A NOTE ON A COMMUTATIVE DIAGRAM CONSISTING OF LOW TERM EXACT SEQUENCES

CHANG LV

Abstract. We establish a useful commutative diagram consisting of low term exact sequences of Grothendieck spectral sequences, which generalize the previous ones appeared in Skorobogatov [4] and [6].

1. Introduction

Consider the

2. The commutative diagram

Suppose that \( \Phi : A \to B \) and \( \Psi_t : B \to C \) are left exact additive functors between abelian categories, \( t = 1, 2, 3 \). Assume that \( A \) and \( B \) has enough injectives and \( \Psi_t \) takes injectives to \( \Phi \)-acyclics. Then for any \( A \in \text{Ob}(A) \), we have Grothendieck spectral sequence

\[
(S)_{\Phi, \Psi_t, A} \quad tE_2^{p,q} = (R^p\Psi)(R^q\Phi)A \Rightarrow R^{p+q}(\Psi\Phi)A
\]

and the low term exact sequence

\[
(E)_{\Phi, \Psi_t, A} \quad 0 \to tE_2^{1,0} \to tE_1 \to tE_2^{0,1} \to tE_2^{2,0} \to tE_0 \to tE_2^{1,1} \to tE_2^{3,0}
\]

attached to them, where \( tE_2^i = \ker(tE^2 \to tE_2^{i,0}) \). Let \( D^+(A) \), \( D^+(B) \), \( D^+(C) \) be the derived category of complexes bounded below, \( R(\Psi), R(\Psi_t) \) the corresponding derived functor and where \( R^\bullet \) stands for the hypercohomology functor.

Proposition 2.1. With the previous notation, suppose that there are morphism of functors \( u : \Psi_1 \to \Psi_2 \) and \( v : \Psi_2 \to \Psi_3 \) such that for any \( F \in D^+(B) \),

\[
(2.2) \quad R\Psi_1(F) \xrightarrow{Ru(F)} R\Psi_2(F) \xrightarrow{Rv(F)} R\Psi_3(F) \xrightarrow{R\Psi_1(F)[1]}
\]

is a distinguished triangle factorial in \( F \).

(i) We have the long exact sequence

\[
\ldots \to tE_2^{i,0} \to tE_2^{i,1} \xrightarrow{tE_2^{i+1,0}} \to \ldots
\]

where \( tE_2^{i,i} = R^i\Psi_t(\tau_{\leq i} R\Phi(A)) \).

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(ii) We have the following commutative diagram with exact rows and columns

\[
\begin{array}{cccccccc}
1E_2^{1.0} & \to & 1E_1^1 & \to & 1E_2^{0.1} & \to & 1E_2^{2.0} & \to & 1E_2^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
2E_2^{1.0} & \to & 2E_1^1 & \to & 2E_2^{0.1} & \to & 2E_2^{2.0} & \to & 2E_2^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
3E_2^{1.0} & \to & 3E_1^1 & \to & 3E_2^{0.1} & \to & 3E_2^{2.0} & \to & 3E_2^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1E_2^{2.0} & \to & 1E_2^{1.1} & \to & 1E_2^{3.0} & \to & 1E_2^3 & \to & 1E_{\leq 1}^3 \\
\end{array}
\]

where \(1E_{\leq 1}^3\) fits into the exact sequence

\[
0 \to 1E_2^2 \to 1E_2^1 \to 1E_2^{0.2} \to 1E_{\leq 1}^3 \to 1E_3,
\]

the rows are parts of the low term exact sequences \((E)_{\Phi, \Psi, \tau}^t\) with \(t\) numbered on the left upper corner of each object, and the columns are induced by taking cohomology at 1 of (2.2) in which \(F\) is substituted with \(\tau_0 R\Phi(A), \tau_{\leq 1} R\Phi(A), \tau_{[0]} R\Phi(A), \tau_{[0]} R\Phi(A)[1], \tau_{\leq 1} R\Phi(A)[1]\), respectively.

