Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach

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We derive the conservative dynamics of non-spinning binaries to third Post-Minkowskian order, using the Effective Field Theory (EFT) approach introduced in [1] together with the Boundary-to-Bound dictionary developed in [2,3]. The main ingredient is the scattering angle, which we compute to $O(G^3)$ via Feynman diagrams. Adapting to the EFT framework powerful tools from the amplitudes program, we show how all of the associated (massless) master integrals in the potential region are bootstrapped to all orders in velocities via differential equations. Remarkably, the boundary conditions can be reduced to the same master integrals that appear in the EFT with Post-Newtonian sources at ‘two loop’ order. For the sake of comparison, we reconstruct the Hamiltonian and, using the ‘impetus formula’, the classical limit of the (infrared-finite part of the) scattering amplitude at two loops. Our results are in perfect agreement with those in Bern et al. [4,5].

Introduction. The discovery potential heralded by the new era of gravitational wave (GW) science [6,7] has motivated high-accuracy theoretical predictions for the dynamics of binary systems [8,10]. Notably, in parallel with more ‘traditional’ approaches [11–17], ideas from particle physics such as Effective Field Theory (EFT) methods [18–23] and scattering amplitudes, e.g. [24–26], have made key contribution to the present state-of-the-art knowledge for the conservative dynamics of non-spinning binaries both in the Post-Newtonian (PN) and Post-Minkowskian (PM) regimes, computed to 4PN [27–34] and 3PM [4,5,35] orders, respectively [4,5]. Until recently, the EFT formalism introduced in [18] had been restricted to the PN regime, while the scattering amplitude program developed in [4,5,35] naturally finds its habitat in the PM scheme. Building upon the boundary-to-bound (B2B) dictionary in [2,3], the gap was recently closed in [1] with the development of an EFT framework to collect the scattering data to input in the B2B map, and readily implemented to 2PM order. (See e.g. [61,67] for alternative routes using the EOB formalism [68].)

In this letter we report the derivation of the conservative dynamics of binary systems to 3PM in the EFT approach. This entails the calculation of the scattering angle to next-to-next-to-leading order (NNLO) in $G$ (Newton’s constant) via Feynman diagrams. Remarkably, we find that the associated integrals can be bootstrapped from their PN counterparts through differential equations in the velocity, as advocated in [69], paving the way forward to higher order computations. For the sake of comparison, we reconstruct the Hamiltonian as well as the (infrared-finite) amplitude in the classical limit and find complete agreement with the results in [4,5]. Our derivation thus independently confirms the ‘impetus formula’ [2], relating the amplitude to the center-of-mass (CoM) momentum, and the legitimacy of the program to extract classical physics from scattering amplitudes [2,5,35,59,63,69,93]. At the same time, we demonstrate the power of the EFT and B2B machinery [1,3], which by design can be systematized to all PM orders.

The EFT framework. The starting point is the effective action from which we derive the scattering trajectories. We proceed by integrating out the metric field $g_{\mu\nu} = \eta_{\mu\nu} + \eta_{\mu\nu}/M_{P1}$ (with $M_{P1}^2 = \sqrt{32\pi G}$)

$$e^{iS_{eff}} = \int \mathcal{D}h_{\mu\nu} e^{iS_{EH}[h] + iS_{GF}[h] + iS_{pp}[\tau_a, h]}$$ (1)

in the (classical) saddle-point and weak-field approximations. We work with the Einstein-Hilbert action, $S_{EH}$, and the convention $\eta_{\mu\nu} = \text{diag}(+, -,-,-)$. The gauge-fixing, $S_{GF}$, is adjusted to simplify the Feynman rules [1]. We use the (Polyakov) point-particle effective action,

$$S_{pp} = -\sum_{a=1,2} \frac{m_a}{2} \int d\tau_a g_{\mu\nu}(x_a^\alpha) \frac{dx_a^\mu}{d\tau_a} \frac{dx_a^\nu}{d\tau_a} + \cdots$$ (2)

with $\tau_a$ the proper time. The ellipses include higher-derivative terms accounting for finite-size effects and counterterms to remove (classical) ultraviolet divergences [11,18]. As usual, we use dimensional regularization.

