Waves propagation through a localized axisymmetric vortical flow

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Abstract. We propose a particular model for the axisymmetric vortex in two-dimensional space that can be generated under the rotation of the circular cylinder in a viscous fluid. The velocity field induced by such vortex is similar to the velocity field of the point vortex in the limited region and exponentially decays at infinity. We consider the plane acoustic wave propagation through such flow. The numerical solution for the scattered field is obtained using the exact integral representation. The scattered wave in the far field is an outgoing cylindrical wave peaking at small scattering angles.

1. Vortex model
The axisymmetric vortex in two-dimensional space is a flow with the velocity field

$$\mathbf{v} = \frac{\Gamma(r)}{2\pi r} \hat{\theta},$$

where $r$, $\theta$ are polar coordinates, $r$ is the distance from the vortex center and $\theta$ is the angle, $\hat{r} = r/r$, $\hat{\theta}$ are the unit vectors, $\Gamma(r)$ is the circulation around the circle of radius $r$. Well-known examples are the point vortex $\Gamma = \text{const}$, the Rankine vortex $\Gamma \propto r^2$ for $r \leq r_0$ and $\Gamma = \text{const}$ for $r > r_0$, and the Lamb–Oseen vortex $\Gamma \propto 1 - \exp(-r^2/r_0^2)$. Here, $r_0$ is the effective size of the vortex core. The common feature of all the examples is $\Gamma \to \text{const}$ and hence $|\mathbf{v}| \propto 1/r$ at large distances from the vortex center. The kinetic energy of a flow with $|\mathbf{v}| \propto 1/r$ as $r \to \infty$ is infinite:

$$E \propto \frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{v}|^2 d^2r \propto \int_0^{\infty} \frac{dr}{r} = \infty,$$

so that it cannot exist in unbounded space, as noticed in [1], [2].

However, there is an example of the axisymmetric vortex that can be generated in reality. Let an infinitely elongated hollow circular cylinder of radius $r_0$ be placed in an incompressible viscous fluid being at rest and at the moment $t = 0$ start to rotate around its axis with the constant angular velocity $\Gamma_0/2\pi r_0^2$. The flow obeys the Navier–Stokes equations. The continuity equation is satisfied by (1). The momentum equation for an axisymmetric flow is reduced to

$$\frac{\partial \Gamma}{\partial t} = \nu \left( \frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right),$$

where $\nu$ is the kinematic viscosity, $\Gamma(r)$ is the circulation around the circle of radius $r$. This equation can be solved analytically in the vicinity of the vortex center.

The numerical solution for the scattered field is obtained using the exact integral representation. The scattered wave in the far field is an outgoing cylindrical wave peaking at small scattering angles.
where $t$ is time, $\nu$ is kinematic viscosity. The boundary conditions are the unperturbed fluid at infinity

$$\Gamma \to 0 \quad \text{as} \quad r \to \infty, \quad (4)$$

the no-slip condition on the cylinder wall

$$\Gamma = \Gamma_0 \quad \text{for} \quad r = r_0, \quad (5)$$

and the absence of a singularity in the vortex center

$$\Gamma / r < \infty \quad \text{as} \quad r \to 0. \quad (6)$$

The initial condition is the unperturbed fluid

$$\Gamma = 0 \quad \text{for} \quad t = 0. \quad (7)$$

The leading-order solution to the problem (3)–(5), (7) as $r_0 / \sqrt{\nu t} \to 0$ outside the cylinder is

$$\Gamma \sim \left\{ \begin{array}{l} \Gamma_0, \\
\Gamma_0 e^{-r^2/(4\nu t)}, \end{array} \quad \frac{r}{r_0} = O(1), \quad \frac{r}{\sqrt{\nu t}} = O(1), \quad \text{for} \quad r > r_0 \right\} \quad (8)$$

The solution (8) corresponds to the point vortex with the circulation $\Gamma_0$ encircled with a mantle possessing the circulation $-\Gamma_0$ such that the velocity field decays exponentially as $r / \sqrt{\nu t} \to \infty$. It is equivalent to the difference between the point vortex and the Lamb–Oseen vortex, the latter being the instantaneous solution to the problem of the point vortex diffusion in a viscous fluid [3]. For the case of a compressible fluid, the solution was obtained in [4], [5]. In the case of low Mach numbers, the leading-order circulation distribution is close to (8).

