On five-dimensional non-extremal charged black holes and FRW cosmology

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ABSTRACT

We consider static non-extremal charged black hole solutions in the context of $N = 2$ gauged supergravity theories in five dimensions, and we show that they satisfy first-order flow equations. Then we analyze the motion of the dual brane in these black hole backgrounds. We express the entropy in terms of a Cardy-Verlinde-type formula, and we show that the equations describing the FRW cosmology on the brane have a form that is similar to the equations for the entropy and for the Casimir energy on the brane. We also briefly comment on the inclusion of a Gauss-Bonnet term in the analysis.
1 Introduction

In [1] a surprising relation was found between the Friedmann-Robertson-Walker (FRW) equations of standard cosmology describing a universe filled with radiation and the equations describing the entropy and the Casimir energy of the radiation. The connection came about by representing the radiation by a strongly coupled conformal field theory (CFT) with an AdS-dual description in terms of a five-dimensional AdS-Schwarzschild black hole [2]. This connection was further studied in [3] from a brane-world perspective [4, 5], where the universe was represented in terms of a spherical brane moving in the black hole background [6]. The universe, which is described by the standard FRW equations, has a size that corresponds to the distance of the brane to the black hole singularity. The brane first expands until it reaches a maximal radius after which it recontracts and falls through the horizon. It was found [1] that the entropy of the radiation can be written in a form analogous to Cardy’s formula for the entropy of a two-dimensional CFT [7]. This is the so-called Cardy-Verlinde formula. Moreover, the Cardy-Verlinde formula and the equation for the Casimir energy have a form that is similar to the two FRW equations on the brane, and these two sets of equations coincide as the brane crosses the horizon of the black hole. This may indicate [1, 3] that the standard FRW equations and the equations for the entropy and the energy of the CFT have a common origin in string or M-theory.

The Cardy-Verlinde formula for the radiation CFT mentioned above expresses the square of the entropy $S$ in terms of the product of the extensive part $E_e$ of the energy and the Casimir energy $E_c$ on the brane. The Casimir energy is defined as the violation of the thermodynamic Euler relation [1]. The energies $E_e$ and $E_c$ behave as

$$E_e \propto S^{4/3}, \quad E_c \propto S^{2/3},$$  \hspace{1cm} (1.1)\

where $a$ denotes the radius of the spherical brane. The proportionality coefficients in (1.1) are independent of $a$ and $S$. In the following we consider charged AdS black holes in five-dimensional $N = 2$ gauged supergravity theories obtained by gauging the $U(1)$ subgroup of the $SU(2)$-automorphism group of the $N = 2$ supersymmetry algebra [8]. These theories have a potential which is constructed out of a superpotential $W$ that depends on the scalar fields of the theory. It turns out to be convenient to also introduce a ‘dual’ superpotential $\tilde{W}$. We determine two quantities that are of the form (1.1) and that we denote by $E_e$ and $E_c$. The latter is the Casimir energy of the field theory on the brane dual to the charged AdS black hole. They come with proportionality coefficients given by $W$ and $\tilde{W}$ evaluated at the event horizon of the black hole, respectively. We note, however, that $W$ and $\tilde{W}$ do not have a simple dependence on extensive quantities. We express the entropy as a Cardy-Verlinde-type formula in terms of the product of $E_e$ and $E_c$. For a discussion of the Cardy-Verlinde formula in the context of the STU model see also [9, 10]. We show that the two FRW equations describing the motion of the brane in the charged black hole background take a form that is similar to the Cardy-Verlinde-type formula and to the equation for the Casimir energy on the brane, respectively. Then, as the brane crosses the event horizon of the black hole, these
two sets of equations again coincide. This has already been addressed in [11, 12, 13, 14] for the case of a static AdS-Einstein-Maxwell black hole. We also discuss the Bekenstein bound on the entropy [15] for some of the black hole solutions arising in the STU model.

The five-dimensional black hole solutions that we consider in the context of the $N = 2$ gauged supergravity theories mentioned above are the non-extremal electrically charged static AdS black holes of [16], which were constructed by solving the associated equations of motion. Special cases of these black holes have been discussed in [17, 18, 19, 20, 21, 22, 23, 24]. We rederive the black hole solutions of [16] by showing that they satisfy first-order flow equations. These are obtained by rewriting the five-dimensional bulk action in terms of squares of first-order differential equations. Since the black hole solutions are non-extremal, there are additional terms in the action which are not of the square type, but which are consistent with the first-order flow equations. First-order flow equations for non-extremal AdS-Einstein-Maxwell black holes have been discussed in the past in [25] and for extremal AdS black holes supported by one scalar field in [23]. More recently, such equations have also been discussed in [26, 27] in the context of Einstein-dilaton-p-form systems.

The paper is organized as follows. In section 2 we review a few relevant facts about the five-dimensional $N = 2$ gauged supergravity theories that we will be looking at. Then we show that the black hole solutions of [16] satisfy first-order flow equations. We briefly discuss various black hole solutions in the context of the STU model. Then we give the gravitational counterpart of the energies $E_e$ and $E_c$ at the moment when the brane crosses the event horizon of the black hole. We denote these by $\tilde{E}_e$ and $\tilde{E}_c$ respectively, and we discuss their relation to the Smarr formula [28] for charged black holes. The quantities $\tilde{E}_e$ and $\tilde{E}_c$ are proportional to $W$ and $\tilde{W}$ evaluated at the horizon, respectively. The resulting Cardy-Verlinde-type formula, which is expressed in terms of $\tilde{E}_e \tilde{E}_c$, is then proportional to the product of $W$ and $\tilde{W}$ evaluated at the horizon. In section 3 we turn to the discussion of the FRW cosmology on the brane, following [3]. Using [29, 30, 31, 32, 33] we determine the equations describing the motion of the dual brane in the black hole background. We briefly discuss the cosmology on the brane in the background of one of the black hole solutions of the STU model, and we comment on the Bekenstein bound. Then we show that the FRW equations have a form that is similar to the equations for the entropy and the Casimir energy on the brane, and that both sets coincide when the brane crosses the event horizon. In section 4 we conclude with a few comments on the inclusion of a Gauss-Bonnet term in the analysis. See also [34, 35, 36, 37, 38] for related discussions.

2 Non-extremal electrically charged static black holes in $N = 2$ gauged supergravity theories

In the following, we consider $N = 2$ gauged supergravity theories in five dimensions obtained by gauging the $U(1)$ subgroup of the $SU(2)$-automorphism group of the $N = 2$ supersymmetry algebra [8]. The gauging is with respect to a linear combination proportional to $h_A A_M^A$ of $U(1)$ gauge fields (with constant $h_A$), and the coupling constant $g$ is identified with the
inverse of the curvature radius of $AdS_5$, i.e. $g = L^{-1}$.

The relevant part of the five-dimensional action reads

$$ S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R - g_{ij} \partial_M \varphi^i \partial^M \varphi^j - \frac{1}{2} G_{AB} F_M^A F^{BM} - V_{\text{pot}} \right). \quad (2.1) $$

We denote the spacetime metric by $g_{MN}$. The real scalar fields $X^A$ satisfy the constraint

$$ \frac{1}{6} C_{ABC} X^A X^B X^C = 1. \quad (2.2) $$

The metric $G_{AB}$ is given by

$$ G_{AB} = -\frac{1}{2} C_{ABC} X^C + \frac{9}{2} X_A X_B, \quad (2.3) $$

where

$$ X_A = \frac{1}{6} C_{ABC} X^B X^C. \quad (2.4) $$

Observe that $X^A X_A = 1$ in view of $(2.2)$. In addition,

$$ X_A \partial_i X^A = 0, \quad (2.5) $$

where $X^A = X^A(\varphi^i)$ and $\partial_i X^A(\varphi) = \partial X^A / \partial \varphi^i$. Here the $\varphi^i$ denote the physical scalar fields with target-space metric

$$ g_{ij} = G_{AB} \partial_i X^A \partial_j X^B. \quad (2.6) $$

A useful relation is

$$ G_{AB} \partial_i X^B = -\frac{3}{2} \partial_i X_A. \quad (2.7) $$

The potential $V_{\text{pot}}$, which is expressed in terms of the superpotential

$$ W = h_A X^A, \quad (2.8) $$

reads

$$ V_{\text{pot}} = g^2 \left( g^{ij} \partial_i W \partial_j W - \frac{4}{3} W^2 \right) = g^2 \left( h_A G^{AB} h_B - 2 W^2 \right), \quad (2.9) $$

where in the second step we used

$$ g^{ij} \partial_i X^A \partial_j X^B = G^{AB} - \frac{2}{3} X^A X^B. \quad (2.10) $$

