NARROW COVERINGS OF ω-ARY PRODUCT SPACES

 RANDALL DOUGHERTY
 Ohio State University

July 1, 1996

ABSTRACT. Results of Sierpiński and others have shown that certain finite-dimensional product sets can be written as unions of subsets, each of which is “narrow” in a corresponding direction; that is, each line in that direction intersects the subset in a small set. For example, if the set ω × ω is partitioned into two pieces along the diagonal, then one piece meets every horizontal line in a finite set, and the other piece meets each vertical line in a finite set. Such partitions or coverings can exist only when the sets forming the product are of limited size.

This paper considers such coverings for products of infinitely many sets (usually a product of ω copies of the same cardinal κ). In this case, a covering of the product by narrow sets, one for each coordinate direction, will exist no matter how large the factor sets are. But if one restricts the sets used in the covering (for instance, requiring them to be Borel in a product topology), then the existence of narrow coverings is related to a number of large cardinal properties: partition cardinals, the free subset problem, nonregular ultrafilters, and so on.

One result given here is a relative consistency proof for a hypothesis used by S. Mrówka to construct a counterexample in the dimension theory of metric spaces.

1. Introduction

The set ω × ω can be partitioned along the diagonal into two pieces \( \{(m, n) : m < n\} \) and \( \{(m, n) : m \geq n\} \). The first of these pieces has a property which might be called “narrowness in the first coordinate”: for each \( n \), there are only finitely many \( m \)’s such that \( (m, n) \) is in the set. (In other words, each “line in the direction of the first coordinate axis” has a relatively small intersection with the set.) And the second piece is “narrow in the second coordinate.” Similarly, if \( \omega_1 \times \omega_1 \) is divided into two pieces in this way, then each piece contains only countably many points along each line in the corresponding coordinate direction. But \( \omega_1 \times \omega_1 \) turns out to be too large to partition into two pieces which are narrow in the finite sense.

This work was partially supported by grants from the National Science Foundation and the Alfred P. Sloan Foundation.
By a more complicated construction, one can partition the set $\omega_1 \times \omega_1 \times \omega_1$ into three pieces, each of which is narrow in one of the three coordinates, in the sense of only containing finitely many points on each line in the corresponding coordinate direction. If one allows the narrow sets to contain countably many points on each such line, then a suitable partition exists for the set $\omega_2 \times \omega_2 \times \omega_2$. The $\omega_1$ and $\omega_2$ are largest possible for the respective partitions to exist. This is part of a large collection of results proven by many people over the past eighty years. A few more details are given in section 2 of this paper; for a much more thorough presentation of the subject, see Simms [20].

The purpose of the present paper is to investigate the problem of expressing an infinitary product as a union of subsets, each of which is narrow in some coordinate direction. More specifically, given a set $X$ and a cardinal $\lambda$, can the $\omega$-dimensional product $\omega X$ be covered by (written as a union of) sets $A_n$ ($n < \omega$), where $A_n$ is $\lambda$-narrow in the $n$'th coordinate direction (i.e., each line parallel to the $n$'th coordinate axis meets $A_n$ in fewer than $\lambda$ points)? Stated this way, the answer turns out to be ‘yes’ no matter how large $X$ is, for any $\lambda \geq 2$. But if one puts further restrictions on the sets $A_n$ (e.g., that they be Borel in the product topology on $\omega X$ with $X$ discrete), then one gets a number of interesting questions related to several other well-known concepts — partition cardinals, the free subset problem, nonregular ultrafilters, and so on.

The problem arose from a construction in dimension theory: S. Mrówka [15,14] has shown that a hypothesis called $\langle A_{\aleph_0} \rangle$ or $S(\aleph_0)$ implies the existence of a metrizable space with zero inductive dimension whose completions (under all possible metrics) have nonzero inductive dimension. (A topological space has zero inductive dimension iff its topology has a basis of clopen sets.) The statement of $S(\aleph_0)$ is: if $X$ has size $2^{\aleph_0}$, then $\omega X$ cannot be written as a union of sets $A_n$ ($n < \omega$) where $A_n$ is $\aleph_1$-narrow in the $n$'th coordinate and is $F_\sigma$ in the product topology on $\omega X$ with $X$ discrete. Here we will show that $S(\aleph_0)$ is consistent relative to a large cardinal (the partition cardinal $\kappa \rightarrow (\omega_1 + \omega)^{\aleph_0}$), and that, conversely, consistency of $S(\aleph_0)$ implies consistency of a slightly smaller large cardinal ($\kappa \rightarrow (\omega)^{\aleph_0}$). So a large cardinal well below the level of a measurable cardinal suffices for the construction of Mrówka’s example.

The organization of the present paper is as follows. Section 2 gives notational conventions, the main definitions of terms including those used informally above, and some basic results. Section 3 gives connections between narrow coverings, indiscernibles, and the free subset problem, thus showing that large cardinals are necessary to get the nonexistence of narrow coverings, and that slightly larger cardinals are sufficient. Section 4 shows that some of these nonexistence results are preserved under forcing which adds Cohen or random reals; this suffices to prove the relative consistency of Mrówka’s hypothesis $S(\aleph_0)$. Section 5 gives a method for using ultrafilters to prove results about Borel sets (an approach previously taken by Louveau [11]), and Section 6 uses this method to get results about narrow coverings using suitably nonregular ultrafilters. Section 7 considers the question of how complicated a clopen narrow covering has to be when it does exist; this leads to the study of ranks of trees of finite free sequences. Section 8 lists some of the more interesting questions which remain open. Sections 4 through 7 are independent of each other, except that Section 6 depends on Section 5.
Much of this paper comes from my doctoral dissertation [6]; however, other parts, such as the consistency proof for $S(\aleph_0)$, are new.

I would like to (and hereby do) thank Professors J. Silver and J. Addison for many enlightening discussions, and T. Carlson and H. Friedman for helpful comments.

2. Definitions and Basic Results

Throughout this paper we will be working in ZFC, the usual axioms of set theory including the axiom of choice. Cardinals will be initial ordinals; the cardinal $\aleph_\alpha$ will be denoted by $\omega_\alpha$ when its set or ordinal nature is being emphasized. Since each cardinal is a set of its own cardinality, we will not lose generality by stating many results for cardinals rather than arbitrary sets. Natural numbers are finite ordinals, and each ordinal is the set of its predecessors. The immediate successor (cardinal) of a cardinal $\lambda$ is denoted by $\lambda^+$. The cardinality of a set $S$ is denoted by $|S|$.

For any function $f$ and any set $S$, $f[S]$ and $f^{-1}[S]$ denote the image and inverse image of $S$ under $f$, respectively. The collection of all functions from $X$ to $Y$ is denoted by $X^Y$. A sequence is a function $s$ whose domain is an ordinal; this ordinal is called $\ell(s)$, the length of $s$. The symbol $\cap$ denotes concatenation of sequences. A sequence may be denoted by a list of its members between angle brackets: $\langle \alpha, \beta, \gamma \rangle$, $\langle \gamma \rangle$, $\langle a_n : n < \omega \rangle$, etc.

If sets $S(\beta)$ are defined for all $\beta < \alpha$, then $S(\beta)$ will denote $\bigcup_{\beta < \alpha} S(\beta)$. Variants such as $S(\beta, \gamma)$ are defined similarly.

A tree is a set $T$ of sequences such that any initial segment of a member of $T$ is a member of $T$. If $T$ is a tree of finite sequences, we define $[T]$ to be the set of sequences $s$ of length $\omega$ such that $s|n \in T$ for all $n < \omega$.

Definition 2.1. Let $\mathcal{X}$ be a product of sets. A line parallel to the $n$'th coordinate axis in $\mathcal{X}$ is a subset of $\mathcal{X}$ obtained by allowing the $n$'th coordinate of a point to vary while holding all other coordinates fixed. In other words, the line parallel to the $n$'th coordinate axis in $\mathcal{X}$ through the point $x$ is the set of $y \in \mathcal{X}$ such that $y(i) = x(i)$ for all $i \neq n$.

Definition 2.2.

(a) A subset $A$ of a product set $\mathcal{X}$ is $\lambda$-narrow in the $n$'th coordinate if every line parallel to the $n$'th coordinate axis in $\mathcal{X}$ meets $A$ in fewer than $\lambda$ points.

(b) A $\lambda$-narrow covering of $\mathcal{X}$ is a collection of sets $A_n$, one for each coordinate $n$, such that $\bigcup_n A_n = \mathcal{X}$ and, for each $n$, $A_n$ is $\lambda$-narrow in the $n$'th coordinate.

In particular, $\aleph_0$-narrow means that each line in the relevant direction contains only finitely many points of the set, while $\aleph_1$-narrow means each such line contains countably many points of the set. A $\lambda$-narrow covering of $\mathcal{X}$ can easily be converted into a partition of $\mathcal{X}$ by replacing the sets $A_n$ with the sets $B_n = A_n \setminus \bigcup_{m<n} A_m$, which will still be $\lambda$-narrow.

Clearly, for given $\lambda$ and $d$, the existence of $\lambda$-narrow coverings of the product $d\mathcal{X}$ depends only on the cardinality of the set $\mathcal{X}$. Furthermore, if such a covering exists for $d\mathcal{X}$ (using sets $A_n \subseteq d\mathcal{X}$), then one exists for $d\mathcal{Y}$ for any $\mathcal{Y} \subseteq \mathcal{X}$ (using the sets $A_n \cap d\mathcal{Y}$). So, if such a covering does not exist for $d\mathcal{X}$, then one also does not exist for $d\mathcal{X}'$ whenever $|X'| \geq |X|$.
The existence of narrow coverings for finite products of an infinite set $X$ has been studied by a number of authors; see Simms [20] for a full survey. The main result along this line is Theorem 2.149 of that survey, which comes from Kuratowski [10].

**Theorem 2.3** (Kuratowski). For any natural number $n > 0$, ordinal $\alpha$, and set $X$, there exists an $\aleph_\alpha$-narrow covering of $nX$ if and only if $|X| < \aleph_\alpha + n - 1$.

For the sake of completeness, we can consider the case of $\lambda$-narrow coverings for finite $\lambda$ as well.

**Proposition 2.4.** For any natural numbers $n, m > 0$ and any set $X$, there exists an $m$-narrow covering of $nX$ if and only if $|X| \leq (m - 1)n$.

**Proof.** Let $k = (m - 1)n$. It will suffice to show that $nk$ has an $m$-narrow covering, but $n(k + 1)$ does not.

Define sets $A_j \subseteq nk$ for $j < n$ as follows:

$$x \in A_j \iff (m - 1)j \leq \left( \sum_{i=0}^{n-1} x(i) \right) \text{ mod } k < (m - 1)(j + 1).$$

It is easy to check that the sets $A_j$ form an $m$-narrow covering of $nk$.

On the other hand, a subset of $n(k + 1)$ which is $m$-narrow in any coordinate must contain at most $(k + 1)^{n-1}(m - 1)$ points, so the union of $n$ such sets contains at most $(k + 1)^n k$ points, and hence is not all of $n(k + 1)$. Therefore, $n(k + 1)$ has no $m$-narrow covering.

We now move on to products of infinitely many sets, specifically products of the form $\omega X$.

The preceding results would suggest that a $\lambda$-narrow covering of $\omega X$ exists if $X$ is sufficiently small, but not if $X$ is too large. The following result shows that the actual situation is rather different. This result was proved for $X = \mathbb{R}$ by Bagemihl [1] using methods of Davies; see Theorem 3.60 of Simms [20].

**Theorem 2.5.** For any $X$, there is a 2-narrow covering of $\omega X$.

**Proof.** Define an equivalence relation $\sim$ on $\omega X$ by: $x \sim y$ iff $\{ i : x(i) \neq y(i) \}$ is finite. For each $x$, let $[x]$ be the equivalence class of $x$. Choose a representative $r(c) \in c$ for each equivalence class $c$. Let

$$A_n = \{ x \in \omega X : x(n) = r([x])(n) \}.$$

If $x \in \omega X$, $y \in A_n$, and $y$ is on the line parallel to the $n$’th coordinate axis through $x$, then $y \sim x$, so

$$y(n) = r([y])(n) = r([x])(n),$$

so $y(n)$ is uniquely determined; hence, $A_n$ is 2-narrow in the $n$’th coordinate. Any $x \in \omega X$ is in $A_n$ for all but finitely many $n$, since $x \sim r([x])$, so $\omega X = \bigcup_{n<\omega} A_n$ and we are done.
This sort of proof is commonly referred to as a "blatant application of the Axiom of Choice." (The proof also involves a blatant application of the Axioms of Separation, but people tend to be less concerned about that.) The usual reaction to such a construction is "But is there an example using 'reasonable' sets?" This leads to the following definition, which is stated negatively because we will usually be considering circumstances under which narrow coverings do not exist.

**Definition 2.6.** Given a set $X$, a cardinal $\lambda$, and a property (or collection) $P$ of subsets of $\omega X$, we say that $\text{NNC}(X, \lambda, P)$ holds iff there does not exist a $\lambda$-narrow covering of $\omega X$ using sets satisfying (or in) $P$.

The property $P$ will often be 'open' or 'Borel' or some other property from topology; in these cases, we will assume that the topology on $\omega X$ is the product topology with $X$ discrete.

As noted before, the existence of narrow coverings of $\omega X$ depends only on the cardinality of $X$; hence, we will usually just consider the case where $X$ is itself a cardinal. A narrow covering of $\omega X$ can be cut down to give a narrow covering of $\omega Y$ for any $Y \subseteq X$. Also, if the condition $P$ and the narrowness requirement on the sets in the covering are relaxed, then any narrow coverings that worked for the strict conditions will still work for the relaxed conditions. These two trivial monotonicity properties can be stated together as follows.

**Lemma 2.7.** If $\neg \text{NNC}(X_0, \lambda_0, P_0)$, $X_1 \subseteq X_0$, $\lambda_1 \geq \lambda_0$, and $\{A \cap \omega X_1 : A \in P_0\} \subseteq P_1$, then $\neg \text{NNC}(\kappa_1, \lambda_1, P_1)$. ■

If we have a $\lambda$-narrow covering $\langle A_i : i < n \rangle$ of a finitary product $\omega^n X$, then we can convert it into a $\lambda$-narrow covering $\langle B_i : i < \omega \rangle$ of $\omega X$ by letting $B_i = \emptyset$ for $i \geq n$ and $B_i = \{x : x|n \in A_i\}$ for $i < n$. Since membership of a point $x$ in the sets $B_i$ depends only on the first $n$ coordinates of $X$, these sets are clopen in $\omega X$. Therefore, Theorem 2.3 (with $n = m + 2$) gives the following consequence.

**Corollary 2.8.** For any ordinal $\alpha$ and any $m < \omega$, $\neg \text{NNC}(\aleph_{\alpha + m}, \aleph_{\alpha}, \text{clopen})$. ■

There is no way to extend this result to get $\neg \text{NNC}(\aleph_{\alpha + \omega}, \aleph_{\alpha}, \text{clopen})$, as we will see in the next section.

Sometimes the following slight variant of $\text{NNC}(X, \lambda, P)$ is useful.

**Definition 2.9.** Given a set $X$, a cardinal $\lambda$, and a property (or collection) $P$ of subsets of $\omega X$, we say that $\text{NNC}(X, <\lambda, P)$ holds iff there do not exist sets $A_n \subseteq \omega X$ with property (or in collection) $P$ such that $\bigcup_{n<\omega} A_n = \omega X$ and, for each $n$, $A_n$ is $\lambda'_n$-narrow in the $n$'th coordinate for some $\lambda'_n < \lambda$.

So $\text{NNC}(X, \lambda, P)$ implies $\text{NNC}(X, <\lambda, P)$, which in turn implies $\text{NNC}(X, \lambda', P)$ for all $\lambda' < \lambda$. In fact, if $\text{cf} \lambda > \omega$, then $\text{NNC}(X, <\lambda, P)$ is equivalent to $\langle \forall \lambda' < \lambda \rangle \text{NNC}(X, \lambda', P)$ (because the supremum of the cardinals $\lambda'_n$ from the definition will be a cardinal $\lambda' < \lambda$). But if $\text{cf} \lambda = \omega$, then $\text{NNC}(X, <\lambda, P)$ says a little more.
In this section, we will show that the statement $\text{NNC}(\kappa, \mu^+, \text{open})$ is equivalent to a more familiar assertion, namely that every structure on $\kappa$ with $\mu$ operations has an infinite free subset. In particular, this will show that $\text{NNC}(\kappa, R_1, \text{open})$ implies the large cardinal property $L \models \kappa \to (\omega)_{<\omega}$. On the other hand, a similar but stronger property will be shown to imply $\text{NNC}(\kappa, \lambda, \text{Borel})$. We will start with the latter result, the idea for which was suggested to me by J. Silver.

Recall some definitions from partition theory. For any set $S$ and any natural number $n$, let $[S]^n = \{a \subseteq S: |a| = n\}$; let $[S]^{<\omega} = \bigcup_{n<\omega}[S]^n$. If $\kappa$ and $\lambda$ are cardinals and $\alpha$ is a limit ordinal, then $\kappa \to (\alpha)^\lambda_{<\omega}$ denotes the assertion that, for any $F: [\kappa]^{<\omega} \to \lambda$, there is a set $S \subseteq \kappa$ of order type $\alpha$ such that, for each $n < \omega$, $F$ is constant on $[S]^n$. (We will omit the $\lambda$ in the case $\lambda = 2$.) Jech [8, pp. 392-396] gives a number of facts about this property, among which is the result of Rowbottom that $\kappa \to (\alpha)^{<\omega}_{<\omega}$ implies $\kappa \to (\alpha)^{<\omega}_{<\omega}$.

**Theorem 3.1.** Let $\kappa$, $\lambda$, and $\mu$ be cardinals, and let $S$ be the collection of subsets of $\omega_\kappa$ which can be expressed as Boolean combinations of $\mu$ open subsets of $\omega_\kappa$. If $\kappa \to (\lambda + \omega)^{<\omega}_{2^\mu}$, then $\text{NNC}(\kappa, \lambda, S)$. If $\lambda$ is infinite and $\kappa \to (\lambda)^{<\omega}_{2^\mu}$, then $\text{NNC}(\kappa, <\lambda, S)$.

**Proof.** The case $\mu = 0$ is trivial, so, by the preceding remark, we may assume that $\mu$ is infinite. Let $(A_n: n < \omega)$ be any sequence of sets in $S$ such that $\bigcup_{n<\omega}A_n = \omega_\kappa$. For the first implication, assume $\kappa \to (\lambda + \omega)^{<\omega}_{2^\mu}$; we must find an $n$ such that $A_n$ is not $\lambda$-narrow in the $n$'th coordinate.

Each $A_n$ is a Boolean combination of $\mu$ open sets, so there is a sequence $(G_\alpha: \alpha < \mu)$ of open subsets of $\omega_\kappa$ such that each $A_n$ is a Boolean combination of these open sets. Define a function $F: [\kappa]^{<\omega} \to \nu 2$ as follows: for any strictly increasing sequence $\sigma \in \omega_\kappa$ and any $\alpha < \mu$, let $F(\text{range}(\sigma))(\alpha) = 1$ iff $\sigma \cap s: s \in \omega_\kappa \subseteq G_\alpha$. Since $\kappa \to (\lambda + \omega)^{<\omega}_{2^\mu}$, there is a strictly increasing function $g: \lambda + \omega \to \kappa$ such that $F$ is constant on $[\text{range}(g)]^{\omega_\kappa}$ for each $n < \omega$.

Now, suppose $s$ and $s'$ are strictly increasing sequences of elements of range($g$) of length $\omega$, and $\alpha < \mu$. If $s \in G_\alpha$, then there is $n < \omega$ such that $\{(s|n) \cap t: t \in \omega_\kappa\} \subseteq G_\alpha$, since $G_\alpha$ is open. This gives $F(s|n)(\alpha) = 1$, so $F(s'|n)(\alpha) = 1$, so $\{(s'|n) \cap t: t \in \omega_\kappa\} \subseteq G_\alpha$, so $s' \in G_\alpha$. Conversely, if $s' \in G_\alpha$, then $s \in G_\alpha$ by the same argument. Therefore, if $s \in G_\alpha$ iff $s' \in G_\alpha$ for each $\alpha < \mu$, so, since $A_n$ is a Boolean combination of the sets $G_\alpha$, $s \in A_n$ iff $s' \in A_n$ for each $n < \omega$.

There is at least one $n$ such that $s \in A_n$, so, since $s$ and $s'$ are arbitrary, there is an $n < \omega$ such that, for all strictly increasing $s \in \omega_\kappa$, $s \in A_n$. In particular, if we let $s_\beta = (g|n) \cap (g(n + \beta)) \cap (g(n + \lambda + m): m < \omega)$ for $\beta < \lambda$, we will have $s_\beta \in A_n$ for all $\beta < \lambda$; since $s_\beta(m) \neq s_\gamma(m)$ only if $m = n$, $A_n$ is not $\lambda$-narrow in the $n$'th coordinate.

This completes the proof of the first implication. The proof of the second is similar: Define $F$ as before, and let $g: \lambda \to \kappa$ be increasing with $F$ constant on $[\text{range}(g)]^{\omega_\kappa}$ for
each \(n\). Find \(n\) such that all increasing \(\omega\)-sequences from \([\text{range}(g)]^n\) are in \(A_n\). For any \(\lambda' < \lambda\), we can find in \(\text{range}(g)\) an increasing sequence of \(n\) elements followed by \(\lambda'\) elements followed by \(\omega\) elements; use these elements to form sequences \(s_\beta\) for \(\beta < \lambda'\) in \(A_n\) which differ only at the \(n\)’th coordinate. This shows that \(A_n\) is not \(\lambda'\)-narrow in the \(n\)’th coordinate.

\begin{corollary}
If \(\kappa \rightarrow (\lambda+\omega)^{<\omega}\), then \(\text{NNC}(\kappa, \lambda, \text{Borel})\). If \(\lambda\) is infinite and \(\kappa \rightarrow (\lambda)^{<\omega}\), then \(\text{NNC}(\kappa, <\lambda, \text{Borel})\).
\end{corollary}

Now we give the relation between \(\text{NNC}(\kappa, \lambda, \text{open})\) and the free subset problem, which has been considered in papers by Devlin \([4, \S 4]\), Devlin and Paris \([5]\), Shelah \([18]\), and Koepke \([9]\), among others. The relevant definitions are as follows. If \(S\) is a subset of (the domain of) a structure \(M\), let \(H_M(S)\) be the substructure of \(M\) generated by \(S\). Such a set \(S\) is said to be free for \(M\) iff, for every \(S' \subseteq S\), \((H_M(S')) \cap S = S'\). If \(\kappa, \lambda, \) and \(\mu\) are cardinals, then \(\text{Fr}_\mu(\kappa, \lambda)\) means that every structure of cardinality \(\kappa\) with \(\mu\) operations (possibly including 0-ary operations, i.e., constants) has a free subset of cardinality \(\lambda\).

\begin{theorem}
For any infinite cardinals \(\kappa\) and \(\mu\), \(\text{Fr}_\mu(\kappa, \kappa_0)\) iff \(\text{NNC}(\kappa, \mu, \text{open})\).
\end{theorem}

\begin{proof}
First suppose that \(\text{Fr}_\mu(\kappa, \kappa_0)\) fails, and let \(M\) be a structure with \(\mu\) operations and universe \(\kappa\) which has no infinite free subset. Define subsets \(A_n\) of \(\omega\kappa\) for \(n < \omega\) as follows: for any \(s \in \omega\kappa\), put \(s \in A_n\) iff \(s(n) \in H_M(\{s(m) : m \neq n\})\). If \(s(m) = s(n)\) but \(m \neq n\), then \(s \in A_n\); if \(s\) is one-to-one, then \(s \in A_n\) for some \(n\) since \(M\) has no infinite free subset. Therefore, \(\bigcup_{n<\omega} A_n = \omega\kappa\). Since \(H_M(S)\) is the union of \(H_M(a)\) over all finite \(a \subseteq S\), the sets \(A_n\) are open. Since \(M\) has only \(\mu\) operations, \(|H_M(S)| \leq \mu\) for any countable \(S\), so \(A_n\) is \(\mu^+\)-narrow in the \(n\)’th coordinate. Therefore, \(\text{NNC}(\kappa, \mu^+, \text{open})\) fails.

