Reconciling Conductance Fluctuations and the Scaling Theory of Localization

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We reconcile the phenomenon of mesoscopic conductance fluctuations with the single parameter scaling theory of the Anderson transition. We calculate three averages of the conductance distribution: \[ \exp(\langle \ln g \rangle), \langle g \rangle \text{ and } 1/\langle R \rangle \] where \( g \) is the conductance in units of \( e^2/h \) and \( R = 1/g \) is the resistance and demonstrate that these quantities obey single parameter scaling laws. We obtain consistent estimates of the critical exponent from the scaling of all these quantities.

In the original proposal of the scaling theory of localization the conductance \( g \) (in units of \( e^2/h \)) is the relevant parameter [1]. The scaling of this parameter is deduced by looking at the limiting cases of a good metal, where weak localization theory is applicable, and the strongly localized regime where an exponential dependence on system size is expected. A smooth monotonic interpolation is supposed between the two limits. The central equation of the theory is the \( \beta \) function

\[
\beta(g) \equiv \frac{d \ln g}{d \ln L} \tag{1}
\]

which describes how the conductance of an \( L \times L \times L \) cube is renormalized with system size \( L \).

The scaling theory has been very influential. It is the basis for the predictions that the Anderson transition is continuous and that the lower critical dimension for the Anderson transition is two.

Shortly after its proposal, the pioneering numerical calculations of Pichard [2], and MacKinnon and Kramer [3] provided indirect support for the scaling theory. In these calculations the localization length of electrons on quasi-one dimensional bars was calculated. The theory of finite size scaling was then applied to deduce the critical parameters from the dependence of the localization length on the transverse dimension of the bars. This has proved useful for making quantitative estimates of critical parameters [2,3].

It became clear later, in particular after the investigation of universal conductance fluctuations in mesoscopic systems, that the conductance of a disordered system is a random variable [1,11]. In the vicinity of the Anderson transition the fluctuations in \( g \) are of the same order of magnitude as it’s mean value [11]. The smooth scaling behavior predicted by [1] is clearly inconsistent with the fluctuating behavior of the conductance which occurs in practice and casts strong doubts over the soundness of the scaling theory [1]. There seem to be two principal remedies which we discuss in turn [2,12,13].

First, we could attempt to establish that the distribution of conductance \( p_L(g) \) (in the limit that the system size \( L \) and the correlation length \( \xi \) are much longer than any microscopic lengths) obeys a single parameter scaling law. The precise meaning of this statement is that it should be possible to parameterize the bulk of the conductance distribution with a single parameter. If we denote this parameter by \( X \) then the bulk of the conductance distribution must be of the form

\[
p_L(g) \simeq F(g; X(L)) \tag{2}
\]

and the parameter \( X \) must obey a single parameter scaling law

\[
\frac{d \ln X(L)}{d \ln L} = \beta(X) \tag{3}
\]

Note that \( X \) need not necessarily be one of the moments of the distribution. Indeed, single parameter scaling of a distribution does not necessarily imply scaling of its moments. The moments may be dominated by non-universal tails of the distribution or might not even exist. We shall refer to this first possibility as strong single parameter scaling.

Second, we could attempt to establish that some typical or average conductance obeys a single parameter scaling law. This is a somewhat weaker statement since one parameter scaling of some average or typical quantity does not imply single parameter scaling of the corresponding distribution. For example, it might happen that several independent parameters are needed to describe the distribution. For this reason we shall refer to this as weak single parameter scaling.
Our purpose in this paper is to firmly establish single parameter scaling in the weaker sense given above, for systems near the critical point of the Anderson transition in three dimensions. A secondary objective is to identify which averages or typical values obey single parameter scaling laws. By simulating the conductance distribution for large ensembles of disordered $L \times L \times L$ cubes we have achieved both objectives.

