CONFORMAL FIELD THEORY, 
BOUNDARY CONDITIONS AND 
APPLICATIONS TO STRING THEORY

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Abstract
This is an introduction to two-dimensional conformal field theory and its applications in string theory. Modern concepts of conformal field theory are explained, and it is outlined how they are used in recent studies of D-branes in the strong curvature regime by means of CFT on surfaces with boundary.
1 Introduction and Overview

Conformal field theory in two dimensions (CFT) is an old subject. It is a rich subject, too. Several points of view on CFT are possible. In the present lecture notes, we adopt one perspective that is suitable for the application of conformal field theory to (perturbative) string theory. Thus in particular we will always work with compact world sheets of Euclidean signature. (The signature of the world sheet should not be confused with the signature of space-time; a consistent string theory even requires time-like directions in the target space.)

To understand CFT it is necessary to clearly distinguish between chiral and full conformal field theory. This distinction is, unfortunately, not always respected in the literature. Later in these lectures we will explain in some detail what we understand by chiral and full CFT. For the moment, the reader only needs to retain the following features. The two theories are defined on different types of two-dimensional manifolds:

- **Chiral CFT** is defined on complex curves. It is the theory that appears in the description of the (fractional) quantum Hall effect. And it is also chiral CFT that is related to topological field theory and that, as a consequence, provides invariants of knots and three-manifolds.
- **Full CFT**, in contrast, is defined on real two-dimensional manifolds with a conformal structure, i.e. an equivalence class, with respect to local rescalings, of metrics. The manifold can have boundaries, and need not be orientable. Notice that even when it is orientable, no definite orientation is chosen.

Full CFT has numerous applications. The point that is of prime interest here is that it provides the world sheet theories for string theory. Other applications concern two-dimensional critical systems of classical statistical mechanics and quasi one-dimensional condensed matter physics. Also, CFT models that are based on a Lagrangian are generally models of full CFT. Note that for writing down a Lagrangian, the world sheet must typically have a Hodge star which is needed for the conformal structure, but it is not necessary to have a metric on the world sheet. Frequently the Lagrangian formulation is in the form of a sigma model, i.e. the path integral is taken over maps $X$ from the world sheet to some manifold, the *target space*. (See e.g. [27] for a review of Wess–Zumino–Witten models from this point of view.) From any chiral CFT one can construct a full CFT. Conversely, it can be expected that every consistent conformal field theory in two dimensions comes from an underlying chiral CFT; in all applications of CFT we are aware of, such a relationship is known in detail.

We will be particularly interested in the study of world sheets with boundaries. Open string perturbation theory in the background of certain string solitons, the so-called D-branes, indeed forces one to analyze conformal field theories on surfaces that may have boundaries and/or can be non-orientable. Other applications of CFT on surfaces with boundary – which are, however, beyond the scope of these notes – arise in the study of defects in systems of condensed matter physics, of percolation probabilities, and in dissipative quantum mechanics.

These notes are organized as follows. We first review in Section 2 aspects of two-dimensional manifolds that will enter our discussion. We then turn to the analysis of chiral conformal field theory. After discussing the underlying algebraic structure – vertex operator algebras (VOAs, for short) – in Section 3, we proceed in Section 4 to combine VOAs and the theory of complex curves, which enables us to define conformal blocks. The latter are the central objects in a chiral CFT. Their physical role is twofold. On one hand, they are building blocks for the correlators of full CFT, and on the other hand they are the spaces of physical states in topological field
theories (TFTs) in three dimensions.

After this discussion of chiral CFT, in Section 5 we explain the construction of a full CFT from a chiral CFT. In Section 6 it is shown how to construct (perturbative) string vacua from full CFTs with appropriate properties. The consideration of full CFT on world sheets with boundaries will also give us some insight into what is sometimes called ‘D-geometry’ [11], that is, the physics of strings and branes for finite values of the string length, i.e. in the region where curvature effects become important.

We conclude this introductory section with a word on the citation policy adopted in these lecture notes. We decided to mention mainly review papers or papers that have been helpful for ourselves to improve our understanding of the topics treated here and of their physical and mathematical background. Earlier references are only included when they treat issues for which we do not know of an accessible review; otherwise we refer the reader to those reviews as a guide to original work. (A more extensive bibliography on CFT can be found at http://home.cern.ch/~jfuchs/ref.html; for lack of space we have refrained from including most of those references here.)

## 2 Two-dimensional manifolds

Manifolds of (real) dimension two constitute the ‘arena’ for our studies. In this section we briefly summarize those features that are needed in our discussion. We start with topological aspects, then discuss complex geometry of these manifolds, and finally describe Teichmüller and moduli spaces.

### 2.1 Topological aspects

Real connected compact two-dimensional topological manifolds $\Sigma$ are classified by three non-negative integers: The numbers $g$ of handles, $b$ of boundaries, and $c$ of crosscaps.

By cutting out a disc from a given surface one introduces a new boundary component. (We insist that the boundaries obtained this way are genuine, physical boundaries. They must never be confused with the small discs one often imagines around insertions of fields, which merely serve to specify local coordinates around the insertion points.) Similarly, a crosscap can be inserted in a surface by first cutting out a disc and then identifying opposite points of the boundary of the disc. The insertion of a crosscap makes a manifold unorientable. Two crosscaps are topologically equivalent to a single crosscap plus a handle, so that it is not necessary to consider more than two crosscaps. World sheets with boundaries play an important role in the description of D-branes; world sheets with crosscaps enter in the construction of string theories of ‘type I’.

An important topological quantity is the Euler characteristic $\chi$. It is defined as $\chi = 2 - 2g - b - c$. Its relevance in string perturbation theory stems from the fact that the contribution of a surface of Euler characteristic $\chi$ to the perturbation expansion is weighted with a factor $g_{\text{str}}^{-\chi}$, where $g_{\text{str}}$ is the string coupling constant. The most important manifolds in our discussion will be the ones of Euler characteristic $\chi = 2$, i.e. the two-sphere $S^2$, of $\chi = 1$, i.e. the disc and the real projective space $\mathbb{R}P^2$ (also called the crosscap), and the manifolds with $\chi = 0$, i.e. the torus ($g = 1, b = c = 0$), the annulus ($g = c = 0, b = 2$), the Möbius strip ($g = 0, c = b = 1$), and the Klein bottle ($g = b = 0, c = 2$).
Another notion we will frequently use is the one of the mapping class group $\Omega$ of $\Sigma$. Consider the group $\text{Homeo}(\Sigma)$ of all homeomorphisms of $\Sigma$. It has a normal subgroup $\text{Homeo}_0(\Sigma)$, consisting of those homeomorphisms that are homotopic to the identity. If $\Sigma$ is orientable, we introduce the subgroup $\text{Homeo}^+(\Sigma)$ of orientation preserving homeomorphisms. For orientable surfaces, we define the mapping class group as the quotient

$$\Omega(\Sigma) := \text{Homeo}^+(\Sigma)/\text{Homeo}_0(\Sigma),$$

while for unorientable surfaces we define it to be

$$\Omega(\Sigma) := \text{Homeo}(\Sigma)/\text{Homeo}_0(\Sigma).$$

The mapping class group $\Omega(\Sigma)$ acts in particular on the first homology $H_1(\Sigma, \mathbb{Z})$. For orientable surfaces without boundary $H_1(\Sigma, \mathbb{Z})$ is torsion free and comes with a symplectic form from the intersection of one-cycles. (For unorientable surfaces $H_1(\Sigma, \mathbb{Z})$ typically has a torsion part.) The mapping class group preserves the intersection form, and hence we obtain a natural group homomorphism from $\Omega(\Sigma)$ to the corresponding symplectic group; this homomorphism is actually surjective so that we have

$$\Omega(\Sigma) \to \text{Sp}(2g, \mathbb{Z}).$$

For the torus, we even have identity:

$$\Omega(T^2) = \text{Sp}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z}).$$

This group is called the modular group.

### 2.2 The Schottky double

We now turn to a construction that allows us to restrict our attention to the case when the manifold $\Sigma$ is oriented and has no boundary: The Schottky double $\hat{\Sigma}$ of $\Sigma$. The idea is to double the space, except for the points on the boundary. (This mimics the method of mirror charges in classical electrodynamics.)

For $\Sigma$ a disc, the double $\hat{\Sigma}$ is a sphere, obtained by gluing a disc and its mirror image along their boundaries. Notice that the reflection $\sigma$ about the equator of this sphere is an orientation-reversing involution, $\sigma^2 = \text{id}$. The original surface $\Sigma$ can be obtained as the quotient — or world sheet orbifold, or parameter space orbifold — of the double:

$$\Sigma = \hat{\Sigma}/\sigma.$$  \hspace{1cm} (1)

The fixed point set of $\sigma$ gives precisely the boundary of $\Sigma$. For $\Sigma$ the crosscap $\mathbb{RP}^2$, the double is again the sphere, but $\sigma$ is now the antipodal map.

It is in fact true in general that $\Sigma$ is obtained from its cover $\hat{\Sigma}$ as the fixed point set under an orientation-reversing involution. Furthermore, $\hat{\Sigma}$ has a natural orientation. For example, for $\Sigma$ without boundary, the double is just the total space of the orientation bundle. (The orientation bundle is a $\mathbb{Z}_2$-bundle over $\Sigma$ whose fiber over $p \in \Sigma$ consists of two points, corresponding to the two local orientations at $p$. Thus for orientable boundaryless $\Sigma$ it is a trivial bundle, the total space being the disconnected sum of two copies of $\Sigma$. In the results about complex curves
The structures we have met so far all belong to the realm of two-dimensional topological manifolds. We now turn to aspects of conformal and complex manifolds that we will need for CFT. Thus let $X$ be a two-dimensional conformal manifold, and $\hat{X}$ its (topological) Schottky double. The following idea turns out to be central:

*Full CFT on a conformal manifold $X$ is constructed from chiral CFT on the double $\hat{X}$ of $X$.*

To understand this construction, we will also need tools from complex geometry; they are the subject of the next subsection.

### 2.3 Complex geometry

We will be particularly interested in two-dimensional manifolds that are even complex manifolds, i.e. that possess a holomorphic structure. (A holomorphic structure is the choice of an atlas with maps that take their values in subsets of the complex plane in such a manner that the transition functions are holomorphic.) On such manifolds, all the additional power of complex geometry is at our disposal.

The following feature is special to two dimensions: A holomorphic structure on an orientable manifold $X$ is equivalent to the choice of a conformal structure plus an orientation. This can be seen as follows. A classical theorem asserts that every metric on a real two-dimensional differentiable manifold is locally conformally flat, i.e. we can find charts $U$ such that the metric is of the form $g = \lambda(x, y) \big( dx^2 + dy^2 \big)$ with $\lambda$ a positive real function on $U$. When $X$ is oriented, we can choose the charts to be compatible with the orientation. The transformations between different charts are then oriented and conformal diffeomorphisms, i.e. biholomorphic transformations, so we have obtained a complex structure on $X$; this complex structure depends only on the conformal equivalence class of the metric. Conversely, it is easy to see that a complex structure implies an orientation and a conformal structure. In contrast, higher-dimensional manifolds are not necessarily locally conformally flat. Accordingly, additional integrability conditions must then be met to obtain a complex structure.

Now the double $\hat{X}$ of a conformal manifold $X$ is oriented and inherits a conformal structure from $X$. In other words, $\hat{X}$ always has a complex structure, i.e. the double is a complex curve! This simple observation allows us to construct full CFT on $X$ in terms of chiral CFT on $\hat{X}$. In particular, at the level of chiral CFT the full power of holomorphy is available, even for the study of open strings.

Once we have a complex structure on a two-manifold $X$, we can introduce the holomorphic cotangent bundle $K$, also called the *canonical bundle* of $X$. $K$ is a complex line bundle, and its sections are proportional to $dz$; its powers $K^{\otimes n}$ are complex line bundles as well.
2.4 Teichmüller space and moduli space

One and the same oriented topological manifold \(\hat{\Sigma}\) without boundary can typically be endowed in different, inequivalent ways with a complex structure. In fact, a lot of important information about CFTs is gained by studying how structures change when one varies the underlying curve. Points at which the underlying curve degenerates in not too bad a way are of special importance; they lead to factorization constraints.

We first consider the space \(\tilde{\mathcal{M}}(\hat{\Sigma})\) of all complex structures \(\mathcal{C}\) on an orientable manifold \(\hat{\Sigma}\). On the space \(\tilde{\mathcal{M}}(\hat{\Sigma})\) the group \(\text{Homeo}(\hat{\Sigma})\) acts as follows. The complex structure \(f^*(\mathcal{C})\) is defined by the requirement that the map

\[
f : (\hat{\Sigma}, f^*(\mathcal{C})) \rightarrow (\hat{\Sigma}, \mathcal{C})
\]

is holomorphic if \(f\) preserves the orientation and antiholomorphic if \(f\) reverses the orientation. One then defines the Teichmüller space \(\mathcal{T}(\hat{\Sigma})\) as the quotient

\[
\mathcal{T}(\hat{\Sigma}) := \tilde{\mathcal{M}}(\hat{\Sigma}) / \text{Homeo}_0(\hat{\Sigma}).
\]

One can show that the Teichmüller space is a complex manifold. For \(g = 0\) it is just a point, while for \(g = 1\) it is isomorphic to the complex upper half plane, \(\mathcal{T}_1 = \{\tau \in \mathbb{C} | \text{Im} \tau > 0\}\). For every \(g \geq 2\) the Teichmüller space is isomorphic to \(\mathbb{C}^{3g-3}\). (To be precise, it is isomorphic to \(\mathbb{C}^{3g-3}\) as a topological manifold, and to \(\mathbb{R}^{6g-6}\) as a real analytic manifold, but not to \(\mathbb{C}^{3g-3}\) as a complex analytic manifold.)

On the Teichmüller space, the mapping class group \(\Omega(\hat{\Sigma})\) acts as the full group of holomorphic automorphisms of \(\mathcal{T}(\hat{\Sigma})\). The moduli space \(\mathcal{M}_g(\hat{\Sigma})\) is obtained from the Teichmüller space by dividing out the action of \(\Omega(\hat{\Sigma})\):

\[
\mathcal{M}_g(\hat{\Sigma}) = \mathcal{T}(\hat{\Sigma}) / \Omega(\hat{\Sigma}) = \tilde{\mathcal{M}}(\hat{\Sigma}) / \text{Homeo}^+(\hat{\Sigma}).
\]

If \(\sigma\) is an orientation reversing homeomorphism of \(\hat{\Sigma}\), it induces by the same procedure an antiholomorphic involution \(\sigma^*\) on the Teichmüller space. This will be important in the discussion of the double. One can show that for \(\hat{\Sigma}\) of genus \(g\), there are \([\frac{3g+4}{2}]\) inequivalent involutions.

The Teichmüller space is simply connected and is in fact the universal covering space of moduli space. This implies that the mapping class group is the fundamental group of the moduli space:

\[
\pi_1(\mathcal{M}_g) = \Omega(\hat{\Sigma}). \quad (2)
\]

(As one is dealing with singular spaces, one must be careful with the definition of the fundamental group. For details, see \[30\].) For \(g = 1\), the action of \(\Omega\) is the standard action

\[
\tau \mapsto \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \tau = \frac{a\tau + b}{c\tau + d} \quad (3)
\]

of \(\text{SL}(2, \mathbb{Z})\) on the upper half-plane \(H\). The action of the mapping class group is not free, and as a consequence the moduli space \(\mathcal{M}_g\) has orbifold singularities, which correspond to curves with non-trivial automorphisms. For genus one, these are the points \(\tau = i\) and \(\tau = \exp(2\pi i/3)\). However, these singular points are not the most interesting ones for conformal field theory.\(^1\)

\(^1\) See, however, a recent application of curves with automorphisms to derive constraints on the representation of the mapping class group on conformal blocks \[3\].
Of far more interest are singularities that are cusps like the point $\tau = i\infty$ for $\hat{g} = 1$, where the curve degenerates and where factorization constraints can be formulated.

We also need Teichmüller spaces for two-dimensional manifolds $\Sigma$ that are not oriented. $\Sigma$ can even be unorientable and have a boundary. Without an orientation, the notion ‘holomorphic’ does not make sense, so we replace in the definition of the Teichmüller space $\mathcal{M}(\Sigma)$ by the space of all di-analytic structures. (A di-analytic structure is an atlas in which the transition functions are either holomorphic or anti-holomorphic.) Teichmüller and moduli space are then defined analogously as before.

Consider now the double $\hat{\Sigma}$ of $\Sigma$, together with the orientation reversing involution $\sigma$ such that $\hat{\Sigma}/\sigma = \Sigma$. One can show that every di-analytic structure on $\Sigma$ gives a unique conformal structure on $\hat{\Sigma}$. The Teichmüller space $T(\Sigma)$ can be canonically identified with the fixed point set of $T(\hat{\Sigma})$ under the involution $\sigma^*$:

$$T(\Sigma) \cong T(\hat{\Sigma})_{\sigma^*}.$$

One can show that this space is real-analytically isomorphic to $\mathbb{R}^{3\chi}$ for Euler characteristic $\chi(\Sigma) \leq -1$, isomorphic to $\mathbb{R}$ for Euler characteristic 0, and to a point for $\chi(\Sigma) = 1$. It can also be shown that Teichmüller spaces for surfaces $\Sigma$ with the same Euler characteristic are real-analytically isomorphic.

