GENERALIZED PARAFERMIONS OF ORTHOGONAL TYPE

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ABSTRACT. There is an embedding of affine vertex algebras $V^k(\mathfrak{gl}_n) \hookrightarrow V^k(\mathfrak{sl}_{n+1})$, and the coset $\mathcal{C}^k(n) = \text{Com}(V^k(\mathfrak{gl}_n), V^k(\mathfrak{sl}_{n+1}))$ is a natural generalization of the parafermion algebra of $\mathfrak{sl}_2$. It was called the algebra of generalized parafermions by the third author and was shown to arise as a one-parameter quotient of the universal two-parameter $W_\infty$-algebra of type $W(2,3,\ldots)$. In this paper, we consider an analogous structure of orthogonal type, namely $\mathcal{D}^k(n) = \text{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}$. We realize this algebra as a one-parameter quotient of the two-parameter even spin $W_\infty$-algebra of type $W(2,4,\ldots)$, and we classify all coincidences between its simple quotient $\mathcal{D}^k(n)$ and the algebras $\mathcal{W}_k(\mathfrak{so}_{2m+1})$ and $\mathcal{W}_k(\mathfrak{so}_{2m})^{\mathbb{Z}_2}$. As a corollary, we show that for the admissible levels $k = -(2n-2) + \frac{1}{2}(2n + 2m - 1)$ for $\mathfrak{so}_{2n}$ the simple affine algebra $L_k(\mathfrak{so}_{2n})$ embeds in $L_k(\mathfrak{so}_{2n+1})$, and the coset is strongly rational. As a consequence, the category of ordinary modules of $L_k(\mathfrak{so}_{2n+1})$ at such a level is a braided fusion category.

1. INTRODUCTION

For $n \geq 1$, the natural embedding of Lie algebras $\mathfrak{gl}_n \hookrightarrow \mathfrak{sl}_{n+1}$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & -\text{tr}(a) \end{pmatrix},$$

induces a vertex algebra homomorphism

$$V^k(\mathfrak{gl}_n) \hookrightarrow V^k(\mathfrak{sl}_{n+1}).$$

The coset vertex algebra

$$\mathcal{C}^k(n) = \text{Com}(V^k(\mathfrak{gl}_n), V^k(\mathfrak{sl}_{n+1}))$$

was called the algebra of generalized parafermions in [LIII]. The reason for this terminology is that for $n = 1$, $\mathcal{C}^k(1)$ is isomorphic to the parafermion algebra $N^k(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, \mathfrak{sl}_2)$, where $\mathcal{H}$ denotes the Heisenberg algebra corresponding to the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}_2$.

By Theorem 8.1 of [LIII], $\mathcal{C}^k(n)$ is of type $\mathcal{W}(2,3,\ldots, n^2 + 3n + 1)$, i.e., it has a minimal strong generating set consisting of one field in each weight $2, 3, \ldots, n^2 + 3n + 1$. This generalizes the case $n = 1$, which appears in [DLY]. When $k$ is a positive integer, (1.1) descends to a map of simple affine vertex algebras $L_k(\mathfrak{gl}_n) \hookrightarrow L_k(\mathfrak{sl}_{n+1})$, and the coset $\text{Com}(L_k(\mathfrak{gl}_n), L_k(\mathfrak{sl}_{n+1}))$ coincides with the simple quotient $\mathcal{C}^k(n)$ of $\mathcal{C}^k(n)$. By Theorem 13.1 of [ACL], we have an isomorphism

(1.3) $$\mathcal{C}^k(n) \cong \mathcal{W}_\ell(\mathfrak{sl}_k), \quad \ell = -k + \frac{k + n}{k + n + 1}.$$\

In particular, $\mathcal{C}^k(n)$ is strongly rational, that is, $C_2$-cofinite and rational. This generalizes the case $n = 1$, which was proved earlier in [ALY].

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A useful perspective on $C^k(n)$ is that these algebras all arise in a uniform way as quotients of the universal two-parameter $W_{\infty}$-algebra $W(c, \lambda)$ of type $W(2, 3, \ldots)$; see Theorem 8.2 of [LIII]. This realization gives a nice conceptual explanation for the isomorphisms appearing in (1.3). Each one-parameter quotient of $W(c, \lambda)$ corresponds to an ideal in $\mathbb{C}[c, \lambda]$, or equivalently, a curve in the parameter space $\mathbb{C}^2$ called the truncation curve. The truncation curves for $W^\ell(sl_m)$ and $C^k(n)$ are given by Equations 7.8 and 8.4 of [LIII], and the above isomorphisms correspond to intersection points on these curves.

The algebras $C^k(n)$ appear naturally as building blocks for affine vertex algebras of type $A$. It is convenient to replace $C^k(n)$ with $\tilde{C}^k(n) = \mathcal{H} \otimes \tilde{C}^k(n)$, where $\mathcal{H}$ is a rank one Heisenberg vertex algebra. Then we have

$$\text{Com}(V^k(gl_{n-1}), V^k(gl_n)) \cong \tilde{C}^k(n-1),$$

so $V^k(gl_n)$ can be regarded as an extension of $V^k(gl_{n-1}) \otimes \tilde{C}^k(n-1)$. Iterating this procedure, we see that $V^k(gl_n)$ is an extension of

$$(1.4) \quad \mathcal{H} \otimes \tilde{C}^k(1) \otimes \tilde{C}^k(2) \otimes \cdots \otimes \tilde{C}^k(n-1).$$

Note that if $k$ is a positive integer, the simple quotient $L_k(gl_n)$ is then an extension of

$$\mathcal{H} \otimes \tilde{C}^k(1) \otimes \tilde{C}^k(2) \otimes \cdots \otimes \tilde{C}^k(n-1) \cong W_{\ell_1}(gl_k) \otimes W_{\ell_2}(gl_k) \otimes \cdots \otimes W_{\ell_n}(gl_k),$$

where $\ell_i = -k + \frac{k+n-1}{k+n+1-i}$. In [ACL], this was regarded as a noncommutative analogue of the Gelfand-Isetin subalgebra of $U(gl_n)$. Similarly, we may regard the subalgebra (1.4) as the universal version of this structure.

