"POTATO KUGEL" FOR NUCLEAR FORCES AND A SMALL ONE FOR ACOUSTIC WAVES

N. Kuznetsov
Laboratory for Mathematical Modelling of Wave Phenomena
Institute for Problems in Mechanical Engineering, RAS
V.O., Bol’shoy pr. 61, St Petersburg 199178, Russia
nikolay.g.kuznetsov@gmail.com

The “potato kugel” theorem of Aharonov, Schiffer, and Zalcman, which concerns an inverse property of harmonic functions, is extended to the settings of the modified Helmholtz equation and the Helmholtz equation that describe nuclear forces and acoustic waves respectively. Bibliography: 18 titles.

1 Introduction

Analytic characterization of balls in the Euclidean space $\mathbb{R}^m$ by means of harmonic functions has a long history; it started in the 1960s, in the pioneering notes [1, 2], and shortly afterwards Kuran [3] obtained the following general result.

**Theorem 1.1** (Kuran [3]). Let $D$ be a domain (a connected open set) of finite (Lebesgue) measure in the Euclidean space $\mathbb{R}^m$, where $m \geq 2$. Suppose that there exists a point $P_0$ in $D$ such that, for every function $h$ harmonic in $D$ and integrable over $D$, the volume mean of $h$ over $D$ equals $h(P_0)$. Then $D$ is an open ball (a disk when $m = 2$) centered at $P_0$.

Presumably, the paper [4] was the first one in which this theorem was referred to as the property of harmonic functions inverse of the mean value identity for balls; the term became widely accepted. A slight modification of Kuran’s considerations shows that his theorem is valid even if $D$ is disconnected (see the survey article [5, p. 377], which also contains some improvements of the Kuran theorem, and a discussion of its applications and possible similar results involving certain averages over $\partial D$, when $D$ is a bounded domain).

Another approach to harmonic characterization of balls was developed by Aharonov, Schiffer, and Zalcman [6] (the origin of a rather unusual title of this paper is explained in the comment published in [7, p. 497]). Namely, the following result was proved.

**Theorem 1.2** (Aharonov, Schiffer, and Zalcman [6]). Let $D \subset \mathbb{R}^m$, $m \geq 3$, be a bounded open set. If the equality

$$\int_D |y - x|^{2-m} \, dy = \frac{a}{|x|^{m-2}} + b$$

Translated from Problemy Matematicheskogo Analiza 117, 2022, pp. 79-84.

1072-3374/22/2673-0375 © 2022 Springer Nature Switzerland AG
is valid with suitable real constants $a$ and $b$ for every $x \in \mathbb{R}^m \setminus D$, then $D$ is an open ball centered at the origin, $a = |D|$ and $b = 0$.

Here and below, $|D|$ stands for the volume of $D$ (area if $D \subset \mathbb{R}^2$). Since $|y - x|^{-1}$ is a fundamental solution of the Laplace equation for $m = 3$, this theorem answers in the affirmative to the following question posed to the authors of [6]:

Let $D$ be a homogeneous, compact, connected “potato” in space, which gravitationally attracts each point outside it as if all its mass were concentrated at a single point. Does this guarantee that $D$ is a ball centered at this point?

Later on, Theorem 1.2 was extended to the sub-Laplacian setting in [8], and an improvement of Theorem 1.2 was obtained in the recent article by Cupini and Lanconelli [9]. It relaxes the original restriction imposed on $D$, and a more natural identity is used to guarantee that $D$ is a ball.

**Theorem 1.3** (Cupini and Lanconelli [9]). Let $D \subset \mathbb{R}^m$, $m \geq 3$, be an open set such that $|D| < \infty$. If for some $x_0 \in D$ the identity

$$
|D|^{-1} \int_D |y - x|^{2-m} \, dy = |x_0 - x|^{2-m}
$$

(1.1)

is fulfilled for every $x \in \mathbb{R}^m \setminus D$, then $D$ is an open ball centered at $x_0$.

The recent survey [10] complements [5] providing coverage for other results in this area; in particular, various characterizations of balls via harmonic functions are given as well as characterizations of other domains (strips, annuli etc.). Moreover, a characterization of balls via solutions to the modified Helmholtz equation

$$
\nabla^2 u - \mu^2 u = 0, \quad \mu \in \mathbb{R} \setminus \{0\}.
$$

(1.2)

is considered. Here and below, $\nabla = (\partial_1, \ldots, \partial_m)$, $\partial_i = \partial/\partial x_i$, denotes the gradient operator. The most important application of this equation is in the theory of nuclear forces; it was developed by Yukawa in his Nobel Prize winning paper [11] (see also [12], where (1.2) is referred to as the Yukawa equation and its solutions are called panharmonic functions—the abbreviation used below). Since the mentioned characterization of balls obtained in [13] is closely related to one of the results proved in this paper, we begin with its formulation, but introduce some notation before that.

