ON THE EQUIPARTITION OF ENERGY FOR THE CRITICAL NLW

LUIS VEGA
UNIVERSIDAD DEL PAIS VASCO, APDO. 64
48080 BILBAO, SPAIN
EMAIL: MTPVEGOL@LG.EHU.ES
TEL.:++34-946015475, FAX: ++34-946012516

AND

NICOLA VISCIGLIA
DIPARTIMENTO DI MATEMATICA UNIVERSITÀ DI PISA
LARGO B. PONTECORVO 5, 56100 PISA, ITALY
EMAIL: VISCI@DM.UNIPI.IT
TEL.: ++39-0502212294, FAX: ++39-0502213224

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Abstract. We study some qualitative properties of global solutions to the
following focusing and defocusing critical NLW:
\[ \Box u + \lambda |u|^{2^* - 2} u = 0, \quad \lambda \in \mathbb{R}, \]
\[ u(0) = f \in \dot{H}^1(\mathbb{R}^n), \quad \partial_t u(0) = g \in L^2(\mathbb{R}^n), \]
on \( \mathbb{R} \times \mathbb{R}^n \) for \( n \geq 3 \), where \( 2^* \equiv \frac{2n}{n-2} \). We will consider the global solutions of the
defocusing NLW whose existence and scattering property is shown in [17], [3] and [2], without any restriction on the initial data \((f, g) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\). As well as the solutions constructed in [15] to the focusing
NLW for small initial data and to the ones obtained in [9], where a sharp condition on the
smallness of the initial data is given. We prove that the solution \( u(t, x) \) satisfies
a family of identities, that turn out to be a precise version of the classical
Morawetz estimates (see [13]). As a by–product we deduce that any global
solution to critical NLW belonging to a natural functional space satisfies:
\[
\lim_{R \to \infty} \frac{1}{R} \int_{|x| < R} |\nabla_x u(t, x)|^2 \, dx \, dt = \lim_{R \to \infty} \frac{1}{2R} \int_{|x| < R} (|\nabla_{t,x} u(t, x)|^2 + \frac{2\lambda}{2^*}|u(t, x)|^{2^*}) \, dx \, dt
\]
\[
= \int_{\mathbb{R}^n} (|\nabla_{t,x} u(0, x)|^2 + \frac{2\lambda}{2^*}|u(0, x)|^{2^*}) \, dx.
\]
In this paper we study some qualitative properties of solutions to the following
family of Cauchy problems:
\[
\Box u + \lambda |u|^{2^* - 2} u = 0, \quad \lambda \in \mathbb{R},
\]
\[ u(0) = f \in \dot{H}^1(\mathbb{R}^n), \quad \partial_t u(0) = g \in L^2(\mathbb{R}^n), \]
\[ (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad n \geq 3, \]
where \( 2^* \equiv \frac{2n}{n-2} \) and \( \Box = \partial_t^2 - \sum_{i=1}^{n} \partial_{x_i}^2 \).
Notice that if $\lambda \equiv 0$, then (0.1) reduces to the linear wave equation. Since now on we shall refer to the Cauchy problem (0.1) with $\lambda \geq 0$, as to the defocusing critical $NLW$ (similarly the focusing critical $NLW$ will be the Cauchy problem (0.1) with $\lambda < 0$).

Along this paper we shall work with solutions $u(t, x)$ belonging to the following space:

\[
X \equiv C(\mathbb{R}; \dot{H}^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^n))
\]

\[
\cap L^{\frac{2(n+1)}{n-2}}(\mathbb{R} \times \mathbb{R}^n) \cap L^{\frac{2(n+1)}{n-2}}_{loc}(\mathbb{R}; B^{\frac{2}{2(n+1)}}_{\infty, 1}(\mathbb{R}^n)).
\]

We shall also assume that the conservation of the energy is satisfied by the solutions $u(t, x)$, i.e.

\[
\int_{\mathbb{R}^n} \left( |\nabla_x u(t, x)|^2 + \frac{2\lambda}{2} |u(t, x)|^2 \right) \, dx \equiv \text{const} \quad \forall T \in \mathbb{R}.
\]

Let us recall that the global well-posedness of the defocusing $NLW$ has been studied in [7] provided that the initial data $(f, g)$ are regular.

Actually the global well-posedness of the defocusing Cauchy problem (0.1) has been studied in [17] for initial data $(f, g)$ in the energy space $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In [3] and [2] the same problem has been analysed from the point of view of scattering theory (see also [14]). In particular by combining the results in [3] and [17] it can be shown that for every $\lambda \geq 0$ and for every initial data $(f, g) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, there exists a unique solution $u(t, x)$ to (0.1) that belongs to the space $X$ introduced in (0.2) and moreover (0.3) is satisfied.

Concerning the global well-posedness of the focusing $NLW$, it is well-known that for every initial data $(f, g) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ small enough, i.e.

\[
\int_{\mathbb{R}^n} \left( |\nabla_x f|^2 + |g|^2 \right) \, dx < \epsilon(|\lambda|)
\]

for a suitable $\epsilon(|\lambda|) > 0$, there exists a unique global solution to (0.1) belonging to the space $X$ above and moreover (0.3) is satisfied. For a proof of this fact see [15].

In the paper [9] a much more precised version of the smallness assumption required on the initial data is given in order to guarantee the global well-posedness of the focusing critical $NLW$. In order to describe the result in [9] let us introduce the function $W(x) \in \dot{H}^1(\mathbb{R}^n)$ defined as follows:

\[
W(x) \equiv \frac{1}{(1 + \frac{|x|^2}{m(n-2)})^{\frac{n-2}{2}}}.
\]

Then in [9] it is shown that the Cauchy problem (0.1) with $\lambda = -1$, has a unique global solution in the space $X$ introduced in (0.2), provided that:

\[
\int_{\mathbb{R}^n} \left( |\nabla_x f|^2 + |g|^2 \right) \, dx < \int_{\mathbb{R}^n} \left( |\nabla_x W|^2 - \frac{n-2}{n} |W|^2 \right) \, dx
\]

and

\[
\int |\nabla_x f|^2 \, dx < \int |\nabla_x W|^2 \, dx.
\]

Moreover in [9] it is proved that blow-up occur provided that $f$ and $g$ satisfy (0.4) and $\int |\nabla_x f|^2 \, dx > \int |\nabla_x W|^2 \, dx$.

It is also well-known that (0.3) is satisfied by the solutions constructed in [9].
Our aim in this paper is to analyse some qualitative properties of global solutions to (0.1) in both focusing and defocusing case, provided that such a global solutions exist and belong to the space $X$. We are mainly interested on the asymptotic behaviour for large $R > 0$, of the following localized energies associated to the solutions $u(t, x)$ of (0.1):

\begin{align}
(0.6) \quad & \frac{1}{R} \int_R \int_{|x|<R} |\nabla_x u|^2 \, dxdt, \\
(0.7) \quad & \frac{1}{R} \int_R \int_{|x|<R} |\nabla_{t,x} u|^2 \, dxdt \quad \text{and} \quad \frac{1}{R} \int_R \int_{|x|<R} (|\nabla_{t,x} u|^2 + \frac{2\lambda}{2^*}|u|^{2^*}) \, dxdt.
\end{align}

Let us recall that the localized energies (0.6) were first obtained in [10] following the ideas in [1]. In this work we shall describe the asymptotic behaviour of the energies (0.6) and (0.7) as a consequence of a family of energy identities satisfied by the global solutions to (0.1).

In our opinion those identities have its own interest since they represent a precise version of the classical Morawetz inequalities, first proved in [13].

Since now on we shall denote by $X$ the space defined in (0.2). Next we state the first result of this paper.

**Theorem 0.1.** Let $(f, g) \in \dot{H}^1(R^n) \times L^2(R^n)$ and $\lambda \in R$ be such that there exists a unique global solution $u(t, x) \in X$ to (0.1). Assume moreover that $u(t, x)$ satisfies (0.3). Let $\psi : R^n \to R$ be a radially symmetric function such that:

$$
(\sqrt{1 + |x|^2})\Delta_x \psi, \Delta_x^2 \psi, \frac{\partial^2 \psi}{\partial x_i \partial x_j} \in L^\infty(R^n) \forall i, j = 1, ..., n
$$

and

$$
\lim_{|x| \to \infty} \partial_{|x|} \psi = \psi'(|x|).
$$

Then we have the following identity:

\begin{align}
(0.8) \quad & \int_R \int_{R^n} (\nabla_x u D^2_x \psi \nabla_x u - \frac{1}{4}|u|^2 \Delta_x^2 \psi + \frac{\lambda}{n}|u|^{2^*} \Delta_x \psi) \, dxdt \\
& = \psi'(\infty) \int_{R^n} (|\nabla_x f|^2 + \frac{2\lambda}{2^*}|f|^{2^*} + |g|^2) \, dx.
\end{align}

**Remark 0.1.** Let us point out that the hypothesis of theorem 0.1 are satisfied by the solutions constructed in [17] for defocusing NLW and in [15],[9] for the focusing NLW.

