Abstract. We study the Brieskorn modules associated to a germ of holomorphic function with non-isolated singularities, and show that the Brieskorn module has naturally a structure of a module over the ring of microdifferential operators of nonpositive degree, and that the kernel of the morphism to the Gauss-Manin system coincides with the torsion part for the action of $t$ and also with that for the action of the inverse of the Gauss-Manin connection. This torsion part is not finitely generated in general, and we give a sufficient condition for the finiteness. We also prove a Thom-Sebastiani type theorem for the sheaf of Brieskorn modules in the case one of two functions has an isolated singularity.

Introduction

Let $f$ be a nonconstant holomorphic function on a complex manifold $X$, and $x \in X_0 := f^{-1}(0)$. The notion of Brieskorn module associated to a Milnor fibration of $f$ around $x$ was introduced by Brieskorn [4] in the case of isolated hypersurface singularities in order to calculate the Milnor cohomology and the monodromy in an algebraic way. The generalization to the case of non-isolated hypersurface singularities was done by H. Hamm [10] (unpublished), see also [9]. He showed the coherence modulo torsion of the relative de Rham cohomology sheaf associated to the Milnor fibration (but the torsion part is not finitely generated in general).

To the Milnor fibration, one can also associate the Gauss-Manin system (see [16] for the isolated singularity case). This is by definition the direct image of the structure sheaf $\mathcal{O}_X$ as a $\mathcal{D}_X$-module under the Milnor fibration. It has been known that this Gauss-Manin system is always a coherent (or more precisely, holonomic) $\mathcal{D}$-module even in the non-isolated hypersurface singularity case according to M. Kashiwara. Furthermore, we can easily see that its stalk at the origin defines a constructible sheaf on $X_0$ when $x \in X_0$
varies. We have a natural morphism of the sheaf of the $i$-th Brieskorn modules $\mathcal{H}_i^f$ to the sheaf of the $i$-th Gauss-Manin systems $\mathcal{G}_i^f$ on $X_0$, and these are closely related to the vanishing cohomology sheaf of degree $i - 1$ on $X_0$. Using this morphism, we can get more precise information on the Brieskorn modules, because the Gauss-Manin systems are easier to describe.

Let $C\{\partial_t^{-1}\}$ be the ring of microdifferential operators $\sum_{j\geq 0} a_j \partial_t^{-j}$ with constant coefficients and nonpositive degree, satisfying the convergence condition $\sum_{j\geq 0} |a_j|r^j/j! < \infty$ for some $r > 0$, see [11], [16]. Then both $\mathcal{H}_i^f$ and $\mathcal{G}_i^f$ have naturally a structure of a sheaf of $C\{t\}$-modules and of $C\{\partial_t^{-1}\}$-modules on $X_0$. The latter structure for $\mathcal{H}_i^f$ uses the integration without constant term, which is similar to a canonical action of $\partial_t^{-1}$ on $C\{t\}$ via the indefinite integral from the origin, see the proof of Proposition (2.4). Let $V^\alpha$ denote the $V$-filtration of M. Kashiwara [12] and B. Malgrange [15] on $\mathcal{G}_i^f$ indexed by $Q$.

For $\alpha > -1$, it can be described by using the Deligne extension [5] in this case.

In this paper we show the following (see (2.3–5)):

**Theorem 1.** The $t$-torsion part of $\mathcal{H}_i^f$ coincide with the $\partial_t^{-1}$-torsion part, and also with the kernel of $\mathcal{H}_i^f \to \mathcal{G}_i^f$. Furthermore these are annihilated by $t^p$ and $\partial_t^{-p}$ for $p \gg 0$ locally on $X_0$. The stalks of the image of the morphism $\mathcal{H}_i^f \to \mathcal{G}_i^f$ are finite free modules over $C\{t\}$ and over $C\{\partial_t^{-1}\}$. The image is contained in $V^{>-1}\mathcal{G}_i^f$ and its rank coincides with that of the Milnor cohomology of degree $i - 1$ at each point of $X_0$.

This is a refinement of Hamm’s result [10] mentioned above (see also [9]), and is closely related to positivity (or negativity) theorems in [11], [14]. In general, the torsion part is not finitely generated, see Remark (3.5). However, using the theory of Kiehl-Verdier [13], we can show its finiteness if there is a stratification such that $f$ is locally trivial along its strata, see Theorem (3.3). This was shown in [17].

In the case dim Sing $f = 1$ and $f$ is locally trivial along Sing $f \setminus \{0\}$, it is possible to prove this finiteness using an elementary method, see [1]. There is a sufficient condition for the torsion-freeness of the Brieskorn module, and we have a formula for the dimension of the Milnor cohomology under this condition together with the above conditions on Sing $f$, see [1]. In the case dim $X = 2$ these conditions are always satisfied so that the formula is valid, see [1]. Furthermore, they are stable by replacing $f$ with a function $f + g$ on $X \times Y$ if $g$ is a holomorphic function with an isolated singularity on $Y$, see [1]. In the last situation we have also the Thom-Sebastiani type theorem ([19], [20]) for the sheaf of the reduced Brieskorn modules (corresponding to reduced cohomology), see Theorem (4.2) and also [1] for a special case.

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1. Gauss-Manin systems

1.1. Sheaf of Gauss-Manin systems. Let $f$ be a nonconstant holomorphic function on a complex manifold $X$. We may assume that the image of $f$ is contained in an open disk $S$ by restricting $X$ if necessary. Let $t$ be the coordinate of $S$, and put $\partial_t = \partial/\partial t$. Let
$i_f : X \to X \times S$ be the graph embedding by $f$. Let $\mathcal{B}_f$ be the direct image of $\mathcal{O}_X$ by $i_f$ as a left $\mathcal{D}_X$-module, see e.g. [11], [15], [16]. Then

$$\mathcal{B}_f = \mathcal{O}_X \otimes_C \mathbb{C}[\partial_t],$$

and the action of a vector field $\xi$ on $X$ is given by

$$(1.1.1) \quad \xi(\omega \otimes \partial^i_t) = \xi \omega \otimes \partial^i_t - (\xi_f)\omega \otimes \partial^{i+1}_t \quad \text{for } \omega \in \Omega^i_X.$$

Let

$$\mathcal{K}^\bullet_f = DR_{X \times S/S}(\mathcal{B}_f)[− \dim X].$$

It is a complex whose $i$-th term is $\Omega^i_X \otimes_C \mathbb{C}[\partial_t]$ and whose differential is given by

$$(1.1.2) \quad d(\omega \otimes \partial^i_t) = df \otimes \partial^i_t - df \wedge \omega \otimes \partial^{i+1}_t \quad \text{for } \omega \in \Omega^i_X.$$

Let $X_0 = f^{-1}(0)$, and define

$$\mathcal{G}^i_f = \mathcal{H}^i\mathcal{K}^\bullet_f|_{X_0},$$

where $\mathcal{H}^i$ denotes the cohomology sheaf of a sheaf complex. We call $\mathcal{G}^i_f$ the sheaf of Gauss-Manin systems associated to $f$. It has naturally a structure of $\mathcal{D}_{S,0}$-modules, where $\mathcal{D}_S$ is the sheaf of holomorphic linear differential operators on $S$, i.e. $\mathcal{D}_{S,0} = \mathcal{C}\{t\}(\partial_t)$.

For $i = 1$, we see that $\omega_0 := df \otimes 1 \in \mathcal{H}^1_f$ is annihilated by $\partial_t$ (i.e. $df \otimes \partial_t = 0$ in $\mathcal{H}^1_f$), considering the image of $1 \otimes 1$ by $d$. So $\mathcal{D}_{S,0}\omega_0 = \mathcal{C}\{t\}\omega_0$. This is a free $\mathcal{C}\{t\}$-module of rank 1 using $[\partial_t, t^i] = it^{i-1}$ inductively (or by the involutivity of the characteristic variety.)