Let $x = (x_1, \ldots, x_m)$ be a point in $\mathbb{R}^m$, $m \geq 2$. We denote by $B_r(x) = \{y \in \mathbb{R}^m : |y - x| < r\}$ the open ball of radius $r$ centered at $x$ (just $B_r$ if centered at the origin). The ball is called admissible with respect to a domain $D \subset \mathbb{R}^m$ provided that $B_r(x) \subset D$. If $D$ has finite Lebesgue measure and a function $f$ is integrable over $D$, then

$$
M(f, D) = \frac{1}{|D|} \int_D f(x) \, dx
$$

is its volume mean value over $D$. The volume of $B_r$ is $|B_r| = \omega_m r^m$, where

$$
\omega_m = 2 \pi^{m/2}/[m \Gamma(m/2)]
$$

is the volume of the unit ball in $\mathbb{R}^m$. The most important application of this equation is in the theory of nuclear forces; it was developed by Yukawa in his Nobel Prize winning paper [11] (see also [12], where (1.2) is referred to as the Yukawa equation and its solutions are called panharmonic functions—the abbreviation used below). Since the mentioned characterization of balls obtained in [13] is closely related to one of the results proved in this paper, we begin with its formulation, but introduce some notation before that.
is the volume of the unit ball. Here, $\Gamma$ denotes the Gamma function. A dilated copy of a domain $D$ is $D_r = D \cup \{x \in \partial D : B_r(x)\}$. Thus, the distance from $\partial D_r$ to $D$ is equal to $r$.

Now, we formulate an analogue of Theorem 1.1 proved in [13].

**Theorem 1.4** (Kuznetsov [13]). Let $D \subset \mathbb{R}^m$, $m \geq 2$, be a bounded domain whose complement is connected, and let $r > 0$ be such that $|B_r| = |D|$. If for a point $x_0 \in D$ and some $\mu > 0$ the identity

$$u(x_0)a_m^+ (\mu r) = M(u, D), \quad a_m^+ (t) = \Gamma \left( \frac{m}{2} + 1 \right) \frac{I_{m/2}(t)}{(t/2)^{m/2}},$$

is valid for every positive function $u$ panharmonic in $D_r$, then $D = B_r(x_0)$.

As usual, $I_\nu$ denotes the modified Bessel function of order $\nu$. Its well-known properties (see [14, pp. 79, 80]) imply that $a_m^+ (t)$ increases monotonically for $t \in [0, \infty)$ from $a_m^+ (0) = 1$ to infinity. In the three-dimensional case related to nuclear forces, $a_3^+ (t) = \sqrt{2\pi} I_{3/2}(t)/t^{3/2}$.

It is clear that Theorem 1.4 is inverse of the $m$-dimensional mean value property

$$a_m^+ (\mu r) u(x) = M(u, B_r(x)) \quad (1.3)$$

which is valid for every admissible ball $B_r(x)$ provided that $u \in C^2(D) \cap L^1(D)$ solves (1.2) in $D$. The identity (1.3) was recently obtained by the author [15], but it was a surprise to discover that only mean value formulas for spheres and circles were known before that for panharmonic functions. As early as 1896, Neumann [16, Chapter 9, Section 3] derived the following formula for spheres in $\mathbb{R}^3$:  

$$a_1^+ (\mu r) u(x) = M(u, \partial B_r(x)),$$

where $a_1^+ (\mu r) = \sinh(\mu r)/(\mu r)$. Much later, Duffin [12, pp. 111, 112] independently rediscovered the same proof, but in $\mathbb{R}^2$ with $a_1^+ (t)$ replaced by $a_0^+ (t) = I_0(t)$.

In order to reformulate the question quoted above for nuclear setting, we recall that Yukawa [11, p. 49] described a source of nuclear force located at $y \in \mathbb{R}^3$ with the help of the following potential:

$$E^-_\mu (x, y) = \frac{\exp\{-\mu|x-y|\}}{|x-y|}, \quad \mu > 0, \quad x \in \mathbb{R}^3 \setminus \{y\}, \quad (1.4)$$

which is a nonnegative fundamental solution to Equation (1.2) decaying rapidly with the distance. Another fundamental solution of this equation grows with the distance, namely:

$$E^+_\mu (x, y) = \frac{\exp\{\mu|x-y|\}}{|x-y|}, \quad \mu > 0, \quad x \in \mathbb{R}^3 \setminus \{y\}. \quad (1.5)$$

The existence of two linearly independent fundamental solutions distinguishes (1.2) from the Laplace equation.

