Global classical solution to 3D isentropic compressible Navier-Stokes equations with large initial data and vacuum

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Abstract

In this paper, we investigate the existence of a global classical solution to 3D Cauchy problem of the isentropic compressible Navier-Stokes equations with large initial data and vacuum. In particular, when the far-field density is vacuum ($\tilde{\rho} = 0$), we get the global classical solutions under the assumption that $\left(\gamma - 1\right) E_0^\frac{2}{3\gamma - 1} \mu^{-\frac{1}{2}}$ is suitably small. In the case that the far-field density is away from vacuum ($\tilde{\rho} > 0$), the global classical solutions are obtained when $\left(\gamma - 1\right) E_0^\frac{2}{3\gamma - 1} \mu^{-\frac{1}{2}}$ is suitably small. It is showed that the initial energy $E_0$ can be large if the adiabatic exponent $\gamma$ is near 1 or the viscosity coefficient $\mu$ is taken to be large. These results improve the one obtained by Huang-Li-Xin in [15], where the existence of the classical solution is proved with small initial energy. It should be pointed out that in the theorems obtained in this paper, no smallness restriction is put upon the initial data.

Key Words: Compressible Navier-Stokes equations, global classical solution, vacuum.

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1 Introduction

In this paper, we consider the following isentropic compressible Navier-Stokes system in three-dimensional space

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u, 
\end{cases} \quad x \in \mathbb{R}^3, \quad t > 0, 
\]

(1.1)

with the initial conditions

\[
(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^3,
\]

(1.2)

and the far-field behavior

\[
\rho(x, t) \to \bar{\rho} \geq 0, \quad u(x, t) \to 0, \quad \text{as} \quad |x| \to \infty, \quad \text{for} \quad t \geq 0.
\]

(1.3)

Here \(\rho = \rho(x, t)\) and \(u = u(x, t) = (u_1, u_2, u_3)(x, t)\) represent the density and velocity of the fluid respectively; the pressure \(P\) is given by

\[
P(\rho) = A \rho^\gamma,
\]

with \(\gamma > 1\) the adiabatic exponent, \(A > 0\) a constant. The constant viscosity coefficients \(\mu\) and \(\lambda\) satisfy the following physical restrictions

\[
\mu > 0, \quad \lambda + \frac{2\mu}{3} \geq 0.
\]

(1.4)

There were lots of works on the well-posedness of solutions to (1.1). In the absence of vacuum (vacuum means \(\rho = 0\)), please refer for instance to [4, 9, 19, 23, 24] and references therein. The local existence and uniqueness of (strong) classical solutions with vacuum are known in [1, 2, 3]. The global existence of weak solution for large initial data was first solved by P.L. Lions in [22], where \(\gamma \geq \frac{3N}{N+2}\) for \(N = 2\) or \(3\). E. Feireisl, A. Novotny and H. Petzeltov in [8] extended Lions’s result to the case \(\gamma > \frac{3}{2}\) for \(N = 3\). Jiang and Zhang in [16, 17] proved the global existence of weak solution for any \(\gamma > 1\) for spherical symmetry or axisymmetric initial data. However, the regularity and uniqueness of weak solutions are basically open in general. Recently, Hoff and associates in [10, 11, 12] obtained a new type of global weak solutions with small energy, which have extra regularity compared with those large weak ones constructed by Lions (22) and Feireisl et al. (8).

It seems that one should not expect better regularities of the global solutions in general due to Xin’s results (31) and Rozanova’s results (26). It was proved that there is no global smooth solution in \(C^1([0, \infty); H^m(\mathbb{R}^d))\) \((m > \left\lfloor \frac{d}{2} \right\rfloor + 2)\) to the Cauchy problem of the full compressible Navier-Stokes system, if the initial density is nontrivial compactly supported (31), or the solutions are highly decreasing at infinity (26). Very recently, Xin and Yan in [32] improved the blow-up results in [31] by removing the assumptions that the initial density has compact support and the smooth solution has finite energy.

More recently, Huang-Li-Xin in [15] established the surprising global existence and uniqueness of classical solutions with constant state as far field which could be either vacuum or nonvacuum to 3D isentropic compressible Navier-Stokes equations with small total energy but possibly large oscillations. Motivated by this work, a natural question is whether we could remove the smallest restriction on the initial energy under certain conditions. In this paper, we shall give a definite answer to the question. More precisely, we establish the existence of a classical solution to (1.1)-(1.3) with vacuum at infinity under the assumption that
$$(\gamma - 1) \frac{3}{2} E_0^2 \mu^{-\frac{1}{2}}$$ is suitably small; in particular, the initial energy is allowed to be large when $\gamma$ goes to 1 or $\mu$ is taken to be large.

Before stating our main results, we would like to give some notations which will be used throughout this paper.

**Notations:**

(i) \[ \int_{\mathbb{R}^3} f = \int_{\mathbb{R}^3} f \, dx \quad \text{and} \quad \int_0^T g = \int_0^T g \, dt. \]

(ii) For $1 \leq l \leq \infty$, denote the $L^l$ spaces and the standard Sobolev spaces as follows:
\[ L^l = L^l(\mathbb{R}^3), \quad D^{k,l} = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^3) : \| \nabla^k u \|_{L^l} < \infty \right\}, \]
\[ W^{k,l} = L^l \cap D^{k,l}, \quad H^k = W^{k,2}, \quad D^1 = \left\{ u \in L^6 : \| \nabla u \|_{L^2} < \infty \right\}, \]
\[ \dot{H}^{\beta} = \left\{ u : \mathbb{R}^3 \to \mathbb{R}, \| u \|_{\dot{H}^{\beta}}^2 = \int |\xi|^{2\beta} |\hat{u}(\xi)|^2 \, d\xi < \infty \right\}, \]
\[ \| u \|_{D^k,l} = \| \nabla^k u \|_{L^l}. \]

(iii) $G = (2\mu + \lambda)\text{div}u - P$ is the effective viscous flux.

(iv) $\omega = \nabla \times u$ is the vorticity.

(v) $\dot{h} = h_t + u \cdot \nabla h$ denotes the material derivative.

(vi) $E_0 = \int_{\mathbb{R}^3} \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right)$ is the initial energy, where $G$ denotes the potential energy density given by
\[ G(\rho) \triangleq \rho \int_{\bar{\rho}}^\rho \frac{P(s) - P(\bar{\rho})}{s^2} \, ds. \]

It is clear that
\[ \begin{cases} G(\rho) = \frac{1}{\gamma - 1} P & \text{if } \bar{\rho} = 0, \\ c(\bar{\rho}, \bar{\rho})(\rho - \bar{\rho})^2 \leq G(\rho) \leq c(\bar{\rho}, \bar{\rho})(\rho - \bar{\rho})^2 & \text{if } \bar{\rho} > 0, \ 0 \leq \rho \leq \bar{\rho}, \end{cases} \]

for some positive constant $c(\bar{\rho}, \bar{\rho})$.

Now we state our main results. One of our main results is the following global existence to (1.1)-(1.3), when the far-field density is vacuum ($\bar{\rho} = 0$).

**Theorem 1.1** For any given $M > 0$ (not necessarily small) and $\bar{\rho} \geq 1$, assume that the initial data $(\rho_0, u_0)$ satisfy
\[ \frac{1}{2} \rho_0 |u_0|^2 + \frac{A}{\gamma - 1} \rho_0^\gamma \in L^1, \quad u_0 \in D^1 \cap D^3, \quad (\rho_0, P(\rho_0)) \in H^3, \] \[ 0 \leq \rho_0 \leq \bar{\rho}, \quad \| \nabla u_0 \|_{L^2}^2 \leq M \] \[ (1.5) \]
and the compatibility conditions
\[ -\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div}u_0 + \nabla P(\rho_0) = \rho_0 g, \] \[ (1.7) \]
where $g \in D^1$ and $\frac{1}{2}g \in L^2$. Then there exists a unique global classical solution $(\rho, u)$ in $\mathbb{R}^3 \times [0, \infty)$ satisfying, for any $0 < \tau < T < \infty$,

$$0 \leq \rho \leq 2\bar{\rho}, \quad x \in \mathbb{R}^3, \ t \geq 0,$$

$$
\begin{cases}
(\rho, P) \in C([0, T]; H^3), \\
u \in C([0, T]; D^1 \cap D^3) \cap L^2(0, T; D^4) \cap L^\infty(\tau, T; D^4), \\
u_\tau \in L^\infty(0, T; D^1) \cap L^2(0, T; D^2) \cap L^\infty(\tau, T; D^2) \cap H^1(\tau, T; D^1), \\
\sqrt{\rho} \nu_t \in L^\infty(0, T; L^2),
\end{cases}
$$

provided that

$$
\frac{(\gamma - 1)\frac{1}{6}E_{11}^\frac{3}{2}}{\mu^2} \leq \varepsilon \triangleq \min \left\{ \varepsilon_3, (2C(\bar{\rho}, M))\frac{24\mu^4}{\gamma}, (4C(\bar{\rho}))^{-2} \right\},
$$

where

$$\varepsilon_3 = \min \left\{ (CE_\tau)^{-3}_{(1<\gamma<\frac{3}{2}}), (CE_{11})^{-2}_{(\gamma>\frac{3}{2})}, \varepsilon_1, \varepsilon_2 \right\},$$

$$\varepsilon_2 = \min \left\{ C(\bar{\rho})^{-2}(\gamma - 1)^{-\frac{2}{3}}E_2^{-3}\mu^5_{(1<\gamma<\frac{3}{2})}, C(\bar{\rho})^{-1}\mu^2_{(\gamma>\frac{3}{2})} \right\},$$

$$\varepsilon_1 = \min \left\{ (4C(\bar{\rho}))^{-6}, 1 \right\}.$$ 

Here, $C$ depending on $\bar{\rho}, M$ and some other known constants but independent of $\mu, \lambda, \gamma - 1$ and $t$ (see (3.71), (3.74)). $E_2, E_7$ and $E_{11}$ are defined by (3.29), (3.53) and (3.65) respectively.

Now we briefly outline the main ideas of the proof of Theorem 1.1, some of which are inspired by [15]. The local existence and uniqueness of classical solutions to (1.1)-(1.3) are shown in [2]. Thus, to extend the classical solution globally in time, we need some global a priori estimates on the solutions $(\rho, u)$ in suitable higher norms. In this paper, the time-independent upper bound for the density (see (3.68)) is the key to the proof, and once that is obtained, the proof of Theorem 1.1 follows in the same way as in [15] (see Lemmas 3.7-3.9 and Section 4 in [15]). However, compared with [15], some new ideas are needed to recover all the a priori estimates under only the assumption (1.16) without the smallness of initial energy (see (3.40)-(3.41)).

(1) In [15], the small initial energy is used to get the smallness of $\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2$ (see (3.6) in [15]), which plays a crucial role in the analysis to prove the time-independent lower-order estimates (see Proposition 3.1 in [15]). Here, in order to close the a priori assumptions (3.22), it is necessary to handle the the right-hand side of (3.39) where $|\nabla u|^2, |\nabla u|^3, |\nabla u|^4, P|\nabla u|^2$ and $|P\nabla u|^2$ are involved. But the smallness of $\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2$ is not valid due to the lack of small initial energy (see Lemma 3.4). The crucial ingredient to encounter this difficulty is that we have a new observation that $|\nabla u|^2$ is in the form of $\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2$ when it appears by itself. And we can just obtain the smallness of $\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2$ under the assumption
that \((\gamma - 1) \frac{1}{\mu} E_0^\frac{1}{2} \mu^{\frac{1}{2}}\) is suitably small (see (3.7)). This combined with the estimates (3.18)-(3.21) implies Lemma 3.8. With these estimates and Zlotnik’s inequality, we can obtain the time-independent upper bound of \(\rho\). Thus, the Proposition 3.1 is proved.

(2) From the proof of Proposition 3.1, we know that it is important to find a suitable match for \(\mu\), \((\gamma - 1)\) and \(E_0\). That means much more complicated estimates than that in [15] are needed. To do this, we derive some more sophisticated inequalities about \(\mu\) (see Lemma 2.2). For more details, please see the proof of Proposition 3.1.

Concerning the global classical solutions for \((1.1)-(1.3)\) in the case that the far-field density is away from vacuum \((\bar{\rho} > 0)\), we have

**Theorem 1.2** For any given \(M > 0\) (not necessarily small) and \(\bar{\rho} \geq \bar{\rho} + 1\), assume that the initial data \((\rho_0, u_0)\) satisfy

\[
\frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \in L^1, \quad u_0 \in H^1 \cap D^3, \quad (\rho_0 - \bar{\rho}, P(\rho_0) - P(\bar{\rho})) \in H^3, \quad (1.11)
\]

\[
0 \leq \rho_0 \leq \bar{\rho}, \quad \|u_0\|_L^2 \leq E_0, \quad \|\nabla u_0\|_L^2 \leq M \quad (1.12)
\]

and the compatibility conditions

\[-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla P(\rho_0) = \rho_0 g, \quad (1.13)\]

where \(g \in D^1\) and \(\frac{1}{2} g \in L^2\). Then there exists a unique global classical solution \((\rho, u)\) in \(\mathbb{R}^3 \times [0, \infty)\) satisfying, for any \(0 < \tau < T < \infty\),

\[
0 \leq \rho \leq 2\bar{\rho}, \quad x \in \mathbb{R}^3, \quad t \geq 0, \quad (1.14)
\]

\[
\begin{cases}
(\rho - \bar{\rho}, P - P(\bar{\rho})) \in C([0, T]; H^3), \\
 u \in C([0, T]; D^1 \cap D^3) \cap L^2(0, T; D^4) \cap L^\infty(\tau, T; D^4), \\
u_t \in L^\infty(0, T; D^1) \cap L^2(0, T; D^2) \cap L^\infty(\tau, T; D^2) \cap H^1(\tau, T; D^1),
\end{cases}
\]

provided that \(\frac{(\gamma - 1) \frac{1}{\mu} E_0^\frac{1}{2}}{\mu^{\frac{1}{2}}} \leq \frac{\bar{\rho}}{2C}\) and

\[
\left(\frac{(\gamma - 1) \frac{1}{\mu} E_0^\frac{1}{2}}{\mu^{\frac{1}{2}}} \right)^{\frac{1}{2}} \leq \varepsilon \equiv \min \left\{ \varepsilon_0, \frac{4C(\bar{\rho}, M)}{(\gamma - 1)^{\frac{1}{2}}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}, \quad (1.16)
\]

where

\[
\varepsilon_6 = \min \left\{ C (E_{18} + E_{19} + E_{20})^{17}, C (E_{18} + E_{19} + E_{21})^{18}, \varepsilon_5 \right\},
\]

\[
\varepsilon_5 = \min \left\{ C (E_{15} E_{17} + E_{16})^{4}, \varepsilon_4 \right\},
\]

\[
\varepsilon_4 = \min \left\{ (4C(\bar{\rho}))^{-6}, 1 \right\}.
\]

Here \(C\) denotes a generic positive constant depending on \(\bar{\rho}, M\) and some other known constants but independent of \(\mu, \lambda, \gamma - 1, \bar{\rho}\), and \(t\).
We introduce the main ideas of the proof of Theorem 1.2 some of which are inspired by the arguments in [15] and the proof of Theorem 1.1. Compared with [15] and the proof of Theorem 1.1, some new difficulties occur.

1. In [15], the smallness of $\|\rho - \tilde{\rho}\|_{L^2}$ is easy to get because of the small initial energy. And in the proof of Theorem 1.1 where $\tilde{\rho} = 0$, $\|\rho\|_{L^2}$ can be small when $\gamma$ is near 1. But the smallness of $\|\rho - \tilde{\rho}\|_{L^2}$ would not stand in the case of $\tilde{\rho} > 0$, even as $\gamma \to 1$. To overcome this difficulty, based on elaborate analysis on the potential energy density $G(\rho)$, we succeed in deriving a new estimate of $\rho - \tilde{\rho}$ which shows that the $L^3$-norm of $\rho - \tilde{\rho}$ can be small when $\gamma \to 1$ (see Lemma 4.2). This estimate will play a crucial role in the analysis of this paper.

2. The second main difficulty is to obtain the smallness of $\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2$ under the the conditions of Theorem 1.2. The method used in Theorem 1.1 is no longer applicable here, we need some new estimates to obtain the smallness of $\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2$ (see Lemma 4.3).

3. The third main difficulty is to close the a priori assumptions on $A_1(T)$ and $A_2(T)$. In [15] and the proof of Theorem 1.1, the a priori assumptions on $A_1(T)$ and $A_2(T)$ can be closed together. But, this’s actually not applicable here because the the smallness of $\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2$ and $\|\rho - \tilde{\rho}\|_{L^2}$ is not valid. To overcome this difficulty, we have a observation that $A_2(T)$ can be controlled by the boundedness of $A_1(\sigma(T))$ and some other terms (see (3.11)). Therefore, to close the a priori assumptions on $A_1(T)$ and $A_2(T)$, the first step is to estimate $A_1(\sigma(T))$ (see (3.55)). Next, we can bounded $A_2(T)$ in (4.54). Finally, the estimate of $A_1(T)$ is obtained in (4.67).

Remark 1.3 In addition to the conditions of Theorem 1.1 and Theorem 1.2, if we assume further that $u \in H^3(\beta \in [\frac{1}{2}, 1])$ and replace $\|\nabla u_0\|_{L^2}^2 \leq M$ with $\|u\|_{H^{3\beta}} \leq M$, the conclusions in Theorem 1.1 and Theorem 1.2 will still hold, and the $\varepsilon$ will also depend on $M$ instead of $\tilde{M}$ accordingly. This can be achieved by a similar way as in [15].

Remark 1.4 The results in this paper generalize the results in [15] where small energy is required. More accurately, in the case of $\tilde{\rho} = 0$ and $\tilde{\rho} > 0$, the existence of classical solutions to (1.1)-(1.3) are obtained respectively; in particular, the initial energy is allowed to be large when $\gamma$ is near 1 or $\mu$ is taken to be large. It should be emphasized that Theorems 1.1 and 1.2 are still applicable to the case that initial energy $E_0$ is small for any fixed $\gamma$ and $\mu$.

Remark 1.5 It is worth noting that the requirement of small energy in [15] is equivalent to both the kinetic energy and the potential energy density ($G(\rho)$) being suitably small. However, the potential energy density could be always large through this article.

Remark 1.6 For the one-dimensional isentropic gas flow, Nishida and Smoller in [25] proved that the Cauchy problem for 1D isentropic Euler equations has a global solution provided that $(\gamma - 1)T.V.\{u_0, \rho_0\}$ is sufficiently small. This means that when $\gamma$ is near 1, one can allow large data, and conversely, as $\gamma$ increases, one must take correspondingly smaller data. Inspired by this work, we look for the large solutions to (1.1)-(1.3) under the assumption that $\gamma$ is near 1. On the other hand, from physical viewpoint, it is very nature to obtain a classical solution to (1.1)-(1.3) in general initial data, when the viscosity coefficient is sufficiently large. Note that the coefficients of viscosity are only required to satisfy the physical restriction (1.4) in the present paper.

Remark 1.7 The initial data can be large if the adiabatic exponent $\gamma$ goes to 1 or the viscosity coefficient $\mu$ is taken to be large, it is still unknown whether the global classical solution exists when the initial data is large for any fixed $\gamma$ and $\mu$. It should be noted that...
the similar question of whether there exists a global smooth solution of the three-dimensional incompressible Navier-Stokes equations with smooth initial data is one of the most outstanding mathematical open problems ([7]). Motivated by this, some blow-up criterions of strong (classical) solutions to \( (1.1) \) have been studied, please refer for instance to [13, 14, 27, 30] and references therein. In fact, for initial-boundary-value problems or periodic problems of compressible Navier-Stokes equations with vacuum in one dimension, or in two dimensions for isentropic flow, or in higher dimensions with symmetric initial data, the existence of global large regular solutions has been obtained, please refer to [5, 6, 18, 28, 29].

Remark 1.8 In this paper, \( \gamma \) may go to 1 or \( \mu \) could be sufficiently large if necessary. But the cases for \( \gamma \) is arbitrary large and \( \mu \) could vanish are not under consideration.

The rest of the paper is organized as follows. In Section 2, we collect some known inequalities and facts which will be frequently used later. In Section 3, we obtain the proof of Theorem 1.1. Then the proof of Theorem 1.2 is completed in Section 4.