We define the sheaf of reduced Gauss-Manin systems $\mathcal{\tilde{G}}^i_f$ by

$$(1.1.3) \quad \mathcal{\tilde{G}}^i_f = \mathcal{G}^i_f \text{ if } i \neq 1, \text{ and } \mathcal{G}^1_f/C\{t\}\omega_0 \text{ if } i = 1.$$

1.2. Nearby and vanishing cycles. We will denote by $\psi_f \mathcal{C}_X$ and $\varphi_f \mathcal{C}_X$ the nearby and vanishing cycle sheaves, see [7]. Their cohomology sheaves $\mathcal{H}^i\psi_f \mathcal{C}_X$ and $\mathcal{H}^i\varphi_f \mathcal{C}_X$ are constructible sheaves of $\mathcal{C}$-vector spaces. They have naturally the action of the monodromy $T$. By definition we have a distinguished triangle

$$(1.2.1) \quad \mathcal{C}_{X_0} \to \psi_f \mathcal{C}_X \xrightarrow{\text{can}} \varphi_f \mathcal{C}_X \xrightarrow{+1}$$

Let $F_x$ denote the Milnor fiber of $f$ around $x \in X_0$. Then there are isomorphisms compatible with the action of the monodromy $T$

$$(1.2.2) \quad (\mathcal{H}^i\psi_f \mathcal{C}_X)_x = H^i(F_x, \mathcal{C}), \quad (\mathcal{H}^i\varphi_f \mathcal{C}_X)_x = \tilde{H}^i(F_x, \mathcal{C}),$$

where $\tilde{H}^i(F_x, \mathcal{C})$ is the reduced cohomology.

1.3. Regular holonomic $\mathcal{D}$-modules in one variable. Let $M$ be a regular holonomic $\mathcal{D}_{S,0}$-module. For $\alpha \in \mathcal{C}$, let

$$M^\alpha = \bigcup_{i > 0} \ker((t\partial_t - \alpha)^i : M \to M).$$
We have the infinite direct sum decomposition

\[(1.3.1) \quad M = \hat{\bigoplus}_\alpha M^\alpha,\]

where \(\hat{\bigoplus}\) means the completion by an appropriate topology (similar to the completion \(\mathbb{C}\{t\}\) of \(\mathbb{C}[t]\) or \(\mathbb{C}\{\{\partial_t^{-1}\}\} \) of \(\mathbb{C}[\partial_t^{-1}]\)). For simplicity, we assume that the direct sum is indexed by \(\alpha \in \mathbb{Q}\), i.e. \(M\) has quasi-unipotent monodromy. We define the nearby and vanishing cycles by

\[\psi_t M = \bigoplus_{-1 < \alpha \leq 0} M^\alpha, \quad \varphi_t M = \bigoplus_{-1 < \alpha < 0} M^\alpha.\]

It has the action of \(T\) defined by \(\exp(-2\pi it\partial_t)\) on \(M^\alpha\). We have the decompositions \(\psi_t M = \psi_{t,1} M \oplus \psi_{t,\neq 1} M\), \(\varphi_t M = \varphi_{t,1} M \oplus \varphi_{t,\neq 1} M\) such that

\[\psi_{t,1} M = M^0, \quad \varphi_{t,1} M = M^{-1}, \quad \psi_{t,\neq 1} M = \varphi_{t,\neq 1} M = \bigoplus_{-1 < \alpha < 0} M^\alpha.\]

We have a morphism

\[(1.3.2) \quad \text{can} : \psi_t M \to \varphi_t M,\]

whose restriction to \(\psi_{t,1} M\) and \(\psi_{t,\neq 1} M\) is given respectively by \(\partial_t\) and the identity. Note that \(M\) is determined by \(\psi_t M, \varphi_t M\) together with the action of \(t, \partial_t\).

The filtration \(V\) of Kashiwara [12] and Malgrange [15] is given by

\[(1.3.3) \quad V^\alpha M = \hat{\bigoplus}_{\beta \geq \alpha} M^\beta, \quad V^{>\alpha} M = \hat{\bigoplus}_{\beta > \alpha} M^\beta,\]

and the above definition of \(\psi, \varphi\) is compatible with the usual one.

We say that a \(\mathbb{C}\{t\}\)-submodule \(L\) of \(M\) is a \textit{lattice} if it is finite over \(\mathbb{C}\{t\}\) and generates \(M\) over \(\mathcal{D}_{S,0}\). The first condition is equivalent to \(L \subset V^\alpha\) for some \(\alpha \in \mathbb{Q}\), and the second implies that \(L \supset V^\beta\) for some \(\beta \in \mathbb{Q}\). The equivalence holds for the last if the above morphism \(\text{can} : \psi_t M \to \varphi_t M\) is surjective; for example, if the action of \(\partial_t\) is bijective. In the last case, \(V^\alpha M\) and \(M\) are finite free modules over \(\mathbb{C}\{\{\partial_t^{-1}\}\}\) and \(\mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]\) respectively. (This can be reduced to the fact that \(\mathbb{C}\{t\} t^\alpha (\alpha > 0)\) is a free \(\{\{\partial_t^{-1}\}\}\)-module of rank 1 using a filtration.) We have the notion of lattice with \(\mathbb{C}\{t\}\) replaced by \(\mathbb{C}\{\{\partial_t^{-1}\}\}\), and \(\mathcal{D}_{S,0}\) by \(\mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]\) or by the ring of microdifferential operators, see [11], [16].

Shrinking \(S\) if necessary, we may assume that \(M\) is the restriction of a regular holonomic \(\mathcal{D}_S\)-module, which is also denoted by \(M\), and that the restriction of \(M\) to \(S \setminus \{0\}\) is coherent over \(\mathcal{O}_S\). Note that \(V^\alpha M\) for \(\alpha > -1\) is identified with the Deligne extension (see [5]) such that the eigenvalues of the residue of \(t\partial_t\) are contained in \([\alpha, \alpha + 1]\).

The de Rham complex of \(M\) is defined by

\[\text{DR}_S(M) = C(\partial_t : M \to M).\]

It is well-known that we have canonical isomorphisms compatible with the action of \(T\)

\[(1.3.4) \quad \psi_t \text{DR}_S(M)[-1] = \psi_t M, \quad \varphi_t \text{DR}_S(M)[-1] = \varphi_t M.\]
(This is a special case of [12], [15].) Indeed, the assertion for $\psi_t$ is quite classical (see [5] for a proof using a modern language), and the assertion for $\varphi_t$ follows from it using a distinguished triangle similar to (1.2.1).

Note that the argument in this subsection (1.3) can be applied to a constructible sheaf of regular holonomic $\mathcal{C}\{t\}\langle\partial_t\rangle$-modules on $X_0$, where $\mathcal{C}\{t\}\langle\partial_t\rangle = \mathcal{D}_{S,0}$. In this case, each direct factor $M^\alpha$ of the decomposition (1.3.1) is a constructible sheaf of $\mathbb{C}$-vector spaces on $X_0$.

1.4. Proposition. The sheaves $\mathcal{G}_j^i$, $\tilde{\mathcal{G}}_j^i$ are constructible sheaves of regular holonomic $\mathcal{D}_{S,0}$-modules, and there are canonical isomorphisms of constructible sheaves of $\mathbb{C}$-modules

\begin{equation}
\psi_t\mathcal{G}_j^{i+1} = \mathcal{H}^i\psi_f\mathcal{C}_X, \quad \varphi_t\mathcal{G}_j^{i+1} = \mathcal{H}^i\varphi_f\mathcal{C}_X,
\end{equation}

which are compatible with the action of $T$ and the morphism can. In particular, $\partial_t : \mathcal{G}_j^i \to \mathcal{G}_j^i$ is bijective for $i \neq 1$ and surjective for $i = 1$ so that the kernel is the constant sheaf $\mathcal{C}_{X_0}$. On the reduced Gauss-Manin system $\tilde{\mathcal{G}}_j^i$, the action of $\partial_t$ is bijective for any $i$.