**Remark 0.2.** Let us underline that the identity (0.8) represents a precise version of an inequality proved in [13], where (0.8) is stated as an inequality and not as an identity in the special case $\psi \equiv |x|$.

**Remark 0.3.** The same type of identities as in theorem 0.1, have been proved in the context of the linear Schrödinger equation in [19] and [21] respectively in the free and in the perturbative case. The $L^2$-critical NLS has been analysed from the same point of view in [20]. However the result stated for the critical NLW in theorem 0.1 is much more precise compared with the one in [20] for NLS.

One of the main differences between NLS and NLW is that in the former case an explicit representation of the asymptotic behavior of the solutions to the free Schrödinger equation is involved in the argument, while in the case of NLW it is
not necessary. In this case one of the fundamental ingredients is the equipartition of energy, see Proposition 2.1 below.

Remark 0.4. Another difference between NLW and NLS, is that on the r.h.s. in (0.8) we get a quantity that is preserved along the evolution for NLW, while in case of NLS we get the \( \dot{H}^{\frac{1}{2}} \)–norm of the initial data, that is not preserved along the evolution for NLS.

Due to the freedom allowed to the function \( \psi \) in theorem 0.1, we can deduce the following result.

**Theorem 0.2.** Let \((f,g) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \lambda \in \mathbb{R} \) and \( u(t,x) \in X \) be as in theorem 0.1. Then we have:

\[
\lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{R}} \int_{|x| < R} \left| \partial_x u \right|^2 \, dx \, dt = \int_{\mathbb{R}^n} \left( |\nabla_x f|^2 + \frac{2\lambda}{2^*} |f|^{2^*} + |g|^2 \right) \, dx.
\]

Moreover

\[
\lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{R}} \int_{|x| < R} \left| \nabla_x u \right|^2 \, dx \, dt = \lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{R}} \int_{|x| < R} \left| u \right|^{2^*} \, dx \, dt = 0,
\]

where \( \partial_x \) and \( \nabla_x \) represent the radial derivative and the tangential part of the gradient respectively.

Notice that theorem 0.2 concerns mainly the concentration of the spatial gradient of the solution. Next we shall present another family of identities that will allow us to study also the behaviour of the energies connected with the time derivative of \( u(t,x) \).

**Theorem 0.3.** Let \((f,g) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \lambda \in \mathbb{R} \) and \( u(t,x) \in X \) be as in theorem 0.1. Let \( \phi : \mathbb{R}^n \to \mathbb{R} \) be a function that satisfies the following conditions:

\[
\Delta_x \phi, \langle x \rangle \phi \in L^\infty(\mathbb{R}^n).
\]

Then the following identity holds

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left( \left| \partial_t u \right|^2 - \left| \nabla_x u \right|^2 - \lambda \left| u \right|^{2^*} \right) \phi + \frac{1}{2} \left| u \right|^2 \Delta_x \phi \, dx \, dt = 0.
\]

In particular we get:

\[
\lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{R}} \int_{|x| < R} \left( \left| \nabla_x u \right|^2 - \left| \partial_t u \right|^2 \right) \, dx \, dt = 0,
\]

and

\[
\lim_{R \to \infty} \frac{1}{R} \int_{\mathbb{R}} \int_{|x| < R} \left( \left| \nabla_{t,x} u \right|^2 + \frac{2\lambda}{2^*} \left| u \right|^{2^*} \right) \, dx \, dt = 2 \int_{\mathbb{R}^n} \left( |\nabla_x f|^2 + \frac{2\lambda}{2^*} |f|^{2^*} + |g|^2 \right) \, dx.
\]

Remark 0.5. Notice that (0.12) can be considered as a different version of the classical equipartition of the energy (see [4]), whose classical version can be stated as follows:

\[
\lim_{t \to \pm \infty} \int_{\mathbb{R}^n} \left( \left| \partial_t u(t,x) \right|^2 - \left| \nabla_x u(t,x) \right|^2 \right) \, dx = 0.
\]
Next we shall fix some notation.

**Notation.**

For any $1 \leq p, q \leq \infty$

$$L^p_x \text{ and } L^q_x$$

denote the Banach spaces

$$L^p(R^n) \text{ and } L^p(R; L^q(R^n)),$$

and in the specific case $p = q$ we also use the notation

$$L^p_{x,t} \equiv L^p(R; L^p(R^n)).$$

We shall denote by $L^{p,q}(R^n)$ the usual Lorentz spaces and by $\dot{B}^s_{p,2}(R^n)$ the Besov spaces.

For every $1 \leq p \leq \infty$ we shall use the following mixed norm for functions defined on $\mathbb{R}^3$:

$$
\|f\|_{L^p_{x,t} - L^q_s}^p \equiv \sup_{r > 0} \int_{S^2} |u(r\omega)|^p \, d\omega
$$

where

$$S^2 \equiv \{\omega \in \mathbb{R}^3 ||\omega|| = 1\}$$

and $d\omega$ denotes the volume form on $S$.

We shall denote by $H^1_x$ the homogeneous Sobolev space $\dot{H}^1(R^n)$.

Given any couple of Banach spaces $Y$ and $Z$, we shall denote by $\mathcal{L}(Y, Z)$ the space of linear and continuous functionals between $Y$ and $Z$.

We shall denote by

$$\mathcal{C}_t(Y) \text{ and } \mathcal{C}^1_t(Y)$$

respectively the spaces

$$\mathcal{C}(R; Y) \text{ and } \mathcal{C}^1(R; Y)$$

where $Y$ is a generic Banach space.

We shall denote by $L^p_x(Y)$ the space of $L^p$ functions defined on $\mathbb{R}$ and valued in $Y$.

We shall denote by $X$ the functional space introduced in (0.2).

Given a space–time dependent function $w(t,x)$ we shall denote by $w(t_0)$ the trace of $w$ at fixed time $t \equiv t_0$, in case that it is well–defined.

We shall denote by $\int ... \, dx$, $\int ... \, dt$ and $\int \int ... \, dx dt$ the integral of suitable functions with respect to space, time, and space–time variables respectively.

When it is not better specified we shall denote by $\nabla v$ the gradient of any time–dependent function $v(t,x)$ with respect to the space variables. Moreover $\nabla_x$ and $\partial_{|x|}$ shall denote respectively the angular gradient and the radial derivative.

If $\psi \in C^2(R^n)$, then $D^2\psi$ will represent the hessian matrix of $\psi$.

Given a set $A \subset R^n$ we denote by $\chi_A$ its characteristic function.

The ball of radius $R > 0$ in $\mathbb{R}^n$ shall be denoted as $B_R$.

We shall use the function

$$\langle x \rangle \equiv \sqrt{1 + |x|^2}.$$
1. On the Strichartz estimates for critical NLW

Recall that by combining the papers \[17\] and \[3\], it follows that the defocusing \(\text{NLW}\) is globally well-posed in the Banach space introduced in (0.2) and moreover the following properties hold:

\[
\lim_{t \to \pm \infty} \|u(t)\|_{L^2_x} = 0
\]

and

\[
u(t, x) \in L^{\frac{2(n+1)}{n-1}}_t B^\frac{1}{2} 2(n+1)_{n-1} (\mathbb{R}^n).
\]

In the next proposition we gather some known facts that we shall use later on. The main point is that it applies to both focusing and defocusing \(\text{NLW}\).

**Proposition 1.1.** Let \((f, g) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) and \(\lambda \in \mathbb{R}\) be such that there exists a unique global solution \(u(t, x) \in X\) to (0.1). Then we have:

\[
(1.1) \quad u(t, x) \in L^\infty_t \dot{H}^1_x;
\]

\[
(1.2) \quad \lim_{t \to \pm \infty} \|u(t)\|_{L^2_x} = 0;
\]

\[
(1.3) \quad u(t, x) \in L^{\frac{2(n+1)}{n-1}}_t B^\frac{1}{2} 2(n+1)_{n-1} (\mathbb{R}^n).
\]

**Proof.** For simplicity we shall prove (1.2) only in the case \(t \to \infty\) and we shall also show boundedness of \(\|u(t)\|_{\dot{H}^1_x}\) only for \(t > 0\). The other cases can be treated in a similar way.

