TOWARD A QUASI-MÖBIUS CHARACTERIZATION OF INVERTIBLE HOMOGENEOUS METRIC SPACES

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ABSTRACT. We study locally compact metric spaces that enjoy various forms of homogeneity with respect to Möbius self-homeomorphisms. We investigate connections between such homogeneity and the combination of isometric homogeneity with invertibility. In particular, we provide a new characterization of snowflakes of boundaries of rank-one symmetric spaces of non-compact type among locally compact and connected metric spaces. Furthermore, we investigate the metric implications of homogeneity with respect to uniformly strongly quasi-Möbius self-homeomorphisms, connecting such homogeneity with the combination of uniform bi-Lipschitz homogeneity and quasi-invertibility. In this context we characterize spaces containing a cut point and provide several metric properties of spaces containing no cut points. These results are motivated by a desire to characterize the snowflakes of boundaries of rank-one symmetric spaces up to bi-Lipschitz equivalence.

1. INTRODUCTION

This paper contributes to the metric characterization of boundaries of rank-one symmetric spaces of non-compact type. Such symmetric spaces are Gromov hyperbolic and therefore possess boundaries at infinity, which we view as metric spaces equipped with visual distances. Such distances are non-Riemannian, unless the symmetric space is real hyperbolic. On this subject there have been several contributions: [Ham91], [Bou95], [Bon96], [FLS07], [FLS07a], [FLS11], [FS12], [Pla13], [BS14], [BS15], [PS17].

In the present paper, we focus on the fact that boundaries at infinity of rank-one symmetric spaces of non-compact type (and their snowflakes) enjoy an abundance of metric homogeneity. In fact, when equipped with a visual distance with base point at infinity they are isometrically homogeneous and admit an inversion (as defined below). Furthermore, the compact boundary is 2-point Möbius homogeneous. One of our main results provides a metric characterization of such spaces in terms of these properties (see Theorem 1.2).

Looking beyond characterizations up to isometric equivalence, we also work towards a characterization of snowflakes of boundaries of non-compact rank-one symmetric spaces up to bi-Lipschitz equivalence. Thus we investigate spaces that are uniformly bi-Lipschitz homogeneous and admit quasi-inversions (see Section 1.2 for relevant definitions). We point out that our results in this area fit into a progression of ongoing study. For example, the authors of the present paper have previously studied bi-Lipschitz homogeneity in the context of curves, surfaces, and more general spaces (see [FH10], [Fre10], [LD10], [LD11]). Quasi-invertibility and bi-Lipschitz homogeneity have been studied in [BB05], [BH05], [DCL17], [Fre12], [Fre14]. Our main goal in continuing this line of investigation is to uncover metric and geometric implications of such homogeneity and/or invertibility in a rather general metric setting. We articulate a specific version of this goal in Conjecture 1.1 below. Before stating this conjecture, we explain a bit of terminology.

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We denote by $\mathcal{A}$ the collection of normed division algebras $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Here $\mathbb{R}$ denotes the real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{H}$ the quaternions, and $\mathbb{O}$ the octonions. Using this notation, we write $\mathbb{H}_K^n$ (with $K \in \mathcal{A}$ and $n \in \mathbb{N}$) to denote the Lie group obtained as the stereographic projection of the visual boundary of the $K$-hyperbolic space of dimension $n + 1$ over $K$. We call $\mathbb{H}_K^n$ the $n$-th $K$-Heisenberg group, and denote by $\rho$ its visual distance with base point at infinity; see (2.2), for the formal definition. Classically, it is known that these metric spaces are isometrically homogeneous and invertible. In particular, they are bi-Lipschitz homogeneous and quasi-invertible in the sense of Section 1.2.2. With this terminology in hand we state the following conjecture.

Conjecture 1.1. A metric space $X$ is bi-Lipschitz equivalent to $(\mathbb{H}_K^n, \rho^\alpha)$, for some $\alpha \in (0,1]$, if and only if $X$ is locally compact, connected, bi-Lipschitz homogeneous, and quasi-invertible.

A significant issue in the study of ideal boundaries is that, in general, the boundary of a Gromov-hyperbolic space is not connected. Even if the boundary is connected it may not contain any non-degenerate rectifiable curves, thus precluding the possibility of any geometric analysis. In this connection we remind the reader that any snowflake of a visual distance remains a visual distance which allows for non-degenerate rectifiable curves. In Theorem 1.8 under the aforementioned homogeneity assumptions, we consider a dichotomy: either the space contains a cut point, or it does not. In the first case, we show that such a space is bi-Lipschitz homeomorphic to a snowflake of the Euclidean line. Thus we provide a complete characterization of spaces containing a cut point. In the second case, when the space does not contain a cut point, we prove several properties that are useful for developing analysis on these metric spaces and point in the direction of Conjecture 1.1.

1.1. Results. Here we summarize the main results of the present paper.

1.1.1. Möbius homogeneity. We first present results addressing various forms of Möbius homogeneity in connected and locally compact metric spaces. In the following we denote by $\hat{X} := X \cup \{\infty\}$ the compactification of $X$ equipped with its natural Möbius structure, see Section 1.2.

Theorem 1.2. Suppose $X$ is an unbounded, locally compact, complete, and connected metric space. For $\mathcal{A} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, the following statements are equivalent:

1. For some $n \in \mathbb{N}$, $K \in \mathcal{A}$, and $\alpha \in (0,1]$, the space $X$ is Möbius homeomorphic to $(\mathbb{H}_K^n, \rho^\alpha)$.
2. For some $n \in \mathbb{N}$, $K \in \mathcal{A}$, and $\alpha \in (0,1]$, the space $X$ is isometric to $(\mathbb{H}_K^n, \rho^\alpha)$.
3. The space $X$ is isometrically homogeneous and invertible.
4. The space $\hat{X}$ is 2-point Möbius homogeneous.

The main content of the above theorem is the implication (3) $\implies$ (2). The style and conclusion of this result is similar to the main result of [BS14]: Let $X$ be a compact Ptolemy space possessing at least one Ptolemy circle and for which there exists a unique space inversion with respect to any two distinct points $\omega, \omega'$ of $X$ and any sphere between $\omega, \omega'$. Under these assumptions, the space $X$ is Möbius equivalent to the boundary at infinity of a rank-one symmetric space of non-compact type taken with the canonical Möbius structure. Amidst the apparent similarities between Theorem 1.2 and the result of [BS14], we point out some key differences. A space inversion (in the sense of [BS14]) must be an involution that is fixed point free. The inversions of Theorem 1.2 need not possess these properties. Moreover, we do not assume presence of a Ptolemy circle.

Given a Carnot group $G$ equipped with a Carnot-Carathéodory distance $d$ (or any comparable distance), the compactification $\hat{G}$ may exhibit unique geometry at the point at $\infty$. In particular, it may be the case that certain classes of mappings on $\hat{G}$ must fix this point; see, for example, [Fre14, page 249]. Or, for another example along this line, note the Pointed Sphere Conjecture of Yves de Cornulier recorded in [dC13, Conjecture 19.104]. In light of these observations, the homogeneity assumptions on the sphericalization of $X$ in Theorem 1.2 may be seen as a natural way to restrict the geometric (and resulting algebraic) structure of $X$ itself.
Conjecture 1.1 is open even in the case that the space $X$ is a Carnot group. In particular, it is an open question whether the complexified Heisenberg group $\mathbb{H}_\mathbb{C}^1$ and the direct product $\mathbb{H}_\mathbb{C}^1 \times \mathbb{H}_\mathbb{C}^1$ admit a quasi-inversion. However, there is some relevant work by Xie [Xie13].

In order to provide additional context for Theorem 1.2, we record the following immediate corollary. This result should be seen as a rephrasing of the result in [KLD16].

Corollary 1.3. Suppose $X$ is an unbounded, locally compact, and connected metric space. There exists $n \in \mathbb{N}$ and $\alpha \in (0, 1]$ such that $X$ is isometric to $(\mathbb{R}^n, |\cdot|^\alpha)$ if and only if $\hat{X}$ is 3-point Möbius homogeneous.

Indeed, one can verify (via Theorem 1.2 and results from Section 2.1) that the 3-point Möbius homogeneity of $\hat{X}$ implies 2-point isometric homogeneity of $X$. In other words, we can conclude that, given pairs $\{x, y\}$ and $\{a, b\}$ of distinct points from $X$ such that $d(x, y) = d(a, b)$, there exists $f \in \text{Isom}(X)$ with $f(x) = a$ and $f(y) = b$. The only space $\mathbb{H}^n_\mathbb{R}$ whose snowflakes enjoy this property is $\mathbb{H}^n_\mathbb{R} = \mathbb{R}^n$.

In light of Theorem 1.2 and Corollary 1.3, we observe that 2-point and 3-point Möbius homogeneity in locally compact and connected metric spaces provide metric analogues to algebraic results about 2-point and 3-point topological homogeneity in such spaces (see, for example, [Kra03, Theorem 3.3 and Corollary 3.4]).

We also explore the metric consequences of 1-point Möbius homogeneity. In order to compensate for this weaker homogeneity assumption we work under stronger connectivity assumptions. Under such assumptions, one can prove that the geometry of the metric space under scrutiny is sub-Riemannian in nature (at least, up to bi-Lipschitz distortion).

Theorem 1.4. Let $X$ be a compact and quasi-convex metric space of finite topological dimension. If $X$ is Möbius homogeneous, then $X$ is bi-Lipschitz homeomorphic to a sub-Riemannian manifold.

We stress that the boundary of a CAT($-1$)-space is naturally equipped with special visual distances: either Bourdon distances or Hamenstädt distances; see Section 3.3. With such metrics, the boundaries become Ptolemy spaces by [FS11]. With this in mind, a consequence of the above theorem is the following.

Corollary 1.5. Let $X$ be the boundary of a CAT($-1$)-space. If $X$ is Möbius homogeneous, of finite topological dimension, and connected by Möbius circles, then $X$ is bi-Lipschitz homeomorphic to a sub-Riemannian manifold.

1.1.2. Uniformly strongly quasi-Möbius homogeneity. Having discussed various forms of homogeneity with respect to Möbius maps, we now present a few results concerning homogeneity with respect to uniformly strongly quasi-Möbius maps. We refer the reader to Section 1.2.2 for relevant terminology.

We start with the coarse analogue of the equivalence (3) $\iff$ (4) of Theorem 1.2.

Proposition 1.6. A proper and unbounded metric space $X$ is uniformly bi-Lipschitz homogeneous and quasi-invertible if and only if $\hat{X}$ is 2-point uniformly strongly quasi-Möbius homogeneous.

The reader may consult Proposition 1.7 for additional characterizations of spaces that are both uniformly bi-Lipschitz homogeneous and quasi-invertible.

For the coarse analogue of the equivalence (1) $\iff$ (2) of Theorem 1.2 we show the following.

Proposition 1.7. A homeomorphism $f : X \to Y$ between proper and unbounded metric spaces is strongly quasi-Möbius if and only if it is bi-Lipschitz. Furthermore, $f$ is Möbius if and only if $f$ is a similarity.

It is straightforward to verify that if $X$ is bi-Lipschitz equivalent to some $(\mathbb{H}^n_\mathbb{R}, \rho^n)$, then $X$ is uniformly bi-Lipschitz homogeneous and quasi-invertible. Conjecture 1.1 claims the converse for connected spaces: the coarse analogue of (3) $\implies$ (2) of Theorem 1.2. Our main contribution towards this conjecture is as follows.
Theorem 1.8. Suppose $X$ is an unbounded locally compact metric space that is uniformly bi-Lipschitz homogeneous, quasi-invertible, and contains an non-degenerate curve.

1. The metric space $X$ is path connected, locally path connected, proper, and Ahlfors regular.
2. If $X$ has a cut point, then, for some $\alpha \in (0, 1)$, $X$ is bi-Lipschitz homeomorphic to $\mathbb{R}, |\cdot|^\alpha$.
3. If $X$ has no cut points, then $X$ is linearly locally connected. Furthermore,
   a. If $X$ contains a non-degenerate rectifiable curve, then $X$ is annularly quasi-convex.
   b. If $X$ does not contain a non-degenerate rectifiable curve, then, for some $\alpha \in (0, 1)$, $X$ is bi-Lipschitz homeomorphic to an $\alpha$-snowflake.

We point out that Theorem 1.8 affirms Conjecture 1.1 in the case that $X$ is path connected and contains a cut point.

At the present time we are unable to provide a coarse analogue of Corollary 1.3. In particular, we do not have answers to the following questions.

Question 1.9. Is the compactified Heisenberg group $\text{Sph}_e(\mathbb{H}^1_\ast)$ 3-point uniformly strongly quasi-Möbius homogeneous?

Question 1.10. If a metric space is 3-point uniformly strongly quasi-Möbius homogeneous and homeomorphic to $S^n$, is it bi-Lipschitz homeomorphic to a snowflake of the round $n$-sphere?

This last question, even in the cases $n = 2$ or $3$, seems challenging. It is related to other open problems (such as [HS97, Question 5]) about structures on spheres that are 3-point quasi-symmetrically homogeneous.

1.1.3.Disconnected spaces. Finally, we present two results pertaining to unbounded, proper, and disconnected metric spaces.

The standard examples of disconnected, isometrically homogeneous, and invertible metric spaces are given by the boundaries at infinity of non-rooted regular trees. Indeed, let $T_N$ denote the $(N+1)$-regular tree equipped with the path distance for which each edge has length 1. The metric space $T_N$ is CAT($-1$). Furthermore, there is a notion of geodesic inversion on it, and in this sense $T_N$ is a sort of non-Riemannian symmetric space. Therefore, the parabolic visual boundary $C_N := \partial_\infty T_N$, equipped with the parabolic visual distance $\rho_s$ of parameter $s$ (see Section 5.1) is disconnected, (2-point) isometrically homogeneous, and invertible. Moreover, it is an ultrametric space. We refer the reader to [BS17], for example, for more information about the boundaries of trees.

In parallel with Theorem 1.2, one might expect that the spaces $(C_N, \rho_s)$, with $N \geq 2$ and $s > 1$, are in some sense the only unbounded and disconnected metric spaces possessing the above homogeneity and invertibility properties. The following theorem tells us that this is indeed the case, up to bi-Lipschitz homeomorphisms. We refer the reader to Section 5 for relevant definitions.

Theorem 1.11. Suppose $X$ is a disconnected, unbounded, locally compact, and isometrically homogeneous metric space. If $X$ is invertible, then there exists a positive integer $N \geq 2$ and $s > 1$ such that $X$ is bi-Lipschitz homeomorphic to $(C_N, \rho_s)$.

Theorem 1.11 is sharp in the sense that, in general, a space $X$ satisfying the assumptions of Theorem 1.11 might not be isometric to any $(C_N, \rho_s)$. This is demonstrated in Example 5.3. Furthermore, in parallel with Corollary 1.3 we have the following characterization.

Theorem 1.12. Suppose $X$ is a disconnected, unbounded, and locally compact metric space. There exists a positive integer $N \geq 2$ and $s > 1$ such that $X$ is isometric to $(C_N, \rho_s)$ if and only if $X$ is three-point Möbius homogeneous.

In the next (and last) theorem, we demonstrate that the structure of the boundary of a tree can still be recovered under the weaker assumptions of uniform bi-Lipschitz homogeneity and quasi-invertibility, at least up to quasi-Möbius homeomorphisms.
**Theorem 1.13.** Suppose $X$ is a disconnected, unbounded, locally compact, and uniformly bi-Lipschitz homogeneous metric space. If $X$ is quasi-invertible, then $X$ is quasi-Möbius homeomorphic to $(C_2, \rho_2)$. 

1.1.4. **Structure of the paper.** The remainder of the Introduction provides terminology and notation for use throughout the paper. In Section 2, we investigate the metric geometry of space that are both isometrically homogeneous and invertible. We also study certain metric Lie groups, and provide a proof of Theorem 1.2. In Section 3, we use results of Montgomery-Zippin pertaining to the structure of locally compact groups to prove Theorem 1.4 and Corollary 1.5. In Section 4, we study spaces that are uniformly bi-Lipschitz homogeneous and quasi-invertible. In particular, we investigate the relationship between quasi-invertibility and quasi-dilation invariance. We also prove Propositions 1.6 and 1.7. Before proving Theorem 1.8, we provide additional characterizations of quasi-invertibility under the assumption of uniform bi-Lipschitz homogeneity in Proposition 4.7. In Section 5, we prove a dichotomy between connectedness and uniform disconnectedness for uniformly bi-Lipschitz homogeneous and quasi-invertible spaces (see Lemma 5.1). Then we illustrate examples of disconnected homogeneous invertible spaces. Finally, we prove Theorems 1.11, 1.12, and 1.13.

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1.2. **Terminology.** We now explain the terminology used in this introduction. In this paper, given a point $p$ in a set $X$, we define

$$X_p := X \setminus \{p\} \quad \text{and} \quad \hat{X} := X \cup \{\infty\}.$$
Thus $\hat{X}_p = (X \cup \{\infty\}) \setminus \{p\}$.

We also make use of the following standard metric-space notation. Given $x \in X$ and $r > 0$, we write $B(x; r)$ to denote the open ball $\{z \in X \mid d(x, z) < r\}$ centered at $x$ of radius $r$. Given $R \geq r$, we write $A(x; r, R)$ to denote the open annulus $\{z \in X \mid r < d(x, z) < R\}$ centered at $x$. Given a subset $U$ of a topological space, we write $\overline{U}$ to denote its topological closure.

1.2.1. Inversions, sphericalizations, and Möbius maps. We say that a metric space $X$ is invertible provided it is unbounded and it admits an inversion at some point $p \in X$. That is, there exists a homeomorphism $\tau_p : X_p \to X_p$ such that, for $x, y \in X_p$, we have

$$d(\tau_p(x), \tau_p(y)) = \frac{d(x, y)}{d(x, p) d(y, p)}.$$

Inversions are closely related to the concept of the inverted space of $X$ at $p$, denoted as $\text{Inv}_p(X)$. This inverted space $\text{Inv}_p(X)$ is given by $(\hat{X}_p, i_p)$, where $i_p$ is the quasi-distance defined by

$$(1.1) \quad i_p(x, y) := \frac{d(x, y)}{d(x, p) d(y, p)} \quad \text{and} \quad i_p(x, \infty) := \frac{1}{d(x, p)}, \quad \text{for } x, y \in X_p.$$

Here we use the term quasi-distance to describe a positive definite and symmetric function $\delta$ on a product $Z \times Z$ such that, for any $x, y, z \in Z$, we have $\delta(x, y) \leq C(\delta(x, z) + \delta(z, y))$; see [BHX08, Section 3.8] for further discussion of the quasi-distance $i_p$. Quasi-distances are sometimes referred to as quasi-metrics in the literature; thus we refer to $\text{Inv}_p(X)$ as a quasi-metric space.

Following [FS11], we say that a metric space $X$ is a Ptolemy space if Ptolemy’s inequality holds. That is, for all $w \in X$, the function $i_p$ from (1.1) is a distance; i.e., it satisfies the triangle inequality (cf. [FS11] Remark 2.6). Observe that an inversion $\tau_p$ is an isometry from $\text{Inv}_p(X)$ onto $X$, where we take $\tau_p(\infty) = p$. In particular, the existence of an inversion on $X$ implies that $X$ is a Ptolemy space.

Given a point $p \in X$, and $x, y \in X$, we define

$$(1.2) \quad s_p(x, y) := \frac{d(x, y)}{(1 + d(x, p))(1 + d(y, p))} \quad \text{and} \quad s_p(x, \infty) := \frac{1}{1 + d(x, p)},$$

and we call $\text{Sph}_p(X) = (\hat{X}, s_p)$ the sphericalized space of $X$ at $p$. In general, the function $s_p$ is a quasi-distance. The topology induced by $s_p$ on $\hat{X}$ agrees with the one-point compactification of $X$ when $X$ is non-compact and proper. As in the case of $\text{Inv}_p(X)$, it is straightforward to verify that when $X$ is a Ptolemy space the function $s_p$ satisfies the triangle inequality and so $\text{Sph}_p(X)$ is a metric space.

A homeomorphism $f : X \to X$ between (quasi-)metric spaces is said to be Möbius provided that, for all quadruples $(a, b, c, d)$ of distinct points in $X$, we have

$$\frac{d(f(a), f(c)) d(f(b), f(d))}{d(f(a), f(d)) d(f(b), f(c))} = \frac{d(a, c) d(b, d)}{d(a, d) d(b, c)}.$$

We denote the group of all Möbius self-homeomorphisms of a space $X$ with the notation $\text{Möb}(X)$. We remark that, for any $p \in X$, we have

$$\text{Möb}(X \cup \{\infty\}) = \text{Möb}(\text{Sph}_p(X)) = \text{Möb}(\text{Inv}_p(X) \cup \{p\}).$$

A metric space $X$ is 2-point Möbius homogeneous if for every two pairs $\{x, y\}$ and $\{a, b\}$ of distinct points in $X$ there exists a Möbius self-homeomorphism of $X$ for which $f(x) = a$ and $f(y) = b$.

1.2.2. Quasi-inversions, quasi-sphericalizations, quasi-dilations, and quasi-Möbius maps. In the sequel we shall use the symbol $a \simeq_L b$ to mean $L^{-1}b \leq a \leq Lb$ for some constant $L \geq 1$.

