The criteria for a solution of the field equations to be a classical limit of a quantum cosmology

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Abstract. If the gravitational field is quantized, then a solution of Einstein’s field equations is a valid cosmological model only if it corresponds to a classical limit of a quantum cosmology. To determine which solutions are valid requires looking at quantum cosmology in a particular way. Because we infer the geometry by measurements on matter, we can represent the amplitude for any measurement in terms of the amplitude for the matter fields, allowing us to integrate out the gravitational degrees of freedom. Combining that result with a path-integral representation for quantum cosmology leads to an integration over 4-geometries. Even when a semiclassical approximation for the propagator is valid, the amplitude for any measurement includes an integral over the gravitational degrees of freedom. The conditions for a solution of the field equations to be a classical limit of a quantum cosmology are: (1) The effect of the classical action dominates the integration, (2) the action is stationary with respect to variation of the gravitational degrees of freedom, and (3) only one saddlepoint contributes significantly to each integration.

Key words: cosmology:theory — large-scale structure of universe

1. Introduction

We normally consider all solutions of Einstein’s field equations to be valid cosmological models. However, this may not be true if a valid cosmological model is required to be the classical limit of a quantum cosmology.

Section 2 points out that we infer the gravitational field from measurements on matter. Therefore, in comparing measurements with theory, it is sufficient to consider the amplitudes for matter fields only, allowing us to integrate over the gravitational degrees of freedom (an integration on a spacelike three-dimensional hypersurface).

Section 3 points out that a path-integral representation of the wave function involves an integration over all 3-geometries on an initial spacelike hypersurface. Section 4 replaces the integrations over 3-geometries on the two spacelike hypersurfaces by the equivalent integration over the 4-geometries connecting those two hypersurfaces.

Section 5 considers the semiclassical approximation for the propagator that takes the wave function for 3-geometries and matter fields from one spacelike hypersurface to another. In that approximation, the propagator depends on only one solution of the field equations. Solutions to the field equations fall into two categories: 1. The action for the propagator dominates the behavior of the path integral and a saddlepoint approximation is valid for each integration in the path integral over the geometry. 2. The action for the propagator does not dominate the behavior of the path integral or a saddlepoint approximation is invalid for at least one of the integrations in the path integral over the geometry.

Section 6 considers the former case. Section 7 considers the latter case. Section 8 interprets these examples. ¹

2. Measurements in quantum cosmology

We can represent a quantum cosmology by \( \langle g, \phi, S | \psi \rangle \), which is the amplitude that on a spacelike hypersurface

¹ This paper considers the meaning of quantum gravity, especially with regard to interpreting measurements, but does not discuss theories of quantum gravity.
S, the 3-geometry is g and the matter fields are φ. This representation is implicit in the path integral approach to quantum gravity (Hawking 1979).

To relate this amplitude to a measurement of the geometry, we notice that we do not measure the geometry directly. We infer the geometry from measurements using matter objects, that is, from measurements on the matter. This allows us to represent any measurement by integrating over the gravitational degrees of freedom to give

$$\langle \phi, S | \psi \rangle = \int \langle g, \phi, S | \psi \rangle D(g),$$

(1)

the amplitude that on a spacelike hypersurface S, the matter fields are φ, where D(g) is the measure on g.

3. Path-integral representation

The wave function over 3-geometries g2 and matter fields φ2 on one 3-dimensional spacelike hypersurface S2 is related to the wave function over 3-geometries g1 and matter fields φ1 on another 3-dimensional spacelike hypersurface S1 by an extension of the path-integral (Feynman & Hibbs 1965) formulation of quantum cosmology (Hawking 1979) to give

$$\langle \phi_2, S_2 | \psi \rangle = \int \langle g_2, \phi_2, S_2 | g_1, \phi_1, S_1 \rangle \langle g_1, \phi_1, S_1 | \psi \rangle D(g_1)D(\phi_1),$$

