Timelike Boundary Sine-Gordon Theory and Two-Component Plasma

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Abstract

It has long been known that there is a relation between boundary sine-Gordon theory and thermodynamics of charge neutral two-component Coulomb plasma on a unit circle. On the other hand, recently it was found that open string worldsheet description of brane decay can be related to a sequence of points of thermodynamic equilibrium of one-component plasma. Here we consider a different decay process which is specifically described by the timelike boundary sine-Gordon theory. We find time evolution to be mapped to a one-dimensional curve in the space of points of thermal equilibrium of a non-neutral two-component Coulomb plasma. We compute the free energy of the system and find that along the curve it is monotonously decreasing, defining a thermodynamic arrow of time.

1 Introduction and summary

There is an interesting proposal [1] to use the value of a tachyon field rolling down in its effective potential as a “clock”, when the tachyonic system is coupled to gravity, to give a definition of time. An interesting a priori unrelated question is whether the concept of time could be “emergent”, and associated with a large N limit.

Previous work on tree level string worldsheet correlation functions in (open string) rolling tachyon background unraveled (in certain case) $U(N)$ matrix structures [2–7]. It was also understood that correlation functions can be related to expectation values of periodic functions in an ensemble of random $U(N)$ matrices of varying rank [5, 7]. The ensemble can alternatively be interpreted as a grand canonical ensemble of point charges on a unit circle, the Dyson gas, with different points of thermal equilibrium labeled by different values of the chemical potential corresponding to different instants of time [5, 8]. Later times turn out also to be related to larger values of the average

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number $\tilde{N}$ of point charges. Worldsheet correlation functions can then be related to thermal expectation values of moments of the electrostatic potential in the presence of test charges. Finally, embedding the system in a curved two-dimensional plane, at the critical (scale invariant point) one can reinterpret the field equations arising from the worldsheet beta functions. In particular, time evolution equations turn out to correspond to differential equations relating thermal expectation values at neighboring points of thermal equilibrium, at different chemical potential.

The $U(N)$ matrix structure was associated to a specific choice of a rolling tachyon background, the “half S-brane” profile $T(X^0) = \exp(X^0)$ on a bosonic open string theory worldsheet. Other choices lead to other structures. For example, generalization to superstrings leads to $U(N) \times U(M)$ type random matrix ensembles [6, 7]. However, the language of random matrices may be too restrictive. For example, the random matrix interpretation of Dyson gas at generic (inverse) temperature was found only relatively recently [9]. It is easier to study what thermodynamic ensembles correspond to different types of tachyon condensation.

This letter is a companion to [8] and focuses on identifying and studying the corresponding thermodynamic ensemble for the “full S-brane” rolling tachyon profile $T(X^0) = \cosh(X^0)$. It turns out to be the grand canonical ensemble of two-component plasma of equal but opposite charges confined to a unit circle on a two-dimensional plane. In this case there is no known random matrix interpretation. An interesting feature of the system is that both components of the plasma a priori have their own chemical potentials. Time evolution of the brane traces out a curve on the chemical potential plane. We evaluate the Helmholtz free energy along the curve and find it to be monotonously decreasing along the direction corresponding to later times. This “thermodynamic arrow of time” is a generalization of the one previously found in [8]; we also show how the latter is recovered in the appropriate scaling limit. We are left wondering whether there could be a more general relation: if a grand canonical statistical mechanical system with a space of points of scale invariant thermal equilibrium has a curve along which the free energy is monotonously decreasing, the curve turns out to correspond to the time evolution of some system in string theory.

## 2 The full S-brane and the disk partition function

We consider a rolling tachyon deformation of the open string worldsheet theory [10], called the full S-brane background. The non-trivial part of the action is given by the timelike boundary sine-Gordon theory (TBSG) [4]

$$S_0 + \delta S_{\text{bdry}} = -\frac{1}{2\pi} \int_\text{disk} \partial X^0 \bar{\partial} X^0 + \lambda_0 \int dt \cosh(X^0(t)) \; .$$  \hfill (1)

\[^1\text{At least not to the authors’ knowledge.}\]
As was discussed in [8], the basic quantity for identifying the statistical mechanical system is the disk partition function (separating out the zero mode $X^0 = x^0 + X^0$ and leaving it unintegrated),

$$Z_{\text{disk}}(x^0) = \int \mathcal{D}X^0 e^{-S_0} e^{-\lambda_0 \oint dt \cosh(X^0(t))}.$$  \hspace{1cm} (2)

