Chi-square and normal inference in high-dimensional multi-task regression

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Abstract: The paper proposes chi-square and normal inference methodologies for the unknown coefficient matrix $B^*$ of size $p \times T$ in a multi-task linear model with $p$ covariates, $T$ tasks and $n$ observations under a row-sparse assumption on $B^*$. The row-sparsity $s$, dimension $p$ and number of tasks $T$ are allowed to grow with $n$. In the high-dimensional regime $p \gg n$, in order to leverage the row-sparsity [33, 42], the multi-task Lasso is considered.

We build upon the multi-task Lasso with a de-biasing scheme to correct for the bias induced by the penalty. The de-biasing scheme requires the introduction of a new data-driven object, coined the interaction matrix, that captures the effective correlations between noise vector and residuals on different tasks. The interaction matrix is symmetric positive semi-definite, of size $T \times T$ and can be computed efficiently.

The interaction matrix lets us derive asymptotic normal and asymptotic $\chi^2_T$ results under general Gaussian design and the rate condition $sT + s \log(p/s)/n \to 0$ which corresponds to consistency in Frobenius norm of the multi-task Lasso. These asymptotic distribution results yield valid confidence intervals for single entries of $B^*$ and valid confidence ellipsoids for single rows of $B^*$. If the covariance of the design is unknown, a modification of the multi-task de-biasing scheme using the nodewise Lasso provides comparable confidence intervals and confidence ellipsoids for the $j$-th row of $B^*$, provided that the $j$-th column of the precision matrix $\Sigma^{-1}$ is sufficiently sparse.

While previous proposals in grouped-variables regression require row-sparsity $s \lesssim \sqrt{n}$ up to constants depending on $T$ and logarithmic factors in $(n,p)$ for unknown $\Sigma$, the de-biasing scheme using the interaction matrix provides confidence intervals and $\chi^2_T$ confidence ellipsoids under the conditions $\min(T^2, \log^2 p)/n \to 0$, allowing for row-sparsity $s \gg \sqrt{n}$ when $\|\Sigma^{-1}e_j\|_0 \log p \to 0$, up to logarithmic factors.

1. Introduction

1.1. Model

We consider a multi-task linear regression model with $T$ tasks, with $n$ i.i.d. observations $(x_i, y^{(1)}_i, ..., y^{(T)}_i)$, where $x_i \in \mathbb{R}^p$ is a random feature vector and $y^{(1)}_i, ..., y^{(T)}_i$ are $T$ different scalar responses. We assume that on each task $t = 1, ..., T$, the response $y^{(t)}_i$ satisfies a linear model

\begin{equation}
    y^{(t)}_i = x^{\top}_i \beta^{(t)} + \epsilon^{(t)}_i, \quad t = 1, ..., T
\end{equation}

where $\beta^{(t)} \in \mathbb{R}^p$ is the unknown coefficient vector on the task $t$. Throughout, $X \in \mathbb{R}^{n \times p}$ is the design matrix with $n$ rows $x^{\top}_1, ..., x^{\top}_n$. The linear models (1.1) may be rewritten in vector and matrix form

\begin{equation}
    y^{(t)} = X \beta^{(t)} + \epsilon^{(t)}, \quad Y = XB^* + E
\end{equation}
where \( y(t) = (y_1(t), ..., y_n(t))^\top \) and \( e(t) = (e_1(t), ..., e_n(t))^\top \) are vectors in \( \mathbb{R}^n \), \( Y \in \mathbb{R}^{n \times T} \) is the response matrix with columns \( y(1), ..., y(T) \), \( E \in \mathbb{R}^{n \times T} \) is a noise matrix with columns \( e(1), ..., e(T) \), and \( B^* \in \mathbb{R}^{p \times T} \) is an unknown coefficient matrix with columns \( \beta^{(1)}, ..., \beta^{(T)} \).

Estimation of \( B^* \) in the above multi-task model has been well studied during the last decade in the high-dimensional regime where \( p \gg n \), see for instance [33]. This literature on multi-task learning suggests to use a joint convex optimization problem over the tasks in order to estimate \( B^* \), namely

\[
\hat{B} = \arg \min_{B \in \mathbb{R}^{p \times T}} \left[ \frac{1}{2nT} \| Y - XB \|_F^2 + g(B) \right] = \arg \min_{B \in \mathbb{R}^{p \times T}} \left[ \frac{1}{2nT} \sum_{t=1}^T \sum_{i=1}^n (y_i(t) - x_i^\top B e_i)^2 + g(B) \right]
\]

where \( e_i \in \mathbb{R}^T \) is the \( t \)-th canonical basis vector, \( \| \cdot \|_F \) is the Frobenius norm of matrices and \( g : \mathbb{R}^{p \times T} \to \mathbb{R} \) is a convex penalty function. The role of the convex penalty \( g \) is to promote a shared structure on the coefficient vectors \( \beta^{(1)}, ..., \beta^{(T)} \). The most common shared structure is that of row-sparsity where one assumes that only a few features are relevant across all tasks: there is a support set \( S \subset \{1, ..., p\} \) of small cardinality (relatively to \( n, p \)) such that for every task \( t = 1, ..., T \), \( \beta^{(t)}_j = 0 \iff j \notin S \). Equivalently, \( e_j^\top B^* = 0_{n \times T} \) if and only if \( j \notin S \), i.e., only \( |S| \) rows of \( B^* \) are nonzero. In this case, the sparsity pattern encoded by \( S \subset \{1, ..., p\} \) is shared on all tasks, and previous literature on estimation in this setting uses a penalty proportional to the \( \ell_2,1 \) norm, \( g(B) = \lambda \sum_{j=1}^p \| B^\top e_j \|_2 \), or alternatively its Elastic-Net version \( g(B) = \lambda \sum_{j=1}^p \| B^\top e_j \|_2 + \mu \| B \|_F^2 \) for non-negative tuning parameters \( \lambda, \mu \geq 0 \). If the row-sparsity assumption holds and such \( \ell_2,1 \) penalty is used, estimation of \( B^* \) by \( \hat{B} \) is improved compared to estimating \( \beta^{(1)}, ..., \beta^{(T)} \) separately [33].

1.2. Noise and residuals: non-trivial correlations for non-separable penalties

Classical multivariate statistics studies the least-squares estimate \( \hat{B}^{(ls)} = (X^\top X)^{-1} X^\top Y \), which corresponds to \( g(\cdot) = 0 \) in the above minimization problem. Here, the estimation on two tasks is independent, as on the \( t \)-th task for \( t = 1, ..., T \) we have \( \hat{B}^{(ls)} e_t = (X^\top X)^{-1} X^\top y(t) \) for the \( t \)-th canonical basis vector \( e_t \in \mathbb{R}^T \): the estimator \( \hat{B}^{(ls)} e_t \) of the unknown regression vector \( \beta^{(t)} \) on the \( t \)-th task only depends on the \( t \)-th response \( y(t) \), and is independent of the other responses \((y^{(t')})_{t' \in \{1, ..., T\} \setminus \{t\}}\). By independence, if the noise \( E \) has i.i.d. mean-zero entries, then

\[
E[e(t)^\top (Y - X \hat{B}^{(ls)})] = 0_{n \times n} \quad \forall t \neq t' ,
\]

i.e., residual and noise on two different tasks are uncorrelated. A similar story holds for multi-task Ridge regression, which corresponds to \( g(B) = \mu \| B \|_F^2 \) in the above minimization problem. The optimization problem is separable in the sense that

\[
\hat{B}^{(R)} = \arg \min_{B \in \mathbb{R}^{p \times T}} \frac{\| Y - XB \|_F^2}{2nT} + \mu \| B \|_F^2 \quad \text{and} \quad \hat{B}^{(R)} e_t = \arg \min_{b \in \mathbb{R}^p} \frac{\| y(t) - Xb \|_2^2}{2nT} + \mu \| b \|_2^2
\]

equivalently define \( \hat{B}^{(R)} \). It follows again that \( \hat{B}^{(R)} e_t \) only depends on the \( t \)-th response \( y(t) \), and if \( E \) has i.i.d. mean-zero entries then (1.3) holds also for \( \hat{B}^{(R)} \) by independence.

The situation is more complex for non-separable penalty functions, for instance if the penalty is proportional to the \( \ell_2,1 \) norm, \( g(B) = \lambda \sum_{j=1}^p \| B^\top e_j \|_2 \) where \( e_j \in \mathbb{R}^p \) is the \( j \)-th canonical basis vector. The corresponding estimator studied throughout the paper is the multi-task Lasso

\[
\hat{B} = \arg \min_{B \in \mathbb{R}^{p \times T}} \left( \frac{1}{2nT} \| Y - XB \|_F^2 + \lambda \| B \|_{2,1} \right) \quad \text{where} \quad \| B \|_{2,1} = \sum_{j=1}^p \| B^\top e_j \|_2 .
\]
The estimate $\hat{B} e_t$ of the unknown vector $\beta^{(t)}$ on the $t$-th task depends in an intricate way on all the responses including $(y^{(t′)})_{t′ \in \{1, ..., T\}\setminus\{t\}}$. Note that this dependence of $\hat{B} e_t$ on all responses is purposeful: we hope to leverage a shared pattern on all tasks (e.g., if $B^*$ is row-sparse and a sparsity pattern is shared by all $\beta^{(t)}, t = 1, ..., T$) in order to improve estimation compared to $B^{(t_1)}$ or $B^{(t_2)}$. In this case, however, (1.3) does not hold and the correlation between the residual on task $t$ and the noise on task $t′$ is non-trivial. Our results below (specifically Lemma F.1) reveal that for $t, t′ \in [T]$,

$$
((Y - X \hat{B}) e_t)^T e(t′) \approx \begin{cases} 
\sigma^2 (n - \hat{A}_{tt′}) & \text{if } t = t′ \\
-\sigma^2 \hat{A}_{tt′} & \text{if } t \neq t′ 
\end{cases}
$$

when the noise $E$ has i.i.d. $\mathcal{N}(0, \sigma^2)$ entries and $\hat{A}_{tt′}$ is the $(t, t′)$ entry of a symmetric matrix $\hat{A} \in \mathbb{R}^{T \times T}$ defined in Section 2. This matrix plays a central role in the present paper to derive asymptotic normality and asymptotic $\chi^2$ results.

1.3. Confidence intervals for linear functionals of $\beta^{(1)}$

A first goal of the present paper is to provide confidence intervals for linear functionals of the regression vector on the first task. Throughout the paper, regarding asymptotic normality and confidence intervals, $a \in \mathbb{R}^p$ is a fixed direction of interest and we wish to construct confidence intervals for $a^\top \beta^{(1)}$. For instance, the direction $a \in \mathbb{R}^p$ may be of the following form.

(i) a canonical basis vector $e_j \in \mathbb{R}^p$. For $a = e_j$, the goal is to construct confidence intervals for $a^\top \beta^{(1)} = \beta_j^{(1)}$, the coefficient of the $j$-th feature on the first task. This is the classical goal in statistics where one wishes to provide inference on the effect of the $j$-th covariate.

(ii) a new feature vector $x_{new} \in \mathbb{R}^p$, that may for instance correspond to the characteristics of a new subject whose responses $(y_{new}^{(1)}, ..., y_{new}^{(T)})$ are not known yet. The goal is to provide a confidence interval for $a^\top \beta^{(1)}$ which corresponds to the expected response of $Y_{new}$ conditionally on the feature vector $x_{new}$.

We stress here that the first task ($t = 1$) has a special role: the unknown parameter $a^\top \beta^{(1)}$ only involves the first unknown coefficient vector $\beta^{(1)}$ and not the other coefficient vectors $\beta^{(t)}, t = 2, ..., T$. If a single linear model $y^{(1)} = X \beta^{(1)} + \varepsilon^{(1)}$ is observed, the construction of confidence intervals for $a^\top \beta^{(1)}$ has been extensively studied. Most related to the present paper, [56, 51, 26, 27] initially provided methodologies for de-biasing (or de-sparsifying) the Lasso for construction of confidence intervals in a canonical basis direction $a = e_j$ for sparsity $s \lesssim \sqrt{n}/\log p$. [28] extended the sparsity requirement to $s \lesssim n/(\log p)^2$, [58, 11, 13, 14, 59, 5] studied estimation and construction of confidence intervals in dense direction $a \in \mathbb{R}^p$, and [6] extended the de-biasing methodologies to arbitrary convex penalties.

Of course, one could throw away the responses $y^{(2)}, ..., y^{(T)}$ and use only the response $y^{(1)}$ with the aforementioned methodologies, since our goal is to construct confidence intervals for $a^\top \beta^{(1)}$. However, throwing away the responses on tasks $2, ..., T$ should intuitively lead to information loss and is not desirable.

1.4. Asymptotic $\chi^2$ results and confidence ellipsoids for rows of $B^*$

The second goal of the paper is to develop confidence ellipsoids for whole rows of the unknown matrix $B^*$. The $j$-th row of $B^*$ is the vector $(B^*)^\top e_j$ in $\mathbb{R}^T$ where $e_j \in \mathbb{R}^p$ is the $j$-th canonical
vector. Given a confidence level $\alpha \in (0, 1)$, a confidence ellipsoid for $(B^*)^\top e_j$ is a subset $\hat{\mathcal{E}}_\alpha$ of $\mathbb{R}^T$ constructed from the data such that

$$\mathbb{P}((B^*)^\top e_j \in \hat{\mathcal{E}}_\alpha) \geq 1 - \alpha - o(1)$$

where $o(1)$ converges to 0 as $n \to +\infty$. Ideally, the confidence ellipsoid enjoys the exact nominal coverage probability $1 - \alpha$ asymptotically in the sense that

$$(1.5) \quad |\mathbb{P}((B^*)^\top e_j \in \hat{\mathcal{E}}_\alpha) - (1 - \alpha)| \to 0$$

as $n \to +\infty$. Note that one could also consider confidence sets $\hat{\mathcal{E}}_\alpha$ that are not ellipsoids (e.g., hyperrectangles); we focus here on ellipsoids as they are the natural confidence sets stemming from $\chi^2$-distributed pivotal quantities. As in classical multivariate statistics, an advantage of confidence ellipsoids is that they provide simultaneous confidence intervals for every direction $b \in \mathbb{R}^T$, that is, $\mathbb{P}(\forall b \in \mathbb{R}^T, e_j^\top B^* b \in \{b^\top u, u \in \hat{\mathcal{E}}_\alpha\}) \to 1 - \alpha$ when (1.5) holds and $\hat{\mathcal{E}}$ is closed and convex.

Such a confidence ellipsoid allows to perform hypothesis tests of

$$(1.6) \quad H_0 : (B^*)^\top e_j = 0_{T \times 1} \quad \text{against} \quad H_1 : \|(B^*)^\top e_j\|_2 \geq \rho,$$

where the null hypothesis corresponds to the signal $Y$ being independent of the $j$-th feature $X e_j$, and $\rho > 0$ is a separation radius. If a single task is observed ($T = 1$), it is impossible to distinguish between the null $\beta_j = 0$ and the alternative $\beta_j \neq 0$ with constant type I and type II errors unless $|\beta_j| \geq c n^{-1/2}$ for some constant $c > 0$. This follows by noting that the total variation distance between $Y H_0 = X \beta H_0 + \varepsilon$ and $Y H_1 = X \beta H_1 + \varepsilon$ converges to 0 if $\beta H_0, \beta H_1$ are the same except on coordinate $j$ where $|\beta_j H_0 - \beta_j H_1| = a_n$ with $a_n = o(\sigma n^{-1/2})$

$\|X e_j\|^2 / n \to 1$ and $\varepsilon \sim N_n(0, I_{n \times n})$, for instance by Pinsker’s inequality and a standard bound on the Kullback-Leibler divergence of two multivariate normals. If several tasks are observed as in the setting of interest here, we will see that it is possible to perform the hypothesis test (1.6) in situations where all nonzero coefficients of $(B^*)^\top e_j$ are of order $o(\sigma n^{-1/2})$, i.e., of indistinguishable order when a single task is observed.

If asymptotic normality results are available for each of the $T$ individual coefficients of $(B^*)^\top e_j$ (for instance such as those described in the previous subsection), a natural strategy to construct confidence ellipsoids is to sum the square of the $T$ asymptotically normal random variables and hope that the resulting sum has approximately the $\chi^2$ distribution with $T$ degrees-of-freedom. However, throughout the paper the number of tasks $T$ is allowed to grow to infinity with $n$ which results in some challenges regarding this strategy, as pointed out by [41]. For the sake of illustrating the resulting difficulty, assume that we have established the asymptotic normality of $T$ pivotal random variables $U_1, \ldots, U_T$ by proving decompositions of the form $U_t = (\tilde{\sigma} / \sigma) Z_t + B_t$ where $Z_t \sim N(0, 1)$ and the convergence in probability $\tilde{\sigma} / \sigma \to 1$ and $B_t \overset{p}{\to} 0$ hold, so that Slutsky’s theorem ensures that the pivotal quantities are asymptotically normal with $U_i \overset{d}{\to} N(0, 1)$. Denoting by $\chi^2_T = \sum_{t=1}^T Z_t^2$, summing the squares of the pivotal quantities and applying the triangle inequality for the Euclidean norm on $\mathbb{R}^T$ yields

$$(1.7) \quad |\sqrt{T} \sum_{t=1}^T U_t^2 - \sqrt{T} \chi^2_T| \leq |\tilde{\sigma} / \sigma - 1| \sqrt{\chi^2_T} + \sqrt{T} \sum_{t=1}^T B_t^2.$$ 

While $\mathbb{E}(\chi^2_T)^{1/2}$ is of order $\sqrt{T}$, the variance and quantiles of $(\chi^2_T)^{1/2}$ are of constant order (specifically, $\mathbb{P}(1/2 - \sqrt{T} \leq \chi^2 / \sqrt{2}) \to 1 - \alpha$ holds by (F.17) below, and $\text{Var}((\chi^2_T)^{1/2}) \to 1/2$ by [25]). This implies that a sufficient condition that ensures that $(\sum_{t=1}^T U_t^2)^{1/2}$ and $(\chi^2_T)^{1/2}$ asymptotically share the same quantiles is that $\sum_{t=1}^T B_t^2 \overset{p}{\to} 0$ and $\sqrt{T} |\tilde{\sigma} / \sigma - 1| \overset{p}{\to} 0$. While $B_t \overset{p}{\to} 0$
and $\hat{\sigma}/\sigma \xrightarrow{p} 1$ are sufficient to grant asymptotic normality for $U_1$ on the first task, the conditions $\sqrt{T}(\hat{\sigma}/\sigma - 1)\zeta_0$ and $(\sum_{t=1}^T B_t^2)^{1/2}\zeta_0$ are much more stringent as they involve the number of tasks $T$.

### 1.5. Asymptotics and assumptions

We will derive asymptotic normality and asymptotic $\chi^2$ results for a sequence of multi-task regression problems of increasing dimensions. For each $n$, we consider the multi-task linear model (1.2) and the multi-task Lasso estimate $\hat{B}$ in (1.4) where $B^*$, the number of tasks $T$, dimension $p$, tuning parameter $\lambda$ and row-sparsity $s$ are all functions of $n$. The dependence in $n$ is implicit and will be omitted to avoid notational burden. We will assume that the sequence of regression problems satisfies the following.

**Assumption 1.1.**

(i) $X \in \mathbb{R}^{n \times p}$ is a Gaussian design matrix with i.i.d. $N_p(0, \Sigma)$ rows;

(ii) $E \in \mathbb{R}^{p \times T}$ is a row-sparse unknown matrix with at most $s$ nonzero rows;

(iii) $E$ is a Gaussian noise matrix with i.i.d. $N(0, \sigma^2)$ entries;

(iv) $\{s, n, T, p\}$ are positive and satisfy $\sqrt{s}(T + \log \frac{p}{n}) \rightarrow 0$ and $n \leq p$, this implies $\frac{s}{p} \sqrt{\frac{T}{n}} \rightarrow 0$;

(v) The spectrum of $\Sigma$ is bounded: $C_{\min} \leq \phi_{\min}(\Sigma) \leq \phi_{\max}(\Sigma) \leq C_{\max}$ for some constants $0 < C_{\min} \leq C_{\max}$ which are independent of $n, p, s, T$;

(vi) $\Sigma$ satisfies $\max_{j=1, \ldots, p} \sigma_{jj} \leq 1$;

(vii) For two constants $\eta_1, \eta_2 > 0$, the tuning parameter $\lambda$ in (1.4) is given by

\begin{equation}
\lambda = (1 + \eta_2)\lambda_0, \quad \text{where} \quad \lambda_0 = \left( \max_{j=1, \ldots, p} \Sigma_{jj}^{1/2} \right) \frac{(1 + \eta_1)}{\sqrt{nT}} \left(1 + \sqrt{(2/T)\log(p/s)}\right).
\end{equation}

### 1.6. Related literature

For integers $\bar{n}, \bar{p} \geq 1$, the multi-task setting above bears resemblance with the single-response linear model of the form

\begin{equation}
\bar{y} = \bar{X}\bar{\beta} + \bar{\varepsilon}
\end{equation}

where $\bar{y} \in \mathbb{R}^{\bar{n}}$, $\bar{\varepsilon} \in \mathbb{R}^{\bar{n}}$, $\bar{X} \in \mathbb{R}^{\bar{n} \times \bar{p}}$, and the features $\{1, \ldots, \bar{p}\}$ are partitioned into $p$ groups with equal sizes. Indeed, with $\bar{p} = pT$, $\bar{n} = nT$ and by vectorizing the matrices in (1.1), our multi-task setting is in one-to-one correspondence with the single-response linear model (1.9) with $\bar{y} = \text{vec}(Y)$, $\bar{\varepsilon} = \text{vec}(E)$, $\bar{X}$ block diagonal with $T$ blocks each equal to $X$, and the partition $(G_1, \ldots, G_p)$ of $\{1, \ldots, \bar{p}\}$ into $p$ groups is given by $G_j = \{j + (t-1)p, t = 1, \ldots, T\}$. With this correspondence, the estimator $\hat{B}$ is the group Lasso $\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \|\bar{y} - \bar{X}\beta\|^2/(2\bar{n}) + \lambda\|\beta\|_2^2$, where $\|\beta\|_2 = \sum_{j=1}^p \|\beta_{G_j}\|$. Inference for grouped variables in a single-response linear model (1.9) focuses on estimation, hypothesis tests or confidence sets for the vector $\hat{\beta}_{G_j}$ for a group $G_j \subset \{1, \ldots, \bar{p}\}$ of interest. In the single task setting (1.9) with grouped variables, [41] extends the de-biasing methodology in [56, 51] to inference for grouped variables and provides $\chi^2$ asymptotic distribution results. The paper [41] already describes some challenges of chi-square inference in high-dimension (cf. the discussion after (1.7)); the multi-task problem of the present paper shares some of these challenges, however our approach and proofs have no overlap with that of [41]. The papers [47, 52] give a different extension of the de-biasing methodology of [56, 51] to the group setting, again based on the group Lasso, but here by estimation of the inverse covariance matrix restricted to the group of interest with a multi-task estimator penalized by the
nuclear norm. False Discovery Rate control in single-task linear models with grouped variables has been studied in [12] with a group SLOPE estimator. Under weak assumptions (in particular, no assumption on \( X \)), [37] provides an approach to inference for grouped variables, although the resulting confidence regions are conservative. The papers [34, 35] study group inference in a sequence rejection fashion when the groups are hierarchically ordered. Bootstrap methods based on the group Lasso are studied in [57], without trying to remove the bias. The paper [20] develops conservative inference methods for quantities of the form \((\hat{\beta}_{G_j} - \text{init})^\top A \hat{\beta}_{G_j}\) for a group \(G_j \subset [p]\) of interest and a given positive definite matrix \(A \in \mathbb{R}^{[G_j] \times |G_j|}\), based on the quadratic program de-biasing methodology given in [56, 26]. Finally, [6] introduces a degrees-of-freedom adjustment for the group Lasso to perform inference on a single coordinate or linear form of the unknown regression vector in (1.9).

Some papers focus on estimation and inference in the multi-task model (1.2). The papers [52, 8] study multi-task models of the form (1.2) where the noise \( E \in \mathbb{R}^{n \times T}\) has i.i.d. rows, and the entries within each row are correlated. A multi-task extension of the square-root Lasso is developed to concurrently estimate \(B^*\) and the correlations in the noise \(E\). Such results on estimating the correlations of the entries in \(E\) are useful to de-bias the group Lasso in the single-task model [52]. Support recovery through bounds on the group norm \(\|B\|_{2,\infty} = \sup_{j \in [p]} \|E_j^\top e_j\|_2\) is studied in [36] under a mutual incoherence assumption on \(X\). The mutual incoherence assumption requires a row-sparsity level \(s \lesssim \sqrt{n}\) if \(X\) has i.i.d. entries. Closest to the setup and goals of the present paper, [17] extends the de-biasing methodology of [56, 51] to the multi-task setting, using the nodewise Lasso to estimate a column of the precision matrix of the design. This approach requires row-sparsity of \(B^*\) of order \(s \lesssim \sqrt{n}\) up to logarithmic factors. Although our approach also involves the nodewise Lasso to estimate columns of the precision matrix, the de-biasing methodology significantly differs from [17] and cannot be seen as a straightforward extension of [56, 51]: our approach requires the introduction of a data-driven symmetric matrix \(\hat{A}\) of size \(T \times T\) which captures the interactions between the residuals on different tasks. Introduction of this novel object lets us significantly relax the requirement on the row-sparsity of \(B^*\) while obtaining normal and \(\chi^2_T\) inference results, that are proved to be non-conservative under some assumption on \(T, s, n, p\).

### 1.7. Adjustments in high-dimensional inference

In single-task models, recent literature on high-dimensional inference has highlighted the necessity to adjust classical inference principles with scalar adjustments. To describe such adjustments consider a single-task linear model \(y = X\beta + \varepsilon\) with \(\beta \in \mathbb{R}^p\), Gaussian noise \(\varepsilon \sim N_{\text{iid}}(0, \sigma^2 I_{n \times n})\) and \(X\) with i.i.d. \(N_{\text{iid}}(0, \Sigma)\) rows, where an initial estimator \(\hat{\beta}^{\text{init}}\) is available. If one is interested in confidence intervals for the projection \(a^\top \beta\) in some direction \(a\) normalized with \(\|\Sigma^{-1/2}a\|_2 = 1\), a 1-step MLE correction in direction \(\Sigma^{-1}a\) [54], i.e., maximizing the likelihood over the one-dimensional model \(\{\beta^{\text{init}} + u\Sigma^{-1}a, u \in \mathbb{R}\}\) yields the corrected estimate

\[
(1.10) \quad a^\top \beta^{\text{init}} + z_0^\top (y - X\hat{\beta}^{\text{init}})z_0^{-2}
\]

where \(z_0 = X\Sigma^{-1}a\) when \(\|\Sigma^{-1/2}a\|_2 = 1\); and the direction \(\Sigma^{-1}a\) is the one that maximizes the Fisher information [54]. (Since \(\|z_0\|^2 \gtrsim \chi^2_n\) concentrates around \(n\), we allow ourselves to replace \(\|z_0\|^2\) by \(n\) in (1.10) in this informal discussion). In high dimensions, this general principle requires a modification that accounts for the degrees-of-freedom of \(\hat{\beta}^{\text{init}}\): [27, 5] for the Lasso and [6] for general penalty suggest to amplify the correction with the degrees-of-freedom adjustment \((1 - df/n)^{-1}\) and to use the estimate

\[
(1.11) \quad a^\top \beta^{\text{init}} + (1 - df/n)^{-1}z_0^\top (y - X\hat{\beta}^{\text{init}})n^{-1}
\]
instead of (1.10). If $\hat{\beta}^{\text{init}}$ is the Lasso, the adjustment $(1 - \hat{d}/n)^{-1}$ is required for efficiency for large sparsity levels [5]. For the Lasso, the data-driven adjustment $(1 - \hat{d}/n)^{-1}$ may be replaced by a deterministic scalar adjustment, i.e.,

\begin{equation}
(1.12) \quad a^T \hat{\beta}^{\text{init}} + (1 - \delta^{-1}s_*)^{-1} \varepsilon_0^T (y - X\hat{\beta}^{\text{init}}) n^{-1}
\end{equation}

where $\delta = n/p$ and $s_*$ is the scalar parameter obtained after solving the system of two equations with two unknowns in [40, Proposition 3.1]. The correspondence between $\hat{d}/n$ and $s_*$ can be seen in [40, Theorem F.1] or [16, Section 3.3]. This system of two nonlinear equations first appeared in [1] for the Lasso and can be extended to permutation invariant penalty functions (see [15] and the references therein) and robust M-estimators [50].

We are not aware of previous proposals to study such high-dimensional adjustments in the multi-task setting, e.g., by extending the data-driven adjustment in (1.11) or the deterministic one in (1.12). One goal of the paper is to fill this gap.

### 1.8. Contributions

To summarize Sections 1.3 and 1.4, the inferential goals of the paper are twofold:

(i) To construct valid confidence intervals for a linear functional $a^T \beta^{(1)}$ of the unknown coefficient on the first task, by leveraging responses on all tasks simultaneously.

(ii) To construct valid confidence ellipsoids for rows $e_j^T B^* \in \mathbb{R}^{1 \times T}$ of the unknown coefficient matrix $B^*$, for instance to provide hypothesis tests on the nullity of the $j$-th row of $B^*$, or equivalently testing that the signal does not depend on the $j$-th covariate.

In order to achieve these statistical goals, we introduce a new object, the data-driven symmetric matrix $\hat{A} \in \mathbb{R}^{T \times T}$. Introduction of the matrix $\hat{A}$ is key to equip the estimator $\hat{B}$ with the inference capabilities (i) and (ii) above, as the theory and simulations of the next sections will show. This data-driven matrix $\hat{A}$ generalizes, to the multi-task setting, the effective degrees-of-freedom and other scalar adjustments in single-task linear models discussed in the previous subsection. Since $\hat{A}$ is symmetric, $T(T+1)/2$ scalar adjustments are necessary in the multi-task setting and that number of adjustments is unbounded if $T \to +\infty$ as a function of $n$. The fact that a growing, unbounded number of scalar adjustments would be necessary to achieve the above inference capabilities in the multi-task setting was surprising—at least to us—, since existing works on adjustments in high-dimensional statistics so far only require a bounded number of scalar adjustments.

The paper also includes contributions related to the performance of the multi-task estimator $\hat{B}$ in (1.4). We improve the logarithmic dependence in tuning parameter $\lambda$ and the known upper bounds on $\|\hat{B} - B^*\|_F$ and $\|X(\hat{B} - B^*)\|_F$ compared to [33]. We also develop tools to show that the random matrix $X$ enjoys a multi-task Restricted Eigenvalue (RE) condition from [9]. Although the single-task case follows in a straightforward manner from Gordon’s escape through a mesh theorem (e.g., [44]), the multi-task version of the RE condition for the random matrix $X$ requires different tools.

### 1.9. Organization

The rest of the paper is organized as follows. The next section summarizes notation. Section 2 describes a new quantity, the interaction matrix $\hat{A}$ that plays a major role in our estimates and confidence intervals. Section 3.1 constructs confidence intervals for $a^T \beta^{(1)}$ when the covariance matrix $\Sigma$ of the design is known. Section 3.2 extends these results and methodologies when
We define the vectorization \( \text{vec} \). Section 4 develops confidence ellipsoids for rows of \( B^* \). Section 5 provides an efficient way of computing the interaction matrix. Section 6 presents numerical experiments that corroborate our theoretical findings. The proofs are deferred to appendices and some intuition behind the main technical argument is given in Appendix A.

1.10. Notation

Throughout the paper, the linear model vector and matrix notation (1.2) holds. \( T, p \) and \( s \) are all non-decreasing functions of \( n \). In all the displays of convergence (e.g., \( \rightarrow, \lim, o(\cdot), O(\cdot) \)), we implicitly mean that \( n \) goes to \( \infty \). Convergence in distribution and in probability are denoted by \( \xrightarrow{d} \) and \( \xrightarrow{p} \).

Estimators of the unknown \( B^* \) are denoted by \( \hat{B} \). For any real \( a, a_+ = \max(0, a) \) and \([k] = \{1, \ldots, k\} \) for any integer \( k \), e.g., \([n], [p], [T] \). We use indices \( i, i', i_1, i_2, \ldots \) to sum or loop over \([n] \) (i.e., over the \( n \) observations), indices \( t, t', t_1, t_2, \ldots \) to sum or loop over \([T] \) (i.e., over the \( T \) tasks), indices \( j, j', j_1, j_2, \ldots \) to sum or loop over \([p] \) (i.e., the \( p \) covariates). The vectors \( e_j \in \mathbb{R}^p, e_t \in \mathbb{R}^T, e_i \in \mathbb{R}^n \) denote the canonical basis vector of the corresponding index; the size of such canonical vector will be made explicit if it is not clear from context. The identity matrices of sizes \( p \times p, n \times n, T \times T \) are \( I_{p \times p}, I_{n \times n} \) and \( I_{T \times T} \) respectively and \( 0_{k \times q} \) is the zero matrix with \( k \) rows and \( q \) columns.