**First step:** \(u(t, x) \in L^\infty_t \dot{H}^1_x\)

Since we are assuming

\[
u(t, x) \in X \subset L^{\frac{2(n+1)}{n-2}}_t,
\]

we can deduce by standard techniques in nonlinear scattering that \(u(t, x)\) is asymptotically free. This means that there exists \((f^+, g^+) \in \dot{H}^1_x \times L^2_x\) such that:

\[
(1.4) \quad \lim_{t \to \infty} \|u(t) - u^+(t)\|_{\dot{H}^1_x} + \|\partial_t u(t) - \partial_t u^+(t)\|_{L^2_x} = 0,
\]

where

\[
\Box u^+ = 0
\]

\[
u^+(0) = f^+, \partial_t u^+(0) = g^+.
\]

The following computation is trivial:

\[
(1.5) \quad \sup_{t \in \mathbb{R}} (\|\nabla_x u(t)\|_{L^2_x} + \|\partial_t u(t)\|_{L^2_x})
\]

\[
\leq \sup_{t \in \mathbb{R}} \|\nabla_x u(t) - \nabla_x u^+(t)\|_{L^2_x} + \sup_{t \in \mathbb{R}} \|\nabla_x u^+(t)\|_{L^2_x}
\]

\[
+ \sup_{t \in \mathbb{R}} \|\partial_t u(t) - \partial_t u^+(t)\|_{L^2_x} + \sup_{t \in \mathbb{R}} \|\partial_t u^+(t)\|_{L^2_x} < \infty,
\]

where at the last step we have used (1.4) and the conservation of the energy for solutions to free wave equation.

**Second step:** \(u(t, x) \in L^\infty_t L^2_x\) and proof of (1.2).

By combining the previous step with the Sobolev embedding:

\[
\dot{H}^1_x \subset L^2_x,
\]

\[
\text{and the conservation of the energy for solutions to free wave equation.}
\]
we deduce that
\[ u(t, x) \in L^\infty_t L^2_x. \]
On the other hand by combining again the Sobolev embedding with (1.4) we get:
\[
\lim_{t \to \infty} \| u(t) \|_{L^2_x} \leq \lim_{t \to \infty} (\| u(t) - u^+(t) \|_{L^2_x} + \| u^+(t) \|_{L^2_x}) = 0,
\]
where at the last step we have used proposition 6.1 in appendix I.

Third step: proof of (1.3)
Once (1.2) has been shown, then the proof of (1.2) follows as in [3].

\[ \square \]

2. ON THE ASYMPTOTIC BEHAVIOUR OF FREE WAVES

First we present a proposition whose content is well–known in the literature. However in Appendix II we shall present a self–contained proof.

**Proposition 2.1.** Let \( u(t, x) \in C_t(\dot{H}^1_x) \cap C^1_t(L^2_x) \) be the unique solution to:
\[
\Box u = 0 \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, n \geq 3
\]
\[ u(0) = f \in \dot{H}^1_x, \partial_t u(0) = g \in L^2_x. \]
Then the following facts occur:
\[ \| \partial_t u(t) \|_{L^2_x} = o(1) \text{ as } t \to \pm \infty, \]
\[
\lim_{t \to \pm \infty} \int |\nabla_x u(t)|^2 \, dx = \lim_{t \to \pm \infty} \int |\partial_t u(t)|^2 \, dx
\]
\[ = \frac{1}{2} \int \left( |\nabla_x f|^2 + |g|^2 \right) \, dx; \]
\[ \int_{2|x|<|t|} (|\partial_t u(t)|^2 + |\nabla_x u(t)|^2) \, dx = o(1) \text{ as } t \to \pm \infty; \]
\[ \int |\nabla_x u(t)|^2 \, dx = o(1) \text{ as } t \to \pm \infty; \]
In particular, in the case \((f, g) \in C_\infty(\mathbb{R}^n) \times C_\infty(\mathbb{R}^n)\) we get the following stronger version of (2.1) and (2.3):
\[ \| \partial_t u(t) \|_{L^2_x} = 0 \left( \frac{1}{|t|} \right) \text{ as } t \to \pm \infty, \]
and
\[
\int_{2|x|<|t|} \left( |\partial_t u(t)|^2 + |\nabla_x u(t)|^2 \right) \, dx = 0 \left( \frac{1}{|t|^2} \right) \text{ as } t \to \pm \infty.
\]

**Proof.** See Appendix II.

Next we shall study some asymptotic expressions involving solutions to the free wave equation with initial data in the energy space \( \dot{H}^1_x \times L^2_x \). Those expressions will play a fundamental role in the sequel.
Lemma 2.1. Assume that \( u(t, x) \in C_t(H^1 \cap L^2_x) \) solves:

\[
\square u = 0
\]

\( u(0) = f \in H^1 \), \( \partial_t u(0) = g \in L^2_x \).

Let \( \psi : \mathbb{R}^n \to \mathbb{R} \) be a radially symmetric function such that the following limit exists:

\[
\lim_{|x| \to \infty} \partial_{|x|} \psi = \psi'(\infty) \in (0, \infty).
\]

Then

\[
\lim_{t \to -\infty} \int_{|x| < t} \partial_t u(t) \nabla_x u(t) \cdot \nabla_x \psi \, dx = \frac{1}{2} \psi'(\infty) \int \left( |\nabla_x f|^2 + |g|^2 \right) \, dx.
\]

Proof. We shall study only the case \( t \to \infty \) (in fact the case \( t \to -\infty \) reduces to the previous one since \( v(t, x) \equiv u(-t, x) \) is still a solution to the free wave equation and its behaviour at infinity is related to the behaviour of \( u(t, x) \) as \( t \to -\infty \)).

Notice that we have:

\[
\lim_{|x| \to \infty} \partial_{|x|} \psi = \psi'(\infty) \in (0, \infty).
\]

that due to (2.3) implies:

\[
\lim_{t \to -\infty} \int_{|x| < t} \partial_t u(t) \nabla_x u(t) \cdot \nabla_x \psi \, dx = 0.
\]

Next notice that due to (2.1) we have

\[
\lim_{t \to -\infty} \int_{|x| > t} \partial_t u(t) \nabla_x u(t) \cdot \nabla_x \psi \, dx
\]

\[
= \lim_{t \to -\infty} \int_{|x| > t} \partial_t u(t) \partial_{|x|} u(t) \partial_{|x|} \psi \, dx = - \lim_{t \to -\infty} \int_{|x| > t} |\partial_t u(t)|^2 \partial_{|x|} \psi \, dx.
\]

On the other hand we have

\[
\inf_{2|\psi| > t} (\partial_{|x|} \psi) \int_{2|\psi| > t} |\partial_t u(t)|^2 \, dx \leq \int_{2|\psi| > t} |\partial_t u(t)|^2 \partial_{|x|} \psi \, dx
\]

\[
\leq \sup_{2|\psi| > t} (\partial_{|x|} \psi) \int_{2|\psi| > t} |\partial_t u(t)|^2 \, dx.
\]

Then due to the assumption done on \( \partial_{|x|} \psi \) implies

\[
\lim_{t \to -\infty} \int_{2|\psi| > t} |\partial_t u(t)|^2 \partial_{|x|} \psi \, dx = \psi'(\infty) \lim_{t \to -\infty} \int_{2|\psi| > t} |\partial_t u(t)|^2 \, dx
\]

provided that the last limit exists. By combining (2.9), (2.10) and (2.11) we deduce that the proof will be concluded provided that we can show

\[
\lim_{t \to -\infty} \int_{2|\psi| > t} |\partial_t u(t)|^2 \, dx = \frac{1}{2} \int \left( |\nabla_x f|^2 + |g|^2 \right) \, dx.
\]

On the other hand we have

\[
\lim_{t \to -\infty} \int_{2|\psi| > t} |\partial_t u(t)|^2 \, dx
\]
\[
\lim_{t \to \infty} \int |\partial_t u(t)|^2 \, dx - \lim_{t \to \infty} \int_{|x| < t} |\partial_t u(t)|^2 \, dx = \frac{1}{2} \int (|\nabla_x f|^2 + |g|^2) \, dx
\]
where we have used (2.2) and (2.3).

\[\Box\]

**Lemma 2.2.** Assume that \(u(t, x) \in C_t(H^1_x) \cap C^1_t(L^2_x)\) solves (2.7). Let \(\phi : \mathbb{R}^n \to \mathbb{R}\) be a radially symmetric function such that:

\(\langle x \rangle \phi \in L^\infty_x\).