Finally, using the action of the mapping class group $\Omega(\Sigma)$ on $T(\hat{\Sigma})$ one can canonically identify $\Omega(\Sigma)$ with the commutant $\Omega_\sigma(\hat{\Sigma})$ of $\sigma^*$ in $\Omega(\Sigma)$. This subgroup is also called the relative modular group. We can therefore identify the moduli space $\mathcal{M}(\Sigma)$ with the quotient

$$\mathcal{M}(\Sigma) = T(\hat{\Sigma})_{\sigma^*}/\Omega_\sigma(\hat{\Sigma}).$$

### 2.5 First remarks on strings

We are now in a position to describe briefly the central goal of (perturbative) string theory: Construct consistent quantum field theories by formulating a perturbation expansion as a sum over two-dimensional manifolds, rather than use (as in Lagrangian quantum field theory) Feynman diagrams, which are graphs, i.e. singular one-dimensional manifolds. What is truly remarkable about string theory is that this idea is applicable even for theories that include gravity and do not possess a well-behaved Feynman perturbation expansion, see e.g. [37].

In the same way we would not assign any physical reality to the lines of a Feynman diagram, we should not directly assign a physical meaning to the string world sheet. As a consequence, a basic theme of string theory can be formulated as “get rid of the world sheet”. This is done by taking first the cohomology with respect to the (super-)Virasoro algebra, then one eliminates the conformal structure by integrating over it, and finally, one sums over all genera, weighted by the string coupling constant $g_{\text{str}}$, to even get rid of topological aspects of the world sheet. Yet, after all these manipulations the string theory one obtains still strongly depends on the conformal field theory model one started with.

If string theory indeed provides a perturbative quantization of certain classical field theories including gravitation, then the following question arises. These field theories possess non-trivial classical solutions [12]. Among these solutions there are configurations that generalize the Schwarzschild solution of Einstein gravity in the sense that they not just describe a black
hole, i.e. a singularity of signature $1+0$, but also higher-dimensional singularities of signature $1+p$, so-called black $p$-branes.

In certain cases it is known how to set up ordinary field theoretic perturbation theory around such solitonic solutions, e.g. around instanton solutions of four-dimensional Yang–Mills theory. It was a major breakthrough for string theory when Polchinski discovered a prescription for string perturbation theory in the background of $p$-branes. His idea was to use open strings that are constrained to end on the black brane. This idea triggered enormous progress in the understanding of (some) non-perturbative sectors of string theory. For us, this constitutes a strong motivation to study the corresponding world sheet theories, that is, conformal field theory on surfaces with boundaries. String theory also has other classical solutions than D-branes, like the Neveu–Schwarz five-brane or the fundamental string solution. For the five-brane a string perturbation theory has been proposed in terms of a different chiral CFT than the one of the flat background. This proposal is rather different in structure from the one for string perturbation theory in the background of D-branes, where the same chiral CFT as for the configuration without branes is used.

3 Algebraic aspects of chiral CFT: Vertex operator algebras

This section is devoted to the study of algebraic and representation theoretic aspects of chiral CFT. The main notion we will introduce is the one of a vertex operator algebra (VOA). It formalizes the notion of a chiral symmetry algebra.\footnote{Unfortunately, the term chiral algebra has been used in the mathematics literature for different, though related, objects, see e.g. \cite{26}.}

3.1 The Virasoro algebra and its super-extensions

First, however, we need a few facts about infinitesimal conformal symmetries in two dimensions and its super-extensions. Conformal symmetry is encoded in the Virasoro algebra. This is the central extension of the Lie algebra of vector fields on a circle $S^1$; thus it is the infinite-dimensional Lie algebra that is spanned by generators $L_n$ with $n \in \mathbb{Z}$ and a central element $C$, subject to the relations

\begin{align}
[L_n, L_m] &= (n-m) L_{n+m} + \frac{1}{12} (n^3 - n) \delta_{n+m,0} C, \\
[L_n, C] &= 0. 
\end{align}

If $v$ is an eigenvector of $L_0$ in a representation of the Virasoro algebra, then its eigenvalue is called the conformal weight of $v$ and denoted by $\Delta_v$. Using a formal variable $z$, we can combine the generators into a ‘field’

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

called the chiral stress-energy tensor. The formal variable $z$ allows us to characterize the Virasoro algebra also as a central extension of the Lie algebra of derivations of the field of rational functions in $z$.\footnote{Unfortunately, the term chiral algebra has been used in the mathematics literature for different, though related, objects, see e.g. \cite{26}.}
There are various super-extensions of this construction. The simplest one, the $N=1$ super-conformal algebra (or $N=1$ Virasoro algebra), is described by the superfield

$$T(z, \vartheta) = \frac{1}{2} G(z) + \vartheta T(z),$$

whose first component $G(z)$ has the expansion

$$G(z) = \sum_{r \in \mathbb{Z}+\epsilon} G_r z^{-r-3/2}.$$  

The parameter $\epsilon \in \{0, \frac{1}{2}\}$ depends on the ‘sector’: $\epsilon = 0$ in the Ramond sector, while $\epsilon = 1/2$ in the Neveu–Schwarz sector. For later purposes one should keep in mind that Ramon d and Neveu–Schwarz sector are distinguished by the monodromies of a specific field. The commutation relations among $G$ and $T$ read

$$[L_n, G_r] = \left( \frac{1}{2} n - r \right) G_{n+r},$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4}) \delta_{r,s,0} C.$$  

The $N=1$ Virasoro algebra also possesses a geometric interpretation. Consider the punctured superdisc $(\mathbb{C}^\times)^1$. This space is characterized by the space of functions on it, which is $C[z, z^{-1}, \vartheta]$, i.e. consists of Laurent polynomials in a usual bosonic variable $z$ and a Grassmann variable $\vartheta$. It possesses a contact structure, given by $dz - \vartheta d\vartheta$, and the Lie algebra of derivations of $C[z, z^{-1}, \vartheta]$ that preserve this contact structure is precisely the $N=1$ Virasoro algebra.

For the applications we have in mind we need a non-minimal supersymmetric extension of the Virasoro algebra – the $N=2$ superconformal algebra (or $N=2$ Virasoro algebra), which apart from $T(z)$ contains two supercurrents $G^\pm(z)$ as well as an abelian current $J(z)$. Their non-vanishing (anti-) commutation relations read

$$\{G^+_r, G^-_s\} = 2L_{r+s} + (r-s) J_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4}) \delta_{r+s,0} C,$$

$$[J_n, J_m] = \frac{1}{3} \delta_{n+m,0} C,$$

$$[J_n, G^\pm_r] = \pm G^\pm_{n+r}.$$  

(6)

It will be important that this algebra admits a family of automorphisms, indexed by a parameter $\theta \in \frac{1}{2}\mathbb{Z}$ and known as spectral flow. These automorphisms act as

$$\omega_\theta(L_n) = L_n - \theta J_n + \frac{C}{6} \theta^2 \delta_{n,0},$$

$$\omega_\theta(J_n) = J_n - \frac{C}{3} \theta \delta_{n,0},$$

$$\omega_\theta(G^\pm_r) = G^\pm_{r+\theta}.$$  

The $N=2$ super Virasoro algebra also admits another family of automorphisms, called odd spectral flow. It is again indexed by a parameter $\theta \in \frac{1}{2}\mathbb{Z}$:

$$\alpha_\theta(L_n) = L_n + \theta J_n + \frac{C}{6} \theta^2 \delta_{n,0},$$

$$\alpha_\theta(J_n) = -J_n - \frac{C}{3} \theta \delta_{n,0},$$

$$\alpha_\theta(G^\pm_r) = G^\mp_{r+\theta}.$$  

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While the ordinary spectral flow provides a group of automorphisms, satisfying $\omega_{\theta_1} \circ \omega_{\theta_2} = \omega_{\theta_1 + \theta_2}$, the odd spectral flow obeys the composition law $\alpha_{\theta_1} \circ \alpha_{\theta_2} = \omega_{\theta_2 - \theta_1}$; in particular all $\alpha_{\theta}$ are involutions. The two types of flows are related by

$$\alpha_{\theta_2} \circ \omega_{\theta_1} = \alpha_{\theta_2 - \theta_1} \quad \text{and} \quad \omega_{\theta_2} \circ \alpha_{\theta_1} = \alpha_{\theta_1 + \theta_2}.$$ 

### 3.2 Chiral algebras

We are now ready to introduce the notation of a chiral algebra. The following structure is expected to be present in every chiral CFT: First, a vacuum vector $v_\Omega$ and its 'descendants’, which span a vector space $\mathcal{H}_\Omega$; they should somehow encode the chiral symmetries of the theory. And second, a field-state correspondence $Y$ which associates fields to states in such a way that applying the field to the vacuum $v_\Omega$ gives back the state – a structure familiar from general quantum field theory.

All these ideas are formalized in the notion of a vertex operator algebra (VOA). A VOA consists of the following data:

- a space of states – a graded vector space

$$\mathcal{H}_\Omega = \bigoplus_{n=0}^{\infty} \mathcal{H}(n)$$

whose homogeneous subspaces $\mathcal{H}(n)$ are finite-dimensional, $\dim \mathcal{H}(n) < \infty$;

- a vacuum vector $v_\Omega \in \mathcal{H}_\Omega$;

- a shift operator

$$T : \mathcal{H}_\Omega \to \mathcal{H}_\Omega;$$

- and a field-state correspondence $Y$ involving a formal variable $z$:

$$Y : \mathcal{H}_\Omega \to \text{End}(\mathcal{H}_\Omega)[[z, z^{-1}]]; \quad (7)$$

$Y(v, \cdot)$ is also called the vertex operator for the vector $v \in \mathcal{H}_\Omega$.

These data are subject to the conditions

- that the field for the vacuum $v_\Omega$ is the identity, $Y(v_\Omega, z) = \text{id}_{\mathcal{H}_\Omega};$
- that the field-state correspondence respects the grading, i.e. if $v \in \mathcal{H}(n)$, then all endomorphisms $v_m$ appearing in $Y(v, z) = \sum_{m \in \mathbb{Z}} v_m z^{-n-m}$ have grade $m$: $v_m(\mathcal{H}(p)) \subseteq \mathcal{H}(p+n);$ 
- that one recovers states by acting with the corresponding fields on the vacuum and ‘sending z to zero’, or more precisely,

$$Y(v, z)v_\Omega \in v + z \mathcal{H}_\Omega[[z]]$$

for every $v \in \mathcal{H}_\Omega$;
- that $T$ implements infinitesimal translations,

$$[T, Y(v, z)] = \partial_z Y(v, z); \quad (8)$$
and that the vacuum is translation invariant, $T v_\Omega = 0$.

Finally, the most non-trivial constraint – called *locality* – is that commutators of fields have poles of at most finite order. More precisely, for any two $v_1, v_2 \in H_\Omega$ there must exist a number $N = N(v_1, v_2)$ such that

$$(z_1 - z_2)^N [Y(v_1, z_1), Y(v_2, z_2)] = 0.$$  \(9\)

This is a kind of weak commutativity. Note that this constraint only makes sense because we consider formal series in the $z_i$, which can extend to both arbitrarily large positive and negative powers. Had we restricted ourselves to ordinary Laurent series, i.e. series without arbitrarily large negative powers, (9) would already imply that the commutator vanishes.

We have motivated the structure of a VOA by physical requirements on a chiral symmetry algebra. There have also been attempts to understand this structure from a purely mathematical point of view, by regarding VOAs as (singular) rings in suitable categories. The aim of these constructions has been to better understand quantum affine Lie algebras [45] and to obtain a generalization of VOAs for conformal field theories in higher dimensions [8]; these issues are beyond the scope of the present lectures.

For the application to CFT one needs a special class of VOAs, namely *conformal* VOAs. A VOA is called conformal, of central charge $c$, if there exists a complex number $c$ and a vector $v_{\text{Vir}} \in H_{(2)}$ such that operators appearing in the mode expansion of

$$T(z) := Y(v_{\text{Vir}}, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

possess the following properties. $L_{-1} = T$ gives the translations; $L_0$ is semisimple and reproduces the grading of $H_\Omega$; $L_0$ acts as $n \text{id}$ on $H_{(n)}$; and finally,

$$L_2 v_{\text{Vir}} = \frac{1}{2} c v_\Omega.$$  \[\text{This axiom excludes so-called logarithmic CFTs from our considerations.}\]

These axioms imply that the modes $L_n$ span a Virasoro algebra of central charge $c$.

Let us also mention that the choice of $v_{\text{Vir}}$ for a given VOA and a given shift operator $T$ is not necessarily unique; it is sometimes also called the choice of a ‘conformal structure’. This should not be confused with the conformal structure of the world sheet; it is, rather, related to (chiral) properties of the target space.

For string theory, we will need VOAs with even more structure. We start with the notion of a *graded VOA*, which is a VOA with an additional grading over $\mathbb{Z}$:

$$H_\Omega = \bigoplus_{n=0}^{\infty} \left( \bigoplus_{g \in \mathbb{Z}} H_{(n)}^{(g)} \right).$$

We call $g$ the *ghost number* and require that the field-state correspondence $Y$ has ghost degree 0. We would also like to have an operator on $H_\Omega$ whose eigenvalue gives the ghost number. Rather than to introduce that operator directly, we will adhere to the following principle: Avoid introducing operators on $H_\Omega$ by hand, rather obtain them via the field-state correspondence.
from states in $\mathcal{H}_\Omega$. Thus for a graded VOA we require in addition that there is a state $v_F \in \mathcal{H}_{(1)}^{(0)}$ such that the eigenvalues of

$$F_0 := \text{Res}_{z=0} Y(v_F, z)$$

gives the ghost number. The field $Y(v_F, z)$ is also called ghost number current.

A particularly important case of graded VOAs are topological VOAs, TVOAs for short. Again we introduce more structure by requiring the existence of vectors in $\mathcal{H}_\Omega$: We demand that there are $v_B \in \mathcal{H}_{(2)}^{(-1)}$ and $v_G \in \mathcal{H}_{(1)}^{(1)}$ such that $Q := \text{Res}_{z=0} Y(v_G, z)$ is nilpotent, $Q^2 = 0$. $Q$ is called BRST charge, and the field $Y(v_G, z)$ is called BRST current.

The stress-energy tensor is cohomologically trivial,

$$\{Q, Y(v_B, z)\} = Y(v_{\text{Vir}}, z) \equiv T(z) \quad (10)$$

A few statements about TVOAs are immediate consequences of these definitions:

- It follows from (10) that the Virasoro algebra is $Q$-exact, $[Q, L_n] = 0$. Together with the Jacobi identity and the nilpotency of $Q$ this implies that

$$[L_n, L_m] = [L_n, \{Q, b_m\}] = \{Q, [L_n, b_m]\} = (n-m)\{Q, L_{n+m}\} = (n-m)L_{n+m},$$

i.e. the Virasoro algebra has vanishing central charge.

- The nilpotency of $Q$ allows us to define its cohomology $H_Q$. Translations in $z$ are represented by $L_{-1}$, which is cohomologically trivial; as a consequence, correlators of closed states do not depend on $z$. Thus the cohomology $H_Q$ carries the structure of a two-dimensional topological field theory.

It is easy to give examples of TVOAs. Every $N=2$ VOA gives in fact rise to two different TVOAs. Indeed, the vectors $(L_{-2} \pm \frac{1}{2}J_{-2}) v_\Omega$ provide two different conformal structures on $\mathcal{H}_\Omega$ with vanishing central charge. The corresponding stress-energy tensors read

$$T^\pm(z) = T(z) \pm \frac{1}{2} \partial J(z);$$

this ‘modification’ of $T(z)$ is also known as a topological twisting. With respect to $T^+$, the supercurrent $G^+$ has conformal weight 1 and provides the BRST current, while $G^-$ has conformal weight 2 and gives the antighost current; with respect to $T^-$, the roles of $G^+$ and $G^-$ are interchanged. The ghost number grading is provided by the $U(1)$ current $\pm J(z)$.

