The impact of S-wave thresholds $D_{s1}\bar{D}_s + \text{c.c.}$ and $D_{s0}\bar{D}_s^* + \text{c.c.}$ on vector charmonium spectrum

Zheng Cao$^{1,2,*}$ and Qiang Zhao$^{1,2,3,†}$

1 Institute of High Energy Physics and Theoretical Physics Center for Science Facilities, Chinese Academy of Sciences, Beijing 100049, China
2 School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China and
3 Synergetic Innovation Center for Quantum Effects and Applications (SICQEA), Hunan Normal University, Changsha 410081, China

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By investigating the very closely lied $D_{s1}\bar{D}_s + \text{c.c.}$ and $D_{s0}\bar{D}_s^* + \text{c.c.}$ thresholds at about 4.43 GeV we propose that the $\psi(4415)$ and $\psi(4160)$ can be mixing states between the dynamic generated states of the strong S-wave $D_{s1}\bar{D}_s + \text{c.c.}$ and $D_{s0}\bar{D}_s^* + \text{c.c.}$ interactions and the quark model states $\psi(4S)$ and $\psi(2D)$. We investigate the $J/\psi K\bar{K}$ final states and invariant mass spectrum of $J/\psi K$ to demonstrate that nontrivial lineshapes can arise from such a mechanism. This process, which goes through triangle loop transitions, is located in the vicinity of the so-called “triangle singularity (TS)” kinematics. As a result, it provides a special mechanism for the production of exotic states $Z_{cs}$, which is the strange partner of $Z_c(3900)$, but with flavor contents of $c\bar{c}q\bar{q}$ (or $c\bar{c}q\bar{q}$) with $q$ denoting u/d quarks. The lineshapes of the $e^+e^\to J/\psi K\bar{K}$ cross sections and $J/\psi K$ ($J/\psi K$) spectrum are sensitive to the dynamically generated state, and we demonstrate that a pole structure can be easily distinguished from open threshold CUSP effects if an exotic state is created. A precise measurement of the cross section lineshapes can test such a mixing mechanism and provide navel information for the exotic partners of the $Z_c(3900)$ in the charmonium spectrum.

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I. INTRODUCTION

During the past decade the observations of a large number of hadronic exotic candidates have initiated tremendous activities and efforts on understanding their dynamic nature in both experiment and theory. Most of these heavily flavored states which are tentatively named by “$XYZ$” are intimately related to some nearby S-wave thresholds. This seems to provide important clues for understanding their intrinsic structures. Typical examples include $X(3872)$ and $Z_c(3900)$ $^1$ which are close to the $DD^* + \text{c.c.}$ threshold, and $Z_c(4020)$ $^2$ to the $D^*\bar{D}^* + \text{c.c.}$ threshold. Their bottomed correspondences are $Z_{bc}(10610)$ and $Z_{bc}(10650)$ $^3$ which are located at the $BB^* + \text{c.c.}$ and $B^*\bar{B}^* + \text{c.c.}$ thresholds, respectively. In the vector charmonium spectrum the mysterious $Y(4260)$ seems to be closely related to the S-wave $D_1(2420)\bar{D} + \text{c.c.}$ threshold in order to understand many new experimental observations of its exclusive decays. Recent studies indicate strong evidence for the hadronic molecule component of $D_1(2420)\bar{D} + \text{c.c.}$ in its wavefunction while a compact core should also be present as the consequence of heavy quark spin symmetry (HQSS) breaking effects $^4$ $^6$.

Following these interesting discoveries, many theoretical interpretations are proposed in the literature. Several recent review articles have given detailed discussions on the experimental status and theoretical models for these exotic candidates (see e.g. Ref. $^8$ for a review of hadronic molecules, Ref. $^6$ for a review of open charm/bottom system, Ref. $^{10}$ for a review of newly discovered states and the comparison with theoretical expectations, and Ref. $^{11}$ for a review of different progresses made in the heavy-quark exotics field). For the heavy quarkonium exotic meson candidates typical scenarios include hadrocharmonium $^{12}$, tetraquarks $^{13}$ $^{14}$, loosely bound molecules $^{15}$ or hybrids $^{16}$. Different kinds of kinematic effects related to these states were also discussed in the literature, such as CUSP effects $^{17}$ $^{20}$, and triangle singularities $^{1}$ $^{21}$ $^{22}$. In Ref. $^{26}$, it was demonstrated that though CUSP effects can result in some structures, it is still not possible to produce pronounced, narrow near-threshold peaks without introducing physical poles. In contrast, the triangle singularity mechanism makes it possible to enhance threshold structures on top of a pole. Special features arising from such a mechanism have attracted a lot of attention in the understanding of many threshold phenomena $^{1}$ $^{27}$ $^{31}$. In Refs. $^{32}$ $^{33}$, a practical parametrization for the line shapes of the near-threshold states is proposed. Based on the Lippmann-Schwinger equations for the coupled channel problem, this approach incorporates the inelastic channels additively with the unitarity and analyticity constraints for the $t$ matrix.