Then we have

\[
(2.14) \quad \lim_{t \to \pm \infty} \int \partial_t u(t) u(t) \phi \, dx = 0.
\]

**Proof.** As in the proof of lemma 2.1 it is sufficient to consider the limit for \(t \to \infty\).

First notice that by combining the decay assumption done on \(\phi\) with the Hardy inequality we get

\[
|\int \partial_t u(t) u(t) \phi \, dx| \leq \|\partial_t u(t)\|_{L^2_x} \|u(t)\|_{L^2_x} \leq C(\|\partial_t u(t)\|_{L^2_x}^2 + \|\nabla_x u(t)\|_{L^2_x}^2) \equiv \text{const} \ \forall t \in \mathbb{R}.
\]

Due to this fact it is easy to show that by a density argument it is sufficient to prove (2.14) under the assumptions that \((f, g) \in C^\infty_0(\mathbb{R}^n) \times C^\infty_0(\mathbb{R}^n)\). Notice that if \(u(t, x)\) is a regular solution to (2.7), then we have

\[
\frac{d^2}{dt^2} \int |u(t)|^2 \, dx = 2 \int (|\partial_t u(t)|^2 - |\nabla_x u(t)|^2) \, dx,
\]
that due to (2.2) implies

\[
\frac{d^2}{dt^2} \int |u(t)|^2 \, dx = o(1) \text{ as } t \to \infty.
\]

After integration of this identity we get:

\[
\int |u(t)|^2 \, dx = \int |f|^2 \, dx + 2t(\int fg \, dx) + o(t^2),
\]
and hence

\[
(2.15) \quad \int |u(t)|^2 \, dx = o(t^2) \text{ as } t \to \infty.
\]

Next notice that we have:

\[
\int \partial_t u(t) u(t) \phi \, dx = I(t) + II(t),
\]
where

\[
I(t) = \int_{|x| > t} \partial_t u(t) u(t) \phi \, dx,
\]
and

\[
II(t) = \int_{|x| < t} \partial_t u(t) u(t) \phi \, dx.
\]
Notice that due to the decay assumption done on \( \phi \) we have

\[
I(t) \leq \frac{C}{t} \int_{|x| > t} |u(t)\partial_t u(t)| \, dx
\]

\[
\leq \frac{C}{t} \|u(t)\|_{L^2_x} \|\partial_t u(t)\|_{L^2_x} = o(1) \|\partial_t u(t)\|_{L^2_x},
\]

where we have used (2.15). In particular we get:

\[
\lim_{t \to \infty} |I(t)| = 0.
\]

On the other hand due to (2.15) and (2.6) we have:

\[
|II(t)| \leq \|\phi\|_{L^\infty_x} \left( \int_{|x| < t} |u(t)|^2 \, dx \right)^\frac{1}{2} \left( \int_{|x| < t} |\partial_t u(t)|^2 \, dx \right)^\frac{1}{2}
\]

\[
= Co(t) 0 \left( \frac{1}{t} \right),
\]

and hence

\[
\lim_{t \to \infty} |II(t)| = 0.
\]

The proof is complete.

3. PROOF OF THEOREM 0.1

We shall need the following propositions.

**Proposition 3.1.** Assume that \( u(t,x) \in X \) is a global solution to (0.1) in dimension \( n \geq 3 \) for some \( \lambda \in \mathbb{R} \) and \( (f,g) \in \dot{H}^1_x \times L^2_x \). Assume moreover that \( u(t,x) \) satisfies (0.3). Then we have:

\[
\lim_{t \to \infty} \int \partial_t u(t) \nabla_x u(t) \cdot \nabla_x \psi(x) \, dx
\]

\[
= -\frac{1}{2} \psi'(\infty) \int_{\mathbb{R}^n} (|\nabla_x f|^2 + \frac{2\lambda}{2^*}|f|^{2^*} + |g|^2) \, dx,
\]

where \( \psi : \mathbb{R}^n \to \mathbb{R} \) is a radially symmetric function such that the following limit exists

\[
\lim_{|x| \to \infty} \partial_x \psi = \psi'(\infty) \in (0,\infty).
\]

**Proof.** As in the proof of proposition 1.1 we only treat the case \( t \to \infty \). Let \( u^+(t,x), f^+, g^+ \) be as in the proof of proposition 1.1.

As a consequence of (1.4) we deduce that:

\[
(3.1) \quad \lim_{t \to \infty} \int \partial_t u(t) \nabla_x u(t) \cdot \nabla_x \psi \, dx
\]

\[
= \lim_{t \to \infty} \int \partial_t u^+(t) \nabla_x u^+(t) \cdot \nabla_x \psi \, dx
\]

\[
= \lim_{t \to \infty} -\frac{1}{2} \psi'(\infty) \int (|\nabla_x f|^2 + |g^+|^2) \, dx
\]

\[
= \lim_{t \to \infty} -\frac{1}{2} \psi'(\infty) \int (|\nabla_x u^+(t)|^2 + |\partial_t u^+(t)|^2) \, dx,
\]
where at the last step we have used (2.14). By combining (3.4) and (3.5) we deduce that

$$\lim_{t \to \infty} \int \partial_t u(t) \nabla_x u(t) \cdot \nabla_x \psi \, dx$$

$$= \lim_{t \to \infty} -\frac{1}{2}\psi'(\infty) \int (|\nabla_x u^+(t)|^2 + |\partial_t u^+(t)|^2 + \frac{2\lambda}{2^*}|u^+(t)|^{2^*} - \frac{2\lambda}{2^*}|u^+(t)|^{2^*}) \, dx$$

$$= \lim_{t \to \infty} -\frac{1}{2}\psi'(\infty) \int (|\nabla_x u(t)|^2 + |\partial_t u(t)|^2 + \frac{2\lambda}{2^*}|u(t)|^{2^*} - \frac{2\lambda}{2^*}|u(t)|^{2^*}) \, dx$$

$$= -\frac{1}{2}\psi'(\infty) \int (|\nabla_x f|^2 + |g|^2 + \frac{2\lambda}{2^*}|f|^{2^*}) \, dx + \frac{\lambda}{2^*}\psi'(\infty) \lim_{t \to \infty} \int |u^+(t)|^{2^*} \, dx$$

$$= -\frac{1}{2}\psi'(\infty) \int (|\nabla_x f|^2 + |g|^2 + \frac{2\lambda}{2^*}|f|^{2^*}) \, dx,$$

where we have used lemma 2.8 and the property

$$\lim_{t \to \infty} |u^+(t)|^{2^*} \, dx = 0.$$ (3.2)

The proof of (3.2) can be found in Appendix I.

**Proposition 3.2.** Assume that $u(t,x) \in X$ is a global solution to (0.1) in dimension $n \geq 3$ for some $\lambda \in \mathbb{R}$ and $(f,g) \in H^1_x \times L^2_x$. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a radially symmetric function such that

$$\langle x \rangle \phi \in L^\infty_x,$$

then we have

$$\lim_{t \to \pm \infty} \int \partial_t u(t)u(t)\phi \, dx = 0.$$ (3.3)

**Proof.** As in proposition 1.1 it is sufficient to treat the case $t \to \infty$.

Let $u^+(t,x), f^+(x)$ and $g^+(x)$ be as in the proof of proposition 1.1. Notice that due to the decay assumption done on $\phi$ and due to the Hardy inequality we deduce that

$$\lim_{t \to \infty} \|\phi(u(t) - u^+(t))\|_{L^2_x}$$

$$\leq C \lim_{t \to \infty} \|\nabla u(t) - \nabla u^+(t)\|_{L^2_x} = 0$$

where at the last step we have used (1.4). On the other hand due again to (1.4) we have:

$$\lim_{t \to \infty} \|\partial_t u(t) - \partial_t u^+(t)\|_{L^2_x} = 0.$$ (3.5)

By combining (3.4) and (3.5) we deduce that

$$\lim_{t \to \infty} \int \partial_t u(t)u(t)\phi \, dx = \lim_{t \to \infty} \int \partial_t u^+(t)u^+(t)\phi \, dx = 0$$

where at the last step we have used (2.14).
Proof of theorem 0.1. Following [16] we multiply the equation (0.1) by $\nabla_x \psi \cdot \nabla_x u + \frac{1}{2} \Delta_x \psi u$ in order to get after integration by parts:

$$\int_{-T}^{T} \int_{\mathbb{R}^n} \nabla_x u D_x^2 \psi \nabla_x u - \frac{1}{4} |u|^2 \Delta_x^2 \psi + \frac{\lambda}{n} |u|^{2^*} \Delta_x \psi \, dx \, dt$$

$$= \sum_{\pm} \left( \pm \int_{\mathbb{R}^n} \partial_t u(\pm T) \nabla_x u(T) \cdot \nabla_x \psi + \frac{1}{2} \partial_t u(\pm T) u(\pm T) \Delta_x \psi \, dx \right).$$

The proof can be completed by taking the limit as $T \to \infty$ and by using propositions 3.1 and 3.2. □

4. Proof of theorem 0.2

We start this section with some preliminary results that will be useful along the proof of theorem 0.2.

