Violation of the Second Fluctuation-dissipation Relation and Entropy Production in Nonequilibrium Medium

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Abstract
We investigate a class of nonequilibrium media described by Langevin dynamics that satisfies the local detailed balance. For the effective dynamics of a probe immersed in the medium, we derive an inequality that bounds the violation of the second fluctuation-dissipation relation (FDR). We also discuss the validity of the effective dynamics. In particular, we show that the effective dynamics obtained from nonequilibrium linear response theory is consistent with that obtained from a singular perturbation method. As an example of these results, we propose a simple model for a nonequilibrium medium in which the particles are subjected to potentials that switch stochastically. For this model, we show that the second FDR is recovered in the fast switching limit, although the particles are out of equilibrium.

Keywords
Nonequilibrium medium · Fluctuation-dissipation relation · Stochastic thermodynamics · Singular perturbation method

1 Introduction
The properties of a system can be investigated by observing the response of the system against external stimuli. The first fluctuation-dissipation relation (FDR) states that, for equilibrium systems, the same information as such a response is carried by an equilibrium correlation function [1, 2]. By contrast, in nonequilibrium systems, the first FDR is violated. Even for this case, there are phenomenological relations that connect the violation of the first FDR to energy dissipation [3–7]. In particular, the Harada-Sasa equality [3–6] enables us to measure energy dissipation from experimentally accessible quantities and has been applied to various systems from molecular motors [8, 9] to turbulence [10, 11].

These phenomenological relations that extend the first FDR to nonequilibrium systems are based on the second FDR, which expresses the balance between the friction and noise intensity in the sense that they are compatible with equilibrium statistics. The second FDR requires the assumption that the nonequilibrium condition imposed on the system does not directly
affect the environments, i.e., the environments are quickly equilibrated [5, 12]. Indeed, it can be derived by imposing the local detailed balance (LDB) condition [5, 12–16]. Therefore, the second FDR can be violated if the environment itself is out of equilibrium. Such a nonequilibrium environment can be found in various systems, particularly biological systems [17–19]. Nonequilibrium fluctuations generated by these environments can induce a variety of rich phenomena that cannot be found in equilibrium systems. For example, the speeds of cargos transported by kinesin in cells are much faster than in vitro although the cell interior is crowded and viscous [20]. In this regard, Ariga et al. have recently shown that kinesin is accelerated by nonthermal fluctuations [21]. It is thus desirable to characterize and classify nonequilibrium environments to deepen our understanding of the phenomena induced by nonequilibrium fluctuations.

As a first step toward this end, we investigate a simple class of nonequilibrium media and seek universal relations on the violation of the second FDR. Specifically, we consider a system consisting of three levels of description: probe, driven particles (nonequilibrium medium), and equilibrium thermal bath. We focus on a class of nonequilibrium media described by Langevin dynamics that satisfies the LDB. Such a formulation has been used in several works to investigate the effective dynamics of a probe immersed in nonequilibrium media [12, 22–26]. For this setup, we derive the effective dynamics of the probe by using nonequilibrium linear response theory [12, 15, 23, 25–30] and investigate the violation of the second FDR. In this paper, we derive an inequality that bounds the violation of the second FDR. This inequality states that the violation of the second FDR is bounded by the fluctuation of the “response” of the total stochastic entropy production in the nonequilibrium medium against a perturbation of the probe position. We also discuss the validity of the effective dynamics. In particular, we show that the effective dynamics obtained from a singular perturbation method corresponds to that obtained from nonequilibrium linear response theory in the Markovian limit. As a simple example of these results, we introduce a potential switching medium, the particles of which are described by the so-called potential switching model, i.e., overdamped Langevin dynamics with a stochastically switching potential [31–33]. For this simple linear system, all relevant quantities can be calculated explicitly. We show that the standard second FDR is recovered in the fast switching limit, although the driven particles are out of equilibrium because of the so-called hidden entropy [33–35]. Correspondingly, we show that the upper bound of the inequality for the violation of the second FDR goes to zero in this limit.

This paper is organized as follows. In Sect. 2, we explain the setup. In Sect. 3, we present the effective dynamics of the probe, in which the second FDR is violated in general. Then, we explain the inequality that bounds the violation of the second FDR, which is our first main result. In Sect. 4, we review the derivation of the effective dynamics based on nonequilibrium linear response theory. Then, we derive the inequality for the violation of the second FDR. The validity of the effective dynamics is discussed in Sect. 5. We show that the effective dynamics is consistent with the result obtained by using a singular perturbation method. In Sect. 6, we introduce the potential switching medium as a simple example. Concluding remarks are provided in Sect. 7.

2 Setup

In this section, we explain the setup, which consists of three levels of description: probe, driven particles (nonequilibrium medium), and equilibrium thermal bath. While we consider one-dimensional systems in the following, the results can be generalizable to higher dimensions.
Let $X_t$ be the position of a probe with mass $M$ at time $t$. The probe is in contact with both an equilibrium thermal bath at temperature $T$ and a nonequilibrium medium that consists of $N$ particles, the positions of which are denoted by $x_j^j$ ($j = 1, 2, \ldots, N$). We denote the collection of $x_j^j$ as $x := \{x^1, x^2, \ldots, x^N\}$. The time evolution of $X_t$ is given by the following underdamped Langevin equation:

\begin{equation}
M \ddot{X}_t = \Phi(x_t, X_t) - \Gamma \dot{X}_t + \sqrt{2B} \xi_t.
\end{equation}

Here, $\Phi(x_t, X_t)$ represents the interaction force between the probe and the particles described by the coupling potential $V(x, X)$:

\begin{equation}
\Phi(x_t, X_t) := -\lambda \frac{\partial}{\partial X_t} V(x_t, X_t),
\end{equation}

where $\lambda$ denotes the dimensionless coupling constant, which can be scaled with $N$. The second and third terms on the right-hand side of (1) represent the coupling with the equilibrium thermal bath, where $\Gamma$ denotes the friction coefficient and $\xi_t$ is the zero-mean white Gaussian noise that satisfies

\begin{equation}
\langle \xi_t \xi_s \rangle = \delta(t-s).
\end{equation}

The noise intensity $B$ is related to the friction coefficient $\gamma$ and temperature $T$ through the second FDR: $B = \Gamma k_B T$.

The dynamics of the particles is described by the following overdamped Langevin equation:

\begin{equation}
\gamma \dot{x}_j^j = F_j^j(x_t) - \lambda \frac{\partial}{\partial x_j^j} V(x_t, X_t) + \sqrt{2B_m} \xi_j^j.
\end{equation}

Here, $F_j^j(x)$ denotes the force acting on the $j$-th particle, generally consisting of nonconservative forces and interactions between particles. The second term on the right-hand side of (4) represents the interaction with the probe. The last term represents the thermal noise: $\xi_j^j$ is the zero-mean white Gaussian noise that satisfies

\begin{equation}
\langle \xi_j^j \xi_k^k \rangle = \delta_{jk} \delta(t-s),
\end{equation}

and the noise intensity $B_m$ is related to $\gamma$ and $T$ through the second FDR: $B_m = \gamma k_B T$. We remark that the following results are valid even in the case where $F_j^j(x)$ includes additional random forces as long as the LDB is satisfied for the additional degrees of freedom (see, e.g., Sect. 6).

We are interested in the regime where the motion of the probe is much slower than that of the particles so that the probe dynamics can be described by some effective model. This assumption will be described more explicitly in the singular perturbation method described in Sect. 5.

\section{Effective Dynamics and the Bound on the Violation of the Second FDR}

Under the setup described in Sect. 2, we can derive the effective dynamics of the probe by eliminating the degrees of freedom of the nonequilibrium medium. The resulting effective dynamics does not generally satisfy the second FDR. In this section, we summarize the effective dynamics of the probe and present our first main result on the violation of the second FDR. The derivation of these results is provided in the next section.
### 3.1 Effective Dynamics of the Probe

The effective dynamics of the probe is described by the following generalized Langevin-type equation:

\[
M \ddot{X}_t = G(X_t) - \Gamma \dot{X}_t - \int_{-\infty}^{t} ds \gamma(t - s) \dot{X}_s + \sqrt{2 \Gamma k_B T} \xi_t + \eta_t.
\]  

(6)

This result states that the interaction force \( \Phi \) is decomposed into three parts: the streaming term \( G(X_t) \), the friction force with the memory kernel \( \gamma(t - s) \), and the zero-mean colored noise \( \eta_t \). The streaming term \( G(X_t) \) is given by

\[
G(X_t) := \langle \Phi(x_t, X_t) \rangle_{X_t},
\]

(7)

where \( \langle \cdot \rangle_{X_t} \) denotes the average with respect to the stationary distribution \( P_{ss}^{X_t}(x) \) for the particle dynamics (4) with \( X_t \) held fixed. The friction kernel is given by

\[
\gamma(t - s) := \frac{1}{2k_B T} \int_{-\infty}^{s} du \left[ \frac{d}{du} \langle \Phi(x_u^-, X_t); \Phi(x_t, X_t) \rangle_{X_t} \right. \\
\left. - \langle \mathcal{L}_u^\dagger \Phi(x_u^-, X_t); \Phi(x_t, X_t) \rangle_{X_t} \right], \quad \text{for} \quad t \geq s.
\]

(8)

Here, \( \langle f; g \rangle_{X_t} := \langle f \rangle_{X_t} - \langle f \rangle_{X_t} \langle g \rangle_{X_t} \) and \( \mathcal{L}_u^\dagger \) denotes the backward operator of the dynamics (4) with \( X_t \) held fixed:

\[
\mathcal{L}_u^\dagger := \sum_j \left[ \frac{1}{\gamma} \left( F^j(x_u) - \lambda \frac{\partial}{\partial x_u^j} V(x_u, X_t) \right) \frac{\partial}{\partial x_u^j} + k_B T \frac{\partial^2}{\partial (x_u^j)^2} \right].
\]

(9)

The noise term is expressed as \( \eta_t = \Phi(x_t, X_t) - \langle \Phi(x_t, X_t) \rangle_{X_t} \), and its correlation is related to the friction kernel \( \gamma(t - s) \) in the following form:

\[
\langle \eta_t \eta_s \rangle_{X_t} = k_B T \left[ \gamma(t - s) + \gamma_{ex}(t - s) \right],
\]

