Missed topological phase transition in the forced Kuramoto Model

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(Dated: 17 December 2020)

A complete bifurcation analysis of explicit dynamical equations for the periodically forced Kuramoto model was performed in [L. M. Childs and S. H. Strogatz. Chaos 18, 043128 (2008)], identifying all bifurcations within the model. However, our numerical analysis of the equations reveals that the model can undergo an abrupt phase transition from oscillations to wobbly rotations of the order parameter under increasing field frequency or decreasing field strength. This transition was not revealed by bifurcation analysis because it is not caused by a bifurcation, and can neither be classified as first nor second order since it does not display critical phenomena characteristic of either transition. We discuss the topological origin of this transition and show that it is determined by a singular point in the order-parameter space.

The Kuramoto model is a representative model of synchronization between heterogeneous oscillators, and has attracted much attention not only due to a wide range of applications in physics, neuroscience, biology, and the social sciences but also due to the diversity of collective phenomena that it demonstrates including exotic collective states such as chimera states5 and Bellerophon states6. The Kuramoto model also demonstrates well-known first and second order phase transitions7-10 and an unusual hybrid phase transition.11 Here we report a topological phase transition in the driven Kuramoto model.

I. INTRODUCTION

The Kuramoto model describes synchronization between coupled phase oscillators, and has been studied extensively since it was first introduced.1,2 Among the different lines of enquiry, we highlight the problem of entrainment by an external periodic field. The phase diagram for a system of uniformly coupled Kuramoto oscillators in an external periodic field was first reported in12. More recently, Childs and Strogatz performed a complete bifurcation analysis of explicit dynamical equations for the periodically forced Kuramoto model, revealing the exact locations of all bifurcations.13 In this work, we report a missed phase transition in the periodically forced Kuramoto model, based on numerical and analytical analysis of the explicit dynamical equations derived in12. We show that this transition was missed by the bifurcation analysis because it is not caused by a bifurcation. Moreover, it cannot be categorized as a first or second order phase transition since it does not display critical phenomena characteristic of either transition. Our analysis reveals the topological character of this transition, and the crucial role of a singular point in the order-parameter space. Topological phase transitions have been observed and studied in quantum systems such as fractional quantum Hall liquids and topological insulators11. Our work shows that a topological transition can also occur in a classical system such as the forced Kuramoto model.

In the next section, we introduce the forced Kuramoto model, the order-parameter space, and a brief summary of the physical behavior displayed by the model. In Section II, we present an updated phase diagram and our analysis of the missing transition. Finally, we discuss our findings in Section IV.
frame rotating with frequency \( \omega_0 \), where the field frequency is \( \sigma - \omega_0 \).

Eqs. (3) and (4) describe the motion of the system in a two-dimensional order-parameter space \((\Re(z), \Im(z)) = (\rho \cos \psi, \rho \sin \psi)\) where positive rotation of \( \psi \) is counterclockwise. It is important to note that the state \( z = 0 \) is a singular point in this space because \( \psi \) becomes undefined when \( \rho = 0 \). Since any trajectory through this order-parameter space requires the function \( \psi(t) \) to be analytic at every time \( t \), e.g. the first derivative defines angular velocity, any trajectory through the singular point \( z = 0 \) is forbidden.

The complete bifurcation analysis of Eqs. (3) and (4) was performed by Childs and Strogatz in the paper[18] where one can find the explicit stability diagram, all bifurcations curves, and phase portraits of dynamical states in the forced Kuramoto model at different model parameters \( F, \Omega \) and \( K \). In short, the model demonstrates the following physical behavior. First, in the absence of the field, when the coupling \( K \) between phase oscillators is larger than the critical coupling \( K_c = 2 \), the interaction results in a synchronous rotation of phase oscillators with the group velocity \( \omega_0 \) and order parameter \( \rho = \sqrt{(K-2)/K} \). The rotational symmetry is spontaneously broken and the phases \( \theta_i \) of a finite fraction of oscillators become aligned along an arbitrary direction \( \psi \). Second, upon applying a field with strength \( F \) at detuning \( |\Omega| \) sufficiently smaller than \( F \), the synchronized group becomes entrained by the field, for a broad range of model parameters (see phase I in Fig. 1). In the original non-rotating frame, the entrained state is phase- and frequency-locked to the field and a finite fraction of phase oscillators rotates synchronously with the field frequency \( \sigma \). If \( \Omega > 0 \), the order parameter \( z \) lags behind the field, i.e., the phase \( \psi \) is negative and the order parameter \( z \) lies in the lower half-plane of the order-parameter space. If \( \Omega < 0 \), then the order parameter \( z \) is ahead the field, the phase \( \psi \) is positive and \( z \) lies in the upper half-plane. The latter symmetry follows from the symmetry of Eqs. (3) and (4) with respect to the replacement \( \Omega \rightarrow -\Omega \) and \( \psi \rightarrow -\psi \). Third, increasing \( \Omega \) at fixed \( F \) disrupts the entrained state and leads to periodic dynamics in the rotating frame, as the system undergoes a SNIPER, saddle-node, or Hopf bifurcation[19].

