INTEGRAL LAPLACIAN GRAPHS WITH A UNIQUE DOUBLE LAPLACIAN EIGENVALUE, II

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Abstract. The set $S_{(i,j)}^m = \{0,1,2,\ldots,m-1,m,m+1,\ldots,n-1,n\} \setminus \{i,j\}$, $0 < i < j \leq n$, is called Laplacian realizable if there exists a simple connected graph $G$ whose Laplacian spectrum is $S_{(i,j)}^m$. In this case, the graph $G$ is said to realize $S_{(i,j)}^m$. In this paper, we completely describe graphs realizing the sets $S_{(i,j)}^m$ with $m = 1, 2$ and determine the structure of these graphs.

1. Introduction

In the present paper, we continue our previous work [12] where we studied graphs whose Laplacian spectrum is

$$S_{(i,j)}^m = \{0,1,2,\ldots,m-1,m,m+1,\ldots,n-1,n\} \setminus \{i,j\}, \quad 0 < i < j \leq n,$$

with $m = n-1, n$. Here we cover the cases $m = 1, 2$ and give a complete description of the correspondent graphs but certain cases directly related to the so-called $S_{n,n}$-conjecture [7]. In this work we follow the notations and preliminaries of the work [12], however, some of them will be stated here for convenience of the reader.

Let $G = (V(G), E(G))$ be a simple graph (without loops or multiple edges) where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is its vertex set and $E(G) = \{e_1, e_2, \ldots, e_r\}$ its edge set. The entries of its Laplacian matrix are defined as follows

$$l_{ij} = \begin{cases} 
  d_i, & \text{if } i = j, \\
  -1, & \text{if } i \neq j \text{ and } v_i \sim v_j, \\
  0, & \text{otherwise},
\end{cases}$$

where $v_i \sim v_j$ means that the vertices $v_i$ and $v_j$ are adjacent.

The Laplacian matrix $L(G)$ is positive semidefinite and singular, see e.g. [27]. A graph $G$ whose Laplacian matrix has integer eigenvalues is called Laplacian integral. As we noticed in [12], there are many famous families of Laplacian integral graphs and we refer the reader to the works [1, 5, 7, 10, 13, 15, 16, 17, 18, 23, 24, 25] and references therein.

One of the most interesting families of Laplacian integral graphs considered by S. Fallat et al. [7] is defined as follows: The set

$$S_{i,n} = \{0,1,2,\ldots,n-1,n\} \setminus \{i\}, \quad i \leq n,$$

is called Laplacian realizable if there exists a simple connected graph $G$ whose Laplacian spectrum is $S_{i,n}$. We also say that $G$ realizes $S_{i,n}$. In [7] the authors established realizability of sets and completely described the graphs realizing $S_{i,n}$. In addition, it is also conjectured in [7] that the set $S_{n,n}$ is not Laplacian realizable for any $n \geq 2$. This problem is now known as the $S_{n,n}$-conjecture and states that $S_{n,n}$ is not Laplacian realizable for every $n \geq 2$. This conjecture was proved for $n \leq 11$, for prime $n$, and for $n \equiv 2, 3 \pmod{4}$ in [7]. Later, Goldberger and Neumann [9] showed that the conjecture is true for $n \geq 6, 649, 688, 933$. The authors of the present work established [11] that if a graph is the Cartesian product of two other graphs, then it does not realize $S_{n,n}$.

Key words and phrases. Laplacian Integral graph, Laplacian matrix, Laplacian spectrum, integer eigenvalues.
As a way of investigating the class of Laplacian integral graphs, the authors of the present work extended this concept further and studied a certain class of Laplacian integral graphs introduced in [12] and is defined as follows.

**Definition 1.1.** A graph $G$ is said to realize the set

$$S_{ij} = \{0, 1, 2, \ldots, m - 1, m, m + 1, \ldots, n - 1, n\} \setminus \{i, j\}$$

for some $i$ and $j$, $i < j \leq n$, if its Laplacian spectrum is the set $S_{ij}$. In this case, the set $S_{ij}$ is called Laplacian realizable. So the set $S_{ij}$ does not contain the numbers $i$ and $j$, while some number $m$ (and only this number) is doubled.

The present work is the second part of our research on the set $S_{ij}$. In the first part [12], we considered graphs realizing the sets $S_{ij}$ for $m = n - 1$ and $m = n$, and completely described them. Moreover, we conjectured that the graphs realizing sets $S_{ij}$ may not exist for large $n$. In particular, we believe that for $n \geq 9$ the sets $S_{ij}$ are not Laplacian realizable. In this paper, we continue our study and consider the cases $m = 1$ and $m = 2$.

First, we note that graphs realizing sets $S_{1i}$ and $S_{ij}$ exist for small $n$. The Laplacian spectra of all the graphs of order up to 5 are listed in [8, p. 286–289]. From that list it follows that for $n \leq 5$ the only Laplacian realizable $S_{1i}$ sets are $S_{2,3}$ and $S_{2,4}$. In Figure 1, the graph $G_1$ is the star graph $K_{1,3}$ on 4 vertices realizing $S_{2,3}$ (see Table 1 in Appendix). The graph $G_2$ realizing the set $S_{2,4}$ is of the form $(K_2 \cup 2K_1) \vee K_1$ (see Table 1). Similarly, for $n \leq 5$, the only Laplacian realizable $S_{ij}$ sets are $S_{1,3}$ and $S_{1,4}$. In Figure 2, the graphs $G_3$ and $G_4$ are the complete bipartite graphs $K_{2,2}$ (or the cycle $C_4$) and $K_{2,3}$ on 4 and 5 vertices respectively, realizing $S_{1,3}$ and $S_{1,4}$ respectively (see Table 2).

![Figure 1](image1.png)

**Figure 1.** Graphs $G_1$ and $G_2$ realizing $S_{2,3}$ and $S_{2,4}$, respectively.

![Figure 2](image2.png)

**Figure 2.** Graphs $G_3$ and $G_4$ realizing $S_{1,3}$ and $S_{1,4}$, respectively.

Considering the case $m = 1$, we show that the set $S_{ij}$ is Laplacian realizable only if $j = n - 1$, that is, only $S_{i,n-1}$ is Laplacian realizable for certain $i$, Theorem 2.3. Further, we list all such $i$ for fixed $n$ and $j = n - 1$, i.e., we find all the Laplacian realizable sets of kind $S_{i,n-1}$, Theorem 3.2. We also present an algorithm for constructing graphs realizing the sets $S_{i,n-1}$, Theorem 3.3. For the case $m = 2$, we
show that if \( i > 1 \), then \( j > n - 3 \), Theorem 1.1 and list all such \( i \) for given \( j \) considering \( j = n - 2 \) and \( n - 1 \) separately. Theorems 1.3 and 1.7, Theorems 4.1 and 4.8 describe the structure of graphs realizing the sets \( S_{(i,j)}^n \) for \( j = n - 2, n - 1 \). If \( i = 1 \) then for all admissible \( j \), the set \( S_{(1,j)}^n \) is Laplacian realizable only if \( G = K_1 \lor F \), where the graph \( F \) realizes \( S_{(j-1,n-1)}^{n-1} \), Theorem 4.9. However, for such case we believe that it may not exists for large \( n \), \( n \geq 6 \). Tables 1 and 2 in Appendix A illustrate the cases \( m = 1 \) and \( m = 2 \).