The algebras $C^k(n)$ also appear as building blocks for various $W$-superalgebras. For example, an important conjecture of Ito [I] asserts that the principal $W$-algebra $W^\ell(sl_{n+1}|n)$ has a coset realization as

$$(1.5) \quad \text{Com}(V^{k+1}(gl_n), V^k(sl_{n+1}|n) \otimes \mathcal{F}(2n)), $$

where $\mathcal{F}(2n)$ denotes the rank $2n$ free fermion algebra, and $(\ell + 1)(k + n + 1) = 1$. Ito’s conjecture was stated in this form in [ACL], and these algebras have the same strong generating type by Lemma 7.12 of [CLII]. In the case $n = 1$, the conjecture clearly holds because both sides are isomorphic to the $N = 2$ superconformal algebra. The first nontrivial case $n = 2$ was proven in [GL]. It was also shown in [GL] that the coset (1.5) is naturally an extension of $W^r(gl_n) \otimes C^k(n)$ for $r = -n + \frac{n+k}{n+k+1}$. An important ingredient in the proof of Ito’s conjecture will be to show that $W^\ell(sl_{n+1}|n)$ is indeed an extension of $W^r(gl_n) \otimes C^k(n)$. Note that $C^k(n)$ is itself a subalgebra of a $W$-superalgebra of $sl_{n+1}|n$ corresponding to a small hook-type nilpotent element [CLIII].

**Generalized parafermion algebras of orthogonal type.** There are two different analogues of $C^k(n)$ in the orthogonal setting. We have natural embeddings $so_{2n} \hookrightarrow so_{2n+1} \hookrightarrow so_{2n+2}$, which induce homomorphisms of affine vertex algebras

$$(1.6) \quad V^k(so_{2n}) \hookrightarrow V^k(so_{2n+1}) \hookrightarrow V^k(so_{2n+2}).$$

The cosets $\text{Com}(V^k(so_{2n}), V^k(so_{2n+1}))$ and $\text{Com}(V^k(so_{2n+1}), V^k(so_{2n+2}))$ both have actions of $\mathbb{Z}_2$, and we define

$$(1.7) \quad D^k(n) = \text{Com}(V^k(so_{2n}), V^k(so_{2n+1}))^{\mathbb{Z}_2}, \quad E^k(n) = \text{Com}(V^k(so_{2n+1}), V^k(so_{2n+2}))^{\mathbb{Z}_2}. $$

Both these algebras arise as one-parameter quotients of the universal even spin $W_{\infty}$-algebra $W_{\infty}^{\infty}(c, \lambda)$ constructed recently by Kanade and the third author in [KL]. Such
quotients of $W_{\text{ev}}(c, \lambda)$ are in bijection with a family of ideals $I$ in the polynomial ring $\mathbb{C}[c, \lambda]$, or equivalently, the truncation curves $V(I) \subseteq \mathbb{C}^2$. The main result in this paper is the explicit description of the truncation curve for $D^k(n)$ for all $n$; see Theorem 3.3. The proof is based on the coset realization of principal $W$-algebras of type $D$ and a certain level-rank duality appearing in [ACL], which implies that

$$D_{2m}(n) \cong W_\ell(\mathfrak{so}_{2m})^\mathbb{Z}_2, \quad \ell = -(2m - 2) + \frac{2m + 2n - 2}{2m + 2n - 1}. \quad (1.8)$$

Here $D_{2m}(n)$ denotes the simple quotient of $D^{2m}(n)$. This is analogous to the isomorphisms (1.3) in type $A$. Since a similar coset realization of type $B$ principal $W$-algebras is not available, we are currently unable to obtain an explicit description of $E^k(n)$, and in this paper we only study $D^k(n)$.

As in type $A$, there is a similar description of affine vertex algebras of orthogonal type as extensions of Gelfand-Tsetlin type subalgebras. Clearly $V^k(\mathfrak{so}_{2n+2})$ is an extension of

$$\mathcal{H} \otimes D^k(1) \otimes E^k(1) \otimes D^k(2) \otimes E^k(2) \otimes \cdots \otimes D^k(n - 1) \otimes E^k(n - 1) \otimes D^k(n).$$

and similarly, $V^k(\mathfrak{so}_{2n+1})$ is an extension of

$$\mathcal{H} \otimes D^k(1) \otimes E^k(1) \otimes D^k(2) \otimes E^k(2) \otimes \cdots \otimes D^k(n - 1) \otimes E^k(n - 1) \otimes D^k(n).$$

Additionally, $D^k(n)$ is a building block for various $W$-(super)algebras. For example, consider the principal $W$-superalgebra $W^\ell(\mathfrak{osp}_{2n}|2n)$ where $(\ell + 1)(k + 2n - 1) = 1$. Note that $1$ and $2n - 1$ are the dual Coxeter numbers of $\mathfrak{osp}_{2n}|2n$ and $\mathfrak{so}_{2n+1}$, respectively. The free fermion algebra $\mathcal{F}(2n)$ carries an action of $L_1(\mathfrak{so}_{2n})$, and it is expected that

$$W^\ell(\mathfrak{osp}_{2n}|2n) \cong \text{Com}(V^{k+1}(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n)). \quad (1.9)$$

This algebra appears in physics in the duality of $N = 1$ superconformal field theories and higher spin supergravities [CHR, CV], and this conjecture appeared in this context. Note that central charges coincide. It is apparent that the coset appearing in (1.9) is an extension of $W^r(\mathfrak{so}_{2n}) \otimes D^k(n)$ where $r = -(2n - 2) + \frac{k+2n-2}{k+2n-1}$. As in the case of Ito’s conjecture, an important step in the proof of (1.9) will be to show that $W^\ell(\mathfrak{osp}_{2n}|2n)$ is also an extension of this structure.