Next one would expand the boundary perturbation in power series, work out the contractions between exponentials $e^{\pm X^0(t)}$ and sum up the series in the end. A subtle issue arises from the Wick ordering of the terms in the correlators, which require contractions of opposite types of exponentials, and cause renormalization of the coupling constant. Exploiting the underlying $SU(2)$ current algebra structure [11] or proceeding via fermionization [12] shows that the bare coupling $\lambda_0$ gets renormalized to

$$\lambda = \sin(\pi \lambda_0).$$  \hspace{1cm} (3)

After Wick ordering, all correlators are of the type

$$G_2(t_1, \ldots, t_{N+}; \tau_1, \ldots, \tau_{N-}) \equiv \left\langle \prod_{i=1}^{N_+} e^{X^0(t_i)} \prod_{m=1}^{N_-} e^{-X^0(\tau_m)} \right\rangle \hspace{1cm} (4)$$

and

$$= \frac{\prod_{1 \leq i < j \leq N_+} |e^{it_i} - e^{it_j}|^2 \prod_{1 \leq m < n \leq N_-} |e^{i\tau_m} - e^{i\tau_n}|^2}{\prod_{i=1}^{N_+} \prod_{m=1}^{N_-} |e^{it_i} - e^{i\tau_m}|^2}.$$  \hspace{1cm} (4)

The final form for the disk partition function reads [10]

$$Z_{\text{disk}}(x^0) = \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{(-\lambda e^{x^0})^{N_+}(-\lambda e^{-x^0})^{N_-}}{N_+!N_-!}$$

$$\cdot \frac{1}{(2\pi)^{N_+ + N_-}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} dt_1 \cdots dt_{N_+} d\tau_1 \cdots d\tau_{N_-} G(t_1, \ldots, t_{N+}; \tau_1, \ldots, \tau_{N-})$$

$$= \frac{1}{1 + \lambda e^{x^0}} + \frac{1}{1 + \lambda e^{-x^0}} - 1.$$  \hspace{1cm} (5)

### 2.1 The two-component plasma

Upon analytic continuation $X^0 \rightarrow iX$, the action (11) is related to the ordinary spacelike boundary sine-Gordon theory (SBSG). More precisely, it is related to the case $\beta = 2$ of the family

$$S_{SG,\beta} = \frac{1}{2\pi} \int_{\text{disk}} \partial X \bar{\partial} X + \lambda_0 \oint dt \cos \left( \frac{\beta}{2} X(t) \right).$$  \hspace{1cm} (6)

It has been known for a long time that the tree level partition function of (5) can be interpreted as the partition function of a two-component plasma at inverse temperature $\beta$, confined to a unit circle in two dimensions (see e.g. [15] for a review).

\[\footnote{See also [13, 14] for additional recent discussion.}
However, in our case we are interested in the thermodynamic system corresponding to the timelike theory \( [1] \) rather than the spacelike theory \( [6] \). The key difference is that in the spacelike theory the dependence on the zero mode \( x = X - X' \) is oscillatory, and was trivially integrated out in previous investigations. Integration over the zero mode enforced charge neutrality, and [16] related the full partition function of the boundary sine-Gordon theory to a grand canonical partition function of an overall charge neutral two-component plasma, summing over the particle number of electrically neutral configurations only. In contrast, in the timelike theory, the zero mode dependence is important. Physically, in the case of D-brane decay, it reflects the presence of a time-dependent source; for example the disk partition function gives the evolution of pressure in spacetime \( [10] \). Integration of the zero mode is associated with Fourier transform to the canonical conjugate variable, the total energy.

The zero mode dependence necessitates a more general analysis of the associated thermodynamic system. Overall charge neutrality is no longer enforced, hence the grand canonical ensemble of the two-component plasma is extended to include populations with net electric charge. Each particle component now has its own chemical potential, to be related to the zero mode dependence. Since there is only one zero mode, it turns out to parameterize a one-dimensional curve in the chemical potential plane. We begin by reviewing the details of the grand canonical ensemble.

