A Chiral Schwinger Model, 
its Constraint Structure and 
Applications to its Quantization

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Abstract

The Jackiw-Rajaraman version of the chiral Schwinger model is studied as a function of the renormalization parameter. The constraints are obtained and they are used to carry out canonical quantization of the model by means of Dirac brackets. By introducing an additional scalar field, it is shown that the model can be made gauge invariant. The gauge invariant model is quantized by establishing a pair of gauge fixing constraints in order that the method of Dirac can be used.

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Quantization of a theory, its effects on the classical symmetries and the mechanisms for mass generation of particles in a quantum field theory are subjects that continue to be of interest. Many important effects are already visible in the context of a smaller model with several degrees of freedom. Electrodynamics and its nonabelian extensions in one-space and one-time dimension with massless fermions is of great interest for many reasons, one of which is that the quantization of the theory can be studied from various points of view. The Schwinger model describes a massless Dirac field in two-dimensions with both chiral components coupled to a $U(1)$ gauge field [1]. The Jackiw-Rajaraman model [2,3] or chiral Schwinger model is related to this model but has coupling to only one chiral component, and it is found to depend on a regularization parameter. Thus there is an anomaly in the singly-coupled model, which cancels against a similar but sign reversed anomaly in the doubly-coupled Schwinger model. Moreover, this parameter, although arbitrary, has a very significant effect on the structure of the model, especially the constraint equations and the nature of its field equations. Quantum field theories with gauge couplings to chiral fermions have the property that there is an anomalous nonconservation of the gauge current. This is an interesting characteristic in itself, but of even more significance is that gauge invariance may be lost as well as the existence of a consistent theory [4]. Since most of the interest in gauge theories in general arises from the fact that they are both renormalizable and unitary, this is a serious problem. Consequently, to ensure that renormalizability and unitarity are not threatened, the structure of a theory may be modified or extended. This is evident in the case of gauge theories in which the gauge group is adjusted so that the fermion content of the theory satisfies a specified rule, for example, the number of quarks equals the number of leptons in the model. Consequently, there can be an anomaly for gauge symmetry in a subsector of the theory, but all the individual contributions must cancel.

The purpose here is to investigate the chiral Schwinger model and show how it corresponds to the bosonized chiral Schwinger model in one-space and one-time dimension which is due to Jackiw and Rajaraman [2]. It will be seen that the structure of the model, especially the structure of the
constraints, depends on the value assigned to the regularization parameter. The structure of the system of constraints will be obtained for several different cases of this parameter and discussed in detail. It is shown how the theory corresponding to the classical Hamiltonian can be quantized based on the Dirac bracket [5]. The study of the constraints is important as far as path integral quantization is concerned, especially when gauge fixing constraints must be added to convert a set of first class constraints into a set of second class constraints [6,7]. The topic of constraints has been of interest recently [8,9].

Gauge invariance may be lost in this process, but it will be shown in a case in which this takes place that, by including a type of Wess-Zumino term [10], gauge invariance can be restored. To accomplish this however a new field must be introduced. Although this effectively enlarges the Hilbert space, gauge invariance is retained and the field appears in the Hamiltonian in a way analogous to the scalar boson field already present. This is in contrast to an alternative procedure which is to use BRST quantization of a gauge invariant theory [11]. To do this, the theory is rewritten as a quantum system that possesses a generalized gauge invariance, and require that the Hilbert space of the gauge invariant theory be enlarged. In this formulation, the gauge-invariant theory replaces the gauge transformation by a BRST transformation. This transformation mixes operators having different statistics, and as with the Wess-Zumino field, the corresponding Hilbert space is enlarged.

It will be seen that gauge invariance is restored using this Wess-Zumino term when the parameter is one. Two gauge fixing constraints are introduced into the theory which serve to establish a gauge. Using all the constraints, a quantization of the theory can be performed [12,13]. Finally a path integral quantization will be outlined at the end [14].

2. Introduction and Properties of the Model

The Lagrangian density for the chiral Schwinger model is given explicitly by

\[ \mathcal{L}_S = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} [i \partial + e \sqrt{\pi} A (1 + i \gamma_5)] \psi, \]

(2.1)

where \( \gamma_5 = i \gamma^0 \gamma^1 \). At the classical level, the Lagrangian (2.1) is invariant under the local gauge
transformations

\[ A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x), \quad \psi(x) \rightarrow e^{2ie\sqrt{\pi}\alpha(x)P_+} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{-2ie\sqrt{\pi}\alpha(x)P_-}, \]

where \( P_\pm = \frac{1}{2}(1 + i\gamma_5) \). The fermion determinant for this two-dimensional system can be evaluated in closed form and yields an effective action of the form

\[ S_e = \int dt\, dx \left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{2}A_\mu(\alpha g^{\mu\nu} - (g^{\mu\alpha} - \epsilon^{\mu\alpha}) \partial_\alpha \partial_\beta (g^{\beta\nu} - \epsilon^{\beta\nu}) A_\nu \right\}. \tag{2.2} \]

The quantity \( a \) in (2.2) is a constant which is not uniquely determined by the different procedures for calculating the fermionic determinant. Its value would be fixed by gauge invariance were it not for the fact that the model has an anomaly \([15]\). However, \( a \) may be allowed to be arbitrary, but the domain of \( a \) will be restricted to a particular subset of values. The two cases \( a > 1 \) and \( a = 1 \) will be of interest here and studied separately, and we will take \( \hbar = 1 \) in what follows.

