Preheating and the Einstein Field Equations

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We inaugurate a framework for studying preheating and parametric resonance after inflation without resorting to any approximations, either in gravitational perturbation theory or in the classical evolution of the field(s). We do this by numerically solving the Einstein field equations in the post-inflationary universe. In this paper we show how to compare our results to those of gauge invariant perturbation theory. We then verify Finelli and Brandenberger’s analysis of super-horizon modes in $m^2\phi^2$ inflation, showing that they are not amplified by resonant effects. Lastly, we make a preliminary survey of the nonlinear couplings between modes, which will be important in models where the primordial metric perturbations undergo parametric amplification.

I. Introduction

Postulating a period of nearly exponential growth in the primordial universe, inflationary cosmology solved many problems which plagued previous models of the big bang. Immediately after inflation the universe is dominated by the inflaton, the scalar field whose evolution controls the dynamics of the inflationary era. The inflaton energy density is then converted into thermalized, “normal” matter, a process called reheating. The first predictions were that the universe would be reheated to a temperature several orders of magnitude below the inflationary scale. Subsequently, it has been realized that non-equilibrium, non-perturbative processes can drive explosive particle production at the end of inflation. This initial era, dubbed preheating, is governed by Mathieu-like equations, and the rapid particle creation associated with the instability bands of these equations is often termed parametric resonance.

Developing a full understanding of preheating has been a long and arduous task. There were two main complications. Firstly the difference between parametric resonance in Minkowski space and in an expanding universe is profound. In the latter case, the instability bands may be time-dependent and so a particular mode may pass in and out of resonance. Secondly, the back-reaction of the created particles onto the other fields must become significant, and indeed the parametric resonance will eventually terminate. With a few exceptions though, (see e.g. [24, 25]), spacetime was described by the usual Friedmann Robertson Walker metric. Although the scale factor could be determined self-consistently (see e.g. [13]), the possibility of resonant growth of metric perturbations was not considered. However, in a recent paper Bassett et al. [22] highlight the importance of couplings between perturbations in the fields and in the metric. As they point out, metric perturbations are sources for the field perturbations, and vice versa, so it cannot be assumed that the role of metric perturbations in preheating is only slight (see also [24,25,23]). During preheating, in this framework, parametric resonance will necessarily amplify metric perturbations and so nonlinear gravitational effects will become significant.

In this paper, we abandon the perturbative approach entirely and study the post-inflationary era by numerically solving the inhomogeneous, nonlinear Einstein field equations, and derive the connection between our results and the predictions of perturbation theory. We then confirm the conclusions of Finelli and Brandenberger who use gauge invariant perturbation theory to show that super-horizon modes are not parametrically amplified after $m^2\phi^2$ inflation. Lastly, we use the $m^2\phi^2$ theory as a toy model to explore the consequences of having significantly amplified modes in the post-inflationary universe. In particular, we see how power can be transferred from modes with large amplitudes to all other modes. The mode-mode coupling thus appears to broaden the impact of resonance in the early universe. This is the first time in the study of preheating that the mode-mode coupling of the field and metric perturbations has been calculated in a fully relativistic manner. In $m^2\phi^2$ inflation, metric perturbations are not amplified so we inject a large perturbation via our initial conditions. In future work we will give a nonlinear treatment of inflationary models such as $\lambda \phi^4$ and $g^2 \phi^2 \chi^2$, where the field perturbations are amplified by resonant effects, and study the role of mode-mode coupling in these systems.

II. Metric and Field Equations

We consider scenarios whose initial state corresponds to a spatially flat Friedman Robertson Walker universe with a perturbation consisting of a single Fourier mode. The wave-vector, $\mathbf{k}$, of this mode can be assumed to be parallel to the $z$ direction, so we adopt a metric where the $x$ and $y$ directions are homogeneous and the inhomogeneous contributions are functions of $z$ alone. The simplest choice of metric which expresses this constraint is

$$ds^2 = dt^2 - A^2(t,z)dz^2 - B^2(t,z)(dx^2 + dy^2).$$

A single scalar field has the Lagrangian

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{\phi^2}{2A^2} - V(\phi)$$

and equation of motion

$$\ddot{\phi} + \frac{1}{2} \frac{\dot{A}}{A} \dot{\phi} - \frac{\dot{V}}{V} \phi = 0.$$
Overdots indicate derivatives with respect to \( t \), and primes with respect to \( z \). The non-zero components of the Einstein field equations, \( G_{\mu\nu} = -\kappa^2 T_{\mu\nu} \), are therefore

\[
\ddot{A} = -\frac{\ddot{B}}{B} + \frac{3B''}{2B} + \frac{B''B'}{AB} - \frac{B'^2}{2AB^2} - \frac{A'B'}{A^2B} - \frac{\kappa^2 A}{2} \left( \frac{\dot{\phi}^2}{2} - \frac{3\phi'^2}{2A^2} - V \right),
\]

Equations (4) and (5) are constraints which must be satisfied by the initial conditions, and are then used to check the accuracy of our numerical integration.