(10)

where \( \gamma_{ex}(t - s) \) denotes the excess friction kernel defined as

\[
\gamma_{ex}(t - s) := \frac{1}{2k_B T} \int_{-\infty}^{s} du \left[ \frac{d}{du} \langle \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t} \right. \\
\left. + \langle \mathcal{L}_u^\dagger \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t} \right].
\]

(11)

Because of this excess friction kernel, the second FDR is generally violated. In the equilibrium case, however, one can immediately confirm that the standard second FDR holds: from the time-reversal symmetry,

\[
\langle \mathcal{L}_u^\dagger \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t} = \langle \mathcal{L}_u^\dagger \Phi(x_t, X_t); \Phi(x_u, X_t) \rangle_{X_t} = 0,
\]

(12)

and thus \( \gamma_{ex}(t - s) = 0 \). Note that the noise \( \eta_t \) need not be Gaussian nor white. The Gaussian noise may be obtained by taking the limit \( N \to \infty \) combined with the weak coupling limit \( \lambda \to 0 \) [30, 36, 37].
In the Markov approximation, the friction kernel is approximated as
\[ \gamma(t-s) = 2\gamma_{\text{eff}} \delta(t-s), \]  
where \( \gamma_{\text{eff}} \) denotes the effective friction coefficient:
\[ \gamma_{\text{eff}} := \int_0^\infty dt \gamma(t). \]
Similarly, the excess friction kernel becomes
\[ \gamma_{\text{ex}}(t-s) = 2\gamma_{\text{ex}} \delta(t-s) \]
with
\[ \gamma_{\text{ex}} := \int_0^\infty dt \gamma_{\text{ex}}(t). \]
Thus, in the Markov approximation, the effective generalized Langevin-type equation (6) becomes
\[ M \ddot{X}_t = G(X_t) - (\Gamma + \gamma_{\text{eff}}) \dot{X}_t + \sqrt{2(\Gamma + \gamma_{\text{eff}} + \gamma_{\text{ex}}) k_B T} \Xi_t. \]

### 3.2 Bound on the Violation of the Second FDR

Let \( \Delta s_{\text{tot}}^{X_t} \) be the total stochastic entropy production up to time \( s \) of the nonequilibrium medium in the nonequilibrium steady state (NESS) with \( X_t \) held fixed. The first main result of this paper is the following inequality, which connects the violation of the second FDR with the entropy production of the nonequilibrium medium:
\[ \left| \gamma(t-s) - \frac{1}{k_B T} \langle \eta_t \eta_s \rangle^{X_t} \right| \leq \sqrt{\langle \Phi^2 \rangle^{X_t}} \sqrt{\text{Var} \left[ \partial_{X_t} \Delta s_{\text{tot}}^{X_t} \right]} . \]  

Here, \( \text{Var}[\cdot] \) denotes the variance with respect to the stationary distribution \( P^{X_t}_{\text{ss}} \).

If we interpret \( \partial_{X_t} \Delta s_{\text{tot}}^{X_t} \) as the “response” of the total stochastic entropy production of the nonequilibrium medium to a perturbation of the probe position, (18) states that the violation of the second FDR is bounded by the fluctuation of the “response.” Hence, if the total stochastic entropy production is “robust” against the perturbation, i.e., \( \text{Var}[\partial_{X_t} \Delta s_{\text{tot}}^{X_t}] = 0 \), the standard second FDR is recovered. In particular, we can easily see that the standard second FDR holds in the equilibrium case because \( \Delta s_{\text{tot}}^{X_t} = 0 \).

### 4 Derivation

In this section, we derive the results presented in Sect. 3. In particular, we review the derivation of the effective probe dynamics based on nonequilibrium linear response theory [12, 15, 23, 25–30]. In Sect. 5, we discuss the validity of this approach and show that the effective dynamics obtained from nonequilibrium linear response theory is consistent with that obtained from a singular perturbation method.
4.1 Derivation of the Effective Dynamics Based on Nonequilibrium Linear Response Theory

Since we are now interested in the regime where the motion of the probe is much slower than that of the particles, we regard the probe motion as a time-dependent perturbation on the particle dynamics. In other words, we regard the dynamics (4) with $X_t$ held fixed as the unperturbed dynamics:

$$\gamma \dot{x}_s^j = F^j(x_s) - \lambda \frac{\partial}{\partial x_s^j} V(x_s, X_t) + \sqrt{2\gamma k_B T} \xi_s^j, \quad \text{for } s \leq t.$$  \hspace{1cm} (19)

Let $P([x]|X_t)$ be the probability density of a trajectory $[x] := \{x_s|s \leq t\}$ of this unperturbed dynamics. Similarly, let $P([x]|X)$ be the probability density of the original dynamics (4) conditioned on an arbitrary probe trajectory up to time $t$, $[X] := \{X_s|s \leq t\}$. To investigate the response of particles against the probe motion, we first compare the response of particles against the probe motion, we first compare

$$P([x]|X) = \exp(-A([x]|X))P([x]|X_t).$$ \hspace{1cm} (20)

Here, the excess action $A([x]|X)$ is given by

$$-A([x]|X) := \sum_j \left[ -\frac{1}{2k_B T} \int_{-\infty}^{t} ds \frac{\partial}{\partial x_s^j} (V(x_s, X_s) - V(x_s, X_t)) \circ \dot{x}_s^j + \frac{1}{2} \int_{-\infty}^{t} ds \frac{\partial}{\partial x_s^j} (V(x_s, X_s) - V(x_s, X_t)) \right]$$

$$-\frac{1}{4} \int_{-\infty}^{t} ds \left( \left( \frac{\partial}{\partial x_s^j} V(x_s, X_s) \right)^2 - \left( \frac{\partial}{\partial x_s^j} V(x_s, X_t) \right)^2 \right)$$

$$+ \frac{1}{2} \int_{-\infty}^{t} ds \frac{\partial^2}{\partial (x_s^j)^2} (V(x_s, X_s) - V(x_s, X_t)) \right],$$ \hspace{1cm} (21)

where the symbol $\circ$ denotes the multiplication in the sense of Stratonovich [39]. By using $V(x_s, X_s) = V(x_s, X_t) + (X_s - X_t) \partial_X V(x_s, X_t) + O((X_s - X_t)^2)$, we obtain to first order in $X_s - X_t$,

$$-A([x]|X) \simeq \frac{1}{2k_B T} \left[ \int_{-\infty}^{t} ds (X_s - X_t) \sum_j \frac{\partial}{\partial x_s^j} \Phi(x_s, X_t) \circ \dot{x}_s^j - \int_{-\infty}^{t} ds (X_s - X_t) L_{\Phi}^j \Phi(x_s, X_t) \right],$$ \hspace{1cm} (22)

where $L_{\Phi}^j$ denotes the backward operator (9). We remark that the first term on the right-hand side of (22) is the entropic part while the second term is the so-called frenetic part [40].

We now decompose the interaction force $\Phi$ into a deterministic part $\langle \Phi | [X] \rangle$ and a fluctuating part $\eta_t := \Phi - \langle \Phi | [X] \rangle$, where $\langle \cdot | [X] \rangle$ denotes the average with respect to $P([x]|X)$:

$$M \ddot{X}_t = \langle \Phi (x_t, X_t) | [X] \rangle - \Gamma \dot{X}_t + \sqrt{2\Gamma k_B T} \xi_t + \eta_t.$$ \hspace{1cm} (23)
The deterministic part can be further decomposed into a streaming term and friction term as follows. By using the relation (20), we obtain

\[
\langle \Phi (x_t, X_t) | [X] \rangle - \langle \Phi (x_t, X_t) \rangle_{X_t} = - (\Phi (x_t, X_t); A)^{X_t} + O((X_s - X_t)^2). \tag{24}
\]

Here, we have used \( \langle A \rangle_{X_t} = 0 \) to first order in \( X_s - X_t \), which follows from the normalization condition \( \langle \exp (-A) \rangle_{X_t} = 1 \). By substituting (22) into (24), we obtain

\[
\langle \Phi (x_t, X_t) | [X] \rangle - \langle \Phi (x_t, X_t) \rangle_{X_t} = \int_{-\infty}^{t} ds (X_s - X_t) R_{\Phi \Phi} (t - s) + O((X_s - X_t)^2), \tag{25}
\]

where

\[
R_{\Phi \Phi} (t - s) := \frac{1}{2k_BT} \left[ \frac{d}{ds} \langle \Phi (x_s, X_t); \Phi (x_t, X_t) \rangle_{X_t} - \langle L_s^v \Phi (x_s, X_t); \Phi (x_t, X_t) \rangle_{X_t} \right]. \tag{26}
\]

The function \( R_{\Phi \Phi} (t - s) \) corresponds to the response function of the interaction force \( \Phi \) against the perturbation of the probe position \( X_s - X_t \):

\[
R_{\Phi \Phi} (t - s) = \frac{\delta \langle \Phi (x_t, X_t) | [X] \rangle}{\delta X_s} \bigg|_{X_s=X_t}. \tag{27}
\]

We introduce \( \gamma (t - s) \) via

\[
\gamma (t - s) := \int_{-\infty}^{s} du R_{\Phi \Phi} (t - u), \quad \text{for} \quad t \geq s. \tag{28}
\]

Then, (25) can be expressed as

\[
\langle \Phi (x_t, X_t) | [X] \rangle - \langle \Phi (x_t, X_t) \rangle_{X_t} = - \int_{-\infty}^{t} ds \gamma (t - s) \dot{X}_s + O((X_s - X_t)^2). \tag{29}
\]

Thus, the deterministic part \( \langle \Phi (x_t, X_t) | [X] \rangle \) is decomposed into the streaming term \( G (X_t) := \langle \Phi (x_t, X_t) \rangle_{X_t} \) and friction term with the kernel \( \gamma (t - s) \).

Next, we specify the statistical property of the fluctuating part \( \eta_t \). To leading order, it is given by

\[
\eta_t \cong \eta_t^{(0)} := \Phi (x_t, X_t) - \langle \Phi (x_t, X_t) \rangle_{X_t}. \tag{30}
\]

The mean and two-time correlation of the noise \( \eta_t^{(0)} \) read

\[
\langle \eta_t^{(0)} \rangle_{X_t} = 0, \tag{31}
\]

\[
\langle \eta_t^{(0)} \eta_s^{(0)} \rangle_{X_t} = \langle \Phi (x_t, X_t); \Phi (x_s, X_t) \rangle_{X_t}. \tag{32}
\]

Because the friction kernel \( \gamma (t - s) \) can be expressed as

\[
\gamma (t - s) = \frac{1}{k_BT} \langle \Phi (x_t, X_t); \Phi (x_s, X_t) \rangle_{X_t} - \frac{1}{2k_BT} \int_{-\infty}^{s} du \left[ \frac{d}{du} \langle \Phi (x_u, X_t); \Phi (x_t, X_t) \rangle_{X_t} + (L_u^v \Phi (x_u, X_t); \Phi (x_t, X_t) \rangle_{X_t} \right], \tag{33}
\]

the two-time correlation can be represented in terms of the friction kernel \( \gamma (t - s) \) as

\[
\langle \eta_t^{(0)} \eta_s^{(0)} \rangle_{X_t} = k_BT \left[ \gamma (t - s) + \gamma_{cs} (t - s) \right], \quad \text{for} \quad t \geq s. \tag{34}
\]
where $\gamma_{\text{ex}}(t-s)$ is the excess friction kernel defined by

$$
\gamma_{\text{ex}}(t-s) := \frac{1}{2k_B T} \int_{-\infty}^{s} du \left[ \frac{d}{du} \langle \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t}^X, \\
+ \langle \mathcal{L}_u \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t}^X \right].
$$

To summarize, the effective dynamics of the probe is given by

$$
M \ddot{X}_t = G(X_t) - \int_{-\infty}^{t} ds \gamma(t-s) \dot{X}_s - \Gamma \dot{X}_t + \sqrt{2k_B T} \varepsilon_t + \eta_t,
$$

where

$$
G(X_t) := \langle \Phi(x_t, X_t) \rangle_{X_t}^X,
$$

$$
\gamma(t-s) := \frac{1}{2k_B T} \int_{-\infty}^{s} du \left[ \frac{d}{du} \langle \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t}^X, \\
- \langle \mathcal{L}_u \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t}^X \right],
$$

for $t \geq s$.

and we have rewritten $\eta^{(0)}_t$ as $\eta_t$ for notational simplicity. The noise intensity is related to the friction kernel in the following form:

$$
\langle \eta_t \eta_s \rangle_{X_t}^X = k_B T \left[ \gamma(t-s) + \gamma_{\text{ex}}(t-s) \right], \quad \text{for } t \geq s,
$$

with the excess friction kernel $\gamma_{\text{ex}}(t-s)$ given by

$$
\gamma_{\text{ex}}(t-s) := \frac{1}{2k_B T} \int_{-\infty}^{s} du \left[ \frac{d}{du} \langle \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t}^X, \\
+ \langle \mathcal{L}_u \Phi(x_u, X_t); \Phi(x_t, X_t) \rangle_{X_t}^X \right].
$$

4.2 Derivation of the Inequality that Bounds the Violation of the Second FDR

Here, we derive our first main result (18). We use the fact that the violation of the second FDR is originated from that of the first FDR. The excess friction kernel $\gamma_{\text{ex}}(t-s)$ can be expressed as

$$
\gamma_{\text{ex}}(t-s) = \frac{1}{k_B T} \langle \eta_t \eta_s \rangle_{X_t}^X - \gamma(t-s)
$$

$$
= \int_{-\infty}^{s} du \left[ \frac{1}{k_B T} \partial_u C_{\Phi,\Phi}(t-u) - R_{\Phi,\Phi}(t-u) \right],
$$

where $C_{\Phi,\Phi}(t-s)$ denotes the connected correlation function,

$$
C_{\Phi,\Phi}(t-s) := \langle \Phi(x_t, X_t); \Phi(x_s, X_t) \rangle_{X_t}^X.
$$

While the standard first FDR holds in equilibrium, $k_B T R_{\Phi,\Phi}(t-s) = \partial_s C_{\Phi,\Phi}(t-s)$ [2, 41], the following Seifert-Speck generalized FDR holds in the nonequilibrium steady state [42]:

$$
R_{\Phi,\Phi}(t-s) - \frac{1}{k_B T} \partial_s C_{\Phi,\Phi}(t-s) = -\partial_s \left( \Phi(x_t, X_t) \frac{\partial}{\partial X_t} \Delta s_{\text{tot}}^X \right)^{X_t},
$$

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where $\Delta s_{xt}^{X_t}$ denotes the total stochastic entropy production up to time $s$ of the nonequilibrium medium in the NESS with $X_t$ held fixed [43–45]:

$$
\Delta s_{xt}^{X_t} = -\ln P_{ss}^{X_t}(x_u)\bigg|_{-\infty}^s + \int_{-\infty}^s du \frac{1}{k_B T} \sum_j \left[ F_j(x_u) - \lambda \frac{\partial}{\partial x_j} V(x_u, X_t) \right] \circ \dot{x}_u^j.
$$

(44)

Here, the first term represents the stochastic Shannon entropy difference, while the last term represents the stochastic entropy production of the equilibrium thermal bath. From this expression, \[ \partial_{X_t} \Delta s_{xt}^{X_t} \] reads

$$
\partial_{X_t} \Delta s_{xt}^{X_t} = -\partial_{X_t} \ln P_{ss}^{X_t}(x_u)\bigg|_{-\infty}^s + \int_{-\infty}^s du \frac{1}{k_B T} \sum_j \partial_{x_j} \Phi(x_u, X_t) \circ \dot{x}_u^j.
$$

(45)

Hence, \[ \langle \partial_{X_t} \Delta s_{xt}^{X_t} \rangle_{X_t} = 0 \]. By substituting (43) into (41) and using the Cauchy-Schwarz inequality, we thus obtain

$$
|\gamma_{\text{ex}}(t - s)| = \left| \int_{-\infty}^s du \partial_u \left( \Phi(x_t, X_t) \frac{\partial}{\partial X_t} \Delta s_{xt}^{X_t} \right) \right| X_t
$$

$$
= \left| \left( \Phi(x_t, X_t) \frac{\partial}{\partial X_t} \Delta s_{xt}^{X_t} \right) \right| X_t
$$

$$
\leq \sqrt{\langle (\Phi(x_t, X_t))^2 \rangle_{X_t}} \sqrt{\text{Var} \left[ \partial_{X_t} \Delta s_{xt}^{X_t} \right]}.
$$

(46)

### 5 Validity of the Effective Dynamics

In this section, we discuss the validity of the effective probe dynamics (6), which is derived by using nonequilibrium linear response theory. There are mainly two subtle points in the derivation. First, it is not clear whether the noise term $\eta_t = \Phi(x_t, X_t) - \langle \Phi(x_t, X_t) \rangle^{X_t}$ and friction kernel $\gamma(t - s)$ exclude the slow modes associated with the probe motion [46–50]. It is also unclear whether the noise and friction kernel include the effects of hydrodynamic fields [51], the properties of which have been well investigated in equilibrium systems [52–56]. In this regard, we conjecture that nonequilibrium linear response theory can describe such hydrodynamic effects because the response function $R_{\Phi\Phi}$ may exhibits long time tails if the back reaction of the particles against the probe motion propagates with slow time scales. Second, the condition for the time-scale separation between the probe and the particles is ambiguous. Specifically, the validity of the expansion of the excess action $A([x][X])$ in terms of $X_s - X_t$, (22), is unclear because the excess action includes the integral from time $t = -\infty$.

The guiding principle here is that if a system allows a phenomenological description at the mesoscale, then it should be uniquely determined. We thus aim to verify the validity of the effective probe dynamics (6) (or (17)) by deriving it through other methods. One of the most frequently used methods is the projection operator method [57]. This method is based on the natural idea of singling out slow degrees of freedom and has recently been applied even to nonlinear lattices to derive fluctuating hydrodynamics [58]. Because various
representations can be obtained depending on the definitions of the projection operator, the most crucial point in this approach is to clarify which definition is consistent with the description at the mesoscale. However, it is generally difficult to provide the condition that uniquely characterizes the projection operator corresponding to the mesoscopic description. In this regard, adiabatic perturbation theory is a systematic method based on more explicit assumptions [51, 59, 60]. In this theory, the probability distribution of the fast variables is expanded around a steady state in terms of the time derivative of the slow variables. A derivation based on the linearized Dean equation has also been recently proposed [61].

While all of these methods are expected to provide results consistent with nonequilibrium linear response theory, here we use the singular perturbation method developed in [62]. This approach is based on a clear assumption about the separation of time scales and thus allows us to derive slow dynamics systematically. In the following, we explain the details of the derivation of the effective dynamics using the singular perturbation method and show that the resulting effective dynamics corresponds to (17).

5.1 Singular Perturbation Method

We first rewrite the model (1) and (4) in the following form:

$$\dot{x}_t = \frac{p_t}{M},$$

$$\dot{p}_t = \Phi(x_t, X_t) - \Gamma \frac{p_t}{M} + \sqrt{2k_B T} \xi_t,$$

$$\dot{x}_t^j = \frac{1}{\gamma} \left[ F^j(x_t) - \lambda \frac{\partial}{\partial x_t^j} V(x_t, X_t) \right] + \sqrt{\frac{2k_B T}{\gamma}} \xi_t^j,$$

where $p_t$ denotes the momentum of the probe. The corresponding Fokker-Planck equation for the probability density $\rho_t(X, P, x)$ reads

$$\frac{\partial}{\partial t} \rho_t = \frac{p_t}{M} \frac{\partial}{\partial X} \rho_t + \frac{\partial}{\partial P} \left[ \left( -\Phi(x, X) + \Gamma \frac{P}{M} \right) \rho_t \right] + \Gamma k_B T \frac{\partial^2}{\partial P^2} \rho_t$$

$$+ \sum_j \frac{1}{\gamma} \frac{\partial}{\partial x_t^j} \left[ \left( -F^j(x) + \lambda \frac{\partial}{\partial x_t^j} V(x, X) \right) \rho_t \right] + \frac{k_B T}{\gamma} \frac{\partial^2}{\partial (x_t^j)^2} \rho_t. \quad (50)$$

We are now interested in the regime where the motion of the probe is much slower than that of the particles. To describe this regime more explicitly, we first explain several characteristic time scales for this system. Let $\ell$ be the characteristic length scale associated with the coupling potential $V$. Here, we have in mind a case where $V$ is a confining potential. Otherwise, hydrodynamic modes may emerge, of which time scales comparable to the time scales of the probe. The probe has two time scales: the characteristic time scale for the probe to relax in the coupling potential, $\tau_X := \sqrt{M \ell^2 / k_B T}$, and the momentum relaxation time, $\tau_p := M / \Gamma$. Similarly, the particles diffuse in the coupling potential with the time scale $\tau_c := \gamma \ell^2 / k_B T$. We denote by $\tau_m$ the time scale for the particles to relax to the steady state. Specifically, $\tau_m$ is defined through the spectral gap $\Delta_m$ between the first two largest eigenvalues of the Fokker-Planck operator for the particles (see (55), below). That is, if we denote by $\Lambda_0$ and $\Lambda_1$ the first two largest eigenvalues, the spectral gap is given by $\Delta_m := -\Re[\Lambda_1] > 0$, because $\Lambda_0 = 0$ and $\Re[\Lambda_1] < 0$ from the Perron-Frobenius theorem. Note that $\tau_m$ may correspond to the time scale associated with the hydrodynamic modes, especially when the
coupling potential $V$ is not a confining potential. The most crucial assumption in the singular perturbation method is the separation of time scales:

$$\tau_p \sim \tau_X \gg \tau_c \sim \tau_m.$$  \hfill (51)

This condition implies that there are no slow modes associated with the particles that are comparable to the motion of the probe. Hereafter, we consider the dynamics on the fast time scale $\tau := t/\tau_c$.

To identify small parameters in (50), we introduce dimensionless variables. We define $\tilde{X} := X/\ell$, $\tilde{x}^j := x^j/\ell$, and $\tilde{P} := P/\sqrt{MK_BT}$. We also define the dimensionless potential and force as $\tilde{\Phi}(\tilde{x}, \tilde{X}) := -\lambda \partial \bar{V}(\tilde{x}, \tilde{X})/\partial \tilde{X}$ with $\bar{V}(\tilde{x}, \tilde{X}) := V(x, X)/k_BT$ and $\bar{F}(\tilde{x}) := F(x)\ell/k_BT$. Correspondingly, we write the probability density as $\tilde{\rho}_\tau(\tilde{X}, \tilde{P}, \tilde{x}) := \rho_{\tau_c\tau}(\ell \tilde{X}, \sqrt{MK_BT} \tilde{P}, \ell \tilde{x})$. Then, (50) can be rewritten as

$$\frac{\partial}{\partial \tau} \tilde{\rho}_\tau = -\frac{\tau_c}{\tau_X} \frac{\partial}{\partial \tilde{X}} \tilde{\Phi} \tilde{\rho}_\tau + \frac{\tau_c}{\tau_X} \frac{\partial}{\partial \tilde{P}} \left[ -\tilde{\Phi}(\tilde{x}, \tilde{X}) \tilde{\rho}_\tau \right] + \frac{\tau_c}{\tau_P} \frac{\partial}{\partial \tilde{P}} \left( \tilde{P} \tilde{\rho}_\tau \right) + \frac{\tau_c}{\tau_P} \frac{\partial^2}{\partial \tilde{P}^2} \tilde{\rho}_\tau + \sum_j \left[ \frac{\partial}{\partial \tilde{x}^j} \left( -\bar{F}(\tilde{x}) + \frac{\partial}{\partial \tilde{x}^j} \bar{V}(\tilde{x}, \tilde{X}) \right) \tilde{\rho}_\tau \right] + \frac{\partial^2}{\partial (\tilde{x}^j)^2} \tilde{\rho}_\tau. \hfill (52)$$

From the condition (51), (52) can be expressed in terms of the small parameter $\epsilon := \tau_c/\tau_X \ll 1$ as

$$\frac{\partial}{\partial \tau} \tilde{\rho}_\tau = (\epsilon L_{pb} + L_m) \tilde{\rho}_\tau,$$  \hfill (53)

where $L_{pb}$ and $L_m$ denote the Fokker-Planck operators for the probe and the nonequilibrium medium, respectively:

$$L_{pb} := -\tilde{P} \frac{\partial}{\partial \tilde{X}} \tilde{\Phi} - \frac{\partial}{\partial \tilde{P}} \tilde{\Phi} \tilde{X} + \frac{\tau_X}{\tau_P} \frac{\partial}{\partial \tilde{P}} \tilde{P} + \frac{\tau_X}{\tau_P} \frac{\partial^2}{\partial \tilde{P}^2},$$  \hfill (54)

$$L_m := \sum_j \left[ \frac{\partial}{\partial \tilde{x}^j} \left( -\bar{F}(\tilde{x}) + \frac{\partial}{\partial \tilde{x}^j} \bar{V}(\tilde{x}, \tilde{X}) \right) + \frac{\partial^2}{\partial (\tilde{x}^j)^2} \right].$$  \hfill (55)

The form of (53) implies that the system first relaxes toward the slow manifold characterized by $\tau \sim 1$ and then evolves slowly on the slow manifold. The motion on the slow manifold is characterized by the following equation for the reduced probability density $R_\tau(\tilde{X}, \tilde{P}) := \int \prod_j d\tilde{x}^j \tilde{\rho}_\tau(\tilde{X}, \tilde{P}, \tilde{x})$, which is obtained by integrating out $\tilde{x}$ in (53):

$$\frac{\partial}{\partial \tau} R_\tau = \epsilon \left[ -\tilde{P} \frac{\partial}{\partial \tilde{X}} R_\tau - \frac{\partial}{\partial \tilde{P}} \int \prod_j d\tilde{x}^j \tilde{\Phi}(\tilde{x}, \tilde{X}) \tilde{\rho}_\tau + \frac{\tau_X}{\tau_P} \frac{\partial}{\partial \tilde{P}} (\tilde{P} R_\tau) + \frac{\tau_X}{\tau_P} \frac{\partial^2}{\partial \tilde{P}^2} R_\tau \right].$$  \hfill (56)

Because $R_\tau$ evolves slowly, secular terms arise in the naive perturbation expansion $\tilde{\rho}_\tau = \tilde{\rho}_\tau^{(0)} + \epsilon \tilde{\rho}_\tau^{(1)} + \cdots$. Therefore, to describe the dynamics on the slow manifold, we assume that the $\tau$-dependence of $\tilde{\rho}_\tau$ is expressed in terms of the $\tau$-dependent operator $M_\tau$ that acts on the reduced probability density $R_\tau$:

$$\tilde{\rho}_\tau(\tilde{X}, \tilde{P}, \tilde{x}) = M_\tau[R_\tau](\tilde{X}, \tilde{P}, \tilde{x}).$$  \hfill (57)
From this functional ansatz, we can decompose the \( \tau \)-dependence of \( \tilde{\rho}_\tau \) into its explicit and implicit parts through \( R_\tau \). Correspondingly, we introduce \( \Omega_\tau \) as the \( \tau \)-dependent operator that represents the slow dynamics:

\[
\Omega_\tau[R_\tau](\tilde{X}, \tilde{P}) := \epsilon \left[ -\tilde{P} \frac{\partial}{\partial \tilde{X}} R_\tau - \frac{\partial}{\partial \tilde{P}} \int \prod_j d\tilde{x}_j \Phi(\tilde{x}_j, \tilde{X}) M_\tau[R_\tau] \right] + \frac{\tau x}{\tau P} \frac{\partial}{\partial \tilde{P}} (\tilde{P} R_\tau) + \frac{\tau x}{\tau P} \frac{\partial^2}{\partial \tilde{P}^2} R_\tau. \tag{58}
\]

In terms of \( M_\tau \) and \( \Omega_\tau \), \( (53) \) can be expressed as

\[
\frac{\partial}{\partial \tau} M_\tau[R_\tau] + \int \frac{\delta M_\tau[R_\tau]}{\delta R_\tau} \Omega_\tau[R_\tau] d\tilde{X} d\tilde{P} = (\epsilon L_{pb} + L_m) M_\tau[R_\tau]. \tag{59}
\]

We now assume that \( M_\tau \) and \( \Omega_\tau \) have asymptotic expansions in terms of the asymptotic sequences \( \{\epsilon^n\}_{n=0}^\infty \) as \( \epsilon \to 0 \):

\[
M_\tau = M^{(0)}_\tau + \epsilon M^{(1)}_\tau + \epsilon^2 M^{(2)}_\tau + \cdots, \tag{60}
\]

\[
\Omega_\tau = \epsilon \Omega^{(1)}_\tau + \epsilon^2 \Omega^{(2)}_\tau + \cdots. \tag{61}
\]

Note that \( \Omega^{(0)}_\tau \) is set to zero because of the form \( (58) \). The leading order of \( (59) \) gives

\[
\frac{\partial}{\partial \tau} M^{(0)}_\tau[R_\tau] = L_m M^{(0)}_\tau[R_\tau]. \tag{62}
\]

From this equation, it follows that

\[
M^{(0)}_\tau[R_\tau](\tilde{X}, \tilde{P}, \tilde{x}) \simeq R_\tau(\tilde{X}, \tilde{P}) Q_{ss}(\tilde{x} | \tilde{X}) \tag{63}
\]

for \( \tau \gg 1 \), where \( Q_{ss}(\tilde{x} | \tilde{X}) \) denotes the stationary distribution for \( \tilde{x} \) under the condition that \( \tilde{X} \) is held fixed:

\[
L_m Q_{ss} = 0. \tag{64}
\]

Here, we have imposed the condition

\[
R_\tau = \int \prod_j d\tilde{x}_j M^{(0)}_\tau[R_\tau]. \tag{65}
\]