Applications. The first application of our main result is to classify all isomorphisms between the simple quotient $D_k(n)$ and the simple algebras $W_\ell(\mathfrak{so}_{2m+1})$ and $W_\ell(\mathfrak{so}_{2m})^\mathbb{Z}_2$. Using results of [KL], this can be achieved by finding the intersection points between the truncation curve for $D^k(n)$, and the truncation curves for $W^\ell(\mathfrak{so}_{2m+1})$ and $W^\ell(\mathfrak{so}_{2m})^\mathbb{Z}_2$, respectively. In the type $A$ case, we find only one family of points where $C_k(n)$ is isomorphic to a strongly rational $W$-algebra of type $A$; these appear in (1.3). In the orthogonal setting, the situation is more interesting. In addition to the isomorphisms (1.3) when $k$ is a positive integer, we also find that for $k = -(2n - 2) + \frac{1}{2}(2n + 2m - 1)$, we have an embedding of simple affine vertex algebras $L_k(\mathfrak{so}_{2n}) \rightarrow L_k(\mathfrak{so}_{2n+1})$, and an isomorphism

$$D_k(n) = \text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^\mathbb{Z}_2 \cong W_\ell(\mathfrak{so}_{2m+1}), \quad \ell = -(2m - 1) + \frac{2m + 2n - 1}{2m + 2n + 1}.$$
This coset is also closely related to level-rank duality. Recall that $2n(2m+1)$ free fermions carry an action of $L_{2n}(\mathfrak{so}_{2m+1}) \otimes L_{2m+1}(\mathfrak{so}_{2n})$. The levels shifted by the respective dual Coxeter numbers are $2n + 2m - 1$ in both cases. Therefore $L_k(\mathfrak{so}_{2n+1})$ is an extension of $L_k(\mathfrak{so}_{2n}) \otimes \mathcal{W}_l(\mathfrak{so}_{2m+1})$, where $l = -(2m-1) + \frac{2m+2n-1}{2m+2n+1}$, i.e., both levels $k$ and $l$ shifted by the respective dual Coxeter numbers are of the form $(2m + 2n - 1)/v$ for $v = 2$ and $v = 2 + 2m + 2n - 1$. In particular, the shifted levels have the same numerator as the original level-rank duality and the two denominators only differ by a multiple of the numerator. Note that under certain vertex tensor category assumptions the tensor product of two vertex algebras can be extended to a larger vertex algebra with a certain multiplicity freeness condition if and only if the two vertex algebras have subcategories that are braid-reversed equivalent, see [CKMII, Main Thm. 3] for the precise statement. Applied to our setting, this means that there are vertex algebra extensions of $L_k(\mathfrak{so}_{2n})$ and $\mathcal{W}_l(\mathfrak{so}_{2m+1})$ that have subcategories of modules that are braid-reversed equivalent.

The theory of vertex algebra extensions, especially [CKMII, Thm. 5.12], then implies that the category of ordinary modules of $L_k(\mathfrak{so}_{2n+1})$ at level $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$ is fusion, i.e. a rigid braided semisimple tensor category. This proves special cases of Conjecture 1.1 of [CHY].

Finally, our rationality results for $\mathcal{D}_k(n)$ suggest the existence of a new series of principal $\mathcal{W}$-superalgebras of $\mathfrak{osp}_{2n|2n}$ which are strongly rational. By Corollary 14.2 of [ACL], the coset $\text{Com}(L_{k+1}(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n))$ is strongly rational when $k$ is a positive integer. In view of the conjectured isomorphism (12), this implies that for $k$ a positive integer and $\ell$ satisfying $(\ell + 1)(k + 2n - 1) = 1$, $\mathcal{W}_l(\mathfrak{osp}_{2n|2n})$ is strongly rational. Similarly, it follows from Corollary 1.1 of [CKMII] that for $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$ and $\ell$ satisfying $(\ell + 1)(k + 2n - 1) = 1$, the coset $\text{Com}(L_{k+1}(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n))$ is again strongly rational. This motivates the following

**Conjecture 1.1.** For $k = -(2n-2) + \frac{1}{2}(2n+2m-1)$ and $\ell$ satisfying $(\ell + 1)(k + 2n - 1) = 1$, $\mathcal{W}_l(\mathfrak{osp}_{2n|2n})$ is strongly rational.

The conjecture is true for the $N = 2$ super Virasoro algebra, i.e. the case $n = 1$. Otherwise strong rationality for principal $\mathcal{W}$-superalgebras of orthosymplectic type is completely open. There is, however, a $C_2$-cofiniteness results in the case of $\mathfrak{osp}_{2|2n}$ [CGN, Cor. 5.19].

## 2. Vertex Algebras

We shall assume that the reader is familiar with vertex algebras, and we use the same notation and terminology as the papers [LIII, KL]. We first recall the universal two-parameter vertex algebra $\mathcal{W}_\text{ev}(c, \lambda)$ of type $\mathcal{W}(2,4,\ldots)$, which was recently constructed in [KL]. It is defined over the polynomial ring $\mathbb{C}[c, \lambda]$ and is generated by a Virasoro field $L$ of central charge $c$, and a weight 4 primary field $W^4$, and is strongly generated by fields \{$L, W^{2i} | i \geq 2\}$ where $W^{2i} = W^4 W^{2i-2}$ for $i \geq 3$. The idea of the construction is as follows.

(1) All structure constants in the OPEs of $L(z)W^{2i}(w)$ and $W^{2j}(z)W^{2k}(w)$ for $2i \leq 12$ and $2j + 2k \leq 14$, are uniquely determined as elements of $\mathbb{C}[c, \lambda]$ by imposing the Jacobi identities among these fields.
(2) This data uniquely and recursively determines all OPEs $L(z)W^{2i}(w)$ and $W^{2j}(z)W^{2k}(w)$ over the ring $\mathbb{C}[c,\lambda]$ if a certain subset of Jacobi identities are imposed.

(3) By showing that the algebras $W^k(\mathfrak{sp}_{2m})$ all arise as one-parameter quotients of $W^{ev}(c,\lambda)$ after a suitable localization, we show that all Jacobi identities hold. Equivalently, $W^{ev}(c,\lambda)$ is freely generated by the fields $\{L, W^{2i} | i \geq 2\}$, and is the universal enveloping algebra of the corresponding nonlinear Lie conformal algebra $[DSK]$.

$W^{ev}(c,\lambda)$ is simple as a vertex algebra over $\mathbb{C}[c,\lambda]$, but there is a certain discrete family of prime ideals $I = (p(c,\lambda)) \subseteq \mathbb{C}[c,\lambda]$ for which the quotient

$$W^{ev,I}(c,\lambda) = W^{ev}(c,\lambda)/I \cdot W^{ev}(c,\lambda),$$

is not simple as a vertex algebra over the ring $\mathbb{C}[c,\lambda]/I$. We denote by $W^{ev,I}(c,\lambda)$ the simple quotient of $W^{ev,I}(c,\lambda)$ by its maximal proper graded ideal $I$. After a suitable localization, all one-parameter vertex algebras of type $W(2,4,\ldots,2N)$ for some $N$ satisfying some mild hypotheses, can be obtained as quotients of $W^{ev}(c,\lambda)$ in this way. This includes the principal $W$-algebras $W^k(\mathfrak{so}_{2m+1})$ and the orbifolds $W^k(\mathfrak{so}_{2m})^{Z_2}$. The generators $p(c,\lambda)$ for such ideals arise as irreducible factors of Shapovalov determinants, and are in bijection with such one-parameter vertex algebras.