Proof. If $X_0$ is a divisor with normal crossings, the assertion is well-known and easy to prove, see also [18]. Let $\pi : \tilde{X} \to X$ be an embedded resolution of $X_0$, i.e. the pull-back of $X_0$ is a divisor with simple normal crossings. Then $\mathcal{O}_X$ is a direct factor of the direct image $\pi_+\mathcal{O}_{\tilde{X}}$ of $\mathcal{O}_{\tilde{X}}$ as a $\mathcal{D}$-module, and the complement is supported on $X_0$. Indeed, we may assume that $\pi$ is an iteration of blowing-ups along smooth centers by Hironaka, and the assertion is easily verified for such morphisms. (This is a special case of the decomposition theorem [2] in the algebraic case.)

Let $f = f\pi$. Then the above arguments imply that $\mathcal{K}_f^\bullet$ is a direct factor of $R\pi_*\mathcal{K}_{\tilde{f}}^\bullet$ and the cohomology sheaves of its complement are $t$-torsion. Furthermore, $R\pi_*\mathcal{K}_{\tilde{f}}^\bullet|_{X_0}$ has constructible cohomology, using the spectral sequence

\begin{equation}
E_2^{p,q} = R^p\pi_* (\mathcal{H}^q\mathcal{K}_{\tilde{f}}^\bullet|_{X_0}) \Rightarrow R^{p+q}\pi_*\mathcal{K}_f^\bullet|_{X_0},
\end{equation}

which is compatible with the corresponding spectral sequences for the nearby and vanishing cycles. So $\mathcal{K}_f^\bullet|_{X_0}$ has constructible cohomology sheaves of regular holonomic $\mathcal{D}_{S,0}$-modules. Thus we get (1.4.1) for $\psi_t$, and it remains to show the assertion on the action of $\partial_t$ on $\mathcal{G}_j^i$, because $\partial_t$ induces the morphism $\text{can} : \psi_t \to \varphi_t$ in (1.3.2).

We have a short exact sequence

$$0 \to \mathcal{K}_f^\bullet \xrightarrow{\partial_t} \mathcal{K}_f^\bullet \to \Omega_X^\bullet \to 0,$$

and it induces a long exact sequence

$$\to \mathcal{H}^{i-1}\Omega_X^\bullet|_{X_0} \to \mathcal{G}_j^i \xrightarrow{\partial_t} \mathcal{G}_j^i \to \mathcal{H}^i\Omega_X^\bullet|_{X_0} \to,$$

where $\mathcal{H}^i\Omega_X^\bullet|_{X_0} = \mathcal{C}_{X_0}$ for $i = 0$, and $\mathcal{H}^i\Omega_X^\bullet|_{X_0} = 0$ otherwise. So the assertion follows.

1.5. Remark. Assume $f : X \to S$ is a Milnor representative at $x \in X_0$, i.e. $X$ is the intersection of $f^{-1}(S)$ with an open $\varepsilon$-ball around $x$, and $S$ is an open disk of radius $\delta$.
with $0 < \delta \ll \varepsilon \ll 1$. Then it is known that the Gauss-Manin system $R^i f_* \mathcal{K}_f$, which is the direct image sheaf of $\mathcal{O}_X$ by $f$ as a $\mathcal{D}$-module, is a regular holonomic $\mathcal{D}_S$-module and its stalk at the origin is independent of $\varepsilon$. This assertion follows from the theory of the direct images of $\mathcal{D}_X$-modules (see [11]), according to Kashiwara. It can be verified also by using a resolution of singularities $\pi: \tilde{X} \to X$ as in the proof of (1.4), because we see that $R^i(f\pi)_* \mathcal{K}_f$ is regular holonomic for $0 < \delta \ll \varepsilon \ll 1$.

1.6. Lemma. For $i = 1$, the short exact sequence

\[(1.6.1) \quad 0 \to \mathbb{C}\{t\} \omega_0 \to G^1_f \to \tilde{G}^1_f \to 0\]

splits canonically, and we have in the notation of (1.3.1)

\[(1.6.2) \quad (\tilde{G}^1_f)^\alpha = 0 \quad \text{for } \alpha \in \mathbb{Z}.\]

Proof. It is well known that the action of the monodromy on the 0-th Milnor cohomology $H^0(F_x, \mathbb{C})$ is semisimple, and the multiplicity of every eigenvalue is 1. (This can be reduced to the normal crossing case easily.) So we get (1.6.2). This implies (1.6.1) because $\mathbb{C}\{t\}^\alpha = 0$ for $\alpha \notin \mathbb{Z}$.

As a corollary, we get the following

1.7. Proposition. There is a canonical action of $\mathbb{C}\{\{\partial_t^{-1}\}\}$ on $G^i_f$ for any $i$. For $i = 1$, it is compatible by (1.6.1) with the canonical action of $\mathbb{C}\{\{\partial_t^{-1}\}\}$ on $\mathbb{C}\{t\}$ which is defined by the integration without constant term.

Proof. This follows from Proposition (1.4) if $i \neq 1$, and we use also Lemma (1.6) for $i = 1$.

2. Brieskorn modules

2.1. Sheaf of Brieskorn modules. With the notation of (1.1), let $\mathcal{A}_f^*$ be the complex whose $i$-th term is $\text{Ker}(df \wedge: \Omega_X^i \to \Omega_X^{i+1})$ and whose differential is induced by $d$, see [4], [10]. There is a natural inclusion

\[(2.1.1) \quad \mathcal{A}_f^* \to \mathcal{K}_f^*.\]

Let

\[\mathcal{H}_f^i = \mathcal{H}_f^i \mathcal{A}_f^*|_{X_0}.\]

Note that $\mathcal{H}_f^i \mathcal{A}_f^*$ is supported on $X_0$ for $i \neq 0$, and we have to take the restriction to $X_0$ in order to define the action of $\partial_t^{-1}$ on $\mathcal{H}_f^i \mathcal{A}_f^*$. We will call $\mathcal{H}_f^i$ the sheaf of Brieskorn modules associated to $f$. By (2.1.1) we have natural morphisms

\[(2.1.2) \quad \mathcal{H}_f^i \to G_f^i.\]
Since the differential is \( f^{-1}\mathcal{O}_S \)-linear, \( \mathcal{H}^i_f \) has a structure of a \( \mathbb{C}\{t\} \)-module. We have the action of \( \partial_i^{-1} \) on \( \mathcal{H}^i_f \) by

\[
(2.1.3) \quad \partial_i^{-1} \omega = df \wedge \eta \quad \text{with} \quad d\eta = \omega.
\]

This action is well defined by an argument similar to [4]. Indeed, for \( i \neq 1 \), the ambiguity of \( \eta \) is given by \( d\eta' \) and \( df \wedge d\eta' = -d(df \wedge \eta') \). For \( i = 1 \), the ambiguity of \( \eta \in \mathcal{O}_X \) is given by \( \mathbb{C} \), and we have a canonical choice of \( \eta \) assuming that the restriction of \( \eta \) to \( X_0 \) vanishes (this is allowed because \( df \wedge d\eta = 0 \)).

For \( i = 1 \), we see that \( \omega_0 := df \otimes 1 \) in (1.1) belongs to \( \mathcal{H}^1_f \). Here we can easily verify that \( \mathcal{H}^1_f \rightarrow \mathcal{G}^1_f \) is injective, see (2.4) below. We define the sheaf of reduced Brieskorn modules \( \tilde{\mathcal{H}}^i_f \) by

\[
(2.1.4) \quad \tilde{\mathcal{H}}^i_f = \mathcal{H}^i_f \quad \text{if} \quad i \neq 1, \quad \text{and} \quad \mathcal{H}^i_f / \mathbb{C}\{t\} \omega_0 \quad \text{if} \quad i = 1.
\]

Let \( \Omega^\bullet_{X/S} \) denote the sheaf complex of relative differential forms in the usual sense, i.e. \( \Omega^i_{X/S} = \Omega^i_X / df \wedge \Omega^{i-1}_X \). Then we have a canonical \( f^{-1}\mathcal{O}_S \)-linear morphism

\[
(2.1.5) \quad df \wedge : \Omega^\bullet_{X/S} \rightarrow \mathcal{A}^\bullet_f [1],
\]

which induces an isomorphism of complexes after the localization by \( f \). Note that the image of 1 by this morphism is \( \omega_0 \).