$^*$ Email address: caco@ihep.ac.cn
$^†$ Email address: zhaoq@ihep.ac.cn
In this work, we investigate the very closely lied $D_{s1}\bar{D}_s + c.c.$ and $D_{s0}D^*_s + c.c.$ thresholds which are located between two nearby charmonia states in the quark model. For the convenience we note these two thresholds by $D_{s1}\bar{D}_s$ and $D_{s0}D^*_s$ as follows in this work. We study the mixing mechanism between two nearby quark model states through these two thresholds and the dynamically generated states which can be possibly related to $\psi(4115)$ and $\psi(4160)$ [46].

Similar to the production process of $Z_c(3900)$ in $e^+e^- \rightarrow Y(4260) \rightarrow J/\psi\pi\pi$ where the $S$-wave threshold $D_1(2420)\bar{D}_1 + c.c.$ plays a crucial role for understanding the properties of $Y(4260)$ and $Z_c(3900)$, the process $e^+e^- \rightarrow J/\psi K\bar{K}$ around the mass region of the thresholds of $D_{s1}\bar{D}_s$ and $D_{s0}D^*_s$ may provide important clues for understanding the nearby $\psi(4415)$. In 2007, Belle Collaboration investigated the $J/\psi K^+K^-$ final states in $e^+e^-$ annihilations via the initial-state radiation (ISR) from threshold to the center of mass (c.m.) energy of 6.0 GeV [34]. The measured cross sections seemed to be improved with the inclusion of a coherent $\psi(4415)$. However, the limited statistics did not allow a conclusion on the detailed properties of $\psi(4415)$. With the possible correlations with the $S$-wave $D_{s1}\bar{D}_s$ and $D_{s0}D^*_s$ thresholds, the $J/\psi K\bar{K}$ decay channel may shed a light on the structure of $\psi(4415)$.

Interesting issues that can also be investigated in $e^+e^- \rightarrow J/\psi K\bar{K}$ are the role played by the triangle singularity (TS) mechanism, and possible production of exotic states which can couple to $D_s\bar{D}^* + c.c.$ and $D^*_s\bar{D} + c.c.$ and contain at least four quarks in their wavefunctions. This is a mechanism similar to the production of $Z_c(3900)$ as proposed in Ref. [4]. In Ref. [25], the TS mechanism corresponding to similar charmed-strange meson thresholds but with final states of $J/\psi$ and a hidden $s\bar{s}$ is also investigated. To be more specific, given that the initial vector states can first couple to $D_{s1}\bar{D}_s$ or $D_{s0}\bar{D}^*_s$, the intermediate $D_{s1}$ or $D_{s0}$ can then rescatter against $D_s$ or $\bar{D}_s$ by exchanging $D^*$ or $D$, respectively, before converting into a Kaon, and then the interactions between the exchanged $D^*$ (or $D$) and $D_s$ (or $\bar{D}_s$) will form $J/\psi$ and an anti-Kaon. Such a transition is via a triangle diagram, and for specific kinematics all these three internal particles may approach their on-shell conditions simultaneously. Such a kinematic condition is called the TS condition and it brings the leading singular amplitude to the loops. Actually, around the mass region of $\psi(4415)$, the kinematics are close to the TS condition and special phenomena are expected to show up that can be explored in experiment. Moreover, in case that exotic states can be formed by the $S$-wave interaction between $D_s\bar{D}^* + c.c.$ (and/or $D^*_s\bar{D} + c.c.$) meson pairs, nontrivial linedhapse in the invariant mass spectrum of $J/\psi K$ ($J/\psi\bar{K}$) is also expected.

As follows, we first present the formalism for the dynamically generated states due to the strong $S$-wave couplings to $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}^*_s$ in Section II. We then analyze the kinematics of the triangle loops in $e^+e^- \rightarrow J/\psi K\bar{K}$ and present the calculation results in Section III with discussions. A brief summary will be given in the last Section.