It will now be shown that an auxiliary scalar field \( \varphi(x) \) can be introduced into the formalism, which links (2.1) to the bosonized version of the model, by introducing a path integral with respect to the scalar field which can be done in closed form as follows

\[ \exp[iS_e(A)] = \int D\varphi \exp[iS(A, \varphi)]. \tag{2.3} \]

The action on the right-hand side of (2.3) is modified from (2.2) to include the new scalar field \( \varphi \) as

\[ S(A, \varphi) = \int dt\, dx \mathcal{L}(A, \varphi), \tag{2.4} \]

where the Lagrangian with the scalar field is

\[ \mathcal{L}(A, \varphi) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) + e(g^{\mu\nu} - \epsilon^{\mu\nu}) A_\mu \partial_\nu \varphi + \frac{1}{2} a e^2 A_\mu A^\mu. \tag{2.5} \]

In (2.4), \( g^{\mu\nu} \) is the Lorentz metric,

\[ g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g_{\mu\nu}, \]

and \( \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}. \)
To see that (2.4) is the bosonized version of the fermion action, note that the $\varphi$ integration is independent of the field $A_\mu$. By using (2.4) and (2.5) in (2.3), then separating the $\varphi$-dependent terms in the path integral as

$$
\int D\varphi \exp\left[i \int d^2 x \left\{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) + e(g_{\mu\nu} - e^{\mu\nu}) A_\nu \partial_\mu \varphi + \frac{1}{2} a e^2 A_\mu A^\mu\right]\right]
$$

$$
= \exp\left[i \int d^2 x \left\{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} a e^2 A_\mu A^\mu\right]\right] \int D\varphi \exp\left[i \int d^2 x \left\{-\frac{1}{2} \varphi \Box \varphi - e(g_{\mu\nu} - e^{\mu\nu}) \partial_\mu A_\nu \varphi\right]\right].
$$

Now the integral with respect to $\varphi$ can be done by completing the square and up to an irrelevant multiplicative determinant factor, this matches $S_e$ in (2.2). The presence of the scalar field in (2.5) also allows another interpretation of the parameter $a$, namely, it reflects the degree of bosonization in the model. Once the action has been determined, as in (2.4), it is straightforward to determine the field equations for both the boson field $\varphi$ and the vector potential $A_\mu$ using the Euler-Lagrange equations [16]

$$
\partial_\mu \frac{\partial L}{\partial \varphi} - \frac{\partial L}{\partial \varphi} = 0,
$$

where $Q_r$ stands for either of the two fields $\varphi$ or $A_\mu$. The following two equations are obtained from (2.5) [17]

$$
\Box \varphi + e(g_{\mu\nu} - e^{\mu\nu}) A_\nu = 0, \quad (2.6)
$$

$$
\partial_\mu F^{\mu\nu} + e(g^{\nu\alpha} - e^{\nu\alpha}) \partial_\alpha \varphi + a e^2 A^\nu = 0. \quad (2.7)
$$

Note that $\varphi$ can be obtained explicitly in terms of $A_\nu$ from (2.6)

$$
\varphi = -e(g^{\mu\nu} - e^{\mu\nu}) \frac{\partial_\mu A_\nu}{\Box}. \quad (2.8)
$$

Substituting $\varphi$ into (2.7), there results the expression

$$
\partial_\mu F^{\mu\nu} + a e^2 A^\nu - e^2 (g^{\nu\alpha} - e^{\nu\alpha}) \partial_\alpha \partial^3 (g_{\beta\mu} - e^{\beta\mu}) A_\mu = 0. \quad (2.9)
$$

It will be shown that a solution to system (2.6)-(2.7) is determined by taking

$$
A^\mu = -\frac{1}{ae}[\partial^\mu \varphi + (1 - a)e^{\mu\nu} \partial_\nu \varphi - a e^{\mu\nu} \partial_\nu h], \quad (2.10)
$$

where $h$ is an arbitrary function that satisfies the wave equation

$$
\Box h = 0, \quad (2.11)
$$
and the function \( \varphi + h \) satisfies the Klein-Gordon equation

\[
\Box (\varphi + h) + \frac{e^2a^2}{a-1}(\varphi + h) = 0. \tag{2.12}
\]

To show that (2.6) is satisfied, we calculate \( g^{\mu\nu} \partial_\mu A_\nu \) using the antisymmetry of \( \epsilon^{\mu\nu} \)

\[
g^{\mu\nu} \partial_\mu A_\nu = -\frac{1}{ae} [\Box \varphi + (1-a)\epsilon^{\sigma\tau} \partial_\sigma \partial_\tau \varphi - a \epsilon^{\mu\nu} \epsilon^{\sigma\tau} \partial_\sigma \partial_\tau h] = -\frac{1}{ae} \Box \varphi,
\]

and moreover, we obtain

\[
e^{\mu\nu} \partial_\mu A_\nu = -\frac{1}{ae} [(1-a)g^{\sigma\tau} \epsilon^{\mu\nu} \epsilon^{\sigma\tau} \partial_\sigma \partial_\tau \varphi - a \epsilon^{\mu\nu} \epsilon^{\sigma\tau} \partial_\sigma \partial_\tau h] = -\frac{1}{ae} [(1-a) \Box \varphi - a \Box h].
\]