We also consider $W^{ev,I}(c,\lambda)$ for maximal ideals

$$I = (c - c_0, \lambda - \lambda_0), \quad c_0, \lambda_0 \in \mathbb{C}.$$ 

Then $W^{ev,I}(c,\lambda)$ and its quotients are vertex algebras over $\mathbb{C}$. Given maximal ideals $I_0 = (c - c_0, \lambda - \lambda_0)$ and $I_1 = (c - c_1, \lambda - \lambda_1)$, let $W_0$ and $W_1$ be the simple quotients of $W^{ev,I_0}(c,\lambda)$ and $W^{ev,I_1}(c,\lambda)$. Theorem 8.1 of [KL] gives a simple criterion for $W_0$ and $W_1$ to be isomorphic. Aside from a few degenerate cases, we must have $c_0 = c_1$ and $\lambda_0 = \lambda_1$. This implies that aside from the degenerate cases, all other coincidences among the simple quotients of one-parameter vertex algebras $W^{ev,I}(c,\lambda)$ and $W^{ev,J}(c,\lambda)$, correspond to intersection points of their truncation curves $V(I)$ and $V(J)$.

We shall need the following result which is analogous to Theorem 6.2 of [LIII].

**Theorem 2.1.** Let $W$ be a vertex algebra of type $W(2,4,\ldots,2N)$ which is defined over some localization $R$ of $\mathbb{C}[c,\lambda]/I$, for some prime ideal $I$. Suppose that $W$ is generated by the Virasoro field $L$ and a weight $4$ primary field $W^4$. If in addition, the graded character of $W$ agrees with that of $W^{ev}(c,\lambda)$ up to weight $13$, then $W$ is a quotient of $W^{I}(c,\lambda)$ after localization.

**Proof.** First, note that Theorem 3.10 of [KL] holds without the simplicity assumption; see Remark 5.1 of [LIII] for a similar statement in the case of the algebra $W(c,\lambda)$ of type $W(2,3,\ldots)$. By Theorem 3.10 of [KL], it suffices to prove that the OPEs $L(z)W^{2i}(w)$ and $W^{2j}(z)W^{2k}(w)$ for $2i \leq 12$ and $2j + 2k \leq 14$ in $W$ are the same as the corresponding OPEs in $W^{ev}(c,\lambda)$ if the structure constants are replaced with their images in $R$. In this notation, $W^{2i} = W^4(1)W^{2i-2}$ for $i \geq 3$. But this is automatic because the graded character assumption implies that there are no null vectors of weight $w \leq 13$ in the (possibly degenerate) nonlinear conformal algebra corresponding to $\{L, W^{2i} | 2 \leq i \leq N\}$. \qed
For \( n \geq 1 \), the natural embedding \( \mathfrak{so}_{2n} \hookrightarrow \mathfrak{so}_{2n+1} \) induces a vertex algebra homomorphism

\[
V^k(\mathfrak{so}_{2n}) \to V^k(\mathfrak{so}_{2n+1}).
\]

The action of \( \mathfrak{so}_{2n} \) on \( V^k(\mathfrak{so}_{2n+1}) \) given by the zero modes of the generating fields integrates to an action of the orthogonal group \( \text{O}_{2n} \). Therefore the coset

\[
\text{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1})) = V^k(\mathfrak{so}_{2n+1})^{\mathfrak{so}_{2n}[t]}
\]

has a nontrivial action of \( \mathbb{Z}_2 \). We define

\[
\mathcal{D}^k(n) = \text{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}.
\]

It has Virasoro element \( L^{\mathfrak{so}_{2n+1}} - L^{\mathfrak{so}_{2n}} \) with central charge

\[
c = \frac{kn(2k + 2n - 3)}{(k + 2n - 2)(k + 2n - 1)}
\]

Note that in the case \( n = 1 \), \( \mathcal{D}^k(n) \cong N^k(\mathfrak{sl}_2)^{\mathbb{Z}_2} \) which is of type \( W(2, 4, 6, 8, 10) \) by Theorem 10.1 of [KL].

**Lemma 3.1.** For all \( n \geq 1 \), \( \mathcal{D}^k(n) \) is of type \( W(2, 4, 6, 8, 10, 2N) \) for some \( N \) satisfying \( N \geq 2n^2 + 3n \).

We conjecture, but do not prove, that \( N = 2n^2 + 3n \). Moreover, for generic values of \( k \), \( \mathcal{D}^k(n) \) is generated by the weight 4 primary field \( W^4 \).

**Proof.** By Theorem 6.10 of [CLI], we have

\[
\lim_{k \to \infty} \mathcal{D}^k(n) \cong \mathcal{H}(2n)^{\text{O}_{2n}},
\]

and a strong generating set for \( \mathcal{H}(2n)^{\text{O}_{2n}} \) corresponds to a strong generating set for \( \mathcal{D}^k(n) \) for generic values of \( k \). Here \( \mathcal{H}(2n) \) denotes the rank \( 2n \) Heisenberg vertex algebra. It was shown in [LI], Theorem 6.5, that \( \mathcal{H}(2n)^{\text{O}_{2n}} \) has the above strong generating type. By Lemma 4.2 of [LI], the weights 2 and 4 fields generate \( \mathcal{H}(2n)^{\text{O}_{2n}} \). In fact, it is easy to check that only the weight 4 field is needed, and that it can be replaced with a primary field which also generates the algebra. Finally, the statement that \( \mathcal{D}^k(n) \) inherits these properties of \( \mathcal{H}(2n)^{\text{O}_{2n}} \) for generic values of \( k \) is also clear; the argument is similar to the proof of Corollary 8.6 of [CLI]. \( \square \)

**Corollary 3.2.** For all \( n \geq 1 \), there exists an ideal \( K_n \subseteq \mathbb{C}[c, \lambda] \) and a localization \( R_n \) of \( \mathbb{C}[c, \lambda]/K_n \) such that \( \mathcal{D}^k(n) \) is the simple quotient of \( \mathcal{W}^\text{ev,K}_n(c, \lambda) \).

