The Phase Space of the Wess-Zumino-Witten Model

G. Papadopoulos

Department of Physics
Queen Mary and Westfield College
London E1 4NS

and

B. Spence

Blackett Laboratory
Imperial College
London SW7 2BZ

Abstract

We prove that the covariant and Hamiltonian phase spaces of the Wess-Zumino-Witten model on the cylinder are diffeomorphic and we derive the Poisson brackets of the theory.

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1 Address from Oct. 1st 1992: Dept. Mathematics, Kings College, London WC2R 2LS.
2 Address from Oct. 1st 1992: Dept. Physics, University of Melbourne, Victoria 3052 Australia.
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1. Introduction

The Wess-Zumino-Witten (WZW) model [1] is a two-dimensional non-linear sigma model with Wess-Zumino term whose target manifold is a compact, connected Lie group $G$, and it is a fundamental conformal field theory. Recently there have been a number of proposals for deriving a quantum group structure in this model [2], [3], [4], [5], [6], [7], [8], [9]. As a consequence of this work, there has been interest in the phase space structure of the WZW model [8], [9], [10]. For example, the authors of Ref. [9] have argued that the quantisation of the Poisson bracket algebra of a set of variables leads to an exchange algebra. However, there has been little study of the global structure of the phase space of the WZW model. This global structure is crucial to the definition of the symplectic form and the derivation of the Poisson brackets of the theory. We will discuss this in the following. Some of this work has also been reported in Ref. [11].

There are two approaches to the definition of the phase space of a field theory. The first is the Hamiltonian definition. In this approach, the space-time is a topological product $M = \Sigma \times \mathbb{R}$ and the momentum of the theory is defined as follows:

$$P_i = \frac{\partial L(X)}{\partial \partial_t X^i}. \quad (1)$$

$X$ are the fields and $L$ is the Lagrangian of the theory quadratic in the time $t$ derivatives of the fields. If there are no constraints in the theory, the Hamiltonian phase space $P_H$ is the co-tangent bundle $T^*Q$ of the configuration space $Q$ of the model. The Poisson brackets of the theory are

$$\{X^i(x), P_j(y)\} = \delta^i_j(x, y), \quad (2)$$

where $x, y \in \Sigma$. These brackets correspond to the standard symplectic structure of $T^*Q$. To describe the other definition of the phase space of a field theory, we start from the Lagrangian $L$ and define the symplectic current

$$S^\mu = \delta X^i \delta \frac{\partial L}{\partial \partial_\mu X^i}. \quad (3)$$

This current is conserved subject to the Lagrangian equations of motion of the theory. The symplectic form is

$$\Omega = \int_\Sigma S_\mu \, d\Sigma^\mu. \quad (4)$$

$\Omega$ is closed and it does not depend on the choice of the Cauchy surface $\Sigma$ (see Ref. [12] for discussions of the Lagrangian approach to the phase space). If the theory does not
have constraints, it is possible to relate the Lagrangian and Hamiltonian definition of the symplectic form by a Legendre transform. One way to parameterise the symplectic form \( \Omega \) is in terms of the initial data \( f(x) = X(x, 0) \) and \( w(x) = (\partial_t X)(x, 0) \). However, if the solutions of the Lagrangian equations of motion are known, then it is possible to parameterise \( \Omega \) in terms of the parameters of the solutions of the theory. The space of solutions of the Lagrangian equations of motion of a theory equipped with the symplectic form of eqn (4) is called the covariant phase space \( PC \) (see Ref. [13] for recent discussions). The covariant phase space is particularly suited to the study of field theories where we know the space of classical solutions, such as in the case of the WZW model.

In the following, we will give a new parameterisation of the space of solutions of the WZW model, and use this to prove that the Hamiltonian and covariant phase spaces are diffeomorphic. The Poisson brackets of the theory will also be derived. Finally, we will compare our approach with other formulations that have appeared in the literature.

2. The WZW Model

In the Hamiltonian approach, one may consider the WZW model as a two-dimensional non-linear sigma model with Wess-Zumino term, whose target space is a group manifold \( G \). Applying the usual Hamiltonian analysis to this sigma model action, one finds directly that the phase space \( PH \) of the WZW model is the co-tangent bundle of its configuration space \( LG \), i.e. \( PH = T^*LG \) (see Ref. [14]).

