Toward the optimization of electronic refrigerators

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Abstract. Normal metal – insulator – superconductor (NIS) junction can act as refrigerator of electron gas in a normal metal electrode under certain conditions. If majority of high energy electrons in a normal metal is moved with a bias voltage to energy levels above the superconductor’s energy gap, their extraction from a normal metal is enabled by tunnelling. Their substitution with lower energy electrons results in lowering of the normal metal electron gas energy. Due to excess quasiparticle density in the superconductive electrode in vicinity of the junction, which is a result of tunnelling of quasiparticles from the normal metal and low capability of removal of these particles from the junction area the cooling capability of junctions deteriorates at lower environment temperatures. This is mainly the results of back tunnelling of excess quasiparticles to a normal metal and of dissipated energy, released by Cooper pair formation in proximity of the junction. Quantitative description of quasiparticle behavior in the superconductive electrode of the NIS junction is given. Energy dependent diffusion of quasiparticles in superconductor, their consumption due to Cooper pair creation and inflow of additional quasiparticles from a normal metal are considered. The model results in position dependence of quasiparticles’ density in the junction proximity.

1. Introduction
In the junction normal metal – insulator - superconductor (NIS junction) the normal metal electron energy levels can be voltage biased with respect to energy levels in the superconductor. This enables tunnelling of high energy electrons from the tail of the Fermi Dirac distribution in the normal metal to empty quasiparticle states in the superconductor (figure 1). With voltage bias satisfying $0 < U < \Delta/e$ the normal metal electrons with energies $E$, higher than Fermi energy $E_F$, can tunnel to the superconductor [1]. Here $\Delta$ is half of the superconductor energy gap and $e$ is unit charge. To enable cooling removed electrons have to be substituted with electrons with lower energies. This can be achieved by adding another superconductive contact to the normal metal electrode. The Andreev reflection of quasiparticles flowing from the superconductive contact to the normal metal at the SN junction of such SNIS junction takes place [2]. It enables occupying of empty electron states in the normal metal with electrons of an average energy equal to Fermi energy. The net energy loss in the electronic gas of the normal metal is in such case equal to $E-E_F$. The total amount of energy in the normal metal electron system is lower and consequently the electron temperature $T_e$ decreases.

In the temperature range of several 100 mK measurements show cooling of electrons as predicted by models. On the other hand, by additional lowering of the environmental temperature the NIS junction starts to behave like a heater: for all values of biasing voltage the electron temperature in a normal metal is higher than the environmental temperature $T_0$ [3]. In spite of that, the qualitative shape of the $T_e(U)$ curve remains similar to the theoretically predicted. These results anticipate the existence of additional, parasitic heat sources, which increase the normal metal electron temperature. Models
mainly describe three important contributions: subgap electric current, Cooper pair formation and back tunneling [4].

Subgap electric current extracts low energy electrons from the normal metal. The mentioned net energy difference \( E - E_F \) is in such case negative and reduces cooling capability of the junction. The effect is modeled with dynamic junction resistance [5].

On the other hand, both Cooper pair formation which dissipates the released energy in close proximity of junction and back tunneling depend on quasiparticle density near the junction. The aim of this work is to analyze quantitatively the position dependence of quasiparticle density in proximity of the junction. The suggested mechanisms which remove quasiparticles from the junction area are diffusion and Cooper pair creation. On the other hand, the increase in number of quasiparticles is due to tunnelling of quasiparticles from the normal metal.

\[ \text{Figure 1. Normal metal – insulator – superconductor junction.} \]

2. Model

In the model a description of quasiparticle density in the NIS junction proximity is given. The model is based on equilibrium of quasiparticle volume and energy density in a stationary state. This is described by the continuity equation [6] for quasiparticles in the superconductor in the immediate vicinity of the junction:

\[ \frac{\partial n}{\partial t} - \nabla \vec{j}(\vec{r}) = 0 \]

Here \( n \) is the number of quasiparticles per unit volume per unit energy and \( \vec{j}(\vec{r}) \) is the volume density of quasiparticles’ flow per unit energy.