Therefore,

\[
(g^{\mu\nu} - e^{\mu\nu}) \partial_\mu A_\nu = -\frac{1}{ae} \Box \varphi + \frac{1}{ae} (1-a) \Box \varphi = -\frac{1}{e} \Box \varphi.
\]

This is exactly (2.6). Similarly, we calculate

\[
\Box A^\mu = -\frac{1}{ae} [\partial^\mu \varphi + (1-a)\epsilon^{\mu\nu} \Box \varphi],
\]

and

\[
\partial^\mu \partial_\nu A^\nu = -\frac{1}{ae} \Box \partial^\mu \varphi.
\]

Using these, we find that

\[
\Box A^\mu - \partial^\mu \partial_\nu A^\nu + ae^2 A^\mu
\]

\[
= -\frac{(1-a)}{ae} \epsilon^{\mu\nu} \partial_\nu \Box \varphi - e \partial^\mu \varphi - e (1-a) \epsilon^{\mu\nu} \partial_\nu \varphi + ae \epsilon^{\mu\nu} \partial_\nu h
\]

\[
= \frac{(1-a)}{ae} \epsilon^{\mu\nu} \partial_\nu (\Box (\varphi + h) + \frac{ae^2}{a-1} (\varphi + h)) - e (g^{\mu\nu} + \epsilon^{\mu\nu}) \partial_\nu \varphi.
\]

If it is required that \( \sigma = \varphi + h \) satisfy the additional equation

\[
\Box \sigma + \frac{a^2e^2}{a-1} \sigma = 0, \tag{2.13}
\]

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then (2.7) holds. When \( a \neq 1 \), equation (2.13) is a Klein-Gordon equation which describes a field of mass

\[
m^2 = \frac{a^2 e^2}{a - 1},
\]

and \( m^2 > 0 \) when \( a > 1 \). As long as \( a > 1 \), this system consists of a free massive degree of freedom described by the field \( \sigma \) such that harmonic excitations propagate along the light cone described by the field \( h \).

Moreover, the quantity \( F = \epsilon^{\mu\nu} \partial_{\mu} A_{\nu} \) obeys the same free massive Klein-Gordon equation (2.13) satisfied by \( \sigma \). This can be shown by simplifying \( F \) as

\[
\epsilon^{\mu\nu} \partial_{\mu} A_{\nu} = -\frac{1}{ae} [(1 - a)\Box \varphi - a \Box h] = \frac{(a - 1)}{ae} \Box \varphi = -ae(\varphi + h).
\]

When \( \sigma = \varphi + h \) satisfies Klein-Gordon equation (2.13), we find that

\[
(\Box + m^2) F = -ae(\Box + m^2)(\varphi + h) = -ae(\Box + m^2)\sigma = 0.
\]

This proves the claim.

**3. Canonical Quantization of the Theory for \( a > 1 \).**

The Lagrangian (2.5) can be put in the form

\[
\mathcal{L} = -\frac{1}{2} F_{01} F^{01} + \frac{1}{2} (\partial_0 \varphi)^2 - \frac{1}{2} (\partial_1 \varphi)^2 + e(g^{0\nu} - \epsilon^{0\nu}) \partial_0 \varphi A_{\nu} + e(g^{1\nu} - \epsilon^{1\nu}) \partial_1 \varphi A_{\nu} + \frac{1}{2} ae^2 A_{\mu} A^\mu.
\]  

(3.1)

From the Lagrangian in this form, the canonical momenta are found by calculating

\[
\pi_r(x,t) = \frac{\partial L}{\partial (\partial_0 Q_r)}.
\]

(3.2)

Replacing \( Q_r \) by \( A_0, A_1 \) and \( \varphi \) respectively, we obtain the momenta \( \pi_0, \pi_1 \) and \( \pi \)

\[
\pi_0 = \frac{\partial L}{\partial A_0} = 0,
\]

(3.3)

\[
\pi_1 = \frac{\partial L}{\partial A_1} = -(\partial^0 A^1 - \partial^1 A^0) = -F^{01} = F_{01},
\]

(3.4)
\[ \pi = \frac{\partial L}{\partial \dot{\varphi}} = \partial_0 \varphi + e(g_{0\mu} - \epsilon_{0\mu})A^\mu. \]  

The Hamiltonian density and Hamiltonian can be determined from the momenta and the Lagrangian density as

\[ \mathcal{H} = \pi \dot{\varphi} + \pi_1 \dot{A}_1 + \pi_0 \dot{A}_0 - \mathcal{L} \]

\[ = \frac{1}{2} \pi_1^2 + \frac{1}{2} \pi_0^2 + \frac{1}{2} (\partial_1 \varphi)^2 + \pi_1 \partial^1 A_0 - \frac{1}{2} ae^2 A_\mu A^\mu + \frac{1}{2} e^2 (g_{0\mu} - \epsilon_{0\mu})(g_{0\nu} - \epsilon_{0\nu})A^\mu A^\nu \]

\[ - e\pi (g_{0\mu} - \epsilon_{0\mu})A^\mu - e(g_{1\nu} - \epsilon_{1\nu})\partial^1 \varphi A^\nu. \]  