Now we consider the covariant approach to the phase space. The symplectic form of the WZW model is (see Ref. [9], for example)

\[
\Omega = -\frac{\kappa}{8\pi} \int_0^1 dx \text{tr} \left( (g^{-1}\delta g)\partial_+(g^{-1}\delta g) - (\delta gg^{-1})\partial_-(\delta gg^{-1}) \right),
\]

where \( \kappa \) is the coupling constant of the model, \( (x, t), 0 \leq x < 1, -\infty < t < +\infty \) are the co-ordinates of the cylinder \( S^1 \times \mathbb{R} \) and \( x^\pm = x \pm t \). This symplectic form is closed and time independent (we take \( t = 0 \) in the following).

One way to parameterise \( \Omega \) is in terms of the initial conditions \( f(x) = g(x, 0) \) and \( w(x) = (g^{-1}\partial_t g)(x, 0) \) on the Cauchy surface \( t = 0 \). In terms of the functions \( f \) and \( w \), the symplectic form \( \Omega \) becomes

\[
\Omega = -\frac{\kappa}{8\pi} \int_0^1 dx \text{tr} \left( \frac{1}{2}f^{-1}\delta f \partial_x(f^{-1}\delta f) - \frac{1}{2}\delta ff^{-1}\partial_x(\delta ff^{-1}) \right) + (f^{-1}\delta f)^2 w + f^{-1}\delta f \delta w.
\]
To construct the covariant phase space $P_C$, we should introduce a parameterisation of the space of solutions of the WZW model. The equations of motion are

$$\partial_-(\partial_+ gg^{-1}) = 0.$$  \hspace{1cm} (7)

The equations of motion are invariant under the semi-local transformations $g \rightarrow l(x^+)g r^{-1}(x^-)$ and the corresponding currents are

$$J_+ = -\frac{\kappa}{4\pi} \partial_+ gg^{-1}, \quad J_- = \frac{\kappa}{4\pi} g^{-1} \partial_- g.$$ \hspace{1cm} (8)

There are several suggestions in the literature as to how to parameterise the space of solutions of the WZW model. In the following section we will discuss the parameterisation of Ref. [11], and in the last section we will compare this parameterisation with others in the literature.

3. The Poisson brackets

In Ref. [11], we have parameterised the space of solutions to the field equations of the WZW model as follows

$$g(x, t) = U(x^+) \mathcal{W}(A; x^+, x^-) V(x^-),$$

$$\mathcal{W}(A; x^+, x^-) = P \exp \int_{x^-}^{x^+} A(s) ds,$$ \hspace{1cm} (9)

where $U$ and $V$ are periodic maps from the real line $\mathbb{R}$ to the group $G$, and the field $A$ in the path-ordered exponential is a $(\text{Lie}G)^*$-valued periodic one-form on the real line. The expression for $g(x, t)$ in Eqn. (9) is then periodic in $x$ and solves the field equations (7). To prove the periodicity of $g$, it is enough to show that $\mathcal{W}(A; x^+ + 1, x^- + 1) = \mathcal{W}(A; x^+, x^-)$. One way to verify this is to use the power series expansion of $\mathcal{W}$ and change variables. This gives $\mathcal{W}(A; x^+ + 1, x^- + 1) = \mathcal{W}(A; x^+, x^-)$ where $\hat{A}(x) = A(x + 1)$. Using the periodicity of $A$ we can then prove that $g$ is periodic. An alternative way is to use the formula $\mathcal{W}(A; x^+ + 1, x^- + 1) = m(x^+) \mathcal{W}(A; x^+, x^-) m^{-1}(x^-) = \mathcal{W}(A^m; x^+, x^-)$, where $m(x) = \mathcal{W}(A; x + 1, x)$ and $A^m = \partial_x m(x) m^{-1}(x) + m(x) A(x) m^{-1}(x)$. However $\partial_x m(x) = A(x + 1) m(x) - m(x) A(x)$, and from the periodicity of $A$ we get $A^m = A$. This again proves the periodicity of $g$.

Next, using the parallel transport equation

$$\partial_s \mathcal{W}(A; s, x^-) = A(s) \mathcal{W}(A; s, x^-),$$ \hspace{1cm} (10)
we can prove that \( g \) in eqn. (9) satisfies the equations of motion of the WZW model. Choosing a point \( x_0 \) on the real line, we can write the solution given in eqn. (9) in a chirally factorised form \( g(x, t) = u(x^+, x_0) v(x^-, x_0) \), by using the identity \( \mathcal{W}(A; x^+, x^-) = \mathcal{W}(A; x^+, x_0) \mathcal{W}(A; x_0, x^-) \). However, this factorisation depends on the choice of the point \( x_0 \).