For the converse, suppose \(\text{NNC}(\kappa, \mu^+, \text{open})\) fails. Let \(\{A_n : n < \omega\}\) be a collection of open sets with union \(\omega\kappa\) such that \(A_n\) is \(\mu^+\)-narrow in the \(n\)’th coordinate. For each triple \((\alpha, m, n)\) with \(\alpha < \mu\) and \(m < n < \omega\), we will define a function \(f_{amn} : n^{-1} \kappa \rightarrow \kappa\). Given \(g, x,\) and \(y\) such that \(g\) is a function with \(x\) in its domain, let \(g(x/y)\) be the function obtained from \(g\) by replacing the value at \(x\) with \(y\); that is, \(g(x/y) = (g\{\{x, g(x)\}\}) \cup \{(x, y)\}\). Now suppose \(m < n < \omega\) and \(\sigma \in \omega\kappa\). Let \(\sigma' \in n^{-1} \kappa\) be the sequence obtained from \(\sigma\) by deleting the \(m\)’th coordinate. Since \(A_m\) is \(\mu^+\)-narrow in the \(m\)’th coordinate, we can choose a sequence \(\langle \beta_\alpha : \alpha < \mu \rangle\) of elements of \(\kappa\) (depending only on \(\sigma'\) and \(m\), not on \(\sigma(m)\)) which includes every \(\beta < \kappa\) such that \(\{\sigma(m/\beta) \cap s : s \in \omega\kappa\} \subseteq A_m\). Let \(f_{amn}(\sigma') = \beta_\alpha\). Now let \(M\) be the structure \((\kappa, (f_{amn})_{\alpha < \mu, m < n < \omega})\); clearly \(M\) has \(\mu\) operations and cardinality \(\kappa\). Let \(S\) be any infinite subset of \(\kappa\), and choose a one-to-one \(s \in \omega\kappa\). There is an \(m < \omega\) such that \(s \in A_m\); since \(A_m\) is open, there is an \(n < \omega\) such that \(\{(s[t]) : t \in \omega\kappa\} \subseteq A_m\), and we may assume \(n > m\). Let \(\sigma' = s[n] \in \omega\kappa\) with coordinate \(m\) deleted. By the definition of \(f_{amn}\), there must be an \(\alpha < \mu\) such that \(f_{amn}(\sigma') = s(m)\). But \(\sigma'\) is a sequence of elements of \(S' \setminus \{s(m)\}\), so \(S\) cannot be free for \(M\). Therefore, \(M\) has no infinite free subset, so \(\text{Fr}_\mu(\kappa, \kappa_0)\) fails.
\end{proof}

This equivalence allows us to translate several results of Devlin and Paris on the free subset problem into results about \(\text{NNC}\):
Corollary 3.4.

(a) If $\kappa$ is real-valued measurable, then NNC($\kappa, \lambda$, open) for all $\lambda < \kappa$.

(b) The statement NNC($\kappa, \mu^+$, open) (as an assertion about $\kappa$ and $\mu$) is absolute downward for transitive models of ZFC, and is preserved under forcing extensions which satisfy the countable chain condition.

(c) If $\kappa \rightarrow (\omega)_{2\mu}^{<\omega}$, then NNC($\kappa, \mu^+$, open).

(d) If $\kappa$ is the least cardinal such that $\kappa \rightarrow (\omega)^{<\omega}$, then NNC($\kappa, \lambda$, open) for all $\lambda < \kappa$.

(e) If $V = L$ or $V = L[D]$ where $D$ is a normal ultrafilter over a measurable cardinal, then NNC($\kappa, \aleph_1$, open) iff $\kappa \rightarrow (\omega)^{<\omega}$.

Proof. (a) Devlin [4, p. 315]. (b) Devlin [4, pp. 314–316]. (c) Any homogeneous set for a structure is free for that structure [4, p. 314]. (d) This follows from (c) and the fact that this $\kappa$ is a strong limit cardinal satisfying $\kappa \rightarrow (\omega)_{\mu}^{<\omega}$ for all $\mu < \kappa$ (Silver; see Jech [8, Lemma 32.9]). (e) Devlin and Paris [5, pp. 334-335].

Therefore, NNC($\kappa, \aleph_1$, open) implies $L \models \kappa \rightarrow (\omega)^{<\omega}$. So the consistency strength of NNC($\kappa, \aleph_1$, open) is the same as that of $\kappa \rightarrow (\omega)^{<\omega}$, while the consistency strength of NNC($\kappa, \aleph_1$, Borel) lies somewhere between that of $\kappa \rightarrow (\omega)^{<\omega}$ and that of $\kappa \rightarrow (\omega+1)^{<\omega}$.

Koepke [9] uses a measurable cardinal to construct a model in which Fr$_{\aleph_0}$($\aleph_0, \aleph_0$) (equivalently, NNC($\aleph_0, \aleph_1$, open)) holds. In fact, the properties he proves about this model imply a stronger result:

Theorem 3.5. If “there is a measurable cardinal” is consistent with ZFC, then so is NNC($\aleph_0, <\aleph_0$, Borel).

Proof. Let $\kappa = \aleph_0$. In the generic extension constructed by Koepke [9], the following property holds: for any $f$: $[\kappa]^{<\omega} \rightarrow 2$, there is a sequence $\langle C_i : i < \omega \rangle$ such that $C_i$ is a cofinal subset of $\omega_{2i+2}$ and, for any finite sequences $\langle i_n : n < N \rangle$, $\langle \alpha_n : n < N \rangle$, and $\langle \beta_n : n < N \rangle$ such that $i_0 < i_1 < \cdots < i_{n-1} < \omega$ and $\alpha_m, \beta_m \in C_{i_m}$, we have $f(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) = f(\beta_0, \beta_1, \ldots, \beta_{n-1})$. The same argument as for Rowbottom’s result that $\kappa \rightarrow (\alpha)^{<\omega}$ implies $\kappa \rightarrow (\alpha)^{<\omega}$ [8, Lemma 32.8] can be used to show that the above property actually holds for any $f$: $[\kappa]^{<\omega} \rightarrow \omega^2$.

Now suppose we have Borel sets $A_n \subseteq \omega_\kappa$ for $n < \omega$ with union $\omega_\kappa$, and natural numbers $k_n$ for $n < \omega$; we must show that, for some $n$, $A_n$ is not $\aleph_{k_n}$-narrow in the $n$’th coordinate. We may assume $k_0 < k_1 < k_2 < \ldots$. There is a sequence $\langle G_m : m < \omega \rangle$ of open subsets of $\omega_\kappa$ such that each of the sets $A_n$ is a Boolean combination of the sets $G_m$, $m < \omega$. Define $f$: $[\kappa]^{<\omega} \rightarrow \omega$ by: $f(\sigma)(m) = 1$ iff $\{\sigma \cap s : s \in \omega_\kappa \} \subseteq G_m$. Since $G_m$ is open, for any $s \in \omega_\kappa$, we have $s \in G_m$ iff there is an $n$ such that $f(s\cap n)(m) = 1$. Find $\langle C_i : i < \omega \rangle$ as in the preceding paragraph. Then, if $s$ and $s'$ are sequences of length $\omega$ such that $s(i), s'(i) \in C_k$, for each $i < \omega$, then $f(s\cap n) = f(s'\cap n)$ for all $n$, so $\{m : s \in G_m \} = \{m : s' \in G_m \}$, and since the $A_n$’s are Boolean combinations of the $G_m$’s, $\{n : s \in A_n \} = \{n : s' \in A_n \}$. Hence, there is a fixed $n$ such that $s \in A_n$ for all such $s$; since there is a collection of $\aleph_{2k_n+2} > \aleph_{k_n}$ such $s$’s which differ only at coordinate $n$, $A_n$ is not $\aleph_{k_n}$-narrow in the $n$’th coordinate, and we are done.
Note that the argument here actually gives NNC(\(\aleph_\omega\), \(<\aleph_\omega\), \(S\)) where \(S\) is the collection of sets which are expressible as Boolean combinations of countably many open sets; this collection includes the Borel sets and many other sets as well.

By the way, standard chain-condition and closure arguments (see Shelah’s version [18]) show that \(\aleph_\omega\) is a strong limit cardinal in this model.

### 4. Forcing and Narrow Coverings

In this section, we will show that, at least for most \(\kappa\) and \(\lambda\), the properties NNC(\(\kappa, \lambda, F_\sigma\)) and NNC(\(\kappa, \lambda, \text{Borel}\)) are preserved under forcing to add any number of Cohen reals or random reals. This will prove the consistency of Mrówka’s hypothesis \(S(\aleph_0)\), given a suitable large cardinal.

**Theorem 4.1.** Let \(M[G]\) be a generic extension of a ground model \(M\) of ZFC, obtained by the standard forcing to add either any number of Cohen reals or any number of random reals. Let \(\kappa\) and \(\lambda\) be cardinals in \(M\), with \(\text{cf}\ \lambda > \omega\). If NNC(\(\kappa, \lambda, F_\sigma\)) is true in \(M\), then it is true in \(M[G]\). The same holds for NNC(\(\kappa, \lambda, \text{Borel}\)).

**Corollary 4.2.** If \((\exists \kappa)(\kappa \rightarrow (\omega_1 + \omega)^{<\omega})\) is consistent with ZFC, then so are NNC(\(2^{\aleph_0}, \aleph_1, F_\sigma\)) (i.e., \(S(\aleph_0)\)) and NNC(\(2^{\aleph_0}, \aleph_1, \text{Borel}\)).

**Proof.** Start with a model where \(\kappa\) has the specified partition property, so that Corollary 3.2 applies, and add \(\kappa\) Cohen or random reals.

Note that, if we start with a measurable cardinal \(\kappa\) and add \(\kappa\) random reals, we get a model where \(\kappa\) is real-valued measurable and NNC(\(\kappa, <\kappa, \text{Borel}\)) holds. It is still open whether NNC(\(\kappa, <\kappa, \text{Borel}\)) actually follows from real-valued measurability of \(\kappa\).

**Corollary 4.3.** If “there is a measurable cardinal” is consistent with ZFC, then so is \((2^{\aleph_0} = \aleph_{\omega_1} + ) + \text{NNC}(\aleph_\omega, <\aleph_\omega, \text{Borel})\).

**Proof.** Start with a model obtained from Theorem 3.5, and add \(\aleph_{\omega_1}\) Cohen or random reals.

So we have a model where \(S(\aleph_0)\) holds and \(2^{\aleph_0} = \aleph_{\omega_1}\). Note that \(\aleph_{\omega_1}\) is the smallest possible value for \(2^{\aleph_0}\) in a model of \(S(\aleph_0)\), since, by Corollary 2.8, NNC(\(\kappa, \aleph_1, \text{clop}\)) cannot hold for \(\kappa < \aleph_\omega\) (and since König’s theorem implies that \(2^{\aleph_0}\) cannot be equal to \(\aleph_\omega\)).

The proof of Theorem 4.1 for random reals is somewhat simpler than that for Cohen reals, so it will be given first. In both cases the \(F_\sigma\) version is given separately because the full Borel version requires additional work.

All of the arguments below are carried out within the ground model \(M\). The forcing partial orders will be written so that \(p \leq q\) means that \(p\) is a stronger condition than \(q\).

The idea of the proof is to show that a counterexample to NNC(\(\kappa, \lambda, S\)) (where \(S\) is ‘\(F_\sigma\)’ or ‘\(\text{Borel}\)’) in the generic extension can be turned into a counterexample in the ground model. To say that there is a counterexample in the extension means that there exist names \(\hat{A}_n\) for \(n < \omega\) and a forcing condition \(p_0\) (in the generic filter) such that

\[
(4.1) \quad p_0 \models \bigcup_{n < \omega} \hat{A}_n = \omega \kappa
\]
and, for each $n < \omega$,

\[(4.2) \quad p_0 \Vdash \dot{A}_n \text{ has property } S \text{ and is } \lambda\text{-narrow in the } n\text{'th coordinate.} \]

One could get a narrow covering of the $\omega \kappa$ of the ground model by simply restricting the sets $\dot{A}_n$ to this space, but the resulting sets would probably not be in the ground model. However, given a name $\dot{A}$, we can define in the ground model a set which will definitely include the set named by $\dot{A}$:

**Definition 4.4.** Given a name $\dot{A}$ and a forcing condition $p_0$, the set of potential members of $\dot{A}$ (assuming $p_0$) is the set of all $x$ (in the ground model) such that there exists $p \leq p_0$ such that $p \Vdash x \in \dot{A}$.

The “(assuming $p_0$)” will usually be omitted since $p_0$ will be clear from the context.

Suppose we have $p_0$ and $\dot{A}_n$ satisfying (4.1) and (4.2). Let $B_n$ be the set of potential members of $\dot{A}_n$. Then $B_n \subseteq \omega \kappa$ for each $n$. Also, for any $s \in \omega \kappa$, we have $p_0 \Vdash (\exists n) s \in \dot{A}_n$, so, for some $n$ and some $p \leq p_0$, $p \Vdash s \in \dot{A}_n$. Therefore, $\bigcup_{n<\omega} B_n = \omega \kappa$. We next show that the set $B_n$ is $\lambda$-narrow in the $n$'th coordinate.

**Lemma 4.5.** Let $P$ be a notion of forcing (partial ordering) with the countable chain condition, and let $\kappa$ and $\lambda$ be cardinals such that $\text{cf } \lambda > \omega$. Suppose that $p_0 \in P$ and $\dot{A}$ is a $P$-name such that $p_0 \Vdash \dot{A} \subseteq \omega \kappa$. If

\[p_0 \Vdash \dot{A} \text{ is } \lambda\text{-narrow in the } n\text{'th coordinate,} \]

then the set of potential members of $\dot{A}$ is $\lambda$-narrow in the $n$'th coordinate.

**Proof.** Let $B$ be the set of potential members of $\dot{A}$. Let $s$ be a member of $\omega \kappa$; we must see that $B$ contains fewer than $\lambda$ points on the line

\[\{ s' \in \omega \kappa : s(m) = s'(m) \text{ for } m \neq n \}. \]

In other words, letting $s(n/\alpha)$ denote the sequence $s$ with entry number $n$ replaced with $\alpha$ (as in Section 3), we must show that $\{ \alpha < \kappa : s(n/\alpha) \in B \}$ has size less than $\lambda$.

Since $p_0 \Vdash \dot{A}$ is $\lambda$-narrow in the $n$'th coordinate, there exist $P$-names $\dot{\beta}$ and $\dot{f}$ such that $p_0$ forces that $\dot{\beta} < \lambda$ and $\dot{f}$ is a function with domain $\dot{\beta}$ enumerating the ordinals $\dot{\alpha}$ such that $s(n/\dot{\alpha}) \in \dot{A}$. By the usual countable chain condition argument (choosing a maximal antichain of conditions below $p_0$ which decide the value of $\dot{\beta}$), there is a countable set $S$ of ordinals less than $\lambda$ such that $p_0 \Vdash \dot{\beta} \in S$. Let $\beta_0$ be the least upper bound of $S$; since $\lambda$ has uncountable cofinality, $\beta_0 < \lambda$.

By the same argument, for each $\gamma < \beta_0$, there is a countable set $W_\gamma \subset \kappa$ such that $p_0$ forces $\dot{f}(\gamma)$, if it exists, to be in $W_\gamma$. Let $W = \bigcup_{\gamma<\beta_0} W_\gamma$. Then, for any ordinal $\alpha < \kappa$, if $\alpha \notin W$, then $p_0$ forces that $\alpha$ is not in the range of $\dot{f}$, so $p_0 \Vdash s(n/\alpha) \notin \dot{A}$, so $s(n/\alpha) \notin B$. Therefore, $\{ \alpha < \kappa : s(n/\alpha) \in B \} \subseteq W$; since $|W| \leq |\beta_0| \cdot \aleph_0 < \lambda$, we are done. \[\blacksquare\]
So the sets $B_n$ form a $\lambda$-narrow covering of $\omega\kappa$ (in the ground model). If we can show that

$$\text{(4.3)} \quad (p_0 \forces \dot{A}_n \text{ has property } S) \implies B_n \text{ has property } S$$

(where $S$ is ‘$F_\sigma$’ or ‘Borel’), then we will have completed the proof that a counterexample to $\text{NNC}(\kappa, \lambda, S)$ in the generic extension gives a counterexample to $\text{NNC}(\kappa, \lambda, S)$ in the ground model.

We first consider the case of random real forcing. Actually, the argument applies more generally to any forcing notion which is a measure algebra. (A measure algebra is a complete Boolean algebra with an associated nonzero $\sigma$-additive probability function; see Jech [8, p. 421] for details. In particular, random real forcing is given by a measure algebra, and any measure algebra has the countable chain condition.) But Maharam [12] has shown that this is not much of a generalization.

Since we are using a complete Boolean algebra as the forcing notion, every sentence $\varphi$ of the forcing language has an associated Boolean value $\|\varphi\|$.

In the usual way, any closed set $F \subseteq \omega\kappa$ can be expressed in the form $[T]$, the set of infinite branches through some tree $T \subseteq <\omega\kappa$: given $F$, let $T$ be the set of finite sequences $\sigma$ such that some member of $F$ starts with $\sigma$. Conversely, any set of the form $[T]$ is closed.

**Lemma 4.6.** Let $P$ be a notion of forcing obtained from a measure algebra. Suppose that $p_0 \in P$ and $\dot{A}$ is a $P$-name such that $p_0 \forces \dot{A} \subseteq \omega\kappa$. If $p_0 \forces \dot{A}$ is $F_\sigma$, then the set of potential members of $\dot{A}$ is $F_\sigma$.

**Proof.** Let $B$ be the set of potential members of $\dot{A}$, and let $\mu$ be the probability function for the measure algebra. Since every nonzero member of the Boolean algebra is given nonzero measure by $\mu$, we can rewrite the definition of $B$ as follows:

$$B = \{ s \in \omega\kappa : \mu(p_0 \cdot \|s \in \dot{A}\|) > 0 \}.$$

We must see that this set is $F_\sigma$.

Any $F_\sigma$ subset of $\omega\kappa$ is a countable union of closed sets, each of which can be expressed in the form $[T]$ for some tree $T \subseteq <\omega\kappa$; furthermore, we may assume that the union is an increasing union. Therefore, there are $P$-names $\dot{T}_m$ for $m < \omega$ such that

$$p_0 \forces \dot{T}_m \text{ is a tree, } [\dot{T}_m] \subseteq [\dot{T}_{m+1}], \text{ and } \dot{A} = \bigcup_{m < \omega} [\dot{T}_m].$$

The Boolean value $p_0 \cdot \|s \in \dot{A}\|$ is the sum (least upper bound) of the Boolean values $p_0 \cdot \|s \in [\dot{T}_m]\|$, which form an increasing sequence; since the $\sigma$-additivity of $\mu$ implies continuity with respect to increasing limits, we get

$$\mu(p_0 \cdot \|s \in \dot{A}\|) = \sup_m \mu(p_0 \cdot \|s \in [\dot{T}_m]\|).$$
Similarly, \( p_0 \cdot \|s \in [\hat{T}_m]\| \) is the decreasing limit of the Boolean values \( p_0 \cdot \|s|k \in \hat{T}_m\| \), so
\[
\mu(p_0 \cdot \|s \in [\hat{T}_m]\|) = \inf_k \mu(p_0 \cdot \|s|k \in \hat{T}_m\|).
\]

Therefore,
\[
s \in B \iff \mu(p_0 \cdot \|s \in \hat{A}\|) > 0 \\
\implies \sup_m \inf_k \mu(p_0 \cdot \|s|k \in \hat{T}_m\|) > 0 \\
\implies (\exists m) \inf_k \mu(p_0 \cdot \|s|k \in \hat{T}_m\|) > 0 \\
\implies (\exists m)(\exists \varepsilon)(\forall k) \mu(p_0 \cdot \|s|k \in \hat{T}_m\|) > \varepsilon,
\]
where \( \varepsilon \) varies over the positive rational numbers. Since the condition \( \mu(p_0 \cdot \|s|k \in \hat{T}_m\|) > \varepsilon \) depends only on \( s|k \), the set of \( s \) satisfying this condition is clopen. Therefore, \( B \) is \( F_\alpha \), as desired.

This completes the proof of Theorem 4.1 for the case of \( F_\alpha \) sets and random real forcing, which suffices for the relative consistency of \( S(\aleph_0) \).

In order to do the case of Borel sets and random real forcing (in fact, measure algebra forcing), we will need to work with codes of Borel sets, and it will be convenient to work with these codes in a slightly more restrictive way than usual.

Define the Borel hierarchy as usual: \( \Sigma^0_1 \) sets are open sets, \( \Pi^0_1 \) sets are closed sets, \( \Sigma^0_\alpha \) sets for \( 1 < \alpha < \omega_1 \) are countable unions of \( \Pi^0_\beta \) sets for (possibly varying) \( \beta < \alpha \), and \( \Pi^0_\alpha \) sets for \( 1 < \alpha < \omega_1 \) are countable intersections of \( \Sigma^0_\beta \) sets for \( \beta < \alpha \). So a \( \Pi^0_\alpha \) set is just the complement of a \( \Sigma^0_\alpha \) set.

Every closed set \( F \subseteq \omega^\omega \) is a countable intersection of clopen sets: if \( F = [T] \), then \( F = \bigcap_{n \in \omega} C_n \) where \( C_n = \{ s \in \omega^\omega : s\upharpoonright n \in T \} \). Similarly, every open set is a countable union of clopen sets \( C_n \). From these facts, one can inductively prove the usual inclusions: \( \Sigma^0_\beta \cup \Pi^0_\beta \subseteq \Sigma^0_\alpha \cap \Pi^0_\alpha \) for \( \beta < \alpha \). Also, the collections \( \Sigma^0_\alpha \) and \( \Pi^0_\alpha \) are closed under finite unions and intersections.

Let \( n, m \mapsto (n, m) \) be a one-to-one function from \( \omega \times \omega \) to \( \omega \setminus \{0\} \) such that \( (n, m) \) increases with \( m \) for each fixed \( n \). As usual, this allows us to code up infinitely many \( \omega \)-sequences into one, and conversely extract from one sequence \( s \) the infinitely many subsequences \( (s)_n \) defined by \( (s)_n(m) = s((n, m)) \).

**Definition 4.7.**

(a) A **Borel code** (of level \( \alpha \)) is a sequence \( c \in \omega^\omega \) such that either \( c(0) \geq 2 \) or, for all \( n \), \( (c)_n \) is a Borel code (of level \( < \alpha \)).

(b) Given a Borel code \( c \) and a sequence of sets \( \langle Z_m : m < \omega \rangle \), define the set \( c(\langle Z_m : m < \omega \rangle) \) as follows: if \( c(0) \geq 2 \), then
\[
c(\langle Z_m : m < \omega \rangle) = Z_{c(1)};
\]
if \( c(0) = 0 \), then
\[
c(\{Z_m : m < \omega\}) = \bigcup_{n<\omega} (c)_n(\{Z_m : m < \omega\});
\]
if \( c(0) = 1 \), then
\[
c(\{Z_m : m < \omega\}) = \bigcap_{n<\omega} (c)_n(\{Z_m : m < \omega\}).
\]

(c) Given \( c \) and \( \langle Z_m : m < \omega \rangle \) as above, where \( Z_m \subseteq \omega \), say that \( \langle Z_m : m < \omega \rangle \) is good for \( c \) iff:

(1) for all \( m \), \( Z_m \) is a clopen set for which membership depends only on the first \( m \) coordinates (i.e., if \( s \in Z_m \) and \( s|m = s'|m \), then \( s' \in Z_m \)); and
(2) during the recursive computation of \( c(\langle Z_m : m < \omega \rangle) \), all unions are increasing unions and all intersections are decreasing intersections.