II. DYNAMICALLY GENERATED STATES

The mass thresholds for both $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}^*_s$ lie at about 4.43 GeV (which are 4.428 GeV and 4.429 GeV, respectively), implying the nearly equal spin splitting of mass in the $(1/2)^+$ and $(1/2)^-$ doublets which also happens in the beauty-strange excited meson pairs [35]. Two charmonia, $\psi(4S)$ and $\psi(2D)$, in the potential quark model with the masses close to these two thresholds can couple to them via an $S$-wave interaction. Given sufficiently strong couplings, it may dynamically generate pole states near these thresholds and result in mixings between the quark model states and the dynamically generated states through the intermediate $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}^*_s$ bubbles as shown in Fig. 1. To investigate such a possible scenario, we construct the propagators of $\psi(4S)$ and $\psi(2D)$ in a coupled-channel approach [34] as the following:

$$G = \frac{1}{|D_1D_2 - |D_{12}|^2|} \begin{pmatrix} D_1 & D_{12} \\ D_{21} & D_2 \end{pmatrix}$$

(1)

$$G = \frac{G_{12}}{\det[G_{12}]}.$$}

(2)
where $D_1$ and $D_2$ are the denominators of the single propagator of $\psi(4S)$ and $\psi(2D)$, respectively, and $D_{12}$ is the mixing term between them through the $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}_s^*$ bubble diagrams. So here we have

$$D_1 = m_a^2 - s - iB_{11},$$
$$D_2 = m_b^2 - s - iB_{22},$$
$$D_{12} = iB_{12},$$

where $B$ is the sum of the two amplitudes of the bubble diagrams of $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}_s^*$ between two states. Since $\psi(4S)$ and $\psi(2D)$ both couple to $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}_s^*$ in an $S$-wave, we have

$$B_{11} = 2g_1^2(I_{20}(P, m_{D_{s1}}, m_{D_s}) + I_{20}(P, m_{D_{s0}}, m_{D_s})), \quad (3)$$
$$B_{22} = 2g_2^2(I_{20}(P, m_{D_{s1}}, m_{D_s}) + I_{20}(P, m_{D_{s0}}, m_{D_s})), \quad (4)$$
$$B_{12} = 2g_1g_2(I_{20}(P, m_{D_{s1}}, m_{D_s}) + I_{20}(P, m_{D_{s0}}, m_{D_s})), \quad (5)$$

where $g_1$ ($g_2$) is the bare coupling for $\psi(4S)$ ($\psi(2D)$) to these two thresholds, $D_{s1}\bar{D}_s$ or $D_{s0}\bar{D}_s^*$; $I_{20}(P, m_a, m_b)$ is the two-point loop integral with the initial energy $P$ and intermediate particle masses $m_a$ and $m_b$. To remove the divergent part of this integral, we adopted an exponential momentum-dependent form factor $\exp(-2\bar{P}^2/\Lambda^2)$ where $\bar{P}$ is the momentum of the particles in the loop. More details about the integral can be found in Appendix A.

The corresponding physical states $|A\rangle$ and $|B\rangle$ can be expressed as mixtures of the quark model states $|a\rangle$ and $|b\rangle$ with a mixing matrix, i.e.

$$\left( \begin{array}{c} |A\rangle \\ |B\rangle \end{array} \right) = \left( \begin{array}{cc} \cos\theta & -\sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & \cos\theta \end{array} \right) \left( \begin{array}{c} |a\rangle \\ |b\rangle \end{array} \right) \equiv R(\theta, \phi) \left( \begin{array}{c} |a\rangle \\ |b\rangle \end{array} \right),$$

(9)

With the mixing matrix $R(\theta, \phi)$ the physical propagator matrix $G_{12p}$ can be related to $G_{12}$ by

$$G_{12p} = RG_{12}R^\dagger.$$  

(11)

The physical propagator matrix $G_{12p}$ should be a diagonal matrix. So we can search for the physical poles in the propagator matrix $G$ by requiring $\det[G_{12}] = 0$.

The coupling constants $g_1$ and $g_2$ are unknown parameters in matrix $G_{12}$ which can be determined by requiring the physical poles located at the masses of the observed states. At this moment, we treat $g_1 = g_2 \equiv g$ for simplicity. It has been studied in the literature that the $D_{s1}(2460)$ is a mixed state of $^1P_1$ and $^3P_1$ with compatible strength. This allows that the $D_{s1}\bar{D}_s$ pair can couple to the $S$ and $D$-wave charmonia with similar coupling strengths in the $S$ wave. This argument, in principle, does not apply to the $\psi(2D)$ coupling to $D_{s0}\bar{D}_s^*$ which will be suppressed by the HQSS. However, as investigated broadly in the charmonium mass region, the HQSS is rather apparently broken [6]. Therefore, it is still reasonable to treat the coupling in an $S$ wave as a leading approximation.