A homeomorphism $f : X \to Y$ is $L$-bi-Lipschitz, for some $L \geq 1$, if, for any $x, y \in X$, we have $d(f(x), f(y)) \geq L d(x, y)$. We say that a metric space $X$ is uniformly bi-Lipschitz homogeneous if there exists $L \geq 1$ such that for any $x, y \in X$, there exists an $L$-bi-Lipschitz homeomorphism $f : X \to X$ for which $f(x) = y$. 
We say that a metric space $X$ is quasi-invertible provided it admits a quasi-inversion at some point $p \in X$. That is, there exists $L \geq 1$ and a homeomorphism $\sigma_p : X_p \to X_p$ such that, for $x, y \in X_p$, we have

$$d(\sigma_p(x), \sigma_p(y)) \simeq_L \frac{d(x, y)}{d(x, p)d(y, p)}.$$ 

Quasi-inversions $\sigma_p$ are closely related to the notion of the quasi-inverted space of $X$ at $p$, denoted by $\text{inv}_p(X)$, which is the metric space $(\hat{X}_p, \hat{d}_p)$, where $\hat{d}_p$ is a distance satisfying

$$(1.3) \quad \frac{1}{4} i_p(x, y) \leq \hat{d}_p(x, y) \leq i_p(x, y).$$

See [BH08] for the construction of such a distance (this notion is referred to as flattening in [LS15]). This distance can be continuously extended to $\hat{X}_p$, and one can use the triangle inequality to verify that, for any point $x \in X$, one has $d_p(x, \infty) = 1/d(x, p)$.

The quasi-sphericalized space of $X$ at $p$ is denoted by $\text{sph}_p(X)$. This is the metric space $(\hat{X}, \hat{d}_p)$, where $\hat{d}_p$ is a distance satisfying

$$(1.4) \quad \frac{1}{4} s_p(x, y) \leq \hat{d}_p(x, y) \leq s_p(x, y).$$

We again refer the reader to [BH08] or [LS15] for the construction of such a distance. This distance can be continuously extended to $\hat{X}$ such that, for $x \in X$, we have $d_p(x, \infty) = 1/(1 + d(x, p))$.

Given $q \in X$, $\lambda > 0$, and $L \geq 1$, a homeomorphism $f : (X, d) \to (X, d)$ is said to be a $(\lambda, L)$-quasi-dilation at $q$ provided that $f(q) = q$ and, for all $x, y \in X$,

$$d(f(x), f(y)) \simeq_L \lambda d(x, y).$$

In particular, a $(\lambda, 1)$-quasi-dilation is a dilation of factor $\lambda$. A metric space $X$ is uniformly quasi-dilation invariant at $q$ provided that there exists $L \geq 1$ such that for all $\lambda > 0$ the space $X$ admits a $(\lambda, L)$-quasi-dilation at $q$.

Given a homeomorphism $\theta : [0, +\infty) \to [0, +\infty)$, a homeomorphism $f : X \to Y$ between metric spaces is said to be a $\theta$-quasi-Möbius map provided that, for all quadruples of distinct points $a, b, c, d \in X$, we have

$$\frac{d(f(a), f(c)) d(f(b), f(d))}{d(f(a), f(d)) d(f(b), f(c))} \leq \theta \left( \frac{d(a, c) d(b, d)}{d(a, d) d(b, c)} \right).$$

When there exists $C \geq 1$ such that $\theta(t) = Ct$, then we say that $f$ is a $C$-strongly quasi-Möbius map. A metric space $X$ is said to be 2-point uniformly strongly quasi-Möbius homogeneous provided there exists $C \geq 1$ such that, given any two pairs $\{x, y\}$ and $\{a, b\}$ of distinct points in $X$, there exists a $C$-strongly quasi-Möbius self-homeomorphism of $X$ for which $f(x) = a$ and $f(y) = b$.

1.2.3. Additional terminology. Given any metric space $(X, d)$ and $\alpha \in (0, 1]$, the $\alpha$-snowflake of $(X, d)$ is the metric space $(X, d^\alpha)$. A metric space $(X, d)$ is called an $\alpha$-snowflake if it is isometric to an $\alpha$-snowflake of a metric space, or, equivalently, if $d^{1/\alpha}$ satisfies the triangle inequality. When we want to emphasize that $\alpha < 1$, we say that $X$ is a non-trivial snowflake.

Given $L \geq 1$, a space $X$ is said to be $L$-quasi-convex if, for any two points $x, y \in X$, there exists a rectifiable curve $\gamma$ joining $x$ to $y$ satisfying $\text{Length}(\gamma) \leq Ld(x, y)$. Such a curve $\gamma$ is said to be an $L$-quasicontinuous curve. Given $z \in X$, if every pair of points in the annulus $A(z; r, 2r)$ can be joined by an $L$-quasi-convex curve contained in $A(z; r/L, 2Lr)$, then we say that $X$ is $L$-annularly quasi-convex at $z$. If $X$ is $L$-annularly quasi-convex at every point, we say that $X$ itself is $L$-annularly quasi-convex. We remark that if a space is annularly quasi-convex, then it is linearly locally connected (in the sense of [BK02]) and it is quasi-convex. Moreover, every proper quasi-convex space is bi-Lipschitz equivalent to a geodesic space.

A metric space $X$ is said to be linearly locally connected if there exists a constant $C \geq 1$ such that the following two conditions are satisfied:
(LLC$_1$) For any $x \in X$, $r > 0$, and points $u, v \in B(x; r)$, there exists a continuum $E \subset B(x; Cr)$ containing $u$ and $v$.

(MLC$_2$) For any $x \in X$, $r > 0$, and points $u, v \in X \setminus B(x; r)$, there exists a continuum $E \subset X \setminus B(x; r/C)$ containing $u$ and $v$.

We say that $(X, d)$ is $C$-uniformly perfect, for some $C \geq 1$, provided that, for every $x \in X$ and $r > 0$ such that $B(x; r) \subseteq X$, we have $B(x; r) \setminus B(x; r/C) \neq \emptyset$.

2. Isometric Homogeneity and Invertibility

In this section we prove Theorem 1.2. We begin with a discussion of relevant definitions in connection with certain isometrically homogeneous metric spaces and metric Lie groups, respectively.

2.1. Isometrically homogeneous metric spaces. In this subsection, we prove a few useful results about metric spaces that are both isometrically homogeneous and invertible.

Proposition 2.1. If $X$ is isometrically homogeneous and invertible, then

1. for any $p \in X$, the space $Sph_p(X)$ is 2-point Möbius homogeneous, and
2. for any $x, y \in X$, the space $X$ admits a dilation of factor $d(x, y)^2$ at $p$.

Proof. To prove (1), let $\tau_p$ be an inversion at some $p \in X$. We show that every point $(a, b) \in (Sph_p(X) \times Sph_p(X)) \setminus \Delta$ can be mapped to $(\infty, p)$. Here $\Delta \subset Sph_p(X) \times Sph_p(X)$ denotes the diagonal. If $a = \infty$, then one uses an element in $\text{Isom}(X) \subset \text{Möb}(X) \subset \text{Möb}(Sph_p(X))$ mapping $b$ to $p$. If $a \neq \infty$, then one first uses an element in $\text{Isom}(X)$ mapping $a$ to $p$, then the map $\tau_p \in \text{Möb}(Sph_p(X))$, to be back in the case $a = \infty$.

To prove (2), let $\tau_p$ denote an inversion at $p \in X$. Choose $x \in X_p$, and define the map $g : X \to X$ as $g := f_3 \circ \tau_p \circ f_2 \circ \tau_p \circ f_1 \circ \tau_p$. Here $f_1 : X \to X$ is an isometry such that $f_1(p) = x$, $f_2 : X \to X$ is an isometry such that $f_2(\tau_p(x)) = p$, and $f_3 : X \to X$ is an isometry such that $f_3(\tau_p(f_2(p))) = p$. We then observe that $g(p) = p$, and that, for any $a, b \in X$, we have

$$d(g(a), g(b)) = \frac{d(\tau_p(f_1(\tau_p(a))), \tau_p(f_1(\tau_p(b))))}{d(f_2(\tau_p(f_1(\tau_p(a)))), p) d(f_2(\tau_p(f_1(\tau_p(b)))), p)}$$

$$= \frac{d(f_2(\tau_p(f_1(\tau_p(a)))), f_2(\tau_p(x))) d(f_2(\tau_p(f_1(\tau_p(b)))), f_2(\tau_p(x)))}{d(\tau_p(a), \tau_p(b))}$$

$$= \frac{d(f_1(\tau_p(a)), p) d(f_1(\tau_p(b)), p)}{d(\tau_p(a), \tau_p(b))} \cdot \frac{d(f_1(\tau_p(a)), x)}{d(\tau_p(a), x)} \cdot \frac{d(f_1(\tau_p(b)), p)}{d(\tau_p(b), x)}$$

$$= \frac{d(a, b) d(x, p)}{d(a, p) d(b, p)} \cdot \frac{d(a, p)}{d(a, b)}$$

$$= d(x, p)^2 d(a, b)$$

Thus $g : (X, d) \to (X, d)$ is a dilation of factor $d(x, p)^2$ at $p$. \qed

In the following lemma, we say that a bijection $h : X \to X$ is a similarity provided that there exists $\lambda > 0$ such that, for any $x, y \in X$, we have $d(f(x), f(y)) = \lambda d(x, y)$. Hence, within this paper, the difference between a similarity and a dilation is that the latter requires the presence of a fixed point while the former does not. Although the following result is contained in the proof of Proposition 1.7, it is included to provide a convenient reference. For a similar result, the reader is pointed to [PS17] Proposition 2.4.

Lemma 2.2. Suppose $X$ and $Y$ are unbounded. If $h : X \to Y$ is a Möbius homeomorphism such that both $h$ and $h^{-1}$ send bounded sets to bounded sets, then $h : X \to Y$ is a similarity.
Proposition 2.4. Suppose $Sph$ the space $h$ such that $x, y \in Sph$ is M"obius and both $h$ and $h^{-1}$ send bounded sets to bounded sets.

**Remark 2.3.** As consequence of Lemma 2.2 we note that, for unbounded spaces $X$ and $Y$, any M"obius homeomorphism $h : Sph_p(X) \to Sph_p(Y)$ fixing $\infty$ is a similarity map from $X$ to $Y$. Indeed, $Sph_p(X) \setminus \{\infty\}$ and $Sph_p(Y) \setminus \{\infty\}$ are M"obius equivalent to $X$ and $Y$, respectively. Therefore, $h : X \to Y$ is M"obius and for some $c > 0$, the space $(X, c d)$ is invertible.

**Proof.** We first prove that $X$ is isometrically homogeneous. By Remark 2.3 every M"obius map $h : Sph_p(X) \to Sph_p(X)$ fixing $\infty$ yields a map $h : X \to X$ that is a $\lambda$-similarity for some $\lambda = \lambda(h) > 0$. Therefore, given any $x, y \in X$, our assumptions on $Sph_p(X)$ ensure the existence of a $\lambda$-similarity $h : X \to X$ such that $h(x) = y$. If $\lambda = 1$ we have an isometry of $X$ sending $x$ to $y$. If $\lambda \neq 1$, then, since $X$ is complete, by the Banach Fixed Point Theorem, there exists $o \in X$ such that $h(o) = o$. Again invoking our assumptions on $Sph_p(X)$ and Remark 2.3 there exists a $\mu$-similarity $g : X \to X$ such that $g(o) = y$. Here $\mu = \mu(g) > 0$. We claim that the composition $g \circ h \circ g^{-1} \circ h$ is an isometry of $X$ that sends $x$ to $y$. Indeed, such a map is a similarity of factor $\mu \lambda^{-1} \mu^{-1} = 1$, and

$$g(h^{-1}(g^{-1}(h(x)))) = g(h^{-1}(g^{-1}(y))) = g(h^{-1}(o)) = g(o) = y.$$  

Since $x, y \in X$ were arbitrary, it follows that $X$ is isometrically homogeneous.

Next, we prove that $X$ admits an inversion, up to a rescaling of its distance. To this end, let $f : Sph_p(X) \to Sph_p(X)$ denote a M"obius map such that $f(p) = \infty$ and $f(\infty) = p$. Then, for any $a, b \in X$ such that $a \neq b$, we have

$$d(p, f(a)) = s_p(p, f(a))(1 + d(p, f(a)))$$

$$= s_p(p, f(a))s_p(\infty, f(b))(1 + d(p, f(a)))$$

$$= s_p(\infty, a)s_p(p, b) \frac{1 + d(p, f(a))}{s_p(\infty, f(b))} s_p(p, f(b))s_p(f(a), \infty)$$

$$= \frac{1}{d(a, p)} \frac{(1 + d(p, f(a)))(1 + d(a, p))(1 + d(b, p))}{d(a, p)(1 + d(a, p))(1 + d(b, p))} \frac{1 + d(f(b), p)}{1 + d(f(b), p)}$$

$$= \frac{d(p, b)}{d(a, p)}.$$  

Since the above equalities hold for any $b \neq a$ in $X$, we conclude that there exists a constant $r > 0$ such that, for any $b \in X$, we have

$$(2.1) \quad d(f(b), p) = r \cdot d(b, p)^{-1}.$$  

Now let $a, b \in X$, be such that $a \neq b$. Using the same function $f$ as above, we observe that

$$d(f(a), f(b)) = s_p(f(a), f(b))(1 + d(f(a), p))(1 + d(f(b), p))$$

$$= s_p(f(a), f(b))s_p(f(p), f(\infty))(1 + d(f(a), p))(1 + d(f(b), p))$$

$$= s_p(a, b)s_p(p, \infty) s_p(a, \infty)s_p(b, \infty)s_p(f(a), p)s_p(f(b), \infty)(1 + d(f(a), p))(1 + d(f(b), p))$$

$$= \frac{d(a, b)d(f(a), p)}{d(b, p)} \frac{(1 + d(f(a), p))(1 + d(f(b), p))(1 + d(a, p))(1 + d(b, p))}{d(a, p)(1 + d(a, p))(1 + d(b, p))}$$

$$= \frac{r \cdot d(a, b)}{d(a, p)d(b, p)}.$$  

Here we note that the final equality follows from (2.1).
Set $c := 1/\sqrt{r}$. Therefore the last formula becomes
\[
c d(f(a), f(b)) = \frac{1}{\sqrt{r}} d(f(a), f(b)) = \frac{\sqrt{r}}{d(a, p) d(b, p)} d(a, b) = \frac{c d(a, b)}{c d(a, p) c d(b, p)}.
\]

Therefore, $f$ satisfies the definition of an inversion for $(X, c d)$. \hfill \Box

2.2. **Metric lie groups.** For the purposes of this paper, we refer to Lie groups equipped with left-invariant distance functions that induce the manifold topology as metric Lie groups. Thus our terminology aligns with that of [CKLD+17]. Important examples of metric Lie groups are provided by groups referred to as generalized Heisenberg groups (as in [Fre14]) or K-Heisenberg groups (as in [PS17]). Here $K$ denotes a real normed division algebra: either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, or the octonions $\mathbb{O}$. These groups can be defined as follows.

- Given $n \in \mathbb{N}$, the $n$-th R-Heisenberg group, or a real Heisenberg group $\mathbb{H}_R^n$, is $\mathbb{R}^n$.
- Given $n \in \mathbb{N}$, the $n$-th C-Heisenberg group, or a complex Heisenberg group $\mathbb{H}_C^n$, is the Carnot group with step two real Lie algebra $\mathfrak{n} = \mathbb{C} \oplus \mathfrak{j}$, where $\mathfrak{j} := \text{Span}\{X_i : 1 \leq i \leq n\}$. Equip $\mathfrak{n}$ with an inner product such that $\langle X_i, Y_i \rangle = 1$ for $1 \leq i \leq n$ is an orthonormal basis. The only non-trivial bracket relations are $[X_i, Y_i] = Z_i = [W_i, Y_i]$, for $1 \leq i \leq n$.
- Given $n \in \mathbb{N}$, the $n$-th H-Heisenberg group, or a quaternionic Heisenberg group $\mathbb{H}_Q^n$, is the Carnot group with step two real Lie algebra $\mathfrak{n} = \mathbb{H} \oplus \mathfrak{j}$, where $\mathfrak{j} := \text{Span}\{X_i, Y_i, W_i, Z_i : 1 \leq i \leq n, 1 \leq k \leq 3\}$. Equip $\mathfrak{n}$ with an inner product such that $\langle X_i, Y_i, W_i, Z_i \rangle = 1$ for $1 \leq i \leq n, 1 \leq k \leq 3$ is an orthonormal basis. The only non-trivial bracket relations are $[X_i, Y_i] = Z_i = [W_i, Y_i]$, $[X_i, W_i] = Z_3 = [Y_i, W_i]$.
- The O-Heisenberg group, or the octonionic Heisenberg group $\mathbb{H}_O^n$, is the Carnot group with step two real Lie algebra $\mathfrak{n} = \mathbb{O} \oplus \mathfrak{j}$, where $\mathfrak{j} := \text{Span}\{X_i, Y_i, W_i, Z_i : 1 \leq i \leq n, 1 \leq k \leq 7\}$. Equip $\mathfrak{n}$ with an inner product such that $\langle X_i, Z_k \rangle = 0$ for $1 \leq i \leq n, 1 \leq k \leq 7$ is an orthonormal basis. The only non-trivial bracket relations are $[X_i, Y_j] = Z_k$ for $1 \leq k \leq 7$ and $[X_i, X_j] = \epsilon_{ijk} Z_k$, for $1 \leq i, j, k \leq 7$. Here $\epsilon$ is a completely antisymmetric tensor whose value is $+1$ when $ijk = 124, 137, 156, 235, 267, 346, 457$.

For our purposes, it is sufficient to define K-hyperbolic space via the results of [CDKR98] and [CDKR91]. In particular, we may view the K-hyperbolic spaces as the rank-one symmetric spaces of non-compact type, and thus the K-Heisenberg groups described above can be viewed as the boundaries at infinity of the K-hyperbolic spaces. For more detailed information about K-Heisenberg groups in relation to K-hyperbolic space the reader may consult [Pla13].

Given $(x, z) = \exp(X + Z) \in \mathbb{H}_K^n$, where $X \in \mathfrak{v}$ and $Z \in \mathfrak{j}$, we define
\[
\| (x, z) \| := \left( \frac{|X|^4}{16} + |Z|^2 \right)^{1/4}.
\]
Here $| \cdot |$ denotes the norm obtained from the inner product on $\mathfrak{n}$ described above. We then define the **parabolic visual distance** $\rho$ on $\mathbb{H}_K^n$ as
\[
(2.2) \quad \rho((x, z), (x', z')) := \| (x', z')^{-1} (x, z) \|.
\]
This distance (or a rescaling thereof) is sometimes referred to as the Korányi-Cygan distance, or simply the Korányi distance (cf. [CDPT07, page 18]). Via the exponential map, an inversion of the metric Lie group $(\mathbb{H}_K^n, \rho)$ is given by
\[
(2.3) \quad \sigma(X, Z) := - \left( \frac{|X|^2}{4} + J_Z \right)^{-1} X - \left( \frac{|X|^4}{16} + |Z|^2 \right)^{-1} Z.
\]
Here $J : z \to \operatorname{End}(v)$ is defined via the formula $\langle J_z X, Y \rangle = \langle Z, [X, Y] \rangle$. See \cite{CDKR91} for a detailed treatment of the map $\sigma$.

One of the primary theoretical tools we shall employ in the proof of Theorem 1.2 is provided by the following version of results from \cite{Kra03}.

**Fact 2.5.** Suppose $G$ is a locally compact and $\sigma$-compact topological group acting continuously, effectively, and 2-transitively on the sphere $S^m$. In this case, $G$ can be given the structure of a Lie group and the identity component $G^0$ is simple, non-compact, and of real rank 1. Furthermore, the action of $G$ on $S^m$ is isomorphic to the action of $G_K$ or $G_K^0$ on the (compact) boundary at infinity of $K$-hyperbolic space. Here $G_K$ denotes the isometry group of $K$-hyperbolic space: If $K = \mathbb{R}$, then $G_K = PO(n, 1)$ for $m = n - 1 \geq 1$. If $K = \mathbb{C}$, then $G_K = PU(n, 1) \times \mathbb{Z}_2$ for $m = 2n - 1$. If $K = H$, then $G_K = PSp(n, 1)$ for $m = 4n - 1$. If $K = O$, then $G_K = F_{4(-20)}$ for $m = 15$.

The above fact follows immediately from \cite[Theorem 3.3(a)]{Kra03} and \cite[Proposition 7.1]{Kra03}. Indeed, by \cite[Theorem 3.3(a)]{Kra03}, we conclude that $G$ is a Lie group with simple and non-compact connected component. Furthermore, $G$ is isomorphic to either $G_K$ or $G_K^0$ for some $K \in \{\mathbb{R}, \mathbb{C}, H, O\}$, as described above. We also point out \cite[Proposition 7.1]{Kra03}, which affirms that the action of $G$ on $S^m$ is the standard action of $G_K$ on the boundary of its corresponding symmetric space, namely $H_K/B$. Here $H_K$ denotes $G_K^0$ and $B = NA M$, where $H_K = NAK$ is the Iwasawa decomposition of $H_K$. $M$ is the centralizer of $A$ in $K$, and $N$ is isomorphic to $\mathbb{H}^n_R$.

Another tool we employ in the proof of Theorem 1.2 is the following version of results from \cite{PS17}.

**Theorem 2.6.** Suppose $d$ is a metric on $\mathbb{H}^n_R$ such that both $d$ and $\rho$ induce the same topology. If $\operatorname{Möb}(\operatorname{Sp}(\mathbb{H}^n_R, \rho))^0 \subset \operatorname{Möb}(\operatorname{Sp}(\mathbb{H}^n_R, d))^0$, then there exists $c > 0$ and $0 < \alpha \leq 1$ such that $d = c \cdot \rho^\alpha$.

