ is singular, as for instance the fix-locus of any transposition in $G_n$ has codimension 2. On the other hand by $[22]$ $S^{[n]}$ is smooth and connected of dimension $2n$ and $\rho : S^{[n]} \to S^{(n)}$ is a resolution of singularities. In fact this is a particularly nice resolution: If $Y$ is a Gorenstein variety, i.e. the dualizing sheaf is a line bundle $K_Y$, a resolution $f : X \to Y$ of singularities is called crepant if it preserves the canonical divisor, that is $f^∗K_Y = K_X$. It is easy to see
that $\rho : S[n] \to S^{(n)}$ is crepant. In the special case that $S$ is an abelian surface or a K3 surface one can get a better result: A complex manifold $X$ is called holomorphic symplectic if there exists an everywhere non-degenerate holomorphic 2-form $\phi$ on $X$. If furthermore $\phi$ is unique up to scalar, $X$ is called irreducible holomorphic symplectic. A Kähler manifold $X$ of real dimension $4n$ is called hyperkähler if its holonomy group is $Sp(n)$. Compact complex manifolds are holomorphic symplectic if and only if they admit a hyperkähler metric. In [7] it is shown that for a K3 surface $P$ the set $\mathbb{H}$ of hyperkähler structures is infinite-dimensional. For many questions about the Hilbert schemes $S[n]$ one should look at all $n$ at the same time. The first instance of this are the Betti numbers and Euler numbers, elliptic genus and by

\begin{align}
\sum_{n \geq 0} p(S[n], z) t^n = \prod_{k \geq 1} \prod_{i=0}^{4} (1 - z^{2k-2+i} t^k)(-1)^{i+1} b_i(S). \tag{2.1}
\end{align}

In particular, $\sum_{n \geq 0} e(S[n]) q^{\frac{n-\chi(S)}{24}} = \eta(\tau)^{-\chi(S)}$.

This was first shown in [19] in the case of the projective plane and of Hirzebruch surfaces using a natural $\mathbb{C}^*$ action. The proof in [24] uses the Weil Conjectures. An important role in this proof as in all subsequent generalizations and refinements is played by the following natural stratification of $S[n]$ and $S^{(n)}$ parametrized by the set $P(n)$ of partitions of $n$. For a partition $\alpha = (n_1, \ldots, n_r) \in P(n)$, the corresponding locally closed stratum $S_\alpha$ of $S^{(n)}$ consists of the set of zero cycles $n_1[x_1] + \cdots + n_r[x_r]$ with $x_1, \ldots, x_r$ distinct points of $S$. We put $S_\alpha = \rho^{-1}(S_\alpha^{(n)})$.

2. Betti number, Euler numbers, elliptic genus

For many questions about the Hilbert schemes $S[n]$ one should look at all $n$ at the same time. The first instance of this are the Betti numbers and Euler numbers, for which we can find generating functions in terms of modular forms. Let $\mathcal{H} := \{ \tau \in \mathbb{C} \mid \Im(\tau) > 0 \}$. A modular form of weight $k$ on $\text{Sl}(2, \mathbb{Z})$ is a function $f : \mathcal{H} \to \mathbb{C}$ s.t.

$$f \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{Sl}(2, \mathbb{Z}).$$

Furthermore, writing $q = e^{2\pi i \tau}$, we require that, in the Fourier development $f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n$, all the negative Fourier coefficients vanish. If also $a_0 = 0$, $f$ is called a cusp form. The most well-known modular form is the discriminant $\Delta(\tau) := q \prod_{n > 0} (1 - q^n)^2$, the unique cusp form of weight 12. The Dirichlet eta function is $\eta = \Delta^{1/24}$.

For a manifold $X$ we denote by $p(X, z) := \sum\{(-1)^i b_i(X) z^i\}$ the Poincaré polynomial and by $e(X) = p(X, 1)$ the Euler number. The Betti numbers and Euler numbers of the $S[n]$ have very nice generating functions [24]:

$$\sum_{n \geq 0} p(S[n], z) t^n = \prod_{k \geq 1} \prod_{i=0}^{4} (1 - z^{2k-2+i} t^k)(-1)^{i+1} b_i(S). \tag{2.1}$$

In particular, $\sum_{n \geq 0} e(S[n]) q^{\frac{n-\chi(S)}{24}} = \eta(\tau)^{-\chi(S)}$. This was first shown in [19] in the case of the projective plane and of Hirzebruch surfaces using a natural $\mathbb{C}^*$ action. The proof in [24] uses the Weil Conjectures. An important role in this proof as in all subsequent generalizations and refinements is played by the following natural stratification of $S[n]$ and $S^{(n)}$ parametrized by the set $P(n)$ of partitions of $n$. For a partition $\alpha = (n_1, \ldots, n_r) \in P(n)$, the corresponding locally closed stratum $S_\alpha$ of $S^{(n)}$ consists of the set of zero cycles $n_1[x_1] + \cdots + n_r[x_r]$ with $x_1, \ldots, x_r$ distinct points of $S$. We put $S_\alpha = \rho^{-1}(S_\alpha^{(n)})$. 

