CMA-ES with Margin: Lower-Bounding Marginal Probability for Mixed-Integer Black-Box Optimization

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ABSTRACT
This study targets the mixed-integer black-box optimization (MI-BBO) problem where continuous and integer variables should be optimized simultaneously. The CMA-ES, our focus in this study, is a population-based stochastic search method that samples solution candidates from a multivariate Gaussian distribution (MGD), which shows excellent performance in continuous BBO. The parameters of MGD, mean and (co)variance, are updated based on the evaluation value of candidate solutions in the CMA-ES. If the CMA-ES is applied to the MI-BBO with straightforward discretization, however, the variance corresponding to the integer variables becomes much smaller than the granularity of the discretization before reaching the optimal solution, which leads to the stagnation of the optimization. In particular, when binary variables are included in the problem, this stagnation more likely occurs because the granularity of the discretization becomes wider, and the existing modification to the CMA-ES does not address this stagnation. To overcome these limitations, we propose a simple modification of the CMA-ES based on lower-bounding the marginal probabilities associated with the generation of integer variables in the MGD. The numerical experiments on the MI-BBO benchmark problems demonstrate the efficiency and robustness of the proposed method.

CCS CONCEPTS
• Mathematics of computing → Mixed discrete-continuous optimization.

KEYWORDS
covariance matrix adaptation evolution strategy, mixed-integer black-box optimization
The proposed adaptive discretization process can be represented as variables in the CMA-ES, and Miyagi et al. [13] used this modification that it is not suitable for binary variables or \( k \)-ary integers.

The most straightforward way to apply the CMA-ES to the MI-BBO is to discretize some of the sampled continuous vector, e.g., [14]. However, because of the plateau caused by the discretization, this simple method may not change the evaluation value by small variations in elements corresponding to the integer variables. Specifically, as pointed out in [5], this stagnation occurs when the sample standard deviation of the dimension corresponding to an integer variable becomes much smaller than the granularity of the discretization. In particular, the step-size tends to decrease with each iteration, which promotes trapping on the plateau of the integer variables. To address this plateau problem in the integer variable treatment, Hansen [5] proposed the injection step-size adaptation, rank-one covariance matrix update, and a covariance matrix of the identity matrix, although we consider the de facto standard CMA-ES [6], which combines the weighted recombination, cumulative step-size adaptation, rank-one covariance matrix update, and rank-\( \mu \) update. We use the default parameters proposed in [6] and listed them in Table 1. The CMA-ES repeats the following steps until a termination criterion is satisfied.

### 2 CMA-ES AND MIXED INTEGER HANDLING

#### 2.1 CMA-ES

Let us consider the black-box minimization problem in the continuous search space for an objective function \( f : \mathbb{R}^N \rightarrow \mathbb{R} \). The CMA-ES samples an \( N \)-dimensional candidate solution \( x \in \mathbb{R}^N \) from an MGD \( \mathcal{N}(\mathbf{m}, \sigma^2 \mathbf{C}) \) parameterized by the mean vector \( \mathbf{m} \in \mathbb{R}^N \), covariance matrix \( \mathbf{C} \in \mathbb{R}^{N \times N} \), and step-size \( \sigma \in \mathbb{R}_{>0} \). The CMA-ES updates the distribution parameters based on the objective function value \( f(x) \). There are several variations in the update methods of distribution parameters, although we consider the de facto standard CMA-ES [6], which combines the weighted recombination, cumulative step-size adaptation, rank-one covariance matrix update, and rank-\( \mu \) update. We use the default parameters proposed in [6] and listed them in Table 1. The CMA-ES repeats the following steps until a termination criterion is satisfied.

#### Sample and Evaluate Candidate Solutions

In the \( t \)-th iteration, the \( \lambda \) candidate solutions \( x_i (i = 1, 2, \ldots, \lambda) \) are sampled independently from the MGD \( \mathcal{N}(\mathbf{m}(t), (\sigma(t))^{2} \mathbf{C}(t)) \) as follows:

\[
y_i = (\sigma(t))^{1/2} \xi_i ,
\]

\[
x_i = \mathbf{m}(t) + \sigma(t) y_i ,
\]

where \( \xi_i \sim \mathcal{N}(0, I) \) represents a random vector with zero mean and a covariance matrix of the identity matrix, and \( (\sigma(t))^{1/2} \) is the square root of the covariance matrix \( \mathbf{C}(t) \) that is the symmetric and positive definite matrix satisfying \( \mathbf{C}(t) = (\sigma(t))^{2} \mathbf{C}(t) \). The candidate solutions \( \{x_1, x_2, \ldots, x_\lambda\} \) are evaluated by \( f \) and sorted by ranking. Let \( x_{i,\lambda} \) be the \( i \)-th best candidate solution; then, \( f(x_{1,\lambda}) \leq f(x_{2,\lambda}) \leq \cdots \leq f(x_{\lambda,\lambda}) \) and let \( y_{i,\lambda} \) be the random vector corresponding to \( x_{i,\lambda} \).

#### Update Mean Vector

The mean vector update uses the weighted sum of the best \( \mu < \lambda \) candidate solutions and updates \( \mathbf{m}(t) \) as follows:

\[
\mathbf{m}(t+1) = \mathbf{m}(t) + \sigma(t) \mu \sum_{i=1}^{\mu} w_i(x_{i,\lambda} - \mathbf{m}(t)) ,
\]

where \( c_m \) is the learning rate for the mean vector, and the weight \( w_i \) satisfies \( w_1 \geq w_2 \geq \cdots \geq w_\mu > 0 \) and \( \sum_{i=1}^{\mu} w_i = 1 \).

#### Compute Evolution Paths

For the step-size adaptation and the rank-one update of the covariance matrix, we use evolution paths that accumulate an exponentially fading pathway of the mean vector in the generation sequence. Let \( \mathbf{p}_\sigma \) and \( \mathbf{p}_c \) describe the evolution paths for the step-size adaptation and rank-one update, respectively; then, \( \mathbf{p}_\sigma \) and \( \mathbf{p}_c \) are updated as follows:

\[
\mathbf{p}_\sigma(t+1) = (1 - c_\sigma) \mathbf{p}_\sigma(t) + \sqrt{c_\sigma(2 - c_\sigma)} \mu^{1/2} w_i y_{i,\lambda} \mathbf{C}^{-1/2} \mathbf{C}(t)^{-1/2} ,
\]

\[
\mathbf{p}_c(t+1) = (1 - c_c) \mathbf{p}_c(t) + h_\sigma \sqrt{c_c(2 - c_c)} \mu w_i y_{i,\lambda} \mathbf{C}(t) ,
\]

where \( c_\sigma \) and \( c_c \) are cumulative rates, and

\[
h_\sigma = \mathbb{E} \left[ ||\mathbf{p}_\sigma(t)|| \right] < \sqrt{1 - (1 - c_\sigma)^2} \left( 1.4 + \frac{2}{N+1} \right) \mathbb{E}[(||\mathcal{N}(0, I)||)]
\]
is an indicator function used to suppress a rapid increase in $p_c$, where $\mathbb{E}[\|N(0, I)\|] \approx \sqrt{\lambda} (1 - \frac{1}{N} + \frac{1}{2 2^{1/2}})$ is the expected Euclidean norm of the sample from a standard Gaussian distribution.