### 2.1 First-order flow equations

The equations of motion derived from $(2.1)$ allow for various classes of solutions that have a description in terms of first-order flow equations. In the ungauged case ($g = 0$) one such class consists of electrically charged static extremal black hole solutions with line element

$$ ds_5^2 = -e^{-4U} dt^2 + e^{2U} dr^2 + e^{2U} r^2 d\Omega_3^2. \quad (2.11) $$
The metric factor $e^{2U}$ and the scalar fields $\varphi^i$ supporting the spherically symmetric black hole solution only depend on the radial coordinate $r$. They satisfy the first-order flow equations

$$\frac{de^{2U}}{d\xi} = \frac{1}{3} Z,$$
$$\frac{d\varphi^i}{d\xi} = -\frac{1}{2} e^{-2U} g^{ij} \partial_j Z,$$  \hspace{1cm} (2.12)

where $\xi$ denotes the variable $\xi = 1/r^2$ and where $Z = q_A X^A$. These flow equations can be combined into

$$X'_A + 2U' X_A = -\frac{2}{3} e^{-2U} q^A r^3,$$  \hspace{1cm} (2.13)

where $'= d/dr$. Indeed, contracting (2.13) with $X^A$ results in the flow equation for $e^{2U}$, while contracting with $\partial_j X^A$ yields the flow equation for $\varphi^i$ in view of the very special geometry relations (2.5) and (2.7).

The flow equations (2.12) are solved in terms of harmonic functions $H_A$,

$$e^{2U} = \frac{1}{3} H_A X^A,$$
$$e^{2U} X_A = \frac{1}{3} H_A,$$  \hspace{1cm} (2.14)

where $H_A = c_A + q_A/r^2$, and where the $c_A$ denote arbitrary integration constants.

In the gauged case ($g \neq 0$) a well-known class of solutions admitting a description in terms of first-order flow equations are flat domain wall solutions with line element [42, 43, 44, 45, 46, 47, 48, 49, 50, 51]

$$ds_5^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu + dp^2.$$  \hspace{1cm} (2.15)

The metric factor $e^{2A}$ and the scalar fields $\varphi^i$ supporting the domain wall solution only depend on the radial coordinate $\rho$. They satisfy the first-order flow equations

$$\frac{dA}{d\rho} = \frac{1}{3} g W,$$
$$\frac{d\varphi^i}{d\rho} = -g g^{ij} \partial_j W.$$  \hspace{1cm} (2.16)

Changing the radial variable from $\rho$ to $r$ such that $d\rho/dr = (g r e^{2U(r)})^{-1}$, with $e^{A(\rho)} = g r e^{U(r)}$, yields the line element in the form

$$ds_5^2 = e^{-4U} f \eta_{\mu\nu} dx^\mu dx^\nu + e^{2U} f^{-1} dr^2,$$  \hspace{1cm} (2.17)

where

$$f = g^2 r^2 e^{6U}.$$  \hspace{1cm} (2.18)

The flow equations (2.16) now take the form

$$U' r = \frac{1}{3} e^{-2U} W - 1,$$
$$X'^A = \frac{e^{-2U}}{r} \left( \frac{2}{3} W X^A - G^{AB} h_B \right).$$  \hspace{1cm} (2.19)
where \( t = \frac{d}{dr} \). Here we have displayed the flow equation for the \( X^A \). The flow equation for the \( \varphi^i \),
\[
\varphi'^i = -\frac{e^{-2U}}{r} g^{ij} \partial_j W ,
\]
follows from the flow equation for \( X^A \) by contracting it with \( G_{AB} \partial_j X^B \).

The electrically charged black hole solutions of ungauged supergravity described above satisfy first-order flow equations based on \( Z = q_A X^A \), whereas the flat domain wall solutions of gauged supergravity just described satisfy first-order flow equations based on \( W = h_A X^A \). One may ask whether there exist charged solutions to gauged supergravity satisfying both sets of first-order flow equations \( (2.13) \) and \( (2.19) \) (common features of black hole and domain wall solutions were recently discussed in [52]). That there exist charged extremal solutions in gauged supergravity based on first-order flow equations was demonstrated in [23], where various examples with one real scalar field and one abelian gauge field were discussed. Non-extremal electrically charged static black hole solutions were constructed in [16] by solving the equations of motion. Here we show that these solutions have a first-order flow description based on the two sets \( (2.13) \) and \( (2.19) \), by rewriting the five-dimensional action \( (2.1) \) in terms of these equations. Then, the compatibility of the flow equations \( (2.13) \) and \( (2.19) \) requires identifying the integration constants \( c_A \) appearing in the solution \( (2.14) \) with \( h_A \), so that now
\[
H_A = h_A + \frac{q_A}{r^2} .
\]

Following [16] we consider non-extremal electrically charged static black hole solutions with line element
\[
\begin{align*}
  ds_5^2 &= -e^{-4U} f dt^2 + e^{2U} f^{-1} dr^2 + e^{2U} r^2 d\Sigma_k^2 , \\
  f &= k - \frac{\mu}{r^2} + g^2 r^2 e^{6U} ,
\end{align*}
\]
where \( U = U(r) \), \( f = f(r) \). Here \( d\Sigma_k^2 \) denotes the line element of a three-dimensional space of constant curvature with metric \( \eta_{\alpha\beta} \), either flat space \( (k = 0) \), hyperbolic space \( (k = -1) \) or a unit three-sphere \( S^3 \) \( (k = 1) \). The presence of a non-vanishing parameter \( \mu \) is necessary in order for the solutions to have a horizon. Observe that the line elements \( (2.11) \) and \( (2.17) \) are special cases of \( (2.22) \). In the following, we will always consider the case \( k = 1 \), but we keep \( k \) in the formulae as a book-keeping device. The scalar fields and the gauge fields supporting the solutions are taken to be functions of \( r \), only. Inserting the line element \( (2.22) \) into the action \( (2.1) \) yields
\[
16\pi G_5 S = S_0 + S_2 + S_{td} ,
\]
where \( S_0 \) and \( S_2 \) comprise the contributions to order \( g^0 \) and \( g^2 \), respectively, and where \( S_{td} \)
contains total derivative terms. \( S_0 \) and \( S_2 \) read

\[
S_0 = \int d^5 x \sqrt{\eta} \left[ 3 \mu \frac{e^{-2U}}{r^3} q_A G^{AB} \left( 2X_B - \frac{1}{3} e^{-2U} H_B \right) - \frac{9}{4} \left( k - \frac{\mu}{r^2} \right) r^3 \left( X'_A + 2U' X_A + \frac{2}{3} e^{-2U} \frac{q_A}{r^3} \right) G^{AB} \left( X'_B + 2U' X_B + \frac{2}{3} e^{-2U} \frac{q_B}{r^3} \right) + r^3 e^{4U} \left( F'_{tr} - e^{-4U} \frac{G^{AC} Q_C}{r^3} \right) G_{AB} \left( F'_{tr} - e^{-4U} \frac{G^{BD} Q_D}{r^3} \right) \right],
\]

\[
S_2 = g^2 \int d^5 x \sqrt{\eta} \left[ \frac{4}{3} r^3 e^{2U} (W - 3 e^{2U} (U' r + 1))^2 - r^5 e^{6U} \left( X'^A - \frac{e^{-2U}}{r} \left( \frac{2}{3} W X^A - G^{AC} h_C \right) \right) G_{AB} \left( X'^B - \frac{e^{-2U}}{r} \left( \frac{2}{3} W X^B - G^{BD} h_D \right) \right) \right],
\]

where \( \prime = d/dr \). In the expression for \( S_0 \), the physical electric charges \( Q_A \) are related to the \( q_A \) by

\[
Q_A G^{AB} Q_B = k q_A G^{AB} q_B + \mu q_A G^{AB} h_B,
\]

and \( H_A \) is given by (2.21). \( S_{td} \) reads

\[
S_{td} = \int d^5 x \sqrt{\eta} \left[ 2 Q_A F'_{tr} + \left( 6 \mu U - 2(k - \frac{\mu}{r^2}) (r^3 U' + e^{-2U} q_A X^A) \right) \right] + g^2 \left( -8r^5 e^{6U} U' - 8r^4 e^{6U} + 2r^4 e^{4U} W \right). \]

Observe that \( S_0 \) and \( S_2 \) are given in terms of squares of first-order differential equations, with the exception of the first line of \( S_0 \), which is proportional to the parameter \( \mu \). Under variation with respect to \( U \) or with respect to \( X^A \), the contribution from the first line of \( S_0 \) vanishes provided that

\[
X_A = \frac{1}{3} e^{-2U} H_A.
\]

Thus, extremizing \( S_0 \) and \( S_2 \) with respect to the fields \( U \), \( X^A \) and \( F'_{tr} \) yields the relation (2.27), the first-order flow equations (2.19) and (2.19) as well as

\[
F'_{tr} = e^{-4U} \frac{G^{AC} Q_C}{r^3}.
\]

The flow equations (2.13) and (2.19) are solved by (2.14) with \( c_A = h_A \), as discussed above, and the solution agrees with (2.27).