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A partition \( \alpha = (n_1, \ldots, n_r) \in P(n) \) can also be written as \( \alpha = (1^{n_1}, \ldots, n^{n_r}) \), where \( n_i \) is the number of occurrences of \( i \) in \( (n_1, \ldots, n_r) \). We put \( |\alpha| = r = \sum \alpha_i \). Then (2.1) can be reformulated as

\[
p(S[n], z) = \sum_{\alpha \in P(n)} p(S^{(\alpha_1)} \times \ldots \times S^{(\alpha_n)}, z)z^{2(|\alpha|)}.
\]

This result has been refined to Hodge numbers in [30], [11] and this was generalized in [13] to the Douady space of a complex surface. It has been further refined to determine the motive and the Chow groups [14] and the element in the Grothendieck group of varieties of \( S[n] \) [28].

Partially motivated by (2.1) and using arguments from physics in [15] a conjectural refinement to the Krichever-Höhn elliptic genus is given. We restrict our attention to the case that \( K_X = 0 \) when the elliptic genus is a Jacobi form. For a complex vector bundle \( E \) on a complex manifold \( X \) and a variable \( t \) we put

\[
\Lambda_t(E) := \bigoplus_{k \geq 0} \Lambda^k(E)t^k, \quad S_t(E) := \bigoplus_{k \geq 0} S^k(E)t^k.
\]

For the holomorphic Euler characteristic we write \( \chi(X, \Lambda_t(E)) := \sum \chi(X, \Lambda^k(E))t^k \) and similarly for \( S_t(E) \). Then the elliptic genus is defined by

\[
\phi(X, q, y) := \chi \left( X, \prod_{m \geq 1} \Lambda_{-y^{-1}q^m} T_X \otimes \Lambda_{-yq^{m-1}} T_X^* \otimes S_{q^m}(T_X \oplus T_X^*) \right).
\]

Writing \( \phi(S) := \sum_{m \geq 0, l} c(m, l)q^my^l \), the conjecture is

\[
\sum_{N \geq 0} \phi(S[n])p^N = \prod_{n > 0, m \geq 0, l} \frac{1}{(1 - p^n q^m y^l)^c(mn, l)}.
\]

3. Infinite dimensional Lie algebras and the cohomology ring

We saw that one gets nice generating functions in \( n \) for the Betti numbers of the \( S[n] \). Now we shall see that the direct sum of all the cohomologies of the \( S[n] \) carries a new structure which governs the ring structures of the Hilbert schemes. We only consider cohomology with rational coefficients and thus write \( H^*(X) \) for \( H^*(X, \mathbb{Q}) \). We write \( H := H^*(S) \) for \( n > 0 \) let \( H_n := H^*(S[n]) \), and \( H := \bigoplus_{n \geq 0} H_n \). We shall see that \( H \) is an irreducible module under a Heisenberg algebra. This was conjectured in [54] and proven in [15], [32]. \( H \) contains a distinguished element \( 1 \in H_0 = \mathbb{Q} \). We denote by \( \int_S \) and \( \int_{S[n]} \) the evaluation on the fundamental class of \( S \) and \( S[n] \). Define for \( n > 0 \) the incidence variety

\[
Z_{t,n} := \{(Z, x, W) \in S[t] \times S \times S[t+n] \mid Z \subset W, \rho(W) - \rho(Z) = n[x]\},
\]
and use this to define operators

\[ p_n : H \to \text{End}(H); \quad p_n(\alpha)(y) := pr_3(\alpha \cdot pr_2(y) \cap [Z, \alpha]). \]

Let \( p_n(\alpha) := (-1)^n p_\nu(\alpha) \), where \( \nu \) denotes the adjoint with respect to \( \int_{S^n} \), and \( p_0(\alpha) := 0 \). By [35, 32] the \( p_n(\alpha) \) fulfill the commutation relations of a Heisenberg algebra:

\[ [p_n(\alpha), p_m(\beta)] = (-1)^{n-1} n \delta_{n,-m} \left( \int_S \alpha \cdot \beta \right) id_H, \quad n, m \in \mathbb{Z}, \ \alpha, \beta \in H. \quad (3.1) \]