2 Preliminaries

If the solutions are regular enough (such as strong solutions and classical solutions), \( (1.1) \) is equivalent to the following system

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u + \nabla P(\rho) &= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u.
\end{aligned}
\]  

System \( (2.1) \) is supplemented with initial conditions

\[
(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^3,
\]  

and the far-field behavior

\[
\rho(x, t) \to \bar{\rho} \geq 0, \quad u(x, t) \to 0, \quad \text{as} \quad |x| \to \infty, \quad \text{for} \quad t \geq 0.
\]  

Since the exact value of \( A \) in the pressure \( P \) doesn’t play a role in this paper, we henceforth assume \( A = 1 \). Next, we will list several facts which will be used in the proof of the main results. The first one is the well-known Gagliardo-Nirenberg inequality (see [20]).

Lemma 2.1 For any \( p \in [2, 6], q \in (1, \infty) \) and \( r \in (3, \infty) \), there exist some generic constants \( C > 0 \) that may depend on \( q \) and \( r \) such that for \( f \in H^1(\mathbb{R}^3) \) and \( g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3) \), we have

\[
\|f\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)} \|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{3p-6}{3p}},
\]  

\[
\|g\|_{C(\mathbb{R}^3)} \leq C \|g\|_{L^q(\mathbb{R}^3)} \|\nabla g\|_{L^r(\mathbb{R}^3)}^{\frac{3r}{3r + q(r-3)}}.
\]  

We now state some elementary estimates that follow from \( (2.4) \) and the standard \( L^p \)-estimate for the following elliptic system derived from the momentum equation in \( (1.1) \):

\[
\Delta G = \text{div}(\rho \dot{u}), \quad \mu \Delta \omega = \nabla \times (\rho \dot{u}).
\]
Lemma 2.2 Let \((\rho, u)\) be a smooth solution of (2.1), (2.3). Then there exists a generic positive constant \(C\) such that for any \(p \in [2, 6]\)

\[
\|\nabla G\|_{L^p} \leq C\|\rho \dot{u}\|_{L^p}, \quad \|\nabla \omega\|_{L^p} \leq \frac{C}{\mu} \|\rho \dot{u}\|_{L^p},
\]

(2.7)

\[
\|G\|_{L^p} \leq C\left( (2\mu + \lambda)\|\nabla u\|_{L^2} + \|P - P(\bar{\rho})\|_{L^2} \right)^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}},
\]

(2.8)

\[
\|G\|_{L^p} \leq C\left( (2\mu + \lambda)\|\nabla u\|_{L^3} + \|P - P(\bar{\rho})\|_{L^3} \right)^{\frac{6-p}{p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}},
\]

(2.9)

\[
\|w\|_{L^p} \leq C\left( \frac{1}{\mu} \right)^{\frac{3p-6}{2p}} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}}
\]

(2.10)

and

\[
\|\nabla u\|_{L^p} \leq CN_p(2\mu + \lambda)^{\frac{6-p}{2p}} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}}
\]

\[+ \frac{C}{2\mu + \lambda} \left( \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \|P - P(\bar{\rho})\|_{L^2}^{\frac{6-p}{2p}} + \|P - P(\bar{\rho})\|_{L^p} \right), \]

(2.11)

where \(N_p = 1 + \left( 2 + \frac{\lambda}{\mu} \right)^{\frac{3p-6}{2p}}\).

**Proof.** The standard \(L^p\)-estimate for elliptic system (2.6), yields (2.7). Using (2.4), we obtain

\[
\|G\|_{L^p} \leq C\|G\|_{L^2}^{\frac{6-p}{2p}} \|G\|_{L^6}^{\frac{3p-6}{2p}} \leq C\|G\|_{L^2}^{\frac{6-p}{2p}} \|\nabla G\|_{L^2}^{\frac{3p-6}{2p}}
\]

\[
\leq C\left( \|\nabla u\|_{L^2} + \|P - P(\bar{\rho})\|_{L^2} \right)^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}},
\]

(2.12)

\[
\|G\|_{L^p} \leq C\|G\|_{L^3}^{\frac{6-p}{p}} \|G\|_{L^6}^{\frac{2p-6}{p}} \leq C\|G\|_{L^3}^{\frac{6-p}{p}} \|\nabla G\|_{L^2}^{\frac{2p-6}{p}}
\]

\[
\leq C\left( \|\nabla u\|_{L^3} + \|P - P(\bar{\rho})\|_{L^3} \right)^{\frac{6-p}{p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}}
\]

(2.13)

and

\[
\|w\|_{L^p} \leq C\|w\|_{L^2}^{\frac{6-p}{2p}} \|w\|_{L^6}^{\frac{3p-6}{2p}} \leq C\|w\|_{L^2}^{\frac{6-p}{2p}} \|\nabla w\|_{L^2}^{\frac{3p-6}{2p}}
\]

\[\leq C\left( \frac{1}{\mu} \right)^{\frac{3p-6}{2p}} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2p}}.
\]

(2.14)

Note that \(-\Delta u = -\nabla \text{div } u + \nabla \times \omega\), which implies that

\[
\nabla u = -\nabla (-\Delta)^{-1} \text{div } u + \nabla (-\Delta)^{-1} \nabla \times \omega.
\]
Thus the standard $L^p$-estimate shows that
\[
\|\nabla u\|_{L^p} \leq C \left( \|\text{div } u\|_{L^p} + \|\omega\|_{L^p} \right) \quad \text{for } p \in [2, 6],
\]
which together with (2.8), (2.10) and the definition of $G$, give
\[
\|\nabla u\|_{L^p} \leq C \|\text{div } u\|_{L^p} + C \|\text{curl } u\|_{L^p}
\]
\[
\leq \frac{C}{2\mu + \lambda} \left( \|G\|_{L^p} + \|P - P(\bar{\rho})\|_{L^p} \right) + C \|w\|_{L^p}
\]
\[
\leq C \left( (2\mu + \lambda)^{\frac{6-3p}{2p}} + \mu^{\frac{6-3p}{2p}} \right) \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho\|_{L^2}^{\frac{3p-6}{2p}}
\]
\[
+ \frac{C}{2\mu + \lambda} \left( \|\rho\|_{L^2}^{\frac{3p-6}{2p}} \|P - P(\bar{\rho})\|_{L^2}^{\frac{6-p}{2p}} + \|P - P(\bar{\rho})\|_{L^p} \right)
\]
\[
\leq CN_p (2\mu + \lambda)^{\frac{6-3p}{2p}} \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\rho\|_{L^2}^{\frac{3p-6}{2p}}
\]
\[
+ \frac{C}{2\mu + \lambda} \left( \|\rho\|_{L^2}^{\frac{3p-6}{2p}} \|P - P(\bar{\rho})\|_{L^2}^{\frac{6-p}{2p}} + \|P - P(\bar{\rho})\|_{L^p} \right),
\]
(2.15)
\[\square\]

**Lemma 2.3** (#33) *Let the function $y$ satisfy*
\[
y'(t) = g(y) + b'(t) \quad \text{on } [0, T], \quad y(0) = y^0,
\]
*with $g \in C(\mathbb{R})$ and $y, b \in W^{1,1}(0, T)$. If $g(\infty) = -\infty$ and*
\[
b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1),
\]
*for all $0 \leq t_1 \leq t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then*
\[
y(t) \leq \max\{y^0, \xi\} + N_0 < \infty \quad \text{on } [0, T],
\]
*where $\xi$ is a constant such that*
\[
g(\xi) \leq -N_1, \quad \text{for } \xi \geq \bar{\xi}.
\]

### 3 The proof of Theorem 1.1

In this section, we will first establish the time-independent upper bound of the density. Assume that $(\rho, u)$ is a smooth solution to (1.1)-(1.3) on $\mathbb{R}^3 \times (0, T)$ for some positive time $T > 0$. Set $\sigma = \sigma(t) = \min\{1, t\}$ and denote
\[
\begin{aligned}
A_1(T) &= \sup_{0 \leq t \leq T} \sigma \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^T \int_{\mathbb{R}^3} \sigma \frac{|\rho|^2}{\mu}, \\
A_2(T) &= \sup_{0 \leq t \leq T} \sigma^2 \int_{\mathbb{R}^3} \frac{|\rho|^2}{\mu} + \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^2, \\
A_3(T) &= \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \frac{|\rho|^3}{\mu^3}.
\end{aligned}
\]
(3.1)

The following proposition plays a crucial role in this section.
**Proposition 3.1** Assume that the initial data satisfies (1.5), (1.6) and (1.7). If the solution \((\rho, u)\) satisfies

\[
A_1(T) + A_2(T) \leq \frac{2(\gamma - 1)\frac{3}{2}E_0}{\mu^2}, \quad A_3(\sigma(T)) \leq \frac{2(\gamma - 1)\frac{3}{2}E_0}{\mu^2}, \quad 0 \leq \rho \leq 2\bar{\rho},
\]

then

\[
A_1(T) + A_2(T) \leq \frac{(\gamma - 1)\frac{3}{2}E_0^\frac{1}{2}}{\mu^2}, \quad A_3(\sigma(T)) \leq \frac{(\gamma - 1)\frac{3}{2}E_0^\frac{1}{2}}{\mu^2}, \quad 0 \leq \rho \leq \frac{7}{4}\bar{\rho},
\]

\((x, t) \in \mathbb{R}^3 \times [0, T]\), provided

\[
0 \leq \rho \leq 2\bar{\rho}, \quad \frac{(\gamma - 1)\frac{3}{2}E_0^\frac{1}{2}}{\mu^2}, \quad A_3(\sigma(T)) \leq \frac{(\gamma - 1)\frac{3}{2}E_0^\frac{1}{2}}{\mu^2}, \quad 0 \leq \rho \leq \frac{7}{4}\bar{\rho},
\]

\((x, t) \in \mathbb{R}^3 \times [0, T]\), provided \(\frac{(\gamma - 1)\frac{3}{2}E_0^\frac{1}{2}}{\mu^2} \leq \varepsilon\). Here

\[
\varepsilon = \min \left\{ \varepsilon_3, (2C(\bar{\rho}, M))^{-\frac{16}{3}}\mu^4, (4C(\bar{\rho}))^{-2} \right\},
\]

where

\[
\varepsilon_3 = \min \left\{ (CE_7)^{-3} \mid_{1 < \gamma \leq \frac{3}{2}}, (CE_{11})^{-2} \mid_{\gamma > \frac{3}{2}}, \varepsilon_1, \varepsilon_2 \right\},
\]

\[
\varepsilon_2 = \min \left\{ C(\bar{\rho})^{-2}(\gamma - 1)^{-2}E_2^{-\frac{5}{2}}\mu^5 \mid_{1 < \gamma \leq \frac{3}{2}}, C(\bar{\rho})^{-1}\mu^2E_2^{-\frac{3}{2}} \mid_{\gamma > \frac{3}{2}} \right\},
\]

\[
\varepsilon_1 = \min \left\{ (4C(\bar{\rho}))^{-6}, 1 \right\}.
\]

Here, \(C\) depending on \(\bar{\rho}, M\) and some other known constants but independent of \(\mu, \lambda, \gamma - 1\) and \(t\) (see (3.71), (3.74)). \(E_2, E_7\) and \(E_{11}\) are defined by \(3.29, 3.33\) and \(3.65\) respectively.

**Proof.** Proposition 3.1 can be derived from Lemmas 3.2-3.9 below.

**Lemma 3.2** Under the conditions of Proposition 3.1 it holds that

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} P \leq (\gamma - 1)E_0,
\]

\[
\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{E_0}{\mu}.
\]

**Proof.** Multiplying (2.1) by \(G'(\rho)\) and (2.1) by \(u\) and integrating, then using (2.3), one gets

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \left( \frac{1}{2} \rho |u|^2 + G(\rho) \right) + \int_0^T \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2) \leq E_0,
\]

which gives (3.4) and (3.5). 

\(\square\)
Lemma 3.3 Under the conditions of Proposition 3.1, assume further that \(1 < \gamma \leq \frac{3}{2}\), we have

\[
\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{(\gamma - 1)^{\frac{2}{3}} E_0^{\frac{4}{3}}}{\mu} E_1,
\]

where \(E_1 = C(\overline{\rho}, M) \left( 1 + \frac{(\gamma - 1)^{\frac{2}{3}} E_0^{\frac{4}{3}}}{\mu^2} \right)\).

Proof. First, assume that \(\frac{(\gamma - 1)^{\frac{2}{3}} E_0^{\frac{1}{3}}}{\mu^2} \leq 1\). Multiplying (2.1) by \(u\) and then integrating the resulting equality over \(\mathbb{R}^3\), and using integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^2 + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2) = \int_{\mathbb{R}^3} P \text{div} u.
\]

Integrating (3.8) over \([0, \sigma(T)]\), and using (3.4) and Cauchy inequality, we have

\[
\sup_{0 \leq t \leq \sigma(T)} \frac{1}{2} \int_{\mathbb{R}^3} \rho |u|^2 + \int_{\mathbb{R}^3}^{\sigma(T)} \left( \frac{\mu}{2} |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2 \right)
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}^3} \rho_0 |u_0|^2 + \frac{C}{\mu} \int_{0}^{\sigma(T)} |P|^2
\]

\[
\leq \frac{1}{2} \|\rho_0\|_{L^\frac{1}{2}} \|u_0\|^2 + \frac{C(\overline{\rho})(\gamma - 1)E_0}{\mu}
\]

\[
\leq C(\overline{\rho}, M)(\gamma - 1)\frac{2}{3} E_0^{\frac{2}{3}} \left( 1 + \frac{(\gamma - 1)^{\frac{2}{3}} E_0^{\frac{4}{3}}}{\mu^2} \right)
\]

\[
\leq C(\overline{\rho}, M)(\gamma - 1)\frac{2}{3} E_0^{\frac{2}{3}} \left( 1 + \frac{(\gamma - 1)^{\frac{2}{3}} E_0^{\frac{4}{3}}}{\mu^2} \right),
\]

where \(1 < \gamma \leq \frac{3}{2}\) has been used. This completes the proof of Lemma 3.2.

Lemma 3.4 Under the conditions of Proposition 3.1, it holds that

\[
A_1(T) \leq \frac{C(2\mu + \lambda)}{\mu} \int_0^T \sigma \|\nabla u\|^3_{L^3} + \frac{C\gamma}{\mu} \int_0^T \sigma P \|\nabla u\|^2
\]

\[
+ \frac{C(2\mu + \lambda)}{\mu} \int_0^{\sigma(T)} \|\nabla u\|^2_{L^2} + \frac{C(\gamma - 1)E_0}{\mu^2}
\]

(3.10)

and

\[
A_2(T) \leq CA_1(T) + \frac{C\gamma^2}{\mu^2} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P \nabla u|^2 + C \left( \frac{2\mu + \lambda}{\mu} \right)^2 \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4.
\]

(3.11)
Proof. The proof of (3.10) and (3.11) is due to Hoff [9] and Huang-Li-Xin [15]. For \( m \geq 0 \), multiplying (2.1) by \( \sigma^m u \), integrating the resulting equality over \( \mathbb{R}^3 \), we have

\[
\int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2 = \int_{\mathbb{R}^3} (-\sigma^m \dot{u} \cdot \nabla P + \mu \sigma^m \Delta u \cdot \dot{u} + (\lambda + \mu) \sigma^m \nabla \text{div} \cdot \dot{u})
\]

\[
= \sum_{i=1}^3 I_i. \tag{3.12}
\]

Integrating by parts gives

\[
I_1 = - \int_{\mathbb{R}^3} \sigma^m \dot{u} \cdot \nabla P
\]

\[
= \int_{\mathbb{R}^3} \sigma^m \text{div} u \cdot \nabla P + \int_{\mathbb{R}^3} \sigma^m \text{div} (u \cdot \nabla) P
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \sigma^m \text{div} u P - m \int_{\mathbb{R}^3} \sigma^m \text{div} P - \int_{\mathbb{R}^3} \sigma^m \text{div} P \rho_t + \int_{\mathbb{R}^3} \sigma^m \text{div} (u \cdot \nabla) P
\]

\[
\leq \left( \int_{\mathbb{R}^3} \sigma^m \text{div} u P \right)_t - m \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \text{div} P
\]

\[
+ (\gamma - 1) \sigma^m \int_{\mathbb{R}^3} P |\text{div}|^2 + C \int_{\mathbb{R}^3} \sigma^m P |\nabla u|^2, \tag{3.13}
\]

\[
I_2 = \int_{\mathbb{R}^3} \mu \sigma^m \Delta u \cdot \dot{u}
\]

\[
= - \int_{\mathbb{R}^3} \mu \sigma^m \nabla u \cdot \nabla u_t + \int_{\mathbb{R}^3} \mu \sigma^m \Delta u (u \cdot \nabla)
\]

\[
\leq - \mu \left( \int_{\mathbb{R}^3} \sigma^m |\nabla u|^2 \right)_t + \frac{\mu m}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} |\nabla u|^2 + C \mu \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3 \tag{3.14}
\]

and

\[
I_3 = \int_{\mathbb{R}^3} (\lambda + \mu) \sigma^m \nabla \text{div} u \cdot \dot{u}
\]

\[
\leq - \mu + \lambda \left( \sigma^m \int_{\mathbb{R}^3} |\text{div}|^2 \right)_t + \frac{m(\mu + \lambda)}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} |\text{div}|^2
\]

\[
+ C(\mu + \lambda) \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3. \tag{3.15}
\]

Substituting (3.13)-(3.15) into (3.12) shows that

\[
\frac{d}{dt} \left( \frac{\mu}{2} \sigma^m |\nabla u|^2_{L^2} + \frac{(\lambda + \mu)}{2} \sigma^m |\text{div} u|^2_{L^2} - \sigma^m \int_{\mathbb{R}^3} \text{div} u P \right) + \int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2
\]
Integrating by parts leads to the resulting equation over $\mathbb{R}$ choosing $m$, $T$.

Integrating (3.16) over $(0, T)$, using (3.14), we get

\[
\sup_{0 \leq t \leq T} \left( \frac{\mu}{4} \sigma^m \| \nabla u \|^2_{L^2} + \frac{(\lambda + \mu)}{2} \sigma^m \| \text{div} u \|^2_{L^2} \right) + \int_0^T \int_{\mathbb{R}^3} \sigma \rho |\dot{u}|^2
\]

\[\leq C(\gamma - 1) E_0 \mu + C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \|P\| |\text{div} u| + \frac{m(4\mu + 3\lambda)}{2} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2
\]

\[+ C(2\mu + \lambda) \int_0^T \sigma \int_{\mathbb{R}^3} |\nabla u|^3 + (3\gamma - 2) \int_0^T \int_{\mathbb{R}^3} \sigma \rho |\dot{u}|^2
\]

\[\leq C(\gamma - 1) E_0 \mu + C(2\mu + \lambda) \int_0^{\sigma(T)} \|P\| |\nabla u|^2 + C(2\mu + \lambda) \int_0^T \sigma \int_{\mathbb{R}^3} |\nabla u|^3
\]

\[+ C \int_0^T \int_{\mathbb{R}^3} \sigma \rho |\nabla u|^2, \]  

(3.17)

choosing $m = 1$, then one gets (3.10).

