We illustrate the connection between electron and neutrino scattering off nuclei and show how the former process can be used to constrain the description of the latter. After reviewing some of the nuclear models commonly used to study lepton-nucleus reactions, we describe in detail the SuSAv2 model and show how its predictions compare with the available electron- and neutrino-scattering data over the kinematical range going from the quasi-elastic peak to pion-production and highly inelastic scattering.
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I. Introduction

Electron scattering represents the most powerful probe to study the internal structure of atomic nuclei, due to the weakness of the electromagnetic interaction as compared to the strong forces which govern the nuclear dynamics.\(\text{[1–3]}\). With this motivation, many electron-nucleus experiments were performed in the past, unraveling several interesting features of the many-body nuclear problem, although still leaving some open questions which would deserve further exploration. The field has seen renewed interest in the last decade due to its connection to neutrino-nucleus scattering\(\text{[4]}\) and the importance of nuclear physics inputs for the interpretation of long baseline neutrino oscillation experiments.\(\text{[5, 6]}\). The main aim of these experiments, both past or ongoing (MiniBooNE, T2K, NOvA) and future (DUNE and T2HK), is to improve the precision on our knowledge of the Pontecorvo-Maki-Nakagawa-Sakata matrix, in particular the CP-violating phase which is related to the matter-antimatter asymmetry in the Universe. The achievement of this goal requires an extremely good control of the systematic uncertainties, which are mostly coming from the modelling of neutrino-nucleus interactions. Hence the accurate description of these reactions has become one of the top challenges for theoretical nuclear physics with electroweak probes.\(\text{[7]}\).

In the energy regime relevant for the above mentioned experiments, going from about 100 MeV up to a few GeV, several mechanisms contribute to the nuclear response: from the excitation of nuclear collective states in the lowest energy part of the spectrum, up to the deep inelastic scattering at the highest energy transfers, encompassing the quasi-elastic region, corresponding to one-nucleon knockout, and the resonance region, corresponding to the excitation of nucleon resonances followed by their decay and subsequent production of pions and other mesons. Nuclear models used to analyze and interpret the data should be able to describe, as consistently as possible, all these processes with a few percent accuracy, a very challenging task which has triggered a lot of recent activity. Moreover, these models should be relativistic, as requested by the high energies involved in the experimental setup.

Due to the rareness of previous neutrino-nucleus experiments in this energy regime, nuclear models must be validated against other experiments. In particular, electron scattering is very closely related to neutrino scattering and therefore constitutes the ideal testing ground for nuclear models. In this paper we focus on the connection between these two processes, illustrating and inter-comparing some of the models used to describe them simultaneously.

The paper is organized as follows: in Sect. II we introduce the general formalism for lepton-nucleus scattering and we comment on the similarities and differences between the electron and neutrino cases. In Sect. III we present different models for the quasielastic (QE) reaction, going from the simplest Relativistic Fermi Gas to more realistic descriptions of the nucleus. Then we discuss relativistic models for the reactions occurring at higher energy transfer: two-particle-two-hole (2p2h) excitations (Sect. IV), pion production (Sect. V), higher inelastic and deep inelastic scattering (Sect. VI). In Sect. VII we concentrate on the SuSAv2 model, developed by our group for both electron and neutrino reactions, collecting in a coherent way the main results obtained in our past work and illustrating the possible implementation of the model in Monte Carlo generators. Finally, in Sect. VIII we draw our conclusions and outline the future developments of our approach.

II. Connecting electron and neutrino scattering

In order to emphasize the similarities and differences between electron- and (anti)neutrino-nucleus scattering, here we present in parallel the general formalism for the two processes. Let us consider the scattering problem

\[ l + A \rightarrow l' + X + B \]

where an incident lepton \(l\) of mass \(m\) and 4-momentum \(K^\mu = (\epsilon, \mathbf{k})\) scatters off a nuclear target of mass number \(A\) (a nucleus or a single nucleon) and a lepton \(l'\) of mass \(m'\) and 4-momentum \(K'^\mu = (\epsilon', \mathbf{k}')\) emerges, with \(\epsilon = \sqrt{k^2 + m^2}\) and \(\epsilon' = \sqrt{k'^2 + m'^2}\), together with a system \(X\) and residual nucleus \(B\).

For electron scattering \(m = m' = m_e\), whereas for neutrino scattering \(m = m_\nu\) and \(m' = m_l\) for charged-current (CC) processes, where a \(W\) boson is exchanged, or \(m' = m_\nu\) for neutral-current (NC) processes, associated to the exchange of a \(Z^0\) boson.

Let us first remind some basic definitions, commonly used in the nuclear physics language.

- The \textit{inclusive} cross section, indicated by the notation \((l, l')\), where only the outgoing lepton is detected, corresponds to the integration over all the final particles except the outgoing lepton and a sum over all open channels compatible with the kinematics.

\footnote{In this paper we shall use the units \(\hbar = c = 1\).}
- The *semi-inclusive* cross section, \((l,l'X)\), corresponds to the situation in which the lepton \(l'\) and a particle (or a system of particles) \(X\) are detected in coincidence in the final state. Note that \(X\) can be any kinematically allowed system (a nucleon, a photon, a meson, a nucleus, etc.), but in general it is not the complete final state: depending on the kinematics, many different channels can be open, and the semi-inclusive cross section is the sum (or integral) over all the unobserved particles.

- The *exclusive* cross section corresponds to the simultaneous detection of *all* the final scattering products.

Exclusive processes are more challenging from both experimental and theoretical points of view: they are more difficult to measure and more sensitive to the details of the nuclear model description. On the other hand, different nuclear models can describe equally well inclusive data, while exclusive or semi-inclusive data constitute a more stringent test for the validity of a model and can better discriminate between different theories.

In this article we mainly focus on the easier case of inclusive reactions, where more data are available and nuclear models are more solid. It should be kept in mind, however, that the validity of nuclear models is better tested against semi-inclusive and exclusive data, and that a good description of the latter is desirable for models to be implemented in Monte Carlo generators used to analyze neutrino oscillation experiments.

The general formalism for electron and (anti)neutrino reactions is developed in great detail in Refs. [3, 8] and [9], respectively. Here we outline some basic features which will be useful for further discussion.

The lepton-nucleus differential cross section \(d\sigma\) is proportional to the contraction of the leptonic \((\eta_{\mu\nu})\) and hadronic \((W_{\mu\nu})\) tensors, defined as

\[
\eta_{\mu\nu}(K, K') = 2m_m \sum_{j,j'} \eta_{\mu\nu}^j(K' s', K s) j_{\mu}^j(K' s', K s)
\]

\[
W_{\mu\nu} = \sum_{SS} J_{SS}^\mu J_{SS}^\nu,
\]

where the symbol \(\sum\) represents average and sum over the appropriate initial and final leptonic or hadronic spin quantum numbers, which will depend on whether or not the initial leptons and hadrons are polarized and on the possible measurement of the final polarizations. \(J_{SS}^\mu\) and \(J_{SS}^\mu\) are the leptonic and hadronic 4-current matrix elements, respectively.

**Leptonic tensor**

In the case of unpolarized electron or (anti)neutrino scattering the leptonic current entering the tensor \(\eta_{\mu\nu}\) can be written as

\[
j^\mu(K' s', K s) = \bar{u}(K', s')(a_V \gamma^\mu + a_A \gamma^\mu \gamma^5) u(K, s),
\]

where \(u(K, s)\) is the lepton Dirac spinor and, according to the Standard Model of electroweak interactions, \(a_V = 1\) while \(a_A = 0\) for unpolarized electron scattering (assuming only pure electromagnetic interaction \(^2\)) and \(a_A = \mp 1\) in the neutrino (−) or antineutrino (+) case. By inserting Eq. (3) into (1) and performing the traces one gets

\[
\eta_{\mu\nu}(K, K') = \eta_{\mu\nu}^{VV} + \eta_{\mu\nu}^{AA} + \eta_{\mu\nu}^{VA}
\]

with

\[
\eta_{\mu\nu}^{VV} = a_V^2 \left[ K_\mu K_\nu + K'_\mu K'_\nu - g_{\mu\nu} (K \cdot K' - m_m') \right]
\]

\[
\eta_{\mu\nu}^{AA} = a_A^2 \left[ K_\mu K_\nu + K'_\mu K'_\nu - g_{\mu\nu} (K \cdot K' + m_m') \right]
\]

\[
\eta_{\mu\nu}^{VA} = -2ia_V a_A \epsilon_{\alpha\beta\gamma\delta} K^\alpha K'^\beta
\]

where the upper labels correspond to the vector and axial components of the two leptonic currents.

---

\(^2\) Electrons can also interact weakly: in this case \(a_V = 4\sin^2\theta_W - 1\) and \(a_A = -1\).
Hadronic tensor: response functions

The number of variables upon which the hadronic tensor $W^{\mu\nu}$, and hence the differential cross section, depend varies of course with the type of process: the inclusive $(l, l')$ cross section is a function of only 2 independent kinematic variables, e.g. the energy and scattering angle of the outgoing lepton; the semi-inclusive $(l, l' N)$ cross section, where the final lepton and a nucleon $N$ are detected in coincidence, depends on 5 variables, and so on. For semi-inclusive processes in which $n$ final particles are detected beyond the lepton, the kinematical variables are $2+3n$. Therefore the degree of complexity of nuclear response increases with the "exclusiveness" of the process.

The number of components of the complex, hermitian ($W^{\mu\nu*} = W^{\nu\mu}$), hadronic matrix contributing to the cross section also depends on the specific process. In the most general case, the contraction $\eta_{\mu\nu} W^{\mu\nu}$ gives rise to 16 independent terms. These are usually organized into real linear combinations of the hadronic tensor components, the nuclear response functions

$$R_K, \quad K = CC, CL, LL, T, TT, TC, TL, T', TC', TL', TT', TC', TL',$$

where the labels $C$, $L$, $T$ and $T'$ correspond to a reference frame where the $z$-axis ($\mu = 3$) is parallel to the momentum transfer $q = k - k'$ and refer to the charge ($\mu = 0$), longitudinal ($\mu = 3$) and transverse ($\mu = 1, 2$) projections of the two currents building up the response. A similar decomposition can be performed for the leptonic tensor, in such a way that the tensor contraction can be written as

$$\eta_{\mu\nu} W^{\mu\nu} = v_0 \sum_K V_K R_K,$$

where the coefficients $v_0 V_K$ are linear combinations of the leptonic tensor components. Their explicit expressions are given in Sect. [III].

In the specific case of semi-inclusive neutrino scattering 6 of these coefficients vanish, and therefore only 10 response functions contribute to the cross section:

$$d\sigma(v_l, l' N) \propto V_{CC} R_{CC} + 2V_{CL} R_{CL} + V_{LL} R_{LL} + V_T R_T + V_{TT} R_{TT} + V_{TC} R_{TC} + V_{TL} R_{TL}$$
$$+ V_T' R_T' + V_{TC'} R_{TC'} + V_{TL'} R_{TL'}.$$

In the case of antineutrino the "primed" terms, originating from the axial current, change sign.

In the case of semi-inclusive unpolarized electron scattering (pure electromagnetic interaction), due to absence of axial current and to the conservation of the vector current (CVC), the response functions reduce to 4:

$$d\sigma(e, e' N) \propto V_L R_L + V_T R_T + V_{TT} R_{TT} + V_{TL} R_{TL}.$$  \hspace{1cm} (11)

Obviously $V_K$ and $R_K$ are not the same in Eqs. (10,11) because the electromagnetic (e.m.) and weak currents are different. In particular, the e.m. currents are purely vector, while the weak ones contain an axial component. The vector e.m. and weak currents are simply related by an isospin rotation.

In the case of inclusive lepton scattering, as a consequence of the integration over the full phase space of the outgoing hadron, some of the responses do not contribute and one has:

$$d\sigma(v_l, l') \propto V_{CC} R_{CC} + 2V_{CL} R_{CL} + V_{LL} R_{LL} + V_T R_T + V_T' R_T'$$
$$d\sigma(e, e') \propto V_L R_L + V_T R_T.$$  \hspace{1cm} (12,13)

In the neutrino case, each response can be decomposed as

$$R_K = R_K^{VV} + R_K^{AA} + R_K^{VA}$$

where the upper labels denote the vector or axial nature of the two hadronic currents entering the response in Eq. (2).

The above decomposition into response functions is valid in all kinematical regions (elastic, quasielastic, $2p2h$ excitations, inelastic), characterized by different nuclear currents. Further details for the response functions will be illustrated in Section [III] where calculations of the quasielastic electromagnetic and weak cross sections will be presented in some specific nuclear models.

What can we learn from electron scattering?

The formalism presented above shows that electron and neutrino scattering are closely related and validation against electron scattering data constrains models to be applied to neutrino scattering. It also shows that a model suitable
describe inclusive reactions is not necessarily a good model for exclusive processes. More nuclear responses are needed to describe the latter, and some details of the nuclear model can be washed out by the extra-integrations involved in the calculation of inclusive cross sections.

Comparison with inclusive electron scattering data is a necessary, but not sufficient, test for nuclear models, because of two main differences:

- The weak current carried by neutrinos has a vector and an axial component, while the electromagnetic current is purely vector. As a consequence, neutrinos can probe the axial nuclear response, not accessible via unpolarized electron scattering. This introduces uncertainties related to the knowledge of the axial and pseudoscalar form factors. In principle valuable information on the axial response could also be extracted from parity-violating (PV) electron scattering off nuclei [10, 12], which could also provide important complementary information on nuclear correlations [13–16] and on the radiative corrections entering in the isovector axial-vector sector [17]. However, few PV data on medium-heavy nuclei exist and are mostly limited to the elastic part of the spectrum.

- In typical electron scattering experiments the incident beam energy is known with good accuracy, hence the transferred energy, \( \omega \), and momentum, \( q \), can be precisely determined by measuring the outgoing lepton kinematics. In long baseline neutrino experiments the beam is not monochromatic: neutrinos are produced from meson decay with a more or less - depending on the experiment - broad distribution around an average value. This implies that each kinematic (momentum and scattering angle) of the outgoing lepton corresponds in general to a finite range of \( \omega \) and \( q \), which can in turn correspond to various overlapping processes. An important example of this difference is represented by the mixing of 1p1h and 2p2h excitations: in the \((e,e')\) the former correspond to the quasielastic peak, while the latter are peaked in the ”dip” region between the QE and the \( \Delta \) resonance peaks. In \((\nu l, l)\) data there is no way to disentangle the two channels and one has to rely on nuclear models to reconstruct the neutrino energy from lepton kinematics.

A very large set of high quality inclusive electron-nucleus scattering data has been collected in the past (see Ref. [18] for an exhaustive collection), covering a wide energy range (from 100 MeV to several GeV) and various nuclei (from \( ^3\text{He} \) up to \( ^{238}\text{U} \)). Recently [19, 20] the \((e,e')\) cross section on argon and titanium has been measured at JLab, with the specific purpose of constraining models used in the analysis of neutrino-nucleus scattering. Some data are also available for the separated longitudinal and transverse responses. These in principle allow for a more stringent test of the models, since nuclear effects are different in the two channels.

Inclusive electron-nucleus scattering and Superscaling

A powerful mean of extracting information on nuclear effects from \((e,e')\) data at different beam energies and on different nuclei is represented by the superscaling analysis proposed in [21–23]. This approach allows for the factorization of the cross section into a single-nucleon term times a function - the scaling function - which embodies the nuclear dynamics. Assuming that the latter is independent of the specific probe, scaling allows one to predict inclusive neutrino-nucleus cross sections using electron-nucleus data [24].

Scaling phenomena occur in various different fields, including atomic, nuclear and hadronic physics, whenever a new scale is probed in a certain process. For example, \( x \)-scaling occurs in lepton-nucleus scattering when the lepton, at high values of \( Q^2 \), is resolving the inner structure of the proton (or neutron), interacting with its constituents, the quarks. The typical manifestation of scaling is that a function of two variables (in this case the nucleon’s structure function \( F_{1,2}(Q^2, \nu) \) becomes a function of only one variable (in this case the Bjorken variable \( x \)). In a similar way, in lepton-nucleus scattering, scaling occurs when the nuclear response, depending in general on the two variables \( q, \omega \), becomes function of only one scaling variable, indicating that the probe interacts with the nucleus’ constituents, the nucleons. Since these, unlike the quarks, are not point-like, the cross section has to be ”reduced”, i.e., divided by a single-nucleon function which takes into account the internal structure of the nucleon. In this context the scaling function is defined as (see Appendix A for details)

\[
F(q, \omega) = \frac{d^2 \sigma / d\Omega \, d\omega}{\sigma_{eN}},
\]  

(15)

the ratio between the double differential cross section and the half-off-shell single-nucleon cross section \( \sigma_{eN} \) evaluated at the smallest possible values of the missing momentum and missing energy compatible with the kinematics and averaged over the ejected nucleon’s azimuthal angle. The corresponding scaling variable is a particular combination of \( q \) and \( \omega \) whose choice is not unique and may involve some approximation. The common choices for the scaling variables in the quasielastic domain are \( y \) and \( \psi \), defined in detail in Appendix A.
satisfied when

\[ F(q, \omega) \to F(y). \] (16)

This occurs for high enough values of the momentum transfer \( q \), able to resolve the internal structure of the nucleus. Quantitavely this corresponds to the region \( q \gtrsim 300\text{--}400 \text{ MeV}/c \), where collective effects become negligible. A further difference between \( x \)- and \( y \)-scaling resides in the fact that, while at high \( Q^2 \) quarks become asymptotically free, nucleons are always interacting inside the nucleus, and the scaling function keeps track of these correlations, being different from that of a collection of free nucleons.

The analysis of \((e,e')\) experimental data performed in Refs. \[21, 22, 25\] shows that scaling is satisfied with good accuracy in the "scaling region" \( \omega < \omega_{QEP} \), where \( \omega_{QEP} \) corresponds to the maximum of the quasielastic peak. It is broken at higher energies loss, indicating that processes other than QE scattering, like pion production and excitation of two-particle two-hole (2p2h) states contribute to the cross section.

The above described \( q \)-independence of the scaling function \( F \) when expressed in terms of \( y \) is also called scaling of first kind, in order to distinguish it from scaling of second kind, concerning the dependence of the function \( F \) upon the specific nucleus. Scaling of second kind is formulated in terms of the scaling variable \( \psi \) (see Appendix A) and consists in the fact that the function

\[ f(q, \omega) = k_F F(q, \omega) \] (17)

only depends on \( \psi \)

\[ f(q, \omega) \to f(\psi). \] (18)

In other words, the reduced cross section \( F \) scales with the Fermi momentum as \( 1/k_F \). This hypothesis is motivated by the RFG model (see Appendix C) where it is exactly fulfilled, and the analysis of data on different nuclei (with mass number \( A \) ranging from 4 to 197) proves that it is respected with very good accuracy in the scaling region.

Superscaling is the simultaneous occurrence of scaling of first and second kind and the function \( f \) is called super-scaling function.

The superscaling analysis can be extended to the separate longitudinal and transverse (relative to the transferred momentum \( q \)) responses. In this case two superscaling functions are introduced:

\[ f_L = k_F \frac{R_L(q, \omega)}{G_L(q, \omega)}, \] (19)

\[ f_T = k_F \frac{R_T(q, \omega)}{G_T(q, \omega)}, \] (20)

where \( R_L \) and \( R_T \) are the longitudinal and transverse response functions and the dividing factors \( G_L, G_T \) are defined in Appendix A.

The analysis of the separated \( L \) and \( T \) responses shows that scaling violations mainly reside in the transverse channel, whereas in the longitudinal one scaling works quite well in the full quasielastic region, i.e. also at \( \omega > \omega_{QEP} \). The reason for this difference is that the main contributions which violate scaling, namely the 2p2h and \( \Delta \) resonance excitations, are essentially transverse, with small longitudinal contamination of relativistic origin. From the analysis of the \( L/T \) separated \((e,e')\) data a phenomenological longitudinal superscaling function has been extracted \[26\]. The agreement with this function represents a strong constrain for nuclear models used in neutrino-nucleus scattering simulations. This topic is examined in detail in subsequent sections. For example, in Fig. 1 the comparison of the experimental superscaling function with the relativistic Fermi gas prediction is shown. The variable \( \psi' \) differs from \( \psi \) by an energy shift related to the nucleon separation energy (see later). Note the striking difference between the RFG result and the experimental data: the RFG predicts a symmetric function around \( \psi = 0 \), with a maximum value of 0.75 (see Eq. (16)), while the data display a pronounced asymmetric tail at large \( \psi \) (large \( \omega \)) and a maximum of \( \sim 0.6 \).
Semi-inclusive electron-nucleus scattering

The study of \((e,e'N)\) reactions on nuclei provides essential information about nuclear structure and dynamics, not only on single-particle properties such as the spectral function and the nucleon’s momentum distribution, but also on more complex mechanisms as nucleon-nucleon correlations and meson-exchange currents [1, 2, 4]. These are essential inputs for the simulation of \(\nu-A\) scattering and therefore neutrino energy reconstruction.

The theoretical prediction of semi-inclusive reactions is a much harder task than modeling the inclusive process since the related observables are far more sensitive to the details of the dynamics. On the other hand the richer structure of the cross section (see Eqs. (10,11)) allows one to better discriminate among different models.

In Plane Wave Impulse Approximation (PWIA) the 6-th differential \((e,e'N)\) cross section can be written as [27]

\[
\frac{d^6\sigma}{dp_Nd\Omega_Ndk'_e/d\Omega_e} = \frac{p_NM_NM_{A-1}}{\sqrt{(M_{A-1})^2 + p^2}} \sigma_{eN}(q,\omega;p,E)S(p_m,E_m),
\]

where \(p_N, \Omega_N, k'_e, \Omega_e\) are the momenta and solid angles of the outgoing nucleon and electron, respectively, \(M_{A-1}\) is the ground state mass of the recoiling system, \(p_m = p_{A-1} - p = q - p_N\) is the missing momentum, \(E_m = W_{A-1} - m_N - M_A\) is the missing energy, \(E = \sqrt{W_{A-1}^2 + p^2 - M_{A-1}^2 + p^2}\) is the excitation energy of the residual nucleus having invariant mass \(W_{A-1}\), related to the missing energy \(E_m\) and to the separation energy \(E_s = M_{A-1} + m_N - M_A\) by \(E \approx E_m - E_s\). Finally, \(\sigma_{eN}\) is the half-off-shell single-nucleon cross section [28] and \(S(p_m,E_m)\) the nuclear spectral function, which yields the probability of removing a nucleon of momentum \(p = p_m\) from the nuclear ground state leaving the residual system with excitation energy \(E\). The spectral function is related to the nucleon’s momentum distribution

\[
n(p) = \int_0^\infty S(p,E) dp.
\]

The factorization [21] breaks if effects beyond the PWIA, as Final State Interactions (FSI) of the knocked-out nucleon with the residual nucleus and effects of two- or many-body currents, are taken into account. Moreover, even in the plane wave limit, the factorized expression [21] no longer holds if dynamical relativistic effects in the bound nucleons, arising from the lower components in the relativistic wave functions, are incorporated. As long as all these effects are properly taken into account in a Relativistic Distorted Wave Impulse Approximation (RDWIA) framework, which describes the distortion of the ejected proton wave function, the experimental data for \((e,e'p)\) can provide reliable information on the nuclear spectral function.
Several \((e,e'p)\) experiments were performed in the past in various laboratories (Saclay, NIKHEF, MIT/Bates, Mainz, JLab) at different kinematic conditions and on a variety of nuclei, from \(^2\text{H}\) to \(^{208}\text{Pb}\). New data on \(^{40}\text{Ar}\) have recently been taken at JLab \cite{29} and will provide a valuable input for the analyses of neutrino experiments. The comparison of existing data with theoretical models points to a systematic overestimation of the data from shell model based calculations. This discrepancy is interpreted as a probe of the limitations of the mean field approach and of the importance of NN correlations in the nuclear wave function. It is often quantified in terms of spectroscopic factors, \(Z_\alpha\), which measure the actual occupancy (different from 1) of each shell (being \(\alpha\) the set of quantum numbers characterizing a given orbital). The spectroscopic factors extracted from \((e,e'p)\) data are typically of the order of 0.5-0.7, depending on the orbital and on the nucleus. The knowledge of these factors, or better the understanding of their microscopic origin, is an important ingredient of the modelling of semi-inclusive reactions.

Summarizing, electron-nucleus scattering provides an essential input for the description of neutrino-nucleus reactions. The huge amount of high precision electron scattering data allows to discriminate between theoretical models much better than the comparison with neutrino scattering data: the latter have bigger errors which, most importantly, are largely due to nuclear model uncertainties. Therefore, it is mandatory that any model used in the analysis of neutrino oscillation experiment is first validated against electron scattering data. The comparison with inclusive \((e,e')\) data will be illustrated in the next sections for the models developed by our group. The next, more challenging, task, will be to develop reliable models for the description of semi-inclusive \((\nu_l, l\overrightarrow{N})\) reactions, which require a more detailed knowledge of the nuclear structure and dynamics.

III. Models for QE in the Impulse Approximation

A. Relativistic Fermi Gas (RFG)

We follow closely the notation introduced in Ref. \cite{23}. The quasielastic electroweak cross section is proportional to the hadronic tensor or response function for single-nucleon excitations transferring momentum \(q\) and energy \(\omega\). The relativistic Fermi gas (RFG) gives the simplest approach to a fully relativistic nuclear system whose response to the hadronic tensor or response function for single-nucleon excitations transferring momentum \(q\) and energy \(\omega\) is given in this model by

\[
W^{\mu\nu}(q, \omega) = \sum_{\mathbf{p}, s, s'} \frac{\delta(E' - E - \omega - m_N^2)}{E^2} J_{s,s'}^{\mu\nu}(\mathbf{p'}, \mathbf{p}) \frac{1}{E^2} J_{s,s'}^{\nu\mu}(\mathbf{p'}, \mathbf{p}) \theta(k_F - p) \theta(p' - k_F) \tag{24}
\]

where \(J^\mu\) is the electroweak current element while \(E = \sqrt{p^2 + m_N^2}\) is the initial nucleon energy in the Fermi gas. The final momentum of the nucleon is \(\mathbf{p'} = \mathbf{p} + \mathbf{q}\) and its energy is \(E' = \sqrt{p'^2 + m_N^2}\). Note that initial and final nucleons have spin component \(s\) and \(s'\), respectively.

In the thermodynamic limit, \(V \to \infty\) we substitute the sums by momentum integrations. Then the volume \(V = \frac{3\pi^2 N}{k_F^3}\) of the system is related to the Fermi momentum \(k_F\) and proportional to the number \(N\) of protons and/or neutrons participating in the process:

\[
W^{\mu\nu}(q, \omega) = \frac{V}{(2\pi)^3} \int d^3p \frac{E^2}{m_N^2} \frac{1}{E^2} 2w^{\mu\nu}_{s,s'}(\mathbf{p'}, \mathbf{p}) \theta(k_F - p) \theta(p' - k_F) , \tag{25}
\]

where we have defined the single-nucleon tensor for the 1p1h excitation

\[
w^{\mu\nu}_{s,s'}(\mathbf{p'}, \mathbf{p}) = \frac{1}{2} \sum_{s,s'} J^\mu_{s,s'}(\mathbf{p'}, \mathbf{p}) J^\nu_{s',s}(\mathbf{p'}, \mathbf{p}) \tag{26}
\]

In the case of electron scattering the electromagnetic current matrix element is given by

\[
J^\mu_{s,s'}(\mathbf{p'}, \mathbf{p}) = \pi_{s'}(\mathbf{p'}) \left[ F_1(Q^2) \gamma^\mu + F_2(Q^2) i\sigma^\mu\nu \frac{Q^\nu}{2m_N} \right] u_s(\mathbf{p}) , \tag{27}
\]
where $F_1$ and $F_2$ are, respectively, the Dirac and Pauli electromagnetic form factors of proton or neutron.

In the case of neutrino or antineutrino CC scattering the weak current matrix element is the sum of vector and axial-vector terms $J^\mu = V^\mu - A^\mu$, where the vector current is

$$V^\mu_{s,s'}(p', p) = \overline{u}_{s'}(p') \left[ 2F_1^V \gamma^\mu + 2F_2^V i\sigma^{\mu \nu} \frac{Q_\nu}{2m_N} \right] u_s(p)$$

(28)

being $F_i^V = (F_s^P - F_s^N)/2$ the isovector form factors of the nucleon. The axial current is

$$A^\mu_{s,s'}(p', p) = \overline{u}_{s'}(p') \left[ G_A \gamma^\mu \gamma_5 + G_P \frac{Q_\mu}{2m_N} \gamma_5 \right] u_s(p),$$

(29)

where $G_A$ is the nucleon axial-vector form factor and $G_P$ is the pseudo-scalar axial form factor. It is usual to assume the dipole parametrization of the axial form factor, with the axial mass $M_A = 1.032$ GeV. From partial conservation of the axial current (PCAC) and pion-pole dominance, they are related by

$$G_P = \frac{4m_N^2}{m_A^2 + |Q|^2} G_A,$$

(30)

where $Q^2 = \omega^2 - q^2 < 0$.

The single nucleon tensor for the electroweak current is computed by performing the traces in Appendix [B].

To obtain the quasielastic cross section for $(e, e')$ reaction we use the standard expansion in terms of response functions

$$\frac{d\sigma}{dE'\,d\Omega} = \sigma_{Mott} \left( v_L R_L^{e,m} + v_T R_T^{e,m} \right)$$

(31)

where $\sigma_{Mott}$ is the Mott cross section, $v_L = Q^2/q^4$ and $v_T = \tan^2(\theta/2) - Q^2/2q^2$, with $\theta$ the scattering angle. The electromagnetic longitudinal and transverse response functions are the following components of the e.m. hadronic tensor in the scattering coordinate system with the $z$-axis in the $q$ direction (longitudinal)

$$R_L^{e,m}(q, \omega) = W_{e.m.}^{00}$$

(32)

$$R_T^{e,m}(q, \omega) = W_{e.m.}^{11} + W_{e.m.}^{22}$$

(33)

Analogously the $(\nu, l^-)$ CCQE cross section for neutrino energy $E_\nu = \epsilon$ and final lepton energy $E' = E_l$, has been expanded in terms of five response functions. If the lepton scattering angle is $\theta_l$, the double-differential cross section can be written as [24] [30]

$$\frac{d^2\sigma}{dE_l d\cos\theta_l} = \sigma_0 \left\{ V_{CC}R_{CC} + 2V_{CL}R_{CL} + V_{LL}R_{LL} + V_T R_T \pm 2V_{T'} R_{T'} \right\}$$

(34)

where we have defined the cross section

$$\sigma_0 = \frac{G^2 \cos^2 \theta_l}{4\pi} \frac{k'}{\epsilon v_0}$$

(35)

In [35] the Fermi weak constant is $G = 1.166 \times 10^{-11}$ MeV$^{-2} \sim 10^{-5}/m^2$, the Cabibbo angle is $\cos \theta_c = 0.975$, $k'$ is the final lepton momentum, and we have defined the factor $v_0 = (\epsilon + k')^2 - q^2$. Note that the fifth response function $R_{T'}$ is added (+) for neutrinos and subtracted (−) for antineutrinos. The lepton $V_K$ coefficients depend only on the lepton kinematics and they are defined by

$$V_{CC} = 1 - \delta^2 \frac{|Q|^2}{v_0}$$

(36)

$$V_{CL} = \frac{\omega}{q} + \frac{\delta^2}{\rho'} \frac{|Q|^2}{v_0}$$

(37)

$$V_{LL} = \frac{\omega^2}{q^2} + \left( 1 + \frac{2\omega}{q\rho'} + \rho\delta^2 \right) \frac{\delta^2 |Q|^2}{v_0}$$

(38)

$$V_T = \frac{|Q|^2}{v_0} + \frac{\rho}{2} \left( \frac{\omega}{q} + \frac{1}{2} \rho\delta^2 \right) \frac{|Q|^2}{v_0}$$

(39)

$$V_{T'} = \frac{1}{\rho'} \left( 1 - \frac{\omega}{q} \delta^2 \right) \frac{|Q|^2}{v_0}.$$
Here we have defined the dimensionless factors $\delta = m_l/\sqrt{|Q^2|}$, proportional to the charged lepton mass $m_l$, $\rho = |Q^2|/q^2$, and $\rho' = q/(\epsilon + \epsilon')$.

The five nuclear response functions $R_K, K = CC, CL, LL, T, T'$ ($C =$Coulomb, $L =$longitudinal, $T =$transverse) in the same scattering coordinate system as for electrons are then given by the following components of the hadronic tensor:

\[
R_{CC} = W^{00},
\]

\[
R_{CL} = -\frac{1}{2}(W^{03} + W^{30}),
\]

\[
R_{LL} = W^{33},
\]

\[
R_T = W^{11} + W^{22},
\]

\[
R_{T'} = -\frac{i}{2}(W^{12} - W^{21}).
\]

It is convenient to introduce dimensionless variables measuring the energy and momentum in units of $m_N$, namely $\lambda = \omega/2m_N$, $\kappa = q/2m_N$, $\tau = \kappa^2 - \lambda^2$, $\eta_F = k_F/m_N$, and $\xi_F = \sqrt{1 + \eta_F^2} - 1$.