**Update Step-size and Covariance Matrix.** Using the evolution paths computed in the previous step, we update $C(t)$ and $\sigma(t)$ as follows:

$$C(t+1) = \begin{cases} 1 - c_1 - c_\mu \sum_{i=1}^{\lambda} w_i (1 - h_\sigma) c_1 c_2 (2 - c_\sigma) C^{(t)} & \text{rank-one update} \\ + c_\mu \frac{C^{(t+1)} p_c^{(t+1)}}{C^{(t)}} + c_\mu \sum_{i=1}^{\lambda} w_i^c y_i^c y_i^c & \text{rank-}\mu \text{ update} \end{cases} \tag{6}$$

$$\sigma(t+1) = \sigma(t) \exp \left( \frac{c_\sigma}{d_\sigma} \left[ \frac{\|p_{c}^{(t+1)}\|}{\mathbb{E}[\|N(0, I)\|]} - 1 \right] \right), \tag{7}$$

where $w_i^c := w_i \cdot (1 - \mathbb{1}_{w_i \geq 0} \mathbb{N}[\|C^{(t)}\|^{-1} 2^{1/2} y_i^c])$, $c_1$ and $c_\mu$ are the learning rates for the rank-one and rank-\mu updates, respectively. Additionally, $d_\sigma$ is a damping parameter for the step-size adaptation.

### 2.2 CMA-ES with Mixed-Integer Handling

In [5], several steps of the CMA-ES are modified to handle the integer variables. To explain this modification, we apply notations $\langle \cdot \rangle$, where the former denotes the $j$-th element of an argument vector and the latter denotes the $j$-th diagonal element of an argument matrix. We denote the number of dimensions as $N = N_{co} + N_{bi}$, where $N_{co}$ and $N_{bi}$ are the numbers of the continuous and integer variables, respectively. More specifically, the $1$st to $N_{co}$th elements and $(N_{co} + 1)$-th to $N$-th elements of the candidate solution are the elements corresponding to the continuous and integer variables, respectively.

**Inject Integer Mutation.** For the element corresponding to the integer variable, stagnation occurs when the sample standard deviation becomes much smaller than the granularity of the discretization. The main idea to solve this stagnation in [5] is to inject the *integer mutation vector* $r_{int}^{(t)} \in \mathbb{Z}^{N_{bi}}$ into the candidate solution, which is given by

$$x_i = m^{(t)} + o^{(t)} y_i + S^{(t)} r_{int}^{(t)}, \tag{8}$$

where $S^{int}$ is the diagonal matrix whose diagonal elements indicate the variable granularities, which is $S^{int}_{ij} = 1$ if $N_{co} + 1 \leq j \leq N$; otherwise $S^{int}_{ij} = 0$ in usual case. The integer mutation vector $r_{int}^{(t)}$ is sampled as follows:

**Step 1.** Set up a randomly ordered set of elements indices $J^{(t)}$ satisfying $2e^{(t)} C^{(t)}_{jj} < S^{int}_{jj}$.

**Step 2.** Determine the number of candidate solutions into which the integer mutation is injected as follows:

$$\lambda_{int}^{(t)} = \begin{cases} \min(\lambda/10 + |J^{(t)}|, [\lambda/2] - 1) & (|J^{(t)}| = 0) \\ (0 < |J^{(t)}| < N) \\ [\lambda/2] & (|J^{(t)}| = N) \end{cases}$$

**Step 3.** $R_{ij}^{(t)} = 1$ if the element indicate $j$ is equal to mod($i$ – 1, $|J^{(t)}|$)-th element of $J^{(t)}$, otherwise $R_{ij}^{(t)} = 0$.

**Step 4.** $R_{ij}^{(t)}$ is sampled from a geometric distribution with the probability parameter $p = 0.7\exp(-|J^{(t)}|)$ if $j \in J^{(t)}$, otherwise $R_{ij}^{(t)} = 0$.

**Step 5.** $r_{int}^{(t)} = \pm (R_{.i}^{(t)} + R_{i.}^{(t)})$ with the sign-switching probability 1/2 if $i \leq \lambda_{int}^{(t)}$, otherwise $r_{int}^{(t)} = 0$.

**Step 6.** If $\lambda_{int}^{(t)} > 0$, $r_{int}^{(t)} = \pm \left( \frac{x_{t,j}}{\sigma_{t,j}} - \frac{m_{t,j}}{\sigma_{t,j}} \right)$ with the sign-switching probability 1/2 if $S^{int}_{.j} > 0$, otherwise $r_{int}^{(t)} = 0$. This is a modified version of [5] and introduced in [13].

**Modify Step-size Adaptation.** If the standard deviation of the elements corresponding to the integer variables is much smaller than the granularity of the discretization, then the step-size adaptation rapidly decreases the step-size. To address this problem, [5] proposed a modification of the step-size adaptation to remove the elements corresponding to integer variables with considerably smaller standard deviations from the evolution path $p_{c}^{(t+1)}$ when updating the step-size as follows:

$$\sigma(t+1) = \sigma(t) \exp \left( \frac{c_\sigma}{d_\sigma} \left[ \frac{\|I_{\sigma}^{(t+1)} p_{c}^{(t+1)}\|}{\mathbb{E}[\|N(0, I_{\sigma}^{(t+1)})\|]} - 1 \right] \right), \tag{9}$$

where $I_{\sigma}^{(t+1)}$ is the diagonal masking matrix, and $\langle I_{\sigma}^{(t+1)} \rangle_{\text{ij}} = 0$ if $5\sigma(C^{(t)})_{ij} < \sqrt{\lambda} < S^{int}_{ij}$; otherwise, $\langle I_{\sigma}^{(t+1)} \rangle_{\text{ij}} = 1$. The expected value $\mathbb{E}[\|N(0, I_{\sigma}^{(t+1)})\|]$ is approximated by $\sqrt{M} \left( 1 - \frac{1}{N_{co} + 1} \right)$, where $M$ is the number of non-zero diagonal elements for $I_{\sigma}^{(t+1)}$.