For any \( q \geq 1 \), \( \| \cdot \|_q \) is the \( \ell_q \)-norm of vector, e.g., \( \| \cdot \|_2 \) is the Euclidean norm. For any matrix \( M, \| M \|_F \) is the Frobenius norm and \( \| M \|_{op} = \sup_{\|u\|_2 = 1} \| Mu \|_2 \) the operator norm, also known as the spectral norm. If \( M \) is symmetric, \( \phi_{\min}(M) \) (resp. \( \phi_{\max}(M) \)) denotes the smallest (resp. largest) eigenvalue of \( M \). The Moore-Penrose pseudoinverse of matrix \( M \) is denoted by \( M^\dagger \). The Kronecker product between two matrices \( U, V \) with \( U \in \mathbb{R}^{k \times q} \) is

\[
U \otimes V := \begin{pmatrix} u_{11}V & \cdots & u_{1q}V \\ \vdots & \ddots & \vdots \\ u_{k1}V & \cdots & u_{kq}V \end{pmatrix}
\]

so that \( I_{T \times T} \otimes X = \begin{pmatrix} X & 0_{n \times p} & \cdots & 0_{n \times p} \\ 0_{n \times p} & X & \cdots & 0_{n \times p} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n \times p} & \cdots & 0_{n \times p} & X \end{pmatrix} \)

for \( X \in \mathbb{R}^{n \times p} \). We will use the mixed product property of Kronecker products,

\[
(U \otimes V)(P \otimes Q) = (UP) \otimes (VQ),
\]

whenever the dimensions are such that the matrix products \( UP \) and \( VQ \) make sense. The following trace property also holds

\[
\text{Tr}[U \otimes V] = \text{Tr}[U] \text{Tr}[V].
\]

If \( \| \cdot \| \) denotes a Schatten norm (e.g., Frobenius or spectral norm), then for any \( U, V \) we have

\[
\| U \otimes V \| = \| U \| \| V \|.
\]

We define the vectorization \( \text{vec}(U) \) of any matrix \( U \in \mathbb{R}^{m \times q} \) by stacking vertically the columns of \( U \) into a column vector in \( \mathbb{R}^{m \times 1} \), i.e.,

\[
\text{vec}(A)^\top = (u_{11} \quad u_{21} \quad \cdots \quad u_{m1} \quad u_{12} \quad u_{22} \quad \cdots \quad u_{m2} \quad u_{1q} \quad u_{2q} \quad \cdots \quad u_{mq}).
\]

For any three matrices \( A, B, C \) such that the matrix product \( ABC \) makes sense, the above vectorization operator satisfies

\[
\text{vec}(ABC) = (C^\top \otimes A) \text{vec}(B).
\]
These many properties of Kronecker products are referenced in Section 4.2 of [23].

We consider restrictions of vectors (respectively matrices) by zeroing the corresponding entries (respectively columns). More precisely, if $v \in \mathbb{R}^p$ and $B \subset [p]$ then $v_B \in \mathbb{R}^p$ is the vector with $(v_B)_j = 0$ if $j \notin B$ and $(v_B)_j = v_j$ if $j \in B$. If $X \in \mathbb{R}^{n \times p}$ and $B \subset [p]$, $X_B \in \mathbb{R}^{n \times p}$ is a matrix of the same dimension as $X$ such that $(X_B)e_j = 0$ if $j \notin B$ and $(X_B)e_j = Xe_j$ if $j \in B$, i.e., $X_B$ is a copy of $X$ after having zeroed the columns not indexed in $B$. Finally, $I \{ i \in B \} = 1$ if $i \in B$ and $I \{ i \in B \} = 0$ if $i \notin B$.

2. The interaction matrix $\hat{A}$ of the Multi-Task Lasso estimator

We consider the multi-task Lasso estimator, with $\ell_{2,1}$ penalty, given (1.4) for some tuning parameter $\lambda > 0$. Let $\hat{S} = \{ j \in [p] : \hat{B}^\top e_j \neq 0 \}$ denote the set of nonzero rows of $\hat{B}$. We will refer to $\hat{S}$ as the support of $\hat{B}$ and denote by $|\hat{S}|$ its cardinality. The above estimator is the one commonly used in the multi-task learning literature under a row-sparsity assumption on $B^*$, see, e.g., [33]. Recall that $X_{\hat{S}} \in \mathbb{R}^{n \times p}$ is a copy of $X$ obtained after zeroing the columns not belonging to $\hat{S}$. Define $\hat{X} := I_{T \times T} \otimes X_{\hat{S}}$ where $\otimes$ denotes the Kronecker product defined in Section 1.10, so that $\hat{X} \in \mathbb{R}^{nT \times pT}$ is block-diagonal with $T$ blocks, each equal to $X_{\hat{S}}$. Consequently $\hat{X}^\top \hat{X} = I_{T \times T} \otimes (X_{\hat{S}}^\top X_{\hat{S}}) \in \mathbb{R}^{(pT) \times (pT)}$ is also block-diagonal with $T$ blocks equal to $X_{\hat{S}}^\top X_{\hat{S}}$. For any $j \in \hat{S}$, define the matrix

$$H^{(j)} := \lambda \| \hat{B}^\top e_j \|_2^{-1} \left( I_{T \times T} - \hat{B}^\top e_j e_j^\top \hat{B} \| \hat{B}^\top e_j \|_2^{-2} \right) \in \mathbb{R}^{T \times T}$$

and note that $H^{(j)}$ is proportional to an orthogonal projection of rank $T - 1$. The matrix $H^{(j)}$ is the Hessian of $u \mapsto \lambda \| u \|_2$ at $u = \hat{B}^\top e_j$. Finally, let $\hat{H} \in \mathbb{R}^{(pT) \times (pT)}$ be the matrix defined by $\hat{H} := \sum_{j \in \hat{S}} H^{(j)} \otimes (e_j e_j^\top)$.

**Definition 1.** The interaction matrix $\hat{A} \in \mathbb{R}^{T \times T}$ of the estimator $\hat{B}$ in (1.4) is defined entrywise by

$$\hat{A}_{tt'} := \text{Tr} \left[ \left[ 0_{n \times p(t-1)} \mid X_{\hat{S}} \mid 0_{n \times p(T-t)} \right] \left[ \hat{X}^\top \hat{X} + n \hat{T} \hat{H} \right]^{\dagger} \left[ \begin{array}{c} 0_{p(T'-1) \times n} \\ (X_{\hat{S}})^\top \\ 0_{p(T-T') \times n} \end{array} \right] \right]$$

for all $t, t' \in [T]$, where $\dagger$ denotes the Moore-Penrose inverse. Equivalently, if $u, v \in \mathbb{R}^T$ then

$$u^\top \hat{A} v = \text{Tr} \left( \left[ u_1 X_{\hat{S}} \mid u_2 X_{\hat{S}} \mid \ldots \mid u_T X_{\hat{S}} \right] \left[ \hat{X}^\top \hat{X} + n \hat{T} \hat{H} \right]^{\dagger} \left[ v_1 X_{\hat{S}} \mid v_2 X_{\hat{S}} \mid \ldots \mid v_T X_{\hat{S}} \right]^\top \right),$$

or with Kronecker product notation,

$$u^\top \hat{A} v = \text{Tr} \left[ (u^\top \otimes X_{\hat{S}}) [\hat{X}^\top \hat{X} + n \hat{T} \hat{H}]^{\dagger} (v \otimes (X_{\hat{S}})^\top) \right].$$

Observe that $\sum_{j \in \hat{S}} (e_j e_j^\top) \otimes H^{(j)}$ is a block-diagonal matrix with $p$ diagonal blocks equal to $I \{ j \in \hat{S} \} H^{(j)}$. For $A, B$ any square matrices, $A \otimes B = P(B \otimes A)P^\top$ holds for a permutation matrix $P$ that only depends on the dimensions of $A$ and $B$. This permutation $P$ is particularly simple and known as a perfect shuffle. It follows that $P \hat{H} P^\top$ is block diagonal with $p$ diagonal blocks for some permutation matrix $P \in \mathbb{R}^{pT \times pT}$. Thus the matrix

$$\hat{X}^\top \hat{X} + n \hat{T} \hat{H} \in \mathbb{R}^{pT \times pT}$$
appearing in (2.2)-(2.3) is the sum of two matrices of size \( pT \times pT \), each summand being block diagonal but in a different basis. If \( \lambda = 0 \) then \( H = 0 \) and \( \hat{A} \) is diagonal as \( \hat{X}^\top \hat{X} + nT \hat{H} \) can be inverted by block. This corresponds to the unregularized least-squares estimate \( \hat{B}^{(0)} \) discussed in (1.2) with \( \hat{B}^{(0)}e_t \) depending on the \( t \)-th response \( y^{(t)} \) only. In the case \( \lambda > 0 \) of interest here, the matrix \( H \) induces nonzero entries outside of the \( T \) diagonal blocks of \( \hat{X}^\top \hat{X} \), the matrix (2.4) is not diagonal by block and the resulting matrix \( \hat{A} \) is not diagonal. Additional structure in (2.4) and \( \hat{A} \) is studied in Section 5, which yields an efficient and practical algorithm to compute \( \hat{A} \).

The interaction matrix plays a major role in the construction of our confidence intervals for \( a^\top \beta^{(1)} \) as well as for chi-square inference regions for rows of \( B^* \). A high-level interpretation of its role is that \( \hat{A} \) captures the correlation between the residuals on different tasks. The following proposition summarizes some useful properties of \( \hat{A} \). Result (iii) is important as our confidence interval for \( a^\top \beta^{(1)} \) defined in the next section will involve the inverse of \( I_{T \times T} - \hat{A}/n \). Proposition 2.1 is proved in Appendix B of the supplement.

**Proposition 2.1.** Let \( \hat{A} \) be defined by (2.2). Then

(i) \( \hat{A} \) is symmetric and positive semi-definite.

(ii) If \( X_{\hat{S}} \) is rank \( |\hat{S}| \) then the spectral norm of \( \hat{A} \) is bounded from above as \( \|\hat{A}\|_{op} \leq |\hat{S}| \).

(iii) If \( X_{\hat{S}} \) is rank \( |\hat{S}| \) and \( |\hat{S}|/n < 1 \) then \( I_{T \times T} - \hat{A}/n \) is positive-definite and \( \|I_{T \times T} - (I_{T \times T} - \hat{A}/n)^{-1}\|_{op} \leq (|\hat{S}|/n)/(1 - |\hat{S}|/n) \).

### 3. Asymptotic normality and confidence intervals in the multi-task setting

#### 3.1 Known \( \Sigma \): Pivotal random variable, asymptotic normality and confidence intervals

We assume throughout this section that the direction \( a \) of interest is normalized with \( \|\Sigma^{-1/2}a\|_2 = 1 \). This normalization assumption is relaxed in the next Section 3.2 where we develop a methodology for unknown \( \Sigma \). If \( \Sigma \) is known, our main result is the following where \( \hat{A} \) denotes the interaction matrix (2.2).

**Theorem 3.1.** Let Assumption 1.1 be fulfilled. Assume that \( \|\Sigma^{-1/2}a\|_2 = 1 \). If \( z_0 = X\Sigma^{-1}a \) then

\[
na^\top(\hat{B} - B^*)b + z_0^2(Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1}b \quad \|\!(Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1}b\!\|_2 \xrightarrow{d} \mathcal{N}(0,1)
\]

for any \( b \in \mathbb{R}^T \). Hence for \( b = e_1 \in \mathbb{R}^T \), the parameter \( a^\top \beta^{(1)} \) of interest satisfies

\[
na^\top\hat{B}e_1 - a^\top\beta^{(1)} + z_0^2(Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1}e_1 \quad \|\!(Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1}e_1\!\|_2 \xrightarrow{d} \mathcal{N}(0,1).
\]

Theorem 3.1 is proved in Appendix E. The left-hand sides of both displays in Theorem 3.1 can be interpreted as \( Z \)-scores that have asymptotically standard normal distribution. In the second display, the only unknown quantity on the left hand side is \( a^\top \beta^{(1)} \), the parameter of interest (while in the first display, the only unknown quantity is the scalar \( a^\top B^*b \)). Consequently if \( z_{\alpha/2} \) is the \( 1 - \alpha/2 \) quantile of the standard normal distribution such that \( \mathbb{P}(|\mathcal{N}(0,1)| \leq z_{\alpha/2}) = 1 - \alpha \), an asymptotic \( 1 - \alpha \) confidence interval for \( a^\top \beta^{(1)} \) is given by \([L_+^\alpha, L_-^\alpha]\) where

\[
L_\pm^\alpha = \frac{a^\top\hat{B}e_1}{\text{initial estimate}} \pm \frac{z_{\alpha/2}((Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1}e_1)}{n} \quad \text{bias correction using the interaction matrix confidence interval half-length}.
\]
Theorem 3.1 states that \( P(\mathbf{a}^\top \beta^{(1)}) \in [L_0^+, L_0^-]) \to (1 - \alpha) \) as \( n, p \to +\infty \).

The confidence interval is centered at \( \mathbf{a}^\top \hat{B} \mathbf{e}_1 \) (which can be interpreted as the initial estimate of \( \mathbf{a}^\top \beta^{(1)} \) given by the estimator \( \hat{B} \) in (1.4)) plus a de-biasing correction \( z_0^\top (Y - X \hat{B}) (I_{T \times T} - \hat{A}/n)^{-1} \) that involves the interaction matrix \( \hat{A} \) through the matrix inverse

\[
(I_{T \times T} - \hat{A}/n)^{-1}.
\]

The fact that penalized estimators such as (1.4) require a de-biasing correction should be expected since it is already the case for \( T = 1 \) for the Lasso \([56, 51, 26, 27, 28, 5]\) and any regularized least-squares \([6]\). However, the apparition in the de-biasing correction of the interaction matrix through the matrix inverse (3.3) is surprising at least to us: we did not expect the multi-task de-biasing correction to require a matrix inversion such as (3.3) when initially tackling this problem. The length of the confidence interval above is \( 2z_{1/2}n^{-1} ||(Y - X \hat{B}) (I_{T \times T} - \hat{A}/n)^{-1} \mathbf{e}_1 ||_2 \) when \( \mathbf{b} = \mathbf{e}_1 \), and an estimate of this norm is given by the following theorem.

**Theorem 3.2.** Let the assumptions and setting of Theorem 3.1 be fulfilled. Then

\[
||(Y - X \hat{B}) (I_{T \times T} - \hat{A}/n)^{-1} \mathbf{b}||_2^2/n \xrightarrow{P} \sigma^2 \text{ when } ||\mathbf{b}||_2 = 1.
\]

Consequently the length of the confidence interval is approximately \( 2z_{1/2}/n \sigma n^{-1/2} \) which is the typical length for two-sided confidence intervals for an unknown mean \( \mu \) when observing i.i.d. \( Y_1, \ldots, Y_n \) with \( \mathbb{E}[Y_i] = \mu, \text{Var}[Y_i] = \sigma^2 \). Theorems 3.1 and 3.2 are proved together in Appendix E.

**Comparison with single-task Lasso on the first task.** It is instructive to compare the above confidence interval with the confidence interval induced by a single-task Lasso estimator computed on \((\mathbf{X}, \mathbf{y}^{(1)})\), i.e., when throwing away the responses \( \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(T)} \) on tasks \( 2, \ldots, T \). This is also a good opportunity to analyse the form of \( \hat{A} \) and the matrix inversion (3.3) in the degenerate case where a single task is observed.

For \( T = 1 \), a response vector \( \mathbf{y}^{(1)} = \mathbf{X} \beta^{(1)} + \mathbf{e}^{(1)} \) in \( \mathbb{R}^n \) is observed and the estimator (1.4) reduces to the usual Lasso with response vector \( \mathbf{y}^{(1)} \),

\[
\hat{\beta}^L = \arg\min_{\mathbf{b} \in \mathbb{R}^p} ||\mathbf{y}^{(1)} - \mathbf{X} \mathbf{b}||^2/(2n) + \lambda ||\mathbf{b}||_1.
\]

The asymptotic normality result in Theorem 3.1 for \( \mathbf{b} = \mathbf{1} \) asserts that

\[
\frac{n \mathbf{a}^\top (\hat{\beta}^L - \beta^{(1)}) + (1 - \hat{A}_{11}/n)^{-1} z_0^\top (\mathbf{y}^{(1)} - \mathbf{X} \hat{\beta}^L)}{(1 - \hat{A}_{11}/n)^{-1} ||\mathbf{y}^{(1)} - \mathbf{X} \hat{\beta}^L||_2} \xrightarrow{d} \mathcal{N}(0, 1).
\]

In the degenerate case \( T = 1 \), the matrices in (2.1) are all zeros and the matrix \( \hat{A} \) reduces to a scalar \( \hat{A}_{11} \) equal to \( \text{Tr}[\mathbf{X} (\mathbf{S}_{L}^L \mathbf{X}^\top) \mathbf{X}^\top] = |\mathbf{S}_{L}^L| \) where \( \mathbf{S}_{L}^L \) is the support of the Lasso \( \hat{\beta}^L \). Here \( \hat{A}_{11} \) is the usual effective degrees-of-freedom for the Lasso. The factor \( (1 - \hat{A}_{11}/n) = (1 - |\mathbf{S}_{L}^L|/n)^{-1} \) in (3.4) is the degrees-of-freedom adjustment for the Lasso studied in [5], which is required for the asymptotic normality result (3.4) when \( s \gtrsim n^{2/3} \) [5]. So Theorem 3.1 reduces to the asymptotic normality result of [5] in the degenerate case \( T = 1 \), and in this case the matrix inversion (3.3) reduces to a degrees-of-freedom adjustment through the scalar multiplication by \( (1 - |\mathbf{S}_{L}^L|/n)^{-1} \). The length of the resulting confidence interval for \( \mathbf{a}^\top \beta^{(1)} \) when \( T = 1 \) (or when the tasks \( 2, \ldots, T \) are thrown away) is then

\[
2z_{1/2}n^{-1} ||\mathbf{y}^{(1)} - \mathbf{X} \hat{\beta}^L||_2 (1 - |\mathbf{S}_{L}^L|/n)^{-1}.
\]

We may compare the lengths of the two confidence intervals:
• The confidence interval \([L_1^1, L_2^1]\) based on (3.2) using the responses on all tasks \(1, ..., T\) with length \(2n^{-1} z_{\alpha/2} \|(Y - \hat{X}\hat{B})(I_{T \times T} - \hat{A}/n)^{-1} e_1\|_2\), and
• The confidence interval based on (3.4) obtained by throwing away the responses on tasks \(2, ..., T\) with length (3.5).

The length of the confidence interval based on \(\hat{B}\) and the responses on all tasks \(1, ..., T\) is smaller than the length (3.5) only when

\[
(3.6) \quad \|y^{(1)} - \hat{X}\hat{\beta}^{L}\|_2 (1 - |\hat{s}^{L}|/n)^{-1} > \|(Y - \hat{X}\hat{B})(I_{T \times T} - \hat{A}/n)^{-1} e_1\|_2.
\]

Our simulations in Section 6 (see Figure 5) reveal that (3.6) holds, in some situations with significant margins, when \(s\) is not too large. Since the comparison (3.6) can be performed by looking at the data, the practitioner should choose the multi-task confidence interval based on (3.4) only when (3.6) holds. When performing this comparison, two tests are constructed which calls for a Bonferroni correction to avoid invalid coverage due to multiple testing.

3.2. Unknown \(\Sigma\): Pivotal random variable, asymptotic normality and confidence intervals

The knowledge of \(\Sigma\) is not available in most practical situations and the methodology of the previous subsection cannot be applied. Indeed the left hand sides in Theorem 3.1 involve \(z_0 = X\Sigma^{-1}a\) which cannot be directly constructed from the data when \(\Sigma\) unknown. Another issue that arises when \(\Sigma\) is unknown is that one cannot verify the normalization \(\|\Sigma^{-1/2}a\|_2 = 1\) required in Theorem 3.1. Intuitively, though, if it was possible to estimate both \(z_0 = X\Sigma^{-1}a\) and \(\|\Sigma^{-1/2}a\|_2\) fast enough, replacing these quantities by their estimates in (3.2) should not break asymptotic normality. Following ideas from the early de-biasing literature \([56, 27, 51]\), we consider a direction

\[
(3.7) \quad a = e_j
\]

for some fixed covariate \(j \in \{1, ..., p\}\) and compute the nodewise Lasso

\[
(3.8) \quad \hat{\gamma}^{(j)} = \arg\min_{\gamma \in \mathbb{R}^p} \|X e_j - X_{-j} \gamma\|_2^2/(2n) + \hat{\tau}_j(1 + \eta)\sqrt{(2/n)\log p}\|\gamma\|_1
\]

for regressing \(X e_j\) on \(X_{-j}\), where \(X_{-j} \in \mathbb{R}^{n \times p}\) is the matrix \(X\) with \(j\)-th column replaced by a column of zeros, \(\hat{\tau}_j\) is a consistent estimate of \(\|\Sigma^{-1/2}e_j\|_2^{-1}\) and \(\eta > 0\) is a small constant. Alternatively, one may use the scale invariant version of (3.8) again for regressing \(X e_j\) on \(X_{-j}\),

\[
(3.9) \quad \hat{\gamma}^{(j)} = \arg\min_{\gamma \in \mathbb{R}^p; \gamma_j = 0} \left(\|X e_j - X_{-j} \gamma\|_2^2/(2n)\right)^{1/2} + (1 + \eta)\sqrt{(2/n)\log p}\|\gamma\|_1,
\]

known as Scaled lasso \([48]\) or square-root Lasso \([7]\), and (3.9) is equal to (3.8) with \(\hat{\tau}_j = \|X e_j - X_{-j} \hat{\gamma}^{(j)}\|_2/\sqrt{n}\). We finally set

\[
(3.10) \quad \hat{z}_j = X e_j - X_{-j} \hat{\gamma}^{(j)}.
\]

This corresponds to the residuals of the estimator \(\hat{\gamma}^{(j)}\) in the linear model

\[
(3.11) \quad X e_j = X_{-j} \gamma^{(j)} + \varepsilon^{(j)}
\]
with response vector $Xe_j \in \mathbb{R}^n$, design matrix $X_{-j}$, true regression vector $\gamma^{(j)} := -\Sigma^{-1/2}e_j\Sigma^{-1}e_j$ (so that $e_j^T\gamma^{(j)} = 0$ and $e_j^T\Sigma^{-1}\gamma^{(j)} = -(\Sigma^{-1})_{jj}^{-1}$ for $k \in [p] \setminus \{j\}$), and Gaussian noise vector $e^{(j)} := \|\Sigma^{-1/2}e_j\|_2^2 X\Sigma^{-1}e_j$ independent of $X_{-j}$ with distribution $e^{(j)} \sim N(0, \tau_j^2 I_{n \times n})$ where $\tau_j^2 := \|\Sigma^{-1/2}e_j\|_2^2 = (\Sigma^{-1})_{jj}^{-1}$. The relationship between $\Sigma^{-1}$ and $(\gamma^{(j)}, \tau_j)$ is the well known connection between precision matrix and linear regression for multivariate normal random vectors (see, e.g., [38, 49]).

The estimators $\hat{\gamma}^{(j)}$ in (3.8) and (3.9) both satisfy inequalities
\begin{align}
\|X^T_{-j}(Xe_j - X_{-j}\hat{\gamma}^{(j)})\|_\infty &= \|X^T_{-j}\hat{\gamma}_j\|_\infty \leq O_{p}(1)\tau_j \sqrt{n \log p}, \\
\|\hat{\gamma}^{(j)} - \gamma^{(j)}\|_1 &\leq O_{p}(1)\|\Sigma^{-1}\|_{op}\|\gamma^{(j)}\|_0 \tau_j \sqrt{\log(p)/n}
\end{align}
provided that $\|\Sigma^{-1}\|_{op}\|\gamma^{(j)}\|_0 \log(p)/n \to 0$. Inequality (3.13) is the usual $\ell_1$ estimation rate for the Lasso [9] or the Scaled Lasso [49, 7], and $\|\Sigma^{-1}\|_{op}$ represents a high-probability lower bound on the restricted eigenvalue in the linear model (3.11) [44]. Inequality (3.12) follows from the KKT conditions of (3.8) for the Lasso, and from the KKT conditions of (3.9) combined with $\hat{\tau}_j/\tau_j \to 1$ which holds thanks to properties of the Scaled or square root Lasso [49, 7]. Inequalities (3.12)-(3.13) are the only properties of $\hat{\gamma}^{(j)}$ that we will use in the proof of the following result. Other estimators $\hat{\gamma}^{(j)}$ could be used, for instance ones based on the Dantzig selector, as long as (3.12)-(3.13) are satisfied.

**Theorem 3.3.** Consider a canonical basis direction $e_j \in \mathbb{R}^p$ for some $j \in [p]$ and let Assumption 1.1 be fulfilled. Additionally assume that the sparsity of $\Sigma^{-1}e_j$ satisfies either
\begin{align}
n^{-1/2}\|\Sigma^{-1}e_j\|_0 \sqrt{T + \log(p/s)} \log p \to 0. \\
or
\|\Sigma^{-1}e_j\|_0 \log(p)/n \to 0 \quad \text{and} \quad \sqrt{s \log(p)[T + \log(p/s)]/n} \to 0.
\end{align}

Then for any estimator $\hat{\tau}^{(j)}$ satisfying (3.12)-(3.13) and every fixed $b \in \mathbb{R}^T$ we have
\begin{equation}
\frac{ne_j^T(\hat{B} - B^*)b + n(\hat{z}_j^TXe_j)^{-1}\hat{z}_j^T(Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1}b}{(\hat{\tau}_j)^{-1}\|((Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1}b)\|_2} \overset{d}{\to} \mathcal{N}(0, 1).
\end{equation}

Asymptotic normality (3.16) still holds if $\tau_j$ in the denominator is replaced by either $(\hat{z}_j^TXe_j/n)^{1/2}$ or $\hat{\tau}_j = (\|\hat{z}_j\|_2/\sqrt{n})$.

Theorem 3.3 is proved in Appendix G.1.

4. Confidence ellipsoids for rows of $B^*$

4.1. Known $\Sigma$

We first construct confidence ellipsoids with the knowledge of $\Sigma$.

**Theorem 4.1.** Define the observable positive semi-definite matrix $\hat{\Gamma} = (Y - X\hat{B})^T(Y - X\hat{B}) \in \mathbb{R}^{T \times T}$ as well as
\begin{equation}
x = (Y - X\hat{B})^Tz_0 + (nI_{T \times T} - \hat{A})(\hat{B} - B^*)^Ta.
\end{equation}
Then under Assumption 1.1, there exists a random variable $\chi^2_T$ with chi-square distribution with $T$ degrees of freedom such that
\[
\sqrt{1 - \frac{T}{n}} \|\hat{\Gamma}^{-1/2} \xi\|_2 - \sqrt{\chi^2_T} \leq o_p(1) + O_p\left(\min\left\{\frac{T}{\sqrt{n}}, \frac{s^2 \log^2(p/s)}{n \sqrt{T}}\right\}\right)
\]
as well as
\[
-o_p(1) - O_p\left(\frac{T}{\sqrt{n}} + \frac{sT + s \log(p/s)}{\sqrt{n}}\right) \leq \sqrt{1 - \frac{T}{n}} \|\hat{\Gamma}^{-1/2} \xi\|_2 - \sqrt{\chi^2_T}.
\]
Consequently,
\[
(i) \quad (1 - \frac{T}{n})^{1/2} \|\hat{\Gamma}^{-1/2} \xi\|_2 - (\chi^2_T)^{1/2} \leq o_p(1) \text{ holds if additionally } \min\left\{\frac{T^2}{n}, \frac{\log^2 p}{n}\right\} \to 0, \text{ and}
(ii) \quad (1 - \frac{T}{n})^{1/2} \|\hat{\Gamma}^{-1/2} \xi\|_2 - (\chi^2_T)^{1/2} \geq o_p(1) \text{ holds if additionally } \frac{T^2}{n} + \frac{sT + s \log(p/s)}{n} \sqrt{T} \to 0.
\]

Theorem 4.1 is proved in Appendix F. The following proposition with Proposition 4.2 is proved in Appendix F. If $\alpha \in (0, 1)$ is a fixed constant not depending on $n, T$ and $q_{T,\alpha} > 0$ is the quantile defined by $\mathbb{P}(\chi^2_T)^{1/2} \leq q_{T,\alpha} = 1 - \alpha$ then
\[
(i) \quad W_n - (\chi^2_T)^{1/2} \leq o_p(1) \implies \mathbb{P}(W_n \leq q_{T,\alpha}) \geq 1 - \alpha - o(1)
\]
and
\[
(ii) \quad W_n - (\chi^2_T)^{1/2} \geq -o_p(1) \implies \mathbb{P}(W_n \leq q_{T,\alpha}) \leq 1 - \alpha + o(1).
\]

Proposition 4.2 is proved in Appendix F. If $T \to +\infty$, the order of $q_{T,\alpha}$ is given by
\[
q_{T,\alpha} - \sqrt{T} \to z_\alpha / \sqrt{2}
\]
where $z_\alpha$ is the standard normal quantile defined by $\int_{-\infty}^{z_\alpha} (\sqrt{2\pi})^{-1} e^{-u^2/2} du = 1 - \alpha$. A short proof of (4.2) is given around (F.17); see [41] for related discussions. However, using $q_{T,\alpha}$ itself to construct confidence sets should be preferred in practice to avoid the approximation error in (4.2).

Combining the above two results provides confidence ellipsoids for the rows of $B^*$, or more generally for the unknown vector $(B^*)^\top a \in \mathbb{R}^T$ for a fixed direction $a \in \mathbb{R}^p$ of interest. Let $\hat{\mathcal{E}}_\alpha$ be the subset of $\mathbb{R}^T$ defined by
\[
\hat{\mathcal{E}}_\alpha := \left\{\theta \in \mathbb{R}^T : (1 - \frac{T}{n})^{1/2} \|\hat{\Gamma}^{-1/2} [(Y - X \hat{B})^\top z_0 + (nI_{T \times T} - \hat{A})(\hat{B}^\top a - \theta)]\|_2 \leq q_{T,\alpha}\right\}.
\]
Since $\hat{\mathcal{E}}_\alpha = \{\theta \in \mathbb{R}^T : (\theta - u)^\top C(\theta - u) \leq 1\}$ where $C = (q_{T,\alpha})^{-2}(1 - \frac{T}{n})(nI_{T \times T} - \hat{A})\hat{\Sigma}^{-1}(nI_{T \times T} - \hat{A})$ and $u = \hat{B}^\top a + (nI_{T \times T} - \hat{A})^{-1}(Y - X \hat{B})^\top z_0$, this set is an ellipsoidal region with center $u$. If
\[
\min\left\{\frac{T^2}{n}, \frac{\log^2 p}{n}\right\} \to 0
\]
additionally to Assumption 1.1 as required in case (i) of Theorem 4.1, then $\mathbb{P}[(B^*)^\top a \in \hat{\mathcal{E}}_\alpha] \geq 1 - \alpha - o(1)$. If additionally
\[
T^2 / n + \sqrt{T} (sT + s \log(p/s)) / n \to 0
\]
as required in case (ii) for the lower bound, then $P[(B^*)^\top a \in \hat{E}_\alpha] \to 1 - \alpha$ and the above confidence ellipsoid provides the exact nominal coverage (i.e., it is provably non-conservative). Note that the upper bound (i) is more important than the lower bound (ii) since the upper bound (i) guarantees that the type I error in the hypothesis test (1.6) is at most $\alpha$, i.e., $P[(B^*)^\top a \in \hat{E}_\alpha] \to 1 - \alpha - o(1)$. It is thus fortuitous that only the weak additional condition (4.3) is required for the upper bound (i) to guarantee the desired type I error, while the more stringent condition (4.4) is only required to prove non-conservativeness.

The additional assumption (4.3) is satisfied for a large class of growths of $(T, n, p)$. For instance it holds under polynomial growth $p \asymp n^\gamma$ or exponential growth of the form $p \lesssim \exp((n^{1/8-\gamma'})$ for constants $\gamma, \gamma' > 0$, as $\log_p n \to 0$ is then satisfied. Although we believe that the mild condition (4.3) is an artefact of the proof, it is unclear at this point how to relax (4.3) unless a different condition (4.4) is required for the upper bound (i) to guarantee the desired type I error, while the more stringent condition (4.4) is only required to prove non-conservativeness.