(iii) Suppose we are given \(\beta \in 3E_2^1\) and \(\gamma \in 2E_2^{0.1}\) such that they map to the same element in \(3E_2^{0.1}\). Then there exists \(\alpha \in 1E_2^{2.0}\) such that \(\alpha, \beta\) map to the same element in \(1E_2^3\) and \(-\alpha, \gamma\) map to the same element in \(2E_2^{2.0}\). In other words, we have the diagram

\[
\begin{array}{cccc}
\alpha & \to & b & \to & 1E_2^{2.0} \\
\downarrow & & \downarrow & & \downarrow \\
\beta & \to & 2E_2^{0.1} & \to & 2E_2^{2.0} \\
\downarrow & & \downarrow & & \downarrow \\
\gamma & \to & 3E_1^1 & \to & 3E_2^{0.1} \\
\end{array}
\]

(iv) The statement of (iii) is also correct if we move our focus one step right. That is, we are given \(\beta \in 3E_2^{0.1}\) and \(\gamma \in 2E_2^{2.0}\), and so on.

**Proof.** The proof is a generalization of [1] Lem. 3 and [2] Prop. 1.1. For any \(F \in D^+(A)\), the truncation functors determine the distinguished triangle

\[
\tau_{\leq 0} F \to F \to \tau_{\geq 1} F \to (\tau_{\leq 0} F)[1].
\]

Note that \(R\Psi_t, t = 1, 2, 3\) are triangulated. Along with the factorial distinguished triangles (2.2) in which \(F\) is substituted with \(\tau_{\leq 0} F, F\) and \(\tau_{\geq 1} F\) respectively, we obtain the following commutative
Let $F = \tau_{\leq 1} R \Phi(A) = \tau_{0,1} R \Phi(A)$ and take cohomology at 1, the diagram become

(2.7)

$R^i \Psi_1(\tau_{\leq 1} R \Phi(A)) \rightarrow R^i \Psi_1(\tau_{\leq 1} R \Phi(A)) \rightarrow R^i \Psi_1(\tau_{\leq 1} R \Phi(A)) \rightarrow R^i \Psi_1(\tau_{\leq 1} R \Phi(A)) \rightarrow \cdots$

with exact rows and columns.

We now identify the objects appearing in (2.7) with the ones in (2.8). Clearly for any $t = 1, 2, 3$ and $i, j, k \in \mathbb{Z}$ with $j, i - j + k \geq 0$, we have

$R^i \Psi_1(\tau_{\leq 1} R \Phi(A))[k] = (R^i - j + k \Psi)(R^i \Phi)(A) = tE_{i-j+k}^j.$

It remains to identify $R^i \Psi_1(\tau_{\leq 1} R \Phi(A))$ with $E^1_1$ and $R^i \Psi_1(\tau_{\leq 1} R \Phi(A))[1]$ with $E^2_1$.

For any $F$ consider the distinguished triangle

$\tau_{\leq 1} F \to F \to \tau_{\geq 2} F \to (\tau_{\leq 1} F)[1]$}

and note that $\Psi_1(\tau_{\geq 2} F)$ is acyclic in 0 and 1. Then we have the long exact sequences

$R^0 \Psi_1(\tau_{\geq 2} F) \to R^1 \Psi_1(\tau_{\leq 1} F) \to R^1 \Psi_1(F) \to R^1 \Psi_1(\tau_{\geq 2} F)$

$\rightarrow R^2 \Psi_1(\tau_{\leq 1} F) \to R^2 \Psi_1(F) \to R^2 \Psi_1(\tau_{\geq 2} F)$

$\rightarrow R^3 \Psi_1(\tau_{\leq 1} F) \to R^3 \Psi_1(F)$

(2.8)}

where $R^i \Psi_1(\tau_{\geq 2} F) = 0$ with $t = 1, 2, 3$ and $i = 0, 1$. Now we take $F = R \Phi(A)$. It follows that