### 3 PRELIMINARY EXPERIMENT: WHY IS IT DIFFICULT TO OPTIMIZE BINARY VARIABLES?

It is known that the integer variable handling of CMA-ES [5] does not work well for binary variables. However, the reasons for this have not been well explored. We then empirically check why this integer handling fails to optimize binary variables.

We consider the function $\text{ENCODING}_{f}(x_i)$ to binarize the elements of the candidate solution corresponding to the binary variables, and the dimension $N = N_{co} + N_{bi}$, where $N_{bi}$ is the number of binary variables. We define $\text{ENCODING}_{f}(x_i) : \mathbb{R}^{N_{co} + N_{bi}} \mapsto \mathbb{R}^{N_{co} \times \{0, 1\}^{N_{bi}}}$ as

$$\text{ENCODING}_{f}(x_i) = \begin{cases} [x_{ij}]_{j} & (1 \leq j \leq N_{co}) \\ \mathbb{1}_{\{[x_{ij}]_{j} > 0\}} (N_{co} + 1 \leq j \leq N) \end{cases}. \tag{10}$$

The partially discretized candidate solution obtained by (10) is denoted by $\tilde{x}_i = \text{ENCODING}_{f}(x_i)$.

Compared to the integer variables, binary variables have a much wider interval, where the same binary variables can be taken after binarization. Therefore, if the variance decreases while the mean vector is so far from the threshold zero at which the binary variable changes, the optimization of the variable fails.
We check the behavior using the CMA-ES with integer variable handling introduced in Section 2.2. Additionally, we use this CMA-ES variant with a box constraint \([x_i]_j \in [-1, 1]\) corresponding to the binary variables. When the box constraint is used, the penalty \(||x^{feas}_i - x_i||^2_2 / N\) is added to the evaluation value, where \(x^{feas}_i\) is the nearest-neighbor feasible solution to \(x_i\). The number of dimensions \(N\) is set to 40, and \(N_{co} = N_{bi} = N/2 = 20\).

The initial mean vector \(m^{(0)}\) is set to uniform random values in the range \([1, 3]\) for continuous variables and 0 for the binary variables, respectively. The covariance matrix and step-size are initialized with \(C^{(0)} = I\) and \(\sigma^{(0)} = 1\), respectively. The optimization is successful when the best-evaluated value is less than \(10^{-10}\), and the optimization is stopped when the minimum eigenvalue of \(\sigma^2 C\) is less than \(10^{-30}\).

Result and Discussion. For the coordinate-wise mean \(m_i\), the coordinate-wise standard deviation \(\sigma(C)_i\), step-size \(\sigma\), and best-evaluated value, the upper and lower sides of the Figure 1 show the transitions of a single typical run of the optimization failure for the CMA-ES with the integer mutation and modification of the step-size adaptation (denoted by CMA-ES-IM) and the CMA-ES-IM with the box constraint. The CMA-ES-IM decreases the coordinate-wise standard deviations for binary variables with the step-size. In contrast, coordinate-wise mean is far from the threshold value of zero. In this case, the integer mutation provided in Step 1 to Step 5 in Section 2.2 is not effective to improve the evaluation value because in the dimension corresponding to the binary variable, \(\delta^{int}_{i} + \overline{\lambda}_{int} \) are smaller than the distances between \(m^{(t)} + \sigma^{(t)} \gamma_i\) and the threshold value of zero. Moreover, when \(\delta^{int}_{A}\) calculated in Step 6 also becomes small, the mutation no longer affects the candidate solutions at all (after 500 iterations in Figure 1).

To introduce this margin correction to the CMA-ES, we define a diagonal matrix \(A\) whose initial value is given by the identity matrix and redefine the MGD that generates the samples as \(N(m, \sigma^2 AC\top)\). The margin correction is achieved by correcting \(A\) and \(m\) so that the probability of the integer variables being generated outside the dominant values is maintained above a certain value \(\alpha\). Because the sample generated from \(N(m, \sigma^2 AC\top)\) is equivalent to applying the affine transformation of \(A\) to the sample generated from \(N(m, \sigma^2 C)\), we can separate the adaptation of the covariance and the update of \(A\). Consequently, the proposed method.

4 PROPOSED METHOD

In this section, we propose a simple modification of the CMA-ES in the MI-BBO. The basic idea is to introduce a lower bound on the marginal probability referred to as the margin, so that the sample is not fixed to a single integer variable. The margin is a common technique in the estimation of distribution algorithms (EDAs) for binary domains to address the problem of bits being fixed to 0 or 1. In fact, the population-based incremental learning (PBIL) [2], a binary variable optimization method based on Bernoulli distribution, restricts the updated marginals to the range \([0, N], 1 - N/\). This prevents the optimization from stagnating with the distribution converging to an undesirable direction before finding the optimum.

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modification can be represented as the affine transformation of the samples used to evaluate the objective function, without making any changes to the updates in CMA-ES. It should be noted that although the mean vector can also be corrected by the affine transformation, we directly correct it to avoid the divergence of $\boldsymbol{m}$.

In this section, we first redefine $\text{ENCODING}_f$ to facilitate the introduction of the margin. Next, we show the process of the CMA-ES with the proposed modification. Finally, we explain the margin correction, namely, the updates of $A$ and $\mathbf{m}$, separately for the cases of binary and integer variables.

The detailed algorithm of the proposed CMA-ES with Margin appears in the supplementary material, and the code is available at https://github.com/EvoConJP/CMA-ES_with_Margin.

4.1 Definition of $\text{ENCODING}_f$ and Threshold $\ell$

Let $z_{j,k}$ be the $k$-th smallest value among the discrete values in the $j$-th dimension, where $N_{co} + 1 \leq j \leq N$ and $1 \leq k \leq K_j$. It should be noted that $K_j$ is the number of candidate integers for the $j$-th variable $z_j$. Under this definition, the binary variable can also be represented as, e.g., $z_{j,1} = 0$, $z_{j,2} = 1$. Moreover, we introduce a threshold $\ell$ for encoding continuous variables into discrete variables.