The radius of $\hat{E}_\alpha$ i.e., the half-length of its largest axis is given by

$$\phi_{\min}(C)^{-1/2} = (1 - T/n)^{-1/2}q_{T,\alpha}\lVert (nI_{T \times T} - \hat{A})^{-1}\rVert_{op}. \tag{4.5}$$

Since $\lVert I_{T \times T} - (I_{T \times T} - \hat{A}/n)^{-1}\rVert_{op} = o_p(1)$ by Proposition 2.1 and Lemma C.3 on the one hand, and all eigenvalues of $\hat{A}$ are of order $\sigma^2 n(1 + o_p(1))$ by the arguments in the proof of Lemma F.3 on the other hand, the radius (4.5) is $q_{T,\alpha}\sigma n^{-1/2}(1 + o_p(1))$ which is of order $\sigma\sqrt{T/n}$ by (4.2).

The random vector (4.1) involves multiplication by $(nI_{T \times T} - \hat{A})$ which differs from the pivotal quantity in the asymptotic normality result (3.1). However, Theorem 4.1 still holds with $\xi$ in (4.1) replaced by

$$\tilde{\xi} = (I_{T \times T} - \hat{A}/n)^{-1}(Y - X\hat{B})^\top z_0 + n(\hat{B} - B^*)^\top a. \tag{4.6}$$

Indeed, with

$$\lVert \hat{\Gamma}^{-1/2} \tilde{\xi} - \hat{\Gamma}^{-1/2} \xi \rVert_2 \leq \lVert \hat{\Gamma}^{-1/2} (\tilde{\xi} - \xi) \rVert_2 = \lVert \hat{\Gamma}^{-1/2} ((I_{T \times T} - \hat{A}/n)^{-1} - I_{T \times T})\xi \rVert_2.$$

Since the eigenvalues of $\hat{\Gamma}$ are all of order $\sigma^2 n(1 + o_p(1))$, since $\lVert (I_{T \times T} - \hat{A}/n)^{-1} - I_{T \times T}\rVert_{op} \leq (1 + o_p(1))\|S\|/n$ by Proposition 2.1 and since $\|\xi\|_2 = O_p(\sqrt{\sigma^2 nT})$ by Theorem F.2, the previous display is $O_p(\sqrt{T}s/n)$ and converges to 0 in probability by Assumption 1.1. Under Assumption 1.1, Theorem 4.1 thus holds for $\xi$ in (4.1) if and only if it holds for $\tilde{\xi}$. Furthermore the corresponding ellipsoid,

$$\hat{E}_\alpha = \{ \theta \in \mathbb{R}^T : (1 - \frac{T}{n})^{1/2}\| \hat{\Gamma}^{-1/2} [(I_{T \times T} - \hat{A}/n)^{-1}(Y - X\hat{B})^\top z_0 + n(\hat{B} - B^*)^\top a - \theta] \|_2 \leq q_{T,\alpha} \}$$

enjoys the same properties as $\hat{E}_\alpha$: Type I error guarantees $P[(B^*)^\top a \in \hat{E}_\alpha] \geq 1 - \alpha - o(1)$ under (4.3), and non-conservativeness $P[(B^*)^\top a \in \hat{E}_\alpha] \to 1 - \alpha$ under (4.4).

### 4.2. Unknown $\Sigma$

A similar result is available if $\Sigma$ is unknown. Consider the notation (3.10) from Section 3.2.

**Theorem 4.3.** Consider a canonical basis direction $e_j \in \mathbb{R}^p$ for some $j \in [p]$ and let Assumption 1.1 be fulfilled. Additionally assume that either (3.14) or (3.15) holds. Then for any
estimator $\hat{\gamma}^{(j)}$ satisfying (3.12)-(3.13),

$$
\sqrt{n-T} \frac{1}{\|\hat{z}_j\|_2} \tilde{\Gamma}^{-1/2} \left[ (Y - X\hat{B})^\top \hat{z}_j + \frac{(nI_{T \times T} - \hat{A})(\hat{B} - B^*)^\top e_j}{n(\hat{z}_j^\top X e_j)^{-1}} \right]_2 \leq \sqrt{\chi^2_T} + o_p(1) + O_p(\min\{\frac{T}{\sqrt{n}}, \frac{\log^2 p}{nT}\})
$$

where $\chi^2_T$ is a random variable with chi-square distribution with $T$ degrees-of-freedom.

Theorem 4.3 is proved in Appendix G.2. The corresponding confidence ellipsoid for the $j$-th row $(B^*)^\top e_j$ of $B^*$ is

$$\hat{e}_\alpha^j = \left\{ \theta_j \in \mathbb{R}^T : \frac{1}{\|\hat{z}_j\|_2} \|\tilde{\Gamma}^{-1/2} \left[ (Y - X\hat{B})^\top \hat{z}_j + \frac{(nI_{T \times T} - \hat{A})(\hat{B}^\top e_j - \theta_j)}{n(\hat{z}_j^\top X e_j)^{-1}} \right]_2 \leq \sqrt{\chi^2_T} \right\}.$$

If either one of the condition (3.14) or (3.15) holds on the growth of the sparsity of $\Sigma^{-1} e_j$, this confidence ellipsoid does not require the knowledge of $\Sigma$ and has the same guarantees as those of the previous section.

### 4.3. Relaxing the additional assumptions (4.3) and (4.4)

Instead of normalizing using $\tilde{\Gamma}^{-1/2}$ as in the previous sections, a simple estimate of $\sigma^2$ lets us relax the conditions (4.3) and (4.4) that are required in the previous section to ensure $\|\hat{\Gamma}^{-1/2}\xi\|_2 = (\chi^2_T)^{1/2} + o_p(1)$.

**Theorem 4.4.** Let $\xi, \hat{\xi}$ be defined in (4.1) and (4.6) respectively, and let $\hat{\sigma}^2 = \|Y - X\hat{B}\|_F^2/(nT)$. Then under Assumption 1.1, there exists a random variable $\chi^2_T$ with chi-square distribution with $T$ degrees of freedom such that

$$
(\hat{\sigma}^2 n)^{-1/2} \|\xi\|_2 = (\chi^2_T)^{-1/2} + o_p(1), \quad (\hat{\sigma}^2 n)^{-1/2} \|\hat{\xi}\|_2 = (\chi^2_T)^{-1/2} + o_p(1).
$$

**Theorem 4.5.** Consider a canonical basis direction $e_j \in \mathbb{R}^p$ for some $j \in [p]$ and let Assumption 1.1 be fulfilled. Additionally assume that either (3.14) or (3.15) holds. Then for any estimator $\hat{\gamma}^{(j)}$ satisfying (3.12)-(3.13),

$$
\frac{1}{\|\hat{z}_j\|_2} \left\| (Y - X\hat{B})^\top \hat{z}_j + \frac{(nI_{T \times T} - \hat{A})(\hat{B}^\top e_j - \theta_j)}{n(\hat{z}_j^\top X e_j)^{-1}} \right\|_2 = \sqrt{\chi^2_T} + o_p(1)
$$

where $\chi^2_T$ is a random variable with chi-square distribution with $T$ degrees-of-freedom.

The above asymptotic chi-square results hold under the same assumptions as Theorem 3.1 and Theorem 3.3. The reason for the success of these estimates is that $\hat{\sigma}$ estimates $\sigma$ at a rate faster than $T^{-1/2}$; we have $|\hat{\sigma}/\sigma - 1| = o_p(T^{-1/2})$ by Theorem F.2. However, simulations in Section 6 reveal that the asymptotic $(\chi^2_T)^{1/2}$ estimates of the previous subsections involving the matrix $\tilde{\Gamma}^{-1/2}$ are more robust to larger sparsity levels, although Assumption 1.1 is oblivious to this phenomenon.

The corresponding $1 - \alpha$ confidence ellipsoid for $(B^*)^\top a$ based on (4.8) and $\hat{\xi}$ is

$$
\hat{\xi}_{\sigma, \alpha} = \left\{ \theta \in \mathbb{R}^T : \frac{1}{\sigma \sqrt{n}} \left\| (I_{T \times T} - \hat{A}/n)^{-1} (Y - X\hat{B})^\top z_0 + n(\hat{B}^\top a - \theta) \right\|_2 \leq q_{T, \alpha} \right\}
$$

and satisfies $\mathbb{P}(B^*)^\top a \in \hat{\xi}_{\sigma, \alpha} \rightarrow 1 - \alpha$ under Assumption 1.1. Similar confidence ellipsoids based on (4.9) can be readily constructed.
4.4. Hypothesis testing

We now turn to type II error for the testing problem

\[ H_0 : (B^*)^\top a = 0_{T \times 1} \quad \text{against} \quad H_1 : \| (B^*)^\top a \|_2 \geq \rho_n \]

where \( \rho_n > 0 \) is a separation radius. The hypothesis test (4.11) at level \( 1 - \alpha \) is naturally achieved by rejecting \( H_0 \) if and only if \( 0_{T \times 1} \notin \mathcal{E}_{\alpha} \) for the ellipsoid in (4.10). Similar rejection procedures can be obtained with \( \mathcal{E}_{\alpha} \) or \( \mathcal{E}_\alpha^f \) for the confidence ellipsoids defined in Sections 4.1 and 4.2.

We can also determine the separation radius \( \rho_n \) required so that this testing procedure has nontrivial power (type II error). Focusing here on \( \mathcal{E}_{\alpha} \) in (4.10), rejection happens if and only if the following quantity is positive

\[
(\hat{\sigma}^2 n)^{-1} \| (I_{T \times T} - \hat{A}/n)^{-1} (Y - X\hat{B})^\top z_0 + n\hat{B}^\top a \|^2_2 - q_{T,a}^2
\]

\[
= W_n^2 - q_{T,a}^2 + (\hat{\sigma}^2 n)^{-1} n(B^*)^\top a \|^2_2
\]

\[
+ 2(\hat{\sigma}^2 n)^{-1/2} q_{T,a} a^\top (B^*)^\top \xi.
\]

where \( W_n^2 = (\hat{\sigma}^2 n)^{-1} \| \xi \|^2_2 \) and \( W_n = (\chi^2_T)^{1/2} + \sigma(1) \) by (4.8). By Theorem 3.1 applied to \( b = (B^*)^\top a \| (B^*)^\top a \|^2_2^{-1} \), the last line is of the form \( 2\hat{\sigma}^{-2} \| (B^*)^\top a \|^2_2 N(0, \sigma^2) \) so that it is of order \( \| (B^*)^\top a \|^2_2 O(\sigma(1)) \). The second line is positive, of order \( \sigma^{-2} n(1 + o(1)) \| (B^*)^\top a \|^2_2 \); this is the quantity that should dominate in order to ensure that the above display is positive.

Since the first line \( W_n^2 - q_{T,a} = (W_n - q_{T,a})(W_n + q_{T,a}) \) is positive with probability at least \( \alpha - o(1) \) by Proposition 4.2, we obtain that if \( \| (B^*)^\top a \|^2_2 \geq \rho_n \) for \( \rho_n \rightarrow +\infty \), then the type II error is at most \( 1 - \alpha + o(1) \). Although this type II error is typically a constant close to 1 (e.g. if \( \alpha = 0.05 \)), this shows that the above test has at most constant type II error as long as the separation radius satisfies \( \rho_n \gg \sigma n^{-1/2} \). We can also find conditions on \( \rho_n \) that ensures that the type II error is smaller than any constant. The first line above is of order \( (W_n - q_{T,a})O(\sqrt{T}) = O\sqrt{T} + q_{T,a} \) and \( q_{T,a} = \sqrt{T} + O(1) \) by Proposition 4.2 and (4.2). Thus \( \rho_n \gg T^{1/4} \sigma n^{-1/2} \) is sufficient in order for \( (\hat{\sigma}^2 n)^{-1} \| (B^*)^\top a \|^2_2 \) to dominate both the first and third lines with probability approaching one. In summary, \( \rho_n \gg \sigma n^{-1/2} \) is sufficient to achieve a constant type II error, while \( \rho_n \gg T^{1/4} \sigma n^{-1/2} \) is sufficient to grant a vanishing type II error.

In single task models, coefficients \( B^*_{jt} \) of order \( o(\sigma n^{-1/2}) \) cannot be detected, cf. the discussion after (1.6). Here on the other hand in the multi-task setting with \( T \rightarrow +\infty \), detection of non-zero vector \( (B^*)^\top e_j \) is possible with constant power even if the individual coefficients in \( (B^*)^\top e_j \) are \( u_n \sigma(Tn)^{-1/2} \) for any slowly increasing \( u_n \) with \( u_n \rightarrow +\infty \). If \( u_n = o(\sqrt{T}) \), the coefficients \( (B^*_{jt})_{t=1,...,T} \) are individually impossible to detect, while detection of the row vector \( (B^*)^\top e_j \) is possible with constant type I and type II errors. Similarly, if the individual coefficients \( (B^*_{jt})_{t=1,...,T} \) are of order \( u_n \sigma T^{-1/4} n^{-1/2} \) for any slowly increasing \( u_n \) with \( u_n \rightarrow +\infty \), the above testing procedure for the row vector \( (B^*)^\top e_j \) has vanishing type II error.

5. Computing the interaction matrix efficiently

Equation (2.2) which defines \( \hat{A} \) is convenient for theoretical purposes, as the pseudoinverse suppresses invertibility issues and the form (2.2) naturally arises in the proofs, see for instance Lemmas C.7 to C.9. However, (2.2) is not computationally tractable as it involves computing a
pseudoinverse of size $pT \times pT$. The goal of this section is to provide a computationally tractable representation for $\mathbf{A}$; in particular we will see that one only needs to compute inverses of matrices of size $|\tilde{S}| \times |\tilde{S}|$. A first step when implementing is to remove all covariates $j \in \{1, ..., p\}$ such that $\mathbf{B}^T e_j = 0$, as dropping those indices and the corresponding columns of $\mathbf{X}$ does not change the value of $\mathbf{A}$ in (2.2). For the purpose of this section and only in this section, we assume without loss of generality that $\tilde{S} = [p]$ and that all variables $j \in [p]$ are such that $\mathbf{B}^T e_j \neq 0$. However, we will keep the notation $\mathbf{X}_{\tilde{S}}$ and use summation $\sum_{j \in \tilde{S}}$ to emphasize that the indices $j \notin \tilde{S}$ and corresponding columns of $\mathbf{X}$ have been dropped.

Before stating a formal proposition with a computationally friendly representation of the matrix $\mathbf{A}$, we explain the crux of the argument, which relies on the Sherman-Morrison-Woodbury inversion formula. Recall that $\mathbf{X}^T \mathbf{X} = (I_{T \times T} \otimes \mathbf{X}_{\tilde{S}}^T \mathbf{X}_{\tilde{S}})$ and that for every $j \in \tilde{S}$

$$H^{(j)} := \lambda \|\mathbf{B}^T e_j\|_2^{-1} (I_{T \times T} - \mathbf{B}^T e_j \mathbf{B} \|\mathbf{B}^T e_j\|_2^{-2}) \in \mathbb{R}^{T \times T},$$

as well as $\mathbf{H} = \sum_{j \in \tilde{S}} H^{(j)} \otimes (e_j e_j^T)$. By splitting the part of $H^{(j)}$ proportional to the identity and the rank one part, we find

$$\mathbf{X}^T \mathbf{X} + nT \mathbf{H} = (I_{T \times T} \otimes \mathbf{X}_{\tilde{S}}^T \mathbf{X}_{\tilde{S}}) + nT \lambda \sum_{j \in \tilde{S}} I_{T \times T} \otimes e_j e_j^T \|\mathbf{B}^T e_j\|_2^{-1} - \left[\mathbf{B}^T e_j e_j^T \mathbf{B} \|\mathbf{B}^T e_j\|_2 \otimes (e_j e_j^T)\right],$$

where $\mathbf{v} \in \mathbb{R}^{|\tilde{S}|}$ is the vector with $v_j = nT \lambda \|\mathbf{B}^T e_j\|_2^{-1}$ and $\text{diag}(\mathbf{v})$ is the square diagonal matrix with $\mathbf{v}$ as its diagonal. By the mixed product property (1.13) we have

$$(\mathbf{B}^T e_j e_j^T \mathbf{B}) \otimes (e_j e_j^T) = (\mathbf{B}^T e_j \otimes e_j)(\mathbf{B}^T e_j \otimes e_j)^T$$

so that, with $b^{(j)} = (nT \lambda \|\mathbf{B}^T e_j\|_2^{-3})^{1/2} \mathbf{B}^T e_j \in \mathbb{R}^T$ we obtain

$$\mathbf{X}^T \mathbf{X} + nT \mathbf{H} = \left(I_{T \times T} \otimes (\mathbf{X}_{\tilde{S}}^T \mathbf{X}_{\tilde{S}} + \text{diag}(\mathbf{v}))\right) - \sum_{j \in \tilde{S}} (b^{(j)} \otimes e_j)(b^{(j)} \otimes e_j)^T$$

$$= \left(I_{T \times T} \otimes (\mathbf{X}_{\tilde{S}}^T \mathbf{X}_{\tilde{S}} + \text{diag}(\mathbf{v}))\right) - \mathbf{U} \mathbf{U}^T$$

$$= \mathbf{M} - \mathbf{U} \mathbf{U}^T,$$

where $\mathbf{U} \in \mathbb{R}^{(|\tilde{S}|T) \times |\tilde{S}|}$ has columns $(b^{(j)} \otimes e_{j})_{j \in \tilde{S}}$ and $\mathbf{M} = I_{T \times T} \otimes (\mathbf{X}_{\tilde{S}}^T \mathbf{X}_{\tilde{S}} + \text{diag}(\mathbf{v}))$. If $\mathbf{M}$ is invertible and its inverse can be computed efficiently, the inverse of the above display is given by the Sherman-Morrison-Woodbury formula [24]; if the matrix $-I_{|\tilde{S}| \times |\tilde{S}|} + \mathbf{U}^T \mathbf{M}^{-1} \mathbf{U}$ is invertible then $\mathbf{M} - \mathbf{U} \mathbf{U}^T$ is also invertible and

$$(\mathbf{M} - \mathbf{U} \mathbf{U}^T)^{-1} = \mathbf{M}^{-1} - \mathbf{M}^{-1} (I_{|\tilde{S}| \times |\tilde{S}|} + \mathbf{U}^T \mathbf{M}^{-1} \mathbf{U})^{-1} \mathbf{U}^T \mathbf{M}^{-1}.$$

Since $\mathbf{v}$ has positive entries, $\mathbf{X}_{\tilde{S}}^T \mathbf{X}_{\tilde{S}} + \text{diag}(\mathbf{v})$ is always invertible and so is $\mathbf{M}$, with

$$\mathbf{M}^{-1} = I_{T \times T} \otimes (\mathbf{X}_{\tilde{S}}^T \mathbf{X}_{\tilde{S}} + \text{diag}(\mathbf{v}))^{-1}. $$

Hence we only need to perform two inversions of matrices of size $|\tilde{S}| \times |\tilde{S}|$: the inversion of $\mathbf{X}_{\tilde{S}}^T \mathbf{X}_{\tilde{S}} + \text{diag}(\mathbf{v})$ and of $-I_{|\tilde{S}| \times |\tilde{S}|} + \mathbf{U}^T \mathbf{M}^{-1} \mathbf{U}$. 


Proposition 5.1. With the above notation for $v \in \mathbb{R}^{|\hat{S}|}$ and $b^{(j)} \in \mathbb{R}^T$ for each $j \in \hat{S}$, if the matrix $P$ defined entrywise by

$$P = \{P_{jk}\}_{(j,k) \in \hat{S} \times \hat{S}}, \quad P_{jk} = -I\{j = k\} + (b^{(j)\top}b^{(k)}) (e_j^{\top}(X_{\hat{S}}^\top X_{\hat{S}} + \text{diag}(v))^{-1}e_k)$$

is invertible then

$$\hat{A} = \text{Tr}\left[ X_{\hat{S}}^\top X_{\hat{S}}(X_{\hat{S}}^\top X_{\hat{S}} + \text{diag}(v))^{-1} \right] I_{T \times T} - \sum_{j \in \hat{S}} b^{(j)} \sum_{k \in \hat{S}} (e_j^{\top}Qe_k)(e_j^{\top}P^{-1}e_k)b^{(k)\top}$$

is equivalent to

$$\left( X_{\hat{S}}^\top X_{\hat{S}} + \text{diag}(v)\right)^{-1}X_{\hat{S}}^\top X_{\hat{S}}\left( X_{\hat{S}}^\top X_{\hat{S}} + \text{diag}(v)\right)^{-1}.$$ 

Proof. By definition of $\hat{A}$ and using the above Sherman-Morrison-Woodbury identity

$$\hat{A}_{t,t'} = \text{Tr}\left[ (e_t^{\top} \otimes X_{\hat{S}})(X_{\hat{S}}^\top X_{\hat{S}} + nT \tilde{\mathbf{H}}) (e_{t'}^{\top} \otimes X_{\hat{S}}^{\top}) \right]$$

$$= \text{Tr}\left[ (e_t^{\top} \otimes X_{\hat{S}})M^{-1}(e_{t'}^{\top} \otimes X_{\hat{S}}^{\top}) \right]$$

$$- \frac{1}{n} \text{Tr}\left[ (e_t^{\top} \otimes X_{\hat{S}})M^{-1}U(-I_{|\hat{S}| \times |\hat{S}|} + U^{\top}M^{-1}U)^{-1}U^{\top}M^{-1}(e_{t'}^{\top} \otimes X_{\hat{S}}^{\top}) \right]$$

$$= \text{Tr}\left[ (e_t^{\top} e_{t'} \otimes (X_{\hat{S}}(X_{\hat{S}}^\top X_{\hat{S}} + \text{diag}(v))^{-1}X_{\hat{S}}^{\top}) \right]$$

$$- \text{Tr}\left[ (U^{\top}M^{-1})(e_{t'}^{\top} e_{t'} \otimes X_{\hat{S}}^\top X_{\hat{S}})M^{-1}U(-I_{|\hat{S}| \times |\hat{S}|} + U^{\top}M^{-1}U)^{-1} \right].$$

By (1.14), the first term equals $I\{t = t'\} \text{Tr}[X_{\hat{S}}(X_{\hat{S}}^\top X_{\hat{S}} + \text{diag}(v))^{-1}X_{\hat{S}}^{\top}]$ which gives the first term in the proposition, proportional to $I_{T \times T}$. Using again the structure of $M^{-1}$ in (5.1), the second summand in the previous display is equal to

$$- \text{Tr}[U^{\top}(e_{t'}^{\top} \otimes Q)U P^{-1}]$$

where $P$ and $Q$ are given in the proposition, after noting that the definition of $P$ is equivalent to $P = -I_{|\hat{S}| \times |\hat{S}|} + U^{\top}M^{-1}U$. Since $U$ has columns $b^{(j)} \otimes e_j$, the entry $(j, k) \in \hat{S} \times \hat{S}$ of the matrix $U^{\top}(e_{t'}^{\top} e_t^{\top} \otimes Q)U$ is equal to

$$(b^{(j)\top} e_t^{\top} b^{(k)}) (e_j^{\top} Q e_k) = (e_j^{\top} b^{(j)}) (e_j^{\top} b^{(k)}) (e_j^{\top} Q e_k).$$

Since $\text{Tr}[AB] = \sum_{j,k} A_{jk}B_{jk}$ for two symmetric matrices of the same size, we obtain

$$\text{(5.2)} = - \sum_{j \in \hat{S}} \sum_{k \in \hat{S}} (e_t^{\top} b^{(j)}) (e_j^{\top} b^{(k)}) (e_j^{\top} Q e_k)(e_j^{\top} P^{-1}e_k)$$

$$= e_t^{\top} \left[ - \sum_{j \in \hat{S}} b^{(j)} \sum_{k \in \hat{S}} (e_j^{\top} Q e_k)(e_j^{\top} P^{-1}e_k)b^{(k)\top} \right] e_t.$$ 

On the last line, the matrix in bracket is the second matrix in the expression of $\hat{A}$. 

We now turn to implementation details. We recommend an approach that makes use of optimized vectorized code as often as possible to compute the quantities in Proposition 5.1, and if available to use a library with Einstein summation routine as this allows the code to mimic the mathematical notation in Proposition 5.1. For concreteness, the following code lets us efficiently compute $\hat{A}$ with the Python library Numpy \cite{22}, and the Einstein summation function \texttt{numpy.einsum} which comes in handy. Assume that $B$ has been computed, the rows in $[p] \setminus \hat{S}$ removed and the result stored in an array $B_{\hat{S}}$ of size $|\hat{S}| \times T$, that $X$ with the columns in $[p] \setminus \hat{S}$ removed is stored in an array $X_{\hat{S}}$ of size $n \times |\hat{S}|$, and that the scalar $nT \lambda$ is stored in variable \texttt{nTlambda}. Then the vector $v$ and matrix with columns $(b^{(j)})_{j \in \hat{S}}$ in variable $b$ can be computed as follows:
import numpy as np

norms = np.linalg.norm(B_S, axis=-1)  # shape (|^S|, )
v = nTlambda * norms**(-1)  # shape (|^S|, )
b = nTlambda * 0.5 * np.einsum("j,jt->jt", norms**(-3/2), B_S)  # shape (|^S|, T)

Finally, matrices \((X_S^TX + \text{diag}(v))^{-1} \text{ and } Q\) are computed using built-in symmetric matrix inversion, while computation of \(P\) and \(\hat{A}\) again resorts to using \texttt{np.einsum}:

\[
\text{gram} = \text{X.S.T} \otimes \text{X.S}  \\
\text{inverse} = \text{np.linalg.inv}(\text{gram} + \text{np.diag}(v))  \\
Q = \text{inverse} \otimes \text{gram} \otimes \text{inverse}  \\
P = -\text{np.eye}(p) + \text{np.einsum}("jt,kt,jk->jk", b, b, inverse)  \\
A = \text{np.eye}(T) + \text{np.einsum}("jk,jk->ju", \text{gram}, \text{inverse}) \\
- \text{np.einsum}("jt,ku,jk->tu", b, b, Q, \text{np.linalg.inv}(P))  \\
\]

In \texttt{einsum}, we use indices \(t\) and \(u\) to loop over \([T]\), and indices \(j\) and \(k\) to loop over \(\hat{S}\). All calls to \texttt{einsum} can be further optimized by pre-computing the optimal order in which tensor contractions should be performed (see \texttt{numpy.einsum_path}).

Empirically, we have observed that this implementation using the Sherman-Morrison-Woodbury identity and the above code is several orders of magnitude faster than a naive one involving sparse matrices and the full inversion of \(X^T\hat{X} + nTH\).

6. Numerical experiments

We run simulations to illustrate the theorems proved in Sections 3 and 4. The values of the parameters are fixed to \(n = 2000, p = 6000, T = 10, \eta_1 = \eta_2 = 0, \sigma^2 = 1.0\). The tuning parameter is \(\lambda = \max_j \sum_{j=0}^{1/2} \frac{1}{\sqrt{nT}} \left(1 + \sqrt{\frac{2}{\log \frac{2}{\xi}}}ight)\) (we explain below how \(\Sigma\) is constructed). The directions of interest are \(a = e_j \in \mathbb{R}^p\) and \(b = e_1 \in \mathbb{R}^T\).

**Quantile-quantile plots of the pivotal quantities**

The goal is to assess how the sparsity of \(B^*\) and \(\Sigma^{-1}e_1\) influence the convergence in Theorems 3.1, 3.3, 4.1, 4.3 and 4.5. Denote by \(s\) and \(s_\Omega\) the respective sparsity parameters that will vary in the experiments. Given a target tuple \((s, s_\Omega)\) we generate \(B^*\) with exactly \(s\) non-zero rows and \(\Sigma\) with exactly \(s_\Omega\) non-zero entries on the first column of \(\Sigma^{-1}\), so that \(s_\Omega = ||\Sigma^{-1}e_1||_0\).

We explain first how \(\Sigma\) is constructed so that it satisfies the constraints in Assumption 1.1 as well as the sparsity requirement on \(\Sigma^{-1}e_1\). Start by sampling \(M\), a \((p-1) \times (p-1)\) matrix with i.i.d. \(\mathcal{N}(0,1)\) entries. Then perform the QR decomposition of \(M\) to obtain an orthogonal matrix \(Q\), the distribution of which is uniform in the sense of Haar measure on the orthogonal group \(O(p-1)\). Next, consider \(D\), the diagonal \((p-1) \times (p-1)\) matrix with entries \(\{1 + j/(p-2) : j \in \{0,\ldots,p-2\}\}\) and set \(\Lambda = QDQ^T\). Define the block matrix

\[
\tilde{\Lambda} = \begin{bmatrix} 3/2 & v^T \\ v & A \end{bmatrix}
\]

where \(v \in \mathbb{R}^{p-1}\) is a vector with sparsity \(||\Sigma^{-1}e_1||_0 - 1\) and norm \(||v||_2 = 1\). This ensures boundedness of the spectrum as the smallest eigenvalue of \(\tilde{\Lambda}\). to satisfies the lower bound

\[
\lambda_{\min}(\tilde{\Lambda}) \geq \lambda_{\min} \left( \begin{bmatrix} 3/2 & -||v||_2 \\ -||v||_2 & \lambda_{\min}(A) \end{bmatrix} \right) = \frac{5 - \sqrt{17}}{4} \gtrsim 0.219,
\]

where the last equality follows from \(\lambda_{\min}(A) = 1\) and \(||v||_2 = 1\). Similarly, the largest eigenvalue of \(\tilde{\Lambda}\) can be bounded above by \(\lambda_{\max}(\tilde{\Lambda}) = \frac{7 + \sqrt{17}}{4} \lesssim 2.8\). Finally set \(\Sigma = \alpha^{-1}\tilde{\Lambda}^{-1}\) where \(\alpha\) is
the greatest diagonal entry of $\Lambda^{-1}$ so that $\max\{\Sigma_{jj}, j = 1, ..., p\} = 1$. This construction leads to $\lambda_{\min}(\Sigma) \approx 0.32$, $\lambda_{\max}(\Sigma) \approx 1.76$ and $(\Sigma^{-1})_{jj} \approx 1.85$.

The row-sparse matrix $B^*$ is constructed as follows. Initialize $B^*$ as a matrix filled with $\lambda$’s and alter it in two different ways:

(i) **Setting with overlapping supports.** In the first setting, we zero out rows of $B^*$ while forcing an overlap of the supports of $B^*$ and $\Sigma^{-1}e_1$ (either $\text{supp}(\Sigma^{-1}e_1) \subset \text{supp}(B^*)$ or the reverse inclusion). The intuition is that this makes inequality (G.4) tight. This constraint is therefore expected to slow down convergence.

(ii) **No-overlap setting.** In the second setting this constraint is removed and the support of $B^*$ is picked uniformly at random as a subset of $\{1, ..., p\} \setminus \text{supp}(\Sigma^{-1}e_1)$.

Assume that the tuple $(s, s_{\Omega})$ is fixed. We sample $N_{\text{sim}} = 128$ instances of $(X, E)$. For each sample, we compute the estimator $\hat{B}$ using the function `MultiTaskElasticNet` from the Python library Scikit-learn [43], build the interaction matrix $\hat{A}$ using the implementation from Section 5 and collect the pivotal quantities appearing in the Theorems. The Q-Q plots and histograms for different pairs $(s, s_{\Omega})$ are then reported in Figures 1 and 2 for the overlapping supports setting (i) and Figures 3 and 4 for the no-overlap setting (ii).

Asymptotic normality is observed empirically on Figure 3 in the no-overlap setting, both when $\Sigma$ is known (blue) and unknown (green). The convergence holds up well across a wide range of sparsity levels. In the overlapping supports setting of Figure 1, convergence is maintained if $\Sigma$ is known, but in the unknown $\Sigma$ case it deteriorates fast when $\|\Sigma^{-1}e_1\|_0$ grows. This suggests that condition (3.14) is not an artefact of the proof.

The picture is different with chi-square results. In the no-overlap setting of Figure 4, convergence is observed across all sparsity levels for pivotal quantities in Theorem 4.1 (known $\Sigma$) and Theorem 4.3 (unknown $\Sigma$) whereas an increase in $s$ slows down convergence in Theorem 4.5 (unknown $\Sigma$). In the overlapping supports setting (i) of Figure 2, pivotal quantities in Theorems 4.1 and 4.5 exhibit the same behavior as in the previous setting whereas the one from Theorem 4.3 shows increasingly slower convergence as $\|\Sigma^{-1}e_1\|_0$ grows. Again, this suggests that condition (3.14) is not an artefact of the proof.