$R^1 \Psi_1(\tau_{\leq 1} R \Phi(A)) = R^1 \Psi_1(R \Phi(A)) = tE^1_1$

where the last equality follows from the isomorphism of functors

$R^i \Psi_1(R \Phi) \cong R^i \Psi \Phi$.
In a same manner, (2.8) and (2.9) yield
\[ R^1\Psi_t(\tau_{\leq 1}R\Phi(A))[1] = \text{ker}(R^2\Psi_t(R\Phi(A)) \to R^2\Psi_t(\tau_{\geq 2}R\Phi(A))) = \text{ker}(R^2(\Psi_t\Phi)A \to R^2\Psi_t(\tau_{\geq 2}R\Phi(A))). \]

Then the low term exact sequence attached to the hypercohomology spectral sequence [3, Appendix C (g)]
\[ E^{p,q}_2 = R^pg(H^q(F)) \Rightarrow R^{p+q}g(F) \]
gives the isomorphism when \( F = \tau_{\geq 2}R\Phi(A) \) and \( g = \Psi_t \)
\[ R^2\Psi_t(\tau_{\geq 2}R\Phi(A)) \cong \text{ker}(R^2(\Psi_t\Phi)A \to R^2\Psi_t(\tau_{\geq 2}R\Phi(A))). \]

Similarly,
\[ R^1\Psi_t(\tau_{\leq 1}R\Phi(A))[1] = \text{ker}(R^2(\Psi_t\Phi)A \to \Psi_t(R^2\Phi(A))) = tE^2_1. \]
and the exact sequence (2.4) is also deduced from (2.8) and (2.9). This completes the identification of objects.

Finally, in diagram (2.7), the identification of the vertical arrows is clear. For that of the horizontal ones, it follows a general fact for such spectral sequences. See, for example, [4, Appendix B], which shows that \( R^1\Psi_t(\tau_{\leq 1}R\Phi(A)) \to R^1\Psi_t(R^2\Phi(A)) = tE^2_1 \).

Consider the subdiagram of (2.6)
\[
\begin{array}{cccccc}
R\Psi_2(F) & \to & R\Psi_2(\tau_{\geq 1}F) & \to & R\Psi_2(\tau_{\leq 0}F)[1] & \\
\downarrow & & \downarrow & & \downarrow & \\
R\Psi_3(F) & \to & R\Psi_3(\tau_{\geq 1}F) & \to & R\Psi_3(\tau_{\leq 0}F)[1] & \\
\downarrow & & \downarrow & & \downarrow & \\
R\Psi_1(F)[1] & \to & R\Psi_1(\tau_{\geq 1}F)[1] & \to & R\Psi_1(\tau_{\leq 0}F)[2] & \\
\downarrow & & \downarrow & & \downarrow & \\
& & & & & \\
\end{array}
\]
whose rows and columns are all distinguished triangles. Since up to an isomorphism, every distinguished triangle in a derived category arises from some a short exact sequence of complexes [2, Chap. IV.2 8. Prop.], we may view (2.10) as a commutative diagram consisting of three rows and three columns of short exact sequences of complexes. Then the result follows from [3 Lem. 4.3.2] by taking \( i = 1 \). This completes the proof of (iii) as well as (i).

(iii) is the same as (ii). The proof is complete. \( \square \)

Next we describe a variant of Proposition 2.1.

**Proposition 2.11.** Keeping assumptions in Proposition 2.11, suppose that there are \( A \in D^+(A) \) and \( B \in D^+(B) \) with a morphism
\[ f : B \to \tau_{\leq 1}R\Phi(A). \]
Let $\Delta = \Delta(\Phi, A, B, f)$ be the cone of $-f[1]$. Denote $^1F^p = \mathbb{R}^p\Psi_t B$ and $^1G^p = \mathbb{R}^p\Psi_t \Delta$. Then we have the long exact sequence