We consider particles brought to the system by tunneling of quasiparticles from the normal metal electrode as a contribution to the increased density in the superconductive part of the junction. On the other hand, besides the flow of quasiparticles out of the junction region (equation (1)), the density of quasiparticles is decreased due to Cooper pair formation:

\[ \frac{\partial n}{\partial t} = \frac{\partial n_{\text{tunneling}}}{\partial t} - \frac{\partial n_{\text{CP}}}{\partial t} \]

2.1. Diffusion

Since in the superconductor in equilibrium there is no electric field, the quasiparticles’ flow is determined by the diffusion:
\[ \vec{j}(\vec{r}) = -D(E)\nabla n \]  

(3)

It can be shown that the diffusion constant in the superconductor \( D(E) \) is connected with the diffusion constant in the same material in non-superconductive state \( D_N(E) \) [7]:

\[ D(E) = \sqrt{1 - \frac{\Delta^2}{E^2}} D_N(E) \]  

(4)

The partial diffusivity in normal state for the electrons with specified energy is given in equation (5), with \( v \) representing velocity of quasiparticle and \( \tau_n \) representing the transport relaxation time, both for particles with energy \( E \).

\[ D_N(E) = \frac{1}{3} v^2 \tau_n \]  

(5)

After considering electron – electron scattering in the electron gas in the normal metal an estimation for the transport relaxation time can be given [8]:

\[ \tau_n \propto \frac{\hbar E_F}{(E - E_F)^2} \]  

(6)

From equations (5) and (6) the following estimation for the energy dependence of the diffusion constant in the superconductor in the normal state can be obtained:

\[ D_N(E) = \frac{2h}{3m} \frac{E_F}{E - E_F} \]  

(7)

The quasiparticles’ flow as follows from equations (3), (4) and (7) is:

\[ \vec{j}(\vec{r}) = -\frac{2h}{3m} \sqrt{1 - \frac{\Delta^2}{E^2}} \frac{E_F}{E - E_F} \nabla n \]  

(8)

2.2. Tunnelling contribution

The number of quasiparticles with specified energy \( E \) which tunnel from the normal metal to the superconductor is obtained from the semiconductor representation of quasiparticles’ flows. Tunnelling \( (j^a, j^b) \) and back tunnelling \( (j^{ba}, j^{bb}) \) flows’ components of quasiparticles above and below the gap in directions normal to the junction area are considered. The net flow is shown in equations (9) and (10), where with \( R_N \) the resistance of the junction in the normal state and with \( f_N(E) \) and \( f_S(E) \) Fermi-Dirac function at energy \( E \) in the normal metal and in the superconductor, respectively, are denoted.

\[ \frac{\partial n_{tunnel}}{\partial t} = j^a - j^b + j^{ba} - j^{bb} \]  

(9)

\[ \frac{\partial n_{tunnel}}{\partial t} = \frac{1}{R_N e^2 \sqrt{E^2 - \Delta^2}} \left( f_N(E + eU) + f_N(E - eU) - 2 f_S(E) \right) \]  

(10)

2.3. Cooper pairs creation

As for each Cooper pair two quasiparticles are needed the rate of quasiparticles which can condensate into Cooper pairs in time unit is proportional to the product of available quasiparticles at specified time:

\[ \frac{\partial N_{CP}}{\partial t} = -\Gamma_{CP} N_S (N_S - 1) \]  

(11)

Due to large number of quasiparticles the product \( N_S (N_S - 1) \) can be simplified into \( N_S^2 \). The rate \( \Gamma_{CP} \) is material parameter. It is obtained from various material constants [9]. Usually the value \( \Gamma_{CP} V_S \)
is determined as it is connected with quasiparticle density instead with number of quasiparticles [10]. Equation (11) can be rewritten into the form compliant with equation (2):

$$\frac{\partial n_{CP}}{\partial t} = -\Gamma_{CP} V_S n^2$$  \hspace{1cm} (12)

### 3. Quasiparticle density

Combining equations (1) and (2) with the derived results for diffusion (equation (8)), for the tunnelling contribution (equation (10)) and for Cooper pairs creation (equation (12)), we obtain equation (13) which is fundamental for calculating the number of quasiparticles per unit volume per unit energy:

$$-\frac{2\hbar}{3m} \sqrt{1 - \frac{\Delta^2}{E^2}} \frac{E_F}{E - E_F} \nabla^2 n(\vec{r}) - \Gamma_{CP} V_S \left(n(\vec{r}) + n_{th}\right)^2 + g(\vec{r}) \frac{1}{R_N e^2} \sqrt{E^2 - \Delta^2} \left(f_N (E + eU) + f_N (E - eU) - 2f_S (E)\right) = 0$$  \hspace{1cm} (13)

The total quasiparticle density is divided into two parts: with $n_{th}$ the equilibrium quasiparticle density at the environmental temperature is marked, whereas with $n(\vec{r})$ the excessive density at position $\vec{r}$ is denoted ($n = n_{th} + n(\vec{r})$). As direct injection of quasiparticles into the superconductor takes place only in the volume of superconductor which is in contact with the normal metal, a multiplier $g(\vec{r})$ is introduced. It describes the difference between the part of superconductive electrode near the NIS junction and the part where no direct quasiparticles injection takes place. In the contact region its value is $V_S^{-1}$, everywhere else it is equal to 0.