The leading-order solution to the problem (3), (5)–(7) as $r_0 / \sqrt{\nu t} \to 0$ inside the cylinder represents the solid-body rotation flow

$$\Gamma \sim \Gamma_0 (r/r_0)^2 \quad \text{for} \quad r \leq r_0. \quad (9)$$

The total flow (8)–(9) is the Rankine vortex encircled with a mantle possessing the opposite circulation.

2. Scattering problem

2.1. Statement of the problem

Let the plane acoustic wave of the wavelength $\lambda$ propagate in an ideal gas through the cylindrical vortex with the circulation distribution $\Gamma(r) = \Gamma_0 e^{-r^2/L^2}$, which corresponds to (8) for $r_0 = 0$ at the moment $t = L^2/(4\nu)$. We assume that the characteristic time of the vortex diffusion is long compared to the characteristic time of the wave propagation through the vortex and hence that the flow obeys the Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (10)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\nabla p}{\rho} = 0, \quad (11)$$

$$\frac{\partial}{\partial \rho^2} \frac{p}{\rho^2} + \mathbf{v} \cdot \nabla \frac{p}{\rho^2} = 0. \quad (12)$$

The goal is to determine the density $\rho$, velocity $\mathbf{v}$ and pressure $p$ time-periodic fields associated with the scattered wave. We introduce the dimensionless variables

$$\bar{\rho} = \rho / \rho_\infty, \quad \bar{p} = p / p_\infty, \quad \bar{\mathbf{v}} = \mathbf{v} / c_\infty, \quad \bar{t} = \omega t, \quad \bar{\mathbf{r}} = k \mathbf{r}, \quad (13)$$
where $\rho_\infty$, $p_\infty$, $c_\infty = \sqrt{\gamma p_\infty/\rho_\infty}$ are the unperturbed gas density, pressure and sound speed, $\gamma$ is the heat capacity ratio, $\omega$ is the angular frequency and $k = \omega/c_\infty = 2\pi/\lambda$ is the wavenumber of the incident wave. The overlines in (13) are dropped further. The equations (10)–(12) are rewritten as

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
$$

$$
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\gamma} \nabla p = 0,
$$

$$
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = 0,
$$

The total flow field is represented as

$$
\mathbf{F} = \mathbf{F}_0(r) + ae^{-it} \mathbf{F}'(r) + M^2 e^{-it} \mathbf{F}''(r) + o(M),
$$

where $\mathbf{F} = \{\rho, p, \mathbf{v}\}$, $M = kL_0/(2\pi c_\infty) << 1$ is the Mach number of the vortex velocity field at the acoustic wavelength scale, $a << 1$ is the dimensionless amplitude of the incident wave. The singularity in the vortex velocity field as $r \to 0$ does not actually affect the solution [6]. The term $\mathbf{F}_0$ represents the mean flow field

$$
\rho_0 = 1 + O(M^2), \quad p_0 = 1 + O(M^2), \quad \mathbf{v}_0 = M e^{-\gamma k^2 r^2} \hat{\mathbf{r}} + O(M^3),
$$

where

$$
\epsilon = \frac{1}{kL}.
$$

The solution at $O(a)$ represents the incident plane wave

$$
\rho' = e^{i\hat{k} \cdot \mathbf{r}}, \quad p' = \gamma e^{i\hat{k} \cdot \mathbf{r}}, \quad \mathbf{v}' = \hat{k} e^{i\hat{k} \cdot \mathbf{r}},
$$

where $\hat{k} = k/k$ is the unit wavevector.