$$\ldots \to F^i \to \ ^1E_{i\leq 1} \to G^{i-1} \to F^{i+1} \to \ldots$$

where $^1E_{i\leq 1} = \mathbb{R}^1\Psi_t(\tau_{\leq 1}\Phi(A))$ and the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
1F^1 & \longrightarrow & 1E^1 & \longrightarrow & 1G^0 & \longrightarrow & 1F^2 & \longrightarrow & 1E^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
2F^1 & \longrightarrow & 2E^1 & \longrightarrow & 2G^0 & \longrightarrow & 2F^2 & \longrightarrow & 2E^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
3F^1 & \longrightarrow & 3E^1 & \longrightarrow & 3G^0 & \longrightarrow & 3F^2 & \longrightarrow & 3E^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1F^2 & \longrightarrow & 1E^2 & \longrightarrow & 1G^1 & \longrightarrow & 1F^3 & \longrightarrow & 1E^3 \\
\end{array}
\]

where $^1E_{3\leq 1}^3$ fits into the exact sequence

$$0 \to \ ^1E_1^3 \to ^1E_2^3 \to ^1E_2^{0,3} \to ^1E_{3\leq 1}^3 \to ^1E_3^3.$$  

Moreover, similar statements as (iii), (iv) in Proposition 2.1 hold. That is, if we are given $\beta \in ^3E_1^1$ (resp. $^3G^0$) and $\gamma \in ^2G^0$ (resp. $^2F^2$), then the corresponding diagram as in Proposition 2.1 is correct.

**Proof.** The same as Proposition 2.1 except that in the diagram (2.7) we replace the distinguished triangle

\[
\tau_{[0]}\Phi(A) \to \tau_{\leq 1}\Phi(A) \to \tau_{[1]}\Phi(A) \to \tau_{[0]}\Phi(A)[1]
\]

by

$$B \to \tau_{\leq 1}\Phi(A) \to \Delta[-1] \to B[1].$$

□

**Remark 2.14.** (a) Obviously, Proposition 2.1 is the special case of Proposition 2.11 where $f$ is the canonical map $\tau_{[0]}\Phi(A) \to \tau_{\leq 1}\Phi(A)$.

(b) If we replace the distinguished triangle (2.13) by

$$\tau_{[0]}\Phi(A) \to \Phi(A) \to \tau_{\geq 1}\Phi(A) \to \tau_{[0]}\Phi(A)[1]$$

we also have the long exact sequence

$$\ldots \to \ ^iE_{2,0}^i \to \ ^iE^i \to \ ^iE_{1,0}^i \to \ ^iE_{2,0}^{i+1} \to \ldots$$

where $^iE_{2,0}^i = \mathbb{R}^i\Psi_t(\tau_{\geq 1}\Phi(A))$.  

3. Applications

Suppose that \( f_* : \mathcal{B} \to \mathcal{A} \) is a left exact additive functor between two abelian categories which has a left adjoint \( f^* \). Assume that \( \mathcal{A} \) and \( \mathcal{B} \) has enough injectives and \( f^* \) is exact. Let \( M \in \text{Ob}(\mathcal{A}) \) and \( N \in \text{Ob}(\mathcal{B}) \). Then \( (S)_{d, \Psi, \psi, A} \) will be

\[
M^p q = \text{Ext}_A^p(M, R^q f_* N) = \text{Ext}_B^{p+q}(f^* M, N).
\]

For simplicity, we omit the category letter in Ext’s if it does not cause a confusion.

**Corollary 3.2.** Let \( C, A \in \text{Ob}(\mathcal{A}) \) and \( u \in \text{Ext}^1(C, A) \) be the element representing the extension

\[
0 \longrightarrow A \overset{i}{\longrightarrow} B \overset{j}{\longrightarrow} C \longrightarrow 0.
\]

Define

\[
\text{Ext}_1^2(f^* M, N) = \ker(\text{Ext}^2(f^* M, N) \to \text{Hom}(M, R^2 f_* N)).
\]