In Appendix C we calculate analytically the response functions of the RFG by integrating Eq. (25) for the different components of the single nucleon tensor computed in Appendix B. A good property of the RFG model is the factorization of the response function as a product of a “single nucleon” response multiplied by an universal function, called superscaling function:

\[
f(\psi) = \frac{3}{4}(1 - \psi^2)\theta(1 - \psi^2).
\]

As shown in Appendix C, the scaling variable, $\psi$, is related to the minimum kinetic energy allowed for the initial nucleon to absorb the momentum and energy transfer $(q, \omega)$, given by

\[
\varepsilon_0 = \text{Max}\left\{\kappa\sqrt{1 + \frac{1}{\tau}} - \lambda, \varepsilon_F - 2\lambda\right\},
\]

where $\varepsilon_F = \sqrt{1 + \eta_F^2}$ is the Fermi energy in units of $m_N$. The scaling variable is defined by

\[
\psi = \sqrt{\frac{\varepsilon_0 - 1}{\varepsilon_F - 1}}\text{sgn}(\lambda - \tau).
\]

Note that $\psi < 0$ for $\lambda < \tau$ (on the left side of the quasielastic peak). The meaning of $\psi^2$ is the following: it is the minimum kinetic energy of the initial nucleon divided by the kinetic Fermi energy.

Using this definition, the electromagnetic responses can be written as

\[
R_{K}^{e.m.} = G_K f(\psi),
\]

\[
G_K = \Lambda(ZU^p_K + NU^n_K),
\]

where $Z$ ($N$) is the proton (neutron) number and

\[
\Lambda = \frac{\xi_F}{m_N\eta_F^2\kappa}.
\]

Analogously, for the weak responses

\[
R_K = \mathcal{N}\Lambda U_K f(\psi),
\]

where $\mathcal{N} = N$ ($Z$) for neutrino (antineutrino) scattering. This factorization of the scaling function inspires the superscaling models discussed in Sect. III E using a phenomenological scaling function instead of the universal superscaling function of the RFG.

The remaining quantities are the integrated single nucleon response functions. The electromagnetic ones, for protons or neutrons, are given by

\[
U_L^{p,n} = \frac{\kappa^2}{\tau}\left[(G_E^{p,n})^2 + \frac{(G_E^{p,n})^2 + \tau(G_M^{p,n})^2}{1 + \tau}\Delta\right],
\]

\[
U_T^{p,n} = 2\tau(G_M^{p,n})^2 + \frac{(G_E^{p,n})^2 + \tau(G_M^{p,n})^2}{1 + \tau}\Delta.
\]
where the quantity $\Delta$ has been introduced
\[
\Delta = \frac{\tau}{\kappa^2} \left[ -\frac{(\lambda - \tau)^2}{\tau} + \xi_F((1 + \lambda)(1 + \psi^2) + \frac{\xi_F}{3}(1 + \psi^2 + \psi^4)) \right].
\] (55)

In what follows we present the explicit expressions for the weak integrated single-nucleon responses.

The $U_{\text{CC}}$ is the sum of vector and axial contributions. The vector part implements the conservation of the vector current (CVC). The axial part can be written as the sum of conserved (c.) plus non conserved (n.c.) parts. Then
\[
U_{\text{CC}} = U_{\text{CC}}^V + (U_{\text{CC}}^A)_{\text{c}} + (U_{\text{CC}}^A)_{\text{n.c.}}.
\] (56)

For the vector CC response we have
\[
U_{\text{CC}}^V = \frac{\kappa^2}{\tau} \left[ (2G_E^V)^2 + \frac{(2G_E^V)^2 + \tau(2G_M^V)^2}{1 + \tau} \Delta \right].
\] (57)

where $G_E^V$ and $G_M^V$ are the isovector electric and magnetic nucleon form factors
\[
G_E^V = F_1^V - \tau F_2^V
\] (58)
\[
G_M^V = F_1^V + F_2^V.
\] (59)

The axial-vector CC responses are
\[
(U_{\text{CC}}^A)_{\text{c}} = \frac{\kappa^2}{\tau} G_A^2 \Delta
\] (60)
\[
(U_{\text{CC}}^A)_{\text{n.c.}} = \frac{\lambda^2}{\tau} (G_A - \tau G_P)^2.
\] (61)

Using current conservation for the conserved part of the integrated single-nucleon responses $K = CL, LL$ we can express them as
\[
U_{CL} = -\frac{\lambda}{\kappa} [U_{\text{CC}}^V + (U_{\text{CC}}^A)_{\text{c}}] + (U_{CL}^A)_{\text{n.c.}}
\] (62)
\[
U_{LL} = \frac{\lambda^2}{\kappa^2} [U_{\text{CC}}^V + (U_{\text{CC}}^A)_{\text{c}}] + (U_{LL}^A)_{\text{n.c.}},
\] (63)

where the n.c. parts are
\[
(U_{CL}^A)_{\text{n.c.}} = -\frac{\lambda \kappa}{\tau} (G_A - \tau G_P)^2
\] (64)
\[
(U_{LL}^A)_{\text{n.c.}} = \frac{\kappa^2}{\tau} (G_A - \tau G_P)^2.
\] (65)

Finally, the transverse responses are given by
\[
U_T = U_T^V + U_T^A
\] (66)
\[
U_T^V = 2\tau(2G_M^V)^2 + \frac{(2G_E^V)^2 + \tau(2G_M^V)^2}{1 + \tau} \Delta
\] (67)
\[
U_T^A = G_A^2 [2(1 + \tau) + \Delta]
\] (68)
\[
U_T' = 2G_A(2G_M^V)^2 \sqrt{\tau (1 + \tau) [1 + \tilde{\Delta}]}
\] (69)

with
\[
\tilde{\Delta} = \frac{1}{\sqrt{\tau (1 + \tau)}} \left[ \frac{\tau}{\kappa}(1 + \lambda) - \sqrt{\tau (\tau + 1)} + \frac{\tau}{\kappa} \frac{1}{2} \xi_F(1 - \psi^2) \right].
\] (70)
B. Semi-relativistic shell models

Nuclear models based on the Fermi gas describe the single nucleon wave functions as plane waves, thus neglecting the finite size of the nucleus, assuming that the response per particle is the same as for infinite nuclear matter. The validity of this assumption has been investigated theoretically in the nuclear shell model (SM) for the kinematics of interest \([31,33]\). It is obvious that the Fermi gas cannot describe very low energy transfer region of the nuclear response, dominated by excitation of discrete states and giant resonances. Besides, Pauli blocking, for small momentum transfer \(q < 2k_F\), forbids emission from a subset of occupied momentum states in the Fermi gas, even if there is enough energy transfer, by momentum conservation. In a real nucleus, however, the nucleons are not plane waves, and the ejected nucleons are plane waves only asymptotically, \(r \to \infty\). This makes the RFG inappropriate to precise modeling of the nuclear response at low momentum transfer \(q < 500\) MeV/c \([31]\).

In ref. \([31]\) the finite size effects on the electromagnetic QE response functions were studied using a continuum shell model (CSM) in the non relativistic regime for \(q \leq 550\) MeV/c. In the CSM the single nucleon wave functions were obtained by solving the Schroedinger equation with a mean field potential \(V(r)\) of Woods-Saxon type, including central and spin orbit interactions, plus a Coulomb part for protons

\[
V(r) = \frac{-V_0}{1 + e^{(r-R)/a}} + \frac{1}{m_e^2} \int r \, dr \left( \frac{-V_{ls}}{1 + e^{(r-R)/a}} \right) \cdot \sigma + V_{Coul}(r).
\]  

This potential is solved numerically for negative and positive energies, and the parameters, \(V_0, V_{ls}, R, a\) are fitted to the experimental energies of the valence shells for protons and neutrons, obtained from the masses of the neighboring nuclei. The SM is the simplest model where the response of a confined quantum system of nucleons can be evaluated, and thus, it allows us to explore the importance of finite size against the infinite FG model. This study was made theoretically by comparing the separate longitudinal and transverse response functions. While the FG model reproduces the position and width of the separate SM responses, it cannot fully account for the detailed energy dependence. Specifically, the SM responses present tails in the low and high ends of the energy transfer, reminiscent of the momentum distribution of a finite system.

The finite size effects are often simulated by the local density approximation (LDA), which consists in replacing the Fermi momentum by a local value, \(k_F \to k_F(r)\), depending on the nuclear density \(\rho(r)\). These local responses per unit volume are then integrated over the full nuclear volume to obtain the LDA responses. In \([31]\) it was shown that, while the LDA responses present in fact short tails closer to the SM, they fail to describe the SM responses around the center of the peak. As a result a bare Fermi gas with constant \(k_F\) is more appropriate to describe the responses of a finite nucleus.

In Ref. \([32]\) the shell model was extended to a semi-relativistic (SR) shell model by implementing relativistic kinematics and using a semi-relativistic expansion of the electromagnetic current. The relativistic electroweak current was expanded in powers of the initial nucleon momentum, while treating exactly the relativistic kinematics of the final ejected particle. The SR expansion of the current and relativistic kinematics was tested in the Fermi gas model by comparison of the RFG and SRFG response functions. The two models are in great accord for all values of the momentum transfer. This is seen in Fig. 2, where results for the semi relativistic shell model (SRSM) are compared to the RFG and the SRFG models for the reaction \(^{12}\)C(\(\nu_{\mu}, \mu^{-}\)) \([30]\). Two incident neutrino energies, \(\epsilon = 1\) and 1.5 GeV, and two scattering angles, \(\theta = 45^\circ, 135^\circ\), are shown. The cross section is plotted as a function of the final muon energy \(\epsilon'\). Therefore a range of \(q\) and \(\omega\) values, which can be high or low depending on the kinematics, is being explored. The RFG and SRFG results are essentially equal, while the SRSM also gives similar results, with the exception of the characteristic finite size tails and a small shift of the maximum.

The semi-relativistic shell model was extended in \([33]\) to improve the description of the final-state interaction of relativistic nucleons. The Dirac-equation-based potential plus the Darwin factor (DEB+D) was used instead of the non-relativistic Woods-Saxon potential of the SRSM. The DEB potential was taken from the relativistic mean field (RMF) model of ref. \([30]\) by a Schrödinger reduction of the Dirac equation with scalar and vector potentials. The resulting DEB potential is local and energy-dependent. The solution of the Schrödinger equation, multiplied by the nonlocality Darwin factor, corresponds to the upper component of the Dirac spinor of the RMF. The Relativistic Mean Field model applied to electron and neutrino-nucleus scattering reactions is discussed in detail in next section. Here we simply summarize those aspects of the model that result of interest for the comparison with the semi-relativistic predictions discussed above.

Results for the DEB+D potential in the semi-relativistic shell model are presented in Fig. 3, where we show the longitudinal and transverse scaling functions for electron scattering for three values of the momentum transfer \(q = 0.5, 0.7\) and 1 GeV/c. We compare with the results of the RMF model with the CC2 current operator (see details in next section) and with the SR shell model with Woods-Saxon potential. The SR-DEB+D scaling functions are similar to the fully relativistic ones. They present the same amount of asymmetry with a long tail in the \(\psi > 1\) region,
while the maximum of the scaling functions is shifted to the right with respect to the SRSM. Notice that the SR-DEB+D shell model contains the same physical ingredients in the FSI as the full RMF, except for the genuine off-shell properties of the CC2 current (see next section), not included in the SR current, which was obtained assuming on-shell (relativistic) kinematics. This explains the differences between the results obtained with the two models, particularly in the transverse scaling function. As shown in next section, where a detailed discussion on the RMF results is presented, these differences are of the same order as the ones found in the RMF between the several prescriptions of the electromagnetic current.

### C. Relativistic Mean Field (RMF)

The kinematics involved in electroweak reactions with very high energy and momentum values transferred in the processes implies the need of using a fully relativistic formalism. This means not only describing relativistically the kinematics but also the nuclear dynamics. This was the basic motivation in the past to introduce the Relativistic Mean Field (RMF) approach and to apply it to the description of electron and neutrino scattering reactions with nuclei. In this section we introduce the basic ideas embodied in the RMF approach as developed in detail in previous works [37–45], and show its capability to describe successfully observables for different type of reactions, namely, inclusive and semi-inclusive processes. The RMF model, as many other approaches applied to electroweak reactions, is used within the framework of the Impulse Approximation (IA). This means that the complex scattering process of a nucleus is simply described as an incoherent sum of single-nucleon scattering reactions. Although this is an oversimplified description of the real process, it accounts for the main effects that emerge in the case of the quasielastic (QE) kinematics. Obviously, ingredients beyond the IA, such as nucleon correlations, meson exchange currents (MEC),
etc. should be taken into account for a proper description of the data in the whole range of kinematics. This topic is discussed at length in subsequent sections.

As already mentioned, the analysis of the world inclusive QE ($e,e'$) shows without ambiguity that scaling is fulfilled at high level. Thus any theoretical model aiming to explain electron scattering is constrained to satisfy scaling properties. Not only the theoretical scaling function should not depend on the momentum transfer $q$, neither on the nuclear system, but also its particular shape, with a long tail extended to high values of the energy transfer (high positive values of the scaling variable), should emerge. Most of the models based on the IA satisfy scaling; this is the case of the RFG, the semirelativistic approach presented in the previous section, and diverse non relativistic approaches based on the use of the spectral function. However, most of them lead to symmetrical scaling functions that depart significantly from the data analysis, missing the tail at high $\omega$. In this section we discuss the basic ingredients of the Relativistic Mean Field (RMF) approach. As it will be shown, this model, even being based on the IA, leads to an asymmetrical superscaling function in accordance with data. This proves the capability of the RMF model to describe inclusive electron scattering reactions and its extension to neutrino processes. Not only that, the model has been also applied to semi-inclusive ($e,e'p$) reactions providing a good description of the reduced cross section (distorted momentum distribution), as well as the spectroscopic factors.

The main ingredient needed to evaluate the electromagnetic and weak tensor is the single-nucleon current matrix element,

$$ \langle J^\mu(Q) \rangle = \int d\mathbf{r} e^{i\mathbf{q} \cdot \mathbf{r}} \bar{\Psi}_F(p_F, \mathbf{r}) \hat{\Gamma}^\mu \psi^m_B(\mathbf{r}) $$

with $\hat{\Gamma}^\mu$ the corresponding single-nucleon current operator for electron or weak neutrino scattering, and $\psi^m_B(\Psi_F(p_F, \mathbf{r}))$ the wave function for the initial bound (emitted) nucleon. The RMF model incorporates a fully relativistic description of the scattering reaction based on the impulse approximation. We use the relativistic free
nucleon expressions for the current operators corresponding to the usual options, denoted as CC1 and CC2,

\[
\hat{\Gamma}^\mu_{\text{CC1}} = (F_1 + F_2)\gamma^\mu - \frac{F_2}{2m_N}(P + P_F)^\mu,
\]

\[
\hat{\Gamma}^\mu_{\text{CC2}} = F_1\gamma^\mu + \frac{iF_2}{2m_N}\sigma^\mu\nu Q^\nu,
\]

where \(F_1\) and \(F_2\) are the Pauli and Dirac nucleon form factors, respectively, that depend only on \(Q^2\). We have introduced the on-shell four-momentum \(P^\mu = (E, p)\) with \(E = \sqrt{p^2 + m^2_N}\) and \(p\) the bound nucleon momentum. Notice that the two operators are equivalent for free on-shell nucleons, as occurs in the RFG model. However, in general one deals with off-shell bound and ejected nucleons, hence the \(CC1\) and \(CC2\) operators lead to different results. Moreover, the current is not strictly conserved and uncertainties linked to the election of the gauge also occur \([39, 46]\). In the case of neutrino scattering processes the current operator also includes axial and pseudoscalar terms,

\[
\hat{\Gamma}^\mu_A = \left[ G_A\gamma^\mu + \frac{G_P}{2m_N}Q^\mu \right] \gamma^5
\]

with \(G_A\) and \(G_P\) the axial-vector and pseudoscalar nucleon form factors, respectively.

Concerning the nucleon wave functions, they are given as solutions of the Dirac equation in presence of phenomenological relativistic potentials with scalar and vector terms. We describe the bound nucleon states, \(\psi_{jm}^n\), as self-consistent Dirac-Hartree solutions, derived within a Relativistic Mean Field (RMF) approach using a Lagrangian with local potentials fitted to saturation properties of nuclear matter, radii and nuclear masses. In Fig. 4 we present the potentials corresponding to the cases of \(^{12}\text{C}\) and \(^{16}\text{O}\). As shown, large scalar (attractive) and vector (repulsive) potentials, that do not depend on the energy, are present. It is important to point out that the RMF does get saturation, even with no Fock terms neither explicit nucleon correlations included. This comes from the combination of the strong scalar and vector potentials that incorporate repulsive and attractive interactions, and it makes a big difference with most non-relativistic approaches where correlations are needed in order to get saturation. The use of strong local scalar and vector terms in the relativistic potentials leads to the presence of non-local terms when performing a non-relativistic reduction (the so-called Dirac equation based (DEB) approach. See ref. \([44]\) for details and the general discussion and results presented in the previous section for the SR model with DEB+D potential).

Different options, that depend on the particular scattering process analyzed, have been considered to describe the ejected nucleon state. In what follows we describe briefly the model applied to semi-inclusive \((e, e'p)\) and inclusive \((e, e')\) and \((\nu, \ell)\) reactions. In the former FSI are described using phenomenological energy-dependent complex optical potentials fitted to elastic proton-nucleus scattering data. The presence of an imaginary term produces absorption, namely, flux lost into the unobserved non-exclusive channels. In Fig. 5 we present the real and imaginary (scalar and vector) terms corresponding to different models: EDAIO, EDAD1, EDAD2 and EDAD3 (see refs. \([40, 41, 43, 45]\) for details), as well as their dependence with the energy of the outgoing nucleon. The RMF model with complex...
optical potentials yields very good agreement with $(e,e'p)$ data. Not only reasonable values for the spectroscopic factors are given \[44, 45\] but also the shape of the reduced cross section fits the behavior of data even at high values of the missing momentum \[43\]. It should be emphasized that this region (high $p$-values) is very sensitive to theoretical models, and in particular, to the strong enhancement of the lower components in the Dirac wave functions produced by the relativistic potentials mainly in the final scattering state. It is remarkable the agreement between the RMF predictions and data for the reduced cross sections (left panel in Fig. 6), and particularly, for the interference longitudinal-transverse response and $TL$ asymmetry with the oscillations observed at moderate-to-high missing momentum values (middle and right panels in Fig. 6). This is a clear signal of the presence and crucial role played by dynamical effects of relativity affecting the lower components in lepton-nucleus scattering \[49\].

The extension of the RMF approach to inclusive processes such as $(e,e')$ and $(\nu_\ell, \ell)$, requires to retain the contributions from the inelastic channels. A simple way of obtaining the inclusive strength is to use purely real potentials. We have considered in the past two approaches. The first uses the same phenomenological relativistic optical potentials considered in semi-inclusive reactions, but with their imaginary terms set to zero. This is known as rROP, and it clearly does not preserve orthogonalization as the bound and ejected nucleon states are evaluated using different potentials. A similar comment applies to the particular case in which FSI are turned off, that is, the Relativistic Plane Wave Impulse Approximation (RPWIA). Finally, a different approach consists of describing the scattered states as solutions of the Dirac equation with the same potential considered for the initial bound nucleon states, namely, the real energy-independent RMF potential. This model preserves orthogonality, verifies continuity equation and current conservation, and fulfils the dispersion relationship. In what follows we use RMF to name the previous model, that is, the use of the same RMF potential to describe both the initial bound and final scattered nucleon wave functions.

In Fig. 7 we show the differential cross section for $(e,e')$ scattering on $^{12}$C at electron beam energy equal to 1 GeV. Results are presented for the two usual current operators, CC1 (bottom panel) and CC2 (top panel). In each case we compare the predictions provided by the RPWIA, rROP and RMF models. As noticed, the RMF-CC1 cross section is significantly larger than the result for CC2. This is a consequence of the extremely large effects introduced by the lower components of the nucleon wave functions, particularly the scattered one, when using the CC1 operator. This topic was studied in detail in \[39\]. A similar comment also applies to the rROP-CC1 versus rROP-CC2, although...
FIG. 6. (Left panel) Reduced cross sections versus missing momentum for the shells $3s_{1/2}$ and $2d_{5/2}$ of $^{208}$Pb. Small circles with error bars are data from Ref. [47]. In the high missing momentum region the relativistic results obtained with the currents CC2 (solid line) and CC1 (long-dashed lines), as well as the nonrelativistic results (short-dashed lines) are compared with data from Ref. [48]. Figure taken from Ref. [43]. (Middle and right panels) Interference longitudinal-transverse response $R_{TL}$ and $TL$ asymmetry $A_{TL}$ for proton knockout from $^{16}$O for the $1p_{1/2}$ (top panels) and $1p_{3/2}$ (bottom panels) orbits. Results correspond to a fully relativistic calculation using the Coulomb gauge and the current operator CC2 (solid line), a calculation performed by projecting the bound and scattered proton wave functions over positive-energy states (short-dashed line) and two nonrelativistic calculations with (long-dashed) and without (dotted) the spin-orbit correction term in the charge density operator. Here the relative effects are weaker. Notice that the three models get the same value for the maximum in the cross section in the case of CC1, whereas a large discrepancy is observed for CC2 being the RMF prediction the lowest. Finally, only the RMF model leads to a significant tail extended to large values of the energy transfer, $\omega$.

The RMF model has proved its capability to describe successfully inclusive electron scattering data in the QE domain. Not only superscaling emerges from the calculations, but also the specific shape of the scaling function with a long tail extended to high $\omega$ (large positive values of the scaling variable $\psi$) is present. However, in spite of its merits, the RMF also presents some drawbacks for increasing values of the momentum transfer $q$. The strong energy-independent scalar and vector potentials involved in RMF lead to a very significant shift of the scaling functions to higher $\omega$, and correspondingly, too severe a decrease in the maxima. In the text that follows we discuss some representative results for the scaling function $f(\psi)$ and the separate $L$ and $T$ contributions (see section III E for explicit expressions) obtained with the different approaches discussed above, i.e., RPWIA, RMF and rROP.

First in Fig. 8 we show the longitudinal scaling function for $(e,e')$ processes evaluated at several values of the momentum transfer ranging from low $q = 0.2$ GeV/c up to $q = 0.7$ GeV/c. In each panel we compare the predictions corresponding to the RMF, RPWIA and rROP models. Also included for reference two fits used in previous works [21, 22, 50]. As observed, scaling clearly breaks at low $q$-values with very different results provided by the models. On the contrary, at $q \geq 0.4 - 0.5$ GeV/c scaling works properly although the three models considered, namely, RPWIA, rROP and RMF, provide different scaling functions with a long tail extended to large positive $\psi$-values being only present in the RMF result. This is consistent with the previous discussion presented for the cross section.

To make clearer how scaling of the first kind works, i.e., independence of the momentum transfer, in Fig. 9 we compare directly the longitudinal scaling function evaluated with the RMF approach for different values of the momentum transfer ranging from 0.5 to 1 GeV/c. A comparison with data is also provided. Notice the shift in the RMF results to larger positive $\psi$-values as the momentum transfer gets higher. On the contrary, although not shown here for brevity, scaling of first kind works extremely well for the RPWIA and rROP results. To complete the
FIG. 7. Double differential cross section for $(e,e')$ scattering on $^{12}$C. Results correspond to the electron beam energy fixed to 1 GeV and scattering angle $\theta = 45^\circ$. Top (bottom) panel refers to results obtained with the CC2 (CC1) current operators. Coulomb gauge has been considered.

discussion on the scaling properties for $(e,e')$ processes we analyze in Fig. 10 (top panels) the off-shell effects in the global scaling function (including the contribution of both the longitudinal and transverse channels) evaluated with the RMF approach. We consider the two current operators, CC1 (right) and CC2 (left), and show results for three different gauges: Coulomb, Lorentz and Weyl (see [39, 46, 52–54] for details). As observed, the CC2 current leads to very similar results in the three gauges whereas the Weyl prediction for the CC1 operator deviates very significantly from the others. This gives us confidence in the use of the CC2 current. The bottom panels of Fig. 10 show the $(e,e')$ predictions of the scaling function for $^{12}$C, $^{16}$O and $^{40}$Ca. Results correspond to the RMF model for the two current operators: CC2 (left panel) and CC1 (right). As observed, scaling of second kind, i.e., independence of the scaling function on the nucleus, is highly satisfied in the case of the CC2, while some discrepancies, mainly connected with $^{40}$Ca, are present with the CC1 current. Again, this reinforces our confidence in the use of the CC2 current operator.

The capability of the RMF model to provide a successful description of the world QE $(e,e')$ data constitutes a benchmark in its extension to the study of neutrino-nucleus reactions. This has been the topic in a large set of previous works [34–36, 39, 51, 55–57]. Here we simply summarize the basic findings and show some illustrative results. The differential cross sections for CCQE neutrino scattering on $^{12}$C is illustrated in Fig. 11 where we compare the results corresponding to different models. The energy of the neutrino beam has been fixed to $\varepsilon_\nu = 1$ GeV and the muon scattering angle to $\theta_\mu = 45^\circ$. We compare the RPWIA and RMF predictions with the ones corresponding to the semi-relativistic analysis presented in [30] with the WS potential (see discussion in the previous section). It is important to point out the similarity between the RPWIA and SR-PWIA results. This shows that, within the plane wave approach for the final state, the particular description of the initial bound states as well as the current operator (relativistic versus nonrelativistic) leads to almost identical cross sections. On the contrary, the effects ascribed to final state interactions, RMF versus SR-FSI (with WS potential), produce cross sections that differ significantly. Not only the strength in the maximum is very different, but also the shape of the cross section with the long tail at smaller values of the final muon energy being present in the RMF case.

In Fig. 12 we present the scaling functions obtained from the calculation of the inclusive charged-current quasielastic (CCQE) differential cross section divided by the appropriate weak single-nucleon cross sections (see [58] for details and explicit expressions). In the panels on the left we study scaling of the first kind for the three models considered: RPWIA, rROP and RMF. Panels on the right refer to scaling of second kind by showing the results for carbon, oxygen and calcium. As noticed, both kinds of scaling are fulfilled at high level except for the first kind within the
FIG. 8. Longitudinal scaling function for $^{12}$C(e,e$'$) evaluated at different values of the momentum transfer. Results are shown for the three models discussed in the text: RPWIA (blue dot-dashed line), rROP (green dashed) and RMF (red solid). Also included for reference two fits (see [21, 22, 50] for details).

FIG. 9. Same as Fig. 8 but restricted to the RMF model.
FIG. 10. Global scaling function corresponding to \((e, e')\) evaluated with the RMF model. Results are presented for the two current operators: CC2 (left panels) and CC1 (right panels). Off-shell effects are shown in the top panels where results are presented for \(^{12}\)C and three gauges: Lorentz, Coulomb and Weyl. Bottom panels refer to the analysis of second kind scaling comparing the results obtained for carbon, oxygen and calcium.

FIG. 11. Double differential cross section for charged-current neutrino scattering on \(^{12}\)C. Results correspond to fixed values of the neutrino energy and muon scattering angle: \(\varepsilon_\nu = 1\) GeV and \(\theta_\mu = 45^\circ\). Predictions of the RMF and RPWIA are compared with the results of a semirelativistic calculation (see text and Ref. [30] for details).

The RMF model where a shift to higher values of the scaling variable is clearly observed as the energy of the neutrino beam increases. This was also the case for \((e, e')\) reactions. Thus we conclude that, within the present models, scaling works at the same level of precision for electron and neutrino scattering reactions with nuclei. In Fig. 13 we compare directly the scaling functions corresponding to the two processes. In the case of electron scattering we show separately the contributions ascribed to the two channels, longitudinal and transverse, and compare with the results for CC muon neutrinos (antineutrinos) processes. Notice that the scaling function for neutrinos (antineutrinos), even being basically transverse (the longitudinal channel is negligible), is more similar to the longitudinal \((e, e')\) result than to the transverse one. This is connected with the isoscalar/isovector contributions in electron scattering processes compared with the pure isovector character of CC neutrino reactions [51]. Moreover, the enhancement shown by the
FIG. 12. Scaling function evaluated from $(\nu, \mu^-)$ on $^{12}$C. Results are shown for RPWIA (top panels), rROP (middle) and RMF (bottom). Scaling of the first kind is analyzed in the graphs on the left that compare results for different values of the energy beam. Scaling of the second kind is shown in the right panels that contain the scaling functions corresponding to $^{12}$C, $^{16}$O and $^{40}$Ca.

FIG. 13. (Color online) Longitudinal and transverse scaling functions for $(e, e')$ compared with $f(\psi')$ evaluated from $(\nu, \mu^-)$ and $(\mathcal{P}, \mu^+)$, All results correspond to the RMF approach using the CC2 current operator. The kinematics selected corresponds to fixed values of the incident lepton energy, $\varepsilon = 1$ GeV, and momentum transfer, $q = 0.7$ GeV/c. The averaged experimental function extracted from longitudinal electron scattering data is also shown [50]. Figure taken from Ref. [51].

$T$ contribution in $(e, e')$ is a consequence of the role played by the lower components of the relativistic nucleon wave functions. This effect is in accordance with the analysis of the separate longitudinal/transverse data.
D. Other models: RPA, CRPA, RGF, SF, GFMC

Other theoretical models have been used to describe inclusive QE electron scattering reactions. These models have been later extended to the analysis of charged and neutral current QE neutrino-nucleus processes. In this section we simply enumerate some of them summarizing their basic ingredients and properties as they are described in detail in other communications published in this special number of JPG. We also illustrate the merits of the different models comparing their predictions with data in some particular situations.

An alternative, relativistic approach to FSI in $(e,e')$ and $(\nu_\ell,\ell)$ reactions is provided by the Relativistic Green Function (RGF) model. Here the description of the initial nucleon states is analogous to the RMF model presented in the previous section, but the scattered nucleon wave function is given by making use of Green’s function techniques. This formalism starts with a complex relativistic (scalar and vector) potential, that describes elastic scattering data. In the RGF approach the components of the nuclear responses can be written in terms of the single-particle optical model Green’s function within a fully relativistic formalism. This method allows one to perform calculations treating FSI consistently in the inclusive and exclusive channels. In the RGF the imaginary part of the optical potential redistributes the flux lost in a channel in the others, so the total flux is conserved in the inclusive process. The RGF has been applied to electron and neutrino-nucleus scattering reactions comparing its predictions with other models, i.e., RMF, rROP and RPWIA, as well as with data. An exhaustive analysis of scaling properties has been also presented in previous works. The reader interested can go to \[59–66\]. As an illustration we present in Fig. 14 the inclusive $^{12}\text{C}(e,e')$ double differential cross section at two representative values of the momentum transfer, $q = 500 \text{ MeV}/c$ and $q = 1 \text{ GeV}/c$. The electron energy beam has been fixed to $\varepsilon = 1 \text{ GeV}$. In each panel we compare the RGF predictions, denoted as GF1 and GF2, with the results corresponding to the RMF, rROP and RPWIA approaches. As signaled in the caption, GF1 and GF2 refer to RGF calculations performed with two different complex optical potentials, EDAD1 and EDAD2, respectively. As observed, the particular election of the potential makes an important difference that depends on the kinematics. Furthermore, the GF results also deviate significantly from the other models. Particularly interesting is the large asymmetry shown by the RMF prediction at $q = 1 \text{ GeV}/c$, while the remaining approaches lead to more symmetrical results with values of the maximum much higher than the RMF one.
FIG. 15. Double differential cross section of the $^{16}\text{O}(e,e')$ reaction at electron beam energy 700 MeV and electron scattering angle $32^\circ$. The SF calculation including FSI (solid line) is compared with the IA (no FSI) calculation (dot-dashed line) and the Fermi Gas (FG) model with $k_F = 225$ MeV and a shift energy $\epsilon = 25$ MeV. Data from Ref. [67]. Figure taken from [68].

FIG. 16. Results for the superscaling function for $^{12}\text{C}$ obtained using harmonic oscillator (HO) and natural orbitals (NO) approaches with FSI (see text for details). Comparison with the RFG and Superscaling (SUSA) predictions, as well as with the longitudinal experimental data. Figure taken from [69].