As a result of our investigation on the set \( S_{(i,j)}^m \), we conclude that the values of \( m \) are closely related to one another. For instance, if \( G \) realizes \( S_{(i,j)}^m \) \((m = n)\) then one can obtain graphs realizing the sets \( S_{(i,j)}^{m-1} \) \((m = n-1)\) by using certain graph operations such as union, join and complement. Similarly, the case \( m = n - 2 \) can be obtained from the case \( m = n - 1 \) and so on. On the other hand, if \( G \) realizes \( S_{(i,j)}^m \) for \( m = n \), then using certain graph operations, we obtain the graph realizes \( S_{(i,j)}^m \) for \( m = 1 \). Similarly, \( m = 2 \) can be obtained from the case \( m = n - 1 \). So, either of the cases can be obtained by using graph operations. However, it is not clear whether the operations on graphs cover all the sets realizing \( S_{(i,j)}^m \) for particular value of \( m \).

The paper is organized as follows. In Section 2 we introduce some basic definitions and review some note worthy results from the literature in this work. We also probe some auxiliary theorems. In Section 3 a complete characterization of all the graphs with double Laplacian eigenvalue \( m = 1 \) is given. The graphs with double Laplacian eigenvalue \( m = 2 \) are discussed in Section 4. Finally, in Appendix A we list all the Laplacian realizable sets \( S_{(i,j)}^n \) and \( S_{(i,j)}^2 \) for \( n = 4, 5, 6, 7, 8 \). The associated graphs realizing these sets are presented.

2. Preliminaries

An isolated vertex is a vertex of degree zero denoted by \( K_1 \), while a pendant vertex is a vertex of degree one. The complement of a simple undirected graph \( G \) denoted by \( \overline{G} \) is a simple graph on the same set of vertices as \( G \) in which two vertices are adjacent if and only if they are not adjacent in \( G \). Given two disjoint graphs \( G_1 \) and \( G_2 \), the union of these graphs, \( G_1 \lor G_2 \), is the graph formed from the unions of the edges and vertices of the graphs \( G_1 \) and \( G_2 \). The join of the graphs \( G_1 \) and \( G_2 \), \( G_1 \lor G_2 \), is the graph formed from \( G_1 \lor G_2 \) by adding all possible edges between vertices in \( G_1 \) and vertices in \( G_2 \), that is, \( G_1 \lor G_2 = (G_1 \lor G_2) \).

We denote by \( 0 = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \) the Laplacian eigenvalues of a graph \( G \). It is easy to see from the form of the Laplacian matrix that the Laplacian spectrum of the union of two graphs is the union of their Laplacian spectra. The largest eigenvalue of the Laplacian matrix, is denoted by \( \rho(G) \). The second smallest eigenvalue \( \mu_2 \) of \( L(G) \) is usually known as the algebraic connectivity of \( G \) denoted by \( a_G \). The vertex connectivity of a connected graph \( G \) is the minimum number of vertices whose removal disconnect \( G \).

The following facts provide information on the Laplacian largest eigenvalue and the Laplacian spectrum of the complement of graph.

**Theorem 2.1** ([22]). Let \( G \) be a simple graph on \( n \) vertices. Then \( \rho(G) \leq n \).

**Theorem 2.2** ([3][22]). Let \( G \) be a graph with \( n \) vertices with Laplacian eigenvalues

\[
0 = \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \leq \mu_{n-1} \leq \mu_n
\]

Then the Laplacian eigenvalues of the complement of \( G \) are the following

\[
0 \leq n - \mu_n \leq n - \mu_{n-1} \leq \cdots \leq n - \mu_3 \leq n - \mu_2.
\]

The Laplacian spectra of the disjoint union and join of graphs are stated in the following theorems.

**Theorem 2.3** ([3]). If \( G \) is the disjoint union of graphs \( G_1, G_2, \ldots, G_k \), then it’s Laplacian characteristic polynomial is

\[
\chi(G, \mu) = \prod_{k=1}^{n} \chi(G_k, \mu)
\]
Theorem 2.4 (Kelman’s). Let $G$ and $H$ be two graphs of order $n$ and $m$, respectively. Suppose that the Laplacian eigenvalues of $G$ and $H$ are of the form

$$0 = \mu_1 \leq \mu_2 \leq \mu_3 \ldots \leq \mu_{n-1} \leq \mu_n$$

and

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \leq \lambda_{m-1} \leq \lambda_m$$

respectively

then the Laplacian spectrum of $G \vee H$ is of the form

$$\{0, m + \mu_2, m + \mu_3, \ldots, m + \mu_{n-1}, m + \mu_n, n + \lambda_2, n + \lambda_3, \ldots, n + \lambda_{m-1}, n + \lambda_m, n + m\}.$$ (2.1)

We cite the above theorem in the form given in [20]. Note that the eigenvalues in (2.1) are not in increasing order, generally speaking.

The following theorem provides a necessary and sufficient condition for a graph to have $n$ as one of its eigenvalues.

Theorem 2.5 ([26]). Let $G$ be a connected graph of order $n$. Then $n$ is a Laplacian eigenvalue of $G$ if and only if $G$ is the join of two graphs.

If the order $n$ is a double Laplacian eigenvalue, then the following theorem holds.

Theorem 2.6 ([12]). Let $G$ be a connected graph of order $n$, and let $n$ be the Laplacian eigenvalue of $G$ of multiplicity 2. Then $G = F \vee H$ where $F$ is a join of two graphs, while $H$ is not a join. Moreover, the eigenvalue 1 is not in the Laplacian spectrum of $G$.

The next proposition provides a necessary and sufficient condition for a graph to have 1 as one of its Laplacian eigenvalues.

Proposition 2.7 ([11]). Let a graph $G$ be a join. The number 1 is a Laplacian eigenvalue of $G$ if and only if $G = F \vee K_1$ where $F$ is a disconnected graph of order at least 2.

To complement to the previous result, we establish the following.

Theorem 2.8. Let $G$ be a connected graph of order $n$ that is a join. Let the number 1 be the Laplacian eigenvalue of $G$ of multiplicity 2. Then $G = (H_1 \cup 2K_1) \vee K_1$ where $H_1$ is a connected graph of order $n - 3$. Moreover, the number $n - 1$ is not in the Laplacian spectrum of $G$.

Proof. Let $G$ be a join that has a Laplacian eigenvalue 1 of multiplicity 2. According to Theorem 2.5, $G = F \vee H$ where the graph $F$ is of order $p$ while the graph $H$ is of order $n - p$ for some $1 \leq p \leq n - 1$.

Let us denote the Laplacian spectra of the graphs $F$ and $H$, respectively, as follows

$$0 = \mu_1 \leq \mu_2 \leq \mu_3 \ldots \leq \mu_p - 1 \leq \mu_p$$

and

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \leq \lambda_{n-p} \leq \lambda_{n-p}.$$ (2.2)

Then by Theorem 2.3, the Laplacian spectrum of $G$ has the form

$$\{0, (n - p) + \mu_2, \ldots, (n - p) + \mu_p, p + \lambda_2, \ldots, p + \lambda_{n-p}, n\}.$$ (2.2)

We remind the reader that the eigenvalues are not in the increasing order here.

Since $G$ has exactly one multiple eigenvalue 1 by assumption, (2.2) provides only two possible situations.

1) If $n - p + \mu_2 + \mu_3 = 1$, then $n - p = p = 1$, so both graphs $F$ and $H$ are single isolated vertices. Thus, $G = K_1 \vee K_1$ and the Laplacian spectrum of $G$ is equal to $\{0, 2\}$, a contradiction.