(i) We have the following commutative diagram with exact rows and columns

\[
\begin{array}{cccccccc}
\text{Ext}^1(C, f_* N) & \longrightarrow & \text{Ext}^1(f^* C, N) & \longrightarrow & \text{Hom}(C, R^1 f_* N) & \longrightarrow & \text{Ext}^2(C, f_* N) & \longrightarrow & \text{Ext}^2(f^* C, N) \\
 \downarrow j^* & & \downarrow f^*(j^*) & & \downarrow j^* & & \downarrow j^* & & \downarrow f^*(j^*) \\
\text{Ext}^1(B, f_* N) & \longrightarrow & \text{Ext}^1(f^* B, N) & \longrightarrow & \text{Hom}(B, R^1 f_* N) & \longrightarrow & \text{Ext}^2(B, f_* N) & \longrightarrow & \text{Ext}^2(f^* B, N) \\
 \downarrow i^* & & \downarrow f^*(i^*) & & \downarrow i^* & & \downarrow i^* & & \downarrow f^*(i^*) \\
\text{Ext}^1(A, f_* N) & \longrightarrow & \text{Ext}^1(f^* A, N) & \longrightarrow & \text{Hom}(A, R^1 f_* N) & \longrightarrow & \text{Ext}^2(A, f_* N) & \longrightarrow & \text{Ext}^2(f^* A, N) \\
 \downarrow u \cup j & & \downarrow f^*(u \cup j) & & \downarrow u \cup j & & \downarrow u \cup j & & \downarrow f^*(u \cup j) \\
\text{Ext}^2(C, f_* N) & \longrightarrow & \text{Ext}^2(f^* C, N) & \longrightarrow & \text{Ext}^1(C, R^1 f_* N) & \longrightarrow & \text{Ext}^3(C, f_* N) \\
\end{array}
\]

where the rows are parts of the low term exact sequences attached to \( A E^p q \) and \( C E^p q \) defined in (3.1).

(ii) The statement of Proposition 2.1 (iii) (resp. (iv)) is also correct if we put \( \beta, \gamma \) in the corresponding positions. That is, we are given \( \beta \in \text{Ext}^1(f^* A, N) \) (resp. \( \text{Hom}(A, R^1 f_* N) \)) and \( \gamma \in \text{Hom}(A, R^1 f_* N) \) (resp. \( \text{Ext}^2(B, f_* N) \)), and so on.

**Proof.** We shall use Proposition 2.1. Let

\[
A \to B \to C \to A[1]
\]

be the distinguished triangle in \( \mathcal{D}^+(\mathcal{A}) \) determined by (5.3). Take \( \Psi_1 = \text{Hom}(C, -) \), \( \Psi_2 = \text{Hom}(B, -) \), \( \Psi_3 = \text{Hom}(A, -) \) and \( \Phi = f_* \), which clearly satisfy the assumptions in Proposition 2.1. See [7], Thm. 10.7.4. Then the result follows.

Let \( k \) be a field with characteristic 0 and \( \Gamma = \text{Gal}(\overline{k}/k) \) where \( \overline{k} \) is a fixed algebraic closure of \( k \). Let \( p : X \to \text{Spec} k \) be a \( k \)-variety and \( \overline{X} = X \times_k \overline{k} \). In Corollary 4.2, take \( \mathcal{A} \) be the category of discrete \( \Gamma \)-modules, \( \mathcal{B} \) the category of étale sheaves on \( X \). We write \( \text{Ext}^1_k \) for \( \text{Ext}_A, \text{Ext}_X \) for \( \text{Ext}_B \) and

\[
\text{Ext}^1_k(p^* T, G_m) = \ker(\text{Ext}^1_X(p^* T, G_m) \to \text{Hom}_k(T, \text{Br} \overline{X})) \quad \text{for } T = M \text{ or } S.
\]

Note that \( \text{Ext}^1_k(\mathbb{Z}, -) = H^1(k, -) \).
Theorem 3.4. With previous notation, let \( u \in H^1(k, M) \) be the element representing the extension

\[
0 \rightarrow M \xrightarrow{i} S \xrightarrow{j} Z \rightarrow 0.
\]