Consider two species of particles, \( N_+ \) particles carrying positive unit charge and \( N_- \) particles carrying negative unit charge, which are confined on a unit circle on a two-dimensional plane with positions \( e^{it_i} \) and \( e^{i\tau_m} \). They interact via the 2-body Coulomb potential,

\[
V(\phi_1, \phi_2) = -\log |e^{i\phi_1} - e^{i\phi_2}|,
\]

the interaction is two-dimensional while the particles are confined to one-dimensional motion.

Let this system be immersed in a reservoir at inverse temperature \( \beta = 1/T \). In the canonical partition function, kinetic energy contributes the usual Gaussian integral over the canonical momenta. We focus on\(^3\) the non-trivial contribution to the partition function from the potential energy,

\[
Z_{N_+,N_-}(\beta) = \frac{1}{(2\pi)^{N_+ + N_-}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{i=1}^{N_+} dt_i \prod_{m=1}^{N_-} d\tau_m e^{-\beta \left[ \sum_{i<j}^{N_+} V(t_i, t_j) + \sum_{m<n}^{N_-} V(t_i, \tau_m) - \sum_{i}^{N_+} \sum_{m}^{N_-} V(t_i, \tau_m) \right]}
\]

\[
= \frac{1}{(2\pi)^{N_+ + N_-}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{i=1}^{N_+} dt_i \prod_{m=1}^{N_-} d\tau_m G^\beta(t_1, \ldots, t_{N_+}; \tau_1, \ldots, \tau_{N_-}),
\]

\(^3\)The partition function factorizes, and the contribution from the kinetic energy is trivial to add in if necessary, hence the literature focuses on the potential energy contribution. Further, often the kinetic contribution is eliminated by working in the limit of infinitely massive particles.
where
\[ G_\beta(t_1, \ldots, t_{N_+}, \tau_1, \ldots, \tau_{N_-}) = \frac{\prod_{1 \leq i < j \leq N_+} |e^{it_i} - e^{it_j}|^\beta \prod_{1 \leq m < n \leq N_-} |e^{i\tau_m} - e^{i\tau_n}|^\beta}{\prod_{i=1}^{N_+} \prod_{m=1}^{N_-} |e^{it_i} - e^{i\tau_m}|^\beta}. \] (9)

The complicated integrals can be evaluated with the help of expansion in Jack polynomials [16]. We consider first the charge neutral case \( N \equiv N_+ = N_- \). The integral was found to be convergent for \( \beta < 1 \) so at \( \beta = 2 \) it should diverge. On the other hand, the grand canonical partition function
\[ Z_{\text{neutral}}(z, \beta) = \sum_{N=0}^{\infty} \frac{z^{2N}}{N!N!} Z_N(\beta) \] (10)
was found to have a distinct feature at \( \beta = 2 \). Previous investigations [17] had argued that the overall charge neutral system undergoes a phase transition from an insulating phase for \( \beta > 2 \) (low temperatures) to a conducting phase for \( \beta < 2 \) (high temperatures). In the low temperature phase, the opposite charges tend to form dipoles and hence become insulating. As one quantitative test [17], it was argued that in the conducting phase, the pressure of the system is non-analytic around \( z = 0 \) in the complex fugacity plane, while in insulating phase it is analytic around \( z = 0 \).

Ref. [16] considered the system at high temperatures, and by a saddle point analysis, they found that the grand partition function has an essential singularity at \( \beta = 2 \),
\[ Z_{\text{neutral}}(z, \beta) \sim \exp \left( \frac{z^{2/(2-\beta)}}{2-\beta} \right). \] (11)

However, to our knowledge a detailed understanding of the phase structure of the system is still lacking.