Inserting the solution (9) into the symplectic form (5) gives
\[
\Omega = -\kappa \int_0^1 dx \, \text{tr} \left( (U^{-1} \delta U) \partial_x (U^{-1} \delta U) + 2(U^{-1} \delta U)^2 A + 2(U^{-1} \delta U) \delta A + 2(\delta V V^{-1}) \partial_x (\delta V V^{-1}) - 2(\delta V V^{-1})^2 A + 2(\delta V V^{-1}) \delta A \right).
\]

The solution \( g \) of the WZW equations of motion given in the parameterisation (9) is invariant under the transformations
\[
U(x) \to U(x) h(x), \quad V(x) \to h^{-1}(x) V(x),
A(x) \to -h^{-1}(x) \partial_x h(x) + h^{-1}(x) A(x) h(x),
\]
where \( h \in LG \). To prove this, we observe that under these transformations \( \mathcal{W}(A; x^+, x^-) \to h^{-1}(x^+) \mathcal{W}(A; x^+, x^-) h(x^-) \). The phase space \( P_G \) of the WZW model is then the space of fields \( \{U, V, A\} \), modulo the transformations (12). This is \( LG \times LG \times A \) where \( A \) is the space of \( G \)-connections over the circle. This is diffeomorphic to \( T^*LG \), i.e. it is the same as the phase space \( P_H \) derived from the Hamiltonian treatment of the theory.

The symplectic form (11) is degenerate along the directions of the action (12) of the loop group \( LG \). We may deal with this by gauge-fixing or by enhancing the phase space of the theory and then imposing constraints (11). In the following, we will use the gauge-fixing method. We may choose as a gauge fixing condition \( U = e \) where \( e \) is the identity element of the loop group \( LG \). This is a good gauge choice, as \( LG \) acts freely and transitively on the space of \( U \)'s, \( \{U\} \). The symplectic form (11) then becomes
\[
\Omega = -\kappa \int_0^1 dx \, \text{tr} \left( - (\delta V V^{-1}) \partial_x (\delta V V^{-1}) - 2(\delta V V^{-1})^2 A + 2(\delta V V^{-1}) \delta A \right).
\]

This symplectic form is not degenerate and is invertible.

4 There two ways to think about the field \( A \). The first is as a periodic one-form on the real line, valued in \((\text{Lie}G)^*\). The second is as a connection over the circle \( S^1 \). We have identified \( \text{Lie}G \) with \((\text{Lie}G)^*\) using the invariant metric.

5 Notice that if we set \( A = \frac{1}{2}(\partial_x f f^{-1} + f w f^{-1}) \) and \( V = f \) in the symplectic form (11) then we get the symplectic form (13).
The simplest way to invert the form (13) is to first rewrite it in terms of a local parameterisation $X^i(x)$ for the maps $V (V = V(X))$. This gives

$$
\Omega = -\frac{\kappa}{8\pi} \int_0^1 dx \left( - \left( R^a_i \delta X^i \right) \partial_x (R^a_j \delta X^j) - f_{ab}^c R^a_i R^b_j A_c \delta X^i \delta X^j + 2 R^a_i \delta X^i \delta A^a \right),
$$

(14)

where $\delta V V^{-1} = R^a t_a$. The remarkable feature of this expression for the form $\Omega$ is that one does not need to invert any differential operator in order to invert the form (c.f. Refs. [9], [10], where in order to invert the symplectic form it was necessary to find the inverse of the operator $\partial_x$ on the circle). The gauge $U = e$ parameterises the symplectic form on $T^* LG$ in terms of the right trivialisation of $T^* LG$ and the gauge $V = e$ parameterises the same symplectic form in terms of the left trivialisation. The inversion of the form (14) is straightforwardly carried out, and leads to the Poisson brackets ($\beta = -\frac{4\pi}{\kappa}$)

$$\{ X^i(x), X^j(y) \} = 0,$$

$$\{ X^i(x), A_a(y) \} = \beta R^i_a X(x) \delta(x, y),$$

$$\{ A_a(x), A_b(y) \} = \beta \left( \delta_{ab} \partial_x + f_{ab}^c A_c(x) \right) \delta(x, y),$$

(15)

where $\delta(x, y)$ is the delta function on $S^1$.

Using Eqn. (15), we can calculate Poisson brackets involving $V$ and $A$ – for example $\{ V(x) \otimes V(y) \} = 0$, $\{ V(x), A_a(y) \} = \beta V(x) t_a \delta(x, y)$. In this gauge, the WZW currents (8) become $J_+ = -\frac{\kappa}{4\pi} A$, $J_- = \frac{\kappa}{4\pi} (V^{-1} \partial_x V - V^{-1} A V)$, and it can be verified by a straightforward calculation that their Poisson bracket algebra is isomorphic to two commuting copies of a Kac-Moody algebra with a central extension.