**Proof.** This holds for \( n = 1 \) by Theorem 10.1 of [KL]. For \( n > 1 \), the simplicity of \( \mathcal{D}^k(n) \) as a vertex algebra over a localization of \( \mathbb{C}[k] \) follows from the simplicity of \( \mathcal{H}(2n)^{\text{O}_{2n}} \), which follows from [DLM]. In view of Theorems 2.1 and 3.1 it then suffices to show that the graded characters of \( \mathcal{D}^k(n) \) and \( \mathcal{W}^\text{ev,C} \) agree up to weight 13. This follows from Weyl’s second fundamental theorem of invariant theory for \( \text{O}_{2n} \) [W], since there are no relations among the generators of weight less than \( 4n^2 + 6n + 2 \). \( \square \)

**Theorem 3.3.** For all \( n \geq 2 \), \( \mathcal{D}^k(n) \) is isomorphic to a localization of the quotient \( \mathcal{W}^\text{ev,K}_n(c, \lambda) \), where the ideal \( K_n \subseteq \mathbb{C}[c, \lambda] \) is described explicitly via the parametrization \( k \mapsto (c_n(k), \lambda_n(k)) \).
given by
\[
c_n(k) = \frac{kn(2k + 2n - 3)}{(k + 2n - 2)(k + 2n - 1)}, \quad \lambda_n(k) = \frac{(k + 2n - 2)(k + 2n - 1)p_n(k)}{7(k - 2)(k + n - 1)(2n - 1)q_n(k)r_n(k)},
\]
\[
p_n(k) = -112 + 188k - 62k^2 - 26k^3 + 12k^4 + 744n - 1336kn + 857k^2n - 252k^3n
\]
\[
+ 36k^4n - 1720n^2 + 2534kn^2 - 1198k^2n^2 + 188k^3n^2 + 1632n^3 - 1544kn^3
\]
\[
+ 304k^2n^3 - 544n^4 + 152kn^4,
\]
\[
q_n(k) = 20 - 19k + 6k^2 - 42n + 28kn + 28n^2,
\]
\[
r_n(k) = 44 - 66k + 22k^2 - 132n + 73kn + 10k^2n + 88n^2 + 10k^2n.
\]

Proof. Let \(n\) be fixed. In view of Corollary 3.2 and the fact that all structure constants in \(D^k(n)\) are rational functions of \(k\), there is some rational function \(\lambda_n(k)\) of \(k\) such that \(D^k(n)\) is obtained from \(W^{\text{ev}}(c, \lambda)\) by setting \(c = c_n(k)\) and \(\lambda = \lambda_n(k)\), and then taking the simple quotient. It is not obvious yet that \(\lambda_n(k)\) is a rational function of \(n\) as well.

For \(k\) a positive integer, it is well known [KW] that the map \(V^k(\mathfrak{so}_{2n}) \to V^k(\mathfrak{so}_{2n+1})\) descends to a homomorphism of simple algebras \(L_k(\mathfrak{so}_{2n}) \to L_k(\mathfrak{so}_{2n+1})\). Letting \(D_k(n)\) denote the simple quotient of \(D^k(n)\), it is apparent from Lemma 2.1 of [ACKL] and Theorem 8.1 of [CLII] that \(\text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))\) is simple and coincides with the simple quotient of \(\text{Com}(V^k(\mathfrak{so}_{2n}), V^k(\mathfrak{so}_{2n+1}))\). Moreover, taking \(\mathbb{Z}_2\)-invariants preserves simplicity, hence
\[
D_k(n) \cong \text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^{\mathbb{Z}_2}.
\]

Next, by Corollary 1.3 of [ACL], for all \(n \geq 1\) and \(m \geq 2\), we have an isomorphism
\[
(\mathbb{L}_{2m}(\mathfrak{so}_{2n+1}) \oplus \mathbb{L}_{2m}(2m\omega_1))^{\mathfrak{so}_{2n}[\ell]} \cong \mathcal{W}_\ell(\mathfrak{so}_{2m}), \quad \ell = -(2m - 2) + \frac{2n + 2m - 2}{2n + 2m - 1}.
\]
In this notation, \(\omega_1\) denotes the first fundamental weight of \(\mathfrak{so}_{2n+1}\) and \(\mathbb{L}_{2m}(2m\omega_1)\) denotes the simple quotient of the corresponding Weyl module.

Note that \((\mathbb{L}_{2m}(\mathfrak{so}_{2n+1})^{\mathfrak{so}_{2n}[\ell]})^{\mathbb{Z}_2} = D_{2m}(n)\) is manifestly a subalgebra of the left hand side of (3.4). Also, the lowest-weight component of \(\mathbb{L}_{2m}(2m\omega_1)\) has conformal weight \(m\). If \(m > 4\), the left-hand side then has a unique primary weight \(4\) field which lies in \(D_{2m}(n)\).
Similarly, since \(W_\ell(\mathfrak{so}_{2m})\) has strong generators in weights \(2, 4, \ldots, 2m\) and \(m\), for \(m > 4\) the right hand side has a unique primary weight \(4\) field, which lies in the \(\mathbb{Z}_2\)-orbifold \(W_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}\).

Since \(D^k(n)\) is generated by the weight \(4\) field as a one-parameter vertex algebra, the weight \(4\) field must generate \(D_{2m}(n)\) for all \(m\) sufficiently large. By Corollary 6.1 of [KL], \(W_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}\) is generated by the weight \(4\) field as a one-parameter vertex algebra; equivalently, this holds for generic values of \(\ell\). By the same argument as Proposition A.4 of [ALY], the vertex Poisson structure on the associated graded algebra \(\text{gr} W_\ell(\mathfrak{so}_{2m})\) with respect to Li’s canonical filtration, is independent of \(\ell\) for all noncritical values of \(\ell\). In particular this holds for the subalgebra \((\text{gr} W_\ell(\mathfrak{so}_{2m}))^{\mathbb{Z}_2} = \text{gr}(W_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2})\). It follows from the same argument as Proposition A.3 of [ALY] that \(W_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}\) is generated by the weights \(2\) and \(4\) fields for all noncritical values of \(\ell\), and the same therefore holds for the simple quotient \(W_\ell(\mathfrak{so}_{2m})^{\mathbb{Z}_2}\). Finally, for \(\ell = -(2m - 2) + \frac{2n + 2m - 2}{2n + 2m - 1}\), it is straightforward to verify that the Virasoro field can be generated from the weight \(4\) field, so the weight \(4\) field generates the whole algebra.
Therefore if \( m \) is sufficiently large, we obtain
\[
(3.5) \quad D_{2m}(n) \cong \mathcal{W}_t(\mathfrak{so}_{2m})^{\mathbb{Z}_2}, \quad \ell = -(2m - 2) + \frac{2m + 2n - 2}{2m + 2n - 1}.
\]

In fact, we will see later (Theorem 4.1) that this holds for all \( m \geq 2 \).