6.1. The advantage of multi-task learning for narrower confidence intervals

In Figure 5 we illustrate the discussion around (3.6) by comparing the lengths of 95% confidence intervals obtained via multi-task Lasso and single-task Lasso. $\|\Sigma^{-1}e_1\|_0$ is set to 5 and the pair $(T, s)$ varies. For a given $(T, s)$ and a sampled $(X, E)$ we compute the relative change $(\text{length}_{\text{multi}} - \text{length}_{\text{single}})/\text{length}_{\text{single}}$. We collect these values over $N_{\text{sim}} = 128$ samples and obtain the bottom figure. Since the results with or without the overlap constraint in the supports are similar, only the no-overlap setting (ii) is shown. In the upper figure, multi-task confidence interval lengths are pooled together over the samples and we compare them to the aggregate single-task lengths. As a sanity check we observe that multi-task and single-task Lasso coincide when $T$ is equal to 1. For $s = 15$, $\hat{A}$-based confidence intervals always have smaller length, which shrinks as $T$ increases. When $T = 20$ we observe a 40% average gain in the width. Exploiting several tasks thus provides better estimates than intervals based on the first task. However, as $s$ grows, this effect fades gradually and when $s = 100$ it is counterbalanced by high variance in the multi-task lengths.
Fig 1: QQ-plots and histograms in the unfavorable setting (i) for pivotal quantities in Theorem 3.1 (blue), Theorem 3.3 (green).
\[ \| \Sigma^{-1} e_j \|_0 = 1 \]
\[ \| \Sigma^{-1} e_j \|_0 = 2 \]
\[ \| \Sigma^{-1} e_j \|_0 = 3 \]
\[ \| \Sigma^{-1} e_j \|_0 = 5 \]
\[ \| \Sigma^{-1} e_j \|_0 = 10 \]

\( s = 25 \)
\( s = 150 \)
\( s = 300 \)

Fig 2: QQ-plots and histograms in the unfavorable setting (i) for pivotal quantities in Theorem 4.1 (blue), Theorem 4.3 (green), Theorem 4.5 (orange).
Fig 3: QQ-plots and histograms in the favorable setting (ii) for pivotal quantities in Theorem 3.1 (blue), Theorem 3.3 (green).
Fig 4: QQ-plots and histograms in the favorable setting (ii) for pivotal quantities in Theorem 4.1 (blue), Theorem 4.3 (green), Theorem 4.5 (orange).
Fig 5: Top: boxplots for lengths of 95% confidence intervals using multi-task Lasso (blue) and single-task Lasso on the first task (orange). Bottom: boxplots for relative change in length with single-task as reference. Only the no-overlap setting (ii) is shown and $\|\Sigma^{-1}e_1\|_0$ is set to 5.
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SUPPLEMENT

Appendix A: Intuition

Let us give some rationale behind the pivotal quantities stated in the main theorems. In this paragraph and only in this paragraph for the sake of providing some intuition, we assume that $a = e_j$ for some canonical basis vector in $\mathbb{R}^p$ and that $\Sigma = I_{p \times p}$ so that entries of $X$ are i.i.d. $\mathcal{N}(0, 1)$. In this setting, the random vector $z_0 = Xe_j$ has i.i.d. $\mathcal{N}(0, 1)$ entries and is independent of $X_{-j}$, the matrix $X$ with $j$-th column removed. Since $z \sim \mathcal{N}_n(0, I_{n \times n})$, Stein’s formula [45, 46] states that $E[z_0^t f(z_0)] = E[\sum_{i=1}^n (\partial/\partial z_0) f_i(z_0)]$ for any differentiable vector field $f = (f_1, ..., f_n)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$, under integrability conditions. For the sake of the current informal argument, assume that Stein’s formula provides reasonable approximation. Then applying Stein’s formula to $f = (Y - X\hat{B})e_t$ for each task $t = 1, ..., T$ (here, $e_t$ is the $t$-th canonical basis vector in $\mathbb{R}^T$), by nontrivial computations that are made rigorous in the proofs given in the supplement, the approximations

$$
\begin{align*}
\hat{z}_0^T (Y - X\hat{B})e_1 & \approx na^\top (\hat{B} - B^*)e_1 - \sum_{t=1}^T \hat{A}_{1t} a^\top (\hat{B} - B^*)e_t, \\
\hat{z}_0^T (Y - X\hat{B})e_2 & \approx na^\top (\hat{B} - B^*)e_2 - \sum_{t=1}^T \hat{A}_{2t} a^\top (\hat{B} - B^*)e_t, \\
& \vdots \\
\hat{z}_0^T (Y - X\hat{B})e_T & \approx na^\top (\hat{B} - B^*)e_T - \sum_{t=1}^T \hat{A}_{Tt} a^\top (\hat{B} - B^*)e_t
\end{align*}
$$

(A.1)

hold up to smaller order terms, where $\hat{A}$ is the interaction matrix in Equation (2.2). By viewing (A.1) as a linear system with $T$ equations and the $T$ unknowns $(a^\top (\hat{B} - B^*)e_i)_{i=1, ..., T}$, and assuming that solving the linear system maintains the approximations, we obtain that

$$
\begin{pmatrix}
a^\top (\hat{B} - B^*)e_1 \\
a^\top (\hat{B} - B^*)e_2 \\
\vdots \\
a^\top (\hat{B} - B^*)e_T
\end{pmatrix}
\approx
\begin{pmatrix}
(nI_{T \times T} - \hat{A})^{-1} \\
\hat{z}_0^T (Y - X\hat{B})e_1 \\
\hat{z}_0^T (Y - X\hat{B})e_2 \\
\vdots \\
\hat{z}_0^T (Y - X\hat{B})e_T
\end{pmatrix}
$$

or equivalently $(\hat{B} - B^*)^\top a = (nI_{T \times T} - \hat{A})^{-1} (Y - X\hat{B})^\top z_0$. Thus the matrix product of $(nI_{T \times T} - \hat{A})^{-1}$ times the residuals projected onto $z_0$ provides us with estimates of the bias of $\hat{B}$ on the direction $a \in \mathbb{R}^p$. This informal argument is the crux of the rigorous methodology developed in the next subsections. In the sequel, we drop the assumption that $\Sigma = I_{p \times p}$. When $\Sigma \neq I_{p \times p}$ is known as in Section 3.1, the score vector $z_0$ in (A.1) has to be replaced by a random vector proportional to $X\Sigma^{-1} a$. When $\Sigma$ is unknown as in Section 3.2, the score vector has to be estimated.

Appendix B: Proof of Proposition 2.1

We restate the proposition for convenience.

Proposition 2.1. Let $\hat{A}$ be defined by (2.2). Then

(i) $\hat{A}$ is symmetric and positive semi-definite.

(ii) If $X_S$ is rank $|S|$ then the spectral norm of $\hat{A}$ is bounded from above as $\|\hat{A}\|_{op} \leq |S|$.
(iii) If \( X_S \) is rank \(|\hat{S}| \) and \(|\hat{S}|/n < 1 \) then \( I_{TXT} - \hat{A}/n \) is positive-definite and \\
\[ \| I_{TXT} - (I_{TXT} - \hat{A}/n)^{-1} \|_{op} \leq (|\hat{S}|/n)/(1 - |\hat{S}|/n). \]

Proof. (i) We have the following equalities:

\[
\begin{align*}
  u^\top \hat{A} v & \overset{(i)}{=} \text{Tr} \left[ (u^\top \otimes X_S) [\hat{X}^\top \hat{X} + nT \hat{H}]^\dagger (v \otimes (X_S)^\top) \right] \\
  & \overset{(ii)}{=} \text{Tr} \left[ (vu^\top \otimes (X_S)^\top) [\hat{X}^\top \hat{X} + nT \hat{H}]^\dagger \right] \\
  & \overset{(iii)}{=} \text{Tr} \left[ [\hat{X}^\top \hat{X} + nT \hat{H}]^\dagger (vu^\top \otimes (X_S)^\top) \right] \\
  & \overset{(iv)}{=} \text{Tr} \left[ [(u^\top v)^\top \otimes (X_S)^\top] [\hat{X}^\top \hat{X} + nT \hat{H}] \right] \\
  & \overset{(v)}{=} \text{Tr} \left[ (u^\top v)^\top \otimes (X_S)^\top [\hat{X}^\top \hat{X} + nT \hat{H}]^\dagger \right] = v^\top \hat{A} u
\end{align*}
\]

where (i) follows from (2.3), (ii) is a consequence of \( \text{Tr}[M_1 M_2] = \text{Tr}[M_2 M_1] \) and the mixed product property (1.13), (iii) and (v) follow from \( \text{Tr}[M] = \text{Tr}[M^\top] \), (iv) holds because the pseudoinverse preserves symmetry.

This proves that \( \hat{A} \) is symmetric. Since the pseudoinverse of a positive semi-definite matrix is positive semi-definite as well, we also have

\[
(B.1) \quad u^\top \hat{A} u = \|[(\hat{X}^\top \hat{X})^\dagger]^{1/2} (u \otimes (X_S)^\top) \|_F^2 \geq 0
\]

so that \( \hat{A} \) is positive semi-definite.

(ii) Recall that \( \hat{X} = I_{TXT} \otimes X_S \). By properties of Gram matrices, \( \text{rank}(\hat{X}^\top \hat{X}) = \text{rank}(\hat{X}) = |\hat{S}|T \), hence by the rank-nullity theorem, \( \ker(\hat{X}^\top \hat{X}) \) has dimension \( (p - |\hat{S}|)T \). By definition of \( X_S \), each vector \( e_i \otimes e_j \) is in the kernel of \( \hat{X} \) for \( j \notin \hat{S} \) and \( t \in [T] \), hence in \( \ker(\hat{X}^\top \hat{X}) \). These \( (p - |\hat{S}|)T \) vectors are linearly independent, so they form a basis of \( \ker(\hat{X}^\top \hat{X}) \).

Besides, since \( \hat{H} = \sum_{k \in \hat{S}} H^{(k)} \otimes (e_k e_k^\top) \), the mixed product property of Kronecker products (1.13) implies that \( H(e_i \otimes e_j) = 0 \) for \( j \notin \hat{S} \) and \( t \in [T] \), hence \( \ker(\hat{X}^\top \hat{X}) \subset \ker(\hat{X}^\top \hat{X} + nT \hat{H}) \).

Since \( \hat{H} \) is positive semi-definite, \( \hat{X}^\top \hat{X} \preceq \hat{X}^\top \hat{X} + nT \hat{H} \) holds in the sense of the positive semi-definite order, and

\[
(B.2) \quad (\hat{X}^\top \hat{X} + nT \hat{H})^\dagger \preceq (\hat{X}^\top \hat{X})^\dagger
\]

holds because the two matrices have the same kernel, see [29]. Next, using (B.1),

\[
\begin{align*}
  u^\top \hat{A} u & = \|[(\hat{X}^\top \hat{X})^\dagger]^{1/2} (u \otimes (X_S)^\top) \|_F^2 + \text{Tr}[(u^\top \otimes X_S) [\hat{X}^\top \hat{X} + nT \hat{H}]^\dagger - (\hat{X}^\top \hat{X})^\dagger] (u \otimes (X_S)^\top)] \\
  & \leq \|[(\hat{X}^\top \hat{X})^\dagger]^{1/2} (u \otimes (X_S)^\top) \|_F^2 + \text{Tr}[(u^\top \otimes X_S) (I_{TXT} \otimes (X_S)^\top) (u \otimes (X_S)^\top)] \\
  & = (u^\top u) \text{Tr}[X_S(X_S^\top X_S)^\dagger (X_S)^\top] \\
  & = \|u\|^2 |\hat{S}|,
\end{align*}
\]

where the first inequality follows from (B.2) and the third and fourth line follow respectively from \( \hat{X}^\top \hat{X} = I_{TXT} \otimes X_S \) and the mixed product property (1.13). The last line stems from the fact that \( X_S(X_S^\top X_S)^\dagger X_S^\top \) is a projection matrix of rank \(|\hat{S}| \) when \( \text{rank}(X_S) = |\hat{S}| \).

(iii) Since \( X_S \) has rank \(|\hat{S}| \), we have by (ii) that \( \|\hat{A}\|_{ap} \leq |\hat{S}| < n \). Since \( \hat{A} \) is positive-semi definite, its spectral norm is its largest eigenvalue, hence all the eigenvalues of \( \hat{A}/n \) are
< 1, and \( I_{T \times T} - \hat{A}/n \) is positive definite. For any \( M \in \mathbb{R}^{T \times T} \) with \( \|M\|_{op} < 1 \) we have \( (I_{T \times T} - M)^{-1} = \sum_{k=0}^{\infty} M^k \). By the triangle inequality and the submultiplicativity of the operator norm,

\[
\| (I_{T \times T} - M)^{-1} - I_{T \times T} \|_{op} \leq \|M\|_{op} \sum_{k=1}^{\infty} \|M\|_{op}^{k-1} = \|M\|_{op}/(1 - \|M\|_{op}).
\]

\( \square \)

Appendix C: Preliminaries

In this section we develop a series of technical lemmas that will be used for proving Sections 3 and 4. We consider model (1.2) and the estimator \( \hat{B} \) from (1.4). Let \( \eta_1 > 0, \eta_2 \geq 2, \eta_3, \eta_4 \in (0, 1) \) and set \( \lambda, \lambda_0 \) as in (1.8). Define the sparsity level

\[
(C.1) \quad \bar{s} = s \left( 1/T + 4\|\Sigma\|_{op}(1 + \eta_4)^2 (2 + \eta_2 + 1/\sqrt{T})^2 \right)^{2/(\lambda/\lambda_0 - 1)^2}
\]

and note that \( \bar{s} \) is of the same order as \( s \) when the spectrum of \( \Sigma \) is bounded away from 0 and infinity as in Assumption 1.1. Let \( C = \{U \in \mathbb{R}^{p \times T} : \|U\|_{2,1} \leq 3\sqrt{s}\|U\|_F, \kappa = (1 - \eta_3)\phi_{\min}(\Sigma)^{1/2} \} \) and define the events

\[
\Omega_1 = \left\{ \max_{U \in \mathbb{C}^T, U \neq 0} \frac{\|XU\|_F}{\|\Sigma^{1/2}U\|_{F \sqrt{n}}} - 1 \right\}, \quad \Omega_2 = \left\{ \sum_{j=1}^{p} (\|E_j X e_j\|_2 - nT \lambda_0)^2 < s n^2 T \lambda_0^2 \right\},
\]

\[
\Omega_3 = \left\{ \max_{B \subset [p]: |B| \leq s + 2 \bar{s} + 1} \left( \max_{v \in \supp(B)} \sqrt{n} \frac{\|Xv\|_2}{\|\Sigma^{1/2}v\|_2} - 1 \right) \right\}, \quad \Omega_4 = \left\{ \|E\|_{op} < \sigma(2\sqrt{n} + \sqrt{T}) \right\}
\]

as well as

\[
(C.2) \quad \Omega_* = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4.
\]

Since the only randomness is with respect to \( (X, E) \), we view the underlying probability space as \( \Omega = (\mathbb{R}^{n \times p}) \times (\mathbb{R}^{n \times T}) \) and \( \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_* \) as subsets of \( \Omega \) so that \( \Omega_i \) occurs if and only if \( (X, E) \in \Omega_i \) for each \( i = 1, 2, 3 \).

Lemma C.1. Let Assumption 1.1 be fulfilled. Then \( P(\Omega_* \rightarrow 1 \).

Lemma C.2. On \( \Omega_* \) we have:

(i) \( \hat{B} - B^* \in C \),

(ii) \( n^{-1/2}\|X(\hat{B} - B^*)\|_F \leq (1 - \eta_3)\hat{R} \),

(iii) \( \|\Sigma^{1/2}(\hat{B} - B^*)\|_F \leq \hat{R} \),

(iv) \( \|B - B^*\|_{2,1} \leq 3\sqrt{s}\|\hat{B} - B^*\|_F \leq 3\sqrt{s}\phi_{\min}(\Sigma)^{-1/2}\hat{R} \),

(v) \( \|Y - XB\|_F^2 \leq 8\sigma^2 n T + 2(1 - \eta_3)^2 n \hat{R}^2 \),

where

\[
\hat{R} := (1 - \eta_3)^{-1} \kappa^{-2} (1 + \eta_1)(3 + \eta_2)\sigma \max_j \Sigma_{jj}^{1/2} \sqrt{sT/n} \left( 1 + \sqrt{(2/T) \log(p/s)} \right).
\]

Moreover, \( \hat{R} \underset{n \rightarrow \infty}{\longrightarrow} 0 \) under Assumption 1.1.
Lemma C.3. On $\Omega_*$, inequality $|\hat{S}| \leq \bar{s}$ holds with $\bar{s}$ in (C.1).

Lemma C.4. On $\Omega_*$ we have $\text{rank}(X_\bar{S}) = |\hat{S}|$.

Lemma C.5. For almost every $(X, E)$, the KKT conditions of $\hat{B}$ in (1.4) hold strictly in the sense that $\mathbb{P}(\max_{j \in S} \|Y - X\hat{B}\|_2 < nT^\lambda) = 1$.

Lemma C.6. Given the noise matrix $E$ and two design matrices $X, \tilde{X}$ define $\hat{B}$ in (1.4) and $\tilde{B}$ by

$$\tilde{B} = \arg\min_{B \in \mathbb{R}^{p \times T}} \left( \frac{1}{2\pi T} \|E + \tilde{X}(B^* - B)\|_F^2 + \lambda \|B\|_{2,1} \right).$$

If $X, \tilde{X}, E$ are such that both $\{(X, E), (\tilde{X}, E)\} \subset \Omega_*$ then

$$n^{1/2} \|\Sigma^{1/2}(\hat{B} - \tilde{B})\|_F \leq C_1(\eta_4)(\bar{R} + \|E\|_{op} n^{-1/2}) \|(X - \tilde{X})\Sigma^{-1/2}\|_F,$$

$$\|\tilde{X}(\tilde{B} - B^*) - X(\hat{B} - B^*)\|_F \leq C_2(\eta_4)(\bar{R} + \|E\|_{op} n^{-1/2}) \|(X - \tilde{X})\Sigma^{-1/2}\|_F$$

for some constants that depend on $\eta_4$ only and $\bar{R}$ is defined in Lemma C.2.

Lemma C.7. For almost every $(X, E)$ in the open set $\Omega_1 \cap \Omega_2 \cap \Omega_3$, $\hat{B}$ is a Fréchet differentiable function of $X$. For almost every $(X, E)$ in $\Omega_1 \cap \Omega_2 \cap \Omega_3$, if

$$\hat{B}(w) = \arg\min_{B \in \mathbb{R}^{p \times T}} \left( \frac{1}{2\pi T} \|E + (X + wa^\top)(B^* - B)\|_F^2 + \lambda \|B\|_{2,1} \right)$$

is the estimate (1.4) with $X$ replaced by the perturbed design $X + wa^\top$, then for any $b \in \mathbb{R}^T$

$$(X + wa^\top)(\hat{B}(w) - B^*)b - \left((X(\hat{B} - B^*))b = (D(b))w + o(\|w\|)$$

as $\|w\| \to 0$, where $D : \mathbb{R}^T \to \mathbb{R}^{n \times n}$ is a linear map given by $D(b) = D^*(b) + D^{**}(b)$ with

$$D^*(b) = (a^\top(\hat{B} - B^*)b)I_{n \times n} - (b^\top \otimes X_\bar{S})(\tilde{X}^\top \tilde{X} + nT\hat{H})^{-\frac{1}{2}}((\hat{B} - B^*)^\top a) \otimes X_\bar{S}^\top$$

$$= (a^\top(\hat{B} - B^*)b)I_{n \times n} - (b^\top \otimes X_\bar{S}) \left(\tilde{X}^\top \tilde{X} + nT\hat{H}\right)^{-\frac{1}{2}} \left(\begin{matrix} a^\top(\hat{B} - B^*)e_1X_\bar{S}^\top \\ \vdots \\ a^\top(\hat{B} - B^*)e_TX_\bar{S}^\top \end{matrix} \right),$$

$$D^{**}(b) = (b^\top \otimes X_\bar{S})(\tilde{X}^\top \tilde{X} + nT\hat{H})^{-\frac{1}{2}}((Y - X\hat{B})^\top \otimes a_\bar{S})$$

$$= (b^\top \otimes X_\bar{S}) \left(\tilde{X}^\top \tilde{X} + nT\hat{H}\right)^{-\frac{1}{2}} \left(\begin{matrix} a_\bar{S}e_1^\top(Y - X\hat{B})^\top \\ \vdots \\ a_\bar{S}e_T^\top(Y - X\hat{B})^\top \end{matrix} \right)$$

for all $b \in \mathbb{R}^T$ and $w \in \mathbb{R}^n$. Note that $D, D^*$ and $D^{**}$ implicitly depend on $(X, E)$. Hence the matrix $D(b)$ of size $n \times n$ is the Jacobian of the map $w \mapsto (X + wa^\top)(\hat{B}(w) - B^*)b$ at $w = 0$.

Lemma C.8. For any $b \in \mathbb{R}^T$ we have on $\Omega_*$

(C.3) \[ \text{Tr}[D^*(b)] = b^\top(nI_{T \times T} - \hat{A})(\hat{B} - B^*)^\top a, \]

(C.4) \[ \sum_{i=1}^{T} \left(\text{Tr}[D^{**}(e_i)]\right)^2 \leq C_3(\Sigma)\sigma^2 sT \]

for some constant depending on $\Sigma$ and $\eta_1, ..., \eta_4$ only.
Lemma C.9. Under Assumption 1.1, as \( n, p \to +\infty \) we have
\[
\frac{1}{\sigma^2 n} \mathbb{E} \left[ I(\Omega_*) \sum_{t=1}^T \left( \tilde{z}_0^T \mathbf{X}(\hat{B} - B^*) e_t - \text{Tr}[D(e_t)] \right)^2 \right] \to 0.
\]

Since \( \Omega_* \) has probability approaching one, this implies that \( \frac{1}{\sigma^2 n} \sum_{t=1}^T (\tilde{z}_0^T \mathbf{X}(\hat{B} - B^*)_e - \text{Tr}[D(e_t)])^2 \) converges to 0 in probability.

We now prove each lemma. The lemmas are restated before their proofs for convenience.

Lemma C.1. Let Assumption 1.1 be fulfilled. Then \( \mathbb{P}(\Omega_*) \to 1 \).

Proof of Lemma C.1. The fact that \( \Omega_* = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 \) has probability approaching one under Assumption 1.1 follows from the propositions in Appendix D: Proposition D.1 (iii) with \( k = 9s \) and \( x = \log n \), Proposition D.2, Proposition D.3 applied with \( k = s + 2s + 1 \), and \( \mathbb{P}(\Omega_4) \geq 1 - e^{-n/2} \) by [18, Theorem I.13].

Lemma C.2. On \( \Omega_* \) we have:

(i) \( \hat{B} - B^* \in \mathcal{C} \),
(ii) \( n^{-1/2} \| \mathbf{X} (\hat{B} - B^*) \|_F \leq (1 - \eta_3) \hat{R} \),
(iii) \( \| \Sigma^{1/2} (\hat{B} - B^*) \|_F \leq \hat{R} \),
(iv) \( \| \hat{B} - B^* \|_{2,1} \leq 3 \sqrt{s} \| \hat{B} - B^* \|_F \leq 3 \sqrt{s} \phi_{\min}(\Sigma)^{-1/2} \hat{R} \),
(v) \( \| Y - X \hat{B} \|_F^2 \leq 8 \sigma^2 nT + 2(1 - \eta_3)^2 n \hat{R}^2 \),

where
\[
\hat{R} := (1 - \eta_3)^{-1} \kappa^{-2}(1 + \eta_1)(3 + \eta_2)\sigma \max_j \Sigma_{jj}^{1/2} \sqrt{sT/n} \left( 1 + \sqrt{(2/T) \log(p/s)} \right).
\]

Moreover, \( \hat{R}_{n \to \infty} \to 0 \) under Assumption 1.1.

Proof of Lemma C.2. In the whole proof we place ourselves on the event \( \Omega_* \). We prove first that \( \hat{B} - B^* \in \mathcal{C} \).

By the definition of \( \hat{B}, 1 - n^{-1} \| \mathbf{X} \hat{B} - Y \|_F^2 + 2\lambda \| \hat{B} \|_{2,1} \leq \frac{1}{n \sigma^2} \| \mathbf{X} B^* - Y \|_F^2 + 2\lambda \| B^* \|_{2,1} \).

Rewriting the LHS as \( \frac{1}{n \sigma^2} \mathbf{X} (\hat{B} - B^*) + (\mathbf{X} B^* - Y) \| F_2^2 + 2\lambda \| B^* \|_{2,1} \) and expanding the square yields
\( \| \mathbf{X} (\hat{B} - B^*) \|_F^2 \leq 2(\mathbf{X} (\hat{B} - B^*), E) F_2 + 2nT \lambda (\| B^* \|_{2,1} - \| \hat{B} \|_{2,1}) \).

The following chain of inequalities holds
\[
\langle \mathbf{X} (\hat{B} - B^*), E \rangle_F \overset{(i)}{\leq} \sum_{j=1}^p \| (\hat{B} - B^*)^T e_j \|_2 \cdot \| (\mathbf{X}^T E)^T e_j \|_2
\]
\[
\overset{(ii)}{\leq} \sum_{j=1}^p \| (\hat{B} - B^*)^T e_j \|_2 \left[ \| \mathbf{X}^T e_j \|_2 - nT \lambda_0 \right]_+ + nT \lambda_0
\]
\[
\overset{(iii)}{\leq} \| \hat{B} - B^* \|_F \left( \sum_{j=1}^p \| \mathbf{X}^T e_j \|_2 - nT \lambda_0 \right)_+^{1/2} + nT \lambda_0 \| \hat{B} - B^* \|_{2,1}
\]
\[
\overset{(iv)}{\leq} \| \hat{B} - B^* \|_F \sqrt{s} \sqrt{nT} \lambda_0 + nT \lambda_0 \| \hat{B} - B^* \|_{2,1}.
\]

(i) and (iii) follow from Cauchy-Schwarz inequality, (ii) stems from the inequality \( a \leq (a-b)_+ + b \) and (iv) holds on \( \Omega_2 \). Thus
\[
\text{(C.5)} \quad \| \mathbf{X} (\hat{B} - B^*) \|_F^2 \leq 2 \| \hat{B} - B^* \|_F \sqrt{s} nT \lambda_0 + 2nT \left[ \lambda_0 \| \hat{B} - B^* \|_{2,1} + \lambda (\| B^* \|_{2,1} - \| \hat{B} \|_{2,1}) \right].
\]
Besides, the quantity inside the bracket on the right hand side satisfies

\[
\lambda_0 \| \hat{B} - B^* \|_{2,1} + \lambda (\| B^* \|_{2,1} - \| \hat{B} \|_{2,1})
\]

\[
(\sum_{j \in S} \| (\hat{B} - B^*)^\top e_j \|_2 + \lambda_0 \sum_{j \notin S} \| \hat{B}^\top e_j \|_2 + \lambda (\sum_{j \in S} \| B^* e_j \|_2 - \| \hat{B}^\top e_j \|_2) - \lambda \sum_{j \notin S} \| \hat{B}^\top e_j \|_2
\]

\[
\leq \lambda_0 \sqrt{s} \| \hat{B} - B^* \|_F + \lambda_0 \sum_{j \notin S} \| \hat{B}^\top e_j \|_2 + \lambda \sum_{j \in S} \| (B^* - \hat{B})^\top e_j \|_2 - \lambda \sum_{j \notin S} \| \hat{B}^\top e_j \|_2
\]

\[
(\sum_{j \in S} \| (\hat{B} - B^*)^\top e_j \|_2 + \lambda_0 \sum_{j \notin S} \| \hat{B}^\top e_j \|_2 + \lambda (\sum_{j \in S} \| B^* e_j \|_2 - \| \hat{B}^\top e_j \|_2) - \lambda \sum_{j \notin S} \| \hat{B}^\top e_j \|_2
\]

where \((ii)\) follows from Cauchy-Schwarz and the reverse triangle inequality applied respectively on the first and third summands of \((i)\), whereas \((iii)\) is a consequence of Cauchy-Schwarz.

Combining this bound with \((C.5)\) and plugging in the value \(\lambda = (1 + \eta_2)\lambda_0\) yields

\[
(C.6) \quad \| X (\hat{B} - B^*) \|_F^2 \leq 2 n T \sqrt{s} \lambda_0 \left( 2 + \eta_2 + \frac{1}{\sqrt{\eta_2}} \right) \| \hat{B} - B^* \|_F^2 - 2 n T \eta_2 \lambda_0 \sum_{j \notin S} \| \hat{B}^\top e_j \|_2.
\]

Non-negativity of the LHS, the equality \(\sum_{j \notin S} \| \hat{B}^\top e_j \|_2 = \| \hat{B} - B^* \|_{2,1} - \sum_{j \in S} \| (\hat{B} - B^*)^\top e_j \|_2\) and Cauchy-Schwarz lead to \(\| \hat{B} - B^* \|_{2,1} \leq (1 + \frac{1}{\eta_2} + \frac{1}{\eta_2 \sqrt{T}}) \sqrt{s} \| \hat{B} - B^* \|_F\).

Since \(T \geq 1\) and \(\eta_2 \geq 2\), we get \(\| \hat{B} - B^* \|_{2,1} \leq 3 \sqrt{s} \| \hat{B} - B^* \|_F\), that is \(\hat{B} - B^* \in \mathcal{C}\).

The inequality

\[
(C.7) \quad \| \hat{B} - B^* \|_F \leq \| \Sigma^{-1/2} \|_{op} \| \Sigma^{1/2} (\hat{B} - B^*) \|_F = \phi_{\min}(\Sigma)^{-1/2} \| \Sigma^{1/2} (\hat{B} - B^*) \|_F
\]

combined with \((C.6)\) and the event \(\Omega_1\) yields

\[
\| X (\hat{B} - B^*) \|_F \leq 2 \kappa^{-1} \left( 2 + \eta_2 + T^{-1/2} \right) \sqrt{n T \lambda_0} \sqrt{s}
\]

\[
\leq 2 \kappa^{-1} (1 + \eta_1) (3 + \eta_2) \sigma \max_j \Sigma_{jj}^{1/2} \sqrt{s} T \left( 1 + \sqrt{(2/T) \log(p/s)} \right)
\]

\[
= \sqrt{\eta_1} (1 - \eta_3) \tilde{R}.
\]

Reusing \(\Omega_1\), we obtain \(\| \Sigma^{1/2} (\hat{B} - B^*) \|_F \leq (\sqrt{\eta_1} (1 - \eta_3)^{-1}) \| X (\hat{B} - B^*) \|_F \leq \tilde{R}\).

Combining this last bound with \((C.7)\) yields \(\| \hat{B} - B^* \|_F \leq \phi_{\min}(\Sigma)^{-1/2} \tilde{R}\), hence \((iv)\).