A different approach to the analysis of electroweak scattering reactions on nuclei consists of using the general formalism of the nuclear spectral function. This approach is based on the factorization ansatz, which amounts to writing the inclusive scattering cross section (likewise the nuclear response functions) as the product of the single-nucleon cross section (single-nucleon responses) and the spectral function that gives the probability of removing a nucleon from the target nucleus with certain values of momentum and energy. Although factorization does not hold in general, it has been proved to work well in most kinematical situations corresponding to the QE regime. Contrary to the independent particle picture where the bound nucleon shells correspond to discrete energy eigenvalues, the formalism of the spectral function introduces dynamical correlations induced by nucleon-nucleon (NN) force, whose effect is not taken into account in principle in the independent particle model. As a result, the spectral function acquires tails that extend to large energy and momentum. In spite of these undeniable merits, the spectral function formalism is entirely based on a non-relativistic formalism. This can introduce some doubts on its application to processes where the energy and momentums transfers can be very high, i.e., the relativistic formalism results imperative.
The existence of final state interactions between the ejected nucleon and the residual nucleus has long been experimentally established. However, the factorization assumption, implicit in the spectral function formalism, is based on the Plane Wave Impulse Approximation (PWIA). Different approaches have been used in the past to incorporate FSI in the spectral function formalism. In most of the cases the cross section (response functions) is finally written in terms of the impulse approximation (IA) cross section convoluted with a folding function that embodies FSI effects. A very detailed study on this topic can be found in [18, 68, 70–77]. We present in Fig. 15 an illustrative example where it is clearly shown that FSI produce a shift and a redistribution of the strength leading to a quenching of the peak and to an enhancement of the tail. To complete this discussion we make contact with the studies presented in [69, 78, 79], that are also based on the use of the spectral function. Here the spectral function is constructed starting with the independent particle model and introducing a Lorentzian function to take care of the energy dependence. Finally, FSI are also incorporated by means of a time-independent optical potential (see Refs. 80, 81 for details). In Fig. 16 we present some illustrative results for the superscaling function evaluated from the double differential cross section. Two models have been selected for the nucleon wave functions denoted as harmonic oscillator (HO) and natural orbitals (NO). Both of them make use of a spectral function and include FSI. As noticed, the role of FSI leads to a redistribution of the strength, with lower values of the scaling function at the maximum and an asymmetric shape around the peak position.

Other descriptions of the inclusive responses include the effects of random phase approximation (RPA) correlations within the framework of different models as the local Fermi Gas approximation and Hartree-Fock (HF). In the former, RPA correlations are introduced through a Landau Migdal residual interaction parametrized in terms of pion and rho exchange. This is the general strategy followed in a long series of works performed by Martini et al. and Nieves and collaborators. These calculations include contributions arising from many-particle, many-hole (up-nh) excitations, and go beyond the impulse approximation-based models considered in this section. A detailed discussion and references are given in the next section focused on Meson Exchange Currents (MEC) and 2p2h contributions. A different approach to the problem is provided by the model developed by the Gent group. Here one uses bound and scattered nucleon wave functions obtained with a self-consistent Hartree-Fock model using a Skyrme-type nucleon-nucleon interaction. This mean-field picture, contrary to the RMF one described in the previous section, is a non-relativistic calculation where the nuclear current is derived from the standard non-relativistic reduction of the single-nucleon current. The HF model is later extended with collective excitations described through a continuum random phase approximation (CRPA). Although inherently non-relativistic, the HF-CRPA model incorporates important relativistic ingredients [82, 83] providing reliable results for the scattering observables from very low momentum/energy transfers, where long-range correlations play an essential role, to moderate values of \( q \) and \( \omega \). The reader interested in more details on the HF-CRPA model can go to Refs. 84–89.

To conclude, some brief words on the Green’s Function Monte Carlo (GFMC) model. This is an ab initio method that allows for a very accurate description of the dynamics of constituent nucleons in nuclei. In spite of some recent remarkable results concerning the electromagnetic responses in QE inclusive \((e,e')\) reactions, the computational complexity of such an approach is currently limited to light nuclei (\(^{12}\)C). Not only the cost of calculation increases exponentially with the number of nucleons, but also the need to include relativistic kinematics and baryon resonance production involve non-trivial difficulties. Detailed studies on the electromagnetic responses and their scaling behavior, as well as the extension to neutrino-nucleus scattering processes have been completed by different groups and presented in several publications [72, 90–97].

### E. Superscaling models: SuSA and SuSAv2

**Scaling functions**

In the previous expressions for the nuclear responses, (49) and (52), and the subsequent differential cross sections, we have observed that the RFG scaling functions were the same for all longitudinal and transverse channels. While this is true for the RFG model, where a universal scaling function emerges from the calculation, the scaling functions that can be extracted from experimental data or different nuclear models can differ between the different channels. In this sense, the general procedure used to define scaling functions consists of constructing the inclusive cross section and the response functions within a particular model (or just taking experimental data) and divide them by the corresponding single-nucleon quantities computed within the RFG model. Thus, as already introduced briefly in Sect. 4 we define a global scaling function as well as a specific one for the different channels:
• Scaling functions obtained from the cross section:

\[ f^{QE}(e,e') = k_F \frac{d^2\sigma}{d\Omega_{ee'}dw} \sigma_{Mott}(v_L G_{ee'}^L + v_T G_{ee'}^T) \quad \text{for } (e,e') \quad (76) \]

\[ f^{QE}(\nu) = k_F \frac{d^2\sigma}{d\Omega_{ll}dw} \sigma_0(\nu L G_{VV}^L + V_{CC} G_{CC}^A + 2V_{CL} G_{CL}^A + V_{LL} G_{LL}^A + v_T G_T + \chi v_T G_{TT}) \quad \text{for CC } (\nu,l^-); (\nu',l^+) \quad (77) \]

• Specific scaling functions for the individual channels:

\[ f_K = k_F \frac{R_K}{G_K} \quad (78) \]

Likewise, the proton (p) and neutron (n) scaling functions can be also isolated as well as the isovector (1) and isoscalar (0) ones:

\[ f^{p,n}_K = k_F \frac{R^{p,n}_K}{G^{p,n}_K} \quad ; \quad f^{(0),(1)}_K = k_F \frac{R^{(0),(1)}_K}{G^{(0),(1)}_K} \quad (79) \]

**SuperScaling Approach: a semiphenomenological model**

Making use of the previous description of the scaling formalism, our aim is to achieve a complete theoretical description of neutrino-nucleus reactions that can be applied up to very high energies. This description has to fulfill two basic requirements: it has to be relativistic and it must describe QE electron scattering data from low-intermediate up to high energies. In this sense, the analysis of the large amount of existing (e,e') data is taken as a solid benchmark to test the validity of the model for neutrino reactions at different kinematics and for several nuclei.

Accordingly, the SuSA model, developed in previous works \[24, 35\], is based on the existence of a superscaling function extracted from the analysis of QE electron scattering data. Hence nuclear effects can be analyzed through a semiphenomenological scaling function \(f(\psi') = f_{\text{SuSA}}(\psi')\) extracted from the ratio between the experimental QE cross section and the appropriate single-nucleon one \[98–100\] (see Sect. [14] and Appendix [A] for definitions):

\[ f(\psi') = k_F \left( \frac{d^2\sigma}{d\Omega_{ee'}dw} \right)_{\text{exp}} \frac{\sigma_{Mott}(v_L G_{ee'}^L + v_T G_{ee'}^T)}{\sigma_{Mott}(v_L G_{ee'}^L + v_T G_{ee'}^T)} \quad (80) \]

The scaling function extracted from the analysis of inclusive electron data at different kinematics and for several nuclei (see Fig. [17]) shows a spread whose width is rather narrow in the region of \(\psi' < 0\) that corresponds to low-intermediate kinematics (below the QE peak). On the contrary, scaling does not work properly in the region above the QE peak where other contributions can play also an important role. Notice the wide spread of data as \(\psi'\) increases.
Nevertheless, the scaling behavior becomes particularly clear if one studies the experimental cross section separated into its longitudinal $f_L(\psi')$ and transverse $f_T(\psi')$ contributions, as shown in Fig. 18. The isolated channel contributions show that the longitudinal experimental data superscale throughout the whole region of the QE peak, whereas the transverse data do not scale, being scaling violations more prominent in the region above the QE peak ($\omega > \omega_{QEP}$, i.e., $\psi' > 0$). Scaling violations at high $\omega$ occur because of other non-QE processes, such as meson production and resonance excitations, which are predominantly transverse, come into play. At very high transfer energies (i.e. high $\psi'$-values) deep inelastic scattering starts to be relevant. Likewise, 2p-2h states induced by meson-exchange currents are known to have a relevant contribution in the “dip” region between the QE and the $\Delta$ peaks. As we detail in section IV, 2p-2h MEC contributions are essentially transverse for electromagnetic interactions. All these contributions beyond the IA are responsible for scaling violations.

Other mechanisms, not included in the present work, can introduce some effects in the general discussion. This is the case of RPA long-range correlations studied in [101–103] that can modify the longitudinal and transverse responses because of the very different isospin character of the two channels. However, RPA effects are generally assumed to be relevant for low energy/momentum transfers and they are expected to give a small contribution in the scaling domain.

The prescription adopted in the first version of the SuperScaling Approach (SuSA) has been to employ the experimental longitudinal responses to define a general scaling function for both electromagnetic longitudinal and transverse channels. This implies that both $L$ and $T$ scaling functions are roughly the same after subtracting the non-scaling contributions related to processes beyond the QE regime, that is, zeroth kind scaling is fulfilled. As we shall illustrate, the most modern version of the model (SuSAv2) contains corrections to this assumption based on the Relativistic Mean Field theory.
FIG. 18. Scaling function, $f_L(\psi')$ and $f_T(\psi')$, from the longitudinal and transverse response, respectively, as a function of $\psi'$ for different nuclei ($A \geq 12$) and for different values of $q$ (in MeV/c). Data taken from [26].

From the analysis of the separate longitudinal data shown in Fig. 18 (left panel), a “universal” phenomenological $L$ superscaling function $f_L$ has been extracted. The analysis of these data leads to the results already shown in Fig. 1 where the solid line corresponds to a fit of data given by the following parametrization (denoted as SuSA scaling function),

$$f_{SuSA}(\psi') \equiv f_L(\psi') = \frac{p_1}{1 + p_2(\psi' - p_3)^2}(1 + e^{p_4\psi'})$$

with $p_1 = 2.9883$, $p_2 = 1.9438$, $p_3 = 0.67310$ and $p_4 = -3.8538$. As already signaled in Sect. II, the RFG scaling function is also shown for reference in Fig. 1 (dashed line). The behavior of data leads to the striking difference between $f_{SuSA}(\psi')$ and the prediction provided by RFG, the former with a pronounced asymmetrical tail extended towards positive values of $\psi'$, i.e., $\omega > \omega_{QEP}$, and the maximum at $\sim 0.6$.

The superscaling behavior shown in Fig. 1 sets a strong constraint to any model aimed to describe electron-nucleus scattering processes. In fact, different models based on the harmonic oscillator (HO+FSI) and the relativistic Fermi gas (RFG+FSI) that incorporate different descriptions of FSI have been applied to the problem (see [104] for details).

Although some level of asymmetry in the corresponding scaling functions emerges from these calculations, mainly due to FSI, the theoretical predictions still differ significantly from the data. On the contrary, as shown in Sect. III C, the RMF model provides a longitudinal scaling function with the right amount of asymmetry in accordance with data.

Moreover, in spite of the difficulty in analyzing the transverse scaling function, previous studies [105] based on the modeling of the QE longitudinal response and contributions from non-QE channels have provided some evidence that scaling of zeroth kind is not fully satisfied by data. In particular, these studies have found $f_{ee}^{\nu}, exp > f_{ee}^{\nu}, L$ from $(e,e'\nu)$ scaling functions evaluated with the RMF model (see the general discussion presented in Sect. III C).

Extension of the Superscaling Approach from Relativistic Mean Field Theory: the SuSAv2 Model

Contrary to the SuSA model, based on the use of a universal scaling function extracted from the analysis of the longitudinal $(e,e')$ data and applicable not only to the electromagnetic transverse channel but also to charged-current quasielastic (CCQE) neutrino- and antineutrino-nucleus cross sections [33], the new SuSAv2 model is based on the RMF findings, i.e., 0th-kind scaling is mildly broken for momentum transfers in the 1 GeV region being $f_T(\psi') > f_L(\psi')$. Moreover, the extraction of $f_{T,exp}^{ee'}$ entails the analysis of the purely-vector longitudinal $(e,e')$ nuclear response, which combines isoscalar+isovector contributions. In contrast, CC neutrino-nucleus reactions involve only isovector couplings and are mainly dominated by purely transverse responses ($T_{VV} + T_{AA}$ and $T_{VA}'$). This subject was studied in [51] by analyzing the scaling functions evaluated with the RMF model (see also discussion in Sect. III C), which has the merit of accounting for the difference between isoscalar and isovector contributions as well as for the separate vector-vector, axial-axial and vector-axial channels. There, it was found that, contrary to what one might expect, the $(e,e')$ longitudinal scaling function agrees with the total $(\nu, l^-)$ one (which is mainly transverse) better than does the transverse scaling function from $(e,e')$ (see Fig. 13). This result is explained by the different roles played by the isovector and isoscalar nucleon form factors in each process (see [51] for details).
The RMF calculations, even being computationally very expensive, reproduce reasonably well the experimental longitudinal scaling function in a wide kinematical region (see [106], providing an excellent accordance with the magnitude and shape of $f_{L,\text{exp}}^{ee'}$ and producing an enhancement of the transverse response, i.e., $f_T^{ee'} > f_T^{ee'}$, in agreement with data.

However, RMF also presents some drawbacks linked to the strong relativistic energy-independent scalar and vector potentials involved in the model. For increasing values of $q$ the RMF presents: i) a strong shift of the scaling functions to higher $\omega$ values, ii) too much enhancement of the area under the tail of the functions, and iii) correspondingly too severe a decrease in the maximum of the scaling functions. Hence in what follows, after correcting for the too strong $q$-dependence of the RMF model, we shall implement the main features of the model in a new version of the superscaling approach, called “SuSAv2”, that makes it possible to obtain numerical predictions to compare with data using fast codes, yet retaining the basic physics of the RMF.

In summary, the SuSAv2 model extends the original SuSA one by incorporating in its formalism information from the RMF theory. We build SuSAv2 in such a way that it reproduces the experimental longitudinal scaling function, produces $f_T^{ee'} > f_T^{ee'}$, takes into account the differences in the isoscalar/isovector scaling functions and avoids the problems of the RMF model in the region of high momentum transfer, where FSI are negligible, by introducing the RPWIA prescription. More details about the scaling behavior of the RMF and RPWIA models, Pauli blocking effects together with the shape of their corresponding scaling functions and the parameters that modulate binding energy effects in this approach can be found in [106].

Following the general discussion presented in the previous paragraphs, the SuSAv2 model is based on the following four assumptions:

1. $f_L^{ee'}$ superscales, i.e., it is independent of the momentum transfer (scaling of first kind) and of the nuclear species (scaling of second kind). It has been proved that $f_L^{ee'}$ superscales even for a range of $q$ relatively low ($300 < q < 570$ MeV/c), see [106]. As in the original SuSA model, here we assume that superscaling is fulfilled by nature.

2. $f_T^{ee'}$ superscales. It has been shown that $f_T^{ee'}$ approximately superscales in the region $\psi < 0$ for a wide range of $q$ ($400 < q < 4000$ MeV/c), see [107]. However we assume that once the contributions from non-QE processes are removed (MEC, $\Delta$-resonance, DIS, etc.) the superscaling behaviour could be extended to the whole range of $\psi$.

3. The RMF model reproduces quite well the relationships between all scaling functions in the \textit{whole} range of $q$. This assumption is supported by the fact that RMF model is able to reproduce the experimental scaling function, $f_{L,\text{exp}}^{ee'}$, and the fact that it naturally yields the inequality $f_T^{ee'} > f_T^{ee'}$.

4. At very high $q$ the effects of FSI disappear and all scaling functions must approach the RPWIA results.

Contrary to what is assumed in the SuSA model, where only $f_{L,\text{exp}}^{ee'}$ is used as \textit{reference} scaling function to build all nuclear responses, within SuSAv2 we use three RMF-based \textit{reference} scaling functions (which will be indicated with the symbol $f$): one for the transverse set, one for the longitudinal isovector set and another one to describe the longitudinal isoscalar scaling function in electron scattering.

We employ the experimental scaling function $f_{L,\text{exp}}^{ee'}$ as guide in our choices for the \textit{reference} ones. In Fig. [19] we display the RMF longitudinal scaling function, $f_L$, for several representative values of $q$. Notice that the functions have been relocated by introducing an energy shift (see [106]) so that the maximum is at $\psi' = 0$. It appears that scaling of first kind is not perfect and some $q$-dependence is observed. Although all the curves are roughly compatible with the experimental error bars, the scaling function that produces the best fit to the data corresponds to $q \approx 650$ MeV/c. This is the result of a $\chi^2$-fit to the 25 experimental data of $f_{L,\text{exp}}^{ee'}$, as illustrated in the inner plot in Fig. [19].

According to this result, we identify the reference scaling functions with $f_L^{T=1,ee'}$, $f_L^{T=0,ee'}$ and $f_L^{T=1,ee'}$ evaluated within the RMF model at $q = 650$ MeV/c and relocated so that the maximum is at $\psi' = 0$ (we will account for the energy shift later):

$$
\bar{f}_T = f_T^{T=1,ee'} \bigg|_{q=650} \approx f_T^{ee'} \bigg|_{q=650} 
$$

(82)

$$
\bar{f}_L^{T=1} = f_L^{T=1,ee'} \bigg|_{q=650} \approx f_L^{ee'} \bigg|_{q=650} 
$$

(83)

$$
\bar{f}_L^{T=0} = f_L^{T=0,ee'} \bigg|_{q=650} \approx f_L^{ee'} \bigg|_{q=650} 
$$

(84)

Thus, by construction, the $(e, e')$ longitudinal scaling function built within SuSAv2 is $f_L^{SuSAv2} = f_L^{RMF} \bigg|_{q=650} \approx f_{L,\text{exp}}^{ee'}$.

In order to work with these reference scaling functions we need analytical expressions for them. To that end, we have used a skewed-Gumbel function which depends on four parameters. The expressions that parametrize the reference scaling functions are presented in [106].
The rest of scaling functions necessary to define the nuclear responses (see also Section II) are defined from the reference ones:

\[
\begin{align*}
\mu_{V,\nu}^T(q) & = \frac{f_{V,\nu}^T(q)}{f_{T=1,ee'}^T(q)} \\
\mu_{A,\nu}^T(q) & = \frac{f_{A,\nu}^T(q)}{f_{T=1,ee'}^T(q)} \\
\mu_{V,\nu}^L(q) & = \frac{f_{V,\nu}^L(q)}{f_{T=1,ee'}^T(q)} \\
\mu_{A,\nu}^L(q) & = \frac{f_{A,\nu}^L(q)}{f_{T=1,ee'}^T(q)} \\
\end{align*}
\]

where we have introduced the ratios \( \mu \) (see \[106\] for details) defined for the transverse and longitudinal sets as:

\[
\begin{align*}
\mu_{T,\nu}^V(q) & = \frac{f_{V,\nu}^T(q)}{f_{T=1,ee'}^T(q)} \\
\mu_{T,\nu}^A(q) & = \frac{f_{A,\nu}^T(q)}{f_{T=1,ee'}^T(q)} \\
\mu_{L,\nu}^V(q) & = \frac{f_{V,\nu}^L(q)}{f_{T=1,ee'}^T(q)} \\
\mu_{L,\nu}^A(q) & = \frac{f_{A,\nu}^L(q)}{f_{T=1,ee'}^T(q)} \\
\end{align*}
\]

Finally, in order to implement the approaching of the RMF results to the RPWIA ones at high kinematics, that is, the disappearance of FSI at high \( q \), we build the SuSAv2 \( L \) and \( T \) scaling functions as linear combinations of the RMF-based and RPWIA reference scaling functions:

\[
\begin{align*}
\mathcal{F}_{L,T=0.1} & = \cos^2 \chi(q) \tilde{f}_{L,T=0.1}^R + \sin^2 \chi(q) \tilde{f}_{T,RPWIA} \\
\mathcal{F}_T & = \cos^2 \chi(q) \tilde{f}_T + \sin^2 \chi(q) \tilde{f}_{T,RPWIA},
\end{align*}
\]

where \( \chi(q) \) is a \( q \)-dependent angle given by

\[
\chi(q) = \frac{\pi}{2} \left( 1 - \left[ 1 + e^{\frac{q-q_0}{w_0}} \right]^{-1} \right)
\]

with \( q_0 \) and \( w_0 \) the transition parameters between RMF and RPWIA prescriptions which are defined in Refs. \[106, 108\]. The reference RPWIA scaling functions, \( \tilde{f}_{T,RPWIA} \), are evaluated at \( q = 1100 \) MeV/c, where FSI effects are assumed to be negligible, while the reference RMF scaling functions, \( \tilde{f}_T \), are evaluated at \( q = 650 \) MeV/c and are shown in
Fig. 20. The explicit expressions of the RMF ($\tilde{f}_K$) and RPWIA ($\tilde{f}_K^{RPWIA}$) scaling functions are given in [106]. With this procedure we get a description of the responses based on RMF behavior at low-intermediate $q$ values while for higher momentum transfers it mimics the RPWIA trend. The transition between RMF and RPWIA behaviors occurs at intermediate $q$-values, namely, $\sim q_0$, in a region of width $\sim w_0$.

The response functions are simply built as:

$$R_{\text{ee}}^T(q, \omega) = \frac{1}{k_F} \left[ F_T^{T=1}(\psi') G_L^{T=1}(q, \omega) + F_L^{T=0}(\psi') G_T^{T=0}(q, \omega) \right]$$  \hspace{1cm} (96)

$$R_{\text{ee}}^L(q, \omega) = \frac{1}{k_F} F_T(\psi') [G_T^{T=1}(q, \omega) + G_T^{T=0}(q, \omega)]$$ \hspace{1cm} (97)

$$R_{\text{VV},\nu}^V(q, \omega) = \frac{1}{k_F} F_T^{T=1}(\psi') G_L^{VV}(q, \omega)$$ \hspace{1cm} (98)

$$R_{\text{AA},\nu}^{A,A}(q, \omega) = \frac{1}{k_F} F_T^{T=1}(\psi') G_L^{AA}(q, \omega)$$ \hspace{1cm} (99)

$$R_{\text{AA},\nu}^{A,L}(q, \omega) = \frac{1}{k_F} F_T^{T=1}(\psi') G_L^{AL}(q, \omega)$$ \hspace{1cm} (100)

$$R_{\text{LL},\nu}^{L,L}(q, \omega) = \frac{1}{k_F} F_T^{T=1}(\psi') G_L^{LL}(q, \omega)$$ \hspace{1cm} (101)

$$R_{\text{T}}^{\nu}(q, \omega) = \frac{1}{k_F} F_T(\psi') [G_T^{VV}(q, \omega) + G_T^{AA}(q, \omega)]$$ \hspace{1cm} (102)

$$R_{\text{T}}^{\nu'}(q, \omega) = \frac{1}{k_F} F_T(\psi') G_T^{A,A}(q, \omega).$$ \hspace{1cm} (103)

Furthermore, in order to reproduce the peak position of RMF and RPWIA scaling functions within SuSAv2 we consider a $q$-dependent energy shift, namely, $E_{\text{shift}}(q)$. This quantity modifies the scaling variable $\psi(q, \omega) \rightarrow \psi'(q, \omega, E_{\text{shift}})$. In particular, we build this function $E_{\text{shift}}(q)$ from the results of the RMF and RPWIA models presented in [106]. The choice of $E_{\text{shift}}(q)$ depending on the particular $q$-domain region considered is solely based on the behavior of the experimental cross sections and their comparison with our theoretical predictions (see results in Section VII). In the past we have considered a fixed value of $E_{\text{shift}}$ [107] (different for each nucleus) to be included within the SuSA model in order to fit the position of the QE peak for some specific $q$-intermediate values. On the other hand, the RMF model leads the cross section to be shifted to higher values of the transferred energy. This shift becomes increasingly larger for higher $q$-values as a consequence of the strong, energy-independent, highly repulsive potentials involved in the RMF model. Comparison with data (see the results in [106]) shows that the shift produced by RMF is too large. Moreover, at very high $q$-values, one expects FSI effects to be less important and lead to results that are more similar to those obtained within the RPWIA approach. This is the case when FSI are described...
through energy-dependent optical potentials \cite{55, 109}. Therefore, as already mentioned, our choice for the functional
dependence of $E_{\text{shift}}(q)$ is motivated as a compromise between the predictions of our models and the comparisons
with data.

In Section \textbf{VII} we will show that the SuSAv2 model describes more accurately the neutrino induced reactions than
the semiphenomenological SuSA approach. This result mainly comes from the $q$-dependence on the energy shift, the
enhancement on the transverse response via RMF prescriptions as well as the possibility of separating scaling functions
into isoscalar and isovector contributions.

\section{IV. Relativistic model for CC MEC and 2p2h responses}

In this section we present a fully relativistic model of meson exchange currents (MEC) to describe the inclusive
neutrino scattering in the two-nucleon emission channel (2p-2h). The present approach was developed in the RFG
model of ref. \cite{110}, where the difficulties of the relativistic reaction are overcome thanks to the plane waves used
for the single-nucleon states. However this channel is cumbersome on the computational side, if compared to the
analytical simplicity of the 1p-1h responses.

The 2p-2h hadronic tensor is given by sumation (integration) over all the 2p-2h excitations of the RFG with two
holes $h_1$, $h_2$, and two particles $p_1'$, $p_2'$, in the final state, with $h_i < k_F$ and $p_i' > k_F$

\[
W_{\mu\nu}^{2p2h} = \frac{V}{(2\pi)^9} \int d^3p_1d^3p_2'd^3h_1d^3h_2 \frac{m_N^4}{E_1E_2E_1'E_2'} \times \omega^{\mu\nu}(p_1', p_2', h_1, h_2) \delta(E_1' + E_2' - E_1 - E_2 - \omega) \times \Theta(p_1', h_1)\Theta(p_2', h_2)\delta(p_1' + p_2' - q - h_1 - h_2),
\]

where the Pauli blocking function $\Theta$ is defined as the product of step-functions

\[
\Theta(p', h) = \theta(p' - k_F)\theta(k_F - h).
\]

The 2p-2h equivalent to the single-nucleon hadronic tensor inside the integral is called $\omega^{\mu\nu}(p_1', p_2', h_1, h_2)$, and
describes two-nucleon transitions with given initial and final momenta, summed up over spin and isospin,

\[
u^{\mu\nu}(p_1', p_2', h_1, h_2) = \frac{1}{4} \sum_{s_1s_2s_1's_2'} \sum_{t_1t_1't_1't_2'} j^\mu(1', 2', 1, 2)A_j^{(1', 2', 1, 2)}.
\]

Here the 2p-2h current function, in short-hand notation for the full spin-isospin-momentum dependence, is

\[
\frac{1}{4} \sum_{s_1s_2s_1's_2'} \sum_{t_1t_1't_1't_2'} j^\mu(1', 2', 1, 2)A_j^{(1', 2', 1, 2)}
\]

and $j^\mu(1', 2', 1, 2)_A$ refers to the direct minus exchange matrix elements

\[
\frac{1}{4} \sum_{s_1s_2s_1's_2'} \sum_{t_1t_1't_1't_2'} j^\mu(1', 2', 1, 2) - j^\mu(1', 2', 2, 1).
\]

The factor $1/4$ in Eq. \textbf{(105)} accounts for the antisymmetry of the two-body wave function, to avoid double counting in
the number of final 2p-2h states. The exchange $1 \leftrightarrow 2$ in the second term implies implicitly the exchange of momenta,
spin and isospin quantum numbers.

Momentum conservation allows to integrate over $p_2'$ to compute the 2p-2h response functions as an 9-dimensional
integral

\[
R_{2p2h}^{K} = \frac{V}{(2\pi)^9} \int d^3p_1d^3h_1d^3h_2 \frac{m_N^4}{E_1E_2E_1'E_2'} \Theta(p_1', h_1)\Theta(p_2', h_2) \times \omega^{\mu\nu}(p_1', p_2', h_1, h_2) \delta(E_1' + E_2' - E_1 - E_2 - \omega),
\]

where $p_2' = h_1 + h_2 + q - p_1'$. The five elementary response functions for a 2p2h excitation, $r^K$, are defined in terms
of the elementary hadronic tensor $\omega^{\mu\nu}$, for the five indices $K = CC, CL, LL, T, T'$.

Due to azimuthal symmetry of nucleon emission around the $z$ axis, defined by $q$, we can fix the azimuthal angle of
particle 1', by setting $\phi_1' = 0$, and multiplying by a factor $2\pi$. Finally, the Dirac delta-function enables us to integrate
over $p_1'$, and so the integral in Eq. \textbf{109} can be reduced to seven dimensions. The details of the integration method
can be found in Ref. \cite{111}.

1. Ref. [111].
In these equations we use the following notation. The momentum transfer to interaction and the pion propagator are contained in the following spin-independent function:

\[ \text{Model of [112].} \]

The different contributions have been taken from the pion production diagrams depicted in Fig. 21. The different contributions have been taken from the pion production model of [110].

Our MEC are given as the sum of four two-body currents:

\[ j_{\mu}(1', 2, 1, 2) = j_{\mu}^{\text{sea}} + j_{\mu}^\rho + j_{\mu}^{\pi} + j_{\mu}^{\Delta}, \]  

(110)

corresponding in Fig. 21 to the seagull (diagrams a,b), pion in flight (c), pion-pole (d,e) and \( \Delta(1232) \) excitation (f,g,h,i). Their expressions are given by

\[
j_{\text{sea}} = \left[ I_{V}^{\pm} \right]_{1'2',12} \frac{f^2}{m_{\pi}^2} V_{\pi NN}(p_{1}', h_1) \times \bar{u}_{s_2}(p_2) \left[ F_{1V}(Q^2) \gamma_5 \gamma^\mu + \frac{F_{\rho}(k_2^2)}{g_A} \gamma^\mu \right] u_{s_2}(h_2) + (1 \leftrightarrow 2) \]

(111)

\[
j_{\rho} = \left[ I_{V}^{\pm} \right]_{1'2',12} \frac{f^2}{m_{\pi}^2} F_{V}(Q^2) V_{\pi NN}(p_{1}', h_1) V_{\pi NN}(p_{2}', h_2) (k_1^\mu - k_2^\mu) \]

(112)

\[
j_{\pi}^{\mu} = \left[ I_{V}^{\pm} \right]_{1'2',12} \frac{f^2}{m_{\pi}^2} F_{\rho}(k_2^2) \frac{Q^\mu \bar{u}_{s_1}(p_1')}{Q^2 - m_{\pi}^2} V_{\pi NN}(p_{2}', h_2) + (1 \leftrightarrow 2) \]

(113)

\[
j_{\Delta} = \left[ I_{V}^{\pm} \right]_{1'2',12} \frac{f^2}{m_{\pi}^2} V_{\pi NN}(p_{2}', h_2) \bar{u}_{s_1}(p_1') \left\{ \left[ U_{V}^{\pm} \right]_{1'2',12} k_{2}^5 G_{\alpha\beta}(h_1 + Q) \Gamma^{\beta\mu}(h_1, Q) \right. \]

\[ + \left. \left[ U_{B}^{\pm} \right]_{1'2',12} k_{2}^5 \Gamma^{\mu\alpha}(p_1', Q) G_{\alpha\beta}(p_1' - Q) \right\} u_{s_1}(h_1) + (1 \leftrightarrow 2). \]

(114)

In these equations we use the following notation. The momentum transfer to ith nucleon is \( k_i^\mu = p_i'^\mu - h_i^\mu \). The \( \pi NN \) interaction and the pion propagator are contained in the following spin-dependent function:

\[
V_{\pi NN}(p_{1}', h_1) = \frac{\bar{u}_{s_1}(p_1') \gamma_5 k_1 u_{s_1}(h_1)}{k_1^2 - m_{\pi}^2}. \]

(115)
We have also defined the isospin operators
\[ I_V^\pm = (I_V)_x \pm i(I_V)_y \]
\[ I_V = i\tau(1) \times \tau(2), \]
where the \( +(-) \) sign refers to neutrino (antineutrino) scattering. In the \( \Delta \) current, the forward, \( U_F^\pm = U_{Fx} \pm iU_{F\gamma} \), and backward, \( U_B^\pm = U_{Bx} \pm iU_{B\gamma} \), isospin transition operators are obtained from the cartesian components of the operators
\[ U_{Fj} = \sqrt{\frac{3}{2}} \sum_i (T_i T_j) \otimes \tau_i \]
\[ U_{Bj} = \sqrt{\frac{3}{2}} \sum_i (T_j T_i^\dagger) \otimes \tau_i, \]
where \( T^\dagger \) is an isovector transition operator from isospin \( \frac{3}{2} \) to \( \frac{1}{2} \), normalized as
\[ T_i T_j^\dagger = \frac{2}{3} \delta_{ij} - \frac{i}{3} \epsilon_{ijk} \tau_k. \]

The coupling constants in the MEC are \( f = 1, g_A = 1.26 \), and \( f^* = 2.13 \). The electroweak form factors \( F_{1V}^\pi \) and \( F_{2\rho}^\pi \) in the seagull and pionic currents are taken from Ref. \[112\]. To take into account the finite size of the hadrons, we apply phenomenological strong form factors (not written explicitly in the MEC) in all the \( NN\pi \) and \( N\Delta\pi \) vertices
\[ F_{\pi NN}(k) = \frac{A^2_\pi - m^2_\pi}{A^2_\pi - k^2} \]
\[ F_{\pi N\Delta}(k) = \frac{A^2_\Delta - k^2}{A^2_\Delta - k^2} \]
where the cutoff constants are \( \Lambda_\pi = 1.3 \text{ GeV} \) and \( \Lambda_\Delta = 1.15 \text{ GeV} \). These values of the strong form factors in the MEC are similar to past works on the two nucleon emission responses for electron scattering both in the non-relativistic \[113\] and relativistic Fermi gas model \[114\]. In the formalism presented here, which is an extension of the Torino model \[115\], we use still the same strong form factors for consistency with previous modeling.