Finally, the truncation curve that realizes \( \mathcal{W}_t(\mathfrak{so}_{2m})^{\mathbb{Z}_2} \) as a quotient of \( \mathcal{W}^{\text{ev}}(c, \lambda) \) is given by Theorem 6.3 of [KL], and in parametric form by Equation (B.1) of [KL]. In view of (3.5), we must have \( \lambda_n(2m) = \lambda_m(\ell) \) for \( \ell = -(2m - 2) + \frac{2n + 2m - 2}{2n + 2m - 1} \) for \( m \) sufficiently large, where \( \lambda_m(\ell) \) is given by Equation (B.1) of [KL]. It follows that for infinitely many values of \( k, \lambda_n(k) \) is given by the above formula (3.3). Since \( \lambda_n(k) \) is a rational function of \( k \), this equality holds for all \( k \) where it is defined. This completes the proof. \( \Box \)

4. Coincidences

In this section, we shall use Theorem 3.3 to classify all coincidences between the simple quotient \( D_k(n) \) and the \( \mathbb{Z}_2 \)-orbifold \( \mathcal{W}_t(\mathfrak{so}_{2m})^{\mathbb{Z}_2} \), as well as \( \mathcal{W}_t(\mathfrak{so}_{2m+1}) \). We also classify all coincidences between \( D_k(n) \) and \( D_\ell(m) \) for \( m \neq n \).

**Theorem 4.1.** For \( n \geq 1 \) and \( m \geq 2 \), aside from the critical levels \( k = -2n + 2 \) and \( k = -2n + 1 \), and the degenerate cases \( c = \frac{1}{2}, -24 \), all isomorphisms \( D_k(n) \cong \mathcal{W}_t(\mathfrak{so}_{2m})^{\mathbb{Z}_2} \) appear on the following list:

1. \( k = 2m, \quad \ell = -(2m - 2) + \frac{2n + 2m - 2}{2n + 2m - 1} \)
2. \( k = -(2n - 2) - \frac{2n - 1}{2(m - 1)}, \quad \ell = -(2m - 2) + \frac{2m - 2n - 1}{2(m - 1)} \)
3. \( k = -(2n - 2) + \frac{n - m}{m}, \quad \ell = -(2m - 2) + \frac{m - n}{m} \)

**Proof.** Recall first that \( \mathcal{W}_t(\mathfrak{so}_{2m})^{\mathbb{Z}_2} \) is realized as the simple quotient of \( \mathcal{W}^{\text{ev}, J_m}(c, \lambda) \), where the ideal \( J_m \subseteq \mathbb{C}[c, \lambda] \) is given in parametrized form by Equation (B.1) of [KL]. First, we exclude the values of \( k \) and \( \ell \) which are poles of the functions \( \lambda_n(k) \) given by (3.3), and \( \lambda_m(\ell) \) given by Equation (B.1) of [KL], since at these values, \( D^k(n) \) and \( \mathcal{W}_t(\mathfrak{so}_{2m})^{\mathbb{Z}_2} \) are not quotients of \( \mathcal{W}^{\text{ev}}(c, \lambda) \). For all other noncritical values of \( k \) and \( \ell \), \( D^k(n) \) and \( \mathcal{W}_t(\mathfrak{so}_{2m})^{\mathbb{Z}_2} \) are obtained as quotients of \( \mathcal{W}^{\text{ev}, J_m}(c, \lambda) \) and \( \mathcal{W}^{\text{ev}, J_m}(c, \lambda) \), respectively. By Corollary 8.2 of [KL], aside from the degenerate cases given by Theorem 8.1 of [KL], all other coincidences \( D_k(n) \cong \mathcal{W}_t(\mathfrak{so}_{2m})^{\mathbb{Z}_2} \) correspond to intersection points on the truncation curves \( V(K_n) \) and \( V(J_m) \). A calculation shows that \( V(K_n) \cap V(J_m) \) consists of exactly five points \( (c, \lambda) \), namely,
\[
(4.1) \quad \left( -24, -\frac{1}{245} \right), \quad \left( \frac{1}{2}, -\frac{2}{49} \right), \quad \left( \frac{mn(4m + 2n - 3)}{(m + n - 1)(2m + 2n - 1)}, \lambda_1 \right), \quad \left( \frac{2mn(3 - 4m - 2n + 4mn)}{2m - 2n - 1}, \lambda_2 \right), \quad \left( \frac{-(2mn + m - 2n)(2mn - m - n)}{m - n}, \lambda_3 \right).
\]
Here
\[
\lambda_1 = \frac{(m + n - 1)(2m + 2n - 1)g}{7(m - 1)(2m + n - 1)(2n - 1)gh},
\]
\[
f = -28 + 94m - 62m^2 - 52m^3 + 48m^4 + 186n - 668mn + 857m^2n - 504m^3n
+ 144m^4n - 430n^2 + 1267mn^2 - 1198m^2n^2 + 376m^3n^2 + 408n^3 - 772mn^3
+ 304m^2n^3 - 136n^4 + 76mn^4,
\]
\[
g = 10 - 19m + 12m^2 - 21n + 28mn + 14n^2,
\]
\[
h = 22 - 66m + 44m^2 - 66n + 73mn + 20m^2n + 44n^2 + 10mn^2.
\]