For every $r > 0$ and arbitrary $x_0 \in \mathbb{R}^3$ these fundamental solutions define two families of integrable panharmonic functions

$$B_r(x_0) \ni y \mapsto E^\pm_\mu (y, x) \text{ parametrized by } x \in \mathbb{R}^3 \setminus B_r(x_0).$$

The mean value property (1.3) yields that

$$a_3^+ (\mu r) E^\pm_\mu (x, x_0) = M(E^\pm_\mu (\cdot, x), B_r(x_0)) \quad (1.6)$$

for every element of these families. This identity is analogous to (1.1) with $D$ changed to $B_r(x_0)$, but involves the factor $a_3^+ (\mu r) > 1$ on the left-hand side. Now, in view of Theorem 1.3 and the identity (1.6), it is natural to expect that the following assertion is true.

377
**Theorem 1.5.** Let “potato” occupy a bounded domain \( D \subset \mathbb{R}^3 \) whose complement is connected, and let \( r \) be such that \(|B_r| = |D|\). If for some \( x_0 \in D \) the mean value identity
\[
a_+^\lambda (\mu r) E_\mu^\pm (x, x_0) = M(E_\mu^\pm (\cdot, x), D) \tag{1.7}
\]
is fulfilled for every \( x \notin D \) and each fundamental solution of Equation (1.2), then \( D = B_r(x_0) \).

It occurs that a characterization of balls analogous to Theorem 1.4 is valid for metaharmonic functions; the term is just an abbreviation for a solution to the Helmholtz equation
\[
\nabla^2 u + \lambda^2 u = 0, \quad \lambda \in \mathbb{R} \setminus \{0\}. \tag{1.8}
\]

Vekua coined it in 1943 in his still widely cited article, which was also published as Appendix 2 to the monograph [17]. In the following assertion proved recently, \( J_\nu \) is the Bessel function of order \( \nu \), where its \( n \)th positive zero is denoted by \( j_{\nu,n} \).

**Theorem 1.6 (Kuznetsov [18]).** Let \( D \subset \mathbb{R}^m, m \geq 2 \), be a bounded domain whose complement is connected, and let \( r > 0 \) be such that \(|B_r| = |D|\). Suppose that there exists a point \( x_0 \in D \) such that for some \( \lambda > 0 \) the identity
\[
M_\mu^\pm \lambda u(x_0) a_m^\pm (\lambda r) = M(u, D), \quad a_m^\pm (t) = \Gamma\left(\frac{m}{2} + 1\right) \frac{J_{m/2}(t)}{(t/2)^{m/2}},
\]
is valid for every function \( u \) metaharmonic in \( D_r \). If also
\[
D \subset B_{r_0}(x_0), \quad \lambda r_0 = j_{\nu,m}, \tag{1.9}
\]
then \( D = B_r(x_0) \).

**Remark 1.1.** For a fixed \( \lambda > 0 \) the assertion is applicable only to domains whose volume is less than or equal to \(|B_{r_0}|\), where \( \lambda r_0 = j_{\nu,m} \). Indeed, every such domain must lie within a ball of radius \( r_0 \), and this distinguishes Theorem 1.6 from Theorem 1.4, imposing no restriction on the domain volume.

The reason for the restriction (1.9) is as follows. The function \( a_{m-2}^-(t) \) used in the proof of Theorem 1.6 oscillates about zero, and so only an interval near the origin, where \( a_{m-2}^- \) decreases monotonically, can be used. On the opposite, \( a_{m-2}^+(t) \) used in the similar proof of Theorem 1.4 is greater than or equal to unity and increases monotonically.

Theorem 1.6, like Theorem 1.4, is inverse of the \( m \)-dimensional mean value property
\[
a_m^-(\lambda r) u(x) = M(u, B_r(x)) \tag{1.10}
\]
which holds for every admissible ball \( B_r(x) \) provided \( u \in C^2(D) \cap L^1(D) \) solves (1.8) in \( D \). The identity (1.10) was also obtained by the author [15] only recently.