In our approach the weak \( N \to \Delta \) transition vertex tensor in the forward current, \( \Gamma^{\beta\mu}(P, Q) \) is given by
\[ \Gamma^{\beta\mu}(P, Q) = \frac{C^V}{m_N} (g^{\beta\mu} Q - Q^{\beta\mu}) \gamma_5 + C^A_5 g^{\beta\mu}. \]

We have kept only the \( C^V_3 \) and \( C^A_5 \) form factors and neglected the smaller contributions of the others. They are taken from \[112\]. The additional terms have been neglected because they are of order \( 1/m^2_N \) or negligible for the intermediate energy kinematics of interest for the quasielastic regime. Besides the vector part of the vertex \[123\] is equivalent to the \( \Delta \) operator used in ref \[115\] for electron scattering.

The corresponding vertex tensor in entering in the backward current is defined by
\[ \hat{\Gamma}^{\mu\alpha}(P', Q) = \gamma^0 [\Gamma^{\alpha\mu}(P', -Q)]^\dagger \gamma^0. \]

The \( \Delta \)-propagator takes into account the finite decay width of the \( \Delta \) (1232) by the prescription
\[ G_{\alpha\beta}(P) = \frac{P_{\alpha\beta}(P)}{P^2 - M^2_\Delta + iM_\Delta \Gamma_\Delta + \frac{i^3}{4}}. \]

where the width \( \Gamma_\Delta(P^2) \) depends only on the \( \Delta \) invariant mass. The explicit dependence is taken from \[114\].

The projector \( P_{\alpha\beta}(P) \) over spin-\( \frac{3}{2} \) particles is given by
\[ P_{\alpha\beta}(P) = -(P + M_\Delta) \left[ g_{\alpha\beta} - \frac{1}{3} \gamma_\alpha \gamma_\beta - \frac{2}{3} \frac{P_\alpha P_\beta}{M_\Delta^2} + \frac{1}{3} \frac{P_\alpha \gamma_\beta - P_\beta \gamma_\alpha}{M_\Delta} \right]. \]

In the present approach of MEC we do not consider the nucleon pole (NP) terms, also called nucleon correlation current, with an intermediate nucleon propagation. These contributions have been evaluated in some works for electron scattering \[113\], and for neutrino scattering \[116,117\], and later in \[118\] for electron and \[119\] for neutrino
scattering. But these approaches are problematic from the theoretical point of view due to the presence of double poles in the hadronic tensor, that need to be regularized in some way not free from ambiguities. Alternative approaches include these contributions in the nuclear wave function, in form of Jastrow correlations, for instance [120]. A third possibility is to include the nuclear correlation effects, at least partially, by using a phenomenological scaling function extracted from \((e,e')\) data, as discussed in Sect. III.

Note also that our MEC do not include heavier mesons such as \(\rho\)-exchange, which is expected to provide a smaller contribution than the leading pion-exchange for the energies of interest.

The computation of the 2p-2h responses in the RFG model presents two main difficulties: the evaluation of the 2p-2h hadronic tensor by performing the spin and isospin traces, and the evaluation of the 7D integral over the phase space. The first problem in fact can be faced with analytical results for the 2p-2h tensor \(w^{\mu\nu}(p'_1, p'_2, h_1, h_2)\), but hundreds of thousands terms arise already in the electron scattering case [113] and many more are present for neutrino scattering. We then follow the alternative and easier approach of computing numerically the Dirac matrix elements of the currents and then summing over spins.

The second problem consists on computing efficiently the 7D integral of the MEC. This can be done in several ways. The integration in the Laboratory system was studied in Ref. [111] in the context of the phase space integral

\[
F(q, \omega) = \int d^3p'_1 d^3h_1 d^3h_2 \frac{m_N}{E_1 E_2 E'_1 E'_2} \Theta(p'_1, h_1) \Theta(p'_2, h_2) \delta(E'_1 + E'_2 - E_1 - E_2 - \omega).
\]

The study of this universal function of the RFG was done in detail in Ref. [111] for relativistic and non relativistic kinematics.

One important conclusion of that study is presented in Fig. 22 where we compare the relativistic and non relativistic phase space functions. The latter uses non-relativistic kinematics in the delta of energies and without the boost factors \(m_N/E\) inside the integral. As expected the relativistic effect increases with the momentum transfer. We also show that using relativistic kinematics without the boost factors worsens the results. This shows that this procedure of “relativizing” a non-relativistic MEC model using relativistic kinematics is not advisable.

In ref. [121] we followed the alternative approach of computing the 7D integral going to the center of mass frame of the final two-nucleon system. There analytical results were found for the phase space function in the non-Pauli blocking regime. The angular distribution of the final particles in the phase space integral in the CM system was found to be isotropic (independent on the emission angle), as it is assumed in the Monte Carlo analyses of two nucleon emission for neutrino scattering [122]. However, these distributions are going to change after including the effects of the MEC, because the angular distribution for fixed momenta \(h_1\) and \(h_2\), is multiplied by the 2p-2h hadronic tensor. Further studies are needed to correct the angular distribution from the isotropic assumption, to reduce the systematic uncertainties of neutrino event generators in the 2p-2h channel.

The 2p-2h inclusive cross section requires to compute two electromagnetic response functions, \(R_{T,L}\), for \((e,e')\) reactions and and five weak response functions, \(R_{CC,CL,LL,TT,TL}\) for \((\nu_\mu, \mu^-)\). All these responses were computed and analyzed in ref. [110].

The numerical results of our model are in agreement with the calculations of [115] and with the much more recent and independent calculation of [77], using the same theoretical MEC model. These results for the response functions have been parametrized in the microscopic range relevant for electron scattering [123], and for the current neutrino quasielastic scattering experiments [124]. The parametrization is convenient because in neutrino scattering there is an additional integration over the incident neutrino flux to obtain the cross section. The parametrization is available from the authors, and it has been already used in other analyses [125] and has been implemented in Monte Carlo event generator GENIE [126].

To illustrate the qualitative predictions of the RFG model, in Figs. 23 and 24 we compare the 1p-1h and 2p-2h neutrino responses for two values of the momentum transfer for electron and neutrino scattering, respectively. The 1p-1h responses only contain the one-body (OB) current. For these values of \(q\) large MEC effects are found. The 2p-2h responses present a maximum due to the presence of the \(\Delta\) propagator in the \(\Delta\) current. The MEC effects are similar in the \(T\) and \(T'\) responses. The 2p-2h contribution to \((e,e')\) is predominantly transverse, due to dominance of the \(\Delta\) excitation in the transverse vector current. The MEC effects in the \(CC\) response are relatively much larger than in the transverse ones. This comes from a large longitudinal contribution of the axial MEC. However, each response function appears in the cross section multiplied by a kinematic factor, which slightly alters the relative contribution to the cross section. In particular, the three responses \(R_{CC}, R_{CL}\) and \(R_{LL}\) largely cancel each other, yielding a small charge/longitudinal cross section. In figs. 23 and 24 only the real part of the denominator of the \(\Delta\) propagator has been considered in the calculation of the MEC, a prescription taken also in refs. [110, 113].

In our model the elementary 2p-2h response functions contain direct-direct, exchange-exchange and direct-exchange contributions due to the antisymmetry of the 2p-2h states. The exchange-exchange term is identical to the direct-direct for the inclusive reactions considered here. Therefore one can write the elementary response functions as direct...
FIG. 22. Effect of implementing relativistic kinematics in a non-relativistic calculation of $F(q, \omega)$. No rel: non-relativistic result. Rel kin: relativistic kinematics only without the relativistic factors $m_N/E$. Rel: fully relativistic result. [111].

minus direct-exchange contribution. For instance for the $T$ responses of neutrino scattering in the pp emission channel we have

$$r_{pp}^T = 4 \sum_{\mu=1}^2 \sum_{s_1 s_2 s'_1 s'_2} \left\{ |J_{pp}^\mu(1'2';12)|^2 - \text{Re} \, J_{pp}^\mu(1'2';12)^* J_{pp}^\mu(2'1';12) \right\},$$

where $J_{pp}^\mu(1'2';12)$ is the matrix element for pp emission with neutrinos. The first term in Eq. (128) is the “direct” contribution, and the second one is the “exchange” contribution, actually being the interference between the direct and exchange matrix elements. There are similar expressions for the remaining response functions (see ref. [110] for details).

In other models of 2p-2h based on the Fermi gas, the exchange interference contributions are disregarded. Specifically they are neglected in the models of Lyon [116, 117] and Valencia [119, 128], because they involve higher dimensional integrals than the direct terms, which can be reduced to low dimensions within some assumptions.

To quantify the importance of the exchange, this contribution was separated in our model in ref. [110]. It was found to be typically about 25% of the 2p-2h inclusive cross section. Later, it was separated in the individual channels of neutron-proton (np) and proton-proton (pp) emission in refs. [127, 129]. An example is given in Fig. 25, where we see the transverse, $T$ and $T'$, response functions for np and pp emission with neutrinos. In the np case the exchange interference reduces the strength by a 50%. This example shows that the ratio pp/np critically depends on the
treatment of the exchange interference.

In other works [130, 131] we have developed some assumptions which simplify and reduce the number of nested integrations to calculate the inclusive 2p-2h responses. In the first of them, Ref. [130], we used the "frozen nucleon approximation" to reduce the 7-dimensional integral to just a single integration, by assuming the initial nucleons to be at rest. The strong dependence on the kinematics near the Δ peak arising in the frozen approximation was avoided by using a smeared Δ propagator that was parametrized for the relevant values of the momentum transfer, in order to make these approximate results to be consistent with those calculated with the full integration procedure. In the second work, Ref. [131], which is in much sense very similar in spirit to that of the IFIC-Valencia model [119, 128], we developed a convolution model of the elementary 2p-2h responses, given in eq. (106), with two 1p-1h responses (Linhard functions) sharing the energy and momentum transfer from the electroweak probe. The importance of these works is two-fold: from the computational point of view, reducing the time to evaluate the inclusive 2p-2h responses makes these models suitable to be incorporated in the Monte Carlo event generators; additionally, with the second work [131], we also can incorporate the direct-exchange terms, that appear in eq. (106), in the calculation of the inclusive 2p-2h response functions in the same or similar fashion as it is done in the Valencia model [119]. But this is even more than what is included in the original Valencia model [119] in this respect, because in this last model those terms were not considered because their calculation was extremely difficult with the technique of the Cutkosky rules.

Finally, it is worth mentioning that, in Ref. [132], the density dependence (or $k_F$ dependence) of the 2p-2h MEC responses was studied, what makes easier the translation of the results of these responses between different nuclear species if one has the responses for one of them.

V. Pion production in the resonance region

There are several reasons that make pion production relevant for present and future accelerator-based neutrino oscillation experiments. Neutral-current $\pi^0$ production is an important background in the electron neutrino and antineutrino appearance analyses due to the misidentification of the photons from the $\pi^0$ decay with the electron or positron signal. Also, the final state for events with a pion produced in the primary vertex may mimic that of a QE one i) if the pion is not detected, and ii) if due to FSI the pion is absorbed. Finally, in the few-GeV energy region, the number of events observed in the far and near detectors arising from neutrino-induced pion production competes with those from the quasielastic process. Since pions can be detected, for example, in coincidence with the final lepton and
FIG. 24. Comparison between 1p-1h and 2p-2h response functions for CC neutrino scattering off $^{12}$C for two values of the momentum transfer. The Fermi momentum of the RFG is $k_F = 228$ MeV/c \cite{110}.

possibly other hadrons, a good understanding of the pion production mechanisms could help to increase the statistics of events that are selected for the reconstruction of the neutrino energy.

To illustrate the relative magnitude of QE and non-QE processes, in Fig. 26(a) we show the $\nu_e$-$^{12}$C inclusive $Q_{QE}^2$ differential cross section averaged with the T2K flux, and in Fig. 26(b) the $\nu_\mu$ total cross section on C$_8$H$_8$ target. The theoretical calculations shown in this figure were presented in Ref. \cite{133}. Three different contributions are shown: i) QE scattering computed within the SuSAv2 model (Sect. III E), ii) the vector part of the 2p2h MEC (Sect. IV), and iii) an inelastic contribution containing mainly one-pion production (labelled as 1\pi), that will be described below. One observes that for the $\nu_e$ T2K flux the QE and 1\pi contributions are similar in magnitude. This should not be surprising since, even though the T2K flux peaks at around $E_\nu = 0.5$ GeV, it has a long tail extending up to 10 GeV \cite{134}. For the total cross section, Fig. 26(b), the QE contribution dominates up to $E_\nu \approx 1$ GeV, from there the QE and 1\pi contributions are comparable.

The motivation of the non-QE SuSAv2 model used in Fig. 26 is to extend the superscaling arguments, previously applied in the QE domain, to the $\Delta$ resonance region \cite{58,130}. Hence, the underlying idea is that pion production in the $\Delta$ region is indeed strongly dominated by the excitation of the $\Delta$ resonance. The first step is to define a experimental scaling function in this inelastic region, $f^{\text{non-QE}}$. For that we subtract the QE and 2p2h MEC
FIG. 25. Separate pp and np contributions to the neutrino $T$ and $T'$ 2p-2h response functions of $^{12}$C, for $q = 600$ MeV/c, compared to the direct contributions obtained by neglecting the direct-exchange interference. The value of the Fermi momentum $k_F = 228$ MeV/c \[127\]

FIG. 26. (a) We show the CC-inclusive T2K flux-folded $\nu_e$-$^{12}$C $Q_E^2$ differential cross section per nucleon. (b) The CC $\nu_\mu$ total cross section on C is presented. Experimental data are from \[134\] (a) and \[135\] (b). Theoretical predictions for QE, non-QE (1π) and the vector part of the 2p2h MEC are shown separately. These plots are taken from Ref. \[133\].

Contributions to the inclusive electron scattering experimental cross section:

\[
\left( \frac{d^2\sigma}{d\Omega d\omega} \right)_{\text{non-QE}} \equiv \left( \frac{d^2\sigma}{d\Omega d\omega} \right)_{\text{exp}} - \left( \frac{d^2\sigma}{d\Omega d\omega} \right)_{\text{1p1h}}^{\text{QE,SuSAv}} - \left( \frac{d^2\sigma}{d\Omega d\omega} \right)_{\text{2p2h}}^{\text{MEC}}.
\]

Then the superscaling function is constructed as

\[
f^{\text{non-QE}}(\psi_\Delta) = k_F \frac{\left( \frac{d^2\sigma}{d\Omega d\omega} \right)_{\text{non-QE}}}{\sigma_M(\nu L G^\Delta_L + \nu T G^\Delta_T)}.
\]

where $G^\Delta_L,T$ are the single-nucleon functions for the $\gamma N \Delta$ vertex and $\psi_\Delta$ is the $\Delta$ scaling function (the explicit expressions can be found in Refs. \[58, 136, 137\]).

The function $f^{\text{non-QE}}(\psi_\Delta)$ is represented in Fig. \[27\](a) for a large set of inclusive electron-$^{12}$C scattering data. One observes that in the region $\psi_\Delta < 0$ scaling occurs but it is not as good as in the QE case. For that reason, the scaling function is parametrized as a broad band that accounts for the spread of the data. This band can be understood as the systematic error attached to the model. Notice that in the region $\psi_\Delta > 0$, as expected, the scaling breaks because contributions from processes beyond the delta production start to be important. Thus, the predictions of this model beyond the delta peak should be taken with caution. In Fig. \[27\](b) we show that the model works well for inclusive electron scattering (see more in \[133\]).

Inclusive models like superscaling-based approaches, or the model of Ref. \[101\], do not provide any information about the final state hadrons. For that, one normally uses theoretical approaches that focus on one particular reaction.
FIG. 27. (a) We show the $f^{\text{non-QE}}(\psi_{\Delta})$, the experimental data can be found in [138]. (b) We compare inclusive electron-$^{12}\text{C}$ data [139] with the predictions of the QE and non-QE SuSAv2, and the 2p2h MEC contribution. These plots are taken from Ref. [133].

channel. The inclusive signal should be recovered by adding all channels allowed for the given energy and momentum transferred and integrating over the hadronic variables. This implies, therefore, the non-trivial task of modelling all those possible reaction channels.

In an effort to improve our current knowledge on the neutrino-induced pion production, in the recent years the MiniBooNE [140–142], MINERvA [143–147] and T2K [148] collaborations have reported (pion detected) differential cross sections for NC and, mainly, CC neutrino-induced pion production on different nuclear targets. From the theoretical side, we aim at providing support to improve our understanding of these and forthcoming data in a more consistent way. Accordingly, in this section we focus on the single-pion production (SPP) process, which is the dominant one in the resonance region and is typically characterized by invariant masses going from the pion threshold to $W < 1.8 - 2$ GeV. The strategy shared by most approaches is based on the idea that the lepton interacts with only one nucleon in the nucleus. This allows one to decompose this complex problem into different pieces: 1) the elementary vertex, 2) the nuclear framework, including the description of the initial and final nuclear states, and 3) the final-state interactions. In what follows we review some of the features of this complex problem. For further reading we refer to the recent review articles [149–152].

A. Elementary vertex

One finds the following reaction channels for SPP off a free nucleon induced by neutral current interactions (EM or WNC):

$$\gamma + p \rightarrow \pi^+ + n, \quad \gamma + n \rightarrow \pi^- + p, \quad \gamma + p \rightarrow \pi^0 + p, \quad \gamma + n \rightarrow \pi^0 + n.$$  (131)

$\gamma$ represent a real (photoproduction) or virtual (electroproduction) photon. For WNC interaction, the virtual photon is replaced by a $Z^0$ boson. In the case of neutrino ($W^+$ exchanged) and antineutrino ($W^-$) CC interactions, one has:

$$W^+ + p \rightarrow \pi^+ + p, \quad W^- + n \rightarrow \pi^- + n, \quad W^+ + n \rightarrow \pi^0 + p, \quad W^- + p \rightarrow \pi^0 + n,$$  (132)

$$W^+ + n \rightarrow \pi^+ + n, \quad W^- + p \rightarrow \pi^- + p.$$

The process is depicted in Fig. 28(a), and it will be discussed in what follows.

a. Cross Section

The differential cross section for SPP off the nucleon in an arbitrary reference frame is:

$$\frac{d\sigma}{dk_f dk_{\pi}} = \frac{K}{\phi} \frac{R_X}{(2\pi)^5} \eta_{\mu\nu} \eta_{\rho\sigma} \delta(\epsilon_i + E_i - \epsilon_f - E_{\pi} - E_f).$$  (133)

\footnote{The invariant mass $W$ is defined as $W^2 = s = (Q + P)^2 = (K_\pi + P_f)^2$, with the four vectors defined in Fig. 28(a).}
with

\[ K = \frac{1}{2 \varepsilon_i \varepsilon_f} \frac{1}{2E_i E_f} \frac{1}{2E_{\pi}}, \quad \phi = \frac{|k_i - p_i|}{E_i} \]  

(134)

This expression is valid for EM, CC and WNC interactions. The factor \( R_X \), where the subscript \( X \) refers to the type of interaction, include the boson propagator as well as the coupling constants at the leptonic and hadronic vertices:

\[ R_{EM} = \left( \frac{4\pi\alpha}{Q^2} \right)^2, \quad R_{CC} = (G_F \cos \theta_W)^2, \quad R_{WNC} = G^2_F. \]  

(135)

\( \eta_{\mu\nu} \) is the lepton tensor discussed in Sect. II. We define the hadronic tensor as

\[ h^{\mu\nu} = 2M_f M_i \sum_{S_i, S_f} (J^{\mu})^* J^{\nu}. \]  

(136)

with the hadronic current

\[ J^{\mu} = \bar{u}(p_f, S_f) \mathcal{O}_{1\pi}^{\mu} u(p_i, S_i). \]  

(137)

\( u(p_i, f, S_i, f) \) represent the free Dirac spinors for the initial and final state nucleons with spin projection \( S_i \) and \( S_f \). \( \mathcal{O}_{1\pi}^{\mu} \) is the transition operator between the initial one-nucleon state and the final one-nucleon one-pion \((N \pi)\) state.

We can use the energy-conservation delta function to integrate over \( k_\pi \). This leads to

\[ \frac{d^5 \sigma}{d\varepsilon f d\Omega f d\Omega_{\pi}} = \frac{K \varepsilon_f k_f E_{\pi} k_\pi}{\phi f(1)} \frac{R_X}{(2\pi)^5} \eta_{\mu\nu} h^{\mu\nu}, \]  

(138)

where we have introduced the nucleon recoil function

\[ f(1) = 1 + \frac{E_{\pi}}{E_f} \left( 1 - \frac{k_\pi \cdot (q + p_i)}{k_\pi^2} \right) \]  

(139)

Alternatively, if we want to study distributions as a function of the pion energy, we can integrate over \( k_f \). In this case one obtains

\[ \frac{d^5 \sigma}{d\Omega_f dE_{\pi} d\Omega_{\pi}} = \frac{K \varepsilon f k_f E_{\pi} k_\pi}{\phi f(1)} \frac{R_X}{(2\pi)^5} \eta_{\mu\nu} h^{\mu\nu}, \]  

(140)

with

\[ f(1) = 1 + \frac{\varepsilon_f}{E_f} \left( 1 - \frac{k_f \cdot (k_i + p_i - k_\pi)}{k_f^2} \right). \]  

(141)

Notice that by choosing a reference frame in which \( k_f \) has no \( \hat{y} \) component, the cross sections in eqs. 138 and 140 will not depend on the azimuthal angle \( \phi_f \). Alternatively, if one chooses a reference frame in which \( k_\pi \) has no \( \hat{y} \) component, the cross sections will be independent on the azimuthal angle \( \phi_\pi \).

\[ \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

---

FIG. 28. (a) Feynman diagram representing the single-pion production off a free nucleon. (b) Representation of the hadronic subprocess in the center of momentum frame.

---

\[ \psi(x, p) = \sqrt{\frac{N}{V E}} u(p, s) e^{i x \cdot p}, \quad \text{with} \quad u(p, s) = \sqrt{\frac{M + E}{2M}} \left( \chi_s \right) \frac{\sigma_{\pi}}{E + M} \chi_s, \]  

\( \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \)
It is important to stress that the cross section formulas in eqs. (133), (138) and (140) are valid in any reference frame.

The leptonic part has been discussed in Sect. II. We will focus now on the description of the hadronic current. In the center of momentum system (CMS) of the hadronic subprocess, Fig. 28(b), the hadronic current depends on only three independent variables.

A usual choice for these are the Lorentz invariants \(Q^2, s\) and \(t\), with
\[
t = (Q - K_\pi)^2 = m_\pi^2 - |Q^2| - 2(QE_\pi - qk_\pi \cos \theta_\pi),
\]
where
\[
\omega = \frac{s - M^2 - |Q^2|}{2W}, \quad E_\pi = \frac{s + m_\pi^2 - M^2}{2W}.
\]
The maximum and minimum \(t\) values allowed by energy-momentum conservation correspond to the values \(\cos \theta_\pi = -1\) and \(\cos \theta_\pi = 1\). Thus, a possible strategy is to compute the hadronic current \(J^{\mu}(Q^2, s, t)\) in the CMS and then bring it back to the lab frame where the experiments are performed. This would allow one to save important computational time when sophisticated models are employed in the calculation of the hadronic current.

b. Nucleon Resonances

SPP in the resonance region is dominated by the excitation and de-excitation of nucleon resonances. The properties of the resonances (spin, isospin, mass, decay width, branching ratios, etc.) lying in the region \(W < 2\ \text{GeV}\) are, in general, known with great precision thanks to partial waves analyses of (mainly) elastic pion-nucleon scattering data. Information on their electroweak properties, on the other hand, can be extracted from analysis of photon- and lepton-induced single-pion production data.

The Feynman diagram representing the electroweak resonant mechanism is illustrated in Fig. 29(a). The cross resonance term, which is sometimes considered to maintain crossing symmetry [154, 155], is shown in Fig. 29(b). In these diagrams we can identify three pieces: the electroweak vertex \(\Gamma^{\mu\nu}_{\text{QNR}}\), the propagator of the resonance \(S^{\mu\nu}_{\text{RNR}}\), and the hadronic vertex \(\Gamma^{\mu\nu}_{\text{R}\pi N}\). The amplitude for these diagrams is given by the contraction of them ‘sandwiched’ between the nucleon spinors. For example, for a spin-3/2 resonance, such as the Delta, one finds:
\[
\begin{align*}
J^{\mu}_{(a)} & \propto \bar{u}(p_f, s_f) \Gamma^{\alpha\beta}_{\text{R}\pi N} S^{\alpha\beta}_{\text{RNR}} u(p_i, s_i), \\
J^{\mu}_{(b)} & \propto \bar{u}(p_f, s_f) \Gamma^{\mu\nu}_{\text{QNR}} S^{\alpha\beta}_{\text{RNR}} \Gamma^{\beta\gamma}_{\text{R}\pi N} u(p_i, s_i),
\end{align*}
\]
with \(\Gamma^{\mu\nu}_{\text{QNR}}(P_f, Q) \equiv \gamma^0[\Gamma^{\mu\nu}_{\text{QNR}}(P_f, -Q)]\gamma^0\). The three pieces are described below for the case of a spin-3/2 resonance, and within a Rarita-Schwinger formalism [156].

1. The electroweak vertex is characterized by the properties of the resonance (spin, isospin, and parity) and the electroweak \(Q^2\)-dependent form factors. It is parametrized by [157]:
\[
\Gamma^{\mu\nu}_{\text{QNR}} = \left(\Gamma^{\mu\nu}_{\text{QNR,V}} + \Gamma^{\mu\nu}_{\text{QNR,A}}\right) \gamma^5,
\]
with \(\gamma^5 = 1\) for even parity resonances, and \(\gamma^5 = \gamma^5\) for odd ones. The vector contribution is
\[
\Gamma^{\mu\nu}_{\text{QNR,V}} = \left[\frac{C_3^V}{M} (g^{\mu\nu} Q - Q^{\mu} Q^{\nu}) + \frac{C_5^V}{M^2} (g^{\mu\nu} Q \cdot K_R - Q^{\mu} K^{\nu}_R) + \frac{C_6^V}{M^2} (g^{\mu\nu} Q \cdot P - Q^{\mu} P^{\nu} + C_6^{\prime} g^{\mu\nu})\right] \gamma^5,\tag{146}
\]
and the axial one is
\[
\Gamma^{\mu\nu}_{\text{QNR,A}} = \frac{C_3^A}{M} (g^{\mu\nu} Q - Q^{\mu} Q^{\nu}) + \frac{C_5^A}{M^2} (g^{\mu\nu} Q \cdot K_R - Q^{\mu} K^{\nu}_R) + C_6^{\prime} g^{\mu\nu} + \frac{C_6^A}{M^2} Q^{\mu} Q^{\nu}.
\tag{147}
\]

\(K^{\mu}_R\) stands for \(K^{\mu}_R = P^{\mu} + Q^{\mu}\) and \(K^{\mu}_R = P^{\mu} - K^{\mu}_R\) for the direct and cross channels respectively. The resonance form factors \(C_i^{V,A}\) account for the internal structure, whose dynamics is explored with a resolution \(Q^2\).

The vector form factors can be extracted from helicity amplitude analyses of single-pion electro-production data [153, 158, 160]. In the axial sector, however, the situation is quite different. While the couplings may be guided by the PCAC hypothesis (see e.g. Appendix C in Ref. [160]), the rather scarce experimental data does not allow to constrain the \(Q^2\) dependence, which is usually taken as dipole shapes with sets of parameters inspired by the vector and QE cases.
2. The propagator is
\[
S_{\alpha\beta} = \frac{-\left(K_R + M_R\right)}{K_R^2 - M_R^2 + iM_R\Gamma_{\text{width}}(W)} \left( g_{\alpha\beta} - \frac{1}{3} \gamma_\alpha \gamma_\beta - \frac{2}{3M_R} K_{R,\alpha} K_{R,\beta} - \frac{2}{3M_R} (\gamma_\alpha K_{R,\beta} - K_{R,\alpha} \gamma_\beta) \right),
\]  
(148)
with \(\Gamma_{\text{width}}(W)\) the resonance decay width (see for instance Ref. 160).

3. The decay vertex is
\[
\Gamma^\alpha_{R\pi N} = \frac{\sqrt{2} f_{\pi NR}}{m_\pi} K_{R,\gamma}^5,
\]  
(149)
with \(f_{\pi NR}\) the strong coupling constant 160, 161.

In Ref. 162, it is proposed an alternative description of the \(\Delta\pi N\) vertex that remove the ‘unwanted’ coupling to the spin-1/2 component of the delta propagator in eq. (148), some results comparing the two vertices were shown in Ref. 163.

![FIG. 29. Diagrams contributing to the SPP. Resonance contribution: Direct (a) and cross (b) channel terms. Background contributions: Direct (c) and cross (d) nucleon pole, contact term (e), pion pole (f), and pion-in-flight (g) term.]

More details on nucleon resonances in neutrino-induced processes can be found in 158, 159, 161, 164–167 and references therein.

c. Models

Resonance contributions alone are not sufficient to reproduce the experimentally observed photo-, electro- and neutrino-production data. Background contributions arising from different mechanisms are necessary, e.g. those in Fig. 29(c)-(g). This is especially evident at the pion threshold, far from the delta peak, where this background is fully determined by chiral symmetry 168. In Ref. 164, the leading order terms of a chiral perturbation theory Lagrangian (ChPT) for the \(\pi\)-nucleon system were computed \(^5\), the diagrams contributing are shown in Fig. 29(c)-(g), while the expressions for their amplitudes can be found in 161, 164, 171, 172.

Strictly speaking, the applicability of the background model 164 is limited to the pion production threshold, however, it has been shown that the incorporation of form factors and the delta terms, Fig. 29(a) and (b), results in an effective model that provides a reasonably good agreement with photon-, electron-, and neutrino-induced SPP data in the region \(W < 1.4\ GeV\) (see 164, 163, 171, 172 and references therein). This approach has been employed by different groups 161, 167, 171, 174 in the description of neutrino-induced SPP. Its strength lies in its relative simplicity and that it is easily exportable to different nuclear frameworks.

To summarize the different models for SPP in the resonance region, that have been applied to electron and neutrino interactions, we use the classification given in Refs. 151, 173. The first kind of models are those that include only resonance amplitudes, e.g. Ref. 158. Models of the second kind combine resonances and background contributions 160, 164, 173, 176, 178. First and second kind, they both have in common that do not respect the unitarity condition of the amplitude. In the third kind we have the dynamical coupled-channels (DCC) model 173, 179, 180, which in words of the authors 151 is an extension of the Sato and Lee (SL) model 181, 182. The DCC model goes a few step further than the first and second kind: i) it accounts for the re-scattering between the hadrons, ii) it includes coupled channels that open for increasing energy transfer, such as \(\eta N\), \(K\Lambda\), \(K\Sigma\) and \(\pi\pi N\), and iii) the

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5. Recently, in Refs. 164, 170 it has been presented a study on the weak pion production in ChPT up to next-to-next-to-leading order, using pions, nucleons, and delta degrees of freedom.
unitarity condition is fulfilled by construction. Similar degree of sophistication is achieved by the MAID unitary isobar model\textsuperscript{183}, unfortunately, to our knowledge it has not been consistently extended to neutrino-induced reactions yet.

It is worth mentioning that unitarity was partially restored in the model of Ref.\textsuperscript{164} by incorporating the relative phases between the background diagrams and the dominant partial wave of the Delta pole\textsuperscript{184}. On top of that, in Ref.\textsuperscript{185}, an additional contact term was added, aiming at improving the description of the $\nu_e n \rightarrow \mu^- \pi^+$ channel and, at the same time, removing spurious spin-1/2 contributions in the Delta diagrams. Recently, in Ref.\textsuperscript{186}, this model, the DCC and the SL models were compared for a variety of reaction channels and kinematical scenarios. The authors concluded that, in the region $W < 1.4$ GeV, the simplest model\textsuperscript{185} is able to reproduce the bulk results from the other more sophisticated ones.

It should be stressed that, for the axial sector, all models rely, in one way or another, on the neutrino-deuterium and neutrino-hydrogen pion-production data from the old bubble chamber BNL and ANL experiments\textsuperscript{187,189}. On top of poor statistics and large systematic errors, the interpretation of these data has the additional challenge of describing the reaction on a deuterium target\textsuperscript{190,191}. Therefore, it is clear that new data on neutrino-hydrogen SPP would be of great value to move forward on a firmer basis.

d. SPP in the high-$W$ region

All the models mentioned describe the amplitude using lowest order interaction terms. For increasing $W$, higher-order contributions are needed, and hence, the predictions of these low-energy models (LEM) beyond $W = 1.4 - 1.5$ GeV should be taken with care. The DCC model can provide reliable predictions for somewhat larger invariant masses ($W \lesssim 2$ GeV) thanks to the incorporation of the relevant resonances up to $W \approx 2$ GeV along with the unitarization of the amplitude.