### 2.2. Solution

The scattered wave is the addition to the superposition of the mean flow (18) and the incident wave (20). In the leading-order approximation it is represented by the solution at $O(Ma)$. Henceforth we restrict our study to the Born approximation, meaning that the amplitude of the scattered wave is assumed much smaller than one of the incident wave $a$. Substituting (17) into (14)–(16), by certain calculations we obtain the inhomogeneous Helmholtz equation for the density perturbations:

$$
\Delta \rho'' + \rho'' = q,
$$

with the source term

$$
q = -2\nabla \cdot (\hat{\mathbf{v}}_0 \cdot \nabla \mathbf{v}') = -2i\hat{k} \cdot \nabla \left[ \frac{\hat{z} \cdot (\mathbf{r} \times \hat{k}) e^{-\gamma k^2 r^2} e^{i\hat{k} \cdot \mathbf{r}}}{r^2} \right],
$$

where

$$
\hat{\mathbf{v}}_0 = \mathbf{v}_0 / M = \frac{e^{-\gamma k^2 r^2}}{r} \hat{\theta}.
$$
The pressure and velocity scattered fields are expressed through the density using (14)–(16):

\[ p'' = \gamma \rho'', \quad \mathbf{v}'' = -i(\nabla \rho'' + \mathbf{v}_0 \cdot \nabla \mathbf{v} + \mathbf{v}' \cdot \nabla \mathbf{v}_0). \] (24)

As the source is localized within a limited region \( \varepsilon r = O(1) \), the solution must fulfill the Sommerfeld radiation condition:

\[ \rho'' = O(r^{-1/2}) \quad \text{as} \quad r \to \infty, \] (25)

\[ \frac{\partial \rho''}{\partial r} - i\rho'' = o(r^{-1/2}) \quad \text{as} \quad r \to \infty. \] (26)

The conditions (25)–(26) imply that only waves outcoming from the flow region to infinity exist in the far field, while waves incoming from infinity are absent. The solution to the boundary-value problem (21), (25)–(26) is exactly represented as convolution of the source with Green’s function of the Helmholtz equation:

\[ \rho'' = -\frac{i}{4} \int_{\mathbb{R}^2} q(r') H_0^{(1)}(|r - r'|) \, d^2r', \] (27)

where \( H_0^{(1)}(r) \) is the Hankel function of the first kind. The integral (27) for the source (22) is absolutely convergent. Note that in the scattering problem for the case of the point vortex [6], [7], [8], the Rankine vortex [9] or the Lamb–Oseen vortex [10] the integral (27) is divergent because of the slow decay of the vortex velocity field \( |\mathbf{v}_0| \propto 1/r \).

We integrate (27) numerically, for any \( r \) considering the contribution of sources within the flow region \( \varepsilon r' = O(1) \). The case \( \varepsilon << 1 \), in which the vortex effective radius is long compared to the wavelength, is of particular interest. The real part of \( \rho'' \) in the case \( \varepsilon = 0.05 \) is shown in figure 1. The angular peak is reached at small scattering angles.

![Figure 1](image.png)  
**Figure 1.** The numerical solution: the real part of \( \rho'' \) for \( \varepsilon = 0.05 \). The distance from the vortex center to the domain right boundary is \( 10\varepsilon^{-1} \). The range displayed is \(-2.63 < \text{Re}(\rho'') < 2.63\).

The far-field solution that agrees with the boundary condition (25)–(26) must have the form

\[ \rho'' \sim e^{i r - i \pi/4} f_0(\theta). \] (28)

The radial dependence of \( |\rho''| \) in (28) agrees with the conservation of the scattered energy flux

\[ W \propto \int_{-\pi}^{\pi} |\mathbf{v}''|^2 r \, d\theta \propto \int_{-\pi}^{\pi} f_0^2(\theta) \, d\theta \] (29)
in the region where there is no sound interaction with the flow. The factor $e^{-i\pi/4}$ in (28) is necessary to make the scattering amplitude $f_0(\theta)$ real-valued.

It is useful to visualize the solution (27) in terms of the scattering amplitude. For this, let us generalize the scattering amplitude by

$$ \rho'' = \frac{e^{ir-i\pi/4}}{\sqrt{r}} f(r). \tag{30} $$

The scattering amplitude $f(r)$ is the complex-valued function. The dependence of its real part on the angle $\theta$ is shown in figure 2 for several distances $r$. This demonstrates that the limiting scattering pattern is reached in the far field: $f(r) \to f_0(\theta)$ as $r \to \infty$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{The real part of the scattering amplitude for $\varepsilon = 0.05$ on $r = 40$ (red), $r = 80$ (orange), $r = 120$ (green), $r = 200$ (blue), $r = 400$ (purple).}
\end{figure}

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