Now a very slight variation of a standard argument gives:

**Lemma 4.8.** For each nonzero \( \alpha < \omega_1 \), there is a universal \( \Sigma^0_\alpha \) code, i.e., a Borel code \( c \) such that every \( \Sigma^0_\alpha \) subset of \( \omega \) is of the form \( c(\langle Z_m : m < \omega \rangle) \) for some sequence \( \langle Z_m : m < \omega \rangle \) which is good for \( c \) (and the converse: \( c(\langle Z_m : m < \omega \rangle) \) is \( \Sigma^0_\alpha \) for any clopen sets \( Z_m \)). Similarly, for each \( \alpha \) there is a universal \( \Pi^0_\alpha \) code.

**Proof.** To get a universal \( \Pi^0_\alpha \) code, just define \( c \) so that \( c(0) = 1 \), \( (c)_n(0) = 2 \), and \( (c)_n(1) = n \) for all \( n \); this gives \( c(\langle Z_m : m < \omega \rangle) = \bigcap_{m<\omega} Z_m \). This works because, given any closed set \( [T] \), we can let \( Z_m = \{ s : s|m \in T \} \) to generate \( [T] \) from \( c \). A similar argument with the complements gives a universal \( \Sigma^0_1 \) code — just let \( c(0) = 0 \) instead of 1.

Now suppose \( \alpha > 1 \). If \( \alpha \) is a limit ordinal, let \( \alpha_0, \alpha_1, \ldots \) be a strictly increasing sequence of ordinals converging to \( \alpha \); if \( \alpha = \beta + 1 \), let \( \alpha_n = \beta \) for all \( n \). Apply the inductive hypothesis to get a universal \( \Pi^0_\alpha \) code \( c_n \) for each \( n \). Let \( c'_n \) be \( c_n \) with all references to the \( m \)'th given clopen set replaced with references to the \( (n, m) \)'th clopen set, so that
\[
c'_n(\langle Z_m : m < \omega \rangle) = c_n(\langle Z_{(n,m)} : m < \omega \rangle)
\]
for any sets \( Z_m \). Now we can find \( c \) so that \( c(0) = 0 \) and \( (c)_n = c'_n \) for all \( n \).

This \( c \) is a universal \( \Sigma^0_\alpha \) code. Given any \( \Sigma^0_\alpha \) set \( A \), find sets \( B_j \) for \( j < \omega \) with union \( A \) so that each \( B_j \) is \( \Pi^0_\beta \) for some \( \beta < \alpha \). Then \( A \) is the increasing union of the sets \( B'_k = \bigcup_{j<k} B_j \), and each \( B'_k \) is also \( \Pi^0_\beta \) for some \( \beta < \alpha \) (and \( B'_0 = \emptyset \)). We can find a nondecreasing sequence \( k_0, k_1, k_2, \ldots \) of natural numbers tending to infinity so slowly that \( B'_{k_n} \) is a \( \Pi^0_\alpha \) set for all \( n \). For each \( n \), choose a sequence \( \langle Z^{(n)}_m : m < \omega \rangle \) which is good for \( c_n \) so that \( c_n(\langle Z^{(n)}_m : m < \omega \rangle) = B'_{k_n} \). Define \( \langle Z_m : m < \omega \rangle \) so that \( Z_{(n,m)} = Z^{(n)}_m \) for all \( m \) and \( n \), and \( Z_k = \emptyset \) for all remaining \( k \); then \( \langle Z_m : m < \omega \rangle \) is good for \( c \) (here we use the fact that \( (n, m) \) increases with \( m \), so that \( (n, m) \geq m \)) and \( c(\langle Z_m : m < \omega \rangle) = A \), as desired.

The argument for \( \Pi^0_\alpha \) is the same. \( \blacksquare \)
Note that the construction of the universal $\Sigma^0_\alpha$ or $\Pi^0_\alpha$ code $c$ is very absolute, once one has chosen a cofinal $\omega$-sequence for each limit ordinal $\leq \alpha$. In particular, if $c$ is constructed for $\alpha$ in a ground model $M$, then the same $c$ will work in any extension $M[G]$ of $M$, although there will probably be more good sequences to apply it to.

**Lemma 4.9.** Let $P$ be a notion of forcing obtained from a measure algebra, with associated probability function $\mu$. Suppose that $p_0 \in P$ and $\dot{A}$ is a $P$-name such that $p_0 \Vdash \dot{A} \subseteq \omega_\kappa$. If $p_0 \Vdash \dot{A}$ is Borel, then the function $s \mapsto \mu(p_0 \cdot \| s \in \dot{A} \|)$ is a Borel-measurable function from $\omega_\kappa$ to $[0, 1]$.

**Proof.** We know that

$$p_0 \Vdash (\dot{A} \text{ is } \Sigma^0_\alpha \text{ for some } \dot{\alpha} < \omega_1).$$

By the usual countable chain condition argument, the set of $\beta < \omega_1$ such that $(\exists p \leq p_0) p \Vdash \dot{\alpha} = \beta$ is countable, and if we choose $\gamma < \omega_1$ to be greater than all such $\beta$, then we will have $p_0 \Vdash \dot{A}$ is $\Sigma^0_\gamma$. Let $c$ be a universal $\Sigma^0_\gamma$ code (in the ground model); then there exist names $\dot{Z}_m$ for $m < \omega$ such that

$$p_0 \Vdash (\langle \dot{Z}_m : m < \omega \rangle \text{ is good for } c \text{ and } c(\langle \dot{Z}_m : m < \omega \rangle) = \dot{A}).$$

So we must show: if we have a Borel code $c$ and names $\dot{Z}_m$ so that (4.4) holds, then the function $f$ defined by $f(s) = \mu(p_0 \cdot \| s \in \dot{A} \|)$ is Borel-measurable. The proof of this is by induction on the complexity of $c$. If $c(0) \geq 2$, then $\dot{A} = c(\langle \dot{Z}_m : m < \omega \rangle)$ is just $\dot{Z}_{c(1)}$. By the goodness assumption, membership of $s$ in $\dot{Z}_{c(1)}$ depends only on $s\upharpoonright (c(1))$, so $f(s)$ depends only on $s\upharpoonright (c(1))$ and hence is a Borel-measurable (even clopen-measurable) function of $s$.

If $c(0) = 0$, then $\dot{A}$ is the increasing union of the sets $\dot{A}_n = (c)_n(\langle \dot{Z}_m : m < \omega \rangle)$, so the Boolean value $p_0 \cdot \| s \in \dot{A} \|$ is the increasing limit of the Boolean values $p_0 \cdot \| s \in \dot{A}_n \|$. Hence, $f(s)$ is the increasing limit of the numbers $f_n(s) = \mu(p_0 \cdot \| s \in \dot{A}_n \|)$; the functions $f_n$ are Borel-measurable by the inductive hypothesis, so $f$ is Borel-measurable.

Similarly, if $c(0) = 1$, then $f$ is a decreasing limit of a sequence of Borel-measurable functions, so $f$ is Borel measurable. $\blacksquare$

In particular, the set of potential members of $\dot{A}$ is Borel, since this set is just $\{ s : \mu(p_0 \cdot \| s \in \dot{A} \|) > 0 \}$. This shows that (4.3) holds for $S = \text{Borel}$, which completes the proof of the random real version of Theorem 4.1.

(If one keeps track of the Borel levels in Lemma 4.9, one finds: if $p_0 \Vdash \dot{A}_n$ is $\Sigma^0_\alpha$, then $\{ s : \mu(p_0 \cdot \| s \in \dot{A} \|) > r \}$ is $\Sigma^0_\alpha$; if $p_0 \Vdash \dot{A}_n$ is $\Pi^0_\alpha$, then $\{ s : \mu(p_0 \cdot \| s \in \dot{A} \|) > r \}$ is $\Sigma^0_{\alpha + 1}$. Hence, the property $\text{NNC}(\kappa, \lambda, \Sigma^0_\alpha)$ is preserved by measure algebra forcing if $\text{cf} \lambda > \omega$.)

Now let $P \in M$ be the forcing notion for adding a certain number of Cohen reals. We may take $P$ to be the set of all finite partial functions from some ordinal $\theta \in M$ to $\{0, 1\}$, where, given two such functions $p, q$, we have $p \leq q$ iff $q \subseteq p$. This $P$ is called $\text{Fn}(\theta, 2)$.

Again, for the proof that $\text{NNC}(\kappa, \lambda, S)$ (where $S$ is ‘$F_\sigma$’ or ‘Borel’) is preserved under forcing with $P$, suppose that we have a condition $p_0$ and names $\dot{A}_n$ for $n < \omega$ such that
(4.1) and (4.2) hold. Let $B_n$ be the set of potential members of $A_n$. Then the sets $B_n$ form a $\lambda$-narrow covering of $\omega\kappa$ as before, and it remains to show that (4.3) holds in order to get a counterexample to $N.N.C(\kappa, \lambda, S)$ in the ground model.

For any statement $\varphi$ in the forcing language for $P$, one can find a maximal antichain $D$ of conditions in $P$ which either force $\varphi$ or force $\neg\varphi$. Since $P$ has the countable chain condition, $D$ is countable. Let $C$ be the union of the domains of the members of $D$; then $C$ is a countable subset of $\theta$. Now, for any condition $q$, $q \Vdash \varphi$ if and only if $q$ is incompatible with all members of $D$ which force $\neg\varphi$; it follows that $q \Vdash \varphi$ iff $q\lvert C \Vdash \varphi$. Call a set $C \subseteq \theta$ with this property a support of $\varphi$. Note that, if $C \subseteq C' \subseteq \theta$ and $S$ is a support of $\varphi$, then $C'$ is a support of $\varphi$ (since $q\lvert C' \leq q\lvert C$).

For each $\varphi$, let $\text{Supp}(\varphi)$ be a countable support of $\varphi$; it does not matter which one is chosen. (One can just take the first one in some fixed well-ordering of the power set of $\theta$.

Or, in fact, one can show that, for the case of this particular forcing notion, each $\varphi$ has a unique minimal support, which can be chosen as $\text{Supp}(\varphi)$. We will assume that, if $\varphi$ and $\varphi'$ are equivalent (i.e., for all $p$, $p \Vdash \varphi \leftrightarrow \varphi'$), then $\text{Supp}(\varphi) = \text{Supp}(\varphi')$.

For any set $C \subseteq \theta$, let $P\lvert C$ be the set of members of $P$ whose domains are subsets of $C$. Note that, if $C$ is countable, then $P\lvert C$ is countable.

For any $m < \omega$, the sets $\{s \in \omega\kappa: \tau \subseteq s\}$ for $\tau \in m\kappa$ form a partition of $\omega\kappa$ into clopen pieces. Hence, a subset of $\omega\kappa$ is closed if and only if its intersection with each of these pieces is closed, and the same holds for $F_\sigma$. In other words, if we define $X\lvert \tau$ (for $X \subseteq \omega\kappa$) to be $\{s \in \omega\kappa: \tau \cap s \in X\}$, then $X$ is $F_\sigma$ if and only if $X\lvert \tau$ is $F_\sigma$ for all $\tau \in m\kappa$.

**Lemma 4.10.** Let $P = \text{Fn}(\theta, 2)$. Suppose that $p_0 \in P$ and $\dot{A}$ is a $P$-name such that $p_0 \Vdash \dot{A} \subseteq \omega\kappa$. If $p_0 \Vdash (\dot{A} = F_\sigma)$, then the set of potential members of $\dot{A}$ is $F_\sigma$.

**Proof.** Let $B$ be the set of potential members of $\dot{A}$. As in Lemma 4.6, there are $P$-names $\dot{T}_m$ for $m < \omega$ such that

$$p_0 \Vdash \dot{T}_m \text{ is a tree, } [\dot{T}_m] \subseteq [\dot{T}_{m+1}], \text{ and } \dot{A} = \bigcup_{m < \omega} [\dot{T}_m].$$

For each finite sequence $\sigma \in \omega\kappa$, define a set $S_\sigma \subseteq \theta$ as follows:

$$S_\sigma = \text{domain}(p_0) \cup \bigcup_{k \leq \ell(\sigma)} \bigcup_{m < \omega} \text{Supp}(\sigma \lvert k \in \dot{T}_m).$$

So $S_\sigma$ is countable, and $S_\sigma \subseteq S_\tau$ if $\sigma \subseteq \tau$.

If $s \in B$, then there is a condition $p' \leq p_0$ such that $p' \Vdash s \in \dot{A}$. Then there must exist $p \leq p'$ and a specific $m < \omega$ such that $p \Vdash s \in [\dot{T}_m]$. Equivalently, $p \Vdash s \lvert k \in \dot{T}_m$ for all $k < \omega$. Now, if $C$ is the countable set $\bigcup_{j < \omega} S_{s \lvert j}$, then $C$ is a support of $(s \lvert k \in \dot{T}_m)$ for all $k$, so $p\lvert C \Vdash s \lvert k \in \dot{T}_m$ for all $k$. Since $C$ is the increasing union of the sets $S_{s \lvert j}$, and the domain of $p$ is finite, we actually have $p\lvert C = p\lvert S_{s \lvert j}$ for some $j$. Also, since $S_{s \lvert j}$ includes the domain of $p_0$, we still have $p\lvert S_{s \lvert j} \leq p_0$. Therefore, if $s \in B$, then there exist $m, j < \omega$
and \( p \in P|S_{s|j} \) such that \( p \leq p_0 \) and \((\forall k) p \Vdash s|k \in \hat{T}_m\). The converse of this statement is clearly true as well. So \( B = \bigcup_{m,j<\omega} \hat{B}_{m,j} \), where

\[
\hat{B}_{m,j} = \{ s : (\exists p \in P|S_{s|j}) p \leq p_0 \land (\forall k) p \Vdash s|k \in \hat{T}_m \}.
\]

Now, given \( m < \omega \) and \( \tau \in ^{<\omega}\kappa \), let

\[
\hat{B}_m^{(\tau)} = \{ s : (\exists p \in P|S_\tau) p \leq p_0 \land (\forall k) p \Vdash s|k \in \hat{T}_m \}.
\]

Then \( \hat{B}_m^{(\tau)} \) is explicitly a countable union (over \( p \)) of a countable intersection (over \( k \)) of clopen sets, so it is an \( F_\sigma \) set. But, for all \( \tau \in ^j\kappa \), we have \( \hat{B}_{m,j}^{(\tau)} = \hat{B}_m^{(\tau)} \), so \( \hat{B}_{m,j}^{(\tau)} \) is \( F_\sigma \). Therefore, \( B \) is \( F_\sigma \).

\[\square\]

So (4.3) holds for \( S = F_\sigma \).

For the Borel case, it will be convenient to change the Borel coding definitions given earlier (Definition 4.7) so as to use intersections and complements instead of intersections and unions. This means that, when \( c(0) = 0 \), we will have

\[c((Z_{m} : m < \omega)) = ^\omega\kappa \setminus c_0((Z_{m} : m < \omega)).\]

The results proved earlier about Borel codes, such as the existence of universal \( \Sigma^0_\alpha \) codes, go through as before.

As we did for \( F_\sigma \) sets, we can show that, for any set \( X \subseteq ^\omega\kappa \) and any \( \alpha \) and \( n \), \( X \) is \( \Sigma^0_\alpha \) if and only if \( X|\tau \) is \( \Sigma^0_\alpha \) for all \( \tau \in ^n\kappa \); the same holds for \( \Pi^0_\alpha \). (This is proved by induction on \( \alpha \), with a little care at limit stages. Alternatively, one can show easily by induction on Borel codes \( c \) that

\[c((Z_{m} : m < \omega))|\tau = c((Z_{m} : m < \omega))\]

for any \( \tau \) and any sets \( Z_{m} \); then apply this to the case of a universal \( \Sigma^0_\alpha \) or \( \Pi^0_\alpha \) code.)

Just as for the random real case, we see that, if

\[p_0 \Vdash \hat{A} \text{ is a Borel subset of } ^\omega\kappa,\]

then we can find a Borel code \( c \) (in the ground model) and a sequence of names \( \hat{Z}_{m} \) such that

\[p_0 \Vdash \langle \hat{Z}_{m} : m < \omega \rangle \text{ is good for } c \text{ and } c((\hat{Z}_{m} : m < \omega)) = \hat{A}.\]

In fact, we can ensure, by modifying the names \( \hat{Z}_{m} \) if necessary, that \( \emptyset \) (the weakest condition in \( P \)) forces “\( \langle \hat{Z}_{m} : m < \omega \rangle \) is good for \( c \).” It follows that \( \text{Supp}(s \in \hat{Z}_{m}) \) depends only on \( s|m \), not on the rest of \( s \).

**Lemma 4.11.** Let \( P = \text{Fn}(\theta,2) \). Suppose that \( c \) is a Borel code (in terms of intersections and complements, as above) and \( \langle \hat{Z}_{m} : m < \omega \rangle \) is a sequence of names for subsets of \( ^\omega\kappa \) such that

\[\emptyset \Vdash \langle \hat{Z}_{m} : m < \omega \rangle \text{ is good for } c.\]
Then:

(a) For any $s \in \omega \kappa$, $C_s = \bigcup_{j<\omega} S_{s\mid j}$ is a support for $(s \in c(\dot{Z}_m : m < \omega))$, where

$$S_{s\mid j} = \bigcup_{m \leq j} \text{Supp}(s \in \dot{Z}_m).$$

(b) There is an ordinal $\alpha < \omega_1$ such that, for each $p \in P$, the set $\{ s \in \omega \kappa : p \vdash s \in c(\dot{Z}_m : m < \omega) \}$ is $\Pi^0_\alpha$.

(The notation $S_{s\mid j}$ makes sense because $\text{Supp}(s \in \dot{Z}_m)$ depends only on $s\mid m$; in other words, $S_s$ is well-defined for $\tau \in \check{\omega}_\kappa$.)

Proof. Induct on $c$. If $c(0) \geq 2$, then $c(\dot{Z}_m : m < \omega)$ is just $\dot{Z}_m$ for $m = c(1)$, so (a) is obvious; for (b), the specified set is actually clopen (and hence $\Pi^0_\alpha$) since membership of $s$ in $\dot{Z}_m$ depends only on $s\mid m$.

If $c(0) = 1$, then $c(\dot{Z}_m : m < \omega)$ is the intersection of the sets $(c)_n(\dot{Z}_m : m < \omega))$, so $p \vdash s \in c(\dot{Z}_m : m < \omega)$ if and only if $p \vdash s \in (c)_n(\dot{Z}_m : m < \omega))$ for all $n$. Now (a) and (b) for $c$ follow easily from the corresponding facts for $(c)_n$. (The $\alpha$ for $c$ is the supremum of the corresponding ordinals for $(c)_n$.)

Now suppose $c(0) = 0$, so $c(\dot{Z}_m : m < \omega)$ is the complement of $(c)_0(\dot{Z}_m : m < \omega))$. The induction hypothesis states that (a) and (b) hold for $(c)_0$. We now get

$$p \vdash s \in c(\dot{Z}_m : m < \omega))$$

$$\iff (\forall q \leq p) \ q \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega)$$

$$\iff (\forall q \leq p) \ q \upharpoonright C_s \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega)$$

$$\iff (\forall q \leq p \upharpoonright C_s) \ q \upharpoonright C_s \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega)$$

$$\iff (\forall q \leq p \upharpoonright C_s) \ q \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega)$$

$$\iff p \upharpoonright C_s \not\vdash s \in c(\dot{Z}_m : m < \omega))$$

So (a) holds for $c$.

Let $\alpha_0$ be the ordinal given by (b) for $(c)_0$, and let $\alpha = \alpha_0 + 1$. Then (b) holds for $c$ for this value of $\alpha$. To see this, let $B$ be the desired set $\{ s \in \omega \kappa : p \vdash s \in c(\dot{Z}_m : m < \omega)) \}$. By the inductive hypothesis, $C_s$ is a support for $(s \in (c)_0(\dot{Z}_m : m < \omega)))$. Since conditions in $P$ are finite and $C_s$ is the increasing union of the sets $S_{s\mid j}$, we have $P \upharpoonright C_s = \bigcup_{j<\omega} P \upharpoonright S_{s\mid j}$. Hence,

$$p \vdash s \in c(\dot{Z}_m : m < \omega))$$

$$\iff (\forall q \leq p) \ q \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega))$$

$$\iff (\forall q \text{ compatible with } p) \ q \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega))$$

$$\iff (\forall q \text{ compatible with } p) \ q \upharpoonright C_s \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega))$$

$$\iff (\forall q \in P \upharpoonright C_s \text{ compatible with } p) \ q \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega))$$

$$\iff (\forall j)(\forall q \in P \upharpoonright S_{s\mid j} \text{ compatible with } p) \ q \not\vdash s \in (c)_0(\dot{Z}_m : m < \omega)).$$
So $B$ is the intersection over $j$ of the sets

$$
\hat{B}_j = \{ s : (\forall q \in P \exists S_s \text{ compatible with } p) \ q \not\models s \in (c)_0(\langle Z_m : m < \omega \rangle) \}.
$$

If we let

$$
\hat{B}^{(\tau)} = \{ s : (\forall q \in P \exists S_{\tau} \text{ compatible with } p) \ q \not\models s \in (c)_0(\langle Z_m : m < \omega \rangle) \},
$$

then $\hat{B}^{(\tau)}$ is a countable intersection (over $q$) of complements of sets that are $\Pi^0_3$ by the induction hypothesis, so $\hat{B}^{(\tau)}$ is $\Pi^0_3$. But $\hat{B}_j|\tau = \hat{B}^{(\tau)}|\tau$ for all $\tau \in j\kappa$, so $\hat{B}_j$ is $\Pi^0_3$. Therefore, $B$ is $\Pi^0_3$, as desired. 

We can now prove (4.3) for $S = \text{Borel}$. Given $p_0$ and $\hat{A}_n$, find $c$ and $\langle \hat{Z}_m : m < \omega \rangle$ as above for the complement of $\hat{A}_n$. Then we find that the set $B_n$ of potential members of $\hat{A}_n$ is just $\{ s : p_0 \not\models s \in c(\langle \hat{Z}_m : m < \omega \rangle) \}$. By Lemma 4.11, the complement of $B_n$ is Borel, so $B_n$ is Borel. This completes the proof of Theorem 4.1.

Again, more careful accounting of Borel levels shows that, if $p_0 \vdash (\hat{A} \sqsubseteq \Sigma^0_3)$, then the set of potential members of $\hat{A}$ is $\Sigma^0_3$. Hence, the property $NNC(\kappa, \lambda, \Sigma^0_3)$ is preserved by forcing to add Cohen reals (assuming cf $\lambda > \omega$).

The proof of Theorem 4.1 does not go through for arbitrary forcing notions with the countable chain condition. In fact, one can show that an arbitrary subset of $\omega^\kappa$ in the ground model can be expressed as the set of “potential members” of a closed subset of $\omega^\kappa$ in a c.c.c. forcing extension. (For $s \in \omega^\kappa$, let $s^*$ be the set of finite initial segments of $s$. Given $B \subseteq \omega^\kappa$, let $P$ be the poset of partial functions $p$ from $\omega^\kappa$ to $\{0, 1\}$ such that the domain of $p$ is the union of a finite set and finitely many sets $s^*$ for $s \in B$, and $p(\sigma) = 0$ for only finitely many $\sigma$. If $G : \omega^\kappa \rightarrow \{0, 1\}$ is the resulting generic function and $\hat{A}$ is a name for the closed set $\{ s \in \omega^\kappa : (\forall n) G(s|n) = 1 \}$, then $B$ is the set of potential members of $\hat{A}$.) So it is still open whether $NNC(\kappa, \lambda, \text{Borel})$ is always preserved by c.c.c. forcing.

5. $U$-MEASURABILITY

Throughout this section and the next, the letter $U$ will denote an ultrafilter, usually over the cardinal $\kappa$. We recall several definitions pertaining to ultrafilters: given cardinals $\kappa$, $\lambda$, and $\mu$, an ultrafilter $U$ over $\kappa$ is uniform iff every member of $U$ has cardinality $\kappa$; $U$ is $\lambda$-complete iff the intersection of any collection of fewer than $\lambda$ members of $U$ is a member of $U$; $U$ is $\lambda$-indecomposable iff every set of cardinality $\lambda$ whose union is in $U$ has a subset of cardinality less than $\lambda$ whose union is in $U$; $U$ is $(\lambda, \mu)$-regular iff there is a collection $\{ Y_\alpha : \alpha < \lambda \}$ of elements of $U$ such that, for any $S \subseteq \lambda$ of cardinality $\mu$, $\bigcap_{\alpha \in S} Y_\alpha = \emptyset$; and $U$ is $(\lambda, \mu)$-nonregular iff $U$ is not $(\lambda, \mu)$-regular.