2) Let $n - p + \mu_2 + \mu_3 = n - p + \mu_3 = 1$ or $p + \lambda_2 = p + \lambda_3 = 1$. Without loss of generality, we can suppose that $n - p + \mu_2 = n - p + \mu_3 = 1$, so $\mu_2 = \mu_3 = 0$ and $n - p = 1$. As well, the inequality $n - p + \mu_4 > 1$ gives us $\mu_4 > 0$. Consequently, the graph $F$ is a disconnected graph of the form $F = H_1 \cup 2K_1$, whereas the graph $H$ is an isolated vertex $K_1$. Thus, $G = (H_1 \cup 2K_1) \vee K_1$ where $H_1$ is a connected graph of order $n - 3$.

Now to prove that $n - 1$ is not in the Laplacian spectrum of $G$, it is enough to show that $n - p + \mu_p \neq n - 1$. Let us suppose that $n - p + \mu_p = n - 1$. Then $\mu_p = n - 2$, and so the Laplacian spectrum of the graph $F$
Suppose Theorem 2.9. the eigenvalue of order \( n \) contains the eigenvalue 1 and the double eigenvalue \( n \). This contradicts Theorem 2.6. Therefore, the eigenvalue \( n - 1 \) is not in the Laplacian spectrum of \( G \).

Now, we remind the reader some results on the sets \( S_{i,n} \) established in [7].

**Theorem 2.9.** Suppose \( n \geq 2 \).

(i) If \( n \equiv 0 \mod 4 \), then for each \( i = 1, 2, 3, \ldots, \frac{n-2}{2} \), \( S_{2i,n} \) is Laplacian realizable;

(ii) If \( n \equiv 1 \mod 4 \), then for each \( i = 1, 2, 3, \ldots, \frac{n-1}{2} \), \( S_{2i-1,n} \) is Laplacian realizable;

(iii) If \( n \equiv 2 \mod 4 \), then for each \( i = 1, 2, 3, \ldots, \frac{n-2}{2} \), \( S_{2i-1,n} \) is Laplacian realizable;

(iv) If \( n \equiv 3 \mod 4 \), then for \( i = 1, 2, \ldots, \frac{n-1}{2} \), \( S_{2i,n} \) is Laplacian realizable.

**Proposition 2.10.** Suppose that \( n \geq 6 \) and that \( G \) is a graph on \( n \) vertices. Then \( G \) realizes \( S_{i,n} \) if and only if \( G \) is formed in one of the following two ways:

(i) \( G = (K_1 \cup K_1) \cap (K_1 \cup G_1) \), where \( G_1 \) is a graph on \( n-3 \) vertices that realizes \( S_{n-4,n-3} \);

(ii) \( G = K_1 \cup H \), where \( H \) is a graph on \( n-1 \) vertices that realizes \( S_{n-1,n-1} \).

In the sequel, we use also the following results established in [12].

**Theorem 2.11** ([12]). If \( S_{\{i,j\}_n^{n-1}} \) is Laplacian realizable then the number \( i \) is either 1 or 2.

**Theorem 2.12** ([12]). Suppose that \( n \geq 3 \) and \( G \) is a graph of order \( n \) realizing \( S_{\{i,j\}_n^m} \) for \( i < j \). Then

(i) for \( n \equiv 0 \) or 3 \( \mod 4 \), the numbers \( (i+j) \) and \( m \) are of the same parities;

(ii) for \( n \equiv 1 \) or 2 \( \mod 4 \), the numbers \( (i+j) \) and \( m \) are of opposite parities.

**Proposition 2.13** ([12]). The graph \( G \) realizes \( S_{\{1,2\}_n^m} \) if and only if \( G \) is formed in one of the following two ways:

(i) \( G = P_3 \cap (K_1 \cup H) \), where \( H \) realizes \( S_{n-3,n-4} \) and \( P_3 \) is the path graph on 3 vertices;

(ii) \( G = K_2 \cap H \), where \( H \) realizes \( S_{n-2,n-2} \).

**Theorem 2.14** ([12]). Let \( G \) be a graph of order \( n \), \( n \geq 5 \).

(a) The graph \( G \) realizes \( S_{\{1,2\}_n^m} \) if and only if \( G \) is formed in one of the following two ways:

(i) \( G = P_3 \cap (K_1 \cup H) \), where \( H \) realizes \( S_{n-3,n-4} \) and \( P_3 \) is the path graph on 3 vertices;

(ii) \( G = K_2 \cap H \), where \( H \) realizes \( S_{n-2,n-2} \).

(b) If \( 3 \leq j \leq n-2 \), then \( G \) realizes \( S_{\{1,j\}_n^m} \) if and only if \( G = K_2 \cap (K_1 \cup H) \), where the graph \( H \) realizes \( S_{n-2,n-3} \).

(c) The graph \( G \) realizes \( S_{\{1,n-1\}_n^m} \) if and only if \( G \) is formed in one of the following two ways:

(i) \( G = K_2 \cap (K_2 \cup H) \), where \( H \) realizes \( S_{2,n-4} \);

(ii) \( G = K_2 \cap (K_1 \cup H) \), where \( H \) realizes \( S_{n-3,n-3} \).

**Theorem 2.15** ([12]). Let \( G \) be a simple connected graph of order \( n \), \( n \geq 6 \).

(i) For \( n \equiv 0 \) or 1 \( \mod 4 \), the set \( S_{\{1,j\}_n^{n-1}} \) is Laplacian realizable if and only if \( j = 2 \).

(ii) For \( n \equiv 2 \) or 3 \( \mod 4 \), the set \( S_{\{1,j\}_n^{n-1}} \) is Laplacian realizable if and only if \( j = 3 \).

**Theorem 2.16** ([12]). Let \( G \) be a graph of order \( n \), \( n \geq 6 \).

(a) The graph \( G \) realizes \( S_{\{1,2\}_n^{n-1}} \) if and only if \( n \equiv 0 \) or 1 \( \mod 4 \), and \( G = (K_1 \cup K_2) \cap (K_1 \cup H) \), where \( H \) realizes \( S_{n-6,n-4} \);
(b) The graph $G$ realizes $S_{\{1,3\}}^{n-1}$ if and only if $n \equiv 2$ or $3 \mod 4$, and $G$ is formed in one of the following two ways:

(i) $G = (K_1 \cup K_1) \lor (K_1 \cup H)$, where the graph $H$ realizes $S_{\{1,n-4\}}^{n-3}$;

(ii) $G = K_1 \lor F$, where the graph $F$ realizes $S_{\{n-1\}}^{n-2}$.

3. Graphs realizing the sets $S_{\{i,j\}}^{1}$

In this section, we describe the graphs realizing the sets $S_{\{i,j\}}^{1}$, and present an algorithm for constructing graphs realizing $S_{\{i,j\}}^{1}$. As we mentioned in Introduction, in [3, p. 286–289] the authors listed the Laplacian spectra of all the graphs of order up to 5. Thus, it follows that for $n \leq 5$ the only Laplacian realizable $S_{\{i,j\}}^{1}$ sets are $S_{\{2,3\}}^{1}$ and $S_{\{2,4\}}^{1}$, see Table 1 in Appendix A. So in what follows, we consider $n \geq 6$.

First we present the following auxiliary fact.

Lemma 3.1. If $S_{i,n}$ is Laplacian realizable, then so is $S_{\{i+1,n+2\}}^{1}$. Indeed, if $G$ is a connected graph of order $n$ realizing $S_{i,n}$, then the graph $(G \lor 2K_1) \lor K_1$ realizes $S_{\{i+1,n+2\}}^{1}$.