(4.2)
\[
\lambda_2 = \frac{(1 - 2m + 2n)f}{7(1 - 2m + 2mn)(-1 - 2n + 4mn)gh},
\]
\[
f = 14 - 33m - 2m^2 + 24m^3 + 74n - 404mn + 873m^2n - 696m^3n + 144m^4n
+ 80n^2 - 178mn^2 - 260m^2n^2 + 452m^3n^2 - 112m^4n^2 - 24n^3 + 264mn^3
- 348m^2n^3 + 256m^3n^3 - 64m^4n^3 + 72mn^4 - 128m^2n^4 - 48m^3n^4 + 32m^4n^4,
\]
\[
g = -10 + 19m - 12m^2 - 2n + 22mn - 8m^2n - 12n^2 - 8m^2n^2 + 8m^2n^2,
\]
\[
h = 11 - 22m + 22n + 15mn - 20m^2n - 10mn^2 + 20m^2n^2.
\]

(4.3)
\[
\lambda_3 = \frac{(n - m)f}{7(m - 1)(2n - 1)(m - n + 2mn)gh},
\]
\[
f = -34m^3 + 19m^4 + 68m^2n - 38m^3n - 22mn^2 - 185m^2n^2 + 302m^3n^2 - 80m^4n^2
- 12n^3 + 204mn^3 - 302m^2n^3 + 80m^3n^3 - 36n^4 + 100mn^4 - 40m^2n^4 - 40m^3n^4
+ 16m^4n^4,
\]
\[
g = -7m^2 + 7mn - 6n^2 - 4mn^2 + 4m^2n^2,
\]
\[
h = -22m - 5m^2 + 22n + 5mn + 10n^2 - 30mn^2 + 20m^2n^2.
\]

By Theorem 8.1 of [KL], the first two intersection points occur at degenerate values of \(c\).
By replacing the parameter \(c\) with the levels \(k\) and \(\ell\), we see that the remaining intersection points yield the nontrivial isomorphisms in Theorem 4.1. Moreover, by Corollary 8.2 of [KL], these are the only such isomorphisms except possibly at the values of \(k, \ell\) excluded above.

Finally, suppose that \(k\) is a pole of the function \(\lambda_n(k)\) given by (3.3). It is not difficult to check that the corresponding values of \(\ell\) for which \(c_n(k) = c_m(\ell)\), are not poles of \(\lambda_m(\ell)\). As above, \(c_n(k)\) and \(\lambda_n(k)\) are given by (3.3), and \(c_m(\ell)\) and \(\lambda_m(\ell)\) are given by Equation (B.1) of [KL]. It follows that there are no additional coincidences at the excluded points. \(\square\)

Next, we classify the coincidences between \(\mathcal{D}_k(n)\) and \(\mathcal{W}_l(\mathfrak{so}_{2m+1})\).

**Theorem 4.2.** For \(n \geq 1\) and \(m \geq 2\), aside from the critical levels \(k = -2n + 2\) and \(k = -2n + 1\), and the degenerate cases \(c = \frac{1}{2}, -24\), all isomorphisms \(\mathcal{D}_k(n) \cong \mathcal{W}_l(\mathfrak{so}_{2m+1})\) appear on the following list:

1. \(k = -(2n - 2) + \frac{1}{2}(2n + 2m - 1), \quad \ell = -(2m - 1) + \frac{2m + 2n - 1}{2m + 2n + 1}\)
would be simple as well. Additionally, Theorem 8.1 of \[ CLII\] would imply that \( V \) points between the truncation curves is realized explicitly by Equation (A.3) of \[ KL\]. The above isomorphisms all arise from the intersection positions. In general, it is a difficult and important problem to determine when \( k \) is simple, and hence its orbifold Com

\[
(2) \quad k = -(2n - 2) + \frac{2n - 2m - 1}{2m + 2}, \quad \ell = -(2m - 1) + \frac{2m - 2n + 1}{2m + 2},
\]

\[
(3) \quad k = -(2n - 2) - \frac{n}{m}, \quad \ell = -(2m - 1) + \frac{m - n}{m},
\]

\[
(4) \quad k = -(2n - 2) - \frac{m}{2(n - 1)}, \quad \ell = -(2m - 1) + \frac{2m - 1}{2m - 2n + 1}.
\]

\[
(5) \quad k = -(2n - 2) + \frac{2(n - m - 1)}{2m + 1}, \quad \ell = -(2m - 1) + \frac{2m + 1}{2(m - n + 1)}.
\]

**Proof.** The argument is the same as the proof of Theorem [4.1]. First, \( W_k(\mathfrak{so}_{2m+1}) \) is realized as the simple quotient of \( W^\text{ev,1}_m(c, \lambda) \) where the ideal \( I_m \subseteq \mathbb{C}[c, \lambda] \) is parametrized explicitly by Equation (A.3) of [KL]. The above isomorphisms all arise from the intersection points between the truncation curves \( V(K_n) \) for \( D^k(n) \) and \( V(I_m) \) for \( W_k(\mathfrak{so}_{2n+1}) \). A calculation shows that there are exactly 7 intersection points: the degenerate points \((\frac{1}{2}, -\frac{2}{39})\) and \((-24, -\frac{1}{245})\), and the five nontrivial ones appearing above. One then has to rule out additional coincidences at the points where \( D_k(n) \) does not arise as a quotient of \( W^\text{ev}(c, \lambda) \), namely, the poles of \( \lambda_n(k) \). The details are straightforward and are left to the reader. \( \square \)

Finally, we classify all isomorphisms \( D_k(m) \cong D_\ell(n) \) for \( n \neq m \).

**Theorem 4.3.** For \( m, n \geq 1 \) and \( n \neq m \), aside from the degenerate cases \( c = \frac{1}{2}, -24 \) and poles of \( c_n(k), \lambda_n(k) \) and \( c_m(k), \lambda_m(k) \) the complete list of isomorphisms \( D_k(m) \cong D_\ell(n) \) is the following:

\[
(1) \quad k = -(2m - 2) + \frac{2(m - 1)}{1 + 2n}, \quad \ell = -(2n - 2) - \frac{2m + 2n - 1}{2(m - 1)},
\]

\[
(2) \quad k = -(2m - 2) - \frac{2m + 2n - 1}{2(n - 1)}, \quad \ell = -(2n - 2) + \frac{2(n - 1)}{1 + 2m}.
\]

The proof is similar to the proof of Theorem 4.1 and is omitted.