Next, for $m \geq 0$, operating $\sigma^m \dot{u}^j [\partial_j / \partial t + \text{div}(u \cdot \cdot)]$ on (2.1), summing over $j$, and integrating the resulting equation over $\mathbb{R}^3 \times [0, T]$, we obtain after integration by parts

\[
\sup_{0 \leq t \leq T} \left( \frac{1}{2} \sigma^m \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right) - \frac{m}{2} \int_0^T \sigma^{-1} \int_{\mathbb{R}^3} \rho |\dot{u}|^2
\]

\[= - \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j [\partial_j P_t + \text{div}(\partial_j Pu)] + \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j \left[ \Delta u^j + \text{div}(u \Delta u^j) \right]
\]

\[+(\lambda + \mu) \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j [\partial_j \partial_j \text{div} u + \text{div}(u \partial_j \text{div} u)]
\]

\[= \sum_{i=1}^3 II_i. \]  

(3.18)

Integrating by parts leads to

\[II_1 = - \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j \left[ \partial_j P_t + \text{div}(\partial_j Pu) \right]
\]

\[= \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_j \dot{u}^j P_t + \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_j \dot{u}^j \partial_j Pu^k
\]

\[= - \int_0^T \int_{\mathbb{R}^3} \sigma^m \text{div} \dot{u} P^r (\rho \text{div} u + u \cdot \nabla \rho) + \int_0^T \int_{\mathbb{R}^3} \sigma^m \text{div} \dot{u} \text{div} u P
\]

\[+ \int_0^T \int_{\mathbb{R}^3} \sigma^m \text{div} \dot{u} \nabla P - \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_j u^k P
\]
\[
\leq C\gamma \int_0^T \int_{\mathbb{R}^3} \sigma^m P|\nabla \dot{u}| |\nabla u| \\
\leq \frac{\mu}{4} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + \frac{C\gamma^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |P\nabla u|^2,
\]
(3.19)

\[
II_2 = \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j \left[ \Delta u^j_t + \text{div}(u\Delta u^j) \right]
\]
\[
= -\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\partial_i \dot{u}^j|^2 + \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_i \partial_j (u^k \partial_k u^j) \\
- \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_i (u^k \partial_i u^j) - \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_i \dot{u}^j \partial_i (u^k \partial_k u^j) \\
\leq -\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + C\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}||\nabla u|^2 + C\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 \\
\leq -\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + C\mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4.
\]
(3.20)

Similarly
\[
II_3 \leq -\frac{\mu + \lambda}{2} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\text{div} \dot{u}|^2 \\
+ C(\mu + \lambda) \left( 1 + \frac{\lambda}{\mu} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 + \frac{\mu}{4} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2.
\]
(3.21)

Substituting (3.19)–(3.21) into (3.18) shows that
\[
\sigma^m \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + (\mu + \lambda) \int_0^T \int_{\mathbb{R}^3} \sigma^m |\text{div} \dot{u}|^2 \\
\leq C \int_0^T \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \frac{C\gamma^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |P\nabla u|^2 \\
+ \frac{C(2\mu + \lambda)^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4,
\]
(3.22)

where (1.4) has been used. Taking \( m = 2 \), we immediately obtain (3.11). The proof of Lemma 3.4 is completed. \( \square \)

**Lemma 3.5** Under the conditions of Proposition 3.1, it holds that
\[
\sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} \leq E_2
\]
(3.23)
and

$$\sup_{0 \leq t \leq \sigma(T)} t \int_{\mathbb{R}^3} \frac{\rho|\dot{u}|^2}{\mu} + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} t|\nabla \dot{u}|^2 \leq E_3,$$

(3.24)

provided

$$\frac{(\gamma - 1)^{\frac{1}{\gamma}} E_0^\frac{1}{\gamma}}{\mu^2} \leq \min \left\{ \left( 4C(\bar{\rho}) \right)^{-6}, 1 \right\} \triangleq \varepsilon_1.$$

**Proof.** First, we assume that

$$\frac{(\gamma - 1)^{\frac{1}{\gamma}} E_0^\frac{1}{\gamma}}{\mu^2} \leq 1.$$ Multiplying (2.1) by $u_t$, integrating the resulting equality over $\mathbb{R}^3$ and using (2.4), we have

$$\frac{d}{dt} \left( \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \|\text{div}u\|_{L^2}^2 - \int_{\mathbb{R}^3} \text{div}uP \right) + \int_{\mathbb{R}^3} \rho|\dot{u}|^2$$

$$= \int_{\mathbb{R}^3} \rho \ddot{u}(u \cdot \nabla u) - \int_{\mathbb{R}^3} \text{div}uP$$

$$\leq C(\bar{\rho}) \left( \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \rho|u|^3 \right)^{\frac{1}{3}} \|\nabla u\|_{L^6} + \int_{\mathbb{R}^3} \text{div}u(Pu) + (\gamma - 1) \int_{\mathbb{R}^3} P|\text{div}u|^2$$

$$\leq \frac{C(\bar{\rho})}{2\mu + \lambda} \left( \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \rho|u|^3 \right)^{\frac{1}{3}} \left( (CN_6 + 1)\|\rho \ddot{u}\|_{L^2} + \|P\|_{L^6} \right)$$

$$- \int_{\mathbb{R}^3} Pu \cdot \nabla u + C(\bar{\rho})(\gamma - 1) \int_{\mathbb{R}^3} |\nabla u|^2$$

$$\leq \frac{C(\bar{\rho})\mu(CN_6 + 1)}{2\mu + \lambda} \left( \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \right) A_3^{\frac{1}{2}}(\sigma(T)) + \frac{C(\bar{\rho})\mu}{2\mu + \lambda} \left( \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \right) \frac{1}{2} A_3^{\frac{1}{2}}(\sigma(T)) \|P\|_{L^6}$$

$$- \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} Pu \cdot \nabla G + \frac{1}{2(2\mu + \lambda)} \int_{\mathbb{R}^3} \text{div}uP^2 + C(\bar{\rho})(\gamma - 1)\|\nabla u\|_{L^2}^2$$

$$\leq C(\bar{\rho}) \left( \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \right) A_3^{\frac{1}{2}}(\sigma(T)) + C(\bar{\rho})A_3^{\frac{1}{2}}(\sigma(T)) \|P\|_{L^6}^2 + C(\bar{\rho})(\gamma - 1)\|\nabla u\|_{L^2}^2$$

$$+ \frac{C}{2\mu + \lambda} \|P\|_{L^3}\|\nabla u\|_{L^2}\|\rho \ddot{u}\|_{L^2} + \frac{C}{2\mu + \lambda} (\|\nabla u\|_{L^2}^2 + \|P\|_{L^1}^2)$$

$$\leq C(\bar{\rho}) \left( \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \right) A_3^{\frac{1}{2}}(\sigma(T)) + C(\bar{\rho})A_3^{\frac{1}{2}}(\sigma(T))(\gamma - 1)\frac{1}{2} E_0^{\frac{3}{2}} + C(\bar{\rho})(\gamma - 1)\|\nabla u\|_{L^2}^2$$

$$+ \frac{1}{4}\|\sqrt{\rho} \ddot{u}\|_{L^2}^2 + \frac{C(\bar{\rho})}{(2\mu + \lambda)^2} (\gamma - 1)^{\frac{1}{2}} E_0^{\frac{3}{2}} \|\nabla u\|_{L^2}^2 + \frac{C}{2\mu + \lambda} \|\nabla u\|_{L^2}^2 + \frac{C(\bar{\rho})}{2\mu + \lambda} (\gamma - 1)E_0$$

$$\leq C(\bar{\rho}) \left( \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \right) A_3^{\frac{1}{2}}(\sigma(T)) + C(\bar{\rho})A_3^{\frac{1}{2}}(\sigma(T))(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{3}{2}} + C(\bar{\rho})(\gamma - 1)\|\nabla u\|_{L^2}^2.$$
where we have used \((2.11)\).

Integrating \((3.25)\) over \((0, \sigma(T))\), we obtain that

\[
\frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \|\text{div}u\|_{L^2}^2 - \int_{\mathbb{R}^3} \text{div}uP + \frac{1}{2} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \leq C(\bar{\rho}) A_3^\frac{1}{3}(\sigma(T))(\gamma - 1)^\frac{1}{3} E_0^\frac{1}{3} + C(\bar{\rho})(\gamma - 1) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2
\]

\[
+ \frac{C(\bar{\rho})}{(2\mu + \lambda)^2}(\gamma - 1)^\frac{2}{3} E_0^\frac{2}{3} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 + \frac{C}{2\mu + \lambda} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2
\]

\[+ \frac{C(\bar{\rho})}{2\mu + \lambda}(\gamma - 1)E_0 + C\mu M, \quad (3.26)\]

provided \(\frac{(\gamma - 1)^\frac{1}{3} E_0^\frac{1}{3}}{\mu^\frac{1}{4}} \leq \left(4C(\bar{\rho})\right)^{-6} \).

For \(1 < \gamma \leq \frac{3}{2}\), using \((3.7)\), we get

\[\sup_{0 \leq t \leq \sigma(T)} \left\{ \|\nabla u\|_{L^2}^2 + (\lambda + \mu)\|\text{div}u\|_{L^2}^2 \right\} + \frac{1}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \leq E_2^1, \quad (3.27)\]

where

\[E_2^1 = \frac{C(\gamma - 1)^\frac{7}{4}}{\mu^\frac{1}{4}} + \frac{C(\gamma - 1)^\frac{1}{3}}{\mu^\frac{1}{3}} E_1 + \frac{C(\gamma - 1)^\frac{7}{4}}{\mu^\frac{1}{4}} E_1 + C(\gamma - 1)^\frac{1}{3} E_1 + C(\gamma - 1)^\frac{7}{4} + CM\]

and we have also used the facts that \(\frac{(\gamma - 1)^\frac{1}{3} E_0^\frac{1}{3}}{\mu^\frac{1}{2}} \leq 1\) and \(\mu + \lambda > 0\).

Similarly, for \(\gamma > \frac{3}{2}\), using \((3.5)\), we have

\[\sup_{0 \leq t \leq \sigma(T)} \left\{ \|\nabla u\|_{L^2}^2 + (\lambda + \mu)\|\text{div}u\|_{L^2}^2 \right\} + \frac{1}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \leq E_2^2, \quad (3.28)\]

where

\[E_2^2 = \frac{C(\gamma - 1)^\frac{7}{4}}{\mu^\frac{1}{4}} + \frac{C(\gamma - 1)^\frac{1}{3}}{\mu} E_1 + \frac{C(\gamma - 1)^\frac{7}{4}}{\mu^\frac{1}{4}} + \frac{C}{\mu^2} + C(\gamma - 1)^\frac{1}{3} + CM.\]

Combining \((3.27)\)-\((3.28)\), the result leads to \((3.23)\), where

\[E_2 = \max \left\{ E_2^1, E_2^2 \right\}. \quad (3.29)\]

Note that \(\mu\) may be different in \(E_2^1\) and \(E_2^2\).
Taking $m = 1$ in (3.22), we obtain that
\[
\sigma \int_{\mathbb{R}^3} \rho|\dot{u}|^2 + \mu \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma|\nabla \dot{u}|^2 + (\mu + \lambda) \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma|\text{div}\dot{u}|^2
\leq \mu E_2 + \frac{C \gamma^2}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma|\dot{P} u|^2 + \frac{C(2\mu + \lambda)^2}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma|\nabla u|^4
\]
\[
\leq \mu E_2 + \frac{C \gamma^2}{\mu} \left( \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho^4 \right)^{\frac{1}{2}} \left( \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma^2|\nabla u|^4 \right)^{\frac{1}{2}} + \frac{C(2\mu + \lambda)^2}{\mu} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma|\nabla u|^4
\]
\[
\leq \mu E_2 + \frac{C \gamma^2(\gamma - 1)^{\frac{1}{2}} E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{C \gamma^2(\gamma - 1)^{\frac{1}{2}} E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + C \left( \frac{\lambda}{\mu} \right)^2 \left( \frac{E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{(\gamma - 1)^{\frac{1}{2}} E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} \right)
\]
\[
\leq \mu E_3,
\]
where
\[
E_3 = E_2 + \frac{C \gamma^2(\gamma - 1)^{\frac{1}{2}} E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{C \gamma^2(\gamma - 1)^{\frac{1}{2}} E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + C \left( \frac{\lambda}{\mu} \right)^2 \left( \frac{E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{(\gamma - 1)^{\frac{1}{2}} E_2^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} \right).
\]
To get (3.30), we have used the following estimate
\[
\int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma|\nabla u|^4
\]
\[
\leq \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2} \int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^6}^3
\]
\[
\leq \frac{E_2^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} \int_0^{\sigma(T)} \sigma \left( \frac{(4\mu + \lambda)^3}{(2\mu + \lambda)^3} \|\rho\dot{u}\|_{L^2}^3 + \frac{\|P\|_{L^6}^3}{(2\mu + \lambda)^3} \right)
\]
\[
\leq \frac{CE_2^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} \sup_{0 \leq t \leq \sigma(T)} \sigma \|\rho\dot{u}\|_{L^2} \int_0^{\sigma(T)} \|\rho\dot{u}\|_{L^2}^2 + \frac{CE_2^{\frac{1}{2}}(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu^{\frac{3}{2}}}
\]
\[
\leq \frac{E_2^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} \left( \mu A_2(T) \right)^{\frac{1}{2}} \mu E_2 + \frac{CE_2^{\frac{1}{2}}(\gamma - 1)^{\frac{1}{2}}}{\mu^{\frac{3}{2}}}
\]
\[
\leq \frac{CE_2^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} + \frac{CE_2^{\frac{1}{2}}(\gamma - 1)^{\frac{1}{2}}}{\mu^{\frac{3}{2}}},
\]
(3.31)
due to Hölder inequality, (2.11), (3.2), the facts that $\frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu^{\frac{3}{2}}} \leq 1$ and $\mu + \lambda > 0$. The proof of Lemma 3.5 is completed. \qed
Lemma 3.6 Under the conditions of Proposition 3.1, we have

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^T \int_{\mathbb{R}^3} \frac{\rho|\dot{u}|^2}{\mu} \leq C(E_2 + 1)
\]  

(3.32)

and

\[
\sup_{0 \leq t \leq T} \sigma \int_{\mathbb{R}^3} \frac{\rho|\dot{u}|^2}{\mu} + \int_0^T \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 \leq C(E_2 + 1),
\]

(3.33)

provided \(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \varepsilon_1\).

Proof. By (3.2) and Lemma 3.5, we immediately get Lemma 3.6. \(\square\)

Next, we will close the a priori assumption on \(A_3(T)\).

Lemma 3.7 Under the conditions of Proposition 3.1, it holds that

\[
A_3(\sigma(T)) \leq \left( \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{3}{4}},
\]

(3.34)

provided

\[
\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ C(\bar{\rho})^{-2} (\gamma - 1)^{-\frac{2}{3}} E_2^{-3} \mu^5 |_{1 < \gamma \leq \frac{3}{2}}, \ C(\bar{\rho})^{-1} \mu^{\frac{2}{3}} E_2^{-\frac{3}{2}} |_{\gamma > \frac{3}{2}} \right\}
\]

\(\triangleq \varepsilon_2\). (3.35)

Proof. For \(1 < \gamma \leq \frac{3}{2}\),

\[
A_3(\sigma(T)) = \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \frac{\rho|u|^3}{\mu^3} \leq \frac{1}{\mu^3} \sup_{0 \leq t \leq \sigma(T)} \|\rho\|_{L^2} \|u\|_{L^6}^3
\]

\[
\leq \frac{C(\bar{\rho})}{\mu^3} (\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} E_2^{\frac{3}{2}}
\]

\[
\leq \left( \frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{3}{4}},
\]

(3.36)

provided \(\frac{(\gamma - 1)^{\frac{1}{6}} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq C(\bar{\rho})(\gamma - 1)^{-\frac{2}{3}} E_2^{-3} \mu^5\), where we have used Hölder inequality, Sobolev inequality and Lemma 3.5.

For \(\gamma > \frac{3}{2}\), using Hölder inequality, (3.6), (3.23) and \(2(\gamma - 1) \geq 1\), we obtain

\[
A_3(\sigma(T)) = \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \frac{\rho|u|^3}{\mu^3} \leq \frac{C(\bar{\rho})}{\mu^3} \sup_{0 \leq t \leq \sigma(T)} \|\rho^{\frac{3}{2}} u\|_{L^2} \|u\|_{L^6}^3
\]
\[
\leq \frac{C(\bar{\rho})(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{3}{2}} E_2^{\frac{3}{2}}}{\mu^2}
\]

\[
\leq \left( \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu^2} \right)^{\frac{1}{2}}, \quad (3.37)
\]

provided \( \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu^2} \leq C(\bar{\rho})\mu^2 E_2^{\frac{1}{2}} \).

\[\square\]

Lemma 3.8 Under the conditions of Proposition 3.1, it holds that

\[
A_1(T) + A_2(T) \leq \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu^2}, \quad (3.38)
\]

provided

\[
\frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu^2} \leq \min \left\{ (CE_7)^{-3} \right|_{1<\gamma \leq \frac{3}{2}}, (CE_{11})^{-2} \right|_{(\gamma > \frac{3}{2}), \varepsilon_1, \varepsilon_2} \right\} \triangleq \varepsilon_3.
\]

**Proof.** First we assume that \( \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu^2} \leq 1 \). It follows from Lemma 3.4 that

\[
A_1(T) + A_2(T) \leq \frac{C(\gamma - 1)E_0}{\mu^2} + \frac{C(2\mu + \lambda)^2}{\mu^2} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 + \frac{C(2\mu + \lambda)}{\mu} \int_0^T \sigma \|\nabla u\|_L^3
\]

\[
+ \frac{C\gamma^2}{\mu^2} \int_0^T \int_{\mathbb{R}^3} \sigma |\nabla u|^2 + \frac{C\gamma}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma P |\nabla u|^2 + C \left( \frac{2\mu + \lambda}{\mu} \right) \int_0^T \sigma |\nabla u|^2 \|u\|_{L^2}^2
\]

\[
\leq \frac{C(\gamma - 1)E_0}{\mu^2} + \sum_{i=1}^{8} II_i, \quad (3.39)
\]

Now we estimate the terms on the right hand side of (3.39).

**Case 1:** \( 1 < \gamma \leq \frac{3}{2} \).

For \( II_4 \), due to (2.11), we just estimate \( \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \) as follows

\[
\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq C N_i^4 (2\mu + \lambda)^{-3} \int_0^T \sigma^2 \|\rho \dot{u}\|_{L^2}^3 \|\nabla u\|_{L^2}
\]

\[
+ C(2\mu + \lambda)^{-4} \int_0^T \sigma^2 \|\rho \dot{u}\|_{L^2}^3 \|P\|_{L^2}
\]

\[
+ C(2\mu + \lambda)^{-4} \int_0^T \sigma^2 \|P\|_{L^4}^4
\]

\[
= \sum_{i=1}^3 III_i, \quad (3.40)
\]
Using Hölder inequality, (3.2), (3.4) and Lemma 3.6 we have

\[ III_1 \leq CN_1^4(2\mu + \lambda)^{-3} \sup_{0 \leq t \leq T} \sigma^2 \|\rho \dot{u}\|_{L^2}^2 \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \]

\[ \leq CN_1^4(2\mu + \lambda)^{-3} \mu \frac{A_2}{T} \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \int_0^T \sigma \|\rho\|_{L^3}^2 \|\nabla \dot{u}\|_{L^2}^2 \]

\[ \leq CN_1^4(2\mu + \lambda)^{-3} \mu \frac{A_2}{T} (T)(\gamma - 1)^{\frac{3}{2}} \frac{E_0^2}{E_2 + 1} \left( E_3 + 1 \right) \]

\[ \leq CN_1^4(2\mu + \lambda)^{-3} \mu \frac{1}{T} \left( \gamma - 1 \right)^{\frac{3}{2}} \frac{E_0^{17}}{E_2 + 1} \left( E_3 + 1 \right). \]

Next, it follows from Hölder inequality, (3.2), (3.4) and Lemma 3.6 that

\[ III_2 \leq C(2\mu + \lambda)^{-4} \sup_{0 \leq t \leq T} \sigma^2 \|\rho \dot{u}\|_{L^2}^2 \int_0^T \sigma \|\rho \dot{u}\|_{L^2} \|P\|_{L^2} \]

\[ \leq C(2\mu + \lambda)^{-4} \mu \frac{A_2}{T} \sup_{0 \leq t \leq T} \|P\|_{L^2} \int_0^T \sigma \|\rho\|_{L^3} \|\nabla \dot{u}\|_{L^2}^2 \]

\[ \leq C(2\mu + \lambda)^{-4} \mu \frac{1}{T} \left( \gamma - 1 \right)^{\frac{3}{2}} \frac{E_0^{17}}{E_2 + 1} \left( E_3 + 1 \right). \]

Thus, one gets from (2.8), (3.41) and (3.42) that

\[ \int_0^T \sigma^2 \|G\|_{L^4}^4 \leq C \left( (2\mu + \lambda) \|\nabla u\|_{L^2} + \|P\|_{L^2} \right) \|\rho \dot{u}\|_{L^2}^3 \]

\[ \leq C(2\mu + \lambda) \mu \frac{1}{T} \left( \gamma - 1 \right)^{\frac{3}{2}} \frac{E_0^{17}}{E_2 + 1} \left( E_3 + 1 \right) \]

\[ + C \mu \frac{1}{T} \left( \gamma - 1 \right)^{\frac{3}{2}} \frac{E_0^{17}}{E_2 + 1} \left( E_3 + 1 \right) \]

\[ \leq (2\mu + \lambda) \mu \frac{1}{T} \left( \gamma - 1 \right)^{\frac{3}{2}} \frac{E_0^{17}}{E_2 + 1} \frac{E_4}{E_2 + 1}, \]

where \( E_4 = C \left( E_2 + 1 \right)^{\frac{1}{2}} \left( E_3 + 1 \right) + C \left( \gamma - 1 \right)^{\frac{3}{2}} \mu \frac{1}{T} \left( E_3 + 1 \right) \). Here we have used the facts that \( \frac{1}{\sqrt{\mu}} \leq 1 \) and \( \mu + \lambda > 0 \).