We conclude this section with a word of warning. The notion of a VOA is a mathematical formalization of a physical concept, the algebra of chiral symmetries. It is, however, not the only available formalization of that physical concept. For instance, there is another formalization which treats $z$ as a complex number rather than as a formal variable \cite{Friedan:1985ge}; in this framework one attempts to reconstruct the chiral algebra from the values of the $n$-point blocks on a finite-dimensional subspace of $\mathcal{H}_\Omega$. A rather different approach, based on the notion of local observables \cite{LeClair:1989rs}, describes the chiral symmetries in terms of nets of von Neumann algebras over a circle $S^1$ (for details see e.g. \cite{Segal:1974db}). The relation between these different incarnations of chiral symmetries remains to be clarified.
3.3 Examples

It is time for presenting examples of VOAs. We start with the chiral CFT for a single free boson; it is based on the Heisenberg algebra, which has generators $b_n$ with $n \in \mathbb{Z}$ and relations

$$[b_n, b_m] = n \delta_{n+m,0}.$$ 

In this case $\mathcal{H}_\Omega$ is nothing but a Fock space, and $v_\Omega$ is the ground state in this Fock space. To define the field-state correspondence one introduces abelian currents

$$J(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}$$

and identifies $Y(b_{-1}v_\Omega, z) = J(z)$. More generally, one sets

$$Y(b_{n_1} \cdots b_{n_k} \Omega, z) = \frac{1}{(n_1-1)! \cdots (n_k-1)!} \partial_z^{n_1-1} J(z) \cdots \partial_z^{n_k-1} J(z),$$

where the colons indicate a normal ordering. This prescription indeed yields the structure of a VOA. (It is not a trivial exercise, though, to check that this works out.)

This example has an important generalization. Let $L$ be a lattice, and $V = L \otimes \mathbb{Z} \mathbb{R}$ be the associated real vector space with basis $\{b^{(i)}\}$. Suppose that $V$ has a non-degenerate bilinear form $\kappa$. To the infinite-dimensional Lie algebra with basis $b^{(i)}_n$, $n \in \mathbb{Z}$, and relations

$$[b^{(i)}_n, b^{(j)}_m] = n \kappa(b^{(i)}, b^{(j)}) \delta_{n+m,0}$$

one associates a Fock space $\mathcal{H}_\Omega$. One checks that it carries again the structure of a VOA. This structure can be further generalized: Suppose that the bilinear form is such that $L$ is an even lattice, i.e. $\kappa(v, w) \in \mathbb{Z}$ and $\kappa(v, v) \in 2\mathbb{Z}$ for all $v, w \in L$. Then the space

$$\mathcal{H}_\Omega \otimes \mathbb{C}[L],$$

where $\mathbb{C}[L]$ is the group algebra of $L$, has the structure of a VOA, too. It is called the lattice VOA for $L$ and describes $\dim V = \text{rank } L$ many compactified chiral free bosons.

Similar constructions are possible when choosing other infinite-dimensional Lie algebras in place of the Heisenberg algebra. One can e.g. take the Virasoro algebra itself; then $T(z)$ roughly plays the role of the abelian current $J(z)$. When the chiral algebra is generated solely from the Virasoro algebra, then the model is called a Virasoro minimal model. Another important class of examples is furnished by untwisted affine Lie algebras, where non-abelian currents $J^a(z)$ (a an adjoint label of the underlying finite-dimensional simple Lie algebra) take over the role of $J(z)$; the models obtained this way are known as Wess–Zumino–Witten (WZW) models. The Heisenberg, Virasoro and affine Lie algebras belong to the so-called Lie algebras of formal distributions; these are Lie algebras $\mathfrak{g}$ that are spanned over $\mathbb{C}$ by the coefficients of a collection of $\mathfrak{g}$-valued mutually local formal distributions $\{a^\alpha(z)\}$.

The previous construction of a VOA for the free boson can be generalized to arbitrary such Lie algebras $\mathfrak{g}$. Namely, to $\mathfrak{g}$ together with a vector space endomorphism $T$ of $\mathfrak{g}$ such that

\[4\] The term ‘minimal model’ is sometimes also used for rational chiral CFTs whose modules are finitely reducible in terms of modules over the Virasoro algebra.
\[ Ta^\alpha(z) = \partial a^\alpha(z) \] and a weight on the positive subalgebra of \( g \), one can construct in a canonical way a VOA. For more information, we refer to sections 2.7 and 4.7 of [31].

Our last example are the so-called first order systems. This is a family of VOAs, labelled by two parameters \( \lambda \in \mathbb{Z}/2 \) and \( \eta = \pm 1 \). One starts with a Lie algebra generated by two formal distributions

\[
\begin{align*}
    b(z) &= \sum_{n \in \epsilon + \mathbb{Z}} b_n z^{-n-\lambda}, \\
    c(z) &= \sum_{n \in \epsilon + \mathbb{Z}} c_n z^{-n-(1-\lambda)}
\end{align*}
\]

of conformal weight \( \lambda \) and \( 1 - \lambda \), respectively. These fields are bosonic for \( \eta = -1 \) and fermionic for \( \eta = 1 \). In the former case, \( \epsilon \) is zero, while in the latter it takes the values 0 for the Ramond sector and 1/2 for the Neveu–Schwarz sector. The modes of \( b \) and \( c \) obey the (anti-)commutation relations \( \{ c_m, b_n \}_\eta = \delta_{n+m,0} \). The VOA is defined on a Fock space, built on a highest weight vector \( v_\Omega \) with relations

\[
    b_n v_\Omega = 0 \quad \text{for} \quad n \geq 1 - \lambda, \quad b_n v_\Omega = 0 \quad \text{for} \quad n > \lambda.
\]

As the stress-energy tensor one takes

\[
    T(z) = -\lambda :b \partial c: - (1-\lambda) :\partial b c: ;
\]

the Virasoro central charge is \( c = 1 - 3Q^2 \) with \( Q := \epsilon(1-2\lambda) \). There is a U(1) current with modes \( j_n = \sum_{m \in \mathbb{Z}+\epsilon} :c_{n-m} b_m: \) which is, however, anomalous:

\[
    [L_n, j_m] = \frac{1}{2} Q n(n+1) \delta_{n+m,0} - m j_{n+m}.
\]

First order systems are of particular interest for the following values: \( \lambda = 2, \eta = 1 \) yields the ghosts for bosonic reparametrizations of the string world sheet, \( \lambda = 3/2, \eta = -1 \) gives the ones for gauging the fermionic operators of an \( N = 1 \) superconformal symmetry on the world sheet, and \( \lambda = 1/2, \eta = 1 \) is just a complex free fermion.

We finally mention a recent development in mathematics, the construction of the chiral de Rham complex [34]. This allows to associate a vertex operator algebra to every complex variety. For many more details about VOAs, see e.g. the recent review [14], which we have followed in this section.

### 3.4 Representations

Recall that the VOA is formalizing aspects of a chiral symmetry algebra, and symmetries in a quantum theory should be represented on the space of states. We are thus lead to study the representation theory of VOAs.

A representation of a VOA is a graded vector space

\[
    M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n)
\]

with

- a translation operator \( T_M \) : \( M \rightarrow M \) of degree 1 and
- a representation map

\[
    Y_M : \quad \mathcal{H}_\Omega \rightarrow \text{End}(M)[[z, z^{-1}]],
\]

In the context of these VOAs, the term ‘chiral algebra’ is sometimes also used to refer to the Lie algebra of formal distributions, rather than to the VOA itself.
such that
- for $v \in \mathcal{H}^{(n)}$ all components of $Y_M(v, z)$ are endomorphisms of $\mathcal{H}_\Omega$ of degree $-n$;
- $Y_M(v_\Omega) = \text{id}_M$;
- $[T_M, Y_M(v, z)] = \partial_z Y_M(v, z)$;
- and

$$Y_M(v_1, z_1)Y_M(v_2, z_2) = Y_M(Y(v_1, z_1-z_2)v_2, z_2). \quad (11)$$

One can show that the VOA furnishes a representation of itself; this is called the \textit{vacuum representation}. This implies that the identity (11) is in particular valid for $\mathcal{H}_\Omega$, i.e. (11) remains true when $Y_M$ is replaced by $Y$. This expresses a kind of associativity of the VOA. Thus for VOAs ‘associativity’ in the sense of (11) is a consequence of ‘commutativity’ in the sense of (9).

Once we know what a representation is, we can consider equivalence classes of isomorphic irreducible representations. They form a set $I$. We will call the elements of $I$ \textit{labels} or also, by an abuse of terminology, \textit{primary fields} or \textit{sectors}. We use lower case greek letters for elements in $I$, and denote by $H_\mu$ the underlying vector space for a representation isomorphic to $\mu \in I$.

When the VOA is conformal, every module $M$ is in particular, by restriction, a module over the Virasoro algebra $\mathfrak{f}$. It follows directly from the definition of a conformal VOA that in each CFT model the central element $C$ of the Virasoro algebra acts as $C = c \text{id}$ with one and the same value of the number $c$ in every irreducible representation that occurs in the model. Also recall that when $v$ is an eigenstate of the Virasoro zero mode $L_0$, its eigenvalue $\Delta_v$ is called the \textit{conformal weight} of $v$. The conformal weights of different vectors in the same irreducible module differ by integers, or in other words, $e^{2\pi i L_0}$ acts as a multiple of the identity on every irreducible module. We will therefore refer, somewhat abusing terminology, to the \textit{fractional part} of the conformal weight of any eigenvector of $L_0$ in an irreducible module $M$ as the conformal weight $\Delta_M$ of the module $M$.

A vertex operator algebra is called \textit{rational} if every module is fully reducible. Rational VOAs possess two surprising properties [10]: The homogeneous subspaces $M^{(n)}$ of the representation spaces $M$ are finite-dimensional, and there are only finitely many inequivalent irreducible representations. Notice, however, that not every VOA that has only finitely many inequivalent irreducible representations is rational in this strong sense. For instance, logarithmic CFTs have reducible, but not fully reducible modules, although there may be only finitely many inequivalent irreducible modules. An example of a non-rational VOA is provided by the one for a single free boson that has been discussed in Section 3.3. It has infinitely many inequivalent irreducible representations, one for each real number $q$. It also has a reducible but not fully reducible modules. Namely, start with a finite-dimensional vector space $V$ on which $b_0$ has non-trivial Jordan decomposition and cannot be diagonalized. It gives rise to a module over the Heisenberg algebra in the same way as the ordinary Fock space is generated from a one-dimensional space. This module is also a module of the VOA, and it is reducible, but not fully reducible.

The above results on rational VOAs imply that for rational VOAs we can define the character of a representation as a formal function in $\tau$:

$$\chi_M(\tau) := \text{tr}_M \exp(2\pi i \tau(L_0 - c/24)). \quad (12)$$

Another remarkable result [10] is that under a certain (technical) finiteness condition the characters are holomorphic functions in $\tau$ on the complex upper half-plane $H$. Moreover, the finite
set of characters of all inequivalent irreducible modules carries an action of $SL(2, \mathbb{Z})$. Quite generally, it is expected that the set of representations of a rational VOA carries even more structure, namely the one of a modular tensor category. This formalizes the tensor product of modules as well as the so-called Moore-Seiberg data like braiding, fusing and the modular transformations of the characters. For more details about modular tensor categories, we refer to [2].

3.5 Remarks on models with extended supersymmetry

We finally discuss some specific aspects of models with extended ($N=2$) superconformal symmetry. (For a review, see also [18].) Unitary theories of this type exist for Virasoro central charges $c \geq 3$ and for $c = 3k/(k+2)$ with $k$ a positive integer. The simplest realization with central charge $c = 3$ is constructed from two real chiral bosons $X^1(z)$, $X^2(z)$ and two real chiral fermions $\psi^1(z)$, $\psi^2(z)$. We may think of these fields as (super-)coordinates of a target space. We endow this target space with a complex structure and consider complex fields

$$X^\pm = \frac{1}{\sqrt{2}} (X^1 \pm iX^2) \quad \text{and} \quad \psi^\pm = \frac{1}{\sqrt{2}} (\psi^1 \pm i\psi^2).$$

The generators of the $N=2$ algebra can be expressed through these fields as

$$T(z) = -\partial_z X^+ \partial_z X^- (z) - \frac{1}{2} (\partial \psi^+ \psi^- (z) + \partial \psi^- \psi^+ (z));$$
$$G^+ (z) = -\frac{1}{2} \psi^+ \partial_z X^- (z), \quad G^- (z) = -\frac{1}{2} \psi^- \partial_z X^+ (z);$$
$$J(z) = :\psi^- \psi^+ (z):.$$

This example is easily generalized to $2k$ bosons and fermions. Any complex structure on a real $2k$-dimensional vector space then gives a realization of the $N=2$ algebra with central charge $c = 3k$.

Our next example are $N=2$ minimal models. They are similar to the Virasoro minimal models, whose chiral algebra is generated by the Virasoro algebra alone. Namely, the chiral algebra of an $N=2$ minimal model is generated by the bosonic subalgebra of the $N=2$ superconformal algebra. (Here it is essential that we regard the chiral algebra as a VOA, and not as a so-called super-VOA.)

The $N=2$ minimal models come in a series, and theories are parameterized by a non-negative integer $k$, the level. Their Virasoro central charge is $c = 3k/(k+2)$. They can be realized by a coset construction,

$$su(2)_k \oplus u(1)/u(1),$$

and accordingly the irreducible representations can conveniently be labelled as $\Phi^{l,s}_m$, with $l \in \{0, 1, \ldots, k\}$, $s \in \mathbb{Z}/4\mathbb{Z}$ and $m \in \mathbb{Z}/(2k+4)\mathbb{Z}$. (Only triples with $l+s+m \in 2\mathbb{Z}$ are allowed, and $\Phi^{l,s}_m$ and $\Phi^{k-l,s+2}_{m+(k+2)}$ refer to isomorphic irreducible representations.)

We call a state $v$ in the NS sector $N=2$-primary if there are numbers $\Delta$ and $q$ such that

$$G^\pm v = 0 \quad \text{for} \quad r \geq 1/2,$$
$$L_n v = J_n v = 0 \quad \text{for} \quad n \geq 0,$$
$$L_0 v = \Delta v \quad \text{and} \quad J_0 v = q v.$$
Chiral primaries are $N = 2$-primary states which in addition obey $G^+_{-1/2} v = 0$, while anti-chiral primaries obey $G^-_{1/2} v = 0$. The reader should be warned that the qualification “chiral” is meant here in a different sense than in the term chiral CFT; it refers to the fact that a state is annihilated by half of the supersymmetry charges.

In unitary $N = 2$ theories the identity

$$0 \leq \langle v, \{G^+_{-1/2}, G^-_{1/2}\} v \rangle = (2\Delta_v + q_v) \langle v, v \rangle$$

holds; this gives immediately the unitarity bound

$$\Delta_v \geq \frac{1}{2} |q_v|.$$ 

States with positive charge $q_v$ saturate this bound if and only if they are chiral primaries; similarly, anti-chiral primaries are precisely the states of negative charge saturating the bound. Analogous arguments show that in the Ramond sector the conformal weight is bounded from below by $c/24$; states satisfying this bound are called Ramond ground states.

It turns out that the space of all (anti-)chiral primary states is finite-dimensional; each of these two spaces can be endowed with a product by the prescription

$$v_1 \star v_2 := \text{Res}_{z=0} \left( \frac{1}{z} Y(v_1, z) v_2 \right).$$

(13)

Using standard properties of VOAs, one can show that this product is associative and commutative. This way one obtains the so-called chiral and the anti-chiral ring; actually these are not rings, but rather algebras over the complex numbers. This algebra is not semi-simple; on the contrary, owing to charge conservation in the product (13), all generators except for the identity are nilpotent. In the case of an $N = 2$ minimal model at level $k$, the chiral ring is isomorphic to $\mathbb{C}[x]/(x^{k+1})$, with $x^l \cong \Phi_l^0$. One can show that spectral flow maps anti-chiral primaries to Ramond ground states, and the latter to chiral primaries. Finally we mention that there is a Hodge-type decomposition theorem: Every vector $v$ in an $N = 2$ superconformal theory can be written in the form

$$v = v_0 + G^+_{-1/2} v' + G^-_{1/2} v''$$

with $v_0$ a chiral primary. Chiral primaries can therefore be thought of as analogues of (one half of) harmonic forms.

4 Geometric aspects of chiral CFT: Conformal blocks

We now to combine the subjects of the two previous sections, VOAs and complex curves. This will lead us to the central notion of a conformal block.

4.1 Conformal blocks

Conformal blocks formalize the following physical idea. To any vector $v \in \mathcal{H}_\mu$, we would like to associate a “field” $\Phi(v, \cdot)$ and, given a complex curve $\hat{X}$ and $m$ non-coinciding points $p_i$ on $\hat{X}$, we would like to associate a “correlator” $\langle \Phi(v_1, p_1) \cdots \Phi(v_m, p_m) \rangle$ to products of such fields. We have put the words “field” and “correlator” in quotation marks, for, as we will see, the
quantities we will obtain do not enjoy all properties we usually expect for correlators in a local quantum field theory.

We want to characterize the “correlators” by Ward identities which express the chiral symmetries globally on the curve $\hat{X}$. To write down these identities in a compact form, it helps to adjust the notation and to write the “correlator” as

$$\langle \Phi(v_1, p_1) \cdots \Phi(v_m, p_m) \rangle = \beta_{\hat{p}, \hat{X}}(v_1 \otimes \cdots \otimes v_m),$$

i.e. interpret it as a linear functional

$$\beta_{\hat{p}, \hat{X}} \in (\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_m})^*$$

that depends on the curve $\hat{X}$ and on the positions $\hat{p} = (p_1, p_2, \ldots, p_m)$ of the field insertions.