Note that, in the approximation in \( (63) \), the additional terms are ignored because they decay exponentially with the time scale of \( \tau_c \). By substituting \( (63) \) into \( (58) \), we obtain

\[
\Omega^{(1)}_\tau[R_\tau] \simeq -\tilde{P} \frac{\partial}{\partial \tilde{X}} R_\tau - \frac{\partial}{\partial \tilde{P}} \left( \langle \Phi(\tilde{x}_j, \tilde{X}) \rangle X R_\tau \right) + \frac{\tau x}{\tau P} \frac{\partial}{\partial \tilde{P}} (\tilde{P} R_\tau) + \frac{\tau x}{\tau P} \frac{\partial^2}{\partial \tilde{P}^2} R_\tau \tag{66}
\]

for \( \tau \gg 1 \), where \( \langle \cdot \rangle X \) denotes the average with respect to \( Q_{ss} \). The subleading order of \( (59) \) gives

\[
\frac{\partial}{\partial \tau} M^{(1)}_\tau[R_\tau] + \int \frac{\delta M^{(0)}_\tau[R_\tau]}{\delta R_\tau} \Omega^{(1)}_\tau[R_\tau] d\tilde{X} d\tilde{P} = L_{pb} M^{(0)}_\tau[R_\tau] + L_m M^{(1)}_\tau[R_\tau]. \tag{67}
\]
For $\tau \gg 1$, we obtain the first-order solution $M^{(1)}_\tau$ as

$$M^{(1)}_\tau[R_\tau] \simeq \mathcal{L}^{-1} \left[ Q_{ss} \Omega^{(1)}_\tau[R_\tau] - \mathcal{L}_{pb} M^{(0)}_\tau[R_\tau] \right]$$

$$= - \int_0^\infty ds e^{s \mathcal{L}_m} \left\{ Q_{ss} \frac{\partial}{\partial P} \left[ (\bar{\Phi}(\bar{x}, \bar{X}) - \langle \bar{\Phi}(\bar{x}, \bar{X}) \rangle^X) R_\tau \right] + \bar{P} R_\tau x \frac{\partial}{\partial X} Q_{ss} \right\}$$

$$= - \left( \frac{\partial}{\partial P} \bar{R}_\tau \right) \int_0^\infty ds e^{s \mathcal{L}_m} \left\{ Q_{ss} \left( \bar{\Phi}(\bar{x}, \bar{X}) - \langle \bar{\Phi}(\bar{x}, \bar{X}) \rangle^X \right) \right\}$$

$$- \bar{P} R_\tau \int_0^\infty ds e^{s \mathcal{L}_m} \frac{\partial}{\partial X} Q_{ss}.$$

(68)

We note that in order for the above expression for $M^{(1)}_\tau$ to be well-defined, it is necessary that $Q_{ss} \Omega^{(1)}_\tau[R_\tau] - \mathcal{L}_{pb} M^{(0)}_\tau[R_\tau]$ does not include the zero eigenfunction of $\mathcal{L}_m$. This solvability condition is nothing but (66). By substituting (68) into (58), we obtain

$$\Omega^{(2)}_\tau[R_\tau] = - \frac{\partial}{\partial P} \int \prod_j d\tilde{x}^j \bar{\Phi}(\tilde{x}, \tilde{X}) M^{(1)}_\tau[R_\tau]$$

$$= \int_0^\infty ds \langle \bar{\Phi}(\tilde{x}_s, \tilde{X}) ; \bar{\Phi}(\tilde{x}_0, \tilde{X}) \rangle^X \frac{\partial^2}{\partial \bar{P}^2} R_\tau$$

$$+ \frac{\partial}{\partial \bar{P}} \left( \int_0^\infty ds \left\{ \bar{\Phi}(\tilde{x}_s, \tilde{X}) \frac{\partial}{\partial \tilde{X}} \ln Q_{ss}(\tilde{x}_0|\tilde{X}) \right\}^X \bar{P} R_\tau \right).$$

(69)

From (58), (66), and (69), the effective dynamics for the slow variable $R_\tau$ is given by

$$\frac{\partial}{\partial \tau} R_\tau \simeq \epsilon \left[ -\bar{P} \frac{\partial}{\partial \bar{X}} R_\tau - \frac{\partial}{\partial \bar{P}} \left( \langle \bar{\Phi}(\bar{x}, \bar{X}) \rangle^X R_\tau \right) + \frac{\tau_X}{\tau_P} \frac{\partial}{\partial \bar{P}} (\bar{P} R_\tau) + \frac{\tau_X}{\tau_P} \frac{\partial^2}{\partial \bar{P}^2} R_\tau \right]$$

$$+ \epsilon^2 \left[ \int_0^\infty ds \langle \bar{\Phi}(\tilde{x}_s, \tilde{X}) ; \bar{\Phi}(\tilde{x}_0, \tilde{X}) \rangle^X \frac{\partial^2}{\partial \bar{P}^2} R_\tau \right]$$

$$+ \frac{\partial}{\partial \bar{P}} \left( \int_0^\infty ds \left\{ \bar{\Phi}(\tilde{x}_s, \tilde{X}) \frac{\partial}{\partial \tilde{X}} \ln Q_{ss}(\tilde{x}_0|\tilde{X}) \right\}^X \bar{P} R_\tau \right) \right]$$

$$= \tau_c \left[ -\frac{P}{M} \frac{\partial}{\partial \bar{X}} R_\tau - \frac{\partial}{\partial \bar{P}} \left( \langle \Phi(x_s, X) \rangle^X - \frac{\Gamma}{M} P \right) R_\tau \right]$$

$$+ \frac{\partial}{\partial \bar{P}} \left( \int_0^\infty dt \left\{ \Phi(x_t, X) \frac{\partial}{\partial \tilde{X}} \ln P^X_{ss}(x_0) \right\}^X \frac{P}{M} R_\tau \right) \right],$$

(70)

where $P^X_{ss}(x) := Q_{ss}(\ell^{-1} x|\ell^{-1} X)$. Therefore, the effective Langevin equation for the probe reads

$$\dot{X}_t = \frac{P_t}{M},$$

(71)

$$\dot{P}_t = G(X_t) - (\Gamma + \gamma_{eff}) \frac{P_t}{M} + \sqrt{2(\Gamma + \gamma_{eff} + \gamma_{ex}) k_B T} \xi_t.$$

(72)

Here, $G(X_t)$ denotes the streaming term,

$$G(X_t) := \langle \Phi(x_t, X_t) \rangle^X_t,$$

(73)
and $\gamma_{\text{eff}}$ denotes the effective friction coefficient,

$$\gamma_{\text{eff}} := \int_0^\infty ds \left\langle \Phi(x_s, X_t) \frac{\partial}{\partial X_t} \ln P^X_{ss}(x_0) \right\rangle^{X_t}.$$  \hfill (74)\hfill

Note that the integrand in the above expression can be expressed in terms of the response function $R_{\Phi\Phi}(t - u)$, (27), by using the Seifert-Speck generalized FDR (43) [42]:

$$\left\langle \Phi(x_s, X_t) \frac{\partial}{\partial X_t} \ln P^X_{ss}(x_0) \right\rangle^{X_t} = \int_{-\infty}^0 du R_{\Phi\Phi}(s - u),$$  \hfill (75)\hfill

where

$$R_{\Phi\Phi}(s - u) = \frac{\partial}{\partial u} \left\langle \Phi(x_s, X_t) \frac{\partial}{\partial X_t} \ln P^X_{ss}(x_u) \right\rangle^{X_t}.$$  \hfill (76)\hfill

Finally, $\gamma_{\text{ex}}$ denotes the excess friction coefficient,

$$\gamma_{\text{ex}} := \frac{1}{k_B T} \int_0^\infty ds \left\langle \Phi(x_s, X_t); \Phi(x_0, X_t) \right\rangle^{X_t} - \gamma_{\text{eff}}$$

$$= \frac{1}{k_B T} \int_0^\infty ds \left\langle \eta^{(0)}_s \eta^{(0)}_0 \right\rangle^{X_t} - \gamma_{\text{eff}},$$  \hfill (77)\hfill

where $\eta^{(0)}$ is defined by (30). Therefore, (71) and (72) exactly correspond to (17), and thus the singular perturbation method and nonequilibrium linear response theory give the same result.

### 6 Example: Potential Switching Medium

We here present a simple model for a nonequilibrium medium as an example of the previous results. In this model, the particles are driven by potentials that switch stochastically. We can confirm that the effective dynamics is consistent with the exact solution because all relevant quantities can be calculated explicitly. Furthermore, in the fast switching limit, this model provides an example of a nonequilibrium medium where the second FDR holds. We can show that the upper bound of the inequality (18) goes to zero in this limit.

#### 6.1 Model

The time evolution of $X_t$ is given by the following underdamped Langevin equation:

$$M \dddot{X}_t = \Phi(x_t, X_t) - \Gamma \dot{X}_t + \sqrt{2\Gamma k_B T} \xi_t,$$  \hfill (78)\hfill

with $\Phi(x_t, X_t) := -\lambda \partial V(x_t, X_t)/\partial X_t$. We suppose that the probe is linearly coupled to the particles. That is, $V(x, X)$ is a harmonic potential with the spring constant $\kappa_c$:

$$\Phi(x_t, X_t) = -\lambda \kappa_c \sum_j (X_t - x^j_t).$$  \hfill (79)\hfill

The particles are described by the so-called potential switching model, i.e., they are subjected to potentials that switch stochastically [31–33]:

$$\gamma \dddot{x}^j_t = -\kappa_b (x^j_t - \sigma^j_t) - \lambda \kappa_c (x^j_t - X_t) + \sqrt{2\gamma k_B T} \xi^j_t.$$  \hfill (80)\hfill
The first term on the right-hand side of (80) denotes the force induced by the switching potential with the spring constant $\kappa_b$ and switching width $L$. Here, $\sigma^j \in \{0, 1\}$ denotes the potential state of the $j$-th particle at time $t$, which switches stochastically between 0 and 1 at a rate $r$ independently for each particle. We note that this model is an example of (4) with $F^j(x) = -\kappa_b(x^j - \sigma^j L)$ and $V(x, X) = \sum_j \kappa_c(x^j - X)^2/2$. Since the transition rates are equal for the transitions from 0 to 1 and 1 to 0, there is no entropy production associated with these transitions. Therefore, this model satisfies the LDB for the combined set of variables $(x^j, \sigma^j)$. We remark that this model can also be regarded as a run-and-tumble model [63].