We can interpret this as follows. Let \( H = H_+ \oplus H_- \) be the decomposition into even and odd cohomology. Put \( S^*(H) := \bigoplus_{i \geq 0} S^i(H^+ \otimes \bigoplus_{i \geq 0} \Lambda^i(H_-)) \). The Fock space associated to \( H \) is \( F(H) := S^*(H \otimes \mathbb{C}[t]) \). Using the above theorems one readily shows that there is an isomorphism of graded vector spaces \( F(H) \to H \). With this \( H \) becomes an irreducible module under the Heisenberg-Clifford algebra.

The ring structure of the \( H^*(S^n) \) is connected to the Heisenberg algebra action. Given an action of a Heisenberg algebra, a standard construction gives an action of the corresponding Virasoro algebra. The important fact however, proven in [37] is that the Virasoro algebra generators have a geometrical interpretation tying them to the ring structure of the cohomology of the \( S^n \). Let \( \delta : S \to S \times S \) be the diagonal embedding, and let \( \delta_s : H^*(S) \to H^*(S \times S) \) be the corresponding pushforward. Let \( p_\nu p_{n-\nu} \delta_s(\alpha) : H^*(S) \to \text{End}(H) \) be defined as \( p_\nu p_{n-\nu}(\beta \times \gamma) := p_\nu(\beta)p_{n-\nu}(\gamma) \) applied to \( \delta_s(\alpha) \in H \times H \). For \( n \neq 0 \) define \( L_n(\alpha) := \sum_{\nu \in \mathbb{Z}} p_\nu p_{n-\nu} \delta_s(\alpha) \), and \( L_0(\alpha) := \sum_{\nu > 0} p_\nu p_{-\nu} \delta_s(\alpha) \). These operators satisfy the relations of the Virasoro algebra:

\[ [L_n(\alpha), L_m(\beta)] = (n - m)L_{n+m}(ab) + \delta_{n,-m} \frac{n^3 - n}{12} \left( \int_S c_2(S)ab \right) id_H. \quad (3.2) \]

Let \( \partial : H \to H \) be the operator which on each \( H^*(S^n) \) is the multiplication with \( c_1(O[n]) \), where \( O[n] = \pi_*(Z_n(S)) \) is the tautological vector bundle associated to the trivial line bundle on \( S \). The tie given in [37] to the ring structure is:

\[ [\partial, p_n(\alpha)] = n L_n(\alpha) + \left( \begin{array}{c} n \\ 2 \end{array} \right) p_n(K_S a), \quad n \geq 0, \ \alpha \in H^*(S). \quad (3.3) \]

In [32], for each \( \alpha \in H^*(S) \), classes \( \alpha^n \in H^*(S^n) \) are defined as generalizations of the Chern characters \( ch(F^n) \) of tautological bundles, which are studied in [37]. The homogeneous components of the \( \alpha^n \) generate the ring \( H^*(S^n) \). [37, 42] relate the multiplication by the \( \alpha^n \) to the higher order commutators with \( \partial \). Let \( \alpha^s \) be the operator which on every \( H^*(S^n) \) is the multiplication with \( \alpha^n \), then

\[ [\alpha^s, p_1(\beta)] = \exp(ad(\partial))p_1(\alpha \beta), \quad (3.4) \]

where for an operator \( A : H \to H \), \( ad(\partial)A = [\partial, A] \).

[3.2, 3.3, 3.4] determine the cohomology rings of the \( S^n \). In case \( K_S = 0 \) this is used in [38, 39] to give an elementary description of the cohomology rings \( H^*(S^n) \) in terms of the symmetric group, which we will relate below to orbifold cohomology rings.
4. Orbifolds and orbifold cohomology

Let $X$ be a compact complex manifold with an action of a finite group $G$ and assume that for all $1 \neq g \in G$ the fixlocus $X^g$ has codimension $\geq 2$. The quotient $X/G$ will usually be singular, but the stack quotient $[X/G]$ is a smooth orbifold. In physics \cite{10, 17} the following orbifold Euler characteristic has been introduced

$$e(X, G) := \sum_{gh=hg\in G} e(X^{g, h}) = \sum_{[g]\subset G} e(X^g/C(g)).$$

Here the first sum runs over all commuting pairs in $G$ and $X^{g, h}$ is the set of common fixpoints; the second sum runs over the conjugacy classes $[g]$ of elements in $G$ and $C(g)$ is the centralizer of $g$. If $Y \to X/G$ is a crepant resolution, then it was expected that $e(X, G) = e(Y)$. As the conjugacy classes of the symmetric group $G_n$ correspond to the partitions of $n$, one can see \cite{63} using formula (2.2) that this is true for the resolution of $S^{(n)}$ by $S[n]$, which was an important check for this conjecture.