(2)

where $$\langle g_2, \phi_2, S_2 | g_1, \phi_1, S_1 \rangle$$ is the propagator (that is, the amplitude to go from a state with 3-geometry g1 and matter fields φ1 on hypersurface S1 to a state with 3-geometry g2 and matter fields φ2 on hypersurface S2), $$\langle g_1, \phi_1, S_1 | \psi \rangle$$ is the wave function over 3-geometries g1 and matter fields φ1 on a spacelike hypersurface S1, D(g1) is the measure on g1, and D(φ1) is the measure on φ1. The integration is over all 3-geometries g1 and matter fields φ1 for which the integral is defined.3

Substituting (2) into (1) gives

$$\langle \phi_2, S_2 | \psi \rangle = \int \langle g_2, \phi_2, S_2 | g_1, \phi_1, S_1 \rangle \langle g_1, \phi_1, S_1 | \psi \rangle D(g_1)D(\phi_1)D(g_2).$$

(3)

4. Integration over 4-geometries

Because (3) involves an integration over all 3-geometries g1 and g2 on S1 and S2, it is equivalent to an integration over all 4-geometries that connect S1 and S2. Thus, (3) can be written as

$$\langle \phi_2, S_2 | \psi \rangle = \int \langle g_2(4), \phi_2, S_2 | g_1(4), \phi_1, S_1 \rangle \langle g_1(4), \phi_1, S_1 | \psi \rangle D(g_4)D(\phi_1),$$

(4)

where D(g(4)) is the measure on the 4-geometry g(4). Of course, until we have a full theory of quantum gravity, we do not have formulas to give most of the functions in these integrals. We can, however, make some semiclassical approximations without having a full theory. To justify replacing (3) by (4), we notice that the integration in (3) is an integration over all 4-geometries that connect S1 and S2, as is the integration in (4).

5. Semiclassical approximation for the propagator

Making the semiclassical approximation4 for the propagator gives (Gerlach 1969)

$$\langle g_2(g(4)), \phi_2, S_2 | g_1(g(4)), \phi_1, S_1 \rangle \approx f_s(g_2(g(4)), S_2; g_1(g(4)), \phi_1, S_1) e^{i I_{\text{classical}}[g_2(g(4)), S_2; g_1(g(4)), \phi_1, S_1]} D(g_1)D(\phi_1)D(g_2),$$

(5)

where I_{\text{classical}}[g_2(g(4)), S_2; g_1(g(4)), \phi_1, S_1] is the action for the classical spacetime bounded by the two 3-geometries that satisfies the field equations and f_s(g_2(g(4)), S_2; g_1(g(4)), \phi_1, S_1) is a slowly varying function. Explicit dependence on φ2 is not shown, because for classical solutions to the field equations, φ2 is determined from φ1 and g(4). Thus, substituting (5) into (4) gives

$$\langle \phi_2, S_2 | \psi \rangle \approx \int f_s(g_2(4), \phi_1) e^{i I_{\text{classical}}[g(4), \phi_1]} \langle g_1(4), \phi_1, S_1 | \psi \rangle D(\phi_1)D(g_2),$$

(6)

where f_s(g(4), φ1) is a slowly varying function and the integration is over all classical 4-geometries that connect S1 and S2.

The number of functions being integrated over to represent the 4-geometry g(4) is probably an order of infinity greater than that of the real numbers. To test the validity as a cosmology of a given 4-geometry, it is sufficient to restrict consideration to a small subset of cases, such as a family of known exact solutions. This allows us to represent the integration over 4-geometries in (6) more explicitly. Solutions to the field equations can be represented by a number of parameters a_i. These are the parameters that specify the 4-geometry that are not constrained by the matter distribution φ1 on the hypersurface S1. The number of these parameters is usually finite, and in most cases, at least countable. I shall assume here, that they are finite, and that there are N of these parameters, although I think the development could be extended to even the uncountable case. Thus, we may rewrite (6) more explicitly as

$$\langle \phi_2, S_2 | \psi \rangle \approx \int f_s(a_1, \phi_1) e^{i I_{\text{classical}}[a_1, \phi_1]} \langle g_1(a_1), \phi_1, S_1 | \psi \rangle D(\phi_1) d^N a_1,$$

(7)

4 A semiclassical approximation for the propagator is not always valid. Here, we consider only cases where it is valid.
where \( f_c(a_i, \phi_1) \) is a slowly varying function that depends explicitly on the parameters \( a_i \) that define the 4-geometry, and now we are left with an ordinary Nth order integral to define the integration over the 4-geometries.