Let $\ell_{j,k|k+1}$ be the threshold of two discrete variables $z_{j,k}$ and $z_{j,k+1}$: it is given by the midpoint of $z_{j,k}$ and $z_{j,k+1}$, namely, $\ell_{j,k|k+1} := (z_{j,k} + z_{j,k+1})/2$. We then redefine $\text{ENCODING}_f$ when $N_{co} + 1 \leq j \leq N$ as follows:

$$
\text{ENCODING}_f([\mathbf{x}_i]_j) = \begin{cases} 
    z_{j,1} & \text{if } |\mathbf{x}_i|_{j} \leq \ell_{j,1|2} \\
    z_{j,k} & \text{if } \ell_{j,k-1|k} < |\mathbf{x}_i|_{j} \leq \ell_{j,k|k+1} \\
    z_{j,K_j} & \text{if } \ell_{j,K_j-1|K_j} < |\mathbf{x}_i|_{j}
\end{cases}
$$

Moreover, if $1 \leq j \leq N_{co}$, $|\mathbf{x}_i|_j$ is isometrically mapped as $\text{ENCODING}_f([\mathbf{x}_i]_j) = |\mathbf{x}_i|_j$. Then, the discretized candidate solution is denoted by $\tilde{\mathbf{x}}_i = \text{ENCODING}_f(\mathbf{x}_i)$. The set of discrete variables $\mathbf{z}_j$ is not limited to consecutive integers such as $\{0, 1, 2\}$, but can handle general discrete variables such as $\{1, 2, 4\}$ and $\{0.01, 0.1, 1\}$.

4.2 CMA-ES with the Proposed Modification

Given $\lambda^{(t)}$ as an identity matrix $I$, the update of the proposed method, termed CMA-ES with margin, at the iteration $t$ is given in the following steps.

Step 1. The $\lambda$ candidate solutions $\mathbf{x}_i$ ($i = 1, 2, \ldots, \lambda$) are sampled from $N(\mathbf{m}^{(t)}, (\sigma^{(t)})^2 \mathbf{C}^{(t)})$ as $\mathbf{x}_i = \mathbf{m}^{(t)} + \sigma^{(t)} \mathbf{y}_i$, where $\mathbf{y}_i \sim N(0, \mathbf{C}^{(t)})$ for $i = 1, 2, \ldots, \lambda$.

Step 2. The affine transformed solutions $\mathbf{v}_i$ ($i = 1, 2, \ldots, \lambda$) are calculated as $\mathbf{v}_i = \mathbf{m}^{(t)} + \sigma^{(t)} A^{(t)} \mathbf{y}_i$ for $i = 1, 2, \ldots, \lambda$.

Step 3. The discretized $\mathbf{v}_i$, i.e., $\tilde{\mathbf{v}}_i$ ($i = 1, 2, \ldots, \lambda$) are evaluated by $f$ and sort $\{\tilde{x}_{1|\lambda}, \tilde{x}_{2|\lambda}, \ldots, \tilde{x}_{\lambda|\lambda}\}$ and $\{|\mathbf{y}_{1|\lambda}, |\mathbf{y}_{2|\lambda}, \ldots, |\mathbf{y}_{\lambda|\lambda}\}$ so that the indices correspond to $f(\tilde{\mathbf{v}}_{1|\lambda}) \leq f(\tilde{\mathbf{v}}_{2|\lambda}) \leq \ldots \leq f(\tilde{\mathbf{v}}_{\lambda|\lambda})$.

Step 4. Based on (3) to (7), update $\mathbf{m}^{(t)}$, $\mathbf{C}^{(t)}$, and $\sigma^{(t)}$ using $\mathbf{x}$ and $\mathbf{y}$.

Step 5. Modify $\mathbf{m}^{(t+1)}$ and update $A^{(t)}$ based on Section 4.3 and Section 4.4.

It should be noted that the algorithm based on the above is consistent with the original CMA-ES if no corrections are made in the stepwise model. In particular, the configuration $\mathbf{A}^{(t)}$ in Section 4.3 and Section 4.4, and the more significant the modification, the closer the above algorithm is to the original CMA-ES, and the more significant the modification, the closer the algorithm is to the original CMA-ES. Moreover, the update of the variance-covariance has not been modified, which facilitates the smooth consideration of the introduction of the CMA-ES properties, e.g., step-size adaptation methods other than CSA.

4.3 Margin for Binary Variables

Considering the probability that a binarized variable $|\tilde{\mathbf{v}}|_j$ is 0 and the probability that it is 1, the following conditions should be satisfied after the modification by the margin:

$$
\min \{ \Pr(|\tilde{\mathbf{v}}|_j = 0), \Pr(|\tilde{\mathbf{v}}|_j = 1) \} \geq \alpha
$$

$$
\Leftrightarrow \min \{ \Pr(|\mathbf{v}|_j < 0.5), \Pr(|\mathbf{v}|_j \geq 0.5) \} \geq \alpha
$$

Figure 2: Example of MGD followed by $\nu$ and its marginal probability. The dashed red ellipse corresponds to the MGD before the correction, whereas the solid one corresponds to the MGD after the correction for the binary variable.

Step 5. In other words, the smaller the margin parameter $\alpha$, described in Section 4.3 and Section 4.4, and the more significant the modification, the closer the above algorithm is to the original CMA-ES. Moreover, the update of the variance-covariance has not been modified, which facilitates the smooth consideration of the introduction of the CMA-ES properties, e.g., step-size adaptation methods other than CSA.

It should be noted that the binarize threshold $\ell_{j,1|2}$ is equal to 0.5.