**Proof.** We note that orientation-preserving similarity mappings of $(\mathbb{H}^n_R, \rho)$ are contained in the identity component $\operatorname{Möb}(\operatorname{Sp}(\mathbb{H}^n_R, \rho))^0$. Therefore, when $K = \mathbb{R}$, we reach the desired conclusion via \cite[Theorem 1.1(a)]{PS17}. In the cases that $K \neq \mathbb{R}$, we note that the inversion $\sigma$ defined in (2.2) is contained in $\operatorname{Möb}(\operatorname{Sp}(\mathbb{H}^n_R, \rho))^0$. Via \cite[Theorem 1.2]{PS17}, we are done. \hfill \Box

In connection with Theorem 2.6, we point out that the norm utilized in the present paper to define the visual distance on $K$-Heisenberg groups differs slightly from the norm defined in \cite[page 358]{PS17}. This is due to a different choice of coordinates for $\mathbb{H}^n_R$. Nevertheless, up to corresponding alterations in the definition of the canonical inversion map (compare \cite[page 363]{PS17} with (2.3)), the proofs of \cite{PS17} yield Theorem 2.6.

**Remark 2.7.** Suppose $X$ is a proper and connected metric space. As a consequence of \cite[Theorem 1.4]{CKLD+17}, we find that if the action of $\operatorname{Möb}(X)$ on $X$ is transitive and not proper, then $X$ has the structure of self-similar metric Lie group in the sense of \cite{CKLD+17}. Indeed, by Lemma 2.2, the group $\operatorname{Möb}(X)$ is precisely the group of similarities. Therefore, if its action is not proper, then $\operatorname{Möb}(X)$ must contain a similarity that is not an isometry.

**2.3. Proof of Theorem 1.2** Before beginning the proof of Theorem 1.2 we first prove a lemma regarding compactness properties of $\operatorname{Möb}(\operatorname{Sp}(X))$, where we remind the reader that, in general, $\operatorname{Sp}(X)$ is a quasi-metric space when equipped with the quasi-distance $s_p$.

If $X$ is an unbounded, proper (i.e., boundedly compact), and connected metric space, then the topology induced on $X$ by the distance $d_p$ from (1.3) coincides with that of the one-point compactification of $X$. Since the distance $d_p$ is bi-Lipschitz equivalent to the quasi-distance $s_p$ on $X$, this topology coincides with the topology on $X$ generated by open balls with respect to $s_p$. Thus we may speak of continuous self-mappings of $\operatorname{Sp}(X)$ with respect to this topology. We then define a quasi-distance

$$s^*_p(f, g) := \sup \{ s_p(f(x), g(x)) \mid x \in \bar{X} \}$$

on the set of continuous mappings of the quasi-metric space $\operatorname{Sp}(X)$. We refer to the topology induced on $\operatorname{Möb}(\operatorname{Sp}(X))$ (a group of continuous mappings of $\operatorname{Sp}(X)$) by the quasi-distance $s^*_p$ as
the topology of uniform convergence. It is straightforward to check that the action of \( \text{M"ob}(\text{Sph}_p(X)) \) on \( \text{Sph}_p(X) \) is a continuous action with respect to these topologies.

Recall from Section 2.2.2 that the quasi-metric space \( \text{Sph}_p(X) \) is bi-Lipschitz equivalent to the metric space \( \text{sph}_p(X) \) via the identity. Hence, the group \( G := \text{M"ob}(\text{Sph}_p(X)) \) acts on the metric space \( \text{sph}_p(X) \) by uniformly strongly quasi-M"obius mappings. Furthermore, the topology of uniform convergence induced on \( G \) by \( s_p \) coincides with the topology of uniform convergence induced on \( G \) by the distance \( d_p \). Via [Fre14, Lemma 4.4], this last observation yields the following lemma.

**Lemma 2.8.** Given an unbounded, proper, and connected metric space \( X \), the group \( \text{M"ob}(\text{Sph}_p(X)) \) is locally compact and \( \sigma \)-compact in the topology of uniform convergence.

**Proof of Theorem 1.2** We prove (4) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (1).

We begin with (4) \( \Leftrightarrow \) (3). By Proposition 2.4 assuming (4) for \( (X,d) \) implies that \( (X,d) \) is isometrically homogeneous and \( (X,c\rho) \) admits an inversion \( \tau \), where \( c \) is some positive constant. Furthermore, by Proposition 2.4 there exists a dilation \( f \) of \( (X,d) \) at \( p \in X \) with dilation factor \( c^{-2} \). We then observe that \( \tau \circ f \) is an inversion of \( (X_p,\rho) \), where \( \tau \) denotes an inversion of \( (X_p,\rho) \) at \( p \). Therefore, (4) \( \Rightarrow \) (3). To see that (3) \( \Rightarrow \) (4), we first note that the combination of isometric homogeneity and local compactness implies that \( X \) is complete. To confirm that \( \text{M"ob}(X) \) acts 2-transitively on \( X \), we refer to Proposition 2.1.

We now prove the main implication (3) \( \Rightarrow \) (2). We claim that \( X \) admits dilations of all factors at \( p \). Indeed, fixing \( y \in X \), the distance function \( x \in X \mapsto d(x,y) \in [0,\infty) \) is continuous and unbounded, since \( X \) is assumed unbounded. Thus \( \Lambda := \{d(x,y) : x \in X\} \) is a closed and unbounded set that contains 0. Since \( X \) is connected, \( \Lambda = [0,\infty) \). By Proposition 2.6, our claim is verified.

Since \( X \) is assumed to be connected and locally compact, by [CKLD’17, Theorem 1.4], we conclude that \( X \) may be given the structure of a metric Lie group for which every dilation fixing the identity element is an automorphism. It then follows from results in [Ste86] that \( X \) is nilpotent and simply connected. In particular, the space \( X \) is homeomorphic to \( \mathbb{R}^n \), for some \( n \in \mathbb{N} \). In addition, since \( X \) is locally compact and admits dilations, it is proper. Consequently, the one-point compactification of \( X \) coincides with the topology of \( \text{Sph}_p(X) \) induced by the quasi-distance \( s_p \).

Therefore, the space \( \text{Sph}_p(X) \) is homeomorphic to the topological sphere \( S^n \). Via Lemma 2.8, we know that \( G = \text{M"ob}(\text{Sph}_p(X)) \) is locally compact and \( \sigma \)-compact. Since \( G \) acts continuously, effectively, and 2-transitively on the topological sphere \( \text{Sph}_p(X) \), by Fact 2.5 we conclude that the action of \( G \) on \( \text{Sph}_p(X) \) is isomorphic to the standard action of either \( G_K \) or \( G_{\mathbb{R}^n} \) on the (compact) boundary at infinity of some \( K \)-hyperbolic space. We recognize this boundary as \( \mathbb{H}^n_{\mathbb{R}}, \) for some \( n \in \mathbb{N} \) via the approach of [CDKR98, and we emphasize that \( G_K \) acts by M"obius mappings on \( \mathbb{H}^n_{\mathbb{R}} \). Thus we identify \( X \) with \( \mathbb{H}^n_{\mathbb{R}} \), identifying \( p \) with the identity element \( e \) of \( \mathbb{H}^n_{\mathbb{R}} \) and \( \infty \) with \( \infty \). Also, we identify the action of \( G \) on \( \text{Sph}_p(\mathbb{H}^n_{\mathbb{R}},d) \) with the action of either \( G_K \) or \( G_{\mathbb{R}^n} \) on \( \text{Sph}_p(\mathbb{H}^n_{\mathbb{R}},\rho) \). In particular, we have

\[
\text{M"ob}(\text{Sph}_p(\mathbb{H}^n_{\mathbb{R}},\rho))^{\circ} \subset \text{M"ob}(\text{Sph}_p(\mathbb{H}^n_{\mathbb{R}},d)).
\]

By Theorem 2.4, we have \( d = c\rho^\alpha \) for some \( c > 0 \) and \( \alpha \in (0,1] \). We conclude by noticing that any dilation (with respect to \( \rho \)) of factor \( c^{1/\alpha} \) provides an isometry from \( (\mathbb{H}^n_{\mathbb{R}},d) \) to \( (\mathbb{H}^n_{\mathbb{R}},\rho^\alpha) \), and thus (3) \( \Rightarrow \) (2).

We next prove (2) \( \Rightarrow \) (3). Clearly, \( (\mathbb{H}^n_{\mathbb{R}},\rho) \) is isometrically homogeneous (since the distance \( \rho \) is left-invariant) and invertible (since it is the boundary of a symmetric space); see [CDKR98] for these classical facts. It is clear that the same is true of its snowflakes.

Finally, we prove (2) \( \Leftrightarrow \) (1). The implication (2) \( \Rightarrow \) (1) is trivial, since the identity map from \( (X,d) \) to \( (X,c\rho) \) is a Möbius homeomorphism. Conversely, suppose that \( f : (\mathbb{H}^n_{\mathbb{R}},\rho^\alpha) \to X \) is a Möbius homeomorphism. For any \( s > 0 \), write \( \delta_s : \mathbb{H}^n_{\mathbb{R}} \to \mathbb{H}^n_{\mathbb{R}} \) to denote the standard automorphic dilation of factor \( s \) with respect to the distance \( \rho \). Given a point \( z_0 \in \mathbb{H}^n_{\mathbb{R}} \) such that \( \rho(e,z_0) = 1, \)
write $\gamma(z_0)$ to denote the closure of the curve

$$\gamma'(z_0) := \{d_s(z_0) \mid s > 0\}.$$  

Thus $\gamma(z_0) = \gamma'(z_0) \cup \{e\}$, where $e$ denotes the identity element of $\mathbb{H}_K^n$. Since $\mathbb{H}_K^n = \cup_{d(z, z')}=1 \gamma(z)$, $\mathbb{H}_K^n$ is proper, $X$ is unbounded, and $f$ is a homeomorphism, we may assume that $f(\gamma(z_0))$ is unbounded.

Write $p := f(e) \in X$. We claim that $d(p, f(\delta_t(z_0))) \to +\infty$ as $t \to +\infty$. Indeed, since $\mathbb{H}_K^n$ is proper and $f(\gamma(z_0))$ is unbounded, there exists a sequence of real numbers $(t_n)_{n \in \mathbb{N}}$ such that $t_n \to +\infty$ and $d(p, f(\delta_{t_n}(z_0))) \to +\infty$. Suppose, by way of contradiction, that there also exists a sequence of real numbers $(s_n)_{n \in \mathbb{N}}$ such that $s_n \to +\infty$ and $\{f(\delta_{s_n}(z_0))\}$ is bounded. Up to a subsequence, we may assume that

$$t_n \geq s_n.$$  

Since $f$ is Möbius and $\rho(e, z_0) = 1$, we have

$$\frac{\rho(\delta_{s_n/t_n}(z_0), z_0)}{\rho(z_0, \delta_{s_n}(z_0))} = \frac{\rho(\delta_{s_n}(z_0), \delta_{t_n}(z_0))}{t_n\rho(z_0, \delta_{s_n}(z_0))} = \frac{\rho(e, z_0) \rho(\delta_{s_n}(z_0), \delta_{t_n}(z_0))}{\rho(e, \delta_{t_n}(z_0)) \rho(z_0, \delta_{s_n}(z_0))} = \frac{d(p, f(z_0)) d(f(\delta_{s_n}(z_0)), f(\delta_{t_n}(z_0)))}{d(p, f(\delta_{s_n}(z_0))) d(f(z_0), f(\delta_{t_n}(z_0)))}.$$  

Since $\{f(\delta_{s_n}(z_0))\}$ is bounded, it is straightforward to verify via the triangle inequality that there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, we have

$$\frac{d(p, f(z_0)) d(f(\delta_{s_n}(z_0)), f(\delta_{t_n}(z_0)))}{d(p, f(\delta_{s_n}(z_0))) d(f(z_0), f(\delta_{t_n}(z_0)))} \leq 2 \frac{d(p, f(z_0))}{d(f(z_0), f(\delta_{t_n}(z_0)))}.$$  

Since $f^{-1}$ is continuous, there exists $c > 0$ such that, for all $n \geq N$, we have $d(f(z_0)), f(\delta_{s_n}(z_0)) \geq c$. Therefore, for all $n \geq N$, we have

$$\frac{d(p, f(z_0))}{d(f(z_0), f(\delta_{s_n}(z_0)))} \in [C^{-1}, C],$$  

for some $C \in (1, +\infty)$. By combining (2.4), (2.5), and (2.6), for any $n \geq N$, we have

$$\rho(z_0, \delta_{s_n}(z_0)) \leq 2C \rho(z_0, \delta_{s_n/t_n}(z_0)) \leq C'$$  

for some $C' < +\infty$. The constant $C'$ arises from the fact that $\{\delta_s(z_0) \mid s \in (0, 1] \cup \{e\}$ is compact. This inequality contradicts the fact that $s_n \to +\infty$. From this contradiction it follows that $d(p, f(\delta_t(z_0))) \to +\infty$ as $t \to +\infty$.

Choose $\lambda > 1$. We claim that the homeomorphism $g_\lambda := f \circ \delta_\lambda \circ f^{-1} : X \to X$ is a dilation of $X$ at $p$. To verify this claim, write $z_n := f(\delta_{\lambda}^n(z_0))$, for $n \in \mathbb{N}$. We then note that $g_\lambda(z_n) = f(\delta_{\lambda}^{n+1}(z_0)) = z_{n+1}$. By the previous paragraph, both $d(p, z_n) \to +\infty$ and $d(p, g_\lambda(z_n)) \to +\infty$ as $n \to +\infty$. Since $g_\lambda$ is a Möbius map, for any $x, y \in X$, we have

$$\frac{d(g_\lambda(x), g_\lambda(y)) d(g_\lambda(z_n), g_\lambda(p))}{d(g_\lambda(x), g_\lambda(p)) d(g_\lambda(z_n), g_\lambda(y))} = \frac{d(x, y) d(z_n, p)}{d(x, p) d(z_n, y)}.$$  

Taking a limit as $n \to +\infty$, we obtain

$$\frac{d(g_\lambda(x), g_\lambda(y))}{d(g_\lambda(x), g_\lambda(p))} = \frac{d(x, y)}{d(x, p)}.$$  

Here we note that $g_\lambda(p) = p$, and thus we have

$$\frac{d(g_\lambda(x), g_\lambda(p))}{d(x, y)} = \frac{d(g_\lambda(x), p)}{d(x, p)}.$$
The left side of this equality is symmetric in the variables $x$ and $y$. The right side is independent of $y$. Therefore, we conclude that there exists some number $\beta > 0$ such that $d(g_\lambda(x), g_\lambda(y)) = \beta d(x, y)$. Thus our claim is verified.

Next, we claim that $\beta > 1$. Indeed, for every $n \in \mathbb{N}$, we have

$$d(p, z_n) = d(p, g_n^\alpha(z_0)) = \beta^n d(p, z_0).$$

Since $d(p, z_n) \to +\infty$, we conclude that $\beta > 1$.

Since $X$ is locally compact and admits a dilation of factor $\beta > 1$, it is straightforward to verify that $X$ is proper. Therefore, any homeomorphism between $X$ and $\mathbb{H}_n^\alpha$ preserves bounded sets. The implication $(1) \Rightarrow (2)$ then follows from Lemma 2.2. Indeed, by Lemma 2.2 there exists a constant $c > 0$ such that the Möbius homeomorphism $f^{-1} : X \to (\mathbb{H}_n^\alpha, c \cdot \rho^\alpha)$ is an isometry. Since $(\mathbb{H}_n^\alpha, \rho^\alpha)$ is dilation invariant, we conclude that $(1) \Rightarrow (2)$. □

3. Möbius Homogeneity and Strong Connectivity

This section is devoted to the proof of Theorem 1.4 and Corollary 1.5. The arguments are heavily based on Montgomery-Zippin results about the structure of locally compact groups.

3.1. Proof of Theorem 1.4

We first use theory pertaining to Hilbert’s Fifth Problem to show that, in the setting of Theorem 1.4, the space of Möbius transformations has the structure of a Lie group. As usual, it is topologized via uniform convergence.

Proposition 3.1 (After Montgomery-Zippin). Let $Z$ be a compact, connected, locally connected metric space of finite topological dimension. If Möb($Z$) acts transitively on $Z$, then Möb($Z$) is a Lie group.

Proof. Since $Z$ is compact, $G := \text{Möb}(Z)$ is a separable, locally compact, and metrizable group. Moreover, the standard action $G \times Z \to Z$ is continuous and effective. Following [MZ74] page 238, the locally compact group $G$ has an open subgroup $G' < G$ that is the inverse limit of Lie groups. In the language of [MZ74], $G'$ has property A.

First, we claim that, for any $q \in Z$, the orbit of $q$ under $G'$, denoted by $G' \cdot q$, is open. This is because the projection $G \to G/H$ is open and the orbit action $G/H \to Z$ is a homeomorphism (see [Hel01] page 121, Theorem 3.2). Here $H$ denotes the isotropy subgroup of $G$ at $q$.

Now we show that the $G'$-action is transitive. Indeed, fix a point $p \in Z$, and suppose (by way of contradiction) that $G' \cdot p \neq Z$. Hence,

$$Z = (G' \cdot p) \bigcup \left( \bigcup_{q \in G' \cdot p} G' \cdot q \right)$$

is a disjoint union of two non-empty open sets of $Z$. This contradicts the fact that $Z$ is connected.

Thus $G'$ satisfies the hypotheses of Montgomery-Zippin’s Theorem [MZ74] page 243, so $G'$ is a Lie group. Since $G'$ does not contain small subgroups, neither does $G$. By work of Gleason-Yamabe (cf. [MZ74] Chapter III), $G$ is a Lie group. □

Proof of Theorem 1.4 Since $Z$ is quasi-convex, it is connected and locally connected. Therefore, by Proposition 3.1, we conclude that $G := \text{Möb}(Z)$ is a Lie group. Since the action of $G$ on $Z$ is transitive, the space $Z$ is a manifold homeomorphic to $G/H$ for some closed subgroup $H \subset G$.

Since $Z$ is quasi-convex and compact, up to a bi-Lipschitz change of distance we can assume that the distance $d_Z$ of $Z$ is geodesic. Also, since $Z$ is compact, every Möbius homeomorphism is bi-Lipschitz (see [Kin15] Remark 3.2). Thus $G$ acts on $Z$ by bi-Lipschitz maps. By [LD11] Theorem 1.1 there exists a completely non-holonomic $G$-invariant distribution on $Z$ such that any Carnot-Carathéodory metric coming from it gives a metric that is locally bi-Lipschitz equivalent to $d_Z$. Since $Z$ is compact, the bi-Lipschitz equivalence is global. □
3.2. Möbius circles and Möbius-homogeneity. A metric space $X$ is said to be connected by Möbius circles if, for any $p, q \in X$, there exists a Möbius embedding $\gamma : S^1 \to X$ such that $p, q \in \gamma(S^1)$. Here $S^1 \subset \mathbb{R}^2$ denotes the unit circle. The following lemma confirms that our definition is consistent with the definition of Möbius circles used in [BS14, Section 2.4].

**Lemma 3.2.** Let $S$ be a subset of a metric space $X$. The following are equivalent

- $S$ is the image of a Möbius embedding of $S^1$.
- $S$ is the closure of the image of a Möbius embedding of $\mathbb{R}$.
- $S$ is homeomorphic to $S^1$ and, for every $x, y, z, u$ in order along $S$ we have
  \[ d(x, z)d(y, u) = d(x, y)d(z, u) + d(x, u)d(y, z). \]

**Proof.** The first two characterization are a consequence of the fact that $\mathbb{R}$ and $S^1$ are Möbius equivalent (up to compactification). Observe that the equation of the lemma is equivalent to

\[ 1 = \frac{d(x, y)d(z, u)}{d(x, z)d(y, u)} + \frac{d(x, u)d(y, z)}{d(x, z)d(y, u)}, \]

and the right-hand side is the sum of two cross ratios. Hence it is a Möbius invariant.

Let $\gamma : S^1 \to X$ be a Möbius embedding with $S = \gamma(S^1)$. Fix consecutive points $x, y, z, u$ along $S$ (here the order is inherited from $S^1$). Let $x', y', z', u'$ be the respective points in $S^1$. Up to a Möbius transformation, we may assume that $x' = 0$, $y' = 1$, $z' = c > 1$, and $u' = \infty$. Under this transformation equation \[ (3.1) \]
becomes $c^{-1} + (c - 1)c^{-1} = 1$, which is true.

Conversely, assume points of $S = \gamma(S^1)$ satisfy \[ (3.1) \], where $\gamma$ is some embedding. Fix $u \in S$ and consider the quasi-metric space $\text{Inv}_u(X)$. In $\text{Inv}_u(X)$ equation \[ (3.1) \] becomes $i_u(x, z) = i_u(x, y) + i_u(y, z)$. Hence the curve $\gamma \setminus \{u\}$ is an infinite geodesic in $\text{Inv}_u(X)$, and thus isometric to $\mathbb{R}$. Since $\text{Inv}_u(X) \setminus \{\infty\}$ is Möbius equivalent to $X_u$, we confirm that $S$ is the closure of the image of a Möbius embedding of $\mathbb{R}$. \hfill $\Box$

Before proceeding to the proof of Corollary 1.5 we first demonstrate that Ptolemy spaces connected by Möbius circles are quasi-convex. This fact (and its proof) was suggested to the authors by V. Schroeder.

**Proposition 3.3.** If $X$ is a Ptolemy space that is connected by Möbius circles, then $X$ is K-quasi-convex, for some universal constant $K \leq 144$.

**Proof.** Fix $p, q \in X$. Let $C$ be a Möbius circle through $p$ and $q$. We consider two cases. Either (1) $D := \text{diam}(C) \leq 6d(p, q)$, or (2) $D > 6d(p, q)$.

Case 1: $D \leq 6d(p, q)$. By continuity, choose a point $w \in C$ for which $d(p, w) = d(w, q)$. The triangle inequality yields $2d(w, q) \geq d(p, q)$. Let $\gamma$ denote the sub-arc of $C \setminus \{p, q\}$ that does not contain $w$ and joins $p$ to $q$.