For inequality \((v)\), using \(\Omega_4\), \(a^2 + b^2 \leq 2a^2 + 2b^2\) and \(\| E \|_F^2 \leq \text{rank}(E) \| E \|_{op}^2\), we have

\[
\| Y - X \hat{B} \|_F^2 \leq 2 \| E \|_F^2 + 2 \| X (\hat{B} - B^*) \|_F^2
\]

\[
\leq 2 \text{rank}(E) \| E \|_{op}^2 + 2 n (1 - \eta_3)^2 \tilde{R}^2
\]

\[
\leq 2 \min(n, T) (2 \sigma \max(\sqrt{\eta_1}, \sqrt{T}))^2 + 2 n (1 - \eta_3)^2 \tilde{R}^2
\]

\[
\leq 8 \sigma^2 n T + 2 (1 - \eta_3)^2 n \tilde{R}^2.
\]

Regarding the limit of \(\tilde{R}\), note that \(\tilde{R} \propto \left( \frac{s^2 T}{n} \right)^{1/2} + \left( \frac{s}{n} \log(\frac{p}{s}) \right)^{1/2}\). By Assumption 1.1, each summand goes to 0 as \(n\) goes to infinity.

\[\square\]

**Lemma C.3.** On \(\Omega_*\), inequality \(|\hat{S}| \leq \bar{s}\) holds with \(\bar{s}\) in \((C.1)\).
Proof of Lemma C.3. The KKT conditions of (1.4) are given by

\begin{equation}
(Y - \hat{X}B)^\top Xe_j = nT\lambda \hat{B}^\top e_j \| \hat{B} - B \|_2^2 \quad \text{for all } j \in \hat{S},
\end{equation}

\begin{equation}
\| (Y - \hat{X}B)^\top Xe_j \|_2 \leq nT\lambda \quad \text{for all } j \notin \hat{S}.
\end{equation}

This implies that \( \forall j \in \hat{S}, \| (Y - \hat{X}B)^\top Xe_j \|_2 = nT\lambda \). Since \( \| (Y - \hat{X}B)^\top Xe_j \|_2 \leq \| E^\top Xe_j \|_2 + \| (X (B^* - \hat{B}))^\top Xe_j \|_2 \), by the triangle inequality, we have for any \( j \in \hat{S} \),

\begin{equation}
nT\lambda \leq (\| E^\top Xe_j \|_2 - nT\lambda_0) + nT\lambda_0 + \| (X (B^* - \hat{B}))^\top Xe_j \|_2,
\end{equation}

\begin{equation}
nT(\lambda - \lambda_0) \leq (\| E^\top Xe_j \|_2 - nT\lambda_0) + \| (X (B^* - \hat{B}))^\top Xe_j \|_2.
\end{equation}

Summing the squares of the above inequalities for a subset \( B \subset \hat{S} \) and using \( (a+b)^2 \leq 2a^2 + 2b^2 \), we get

\[
\frac{|B|n^2(\lambda - \lambda_0)^2T^2}{2} \leq \sum_{j \in B} (\| E^\top Xe_j \|_2 - nT\lambda_0)^2 + \text{Tr}\left\{ (X (B^* - \hat{B}))^\top \{ \sum_{j \in B} Xe_j e_j^\top \} \{ X (B^* - \hat{B}) \} \right\}.
\]

The first term is bounded from above by \( s \lambda_0^2/T + \frac{1}{nT^2} \| X (\hat{B} - B^*) \|_F^2 \psi_{\text{max}}(B) \),

where \( \psi_{\text{max}}(B) \) is the largest eigenvalue of \( \frac{1}{n} \sum_{j \in B} Xe_j e_j^\top \), or equivalently the largest eigenvalue of \( \frac{1}{n} (X_B X_B^\top) \), which is also the largest eigenvalue of \( \frac{1}{n} (X_B X_B^\top) \). On the event of \( \Omega_* \), we obtain

\[
\frac{|B|n^2(\lambda - \lambda_0)^2T^2}{2} \leq s \lambda_0^2/T + \frac{\psi_{\text{max}}(B)}{nT^2} \left( \frac{4}{\kappa^2} (2 + \eta_2 + 1/\sqrt{T})^2 nT^2 \lambda_0^2 s, \right.
\]

or equivalently

\[
\frac{|B|n^2(\lambda - \lambda_0)^2T^2}{2} \leq s \left( \frac{1}{T} + \frac{\psi_{\text{max}}(B)}{\kappa^2} \right) \left( 2 + \eta_2 + 1/\sqrt{T} \right)^2.
\]

Let \( \bar{s} \) be as in (C.1) and assume that \( |\hat{S}| \leq \bar{s} \) is violated on \( \Omega_* \). Then on \( \Omega_3 \), any \( B \subset \hat{S} \) with size \( |B| = |\hat{S}| + 1 \) satisfies \( \forall v \in \mathbb{R}^p, \| X_B v_B \|_2 \leq (1 + \eta_2) \sqrt{n} \| \Sigma \|_{op} \| v_B \|_2 \). Squaring yields \( \psi_{\text{max}}(B) \leq \| \Sigma \|_{op} (1 + \eta_4)^2 \). Then

\[
\frac{|B|n^2(\lambda - \lambda_0)^2T^2}{2} \leq s \left( \frac{1}{T} + \frac{4\| \Sigma \|_{op} (1 + \eta_4)^2}{\kappa^2} \right) \left( 2 + \eta_2 + 1/\sqrt{T} \right)^2.
\]

which shows that \( |B| \leq \bar{s} \) by definition of \( \bar{s} \), a contradiction.

Lemma C.4. On \( \Omega_* \) we have \( \text{rank}(X_{\hat{S}}) = |\hat{S}| \).

Proof. By Lemma C.3, we have \( |\hat{S}| \leq \bar{s} \) on \( \Omega_* \). Since \( s \leq s + 2\bar{s} + 1 \), the event \( \Omega_3 \) yields \( \forall v \in \mathbb{R}^p, \text{supp}(v) \subset \hat{S} \Rightarrow (1 - \eta_4) \sqrt{n} \| \Sigma \|_{op} \| v \|_2 \leq \| X_{\hat{S}} v \|_2 \). If \( v \) is such that \( \text{supp}(v) \subset \hat{S} \) and \( X_{\hat{S}} v = 0 \), then we must have \( v = 0 \). Equivalently, the linear span of \( \{ e_j \}_{j \in \hat{S}} \) has intersection \( \{ 0 \} \) with \( \text{ker}(X_{\hat{S}}) \), hence \( \text{ker}(X_{\hat{S}}) \) must be contained in the span of \( \{ e_j \}_{j \notin \hat{S}} \). Thus \( \text{dim ker}(X_{\hat{S}}) \leq p - |\hat{S}| \) and by the rank-nullity theorem, \( \text{rank}(X_{\hat{S}}) \geq |\hat{S}| \). By definition of \( X_{\hat{S}} \), it is also clear that \( \text{rank}(X_{\hat{S}}) \leq |\hat{S}| \), hence the conclusion.
Lemma C.5. For almost every $(\mathbf{X}, \mathbf{E})$, the KKT conditions of $\mathbf{B}$ in (1.4) hold strictly in the sense that $\mathbb{P}(\max_{j \in S} \|Y_j - \mathbf{X}\mathbf{B}_j\|_2 < nT\lambda) = 1$.

Proof of Lemma C.5. This follows from the argument in Lemma 6.4 of [5, arXiv version v1, 24 Feb 2019].

Lemma C.6. Given the noise matrix $\mathbf{E}$ and two design matrices $\mathbf{X}, \mathbf{X}$ define $\mathbf{B}$ in (1.4) and $\mathbf{B}$ by

$$\mathbf{B} = \arg \min_{\mathbf{B} \in \mathbb{R}^{p \times T}} \left( \frac{1}{2n} \| \mathbf{E} + \mathbf{X}(\mathbf{B}^* - \mathbf{B}) \|^2_F + \lambda \| \mathbf{B} \|_{2,1} \right).$$

If $\mathbf{X}, \mathbf{X}, \mathbf{E}$ are such that both $\{(\mathbf{X}, \mathbf{E}), (\mathbf{X}, \mathbf{E})\} \subset \Omega$, then

$$n^{1/2} \| \Sigma^{1/2}(\mathbf{B} - \mathbf{B}) \|_F \leq C_4(\eta_4)(\bar{\mathcal{R}} + \| \mathbf{E} \|_{op} n^{-1/2}) \| (\mathbf{X} - \mathbf{X}) \Sigma^{-1/2} \|_F,$$

$$\| \mathbf{X}(\mathbf{B} - \mathbf{B}^*) - \mathbf{X}(\mathbf{B} - \mathbf{B}^*) \|_F \leq C_5(\eta_4)(\bar{\mathcal{R}} + \| \mathbf{E} \|_{op} n^{-1/2}) \| (\mathbf{X} - \mathbf{X}) \Sigma^{-1/2} \|_F$$

for some constants that depend on $\eta_4$ only and $\bar{\mathcal{R}}$ is defined in Lemma C.2.

Proof of Lemma C.6. By Lemma C.3, $\mathbf{B} - \mathbf{B}$ has at most 25 non-zero rows. $\Omega_3$ applied on each column of $\Sigma^{1/2}(\mathbf{B} - \mathbf{B})$ gives $(1 - \eta_4^2)n \| \Sigma^{1/2}(\mathbf{B} - \mathbf{B}) \|^2_F \leq \| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|^2_F$. Similarly, using $\Omega_3$ with $\mathbf{X}$ and summing the resulting inequality with the previous one yields

$$2(1 - \eta_4^2)n \| \Sigma^{1/2}(\mathbf{B} - \mathbf{B}) \|^2_F \leq \| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|^2_F + \| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|^2_F.$$

Define $\varphi : \mathbf{B} \mapsto \frac{1}{2n} \| \mathbf{E} + \mathbf{X}(\mathbf{B}^* - \mathbf{B}) \|_F^2 + \lambda \| \mathbf{B} \|_{2,1}$, $\psi : \mathbf{B} \mapsto \frac{1}{2n} \| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|_F^2$ and $\gamma : \mathbf{B} \mapsto \varphi(\mathbf{B}) - \psi(\mathbf{B})$. When expanding the squares, it is clear that $\gamma$ is the sum of a linear function of the convex penalty, thus $\gamma$ is convex. Additivity of subdifferentials yields $\partial \varphi(\mathbf{B}) = \partial \gamma(\mathbf{B}) + \partial \psi(\mathbf{B}) = \partial \gamma(\mathbf{B})$. By optimality of $\mathbf{B}$ we have $0_{p \times T} \in \partial \gamma(\mathbf{B})$, thus $0_{p \times T} \in \partial \gamma(\mathbf{B})$. This implies $\gamma(\mathbf{B}) \leq \gamma(\mathbf{B})$. Letting $\mathbf{H} = \mathbf{B} - \mathbf{B}^*$ and $\mathbf{H} = \mathbf{B} - \mathbf{B}^*$, the last inequality rewrites as

$$\| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|^2_F \leq \| \mathbf{E} - \mathbf{X} \mathbf{H} \|^2_F + \| \mathbf{E} - \mathbf{X} \mathbf{H} \|^2_F + g(\mathbf{B}) - g(\mathbf{B}).$$

Summing the similar inequality obtained by replacing $\mathbf{X}$ with $\mathbf{X}$ yields

$$\| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|^2_F + \| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|^2_F \leq \| \mathbf{E} - \mathbf{X} \mathbf{H} \|^2_F + \| \mathbf{E} - \mathbf{X} \mathbf{H} \|^2_F - \| \mathbf{E} - \mathbf{X} \mathbf{H} \|^2_F.$$

Combining the above displays, we obtain

$$2(1 - \eta_4^2)n \| \Sigma^{1/2}(\mathbf{B} - \mathbf{B}) \|^2_F \leq \| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|^2_F + \| \mathbf{X}(\mathbf{B} - \mathbf{B}) \|^2_F \leq \| \mathbf{E} - \mathbf{X} \mathbf{H} \|^2_F + \| \mathbf{E} - \mathbf{X} \mathbf{H} \|^2_F - \| \mathbf{E} - \mathbf{X} \mathbf{H} \|^2_F.$$

The second summand rewrites as $\langle \mathbf{X}(\mathbf{H} - \mathbf{H}), (\mathbf{X} - \mathbf{X})(\mathbf{H} + \mathbf{H}) \rangle_F + \langle (\mathbf{X} - \mathbf{X})(\mathbf{H} - \mathbf{H}), \mathbf{X}(\mathbf{H} + \mathbf{H}) \rangle_F$. By Cauchy-Schwarz and the submultiplicativity of the Frobenius norm, the second summand is bounded above by

$$\| \mathbf{X}(\mathbf{H} - \mathbf{H}) \|_F \| (\mathbf{X} - \mathbf{X}) \mathbf{H}^{-1} \|_F \| \Sigma^{1/2}(\mathbf{H} + \mathbf{H}) \|_F + \| (\mathbf{X} - \mathbf{X}) \mathbf{H}^{-1} \|_F \| \Sigma^{1/2}(\mathbf{H} - \mathbf{H}) \|_F \| \mathbf{X}(\mathbf{H} + \mathbf{H}) \|_F.$$
Since $H - \widetilde{H} = \hat{B} - \overline{B}$ and $H + \widetilde{H} = \hat{B} + \overline{B} - 2B^*$ have respectively at most $2\tilde{s}$ and $2\tilde{s} + s$ non-zero rows, using $\Omega_3$ twice gives the following bound on the second summand:

$$2(1 + \eta_4)\sqrt{n}\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F \|\Sigma^{-1/2}(\hat{X} - \overline{X})\|_F \|\Sigma^{1/2}(\hat{B} + \overline{B} - 2B^*)\|_F.$$ 

Combining the above displays, we find

$$2(1 + \eta_4)^2n\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F^2 \leq 2\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F \|\Sigma^{-1/2}(\hat{X} - \overline{X})\|_F \|\Sigma^{1/2}(\hat{B} + \overline{B} - 2B^*)\|_F.$$ 

Thanks to Lemma C.2 we have $\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F \leq \tilde{R}$ and $\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F \leq \tilde{R}$; this shows that $\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F \leq \|\Sigma^{-1/2}(\hat{X} - \overline{X})\|_F \|\Sigma^{1/2}(\hat{B} + \overline{B} - 2B^*)\|_F$.

hence

$$n^{1/2}\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F \leq 2(1 + \eta_4)(1 - \eta_4)^{-2}(\tilde{R} + \|\Sigma^{1/2}(\hat{B} + \overline{B} - 2B^*)\|_F).$$

We also have by the triangle inequality

$$\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F \leq (1 + \eta_4)n^{1/2}\|\Sigma^{1/2}(\hat{B} - \overline{B})\|_F + \|\Sigma^{-1/2}(\hat{X} - \overline{X})\|_F \|\Sigma^{1/2}(\hat{B} + \overline{B} - 2B^*)\|_F.$$ 

where the last line follows from the inequality $2(1 + \eta_4)(1 - \eta_4)^{-2} \geq 2$ for $\eta_4 \in (0, 1)$. \qed

Lemma C.7. For almost every $(X, E)$ in the open set $\Omega_1 \cap \Omega_2 \cap \Omega_3$, $\hat{B}$ is a Fréchet differentiable function of $X$. For almost every $(X, E)$ in $\Omega_1 \cap \Omega_2 \cap \Omega_3$, if

$$\hat{B}(w) = \arg \min_{B \in \mathbb{R}^p \times T} \left( \frac{1}{2nT} \|E + (X + wa^\top)(B^* - B)\|_F^2 + \lambda \|B\|_{2,1} \right)$$ 

is the estimate (1.4) with $X$ replaced by the perturbed design $X + wa^\top$, then for any $b \in \mathbb{R}^T$

$$(X + wa^\top)(\hat{B}(w) - B^*)b - (X(\hat{B} - B^*))b = (D(b))w + o(\|w\|)$$

as $\|w\| \to 0$, where $D : \mathbb{R}^T \to \mathbb{R}^{n \times n}$ is a linear map given by $D(b) = D^*(b) + D^{**}(b)$ with

$$D^*(b) = (a^\top(\hat{B} - B^*)b)I_{n \times n} - (b^\top \otimes X_S)(X^\top \hat{X} + nT \hat{H})^\dagger ((\hat{B} - B^*)^\top a) \otimes X_S^\top$$

$$= (a^\top(\hat{B} - B^*)b)I_{n \times n} - (b^\top \otimes X_S)(X^\top \hat{X} + nT \hat{H})^\dagger \begin{pmatrix} a^\top(\hat{B} - B^*)e_1 X_S^\top \\ \vdots \\ a^\top(\hat{B} - B^*)e_T X_S^\top \end{pmatrix},$$

$$D^{**}(b) = (b^\top \otimes X_S)(X^\top \hat{X} + nT \hat{H})^\dagger ((Y - X \hat{B})^\top \otimes a_S)$$

$$= (b^\top \otimes X_S)(X^\top \hat{X} + nT \hat{H})^\dagger \begin{pmatrix} a_S e_1^\top (Y - X \hat{B})^\top \\ \vdots \\ a_S e_T^\top (Y - X \hat{B})^\top \end{pmatrix},$$

for all $b \in \mathbb{R}^T$ and $w \in \mathbb{R}^n$. Note that $D, D^*$ and $D^{**}$ implicitly depend on $(X, E)$. Hence the matrix $D(b)$ of size $n \times n$ is the Jacobian of the map $w \mapsto (X + wa^\top)(\hat{B}(w) - B^*)b$ at $w = 0$. 

Proof of Lemma C.7. By Lemma C.6 and Rademacher’s theorem, we know that the Fréchet derivative of $\bar{B}$ with respect to $X$ exists almost everywhere, so that $\mathbb{D}(\bar{b})$ exists for almost every $(X,E) \in \Omega$. By Lemma C.5, we also have that for almost every $(X,E)$, the KKT conditions are strict in the sense given in Lemma C.5. In the following, we consider $(X,E) \in \Omega$ such that $\mathbb{D}(\bar{b})$ exists and such that the KKT conditions are strict; almost every $(X,E) \in \Omega$, satisfy these two conditions.

Since we know that the Jacobian $\mathbb{D}(\bar{b})$ exists by Rademacher’s theorem, it is enough to characterize its value, for instance by computing the directional derivative in any fixed direction $w \in \mathbb{R}^n$. To this end, for a real $u$ in a neighborhood of 0, let $X(u) = X + uw^\top$ and $B(u) = \bar{B}(uw)$ . Define the active set $\hat{S}(u) = \{ j \in [p] : \| B(u) e_j \|_2 > 0 \}$. We also write $X = (d/du)X|_{u=0} = w^\top$, and $B = (d/du)B(u)|_{u=0}$. At 0, we have $X(0) = X$ and $B(0) = \bar{B}$ is the estimator computed at $(X,Y)$ with $Y = XB^* + E$.

As in (C.8), the KKT conditions for $B(u)$ read, for $j \in \hat{S}(u)$ (i.e., $e_j^\top B(u) \neq 0$),

$$e_j^\top X(u)^\top [E - X(u)(B(u) - B^*)] = \frac{nT \lambda}{\| B(u)^\top e_j \|_2} e_j^\top B(u) \in \mathbb{R}^{1 \times T}$$

and for $j \notin \hat{S}(u)$ (i.e., $e_j^\top B(u) = 0$),

$$\| e_j^\top X(u)^\top [E - X(u)(B(u) - B^*)] \|_2 < nT \lambda.$$

By Lipschitz continuity of $u \mapsto B(u)$ established in Lemma C.6, the set $\hat{S}(u)$ is constant in a neighborhood of 0 because the KKT conditions on $\hat{S}(u)^c$ are bounded away from $nT \lambda$ on a neighborhood of 0 by continuity, and because the nonzero rows of $B(u)$ are bounded away from 0 in a neighborhood of 0 again by continuity of $B(u)$. Differentiation of the above display for $j \in \hat{S}(u)$ at $u = 0$ and the product rule yield

$$e_j^\top \left[ X^\top (E - X(\hat{B} - B^*)) - X^\top (\hat{X}(\hat{B} - B^*) + \hat{X} \hat{B}) \right] = nT e_j^\top \hat{B} H^{(j)}$$

with $H^{(j)}$ in (2.1). Rearranging and using $\hat{X} = wa^\top$,

$$e_j^\top \left[ aw^\top (E - X(\hat{B} - B^*)) - X^\top (wa^\top (\hat{B} - B^*)) \right] = e_j^\top \left[ nT \hat{B} H^{(j)} + X^\top X \hat{B} \right] \in \mathbb{R}^{1 \times T}.$$

Let $P_S = \sum_{j \in S} e_j e_j^\top \in \mathbb{R}^{p \times p}$. Multiplying by $e_j$ to the left and summing over $j \in \hat{S}$, we obtain

$$P_S \left[ aw^\top (E - X(\hat{B} - B^*)) - X^\top (wa^\top (\hat{B} - B^*)) \right] = P_S \left[ nT \hat{B} H^{(j)} + X^\top X \hat{B} \right] \in \mathbb{R}^{p \times T}.$$

Since $\hat{S}(u)$ is locally constant for $u$ in a neighborhood of 0, we have $P_S \hat{B} = \hat{B}$ thus $X \hat{B} = X_S \hat{B}$, hence

$$a_S w^\top (E - X(\hat{B} - B^*)) - X_S^\top (wa^\top (\hat{B} - B^*)) = nT \left[ \sum_{j \in S} e_j e_j^\top B H^{(j)} \right] + X_S^\top X_S B I_{T \times T} \in \mathbb{R}^{p \times T}.$$

We now use the relationship between vectorization and Kronecker product (1.16). Applying (1.14) to the previous display for each term, we find

$$((Y - X \hat{B})^\top \otimes a_S) \text{vec}(w)^\top - \left( ((\hat{B} - B^*)^\top a) \otimes X_S^\top \right) \text{vec}(w)$$

$$= \left( nT \sum_{j \in S} (H^{(j)} \otimes e_j e_j^\top) \right) + I_{T \times T} \otimes X_S^\top X_S \text{vec}(\hat{B})$$

$$= \left( \hat{X}^\top \hat{X} + nT \hat{H} \right) \text{vec}(\hat{B}).$$
Since \( \text{vec}(\cdot) \) is always a column vector, \( \text{vec}(w^\top) = \text{vec}(w) = w \). Finally, we have again using (1.16) and the chain rule, for any fixed \( b \in \mathbb{R}^T \),

\[
D(b)w = \frac{d}{dn}X(u)(B(u) - B^*)b \bigg|_{u=0} = wa^\top(B(0) - B^*)b + XBb = wa^\top(B(0) - B^*)b + X_S\tilde{B}b.
\]

By Lemma C.4, \( \text{rank}(X_S) = |\tilde{S}| \). The argument developed in the proof of Proposition 2.1 (ii) shows that the nullspace of the matrix \( \tilde{X}\top \tilde{X} + nT\tilde{H} \) is exactly the linear span of \( \{e_i \otimes e_j, (j, t) \in \tilde{S}^c \times [T] \} \). Because \( P_{\tilde{S}}B = \tilde{B}, \text{vec}(B) \) is in \( \ker(\tilde{X}\top \tilde{X} + nT\tilde{H})^\perp = \text{range}(\tilde{X}\top \tilde{X} + nT\tilde{H}) \). Since for any symmetric matrix \( M, M^\top M \) is the orthogonal projection on the range of \( M \), we have \((nT\tilde{H} + \tilde{X}\top \tilde{X})\) \((nT\tilde{H} + \tilde{X}\top \tilde{X})\) vec\( (B) = \text{vec}(\tilde{B}) \). Since \( X_S\tilde{B}b \) is a column vector, using (1.16) again,

\[
X_S\tilde{B}b = \text{vec}(X_S\tilde{B}b) = (b^\top \otimes X_S) \text{vec}(\tilde{B}) = (b^\top \otimes X_S) (\tilde{X}\top \tilde{X} + nT\tilde{H})^\dagger[(Y - \tilde{X}\tilde{B})^\top \otimes a_S] - ((\tilde{B} - B^*)^\top a) \otimes X_S^\top \bigg| w.
\]

Since this holds for all \( w \), this provides the desired expression for \( D(b) \) for all \( b \).

**Lemma C.8.** For any \( b \in \mathbb{R}^T \) we have on \( \Omega_* \),

\[
\text{(C.3)} \quad \text{Tr}[D^*(b)] = b^\top (nI_{T \times T} - \tilde{A})(\tilde{B} - B^*)^\top a,
\]

\[
\text{(C.4)} \quad \sum_{t=1}^T \left( \text{Tr}[D^{**}(e_t)] \right)^2 \leq C_6(\Sigma)\sigma^2sT
\]

for some constant depending on \( \Sigma \) and \( \eta_1, \ldots, \eta_4 \) only.

**Proof of Lemma C.8.** For the first equality,

\[
\text{Tr}[D^*(b)] = \text{Tr}[a^\top (\tilde{B} - B^*)b]I_{n \times n} - (b^\top \otimes X_S) (\tilde{X}\top \tilde{X} + nT\tilde{H})^\dagger[((\tilde{B} - B^*)^\top a) \otimes X_S^\top]\]

and the conclusion follows from (2.3).

For (C.4), the following bounds will be useful. Inequality \( \|X_S\|_{op}^2 \leq \|\Sigma\|_{op}(1 + \eta_t)^2n \) holds on \( \Omega_* \). Furthermore since \( \ker N = \ker N^\dagger \) for all symmetric matrices \( N \) and since \( \tilde{H} \) is positive semi-definite, on \( \Omega_* \), we find

\[
\|(\tilde{X}\top \tilde{X} + nT\tilde{H})\|^2_{op} = \left[ \min_{u \in \mathbb{R}^n, u \in \text{ker}(\tilde{X}\top \tilde{X} + nT\tilde{H})^\perp} u^\top (\tilde{X}\top \tilde{X} + nT\tilde{H})u \right]^{-2} \leq \left[ \min_{u \in \mathbb{R}^n, u \in \text{ker}(\tilde{X}\top \tilde{X} + nT\tilde{H})^\perp} u^\top (\tilde{X}\top \tilde{X} + nT\tilde{H})u \right]^{-2} \leq \phi_{\text{min}}(\Sigma)^{-2}(1 - \eta_t)^{-4}n^{-2}.
\]

We now work on \( \sum_{t=1}^T \text{Tr}[D^{**}(e_t)]^2 = \|v\|^2 \), the left hand side of (C.4). For brevity, define \( M = (\tilde{X}\top \tilde{X} + nT\tilde{H})^{\dagger}(I_{T \times T} \otimes a_S)(Y - \tilde{X}\tilde{B})^\top \). Then if \( e_t \in \mathbb{R}^T \) and \( e_t \in \mathbb{R}^n \) denote canonical basis vectors, \( \sum_{t=1}^T \text{Tr}[D^{**}(e_t)]^2 = \|v\|^2 \) where \( v \in \mathbb{R}^T \) has components \( v_t = \text{Tr}[D^{**}(e_t)] \) so that

\[
v_t = \sum_{i=1}^n e_T^\top [D^{**}(e_t)]e_i = \frac{n}{n} e_T^\top (e_t^\top \otimes X_S)Me_i = e_T^\top \sum_{i=1}^n (I_{T \times T} \otimes (e_t^\top X_S))Me_i,
\]

for all \( t \in \{1, \ldots, T\} \).
where the last equality stems from two applications of the mixed product property \((1.13)\):
\[
e_i^T(e_i^T \otimes X_S) = (1 \otimes e_i^T)(e_i^T \otimes X_S) = (e_i^T) \otimes (e_i^T I_{TX}) \otimes (1(e_i^T X_S)) = e_i^T(I_{TX} \otimes (e_i^T X_S)).
\]
Thus \(v = \sum_{i=1}^n(I_{TX} \otimes (e_i^T X_S))Me_i\) and since \(\|v\|_2^2 = v^T v = (v^T \otimes 1)v\), it follows that \(\|v\|_2^2 = \sum_{i=1}^n(v^T \otimes (e_i^T X_S))Me_i = \text{Tr}[(v^T \otimes X_S)M]\) as \((1.13)\).

By the definition of \(\hat{M}\), using the commutation property of the trace we have
\[
\|v\|_2^2 = \text{Tr}[(Y - X\hat{B})^T(v^T \otimes X_S)\hat{X}^T \hat{X} + nT\hat{H})^T(I_{TX} \otimes a_S)]).
\]
By the Cauchy-Schwarz inequality for \(\langle \cdot, \cdot \rangle_F\) and using \(\|UV\|_F \leq \|U\|_2\|V\|_F\) twice, we find
\[
\|v\|_2^2 \leq \|(Y - X\hat{B})^T(v^T \otimes X_S)\|_F \|(\hat{X}^T \hat{X} + nT\hat{H})^T(I_{TX} \otimes a_S)\|_F \\
\leq \|(Y - X\hat{B})\|_2 \|v^T \otimes X_S\|_F \|(\hat{X}^T \hat{X} + nT\hat{H})\|_F \|(I_{TX} \otimes a_S)\|_F.
\]
and the second factor equals \(\|v^T \otimes X_S\|_2 = \|v\|_2 \|X_S\|_F\) by \((1.15)\) for the Frobenius norm.

We introduce the notation \(\precsim\) to denote an inequality up to a constant that depends on \(\eta_1, \ldots, \eta_4, \phi_{min}(\Sigma), \phi_{max}(\Sigma)\) only. On \(\Omega_4\), we have the operator norm bound \((C.11)\), the bound \(\|X_S\|_F \leq \sqrt{S^2}^{1/2}\|X_S\|_op \precsim (|S|n)^{1/2}\) as well as \(\|(I_{TX} \otimes a_S)\|_F = \sqrt{T}\|a_S\|_2 \precsim \sqrt{T}\) so that
\[
\|v\|_2 \precsim \|Y - X\hat{B}\|_F \sqrt{npn^{-1}}\sqrt{T}
\]
and \(\|Y - X\hat{B}\|_op \leq \|E\|_op + \|X(B^* - \tilde{B})\|_F \leq \|E\|_op + \|X\| \|B^* - \tilde{B}\|_F \precsim \sigma(\sqrt{T} + 2\sqrt{n}) + \sqrt{n}\tilde{R}\) thanks to \(\Omega_4\) and Lemma \((C.2)\). Since \(T \leq n\) and \(\tilde{R} \leq 1\) under Assumption \((1.1)\), we have proved that \(\|v\|_2 \precsim \sigma\sqrt{sT}\) holds on \(\Omega_*\) which is exactly the desired bound \((C.4)\).

\[\blacksquare\]

**Lemma C.9.** Under Assumption \((1.1)\), as \(n, p \to +\infty\) we have
\[
\frac{1}{\sigma^2 n} \mathbb{E}\left[I(\Omega_4) \sum_{t=1}^T \left(z_0^T X(\hat{B} - B^*)e_t - \text{Tr}[D(e_t)]\right)^2\right] \to 0.
\]
Since \(\Omega_*\) has probability approaching one, this implies that \(\frac{1}{\sigma^2 n} \sum_{t=1}^T (z_0^T X(\hat{B} - B^*)e_t - \text{Tr}[D(e_t)])^2\) converges to \(0\) in probability.

**Proof of Lemma C.9.** Recall that we assume the normalization \(\|\Sigma^{-1/2}a\|_2 = 1\). Following the notation in \([5]\) we define the quantities:
\[
u_0 = \Sigma^{-1}a, \quad z_0 = Xu_0, \quad Q_0 = I_{p \times p} - u_0a^T.
\]
We have the decomposition \(X = XQ_0 + z_0a^T\), the vector \(z_0\) is independent of \(XQ_0\) and \(z_0\) has distribution \(\mathcal{N}_n(0, I_{n \times n})\). Given a value of \((E, XQ_0)\), define the open set
\[
U_0 = \{z_0 \in \mathbb{R}^n : (E, XQ_0 + z_0a^T) \in \Omega_*\} \subset \mathbb{R}^n.
\]
Since \(\Omega_*\) is open, so is the set \(U_0\). Given a value of \((E, XQ_0)\) we also define the function \(U_0 \to \mathbb{R}^{p \times T}\) given by
\[
\hat{B}(z_0) = \text{arg} \min_{B \in \mathbb{R}^{p \times T}} \left(\frac{1}{2mT}\|E + (XQ_0 + z_0a^T)(B^* - B)\|_F^2 + \lambda\|B\|_{2,1}\right)
\]
where \(m\) is a constant depending on \(\Sigma\).
as well as
\[ \mathbf{F} : U_0 \to \mathbb{R}^{n \times T}, \quad \mathbf{F} : z_0 \mapsto (XQ_0 + z_0 a^\top)(\mathbf{B}(z_0) - B^*). \]
Since \( \tilde{R} \to 0 \) under Assumption 1.1 and \( \|\mathbf{E}\|_{op} n^{-1/2} \) is bounded by an absolute constant on \( \Omega_4 \) when \( T \leq n \), Lemma C.6 shows that \( \mathbf{F} \) is \( L \)-Lipschitz for some constant \( L \) of the form \( L = \sigma C_T(\eta_1, ..., \eta_4, \Sigma) \) where the constant depends only on \( \eta_1, ..., \eta_4 \) and the minimal and maximal eigenvalues of \( \Sigma \). By Kirszbraun’s Theorem, there exists an \( L \)-Lipschitz function \( \tilde{F} : \mathbb{R}^n \to \mathbb{R}^{n \times T} \) which is an extension of \( \mathbf{F} \), i.e., it satisfies \( \mathbf{F}(z_0) = \tilde{F}(z_0) \) for all \( z_0 \in U_0 \). Since \( \mathbf{F}(z_0) \) is bounded from above by \( n^{1/2}(1 - \eta_3)\tilde{R} \) in \( U_0 \) by Lemma C.2, we define the function \( \hat{F} : \mathbb{R}^n \to \mathbb{R}^{n \times T} \) by
\[ \hat{F}(z_0) = \Pi \circ \tilde{F}(z_0) \]
where \( \Pi : \mathbb{R}^{n \times T} \to \mathbb{R}^{n \times T} \) is the convex projection onto the Frobenius ball of radius \( n^{1/2}\tilde{R} \) in \( \mathbb{R}^{n \times T} \). Since convex projections are 1-Lipschitz functions, the function \( \hat{F} \) is also an \( L \)-Lipschitz extension of \( \mathbf{F} \).