We take \( h_A \) and \( q_A \) in (2.21) to be positive to ensure that \( H_A > 0 \). We also take \( X^A > 0 \), so that \( e^{2U} > 0 \) along the flow. We impose the normalization \( e^{2U} = 1 \) at \( r = \infty \). The asymptotic value of \( X_A \) is then \( \frac{1}{3} h_A \). Denoting the asymptotic value of the \( X^A \) by \( h^A \), we have \( \frac{1}{3} h^A h_A = 1 \) in view of (2.2). We introduce the ‘dual’ superpotential \( \tilde{W} \) as

\[
\tilde{W} = h^A X_A,
\]

for later convenience.
Summarizing, the line element (2.22) together with (2.14), (2.21), (2.28) and (2.25) describe non-extremal electrically charged static black hole solutions to first-order flow equations. It can be checked that they solve all the equations of motion.

On the solution, the total derivative terms (2.26) can be written as

\[ S_{td} = \int d^5x \sqrt{\eta} \left[ 6 \mu U' - 2 \left( r^3 f U' + r^2 (f - k) \right) \right], \tag{2.30} \]

while the first line of \( S_0 \) yields

\[ - \int d^5x \sqrt{\eta} 6 \mu U'. \tag{2.31} \]

Thus, on the solution, the action (2.1) evaluates to

\[ 16 \pi G_5 S = -2 \int d^5x \sqrt{\eta} \left( r^3 f U' + r^2 (f - k) \right)', \tag{2.32} \]

in agreement with (33).

The electric field \( F_{\tau r}^A \) is determined in terms of the potential \( \phi^A(r) \), i.e. \( F_{\tau r}^A = -\partial_r \phi^A(r) \).

In the following, we compute the contraction \( Q_A \phi^A \). To this end, we differentiate the first equation of (2.12) and obtain

\[ U'' = \frac{1}{3} e^{-2U} \left( -2U' \frac{Z}{r^3} - \frac{3}{r^3} + \frac{q_A X'^A}{r^3} \right). \tag{2.33} \]

Using (2.13) we compute

\[ X'^A = -\frac{3}{2} G^{AB} X_B = 2U' X^A + e^{-2U} \frac{G^{AB} q_B}{r^3}. \tag{2.34} \]

Then, using (2.33) and (2.34) gives

\[ U'' + \frac{3}{r} U' = -\frac{1}{3} e^{-2U} \frac{q_A G^{AB} q_B}{r^6}. \tag{2.35} \]

With the help of (2.12), (2.25) and (2.35) we obtain

\[ -3 \left[ r^3 \left( k - \frac{\mu}{r^2} \right) U'' \right]' = e^{-4U} \frac{Q_A G^{AB} Q_B}{r^3}, \tag{2.36} \]

which equals \( Q_A F_{\tau r}^A \), as can be seen from (2.28). Hence we establish that

\[ Q_A \phi^A = -\left( k - \frac{\mu}{r^2} \right) e^{-2U} q_A X^A + k q_A h^A, \tag{2.37} \]

where we used the first equation of (2.12) once more. We chose the integration constant in such a way that \( Q_A \phi^A \) vanishes at spatial infinity, as in (28). In the context of the AdS/CFT correspondence [53, 54, 55] this means that the potentials \( \phi^A \) associated with the \( U(1) \) charges \( Q_A \) approach the boundary at a vev rate [56, 23]. A different choice of the integration constant can be incorporated into the Cardy-Verlinde-type formula given below by an appropriate shift.
The quantity appearing in the first law of black hole mechanics, \( dM = T_H dS + \phi^A dQ_A \), is not \((2.37)\) but a rescaled one given by \([17]\)

\[
Q_A \phi^A = \frac{2}{3w_5} \left( - \left( k - \frac{\mu}{r^2} \right) e^{-2U} q_A X^A + k q_A \phi^A \right),
\]

where

\[
w_5 = \frac{16\pi G_5}{3 \text{vol}}.
\]

Here, \( \text{vol} = \int d^3x \sqrt{\eta} \) denotes the volume of the three-dimensional space of constant curvature with line element \( d\Sigma_k^2 \). For \( k = 1 \) this space is a unit three-sphere \( S^3 \) with volume \( \text{vol} = \text{vol}(S^3) \). As already stated, we will focus on the case \( k = 1 \) but we will nevertheless keep \( k \) in the formulae as a book-keeping device.

Next, we compute the coefficient of the \( 1/r^2 \)-term in the metric factor \(-e^{-4U} f\) of the line element \((2.22)\). We denote this coefficient by \( w_5 \). Expanding \( e^{2U} = 1 + \kappa/r^2 + \ldots \) as well as \( X^A = h^A + \beta^A/r^2 + \ldots \) and using the first equation of \((2.14)\) results in \( \kappa = \frac{1}{2} \left( h^A q_A + h_A \beta^A \right) \). On the other hand, inserting the expansion of \( X^A \) into \((2.2)\) yields \( h_A \beta^A = 0 \). It follows that the coefficient \( w_5 \) \( M \) is given by

\[
w_5 \ M = \mu + 2k \kappa = \mu + \frac{2}{3} k h^A q_A.
\]

\( M \) denotes the mass of the black hole relative to the asymptotic \( AdS_5 \) spacetime. Pure global \( AdS_5 \) \( (k = 1) \) has a mass given by \( \frac{4}{3} w_5^{-1} g^{-2} \) \([29]\), so that the total energy reads \([29, 33]\)

\[
\frac{1}{w_5} \left( \mu + \frac{2}{3} h^A q_A + \frac{1}{4} g q^2 \right).
\]

\((2.41)\)

The contribution proportional to \( 1/g^2 \) will, however, not play any role in the Cardy-Verlinde formula and in the matching of the FRW equations with thermodynamic equations on the brane.

Next, let us discuss a few black hole solutions explicitly. We specialize \((2.2)\) to the case of the STU model with \( X^1 X^2 X^3 = 1 \). The associated metric \( G_{AB} \) is diagonal. The solution to the equations \((2.14)\) reads \([10]\)

\[
e^{6U} = H_1 H_2 H_3, \quad X^1 = \left( \frac{H_2 H_3}{H_1} \right)^{1/3}, \quad X^2 = \left( \frac{H_1 H_3}{H_2} \right)^{1/3}, \quad X^3 = \left( \frac{H_1 H_2}{H_3} \right)^{1/3}.
\]

\((2.42)\)

The Maxwell case is obtained by setting \( H_1 = H_2 = H_3 = 1 + q/r^2 \). Then \( X^1 = X^2 = X^3 = 1 \) and \( e^{2U} = H_1 \), so that \( W = 3, \bar{W} = 1 \), and \((2.40)\) and \((2.38)\) become (with \( k = 1 \))

\[
w_5 \ M = \mu + 2q, \quad Q_A \phi^A = \frac{2}{w_5} \frac{q (\mu + q)}{q + r^2}.
\]

\((2.43)\)

On the other hand, setting \( H_1 = H_2 = 1 + q/r^2 \) and \( H_3 = 1 \) yields the \( p = \frac{4}{3} \) example discussed in \([23]\) with \( e^{6U} = H_1^2 \) and \( X^1 = X^2 = H_1^{-1/3}, \quad X^3 = H_1^{2/3} \). Then \( W = 2H_1^{-1/3} + H_1^{2/3} \) and \( \bar{W} = \frac{1}{3} \left( 2H_1^{1/3} + H_1^{-2/3} \right) \), and \((2.40)\) and \((2.38)\) become (with \( k = 1 \))

\[
w_5 \ M = \mu + \frac{4}{3} q, \quad Q_A \phi^A = \frac{4}{3w_5} \frac{q (\mu + q)}{q + r^2}.
\]

\((2.44)\)
The existence of a horizon shielding the singularity at $r = 0$ requires taking $\mu > g^2 q^2$, and there is no inner horizon \cite{21,23}.