6. When a saddlepoint approximation is valid

When the behavior of \( e^{I_{\text{classical}}} \) dominates over that of \( \langle g_1(a_i), \phi_1, S_1|\psi \rangle \) and \( f_c(a_i, \phi_1) \) in the integration over each \( a_i \) in (7) and when a saddlepoint approximation for each integration is valid, then we can approximate each of those integrations by a saddlepoint approximation. We analytically continue each function into the complex domain, deform the path of integration in the complex plane, and now we are left with an ordinary Nth order integral to define the integration over the 4-geometries.

for each \( a_i \). For each integration, the path must be deformed (without passing over any non-analytic points) onto a steepest descent path or a stationary phase path. Also, to be a valid approximation, there must not be any non-analytic points too close to the saddlepoint. For stationary phase paths, the saddlepoint approximation gives e.g. (Jeffreys & Jeffreys 1978)

\[
\langle \phi_2, S_2|\psi \rangle \approx \int f_c(a_0, \phi_1) e^{I_{\text{classical}}[a_0, \phi_1]} \langle g_1(a_0), \phi_1, S_1|\psi \rangle \]

\[
e^{N\pi^2/4} \prod_{i=1}^{N} \left| \frac{2\pi}{\partial^2 I/\partial a_i^2} \right|^{1/2} D(\phi_1).
\]

for steepest descent paths, the formula differs only by a phase.

The usual form for the action \( I \) is

\[
I = \int (-|g|^{(4)})^{1/2} L d^4x,
\]

where \( |g| \) is the determinant of the metric tensor \( g_{\mu\nu} \),

\[
L = \frac{R - 2\Lambda}{16\pi} - \frac{\rho_c}{\rho_e} A_\mu U^\mu - \frac{F_{\mu\nu} F^{\mu\nu}}{16\pi}
\]

is the Lagrangian, \( R \) is the Riemann scalar, \( \Lambda \) is the cosmological constant, \( \rho_c \) is the mass density, \( U^\mu \) is the four-velocity, \( \rho_e \) is the electric charge density, \( A_\mu \) is the electromagnetic 4-vector potential, \( F_{\mu\nu} \) is the electromagnetic field tensor, and the usual designation of the four terms is shown.\(^5\)

Because the integration in (10) must consider the light-cone structure of the propagators, it is more appropriate to derive a formula for the amplitude of observing a particular event instead of deriving a general formula for all possible measurements. The integral for the action in (10) must therefore be restricted to the past light cone of the event whose amplitude is being calculated. There is some fuzziness to the light cone,\(^6\) which is taken into account by using the correct propagators (Feynman 1962).

An example of applying such a saddlepoint approximation to a family of solutions to the field equations will be given in a future publication.

7. When a saddlepoint approximation is not valid

We consider here several examples where the saddlepoint approximation is either not valid or not applicable. We take \( \Lambda, F_{\mu\nu} \) and \( A_\mu \) to be zero in these examples. In addition, we take \( R \) and \( \rho \) to be zero except where there are masses.

7.1. Minkowski space

In empty Minkowski space, the Lagrangian is everywhere zero because the scalar curvature \( R \) is zero and the matter density is zero, and therefore, the action \( I_{\text{classical}} \) is zero. Because there is no matter, there is no possibility for measurements, so this case is not applicable.

7.2. Schwarzschild metric

The simplest matter distribution added onto Minkowski space-time gives us the Schwarzschild metric. Normally, we use the Schwarzschild metric to represent the local field around a planet or star or black hole, but not for a whole cosmology, and there may be good reason for that.

There are no gravitational degrees of freedom defining the Schwarzschild metric, so there is no integration over 4-geometries. However, formally, we could write (7) as

\[
\langle \phi_2, S_2|\psi \rangle \approx \int f_c(\phi_1) e^{I_{\text{classical}}[\phi_1]} \langle g_1, \phi_1, S_1|\psi \rangle D(\phi_1).
\]

7.3. Kerr metric

The next simplest model is a symmetric body like a planet that has a rotation rate relative to an inertial frame. We can represent the field outside of the body

\(^5\) All of the examples here use Einstein’s theory of General Relativity, but the procedure applies to nearly any gravitational theory.