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1 Partial results are also known to 5PN (static) [36,37] and 6PN [38,39]; radiation and spin are incorporated in e.g [40-63].
Impulse from Action. From the action we read off the effective Lagrangian at each order in $G$: $\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \cdots$. Although it may be non-local in time when radiation-reaction effects are included [38, 39], it is manifestly local with only potential modes [1]. Using the effective Lagrangian we obtain the trajectories. For scattering events we PM expand the solution as

$$x_n^\mu(\tau_a) = b_n^\mu + u_n^\mu \tau_a + \sum_n \delta^{(n)} x_n^\mu(\tau_a),$$  

with $u_n^\mu$ the velocity at infinity, obeying $u_n^2 = 1$, and $b_n^\mu \equiv b_1^\mu - b_2^\mu$ the impact parameter. For instance, at LO,

$$\delta(1) x_1^\mu(\tau_1) = -\frac{m_2}{8\pi^4} (2\gamma^2 - 1) \eta^{\mu\nu} - 2(2\gamma u_2^\mu - u_1^\mu) u_2^\nu$$

$$\times \int k d{k, u_2} e^{ik - \hat{\nu}} \eta^{\mu\nu} + \hat{n}.$$  

We use the notation $\int_k \equiv \int \frac{d{k, \nu}}{2\pi^3}$, $\delta(x) \equiv 2\pi \delta(x)$ and

$$\gamma \equiv u_1 \cdot u_2 = \frac{p_1 p_2}{m_1 m_2} = \frac{E_1 E_2 + p^2}{m_1 m_2},$$  

where $E_a = \sqrt{p^2 + m_a^2}$ and $\pm p$ is the CoM momentum. Notice the factor of $(k \cdot u_1 - i0^+)^{-1}$, with the $i0^+$ to ensure convergence of the time integrals, which resembles the linear propagators appearing in heavy-quark effective theory [94]. The pole shifts to $(k \cdot u_2 + i0^+)^{-1}$ for particle 2. The impulse follows from the effective action,

$$\Delta p^\mu = -\eta^{\mu\nu} \int_{-\infty}^{+\infty} d\tau_a \frac{\partial \mathcal{L}_{\text{eff}}}{\partial x^\nu_a}(x_a(\tau_a)),$$  

where the overall sign is due to our conventions. The impulse can then be solved iteratively, starting with the undeflected trajectory in [3]. Notice that all of the $\mathcal{L}_{k < n}$'s contribute to nPM order, and must be evaluated on the trajectories up to $(n-k)$-th order in $G$. We refer to this procedure as iterations $\textbf{I}$. The scattering angle,

$$\frac{\chi}{2} = \sum_n \lambda_b^{(n)} \left( \frac{GM}{b} \right)^n = \sum_n \lambda_b^{(n)} j_n,$$  

with $1/j = GM \mu / (p_\infty b)$, is obtained from the relation

$$2 \sin \frac{\chi}{2} = 2 \left( \frac{\chi}{2} - \frac{1}{6} \left( \frac{\chi}{2} \right)^3 + \cdots \right) = \sqrt{-\Delta p^2} / p_\infty,$$  

where

$$p_\infty = \mu \sqrt{\gamma^2 - 1}, \quad \Gamma = \frac{E}{M} = \sqrt{1 + 2\nu(\gamma - 1)},$$  

with $E, M$ the total mass and energy, respectively. We use the notation $\mu = m_1 m_2 / M^2$ for the reduced mass, and $\nu = \mu / M$ for the symmetric mass ratio.

The impulse may be further split into a contribution along the direction of the impact parameter as well as a term proportional to the velocities $\textbf{I}$. Due to momentum conservation and the on-shell condition, we have

$$(p_a + \Delta p_a)^2 = p_a^2 \implies 2 p_a \cdot \Delta p_a = -\Delta p_a^\mu.$$  

Moreover, since $\Delta(1)p^\mu \propto b^\mu$ at leading PM order $\textbf{I}$, and $b \cdot u_n = 0$, we can use $\textbf{I}$ to solve iteratively for the component along the velocities. This allows us to restrict the derivation of the impulse to the perpendicular plane $\textbf{I}$. Feynman Integrals. To 3PM order the Feynman topologies are shown in Fig. 1. The computation yields four-dimensional relativistic integrals constrained by a series of $\delta$-functions, $\delta(k \cdot u_a)$, which arise due to the time integration in [3] after inputting [3]. Moreover, in addition to the standard factors of $1/k^2$ from the gravitational field, we have linear propagators, as in [4], which are needed to compute the iterations. As we mentioned, we restrict ourselves to the computation of the impulse in the direction of the impact parameter. The derivation is then reduced to a series of terms proportional to the Fourier transform in the ‘transfer momentum’