The Ward identities make use of the holomorphic structure on $\hat{X}$. We restrict ourselves to the case of WZW theories, where the VOA is generated by non-abelian currents $J^a(z)$ that correspond to a finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$. We first introduce the (associative, commutative) algebra $\mathcal{F}$ of holomorphic functions on $\hat{X} \setminus \hat{p}$ which have at most finite order poles at the points $p_i$. The corresponding algebra $\bar{\mathfrak{g}} \otimes \mathcal{F}$ of $\bar{\mathfrak{g}}$-valued functions has an action on

$$\mathcal{H}_{\hat{X}} := \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_m}.$$

To describe this action, we choose local coordinates $\xi_i$ around the insertion points $p_i$. Given a homogeneous element $J^a \otimes f \in \bar{\mathfrak{g}} \otimes \mathcal{F}$, for each insertion point $p_i$ we consider the expansion of $f$ in a Laurent series in $\xi_i$,

$$f^{(i)}(\xi_i) = \sum_{n \gg -\infty} a^{(i)}_n \xi^n_i.$$

The coefficients $a^{(i)}_n \in \mathbb{C}$ yield an element

$$\sum_{n \gg -\infty} a^{(i)}_n J^a_n$$

of the corresponding untwisted affine Lie algebra $\mathfrak{g} = \bar{\mathfrak{g}}^{(1)}$. The $J^a_n$ are operators acting on the irreducible representation $\mathcal{H}_{\lambda_k}$. The action of $J^a \otimes f$ on $\mathcal{H}_{\hat{X}}$ is then defined as

$$\sum_{i=1}^m \mathbf{1} \otimes \cdots \otimes \left( \sum_{n \gg -\infty} a^{(i)}_n J^a_n \right) \otimes \cdots \otimes \mathbf{1},$$

where the non-trivial factor sits at the $i$th position.

The vector space of our interest consists of all vectors in $(\mathcal{H}_{\hat{X}})^*$ on which the dual action is trivial:

$$V_{\hat{X}}(\hat{X}, \hat{p}) := (\mathcal{H}_{\hat{X}})^*.$$  \hspace{1cm} (14)

This is called the space of conformal blocks. Notice that the action of $\bar{\mathfrak{g}} \otimes \mathcal{F}$ depends on the choice of local coordinates at the insertion points. Hence the conformal blocks depend on that choice as well. For a conformal VOA, however, the Virasoro algebra provides a natural action of the group of changes of local coordinates on the representation spaces. As a consequence, the conformal blocks transform covariantly under such choices.
In all known rational chiral CFTs, in accordance with the Verlinde formula, the space $V_\vec{\lambda}$ is a finite-dimensional vector space; it is a subspace of $(\mathcal{H}_\vec{\lambda})^*$ that depends on the labels $\vec{\lambda}$, on the curve $\hat{X}$, and the positions $\vec{p}$ of the insertion points. These vector spaces also turn out to be the spaces of physical states of certain three-dimensional topological field theories (TFTs), the Chern–Simons theories. It must be emphasized, though, that this relation between TFT and chiral CFT does by no means imply that the two theories are equivalent. Indeed, already the respective spaces of physical states are rather different. For chiral CFT, the state space is provided by the infinite-dimensional graded vector spaces $H_\lambda$ that underly the VOA-modules. The spaces of conformal blocks are of independent interest in mathematics; they provide nonabelian generalizations of theta functions, i.e. they are naturally isomorphic to spaces of sections over moduli spaces of (stable equivalence classes of holomorphic) $G$-bundles over $\hat{X}$, where $G$ is the connected and simply connected complex Lie group whose Lie algebra is $\bar{g}$. For more details, we refer to [47].

It is again time for an example. Take the sphere $\hat{X} = \mathbb{CP}^1$, with the usual (quasi-)global coordinate $z$, and two insertions at $z_1 = 0$ and $z_2 = \infty$. As local coordinates, we choose $\xi_1 = z$ and $\xi_2 = 1/z$. The algebra $\mathcal{F}$ is in this case the algebra of all polynomials in $z$ and $z^{-1}$, $\mathcal{F} = \langle z^n, n \in \mathbb{Z} \rangle$. The element $J^a \otimes z^n$ acts via $J^a_\xi \otimes 1 + 1 \otimes J^a_{-n}$. The two-point blocks are then functionals $\beta \in (\mathcal{H}_\lambda \otimes \mathcal{H}_\mu)^*$ with the property that

$$\beta \circ (J^a_\xi \otimes 1 + 1 \otimes J^a_{-n}) = 0 \quad (15)$$

for all $a = 1, 2, \ldots, \dim \bar{g}$ and all $n \in \mathbb{Z}$. One can show that non-vanishing functionals $\beta$ obeying (15) exist only if $\lambda$ and $\mu$ are conjugate $\bar{g}$-weights, $\mu = \lambda^\dagger$. (An analogous notion of conjugate fields exists in every chiral CFT, see Subsection 4.3 below.)

Note that these linear functionals are in the algebraic dual of $\mathcal{H}_\lambda \otimes \mathcal{H}_\mu$; the Hilbert space dual is too small to contain them. Still, one abuses bra-ket notation and likes to write them as vectors of $\mathcal{H}_\lambda \otimes \mathcal{H}_\mu$. In terms of these “vectors” $|B_\lambda\rangle$, formula (13) is written as

$$\langle J^a_\xi \otimes 1 + 1 \otimes J^a_{-n} | B_\lambda \rangle = 0 \quad (16)$$

The quantities $|B_\lambda\rangle$ show up in various circumstances. In the context of conformally invariant boundary conditions they are also known as Ishibashi states. The reader should keep in mind that these are nothing but two-point blocks on the sphere. It is sometimes possible to write down the Ishibashi state $|B_\lambda\rangle$ explicitly; e.g. for theories based on a free boson, it can be written as a generalized coherent state,

$$|B_\lambda\rangle = \exp \left( - \sum_{n=1}^{\infty} b_{-n} \otimes b_{-n} \right) v_\lambda,$$

where $v_\lambda$ is the highest weight state in the tensor product of Fock spaces. Such a realization is helpful when one is interested in calculating one-point functions on a disc explicitly. It is, however, not necessary to know such an explicit realization if one wants to determine the spectrum of boundary fields. In this case, it is sufficient to know how $|B_\lambda\rangle$ behaves under factorization (see below). The crucial information that allows to calculate concretely with boundary states is the following identity that relates two-point blocks and characters:

$$\chi_\lambda(2\tau) = \langle B_\lambda | e^{2\pi i r (L_0 \otimes 1 + 1 \otimes L_0 - c/12)} | B_\lambda \rangle.$$
We presented the definition of conformal blocks for specific VOAs, the ones that underly WZW theories. A general approach has been developed in [4]; here we just sketch the idea. For WZW theories it was sufficient to take into account only the Ward identities implied by the currents, i.e. by the Virasoro-primary fields of conformal weight $\Delta = 1$. In the general case we must implement more information about the VOA. The idea is to define a bundle $\tilde{\mathcal{H}}_\Omega$ over the curve $\hat{X}$ whose fibers are isomorphic to $\mathcal{H}_\Omega$. Roughly speaking, one wants to have an appropriate copy of the VOA at each point of the curve. (The bundle $\tilde{\mathcal{H}}_\Omega$ is actually twisted in such a way that covariance under changes of local coordinates is implemented; for simplicity, we suppress this point in our discussion.) Similarly one defines a bundle $\tilde{\mathcal{H}}_\lambda$ over $\hat{X}$ for each module $\mathcal{H}_\lambda$, again with the goal to have at one’s disposal, as the fiber of $\tilde{\mathcal{H}}_\lambda$ at $x \in X$, a copy of the module over every point $\hat{x}$ of the curve $\hat{X}$.

The field-state correspondence $Y$ of the VOA then translates into the following structure. Locally on a small disc around every point of the curve we have a section $\gamma$ in the dual bundle $\tilde{\mathcal{H}}_\Omega^*$ with values in the endomorphisms of $\mathcal{H}_\Omega$. This is the analogue to having for each value of the formal coordinate $z$ a map that associates to every vector $v \in \mathcal{H}_\Omega$ an endomorphism of $\mathcal{H}_\Omega$.

The conformal blocks are defined as linear forms $\varphi$ on the tensor product of the vector spaces $\tilde{\mathcal{H}}_{\lambda_i, p_i}$, i.e. the fibers of $\tilde{\mathcal{H}}_\lambda$ over the insertion points $p_i$. Again we must select the conformal blocks by an invariance property that uses global holomorphic features of the curve $\hat{X}$. We require that for every $m$-tuple of vectors $v_i \in \mathcal{H}_{\lambda_i}$ the sections

$$\varphi(v_1 \otimes \cdots \otimes \gamma(\cdot, \cdot) v_i \otimes \cdots \otimes v_m)$$

in the restriction of the bundle $\tilde{\mathcal{H}}_\Omega^*$ over the local discs around the points $p_i$ can be extended to a global holomorphic section on the punctured curve $\hat{X} \setminus \{\hat{p}\}$.

One can apply the section $\varphi$ in $\tilde{\mathcal{H}}_\Omega^*$ in particular to a Virasoro-primary field of conformal weight $\Delta \in \mathbb{Z}_{\geq 0}$. Currents are such fields with $\Delta = 1$, and the resulting Ward identities are expressible in terms of functions, i.e. sections in a power $\mathcal{K}^0 = \mathcal{K}^{1-\Delta}$ of the canonical line bundle of $\hat{X}$. One can show that for arbitrary $\Delta$ the general prescription implies Ward identities that use sections in $\mathcal{K}^{1-\Delta}$.

We finally introduce the notion of a tensor product of two chiral CFTs. This plays a crucial role in the construction of string vacua. Its chiral algebra is a tensor product of two conformal VOAs $(\mathcal{H}_\Omega^{(i)}, \varphi^{(i)}, v^{(i)}_{\text{vir}}, T_i, Y_i)$. Indeed, it is not hard to check that the data

$$\mathcal{H}_\Omega := \mathcal{H}_\Omega^{(1)} \otimes \mathcal{H}_\Omega^{(2)}, \quad v_\Omega := v^{(1)}_{\Omega} \otimes v^{(2)}_{\Omega}, \quad T := T_1 \otimes 1 + 1 \otimes T_2,$$

$$v_{\text{vir}} := v^{(1)}_{\text{vir}} \otimes v^{(2)}_{\text{vir}} + v^{(1)}_{\text{vir}} \otimes v^{(2)}_{\text{vir}}, \quad Y := Y_1 \otimes Y_2$$

define a new conformal VOA of central charge $c = c_1 + c_2$. Its irreducible representations are tensor products of irreducible representations, $\mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2}$, and the conformal blocks are tensor products of vector spaces as well.

We end with a word of warning. In the bosonic language we are using here, the tensor product of superconformal theories is not a superconformal theory any longer. The reason for this is simple: The supercurrent $G$ is not a field in the chiral algebra, since it does not have integral conformal weight. Rather, it corresponds to a state $G_{-3/2} \Omega_\Omega$ in some different sector that we call $\mathcal{H}_v$. For the total supercurrent in a tensor product of two superconformal theories, we would like to take $G(z) \otimes 1 + 1 \otimes G(z)$. But this sum is not well-defined; indeed, the two terms of the sum are in two different superselection sectors $\mathcal{H}_v \otimes \mathcal{H}_\Omega$ and $\mathcal{H}_\Omega \otimes \mathcal{H}_v$ of the tensor product. We will see later how to deal correctly with this situation.
4.2 Moduli spaces

So far, we have kept the curve $\hat{X}$ and the insertion points $\vec{p}$ fixed. Now we investigate what happens when they are varied. The data $\hat{X}$ and $\vec{p}$ determine a point in the moduli space $M_{g,m}$ of curves of genus $g$ with $m$ distinct marked points. We have seen that given an $m$-tuple $\vec{\lambda} = (\lambda_1, \ldots, \lambda_m)$, we can associate to every point of $M_{g,m}$ a vector space, the space $V_{\vec{\lambda}}(\hat{X},\vec{p})$ of conformal blocks.

A rather non-trivial property of these vector spaces is that they fit together into the total space of a vector bundle over $M_{g,m}$. In particular, the dimension of the spaces of conformal blocks is equal to the rank of the vector bundle, so it depends only on the genus $g$ of the curve (but not on the specific holomorphic structure), on the number $m$ of insertions (but not on their positions), and on the labels $\vec{\lambda}$. Furthermore, the vector bundle of conformal blocks (sometimes called Friedan–Shenker bundle) comes with additional structure, a projectively flat connection, known as the Knizhnik–Zamolodchikov connection. It is actually a consequence the existence of a flat connection $\nabla = d + L_{-1} \otimes dz$ on the bundles $\mathcal{H}_\lambda$ over $M$. The existence of this projectively flat connection immediately implies a (projective) action of the fundamental group of $M_{g,m}$ on each fiber, i.e. an action of the mapping class group (cf. (2)) on the vector spaces of conformal blocks. (For more details and the relation to twisted $D$-modules, we refer to chapter 6 of [2].)

Sometimes the word “conformal block” is also used for (locally defined) horizontal sections in these vector bundles. These bundles are generically non-trivial, i.e. the conformal blocks are multivalued functions of the insertion points. So they are not correlation functions; this is the reason why above we used the words “fields” and “correlation functions” with quotation marks only. The condition that these sections are horizontal implies that they satisfy a first order differential equation. This equation is known as the Knizhnik–Zamolodchikov equation.

For genus $g = 1$ and one insertion of the vacuum $\Omega$, the (orbifold-)fundamental group is $SL(2,\mathbb{Z})$ [30]; one thus obtains a representation of the modular group $SL(2,\mathbb{Z})$ on a complex vector space of dimension $|I|$. In a natural basis, the generator $T$, acting on the complex upper half plane $H$ as $T: \tau \mapsto \tau + 1$ of $SL(2,\mathbb{Z})$, is represented by a unitary diagonal matrix $T_{\lambda,\mu}$, and $S$, acting on $H$ like

$$S: \tau \mapsto -1/\tau,$$

is represented by a unitary symmetric matrix $S_{\lambda,\mu}$. It is an important conjecture that the modular transformations of the characters discussed in Section 3.4 are the same as those of the one-point conformal blocks on the torus. This is the core of the Verlinde conjecture.

Up to this point, the structures we discussed are mathematically essentially under control (see e.g. [4]). There is, however, one further crucial aspect of conformal blocks, known as factorization. So far we have been talking about smooth curves. But actually, all our constructions go through even when the curve $\hat{X}$ is allowed to possess certain mild singularities, so-called ordinary double points. Such singularities can be resolved by “blowing up” the double point $p$; this yields a new curve $\hat{X}'$ with a projection to $\hat{X}$ under which $p$ has two pre-images $p'_{\pm}$. By factorization one means the existence of canonical isomorphisms

$$g_{\hat{X},\hat{X}'}: V_{\hat{X}}(\hat{X}) \cong \bigoplus_{\mu \in I} V_{\hat{X}'}(\mu,\mu^+)(\hat{X}')$$

between the blocks on $\hat{X}$ and $\hat{X}'$. This structure tightly links the system of bundles $\mathcal{V}_{\mu}$ over
the moduli spaces $\mathcal{M}_{g,m}$ for different values of $g$ and $m$. From the field theoretic point of view, factorization is a novel and surprising structure: It relates quantum field theories defined on different manifolds.

Factorization can also be described from a different point of view. One can take the curve $\hat{X}'$, cut out small discs around $p^+$ and $p^-$ and identify the boundaries of these discs. Taking the radius of the disc to zero, we produce the singular curve $\hat{X}$. In other words $\hat{X}$ can be obtained from $\hat{X}'$ by a gluing procedure. To justify factorization, frequently one invokes the physical intuition of ‘inserting a complete system of states’. But to prove factorization properties rigorously is very hard.

One consequence of factorization is that one can express the rank of $V_{\lambda,g}$ for all values of $m$ and $g$ in terms of the matrix $S$ that we encountered in the description of the action of the modular group. This results in the famous Verlinde formula, which reads

$$\text{rank } V_{\lambda,g} = \sum_{\mu \in I} |S_{\Omega,\mu}|^{2-2g} \prod_{i=1}^{m} S_{\lambda_i,\mu} S_{\Omega,\mu}. \quad (18)$$

This implies in particular that the ranks of the bundles of conformal blocks are finite.

Using the assumption that the matrix $S$ also describes the modular transformations of the characters, for concrete models the matrix $S$ can be computed explicitly with tools from representation theory. For WZW models, $S$ is given by the Kac–Peterson formula. The combination of the Kac–Peterson formula for $S$ with the general Verlinde formula (18) then gives the Verlinde formula in the sense of algebraic geometry (for reviews see [6, 46]).