To estimate \( III_3 \), one deduces from (2.1) that \( P \) satisfies

\[ P_t + u \cdot \nabla P + \gamma P \text{div} u = 0. \]

Multiplying (3.44) by \( 3 \sigma^2 P^2 \) and integrating the resulting equality over \( \mathbb{R}^3 \times [0, T] \), one gets that

\[ \frac{3 \gamma - 1}{2\mu + \lambda} \int_0^T \sigma^2 \|P\|_{L^4}^4 \]
\[
= -\sigma^2 \|P\|_{L^3}^3 + 2\sigma' \int_0^T \|P\|_{L^3}^3 - \frac{3\gamma - 1}{2\mu + \lambda} \int_0^T \sigma^2 \int_{\mathbb{R}^3} P^3 G
\]
\[
\leq C\|P\|_{L^3}^3 + \frac{3\gamma - 1}{4\mu + 2\lambda} \int_0^T \sigma^2 \|P\|_{L^4}^4 + \frac{C(3\gamma - 1)}{2\mu + \lambda} \int_0^T \sigma^2 \|G\|_{L^4}^4.
\] (3.45)

The combination of (3.43) with (3.45) implies
\[
\int_0^T \sigma^2 \|P\|_{L^4}^4 \leq C(2\mu + \lambda)(\gamma - 1)E_0 + C(2\mu + \lambda)\mu^{\frac{3}{4}}(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{24}}E_4
\]
\[
\leq (2\mu + \lambda)(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{24}}\mu^{\frac{1}{8}}E_5,
\] (3.46)

where \(E_5 = C(\gamma - 1)^{\frac{3}{4}}\mu^{\frac{1}{8}} + CE_4\).

Substituting (3.41), (3.42) and (3.46) into (3.40) shows that
\[
\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq (2\mu + \lambda)^{-3}\mu^{\frac{3}{4}}(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{12}}E_6,
\] (3.47)

where \(E_6 = CN_4^4E_4 + CE_5\). Thus,
\[
II_4 = C\left(\frac{2\mu + \lambda}{\mu}\right)^{\frac{3}{2}} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4
\]
\[
\leq \frac{(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{12}}E_6}{(2\mu + \lambda)\mu^{\frac{11}{8}}}.
\] (3.48)

To estimate \(II_5\), using Hölder inequality, (3.51) and (3.47), we have
\[
II_5 = \frac{C(2\mu + \lambda)}{\mu} \int_0^T \sigma \|\nabla u\|_{L^3}^3
\]
\[
\leq \frac{C(2\mu + \lambda)}{\mu} \left(\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4\right)^{\frac{1}{2}}
\]
\[
\leq \frac{C(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{23}{24}}E_6^{\frac{1}{4}}}{\mu^{\frac{11}{24}}(2\mu + \lambda)^{\frac{1}{2}}}.
\] (3.49)

For \(II_6\), using Hölder inequality, (3.46) and (3.47), one gets
\[
II_6 = \frac{C\gamma^2}{\mu^2} \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P\nabla u|^2
\]
\[
\leq \frac{C\gamma^2}{\mu^2} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |P|^4\right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4\right)^{\frac{1}{2}}
\]
\[
\leq \frac{C\gamma^2 E_5^{\frac{1}{4}}E_6^{\frac{1}{4}}(\gamma - 1)^{\frac{3}{4}}E_0^{\frac{11}{4}}}{\mu^{\frac{11}{24}}(2\mu + \lambda)}.
\] (3.50)
It holds from Hölder inequality, (3.5) and (3.46)-(3.47) that

\[ II_7 = \frac{C\gamma}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma P |\nabla u|^2 \]
\[ \leq \frac{C\gamma}{\mu} \left( \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P|^4 \right)^{\frac{1}{4}} \left( \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \right)^{\frac{1}{4}} \left( \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \]
\[ \leq \frac{C\gamma E_7^{\frac{1}{4}} E_6^{\frac{1}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{23}{24}}}{\mu^{\frac{11}{3}} (2\mu + \lambda)^{\frac{1}{2}}}. \]  

(3.51)

Finally, it follows from (3.7), (3.39) and (3.48)-(3.51) that

\[ A_1(T) + A_2(T) \leq \frac{C(\gamma - 1) E_0}{\mu^2} + \frac{C(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}} E_6^{\frac{1}{12}}}{(2\mu + \lambda) \mu^{\frac{11}{3}}} + \frac{C(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}} E_6^{\frac{1}{12}}}{(2\mu + \lambda) \mu^{\frac{11}{3}}} \]
\[ + \frac{C(2\mu + \lambda)(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}} E_6^{\frac{1}{12}}}{\mu^{\frac{11}{3}}}. \]
\[ \leq C \left( \frac{(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}}}{\mu^{\frac{11}{3}}} \right)^{\frac{4}{7}} E_7, \]  

(3.52)

where

\[ E_7 = \frac{(\gamma - 1)^{\frac{3}{4}}}{\mu} + \frac{(\gamma - 1)^{\frac{3}{4}}}{\mu^{\frac{11}{12}}} + \frac{(\gamma - 1)^{\frac{3}{4}} E_6^{\frac{1}{12}}}{\mu^{\frac{11}{12}}} + \frac{(\gamma - 1)^{\frac{3}{4}} E_6^{\frac{1}{12}}}{\mu^{\frac{11}{12}}} \]
\[ + \frac{\gamma(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}}}{\mu^{\frac{11}{12}}} + \frac{(2 + \lambda\frac{1}{\mu}) (\gamma - 1)^{\frac{3}{4}} E_1}{\mu^{\frac{11}{3}}}. \]  

(3.53)

and we have also used the facts that \( \frac{(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}}}{\mu^{\frac{11}{3}}} \leq 1 \) and \( \mu + \lambda > 0 \). It thus follows from (3.52) that

\[ A_1(T) + A_2(T) \leq C \left( \frac{(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}}}{\mu^{\frac{11}{3}}} \right)^{\frac{4}{7}} E_7 \leq \frac{(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}}}{\mu^{\frac{11}{3}}}, \]  

(3.54)

provided \( \frac{(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{11}{12}}}{\mu^{\frac{11}{3}}} \leq (CE_7)^{-3} \).

**Case 2:** \( \gamma > \frac{3}{2} \).
Just like Case 1,
\[
\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq \sum_{i=1}^{3} III_i. \tag{3.55}
\]

For \(III_1\), using (3.32) and Lemma 3.6 we have
\[
III_1 \leq CN_4^1(2\mu + \lambda)^{-3} \sup_{0 \leq t \leq T} (\sigma^2 \|\rho \dot{u}\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \leq C N_4^1(2\mu + \lambda)^{-\frac{1}{2}} A_2^{\frac{1}{2}}(T) \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \leq C N_4^1(2\mu + \lambda)^{-\frac{1}{2}} A_2^{\frac{1}{2}}(T) A_1(T)(E_2 + 1)^{\frac{1}{2}} \leq C N_4^1(2\mu + \lambda)^{-\frac{3}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{3}{4}} (E_2 + 1)^{\frac{1}{2}}. \tag{3.56}
\]

It follows from (3.2), (3.4) and Lemma 3.6 that
\[
III_2 \leq C(2\mu + \lambda)^{-4} \sup_{0 \leq t \leq T} (\sigma^2 \|\rho \dot{u}\|_{L^2}^2)^{\frac{1}{2}} \int_0^T \sigma \|\rho \dot{u}\|_{L^2}^2 \|P\|_{L^2} \leq C(2\mu + \lambda)^{-4} \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{3}{4}} (E_2 + 1)^{\frac{1}{2}}. \tag{3.57}
\]

Thus, one gets from (2.8), (3.56) and (3.57) that
\[
\int_0^T \sigma^2 \|G\|_{L^4}^4 \leq (2\mu + \lambda) \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{3}{4}} E_8, \tag{3.58}
\]
where
\[
E_8 = C(E_2 + 1)^{\frac{1}{2}} \left(1 + \frac{(\gamma - 1)^{\frac{3}{4}}}{\mu^{\frac{1}{2}}} \right). \tag{3.59}
\]

Using (3.45), \(\frac{1}{\gamma - 1} < 2\) and \(\frac{(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{3}{4}}}{\mu^{\frac{1}{2}}}\), we obtain
\[
\int_0^T \sigma^2 \|P\|_{L^4}^4 \leq C(2\mu + \lambda)(\gamma - 1) E_0 + C(3\gamma - 1)(2\mu + \lambda) \mu^{\frac{3}{4}} (\gamma - 1)^{\frac{3}{4}} E_0^{\frac{3}{4}} E_8 \leq C(2\mu + \lambda) \mu^{\frac{3}{4}} (\gamma - 1) E_0^{\frac{3}{4}} \left[E_0^{\frac{1}{2}} \mu^{\frac{3}{4}} + (3\gamma - 1)(\gamma - 1)^{-\frac{3}{4}} E_8 \right] \leq C(2\mu + \lambda) \mu^{\frac{3}{4}} (\gamma - 1) E_0^{\frac{3}{4}} \left[E_0^{\frac{1}{2}} \mu^{\frac{3}{4}} + 2^{\frac{3}{4}} (3\gamma - 1) E_8 \right] \leq (2\mu + \lambda) \mu^{\frac{3}{4}} (\gamma - 1) E_0^{\frac{3}{4}} E_9, \tag{3.60}
\]
where \(E_9 = C \mu^{-\frac{1}{2}} + C(3\gamma - 1) E_8\).
Substituting (3.56), (3.57) and (3.60) into (3.55), we get
\[
\int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq (2\mu + \lambda)^{-3} \mu^\frac{9}{7} (\gamma - 1)^\frac{3}{7} E_0^\frac{3}{7} E_{10},
\] (3.61)
where \( E_{10} = CN^\frac{1}{4} (E_2 + 1)^\frac{3}{7} + C \mu^\frac{7}{8} (\gamma - 1)^\frac{8}{7} (E_2 + 1)^\frac{3}{7} + C (\gamma - 1)^\frac{3}{7} E_0 \). Thus,
\[
II_4 \leq C \left( \frac{2\mu + \lambda}{\mu} \right)^2 (2\mu + \lambda)^{-3} \mu^\frac{9}{7} (\gamma - 1)^\frac{3}{7} E_0^\frac{3}{7} E_{10} \leq \frac{C (\gamma - 1)^\frac{3}{7} E_0^\frac{7}{7} E_{10}}{\mu^\frac{9}{7} (2\mu + \lambda)}. \] (3.62)

Similar to (3.49), (3.50) and (3.51), we obtain
\[
\begin{align*}
II_5 & \leq \frac{C (2\mu + \lambda)}{\mu} \left( \frac{E_0}{\mu} \right)^\frac{1}{7} ((2\mu + \lambda)^{-3} \mu^\frac{9}{7} (\gamma - 1)^\frac{3}{7} E_0^\frac{3}{7} E_{10})^\frac{1}{7}, \\
II_6 & \leq \frac{C (\gamma - 1)^\frac{3}{7} E_0^\frac{7}{7} E_{10}}{\mu^\frac{9}{7} (2\mu + \lambda)^\frac{1}{7}}, \\
II_7 & \leq \frac{C (\gamma - 1)^\frac{3}{7} E_0^\frac{7}{7} E_{10}}{\mu^\frac{9}{7} (2\mu + \lambda)^\frac{1}{7}}, \\
II_8 & \leq \frac{C (2\mu + \lambda) E_0}{\mu^2} \leq \frac{C (2\mu + \lambda)(\gamma - 1) E_0}{\mu^2}.
\end{align*}
\] (3.63)

It follows from (3.39), (3.63) and (3.7) that
\[
A_1(T) + A_2(T) \leq \frac{C (\gamma - 1) E_0}{\mu^2} + \frac{C (\gamma - 1)^\frac{3}{7} E_0^\frac{3}{7} E_{10}}{\mu^\frac{9}{7} (2\mu + \lambda)^\frac{1}{7}} + \frac{C (\gamma - 1)^\frac{3}{7} E_0^\frac{7}{7} E_{10}}{\mu^\frac{9}{7} (2\mu + \lambda)^\frac{1}{7}}
\]
\[
+ \frac{C (\gamma - 1)^\frac{3}{7} E_0^\frac{7}{7} E_{10}}{\mu^\frac{9}{7} (2\mu + \lambda)^\frac{1}{7}} + \frac{C (2\mu + \lambda)(\gamma - 1) E_0}{\mu^2}
\]
\[
\leq C \left( \frac{(\gamma - 1)^\frac{3}{7} E_0^\frac{1}{7}}{\mu^\frac{9}{7}} \right)^\frac{2}{3} E_{11} ,
\] (3.64)

where
\[
E_{11} = \frac{\gamma - 1}{\mu} + \frac{E_{10}}{\mu} + \frac{E_{10}^\frac{3}{7}}{\mu^\frac{9}{7}} + \frac{\gamma^2 (\gamma - 1)^\frac{3}{7} E_{10}^\frac{3}{7}}{\mu^\frac{9}{7}} + \frac{\gamma (\gamma - 1)^\frac{3}{7} E_{10}^\frac{7}{7}}{\mu^\frac{9}{7}}
\]
\[
+ \left( 2 + \frac{\lambda}{\mu} \right) (\gamma - 1)^\frac{3}{7}.
\] (3.65)
Here we have used the facts that \( \frac{1}{\gamma - 1} < 2 \), \( \frac{(\gamma - 1)\frac{1}{2}E_{0}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq 1 \) and \( \mu + \lambda > 0 \). It thus follows from (3.64) that

\[
A_{1}(T) + A_{2}(T) \leq C \left( \frac{(\gamma - 1)\frac{1}{2}E_{0}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^{\frac{3}{2}} E_{11} \leq \frac{(\gamma - 1)\frac{1}{2}E_{0}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}},
\]

(3.66)

provided \( \frac{(\gamma - 1)\frac{1}{2}E_{0}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq (CE_{11})^{-2} \).

By (3.54) and (3.66), for Case 1 and Case 2, we conclude that if

\[
\frac{(\gamma - 1)\frac{1}{2}E_{0}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ (CE_{7})^{-3} \left|_{(1<\frac{\gamma}{2})} \right., (CE_{11})^{-2} \left|_{(\gamma>\frac{3}{2})} \right., \varepsilon_{1}, \varepsilon_{2} \right\} \triangleq \varepsilon_{3},
\]

then

\[
A_{1}(T) + A_{2}(T) \leq \frac{(\gamma - 1)\frac{1}{2}E_{0}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}},
\]

(3.67)

Note that \( \mu \) may differ in \( E_{7} \) and \( E_{11} \). The proof of Lemma 3.8 is completed.

Next, we will derive the time-independent upper bound for the density. The approach is motivated by Huang-Li-Xin in [15] and Li-Xin in [21].

**Lemma 3.9** Under the conditions of Proposition 3.1, it holds that

\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq \frac{7\bar{\rho}}{4}
\]

for any \( (x,t) \in \mathbb{R}^{3} \times [0,T] \), provided

\[
\frac{(\gamma - 1)\frac{1}{2}E_{0}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ \varepsilon_{3}, \left(2C(\bar{\rho}, M)\right)^{-\frac{16}{\gamma}} \mu^{4}, (4C(\bar{\rho}))^{-2} \right\} \triangleq \varepsilon.
\]

**Proof.** Denoting \( D_{t}\rho = \rho_{t} + u \cdot \nabla \rho \) and expressing the equation of the mass conservation

\[
D_{t}\rho = g(\rho) + b'(t),
\]

where

\[
g(\rho) \triangleq -\frac{\rho P}{2\mu + \lambda}, \quad b(t) \triangleq -\frac{1}{2\mu + \lambda} \int_{0}^{t} \rho Gd\tau.
\]

For \( t \in [0,\sigma(T)] \), one deduces from that for all \( 0 \leq t_{1} < t_{2} \leq \sigma(T) \),

\[
|b(t_{2}) - b(t_{1})| \leq C \int_{0}^{\sigma(T)} \|||G(\cdot, t)||_{L^\infty} dt
\]

\[
\leq \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{0}^{\sigma(T)} \|G(\cdot, t)||^{\frac{1}{2}}_{L^{6}} \|\nabla G(\cdot, t)||^{\frac{1}{2}}_{L^{2}} dt
\]

\[
\leq \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{0}^{\sigma(T)} \|\rho u||^{\frac{1}{2}}_{L^{2}} \|\nabla u||^{\frac{1}{2}}_{L^{2}} dt
\]

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≤ \frac{C(\bar{\rho})}{2\mu + \lambda} \left( \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 t^{-\frac{3}{4}} dt \right)^{\frac{3}{4}} \left( \int_0^{\sigma(T)} t \|\nabla \dot{u}\|_{L^2}^2 \right)^{\frac{1}{4}}

≤ \frac{C(\bar{\rho}, M)}{2\mu + \lambda} \left( \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 t^{-\frac{3}{4}} dt \right)^{\frac{3}{4}} \left( \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 t^{-\frac{3}{4}} dt \right)^{\frac{3}{4}}

≤ \frac{C(\bar{\rho}, M)}{2\mu + \lambda} \sup_{0 \leq t \leq \sigma(T)} \left( \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 t^{-\frac{3}{4}} dt \right)^{\frac{3}{4}} \left( \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 t^{-\frac{3}{4}} dt \right)^{\frac{3}{4}}

≤ \frac{C(\bar{\rho}, M)}{\mu^\frac{3}{4}} A_1(\sigma(T)) \frac{3}{16}

≤ \frac{C(\bar{\rho}, M)}{\mu^\frac{3}{4}} \left( \frac{\gamma - 1}{\mu^\frac{1}{2}} \right) \frac{3}{16}.

(3.69)

Therefore, for \( t \in [0, \sigma(T)] \), one can choose \( N_0 \) and \( N_1 \) in Lemma 2.3 as follows

\[ N_1 = 0, \quad N_0 = \frac{C(\bar{\rho}, M)}{\mu^\frac{3}{4}} \left( \frac{\gamma - 1}{\mu^\frac{1}{2}} \right) \frac{3}{16}, \]

and \( \bar{\zeta} = 0 \). Then

\[ g(\zeta) = \frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = 0 \quad \text{for all} \quad \zeta \geq \bar{\zeta} = 0. \]

Thus

\[ \sup_{0 \leq t \leq \sigma(T)} \|\rho\|_{L^\infty} \leq \max \left\{ \bar{\rho}, 0 \right\} + N_0 \leq \bar{\rho} + \frac{C(\bar{\rho}, M)}{\mu^\frac{3}{4}} \left( \frac{\gamma - 1}{\mu^\frac{1}{2}} \right) \frac{3}{16} \leq \frac{3\bar{\rho}}{2}, \]

(3.70)

provided

\[ \frac{\gamma - 1}{\mu^\frac{1}{2}} \frac{E_0^\frac{1}{2}}{\mu^\frac{1}{2}} \leq \min \left\{ \varepsilon_3, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^\frac{1}{4} \right\}. \]

(3.71)

On the other hand, for \( t \in [\sigma(T), T] \), one deduces from (2.1), (2.7) and (3.1)

\[ |b(t_2) - b(t_1)| \leq \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{t_1}^{t_2} \|G(\cdot, t)\|_{L^\infty} dt \]

\[ \leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{\sigma(T)}^{t} \|G\|_{L^\infty} dt \]

\[ \leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{\sigma(T)}^{t} \|G\|_{L^6}^2 \|\nabla G\|_{L^6}^2 \]

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\begin{align*}
\leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{\sigma(T)} \|\nabla G\|_{L^2}^2 \|\nabla \dot{u}\|_{L^2}^2 \\
\leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + \frac{C(\bar{\rho})}{2\mu + \lambda} \int_{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 \|\nabla \dot{u}\|_{L^2}^2 \\
\leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) A_2(T) \int_{\sigma(T)} \|\nabla \dot{u}\|_{L^2}^2 \\
\leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) A_2^2(T) \\
\leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) \left( \frac{(\gamma - 1) \frac{1}{2} E_0^{1/2}}{\mu} \right)^2,
\end{align*}

provided \( \frac{(\gamma - 1) \frac{1}{2} E_0^{1/2}}{\mu} \leq \varepsilon_3 \).

