Separation Method of Semi-Fixed Variables Together with Dynamical System Method for Solving Nonlinear Time-Fractional PDEs with Higher-Order Terms

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Research Article

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Separation method of semi-fixed variables together with dynamical system method for solving nonlinear time-fractional PDEs with higher-order terms *

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Abstract

It is well known that methods for solving fractional-order PDEs are grossly inadequate compared with integer-order PDEs. In this paper, a new approach which combined with the separation method of semi-fixed variables and dynamical system method is introduced. As example, a time-fractional reaction-diffusion equation with higher-order terms is studied under the different kinds of fractional-order differential operators. In different parametric regions, phase portraits of systems which derived from the reaction-diffusion equation are presented. Existence and dynamic properties of solutions of this nonlinear time-fractional models are investigated. In some special parametric conditions, some exact solutions of this time-fractional models are obtained. The dynamical properties of some exact solutions are discussed and the graphs of them are illustrated.

Keywords: Separation method of semi-fixed variables; Dynamical system method; Nonlinear time-fractional PDEs; Time-fractional reaction-diffusion equations; Existence and dynamical property of solution.

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1 Introduction

It is well known that the concept of fractional-order derivative was first appeared in a famous correspondence between L’Hôpital and Leibniz in 1695. Since then, an effective theory of fractional-order calculus was gradually established with the joint efforts of many mathematicians during the past three hundred years, so far, there are more than dozen definitions on fractional-order derivative. Although the concepts of both fractional-order calculus and integer-order calculus originated in the Leibniz’s era, their developments were uneven. