Chiral Anomaly in Lattice QCD with Twisted Mass Wilson Fermion

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Abstract

The flavour singlet axial Ward identity with Osterwalder-Seiler twisted mass Wilson fermion action is studied on a finite lattice, with finite fermion mass and the Wilson parameter \( r \) up to 1. Approach to the infinite volume chiral limit and emergence of the anomaly is significantly better than that obtained with \( \mathcal{O}(a) \) and \( \mathcal{O}(a^2) \) improved fermion actions. We have shown explicitly that up to \( \mathcal{O}(g^2) \), parity violating terms cancel in the Ward identity even at finite volume and finite lattice spacing.

Key words: chiral anomaly, Wilson fermions, twisted mass, improved actions

PACS: 11.15.Ha, 11.30.Rd, 11.40.Ha, 12.38.Gc

1. Introduction

It is well known that the naive Wilson fermion [1] reproduces the chiral anomaly in lattice QCD in the continuum limit which was demonstrated analytically in weak coupling lattice perturbation theory by Karsten and Smit [2] who also showed that the result is independent of the Wilson parameter \( r \). Subsequently, Kerler [3] starting from the flavor singlet axial Ward identity also showed the emergence of the anomaly term in the continuum limit for small values of \( r \).

The Wilson fermions have a dimension five chiral symmetry breaking term which leads to complications in lattice simulations, such as additive quark mass renormalization, axial current renormalization and nontrivial mixing between operators. Even though the \( \mathcal{O}(a) \) lattice artifacts associated with Wilson fermions go to zero in the continuum limit, the technical difficulties are significant in actual simulations which are performed in a finite volume with lattice spacing and quark mass both being non-zero.

Currently, twisted mass Wilson QCD (tm WQCD) (for reviews, see, for example, [4, 5, 6]) has become a popular way of simulating lattice QCD. In addition to the absence of unphysical zero modes of the Wilson-Dirac operator, the ability to overcome
some of the lattice renormalization problems and the automatic $\mathcal{O}(a)$ improvement are its obvious advantages.

In the early days of lattice QCD, Osterwalder and Seiler \cite{7} introduced a chirally twisted Wilson term (OStm WQCD) to avoid doublers. Compared to OStm WQCD, tm WQCD has a nontrivial flavour dependence which leads to flavour and parity breaking in the lattice theory which is expected to disappear in the continuum limit. On the other hand, parity violating effects of OStm WQCD may survive the continuum limit non-perturbatively.

In this work we explicitly show that the parity violating terms are absent up to $\mathcal{O}(g^2)$ in the flavour singlet axial Ward Identity on the finite lattice. We study the dependence of the chiral anomaly with OStm WQCD on the Wilson parameter $r$ as a function of the lattice quark mass $a m$ and show the independence as one approaches the chiral limit. We also study the volume dependence of the result as one approaches the chiral limit.

The chiral anomaly on the lattice with finite volume and non-zero quark mass actually provides an excellent laboratory to compare the OStm WQCD with unimproved as well as $\mathcal{O}(a)$ and $\mathcal{O}(a^2)$ improved actions with Wilson fermions. Our detailed investigation provides a quantitative measure of the effectiveness of the twisted mass Wilson fermions in further reducing finite lattice spacing artifacts compared to $\mathcal{O}(a)$ and $\mathcal{O}(a^2)$ \cite{8,9,10} improved Wilson fermions.

Two comments, however, are in order: (i) The weak coupling perturbative analysis is not sensitive to some lattice artifacts that may be present in numerical simulations (for example, see \cite{11}), (ii) the positivity property of the quark determinant is not satisfied for the OStm WQCD action \cite{12} and hence numerical simulation with dynamical quarks with this action is not feasible. Numerical simulation is nevertheless possible with a mixed action (the OStm WQCD action for the valence quarks and the tm WQCD action for the sea quarks).

