The Casimir force on two-dimensional pistons for massive scalar fields with both Dirichlet and hybrid boundary conditions is computed. The physical result is obtained by making use of generalized ζ-function regularization technique. The influence of the mass and the position of the piston in the force is studied graphically. The Casimir force for massive scalar field is compared to that for massless scalar field.

Keywords: Casimir force; piston; generalized ζ-function.

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About 60 years ago Casimir gave the prediction that an attractive force should act between two plate-parallel uncharged perfectly conducting plates in vacuum[1]. Especially in recent 10 years, the effect has been paid more attention because of the development of precise measurements[2]. At the same time, Casimir energies and forces have been calculated theoretically in various different configurations. Different properties of the Casimir force (attractive or repulsive) can be obtained for different boundary conditions and different geometries[3, 4]. For example, it has been claimed that the Casimir energy inside rectangular cavities can be either positive or negative depending on the ratio of the sides[5, 6, 7, 8, 9]. But the conclusion is worth suspecting because the calculations ignore the divergent term associated with the boundaries and the nontrivial contribution from the outside region of the box[10, 11, 12]. Recently, a modification of the rectangle—"Casimir piston" was introduced to avoid the above problems[13]. The Casimir force on the piston is a well-defined force because the position of the piston is independent of the divergent terms in the interval vacuum energy and external region. Successively, the study of this geometry attracted a lot of interests. The Casimir force on the piston was studied for different dimensions, different fields and different boundary conditions[14, 15, 16, 17, 18, 19, 20]. The results indicate that the Casimir force on the piston can be attractive or repulsive for different cases. The repulsive Casimir force has special importance in that it can be applied to microelectromechanical systems (MEMS) [21, 22]. We discussed Casimir pistons for a massless scalar field with hybrid boundary conditions and obtained the repulsive Casimir force on the piston[23].

On the other hand, the Casimir effect for the massive scalar field also studied by some authors[24, 25, 26]. As is known that the Casimir effect disappears as the mass of the field goes to infinity since there are no more quantum fluctuations in the limit. But the precise way the Casimir energy varies as the mass changes is worth studying[27]. In this paper, we consider the Casimir force on the piston for the massive scalar field with two types of boundary conditions, that is, Dirichlet and hybrid. We obtain our physical results using ζ-function regularization technique. We discuss the influence of the mass and the ratio of the sides graphically. The results tell us the expectable properties of the force on the piston as is for massless scalar field and also tell us the variation of the force as the mass changes.

FIG. 1: Casimir piston in two dimensions.

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for different ratio of the sides. 

According to [23], the principal features of a 2-dimensional piston can represent those of n-dimensional ones. Therefore, we discuss 2-dimensional cases in detail. (see Fig. 1). The piston divides the rectangle cavity into two parts labeled A and B and a quantized massive scalar field is constrained in the interval L. We need to calculate the Casimir force on the piston when L approaches infinity. The total energy of the vacuum for the system can be written as the sum of three terms:

\[ E = E^A(a, b) + E^B(L - a, b) + E^{\text{out}} \]  

where \( E^A(a, b) \) and \( E^B(L - a, b) \) are given by the result through cut-off technique, which consist of divergent terms and finite terms, where the finite terms are the same as what are obtained by \( \zeta \)-function regularization denoted \( E_R^A(a, b) \) and \( E_R^B(L - a, b) \). The divergent terms and the energy from the exterior region in the total energy are independent of the position of the piston [13], so the Casimir force on the piston is as follows:

\[ F(a) = -\frac{\partial}{\partial a} [E_R^A(a, b) + E_R^B(L - a, b)] \]  

In order to calculate the Casimir energy and Casimir force, we have to fix the boundary conditions. We consider two types of boundary conditions. One is Dirichlet(D.B.), that is, the boundary conditions on all surfaces are Dirichlet. The other is hybrid(H.B.), that is, the boundary condition on the piston is Neumann, and the ones on other surfaces are Dirichlet.