We choose \( A, B \) and \( \phi \) to be independent of \( z \) at \( t = 0 \), so the initial inhomogeneity is expressed by \( \dot{A}, \dot{B} \) and \( \dot{\phi} \). This makes it easier to ensure that the initial data satisfies the constraints, but does not involve an unacceptable loss of generality. With \( A'(0, z) = B'(0, z) = \phi'(0, z) = 0 \), the first constraint, equation (4), is trivial. Rescaling the spatial co-ordinates fixes \( A(0, z) = B(0, z) = 1 \), while we denote \( \phi(0, z) = \phi_0 \). Equation (5) requires

\[
A(0, z) = \frac{\kappa^2}{2C} \left( \frac{\dot{\phi}^2}{2} + V(\phi_0) \right) - C, \quad B(0, z) = \sqrt{\frac{\kappa^2}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi_0) \right) - C},
\]

Equations (11) and (12) are solutions of the background universe, defined by \( b(t, z) = 0, \dot{b}(t, z) = 0 \). Comparing with equation (4), we deduce that

\[
\varphi = 0, \quad B = a(t)(1 - 2\psi + 2E''), \quad A^2 = a^2(t)(1 - 2\psi + 2E'),
\]

where \( Z \) is the length of our “box”. This definition rescales \( k \) in units of \( Z \), and we use this form throughout the rest of this paper. The magnitude of the perturbation is fixed by \( \epsilon \). Most inflationary models predict that \( \epsilon \ll 1 \), but we can consider an arbitrary \( \epsilon \), since we are working with the full Einstein equations.

III. Testing Perturbation Theory Our first goal is to test the perturbative analysis of the interaction between the field and background perturbations in an inflationary model with the potential, \( V(\phi) = m^2 \phi^2/2 \). In particular, Finelli and Brandenberger [23] showed that first order gravitational perturbation theory does not predict the parametric amplification of scalar perturbations with wavelengths considerably larger than the Hubble radius, as was suggested by Bassett et al. [22].

To independently check Finelli and Brandenberger’s prediction and to confirm that perturbation theory gives an accurate description of the universe at the end of inflation, we use our nonlinear code to analyze the same system. Consequently, we must relate our metric, which corresponds to a particular gauge choice, to gauge invariant cosmological perturbations.

Working in physical time, with our symmetry constraint and choice of flat spatial sections in the unperturbed universe, the general scalar perturbation is

\[
ds^2 = (1 + 2\varphi(t, z))dt^2 - 2a(t)b(t, z),dx^i dx^i \quad - a^2(t) [(1 - 2\psi(t, z))\delta_{ij} + 2E(t, z),ij] \quad dx^i dx^j dx^i dx^j.
\]

where \( \varphi \) and \( b \) denote ordinary derivatives. The background solution, \( a(t) \), is defined by the FRW universe with our chosen initial conditions and \( \epsilon = 0 \). Comparing with equation (4), we deduce that

\[
\varphi = 0, \quad b = 0
\]

\[
A^2 = a^2(t)(1 - 2\psi + 2E''), \quad B^2 = a^2(t)(1 - 2\psi).
\]

We are now in a position to calculate the gauge invariant metric perturbation \( \Phi \).
\[ \Phi = \varphi + \frac{\partial}{\partial t} \left[ a(b - a\dot{E}) \right] = -\frac{\partial}{\partial t} (a^2 \dot{E}), \]  
(15)

where we have set integration constants in the final version to ensure that \( \Phi \) vanishes when there is no perturbation. We can express \( \Phi'' = \dot{BB} + \dot{B} - \dot{AA} - \dot{A}^2 + \) \( \dot{H}(A^2 - B^2) + 2H(\dot{AA} - \dot{BB}), \)  
(16)

where \( H = \dot{a}/a \). Perturbation theory predicts the Fourier modes, \( \Phi_k \), of \( \Phi \). Recalling the Fourier transform, \( \mathcal{F} \), of a derivative, we obtain \( \Phi_k = -k^{-2} \mathcal{F}(\Phi'')_k \). However, in linear perturbation theory \( \Phi_k \) is obtained from Mukhanov’s variable, \( Q_k \), which satisfies \( \dot{Q}_k = \dot{\phi}_k/k \) \( \dot{\phi} \) \( \dot{\phi}_k/k \) \( Q_k = k \dot{\phi}_k/k = \frac{k^2}{2} \frac{\dot{\phi}_k}{H} \) \( \dot{Q}_k = 3HQ_k + \) \( \left[ \frac{k^2}{a^2} + \frac{d^2V}{d\phi^2} + 2 \frac{d}{dt} \left( \frac{\dot{H}}{H} + 3H \right) \right] Q_k = 0, \)  
(18)