If \( D(b) \) denotes the Jacobian such that \( \hat{F}(w)b - \hat{F}(0)b = D(b)w + o(\|w\|) \) for all \( b \in \mathbb{R}^T \), then \( \hat{D}(b) = D(b) \) on \( U_0 \) because two functions that coincide on an open set have the same gradient on this open set. This implies
\[
E \left[ I\{\Omega_*\} \sum_{t=1}^T \left( z_0^\top \hat{F}(z_0)e_t - \text{Tr}[D(e_t)] \right)^2 \right] = E \left[ I\{\Omega_*\} \sum_{t=1}^T \left( z_0^\top \tilde{F}(z_0)e_t - \text{Tr}[\hat{D}(e_t)] \right)^2 \right] 
\leq E \left[ \sum_{t=1}^T \left( z_0^\top \tilde{F}(z_0)e_t - \text{Tr}[\hat{D}(e_t)] \right)^2 \right]
\]
where the second display simply follows from \( I\{\Omega_*\} \leq 1 \). By the main result of [4] we find
\[
E \left[ (z_0^\top \hat{F}(z_0)e_t - \text{Tr}[\hat{D}(e_t)])^2 \right] = E \left[ \|\hat{F}(z_0)e_t\|_F^2 + \text{Tr}(\{\hat{D}(e_t)\}^2) \right] 
\leq E \left[ \|\hat{F}(z_0)e_t\|_F^2 + \|\hat{D}(e_t)\|_F^2 \right]
\]
for each \( t = 1, ..., T \) since \( z_0 \sim \mathcal{N}_n(0, I_{n \times n}) \). Summing this inequality over \( t = 1, ..., T \) yields
\[
\frac{1}{n\sigma^2} E \left[ \sum_{t=1}^T \left( z_0^\top \hat{F}(z_0)e_t - \text{Tr}[\hat{D}(e_t)] \right)^2 \right] 
\leq \frac{1}{n\sigma^2} E \left[ \|\hat{F}(z_0)\|_F^2 + \sum_{t=1}^T \|\hat{D}(e_t)\|_F^2 \right] 
\leq \frac{\tilde{R}^2}{\sigma^2} + \frac{1}{n\sigma^2} E \left[ \sum_{t=1}^T \|\hat{D}(e_t)\|_F^2 \right]
\leq \frac{\tilde{R}^2}{\sigma^2} + \frac{1}{n\sigma^2} E \left[ I\{\Omega_*\} \sum_{t=1}^T \|\hat{D}(e_t)\|_F^2 \right] + \frac{1}{n\sigma^2} E \left[ I\{\Omega_*\} \sum_{t=1}^T \|\hat{D}(e_t)\|_F^2 \right].
\]
Note that the first term, \( \tilde{R}^2/\sigma^2 \), converges to 0, as stated in Lemma C.2. We now bound the third term, on \( \Omega_* \). The quantity \( \sum_{t=1}^T \|\hat{D}(e_t)\|_F^2 \) is exactly the squared Frobenius norm of the Jacobian of the map \( \hat{F} : \mathbb{R}^n \to \mathbb{R}^{n \times T} \) (this Jacobian has dimensions \( (nT) \times n \) but we do not need to write it explicitly or choose a specific vectorization of \( \mathbb{R}^{n \times T} \) into \( \mathbb{R}^{nT} \)). Since \( \hat{F} \) is \( L \)-Lipschitz, the operator norm of the Jacobian is at most \( L \). Since the rank of the Jacobian of
a map from $\mathbb{R}^n$ to any other linear space is at most $n$, the rank of the Jacobian is at most $n$. If follows from $\|J\|_F^2 \leq \text{rank}(J)\|J\|_{op}^2$ with $J \in \mathbb{R}^{(nT) \times n}$ the Jacobian of $F$ that

$$\sum_{t=1}^T \|\mathbb{D}(e_t)\|_F^2 = \|J\|_F^2 \leq nL^2$$

so that $\frac{1}{nT}\mathbb{E}[\Omega_*^t] \sum_{t=1}^T \|\mathbb{D}(e_t)\|_F^2 \leq \mathbb{P}(\Omega_*^t)L^2/\sigma^2$ which converges to 0 under Assumption 1.1 thanks to $\mathbb{P}(\Omega_*^t) \to 1$ in Lemma C.1.

It remains to show that $\frac{1}{nT}\mathbb{E}[\Omega_*] \sum_{t=1}^T \|\mathbb{D}(e_t)\|_F^2$ converges to 0. This quantity is equal to $\frac{1}{nT}\mathbb{E}[\Omega_*] \sum_{t=1}^T \|\mathbb{D}(e_t)\|_F^2$ since the derivatives of $\tilde{F}$ and $F$ coincide on $U_0$. To bound this quantity, we use the explicit formulæ obtained in Lemma C.7 with $\|\mathbb{D}(e_t)\|_F^2 \leq 2\|\mathbb{D}^*(e_t)\|_F^2 + 2\|\mathbb{D}^{**}(e_t)\|_F^2$. We can use the following property of Kronecker products. If $M, Q$ are two matrices, and $e_t$ is the $t$-th canonical basis vector in $\mathbb{R}^T$, then by the mixed product property (1.13)

$$\sum_{t=1}^T \|(e_t^T \otimes M)Q\|_F^2 = \sum_{t=1}^T \text{Tr}[Q^\tau (e_t \otimes M^\tau)(e_t^T \otimes M)Q]$$

$$= \text{Tr}[Q^\tau \sum_{t=1}^T (e_t \otimes M^\tau)(e_t^T \otimes M)]Q]$$

$$= \text{Tr}[Q^\tau (I_{T \times T} \otimes M)Q]$$

$$= \|(I_{T \times T} \otimes M)Q\|_F^2. \tag{C.12}$$

Since $\|\mathbb{D}^*(e_t)\|_F^2 \leq 2(a^T(\tilde{B} - B^*)e_t)^2\|I_{n \times n}\|_F^2 + 2\|(e_t^T \otimes X_S)(\tilde{X}^T \tilde{X} + nT\tilde{H})^\dagger(((\tilde{B} - B^*)^T a) \otimes X_S^\top)\|_F^2$, thanks to (C.12) with $M = \tilde{X}_S$ and $Q = (\tilde{X}^T \tilde{X} + nT\tilde{H})^\dagger(((\tilde{B} - B^*)^T a) \otimes X_S^\top)$ for the second term we find

$$\sum_{t=1}^T \|\mathbb{D}^*(e_t)\|_F^2 \leq 2n\|((\tilde{B} - B^*)^T a)\|_F^2 + 2\|(I_{T \times T} \otimes \tilde{X}_S)(\tilde{X}^T \tilde{X} + nT\tilde{H})^\dagger(((\tilde{B} - B^*)^T a) \otimes \tilde{X}^\top_S)\|_F^2.$$

The first summand is bounded by $2n\|((\tilde{B} - B^*)^T a)\|_F^2 \leq 2n\phi_{\min}(\Sigma)^{-1/2}R \phi_{\max}(\Sigma)$ and the second summand by

\begin{align*}
(i) & \leq 2\|(I_{T \times T} \otimes \tilde{X}_S)\|_{op}^2 \|(\tilde{X}^T \tilde{X} + nT\tilde{H})\|_{op}^2 \|(\tilde{B} - B^*)^T a) \otimes \tilde{X}^\top_S\|_F^2.
(ii) & \leq 2\|\tilde{X}_S\|_{op}^2 \|(\tilde{X}^T \tilde{X} + nT\tilde{H})\|_{op}^2 \|(\tilde{B} - B^*)^T a\|_F^2 \|\tilde{X}_S\|_F^2.
(iii) & \leq 2(\phi_{\max}(\Sigma))^{(1 + \eta_k)n^2}(\phi_{\min}(\Sigma)^{-2}(1 - \eta_k)^{-4}n^{-2})(\phi_{\min}(\Sigma)^{-1/2}R \phi_{\max}(\Sigma))\tilde{s}.
\end{align*}

Above, (i) follows from $\|MN\|_F \leq \|M\|_{op} \|N\|_{op} \|U\|_F$, (ii) is a consequence of (1.15) and (iii) holds on $\Omega_\ast$. Thus $\sum_{t=1}^T \|\mathbb{D}^*(e_t)\|_F^2 \leq n\tilde{R}.$
Likewise,
\[ \sum_{t=1}^{T} \|D^{**}(e_t)\|_F^2 \leq \|(I_{T \times T} \otimes X_\delta)(\hat{X}^T \hat{X} + nT \hat{H})^{-1}(Y - X\hat{B})^T \otimes a_\delta)\|_F^2 \]
\[ \leq (\phi_{\text{max}}(\Sigma)(1 + \eta_4^2)n)(\phi_{\text{min}}(\Sigma)^{-2}(1 - \eta_4)^{-4}n^{-2})(8\sigma^2nT + 2(1 - \eta_3)^2nR^2)\phi_{\text{max}}(\Sigma) \]
\[ \lesssim \sigma^2 T \]

Thus \( \frac{1}{n\sigma^2} \mathbb{E}[I(\Omega_\ast)] \sum_{t=1}^{T} \|D(e_t)\|_F^2 \lesssim \frac{\hat{R}}{n} + \frac{T}{\sigma} \) and the right hand side converges to 0 under Assumption 1.1.

\[ \Box \]

**Appendix D:** Proof that \( \mathbb{P}(\Omega_\ast) \to 1 \)

**D.1: Restricted Eigenvalues for random matrices in multi-task learning**

**Proposition D.1.** Let \( G \in \mathbb{R}^{n \times p} \) be a random matrix with i.i.d. \( \mathcal{N}(0,1) \) entries and let \( A \) be a subset of \( \mathbb{R}^{p \times T} \) with \( \|B\|_F = 1 \) for all \( B \in A \).

(i) For any two \( A, B \in A \), \( \mathbb{P}(\|GA\|_F - \|GB\|_F \geq C_8 \sqrt{x}\|B - A\|_F) \leq 6e^{-x} \) for all \( x > 0 \).

(ii) \( \sup_{A,B \in A} (\|GA\|_F - \|GB\|_F) \leq C_9 \mathbb{E} \sup_{B \in A} |\text{Tr}(B^T G')| + C_{10} \sqrt{x} \) with probability at least \( 1 - e^{-x} \), where \( G' \in \mathbb{R}^{p \times T} \) has i.i.d. \( \mathcal{N}(0,1) \) entries.

(iii) If \( X \) has i.i.d. \( \mathcal{N}(0, \Sigma) \) rows with \( \max_{j \in [p]} \Sigma_{jj} \leq 1 \) and

\[ C = \{ A \in \mathbb{R}^{p \times T} : \|A\|_{2,1} \leq \sqrt{k}\|A\|_F \}, \]

then with probability at least \( 1 - 3e^{-x} \),

\[ \sup_{A \in C : \|\Sigma^{1/2} A\|_F = 1} n^{-1/2} \|X \Sigma^{-1/2} A\|_F - 1 = \sup_{B \in \mathbb{R}^{p \times T} : \|\Sigma^{1/2} B\|_F = 1} n^{-1/2} \|X \Sigma^{-1/2} B\|_F - 1 \]
\[ \leq C_{13} \sqrt{x} / n + C_{14} n^{-1/2} \mathbb{E} \sup_{B \in \mathbb{R}^{p \times T} : \|\Sigma^{1/2} B\|_F = 1} |\text{Tr}(B^T G')| \]
\[ \leq C_{15} \sqrt{x} / n + C_{16} \sqrt{\max\{kT + k \log(p/k)\} / (\phi_{\text{min}}(\Sigma))} \]

This implies that for any constant \( \eta_3 \in (0,1) \), if \( \{kT + k \log(p/k)\} / (n\phi_{\text{min}}(\Sigma)) \to 0 \) then \( \mathbb{P}(\max_{A \in C : \|\Sigma^{1/2} A\|_F = 1} n^{-1/2} \|X \Sigma^{-1/2} A\|_F - 1 \leq \eta_3) \to 1 \).

The proof follows the argument from [32], adapted to the multi-task setting.

**Proof of (i).** We distinguish two cases.

**Case (a):** \( \sqrt{kT} n \leq n/4 \). In this case we use that

\[ \|GA\|_F - \|GB\|_F \leq \|G(A - B)\|_F = \left( \sum_{i=1}^{n} \| (A - B) G^T e_i \|_2^2 \right)^{1/2} \]

and we apply [53, Theorem 6.3.2] to the vector \( \text{vec}(G^T) \in \mathbb{R}^{np \times 1} \) and the block diagonal matrix with \( n \) blocks, each block being \( (A - B)^T \). This yields

\[ \mathbb{P}(\|GA\|_F - \|GB\|_F \geq \sqrt{x}\|B - A\|_{op} + \sqrt{n}\|B - A\|_F) \leq 2e^{-C_{17}x}. \]
Here, $\sqrt{n} \leq 4\sqrt{x}$ and we can bound from above the first term to obtain the desired bound.

Case (b): $\sqrt{\frac{n}{x}} \leq n/4$. Write $\|GA\|_F - \|GB\|_F = \frac{\|GA\|_2^2 - \|GB\|_2^2}{\|GA\|_F + \|GB\|_F}$. We will use repeatedly the following concentration bounds: if $z \sim \mathcal{N}_q(0, I_{q \times q})$ and $M \in \mathbb{R}^{q \times q}$ is symmetric positive semi-definite, then

$$
\mathbb{P}(z^\top M z - \text{Tr } M < 2\sqrt{x}\|M\|_F) \leq e^{-x}.
$$

This is a straightforward consequence of [31, Lemma 1] after diagonalizing the symmetric positive semi-definite matrix $M$. Furthermore, for any $M \in \mathbb{R}^{q \times q}$,

$$
\mathbb{P}(z^\top M z - \text{Tr } M > 2\sqrt{x}\|M\|_F + 2x\|M\|_{op}) \leq e^{-x}
$$

see for instance [10, Example 2.12] or [2, Lemma 3.1].

If $g_1^\top, \ldots, g_n^\top$ are the rows of $G$ then $\|GA\|_2^2 = \sum_{i=1}^n g_i^\top AA^g g_i$ is of the above form with $q = np$ and $M$ is block diagonal with $n$ blocks equal to $AA^\top \in \mathbb{R}^{p \times p}$. Thus $\|GA\|_F^2 \geq n\|A\|_F^2 - 2\sqrt{\frac{n}{x}}\|AA^\top\|_F \geq n - 2\sqrt{n}$ with probability at least $1 - e^{-x}$ by (D.2) and thanks to $\|A\|_F = 1$. The same holds for a lower bound on $\|GB\|_F^2$. For the numerator, thanks to (D.3), with probability at least $1 - e^{-x}$:

$$
\|GA\|_F^2 - \|GB\|_F^2 = \sum_{i=1}^n g_i^\top (A - B)(A + B)^\top g_i \\
\leq 2\sqrt{\frac{n}{x}}\|A - B\|(A + B)^\top_{op} + 2x\|A - B\|(A + B)^\top_{op}.
$$

By the union bound, with probability at least $1 - 3e^{-x}$,

$$
\|GA\|_F - \|GB\|_F \leq \frac{2\sqrt{\frac{n}{x}}\|A - B\|(A + B)^\top_{op} + 2x\|A - B\|(A + B)^\top_{op}}{2(n - 2\sqrt{n})_{+}^{1/2}}.
$$

Since here $\sqrt{\frac{n}{x}} \leq n/4$, the denominator is at least $2(n/2)^{1/2}$ and using the submultiplicativity of the Frobenius norm with $\|A + B\|_F \leq 2$ for the numerator we find

$$
\frac{\|GA\|_F - \|GB\|_F}{\|A - B\|_F} \leq 2\sqrt{\frac{n}{x}} + x \leq (n/2)^{1/2} \leq C_{18}\sqrt{x}.
$$

\[\square\]

**Proof of (ii).** Since (i) proves that the process $Z_A = \|GA\|_F$ has subgaussian increment with respect to the Frobenius norm, (ii) follows by Talagrand Majorizing Measure theorem, for example as stated in [32, Theorem 4.1].

The second statement follows by taking a fixed $B \in A$ and using $|\sqrt{n} - \|GB\|_F| \leq C_{19}\sqrt{x}$ with probability at least $1 - 2e^{-x}$ by [53, Theorem 6.3.2] applied to the block diagonal matrix with $n$ blocks, each block being $B^\top$.

\[\square\]

**Proof of (iii).** Recall that $G' \in \mathbb{R}^{p \times T}$ has i.i.d. $\mathcal{N}(0, 1)$ entries. By application of (ii), it is sufficient to control the Gaussian width

$$
\mathbb{E}_{B \in \mathbb{R}^{p \times T}: \Sigma^{-1/2}B \in C, \|B\|_F = 1} \sup \left| \text{Tr}[B^\top G'] \right| = \mathbb{E}_{A \in C: \Sigma^{-1/2}A \|F = 1} \sup \left| \text{Tr}[A^\top \Sigma^{1/2}G'] \right|.
$$

Let $A \in C$ and let $g_1^\top, \ldots, g_p^\top$ be the rows of $\Sigma^{1/2}G'$. For any fixed $j \in [p]$, the random vector $g_j \in \mathbb{R}^{T \times 1}$ has $\mathcal{N}_T(0_{1 \times T}, \Sigma_{jj}I_{T \times T})$ distribution. By the triangle inequality and the
Cauchy-Schwarz inequality we have for some \( m, t > 0 \)

\[
|\text{Tr} [A^T \Sigma^{1/2} G^*]| \leq \sum_{j=1}^{p} \| A^T e_j \|_2 \| g_j \|_2 = \| A \|_{2,1} (m + t) + \sum_{j=1}^{p} \| A^T e_j \|_2 (\| g_j \|_2 - m - t) \\
\leq \| A \|_F \sqrt{k} (m + t) + \| A \|_F \left( \sum_{j=1}^{p} (\| g_j \|_2 - m - t)^2 \right)^{1/2}
\]

where for the second line we used that \( A \in C \). We have \( \| A \|_F \leq \| \Sigma^{-1/2} \|_{op} \) if \( \| \Sigma^{1/2} A \|_F = 1 \).

Next, we now define \( m \) such that \( m^2 \) is the median of the \( \chi^2_T \) distribution, and \( t = \sqrt{2\log(p/k)} \).

As explained in the proof of Proposition D.2 around (D.6) we have \( m \leq \sqrt{T} \) [39] as well as \( \mathbb{E} \sum_{j=1}^{p} (\| g_j \|_2 - m - t)^2 \leq k \). By the inequality \( \sqrt{a} + \sqrt{b} \leq \sqrt{2(a + b)} \), (D.4) is bounded from above by \( \| \Sigma^{-1/2} \|_{op} (\sqrt{2k(T + 2\log(p/k)) + \sqrt{k}}) \leq \| \Sigma^{-1/2} \|_{op} \sqrt{8k(T + \log(p/k))} \) and the proof is complete.

\[\square\]

**D.2. \( \Omega_2: \) Control of the noise**

**Proposition D.2.** Let \( \alpha_+ = \max(0, \alpha) \). If \( E \in \mathbb{R}^{n \times T} \) has i.i.d. \( N(0, \sigma^2) \) entries and \( X \in \mathbb{R}^{n \times p} \) has i.i.d. \( \mathcal{N}_p(0, \Sigma) \) rows independent of \( E \), then

\[
\sum_{j=1}^{p} \left( \frac{\|E^T X e_j\|_2}{\sigma(1 + \eta_1) \sqrt{n} \Sigma_{jj}} - \sqrt{T} - \sqrt{2\log(p/s)} \right)^2 \leq \sum_{j=1}^{p} \left( \frac{\|E^T X e_j\|_2}{\sigma\|X e_j\|_2} - \sqrt{T} - \sqrt{2\log(p/s)} \right)^2 \leq s
\]

(D.5)

with probability at least \( 1 - 4/(\{2\log(p/s) + 2\}(4\pi \log(p/s) + 4)^{1/2}) - p e^{-n \eta_1^{1/2}} \). Consequently, on the same event with

\[
\lambda_0 = \left( \max_{j=1, \ldots, p} \Sigma_{jj}^{1/2} \right) \frac{\sigma(1 + \eta_1)}{\sqrt{n} \sqrt{T}} \left( 1 + \sqrt{(2/T) \log(p/s)} \right)
\]

we have \( \sum_{j=1}^{p} (\|E^T X e_j\|_2 - n T \lambda_0)^2 \leq \sigma^2(1 + \eta_1)^2 n \max_j \Sigma_{jj} s \leq 2n T \lambda_0^2 \).

**Proof.** Since \( E e_j \) has i.i.d. \( \mathcal{N}(0, \Sigma_{jj}) \) entries, \( \mathbb{P}(\|X e_j\|_2 \geq \Sigma_{jj}^{1/2}(\sqrt{n} + t)) \leq e^{-t^2/2} \) holds by standard bounds on \( \chi^2_n \) random variables, e.g., as a consequence of [10, Theorem 5.5]. The choice \( t = \eta_1 \sqrt{n} \) and the union bound over \( \{1, \ldots, p\} \) provides the first inequality in (D.5).

Since \( E \) is independent of \( X \), conditionally on \( X \) the random variable \( g_j := E^T X e_j/\sigma\|X e_j\|_2 \) has standard normal distribution \( \mathcal{N}_T(0, I_{T \times T}) \). Since the conditional distribution does not depend on \( X \), the unconditional distribution of \( g_j \) is also \( \mathcal{N}_T(0, I_{T \times T}) \). By [10, Theorem 10.17] applied to the 1-Lipschitz function \( g_j \mapsto \|g_j\|_2 \), inequality \( \mathbb{P}(\|g_j\|_2 \geq m_j + t) \leq \mathbb{P}(Z_j \geq t) \) holds, where \( Z_j \sim \mathcal{N}(0, 1) \) and \( m_j \) is the median of the random variable \( \|g_j\|_2 \).

It follows that for any \( t > 0 \)

\[
W := \sum_{j=1}^{p} (\|g_j\|_2 - m_j - t)^2 \quad \text{satisfies} \quad \mathbb{E}[W] \leq \sum_{j=1}^{p} \mathbb{E}(Z_j - t)^2 \leq \frac{4pe^{-t^2/2}}{(t^2 + 2)(2\pi t^2 + 4)^{1/2}},
\]

where the second inequality follows from [4, Lemma G.1]. By the argument in [39], the median of the \( \chi^2_T \) distribution is smaller than \( T \) so that \( m_j \leq \sqrt{T} \). Furthermore, for \( t = (2\log(p/s))^{1/2} \) we have \( \mathbb{E}[W] \leq sq \) where \( q^{-1} = (t^2 + 2)(2\pi t^2 + 4)^{1/2}/4 > 1 \). The second inequality in (D.5) thus holds with probability at least \( 1 - q \) by Markov’s inequality \( \mathbb{P}(W > \mathbb{E}[W]q^{-1}) \leq q \). \[\square\]
D.3. $\Omega_3$: Restricted Isometry Properties

The following bound is well known in the literature on the RIP property for Gaussian matrices, as a consequence of Gordon’s Lemma, see, e.g., [55]. We provide the argument here for completeness.

**Proposition D.3** (Bound on upper sparse eigenvalues of random matrices, Gordon’s lemma). Let $p \geq n$. If $X \in \mathbb{R}^{n \times p}$ has i.i.d. $\mathcal{N}(0, \Sigma)$ rows, then

(i) for any set $B \subset [p]$ we have

$$
P \left( \max_{v \in \mathbb{R}^p, \supp(v) \subset B} \left| \frac{\|Xv\|}{\sqrt{n} \|\Sigma^{1/2}v\|} - 1 \right| \leq \sqrt{|B|/n} + t \right) \geq 1 - 2e^{-nt^2/2}$$

by Gordon’s escape through the mesh theorem and its consequence, cf. for instance in [18, Theorem II.13] applied to the Gaussian matrix $X\Sigma^{-1/2}$ and the intersection of the unit ball with the $|B|$ dimensional linear span of $\{\Sigma^{1/2}e_j, j \in B\}$.

(ii) Let $\eta_4 \in (0, 1)$ be a constant. If $k$ is such that $\sqrt{k/n} \leq \eta_4/2$ and $k \log(ep/k)/n \leq \eta_4^2/16$, then simultaneously for all $B$ with $|B| \leq k$

$$
P \left( \max_{B \subset [p]: |B| \leq k} \left( \max_{v \in \mathbb{R}^p, \supp(v) \subset B} \left| \frac{\|Xv\|}{\sqrt{n} \|\Sigma^{1/2}v\|} - 1 \right| > \eta_4 \right) \right) \leq 2 \exp(-n\eta_4^2/16).$$

**Proof.** For (ii), by the union bound with $t = \eta_4/2$ we have

$$
P \left( \max_{B \subset [p]: |B| \leq k} \left( \max_{v \in \mathbb{R}^p, \supp(v) \subset B} \left| \frac{\|Xv\|}{\sqrt{n} \|\Sigma^{1/2}v\|} - 1 \right| \right) > \eta_4 \right) \leq 2 \left( \frac{p}{k} \right) e^{-n\eta_4^2/8}.$$

Since $\log \left( \frac{p}{k} \right) \leq k \log(ep/k)$, the right hand side is bounded from above by $2 \exp(-n\eta_4^2/16)$ by assumption on $k$. \qed

**Appendix E: Proof of Theorems 3.1 and 3.2**

**Proof.** By replacing $b$ by $b/\|b\|_2$ if necessary, we assume that $\|b\|_2 = 1$ without loss of generality. The proof is based on the decomposition

$$
(n\sigma^2)^{-1/2} (na^T (\hat{B} - B^*) b + z_0^T (Y - X\hat{B})(I_{TX} - \hat{A}/n)^{-1} b) = (n\sigma^2)^{-1/2} z_0^T E b + r^T b + \tilde{r}^T b
$$

with the remainder terms $r^T b$ and $\tilde{r}^T b$ defined by the random vectors $r, \tilde{r} \in \mathbb{R}^T$

$$
r^T = (n\sigma^2)^{-1/2} z_0^T \left[ (I_{TX} - \hat{A}/n)^{-1} - (I_{TX}) \right],
$$

$$
\tilde{r}^T = (n\sigma^2)^{-1/2} \left[ a^T (\hat{B} - B^*) (nI_{TX} - \hat{A}) - z_0^T X (\hat{B} - B^*) \right] (I_{TX} - \hat{A}/n)^{-1}.
$$

Since $z_0 \sim \mathcal{N}_n(0, I_n)$ is independent of $Eb \sim \mathcal{N}_n(0, \sigma^2 I_n)$ we have $z_0^T E b / \|z_0\|_2 \sim \mathcal{N}(0, \sigma^2)$. Since $\|z_0\|_2^2 n^{-1/2} \rightarrow 1$ by the law of large numbers, we obtain that $(n\sigma^2)^{-1/2} z_0^T E b \overset{d}{\rightarrow} \mathcal{N}(0, 1)$ by Slutsky’s theorem. To conclude with another application of Slutsky’s theorem, it remains to prove that $\|r\|_2$ and $\|\tilde{r}\|_2$ both converge to 0 in probability, and to prove that for the denominator, $(n\sigma^2)^{-1/2} \| (Y - X\hat{B}) (I_{TX} - \hat{A}/n)^{-1} b \|_2 \overset{p}{\rightarrow} 1$. 

For $\mathbf{r}$, on $\Omega_*$ we have $\|(I_{T \times T} - \hat{A}/n)^{-1} - (I_{T \times T})\|_{op} \leq \bar{s}/(n - \bar{s})$ by Proposition 2.1(iii) and Lemma C.3. It follows that

$$
E[\min(1, \|\mathbf{r}\|_2)] \leq \mathbb{P}(\Omega_*) + \mathbb{E}[I\{\Omega_*\}(n\sigma^2)^{-1/2}\|E^\top\mathbf{z}_0\|_2\| (I_{T \times T} - \hat{A}/n)^{-1} - (I_{T \times T})\|_{op}]
$$

$$
\leq \mathbb{P}(\Omega_*) + (n\sigma^2)^{-1/2}(\bar{s}/(n - \bar{s}))E[\|E^\top\mathbf{z}_0\|_2]
$$

$$
\leq \mathbb{P}(\Omega_*) + (n\sigma^2)^{-1/2}(\bar{s}/(n - \bar{s}))\sqrt{nT\sigma^2}
$$

$$
= \mathbb{P}(\Omega_*) + (\bar{s}/(n - \bar{s}))\sqrt{T}
$$

by Jensen’s inequality and $\mathbb{E}[\|E^\top\mathbf{z}_0\|_2^2] = nT\sigma^2$. The last line converges to 0 by Lemma C.1 and Assumption 1.1. Since $W_n \xrightarrow{p} 0$ if and only if $E[\min(1, |W_n|)] \to 0$, this proves the convergence $\|\mathbf{r}\|_2 \xrightarrow{p} 0$.

For $\tilde{\mathbf{r}}$, we use that

$$
E[\min(1, \|\tilde{\mathbf{r}}\|_2)] \leq \mathbb{P}(\Omega_*) + \mathbb{E}[I\{\Omega_*\}\|\tilde{\mathbf{r}}\|_2]
$$

with $\mathbb{P}(\Omega_*) \to 0$ as above. For the second term, on $\Omega_*$ we have $\|(I_{T \times T} - \hat{A}/n)^{-1}\|_{op} \leq \|(I_{T \times T})\|_{op} + \|(I_{T \times T} - (I_{T \times T} - \hat{A}/n)^{-1})\|_{op} \leq 1 + \bar{s}/(n - \bar{s}) = (1 - \bar{s}/n)^{-1}$ by Proposition 2.1 and Lemma C.3. It follows that

$$
I\{\Omega_*\}\|\tilde{\mathbf{r}}\|_2 \leq I\{\Omega_*\}\frac{1}{\sqrt{n}}(1 - \frac{1}{n})^{-1}\|(nI_{T \times T} - \hat{A})(\hat{B} - B^*)^\top\mathbf{a} - (\hat{B} - B^*)^\top\mathbf{X}_{T}^\top\mathbf{z}_0\|_2
$$

$$
= I\{\Omega_*\}\frac{1}{\sqrt{n}}(1 - \frac{1}{n})^{-1}\left[\sum_{t=1}^{T}\left(\text{Tr}[[D^*(e_t)] - \mathbf{z}_0^\top\mathbf{X}(\hat{B} - B^*)e_t]\right)^2\right]^{1/2}
$$

where the equality is a consequence of Lemma C.8. Since $D^* = D - D^{**}$, and using the inequalities $(a + b)^2 \leq 2a^2 + 2b^2$ and $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$,

$$
\mathbb{E}\left[I\{\Omega_*\}\sum_{t=1}^{T}\left(\mathbf{z}_0^\top(\hat{B} - B^*)\mathbf{X}e_t - \text{Tr}[D^*(e_t)]\right)^2\right]^{1/2}
$$

$$
\leq \left[2\mathbb{E}\left[I\{\Omega_*\}\sum_{t=1}^{T}[\mathbf{z}_0^\top(\hat{B} - B^*)\mathbf{X}e_t - \text{Tr}[D(e_t)]]\right]^2\right]^{1/2}
$$

$$
\leq o((n\sigma^2)^{1/2}) + o(\sigma \min(T, (sT)^{1/2}))
$$

by Lemma C.9 and inequality (C.4) in Lemma C.8. Combining the above displays yields

$$
E[\min(1, \|\tilde{\mathbf{r}}\|_2)] \leq \mathbb{P}(\Omega_*) + (n\sigma^2)^{-1/2}(1 - \frac{1}{n})^{-1}\left[o((n\sigma^2)^{1/2}) + O(\sigma (sT)^{1/2})\right] = o(1),
$$

or equivalently $\|\tilde{\mathbf{r}}\|_2 \xrightarrow{p} 0$.

Let us prove Theorem 3.2, that is $(n\sigma^2)^{-1/2}\|((Y - \hat{X}\hat{B})(I_{T \times T} - \hat{A}/n)^{-1}\mathbf{b})\|_{2} \xrightarrow{p} 1$. By the law of large numbers, we have $\|\mathbf{E}\mathbf{b}\|_2^2/(n\sigma^2) \xrightarrow{p} 1$, so it suffices to show that

$$(n\sigma^2)^{-1/2}\mathbb{E}[\|I_{T \times T} - \hat{A}/n\|^{-1} - I_{T \times T}\|\|\mathbf{b} - X(B^* - \hat{B})(I_{T \times T} - \hat{A}/n)^{-1}\|_{2} \xrightarrow{p} 0].$$

Techniques similar to those above show that $(n\sigma^2)^{-1/2}\|\mathbb{E}[\|I_{T \times T} - \hat{A}/n\|^{-1} - I_{T \times T}\|\|\mathbf{b} \xrightarrow{p} 0$ by Proposition 2.1(iii), and that $(n\sigma^2)^{-1/2}\|X(B^* - \hat{B})(I_{T \times T} - \hat{A}/n)^{-1}\|_{2} \xrightarrow{p} 0$ by Lemma C.2 and $R \to 0$.