To be more specific with our study here, we have assumed that the $D_{s1}\bar{D}_s$ pair can couple to the nearby $S$-wave thresholds are strong enough for dynamically generating the

FIG. 2: Pole structures determined by $\det[G_{12}] = 0$ in the propagator matrix. The thick grey vertical line indicates the threshold of $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}_s^*$ which are degenerate with each other. The two thin grey lines denote the masses of $\psi(4S)$ and $\psi(2D)$ in the potential quark model, respectively.
physical states $\psi(4160)$ and $\psi(4415)$. Whether this is a reasonable assumption can be examined by two aspects. The first one is the cross section lineshape in the vicinity of these two thresholds. Given the strong $S$-wave interactions with the nearby quark model states, the propagators cannot be described by a simple Breit-Wigner form. Thus, the cross section lineshape will appear to be nontrivial. The second aspect is the decay modes of such dynamically generated states. They will favor decay channels correlated with the threshold interactions. In this case, the reaction channel of $e^+e^- \rightarrow J/\psi KK$ will be extremely interesting.

Since we still lack experimental data for $e^+e^- \rightarrow J/\psi KK$ in the vicinity between $\psi(4160)$ and $\psi(4415)$, the following strategy is adopted for investigating the underlying dynamics. By examining the movement of the pole positions of the physical states in terms of coupling $g$ from 1 to 10 GeV$^{-1/2}$ which is the typical coupling range for the $S$-wave coupling of charmonium-like states to heavy-light $D$ mesons, we identify poles which can match the nearby charmonium states $\psi(4160)$ and $\psi(4415)$ with a reasonable coupling strength for $g$. We find that with $g = 7$ GeV$^{-1/2}$ the obtained pole masses by diagonalizing the propagator matrix $G_{12}$ lie at about 4.41 and 4.17 GeV which are very close to the masses of $\psi(4415)$ and $\psi(4160)$, respectively, as showed in Fig. 2. It suggests the possibility of regarding $\psi(4415)$ and $\psi(4160)$ as the mixtures of dynamically generated states by the coupled-channel interactions with the quark model states (\psi(4S) and \psi(2D)), near the $S$-wave thresholds, i.e., $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}_s^*$.

For the physical propagator of $\psi(4415)$ we need to sum over all the combination of bubbles and bare propagators like Fig. 3. So the physical propagator can be written as

$$i \frac{N_{1p}}{N_{1}} = i \frac{N_{1}}{N_{1}} G \frac{i}{N_{s}} + i \frac{N_{1}}{N_{s}} G \frac{i}{N_{s}} + ..., \tag{12}$$

where $i/N_s = i/N_1 + i/N_2$ is the sum of the two bare propagators of $\psi(4S)$ and $\psi(2D)$ and $G(E)$ is the sum of the two kinds of bubbles of $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}_s^*$ with bare couplings. So we have

$$i \frac{N_{1p}}{N_{1}} = i \frac{N_{1}}{N_{1}} G \frac{i}{N_{s}} + i \frac{N_{1}}{N_{s}} G \frac{i}{N_{s}} + ... \tag{13}$$

$$= \frac{i}{N_{1}} / (1 - G \frac{i}{N_{s}}) \tag{14}$$

$$= \frac{i}{2m_1(E - m_1 - \Sigma_1(E)),} \tag{15}$$

where $\Sigma_1 = iG(E)(N_1 + N_2)/(2N_2m_1)$. $E$ is the initial energy of $\psi(4415)$ and $m_1$ is the bare mass of $\psi(4S)$. By expanding the denominator of the propagator near the physical mass $\Sigma_1$ we have

$$\frac{2m_1i}{N_{1p}} = \frac{i}{E - m_1 - \Sigma_1(E)} \tag{16}$$

$$= \frac{iZ}{E - m_1p - Z\Sigma_1(E)} \tag{17}$$

where $\Sigma_1(E) = \Sigma_1(E) - Re[\Sigma_1(m_{1p})] - (E - m_{1p})Re[\partial_E \Sigma_1(m_{1p})]$ and $Z = 1/\sqrt{1 - Re(\partial_E \Sigma_1(m_{1p}))}$ is the wavefunction renormalization constant, $m_{1p}$ is the physical mass of $\psi(4415)$ and $m_1 = m_{1p} - Re[\Sigma_1(m_{1p})]$.

III. MANIFESTATION OF THE DYNAMICALLY GENERATED STATE IN $e^+e^- \rightarrow J/\psi KK$

With this physical propagator of $\psi(4415)$ and its strong coupling to the $D_{s1}\bar{D}_s$ and $D_{s0}\bar{D}_s^*$ thresholds we consider the $e^+e^- \rightarrow J/\psi KK$ through triangle loops showed in Fig. 3(a) and (b). In these diagrams, the $D_{s1}$ decays into $D^*K$ and $D_{s0}$ decays into $DK$ both in a relative $S$ wave. Also, the scatterings of $D_s D^*$ and $D_s^* D$ to $J/\psi K$ are also via an $S$ wave. As pointed out earlier, the triangle transition is located in the vicinity of the kinematic condition for

\[ ... \]

\[ ... \]

\[ ... \]

\[ ... \]

\[ ... \]

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the TS. Therefore, it is necessary to investigate the kinematic effects arising from the TS mechanism and identify the dynamically generated states via the S-wave interactions with the nearby open thresholds.