The procedure of adding higher order contributions soon becomes unfeasible. Regge phenomenology\textsuperscript{192,194} is an alternative approach that provides the correct $s$-dependence of the hadronic amplitude in the high-$s$ small-$t$ limit. The method, first proposed in\textsuperscript{195} and further developed in\textsuperscript{190}, boils down to replacing the $t$-channels meson-exchange diagrams by the corresponding Regge amplitudes, which includes an entire family of hadrons. In the context of\textsuperscript{195}, the $s$- and $u$-channel Born diagrams are kept in order to maintain CVC. As mentioned, Regge phenomenology only provides the $s$-dependence of the amplitude, but not the $t$- or $Q^2$-dependence. The $t$-dependence of the Regge amplitude is determined by the background terms of the LEM. The idea behind this is that the physical region of the reaction is not too far from the nearest pole of the Regge trajectory.\textsuperscript{6} This Regge-based $t$-channel approach\textsuperscript{196} compares well to photoproduction and low-$Q^2$ electroproduction data, but it fails to reproduce the higher $Q^2$ data and the transverse component of the electroproduction cross section. To remedy this, in Ref.\textsuperscript{201}, this Regge-based model was supplemented with hadronic form factors, that are included in the $s$- and $u$-channel Born terms, and effectively account for the contributions of nucleon resonances in the high-$Q^2$ sector.

An extension to neutrino-induced reactions of the models of Refs.\textsuperscript{196,201} was recently developed in\textsuperscript{161}. The model presented in Ref.\textsuperscript{161} consists in a low-energy model (LEM) and a high-energy model (HEM) that are combined by a smooth $W$-dependent transition function. This results in a hybrid model that safely can be applied over a large $W$ region. In particular, the LEM is built up from the coherent sum of background terms\textsuperscript{164} and the $s$- and $u$-channel diagrams for the $P_{33}(1232)$ (Delta), $D_{13}(1520)$, $S_{11}(1535)$ and $P_{11}(1440)$ resonances. The parametrization of the Olsson phases from Ref.\textsuperscript{152}, as well as the corresponding delta form factors, were used. The HEM is obtained by Reggeizing the background terms\textsuperscript{160} and introducing hadronic form factors\textsuperscript{201,202}. This HEM is compared with the LEM and with exclusive $p(e,e'\pi^+)n$ data in Fig.\textsuperscript{30} The results show that the LEM clearly overshoots the data while the HEM provides a better description of them. Finally, the $s$- and $u$-channel resonance diagrams are regularized by using phenomenological cut-off form factors, whose role is just to eliminate their contributions far from the resonance peak $W \approx M_R$. More details can be found in\textsuperscript{161}.

Event generators, such as NuWro\textsuperscript{204,206} and GiBUU\textsuperscript{206,207}, use DIS-based formalisms\textsuperscript{208,209} and PYTHIA hadronization routines\textsuperscript{210} to describe neutrino-induced SPP in the high energy region ($W > 1.5 - 2$ GeV). The transition from the resonance region to the high-$W$ regime is done by a transition function that smoothly, for increasing $W$ values, switches on (off) the high-(low-)energy components of the model.\textsuperscript{6} The validity of these DIS-based approaches is not clear at small $Q^2$, where it concentrates most of the strength of SPP\textsuperscript{201}. This could be tested with the wide collection of electron-nucleon exclusive SPP data in the high-$W$ region\textsuperscript{203,211,220}.

e. Some results

In Fig.\textsuperscript{31} we show inclusive electron-proton scattering cross sections for three different energies and scattering angles. The curves labelled as “LEM w/o ff” and “LEM w/ ff” correspond to the LEM without and with these

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\textsuperscript{6} For a first introduction into the Regge phenomenology applied to meson production we recommend the Refs.\textsuperscript{192,201}. For a more advanced reading Ref.\textsuperscript{193,194}.

\textsuperscript{7} The results of the hybrid model and NuWro for neutrino-induced SPP on the nucleon were compared in Ref.\textsuperscript{161}.
resonance form factors, respectively. In both cases the LEM background \[164\] is used. The model labelled as “Hybrid” includes the resonance form factors and the transition to the Regge-based model \[161\]. The theoretical predictions include only the single-pion production channel, therefore, they should underestimate the inclusive data. By construction, the three models provide essentially the same results for \( W < 1.3 \text{ GeV} \). The need of cut-off resonance form factors and a high-energy model becomes evident if one moves towards higher \( W \).

The predictions of the hybrid and DCC model \[175\] for SPP are shown in Fig. 32 for the same process and kinematics as in Fig. 51. The inclusive DCC predictions, which include other reaction channels such as two-pion production, is also shown. Up to the delta peak, the DCC and hybrid models, and the data are comparable. Above the delta peak, it is evident that the basic ingredients in the hybrid model are not enough to reproduce the strength of the SPP channel predicted by the DCC model. In particular, the hybrid model underpredicts the ‘dip’ between the delta peak and the second resonance region. This may be related with the lack of other processes, such as \( \rho \)- and \( \omega \)-exchanged contributions, the lack of pion-nucleon re-scattering, and the use of too strong cut-off resonance form factors. The third resonance region (third peak in the spectra) is also underpredicted by the hybrid model. This could be improved by including the relevant resonances in that region.

Finally, in Fig. 53 we compare the ANL and BNL data for CC neutrino-induced SPP with the low energy and hybrid models. In the region \( E_\nu > 2 \text{ GeV} \) the differences between the three models are important. Since the predictions of the LEMs cannot be considered realistic in this region, these results point to the need of a HEM. Also, the apparent agreement of the LEMs with data in the region \( E_\nu > 1 - 1.5 \text{ GeV} \) should be interpreted with care.

### B. Nuclear framework

The elementary reaction described above for SPP can be incorporated in a nuclear framework using the impulse approximation, i.e., considering that the electroweak boson interacts with only one nucleon in the nucleus. The simplest choice is the global relativistic Fermi gas (RFG) model which, in spite of its simplicity, is still employed by MC generators in the analyses of data \[140, 142, 144, 145, 148\]. The global RFG is a fully relativistic model, it respects the fundamental symmetries of a relativistic quantum field theory and it is able to capture the gross features of the nuclear response. Though, it is quite far from the level of precision required in the new era of the neutrino-oscillation
FIG. 31. Inclusive electron-proton scattering cross sections data [221] are compared with the SPP results of the low-energy (LEM) model (with and without resonance form factors) and the hybrid model of Ref. [161].

FIG. 32. Inclusive electron-proton scattering cross sections data [221] are compared with the SPP results from the DCC and Hybrid models. The inclusive DCC results (full-DCC) are also shown. The results of the DCC model have been taken from Ref. [175].

FIG. 33. Total cross sections for CC neutrino-induced SPP. The recent analysis [222] of the ANL and BNL data [187, 189] is compared with the low-energy and hybrid models (data without cuts in the invariant mass). Deuteron effects are not considered. The figure was adapted from [161].

a. What Mean-Field models can offer to the neutrino interaction community

Mean-field models are able to capture a good part of the nuclear dynamics by describing the ground state nucleus as a set of independent-particle nucleon wave functions that are solutions of the mean-field equations. In previous sections we already discussed these models applied to electron and neutrino scattering reactions. Here we simply summarize the basic points of interest for the discussion that follows.
As known, the case of inclusive processes can be efficiently modelled by describing the final states as scattering solutions of the corresponding wave equation: Dirac or Schrödinger equation for the nucleons and Klein-Gordon equation for the pions. Since the flux has to be conserved, one can use mean-field potentials with only real part or full complex optical potentials, in which the flux lost (transferred to inelastic channels) due to the imaginary term is recovered by a summation over those channels, as done in the Relativistic Green Function (RGF) model. On the contrary, in the exclusive case one needs to account for the flux moved to the inelastic channels (absorption, multi-particle emission, charge exchange, etc.). This can be accounted for by using phenomenological complex optical potentials (see discussion in Sect. III C) fitted to elastic nucleon-nucleus scattering or elastic pion-nucleus scattering data, and correcting the results with spectroscopic factors.

In neutrino experiments, however, fully exclusive conditions are never satisfied: the neutrino energy is unknown and the limited acceptances of the detectors make it impossible to detect the complete final state. Therefore, Monte Carlo (MC) neutrino event generators have to deal with inclusive and semi-inclusive events and, by combining the available experimental information with the nuclear theory, reconstruct the neutrino energy for every selected event. Modeling all possible semi-inclusive scenarios in a consistent way means to solve an extremely complicated many-body coupled-channel problem. A situation still far from being resolved. In spite of that, one can try to minimize the systematic errors that propagate to oscillation analyses by improving the theoretical nuclear models implemented in the MC’s.

In MC event generators there are two clearly separated steps: i) the elementary vertex, and ii) the propagation in the nuclear medium of the created hadrons, using a classical cascade model that generates the complete final state. Due to this factorization, ‘elementary vertex x hadron propagation’, the inclusive cross section will not be affected by the cascade process. Therefore, the primary model (the one that describes the elementary vertex) should be able to provide a good inclusive response. In summary, the primary model puts 100% of the strength in the elastic channel, while the cascade takes care of splitting the flux into the different inelastic channels. It is, therefore, preferable to use primary models that provide information not only on the final lepton (inclusive models) but also on the hadrons, so that they can be used as the ‘seed’ for the cascade in a more consistent way. The mean-field models discussed here can satisfy those requirement: full hadronic information and good inclusive results.

b. Cross section

Given the momentum $k_f$ and $p_A$ of the incoming lepton and target nucleus, the scattering process is completely determined by 9 independent variables. The nucleon wave functions are computed in the center of mass of the target nucleus, therefore, it is natural to work in the laboratory reference frame where the target nucleus is at rest ($p_A = 0$). Hence, we use the laboratory variables $\varepsilon_f$, $\theta_f$, $\phi_f$, $E_\pi$, $\theta_\pi$, $\phi_\pi$, $\theta_N$, $\phi_N$, and the missing energy $E_m$, as independent variables (see Fig. 34). In terms of these, the differential cross section for the process in Fig. 34 reads:

$$
\frac{d^9\sigma}{d\varepsilon_f d\theta_f d\phi_f d\varepsilon_\pi d\theta_\pi d\phi_\pi dE_m} = \frac{R_X}{(2\pi)^8} \frac{k_f p_N E_N E_\pi k_\pi}{2\varepsilon_f f_{rec}} \eta_{\mu\nu} h_{(\kappa)}^{\mu\nu} \rho_B(E_m). \tag{150}
$$

The function $f_{rec}$ accounts for the recoil of the residual nucleus and is given by:

$$
f_{rec} = \left[ 1 + \frac{E_N}{E_B} \left( 1 + \frac{p_N \cdot (k_e - q)}{p_N^2} \right) \right]. \tag{151}
$$

The energy of the outgoing nucleon $E_N$ and the residual nucleus $E_B$ are obtained from the independent variables.

The factor $R_X$ was defined in Eq. (135) and the leptonic tensor $\eta_{\mu\nu}$ in Sect. III. The hadronic tensor for one nucleon knock out from a $\kappa$ shell is given by:

$$
h_{(\kappa)}^{\mu\nu} = \frac{1}{2j+1} \sum_{m_j} \sum_{s_N} (J^\mu) J^\nu, \tag{152}
$$

we sum over the spin projections of the knockout nucleon ($\sum_{s_N=\pm1/2}$); and average over the bound nucleons of a $\kappa$ shell ($\sum_{m_j}$). $j$ is the total angular momentum and $m_j$ its third component.

The fact that the residual nucleus may be in an excited state is introduced by the function $\rho_B(E_m)$, that represents the density of final states for the residual nucleus. Its mass, $M_B$, is determined from the energy-conservation relation $M_B = M_A + E_m - M$, with $M$ the free nucleon mass. Within a pure shell model:

$$
\rho_B(E_m) = \sum_\kappa \delta(E_m - E_m^\kappa), \tag{153}
$$

---

8 We note that there is a typo in the expression of $f_{rec}$ given in Ref. [229].
where $E_n^m$ is a fixed value for each shell.

c. Hadron current

In this framework, the most general hadronic current (for given $\kappa$, $m_j$ and $s_N$) in momentum space reads

$$J^\mu = \frac{1}{(2\pi)^{3/2}} \int dp' N \int dp \bar{\psi}^{s_N}(p', p) \phi^*(k_{\pi}, k_{\pi}) \mathcal{O}_{1\pi}^\mu(Q, P, K_{\pi}', P_{N}') \psi^{m_j}(p),$$

with $K_{\pi}' = Q + P - P_{N}'$. $\psi^{m_j}(p)$ is the Dirac spinor of the bound nucleon (see Ref. [46] for details), and $P$ represents its four momentum. $\psi^{s_N}(p', p_N)$ and $\phi(k_{\pi}, k_{\pi})$ represent the wave functions of the final nucleon and pion (see Fig. 34). In deriving Eqs. (150) and (154), it has been considered that all the particles have well defined energy in all steps of the process, so that the energy of the bound nucleon $E$, which is needed to build the operator $\mathcal{O}_{1\pi}^\mu$, can be determined by energy conservation $\omega + E = E_N + E_\pi$. Note that the transition operator $\mathcal{O}_{1\pi}^\mu(Q, P, K_{\pi}', P_{N}')$ has to be evaluated for every $p$ and $p_N'$. All this, together with the large phase space for this process (4 particles in the final state), represents a non trivial problem from the computational point of view.

![FIG. 34. Representation of the electroweak SPP on a nucleus within the impulse approximation. An incoming lepton $K_i$ scatters on a nucleus $P_A$. A single boson $Q$ is exchanged. The final state consists in a scattered lepton $K_f$, the residual system $P_B$, and a knockout nucleon and pion with asymptotic four momentum $P_N$ and $K_{\pi}$, respectively. Inside the nucleus the momenta of the initial and final nucleons ($p$ and $p_N'$) and the pion ($k_{\pi}'$) are given by probability distributions. See text for details.](image)

In what follows we describe some approximations that simplify the problem and have allowed us to obtain numerical results that can be compared with cross section data:

1. Using the asymptotic values for the four momenta of the final nucleon and pion in the operator $\mathcal{O}_{1\pi}^\mu$, i.e.:

$$\mathcal{O}_{1\pi}^\mu(Q, P, K_{\pi}', P_{N}') \rightarrow \mathcal{O}_{1\pi}^\mu(Q, P, K_{\pi}, P_{N})$$

with $P = P_N + K_{\pi} - Q$.

In this way, $\mathcal{O}_{1\pi}^\mu$ can be evaluated out of the integrals in Eq. (154).

2. Describing the pion as a plane wave. In this case, Eq. (154) reduces to:

$$J^\mu = \frac{1}{\sqrt{2E_\pi}} \int dp \bar{\psi}^{s_N}(p', p_N) \mathcal{O}_{1\pi}^\mu(Q, P, K_{\pi}, P_{N}) \psi^{m_j}(p),$$

with $p_N' = Q + P - K_{\pi}$. Notice that if the approximation (155) is not employed, the operator should be calculated for every $p$ value.

3. Describing the pion and the outgoing nucleon as plane waves. Thus, Eq. (156) reads:

$$J^\mu = (2\pi)^{3/2} \sqrt{\frac{M}{2E_\pi E_N}} \bar{\psi}(p_N, s_N) \mathcal{O}_{1\pi}^\mu(Q, P, K_{\pi}, P_{N}) \psi^{m_j}(p),$$

with $P = P_N + K_{\pi} - Q$. In this case, the momenta of all particles are well defined, so that the approximation (155) does not apply.
4. Describing the initial and final nucleon, and the pion as plane waves. The hadronic current results

\[ J^\mu = (2\pi)^3 \delta^3(p_N + k_x - q - p) \sqrt{\frac{M^2}{2E_n E_N E}} \mathcal{M}(p_N, s_N) \mathcal{O}_{1\pi}^\mu(Q, P, K_x, P_N, u(p, s)), \]  

(158)

the replacement \( \frac{1}{2} \sum_{j=1}^N \to \frac{1}{2} \sum_{s} \) in Eq. (152) is assumed. This is the hadronic current in the RFG model. From Eqs. (150) and (158), it is straightforward to obtain numerical results within the RFG; additionally, the free nucleon case can be recovered as the limit \( p_F \to 0 \), with \( p_F \) the Fermi momentum.

The impact of the approximations described above will depend on the particular case under study, but in general we can say that the more exclusive conditions (smaller phase space) and the further we are from the maximum of the distributions, the greater the dependence on these nuclear effects. In Fig. 35(a), we show the \( Q^2 \) differential cross section (per nucleon) for the process \( \nu_\mu + ^{12}\mathrm{C} \to \mu^- + ^{11}\mathrm{B} + p + \pi^+ \) computed with three different approaches: i) considering a free proton target, ii) the RFG, and iii) the RPWIA. This comparison allows us to estimate the effect of Fermi motion and binding energy as well as the impact of using a distribution instead of a Dirac delta in the initial state (compare Eqs. (157) and (158)). One observes that the free-nucleon model clearly departs from the RPWIA and RFG results, while the latter are almost indistinguishable. As explained in the original paper [163], this is the case for sufficiently high energy and momentum transfer and for a RFG model with an appropriate binding-energy correction.

In Fig. 35(b) we show the inclusive \( ^{12}\mathrm{C}(e, e') \) electron scattering cross section computed with two approaches based on Eqs. (156) and (157), i.e.: the pion is described as a plane wave and the initial nucleon is computed within the RMF model, the final nucleon is i) a plane wave (Eq. (157), labelled as RPWIA), or ii) it is a continuum relativistic distorted wave (Eq. (156), labelled as ED-RMF). The results show that in spite of the large phase space that contribute (integrals over the nucleon and pion variables have been performed), the nuclear effects, such as Pauli blocking and distortion, that are incorporated in the ED-RMF model and not in the RPWIA, remain quite important (see Ref. [231] for further details).

The approximation defined in Eq. (155) was considered in the ED-RMF calculations shown in Fig. 35(b), and in those of Ref. [231]. The impact of this approximation has been studied in the past for different reactions, in Ref. [242] for pion photoproduction and in Ref. [243] for coherent pion production. Its effect in lepton-induced SPP within the RMF framework will be the subject of future work.

Finally, describing the pion as a plane wave (Eq. (156)) is clearly an oversimplification of the problem, especially if we aim at describing semi-inclusive results in which the pion is detected. This is further discussed in Sect. V C.

d. In-medium modification of the delta resonance

The properties of the resonances are modified in the nuclear medium. Due to the dominant role of the delta resonance, the in-medium modification of its decay width is one of the main nuclear effects in neutrino-induced SPP [244, 245]. The procedure developed in Refs. [246, 248] consists in modifying the delta decay width by the complex part of the delta self-energy calculated in the nuclear medium, as well as accounting for Pauli blocking that reduces the phase
space available for the delta to decay. It has been discussed in many references [165, 171, 249–251], so we do not repeat it here. The uncertainties and inconsistencies of using this method, which was developed within a Fermi gas model, in the framework of our relativistic mean-field model were discussed in Ref. [229, 230]. Its effect in the cross section is shown in Fig. 36 as a red band, that represents the uncertainty attached to this nuclear effect.

Other studies on the medium modification of the delta width have been presented in Refs. [252–256].

### C. Pion final-state interactions

In Refs. [229, 230] theoretical SPP cross sections were compared with different sets of 1π-detected data reported by the MiniBooNE [140–142], MINERvA [143–146] and T2K [148] experiments. Due to the semi-inclusive nature of these experiments, a meaningful comparison with these data requires to account for the pion final-state interactions. The pions produced in the primary vertex may suffer elastic rescattering, charge exchange, be absorbed, create new pions, etc. Indeed, the pion that reaches the detector may not be the one produced in the primary vertex but another one originated in a secondary interaction.

In Fig. 36 we present a selection of the results presented in Refs. [229, 230]. On the one hand, we show the predictions of the hybrid-RPWIA model. In this approach, the hybrid model described in Sect. V A 0d was used for the current operator, and the RPWIA was used for the nuclear part (Eq. 154). Thus, final-state interactions (FSI) are neglected. On the other hand, we show the results of the MC neutrino event generator NuWro [204, 257]. In NuWro the elementary vertex is described in a somewhat simpler way: the delta resonance and a background, which is an extrapolation of the DIS contribution to the low-W region [205]. On the contrary, NuWro accounts for FSI using a sophisticated cascade model [257, 258] (see Ref. [259] for a recent study in which the model is updated and benchmarked), so it gives us a quantitative estimate of the effect of FSI on the pion distributions. In the calculations shown in Fig. 36 red bands correspond to the Hybrid-RPWIA with (lower line) and without (upper line) medium modification of the delta width. The blue-solid lines are NuWro results using the same definition of the signal as in the experiment. The blue-dashed lines are NuWro but without FSI. The orange dash-dotted lines are NuWro results without FSI and selecting those events in which only one pion and one nucleon exit the nucleus. These latter results correspond to the elementary SPP process as predicted by NuWro, and they could be compared with the hybrid-RPWIA results.

![FIG. 36. CC \( \nu_\mu \)-induced one pion production on the nucleus. Single differential cross sections are represented as a function of the kinetic energy \( T_\pi \) or the momentum \( p_\pi \) of the pion. Panels (a), (b) and (c): MiniBooNE [141], MINERvA [144, 145] and T2K [148] \( 1\pi^+ \) production on CH\(_2\), CH and H\(_2\)O, respectively. Panels (d) and (e): MiniBooNE [142] and MINERvA [143] \( 1\pi^0 \) production on CH\(_2\) and CH, respectively. Figures adapted from Refs. [229, 230].](image)

Finally, the effect of the pion FSI in Fig. 36 can be understood attending to the mechanisms that are implemented in
the cascade, i.e.: secondary interactions slow down the pions, which shift the strength towards lower \( T_\pi \) regions; pion absorption reduces the total strength; charge-exchange reactions moves flux from one channel to another, but since the \( 1\pi^+ \) channel is dominant in CC \( \nu_\mu \)-induced pion production, the net effect is to reduce (increase) the magnitude of the \( 1\pi^+ \) (\( 1\pi^0 \)) distributions.

VI. Higher Inelastic and DIS

A. Higher lying resonance contributions and models

Beyond the excitation of \( \Delta(1232) \) resonance, other higher lying resonances (\( W > 1.4 \) GeV) have to be taken into account in a description based on resonance production followed by their subsequent decays, either on specific channels (\( \pi N, \pi N, \eta N, KY... \)), or summing over all of them for inclusive (\( e,e' \)) or (\( \nu, l \)) reactions at higher energy transfers. This region of invariant masses \( W > 1.4 \) GeV is usually called the second resonance region. The most relevant and theoretically studied resonances in this region are the three isospin 1/2 (\( \theta \)) angle, which is much smaller, sin of the 1\( \pi \) absorption reduces the total strength; charge-exchange reactions moves flux from one channel to another, but since the \( 1\pi^+ \) channel is dominant in CC \( \nu_\mu \)-induced pion production, the net effect is to reduce (increase) the magnitude of the \( 1\pi^+ \) (\( 1\pi^0 \)) distributions.

Most of the information about the electromagnetic and weak resonance transition vector form factors comes from the studies of pion electro and photo-production data off nucleons by including these resonances in the analyses \([261–264]\). The electromagnetic nucleon-to-resonance form factors can be usually obtained from the helicity amplitudes, in the same fashion as the MAID model \([265]\). The corresponding weak vector transition form factors can then be related to the electromagnetic ones by assuming that the weak vector charged current belongs to the same triplet of current operators as the isovector component of the electromagnetic current (for a very detailed description on how to work out this, please see the appendices of Ref. \([266]\)). Normally, the transition axial form factors are much more difficult to determine because of the lack of guidance from the electromagnetic interactions. The usual trick is to use PCAC to relate two of the axial form factors, and the coupling of one of them at the partial decay width of the particular resonance into the \( \pi N \) channel \([264, 267]\). This is done in the same fashion as when we relate the nucleon pseudo-scalar and axial form factors, while the axial coupling is obtained by assuming pion-pole dominance and to be related to the strong \( \pi N \) coupling through the Goldberger-Treiman relation.

The aforementioned procedure is general and can be applied to the neutrino-production of many resonances in the range of invariant masses between 1 and 2 GeV. It has been applied, for instance, in Refs. \([260, 264, 268, 269]\) to the study of electron and neutrino-nucleus scattering in the resonance region, including the \( \Delta \) region as well. For more recent works where the inclusion of higher lying baryon resonances is taken into account for the study of single pion production driven by weak and neutral currents, the reader is referred to Ref. \([270]\). Other works where the excitation of the \( N^*(1535) \) and \( N^*(1650) \) resonances is taken into account for the calculation of \( \eta \)-meson production cross sections off nucleons can be found in \([271, 272]\).

One of the most relevant works in the analysis of single pion neutrino-production is that of Rein and Sehgal \([273]\). They included all the relevant resonances below 2 GeV for the analysis of both charged and neutral current single pion production data by then. In this latter work, however, the helicity amplitudes are not phenomenologically obtained, but theoretically calculated within the relativistic harmonic oscillator quark model of Feynman, Kislinger and Ravndal \([274]\). The Rein-Sehgal model is widely used in some of the Monte Carlo event generators used for the analyses and simulations of single pion production events, namely GENIE, Neut and Nuance \([273, 274]\). This significant model has been also extended to account for the final lepton mass and spin in lepton polarization studies in neutrino-nucleon scattering \([280, 281]\).

It is also worth mentioning that some specific channels, such as the two pion production \( \pi \pi N \) one, have been very scarcely studied from the experimental and theoretical side. In Ref. \([282]\) it is shown that the inclusion of the Roper resonance, \( N^*(1440) \), is relevant for this reaction channel near its production threshold.

Up to now we have only mentioned the excitation of non-strange resonances. However the weak charged current can induce strangeness-changing transitions \( |\Delta S| = 1 \) as well, although these are highly suppressed with respect to their strangeness-preserving \( |\Delta S| = 0 \) counterparts because the amplitudes for the latter are proportional to the cosine of the Cabibbo angle, \( \cos \theta_c \sim 0.974 \), while those for strangeness-changing are proportional to the sine of the same angle, which is much smaller, \( \sin \theta_c \sim 0.226 \). This kind of processes normally requires the inclusion of strange baryon resonance degrees of freedom, normally the \( \Sigma^* \) (1385), which belongs to the same SU(3) multiplet as the \( \Delta(1232) \). Quite recent works studying the production of strange particles induced by electrons, neutrinos and anti-neutrinos through the strangeness-changing part of the weak charged current can be found in Refs. \([283, 284]\).
B. Inelastic contribution within the scaling formalism

Within the scaling formalism, in Ref. [287], an extension of the RFG model to the inelastic region was carried out by allowing the reached final invariant masses to be larger than in the QE case \((m_N+m_\pi \leq W_X \leq m_N+\omega-E_S,\) with \(E_S\) the single nucleon separation energy between the initial and residual nuclei), and then performing an integration over all the allowed invariant masses with an "spectral function" \((\text{the inelastic RFG scaling function in terms of the inelastic scaling variable } \psi_X)\) accounting for the energy-momentum distribution of nucleons in the RFG description of the initial nuclear state. To apply this formalism is completely necessary to resort to phenomenological parameterizations of the inelastic single-nucleon structure functions, extracted mainly from DIS \(e-N\) scattering, but also incorporating resonances in the \(W < 2 \text{ GeV} \) region [288, 291]. More recent and complete parameterizations of these inelastic single-nucleon structure functions extracted from inelastic electron-proton and electron-deuteron scattering experiments can be found in Refs. [292, 293].

In other more recent work [105], the authors analysed the scaling and super-scaling properties of the non-QE experimental data within the framework of several theoretical models. To this end, they basically subtracted the QE contribution to the inclusive \((e,e')\) experimental data by assuming an universal phenomenological longitudinal scaling function fitted in Refs. [21, 22]. They also assumed that the QE transverse and longitudinal scaling functions were equal \((f^\text{LQE}_\nu(\psi_{\text{QE}}) = f^\text{QE}_T(\psi_{\text{QE}}))\) in order to calculate the theoretical QE cross section to be subtracted from the inclusive \((e,e')\) data to obtain the non-QE contribution. After that, this non-QE cross section was translated into a non-QE scaling function, \(f^{\text{non-QE}}_\nu(\psi_\Delta)\), by dividing by the single-hadron \(N \to \Delta\) cross section, whose explicit expressions can be found in detail in Refs. [24, 294]. These residual data were mainly compared with two different theoretical models: the first one was basically that of Ref. [287], already discussed in the previous paragraph, but with updated inelastic single-nucleon structure functions taken from [292, 293]; the other model used to analyze the data was that of Ref. [294], which is based on the assumption of the \(\Delta\) dominance in the inelastic region, but taking into account the finite \(\Delta\) width. One of the main conclusions of this work [105] is that, besides the importance of non-impulsive mechanisms (2p-2h, correlations...) to be incorporated consistently in any model, there were kinematics much better described by the inelastic model of Ref. [287] than by the \(\Delta\)-dominance model of [294], thus indicating the importance of higher inelastic contributions beyond the \(\Delta\) at those kinematics (mainly corresponding to larger incident electron energies and medium/large scattering angles, thus linked to higher momentum transfers as well).

C. Quark-hadron duality and transition to DIS

The resonance electroproduction data \((1.1 \leq W \leq 2 \text{ GeV})\) were seen soon to be linked to Deep Inelastic Scattering (DIS) data at higher final invariant masses \(W \geq 2 \text{ GeV}\) at the end of the 60's, within the framework of duality ideas [295, 301]. Historically, these ideas were borrowed from the hadron physics world, and they basically related the high energy hadron-hadron cross section behaviors to the presence (or not) and behavior of low energy resonances in the \(s\)-channel. The paradigmatic examples are those of the \(K^\pm p\) and \(K^0 p\) total cross sections, whose behaviors are different at high energies and, while the \(K^+ p\) system has no resonances at low energy, the \(K^0 p\) system has a lot of strange \(Y^*\) hyperon resonances. The same happens for the \(p p\) and \(\bar{p} p\) total cross sections at high energies, and its discussion in terms of low energy strong resonances in the \(s\)-channel is alike to the \(K N\) case (see Refs. [296, 301, 302] for more detailed explanations).

The success of the work of Bloom and Gilman [300, 301] (nowadays known as Bloom-Gilman or quark-hadron duality) was to realize that the resonance electroproduction data off nucleons, with final invariant masses in the range \(1.1 \leq W \leq 2 \text{ GeV}\), merged into the inelastic structure function \(\nu W_2(\nu, Q^2)\) in the scaling region (corresponding to the limiting case \((Q^2, \nu) \to \infty\), but finite Bjorken scaling variable \(x = Q^2/(2m_N\nu)\), where \(\nu\) is the energy transfer to the nucleon in the Lab frame) when \(Q^2\) increases. Note that the scaling region and that of the resonance electroproduction correspond to different regions of the phase space: resonance electroproduction data occur when the final hadron invariant masses are within the resonance region \((1.1 \leq W \leq 2 \text{ GeV})\) irrespective of the \(Q^2\) value; however, the scaling region happens for \(Q^2 \geq 1 \text{ GeV}^2\) and hadron invariant masses outside the resonance region, \(W \geq 2 \text{ GeV}\). The non-trivial feature of the inelastic structure functions \(2m_N W_1(\nu, Q^2)\) and \(\nu W_2(\nu, Q^2)\) in the scaling (or DIS) region is that they become functions of a single variable instead of being functions of two independent variables \(a \text{ priori}\). Normally, the usual variable to represent the scaling property is chosen to be the Bjorken \(x\) scaling variable [303], although others have also been used [300, 301, 304] and they reduce to Bjorken \(x\) in some limits.

Notice that what is not obvious is that the resonance electroproduction data, which correspond to a low-\(W\) region of the phase space, merge into the scaling curve (which has been fitted to other data) as long as \(Q^2\) increases. This was the observation made in Refs. [300, 301] for the first time. Laterly, with the advent of more precise data, this duality between the behavior of electroproduction data at high \(Q^2\) in the resonance region and the description based on parton distribution functions (PDFs) in the DIS region has been more precisely verified [305, 307] in electron
scattering experiments.

Although a similar behavior is expected to hold for the neutrino/anti-neutrino scattering case, the scarcity of resonance production data and the poor knowledge of the axial form factors in this resonance region, specially beyond the \( \Delta(1232) \) case, have prevented almost any experimental test. Nonetheless, there are theoretical studies \([308]\) that point in the direction that quark-hadron duality also holds for the neutrino structure functions when averaged over proton and neutron targets. However, other theoretical studies on quark-hadron duality for the nuclear structure functions point out differences between nuclear effects in the resonance and DIS regimes \([309]\) and it concludes pointing in the direction that quark-hadron duality holds much worse for nuclei than it does for nucleons.

D. DIS at nucleon level

At high four-momentum transfers, \( Q^2 \gg 1 \text{ GeV}^2 \), the electroweak probe has enough resolving power to distinguish the substructure of the nucleons, called initially partons \([310, 311]\), and afterwards identified with the elementary constituents of hadrons, namely quarks and gluons.