We supposed perfect leads were attached to a pair of opposite sides of each cube and used the Landauer formula to relate the (Landauer) conductance $g_L$ to the transmission matrix $t$ which describes the transmission of electrons from one lead to the other:

$$g_L = 2 \text{tr } tt^\dagger$$  \hspace{1cm} (4)

We use the notation $g_L$ to emphasize that this is the conductance that would be measured in a two probe measuring geometry. In the original work on the scaling theory the Thouless conductance $g = E_C/\Delta$ was considered where $E_C$ is the Thouless energy and $\Delta$ the mean energy level spacing. The Landauer conductance and the Thouless conductance are not completely equivalent. Their relation has been considered in detail by Braun et al [15] who state that the contact resistance, which is always present in a two terminal measurement, should be subtracted from $g_L$. Hence we study the statistics of

$$1/g = 1/g_L - 1/2N$$  \hspace{1cm} (5)

Here $N \equiv N(E_F)$ is the number of propagating channels in the contacts at Fermi energy $E_F$ and $1/2N$ is the contact resistance appropriate for the situation we have simulated.

The motion of the electrons in the system is described by the Anderson model

$$H = V \sum_{<i,j>} C_i^\dagger C_j + \sum_i W_i C_i^\dagger C_i,$$  \hspace{1cm} (6)

where $C_i^\dagger (C_i)$ denotes the creation (annihilation) operator of an electron at site $i$ of a 3D cubic lattice. The amplitude of the random potential at site $i$ is $W_i$. Hopping is restricted to nearest neighbors and its amplitude is taken as the unit of energy, $V = 1$. We supposed a box distribution with each $W_i$ uniformly distributed on the interval $[-W/2, W/2]$. Previous work has verified the universality of the critical behavior in this model with respect to the choice of distribution of the random potential [14][16][22]. We imposed fixed boundary conditions in the transverse directions since we have found in previous work that corrections to scaling vanish more quickly with system size in this case [14][16]. The Hamiltonian [1] has both time reversal and spin rotation symmetries so that the observed critical behavior should be that of the orthogonal universality class. We used the method of Pendry et al [18] to calculate the transmission matrix which appears in [10]. We set the Fermi energy $E_F = 0.5V$ and for each combination of disorder $W$ and system size $L$ we generated an ensemble of 1,000,000 samples (except for $L = 16$ where 500,000 were generated). This allowed us to estimate the various averages of the conductance to roughly an accuracy of 0.1%.

We examined the behavior of three different averages of the conductance distribution: $\exp(\langle \ln g \rangle)$, $\langle g \rangle$ and $1/(R)$ where $R = 1/g$ is the resistance. Note that for each average it is possible to define a different $\beta$ function. For example, for the typical conductance $\exp(\langle \ln g \rangle)$

$$\beta(\exp(\langle \ln g \rangle)) \equiv \frac{d \langle \ln g \rangle}{d \ln L},$$  \hspace{1cm} (7)

In the critical region the conductance fluctuations are of the same order of magnitude as the mean conductance so that these averages are not at all equivalent.

We fitted the disorder and system size dependence of the averages to the standard scaling form. Taking the typical conductance as an example we supposed that

$$\langle \ln g \rangle = F_0(\psi^{1/\nu}, \phi L^y),$$  \hspace{1cm} (8)

where $\psi$ is a relevant scaling variable and $\phi$ is an irrelevant scaling variable which allows us to take account of corrections to scaling. Such deviations from perfect scaling are always present in a simulation of a finite system and it is necessary to have some means of accounting for them. We approximated this scaling function by its first order expansion in the irrelevant scaling variable and fitted the numerical data to the form

$$\langle \ln g \rangle = F_0(\psi^{1/\nu}) + \phi L^y F_1(\psi^{1/\nu}).$$  \hspace{1cm} (9)

The scaling variables were approximated by expansions in terms of the dimensionless disorder $w = (W_c - W)/W_c$ where $W_c$ is the critical disorder separating the insulating and metallic phases.

$$\psi = \psi_1 w + \psi_2 w^2, \quad \phi = \phi_0.$$  \hspace{1cm} (10)

The critical exponent $\nu$ describes the divergence of the localization (correlation) near the critical point

$$\xi = \xi_0 |\psi|^{-\nu}.$$  \hspace{1cm} (11)

The absolute scale of the localization length $\xi_0$ cannot be determined from this fit. The system size dependence of the irrelevant scaling variable is described by an exponent $y < 0$. The functions $F_0$ and $F_1$ were expanded to third order in $w$. We found that this fitting scheme was the simplest which still allowed for goodness of fit probabilities in excess of 0.1.