We claim that $\text{Length}_{d}(\gamma) \leq Kd(p, q)$, for $K = 144$. Indeed, since $X$ is assumed to be Ptolemy, we consider the metric space $\text{Inv}_w(X)$. In this space, the set $C \setminus \{w\}$ is an infinite geodesic Möbius equivalent to $\mathbb{R}$ (see Lemma 3.2). Therefore,

\[ \text{Length}_{i_w}(\gamma) = i_w(p, q) = \frac{d(p, q)}{d(p, w)d(q, w)} = \frac{d(p, q)}{d(w, q)^2} \leq \frac{4d(p, q)}{d(p, q)^2} = \frac{4}{d(p, q)}. \]

For $x, y \in \gamma \subset C$, we have

\[ d(x, y) = d(x, w)d(y, w)i_w(x, y) \leq D^2i_w(x, y). \]

Therefore, since we are in the case that $D \leq 6d$, we conclude that

\[ \text{Length}_{d}(\gamma) \leq 36d(p, q)^2\text{Length}_{i_w}(\gamma) \leq \frac{4}{d(p, q)}36d(p, q)^2 = Kd(p, q). \]
Case 2: $D \geq 6d(p, q)$. We claim that by continuity there is a point $w \in C$ such that $d(p, w) = D/3$. If not, we would have $C \subset B(p; D/3)$, and thus arrive at the contradiction $D \leq 2D/3 < D$. Thus we fix $w \in C$ such that $d(p, q) = D/3$. Via the assumption that $D \geq 6d(p, q)$, we have

$$d(q, w) \geq d(p, w) - d(p, q) = \frac{D}{3} - d(p, q) \geq \frac{D}{3} - \frac{D}{6} = \frac{D}{3}.$$ 

As in Case 1, let $\gamma$ denote the sub-arc of $C$ not containing $w$ and joining $p$ to $q$. Then

$$\text{Length}_{i_w}(\gamma) = \frac{d(p, q)}{d(p, w)d(q, w)} \leq \frac{d(p, q)}{(D/3)(D/3)} = \frac{9d(p, q)}{D^2}.$$ 

As before, for $x, y \in \gamma$, we have $d(x, y) \leq D^2i_w(x, y)$. Therefore, we conclude that

$$\text{Length}_d(\gamma) \leq D^2\text{Length}_{i_w}(\gamma) \leq 9d(p, q).$$

□

3.3. Proof of Corollary 1.5. Given a pointed metric space $(X, d, o)$ one considers the visual function

$$\rho^o_d(x, y) = \exp(-(x, y)_o),$$

where $(x, y)_o$ denotes the Gromov product in $(X, d)$. Bourdon proved in [Bou95] that, on every CAT($-1$) space $X$, the function $\rho^o_d$ satisfies the triangle inequality and the visual boundaries $(\partial\infty X, \rho^o_d)$ corresponding to different base points $o, o' \in X$ are Möbius equivalent. Thus we refer to $\rho^o_d$ as the Bourdon distance, based at $o$.

In [Ham89] Hamenstädt studied similar distances where the point $o$ is replaced with a point in the boundary. We refer to such distances as Hamenstädt distances. In [FS11, BS07], simple arguments are presented which demonstrate that these distances are Möbius equivalent.

Proof of Corollary 1.5. Suppose the boundary $X$ of a CAT($-1$)-space endowed with a Bourdon distance $d$ is bi-Lipschitz homeomorphic to a sub-Riemannian manifold. If $X$ is equipped with a Hamenstädt distance $d'$, then $(X, d')$ is locally uniformly bi-Lipschitz homeomorphic to a sub-Riemannian manifold. Up to changing the sub-Riemannian metric, we conclude that this bi-Lipschitz equivalence is global.

By [FS11, Theorem 1], if $X$ is the boundary of a CAT($-1$)-space endowed with a Bourdon distance, then $X$ is a Ptolemy space. By Proposition 3.3, the space $X$ is quasi-convex. We then obtain the desired conclusion via Theorem 1.4. □

4. Bi-Lipschitz Homogeneity and Quasi-Invertibility

4.1. Quasi-inversions and quasi-dilation invariance. In this subsection we prepare for the proof of Proposition 1.2 by investigating the relationship between quasi-inversions and quasi-dilations in a uniformly bi-Lipschitz homogeneous metric space. The reader can find the definitions of these terms along with the definition of quasi-dilation invariance in Section 1.2.2. The definition of uniform perfectness is provided in Section 1.2.3.

Lemma 4.1. Suppose $X$ is a uniformly $L$-bi-Lipschitz homogeneous metric space. If there exists a point $p \in X$ at which $X$ is $M$-quasi-invertible, then, for any $x \in X_p$, the space $X$ admits a $(C, r)$-quasi-dilation at $p$, where $r = d(x, p)^2$ and $C = C(L, M)$. Furthermore:

1. If $X$ is $N$-uniformly perfect, then $X$ is $K$-quasi-dilation invariant, with $K = K(L, M, N)$.
2. If $X$ is connected, then $X$ is $K$-quasi-dilation invariant, with $K = K(L, M)$.

Proof. Let $\sigma_p$ denote an $M$-quasi-inversion of $X$ at $p$. Choose $x \in X_p$, and define the map $g : X \to X$ as $g := f_3 \circ \sigma_p \circ f_2 \circ \sigma_p \circ f_1 \circ \sigma_p$. Here $f_1 : X \to X$ is an $L$-bi-Lipschitz map such that $f_1(p) = x$, $f_2 : X \to X$ is an $L$-bi-Lipschitz map such that $f_2(\sigma_p(x)) = p$, and $f_3 : X \to X$ is an $L$-bi-Lipschitz
map such that \( f_2(\sigma_p(f_2(p))) = p \). We then observe that \( g(p) = p \), and that, for any \( a, b \in X \), we have

\[
d(g(a), g(b)) \simeq \frac{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))} \frac{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))} \cdot \frac{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}
\]

\[
= \frac{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))} \cdot \frac{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))} \cdot \frac{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}
\]

\[
\simeq \frac{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))} \cdot \frac{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}{d(\sigma_p(f_1(\sigma_p(a))), \sigma_p(f_1(\sigma_p(b))))}
\]

Thus \( g : X \to X \) is a \((K, d(x, p)^2)\)-quasi-dilation at \( p \), where \( K = K(L, M) \).

To verify (2), assume that \( X \) is connected. It follows that, for all \( r > 0 \), there exists \( x \in X \) such that \( d(x, p)^2 = r \) and a \((K, r)\)-quasi-dilation as constructed above.

To verify (1), assume that \( X \) is \( N \)-uniformly perfect. By definition, for all \( r > 0 \), there exists a point \( x \in X \) such that \( \sqrt{2r}/N \leq d(x, p) < \sqrt{2r} \). Therefore, there exists a \((K, s)\)-quasi-dilation \( f : X \to X \) as constructed above, where \( 2r/N^2 \leq s < 2r \). Thus, for every \( a, b \in X \), we have

\[
\frac{2r}{KN^2}d(a, b) \leq \frac{s}{K}d(a, b) \leq d(f(a), f(b)) \leq Ksd(a, b) \leq 2Krd(a, b).
\]

Therefore, \( f : X \to X \) is a \((2KN^2, r)\)-quasi-dilation. \( \square \)

We distinguish between the connected and disconnected cases in Lemma 4.1 in order to clarify quantitative dependence of the conclusions on the parameters pertaining to the assumptions. In a qualitative sense, a space \( X \) satisfying the assumptions of Lemma 4.1 is always uniformly perfect. This is the content of the following lemma.

**Lemma 4.2.** Suppose that \( X \) is an unbounded metric space. If \( X \) is \( L \)-uniformly bi-Lipschitz homogeneous and \( M \)-quasi-invertible, then \( X \) is uniformly perfect and, in particular, it has no isolated points.

**Proof.** First we prove that \( X \) does not contain any isolated points. Let \( \sigma_p : X_p \to X_p \) denote a quasi-inversion at \( p \in X \), and let \((x_i)_{i=0}^{+\infty} \) denote a sequence of points in \( X \) such that \( d(p, x_i) \to +\infty \). It follows from the definition of a quasi-inversion that \( d(\sigma_p(x_i), p) \to 0 \). Therefore, we conclude that \( p \) is not an isolated point of \( X \). By uniform bi-Lipschitz homogeneity, no point of \( X \) is isolated.

Suppose \( X \) is not uniformly perfect. Via uniform bi-Lipschitz homogeneity, we can assume that there exist positive numbers \( r_k > 0 \) and \( C_k \to +\infty \) such that, for each \( k \in \mathbb{N} \), we have

\[
(4.1) \quad A(p; r_k, C_k r_k) = \emptyset.
\]

If there exists \( 1 \leq C < +\infty \) such that, for all \( k \in \mathbb{N} \), we have \( r_k \in [C^{-1}, C] \), then \( X \) is bounded, in contradiction to the assumption that \( X \) is unbounded. Therefore, we may assume that there exists a subsequence of \((r_k)_{k \in \mathbb{N}}\) either converging to 0 or diverging to \( +\infty \). If there exists a subsequence \( r_{n_k} \to +\infty \), then we may use the \( M \)-quasi-inversion at \( p \) to ensure that, for each \( k \in \mathbb{N} \), we have

\[
A(p; M(C_{n_k} r_{n_k})^{-1}, (M r_{n_k})^{-1}) = \emptyset.
\]

By the above paragraph, we may assume that there exist sequences \( r_k \to 0 \) and \( C_k \to +\infty \) such that, for each \( k \in \mathbb{N} \), we have \((4.1) \). Since \( X \) is unbounded and contains no isolated points, we may assume that these empty annuli are maximal in the sense that there exist \( x_k, y_k \in X \) such that \( d(p, x_k) = r_k \) and \( d(p, y_k) = C_k r_k \). Therefore, up to a subsequence, for each \( k \in \mathbb{N} \) we have \( C_{k+1} r_{k+1} \leq r_k \), and so \( C_k r_k \to 0 \). Fix \( x \in X \) such that \( r := d(p, x) \) satisfies \( r^2 > L \). Note that this
is possible because $X$ is unbounded. By Lemma 4.1 there exists an $(L, r^2)$-quasi-dilation $f : X \to X$ at $p$. Therefore, for every $k \in \mathbb{N}$, we have

$$A(p; Lr^2 r_k, L^{-1} C k r^2 r_k) = \emptyset.$$ 

Since $L^{-1} r^2 > 1$ and $d(p, y_k) = C k r_k$, it follows that, for every $k \in \mathbb{N}$, we have $L r^2 > C k$. Since $C k \to +\infty$, this is a contradiction. This contradiction reveals that $X$ must be uniformly perfect. □

4.2. Proof of Propositions 1.6 and 1.7

Remark 4.3. The inverse of a $\theta$-quasi-Möbius map is $\theta'$-quasi-Möbius, where $\theta'(t) = \theta^{-1}(t^{-1})^{-1}$ (see [Vä85, pg. 219]). Therefore, for use below, we remark that the inverse of a $C$-strongly quasi-Möbius map is $C$-strongly quasi-Möbius.

Proof of Proposition 1.6. We first prove sufficiency. To verify that $X$ is uniformly bi-Lipschitz homogeneous, we proceed as in Proposition 4.3. Let $h : \text{sph}_p(X) \to \text{sph}_p(X)$ denote a $C$-strongly quasi-Möbius map such that $h(\infty) = \infty$. We claim that $h$ is a quasi-similarity mapping of $X$. In other words, there exists $L = L(C) > 0$ such that, for any $a, b \in X$, we have

$$d(h(a), h(b)) \approx_L \lambda d(a, b).$$

To verify this claim, let $a, b, c \in X$ be a triple of distinct points. Then we have

$$d(h(a), h(b)) \approx_4 \hat{d}_p(h(a), h(b))(1 + d(h(a), p))(1 + d(h(b), p))$$

$$= \frac{\hat{d}_p(h(a), h(b)) \hat{d}_p(h(c), \infty)}{\hat{d}_p(h(c), \infty)} (1 + d(h(a), p))(1 + d(h(b), p))$$

$$\approx_C \frac{\hat{d}_p(a, b) \hat{d}_p(c, \infty) (1 + d(h(a), p))(1 + d(h(b), p))}{\hat{d}_p(a, \infty) \hat{d}_p(b, c) \hat{d}_p(h(c), \infty)} \hat{d}_p(h(a), \infty) \hat{d}_p(h(b), h(c))$$

$$\approx_4 \frac{d(h(a), h(b))}{d(a, b)}.$$ 

Here we have used Remark 4.3 and omitted some of the straightforward calculations. Since the above comparability statements hold for any triple of distinct points $a, b, c \in X$, we conclude that $d(h(b), h(c)) \approx_L \lambda d(b, c)$ for some $L = L(C)$ and $\lambda > 0$. Therefore, any $C$-strongly quasi-Möbius map of $\text{sph}_p(X)$ fixing $\infty$ is quasi-similarity mapping of $X$.

Given any $a \in X$, let $h : \text{sph}_p(X) \to \text{sph}_p(X)$ denote a $C$-strongly quasi-Möbius map fixing $\infty$ such that $h(a) = p$. Let $\lambda > 0$ and $L = L(C)$ denote the corresponding constants such that, for any $x, y \in X$, we have $d(h(x), h(y)) \approx_L \lambda d(x, y)$. If $L^{-1} \leq \lambda \leq L$, then we conclude that $h$ is $L^2$-bi-Lipschitz. If $\lambda < L^{-1}$ (or $\lambda > L$) then $h$ (or $h^{-1}$) is a strict contraction mapping $X$ to itself. Since $X$ is proper, it is complete. Therefore, by the Banach Fixed Point theorem, there exists a point $o \in X$ such that $h(o) = o$. Now let $g : \text{sph}_p(X) \to \text{sph}_p(X)$ denote a $C$-strongly quasi-Möbius map fixing $\infty$ and sending $o$ to $p$. Write $\mu > 0$ and $M = M(C)$ to denote constants such that, for any $x, y \in X$, we have $d(g(x), g(y)) \approx_M \mu d(x, y)$. We consider the map $g \circ h^{-1} \circ g^{-1} \circ h$. First, we note that this map sends $a$ to $p$. Then, we note that this map is $(ML)^2$-bi-Lipschitz. It follows that $X$ is uniformly bi-Lipschitz homemogeneous.

Next, we demonstrate that $X$ admits a quasi-inversion. To this end, let $f : \text{sph}_p(X) \to \text{sph}_p(X)$ denote a $C$-strongly quasi-Möbius map such that $f(p) = \infty$ and $f(\infty) = p$. Then, for any $a, b \in X_p$,
such that $a \neq b$, we have

$$
\begin{align*}
    d(p, f(a)) & \simeq_4 \frac{\hat{d}_p(p, f(a))(1 + d(p, f(a)))}{\hat{d}_p(\infty, f(b))} \\
    & = \frac{\hat{d}_p(p, f(a))\hat{d}_p(\infty, f(b))(1 + d(p, f(a)))}{\hat{d}_p(\infty, f(b))} \\
    & \simeq_C \frac{\hat{d}_p(\infty, a)\hat{d}_p(p, b) + d(p, f(a))}{\hat{d}_p(\infty, b)\hat{d}_p(a, p)} \\
    & \simeq_4 \sqrt{\frac{1}{\hat{d}(a, p)}} \frac{(1 + d(p, f(a)))(1 + d(a, p))(1 + d(b, p))(1 + d(f(b), p))}{\hat{d}(p, b)\hat{d}(p, f(b))} \\
    & = \frac{\hat{d}(a, b)\hat{d}(f(a), p)(1 + d(a, p))(1 + d(b, p))}{\hat{d}(b, p)}. \\
\end{align*}
$$

The above statement again utilizes Remark 4.3. Since the above comparabilities hold for any $b \neq a$ in $X$, we conclude that there exist constants $L = L(C)$ and $r > 0$ such that, for any $b \in X$, we have

$$
(4.2) 
\quad d(f(b), p) \simeq_L r \cdot d(p, b)^{-1}.
$$

Now let $a, b \in X_p$ be such that $a \neq b$. Using the same function $f$ as above, we observe that

$$
\begin{align*}
    d(f(a), f(b)) & \simeq_4 \frac{\hat{d}_p(\infty, \hat{d}_p(p, f(a))(1 + d(f(a), p))(1 + d(f(b), p))}{\hat{d}_p(\infty, \hat{d}_p(p, f(b)))} \\
    & = \frac{\hat{d}_p(a, b)\hat{d}_p(p, f(a))\hat{d}_p(\infty, \hat{d}_p(p, f(b)))}{\hat{d}_p(a, \infty)\hat{d}_p(b, p)} \\
    & \simeq_4 \sqrt{\frac{d(a, b)\hat{d}(f(a), p)(1 + d(a, p))(1 + d(b, p))}{\hat{d}(b, p)}} \\
    & \simeq_L \sqrt{\frac{\hat{d}(a, b)}{\hat{d}(b, p)}}, \\
\end{align*}
$$

where the final comparison follows from (4.2). Therefore, $f$ is a quasi-inversion of $X$.

To prove necessity, we assume that $X$ is uniformly $L$-bi-Lipschitz homogeneous and admits a $K$-quasi-inversion at some point $p \in X$. To confirm that $\text{sph}_p(X)$ is 2-point uniformly strongly quasi-Möbius homogeneous, we mimic the proof of Proposition 2.1. Given $p \in X$, let $\sigma_p$ denote a $K$-quasi-inversion of $X$ at $p$. We show that every point $(a, b) \in (\text{sph}_p(X) \times \text{sph}_p(X)) \setminus \Delta$ can be mapped to $(\infty, p)$ via a uniformly strongly quasi-Möbius map of $\text{sph}_p(X)$. If $a = \infty$, then simply map $p$ to $p$ via an $L$-bi-Lipschitz map of $X$. Here we note that any $L$-bi-Lipschitz map of $X$ is an $L^4$-strongly quasi-Möbius map of $\text{sph}_p(X)$. If $a \neq \infty$, then we map $a$ to $p$ via an $L$-bi-Lipschitz map of $X$ before applying $\sigma_p$. This composition is an $(L)K^4$-strongly quasi-Möbius map of $\text{sph}_p(X)$. Thus we return to the case that $a = \infty$. \hfill \Box

Proof of Proposition 4.7 Assume $f : X \to Y$ is $L$-bi-Lipschitz, then $f$ is $L^4$-strongly quasi-Möbius. Furthermore, we note that if $f$ is a similarity mapping, then $f$ is Möbius.

Conversely, assume that $h : X \to Y$ is $C$-strongly quasi-Möbius. We first claim that $h$ extends homeomorphically to $h : \hat{X} \to \hat{Y}$ such that $h(\infty) = \infty$. Indeed, because $X$ and $Y$ are proper, both $h$ and $h^{-1}$ must send bounded sets to bounded sets. The claim follows. Therefore, we may view $h$ as a $C$-strongly quasi-Möbius map $h : \text{Sph}_p(X) \to \text{Sph}_q(Y)$ for some points $p \in X$ and $q \in Y$. 

Let $a, b, c \in X$ be a triple of distinct points. We observe that

\[
d(h(a), h(b)) = s_p(h(a), h(b))(1 + d(h(a), p))(1 + d(h(b), p))
\]

\[
= \frac{s_p(h(a), h(b))s_p(h(c), \infty)}{s_p(h(c), \infty)}(1 + d(h(a), p))(1 + d(h(b), p))
\]

\[
\leq C \frac{s_p(a, b)s_p(c, \infty)}{s_p(a, \infty)s_p(b, c)}(1 + d(h(a), p))(1 + d(h(b), p))
\]

\[
= \frac{d(a, b)d(h(b), h(c))}{d(b, c)}(1 + d(a, p))(1 + d(b, p))(1 + d(c, p))
\]

\[
\leq \frac{(1 + d(h(a), p))(1 + d(h(b), p))(1 + d(h(c), p))}{(1 + d(h(a), p))(1 + d(h(b), p))(1 + d(h(c), p))}
\]

\[
= d(a, b) \frac{d(h(b), h(c))}{d(b, c)}.\]

Since the above equalities hold for any triple of distinct points $a, b, c \in X$, we conclude that there exists $\lambda > 0$ such that, for any $a, b \in X$, we have $d(h(a), h(b)) \approx_C \lambda \cdot d(a, b)$. Therefore, $h$ is $(C\lambda)$-bi-Lipschitz. When $C = 1$, the map $h$ is a $\lambda$-similarity. \hfill \square

4.3. Characterizing quasi-invertibility. This subsection records a few useful technical results and culminates in the statement and proof of Proposition 4.7. We begin with the following lemma, which extends [BHX08, Lemma 3.2] in the case of quasi-sphericization.

Lemma 4.4. Let $f : X \to X$ be a homeomorphism of a metric space, and let $p \in X$. If $f$ is $L$-bi-Lipschitz, then $f \circ \text{sph}_p(X) \to \text{sph}_p(X)$ is $C$-bi-Lipschitz, where $C = C(L, d(f(p), p))$.

Proof. Given $a \in X$, we first note that, for $C_0 := L(1 + d(f(p), p))$,

\[
n + d(f(a), p) \leq 1 + d(f(a), f(p)) + d(f(p), p)
\]

\[
\leq 1 + Ld(a, p) + d(f(p), p) \leq C_0(1 + d(a, p)).
\]

Therefore, given two points $a, b \in X$, we have

\[
\hat d_p(f(a), f(b)) \geq \frac{d(a, b)}{4LC_0^2(1 + d(a, p))(1 + d(b, p))} \geq \frac{\hat d_p(a, b)}{4LC_0^2}
\]

To obtain a relevant upper bound on $\hat d_p(f(a), f(b))$, we consider two cases.