From (3.3), we introduce the first class constraint \( \Omega_1 = \pi_0 \approx 0 \) and incorporate \( \Omega_1 \) into the total Hamiltonian \( H_T \) by means of a Lagrange multiplier \( \lambda_0(x,t) \)

\[ H_T = H + \int dx \lambda_0 \pi_0, \]  

where

\[ H = \int dx \left( \frac{1}{2} \pi_1^2 + \frac{1}{2} \pi_0^2 - A_0 \partial^1 \pi_1 + \frac{1}{2} (\partial_1 \varphi)^2 - \frac{1}{2} ae^2 A_\mu A^\mu + \frac{1}{2} e^2 (g_{0\mu} - \epsilon_{0\mu})(g_{0\nu} - \epsilon_{0\nu})A^\mu A^\nu \right. \]

\[ - \left. e\pi (g_{0\mu} - \epsilon_{0\mu})A^\mu - e(g_{1\nu} - \epsilon_{1\nu})\partial^1 \varphi A^\nu. \right] \]  

Now it is required that the primary constraint \( \Omega_1 \) be preserved in time under the action of the Hamiltonian \( H \),

\[ \hat{\pi}_0 = \{ \pi_0, H \}, \]  

where these brackets denote the standard Poisson bracket defined as,

\[ \{ f_1(y), f_2(x) \} = \int d\tau \sum_j \left[ \frac{\partial f_1(y)}{\partial q_j(\tau)} \frac{\partial f_2(x)}{\partial p_j(\tau)} - \frac{\partial f_1(y)}{\partial p_j(\tau)} \frac{\partial f_2(x)}{\partial q_j(\tau)} \right]. \]

This requirement leads to the existence of a second-class constraint, namely,

\[ \Omega_2 \equiv \partial^1 \pi_1 + ae^2 A^0 - e^2 (g_{0\nu} - \epsilon_{0\nu})A^\nu + e\pi - e\partial^1 \varphi \approx 0. \]  

For the case here in which \( a > 1 \), no new constraints are generated by requiring the persistence in time of \( \Omega_2 \) in (3.11). Since the Poisson bracket

\[ \{ \Omega_1(y), \Omega_2(x) \} = \{ \pi_0, \partial^1 \pi_1 + ae^2 A_0 - e^2 A_0 - e^2 (g_{0\nu} - \epsilon_{0\nu})A^1 + e\pi - e\partial^1 \varphi \} \]
\[= -(a^2 - 1)e^2\delta(y - x) \quad (3.12)\]
does not vanish for \(a > 1\), the constraints are second-class. Hence, requiring that \(\dot{\Omega}_2 = 0\) only acts to determine the Lagrange multiplier \(\lambda_0\) in (3.7). The nonvanishing of the bracket implies that the local gauge invariance has been broken at the level of the effective Lagrangian.

The matrix of Poisson brackets which is based on the constraints \(\Omega_\alpha\) is 2 \(\times\) 2 and has the form

\[
\Delta_{\alpha\beta}(y, x) = \{\Omega_\alpha(y), \Omega_\beta(x)\} = \begin{pmatrix} 0 & -(a^2 - 1)e^2\delta(y - x) \\ (a - 1)e^2\delta(y - x) & 0 \end{pmatrix}. \quad (3.13)
\]

This is a nonsingular matrix, and its inverse is required to evaluate the Dirac brackets for this case. The inverse matrix is given by

\[
\Delta^{-1}_{\alpha\beta}(y, x) = \begin{pmatrix} 0 & \frac{1}{e^2(a - 1)}\delta(y - x) \\ -\frac{1}{e^2(a - 1)}\delta(y - x) & 0 \end{pmatrix}. \quad (3.14)
\]

It can be verified that \(\Delta^{-1}_{\alpha\beta}\) satisfies the condition

\[
\int d\tau \Delta(y, \tau)\Delta^{-1}(\tau, x) = 1\delta(y - x). \quad (3.15)
\]

The Dirac brackets can be evaluated by means of the matrix elements of \(\Delta^{-1}\) given that the canonically conjugate pairs are \((\varphi, \pi), (A_0, \pi_0)\) and \((A_1, \pi_1)\). Once these brackets are known, Dirac’s algorithm generates a quantization scheme. In terms of the constraints \(\Omega_\alpha\), the Dirac bracket [5] is given by

\[
[f_1, f_2]_D = \{f_1, f_2\} - \{f_1, \Omega_s\}\Delta^{-1}_{ss'}\{\Omega_{s'}, f_2\}. \quad (3.16)
\]

For example, the following bracket yields

\[
[A_1(y), \pi_1(x)]_D = \int dz\delta(y - z)\delta(z - x) - \int dz'dz'\{A_1(y), \pi_0(z)\}\Delta^{-1}_{12}\{\Omega_2(z'), \pi_1(x)\}
\]

\[
-\{A_1(y), \Omega_2(z)\}\Delta^{-1}_{21}\{\pi_0(z'), \pi(x)\} = \delta(y - x),
\]

since both off-diagonal elements of \(\Delta^{-1}\) are zero.