Proposition 4.1. Let $(f, g) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$ be such that there exists a unique global solution $u(t, x) \in X$ to (0.1) for $n \geq 3$. Then:

$$\int \int 1_{\langle x \rangle} |u|^{2^*} \, dx \, dt < \infty,$$

and in particular:

$$\lim_{R \to \infty} \frac{1}{R} \int \int_{B_R} |u|^{2^*} \, dx \, dt = 0.$$

Proof. Notice that due to the Hölder inequality in Lorentz spaces we get:

$$\int 1_{\langle x \rangle} |u|^{2^*} \, dx \leq \left\| 1_{\langle x \rangle} \right\|_{L^{\frac{2^*}{n}}(\mathbb{R}^n)} \left\| u \right\|_{L^{\frac{2^*}{n-1}}(\mathbb{R}^n)}^{2^*} \leq C \left\| u \right\|_{L^{2^*}(\mathbb{R}^n)}^{2^*} \left\| u \right\|_{L^{2^*(1-\theta)}(\mathbb{R}^n)}^{2(1-\theta)},$$

where:

$$\theta = \frac{(n+1)(n-2)}{n(n-1)},$$

i.e.

$$\theta = \frac{(n+1)(n-2)}{n(n-1)}.$$

By combining the previous inequality with the Sobolev embedding:

$$\dot{H}^1_x \subset L^{2^*}^{2^*}(\mathbb{R}^n)$$

and

$$\dot{B}^{\frac{n+1}{2(n+1)}}_{n+1}(\mathbb{R}^n) \subset L^{\frac{2n(n+1)}{n^2-2n-1}} \cdot 2^{\frac{n-1}{2}}(\mathbb{R}^n),$$

we get:

$$\int \int 1_{\langle x \rangle} |u|^{2^*} \, dx \leq C \left\| u \right\|_{L^{\frac{2(n+1)}{n-1}} B^{\frac{n+1}{2(n+1)}}_{n+1}(\mathbb{R}^n)}^{\frac{4}{n-1}} \left\| u \right\|_{L^{\frac{n-1}{2}} H^1_x}^{\frac{4}{n-1}} < \infty,$$

where at the last step we have used (1.1) and (1.3). The proof of (4.1) is complete. Notice that (4.2) follows by combining (4.1) with the dominated convergence theorem. □
Proposition 4.2. Let \((f, g) \in \dot{H}_x^1 \times L^2_x\) and \(\lambda \in \mathbb{R}\) be such that there exists a unique global solution \(u(t, x) \in X\) to (0.1) with \(n \geq 3\). Then we have:

\[
\lim_{R \to \infty} \int \int |\Delta_x \phi_R||u|^2^* \, dx \, dt = 0,
\]

where \(\phi\) is a radially symmetric function such that

\[
|\Delta_x \phi| \leq \frac{C}{|x|}\]

and \(\phi_R = R \phi\left(\frac{x}{R}\right)\).

**Proof.** By assumption we have

\[
\int \int |\Delta_x \phi_R||u|^2^* \, dx \, dt \leq C \int \int \frac{C}{R + |x|}|u|^2^* \, dx \, dt \to 0 \text{ as } R \to \infty,
\]

where at the last step we have combined the dominated convergence theorem with (4.1).

\[
\square
\]

Proposition 4.3. Let \((f, g) \in \dot{H}_x^1 \times L^2_x\) and \(\lambda \in \mathbb{R}\) be such that there exists a unique global solution \(u(t, x) \in X\) to (0.1) for \(n \geq 3\). Then \(u(t, x)\) satisfies:

\[
\lim_{R \to \infty} \frac{1}{R^3} \int \int_{B_R} |u|^2 \, dx \, dt = 0.
\]

In order to prove proposition 4.3 we shall need some lemma. In particular next result will be particularly useful along the proof of proposition 4.3 in the case \(n = 3\).

Lemma 4.1. Let \(u(t, x) \in \mathcal{C}_t(\dot{H}_x^1) \cap \mathcal{C}_t^1(L^2_x)\) be the unique solution to

\[
\Box u = F \in L^1_t L^2_x, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3
\]

\[
u(0) = f \in \dot{H}_x^1, \quad \partial_t u(0) = g \in L^2_x.
\]

For every \(1 \leq p < \infty\) there exists a constant \(C = C(p) > 0\) such that the following a–priori estimate holds:

\[
\|u\|_{L^p_t L^\infty_x L^p} \leq C(\|f\|_{\dot{H}_x^1} + \|g\|_{L^2_x} + \|F\|_{L^1_t L^2_x}).
\]

**Proof.** In [12] it is proved the following estimate for every \(1 \leq p < \infty\):

\[
\|u\|_{L^p_t L^\infty_x L^p} \leq C(\|f\|_{\dot{H}_x^1} + \|g\|_{L^2_x})
\]

where \(u(t, x) \in \mathcal{C}_t(\dot{H}_x^1) \cap \mathcal{C}_t^1(L^2_x)\) is the unique solution to:

\[
\Box u = 0
\]

\[
u(0) = f \in \dot{H}_x^1, \quad \partial_t u(0) = g \in L^2_x.
\]

The proof of lemma 4.1 in the case \(F(t, x) \neq 0\) follows easily by combining the previous estimate with the Minkowski inequality and the Duhamel formula.

\[
\square
\]
Lemma 4.2. Let \((f,g) \in \dot{H}_x^1 \times L_x^2\) and \(\lambda \in \mathbb{R}\) be such that there exists a unique global solution \(u(t,x) \in X\) to (0.1) for \(n \geq 3\). Then we have:

\[(4.6)\quad \|u(t,x)\|_{L_t^2 L_x^{\frac{2n}{n-2}}} < \infty \text{ when } n \geq 4\]

and

\[(4.7)\quad \|u\|_{L_t^2 L_x^{\infty} L_g^p} < \infty \quad \forall 1 \leq p < \infty \text{ when } n = 3.\]

Proof. We split the proof in two parts.

Proof of (4.6)

Notice that by combining the Sobolev embedding:

\[B_{\frac{2(n+1)}{n}}^{\frac{1}{2}}(\mathbb{R}^n) \subset L_{x}^{\frac{2n}{n-2(n+1)}}\]

with (1.3) we get

\[\|u(t,x)\|_{L_t^{\frac{2(n+1)}{n-2}} L_x^{\frac{2n}{n-2(n+1)}}} < \infty.\]

On the other hand (1.1) implies

\[\|u(t,x)\|_{L_t^{\infty} L_x^{2^*}} < \infty\]

and hence by interpolation

\[(4.8)\quad \|u(t,x)\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n-2(n+2)}}} < \infty.\]

Recall that \(u(t,x)\) solves:

\[\Box u = -\lambda |u|^{2^*-2}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \quad n \geq 4\]

\[u(0) = f \in \dot{H}_x^1(\mathbb{R}^n), \quad \partial_t u(0) = g \in L_x^2(\mathbb{R}^n),\]

and hence by Strichartz estimates (see [6]) we deduce:

\[(4.9)\quad \|u(t,x)\|_{L_t^4 L_x^{12}} \leq C \left( \|f\|_{\dot{H}_x^1} + \|g\|_{L_x^2} + |\lambda| \|u^{\frac{n+2}{n-2}}\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \right).\]

By combining this estimate with (4.8) we get (4.6).

Proof of (4.7)

Notice that the proof of (4.6) fails in dimension \(n = 3\) since in this case the end–point Strichartz estimate (i.e. a version of (4.9) for \(n = 3\)) is false. Next we shall overcome this difficulty by using lemma 4.1. By combining (1.3) (where we choose \(n = 3\)) with the Sobolev embedding:

\[B_{\frac{1}{4}}^{\frac{1}{2}}(\mathbb{R}^3) \subset L_{x}^{12}(\mathbb{R}^3),\]

we deduce that

\[(4.10)\quad u(t,x) \in L_t^4 L_x^{12}.\]

On the other hand due to (1.1) and due to the Sobolev embedding \(\dot{H}_x^1(\mathbb{R}^3) \subset L_x^6(\mathbb{R}^3))\), we get

\[\|u(t,x)\|_{L_t^{\infty} L_x^6} < \infty.\]

Hence by interpolation we get:

\[(4.11)\quad \|u\|_{L_t^5 L_x^{10}} < \infty.\]
Next notice that \( u(t, x) \in X \) solves the following Cauchy problem with forcing term:

\[
\Box u = -\lambda u^5, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3 \\
 u(0) = f \in H^1_x, \partial_t u(0) = g \in L^2_x.
\]

By combining this fact with lemma (4.1) (where we choose \( F = -\lambda u^5 \)) and (4.11) we deduce:

\[
\|u\|_{L^5_t L^\infty^x L^6_x} \leq C(\|f\|_{H^1_x} + \|g\|_{L^2_x} + |\lambda|\|u\|^5_{L^6_t L^\infty^x L^6_x}) < \infty.
\]

**Proof of proposition 4.3**

We shall prove proposition 4.3 in dimension \( n = 3 \) by using as a basic tool (4.7). It will be clear that the same argument works in dimension \( n > 3 \) provided that we use (4.6) instead of (4.7).