Case 1: $d(f(a), p) \leq 1$. In this case, we note that, for $C_1 := (1 + L + d(f(p), p))$,

\[
n + d(a, p) \leq 1 + Ld(a, p) \leq 1 + Ld(f(a), f(p)) \leq 1 + Ld(f(a), p) + d(f(p), p)
\]

\[
\leq 1 + Ld(f(p), p) \leq C_1(1 + d(f(a), p)).
\]

Case 2: $d(f(a), p) > 1$. We consider two subcases. First, suppose that $d(f(a), p) \geq (2L)^{-1}d(a, p)$. Then we note that, for $C_2 := 2L$,

\[
n + d(a, p) \leq 1 + 2Ld(f(a), p) \leq C_2(1 + d(f(a), p)).
\]

Next, suppose that $d(f(a), p) < (2L)^{-1}d(a, p)$. Then we note that

\[
d(f(p), p) \geq d(f(p), f(a)) - d(f(a), p) \geq L^{-1}d(a, p) - d(f(a), p) \geq (2L)^{-1}d(a, p).
\]

Therefore, $d(a, p) \leq 2Ld(f(p), p) \leq 2Ld(f(p), p)d(f(a), p)$, and so, for $C_3 := 2Ld(f(p), p) > 1$,

\[
n + d(a, p) \leq C_3(1 + d(f(a), p)).
\]

Considering Case 1 and Case 2 together, we conclude that, for any two points $a, b \in X$, we have

\[
\hat d_p(f(a), f(b)) \leq \frac{LC_4^2 d(a, b)}{(1 + d(a, p))(1 + d(b, p))} \leq 4LC_4^2 \hat d_p(a, b),
\]

where $C_4 = \max\{C_1, C_2, C_3\}$. Combining (4.3) and (4.4), we reach the desired conclusion. \hfill \square
Lemma 4.5. Suppose $X$ is an $L$-bi-Lipschitz homogeneous metric space. For $p, x \in X$, any $M$-quasi-inversion $\sigma_x : X_x \to X_x$ is a $C$-bi-Lipschitz self-homeomorphism of $\text{sph}_p(X)$, with $C = C(L, M, d(p, x))$. If $p = x$, then we reach the same conclusion with $C = 4M^3$.

**Proof.** Let $x, y \in X$ be given, and fix a point $p \in X$. Then

$$\hat{d}_p(\sigma_p(x), \sigma_p(y)) \leq \frac{d(\sigma_p(x), \sigma_p(y))}{(1 + d(\sigma_p(x), p))(1 + d(\sigma_p(y), p))} \leq M \frac{d(x, y)}{d(p, x) + (M d(p, x))^{-1}} \leq 4M^3 \hat{d}_p(x, y).$$

On the other hand, we have

$$\hat{d}_p(\sigma_p(x), \sigma_p(y)) \geq \frac{d(x, y)}{4M d(p, x) d(p, y) (1 + M d(p, x))^{-1}(1 + M d(p, y))^{-1}} \geq M \frac{d(x, y)}{d(p, x) + M(d(p, y) + M^{-1})} \geq \frac{d_p(x, y)}{4M^3}.$$
To prove (2), let \(d''\) denote the quasi-sphericalized distance \(\hat{d}_p\) on \(\hat{X}\). For any \(x, y \in X\), we have
\[
d''(x, y) \simeq \frac{d_p(x, y)}{(1 + d_p(x, \infty))(1 + d_p(y, \infty))}
\]
which, under \(1.3\), yields
\[
d''(x, y) = \frac{d(x, y)}{d(x, p)d(y, p)(1 + d(x, p)^{-1})(1 + d(y, p)^{-1})}
\]
\[
\simeq \frac{d(x, y)}{(1 + d(x, p))(1 + d(y, p))} \simeq \hat{d}_p(x, y).
\]
If \(y = \infty\), similar calculations reveal that \(d''(x, \infty) \simeq \hat{d}_p(x, \infty)\). □

At this point we are ready to state and prove Proposition 4.7. As stated above, the purpose of this result is to provide equivalent characterizations of quasi-invertibility under the assumption that \(X\) is uniformly bi-Lipschitz homogeneous.

**Proposition 4.7.** Suppose \(X\) is an unbounded and uniformly bi-Lipschitz homogeneous metric space. Given any point \(p \in X\), the following statements are equivalent:

1. \(X\) admits a quasi-inversion at \(p\).
2. \(X\) is bi-Lipschitz equivalent to \(\text{inv}_p(X)\).
3. \(\text{inv}_p(X)\) is uniformly bi-Lipschitz homogeneous.
4. \(\text{sph}_p(X)\) is uniformly bi-Lipschitz homogeneous.

**Proof.** We prove (3) ⇒ (1) ⇒ (4) ⇒ (2) ⇒ (3).

Suppose first that \(\text{inv}_p(X)\) is uniformly bi-Lipschitz homogeneous, and fix some \(q \in X_p\). Let \(f : X \to X\) denote a bi-Lipschitz map such that \(f(p) = q\). Let \(g : \text{inv}_p(X) \to \text{inv}_p(X)\) denote a bi-Lipschitz map such that \(g(q) = \infty\). Lastly, let \(h : X \to X\) denote a bi-Lipschitz map such that \(h(g(\infty)) = p\). We claim that the composition \(h \circ g \circ f : X_p \to X_p\) is a quasi-inversion. Indeed, we first note that \(h(g(f(p))) = \infty\) and \(h(g(f(\infty))) = p\). Furthermore, for any \(x, y \in X\), we have
\[
d(h(g(f(x))), h(g(f(y)))) \simeq d(g(f(x)), g(f(y)))
\]
\[
\simeq \frac{d(f(x), f(y)) d(g(f(x)), p) d(g(f(y)), p)}{d(f(x), p) d(f(y), p)}
\]
\[
\simeq \frac{d(x, y) d(f(x), p) d(f(y), p)}{d(f(x), p) d(f(y), p)}
\]
Here we use (3). We then note that
\[
\frac{1}{d(g(f(x)), p)} = d_p(g(f(x)), \infty) = d_p(g(f(x)), g(q)) \simeq d_p(f(x), q)
\]
\[
d_p(f(x), f(p)) \simeq \frac{d(x, p)}{d(f(x), p) d(q, p)}.
\]
It follows that
\[
d(h(g(f(x))), h(g(f(y)))) \simeq \frac{d(x, y) d(f(x), p) d(f(y), p)d(q, p)^2}{d(x, p) d(y, p) d(f(x), p) d(f(y), p)}
\]
\[
\simeq \frac{d(x, y)}{d(x, p) d(y, p)}.
\]
Here we note that the final comparability depends on the quantity \(d(q, p)\). We also note that our claim regarding \(h \circ g \circ f\) has been verified. Therefore, we conclude that (3) ⇒ (1).

Now we suppose that \(X\) admits an \(M\)-quasi-inversion \(\sigma_p\). We claim there exists \(C \geq 1\) such that any point \(q \in \text{sph}_p(X)\) can be mapped to \(p\) by an \(C\)-bi-Lipschitz self-homeomorphism of \(\text{sph}_p(X)\). To verify this claim, we first assume that \(q \in B(p, 1) \subset X\). Based on the assumption that \(X\) is \(L\)-bi-Lipschitz homogeneous, for some \(L \geq 1\), let \(f : X \to X\) denote an \(L\)-bi-Lipschitz map such
that $f(q) = p$. By Lemma 4.4, we conclude that $f : \text{sph}_p(X) \to \text{sph}_p(X)$ is $K_1$-bi-Lipschitz, where $K_1 = K_1(L, d(f(p), p))$. Since $d(f(p), p) = d(f(p), f(q)) \leq L$, we have $K_1 = K(L)$. Next, we assume that $q \notin B(p; 1)$. Then $q' := \sigma_p(q)$ satisfies $d(p, q') \leq M$. Letting $g : X \to X$ denote an $L$-bi-Lipschitz map such that $g(q') = p$, Lemma 4.4 and Lemma 4.5 allow us to conclude that $g \circ \sigma_p : \text{sph}_p(X) \to \text{sph}_p(X)$ is $K_2$-bi-Lipschitz, with $K_2 = K_2(L, M)$. It follows that $\text{sph}_p(X)$ is $C^2$-bi-Lipschitz homogeneous, with $C = \max\{K_1, K_2\}$. Thus we prove (1) $\Rightarrow$ (4).

Next, suppose $\text{sph}_p(X)$ is uniformly bi-Lipschitz homogeneous. Therefore, there exists a bi-Lipschitz homeomorphism $f : \text{sph}_p(X) \to \text{sph}_p(X)$ such that $f(\infty) = p$. By [BHx08 Lemma 3.2], we conclude that $\text{inv}_\infty(\text{sph}_p(X))$ is bi-Lipschitz homeomorphic to $\text{inv}_p(\text{sph}_p(X))$. By [BHx08 Proposition 3.4], we conclude that $\text{inv}_\infty(\text{sph}_p(X))$ is bi-Lipschitz homeomorphic to $X$, and, by Lemma 4.6(1), we conclude that $\text{inv}_p(\text{sph}_p(X))$ is bi-Lipschitz homeomorphic to $\text{inv}_p(X)$. Thus $X$ is bi-Lipschitz homeomorphic to $\text{inv}_p(X)$, and we establish (4) $\Rightarrow$ (2).

Lastly, we note that (2) $\Rightarrow$ (3) is almost immediate. Indeed, if $X$ is $L$-bi-Lipschitz homogeneous and $M$-bi-Lipschitz equivalent to $\text{inv}_p(X)$, for some numbers $L, M \geq 1$, then $\text{inv}_p(X)$ is $LM^2$-bi-Lipschitz homogeneous. Thus (2) $\Rightarrow$ (3). □

**Remark 4.8.** We note that Proposition 4.7 clarifies the relationship between the assumptions of uniform bi-Lipschitz homogeneity and quasi-invertibility with the terminology inversion invariant bi-Lipschitz homogeneity as used, for example, in [Fre12].

### 4.4. Additional consequences of bi-Lipschitz homogeneity

Given a proper, uniformly bi-Lipschitz homogeneous metric space $X$ and a compact subset $K \subset X$, the next lemma demonstrates that one can map a point $x \in K$ to a point $y \in X$ using a bi-Lipschitz map that almost fixes points of $K$, provided that $x$ and $y$ are near enough to each other.

**Lemma 4.9.** Suppose $X$ is a proper and $L$-bi-Lipschitz homogeneous metric space. For every $x \in X$, $\varepsilon > 0$, and compact set $K \subset X$ containing $x$, there exists $\delta > 0$ such that, for any $y \in B(x; \delta)$ there exists an $L^2$-bi-Lipschitz homeomorphism $h : X \to X$ such that $h(x) = y$ and $\sup_{z \in K} d(h(z), z) < \varepsilon$.

**Proof.** Let $x \in X$, $\varepsilon > 0$, and a compact set $K \subset X$ containing $x$ be fixed. For a given $n \in \mathbb{N}$, set $K_n := \overline{B(x; n)}$, the closure of the ball of radius $n$. Let $(x_m)_{m=1}^{+\infty}$ denote any sequence of points in $X$ such that $d(x_m, x) \to 0$. For each $m$, write $x_{1,m} := x_m$. Suppose there exists a sequence of $L$-bi-Lipschitz homeomorphisms $f_{1,m} : X \to X$ such that $f_{1,m}(x) = x_{1,m}$. Since $X$ is proper, we can assume (up to a subsequence) that $f_{1,m}$ uniformly converges on $K_1$ to an $L$-bi-Lipschitz embedding $f_1 : K_1 \to X$ such that $f_1(x) = x$. Inductively define sequences of points $(x_m, m=1)_{m=1}^{+\infty}$ such that, for $n \geq 2$, each $(x_{n,m})_{m=1}^{+\infty}$ is a subsequence of $(x_{n-1,m})_{m=1}^{+\infty}$. Furthermore, define sequences $(f_{n,m})_{m=1}^{+\infty}$ of $L$-bi-Lipschitz self-homeomorphisms of $X$ such that, for $n \geq 2$, each $(f_{n,m})_{m=1}^{+\infty}$ is a subsequence of $(f_{n-1,m})_{m=1}^{+\infty}$ such that $f_{n,m}(x) = x_{n,m}$. We can also assume that $(f_{n,m})_{m=1}^{+\infty}$ converges uniformly on $K_n$ to an $L$-bi-Lipschitz embedding $f_n : K_n \to X$ such that $f_n(x) = x$. Note also that $f_n = f_{n-1}$ when restricted to $K_{n-1}$. The sequence $(f_n)_{n=1}^{+\infty}$ locally uniformly converges to an $L$-bi-Lipschitz homeomorphism $f : X \to X$ such that $f(x) = x$. Fix $N \in \mathbb{N}$ such that $K \subset K_N$. For $n \geq N$, define $g_n := f_n \circ f^{-1}$. Then $g_n(x) = x_{n,n}$, and $g_n$ uniformly converges to the identity map on $K$.

The above paragraph allows us to conclude that, up to a subsequence, for any sequence of points $x_n \to x$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, there exists an $L^2$-bi-Lipschitz map $g_n : X \to X$ such that $g_n(x) = x_n$ and $\max_{z \in K} d(g_n(z), z) < \varepsilon$. This implies the existence of $\delta > 0$ such that, for any $y \in B(x; \delta)$, there exists an $L^2$-bi-Lipschitz map $h : X \to X$ such that $h(x) = y$ and $\max_{z \in K} d(h(z), z) < \varepsilon$. □

Regarding the next lemma, we recall that a point $x \in X$ is called a strong cut point if $X \setminus \{x\}$ has exactly two connected components.

**Lemma 4.10.** Let $X$ be a proper and $L$-bi-Lipschitz homogeneous metric space. Assume that $X$ is path connected and locally path connected. Then any cut point of $X$ is a strong cut point.
Proof. Suppose $x$ is a cut point of $X$. Suppose, by way of contradiction, that $x$ is not a strong cut point. In other words, suppose there exist three points $z_1$, $z_2$, and $z_3$ in three different connected components $X_1$, $X_2$, and $X_3$ of $X \setminus \{x\}$, respectively. Since $X$ is path connected, there exist curves $\gamma_i$ joining $z_i$ to $x$, for $i = 1$, $2$, $3$, and we may further assume that $\gamma_i := \gamma_i \setminus \{x\}$ is connected. Note that $\gamma_1^*, \gamma_2^*$ and $\gamma_3^*$ are contained in different components of $X \setminus \{x\}$, and are therefore pairwise disjoint. Let $U_2$ and $U_3$ denote path connected neighborhoods of $z_2$ and $z_3$, respectively, that do not contain $x$. Hence, we have $U_2 \subset X_2$ and $U_3 \subset X_3$.

Choose $\varepsilon > 0$ such that $B(z_i; \varepsilon) \subset U_i$, for $i = 2$, $3$. Apply Lemma 4.10 with $K := \{x, z_2, z_3\}$. Thus, there exists $\delta > 0$ such that, for any $y \in \gamma_1 \cap B(x; \delta)$, there exists a $L^2$-bi-Lipschitz homeomorphism $h : X \to X$ such that $h(x) = y$ and $h(z_i) \in B(z_i; \varepsilon)$, for $i = 2$, $3$.

By the construction of $U_2$ and $U_3$, there exist curves $\eta_i \subset U_i$ joining $z_i$ to $h(z_i)$, for $i = 2$, $3$. Therefore, the connected set $\eta_1 := \eta_1 \cup h(\gamma_1) \cup \gamma_1^*$ contains both $z_1$ and $z_2$. Notice that $\eta \cup \eta_2 \cup \gamma_1^*$ does not contain $x$. Therefore $x \in h(\gamma_2) \cap h(\gamma_3) = h(\gamma_2 \cap \gamma_3) = h(x)$. However, $h(x) \neq x$. The contradiction ends the proof. \qed

Our next step is to prove that, given two points $x, y \in X$ along with a compact neighborhood $K$ containing both $x$ and $y$, one can find a map that is bi-Lipschitz on $K$, fixes $x$, and sends $y$ to any point within a small enough neighborhood of $y$.

Lemma 4.11. Suppose $X$ is unbounded, proper, $L$-bi-Lipschitz homogeneous, and $M$-quasi-invertible. Let $x \in X$ and $0 < R < \infty$. There exists $C = C(L, M, R)$ such that, for any $y \in B(x; R) \setminus \{x\}$, there exists $\delta > 0$ such that, for any point $u \in B(y; \delta)$, there exists a homeomorphism $f : \text{sph}_x(X) \to \text{sph}_x(X)$ such that, for any $a, b \in B(x; R)$, we have $d(f(a), f(b)) \leq C \cdot d(a, b)$. Moreover, $f(x) = x$ and $f(y) = u$.

Proof. Fix distinct points $x, y \in X$ and $R > 0$. We claim there exist constants $C = C(L, M) < +\infty$ and $\delta > 0$ such that, for any $y \in B(y; \delta)$, there exists a $C$-bi-Lipschitz homeomorphism of $f : \text{sph}_x(X) \to \text{sph}_x(X)$ such that $f(x) = x$, $f(y) = u$, and $\tilde{d}_x(f(\infty), \infty) \leq (2C(1 + R))^{-1}$.

To verify this claim, choose $\varepsilon \in (0, 1)$ (whose value is to be determined below) and $N \in \mathbb{N}$ such that $x \in K := B(v; N)$, where $v := \gamma(x)$. By Lemma 4.9, the map $\sigma_\varepsilon : \text{sph}_x(X) \to \text{sph}_x(X)$ is $C_1$-bi-Lipschitz, with $C_1 = C_1(M)$. Therefore, for any $a, b \in X$, if $\tilde{d}_x(a, \sigma_\varepsilon(b)) < \varepsilon/C_1$, then $\tilde{d}_x(x, b) < \varepsilon$.

By Lemma 4.9, there exists $\delta_1 > 0$ such that, for any $u \in X$ satisfying $\sigma_\varepsilon(u) \in B(v; \delta_1)$, there exists an $L^2$-bi-Lipschitz homeomorphism $h_u : X \to X$ such that $h_u(v) = \sigma_\varepsilon(u)$ and $\max_{a \in K} d(h_u(a), a) < \varepsilon/C_1$.

In particular, $d(h_u(x), x) < 1$. By Lemma 4.4, we conclude that $h_u : \text{sph}_x(X) \to \text{sph}_x(X)$ is $C_2$-bi-Lipschitz, with $C_2 = C_2(L)$.

For each $u$ such that $\sigma_\varepsilon(u) \in B(v; \delta_1)$, define $g_u := \sigma_\varepsilon^{-1} \circ h_u \circ \sigma_\varepsilon$. Choose $\delta_2 > 0$ small enough to ensure that $\sigma_\varepsilon(B(y; \delta_2)) \subset B(h_u(x); \varepsilon/C_1)$. By the two preceding paragraphs, $\{g_u \mid u \in B(y; \delta_2)\}$ is a collection of uniformly $C_3$-bi-Lipschitz self-homeomorphisms of $\text{sph}_x(X)$, where $C_3 = C_3(L, M)$.

Here we homeomorphically extend $h_u$ such that $h_u(\infty) = \infty$. Thus we have $g_u(x) = x$ and $g_u(y) = u$, and we note that $\tilde{d}_x(h_u(x), x) \leq d(h_u(x), x) < \varepsilon/C_1$. Therefore, $\tilde{d}_x(g_u(\infty), \infty) < \varepsilon$, and, if we choose $\varepsilon = (2C_3(1 + R))^{-1} < 1$, then our claim is verified.

To conclude the proof of the lemma, choose $x \in X \cap R > 0$. Then choose $y \in B(x; R) \setminus \{x\}$. By the above claim, there exist constants $C = C(L, M)$ and $\delta > 0$ such that, for any $u \in B(y; \delta)$, there exists a $C$-bi-Lipschitz homeomorphism $g_u : \text{sph}_x(X) \to \text{sph}_x(X)$ such that $g_u(x) = x$, $g_u(y) = u$, and $\tilde{d}_x(g_u(\infty), \infty) < (2C(1 + R))^{-1}$. Here we may assume that $\delta$ is small enough to ensure that $B(y; \delta) \subset B(x; R)$. For any $a \in B(x; R)$, it follows from the triangle inequality and the properties of $g_u$ that

$$d(g_u(a), x) = \frac{1}{\tilde{d}_x(g_u(a), \infty)} - 1 \leq 2C(1 + R) - 1 \leq 2C(1 + R).$$

(4.5)
Set $C_4 := 2C(1 + R)$. Via (4.3), for any $a, b \in B(x; R)$, we have
\[
d(g_u(a), g_u(b)) \leq 4\tilde{d}_x(g_u(a), g_u(b))(1 + d(g_u(a), x))(1 + d(g_u(b), x)) \\
\leq 4C\tilde{d}_x(a, b)(1 + C_4)^2 \leq 4C(1 + C_4)^2d(a, b).
\]
On the other hand, we have
\[
d(g_u(a), g_u(b)) \geq \frac{d(a, b)}{C} \geq \frac{d(a, b)}{4C(1 + d(a, x))(1 + d(b, x))} \geq \frac{d(a, b)}{4C(1 + R)^2} \geq \frac{d(a, b)}{4C(1 + C_4)^2}.
\]
Defining $C_5 := 4C(1 + C_4)^2, \forall a, b \in B(x; R)$ we have $d(g_u(a), g_u(b)) \simeq_{C_5} d(a, b)$.

4.5. Proof of Theorem 1.8. In this section we prepare for and present the proof of Theorem 1.8.

We begin by establishing a few technical results. The first of these lemmas is of a general nature.