To proceed, we first give the corresponding Lagrangians for the S-wave coupling:

$$\mathcal{L}_{\Psi SH} = \langle \overline{\Psi} \tilde{S}_a H_a + \Psi \tilde{H}_a \overline{S}_a \rangle,$$

$$\mathcal{L}_{SHA} = i \hbar \langle \overline{H}_a S_b \gamma_\mu \gamma_5 A^\mu_{ba} \rangle,$$

$$\mathcal{L}_{HH\Psi A} = C \langle \overline{\Psi} \tilde{H}_a \gamma_\mu \gamma_5 H_a A^\mu_{ba} \rangle,$$

(18)

where $S$ and $H$ represent the positive and negative parity charmed mesons, respectively; while $\Psi$ is the field of the vector charmonium states and $A$ is the chiral field. The explicit form of each field can be found in Appendix B. As emphasized before, we require that these couplings are within the natural scale and more stringent constraints can be imposed by future experimental measurements.

Typical triangle diagrams for charmonium decays into $J/\psi KK$ are plotted in Fig. 4. For Fig. 4 (a), the intermediate $D_{s1}$ plays a key role since it has a strong S-wave coupling to $D^* K$. As broadly studied in the literature (see e.g. Ref. [8] and references therein for a recent review of hadronic molecules), the $D_{s1}$ has been an ideal candidate for a $D^* K$ molecule. Although the mass of $D_{s1}$ is slightly lower than the mass threshold of $D^* K$ which is 2.55 GeV, it has approached the TS kinematics closely with the initial energy also approaching the $D_{s1} \bar{D}_s$ threshold. The presence of the TS also indicates that the dominant contributions from the triangle loop come from the kinematic region where all the internal states are approaching their on-shell condition simultaneously.

One feature arising from the specific process under discussion is that the physical kinematic region for the TS is quite limited. As analyzed in Ref. [24], the physical region for the TS is related to the phase space of the intermediate state two-body decay, i.e. $D_{s1} \rightarrow D^* K$. Since the mass of $D_{s1}$ is slightly lower than the $D^* K$ threshold, the contributions from the TS mechanism will be limited to a rather narrow kinematic region. But still, abnormal lineshape can be expected.

Similar phenomenon happens with the $D_{s0} \bar{D}_s^*$ loop of Fig. 4 (b). Also, it should be mentioned that $D_{s0}(2317)$ is an ideal candidate for the $DK$ molecule (see Ref. [8] for a detailed review). Because of the lack of phase space for $D_{s0} \rightarrow DK$, the TS kinematics will be restricted within a narrow physical region. But still, observable effects can be expected.

The explicit amplitudes of the diagrams in Fig. 4 can be found in Appendix B. Before we come to the calculation results for the final state invariant mass spectra, we first examine the cross section lineshape for $e^+e^- \rightarrow J/\psi KK$ around the mass of $\psi(4415)$. In Fig. 5 the calculated cross sections are compared with the experimental data from
the Belle Collaboration \[34\]. The \(\psi(4415)\) as the dynamically generated state which mixes with the quark model state has a lineshape which is apparently deviated from the symmetric Breit-Wigner distribution. Although it is not conclusive from the present data quality, such an effect can be investigated at BESIII or future Belle-II.

The invariant mass spectrum of the \(J/\psi K\) is generally sensitive to the TS mechanism. We plot the \(J/\psi K\) spectra in Fig. 6(a) where contributions from Fig. 4(a) and (b) are both included. Also, in order to see the evolution of the TS contributions in terms of the initial energy, we plot the spectra at several energy points from 4.5 to 4.8 GeV. One can see that a CUSP structure, which is located at the common threshold of \(D_sD^*\) and \(\bar{D}_s^*D\), appears in the \(J/\psi K\) invariant mass spectrum. It is difficult to find very clear pole-like structure when the initial energy of \(\sqrt{s}\) is just above the threshold of \(D_{s1}\bar{D}_s\) or \(D_{s0}\bar{D}_s^*\). But as the initial energy of \(\psi(4415)\) increases from 4.5 GeV to 4.8 GeV, a peak-like structure near the threshold of \(D_sD^*\) (\(\bar{D}_s^*D\)) indeed becomes more obvious. Since the mass of \(D_{s1}\) is so close to the threshold of \(D^*K\), we also discussed the behavior of the spectrum when the mass of \(D_{s1}\) is shifted a little bit in Appendix C. In Ref. \[26\], it has been shown that lower order singularities than the TS would not produce narrow and pronounced peaks if the interactions between the rescattering hadrons are not strong enough. Similar phenomenon is observed here as shown by Fig. 6. Because of the limited phase space, the TS condition cannot be fully satisfied, thus, the nontrivial threshold structure appears as a CUSP effect instead of the typical narrow peak \[47\]. Thus, this process can serve as an ideal channel for the search for possible exotic candidates without ambiguities from the kinematic effects.