**2.2. Proposition.** Assume \( X_0 \) is a divisor with normal crossings. Then \( \mathcal{H}^i_f \) and \( \tilde{\mathcal{H}}^i_f \) are constructible sheaves of finite free \( \mathbb{C}\{t\} \)-modules which are stable by the action of \( t\partial_i \) and the eigenvalues of the residue of \( t\partial_i \) are contained in \((-1,0] \), see [5]. Furthermore, the canonical morphism \( \mathcal{H}^1_f \rightarrow \mathcal{G}^1_f \) induces isomorphisms compatible with the action of \( \mathbb{C}\{t\} \)

\[
(2.2.1) \quad \mathcal{H}^i_f = V^{> -1} \mathcal{G}^i_f, \quad \tilde{\mathcal{H}}^i_f = V^{> -1} \tilde{\mathcal{G}}^i_f,
\]

where \( V \) is as in (1.3.3).

**Proof.** We first recall the proof of the corresponding assertion for the relative logarithmic de Rham complex \( \Omega^\bullet_{X/\text{proj}}(\log X_0) \), see [21]. Let \( (x_1, \ldots, x_n) \) be a local coordinate system such that \( f = \prod_{i=1}^r x_i^{m_i} \). Then

\[
\frac{df}{f} = \sum_{i=1}^r m_i dx_i / x_i,
\]

and hence \( (\Omega_{X,0}^\bullet(\log X_0), df \wedge) \) is acyclic, because it is identified with the Koszul complex associated to the morphisms \( m_i : \mathbb{C}\{x\} \rightarrow \mathbb{C}\{x\} \) for \( 1 \leq i \leq n \) where we put \( m_i = 0 \) for \( i > r \) (and \( m_i \neq 0 \) for \( i \leq r \)).

Let \( \partial_i = \partial / \partial x_i, \eta_i = dx_i / x_i, \) and \( \eta_I = \eta_{i_1} \wedge \cdots \wedge \eta_{i_p} \) for \( I = \{i_1, \ldots, i_p\} \subset \{2, \ldots, n\} \). Since \( \eta_i (i > 1) \) and \( df / f \) form a basis of \( \Omega_{X,0}^\bullet(\log X_0) \), we see that the \( \eta_I \) for \( I \subset
\{2, \ldots, n\} form a basis of $\Omega^k_{X/S,0}(\log X_0)$ over $\mathbb{C}\{x\}$. Since $(\Omega^*_{X,0}(\log X_0), d)$ is identified with the Koszul complex for the morphisms $x_i \partial_i (i \leq r)$, $\partial_i (i > r)$, we can identify $(\Omega^*_{X/S,0}(\log X_0), d)$ with the Koszul complex for

$$x_i \partial_i - (m_i/m_1) x_1 \partial_1 \ (2 \leq i \leq r), \ \partial_i \ (i > r).$$

The last complex has a structure of double complex, and we may assume $r = n$ by taking first the cohomology of the differential defined by $\partial_i (i > r)$.

Let $e$ be the greatest common divisor of the $m_i (1 \leq i \leq r)$, and put $\mu_i = m_i/e$. Then the morphisms $x_i \partial_i - (m_i/m_1) x_1 \partial_1$ preserve $\mathbb{C}x^\nu$ for any $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{N}^r$, and $H^p\Omega^*_{X/S,0}(\log X_0)$ is a free $\mathbb{C}\{t\}$-module generated by

$$(2.2.2) \quad x^k \eta_I \ (0 \leq k < e, \ |I| = p).$$

For $\omega \in H^p\Omega^*_{X/S,0}(\log X_0)$, put $\eta = t \partial_\omega$. It is the image of $\omega$ by the logarithmic Gauss-Manin connection, and $d\omega = f^{-1} df \wedge \eta$ by definition. Let $\xi$ be a holomorphic vector field such that $\xi f = f$. Let $L_\xi$ and $i_\xi$ denote respectively the Lie derivation and the interior product. Then

$$(2.2.3) \quad L_\xi \omega = t \partial_\omega \ \text{in} \ H^p\Omega^*_{X/S,0}(\log X_0),$$

because $L_\xi \omega - d i_\xi \omega = \partial_\xi d\omega = \partial_\xi (f^{-1} df \wedge \eta) = \eta - f^{-1} df \wedge i_\xi \eta$. (This holds for any holomorphic function $f$ having a vector field $\xi$ such that $\xi f = f$.) So we get the assertion for the relative logarithmic complex.

Let $\tilde{A}_f^i = \text{Ker}(f^{-1} df \wedge : \Omega^i_X(\log X_0) \to \Omega^{i+1}_X(\log X_0))$. By the acyclicity of the Koszul complex $(\Omega^*_X(\log X_0), f^{-1} df \wedge)$, we have an isomorphism of complexes

$$f^{-1} df \wedge : \Omega^*_X(\log X_0) \to \tilde{A}_f^i[1],$$

and the cohomology of $\tilde{A}_f^i$ gives the Deligne extension such that the eigenvalues of the residue of the connection are contained in $[-1, 0)$. Put $g = \prod_{i=1}^n x_i$. By the above calculation of the relative logarithmic complex in (2.2.2–3), it is enough to show

$$A_f^i = g \tilde{A}_f^i.$$

Here the inclusion $A_f^i \supset g \tilde{A}_f^i$ is clear. The opposite inclusion follows from the fact that a meromorphic form $\omega$ is logarithmic if $g \omega$ is holomorphic and $f^{-1} df \wedge \omega$ is logarithmic. (Indeed, we have $\omega = \sum a_I \eta_I$ where $a_I \in g^{-1} \mathbb{C}\{x\}$ and $a_I$ does not have a pole along $x_i = 0$ for $i \in I$. If $a_I$ has a pole along $x_j = 0$ with $j \notin I$, then $f^{-1} df \wedge a_I \eta_I$ is not logarithmic along $x_j = 0$, and the highest order part of the pole of $x_j^{-1} dx_j \wedge a_I \eta_I$ along $x_j = 0$ does not come from other $f^{-1} df \wedge a_J \eta_I$ with $J \subset I \cup \{j\}$.)

The compatibility with $\partial_i^{-1}$ is clear by the definition of the differential of $K_f^i$. This completes the proof of Proposition (2.2).
The following gives a refinement of Hamm’s result [10] and is closely related to positivity (or negativity) theorems in [11], [14].

2.3. Theorem. The $t$-torsion part of $\mathcal{H}^i_f$ is annihilated by $t^p$ for $p \gg 0$ locally on $X_0$, and coincides with the kernel of the canonical morphism $\mathcal{H}^i_f \to \mathcal{G}^i_f$. It coincides further with the $\partial^{-1}_t$-torsion part of $\mathcal{H}^i_f$ for any $i$, and vanishes for $i = 1$. The stalks of the image of the morphism $\mathcal{H}^i_f \to \mathcal{G}^i_f$ are finite free modules over $\mathbb{C}\{t\}$, which are contained in $V^{\geq -1}\mathcal{G}^i_f$ and contain $V^{\alpha}\mathcal{G}^i_f$ for $\alpha \gg 0$.