6.2 Effective Dynamics

Even in this model, we can derive the effective dynamics of the probe by using nonequilibrium linear response theory (see Appendix 1 for the detailed derivation). The effective dynamics of the probe is given by the following generalized Langevin-type equation:

$$M \ddot{X}_t = G(X_t) - \Gamma \dot{X}_t - \int_{-\infty}^{t} ds \gamma(t-s) \dot{X}_s + \sqrt{2 \Gamma k_B T} \Xi_t + \eta_t.$$  \hspace{1cm} (81)

The streaming term $G(X_t) = \langle \Phi(x_t, X_t) \rangle_{X_t}$ is expressed as

$$G(X_t) = -\frac{N \lambda \kappa_c \kappa_b}{\kappa_b + \lambda \kappa_c} \left( X_t - \frac{L}{2} \right),$$  \hspace{1cm} (82)

where $\langle \cdot \rangle_{X_t}$ denotes the average with respect to the stationary distribution $P_{ss}^{X_t}(x, \sigma)$ for the particle dynamics (80) with $X_t$ held fixed, where $\sigma := \{\sigma^1, \sigma^2, \ldots, \sigma^N\}$. The friction kernel is given by

$$\gamma(t-s) = \frac{N \lambda^2 \kappa_c^2}{\kappa_b + \lambda \kappa_c} e^{-\frac{t-s}{\tau_x}}, \quad \text{for } t \geq s,$$  \hspace{1cm} (83)

where $\tau_x := \gamma / (\kappa_b + \lambda \kappa_c)$ denotes the characteristic time scale for the particles to relax in the coupling and switching potentials. The expression (83) states that dissipation happens on the time scale $\tau_x$. By contrast, the noise correlation $\langle \eta_t \eta_s \rangle_{X_t}$ additionally includes the switching time scale $\tau_r := 1/2r$:

$$\langle \eta_t \eta_s \rangle_{X_t} = k_B T \left[ \gamma(t-s) + \gamma_{\text{ex}}(t-s) \right],$$  \hspace{1cm} (84)

where $\gamma_{\text{ex}}(t-s)$ denotes the excess friction kernel

$$\gamma_{\text{ex}}(t-s) = \frac{1}{k_B T} \frac{N \lambda^2 \kappa_c^2}{\kappa_b + \lambda \kappa_c} \frac{L^2}{\gamma^2} \left( e^{-\frac{|t-s|}{\tau_x}} - \frac{2r \gamma}{\kappa_b + \lambda \kappa_c} e^{-\frac{|t-s|}{\tau_x}} \right).$$  \hspace{1cm} (85)

In the Markov approximation, the friction kernel can be approximated as

$$\gamma(t-s) = 2 \gamma_{\text{eff}} \delta(t-s),$$  \hspace{1cm} (86)

where $\gamma_{\text{eff}}$ denotes the effective friction coefficient:

$$\gamma_{\text{eff}} := \int_0^\infty dt \gamma(t) = \frac{N \lambda^2 \kappa_c^2}{(\kappa_b + \lambda \kappa_c)^2} \gamma.$$  \hspace{1cm} (87)
For the excess friction kernel, we can show that
\[
\gamma_{\text{ex}}(t - s) = 2\gamma_{\text{ex}}\delta(t - s)
\]  
with
\[
\gamma_{\text{ex}} := \int_0^\infty dt \gamma_{\text{ex}}(t) = \frac{1}{k_B T} \frac{N \lambda^2 \kappa_c^2 \kappa_b^2 L^2}{8r (\kappa_b + \lambda \kappa_c)^2}.
\]

Thus, in the Markov approximation, the effective generalized Langevin-type equation (81) becomes
\[
M \ddot{X}_t = G(X_t) - (\Gamma + \gamma_{\text{eff}}) \dot{X}_t + \sqrt{2(\Gamma + \gamma_{\text{eff}} + \gamma_{\text{ex}}) k_B T \Xi_t}.
\]

Note that \(\gamma_{\text{ex}} \geq 0\). This implies that stochastic switching enhances the noise intensity. We remark that if the coupling constant is rescaled as \(\lambda = \lambda_0 / N^{1/2}\), \(\gamma_{\text{eff}}\) and \(\gamma_{\text{ex}}\) are finite even in the limit \(N \to \infty\) [12].

### 6.3 Validity of the Effective Dynamics

Because (80) is linear with respect to \(x^j_t\), it can be solved exactly. The stationary solution reads
\[
x^j_t = \frac{\lambda \kappa_c}{\kappa_b + \lambda \kappa_c} X_t + \frac{\kappa_b}{\kappa_b + \lambda \kappa_c} L + \int_{-\infty}^t dse^{-\frac{t-s}{\tau_x}} \left[ -\frac{\lambda \kappa_c}{\kappa_b + \lambda \kappa_c} \dot{X}_s + \frac{\kappa_b}{\gamma} \left( \sigma^j_s - \frac{1}{2} \right) L \right. \\
+ \sqrt{\frac{2k_B T}{\gamma} \xi^j_s} \right],
\]

with \(\tau_x = \gamma / (\kappa_b + \lambda \kappa_c)\). By substituting (91) into (78), we obtain
\[
M \ddot{X}_t = -\frac{N \lambda \kappa_c \kappa_b}{\kappa_b + \lambda \kappa_c} (X_t - \frac{L}{2}) - \Gamma \dot{X}_t - \int_{-\infty}^t dse^{-\frac{t-s}{\tau_x}} \left[ -\frac{N \lambda^2 \kappa_c^2 \kappa_b^2 L^2}{8r (\kappa_b + \lambda \kappa_c)^2} \dot{X}_s + \sqrt{2\Gamma k_B T} \Xi_t \\
+ \sum_j \lambda \kappa_c \int_{-\infty}^t dse^{-\frac{t-s}{\tau_x}} \left[ \frac{\kappa_b}{\gamma} \left( \sigma^j_s - \frac{1}{2} \right) L + \sqrt{\frac{2k_B T}{\gamma} \xi^j_s} \right] \right].
\]

Note that the last term corresponds to the noise term \(\eta_t = \lambda \kappa_c \sum_j (x^j_t - \langle x^j_t \rangle X_t)\) in (81) because the exact solution of (80) with \(X_t\) held fixed reads
\[
x^j_t = \frac{\lambda \kappa_c}{\kappa_b + \lambda \kappa_c} X_t + \frac{\kappa_b}{\kappa_b + \lambda \kappa_c} L + \int_{-\infty}^t dse^{-\frac{t-s}{\tau_x}} \left[ \frac{\kappa_b}{\gamma} \left( \sigma^j_s - \frac{1}{2} \right) L + \sqrt{\frac{2k_B T}{\gamma} \xi^j_s} \right] \\
= \langle x^j_t \rangle X_t + \int_{-\infty}^t dse^{-\frac{t-s}{\tau_x}} \left[ \frac{\kappa_b}{\gamma} \left( \sigma^j_s - \frac{1}{2} \right) L + \sqrt{\frac{2k_B T}{\gamma} \xi^j_s} \right],
\]

where \(\langle x^j_t \rangle X_t = (\lambda \kappa_c X_t + \kappa_b L / 2) / (\kappa_b + \lambda \kappa_c)\). Therefore, (92) exactly corresponds to (81). We remark that (90) can also be obtained by using the singular perturbation method described in Sect. 5.
6.4 Fast Switching Limit

In the fast switching limit $\tau_r/\tau_x \to 0 \ (r \to \infty)$, we can easily see that the excess friction kernel (85) goes to zero:

$$\frac{1}{k_B T} \langle \eta_t \eta_s \rangle X_t - \gamma (t - s) = \gamma (t - s) \to 0,$$

and thus the standard second FDR is recovered. Correspondingly, we can show that the upper bound of the inequality for the violation of the second FDR (18) also goes to zero, as derived below:

$$\left| \gamma (t - s) - \frac{1}{k_B T} \langle \eta_t \eta_s \rangle X_t \right| \leq \sqrt{\langle \Phi^2 \rangle X_t} \sqrt{\text{Var} \left[ \partial X_t \Delta s_{\text{tot}} \right]} \to 0.$$

The important point here is that, even in this limit, the particles are out of equilibrium. In fact, if we denote by $\dot{s}_{\text{env}}^{X_t}$ the stochastic entropy production rate of the equilibrium thermal bath for the dynamics (80) with $X_t$ held fixed, it can be shown that

$$\langle \dot{s}_{\text{env}}^{X_t} \rangle X_t = \frac{1}{k_B T} \sum_j \langle \dot{x}_j \rangle X_t \circ \left[ -\kappa_b (x_j^i - \sigma_i^j L) \right] X_t = \frac{N \gamma}{k_B T} \frac{\kappa_b^2 L^2}{\gamma k_B T 2(\kappa_b + \lambda \kappa_c) / \gamma' + 4} \to \frac{N \kappa_b^2 L^2}{4 \gamma} \text{ as } \tau_r/\tau_x \to 0. \quad (96)$$

Thus, the potential switching medium (80) in the fast switching limit provides an example of a nonequilibrium medium where the standard second FDR holds.

Note that the entropy production $\langle \dot{s}_{\text{env}}^{X_t} \rangle X_t$ is induced by the fast degrees of freedom $\sigma^j$, which does not appear in the dynamics in the fast switching limit:

$$\gamma \dot{x}_j^i = -\kappa_b \left( x_j^i - \frac{L}{2} \right) - \lambda \kappa_c (x_j^i - X_t) + \sqrt{2 \gamma k_B T} \xi_t^j. \quad (97)$$

(97) can, for example, be obtained by using the singular perturbation method as described in Sect. 5. Therefore, the potential switching medium appears to be just an equilibrium thermal bath, and thus the standard second FDR holds. We remark that $\langle \dot{s}_{\text{env}}^{X_t} \rangle X_t$ is an example of hidden entropy, i.e., an entropy production invisible from the coarse-grained dynamics [33–35].