Here, Figure 2 shows an example of the updated MGD followed by $\nu$ and the marginal probabilities. The MGD before the margin correction (dashed red ellipse) shows that its marginal probability $\Pr(|\mathbf{v}|_1 \geq 0.5)$ is smaller than the margin parameter $\alpha$. In this case, by modifying the element of the mean vector $\mathbf{m}^{(t+1)}$, we can correct the marginal probability $\Pr(|\mathbf{v}|_1 \geq 0.5)$ to $\alpha$ without affecting the other dimensions. To calculate the amount of the correction for this mean vector, we consider the confidence interval of the probability $1 - 2\alpha$ in the marginal distribution of the $j$-th dimension. The confidence interval of the $j$-th dimension is represented by

$$
[\mathbf{m}^{(t+1)}]_j \in [CI]^{(t+1)}_j(1 - 2\alpha), [\mathbf{m}^{(t+1)}]_j + CI^{(t+1)}_j(1 - 2\alpha)
$$


It should be noted that $\text{CI}_j^{(t+1)}(1 - 2\alpha)$ is defined as

$$
\text{CI}_j^{(t+1)}(1 - 2\alpha) := \sqrt{\frac{\chi^2_{ppf}(1 - 2\alpha)\sigma(t+1)^2}{2}} \left( A^{(t)} C^{(t+1)} A^{(t)^\top} \right)_{jj},
$$

where $\chi^2_{ppf}(\cdot)$ is the function that, given the lower cumulative probability, returns the percentage point in the chi-squared distribution with 1 degree of freedom. If the threshold $\ell_{j,1:2} = 0.5$ is outside this confidence interval, the marginal probability to be corrected is less than $\alpha$. Given the encoding threshold closest to $[m^{(t+1)}]_j$ as $\ell([m^{(t+1)}]_j)$, which is equal to 0.5 in the binary case, the modification for the $j$-th element of the mean vector can be denoted as

$$
[m^{(t+1)}]_j \leftarrow \ell([m^{(t+1)}]_j) + \text{sign}([m^{(t+1)}]_j - \ell([m^{(t+1)}]_j)) \cdot \min \left(\left| [m^{(t+1)}]_j - \ell([m^{(t+1)}]_j) \right|, \text{CI}_j^{(t+1)}(1 - 2\alpha) \right).
$$

(13)

Additionally, no changes are made to $(A^{(t)})_j$, namely,

$$
(A^{(t)})_j \leftarrow (A^{(t)})_j.
$$

(14)

As shown in the solid red line in Figure 2, the marginal probability after this modification is lower-bounded by $\alpha$.

### 4.4 Margin for Integer Variables

First, we consider the cases where the $j$-th element of the mean vector satisfies $[m^{(t+1)}]_j \leq \ell_{j,1:2}$ or $\ell_{j,K_j-1:[K_j]} < [m^{(t+1)}]_j$. In these cases, the integer variable $[v]_j$ may be fixed to $z_{j,1}$ or $z_{j,K_j}$, respectively. Thus, we correct the marginal probability of generating one inner integer variable, i.e., $z_{j,1}$ or $z_{j,K_j-1}$ for $z_{j,K_j}$, to maintain it above $\alpha$, respectively. This correction is achieved by updating $[m^{(t+1)}]_j$ and $(A^{(t)})$ based on (13) and (14).

Next, we consider the case of other integer variables. Figure 3 shows an example of the updated MGD followed by $v$ and its marginal probability. The dashed red ellipse corresponds to the MGD before the correction, whereas the solid one corresponds to the MGD after the correction for the integer variable.

![Figure 3: Example of MGD followed by $v$ and its marginal probability.](image)

Next, we restrict $p_{\text{low}}$, $p_{\text{up}}$, and $p_{\text{mid}}$ as follows.

$$
p_{\text{low}}^{\prime} \leftarrow \max \left\{ \alpha/2, p_{\text{low}} \right\}
$$

(20)

$$
p_{\text{up}}^{\prime} \leftarrow \max \left\{ \alpha/2, p_{\text{up}} \right\}
$$

(21)

$$
p_{\text{low}}^{\prime\prime} \leftarrow p_{\text{low}}^{\prime} + \frac{1 - p_{\text{low}}^{\prime} - p_{\text{up}}^{\prime} - p_{\text{mid}}}{p_{\text{low}}^{\prime} + p_{\text{up}}^{\prime} + p_{\text{mid}} - 3 \cdot \alpha/2} (p_{\text{low}}^{\prime} - \alpha/2)
$$

(22)

$$
p_{\text{up}}^{\prime\prime} \leftarrow p_{\text{up}}^{\prime} + \frac{1 - p_{\text{low}}^{\prime} - p_{\text{up}}^{\prime} - p_{\text{mid}}}{p_{\text{low}}^{\prime} + p_{\text{up}}^{\prime} + p_{\text{mid}} - 3 \cdot \alpha/2} (p_{\text{up}}^{\prime} - \alpha/2)
$$

(23)

The equations (22) and (23) ensure $p_{\text{low}}^{\prime\prime} + p_{\text{up}}^{\prime\prime} + p_{\text{mid}}^{\prime\prime} = 1$, while keeping $p_{\text{low}}^{\prime\prime} \geq \alpha/2$ and $p_{\text{up}}^{\prime\prime} \geq \alpha/2$, where $p_{\text{mid}}^{\prime\prime} = 1 - p_{\text{low}}^{\prime\prime} - p_{\text{up}}^{\prime\prime}$. This handling method is also adopted in [1, Appendix D]. We update $[m^{(t+1)}]$ and $(A^{(t)})$ so that the corrected marginal probabilities $\Pr([v]_j \leq \ell_{\text{low}}([m^{(t+1)}]_j))$ and $\Pr([v]_j > \ell_{\text{up}}([m^{(t+1)}]_j))$ are $p_{\text{low}}^{\prime\prime}$ and $p_{\text{up}}^{\prime\prime}$, respectively. The conditions to be satisfied are as follows.

$$
\begin{align*}
&\ell_{\text{low}}([m^{(t+1)}]_j) - \ell_{\text{low}}([m^{(t+1)}]_j) = C_j^{(t+1)}(1 - 2p_{\text{low}}^{\prime\prime}) \\
&\ell_{\text{up}}([m^{(t+1)}]_j) - [m^{(t+1)}]_j = C_j^{(t+1)}(1 - 2p_{\text{up}}^{\prime\prime})
\end{align*}
$$

(24)

Finally, the solutions of the simultaneous linear equations for $[m^{(t+1)}]_j$ and $(A^{(t)})_j$ are applied to the updated $[m^{(t+1)}]_j$ and $(A^{(t)})_j$. Correcting $m^{(t+1)}$ and $(A^{(t)})$ in this way bounds both $\Pr([v]_j \leq \ell_{\text{low}}([m^{(t+1)}]_j))$ and $\Pr([v]_j > \ell_{\text{up}}([m^{(t+1)}]_j))$ above $\alpha/2$, as indicated by the solid line in Figure 3. Moreover, we note that there are cases where $p_{\text{mid}}$, $\Pr(0.5 < [m^{(t+1)}]_j \leq 1.5)$ in Figure 3, is less than $\alpha/2$ even with the margin. In that case, the variance is sufficiently large that no fixation of the discrete variable occurs in the corresponding dimension.