Finally, setting $H_1 = 1 + q/r^2$ and $H_2 = H_3 = 1$ yields the $p = \frac{1}{3}$ example discussed in \cite{23} with $e^{6U} = H_1$ and $X^1 = H_1^{-2/3}$, $X^2 = X^3 = H_1^{1/3}$. Then $W = H_1^{-2/3} + 2H_1^{1/3}$ and $\tilde{W} = \frac{1}{3} \left( H_1^{2/3} + 2H_1^{-1/3} \right)$, and \eqref{2.40} and \eqref{2.38} become (with $k = 1$)

$$w_5 M = \mu + \frac{2}{3} q , \quad Q_A \phi^A = \frac{2}{3w_5} \frac{q(\mu + q)}{q + r^2} . \quad \text{(2.45)}$$

The solution has a single horizon shielding the singularity at $r = 0$ whenever $\mu > 0$ \cite{21,23}.

### 2.2 A Cardy-Verlinde-type formula for charged black holes

Now we will show that for any model \eqref{2.2} the entropy of a charged black hole \eqref{2.22} can be written as a Cardy-Verlinde-type formula \cite{1}. This has already been discussed in \cite{9,11,10,13,14} for various black holes in the context of the STU model.

In the coordinates \eqref{2.22}, the exterior horizon $r_h$ of the black hole is located at the largest real positive root of (we assume $e^{-4U(r_h)} \neq 0$)

$$f(r_h) = 0 . \quad \text{(2.46)}$$

Its Hawking temperature $T_H$ is given by \cite{33}

$$T_H = \frac{1}{4\pi} f'(r_h) e^{-3U(r_h)} . \quad \text{(2.47)}$$

In the following, we consider a black hole with a spherical horizon ($k = 1$). The Bekenstein-Hawking entropy of the black hole is given by a quarter of the area of the event horizon, a three-sphere with radius $a_h = r_h e^{U(r_h)}$,

$$S = \frac{\text{vol}(S^3)}{4 G_5} = \frac{4\pi}{3w_5} a_h^3 , \quad \text{(2.48)}$$

with $w_5$ given by \eqref{2.39}. Hence

$$T_H S = \frac{1}{3w_5} f'(r_h) r_h^3 , \quad \text{(2.49)}$$

which we now compute. Using the first equation of \eqref{2.12} we obtain

$$f'(r) r^3 = 2\mu + 2 g^2 r^4 e^{6U} \left( 1 - e^{-2U} \frac{q A X^A}{r^2} \right) . \quad \text{(2.50)}$$

At the horizon, it follows from \eqref{2.46} that

$$\mu - k r_h^2 = g^2 r_h^4 e^{6U(r_h)} , \quad \text{(2.51)}$$

and hence

$$T_H S = \frac{2}{3w_5} \left[ 2\mu - k r_h^2 + \left( \frac{\mu}{r_h^2} \right) e^{-2U(r_h)} q A X^A \right] . \quad \text{(2.52)}$$
Next, let us consider the Smarr-type combination

\[ \frac{4}{3} M - T_H S - Q_A \phi_h^A , \]  

(2.53)

with \( Q_A \phi_h^A \) given by (2.38) and evaluated at the horizon. This combination is the gravitational counterpart of the Casimir energy \( E_c/3 \) on the brane. \( E_c \) is defined as the violation of the thermodynamic Euler relation \([1]\), as we will briefly review in the next section. The combination (2.53) can also be motivated by exhibiting its relation to the Smarr formula, as follows. In the absence of charges, the area \( A \) of the event horizon is determined in terms of the mass parameter \( \mu \) and \( g \) using (2.51). We can view this as a relation \( \mu = \mu(A, g) \).

Under the simultaneous rescaling \( r_h \rightarrow \lambda r_h \) and \( g \rightarrow \lambda^{-1} g \) we have \( A \rightarrow \lambda^3 A \) as well as \( \lambda^2 \mu = \mu(\lambda^3 A, \lambda^{-1} g) \). Differentiating with respect to \( \lambda \), setting \( \lambda = 1 \) and multiplying with \( w_5 \) results in

\[ \frac{2}{3} M = 3 T_H S - g \frac{\partial M}{\partial g} , \]  

(2.54)

where we used the first law of thermodynamics, \( dM = T_H dS \). Using (2.51), we compute

\[ g \frac{\partial M}{\partial g} = 2 M - \frac{2k}{w_5} r_h^2 . \]  

(2.55)

Inserting (2.55) into (2.54) we obtain

\[ \frac{4}{3} M - T_H S = \frac{2k}{3w_5} r_h^2 . \]  

(2.56)

This is the result for the Smarr-type combination (2.53) for uncharged black holes. In the ungauged case \( (g = 0) \) we have \( k r_h^2 = \mu \) and (2.56) yields the Smarr formula \( \frac{2}{3} M = T_H S \) \([57]\).

In analogy to \([1]\), we will denote the Smarr-type combination (2.53) by \( \tilde{E}_c/3 \). Using (2.40), (2.52) and (2.38) we obtain

\[ \tilde{E}_c = \frac{2k}{w_5} \left( r_h^2 + \frac{1}{3} h^A q_A \right) . \]  

(2.57)

On the other hand, contracting (2.27) with \( h^A \) gives

\[ r^2 e^{2U} h^A X_A = r^2 + \frac{1}{3} h^A q_A , \]  

(2.58)

and hence

\[ \tilde{E}_c = \frac{2k}{w_5} \tilde{W}_h a_h^2 , \]  

(2.59)

where \( \tilde{W}_h \) denotes the ‘dual’ superpotential (2.29) evaluated at the horizon. The quantity \( \tilde{E}_c \) is thus non-vanishing for a horizon of spherical topology. Observe that \( a_h^2 \propto S^{2/3} \) (cf. (1.1)). In the ungauged case \( (g = 0) \) we have \( k r_h^2 = \mu \) as well as \( \tilde{E}_c = 2 M - Q_A \phi_h^A \), and (2.53) yields the Smarr formula \( \frac{2}{3} M = T_H S + \frac{2}{3} Q_A \phi_h^A \) \([57] [28]\).

In the gauged case \( (g \neq 0) \) the combination \( 2 M - Q_A \phi_h^A - \tilde{E}_c \) is no longer vanishing. We find

\[ 2 M - Q_A \phi_h^A - \tilde{E}_c = \frac{2}{3w_5} \left( \mu - k r_h^2 \right) \left( 3 - e^{-2U(r_h)} \frac{q_A X_h^A}{r_h^2} \right) , \]  

(2.60)
where we used (2.40), (2.38) and (2.57). With the help of the first equation of (2.14) and (2.8) we have

\[ \frac{qA X_A}{\gamma^2} = 3e^{2U} - W , \]  

(2.61)

so that

\[ 2M - Q_A \phi_h^A - \tilde{E}_c = \frac{2}{3w_5} (\mu - k r_h^2) e^{-2U(r_h)} W_h , \]  

(2.62)

where \( W_h \) denotes (2.8) evaluated at the horizon. Using (2.51) this can be written as

\[ 2M - Q_A \phi_h^A - \tilde{E}_c = \frac{2}{3w_5} g^2 W_h a_h^4 , \]  

(2.63)

which is positive. Observe that \( a_h^4 \propto S^{4/3} \) (cf. (1.1)). We denote this combination by \( 2 \tilde{E}_c \), and we note that \( 2M = 2 \tilde{E}_c + \tilde{E}_c + Q_A \phi_h^A \).