As discussed below, we found that coupled oscillators can both oscillate synchronously with respect to the field direction (phase II in Fig. 1) or simply drift at a frequency other than the field frequency (phase III in Fig. 1).

### III. SPOR Transition

Our numerical analysis of Eqs. (3) and (4) revealed that the phase diagram[18] is incomplete since it misses a phase transition from the phase with oscillations (phase II in Fig. 1) into a phase where the phase oscillators demonstrate a wobbling rotation around the singular point \( z = 0 \) with an angular frequency different from the field frequency (phase III in Fig. 1). In order to understand the origin and properties of this transition, which we have tentatively named a Singular Point Oscillation-to-Rotation (SPOR) transition, and the reasons why this transition was missed, we performed a detailed numerical analysis of Eqs. (3) and (4).

First, we studied the oscillations in phase II. We fixed the field strength \( F \) and increased \( \Omega \). The non-uniform dynamics of \( z \) in phase II are typified in Fig. 2(b), for oscillations born above the Hopf bifurcation. The order parameter \( z \) describes a limit cycle in the lower half plane of the order-parameter space. The amplitude of the oscillations of \( \psi \) between \( \psi_{\text{max}} \) and \( \psi_{\text{min}} \) increases with the detuning \( \Omega \), so that the amplitude \( \psi_{\text{max}} - \psi_{\text{min}} \) tends to \( \pi \) at a critical detuning \( \Omega_c \) that determines the boundary between phases II and III. Fig. 2(b) shows that over one period of oscillation, the order parameter slowly falls behind the field (\( \psi \) moves slowly from 0 to \( -\pi \)) and then quickly catches up (\( \psi \) quickly moves from \( -\pi \) to 0). The sharp increase of \( \psi \) from a value a little bit above \( -\pi \) to a value a little bit below 0 corresponds to fast motion of \( z \) along the upper part of the limit cycle, due to high angular velocity \( d\psi/dt \) caused by the \( 1/\rho \) singularity in Eq. (4). Finally, we note that in the rotating frame the angular velocity \( d\psi/dt \) averaged over the period of oscillations is zero. Thus, \( z \) is on average frequency-locked to the field.

![FIG. 1. Partial phase diagram for the Kuramoto model in a homogeneous field with strength \( F \) and detuning \( \Omega \) at \( K = 5 \). Each phase is characterized by distinct dynamics of \( \psi(t) \). Phase I corresponds to the entrained phase. Phase II is the oscillating phase. Phase III, is the rotating phase. Full circles (blue) define a boundary (the SPOR transition) separating phases II (shadowed region) and III. The upper boundary of phase I was calculated following[19] the dash-dotted line (black) indicates a SNIPER bifurcation, which becomes a Saddle-Node bifurcation (full black line) at the triangular marker, which in turn becomes a Hopf bifurcation (dotted green line) at the square marker. A detailed phase diagram in the region I is presented in[20].](image-url)
\(\Omega_c\) (see Figs. 2(b) and 2(c)).

\[
\kappa(\psi) = \frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{[\rho^2 + (\rho')^2]^{3/2}},
\]