Proof. Let $\pi : \tilde{X} \to X$ be an embedded resolution of $X_0$ as in the proof of (1.4). Put $\tilde{f} = f\pi$. By (2.2) together with a spectral sequence, the $R^i\pi_*\mathcal{A}^*_{\tilde{f}}$ are constructible sheaves of finite free $\mathbb{C}\{t\}$-modules which are stable by the action of $t\partial_t$ and the eigenvalues of the residue of $t\partial_t$ are contained in $(-1, 0]$. Furthermore the kernel and the cokernel of the canonical morphism

$$\mathcal{H}^i_f = \mathcal{H}^i\mathcal{A}^*_{\tilde{f}}|_{X_0} \to R^i\pi_*\mathcal{A}^*_{\tilde{f}}|_{X_0}$$

are annihilated by $t^j$ for $j \gg 0$ locally on $X_0$, because the cohomology sheaves of the mapping cone of $\mathcal{A}^i_{\tilde{f}} \to R^i\pi_*\mathcal{A}^i_{\tilde{f}}$ for each $i$ are annihilated by $t^j$ for $j \gg 0$ locally on $X_0$, using the coherence of the direct image sheaves. So the $t$-torsion part of $\mathcal{H}^i_f$ is annihilated by a high power of $t$ locally on $X_0$, and the free part of $\mathcal{H}^i_{f,x}$ is a finite free $\mathbb{C}\{t\}$-module contained in $(R\pi_*\mathcal{A}^i_{\tilde{f}})_{x}$. (This is essentially the same argument as in [10], see also [14].)

Furthermore, $(R\pi_*\mathcal{A}^i_{\tilde{f}})_{x}$ is canonically isomorphic to $V^{> -1}\mathcal{G}^i_{f,x}$ by the theory of Deligne extension [5] using a Milnor representative as in (1.5) (or a spectral sequence as in (1.4.2)).

For $i \neq 1$, the action of $\partial_t$ is bijective on $\mathcal{G}^i_{f,x}$ by Proposition 1.4, and $V^{> -1}\mathcal{G}^i_{f,x}$ and $\mathcal{G}^i_{f,x}$ are finite free modules over $\mathbb{C}\{\partial^{-1}_t\}$ and $\mathbb{C}\{\partial^{-1}_t\}[\partial_t]$ respectively, see (1.3). To show that the kernel of $\mathcal{H}^i_f \to \mathcal{G}^i_f$ coincides with the $\partial^{-1}_t$-torsion part, we have

$$d(\sum_{j=0}^{r} \eta_j \otimes \partial^j) = \omega \otimes 1,$$

if and only if $d\eta_0 = \omega$, $df \wedge \eta_j = d\eta_{j+1} (0 \leq j < r)$ and $df \wedge \eta_r = 0$. Note that the first two equalities mean that $df \wedge \eta_r$ represents $\partial^{-r}_t\omega$ by (2.1.3). These imply that the kernel of $\mathcal{H}^i_f \to \mathcal{G}^i_f$ is contained in the $\partial^{-1}_t$-torsion part. Conversely, if the above first two equalities hold and $df \wedge \eta_r$ vanishes in $\mathcal{H}^i_f$, then there exists $\eta_{r+1}$ such that $df \wedge \eta_r = d\eta_{r+1}$ and $df \wedge \eta_{r+1} = 0$ by the definition of $\mathcal{H}^i_f$.

For $i = 1$, a similar argument shows that the canonical morphism $\mathcal{H}^1_f \to \mathcal{G}^1_f$ is injective and the action of $\partial^{-1}_t$ on $\mathcal{H}^1_f$ is injective, because $df \wedge : \mathcal{O}_X \to \Omega^1_X$ is injective. Then $\mathcal{H}^1_f$ is $t$-torsion-free by the second assertion of this theorem. This completes the proof of Theorem (2.3).

2.4. Proposition. The action of $\partial^{-1}_t$ on $\mathcal{H}^1_f$ is naturally extended to that of $\mathbb{C}\{\partial^{-1}_t\}$ on $\mathcal{H}^1_f$. This is compatible by the morphism $\mathcal{H}^1_f \to \mathcal{G}^1_f$ with the action of $\mathbb{C}\{\partial^{-1}_t\}$ on $\mathcal{G}^1_f$ in (1.7).
Proof. Assume first $i \neq 1$ so that the action of $\partial_t$ on $\mathcal{G}_j^i$ is bijective. Then the canonical morphism $\mathcal{H}_j^i \to \mathcal{G}_j^i$ is compatible with the action of $\partial_t^{-1}$ by definition, and its image is stable by the action of $\partial_t^{-1}$ by Lemma (2.5) below, there is a positive integer $p$ such that the torsion part of $\mathcal{H}_j^i$ is annihilated by $\partial_t^{-p}$, and we have a canonical action of $\partial_t^{-p} \mathcal{C}\{\partial_t^{-1}\}$ on $\mathcal{H}_j^i$ compatible with the action of $\mathcal{C}[\partial_t^{-1}]$ and also with the change of $p$. Indeed, for $P \in \mathcal{C}\{\partial_t^{-1}\}$ and $\omega \in \mathcal{H}_j^i$, consider the image $\omega'$ of $\omega$ in $\mathcal{G}_j^i$, take a lifting of $P\omega'$ to $\mathcal{H}_j^i$, and then multiply it by $\partial_t^{-p}$ so that the ambiguity of the lifting is annihilated. The compatibility follows from that of $\partial_t^{-1}$ with $\mathcal{H}_j^i \to \mathcal{G}_j^i$.

Now assume $i = 1$. With the notation of (1.3.3), we have

\begin{equation}
\partial_t^{-1} \mathcal{G}_j^1 \subset V^>0 \mathcal{G}_j^1 + \tilde{\mathcal{G}}_j^1,
\end{equation}

using the canonical splitting of (1.6.1). On the other hand, we have

\begin{equation}
\partial_t^{-1} \mathcal{H}_j^1 \subset V^>0 \mathcal{G}_j^1,
\end{equation}

by the definition of $\partial_t^{-1}\omega$ in (2.1.3) which assumes the condition $\eta|_{X_0} = 0$. Indeed, (2.4.2) is proved, for example, by using the period integral of $\eta$ along a horizontal family of topological 0-cycles, see [14]. Note that the period integral of $\omega \in \mathcal{H}_j^1$ is defined after dividing $\omega$ by $df$, i.e., by taking the inverse image by the morphism (2.1.5).

Since the intersection of $V^>0 \mathcal{G}_j^1 + \tilde{\mathcal{G}}_j^1$ with $\mathcal{C}\omega_0$ vanishes, and the ambiguity of $\partial_t^{-1}$ is given by $\mathcal{C}\omega_0$, the assertion follows. Thus the proof of Proposition (2.4) is reduced to the following.

2.5. Lemma. The $\partial_t^{-1}$-torsion part of $\mathcal{H}_j^i$ is annihilated by $\partial_t^{-p}$ for $p \gg 0$ locally on $X_0$.

Proof. By Theorem (2.3), this is reduced to the following:

(A) Let $M$ be a torsion-free abelian group with an action of $t$, $\partial_t^{-1}$ satisfying the relation: $[t, \partial_t^{-1}] = \partial_t^{-2}$. Assume $t^p M = 0$ for a positive integer $p$. Then $\partial_t^{-2p} M = 0$.

For this, we prove the following by increasing induction on $q > 0$:

(A') If $m \in M$ satisfies $t^q m = 0$ and $t^p M = 0$, then $\partial_t^{-(p+q)} m = 0$.

Indeed, $t^{p+q-1} \partial_t^{-1} m$ is a linear combination of $\partial_t^{-(p+q-i)} t^i m$ with $0 \leq i \leq p + q - 1$, and the coefficient of $\partial_t^{-(p+q-i)} m$ is $(p+q-1)!$ because $[t, \partial_t^{-j}] = j \partial_t^{-j-1}$ for $j > 0$. Then we can apply the inductive hypothesis to $t^i m$ with $q$ replaced by $q - i$ so that $\partial_t^{-(p+q-i)} t^i m = 0$ for $i > 0$. Thus the assertion is proved.