The canonical quantization of the theory is achieved by abstracting the equal-time commutators from the corresponding Dirac brackets. The quantum theory is obtained by taking the commutation relations to correspond to these new bracket relations. Thus the Dirac brackets
are replaced by commutators and a multiplicative factor of $i$ is placed with what results on the right-hand side. Nonvanishing equal time commutators are presented here

$$[\varphi(y), \pi(x)] = i\delta(y-x),$$

$$[A_1(y), \pi_1(x)] = i\delta(y-x),$$

$$[A_0(y), A_1(x)] = \frac{i}{e^2(a-1)} \partial_y \delta(y-x),$$

$$[A_0(y), \varphi(x)] = \frac{i}{e(a-1)} \delta(y-x),$$

$$[A_0(y), \pi(x)] = -\frac{i}{a-1} \delta(y-x),$$

$$[A_0(y), \pi(x)] = \frac{i}{e(a-1)} \partial_y \delta(y-x).$$

\textbf{4. The Gauge-Noninvariant Theory For $a = 1$}

Many of the commutators in (3.16) obtained for $a > 1$ become singular as $a$ approaches one and the structure of the constraint $\Omega_2$ in (3.11) changes significantly when $a = 1$. The theory will be studied in more detail for this case. The constraints become more complicated and so to simplify we set $e = 1$ and also $a = 1$ in the Lagrangian density. It is given by

$$L = \frac{1}{2}(\dot{\varphi}^2 - \varphi'^2) + (\varphi + \varphi')(A_0 - A_1) + \frac{1}{2}(\dot{A}_1 - A'_0)^2 + \frac{1}{2}(A_0^2 - A_1^2).$$

(4.1)

To simplify $L$, it has been expanded out in detail and dot and prime denote time and space derivatives, respectively. The terms in (4.1) have interpretations. The first term corresponds to a massless boson the second represents the chiral coupling of $\varphi$ to the electromagnetic field $A_\mu$, the third term is the kinetic energy of the electromagnetic field, and the last term is associated with the mass for the vector particle.

The canonical momenta are determined to be

$$\pi_0 = \frac{\partial L}{\partial \dot{A}_0} = 0,$$

$$\pi_1 = \frac{\partial L}{\partial \dot{A}_1} = \dot{A}_1 - A'_0.$$
\[ \pi = \frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi} + A_0 - A_1. \]

The Hamiltonian density can be determined using these momenta from

\[ H = \pi \dot{\varphi} + \pi_1 \dot{A}_1 + \pi_0 \dot{A}_0 - L. \]

It is determined to be

\[ H = \frac{1}{2} \pi^2 + \frac{1}{2} \pi_1^2 + \frac{1}{2} \varphi'^2 + \pi_1 A_0' + (\pi + \varphi' + A_1)(A_1 - A_0), \tag{4.2} \]

and the Hamiltonian \( H \) is the integral of \( H \) over the space variable. The canonically conjugate pairs can then be summarized as \((\varphi, \pi), (A_0, \pi_0)\) and \((A_1, \pi_1)\). The Lagrangian density in (4.1) possesses the following four second-class constraints

\[ \Omega_1 = \pi_0 \approx 0, \]

\[ \Omega_2 = \pi_1' + \varphi' + \pi + A_1 \approx 0, \]

\[ \Omega_3 = \pi_1 \approx 0, \]

\[ \Omega_4 = -\pi - \varphi' - 2A_1 + A_0 \approx 0. \tag{4.3} \]

In (4.3), \( \Omega_1 \) is a primary constraint and \( \Omega_2, \Omega_3 \) and \( \Omega_4 \) are secondary constraints.

To prove this we proceed as follows. The momentum \( \pi_0 \) is seen to vanish hence \( \pi_0 \approx 0 \) is a primary constraint. Now it is required that this constraint be invariant under the action of the Hamiltonian. By calculating the Poisson bracket \( \{ \pi_0(y), H(x) \} \) and requiring that this be zero generates a new constraint, \( \Omega_2 \). Proceeding in a similar way, the remaining two constraints are obtained. Using the constraint \( \Omega_\alpha \), the matrix of Poisson brackets can be calculated using (3.10) explicitly, and it is given by

\[ \Delta_{\alpha\beta}(y, x) = \begin{pmatrix} 0 & 0 & 0 & -\delta(y - x) \\ 0 & 0 & \delta(y - x) & 0 \\ 0 & -\delta(y - x) & 0 & 2\delta(y - x) \\ \delta(y - x) & 0 & -2\delta(y - x) & 2\partial_y \delta(y - x) \end{pmatrix} \tag{4.4} \]

This matrix is nonsingular and has an inverse \( \Delta_{\alpha\beta}^{-1} \) which satisfies (3.15) and can be used to calculate the Dirac bracket by means of (3.15). Once these are obtained, Dirac’s algorithm for quantization discussed before can be applied.