### 5 EXPERIMENT AND RESULT

We apply the proposed method to the MI-BBO optimization problem for several benchmark functions to validate its robustness and
efficiency. In Section 5.1, we check the performance changes of the proposed method according to the hyperparameter \( \alpha \). In Section 5.2, we check the difference in the search success rate and the number of evaluations between the proposed method and CMA-ES-IM for several artificial MI-BBO benchmark functions. The definitions of the benchmark functions used in this section are listed as below:

- **SphereOneMax(\(\bar{x}\)) = \sum_{j=1}^{N} |x_j|^2 + N_{\text{bi}} - \sum_{k=K_{\text{co}}+1}^{N} |x_k|
- **SphereLeadingOnes(\(\bar{x}\)) = \sum_{j=1}^{N} |x_j|^2 + N_{\text{bi}} - \sum_{k=K_{\text{co}}+1}^{N} \prod_{i=K_{\text{co}}+1}^{k} |x_i|
- **EllipsoidOneMax(\(\bar{x}\)) = \sum_{j=1}^{N_{\text{co}}} \left( \frac{1}{N_{\text{co}}} \sum_{j=1}^{N_{\text{co}}} |x_j|^2 \right) + N_{\text{bi}} - \sum_{k=K_{\text{co}}+1}^{N} |x_k|
- **EllipsoidLeadingOnes(\(\bar{x}\)) = \sum_{j=1}^{N_{\text{co}}} \left( \frac{1}{N_{\text{co}}} \sum_{j=1}^{N_{\text{co}}} |x_j|^2 \right) + N_{\text{bi}} - \sum_{k=K_{\text{co}}+1}^{N} \prod_{i=K_{\text{co}}+1}^{k} |x_i|
- **SphereInt(\(\bar{x}\)) = \sum_{j=1}^{N_{\text{co}}} |x_j|^2
- **EllipsoidInt(\(\bar{x}\)) = \sum_{j=1}^{N_{\text{co}}} \left( \frac{1}{N_{\text{co}}} \sum_{j=1}^{N_{\text{co}}} |x_j|^2 \right)

In all functions, the first \(N_{\text{co}}\) variables are continuous, whereas the last \(N - N_{\text{co}}\) variables are binary or integer. In all the experiments, we adopted the default parameters of CMA-ES listed in Table 1.

5.1 Hyperparameter Sensitivity for \( \alpha \)

We use the SphereInt function of the objective function and adopt the same initialization for the distribution parameters and termination condition as in Section 3. The number of dimensions \( N \) is set to 20, 40, or 60, and the numbers of the continuous and integer variables are \( N_{\text{co}} = N_{\text{bi}} = N_{\text{int}} = N/2\). The integer variables are assumed to take values in the range \([-10, 10]\). We argue that it is reasonable that the hyperparameter \( \alpha \), which determines the margin in the proposed method, should depend on the number of dimensions \( N \) and the sample size \( \lambda \). In this experiment, we evaluate a total of 48 settings except for \( \alpha = 1 \) which we set as \( \alpha = N^{-m} \lambda^{-n} (m, n \in \{0, 0.5, 1, 1.5, 2, 2.5, 3\}) \). In each setting, 100 trials are performed independently using different seed values.

**Results and Discussion.** Figure 4 shows the success rate and the median evaluation count for successful cases in each setting. For the success rate, when \( \alpha \) is set to a large value as \( N^{-0.5} \lambda^{-0.5} \), all trials fail. Additionally, when \( \alpha \) is set as smaller than \((N \lambda)^{-1}\), the success rate tends to decrease as the number of dimensions \( N \) increases. If \( \alpha \) is too large, the probability of the integer changing is also too large and the optimization is unstable; however, if \( \alpha \) is too small, it is difficult to get out of the stagnation because the conditions under which the mean \([m]\) and affine matrix \(A\) corrections are applied become more stringent. For the median evaluation count, there is no significant difference for any dimension except for \( \alpha = N^{-1} (N \lambda)^{-0.5} \lambda \). Therefore, for robustness and efficiency reasons, we use \( \alpha = (N \lambda)^{-1} \) as a default parameter in the subsequent experiments in this study.

5.2 Comparison of Optimization Performance

We compare the optimization performance of the benchmark functions listed in Section 5 for the proposed method, CMA-ES-IM, and CMA-ES-IM with box constraints. As in Section 5.1, the number of dimensions \( N \) is set to 20, 40, and 60. The number of continuous and integer variables are \( N_{\text{co}} = N_{\text{bi}} = N_{\text{int}} = N/2\), respectively. For CMA-ES-IM with the box constraint, SphereOneMax, SphereLeadingOnes, EllipsoidOneMax, and EllipsoidLeadingOnes functions are given the constraint \([x_j] \in [-1, 1]\) corresponding to the binary variables, and other functions are given the constraint \([x_j] \in [-10, 10]\) corresponding to the integer variables. The calculating method of the penalty for violating the constraints is the same as in Section 3. The optimization is successful when the best-evaluated value is less than \(10^{-10}\), whereas the optimization is stopped when the minimum eigenvalue of \( \alpha^2 C \) is less than \(10^{-30}\) or the condition number of \( C \) exceeds \(10^{14}\).
Table 2: Comparison of evaluation counts and success rate for the benchmark functions. The evaluation counts are reported as the median value of successful trials. The bold fonts represent the best median evaluation counts and the best success rates among the methods. The inside of the parentheses represents the interquartile range (IQR).