## 4.3 Fusion rings

We now use a specific type of conformal blocks, the three-point blocks on $\mathbb{CP}^1$, to define the structure of a fusion ring. For every triple of labels we introduce the non-negative integer

$$N_{\lambda_1,\lambda_2,\lambda_3} := \dim V_{\lambda_1,\lambda_2,\lambda_3}(\mathbb{CP}^1).$$

(A chiral CFT with the property that $\sum_{\lambda_3} N_{\lambda_1,\lambda_2,\lambda_3}$ is finite for all pairs $\lambda_1, \lambda_2$ is called quasi-rational.) Recall that there is a special label, the vacuum $\Omega$. One can show that

$$C_{\lambda_1,\lambda_2} := N_{\lambda_1,\lambda_2,\lambda_3} \Omega. \quad (19)$$

takes only the values 0 or 1 and is a permutation of order two; this is called the conjugation. We write $\lambda^+$ for the field conjugate to $\lambda$ and set $N_{\lambda_1,\lambda_2,\lambda_3} = N_{\lambda_1,\lambda_2,\lambda_3^+}$.

We now consider a ring (over $\mathbb{Z}$) with a basis $B := \{\Phi_{\lambda}\}_{\lambda \in I}$, labelled by $I$ and multiplication

$$\Phi_{\lambda_1} \ast \Phi_{\lambda_2} = \sum_{\lambda_3} N_{\lambda_1,\lambda_2,\lambda_3} \Phi_{\lambda_3}. \quad (20)$$

This ring is called the fusion ring. Factorization implies that it is associative, commutative, semi-simple, and that $\Phi_{\Omega}$ acts as the identity. A fusion ring should not be thought of as just a ring; rather, very much like the representation ring of a Lie group, it is a ring together with a
**distinguished basis.** As a special case of the Verlinde formula (18), the structure constants of the fusion ring can be written as

\[ \mathcal{N}_{\lambda_1,\lambda_2}^{\lambda_3} = \sum_{\mu} S_{\lambda_2,\mu} S_{\lambda_3,\mu}^* S_{\Omega,\mu}. \]

This relation can be interpreted as follows. The matrix \( S \) diagonalizes simultaneously all matrices \( (\mathcal{N}_\lambda)_{\lambda_2}^{\lambda_3} := \mathcal{N}_{\lambda_1,\lambda_2}^{\lambda_3} \); this is sometimes summarized by saying that the \( S \)-matrix diagonalizes the fusion rules. The eigenvalues of \( \mathcal{N}_{\lambda_1} \) are the quotients \( S_{\lambda_1,\mu}/S_{\Omega,\mu} \). The ratio

\[ D_\lambda := \frac{S_{\lambda,\Omega}}{S_{\Omega,\Omega}} \]

is called the *quantum dimension* of \( \lambda \). It is a real number and satisfies \( D \geq 1 \); for \( 1 \leq D < 2 \), \( D \) must be of the form \( D = 2 \cos(2\pi/n) \) for some \( n \in \mathbb{Z}_{\geq 3} \). (For some other aspects of fusion rings, see e.g. [16].)

The vacuum \( \Phi_\Omega \) is an element of the distinguished basis that is invertible, with inverse again in the distinguished basis. Other labels \( J \in I \) with this property are of particular interest. These so-called *simple currents*\(^6\) obey

\[ \Phi_J \star \Phi_J^+ = \Phi_\Omega. \]

Alternatively, they can be characterized by the property that the fusion product with any element \( \Phi_\lambda \) of \( \mathcal{B} \) contains just a single element, which we denote by \( \Phi_{J\lambda} \):

\[ \Phi_J \star \Phi_\lambda = \Phi_{J\lambda}. \quad (21) \]

Yet another characterization of simple currents is that they are precisely the fields in the distinguished basis of quantum dimension 1.

Simple currents display a number of important properties (for a review see [11]). The set of all simple currents of a rational CFT forms a finite abelian group \( \mathcal{G} \), also called the *center* of a CFT. As a consequence of (21), simple currents organize the primary fields into orbits. Every simple current \( J \) allows to assign to each label a number, the *monodromy charge* \( Q_J(\lambda) \in \mathbb{Q}/\mathbb{Z} \). It is defined as the combination

\[ Q_J(\lambda) = \Delta_J + \Delta_\lambda - \Delta_{J\lambda} \mod \mathbb{Z} \]

of conformal weights. The monodromy charge is conserved in the fusion product, in the sense that \( \mathcal{N}_{\lambda_1,\lambda_2,\lambda_3} \neq 0 \) is only possible if \( Q_J(\lambda_1) + Q_J(\lambda_2) + Q_J(\lambda_3) = 0 \mod \mathbb{Z} \) for all simple currents \( J \in \mathcal{G} \). The elements of the matrix \( S \) for labels on the same simple current orbit are equal up to a phase that is determined by the monodromy charge:

\[ S_{J\lambda,\mu} = \exp(2\pi i Q_J(\mu)) S_{\lambda,\mu}. \quad (22) \]

This relation turns out to be central for many applications of simple currents.

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\(^6\) The name simple currents is somewhat of a misnomer. While their fusion rules are simple, and they also correspond to simple superselection sectors in the sense of [29], they are definitely not currents in the sense of VOAs.
In view of the applications to string theory we have in mind, we note that every chiral CFT with at least $N=1$ supersymmetry possesses a non-trivial simple current, namely the primary field $H_v$ that contains the supercurrent $G(z)$. This simple current $v$ has order two and conformal weight $\Delta_v = 3/2$; the associated monodromy charge takes the value $Q_v(\lambda) = 0$ for $\lambda$ a field in the Neveu-Schwarz sector and $Q_v(\lambda) = 1/2$ for $\lambda$ in the Ramond sector. The orbits of $v$ are just the components of a superfield. In $N=2$ minimal models the simple current $v$ is the primary field labelled as $\Phi^{0,2}$.

Superconformal chiral CFTs with $N=2$ supersymmetry possess in addition at least one more simple current, which we call $s$. It is in the Ramond sector and has conformal weight $\Delta_s = c/24$, implying that it is a Ramond ground state. Its order is model dependent, and its orbits are given by spectral flow. The monodromy charge with respect to $s$ equals half the $U(1)$ charge mod $\mathbb{Z}$ with respect to the abelian current in the $N=2$ algebra. In $N=2$ minimal models this simple current is labelled as $\Phi^{0,1}$.

4.4 Simple current extensions

Next we introduce a concept that will be crucial for our construction of string vacua: Simple current extensions. Let $G'$ be a subgroup of simple currents of integral conformal weight $\Delta$. Then a new VOA can be defined on the vector space

$$\mathcal{H}_\Omega := \bigoplus_{J \in G'} \mathcal{H}_J. \quad (23)$$

We call the corresponding chiral CFT the extension of the original model by $G'$; the whole procedure is called a simple current extension.

On the new VOA $\mathcal{H}_\Omega$ the group $G := (G')^*$ dual to $G'$ acts by automorphisms. For a character $\psi: G' \to \mathbb{C}$ the action is defined by

$$R(\psi)|_{\mathcal{H}_J} := \psi(J) \mathrm{id}_{\mathcal{H}_J}. \quad (24)$$

The original VOA $\mathcal{H}_\Omega$ can thus be characterized as the subalgebra that is left pointwise fixed under the action of $G$. Such a subalgebra is called an orbifold subalgebra. Orbifolding by an abelian group of automorphisms is the inverse operation of simple current extension. A word of warning is in order: Geometric orbifolds – that is, sigma models for which the target space has the form of a quotient $M/G$ of a manifold $M$ – can correspond to simple current extensions, to algebraic orbifolds in the sense introduced here, to a combination of both, or even to more general simple current modular invariants (see Section 5.3. below).

The labels and the modular matrix $S$ of the extended theory can be described explicitly $[20]$. To arrive at these explicit formulas we take for every simple current $J$ in $G'$ the matrix $S^J$ that describes the modular transformations of the one-point blocks on the torus with insertion $J$. Just like $S = S_\Omega$, for specific models such as WZW theories or coset models (and thus also for $N=2$ minimal models) these matrices can be computed by representation theoretic techniques $[13]$. For WZW models $S^J$ coincides, up to a computable phase, with the ordinary $S$-matrix of a different affine Lie algebra, the so-called orbit Lie algebra $\hat{g}$; the relevant orbit Lie algebras
are given in the following table.

| $\mathfrak{g}$ | s.c.            | order         | $\tilde{\mathfrak{g}}$ |
|---------------|-----------------|---------------|-------------------------|
| $A_{n}^{(1)}$ | $J^{(n+1)/N}$   | $N < n+1$     | $A_{(n+1)/N}^{(1)}$     |
| $A_{n}^{(1)}$ | $J$             | $n + 1$       | $\{0\}$                |
| $B_{n}^{(1)}$ | $J$             | $2$           | $\tilde{B}_{n-1}^{(2)}$ |
| $C_{2n}^{(1)}$| $J$             | $2$           | $\tilde{B}_{n}^{(2)}$   |
| $C_{2}^{(1)}$ | $J$             | $2$           | $A_{1}^{(2)}$           |
| $C_{2n+1}^{(1)}$| $J$           | $2$           | $C_{n}^{(1)}$           |
| $D_{n}^{(1)}$ | $J_{v}$         | $2$           | $C_{n-2}^{(1)}$         |
| $D_{2n}^{(1)}$| $J_{s}$         | $2$           | $B_{n}^{(1)}$           |
| $D_{2n+1}^{(1)}$| $J_{s}$      | $4$           | $C_{n-1}^{(1)}$         |
| $E_{6}^{(1)}$ | $J$             | $3$           | $G_{2}^{(1)}$           |
| $E_{7}^{(1)}$ | $J$             | $2$           | $F_{4}^{(1)}$           |

Here the following notation is used. $\mathfrak{g} = X_{n}^{(1)}$ is the untwisted affine Lie algebra with horizontal subalgebra $X_{n}$; $\{0\}$ is the trivial zero-dimensional Lie algebra. Only one series of twisted affine Lie algebras, denoted by $\tilde{B}_{n}^{(2)}$, appears: It is the only such series for which the characters of integrable representations span a module of $\mathrm{SL}(2, \mathbb{Z})$. Whenever the simple current is unique up to conjugation, we have denoted it by $J$; conjugated simple currents lead to identical orbit Lie algebras. For $D_{n}$, there is both the vector simple current $J_{v}$, of order 2 and conformal weight $k/2$, and two spinor simple current $J_{s}$ of conformal weight $kn/8$. For $A_{n}$ there are $n+1$ simple currents, which can be written as powers of one of them, which has order $n+1$ and which we denote by $J$.

The simple current symmetry \((22)\) generalizes to

$$S_{K,\lambda,\mu}^{J} = F_{\lambda}(K, J) \exp(2\pi i Q_{K}(\mu)) S_{\lambda,\mu}^{J}. \quad (25)$$

The action of $\mathcal{G}'$ is typically not free. Accordingly one associates to every label $\lambda$ its stabilizer subgroup

$$S_{\lambda} := \{ J \in \mathcal{G}' | \Phi_{J} \ast \Phi_{\lambda} = \Phi_{\lambda} \}. \quad (26)$$

A label with non-trivial stabilizer is also called a fixed point. $S_{\lambda,\mu}^{J}$ can be non-zero only if both $\lambda$ and $\mu$ are fixed points of $J$. The quantities $F_{\lambda}$ appearing in formula \((25)\) can be used to define the subgroup

$$U_{\lambda} := \{ J \in S_{\lambda} | F_{\lambda}(K, J) = 1 \text{ for all } K \in S_{\lambda} \} \quad (26)$$

of the stabilizer $S_{\lambda}$, the so-called untwisted stabilizer. The quantity $F_{\lambda}$ has a cohomological interpretation \[4\], which implies in particular that

$$d_{\lambda} := \sqrt{|S_{\lambda}| / |U_{\lambda}|}$$

\(\footnote{For a list including also the orbit Lie algebras for outer automorphisms of $\mathfrak{g}$ that are not related to simple currents, see table (2.24) of \[13\].} \)
is a non-negative integer.

The labels of the extension are then equivalence classes of pairs \((\rho, \psi)\), where \(\rho\) is a label with \(Q_J(\rho) = 0\) for all \(J \in G'\) and \(\psi\) is a character on the untwisted stabilizer \([26]\). Two labels \((\rho, \psi)\) and \((\rho', \psi')\) are considered to be equivalent if there is a simple current \(J \in G'\) such that \(\rho' = J\rho\) and \(\psi'(K) = F_{\rho}(J, K)\psi(K)\). The fact that a character \(\psi\) must be introduced to label fields with non-trivial untwisted stabilizer is known as fixed point resolution. We emphasize that only labels of vanishing monodromy charge \(Q_J(\rho) = 0\) appear. This property allows us to implement projections in string theory.

The ease with which projections can be implemented with the help of simple currents has also been used in applications of chiral CFT to the fractional quantum Hall effect [15]. In that context the electron corresponds to a simple current with half-integer conformal weight and the constraints stem from the physical locality requirement of Fermi statistics of the electron. (Note, though, that in this application one is not dealing with an extension.)

The characters of these fields are given, as a function of \(\tau \in H\), by

\[
\chi_{[\rho,\psi]}(\tau) = d_\rho \sum_{J \in G'/S_0} \chi_J(\tau).
\]

This determines also the modules of the extension, at least when regarded as modules over the original chiral algebra. Finally, the modular matrix \(\tilde{S}\) of the extension reads

\[
\tilde{S}_{[\lambda\psi],[\lambda'\psi']} = \frac{|G'|}{\sqrt{|S_\lambda||U_\lambda|\sqrt{|S'_{\lambda'}|U'_{\lambda'}}}} \sum_{J \in U_\lambda \cap U'_{\lambda'}} \psi(J) S_{\lambda,\lambda'}^J \psi'(J)^*. \tag{28}
\]

We conclude with a first application of simple extensions that will be quite important for the construction of string vacua. As we have seen, the ordinary tensor product \(C_1 \times C_2\) of superconformal theories is not supersymmetric. We can, however, consider a simple current extension of this tensor product, namely by the tensor product of simple currents \((v_{(1)}, v_{(2)})\). This simple current has integral conformal weight and hence can indeed serve as an extension. In the extended theory, the two fields \(G(z) \otimes 1\) and \(1 \otimes G(z)\) belong to the same module, and it makes sense to consider their sum; this provides a supercurrent for the extended theory. Another effect of the extension is that only fields \((\lambda_1, \lambda_2)\) with vanishing monodromy charge survive. This is equivalent to the statement that the two fields are either both in the Neveu–Schwarz or both in the Ramond sector. Thus the extension aligns the periodicity of fermionic fields on the world sheet. For more than two, say \(r\), factors in a tensor product, we must extend by the \((\mathbb{Z}_2)^{r-1}\) group of simple currents of the form \((v_{(1)}^{\epsilon_1}, v_{(2)}^{\epsilon_2}, \ldots, v_{(r)}^{\epsilon_r})\), where \(\epsilon_i\) takes the values 0 or 1 and \(\sum_i \epsilon_i\) is even.

This prescription can also be understood in the language of super VOAs. These are VOAs equipped with an additional grading over \(\mathbb{Z}_2\), \(\mathcal{H}_\Omega = \mathcal{H}_\Omega^{(0)} \oplus \mathcal{H}_\Omega^{(1)}\), with the requirement that field-state correspondence preserves the parity: \(Y(v, z)\) is a formal series of even operators if \(v\) is in the bosonic subalgebra, \(v \in \mathcal{H}_\Omega^{(0)}\), and of odd operators if \(v\) is in the fermionic part, \(v \in \mathcal{H}_\Omega^{(1)}\). One requires that bosonic states have integral conformal weight and that fermionic states have conformal weight in \(\frac{1}{2} + \mathbb{Z}\). Notice that the bosonic subalgebra is a VOA; this is the VOA we have been considering for supersymmetric theories. \(\mathcal{H}_\Omega^{(1)}\) is a module over this VOA, and for supersymmetric theories this is just the module \(\mathcal{H}_\psi\). The tensor product of two super VOAs
has
\[
\left( \mathcal{H}_\Omega^{(0;1)} \otimes \mathcal{H}_\Omega^{(0;2)} \right) \oplus \left( \mathcal{H}_\Omega^{(1;1)} \otimes \mathcal{H}_\Omega^{(1;2)} \right)
\]
as its bosonic part. This provides yet another explanation for the presence of the simple current extension by $\mathcal{H}_v^{(1)} \otimes \mathcal{H}_v^{(2)}$ in the supersymmetric tensor product.

To summarize: For supersymmetric theories a tensor product should always be accompanied by an appropriate simple current extension.

## 5 Full CFT

We are now finally in a position to address full CFT. The most important point of this section is that the construction of a full CFT from a chiral CFT can be formulated in completely model-independent terms. In this transition chiral data related to the action of the mapping class group on the spaces of conformal blocks – S-matrices, fusing matrices, conformal weights and the like – enter.