In the next theorem, we describe all the Laplacian realizable sets $S_{\{i,j\}}^{1}$.

Theorem 3.2. Suppose $n \geq 6$. The only Laplacian realizable sets $S_{\{i,j\}}^{1}$, $i < j < n$, are the following ones.

(i) If $n \equiv 0 \mod 4$, then for each $k = 1, 2, \ldots, \frac{n-2}{2}$, $S_{\{2k,n-1\}}^{1}$ is Laplacian realizable;

(ii) If $n \equiv 1 \mod 4$, then for each $k = 1, 2, \ldots, \frac{n-3}{2}$, $S_{\{2k,n-1\}}^{1}$ is Laplacian realizable;

(iii) If $n \equiv 2 \mod 4$, then for each $k = 1, 2, \ldots, \frac{n-4}{2}$, $S_{\{2k+1,n-1\}}^{1}$ is Laplacian realizable;

(iv) If $n \equiv 3 \mod 4$, then for each $k = 1, 2, \ldots, \frac{n-5}{2}$, $S_{\{2k+1,n-1\}}^{1}$ is Laplacian realizable.

Proof. According to Theorem 2.8, if $S_{\{i,j\}}^{1}$ is Laplacian realizable, then the eigenvalue $n - 1$ is not in the Laplacian spectrum of $G$, so $j = n - 1$. Consequently, the sets $S_{\{i,j\}}^{1}$ are not Laplacian realizable if $j < n - 1$.

(i) If $n \equiv 0 \mod 4$, then $n - 3 \equiv 1 \mod 4$, so for each $k = 1, 2, \ldots, \frac{n-4}{2}$, $S_{\{2k-1,n-3\}}^{1}$ is Laplacian realizable by Theorem 2.9 (ii). According to Lemma 3.1, $S_{\{2k,n-1\}}^{1}$ is Laplacian realizable for any $k = 1, 2, \ldots, \frac{n-4}{2}$. Moreover, since $n$ is even and $m = 1$, $S_{\{2k+1,n-1\}}^{1}$ is not Laplacian realizable by Theorem 2.12 (i).

Let us now deal with the set $S_{\{n-2,n-1\}}^{1}$. The set $S_{2,n-4}$ is Laplacian realizable by Theorem 2.9 (i). Suppose that a graph $F_1$ realizes $S_{2,n-4}$. If $F_3$ is the path graph on 3 vertices, then $\sigma_L(F_3) = \{0, 0, 2\}$, and $\sigma_L(F_3 \lor F_1) = \{0, 0, 0, 1, 2, \ldots, n-5, n-4\}$. According to Theorem 2.4, the graph $K_1 \lor (F_3 \lor F_1)$ realizes $S_{\{n-2,n-1\}}^{1}$.

(ii) If $n \equiv 1 \mod 4$, then $n - 3 \equiv 2 \mod 4$. Therefore, by Theorem 2.9 (iii), the set $S_{2k-1,n-3}$ is Laplacian realizable for $k = 1, 2, \ldots, \frac{n-3}{2}$. By Lemma 3.1, $S_{\{2k,n-1\}}^{1}$ is Laplacian realizable for any $k = 1, 2, \ldots, \frac{n-3}{2}$. As $n$ is odd and $m = 1$, therefore, by Theorem 2.12 (ii), $S_{\{2k+1,n-1\}}^{1}$ is not Laplacian realizable.

The case (iii) can be proved analogously.

(iv) If $n \equiv 3 \mod 4$, then $n - 3 \equiv 0 \mod 4$. Thus, for $k = 1, 2, \ldots, \frac{n-5}{2}$, the sets $S_{2k,n-3}$ is Laplacian realizable by Theorem 2.9 (i). Now Lemma 3.1 implies that $S_{\{2k+1,n-1\}}^{1}$ is Laplacian realizable for
where the graph $H(a)$ realizes $S$.

(b) Let $S$ be a graph realizing $S_n$. Then for the path graph $P_3$, the graph $K_1 \lor (P_3 \cup F_1)$ realizes $S_{n-2,n-1}^{\lor}$ by Theorem 2.4. Therefore, $S_{n-2,n-1}^{\lor}$ is Laplacian realizable.

\[ \square \]

In the following theorem, we discuss the structure of graphs realizing $S_{i,n}^{\lor}$ for various possible values of $i$. We remind the reader that the sets $S_{i,j}^{\lor}$ for $j < n - 1$ are not Laplacian realizable as we found out above.

**Theorem 3.3.** Let $G$ be a graph of order $n$, $n \geq 6$.

(a) If $2 \leq i \leq n - 3$, then $G$ realizes $S_{i,n}^{\lor}$ if and only if $G = K_1 \lor (2K_1 \cup (K_1 \lor H))$, where the graph $H$ realizes the set $S_{n-i-2,n-4}$.

(b) The graph $G$ realizes $S_{n-2,n-1}^{\lor}$ if and only if $G$ is formed in one of the following two ways:

(i) $G = K_1 \lor (P_3 \cup (K_1 \lor H))$, where the graph $H$ of order $n - 5$ realizes $S_{n-6,n-5}$;

(ii) $G = K_1 \lor (2K_1 \lor H)$, where the graph $H$ realizes $S_{n-3,n-3}$.

**Proof.**

(a) Suppose that $G$ realizes $S_{i,n}^{\lor}$ for $2 \leq i \leq n - 3$. By Theorem 2.2, one has $\sigma_L(G) = \{0\} \cup S_{i,n}^{\lor}$. Therefore, the complement of the graph $G$ has the form $\overline{G} = K_1 \lor \overline{F}$, where $\overline{F}$ is connected and $\sigma_L(\overline{F}) = S_{i,n}^{\lor}$. According to Theorem 2.14 (b), $\overline{F}$ realizes $S_{i,n}^{\lor}$ if and only if $\overline{F} = K_2 \lor (K_1 \lor H)$, where the graph $H$ realizes $S_{n-i-2,n-4}$. Consequently, $F = 2K_1 \lor (K_1 \lor H)$, so $G = K_1 \lor (2K_1 \lor (K_1 \lor H))$.

Conversely, if $G = K_1 \lor (2K_1 \lor (K_1 \lor H))$, where the graph $H$ realizes the set $S_{n-i-2,n-4}$, then from Theorems 2.3 and 2.4, it follows that $G$ realizes $S_{i,n}^{\lor}$.

(b) Let $G$ be a graph realizing $S_{n-2,n-1}^{\lor}$. Then from Theorem 2.2, we obtain $\sigma_L(G) = \{0\} \cup S_{1,2}^{\lor}$. So the complement of the graph $G$ can be represented as follows $\overline{G} = K_1 \lor \overline{F}$, where $\overline{F}$ is connected and $\sigma_L(\overline{F}) = S_{1,2}^{\lor}$. According to Theorem 2.14 (a), the graph $\overline{F}$ must be one of the following form:

(i) $\overline{F} = P_3 \lor (K_1 \lor H)$, where $H$ realizes $S_{n-6,n-5}$ and $P_3$ is the path graph on 3 vertices;

(ii) $\overline{F} = K_2 \lor H$, where $H$ realizes $S_{n-3,n-3}$.