where \(\rho' = d\rho/d\psi\). We found the function \(\rho(\psi)\), which defines the limit cycle, and its derivatives by use of the numerical solution \(\rho(t)\) and \(\psi(t)\). In the tested range of \(\Omega\) below \(\Omega_c\), the limit cycles presented no peculiarity in shape, such as flattening (flattening corresponds to \(\kappa \to 0\)). The curvature is finite and nonzero at all points on the limit cycle, as shown in Fig. 3. For the limit cycle of oscillations at \(\Omega\) very close to \(\Omega_c\). We also extended our numerical analysis to limit cycles above the critical point \(\Omega_c\). At \(\Omega > \Omega_c\), the singular point is encircled by the limit cycle, as shown in Fig. 2(a). The shape of the limit cycle, its curvature, and the tangential velocity are similar to those below \(\Omega_c\). Although we found the angular velocity diverges in the vicinity of the singular point \(z = 0\) as \(\Omega\) tends to \(\Omega_c\), namely that \(\max(d\psi/dt) \to +\infty\) from below and \(\min(d\psi/dt) \to -\infty\) from above, the tangential velocity \((dx/dt, dy/dt)\) is finite at all points on the limit cycle, where \(x = \Re(z)\) and \(y = \Im(z)\). Finally, we analyzed the relaxation time above and below \(\Omega_c\) and found that the relaxation time is finite and does not demonstrate critical behavior such as critical slowing down. Likewise, we found no evidence of hysteresis depending on initial conditions. Thus, we found no evidence suggesting the observed transition is second- or first-order, as would otherwise be suggested by the presence of the above-mentioned phenomena.

In order to understand the origin of the SPOR transition, we analyzed the behavior of the bifurcation point and limit cycle. In phases II (oscillating phase) and III (rotating phase) of the phase diagram there is only one bifurcation point, an unstable spiral. In the oscillating phase, the limit cycle lies in the lower half-plane as shown in Fig. 2(a). Increasing \(\Omega\) moves the system away from the critical boundary with region I, expanding the limit cycle around the bifurcation point, and moving the upper part of the limit cycle towards the singular point \(z = 0\). For \(\Omega < \Omega_c\), the limit cycle remains in the lower half-plane, never crossing the singular point. Increasing \(\Omega\) from below to above \(\Omega_c\) causes the bifurcation point to move continuously towards the singular point \(z = 0\), as depicted in Fig. 2(a). Next, we numerically analyzed the shape and curvature of the limit cycles and found no peculiarities even at \(\Omega\) very close (\(10^{-8}\)) to \(\Omega_c\). We used the well known equation for the curvature \(\kappa\) of a curve, given by a function \(\rho = \rho(\psi)\) in polar coordinates.

![Fig. 2.](image_url)

**Fig. 2.** (a) Orbits of the order parameter \(z\) at different \(\Omega\) and parameters \(F = 3.5\) and \(K = 5\). Dynamics of \(\psi\) and \(\rho\) in (b) the oscillating phase at \(\Omega = 3.90754900\) below the SPOR transition, and (c) the rotating phase at \(\Omega = 3.90754901\) above the SPOR transition. Our estimation of the critical detuning parameter is \(3.90754900 < \Omega_c < 3.90754901\). The origin of the complex plane (singular point) is marked with a star. The remaining markers indicate the unstable fixed point (spiral) and are mapped to each orbit by color. The inset in (a) shows the region of the order parameter space near the singular point, and a clear change in the position of the singular point relative to the orbits for a small change \((10^{-8})\) in \(\Omega\). Arrowheads in (a) indicate the direction of motion along the orbits.

![Fig. 3.](image_url)

**Fig. 3.** Curvature \(\kappa\) versus \(\psi\) of the limit cycle of oscillations near the critical point of the transition to rotation. Parameters: \(K = 5\), \(F = 3.5\), \(\Omega = 3.907549\).

In the original non-rotating frame, the phase of the order parameter \(z\) equals \(\psi = \sigma t + \psi\). Therefore, the average cycling frequency \(f\) of \(z\) over one period \(T\) of oscillations or wobbling rotations is