It follows that we can express the square of the entropy (2.48) as

\[ k g^2 W_h \tilde{W} h S^2 = \frac{8\pi^2}{3} \tilde{E}_c \tilde{E}_c = \frac{4\pi^2}{3} \tilde{E}_c \left( 2M - Q_A \phi_h^A - \tilde{E}_c \right) . \]  

(2.64)

This is a Cardy-Verlinde-type formula for charged AdS black holes, here expressed in terms of gravitational quantities. It makes use of both \( W \) and \( \tilde{W} \) evaluated at the horizon. Using (2.10) we note the relation

\[ W \tilde{W} = 3 \left( 1 + \frac{1}{2} g^{ij} \partial_i W \partial_j \tilde{W} \right) , \]  

(2.65)

which shows that in general \( W_h \tilde{W} h \neq 3 \).

Summarizing, we find that three combinations are naturally expressed in terms of the superpotential quantities \( W \) and \( \tilde{W} \), namely (2.59), (2.63) and (2.64).

3 FRW cosmology on the brane

A radiation dominated FRW universe can be described in terms of a co-dimension-one brane of fixed tension moving in the background of a five-dimensional AdS-Schwarzschild black hole [6, 1, 3]. The radiation is represented by a CFT on the brane, and the induced metric on the brane takes the form of a standard FRW metric. The FRW equations have a structure that is similar to the equations for the entropy and the Casimir energy on the brane, and both sets of equations coincide when the brane crosses the horizon of the black hole [1, 3]. This also holds for the charged static AdS black holes discussed in the previous section, as we will show below.

3.1 Brane motion in the background of an AdS black hole

Following [6, 58, 3] we regard the brane as the boundary of the AdS geometry. The bulk-spacetime action (2.1) needs to be supplemented by a boundary term, the so-called Gibbons-Hawking term [59], and by counterterms [29, 30, 32, 33],

\[ - \frac{1}{8\pi G_5} \int_{\Sigma} d^4x \sqrt{-\gamma} \left( K + g W + \frac{1}{4} g R \right) , \]  

(3.1)
where $\Sigma$ denotes the brane, $\mathcal{R}$ denotes the Ricci scalar on the brane, and $K$ is the Gibbons-Hawking term. Observe that since we focus on the case $k = 1$ for which the boundary geometry is $S^1 \times S^3$, the holographic trace anomaly [60] vanishes and no further counterterms are required [30, 61]. The counterterms ensure that when the brane is moved to infinity the on-shell action of the black hole, which is given by the sum of (2.32) and (3.1), is finite.

The extrinsic curvature $K$ is given by

$$K = \gamma^{MN} K_{MN}, \quad K_{MN} = \gamma^P N Q \nabla_P (\gamma^{nQ}).$$

(3.2)

Here the tensor $\gamma_{MN} = g_{MN} - n_M n_N$ denotes the projection of $g_{MN}$ onto $\Sigma$, so that the induced metric on $\Sigma$ is given by the tangential components of $\gamma_{MN}$. We note that $\gamma^{MN} = g^{MN} - n^M n^N$ and $K = \nabla_M n^M$, where $n^M = g^{MN} n_N$. The vector $n = n^M \partial_M$ is the unit normal to $\Sigma$, i.e. $n^M n_M = 1$.

We view the brane as a dynamical entity [6, 3] moving in a background of the form

$$ds_5^2 = g_{MN} dx^M dx^N = -A(a) dt^2 + B(a) da^2 + a^2 d\Sigma_k^2.$$  

(3.3)

Note that the black hole metric (2.22) is of this type with

$$a = r e^U$$

(3.4)

and

$$A = e^{-4U} f, \quad B = \frac{1}{(1 + r U')^2} f, \quad U' = \frac{dU}{dr},$$

(3.5)

where $A$ and $B$ are related by

$$A = \left( \frac{3}{W} \right)^2 \frac{1}{B}$$

(3.6)

by virtue of the first equation of (2.19).

We take the induced metric $\gamma_{\mu\nu}$ on the brane to have the form of a standard FRW metric with cosmic scale factor $a(\tau)$ [3],

$$ds_4^2 = -d\tau^2 + \gamma_{ij} dx^i dx^j = -d\tau^2 + a^2(\tau) d\Sigma_k^2.$$  

(3.7)

Comparing (3.7) with (3.3) shows that the induced metric is obtained from the bulk metric by requiring

$$-A \left( \frac{dt}{d\tau} \right)^2 + B \left( \frac{da}{d\tau} \right)^2 = -1.$$  

(3.8)

This results in

$$\frac{dt}{d\tau} = A^{-1} \sqrt{A + AB \dot{a}^2}, \quad \dot{a} = \frac{da}{d\tau}.$$  

(3.9)

To compute the associated vector $n^M$, we follow [31] and introduce the velocity vector $v^M$ ($v^M v_M = -1$),

$$v^M = \frac{dx^M}{d\tau} = \left( \frac{dt}{d\tau}, \dot{a}, 0 \right).$$  

(3.10)

Using

$$v^M n_M = 0,$$  

(3.11)
we find that the unit normal vector \( n^M \) is given by

\[
n^M = \pm \frac{1}{\sqrt{AB}} (B \dot{a}, \sqrt{A + AB \dot{a}^2}, 0) .
\]  

(3.12)

Below we will see that we have to take the minus sign for consistency.

Next, proceeding as in [62], we vary the combined action (2.1) and (3.1) with respect to the metric \( g_{MN} \). Setting the variation to zero results in the equations [63, 29, 62]

\[
K_{\mu \nu} - (K + g W) \gamma_{\mu \nu} = 0
\]

(3.13)

and

\[
R_{\mu \nu} - \frac{1}{2} R \gamma_{\mu \nu} = 8\pi G_4 T_{\mu \nu}^{\text{matter}} .
\]

(3.14)

Here we have split the brane equation of motion into two equations, where the first one (3.13) is given in terms of the extrinsic curvature, while the second one (3.14) is given in terms of the Ricci tensor on the brane \( \Sigma \). This splitting can be motivated by noting that the five- and four-dimensional Newton’s constants are related by [6]

\[
G_5 = \frac{1}{2} G_4 L ,
\]

(3.15)

as can be seen from (3.1) (with \( L = g^{-1} \)). Then, the terms in (3.13) stem from those terms in the action (3.1) that are proportional to \( G_4^{-1} \) and hence intrinsically four-dimensional, whereas the terms in (3.13) come from terms in (3.1) that are multiplied by powers of \( G_5 \) and \( L \) in such a way that these factors do not combine into powers of \( G_4 \) only. Thus, \( W \) only contributes to (3.13), and there is no induced cosmological constant on the brane (cf. (3.28)) [3]. We note that in order for the two equations (3.13) and (3.14) to be consistent with one another, we have to supplement the action (3.1) with terms describing the non-gravitational degrees of freedom on the brane. This results in the presence of an energy-momentum tensor \( T_{\mu \nu}^{\text{matter}} \) in (3.14) that is homogeneous and isotropic, i.e. \( T_{\mu \nu}^{\text{matter}} = \text{diag}(-\rho_{\text{eff}}, p_{\text{eff}}, p_{\text{eff}}, p_{\text{eff}}) \), and that is conserved,

\[
\frac{d\rho_{\text{eff}}}{d\tau} = -3H (\rho_{\text{eff}} + p_{\text{eff}}) ,
\]

(3.16)

where

\[
H = \frac{\dot{a}}{a}
\]

(3.17)

denotes the Hubble parameter (\( H \) should not be confused with the \( H_A \) given in (2.21)).

Tracing (3.13) yields

\[
K_{\mu \nu} = -\frac{\Theta}{3} W \gamma_{\mu \nu} ,
\]

(3.18)

which we now evaluate. We first compute the \( ij \)-component of \( K_{\mu \nu} \). Using the definition (3.2) as well as (3.12) we obtain

\[
K_{ij} = \frac{1}{2} n^a \partial_a \gamma_{ij} = a^{-1} n^a \gamma_{ij} = \pm \frac{\sqrt{A + AB \dot{a}^2}}{\sqrt{AB a}} \gamma_{ij} ,
\]

(3.19)

and hence we get from (3.18),

\[
\pm \frac{\sqrt{A + AB \dot{a}^2}}{\sqrt{AB a}} = -\frac{\Theta}{3} W .
\]

(3.20)
Since the right hand side is negative, we take the minus sign in (3.12). Then, squaring (3.20) results in
\[ H^2 = \left( \frac{9}{3} W \right)^2 - \frac{1}{B a^2}. \] (3.21)
This is the Friedmann equation describing the dynamics of the scale factor \( a(\tau) \).