In these processes, called Deep Inelastic Scattering (DIS), the electron or neutrino/antineutrino scatters elastically off the constituent partons of the target that can be probed at an energy scale defined by the four-momentum transfer squared \( Q^2 \) and the energy transfer \( \nu \) in the Lab system. To describe this kind of interactions and following the ideas developed by Bjorken about scaling invariance \([303]\), the dimensionless Bjorken scaling variable is introduced \([9]\),

\[
x = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{2m_N \nu},
\]

where the last equality holds in the nucleon rest frame. This Bjorken scaling variable represents the fraction of the initial nucleon momentum carried by anyone of the partons that can take part in the interaction process.

The key idea behind the DIS processes is that the electroweak probe can scatter elastically and incoherently off all the partons inside the nucleon that carry a fraction \( x \) of the nucleon momentum. This picture of the interaction process is exactly the same as in the impulse approximation approach for the QE scattering, where again the electroweak probe scatters elastically and incoherently off all the nucleons that make up the nucleus. This approach, of course, assumes that there is a probability density for finding parton \( i \) inside the nucleon carrying a fraction \( x \) of its momentum. This is exactly the same, from the theoretical point of view, that assuming a momentum distribution for nucleons inside the nucleus in the impulse approximation approach to QE electron or neutrino scattering. In the DIS formalism, these probability density functions for finding parton \( i \) carrying a fraction \( x \) of the nucleon momentum are called parton distribution functions (PDFs), and are represented by \( q_i(x) \) for quark flavor \( i \), \( \bar{q}_i(x) \) for antiquarks or \( g(x) \) for gluons.

At leading order (LO), the electroweak probe (either a photon or a \( W^\pm \) boson) only couples to the quarks (and not to the gluons) because these have electric and weak charges (see Figure 37). The contribution of gluons to DIS appears at the next order in the QCD corrections to the so-called naive parton model, which is the one being discussed in this

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\[9\] In this subsection we use the usual DIS notation, where \( Q^2 = -q^2 \) is the positive four-momentum transfer squared, and \( p \) and \( q \) are the four-momenta of the initial nucleon and the exchanged virtual photon or intermediate vector boson \( W^\pm \), respectively.
section. For further details on how the gluons contribute to the description of electron and neutrino DIS, the reader is referred to Refs. [312, 313]. The fact that the only partons contributing to electron and neutrino/antineutrino DIS at LO are the quarks means that the inelastic structure functions \( F_1(x) \), \( F_2(x) \) and \( F_3(x) \) in the DIS formalism can be written only in terms of quarks and antiquarks PDFs. These structure functions are related to the usual \( W_i(Q^2, p \cdot q) \) structure functions appearing in the general expansion of the hadron tensor in terms of a suitable tensor basis as in Eq. (139). Their relationships are:

\[
\begin{align*}
F_1(x) &= m_N W_1(Q^2, p \cdot q), \\
F_2(x) &= \nu W_2(Q^2, p \cdot q), \\
F_3(x) &= \nu W_3(Q^2, p \cdot q),
\end{align*}
\]

(160)

and the Bjorken scaling is verified in the limit \( Q^2 \to \infty \) and \( p \cdot q \to \infty \), with the constraint of \( x \), given by eq. (159), remaining finite.

The double differential cross sections for charged lepton and neutrino/antineutrino DIS can be written, in terms of the structure functions \( F_i(x) \), as

\[
\frac{d\sigma}{dx dy} = \frac{4\pi\alpha^2}{Q^4} \left[ x y^2 F_1^2(x) + (1 - y) F_2^2(x) \right]
\]

(161)

\[
\frac{d\sigma_{\nu(\bar{\nu})}}{dx dy} = \frac{G_F^2}{2\pi} \left[ x y^2 F_1^{\nu(\bar{\nu})}(x) + (1 - y) F_2^{\nu(\bar{\nu})}(x) \pm xy \left( \frac{1 - y}{2} \right) F_3^{\nu(\bar{\nu})}(x) \right]
\]

(162)

where \( s \simeq 2m_N E \) (by neglecting the initial masses in the ultra-relativistic limit), \( y = \frac{x}{x+y} \) is the so-called inelasticity, \( E \) is the incident lepton energy in the Lab system, and the plus (minus) sign in eq. (162) applies to neutrino (antineutrino) DIS, respectively.

The appearance of only two structure functions \( F_{1,2}^{\nu}(x) \) in charged lepton DIS, eq. (161), is typical of electron scattering and has to do with Rosenbluth separation. However, in DIS, given the form of the elementary electromagnetic quark currents which couple to the photon in the lower part of a figure similar to Fig. 37 with a \( \gamma \)-exchange, these two structure functions are not independent. They are related through the Callan-Gross relation [316]

\[
F_2^{\nu}(x) = 2xF_1^{\nu}(x) = \sum_i x e_i^2 (q_i(x) + \bar{q}_i(x)).
\]

(163)

Note that, in the above equation, both quarks and antiquarks PDFs contribute being added. The reason is that the quark and antiquark electromagnetic currents, in the limit of massless quarks (as it is the case of the quark parton model assumption, at least for the lightest flavors), give exactly the same contribution when constructing the hadron tensor out of them, and they have exactly the same squared charged \( (e_i^2) \):

\[
\begin{align*}
\bar{u}_{q_i}(p_f)\gamma^\mu u_{q_i}(p_i) &= \bar{u}_{q_i}(p_f)\gamma^\mu u_{q_i}(p_i) \\
\bar{\bar{u}}_{q_i}(p_f)\gamma^\mu v_{q_i}(p_i) &= \bar{\bar{u}}_{q_i}(p_f)\gamma^\mu v_{q_i}(p_i)
\end{align*}
\]

(164)

(165)

Both traces give exactly the same result, as can be deduced either from direct calculation or from inspection of eqs. (3) and (5) with \( a_V = 1, a_A = 0, m = m' = 0 \). In other words, the elementary “quark hadron tensor” is invariant under the simultaneous changes of \( u \leftrightarrow v \) spinors (change of quarks by antiquarks) and \( p_i \leftrightarrow p_f \) (change of initial and final parton momenta). This property will be important when discussing the structure function \( F_3^{\nu/\bar{\nu}}(x) \) in \( \nu/\bar{\nu} \) DIS.

Finally, using eq. (163), we can write the electromagnetic \( F_2^{\nu} \) structure functions in proton and neutron targets by noticing that the quark PDFs are not the same in both targets, but they can be related by assuming exact isospin symmetry. Comparatively, this is the same than saying that the QE electromagnetic response functions are not equal in protons and neutrons because their form factors are not the same.

\[
F_2^{\nu}(x) = x \left( \frac{4}{9} \left( u(x) + \bar{u}(x) + c(x) + \bar{c}(x) \right) + \frac{1}{9} \left( d(x) + \bar{d}(x) + s(x) + \bar{s}(x) \right) \right),
\]

(166)

where the quark PDFs refer to the proton. To obtain the structure function for the neutron, one usually assumes exact isospin symmetry, which amounts to make the simultaneous replacements \( u \leftrightarrow d \) and \( \bar{u} \leftrightarrow \bar{d} \) on eq. (166), leaving the strange and charm flavor content unchanged:

\[
F_2^{\nu}(x) = x \left( \frac{4}{9} \left( d(x) + \bar{d}(x) + c(x) + \bar{c}(x) \right) + \frac{1}{9} \left( u(x) + \bar{u}(x) + s(x) + \bar{s}(x) \right) \right).
\]

(167)

\[\text{Note, however, that in eq. (159) the square of the four-momentum transfer } Q^2 \text{ is negative, while in this section, the same variable is defined positive. Therefore the true expansion, correct for the discussion in this section, is that of eq. (159) with } Q^2 \text{ replaced by } q^2 \text{ and } Q^2 \text{ replaced by } q^2.\]
Normally, it is also given the nucleon structure function $F_2^{in}$, which is the average of neutron and proton $F_2$ structure functions:

$$F_2^{in}(x) = \frac{1}{2} \left( F_2^{np} + F_2^{pn} \right) = x \left( \frac{5}{18} (u + \bar{u} + d + \bar{d}) + \frac{4}{9} (c + \bar{c}) + \frac{1}{9} (s + \bar{s}) \right), \quad (168)$$

which is an invariant function (the $x$-dependence on the PDFs has been dropped for simplicity) under the simultaneous isospin transformations $u \leftrightarrow d$ and $\bar{u} \leftrightarrow \bar{d}$, as it should be for an isoscalar nucleon.

The discussion of the CC neutrino and antineutrino DIS structure functions follows the same lines as those for electromagnetic ones. The only difference is the appearance of a new structure function, $F_3$, which comes out from the interference of the vector and axial-vector parts of the elementary quark current, and the fact that the neutrino and antineutrino structure functions, $F_i^{\nu(\bar{\nu})}$, are not equal because the weak charged current is not hermitic (the exchange of a $W^+$ or $W^-$ boson does not lead to the same quark flavor transitions), contrary to the electromagnetic or weak neutral current.

The best way to approach the neutrino DIS is to write down the effective Lagrangian for the neutrino-quark interaction,

$$\mathcal{L}_{CC} = -\frac{G_F}{\sqrt{2}} f(x) \gamma^\mu (1 - \gamma_5)\nu_\eta(x) \sum_{q_i \bar{q}_j} V_{q_i \bar{q}_j} \bar{q}_j(x) \gamma_\mu (1 - \gamma_5)q_i(x) + h.c., \quad (169)$$

where the effect of the $W^\pm$ intermediate boson propagation has been absorbed into the effective Fermi coupling constant $G_F$, and the $V_{q_i \bar{q}_j}$ are the elements of the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix. The sum in eq. (169) runs over up-type quarks ($q_i = u, c, t$) for $q_i$ and down-type ones for $q_j$ ($q_j = d, s, b$). Note that the configuration of quark and lepton fields in eq. (169) is perfectly suited to calculate the processes depicted in Fig. 37, while those pieces for the antineutrino induced processes are in the h.c. part of eq. (169).

The lepton tensor that can be constructed out of the CC weak lepton currents is basically the same as that of eqs. (3)- (7), with only one difference: in this section we define our lepton tensor as the complex conjugate of that given in eq. (1). This convention only amounts to a change of sign in eq. (7) (notice that we have already used this same convention in eqs. (163) and (165) for the elementary "quark hadron tensors").

If we explicitly calculate the CC weak "quark hadron tensors", the counterparts of eqs. (164) and (165) for the CC weak neutrino-quark processes depicted in Fig. 37 we obtain:

$$\tilde{u}_{\bar{q}_i, q_j}^{\mu \nu} = \sum_{\text{spin}} V_{q_j \bar{q}_i} \bar{u}_q (p_f) \gamma^{\mu} (1 - \gamma_5)u_q (p_i) \left( V_{q_i \bar{q}_j} \bar{q}_j (p_f) \gamma^{\nu} (1 - \gamma_5)u_q (p_i) \right)^* = 2 \left| V_{q_j \bar{q}_i} \right|^2 \text{Tr} \left[ \gamma^{\mu} (1 - \gamma_5)\not{p}_i \gamma^{\nu} \not{p}_f \right] \quad (170)$$

$$\tilde{w}_{\bar{q}_i, q_j}^{\mu \nu} = \sum_{\text{spin}} V_{q_j \bar{q}_i} \bar{v}_q (p_i) \gamma^{\mu} (1 - \gamma_5)v_q (p_f) \left( V_{q_i \bar{q}_j} \bar{q}_j (p_i) \gamma^{\nu} (1 - \gamma_5)v_q (p_f) \right)^* = 2 \left| V_{q_j \bar{q}_i} \right|^2 \text{Tr} \left[ \gamma^{\mu} (1 - \gamma_5)\not{p}_f \gamma^{\nu} \not{p}_i \right] \quad (171)$$

Notice in eqs. (170) and (171), the different positions occupied in the trace by the initial and final parton momenta in the case of quarks and antiquarks. These different positions will not affect the piece coming from the trace of $\gamma^\mu \not{p}_i \gamma^\nu \not{p}_f$, because this trace is invariant under the interchange of initial and final parton momenta, $p_i \leftrightarrow p_f$, and contributes only to the tensor component which goes with the $F_2^{\nu(\bar{\nu})}$ structure function. However, the situation is totally different for the trace of the piece $\gamma^\mu \gamma_5 \not{p}_i \gamma^\nu \not{p}_f$, because the interchange of initial and final parton momenta here amounts to a global change of sign in the trace. This can be checked either by direct calculation or from inspection of eq. (7), where changing $K \leftrightarrow K'$ involves a change of sign due to the anti-symmetry of the Levi-Civita tensor. This latter trace comes from the vector-axial interference of the elementary quark currents and only contributes to the tensor structure that goes with $F_3$. Therefore, for neutrino-quark and neutrino-antiquark scattering, the role of quarks and antiquarks is going to be different at the level of the $F_3$ structure function, where quark PDFs will enter with one sign and antiquark PDFs with the opposite.

Finally, it is worth noticing that we can get rid of the squares of the CKM matrix elements, $\left| V_{q_i \bar{q}_i} \right|^2$, when summing (for a fixed initial flavor) over all the possible final parton flavors, with the aid of the unitarity of the CKM matrix. This is particularly true even for the four lightest quark flavors $(u, d, s, c)$, because the CKM sub-matrix that mixes them, with approximate matrix elements

$$\begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix} \approx \begin{pmatrix} \cos \theta_c & \sin \theta_c \\ -\sin \theta_c & \cos \theta_c \end{pmatrix}, \quad (172)$$

is also approximately unitary, i.e, the lightest flavors are much more uncoupled with the heavier ones $(b, t)$ than they are coupled between themselves. Notice that, when summing for a fixed initial flavor, we have to multiply the
contribution with the probability density of finding that fixed initial flavor in the hadron with fraction $x$ of the total hadron momentum, and this is nothing else than the PDF of the initial flavor. With this, we can write the quark and antiquark content of the neutrino-proton $F_2$ and $F_3$ structure functions:

$$F_2^{2\bar{p}}(x) = 2x(d + s + \bar{u} + \bar{c})$$

$$xF_3^{2\bar{p}}(x) = 2x(d + s - \bar{u} - \bar{c}).$$

The factor 2 in the above equations has its origin in the fact that both vector-vector (VV) and axial-axial (AA) contributions to $F_2$ are equal for CC weak processes (the quark vector and axial charges squared are the same for CC interactions). And vector-axial (VA) and axial-vector (AV) interference terms contribute equally to $F_3$ in CC weak processes as well. In fact, this factor is already present in eqs. (160) and (171), cf. with similar eqs. (164) and (165) for electromagnetic processes.

The usual exact isospin symmetry can be applied to obtain the neutrino-neutron CC weak structure functions, as

$$F_2^{\nu n}(x) = 2x(u + s + \bar{d} + \bar{s})$$

$$xF_3^{\nu n}(x) = 2x(u + s - \bar{d} - \bar{s}).$$

For antineutrinos, besides the change in flavors involved in the antineutrino-quark interaction (because of the exchange of a $W^-$ boson, i.e., we are now working with the h.c. piece of eq. (169)), there is another important difference with respect to neutrinos case: now there is also a change of sign in the VA interference term of the lepton tensor because of the change of $u \leftrightarrow \nu$ lepton spinors, but this sign has been explicitly incorporated in the cross section expression for the $F_3$ term already, as in eq. (162), and it has not to be accounted for again in the quark/antiquark PDFs. With these considerations, we can readily write

$$F_2^{\bar{p}p}(x) = 2x(u + c + \bar{d} + \bar{s})$$

$$xF_3^{\bar{p}p}(x) = 2x(u + c - \bar{d} - \bar{s})$$

$$F_2^{\bar{v}n}(x) = 2x(d + c + \bar{u} + \bar{s})$$

$$xF_3^{\bar{v}n}(x) = 2x(d + c - \bar{u} - \bar{s}).$$

We can also perform here an average of neutrino and antineutrino structure functions over protons and neutrons, as we did for the electromagnetic case in eq. (168):

$$F_2^{\nu N}(x) = \frac{1}{2}(F_2^{2p} + F_2^{\nu n}) = x(u + d + 2s + \bar{u} + \bar{d} + 2\bar{c})$$

$$F_2^{\bar{v} N}(x) = \frac{1}{2}(F_2^{\bar{p}p} + F_2^{\bar{v} n}) = x(u + d + 2c + \bar{u} + \bar{d} + 2\bar{s})$$

$$xF_3^{\nu N}(x) = \frac{1}{2}(xF_3^{2p} + xF_3^{\nu n}) = x(u + d + 2s - \bar{u} - \bar{d} - 2\bar{c})$$

$$xF_3^{\bar{v} N}(x) = \frac{1}{2}(xF_3^{\bar{p}p} + xF_3^{\bar{v} n}) = x(u + d + 2c - \bar{u} - \bar{d} - 2\bar{s}).$$

Notice that, as it is obvious, the above four structure functions are invariant under isospin symmetry transformation (simultaneous change of $u \leftrightarrow d$ and $\bar{u} \leftrightarrow \bar{d}$). Further assumptions can be made regarding the proton content of strangeness and charm in the quark parton model. In particular, sometimes one assumes that $s(x) = \bar{s}(x)$ and $c(x) = \bar{c}(x)$, and then further relationships between CC weak neutrino and antineutrino-nucleon structure functions can be found (see Ref. 314 for details), and even between the neutrino/antineutrino $F_2$ and the electromagnetic one if $s$ and $c$ quark PDFs are neglected. Nonetheless, these assumptions are much stronger than the exact isospin symmetry used to relate the proton and neutron structure functions and some deviations have been found and studied in the past (see 317 and references therein) about these stronger assumptions, related to significant discrepancies found in the extraction of some electroweak parameters, in particular the weak mixing angle, in neutrino/antineutrino DIS experiments 319,321.

The expressions for the neutrino/antineutrino structure functions $F_{2,3}$ shown through Eqs. (173) to (184) are true when it is energetically possible to produce a final charmed quark, i.e., above charm production threshold. If this is no longer true, then the initial down-type quark and antiquark flavors appearing in the above equations have to be multiplied by the corresponding CKM matrix elements squared, implying the impossibility of producing a final charmed quark or antiquark due to its higher energetic threshold. The corresponding expressions can be found, for instance, in Ref. 322 (see also comments made in Ref. 323).

Notice that in eq. (162) only three structure functions appear. However, in general, there are five contributing to neutrino DIS 324. The other two structure functions, $F_{1,5}(x)$, come from the $w_4$ and $w_5$ structure function expansion
of the general hadron tensor shown in eq. (B9). These two structure functions, as they appear multiplying tensor structures proportional to $q^\mu$, when contracted with the lepton tensor, always give rise to terms proportional to the final charged lepton mass squared, giving the probability density of finding an off-shell nucleon with some energy and momentum distribution.

The spectral function. It is in this sense that the spectral function can be regarded as a kind of nuclear wave-function dynamics models \[367–370\], or semiphenomenologically calculated in others \[80, 353, 371, 372\], and it is generically made through a convolution formula of the free (or with some off-shellness dependence \[353, 354\]) nucleon structure functions, where for the first time a substantial disagreement between the ratio of nuclear to nucleon (or deuteron) structure functions was observed with respect to the theoretical predictions accounting for kinematic effects like the Fermi motion (with or without high momentum tails) of the nucleons inside the nuclei \[288, 289\].

This different behavior of the nuclear structure functions with respect to the nucleon (or deuteron) ones has been traditionally interpreted in terms of different models incorporating some genuine nuclear effects. Some models are based on the distortion of the quark momentum distributions in nuclei due to the formation of multi-quark clusters (bag models) \[344–348\]; other models assume that the enhancement of the sea quark distributions is due to an excess of pions surrounding the bound nucleons in nuclei \[344, 351\], and others also attribute this effect to nucleon correlations, nuclear binding and other intrinsic modifications of the nucleon properties in the nuclear medium \[351, 366\]. In the vast majority of the nucleon-meson pictures of the nuclei, the description of the nuclear structure functions is made through a convolution formula of the free (or with some off-shellness dependence) nucleon structure functions with another function carrying information about the energy and momentum distribution of the nucleons (and mesons) inside the nuclear environment. This last function can be either theoretically calculated in light-front dynamics models \[354, 370\], or semiphenomenologically calculated in others \[353, 371, 372\], and it is generically referred as the spectral function. Normally, this spectral function contains a part coming from a mean-field description of the nucleus plus a term describing the nucleon correlations, and therefore it incorporates nucleon high momentum components in the nuclear wave-function.

Anyway, irrespective of how the spectral function is calculated, the convolution formulae to calculate the nuclear structure functions entering into the electron-nucleon or (anti)neutrino-nucleon cross sections have very similar expressions in the literature, see for instance Refs. \[309, 353, 354, 363–365, 373\]. Loosely speaking, the effect of the convolution is to smear out the nucleon structure functions due to the integration, i.e., taking strength from one zone in the nucleon Bjorken variable and transferring it to other zones in the nuclear Bjorken variable. Or said with other words, the nuclear structure function, $F_A(x_A, Q^2)$, acquires contributions from the nucleon structure functions coming from different Bjorken nucleon $x_N = \frac{2(p^2 - P_t^2, q_t^2)}{2p^2}$ values, each one of these contributions being weighted by the spectral function. It is in this sense that the spectral function can be regarded as a kind of nuclear wave-function squared, giving the probability density of finding an off-shell nucleon with some energy and momentum distribution.

Other approaches \[374, 370\] directly employ nuclear parton distribution functions (NPDFs), which are globally fitted to charged lepton-nucleus, (anti)neutrino-nucleus and Drell-Yan data, and from where nuclear correction factors can

\[ E. \text{ Nuclear DIS} \]

Up to now we have been discussing electron-nucleon and (anti)neutrino-nucleon DIS. However, most of the experiments carried out on DIS region with neutrinos/antineutrinos have used nuclear targets such as iron in NuTeV/CCFR \[322, 325\] or CDHS \[335, 336\]; lead in CHORUS \[337, 338\]; and carbon, iron, lead and hydrocarbon in Mini-erva experiment \[339–341\], among other neutrino scattering experiments. For this reason, the nuclear effects present in nuclear DIS are relevant to be briefly discussed here.

Historically, the relevance of nuclear effects in DIS was first observed in connection with the EMC (European Muon Collaboration) effect \[342, 343\], where for the first time a substantial disagreement between the ratio of nuclear to nucleon DIS cross sections \[322, 324\] was observed with respect to the theoretical predictions accounting for the nuclear binding and other intrinsic modifications of the nucleon properties in the nuclear medium \[351–366\]. In

\[ \text{E. Nuclear DIS} \]

11 The $w_5$ structure function appears multiplying a tensor structure of the form $\frac{2p^\mu p^\nu + p_t^\mu q^\nu}{m_N}$, where $p_t^\mu$ is the initial parton four-momentum and $q^\mu$ the four-momentum transfer, called $Q^\mu$ in eq. (B9) and other parts of this review.

12 The appearance of a $w_4$ term in eq. (B9) is due to the form of the CC weak nucleon current of eqs. (169-171), which besides the vector $q^\mu$ and axial $\gamma^\mu\gamma_5$ contributions, also has magnetic $(q^\mu q_\nu)$ and pseudo-scalar $(q^\mu\gamma_5)$ terms that are absent in the elementary quark and lepton currents.

13 Note, however, that the tensor basis used in some references \[322, 324, 325, 327\] to expand the hadron tensor is different from the convention followed here in eq. (B9) and in Ref. 314, where the $w_1$ and $w_2$ structure functions are multiplied by tensor pieces that are already orthogonal to $q^\mu$ and, therefore, explicitly gauge-invariant. This difference in the convention makes the Albright-Jarlskog relationships suitable to be used with the convention used in the above-mentioned references, but not here. Or said in other words, the $F_{4,5}$ structure functions of Refs. \[322, 324, 325, 327\] are not straightforwardly related to our $w_{4,5}$ in eq. (B9), just because the tensor bases used to express the hadron tensor are not the same, and there is a mixture in the definition of the $F_{1,4}$ of those references with the $w_{1,4}$ used here. Exactly the same applies for the $F_{2,5}$ used there and the $w_{2,5}$ here.
be extracted (with their uncertainties) to take into account the nuclear dependence of the PDFs. In this way, these NPDFs can be also applied to other nuclei where no data are present right now. For general reviews on these and related topics, the reader is also referred to Refs. 367, 377, 378.

VII. The SuSAv2-MEC model

In Section III E, a detailed description of the SuSAv2 model was given, that, as known, incorporates the predictions from the RMF theory and a transition to the RPWIA model at high values of the momentum transfer. This transition between the RMF and RPWIA regimes is governed by a blending function whose explicit expression was described in Section III E (see also 106, 123 for details). In order to apply the SuSAv2 model not only to the QE regime but also to the full inelastic spectrum, it is required a good control of the transition parameters \((q_0, \omega_0)\) that determine the relative strength of the RMF and RPWIA responses, and how the transition between them evolves as the transfer momentum varies. Accordingly, the transition parameter, \(q_0\), is expected to increase with \(q\) in such a way that the RMF contribution will be dominant at low kinematics whereas the RPWIA one starts to be relevant at higher energies.

The particular procedure to determine the \(q_0\)-behavior with \(q\) is in accordance to the best fit to a large amount of \((e,e')\) measurements on \(^{12}\text{C}\) in a wide kinematical region, covering from low to high \(q\)-values \((q: 239 – 3432 \text{ MeV}/c)\). For this analysis, \(^{12}\text{C}\) is employed as target reference due to the ample variety of existing data for electron scattering as well as its relevance for neutrino oscillation experiments. The method applied to determine the RMF/RPWIA transition in the SuSAv2 model in both QE and inelastic regimes is based on a reduced-\(\chi^2\) analysis of the data sets (see 124 for details).

The so-called SuSAv2-MEC model adds the 2p2h-MEC contributions described in Section IV to the SuSAv2 approach to produce a model able to describe the CC0 channel in neutrino interactions and the full regime in \((e,e')\) reactions due to the extension of the RMF-based SuSAv2 approach to the inelastic regime. Work is in progress to apply this extension to the weak sector.

A. Comparison with electron scattering data on \(^{12}\text{C}\)

In this Section, we present our analysis for electron-nucleus reactions within the SuSAv2-MEC approach, comparing our theoretical predictions with the existing inclusive \(^{12}\text{C}\) \((e,e')\) experimental data in a wide kinematical region. The QE regime is described in terms of the SuSAv2 model described in Section III E which has been extended to include the complete inelastic spectrum — resonant, nonresonant and deep inelastic scattering — as described in 50, 123, 379. We also discuss the impact of 2p-2h meson-exchange currents following the fully relativistic procedure described in Section IV. Our predictions are also compared with Rosenbluth separated cross section data in terms of longitudinal and transverse contributions. Finally, the extension of the SuSAv2-MEC formalism to other nuclei is also addressed. The capability of the model to describe electron scattering data with accuracy gives us confidence in its subsequent extension, and validity, when applied to recent neutrino oscillation experiments.

In what follows we present the double differential inclusive \(^{12}\text{C}\)\((e,e')\) cross section versus the energy transferred to the nucleus \((\omega)\), confronting our SuSAv2-MEC predictions with the available experimental data 138, 380. Results are shown in Fig. 38 in each panel we show the three separate contributions to the inclusive cross section, namely, quasielastic, 2p-2h MEC and inelastic. We adopt the Bosted and Christy parametrization for the single-nucleon inelastic structure functions 381, 382 which describes DIS, resonant and non-resonant regions, providing a good description of the resonant structures in \((e,e')\) cross sections and covering a wide kinematic region. As shown in 379, the use of other choices such as the Bodek-Ritchie 383, 384 parametrization and models based on Parton Distribution Functions (PDF) 385 leads to very large discrepancies with \((e,e')\) data. For the QE regime, we employ the electromagnetic form factors of the extended Gari-Krumpelmann (GKex) model 386, 388 which improves the commonly used Galster parametrization for \(|Q^2| > 1 \text{ GeV}^2\) (see 371 for details). The sensitivity of the QE results to the different parametrizations of the nucleon form factors has been discussed in 389 and it will be addressed in Section VII F. Additionally, for the Fermi momentum we employ the values shown in 371, namely \(k_F = 228 \text{ MeV}/c\) for \(^{12}\text{C}\).

The comparisons are carried out for a very wide range of kinematics from low-intermediate energies to the highly-inelastic regime. Each panel corresponds to fixed values of the incident electron energy \((E_i)\) and the scattering angle \((\theta_e)\), covering a wide kinematical range. To make it easier to discuss the results to follow, the ordering of the panels has been done according to the corresponding value for the momentum transfer at the quasielastic peak, denoted as \(q_0\). This gives us the value of \(q\) where the maximum in the QE peak appears. However, it is important to point out that as \(\omega\) varies, \(q\) also varies. For completeness, we also include in each panel a curve that shows how the momentum transfer changes with \(\omega\). Results illustrate that at very forward angles the value of \(q\) increases with

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FIG. 38. Comparison of inclusive $^{12}$C(e,e') cross sections and predictions of the QE-SuSAv2 model (long-dashed red line), 2p-2h MEC model (dot-dashed brown line) and inelastic-SuSAv2 model (long dot-dashed orange line). The sum of the three contributions is represented with a solid blue line. The $q$-dependence with $\omega$ is also shown (short-dashed black line). The y-axis on the left represents $d^2\sigma/d\Omega/d\omega$ in nb/GeV/sr, whereas the one on the right represents the $q$ value in GeV/c. 

The energy transfer, whereas this trend tends to reverse at backward angles. Thus for electrons scattered backwards, the $q$-values corresponding to the inelastic process are smaller than those ascribed to the QE regime. However, in this situation the cross section is clearly dominated by the QE peak. On the contrary, at very forward kinematics the inelastic process takes place at larger values of $q$. Thus, the two regimes, QE and inelastic, overlap strongly, the inelastic processes being the main ones responsible for the large cross sections observed at increasing values of $\omega$. Finally, for intermediate scattering angles the behavior of $q$ exhibits a region where it decreases (QE-dominated process), whereas for higher $\omega$ (inelastic regime) the behavior of $q$ reverses and starts to go up. In these situations the QE peak, although significantly overlapped with the inelastic contributions, is clearly visible even for very high electron energies.

The systematic analysis presented in Fig. 38 demonstrates that the present SuSAv2-MEC model provides a very successful description of the whole set of (e,e') data, validating the reliability of our predictions. The positions, widths and heights of the QE peak are nicely reproduced by the model taking into account not only the QE domain but also the contributions given by the 2p-2h MEC terms (around $\sim 10 - 15\%$). Notice also that the dip region is successfully reproduced by the theory. A more detailed analysis of the SuSAv2-MEC model with regard to (e,e') data can be found on [123, 379].

Some comments concerning the “dip” region between the QE and the $\Delta$ peaks are also in order. This is the region where the QE and the inelastic contributions overlap the most and where FSI effects that modify in a significant way the tail of the QE curve at large $\omega$-values can introduce an important impact. Moreover, the role of the 2p-2h MEC effects is essential because its maximum contribution occurs in this region. Thus, only a realistic calculation of these ingredients beyond the IA can describe successfully the behavior of the cross section.

To conclude, the accordance between theory and data in the inelastic regime, where a wide variety of effects are taken into account, also gives us a great confidence in the reliability of our calculations. The inelastic part of the cross section is dominated by the $\Delta$-peak that mainly contributes to the transverse response function. At low electron scattering angles the longitudinal QE response function dominates the cross section and the inelastic
contribution is smaller (as will be shown in Section VII B). The opposite holds at large scattering angles, where the Δ-peak contribution is important. On the other hand, for increasing values of the transferred momentum the peaks corresponding to the Δ and QE domains become closer, and their overlap increases significantly. This general behaviour is clearly shown by our predictions compared with data. In those kinematical situations where inelastic processes are expected to be important, our results for the QE peak are clearly below the data which is compensated by the larger inelastic contribution. On the contrary, when the inelastic contributions are expected to be small, the QE theoretical predictions get closer to data. Note also the excellent agreement in some situations (bottom panels on Fig. 38) even being aware of the limitations and particular difficulties in order to obtain phenomenological fits of the inelastic structure functions, and the reduced cross sections at these kinematics.

B. Separate L/T analysis

The separate analysis of the longitudinal and transverse response functions of $^{12}$C is presented in Fig. 39. We compare our predictions with data taken from Jourdan \cite{26} based on a Rosenbluth separation of the $(e,e')$ world data. In each case we isolate the contributions corresponding to the QE, inelastic and 2p-2h MEC sectors. Three kinematical situations corresponding to fixed values of the momentum transfer have been considered in Fig. 39: $q = 300$ MeV/c (top panel), 380 MeV/c (middle) and 570 MeV/c (bottom). As observed, the longitudinal channel is totally dominated by the QE contribution. Only at very large values of $\omega$ does the inelastic process enter giving rise to a minor response, whereas the effects due to 2p-2h MEC are negligible. This result is in accordance with previous work \cite{26,102,390,391}, and it clearly shows that the longitudinal response is basically due to the IA. On the contrary, the transverse sector shows an important sensitivity to MEC and inelastic processes. Note that the inelastic transverse response gives rise to the high tail shown by data at large $\omega$-values, whereas the 2p-2h MEC can modify significantly the transverse response in the dip region as well as in the maximum of the QE peak. It is also worth mentioning that the natural enhancement in the transverse response arising from the RMF model is necessary in order to reproduce the separate L/T data, rejecting the idea of 0-th kind scaling.

