Towards a more fundamental theory beyond quantum mechanics,
avoiding the Schroedinger paradox

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Abstract

The main distinction between classical mechanics and quantum mechanics is the lack in the latter of a full mechanical determinism: different final states can arise from the same physical state, after the measurement. No hidden variable is supposed to exist, nothing can discriminate two apparently identical states even if they give a different result.

In this paper we try to put the basis for a more fundamental theory that (approximately) coincides with quantum mechanics when comparing statistics, but it is more fundamental, since it mathematically describes measurement processes giving an explicit time evolution of the wave function during the collapse. The theory is deterministic even if the Heisenberg uncertainty principle is still valid. The theory distinguishes physical states that collapse and physical states that do not collapse. The theory can be made compatible with all experiments done in the past, but new phenomena such as violations of the Born law or the superposition principle could transpire. However, even if we have probably shown that it is possible to build ad hoc a theory that can describe both the wave function collapse and the Schroedinger linear evolution, a simple and unified construction is still missing.

1 Introduction

Quantum Mechanics (QM) is a probabilistic theory that predicts the time evolution of any physical system, once some initial conditions have been assigned. Even if these initial conditions appear to be unique\footnote{It is not possible to rule out the existence of hidden variables in the initial conditions, that are physical objects which cannot be directly fixed by any experimental instrument or measurement. But Quantum Mechanics (QM) is a theory that does not consider this possibility.}, QM only provides us with the probability that the physical system ends in one among a large set of possible final states. We also know that the theory must correctly describe experiments that show phenomena of destructive and constructive interference. Equations are linear, and this linearity applies to any physical system, both in first and second quantization\footnote{For a short discussion of theories that violate the superposition principle see \cite{1}, where also potential risks, due to such violations, are mentioned.}. This means that if a physical system is described by the wave function $\psi(x,0) + \chi(x,0)$, then the time evolution is $\psi(x,t) + \chi(x,t)$, where both $\psi(x,t)$ and $\chi(x,t)$ are solutions of the same Schroedinger equation. QM extends this superposition principle to all systems, both microscopic systems like a simple proton and macroscopic system like a gas of several atoms. As a consequence, in real situations, QM predicts that a system quickly evolves toward the linear superposition of extremely different physical states like in the famous Schroedinger paradox.

Presumably, we can strictly apply QM to simple enough physical systems and not any system. Probably the extent of validity of QM does not include physical objects containing several atoms.

In this paper we will try to put the basis for a mechanical and deterministic theory that is approximately equivalent to QM in the statistical sense, \textit{i.e.} it is almost compatible to QM. We require mechanical determinism: \textit{the time evolution of any physical system must satisfy the fundamental principle\footnote{This principle is not true in QM, which predicts that the same state can give different results after a measurement.} that different final states always descend from different initial states (initial conditions).} Therefore it is necessary to assume that additional (hidden) variables distinguish two states that evolve in different
final states. In this work we restrict ourselves to study those situations in first quantization, with the aim to infer
some necessary requirements to any algorithm that simulates QM, both the Schrödinger linear time evolution and
the wave function collapse during the quantum measurement. To this purpose we will make use of some stochastic
methods.

2 The Born rule of Quantum Mechanics and some possible violations

The fundamental axioms of quantum mechanics are a direct consequence of the following requirements

i) A probability distribution $P(x)$ is always positive.

This requirement can be easily realized, by choosing $P(x) = |\psi(x)|^2$, where $\psi(x)$ is an arbitrary complex
function. In general the function $\psi(x,t)$ depends on the time variable $t$, and assuming a linear time evolution we have

$$i \frac{\partial}{\partial t} \psi(x,t) = H \psi(x,t).$$  \hspace{1cm} (1)

For any time $t$, we must have that

ii) The probability distribution always satisfies $\int P(x,t) \, dx = 1$, then $H$ must be Hermitian. The time evolution (1) is correct before any quantum measurement occurs, that is the wave function collapse into an eigenstate of an observable.