As in the preceding section, given $X \subseteq \omega^\kappa$ and $\sigma < \omega^\kappa$, define $X[\sigma]$ to be $\{ s \in \omega^\kappa : \sigma \cap s \in X \}$. Again recall that any closed subset of $\omega^\kappa$ can be expressed in the form $[T]$, the set of infinite branches through some tree $T \subseteq \omega^\kappa$. 

Definition 5.1. Let $U$ be an ultrafilter over $\kappa$.

(a) A tree $T \subseteq <\omega \kappa$ is $U$-branching iff $\langle \rangle \in T$ and, for any $\sigma \in T$, $\{\alpha \in \kappa: \sigma \cap \langle \alpha \rangle \in T\} \in U$.

(b) A set $X \subseteq \omega \kappa$ is $U$-large ($U$-small) iff there is a $U$-branching tree $T$ such that $[T] \subseteq X$ and $([T] \cap X = \varnothing$).

(c) A set $X \subseteq \omega \kappa$ is $U$-determined iff $X$ is either $U$-large or $U$-small.

(d) A set $X \subseteq \omega \kappa$ is $U$-null iff, for each $\sigma \in <\omega \kappa$, $X|\sigma$ is $U$-small.

(e) A set $X \subseteq \omega \kappa$ is $U$-measurable iff, for each $\sigma \in <\omega \kappa$, $X|\sigma$ is $U$-determined.

The remainder of this section is devoted to results on $U$-null and $U$-measurable sets; many of these results are analogous to facts about the standard notion of measurability for subsets of, say, the Cantor space, or the real line. In the next section we will use these results to obtain further information about the property $NNC$.

Louveau [11] gives definitions equivalent to these, for the case $\kappa = \omega$, and uses them to give an alternate proof of the theorem of Silver [19] that all analytic subsets of $\omega \omega$ are Ramsey. Much of the rest of this section appears in another form in Louveau’s paper. (Carlson and Galvin have also done unpublished work along these lines.)

Proposition 5.2. The intersection of two $U$-branching trees is $U$-branching. Hence, the $U$-large sets form a filter over $\omega \kappa$, and the $U$-small sets form the dual ideal.

Proof. Easy. ■

Lemma 5.3. If $X_n \subseteq \omega \kappa$ for $n < \omega$ and, for each $n < \omega$ and $\sigma \in n \kappa$, $X_n|\sigma$ is $U$-small, then $\bigcup_{n<\omega} X_n$ is $U$-small.

Proof. Let $X = \bigcup_{n<\omega} X_n$. For each $n \in \omega$ and each $\sigma \in n \kappa$, choose a $U$-branching tree $T'_\sigma$ such that $(X_n|\sigma) \cap [T'_\sigma] = \varnothing$. Let $T_n = <n \kappa \cup \{\sigma \cap \tau: \sigma \in n \kappa, \tau \in T'_\sigma\}$; then $T_n$ is a $U$-branching tree, $m \kappa \subseteq T_n$ for $m \leq n$, and $[T_n] \cap X_n = \varnothing$. Let $T = \bigcap_{n<\omega} T_n$; then $[T] \cap X = \varnothing$. Clearly $\langle \rangle \in T$, and for any $\sigma \in T$, if $\sigma \in n \kappa$, then

$$\{\alpha \in \kappa: \sigma \cap \langle \alpha \rangle \in T\} = \{\alpha \in \kappa: (\forall m \in \omega) (\sigma \cap \langle \alpha \rangle \in T_m)\}
= \{\alpha \in \kappa: (\forall m \leq n) (\sigma \cap \langle \alpha \rangle \in T_m)\}$$

(since $n+1 \kappa \subseteq T_m$ for $m > n$)

$$= \bigcap_{m \leq n} \{\alpha \in \kappa: \sigma \cap \langle \alpha \rangle \in T_m\},$$

and since each tree $T_m$ is $U$-branching, each set $\{\alpha \in \kappa: \sigma \cap \langle \alpha \rangle \in T_m\}$ is in $U$, so $\{\alpha \in \kappa: \sigma \cap \langle \alpha \rangle \in T\} \in U$. Therefore, $T$ is $U$-branching, so $X$ is $U$-small. ■

Theorem 5.4. The $U$-null sets form a $\sigma$-ideal.

Proof. Clearly any subset of a $U$-null set is $U$-null. Now suppose that we have $U$-null sets $X_n$, $n \in \omega$, and let $X = \bigcup_{n<\omega} X_n$; we must see that $X$ is $U$-null. For each $\sigma \in <\omega \kappa$, we have $X|\sigma = \bigcup_{n<\omega} (X_n|\sigma)$. If $n \in \omega$ and $\tau \in n \kappa$, then $(X_n|\sigma)|\tau = X_n|\sigma \cap \tau$ is $U$-small by hypothesis; therefore, by Lemma 5.3, $X|\sigma$ is $U$-small. Since $\sigma$ was arbitrary, $X$ is $U$-null. ■
Theorem 5.5. Every open subset of $\omega_\kappa$ is $U$-measurable.

Proof. Let $X \subseteq \omega_\kappa$ be open; then $X|\sigma$ is open for each $\sigma \in \omega_\kappa$, so it will suffice to show that $X$ is $U$-determined. Let $S = \{\sigma \in \omega_\kappa: X|\sigma$ is $U$-large$\}$. If $() \in S$, we are done, so assume $(\not\in S$. If $\sigma \in \omega_\kappa$ and $A = \{\alpha \in \kappa: \sigma^\uparrow(\alpha) \in S\} \subseteq U$, then choose a $U$-branching tree $T_\alpha$ for each $\alpha \in A$ such that $[T_\alpha] \subseteq X|\sigma^\uparrow(\alpha)$, and let

$$T' = \{(\not\in\}\cup \{\langle\alpha\rangle^{\uparrow}\tau: \alpha \in A, \tau \in T_\alpha\};$$

it is easy to see that $T'$ is a $U$-branching tree and $[T'] \subseteq X|\sigma$, so $\sigma \in S$. Hence, for any $\sigma \in \omega_\kappa \setminus S$, $\{\alpha \in \kappa: \sigma^\uparrow(\alpha) \not\in S\} \subseteq U$. Now let

$$T = \{\sigma \in \omega_\kappa: (\forall m \leq \ell(\sigma)) \sigma|m \not\in S\};$$

since $\sigma \in T$ and $\sigma^\uparrow(\alpha) \not\in S$ imply $\sigma^\uparrow(\alpha) \in T$, $T$ is a $U$-branching tree. If $s \in [T]$, then $s|n \not\in S$ for all $n \in \omega$, so $X|s|n \neq \omega_\kappa$ for all $n \in \omega$; since $X$ is open, this implies $s \not\in X$. Therefore, $[T] \cap X = \emptyset$, so $X$ is $U$-small.

Theorem 5.6. The $U$-measurable sets form a $\sigma$-algebra of subsets of $\omega_\kappa$.

Proof. Clearly the complement of a $U$-measurable subset of $\omega_\kappa$ is $U$-measurable. Now suppose we have $U$-measurable sets $X_n$, $n \in \omega$; we must see that $X = \bigcup_{n<\omega} X_n$ is $U$-measurable. Again it will suffice to show that $X$ is $U$-determined, since the same will apply to $X|\sigma = \bigcup_{n<\omega} (X_n|\sigma)$ for any $\sigma \in \omega_\kappa$. Let

$$Y = \{s \in \omega_\kappa: (\exists n,m \in \omega) X_n|(s|m) \text{ is } U\text{-large}\}.$$ 

Then $Y$ is open, so by Theorem 5.5 there is a $U$-branching tree $T$ such that either $[T] \subseteq Y$ or $[T] \cap Y = \emptyset$.

Suppose $[T] \subseteq Y$, and let

$$S = \{\sigma \in T: (\exists n \in \omega) (X_n|\sigma \text{ is } U\text{-large}) \text{ and } (\forall n \in \omega) (\forall m \leq \ell(\sigma)) (X_n|\sigma|m \text{ is } U\text{-small})\};$$

then $S$ is an antichain in $T$, and since $[T] \subseteq Y$, for each $s \in [T]$ there is $m \in \omega$ such that $s|m \in S$. For each $\sigma \in S$, choose a $U$-branching tree $T_\sigma$ such that $[T_\sigma] \subseteq X_n|\sigma$ for some $n$. Let

$$T' = \{\sigma^\uparrow\tau: \sigma \in S, \tau \in T_\sigma\} \cup \{\sigma|m: \sigma \in S, m < \ell(\sigma)\}.$$ 

Clearly $\{\sigma|m: \sigma \in S, m < \ell(\sigma)$ is a subtree of $T$ which does not meet $S$; since every infinite branch through $T$ meets $S$, $\{\sigma|m: \sigma \in S, m < \ell(\sigma)$ has no infinite branches. Therefore, for any $s \in [T']$, there are $\sigma \in S$ and $b \in [T_\sigma]$ such that $s = \sigma^\uparrow b$; since $[T_\sigma] \subseteq X|\sigma$ for each $\sigma \in S$, $[T'] \subseteq X$. Now, if $\sigma \in S$ and $\tau \in T_\sigma$, then

$$\{\alpha \in \kappa: \sigma^\uparrow\tau^\uparrow(\alpha) \in T'\} = \{\alpha \in \kappa: \tau^\uparrow(\alpha) \in T_\tau\} \subseteq U;$$

and if $\sigma \in S$ and $m < \ell(\sigma)$, then

$$\{\alpha \in \kappa: (\sigma|m)^\uparrow(\alpha) \in T'\} = \{\alpha \in \kappa: (\sigma|m)^\uparrow(\alpha) \in T\} \subseteq U.$$
(To see this, let $\tau = (\sigma|m) \cap (\alpha)$. Since $S$ is an antichain, no proper initial segment of $\sigma$ is in $S$, so no proper initial segment of $\tau$ is in $S$. This means that, if $\tau \in T'$, then $\tau$ must be a member of $S$ or an initial segment of one, so $\tau \in T$. On the other hand, if $\tau \in T$, then $\tau$ can be extended to some $s \in [T]$, and some initial segment of $s$ must be in $S$, so $\tau$ must be a member of $S$ or an initial segment of one, so $\tau \in T'$.) Therefore, $T'$ is $U$-branching, so $X$ is $U$-large.

Now suppose $[T] \cap Y = \emptyset$. Let $X'_n = X_n \cap [T]$. If $\sigma \in T$, then $X_n|\sigma$ is $U$-small by definition of $Y$, so $X'_n|\sigma$ is $U$-small; if $\sigma \in \kappa^\omega \setminus T$, then $X'_n|\sigma = \emptyset$. Therefore, $X'_n$ is $U$-null, so by Theorem 5.6 $X \cap [T] = \bigcup_{n<\omega} X'_n$ is $U$-null. By Proposition 5.2, $X \subseteq (X \cap [T]) \cup (\kappa^\omega \setminus [T])$ is $U$-small.

**Corollary 5.7.** Every Borel subset of $\kappa^\omega$ is $U$-measurable.

**Lemma 5.8.** For every set $X \subseteq \kappa^\omega$ there is an $F_\sigma$ set $Z \subseteq X$ such that, for each $\sigma \in \kappa^\omega$, if $X|\sigma$ is $U$-large, then $Z|\sigma$ is $U$-large.

**Proof.** For each $\sigma$ such that $X|\sigma$ is $U$-large, choose a $U$-branching tree $T_\sigma$ such that $[T_\sigma] \subseteq X|\sigma$; if $X|\sigma$ is not $U$-large, let $T_\sigma = \emptyset$. Now let $Z = \{\sigma \cap s : \sigma \in \kappa^\omega, s \in [T_\sigma]\}$. Clearly $Z \subseteq X$ and, for all $\sigma$, if $X|\sigma$ is $U$-large, then $Z|\sigma$ is $U$-large. And since the sets $Z_n = \{\sigma \cap s : \sigma \in n^\kappa, s \in [T_\sigma]\}$ for $n \in \omega$ are closed, and $Z = \bigcup_{n<\omega} Z_n$, $Z$ is $F_\sigma$.

**Theorem 5.9.** A set $X \subseteq \kappa^\omega$ is $U$-measurable iff there are sets $Z, Y$ such that $Z \subseteq X \subseteq Y$, $Z$ is $F_\sigma$, $Y$ is $G_\delta$, and $Y \setminus Z$ is $U$-null.

**Proof.** If $Z$ and $Y$ are as above, then $X = Z \cup (X \setminus Z)$, $Z$ is Borel and hence $U$-measurable, and $X \setminus Z$ is $U$-null, so $X$ is $U$-measurable. Conversely, if $X$ is $U$-measurable, then we can find $F_\sigma$ sets $Z \subseteq X$, $Z' \subseteq \kappa^\omega \setminus X$ as in Lemma 5.8. Let $Y = \kappa^\omega \setminus Z'$. For each $\sigma \in \kappa^\omega$, either $X|\sigma$ or $(\kappa^\omega \setminus X)|\sigma$ is $U$-large, so either $Z|\sigma$ or $Z'|\sigma$ is $U$-large, so $(Y \setminus Z)|\sigma$ is $U$-small. Therefore, $Y \setminus Z$ is $U$-null.

**Theorem 5.10.** The collection of $U$-measurable subsets of $\kappa^\omega$ is closed under Suslin’s operation $\mathcal{A}$.

**Proof.** The collection of $U$-null sets is a $\sigma$-ideal over $\kappa^\omega$. The collection of $U$-measurable sets is a $\sigma$-algebra. For every set $X \subseteq \kappa^\omega$, there is a $U$-measurable set $Y \supseteq X$ such that any $U$-measurable subset of $Y \setminus X$ is $U$-null. (Let $Y$ be the complement of the set $Z \subseteq \kappa^\omega \setminus X$ obtained by applying Lemma 5.8 to $\kappa^\omega \setminus X$.) By Theorem 2H.1 of Moschovakis [13], these statements imply the desired result. (Theorem 2H.1 is stated only for certain spaces $\mathcal{X}$, but the proof of the relevant part applies to any set $\mathcal{X}$.)

So, in the case $\kappa = \omega$, we see that all analytic and coanalytic sets, and many others, are $U$-measurable. This does not, however, necessarily extend to all $\Delta^1_2$ (or even $\Delta^1_1$) subsets of $\omega^\omega$ (unless $U$ is principal, in which case every subset of $\omega^\omega$ is $U$-measurable). If $U$ is nonprincipal, then clearly $[T]$ is a perfect set for any $U$-branching tree $T$. Therefore, any subset of $\omega^\omega$ which is a Bernstein set (a set such that neither it nor its complement has a perfect subset; such sets can be constructed using a well-ordering of $\omega^\omega$) cannot be $U$-determined. Well-known results in descriptive set theory [8, §41] show that, in the constructible universe, one can construct a $\Delta^1_2$ Bernstein set.
Louveau’s proof that all analytic sets are Ramsey is completed by the following result.

**Proposition 5.11.** If $U$ is a nonprincipal ultrafilter over $\omega$, and $X \subseteq \omega$ is $U$-determined, then there is an infinite set $H \subseteq \omega$ such that either all strictly increasing $\omega$-sequences from $H$ are in $X$ or all such sequences are in the complement of $X$.

**Proof.** Let $T$ be a $U$-branching tree such that $[T] \subseteq X$ or $[T] \cap X = \emptyset$. It will suffice to construct an infinite set $H$ such that all strictly increasing sequences from $H$ are in $[T]$. To do this, we will recursively choose natural numbers $h_0 < h_1 < h_2 < \ldots$ such that every finite subsequence of $\langle h_0, h_1, h_2, \ldots \rangle$ is in $T$.

Suppose we have $h_i$ for $i < n$. For each subsequence $\sigma$ of $\langle h_0, h_1, \ldots, h_{n-1} \rangle$, since $\sigma \in T$ and $T$ is $U$-branching, the set of $k$ such that $\sigma \cap \langle k \rangle \in T$ is in $U$. There are $2^n$ such subsequences $\sigma$; the intersection of the $2^n$ corresponding sets in $U$ is still in $U$. Therefore, we can choose $h_n$ to be any member of this intersection which (if $n > 0$) is above $h_{n-1}$; then every subsequence of $\langle h_0, h_1, \ldots, h_n \rangle$ will be in $T$, as desired. ■

The strong analogy between $U$-measurability and ordinary measurability suggests the following question: is there a $\sigma$-additive probability measure $m$ (on some $\sigma$-algebra of subsets of $\omega$) such that all $U$-measurable sets are $m$-measurable? The answer is yes if $U$ is $\aleph_1$-complete, because we can define such an $m$ by letting $m(X) = 1$ for all $U$-large sets $X$ and $m(X) = 0$ for all $U$-small sets $X$. On the other hand, if $U$ is not $\aleph_1$-complete, then such an $m$ cannot exist unless its completion is a measure on all subsets of $\omega$; this follows from the following proposition.

**Proposition 5.12.** Let $U$ be an $\aleph_1$-incomplete ultrafilter over $\kappa$. Suppose that $m$ is a ($\sigma$-additive) probability measure on a $\sigma$-algebra of subsets of $\omega$ which includes all clopen subsets of $\omega$. Then there is a $U$-null set $X \subseteq \omega$ such that $m(X) = 1$.

**Proof.** By Theorem 5.4, it suffices to prove that, for any $\varepsilon > 0$, there is a $U$-null set $X$ such that $m(X) \geq 1 - \varepsilon$. Let $\{ S_n : n < \omega \}$ be a collection of sets not in $U$ such that $\bigcup_{n < \omega} S_n = \kappa$. We will define sets $X_n \subseteq \omega$ with $m(X_n) > 1 - \varepsilon$ by recursion on $n$. Let $X_0 = \omega$. Given $X_n$ such that $m(X_n) > 1 - \varepsilon$, let $Y_i = X_n \cap \{ s : s(n) \in S_{<i} \}$ for $i < \omega$. Then $\langle Y_i : i < \omega \rangle$ is an increasing sequence of $m$-measurable sets and $\bigcup_{i < \omega} Y_i = X_n$, so there must be an $i < \omega$ such that $m(Y_i) > 1 - \varepsilon$; let $X_{n+1}$ be $Y_i$ for the least such $i$. This completes the definition of $\langle X_n : n < \omega \rangle$; it is clear that this is a decreasing sequence of sets such that $m(X_n) > 1 - \varepsilon$ for all $n$, but $X_n \cap \sigma$ is $U$-small for all $\sigma \in \mathcal{C}_n$. Therefore, if we let $X = \bigcap_{n < \omega} X_n$, then $X$ will be a $U$-null set such that $m(X) \geq 1 - \varepsilon$, as desired. ■

6. $(\lambda, \mu; \alpha)$-NONREGULARITY

In this section we will apply the results of the previous section to obtain new information about the property $NNC$. In particular, we will define a property of ultrafilters $U$ over $\kappa$ which implies $NNC(\kappa, \mu, U$-measurable), and then give several cases in which this property is satisfied. The definition of this property, $(\lambda, \mu; \omega)$-nonregularity, will be given in somewhat more generality than necessary, in order to show its relation to the usual definition of $(\lambda, \mu)$-nonregularity. We start by generalizing Definition 5.1(a).
Definition 6.1. A tree $T \subseteq \leq \alpha \kappa$ is a closed $U$-branching tree of height $\alpha$ iff:

(a) for every successor $\beta + 1 < \alpha$ and every $s \in T \cap \beta \kappa$, $\{\eta \in \kappa : s \upharpoonright \eta \in T\} \in U$;

(b) for every non-successor $\beta < \alpha$ and every $s \in \beta \kappa$, $s \in T$ iff $s|\gamma \in T$ for all $\gamma < \beta$.

Note that a tree $T \subseteq \leq \omega \kappa$ is a $U$-branching tree under Definition 5.1(a) iff it is a closed $U$-branching tree of height $\omega$.

Definition 6.2. If $\kappa$, $\lambda$, and $\mu$ are cardinals and $\alpha$ is an ordinal, then an ultrafilter $U$ over $\kappa$ is $(\lambda, \mu; \alpha)$-regular iff there is a family $\{T_\beta : \beta \in \lambda\}$ of closed $U$-branching trees of height $\alpha + 1$ such that no subfamily $\{T_\beta : \beta \in S\}$ with $S \subseteq \lambda$ of cardinality $\mu$ has a common maximal element, i.e. a sequence $s \in \alpha \kappa$ such that $s \in T_\beta$ for all $\beta \in S$. The ultrafilter $U$ is $(\lambda, \mu; \alpha)$-nonregular iff it is not $(\lambda, \mu; \alpha)$-regular.

We start with some easy but useful results.

Proposition 6.3.

(a) If $U$ is a $(\lambda, \mu; \alpha)$-regular ultrafilter, $\lambda' \leq \lambda$, $\mu' \geq \mu$, and $\alpha' \geq \alpha$, then $U$ is $(\lambda', \mu'; \alpha')$-regular.

(b) If $U$ is $(\lambda, \lambda'; \alpha)$-nonregular and $(\lambda', \mu; \alpha')$-nonregular, then $U$ is $(\lambda, \mu; \alpha+\alpha')$-nonregular.

(c) If $U$ is $\mu^+$-complete, then the intersection of $\mu$ closed $U$-branching trees of the same height is a closed $U$-branching tree of that height.

(d) An ultrafilter $U$ is $(\lambda, \mu; \omega)$-regular iff there is a family $\{X_\beta : \beta < \lambda\}$ of $U$-large sets such that any subfamily $\{X_\beta : \beta \in S\}$ with $S \subseteq \lambda$ of cardinality $\mu$ has empty intersection.

Proof. Suppose $U$ is an ultrafilter over $\kappa$. For (a), let $\{T_\beta : \beta < \lambda\}$ be a witness to the $(\lambda, \mu; \alpha)$-regularity of $U$, and let

$$T'_\beta = T_\beta \cup \{s \in \leq \alpha \kappa : \ell(s) > \alpha \text{ and } s|\alpha \in T_\beta\};$$

then $\{T'_\beta : \beta < \lambda'\}$ witnesses the $(\lambda', \mu'; \alpha')$-regularity of $U$. For (b), let $\{T_\beta : \beta < \lambda\}$ be a collection of closed $U$-branching trees of height $\alpha + \alpha' + 1$. Then $\{T_\beta \cap \leq \alpha \kappa : \beta < \lambda\}$ is a collection of closed $U$-branching trees of height $\alpha + 1$, so there is a set $S' \subseteq \lambda$ of cardinality $\lambda'$ such that $\{T_\beta \cap \leq \alpha \kappa : \beta < \lambda\}$ has a common maximal element $s$. Let $T'_\beta = \{t : s^\upharpoonright \beta \in T_\beta\}$ for $\beta \in S'$; $\{T'_\beta : \beta \in S'\}$ is a collection of closed $U$-branching trees of height $\alpha' + 1$, and $S'$ has cardinality $\lambda'$, so there is a set $S \subseteq S'$ of cardinality $\mu$ such that $\{T'_\beta : \beta \in S\}$ has a common maximal element $s'$. Then $s^\upharpoonright s'$ is a common maximal branch of $\{T_\beta : \beta \in S\}$.

For part (c), note that if $s$ is in the intersection and is not maximal in the original trees, then the set of $\beta$ such that $s^\upharpoonright \beta$ is in the intersection is the intersection of $\mu$ members of $U$. Finally, for part (d), note that $F(T) = T \cup [T]$ and $F^{-1}(T) = T \cap \leq \omega \kappa$ define a one-to-one correspondence between the closed $U$-branching trees of height $\omega$ and the closed $U$-branching trees of height $\omega + 1$.