\( k^2 \frac{d}{a^2} \Phi_k = \left( \frac{H}{\dot{\phi}_k}Q_k \right) \) \( \dot{\phi}(0, z) = \dot{\phi}_0 + \epsilon \sin \left( \frac{2\pi k z}{Z} \right) \)  
(19)

We ran our code for an initial perturbation with a wavelength \( Z \) equal to the size of our simulation region, which is \( 10^2 \) larger than the post-inflationary Hubble length, and \( \epsilon = 0.001 \). This value of \( \epsilon \) is somewhat arbitrary, but corresponds to a density contrast of roughly the same magnitude as those observed at COBE scales. We extracted \( \Phi \), as described above, and solved numerically for \( Q \) with initial conditions chosen to match those of the nonlinear simulation. We find that the nonlinear and the perturbative results agree to within approximately 1 part in \( 10^3 \). The evolution of \( \Phi_k \) corresponding to the single excited mode is shown in Fig. 1. This verifies both the perturbative treatment of the post-inflationary era in the absence of resonance and the specific conclusions of Finelli and Brandenberger [24], i.e. that for \( m^2\phi^2 \) inflation, \( \Phi_k \) is constant for super-horizon modes and does not undergo parametric amplification.

**IV. Beyond Linear Gravity** We now consider the role of nonlinear gravitational effects during preheating. A full relativistic treatment (or higher order perturbation theory) couples Fourier modes which are independent at first order. Post-inflationary perturbations are small, but in \( \phi^4 \) inflation (and in more complicated models), parametric resonance can increase \( \Phi \) dramatically. If this growth lasts long enough, nonlinear gravitational effects are unavoidable.

We leave the full numerical analysis of models with more complicated potentials for future work, and use our \( m^2\phi^2 \) system as a toy model to investigate the consequences of one or more modes acquiring a large amplitude. Since we have just shown that super-horizon modes are not amplified with a \( \phi^2 \) potential, we must inject this through our initial conditions.

We examine a perturbation with \( k = 5 \), and \( \epsilon = 0.1 \), which is large in contrast to the primordial COBE scale perturbations, but is not inconsistent with a mode amplified by parametric resonance. Two \( \Phi_k \) are plotted in Fig. 2. Modes for which \( k \) is an integer multiple of the \( k \) of the initial perturbation are excited. The \( k = 5 \) mode is outside the initial Hubble radius, but nonlinear effects cause it to grow, and the perturbative prediction is not applicable. We also observe that in initially over-dense (with respect to a flat, unperturbed universe) regions, \( A \) begins to decrease, signifying the onset of gravitational collapse.

Finally, we consider a more complex case where two modes, whose \( k \) values are not multiples of one another, are initially excited. Specifically, we replace equation \( (10) \) by
Fig. 3. Several modes of Φ are plotted with ɛ = 0.05 and k = 8, and the perturbation given by equation (20). Here, Φ₁ is initially excited, but significant amounts of power are transferred to other modes.

\[ +0.01 \sin \left( \frac{2\pi(k-1)z}{Z} + \frac{\pi}{4} \right) \] (20)

The two modes we initially excite become mixed in the gauge invariant perturbation, so Φ₁ always has a significant amplitude. This can be seen in Fig. 3, which corresponds to the choice ɛ = 0.05 and k = 8. However, unlike the previous case where only modes which were multiples of the initially excited mode were enhanced, we now see a transfer of power between the initially excited “band” and a wide variety of other modes. Consequently, if some modes acquire a significant amplitude through parametric resonance, nonlinear effects can then transfer power to modes which fall outside the resonance bands.

V. Discussion Nonlinear effects may be important in a number of ways not considered here. For instance, when inflation is driven by two interacting scalar fields, the homogeneous post-inflationary universe may exhibit chaos, and the interaction between this chaos and nonlinear inhomogeneous effects is intriguing. Likewise, parametrically enhanced perturbations will grow further due to nonlinear gravitational effects and could eventually undergo gravitational collapse, leading to the formation of primordial black holes in models where their existence is not currently predicted. The scattering of scalar perturbations is also a nonlinear effect and will lead to gravitational wave production.

By abandoning perturbation theory and solving the Einstein field equations directly, we have laid the foundation for a fully nonlinear analysis of preheating in an expanding universe, which will include all possible couplings between perturbations in spacetime and the classical field(s). We verified Finelli and Brandenberger’s analysis of \( m^2\phi^2 \) inflation, showing that the super-horizon modes of the gauge invariant perturbation are constant, and unamplified by any resonant processes. We also made a preliminary exploration of interactions between modes when the perturbation is large, demonstrating the ability of nonlinear effects to couple modes, which allows a transfer of power from modes which are parametrically enhanced to those which are not in resonance at the perturbative level. Consequently, we conjecture that the nonlinear gravitational effects will broaden any resonance bands seen in the conventional perturbative analysis; we are currently investigating these questions in detail.

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