Let us then consider the general case \( N_+ \neq N_- \). The canonical partition function (following [16]) is
\[ Z_{N_+,N_-}(\beta) = Z_{N_+}(\beta) Z_{N_-}(\beta) \left\{ \sum_{\ell(\lambda) \leq \min(N_+,N_-)} N_\lambda(N_+) N_\lambda(N_-) \right\}, \] (12)
where
\[ Z_N(\beta) = \frac{\Gamma(1 + \frac{\beta N}{2})}{\Gamma(1 + \frac{\beta}{2})^N} \] (13)
is the canonical partition function for a single charge Coulomb gas on a circle (the Dyson gas), and
\[ N_\lambda(N) = \prod_{s \in \lambda} \left( \frac{j - 1 + \frac{\beta}{2} (N - i + 1)}{j + \frac{\beta}{2} (N - i)} \right). \] (14)
The sum in (12) is over partitions with an upper bound on length and the product in (14) is over the cells of a partition. Particular cases are \( Z_{N_+,0}(\beta) = Z_{N_+}(\beta), Z_{0,N_-}(\beta) = \)
\(Z_{N_+}(\beta)\). Consider then the special value \(\beta = 2\), where by the example (11) we would expect the partition function (12) to diverge. However, guided by the disk partition function (5), one can adopt a prescription to regulate the integrals, the details are given in Appendix. The regularized partition function simplifies miraculously to

\[
Z_{N_+,N_-}(\beta = 2) = N_+!N_-!(\delta_{N_+,0} + \delta_{N_-,0} - \delta_{N_+,N_-}) .
\]  

Next we allow particle exchange with the reservoir and move to the grand canonical ensemble. Unlike in the previous investigations, we let the opposite charged particles have independent populations with chemical potentials \(\mu_\pm\), so the grand partition function of the system is

\[
Z_{2G}(z_+, z_-, \beta) = \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{z_+^{N_+} z_-^{N_-}}{N_+!N_-!} Z_{N_+,N_-}(\beta) ,
\]  

With the regulated expression (15), at inverse temperature \(\beta = 2\) (16) simplifies to

\[
Z_{2G}(z_+, z_-, \beta = 2) = \frac{1}{1 - z_+} + \frac{1}{1 - z_-} - 1 ,
\]  

resembling the disk partition function (5). Note that away from the poles at \(z_\pm = 1\) the partition function is regular, in contrast to (11).

As in the previous investigation [8], relating the grand partition function of the thermodynamic system to the worldsheet disk partition function will require analytic continuation, to map chemical potential to time (the explicit zero mode of (5)). But now there is a new interesting twist: the two components of the plasma come with two chemical potentials but there is only one physical time. So one must restrict to a subset of points of chemical equilibrium, corresponding to a one-dimensional curve in the two-dimensional chemical potential (or fugacity) plane. Following [8], we start from (17), then analytically continue to negative fugacities, which we identify with real values of time \(x^0\):

\[
z_\pm \rightarrow -z_\pm = -\lambda e^{\pm x^0} ,
\]  

which defines a one-dimensional curve \(z_- = \lambda^2/z_+\) in the fugacity plane.

In this way, the grand partition function (17) is mapped to the disk partition function (5). Note that we need to restrict to values \(|z_\pm| < 1\) which correspond to

\[
|x^0| < \frac{\Delta x^0}{2} \equiv -\log \lambda .
\]  

It is interesting that the real time range \(\Delta x^0\), corresponding to the allowed values of the fugacities, has a very natural physical interpretation: the lifetime of the brane.

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4We thank S. Brodsky for this remark.

5I.e., the time interval during which the unstable brane first forms and then decays, see e.g. [18].
2.2 Thermodynamic arrow of time

In the case of the half S-brane, it was found that \( \bar{N} \), the average number of charges in the Dyson gas, could be used as the time variable, and the free energy of the system was a monotonically decreasing function of \( \bar{N} \), giving an arrow of time. This analysis was done at the temperature \( \beta = 2 \) corresponding to the exactly marginal rolling tachyon deformation [8]. Here we extend the calculations for the full S-brane. Extension to superstrings will be considered in [19].