In Fig. 6(b) we show the calculations at the initial energy of 4.6 GeV but including explicitly a physical pole right at the mass of the threshold of \(D_sD^*\) and \(\bar{D}_s^*D\). The consideration is that if there exists the strange partner \(Z_{cs}\) of \(Z_s(3900)\) as the hadronic molecules of \(D_sD^*\) and \(\bar{D}_s^*D\), the pole structure near the open charm threshold will produce different lineshapes compared with the kinematic effects shown in Fig. 4(a). Similar to the treatment of Refs. \[2, \, 4, \, 7\] the pole structure can be dynamically generated by the strong \(D_sD^*\) and \(\bar{D}_s^*D\) interactions. Although the detailed dynamics need elaborate studies and are not going to be discussed here, we note that if any mechanism allows the formation of the exotic state with quark contents of \(c\bar{c}q\bar{s}\) (\(c\bar{c}\bar{q}\bar{s}\)) in this process, the pole structure will appear explicitly in the \(J/\psi K\) \((J/\psi \bar{K})\) invariant mass spectrum as the signature for a genuine state. The thin and broad solid lines in Fig. 6 (b) correspond to the pole coupled to \(J/\psi K\) with couplings 0.25 and 0.5 of the nature scale, respectively. Compared with the other lines without the pole structure in the \(J/\psi K\) invariant mass spectrum, it shows that the pole contributions and pure TS contributions behave quite differently. In this case, the TS mechanism can produce non-trivial lineshapes, but cannot produce predominant peaks at the threshold of \(J/\psi K\). If narrow and sharp-peakings structures are observed in the invariant mass spectrum of \(J/\psi K\), they can be confidently assigned as signatures for exotics. Also, note that the asymmetric lineshapes are because of the triangle function which will affect the formation of the exotic \(Z_{cs}\) state. In this sense, this channel is ideal for testing the TS mechanism and searching for exotic candidates in \(e^+e^-\) annihilations.

IV. SUMMARY

In this work, we investigate phenomena arising from the possible strong couplings of the degenerate thresholds \(D_{s1}\bar{D}_s\) and \(D_{s0}\bar{D}_s^*\) which may lead to dynamically generated hadronic molecule states and mix with the nearby
conventional charmonia $\psi(4S)$ and $\psi(2D)$. We find that such a mechanism may have observable effects on $\psi(4415)$ with relatively large mixture of the $D_s \bar{D}_s$ and $D_s \bar{D}_s^*$ molecules, while its impact on $\psi(4160)$ is relatively small. With the same coupling of $\psi(4415)$ to $D_s \bar{D}_s$ and $D_s \bar{D}_s^*$ we study the $J/\psi K$ final state invariant mass spectrum in $\psi(4415) \to J/\psi K \bar{K}$. It shows that nontrivial lineshapes can be produced by the molecular nature of $\psi(4415)$ in the invariant mass spectrum of $J/\psi K$ due to the presence of the TS mechanism. However, since the TS kinematic region is limited, it is unlikely that the TS mechanism alone would generate peaking structures near the threshold of $D^* \bar{D}_s + c.c.$ and $DD_s^* + c.c.$ This provides an ideal channel for testing the TS mechanism on the one hand, and on the other hand, pinning down the process which is sensitive to the production of exotic states $Z_{cs}$ near heavy flavor thresholds. We claim that any predominant peaking structure in the invariant mass spectrum of $J/\psi K$ should confirm its being a genuine state instead of kinematic effects. Experimental data from BESIII and Belle-II can help clarify such a phenomenon in the future.