Orbifold Euler numbers have been refined to orbifold cohomology groups \cite{60}. We again take all cohomology with $\mathbb{Q}$ coefficients. Define a rationally graded $\mathbb{Q}$-vector space

$$H^*_{orb}([X/G]) := \bigoplus_{[g]\subset G} H^*(X^g/C(g)).$$

The grading is defined as follows. Assume for simplicity that all $X^g$ are connected. For $x \in X^g$ let $e^{2\pi i r_1}, \ldots, e^{2\pi i r_k}$ be the eigenvalues of $g$ on $T_x x$. Put $a(g) := \sum r_i \in \mathbb{Q}$ where $r_i \in [0, 1)$. This is independent of $x$. For $\alpha \in H^i(X^g/C(g))$ its degree in the $[g]$-th summand of $H^*_{orb}([X/G])$ is $i + 2a(g)$. If $X/G$ is Gorenstein, then it is easy to see that $a(g) \in \mathbb{Z}_{\geq 0}$. For crepant resolutions $Y \to X/G$, it was conjectured that $H^*_{orb}([X/G]) = H^*(Y)$ as graded vector spaces. In the case of $S[n] \to S^{(n)}$ this can again be verified from formula (2.2). In \cite{9} it has been established for all crepant resolutions $Y \to X/G$.

Recently orbifold cohomology rings, i.e. a ring structure on the orbifold cohomology have been defined as a special case of quantum cohomology of orbifolds \cite{12, 2, 1}. In \cite{53} it is conjectured for an orbifold $X$ with a hyperkähler resolution $Y \to X$, i.e. a crepant resolution such that $Y$ is hyperkähler, that the orbifold cohomology ring of $X$ is isomorphic to $H^*(Y)$. The most relevant case of such a resolution is $S[n] \to S^{(n)}$ when $K_S = 0$. This is precisely the case in which \cite{39} gives an elementary description of the cohomology ring of $S[n]$. In \cite{21, 52} an elementary description of the orbifold cohomology of a quotient $[X/G]$ by a finite group is given. We define $H^*(X, G) := \sum_{g \in G} H^*(X^g)$. This carries a $G$-action by $h \cdot \alpha_g = (h \cdot \alpha)_{hg^{-1}}$, and, for a suitable grading on $H^*(X, G)$, it follows that the $G$ invariant part is just $H^*_{orb}([X/G])$ as a graded vector space. In order to define the ring structure on $H^*_{orb}([X/G])$ one therefore defines a ring structure on $H^*(X, G)$ compatible with the $G$-action. In \cite{39} the cohomology ring $H^*(S[n])$ is also described as the $G_n$ invariant part of a ring structure on $H^*(S^n, G_n)$ and one checks that the two ring structures on $H^*(S^n, G_n)$ coincide up to an explicit sign change, thus proving the conjecture of \cite{54} for $S[n]$. 

If \( \pi : Y \to X/G \) is only a crepant resolution but not hyperkähler, then usually \( H^*_\text{orb}([X/G]) \) and \( H^*(Y) \) are not isomorphic as rings. However in \[51\] a precise conjecture is made relating the two: One has to correct \( H^*_\text{orb}([X/G]) \) by Gromov-Witten invariants coming from classes of rational curves \( Y \) contracted by \( \pi \). In the case of the Hilbert scheme these curve classes are the multiples of a unique class. The conjecture was verified for \( S[2] \).