The average particle numbers for positive and negative charges are

\[
\bar{N}_\pm \equiv \frac{1}{Z_{2G}} \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} N_+^z N_-^z \frac{N_+! N_-!}{N_+^z N_-^z} Z_{N_+, N_-} = \frac{1 - z_+ z_-}{1 - \lambda^2} \frac{z_+ z_-}{1 - z_+ z_-} .
\]  

(20)

Restricting to the curve \( z_+ z_- = \lambda^2 \), the numbers \( \bar{N}_\pm \) are related by

\[
\bar{N}_+ \bar{N}_- = \frac{\lambda^2}{(1 - \lambda^2)^2} .
\]  

(21)

To specify the thermodynamics of the system onto the curve, we solve the constraint and express the grand partition function in terms of \( \mu_+ \),

\[
Z_{2G} = \frac{(1 - \lambda^2)e^{\beta \mu_+}}{(1 - e^{\beta \mu_+})(e^{\beta \mu_+} - \lambda^2)} ,
\]  

(22)

Then, on the curve the Legendre transform to the Helmholtz free energy is given by

\[
A_{\text{curve}}(\bar{N}_+) = -\beta^{-1} \log Z_{2G} + \beta^{-1} \mu_+ \frac{\partial}{\partial \mu_+} \log Z_{2G} .
\]  

(23)

At \( \beta = 2 \), the free energy along the curve becomes

\[
A_{\text{curve}}(\bar{N}_+) = \frac{1}{2} \log \bar{N}_+ - \frac{1}{2} \log(1 + (1 - \lambda^2) \bar{N}_+) - \frac{1}{2} \bar{N}_+ \log(1 + (1 - \lambda^2) \bar{N}_+) \\
+ \frac{1}{2 (1 - \lambda^2)^2} \log(1 + (1 - \lambda^2) \bar{N}_+) - \frac{1}{2} \log(\lambda^2 + (1 - \lambda^2) \bar{N}_+) + \log(1 - \lambda^2) \\
+ \frac{1}{2} \bar{N}_+ \log(\lambda^2 + (1 - \lambda^2) \bar{N}_+) - \frac{1}{2} \frac{\lambda^2}{(1 - \lambda^2)^2} \bar{N}_+ \log(\lambda^2 + (1 - \lambda^2) \bar{N}_+) .
\]  

(24)

It is straightforward to check that the free energy \( A_{\text{curve}} \) is indeed monotonously decreasing as a function of the curve parameter \( \bar{N}_+ \). So it can be interpreted as a thermodynamic arrow of time, in the sense of [8].

2.3 Dyson gas as a scaling limit of the two-component plasma

Recall that to get half S-brane, we are shifting the origin of the time coordinate as follows

\[
\lambda_0 \int dt \cosh(X^0(t) + C) = \frac{1}{2} \lambda_0 \int dt (e^{X^0+C} + e^{-X^0-C}) \rightarrow \lambda \int dt e^{X^0} ,
\]  

(25)
where the limit
\[
\lambda_0 \to 0 \ , \ C \to \infty \ , \ \lambda_0 e^C \equiv 2\lambda = \text{fixed} \quad (26)
\]
was applied. This limit is also realized in Dyson gas. Recall the partition function \((17)\). First we implement the shift \(x^0 \to x^0 + C\):
\[
z_{\pm} = \hat{\lambda} e^{\pm(x^0 + C)} . \quad (27)
\]
and use a different notation for the renormalized coupling, \(\hat{\lambda} = \sin(\pi \lambda_0)\). Now we apply the limit \((26)\) for the two-component plasma,
\[
\hat{\lambda} \to \pi \lambda_0 \to 0 \quad (28)
\]
\[
z_{-} = \hat{\lambda} e^{-x^0 - C} \to 0 \quad (29)
\]
\[
z_{+} = \hat{\lambda} e^{x^0 + C} \to \pi \lambda_0 e^{C} e^{x^0} \equiv 2\pi \lambda e^{x^0} = z , \quad (30)
\]
so the population of negative charges goes to zero while the positive charges remain. In more detail,
\[
\bar{N}_{+} \to \frac{z}{1 - z} ; \ \delta N_{+} \to \frac{1}{\sqrt{z}} \quad (31)
\]
while
\[
\bar{N}_{-} \to 0 ; \ \delta N_{-} \to 0 . \quad (32)
\]
Also the Helmholtz free energy \((24)\) reduces to the previous result obtained in the case of the half S-brane
\[
A_{\text{curve}}(\bar{N}_{+}) \to -\frac{1}{2} [ (\bar{N}_{+} + 1) \log(\bar{N}_{+} + 1) - \bar{N}_{+} \log \bar{N}_{+} ] . \quad (33)
\]
In particular, the grand partition function reduces to
\[
Z_{2G} \to \frac{1}{1 - z} = Z_G , \quad (34)
\]
the partition function of the one-component plasma (Dyson gas).