### 5.1 The central task

It should have become apparent that chiral CFT is not a local quantum field theory in the usual sense and that conformal blocks, being multivalued, are not genuine correlation functions. Moreover, we have defined the theory on a complex curve, but for applications we wish to set up full CFT on a two-dimensional conformal manifold $X$.

Suppose we are interested in correlators on $X$ of a full CFT that is based on some given chiral CFT. We want to relate them to conformal blocks on the double $\hat{X}$ of $X$. To this end, we must lift the situation from $X$ to $\hat{X}$. Boundary points on $X$ have a unique pre-image on $\hat{X}$. Thus, in order to define conformal blocks, we need to attach one chiral label to each boundary insertion. A bulk point, in contrast, has two pre-images on $\hat{X}$, and hence requires two chiral labels.

In the case of the sphere, $X = S^2$, the cover $\hat{X}$ consists of two spheres with opposite orientation. As quasi-global complex coordinates on the two components of $\hat{X}$, we choose $z$ and $\bar{z}$. Then the involution $\sigma$ maps the point with coordinate $z$ to a point on the other component with complex conjugate coordinate: $\bar{z} = z^*$. For this reason, the coordinate $\bar{z}$ is more commonly, and more suggestively, denoted by $\bar{z}$. But one should be aware of the fact that in chiral CFT $\bar{z}$ and $z$ are indeed two completely independent complex variables. A frequent description in the literature is to say that “one starts with a single complex variable $z$ and its complex conjugate and then treats the two variables as formally independent”. When it comes to concrete calculations this statement amounts to nothing else than working with the Schottky double to treat the chiral aspects of CFT.

The observations about bulk and boundary insertions supply us with enough structure to define conformal blocks for the double, and we get a vector bundle $\mathcal{V}$ over the moduli space $\mathcal{M}_{\hat{X}}$ of the double. (We will always have in mind surfaces with marked points and associated insertion labels, but for brevity suppress them in our notation.) There is, however, no embedding of the moduli space $\mathcal{M}_X$ of $X$ itself into $\mathcal{M}_{\hat{X}}$; only the Teichmüller space $\mathcal{T}_X$ of $X$ can be embedded into the Teichmüller space $\mathcal{T}_{\hat{X}}$ of the double.
The construction is therefore slightly more involved: Using the fact that the Teichmüller space is a covering of the moduli space, we first pull back the bundle $\mathcal{V}$ by the covering map to $T\hat{X}$ and then restrict it to the submanifold $T_X$.

We have now obtained a vector bundle $\mathcal{V}$ of conformal blocks over the Teichmüller space $T_X$. The central task of full CFT can then be phrased as follows:

Select a (global) horizontal section of this bundle in such a way that it is just the correlation function we are looking for.

In fact, what we want to obtain is not just the correlation function as a function on Teichmüller space. Rather, we want to know the correlator as a function of the moduli of $X$ and of the insertion points, i.e. we are looking for a function on the moduli space $\mathcal{M}_X$. For non-vanishing Virasoro central charge, this is actually too strong a requirement, due to the Weyl anomaly. In string theory, however, the total central charge vanishes; we therefore suppress this subtlety in the present review. Moreover, since in string theory one wishes to integrate CFT amplitudes over the moduli space, one needs a volume form over the moduli space $\mathcal{M}_X$; indeed, a natural measure is provided by the ghost system.

Recall from Section 2.4 that the moduli space $\mathcal{M}_X$ can be obtained from the Teichmüller space $T\hat{X}$ by modding out the relative modular group $\Omega_\sigma(X)$. Correlation functions should be genuine functions on $\mathcal{M}_X$, because this ensures that they are functions of moduli and insertion points. This holds for those sections in the restriction of $\mathcal{V}$ to $T_X$ that are invariant under the action of the relative modular group. Modular invariance is therefore a locality requirement. It is the first condition we impose on correlation functions. But the requirement of locality is too weak to fix them uniquely. One must in addition impose compatibility with the factorization properties (or equivalently, the gluing properties) (17) of the conformal blocks.

There are two types of factorization constraints, corresponding to bulk and boundary insertions, respectively. First we study what happens when we glue together two bulk insertions on a world sheet $X$. On the double, this amounts to a simultaneous gluing of two pairs of insertions. For the correlation functions $C(X)$, one thus requires

$$C(X) = \sum_{\lambda \in I} c_\lambda (g_{\hat{X},\hat{X}})^{-1} C(X'_\lambda),$$

where $c_\lambda$ is the normalization of the bulk two-point function and $g_{\hat{X},\hat{X}}$ is the isomorphism (17). It is important to notice that $c_\lambda$ is non-zero only for those bulk fields that appear in the modular invariant partition function. In this sense the factorization constraint guarantees that only physical fields propagate in ‘loop channels’.

Similarly, one can glue two boundary insertions. In this case one deals with a single gluing on the double. One finds

$$C(X') = \sum_{\lambda \in I} \tilde{c}_\lambda (g_{\hat{X}',\hat{X}})^{-1} C(X'_\lambda),$$

where $\tilde{c}_\lambda$ is the normalization of the two-point functions of boundary fields. At this point, it is far from obvious whether there exists any solution to these constraints, nor whether possible solutions are unique. In fact, only little is known about uniqueness whereas, at least in the simplest cases, the existence of a consistent solution has been established, see below.

We conclude this subsection with two possible generalizations of the construction. The first one concerns the case when we consider full CFT only on closed and orientable surfaces $X$. Then
the double has two disjoint components, corresponding to left movers and right movers. This allows us to consistently use two different chiral conformal field theories on the two components. The models obtained this way are called heterotic; the heterotic string is the most prominent example. Other examples are asymmetric orbifolds. We will see below how simple currents allow to construct also heterotic theories. When one considers non-heterotic CFT on closed oriented surfaces only, the central task has also been formulated as follows: Find hermitian scalar products on all bundles of conformal blocks that are compatible with factorization. These scalar products provide us with genuine functions, the correlation functions.

The second generalization concerns the other extreme: When we allow also for surfaces $X$ with boundaries, we may decide to use on the double $\hat{X}$ of a surface $X$ with non-empty boundary not the conformal blocks for the original chiral algebra, but those for a subalgebra of the chiral algebra. Since the chiral algebra encodes the chiral symmetries, such a setup allows us to describe boundary conditions that preserve only part of the bulk symmetries.

### 5.2 Modular invariance

As an example we consider the torus $X = T^2$ with a single insertion of the vacuum. Its double consists of two disjoint copies of a torus, with opposite orientation: $\hat{X} = (T^2) \cup (-T^2)$. The Teichmüller space of the torus is the complex upper half plane, so the mapping class group of the double is the product of two copies of $\text{SL}(2, \mathbb{Z})$. The double $\hat{X}$ is characterized by the values $(\tau, -\tau^*)$ of the modular parameters of the toroidal connected components. This can be seen as follows: The anticonformal map on the plane from which the torus is obtained by dividing out a lattice is complex conjugation. It maps a lattice with fundamental cell $0, 1, \tau$ and $\tau+1$ to a fundamental cell $0, 1, \tau^*, \tau^*+1$, which is equivalent by reflection about the origin to a torus with parameter $-\tau^*$. As a consequence, the relative modular group is the diagonal subgroup of $\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})$ acting by $(S, S^*)$ and $(T, T^*)$ on the product of the two upper half planes. Once one identifies characters and one-point blocks, invariance under the relative modular group implies ordinary modular invariance

$$[Z, S] = 0 = [Z, T]$$ (29)

for the partition function

$$Z(\tau) = \sum_{\lambda, \tilde{\lambda} \in I} Z_{\lambda \tilde{\lambda}} \chi_\lambda(\tau) \chi_{\tilde{\lambda}}(\tau^*) .$$

Here we used that

$$(\chi_\lambda(\tau))^* = \sum_{n \geq 0} d_n e^{2\pi (-i)\tau (\Delta + n - c/24)} = \chi_{\lambda}(-\tau^*) .$$

The constraints (29) always admit two canonical solutions: The identity $Z_{\lambda \mu} = \delta_{\lambda \mu}$, and the conjugation matrix (19), $Z_{\lambda \mu} = C_{\lambda \mu}$. More generally, together with any $Z_{\lambda \mu}$, $(ZC)_{\lambda \mu} = Z_{\lambda \mu^*}$ is a modular invariant as well.

Modular invariance is a very strong requirement on the completeness of the theory. In closed string theory, it implies that all anomalies of the corresponding space-time theory vanish [32]. The converse is not true (this is not always appreciated); it is e.g. easy to construct anomaly free spectra that violate modular invariance. For us the fact that a consistency condition on the two-dimensional world sheet theory is more restrictive for the space-time theory than ordinary field-theoretic consistency conditions is one of the most amazing features of string theory.
5.3 Simple current modular invariants

Simple currents allow to set up a remarkably simple algorithm \cite{28} for constructing solutions to the modular invariance condition (29). These solutions will enter our discussion of string theory in two different ways: They serve to implement projections in string theory, and they allow to construct the mirror of a Gepner compactification. Let us describe the construction of simple current modular invariants in some detail:

1) Recall that all simple currents form a finite abelian group, the center $G$. The effective center is defined as the subgroup of $G$ consisting of those simple currents $J$ whose conformal weight $\Delta_J$ multiplied by the order $N_J$ of $J$ is integral. The first step is to choose a subgroup $G'$ of the effective center.

2) Like any finite abelian group, $G'$ can be written as a product of cyclic groups, $G' = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_k}$.

We choose generators $J_i$ of order $N_i$ for the cyclic groups $\mathbb{Z}_{N_i}$.

3) Compute the relative monodromy charges:
$$ R_{ik} := Q_{J_k}(J_1) = Q_{J_k}(J_i) \mod \mathbb{Z}. $$

The diagonal elements of the matrix $R = (R_{ik})$ must be defined even modulo $2\mathbb{Z}$. Indeed, one can show that the conformal weight $\Delta(J_k)$ obeys
$$ \Delta(J_k) = (N_k - 1) \frac{R_{kk}}{2} \mod \frac{1}{2} \mathbb{Z}. \quad (30) $$

We define $R_{kk}$ modulo $2\mathbb{Z}$ such that the identity (30) holds modulo $\mathbb{Z}$.

4) Choose a matrix $X$ subject to the two conditions
$$ X + X^t = R, \quad X_{ij} \in \frac{\mathbb{Z}}{\gcd(N_i, N_j)} \mod \mathbb{Z}. \quad (31) $$

Thus the symmetric part of $X$ is fixed by the relative monodromies, while the antisymmetric part is free. The possibilities are in fact in one-to-one correspondence to the elements of the cohomology group $H^2(G', \mathbb{C}^\times)$. By analogy with geometric orbifolds, the antisymmetric part of $X$ has been called discrete torsion \cite{28}. Recall, however, that the relationship between geometric orbifolds and simple current modular invariants is rather subtle in general. Analogous subtleties appear in the connection between the antisymmetric part of $X$ and discrete torsion in geometric orbifolds. Still, these considerations illustrate that discrete torsion is a structure that appears naturally in CFT; it is neither ‘inherently stringy’, nor does it require a geometric interpretation of the CFT. When a sigma model formulation of a CFT is available, then the presence of discrete torsion amounts to having a non-trivial action of the orbifold group on the antisymmetric tensor field \cite{13}.

Given all these data, one can show that the matrix $Z$ whose only non-zero entries read

$$ Z_{\lambda^+, \lambda} = |S_{\lambda}| \prod_{i=1}^{k} \delta^{(2)}(Q_{J_i}(\lambda) + \sum_j X_{ij} \beta_j) \quad (32) $$

\footnote{There are also exceptional modular invariants that cannot be obtained this way. This includes e.g. certain invariants stemming from so-called conformal embeddings, compare e.g. \cite{6, 44}.}
are solutions to \((29)\). Here \(\delta^{(\mathcal{Z})}(x)\), defined for real arguments, takes the value 1 for integral \(x\) and is zero else, and we expressed \(J\) as \(J = \prod_{j=1}^{k} J_j^{\beta_j}\) through the generators \(J_j\).

According to quite general arguments \([35]\), a modular invariant is of the following form. One must choose an extension \(\mathfrak{A}_L\) of the chiral algebra for left movers and a possibly different extension \(\mathfrak{A}_R\) for the right movers. We denote the characters of these extensions by \(\chi_{\lambda,L}\) and \(\chi_{\tilde{\lambda},R}\), respectively. The two extensions must have the property that the two fusion rings are isomorphic, and more specifically, that there is an isomorphism \(\omega\) such that for each \(\lambda\) the difference between the conformal weights of \((\lambda,L)\) and \(\omega(\lambda,L)\) is integral. Every modular invariant is then of the form

\[
Z(\tau) = \sum_{(\lambda,L)} \chi_{(\lambda,L)} \chi_{\omega(\lambda,L)}.
\]

For the modular invariants \((\mathcal{Z})\), both \(\mathfrak{A}_L\) and \(\mathfrak{A}_R\) are simple current extensions. The simple currents appearing in \(\mathfrak{A}_R\) are those in the right kernel and the ones in \(\mathfrak{A}_L\) are those in the left kernel of the matrix \(X\). For a pure extension (all \(J \in \mathcal{G}^\prime\) of integral conformal weight and \(X = 0\)), \(\mathfrak{A}_R\) and \(\mathfrak{A}_L\) coincide, and the prescription \((\mathcal{Z})\) reduces to

\[
Z(\tau) = \sum_{[\mu]} \left| U_\mu \right| \sum_{J \in \mathcal{G}^\prime/S_\mu} \chi_{J\mu}(\tau) \right|^2 = \sum_{[\mu]} \sum_{\psi \in U_\mu^*} \left| \chi_{[\mu,\psi]}(\tau) \right|^2,
\]

where \(U_\mu\) is the untwisted stabilizer \((\mathcal{Z})\) and in the second equality the formula \((\mathcal{Z})\) for the extended characters \(\chi_{[\mu,\psi]}(\tau)\) is used.

### 5.4 Boundary conditions and four-point blocks on \(\mathbb{C}\mathbb{P}^1\)

We now turn to solutions of the factorization constraints including also surfaces with boundaries. As it turns out, even after fixing a modular invariant torus partition function, the solution to these constraints is in general not unique. It is generally expected that the set of solutions can be parametrized by assigning to every boundary segment a boundary condition. When referring to segments of the boundary, also any insertion point on the fixed point set of \(\sigma\) must be regarded as separating the boundary into different segments. One must assign boundary conditions to each of these boundary segments, and to each insertion point one must assign a boundary field. Such a boundary field \(\Psi\) carries, in addition to the chiral label \(\mu\), the two labels \(a, b\) of the adjacent boundary conditions, as well as a degeneracy label \(A\) that distinguishes among inequivalent ways of transforming one of these boundary conditions into the other. The full labelling of a boundary field is therefore \(\Psi^{a,b}_\mu\). In particular, for \(a \neq b\) a boundary field plays the role of changing the boundary condition; on the other hand, boundary fields are not taking part in the characterization of the boundary conditions themselves. We can therefore regard a boundary condition as a solution to the factorization constraints for surfaces \(X\) with a single boundary component and only bulk insertions. Moreover, factorization (e.g. of the Möbius strip to a crosscap plus a disc) allows us to restrict our attention to the case where \(X\) is the disc and where there is a single bulk insertion.

The situation when the bulk partition function \(Z\) is given by conjugation and when the boundary conditions preserve all bulk symmetries (so that one and the same chiral algebra can be used to construct conformal blocks on the covers of all surfaces) has been called the Cardy case. Remarkably, in that case all correlators for any topology of \(X\) can be described
using techniques from topological field theory. In that framework, factorization properties and invariance under the relative modular group can be rigorously proven. (A short review of this approach can be found in [23].)

Thus, to determine individual boundary conditions we should study correlators of bulk fields on a disc. At the chiral level, these correlators correspond to conformal blocks on the sphere \( \mathbb{C}P^1 \), with an even number of insertions. The moduli space of three or less points on a sphere is trivial (see below), so non-trivial constraints arise for the first time from the four-point blocks. These blocks appear also in the familiar case of correlators of four bulk insertions on \( X = S^2 \), as well as for four boundary insertions on the disc. The former case is treated in standard texts on CFT, and we will not discuss it any further, while the latter situation is described in some detail below.

To examine boundary conditions on the disc, we introduce the bulk-boundary operator product

\[
\Phi_{\mu\mu}^+(z) \sim \sum_{\nu \in I} \sum_B (1 - |z|^2)^{-2\Delta_\nu + \Delta_\nu} \mathcal{R}_{\mu\nu}^0 \Psi_{\nu}^{aB} (\arg z) \quad \text{for } |z| \to 1. 
\]  

(33)

This tells us what happens when a bulk field \( \Phi_{\mu\mu}^+ \) approaches the boundary of the disc \( |z| \leq 1 \) with boundary condition \( a \): It creates excitations on the boundary, described by the boundary operators \( \Psi_{\nu}^{aB} \).