5. Moduli of vector bundles

We denote by \( M^H_S(r,c_1,c_2) \) the moduli space of Gieseker \( H \)-semistable coherent sheaves of rank \( r \) on \( S \) with Chern classes \( c_1, c_2 \). Here a sheaf \( \mathcal{F} \) of rank \( r > 0 \) on \( S \) is called semistable, if \( \chi(\mathcal{G} \otimes H^n)/r' \leq \chi(\mathcal{F} \otimes H^n)/r \) for all sufficiently large \( n \) and for all subsheaves \( \mathcal{G} \subseteq \mathcal{F} \) of positive rank \( r' \). As \( M^H_S(1,0,c_2) \approx \text{Pic}^0(S) \times S[\omega] \), the Hilbert scheme of points is a special case. We will often restrict our attention to the case of \( r = 2 \) and write \( M^H_S(c_1,c_2) \). The Hilbert schemes of points are related in several ways to the \( M^H_S(c_1,c_2) \). The most basic tie is the Serre correspondence which says that under mild assumptions rank two vector bundles on \( S \) can be constructed as extensions of ideal sheaves of finite subschemes by line bundles. Related to this is the dependence of the \( M^H_S(c_1,c_2) \) on the ample divisor \( H \) via a system of walls and chambers. This has been studied by a number of authors (e.g. \[10, 23, 18\]). Assume for simplicity that \( S \) is simply connected. A class \( \xi \in H^2(S, \mathbb{Z}) \) defines a wall of type \( (c_1,c_2) \) if \( \xi + c_1 \in 2\mathbb{H}(S, \mathbb{Z}) \) and \( c_1^2 - 4c_2 \leq \xi^2 < 0 \). The corresponding wall is \( W^\xi = \{ \alpha \in H^2(S, \mathbb{R}) \mid \alpha \cdot \xi = 0 \} \). The connected components of the complement of the walls in \( H^2(X, \mathbb{R}) \) are called the chambers of type \( (c_1,c_2) \). If a sheaf \( \mathcal{E} \in M^H_S(c_1,c_2) \) is unstable with respect to \( L \), then there is a wall \( W^\xi \) with \( H\xi < 0 < L\xi \) and an extension

\[
0 \to \mathcal{I}_Z \otimes A \to \mathcal{E} \to \mathcal{I}_W \times B \to 0,
\]

where \( A, B \in \text{Pic}(S) \) with \( A - B = \xi \) and \( \mathcal{I}_Z, \mathcal{I}_W \) are the ideal sheaves of zero dimensional schemes on \( S \). It follows that \( M^H_S(c_1,c_2) \) depends only on the chamber of \( H \) and the set theoretic change under wallcrossing is given in terms of Hilbert schemes of points on \( S \). In the case e.g. of rational surfaces and K3-surfaces, the change can be described as an explicit sequence of blow ups along \( \Gamma_k \) bundles over products \( S[n] \times S[m] \) followed by blow downs in another direction \[23, 18\]. The change of the Betti and Hodge numbers under wallcrossing can be explicitly determined and this can be used e.g. to determine the Hodge numbers of \( M^H_S(c_1,c_2) \) for rational surfaces. For suitable choices of \( H \) one can find the generating functions in terms of modular forms and Jacobi forms \[27\].

The appearance of modular forms is in accord with the S-duality conjectures \[44\] from theoretical physics, which predict that under suitable assumptions the generating functions for the the Euler numbers of moduli spaces of sheaves on surfaces should be given by modular forms. One of the motivating examples for this conjecture is the case that \( S \) is a K3-surface. In this case the conjecture is that, if \( M^H_S(c_1,c_2) \) is smooth, then it has the same Betti numbers as the Hilbert scheme of points on \( S \) of the same dimension. Assuming this, the formula \[24\] for the Hilbert
schemes of points implies that the generating function for the Euler numbers is a modular form. If $c_1$ is primitive this was shown in [29]. The result was shown in general for $M^H_S(r, c_1, c_2)$ with $r > 0$ in $[57],[59]$, by relating the Hilbert scheme and the moduli space via birational correspondences and deformations. One concludes that $M^H_S(r, c_1, c_2)$ has the same Betti numbers as the Hilbert scheme of points of the same dimension, as both spaces are holomorphic symplectic and birational manifolds with trivial canonical class have the same Betti numbers. Similar results are shown in [58] for abelian surfaces. Other motivating examples for the S-duality conjecture were the case of $P_2$ [56] and the blowup formula relating the generating function for the Euler numbers of the moduli spaces of rank 2 sheaves on a surface $S$ to that on the blowup of $S$ in a point, which has since been established ([40],[41], see also [27]).