For the sake of clarity, we discuss the physical process of a measurement in the specific case of a free particle
confined by a wall barrier in a finite region $0 < x < L$. To solve the differential equation (1), we consider a lattice:
we replace the real variable $x$, with an integer $0 < n < L$ and take $L = 6$, to make an analogy with a dice. This
simplification implies that the wave function $\psi(x,t)$ is replaced by a vector with six complex components

$$\vec{\psi}(t) = (\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t), \psi_5(t), \psi_6(t)).$$  \hspace{1cm} (2)

The Schrödinger equation is

$$i \frac{\partial}{\partial t} \psi(x,t) = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t)$$  \hspace{1cm} (3)

that in our lattice approximation becomes

$$i \frac{\partial}{\partial t} \vec{\psi}(t) = \frac{1}{2m} \left( \frac{6}{L} \right)^2 \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix} \vec{\psi}(t)$$  \hspace{1cm} (4)

The equations (3) and (4) are equivalent in the limit of lattice approximation that we are considering.

The equation (4) fixes the time evolution of the vector $\vec{\psi}(t)$; but in what it follows, we replace the time derivative with a finite difference $\vec{\psi}(t+m \Delta) - \vec{\psi}(t+(m-1) \Delta)$: instead of (4), a differential equation, we deal with an algorithm that gives $\vec{\psi}(t+m \Delta)$ in terms of the preceding $\vec{\psi}(t+(m-1) \Delta)$. Any algorithm tends to a differential equation if the limit $\Delta \to 0$ exists. It is also true that any linear Schrödinger equation can be replaced by an algorithm that generates iteratively a sequence of $\psi(t+m \Delta)$ for any integer $m$.

The vector $\vec{\psi}(t)$ contains six complex variables

$$\begin{pmatrix}
|\psi_1| \, e^{i \alpha_1}, |\psi_2| \, e^{i \alpha_2}, |\psi_3| \, e^{i \alpha_3}, |\psi_4| \, e^{i \alpha_4}, |\psi_5| \, e^{i \alpha_5}, |\psi_6| \, e^{i \alpha_6}
\end{pmatrix}$$  \hspace{1cm} (5)

that satisfy the following condition

$$\sum_{i=1}^{6} |\psi_i|^2 = 1.$$
Hereafter we give just one example on how to make it possible, but it is understood that several and alternative mathematical representations can be used to define the state of a physical system.

For example we can assume that a physical state is unambiguously defined once a sequence $\Gamma = \{x_1, \cdots, x_N\}$, of integer numbers $0 < x_n \leq L$ ($L = 6$ in our example) is fixed together with a set of (six) phases $\alpha_i$. The sequence

$$\Gamma = \{x_1, \cdots, x_N\}$$

(6)

plus the phases

$$(\alpha_1, \ldots, \alpha_L)$$

(7)

define only one vector $\bar{\psi}$ through the identification

$$p_i \equiv |\psi_i|^2 \equiv \frac{n_i}{N}$$

(8)

where $n_i$ is the number of times, the integer $i$ appears in the sequence $\Gamma$, and $N$ is the length of $\Gamma$. It is clear that our definition of the physical state contains more (hidden) variables then the vector $\bar{\psi}$.

The time evolution both of the sequence $\Gamma(t)$ and of the phases $\alpha_i(t)$ unambiguously fixes the time evolution of $\bar{\psi}(t)$ (see (5)). If the probability distribution has no peak, and it is small enough the time evolution of $\bar{\psi}(t)$ will probably follow a linear Schroedinger evolution

$$i \frac{\partial}{\partial t} \psi(x,t) = \hat{H} \psi(x,t)$$

(9)

but in some cases, probably when a measurement occurs, the evolution (9) fails, and one is forced to consider the full sequence $\Gamma(t)$ instead of just the vector $\bar{\psi}(t)$ to get the right and exact time evolution. The aim of this work is to outline some crucial ingredients of the time evolution $\Gamma(t)$ that correctly describe the so called Born rule of the quantum mechanics: a measurement of the observable $x$ induces a collapse of the wave function into an eigenstate of the observable $x$ with probability given by the squared wave function $|\psi(x)|^2$.

In our example this means that a measurement of $\psi$ should induce a collapse into any of the following states

$$(0, \ldots, e^{i\alpha_i}, \ldots, 0)$$

(10)

with probability $p_i = |\psi_i|^2$. The following time evolution of $\Gamma(t)$ will satisfy this Born rule.

