Flux fluctuations in a multi-random-walker model and surface growth dynamics

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We study the dynamics of visitation flux in a multi-random-walker model by comparison to surface growth dynamics in which one random walker drops a particle to a node at each time the walker visits the node. In each independent experiment (trial or day) for the multi-random-walker model, the number of walkers are randomly chosen from the uniform distribution \([N_{RW}] = \Delta N_{RW}, \langle N_{RW} \rangle + \Delta N_{RW}\]. The averaged fluctuation \(\sigma(T_{RW})\) of the visitations over all nodes \(i\) and independent experiments is shown to satisfy the power-law dependence on the walk step \(T_{RW}\) as \(\sigma(T_{RW}) \propto T_{RW}^\beta\). Furthermore two distinct values of the exponent \(\beta\) are found on a scale-free network, a random network and regular lattices. One is \(\beta_i\), which is equal to the growth exponent \(\beta\) for the surface fluctuation \(W\) in one-random-walker model, and the other is \(\beta = 1\). \(\beta_i\) is found for small \(\Delta N_{RW}\) or for the system governed by the internal intrinsic dynamics. In contrast \(\beta = 1\) is found for large \(\Delta N_{RW}\) or for the system governed by the external flux variations. The implications of our results to the recent studies on fluctuation dynamics of the nodes on networks are discussed.

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Recent advances of network theories \(^1\) have elucidated that the dynamical structure of many complex systems from the internet through the biosystems to the social systems forms nontrivial networks such as scale-free networks. Therefore the dynamical behaviors of the interacting (linked) units (nodes) on the networks are very important to understand the dynamics of such complex systems. Recently an important theory \(^2\) \(^3\) \(^4\) \(^5\) \(^6\) \(^7\) on the dynamical behaviors on such interacting (correlated) nodes is suggested and confirmed on the internet, World Wide Web(WWW), river networks and etc.

The main theory \(^2\) \(^3\) for the dynamics of the node is that there exists the power-law relation between the average activity or flux \(\langle f_i \rangle\) of each node \(i\) to the flux fluctuation \(\sigma_i\) as

\[
\sigma_i \sim \langle f_i \rangle^\alpha.
\]

Furthermore the theory suggested that there can be two distinct classes of dynamical systems. One class consists of the systems with \(\alpha \simeq 1/2\) and the other consists of those with \(\alpha = 1\). The \(\alpha \simeq 1/2\) systems are those in which the main controlling dynamics is the internal intrinsic dynamics. Examples are signal activity in microprocessors and time-resolved information in internet routers. The \(\alpha = 1\) systems are those in which the main dynamics is controlled by the external variations. Examples are river networks, highway traffic and World Wide Webs. Recently another study \(^4\) suggested a dynamical model which has nonuniversal \(\alpha\) values on complex networks by introducing an impact variable to each node.

To understand the origin of two distinct classes, several dynamical models have been suggested \(^2\) \(^3\). One of them is based on a multi-random-walker model. The details of the multi-random-walker model are as follows. In each experiment or trial \(d\), which was called as “day” in Refs. \(^2\) \(^3\), the number of unbiased random walkers \(N_{RW}\) are randomly chosen from the uniform distribution \([N_{RW}] = \Delta N_{RW}, \langle N_{RW} \rangle + \Delta N_{RW}\). The walkers are initially randomly distributed on a given support (network) with the \(N\) nodes (sites). Then the distributed walkers simultaneously take preassigned fixed \(T_{RW}\) steps in a given experiment \(d\) and count the number of visits (or flux) \(f_{id}\) to the node \(i\) during \(T_{RW}\) steps. By repeating these experiments \(D\) times, one can get a series of data \(\{f_{id}\}\) with \(d = 1,2,.....D\) and \(i = 1,2,....,N\). From the data we can calculate the average flux and the flux fluctuation of a node \(i\) as

\[
\langle f_i(T_{RW}) \rangle = \frac{1}{D} \sum_{d=1}^{D} f_{id}(T_{RW}), \quad (2)
\]

\[
\sigma^2_i(T_{RW}) = \frac{1}{D} \sum_{d=1}^{D} f_{id}(T_{RW}) - \langle f_i(T_{RW}) \rangle^2. \quad (3)
\]

From this model on the scale-free networks with the degree exponent \(\gamma = 3\) the power-law relation \(\sigma_i(T_{RW}) \sim \langle f_i(T_{RW}) \rangle^\alpha\) or Eq. (1) was numerically shown. Furthermore the systems for small \(\Delta N_{RW}\) were shown to belong to \(\alpha \simeq 1/2\) class, while those for large \(\Delta N_{RW}\) were shown to satisfy \(\alpha = 1\). From these results it was explained why \(\alpha = 1\) systems are governed by the external flux variations and the behavior for \(\alpha = 1/2\) is from the internal intrinsic dynamics \(^2\) \(^3\).