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Following the same procedure, the non-vanishing equal-time commutators obtained by the quantization of this system are presented below

\[ [A_0(y), \varphi(x)] = [A_1(y), \varphi(x)] = [\varphi(y), \pi(x)] = i\delta(y - x), \]
\[ [A_0(y), \pi(x)] = [A_1(y), \pi(x)] = -i\partial_y \delta(y - x), \]
\[ [A_0(y), A_0(x)] = [A_0(y), A_1(x)] = [A_1(y), A_1(x)] = 2i\partial_y \delta(y - x). \]

(4.5)

5. The Gauge Invariant Theory.

In constructing a gauge-invariant model corresponding to the Lagrangian in (4.1), a type of Wess-Zumino term is calculated [10]. To do this, the actual Hilbert space of the theory is expanded to include a new field, which we call \( \vartheta \). This is done by redefining the fields \( \varphi \) and \( A^\mu \) in the original Lagrangian density as [18]

\[ \varphi \rightarrow \varphi - \vartheta, \quad A^\mu \rightarrow A^\mu + \partial^\mu \vartheta. \]

(5.1)

Under this replacement, \( \mathcal{L} \) is mapped into \( \mathcal{L}_T \) given by

\[ \mathcal{L}_T = \frac{1}{2}(\dot{\varphi}^2 - \varphi'^2) + (\dot{\vartheta} + \dot{\varphi})(A_0 - A_1) + \frac{1}{2}(\dot{A}_1 - A'_0)^2 + \frac{1}{2}(A_0^2 - A_1^2) \]
\[ + \varphi' \dot{\vartheta} - \varphi \vartheta' + \dot{\vartheta}A_1 - \vartheta A_0 \]
\[ = \mathcal{L} + \mathcal{L}_\vartheta. \]

(5.2)

Here, we have defined

\[ \mathcal{L}_\vartheta = \varphi' \dot{\vartheta} - \varphi \vartheta' + \dot{\vartheta}A_1 - \vartheta A_0. \]

(5.3)

Since the total action is an integral of (5.2) over \((x,t)\), the first two terms in (5.3) could be eliminated from the action using integration by parts. However, they have an effect on the structure of the constraints and should be retained here. The constraint structure is very important as far as quantization is concerned, in particular, as far as path integral quantization is concerned when gauge constraints must be invoked. In fact, \( \mathcal{L}_T \) describes a gauge-invariant theory.
The Euler-Lagrange equations obtained from $L_T$ including the terms $\varphi'\dot{\vartheta} - \dot{\varphi}\vartheta'$ in $L_\vartheta$ are identical to the Lagrange equations without these terms and are given by

\begin{align}
\ddot{\varphi} - \varphi'' &= \dot{A}_1 - A_0' - \dot{A}_0 + A_1', \\
\ddot{A}_1 - A_1' &= \dot{\vartheta} - \dot{\varphi} - \varphi' - A_1, \\
\ddot{A}_1' - A_1'' &= \vartheta' - A_0 - \dot{\varphi} - \varphi', \\
\dot{A}_1 - A_0' &= 0.
\end{align}

(5.4)

Using $L_T$ in (5.2), the canonical momenta for the gauge-invariant theory are calculated to be

\begin{align}
\pi_0 &= \frac{\partial L_T}{\partial \dot{A}_0} = 0, \\
\pi_\vartheta &= \frac{\partial L_T}{\partial \dot{\vartheta}} = A_1 + \varphi', \\
\pi_1 &= \frac{\partial L_T}{\partial \dot{A}_1} = \dot{A}_1 - A_0', \\
\pi &= \frac{\partial L_T}{\partial \dot{\varphi}} = \dot{\varphi} + A_0 - A_1 - \vartheta'.
\end{align}

(5.5)

Thus the theory possesses two primary constraints, each independent of velocity terms

\begin{align}
\psi_1 = \pi_0 \approx 0, \quad \psi_2 = \pi_\vartheta - A_1 - \varphi' \approx 0.
\end{align}

(5.6)

Only the momenta $\pi$ and $\pi_1$ in (5.5) involve time derivatives, and the time derivatives can be solved for explicitly

\begin{align}
\dot{A}_1 = \pi_1 + A_0', \quad \dot{\varphi} = \pi - A_0 + A_1 + \vartheta'.
\end{align}

(5.7)

The canonical Hamiltonian density can be calculated from these as

\begin{align}
\mathcal{H}_T = \pi\dot{\varphi} + \pi_\vartheta \dot{\vartheta} + \pi_1 \dot{A}_1 + \pi_0 \dot{A}_0 - L_T.
\end{align}

(5.8)

Using (5.5) and (5.7), $\mathcal{H}_T$ is found to be

\begin{align}
\mathcal{H}_T = \frac{1}{2}(\pi^2 + \pi_1^2) + \frac{1}{2}(\varphi'^2 + \vartheta'^2) + \pi_1 A_0' + (\pi + \varphi' + A_1 + \vartheta')(A_1 - A_0) + \vartheta'A_0 + \pi\vartheta'.
\end{align}

(5.9)
All of the velocities have been eliminated in obtaining (5.9), and only derivatives with respect to the spatial coordinates remain, even as far as the $\vartheta$ field is concerned.