The results of the analysis are displayed in Table 1 and Figures 1[16][22]. A number of points can be noticed. First,
acceptable fits to the single parameter scaling law are obtained for all three averages. Second, mutually consistent values for the critical exponent are obtained and these estimates are also consistent with estimates on the scaling of the localization length of electrons on bars \[.\] Third, mutually consistent values of the critical disorder are obtained. Finally, the critical values of the quantities considered vary widely, indicative of the breadth of the distribution of the conductance fluctuations in the critical regime. This means that the precise form of the \( \beta \) function is different for each average, though all should have the same slope \((= 1/\nu)\) at their respective zeros \[12\]. Note that the \( \beta \) function will also depend on the boundary conditions since it is known that critical conductance distribution depends strongly on the boundary conditions even in the limit that \( L \to \infty \).

These results firmly establish that the typical conductance, the mean conductance and the mean resistance all obey single parameter scaling in the critical regime.

To what extent do these results also support scaling in the strong sense? Let us consider the vicinity of the metallic, insulating and critical fixed points in turn.

The metallic fixed point can be reached by taking the limit \( L \to \infty \) with \( \xi \) fixed from any starting point on the metallic side of the transition. In this limit \( p^*_L(g) \) approaches a normal distribution with a size independent variance \[12\]. Therefore a single parameter, \( \langle g \rangle \), is sufficient to parameterize the distribution. Further it can be established that \( \langle g \rangle \) obeys a single parameter scaling law using weak localization theory near the metallic fixed point \[1\].

The insulating fixed point can be reached by taking the limit \( L \to \infty \) with \( \xi \) fixed from any starting point on the metallic side of the transition. In this limit \( p^*_L(g) \) approaches a normal distribution with a size independent variance \[12\]. Therefore a single parameter, \( \langle g \rangle \), is sufficient to parameterize the distribution. Further it can be established that \( \langle g \rangle \) obeys a single parameter scaling law using weak localization theory near the metallic fixed point \[1\].

At the critical fixed point \( \xi \) diverges and the single parameter scaling hypothesis predicts a scale independent universal critical conductance distribution \( p_c(g) \). This has been confirmed in numerical simulations \[3,1\] \[2\] \[3\] \[24\].

Thus, it seems likely that the conductance distribution also obeys single parameter scaling in the strong, as well as the weak, senses. However, the available results are not completely conclusive. It seems to us that a demonstration that \( p^*_L(g) \) obeys \[2\] \[3\] in the limit \( L \to \infty \) and \( \xi \to \infty \) with \( L/\xi \) fixed for any value of the ratio \( L/\xi \) is required.

We would like to thank the ISSP and the Slovak Academy of Sciences for the use of their computer facilities. PM would like to thank the Japan Society for the Promotion of Science and Sophia University for their hospitality and VEGA for financial support.

| \( X \) | \( \exp(\langle \ln g \rangle) \) | \( \langle g \rangle \) | \( 1/\langle R \rangle \) |
|---|---|---|---|
| \( \nu \) | 1.57(56,58) | 1.58(57,60) | 1.54(53,56) |
| \( W_c \) | 16.48(47,49) | 16.47(45,48) | 16.49(48,50) |
| \( X_c \) | 0.291(290,293) | 0.573(570,576) | 0.100(099,101) |
| \( Q \) | 0.5 | 0.4 | 0.3 |

**TABLE I.** Results of the scaling analysis for each average \( X \). Estimates of the critical exponent \( \nu \) and the critical disorder \( W_c \) together with 95% confidence intervals are given. \( X_C \) is the value of the relevant statistic at the critical point. The number of data values was 199 and the number of parameters in each fit was 12. The goodness of fit probability \( Q \) is also given.
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FIG. 1. The logarithm of the typical conductance versus the amplitude of the potential fluctuations for system sizes \( L = 4, 6, 8, 10, 12, 14 \) and 16. The solid lines are the fit to the data.

FIG. 2. The same data as in Fig. 1 after corrections to scaling are subtracted and plotted versus \( L/\xi \) to exhibit the single parameter scaling function.
FIG. 3. The same data as in Figure 1 but plotted versus system size.

FIG. 4. The $\beta$ function determined from the fit to the data in Fig. 1.