2.1 An algorithm that simulates the Born law of Quantum Mechanics

For the sake of clarity we will make an analogy with a dice with six faces. Exploiting this similarity we can find a rule that gives the sequence $\Gamma(t + m \Delta)$ from the immediately preceding sequence, that is $\Gamma(t + (m - 1) \Delta)$. Suppose that the six faces of the dice are not equally probable but at any time each face has probability $p_i$, given by $p_i(t + (m - 1) \Delta) = n_i/N$. $n_i$ is the number of times that $i$ appears in the sequence $\Gamma(t + (m - 1) \Delta)$. Then we can get a new sequence $\Gamma(t + m \Delta)$ simply throwing this (non-equally probable) dice $N$ times.

We can repeat these steps, to obtain a new sequence $\Gamma(t + (m + 1) \Delta)$, and taking into account that the dice face probabilities (see also eq.(8))

$$p_i(t + (m \Delta)) = \frac{n_i}{N} \neq p_i(t + (m - 1) \Delta)$$

(11)

have changed since the sequence $\Gamma(t + (m \Delta))$ is changed.

In other words, throwing the dice, we get a sequence $\Gamma$ and, in its turn, the new sequence $\Gamma$ updates and changes the face probabilities $p_i$ of the dice (11). This iteration can be repeated several times: it can be shown that it exist a $M$ large enough that for any $m > M$ we always get

$$p_i(t + M \Delta) = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases}$$

(12)

and $k$ is an integer between 1 and 6. It can be shown that the value of $k$ at the end of the process occurs a fraction of times proportional to the initial probability $p_k(t = 0)$. Therefore this process simulates the Born rule but through a mechanical and deterministic process. In fact each $\Gamma(t + m \Delta)$ derives from $\Gamma(t + (m - 1) \Delta)$ (simply throwing the dice).

\footnote{The converse is not true, since any permutation of the sequence $\Gamma$ gives the same vector $\bar{\psi}$.}

\footnote{But this is not mandatory, since in general hidden variables can play a not negligible role in those situations too. We will not discuss this issue in this paper.}
2.2 The Born rule through random-walk-like algorithms

The simplest and more interesting method to simulate the Born rule in a measurement process is obtained considering processes similar to a random walk. Let us define what we mean for random walk process in this specific context: the face probabilities $p_i(t + m \Delta)$ are obtained from those at the preceding time $p_i(t + (m - 1) \Delta)$ through a (pseudo-)random algorithm, where for any $i$

$$p_i(t + m \Delta) = \begin{cases} p_i(t + (m - 1) \Delta) + d_i g_i & \text{if } 0 < p_i < 1 \\ 0 & \text{if } p_i < 0 \\ 1 & \text{if } p_i > 1 \end{cases}$$

where the $d_i << 1$ are some very small constants, while the $g_i$ are stochastic variables that can take only two values $+1$ or $-1$, with equal probability; they are only subjected to the following requirement

$$\sum_{i=1}^{6} d_i g_i = 0.$$ 

It is possible to show that for any time evolution of this type, it exists a $M$ large enough for which the equation (12) applies and the final value $k$ is statistically distributed as demanded by the Born rule.

2.3 When the wave function collapses into the eigenstate of an observable

In the previous section we reproduced the Born law, but we have not yet clarified when a physical system is correctly described by the Schrödinger equation and when the Schrödinger equation fails and it is replaced by a more complex evolution that induces the wave function collapse. This collapse involves the wave function of the full physical system, including the experimental apparatus. We know that the Schrödinger equation is correct during the time evolution when an atom does not interact with the rest of the environment; on the other side we know that an electron that travel across a bubble chamber leaves a track, and there is an elapsed time during which the electron wave function irreversibly collapses and choose a propagation direction. We talk about entanglement when the electron is not an isolated system, since it interacts with several atoms. In the following we will put the basis for a clearer mathematical distinction of the two systems, in both scenarios described above.

2.3.1 A metric in the configuration space to better define the entanglement

Hereafter we would like to delineate when the linear superposition of two physically stable states does not give a new stable state (as predicted by QM): instead this superposition immediately collapse into one of them, due to a new dynamics that we usually call a measurement process. The superposition principle is violated. In particular we would like to know why states of an electron propagating in different directions can be superposed without inducing a collapse, while the superposition of two states representing a macroscopic object in two completely different configurations inevitably collapses into one of them (e.g., alive or dead are very different configurations in the Schrödinger paradox). Our goal is to define a metric in the configuration space, in order to introduce a distance between any couple of physical states.