\textbf{Lemma 4.12.} Let $X$ denote a proper metric space. Fix constants $C, R < +\infty$ and a point $x \in X$. If each open ball in $X$ has infinite Hausdorff $1$-measure, then there exists $\delta > 0$ such that, for any rectifiable curve $\gamma \subset X$ such that $\text{Length}(\gamma) \leq C$, there exists $y \in B(x; R)$ such that $B(y; \delta) \cap \gamma = \emptyset$.

\textbf{Proof.} By way of contradiction, suppose that there exists a sequence of positive numbers $\delta_n \to 0$ and a sequence of rectifiable curves $\gamma_n \subset X$ such that, for every $q \in B(x; R)$, we have $B(q; \delta_n) \cap \gamma_n \neq \emptyset$. Furthermore, for every $n \in \mathbb{N}$, we have $\text{Length}(\gamma_n) = C_n \leq C$. For each $n \in \mathbb{N}$, we write $\alpha_n : [0, C] \to \gamma_n$ to denote a parametrization such that $\alpha_n|_{[0, C_n]}$ is an arclength parameterization of $\gamma_n$ and $\alpha_n$ is constant on $[C_n, C]$. Thus each $\alpha_n$ is $1$-Lipschitz. Since $X$ is proper and, for every $n \in \mathbb{N}$, we have $\gamma_n \cap B(x; R) \neq \emptyset$, by Arzela-Ascoli we can assume that (up to a subsequence) the maps $\alpha_n$ are uniformly convergent to a $1$-Lipschitz map $\alpha_\infty : [0, C] \to X$. Write $\gamma_\infty = \alpha_\infty([0, C])$. Thus we have $d_H(\gamma_n, \gamma_\infty) \to 0$, where $d_H$ denotes Hausdorff distance. Let $z \in B(x; R)$. For each $n \in \mathbb{N}$, we have $B(z; \delta_n) \cap \gamma_n \neq \emptyset$. Since $\delta_n \to 0$, it follows that $z \in \gamma_\infty$. Therefore, $B(x; R) \subset \gamma_\infty$. Since $\alpha_\infty : [0, C] \to X$ is $1$-Lipschitz, we conclude that $\mathcal{H}^1(B(x; R)) \leq \mathcal{H}^1(\gamma_\infty) \leq C < +\infty$. This contradiction implies the lemma.

\textbf{Proposition 4.13.} Suppose $X$ is unbounded, proper, $L$-bi-Lipschitz homogeneous, and $M$-quasi-invertible. If $X$ contains a non-degenerate rectifiable curve, then $X$ is either bi-Lipschitz homeomorphic to $\mathbb{R}$ or $X$ is annularly quasiconvex.

\textbf{Proof.} The proof will proceed by a bootstrapping argument. In Part 1, we prove that $X$ is rectifiably connected. In Part 2, we prove that $X$ is quasiconvex. Finally, in Part 3, we prove the conclusion of the proposition.

\textbf{Part 1.} For every $x \in X$, let $E(x)$ be the set of all points in $X$ that can be joined to $x$ by a rectifiable curve in $X$. Fix any $x \in X$. By assumption, there is a rectifiable curve in $X$ joining two distinct points; by uniform bi-Lipschitz homogeneity, we may assume that such a curve, denoted by $\gamma$, joins $x$ with some other point $y \neq x$. Since $\gamma$ is compact, there exists $R < +\infty$ such that $\gamma \subset B(x; R)$. By Lemma 4.11 there exists $C_1 = C_1(L, M, R)$ and $\delta_1 > 0$ such that, for any point $u \in B(y; \delta_1)$, there exists a $C_1$-bi-Lipschitz embedding $f : B(x; R) \to X$ such that $f(x) = x$ and $f(y) = u$. In particular, the curve $f(\gamma)$ is rectifiable and joins $f(x) = x$ to $f(y) = u$. Consequently, the set $E(x) \setminus \{x\}$ is open. By symmetry, the point $x$ is in the interior of $E(y)$. In other words, starting from $y$ we can get to an arbitrary point in some neighbourhood of $x$ by a rectifiable curve. Concatenating the curve $\gamma$ (and its reverse parametrization) with these curves, we conclude that $x$ is in the interior of $E(x)$. That is, there exists $\delta_2 > 0$ such that $B(x; \delta_2) \subset E(x)$. Since $X$ is unbounded, Lemma 4.11 implies the existence of $(C_3, R_n)$-quasi-dilations $f_n : X \to X$ fixing $x$. Here $R_n \to +\infty$ and $C_3 = C_3(L, M)$. Given any $z \in X$, there exists $n \in \mathbb{N}$ such that $\delta_2 R_n / C_3 > d(x, z)$,
and thus $z \in f_n(B(x; \delta_2)) \subset E(x)$. Since $z \in X$ was arbitrary, we conclude that $E(x) = X$, and $X$ is rectifiably connected.

**Part 2.** Since $X$ is rectifiably connected, it is connected. Since $X$ is connected and unbounded, there exist points $x, y \in X$ such that $d(x, y) = 1$. Let $\gamma_y$ denote a rectifiable curve joining two such points $x$ and $y$. Choose $R_y > 0$ large enough to ensure that $\gamma_y \subset B(x; R_y)$. By Lemma 4.11, there exists $C_1 = C(L, M, y) < \infty$ and $\delta_y > 0$ such that, for any $u \in B(y; \delta_y)$, there exists a $C_1$-bi-Lipschitz embedding $f : B(x; R_y) \to X$ such that $f(x) = x$ and $f(y) = u$. Therefore, each point in $B(y; \delta_y)$ is connected to $x$ by a rectifiable curve whose length is at most $C_1 \text{Length}(\gamma_y)$.

By Lemma 4.11, the metric space $X$ is $C_2$-uniformly quasi-dilation invariant, for $C_2 = C_2(L, M)$. Since $X$ is proper, the closure of the annulus $A := A(x; C_2^{-1}, C_2)$ is compact. Therefore, the collection of open balls $\{ B(y; \delta_y) | y \in \overline{A} \}$ contains a finite sub-collection whose union covers $\overline{A}$. It follows that there exists $1 \leq C_3 < \infty$ such that, for every $v \in A$, there exists a rectifiable curve $\gamma_v$, joining $x$ to $v$ satisfying $\text{Length}(\gamma_v) \leq C_3 d(x, v)$.

Fix $w \in X \setminus \{x\}$. Since $X$ is $C_2$-uniformly quasi-dilation invariant, there exists a $(C_2, 1/d(w, x))$-quasi-dilation $f : X \to X$ fixing $x$ such that $C_2^{-1} \leq d(f(w), x) \leq C_2$. By the previous paragraph, there exists a rectifiable curve $\gamma_{f(w)}$ joining $x$ to $f(w)$ such that $\text{Length}(\gamma_{f(w)}) \leq C_3 d(x, f(w))$.

Then $\gamma_w = f^{-1}(\gamma_{f(w)})$ is a rectifiable curve joining $x$ to $w$ such that

$$\text{Length}(\gamma_w) \leq C_2 d(x, w) \text{Length}(\gamma_{f(w)}) \leq C_2 C_3 d(x, w) d(x, f(w)) \leq C_2^2 C_3 d(x, w).$$

Therefore, for any $w \in X \setminus \{x\}$, there exists a rectifiable curve $\gamma_w$ joining $x$ to $w$ such that $\text{Length}(\gamma_w) \leq C_4 d(x, w)$, for $C_4 = C_2^2 C_3$. Since $X$ is $L$-bi-Lipschitz homogeneous, it follows that $X$ is $C_5$-quasiconvex, with $C_5 = L^2 C_4$.

**Part 3.** Fix $p \in X$. Assume that, for any $r > 0$ and $z \in X$, we have $\mathcal{H}^1(B(z; r)) < +\infty$. Since sph$_p(X)$ is compact, $X$ is locally uniformly bi-Lipschitz equivalent to sph$_p(X) \setminus \{\infty\}$, and by Proposition 4.7, we know that sph$_p(X)$ is uniformly bi-Lipschitz homogeneous, it follows that $\mathcal{H}^1(\text{sph}_p(X)) < +\infty$. Since sph$_p(X)$ is a connected metric space, we conclude that $1 \leq \dim_H(\text{sph}_p(X)) \leq \dim_H(\text{sph}_p(X)) \leq 1$. Here $\dim_H$ denotes Hausdorff dimension and $\dim_T$ denotes topological dimension. It follows that the Hausdorff and topological dimensions of sph$_p(X)$ agree.

By [Fre14] Theorem 1.3, we conclude that $X$ is bi-Lipschitz homeomorphic to $\mathbb{R}$.

Hereafter, we assume that, for any $r > 0$ and $z \in X$, we have $\mathcal{H}^1(B(z; r)) = +\infty$. Choose $x, y \in A(p; 1/(4L C_1), 2LC_1)$. Here $C_1 = C_1(L, M)$ is the quasi-dilation invariance constant for $X$ provided by Lemma 4.11. By Part 2 of the current proof, there exists $C_2 < +\infty$ such that $X$ is $C_2$-quasiconvex. Let $\gamma$ denote a rectifiable curve joining $x$ to $y$ in $X$ satisfying $\text{Length}(\gamma) \leq C_2 d(x, y)$.

If $d(x, y) < 1/(4LC_1 C_2)$, then $\gamma$ also satisfies

$$\gamma \subset A(p; 1/(2LC_1), 3LC_1).$$

We assume in the sequel that $d(x, y) \geq 1/(4LC_1 C_2)$. For any $a \in \gamma$, we have

$$d(p, a) \leq d(p, x) + d(x, a) \leq 2LC_1 + \text{Length}(\gamma) \leq 2LC_1 + C_2 d(x, y) < 8LC_1 (1 + C_2) =: C_3.$$

Therefore,

$$\gamma \subset B(p; C_3).$$

By Lemma 4.3, there exists $0 < \delta_1 < 1/(4C_2)$ such that, for any $q \in B(p; \delta_1)$, there exists an $L^2$-bi-Lipschitz homeomorphism $h_q : X \to X$ such that $h_q(q) = p$ and $\sup_{z \in B(p; C_3)} d(h_q(z), z) < 1/(4LC_1 C_2^2)$. Since $\text{Length}(\gamma) \leq 4LC_1 C_2$, by Lemma 4.12, there exists $0 < \delta_2 < L/(4C_2)$ and a point $q \in B(p; \delta_1)$ such that $B(q; \delta_2) \cap \gamma = \emptyset$. Write $\gamma_1 := h_q(\gamma)$. By 4.7, we have

$$\gamma_1 \subset B(p; C_3 + 1/(4LC_1 C_2^2)) \subset B(p; 2C_3).$$

Moreover, we note that

$$\gamma_1 \subset A(p; \delta_2/L^2, 2C_3).$$
Let \( \gamma_2 \) and \( \gamma_3 \) denote rectifiable curves joining \( x \) to \( h_q(x) \) and \( y \) to \( h_q(y) \), respectively, such that \( \text{Length}(\gamma_2) \leq C_2d(x, h(x)) \) and \( \text{Length}(\gamma_3) \leq C_2d(y, h(y)) \). Let \( a \) denote any point in \( \gamma_2 \). Then we observe that

\[
d(p, a) < d(p, x) + d(x, a) < 2LC_1 + \text{Length}(\gamma_2) \leq 2LC_1 + 1/4 < 3LC_1.
\]

On the other hand,

\[
d(p, a) \geq d(p, x) - d(x, a) > 1/(LC_1) - \text{Length}(\gamma_2) \geq 1/(LC_1) - 1/(4LC_1) > 1/(3LC_1)
\]

The same argument can be applied to points in \( \gamma_3 \), and thus, for \( i = 2, 3 \), we have

\[
(4.9) \quad \gamma_i \subset A(p; 1/(3LC_1), 3LC_1).
\]

Concatenating the curves \( \gamma_2, \gamma_1, \) and \( \gamma_3 \), we obtain a rectifiable curve \( \gamma_4 \) joining \( x \) to \( y \) such that

\[
\text{Length}(\gamma_4) \leq 1/(2LC_1C_2) + L^2C_2d(x, y) \leq (2 + L^2C_2)d(x, y) = C_4d(x, y).
\]

Here \( C_4 = 2 + L^2C_2 \), and we use the assumption that \( d(x, y) \geq 1/(4LC_1C_2) \). Furthermore, by (4.8) and (4.9) we observe that, for \( C_5 = \max\{2C_3, 3LC_1, L^2/\delta_2\} \), we have

\[
(4.10) \quad \gamma_4 \subset A(p; 1/C_5, C_5).
\]

We summarize our work in Part 3 thus far in order to clarify the roles of various constants. Again writing \( C_1 \) to denote the quasi-dilation invariance constant for \( X \) provided by Lemma 4.4, we have shown that there exists a constant \( C_0 < +\infty \) such that, for any \( x, y \in \overline{A} \), there exists a constant \( C_{x,y} \in (4LC_1, +\infty) \), and a \( C_0 \)-quasi-convex curve \( \gamma_{x,y} \subset A(p; 1/C_{x,y}, C_{x,y}) \) joining \( x \) to \( y \). Here we write \( \overline{A} \) to denote the closure of \( A = A(p; 1/(LC_1), 2LC_1) \).

We note that \( \overline{A} \times \overline{A} \subset X \times X \) is compact. Furthermore, we note that, for any \( x, y \in \overline{A} \), the product \( B(x; c) \times B(y; c) \) is open in \( X \times X \). Here \( c > 0 \) is such that any points of \( \overline{A} \) within distance \( 2c \) of one another can be joined by a \( C_0 \)-quasi-convex curve contained in \( A(p; 1/(4LC_1), 4LC_1) \); see the discussion immediately preceding (4.6). It follows that any pair \( (u, v) \in (B(x; c) \cap \overline{A}) \times (B(y; c) \cap \overline{A}) \) can be joined by a \( (3C_0) \)-quasi-convex curve \( \gamma_{u,v} \) such that \( \gamma_{u,v} \subset A(p; 1/C_{x,y}, C_{x,y}) \). Since \( \overline{A} \times \overline{A} \) is compact, there exists a finite collection of open sets of the form \( B(x; c) \cap \overline{A} \times (B(y; c) \cap \overline{A}) \) whose union covers \( \overline{A} \times \overline{A} \). It follows that there exists a constant \( K < +\infty \) such that, for any points \( x, y \in A = A(p; 1/LC_1, 2LC_1) \), there exists a \( K \)-quasi-convex curve \( \gamma \) joining \( x \) to \( y \) such that \( \gamma \subset A(p; 1/K, K) \).

To conclude Part 3 and the proof of the whole, choose any \( z \in X, r > 0 \), and \( a, b \in A(z; r, 2r) \). Let \( f : X \to X \) denote a \( (C_1, 1/r) \)-quasi-dilation fixing \( z \). Let \( g : X \to X \) denote an \( L \)-bi-Lipschitz homeomorphism such that \( g(z) = p \). Then \( g \circ f(A(z; r, 2r)) \subset A(p; 1/(LC_1), 2LC_1) \). By the preceding paragraph, there exists a \( K \)-quasi-convex curve \( \gamma \) joining \( g(f(b)) \) such that \( \gamma \subset A(p; 1/K, K) \). Writing \( \gamma' := f^{-1}(g^{-1}(\gamma)) \), we observe that \( \gamma' \) is a \( (L^2K)^{-2} \)-quasi-convex curve joining \( a \) to \( b \) such that

\[
\gamma' \subset A \left( z; \frac{r}{(LC_1)^2}, 2(LC_1)^2r \right).
\]

Therefore, \( X \) is \( (LC_1K)^2 \)-annularly quasi-convex. \( \square \)

We conclude this subsection with the following result connecting Laakso’s line-fitting property with the existence of rectifiable curves. Following [TW05], we say that a space is line-fitting provided that, for each \( n \in \mathbb{N} \), there is a distance \( d_n \) on the disjoint union \( X \sqcup [0, 1] \) such that \( d_n \) is the standard Euclidean distance on \([0, 1] \), \( d_n \) is a constant multiple of \( d \) on \( X \), and \([0, 1] \) is contained in the 1/\( n \)-neighborhood of \( X \).

**Lemma 4.14.** Suppose \( X \) is uniformly \( L \)-bi-Lipschitz homogeneous and admits an \( M \)-quasi-inversion. If \( X \) is line-fitting, then \( X \) contains a non-degenerate rectifiable curve.
Proof. For each $n \in \mathbb{N}$, let $d_n$ denote the distance on $X \cup [0, 1]$ given by the assumption that $X$ is line-fitting. For each $n \in \mathbb{N}$, let $\{x_k^n\}_{k=0}^{2^n}$ denote a sequence of points in $X$ such that, for each $0 \leq k \leq 2^n$, we have $d_n(x_k^n, k/2^n) < 1/2^n$. Here $k/2^n \in [0, 1] \subset X \cup [0, 1]$. For each $n \in \mathbb{N}$, let $c_n > 0$ denote the constant such that $d_n = c_n d$, and let $f_n : X \rightarrow X$ denote a $(K, c_n)$-quasi-dilation at $x_0^n$, where $K$ is independent of $n$ (here we use Lemmas 4.3 and 4.2). Define $\{y_k^n\}_{k=0}^{2^n} := \{f_n(x_k^n)\}_{k=0}^{2^n}$, and fix a point $p \in X$. For each $n \in \mathbb{N}$, there exists an $L$-bi-Lipschitz homeomorphism $g_n : X \rightarrow X$ such that $g_n(p) = x_0^n = y_0^n$. Define $\{z_k^n\}_{k=0}^{2^n} := \{g_n^{-1}(y_k^n)\}_{k=0}^{2^n}$, and note that, for each $n \in \mathbb{N}$, we have $p = z_0^n$. Given any $n \in \mathbb{N}$ and $0 \leq k \leq 2^n$, we observe that
\[
d(p, z_k^n) = d(g_n^{-1}(f_n(x_0^n)), g_n^{-1}(f_n(x_k^n))) \leq LKc_n d(x_0^n, x_k^n) = LKd_n(x_0^n, x_k^n) \leq LK(2^{-n} + k2^{-n} + 2^{-n}) \leq 2LK.
\]
Since $X$ is assumed to be proper, and the sequences $\{z_k^n\}_{k=0}^{2^n}$ are all within a bounded distance of $p$, by Blaschke’s Theorem there exists a compact set $E \subset X$ to which the sets $\{z_k^n\}_{n \in \mathbb{N}}$ converge with respect to Hausdorff distance (up to a subsequence).

We claim that $E$ is a non-degenerate rectifiable curve. We first note that the points $z_2^n$ converge (up to a subsequence) to a point $z \in E$ such that $z \neq x$. Indeed, for every $n \in \mathbb{N}$, we have
\[
d(p, z_2^n) = d(g_n^{-1}(f_n(x_0^n)), g_n^{-1}(f_n(x_2^n))) \geq (LK)^{-1}c_n d(x_0^n, x_2^n) = (LK)^{-1}d_n(x_0^n, x_2^n) > (LK)^{-1}(1 - 2^{-n+1}).
\]
Therefore, for every $n \geq 2$, we have $d(p, z_2^n) \geq 1/(2LK) > 0$, and so $z \neq x$. This demonstrates that $E$ is non-degenerate.

To see that $E$ is a curve, for each $n \in \mathbb{N}$, define the map $h_n : \{k/2^n\}_{k=0}^{2^n} \rightarrow X$ as $h_n(k/2^n) = z_k^n$. We note that this sequence of maps $(h_n)_{n=1}^\infty$ is both locally uniformly bounded and locally equicontinuous in the sense of [Her16 Section 5.2]. Therefore, via [Her16 Proposition 5.1] we conclude that the sequence $(h_n)_{n=1}^\infty$ converges locally uniformly (in the sense of [Her16 Section 5.2]) to a continuous map $h : [0, 1] \rightarrow X$. It is straightforward to verify that $h([0, 1]) = E$.

Finally, to see that $E$ is rectifiable, we note that each map $h_n$ is $(3LK)$-Lipschitz. By the remarks immediately following the proof of [Her16 Proposition 5.1], we conclude that $h$ is also Lipschitz. Therefore, $E$ is rectifiable.

With the lemmas established we are ready to finish the proof of Theorem 1.8.

Proof of Theorem 1.8. We begin by confirming (1). Using the argument from Part 1 of the proof of Proposition 4.13 the existence of a non-degenerate arc in $X$ allows us to conclude that $X$ is path connected. In particular, $X$ is connected.

Since $X$ is locally compact, given any point $x \in X$, there exists an open neighborhood $U$ of $x$ contained in a compact subset $E \subset X$. In particular, $\overline{U}$ is compact. Given any $r > 0$, via Lemmas 4.11 and 4.2 there exists a $K$-quasi-dilation $f : X \rightarrow X$ at $x$ of factor $s > 0$ such that $f(B(x; r)) \subset U$. Therefore, $\overline{f(B(x; r))}$ is compact. Since $f$ is a homeomorphism, $\overline{B(x; r)}$ is compact. Since $r > 0$ and $x \in X$ were arbitrary, we have demonstrated the properness of $X$.

To see that $X$ is Ahlfors $Q$-regular, fix $x \in X$. Since $X$ is proper, the ball $B(x; 1)$ can be covered by finitely many balls of radius $1/2$. Using the uniform bi-Lipschitz homogeneity and quasi-dilation invariance of $X$, one can then verify that $X$ is doubling. Via Proposition 4.17 we now satisfy the assumptions of [Pre12 Theorem 1.1], and so $X$ is Ahlfors $Q$-regular, for some $Q \geq 1$.

Via the argument from Part 2 of the proof of Proposition 4.13 the path connectedness of $X$ implies that $X$ is LLC with respect to curves. That is, there exists a constant $C \geq 1$ such that, given $x, y \in X$, there exists a curve $\gamma$ joining $x$ and $y$ such that $\text{Diam}(\gamma) \leq C d(x, y)$. In particular, $X$ is locally path connected, and thus locally connected.
We now prove (2). By Lemma 4.10, the cut point of $X$, given by the assumption, is a strong cut point. Via bi-Lipschitz homogeneity, every point of $X$ is a strong cut point. Since $X$ is proper, it is separable. Therefore, $X$ is a separable, locally connected, locally compact, and Hausdorff space in which each point is a strong cut point. By Ward’s theorem, see [FK71], there exists a homeomorphism $\varphi : R \to X$.