From results in Fig. 39 we observe that the model leads to a reasonable agreement with data in both channels, although some discrepancies also emerge. Notice that the longitudinal prediction at $q = 300$ MeV/c ($q = 380$ MeV/c) overestimates data by $\sim 12\%$ ($\sim 15\%$). It is important to point out that SuSAv2 is based on the existence of the scaling phenomenon for $(e,e')$ data, and this is completely fulfilled when the value of $q$ is large enough ($q \geq 400$ MeV/c). Therefore, the extension of the superscaling approach to low $q$-values is not well established even though a good agreement at low kinematics has been achieved in the previous section. Furthermore, the minor discrepancies observed may also be due to the specific Rosenbluth separation method used in \cite{26}, which introduces some level of model dependence through $y$-scaling assumptions and the treatment of radiative corrections.

C. Extension of the SuSAv2-MEC model to other nuclei

Once analyzed the capability of the SuSAv2 model to reproduce the $^{12}$C$(e,e')$ measurements, we extend the previous formalism to the analysis of electron scattering data on other nuclei. For this purpose, no differences in the scaling functions are assumed for the different nuclei except for the values used for the Fermi momentum and energy shift (see \cite{50} for details). The use of the same scaling functions for different nuclear systems is consistent with the property of scaling of second type, i.e., independence of the scaling function with the nucleus, and it also follows from...
FIG. 40. Analysis of second kind scaling within the RMF model for $^{12}$C, $^{16}$O and $^{40}$Ca.

the theoretical predictions provided by the RMF and RPWIA models on which SuSAv2 relies. This has been studied in detail in previous works (see \[36, 39, 51, 55, 57\]) where the electromagnetic and weak scaling functions evaluated with the RMF and RPWIA approaches have been compared for $^{12}$C, $^{16}$O and $^{40}$Ca. Other theoretical analyses for these nuclei can also be found in \[392\] and \[102\] in the framework of the RGF and CRPA models, respectively. In Figure 40 we compare the general RMF scaling functions for these nuclei, which exhibit no remarkable differences in terms of the nuclear species. Therefore, we apply the reference scaling functions for $^{12}$C to the analysis of QE and inelastic regimes in other nuclei. The extension of the 2p-2h MEC contributions to other targets was previously described in Section IV.

In the case of $^{16}$O, the $k_F$-value selected ($k_F = 230$ MeV/c) is also consistent with the general trend observed in \[107\], i.e., an increase of the Fermi momentum with the nuclear density. Additionally, an offset of -4 MeV is applied with regard to the $^{12}$C $E_{\text{shift}}$-values. The choice of the $k_F$- and $E_{\text{shift}}$-values provides a consistent analysis of the superscaling behavior in the deep scaling region below the QE peak, as shown in \[379\], and more importantly, it also provides a good agreement with electron scattering data.

Thus, in accordance with the previous analysis, we show in Fig. 41 the predictions of the SuSAv2-MEC model for six different kinematical situations, corresponding to the available $(e,e')$ data on $^{16}$O. In all the cases we present the separate contributions for the QE, 2p-2h MEC and inelastic regimes. The 2p-2h MEC responses are extrapolated from the exact calculation performed for $^{12}$C assuming the scaling law $R_{2p2h} \sim A k_F^2$ deduced in Ref. 393 as well as in Section IV. The inclusive cross sections are given versus the transferred energy ($\omega$), and each panel corresponds to fixed values of the incident electron energy ($E_i$) and the scattering angle ($\theta$). Whereas the latter is fixed to 32° except for one case (center panel on the top, i.e, $\theta = 37.1^\circ$ \[394\]), the electron energy values run from 700 MeV (left-top panel), where the QE peak dominates, to 1500 MeV (right-bottom) with the inelastic channel giving a very significant contribution. This is due to the values of the transferred momentum $q$ involved in each situation. Although $q$ is not fixed in each of the panels, i.e., it varies as $\omega$ also varies, the range of $q$-values allowed by the kinematics increases very significantly as the electron energy grows up (for fixed scattering angles). Thus, for higher $E_i$ the two regimes, QE and inelastic, overlap strongly, the inelastic processes being responsible for the large cross sections at increasing values of $\omega$. This different range of $q$-values spanned in each panel also explains the relative role played by the RMF versus the RPWIA approaches.

As observed, the SuSAv2-MEC predictions are in very good accordance with data for all kinematical situations. Although the relative role of the 2p2h-MEC effects is rather modest compared with the QE and inelastic contributions, at the peak of the 2p2h response the three contributions are comparable in size, as also observed for $^{12}$C.

For completeness, we also present in Figure 42 the calculations for the heavier target $^{40}$Ca ($k_F = 241$ MeV/c, $E_{\text{shift}} = 28$ MeV within the RFG formalism \[102\]) where the comparison with data is again very precise from forward to very backward angles. The analysis of these results is relevant because of the similarity with $^{40}$Ar, a target of interest for recent and forthcoming neutrino oscillation experiments. Note also that the 2p-2h MEC contributions are more prominent for $^{40}$Ca than for $^{12}$C and $^{16}$O with respect to the QE regime due to the different $k_F$ dependence of the QE and 2p-2h MEC contributions, $A / k_F$ and $A k_F^2$, respectively 395.
FIG. 41. Comparison of inclusive $^{16}\text{O} (e,e')$ cross sections and predictions of the SuSAv2-MEC model. The separate contributions of the pure QE response (dashed line), the 2p-2h MEC (dot-dashed), inelastic (double-dot dashed) are displayed. The sum of the three contributions is represented with a solid blue line. The $y$ axis represents $d^2\sigma/d\Omega/d\omega$ in nb/GeV/sr. Data from Refs. [67] and [394].

FIG. 42. Comparison of inclusive $^{40}\text{Ca} (e,e')$ cross sections and predictions of the SuSAv2-MEC model. The separate contributions of the pure QE response (dashed line), the 2p-2h MEC (dot-dashed), inelastic (double-dot dashed) are displayed. The sum of the three contributions is represented with a solid blue line. The $y$ axis represents $d^2\sigma/d\Omega/d\omega$ in nb/GeV/sr. Data from Refs. [396] and [397].
D. Comparison with recent JLab data

In addition to the previous analyses, we have also tested the validity of the SuSAv2-MEC model and the scaling rules applied when extrapolating to different nuclei through the analysis of the recent JLab data \[398\] for inclusive electron scattering data on three different targets (C, Ar and Ti) \[399\]. As observed in Fig. 43, the agreement is very good over the full energy spectrum, with some discrepancy seen only in the deep inelastic region. The 2p2h response, peaked in the dip region between the QE and \(\Delta\)-resonance peak, is essential to reproduce the data. We also analyze in Fig. 44 the \(k_F\) (Fermi momentum) dependence of the data in terms of scaling of second kind, showing that the 2p2h response scales very differently from the quasielastic one, in full accord with what is predicted by the model. As observed, scaling of second kind only works in the QE peak, then it breaks down because non-impulsive contributions (2p2h) and inelastic channels come into play, being consistent with previous analyses \[21, 22\]. We also present the superscaling function \(f(\psi')\) but divided by \(\eta_F^3\), where \(\eta_F \equiv k_F/m_N\). As shown, scaling is highly broken in the QE peak (results for carbon are significantly higher), however, results collapse into a single curve within the dip region. This result confirms our previous study in \[132\], where we predicted that 2p2h response scales as \(k_F^3\). It is important to point out that the minimum in the cross section shown in Fig. 43 corresponds to the maximum in the 2p2h contribution (see Fig. 43). Although contributions from the QE and inelastic domains also enter in the dip region, and this may at some level break the scaling behavior, results in bottom panels of Fig. 44 strongly reinforce our confidence in the validity of our 2p2h-MEC model, whose predictions are very successfully confirmed by the experimental data. It is also interesting to note that the same type of scaling, i.e., \(f(\psi')/\eta_F^3\), seems to work reasonably well not only in the dip region but also in the resonance and DIS domains. Further studies on the origin of this behavior are underway. In overall, these results represent a valuable test of the applicability of the model to neutrino scattering processes on different nuclei. A more detailed analysis about these data can be found in \[399\].

E. Comparison with neutrino scattering data

In this Section we present our theoretical predictions compared with charged-current neutrino scattering data from different collaborations, mainly T2K and MINERvA but also analyzing MiniBooNE kinematics. The SuSAv2-MEC model, that has already proven to describe accurately \((e, e')\) data (see previous section), is here applied to the analysis...
FIG. 45. The T2K flux-integrated CCQE and 2p2h double-differential cross-section for neutrino for scattering on $^{12}\text{C}$, within the SuSAv2-MEC model at T2K kinematics in units of $10^{-39}\text{cm}^2/\text{GeV per nucleon}$. The CC0π T2K data are from Ref. [400].

of recent neutrino data with the aim of showing its capability to describe successfully a large variety of experimental measurements covering a wide range of kinematics. Our study is mainly restricted to the “quasielastic-like” regime where the impulse approximation used to describe the one-nucleon knockout process in addition to the effects linked to the 2p-2h meson-exchange currents play a major role.

In Figs. 45 and 46 we explore the similarities and differences between the T2K CC0π (anti)neutrino scattering on $^{12}\text{C}$ and $^{16}\text{O}$ and their comparison with the SuSAv2-MEC model. The CC0π scattering is defined, equivalently to the CCQE-like one, as the process where no pions are detected in the final state. As already explained in previous sections, quasielastic (QE) scattering and multi-nucleon excitations dominated by 2p-2h MEC contributions should be included in the analysis together with other minor effects such as pion-absorption processes in the nucleus that can mimic a CCQE-like event. However, as it will be shown in Section VIII the pion-absorption effects at T2K kinematics are not particularly relevant at T2K kinematics. Thus, these two main mechanisms, QE and 2p2h, have in general a different dependence upon the nuclear species, namely they scale differently with the nuclear density, as previously analyzed.

As observed in Figs. 45 and 46, the SuSAv2-MEC model is capable of reproducing the T2K data for both nuclei which can help to disentangle how nuclear effects enter in the analysis of the T2K experiment as well as to reduce nuclear-medium uncertainties. A more detailed exploration of the C/O differences will be carried out soon with the forthcoming T2K data analysis on neutrino-antineutrino asymmetry for carbon and oxygen.

However, due to the present level of experimental accuracy and the large error bands shown by T2K data in most of the kinematical situations, both the pure QE as well as the total, QE+2p-2h MEC, predictions are in accordance with the experiment. It is interesting to point out the results for the most forward angles, i.e., the panel on the right-bottom corner. Notice that the QE and 2p-2h MEC contributions are stabilized to values different from zero for increasing muon momenta as a consequence of the high energy tail of the T2K neutrino flux. This is at variance with all remaining situations where the cross sections decrease significantly as the muon momentum $p_\mu$ goes up.

Next, in Fig. 48 we compare the results of the SuSAv2 model including meson-exchange currents (MEC) with the recent measurement of the quasielastic-like double differential antineutrino cross section on hydrocarbon (CH) performed by the MINERvA Collaboration. The relativistic nature of the model makes it suitable to describe these data, which correspond to a mean beam energy of 3.5 GeV. The standard SuSAv2 model predictions agree well with the data without needing any additional or tuned parameter.

Going into detail, in Fig. 48 we show the double differential cross section of muonic antineutrino on hydrocarbon as a function of the transverse (with respect to the antineutrino beam) momentum of the outgoing muon, in bins of the
muon longitudinal momentum. For the data we use the same nomenclature employed in the experimental paper \cite{403}. The “QE-like” experimental points include, besides pure quasielastic contributions, events that have post-FSI final states without mesons, prompt photons above nuclear de-excitation energies, heavy baryons, or protons above the proton tracking kinetic energy threshold of 120 MeV, thus including zero-meson final states arising from resonant pion production followed by pion absorption in the nucleus and from interactions on multinucleon states. This is similar to the so-called CC0π definitions used by other experiments as T2K. On the contrary, the “CCQE” signal
(also defined in other experiments as “CCQE-like”) corresponds to events initially generated in the GENIE neutrino interaction event generator \([276]\) as quasi-elastic (that is, no resonant or deep inelastic scatters, but including scatters from nucleons in correlated pairs with zero-meson final states), regardless of the final-state particles produced, thus including CCQE and 2p2h interactions. The difference between the two data sets, mainly due to pion production plus re-absorption, varies between \(\sim 15\%\) and \(\sim 5\%\) depending on the kinematics. According to MINERvA’s acceptance, the muon scattering angle is limited to \(\theta_\mu < 20^\circ\) as well as the muon kinematics (1.5 GeV < \(p_T\) < 1.5 GeV) in both experimental and theoretical results, leading to a significant phase-space restriction for large energy and momentum transfer to the nuclear target.

The theoretical curves in Fig. 48 correspond to the aforementioned SuSAv2 model and include 2p2h excitations induced by meson exchange currents. The antineutrino hydrogen contribution in the cross sections only enters through the 1p1h channel and has been evaluated by computing the elastic antineutrino-proton cross section. The present calculation does not include processes corresponding to pion emission followed by re-absorption inside the nucleus. Therefore the curves are meant to be compared with the “CCQE” data rather than with the “QE-like” ones. However, we also display the QE-like cross sections, to illustrate MINERvA’s estimation of the magnitude of the QE-like resonance component among other minor effects.

At MINERvA kinematics, the SuSAv2-MEC results seems to be compatible with the MINERvA GENIE simulations although some differences can be observed, as detailed in \([402]\). Overall, the comparison with MINERvA data is not very different for the two models, as we have checked by performing a \(\chi^2\) test in \([402]\) where we have obtained \(\chi^2/d.o.f=1.79\) for SuSAv2-MEC and \(\chi^2/d.o.f=1.58\) for MnvGENIE. Thus, the \(\chi^2/d.o.f\) values obtained using the SuSAv2-MEC model turn out to be compatible with the MnvGENIE ones and with MINERvA data.

Here we also show a former analysis of \(d\sigma/dQ_{QE}^2\) at MINERvA kinematics in Fig. 49. The left panel in Fig. 49 refers to \(\nu_\mu\)–\(^{12}\)C whereas the right panel contains predictions and data for \(\bar{\nu}_\mu\)–CH. The mean energy of the MINER\(\nu\) muonic flux is much higher than the T2K one, about 3.5 GeV for both \(\nu_\mu\) and \(\bar{\nu}_\mu\). As observed, significant contributions of the 2p-2h MEC, of the order of \(\sim 35\%\) - \(\sim 40\%\) \((\sim 25\%\) at the maxima for \(\bar{\nu}_\mu\) \((\bar{\nu}_\mu\)) are needed in order to reproduce the experimental data that correspond to the analysis performed by the MINER\(\nu\)A collaboration \([403, 404]\) in 2016.

For completeness, it is worth mentioning that in the specific conditions of MINER\(\nu\)A, it clearly appears that, even if the neutrino energy is as large as 3 GeV, the process is largely dominated by relatively small energy and momentum transfer, namely, \(\omega < 500\) MeV, \(q < 1000\) MeV, whereas contributions below \(\omega < 50\) MeV, \(q < 200\) MeV govern the lowest \(Q_{QE}^2\) region. More specific details can be found in Ref. \([124, 407]\).
F. Form Factors analysis

As already considered in the analysis of \((e,e')\) reactions, here we also adopt the electromagnetic nucleon form factors of the extended Gari-Krumpelmann (GKeX) model \cite{386,388} for the vector CC current entering into neutrino cross sections. As described in \cite{379}, this prescription improves the commonly used Galster parametrization at \(|Q^2| > 1\) GeV\(^2\). For completeness, we show in Figure 50 the sensitivity of the total CCQE neutrino cross section within the SuSA approach for the different up-to-date parametrizations of the nucleon form factors (see Refs. \cite{12,389} for details) where all of them are essentially equivalent for the MiniBooNE kinematics, while some difference emerges at the energies of the NOMAD experiment, which implies larger \(|Q^2|\) values. Similar comments also apply to the SuSAv2 model.

Regarding the axial contributions, we employ the commonly used dipole axial nucleon form factor described in \cite{379} where a comparison between dipole and monopole axial form factor is shown together with a discussion on the axial coupling \(g_A\) parameter. More recently, we have also compared the widely used dipole axial form factor with other choices based on the so-called the two-component model \cite{408,409}. In this sense, a joint fit to neutrino-nucleon

![Figure 49](image.png)

**FIG. 49.** Flux-folded \(\nu_\mu - ^{12}\text{C}\) CCQE (upper panel) and \(\bar{\nu}_\mu - \text{CH}\) (lower panel) scattering cross section per target nucleon as a function of \(Q^2_{QE}\) and evaluated in the SuSAv2 and SuSAv2-MEC models. MINER\(\nu\)A data are from \cite{405}.

![Figure 50](image.png)

**FIG. 50.** CCQE \(\nu_\mu - ^{12}\text{C}\) cross section per nucleon evaluated in the SuSA model for various parametrizations of the nucleon electromagnetic form factors \cite{389}. A sub-panel zooming in the region near the maximum is inserted on the top.
scattering and pion electroproduction data has been performed to evaluate the nucleon axial form factor in the two-component model consisting of a three-quark intrinsic structure surrounded by a meson cloud. Further constrains on the model are obtained by re-evaluating the electromagnetic form factor using electron scattering data. The results of the axial form factor show sizable differences at some kinematics with respect to the widely used dipole model. The impact of such changes on the CCQE neutrino-nucleus cross-section is evaluated in the SuSAv2 nuclear model in comparison with recent T2K and MINERvA measurements in Figs. 51 and 52. The two-nucleon component approach was successful in describing the four-nucleon electromagnetic form factors (electric and magnetic, for proton and for neutron) [409–411], the strange form factors of the proton [412] and was applied to the deuteron as well [413]. Advantages of this model are that it contains a limited number of parameters and can be applied both in the space- and time-like regions. The extension to axial form factors has been done in [414].

Following Ref. [408], the nuclear axial form factor is parametrized in the two-component approach as:

\[
G_A(Q^2) = G_A(0) g(Q^2) \left[ 1 - \alpha + \alpha \frac{m_A^2}{m_A^2 + Q^2} \right],
\]

\[
g(Q^2) = (1 + \gamma Q^2)^{-2},
\]

(185)

where \( Q^2 > 0 \) in the space-like region and \( \alpha \) is a fitting parameter which corresponds to the coupling of the photon with an axial meson. One can fix \( m_A = 1.230 \) GeV, corresponding to the mass of the axial meson \( a_1(1260) \) with \( J^G(J^{PC}) = 1^- (1^{++}) \). The form factor \( g(Q^2) \) describes the coupling to the intrinsic structure (three valence quarks) of the nucleon. Note that setting \( \alpha = 1 \) and \( \gamma = 0 \), the usual dipole axial functional form is recovered.

The results of fit to pion electroproduction data and neutrino scattering data for different extractions of the pion data can be seen in [415]. Here we will focus on the two most extreme cases, the Soft Pion dataset and the PCAC (Partially Conserved Axial Current) one, also detailed in [415]. Thus the impact of the axial form factor choice on the CCQE-like cross section is illustrated in Figs. 51 and 52 for T2K and MINERvA kinematics, respectively.

In Fig. 51 the T2K neutrino and antineutrino double-differential cross-sections and the corresponding asymmetry

\[
\text{asymmetry} = \frac{d^2\sigma_\nu - d^2\sigma_{\bar{\nu}}}{d^2\sigma_\nu + d^2\sigma_{\bar{\nu}}}
\]

(186)

are shown for different axial form factors. The largest differences between neutrino cross-sections evaluated with different form factors is about 5%. By mapping the muon kinematics \( (p_\mu, \theta_\mu) \) into \( Q^2 \) on the basis of [415], the largest difference with respect to the cross-section with dipole form factor, appears always in correspondence of \( Q^2 \sim 0.5 \text{ GeV}^2 \) for SoftPion. For forward angles, the relevant kinematic region is \( Q^2 \sim 0.5 \text{ GeV}^2 \) where the cross-section is mostly unaffected by form factor differences. For backward angles, the \( Q^2 \sim 0.5 \text{ GeV}^2 \) region corresponds instead exactly to the region of larger cross-section with intermediate muon momentum, thus the impact of form factors difference is larger, as detailed on [415]. In the backward region, the form factor differences can reach 5%. In such region the effect in the antineutrino case is even larger, up to 10%. Still, in the neutrino-antineutrino asymmetry the effect is at % level. The region \( Q^2 > 1 \text{ GeV} \), where the different axial form factors depart from each other sizeably, is negligible in T2K data.

The effects of axial form factors at MINERvA kinematics are shown via the analysis of the double differential cross-section as a function of \( p_T, p_L \) in Fig 52. The region of \( Q^2 \sim 0.5 \text{ GeV}^2 \) shows differences of the order of 5%, similarly to T2K, while in the region of high \( p_T \) and lower cross-section effects up to 10% and above can be observed.

In overall the agreement of the SuSAv2-MEC model with data, using the different axial form factors, is positive but the experimental uncertainties on T2K and MINERvA measurements do not allow yet to clearly discriminate between the various evaluations. It is interesting to note that, in the model considered here, the form factor effects have a different \( Q^2 \) dependence than the one of 2p2h, as well as a different neutrino/antineutrino dependence, making the disentangling of nucleon and nuclear effects feasible in future with higher statistics measurements. The feasibility of this approach relies on the capability of exploiting external data to drive the \( Q^2 \) dependence of the form factor. For this reason, the investigation of the earlier data of pion electro-production, as shown in [415] is of primary importance.

G. Relevance of L/T channels for neutrino reactions

In this section we study in detail the relevance of the different longitudinal and transverse channels that contribute to the QE and 2p-2h MEC at MiniBooNE kinematics, also accounting for the corresponding axial and vector contributions which arise from the hadronic currents. The conclusions extracted from this analysis are roughly extensible to the ones from the T2K and MINERvA experiments, whose relevant kinematic regions are not significantly different.
An analysis on the different channels for the 2p-2h MEC nuclear responses and the total cross section was addressed in [124, 379], showing a predominance of the transverse responses over the longitudinal ones. In the latter the pure vector contributions were negligible in comparison with the axial ones. Moreover, the separate transverse channels, $T_{VV}, T_{AA}$ and $T'_{VA}$, while showing some remarkable differences for different $q$ values, contribute in a similar way to the total cross section. This is due to the relevant kinematic regions explored which goes from 0.3 GeV/c up to 1 GeV/c in $q$ and from 0.3 GeV to 0.8 GeV in $\omega$ [124, 379]. Here we investigate the relevance of the different channels at kinematics relevant for MiniBooNE, T2K and MINERvA experiments. In Fig. 53 the separate 2p-2h MEC contributions to the different channels ($L, T_{VV}, T_{AA}$ and $T'_{VA}$) corresponding to the MiniBooNE double differential cross section at different bins of the muon scattering angle are presented.

Results in Fig. 53 show the differences between the $T_{AA}$ and $T_{VV}$ contributions, the latter being shifted to higher $T_{AA}$ values by about 50 MeV for all angular bins. At very forward angles, i.e., lower $q$-values, the global magnitude of the $AA$ channel is greater than the $VV$ one, in accordance with the results observed in [379], where the $T_{VV}$ and $T_{AA}$ responses differ roughly by a factor 2 at the maximum at $q$ of the order of 400 MeV/c. Concerning the interference $T'_{VA}$ component, its magnitude is not particularly different from the $VV$ and $AA$ ones at very forward angles, being on the contrary the most relevant term at larger angles. Finally, although the longitudinal channel gives the smallest global contribution, its role is essential in order to interpret antineutrino scattering at backward angles. This is a consequence of the negative $T'_{VA}$ term for antineutrino reactions that almost cancels out the $T_{VV} + T_{AA}$ contribution.
The conclusions that can be extracted from these results launch a warning to those 2p2h models which neglect the longitudinal contributions in their analysis as well as to those ones who extrapolate the results from electrons (purely vector responses) to the weak ones (vector+axial), making that approaches rather questionable in some cases. The same conclusions apply to the recent MINERvA results shown in Fig. \ref{fig:MINERvA_results}

The conclusions extracted from the previous analysis on the 2p-2h MEC cross section also apply for the separate QE contributions to neutrino and antineutrino cross sections. The different QE channels are analyzed for the MiniBooNE double differential cross sections in Fig. \ref{fig:MiniBooNE_QE_channels} where the transverse contribution predominates at all kinematics whilst the net longitudinal channel, even not being a relevant contribution, is essential to describe antineutrino data at backward kinematics together with the 2p-2h longitudinal one.

In Fig. \ref{fig:MiniBooNE_total_channels} we show the breakdown of the total integrated neutrino cross sections into $L(= L_{VV} + L_{AA})$, $T(= T_{VV} + T_{AA})$, $T_{VV}$, $T_{AA}$ and $T^\prime_{VA}$ contributions, with the last occurring as a positive (constructive) term in the neutrino cross section and a negative (destructive) term in the antineutrino one. The sign of the $T^\prime_{VA}$ channel represents the main difference between the total neutrino and antineutrino cross sections. In addition to the opposite sign in the $VA$ response, some minor differences between neutrino and antineutrino cross sections arise from the Coulomb distortion of the emitted lepton and the different nuclei involved in the CC neutrino (nitrogen) and antineutrino (boron) scattering processes on carbon. We also notice that below 1 GeV the $T^\prime_{VA}$ response is higher than the $T_{VV}$ one but of the same order as the $T_{AA}$ one. Note that the maximum of the $VA$ channel is around the peak of the MiniBooNE neutrino flux. On the contrary, VA contribution to cross section is negligible at energies above 10 GeV as a consequence of

FIG. 52. MINERvA flux-integrated double-differential cross-section for different axial form factors, as a function of the muon transverse and longitudinal momentum, for neutrino (first row), antineutrino (second row) and the neutrino-antineutrino asymmetry (third row). Double differential cross sections are shown in units of $10^{-39}$ cm$^2$/GeV$^2$ per nucleon.
the small values of the axial form factor $G_A$ and the lepton factor $V_{T'}$ at high $E_\nu$ and $|Q^2|$ (see \cite{379} for details). This is also in agreement with some previous QE results \cite{389}. As a consequence, for very high $\nu_\mu$ ($\bar{\nu}_\mu$) energies (above $\sim 10$ GeV) the total cross section for neutrinos and antineutrinos is very similar. Only the $L$ and $T$ channels contribute for the higher values explored by NOMAD experiment. On the contrary, in the region explored by the MiniBooNE collaboration, the main contributions come from the two transverse $T$ and $T'$ channels.
FIG. 54. (Color online) As Fig.1, but showing the separate contribution of the pure transverse MEC (dashed curves) to also stress the relevance of the longitudinal MEC channel.

FIG. 55. Separation into components of the MiniBooNE CCQE $\nu_{\mu}$ (top panel) and $\bar{\nu}_{\mu}$ (bottom panel) double-differential cross section per nucleon displayed versus $T_{\mu}$ for various bins of $\cos \theta_{\mu}$ within the SuSAv2 approach. The MiniBooNE data \cite{416, 417} are also shown for reference.

H. Implementation of the SuSAv2-MEC and RMF models in MC event generators and extension to semi-inclusive processes

In Section \textbf{VII}, the SuSAv2-MEC model has been shown to be capable of reproducing the nuclear dynamics and superscaling properties observed in ($e, e'$) reactions which serves as a stringent test for nuclear models, whilst also providing an accurate description of existing neutrino data. Up to now, SuSAv2-MEC is the only fully relativistic model that can be extended without kinematical restrictions or further approximations to the full-energy range of interest for present and future neutrino experiments. This has motivated the implementation of SuSAv2-MEC 1p1h and 2p2h contributions in the GENIEv3 MonteCarlo neutrino interaction simulation \cite{419} in order to use it to better characterise nuclear effects in T2K and MINERvA neutrino scattering cross-section measurements.

Accordingly, we present in this section the implementation and validation of the SuSAv2-MEC 1p1h and 2p2h models in the GENIE neutrino-nucleus interaction event generator and a comparison of the subsequent predictions to measurements of lepton and hadron kinematics from the T2K experiment. These predictions are also compared to those of other available models in GENIE. We additionally compare the semi-inclusive predictions of the implemented 1p1h model to those of the microscopic model on which SuSAv2 is based - Relativistic Mean Field (RMF) - to begin to test the validity of widely-used ‘factorisation’ assumptions employed by generators to predict hadron kinematics from inclusive input models. The results highlight that a more precise treatment of hadron kinematics in generators is essential in order to attain the few-% level uncertainty on neutrino interactions necessary for the next generation of accelerator-based long-baseline neutrino oscillation experiments. More details about this analysis and the implementation can be found in \cite{419}. 


FIG. 56. Separation into components of the CCQE $\nu_\mu$ cross section per nucleon on $^{12}$C displayed versus neutrino energy $E_\nu$ within the SuSAv2 approach. The MiniBooNE \[416\] and NOMAD \[418\] data are also shown for reference.

The recent experimental interest on more exclusive measurements is related to the information about the final state nucleons that they provide, such as those which have recently been performed by T2K \[420\] and Minerva \[421\], which have been demonstrated to have a much more acute sensitivity to the different nuclear effects involved in neutrino-nucleus interactions. Unfortunately a comparison of these measurements directly to microscopic models requires semi-inclusive or exclusive predictions which the majority of models are not able to make, as they simplify their calculations by integrating over outgoing nucleon kinematics. An exception to this is the RMF model, used to construct the SuSAv2 predictions, which is capable of semi-inclusive predictions for neutrino reactions.\[14\] As described in \[419\], the simulations used by experiments circumvent this limitation by factorising the leptonic and hadronic components of the interaction. Among other approximations, this approach relies strongly on a semi-classical description of FSI and the distribution of initial state nucleon kinematics seen by the probe being independent of its energy and momentum transfer.

The implementation of the SuSAv2 1p1h model in GENIE provides a first opportunity to test the factorisation approach in event generators as well as to compare with other models available in event generators. In Figs. 58 and 59 we show a comparison of the SuSAv2 and Valencia model predictions (1p1h and 2p2h) as implemented in GENIE on top of GENIE’s Berger-Sehgal pion production prediction for T2K inclusive CC0π and ‘semi-semi-inclusive’ CC0πNp results, being the latter a semi-inclusive cross section where the final state proton is below 500 MeV/c. This clearly shows that the implemented Valencia and SuSA models differ substantially, with only the SuSA model able to describe the very forward data and the Valencia model describing the mid-angle data a little better. The discrepancies between the model and data is consistent between the inclusive and semi-inclusive results, suggesting that they at least partially stem from the underlying inclusive cross section model.

Beyond the inclusive comparison, the ‘semi-semi-inclusive’ predictions within the kinematic region where SuSAv2 is a good description of RMF allows us to study the validity of the factorisation approach used in event generators. Here it can be seen that the implementation with both the kinematic-dependent binding energy and with FSI is closest to reproducing the RMF microscopic model prediction, but still appears to peak at too low muon momentum and also fails to describe the higher momentum region. It can also be seen that variations to the hadronic component of the interaction cause substantial alterations to the predictions, highlighting the role of these nonphysical freedoms available within the factorisation approach. Further work will focus on more stringent tests through the implementation of the RMF spectral function into event generators and by exploring the predictions in a wider region of hadronic kinematic phase-space (ideally using a fully semi-inclusive version of RMF).

\[14\] The RMF model has proven its validity to address exclusive predictions for electron scattering \[41\] and work is underway to fully extend it to neutrino reactions.
FIG. 57. Comparisons of single differential CC0π muon-neutrino cross sections on Carbon at T2K kinematics as a function of the muon kinematics when there are no protons (with momenta above 500 MeV). Two 1p1h predictions are used (one from RMF, the other from SuSAv2 implemented in GENIE), in addition to the SuSAv2 2p2h and Berger-Sehgal pion absorption contributions from GENIE. Goodness of fit are calculated to be $\chi^2_{RMF} = 171.87$ (59 bins) and $\chi^2_{SuSA} = 168.92$ (60 bins), where the latter includes a single extra bin from -1.0 to -0.3 $\cos \theta$ (not shown). The data points are taken from [420].

The T2K semi-inclusive CC0π measurement of interactions with protons less than 500 MeV [420] provides an opportunity to compare the RMF ‘semi-semi-inclusive’ model predictions to data, which is shown in Fig. 57 alongside the SuSA-GENIE predictions using the factorisation approach. In order to make this comparison the RMF predictions are added to the SuSAv2-MEC (2p2h) and pion-absorption predictions from GENIE (for pion production the Berger-Sehgal model was used [422]). In general, we observe a fair agreement of both RMF+GENIE (SuSAv2-2p2h+π-abs) and GENIE (SuSAv2-1p1h+SuSAv2-2p2h+π-abs). The overestimation of data at very forward angles by the SuSAv2-GENIE is ascribed to the aforementioned low energy transfer scaling violations absent in the SuSAv2-model but present in RMF, thereby explaining the better agreement achieved with the latter. On the contrary, the larger results from SuSAv2-1p1h at very backward angles compared to RMF are related to the previously discussed FSI treatment alterations. In general, it is clear that RMF performs better within the most forward angular bins (where additional RMF effects are most important). The recently developed Energy-Dependent RMF (ED-RMF) model [423, 424], which keeps the original RMF potentials at low kinematics but makes them softer for increasing nucleon momenta, following the SuSAv2 approach, will solve the limitations of the SuSAv2 model at forward angles while solving the drawbacks of the RMF at very high kinematics, constituting a promising candidate to be implemented in neutrino event generators. This will help to reduce nuclear-medium uncertainties and to improve systematics in neutrino oscillation experiments.