\[
f = \frac{1}{2\pi T} \int_0^{\pi} d\tau \frac{d\psi}{d\tau} = \frac{\sigma}{2\pi} + \frac{1}{2\pi T} \left[\psi(t+T) - \psi(t)\right].
\]
that \( f = \sigma / (2\pi) \) since \( \psi(t + T) = \psi(t) \), showing the system is on average frequency-locked to the field. In the rotating phase, the average cycling frequency is \( f = \sigma / (2\pi) - 1 / T \) since \( \psi(t + T) = \psi(t) - 2\pi \) for clockwise rotation, and the frequency \( f \) tends to \( \omega_k \) for increasing \( \Omega \) since a high frequency field does not impact coupled phase oscillators. Thus, the SPOR transition from oscillations to wobbling rotations appears as an abrupt drop in the average cycling frequency \( f \) at \( \Omega_C \), and the cycling frequency \( f \) decreases for \( \Omega > \Omega_C \), see Fig. 4(c). The drop equals the inverse of the oscillation period \( T \) at the critical point. Our numerical results allow us to conclude that the only important difference between limit cycles below and above \( \Omega_C \) is in the relative position of the singular point \( z = 0 \). The singular point lies outside the limit cycles at \( \Omega < \Omega_C \) and is encircled by the limit cycles at \( \Omega > \Omega_C \), as shown in Fig. 2(a). As a result, these two types of limit cycles acquire different topological properties. Based on the topological theory of ordered media, we can characterize limit cycles by a winding number \( n \). The field \( s(r) = (\rho \cos \psi, \rho \sin \psi) \) is known on all points \( r \) on a limit cycle [see, for example, vectors A and B in Fig. 4(d)]. We can measure the total angle \( \psi \) when the vector \( s(r) \) turns as \( r \) traverses the complete limit cycle (counterclockwise increments in angle are positive and clockwise increments are negative). Since \( s(r) \) is continuous on the cycle, this angle must be an integral multiple of \( 2\pi \). Since we are mapping the temporal behavior of the order parameter \( z(t) \) to the order-parameter space \((\rho, \psi)\), we parameterize the cycle by time \( t \). The winding number \( n \) of a limit cycle may then be defined as

\[
 n = \frac{1}{2\pi} \int_0^T dt' \frac{d\psi}{dt'} = \frac{1}{2\pi} [\psi(t + T) - \psi(t)],
\]

where \( T \) is the period and \( t \) is an arbitrary point in time. In the case of oscillations, the winding number of the limit cycle is \( n = 0 \), since \( \psi(t + T) = \psi(t) \). For positive \( \Omega > \Omega_C \) (clockwise rotation), rotations are characterized by \( n = 1 \), since \( \psi(t + T) = \psi(t) - 2\pi \). Moreover, by comparing Eqs. (7) and (9), we can immediately see that the abrupt drop in cycling frequency \( f \) measured in the original non-rotating frame is directly related with the winding number \( n \) as follows

\[
f = \frac{\sigma}{2\pi} + \frac{n}{T}.
\]

In summary, our analysis revealed a missing phase transition (the SPOR transition) in the forced Kuramoto model, from oscillations to wobbling rotations of the complex order parameter \( z \). The transition takes place at \( \Omega = \Omega_C \), as the orbit of \( z(\rho, \psi) \) in the rotating frame approaches the singular point \( z = 0 \), such that \( \min(\rho(t)) \to 0 \) as shown in Fig. 4(a). In the non-rotating frame, the transition is marked by an abrupt drop in the average cycling frequency \( f \), as shown in Fig. 4(c). Numerical analysis of the curvature shows the limit cycles immediately below and above \( \Omega_C \) are identical, and related by a simple translation in the order-parameter space, as depicted schematically in Fig. 4(d). However, the existence of the singular point \( z = 0 \) makes this translation impossible. In addition, the oscillating and rotating phases can be distinguished by a topological characteristic, the winding number \( n \) of the limit cycles, as shown in Fig. 4(c). Notably, the average cycling frequency \( f \) measured in the non-rotating frame is directly related with the winding number.

FIG. 4. (a) Minimum value of the amplitude \( \rho(t) \) of the complex order parameter \( z = \rho \exp(i\psi) \) as a function of the detuning \( \Omega \). (b) Winding number \( n \), Eq. 7, of the orbit described by \( z \) in a frame rotating at the field frequency, for both positive and negative \( \Omega \). (c) Average cycling frequency of rotation \( f \) in the original non-rotating frame, for both positive and negative \( \Omega \). The dashed line indicates non-oscillating phases, and \( \Omega_C \) is the critical detuning. (d) Schematic representation of two orbits for detuning \( \Omega \) slightly greater (dashed line) and slightly smaller (solid line) than \( \Omega_C \). These orbits differ simply by the translation represented by the green dashed vector connecting unstable spirals 1 and 2, and the translation is exaggerated for illustrative purposes. The tangential velocity at equivalent points under the translation is the same for both orbits, as shown at points A and B. The red vectors from the origin of the complex plane (marked with a star) to points A and B describe orbits with winding numbers 0 (solid vector) and \(-1 \) (dashed vector).
IV. DISCUSSION

In this paper we found that, by increasing the frequency or decreasing the strength of an external field, the periodically forced Kuramoto model undergoes an abrupt phase transition from a phase with oscillations to a phase with wobbly rotations of the order parameter. We call this a ‘Singular Point Oscillation-to-Rotation’ (SPOR) phase transition. In the original non-rotating frame, the SPOR transition appears as an abrupt drop in average group frequency of phase oscillators.