The existence of two linearly independent fundamental solutions is another feature common to Equations (1.2) and (1.8), but unlike (1.4) and (1.5) the solutions of (1.8), namely,
\[
E_\lambda^\pm (x, y) = \frac{\exp\{\pm i\lambda|x - y|\}}{|x - y|}, \quad \lambda > 0, \quad x \in \mathbb{R}^3 \setminus \{y\}, \tag{1.11}
\]
are complex-valued, thus allowing to describe outgoing and incoming acoustic waves in the time domain (see [17, Appendix 2]).
As in the panharmonic case, for every \( r > 0 \) and arbitrary \( x_0 \in \mathbb{R}^3 \) the fundamental solutions (1.11) define two families of integrable metaharmonic functions

\[ B_r(x_0) \ni y \mapsto E^\pm_\lambda(y, x) \]

for every element of these families. This identity is analogous to (1.1) with \( D \) changed to \( B_r(x_0) \), but involves the factor \( a^{-3}(\lambda r) < 1 \) on the left-hand side, which is positive on the interval \((0, j_{3/2})\) and then changes sign. In view of Theorem 1.3 and the identity (1.12), it is natural to expect that the following assertion is true.

**Theorem 1.7.** Let “potato” occupy a bounded domain \( D \subset \mathbb{R}^3 \) whose complement is connected, and let \( r \) be such that \( |B_r| = |D| \). If for some \( x_0 \in D \) the condition (1.9) is fulfilled for \( m = 3 \) and some \( \lambda > 0 \) and the mean value identity

\[ a^{-3}(\lambda r) E^\pm_\lambda(x, x_0) = M(E^\pm_\lambda(\cdot, x), D) \tag{1.13} \]

holds for every \( x \notin D \) and each fundamental solution of Equation (1.8), then \( D = B_r(x_0) \).

**Remark 1.2.** In the acoustical case with the wave number \( \lambda > 0 \) fixed, the condition (1.9) imposes a restriction on the size of “potato” \( D \). Namely, the volume of \( D \) must be less than or equal to \( |B_{r_0}| \), where \( \lambda r_0 = j_{3/2} \); moreover, every such domain must lie within a ball of radius \( r_0 \).

## 2 Proof of Theorems 1.5 and 1.7

**Proof of Theorem 1.5.** Since \( E^+_\mu(x, x_0) \) and \( E^-_\mu(x, x_0) \) satisfy (1.7) for every \( x \notin D \), the same is true for every linear combination of these fundamental solutions. In particular,

\[ |D| a^+_3(\mu r) \frac{\sinh(\mu|x-x_0|)}{\mu|x-x_0|} = \int_D \frac{\sinh(\mu|x-y|)}{\mu|x-y|} \, dy \quad \forall x \notin D. \]

Moreover, the identity is valid throughout \( \mathbb{R}^3 \) because we have real-analytic functions of \( x \) on both sides (a consequence of the analyticity of \( z^{-1} \sinh z \)), and so we substitute \( x = x_0 \) thus obtaining

\[ |D| a^+_3(\mu r) = \int_D \frac{\sinh(\mu|x_0-y|)}{\mu|x_0-y|} \, dy. \]

Let us relocate, without loss of generality, the domain \( D \) so that \( x_0 \) coincides with the origin, which simplifies the identity to

\[ |D| a^+_3(\mu r) = \int_D U_+(y) \, dy, \quad U_+(y) = \frac{\sinh(\mu|y|)}{\mu|y|}. \tag{2.1} \]

On the other hand, the mean value property (1.3) is valid for \( U_+ \) over \( B_r \):

\[ |B_r| a^+_3(\mu r) = \int_{B_r} U_+(y) \, dy. \tag{2.2} \]
If we assume that $D \neq B_r$, then $G_i = D \setminus \overline{B_r}$ and $G_e = B_r \setminus \overline{D}$ are bounded open sets such that $|G_e| = |G_i| \neq 0$, which follows from the assumptions made about $D$ and $r$. Then, subtracting (2.2) from (2.1), we obtain

$$0 = \int_{G_i} U_+(y) \, dy - \int_{G_e} U_+(y) \, dy > 0.$$ 

Indeed, the difference is positive since $U_+(y)$ (positive and monotonically increasing with $|y|$) is greater than $[U_+(y)]_{|y|=r}$ in $G_i$ and less than $[U_+(y)]_{|y|=r}$ in $G_e$, whereas $|G_i| = |G_e|$. The obtained contradiction proves the result. \qed

**Remark 2.1.** The final part of this proof repeats literally the argument used in the proof of Theorem 1 in [13].

**Proof of Theorem 1.7.** In the acoustical case, we suppose, without loss of generality, that $D$ is located so that $x_0$ coincides with the origin, and consider the following linear combination of $E^{-}(y, 0)$ and $E^{+}(y, 0)$:

$$U_-(y) = \frac{\sin(\lambda |y|)}{\lambda |y|}.$$ 

As in the proof of Theorem 1.5, we arrive at the following consequence of (1.13):

$$|D| a_3^{-}(\lambda r) = \int_{D} U_-(y) \, dy,$$ 

where the condition $U_-(0) = 1$ is taken into account (cf. (2.1)). Again, assuming that $D \neq B_r$, we consider the bounded open sets $G_i = D \setminus \overline{B_r}$ and $G_e = B_r \setminus \overline{D}$ such that $|G_e| = |G_i| \neq 0$; a consequence of the assumptions made about $D$ and $r$.