We now show that the upper bound of the inequality (18) goes to zero in the fast switching limit. To this end, we calculate $\langle (\Phi (x_t, X_t))^2 \rangle X_t$ and $\text{Var} \left[ \partial X_t \Delta s_{\text{tot}} \right]$. By using (93) and (A.12), $\langle (\Phi (x_t, X_t))^2 \rangle X_t$ can be calculated as
\[(\Phi(x_t, X_t))^2 X_t = \lambda^2 \kappa_c^2 \sum_i \sum_j ((x_t^i - X_t)(x_t^j - X_t))^2 \]
\[= \frac{N^2 \lambda^2 \kappa_c^2 \kappa_b^2}{(k_b + \lambda \kappa_c)^2} \left( X_t - \frac{L}{2} \right)^2 \]
\[+ N \lambda^2 \kappa_c \left[ \frac{k_b T}{k_b + \lambda \kappa_c} + \frac{\kappa_b^2 / \gamma^2}{(k_b + \lambda \kappa_c)^2 / \gamma^2 - 4 \kappa_b^2 / 4 \gamma^2} \left( 1 - \frac{2r \gamma}{k_b + \lambda \kappa_c} \right) \right]. \tag{98} \]

From this expression, it follows that
\[(\Phi(x_t, X_t))^2 X_t \rightarrow \frac{N^2 \lambda^2 \kappa_c^2 \kappa_b^2}{(k_b + \lambda \kappa_c)^2} \left( X_t - \frac{L}{2} \right)^2 + N k_b T \frac{\lambda^2 \kappa_c^2}{k_b + \lambda \kappa_c}. \tag{99} \]

in the fast switching limit. \( \text{Var} \left[ \partial X_t \Delta s_{\text{tot}}^X_t \right] \) can be explicitly calculated from the expression (45) as follows:
\[\text{Var} \left[ \partial X_t \Delta s_{\text{tot}}^X_t \right] = 2 \text{Var} \left[ -\partial X_t \ln P_{ss}^X(x_s, \sigma_s) + \frac{1}{k_b T} \Phi(x_s, X_t) \right] \]
\[= 2 N \left( \frac{\lambda \kappa_c}{k_b T} \right)^2 \left[ \frac{k_b T}{k_b + \lambda \kappa_c} + \frac{\kappa_b^2 / \gamma^2}{(k_b + \lambda \kappa_c)^2 / \gamma^2 - 4 \kappa_b^2 / 4 \gamma^2} \left( 1 - \frac{2r \gamma}{k_b + \lambda \kappa_c} \right) \right] \]
\[- 4 N \left( \frac{\lambda \kappa_c}{k_b T} \right)^2 \frac{k_b T}{k_b + \lambda \kappa_c} + 2 I(X_t), \tag{100} \]

where \( I(X_t) \) denotes the Fisher information [64] defined by
\[I(X_t) := \langle (\partial X_t \ln P_{ss}^X(x_t, \sigma_t))^2 \rangle X_t. \tag{101} \]

By noting that, in the fast switching limit, \( P_{ss}^X(x, \sigma) \) is given by
\[P_{ss}^X(x, \sigma) = N \exp \left( -\frac{1}{k_b T} \sum_j \left[ \frac{\kappa_b}{2} \left( x^j - \frac{L}{2} \right)^2 + \frac{\lambda \kappa_c}{2} (x^j - X_t)^2 \right] \right), \tag{102} \]

where
\[N = \frac{1}{2^N} \left( \frac{\kappa_b + \lambda \kappa_c}{2 \pi k_b T} \right)^{N/2} \exp \left( \frac{N}{2 k_b T} \frac{\lambda \kappa_c k_b}{\kappa_b + \lambda \kappa_c} \left( X_t - \frac{L}{2} \right)^2 \right), \tag{103} \]

we obtain
\[I(X_t) = N \left( \frac{\lambda \kappa_c}{k_b T} \right)^2 \left[ \frac{k_b T}{k_b + \lambda \kappa_c} + \frac{\kappa_b^2 / \gamma^2}{(k_b + \lambda \kappa_c)^2 / \gamma^2 - 4 \kappa_b^2 / 4 \gamma^2} \left( 1 - \frac{2r \gamma}{k_b + \lambda \kappa_c} \right) \right]. \tag{104} \]

By substituting (104) into (100), we find that in the limit \( \tau_r / \tau_s \rightarrow 0, \)
\[\text{Var} \left[ \partial X_t \Delta s_{\text{tot}}^X_t \right] = 4 N \left( \frac{\lambda \kappa_c}{k_b T} \right)^2 \left[ \frac{\kappa_b^2 / \gamma^2}{(k_b + \lambda \kappa_c)^2 / \gamma^2 - 4 \kappa_b^2 / 4 \gamma^2} \left( 1 - \frac{2r \gamma}{k_b + \lambda \kappa_c} \right) \right] \]
\[\rightarrow 0. \tag{105} \]

From (99) and (105), we thus find that the upper bound of the inequality (18) goes to zero in the fast switching limit.
7 Concluding Remarks

In summary, we have investigated a class of nonequilibrium media described by Langevin dynamics that satisfies the LDB. For the effective dynamics of a probe immersed in the medium, we have derived an inequality that bounds the violation of the second FDR. The upper bound of the inequality can be interpreted as a measure of robustness of the nonequilibrium medium against perturbation of the probe position. This implies that a nonequilibrium medium may be characterized by robustness against perturbation. We have also discussed the validity of the effective dynamics. In particular, we have shown that the effective dynamics obtained from nonequilibrium linear response theory is consistent with that obtained from the singular perturbation method. As an example of these results, we have proposed the potential switching medium in which the particles are subjected to potentials that switch stochastically. For this model, we have shown that the second FDR is recovered in the fast switching limit, although the particles are out of equilibrium.

Although we have focused on a class of nonequilibrium media described by Langevin dynamics that satisfies the LDB, it is possible to derive effective dynamics for more general nonequilibrium media. For example, Maes has recently derived the effective dynamics of a probe immersed in an active Ornstein-Uhlenbeck (AOU) medium [30, 65]. In that case, the persistence of the medium generates extra mass and additional friction breaking the second FDR. Because this violation of the second FDR also originates from the violation of the first FDR, we expect that a relation similar to (18) still holds even for this case. We also remark that the singular perturbation method described in this paper can be applied to more general nonequilibrium media, including the AOU medium.

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Appendix A: Derivation of the Effective Dynamics for the Potential Switching Medium

A.1 Excess Action

We first confirm that, to first order in $X_s - X_t$, the excess action $A([x, \sigma][X])$ is given by

$$-A([x, \sigma][X]) \simeq \frac{1}{2k_B T} \left[ \int_{-\infty}^{t} ds (X_s - X_t) \sum_j \frac{\partial}{\partial x_j} \Phi(x_s, X_t) \partial x_j^s - \int_{-\infty}^{t} ds (X_s - X_t) L_s^\dagger \Phi(x_s, X_t) \right]$$

(A.1)

with the backward operator for the dynamics (80) with $X_t$ held fixed:

$$L_s^\dagger := \sum_j \left[ \frac{1}{\gamma} (-\kappa_b (x_j^s - \sigma_j^s L) - \lambda \kappa_c (x_j^s - X_t)) \frac{\partial}{\partial x_j^s} + \frac{k_B T}{\gamma} \frac{\partial^2}{\partial (x_j^s)^2} \right].$$

(A.2)
To this end, we calculate \( \mathbb{P}(\{x, \sigma\} | \{X\}) \) and \( \mathbb{P}(\{x, \sigma\} | X_t) \). We first consider a trajectory in the time interval \([0, t]\) and discretized time \( t_n = n \Delta t \in [0, t] \) \((n = 0, 1, \ldots, M)\) with \( t = M \Delta t \). Correspondingly, let \( \{x, \sigma\} := \{(x_0, \sigma_0), (x_1, \sigma_1), \ldots, (x_M, \sigma_M)\} \) be the discretized trajectory, where \((x_n, \sigma_n) := (x_{tn}, \sigma_{tn})\). Suppose that the state \( x_j \) is switched at time intervals with \( n = n^j_1, n^j_2, \ldots, n^j_{k_j} \in \{0, 1, \ldots, M\} \) as
\[
\sigma^j_{n^j_{\ell + 1}} = 1 - \sigma^j_{n^j_{\ell}},
\] 
(A.3)

We denote by \( \Sigma^j_\ell \in \{0, 1\} \) the value of \( \sigma^j_n \) for \( n^j_\ell < n \leq n^j_{\ell + 1} \) with \( n^j_0 := -1 \) and \( n^j_{k_j + 1} := M \). For notational simplicity, we rewrite (80) as
\[
y \dot{x}^j_s = -U'_1(x^j_s, \sigma^j_s) - \lambda V'_1(x^j_s, X_s) + \sqrt{2\gamma k_B T} \xi^j_s = -U'_1(x^j_s, \sigma^j_s) - \lambda V'_1(x^j_s, X_t) + h_s f(x^j_s) + \sqrt{2\gamma k_B T} \xi^j_s.
\] 
(A.4)

where \( U_1(x^j, \sigma^j) := \kappa_b (x^j - \sigma^j L)^2 / 2 \), \( V_1(x^j, X) := \kappa_c (x^j - X)^2 / 2 \), and the prime denotes the derivative with respect to \( x^j \). In the second line, we have introduced a time-dependent amplitude \( h_s := X_t - X_t \) and \( f(x^j) := \partial_j \Phi(x_s, X_t) = \lambda \kappa_c \) to explicitly represent the deviation from the dynamics with \( X_t \) held fixed. Then, the probability density of a trajectory \( \mathbb{P}(\{x, \sigma\} | \{X\}) \) starting from \((x_0, \sigma_0)\) reads [6]
\[
\mathbb{P}(\{x, \sigma\} | \{X\}) = \prod \prod_{n=0}^{n^j_{\ell + 1} - 1} \sqrt{\frac{\gamma}{4\pi k_B T}} \Delta t \exp\left[-\frac{\Delta t}{\gamma k_B T}
\times \left( U_1(x^j_{n^j_{\ell}}, \sigma^j_{n^j_{\ell}}) + \lambda V_1(x^j_{n^j_{\ell}}, X_t) - h_n f(x^j_{n^j_{\ell}}) \right)^2 + \frac{\Delta t}{\gamma k_B T} U_1(x^j_{n^j_{\ell}}, \sigma^j_{n^j_{\ell}}) + \lambda V_1(x^j_{n^j_{\ell}}, X_t) - h_n f(x^j_{n^j_{\ell}}) \right] \Delta t
\] 
(A.5)