Next, we compute the \( \tau\tau \)-component of (3.18) using the method of [64, 62], which we now review. We express \( K_{\tau \tau} \) as
\[ K_{\tau \tau} = K_{MN} v^M \nabla_M v^N = -n_N A^N, \] (3.22)
where we used (3.11) and where \( A^N = v^M \nabla_M v^N \). Since \( v^N v_N = -1 \) we have \( v_N A^N = 0 \), and hence \( A^N \) is proportional to the normal \( n_N \), i.e. \( A^N = \tilde{A} n_N \). To compute \( \tilde{A} \) we use the fact that the black hole metric (2.22) has a timelike Killing vector \( l = l^M \partial_M = \partial_t \).

We compute \( \partial_t (l_M v^M) = v^N \nabla_N (l_M v^M) = l_N A^N = l_N n^N \tilde{A} \), where we used the Killing equation \( \nabla_M l_N = -\nabla_N l_M \) once. Hence [64, 62]
\[ K_{\tau \tau} = \frac{\partial_t (l_M v^M)}{l^N n_N} = -\frac{\partial_t v_t}{n_t}. \] (3.23)
Thus, the \( \tau\tau \)-component of (3.18) yields
\[ \partial_t v_t = \frac{9}{3} W n_t. \] (3.24)
Using (3.6), (3.9) and (3.21), we find that (3.24) results in
\[ \dot{H} + H^2 = \left( \frac{9}{3} W \right)^2 \left( 1 + \frac{a}{W} \frac{\partial W}{\partial a} \right) + \frac{1}{2B^2 a} \frac{\partial B}{\partial a}, \] (3.25)
where \( \dot{H} = dH/d\tau \). This equation is precisely the \( \tau \)-derivative of (3.21).

Finally, we compute \( K_{\tau i} = K_{M i} v^M \) using (3.2) and find \( K_{\tau i} = 0 \), which is consistent with (3.18). Thus, we conclude that the equations (3.18) consistently reduce to the Friedmann equation (3.21) and its \( \tau \)-derivative.

The Friedmann equation (3.21) can be rewritten in terms of the black hole data \( M \) and \( Q_A \phi^A \) as follows. Using (3.4) as well as (2.61) we obtain
\[ e^{2U} = \frac{W}{3 a^2 - q_A \phi^A a^2}. \] (3.26)
Using (3.20), \( Q_A \phi^A \) given in (2.58) becomes
\[ Q_A \phi^A = \frac{2}{3 w_5} \left( -3 k \frac{q_A X_A}{W} + k \frac{(q_A X_A)}{W a^2} + \frac{q_A X_A}{a^2} + k q_A h^A \right). \] (3.27)
Inserting (3.5), (2.19) and (3.26) into (3.21) and using (3.27) yields
\[ H^2 = -k \frac{W W}{3 a^2} + \frac{w_5 W}{3 a^4} M - \frac{w_5 W}{6 a^4} Q_A \phi^A. \] (3.28)
It is instructive to check whether \( H^2 \geq 0 \) along the motion of the brane in the AdS black hole background. Let us therefore consider the black hole background (2.44) for concreteness.
We set $\mu = q = 1$ (in appropriate units). The existence of a horizon then requires taking $g < 1$, as shown in figure 1. Picking the value $g^2 = 0.30$ we find that the horizon is at $r_h = 0.64$ and that $H^2$ vanishes at $r = 1$. In figure 2 $H^2$ is plotted over $r$ using (3.28). We find that $H^2 \geq 0$ in the outside region between the horizon and the turning point at $r = 1$. The brane thus expands until it reaches its maximal radius at $r = 1$ after which it recontacts and falls through the horizon.

3.2 Dual description on the brane

The AdS black hole provides a dual description of the radiation CFT on the brane at finite temperature [2]. Here the mass $M$, the Hawking temperature $T_H$ and the entropy $S$ of the black hole are related to the energy $E$ of the radiation on the brane, to its temperature $T$ and to its entropy $S$. The relation makes use of a conversion factor which is determined by the asymptotic behaviour of $dt/d\tau = A^{-1/2}$ [3]. For the line element (2.22) it is given by $(g_0)^{-1}$. In the charged case, the black hole and field theory data are thus related as in table [2] [3], where $Q_A \phi_h^A$ denotes the horizon value of (2.38) and where we recall that $L = g^{-1}$. 

![Figure 1: $r_h^2$ over $g^2$. The physical range is $g < 1$.](image1)

![Figure 2: $H^2$ over $r$. The physical range is $r \leq 1$.](image2)
The spatial volume of the brane is given by

$$V = a^3 \text{vol} ,$$

(3.29)

with \( \text{vol} \) described below (2.39). The energy density \( \rho \) of the radiation is

$$\rho = \frac{E}{V}$$

(3.30)

and its pressure satisfies \( p = \frac{1}{3} \rho \). The energy \( E \) is not a purely extensive quantity. It contains a sub-extensive part called the Casimir energy defined by \( (3.31) \)

$$E_c = 3 \left( E + p V - T S - Q A \hat{\Phi}^A \right) .$$

(3.31)

Here we have defined \( E_c \) in terms of \( \hat{\Phi}^A \) rather than \( \Phi^A \). The Casimir energy \( (3.31) \) denotes the violation of the thermodynamic Euler relation. The Euler relation for a system based on the first law of thermodynamics \( dE = T dS - p dV + \hat{\Phi}^A dQ_A \) states that if the energy \( E(S, V, Q) \) is extensive, i.e. if it satisfies \( E(\lambda S, \lambda V, \lambda Q) = \lambda E(S, V, Q) \), then the energy takes the form \( E = T S - p V + Q A \hat{\Phi}^A \). This relation is derived by differentiating once with respect to \( \lambda \), then setting \( \lambda = 1 \) and subsequently using the first law of thermodynamics.

The gravitational counterpart of the quantity \( E_e \) and of the Casimir energy \( E_c \) is given in (2.63) and (2.59), respectively. Using table 1, we obtain

$$E_e = \frac{1}{g a} \tilde{E}_e = \frac{1}{4\pi} \left( \frac{2 G_4}{\text{vol}} \right)^{1/3} g^{2/3} W_h \frac{S^{4/3}}{a} ,$$

$$E_c = \frac{1}{g a} \tilde{E}_c = \frac{3k}{2\pi} \left( \frac{\text{vol}}{2 G_4} \right)^{1/3} \tilde{W}_h \frac{S^{2/3}}{a} ,$$

(3.32)

where we employed (3.15) to rewrite (2.39) in terms of \( G_4 \). Observe that \( W_h \) and \( \tilde{W}_h \) do not have a simple scaling behaviour under \( a \rightarrow \lambda^{1/3} a, S \rightarrow \lambda S, Q \rightarrow \lambda Q \).

Then, it follows that the entropy relation (2.64) can be written as

$$k W_h \tilde{W}_h S^2 = \frac{8 \pi^2}{3} a^2 E_e E_c = \frac{4 \pi^2}{3} a^2 E_c \left( 2 E - Q A \hat{\Phi}^A - E_c \right)$$

(3.33)

in terms of field theory data on the brane. This is of the type of a Cardy-Verlinde formula (the higher-dimensional analogue of Cardy’s formula \( S = 2\pi \sqrt{L_0 - \frac{c}{24}} \) for the entropy of a two-dimensional CFT \( [7] \)).

| AdS | CFT |
|-----|-----|
| \( M \) | \( E = M L/a \) |
| \( T_H \) | \( T = T_H L/a \) |
| \( Q_A \phi_h^A \) | \( Q_A \hat{\Phi}^A = Q_A \phi_h^A L/a \) |
| \( Q_A \phi^A \) | \( Q_A \Phi^A = Q_A \phi^A L/a \) |
| \( S \) | \( S \) |

Table 1: Relation between black hole and field theory data.
In [1] it was proposed that the entropy $S$ of a CFT with an AdS-dual description satisfies the bound $S \leq S_B$, where $S_B = \frac{2\pi}{3} a E$ denotes the Bekenstein entropy [15]. To analyze whether this also holds for the expression (3.33) is not straightforward due to the presence of the factor $W_h \tilde{W}_h$, which satisfies the relation (2.65). For the specific STU models (2.44) and (2.45), however, it is possible to check that the bound holds. For these models we have $3W \tilde{W} = 5 + 2H_1^{-1} + 2H_1 = 9 + 2q^2/(r^4 H_1) > 9$, since $H_1 > 0$. It follows that $W_h \tilde{W}_h > 3$. From (2.44) and (2.45) we see that the quantity $Q_A \Phi^A$ is positive. Therefore it follows that $S < \frac{2\pi}{3} a \sqrt{E_c (2E - E_c)}$, which has a maximum at $E = E_c$ for a given energy $E$ [1], and hence the bound $S \leq S_B$ is satisfied for these two models.