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A Partition Function at $\beta = 2$

The purpose of this appendix is to fill in the gaps between (16) and (17). Let us begin from (16). It is straightforward to manipulate

$$
\sum_{N_+ = 0}^{\infty} \sum_{N_- = 0}^{N_+} \frac{(z_+)^{N_+} (z_-)^{N_-}}{N_+! N_-!} Z_{N_+, N_-} = \sum_{N_+ = 1}^{\infty} \sum_{N_- = 1}^{N_+} \frac{(z_+)^{N_+} (z_-)^{N_-}}{N_+! N_-!} Z_{N_+, N_-} + \frac{1}{1 - z_+} + \frac{1}{1 - z_-} - 1 ,
$$

(35)

where

$$
Z_{N_+, N_-} = \frac{1}{(2\pi)^{N_+ + N_-}} \int_{C} \cdots \int_{C} \frac{dz_1 \cdots dz_{N_+}}{z_1 \cdots z_{N_+}} \frac{dw_1 \cdots dw_{N_-}}{w_1 \cdots w_{N_-}} 
\cdot \Pi_{1 \leq i < j \leq N_+} (z_i - z_j) \left( \prod_{1 \leq m < n \leq N_-} (w_m - w_n) \right) \left( \frac{1}{z_i} - \frac{1}{w_m} \right)
\cdot \Pi_{1 \leq i < j \leq N_+} \left( \frac{1}{z_i - z_j} \right) \Pi_{1 \leq m < n \leq N_-} \left( \frac{1}{w_m - w_n} \right)
\cdot \Pi_{1 \leq i < j \leq N_+} \left[ -(z_i - z_j)^2 \right] \Pi_{1 \leq m < n \leq N_-} \left[ -(w_m - w_n)^2 \right]
\cdot \frac{1}{\Pi_{N_+} \Pi_{N_-} \prod_{m=1}^{N_-} \left[ -(z_i - w_m)^2 \right]} ,
$$

(37)

where $C$ is the unit circle (integrated counterclockwise).

Due to the double poles, one needs to properly define a regularization prescription. We use the principal value prescription, defined as

$$
\frac{1}{(z_i - w_m)^2} \to \frac{1}{2 (z_i - C w_m)^2} + \frac{1}{2 (C z_i - w_m)^2} \equiv P \frac{1}{(z_i - w_m)^2} , \quad C = 1 + \epsilon .
$$

(39)

Let us now integrate over one variable, and without any loss of generality we can choose $z_1$, assuming $N_+ \leq N_-$.

Now

$$
Z_{N_+, N_-} \sim \oint_{C} dz_1 \frac{1}{z_1^{N_+ - N_-}} \prod_{j=2}^{N_+} \left[ -(z_1 - z_j)^2 \right] \cdot \prod_{m=1}^{N_-} \left[ -P \frac{1}{(z_1 - w_m)^2} \right]
\equiv \oint_{C} dz_1 g(z_1) \equiv J .
$$

(40)

(41)

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$^6$If $N_+ > N_-$ we will choose $w_1$. 

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At infinity
\[ g(z_1) \sim \frac{1}{z_1^{N_+ - N_-} z_1^{2(N_+ - 1)}} = \frac{1}{z_1^{2 - N_+ N_-}} , \] (42)
hence there is no residue at \( z_1 = \infty \). At zero there is no residue as well, since
\[ g(z_1) \sim z_1^{N_+ - N_-} . \] (43)

We thus conclude that all the residues of \( g \) are lying on the integration contour. The principal value prescription gives one half of each residue contribution,
\[ Z_{N_+ > 0, N_- > 0} \sim J = \pi i \sum \text{Res} \ g , \] (44)
where
\[ \sum \text{Res} \ g = 0 \] (45)
as the sum of the residues of any rational function is zero.

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