Since now on we shall assume \( n = 3 \). Notice that for every \( T > 0 \) we have:

\[
\int_T^\infty \int_{B_R} |u|^2 \, dx \, dt = \int_T^\infty \int_0^R \left( \int_{S^2} |u(t, r\omega)|^2 \, d\omega \right) r^2 \, dr \, dt \\
\leq R^2 \int_T^\infty \int_0^R \left( \int_{S^2} |u(t, r\omega)|^2 \, d\omega \right) r^2 \, dr \, dt \\
\leq R^3 \int_T^\infty \sup_{r \in (0, R)} \left( \int_{S^2} |u(t, r\omega)|^2 \, d\omega \right) r^2 \, dr \, dt \\
= R^3 \int_T^\infty \left( \sup_{r \in (0, R)} \|u(t, r\omega)\|_{L^6_x}^2 \right) \, dt = R^3 \|u\|^2_{L^6_t((T, \infty); L^\infty^x L^6_x)}.
\]

By combining this fact with (4.7) we get the following implication:

\[
\forall \epsilon > 0 \text{ there exists } T_1(\epsilon) > 0 \text{ s.t. } \limsup_{R \to \infty} \frac{1}{R^3} \int_{T_1(\epsilon)}^\infty \int_{B_R} |u|^2 \, dx \, dt \leq \epsilon.
\]

Of course by a similar argument we can prove that:

\[
\forall \epsilon > 0 \text{ there exists } T_2(\epsilon) > 0 \text{ s.t. } \limsup_{R \to \infty} \frac{1}{R^3} \int_{-T_2(\epsilon)}^{-T(\epsilon)} \int_{B_R} |u|^2 \, dx \, dt \leq \epsilon.
\]

In particular, if we choose \( T(\epsilon) = \max\{T_1(\epsilon), T_2(\epsilon)\} \), then we get:

(4.12) \[
\forall \epsilon > 0 \text{ there exists } T(\epsilon) > 0 \text{ s.t. } \\
\limsup_{R \to \infty} \frac{1}{R^3} \int_{R \setminus [-T(\epsilon); T(\epsilon)]} \int_{B_R} |u|^2 \, dx \, dt \leq \epsilon.
\]

Hence the proof of proposition 4.3 (in the case \( n = 3 \)) will follow from the following fact:

(4.13) \[
\forall T > 0 \text{ we have } \limsup_{R \to \infty} \frac{1}{R^3} \int_{-T}^{T} \int_{B_R} |u|^2 \, dx \, dt = 0.
\]

Notice that by using the Hölder inequality we get:

\[
\int_{B_R} |u(t)|^2 \, dx \leq R^2 \|u(t)\|^2_{L^6_x},
\]

and this implies:

\[
\frac{1}{R^3} \int_{-T}^{T} \int_{B_R} |u|^2 \, dx \, dt \leq \frac{C}{R} \int_{-T}^{T} \|u(t)\|^2_{L^6_x} \, dt \leq \frac{2CT}{R} \|u\|^2_{L^\infty_t L^6_x}.
\]
By combining this fact with (1.1) and with the Sobolev embedding $\dot{H}^1_x \subset L^6_x$, we finally get (4.13).

\[ \square \]

Proof of theorem 0.2 First of all let us recall the following identity
\[ \nabla_x \bar{u} D^2_x \psi \nabla_x u = \partial^2_{|x|} \psi |\partial_{|x|} u|^2 + \frac{\partial_{|x|} \psi}{|x|} |\nabla_x u|^2, \]
where $\psi$ is a radially symmetric function. By using this identity and by choosing in the identity (0.8) the function $\psi \equiv \langle x \rangle$, then it is easy to deduce that
\[ \int \int_{|x| > 1} \frac{|\nabla_x u|^2}{|x|} \, dxdt < \infty. \]
In particular we deduce:
\[ \lim_{R \to \infty} \int \int_{|x| > R} \frac{|\nabla_x u|^2}{|x|} \, dx = 0. \]
and
\[ \lim_{R \to \infty} \frac{1}{R} \int \int_{B_R} |\nabla_x u|^2 \, dxdt = 0. \]
By combining (4.2) with (4.16) we get (0.10).

Next we shall prove (0.9). For any $k \in \mathbb{N}$ we fix a function $h_k(r) \in C_0^\infty(\mathbb{R}; [0,1])$ such that:
\[ h_k(r) = 1 \ \forall r \in \mathbb{R} \ \text{s.t.} \ |r| < 1, h_k(r) = 0 \ \forall r \in \mathbb{R} \ \text{s.t.} \ |r| > \frac{k+1}{k}, \]
\[ h_k(r) = h_k(-r) \ \forall r \in \mathbb{R}. \]

Let us introduce the functions $\psi_k(r), H_k(r) \in C^\infty(\mathbb{R})$:
\[ \psi_k(r) = \int_0^r (r-s) h_k(s) \, ds \quad \text{and} \quad H_k(r) = \int_0^r h_k(s) \, ds. \]
Notice that
\[ \psi_k''(r) = h_k(r), \psi_k'(r) = H_k(r) \forall r \in \mathbb{R} \quad \text{and} \quad \lim_{r \to \infty} \partial_r \psi_k(r) = \int_0^\infty h_k(s) \, ds. \]
Moreover an elementary computation shows that:
\[ \Delta_x \psi_k \leq \frac{C}{(x)} \ \forall x \in \mathbb{R}^n, n \geq 3 \]
and
\[ \Delta^2_x \psi_k(x) = \frac{C}{|x|^3} \ \forall x \in \mathbb{R}^n \ \text{s.t.} \ |x| \geq 2 \ \text{and} \ n \geq 4, \]
\[ \Delta^2_x \psi_k(x) = 0 \ \forall x \in \mathbb{R}^3 \ \text{s.t.} \ |x| \geq 2, \]
where $\Delta^2_x$ is the bilaplacian operator. Thus the functions $\phi \equiv \psi_k$ satisfy the assumptions of proposition 4.2. In the sequel we shall need the rescaled functions
\[ \psi_{k,R}(x) \equiv R \psi_k \left( \frac{x}{R} \right) \ \forall x \in \mathbb{R}^n, k \in \mathbb{N} \ \text{and} \ R > 0, \]
where $\psi_k$ is defined in (4.18). Notice that by combining the general identity (4.14) with (0.8), where we choose $\psi = \psi_{k,R}$ defined in (4.22), and recalling (4.19) we get:

$$
\int \int \left( \frac{\partial^2 |x| \psi_{k,R}}{x} |\nabla_x u|^2 + \frac{1}{4} |u|^2 \Delta^2 \psi_{k,R} + \frac{\lambda}{n} |u|^2 \Delta_x \psi_{k,R} \right) dx dt
\quad = \left( \int \left( |\nabla_x f|^2 + \frac{2\lambda}{2^*} |f|^{2^*} + |g|^2 \right) dx \right) \forall k \in \mathbb{N}, R > 0.
$$

Notice also that due to (4.21) we get:

$$
\int \int_{B_R} |\Delta^2 \psi_{k,R}| |u|^2 dx dt \leq C \frac{R^3}{n} \int \int_{B_R} |u|^2 dx dt
$$

provided that $n = 3$, and in particular by using (4.5) we get

$$
\lim_{R \to \infty} \int \int_{B_R} |\Delta^2 \psi_{k,R}| |u|^2 dx dt = 0.
$$

In the case $n \geq 4$ we use (4.20) in order to deduce:

$$
\int \int_{R^n} |\Delta^2 \psi_{k,R}| |u|^2 dx dt
\quad \leq C \left( \frac{1}{R^3} \int \int_{B_R} |u|^2 dx dt + \int \int_{R^n \setminus B_R} \frac{|u|^2}{|x|^3} dx dt \right).
$$

On the other hand an explicit computation shows that if we choose in (0.8) $\psi \equiv \langle x \rangle$, when $n \geq 4$, then we get:

$$
\int \int_{R^n} \frac{|u|^2}{|x|^3} dx dt < \infty \text{ for } n \geq 4,
$$

that in conjunction with the Lebesgue dominated convergence theorem, (4.5) and (4.25) implies:

$$
\lim_{R \to \infty} \int \int_{R^n} |\Delta^2 \psi_{k,R}| |u|^2 dx dt = 0 \text{ for } n \geq 4.
$$

By using (4.24), (4.27), (4.3) and (4.15) we get:

$$
\lim_{R \to \infty} \int \left( \frac{\partial |x| \psi_{k,R}}{x} \left| \frac{\nabla_x u}{|x|} \right|^2 - \frac{1}{4} \Delta^2 \psi_{k,R} |u|^2 + \frac{\lambda}{n} \Delta_x \psi_{k,R} |u|^2 \right) dx dt = 0
$$

for every $k \in \mathbb{N}$ and for every dimension $n \geq 3$. We can combine this fact with (4.23) in order to deduce:

$$
\lim_{R \to \infty} \int \int \partial^2 |x| \psi_{k,R} |\partial |x| u|^2 dx dt
\quad = \left( \int \int h_k(s) ds \right) \int \left( |\nabla_x f|^2 + \frac{2\lambda}{2^*} |f|^{2^*} + |g|^2 \right) dx \forall k \in \mathbb{N}.
$$