We construct a useful parameterization $g : R \to X$ following the method of [GH98, Lemma 2.1]. For $t \geq 0$, define

$$m(t) := \begin{cases} -\mathcal{H}^Q(\varphi([t, 0])) & \text{if } t \leq 0 \\ \mathcal{H}^Q(\varphi([0, t])) & \text{if } t \geq 0 \end{cases}$$

Here we recall that $X$ is Ahlfors $Q$-regular. Due to basic properties of the measure $\mathcal{H}^Q$, the map $m$ is a self-homeomorphism of $R$. Then, for any interval $I \subset R$, it is straightforward to verify that the homeomorphism $g(t) := \varphi(m^{-1}(t))$ from $R$ to $X$ satisfies $\mathcal{H}^Q(g(I)) = \mathcal{H}^Q(I)$.

Given any $x, y \in X$, write $a = g^{-1}(x)$ and $b = g^{-1}(y)$. Suppose $a < b$. Then we observe that

$$|b - a| = \mathcal{H}^1([a, b]) = \mathcal{H}^Q(g([a, b])).$$

We claim that $\mathcal{H}^Q(g([a, b])) \simeq d(x, y)^Q$, up to a constant independent of the points $x$ and $y$. Indeed, via [HN99] Theorem E the space $X$ satisfies a generalized chordarc condition. Since $X$ is Ahlfors $Q$-regular, this generalized chordarc condition is in fact a $Q$-dimensional chordarc condition in the sense of [GH98, Section 4]. This $Q$-dimensional chordarc condition is precisely the desired comparability. Therefore, for any points $x, y \in X$, we have

$$d(x, y) \simeq |g^{-1}(x) - g^{-1}(y)|^{1/Q}.$$ 

In particular, $X$ is bi-Lipschitz homeomorphic to the snowflake $(R, \cdot^{1/Q})$, where $1/Q \in (0, 1]$.

Next, we prove (3). Suppose $X$ contains no cut points. We have already demonstrated in the proof of (1) that $X$ is LLC1. To see that $X$ is also LLC2, and thus linearly locally connected, we cite [Fre12, Theorem 1.2] and Proposition 4.13. If $X$ contains a non-degenerate rectifiable curve, then Proposition 4.13 implies that $X$ is either bi-Lipschitz homeomorphic to $R$ or annularly quasi-convex. Since $X$ contains no cut point, $X$ is annularly quasi-convex. If $X$ does not contain a non-degenerate rectifiable curve, then Lemma 4.14 enables us to conclude that $X$ is not line-fitting. Therefore, by [TW05, Theorem 7.2], the space $X$ is bi-Lipschitz homeomorphic to a (non-trivial) snowflake.

5. Disconnected Spaces

In this final section we prove our results pertaining to disconnected metric spaces. Before proceeding with these proofs we introduce additional of terminology.

Following [DS97, Definition 15.1], given $\alpha \in (0, 1]$, we say that a metric space $X$ is $\alpha$-uniformly disconnected if for every $x \in X$ and $r > 0$ there exists a closed subset $A \subset X$ such that $B(x; r) \subset A \subset B(x; r)$, and $\text{dist}(A, X \setminus A) \geq \alpha r$. For example, an ultrametric space is 1-uniformly disconnected (see [Hee17, MT10]). A sequence of points $\{x_0, \ldots, x_n\} \subset X$ is an $\alpha$-chain if, for $k = 1, \ldots, n$, we have $d(x_{k-1}, x_k) \leq \alpha d(x_0, x_n)$. We say that a space $X$ is uniformly disconnected with respect to $\alpha$-chains if there exist no $\alpha$-chains in $X$.

**Lemma 5.1.** Suppose $X$ is an unbounded, locally compact, uniformly bi-Lipschitz homogeneous, and quasi-invertible metric space. If $X$ is disconnected, then $X$ is uniformly disconnected.

**Proof.** Our first goal is to show that $X$ is totally disconnected, and then we will proceed to show that $X$ is uniformly disconnected. For use later in the proof, we being by observing that $X$ satisfies the assumptions of Lemmas 4.1 and 4.2 and so $X$ is uniformly quasi-dilation invariant. Using this property along with uniform bi-Lipschitz homogeneity it is not hard to confirm that $X$ is proper.
To see that $X$ is totally disconnected, we assume that it is not and proceed by way of contradiction through the following three steps: We first show that each connected component of $X$ is unbounded. Next, we show that each connected component of $X$ is a cut point space in the sense of [HB99]. Finally, in order to obtain the desired contradiction, we show that each connected component of $X$ is not a cut point space.

Step 1: To see that each connected component of $X$ is unbounded, let $X(p)$ denote the connected component of $X$ containing a point $p \in X$. Since we are assuming that $X$ is not totally disconnected, there exists a connected component of $X$ consisting of more than one point. Since $X$ is bi-Lipschitz homogeneous, every connected component of $X$ consists of more than one point. In particular, the cardinality of $X(p)$ is greater than one. It follows from Lemmas 4.1 and 4.2 that $X$ is uniformly quasi-dilation invariant. Therefore, $X(p)$ is unbounded. By uniform bi-Lipschitz homogeneity, every connected component is unbounded.

Step 2: To see $X(p)$ is a cut point space (and thus every connected component is a cut point space), we refer to our assumption that $X$ is not connected to ensure the existence of a connected component $E$ of $X$ such that $E \neq X(p)$. Since $E$ is unbounded, $p$ is an accumulation point of $\sigma_p(E) \subset X$, and so $p \in \sigma_p(E) \subset X$. Since $\sigma_p(E)$ is connected in $X$ and shares a point with the connected set $X(p)$, the union $\sigma_p(E) \cup X(p)$ is also connected in $X$. Since $X(p)$ is a maximal connected subset of $X$, we have $\sigma_p(E) \subset X(p)$ and thus $\sigma_p(E) \subset X(p)$. We also note that $p$ is not an accumulation point of the closed set $E$, and thus $\sigma_p(E)$ is bounded in $X$.

We claim that $p$ is a cut point of the connected set $X(p)$. In other words, $X(p) \setminus \{p\}$ is disconnected. By way of contradiction, we assume that $X(p) \setminus \{p\}$ is connected. First, it is straightforward to verify that, because $p \notin E$ (the connected component of $X$ described above), the set $E$ is also a connected component of the space $X_p = X \setminus \{p\}$. This implies that $\sigma_p(E)$ is also a connected component of $X_p$. Next, we note that since $\sigma_p(E) \subset X(p) \setminus \{p\}$ and $X(p) \setminus \{p\}$ is assumed to be connected, we have $\sigma_p(E) = X(p) \setminus \{p\}$ (else $\sigma_p(E)$ is not maximal). Since $\sigma_p(E)$ is bounded, while $X(p) \setminus \{p\}$ is unbounded, we reach a contradiction. This contradiction confirms that $p$ is a cut point of $X(p)$.

Since bi-Lipschitz self-homeomorphisms permute connected components of $X$, the assumptions on $X$ imply that $X(p)$ is itself $L$-bi-Lipschitz homogeneous. By way of this homogeneity, we conclude that every point of $X(p)$ is a cut point for $X(p)$. In other words, $X(p)$ is a cut-point space. Indeed, every connected component of $X$ is a cut-point space.

Step 3: We now show that $X(p)$ is not a cut-point space. Given the connected component $E \neq X(p)$ as above, it is easy to see that $K := \sigma_p(E) \cup \{p\}$ is closed and bounded in $X(p)$. Since $X(p)$ is a proper metric space, this implies that $K$ is compact. Furthermore, since $K = \overline{\sigma_p(E)}$, and $\sigma_p(E)$ is connected, we conclude that $K$ is also connected. Since $K$ contains more than one point, by [HB99] Theorem 3.9, the set $K$ contains at least two points that are not cut points of $K$. This implies that some point $x \in \sigma_p(E)$ is not a cut point for $K$. Since $X(p)$ is a cut-point space, let $U_1$ and $U_2$ denote disjoint open sets in $X$ such that $X(p) \setminus \{x\} \subset U_1 \cup U_2$. Without loss of generality, $p \in U_1$, and thus $K \cap U_1 \neq \emptyset$. If $K \cap U_2 \neq \emptyset$, then $K \setminus \{x\}$ is separated by $U_1 \cap K$ and $U_2 \cap K$, which contradicts the fact that $x$ is not a cut point for $K$. Therefore, $K \cap U_2 = \emptyset$, and so $\sigma_p(E) \setminus \{x\} \subset U_1$.

Let $E'$ denote any connected component of $X(p) \setminus \{p\}$ such that $E' \neq \sigma_p(E)$. Note that such a component must exist due to the fact that $\sigma_p(E)$ is bounded while $X(p) \setminus \{p\}$ is unbounded. Since $U_1$ is open in $X$, $p \in U_1$, and $p$ is an accumulation point of $E'$, it follows that $U_1 \cap E' \neq \emptyset$. Since $E'$ is connected and $x \notin E'$, we must have $E' \cap U_2 = \emptyset$. Otherwise, $U_1 \cap E'$ and $U_2 \cap E'$ would form a separation of $E'$. This argument indicates that every connected component of $X(p) \setminus \{p\}$ other than $\sigma_p(E)$ is contained in $U_1$.

The previous two paragraphs imply that $X(p) \setminus \{x, p\} \subset U_1$. Since $p \in U_1$, we conclude that $X(p) \setminus \{x\} \subset U_1$. This implies that $U_2 = \emptyset$, and it follows that $X(p) \setminus \{x\}$ is connected. Therefore, $X(p)$ is not a cut-point space.
Combining the conclusions of Steps 2 and 3 above, we reach the desired contradiction to our assumption that $X$ is not totally disconnected. Therefore, $X$ is totally disconnected.

Having demonstrated that $X$ is totally disconnected, we finish the proof by demonstrating that $X$ is uniformly disconnected. By way of contradiction, suppose $\theta_k \to 0$ is a sequence of positive numbers such that, for each $k \in \mathbb{N}$, there exists a $\theta_k$-chain $(x_i^{(k)})_{i=0}^{n_k}$ in $X$. By uniform bi-Lipschitz homogeneity and a quantitatively controlled change the numbers $\theta_k$, we may assume that, for each $k \in \mathbb{N}$, we have $x_0^{(k)} = p$. Furthermore, Lemmas 1.1 and 1.2 yield a constant $M \geq 1$ such that, for each $k \in \mathbb{N}$, we have $E_k := \{x_i^{(k)}\}_{i=0}^{n_k} \subset B(p;M)$. Furthermore, we may assume there exists $j_k \in \{1, \ldots, n_k\}$ such that $M^{-1} \leq d(p, x_{j_k}^{(k)}) \leq M$. Again using the properness of $X$, we may assume, up to a subsequence, that the sets $E_k$ converge to a non-degenerate compact set $E \subset X$ with respect to Hausdorff distance.

We claim that $E$ is connected. Indeed, suppose (by way of contradiction) $E'$ and $E''$ are distinct connected components of $E$. Both $E'$ and $E''$ are closed (in $E$) and bounded. Since $E$ is compact, each of $E'$ and $E''$ is compact. Let $\varepsilon > 0$ be such that $\text{dist}(E', E'') = 3\varepsilon$. Write $U_1$ and $U_2$ to denote $\varepsilon$-neighborhoods of $E'$ and $E''$, respectively. Since $E$ is compact and $U_1 \cup U_2$ is open, there exists $N_1 \in \mathbb{N}$ such that, for all $k \geq N_1$, we have $E_k \subset U_1 \cup U_2$. Furthermore, there exists $N_2$ such that, for all $k \geq N_2$, we have $\theta_k d(p, x_{j_k}^{(k)}) \leq M \theta_k < \varepsilon$. The definition of a $\theta_k$-chain implies that $\varepsilon \leq \text{dist}(U_1, U_2) < \varepsilon$. This contradiction proves that $E$ is connected.

We have shown that if $X$ is not uniformly disconnected, then $X$ contains a non-degenerate continuum. This contradicts the fact that $X$ is totally disconnected. We conclude that $X$ is uniformly disconnected.

\section*{5.1. Examples of disconnected spaces.}

\textbf{Example 5.2.} We present the basic example of a disconnected, isometrically homogeneous, and invertible metric space. In contrast to the brief description provided in Section 1.1, we here provide a more detailed construction. We fix $N \in \mathbb{N}$ with $N \geq 2$ and $s > 1$. Define the metric space $(C_N, \rho_s)$ by considering the set

$$
C_N := \{\xi = (\xi_i)_{i \in \mathbb{Z}} \mid \forall i, \xi_i \in \{1, \ldots, N\} \text{ and } \exists m \in \mathbb{Z} \text{ such that } \forall i \leq m, \xi_i = 1\}
$$

equipped with the distance

$$
\rho_s(\xi, \zeta) := s^{-m(\xi, \zeta)}, \quad \text{where } m(\xi, \zeta) := \sup\{m \in \mathbb{Z} \mid \forall i \leq m, \xi_i = \zeta_i\}.
$$

The metric space $(\hat{C}_N, \rho_s)$, which represents a sphericalization of the metric space $(C_N, \rho_s)$, is defined by the set

$$
\hat{C}_N := \{\xi = (\xi_i)_{i \in \mathbb{Z}} \mid \xi_1 = 1, \xi_2 \in \{1, \ldots, N + 1\}, \text{ and } \forall i \geq 3, \xi_i \in \{1, \ldots, N\}\}
$$

and $\rho_s$ is defined by \eqref{5.1}. Note that for points $\xi, \zeta \in \hat{C}_N$ we have $m(\xi, \zeta) \geq 1$. To see that $\hat{C}_N$ is bi-Lipschitz homeomorphic to a sphericalization of $C_N$, we argue as follows. Write $1 \in C_N$ to denote the constant sequence whose every entry is equal to 1. Given $\xi \in C_N$, define $\hat{\xi} \in \hat{C}_N$ according to the following cases. If $m(\xi, 1) \geq 0$, then $\hat{\xi}_i = \xi_{i-1}$ for all $i \geq 1$. If $m(\xi, 1) = -1$,

$$
\hat{\xi}_i = \begin{cases} 
1 & \text{if } i = 1 \\
N + 1 & \text{if } i = 2 \\
\xi_{i-3} & \text{if } i \geq 3.
\end{cases}
$$

If $m(\xi, 1) \leq -2$,

$$
\hat{\xi}_i = \begin{cases} 
1 & \text{if } i = 1 \\
N + 1 & \text{if } i = 2 \\
1 & \text{if } 3 \leq i \leq 1 - m(\xi, 1) \\
\xi_{i+2m(\xi, 1)-1} & \text{if } i \geq 2 - m(\xi, 1).
\end{cases}
$$

\section{Invertible Homogeneous Metric Spaces}
This establishes a bijection between points in \( \text{Sph}_1(C_N) \) and \( \hat{C}_N \). Here we note that the point at infinity is identified with the point \((1, N+1, 1, \ldots) \in \hat{C}_N \), where the ellipsis indicates a constant sequence of terms equal to 1. Via a tedious but straightforward case analysis, one can verify that, for any \( \xi, \zeta \in C_N \),

\[
\frac{\rho_s(\xi, \zeta)}{(1 + \rho_s(\xi, 1))(1 + \rho_s(\zeta, 1))} \simeq \rho_s(\hat{\xi}, \hat{\zeta}).
\]

Thus we see that \( \hat{C}_N \) is indeed bi-Lipschitz homeomorphic to the sphericalized space \( \text{Sph}_1(C_N) \). We note that when \( N = 2 \) and \( s = 2 \), \( \hat{C}_N \) is the symbolic Cantor set studied in [DS97] Section 2.3.

The function \( \rho_s \) is an ultrametric both on \( C_N \) and in \( \hat{C}_N \). The space \( (C_N, \rho_s) \) is proper, unbounded, two-point isometrically homogeneous, and invertible. We shall prove these properties in Example 5.3 where we construct a slightly more general collection of spaces.

**Example 5.3.** In order to illustrate the sharpness of Theorem 1.11, we provide the following generalization of the construction from Example 5.2. Using the terminology of the previous example, for any \( N, M \in \mathbb{N} \) such that \( N \geq M \), we consider the subset \( C_{N|M} \subset C_N \) defined by

\[
C_{N|M} := \{ \xi \in C_N : \xi_i \in \{1, \ldots, M\}, \forall i \text{ even} \}. \tag{5.2}
\]

Note that \( C_{N|N} = C_N \). We consider \( C_{N|M} \) with the metric \( \rho_s \) given by (5.1). The space \( (C_{N|M}, \rho_s) \) is proper. Indeed, every point has a neighborhood that is topologically a Cantor set.

We claim that \((C_{N|M}, \rho_s)\) is 2-point isometrically homogeneous. To verify this claim, we first demonstrate that \((C_{N|M}, \rho_s)\) is 1-point isometrically homogeneous. Fix \( \xi, \zeta \in C_{N|M} \). For each \( i \in \mathbb{Z} \) chose a permutation \( \iota_i \) of \( \{1, \ldots, M\} \), if \( i \) is even, and of \( \{1, \ldots, N\} \), if \( i \) is odd, such that \( \iota_i(\xi_i) = \zeta_i \). We then define \( f : C_{N|M} \to C_{N|M} \) such that, for any \( \omega \in C_{N|M} \), we have

\[
f(\omega) = \theta \in C_{N|M} \text{ such that } \begin{cases} 
\theta_i = \omega_i & \text{if } \xi_i = \zeta_i \\
\theta_i = \iota_i(\omega_i) & \text{if } \xi_i \neq \zeta_i.
\end{cases} \tag{5.1}
\]

We note that \( f \) is an isometry of \((C_{N|M}, \rho_s)\) such that \( f(\xi) = \zeta \). Therefore, \((C_{N|M}, \rho_s)\) is 1-point isometrically homogeneous.

In light of 1-point isometric homogeneity, it suffices to show that any metric sphere \( S(1; s^k) = \{ \omega \in C_{N|M} : \rho_s(1, \omega) = s^{-k} \} \) is isometric with respect to isometries of \((C_{N|M}, \rho_s)\) fixing 1. To see this, we modify the construction given in (5.2). We define the map \( f_1 \) to be the identity away from \( S(1; s^{-k}) \). Given \( \xi \) and \( \zeta \) in \( S(1; s^{-k}) \), we define \( f_1 \) on \( S(1; s^{-k}) \) as in (5.2) under the additional requirement that \( \iota_k(1) = 1 \). This is additional requirement is possible because neither \( \xi_{k+1} \) nor \( \zeta_{k+1} \) is equal to 1. Furthermore, this requirement ensures that \( f_1 \) is a self-bijection of \( S(1; s^{-k}) \). It is then straightforward to see that \( f_1 \) is an isometry of \((C_{N|M}, \rho_s)\) fixing 1 and sending \( \xi \) to \( \omega \). It follows that \( C_{N|M} \) is 2-point isometrically homogeneous.

Next, we claim that \((C_{N|M}, \rho_s)\) is invertible. Indeed, we define an involutive inversion \( \tau \) as follows. Denote by \( T \) the shift operator \( T(\xi)_i := \xi_{i-1} \), for every \( i \in \mathbb{Z} \). We define an involution \( \tau \) as

\[
\tau : C_{N|M} \setminus \{ 1 \} \to C_{N|M} \setminus \{ 1 \} \quad \text{where } m := m(\xi, 1),
\]

where \( m \) is the function in (5.1). To see that \( \tau \) is indeed an inversion, fix \( \xi \) and \( \zeta \) in \( C_{N|M} \setminus \{ 1 \} \).

We consider two cases.

Case 1: \( m(\xi, 1) = m(\zeta, 1) = m \). In this case, we have

\[
m(\tau(\xi), \tau(\zeta)) = m(\xi, \zeta) - 2m.
\]

Thus,

\[
\rho_s(\tau(\xi), \tau(\zeta)) = \frac{\rho_s(\xi, \zeta)}{\rho_s(\xi, 1)\rho_s(\zeta, 1)}.
\]

Case 2: \( m_1 := m(\xi, 1) < m(\zeta, 1) = m_2 \). Hence we have \( \tau(\xi) = T^{2m_1}(\xi) \) and \( \tau(\zeta) = T^{2m_2}(\zeta) \). Since \( -m_2 < -m_1 \), then for any \( i \leq -m_2 \) we have \( 1 = \tau(\xi)_i = \tau(\zeta)_i \). However, since \( -m_2 + 1 \leq \)
-m_1$, we have $\tau(\zeta)_{m_2+1} \neq 1 = \tau(\xi)_{m_2+1}$. Consequently, we have $m(\tau(\xi), \tau(\zeta)) = -m_2$ and $m(\xi, \zeta) = m_1$. Hence, we observe that

$$m(\tau(\xi), \tau(\zeta)) = -m_2 = m_1 - m_2 = m(\xi, \zeta) - m(\xi, 1) - m(\zeta, 1),$$

and so

$$\rho_s(\tau(\xi), \tau(\zeta)) = \frac{\rho_s(\xi, \zeta)}{\rho_s(\xi, 1)\rho_s(\zeta, 1)}.$$

In both of the above cases we obtain the desired metric behavior for $\tau$. Furthermore, it is straightforward to verify that $\tau : C_{N|M} \setminus \{1\} \to C_{N|M} \setminus \{1\}$ is a homeomorphism. Therefore, $\tau$ satisfies the definition of an inversion at 1.