4. Discussion

We would now like to discuss how our results relate to other work in the literature. In Ref. [1], Witten proposed that the general solution of the WZW equations of motion is $g(x, t) = U(x^+) V(x^-)$ where $U$ and $V$ are maps from the circle into the group $G$. In this parameterisation the covariant phase space is $LG \times LG$ where $LG$ is the loop group of the group $G$. Recently, another parameterisation of the space of solutions of the WZW model was proposed by Chu et al in Ref. [9]. In this parameterisation the solution of the WZW model factorises as $g(x, t) = u(x^+) v(x^-)$. However, the functions $u$ and $v$ are not periodic and satisfy the conditions $u(x + 1) = u(x) M$, $v(x + 1) = M^{-1} v(x)$, where $M \in G$ is the monodromy. In addition, the authors of Ref. [9] observed that
the solution $g = uv$ of the WZW model remains invariant under the action $u(x) \rightarrow u(x)k$, $v(x) \rightarrow k^{-1}v(x)$, $M \rightarrow k^{-1}Mk$ of the group $G$, with $k \in G$. An equivalent way to write the solution $g$ of the WZW model in this parameterisation is

$$g(x, t) = U(x + M^2t)V(x^-),$$

where $U, V$ are periodic. The covariant phase space of the WZW model in the parameterisation of Ref. [9] is

$$LG \times LG \times G.$$ 

Neither this covariant phase space, nor that of Ref. [1], are diffeomorphic to the Hamiltonian phase space $T^*LG$ of the WZW model – for example, $\pi_2(LG \times LG) = \pi_2(LG \times LG \times G) = \mathbb{Z} \oplus \mathbb{Z}$, whereas $\pi_2(T^*LG) = \mathbb{Z}$ (for $G$ simple and simply connected).

To relate the parameterisation of the space of solutions of the WZW model of Ref. [9] to the parameterisation of Section Three, the symmetries of the space of solutions of the WZW model (Eqn. (12)) can be treated by choosing the gauge-fixing conditions to be different from those considered in Section Three above. For example in the parameterisation (9) one can gauge-fix the connection $A$ so that it is a constant connection over the circle. The residual transformations for this gauge-fixing are the constant gauge transformations. The constant gauge transformations are parameterised by the elements of the group $G$ and they act on the parameters of the solutions as $U \rightarrow Uk$, $V \rightarrow k^{-1}V$ and $A \rightarrow k^{-1}Ak$ where $k \in G$ and $A$ is a constant connection. This parameterisation is the one given in Ref. [1]. The resulting phase space of the theory is

$$\frac{LG \times LG \times G}{G},$$ 

if we parameterise the solutions in terms of the monodromy $M$ of the connection $A$. The reason that this phase space is different from $T^*LG$ is that there is a Gribov-Singer ambiguity associated with this gauge fixing; note that this is the one-dimensional analogue of the four-dimensional Yang-Mills Gribov-Singer ambiguity [15]. The $k$-symmetry just mentioned can be further gauge fixed by choosing $A$ to be in the Cartan subalgebra $h$ of Lie $G$, and this parameterisation was used to calculate the Poisson brackets of this theory in Refs. [9], [10]. Finally, the parameterisation of Ref. [1] corresponds to the gauge-fixing choice $A = 0$. However, not all connections can be brought into this form by a gauge transformation.

An alternative way to compare the different parameterisations of the space of solutions of the WZW model is to study the initial values $f(x) = g(x, 0)$ and $w(x) = (g^{-1}\partial_x g)(x, 0)$ of the field $g$ that correspond to these parameterisations. In general $f$ and $w$ are independent functions. It is easy to see that the parameterisation of eqn. (9) corresponds to the most general Cauchy data of the theory. However, the parameterisations of Refs. [9] and [1] are associated to a subset of the available Cauchy data of this theory. The restriction that the parameterisation of Ref. [9] imposes on the Cauchy data is that the holonomy of the connection $\hat{w} = \frac{1}{2}(f^{-1}\partial_x f - w)$ does not depend on the point chosen to evaluate it, i.e.
\[ \partial_x W(\hat{w}; x + 1, x) = 0. \] The constraint that the parameterisation of Ref. [1] imposes on the Cauchy data is that the holonomy of the connection \( \hat{w} \) on the circle \( S^1 \) must be the identity group.

In conclusion, the parameterisation of the solutions of the WZW model given in Section Three (Eqn. (3)) is general in the sense that it is invariant under a larger symmetry than other parameterisations considered in the literature, and the latter can be thought of as locally-valid gauge-fixed versions of it. In our parameterisation, the covariant canonical phase space of the WZW model is diffeomorphic to the Hamiltonian phase space of the theory, and the calculation of the Poisson brackets is straightforward.

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