Now we write the Friedmann equation (3.28) in terms of field theory data (cf. table 1). Introducing the charge density

$$\rho_A = \frac{Q_A}{V}$$

(3.34)

we obtain

$$H^2 = -k \frac{W \tilde{W}}{3 a^2} + 8\frac{\pi G_4}{9} W \left( \rho - \frac{1}{2} \rho_A \Phi^A \right),$$

(3.35)

where we used the relation (3.15) to express the Friedmann equation in terms of four-dimensional quantities. Observe that for any model (2.2), $W$, $\tilde{W}$ and $\Phi^A$ are generically complicated functions of $a$. Defining

$$\rho_{\text{eff}} = \frac{W}{3} \rho - \frac{W}{6} \rho_A \Phi^A + k \frac{(3 - W \tilde{W})}{8\pi G_4 a^2},$$

(3.36)

yields the Friedmann equation in the usual form,

$$H^2 = -\frac{k}{a^2} + \frac{8\pi G_4}{3} \rho_{\text{eff}}.$$

(3.37)

This is the form following from (3.14) with a suitably chosen $T_{\mu\nu}^{\text{matter}}$.

Next, we differentiate the Friedmann equation (3.35) with respect to proper time $\tau$ to obtain

$$\dot{H} = \frac{dH}{d\tau} = k \frac{W \tilde{W}}{3 a^2} - \frac{4\pi G_4 W}{3} (\rho + p) - \frac{k}{6 a} \frac{d}{da} \left( W \tilde{W} \right) + \frac{4\pi G_4 a}{9} \left( \frac{dW}{da} \rho - \frac{1}{2} \frac{d}{da} (W \rho_A \Phi^A) \right),$$

(3.38)

where we used that radiation satisfies the equation

$$\frac{d\rho}{d\tau} = -3H (\rho + p).$$

(3.39)

The equation for $\dot{H}$ can be written in the standard FRW form following from (3.14),

$$\dot{H} = \frac{k}{a^2} - 4\pi G_4 \left( \rho_{\text{eff}} + p_{\text{eff}} \right),$$

(3.40)

with $\rho_{\text{eff}}$ given by (3.36) and with $p_{\text{eff}}$ given by

$$p_{\text{eff}} = \frac{W}{3} \rho + \frac{W}{6} \rho_A \Phi^A + k \frac{1}{24\pi G_4 a^2} \left( a \frac{d}{da} \left( W \tilde{W} \right) + W \tilde{W} - 3 \right) - \frac{a}{9} \left( \frac{dW}{da} \rho - \frac{1}{2} \frac{d}{da} (W \rho_A \Phi^A) \right).$$

(3.41)
Observe that $T_{\mu\nu}^{\text{matter}}$ is, in general, not traceless,

$$ T_{\mu\nu}^{\text{matter}} = -\rho_{\text{eff}} + 3p_{\text{eff}} = \frac{2}{3} W \rho_A \Phi^A + k \frac{(W\tilde{W} - 3)}{4\pi G_4 a^2} + \frac{k}{8\pi G_4 a} \frac{d}{da} \left( W\tilde{W} \right)$$

$$-\frac{a}{3} \left( \frac{dW}{da} \rho - \frac{1}{2} \frac{d}{da} \left( W \rho_A \Phi^A \right) \right). \quad (3.42)$$

In general, the energy density $\rho_{\text{eff}}$ is not of the standard form. Therefore we now check that $\rho_{\text{eff}} > 0$ in the example considered before describing the motion of the brane in the black hole background (2.44) with the parameters $\mu, q$ set to the values $\mu = q = 1$. We find that $\rho_{\text{eff}} > 0$ for $r > 0$, as shown in figure 3. The behaviour of the pressure $p_{\text{eff}}$ for the same values of the parameters is displayed in figure 4. It is positive throughout. We also find that outside of the horizon, the trace of the energy-momentum tensor $a^4 T_{\mu\nu}^{\text{matter}}$ only becomes vanishing asymptotically. This is displayed in figure 5.

### 3.3 Correspondence between the first FRW equation and the entropy

The Friedmann equation (3.35), when written as

$$k W\tilde{W} \left( \frac{9 V^2}{4 G_4 a^2} \right) = \frac{4\pi^2}{3} a^2 \left( k \frac{3 V \tilde{W}}{4\pi G_4 a^2} \right)^2 \left( 2E - Q_A \Phi^A - k \frac{3 V \tilde{W}}{4\pi G_4 a^2} \right), \quad (3.43)$$

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has a structure that is similar to that of the Cardy-Verlinde-type formula (3.33). When the brane crosses the event horizon, both equations coincide. Namely, at the horizon where $f(r_h) = 0$, (3.21) yields

$$H_h^2 = \left( \frac{6}{3} W_h \right)^2,$$

which can be used to express $L = g^{-1}$ in terms of $H_h$ and $W_h$. Then, inserting this relation into the expression for the entropy (2.48) gives

$$S^2 = \frac{9 V_h^2}{4 G_4^2 W_h^2} H_h^2,$$

where $V_h$ denotes the volume (3.29) evaluated at the horizon, and where we used the relation (3.15). On the other hand, at the horizon the Casimir energy $E_c$ can be written as

$$E_{c,h} = k \frac{3 V_h}{4 \pi G_4} \frac{W_h}{a_h^2},$$

by virtue of (3.45). Inserting the expressions (3.45) and (3.46) into (3.43) then yields the Cardy-Verlinde-type formula (3.33).

### 3.4 Correspondence between the second FRW equation and the Casimir energy

Now we rewrite the second FRW equation (3.38) in order to exhibit its similarity with equation (3.31) for the Casimir energy $E_c$. Using the second equation of (2.14) and (3.26) we obtain

$$\tilde{W} = e^{-2U} + \frac{q_A h^A}{3 a^2},$$

$$W \tilde{W} = 3 - \frac{q_A X^A}{a^2} + \frac{W}{3} \frac{q_A h^A}{a^2},$$

while using (2.35) and (3.26) we get

$$\frac{d}{da} \left( \rho_A \Phi^A \right) = -\frac{6}{a} \rho_A \Phi^A + \frac{1}{4 \pi G_4 a^5} \left( 2k q_A h^A - 6k q_A X^A \frac{W}{a^2} \right) \left( k - \frac{\mu}{a^2} e^{2U} \right) e^{-2U} a q_A \frac{dX^A}{da}.$$
Inserting (3.47) and (3.48) into (3.38) and rearranging the terms we find that the second FRW equation (3.38) can be written as

$$k \frac{3 V W}{4 \pi G_4 a^2} = 3 \left( E + p V - Q_A \phi^A + \frac{3 V}{4 \pi G_4 a^4 W} \left( \frac{1}{9 a} \left( k - \frac{\mu}{a^2} e^{2U} \right) e^{-4U} W \frac{dW}{da} + \dot{H} \right) \right)$$

(3.49)

by virtue of the relation $H_A dX^A / da = 0$, which holds due to the second equation of (2.14) and the very special geometry relation (2.5).