Therefore, one can choose \( N_1 \) and \( N_0 \) in Lemma 2.3 as

\[ N_1 = \frac{1}{2\mu + \lambda}, \quad N_0 = C(\bar{\rho}) \left( \frac{(\gamma - 1) \frac{1}{2} E_0^{1/2}}{\mu} \right)^2. \]

Note that

\[ g(\zeta) = -\frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = -\frac{1}{2\mu + \lambda} \text{ for all } \zeta \geq 1, \]

one can set \( \bar{\zeta} = 1 \). Thus

\[ \sup_{\sigma(T) \leq s \leq T} \|\rho\|_{L^\infty} \leq \max \left\{ \frac{3}{2\bar{\rho}}, 1 \right\} + N_0 \leq \frac{3}{2\bar{\rho}} + C(\bar{\rho}) \left( \frac{(\gamma - 1) \frac{1}{2} E_0^{1/2}}{\mu} \right)^2 \leq \frac{7\bar{\rho}}{4}, \]

provided

\[ \frac{(\gamma - 1) \frac{1}{2} E_0^{1/2}}{\mu} \leq \min \left\{ \varepsilon_3, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}. \]

The combination of (3.70) and (3.73) completes the proof of Lemma 3.9. \( \square \)

In the following, we will prove the main result of this paper. First, we derive the time-dependent higher norm estimates of the smooth solution \((\rho, u)\). From now on, we will always assume (3.74) holds and denote the positive constant by \( C \) which will depends only on

\[ T, \quad \|\rho^0\|_{L^2}, \quad \|\nabla g\|_{L^2}, \quad \|\nabla u_0\|_{H^2}, \quad \|\rho_0\|_{H^3}, \quad \|P_0\|_{H^3}, \]

besides \( \mu, \lambda, \gamma \) and \( \bar{\rho} \), where \( g \) is as in (1.7). The higher-order estimates have been proved in [15]. For the convenience, we only list them without any proof.
Lemma 3.10  The following estimates hold:

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho u^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \leq C, \tag{3.75}
\]

\[
\sup_{0 \leq t \leq T} \left( \|\nabla \rho\|_{L^2 \cap L^6} + \|\nabla u\|_{H^1} \right) + \int_0^T \|\nabla u\|_{L^\infty} \leq C, \tag{3.76}
\]

\[
\sup_{0 \leq t \leq T} \left( \|\rho\|_{H^1} + \|P_t\|_{H^1} \right) + \int_0^T \left( \|\rho u_t\|_{L^2} + \|P u_t\|_{L^2} \right) \leq C, \tag{3.77}
\]

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\nabla u_t|^2 + \int_0^T \int_{\mathbb{R}^3} \rho u_{tt}^2 \leq C, \tag{3.78}
\]

\[
\sup_{0 \leq t \leq T} \left( \|\rho\|_{H^3} + \|P\|_{H^3} \right) \leq C, \tag{3.79}
\]

\[
\sup_{0 \leq t \leq T} \left( \|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^2} \right) + \int_0^T \left( \|\nabla u_t\|_{H^1} + \|\nabla u_{tt}\|_{H^3} \right) \leq C, \tag{3.80}
\]

for any \((x, t) \in \mathbb{R}^3 \times [0, \infty)\).

\[
\sup_{\tau \leq t \leq T} \left( \|\nabla u_t\|_{H^1} + \|\nabla^4 u\|_{L^2} \right) + \int_\tau^T \|\nabla u_t\|_{L^2} \leq C \tag{3.81}
\]

for any \(\tau \in (0, T)\), and \(C\) depends on \(\tau\).

With all the \textit{a priori} estimates above, we can prove the Theorem 1.1. In fact, the process of the proof is similar with [15], for simplicity, we omit it here. Interested readers can refer to [15].

4  The proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Recalling that

\[
\begin{align*}
A_1(T) &= \sup_{0 \leq t \leq T} \sigma \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^T \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu}, \\
A_2(T) &= \sup_{0 \leq t \leq T} \sigma^3 \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} + \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^2, \\
A_3(T) &= \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \frac{\rho |u|^3}{\mu^3}.
\end{align*}
\tag{4.1}
\]

Throughout the rest of the paper, we denote generic constant by \(C\) depending on \(\bar{\rho}, M\) and some other known constants but independent of \(\mu, \lambda, \gamma - 1, \) and \(t, \) and we write \(C(\alpha)\) to emphasize that \(C\) may depend on \(\alpha.\) For simplicity of presentation, we shall assume that

\[
\frac{(\gamma - 1 + \bar{\rho})E_0^\alpha}{\mu^3} \leq 1, \tag{4.2}
\]
where \( \frac{1}{10} \leq \alpha \leq 100, \frac{2}{3} \leq \beta \leq 12 \). It’s worth noting that the assumption (4.2) is not essential for our paper. Without this assumption, \( E_{12}E_{21} \) in the proof of Theorem 1.2 should be more complex.

The following proposition plays a crucial role in this section.

**Proposition 4.1** Assume that the initial data satisfies (1.11), (1.12) and (1.13), \( 1 < \gamma < 2 \).

If the solution \((\rho, u)\) satisfies

\[
\begin{align*}
0 \leq \rho & \leq 2\bar{\rho}, \\
A_1(T) & \leq 2 \left\{ \frac{\left( (\gamma - 1) \frac{3}{\omega} + \frac{1}{\omega} \right) E_0^\frac{\gamma}{\omega}}{\mu^\frac{1}{\omega}} \right\}^\frac{3}{\gamma}, \\
A_2(T) & \leq 2 \left( (\gamma - 1) \frac{1}{\omega} + \bar{\rho} \right) E_0^\frac{1}{\omega}, \\
A_3(\sigma(T)) & \leq 2 \left\{ \frac{\left( (\gamma - 1) \frac{1}{\omega} + \bar{\rho} \right) E_0^\frac{1}{\omega}}{\mu^\frac{1}{\omega}} \right\}^\frac{1}{\gamma},
\end{align*}
\]

(4.3)

then

\[
\begin{align*}
0 \leq \rho & \leq \frac{7}{4} \bar{\rho}, \\
A_1(T) & \leq \left\{ \frac{\left( (\gamma - 1) \frac{3}{\omega} + \frac{1}{\omega} \right) E_0^\frac{\gamma}{\omega}}{\mu^\frac{1}{\omega}} \right\}^\frac{3}{\gamma}, \\
A_2(T) & \leq \left( (\gamma - 1) \frac{1}{\omega} + \bar{\rho} \right) E_0^\frac{1}{\omega}, \\
A_3(\sigma(T)) & \leq \left\{ \frac{\left( (\gamma - 1) \frac{1}{\omega} + \bar{\rho} \right) E_0^\frac{1}{\omega}}{\mu^\frac{1}{\omega}} \right\}^\frac{1}{\gamma},
\end{align*}
\]

(4.4)

\((x,t) \in \mathbb{R}^3 \times [0,T], \) provided \( \frac{(\gamma - 1) \frac{3}{\omega} E_0^\frac{1}{\omega}}{\mu^\frac{1}{\omega}} \leq \frac{\bar{\rho}}{2C} \) and \( \frac{(\gamma - 1) \frac{1}{\omega} + \bar{\rho}}{\mu^\frac{1}{\omega}} \leq \varepsilon \). Here

\[
\varepsilon = \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{\omega}} \mu^4, (4C(\bar{\rho}))^{-2} \right\},
\]

where

\[
\varepsilon_6 = \min \left\{ \left( C(E_{18} + E_{19} + E_{20}) \right)^{-17}, \left( C(E_{18} + E_{19} + E_{21}) \right)^{-8}, \varepsilon_5 \right\},
\]

\[
\varepsilon_5 = \min \left\{ \left( C(E_{15}E_{17} + E_{16}) \right)^{-4}, \varepsilon_4 \right\},
\]

\[
\varepsilon_4 = \min \left\{ \left( 4C(\bar{\rho}) \right)^{-6}, 1 \right\}.
\]

**Proof.** Proposition 4.1 follows from Lemmas 4.2-4.8 below.

**Lemma 4.2** Let \((\rho, u)\) be a smooth solution of (1.1)-(1.3) with \( 0 \leq \rho \leq 2\bar{\rho} \) and \( 0 < \gamma - 1 < 1 \).

Then there exists a positive constant \( C \) such that

\[
\int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 \leq C(\bar{\rho})(\gamma - 1)\frac{\gamma}{\omega} E_0.
\]

(4.5)
Proof. A straightforward calculation implies that
\[
G(\rho) = \rho \int_{\rho}^{\hat{\rho}} \frac{P(s) - P(\hat{\rho})}{s^2} ds = \frac{1}{\gamma - 1} [\rho^\gamma - \hat{\rho}^\gamma - \gamma \hat{\rho}^{\gamma - 1}(\rho - \hat{\rho})].
\]

Next, we want to show that
\[
G(\rho) \geq \begin{cases} 
(\gamma - 1)^{-\frac{1}{2}} |\rho - \hat{\rho}|^{\gamma - 1}, & |\rho - \hat{\rho}| > (\gamma - 1)^{\frac{1}{2}}, \\
(\gamma - 1)^{-\frac{1}{4}} |\rho - \hat{\rho}|^{3}, & |\rho - \hat{\rho}| \leq (\gamma - 1)^{\frac{1}{2}}.
\end{cases}
\]
(4.6)

Case 1: $|\rho - \hat{\rho}| \geq (\gamma - 1)^{\frac{1}{2}}$

Without loss of generality, we only consider the case of $\rho - \hat{\rho} \geq (\gamma - 1)^{\frac{1}{2}}$. Now we define
\[
f(\rho) = \rho^\gamma - \hat{\rho}^\gamma - \gamma \hat{\rho}^{\gamma - 1}(\rho - \hat{\rho}) - (\gamma - 1)^{\frac{3}{2}} (\rho - \hat{\rho})^{\gamma - 1},
\]
and note that
\[
f(\hat{\rho}) = 0, \quad f'(\rho) = \gamma \rho^{\gamma - 1} - \gamma \hat{\rho}^{\gamma - 1} - (\gamma - 1)^{\frac{3}{2}} (\rho - \hat{\rho})^{\gamma - 2},
\]
thus
\[
f(\rho) = f'(\xi)(\rho - \hat{\rho}),
\]
where $\xi = (1 - \theta)\rho + \theta \hat{\rho}$ ($0 < \theta < 1$). Then
\[
f'(\xi) = \gamma \xi^{\gamma - 1} - \gamma \hat{\rho}^{\gamma - 1} - (\gamma - 1)^{\frac{3}{2}} (\xi - \hat{\rho})^{\gamma - 2}
\]
\[= \gamma(\gamma - 1)(\xi - \hat{\rho}) [(1 - \theta_1)\xi + \theta_1 \hat{\rho}]^{\gamma - 2} - (\gamma - 1)^{\frac{3}{2}} (\xi - \hat{\rho})^{\gamma - 2} \quad (0 < \theta_1 < 1)
\]
\[= (\gamma - 1)(1 - \theta)(\rho - \hat{\rho}) \left[ \gamma((1 - \theta_1)\xi + \theta_1 \hat{\rho})^{\gamma - 2} - (\gamma - 1)^{\frac{3}{2}} (\xi - \hat{\rho})^{\gamma - 3} \right]
\]
\[> (\gamma - 1)(1 - \theta)(\rho - \hat{\rho}) \left[ C\hat{\rho}^{\gamma - 2} - C(\gamma - 1)^{\frac{3}{2}} (\gamma - 1)^{\frac{3}{2}} \right]
\]
\[> (\gamma - 1)(1 - \theta)(\rho - \hat{\rho}) \left[ C\hat{\rho}^{\gamma - 2} - C(\gamma - 1)^{\frac{1}{2}} \right] \quad (1 < \gamma < 2)
\]
\[> 0,
\]
when $\gamma - 1 \leq \min \left\{ C\hat{\rho}^{12(\gamma - 2)}, \frac{1}{2} \right\} = \min \left\{ C\hat{\rho}^{-12}, C\hat{\rho}^{-6}, \frac{1}{2} \right\} = \eta$. This together with (4.7) implies the first inequality of (4.6).

Case 2: $|\rho - \hat{\rho}| < (\gamma - 1)^{\frac{1}{2}}$

Similar to case 1, we only consider the case of $0 \leq \rho - \hat{\rho} < (\gamma - 1)^{\frac{1}{2}}$. Now let
\[
g(\rho) = \rho^\gamma - \hat{\rho}^\gamma - \gamma \hat{\rho}^{\gamma - 1}(\rho - \hat{\rho}) - (\gamma - 1)^{\frac{3}{2}} (\rho - \hat{\rho})^{3},
\]
and note that
\[
g'(\rho) = \gamma \rho^{\gamma - 1} - \gamma \hat{\rho}^{\gamma - 1} - 3(\gamma - 1)^{\frac{3}{2}} (\rho - \hat{\rho})^{2},
\]
and
\[
g''(\rho) = \gamma(\gamma - 1)\rho^{\gamma - 2} - 6(\gamma - 1)^{\frac{3}{2}} (\rho - \hat{\rho}).
\]
Thus $g(\hat{\rho}) = g'(\hat{\rho}) = 0$, so that
\[
g(\rho) = g''(\zeta)(\rho - \hat{\rho})^{2},
\]
(4.8)
where $\zeta = (1 - \theta_2)\rho + \theta_2\bar{\rho}$ (0 < $\theta_2$ < 1). Then
\[
g''(\zeta) = \gamma(\gamma - 1)\zeta^{\gamma - 2} - 6(\gamma - 1)^{\frac{3}{2}}(\zeta - \bar{\rho})
\]
\[
= \gamma(\gamma - 1) [(1 - \theta_2)\rho + \theta_2\bar{\rho}]^{\gamma - 2} - 6(1 - \theta_2)(\gamma - 1)^{\frac{3}{2}}(\rho - \bar{\rho})
\]
\[
> (\gamma - 1) \left[ C\bar{\rho}^{\gamma - 2} - C(\gamma - 1)^{\frac{3}{2}} \right]
\]
\[
> 0,
\]
when $\gamma - 1 \leq \min \left\{ \frac{C\bar{\rho}^{12(\gamma - 2)}}{2}, \frac{1}{2} \right\} = \min \left\{ \frac{C\bar{\rho}^{12}, C\bar{\rho}^{-6}}{2} \right\} = \eta$. This together with (4.8) implies the second inequality of (4.6). We thus obtain (4.6). Combining (3.6) and (4.6), we get
\[
\int_{\Sigma_1} |\rho - \bar{\rho}|^3 \leq C(\bar{\rho}) \int_{\Sigma_1} |\rho - \bar{\rho}|^{\gamma - 1} \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{\gamma}} \int_{\mathbb{R}^3} G(\rho) \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{\gamma}} E_0,
\]
\[
\int_{\Sigma_2} |\rho - \bar{\rho}|^3 \leq (\gamma - 1)^{\frac{1}{\gamma}} \int_{\mathbb{R}^3} G(\rho) \leq (\gamma - 1)^{\frac{1}{\gamma}} E_0,
\]
where
\[
\begin{align*}
\Sigma_1 &= \left\{ x \in \mathbb{R}^3 | \rho(x, t) - \bar{\rho} > (\gamma - 1)^{\frac{1}{\gamma}} \right\}, \\
\Sigma_2 &= \left\{ x \in \mathbb{R}^3 | \rho(x, t) - \bar{\rho} \leq (\gamma - 1)^{\frac{1}{\gamma}} \right\}.
\end{align*}
\]
Thus
\[
\int_{\mathbb{R}^3} |\rho - \bar{\rho}|^3 = \int_{\Sigma_1} |\rho - \bar{\rho}|^3 + \int_{\Sigma_2} |\rho - \bar{\rho}|^3 \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{\gamma}} E_0.
\]
On the other hand, for $1 > \gamma - 1 > \eta$, it is clear that $G(\rho) \geq C(\bar{\rho})(\rho - \bar{\rho})^2$. Then
\[
\int_{\mathbb{R}^3} |\rho - \bar{\rho}|^3 \leq C(\bar{\rho}) \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 \leq C(\bar{\rho})\eta^{-\frac{1}{2}}(\gamma - 1)^{\frac{1}{\gamma}} E_0 \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{\gamma}} E_0.
\]
This completes the prove of the Lemma 4.2.

**Lemma 4.3** Under the conditions of Proposition 4.1, we have
\[
\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{C(\gamma - 1)\frac{1}{\mu} E_0^{\frac{2}{\gamma}} E_{12}}{\mu^{\frac{1}{\gamma}}} + \frac{C\rho E_0}{\mu},
\]
provided $\frac{(\gamma - 1)\frac{1}{\mu} E_0^{\frac{1}{\gamma}}}{\mu^{\frac{1}{\gamma}}} \leq \frac{\bar{\rho}}{2C}$.

**Proof.** Multiplying (2.1) by $u$ and then integrating the resulting equality over $\mathbb{R}^3$, and using integration by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho|u|^2 + \int_{\mathbb{R}^3} (\mu|\nabla u|^2 + (\lambda + \mu)|\text{div} u|^2) = \int_{\mathbb{R}^3} (P - P(\bar{\rho}))\text{div} u.
\]
Now, we turn to estimate the term on the right-hand side of (4.10),
\[
\int_{\mathbb{R}^3} (P - P(\bar{\rho}))\text{div} u \leq \|P - P(\bar{\rho})\|_{L^1} \|\nabla u\|_{L^{\frac{1}{\gamma}}}
\]
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\begin{align}
\leq |P - P(\tilde{\rho})| ||u||_{L^2} \frac{1}{2} \| \nabla u \|_{L^2} \\
\leq \frac{C}{\mu} |P - P(\tilde{\rho})| ||u||_{L^2} \frac{7}{2} \frac{\| \nabla u \|_{L^2}}{L^2} + \frac{\mu}{4} \| \nabla u \|_{L^2}^2. \tag{4.11}
\end{align}

Since

\[
\tilde{\rho} \int_{\mathbb{R}^3} |u|^2 \leq \int_{\mathbb{R}^3} |\rho - \tilde{\rho}| |u|^2 + \int_{\mathbb{R}^3} \rho |u|^2 \\
\leq \left( \int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^3 \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |u|^3 \right)^{\frac{7}{4}} + E_0 \\
\leq C(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} (\mu - \frac{1}{2}) ||u||_{L^2}^2 + \mu \frac{1}{2} \| \nabla u \|_{L^2}^2 + E_0 \\
\leq \frac{C(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} ||u||_{L^2}^2}{\mu^2} + C(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} \| \nabla u \|_{L^2}^2 + E_0 \\
\leq \frac{\tilde{\rho}}{2} ||u||_{L^2}^2 + C(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} \| \nabla u \|_{L^2}^2 + E_0,
\]

provided \( \frac{(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2}}{\mu^2} \leq \frac{\tilde{\rho}}{2C} \). Then

\[
\int_{\mathbb{R}^3} |u|^2 \leq \frac{C(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} \| \nabla u \|_{L^2}^2}{\tilde{\rho}} + CE_0 \\
\leq \frac{C(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} \| \nabla u \|_{L^2}^2}{(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2}} + \frac{\mu}{4} \| \nabla u \|_{L^2}^2. \tag{4.12}
\]

Combining (4.11) and (4.12), we get

\[
\int_{\mathbb{R}^3} (P - P(\tilde{\rho})) \text{div} u \leq \frac{C}{\mu} (\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} \left( \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} \frac{\mu}{\mu^2} \| \nabla u \|_{L^2}^2 + \frac{\mu}{16} \| \nabla u \|_{L^2}^2 \right) + \frac{\mu}{4} \| \nabla u \|_{L^2}^2 \\
\leq \frac{C(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2} \| \nabla u \|_{L^2}^2}{\mu^2} + \frac{(\gamma - 1) \frac{11}{18} E_0^\frac{11}{18}}{\mu^2} + \frac{\mu}{16} \| \nabla u \|_{L^2}^2 \\
\leq \frac{C(\gamma - 1) \frac{1}{\mu^2} E_0^\frac{1}{\mu^2}}{\mu^2} + \frac{C(\gamma - 1) \frac{11}{18} E_0^\frac{11}{18}}{\mu^2} + \frac{\mu}{8} \| \nabla u \|_{L^2}^2. \tag{4.13}
\]

Substituting (4.13) into (4.10) and integrating the resulting inequality over \([0, \sigma(T)]\), we get

\[
\sup_{0 \leq t \leq \sigma(T)} \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 + \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2 \right) \\
\leq \frac{1}{2} \int_{\mathbb{R}^3} (\rho_0 - \tilde{\rho}) |u_0|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \tilde{\rho} |u_0|^2 + \frac{C(\gamma - 1) \frac{7}{54} E_0^\frac{7}{54}}{\mu^\frac{19}{12}} + \frac{C(\gamma - 1) \frac{11}{18} E_0^\frac{11}{18}}{\mu^\frac{8}{12}}.
\]
Lemma 4.4

where (4.2) has been used.