Two factorizations of this amplitude are possible. One may first use the bulk OPE to produce a single bulk field and then consider its bulk-boundary operator product; the resulting expression contains the reflection coefficient \( \mathcal{R}_{\nu}^a \) once. The other factorization is to apply (33) to both bulk fields; then two reflection coefficients \( \mathcal{R}_{\nu}^a \) appear. Comparison of the two factorizations and specialization to the case when the chiral label \( \nu \) of the boundary field is the vacuum \( \Omega \) yields an identity of the form

\[
\mathcal{R}_{\lambda_1\lambda_2}^a \mathcal{R}_{\lambda_3\lambda_3^+}^a = \sum_{\lambda_3 \in I} \tilde{N}_{\lambda_1\lambda_2}^{\lambda_3} \mathcal{R}_{\lambda_3\lambda_3^+}^a \Omega, 
\]  

(34)

where \( \tilde{N}_{\lambda_1\lambda_2}^{\lambda_3} \) is some complicated combination of operator product coefficients and representation matrices for \( \pi_1(\mathcal{M}_{4,0}) \) acting on four-point blocks.

One structural information can immediately be read off (34): For fixed boundary condition \( a \) the quantities \( \mathcal{R}_{\nu}^a \) form, a one-dimensional representation of an associative algebra with structure constants \( \tilde{N}_{\lambda_1\lambda_2}^{\lambda_3} \), this algebra is called the classifying algebra. The problem of classifying boundary conditions is thereby reduced to the problem of studying the representation theory of the classifying algebra.

In the Cardy case, it is found that the structure constants \( \tilde{N}_{\lambda_1\lambda_2}^{\lambda_3} \) are just the fusion rules. The one-dimensional representations of the fusion algebra are the (generalized) quantum dimensions. Indeed, it follows from the Verlinde formula (18) that for each \( a \in I \) there is one irreducible representation \( R_a \), reading

\[
R_a(\Phi_\lambda) = \frac{S_{\lambda,a}}{S_{\Omega,a}}.
\]

It follows in particular that the boundary conditions in the Cardy case are in one-to-one correspondence to the primary fields.

The reflection coefficients \( \mathcal{R} \) also determine the correlation functions of a single bulk field on a disc with boundary condition \( a \). The information on these correlators is frequently summarized in a single quantity, the so-called boundary state \( |a\rangle \rangle \). Just like the Ishibashi states
(10), this is not an element of the space of bulk states, but a bilinear form on that space. The correlator for the bulk field $\Phi(v \otimes \tilde{v}; z=0)$ inserted in the center of the disc $|z| \leq 1$ is given by the value of $|a\rangle$ on $v \otimes \tilde{v}$:

$$\langle \Phi(v \otimes \tilde{v}; z=0) \rangle_a = \langle a | v \otimes \tilde{v} \rangle.$$

(Other positions of the bulk field can be related to this case using the action of the Möbius group $\text{SL}(2, \mathbb{R})$ on the disc.) In the Cardy case, the boundary state is a linear combination of Ishibashi states. With a suitable normalization of bulk fields, it reads

$$|a\rangle = \sum_{\lambda \in I} \frac{S_{\lambda}}{S_{\Omega}} B_{\lambda}.$$

This analysis has been generalized beyond the Cardy case [21]. We do not describe those results here, but indicate the main idea: The fusion rules, which are the structure constants of the classifying algebra in the Cardy case, are the dimensions of the spaces of three-point blocks. A dimension is the trace of the identity operator. Similarly, in the more general situation, the structure constants of the classifying algebra can be obtained as traces of non-trivial operators $\Theta$ on the conformal blocks:

$$\mathcal{N}_{\lambda_1 \lambda_2 \lambda_3} = \text{tr} V_{\lambda_1 \lambda_2 \lambda_3} \Theta.$$

For details, we refer to a recent review [22]. We also remark that the one-point functions on the real projective plane $\mathbb{R}P^2$, the crosscap, are constrained by factorization constraints on four-point blocks as well; for a review of the so-called cross-cap constraint we refer to [38].

Factorization arguments allow to compute the zero-point amplitude of an annulus with boundary conditions $a$ and $b$ from the boundary state:

$$A_{ab}(t) = \langle a | e^{-(\pi i / t)} (L_{a} \otimes 1 \otimes 1 - c/12) | b \rangle.$$

In the Cardy case, this leads to the result

$$A_{ab}(t) = \sum_{\nu \in I} \mathcal{N}_{ab} \chi_{\nu}(\frac{it}{2}),$$

which tells us that the so-called annulus multiplicities are equal to the fusion rules.

Now recall that in the case of the torus, we could regard the zero-point amplitude as a partition function, counting closed string states. The annulus is a one-loop diagram, too, and we will interpret it as the partition function for open string states. Under (non-chiral) field-state correspondence, these states give rise to boundary fields. The partition function (36) tells us in particular what the precise meaning of the degeneracy label $A$ of the boundary fields $\Psi^{ab}_{\lambda}$ is in the Cardy case: It specifies the chiral coupling between $\lambda$, $a$ and $b$ and hence can take the values $A = 1, 2, ..., \mathcal{N}_{ab} \lambda$.

Just like the bulk fields, the boundary fields should satisfy an operator product expansion, which schematically reads

$$\Psi^{ab}_{\lambda}(x) \Psi^{bc}_{\mu}(y) \sim \sum_{\nu} \sum_{L=1}^{\mathcal{N}_{ab} \lambda} \sum_{C=1}^{\mathcal{N}_{bc} \mu} C_{\lambda \mu L \nu}^{ab} \Psi^{ac}_{\nu}(y) + ...$$

where $L \in \{1, 2, ..., \mathcal{N}_{ab} \lambda\}$ labels a basis of the space of chiral couplings from $\lambda$ and $\mu$ to $\nu$. The constants $C_{\lambda \mu L \nu}^{ab}$ can be determined by a similar method as the one that yields the OPE
of bulk fields. The moduli space of four points on \( \mathbb{CP}^1 \) is one-dimensional: The invariance of \( \mathbb{CP}^1 \) under the Möbius group \( \text{SL}(2, \mathbb{C}) \) allows to fix the positions of three points, say to \( z_1 = 0, z_2 = 1 \) and \( z_3 = \infty \). The position \( z \) of the fourth point is the only remaining coordinate. The moduli space \( \mathcal{M}_{0,4} \) has singularities for \( z \to z_1, z_2 \) and \( z_3 \); one calls these three limits the \( s \)-, \( t \)-, and \( u \)-channel, respectively. In these limits \( z \to z_i \), the conformal blocks possess natural expansions, which can be compared with the help of parallel transport with respect to the Knizhnik–Zamolodchikov connection. The comparison of the four-point correlators in the different channels yields restrictions both for four bulk and for four boundary fields. Notice that in both cases the four-point blocks on the sphere are again the crucial ingredient to determine OPEs.

In the Cardy case, it can be shown that for every (rational) conformal field theory, the structure constants \( C_{\lambda \mu \nu}^{a b c} \) are nothing but suitable entries of fusing matrices \( F \):

\[
C_{\lambda \mu \nu}^{a b c} = (F_{L \mu \nu, A b c}^{\lambda})^*.
\]

(Fusing matrices relate the four-point blocks on the sphere in the \( s \)-channel to those in the \( t \)-channel.) The proof relies on the pentagon equation and the tetrahedral symmetry for \( F \). The relation (38) can indeed be generalized beyond the Cardy case: The structure constants of the boundary OPE are fusing matrices of more general fusion rings, see [23]. (For a different approach to boundary OPEs beyond the Cardy case, see [30].)

### 5.5 Boundary conditions in a general simple current modular invariant

We now study the boundary conditions for an arbitrary simple current modular invariant. This will be a crucial input in our discussion of D-branes in Gepner models. We consider a full CFT with the modular invariant given by (32) and wish to describe all boundary conditions for the extended theory that preserve (at least) the original chiral algebra \( \mathfrak{A} \).

We wish to generalize (35). To this end we must give three types of data: The Ishibashi states, the labels \( a \) for the boundary conditions, and a generalization of the quotient of \( S \)-matrix elements. We will see that the latter are again chiral data associated to the original chiral algebra.

The following prescription [17] gives the correct result. The Ishibashi states are the two-point blocks on the sphere for the \( \mathfrak{A} \)-theory. They correspond to terms \( Z_{\lambda \mu} \) in the partition function (32), and they are given by pairs \( (\lambda, J) \) with \( J \in S_\lambda \) and \( Q_{K}(\lambda) + X(K, J) \in \mathbb{Z} \) for all \( K \in \mathcal{G}' \).

The boundary conditions take values in a different set. To define it, we must generalize the untwisted stabilizer (26) to the central stabilizer \( \mathcal{C}_\lambda \):

\[
\mathcal{C}_\lambda := \{ J \in S_\lambda | \phi_\lambda(K, J) = 1 \text{ for all } K \in \mathcal{G}' \}
\]

This construction covers huge class of boundary conditions, but in general it does not exhaust the space of all conformal boundary conditions of a CFT. In particular there exist boundary conditions without automorphism type; known examples include boundary conditions associated to conformal embeddings (for a review see [5], [23] and lectures by J.-B. Zuber in this volume) and certain boundary conditions for the \( \mathbb{Z}_2 \)-orbifold of a compactified free boson [21].
with \[ \phi_\lambda(K, J) := e^{2\pi i X(K,J)} F_\lambda(K, J)^* . \]

This is again a subgroup of the stabilizer \( S_\lambda \). It possesses a cohomological interpretation. Namely, \( \phi_\lambda \) is an alternating bihomomorphism on the stabilizer \( S \). For abelian groups such maps are in one-to-one correspondence to cohomology classes in \( H^2(S_\lambda, U(1)) \). To any such a cohomology class \([\epsilon]\) there is associated the twisted group algebra \( \mathbb{C}\epsilon S_\lambda \); this is a finite-dimensional associative algebra with a basis \( b_J \) labelled by \( S_\lambda \) and multiplication rule

\[ b_{J_1} \times b_{J_2} = \epsilon(J_1, J_2) b_{J_1 J_2} . \]

If the cocycle \( \epsilon \) is cohomologically non-trivial, which is the case if and only if \( \phi \) is not identically 1, i.e. if and only if the central stabilizer \( C_\lambda \) is strictly smaller than the stabilizer \( S_\lambda \), then the twisted group algebra \( \mathbb{C}\epsilon S_\lambda \) is non-abelian. It is a direct sum of \(|C_\lambda|\)-many full matrix algebras of rank

\[ n := \sqrt{|S_\lambda| / |C_\lambda|} . \]

(By the cohomology theory of finite abelian groups, \( n \) is an integer.) The fact that the twisted group algebra can be non-abelian will have important consequences for the gauge symmetries on D-branes.

The boundary labels are now defined as equivalence classes \([\rho, \psi]\) of pairs, where \( \rho \) is any label of \( \mathfrak{A} \) and \( \psi \) is a character of the central stabilizer. The equivalence relation is the same as for the labels of simple current extensions (cf. Section 4.3), but with \( F_\rho \) replaced by \( \phi_\rho \). Accordingly, for boundary conditions the phenomenon of fixed point resolution arises, too.

To each boundary condition \([\rho, \psi]\) for a simple current invariant of extension type one associates its automorphism type \( g_\rho \). This is the character of \( G' \) that is furnished by the monodromy charge:

\[ g_\rho(J) = e^{2\pi i Q(J, \rho)} \]

for \( J \in G' \). Recall that the monodromy charge is preserved in fusion products of \( \mathfrak{A} \). In the annulus coefficients it is preserved as well: Expanding the annulus amplitude \( A_{[\rho_1, \psi_1], [\rho_2, \psi_2]} \) in terms of characters of \( \mathfrak{A} \), only characters of monodromy charge \( Q(\rho_1) - Q(\rho_2) \) contribute.

We are now finally in a position to determine the boundary coefficients \( B_{(\lambda, J), [\rho, \psi]} \) that according to

\[ \langle [\rho, \psi] \rangle = \sum_{(\lambda, J)} B_{(\lambda, J), [\rho, \psi]} \langle (\lambda, J) \rangle \]  

express the boundary states through the Ishibashi states. The formula is of amazing simplicity:

\[ B_{(\lambda, J), [\rho, \psi]} = \sqrt{\frac{|G'|}{|S_\rho| |C_\rho|} \frac{S^J_{\lambda\rho}}{S_{\Omega, \lambda}} \psi(J)^*} . \]  

Notice that the boundary coefficients are expressed in terms of chiral data, in particular the matrices \( S^J \) that describe the modular transformations of the one-point blocks on the torus. The formula (41) encompasses several formulae that were previously known in more special situations; a similar formula can also be derived for the crosscap (17). One can show that the matrix \( B \) is unitary; this implies in particular that the number of boundary conditions equals the number of Ishibashi states. Finally, we mention that from the result (41) one can compute the annulus coefficients and show them to be non-negative integers.
6 Applications to string compactifications

A major role of full CFT is to provide the world sheet theory of strings. We now apply the techniques we have just developed to D-branes in string compactifications. To this end we first study chiral aspects of string compactifications.

6.1 Chiral aspects of string compactifications

We consider compactifications of the string to \( D \) flat space-time dimensions. For simplicity we take \( D \) to be even, which includes the most interesting case \( D = 4 \). String theory requires a time-like coordinate on the target space; for the bosonic string and the usual superstring, one needs Minkowskian signature. We will therefore require one of the flat dimensions to be time-like.

At the level of chiral CFT the flat dimensions are described by the tensor product of \( D \) free bosons and \( D \) free fermions. As we have seen, this tensor product \( \mathcal{C}^\text{st,bos}_D \times \mathcal{C}^\text{st,ferm}_{D/2} \) has \( N = 2 \) supersymmetry (subscripts stand for the Virasoro central charge). The compactification manifold gives rise to another \( N = 2 \) superconformal theory \( \mathcal{C}^\text{inner} \).

Remember that the main idea of (perturbative) string theory is to ‘forget’ the world sheet and that as part of this task we must take the cohomology with respect to an \( N = 1 \) super-Virasoro algebra. We therefore must include in the tensor product also two of the first order systems that have been discussed in Section 3.3: The chiral CFT \( \mathcal{C}^\text{gh}_{-26} \) of ghosts to gauge the stress-energy tensor, and the chiral CFT \( \mathcal{C}^\text{sgh}_{11} \) of superghosts to gauge one \( N = 1 \) supercurrent. We are thus lead to consider the tensor product

\[
\mathcal{C}^\text{st,bos}_D \times \mathcal{C}^\text{st,ferm}_{D/2} \times \mathcal{C}^\text{inner} \times \mathcal{C}^\text{gh}_{-26} \times \mathcal{C}^\text{sgh}_{11}.
\]

A cohomology can only be defined if the BRST operator constructed from symmetry generators and ghosts is nilpotent, or in other words, only if the chiral symmetry in (42) is a topological VOA. This requires that the total Virasoro central charge of the tensor product vanishes. As a consequence, the central charge of the inner sector is fixed to

\[
c = 15 - 3D/2
\]

The tensor product theory can be simplified as follows. We first observe that neither \( \mathcal{C}^\text{st,bos}_D \) nor \( \mathcal{C}^\text{gh}_{-26} \) contain space-time fermions; these factors are therefore easy to understand, so we drop them from our discussion. Next, we notice that the superghosts can be bosonized, giving rise to a free boson with ‘wrong’ sign in the two-point blocks. Also, bosonization of the fermions gives free bosons compactified on the root lattice \( L_{D/2} \) of the Lie algebra \( \mathfrak{so}(D) \). Hence superghosts and space-time fermions can simultaneously be described by free bosons compactified on a lattice \( L_{D/2,1} \) of Lorentzian signature. For all aspects concerning representations of the mapping class group, in particular for the modular transformations of the characters, this chiral non-unitary CFT behaves exactly like the unitary theory of \( \mathfrak{so}(D+6) \). (To get all scalar factors right, one should also tensor with \( E_8 \) at level 1, a subtlety that we will neglect.) In the correspondence between \( L_{D/2,1} \) and \( \mathfrak{so}(D+6) \), scalar and vector representations should be exchanged and the spinorial representations acquire additional minus signs. This mapping goes under the name \textit{bosonic string map}. It also plays in important role in the construction of the heterotic string; for a review see [32].
In conclusion, we can restrict our attention to the tensor product
\[ so(D+6) \times C^{\text{inner}}. \] (43)

It is, however, not simply this tensor product that describes the chiral aspects of a string compactification. We already saw that the alignment of the world sheet fermions forces us to perform a simple current extension; here the relevant simple current is \((v, v)\) where the first field \(v\) is the primary of \(so(D+6)\) that corresponds to the vector conjugacy class and the second \(v\) is the primary field that contains the supercurrent of the inner sector.