2. Osterwalder-Seiler Chirally Twisted Wilson Term

The standard Wilson fermion action

$$S_F[\psi, \overline{\psi}, U] = a^4 \sum_{x,y} \overline{\psi}_x M_{xy} \psi_y = a^4 \sum_{x,y} \overline{\psi}_x \left[ g_\mu D_\mu + W + m \right]_{xy} \psi_y$$

where

$$[D_\mu]_{xy} = \frac{1}{2a} \left[ U_{x,\mu} \delta_{x+\mu,y} - U^\dagger_{x-\mu,y} \delta_{x-\mu,y} \right],$$

$$W_{xy} = \frac{r}{2a} \sum_\mu \left[ 2 \delta_{x,y} - U_{x,\mu} \delta_{x+\mu,y} - U^\dagger_{x-\mu,y} \delta_{x-\mu,y} \right],$$

with the variations $\psi_x \rightarrow \psi_x' = [1 - i g_5 \alpha] \psi_x$, $\overline{\psi}_x \rightarrow \overline{\psi}_x' = \overline{\psi}_x [1 - i g_5 \alpha]$ lead to the flavor singlet axial Ward Identity

$$\langle \Delta^\mu_\mu J_5(x) \rangle = 2m \langle \overline{\psi} g_5 \psi \rangle + \langle \chi \rangle$$
where \( \langle \mathcal{O} \rangle \) denotes the functional average of \( \mathcal{O} \). Explanation of other terms are as follows:

The backward derivative,
\[
\Delta^b_{\mu} f(x) = \frac{1}{a} \left[ f(x) - f(x - \mu) \right],
\]

\[
J_{5\mu}(x) = \frac{1}{2} \left[ \bar{\psi}_x \gamma_\mu \gamma_5 U_{x+\mu} \psi_{x+\mu} + \bar{\psi}_{x+\mu} \gamma_\mu \gamma_5 U_x^\dagger \psi_x \right],
\]

and \( \langle \chi \rangle = -\text{Trace}[\gamma_5 (GW + WG)] \) .

The Green’s function
\[
G(x, y) = \langle x | \frac{1}{[\gamma_\mu D_\mu + W + m]} | y \rangle .
\]

Following the method of Kerler [3], to \( \mathcal{O}(g^2) \), one arrives at

\[
\langle \chi \rangle = 2 g^2 \epsilon_{\mu\nu\rho\lambda} F_{\mu\nu}(x) F_{\rho\lambda}(x) \frac{1}{(2\pi)^4} \sum_p \cos(p_\mu a)\cos(p_\nu a) \cos(p_\rho a) \times W_0(p) \left[ \cos(p_\lambda a)[m + W_0(p)] - 4r\sin^2(p_\lambda a) \right] \langle g_0(p) \rangle^3,
\]

\[
= -\frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\lambda} F_{\mu\nu}(x) F_{\rho\lambda}(x) I(\alpha m, r, L)
\]

Here
\[
W_0(p) = \frac{r}{a} \sum_n [(1 - \cos(\alpha p_\mu))],
\]
\[
\langle g_0(p) \rangle = \left( \frac{1}{a^2} \sum_\mu \sin^2(\alpha p_\mu) + \frac{r}{a} \sum_\mu [1 - \cos(\alpha p_\mu)]^2 \right)^{-1}.
\]

Explicitly, \( \sum_\mu = \left( \frac{1}{a} \right)^4 \sum n_1, n_2, n_3, n_4 \) where \( n_1, n_2, n_3, n_4 = 0, 1, 2, 3, \ldots \). In the infinite volume chiral limit, \( I \rightarrow 1 \). In all our plots it is the function \( I(\alpha m, r, L) \) which we have plotted.

Next we consider the chirally twisted Wilson term introduced by Osterwalder and Seiler [7] which amounts to the replacement \( W \rightarrow R = -i\gamma_5 W \) [3]. (Note that Seiler and Stamatescu [13] introduced the generalization \( -i\gamma_5 \rightarrow \exp(-i\theta\gamma_5) \) so that the chiral angle \( \theta \) can be directly related to the theta vacua of QCD. This generalization is not considered in this work.)

The OStm WQCD leads to the Flavor Singlet Axial Ward Identity

\[
\langle \Delta^b_{\mu} J_{\mu 5}(x) \rangle = -2m \text{Trace} \gamma_5 G^{\alpha\alpha} - \langle \chi^{\alpha\alpha}(x) \rangle
\]

\[
= -2m \text{Trace} \gamma_5 G^{\alpha\alpha} - \text{Trace} \gamma_5 (G^{\alpha\alpha} R +RG^{\alpha\alpha})
\]

with

\[
G^{\alpha\alpha} = \frac{1}{D + m - i\gamma_5 W} = \left( D - m - i\gamma_5 W \right) \frac{1}{(g^{\alpha\alpha})^{-1} - V}
\]
where

\[(g^{os})^{-1} = D^2 - m^2 - W^2 \text{ and } V = V_1 + V_2^{os}\]

with \(V_1 = i \frac{1}{2} \sigma_{\mu \nu} [D_\mu, D_\nu] \) and \(V_2^{os} = -i \gamma_5 [\mathcal{D}, W].\) (11)