Again, the primary constraints can be included in the canonical Hamiltonian density by making use of Lagrange multipliers $\lambda_0$ and $\lambda_1$ as follows

$$\mathcal{H}_E = \mathcal{H}_T + \lambda_0 \pi_0 + \lambda_1 (\pi_0 - A_1 - \varphi')$$

$$= \frac{1}{2} (\pi^2 + \pi_1^2) + \frac{1}{2} (\varphi'^2 + \vartheta'^2) + \pi_1 A'_0 + (\pi + \varphi' + A_1 + \vartheta') (A_1 - A_0) + \vartheta' A_0 + \pi \vartheta'$$

$$+ \pi_0 \lambda_0 + (\pi_\vartheta - A_1 - \varphi') \lambda_1.$$  

The total Hamiltonian is given by the integral of $\mathcal{H}_E$ over $x$. From the total Hamiltonian the set of Hamilton’s equations can be obtained. This will be done since they can be used as an alternative way to determine the evolution of the constraints under the action of the Hamiltonian.

By differentiating the Hamiltonian, we have

$$\dot{\varphi} = \frac{\partial \mathcal{H}_E}{\partial \pi} = \pi + A_1 - A_0 + \vartheta', \quad -\dot{\pi} = \frac{\partial \mathcal{H}_E}{\partial \varphi} = -\varphi'' - A'_1 + A'_0 + \lambda'_1;$$

$$\dot{\lambda}_0 = \frac{\partial \mathcal{H}_E}{\partial \pi_0} = \lambda_0, \quad -\dot{\lambda}_0 = \frac{\partial \mathcal{H}_E}{\partial A_0} = -\pi'_1 - \pi - \varphi' - A_1;$$

$$\dot{\pi}_0 = \frac{\partial \mathcal{H}_E}{\partial A'_0} = \pi_1 + A'_1, \quad -\dot{\pi}_0 = \frac{\partial \mathcal{H}_E}{\partial A_1} = \pi + \varphi' + 2 A_1 + \vartheta' - A_0 - \lambda_1;$$

$$\dot{\vartheta} = \frac{\partial \mathcal{H}_E}{\partial \pi_\vartheta} = \lambda_1, \quad -\dot{\lambda}_1 = \frac{\partial \mathcal{H}_E}{\partial \lambda_1} = -\vartheta'' - A'_1 - \pi';$$

$$\dot{\lambda}_0 = \frac{\partial \mathcal{H}_E}{\partial p_{\lambda_0}} = 0, \quad -\dot{p}_{\lambda_0} = \frac{\partial \mathcal{H}_E}{\partial \lambda_0} = \pi_0;$$

$$\dot{\lambda}_1 = \frac{\partial \mathcal{H}_E}{\partial p_{\lambda_1}} = 0, \quad -\dot{p}_{\lambda_1} = \frac{\partial \mathcal{H}_E}{\partial \lambda_1} = \pi_\vartheta - A_1 - \varphi'.$$

The two primary constraints are $\psi_1$ and $\psi_2$, and it is required that these constraints be preserved in time. Demanding that the primary constraint $\psi_1$ be preserved in time, a secondary constraint is obtained. Using Hamilton’s equations, since $\dot{\pi}_0$ is now known,

$$\{\psi_1, \mathcal{H}_E\} = \dot{\pi}_0 = \pi'_1 + \pi + \varphi' + A_1 \approx 0.$$  

(5.12)
Thus, (5.12) gives a third constraint
\[ \psi_3 = \pi_1' + \pi + \varphi' + A_1 \approx 0. \] (5.13)

The constraint \( \psi_3 \) leads in turn to a fourth constraint \( \psi_4 \). Using Hamilton’s equations (5.14), this is found by evaluating
\[ \{ \pi_0, H_E \} = \pi_1' + \pi + \varphi' + \dot{A}_1 = \pi_1 + \lambda_1'. \] (5.14)

The preservation of \( \psi_2 \) and \( \psi_4 \) in time do not yield further constraints if we make \( \lambda_1 \) independent of \( x \). Thus, the theory is seen to possess four constraints which are summarized below
\[ \psi_1 = \pi_0 \approx 0, \]
\[ \psi_2 = \pi_0 - A_1 - \varphi' \approx 0, \]
\[ \psi_3 = \pi_1' + \pi + \varphi' + A_1 \approx 0, \]
\[ \psi_4 = \pi_1 \approx 0. \] (5.15)

The conjugate pairs for the system now read \((\varphi, \pi), (A_0, \pi_0), (A_1, \pi_1), (\vartheta, \pi_\vartheta), (\lambda_0, p_{\lambda_0})\) and \((\lambda_1, p_{\lambda_1})\). A current can be defined in this case which is conserved and it is given by \( J^\nu = \partial_\mu F^{\mu\nu} \).