An application of Slutsky’s lemma completes the proof of Theorem 3.1. \[ \square \]
Appendix F: Proof for $\chi^2$ limits, and confidence ellipsoid with nominal coverage

Lemma F.1 (Differentiation with respect to $E$). Here, we consider differentiation with respect to $E$ for fixed $X$. We have

$$\mathbb{E}\left[ I(\Omega_n) \left\| E^T X (\hat{B} - B^*) - \sigma^2 \hat{A} \right\|_F^2 \right] \leq \sigma^2 n T \tilde{R}^2 + \sigma^4 n T.$$

Proof. Let $F : \mathbb{R}^{n \times T} \rightarrow \mathbb{R}^{n \times T}$ be the function $F : E \mapsto X(\hat{B} - B^*)$. The function $F$ is 1-Lipschitz by [3, Proposition 3.1]. Furthermore, $\|F\|_F \leq \sqrt{n} \tilde{R}$ on $\Omega_*$ by Lemma C.2, so that if $\Pi : \mathbb{R}^{n \times T} \rightarrow \mathbb{R}^{n \times T}$ is the convex projection onto the Frobenius ball of radius $\sqrt{n} \tilde{R}$, the composition $F = \Pi \circ F$ coincides with $F$ on $\Omega_*$. The function $F$ is also 1-Lipschitz by composition of two 1-Lipschitz functions, and since $\Omega_*$ is open, the derivatives of $F$ and $F$ with respect to $E$ coincide in $\Omega_*$ where the derivatives exist (this existence of the derivatives is granted almost everywhere by Rademacher’s theorem).

For any $t, t' \in [T]$, by the main result of [4] applied to the function $EE_{t'} \mapsto FE_{t}$, we have

$$\mathbb{E}\left[ (e_{t'}^T E_{t}^T F_{t} - \sigma^2 \sum_{i=1}^{n} \frac{\partial e_{t'}^T F_{t}}{\partial E_{t'}})^2 \right]$$

$$= \sigma^2 \mathbb{E}\left[ \| F_{t} \|_F^2 \right] + \sigma^4 \mathbb{E}\left[ \sum_{i=1}^{n} \left( \frac{\partial}{\partial E_{t'}} e_{t'}^T F_{t} \right) \left( \frac{\partial}{\partial E_{t'}} e_{t'}^T F_{t} \right)^T \right]$$

$$\leq \sigma^2 \mathbb{E}\left[ \| F_{t} \|_F^2 \right] + \sigma^4 \mathbb{E}\left[ \sum_{i=1}^{n} \sum_{i'=1}^{n} \left( \frac{\partial}{\partial E_{t'}} e_{t'}^T F_{t} \right) \left( \frac{\partial}{\partial E_{t'}} e_{t'}^T F_{t} \right)^T \right]$$

We now sum the above inequalities for all $t, t' \in [T]$ to find

$$\sum_{t=1}^{T} \sum_{t'=1}^{T} \mathbb{E}\left[ (e_{t'}^T E_{t}^T F_{t} - \sigma^2 \sum_{i=1}^{n} \frac{\partial e_{t'}^T F_{t}}{\partial E_{t'}})^2 \right]$$

$$\leq \sigma^2 T \mathbb{E}\left[ \| F \|_F^2 \right] + \sigma^4 \mathbb{E}\left[ \sum_{t=1}^{T} \sum_{t'=1}^{T} \sum_{i=1}^{n} \sum_{i'=1}^{n} \left( \frac{\partial}{\partial E_{t'}} e_{t'}^T F_{t} \right) \left( \frac{\partial}{\partial E_{t'}} e_{t'}^T F_{t} \right)^T \right]$$

$$\leq \sigma^2 T n \tilde{R}^2 + \sigma^4 n T,$$

where for the last inequality we used that $\| \bar{F} \|_F \leq \tilde{R}\sqrt{n}$ by construction of $\bar{F}$ and that $\bar{F} : \mathbb{R}^{n \times T} \rightarrow \mathbb{R}^{n \times T}$ is 1-Lipschitz, so that the Frobenius norm of the Jacobian of $\bar{F}$ (which is a matrix of size $(nT) \times (nT)$) is at most $\sqrt{nT}$. Finally, on $\Omega_*$, we have $F = \bar{F}$ and their derivatives coincide, and by differentiating the KKT conditions of $\hat{B}$ we find $\sum_{i=1}^{n} \frac{\partial e_{t'}^T F_{t}}{\partial E_{t'}} = \hat{A}_{i'}$ on $\Omega_*$ for $F = X(\hat{B} - B^*)$. This completes the proof. \qed

Theorem F.2. Let $a \in \mathbb{R}^p$ with $\|\Sigma^{-1/2}a\|_2 = 1$. Let $\xi$ be defined in (4.1) and $\hat{\sigma}^2 = \|Y - X\hat{B}\|_F^2/(nT)$. Then under Assumption 1.1, $|\hat{\sigma}/\sigma - 1| = \alpha_T(T^{-1/2})$ as well as

$$\max\{(\sigma^2 n)^{-1/2}, (\hat{\sigma}^2 n)^{-1/2}\} \|\xi - \sqrt{n} E^T z_0\|_2^2 \leq \alpha_T(1).$$

Proof of Theorem F.2. By definition of $\xi$ we have

$$(n\sigma^2)^{-1/2} \|\xi - E^T z_0\|_2 = (n\hat{\sigma}^2)^{-1/2} \|\hat{B} - B^*\|_F^2 X^T z_0 - (nI_{T \times T} - \hat{A})(\hat{B} - B^*)^T a\|_2$$
which converges to 0 in probability by Lemma C.9. Next, with $\chi_T^2 = \sigma^{-2} \| E^T z_0 \|_{2}^{-1} \| z_0 \|_{2}^{-2}$,

\[(n \sigma^2)^{-1/2} \| \sqrt{n} E^T z_0 \|_{2}^{-1} - E^T z_0 \|_{2}^{-1} = (\chi_T^2)^{-1/2} [1 - n \sigma^{-2} \| z_0 \|_{2}^{-2}] \]  

\[(\text{F.2}) \]  

By the Cauchy-Schwarz inequality we have $E[(\chi_T^2)^{-1/2} [1 - n \sigma^{-2} \| z_0 \|_{2}^{-2}]] \leq \sqrt{T/n} E[\| z_0 \|_2 - \sqrt{n}]^{-1/2}$, Combining Theorem 3.1.1 and Equation 2.15 in [53] yields $E[\| z_0 \|_2 - \sqrt{n}]^{-1/2} \leq C$ for some absolute constant $C$. Thus, by Assumption 1.1 we have $T/n \to 0$ so that (F.2) converges to 0 in $L^1$, hence in probability. This proves $(\sigma^2 n)^{-1/2} \| \xi - \sqrt{n} E^T z_0 \|_{2}^{-1} \| z_0 \|_{2}^{-1} = o_P(1)$.

We now prove the same bound with $\sigma^2 n$ replaced by $\tilde{\sigma}^2 n$. Let $\Omega_8 = \{ \| E/F/\sigma - \sqrt{n}T \| \leq \sqrt{\log n} \}$. Then $P(\Omega_8) \to 1$ by [53, Theorem 3.1.1] and

\[I\{ \Omega_8 \cap \Omega_8 \}|\tilde{\sigma}/\sigma - 1| \leq I\{ \Omega_8 \}|X(\hat{B} - B^*)\|_F(\sigma^2 nT)^{-1/2} + I\{ \Omega_8 \}|\sqrt{n}T - \| E/F/\sigma| (nT)^{-1/2} \]

\[(\text{F.3}) \]  

by Lemma C.2 for the first term. This proves that $|\tilde{\sigma}/\sigma - 1| = o_P(T^{-1/2})$, under Assumption 1.1 so that using $\frac{1}{2} \frac{1}{\sigma} - 1 \leq |u - 1|$ for $u \in [\frac{1}{2}, \frac{3}{2}]$ we obtain for $n$ large enough

\[|1/2| I\{ \Omega_8 \cap \Omega_8 \}|\sigma/\tilde{\sigma} - 1| \leq I\{ \Omega_8 \cap \Omega_8 \}|\tilde{\sigma}/\sigma - 1| \leq (F.3). \]

Hence $\sigma/\tilde{\sigma} = 1 + o_P(1)$, thus

\[(n \sigma^2)^{-1/2} \| \sqrt{n} E^T z_0 \|_{2}^{-1} - E^T z_0 \|_{2}^{-1} = (\sigma/\tilde{\sigma})(n \sigma^2)^{-1/2} \| \sqrt{n} E^T z_0 \|_{2}^{-1} - E^T z_0 \|_{2}^{-1} = (1 + o_P(1)) o_P(1) = o_P(1). \]

**Theorem 4.1.** Define the observable positive semi-definite matrix $\hat{\Gamma} = (Y - X\hat{B})^T (Y - X\hat{B}) \in \mathbb{R}^{T \times T}$ as well as

\[\xi = (Y - X\hat{B})^T z_0 + (n I_{T \times T} - \hat{A})(\hat{B} - B^*)^T a. \]

Then under Assumption 1.1, there exists a random variable $\chi_T^2$ with chi-square distribution with $T$ degrees of freedom such that

\[\sqrt{1 - \frac{T}{n}} \| \hat{\Gamma}^{-1/2} \xi \|_2 - \sqrt{\chi_T^2} \leq o_P(1) + o_P \left( \min \left\{ \frac{T}{\sqrt{n}}, \frac{sT \log^2(p/s)}{n \sqrt{T}} \right\} \right) \]

as well as

\[-o_P(1) - O_P \left( \frac{T}{\sqrt{n}} + \frac{sT + s \log(p/s)}{n \sqrt{T}} \right) \leq \sqrt{1 - \frac{T}{n}} \| \hat{\Gamma}^{-1/2} \xi \|_2 - \sqrt{\chi_T^2}. \]

Consequently,

\[(i) \quad \left( 1 - \frac{T}{n} \right)^{1/2} \| \hat{\Gamma}^{-1/2} \xi \|_2 - (\chi_T^2)^{1/2} \leq o_P(1) \text{ holds if additionally } \min \left\{ \frac{T^2}{n}, \frac{T \log^3(p)}{n} \right\} \to 0, \]

\[(ii) \quad \left( 1 - \frac{T}{n} \right)^{1/2} \| \hat{\Gamma}^{-1/2} \xi \|_2 - (\chi_T^2)^{1/2} \geq o_P(1) \text{ holds if additionally } \frac{T^2}{n} + \frac{sT + s \log(p/s)}{n \sqrt{T}} \to 0. \]

**Proof of Theorem 4.1.** Theorem F.2 applied with $z = z_0 \| z_0 \|_2^{-1}$ yields the bound $(\sigma^2 n)^{-1/2} \| \xi - \sqrt{n} E^T z_0 \|_{2}^{-1} \| z_0 \|_{2}^{-2} = o_P(1)$. The proof then follows from Lemma F.3. \(\square\)

**Lemma F.3.** Let Assumption 1.1 be fulfilled. Let $z, \xi$ be random vectors valued in $\mathbb{R}^n$. Assume that $z$ is a measurable function of $X$ with $P(\| z \|_2 = 1) = 1$ and let $P_{z}^\perp = I_n - zz^T$. Then the random variable $F_{n,n-nT} = \frac{nT}{n} \| (E^T P_{z}^\perp E)^{-1/2} E^T z \|_2^2$ has the $F$ distribution with degrees of freedom $T$ and $n - T$, and the following holds:
Consequently, if $\chi^2_T$ is a random variable with chi-square distribution with $T$ degrees of freedom, $\sqrt{T}F_{T,n>T} = \sqrt{\chi^2_T + o_p(1)}$ as $n \to +\infty$ when $T/n \to 0$ where $\chi^2_T$ is a random variable with chi-square distribution with $T$ degrees of freedom.

(iii) $\sqrt{n-T}\|\hat{\Gamma}^{-1/2}E^TZ\|_2 - \sqrt{TF_{T,n>T}} \leq o_p(1) + O_p\left(\frac{T}{\sqrt{n}}\right)$.

(iv) $\sqrt{n-T}\|\hat{\Gamma}^{-1/2}E^TZ\|_2 - \sqrt{TF_{T,n>T}} \geq -o_p(1) - O_p\left(\frac{T}{\sqrt{n}} + \frac{T + s \log(p/s)}{n}\right)$.

(v) $\sqrt{n-T}\|\hat{\Gamma}^{-1/2}E^TZ\|_2 - \sqrt{TF_{T,n>T}} \leq o_p(1) + O_p\left(\frac{(s + T)\log^2(p/s)}{n}\right)$.

Consequently, if $(\sigma^2 n)^{-1/2}\|\xi - \sqrt{n}E^TZ\|_2 = o_p(1)$ then

\begin{align}
\chi^2_T &\leq (\frac{T}{n})^{1/2}\|\hat{\Gamma}^{-1/2}\xi\|_2 + \frac{\sqrt{T}}{\sqrt{n}}\|\xi\|_2 + \sqrt{T}\log(p/s) + O_p\left(\frac{T}{\sqrt{n}} + \frac{\sqrt{T}}{\sqrt{n}}\right)\log^2(p/s) + O_p(\frac{T}{\sqrt{n}}), \\
\chi^2_T &\geq (\frac{T}{n})^{1/2}\|\hat{\Gamma}^{-1/2}\xi\|_2 - \frac{\sqrt{T}}{\sqrt{n}}\|\xi\|_2 - \sqrt{T}\log(p/s) - O_p\left(\frac{T}{\sqrt{n}} + \frac{\sqrt{T}}{\sqrt{n}}\right)\log^2(p/s) - O_p(\frac{T}{\sqrt{n}}),
\end{align}

Proof of Lemma F.3. For (i), we introduce the quantity

\begin{equation}
H := (n-1)\|(E^T P_z^\perp E)^{-1/2}E^TZ\|_2^2 = (n-1)g^TW^{-1}g
\end{equation}

where $g = \sigma^{-1}\sqrt{n}E^TZ$ and $W = \sigma^{-2}E^TP_z^\perp E$. Since $E$ and $Z$ are independent and since $\|Z\|_2 = 1$, $g$ has distribution $N_T(0, I_{T \times T})$. $P_z^\perp$ can be orthogonally diagonalized as $Q(\sum_{i=1}^{n-1} e_i e_i^T)Q^T$ where $Q$ is an $n \times n$ orthogonal matrix, thus $W = \sum_{i=1}^{n-1} n_i n_i^T$ where the random vectors $n_i = \sigma^{-1}\sqrt{n}Q e_i$ are iid with standard normal $N_T(0, I_{T \times T})$ distribution. Therefore $W$ has the Wishart distribution with identity covariance and $n-1$ degrees-of-freedom. Since $E^TZ$ and $(E^T P_z^\perp E)(E^T P_z^\perp E)^{-1} = E^T P_z^\perp E$ are independent. By [21, Theorem 5.8] $H$ has the Hotelling distribution with parameters $T, n-1$, and

\begin{equation}
\frac{n-1-T+1}{n-1} H \sim FT,n-1-T+1 = FT,n-T
\end{equation}

where the right-hand side is the $F$ distribution with degrees-of-freedom $T$ and $n-T$. Furthermore, since $FT,n-T = \chi^2_T/(n-T)$ for some random variables having chi-square distributions with respective parameter $T$ and $n-T$, we have

\begin{equation}
\left|\sqrt{TF_{T,n>T}} - \sqrt{\chi^2_T}\right| = \left|\sqrt{\chi^2_T/(\chi^2_n/(n-T))} - \sqrt{\chi^2_T}\right| = O_p(\sqrt{T})\log^2(p/s).
\end{equation}

Thus $\chi^2_{n-T}/(n-T) = \chi^2_T(1/2) + O_p\left(\frac{T}{n} - 1\right)^{-1/2}$, and since $\frac{T}{n} \to 0$ we have $\frac{\sqrt{TF_{T,n>T}}}{\sqrt{TF_{T,n>T}} \leq q_{r,\alpha} \to 1 - \alpha$ by Proposition 4.2. This proves (i).

Next we establish a lower bound on the eigenvalues of $\hat{\Gamma}$. Let $H = B - B^*$ and consider the decomposition

\begin{equation}
\hat{\Gamma} = E^T E + (XH)^T (XH) - [E^T XH + (XH)^T E]
\end{equation}

Since $(XH)^T (XH)$ is positive semidefinite we have

\begin{equation}
\hat{\Gamma} \succeq E^T E - 2\|E^T XH\|_{op} I_{T \times T}.
\end{equation}
Since $E$ has i.i.d. $\mathcal{N}(0, \sigma^2)$ entries, if $s_{\min}(E)$ and $s_{\max}(E)$ denote the smallest and greatest singular values of $E$ we have $\sigma(\sqrt{n} - \sqrt{T}) \leq \mathbb{E}[s_{\min}(E)] \leq \mathbb{E}[s_{\max}(E)] \leq \sigma(\sqrt{n} + \sqrt{T})$ by [18, Theorem II.13]. Since $s_{\min}(E)$ and $s_{\max}(E)$ are 1-Lipschitz functions of $E$ when considered as a vector in $\mathbb{R}^{n_T}$, Gaussian concentration as stated in [19, Theorem B.6] yields the existence of exponential random variables $Z_1, Z_2 \sim \text{Exp}(1)$ such that almost surely
\[
\sigma(\sqrt{n} - \sqrt{T} - \sqrt{2Z_1}) \leq s_{\min}(E) \leq s_{\max}(E) \leq \sigma(\sqrt{n} + \sqrt{T} + \sqrt{2Z_2}).
\]
Letting $Z = 2 \max(Z_1, Z_2)$, we have
\[
\sigma^2(\sqrt{n} - \sqrt{T} - \sqrt{Z})^2I_{T \times T} \leq E^\top E \preceq \sigma^2(\sqrt{n} + \sqrt{T} + \sqrt{Z})^2I_{T \times T}.
\]
Thanks to (F.8) and the inequality $(1 - x)^2 \geq 1 - 2x$ for $x \geq 0$ we have
\[
(F.9) \quad \hat{\Gamma} \succeq \sigma^2_n[1 - 2(\sqrt{T/n} + \sqrt{Z/n}) - 2\|E^\top XH\|_{op}/(\sigma^2 n)]I_{T \times T}.
\]
On the event $\Omega_0 = \{1 - 2(\sqrt{T/n} + \sqrt{Z/n}) - 2\|E^\top XH\|_{op}/(\sigma^2 n) > 1/2\}$ we have $\lambda_{\min}(\hat{\Gamma}) \geq \lambda_{\min}(E^\top E - 2\|E^\top XH\|_{op}I_{T \times T}) \geq \sigma^2 n/2$. We now proceed to show that $\mathbb{P}(\Omega_0) \to 1$. We have by the triangle inequality for the norm $\mathbb{E}[(\cdot)^2]^{1/2}$ that
\[
(F.10) \quad \mathbb{E}
\left[
\left.
I\{\Omega_\star\}
\left(T/n + \sqrt{Z/n} + \|E^\top XH\|_{op}/(\sigma^2 n)\right)^2
\right.
\right]^{1/2}
\leq \sqrt{T/n + \mathbb{E}[Z]^{1/2}}/\sqrt{n} + \tilde{s}/n + \mathbb{E}[I\{\Omega_\star\}]/\mathbb{E}[E^\top XH - \sigma^2 \hat{\Gamma}_{op}^2/(\sigma^2 n)]]^{1/2}
\leq \sqrt{T/n + \mathbb{E}[Z]^{1/2}}/\sqrt{n} + \tilde{s}/n + [T/n](1 + \hat{R}/\sigma^2)]^{1/2}
\]
where we used Proposition 2.1(ii) and Lemma C.3 to bound $\|\hat{\Gamma}_{op}\|$ from above by $\tilde{s}$ on $\Omega_\star$ for the first inequality, and Lemma F.1 the second inequality. Hence under Assumption 1.1, the previous display converges to 0. Next, $\mathbb{P}(\Omega_0) = \mathbb{P}(\Omega_0^\star \cap \Omega_\star^\circ) + \mathbb{P}(\Omega_0 \cap \Omega_\star)$, Markov’s inequality and an application of Jensen’s inequality yield
\[
\mathbb{P}(\Omega_0) = \mathbb{P}(\Omega_0^\star \cap \Omega_\star^\circ) + \mathbb{P}\left(1/4 \leq I\{\Omega_\star\}(\sqrt{T/n} + \sqrt{Z/n} + \|E^\top XH\|_{op}/(\sigma^2 n))\right)
\leq \mathbb{P}(\Omega_0^\star) + 4\mathbb{E}
\left[
\left.
I\{\Omega_\star\}(\sqrt{T/n} + \sqrt{Z/n} + \|E^\top XH\|_{op}/(\sigma^2 n))\right.
\right]\leq \mathbb{P}(\Omega_0^\star) + 4(F.10)
\]
where $4(F.10)$ refers to four times the quantity $(F.10)$ which converges to 0. Thus the event $\Omega_0$ has probability approaching one and claim (ii) follows.

We now prove (iii)-(v). Let $\Omega(n)$ be a sequence of events with $\mathbb{P}(\Omega(n)) \to 1$, $V_n$ be any sequence of random variables and $a_n$ be any deterministic sequence of real numbers. It is easily seen that $I\{\Omega(n)\}V_n = \sigma \tau(a_n)$ implies $V_n = \sigma \tau(a_n)$ and $I\{\Omega(n)\}V_n = \sigma \tau(a_n)$ implies $V_n = \sigma \tau(a_n)$. This observation will allow us to transition seamlessly from bounds on $I\{\Omega(n)\}V_n$ to bounds on $V_n$ by choosing, e.g., $\Omega(n) = \Omega_\star \cap \Omega_0$ or other events of probability approaching one in our problem. It will be useful to note that by the same argument as above $\phi_{\min}(E^\top P_\perp^2 E) \geq \sigma^2(\sqrt{n - T} - \sqrt{T - \sqrt{2Z_1}})^2$, where $Z_1 \sim \text{Exp}(1)$, so that $\phi_{\min}(E^\top P_\perp^2 E) \geq \sigma^2 n/2$ on an event $\Omega_0$ of probability approaching one. We will use the following fact: if $M, N$ are two positive definite matrices with eigenvalues at least 1/2 then
\[
(F.11) \quad \|M^{-1/2} - N^{-1/2}\|_{op} \leq 2\|M^{1/2} - N^{1/2}\|_{op} \leq \sqrt{2}\|M - N\|_{op}
\]
using the resolvent identity $M^{-1/2} - N^{-1/2} = N^{-1/2}(N^{1/2} - M^{1/2})M^{-1/2}$ for the first inequality and [30] for the second. To prove (iii), we apply (F.11) to $M = (\sigma^2 n)^{-1}E^\top E -
\[ 2\|E^T XH\|_{op} I_{TXT} \] and \( N = (\sigma^2 n)^{-1} E^T P_z E \), both matrices having eigenvalues at least 1/2 on \( \Omega_9 \cap \Omega_9 \). Rewriting (F.8) as \( \Gamma^{-1/2} \leq (\sigma^2 n)^{-1/2} M^{-1/2} \), applying the triangle inequality and (F.11), we have on \( \Omega_9 \cap \Omega_9 \)

\[
\Delta := \sqrt{n - T} \|\tilde{\Gamma}^{-1/2} E^T z\|_2 - \sqrt{T/n - T} \\
= \sqrt{n - T} (\|\tilde{\Gamma}^{-1/2} E^T z\|_2 - (\|E^T P_z E\|^{-1/2} E^T z\|_2) \\
\leq \sqrt{(n - T)/(\sigma^2 n)} \|M^{-1/2} E^T z\|_2 - \sqrt{(n - T)/(\sigma^2 n)} \|N^{-1/2} E^T z\|_2 \\
\leq \sqrt{(n - T)/(\sigma^2 n)} \|(M^{-1/2} - N^{-1/2}) E^T z\|_2 \\
\leq \sqrt{1 - T/n} \sqrt{2} \|\sigma^2 n^{-1} [E^T z - E - 2\|E^TXH\|_{op} I_{TXT}]\|_{op} \|E^T z\|_2 \sigma^{-1}.
\]

The bounds used in (F.10) yield \( I\{\Omega_\epsilon\} \|E^T XH - \sigma^2 \hat{\Lambda}\|_{op}(\sigma^2 n)^{-1} = O_P(\sqrt{T/n}) \) and \( I\{\Omega_\epsilon\} \|\hat{\Lambda}\|_{op} = O_P(\delta_1) \), hence \( \|E^T XH - \sigma^2 \hat{\Lambda}\|_{op} \leq \|E^T XH - \sigma^2 \hat{\Lambda}\|_{op} + \|\hat{\Lambda}\|_{op} = O_P(\sqrt{T/n}) + O_P(\sqrt{T}) \). Furthermore \( \|E^T z\|^2/\sigma^2 \) has \( \chi^2_1 \) distribution, thus \( \|E^T z\|^2/\sigma^2 = O_P(T) \) and we obtain

\[ \Delta \leq \sqrt{1 - T/n} [O_P(\frac{T}{\sqrt{n}}) + O_P(\frac{\sqrt{T}}{\sqrt{n}}) + O_P(\frac{T}{n})] O_P(\sqrt{T}). \]

Since \( \frac{T}{n} \to 0 \), the right-hand side of the equality is \( O_P(\frac{T}{\sqrt{n}}) + O_P(\frac{\sqrt{T}}{\sqrt{n}}) = O_P(\frac{T}{\sqrt{n}}) + o_P(1) \).

For claim (iv), with \( \Delta \) defined in (F.12) a similar argument yields

\[ |\Delta| \leq \sqrt{n - T} \|\tilde{\Gamma}^{-1/2} - (E^T P_z E)^{-1/2}\| E^T z\|_2 \\
\leq \sqrt{2} \sqrt{1 - T/n} (\sigma^2 n)^{-1} \|E^T z\|^2_{op} + 2\|E^T XH\|_{op} + \|E^T XH\|_{op}^2 \|E^T z\|_2/\sigma \\
on \Omega_9 \cap \Omega_9, \text{ thus } |\Delta| \leq \sqrt{1 - T/n} (O_P(\frac{T}{n}) + O_P(\frac{\sqrt{T}}{\sqrt{n}}) + O_P(\frac{T}{n}) + O_P(\tilde{R}) + O_P(\sqrt{T})) = O_P(\tilde{R}^2) \] thanks to Lemma C.2(ii) for the term \( \|XH\|_{op}/(\sigma^2 n) \). This proves (iv).

It remains to prove (v), for which we need a more subtle argument. The important remark is that on the one hand \( E^T P_z E \) is independent of \( E^T z \) because \( H \) has iid \( N(0, \sigma^2) \) entries, while on the other hand \( \tilde{\Gamma} \) is not independent of \( E^T z \). To overcome this lack of independence, we bound \( \hat{\Gamma} \) from below by a positive definite matrix independent of \( E^T z \), as follows. For a fixed subset \( J \subset [p] \), let \( P_J \) be the orthogonal projection matrix onto the linear span of \( \{z_j \cup \{Xe_j, j \in J \} \) so that the rank of \( P_J \) is at most \( |J| + 1 \). Set \( P_J = I_{n \times n} - P_J \). Then in the event

\[
\tilde{S} \cup \supp(B^*) \subset J,
\]

we have \( P_J X(\hat{B} - B^*) = 0 \), hence \( \tilde{\Gamma} \succeq (Y - \hat{X})^T P_J (Y - \hat{X}) = E^T P_J E \), thus

\[ \sqrt{n - T} \|\tilde{\Gamma}^{-1/2} E^T z\|_2 \leq \sqrt{n - T} \|E^T P_J E\|^{-1/2} E^T z\|_2. \]

For a fixed \( J \) and in the event \( \tilde{S} \cup \supp(B^*) \subset J \), we can bound from above \( \Delta \) in (F.12) as

\[
\Delta \leq \sqrt{n - T} \|\tilde{\Gamma}^{-1/2} E^T z\|_2 - \|E^T P_J E\|^{-1/2} E^T z\|_2 \\
\leq \sqrt{n - T} \frac{\|E^T P_J E\|^{-1/2} E^T z\|_2}{\|E^T P_J E\|^{-1/2} E^T z\|_2} \left[ \|E^T P_J E\|^{-1/2} E^T z\|_2^2 - \|E^T P_J E\|^{-1/2} E^T z\|_2^2 \right] \\
= \frac{\sqrt{n - T}}{\|E^T P_J E\|^{-1/2} E^T z\|_2} \left[ g^T \left\{ (E^T P_J E)^{-1} - (E^T P_J E)^{-1}\right\} g \right]_+ \]

where \( g = \left( E^T P_J E \right)^{-1/2} E^T z \) and \( \|g\|_2 \leq \|E^T P_J E\|^{-1/2} E^T z\|_2 \).
where \( g = E^T z \sim N_T(0, \sigma^2 I_{T \times T}) \) as before, the first inequality follows from \( \tilde{\Gamma}^{-1/2} \preceq (E^T P_{1\tilde{\Gamma}} E)^{-1/2} \) and the second from \( \sqrt{a - \sqrt{b}} \leq (a - b)/\sqrt{\sqrt{b}} \). For any \( J \subset [p] \), the null space inclusion ker \( P_J \subset \ker z^2 \) holds and the matrix \( P_{2z}^+ = P_J^+ \) is an orthogonal projection matrix with rank \( r \leq |J| \) so that \( P_{2z}^+ - P_J^+ = Q_J Q_J^\top \) for the matrix \( Q_J \in \mathbb{R}^{n \times r} \) with orthonormal columns given by \( Q_J = \sum_{k=1}^u \mathbf{u}_k \mathbf{e}_k^\top \) where \( \mathbf{u}_k \in \mathbb{R}^n \) are orthonormal eigenvectors of \( P_{2z}^+ - P_J^+ \) corresponding to the non-zero eigenvalues and \( \mathbf{e}_k \) are canonical basis vectors in \( \mathbb{R}^T \). By the Sherman-Morrison-Woodbury identity, the matrix in curly brackets is equal to

\[
M_J := (E^T P_{1\tilde{\Gamma}} E)^{-1} \sigma \sqrt{N} J (I_{r \times T} - Q_J Q_J^\top E (E^T P_{1\tilde{\Gamma}} E)^{-1} E^T Q_J)^{-1} Q_J^\top E (E^T P_{1\tilde{\Gamma}} E)^{-1}.
\]

Applying [18, Theorem II.13] to the Gaussian matrices \( Q_J^\top E \) and \( P_{2z}^+ E \), we find

\[
P(\|E^T Q_J\|_2 \geq \sigma (\sqrt{T} + \sqrt{|J|} + t)) \leq e^{-t^2/2},
\]

\[
P(\phi_{min}(P_{2z}^+ E) \leq \sigma^2 (\sqrt{n-1} - \sqrt{T} - t)) \leq e^{-t^2/2}
\]

for all \( t > 0 \). As long as \( \frac{1}{2} \geq \left( \frac{\sqrt{T} + \sqrt{|J|} + t}{\sqrt{n-1} - \sqrt{T} - t} \right)^2 \) we have

\[
I_{r \times T} - Q_J^\top E (E^T P_{1\tilde{\Gamma}} E)^{-1} E^T Q_J \succeq I_{r \times T}/2
\]

and thus \( g^\top M_J g \leq 2\|Q_J^\top E (E^T P_{1\tilde{\Gamma}} E)^{-1} g\|_2^2 \). Applying Theorem 6.3.2 in [53] and because \( g \) is independent of \( (Q_J^\top E, P_{1\tilde{\Gamma}} E) \), we find

\[
P(\|Q_J^\top E (E^T P_{1\tilde{\Gamma}} E)^{-1} g\|_2 \geq \|Q_J^\top E (E^T P_{1\tilde{\Gamma}} E)^{-1} g\|_F + C t \|Q_J^\top E (E^T P_{1\tilde{\Gamma}} E)^{-1} g\|_F) \leq 2 e^{-t^2/2}
\]

for some absolute constant \( C > 0 \). Combined with \( (F.15) \) and the union bound,

\[
P(g^\top M_J g \geq 2\left( \frac{\sigma^2}{\sqrt{T} + \sqrt{|J|} + t} \right)^2 + C t \left( \sqrt{T} + \sqrt{|J|} + t \right)) \leq 4 e^{-t^2/2}.
\]

By concentration of chi-square distributed random variables with \( Tr \) degrees of freedom (e.g., Theorem 5.6 in [10]), we also have \( P(\sigma^{-1} \|Q_J^\top E\|_F \geq \sqrt{T}|J| + t) \leq e^{-t^2/2} \) since \( Tr \leq |J| \). Let \( s_* = \bar{s} + s \) and note that for \( t \geq 0 \),

\[
P\left\{ \delta \geq \frac{n - T}{\sqrt{T} \bar{s} - s} \left( \frac{\sqrt{T} s_* + t + C t (\sqrt{T} + \sqrt{s_*} + t)}{(\sqrt{n - 1} - \sqrt{T} - t)^2} \right)^2 \right\} \cap \{ \tilde{S} \cup \text{supp}(B^*) \subset J \}
\]

\[
\leq \sum_{J \subset [p]} \sum_{|J| = s} P\left\{ \delta \geq \frac{n - T}{\sqrt{T} \bar{s} - s} \left( \frac{\sqrt{T} |J| + t + C t (\sqrt{|J|} + t)}{(\sqrt{n - 1} - \sqrt{T} - t)^2} \right)^2 \right\} \cap \{ \tilde{S} \cup \text{supp}(B^*) \subset J \}
\]

\[
\leq 5 \left( \frac{p}{s_*} \right) e^{-t^2/2},
\]

where the first inequality holds because \( |\tilde{S} \cup \text{supp}(B^*)| \leq s_* \) on \( \Omega_* \) by Lemma C.3. and the last one is obtained by putting together the previous concentration bounds. Setting \( t = x + (2 \log (p))^{1/2} \), we find that \( \Delta \) is smaller than

\[
\frac{n - T}{\sqrt{T} \bar{s} - s} \left( \frac{\sqrt{T} s_* + \sqrt{2 \log (p)} + x + C (\sqrt{2 \log (p)} + x) (\sqrt{T} + \sqrt{s_*} + \sqrt{2 \log (p)} + x)}{(\sqrt{n - 1} - \sqrt{T} - \sqrt{2 \log (p)} - x)^2} \right)^2.
\]
with probability at least \(1 - 5e^{-x^2/2} - \mathbb{P}(\Omega_n^c)\). Since \(\mathbb{E}[F_{T,n-T}^{-1}] = T/(T-2)\), we have the estimate
\[F_{T,n-T}^{-1} = O_p(1)\]. Under Assumption 1.1(iv) to control the denominator, and by the bound \(\log \left(\frac{p}{n}\right) \leq s \cdot \log \left(\frac{p}{n}\right)\) the above display is thus
\[O_p\left(\frac{n}{\sqrt{T}} \left[ Ts + s \log(p/s) + sT \log(p/s) + s^2 \log^2(p/s) \right] \right) \]
\[= O_p\left(\frac{n}{\sqrt{T}} Ts + s \log(p/s) + \frac{sT \log(p/s)}{n\sqrt{T}} + \frac{s^2 \log^2(p/s)}{n\sqrt{T}} \right) .\]

In the right-hand side, the first term is \(o_p(1)\) thanks to Assumption 1.1(iv). For \(n\) large enough \(\log(p/s) \geq 1\) holds, thus the second and third term are smaller than \(O_p\left(\frac{s(s+T)\log^2(p/s)}{n\sqrt{T}}\right)\). This proves (v).