Next, we will give the details of the calculation of $E_{12}$. In fact

$$
\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{C(\gamma-1)^{\frac{1}{13}} E_0^{\frac{5}{36}}}{\mu^{\frac{12}{11}}} \left( (\gamma-1)^{\frac{1}{13}} E_0^{\frac{5}{36}} + \frac{C(\gamma-1)^{\frac{7}{36}}}{\mu^{\frac{12}{11}}} \right) + \frac{C\bar{\rho} E_0}{\mu},
$$

where

$$
E_{12} = 1 + \frac{1}{\mu} + \frac{(\gamma-1)^{\frac{35}{603}}}{\mu^{\frac{11}{11}}}
$$

Next, we will give the details of the calculation of $E_{12}$. In fact

$$
\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2
\leq \frac{C(\gamma-1)^{\frac{1}{13}} E_0^{\frac{5}{36}}}{\mu^{\frac{12}{11}}} \left( (\gamma-1)^{\frac{1}{13}} E_0^{\frac{5}{36}} + \frac{C(\gamma-1)^{\frac{7}{36}}}{\mu^{\frac{12}{11}}} \right) + \frac{C\bar{\rho} E_0}{\mu}
$$

$$
= \frac{C(\gamma-1)^{\frac{1}{13}} E_0^{\frac{5}{36}}}{\mu^{\frac{12}{11}}} \left( (\gamma-1)^{\frac{1}{13}} E_0^{\frac{5}{36}} + \frac{C(\gamma-1)^{\frac{7}{36}}}{\mu^{\frac{12}{11}}} \right) + \frac{C\bar{\rho} E_0}{\mu}
$$

$$
= \frac{C(\gamma-1)^{\frac{1}{13}} E_0^{\frac{5}{36}}}{\mu^{\frac{12}{11}}} \left\{ \frac{(\gamma-1)E_0^{\frac{5}{36} \times 156}}{\mu^{\frac{12}{11}}} + \frac{(\gamma-1)E_0^{\frac{1}{17} \times 468}}{\mu^{\frac{12}{11}}} \right\} + \frac{C\bar{\rho} E_0}{\mu}
$$

$$
\leq \frac{C(\gamma-1)^{\frac{1}{13}} E_0^{\frac{5}{36}} E_{12}}{\mu^{\frac{12}{11}}} + \frac{C\bar{\rho} E_0}{\mu},
$$

where (4.12) has been used.

\[ \square \]

Lemma 4.4 Under the conditions of Proposition 4.1, we have

$$
A_1(T) \leq \frac{4}{\mu} \int_{\mathbb{R}^3} \sigma \text{div}(P - P(\bar{\rho})) + \frac{C(\gamma - 1 + P(\bar{\rho})) E_0}{\mu}
$$
and

\[ A_2(T) \leq C A_1(\sigma(T)) + C \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\bar{\rho})|^4 \]

\[ + C \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 + \frac{C P(\bar{\rho})^2}{\mu^2} E_0. \]  

**Proof.** The proof of Lemma 4.4 is similar to Lemma 3.4, we just need to deal with the first terms in (3.12) and (3.18) again. Here \( J_1 \) and \( J_2 \) denote \( I_1 \) and \( II_1 \) in Lemma 3.4 respectively. Integrating by parts gives

\[ J_1 = - \int_{\mathbb{R}^3} \sigma^m \dot{u} \cdot \nabla P \]

\[ = \int_{\mathbb{R}^3} \sigma^m \text{div}_t (P - P(\bar{\rho})) + \int_{\mathbb{R}^3} \sigma^m \text{div}(u \cdot \nabla u)(P - P(\bar{\rho})) \]

\[ = \frac{d}{dt} \left( \int_{\mathbb{R}^3} \sigma^m \text{div}(P - P(\bar{\rho})) \right) - m \sigma^{m-1} \sigma \int_{\mathbb{R}^3} \sigma^m \text{div}(P - P(\bar{\rho})) \]

\[ + \int_{\mathbb{R}^3} \sigma^m \left( (\gamma - 1) P(\text{div} u)^2 + (P - P(\bar{\rho})) \partial_i u^j \partial_j u^i + P(\bar{\rho}) (\text{div} u)^2 \right) \]

and

\[ J_2 = - \int_0^T \int_{\mathbb{R}^3} \sigma^m \dot{u}^j \left[ \partial_j P_t + \text{div}(\partial_j Pu) \right] \]

\[ = \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_j \dot{u}^j P_t + \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j (\partial_j P u^k) \]

\[ = (1 - \gamma) \int_0^T \int_{\mathbb{R}^3} \sigma^m \text{div} u \text{div}(P - P(\bar{\rho})) - \gamma P(\bar{\rho}) \int_0^T \int_{\mathbb{R}^3} \sigma^m \text{div} u \text{div} u \]

\[ - \int_0^T \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_j u^k (P - P(\bar{\rho})) \]

\[ \leq \frac{\mu}{4} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + C \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^m |P - P(\bar{\rho})|^4 \]

\[ + C \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 + \frac{C P(\bar{\rho})^2}{\mu} \int_0^T \int_{\mathbb{R}^3} \sigma^m |\nabla u|^2. \]

Then, from (3.17) and (3.22), we have

\[ \sup_{0 \leq t \leq T} \left( \frac{\mu}{4} \sigma^m \|\nabla u\|_{L^2}^2 + \frac{(\lambda + \mu)}{2} \sigma^m \|\text{div} u\|_{L^2}^2 \right) + \int_0^T \int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2 \]

\[ \leq \int_{\mathbb{R}^3} \sigma^m \text{div}(P - P(\bar{\rho})) - \int_0^T \int_{\mathbb{R}^3} m \sigma^{m-1} \text{div}(P - P(\bar{\rho})) \]

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\[ + C \int_{T}^{0} \int_{\mathbb{R}^3} \sigma^m \left( (\gamma - 1)P(\text{div}u)^2 + (P - P(\tilde{\rho}))(\nabla u)^2 \right) \]
\[ + C(2\mu + \lambda) \int_{0}^{T} \sigma^m \int_{\mathbb{R}^3} |\nabla u|^3 \]  
\[
(4.22)
\]

and
\[
\sigma^m \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_{0}^{T} \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + (\mu + \lambda) \int_{0}^{T} \int_{\mathbb{R}^3} \sigma^m |\text{div} \dot{u}|^2 \leq C \int_{0}^{T} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + C \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} \right) \int_{0}^{T} \int_{\mathbb{R}^3} \sigma^m |P - P(\tilde{\rho})|^4 \]
\[ + C \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right) \int_{0}^{T} \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 \]
\[ + \frac{CP(\tilde{\rho})^2}{\mu} \int_{0}^{T} \int_{\mathbb{R}^3} \sigma^m |\nabla u|^2. \]  
\[
(4.23)
\]

choosing \( m = 1 \) in (4.22) and \( m = 3 \) in (4.23), one gets (4.18) and (4.19).

\[ \Box \]

**Lemma 4.5** Under the conditions of Proposition 4.1, it holds that
\[
\sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu} \leq E_{13},
\]
\[
(4.24)
\]
\[
\sup_{0 \leq t \leq \sigma(T)} \sigma \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 \leq E_{14},
\]
\[
(4.25)
\]

provided
\[
\left( \frac{(\gamma - 1)\bar{\mu} + \bar{\rho}^{\frac{1}{2}}}{\mu} \right) E_{0}^{\frac{3}{2}} \leq \min \left\{ \left( 4C(\tilde{\rho}) \right)^{-6}, 1 \right\} \triangleq \varepsilon_{4},
\]

where
\[
E_{13} = C(\tilde{\rho}) + C(M + 1) + C \left( \frac{1}{\mu} + \gamma + P(\tilde{\rho}) \right) (E_{12} + 1)
\]

and
\[
E_{14} = E_{13} + C(\tilde{\rho}) \left( 1 + (\gamma - 1)^2 \right) + C \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right)^2 \frac{E_{13}^3}{\mu^4}
\]
\[ + C \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \frac{E_{13}^3}{\mu^4} + C\tilde{\rho}^{2\gamma - 1}. \]  
\[
(4.26)
\]

**Proof.** By an argument similar to the proof of Lemma 3.4, one can derive from (3.25) that
\[
\frac{d}{dt} \left( \frac{\mu}{2} ||\nabla u||_{L^2}^2 + \frac{(\lambda + \mu)}{2} ||\text{div} u||_{L^2}^2 - \int_{\mathbb{R}^3} \text{div} (P - P(\tilde{\rho})) \right) + \int_{\mathbb{R}^3} \rho |\dot{u}|^2
\]

\[ \Box \]
\[ \int_{\mathbb{R}^3} \rho \hat{u} (u \cdot \nabla u) - \int_{\mathbb{R}^3} \text{div} u P_t \]

\[ \leq C(\bar{\rho}) \left( \int_{\mathbb{R}^3} \rho |\hat{u}|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \rho |u|^3 \right)^{\frac{1}{3}} \|\nabla u\|_{L^6} + \int_{\mathbb{R}^3} \text{div div} ((P - P(\bar{\rho})) u) \]

\[ + (\gamma - 1) \int_{\mathbb{R}^3} P |\text{div} u|^2 + P(\bar{\rho}) \int_{\mathbb{R}^3} |\text{div} u|^2 \]

\[ \leq C(\bar{\rho}) \left( \int_{\mathbb{R}^3} \rho |\hat{u}|^2 \right) A \left( \sigma(T) \right) + C(\bar{\rho}) A \left( \sigma(T) \right) (\gamma - 1) \frac{1}{\mu^3} E_0^\frac{4}{3} \]

\[ + \frac{C}{2\mu + \lambda} \left( \|\nabla u\|^2_{L^2} + \|P - P(\bar{\rho})\|_{L^4}^4 \right) + \frac{C(\bar{\rho})}{(2\mu + \lambda)^2} \|P - P(\bar{\rho})\|^2_{L^2} \|\nabla u\|^2_{L^2} \]

\[ + \frac{1}{4} \rho |\hat{u}|^2_{L^2} + C(\bar{\rho})(\gamma - 1) \|\nabla u\|^2_{L^2} + P(\bar{\rho}) \|\nabla u\|^2_{L^2}. \] (4.27)

Integrating (4.27) over \((0, \sigma(T))\) and using (4.13) give that

\[ \frac{\mu}{2} \|\nabla u\|^2_{L^2} + \frac{1}{2} \int_{\mathbb{R}^3} \text{div} (P - P(\bar{\rho})) \]

\[ \leq C(\bar{\rho}) A \left( \sigma(T) \right) (\gamma - 1) \frac{1}{\mu^3} E_0^\frac{4}{3} + \frac{C}{(2\mu + \lambda)} \int_0^{\sigma(T)} \|P - P(\bar{\rho})\|_{L^4}^4 \]

\[ + C \left( \frac{1}{(2\mu + \lambda)} + \frac{1}{\mu^3} \rho \hat{u}^2 + (\gamma - 1) + P(\bar{\rho}) \right) \int_0^{\sigma(T)} \|\nabla u\|^2_{L^2} + C \mu (M + 1), \]

provided \( \frac{(\gamma - 1) \frac{1}{\mu^3} + \frac{1}{\mu^3}}{\mu^4} E_0^\frac{4}{3} \leq (4C(\bar{\rho}))^{-6} \).

Then, using Lemma 3.2, Lemma 4.2 and (4.3), we have

\[ \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \rho |\hat{u}|^2 \frac{1}{\mu} \]

\[ \leq C(\bar{\rho}) (\gamma - 1) \frac{1}{\mu^3} E_0^\frac{4}{3} \left( \frac{(\gamma - 1) \frac{1}{\mu^3} + \frac{1}{\mu^3}}{\mu^4} E_0^\frac{4}{3} \right)^{\frac{2}{3}} + \frac{C(\gamma - 1) \frac{1}{\mu^3} E_0^\frac{4}{3}}{\mu^2} \]

\[ + \frac{C}{\mu} \left( \frac{1}{(2\mu + \lambda)} + \frac{1}{\mu^3} \rho \hat{u}^2 + (\gamma - 1) + P(\bar{\rho}) \right) \int_0^{\sigma(T)} \|\nabla u\|^2_{L^2} + C(M + 1) \]

\[ \leq C(\bar{\rho}) (\gamma - 1) \frac{1}{\mu^3} E_0^\frac{4}{3} + \frac{C(\gamma - 1) \frac{1}{\mu^3} E_0^\frac{4}{3}}{\mu^2} + C(M + 1) \]
In fact, we just need to deal with $K_3$. Using (2.11) and Cauchy inequality, we get

$$K_3 = C \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right) \int_0^{\sigma(T)} \sigma \| \nabla u \|_{L^2} \| \nabla u \|_{L^6}^3 \leq \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right) \frac{C}{(2\mu + \lambda)^3} \sup_{0 \leq t \leq \sigma(T)} \| \nabla u \|_{L^2} \int_0^{\sigma(T)} \sigma \| \sqrt{\rho} \hat{u} \|_{L^2}^3$$
+ \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu} \right) \frac{C}{(2\mu + \lambda)^3} \sup_{0 \leq t \leq \sigma(T)} \| \nabla u \|_{L^2} \int_{0}^{\sigma(T)} \sigma \| P - P(\bar{p}) \|_{L^6}^3 \right) \nonumber \\
\leq \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right) \frac{C E_0^{1/2}}{(2\mu + \lambda)^3} \sup_{0 \leq t \leq \sigma(T)} \frac{\sigma^2}{\sqrt{\rho}} \| \nabla u \|_{L^2} \int_{0}^{\sigma(T)} \| \sqrt{\rho} u \|_{L^2}^2 \right) \nonumber \\
+ \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right) \frac{C E_0^{1/2}}{(2\mu + \lambda)^3} \int_{0}^{\sigma(T)} \| P - P(\bar{p}) \|_{L^6}^3 \right) \nonumber \\
\leq \frac{1}{4} \sup_{0 \leq t \leq \sigma(T)} \sigma \| \sqrt{\rho} u \|_{L^2}^2 + \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right)^2 \frac{C E_{13}}{(2\mu + \lambda)^6} \left( \int_{0}^{\sigma(T)} \| \sqrt{\rho} u \|_{L^2}^2 \right) \nonumber \\
+ \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right) \frac{C E_0^{1/2}}{(2\mu + \lambda)^3} \frac{(\gamma - 1)^{3/2} E_0^{1/2}}{\rho} \nonumber \\
\leq \frac{1}{4} \sup_{0 \leq t \leq \sigma(T)} \sigma \| \sqrt{\rho} u \|_{L^2}^2 + \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right)^2 \frac{C E_0^{3/2}}{(2\mu + \lambda)^6} \left( \int_{0}^{\sigma(T)} \| \sqrt{\rho} u \|_{L^2}^2 \right) \nonumber \\
+ \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right) \frac{C E_0^{1/2}}{(2\mu + \lambda)^3} \frac{(\gamma - 1)^{3/2} E_0^{1/2}}{\rho} \nonumber \\
\leq E_{13} + C(\bar{\rho}) \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} \right) \left( \gamma - 1 \right) \frac{1}{\rho} E_0 + \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right)^2 \frac{C E_{13}}{(2\mu + \lambda)^6} \left( \int_{0}^{\sigma(T)} \| \sqrt{\rho} u \|_{L^2}^2 \right) \nonumber \\
+ \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \frac{C E_0^{1/2}}{(2\mu + \lambda)^3} \frac{(\gamma - 1)^{3/2} E_0^{1/2}}{\rho} \nonumber \\
\leq E_{13} + C(\bar{\rho}) \left( 1 + (\gamma - 1)^2 \right) \frac{(\gamma - 1) E_0^{1/4}}{\mu^8} + \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right)^2 \frac{C E_0^{3/2}}{\rho^{2\gamma - 1} E_0^{1/5}} \nonumber \\
+ \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \frac{C E_0^{1/2}}{\mu^2} \left( \frac{(\gamma - 1) E_0^{1/4}}{\mu^{12}} \right)^{3/2} \nonumber \\
\leq E_{13} + C(\bar{\rho}) \left( 1 + (\gamma - 1)^2 \right) + C \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right)^2 \frac{E_0^{3/2}}{\mu^5} \nonumber \\
+ C \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \frac{E_0^{1/2}}{\mu^2} + C \bar{\rho}^{2\gamma - 1} = E_{14}, \quad (4.31) \nonumber \

Substituting (4.31) into (4.30), and using (4.28), we have

\sup_{0 \leq t \leq \sigma(T)} \sigma \int_{\mathbb{R}^3} \frac{\rho |\nabla u|^2}{\mu} + \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 \nonumber \\
\leq E_{13} + C(\bar{\rho}) \left( 1 + (\gamma - 1)^2 \right) \left( \frac{(\gamma - 1) E_0^{1/4}}{\mu^8} \right) + \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right)^2 \frac{C E_0^{3/2}}{\rho^{2\gamma - 1} E_0^{1/5}} \nonumber \\
+ \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \frac{C E_0^{1/2}}{\mu^2} \left( \frac{(\gamma - 1) E_0^{1/4}}{\mu^{12}} \right)^{3/2} \nonumber \\
\leq E_{13} + C(\bar{\rho}) \left( 1 + (\gamma - 1)^2 \right) + C \left( \frac{1}{\mu} + \frac{(\gamma - 1)^2}{\mu} + \frac{(2\mu + \lambda)^2}{\mu} \right)^2 \frac{E_0^{3/2}}{\mu^5} \nonumber \\
+ C \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \frac{E_0^{1/2}}{\mu^2} + C \bar{\rho}^{2\gamma - 1} = E_{14}, \quad (4.32) \nonumber 

where (4.22) has been used. \square

Next, we will close the a priori assumption on A_3(\sigma(T)).
Lemma 4.6 Under the conditions of Proposition 4.1, it holds that

\[
A_3(\sigma(T)) \leq \left\{ \left( \frac{(\gamma - 1) \frac{3\mu}{\kappa} + \tilde{\rho} T}{\mu^3} \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right\}^{\frac{1}{2}},
\]

provided

\[
\frac{(\gamma - 1) \frac{3\mu}{\kappa} + \tilde{\rho} T}{\mu^3} \leq \min \left\{ (C(E_{15} E_{17} + E_{16}))^{-4}, \varepsilon_4 \right\} \triangleq \varepsilon_5.
\]

Proof. Multiplying (2.1) by \(3|u|u\) and integrating the resulting equation over \(\mathbb{R}^3\), using Lemma 2.1, Lemma 2.2 and Hölder inequality, we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^3
\]

\[
\leq C\mu \int_{\mathbb{R}^3} |u| |\nabla u|^2 + C \int_{\mathbb{R}^3} |P - P(\bar{\rho})| |u| |\nabla u|
\]

\[
\leq C\mu ||u||_{L^6} ||\nabla u||_{L^2} ||\nabla u||_{L^3} + C ||P - P(\bar{\rho})||_{L^3} ||u||_{L^6} ||\nabla u||_{L^2}
\]

\[
\leq C\mu^{\frac{1}{4}} ||\nabla u||_{L^2}^{\frac{5}{2}} ||\sqrt{\rho} u||_{L^2}^{\frac{3}{2}} + ||P - P(\bar{\rho})||_{L^6}^{\frac{1}{2}} + C ||P - P(\bar{\rho})||_{L^3} ||\nabla u||_{L^2}^{\frac{1}{2}}.
\]

Integrating (4.35) over \((0, \sigma(T))\) and using Hölder inequality, one gets

\[
\sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \rho |u|^3
\]

\[
\leq C\mu^{\frac{1}{2}} \int_0^{\sigma(T)} ||\nabla u||_{L^2}^{\frac{5}{2}} ||\sqrt{\rho} u||_{L^2}^{\frac{3}{2}} + C\mu^{\frac{1}{2}} \int_0^{\sigma(T)} ||\nabla u||_{L^2}^{\frac{5}{2}} ||P - P(\bar{\rho})||_{L^6}^{\frac{1}{2}}
\]

\[
+ C \int_0^{\sigma(T)} ||P - P(\bar{\rho})||_{L^3} ||\nabla u||_{L^2}^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} \rho_0 |u_0|^3
\]

\[
\leq C\mu^{\frac{1}{2}} \left( \int_0^{\sigma(T)} ||\nabla u||_{L^2}^{2} \right)^{\frac{1}{4}} \left( \int_0^{\sigma(T)} ||\sqrt{\rho} u||_{L^2}^{2} \right)^{\frac{1}{4}} \left( \int_0^{\sigma(T)} ||\nabla u||_{L^2}^{2} \right)^{\frac{1}{4}}
\]

\[
+ C\mu^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} ||P - P(\bar{\rho})||_{L^6}^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} ||\nabla u||_{L^2}^{\frac{1}{2}} \int_0^{\sigma(T)} ||\nabla u||_{L^2}^{2}
\]

\[
+ C \sup_{0 \leq t \leq \sigma(T)} ||P - P(\bar{\rho})||_{L^3} \int_0^{\sigma(T)} ||\nabla u||_{L^2}^{2} + \int_{\mathbb{R}^3} |\rho_0 - \bar{\rho}| |u_0|^3 + \int_{\mathbb{R}^3} |\bar{\rho}| |u_0|^3
\]

\[
\leq C\mu^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} ||\nabla u||_{L^2} \left( \int_0^{\sigma(T)} ||\nabla u||_{L^2}^{2} \right)^{\frac{3}{4}} \left( \int_0^{\sigma(T)} ||\sqrt{\rho} u||_{L^2}^{2} \right)^{\frac{1}{4}}
\]

\[
+ C\mu^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} ||P - P(\bar{\rho})||_{L^6}^{\frac{1}{2}} \sup_{0 \leq t \leq \sigma(T)} ||\nabla u||_{L^2}^{\frac{1}{2}} \int_0^{\sigma(T)} ||\nabla u||_{L^2}^{2}
\]