(i) We have the following commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
\text{Br}_1 X & \xrightarrow{j^*} & \text{Ext}_1^2(S, k[X]^\times) \\
\downarrow & & \downarrow p^* \\
\text{Hom}_k(M, \text{Pic}X) & \xrightarrow{\beta} & \text{Ext}_1^2(M, k[X]^\times) \\
\downarrow u \cup - & & \downarrow u \cup - \\
\text{Br}_1 X & \xrightarrow{r} & H^1(k, \text{Pic}X)
\end{array}
\]

where the top, middle and bottom row is a part of the low term exact sequences attached to

\[
\text{Ext}^n(M, R^q p_\ast G_m) \Rightarrow \text{Ext}^{n+q}(p^\ast M, G_m),
\]

\[
\text{Ext}^n(S, R^q p_\ast G_m) \Rightarrow \text{Ext}^{n+q}(p^\ast S, G_m)
\]

and

\[
H^n(k, H^q(X, G_m)) \Rightarrow H^{n+q}(X, G_m)
\]

respectively.

(ii) Let \( \beta \in \text{Hom}_k(M, \text{Pic}X) \) be such that \( u \cup \beta \in \text{im} r \). Then there exists \( \alpha \in \text{Br}_1 X \) and \( \gamma \in \text{Ext}_1^2(S, k[X]^\times) \) such that \( r(\alpha) = u \cup \beta \), \( i^* (\gamma) = \partial(\beta) \) and \( j^* (-\alpha) = p^* (\gamma) \). In other words, we have the diagram

\[
\begin{array}{ccc}
\gamma & \xrightarrow{-\alpha} & \text{Br}_1 X \\
\downarrow & & \downarrow p^*(\gamma) \\
\text{Ext}_1^2(S, k[X]^\times) & \xrightarrow{\partial(\beta)} & \text{Ext}_1^2(p^\ast S, G_m) \\
\downarrow & & \downarrow \\
\text{Hom}(M, \text{Pic}X) & \xrightarrow{\beta} & \text{Ext}_1^2(M, k[X]^\times) \\
\downarrow & & \downarrow \\
\text{Br}_1 X & \xrightarrow{r(\alpha)} & H^1(k, \text{Pic}X)
\end{array}
\]
Proof. (i) In Corollary $3.2$ take $f_\ast = p_\ast$, $N = G_m$ and $\mathfrak{X}$ to be $\mathfrak{X}$, Then we obtain

\[
\begin{array}{c}
\Ext^2_1(p^\ast Z, G_m) \\
\Ext^2_k(S, p_\ast G_m) \xrightarrow{p^\ast} \Ext^2_1(p^\ast S, G_m) \\
\Hom_k(M, R^1p_\ast G_m) \xrightarrow{\partial} \Ext^2_k(M, p_\ast G_m) \xrightarrow{\beta} \Ext^2_1(p^\ast M, G_m) \\
\Ext^2_1(p^\ast Z, G_m) \xrightarrow{r} H^1(k, R^1p_\ast G_m) \xrightarrow{d} H^3(k, G_m)
\end{array}
\]

Then (i) follows from the facts that for $p, q \geq 0$, $R^q p_\ast G_m = H^q(X, G_m)$, and $H^p(X, G_m) = \Ext^p_k(p^\ast Z, G_m)$, for which see [5, p. 23] and [1, Prop. 1.4.1], respectively.

(ii) The existence of $y$ follows from an easy diagram chase. Since $u \cup \beta \in \text{im} \ Br_1 X$,

\[u \cup \partial(\beta) = d(u \cup \beta) = 0.\]

Then there exists $\gamma \in \Ext^2_k(S, \mathbb{X}[X]^{\times})$ such that $i^\ast(\gamma) = d(\beta)$. The result follows from Corollary $3.2$.  

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State Key Laboratory of Information Security, Institute of Information Engineering, Chinese Academy of Sciences, Beijing 100093, P.R. China

Email address: lvchang@amss.ac.cn