Evaluation of \(\langle \chi^{os}(x) \rangle\) following a similar method as above now produces some parity violating terms which are not of the form \(\mathcal{F}\mathcal{F}\) and therefore do not contribute to the anomaly, however, would contribute to the axial Ward identity. Up to \(\mathcal{O}(g^2)\) these parity violating terms arising out of \(\langle \chi^{os}(x) \rangle\) are as follows,

\[\mathcal{O} (g^0) : \quad -2i m \text{ Trace } Wg^{os}\]

\[\mathcal{O} (g^1) : \quad \text{vanishes because of Dirac Trace}\]

\[\mathcal{O} (g^2) : \quad -2i m \text{ Trace } Wg^{os} V_1 g^{os} V_1 g^{os} - 2i m \text{ Trace } Wg^{os} V_2^{os} g^{os} V_2^{os} g^{os}.\] (12)

Contribution to the anomaly from \(\langle \chi^{os}(x) \rangle\) can be calculated by making the following changes from the standard Wilson case:

\[\text{Trace } \gamma_5 (W + m) \mathcal{D} V_1 \gamma_5 \mathcal{D} W \rightarrow \text{Trace } \gamma_5 Wg^{os} V_1 g^{os} V_1 g^{os} W \] (13)

and

\[\text{Trace } \gamma_5 \mathcal{D} V_1 \gamma_5 [\mathcal{D}, W] \gamma_5 W \rightarrow \text{Trace } \gamma_5 \mathcal{D} g^{os} V_1 g^{os} [\mathcal{D}, W] g^{os} W.\] (14)

Thus compared to the expression in the case of standard Wilson term, the expression for anomaly in the case of Osterwalder-Seiler Wilson term has the following features: (i) absence of fermion mass \(m\) in the numerator and (ii) absence of mixing between mass term and Wilson term \((W m)\) in the denominator. Explicitly, in place of Eq. (6) we find the contribution to \(\mathcal{F}\mathcal{F}\) from \(\langle \chi^{os}(x) \rangle\), in the case of the Osterwalder-Seiler twisted Wilson term

\[
\langle \chi^{os}_x \rangle = 2 g^2 \epsilon_{\mu \nu \rho \lambda} F_{\mu \nu}(x) F_{\rho \lambda}(x) \sum_p \cos(p_{\mu} a) \cos(p_{\nu} a) \cos(p_{\rho} a) \times W_0(p) \left[ \cos(p_{\lambda} a) W_0(p) - 4 r \sin^2(p_{\lambda} a) \right] (g^{os}(p))^3.
\] (15)

In the following we refer to the terms proportional to \(\cos(p_{3} a)\) and \(\sin^2(p_{3} a)\) as DD and DW terms respectively since they correspond to contributions from \(V_1 V_1\) and \(V_1 V_2^{os}\) terms in the trace.

In the Axial Ward Identity, the mass term can be written as

\[-2 m \text{ Trace } \gamma_5 G^{os} = -2 m \text{ Trace } \gamma_5 (\mathcal{D} - m - i \gamma_5 W)(g^{os} + g^{os} V g^{os} + g^{os} V g^{os} V g^{os} + \ldots).\] (16)

Up to \(\mathcal{O}(g^2)\), the parity violating contributions from the mass term are

\[\mathcal{O} (g^0) : \quad +2i m \text{ Trace } Wg^{os}\]

\[\mathcal{O} (g^1) : \quad -2i m \text{ Trace } \mathcal{D} g^{os} [\mathcal{D}, W] g^{os}\]

\[\mathcal{O} (g^2) : \quad +2i m \text{ Trace } Wg^{os} V_1 g^{os} V_1 g^{os} + 2i m \text{ Trace } Wg^{os} V_2^{os} g^{os} V_2^{os} g^{os}.\] (17)
Comparing Eqs. (12) and (17), we immediately see that $O(g^0)$ and $O(g^2)$ parity violating terms cancel between mass term and $\langle \chi^o(x) \rangle$.