Solving of equation (13) is nontrivial, mainly because of the part of equation, describing diffusion. However, if a refrigerator is constructed with long and narrow electrodes (figure 2) the 3-d equation can be simplified into a 1-d one. The $n(\vec{r})$ dependence is simplified into $n(x)$ dependence, the gradient $\nabla n(\vec{r})$ is simplified into spatial derivative $n'(x) = \frac{dn}{dx}$.

![](figure2.png)

**Figure 2.** One dimensional junction.

Firstly, the position dependence of the quasiparticle density per unit energy is calculated for different quasiparticles’ energies in region from the edge of the energy gap toward higher energies. Following, results are integrated over the energy region used in the first part of calculation to obtain the position dependent volume density of quasiparticles in the superconductor.

For the calculation of position dependence of the quasiparticle volume density per unit energy adequate boundary conditions are taken into consideration. Due to quasiparticles’ reflection at $x=0$ first spatial derivative of density is equal to 0: $n'(x=0)=0$. At the position $x_{\text{junction}}$ the solutions and the first spatial derivatives for equation which includes quasiparticle tunneling ($g(x)$ is nonzero) and for equation without quasiparticles’ injection ($g(x)=0$) have to match: $n_{\text{tunneling}}(x=x_{\text{junction}}) = n_{\text{no tunneling}}(x=x_{\text{junction}})$ and $n'_{\text{tunneling}}(x=x_{\text{junction}}) = n'_{\text{no tunneling}}(x=x_{\text{junction}})$. A quasiparticle trap is added to the model to fulfill the last boundary condition (figure 2). Here it is set $n(x=x_{\text{trap}})=0$. The solutions are calculated by solving the differential equation by using the Runge Kutta method combined with shooting. The calculation is repeated for different energies starting at the above edge of the
superconductor energy gap and finishing with energies where contribution to the density \( n(x) \) is below the calculation accuracy.

After determining the position dependence of quasiparticle density at different energies, results have to be integrated over the energy. Discrete values for the quasiparticle density per unit energy are multiplied by the width of the energy interval between two calculations.

As an example the quasiparticle density has been calculated in a refrigerator consisting of silver, aluminum oxide and aluminum as a normal metal, insulator and superconductor, respectively [4]. Its geometrical and material constants are described in table 1.

| Table 1. Geometrical and material data of the junction. |
|---------------------------------|
| \( R_N \) | 77 \( \Omega \) |
| \( S_{\text{junction}} \) | \( 15 \times 10 \mu m^2 \) |
| \( V_{\text{superconductor}} \) | \( 350 \mu m^3 \) |
| \( \Delta \) | 183 \( \mu eV \) |
| \( E_F \) | 11.7 \( eV \) |
| \( eU/\Delta \) | 0.9 |
| \( T_0 \) | 210 mK |
| \( T_N \) | 304 mK |
| \( T_S \) | 302 mK |

The calculated results \( n(x) \) are shown at figure 3. The horizontal axis shows the position in the superconductive electrode, whereas on the vertical axis the volume density of quasiparticles is presented. Results, calculated as described above, are shown with diamonds. With equilibrium quasiparticle density at environmental temperature being approximately \( 3 \times 10^5 \mu m^{-3} \) [11] the quasiparticles which tunnel from the normal metal and do not diffuse or condensate into Cooper pairs and thus disappear from the junction region add about one third of excessive quasiparticle density. Such increase in quasiparticle density in the junction proximity leads to a high rate of back tunneling and Cooper pair formation and thus deteriorates the cooling capability of the junction.

![Figure 3. Position dependent volume density of quasiparticles in superconductor.](chart)

Additional information can be obtained from comparison of our calculated results with similar results, obtained from a simpler model with energy independent normal state diffusion constant. The later data is in figure 3 represented with the full squares. In the model with energy dependent normal state diffusion constant (equation 7) a value of diffusion constant is lowered with increasing quasiparticles’ energy. Quasiparticles with higher energies remain in the junction area longer than quasiparticles with energies close to the energy gap. However, as number of high energy quasiparticles
is relatively low due to their Fermi Dirac distribution the rise of quasiparticle density is low as well.
On the other hand the rise of quasiparticle density observed at the results for energy dependent diffusion constant could be important when considering energy balance of the junction. High density of high energy quasiparticles could lead to its deterioration.

In conclusion: the energy dependence of the diffusion constant adds substantial contribution to the excessive quasiparticle density. Although the calculation is subject to approximately 10% error as a result of numerical errors and limited calculation accuracy, it shows that the energy dependence of diffusion constant is not negligible.

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