Using (2.7) with \( a = 1 \), and equations (5.4), we find that
\[ -\partial_\nu (\partial_\mu F^{\mu\nu}) = (\delta^{\nu\alpha} - \epsilon^{\nu\alpha}) \partial_\nu \partial_\alpha \varphi + \partial_\nu A^\nu = \dot{A}_0 - A_1' + \dot{\varphi} - \varphi'' = \dot{A}_0 - A_1' + \dot{A}_1 - A_0' = 0. \]

Thus \( \partial_\nu J^\nu = 0 \) and so the gauge-invariant theory is nonanomalous.

The next step is to work out the matrix of Poisson brackets, which is \( 4 \times 4 \) in this case for the constraints \( \psi_a \). It is found to be a singular matrix with a row and a column of zeros appearing in the matrix. This implies that the set of constraints \( \psi_a \) form a set of first class constraints, and the theory described by the Lagrangian is a gauge-invariant theory.

If the theory is going to be quantized using Dirac’s procedure, the first-class constraints of the theory must be converted into second class constraints. To do this, some additional constraints
are to be imposed arbitrarily on the system. These are what is referred to as a set of gauge-fixing conditions. Suppose we require that the $\vartheta$ field satisfy the special condition given by requiring

$$\partial^\mu \vartheta = 0.$$  \hfill (5.16)

This condition can be satisfied by taking the following pair of equations to hold simultaneously as the two gauge conditions

$$\dot{\vartheta} = 0, \quad -\vartheta' = 0.$$  \hfill (5.17)

Thus, using the fact that $\dot{\pi}_1 \approx 0$, the remaining constraint is taken to be

$$\dot{\vartheta} = -\pi - \varphi' - 2A_1 + A_0 - \vartheta' \approx 0,$$

and the total set of six constraints for the theory in this form is summarized here

$$\chi_1 = \psi_1 = \pi_0 \approx 0,$$

$$\chi_2 = \psi_2 = \pi_0 - A_1 - \varphi' \approx 0,$$

$$\chi_3 = \psi_3 = \pi_1' + \pi + \varphi' + A_1 \approx 0,$$

$$\chi_4 = \psi_4 = \pi_1 \approx 0,$$

$$\chi_5 = G_1 = -\vartheta' \approx 0,$$

$$\chi_6 = G_2 = -\pi - \varphi' - 2A_1 + A_0 + \vartheta' \approx 0.$$  \hfill (5.18)

Due to the presence of the new constraint $\xi_6$, there exists a coupling between $\xi_1$ and $\xi_6$ in the Poisson brackets, since the variables $A_0$ and $\pi_0$ occur in a canonical pair in the matrix of Poisson brackets.

From this collection of Poisson brackets, the corresponding matrix of Poisson brackets based on the six constraints $\chi_a$ can be written down in the following form

$$\Delta_{\alpha\beta}(y, x)$$
This matrix is nonsingular and an inverse $\Delta^{-1}_{\alpha\beta}$ can be calculated which satisfies (3.15). Using the inverse matrix, the Dirac brackets can be calculated and the theory can be quantized in the same way as before using (3.16). This process reproduces the commutators already given in (4.5) and in addition generates a few additional commutators which pertain to the new $\vartheta$ field, namely

$$
[\vartheta(y), \pi_\vartheta(x)] = 2i\delta(y - x), \quad [\varpi_\vartheta(y), \varphi(x)] = -i\delta(y - x),
$$

$$
[A_0(y), \pi_\vartheta(x)] = 2[\varpi_\vartheta(y), \pi(x)] = -[\varpi_\vartheta(y), \varpi_\vartheta(x)] = 2i\partial_y \delta(y - x).
$$

6. Summary and Further Ideas.

The constraint structure for this model has been examined. The method of Dirac brackets provides a well defined strategy for finding a canonical quantization. This has been done for two regimes of the arbitrary renormalization parameter in the model. There are other ways to quantize classical systems, and for comparison and future work, we consider the path integral approach. The path integral also provides a means of quantizing a theory. Since the introduction of the $\vartheta$ term has led to a gauge invariant theory, it would seem appropriate to apply that here. To do this, the two constraints $\chi_1$ and $\chi_2$ in (5.18) are taken with the gauge fixing conditions $G_1$ and $G_2$ and the Poisson brackets $\{\chi_a, G_c\}$ are evaluated then put in a $2 \times 2$ matrix. The determinant can be evaluated, and so the transition amplitude can be expressed in the form of a path integral

$$
\mathcal{A} = \int \prod_t \prod_c \delta(\chi_c) \det \left| \{\chi_a, G_b\} \right| \frac{D\pi D\varphi}{(2\pi)^2} \frac{D\pi_0 DA_0}{(2\pi)^2} \frac{D\pi_1 DA_1}{(2\pi)^2} \frac{D\pi_\vartheta D\vartheta}{(2\pi)^2}
$$

$$
\cdot \frac{D\lambda_c(t)}{2\pi} \exp \left\{ i \int_{t'}^t \left[ \pi_\vartheta \dot{\vartheta} + \pi_0 \dot{A}_0 + \pi_1 \dot{A}_1 + \pi_\vartheta \dot{\vartheta} - H_T - \sum_{a=1}^2 \lambda_a \chi_a \right] \right\}.
$$
The gauge-invariant version of the model can be written as a system that possesses BRST symmetry. This symmetry can be thought of as a generalized gauge invariance. Quantization can also be done in this way as well.

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