On the other hand, due to the properties of $h_k$ (see (4.17)), we get

$$
\frac{1}{R} \int \int_{B_R} |\partial |x| u|^2 dx dt \leq \int \int \frac{\partial^2 |x| \psi_{k,R}}{|x|} |\partial |x| u|^2 dt dx
\quad = \frac{1}{R} \int \int \frac{1}{R} \left( \frac{x}{R} \right) |\partial |x| u|^2 dt dx \leq \frac{1}{R} \int \int \frac{1}{|x| < \frac{k}{R}} |\partial |x| u|^2 dt dx
$$
that due to (4.29) implies:

\begin{equation}
\limsup_{R \to \infty} \frac{1}{R} \int \int_{B_R} |\partial_x u|^2 \, dx \, dt \\
\leq \left( \int_0^\infty h_k(s) \, ds \right) \int (|\nabla_x f|^2 + \frac{2\lambda}{2^*} |f|^{2^*} + |g|^2) \, dx \\
\leq \frac{k+1}{k} \liminf_{R \to \infty} \frac{1}{R} \int \int_{B_R} |\partial_x u|^2 \, dx \, dt \forall k \in \mathbb{N}.
\end{equation}

Since \( k \in \mathbb{N} \) is arbitrary and since the following identity is trivially satisfied:

\[ \lim_{k \to \infty} \int_0^\infty h_k(s) \, ds = 1, \]

we can deduce (0.9) by using (4.30).

The proof is complete. \( \square \)

5. Proof of Theorem 0.3

First step: proof of (0.11)
Following [16] we multiply the equation (0.1) by \( \varphi u \) and integrating the corresponding identity on the strip \((-T, T)\) we get:

\begin{equation}
\int_{-T}^T \int |\partial_t u|^2 - |\nabla_x u|^2 - \lambda |u|^{2^*} \, \varphi + \frac{1}{2} |u|^2 \Delta_x \varphi \, dx \, dt \\
= \sum_{\pm} \int \partial_t u(\pm T) u(\pm T) \varphi \, dx
\end{equation}

By taking the limit as \( T \to \infty \) and by using proposition 3.2 we get (0.11).

Second step: proof of (0.12)

For any \( k \in \mathbb{N} \) we fix a function \( \varphi_k(r) \in C_0^\infty(\mathbb{R}; [0, 1]) \) such that:

\begin{equation}
\varphi_k(r) = 1 \text{ \forall } r \in \mathbb{R} \text{ s.t. } |r| < 1, \varphi_k(r) = 0 \text{ \forall } r \in \mathbb{R} \text{ s.t. } |r| > \frac{k+1}{k},
\end{equation}

\[ \varphi_k(r) = \varphi_k(-r) \text{ \forall } r \in \mathbb{R}. \]

We also introduce the rescaled functions

\[ \varphi_{k,R} \equiv \frac{1}{R} \varphi_k \left( \frac{x}{R} \right). \]

Notice by combining the cut-off property of the functions \( \varphi_k \) with (4.2) and (4.5), we get:

\[ \lim_{R \to \infty} \int \int |u|^{2^*} \varphi_{k,R} \, dx \, dt = \lim_{R \to \infty} \int \int |u|^2 \Delta_x \varphi_{k,R} \, dx \, dt = 0 \forall k \in \mathbb{N} \]

in any dimension \( n \geq 3 \). By using this fact in conjunction with (0.11), where we choose \( \varphi \equiv \varphi_{k,R} \), we get:

\begin{equation}
\lim_{R \to \infty} \int \int (|\partial_t u|^2 - |\nabla_x u|^2) \varphi_{k,R} \, dx \, dt = 0 \forall k \in \mathbb{N}.
\end{equation}

Notice that by combining (5.3) with the cut-off properties of \( \varphi_k \) we get:

\begin{equation}
\forall k \in \mathbb{N} \text{ there exists } R(k) > 0 \text{ s.t.}
\end{equation}
\[
\frac{1}{R} \int \int_{B_R} |\partial_t u|^2 \, dx \, dt \leq \frac{k+1}{k} \frac{1}{R^{(k+1)/k}} \int \int_{B_R^{(k+1)/k}} |\nabla_x u|^2 \, dx \, dt + \frac{1}{k} \forall R > R(k).
\]

By combining (5.4) with (0.9) and (0.10), we get:
\[
\limsup_{R \to \infty} \frac{1}{R} \int \int_{B_R} |\partial_t u|^2 \, dx \, dt \\
\leq \frac{k+1}{k} \int \int_{B_R} (|\nabla_x f|^2 + \frac{2\lambda}{2^*}|f|^{2^*} + |g|^2) \, dx + \frac{1}{k} \forall k \in \mathbb{N}
\]
and in particular
\[
\limsup_{R \to \infty} \frac{1}{R} \int \int_{B_R} |\partial_t u|^2 \, dx \, dt \\
\leq \int (|\nabla_x f|^2 + \frac{2\lambda}{2^*}|f|^{2^*} + |g|^2) \, dx.
\]
Similarly one can show that:
\[
\liminf_{R \to \infty} \frac{1}{R} \int \int_{B_R} |\partial_t u|^2 \, dx \, dt \\
\geq \int (|\nabla_x f|^2 + \frac{2\lambda}{2^*}|f|^{2^*} + |g|^2) \, dx,
\]
and finally we get:
\[
\lim_{R \to \infty} \frac{1}{R} \int \int_{B_R} |\partial_t u|^2 \, dx \, dt = \int (|\nabla_x f|^2 + \frac{2\lambda}{2^*}|f|^{2^*} + |g|^2) \, dx
\]
where at the last step we have combined (0.9) with (0.10). The proof of (0.12) is complete.

Finally notice that by combining (0.9), (0.10) and (0.12), we get (0.13).

\section{Appendix I}

The aim of this appendix is to show that the $L^{2^*}$-norm of the solution to the following Cauchy problem:
\begin{equation}
\square u = 0
\end{equation}
\[u(0) = f \in \dot{H}^1_x, \partial_t u(0) = g \in L^2_x,\]
goes to zero as $t \to \pm \infty$. Notice that this fact represents a slight improvement compared with the usual Strichartz estimate
\[\|u(t, x)\|_{L^{2^*}_t L^{2^*}_x} \leq C(\|f\|_{\dot{H}^1_x} + \|g\|_{L^2_x}).\]
On the other hand in proposition 6.2 we shall show that in general no better result can be expected. In fact we shall show that there cannot exist a–priori any rate on the decay of the $L^2$–norm of the solution to (6.1).

Along this section, when it is not better specified, we shall denote by $T(t)(f, g)$ the solution to the Cauchy problem (6.1) with initial data $(f, g)$ computed at time $t$, i.e.:
\[T(t) : \dot{H}^1_x \times L^2_x \ni (f, g) \to u(t) \in \dot{H}^1_x,\]
where $u(t, x)$ solves (6.1).
In particular by combining (6.3) with (6.4) we deduce:

\[ \lim_{t \to \pm \infty} \|u(t)\|_{L^2_x} = 0. \]

**Proof.** We treat for simplicity the case \( t \to \infty \) (the case \( t \to -\infty \) can be treated in a similar way). Notice that due to the Sobolev embedding and the conservation of the energy we have:

\[ (6.2) \quad \|u(t)\|_{L^2_x}^2 \leq S(\|\nabla_x u(t)\|_{L^2_x}^2 + \|\partial_t u(t)\|_{L^2_x}^2) \]

\[ = S(\|\nabla_x f\|_{L^2_x}^2 + \|g\|_{L^2_x}^2) \forall (f, g) \in \dot{H}^1_x \times L^2_x \]

In particular the operators \( T(t) \) introduced above, are uniformly bounded for every \( t > 0 \) in the space \( \mathcal{L}(\dot{H}^1_x \times L^2_x, L^2_x) \).