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By Lemma 3.2, Lemma 4.2, Lemma 4.3 and Lemma 4.5 we obtain

\[ +C \sup_{0 \leq t \leq \sigma(T)} \| P - P(\bar{\rho}) \|_{L^3} + C \| \rho_0 - \bar{\rho} \|_{L^3} \| u_0 \|_{L^6} \| u_0 \|_{L^4}^2 \]

\[ + C \bar{\rho} \| u_0 \|_{L^3}^3. \]

(4.36)

\[ \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} \rho \| u \|_{L^3}^3 \]

\[ \leq C \left( \int_0^\sigma(T) \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} \left( \frac{E_{13}^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{E_{13}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu} \frac{E_0^{\frac{1}{2}}}{\mu} + \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu} \frac{E_0^{\frac{1}{2}}}{\mu} \right) \]

\[ + \frac{C}{\mu^3} \| \rho_0 - \bar{\rho} \|_{L^3} \| \nabla u_0 \|_{L^2} \| u_0 \|_{L^2} \| u_0 \|_{L^2} \| \nabla u_0 \|_{L^2} \]

\[ \leq C \left( \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{E_0^{\frac{1}{2}}}{\mu} \right) \left( E_{15} + C \left( \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu} \frac{\bar{\rho} E_0^{\frac{1}{2}}}{\mu} \right)^{\frac{1}{2}} \right) \]

\[ \leq C \left( \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{\bar{\rho} E_0^{\frac{1}{2}}}{\mu} \right) \left( E_{15} E_{16} + E_{17} \right) \]

\[ \leq \left( \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{\bar{\rho} E_0^{\frac{1}{2}}}{\mu} \right)^{\frac{1}{2}}, \]

(4.37)

provided \((\gamma - 1)^{\frac{1}{2}} E_0^{\frac{3}{2}} + \frac{\bar{\rho} E_0^{\frac{1}{2}}}{\mu} \leq (C(E_{15} E_{17} + E_{16})))^{-4},\) where

\[ \begin{align*}
E_{15} &= \frac{E_{13}^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{E_{13}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} + \frac{1}{\mu^{\frac{1}{2}}}, \\
E_{16} &= \frac{M_3^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + M_3^{\frac{3}{4}}, \\
E_{17} &= \left( E_{12} + \frac{\bar{\rho}^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu} \right)^{\frac{1}{2}}.
\end{align*} \]

In what follows, we give the details of the calculations of \(E_{15} - E_{17}.\) (4.2) gives

\[ E_{15} = \frac{1}{\mu^{\frac{1}{2}}} \left( \frac{E_{13}^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} + \frac{E_{13}^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu} \frac{E_0^{\frac{1}{2}}}{\mu} + \frac{(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}}{\mu} \frac{E_0^{\frac{1}{2}}}{\mu} \right). \]
\[\begin{align*}
\frac{E_3^4}{\mu^2} + \frac{E_1^4}{\mu^2} + \frac{1}{\mu^3},
\end{align*}\]

\[E_{16} = \frac{(\gamma - 1)E_0^{12\nu} M_2^{\frac{\nu}{2}}}{\mu^{\frac{\nu}{2}}} + \frac{\bar{\rho}^2 E_0^{\frac{12\nu}{2}} M_2^{\frac{\nu}{2}}}{\mu^{\frac{\nu}{2}}} \]

\[= \left( \frac{(\gamma - 1)E_0^{12\nu}}{\mu^{12}} \right)^{\frac{\nu}{2}} M_2^{\frac{\nu}{2}} + M_2^{\frac{\nu}{2}} \left( \frac{\bar{\rho} E_0^{\frac{12\nu}{2}}}{\mu^{\frac{\nu}{2}}} \right)^{\frac{\nu}{2}} \]

\[= M_2^{\frac{\nu}{2}} + M_2^{\frac{\nu}{2}} \]

and

\[C \left( \frac{(\gamma - 1)\frac{1}{\mu^{12}} E_0^{\frac{25}{12}} E_{12}}{\mu^{12}} + \frac{\bar{\rho} E_0}{\mu} \right)^{\frac{3}{4}} \]

\[= C \left( \frac{(\gamma - 1)\frac{1}{\mu^{12}} E_0^{\frac{1}{4}} (\gamma - 1)\frac{12}{\mu^{12}} E_0^{\frac{7}{12}} E_{12}}{\mu^{12}} + \left( \frac{\bar{\rho} E_0^{\frac{1}{4}}}{\mu^{12}} \right)^{\frac{7}{2}} \frac{\bar{\rho} E_0^{\frac{1}{4}}}{\mu^{12}} \right)^{\frac{3}{4}} \]

\[= C \left( \frac{(\gamma - 1)\frac{1}{\mu^{12}} E_0^{\frac{1}{4}} (\gamma - 1)\frac{12}{\mu^{12}} E_0^{\frac{7}{12}} E_{12}}{\mu^{12}} + \left( \frac{\bar{\rho} E_0^{\frac{1}{4}}}{\mu^{12}} \right)^{\frac{7}{2}} \frac{\bar{\rho} E_0^{\frac{1}{4}}}{\mu^{12}} \right)^{\frac{3}{4}} \]

\[\leq C \left( \frac{(\gamma - 1)\frac{1}{\mu^{12}} E_0^{\frac{1}{4}} + \frac{\bar{\rho} E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \bar{\rho} E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \right)^{\frac{3}{4}} \left( E_{12} + \frac{\bar{\rho} E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \right)^{\frac{3}{4}} \]

\[= C \left( \frac{(\gamma - 1)\frac{1}{\mu^{12}} E_0^{\frac{1}{4}} + \frac{\bar{\rho} E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \bar{\rho} E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \right)^{\frac{3}{4}} E_{17}. \]

\[\Box\]

**Lemma 4.7** Under the conditions of Proposition 4.1, it holds that

\[A_1(T) \leq \left( \frac{(\gamma - 1)\frac{1}{\mu^{12}} + \frac{1}{\mu^{\frac{3}{2}}} E_0^{\frac{1}{4}}}{\mu^{\frac{3}{2}}} \right)^{\frac{3}{4}} \]
\[ A_2(T) \leq \frac{(\gamma - 1)^{\frac{1}{2}} + \tilde{p}^i}{\mu^i} E_0^{\frac{1}{4}}, \quad (4.42) \]

provided
\[ \frac{(\gamma - 1)^{\frac{1}{2}} + \tilde{p}^i}{\mu^i} E_0^{\frac{1}{4}} \leq \varepsilon_6, \]

where
\[ \varepsilon_6 = \min \left\{ \left( C(E_{18} + E_{19} + E_{20}) \right)^{-17}, \left( C(E_{18} + E_{19} + E_{21}) \right)^{-8}, \varepsilon_5 \right\} \]

and \( E_{18} - E_{21} \) are given by (4.59), (4.61), (4.63) and (4.71) respectively.

**Proof.** First, we will prove (4.42). Recalling (4.19), we have
\[ A_2(T) \leq CA_1(\sigma(T)) + C \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\tilde{\rho})|^4 \]
\[ + C \left( \frac{1}{\mu^2} + \frac{(\gamma - 1)^2}{\mu^2} + \frac{(2\mu + \lambda)^2}{\mu^2} \right) \int_0^T \int_{\mathbb{R}^3} \sigma^3 \|\nabla u\|^4 + \frac{CP(\tilde{\rho})^2}{\mu^2} E_0. \quad (4.43) \]

Now, we turn to estimate the term \( \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 \). Due to (2.11),
\[ \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4 \]
\[ \leq C \left( \frac{1}{(2\mu + \lambda)^2} + \frac{1}{\mu^2} \right) \int_0^T \sigma^3 \|\nabla u\|^2_{L^6} \|\sqrt{\rho} u\|^2_{L^2} + \frac{C}{(2\mu + \lambda)^4} \int_0^T \sigma^3 \|\sqrt{\rho} u\|^2_{L^6} \|P - P(\tilde{\rho})\|^2_{L^4} \]
\[ + \frac{C}{(2\mu + \lambda)^4} \int_0^T \sigma^3 \|P - P(\tilde{\rho})\|^4_{L^4} \]
\[ = \sum_{i=1}^3 L_i. \quad (4.44) \]

By using Hölder inequality, Young inequality and (4.13), \( L_1 \) can be estimated as follows,
\[ L_1 = C \left( \frac{1}{(2\mu + \lambda)^2} + \frac{1}{\mu^2} \right) \int_0^T \sigma^3 \|\nabla u\|^2_{L^6} \|\sqrt{\rho} u\|^2_{L^2} \]
\[ \leq C \left( \frac{1}{(2\mu + \lambda)^2} + \frac{1}{\mu^2} \right) \int_0^T \sigma^3 \|\nabla u\|^2_{L^6} \|\nabla u\|^\frac{3}{2}_{L^4} \|\sqrt{\rho} u\|^2_{L^2} \]
\[ \leq \frac{C}{\mu^2} \left( \int_0^T \sigma^3 \|\nabla u\|^4_{L^4} \right)^\frac{3}{4} \left( \int_0^T \sigma^3 \|\nabla u\|^2_{L^6} \|\sqrt{\rho} u\|^2_{L^2} \right)^\frac{1}{2} \]
\[ \leq \frac{1}{4} \int_0^T \sigma^3 \|\nabla u\|^4_{L^4} + \frac{C}{\mu^2} \int_0^T \sigma^3 \|\nabla u\|^2_{L^6} \|\sqrt{\rho} u\|^2_{L^2} \]
\[ \leq \frac{1}{4} \int_0^T \sigma^3 \|\nabla u\|^4_{L^4} + \frac{C}{\mu^2} \int_0^T \sigma^3 \|\nabla u\|^2_{L^6} \|\sqrt{\rho} u\|^2_{L^2} \]

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\[
\leq \frac{1}{4} \int_0^T \sigma^3 \| \nabla u \|_{L^4}^4 + \frac{C}{\mu^3} \sup_{0 \leq t \leq T} (\sigma^3 \| \nabla u \|_{L^2}) \sup_{0 \leq t \leq T} \left( \frac{\sigma^2}{2} \| \nabla \hat{u} \|_{L^2} \right) \int_0^T \sigma \| \nabla \hat{u} \|_{L^2}^2
\]
\[
\leq \frac{1}{4} \int_0^T \sigma^3 \| \nabla u \|_{L^4}^4 + \frac{C}{\mu^3} A^3_1(T) A^3_2(T).
\]

It follows from Lemma 4.2 and (4.3) that
\[
L_2 = \frac{C}{(2\mu + \lambda)^3} \int_0^T \sigma^3 \| \sqrt{\rho \hat{u}} \|_{L^2}^2 \| P - P(\bar{\rho}) \|_{L^3}^2
\]
\[
\leq \frac{C}{(2\mu + \lambda)^3} \sup_{0 \leq t \leq T} (\| P - P(\bar{\rho}) \|_{L^3}^2) \int_0^T \sigma^3 \| \sqrt{\rho \hat{u}} \|_{L^2}^2
\]
\[
\leq \frac{C(\gamma - 1)^{\frac{3}{2}} E_0^2 A_1(T)}{(2\mu + \lambda)^4}.
\]

Following a process similar to [15], we focus on estimating the term \( \int_0^T \| P - P(\bar{\rho}) \|_{L^4}^4 \).

One deduces from (2.11) that \( P - P(\bar{\rho}) \) satisfies
\[
(P - P(\bar{\rho}))(t) + u \cdot \nabla (P - P(\bar{\rho})) + \gamma (P - P(\bar{\rho})) \text{div} u + \gamma P(\bar{\rho}) \text{div} u = 0.
\]

Multiplying (4.47) by \( 3\sigma^3(P - P(\bar{\rho}))^2 \) and integrating the resulting equality over \( \mathbb{R}^3 \times [0, T] \), using \( \text{div} u = \frac{1}{2\mu + \lambda} (G + P - P(\bar{\rho})) \), we get
\[
3\gamma - 1 \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\bar{\rho})|^4
\]
\[
= \int_{\mathbb{R}^3} \sigma^3 (P - P(\bar{\rho}))^3 - \frac{(3\gamma - 1)}{2\mu + \lambda} \int_0^T \int_{\mathbb{R}^3} \sigma^3 (P - P(\bar{\rho}))^3 G + 3 \int_0^T \int_{\mathbb{R}^3} \sigma^2 (P - P(\bar{\rho}))^3
\]
\[
- 3\gamma P(\bar{\rho}) \int_0^T \int_{\mathbb{R}^3} \sigma^3 (P - P(\bar{\rho}))^2 \text{div} u
\]
\[
\leq C \sup_{0 \leq t \leq T} (\| P - P(\bar{\rho}) \|_{L^3}^3) + C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \sigma^2 (P - P(\bar{\rho}))^3 + \frac{C(3\gamma - 1)}{2\mu + \lambda} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |G|^4
\]
\[
+ \frac{3\gamma - 1}{2(2\mu + \lambda)} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\bar{\rho})|^4 + \frac{C P(\bar{\rho})^2(2\mu + \lambda)}{3\gamma - 1} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^2
\]
\[
\leq C(\gamma - 1)^{\frac{3}{2}} E_0 + \frac{C}{2\mu + \lambda} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |G|^4 + \frac{3\gamma - 1}{2(2\mu + \lambda)} \int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\bar{\rho})|^4
\]
\[
+ \frac{C(2\mu + \lambda) P^2(\bar{\rho}) E_0}{\mu}.
\]

It follows from (2.33) that
\[
L_3 = C(\gamma - 1)^{\frac{3}{2}} E_0 + \frac{C}{(2\mu + \lambda)^3} \int_0^T \sigma^3 \| \sqrt{\rho \hat{u}} \|_{L^2}^2 \| \nabla u \|_{L^3}^2
\]
\]
+ \frac{C}{(2\mu + \lambda)^4} \int_0^T \sigma^3 \|\sqrt{\rho \dot{u}}\|_{L^2}^2 \|P - P(\bar{\rho})\|_{L^2}^2 + \frac{CP(\bar{\rho})^2 E_0}{(2\mu + \lambda)^2}\mu

\leq C(\gamma - 1)^{\frac{7}{4}}E_0 + \frac{1}{4} \int_0^T \sigma^3 |\nabla u|^4 + \frac{1}{(2\mu + \lambda)^3} \int_0^T \sigma^3 \|\sqrt{\rho \dot{u}}\|_{L^2}^2 \sigma \int_0^T \sigma \|\nabla u\|_{L^2}^2 + \frac{CP(\bar{\rho})^2 E_0}{(2\mu + \lambda)^2}\mu

\leq C(\gamma - 1)^{\frac{7}{4}}E_0 + \frac{1}{4} \int_0^T \sigma^3 |\nabla u|^4 + \frac{1}{(2\mu + \lambda)^4} \sup_{0 \leq t \leq T} (\|P - P(\bar{\rho})\|_{L^3}^2) \int_0^T \sigma^3 \|\sqrt{\rho \dot{u}}\|_{L^2}^2

\leq C(\gamma - 1)^{\frac{7}{4}}E_0 + \frac{1}{4} \int_0^T \sigma^3 |\nabla u|^4 + \frac{(\gamma - 1)^{\frac{7}{4}} E_0^2 \sigma A_1(T)}{(2\mu + \lambda)^4} + \frac{A_3^2(T) A_5^2(T)}{(2\mu + \lambda)^3}

+ \frac{1}{(2\mu + \lambda)^4} \sup_{0 \leq t \leq T} (\|P - P(\bar{\rho})\|_{L^3}^2) \int_0^T \sigma \|\nabla u\|_{L^2}^2 + \frac{CP(\bar{\rho})^2 E_0}{(2\mu + \lambda)^2}\mu.

(4.49)

Substituting (4.45) – (4.49) into (4.44) shows that

\int_0^T \int_{\mathbb{R}^3} \sigma^3 |\nabla u|^4

\leq C(\gamma - 1)^{\frac{7}{4}}E_0 + \frac{CA_1^2(T) A_5^2(T)}{\mu^3} + \frac{C(\gamma - 1)^{\frac{7}{4}} E_0^2 A_1(T)}{\mu^4} + \frac{CP(\bar{\rho})^2 E_0}{\mu^3}.

(4.50)

And also we get

\int_0^T \int_{\mathbb{R}^3} \sigma^3 |P - P(\bar{\rho})|^4 \leq C \mu (\gamma - 1)^{\frac{7}{4}} E_0 + C \mu A_1^2(T) A_5^2(T)

+ C(\gamma - 1)^{\frac{7}{4}} E_0^2 A_1(T) + C(\frac{2\mu + \lambda}{\mu})^2 P^2(\bar{\rho}) E_0.

(4.51)

Next, we turn to estimate \(A_1(\sigma(T))\). (4.18) shows that

\[ A_1(\sigma(T)) \leq \frac{4}{\mu} \int_{\mathbb{R}^3} \sigma |\text{div} u| |P - P(\bar{\rho})| + \frac{C(\gamma - 1) E_0}{\mu} + \frac{CP(\bar{\rho}) E_0}{\mu}

+ \frac{C(2\mu + \lambda)}{\mu} \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^3}^2 + \frac{C}{\mu} \int_0^{\sigma(T)} \sigma |P - P(\bar{\rho})| |\nabla u|^2. \]

(4.52)

Based on Lemma 2.2, Lemma 4.3 and (4.3), the last two terms in the right hand side of (4.52) can be estimated as follows

\[ \frac{C(2\mu + \lambda)}{\mu} \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^3}^2 \]

\[ \leq C \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^6}^4 \]

\[ \leq C \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^6}^4 \]

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Collecting (4.9), (4.43), (4.49), (4.50) and (4.56) implies that
\[ \text{CA} \leq \text{C} \leq \text{C} + \mu \text{A} \\int_0^\infty \sigma \text{div} \rho u \, \text{d}t \leq \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} \\left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{1}{4}} \]
\[ + \frac{C}{\mu^{\frac{3}{4}}} \sup_{0 \leq t \leq T} \| P - P(\bar{\rho}) \|_{L^2} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{1}{4}} \]
\[ \leq \frac{CA_1(T)E_{13}}{\mu^{\frac{3}{4}}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} + \frac{C(\gamma - 1)E_0^{\frac{1}{16}}E_{12}}{\mu^{\frac{3}{2}}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} \]
\[ + \frac{C(\bar{\rho})}{\mu} \int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2. \]  

and
\[ \frac{C}{\mu} \int_0^\infty \int_{\mathbb{R}^3} \sigma |P - P(\bar{\rho})|^2 \leq \frac{C(\bar{\rho})}{\mu} \int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2. \]  

Substituting (4.53)-(4.54) into (4.52), one has
\[ A_1(\sigma(T)) \leq \frac{4}{\mu} \int_{\mathbb{R}^3} \text{div} (P - P(\bar{\rho})) + \frac{C(\gamma - 1)E_0}{\mu} + \frac{C P(\bar{\rho})E_0}{\mu} \]
\[ + \frac{CA_1(T)E_{13}}{\mu^{\frac{3}{4}}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} + \frac{C(\gamma - 1)E_0^{\frac{1}{16}}E_{12}}{\mu^{\frac{3}{2}}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} \]
\[ + \frac{C(\bar{\rho})}{\mu} \int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2. \]  

It follows from (4.13) that
\[ A_1(\sigma(T)) \leq \frac{C(\gamma - 1)E_0^{\frac{12}{11}}}{\mu^{\frac{12}{11}}} + \frac{C(\gamma - 1)E_0^{\frac{25}{19}}}{\mu^{\frac{9}{19}}} + \frac{C(\gamma - 1)E_0}{\mu} + \frac{C P(\bar{\rho})E_0}{\mu} \]
\[ + \frac{CA_1(T)E_{13}}{\mu^{\frac{3}{4}}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} + \frac{C(\gamma - 1)E_0^{\frac{1}{16}}E_{12}}{\mu^{\frac{3}{2}}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} \]
\[ + \frac{C(\bar{\rho})}{\mu} \int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2. \]  

Collecting (4.3), (4.43), (4.49), (4.50) and (4.56) implies that
\[ A_2(T) \leq \frac{C(\gamma - 1)E_0^{\frac{12}{11}}}{\mu^{\frac{12}{11}}} + \frac{C(\gamma - 1)E_0^{\frac{25}{19}}}{\mu^{\frac{9}{19}}} + \frac{C(\gamma - 1)E_0}{\mu} + \frac{C P(\bar{\rho})E_0}{\mu} \]
\[ + \frac{CA_1(T)E_{13}}{\mu^{\frac{3}{4}}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} + \frac{C(\gamma - 1)E_0^{\frac{1}{16}}E_{12}}{\mu^{\frac{3}{2}}} \left( \int_0^\infty \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{4}} \]
\[ + \frac{C(\bar{\rho})}{\mu} \int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2. \]  