Here, \( \tilde{x}_n := (x^j_{n+1} + x^j_n) / 2 \) and \( \tilde{h}_n := (h_{n+1} + h_n) / 2 \). We note that \( \mathbb{P}(\{x, \sigma\} | X_t) \) is immediately obtained from (A.5) by setting \( \tilde{h}_n = 0 \). From these expressions, it follows that
\[
\ln \frac{\mathbb{P}(\{x, \sigma\} | \{X\})}{\mathbb{P}(\{x, \sigma\} | X_t)} = \sum_j \left[ \frac{1}{2k_B T} \sum_{n=0}^{M-1} \tilde{h}_n f(\tilde{x}^j_n) (x^j_{n+1} - x^j_n) \right] - \frac{1}{2\gamma k_B T} \left[ \sum_{n=0}^{n^j_{\ell + 1} - 1} \tilde{h}_n f(\tilde{x}^j_n) \right]
\]
By taking the continuum limit and replacing the time interval from \([0, t]\) to \([−\infty, t]\), we obtain the excess action \(A([x, \sigma]||X])\):

\[
−A([x, \sigma]||X) = \ln \frac{\mathbb{P}([x, \sigma]||X)}{\mathbb{P}([x, \sigma]|X_t)} = \sum_j \left[ \frac{1}{2k_B T} \int_{−\infty}^t ds h_s f(x^j_s) \circ \dot{x}^j_s \right. \\
− \frac{1}{2\gamma} k_B T \int_{−\infty}^t ds h_s f(x^j_s) \left(−U^j_1(x^j_s, \sigma^j_s) − \lambda V^j_1(x^j_s, X_t)\right) \\
− \frac{1}{2\gamma} \int_{−\infty}^t ds h_s \frac{\partial}{\partial x^j_s} f(x^j_s) \right] + O(h^2_X).
\]

\[
\simeq \frac{1}{2k_B T} \left[ \int_{−\infty}^t ds \sum_j \frac{\partial}{\partial x^j_s} \Phi(x_s, X_t) \circ \dot{x}^j_s − \int_{−\infty}^t ds h_s \mathcal{L}^j_\sigma \Phi(x_s, X_t) \right],
\]

where

\[
\mathcal{L}^j_\sigma := \sum_j \left[ \frac{1}{\gamma} \left(−U^j_1(x^j_s, \sigma^j_s) − \lambda V^j_1(x^j_s, X_t)\right) \frac{\partial}{\partial x^j_s} + \frac{k_B T}{\gamma} \frac{\partial^2}{\partial (x^j_s)^2} \right].
\]

### A.1.1 Explicit Calculation of \(G(X_t), \gamma (t - s), \) and \(\gamma_{ex}(t - s)\)

Here, we calculate \(G(X_t), \gamma (t - s), \) and \(\gamma_{ex}(t - s)\) explicitly. The starting point is the stationary solution of (80) with \(X_t\) held fixed (93):

\[
x^j_t = \frac{\lambda k_c}{k_b + \lambda k_c} X_t + \frac{k_b}{k_b + \lambda k_c} \frac{L}{2} + \int_{−\infty}^t ds e^{−\frac{i\xi_s}{\sqrt{\gamma}}} \left[ \frac{k_b}{\gamma} \left(\sigma^j_s − \frac{1}{2}\right) \frac{L}{2} + \sqrt{\frac{2k_B T}{\gamma}} \xi^j_s \right],
\]

\[
= \langle x^j_t \rangle X_t + \int_{−\infty}^t ds e^{−\frac{i\xi_s}{\sqrt{\gamma}}} \left[ \frac{k_b}{\gamma} \left(\sigma^j_s − \frac{1}{2}\right) \frac{L}{2} + \sqrt{\frac{2k_B T}{\gamma}} \xi^j_s \right].
\]

The statistical force is immediately obtained by substituting (A.9) into its definition:

\[
G(X_t) := \langle \Phi(x_t, X_t) \rangle_{X_t} = \left\langle −\lambda k_c \sum_j (X_t - x^j_t) \right\rangle_{X_t} \\
= \frac{N\lambda k_c k_b}{k_b + \lambda k_c} (X_t - \frac{L}{2}).
\]
To calculate the friction kernel, we first calculate the response function \( R_{\Phi \Phi}(t - s) \). The response function \( R_{\Phi \Phi}(t - s) \) is expressed as

\[
R_{\Phi \Phi}(t - s) = \frac{1}{2k_B T} \frac{d}{ds} \langle \Phi(x_s, X_t); \Phi(x_s, X_t) \rangle^{X_t} - \langle \mathcal{L}_s^{X} \Phi(x_s, X_t); \Phi(x_s, X_t) \rangle^{X_t}
\]

\[
= \frac{\lambda^2 \kappa_c^2}{2k_B T} \sum_i \sum_j \left[ \frac{d}{ds} (x_i^t - s_i^t; x_j^t - s_j^t)^{X_t} + \frac{k_b}{\gamma} (x_i^t - \sigma_s^i L; x_j^t - s_j^t)^{X_t} + \frac{\lambda \kappa_c}{\gamma} (x_i^t - s_i^t; x_j^t - s_j^t)^{X_t} \right]
\]

\[
= \frac{\lambda^2 \kappa_c^2}{2k_B T} \sum_i \sum_j \left[ \frac{d}{ds} (x_i^t; x_j^t)^{X_t} + \frac{k_b + \lambda \kappa_c}{\gamma} (x_i^t; x_j^t)^{X_t} - \frac{k_b}{\gamma} L (\sigma_s^i; x_j^t)^{X_t} \right].
\]  

(A.11)

By using (A.9) and the relation

\[
\langle \sigma_s^i \sigma_s^j \rangle = \begin{cases} 
(1 + e^{-2r|t-s|})/4 & \text{for } i = j \\
1/4 & \text{for } i \neq j.
\end{cases}
\]

\( (x_i^t; x_j^t)^{X_t} \) and \( (\sigma_s^i; x_j^t)^{X_t} \) are calculated as

\[
(x_i^t; x_j^t)^{X_t} = \delta_{ij} \left[ \frac{k_b^2/\gamma^2}{(k_b + \lambda \kappa_c)^2/\gamma^2 - 4r^2} \frac{L^2}{4} \left( e^{-2r|t-s|} - \frac{2r \gamma}{k_b + \lambda \kappa_c} e^{-\frac{k_b + \lambda \kappa_c}{\gamma} |t-s|} \right) \right.
\]

\[
+ \frac{k_b T}{k_b + \lambda \kappa_c} e^{-\frac{k_b + \lambda \kappa_c}{\gamma} |t-s|},
\]  

(A.12)

\[
(\sigma_s^i; x_j^t)^{X_t} = \delta_{ij} \left[ \begin{array}{l}
\frac{k_b}{\gamma} \frac{L}{(k_b + \lambda \kappa_c)/\gamma + 2r} e^{-2r|t-s|}, \text{ for } t < s \\
\frac{k_b}{\gamma} \frac{L}{(k_b + \lambda \kappa_c)/\gamma - 2r} e^{-2r|t-s|} - e^{-\frac{k_b + \lambda \kappa_c}{\gamma} |t-s|}, \text{ for } t \geq s.
\end{array} \right.
\]  

(A.13)

Therefore, for \( t \geq s \), the response function is

\[
R_{\Phi \Phi}(t - s) = \frac{N \lambda^2 \kappa_c^2}{\gamma} e^{-\frac{k_b + \lambda \kappa_c}{\gamma} (t-s)}.
\]  

(A.14)

From this result, it follows that

\[
\gamma(t - s) := \int_{-\infty}^s du R_{\Phi \Phi}(t - u)
\]

\[
= \frac{N \lambda^2 \kappa_c^2}{k_b + \lambda \kappa_c} e^{-\frac{k_b + \lambda \kappa_c}{\gamma} (t-s)}, \text{ for } t \geq s.
\]  

(A.15)
To obtain the explicit expression of $\gamma_{\text{ex}}(t-s)$, we calculate the noise correlation $\langle \eta_t \eta_s \rangle^{X_t}$.

By using (A.12), the noise correlation is calculated as

$$
\langle \eta_t \eta_s \rangle^{X_t} = \lambda^2 \kappa_c^2 \sum_i \sum_j \langle (X^i_t - X^i) (X^j_s - X^j) \rangle^{X^i_t} 
$$

$$
= k_B T \gamma(t-s) + \frac{N \lambda^2 \kappa_c^2 \kappa_b^2}{(\kappa_b + \lambda \kappa_c)^2} \frac{L^2}{\gamma^2 - 4r^2} \left( e^{-2r|t-s|} - \frac{2r \gamma}{\kappa_b + \lambda \kappa_c} e^{-\frac{s_b + \lambda \kappa_c}{\gamma} |t-s|} \right). 
$$

(A.16)

Thus, the excess friction kernel $\gamma_{\text{ex}}(t-s)$ is given by

$$
\gamma_{\text{ex}}(t-s) = \frac{1}{k_B T} \frac{N \lambda^2 \kappa_c^2 \kappa_b^2}{(\kappa_b + \lambda \kappa_c)^2} \frac{L^2}{\gamma^2 - 4r^2} \left( e^{-2r|t-s|} - \frac{2r \gamma}{\kappa_b + \lambda \kappa_c} e^{-\frac{s_b + \lambda \kappa_c}{\gamma} |t-s|} \right). 
$$

(A.17)

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