On the other hand we have the following dispersive estimate (see [18]):

\[ (6.3) \quad \|u(t, x)\|_{L^\infty_x} \leq \frac{C}{t^{\frac{1}{2}}} \left( \|f\|_{B^{\frac{n-1}{2}}_{1,1}} + \|g\|_{B^{\frac{n-1}{2}}_{1,1}} \right) \]

(here \( B^{s}_{p,q} \) denotes the standard Besov spaces). Notice also that the Fourier representation of the solution to (6.1) implies:

\[ (6.4) \quad \|u(t)\|_{L^2_x} \leq C(\|f\|_{L^2_x} + \|g\|_{\dot{H}^{-1}_x}). \]

In particular by combining (6.3) with (6.4) we deduce:

\[ \|u(t)\|_{L^2_x} \leq \|u(t)\|_{L^2_x}^{\frac{3}{2}} \|u\|_{L^2_x}^{\frac{1}{2}} \leq \frac{C}{t^{\frac{1}{2}}} \forall (f, g) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n), \]

where \( C \equiv C(f, g) > 0 \). As a consequence we get

\[ (6.5) \quad \lim_{t \to \infty} \|u(t)\|_{L^2_x} = 0 \forall (f, g) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n). \]

It is now easy to remove in (6.5) the regularity assumption \( (f, g) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \) by a classical density argument.

Notice that the previous result represents a slight improvement compared with the usual Strichartz estimate:

\[ \|u(t, x)\|_{L^\infty_x L^2_x} \leq C(\|f\|_{\dot{H}^1_x} + \|g\|_{L^2_x}). \]

On the other hand next proposition shows that in general no better result can be expected, since there cannot exist a-priori any rate on the decay of the \( L^2_x \)-norm of the solution to (6.1).

**Proposition 6.2.** Let \( \gamma \in C([0, \infty); \mathbb{R}) \) be any function such that

\[ \lim_{t \to \infty} \gamma(t) = \infty. \]

Then there exists \( g \in L^2_x \) such that:

\[ \|u(t_n)\|_{L^2_x} > \frac{1}{\gamma(t_n)}, \]

where \( \{t_n\} \) is a suitable sequence that goes to \(+\infty\) and

\[ \Box u = 0 \quad u(0) = 0, \partial_t u(0) = g \in L^2_x. \]
Proof. We claim the following fact:

\[(6.7) \quad \|S(t)\|_{L^2_x(L^2_x^*)} \geq \epsilon_0 > 0 \]

where

\[S(t) : L^2_x \ni g \rightarrow u(t) \in L^2_x^*\]

is the solution operator associated to the Cauchy problem (6.6).

Notice that due to (6.7) we get:

\[\lim_{t \to \infty} \gamma(t) \|S(t)\|_{L^2_x(L^2_x^*)} = \infty,\]

and in particular due to the Banach–Steinhaus theorem the operators \(\gamma(t)S(t)\) cannot be pointwisely bounded, or in an equivalent way there exists at least one \(g \in L^2_x\) such that:

\[(6.8) \quad \sup_{[0, \infty)} \gamma(t) \|S(t)g\|_{L^2_x^*} = \infty.\]

On the other hand the function \(\gamma(t) \|S(t)g\|_{L^2_x^*}\) is bounded on bounded sets of \([0, \infty)\), and hence (6.8) implies that

\[\lim_{t \to \infty} \sup_{[0, \infty)} \gamma(t) \|S(t)g\|_{L^2_x^*} = \infty\]

and it completes the proof.

Next we shall prove (6.7). Let us fix \(h \in L^2_x\) such that:

\[\|h\|_{L^2_x} = 1 \text{ and } \|S(1)h\|_{L^2_x^*} = \|u(1,x)\|_{L^2_x^*} = \eta_0 > 0\]

where \(u(t,x)\) denotes the unique solution to (6.6) with \(g = h\).

A rescaling argument implies that \(u_\epsilon(t,x) \equiv \epsilon^{\frac{d}{2}-1}u(\epsilon t, \epsilon x)\) solves (6.6) with initial data \(g \equiv h_\epsilon \equiv \epsilon^{\frac{d}{2}}h(\epsilon x)\). In particular this implies that:

\[S\left(\frac{1}{\epsilon}\right) h_\epsilon = \epsilon^{\frac{d}{2}-1}u(1,\epsilon x) \text{ and } \|h_\epsilon\|_{L^2_x} = 1,\]

and hence:

\[\|S\left(\frac{1}{\epsilon}\right)\|_{L^2_x(L^2_x^*)} \geq \|S\left(\frac{1}{\epsilon}\right)h_\epsilon\|_{L^2_x^*} = \epsilon^{\frac{d}{2}-1}\|u(1,\epsilon x)\|_{L^2_x^*} = \|u(1)\|_{L^2_x^*} = \|S(1)h\|_{L^2_x^*} = \eta_0 > 0 \forall \epsilon > 0.\]

The proof of (6.7) is complete.

□

7. Appendix II

This section is devoted to the proof of proposition 2.1. Let us underline that its content is well–known in the literature, in particular it contains the equipartition of the energy principle first proved in [4] by using Fourier analysis.

The aim of this section is to present a proof that involves the conformal energy.

Proof of prop. 2.1 First of all notice that (2.1) implies:

\[\lim_{t \to \infty} \int |\partial_t u(t)|^2 = \lim_{t \to \infty} \int |\partial_t |u(t)||^2 \ dx.\]

By combining this fact with (2.2) and with the following trivial identity:

\[|\nabla_x u|^2 = |\partial_x |u|^2 + |\nabla_x u|^2,\]
we can deduce (2.4). Hence it is enough to prove (2.1) and (2.2) in order to deduce (2.4).

Next notice that by a density argument it is sufficient to prove (2.2), (2.5) and (2.6) under the stronger assumption $(f, g) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ in order to deduce (2.1), (2.2) and (2.3) under the weaker assumption $(f, g) \in H^1_\omega \times L^2_\omega$.

Since now on we shall assume that $(f, g) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$. Following [5] we introduce the conformal energy factor

$$[(t^2 + |x|^2)\partial_t u + 2ntx_j \partial_j u].$$

Since $\Box u = 0$ we get for every $T > 0$ the following identity:

$$0 = \sum_{j=1}^n \int_0^T \int [(t^2 + |x|^2)\partial_t u + 2tx_j \partial_j u] \Box u \, dx dt$$

$$= \int \frac{n}{2} (T^2 + |x|^2)(|\partial_t u(T)|^2 + |\nabla_x u(T)|^2) + 2nT r |\partial_t u(T)|^2 + 4T |x| |\partial_t x| u(T) |\partial_t u(T)| \, dx$$

$$- \int \frac{n}{2} |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx$$

$$+ n(n-1) \int_0^T \int t (|\partial_t u|^2 + |\nabla_x u|^2) \, dx dt,$$

where we have used the Stokes formula.

Notice that this identity implies the following inequality:

$$(7.1) \quad \int (T^2 + |x|^2)(|\partial_t u(T)|^2 + |\nabla_x u|^2) + 4T |x| |\partial_t x| u(T) |\partial_t u(T)| \, dx$$

$$\leq \int |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx.$$

On the other hand we have the trivial pointwise inequality $|\partial_t x|^2 \leq |\nabla_x u|^2$ that can be combined with (7.1) in order to give:

$$(7.2) \quad \int (T^2 + |x|^2)(|\partial_t u(T)|^2 + |\partial_t x|^2) + 4T |x| |\partial_t x| u(T) |\partial_t u(T)| \, dx$$

$$\leq \int |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx,$$

and

$$(7.3) \quad \int (T^2 + |x|^2)(|\partial_t u(T)|^2 + |\nabla_x u(T)|^2) - 4T |x| |\nabla_x u(T)| |\partial_t u(T)| \, dx$$

$$\leq \int |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx.$$

Next recall the basic inequality:

$$(a + b)^2 (c + d)^2 + (a - b)^2 (c - d)^2$$

$$\leq 4(a^2 + b^2)(c^2 + d^2) + 16abcd \forall a, b, c, d \in \mathbb{R}$$

whose proof is completely elementary. By combining this inequality with (7.2) and (7.3) we get respectively:

$$\int (T + |x|)^2 |\partial_t u(T) + \partial_{|x|} u(T)|^2 + (T - |x|)^2 |\partial_t u(T) - \partial_{|x|} u(T)|^2 \, dx$$
\[ \leq 4 \int |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx, \]

and

\[ \int (T + |x|)^2 |\partial_t u(T)| - |\nabla_x u(T)|^2 + (T - |x|)^2 |\partial_t u(T)| + |\nabla_x u(T)|^2 \, dx \leq 4 \int |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx. \]

This in turn implies:

(7.4) \[ \int |\partial_t u(T) + \partial_x u(T)|^2 \, dx \leq \frac{4}{T^2} \int |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx, \]

(7.5) \[ \int ||\partial_t u(T)| - |\nabla_x u(T)||^2 \, dx \leq \frac{4}{T^2} \int |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx, \]

and

(7.6) \[ \int_{2|x|<T} ||\partial_t u(T)| - |\nabla_x u(T)||^2 + ||\partial_t u(T)| + |\nabla_x u(T)||^2 \, dx \leq \frac{16}{T^2} \int |x|^2 (|\nabla_x f|^2 + |g|^2) \, dx. \]

The proof is complete. \[ \square \]

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