The $O(g^1)$ term in Eq. (17)

$$\sum (\mathcal{H}_0)_{s_1s_2} (\mathcal{H}_0)_{s_2s_3} ([\mathcal{D}, W])_{s_2s_3} (\mathcal{H}_0)_{s_3s}$$

$$\Rightarrow \sin(ap_{\mu})F_{\mu\rho} \left[ (\cos(ap_{\mu}) + i \sin(ap_{\mu}))(\cos(ap_{\rho}) + i \sin(ap_{\rho})) 
- (\cos(ap_{\mu}) - i \sin(ap_{\mu}))(\cos(ap_{\rho}) - i \sin(ap_{\rho})) 
- (\cos(ap_{\mu}) + i \sin(ap_{\mu}))(\cos(ap_{\rho}) - i \sin(ap_{\rho})) 
+ (\cos(ap_{\mu}) - i \sin(ap_{\mu}))(\cos(ap_{\rho}) + i \sin(ap_{\rho})) \right]$$

$$\Rightarrow 0, \text{ on summation.} \quad (18)$$

Thus we explicitly verify, in the case of Osterwalder-Seiler twisted Wilson term, the cancellation of parity violating terms in the flavour singlet Axial Ward Identity up to $O(g^2)$. 

Figure 1: The function $I(\alpha m, r, L)$ for $r = 1$ for unimproved, $O(a)$ improved, $O(a^2)$ improved, and OSim Wilson fermions for the range of $\alpha m$ between 0.01 and 1.0 at $L = 40$.
3. Comparison with unimproved, $O(a)$ and $O(a^2)$ improved Wilson fermions

Consider the following next to nearest neighbour interaction term added to modify the standard Wilson term [8]:

\[
\Delta S^I = a^4 \sum_{x, \mu} \left\{ \frac{r}{4a} \left[ \overline{\psi}(x) U_{x, \mu} U_{x+\mu, \mu} \psi(x+2\mu) \right] + \overline{\psi}(x+2\mu) U_{x+\mu, \mu} U_{x, \mu} \psi(x) - 2 \overline{\psi}(x) \psi(x) \right\} = a^4 \sum_x \overline{\psi}(x) W^I \psi(x).
\] (19)

The coefficient $r_{8a}$ is chosen so as to cancel $O(a)$ contributions to the tree level fermion propagator and the fermion-gluon vertex. Now the total fermion action is

\[
S = S_F + \Delta S^I.
\]

From the corresponding flavour singlet axial Ward identity, we find the total contribution to the axial anomaly

\[
\langle \chi^I \rangle = 2 \epsilon_{\mu\nu\rho\lambda} g^2 F_{\mu\nu}(x) F_{\rho\lambda}(x) \sum_p \cos(a p_\mu) \cos(a p_\rho) \cos(a p_\lambda) \left[ \mathcal{G}_0^I(p) \right]^3 \left[ \cos(a p_\nu) [m + W_0(p) + W_0^I(p)] - 4r \sin(a p_\nu) (\sin(a p_\nu) - \frac{1}{2} \sin(2a p_\nu)) \right] \left[ W_0(p) + W_0^I(p) \right].
\] (20)

Here

\[
W_0(p) + W_0^I(p) = \left[ \sum_\mu \left\{ \frac{r}{a} (1 - \cos(a p_\mu)) + \frac{r}{4a} (-1 + \cos(2a p_\mu)) \right\} \right],
\]

\[
\mathcal{G}_0^I(p) = \left( \frac{1}{a^2} \sum_\mu \sin^2(a p_\mu) + (m + \sum_\mu \frac{r}{a} [1 - \cos(a p_\mu)] + \frac{r}{4a} [1 - \cos(2a p_\mu)])^2 \right)^{-1}.
\] (21)

Next consider the modification of the kinetic term with the next to nearest neighbour interaction [8][9][10]:

\[
\Delta S^{II} = a^4 \sum_{x, \mu} \left\{ - \frac{1}{16a} \left[ \overline{\psi}(x) \gamma_\mu U_{x, \mu} U_{x+\mu, \mu} \psi(x+2\mu) \right] - \overline{\psi}(x+2\mu) \gamma_\mu U_{x+\mu, \mu} U_{x, \mu} \psi(x) \right\} = a^4 \sum_x \overline{\psi}(x) \gamma_\mu D^I_\mu \psi(x).
\] (22)

so that the total fermion action $S = S_F + \Delta S^I + \Delta S^{II}$ is $O(a^2)$ improved and leads to the
total contribution to the chiral anomaly

\[ \langle \chi^{H} \rangle = 2\varepsilon_{\mu\nu\rho\lambda} g^{2} F_{\mu\nu}(x) F_{\rho\lambda}(x) \sum_{p} \Pi_{\alpha=1}^{H}(\cos(ap_{\lambda}) - \frac{1}{4}\cos(2ap_{\lambda})) \left[ \Phi_{0}^{H}(p) \right]^{3} \]

\[ \times \left[ \frac{3}{a} - (W_{0}(p) + W_{L}(p)) \right] \times \left[ \frac{3}{4}m + \frac{3}{a} - \frac{2}{a} \sum_{\mu} \frac{1}{\Phi} \left[ \sin(ap_{\mu}) - \frac{1}{4}\sin(2ap_{\mu}) \right] \right], \]