5. SOME RATIONAL COSETS

By composing the map \( V^k(\mathfrak{so}_{2n}) \to V^k(\mathfrak{so}_{2n+1}) \) with the quotient map \( V^k(\mathfrak{so}_{2n+1}) \to L_k(\mathfrak{so}_{2n+1}) \), we obtain an embedding

\[
\widehat{V}^k(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n+1}),
\]

where \( \widehat{V}^k(\mathfrak{so}_{2n}) \) denotes the quotient of \( V^k(\mathfrak{so}_{2n}) \) by the kernel \( J_k \) of the above composition. In general, it is a difficult and important problem to determine when \( J_k \) is the maximal proper graded ideal, or equivalently, when \( \widehat{V}^k(\mathfrak{so}_{2n}) = L_k(\mathfrak{so}_{2n}) \). In the case where \( k \) is an admissible level for \( \mathfrak{so}_{2n} \), Lemma 2.1 of [ACKL] would then imply that \( \text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1})) \) is simple, and hence its orbifold \( \text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^\mathbb{Z}_2 \) would be simple as well [DLM]. Additionally, Theorem 8.1 of [CLII] would imply that \( \text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^\mathbb{Z}_2 \) coincides with the simple quotient \( D_k(n) \) of \( D^k(n) \). This is particularly interesting in the cases where \( D_k(n) \) is strongly rational.

We conclude by proving this for first family in Theorem 4.2. These are new examples of cosets of non-rational vertex algebras by admissible level affine vertex algebras, which are strongly rational.
Lemma 5.1. For $n \geq 2$ and $m \geq 0$, we have an embedding of simple affine vertex algebras

$$L_k(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n+1}), \quad k = -(2n - 2) + \frac{1}{2}(2n + 2m - 1).$$

Proof. We proceed by induction on $m$. In the case $m = 0$, we have $k = -n + \frac{3}{2}$, and it is well known that there exists a conformal embedding $L_k(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n+1})$, see e.g. Section 3 of [AKMPP]. Next, we assume the result for $m - 1$, so that $k = -n + \frac{3}{2} + m - 1$. Recall that the rank $2n + 1$ free fermion algebra $\mathcal{F}(2n + 1)$ admits an action of $L_1(\mathfrak{so}_{2n+1})$, as well as an action of $L_1(\mathfrak{so}_{2n})$ via the embedding $L_1(\mathfrak{so}_{2n}) \hookrightarrow L_1(\mathfrak{so}_{2n+1})$. The image of $L_1(\mathfrak{so}_{2n})$ lies in the subalgebra $\mathcal{F}(2n) \subseteq \mathcal{F}(2n + 1)$.

Since $k$ is admissible for $\mathfrak{so}_{2n+1}$, it is known [KW] that we have a diagonal embedding of simple affine vertex algebras

$$L_{k+1}(\mathfrak{so}_{2n+1}) \hookrightarrow L_k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n + 1).$$

By induction, we have the map $L_k(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n+1})$. Then we have an embedding

$$L_{k+1}(\mathfrak{so}_{2n}) \hookrightarrow L_k(\mathfrak{so}_{2n}) \otimes \mathcal{F}(n) \hookrightarrow L_k(\mathfrak{so}_{2n+1}) \otimes \mathcal{F}(2n + 1),$$

where $\mathcal{F}(2n) \hookrightarrow \mathcal{F}(2n + 1)$ is the isomorphism onto the first $2n$ copies. Since the image of $L_{k+1}(\mathfrak{so}_{2n+1})$ lies in the image of (5.1), it follows that $L_{k+1}(\mathfrak{so}_{2n})$ embeds in $L_{k+1}(\mathfrak{so}_{2n+1})$. □

This has the following immediate consequence.

Corollary 5.2. For $n \geq 2$, $m \geq 0$, and $k = -(2n - 2) + \frac{1}{2}(2n + 2m - 1)$, we have an isomorphism

$$\text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^\mathbb{Z}_2 \cong \mathcal{W}_\ell(\mathfrak{so}_{2m+1}), \quad \ell = -(2m - 1) + \frac{2m + 2n - 1}{2m + 2n + 1}.$$ 

In particular, $\text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^\mathbb{Z}_2$ is strongly rational.

Proof. This follows from Theorem 4.2 together with the fact that $\text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^\mathbb{Z}_2$ is simple, and the map $D^k(n) \to \text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))^\mathbb{Z}_2$ is surjective. □

Recall that the category of ordinary modules of an affine vertex algebra at admissible level is semisimple [ArIII] and a vertex tensor category [CHY]. Conjecturally, this category is fusion [CHY] and this has been proven for simply-laced Lie algebras [C]. For type $\mathfrak{so}_{2n+1}$ and level $k = -(2n - 2) + \frac{1}{2}(2n + 2m - 1)$ this conjecture is also true. First, $\text{Com}(L_k(\mathfrak{so}_{2n}), L_k(\mathfrak{so}_{2n+1}))$ is a simple current extension, call it $\mathcal{V}_\ell(\mathfrak{so}_{2m+1})$, of $\mathcal{W}_\ell(\mathfrak{so}_{2m+1})$ and thus rational as well [Li]. It follows that $L_k(\mathfrak{so}_{2n+1})$ is a simple $\mathbb{Z}$-graded extension of $L_k(\mathfrak{so}_{2n}) \otimes \mathcal{V}_\ell(\mathfrak{so}_{2m+1})$ in a rigid vertex tensor category $\mathcal{C}$ of $L_k(\mathfrak{so}_{2n}) \otimes \mathcal{V}_\ell(\mathfrak{so}_{2m+1})$-modules, namely the Deligne product of the categories of ordinary $L_k(\mathfrak{so}_{2n})$-modules and $\mathcal{V}_\ell(\mathfrak{so}_{2m+1})$-modules. Every ordinary module for $L_k(\mathfrak{so}_{2n+1})$ must be an object in this category $\mathcal{C}$. This means that as a braided tensor category the category of ordinary modules of $L_k(\mathfrak{so}_{2n+1})$ is equivalent to the category of local modules for $L_k(\mathfrak{so}_{2n+1})$ viewed as an algebra object in $\mathcal{C}$ [HKL], [CKMII]. All assumptions of Theorem 5.12 of [CKMIII] are satisfied (with $U = \mathcal{V}_\ell(\mathfrak{so}_{2m+1})$ and $V = L_k(\mathfrak{so}_{2n})$) and so

Corollary 5.3. The category of ordinary modules of $L_k(\mathfrak{so}_{2n+1})$ at level $k = -(2n - 2) + \frac{1}{2}(2n + 2m - 1)$ is fusion.
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