Thus, from (i) one gets $F = P_3 \lor (K_1 \lor H)$. According to Theorem 2.2, one has $\sigma_L(H) = \{0\} \cup S_{1,n-6}$. So $H$ can be represented as follows $H = K_1 \lor H_1$, where $H_1$ realizes $S_{1,n-6}$ (for the construction of $H_1$, see Proposition 2.10). Thus, $G = K_1 \lor (P_3 \lor (K_1 \lor H))$, where the graph $H$ of order $n - 5$ realizes $S_{n-6,n-5}$.

Also from (ii), we have $\overline{F} = K_2 \lor H$. Therefore, $F = 2K_1 \lor H$, and the graph $H$ is a connected graph realizing $S_{n-3,n-3}$. Thus, $G = K_1 \lor (2K_1 \lor H)$, where $H$ realizes $S_{n-3,n-3}$.

Conversely, if $G = K_1 \lor (P_3 \lor (K_1 \lor H))$, where the graph $H$ of order $n - 5$ realizes $S_{n-6,n-5}$, then from Theorems 2.3 and 2.4, it follows that $G$ realizes $S_{n-2,n-1}^{\lor}$. Similarly, if $G = K_1 \lor (2K_1 \lor H)$, where $H$ realizes $S_{n-3,n-3}$, then again by Theorems 2.3 and 2.4, the graph $G$ realizes $S_{n-2,n-1}^{\lor}$. \[ \square \]

**Remark 3.4.** In the above theorem, the structure of the graph $G$ shows that there exist at least two pendant vertices in $G$. As well, Theorems 3.2 and 3.3 completely resolve the existence of graphs realizing the spectrum $S_{i,j}^{\lor}$, where $2 \leq i < j < n$. Furthermore, it is easily deduced from Theorem 3.3 that if the sets $S_{n,n}$ were not realizable for any $n$, then there is a unique graph, which realizes $S_{i,j}^{\lor}$, for $2 \leq i < j < n$.

For the case $j = n$, we conjectured that the set $S_{i,n}^{\lor}$, $n \geq 9$, does not exist [12]. However, unless otherwise is proved, we must include this case in our consideration.

**Proposition 3.5.** Let $G$ realize $S_{i,n}^{\lor}$. Then the following holds:
(a) \( n \geq 9 \);
(b) \( n \) is not a prime number;
(c) \( 2 \leq \min_{1 \leq i \leq n} d_i \leq \max_{1 \leq i \leq n} d_i \leq n - 3 \), where \( d_i \) is the degree of vertex \( i \).

**Proof.** Let \( G \) realize \( S_{i,n} \). Then Properties (a) and (b) follow from [12, Proposition 5.2].

If we suppose, on the contrary, that \( G \) has a pendant vertex, then the vertex connectivity of \( G \) equals 1. At the same time, its algebraic connectivity is also 1. So, according to [19, Theorem 2.1] if the algebraic and vertex connectivity are equal, then \( G \) must be a join, a contradiction.

Property (c) in the above proposition coincides with the one of the \( S_{n,n} \)-conjecture proposed in [7, Observation 3.5].

The *Cartesian product* of the graphs \( G_1 \) and \( G_2 \) is the graph \( G_1 \times G_2 \) whose vertex set is the Cartesian product \( V(G_1) \times V(G_2) \), and for \( v_1,v_2 \in V(G_1) \) and \( u_1,u_2 \in V(G_2) \), the vertices \( (v_1,u_1) \) and \( (v_2,u_2) \) are adjacent in \( G_1 \times G_2 \) if and only if either

- \( v_1 = v_2 \) and \( \{u_1,u_2\} \in E(G_1) \);
- \( \{v_1,v_2\} \in E(G_2) \) and \( u_1 = u_2 \).

The authors of the present work established [12] that if \( G = G_1 \times G_2 \) (i.e., \( G \) is a Cartesian product of two graphs) then it does not realize the set \( S_{i,n} \) for \( n \geq 9 \) and any \( m \neq i,n \). According to Proposition 3.5 the set \( S_{i,n} \) is not Laplacian realizable if \( n \) is a prime number, and so the Cartesian product does not realizes the set \( S_{i,n} \) for a prime number \( n \). All these facts motivated us to state the following Conjecture.

**Conjecture 3.6.** For \( n \geq 3 \), the set \( S_{i,n} \) is not Laplacian realizable.

### 4. Graphs realizing the sets \( S_{i,j} \)

This section establishes the existence of graphs realizing the sets \( S_{i,j} \). As we already mentioned in Section 4 for \( n \leq 5 \), the only Laplacian realizable sets of type \( S_{i,j} \) are \( S_{1,3} \) and \( S_{1,4} \), see Table 2 in Appendix A. So throughout this section we consider \( n \geq 6 \). In Sections 4.1, 4.2 the case \( 1 < i < j < n \), while Section 4.3 is devoted to the sets \( S_{i,j} \) (\( i = 1 \)) and \( S_{i,n} \) (\( j = n \)). We believe that these sets are not Laplacian realizable for large \( n \). To proceed with the main results of this section, we first establish a relation between the numbers \( i \) and \( j \) to realize \( S_{i,j} \) when \( i > 1 \) as follows:

**Theorem 4.1.** Let \( G \) realize \( S_{i,j} \). If \( i > 1 \), then \( j > n - 3 \).

**Proof.** Suppose on the contrary that \( S_{i,j} \) is Laplacian realizable for some \( j \leq n - 3 \), and let \( G \) be a graph realizing this set. Clearly, the numbers \( n - 2, n - 1 \) and \( n \) belong to \( \sigma_L(G) \). According to Theorem 2.2

\[
\sigma_L(G) = \{0,0,1,2,\ldots,n-j-1,n-j+1,n-i-1,n-i+1,\ldots,n-2,n-2,n-1\}.
\]

By Theorem 2.1 one of the components of \( G \), say, \( G_2 \) has \( n - 1 \) vertices, so \( G_1 = K_1 \). Now from 4.1 one has

\[
\sigma_L(G_2) = \{0,1,2,\ldots,n-j-1,n-j+1,n-i-1,n-i+1,\ldots,n-2,n-2,n-1\}.
\]

This contradicts Theorem 2.11. Therefore, for \( i > 1 \), the set \( S_{i,j} \) is Laplacian realizable only if \( j > n - 3 \).  

The above theorem claims that for \( i > 1 \) either \( j = n - 2, n - 1 \) or \( n \). In the sequel, we consider these three cases separately.

#### 4.1. Graphs realizing the sets \( S_{i,n-2} \)

We start with the following fact established in [12].

**Proposition 4.2 (12).** Let \( n \geq 3 \), if \( S_{1,n} \) is Laplacian realizable, then so is \( S_{i+1,j+1}^{n+1} \).

The following theorem lists all the Laplacian realizable sets \( S_{i,n-2} \). We remind that \( i > 1 \).

**Theorem 4.3.** Suppose \( n \geq 6 \). The only Laplacian realizable sets \( S_{i,n-2} \) are the following ones.

\[
S_{i,n-2} = \{0,1,2,\ldots,n-j-1,n-j+1,n-i-1,n-i+1,\ldots,n-2,n-2,n-1\}.
\]
(i) If \( n \equiv 0 \mod 4 \), then for each \( k = 1, 2, \ldots, \frac{n - 6}{2} \), \( S_{(2k+2,n-2)} \) is Laplacian realizable;

(ii) If \( n \equiv 1 \mod 4 \), then for each \( k = 1, 2, \ldots, \frac{n - 5}{2} \), \( S_{(2k+2,n-2)} \) is Laplacian realizable;

(iii) If \( n \equiv 2 \mod 4 \), then for each \( k = 1, 2, \ldots, \frac{n - 4}{2} \), \( S_{(2k+1,n-2)} \) is Laplacian realizable;

(iv) If \( n \equiv 3 \mod 4 \), then for each \( k = 1, 2, \ldots, \frac{n - 5}{2} \), \( S_{(2k+1,n-2)} \) is Laplacian realizable.