| Function                    | N  | CMA-ES-IM Evaluation Counts | CMA-ES-IM Evaluation Success Rates | CMA-ES-IM & Box-constraint Evaluation Counts | CMA-ES-IM & Box-constraint Evaluation Success Rates | CMA-ES w. Margin (Proposed) Evaluation Counts | CMA-ES w. Margin (Proposed) Evaluation Success Rates |
|-----------------------------|----|-----------------------------|------------------------------------|-----------------------------------------------|---------------------------------------------------|-----------------------------------------------|--------------------------------------------------|
| SphereOneMax                | 20 | 2964 (174)                  | 83/100                             | 4752 (100)                                   | 100/100                                           | 3876 (435)                                    | 100/100                                         |
|                             | 40 | 5745 (558)                  | 62/100                             | 7575 (543)                                   | 64/100                                            | 7995 (514)                                    | 100/100                                         |
|                             | 60 | 8112 (340)                  | 46/100                             | 12240 (336)                                  | 21/100                                            | 12408 (1012)                                  | 100/100                                         |
| SphereLeadingOnes           | 20 | 2904 (192)                  | 77/100                             | 5124 (296)                                   | 100/100                                           | 4158 (339)                                    | 100/100                                         |
|                             | 40 | 5647 (296)                  | 24/100                             | 8280 (112)                                   | 10/100                                            | 8505 (724)                                    | 100/100                                         |
|                             | 60 | 8816 (12)                   | 8/100                              | 12624 (96)                                   | 2/100                                             | 13424 (1008)                                  | 100/100                                         |
| EllipsoidOneMax             | 20 | 9918 (396)                  | 20/100                             | 26700 (7020)                                 | 96/100                                            | 11172 (666)                                   | 100/100                                         |
|                             | 40 | 35325 (0)                   | 1/100                              | 79912 (28661)                                | 14/100                                            | 40590 (1789)                                  | 100/100                                         |
|                             | 60 | 78560 (0)                   | 5/100                              | 283440 (0)                                   | 1/100                                             | 88064 (3536)                                  | 100/100                                         |
| EllipsoidLeadingOnes        | 20 | 10104 (441)                 | 14/100                             | 24880 (7305)                                 | 92/100                                            | 11454 (876)                                   | 100/100                                         |
|                             | 40 | -                           | 0/100                              | 85807 (13335)                                | 4/100                                             | 41048 (1744)                                  | 100/100                                         |
|                             | 60 | -                           | 0/100                              | -                                            | 0/100                                             | 91496 (3488)                                  | 100/100                                         |
| SphereInt                   | 20 | 5130 (477)                  | 86/100                             | 5280 (744)                                   | 89/100                                            | 3840 (306)                                    | 100/100                                         |
|                             | 40 | 7950 (697)                  | 71/100                             | 8070 (555)                                   | 85/100                                            | 7838 (458)                                    | 100/100                                         |
|                             | 60 | 13184 (816)                 | 41/100                             | 12992 (544)                                  | 34/100                                            | 11512 (544)                                   | 100/100                                         |
| EllipsoidInt                | 20 | 19128 (3192)                | 76/100                             | 19476 (5028)                                 | 73/100                                            | 8418 (837)                                    | 100/100                                         |
|                             | 40 | 43935 (4095)                | 73/100                             | 43440 (4762)                                 | 83/100                                            | 22815 (1733)                                  | 100/100                                         |
|                             | 60 | 89200 (6152)                | 54/100                             | 86848 (6136)                                 | 54/100                                            | 42000 (3320)                                  | 100/100                                         |

Results and Discussion. Table 2 summarizes the median evaluation counts and success rates in each setting. Comparing the proposed method with the CMA-ES-IM with and without the box constraint for the SphereOneMax and SphereLeadingOnes functions, the CMA-ES-IM reaches the optimal solution in fewer evaluation counts. However, the success rate of CMA-ES-IM with and without the box constraint decreases as the number of dimensions increases. However, the success rate of the proposed method remains 100% regardless of the increase in the number of dimensions. For the EllipsoidOneMax and EllipsoidLeadingOnes functions, the CMA-ES-IM without the box constraint fails on most of the trials in all dimensions, and the CMA-ES-IM with box constraint has a relatively high success rate in $N = 20$ but deteriorates rapidly in $N = 40$ or more dimensions. In contrast, the proposed method maintains a 100% success rate and reaches the optimal solution in fewer evaluation counts than the CMA-ES-IM with the box constraint in $N = 20, 40$. For the SphereInt and EllipsoidInt functions, the proposed method successfully optimizes with fewer evaluations in all dimensions than the other methods, maintaining a 100% success rate. These results show that the proposed method can perform the MI-BBO robustly and efficiently for multiple functions with an increasing number of dimensions.

6 CONCLUSION

In this work, we first experimentally confirmed that the existing integer handling method, CMA-ES-IM [5] with or without the box constraint, does not work effectively for binary variables, and then proposed a new integer variable handling method for CMA-ES. In the proposed method, the mean vector and the diagonal affine transformation matrix for the covariance matrix are corrected so that the marginal probability for an integer variable is lower-bounded at a certain level, which is why the proposed method is called the CMA-ES with margin; it considers both the binary and integer variables.

The proposed method has a hyperparameter, $\alpha$, that determines the degree of the lower bound for the marginal probability. We investigated the change in the optimization performance on the SphereInt function with multiple $\alpha$ settings in order to determine the default parameter. With the recommended value of $\alpha$, we experimented the proposed method on several MI-BBO benchmark problems. The experimental results demonstrated that the proposed method is robust even when the number of dimensions increases and can find the optimal solution with fewer evaluations than the existing method, CMA-ES-IM with or without the box constraint.

There are still many challenges left for the MI-BBO; for example, Tušar et al. [15] pointed out the difficulty of optimization for non-separable ill-conditioned convex-quadratic functions, such as the rotated Ellipsoid function. In future, we need to address these issues, which have not yet been addressed by the proposed or existing methods, by considering multiple dimension correlations. Additionally, evaluating the proposed method on real-world MI-BBO problems is also an important future direction.
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A ALGORITHM DETAILS OF THE CMA-ES WITH MARGIN

First, we show the updated $\mathbf{m}^{(t+1)}$ and $(A^{(t+1)})_{j}$ in the margin for the integer variables. Solving the simultaneous linear equations in (24) for $\mathbf{m}^{(t+1)}$ and $(A^{(t)})_{j}$, we obtain $\mathbf{m}^{(t+1)}$ as

$$
\ell_{\text{low}} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) - \ell_{\text{up}} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) \over \sqrt{2 \gamma_{\text{ppf}} \left( 1 - 2 p_{\text{up}}^{(t)} \right)} + \sqrt{2 \gamma_{\text{ppf}} \left( 1 - 2 p_{\text{low}}^{(t)} \right)}
$$

and $(A^{(t)})_{j}$ as

$$
\frac{\ell_{\text{up}} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) - \ell_{\text{low}} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right)}{\sigma^{(t)} \sqrt{\left( C^{(t+1)} \right) \left( \sqrt{2 \gamma_{\text{ppf}} \left( 1 - 2 p_{\text{up}}^{(t)} \right)} + \sqrt{2 \gamma_{\text{ppf}} \left( 1 - 2 p_{\text{low}}^{(t)} \right)} \right)}}
$$

These solutions are applied as updated $\mathbf{m}^{(t+1)}$ and $(A^{(t+1)})_{j}$, respectively.