\(^6\) There is an interesting similarity between the light cone and the event horizon of a black hole. In both cases, travel across the boundary is classically possible in only one direction (into the light cone or into the black hole), but the prohibition is not absolute in either case, because in the quantum situation a particle can temporarily escape by doing a zigzag path in space-time (Feynman 1962), resulting in Hawking radiation in the case of a black hole.
by the exterior Kerr metric. This metric has three gravitational degrees of freedom to characterize the direction and magnitude of the rotation rate (which I shall refer to as $a_1$, $a_2$, and $a_3$ here). Because the scalar curvature and matter density are everywhere zero outside of the body, the only contribution to the action $I_{\text{classical}}$ is from the mass of the body, which does not depend on the rotation rate. Thus, (7) becomes

$$\langle \phi_2, S_2 | \psi \rangle \approx \int e^{+I_{\text{classical}}[\phi_1]} f_d(a_1, a_2, a_3, \phi_1)$$

$$\langle g_1(a_1, a_2, a_3), \phi_1, S_1 | \psi \rangle D(\phi_1) \, da_1 \, da_2 \, da_3,$$

(13)

where $f_d(a_1, a_2, a_3, \phi_1)$ is a slowly varying function. Because the exponential factor does not dominate the integration, we cannot make a saddlepoint approximation for the integration over $a_1, a_2, \text{ and } a_3$. We are left with an integration over various Kerr metrics with various rotation rates. There is no single 4-geometry that dominates the integration.

We normally consider the Kerr metric to represent the local gravitational field around a spinning planet, star, or black hole, rather than for a cosmology. In light of the result here, this seems appropriate.

We want matter in the cosmological model so that we can do measurements. That is, because we cannot directly measure the geometry, we must infer it from measurements on matter. However, the example of a single body represented here by the Kerr metric is not really interesting enough to offer the possibility for measurements on the geometry. If we had a planetary system, we might be able to model possible measurements on the geometry using matter.

### 7.4. Asymptotically flat metrics

Therefore, consider a collection of planets and a star in some star system as the only matter in the universe. We assume we have some solution of the field equations for these. In fact, we will have many solutions, because we have some freedom in applying boundary conditions.

Let us consider a subset of those solutions in which we apply asymptotically flat boundary conditions. Then very far from where all of the matter is concentrated for the star system, the solution will be approximately that of a Kerr metric, in which the solution is characterized by the angular momentum of the matter relative to the flat metric to which the Kerr solution is asymptotic. The angular momentum is characterized by 3 values, say $a_1$, $a_2$, and $a_3$. This leads to the wave function given by (13), but we cannot apply a saddlepoint approximation because the action is independent of $a_1$, $a_2$, and $a_3$.

### 8. Interpretation

In summary, the conditions for a solution of the field equations to be a classical limit of a quantum cosmology are: (1) The effect of the classical action dominates the integration, (2) the action is stationary with respect to variation of the gravitational degrees of freedom, and (3) only one saddlepoint contributes significantly to each integration.

As pointed out earlier, we can always represent a measurement of the geometry in terms of the matter; we infer the geometry from measurements on the matter. So, in the above examples, what geometry would we infer from measurements on the matter?

Measurements on the matter in section 6 would indicate a geometry that was confined within the limits given by $|I_{\text{saddlepoint}} - I_{\text{classical}}| < \hbar$.

On the other hand, measurements on the matter in section 7.4 would indicate an ambiguous geometry. In fact, the system of bodies would seem very nonclassical. There is an aspect of relativity here. Although it is the background geometry that is quantum, we can infer the geometry and matter only relative to each other. More specifically, we can observe directly, only the matter, so it will appear to an observer that the matter is behaving in a quantum manner.

It should be pointed out that there are no new theories or assumptions here. This is simply an application of standard ideas about quantum theory to cosmology. To falsify the results presented here, it would be sufficient to show that our present cosmology does not satisfy the criteria given here for a valid cosmological model. But unless I have made a logical error, that would also invalidate some of our standard ideas about quantum theory.

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