\textbf{Proof.}

(i) If \( n \equiv 0 \mod 4 \), then \( n - 2 \equiv 2 \mod 4 \), then for \( k = 1, 2, \ldots, \frac{n - 6}{2} \), so the sets \( S_{(2k+1,n-3)} \) are Laplacian realizable by Theorem 3.2 (ii). From Proposition 4.2 it follows that \( S_{(2k+2,n-2)} \) is Laplacian realizable for each \( k = 1, 2, \ldots, \frac{n - 6}{2} \). At the same time, the sets \( S_{(2k+1,n-2)} \) are not Laplacian realizable for any \( k \) by Theorem 2.12 (i), since the double eigenvalue \( m = 2 \) is an even number in this case.

(ii) For \( n \equiv 1 \mod 4 \), we have \( n - 2 \equiv 3 \mod 4 \). Thus, for \( k = 1, 2, \ldots, \frac{n - 5}{2} \), the sets \( S_{(2k+1,n-3)} \) are Laplacian realizable by Theorem 3.2 (iv). Now Proposition 4.2 implies that \( S_{(2k+2,n-2)} \) are Laplacian realizable for \( k = 1, 2, \ldots, \frac{n - 5}{2} \). As \( m = 2 \) is even, from Theorem 2.12 (ii) it follows that the sets \( S_{(2k+1,n-2)} \) are not Laplacian realizable for any \( k \).

The cases (iii) and (iv) can be proved analogously with the use of Theorem 3.2 and Proposition 4.2.

\textbf{Theorem 4.4.} Let \( n \geq 6 \), and let \( G \) be a connected graph of order \( n \). Then \( G \) realizes \( S_{(i,n-2)} \) if and only if \( G = K_1 \cup (K_1 \cup H) \), where \( H \) is a graph on \( n - 2 \) vertices realizing \( S_{(i-1,n-3)} \).

\textbf{Proof.} Let \( G \) realize \( S_{(i,n-2)} \). Then \( G = G_1 \cup G_2 \) by Theorem 2.5, so \( G = G_1 \cup \overline{G_2} \). From Theorem 2.2 it follows that \( \sigma_L(G) = \{0\} \cup S_{(2,n-1)} \). Thus, by Theorem 2.1 we obtain that \( G_1 = K_1 \), and \( G_2 \) is of order \( n - 1 \), so that \( \sigma_L(G_2) = S_{(2,n-1)} \). Using Theorem 2.2, we get \( \sigma_L(G_2) = \{0\} \cup S_{(i-1,n-3)} \). Again Theorem 2.1 gives us that \( G_2 = K_1 \cup H \), where \( H \) is a graph on \( n - 2 \) vertices realizing \( S_{(i-1,n-3)} \). Consequently, \( G = K_1 \cup (K_1 \cup H) \), as required.

Conversely, if \( G = K_1 \cup (K_1 \cup H) \), where \( H \) is a graph on \( n - 2 \) vertices realizing \( S_{(i-1,n-3)} \), then from Theorem 2.3 and 4.4 it follows that \( G \) realizes \( S_{(i,n-2)} \).

\textbf{Remark 4.5.} Theorems 4.3 and 4.4 completely resolve the existence of graphs realizing the spectrum \( S_{(i,n-2)} \), where \( 2 \leq i \leq n - 3 \).

\subsection*{4.2. Graphs realizing the sets \( S_{(i,n-1)} \).}

To determine all the graphs realizing the set \( S_{(i,n-1)} \) for various possible values of \( i, i > 1 \), first we establish the following auxiliary lemma.

\textbf{Lemma 4.6.} The set \( S_{(i,j)} \) is Laplacian realizable if and only if the set \( S_{(n-j+1,n-i+1)} \) is Laplacian realizable.

\textbf{Proof.} Let \( S_{(i,j)} \) be Laplacian realizable and let a graph \( G \) realize \( S_{(i,j)} \). Consider the graph \( H = \overline{G} \cup K_1 \). Since the Laplacian spectrum of the graph \( \overline{G} \cup K_1 \) has the form \( \{0\} \cup S_{(i,j)} \), the spectrum of \( H \) is \( S_{(n-j+1,n-i+1)} \) by Theorem 2.2.

Conversely, suppose that \( S_{(n-j+1,n-i+1)} \) is Laplacian realizable, and a graph \( H \) realizes it. According to Theorem 2.2 we have

\begin{equation}
\sigma_L(H) = \{0\} \cup S_{(i,j)}.
\end{equation}
Thus, $\overline{H}$ is the union of two disjoint graphs, say, $\overline{H} = G \cup F$, and the Laplacian spectrum of $\overline{H}$ is the union of the spectra of $G$ and $F$. Consequently, one of these graphs, say, $G$, has $n$ as its Laplacian eigenvalue, so that its order is at least $n$ by Theorem 2.1. Since the order of $\overline{H}$ is $n + 1$, we have $F = K_1$. Therefore, the graph $G$ of order $n$ realizes $S_{\{i,j\}}^n$, according to (1.2).

Now, we are in a position to describe all the Laplacian realizable sets $S_{\{i,n-1\}}^2$. Remind that $i > 1$.

**Theorem 4.7.** Suppose $n \geq 6$. The only Laplacian realizable sets $S_{\{i,n-1\}}^2$ are the following ones.

(i) If $n \equiv 0 \mod 4$, then $S_{\{i,n-1\}}^2$ is Laplacian realizable if and only if $i = n - 3$.

(ii) If $n \equiv 1 \mod 4$, then $S_{\{i,n-1\}}^2$ is Laplacian realizable if and only if $i = n - 2$.

**Proof.** Let $G$ realizes $S_{\{i,n-1\}}^2$.

(i) If $n \equiv 0 \mod 4$, then $n - 1 \equiv 2 \mod 4$, so by Theorem 2.13 (ii), the set $S_{\{1,j\}}^{n-2}$ is Laplacian realizable if and only if $j = 3$. Using Lemma 4.6, the set $S_{\{n-3,n-1\}}^2$ is also Laplacian realizable if and only if $j = 3$. Thus, $S_{\{i,n-1\}}^2$ is Laplacian realizable only, if $i = n - 3$.

(ii) If $n \equiv 1 \mod 4$, then $n - 1 \equiv 0 \mod 4$. Therefore, by Theorem 2.15 (i), the set $S_{\{1,j\}}^{n-2}$ is Laplacian realizable if and only if $j = 2$. Using Lemma 4.6, the set $S_{\{n-3,n-1\}}^2$ is Laplacian realizable only, if $j = 2$. Consequently, $S_{\{i,n-1\}}^2$ is Laplacian realizable only, if $i = n - 2$.

Our next result, discuss the structure of graphs realizing the sets $S_{\{i,n-1\}}^2$, $i > 1$ as follows.