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Next we focus on dealing with $N_1 - N_6$. In fact, (4.2) leads to

$$N_1 + N_2 \leq C \left( \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} + \frac{\bar{E}_0^4}{\mu^4} \right) \left\{ \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} + \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} \right\} + C \left( \frac{\bar{E}_0^4}{\mu^4} \right)^2 \left( \frac{\bar{E}_0^4}{\mu^4} \right)^2 \frac{\rho}{\rho - \frac{\gamma}{2}}$$

$$+ C \left( \frac{\bar{E}_0^4}{\mu^4} \right)^2 \frac{\rho}{\rho - \frac{\gamma}{2}}$$

$$= C \left( \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} + \frac{\bar{E}_0^4}{\mu^4} \right) \left\{ \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} + \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} \right\} + C \left( \frac{\bar{E}_0^4}{\mu^4} \right)^2 \left( \frac{\bar{E}_0^4}{\mu^4} \right)^2 \frac{\rho}{\rho - \frac{\gamma}{2}}$$

$$+ \left( \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} \right) \left\{ \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} + \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} \right\} \left( \frac{\bar{E}_0^4}{\mu^4} \right)^2 \left( \frac{\bar{E}_0^4}{\mu^4} \right) \frac{\rho}{\rho - \frac{\gamma}{2}}$$

$$+ \left( \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} \right) \left\{ \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} + \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} \right\} \left( \frac{\bar{E}_0^4}{\mu^4} \right)^2 \left( \frac{\bar{E}_0^4}{\mu^4} \right) \frac{\rho}{\rho - \frac{\gamma}{2}}$$

$$+ \left( \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} \right) \left\{ \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} + \frac{\gamma - 1}{\mu^4} \frac{\bar{E}_0^4}{\sqrt{\bar{E}_0}} \right\} \left( \frac{\bar{E}_0^4}{\mu^4} \right)^2 \left( \frac{\bar{E}_0^4}{\mu^4} \right) \frac{\rho}{\rho - \frac{\gamma}{2}}$$

(4.57)
\[
\leq C \left( \frac{(\gamma - 1)^{\frac{1}{3\gamma}} E_{0}^{\frac{1}{3}}}{{\mu}^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right)^{\frac{18}{17}} E_{18}, \tag{4.58}
\]

where

\[
E_{18} = \frac{1}{{\mu}^{\frac{1}{3}}} + \frac{1}{{\mu}^{\frac{1}{3}}} + 1 + \frac{E_{12}^{\frac{1}{3}} E_{13}^{\frac{1}{3}}}{{\mu}^{\frac{1}{3}}} + \tilde{\rho}^{\gamma - \frac{2}{3}} + \frac{\tilde{\rho}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}}. \tag{4.59}
\]

\[
N_{3} + N_{4} \leq C \left( \frac{(\gamma - 1)^{\frac{1}{3\gamma}} E_{0}^{\frac{1}{3}}}{{\mu}^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right)^{\frac{18}{17}} \left\{ \left( \frac{(\gamma - 1)^{\frac{223}{1740}} E_{0}^{\frac{443}{1740}}}{{\mu}^{\frac{1}{1742}}} \right) + \left( \frac{(\gamma - 1)^{\frac{263}{321}} E_{0}^{\frac{263}{321}}}{{\mu}^{\frac{321}{321}}} \right) \right\}
\]

\[
+ \left( \frac{(\gamma - 1)E_{0}^{\frac{1}{12}}}{{\mu}^{\frac{1}{12}}} \right) \left( \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right)^{\frac{34}{17}} \left( \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right) \cdot \left( \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right) \right\}
\]

\[
= C \left( \frac{(\gamma - 1)^{\frac{1}{3\gamma}} E_{0}^{\frac{1}{3}}}{{\mu}^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right)^{\frac{18}{17}} \left\{ \left( \frac{(\gamma - 1)E_{0}^{\frac{263}{321}}}{{\mu}^{\frac{321}{321}}} \right) \left( \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right) \right\}
\]

\[
\leq C \left( \frac{(\gamma - 1)^{\frac{1}{3\gamma}} E_{0}^{\frac{1}{3}}}{{\mu}^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right)^{\frac{18}{17}} E_{19}, \tag{4.60}
\]

where

\[
E_{19} = \frac{1}{{\mu}^{\frac{1}{3}}} + \frac{1}{{\mu}^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}}. \tag{4.61}
\]

\[
N_{5} \times N_{6} + N_{7} \leq C \left( 1 + \frac{1}{{\mu}^{\frac{1}{3}}} + \frac{(\gamma - 1)^{2}}{{\mu}^{\frac{1}{3}}} + \frac{(2\mu + \lambda)^{2}}{{\mu}^{\frac{1}{3}}} \right) \left( \frac{(\gamma - 1)^{\frac{1}{3\gamma}} E_{0}^{\frac{1}{3}}}{{\mu}^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right)^{\frac{18}{17}}
\]

\[
\times \left\{ \left( \frac{(\gamma - 1)^{\frac{1}{15}} E_{0}^{\frac{25}}}{\mu^{\frac{1}{17}}} \right) + \frac{(\gamma - 1)^{\frac{1}{70}} E_{0}^{\frac{103}}}{\mu^{\frac{1}{7}}} + \frac{(\gamma - 1)^{\frac{1}{31}} E_{0}^{\frac{412}}}{\mu^{\frac{1}{17}}} \right\}
\]

\[
+ \left( \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right) + \left( \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right) + \left( \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right) \right\}
\]

\[
= C \left( 1 + \frac{1}{{\mu}^{\frac{1}{3}}} + \frac{(\gamma - 1)^{2}}{{\mu}^{\frac{1}{3}}} + \frac{(2\mu + \lambda)^{2}}{{\mu}^{\frac{1}{3}}} \right) \left( \frac{(\gamma - 1)^{\frac{1}{3\gamma}} E_{0}^{\frac{1}{3}}}{{\mu}^{\frac{1}{3}}} + \frac{\tilde{\rho}^{\frac{1}{2}} E_{0}^{\frac{1}{2}}}{{\mu}^{\frac{1}{3}}} \right)^{\frac{18}{17}}
\]
\[
\times \left\{ \left( \frac{(\gamma - 1) E_0^{\frac{10}{\mu}}}{\mu^{\frac{16}{\gamma}}} \right)^{\frac{15}{14}} \frac{1}{\mu^{\frac{14}{\gamma}}} + \left( \frac{(\gamma - 1) E_0^q}{\mu^{12}} \right)^{\frac{10}{7}} \frac{1}{\mu^{\frac{12}{7}}} + \left( \frac{(\gamma - 1) E_0^{\frac{44}{11}}}{\mu^{12}} \right)^{\frac{7}{11}} \right\}
\]

\[ + C \left( \frac{(\gamma - 1) \frac{\mu^{\frac{13}{\gamma}}}{E_0^{\frac{1}{3}}} + \frac{\rho^{\frac{13}{\gamma}} E_0^{\frac{1}{3}}}{\mu^{\frac{13}{\gamma}}} \right) \leq C \left( \frac{(\gamma - 1) \frac{\mu^{\frac{13}{\gamma}}}{E_0^{\frac{1}{3}}} + \frac{\rho^{\frac{13}{\gamma}} E_0^{\frac{1}{3}}}{\mu^{\frac{13}{\gamma}}} \right) E_{20}, \tag{4.62} \]

\[ E_{20} = \left( 1 + \frac{1}{\mu^4} + \frac{(\gamma - 1)^2}{\mu^4} + \frac{2\mu + \lambda}{\mu^8} \right) \left( 1 + \frac{1}{\mu^4} \right) \left( \frac{\gamma}{\mu^4} \right) \frac{\rho^{\frac{13}{\gamma}}}{\mu^{\frac{13}{\gamma}}}. \tag{4.63} \]

It thus follows (4.58), (4.60) and (4.62) that

\[ A_2(T) \leq C \left( \frac{(\gamma - 1) \frac{\mu^{\frac{13}{\gamma}}}{E_0^{\frac{1}{3}}} + \frac{\rho^{\frac{13}{\gamma}} E_0^{\frac{1}{3}}}{\mu^{\frac{13}{\gamma}}} \right) (E_{18} + E_{19} + E_{20}) \]

\[ \leq \left( \frac{(\gamma - 1) \frac{\mu^{\frac{13}{\gamma}}}{E_0^{\frac{1}{3}}} + \frac{\rho^{\frac{13}{\gamma}} E_0^{\frac{1}{3}}}{\mu^{\frac{13}{\gamma}}} \right) \left( C(E_{18} + E_{19} + E_{20}) \right)^{-17}. \tag{4.64} \]

Finally, to finish the proof of lemma 4.7, it remains to prove (4.41). With (4.15) and (4.55) at hand, we just have to estimate the terms \( \frac{2\mu + \lambda}{\mu} \int_{\sigma(T)} \sigma \| \nabla u \|_{L^3}^3 \) and \( \frac{C}{\mu} \int_{\sigma(T)} \int_{\mathbb{R}^3} \sigma |P - P(\bar{p})| \| \nabla u \|^2 \). By Hölder inequality, we have

\[ \frac{2\mu + \lambda}{\mu} \int_{\sigma(T)} \sigma \| \nabla u \|_{L^3}^3 \leq C \left( \int_{\sigma(T)} \| \nabla u \|_{L^2}^2 \right)^{\frac{3}{2}} \left( \int_{\sigma(T)} \| \nabla u \|_{L^4}^4 \right)^{\frac{3}{4}} \]

\[ \leq \left( CE_0 \right)^{\frac{3}{4}} \left( \int_{\sigma(T)} \| \nabla u \|_{L^4}^4 \right)^{\frac{3}{4}} \tag{4.65} \]

and

\[ \frac{C}{\mu} \int_{\sigma(T)} \int_{\mathbb{R}^3} \sigma |P - P(\bar{p})| \| \nabla u \|^2 \]

\[ \leq \frac{C}{\mu} \left( \int_{\sigma(T)} \int_{\mathbb{R}^3} |P - P(\bar{p})|^4 \right)^{\frac{1}{4}} \left( \int_{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^4 \right)^{\frac{1}{4}} \left( \int_{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \tag{4.66} \]
It follows from Lemma 4.3, (4.48), Lemma 4.4, (4.65) and (4.66) that

\[ A_1(T) \leq A_1(\sigma(T)) + \frac{C(2\mu + \lambda)}{\mu} \int_{\sigma(T)}^T \sigma \| \nabla u \|^2_{L^3} + \frac{C}{\mu} \int_{\sigma(T)}^T \int_{\mathbb{R}^3} \sigma |P - P(\tilde{\rho})| \| \nabla u \|^2 \]

\[ \leq \frac{C(\gamma - 1) \frac{7}{36} E_0^{\frac{1}{3}}}{\mu^\frac{1}{2}} + \frac{C(\gamma - 1) \frac{11}{18} E_0^{\frac{1}{3}}}{\mu^\frac{1}{2}} + \frac{C(\gamma - 1) E_0}{\mu} + \frac{C P(\tilde{\rho}) E_0}{\mu} \]

\[ \leq \frac{C A_1(T) E_0^{\frac{1}{2}}}{\mu^\frac{1}{2}} \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{E_0}{\mu} \right) \]

\[ + \frac{C(\gamma - 1) \frac{11}{18} E_0^{\frac{1}{3}}}{\mu^\frac{1}{2}} \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right) \]

\[ + \frac{C(\tilde{\rho}, \tilde{\rho})}{\mu} \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right) \]

\[ + \frac{C E_0^{\frac{1}{2}}}{\mu^\frac{3}{2}} \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right) \]

\[ \leq \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right)^{\frac{18}{17}} E_{18} \]

\[ \leq \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right)^{\frac{1}{2}} E_{18}, \quad (4.68) \]

Following a process similar to \( N_1 - N_6 \), one gets \( N_7 - N_{11} \) as follows

\[ N_7 + N_8 = N_1 + N_2 \leq C \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right)^{\frac{18}{17}} E_{18} \]

\[ \leq \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right)^{\frac{1}{2}} E_{18}, \quad (4.68) \]

\[ N_9 + N_{10} = N_3 + N_4 \leq C \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right)^{\frac{18}{17}} E_{19} \]

\[ \leq \left( \frac{\gamma - 1}{} \frac{11}{18} E_0^{\frac{1}{3}} + \frac{\tilde{E}_0}{\mu} \right)^{\frac{1}{2}} E_{19} \quad (4.69) \]
where

\[ N_{11} \leq C \left( \frac{(\gamma - 1) \frac{1}{36} E_0^{\frac{1}{3}}}{\mu^3} + \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^3} \right) \left\{ \frac{(\gamma - 1) \frac{1}{5} E_0^{\frac{5}{8}}}{\mu^4} \right\} \]

\[ + \left( \frac{(\gamma - 1) \frac{1}{36} E_0^{\frac{1}{3}}}{\mu^3} + \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^3} \right) \left( \frac{E_0^{\frac{1}{2}}}{\mu^2} + \frac{(\gamma - 1) \frac{5}{32} E_0^{\frac{17}{32}} A_1(T)}{\mu^x} + \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^x} \right) \]

\[ \leq C \left( \frac{(\gamma - 1) \frac{1}{36} E_0^{\frac{1}{3}}}{\mu^3} + \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^3} \right) \left\{ \left( \frac{(\gamma - 1) E_0^{\frac{63}{32}}}{\mu^{12}} \right) \left( \frac{1}{\mu^2} \right) + \left( \frac{\tilde{\rho} E_0^{\frac{3}{8}}}{\mu^{24}} \right) \left( \frac{\tilde{\rho}}{\mu^3} \right) \right\} \]

\[ + \left( \frac{\tilde{\rho} E_0^{\frac{1}{4}}}{\mu^{24}} \right) \left( \frac{E_0^{\frac{1}{2}}}{\mu^2} + \left( \frac{\tilde{\rho} E_0^{\frac{3}{8}}}{\mu^{32}} \right) \left( \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^3} \right) \right) \]

\[ \leq C \left( \frac{(\gamma - 1) \frac{1}{36} E_0^{\frac{1}{3}}}{\mu^3} + \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^3} \right)^\frac{1}{2} E_{21}, \quad (4.70) \]

where

\[ E_{21} = \frac{1}{\mu^2} + \frac{1}{\mu^3} + \frac{1}{\mu^1} + \frac{\tilde{\rho}^\frac{1}{2}}{\mu^2}, \quad (4.71) \]

and we have used \( \frac{(\gamma - 1) \frac{1}{36} E_0^{\frac{1}{3}}}{\mu^3} \leq \frac{\tilde{\rho}}{2C} \).

Substituting (4.68), (4.69) and (4.70) into (4.67), we obtain

\[ A_1(T) \leq C \left( \frac{(\gamma - 1) \frac{1}{36} E_0^{\frac{1}{3}}}{\mu^3} + \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^3} \right)^\frac{1}{2} (E_{18} + E_{19} + E_{21}) \]

\[ \leq \left( \frac{(\gamma - 1) \frac{1}{36} E_0^{\frac{1}{3}}}{\mu^3} + \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^3} \right)^\frac{3}{8} , \quad (4.72) \]

provided \( \left( \frac{(\gamma - 1) \frac{1}{36} E_0^{\frac{1}{3}}}{\mu^3} + \frac{\tilde{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^3} \right)^8 \leq \left( C(E_{18} + E_{19} + E_{21}) \right)^{-8} \).

Now we are in a position to close the a priori assumption on \( \rho \).
**Lemma 4.8** Under the conditions of Proposition 4.7, it holds that

\[
\sup_{0 \leq t \leq T} \| \rho \|_{L^\infty} \leq \frac{7 \bar{\rho}}{4} \tag{4.73}
\]

for any \( (x, t) \in \mathbb{R}^3 \times [0, T] \), provided

\[
\left( \frac{\gamma - 1}{\frac{\bar{\mu}}{\mu}} + \frac{\bar{\rho} b}{\mu} \right) E_0^{\frac{1}{2}} \leq \varepsilon \triangleq \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}.
\]

**Proof.** In fact, the proof is similar to the one in (3.68), then we just list some differences. Here we rewrite (3.69) as follows

\[
|b(t_2) - b(t_1)| \leq \frac{C(\bar{\rho})}{\mu^{\frac{3}{2}}} \left( \frac{(\gamma - 1) \frac{\bar{\mu}}{\mu} E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} + \frac{\bar{\rho} \mu^0 E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} \right)^{\frac{3}{16}}.
\]

(4.74)

Therefore, for \( t \in [0, \sigma(T)] \), one can choose \( N_0 \) and \( N_1 \) in Lemma 2.3 as follows

\[
N_1 = 0, \quad N_0 = \frac{C(\bar{\rho})}{\mu^{\frac{3}{2}}} \left( \frac{(\gamma - 1) \frac{\bar{\mu}}{\mu} E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} + \frac{\bar{\rho} \mu^0 E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} \right)^{\frac{3}{16}},
\]

and \( \bar{\zeta} = 0 \). Then

\[
g(\zeta) = -\frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = 0 \quad \text{for all} \quad \zeta \geq \bar{\zeta} = 0.
\]

Thus

\[
\sup_{0 \leq t \leq \sigma(T)} \| \rho \|_{L^\infty} \leq \max\{\bar{\rho}, 0\} + N_0 \leq \bar{\rho} + \frac{C(\bar{\rho})}{\mu^{\frac{3}{2}}} \left( \frac{(\gamma - 1) \frac{\bar{\mu}}{\mu} E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} + \frac{\bar{\rho} \mu^0 E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} \right)^{\frac{3}{16}} \leq \frac{3 \bar{\rho}}{2}, \tag{4.75}
\]

provided

\[
\left( \frac{\gamma - 1}{\frac{\bar{\mu}}{\mu}} + \frac{\bar{\rho} b}{\mu} \right) E_0^{\frac{1}{2}} \leq \varepsilon \triangleq \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{3}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}.
\]

(4.76)

On the other hand, for \( t \in [\sigma(T), T] \), we can rewrite (3.72) as follows

\[
|b(t_2) - b(t_1)| \leq \frac{1}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) \left( \frac{(\gamma - 1) \frac{\bar{\mu}}{\mu} E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} + \frac{\bar{\rho} \mu^0 E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} \right)^2, \tag{4.77}
\]

provided \( \left( \frac{\gamma - 1}{\frac{\bar{\mu}}{\mu}} + \frac{\bar{\rho} b}{\mu} \right) E_0^{\frac{1}{2}} \leq \varepsilon_6 \). Therefore, one can choose \( N_1 \) and \( N_0 \) in Lemma 2.3 as

\[
N_1 = \frac{1}{2\mu + \lambda}, \quad N_0 = C(\bar{\rho}) \left( \frac{(\gamma - 1) \frac{\bar{\mu}}{\mu} E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} + \frac{\bar{\rho} \mu^0 E_0^{\frac{1}{3}}}{\mu^{\frac{3}{2}}} \right)^2.
\]

Note that

\[
g(\zeta) = -\frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = -\frac{1}{2\mu + \lambda} \quad \text{for all} \quad \zeta \geq 1,
\]
one can set $\zeta = 1$. Thus

$$\sup_{\sigma(T) \leq s \leq T} \|\rho\|_{L^\infty} \leq \max \left\{ \frac{3}{2} \bar{\rho}, 1 \right\} + N_0 \leq \frac{3}{2} \bar{\rho} + C(\bar{\rho}) \left( \frac{(\gamma - 1) \frac{3}{2} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} + \frac{\bar{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \right)^2 \leq \frac{7\bar{\rho}}{4},$$

(4.78)

provided

$$\frac{(\gamma - 1) \frac{3}{2} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} + \frac{\bar{\rho}^\frac{1}{2} E_0^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \leq \min \left\{ \varepsilon_6, (2C(\bar{\rho}, M))^{-\frac{16}{9}} \mu^4, (4C(\bar{\rho}))^{-2} \right\}. \quad (4.79)$$

The combination of (4.75) and (4.78) completes the proof of Lemma 4.8.

Now, the proof of Proposition 4.1 is completed. Next, following a process similar to that in the proof of Theorem 1.1, we can prove that the results obtained in Proposition 4.1 still hold in the case of $\gamma \geq 2$. At last, we will derive the time-dependent higher norm estimates of the smooth solution $(\rho, u)$. In fact the proofs are the same as the ones in Theorem 1.1. For the convenience, we omit them here.

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