2.6. Remark. The above lemma follows also from a formula in [1] showing that $\partial_t^{-2p}$ is a linear combination of $\partial_t^{-j} t^p \partial_t^{-(p-j)}$.\]
3. Finiteness property

3.1. Triviality of $f$ along a stratification. Let $f$ be as in (1.1). Let

$$\Theta_f = \{ \xi \in \Theta_X : \xi f = 0 \},$$

where $\Theta_X$ is the sheaf of holomorphic vector fields. Let $\delta_x$ be the dimension of the image of $\Theta_{f,x}$ in the tangent space $T_{X,x}$. Note that the restriction of $H^i_f$ to an integral curve of a nowhere vanishing vector field $\xi \in \Theta_f$ is locally constant, see Lemma (3.2) below. Let

$$S_i = \{ x \in X : \delta_x = i \}.$$

We have $\dim S_i \geq i$ in general (by induction on $\dim X$ considering the integral curves of a vector field). We say that $f$ is locally trivial along a stratification if

$$(C') \quad \dim S_i = i \quad \text{for any } i.$$

In this case, there is Whitney stratification $\{ S'_i \}$ such that the tangent space of $S'_{i,x}$ is contained in the image of $\Theta_{f,x}$. (This condition was considered in [17].)

Let $x \in X$, and take a local coordinate system $(z_1, \ldots, z_n)$ around $x$. We have the distance function $\rho$ from $x$ defined by $(\sum_i |z_i|^2)^{1/2}$ as usual. If condition $(C)$ is satisfied, we see that the following condition is satisfied (applying a standard argument to the above stratification and $\rho$):

$$(C') \quad \text{For any } y \in X \setminus \{ x \} \text{ near } x, \text{ there exists } \xi_y \in \Theta_{f,y} \text{ such that } \langle \xi_y, d\rho \rangle_y \neq 0.$$

This means that the integral curve of $\xi_y$ intersects transversely the sphere $S_{x,\varepsilon}$ with center $x$ and containing $y$ (i.e. $\varepsilon = \rho(y)$).

3.2. Lemma. Assume $f = \pi^* g$ with $\pi : X \to Y$ a smooth morphism of complex manifolds (i.e. the differential has the maximal rank). Then $H^i_f = \pi^{-1} H^i_g$.

Proof. Since the assertion is local, we may assume $X = Y \times Z$ for a complex manifold $Z$. The complexes $(\Omega^\bullet_X, df \wedge)$, $(\Omega^\bullet_Y, d)$ and $(A^\bullet_f, d)$ have structures of double complexes, and the differential for the second component of $(\Omega^\bullet_X, df \wedge)$ vanishes. So $(A^\bullet_f, d)$ is the external product of $(A^\bullet_g, d)$ and $(\Omega^\bullet_Z, d)$. Then, taking first the cohomology of the differential for the second component and using the spectral sequence associated to the double complex, the assertion follows.

The following was essentially shown in [17] with $A^\bullet_f$ replaced by $\Omega^\bullet_{X/S}$.

3.3. Theorem. With the above notation, assume condition $(C)$ is satisfied. Then $H^i_f$ is a constructible sheaf of finite modules over $C\{ t \}$ and over $C\{ \{ \partial^{-1} \} \}$. In particular, its torsion part is a constructible sheaf of finite dimensional $C$-vector spaces.

Proof. By Lemma (3.2), $H^i_f$ is a constructible sheaf. To show the finiteness, we may assume that $f : X \to S$ is a Milnor representative as in the proof of (1.4) (in particular, $X$ is Stein). Since the $A^i_f$ are coherent $O_X$-modules and the differential is $f^{-1} O_S$-linear, it is enough to show that $R^i f_* A^\bullet_f$ is independent of $\varepsilon$ (shrinking $\delta$ if necessary) using the theory of Kiehl-Verdier [13], see also [8]. So the assertion follows from the constructibility.
3.4. Remark. We can show the finiteness of $\mathcal{H}^i_{f,x}$ under the assumption $(C')$.

3.5. Remark. Without condition $(C)$ or $(C')$, the torsion part of $\mathcal{H}^i_{f,x}$ is not a finite dimensional $\mathbb{C}$-vector space in general. For example, consider $f = x^5 + y^5 + x^3y^3z$, or rather $f = x^5/5 + y^5/5 + x^3y^3z/3$ replacing the coordinates. Then

$$f_x = x^4 + x^2y^3z = x^2(x^3 + y^3 z),$$

$$f_y = y^4 + x^3y^2z = y^2(y^2 + x^3 z),$$

$$f_z = x^3y^3/3,$$

and $\text{Sing } f = \{x = y = 0\}$. Furthermore, $f$ is a topologically trivial deformation of holomorphic functions of two variables, and the highest Milnor cohomology $H^2(F_0, \mathbb{C})$ vanishes. So it is enough to show that $\mathcal{H}^3_{f,0}$ is not finite dimensional.

First we have to calculate $A^2_{f,0}(= \text{Ker } df \wedge)$. This is equivalent to the determination of the intersection of the two ideals $(f_x, f_y)$ and $(f_z)$. Since $f$ is weighted-homogeneous of weight $(1,1,−1)$, it is sufficient to consider only monomials $x^i y^j z^k$ with $i + j - k = c$ for some constant $c$. So we may substitute $z = 1$ for the calculation of the intersection of the ideals (after taking the partial derivatives), and conclude that $A^2_{f,0}$ is generated over $\mathbb{C}\{x, y, z\}$ by

$$xy^3 dy \wedge dz - 3(x^2 + y^3 z) dx \wedge dy,$$

$$x^3 y dx \wedge dz + 3(y^2 + x^3 z) dx \wedge dy,$$

$$x^2 y^2 z dy \wedge dz + x^3 dx \wedge dz + 3(y - xy^2 z^2) dx \wedge dy,$$

$$x^2 y^2 z dx \wedge dz + y^3 dy \wedge dz - 3(x - x^2 y^2 z^2) dx \wedge dy.$$

Then we see that the image of $dA^2_{f,0}$ is contained in the submodule $x\Omega^3_{X,0} + y\Omega^3_{X,0}$ which is identified with the ideal $(x, y)$ of $\mathbb{C}\{x, y, z\}$. Since the action of $f$ on $\Omega^3_{X,0}/(x\Omega^3_{X,0} + y\Omega^3_{X,0})$ is the multiplication by 0, it implies that the torsion part of $\mathcal{H}^i_{f,0}$ is infinite dimensional. (A similar argument would apply also to $f = x^5 + y^5 + x^2y^2z$.)

3.6. Remark. In [1], the following condition was considered for $i \geq 2$:

$$(P') \quad d(\text{Ker } df \wedge) \cap \text{Im } df \wedge = \text{Im } df \wedge d \quad \text{in } \Omega^i_{X,x}.$$

We can easily show that this is a necessary and sufficient condition for the torsion-freeness of $\mathcal{H}^i_{f,x}$. A stronger condition $(P)$ was also considered there in order to prove a formula for the dimension of the Milnor cohomology. In the case $\text{dim } X = 2$, it was proved in loc. cit. that condition $(P)$ always holds, and hence also $(P')$ does:

3.7. Proposition [1]. Assume $\text{dim } X = 2$. Then condition $(P)$ always holds and hence the $\mathcal{H}^i_f$ are torsion-free.

3.8. Remark. We can show $(P')$ in the case $\text{dim } X = 2$, using a resolution of singularities as follows. We may assume $i = 2$. Since $df \wedge$ is acyclic after localized by $f$, any element of $\text{Ker } df \wedge$ is uniquely written as $hdf/f$ with $h \in O_X$ such that $h|_{X_0} = 0$. Let $h_x, h_y$ denote the partial derivatives by local coordinates $x, y$. Assume

$$(3.8.1) \quad d(hdf/f) = dh \wedge df/f = (h_x f_y/f - h_y f_x/f) dx \wedge dy \in \text{Im } df \wedge.$$
We will show that \( h/f \) belongs to \( \mathcal{O}_X \) modifying \( h \) by a linear combination of \( f^\alpha \) where the \( \alpha \) are rational numbers such that \( f^\alpha \) is holomorphic (i.e. for any irreducible component \( D_i \) of \( f^{-1}(0) \) with multiplicity \( m_i \), we have \( m_i \alpha \in \mathbb{N} \)). Note that \( df^\alpha \wedge df/f = 0 \).