Trivially, any ultrafilter is $(\lambda, \mu; \alpha)$-regular if $\mu > \lambda$. On the other hand, if $\mu$ is finite, then every ultrafilter $U$ is $\mu^+$-complete; hence, by Proposition 6.3(c), $U$ is $(\lambda, \mu; \alpha)$-nonregular for any $\lambda \geq \mu$ and any $\alpha$. ■
The next result gives the motivation for the term ‘\((\lambda, \mu; \alpha)\)-regular,’ and shows that \((\lambda, \mu; \alpha)\)-nonregularity gives a family of properties between \((\lambda, \mu)\)-nonregularity and \(\mu^+\)-completeness.

**Proposition 6.4.** Let \(U\) be an ultrafilter.

(a) If \(\lambda \geq \mu\), then \(U\) is \((\lambda, \mu; 0)\)-nonregular.

(b) \(U\) is \((\lambda, \mu; 1)\)-regular iff \(U\) is \((\lambda, \mu)\)-regular.

(c) If \(|\alpha| \geq \lambda \geq \mu \geq \aleph_0\), then \(U\) is \((\lambda, \mu; \alpha)\)-nonregular iff \(U\) is \(\mu^+\)-complete.

**Proof.** Parts (a) and (b) are easy. For part (c), first suppose that \(U\) is \(\mu^+\)-complete; then Proposition 6.3(c) easily implies that \(U\) is \((\lambda, \mu; \alpha)\)-nonregular. To prove the other direction of part (c), we first need a lemma.

**Lemma 6.5.** For any infinite cardinals \(\lambda \geq \mu\) such that \(\mu\) is regular, there is a function \(F: \lambda \times \lambda \to \mu\) such that, for any \(S \subseteq \lambda\) of cardinality \(\mu\), there is \(\alpha \in \lambda\) such that \(\{F(\alpha, \beta): \beta \in S\}\) has cardinality \(\mu\).

**Proof.** We prove this for all ordinals \(\lambda \geq \mu\), by induction on \(\lambda\). If \(\lambda = \mu\), we simply let \(F(\alpha, \beta) = \beta\). If \(\lambda\) is not a cardinal, then \(\lambda > |\lambda| \geq \mu\), so let \(f: \lambda \to |\lambda|\) be a bijection and let \(F': |\lambda| \times |\lambda| \to \mu\) be obtained from the induction hypothesis; then the function \(F: \lambda \times \lambda \to \mu\) defined by \(F(\alpha, \beta) = F'(f(\alpha), f(\beta))\) has the required properties. Now suppose that \(\lambda\) is a cardinal greater than \(\mu\). Let \((\lambda_\alpha: \alpha < cf \lambda)\) be a strictly increasing sequence of ordinals with limit \(\lambda\), such that \(\lambda_0 \geq \mu\). For each \(\alpha < cf \lambda\), obtain a function \(F_\alpha: \lambda_\alpha \times \lambda_\alpha \to \mu\) from the induction hypothesis. Let \(\delta_\alpha = 1 + \sum_{\beta < \alpha} \lambda_\beta\) for \(\alpha < cf \lambda\); since \(\lambda\) is a cardinal, it is easy to see that \(\delta_\alpha < \lambda\). Now define a function \(F: \lambda \times \lambda \to \mu\) by:

\[
\begin{align*}
F(\delta_\alpha + \gamma, \beta) &= F_\alpha(\gamma, \beta) & \text{if } \gamma, \beta < \lambda_\alpha; \\
F(0, \delta_\alpha + \gamma) &= \alpha & \text{if } cf \lambda = \mu \text{ and } \gamma < \lambda_\alpha; \\
F(\gamma, \beta) &= 0 & \text{for all other } (\gamma, \beta).
\end{align*}
\]

To see that this works, let \(S\) be any subset of \(\lambda\) of cardinality \(\mu\). If there is \(\beta < \lambda\) such that \(|S \cap \beta| = \mu\), then there is \(\alpha < cf \lambda\) such that \(|S \cap \lambda_\alpha| = \mu\); if we choose \(\gamma < \lambda_\alpha\) such that \(|\{F_\alpha(\gamma, \beta): \beta \in S \cap \lambda_\alpha\}| = \mu\), then we will have \(|\{F(\delta_\alpha + \gamma, \beta): \beta \in S\}| = \mu\). Now suppose \(|S \cap \beta| < \mu\) for all \(\beta < \lambda\). This clearly implies cf \(\lambda = cf \mu\), so since \(\mu\) is regular we get \(|\{F(0, \beta): \beta \in S\}| = \mu\). This completes the induction. \(\blacksquare\)

To finish the proof of Proposition 6.4, suppose that \(|\alpha| \geq \lambda \geq \mu \geq \aleph_0\) and \(U\) is an ultrafilter over \(\kappa\) which is not \(\mu^+\)-complete; we must see that \(U\) is \((\lambda, \mu; \alpha)\)-regular. Let \(\mu'\) be the least cardinal such that \(U\) is not \(\mu'^+\)-complete; then \(\mu \geq \mu' \geq \aleph_0\), so by Proposition 6.3(a) we may assume that \(\mu = \mu'\). It is well-known [8, §27, p. 299] that \(\mu'\) must be either \(\aleph_0\) or a measurable cardinal, so \(\mu\) is regular. Let \(\{W_\beta: \beta < \mu\}\) be a family of sets in \(U\) which has empty intersection, and let \(Z_\beta = \bigcap_{\gamma < \beta} W_\gamma\). By our assumption, \(Z_\beta \in U\) for all \(\beta < \mu\), but, for any \(S \subseteq \mu\) of cardinality \(\mu\), \(\bigcap\{Z_\beta: \beta \in S\} = \emptyset\). Choose \(F: \lambda \times \lambda \to \mu\) satisfying the conclusion of Lemma 6.5. Now, for each \(\beta < \lambda\), define a closed \(U\)-branching
tree $T_\beta$ of height $\alpha + 1$ as follows: for any $s \in \leq^\alpha \kappa$, $s \in T_\beta$ iff, for each $\gamma < \min(\ell(s), \lambda)$, $s(\gamma) \in Z_{F(\gamma, \beta)}$. Let $S$ be any subset of $\lambda$ of cardinality $\mu$; we must see that $\{T_\beta : \beta \in S\}$ has no common maximal element. The choice of $F$ guarantees that there is a $\gamma < \lambda$ such that $\{F(\gamma, \beta) : \beta \in S\}$ has cardinality $\mu$, and hence $\bigcap\{Z_{F(\gamma, \beta)} : \beta \in S\} = \emptyset$. But if $s$ were a common maximal branch of $\{T_\beta : \beta \in S\}$, we would have $s(\gamma) \in Z_{F(\gamma, \beta)}$ for each $\beta \in S$, which is impossible. Therefore, $\{T_\beta : \beta \in S\}$ has no common maximal element; since $S$ was arbitrary, $U$ is $\langle \lambda, \mu; \alpha \rangle$-regular.

We now give the reason for studying $\langle \lambda, \mu; \omega \rangle$-nonregularity here.

**Proposition 6.6.** Let $U$ be an ultrafilter over $\kappa$ such that every set in $U$ has cardinality at least $\lambda$. If $U$ is $\langle \lambda, \mu; \omega \rangle$-nonregular, then $\text{NNC}(\kappa, \mu, U$-measurable). If $U$ is $\langle \lambda, \mu'; \omega \rangle$-nonregular for all $\mu' < \mu$, then $\text{NNC}(\kappa, <\mu, U$-measurable).

**Proof.** We prove the second implication; the proof of the first is the same (or one can easily deduce the first from the second). Let $U$ be $\langle \lambda, \mu'; \omega \rangle$-nonregular for all $\mu' < \mu$. Suppose $\{A_n : n < \omega\}$ is a family of $U$-measurable sets with union $\omega;\kappa$; we must show that there is an $n$ such that, for all $\mu' < \mu$, $A_n$ is not $\mu'$-narrow in the $n$'th coordinate. By Lemma 5.3, there exist $n < \omega$ and $\sigma \in {\kappa}_n$ such that $A_n|\sigma$ is not $U$-small, and hence is $U$-large. Let $T$ be a $U$-branching tree (of height $\omega$) such that $[T] \subseteq A_n|\sigma$. The set $\{\gamma : \langle \gamma \rangle \in T\}$ is in $U$, so there exist distinct $\gamma_\beta$, $\beta < \lambda$, such that $\langle \gamma_\beta \rangle \in T$ for all $\beta$. For each $\beta < \lambda$, let $T_\beta = \{\tau : \langle \gamma_\beta \rangle \cap [T] \subseteq A_n|\sigma\}$. Then $T_\beta$ is a $U$-branching tree, so $T_\beta \cup [T_\beta]$ is a closed $U$-branching tree of height $\omega + 1$. For any $\mu' < \mu$, since $U$ is $\langle \lambda, \mu'; \omega \rangle$-nonregular, there is $S \subseteq \lambda$ of cardinality $\mu'$ such that $\{T_\beta \cup [T_\beta] : \beta \in S\}$ has a common maximal element $z$. For each $\beta \in S$ we have $z \in [T_\beta]$, so $\langle z \rangle$ is of cardinality $\omega_\alpha\kappa$, $z \subseteq A_n|\sigma$, so $\sigma \cap \langle z \rangle$ is not $\mu'$-narrow in the $n$'th coordinate. Since $\mu'$ was arbitrary, we are done.

This proposition, together with Propositions 6.4(c) and 6.3(a), immediately gives:

**Theorem 6.7.** If $U$ is a non-principal $\lambda$-complete ultrafilter over $\kappa$, then $\text{NNC}(\kappa, <\lambda, U$-measurable).

**Corollary 6.8.** If $\kappa$ is $\aleph_0$ or a measurable cardinal, then $\text{NNC}(\kappa, <\kappa, \text{Borel})$.

Corollary 6.8 for measurable cardinals also follows from Theorem 3.1, but Theorem 6.7 gives more information for this case. In fact, if $U$ is a $\kappa$-complete ultrafilter over $\kappa$, then Proposition 6.3(c) easily implies that the collection of $U$-measurable sets is closed under unions and intersections of fewer than $\kappa$ sets. Now, any Boolean combination of certain sets can be written as a union of intersections of these sets and their complements; since $\kappa$ is a strong limit cardinal, we see that any Boolean combination of fewer than $\kappa$ open subsets of $\omega;\kappa$ is $U$-measurable. So, for the measurable cardinal case, the conclusion of Theorem 6.7 subsumes that of Theorem 3.1. In fact, the conclusion of Theorem 6.7 is strictly stronger:

**Proposition 6.9.** If $U$ is a $\kappa$-complete ultrafilter over the measurable cardinal $\kappa$, then there are $U$-measurable subsets of $\omega;\kappa$ which cannot be expressed as Boolean combinations of fewer than $\kappa$ open sets.
Proof. First we show that, for any \( \lambda < \kappa \), there is a subset of \( \omega \kappa \) which is a Boolean combination of fewer than \( \kappa \) open sets but not a Boolean combination of \( \lambda \) open sets. To see this, let \( \mu \) be a strong limit cardinal of cofinality \( \omega \) such that \( \lambda < \mu < \kappa \); then \( 2^\mu = \mu^{\aleph_0} \) and \( 2^\lambda < \mu \). This implies that, for any sequence \( \langle X_\alpha : \alpha < \lambda \rangle \) of subsets of \( \omega \mu \) and any \( X \subseteq \omega \mu \) of cardinality \( \mu^{\aleph_0} \), there are distinct \( x \) and \( y \) in \( X \) such that \( \{ \alpha < \lambda : x \in X_\alpha \} = \{ \alpha < \lambda : y \in X_\alpha \} \). Let \( \delta = \mu^{\aleph_0} \). Then the number of \( \lambda \)-sequences of open subsets of \( \omega \mu \) is \( (2^\mu)^\lambda = 2^\mu = \mu^{\aleph_0} = \delta \), so we can enumerate all such sequences in a sequence of length \( \delta \). Now an easy recursive construction gives one-to-one sequences \( \langle x_\beta : \beta < \delta \rangle \) and \( \langle y_\beta : \beta < \delta \rangle \) of elements of \( \omega \mu \) such that \( \{ x_\beta : \beta < \delta \} \cap \{ y_\beta : \beta < \delta \} = \emptyset \) and, for any sequence \( \langle X_\alpha : \alpha < \lambda \rangle \) of open subsets of \( \omega \mu \), there is \( \beta < \delta \) such that \( \{ \alpha < \lambda : x_\beta \in X_\alpha \} = \{ \alpha < \lambda : y_\beta \in X_\alpha \} \). So \( \{ x_\beta : \beta < \delta \} \) is not a Boolean combination of \( \lambda \) open subsets of \( \omega \mu \); since the intersection of \( \omega \mu \) with an open subset of \( \omega \kappa \) is an open subset of \( \omega \mu \), \( \{ x_\beta : \beta < \delta \} \) is not a Boolean combination of open subsets of \( \omega \kappa \). But any one-element subset of \( \omega \kappa \) is an intersection of \( \aleph_0 \) clopen subsets of \( \omega \kappa \), so \( \{ x_\beta : \beta < \delta \} \) is a Boolean combination of \( \delta < \kappa \) open subsets of \( \omega \kappa \).

This easily implies that the collection of Boolean combinations of fewer than \( \kappa \) open subsets of \( \omega \kappa \) is not closed under the \( \kappa \)-ary operation which takes \( \langle X_\alpha : \alpha < \kappa \rangle \) to \( \langle (\alpha) \cap s : s \in X_\alpha \rangle \), while the collection of \( U \)-measurable sets is easily seen to be closed under this operation, so the latter collection contains sets not in the former collection, as was to be shown.

(Another way to see that not every \( U \)-measurable set is such a Boolean combination is to construct a \( U \)-measurable set which is not \( V \)-measurable for some other nonprincipal \( \kappa \)-complete ultrafilter \( V \) over \( \kappa \). To do this, choose \( S \in V \setminus U \), and note that \( Z = \{ s \in \omega \kappa : (\exists m < \omega)(\forall n > m) s(n) \in S \} \) is \( U \)-null but its complement is \( V \)-null. By Theorems 6.7 and 2.5, there is a set \( Y \subseteq \omega \kappa \) which is not \( V \)-measurable; then \( Y \cap Z \) is \( U \)-measurable but not \( V \)-measurable.)

In the case \( \kappa = \aleph_0 \), Theorem 6.7 again says more than Corollary 6.8. For one thing, there are \( 2^{\aleph_0} \) \( U \)-measurable sets but only \( 2^{\aleph_0} \) Borel sets. Also, recall the remarks after the proof of Proposition 5.10. We now have \( NNC(\aleph_0, <\aleph_0, \text{analytic}) \) and more. But we saw that not all \( \Delta^1_2 \) sets are \( U \)-measurable; in fact, the proof of Theorem 2.5, done carefully using a \( \Sigma^1_2 \)-good well-ordering of \( \omega \) (see Moschovakis [13, §5A]), shows that \( NNC(\aleph_0, 2, \Delta^1_2) \) fails in the constructible universe.

(Since this proof of \( NNC(\aleph_0, <\aleph_0, \text{analytic}) \) uses a nonprincipal ultrafilter over \( \omega \), it would appear to need more of the Axiom of Choice than most proofs of similar results in descriptive set theory. However, one can modify the proof so that it only needs a weaker form of Choice, such as the Axiom of Dependent Choices. This is done by proving versions of the results in Section 5 using the concept of \( F \)-measurability where \( F \) is a filter rather than an ultrafilter, and \( F \) is enlarged as necessary so as to make the relevant sets measurable. For instance, the modified version of Corollary 5.7 states that, for any Borel set \( B \subseteq \omega \kappa \) and any filter \( F \) over \( \kappa \), there is a filter \( F' \supseteq F \) such that \( B \) is \( F' \)-measurable.)

To apply Proposition 6.6, we need to find ultrafilters which are \( (\lambda, \mu; \omega) \)-nonregular. The remainder of this section will give cases in which such ultrafilters can (or cannot) be
found.

**Proposition 6.10.** If \( \lambda \) is an infinite cardinal and \( U \) is an ultrafilter over \( \kappa \) which is \((\lambda, \lambda; \alpha_n)\)-nonregular for each \( n < \omega \), then \( U \) is \((\lambda, \aleph_0; \sum_{n<\omega} \alpha_n)\)-nonregular.

**Proof.** Let \( \{T_{\beta} : \beta \in \lambda \} \) be a collection of closed \( U \)-branching trees of height \((\sum_{n<\omega} \alpha_n)+1\). We will recursively construct \( S_n, \gamma_n, s_n, \) and \( \{T_{\beta}^{(n)} : \beta \in S_n \} \) for \( n < \omega \) so that \( \langle \gamma_n : n \in \omega \rangle \) is a sequence of distinct elements of \( \lambda \) and \( \{T_{\gamma_n} : n < \omega \} \) has a common maximal element; this suffices to show that \( U \) is \((\lambda, \aleph_0; \sum_{n<\omega} \alpha_n)\)-nonregular.

Let \( S_0 = \lambda, s_0 = \langle \rangle \), and \( T_{\beta}^{(0)} = T_{\beta} \) for \( \beta \in \lambda \). Now suppose we are given \( S_n \subseteq \lambda \) of cardinality \( \lambda \), a sequence \( s_n \), and a collection \( \{T_{\beta}^{(n)} : \beta \in S_n \} \) of closed \( U \)-branching trees of height \((\sum_{n \leq m < \omega} \alpha_m)+1\). Let \( \gamma_n \) be the least member of \( S_n \). Proposition 6.3(c) for \( \mu = 2 \) implies that, for each \( \beta \in S_n \), \( T_{\beta}^{(n)} \cap T_{\gamma_n}^{(n)} \) is a closed \( U \)-branching tree of height \((\sum_{n \leq m < \omega} \alpha_m)+1\), and therefore \( T_{\beta}^{(n)} \cap T_{\gamma_n}^{(n)} \cap \{T_{\kappa}^{(n)} : \kappa \leq \alpha_n \} \) is a closed \( U \)-branching tree of height \( \alpha_n + 1 \). Since \( S_n \setminus \{\gamma_n\} \) has cardinality \( \lambda \) and \( U \) is \((\lambda, \lambda; \alpha_n)\)-nonregular, we can find a set \( S_{n+1} \subseteq S_n \setminus \{\gamma_n\} \) of cardinality \( \lambda \) such that \( \{T_{\beta}^{(n)} \cap T_{\gamma_n}^{(n)} \cap \{T_{\kappa}^{(n)} : \kappa \leq \alpha_n \} : \beta \in S_{n+1}\} \) has a common maximal element \( s \). For each \( \beta \in S_{n+1} \), let \( T_{\beta}^{(n+1)} = \{t : s \cap t \in T_{\beta}^{(n)} \cap T_{\gamma_n}^{(n)}\} \); then \( T_{\beta}^{(n+1)} \) is a closed \( U \)-branching tree of height \((\sum_{n < m < \omega} \alpha_m)+1\). Let \( s_{n+1} = s_n \cap s \). This completes the recursive definition.

Clearly \( s_m \subseteq s_n \) for \( m < n < \omega \); also, \( \gamma_m \in S_m \) but \( \gamma_n \notin S_m \) for \( n > m \), so \( \gamma_m \neq \gamma_n \) for \( m < n < \omega \). It is easy to show by induction on \( m \) that

\[
T_{\beta}^{(m)} = \{t : s_m \cap t \in T_{\beta} \cap \bigcap_{n < m} T_{\gamma_n}\}
\]

for each \( \beta \in S_m \); it follows immediately that \( s_m \in T_{\gamma_n} \) for all \( m, n \in \omega \). Therefore, if we let \( s = \bigcup_{m < \omega} s_m \), then \( s \) will be a common maximal element of \( \{T_{\gamma_n} : n < \omega\} \), as desired. \( \Box \)

**Corollary 6.11.** If \( \lambda \) is infinite, then any \((\lambda, \lambda)\)-nonregular ultrafilter is \((\lambda, \aleph_0; \omega)\)-nonregular. \( \Box \)

We will see later that \((\lambda, \aleph_1; \omega)\)-nonregularity does not follow from \((\lambda, \lambda)\)-nonregularity. As to the problem of finding \((\lambda, \lambda)\)-nonregular ultrafilters, Silver has shown [8, Ex. 34.4, p. 426] that, if \( \lambda \) is regular, any \( \lambda \)-saturated \( \lambda^+ \)-complete ideal \( I \) over \( \kappa \) has the property that any collection of \( \lambda \) subsets of \( \kappa \) not in \( I \) has a subcollection of size \( \lambda \) with nonempty intersection; this property clearly implies that any ultrafilter over \( \kappa \) disjoint from \( I \) (i.e., extending the filter dual to \( I \)) is \((\lambda, \lambda)\)-nonregular. It follows that if \( \kappa \) is a cardinal carrying a nonprincipal \( \lambda \)-saturated \( \lambda^+ \)-complete ideal, then \( NNC(\kappa, \aleph_0, \text{Borel}) \); in particular, if there is a real-valued measurable cardinal, then \( NNC(2^{\aleph_0}, \aleph_0, \text{Borel}) \).

To get \( NNC(\kappa, \lambda, \text{Borel}) \) for uncountable \( \lambda \) by this method, we need ultrafilters with stronger properties.
Proposition 6.12. Suppose $\kappa$, $\lambda$, and $\mu$ are cardinals, and $U$ is an ultrafilter over $\kappa$ with the following property: for any $S \subseteq U$ of cardinality at most $\lambda$, there is a set $X$ of cardinality at most $\mu$ such that $X \cap A \neq \emptyset$ for all $A \in S$. Then, for any $\alpha$ such that $\mu^{[\alpha]} < \lambda$, $U$ is $(\lambda, \lambda'; \alpha)$-nonregular for all $\lambda' < \lambda$; if $\mu^{[\alpha]} < \text{cf} \lambda$, then $U$ is $(\lambda, \lambda; \alpha)$-nonregular.

Proof. We may assume $\lambda$ is infinite, since otherwise any ultrafilter is $(\lambda, \lambda; \alpha)$-nonregular. Fix $\alpha$, and let $\{T_{\beta} : \beta < \lambda\}$ be a collection of closed $U$-branching trees of height $\alpha + 1$. For each $\beta < \lambda$ we will define a maximal branch $s_{\beta}$ of $T_{\beta}$. The definition will be by simultaneous recursion on the length of the sequences. So suppose $\gamma < \alpha$ and we have defined $s_{\beta}|\gamma \in T_{\beta}$ for each $\beta < \lambda$. Fix $s \in \gamma\kappa$, and let $S = \{\beta < \lambda : s_{\beta}|\gamma = s\}$. Then $\{\{\delta : s^\delta \cap \delta \in T_{\beta}\} : \beta \in S\}$ is a collection of at most $\lambda$ members of $U$, so there is a set $X \subseteq \kappa$ of cardinality at most $\mu$ which has nonempty intersection with each member of this collection. For each $\beta \in S$, define $s_{\beta}(\gamma)$ to be the least member of $X \cap \{\delta : s^\delta \cap \delta \in T_{\beta}\}$. Do this for all $s \in \gamma\kappa$ to define $s_{\beta}(\gamma)$ for all $\beta < \lambda$. This completes the recursion.

Clearly $s_{\beta}$ is a maximal element of $T_{\beta}$ for each $\beta < \lambda$. It is clear from the definition of the sequences $s_{\beta}$ that, for any $s$, there are at most $\mu$ $\delta$’s such that $s^\delta \cap \delta$ is an initial segment of some $s_{\beta}$. It follows easily that, for each $\gamma \leq \alpha$, $|s_{\beta}|\gamma : \beta < \lambda| \leq \mu^{[\gamma]}$. In particular, $|s_{\beta} : \beta < \lambda| \leq \mu^{[\alpha]}$. Now, if $\mu^{[\alpha]} < \lambda$ and $\lambda' < \lambda$, then there must be an $s$ such that $|\{\beta < \lambda : s_{\beta} = s\}| > \lambda'$, since otherwise we would have expressed $\lambda$ as the union of at most $\mu^{[\alpha]}$ sets each of cardinality at most $\lambda'$, which is impossible. Similarly, if $\mu^{[\alpha]} < \text{cf} \lambda$, then there must be an $s$ such that $|\{\beta < \lambda : s_{\beta} = s\}| = \lambda$. Since $\{T_{\beta} : \beta < \lambda\}$ was arbitrary, we are done.