To obtain a contradiction, we write the mean value property (1.10) for $U_-$ over $B_r$:

$$|B_r| a_3^{-}(\lambda r) = \int_{B_r} U_-(y) \, dy.$$ 

Subtracting (2.4) from (2.3) and using the definition of $r$, we obtain

$$0 = \int_{G_i} U_-(y) \, dy - \int_{G_e} U_-(y) \, dy < 0.$$ 

Indeed, $U_-(y)$ monotonically decreases with $|y|$ in the whole $D$ because $D \subset B_{r_0}$. Therefore, the difference is negative because $U_-(y)$ is strictly greater than $[U_-(y)]_{|y|=r}$ in $G_e$ and strictly less than this value in $G_i$, whereas $|G_i| = |G_e|$. The obtained contradiction proves the theorem. \qed

**Remark 2.2.** Here, the argument is similar to that in the proof of Theorem 1.5. Indeed, both rely on monotonicity of a certain solution to the corresponding equation. However, there is an essential distinction between the two theorems concerning the size of a domain. Indeed, no restriction on the size is imposed in Theorem 1.5. However, the radially symmetric function $U_-$ decreases monotonically near the origin, but only when $\lambda |y|$ belongs to a bounded interval adjacent to zero, for which reason the condition (1.9) is imposed in Theorem 1.7.

380
References

1. B. Epstein, “On the mean-value property of harmonic functions,” *Proc. Am. Math. Soc.* **13**, 830 (1962).

2. B. Epstein and M. M. Schiffer, “On the mean-value property of harmonic functions,” *J. Anal. Math.* **14**, 109–111 (1965).

3. Ü. Kuran, “On the mean value property of harmonic functions,” *Bull. Lond. Math. Soc.* **4**, 311–312 (1972).

4. W. Hansen and I. Netuka, “Inverse mean value property of harmonic functions,” *Math. Ann.* **297**, No. 1, 147–156 (1993); Corrigendum: *Math. Ann.* **303**, No. 2, 373–375 (1995).

5. I. Netuka and J. Veselý, “Mean value property and harmonic functions,” In: *Classical and Modern Potential Theory and Applications*, pp. 359–398, Kluwer, Dordrecht (1994).

6. D. Aharonov, M. M. Schiffer, and L. Zalcman, “Potato Kugel,” *Isr. J. Math.* **40**, 331–339 (1981).

7. M. M. Schiffer, *Selected Papers. Vol. 2*, Springer, New York (2014).

8. E. Lanconelli, “Potato kugel’ for sub-Laplacians,” *Isr. J. Math.* **194**, Part A, 277–283 (2013).

9. G. Cupini and E. Lanconelli, “On the harmonic characterization of domains via mean value formulas,” *Matematiche* **75**, No. 1, 331–352 (2020).

10. N. Kuznetsov, “Inverse mean value properties (a survey),” *J. Math. Sci.* **262**, No. 3, 275–290 (2022).

11. H. Yukawa, “On the interaction of elementary particles. I,” *Proc. Phys.-Math. Soc. Japan (3)* **17**, 48–57 (1935).

12. R. J. Duffin, “Yukawan potential theory,” *J. Math. Anal. Appl.* **35**, 105–130 (1971).

13. N. Kuznetsov, “Characterization of balls via solutions of the modified Helmholtz equation,” *C. R., Math., Acad. Sci. Paris* **359**, No. 8, 945–948 (2021).

14. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge (1944).

15. N. Kuznetsov, “Mean value properties of solutions to the Helmholtz and modified Helmholtz equations,” *J. Math. Sci.* **257**, No. 5, 673–683 (2021).

16. C. Neumann, *Allgemeine Untersuchungen über das Newtonsche Prinzip der Fernwirkungen*, Teubner, Leipzig (1896).

17. I. N. Vekua, *New Methods for Solving Elliptic Equations*, North Holland, Amsterdam etc. (1967).

18. N. Kuznetsov, “Inverse mean value property of metaharmonic functions,” *J. Math. Sci.* **264**, No. 5, 603–608 (2022).

Submitted on July 20, 2022