$$\int_q \delta(q \cdot u_1) \delta(q \cdot u_2) iq^\mu \tau^s M_{n_1 n_2 i_1 \cdots i_5}^{(a, \bar{a})}(q, \gamma) e^{i q^\nu b},$$  

where the factor of $t^s$, with $t = -q^2$, depends on the tensor reduction of the given diagram. We find the following (cut) ‘two loop’ integrals $\textbf{I}$

$$M_{n_1 n_2 i_1 \cdots i_5}^{(a, \bar{a})}(q, \gamma) \equiv \int_{k_1, k_2} \delta(k_1 \cdot u_a) \delta(k_2 \cdot u_{\bar{a}})$$

$$A_{1, \bar{a}} = k_1 \cdot u_{\bar{a}}, \quad A_{2, \bar{a}} = k_2 \cdot u_{\bar{a}}, \quad D_1 = k_1^2, \quad D_2 = k_2^2,$$

$$D_3 = (k_1 + k_2 - q)^2, \quad D_4 = (k_1 - q)^2, \quad D_5 = (k_2 - q)^2.$$  

All the integrals we encounter in our computation, including the iterations, can be embedded into the family in [12] with different choices of $(a, \bar{a})$. The $i0^+$-prescription is such the $u_{1, 2}$ are always accompanied by $\mp i0^+$, as in [4]. The other cases are obtained by different symmetrizations $\textbf{I}$. We keep only non-analytic terms in $t$ which yield long-range interactions $\textbf{I}$. We outline the integration procedure momentarily. The outcome is the scaling

$$t^s M_{n_1 n_2 i_1 \cdots i_5}^{(a, \bar{a})} \propto t^{-2 \epsilon} / \epsilon,$$  

with $\epsilon = (4 - D)/2$, which gives for the impulse in $\textbf{I}$ the expected $b^\mu / b^4$ in $D = 4$. The poles (and log $\mu$'s) in dimensional regularization accompanying the log $t$ produce contact terms that neatly drop out without referring to subtraction schemes $\textbf{I}$. 
Potential Modes. In the framework of the PN expansion, the integrals would be performed using a mode factorization into potential ($k_0 \ll |k|$) and radiation ($k_0 \sim |k|$) modes, while keeping manifest power counting in the velocity [18, 96]. The computation with potential modes then reduces to a series of three-dimensional (massless) integrals [23]. In contrast, in the PM scheme we sought to keep the propagators fully relativistic. The associated Feynman integral still receive contributions from both potential and radiation modes (yielding real and imaginary parts). We are interested here in the conservative sector, and we ignore for now radiation-reaction effects. As discussed in [1], to isolate the potential modes we adapt to our EFT framework the powerful tools developed in [4, 5, 69]. Notably, we make use of the methodology of differential equations using boundary conditions from the (static) limit $\gamma \to 1$ [69].

On the one hand, for diagrams (c) and (d) in Fig. 1 only the $M^{(1,1)}_{n_1,n_2...}$ in (12) are needed, with $(n_1,n_2) \leq 0$, plus mirror images. These integrals, which contribute to the one-point function of a (boosted) Schwarzschild background, can be computed in the rest frame

$$u_1 = (1, 0, 0, 0), \quad u_2 = (\gamma, \gamma \beta, 0, 0),$$

with $\beta \gamma = \sqrt{\gamma^2 - 1}$ [1]. At the end of the day, they turn into the same type that appear in the static limit of the PN expansion, see e.g. [27]. For diagrams (e), (f) and (g) in Fig. 1, on the other hand, the $M^{(1,2)}_{n_1,n_2...}$ are required instead, also with $(n_1,n_2) \leq 0$. Remarkably, the associated integrals for all these diagrams can be decomposed into a basis involving only the $M^{(1,2)}_{00...}$ subset [95]. Furthermore, using integration by part (IBP) relationships [97, 98], the contribution from diagrams (e) and (f) in Fig. 1 reduces to integrals with $i_3 = 0$. It is then straightforward to show that both diagrams vanish in $D = 4$. (This is reminiscent of the fact that they do not enter at 2PN either [34].) Using the IBP relations and the aid of FIRE6 [99] and LiteRed [100], as well as symmetry arguments, the calculation of the remaining (so-called H) diagram in Fig. 1(g) is reduced to the following basis [97]