Proposition 6.13. If $U$ is an ultrafilter over $\kappa$ which is $\nu$-indecomposable for all $\nu$ such that $\mu < \nu \leq 2^\lambda$, then $U$ has the property in the hypothesis of Proposition 6.12.

Proof. Suppose $S \subseteq U$ and $|S| \leq \lambda$. Define an equivalence relation $\sim$ on $\kappa$ as follows: for any $\beta, \gamma < \kappa$, $\beta \sim \gamma$ iff, for all $A \in S$, we have $\beta \in A \iff \gamma \in A$. Clearly there are at most $2^\lambda \sim$-equivalence classes, and the union of these classes is $\kappa$, so the indecomposability of $U$ implies that there is a set $Z$ of at most $\mu$ $\sim$-equivalence classes such that $\bigcup Z \in U$. Let $X \subseteq \bigcup Z$ be a set which contains exactly one member of each set in $Z$, then $|X| \leq \mu$. If $A \in S$, then $A \in U$, so $A \cap \bigcup Z \neq \emptyset$. If $y \in A \cap \bigcup Z$, then there is $x \in X$ such that $x \sim y$; since $y \in A$, $x \in A$, so $X \cap A \neq \emptyset$. Since $A$ was arbitrary, $X$ is the desired set.

These two propositions show that $\text{NNC}(\kappa, \lambda, \text{Borel})$ follows from the existence of sufficiently indecomposable ultrafilters over $\kappa$; in particular, if $\kappa$ is a strong limit cardinal carrying a uniform ultrafilter $U$ which is $\nu$-indecomposable for all sufficiently large $\nu < \kappa$, then $\text{NNC}(\kappa, <\kappa, U$-measurable) (and hence $\text{NNC}(\kappa, <\kappa, \text{Borel})$).

One application of these propositions is to show that certain cardinals $\kappa$ which satisfy $\text{NNC}(\kappa, \lambda, \text{Borel})$ for all $\lambda < \kappa$ for trivial reasons (they are limits of smaller cardinals with this property) actually satisfy the stronger statement $\text{NNC}(\kappa, <\kappa, \text{Borel})$ less trivially.

Corollary 6.14. If $\kappa$ is a limit of an $\omega$-sequence of measurable cardinals, or if $\kappa$ is the cardinal obtained by adjoining a Prikry sequence through a measurable cardinal, then $\text{NNC}(\kappa, <\kappa, \text{Borel})$. 
On the other hand, the hypothesis of Proposition 6.15 cannot hold if there is a real-valued ultrafilter over \( \kappa \). Proposition 6.16. The cardinality hypotheses of Propositions 6.12 and 6.13 prevent us from applying them to get new results about cardinals \( \kappa \leq 2^{\aleph_0} \). The next two propositions (6.16 in particular) will show that even the assumption that \( \kappa \) is real-valued measurable is not strong enough to get a uniform \( (\kappa, \aleph_1; \omega) \)-nonregular ultrafilter over \( \kappa \).

**Proposition 6.15.** If \( \lambda \) and \( \mu \) are cardinals and there is a set \( S \subseteq \omega_\omega \) of cardinality \( \lambda \) such that, for any compact \( C \subseteq \omega_\omega \), \( |S \cap C| \leq \mu \), then any \( \aleph_1 \)-incomplete ultrafilter is \( (\lambda, \mu; \omega) \)-regular.

**Proof.** Fix such an \( S \), say \( S = \{ x_\beta : \beta < \lambda \} \) with \( x_\beta \neq x_\gamma \) for \( \beta \neq \gamma \), and fix an \( \aleph_1 \)-incomplete ultrafilter \( U \) over \( \kappa \). Let \( (Y_n : n < \omega) \) be a sequence of sets in \( U \) such that \( \bigcap_{n<\omega} Y_n \notin U \); we may assume that \( \bigcap_{n<\omega} Y_n = \emptyset \) and that \( Y_n \supseteq Y_{n+1} \) for \( n < \omega \). Define a sequence \( (T_\beta : \beta < \lambda) \) of closed \( U \)-branching trees of height \( \omega + 1 \) as follows: for any \( \beta < \lambda \) and any \( s \in \omega_\kappa \), put \( s \in T_\beta \) iff, for each \( n < \ell(s) \), \( s(n) \in Y_{x_\beta(n)} \). To see that the trees \( T_\beta \) have the required properties, let \( s \) be any element of \( \omega_\kappa \), and let \( Z = \{ \beta < \lambda : s \notin T_\beta \} \); we must see that \( |Z| < \mu \). Define \( t \in \omega_\omega \) so that \( t(n) \) is the least \( m \) such that \( s(n) \notin Y_m \); since \( (Y_m : m < \omega) \) is a decreasing sequence, it is easy to see that \( x_\beta(n) \neq t(n) \) for all \( n < \omega \) and \( \beta \in Z \). Let \( C = \{ x \in \omega_\omega : (\forall n<\omega) x(n) < t(n) \} \); then \( C \) is a compact set, so \( |Z| = |\{ x_\beta : \beta \in Z \} \| \leq |S \cap C| < \mu \), as desired. \[\blacksquare\]

Prikry [16] (see also Jech [8, pp. 425-426]) has shown that, in the model obtained by adding \( \lambda \) Cohen-generic reals to a model containing a measurable cardinal \( \kappa \), \( \kappa \) carries a \( \kappa \)-complete \( \aleph_1 \)-saturated ideal. However, he has also shown that this model satisfies the hypothesis of Proposition 6.15 with \( \mu = \aleph_1 \); therefore, the existence of a \( \kappa \)-complete \( \aleph_1 \)-saturated ideal over \( \kappa \) does not imply the existence of a \( (\kappa, \aleph_1; \omega) \)-nonregular ultrafilter. On the other hand, the hypothesis of Proposition 6.15 cannot hold if there is a real-valued measurable cardinal \( \kappa \) such that \( \lambda \geq \kappa \geq \mu \geq \aleph_1 \); this limitation does not apply to the following proposition.

**Proposition 6.16.** If \( \kappa \), \( \lambda \), and \( \mu \) are cardinals and there is a sequence \( (A_\beta : \beta < \lambda) \) such that, for any infinite \( S \subseteq \kappa \), \( |\{ \beta < \lambda : S \subseteq A_\beta \text{ or } S \cap A_\beta = \emptyset \}| < \mu \), then any nonprincipal ultrafilter over \( \kappa \) is \( (\lambda, \mu; \omega) \)-regular.

**Proof.** Assume the hypothesis, and let \( U \) be a nonprincipal ultrafilter over \( \kappa \). Define closed \( U \)-branching trees \( T_\beta \) of height \( \omega + 1 \) for \( \beta < \lambda \) as follows: for any \( \beta < \lambda \) and any \( s \in \omega_\kappa \), put \( s \in T_\beta \) iff \( s \) is one-to-one and \( s(n) \in B_\beta \) for each \( n < \ell(s) \), where \( B_\beta \) is that one of
\[ \kappa \cap A_\beta \text{ and } \kappa \setminus A_\beta \text{ which is in } U. \] To see that these trees have the required property, let \( s \) be any element of \( \omega \kappa \), and let \( Z = \{ \beta < \lambda : s \in T_\beta \} \); we must see that \( |Z| < \mu \). We may assume that \( s \) is one-to-one, since otherwise \( Z = \emptyset \). Let \( S = \{ s(n) : n < \omega \} \); then \( S \) is infinite and, for each \( \beta \in Z \), \( S \subseteq B_\beta \), so \( S \subseteq A_\beta \) or \( S \cap A_\beta = \emptyset \). Hence, \( |Z| < \mu \), as desired.

If \( M \) is a model obtained by adding a sequence \( \langle x_\beta : \beta < \lambda \rangle \) of Cohen-generic members of \( \omega^2 \) (or \( \omega^\omega \)) to some ground model, then \( M \) satisfies the hypothesis of Proposition 6.16 for \( \aleph_0 \leq \kappa \leq \lambda \) and \( \mu = \aleph_1 \). To define the sequence \( \langle A_\beta : \beta < \lambda \rangle \), let \( f : \lambda \times \lambda \to \lambda \) be a bijection which is in the ground model, and let

\[ A_\beta = \{ \gamma < \lambda : f(\beta, \gamma)(0) = 1 \}. \]

To see that this works, let \( S \) be an infinite subset of \( \lambda \) in \( M \); since the desired property for \( S \) follows from that property for some infinite subset of \( S \), we may assume that \( S \) is countable. Since the forcing notion has the countable chain condition, there is a countable set \( S' \) in the ground model such that \( S \subseteq S' \subseteq \lambda \). For each \( \alpha \in S' \), choose a maximal antichain (in the ground model) of conditions which decide whether \( \alpha \in S \); each of these antichains is countable. Hence, if \( W \) is the set of \( \beta < \lambda \) such that some element of one of these antichains gives some information about \( x_\beta \), then \( W \) is countable, and so is \( Z = \{ \beta < \lambda : (\exists \gamma < \lambda) f(\beta, \gamma) \in W \} \). An easy genericity argument shows that, for any \( \beta \in \lambda \setminus Z \), \( S \cap A_\beta \neq \emptyset \) and \( S \setminus A_\beta \neq \emptyset \); since \( Z \) is countable, we are done.

This gives another proof that \( \kappa \) can carry a \( \kappa \)-complete \( \aleph_1 \)-saturated ideal without carrying a \( (\kappa, \aleph_1; \omega) \)-nonregular ultrafilter. In this case, however, an analogous proof works to give a real-valued measurable cardinal \( \kappa \) carrying no \( (\kappa, \aleph_1; \omega) \)-nonregular ultrafilter. The model \( M \) for this case is obtained by forcing to add \( \lambda \) random reals; specifically we define this forcing notion using as conditions the subsets of \( \lambda^2 \) of positive measure in the symmetric product measure on \( \lambda^2 \). Let \( G \) be the generic set, and find \( f \) in the ground model as in the preceding paragraph; then let

\[ A_\beta = \{ \gamma < \lambda : (s(f(\beta, \gamma)) = 1) \in G \}. \]

The proof that this works is the same as before, once we note that each measurable set has countable support. But any measurable cardinal in the ground model is real-valued measurable in \( M \).

The preceding two propositions and the associated remarks do not preclude the existence of a cardinal \( \kappa \leq \aleph_0 \) carrying a \( (\kappa, \aleph_1; \omega) \)-nonregular ultrafilter. In particular, the hypotheses of both propositions contradict Martin’s Axiom (MA), given some mild hypotheses (namely \( \mu \leq \lambda \leq \aleph_0 \), \( \mu < \aleph_0 \), \( \mu \leq \omega \), and \( \kappa \geq \aleph_0 \)). For 6.15 we recall that MA implies that for any set \( S \subseteq \omega_1 \) of cardinality \( \mu \) there is \( g \in \omega_1 \) such that, for each \( f \in S \), \( f(n) \leq g(n) \) for all sufficiently large \( n \) [8, p. 261]. Hence, \( S \) is contained in the union of \( \aleph_0 \) compact sets, namely \( \{ f \in \omega_1 : (\forall n < \omega) f(n) \leq h(n) \} \) for all \( h \in \omega_1 \) such that \( h(n) = g(n) \) for all sufficiently large \( n \), so one of these compact sets must contain \( \mu \) members of \( S \). For 6.16, we use the following argument, which Baumgartner and Hajnal
\cite[p. 196]{3} attribute to Solovay. Let \( \langle A_\beta : \beta < \lambda \rangle \) be arbitrary. Let \( U \) be a nonprincipal ultrafilter over \( \omega \); for each \( \beta < \mu \), let \( B_\beta \) be that one of \( \omega \cap A_\beta \) and \( \omega \setminus A_\beta \) which is in \( U \). Define a forcing notion \( P \) as follows: a condition is a pair \( (a, b) \) where \( a \subseteq \omega \) and \( b \subseteq \mu \) are finite; \((c, d)\) is stronger than \((a, b)\) iff \( a \subseteq c \), \( b \subseteq d \), and, for each \( n \in c \setminus a \) and each \( \beta \in b \), \( n \in B_\beta \). Since any two conditions \((a, b),(c, d)\) with \( a = c \) are compatible, \( P \) has the countable chain condition. Using the definition of \( B_\beta \), we easily see that the sets \( D_n = \{(a, b) \in P : |a| \geq n\} \) and \( E_\beta = \{(a, b) \in P : \beta \in b\} \) are dense in \( P \) (for \( n < \omega \), \( \beta < \mu \)). Now apply MA to get a filter \( G \) on \( P \) which meets each of these dense sets. Let \( S = \bigcup\{(a, b) : (a, b) \in G\} \); then \( S \) is infinite and, for each \( \beta < \mu \), \( S \setminus B_\beta \) is finite. Since \( S \) has only countably many finite subsets, there must be a finite \( z \subseteq S \) such that \(|\{\beta < \mu : S \setminus z \subseteq B_\beta\}| = \mu \), and hence \(|\{\beta < \lambda : S \setminus z \subseteq A_\beta \text{ or } (S \setminus z) \cap A_\beta = \emptyset\}| \geq \mu \).

But starting with a model containing a cardinal \( \kappa \) carrying a \( \kappa \)-complete \( \aleph_1 \)-saturated ideal, one can obtain a model of \( \text{MA} + \text{“}2^{\aleph_0} \text{ is large} \) by a c.c.c. forcing extension \cite[§23]{8}, and \( \kappa \) will still carry a \( \kappa \)-complete \( \aleph_1 \)-saturated ideal in the extension \cite[Ex. 34.5, p. 426]{8}. It is quite possible that this model, or a model obtained by some more specialized c.c.c. forcing notion over a model with a measurable cardinal, will contain a cardinal \( \kappa \) carrying a nontrivial \((\kappa, \aleph_1; \omega)\)-nonregular ultrafilter. Another possibility is that \( \text{NNC}(\kappa, \aleph_1, \text{Borel}) \) will follow from the real-valued measurability of \( \kappa \) by a different argument. (By the random-real case of Theorem 4.1, we know that \( \text{NNC}(\kappa, \aleph_1, \text{Borel}) \) is at least relatively consistent with the real-valued measurability of \( \kappa \).)

7. The Complexity of Narrow Clopen Partitions

In this section, we consider a slightly different question. Let \( \kappa \) and \( \lambda \) be infinite cardinals. Suppose that there does exist a \( \lambda \)-narrow covering of \( \omega \kappa \) by open sets. Must such a covering be complicated?

Of course, we cannot ask this without a suitable measure of complexity of open coverings of \( \omega \kappa \). We can get such a measure by considering trees associated with the coverings.

For any finite sequence \( \sigma \in \omega \kappa \), let \( N_\sigma \) be the basic open subset of \( \omega \kappa \) consisting of those infinite sequences that begin with \( \sigma \). Now, given an open covering of \( \omega \kappa \), let \( T \) be the set of all \( \sigma \in \omega \kappa \) such that \( N_\sigma \) is not a subset of any member of the covering. Clearly \( T \) is a tree, since \( N_\tau \subseteq N_\sigma \) whenever \( \sigma \subseteq \tau \). Furthermore, \(|T|\) is empty: any \( s \in \omega \kappa \) is in some member \( A \) of the covering, and \( A \) is open, so some basic neighborhood \( N_\sigma \) of \( s \) is included in \( A \), which gives \( \sigma \subseteq s \) and \( \sigma \notin T \), so \( s \notin [T] \).

We recall some basic definitions in order to fix notation. A tree \( T \subseteq \omega \kappa \) is well-founded if \(|T| = \emptyset \). For any well-founded tree, we define a rank function \( \text{rk}_T \) mapping \( T \) to the ordinals by well-founded recursion as follows: if \( \sigma \in T \), then \( \text{rk}_T(\sigma) \) is the least ordinal greater than \( \text{rk}_T(\sigma^\frown \langle \beta \rangle) \) for all \( \beta \) such that \( \sigma^\frown \langle \beta \rangle \in T \). For \( \sigma \notin T \) we put \( \text{rk}_T(\sigma) = -1 \). Define \( \text{rk}(T) \), the rank of the well-founded tree \( T \), to be \( 1 + \text{rk}_T(\langle \rangle) \).

Now we can define the complexity (or rank) of an open covering of \( \omega \kappa \) to be the rank of the associated well-founded tree. This will be an ordinal less than \( \kappa^+ \).

This may be slightly clearer when the open covering is actually a partition of \( \omega \kappa \) into open sets. In this case the sets are necessarily clopen, since the complement of one set is
the union of the others. And the tree $T$ can be defined to be the set of all $\sigma$ such that $N_\sigma$ meets more than one set in the partition.

By a standard argument, any narrow covering by open sets can be reduced to a narrow partition:

**Proposition 7.1.** If there exists a $\lambda$-narrow covering of $\omega^\kappa$ using open sets, then there exists a $\lambda$-narrow partition of $\omega^\kappa$ using open (and hence clopen) sets.

**Proof.** Let $\langle A_n : n < \omega \rangle$ be such a covering. Since $A_n$ is open, we have $A_n = \bigcup_{m<\omega} B_{nm}$, where

$$B_{nm} = \bigcup \{ N_\sigma : \sigma \in m^\kappa, N_\sigma \subseteq A_n \}.$$

The sets $B_{nm}$ are clopen; in fact, membership of $s \in \omega^\kappa$ in $B_{nm}$ depends only on $s|m$. Hence, the sets

$$B'_{nm} = B_{nm} \setminus \left( \bigcup_{n'<n} B_{n'm} \cup \bigcup_{m'<m} B_{n'm'} \right)$$

are also clopen. The sets $B'_{nm}$ are disjoint, and we have $B'_{nm} \subseteq B_{nm}$ and $\bigcup_{n,m<\omega} B'_{nm} = \bigcup_{n,m<\omega} B_{nm} = \omega^\kappa$. Therefore, the open sets $A'_n = \bigcup_{m<\omega} B'_{nm}$ form a partition of $\omega^\kappa$; since $A'_n \subseteq A_n$, $A'_n$ is $\lambda$-narrow, as desired. $\blacksquare$

It is easy to see that the tree associated with the clopen partition constructed above is the same as the tree associated with the original open covering, so the reduction process does not change the complexity of the covering. Hence, we may restrict ourselves to clopen partitions when trying to find the minimum complexity of a $\lambda$-narrow open covering of $\omega^\kappa$. (However, often it will be just as convenient to work with the open coverings.)

This notion of complexity, for the case of individual clopen subsets of the Baire space $\omega^\omega$, is called the Kalmar rank; see Barnes [2].

If $n$ is finite, then a clopen partition of $\omega^\kappa$ has rank at most $n$ if and only if, for every $s \in \omega^\kappa$, the piece of the partition that contains $s$ is determined by $s|n$. Such a partition is essentially a partition of the finite-dimensional product $^n\kappa$. Also, if a set in such a partition contains a point $s$, it must contain all points on the line through $s$ parallel to the $j$’th coordinate axis, for any $j \geq n$. Therefore, if the clopen sets $A_j$ for $j < \omega$ form a $\lambda$-narrow partition of $\omega^\kappa$ of rank at most $n$, where $\lambda \leq \kappa$, then necessarily $A_j = \emptyset$ for $j \geq n$, and the sets $A_j$ for $j < n$ are determined by a $\lambda$-narrow partition of $^n\kappa$. Hence, Theorem 2.3 tells us when such partitions exist:

**Proposition 7.2.** For any natural number $n > 0$, ordinal $\alpha$, and cardinal $\kappa$, there exists an $\aleph_\alpha$-narrow clopen partition of $\omega^\kappa$ of complexity at most $n$ if and only if $\kappa < \aleph_\alpha + n - 1$. $\blacksquare$

Similarly, one can translate Proposition 2.4 into a statement about finite-rank narrow clopen partitions of $\omega^\kappa$ for finite $\kappa$.

The next case to consider is $\kappa = \aleph_\omega$. Here Proposition 7.2 tells us that, for any $\lambda < \aleph_\omega$, a $\lambda$-narrow clopen partition of $\omega^\kappa$, if it exists, must have rank at least $\omega$. However, we will see that the rank must actually be much higher than this.
Such a partition might not exist at all; see Theorem 3.5. On the other hand, there are models in which such partitions do exist; for instance, Corollary 3.4(e) (along with Proposition 7.1) tells us that, if $V = L$, then an $\aleph_1$-narrow clopen partition of $\omega\kappa$ exists. So, in such a model, one can try to find the least possible complexity of such a partition.

It is convenient to reformulate this question in terms of free subsets of algebras, as in Theorem 3.3. Given a structure $M$, one can form the tree $T_M$ of all finite sequences $\sigma$ of members of $M$ which are free for $M$ (i.e., $\sigma$ is one-to-one and the range of $\sigma$ is a free subset of $M$). If $M$ has no infinite free subset, then there can be no infinite branch through $T_M$, so $T_M$ is well-founded, and one can compute its rank.

**Proposition 7.3.** Let $\kappa$ and $\mu$ be infinite cardinals, with $\kappa > \mu$, such that $\text{NNC}(\kappa, \mu^+, \text{open})$ does not hold. Then the least possible complexity of a $\mu^+$-narrow covering of $\omega\kappa$ by open sets is equal to the least possible rank of the tree $T_M$ of finite free sequences for an algebra $M$ of size $\kappa$ with $\mu$ operations and no infinite free subset.

**Proof.** Given a $\mu^+$-narrow covering $\langle A_n : n < \omega \rangle$ of $\omega\kappa$ by open sets, let $T$ be the associated tree. Define a structure $M$ from the covering as in the second part of the proof of Theorem 3.3. If $\sigma \in {}^{<\omega}\kappa$ is not in $T$, then $N_\sigma \subseteq A_m$ for some $m$. Let $n = \ell(\sigma)$. Since $\kappa \geq \mu^+$, $N_\sigma$ is not $\mu^+$-narrow in the $j$'th coordinate for $j \geq n$, so we must have $m < n$. Therefore, if $\sigma'$ is $\sigma$ with the $m$'th coordinate deleted, then $\sigma(m) = f_{\alpha\beta}(\sigma')$ for some $\alpha < \mu$, so $\sigma \notin T_M$. This proves that $T_M \subseteq T$, so $\text{rk}(T_M) \leq \text{rk}(T)$.

Conversely, suppose we have a structure $M$ with universe $\kappa$ which has $\mu$ operations and no infinite free subset. Define a corresponding $\mu^+$-narrow open covering $\langle A_n : n < \omega \rangle$ of $\omega\kappa$ as in the first part of the proof of Theorem 3.3, and let $T$ be the associated tree. If $\sigma \in {}^{<\omega}\kappa$ is not in $T_M$, then, for some $m < \ell(\sigma)$, $\sigma(m)$ is generated in $M$ from $\{\sigma(j) : j \neq m\}$. This implies $N_\sigma \subseteq A_m$, so $\sigma \notin T$. Therefore, $T \subseteq T_m$, so $\text{rk}(T) \leq \text{rk}(T_M)$. □

So we can study trees associated with coverings or trees of finite free sequences, whichever is more convenient at the time.

We will now see that, when $\kappa \geq \aleph_\omega$, the trees above must have rank much higher than the finite ranks produced in Proposition 7.2.

**Theorem 7.4.** Let $\kappa$ be an uncountable limit cardinal. If $M$ is an algebra with universe $\kappa$ which has fewer than $\kappa$ operations and no infinite free subset, then $\text{rk}(T_M) \geq \kappa$.

**Proof.** First note that, if $T$ is a well-founded tree of rank $\alpha$, then there is a subtree $T' \subseteq T$ such that $\text{rk}(T') = \alpha$ and $|T'| = |\alpha|$. This is proved by induction on $\alpha$; it is trivial for $\alpha \leq 1$. Assume it is true for all $\beta < \alpha$, and let $T$ be a tree of rank $\alpha$, where $\alpha > 1$. If $\alpha = \beta + 1$, choose $c$ such that $\langle c \rangle \in T$ and $\text{rk}_T(\langle \rangle) = \text{rk}_T(\langle c \rangle) + 1$. Let $T_c = \{\sigma : \langle c \rangle \cap \sigma \subseteq T\}$. Then $\text{rk}(T_c) = \beta$, so we can apply the induction hypothesis to get $T'_c \subseteq T_c$ with $\text{rk}(T'_c) = \beta$ and $|T'_c| = |\beta|$. Let $T' = \{\sigma\} \cup \{\langle c \rangle \cap \sigma : \sigma \in T'_c\}$; then $T'$ is the desired subtree of $T$. If $\alpha$ is a limit ordinal, choose a set $C$ of size at most $|\alpha|$ such that $\{\text{rk}_T(\langle c \rangle) : c \in C\}$ is cofinal in $\alpha$. Apply the induction hypothesis to each $T_c$ to get $T'_c$ as above; then the tree $T' = \{\langle \rangle\} \cup \{\langle c \rangle \cap \sigma : c \in C, \sigma \in T'_c\}$ will be as desired.