In the multi-random-walk model, the flux configuration \(\{f_{id}(T_{RW})\}\) exactly corresponds to the height configuration \(\{h_{id}(T_{RW})\}\) in the following surface growth model. In the growth model each of \(N_{RW}\) random walkers drops a particle on the node \(i\) whenever it visits the node \(i\) and thus \(f_{id}(T_{RW}) = h_{id}(T_{RW})\). Recently this kind of surface growth model in which \(N_{RW}\) is fixed (not-varied) in any experiment \(d\) is studied to know the effect of random-walk-like colored noises \(^7\) on the dynamical scaling properties of the surface roughening \(^8\). In the study \(^7\) the support is an ordinary one-dimensional lat-
tice and the dynamical scaling of the surface width $W$ satisfies the conventional scaling as

$$W = L^{\alpha_s} f(T_{RW}/L^z) = \begin{cases} T_{RW}^{\alpha_s}, & T_{RW} \ll L^z \\ L^{\alpha_s}, & T_{RW} \gg L^z \end{cases}$$

with $z \equiv \alpha_s/\beta$.

In this paper we want to introduce another method to discriminate the systems governed by the internal dynamics from those derived by external input flux variations by comparison to corresponding surface growth dynamics. The main quantity for the comparison is the averaged $\sigma_i^2(T_{RW})$ over the node $i$, which is defined as

$$\overline{\sigma}^2(T_{RW}) = \frac{1}{N} \sum_{i=1}^N \sigma_i^2(T_{RW}). \tag{5}$$

As we shall see, $\overline{\sigma}(T_{RW})$ for the multi-random-walker model shows the power-law dependence on the walk step $T_{RW}$ as $\overline{\sigma}(T_{RW}) \approx T_{RW}^{\beta}$. Furthermore, if the governing dynamics is not anomalous but the internal intrinsic dynamics, then $\overline{\sigma}(T_{RW})$ essentially has the same dynamical behavior as that of the mean-square surface fluctuation (width) $W(T_{RW})$ in one-random-walker surface growth model. Then $T_{RW}$ is (acts as) the growth time and the exponent $\beta$ is (or act as) the growth exponent in dynamical surface scaling. If $W(T_{RW})$ in one-random-walker model dynamically behaves as $W(T_{RW}) \approx T_{RW}^{\beta}$, then $\overline{\sigma}(T_{RW})$ for the systems governed by the internal intrinsic dynamics satisfies $\overline{\sigma}(T_{RW}) \approx T_{RW}^{\beta}$. If $\beta \neq \beta_i$, then governing dynamics should come from the external flux variations.

We first want to show the exact mathematical relation of $\overline{\sigma}$ to the surface fluctuation $W$. From Eq. (5),

$$\overline{\sigma}^2(T_{RW}) = \frac{1}{N} \sum_{i=1}^N \sigma_i^2(T_{RW}) = \frac{1}{N} \sum_{i} \left[ \frac{1}{D} \sum_{d=1}^D f_i d - \left( \frac{1}{D} \sum_{d=1}^D f_i \right)^2 \right]. \tag{6}$$

Since $f_i = h_{id}$ in the corresponding surface growth model, the mean-square surface width $W^2$ is

$$W^2(T_{RW}) = \frac{1}{D} \sum_{d=1}^D W_d^2(T_{RW}) = \frac{1}{D} \sum_{d} \left[ \frac{1}{N} \sum_{i=1}^N f_i d - \left( \frac{1}{N} \sum_{i=1}^N f_i \right)^2 \right]. \tag{7}$$

Thus

$$\overline{\sigma}^2(T_{RW}) - W^2(T_{RW}) = \frac{1}{D} \sum_{d} \left( \frac{1}{N} \sum_{i=1}^N f_i d \right)^2 - \frac{1}{N} \sum_{i} \left( \frac{1}{D} \sum_{d=1}^D f_i \right)^2. \tag{8}$$

The difference between $\overline{\sigma}^2$ and $W^2$ for the given step $T_{RW}$ comes from the different orders of taking the average over nodes (i.e., $1/N \sum f_i(T_{RW})$) and that over experiments (or trials) (i.e., $1/D \sum f_i(T_{RW})$).

In normal processes where any particular node does not receive extraordinarily smaller or larger flux than any other regions, then one can expect that the order of taking averages makes no anomalous effects and thus $1/N \sum f_i(T_{RW}) \simeq 1/D \sum f_i(T_{RW})$. Then the difference $\overline{\sigma}^2(T_{RW}) - W^2(T_{RW})$ becomes negligible or $\overline{\sigma}(T_{RW}) \approx W(T_{RW})$. Therefore the possible origins which make the physically important difference comes from the breakdown of the spatial or temporal symmetry of the given system. Typical examples for such breakdowns are the local columnar defect and the temporally colored noises. When the input flux fluctuation becomes large or $\Delta N_{RW}$ is large in the multi-random-walker model, it is also possible that the input flux to some localized region of the support can become anomalously larger or smaller and then the difference $\overline{\sigma}(T_{RW}) - W(T_{RW})$ can have some crucial effects and remains for a certain length time.

To see the explained effects numerically, we analyze the numerical data from the simulations for the random-walker models. To see the universal features on homogeneous networks as well as inhomogeneous networks, we use regular lattices of one and two dimensions as well as random network(RN) and scale-free network (SFN) with $\gamma = 3$ as supports in the simulations. Used size or the number $N$ of nodes (sites) of RN, SFN, and one-dimensional lattice are $10^4$, $10^4$, and $10^3$, respectively. The used size of the two-dimensional square lattice is $100 \times 100$. Each experiment (trial) is repeated over $D = 100$ times. Every walker in each experiment $d$ takes $10^5$ steps. Data for the flux $\{f_i(T_{RW})\}$ are obtained for the walk steps $T_{RW} = 1, 2, \ldots, 10^4$.

The first simulation data to analyze are those for one-random-walker model in which only one random walker is initially dropped on a randomly chosen node (site) of RN, SFN, and an one-dimensional lattice are $10^4$, $10^4$, and $10^3$, respectively. The used size of the two-dimensional square lattice is $100 \times 100$. Each experiment (trial) is repeated over $D = 100$ times. Every walker in each experiment $d$ takes $10^5$ steps. Data for the flux $\{f_i(T_{RW})\}$ are obtained for the walk steps $T_{RW} = 1, 2, \ldots, 10^4$.

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mensional lattices. On the inhomogeneous networks such as SFN and RN, $\beta_i$ is very close to $\beta = 1/2$, which is the value of $\beta$ of random ballistic depositions on any support. In contrast on one and two dimensional lattices $\beta_i$ is quite different from $1/2$.

In the simulations for multi-random-walker models, the number $N_{RW}$ of walkers in an experiment $d$ is selected randomly among the uniform distribution $[(N_{RW}) - \Delta N_{RW}, (N_{RW}) + \Delta N_{RW}]$. Then the $N_{RW}$ walkers are initially randomly distributed over the nodes (sites) of the support. In the simulations $\langle N_{RW} \rangle = 10^4$ is always imposed and $\Delta N_{RW}$ is varied as $\Delta N_{RW} = 0, 20, 40, 80, 100, 200, 800, 10^3, 4 \times 10^3, 10^4$. Other conditions are exactly the same as those in the simulations for one-random-walker model. The results for the dependence of $\sigma$ on $T_{RW}$ in multi-random-walker model for various $\Delta N_{RW}$ are also shown in upper plots of Figs. 1(a)-(d). From the data and the relation $\sigma \sim T_{RW}^{\beta_i}$ the obtained values of $\beta$ for $\Delta N_{RW}$’s are displayed in the lower plots of Figs. 1(a)-(d), where the dotted line denotes the values of $\beta_i$ in one-random-walker model on the corresponding support. As can be seen from the lower plots, $\beta$ value for small $\Delta N_{RW}$ or $\Delta N_{RW} \leq 200$ is nearly equal to $\beta_i$ (i.e., $\beta = \beta_i$) on both homogenous and inhomogeneous networks. On the other hand $\beta$ for large $\Delta N_{RW}$ or $\Delta N_{RW} \geq 4 \times 10^3$ approaches to $\beta = 1$ (or $\beta \geq 0.9$).

\[
\beta \text{ for crossover values of } \Delta N_{RW} (\text{or } \Delta N_{RW} = 800 \text{ or } 1000) \text{ has crossover values between } \beta_i \text{ and } 1.
\]

The results for the multi-random-walker models are as follows. For small $\Delta N_{RW}$ or in the systems where the internal intrinsic dynamics is dominant, $\sigma(T_{RW})$ follows the behavior of $W(T_{RW}) \sim T_{RW}^{\beta_i}$ in one-random-walker model quite well. For large $\Delta N_{RW}$ or in the systems where the external flux fluctuation is dominant for dynamical behaviors, $\sigma(T_{RW})$ follows the behavior $\sigma(T_{RW}) \sim T_{RW}^\beta$ with $\beta = 1$.

\[
\text{Final comments are on the nonuniversal values of exponent } \alpha \text{ for Eq. (1) in regular lattices or homogeneous net-}
\]

FIG. 1: Triangles (▲) and straight lines in the upper plots of figures (a), (b), (c), and (d) represent the data for the average flux fluctuation $\sigma(T_{RW})$ over nodes and the surface fluctuation $W(T_{RW})$ in the one-random-walker model, respectively. Other symbols in the upper plots represent $\sigma(T_{RW})$ for $\Delta N_{RW} = 0, 20, 40, 80, 100, 200, 800, 1000, 4000, 10000$ from bottom to top in the multi-random walker model. The lower plots show the values of exponent $\beta$ for various $\Delta N_{RW}$ obtained by fitting the data in upper plots to the relation $\sigma(T_{RW}) \sim T_{RW}^{\beta_i}$. The dotted lines in the lower plots denote the values of $\beta_i$ obtained from the relation $W(T_{RW}) \sim \sigma(T_{RW}) \sim T_{RW}^{\beta_i}$ in one-random-walker model. (a) on Scale-free network with $\gamma = 3$. (b) on Random network. (c) on one dimensional lattice. (d) on two dimensional lattice.
works. Recently a study [4] suggested a model in which nonuniversal $\alpha$ values (i.e., $\alpha \neq 1$ or $1/2$) for the relation $\sigma_i(T_{RW}) \simeq \langle f_i(T_{RW}) \rangle^\alpha$ with the fixed $T_{RW}$. By using the multi-random-walker model with the quenched impact variable $V(i)$ for each node $i$, which is dependent on the degree $k(i)$, the study [4] showed nonuniversal $\alpha$ values (i.e. $1/2 < \alpha < 1$) on scale-free networks. Nonuniversal $\alpha$ values are also possible on the homogeneous networks or regular lattices without the impact variable $V(i)$. By using the relation $\sigma_i(T_{RW}) \simeq \langle f_i(T_{RW}) \rangle^\alpha$ with the fixed $T_{RW}$ on one and two dimensional lattices, we obtain $\alpha$ values for various $\Delta N_{RW}$. The results are shown in Fig. 2. The simulation conditions except $T_{RW}$ for Fig. 2 are exactly the same as those for Figs. 1(c) and (d). $T_{RW}$ for each experiment (day) is fixed as $T_{RW} = 100$. For $\Delta N_{RW} < 500$ where the internal intrinsic dynamics are dominant, $\alpha$ values are nearly constant. The values of the exponent $\alpha$ for small $\Delta N_{RW}$ are $\alpha = 0.77(1)$ in one dimension and $\alpha = 0.60(2)$ in two dimension, respectively. The behavior $\alpha > 1/2$ for internal intrinsic dynamics should come from the reentrant properties of random walks. As explained before the reentrant property makes the power-law correlation in time for the probability to visit a certain site again. This kind of temporal correlation effects makes the probability distribution for $\langle f_i \rangle$ deviate from the Gaussian and thus gives nonuniversal $\alpha$ values for small $\Delta N_{RW}$. For large $\Delta N_{RW}$ $\alpha$ approaches 1 even for homogeneous networks as can be seen from data for $\Delta N_{RW} = 4000, 10000$ in Fig. 2.

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