(23)

with \((D + D^{I})_{0\xi}(p) = \frac{i}{a}[\sin(p_{\lambda}a) - \frac{1}{8}\sin(2p_{\lambda}a)]\) and

\[ \Phi_{0}^{H}(p) = -\left[ \frac{1}{a} \sum_{\mu} \{(D + D^{I})_{0\mu}(p)\}^{2} + \left\{ \frac{3}{4}m + W_{0}(p) + W_{L}(p) \right\}^{2} \right]^{-1}. \]

Fig. \[\text{1}\] compares the function \(I(am, r, L)\) for \(r = 1.0\) for unimproved, \(O(a)\) improved, \(O(a^2)\) improved and OStm Wilson fermions for the range of \(am\) between 0.01 and 1.0 at \(L = 40\). Note that for clarity, \(am\) is plotted in the logarithmic scale. The figure shows that in the range of \(am\) studied, \(O(a^2)\) corrections are relatively small compared to the \(O(a)\) corrections and the approach to the chiral limit is the fastest and the flattest for the OStm Wilson fermions.

4. Numerical Results for Osterwalder-Seiler twisted mass Wilson Fermion

In this section we present numerical results for the function \(I(am, r, L)\) for Osterwalder-Seiler twisted mass Wilson fermions. Since our main concern is the approach to the infinite volume chiral limit in the continuum, we study in detail the quark mass \((am)\), the Wilson parameter \((r)\) and the finite volume \((L)\) dependence. Since a positive semi-definite transfer matrix is guaranteed only for \(r \leq 1\), we restrict our study to the range \(0 \leq r \leq 1\).

4.1. Quark mass dependence

In Fig. \[\text{2}\] we show the function \(I(am, r, L)\) for OStm Wilson fermion for the range of \(am\) between 0.1 and 1.0 at \(L = 40\). We find that for most values of \(am\), DW term overshoots the answer, which is compensated by the DD term. Cutoff effects are present in the mass range \(am > 0.1\), but the result is very close to the continuum chiral answer for \(am < 0.1\).

4.2. Wilson parameter \(r\) dependence

For unimproved, \(O(a)\) and \(O(a^2)\) improved Wilson fermions, it has been demonstrated that in the chiral limit, the function \(I(am, r, L)\) is independent of the Wilson parameter \(r\). For the OStm Wilson fermions, it is also interesting to check the \(r\) dependence for non-zero \(am\). In Fig. \[\text{3}\] we show the \(r\) dependence of terms with integrands proportional to \(\cos(p_{\lambda}a)\) (DD) and \(\sin^2(p_{\lambda}a)\) (DW) in Eq. \[\text{15}\] and the sum of the two
contribution to the function $I(am, r, L)$ (DD and DW) for $am = .01$ (right) and $am = .1$ (left) for OStm Wilson fermions at $L = 80$. We note that the first term (DD) is dominant at small $r$ and the second term (DW) is dominant at large $r$. Even though the two individual contributions have very strong $r$ dependence, the sum is seen to be independent of $r$ to a good numerical accuracy for $am = .01$.

4.3. Finite volume dependence

In Fig. 4 we compare the finite volume dependence of the function $I(am, r, L)$ for OStm Wilson fermions at $r = 1$ and range of $am$ between 0.01 and 1.0 for $L=40$. In this range of $am$, convergence is satisfactory as volume increases but the convergence rate becomes slower if the value of $am$ is smaller.

In conclusion, we have shown that parity violating terms do not arise in the flavour singlet axial vector Ward identity up to $O(g^2)$ for the OStm Wilson fermions and the approach to the chiral limit in comparison with $O(a)$, and $O(a^2)$ improved Wilson fermions is very satisfactory.

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Figure 3: The function $I(\text{am}, r, L)$ for OSStm Wilson fermions for the range of $r$ between 0.1 and 1.0 for $\text{am}$ 0.01 (right) and 0.1 (left) at $L = 80$. The contributions from DD and DW terms are also shown separately.

Figure 4: The function $I(\text{am}, r, L)$ for OSStm Wilson fermions for the ranges of $\text{am}$ between 0.01 and 0.1 for different $L$. 


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