We now prove the theorem by showing by induction on ordinals $\alpha < \kappa$ that, if $M$ is an algebra with universe $\kappa$ which has fewer than $\kappa$ operations and no infinite free subset, then
rk\(T_M\) > \(\alpha\). Suppose this is true for all \(\alpha' < \alpha\). Let \(M\) be such an algebra. Let \(\lambda\) be an uncountable regular cardinal less than \(\kappa\) but greater than \(\alpha\) and greater than the number of operations of \(M\). Let \(M'\) be \(M\) with an additional constant function \(c_\gamma\) with value \(\gamma\) for each \(\gamma < \lambda\). By the induction hypothesis, \(\text{rk}(T_{M'})\) is greater than \(\alpha'\) for all \(\alpha' < \alpha\), so \(\text{rk}(T_{M'}) \geq \alpha\). If \(\text{rk}(T_{M'}) > \alpha\), then \(\text{rk}(T_M) > \alpha\) as desired because \(T_{M'} \subseteq T_M\), so suppose \(\text{rk}(T_{M'}) = \alpha\). Let \(T'\) be a subtree of \(T_{M'}\) which has rank \(\alpha\) and cardinality \(|\alpha|\). Let \(S\) be the set of members of \(\kappa\) which are mentioned in \(T'\). Then \(|S| < \lambda\) and \(M\) has fewer than \(\lambda\) operations, so \(|H_M(S)| < \lambda\) (recall that \(H_M(S)\) is the subalgebra of \(M\) generated by \(S\)). Choose \(\gamma < \lambda\) which is not in \(H_M(S)\). Then, for every \(\sigma \in T'\), \(\langle \gamma \rangle^\sigma\) is free for \(M\). (By choice of \(\gamma\), \(\gamma\) is not generated by the members of \(\sigma\); and no member of \(\sigma\) is generated from \(\gamma\) and the other members of \(\sigma\) because \(\sigma\) is free for \(M'\).) Therefore, \(\text{rk}_{T_M}(\langle \gamma \rangle) \geq \text{rk}_{T'}(\langle \rangle)\), so \(\text{rk}(T_M) > \text{rk}(T') = \alpha\), as desired. This completes the induction. 

This argument for limit cardinals produces very little when applied to successor cardinals; in fact, the following proposition shows that the ranks obtained from algebras of successor cardinal size are only slightly higher than those obtained from the preceding limit cardinal.

**Proposition 7.5.** Let \(\mu\) and \(\kappa\) be infinite cardinals with \(\mu \leq \kappa\). If there is an algebra on \(\kappa\) with \(\mu\) operations and no infinite free subset, then there is such an algebra on \(\kappa^+\) as well. Furthermore, if \(\alpha_0\) is the least possible rank for the tree of finite free sequences for such an algebra on \(\kappa\), and \(\alpha_1\) is the corresponding least possible rank for \(\kappa^+\), then \(\alpha_0 + 1 \leq \alpha_1 \leq 2 \cdot \alpha_0 + 1\).

**Proof.** Let \(M_0\) be an algebra on \(\kappa\) with \(\mu\) operations and no infinite free subset, such that \(\text{rk}(T_{M_0}) = \alpha_0\). Also, for each ordinal \(\xi < \kappa^+\), let \(g_\xi\) be a bijection between \(\xi + 1\) and some ordinal \(\leq \kappa\). Let \(M_1\) be an algebra on \(\kappa^+\) with \(\mu\) operations which include: all of the operations of \(M_0\), extended in some arbitrary manner to operations on \(\kappa^+\); a binary operation \(G\) such that \(G(\xi, \eta) = g_\xi(\eta)\) whenever \(\eta \leq \xi\); and a binary operation \(G'\) such that \(G'(\xi, \eta) = g_\xi^{-1}(\eta)\) whenever \(\eta \in \text{range}(g_\xi)\). We will see that \(\text{rk}(T_{M_1}) \leq 2 \cdot \alpha_0 + 1\).

Given two ordinals \(\beta, \gamma < \kappa^+\), we can produce an ordinal \(\delta < \kappa\) by letting \(\delta = G(\max(\beta, \gamma), \min(\beta, \gamma))\). On the other hand, if we are given \(\delta\) and the larger of \(\beta\) and \(\gamma\), we can recover the other ordinal in the pair \(\{\beta, \gamma\}\), since \(\min(\beta, \gamma) = G'(\max(\beta, \gamma), \delta)\). Now, given a finite sequence \(\sigma \in <\omega \kappa\) of length \(n\), we can produce a finite sequence \(h(\sigma) \in <\omega \kappa\) of length \(n/2\) (rounded down) by applying the above procedure to the pairs \(\{\sigma(0), \sigma(1)\}, \{\sigma(2), \sigma(3)\}\), and so on.

If \(\sigma\) is such that \(h(\sigma) \notin T_{M_0}\), then there is \(k < \ell(h(\sigma))\) such that \(h(\sigma)(k)\) is obtainable from the other coordinates of \(h(\sigma)\) using the operations of \(M_0\). Let \(j\) be whichever of \(2k\) and \(2k + 1\) has the smaller coordinate of \(\sigma\). Then \(\sigma(j)\) is obtainable from the other coordinates of \(\sigma\) using the operations of \(M_1\): use \(G\) to obtain \(h(\sigma)(i)\) for \(i \neq k\), then use the operations of \(M_1\) extending those of \(M_0\) to obtain \(h(\sigma)(k)\), then apply \(G'\) to \(\sigma(4k + 1 - j)\) and \(h(\sigma)(k)\) to get \(\sigma(j)\). Therefore, \(\sigma \notin T_{M_1}\).

Now a straightforward induction on \(\text{rk}_{T_{M_0}}(h(\sigma))\) shows that, for any \(\sigma \in T_{M_1}\), if \(\ell(\sigma)\) is
odd, then \( \text{rk}_{M_0}(\sigma) \leq 2 \cdot \text{rk}_{M_0}(h(\sigma)) \), and if \( \ell(\sigma) \) is even, then \( \text{rk}_{M_0}(\sigma) \leq 2 \cdot \text{rk}_{M_0}(h(\sigma)) + 1 \). Therefore, \( \text{rk}(T_{M_1}) \leq 2 \cdot \text{rk}(T_{M_0}) + 1 \), as desired.

For the other direction, let \( M \) be an algebra on \( \kappa^+ \) with \( \mu \) operations and no infinite free subset such that \( \text{rk}(T_M) = \alpha_1 \). The subalgebra of \( M \) generated by the set \( \kappa \subseteq \kappa^+ \) has size \( \kappa \), so we can choose \( \gamma < \kappa^+ \) which is not in this subalgebra. Let \( M' \) be \( M \) with an additional constant operation with value \( \gamma \). Now let \( M'' \) be an algebra on \( \kappa \) with \( \mu \) operations such that, for each operation \( f \) on \( \kappa^+ \) which is a composition of operations of \( M' \), there is an operation \( \tilde{f} \) of \( M'' \) such that, for any \( \beta_0, \ldots, \beta_{n-1} < \kappa \), if \( f(\beta_0, \ldots, \beta_{n-1}) < \kappa \), then \( \tilde{f}(\beta_0, \ldots, \beta_{n-1}) = f(\beta_0, \ldots, \beta_{n-1}) \). Any free set for \( M'' \) will also be free for \( M \), so \( M'' \) has no infinite free subset. The tree \( T_{M''} \) must have rank at least \( \alpha_0 \). But for any \( \sigma \in T_{M''} \), \( \langle \gamma \rangle \cdot \sigma \) must be in \( T_M \) (as in the proof of the preceding proposition), so \( \text{rk}(T_M) > \text{rk}(T_{M''}) \), so \( \alpha_1 \geq \alpha_0 + 1 \).

Note that this multiplication on the left by 2 has no effect on the limit part of the ordinal \( \alpha_0 \). Hence, if \( \gamma \) is a limit ordinal, \( \mu \) is an infinite cardinal less than \( \aleph_\gamma \), and the least possible rank for the tree of finite free sequences for an algebra of size \( \aleph_\gamma \) with \( \mu \) operations is \( \delta + m \) where \( \delta \) is a limit ordinal and \( m \) is finite, then, for any finite \( n \), one can apply Proposition 7.5 \( n \) times to show that the least possible rank \( \alpha \) for the tree of finite free sequences for an algebra of size \( \aleph_{\gamma+n} \) with \( \mu \) operations must satisfy \( \delta + m + n \leq \alpha \leq \delta + (m+1)2^n - 1 \). One can instead use a direct argument, rather than an \( n \)-fold iteration, to reduce this upper bound to \( \delta + (m+1)(n+1) - 1 \). This will suffice to determine \( \alpha \) completely if \( m \) happens to be 0.

This shows that the main case of interest for the problem of free-sequence tree ranks, or for complexity of open narrow coverings, is the case of limit cardinals \( \kappa \). Here Theorem 7.4 gives a lower bound of \( \kappa \), but it is quite possible that this bound can be improved; the only obvious upper bound is \( \kappa^+ \) (assuming that a suitable algebra or narrow covering exists at all). In the rest of this section, we will see that, for the particular case where \( \kappa \) is an uncountable strong limit cardinal of cofinality \( \omega \), the lower bound can indeed be substantially improved.

For the rest of this section, we will make the following definitions and assumptions:

Let \( \kappa \) be an strong limit cardinal of cofinality \( \omega \). Assume that we have (not necessarily fixed) sequences \( \langle \kappa_n : n < \omega \rangle \) and \( \langle \lambda_n : n < \omega \rangle \) of infinite cardinals such that \( \kappa_n \leq \lambda_n \), \( \kappa_{n+1} = (2^{\lambda_n})^+ \), and \( \lim_{n \to \infty} \kappa_n = \kappa \). Also, in order to make \( \kappa_0 \) have the same properties as the other cardinals \( \kappa_n \), assume that we have infinite cardinals \( \kappa_{-1} \leq \lambda_{-1} \) such that \( \kappa_{-1} = (2^{\lambda_{-1}})^+ \).

For each \( n \), let \( P_n \) be the \( n \)-fold Cartesian product \( \prod_{i=0}^{n-1} \kappa_i \) (not the cardinal product, which would just be \( \kappa_{n-1} \)).

We will be using primarily the cardinals \( \kappa_n \); the separate cardinals \( \lambda_n \) are only needed in order to allow the sequence \( \langle \kappa_n : n < \omega \rangle \) to be cofinal in \( \kappa \) even when \( \kappa \) is a limit of strong limit cardinals. If \( \kappa = \aleph_\omega \), we can just let \( \lambda_n = \kappa_n \).

We will show that any narrow open covering of \( \omega \cdot \kappa \) must have high complexity by establishing two facts: the tree associated with a narrow open covering must meet all ‘large’ subproducts of the product sets \( P_n \), and a tree of small rank cannot meet all such
subproducts.

**Definition 7.6.** A finite sequence $\vec{Y} = \langle \vec{Y}(i) : i < n \rangle$ with $\vec{Y}(i) \subseteq \kappa_i$ for all $i < n$ is a large sequence if $|\vec{Y}(i)| = \kappa_i$ for all $i$.

If $T \subseteq \kappa$ is a tree and $\vec{Y}$ is a large sequence, then $T \upharpoonright \vec{Y}$ is the subtree of $T$ consisting of all $\sigma \in T$ such that $\sigma(i) \in \vec{Y}(i)$ for all $i < \min(\ell(\vec{Y}), \ell(\sigma))$. Also, we say that $T$ avoids $\vec{Y}$ if $T \cap \prod_{i < \ell(\vec{Y})} \vec{Y}(i) = \emptyset$.

If $\vec{Y}$ and $\vec{Z}$ are large sequences, then $\vec{Z} \preceq \vec{Y}$ means that $\ell(\vec{Z}) \geq \ell(\vec{Y})$ and $\vec{Z}(i) \subseteq \vec{Y}(i)$ for all $i < \ell(\vec{Y})$.

Easily, if $\vec{Z} \preceq \vec{Y}$ and the tree $T$ avoids $\vec{Y}$, then $T$ avoids $\vec{Z}$. Also, $T$ avoids $\vec{Y}$ if and only if $\text{rk}(T \upharpoonright \vec{Y}) \leq \ell(\vec{Y})$.

**Lemma 7.7.** Let $F$ be a function from $P_{n+1}$ to $S$, where $|S| < \kappa_n$. Then there is a set $Z \subseteq \kappa_n$ of size $\kappa_n$ such that $F(\sigma)$ depends only on $\sigma \upharpoonright n$ if $\sigma(n) \in Z$ (i.e., if $\sigma \upharpoonright n = \sigma' \upharpoonright n$ and $\sigma(n), \sigma'(n) \in Z$, then $F(\sigma) = F(\sigma')$). Furthermore, if $Y$ is a given subset of $\kappa_n$ of size $\kappa_n$, then $Z$ can be taken to be a subset of $Y$.

**Proof.** For each $\beta < \kappa_n$, define $f_\beta : P_n \to S$ by $f_\beta(\sigma) = F(\sigma \upharpoonright \beta)$. Since $|S| < \kappa_n$, $|S| \leq 2^\lambda_{n-1}$, so the number of possible functions $f_\beta$ is at most

$$|S|^{\kappa_0 \cdot \kappa_1 \cdots \kappa_{n-1} - (2^\lambda_{n-1})^\lambda_{n-1} = 2^\lambda_{n-1}.$$  

Since there are $\kappa_n = (2^\lambda_{n-1})^+$ ordinals $\beta$ in $Y$ (let $Y$ be $\kappa_n$ if no $Y$ is given), there must be a set $Z \subseteq Y$ of cardinality $\kappa_n$ such that $f_\beta = f_\gamma$ for all $\beta, \gamma \in Z$. This $Z$ satisfies the conclusion of the lemma. \hfill \blacksquare

If we have a function $F : P_{n+m} \to S$ where $|S| < \kappa_n$, then we can apply Lemma 7.7 repeatedly to restrict $F$ to a subdomain on which $F(\sigma)$ depends only on $\sigma \upharpoonright n$. This can be stated in terms of large sequences as follows:

**Lemma 7.8.** Let $F$ be a function from $P_{n+m}$ to $S$, where $|S| < \kappa_n$, and let $\vec{Y}$ be a large sequence. Then there is a large sequence $\vec{Z} \preceq \vec{Y}$ of length at least $n + m$ such that $\vec{Z}(i) = \vec{Y}(i)$ for $i < \min(n, \ell(\vec{Y}))$ and, for $\sigma \in \prod_{i < n + m} \vec{Z}(i)$, $F(\sigma)$ depends only on $\sigma \upharpoonright n$.

**Proof.** If the given $\vec{Y}$ has length less than $n + m$, then extend it to length $n + m$ by letting $\vec{Y}(i) = \kappa_i$ for larger values of $i$. We now define $\vec{Z} \preceq \vec{Y}$ of the same length as $\vec{Y}$ as follows.

Let $\vec{Z}(i) = \vec{Y}(i)$ if $i < n$ or $i \geq n + m$. Also, let $F_m = F$. If $i < m$ and we have a function $F_{i+1} : P_{n+i+1} \to S$, then by Lemma 7.7 we can find $\vec{Z}(i) \subseteq \vec{Y}(i)$ and $F_i : P_{n+i} \to S$ such that $|\vec{Z}(i)| = \kappa_{n+i}$ and $F_i(\sigma) = F_{i+1}(\sigma \upharpoonright \beta)$ for all $\sigma \in P_{n+i}$ and $\beta \in \vec{Z}(i)$. Do this successively for $i$ from $m - 1$ down to 0 to finish defining the required large sequence $\vec{Z}$.

\hfill \blacksquare
In the case \( n = 0 \), the conclusion of Lemma 7.8 is that \( F \) is constant on the part of its domain specified by the large sequence \( \vec{Z} \).

Using Lemma 7.8, we can prove one of the two facts mentioned earlier:

**Proposition 7.9.** If \( T \) is the tree associated with a \( \kappa_0 \)-narrow open covering of \( \omega \kappa \), then \( T \) does not avoid any large sequence.

*Proof.* Let \( \langle A_n : n < \omega \rangle \) be the narrow open covering, and suppose that \( \vec{Y} \) is a large sequence which is avoided by \( T \). Then, for each \( \sigma \in \prod_{i < \ell(\vec{Y})} \vec{Y}(i) \), since \( \sigma \notin T \), there exists \( n < \omega \) such that \( N_\sigma \subseteq A_n \); let \( F(\sigma) \) be the least such \( n \). This defines a function \( F : \prod_{i < \ell(\vec{Y})} \vec{Y}(i) \to \omega \). Apply Lemma 7.8 to get \( \vec{Z} \preceq \vec{Y} \) such that \( F \) is constant on \( \prod_{i < \ell(\vec{Y})} \vec{Z}(i) \), say with value \( \bar{n} \). This means that any \( s \in \omega \kappa \) such that \( s(i) \in \vec{Z}(i) \) for all \( i < \ell(\vec{Z}) \) is in \( A_{\bar{n}} \). But clearly we can fix all coordinates of such an \( s \) other than the \( \bar{n} \)'th coordinate, which we allow to vary, to get \( \kappa_{\bar{n}} \) points in \( A_{\bar{n}} \) on the same line parallel to the \( \bar{n} \)'th coordinate axis. Therefore, \( A_{\bar{n}} \) is not \( \kappa_0 \)-narrow in the \( \bar{n} \)'th coordinate, which is a contradiction. \( \blacksquare \)

It now remains to prove the other fact, that a tree of low rank must avoid some large sequence. This will be proved by induction on the rank of the tree. We will give two versions of the inductive argument; the second version will be more complicated, but will attain a better result.

**Proposition 7.10.** If \( T \subset \omega \kappa \) is a well-founded tree of rank less than \( \kappa \cdot \kappa_0 \) (ordinal multiplication), and \( \vec{Y} \) is a large sequence, then there is a large sequence \( \vec{Z} \preceq \vec{Y} \) such that \( T \) avoids \( \vec{Z} \).

*Proof.* By induction on \( \text{rk}(T) \). Suppose that the result is already known for trees of rank less than \( \text{rk}(T) \). We consider three cases.

**Case 1:** \( \text{rk}(T) < \kappa \). Choose \( n \) such that \( \text{rk}(T) < \kappa_n \). It follows that the range of the function \( \text{rk}_T \) has size less than \( \kappa_n \).

Find large sequences \( \vec{Y}_0 \succeq \vec{Y}_1 \succeq \vec{Y}_2 \succeq \cdots \) as follows. Let \( \vec{Y}_0 \) be \( \vec{Y} \), extended arbitrarily if necessary so as to have length at least \( n \). Given \( \vec{Y}_{m-1} \), apply Lemma 7.8 to the function \( \text{rk}_T|P_{n+m} \) to get \( \vec{Y}_m \succeq \vec{Y}_{m-1} \) such that \( \vec{Y}_m|n = \vec{Y}_{m-1}|n \) and, for \( \sigma \in \prod_{i < n+m} \vec{Y}_m(i) \), \( \text{rk}_T(\sigma) \) depends only on \( \sigma|n \).

Now, for any \( \tau \in \prod_{i < n} \vec{Y}_0(i) \) and any \( m < \omega \), let \( F_m(\tau) \) be the common value of \( \text{rk}_T(\sigma) \) for \( \sigma \in \prod_{i < n+m} \vec{Y}_m(i) \) extending \( \tau \). We also have \( F_{m-1}(\tau) = \text{rk}_T(\sigma|n + m - 1) \) for such \( \sigma \); hence, either \( F_m(\tau) < F_{m-1}(\tau) \) or \( F_m(\tau) = F_{m-1}(\tau) = -1 \). Since there is no infinite descending sequence of ordinals, for each \( \tau \) there must be an \( m \) such that \( F_m(\tau) = -1 \); let \( G(\tau) \) be the least such \( m \).

Apply Lemma 7.8 again to get \( \vec{Z}_0 \succeq \vec{Y}_0 \) such that \( G \) is constant on \( \prod_{i < n} \vec{Z}_0(i) \); let \( \bar{m} \) be the constant value of \( G \) on this set. Define the large sequence \( \vec{Z} \) of length \( \ell(\vec{Y}_{\bar{m}}) \) by letting \( \vec{Z}(i) = \vec{Z}_0(i) \) for \( i < n \) and \( \vec{Z}(i) = \vec{Y}_{\bar{m}}(i) \) for \( i \geq n \). Then we have \( \text{rk}_T(\sigma) = F_{\bar{m}}(\sigma|n) = -1 \) for all \( \sigma \in \prod_{i < n+\bar{m}} \vec{Z}(i) \), so \( T \) avoids \( \vec{Z} \).
Case 2: \( \text{rk}(T) \) is of the form \( \kappa \cdot \alpha + \beta \), where \( \alpha > 0 \) and \( \omega \leq \beta < \kappa \). Let \( T' = \{ \sigma \in T : \text{rk}_T(\sigma) \geq \kappa \cdot \alpha \} \). Then \( T' \) is a subtree of \( T \), and an easy induction shows that \( \text{rk}_T(\sigma) = \kappa \cdot \alpha + \text{rk}_T(\sigma) \) for any \( \sigma \in T' \). Hence, \( \text{rk}(T') = \beta < \text{rk}(T) \), so, by the induction hypothesis, there exists a large sequence \( \vec{Z}' \preceq \vec{Y} \) such that \( T' \) avoids \( \vec{Z}' \). Now let \( T'' = T \models \vec{Z}' \); it is easy to see that \( \text{rk}(T'') < \kappa \cdot \alpha + \ell(\vec{Z}') \), so we can again apply the induction hypothesis to get \( \vec{Z} \preceq \vec{Z}' \) such that \( T'' \) avoids \( \vec{Z} \). It follows that \( T \) avoids \( \vec{Z} \).

Case 3: \( \text{rk}(T) \) is of the form \( \alpha + n \), where \( \alpha \) is a limit ordinal, \( n < \omega \), and \( cf \alpha < \kappa_n \).

Let \( \langle \alpha_\beta : \beta < \delta \rangle \) be a strictly increasing sequence of ordinals which converges to \( \alpha \), where \( \delta < \kappa_n \). For each \( \sigma \in P_{n+1} \), we must have \( \text{rk}_T(\sigma) < \alpha \); hence, we can define a function \( F : P_{n+1} \rightarrow \delta \) by: \( F(\sigma) \) is the least \( \beta \) such that \( \text{rk}_T(\sigma) < \alpha_\beta \). By Lemma 7.8, there is a large sequence \( \vec{Y}' \preceq \vec{Y} \) of length at least \( n + 1 \) such that, for \( \sigma \in \prod_{i<n+1} \vec{Y}'(i) \), \( F(\sigma) \) depends only on \( \sigma \mid n \). Let \( T' = T \models \vec{Y}' \). Clearly \( \text{rk}_T'(\sigma) \leq \text{rk}_T(\sigma) \) for all \( \sigma \in \vec{Y}' \). If \( \sigma \in T' \) is of length \( n \), and \( \beta \) is the common value of \( F(\sigma \cap \gamma) \) for \( \gamma \in \vec{Y}'(n) \), then \( \text{rk}_T'(\sigma \cap \gamma) < \alpha_\beta \) for all \( \gamma \in \kappa_n \), so \( \text{rk}_T'(\sigma) \leq \alpha_\beta < \alpha \); this implies that \( \text{rk}_T'(\langle \rangle) \leq \alpha + n - 1 \), so \( \text{rk}(T') < \text{rk}(T) \). Apply the induction hypothesis to \( T' \) to get \( \vec{Z} \preceq \vec{Z}' \) such that \( T' \) avoids \( \vec{Z} \); then \( T \) also avoids \( \vec{Z} \).