**Theorem 4.8.** Let $G$ be a graph of order $n$, $n \geq 6$.

(a) The graph $G$ realizes $S_{\{n-3,n-1\}}^2$ if and only if $n \equiv 0 \mod 4$, and $G$ is formed in one of the following two ways:

(i) $G = K_1 \cup (K_2 \cup (K_1 \cup (2K_1 \cup H)))$, where the graph $H$ of order $n - 6$ realizes $S_{1,n-6}$;

(ii) $G = K_1 \cup (K_2 \cup \overline{F})$, where the graph $F$ realizes $S_{\{2,n-2\}}^{n-2}$.

(b) The graph $G$ realizes $S_{\{n-2,n-1\}}^2$ if and only if $n \equiv 1 \mod 4$, and $G = K_1 \cup (P_3 \cup (K_1 \cup \overline{F}))$, where $H$ realizes $S_{n-7,n-5}$.

**Proof.** (a) Let $G$ realizes $S_{\{n-3,n-1\}}^2$, and let $n \equiv 0 \mod 4$. By Theorem 2.5, $G = G_1 \cup G_2$, so $\overline{G} = \overline{G_1} \cup \overline{G_2}$. From Theorem 2.2 it follows that $\sigma_L(\overline{G}) = \{0\} \cup S_{\{1,3\}}^{n-2}$. Thus, by Theorem 2.1 we obtain that $G_1 = K_1$, and $G_2$ is of order $n - 1$, so that $\sigma_L(\overline{G_2}) = S_{\{1,3\}}^{n-2}$. According to Theorem 2.16, the graph $\overline{G_2}$ can be formed in one of the following two ways.

(i) $\overline{G_2} = (K_1 \cup K_1) \cup (K_1 \cup H)$, where the graph $H$ realizes $S_{\{1,n-4\}}^{n-3}$;

(ii) $\overline{G_2} = K_1 \cup F$, where the graph $F$ realizes $S_{\{2,n-2\}}^{n-3}$.

Thus, from (i) one gets $G_2 = K_2 \cup (K_1 \cup \overline{H})$. According to Theorem 2.2, one has $\sigma_L(\overline{H}) = \{0,0\} \cup S_{1,n-5}$. Thus, $\overline{H}$ can be represented as follows $\overline{H} = 2K_1 \cup H_1$, where $H_1$ realizes $S_{1,n-5}$ (for the construction of $H_1$ see Proposition 2.10). Thus, $G = K_1 \cup (K_2 \cup (K_1 \cup (2K_1 \cup H_1)))$, where the graph $H_1$ of order $n - 6$ realizes $S_{1,n-6}$.

Also from (ii), we have $G_2 = K_1 \cup F$. According to Theorem 2.2, one has $\sigma_L(\overline{F}) = S_{\{n-4,n-2\}}^{n-2}$. Thus, $G = K_1 \cup (K_1 \cup \overline{F})$ where $F$ realizes $S_{\{2,n-2\}}^{n-3}$ (for the construction of $H_1$ see Proposition 2.10).

Conversely, if $G = K_1 \cup (K_2 \cup (K_1 \cup (2K_1 \cup H_1)))$, the graph $H_1$ of order $n - 6$ realizes $S_{1,n-6}$. Then, from Theorem 2.3 and 2.4, it follows that $G$ realizes $S_{\{n-3,n-1\}}^2$. Similarly, if $G = K_1 \cup (K_1 \cup \overline{F})$ where the graph $F$ realizes $S_{\{2,n-2\}}^{n-3}$, then again from Theorem 2.3 and 2.4, the graph $G$ realizes $S_{\{n-3,n-1\}}^2$. 

10 A. HAMEED AND M. TYAGLOV
(b) Let $G$ realizes $S_{(n-2,n-1)}^2$ and let $n \equiv 1$ or 2 mod 4. Then $G = G_1 \vee G_2$ by Theorem 2.5 so $\overline{G} = \overline{G}_1 \cup \overline{G}_2$. From Theorem 2.2 it follows that $\sigma_L(\overline{G}) = \{0\} \cup S_{(1,2)}^{n-2}$. Thus, by Theorem 2.1 we obtain that $G_1 = K_1$, and $G_2$ is of order $n - 1$, so that $\sigma_L(\overline{G_2}) = S_{(1,2)}^{n-2}$. According to Theorem 2.16 the graph $\overline{G}_2$ is of the following form. $\overline{G}_2 = (K_1 \cup K_2) \vee (K_1 \cup H)$, where $H$ realizes $S_{n-7,n-5}$. Thus, $G_2 = P_3 \cup (K_1 \vee \overline{H})$. According to Theorem 2.2, one has $\sigma_L(\overline{H}) = \{0\} \cup S_{2,n-6}$. Thus, $\overline{H}$ can be represented as follows $\overline{H} = K_1 \cup H_1$, where $H_1$ realizes $S_{2,n-6}$ (for the construction of $H_1$ see Proposition 2.10). Consequently, $G = K_1 \vee (P_3 \cup (K_1 \vee \overline{H}))$, where $H$ realizes $S_{n-7,n-5}$. Conversely, if $G = K_1 \vee (P_3 \cup (K_1 \vee \overline{H}))$, where $H$ realizes $S_{n-7,n-5}$. Then, from Theorem 2.3 and 2.4 it follows that $G$ realizes $S_{(n-2,n-1)}^2$.

4.3. Graphs realizing the sets $S_{(1,j)}^{n-2}$ and $S_{(i,n)}^{n-2}$. Now we are in a position to study graphs realizing the sets $S_{(1,j)}^{n-2} (i = 1)$ and $S_{(i,n)}^{n-2} (j = n)$. Since all the correspondent graphs on less than 6 vertices have been listed, see Appendix A in what follows we consider $n \geq 6$. The structure of graph realizing the set $S_{(1,j)}^{n-2}$ is described by the following theorem.

**Theorem 4.9.** Let $G$ be a graph of order $n$. Then for each admissible $j$, $1 < j < n$ the graph $G$ realizes $S_{(1,j)}^{n-2}$ if and only if $G = K_1 \vee F$ where $F$ realizes $S_{(j-1,n-1)}^{n-2}$.

**Proof.** Let $S_{(1,j)}^{n-2}$ for $1 < j < n$ be Laplacian realizable by a graph $G$. Then Theorem 2.2 implies

\[(4.3) \quad \sigma_L(\overline{G}) = \{0, 0, 1, 2, \ldots, n - j - 1, n - j + 1, \ldots, n - 2, n - 2\}.
\]

Here $\overline{G}$ is a disconnected graph of the form $\overline{G} = \overline{G}_1 \cup \overline{G}_2$. By Theorem 2.1 one of the components, say, $\overline{G}_2$, must be of order at least $n - 2$.

If $\overline{G}_2$ has $n - 2$ vertices, then $\overline{G}_1$ has two vertices, and therefore $\sigma_L(\overline{G_1}) = \{0, 2\}$. By Theorem 2.2 we obtain $\sigma_L(G_1) = \{0, 2\}$, so $G_1 = K_1$. Now from (4.3), we have

\[\sigma_L(\overline{G}(2)) = \{0, 1, \ldots, n - j - 1, n - j + 1, \ldots, n - 2, n - 2\}.
\]

This contradicts Theorem 2.6.

If $\overline{G}_2$ has $n - 1$ vertices, then $\overline{G}_1 = K_1$. Again from (4.3)

\[\sigma_L(\overline{G}(2)) = \{0, 1, 2, \ldots, n - j - 1, n - j + 1, \ldots, n - 2, n - 2\},
\]

by Theorem 2.2

\[\sigma_L(G_2) = \{0, 1, 1, 2, \ldots, j - 2, j, \ldots, n - 3, n - 2\}.
\]

Here the graph $G_2$ is of order $n - 1$ realizes $S_{(j-1,n-1)}^{n-2}$. Thus, $G = K_1 \vee F$, where $F = G_2$ realizes $S_{(j-1,n-1)}^{n-2}$.