$$\{I_{11111}, I_{11211}, I_{01101}, I_{11011}, I_{00211}, I_{00112}, I_{00011}\},$$

with $I_{11...i_5} \equiv M^{(1,2)}_{00...i_5}$. For the computation we follow [101] and various tools, e.g. epsilon [102], to construct a canonical basis $\vec{h} = \{h_{n=1...7}\}$ such that the velocity dependence is obtained via differential equations,

$$\partial_x \vec{h}(x, \epsilon) = \epsilon \mathcal{M}(x) \vec{h}(x, \epsilon)$$

with $\gamma = (x^2 + 1)/(2x)$, as advocated in [69]. Because the set in (16) contains up to five (quadratic) propagators only, the associated boundary conditions in our case are then reduced to the same type of integrals that appear in the PN regime at two loops (Kite diagrams, e.g. [31]). It turns out only a handful contribute to the $H$ diagram in $D = 4$, featuring the much anticipated factor of $\log x$ observed in [4, 5, 69].

To complete the derivation we have to include the iterations. Surprisingly, the set in (16) is (almost) sufficient for all the contributions. For instance, iterations involving the deflection due to Fig. 1(a) at LO order for the impulse due to Fig. 1(b), and vice versa, follow from (16). Yet, for the deflection from Fig. 1(a) to NLO additional integrals are needed, resembling other (cut) topologies in [5, 69]. In our case, we need the following two

$$\{M^{(1,1)}_{11;111100}, M^{(1,2)}_{11;1111100}\}.$$  

Due to the presence of divergences, however, their computation is somewhat subtle. For the first one we can readily go to the rest frame in (15) producing a $D - 1$ integral. We then use the symmetrization described in [69]. Alternatively, it may be computed using the prescription in [4, 5, 35] in the $u_2$-frame. Both can be adapted to all $\pm 0$ choices. The result is proportional to (twice) the standard one loop bubble integrals with static PN sources [27], although in $D - 2$ dimensions. The same trick does not apply to the latter, but it can be easily incorporated into the canonical basis to obtain its $\gamma$-dependence. Yet, due to a divergence in the static limit, we need some care with the boundary condition. This is accounted for in the canonical basis by pulling out the relevant factor of $\beta$ (and $\epsilon$). Once again we perform the integral in the rest frame, expand in small velocity and retain the leading term in $1/\beta$. In this limit, the $M^{(1,2)}_{11;...}$ integral turns out to be equivalent (modulo different $\pm 0$ choices) to the $M^{(1,1)}_{11;...}$ counterpart. We have checked all these relationships explicitly via a standard $\alpha$-parameterization [103]. At the end, as expected, the associated divergences cancel out in the final answer without subtractions.

The above steps culminate the derivation of the master integrals in the potential region via differential equations. Using various arguments, the boundary conditions are reduced to the master integrals that appear in the static limit of the PN expansion at the same loop order. See [69] for a more detailed discussion.

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2 Hereditary tail effects, which enter in the conservative dynamics through a non-local contributions to the effective action e.g. [19, 60], first appear at $\mathcal{O}(G^2 a^2 v^2) \sim \mathcal{O}(G^4 v^2)$ [34], namely 4PML.

3 In principle we find all $\pm 0$ combinations. Naively, due to the lack of ‘crossing’ (e.g. $u_1 \rightarrow -u_1$) in the potential region, the connection between them is not obvious, see [60]. Yet, we can show these integrals are related in the static limit (see text). The upshot is that various $\pm 0$ choices differ by relative factors of 2. (We thank Julio Parra-Martinez and Mao Zeng for discussions about this point.) These turn out to be crucial to ensure the cancellation of intermediate spurious infrared poles $\propto t^{-\delta}/c^2$ [69].
Scattering data. The result for the impulse now follows from basic algebraic manipulations, and we arrive at

\[ \Delta^{(3)}p'_1 = \frac{G^3b^\mu}{|b|^2} \left( \frac{16m^2_1m^2_2(4\gamma^4 - 12\gamma^2 - 3)\sinh^{-1}\sqrt{\gamma^2 - 1}}{(\gamma^2 - 1)} \right. \]

\[ - \frac{4m^2_1m^2_2\gamma(52\gamma^6 - 138\gamma^4 + 144\gamma^2 - 57)}{3(\gamma^2 - 1)^{5/2}} \]

\[ - \frac{2m_1m_2(m^2_1 + m^2_2)(80\gamma^6 - 144\gamma^4 + 72\gamma^2 - 7)}{3(\gamma^2 - 1)^{5/2}} \]

\[ + \frac{3\pi}{2} \left( \frac{2\gamma^2 - 1}{(\gamma^2 - 1)^{5/2}} \right) G^3M^2\mu \]

\[ \times \left( (\gamma m_2 + m_1)u^\mu_2 - (\gamma m_1 + m_2)u^\mu_1 \right). \tag{19} \]