The moduli spaces $M^H_S(c_1, c_2)$ can be used to compute the Donaldson invariants of $S$. In case $p_g = 0$ these depend on a metric, corresponding to the dependence of $M^H_S(c_1, c_2)$ on $H$. For rational surfaces one can use the above description of the wallcrossing for the $M^H_S(c_1, c_2)$ to determine the change of the Donaldson invariants in terms of Chern numbers of generalizations of the tautological sheaves $L^{[n]}$ on products $S^{[n]} \times S^{[m]}$ of Hilbert schemes of points [18],[23]. The leading terms of these expressions can be explicitly evaluated. The wallcrossing of Donaldson invariants has also been studied in gauge theory (e.g. [35],[36]). There a conjecture about the structure of the wallcrossing formulas is made. Assuming this conjecture one can determine the generating functions for the wallcrossing in terms of modular forms ([25],[31]).

### 6. Enumerative geometry of curves

Now we want to see some striking relations between the Hilbert schemes $S^{[n]}$ and the enumerative geometry of curves on $S$. First let $S$ be a K3 surface and $L$ a primitive line bundle on $S$. Then $L^2 = 2g - 2$, where the linear system $|L|$ has dimension $g$ and a smooth curve in $|L|$ has geometric genus $g$. As a node imposes one linear condition, one expects a finite number of rational curves (i.e. curves of geometric genus 0) in $|L|$. Partially based on arguments from physics, a formula is given in [54] for the number of rational curves in $|L|$ and in [8] this made mathematically precise. Writing $n_g$ for the number of rational curves in $|L|$ with $L^2 = 2g - 2$ (counted with suitable multiplicities), the formula is

$$\sum_{g \geq 0} n_g q^g = \frac{q}{\Delta},$$

where $\Delta$ is again the discriminant. By [24] this implies the surprising fact that $n_g$ is just the Euler number of $S^{[g]}$. In fact the argument relates the number of curves to $S^{[g]}$: Let $C \to |L|$ be the universal curve and let $J \to |L|$ be the corresponding relative compactified Jacobian, whose fibre over the point corresponding to a curve $C$ is the compactified Jacobian $J(C)$. One can show that $e(J(C)) = 0$ unless $g(C) = 0$. It follows that $e(J)$ is the sum over the $e(J(C))$ for $C \in |L|$ with $g(C) = 0$. It is not difficult to show that $S^{[g]}$ and $J$ are birational. $J$ is also smooth
and hyperkähler as a moduli space of sheaves on a K3 surface \[44\]. As already used in the section on vector bundles, birational manifolds with trivial canonical bundle have the same Betti numbers \(5\). Thus \(J\) and \(S^g\) have the same Euler numbers.

This shows (6.1), where the multiplicity of a rational curve \(C\) is \(e(J(C))\). By \(20\) this multiplicity is the multiplicity of the corresponding moduli space of stable maps, in particular it is always positive. In \(20\) a conjectural generalization of \(6.1\) to arbitrary surfaces \(S\) is given.

**Conjecture 6.1**

1. For all \(\delta \geq 0\), there exists a universal polynomial \(T_\delta(x, y, z, w)\), such that for all projective surfaces \(S\) and all sufficiently ample line bundles \(L\) on \(S\) the number of \(\delta\)-nodal curves in a general \(\delta\)-dimensional linear subspace of \(|L|\) is \(T_\delta(\chi(L), \chi(\mathcal{O}_S), LK_S, K_S^2)\).

2. There are universal power series \(B_1, B_2 \in \mathbb{Z}[[q]]\) whose coefficients can be explicitly determined, such that

\[
\sum_{\delta \geq 0} T_\delta(\chi(L), \chi(\mathcal{O}_S), LK_S, K_S^2)(DG_2)^\delta = \frac{(DG_2/q) (\chi(L) B_1 LK_S B_2 K_S^2)}{(\Delta D^2 G_2/q^2) \chi(\mathcal{O}_S)/2}.
\]

Here \(D = q \frac{D^2}{dq}\) and \(G_2 = -\frac{1}{24} \frac{D^4}{D^2}\).

The expectation that universal polynomials should exist is implicit in \(53\), \(34\); where the \(T_\delta\) are determined for \(\delta \leq 8\). In \(26\) also another tie of the conjecture to the Hilbert scheme of points is given: conjecturally the numbers \(T_\delta(\chi(L), \chi(\mathcal{O}_S), LK_S, K_S^2)\) are suitable intersection numbers on the Hilbert scheme \(S^{[3]}\) of 3\(\delta\) points of \(S\). If \(S\) is a K3 surface or an abelian surface, then the conjecture predicts that the generating function can be written in terms of modular forms. In this case a modified version of Conjecture 6.1 was proven for primitive line bundles in \(9\) and \(11\), replacing the numbers of \(\delta\)-nodal curves with the corresponding modified Gromov-Witten invariants. In \(43\) a proof of the conjecture is published.

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