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Appendix A: Two-point loop integral

The $I_{20}$ in Section II can be written in the form

$$I_{20}(P, m_1, m_2) = \int \frac{d^4l}{(2\pi)^4} \frac{\exp(-2l^2/\Lambda^2)}{(l^2 - m_1^2 + i\epsilon)((P - l)^2 - m_2^2 + i\epsilon)}.$$  \hspace{1cm} (A1)

And this integral can be calculated analytically:

$$I_{20} = \int \frac{d^4l}{(2\pi)^4} \frac{\exp(-2l^2/\Lambda^2)}{(l^2 - m_1^2 + i\epsilon)((P - l)^2 - m_2^2 + i\epsilon)}$$

$$= \frac{i}{4m_1m_2} \frac{4\pi}{(2\pi)^3} \int_0^\infty dl \frac{l^2 \exp(-2l^2/\Lambda^2)}{P - m_1 - m_2 - l^2/2\mu_{12}}$$

$$= \frac{i}{4m_1m_2} \left\{ -\frac{\mu \Lambda}{(2\pi)^3/2} + \frac{\mu k}{2\pi} e^{-2k^2/\Lambda^2} \left[ \text{erfi} \left( \frac{\sqrt{2}k}{\Lambda} \right) - i \right] \right\},$$  \hspace{1cm} (A2)

where $k = \sqrt{2\mu(M - m_1 - m_2)}$ and $\mu_{ij}$ is the reduced mass of the intermediate particles which are labeled as $i$ and $j$. The imaginary error function is defined as

$$\text{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^t dt.$$  \hspace{1cm} (A3)

Appendix B: Triangle diagram amplitude

The fields in the Lagrangians in Eq. (18) can be written in the form of

$$H_a = \frac{1 + \gamma_5}{2} (D^*_a \gamma^\mu - D_a \gamma^\mu),$$  \hspace{1cm} (B1)

$$S_a = \frac{1 + \gamma_5}{2} (D^\mu_a \gamma_5 \gamma^\mu - D^\mu_a \gamma_5),$$  \hspace{1cm} (B2)

$$\Psi = \frac{1 + \gamma_5}{2} (\psi(n S)^\mu \gamma^\mu) \frac{1 - \gamma_5}{2},$$  \hspace{1cm} (B3)
FIG. 7: Typical 3-body decays via a triangle diagram. Here, we define $s_2 = (p_b + p_c)^2$ and $s_3 = p_a^2$.

FIG. 8: The invariant mass spectrum of $J/\psi K^-$ calculated for Fig. 4(a) at the initial energy $\sqrt{s}$ near the three body on-shell condition with $m_2 = \Delta m + m_{D^*}$, and $\Delta m = 0, 30, 40, 50$ MeV for the solid, dotted, dashed and dot-dashed curve, respectively.

where $a$ is light flavor index. We can then write the amplitude for both triangle diagrams in Fig. 4 as follows in the non-relativistic limit of the heavy mesons

$$iM = \tilde{g} \epsilon_{\psi} \cdot \epsilon_{J/\psi} E_{1K} E_{2K} \times [I^{(0)}(m_{D_0}, m_{D_0}, m_{D^*}, P, m_{K^+}, m_{J/\psi K^-}) + I^{(0)}(m_{D_0}, m_{D^*}, P, m_{K^+}, m_{J/\psi K^-})$$
$$+ I^{(0)}(m_{D_0}, m_{D^*}, P, m_{K^+}, m_{J/\psi K^-}) + I^{(0)}(m_{D_0}, m_{D^*}, P, m_{K^+}, m_{J/\psi K^+})],$$

where $\tilde{g}$ denotes the product of all the coupling constants from the vertices in the triangle diagram, and $I^{(0)}$ is the triangle diagram integral:

$$I^{(0)}(l_1, l_2, l_3, M_1, M_2) = i \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - m^2_1 + i\epsilon)(l^2 - m^2_2 + i\epsilon)}.$$

By defining $s = P^2$, $s_1 = m_{K^+ K^-}^2$ and $s_2 = m_{J/\psi K^-}^2$, we will have

$$m_{J/\psi K^+}^2 = s + m_{J/\psi}^2 + 2m_K^2 - s_1 - s_2,$$

and $E_{1K} \equiv (s + m_K^2 - s_2)/(2\sqrt{s})$ is the energy of the individual Kaon at the $SHA$ vertex, $E_{2K} \equiv (s + m_K^2 - s - 2m_K^2 - m_{J/\psi K}^2 + s_1 + s_2)/(2\sqrt{s})$ is the energy of the Kaon at the $J/\psi K$ vertex. The amplitude is a function depending on $s$, $s_1$ and $s_2$, with $s$ the initial energy squared. The total cross section can be obtained by integrating over $s_1$ and $s_2$ in their phase spaces, and the invariant mass spectrum of $J/\psi K^-$ can be obtained by only integrating over $s_1$.