The last term, which does not feature in the deflection angle at this order, is obtained from (10) and the result in [1]. Hence, using [2], the 1PM angle (cube) and the 2PM impulse along the velocities in [1], we find

\[ \frac{\chi^{(3)}_b}{\Gamma} = \frac{1}{(\gamma^2 - 1)^{5/2}} \left[ - \frac{4\gamma}{3} \gamma \sqrt{1 + (14\gamma^2 + 25)} \right. \]

\[ + \frac{64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5}{(\gamma^2 - 1)^{5/2}} \]

\[ - 8\nu(4\gamma^4 - 12\gamma^2 - 3)\sinh^{-1}\sqrt{\gamma - 1}\left( \frac{\gamma}{2} \right), \tag{20} \]

which, using \( \chi^{(3)}_b = (p_\infty/\mu)^3\chi^{(3)}_b = \left( \sqrt{\gamma^2 - 1}/\Gamma \right)^3 \chi^{(3)}_b \), is in agreement with the derivation in [1, 2, 3], see also [104].

B2B map. The scattering data allows us to construct the (reduced) radial action [2, 3]

\[ i_r = \frac{p_\infty}{\sqrt{-p^2_\infty}} \chi^{(1)}_b - j \left( 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\chi^{(2n)}_j}{(1 - 2n)^{2n}} \right), \tag{21} \]

via analytic continuation to \( \gamma < 1 \). As we discussed in [2, 3], the natural power counting in \( 1/j \) in the PM expansion requires the (so far unknown) \( \chi^{(4)}_j \) coefficient. The latter can be written, using the results in [2, 3], as

\[ \chi^{(4)}_j = \frac{3\pi}{8M^4\mu^4} \left( P_1P_3 + \frac{1}{2} P^2_2 + p^2_\infty P_4 \right), \tag{22} \]

with the \( P_n \)'s from the expansion of the CoM momentum

\[ p^2 = p^2_\infty + \sum_{n=1}^{\infty} P_n(E) \left( \frac{G}{r} \right)^n. \tag{23} \]

The \( P_n \)'s can also be obtained from the scattering angle, as described in [2, 3]. For instance, inverting the relation

\[ \chi^{(3)}_j = \frac{1}{M^4\mu^4p^2_\infty} \left( - \frac{P^3}{24} + p^3_\infty + \frac{P_1P_2}{2} + p^3_\infty P_3 \right), \tag{24} \]

together with (20) and the results in [1], yields

\[ \frac{P_3}{M^3\mu^2} = \left( \frac{18\gamma^2 - 1}{2\nu} + \frac{8\nu}{\Gamma}(3 + 12\gamma^2 - 4\gamma^4) \right) \sinh^{-1}\sqrt{\gamma^2 - 1} + \frac{\nu}{6\Gamma} \left( 6 - 20\gamma - 10\gamma^2 - 24\gamma^3 + \frac{18\Gamma(1 - 2\gamma^2)(1 - 5\gamma^2)}{(1 + \Gamma)(1 + \gamma)} \right). \tag{25} \]

This compact expression encodes all the information at 3PM order. It can be analytically continued to negative binding energies (\( \gamma < 1 \)) to compute observables for binary systems via the B2B map. Because of the factor of \( \gamma < 1 \) in [22], and the fact that [23] has a well-defined static limit, the contribution from the \( P_4 \) term to the scattering angle becomes subleading in the PN expansion. This allows us to perform a consistent PN-truncation of [21], by keeping the \( P_{n\leq3} \) terms in [22]. This is carried out in detail in [24, 3], and shown to agree with the literature in the overlapping regime of validity.