Our analysis of the dynamical behavior of the forced Kuramoto model shows that the SPOR transition can neither be classified as a first nor as a second order phase transition since it does not display critical phenomena characteristic of either transition. First, according to the classification of phase transitions in Landau’s phenomenological theory, second order (continuous) phase transitions are accompanied by symmetry breaking across the transition and a gradual increase of the order parameter in the ordered phase. In the case of the SPOR transition, symmetry breaking is absent. Second, the proximity to the critical point of a continuous transition is signalled by an increase of critical fluctuations that result in critical phenomena, such as a strong increase in susceptibility, correlation length, relaxation rate (known as critical slowing down of dynamics), etc. Our detailed analysis of the dynamical behavior of the forced Kuramoto model near the SPOR transition revealed no anomaly in behavior in the corresponding fixed point, the curvature of the limit cycle, and the relaxation rate. Third, first order (discontinuous) phase transitions are characterized by an abrupt appearance of order, hysteresis, and a region of metastable states. Moreover, critical phenomena occur close to the critical boundary of metastable states. In our case, hysteresis and metastable states are absent. Fourth, the absence of critical correlations also suggests the SPOR transition is not a hybrid phase transition, which combines the abrupt appearance of order, as in first order phase transitions, with the absence of hysteresis and the presence of critical fluctuations, as in second order phase transitions (for an example, consider the hybrid transition in the Kuramoto model with frequency-degree correlation). Fifth, despite the absence of symmetry breaking, states above and below the critical point, the curvature of the limit cycle, and the relaxation rate. Based on this analysis, we conclude that the SPOR transition is a topological phase transition between states with winding numbers 0 and ±1. Since the transition is topological in origin, it is not caused by a specific bifurcation, and was therefore missed by previous studies based on bifurcation theory. A similar phase transition has also been reported in the Kuramoto model with interacting identical phase oscillators subjected to white noise.

As mentioned in the Introduction, topological phases of matter have been found in quantum systems, namely in fractional quantum Hall liquids and topological isolators, see, for example, reviews. There is interesting similarity between the SPOR transition in the classical forced Kuramoto model and a transition from the flipper phase to the spinner phase in a mechanical model of topological metamaterials (compare our Fig. 2 and Fig. 4(c) in paper). Topological states with winding number +1 and 0 on Fig. 2 are topologically similar to topological edge states in two-dimensional topological circuits, where bound two photon topological states appear if the topoelectrical circuit has a unit cell characterized by winding number 1, but are absent if the winding number is zero (see Fig. 3 in paper). Note that these topological metamaterials and topoelectrical circuits are classical analogs of quantum systems with topological edge states.

The SPOR transition is an example of a topological transition between two distinct topological phases with winding numbers ±1 and zero in a classical system of interacting phase oscillators with a singular point in the order-parameter space. We believe this kind of topological transition may be found in other classical systems. As an illustrative example of a real-world SPOR transition, we may consider a motorboat making circles on the open sea. All circles are topologically equivalent independently on their position. Let us now introduce an observer in the water. The situation becomes crucially different. If the observer is placed outside these circles, then the motorboat oscillates with respect to the observer. The observer can track the motorboat just by turning their neck. If the motorboat encircles the observer, rotating around them, the observer must also rotate their body to track the motorboat. The critical circle corresponds to the situation when the motorboat passes through the observer. Assuming the person steering the motorboat wishes to avoid committing a crime, passing through the observer is forbidden. In this model, the observer plays the role of the singular point. States with the observer outside and inside the closed tracks created by the motorboat have different topological properties with respect to the observer, i.e. winding numbers 0 and ±1, respectively. A transition from one state to another is an abrupt topological
ACKNOWLEDGMENTS

We thank Ricardo Guimarães Dias for useful discussions. This work is funded by national funds (OE), through Portugal’s FCT Fundação para a Ciência e Tecnologia, I.P., within the scope of the framework contract foreseen in paragraphs 4, 5 and 6 of article 23, of Decree-Law 57/2016, of August 29, and amended by Law 57/2017, of July 19. E. A. P. W. acknowledges the financial support provided by FCT under PhD grant SFRH/BD/121331/2016.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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