Let \( \pi : \tilde{X} \to X \) be an embedded resolution of singularities of \( f^{-1}(0) \). Put \( \tilde{f} = f \pi, \tilde{h} = h \pi \) and \( \tilde{D} = \tilde{f}^{-1}(0) \). Note that the above conditions are compatible with the pull-back, and it is sufficient to show that \( \tilde{h}/\tilde{f} \) is holomorphic on \( \tilde{X} \), modifying \( \tilde{h} \) as above.

For a singular point 0 of \( \tilde{D}_{\text{red}} \), let \( x, y \) be local coordinates such that \( \tilde{f} = x^p y^q \) around this point. Let \( V \) be the filtration on \( \mathcal{O}_{\tilde{X},0}[\tilde{f}^{-1}] = \mathbb{C}\{x, y\}[x^{-1} y^{-1}] \) such that \( V^\alpha \) for \( \alpha \in \mathbb{Q} \) is generated over \( \mathcal{O}_X \) by \( x^i y^j \) with \( i/p \geq \alpha, \ j/q \geq \alpha \). This can be defined also for a smooth point of \( \tilde{D}_{\text{red}} \) by considering only the condition \( i/p \geq \alpha \) if \( \tilde{f} = x^p \) locally. So the filtration \( V \) is globally well-defined on \( \mathcal{O}_{\tilde{X}}[\tilde{f}^{-1}] \), and \( V^1 \) is generated by \( \tilde{f} \) over \( \mathcal{O}_{\tilde{X}} \).

Let \( V^{>\alpha} = \bigcup_{\beta > \alpha} V^\beta \). The restriction of the graded pieces \( \text{Gr}_V^\alpha := V^\alpha / V^{>\alpha} \) to the smooth points of \( \tilde{D} \) is a line bundle if it does not vanish.

At a singular point of \( \tilde{D}_{\text{red}} \), let \( g = xy(h_x f_y/\tilde{f} - h_y f_x/\tilde{f}) \). Then \( gdx \wedge dy/xy \in \text{Im } \tilde{f} \wedge \) by (3.8.1) on \( \tilde{X} \), and hence \( g \) belongs to \( V^1 \) because \( xy(\partial f) \subset V^1 \) where \( (\partial f) \) denotes the Jacobian ideal. So we get

\[
(3.8.2) \quad q x \tilde{h}_x - p y \tilde{h}_y \in V^1.
\]

Since the differential operator \( q x \frac{\partial}{\partial x} - p y \frac{\partial}{\partial y} \) preserves \( \mathbb{C} x^i y^j \) and \( V^\alpha \), we see that if \( \tilde{h} \in V^\alpha \)

with \( 0 < \alpha < 1 \) and (3.8.2) holds at every singular point of \( \tilde{D}_{\text{red}} \), then there is a unique decomposition

\[
(3.8.3) \quad \tilde{h} = \tilde{h}' + c \tilde{f}^\alpha \quad \text{with} \quad \tilde{h}' \in V^{>\alpha}, \ c \in \mathbb{C},
\]

where \( c = 0 \) unless \( \tilde{f}^\alpha \) exists globally, i.e. unless \( m \alpha \in \mathbb{N} \) with \( m \) the greatest common divisor of the multiplicities \( m_i \) of \( \tilde{f} \) along the irreducible components \( \tilde{D}_i \) of \( \tilde{D} \). (Indeed, at singular points of \( \tilde{D}_{\text{red}} \), this follows easily from (3.8.2). It implies that if the image of \( \tilde{h} \) in \( \text{Gr}_V^\alpha \) does not vanish generically on an irreducible component \( \tilde{D}_i \) of \( \tilde{D} \) and hence \( m_i \alpha \) is an integer, then this holds also for the other components \( \tilde{D}_j \) intersecting \( \tilde{D}_i \) because otherwise \( c = 0 \) at the intersection point. Here we use analytic continuation on the smooth points of \( \tilde{D}_{\text{red}} \) together with the connectivity of \( \pi^{-1}(0) \).) So we may replace \( \tilde{h} \) with \( \tilde{h}' \), and the assertion follows by increasing induction on \( \alpha \).

4. Thom-Sebastiani type theorem

4.1. External product. Let \( f, g \) be nonconstant holomorphic functions on complex manifolds \( X \) and \( Y \) respectively. Define \( h = f + g \) on \( Z := X \times Y \). Let \( z = (x, y) \in X_0 \times Y_0 \), and take \( \omega \in \Omega_{X,x} \) such that \( df \wedge \omega = d\omega = 0 \) with \( i > 0 \). Then we have the morphisms of complexes

\[
\mathcal{A}^\bullet_{g,y} \to \mathcal{A}^\bullet_{h,z}[i], \quad \mathcal{K}^\bullet_{g,y} \to \mathcal{K}^\bullet_{h,z}[i],
\]
These are defined by \( \eta \) and \( \sum_k \eta_k \otimes \partial_t^{-k} \rightarrow \sum_k \omega \wedge \eta_k \otimes \partial_t^{-k} \). They induce morphisms

\begin{equation}
H^j_{g,y} \rightarrow H^{i+j}_{h,z}, \quad G^j_{g,y} \rightarrow G^{i+j}_{h,z}.
\end{equation}

These are \( \partial_t^{-1} \)-linear if \( j > 1 \). For \( j = 1 \), we can show by increasing induction on \( k \)

\begin{equation}
\omega \wedge g^k dg = 0 \quad \text{in } H^{i+j}_h.
\end{equation}

Indeed, we have \( \omega = d\omega' \) with \( \omega' \in \Omega^{i-1}_X \), and

\[
d(\omega' \wedge h^k dh) = \omega \wedge h^k dh = \sum_{0 \leq i \leq k} \binom{k}{i} f^i \omega \wedge g^{k-i} dg.
\]

So we can apply the inductive hypothesis to \( f^i \omega \) (\( 1 \leq i \leq k \)) instead of \( \omega \).

The assertion (4.1.2) means that the restriction of (4.1.1) to \( C\{t\}dg \) vanishes. It implies that its restriction to \( C\{t\}dg \) also vanishes, because the action of \( \partial_t^{-1} \) on \( C\{t\}/C[t] \) is bijective. (Indeed, the last bijectivity implies that any \( e \in C\{t\}/C[t] \) is infinitely \( \partial_t^{-1} \)-divisible, i.e. for any \( r \in \mathbb{N} \), there exists \( e' \in C\{t\}/C[t] \) such that \( \partial_t^{-r} e' = e \). But this property does not hold for any nonzero element of \( H^{i+j}_h \) by (2.3) and (2.5).) Thus (4.1.1) induces morphisms

\begin{equation}
\tilde{H}^j_{g,y} \rightarrow \tilde{H}^{i+j}_{h,z}, \quad \tilde{G}^j_{g,y} \rightarrow \tilde{G}^{i+j}_{h,z},
\end{equation}

and these are \( C\{\partial_t^{-1}\} \)-linear and \( C\{\partial_t^{-1}\}[\partial_t] \)-linear respectively. (Indeed, (2.3–5) implies that \( \bigcap_{k \geq 0} \partial_t^{-k}\tilde{H}^j_{h,z} = 0 \).

We have an assertion similar to (4.1.3) with \( \tilde{H}^j_{g,y}, \tilde{G}^j_{g,y} \) replaced by \( \tilde{H}^j_{f,x}, \tilde{G}^j_{f,x} \), if we take \( \omega' \in \Omega^j_{Y,y} \) such that \( dg \wedge \omega' = d\omega' = 0 \) with \( j > 0 \). So we get well-defined morphisms

\begin{equation}
\tilde{H}^j_{f,x} \otimes C\{\partial_t^{-1}\} \tilde{H}^j_{g,y} \rightarrow \tilde{H}^{i+j}_{h,z},
\end{equation}

\[
\tilde{G}^j_{f,x} \otimes C\{\partial_t^{-1}\}[\partial_t] \tilde{G}^j_{g,y} \rightarrow \tilde{G}^{i+j}_{h,z},
\]

such that the action of \( t \otimes 1 + 1 \otimes t \) on the left-hand side corresponds to that of \( t \) on the right-hand side.