Amplitude & Hamiltonian. It is instructive to use the B2B dictionary to also reconstruct both, the classical limit of the scattering amplitude as well as the Hamiltonian for the two-body system in the CoM (isotropic) frame. Using the impetus formula for potential modes [2],

\[ p^2 = p^2_\infty + \frac{1}{2\nu} \int d^3r\mathcal{M}(p, q)e^{i\varphi r}, \tag{26} \]

we immediately read off from (25) the (infrared-finite part of the) scattering amplitude in the classical limit, which agrees with the result in [3] (see Eq. (9.3)). For the PM expansion of the Hamiltonian,

\[ H(r, p^2) = \sum_i c_i(p^2) \left( \frac{G}{r} \right)^i, \tag{27} \]

the coefficients can also be expressed iteratively in terms of the \( P_n \)'s in [23, 25]. To 3PM order we find

\[ c_3(p) = \frac{-P_3(E)}{3!} = \frac{-P_3(E)}{2E\xi} + \left( \frac{3\xi - 1}{4E^3\xi^3} - \frac{16E^2\xi^2}{4E^3\xi^3} \right) \frac{P_2(E)P_1(E)P_1(E)}{16E^5\xi^5} \]

\[ - \frac{P_2^2(E)P_1(E)}{16E^3\xi^3} - \frac{P_1(E)P_1(E)P_1(E)}{16E^4\xi^4} \right) \]

\[ - \frac{P^2_2(E)P_1^2(E)}{8E^3\xi^3}, \tag{28} \]

where prime denotes a derivative with respect to \( E \), and \( \xi \equiv E_i E_2/(E_1 + E_2)^2 \). Inputting [25], and \( P_{1,2} \) from the 2PM results [1], we exactly reproduce the \( c_3 \) in [4, 5]. Notice, however, that the relevant PM information to compute observables through the B2B map is (more succinctly) encoded in (25) at two loops, and ultimately the (yet to be computed) scattering angle at 4PM order.
Conclusions. Using the EFT approach and B2B dictionary \cite{4,5}, we derived the conservative dynamics for non-spinning binary systems to 3PM order. Our results, purely within the classical realm, are in perfect agreement with those reported in \cite{4,5}, thus removing the objections raised in \cite{6} against their validity. Even though, unlike the approach in \cite{4,5}, our derivation entails the use of Feynman diagram, because of the simplifications of the EFT/B2B framework just a handful are required (two of which are zero at this order), see Fig. 1. Moreover, only massless integrals appear and, as it was already illustrated in \cite{1}, we do not encounter the (super-classical) infrared singularities which have, thus far, polluted the extraction of classical physics from the amplitudes program. By adapting to our EFT approach the methods in \cite{4,5,35,69}, we have found that the contribution from potential modes to the master integrals can be computed to all orders in velocities using differential equations (without the need of the PN-type resumptions in \cite{4,5}). Remarkably, the boundary conditions are obtained from the knowledge of the same master integrals which appear in the static limit with PN sources to two loops, albeit in $D - 1$ and $D - 2$ dimensions. This implies that the PM dynamics can be bootstrapped from PN information (at least to NNLO). This is not surprising for the evaluation on the unperturbed trajectories, which serves as a stationary limit of the PM regime, but strikingly the same occurs for the iterations. Since master integrals for the PN expansion are known to four loops \cite{31}, bootstrapping integrals through differential equations could potentially give us all that is needed to solve for the dynamics up to 5PM order. We note also that the infusion of data from outside of PN/PM schemes can further simplify the computation. For instance, matching to the dynamics in Schwarzschild would have allowed us to obtain the value of the $M^{(1,1)}$ master integrals which appeared in the iterations. In turn, these are related to the $M^{(1,2)}$ family in the static limit. Hence, this would permit us to read off their boundary condition, and subsequently the entire velocity dependence with the differential equations, directly from the test-body limit. The fact that we get extra mileage from the probe limit is not surprising \cite{2}. After all, the iterations are composed of one-point functions, as seen in diagrams (a) and (b) in Fig. 1. Together with diagrams (c) and (d), all of which have to be symmetrized under $m_1 \leftrightarrow m_2$, they fix the coefficient of the $(m_1^2 + m_2^2)$ term neatly combines the $(m_1^2 + m_2^2)$ term in \cite{19} to reorganize themselves into the first line of \cite{20}. Likewise, information from the gravitational self-force program \cite{106,107} may be used to aid the calculation in the PM expansion, see e.g. \cite{39,64,67,108,111}. Irrespectively of the weapon of choice, the B2B dictionary \cite{2,3} is imploring us to continue to even higher orders. The derivation of the needed 4PM scattering angle is ongoing in the EFT approach, which we have demonstrated here is a powerful framework, not only for PN calculations \cite{18,22}, but also in the PM regime \cite{1}.

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