2.3.2 The time evolution during a measurement: the entanglement and possible violations of the Born law

In this paragraph we will show a new algorithm in order to give a slightly different time law for the sequence $\Gamma(t + m \Delta)$ (defined in (6)). The main difference is the physical variable $0 < f < 1$, that induces the wave function collapse if and only if $f$ is very close to 1. At each step of the iteration we cancel the first variable $x_1$ at the beginning of the sequence $\Gamma$, and we add a new variable $x_{new}$ at the end of $\Gamma$, as follows

$$\Gamma(m \Delta) = (x_1, \cdots, x_N) \Rightarrow \Gamma((m + 1) \Delta) = (x_2, \cdots, x_N, x_{new}).$$

The variable $1 \leq x_{new} \leq 6$ is chosen at each step as follows

$$x_{new} = x_N \quad \text{with probability } f$$

or

$$x_{new} = i \quad \text{with probability } p_i(m\Delta) (1 - f)$$
where we assume that $N$ is very large. $f$ introduces a correlation between $x_m$ and $x_{m-1}$, while this correlation is absent in the previous algorithm described in section 2.1. If $f > 1 - 1/N$ then the sequence $\Gamma$ collapses into a sequence of all equal integers $i$, after a large number of iterations \(^{14}\). The probability to get the specific value $i$ at the end of the collapse is close to $p_i$, the assigned probability at the beginning of the process.

When $f \simeq 0$, the probabilities $p_i$ are stable, $p_i(m\Delta) \simeq p_i((m-1)\Delta)$, and the mechanism that induces a wave function collapse is turned off. This means that the value of $f$ is probably related to what has been discussed in the previous subsection 2.3.1.

### 3 Conclusion

In this paper we have addressed the well known issue in quantum mechanics, concerning the lack of mechanical determinism: i.e. different final states can derive from identical initial quantum states. This lack of determinism is ascribed to the measuring process, when a wave function collapse occurs: one final state is selected among several ones, with apparently no a priori or theoretical reason. However the Heisenberg uncertainty principle does not necessarily imply the absence of a more fundamental deterministic theory. The statistical intrinsic aspect of QM could be due to our ignorance on some hidden variable dynamics. We have put the basis for a theory that embeds quantum mechanics, but it is a more fundamental theory since it satisfies mechanical determinism in all situations: different final states always correspond to different initial conditions, probably due to some hidden variables. Quantum mechanics neglects this hidden variables in the initial condition and only deals with the probability that a certain final state occurs.

First we have changed the definition of a physical state: instead of the usual wave function $\psi(x)$, we have a sequence of real numbers $\Gamma = \{x_1, ..., x_N\}$ plus a phase $0 < \alpha(x) < 2\pi$ as a function of $x$. Once a physical state is assigned according to the previous definition, one (and only one) wave function can be deduced through the following identification

$$\psi(x) \equiv \sqrt{n_x} \ e^{i\alpha(x)}$$

(15)

where $n_x$ is the probability density that a randomly chosen number, extracted from the sequence $\Gamma$, is $x$. The time evolution of the physical state corresponds to the time evolution of the sequence $\Gamma(t)$ and the phase function $\alpha(x,t)$. In normal situations these evolutions imply the Schroedinger equation for the wave function \(^{15}\), but when a measurement occurs then the wave function collapses and the Schroedinger equation fails. The exact dynamics now must take into account the full sequence $\Gamma(t)$.

We have addressed the issue to find few explicit examples where the time law of $\Gamma(t)$ is such that the Born law of quantum mechanics finally holds. We have not explicitly discussed this issue in the paper, but we are assuming that Einstein relativity is wrong and that a preferred reference frame really exists. This seems to be necessary, because we require time causality, a mechanical and deterministic time evolution, during which the wave function collapses. Since the wave collapse is non local, an absolute (and not relative) definition of time seems to be unavoidable \(^6\). However, even if we have probably shown that it is possible to build ad hoc a theory that can describe both the wave function collapse and the Schroedinger linear evolution, a simple and unified construction is still missing. Experimental searches should focus on the violations of the superposition principle and/or the Born law.

### References

[1] F. Caravaglios, Talk given at “Les Rencontres de Physique de la Vallee d’ Aoste”, La Thuile, March (2008). arXiv:0805.2057 [hep-ph].

\(^6\)This is equivalent to assume the existence of a preferred reference frame, where one can define a universal absolute time.