\textbf{4.2. Theorem.} Let \( n = \dim X \). Assume \( f \) has an isolated singularity at \( x \in X_0 \) so that \( \tilde{H}^i_{f,x} = \tilde{G}^i_{f,x} = 0 \) for \( i \neq n \). Then, for \( i = n \) and for any integers \( j \), the morphisms (4.1.4) induces isomorphisms

\begin{equation}
\tilde{H}^n_{f,x} \otimes C\{\partial_t^{-1}\} \tilde{H}^{j-n}_{g} \rightarrow \tilde{H}^{j}_{h\{x\} \times Y},
\end{equation}

\[
\tilde{G}^n_{f,x} \otimes C\{\partial_t^{-1}\}[\partial_t] \tilde{G}^{j-n}_{g} \rightarrow \tilde{G}^{j}_{h\{x\} \times Y},
\]

where \( Y \) is identified with \( \{x\} \times Y \subset Z \).
Proof. Let $V$ be a finite dimensional $\mathbb{C}$-vector subspace of $\Omega^n_{X,x}$ such that the morphism $V \to \Omega^n_{X,x}/df \wedge \Omega^{n-1}_{X,x}$ is bijective. Then

$$\mathcal{H}^n_{j,x} = \mathbb{C}\{\{\partial^{i-1}_j\}\} \otimes \mathcal{C} V, \quad \mathcal{G}^n_{j,x} = \mathbb{C}\{\{\partial^{i-1}_j\}\}[\partial_i] \otimes \mathcal{C} V,$$

and there are injective morphisms of complexes

$$(4.1.2) \quad \gamma : V \otimes \mathcal{C} A^*_{g,y}[-n] \to A^*_{h,z}, \quad \gamma' : V \otimes \mathcal{C} K^*_{g,y}[-n] \to K^*_{h,z}.$$  

We have to show that (4.2.1) induces isomorphisms of reduced Brieskorn modules and of reduced Gauss-Manin systems.

Let $\tilde{A}^*_{h,z}, \tilde{K}^*_{h,z}$ denote the complexes defined by their cokernels respectively. Then the inclusion induces isomorphisms

$$(4.2.2) \quad \beta : \mathcal{H}^i\tilde{A}^*_{h,z} \sim \rightarrow \mathcal{H}^i\tilde{K}^*_{h,z},$$

i.e. $\tilde{K}^*_{h,z}/\tilde{A}^*_{h,z}$ is acyclic. Indeed, $\tilde{K}^*_{h,z}$ has the filtration $F$ defined by

$$F_p\tilde{K}^i_{h,z} = \bigoplus_{k \leq p+i} \Omega^k_{Z,z} \otimes \partial^k,$$

and the graded pieces of its quotient filtration $F$ on $\tilde{K}^*_{h,z}/\tilde{A}^*_{h,z}$ are $\tau$-truncated complexes of $(\Omega^k_{Z,z}/V \otimes \mathcal{C} \Omega^\ast_{Y,y}[-n], dh \wedge)$, where $\tau$ denotes the canonical truncation which is defined by using the kernel of the differential at a certain degree as in [6]. Note that the Koszul complex $(\Omega^\ast_{Z,z}, dh \wedge)$ is the external product of $(\Omega^\ast_{X,x}, df \wedge)$ and $(\Omega^\ast_{Y,y}, dg \wedge)$, and is isomorphic to

$$(\Omega^\ast_{Z,z}/V \otimes \mathcal{C} \Omega^\ast_{Y,y}[-n], dh \wedge) \oplus (V \otimes \mathcal{C} \Omega^\ast_{Y,y}[-n], dg \wedge),$$

because

$$(\Omega^\ast_{X,x}, df \wedge) = (\Omega^\ast_{X,x}/V[-n], df \wedge) \oplus V[-n].$$

Since $(\Omega^\ast_{X,x}/V[-n], df \wedge)$ is acyclic, the above arguments imply the filtered acyclicity of $\tilde{K}^*_{h,z}/\tilde{A}^*_{h,z}$ (i.e. the graded pieces are acyclic).

We now show that (4.2.1) induces isomorphisms

$$(4.2.3) \quad \tilde{\gamma}_j : V \otimes \mathcal{C} \tilde{H}^i_{g,y} \sim \rightarrow \tilde{H}^i_{h,z}, \quad \tilde{\gamma}'_j : V \otimes \mathcal{C} \tilde{G}^i_{g,y} \sim \rightarrow \tilde{G}^i_{h,z}.$$  

By definition, the isomorphism $\beta$ in (4.2.2) is embedded into the commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \text{Coker} \mathcal{H}^i\gamma & \rightarrow & \mathcal{H}^i\tilde{A}^*_{h,z} & \rightarrow & \text{Ker} \mathcal{H}^{i+1}\gamma & \rightarrow & 0 \\
& & \downarrow \beta' & \downarrow \beta & & \downarrow \beta'' & & \downarrow \beta'' & \\
0 & \rightarrow & \text{Coker} \mathcal{H}^i\gamma' & \rightarrow & \mathcal{H}^i\tilde{K}^*_{h,z} & \rightarrow & \text{Ker} \mathcal{H}^{i+1}\gamma' & \rightarrow & 0
\end{array}
$$

By (4.1.2–3) we may replace $\mathcal{H}^\ast\gamma, \mathcal{H}^\ast\gamma'$ in this diagram with $\tilde{\gamma}, \tilde{\gamma}'$ in (4.2.3) (i.e. the Brieskorn modules and the Gauss-Manin systems are replaced with the reduced ones) by modifying $\mathcal{H}^i\tilde{A}^*_{h,z}, \mathcal{H}^i\tilde{K}^*_{h,z}$ appropriately, but the middle vertical morphism $\tilde{\beta}$ is still bijective, where the vertical morphisms $\beta', \beta, \beta''$ for the new diagram are denoted respectively.
by $\tilde{\beta}', \tilde{\beta}, \tilde{\beta}''$. Note that (4.2.3) is equivalent to the vanishing of the bottom row of the new diagram, because $\tilde{\beta}$ is bijective.

Let $\mathcal{C}'$ denote the category of $\mathcal{C}'\{\partial t^{-1}\}$-modules which are annihilated by sufficiently high powers of $\partial t^{-1}$. Considering the commutative diagram modulo $\mathcal{C}'$, we may essentially neglect the torsion part of the Brieskorn modules by (2.5). Note that $\tilde{\gamma}'_j$ in (4.2.3) is obtained by the tensor of $\tilde{\gamma}_j$ with $\mathcal{C}'\{\partial t^{-1}\}[\partial_t] \text{ over } \mathcal{C}'\{\partial t^{-1}\}$, and this tensor is an exact functor, and hence commutes with Ker and Coker. Thus the target of $\tilde{\beta}'$ and $\tilde{\beta}''$ is obtained by the tensor of their source with $\mathcal{C}'\{\partial t^{-1}\}[\partial_t] \text{ over } \mathcal{C}'\{\partial t^{-1}\}$, and their cokernel is obtained by the tensor with $\mathcal{C}'\{\partial t^{-1}\}[\partial_t]/\mathcal{C}'\{\partial t^{-1}\}$ over $\mathcal{C}'\{\partial t^{-1}\}$ using the right exactness of tensor. In particular, the vanishing of the target of $\tilde{\beta}'$, $\tilde{\beta}''$ is reduced to that of their cokernels. Since the cokernels are infinitely $\partial t^{-1}$-divisible and does not belong to $\mathcal{C}'$ if it does not vanish, the snake lemma implies that each term of the bottom exact sequence of the commutative diagram vanishes. So the assertion follows.
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