Now let us consider the Casimir energy $E_c$. Following [3], we first relate the brane temperature $T$ to $\dot{H}$. Using the first equation in (2.19), the Hawking temperature (2.47) can be written as

$$T_H = \frac{1}{4 \pi} \left[ \frac{d f}{da} e^{-4U} W \right]_h.$$

(3.50)

Then, taking the $\tau$-derivative of (3.21) and using that at the horizon $f(r_h) = 0$ as well as (3.44), we obtain for the temperature on the brane,

$$T = T_H \frac{L}{a} = \frac{a_h}{2 \pi a} \left[ \frac{|H|}{W} a \frac{dW}{da} - \frac{\dot{H}}{|H|} \right]_h.$$

(3.51)

Inserting the expressions (3.45), (3.46) and (3.51) into the defining relation (3.31) yields the equation for the Casimir energy in the form

$$k \frac{3 V h \dot{W}_h}{4 \pi G_4 a_h a} = 3 \left( E + p V - Q_A \phi^A + \frac{3 V a_h}{4 \pi G_4 W h a} \left[ \frac{H^2}{W} a \frac{dW}{da} + \dot{H} \right]_h \right).$$

(3.52)

Comparing (3.49) with (3.52) shows that both equations have a similar structure. They coincide at the event horizon of the black hole since

$$\left[ \frac{H^2}{W} a \frac{dW}{da} \right]_h = \frac{1}{9 a} \left( k - \frac{\mu}{a^2} e^{2U} \right) e^{-4U} W \frac{dW}{da} \bigg|_h,$$

(3.53)

where we used (3.44) as well as $f(r_h) = 0$.

4 Final comments

It would be interesting to extend the analysis at the two-derivative level given above and to include higher-curvature terms such as the Gauss-Bonnet combination with a scalar-field dependent coupling function. The presence of these terms leads to further sub-extensive contributions that modify the form of the Cardy-Verlinde-type formula, the Casimir energy and the FRW equations.

A simpler example, on which we will now comment, is provided by a static AdS-Schwarzschild black hole modified by the presence of a Gauss-Bonnet interaction term. The black hole solution is known in closed form [65, 66, 67], and the Friedmann equation for a spherical brane moving in this black hole background has been derived in [68, 69, 70]. The bulk action is given by

$$S = \frac{1}{16 \pi G_5} \int d^5 x \sqrt{-g} \left[ R - 2 \Lambda + \alpha \left( R^2 - 4 R^{MN} R_{MN} + R^{MNPQ} R_{MNPQ} \right) \right],$$

(4.1)
where $\Lambda = -6g^2$, which corresponds to setting $W = 3$ in (2.9). In the context of string theory, $\alpha$ is proportional to $\alpha'$. The associated black hole solution with a spherical horizon has the line element [65, 66, 67] (we use the notation of [70])

$$ds_5^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_3^2,$$  

$$f = 1 + \frac{r^2}{4\alpha} \left( 1 - \sqrt{1 + 8\alpha \left( \frac{g^2}{r^4} - g^2 \right)} \right), \quad (4.2)$$

where $d\Omega_3^2$ denotes the line element of a unit three-sphere. The horizon of the black hole is located at $f(r_h) = 0$, which yields

$$\mu = g^2 r_h^4 + r_h^2 + 2\alpha. \quad (4.3)$$

The mass of the black hole is $M = \mu/w_5$. The Gauss-Bonnet corrected entropy reads [66, 71]

$$S = \frac{4\pi}{3w_5} \left( r_h^3 + 12\alpha r_h \right), \quad (4.4)$$

and the Hawking temperature $T_H$ is given by

$$T_H = \frac{1}{4\pi} f'(r_h) = \frac{1}{2\pi r_h} \left( 1 + 2 \frac{g^2 r_h^4}{r_h^2 + 4\alpha} \right). \quad (4.5)$$

The relation (4.4) can be inverted to express the horizon radius $r_h$ in terms of the entropy,

$$r_h(s) = \frac{-8\alpha + 2^{1/3} \left( s + \sqrt{s^2 + 256\alpha^3} \right)^{2/3}} {2^{2/3} \left( s + \sqrt{s^2 + 256\alpha^3} \right)^{1/3}}, \quad (4.6)$$

where

$$s = \frac{3w_5}{4\pi} S. \quad (4.7)$$

Using (4.3) and (4.6) the mass $M$ can be expressed as a power series in $s$. At quadratic order in $\alpha$, this gives

$$M = \frac{1}{w_5} \left( g^2 s^{4/3} + (1 - 16\alpha g^2) s^{2/3} - 6\alpha (1 - 16\alpha g^2) + 16\alpha^2 s^{-2/3} \right), \quad (4.8)$$

which shows that the Gauss-Bonnet term induces an infinite series of sub-extensive corrections expressed in powers of $\alpha s^{-2/3}$.

As it was the case at the two-derivative level, the bulk action (4.1) needs to be supplemented by both boundary terms [72] and counterterms. The latter are given by [70]

$$-\frac{1}{8\pi G_5} \int d^3 x \sqrt{-\gamma} \left( c_1 + \frac{c_2}{2} R \right), \quad (4.9)$$

where

$$c_1 = \frac{1 + 8\alpha g^2 - \sqrt{1 - 8\alpha g^2}} {2\sqrt{\alpha}} \frac{2\sqrt{1 - \sqrt{1 - 8\alpha g^2}}}{\sqrt{3 - 8\alpha g^2 - 3\sqrt{1 - 8\alpha g^2}}} \left( 1 - \sqrt{1 - 8\alpha g^2} \right)^{3/2}, \quad (4.10)$$

$$c_2 = \frac{\sqrt{\alpha} \left( 3 - 8\alpha g^2 - 3\sqrt{1 - 8\alpha g^2} \right)} {\left( 1 - \sqrt{1 - 8\alpha g^2} \right)^{3/2}},$$
Inspection of (4.9) shows that the five- and four-dimensional Newton’s constants are related by

\[ G_5 = c_2 G_4. \]  

(4.11)

In the limit \( \alpha \to 0 \) one recovers both the counterterms in (3.1) and (3.15).

The equation of motion for the brane moving in the black hole background (4.2) is expressed in terms of both the extrinsic curvature tensor \( K_{MN} \) and the Riemann tensor on the brane [70]. As in the two-derivative case (see (3.13) and (3.14)), we separate the term proportional to the Einstein tensor on the brane from the other terms. We take the induced metric on the brane to have the form of a standard FRW metric (3.7) with scale factor \( a(\tau) = r(\tau) \) and Hubble parameter \( H = \dot{r}/r \), where \( \dot{r} = dr/d\tau \). The resulting Friedmann equation takes the form [68, 69]

\[ \left( H^2 + \frac{f}{r^2} \right) \left( 3 + 8\alpha \left( H^2 + \frac{1}{r^2} \right) + 4\alpha \left( 1 - \frac{f}{r^2} \right) \right)^2 = c_2^2. \]  

(4.12)

It can be checked that there is no induced cosmological constant on the brane. Indeed, dropping all the terms that involve powers of \( \mu/r^4 \) and of \( H^2 + 1/r^2 \) in (4.12) yields a perfect cancellation of all the remaining terms.

At quadratic order in \( \alpha \), (4.12) gives

\[ H^2 = -\frac{1}{r^2} + (1 - 4\alpha g^2 + 8\alpha^2 g^4) \frac{\mu}{r^4} - 2\alpha \left( 1 - 20\alpha g^2 \right) \frac{\mu^2}{r^8} + 8\alpha^2 \frac{\mu^3}{r^{12}}. \]  

(4.13)

The energy \( E \) on the brane is related to the mass \( M \) by a conversion factor which is determined by the asymptotic behaviour of \( dt/d\tau = f^{-1/2} \) [3]. For the line element (4.2) this yields the conversion factor \( (g r)^{-1}(1 - \alpha g^2 - 5\alpha^2 g^4) \) at quadratic order in \( \alpha \). Using this as well as (4.11) the Friedmann equation (4.13) can be expressed in terms of the energy density \( \rho \) (3.30) on the brane,

\[ H^2 = -\frac{1}{r^2} + \frac{8\pi G_4}{3} \rho - 2\alpha \left( 1 - 16\alpha g^2 \right) \left( \frac{8\pi G_4}{3} \rho \right)^2 + 8\alpha^2 \left( \frac{8\pi G_4}{3} \rho \right)^3. \]  

(4.14)

Observe that the coefficient of the term linear in \( \rho \) does not receive \( \alpha \)-corrections. It would be interesting to show that the equations for the entropy on the brane (the Cardy-Verlinde formula and the equation for the Casimir energy) have a structure that is similar to (4.14) and its \( \tau \)-derivative.

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