Finally, the single update in the CMA-ES with Margin is shown in Algorithm 1. Note that here we consider a minimization problem $\min_{\mathbf{x}} f(\mathbf{x})$, where $[\mathbf{x}]_{j}$ ($j = 1, \ldots, N_{\text{co}}$) are continuous variables, $[\mathbf{x}]_{j}$ ($j = N_{\text{co}} + 1, \ldots, N_{\text{co}} + N_{\text{bi}}$) are binary variables, and $[\mathbf{x}]_{j}$ ($j = N_{\text{co}} + N_{\text{bi}} + 1, \ldots, N$) are integer variables.

\begin{algorithm}
\caption{Single update in CMA-ES with Margin for optimization problem $\min_{\mathbf{x}} f(\mathbf{x})$}
\begin{algorithmic}[1]
\STATE given $\mathbf{m}^{(t)} \in \mathbb{R}^{N}$, $\sigma^{(t)} \in \mathbb{R}^{+}$, $C^{(t)} \in \mathbb{R}^{N \times N}$, $\mathbf{p}_{\sigma}^{(t)} \in \mathbb{R}^{N}$, $\mathbf{p}_{c}^{(t)} \in \mathbb{R}^{N}$, and $A^{(t)} \in \mathbb{R}^{N \times N}$ (diagonal matrix)
\FOR{$i = 1, \ldots, \lambda$}
\STATE $y_i \sim N(0, C^{(t)})$
\STATE $x_i \leftarrow \mathbf{m}^{(t)} + \sigma^{(t)} y_i$
\STATE $v_i \leftarrow \mathbf{m}^{(t)} + \sigma^{(t)} A^{(t)} y_i$
\STATE $\tilde{y}_i \leftarrow \text{ENCODING}(v_i)$
\ENDFOR
\STATE Sort $\{x_{1,2}, x_{2,3}, \ldots, x_{N,1}\}$ and $\{y_{1,2}, y_{2,3}, \ldots, y_{N,1}\}$ so that the indices correspond to $f(\tilde{y}_{1,2}) \leq f(\tilde{y}_{2,3}) \leq \ldots \leq f(\tilde{y}_{N,1})$
\STATE $\mathbf{m}^{(t+1)} \leftarrow \mathbf{m}^{(t)} + \mathbf{c}_{\mu} \sum_{i=1}^{\mu} w_i (x_i - \mathbf{m}^{(t)})$
\STATE $\mathbf{p}_{\sigma}^{(t+1)} \leftarrow (1 - c_{\sigma}) \mathbf{p}_{\sigma}^{(t)} + \sqrt{c_{\sigma} (2 - c_{\sigma}) p_{\mu} C^{(t+1)} - \frac{1}{\sigma^{(t)}} \sum_{i=1}^{\mu} w_i y_i}$
\STATE $h_{\sigma} \leftarrow \mathbb{E} \left[ ||\mathbf{p}_{\sigma}^{(t+1)}|| < \sqrt{1 - \left( 1 - c_{\sigma} \right) 2^{(t+1)} \left( 1.4 + \frac{2}{\lambda^{(t+1)}} \right) E[||N(0, I)||]} \right]$
\STATE $\mathbf{p}_{c}^{(t+1)} \leftarrow (1 - c_{c}) \mathbf{p}_{c}^{(t+1)} + h_{\sigma} c_{c} \mathbf{c}_{1} \mathbf{c}_{1}^{\top} + c_{c} \sum_{i=1}^{\lambda} w_{i} y_{i,2}^{\top} y_{i,2}$
\STATE $C^{(t+1)} \leftarrow C^{(t)} + \mathbf{c}_{1} \mathbf{p}_{c}^{(t+1)} \mathbf{p}_{c}^{(t+1)}^{\top} + c_{c} \sum_{i=1}^{\lambda} w_{i} y_{i,2}^{\top} y_{i,2}$
\STATE $\sigma^{(t+1)} \leftarrow \sigma^{(t)} \exp \left( \frac{c_{c}}{c_{\sigma}} \left( \mathbb{E}[||\mathbf{p}_{\sigma}^{(t+1)}||] - 1 \right) \right)$
\STATE // Margin for Binary Variables
\FOR{$j = N_{\text{co}} + 1, \ldots, N_{\text{co}} + N_{\text{bi}}$}
\STATE $\left[ \mathbf{m}^{(t+1)} \right]_{j} \leftarrow \ell \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) + \sigma \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) \text{sign} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} - \ell \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) \right) \min \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} - \ell \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right), C_{1}^{(t+1)} (1 - 2 \alpha) \right)$
\STATE $(A^{(t+1)})_{j} \leftarrow (A^{(t)})_{j}$
\ENDFOR
\STATE // Margin for Integer Variables
\FOR{$j = N_{\text{co}} + N_{\text{bi}} + 1, \ldots, N$}
\IF{$\left[ \mathbf{m}^{(t+1)} \right]_{j} \leq f_{j,1} \text{ or } f_{j,K-1} < \left[ \mathbf{m}^{(t+1)} \right]_{j}$}
\STATE $\left[ \mathbf{m}^{(t+1)} \right]_{j} \leftarrow \ell \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) + \sigma \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) \text{sign} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} - \ell \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) \right) \min \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} - \ell \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right), C_{1}^{(t+1)} (1 - 2 \alpha) \right)$
\STATE $(A^{(t+1)})_{j} \leftarrow (A^{(t)})_{j}$
\ELSE
\STATE $\left[ \mathbf{m}^{(t+1)} \right]_{j} \leftarrow \ell_{\text{low}} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) - \ell_{\text{up}} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) \Theta \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) \left[ \sqrt{2 \gamma_{\text{ppf}} \left( 1 - 2 p_{\text{up}}^{(t)} \right)} + \sqrt{2 \gamma_{\text{ppf}} \left( 1 - 2 p_{\text{low}}^{(t)} \right)} \right)$
\STATE $(A^{(t+1)})_{j} \leftarrow \frac{\ell_{\text{up}} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right) - \ell_{\text{low}} \left( \left[ \mathbf{m}^{(t+1)} \right]_{j} \right)}{\sigma^{(t)} \sqrt{\left( C^{(t+1)} \right) \left( \sqrt{2 \gamma_{\text{ppf}} \left( 1 - 2 p_{\text{up}}^{(t)} \right)} + \sqrt{2 \gamma_{\text{ppf}} \left( 1 - 2 p_{\text{low}}^{(t)} \right)} \right)}}$
\ENDIF
\ENDFOR
\end{algorithm}