Conversely, if $G = K_1 \vee F$, where the graph $F$ of order $n - 1$ realizes $S_{(j-1,n-1)}^{n-2}$. Then from Theorems 2.4 it follows that $G$ realizes $S_{(1,j)}^{n-2}$.

Theorem 4.9 guarantees that if the set $S_{(i,n)}^{n-2}$ is not Laplacian realizable, then no graphs realizing $S_{(1,j)}^{n-2}$ exist. From Theorem 4.9 it follows that for all admissible $j$, the sets $S_{(1,j)}^{n-2}$ are Laplacian realizable only if the Conjecture 3.6 is not true. Moreover, for $n = p + 1$ where $p$ is a prime number, the set $S_{(1,j)}^{n-2}$ is not Laplacian realizable according to Proposition 4.5. For instance, if $G$ is of order $n = 6, 8, 12, 14, 18$ etc., then it does not realize $S_{(1,j)}^{n-2}$.

Finally, we consider the case when $j = n$. Recall that for $n \leq 5$, there are no graphs realizing the set $S_{(1,n)}^{n-2}$, see Appendix A. As well, we conjectured in [12] that graphs realizing the set $S_{(1,n)}^{n-2}$, $n \geq 9$ do not exist. In that paper, it is also shown that the set set $S_{(i,n)}^{n-2}$ is not Laplacian realizable for a prime number $n$, and so the set $S_{(1,n)}^{n-2}$ is not Laplacian realizable if $n$ is prime. Moreover, as we mentioned in Section 3 above, the set $S_{(1,n)}^{n-2}$ is not Laplacian realizable by the Cartesian product for $n \geq 9$. Additionally,
for \( i > 1 \), one can see from Proposition 3.5 that \( G \) has the minimum and maximum degree of the following form:

\[
2 \leq \min_{1 \leq i \leq n} d_i \leq \max_{1 \leq i \leq n} d_i \leq n - 3,
\]

where \( d_i \) is the degree of vertex \( i \). So the minimum degree of graphs realizing the sets \( S_{(i,n)}^{2} \), \( i > 1 \) and, more generally, the sets \( S_{(i,n)}^{m} \) (if any) is greater than or equal to 2 and by graph complement one can obtain the maximum degree \( n - 3 \). If the minimum degree equals 1, then \( G \) is a join, a contradiction. Note that this property is analogous to the one of \( S_{n,n} \)-conjecture proposed in [7, Section 3] the authors showed that for \( n = 8, 9 \) the set \( S_{n,n} \) is not Laplacian realizable. We believe that the same is true for sets \( S_{(i,n)}^{m} \). Thus, if \( S_{n,n} \)-conjecture is true, then our \( S_{(i,n)}^{m} \)-conjecture will also hold true. According to these observations, we believe that the set \( S_{(i,n)}^{2} \), is not Laplacian realizable. For more details on \( S_{(i,n)}^{n} \)-conjecture, we refer the reader to [12, Section 5].

5. Conclusion

In [12], we established the existence of graphs realizing the sets \( S_{(i,j)}^{m} \) for \( m = n \) and \( m = n - 1 \) and completely described them. The present work continues our study on the realizability of sets \( S_{(i,j)}^{m} \) for cases \( m = 1, 2 \). We completely characterized the graphs realizing those sets and developed an algorithm for constructing them but the sets \( S_{(1,j)}^{3} \) which are conjectured to be not Laplacian realizable for large \( n \).

Conjecture 5.1. If \( n \geq 3 \), then the only Laplacian realizable set of kind \( S_{(1,j)}^{2} \) are \( S_{(1,3)}^{2} \) and \( S_{(1,4)}^{2} \).

In addition, we believe that Lemma 4.6 and Proposition 1.2 provide a way forward to find graphs realizing the sets \( S_{(i,j)}^{m} \) for other particular values of \( m \) from already existing values of \( m \). For instance, the case \( m = 1 \) can be obtained from the case \( m = n \) using Lemma 4.6. In a similar way, one can describe graphs realizing \( S_{(i,j)}^{m} \) for \( m = 2 \) from those with \( m = n - 1 \) using Lemma 4.6. As well, Proposition 1.2 can help to find the repeated eigenvalue \( m \) from previously known values of \( m \). However, it is not clear whether the use of Proposition 1.2 and Lemma 4.6 may cover all the sets realizing \( S_{(i,j)}^{m} \) for fixed \( m \) and \( n \).

Appendix A. List of Laplacian integral graphs realizing \( S_{(i,n-1)}^{3} \) up to order 8

In the paper by \( K_n, P_n, K_{1,n-1} \) and \( K_{p,q} \) \((p + q = n)\) we denote the complete graph, the path graph, the star graph and the complete bipartite graph on \( n \) vertices, respectively. For the concepts and results about graphs not presented here, see, e.g., Bondy and Murty [2], and Diestel [6].

In [8, p. 301–304] the authors found the Laplacian spectra of all graphs up to order 5. Also in [3], the authors depicted all the graphs of order 6 without calculating their Laplacian spectra. We lists all the graphs realizing the sets \( S_{(i,j)}^{1} \) and \( S_{(i,j)}^{2} \) up to order 6. Also, we list few graphs of order 7 and 8 of such types. Note that we use the notation \( A_n \) for the anti-regular graph of order \( n \), see, e.g., [23].

| Construction       | Laplacian Spectrum | \( S_{(i,j)}^{m} \) |
|--------------------|---------------------|----------------------|
| \( S_{4} \cong K_{3,1} \) | \( \{0, 1, 1, 4\} \) | \( S_{(2,3)}^{1} \) |
| \( K_{1} \lor K_{1,1,2} \)   | \( \{0, 1, 1, 3, 5\} \) | \( S_{(2,4)}^{1} \) |
| \( K_{1} \lor (2K_{1} \cup P_{3}) \) | \( \{0, 1, 1, 2, 4, 6\} \) | \( S_{(3,5)}^{1} \) |
| \( K_{1} \lor (P_{3} \cup P_{3}) \) | \( \{0, 1, 1, 2, 3, 4, 7\} \) | \( S_{(5,6)}^{1} \) |
Table 2. Laplacian integral graphs realizing $S_{(i,j)}^\alpha_n$ for $n = 4, 5, 6, 7, 8.$

| Construction | Laplacian Spectrum | $S_{(i,j)}^\alpha_n$ |
|--------------|-------------------|---------------------|
| $K_1 \lor [2K_1 \cup (K_1 \lor P_3)]$ | $\{0, 1, 1, 2, 4, 5, 7\}$ | $S_{(3,6)}^\alpha$ |
| $[(2K_1 \lor P_3) \cup 2K_1] \lor K_1$ | $\{0, 1, 1, 3, 4, 5, 6, 8\}$ | $S_{(2,7)}^\alpha$ |
| $(A_5 \cup 2K_1) \lor K_1$ | $\{0, 1, 1, 2, 3, 5, 6, 8\}$ | $S_{(4,7)}^\alpha$ |
| $(P_3 \lor A_4) \lor K_1$ | $\{0, 1, 1, 2, 3, 4, 5, 8\}$ | $S_{(6,7)}^\alpha$ |

APPENDIX B. Acknowledgements

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14

A. HAMEED AND M. TYAGLOV

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