Finally, we point out that $(C_{N|M}, \rho_s)$ is isometric to $(C_{N|M}, \rho_{s'})$ if and only if $N' = N$, $M' = M$ and $s' = s$. In particular, when $N > M$ then $(C_{N|M}, \rho_s)$ is not isometric to any $(C_{N'}, \rho_{s'})$, for $N' > M$ and $s' > 1$. To see this, we first observe that the set of distances in $(C_{N|M}, \rho_s)$ is equal to $\{s^k | k \in \mathbb{Z}\}$. Hence we only need to consider the case $s' = s$. Second, we observe that the metric components of the metric spheres $S(1; s^{-k}) \subset (C_{N|M}, \rho_s)$ characterize $N$ and $M$. We require a bit of terminology: A subset $E \subset X$ is a $\delta$-component if it is a maximal subset with the property that every pair of points from $E$ can be joined with by a sequence of points in $E$ whose consecutive distances are less than $\delta$. Using this terminology, we note that, for each $\delta \in (1/s, 1)$ the number of $\delta$-components in $S(1; 1)$ is exactly $N$, while for $\delta \in (1, s)$ the number of $\delta$-components in $S(1; s)$ is exactly $M$. In conclusion, the values of $N$ and $M$ are metric invariants for $(C_{N|M}, \rho_s)$.

**Remark 5.4.** In light of Theorem 1.12 (proved in the sequel), we note that the spherically homogeneous spaces $Sph_p(C_{N|M})$ are not three-point Möbius homogeneous if $N \neq M$, despite the fact that they are 2-point isometrically homogeneous and invertible. This can be seen in the fact that, via Lemma 5.5, the 3-point Möbius homogeneity of $Sph_p(X)$ implies that $X$ admits dilations of all factors $\lambda \in \{d(x, y) | x, y \in X\}$, while, if $N \neq M$, the space $C_{N|M}$ only admits dilations of factors $\lambda^2$ for $\lambda \in \{\rho_s(x, y) | x, y \in C_{N|M}\}$ (see also Proposition 2.1).

### 5.2. Proofs of Theorems 1.11, 1.12 and 1.13

In order to present the proof of Theorem 1.11 we require the following definitions. Given $\delta > 0$, a sequence of points $\{x_0, \ldots, x_n\} \subset X$ is a $\delta$-sequence if, for $k = 1, \ldots, n$, we have $d(x_{k-1}, x_k) < \delta$. A subset $E \subset X$ is $\delta$-connected provided that any two points $x, y \in E$ can be joined by a $\delta$-sequence such that $x_0 = x$ and $x_n = y$. A $\delta$-component of $X$ is a maximal $\delta$-connected subset of $X$.

**Proof of Theorem 1.11.** Suppose that $X$ is a disconnected, unbounded, locally compact, isometrically homogeneous metric space that admits an inversion $\sigma_p$ at some point $p \in X$. By Lemma 5.1 there exists $\alpha \in (0, 1]$ such that $X$ is $\alpha$-uniformly disconnected. Fix $x \in X$. Let $A' \subset X$ denote a closed set such that $B(x, \alpha) \subset A' \subset B(x; 1)$ and dist$(A', X \setminus A') \geq \alpha$. Let $A$ denote the $\alpha$-component of $X$ containing $x$. Note that $A \subset A' \subset B(x; 1)$. Since Isom$(X)$ acts transitively on $X$, the collection $\mathcal{X}_0 = \{f(A) | f \in \text{Isom}(X)\}$ covers $X$. We also claim that $\mathcal{X}_0$ consists of pairwise disjoint sets in the sense that, for $f, g \in \text{Isom}(X)$, either $f(A) = g(A)$ or $f(A) \cap g(A) = \emptyset$. Indeed, suppose that $f, g \in \text{Isom}(X)$ and there exists a point $z \in f(A) \cap g(A)$. By concatenating the $\alpha$-sequences between $f(x)$ and $z$ and between $z$ and $g(x)$ we obtain a $\alpha$-sequence joining $f(x)$ to $g(x)$. Therefore, given any point $w \in g(A)$, there exists a $\alpha$-sequence joining $f(x)$ to $w$. Since $w$ was an arbitrary point of $g(A)$, it follows that $f^{-1}g(A) \subset A$. By symmetry, $g^{-1}f(A) \subset A$. Therefore, $f(A) = g(A)$.

Choose $s > 1$ such that there exists $y \in X$ satisfying $d(x, y) = \sqrt{s}$. By Proposition 2.1 there exists an $s^{-1}$-dilation $h : X \to X$ at $x$. For each $i \in \mathbb{Z}$, define the set of sets

$$\mathcal{X}_i = h^i(\mathcal{X}_0) = \{h^i \circ f(A) | f \in \text{Isom}(X)\}.$$ 

Here $h^i$ denotes the $i$-fold composition of $h$ with itself. Thus, for any $E \in \mathcal{X}_i$, we have $h(E) \in \mathcal{X}_{i+1}$. We note that the same set $E$ in $\mathcal{X}_i$ may correspond to two different isometries $f, g \in \text{Isom}(X)$, but this will not hinder our use of $\mathcal{X}_i$ in the sequel.
For later use, we write $\mathcal{X}$ to denote the disjoint union $\bigcup_{i \in \mathbb{Z}} \mathcal{X}_i$. Let $N \in \mathbb{N}$ denote the number of distinct sets from $\mathcal{X}_1$ contained in $A$. Since $s^{-1} < 1$, we have $N \geq 2$. Since Isom$(X)$ permutes elements of $\mathcal{X}_0$, $N$ also represents the number of distinct sets from $\mathcal{X}_1$ contained in every element of $\mathcal{X}_0$. Similarly, given any $i \in \mathbb{Z}$, the number $N$ represents the number of distinct sets from $\mathcal{X}_i$ contained in every element of $\mathcal{X}_{i−1}$.

We label each of the $N$ distinct sets from $\mathcal{X}_1$ contained in $A$ using the labels $\{1, 2, \ldots, N\}$. We do this such that $h^i(A) \subset A$ receives the label $1$. For each $i \in \mathbb{Z}$, we use isometries and dilations to transfer this labelling to the $N$ distinct sets from $\mathcal{X}_i$ contained in each element of $\mathcal{X}_{i−1}$. While this labelling is certainly not uniquely determined, we emphasize that, for all $i \in \mathbb{Z}$, we may assume that the set $h^i(A)$ receives the label $1$.

We can obtain a bijection between points of $X$ and certain sequences in $\mathcal{X}$ as follows. For each $i \in \mathbb{Z}$, we denote the collection of distinct (and thus pairwise disjoint) sets in $\mathcal{X}_i$ as $\{E_{i,k}\}_{k \in \mathbb{N}}$. Given any point $z \in X$, there exists a unique sequence $(E_{i,k})_{i \in \mathbb{Z}}$ such that, for each $i \in \mathbb{Z}$, we have $z \in E_{i,k} \in \mathcal{X}_i$, and $E_{i+1,k+i} \subset E_{i,k}$. Since $B(x;\alpha) \subset A$, there exists $M = M(z) \in \mathbb{Z}$ such that, for any $i \leq M(z)$, we have $z \in h^i(A) \in \mathcal{X}_i$. In other words, for $i \leq M(z)$, we have $E_{i,k} = h^i(A)$.

Conversely, given any sequence $(E_{i,k})_{i \in \mathbb{Z}}$ consisting of elements from $\mathcal{X}$ such that, for each $i \in \mathbb{Z}$, we have $E_{i,k} \in \mathcal{X}_i$ and $E_{i+1,k+i} \subset E_{i,k}$, there exists a unique point $z \in X$ such that $\bigcap_{i=0}^M E_{i,k} = \{z\}$ (this is because $X$ is proper, each set $E_{i,k}$ is closed, and Diam$(E_{i,k}) \to 0$ as $i \to +\infty$). As in the preceding paragraph, there exists $M = M(z) \in \mathbb{Z}$ such that, for any $i \leq M(z)$, we have $z \in h^i(A)$ and thus $E_{i,k} = h^i(A)$.

Via the preceding two paragraphs, the labelling of $X$ constructed above yields a bijection between $X$ and $C_N$, as defined in Example 5.2. We denote this bijection by $\varphi : X \rightarrow C_N$.

To see that $\varphi$ is bi-Lipschitz when $C_N$ is equipped with the distance $\rho_s$, we proceed as follows. Choose $\xi = \varphi(u)$ and $\zeta = \varphi(v)$ in $C_N$, and write $m \in \mathbb{Z}$ to denote $m(\xi, \zeta)$, where $m(\xi, \zeta)$ is defined as in (5.1). By the construction of $\varphi$, there exists $E = h^m(f(A)) \in \mathcal{X}_m$ such that $u, v \in E$ but $u$ and $v$ are contained in disjoint elements of $\mathcal{X}_{m+1}$. Note that $f(x) \in E \subset B(f(x); s^{-m})$. Since Isom$(X)$ acts transitively on $X$ and permutes elements of $\mathcal{X}_m$, we conclude that Isom$(X)$ acts transitively on $E$. Therefore, $E \subset B(u; s^{-m})$, and so $d(u, v) < s^{-m}$. On the other hand, distinct sets from $\mathcal{X}_{m+1}$ are separated by a distance of at least $\alpha s^{-m−1}$. Therefore, $d(u, v) \geq \alpha s^{-m−1}$. It follows that

\begin{equation}
(5.3) \quad d(u, v) < \rho_s(\varphi(u), \varphi(v)) \leq \frac{s}{\alpha}d(u, v).
\end{equation}

Thus $\varphi : X \rightarrow C_N$ is $(s/\alpha)$-bi-Lipschitz. \hfill \Box

We shall make use of the following result in the proof of Theorem 1.12. We include the proof for the sake of completeness, noting its similarity to the proof of Proposition 2.4.

**Lemma 5.5.** Suppose $X$ is unbounded. The space $\hat{X}$ is 3-point Möbius homogeneous if and only if the following two statements are true:

1. For any two pairs of distinct points $x, y$ and $u, v$ in $X$, there exists a $\lambda$-similarity $f : X \rightarrow X$ such that $f(x) = u$, $f(y) = v$, and $\lambda = d(u, v)/d(x, y)$.
2. $X$ is invertible.

**Proof.** We first assume that Sph$_p(X)$ is 3-point Möbius homogeneous. Given any two pairs of distinct points $x, y$ and $u, v$ in $X$, let $f : \text{Sph}_pX \rightarrow \text{Sph}_pX$ denote a Möbius map fixing $\infty$ such that $f(x) = u$ and $f(y) = v$. By Remark 2.3, $f$ is a $\lambda$-similarity of $X$. Furthermore,

\[d(u, v) = d(f(x), f(y)) = \lambda d(x, y),\]

and so $\lambda = d(u, v)/d(x, y)$. Thus we confirm (1).

To verify (2), fix any point $a \in X \setminus \{p\}$. Let $f : \text{Sph}_pX \rightarrow \text{Sph}_pX$ denote a Möbius homeomorphism such that $f(p) = \infty$, $f(\infty) = p$, and $f(a) = a$. For any point $x \in X$, we find that

\[d(p, f(x)) = r \cdot d(p, x)^{-1},\]
where \( r = d(p, a)^2 \). Here we follow the calculations utilized in the proof of Proposition 2.4. Continuing these calculations, we find that, for any \( x, y \in X \setminus \{ p \} \), we have
\[
d(f(x), f(y)) = \frac{r \, d(x, y)}{d(x, p) \, d(y, p)}.
\]

By the proof of (1) above, the space \( X \) admits an \( r \)-dilation \( h \) at \( p \) of factor \( r \). Therefore, \( f \circ h : X_p \to X_p \) is an inversion of \( X \) at \( p \).

Conversely, if \( X \) satisfies (1) and (2), then fix a triple \( a, b, \infty \) of distinct points from \( X \). Let \( x, y, z \) denote a second triple of distinct points from \( X \). If \( z = \infty \), then, via (1), there exists a Möbius map \( f : \text{Sph}_p(X) \to \text{Sph}_p(X) \) fixing \( \infty \) such that \( f(x) = a \) and \( f(y) = b \). If \( z \neq \infty \), then we first map \( z \) to \( p \) via an isometry of \( X \) and then, via (2), send \( p \) to \( \infty \) via the inversion of \( X \). Thus we are back in the case that \( z = \infty \), and we confirm that \( \text{Sph}_p(X) \) is 3-point Möbius homogeneous.

\( \square \)

\textbf{Proof of Theorem 1.12.} Via Lemma 5.5, we see that \( X \) admits dilations of arbitrarily large factors. Therefore, since \( X \) is locally compact, it is straightforward to verify that \( X \) is proper. Furthermore, \( X \) satisfies the assumptions of Lemma 5.1, and so \( X \) is uniformly disconnected. We claim that \( X \) is an ultrametric space. By way of contradiction, suppose there exist points \( x, y, z \in X \) such that \( d(x, y) > \max\{d(x, z), d(z, y)\} \). In particular, there exists \( c \in (0, 1) \) such that \( c \, d(x, y) \geq \max\{d(x, z), d(y, z)\} \).

In order to obtain the desired contradiction, we construct a non-degenerate connected subset of \( X \) using a construction from the proof of \([KLD16, \text{Lemma 3.5}]\). We write \( x^{(1)}_0 = x, x^{(1)}_1 = z, \) and \( x^{(1)}_2 = y \). Given a sequence
\[
x = x^{(k)}_0, x^{(k)}_1, \ldots, x^{(k)}_{2^k} = y,
\]
for which pairs of consecutive points are distinct, we form a new sequence
\[
x = x^{(k+1)}_0, x^{(k+1)}_1, \ldots, x^{(k+1)}_{2^{k+1}} = y
\]
by defining
\[
x^{(k+1)}_i = \begin{cases} x^{(k)}_{i/2} & \text{if } i \text{ is even} \\ f^{(k+1)}_i(z) & \text{if } i \text{ is odd.} \end{cases}
\]
Here \( f^{(k+1)}_i \) is a \( \lambda^{(k+1)}_i \)-similarity of \( X \) such that
\[
f^{(k+1)}_i(x) = x^{(k+1)}_{i-1}, \quad f^{(k+1)}_i(y) = x^{(k+1)}_{i+1}, \quad \lambda^{(k+1)}_i = d(x^{(k+1)}_{i-1}, x^{(k+1)}_{i+1})/d(x, y).
\]

For use in the sequel, we also define \( \lambda^{(1)}_1 := 1 \). By construction, we note that pairs of consecutive points in the newly created chain are distinct.

We claim that \( \Lambda_k := \max\{\lambda^{(i)}_i \mid i = 1, \ldots, 2^k\} \) converges to 0 as \( k \to +\infty \). To see this, for any \( k \geq 1 \) and odd integer \( 1 \leq i < 2^{k+1} \), we have
\[
\lambda^{(k+1)}_i = \frac{d(x^{(k)}_{(i-1)/2}, x^{(k)}_{(i+1)/2})}{d(x, y)} = \frac{d(f^{(k)}_j(z), f^{(k)}_j(w))}{d(x, y)}.
\]

Here \( j \in \{(i-1)/2, (i+1)/2\} \) is odd. If \( j = (i-1)/2 \), then \( w = y \). If \( j = (i+1)/2 \), then \( w = x \). Continuing, we find that
\[
\frac{d(f^{(k)}_j(z), f^{(k)}_j(w))}{d(x, y)} \leq \Lambda_k \frac{d(z, w)}{d(x, y)} \leq c\Lambda_k.
\]

Therefore, \( \Lambda_{k+1} \leq c\Lambda_k \). By way of induction, \( \Lambda_k \leq c^{k-1} \). Since \( c \in (0, 1) \), we conclude that \( \Lambda_k \to 0 \) as \( k \to +\infty \).

For \( k \in \mathbb{N} \), write \( x_k \) to denote the collection of points \( \{x^{(k)}_0, \ldots, x^{(k)}_{2^k}\} \) constructed as above. Write \( E \) to denote the closure of the union \( \cup_{k \in \mathbb{N}} x_k \subseteq X \). In order to reach a contradiction and conclude that \( X \) is an ultrametric space, we demonstrate that \( E \) is a non-degenerate connected set. Indeed, \( x, y \in E \) and \( x \neq y \), so \( E \) is non-degenerate. To see that \( E \) is connected, suppose that \( U_1 \cup U_2 \) is a non-trivial separation of \( E \) by disjoint open sets. It is not difficult to verify that \( E \) is bounded,
and so \( E \) is compact. Therefore, we may assume that \( \overline{U}_1 \cap \overline{U}_2 = \emptyset \) and \( \text{dist}(U_1, U_2) > \varepsilon \) for some \( \varepsilon > 0 \). However, since \( \Lambda_k \to 0 \), there exists \( K \in \mathbb{N} \) such that, for any \( k \geq K \), we have \( \Lambda_k < \varepsilon d(x, y) \).

Since consecutive points of each \( x_k \) are within distance of \( \Lambda_k d(x, y) \) of each other, it follows that \( \text{dist}(U_1, U_2) < \varepsilon \). This contradiction demonstrates that \( E \) is connected, which in turn contradicts the fact that \( X \) is uniformly disconnected. Therefore, \( X \) is an ultrametric space.

Given \( r > 0 \) and \( x \in X \), since \( X \) is an ultrametric space, the ball \( B(x; r) \) is closed. Therefore,
\[
s(r) = \max\{d(x, y) \mid y \in B(x; r)\} < r.
\]

If there exist \( \lambda \)-dilations of \( X \) at \( x \) with \( \lambda > 1 \) arbitrarily close to 1, we contradict the definition of \( s(r) \). Therefore,
\[
\lambda_0 := \inf\{\lambda > 1 \mid \exists \lambda \text{-dilations of } X\} = \min\{\lambda \mid \exists \lambda \text{-dilations of } X\} > 1.
\]

Here the properness of \( X \) allows us to replace the infimum by a minimum in the definition of \( \lambda_0 \).

By Lemma 5.5, \((1, \lambda_0) \cap \Delta(X) = \emptyset \) and \((\lambda_0^{-1}, 1) \cap \Delta(X) = \emptyset \). Here
\[
\Delta(X) := \{r \geq 0 \mid \exists x, y \in X \text{ such that } d(x, y) = r\}.
\]

Since there exists a \( \lambda_0 \)-dilation of \( X \), it follows that, for every \( k \in \mathbb{Z} \), we have \((\lambda_0^k, \lambda_0^{k+1}) \cap \Delta(X) = \emptyset \).

Since \( \Delta(X) \neq \emptyset \) and \( X \) contains more than one point, we conclude that \( \Delta(X) = \{\lambda_k^n \mid n \in \mathbb{Z}\} \).

To conclude, we appeal to the proof of Theorem 1.11. Using the methods of this proof, we construct sets
\[
X_i := h^i(\Lambda_0) := \{h^i \circ f(B(x; 1)) \mid f \in \text{Isom}(X)\}.
\]

Here \( h \) is a \( \lambda_0^{-1} \)-dilation of \( X \) at \( x \). We then proceed to construct the bijection \( \varphi : X \to C_N \), where \( N \) is the number of pairwise distinct balls of radius \( \lambda_0^{-1} \) contained in \( B(x; 1) \). As in (5.3), for points \( u, v \in X \), we have
\[
d(u, v) < \rho_{\lambda_0}(\varphi(u), \varphi(v)) \leq \lambda_0 d(u, v).
\]

Here we use the fact that \( X \) is \( \alpha \)-uniformly disconnected with \( \alpha = 1 \). Since both \( d(u, v) \) and \( \rho_{\lambda_0}(\varphi(u), \varphi(v)) \) are integer powers of \( \lambda_0 \), we conclude that \( \rho_{\lambda_0}(\varphi(u), \varphi(v)) = \lambda_0 d(u, v) \). We conclude that \( \varphi \circ h : X \to C_N \) is an isometry. \( \square \)

**Proof of Theorem 1.15.** From Lemmas 4.11 4.12 and 5.1 we conclude that \( X \) is uniformly perfect, uniformly disconnected, proper, and doubling. Here we say that a metric space \( X \) is **doubling** provided that there exists a finite constant \( D \geq 1 \) such that any ball of radius \( r > 0 \) in \( X \) can be covered by at most \( D \) balls of radius \( 2r \). Given \( p \in X \), via [Hee17, Theorem 1.2] we conclude that \( \text{ sph}_p(X) \) is uniformly disconnected. Via [Mey09, Theorem 7.1] we conclude that \( \text{ sph}_p(X) \) is uniformly perfect. Via [Hee17, Theorem 1.1] (see also [LS15, Proposition 3.2.2]) we conclude that \( \text{ sph}_p(X) \) is doubling. In these assertions we are using the facts that the identity map between \( X \) and \( \text{ sph}_p(X) \setminus \{\infty\} \) is strongly quasi-Möbius and that quasi-sphericalization can be viewed as a special case of quasi-inversion (see [BHX08, pg. 847]).

Since \( \text{ sph}_p(X) \) is compact, doubling, uniformly perfect, and uniformly disconnected, by [DS97, Proposition 15.11] we conclude that \( \text{ sph}_p(X) \) is quasi-symmetrically homeomorphic to \( (\mathbb{C}_2, \rho_2) \) (see [DS97, Section 2.3] and Example 5.2). Since \( \text{ sph}_p(X) \setminus \{\infty\} \) is quasi-Möbius homeomorphic to \( X, \mathbb{C}_2 \setminus \{(1, 3, 1, \ldots)\} \) is quasi-Möbius equivalent to \( C_2 \) (see Example 5.2), and all of these spaces are uniformly bi-Lipschitz homogeneous (via Proposition 1.7), it follows that \( X \) is quasi-Möbius homeomorphic to \( (\mathbb{C}_2, \rho_2) \). In fact, \( X \) is quasi-symmetrically homeomorphic to \( (\mathbb{C}_2, \rho_2) \). \( \square \)

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