In order to deduce the upper bound (F.4) from (iii) and (v), it is sufficient to show that
\[(F.16)\quad \min\{T/\sqrt{n}, s(s + T) \log^2(p/s)/(n\sqrt{T})\} = o(1) + o(\log^2(p/s)n^{-1/4})\]
holds under Assumption 1.1. Let \(u_n = sT/n\) and note that \(u_n \rightarrow 0\) by Assumption 1.1. On the one hand, if \(T \leq \max\{\sqrt{n}u_n, s\}\) then \(T/\sqrt{n} \leq \max\{u_n, \sqrt{sT/n}\} = o(1)\). On the other hand, if \(T > \max\{\sqrt{n}u_n, s\}\) then
\[
\frac{s(s + T) \log^2(p/s)}{n\sqrt{T}} \leq \frac{2sT \log^2(p/s)}{n\sqrt{T}} = \frac{2u_n \log^2(p/s)}{\sqrt{T}} \leq \frac{2u_n^{1/2} \log^2(p/s)}{n^{1/4}} = o\left(\frac{\log^2(p/s)}{n^{1/4}}\right) .
\]
This proves (F.16) and completes the proof.

**Proposition 4.2.** Let \((W_n)_{n \geq 1}\) be a sequence of random random variables and \(\chi^2_{T}\) a sequence of random variables with chi-square distribution with \(T\) degrees of freedom, where \(T = T_n\) is function of \(n\) (in particular, \(T \rightarrow +\infty\) as \(n \rightarrow +\infty\) is allowed). If \(\alpha \in (0, 1)\) is a fixed constant not depending on \(n, T\) and \(q_{T,\alpha} > 0\) is the quantile defined by \(\mathbb{P}(\chi^2_{T})^{1/2} \leq q_{T,\alpha} = 1 - \alpha\) then

1. \(W_n - (\chi^2_{T})^{1/2} \leq o_p(1)\) implies that \(\mathbb{P}(W_n \leq q_{T,\alpha}) \geq 1 - \alpha - o(1)\) and
2. \(W_n - (\chi^2_{T})^{1/2} \geq -o_p(1)\) implies that \(\mathbb{P}(W_n \leq q_{T,\alpha}) \leq 1 - \alpha + o(1)\).

**Proof of Proposition 4.2.** We first prove case (i). Then by definition of \(q_{T,\alpha}\) and the union bound, for any constant \(\delta > 0\) not depending on \(n, T,\)
\[
\mathbb{P}(W_n > q_{T,\alpha}) \leq \mathbb{P}(o_p(1) > \delta) + \mathbb{P}(\chi^2_{T})^{1/2} > q_{T,\alpha} - \delta = \mathbb{P}(\chi^2_{T})^{1/2} > q_{T,\alpha} - \delta, q_{T,\alpha})] .
\]

We now bound the third term. Let \(f_T : [0, +\infty) \rightarrow [0, \infty)\) be the probability density function of \((\chi^2_{T})^{1/2}\), which admits the closed form \(f_T(x) = (2^{T/2-1} \Gamma(T/2))^{-1} x^{T-1} e^{-x^2/2}\) for \(x \geq 0\). Then \(\mathbb{P}(\chi^2_{T})^{1/2} \in [q_{T,\alpha} - \delta, q_{T,\alpha}]\) \(= \mathbb{E}_{x > 0} f_T(x)\). The supremum \(\sup_{x > 0} f_T(x)\) is attained at \(x = \sqrt{T - 1}\), the mode of the chi distribution with \(T\) degrees of freedom, so that
\[
\sup_{x > 0} f_T(x) = (2^{T/2-1} \Gamma(T/2))^{-1} (T - 1)^{(T-1)/2} e^{-T^{-1}/2} \rightarrow 1/\sqrt{\pi} as x \rightarrow +\infty.
\]
by Stirling’s formula. Hence there exists an absolute constant \(C_0 > 0\) such that
\[
\mathbb{P}(W_n > q_{T,\alpha}) \leq \mathbb{P}(o_p(1) > \delta) + \alpha + \delta C_0 .
\]
For any \(\epsilon > 0\), let \(\delta = \epsilon/C_0\). Using by the definition of convergence in probability, for \(n\) large enough we have \(\mathbb{P}(o_p(1) > \delta) \leq \epsilon\) so that \(\mathbb{P}(W_n > q_{T,\alpha}) - \alpha \leq 2\epsilon\). Since this holds for any \(\epsilon > 0\), the claim is proved. The same argument can be applied in case (ii) by reversing the inequalities. \(\Box\)
Proof of (4.2). The convergence in distribution

\[(\frac{\chi^2_j}{j})^{1/2} - \sqrt{j} \to_d N(0,1)\]

holds by the Central Limit Theorem for \( (\frac{\chi^2_j}{j} - T) \to_d N(0,1)\), the weak law of large numbers for \( (\frac{\chi^2_j}{j})^{1/2} \to_p 1 \) and Slutsky’s theorem. If \( \Phi(u) = \Phi(N(0,1) \le u) \) is the standard normal cdf, for any subsequence \( (a_T)_T \) of \( a_T = \Phi(\sqrt{2} (q_{T,\alpha} - \sqrt{\bar{T}})) \) converging to an accumulation point \( L \), we have for any \( \epsilon > 0 \) and \( T' \) large enough

\[\mathbb{P} \left[ \Phi \left( (\frac{\chi^2_j}{j})^{1/2} - \sqrt{T'} \right) \right] \le L - \epsilon \le 1 - \alpha \le \mathbb{P} \left[ \Phi \left( (\frac{\chi^2_j}{j})^{1/2} - \sqrt{T'} \right) \right] \le L + \epsilon\]

so that \( L - \epsilon \le 1 - \alpha + o(1) \) and \( 1 - \alpha \le L + \epsilon + o(1) \) by the weak convergence (F.17). It follows that \( L = 1 - \alpha \) is the only accumulation point and \( q_{T,\alpha} - \sqrt{T} \to z_\alpha / \sqrt{2} \), as desired. \( \square \)

Appendix G: Proofs for unknown covariance

G.1. Asymptotic normality

Proof of Theorem 3.3 under assumption (3.14). We will use throughout the proof the notation defined after (3.11) for \( \tau_j, \gamma_j \) and \( e_j \). Define the direction \( \tilde{a}_j = e_j (\Sigma^{-1})_{jj}^{1/2} = \tau_j e_j \) normalized such that \( \|\Sigma^{-1/2} \tilde{a}_j\|_2 = 1 \) by construction, as well as \( z_j = \frac{X_\Sigma^{-1} \tilde{a}_j}{\sim N_n(0, I_n)} \). Next, define \( \xi_j, \tilde{\xi}_j \in \mathbb{R}^T \) by

\[\xi_j = (Y - X \tilde{B})^T \tilde{z}_j + (nI_{T \times T} - \tilde{A}) (\tilde{B} - B^*)^T \tilde{a}_j,\]

\[\tilde{\xi}_j = (Y - X \tilde{B})^T \tilde{z}_j \left[ n(\tilde{z}_j^T X e_j)^{-1} \right] \tau_j + (nI_{T \times T} - \tilde{A}) (\tilde{B} - B^*)^T \tilde{a}_j\]

so that \( \xi_j \) coincides with (4.1) for the normalized direction \( \tilde{a}_j \). Since the second term in \( \xi_j \) is the same as the second term in \( \tilde{\xi}_j \),

\[\|\xi_j - \tilde{\xi}_j\|_2 = \| (Y - X \tilde{B})^T \left\{ \tilde{z}_j \tau_j^{-1} \left[ n\tau_j^2 (\tilde{z}_j^T X e_j)^{-1} \right] - \tilde{z}_j \right\} \|_2.\]

Since \( \gamma_j = -(I_p - e_j e_j^T) \Sigma^{-1} e_j (\Sigma^{-1})_{jj}^{-1} \) in (3.11), or equivalently \( e_j - \gamma_j = \tau_j \Sigma^{-1} e_j \), we have

\[\tilde{z}_j = \tau_j X \Sigma^{-1} e_j = \tau_j^{-1} X (e_j - \gamma_j).\]

Next, \( \tilde{z}_j = X e_j - X_j \tilde{\gamma}_j = X [e_j - \gamma_j] \) since by definition of \( \tilde{\gamma}_j \), the \( j \)-th coordinate of \( \tilde{\gamma}_j \) is zero, so that \( X_j \tilde{\gamma}_j = X \tilde{\gamma}_j \). By inserting \( I_p = \sum_{k=1}^p e_k e_k^T \) in (G.3), using that the KKT conditions of \( \tilde{B} \) imply that \( \max_{k \in [p]} \| (Y - X \tilde{B})^T X e_k \|_2 \le nT \lambda \) and the triangle inequality, we find

\[\|\xi_j - \tilde{\xi}_j\|_2 = \tau_j^{-1} \left\| (Y - X \tilde{B})^T X \sum_{k=1}^p e_k e_k^T \left\{ (e_j - \tilde{\gamma}_j) \left[ n\tau_j^2 (\tilde{z}_j^T X e_j)^{-1} \right] - (e_j - \gamma_j) \right\} \right\|_2\]

\[\le \tau_j^{-1} nT \lambda \sum_{k=1}^p \left( e_k^T \left\{ (e_j - \tilde{\gamma}_j) \left[ n\tau_j^2 (\tilde{z}_j^T X e_j)^{-1} \right] - (e_j - \gamma_j) \right\} \right)\]

\[= \tau_j^{-1} nT \lambda \left\| \left\{ (e_j - \tilde{\gamma}_j) \left[ n\tau_j^2 (\tilde{z}_j^T X e_j)^{-1} \right] - (e_j - \gamma_j) \right\} \right\|_1.\]
There are two errors at this point: the estimation error \( \| \hat{\gamma}^{(j)} - \gamma^{(j)} \|_1 \) and the estimation error \( | n/\tau_j^2 (\hat{z}_j^T X e_j)^{-1} - 1 | \), which corresponds to the relative error of the estimation of the variance \( \tau_j^2 \) by \( n^{-1} \hat{z}_j^T X e_j \) in the linear model (3.11). Keeping these two errors in mind, by the triangle inequality the previous display yields
\[
\| \xi_j - \hat{\xi}_j \|_2 \leq \frac{\tau_j^{-1} n T \lambda}{(\hat{z}_j^T X e_j)/(n \tau_j^2)} \left( \| \gamma^{(j)} - \hat{\gamma}^{(j)} \|_1 + \| e_j - \hat{\gamma}^{(j)} \|_1 \right) \left( 1 - \frac{\hat{z}_j^T X e_j}{n \tau_j^2} \right).
\]

For the first term in the parenthesis, inequality (3.13) holds: this is the usual \( \ell_1 \) estimation rate for the Lasso estimate \( \hat{\gamma}^{(j)} \) for the sparse estimation target \( \gamma^{(j)} \) in the linear model (3.11) with noise variance \( \tau_j^2 \). For the second term, inequality
\[
\tau_j^{-1} \| e_j - \gamma^{(j)} \|_1 = \tau_j \| \Sigma^{-1} e_j \| \leq \tau_j \| \Sigma^{-1} \|_{op} \| e_j \|_2 \leq \| \Sigma^{-1} \|_{op} \| e_j \|_2^{1/2}
\]
holds thanks to the Cauchy-Schwarz inequality and \( \tau_j = \| \Sigma^{-1/2} e_j \|_2^{-1} \). Furthermore, by the triangle inequality, we have
\[
\| 1 - \frac{\hat{z}_j^T X e_j}{n \tau_j^2} \| \leq \left| 1 - \frac{\| e_j \|_2^2}{\tau_j^2} \right| + \left| \frac{\| \Sigma^{-1} \|_{op} \| e_j \|_2^{1/2} \} \right| \frac{\| e_j - \hat{\gamma}^{(j)} \|_1}{\tau_j^2} \leq \frac{\| e_j \|_2^2}{\tau_j^2} \| e_j - \hat{\gamma}^{(j)} \|_1 \| X^{-T} e_j \|_\infty.
\]

Each factor in the right hand side is bounded from above as follows: \( \| \hat{\gamma}^{(j)} - \gamma^{(j)} \|_1 = \tau_j \| \Sigma^{-1} \|_{op} \| \gamma^{(j)} \|_{0} O_p(\sqrt{n \log p}) \) thanks to (3.13) and \( \| X^{-T} e_j \|_\infty = \tau_j O_p(\sqrt{n \log p}) \) because \( X^{-j} \) is independent of \( e_j \) and max\( k \in [p] \setminus \{ j \} \Sigma_{jk} \leq 1 \). This proves that \( \| (e_j - \hat{z}_j) - \hat{\gamma}^{(j)} \|_1 / (n \tau_j^2) \leq \| \Sigma^{-1} \|_{op} \| \gamma^{(j)} \|_{0} O_p(1) \). We also have
\[
\| \hat{z}_j^T (X e_j - \hat{\gamma}^{(j)}) \| = \| \hat{z}_j^T X^{-j} \gamma^{(j)} \| \leq \| \hat{z}_j^T X^{-j} \|_\infty \| \gamma^{(j)} \|_1 \leq O_p(\tau_j \sqrt{n \log p}) \| \gamma^{(j)} \|_1
\]
thanks to Hölder’s inequality and the KKT conditions for \( \hat{\gamma}^{(j)} \) in (3.12) to bound the \( \ell_\infty \) norm. We have \( \| \gamma^{(j)} \|_1 \leq \| \gamma^{(j)} \|_0^{1/2} \| \gamma^{(j)} \|_2 \) and \( \| \gamma^{(j)} \|_2 = \| \tau_j^{-1} \Sigma^{-1} \|_{op} \| e_j \|_2 \leq \tau_j \| \Sigma^{-1/2} \|_{op} \) by definition of \( \gamma^{(j)} \) and the Cauchy-Schwarz inequality. Combining these bounds provide an upper bound on the right hand side of (G.7), so that
\[
\| 1 - \frac{\hat{z}_j^T X e_j}{n \tau_j^2} \| \leq O_p \left( \frac{1}{\sqrt{n}} \right) + \| \Sigma^{-1} \|_{op} \| \gamma^{(j)} \|_0 O_p \left( \frac{\log p}{n} \right) + \| \Sigma^{-1/2} \|_{op} \left( \frac{\| \gamma^{(j)} \|_0 \log p}{n} \right)^{1/2} \leq \| \Sigma^{-1} \|_{op} \left( \| \gamma^{(j)} \|_0 \log(p)/n \right)^{1/2} O_p(1)
\]
where the second line follows by bounding from above the first two terms thanks to assumption (3.14) and \( \| \Sigma^{-1} \|_{op} \geq 1 \) (this is a consequence of \( \Sigma_{jj} \leq 1 \) in Assumption 1.1). The bound (G.8) also provides \( \hat{z}_j^T X e_j / (n \tau_j^2) \overset{p}{\rightarrow} 1 \) and thus \( n^{-1} \| \hat{z}_j^T X e_j \| = O_p(1) \). Using (3.13), (G.6) and (G.8) to bound from above the right hand side of (G.5) we find
\[
\| \xi_j - \hat{\xi}_j \|_2 \leq n T \lambda \left( \| \Sigma^{-1} \|_{op} \| \gamma^{(j)} \|_0 O_p(\sqrt{n^{-1} \log p}) \| \Sigma^{-1/2} \|_{op} \| \Sigma^{-1} \|_{op} \| \gamma^{(j)} \|_0 \right)^{1/2} O_p(\sqrt{n^{-1} \log p}),
\]
Since $\|\gamma(j)\|_0 = \|\Sigma^{-1} e_j\|_0 - 1$, this implies $\|\xi_j - \hat{\xi}_j\|_2 \leq nT\lambda\|\Sigma^{-1}\|_{op}^3/2\|\Sigma^{-1} e_j\|_0 O_p(\sqrt{n^{-1}\log p})$. Thanks to $\lambda = O(\sigma(nT)^{-1/2})(1 + \log(p/s)/T))$ by definition of $\lambda$, we eventually obtain

\begin{equation}
(\sigma^2 n)^{-1/2} ||\hat{\xi}_j - \xi_j||_2 = O_p((\sqrt{T} + \sqrt{\log(p/s)})\|\Sigma^{-1} e_j\|_0 \sqrt{\log(p/n)})
\end{equation}

which converges to 0 in probability thanks to assumption (3.14).

To complete the proof of Theorem 3.3 and prove asymptotic normality for some fixed $b \in \mathbb{R}^T$ with $\|b\|_2 = 1$, notice that

$$\zeta_j := \frac{n\tilde{a}_j^T(\hat{B} - B^*) b + \tilde{z}_j^T (Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b}{\|Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b\|_2}$$

satisfies $\zeta_j \Rightarrow \mathcal{N}(0,1)$ by Theorem 3.1 applied to the normalized direction $\tilde{a}_j$. Furthermore,

$$\left| \zeta_j - \frac{ne_j^T(\hat{B} - B^*) b + n(\tilde{z}_j^T X e_j)^{-1}\tilde{z}_j^T (Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b}{(\tau_j)^{-1} ||(Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b||_2} \right|$$

$$\leq \frac{||\xi_j - \hat{\xi}_j||_2}{(\tau_j)^{-1} ||(Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b||_2} \leq \frac{(\sigma^2 n)^{-1/2} ||\xi_j - \hat{\xi}_j||_2 \|I_{T \times T} - \hat{A}/n\|^{-1} op \left(||(Y - X\hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b||_2^{-1} (\sigma^2 n)^{1/2} \right).$$

In the above display, $(\sigma^2 n)^{-1/2} ||\xi_j - \hat{\xi}_j||_2 \overset{p}{\rightarrow} 0$ when (3.14) holds, $||(I_{T \times T} - \hat{A}/n)^{-1} ||_{op} \overset{p}{\rightarrow} 1$ by Proposition 2.1(iii) and Lemma C.3, and the rightmost factor converges to 1 in probability by Theorem 3.2.

Since $\tau_j = (\Sigma^{-1})_{jj}^{-1/2}$, the last claim follows by $\tilde{z}_j^T X e_j/(n\tau_j^2)^{-1/2}$ by (G.8) and Slutsky’s theorem. We also have $\|\tilde{z}_j\|_2/(\tau_j \sqrt{n}) \overset{p}{\rightarrow} 1$ since, using (3.12) and the triangle inequality,

\begin{equation}
(\tau_j^2 n)^{-1} ||\tilde{z}_j\|_2^2 - \tilde{z}_j^T X e_j = (\tau_j^2 n)^{-1} ||\tilde{z}_j^T X - \gamma(j)\|_2^2
\end{equation}

\begin{align*}
&\leq (\tau_j^2 n)^{-1} O_p(1) \sqrt{n \log p} ||\gamma(j)||_1 \\
&\leq O_p(1) \sqrt{\log(p)/n} ||\tau_j^{-1} (\gamma(j) - \gamma(j))||_1 + ||\gamma(j)||_1 / \tau_j \\
&\leq O_p(1) \sqrt{\log(p)/n} \left[ ||\gamma(j)||_1 \sqrt{\log(p/n) + ||\gamma(j)||_0^{1/2}} \right] \\
&= O_p(1)
\end{align*}

thanks to (3.13) for the first term and the Cauchy-Schwarz inequality for the second. The convergence to 0 in probability in the last line follows from (3.14).

\textit{Proof of Theorem 3.3 under assumption (3.15).} With $\hat{\xi}_j$ in (G.2) and $\hat{\xi}_j := E^T \hat{z}_j [n(\tilde{z}_j^T X e_j)^{-1}] \tau_j$ we have

\begin{equation}
||\hat{\xi}_j - \hat{\xi}_j||_2 = || - \hat{A}(\hat{B} - B^*)^T \tilde{a}_j + \tau_j (B^* - \hat{B})^T \tilde{z}_j [n(\tilde{z}_j^T X e_j)^{-1}] ||_2
\end{equation}

\begin{align*}
&\leq ||\hat{A}||_{op} ||\Sigma^{1/2}(\hat{B} - B^*)||_{op} + \tau_j ||\hat{B} - B^*||_{2,1} ||\tilde{z}_j||_{\infty} ||[n(\tilde{z}_j^T X e_j)^{-1}]||_1 \\
&\leq ||\hat{A}||_{op} ||\Sigma^{1/2}(\hat{B} - B^*)||_{op} + \tau_j (\sqrt{\log(n)} + \sqrt{n\log p}) ||\tau_j^{-1} \hat{z}_j\|_{\infty} ||[n(\tilde{z}_j^T X e_j)^{-1}]||_1
\end{align*}

thanks to $||\Sigma^{-1/2} \tilde{a}_j||_2 = 1$ for the first term and Hölder’s inequality for the second term. Thanks to Lemma C.2(iii), Lemma C.3 and Proposition 2.1 we find $||\hat{A}||_{op} ||\Sigma^{1/2}(\hat{B} - B^*)||_{op} = O_p(\sqrt{sR})$.

For the second term, thanks to (3.12) and Lemma C.2(iv) we have

$$\tau_j ||\hat{B} - B^*||_{2,1} ||\tilde{z}_j||_{\infty} ||[n(\tilde{z}_j^T X e_j)^{-1}]||_1 \leq O_p(\sqrt{sR}) \sqrt{n \log p} ||\tau_j^{-1} \hat{z}_j\|_{\infty} ||[n(\tilde{z}_j^T X e_j)^{-1}]||_1$$

\begin{align*}
&\leq O_p(1) \sqrt{\log(p/n) + ||\gamma(j)||_0^{1/2}} \\
&= O_p(1)
\end{align*}

thanks to (3.13) for the first term and the Cauchy-Schwarz inequality for the second. The convergence to 0 in probability in the last line follows from (3.14).

\hfill \Box
and the bound (G.8) grants \( \frac{\tilde{z}_j^T X e_j}{n \tau_j^2} \xrightarrow{p} 1 \) thanks to the leftmost assumption in (3.15). In summary, \( (\sigma^2 n)^{-1/2} \| \tilde{\xi}_j - \xi_j \| = O_p(n^{-1/2} R + \sqrt{s R \log p}) = O_p(\sqrt{\log p}) \) thanks to \( n^{-1/2} \sqrt{s} \leq 1 \). Hence due to the rightmost assumption in (3.15),

\[
(G.12) \quad (\sigma^2 n)^{-1/2} \| \tilde{\xi}_j - \xi_j \| \xrightarrow{p} 0.
\]

Next, assume without loss of generality that \( \| b \|_2 = 1 \). By definition of \( \tilde{\xi}_j \) in (G.2),

\[
\frac{ne_j^T (B - B^*) b + n(\tilde{z}_j^T X e_j)^{-1} \tilde{z}_j^T (Y - X \hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b}{\sigma \sqrt{n}} = \frac{(\xi_j - \tilde{\xi}_j)^T (I_{T \times T} - \hat{A}/n)^{-1} b}{\sigma \sqrt{n}}.
\]

The first term converges to 0 in probability thanks to the previous paragraph, while the second term is \( O_p(s/n) \| \xi_j \|_2 (\sigma^2 n)^{-1/2} \) by Proposition 2.1 and Lemma C.3. \( \zeta_j : = [n \tau_j^2 (\tilde{z}_j^T X e_j)^{-1}]^{-1} \tau_j \| \tilde{z}_j \|_2 \tilde{\xi}_j \)

has \( N_T(0, \sigma^2 I_{T \times T}) \) distribution by independence of \( E \) and \( X \). Next, \( \| \xi_j \|_2 = \| \tilde{z}_j \|_2 [n \tau_j^2 (\tilde{z}_j^T X e_j)^{-1}] \| \xi_j \|_2 \) and \( \| \tilde{\xi}_j \|_2 = O_p(\sqrt{T}) \) since \( \| \| \tilde{z}_j \|_2 \| = T \). Furthermore, \( n \tau_j^2 (\tilde{z}_j^T X e_j)^{-1} \xrightarrow{p} 1 \) by (G.8). We also have \( \tau_j \sqrt{n} \| \tilde{z}_j \|_2 \xrightarrow{p} 1 \) by (G.10), thanks to the leftmost assumption in (3.15) for the last line in (G.10). This shows that \( \| \tilde{\xi}_j \|_2/(\sqrt{n} \| \xi_j \|_2) \xrightarrow{p} 1 \) and that the second term in (G.13) is \( O_p(s/n) \) and converges to 0 in probability. We conclude by observing that \( \tilde{\xi}_j b/(\sigma \sqrt{n}) \xrightarrow{d} N(0, 1) \) by Slutsky’s theorem thanks to \( n \tau_j^2 (\tilde{z}_j^T X e_j)^{-1} \xrightarrow{p} 1 \) and \( \tau_j \sqrt{n} \| \tilde{z}_j \|_2 \xrightarrow{p} 1 \). In the denominator, \( \| (Y - X \hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b \|_2 \) and \( \sigma \sqrt{n} \) can be used interchangeably, again by Slutsky’s theorem, since \( \| (Y - X \hat{B})(I_{T \times T} - \hat{A}/n)^{-1} b \|_2/(\sigma \sqrt{n}) \xrightarrow{d} 1 \) by Theorem 3.2.

\( G.2. \) Asymptotic \( \chi^2_T \) distribution

**Proof of Theorem 4.3 under assumption (3.15)**. Let \( \tilde{\xi}_j \) and \( \tilde{\xi}_j \) be defined respectively in (G.2) and in the sentence preceding (G.11). Notice that the quantity in the left hand side of (4.7) is equal to \( (1 - T/n)^{1/2} || \tilde{\Gamma}^{-1/2} \xi ||_2 \) where

\[
(x_j^T \| \tilde{z}_j \|_2 [n(\tilde{z}_j^T X e_j)^{-1}]^{-1}) \xi_j.
\]

Set \( z = \tilde{z}_j/\| \tilde{z}_j \|_2 \). For these values of \( \xi \) and \( z \), we have

\[
(\sigma^2 n)^{-1/2} || \xi - \sqrt{n} E^T z ||_2 = (\sigma^2 n)^{-1/2} \| \xi_j \|_2 \| \tilde{z}_j \|_2 \| [n(\tilde{z}_j^T X e_j)^{-1}]^{-1} - E^T \tilde{z}_j \| \tilde{z}_j \|_2 \|_2
\]

\[
= (\sigma^2 n)^{-1/2} \| \tilde{\xi}_j - \xi_j \|_2 \| \tilde{z}_j \|_2 \| [n(\tilde{z}_j^T X e_j)^{-1}]^{-1} - 1.
\]

Hence the above is \( o_p(1) \) by combining (G.12) with (G.8) and (G.10). An application of Lemma F.3 for these values of \( z \) and \( \xi \) yields (F.4) which completes the proof.

**Proof of Theorem 4.3 under assumption (3.14)**. Let \( \xi_j \) be defined in (G.1) Since \( \tilde{a}_j, \tilde{z}_j \) defined in the proof of Theorem 3.3 satisfy the assumptions of Theorem 4.1, we have already established that \( (\sigma^2 n)^{-1/2} \| \xi_j - \sqrt{n} E^T \tilde{z}_j \| \tilde{z}_j \|_2 \|_2 = o_p(1) \), cf. (F.1) with \( a = \tilde{a}_j \) and \( z_0 = \tilde{z}_j \).
We now proceed to show that $(\sigma^2 n)^{-1/2}||\xi - \xi_j||_2 = o_P(1)$ for $\xi$ defined in (G.14). By the triangle inequality and since $\tilde{\xi}_j$ in (G.14) is proportional to $\tilde{\xi}_j$, we have

$$
(\sigma^2 n)^{-1/2}||\xi - \xi_j||_2
= \frac{1}{\sigma \sqrt{n}} \left|\tilde{\xi}_j - \xi_j\right|_2 = \frac{1}{\sigma \sqrt{n}} \left|\tilde{\xi}_j - \xi_j\right|_2 \left|\frac{\tau_j^{-1} \sqrt{n}}{\xi_j} \right|_2 \left|\left[ n(\tilde{z}_j^\top X e_j)^{-1}\right] - 1 \right|_2
\leq \frac{1}{\sigma \sqrt{n}} \left|\tilde{\xi}_j - \xi_j\right|_2 \left|\frac{\tau_j^{-1} \sqrt{n}}{\xi_j} \right|_2 \left|\left[ n(\tilde{z}_j^\top X e_j)^{-1}\right] - 1 \right|_2
$$

For the first term, by (G.9) we already have $(\sigma^2 n)^{-1/2}||\xi_j - \xi||_2 = o_P(1)$. Combined with $\left|\tilde{\xi}_j - \xi_j\right|_2/\left(\tau_j \sqrt{n}\right) \rightarrow 0$ and $n\tau_j^2(\tilde{z}_j^\top X e_j)^{-1} \rightarrow 1$ (see (G.8) and (G.10)), this proves that the first term above is $o_P(1)$. For the remaining terms, $(\sigma^2 n)^{-1/2}||\xi_j||_2 = O_P(\sqrt{T})$ by (F.1), and the question is whether

$$
O_P(\sqrt{T}) \left|\frac{\tau_j^{-1} \sqrt{n}}{\xi_j} \right|_2 \left|\left[ n(\tilde{z}_j^\top X e_j)^{-1}\right] - 1 \right|
$$

converges to 0 using (G.8) and (G.10). With $a_j = \left|\tilde{z}_j\right|_2/\left(\tau_j^2 n\right)$ and $b_j = \tilde{z}_j^\top X e_j/\left(\tau_j^2 n\right)$ for brevity,

$$
\left|\tau_j^{-1} \sqrt{n} \left[ n(\tilde{z}_j^\top X e_j)^{-1}\right] - 1 \right| = a_j^{-1/2} \left|b_j - a_j^{1/2}\right|
\leq a_j^{-1/2} \left(\left|b_j - 1\right| + \left|1 - a_j^{1/2}\right|\right)
= a_j^{-1/2} \left(\left|b_j - 1\right| + \left|1 - a_j\right|\left(1 + a_j\right)^{-1}\right).
$$

We have $\left|a_j - 1\right| + \left|b_j - 1\right| = \sqrt{\|\tau_j^{-1} \sqrt{n} \left[ n(\tilde{z}_j^\top X e_j)^{-1}\right] - 1 \|_0}$ $O_P(1)$ thanks to (G.8) and (G.10). Hence thanks to (3.14), quantity (G.15) is $o_P(1)$. Combining all the pieces, we have proved that

$$
(\sigma^2 n)^{-1/2}||\xi - \sqrt{n}E^\top \tilde{z}_j||_2 \leq (\sigma^2 n)^{-1/2}||\xi - \xi_j||_2 + o_P(1) \leq o_P(1).
$$

Applying Lemma F.3 to $\xi$ in (G.14) and $z = \tilde{z}_j||\tilde{z}_j||_2^{-1}$, conclusion (F.4) completes the proof. \(\square\)