In cavity A, the vacuum energy is given by \((\hbar = c = 1 \text{ where } c \text{ is the speed of light})\)

\[
E^A(a, b) = \begin{cases} 
\frac{1}{2} \sum_{n_1=1}^{\infty} \left[ \left( \frac{n_1 \pi}{a} \right)^2 + \left( \frac{n_2 \pi}{b} \right)^2 + m^2 \right]^2, & \text{for D.B.,} \\
\frac{1}{2} \sum_{n_1=1}^{\infty} \left[ \left( n_1 + \frac{1}{2} \right)^2 + \left( \frac{n_2 \pi}{b} \right)^2 + m^2 \right]^2, & \text{for H.B..} 
\end{cases}
\]  

In order to use \( \zeta \)-function regularization, we need to re-express Eq. (3) as:

\[
E_R^A(a, b) = \begin{cases} 
\mathcal{E}(a, b, m), & \text{for D.B.,} \\
\mathcal{E}(2a, b, m) - \mathcal{E}(a, b, m), & \text{for H.B..} 
\end{cases}
\]  

where

\[
\mathcal{E}(a, b, m) = \frac{1}{8} \left\{ \sum_{n_1=-\infty}^{\infty} \left[ \left( \frac{n_1 \pi}{a} \right)^2 + \left( \frac{n_2 \pi}{b} \right)^2 + m^2 \right]^2 - \sum_{n_2=-\infty}^{\infty} \left[ \left( \frac{n_2 \pi}{b} \right)^2 + m^2 \right]^2 - m \right\} 
\]

Using Epstein \( \zeta \)-function [28]

\[
Z_\nu(s; a_1, \cdots, a_p; m) = \sum_{n_1, \cdots, n_p=-\infty}^{\infty} \left[ a_1 n_1^2 + \cdots + a_p n_p^2 + m^2 \right]^{-s} 
\]

\[
= \frac{\pi^{\frac{\nu}{2}}}{\sqrt{a_1 \cdots a_p}} \frac{\Gamma \left( s - \frac{\nu}{2} \right)}{\Gamma \left( s \right)} m^{p-2s} + \frac{\pi^{\frac{\nu}{2}}}{\sqrt{a_1 \cdots a_p}} \frac{2}{\Gamma \left( s \right)} 
\]

\[
\times \sum_{n_1, \cdots, n_p=-\infty}^{\infty} \left[ n_1^2 \cdots n_p^2 \right]^{\frac{\nu}{2}} \left( s - \frac{\nu}{2} \right) 
\]

\[
\times K_{\frac{\nu}{2}} \left( 2\pi m \left[ \frac{n_1^2}{a_1} + \cdots + \frac{n_p^2}{a_p} \right]^{\frac{\nu}{2}} \right) 
\]

where \( K_\nu(z) \) is the second type of modified Bessel function and the prime on the sum denotes that the term \( n_1 =
\( n_2 = \cdots = n_p \) is omitted, so we can re-express Eq. (5) as

\[
\mathcal{E}(a, b, m) = \frac{1}{8} \left\{ \frac{m^3 ab \Gamma(-\frac{3}{2})}{\pi \Gamma(-\frac{1}{2})} + \frac{m^3 ab}{\pi} \frac{2}{\Gamma(-\frac{1}{2})} \sum_{n_1 n_2 = -\infty}^{\infty} K_\frac{3}{2} \left( \frac{2m \sqrt{n_1^2 a^2 + n_2^2 b^2}}{n_1^2 a^2 + n_2^2 b^2} \right) \right. \\
- \frac{m^2 a \Gamma(-1)}{\pi^{\frac{3}{2}} \Gamma(-\frac{1}{2})} - \frac{2m}{\pi^{\frac{3}{2}} \Gamma(-\frac{1}{2})} \sum_{n_1 = -\infty}^{\infty} K_1(2mn_1) \\
- \frac{m^2 b \Gamma(-1)}{\pi^{\frac{3}{2}} \Gamma(-\frac{1}{2})} - \frac{2m}{\pi^{\frac{3}{2}} \Gamma(-\frac{1}{2})} \sum_{n_2 = -\infty}^{\infty} K_1(2mn_2) - m \right\} 
\]

Instead of \( a \) by \( L - a \), one can obtain the finite energy \( E_B^L(L - a, b) \) in cavity \( B \). Substituting the expression for \( E_B^L(a, b) \) and \( E_B^L(L - a, b) \) into Eq. (2) and letting \( L \to \infty \), the resulting Casimir forces on the piston \( F_D(a, m) \) (for D.B.) and \( F_H(a, m) \) (for H.B.) are:

\[
F_D(a, m) = -\frac{1}{4\Gamma(-\frac{1}{2})} \left\{ \frac{4m^2 b}{\pi} \sum_{n_1 n_2 = 1}^{\infty} \frac{2K_\frac{3}{2} \left( \frac{2m \sqrt{n_1^2 a^2 + n_2^2 b^2}}{n_1^2 a^2 + n_2^2 b^2} \right)}{n_1^2 a^2 + n_2^2 b^2} \\
- \frac{8m^2 a^2 b}{\pi} \sum_{n_1 n_2 = 1}^{\infty} n_1^2 K_\frac{3}{2} \left( \frac{2m \sqrt{n_1^2 a^2 + n_2^2 b^2}}{n_1^2 a^2 + n_2^2 b^2} \right) \right. \\
+ \frac{2m^2 b}{\pi a^2} \sum_{n_1 = 1}^{\infty} \frac{2K_1(2mn_1)}{n_1} - \frac{4m^2 b}{\pi a^2} \sum_{n_1 = 1}^{\infty} K_1(2mn_1) \\
\left. - \frac{2m}{\pi a} \sum_{n_1 = 1}^{\infty} K_1(2mn_1) + 4m^2 \sum_{n_1 = 1}^{\infty} \frac{K_2(2mn_1)}{n_1} \right\} 
\]

and

\[
F_H(a, m) = 2F_D(2, m) - F_D(a, m)
\]

In order to consider the influence of the mass, we write down the Casimir force for massless scalar field on the piston [13, 23] as follows:

\[
F_D(a, 0) = -\frac{b c(3)}{8\pi a^3} + \frac{\pi}{48a^2} - \frac{\zeta(3)}{16\pi b^2} + \frac{\pi b}{a^3} \sum_{n_1, n_2 = 1}^{\infty} n_2^2 K_0 \left( \frac{2\pi n_1 n_2 b}{a} \right)
\]

and

\[
F_H(a, 0) = \frac{3b \zeta(3)}{32\pi a^3} - \frac{\pi}{96a^2} - \frac{\zeta(3)}{16\pi b^2} - \frac{4b}{4a^3} \sum_{n_1, n_2 = 1}^{\infty} n_2^2 \left[ K_0 \left( \frac{\pi n_1 n_2 b}{a} \right) - 4K_0 \left( \frac{2\pi n_1 n_2 b}{a} \right) \right]
\]

we study the Casimir force on the piston for a massive scalar field graphically. We can see the variation of the Casimir force depending on the mass of the scalar field and also the influence of the ratio of the sides \( b/a \). It is worth emphasizing that compared to the case of massless scalar field, the product \( ma \) appears as a variable. We can discuss the influence of the mass through fixing the distance \( a \).

Fig.2 to Fig.4 are the results for D.B. and Fig.5 to Fig.7 are for H.B. In Fig.2 and Fig.5, choosing different ratio of the sides, we plot the dependence of the Casimir force (in units of \( 1/L \)) on the mass for fixed distance. It is clear that the force on the piston is attractive for D.B. and it is repulsive for H.B. and it vanishes as the mass goes to infinity. In Fig.3 and Fig.6, fixing the mass, we give the force (in units of \( m^2 \)) with \( b/a \) increasing, where we choose different value of \( ma \). The graph tells that the force on the piston increases rapidly with the ratio \( b/a \) increasing, which is similar to the result for massless scalar field. In order to find the influence of the mass in the Casimir force more explicitly, we plot the ratio \( F(a, m)/F(a, 0) \) via the product \( ma \) in Fig.4 and Fig.7, where we plot three curves for different ratio of \( b/a \), respectively. From the graphs we can see that the ratio approaches 1 in the limit \( ma = 0 \) and
FIG. 2: Plot of the Casimir force on the piston (in units of \( \frac{1}{a^2} \)) versus \( ma \) for different ratio of \( b/a \) with D.B..

FIG. 3: Plot of the Casimir force on the piston (in units of \( m^2 \)) versus \( b/a \) for different value of \( ma \) for D.B..

FIG. 4: Plot of the ratio of the Casimir force on the piston for a massive scalar field and massless scalar field versus \( ma \) for different ratio of \( b/a \) for D.B..
it tends to zero as \( m \) goes to infinity. Furthermore with \( b/a \) increasing the change of the ratio with \( m \) is insensitive for different \( b/a \).

Our main results are summarized as follows: (i) The Casimir force on the piston for massive scalar field is attractive for D.B. and it is repulsive for H.B., which is the same result as that for massless scalar field. (ii) The Casimir force decreases with the mass increasing. The force vanishes as the mass goes to infinity, while in the limit \( m = 0 \) the force recovers the result for massless scalar field. (iii) For fixed mass the force on the piston increases as the piston moves close to the nearest end, which is also similar to the result for massless scalar field.

For three-dimensional pistons, one can discuss similarly and can get the same result but with more complicated calculations.

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FIG. 7: Plot of the ratio of the Casimir force on the piston for a massive scalar field and massless scalar field versus $ma$ for different ratio of $b/a$ for H.B..

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