VIII. Conclusions

The advent of next generation neutrino oscillation experiments demands more and more sophisticated nuclear modelling, necessary to extract significant information on the properties of neutrinos and physics beyond the Standard Model of electroweak interactions. Nuclear theory plays a crucial role in these analyses, which strongly rely on the accurate description of neutrino interactions with the detector, made of medium/heavy nuclei, implemented in Monte Carlo generators.

Contrary to the case of electron scattering, where many precise data from previous experiments exist, neutrino scattering data are rare and have large error bars. Beyond that, the identification of the primary reaction suffers from the absence of monochromatic neutrino beams. Therefore, electron scattering represents not only a necessary test but
FIG. 58. Comparison of the T2K CC0π measurement of muon-neutrino interactions on Carbon with the SuSAv2 and Valencia models (1p1h+2p2h) implemented in GENIE with additional pion-absorption effects (from GENIE’s Berger-Sehgal model). The top plots are the SuSAv2 predictions whilst the Valencia ones are below. The data points are taken from [420].

FIG. 59. Comparison of the T2K CC0π measurement of muon-neutrino interactions on Carbon where there are no protons above 500 MeV with the SuSAv2 and Valencia models (1p1h+2p2h) implemented in GENIE with additional pion-absorption effects (from GENIE’s Berger-Sehgal model). The top plots are the SuSAv2 predictions whilst the Valencia ones are below. The data points are taken from [420].

also a useful tool to understand and control nuclear effects in oscillation experiments.

In this paper we have reviewed the current status of the description of lepton-nucleus scattering at different kinematics, going from the quasi-elastic up to the deep-inelastic regime. We have focused in particular on the SuSAv2/MEC model, developed and improved by our group, collecting the main results obtained in the last few years. The model is based on the Relativistic Mean Field description of the nucleus, complemented with the contribution of two-body currents. We have shown that the model is capable of describing in a very satisfactory way both electron and neutrino
data in the case of inclusive reactions, in which only the outgoing lepton is detected. This outcome, although reassuring, is not sufficient to guarantee the required precision - of a few % - on the description of nuclear effects. Work is in progress on the study of exclusive reactions, where one or more hadrons are detected in coincidence with the lepton, in the same relativistic framework. The extension of nuclear models to semi-inclusive reactions is a challenge to be faced by theorists working in the field. A proper description of the hadrons and mesons in the final state will be essential for the next-generation of neutrino experiments. This requires having a reasonable control on the reconstruction of the energy neutrino which can be only achieved by analyzing the kinematics of the final particles.

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A. Scaling and superscaling: definitions

In this appendix we recall the definitions of the scaling and superscaling functions, along with various choices for the scaling variable \[21, 23, 25, 426\].

Let us consider inclusive electron scattering off a nucleus, namely the process where only the outgoing electron is detected. We work in the target rest frame and we use fully relativistic kinematics.

- Scaling variables

The commonly used scaling variables in QE lepton-nucleus scattering are \( y \) and \( \psi \). The former has the dimensions of a momentum and is (in a certain limit specified below) independent of the specific nucleus. The latter is dimensionless and varies with the nuclear density.

- \( y \) scaling variable

The scaling variable

\[
y(q, \omega) = \pm p_{\text{min}} \quad \tag{A1}
\]

is (up to a sign) the minimum missing momentum \( p_{\text{min}} \) allowed by given values of the energy-momentum transfer \( (\omega, q) \) for a specific process. If we focus on the quasielastic case this process is the semi-inclusive \((e, e'N)\) reaction, with \( N = p, n \).

The explicit expression for \( y \) is obtained by imposing energy-momentum conservation in the \((e, e'N)\) process

\[
\omega + M_A = E_N + E_{A-1}^* \quad \tag{A2}
\]

\[
q = p_N + p_{A-1} \quad \tag{A3}
\]

where \( E_N = \sqrt{p_N^2 + m_N^2} \) is the on-shell energy of the outgoing nucleon, \( E_{A-1}^* \) is the energy of the residual nucleus, generally left in an excited state, and

\[
p_{A-1} \equiv -p = q - p_N \quad \tag{A4}
\]

is the momentum of the residual nucleus, or missing momentum. In the Impulse Approximation (IA), where the probe interacts with one nucleon inside the nucleus, \( p \) coincides with the struck nucleon momentum, but this assumption is not necessary in the present formalism.

The determination of the minimum value of \( p \) compatible with \[(A2), (A3)\] yields the following exact expression

\[
y = \frac{1}{2W^2} \left\{ (M_A + \omega) \sqrt{W^2 - (M_{A-1} + m_N)^2} \sqrt{W^2 - (M_{A-1} - m_N)^2} - q [W^2 + M_{A-1}^2 - m_N^2] \right\}, \quad (A5)
\]

where \( W = \sqrt{(M_A + \omega)^2 - q^2} \) depends upon the rest masses of the target \((M_A)\) and residual \((M_{A-1})\) nuclei.
In the limit $M_{A-1} \rightarrow \infty$, valid for medium-heavy nuclei, one gets
\[ y(q, \omega) \simeq \sqrt{\omega(\omega + 2m_N)} - q, \]  
(A6)

where $\bar{\omega} = \omega - E_s$, being
\[ E_s = M_{A-1} + m_N - M_A > 0 \]  
(A7)

the separation energy.

The scaling variable $y$ vanishes at the QEP
\[ y = 0 \rightarrow \omega = \sqrt{q^2 + m_N^2} - m_N + E_s \equiv \omega_{QEP}, \]  
(A8)

corresponding to the energy transfer for electron-nucleon elastic scattering in the nucleon rest frame plus the separation energy. For $\omega < \omega_{QEP}$ the scaling variable $y$ is negative (this is the so-called "scaling region"), while $y > 0$ for $\omega < \omega_{QEP}$.

- $\psi$ scaling variable

The scaling variable $\psi$ naturally emerges in the Relativistic Fermi Gas calculation of the quasielastic responses, which will be illustrated in Appendix C. In the RFG model the reduced response functions depend only upon this variable.

The variable $\psi$ can be defined in terms of the lowest kinetic energy $T_{min} \equiv T_{min}(q, \omega)$ of the struck nucleon inside the Fermi sphere at given $q$ and $\omega$ according to the formula
\[ \psi = \pm \sqrt{T_{min}/T_F} = \pm \sqrt{\frac{\epsilon_0 - 1}{\epsilon_F - 1}}, \]  
(A9)

where
\[ \epsilon_0 = \text{Max}\left\{ \kappa \sqrt{1 + \frac{1}{\tau} - \lambda}, \epsilon_F - 2\lambda \right\}, \]  
(A10)

$\epsilon_F = \sqrt{k_F^4 + m_N^2}/m_N$, $k_F$ being the Fermi momentum characterizing the nucleus, and
\[ \lambda = \omega/2m_N, \quad \kappa = q/2m_N, \quad \tau = \kappa^2 - \lambda^2 \]  
(A11)

are the dimensionless energy, momentum and four-momentum transfers, respectively. The second value inside the Max in (A10) takes into account Pauli blocking, which prevents the ejected nucleon from occupying a state with $k < k_F$. For $q > 2k_F$ no Pauli blocking effects are present in the RFG model and $\epsilon_0 = \kappa \sqrt{1 + 1/\tau}$.

Like $y$, the variable $\psi$ vanishes at the QEP and is negative (positive) for transferred energies lower (higher) than $\omega_{QEP}$.

An alternative, and fully equivalent, expression for $\psi$ is
\[ \psi = \frac{1}{\sqrt{\xi_F}} \sqrt{\frac{\lambda - \tau}{(1 + \lambda)\tau + \kappa \sqrt{\tau}(1 + \tau)}}, \]  
(A12)

valid only in the non Pauli-blocked regime (the interesting regime for scaling arguments).

The variable $\psi$ can be extended to the inelastic region, corresponding to the excitation of a resonance $N^*$ of mass $m^*$, by introducing the inelasticity function $\rho$
\[ \rho = 1 + \frac{\mu^2 - 1}{4\tau}, \quad \mu = \frac{m^*}{m_N} \]  
(A13)

through the replacements
\[ \lambda \rightarrow \lambda \rho, \quad \kappa \rightarrow \kappa \rho, \quad \tau \rightarrow \tau \rho^2 \]  
(A14)
in \( \text{(A12)} \). This yields the inelastic scaling variable

\[
\psi^* = \frac{1}{\sqrt{\xi_F}} \frac{\lambda - \tau \rho}{\sqrt{(1 + \lambda \rho)^2 + \kappa \sqrt{\lambda \rho (1 + \tau \rho)^2}}}.
\]  

(A15)

The non-relativistic version of the variable \( \psi \) is

\[
\psi_{nr} = \frac{1}{\eta_F} \left[ \frac{\lambda}{\kappa} - \kappa \right],
\]  

(A16)

which is obtained using the same procedure as \( \psi \) (see Appendix C) but in the non-relativistic Fermi gas framework.

A more accurate approximation of \( \psi \) can be achieved with the simple replacement

\[
\lambda \rightarrow \lambda (\lambda + 1),
\]  

(A17)

which corresponds to use the relativistic energy-momentum dispersion relation, \( E = \sqrt{p^2 + m^2_N} \), for the outgoing nucleon and the non-relativistic one, \( E = m_N + \frac{p^2}{2m_N} \), for the bound nucleon. Then the "semi-relativistic" scaling variable is

\[
\psi_{sr} = \frac{1}{\eta_F} \left[ \frac{\lambda (\lambda + 1)}{\kappa} - \kappa \right].
\]  

(A18)

The relation between the variables \( \psi \) and \( y \) is in general non-linear and depends on \( q \) and \( \omega \). However, around the QEP the following approximate relation holds:

\[
\psi \approx y/k_F \quad \text{for} \quad \omega \approx \omega_{QEP}.
\]  

(A19)

The departure from the linear behavior of Eq. \( \text{(A19)} \) becomes more and more important at energy transfers far from the QEP.

- **Scaling functions**

  The scaling function is defined as

  \[
  F(q, \omega) = \frac{d^2 \sigma / d\Omega_e d\omega}{\sigma_{eN}(q, \omega; p = -y, \mathcal{E} = 0)},
  \]  

  (A20)

  where

  - \( d^2 \sigma / d\Omega_e d\omega \) is the double differential cross section with respect to the outgoing electron solid angle \( \Omega_e \) and energy transfer \( \omega \);

  - the dividing factor

  \[
  \sigma_{eN}(q, \omega; p = -y, \mathcal{E} = 0) = \frac{1}{2\pi} \int_0^{2\pi} \sigma(q, \omega; p = -y, \mathcal{E} = 0; \phi_N) d\phi_N d\phi_N
  \]  

  (A21)

  is the single-nucleon cross section evaluated at the smallest possible values of the struck nucleon momentum, \( p = -y \), and at vanishing excitation energy of the residual nucleus, \( \mathcal{E} = 0 \), with

  \[
  \mathcal{E} \equiv E^*_{A-1} - E_{A-1} = \sqrt{M^2_{A-1} + p^2} - \sqrt{M^2_{A-1} + p^2}.
  \]  

  (A22)

  From energy conservation one also obtains the following relation

  \[
  \mathcal{E} = E_m - E_s,
  \]  

  (A23)

  where \( E_m = E_N - \omega \) is the missing energy and \( E_s \) the separation energy \( \text{(A7)} \).

  In Eq. \( \text{(A21)} \) \( \mathcal{E} = 0 \) is therefore the minimum value of this variable compatible with the kinematics: in this situation the missing energy coincides with the separation energy, \( E_m = E_s \).
The choice \( p = -y \) and \( E_m = E_s \) in the dividing factor of (A20) is motivated by the fact that for these values the underlying nuclear spectral function is expected to give the largest contribution to the cross section. Without this assumption it would be impossible to factorize out the single-nucleon contribution from the inclusive cross section.

Finally, in (A21) an average over the outgoing nucleon azimuthal angle \( \phi_N \) is performed.

Note that \( \tilde{\sigma}_{eN} \) is not the free stationary elastic \( eN \) cross section, because the hit nucleon is moving inside the nucleus and it is off-shell. The treatment of off-shell effects introduces ambiguities which are more or less important depending on the kinematics. The relativistic off-shell cross section used in the scaling analyses of experimental data corresponds to the "CC1" prescription by De Forest [28].

The superscaling function is defined as

\[
f = k_F \times F. \tag{A24}
\]

It normally depends on two independent variables (\( q \) and \( \omega \) or combinations of them) and becomes a function the variable \( \psi \) only if superscaling occurs.

The longitudinal and transverse superscaling functions are

\[
f_{L,T} = R_{L,T}(q,\omega)/G_{L,T}(q,\omega) \tag{A25}
\]

where

\[
G_{L,T} = U_{L,T}/(2\kappa D) \tag{A26}
\]

with

\[
U_L = \frac{\kappa^2}{\tau} \left( \tilde{G}_E^2 + \tilde{W}_2 \Delta \right) \tag{A27}
\]

\[
U_T = 2\tau \tilde{G}_M^2 + \tilde{W}_2 \Delta, \tag{A28}
\]

being

\[
\tilde{G}_E^2 = ZG_{Ep}^2 + NG_{En}^2 \tag{A29}
\]

\[
\tilde{G}_M^2 = ZG_{Mp}^2 + NG_{Mn}^2 \tag{A30}
\]

\[
\tilde{W}_2 = \frac{\tilde{G}_E^2 + \tilde{G}_M^2}{1 + \tau} \tag{A31}
\]

\[
\Delta = \xi_F (1 - \psi^2) \left[ \frac{\sqrt{\tau(1 + \tau)}}{\kappa} + \frac{\xi_F}{3} \frac{\tau}{\kappa^2}(1 - \psi^2) \right] \tag{A32}
\]

\[
D = 1 + \frac{\xi_F}{2} (1 + \psi^2). \tag{A33}
\]

The terms proportional to the Fermi kinetic energy \( \xi_F = T_F/m_N \) in Eqs. (A32, A33) are small and they are usually neglected in the dividing factors.

### B. Single nucleon tensor

In this appendix we derive the single nucleon tensor and responses for a nucleon transition \( p \rightarrow p' \) in the Fermi gas before integration. The starting point is the single nucleon tensor given in Eq. (26). We consider for definiteness the tensor induced by the weak CC interaction, which is the sum of vector and axial currents, given in Eqs. (28,29).

The electromagnetic tensor can be easily obtained from the below expressions by considering only the vector part of the current.

We start, using the properties of the Dirac spinors, writing the sum over spin indices in Eq. (26) as a trace of Dirac matrices

\[
w^{\mu\nu}_{s,n}(p',p) = \frac{1}{2} \text{Tr} \left[ \frac{\not{b} + m_N}{2m_N} (\Gamma_V - \Gamma_A)^{\mu} \right] \frac{\not{b} + m_N}{2m_N} (\Gamma_V - \Gamma_A)^{\nu} \tag{B1}
\]

where \( \Gamma_V^{\mu} \) is the spin matrix of the vector current after using Gordon identity

\[
\Gamma_V^{\mu} = 2G_M^V \gamma^{\mu} - 2F_2^V (p + p')^{\mu}/2m_N, \tag{B2}
\]
while $\Gamma_A^\mu$ is the spin matrix of the axial current

$$\Gamma_A^\mu = G_A \gamma^\mu \gamma^5 + G_F \frac{Q^\mu}{2m_N} \gamma^5$$

(B3)

Finally $\tilde{\Gamma}_A^\mu = \gamma^0 \Gamma_A^{\mu + \gamma^0}$ or

$$\tilde{\Gamma}_A^\mu = G_A \gamma^\mu \gamma^5 - G_F \frac{Q^\mu}{2m_N} \gamma^5$$

(B4)

We see that the $V - A$ dependence of the weak current allows to single out four contributions labeled $VV$, $AA$, $VA$ and $AV$

$$w_{s.n.}^{\mu\nu}(p', p) = w_{VV}^{\mu\nu} + w_{AA}^{\mu\nu} + w_{VA}^{\mu\nu} + w_{AV}^{\mu\nu}$$

(B5)

Performing the corresponding traces, it is straightforward to obtain the result:

$$w_{VV}^{\mu\nu} = \frac{1}{4m_N^2} \left[ (2G_M^V)^2 (Q^2 g^{\mu\nu} - Q^{\mu}Q^{\nu}) + ((2F_M^V)^2 + \tau (2F_M^V)^2) (p + p')^\mu (p + p')^\nu \right]$$

(B6)

$$w_{AA}^{\mu\nu} = \frac{1}{8m_N^2} \left[ 4G_A^2 \left( \frac{1}{2} (p + p')^\mu (p + p')^\nu - \frac{1}{2} Q^{\mu}Q^{\nu} - \left( \frac{2m_N^2 - Q^2}{2} \right) g^{\mu\nu} \right) - \left( 4G_A G_F + G_F^2 \frac{Q^2}{2m_N} \right) Q^{\mu}Q^{\nu} \right]$$

(B7)

$$w_{VA}^{\mu\nu} + w_{AV}^{\mu\nu} = \frac{2G_M^V G_A}{m_N^2} \epsilon^{\alpha\beta\mu\nu} p_\alpha p_\beta$$

(B8)

Note that the VA interference tensor is purely imaginary and antisymmetric and therefore it only contributes to the $T'$ response function, see Eqs. (B9).

Using $p'_\mu = p_\mu + Q_\mu$ we can write the hadronic tensor in the more standard form

$$w_{s.n.}^{\mu\nu} = -w_1 \left( g^{\mu\nu} - \frac{Q^{\mu}Q^{\nu}}{Q^2} \right) + w_2 K^{\mu}K^{\nu} - \frac{i}{m_N} w_3 \epsilon^{\alpha\beta\mu\nu} Q_\alpha K_\beta + w_4 \frac{Q^{\mu}Q^{\nu}}{m_N^2}$$

(B9)

where the invariant structure functions $w_i = w_i(Q^2)$ only depend on $Q^2$, and we have defined the four vector

$$K^{\mu} = \frac{1}{m_N} \left( p_\mu - \frac{p \cdot Q}{Q^2} Q^\mu \right)$$

(B10)

After a straightforward calculation, using Eqs. (B6, B7, B8), we obtain

$$w_1 = \tau (2G_M^V)^2 + (1 + \tau) G_A^2$$

(B11)

$$w_2 = \frac{(2G_M^V)^2 + \tau (2G_M^V)^2}{1 + \tau} + G_A^2$$

(B12)

$$w_3 = 2G_M^V G_A$$

(B13)

$$w_4 = \frac{1}{4\tau} (G_A - \tau G_F)^2$$

(B14)

Finally we extract the suitable components of the hadron tensor to obtain the corresponding single nucleon responses to be used later for computing the nuclear responses in the RFG.

The single-nucleon $CC$ response function, using Eq. (B3), can be written as

$$r_{CC} = w_{s.n.}^{00} = -w_1 \left( 1 - \frac{\omega^2}{Q^2} \right) + w_2 (K^0)^2 + w_4 \frac{\omega^2}{m_N^2}$$

(B15)

Writing it in terms of the dimensionless variables $\lambda, \kappa, \tau$, introduced in Sect. IIIA and the reduced initial nucleon energy $\varepsilon \equiv E/m_N$, we find

$$r_{CC} = -w_1 \frac{\kappa^2}{\tau} + w_2 (\varepsilon + \lambda)^2 + 4w_4 \lambda^2.$$  

(B16)
Analogously, for the case of the CL and LL single-nucleon responses

\[
    r_{CL} = -\frac{1}{2}(w_{s,n}^{03} + w_{s,n}^{30}) = -\frac{\lambda}{\kappa} \left[ -w_1 \frac{\kappa^2}{\tau} + w_2 (\varepsilon + \lambda) \right] - 4w_4 \lambda \kappa \quad (B17)
\]

\[
    r_{LL} = w_{s,n}^{33} = \frac{\lambda^2}{\kappa^2} \left[ -w_1 \frac{\kappa^2}{\tau} + w_2 (\varepsilon + \lambda) \right] + 4w_4 \kappa^2 \quad (B18)
\]

Note that the first term of \( r_{CL} \), between squared parentheses, and multiplied by \(-\lambda/\kappa\), comes from current conservation of the terms with \( w_1 \) and \( w_2 \) of the hadronic tensor. The same comment applies to the first term of the \( r_{LL} \) response, with the factor \( \lambda^2/\kappa^2 \).

The transverse response is the component

\[
    r_T = w_{s,n}^{11} + w_{s,n}^{22} = 2w_1 + w_2 [(K^1)^2 + (K^2)^2] \quad (B19)
\]

Using the definition of the vector \( K^\mu \), Eq. \( (B10) \), we have

\[
    (K^1)^2 + (K^2)^2 = \eta^2 - \eta_3^2 \quad (B20)
\]

where \( \eta_i = p_i/m_N \) is the reduced initial nucleon momentum vector with length \( \eta \). Using the condition on-shell nucleons \( p \cdot q = E\omega + Q^2/2 \) written in terms of dimensionless variables

\[
    \eta_3 = \frac{\varepsilon \lambda - \tau}{\kappa} \quad (B21)
\]

Using \( \eta^2 = \varepsilon^2 - 1 \), we can finally write

\[
    r_T = 2w_1 + w_2 \frac{\varepsilon^2 \tau - \kappa^2 - \tau^2 + 2\varepsilon \lambda \tau}{\kappa^2} \quad (B22)
\]

Finally, the \( r_{T'} \) response comes from the antisymmetric part of the hadronic tensor\textsuperscript{15}

\[
    r_{T'} = -\frac{i}{2}(w_{s,n}^{12} - w_{s,n}^{21}) = -\frac{1}{m_N} w_3 (-qK^0 + \omega K^3) \quad (B23)
\]

Using \( K^0 = \varepsilon + \lambda \) and \( K^3 = \omega K^0/q \), we obtain

\[
    r_{T'} = 2w_3 \frac{\tau}{\kappa} (\varepsilon + \lambda) \quad (B24)
\]

C. Derivation of response functions in the RFG model

We outline here the main steps used to obtain the analytical expression of the response functions in the RFG, as this material only can be found for electron scattering and not for neutrino scattering in such detailed form.

We start with the definition of the response function \( R_K \) for neutrino (antineutrino) CC quasielastic scattering. Taking the corresponding components of the hadronic tensor from Eq. \( (25) \), we have

\[
    R_K(q, \omega) = \frac{3N m_N^2}{4\pi k_F^4} \int d^4 p \delta(E' - E - \omega) \frac{1}{E E'} r_K(p, q, \omega) \theta(k_F - p) \theta(p' - k_F) \quad (C1)
\]

where \( N = N \) for neutrinos and \( Z \) for antineutrinos. The single nucleon responses \( r_K \) have been obtained in the previous section \[ for on shell nucleons with momenta \( p' = p + q \). Note that \( r_K \) depends on \( q, \omega \) via the variables \( \kappa, \lambda, \) and \( \tau \), and that it depends on \( p \) only through the reduced initial energy \( \varepsilon \), see Eqs. \[ B10 \] B17 B18 B22 B24.

To perform the integral over the initial nucleon momentum, we change variables \( (p, \theta, \phi) \rightarrow (E, E', \phi) \), where \( E^2 = m_N^2 + p^2 \) and \( E'^2 = m_N^2 + (p + q)^2 \) are the initial and final nucleon energies. The volume element transforms as

\[
    d^3p = \frac{EE'}{q} dEdE' d\phi \quad (C2)
\]

\textsuperscript{15} Our convention is \( \epsilon_{0123} = 1 \) for the Levi-Civita four tensor.
and the integral with the appropriate integration limits becomes
\[
R_K(q, \omega) = \frac{3N m_N^2}{4 \pi k_F^2} \int_{E_F}^{E_{p+q}} \frac{dE}{q} \int_{E_{p-q}}^{E_F} dE' \int_0^{2\pi} d\phi \delta(E' - E - \omega) r_K(p, q, \omega) \theta(E' - E_F) \tag{C3}
\]
where \( E_F \) is the relativistic Fermi energy, \( E_{p+q} = m_N^2 + (p + q)^2 \), and \( E_{p-q}^2 = m_N^2 + (p - q)^2 \). The integral over \( d\phi \) gives a factor \( 2\pi \). Because the single nucleon responses do not depend on the angle \( \phi \). Integrating the delta function we obtain \( E' = E + \omega \). The result is written as an integral over the initial energy
\[
R_K(q, \omega) = \frac{3N}{4\eta_F^3 km_N^2} \int_{E_F}^{E_{p+q}} \frac{dE}{E} \theta(E_{p+q} - E - \omega) \theta(E + \omega - E_{p-q}) \theta(E' - E_F) r_K(E, q, \omega) \tag{C4}
\]
where we have used the reduced variable \( \eta_F = k_F/m_N \) and \( \kappa = q/2m_N \). Note that the single nucleon responses only depend on the initial energy.

The step functions and the integration limits are equivalent to the following constraints to the energy:
\[
E_{p-q} < E + \omega < E_{p+q} \tag{C5}
E_F < E + \omega < E_F + \omega \tag{C6}
\]
Taking the square of (C5) and introducing the reduced variables: momentum, \( \eta = p/m_N \), energy, \( \varepsilon \) and \( \lambda, \kappa \) and \( \tau \), it is straightforward to obtain
\[
|\varepsilon \lambda - \tau| < \kappa \eta \tag{C7}
\]
which in turn can be squared again, using \( \eta^2 = \varepsilon^2 - 1 \), in terms of \( \varepsilon \), it gives
\[
\kappa^2 \left( 1 + \frac{1}{\tau} \right) < (\varepsilon + \lambda)^2 \tag{C8}
\]
Solving for \( \varepsilon \) we arrive to the following lower bound for given \( q, \omega \)
\[
\kappa \sqrt{1 + \frac{1}{\tau}} - \lambda < \varepsilon \tag{C9}
\]
On the other hand, from (C6), \( \varepsilon \) must verify the Pauli blocking condition as a second lower bound
\[
\varepsilon_F - 2\lambda < \varepsilon \tag{C10}
\]
where \( \varepsilon_F = E_F/m_N \). Both lower bounds for \( \varepsilon \) imply
\[
\text{Max} \left\{ \kappa \sqrt{1 + \frac{1}{\tau}} - \lambda, \varepsilon_F - 2\lambda \right\} \equiv \varepsilon_0 < \varepsilon \tag{C11}
\]
Note now that the following inequality always holds
\[
\kappa \sqrt{1 + \frac{1}{\tau}} - \lambda \geq 1 \tag{C12}
\]
This can be demonstrated by moving \( \lambda \) to the right-hand side and taking the square on both sides of the inequality. Hence \( 1 < \varepsilon_0 < \varepsilon \) and therefore \( \varepsilon \) correspond to an allowed nucleon energy. The response function can thus be written as the following integral over \( \varepsilon \)
\[
R_K(q, \omega) = \frac{3N}{4\eta_F^3 km_N^2} \theta(\varepsilon_F - \varepsilon_0) \int_{\varepsilon_0}^{\varepsilon_F} r_K(\varepsilon, \kappa, \lambda) d\varepsilon \tag{C13}
\]
Note that there is a dependence of the response functions on the variable \( \varepsilon_0 \) through the lower limit of the integral. In the scaling approach \( \varepsilon_0 \) is written in terms of the scaling variable \( \psi \), defined by
\[
\psi = \sqrt{\varepsilon_0 - 1 \over \varepsilon_F - 1} \text{sign}(\lambda - \tau). \tag{C14}
\]
The scaling variable is defined such as it is positive to the right side of the QE peak ($\lambda = \tau$) and negative otherwise. The inverse relation is

$$
\varepsilon_0 = 1 + \psi^2 \xi_F
$$

where $\xi_F = \varepsilon_F - 1$ is the kinetic Fermi energy in units of the nucleon mass. The single nucleon responses, obtained in App. [1] depend on $\varepsilon$ as a second-degree polynomial at most. Therefore, in terms of this scaling variable, the only relevant integrals are the following:

$$
\int_{\varepsilon_0}^{\varepsilon_F} d\varepsilon = \xi_F(1 - \psi^2)
$$

$$
\int_{\varepsilon_0}^{\varepsilon_F} \varepsilon d\varepsilon = \xi_F(1 - \psi^2)(1 + \frac{1}{2} \xi_F(1 + \psi^2))
$$

$$
\int_{\varepsilon_0}^{\varepsilon_F} \varepsilon^2 d\varepsilon = \xi_F(1 - \psi^2)(1 + \xi_F(1 + \psi^2) + \frac{1}{3} \xi_F^2(1 + \psi^2 + \psi^4))
$$

Applying these results to the response $R_{CC}$ we can directly write

$$
R_{CC} = \frac{3N}{4m_N\eta_F^2} \theta(1 - \psi^2)\xi_F(1 - \psi^2) \left\{ -w_1 \frac{\kappa^2}{\tau} + w_2 \lambda^2 + 4w_4 \lambda^2 \\
+ 2w_2 \lambda(1 + \frac{1}{2} \xi_F(1 + \psi^2)) \\
+ w_2(1 + \xi_F(1 + \psi^2) + \frac{1}{3} \xi_F^2(1 + \psi^2 + \psi^4)) \right\}
$$

$$
= N \Lambda f(\psi) U_{CC}
$$

where the RFG scaling function $f(\psi)$ was defined in Eq. [40] and the factor $\Lambda$ was introduced in Eq. [51].

To write this expression in terms of the quantity $\Delta$ introduced in Eq. [52] we add and subtract the elastic limit of the function $U_{CC}$, corresponding to $\eta_F = \varepsilon_F = 0$ and $\lambda = \tau$.

$$
U_{CC}^0 = U_{CC}(\varepsilon_F = 0, \lambda = \tau) = \frac{\kappa^2}{\tau} \left[ -w_1 + (1 + \tau)w_2 + 4\tau w_4 \frac{\lambda^2}{\kappa^2} \right]
$$

The small quantity $\Delta$, reflecting the deviation from the elastic value due to the Fermi motion of the nucleons, is $U_{CC} - U_{CC}^0 = (\kappa^2/\tau)w_2\Delta$. Therefore

$$
U_{CC} = U_{CC}^0 + U_{CC} - U_{CC}^0 = \frac{\kappa^2}{\tau} \left[ -w_1 + (1 + \tau)w_2 + 4\tau w_4 \frac{\lambda^2}{\kappa^2} + w_2\Delta \right]
$$

Finally, we replace the values of the $w_i$ structure functions, given in Eqs. [B11] [B14], obtaining Eqs. [66] [67], with corresponding functions $U_{CL}$ and $U_{LL}$. These functions can be obtained more easily by writing them as sum of two pieces, coming from the conserved and non-conserved parts of the hadron tensor. First note that

$$
U_{CC} = (U_{CC})_c. + (U_{CC})_{n.c.}
$$

where the conserved and non conserved parts are

$$
(U_{CC})_c. = \frac{\kappa^2}{\tau} \left[ -w_1 + (1 + \tau)w_2 + w_2\Delta \right]
$$

$$
(U_{CC})_{n.c.} = 4\lambda^2 w_4
$$

Note that the non conserved part $(U_{CC})_{n.c.}$ is precisely the term of $r_{CC}$, in Eq. [B10], containing the $w_4$ structure function. This is so because this single-nucleon contribution is independent on $\varepsilon$ and it factorizes out of the energy integration. The same can be said for the non conserved parts of $r_{CL}$ and $r_{LL}$ (Eqs. [B17] [B18]).

Consequently, applying current conservation to the conserved part, we can write

$$
(U_{CL})_c. = -\frac{\lambda}{\kappa}(U_{CC})_c.
$$

$$
(U_{LL})_c. = \frac{\lambda^2}{\kappa^2}(U_{CC})_c.
$$
On the other hand for the non conserved part due to the factorization property

\[ (U_{CL})_{n.c.} = -4\lambda \kappa \omega_4 \]  
(C28)

\[ (U_{LL})_{n.c.} = 4\kappa^2 \omega_4. \]  
(C29)

Proceeding in the same lines for the \( T \) response we obtain

\[
U_T = 2w_1 - w_2 \frac{\kappa^2 + \tau^2}{\kappa^2} + w_2 \frac{2\lambda \tau}{\kappa^2} \left[ 1 + \frac{1}{2} \xi F(1 + \psi^2) \right] \\
+ w_2 \frac{\tau}{\kappa^2} \left[ 1 + \xi F(1 + \psi^2) + \frac{1}{2} \xi^2 F(1 + \psi^2 + \psi^4) \right] \\
= 2w_1 + w_2 \Delta
\]
(C31)

where the same term \( \Delta \) as before is obtained as the deviation from the elastic limit for \( k_F = 0 \).

Finally, the \( T' \) response gives

\[
U_{T'} = 2w_3 \frac{\tau}{\kappa} \left[ 1 + \lambda + \frac{1}{2} \xi F(1 + \psi^2) \right] \\
= 2w_3 \sqrt{\tau} (\tau + 1) \left[ 1 + \bar{\Delta} \right]
\]
(C33)

where \( \bar{\Delta} \) was defined in Eq. (70) and results from adding and subtracting the elastic limit for \( k_F = 0 \) and \( \lambda = \tau \).

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