**Appendix C: Triangle singularity condition**

Early studies of the TS can be found in the literature of last 60’s [37–45]. Its manifestations in high-energy reactions are recognized recently thanks to the high-quality experimental data from B-factories, CLEO-c, BESIII, and LHCb in various exclusive processes. A revival of studying the TS can be found in the recent literature [4, 21–25, 27–30]. An up-to-date review can be found in Ref. 8. Here, we only show how the phase space limits the manifestation of the TS.
For a 3-point loop diagram showed in Fig. 7, the location of external momentum variables for different kinds of singularities are determined by the Landau Equation (37). When all the three internal particles get on-shell simultaneously, it pinches the leading singularity of the triangle loop which corresponds to the TS. In Fig. 4 given the initial energy square \( s\), \( s_2 \equiv (p_b + p_c)^2\), and \( s_3 \equiv p_a^2\), the location of the TS can be determined by solving the Landau Equation:

\[
s^\pm = (m_2 + m_3)^2 + \frac{1}{2m_1^2}(m_1^2 + m_2^2 - s)(s_2 - m_1^2 - m_3^2) - 4m_1^2m_2m_3 \\
\pm \lambda^{1/2}(s_2, m_1^2, m_3^2)\lambda^{1/2}(s_3, m_1^2, m_2^2)),
\]

(C1)

with \( \lambda(x, y, z) = (x - y - z)^2 - 4yz \). This is the solutions of \( s \) when we fix the masses of all internal particles and \( s_2, s_3 \). By exchanging \( s_1 \) and \( s_2 \), we can obtain the similar solutions for the TS in \( s_2, i.e.,

\[
s_2^\pm = (m_1 + m_3)^2 + \frac{1}{2m_2^2}(m_1^2 + m_2^2 - s_3)(s_2 - m_1^2 - m_3^2) - 4m_1^2m_2m_3 \\
\pm \lambda^{1/2}(s_2, m_2^2, m_3^2)\lambda^{1/2}(s_3, m_1^2, m_2^2)).
\]

(C2)

With the help of the single dispersion relation for the 3-point function, we learn that only \( s^\pm\) or \( s_2^\pm\) corresponds to the TS solutions within the physical boundary (24). The normal and singular thresholds for \( s \) and \( s_2 \) with \( s_3 \) fixed can be determined as

\[
s_N = (m_2 + m_3)^2, \quad s_C = (m_2 + m_3)^2 + \frac{m_3}{m_1}(m_2 - m_1)^2 - s_3], \\
s_{2N} = (m_1 + m_3)^2, \quad s_{2C} = (m_1 + m_3)^2 + \frac{m_3}{m_2}(m_2 - m_1)^2 - s_3].
\]

(C3)

It describes the motion of the singular thresholds of the TS on the complex plane. Namely, with the fixed \( s_3 \) and internal masses, when \( s \) reaches \( s_N, s_2^\pm \) will access its critical threshold \( s_{2C} \). Then, with the increase of \( s \) from \( s_N \) to \( s_C, s_2^\pm \) will move from \( s_{2C} \) to \( s_{2N} \). This motion will pinch the singularity in the denominator of the dispersion relation, and the range of the motion reflects how significant the TS mechanism can contribute to the loop function. As discussed in Ref. [24], the phase space of internal particle \( m_2 \) decays into \( m_a + m_1 \) is correlated with the magnitude of the TS. In our case one notices that \( m_2 \approx m_a + m_1 \), which means that the TS will be suppressed, or the TS contribution will reduce to a lower order singularity similar to that arising from a two-body cut, i.e, a CUSP effect.

To demonstrate this we plot the invariant mass spectrum for J/ψK in Fig. 8 at the normal threshold of the initial energy \( s = (m_{D_s} + m_K)^2 \) for the contributions from Fig. 4 (a) in e^+e^- \( \rightarrow \) J/ψKK. In order to fulfill the TS condition, we increase the \( D_s \) mass \( m_{D_s} \) by a value of DM to make it close to the critical threshold for \( s_{2C} \), i.e. to satisfy the on-shell condition for the \( D_s \) threshold. As shown by the solid curve in Fig. 8 the kinematics for the TS cannot be fulfilled since the mass of the \( D_s \) is about 41 MeV below the threshold of \( (m_{D_s} + m_K) \). Therefore, the critical threshold \( s_{2C} \) does not show up in the invariant mass spectrum. By increasing the mass of \( D_s \) by \( DM \), the TS will move to the physical kinematic region and the TS effects become more and more important. As shown by the dot-dashed curves in Fig. 8 the TS can produce narrow strong peak at the vicinity of the \( D_s \) threshold. This is a direct demonstration of the role play by the TS when the kinematics are close to the TS condition. It also shows that for the physical case under discussion the observation of predominant peaking structure in the invariant mass spectrum of J/ψK would imply the existence of a genuine threshold state produced via the triangle transition process.

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This scenario will be demonstrated in Appendix C.