Moreover, this is not the only projection we must impose. Space-time supersymmetry requires the GSO projection. After the bosonic string map, the GSO projection corresponds to a simple current extension as well, this time one that is connected to spectral flow. Indeed, the generators for space-time supersymmetry can be constructed from the spectral flow operators. Even a converse of this statement holds [3]: For \(D = 4\), space-time supersymmetry requires \(N = 2\) superconformal symmetry on the world sheet. (But the situation is more involved for \(D = 3\) or 2 [4] which geometrically corresponds to the existence of manifolds with special holonomy.) The extending simple current can be given explicitly: It is the field \((s, s)\), where the first \(s\) refers to the spinor representation of \(so(D+6)\) and the other \(s\) is the spectral flow simple current of \(C^{\text{inner}}\). This simple current has integral conformal weight:
\[
\frac{D + 6}{16} + \frac{c}{24} = \frac{D + 6}{16} + \frac{15 - 3D/2}{24} = 1.
\]
Here we inserted the value \(c\) of the Virasoro central charge for the inner sector; this value is determined by the nilpotency of the BRST operator. It is remarkable that this condition (which does not know anything about space-time supersymmetry) also ensures that we can consistently implement the GSO projection with simple currents.

Let us summarize our discussion. The chiral CFT underlying a string compactification can be formulated as the tensor product (43) together with a specific simple current extension. This point is already important for closed string theory, e.g. for the correct computation of massless spectra. It becomes even more crucial once one considers boundary conditions. Indeed, we learned that chiral data like S-matrix elements enter crucially in the construction of boundary states; these chiral data should be the ones of the simple current extension, not the ones of the naive tensor product (43).

### 6.2 The Gepner construction

We are now ready to examine a specific construction of string vacua that dates back to Gepner. His idea was to use a tensor product
\[ \bigotimes_{\alpha=1}^{r} C^{(\alpha)} \] (44)
of \(N = 2\) minimal models as the starting point for describing the inner sector. The levels \(k_{\alpha}\) of the minimal models must be chosen such that the correct Virasoro anomaly results:
\[
\sum_{\alpha=1}^{r} \frac{3k_{\alpha}}{k_{\alpha} + 2} = 15 - \frac{3D}{2}.
\]
A famous solution for $D=4$ is given by $r=5$ and all levels equal to 3. As it turns out, this model (with charge conjugation modular invariant) corresponds geometrically to a compactification on the so-called Quintic threefold, i.e. the zero-locus of the homogeneous polynomial $\sum_{\alpha=1}^{5} x_{\alpha} = 0$ in the projective space $\mathbb{CP}^{4}$.

We have seen that one cannot simply take the tensor product $(44)$ as the inner sector, but in addition one must extend by the bilinears in the supercurrents so as to obtain a supersymmetric theory. Furthermore, the GSO projection is (after the bosonic string map) a projection to even integral $U(1)$ charges. By comparison with the possible values of the $U(1)$ charge in the flat space-time and superghost parts of the string theory, it follows that to obtain a CFT on which the GSO projection can be performed, we already need to project the tensor product $(44)$ onto integral $U(1)$ charge. (One might refer to this operation as the pre-GSO projection.) Thus, the internal theory used as $C_{\text{inner}}$ in the sense of the previous subsection really is an extension of $(44)$ by the $(\mathbb{Z}_2)^{r-1}$ group of bilinears in supercurrents $v_i, v_j$ and by the additional current 

$$v_1^{D/2-1}(s^2, s^2, \ldots, s^2)$$

with $s = \Phi_1^{0,1}$ as described at the end of Subsection 4.3. Only after this extension can the inner theory be compared with the $\sigma$-model on a Calabi-Yau manifold. Accordingly we call this extension of the tensor product the \textit{Calabi-Yau extension}.

When studying string compactifications in the geometrical framework, one finds that often several different compactification manifolds give physically indistinguishable string theories. In particular, given a string theory compactified on a CY manifold $M$, there is another CY manifold $W$, the \textit{mirror manifold}, that gives the same physical results.

In the conformal field theory description, the transition from $M$ to $W$ amounts to replacing a modular invariant $Z_{\lambda, \mu}$ by the invariant $Z_{\lambda^*, \mu^*}$, where in the Neveu-Schwarz sector the star stands for charge conjugation, i.e. $\lambda^* = \lambda^+$, while in the Ramond sector it slightly differs from conjugation. In the Gepner construction, this can be afforded by taking the simple current modular invariant for a group of simple currents $G_{\text{GP}}$ that beyond the simple currents already present in the CY extension also contains all simple currents of the form

$$(v_1)^{c} \prod_{\alpha=1}^{r} (\Phi_0^{0,2})^{\pi_\alpha}$$

such that

$$\sum_{\alpha=1}^{r} \frac{\pi_\alpha}{k_\alpha + 2} + \frac{c}{2} \in \mathbb{Z}.$$ 

Here the numbers $\pi_\alpha$ are integers, defined modulo $k_\alpha + 2$, and $c$ vanishes for $r + c_{\text{inner}}/3 \in 2\mathbb{Z}$ and can take the values 0 and 1 for $r + c_{\text{inner}}/3 \in 2\mathbb{Z} + 1$. Moreover, an appropriate discrete torsion matrix $X$ (31) must be chosen; for details we refer to \textit{[18]}. This construction goes under the name \textit{Greene-Plesser construction}. It has been most successfully applied in the context of Gepner models (i.e. when one starts from the product of minimal models), where it receives its well-known geometric interpretation for the mirror construction. However, using simple current symmetries to obtain a mirror model should also work for some other cases, such as for certain Kazama-Suzuki models.
6.3 Applications to D-branes

We now apply the results of Section 5.5 on boundary conditions in simple current modular invariants to the study of D-branes in string compactifications. In string theory, the action of the (super-)Virasoro algebra is gauged. As a consequence one is not allowed to break superconformal invariance of the underlying chiral CFT. This implies that the subalgebra of the chiral algebra that is preserved by the boundary conditions must contain at least an \( N=1 \) subalgebra of the total \( N=2 \) algebra as well as the bilinears in the vector currents \( v \). We refer to such boundary conditions as being *super-conformally invariant*.

From the geometric analysis one knows that D-branes in CY compactifications come in two species. Those of A-type are sensitive to the symplectic structure of the CY manifold; they correspond to special Lagrangian submanifolds (together with U(1)-bundle on them). Those of B-type test the holomorphic structure of the compactification space; they correspond to holomorphic vector bundles on submanifolds or, more generally, to coherent sheaves on the CY. Mirror symmetry exchanges D-branes of type A on \( \mathcal{M} \) with D-branes of type B on its mirror \( \mathcal{W} \). (This fact can be used to give the conceptually clearest definition of the mirror \( \mathcal{W} \) as the moduli space of so-called supersymmetric 3-cycles of \( \mathcal{M} \); for a review see [36].)

The algebraic characterization uses the U(1) current \( J(z) \) of the total \( N=2 \) algebra. For the charge conjugation modular invariant, D-branes of type A correspond to (super-)conformally invariant boundary conditions that preserve \( J \), i.e. we have a “Neumann-like” condition on \( J \) which translates into

\[
(J_n \otimes 1 + 1 \otimes J_{-n}) |B\rangle_A = 0 \quad (45)
\]

for the Ishibashi state. Similarly, for B-type one has “Dirichlet-like” boundary conditions:

\[
(J_n \otimes 1 - 1 \otimes J_{-n}) |B\rangle_B = 0 . \quad (46)
\]

Let us briefly discuss boundary conditions of A-type [24]. As we want to preserve world sheet supersymmetry, we must consider boundary conditions that preserve much more symmetry, namely the simple current extension \( \mathfrak{A}_{WS} \) of the tensor product \( (\mathfrak{A}_D \otimes \mathfrak{A}_V) \) by all bilinears of supercurrents \( v \). (The full CY extension is obtained from \( \mathfrak{A}_{WS} \) as a further extension by the cyclic group generated by the spectral flow simple current.)

The following two features of A-type boundary conditions are immediate consequences of our general discussion in Section 5.5:

- A simple current orbit \( [\rho] \) of \( \mathfrak{A}_{WS} \)-primaries does not suffice to specify an irreducible boundary condition. One needs in addition a character of the (central) stabilizer. In other words, the boundary condition associated to an orbit is in general not irreducible and can be split. We will encounter a similar phenomenon for B-type boundary conditions as well. In that case more refined geometrical tools allow us to relate the splitting to the existence of bound states at threshold. By mirror symmetry, we expect the same phenomenon to take place also for A-type branes.

- Boundary fields come in families, labelled by their automorphism type.\(^\text{10}\) This quantum number provides a grading on the annuli. It follows that only such boundary fields \( \Psi^\rho_\lambda \) exist for which the automorphism types are related by \( g_\lambda = g_\rho / g_{\rho'} \).

\(^{10}\) Conformal boundary conditions without automorphism type are of interest, too; they correspond to non-BPS D-branes. In contrast to the situation for free theories \([12, 33]\), so far only little is known about such boundary conditions for Gepner models.
In theories of closed strings, one must choose a modular invariant torus partition function to fix the field content of the theory. This amounts to prescribing a multiplicity \( Z_{\lambda,\mu} \in \mathbb{Z}_{\geq 0} \) for each possible bulk field \( \Phi_{\lambda,\mu} \). For boundary fields in theories that contain open strings, a similar choice must be made. However, this task arises not yet at the conceptual level of full CFT, but only once one specifies the string vacuum: For each boundary condition \( a \) one must choose a multiplicity \( N_a \in \mathbb{Z}_{\geq 0} \), called the Chan–Paton multiplicity of \( a \). Geometrically, this assignment amounts to choosing a brane configuration. At the same time, it might be necessary to include also unoriented strings; these world sheets effectively serve to (anti-)symmetrize the closed and open string spectrum. Insisting that this can be done in a consistent manner results in a number of consistency constraints on the conformal field theory, like relations between amplitudes for different surfaces of Euler characteristic zero; for details, we refer to section 4 of [40]. The assignment of Chan–Paton multiplicities must obey additional consistency constraints, which go under the name of ‘tadpole cancellation’; they ensure in particular the absence of anomalies in the low energy effective action. In fact, anomaly cancellation in these models requires a non-trivial extension of the Green–Schwarz formalism; for a review see [39].

Our previous results already constrain the possible brane configurations. For instance, for a brane configuration with branes of different automorphism type \( g_\rho \neq g_\rho' \), boundary fields with \( g_\lambda \neq 1 \) appear. However, in the present context the monodromy charge is the one of the spectral flow simple current and hence projecting to \( g_\lambda = 1 \) is equivalent to the pre-GSO projection. We conclude that such brane configurations do not respect the pre-GSO projection in the open string sector. As a consequence, typically tachyonic open string modes appear and the whole configuration is unstable. This effect generalizes the behavior of branes at angles in a flat background.

### 6.4 D-branes of B-type

The fact that mirror symmetry exchanges A- and B-type branes allows us to discuss D-branes of B-type in Gepner models by studying D-branes of A-type on the mirror model. The latter can be obtained explicitly from the Greene–Plesser construction.

The analysis shows that in a tensor product of \( r \) minimal models, the boundary conditions \( \rho \) of B-type are (partially) labelled by a collection of integers \( (L_1, L_2, \ldots, L_r) \) such that \( 0 \leq L_\alpha \leq [k_\alpha/2] \). (A further integer \( M \), related to the preserved space-time supersymmetry, is needed as well; we will suppress this in the discussion below.) But this labelling does not yet account for the fixed points under the Greene–Plesser simple current group. Resolving the fixed points correctly, we will be able to give a geometric interpretation both for fixed point resolution itself and for the role of the central stabilizer.

Starting from the boundary states with label \( \rho \) before fixed point resolution, one can compute the number \( \nu \) of vacuum states in the open string sector; the result is

\[
\nu = A_{\rho,\rho^+} = \sum_{J \in \mathcal{G}_{GP}} \mathcal{N}_{\rho^+J}^J = |\mathcal{S}_\rho|.
\]  

(47)

Here \( \mathcal{N} \) are fusion rules of the tensor product theory and \( \mathcal{S} \) denotes the stabilizer in the Greene–Plesser group \( \mathcal{G}_{GP} \). Concretely, denoting by \( \ell \) the number of components \( \alpha \) such that \( L_\alpha = k_\alpha/2 \), one finds \( \nu = 2^{\tilde{\ell}} \), where \( \tilde{\ell} = \ell \) if \( n+r \) is even and \( \tilde{\ell} = \max(\ell-1, 0) \) if \( n+r \) is odd.
The quantity $\nu$ is of direct physical interest because it counts the number of gauge bosons on the brane $\rho$. To learn more about these gauge bosons, we observe that the action of $S_\rho$ on the space whose dimension as given by (47) is typically only \textit{projective}. We already learned how to control the projectivity of such an action by using the central stabilizer. Using also the structure of the corresponding twisted group algebra, we find that the number $\tilde{\nu}$ of U(1) factors in the gauge group is equal to the number of elements in the central stabilizer,

$$\tilde{\nu} = |C_\rho|.$$  

The U(1) factors correspond to center-of-mass degrees of freedom of the brane, and therefore $\tilde{\nu}$ counts the number of constituent branes. Concretely, we find that $U \cong \mathbb{Z}_2$ when either $n+r$ is even and $\ell = 2, 4, \ldots$ or $n+r$ is odd and $\ell = 1, 3, \ldots$. In all other cases the central stabilizer is trivial.

It follows that the underlying twisted algebra is a direct sum of $\tilde{\nu}$ ideals, each of which is a full matrix algebra of rank $N$, where

$$\nu = N^2 \tilde{\nu}.$$  

(48)

We have thus found $\tilde{\nu}$ degenerate branes with SU($N$) gauge symmetry. The central stabilizer therefore provides a new mechanism to produce non-abelian gauge symmetries in type II compactifications.

Equation (48) is strongly reminiscent of similar relations for branes in orbifolds with discrete torsion. Also, the number $N$ describes an $N$-fold wrapping of the brane. A world sheet analysis of global anomalies has shown that such a multiple wrapping can also be the consequence of a flat (but non-trivial) $B$-field on a torsion two-cycle of the CY of order $N$. It is thus tempting to speculate that central stabilizers give a hint on the existence of torsion cycles in the CY.

One can also compute the Ramond-Ramond (RR) charges of the brane $\rho$. The following results have been obtained in [18] for the Fermat point of the K3 surface in the weighted projective space $\mathbb{CP}[1,1,1,3](6)$ which corresponds to a tensor product of three minimal models, all of level $k=4$. Up to normalization, they are given by the one-point correlator of a massless RR bulk-field $\Phi_{RR}$ on a disc with boundary condition $\rho$. This information is encoded in the boundary state:

$$q_{\text{RR}}(\rho) \sim \langle \Phi_{\text{RR}} \rangle = \langle \Phi_{\text{RR}}|\rho\rangle \sim B_{(RR,J),\rho}.$$  

Comparing the trace $\text{tr}_{H_{\rho_1,\rho_2}}(-)^F$ to the intersection form, one can establish the correspondence to the geometric charge lattice and the appropriate normalization. One finds that the RR charge after fixed point resolution is the RR charge of the unresolved, reducible brane, divided by the number $\tilde{\nu}$ of components; in short, RR charge is equally distributed over the resolved fields.

We thus arrive at the following geometrical interpretation. A first hint comes from the observation that the simple currents with fixed points typically have non-integral conformal weight. This suggests that the effect is not related to singularities of the CY manifold (which would also affect the bulk theory), but rather to singularities of the gauge bundle. One therefore expects a relation to an interesting mathematical question, namely the compactification of the moduli space of vector bundles on K3.

To corroborate this conjecture, we study a configuration of RR charge

$$v = (r, c_1, r + \frac{1}{2} c_1^2 - c_2) = (2, 0, 2k),$$  

41
for which we find $\nu = \tilde{\nu} = 2$. It splits into two branes of identical charge $(1, 0, 1-k)$. No line bundle can have such a RR charge, only so-called strictly semi-stable sheaves $E$; these are sheaves $E$ which possess a subsheaf $E'$ of the same slope,

$$\frac{v(E)}{\text{rk } E} = \frac{v(E')}{\text{rk } E'}.$$

Indeed, such sheaves only appear for values of the RR charges for which the moduli space is not compact, and this requires that the greatest common divisor of the RR charges to satisfy $\gcd(q_0, q_2, q_4) > 1$. Thus we have collinear RR charges, which are the central charges in the (space-time) supersymmetry algebra. This is characteristic for bound states at threshold, and establishes our interpretation of fixed point resolution.

## 7 Conclusions

Our conclusions are short. Conformal field theory has many applications. One exciting application is to string theory: CFT gives insight into string theory with branes in the strong curvature regime. In this regime standard geometric tools risk to break down, and the fact that CFT provides independent information is definitely much welcome.

Another point we would like to emphasize is that the understanding of two-dimensional CFT has made a lot of progress in the past few years. New tools have become available which make computations feasible that would have been beyond reach some years ago. We therefore expect that the interplay between CFT and geometric methods in string theory will continue to be fruitful.

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