It is not hard to see that any value for \( \text{rk}(T) \) less than \( \kappa \cdot \kappa_0 \) falls under at least one of these three cases, so the induction is complete.

**Corollary 7.11.** If \( \kappa \) is an uncountable strong limit cardinal of cofinality \( \omega \), and \( \lambda < \kappa \), then any \( \lambda \)-narrow covering of \( \prec \kappa \) using open sets must have complexity at least \( \kappa \cdot \kappa \).

**Proof.** Let \( T \) be the tree associated with such a covering, and suppose that \( \text{rk}(T) < \kappa \cdot \kappa \); then there is \( \alpha < \kappa \) such that \( \text{rk}(T) < \kappa \cdot \alpha \). Choose the cardinals \( \kappa_n \) and \( \lambda_n \) as specified in the global assumptions, so that \( \kappa_0 \) is greater than \( \alpha \) and \( \lambda \). Then Proposition 7.10 (with \( \vec{Y} = \langle \rangle \)) states that there is a large sequence \( \vec{Z} \) such that \( T \) avoids \( \vec{Z} \), while Proposition 7.9 states that there is no such \( \vec{Z} \), so we have a contradiction.

Now we give the second version of the inductive argument. In order to reach higher tree ranks, we work with an entire collection of trees simultaneously. We will show not only that each tree in the collection avoids some large sequence, but that one can find a relatively small number of large sequences such that each tree in the collection avoids at least one of them.

**Proposition 7.12.** Let \( \mu \) be an infinite cardinal less than \( \kappa_0 \). Suppose that \( X \) is a collection of well-founded trees \( T, T \in X \) such that \( \sup \{ \text{rk}(T) : T \in X \} < \kappa^{\mu^+} \) (ordinal exponentiation) and \( |X| \leq \kappa_{\mu^+} \). Finally, suppose that \( \vec{Y} \) is a large sequence. Then there is a collection \( \mathcal{C} \) of large sequences \( \vec{Z} \preceq \vec{Y} \) such that \( |\mathcal{C}| \leq \mu \) and, for every \( T \in X \), there exists \( \vec{Z} \in \mathcal{C} \) such that \( T \) avoids \( \vec{Z} \).

**Proof.** Let \( \alpha_0 \) be the least ordinal which is greater than \( \text{rk}(T) \) for all \( T \in X \); then \( \alpha_0 < \kappa^{\mu^+} \). If \( \alpha_0 < \omega \), the conclusion is trivial: just let \( \mathcal{C} = \{ \vec{Z} \} \) where \( \vec{Z} \) is any large sequence of length at least \( \alpha_0 \) such that \( \vec{Z} \preceq \vec{Y} \). So suppose \( \alpha_0 \geq \omega \). Then there is a unique ordinal \( \theta < \mu^+ \) such that \( \omega \cdot \kappa^\theta \leq \alpha_0 < \omega \cdot \kappa^{\theta+1} \). The proof will be by induction on \( \theta \), simultaneously for
all sequences of cardinals $\kappa_n$ and $\lambda_n$ satisfying the global assumptions. (However, $\kappa$ and $\mu$ will be fixed.)

Suppose the statement is true for all $\theta' < \theta$. For convenience, we divide the induction step into two cases.

Case 1: $\omega \cdot \kappa^\theta \leq \alpha_0 < \omega \cdot \kappa^\theta \cdot \kappa_0$. Let $\delta = \omega \cdot \kappa^\theta$, and choose a strictly increasing and continuous sequence $(\delta_\beta : \beta < \text{cf} \delta)$ converging to $\delta$ such that $\delta_0 = 0$ and $\delta_1 \leq \omega$. Note that $\text{cf} \delta$ is either $\text{cf} \omega$, $\text{cf} \kappa$, or $\text{cf} \theta$, so $\text{cf} \delta \leq \mu$.

We will construct a sequence $\langle \mathcal{C}_k : k < \omega \rangle$ of sets of large sequences with the following properties:

- $\mathcal{C}_0 = \{ \mathcal{Y} \}$;
- $|\mathcal{C}_k| \leq \mu$ for all $k$;
- if $\mathcal{Z}' \in \mathcal{C}_{k+1}$, then there is $\mathcal{Z} \in \mathcal{C}_k$ such that $\mathcal{Z}' \leq \mathcal{Z}$; and
- if $\mathcal{Z} \in \mathcal{C}_k$ and $T \in \mathcal{X}$, then there is $\mathcal{Z}' \leq \mathcal{Z}$ in $\mathcal{C}_{k+1}$ such that either $T$ avoids $\mathcal{Z}'$ or

$$\text{rk}(T \upharpoonright^* \mathcal{Z}') < \text{rk}(T \upharpoonright^* \mathcal{Z}).$$

Once we have this sequence, we can let $\mathcal{C} = \bigcup_{k<\omega} \mathcal{C}_k$. Then $\mathcal{C}$ will be a set of large sequences $\mathcal{Z} \leq \mathcal{Y}$, with $|\mathcal{C}| \leq \mu$. For every $T \in \mathcal{X}$, there will be $\mathcal{Z} \in \mathcal{C}$ such that $T$ avoids $\mathcal{Z}$; if this were not so, one could start with $\mathcal{Z}_0 = \mathcal{Y}$, find $\mathcal{Z}_1 \in \mathcal{C}_1$ such that $\text{rk}(T \upharpoonright^* \mathcal{Z}_1) < \text{rk}(T \upharpoonright^* \mathcal{Z}_0)$, then find $\mathcal{Z}_2 \in \mathcal{C}_2$ such that $\text{rk}(T \upharpoonright^* \mathcal{Z}_2) < \text{rk}(T \upharpoonright^* \mathcal{Z}_1)$, and so on, thus producing an infinite descending sequence of ordinal ranks, which is impossible. Therefore, $\mathcal{C}$ will be as desired.

Given $\mathcal{C}_k$, we will construct $\mathcal{C}_{k+1}$ by examining each large sequence $\mathcal{Z} \in \mathcal{C}_k$ and thereby producing a collection of at most $\mu$ new large sequences to be put into $\mathcal{C}_{k+1}$. So let $\mathcal{Z}$ be an arbitrary member of $\mathcal{C}_k$, and proceed as follows.

For each $T \in \mathcal{X}$, we can express $\text{rk}(T \upharpoonright^* \mathcal{Z})$ in the form $\delta \cdot \gamma_0 + \gamma_1 + n$ where $n < \omega$ and $\gamma_1$ is zero or a limit ordinal less than $\delta$, and this expression is unique. Note that the number of possibilities for $\gamma_0$ is less than $\kappa_0$, since $\alpha_0 < \delta \cdot \kappa_0$. Let $f(T) = (n, \beta, c)$, where $\beta$ is the unique $\beta$ such that $\delta_\beta \leq \gamma_1 < \delta_{\beta+1}$, and $c$ is 0 if $\gamma_0 = 0$, 1 otherwise. Note that the number of possible values for $f(T)$ is at most $|\omega \times (\text{cf} \delta) \times 2| \leq \mu$. We consider each possible triple $(n, \beta, c)$ separately. Fix $(n, \beta, c)$ with $n < \omega$, $\beta < \text{cf} \delta$, and $c < 2$, and let

$$\mathcal{X}_{n^{\beta c}} = \{ T \upharpoonright^* \mathcal{Z} : T \in \mathcal{X} \text{ and } f(T) = (n, \beta, c) \}.$$

We now consider several subcases.

Subcase 1: $c = 0$. Then the trees in $\mathcal{X}_{n^{\beta c}}$ all have rank less than $\delta_{\beta+1} + n$, which is below $\delta$, so we can apply the induction hypothesis to get a collection of at most $\mu$ large sequences $\mathcal{Z}' \leq \mathcal{Z}$ such that every $T' \in \mathcal{X}_{n^{\beta c}}$ avoids at least one of the sequences. It follows that every tree $T \in \mathcal{X}$ such that $f(T) = (n, \beta, c)$ must avoid one of these sequences. Add all of these large sequences to $\mathcal{C}_{k+1}$.

Subcase 2: $c = 1$ and $\beta > 0$. For each $T' \in \mathcal{X}_{n^{\beta c}}$, express $\text{rk}(T')$ in the form $\delta \cdot \gamma_0 + \gamma_1 + n$ as above (where $\gamma_0$ and $\gamma_1$ depend on $T'$), and let $T'_* = \{ \sigma \in T' : \text{rk}_{T'_*}(\sigma) \geq \delta \cdot \gamma_0 \}$. It is
easy to see that \( \text{rk}(T'_\ast) = \gamma_1 + n < \delta_{\beta+1} + n < \delta \) for each \( T' \in X_{n\beta c} \). Therefore, we can apply the induction hypothesis to the set \( \{ T'_\ast : T' \in X_{n\beta c} \} \) to get a collection of at most \( \mu \) large sequences \( \vec{Z}' \preceq \vec{Z} \) such that every such tree \( T'_\ast \) avoids at least one of the sequences. If \( T'_\ast \) avoids \( \vec{Z}' \), then \( \text{rk}(T' \upharpoonright * \vec{Z}') < \delta \cdot \gamma_0 + \ell(\vec{Z}') < \text{rk}(T') \). (Note that, if \( T' = T \upharpoonright * \vec{Z} \) and \( \vec{Z}' \preceq \vec{Z} \), then \( T' \upharpoonright * \vec{Z}' = T \upharpoonright * \vec{Z}' \).) Again, add all of these large sequences \( \vec{Z}' \) to \( \mathfrak{C}_{k+1} \).

Subcase 3: \( c = 1 \) and \( \beta = 0 \). Then, for each \( T' \in X_{n\beta c} \), we can express \( \text{rk}(T') \) in the form \( \delta \cdot \gamma_0 + n \), where \( \gamma_0 < \kappa_0 \) since \( \alpha_0 < \delta \cdot \kappa_0 \). Since \( \text{cf} \delta \leq \mu < \kappa_0 \), we have \( \text{cf}(\delta \cdot \gamma_0) < \kappa_0 \).

Let \( \nu = \text{the \preceq restrictive cardinal of } \kappa_0 \), i.e., \( \kappa \). Then we can partition \( \delta \cdot \gamma_0 \) into sets \( B_\xi^{T'} \), \( \xi < \nu \), none of which is cofinal in \( \delta \cdot \gamma_0 \). Now, for each \( \sigma \in P_{n+1} \), let \( F(\sigma) \) be the function from \( X_{n\beta c} \to \nu \) defined by: \( F(\sigma)(T') = \text{the unique } \xi < \nu \text{ such that } \text{rk}(T') \in B_\xi^{T'} \).

The number of possible functions \( F(\sigma) \) is at most \( \nu^{|X_{n\beta c}|} \); since \( |X_{n\beta c}| \leq |X| \leq \kappa_{-1} \leq \lambda_{-1} \) and \( \nu = 2^{\lambda_{-1}} \), we have \( \nu^{|X_{n\beta c}|} \leq \nu \), so there are fewer than \( \kappa_0 \) possible values of \( F(\sigma) \). We can now apply Lemma 7.8 to get a large sequence \( \vec{Z}' \preceq \vec{Z} \) of length at least \( n + 1 \) such that \( F(\sigma)(T') \) is constant on \( \prod_{i<n+1} \vec{Z}'(i) \).

Add this \( \vec{Z}' \) to \( \mathfrak{C}_{k+1} \).

Once the relevant subcase step has been performed for each \( \vec{Z} \in \mathfrak{C}_k \) and each \( (n, \beta, c) \), the construction of \( \mathfrak{C}_{k+1} \) is complete. We have ensured that \( \mathfrak{C}_{k+1} \) has all of the required properties. This finishes Case 1 of the induction.

Case 2: \( \omega \cdot \kappa^\theta \cdot \kappa_0 \leq \alpha_0 < \omega \cdot \kappa^\theta+1 \). Fix \( n \) such that \( \alpha_0 < \omega \cdot \kappa^\theta \cdot \kappa_0 \). If we let \( \kappa'_m = \kappa_{n+m} \) and \( \lambda'_m = \lambda_{n+m} \) for all \( m \) (including \( -1 \)), and define \( P'_m \) accordingly, then the global assumptions will be satisfied for these new values. Consider the collection

\[ \mathcal{X}' = \{ T_\sigma : \sigma \in P_n, T \in \mathcal{X} \} \]

where \( T_\sigma = \{ \tau : \sigma \cap \tau \in T \} \). Clearly \( \mathcal{X}' \) is a collection of trees \( T' \preceq \langle \omega \cdot \kappa \rangle \), and \( |\mathcal{X}'| \leq |\mathcal{X}| \cdot |P_n| \leq \kappa_{-1} = \lambda_{-1} \). Also, for each \( T \in \mathcal{X} \) and each \( \sigma \in P_n \),

\[ \text{rk}(T_\sigma) \leq \text{rk}(T) < \alpha_0 < \omega \cdot \kappa^\theta \cdot \kappa'_0. \]

Therefore, we can apply Case 1 to get a collection \( \mathcal{C}' \) of at most \( \mu \) large (for the cardinals \( \kappa'_m \)) sequences \( \vec{Z}' \preceq \langle \vec{Y}(n + i) : i < \ell(\vec{Y}) - n \rangle \) such that, for each \( T' \in \mathcal{X}' \), there exists \( \vec{Z}' \in \mathcal{C}' \) such that \( T' \) avoids \( \vec{Z}' \).

For each \( \sigma \in P_n \), define \( F(\sigma) : \mathcal{X} \to \mathcal{C}' \) so that \( F(\sigma)(T') \) is some \( \vec{Z}' \in \mathcal{C}' \) such that \( T_\sigma \) avoids \( \vec{Z}' \). Since \( |\mathcal{C}'| \leq \mu < \kappa_0 \) and \( |\mathcal{X}| \leq \kappa_{-1} \), there are fewer than \( \kappa_0 \) functions from \( \mathcal{X} \) to \( \mathcal{C}' \). Therefore, by Lemma 7.8, there is a large sequence \( Y_\ast \) of length at least \( n \) such that \( \vec{Y}_\ast \preceq \vec{Y}|n \) and \( F \) is constant on \( \prod_{i<n} \vec{Y}_\ast(i) \). Let \( h = F(\sigma) \) for \( \sigma \) in this set, and let \( \mathcal{C} = \{ (\vec{Y}_\ast|n) \cap \vec{Z}' : \vec{Z}' \in \mathcal{C}' \} \). Then \( \mathcal{C} \) is a collection of at most \( \mu \) large sequences \( \vec{Z} \preceq \vec{Y} \), and each \( T \in \mathcal{X} \) avoids some \( \vec{Z} \in \mathcal{C} \), namely \( (\vec{Y}_\ast|n) \cap h(T) \). This completes the induction. ■
Just as for Corollary 7.11, we can apply Proposition 7.12 and Proposition 7.9 to a single given tree $T$ (i.e., let $X = \{T\}$) with the cardinals $\mu$ and $\kappa_n$ chosen as large as necessary below $\kappa$ to get:

**Corollary 7.13.** If $\kappa$ is an uncountable strong limit cardinal of cofinality $\omega$, and $\lambda < \kappa$, then any $\lambda$-narrow covering of $\omega^{\kappa}$ using open sets must have complexity at least $\kappa^\kappa$. ■

There is no reason to believe that the lower bound $\kappa^\kappa$ obtained here is optimal; improvements in the argument might yield better results. The obvious way to provide an upper limit on this ordinal would be to produce an explicit open narrow covering, or an algebra with no infinite free subset, and compute the rank of the corresponding tree. For instance, in the constructible universe, with $\kappa = \aleph_\omega$, one can consider the algebra consisting of all operations (unary, binary, etc.) on $\kappa$ which are definable in $L_{\kappa^+}$; there are countably many of these. Devlin and Paris [5] have shown that this algebra has no infinite free subset. However, their proof gives no information about the rank of the tree of finite free sequences. I do not know of any upper bound for this rank beyond the obvious fact that it is less than $\kappa^+$.

There are other families of sets besides the open sets for which one could make a similar study of complexity of narrow coverings. For instance, one could consider the case of $\Sigma^0_2$ sets. A reduction argument similar to that of Proposition 7.1 shows that, if there is a narrow covering using $\Sigma^0_2$ sets, then there is a narrow partition using $\Sigma^0_2$, and hence $\Delta^0_2$, sets. One can assign an ordinal rank to such a partition in various ways, such as the first level in the difference hierarchy which includes all of the individual $\Delta^0_2$ sets, and then ask what the smallest possible rank for the partition is. However, it does not seem useful to study this question yet, since no case is currently known where a $\Delta^0_2$ narrow partition exists and a clopen narrow partition does not.

8. Open Problems

There are many open questions related to the concepts studied in this paper; here are some of the more interesting ones.

1. Does $NNC(\kappa, \aleph_1, F_\sigma)$ (or even $NNC(\kappa, <\kappa, \text{Borel})$) follow from the real-valued measurability of $\kappa$?
2. What is the exact consistency strength of $NNC(\kappa, \aleph_1, \text{Borel})$? In particular, does it imply the existence of $0^\sharp$?
3. Must the least $\kappa$ satisfying $NNC(\kappa, \lambda, \text{Borel})$ for a given $\lambda$ actually satisfy $NNC(\kappa, <\kappa, \text{Borel})$?
4. Does $NNC(\kappa, \lambda, \text{open})$ always imply $NNC(\kappa, \lambda, \text{Borel})$?
5. Is $NNC(\kappa, \lambda, \text{Borel})$ preserved by any forcing with the countable chain condition?
6. Can a cardinal $\kappa \leq 2^{\aleph_0}$ carry a uniform $(\kappa, \aleph_1; \omega)$-nonregular ultrafilter?
7. Does Projective Determinacy imply that all projective subsets of $\omega^\omega$ are $U$-measurable, where $U$ is a nonprincipal ultrafilter over $\omega$? (Louveau [11] mentions that, if a measurable cardinal exists or if $\text{MA}+\neg\text{CH}$ holds, then there are many ultrafilters $U$ such that all $\Sigma^1_2$ sets are $U$-measurable, but that it is open whether this is so for all $U$ in these cases.) Does Projective Determinacy imply $NNC(\aleph_0, <\aleph_0, \text{projective})$?
8. What is the least possible rank for the tree of all finite free sequences obtained from an algebra of size \( \aleph_\omega \) with fewer than \( \aleph_\omega \) operations and no infinite free subset?

9. Mrówka [15] gives some ostensibly weaker variants of \( \text{NNC}(2^{\aleph_0}, \aleph_1, F_\sigma) \) which would still suffice for his metric space constructions. One such variant is: \( \omega(\omega^2) \) cannot be written as a union of sets \( A_n \ (n < \omega) \) where \( A_n \) is \( \aleph_1 \)-narrow in the \( n \)'th coordinate and \( A_n \) is \( F_\sigma \) in the product topology on \( \omega(\omega^2) \) where the \( n \)'th factor \( \omega^2 \) is given the usual Cantor topology while the other factors are given the discrete topology. Are these variants actually weaker? Can they be attained using weaker large cardinals (or none at all)? Can such a metric space be constructed at all without large cardinals?

10. Mrówka [15] also mentions the statement \( \text{NNC}(2^\kappa, \kappa^+, F_\sigma) \) where \( \kappa \) is an uncountable strong limit cardinal of cofinality \( \omega \). Is this consistent with ZFC? If so, it will require stronger large cardinals than the ones used in this paper: the statement clearly implies \( 2^\kappa > \kappa^+ \), so \( \kappa \) must be a counterexample to the Singular Cardinals Hypothesis, and this entails the consistency of measurable cardinals of high order [7].

11. What happens if one considers products of sets of different sizes? That is, when can one express the infinite product \( \prod_{n<\omega} \kappa_n \) as a union of ‘nice’ sets \( A_n \ (n < \omega) \) such that \( A_n \) is \( \lambda_n \)-narrow in the \( n \)'th coordinate? This is of interest even in the finite-dimensional version with no restrictions on the sets \( A_n \); Simms [20] lists this as Open Problem 2, and cites results of Ristow [17] that settle it assuming a weak form of GCH (every limit cardinal is a strong limit).

In fact, one can ask the same question about finite products of finite sets, but this question has been settled. If \( |X_j| = k_j \) for each \( j < n \), then \( \prod_{j<n} X_j \) can be expressed as the union of sets \( A_j \ (j < n) \), where \( A_j \) is \( l_j \)-narrow in the \( j \)'th coordinate, if and only if \( \sum_{j<n} (l_j - 1)/k_j \geq 1 \). The necessity of this inequality is a simple counting argument. Conversely, if the inequality holds, then one can construct suitable sets \( A_j \) by the following modification of the proof of Proposition 2.4, due to J. Rickard (personal communication): partition the half-open interval \( [0,1) \) into intervals \( [y_j, z_j) \) for \( j < n \) so that \( z_j - y_j \leq (l_j - 1)/k_j \), let \( X_j = \{0, 1, \ldots, k_j - 1\} \), and define sets \( A_j \subseteq \prod_{j<n} X_j \) for \( j < n \) as follows:

\[
x \in A_j \iff y_j \leq \left( \sum_{i=0}^{n-1} x(i)/k_i \right) \mod 1 < z_j.
\]

References

1. F. Bagemihl, A decomposition of an infinite dimensional space, Z. Math. Logik Grundlag. Math. 31 (1985), 479–480.
2. R. Barnes, Jr., The classification of the closed-open and the recursive sets of number-theoretic functions, Doctoral Dissertation, University of California, Berkeley (1966).
3. J. Baumgartner and A. Hajnal, A proof (involving Martin's Axiom) of a partition relation, Fund. Math. 78 (1973), 193–203.
4. K. Devlin, Some weak versions of large cardinal axioms, Ann. Math. Logic 5 (1973), 291–325.
5. K. Devlin and J. Paris, More on the free subset problem, Ann. Math. Logic 5 (1973), 327–336.
6. R. Dougherty, Narrow coverings of \( \omega \)-product spaces, Doctoral Dissertation, University of California, Berkeley (1985).
7. M. Gitik, The strength of the failure of the Singular Cardinal Hypothesis, Ann. Pure Appl. Logic 51 (1991), 215–240.
8. T. Jech, Set theory, Academic Press, New York, 1978.
9. P. Koepke, The consistency strength of the free-subset property for $\omega$, J. Symbolic Logic 49 (1984), 1199–1204.
10. C. Kuratowski, Sur une caractérisation des alephs, Fund. Math. 38, 14–17.
11. A. Louveau, Une méthode topologique pour l'étude de la propriété de Ramsey, Israel J. Math. 23 (1976), 97–116.
12. D. Maharam, On homogeneous measure algebras, Proc. Nat. Acad. Sci. U.S.A. 28 (1942), 108–111.
13. Y. Moschovakis, Descriptive set theory, North-Holland, Amsterdam, 1980.
14. S. Mrówka, $N$-compactness, metrizability, and covering dimension, Rings of continuous functions (C. Aull, ed.), Marcel Dekker, New York and Basel, 1985, pp. 247–275 and 312–314.
15. ______, Small inductive dimension of completions of metric spaces, preprint.
16. K. Prikry, Changing measurable into accessible cardinals, Dissertationes Math. (Rozprawy Mat.) 68 (1970), 5–52.
17. A. Ristow, The existence of certain partitions on Cartesian products, Z. Math. Logik Grundlag. Math. 24 (1978), 325–333.
18. S. Shelah, Independence of strong partition relation for small cardinals, and the free-subset problem, J. Symbolic Logic 45 (1980), 505–509.
19. J. Silver, Every analytic set is Ramsey, J. Symbolic Logic 35 (1970), 60–64.
20. J. Simms, Sierpiński’s theorem, Simon Stevin 65 (1991), 69–163.
21. R. Solovay, Real-valued measurable cardinals, Axiomatic set theory (D. Scott, ed.), Proc. Sympos. Pure Math. 13, vol. I, Amer. Math. Soc., Providence, Rhode Island, 1971, pp. 397–428.

Department of Mathematics, Ohio State University, Columbus, OH 43210
E-mail address: rld@math.ohio-state.edu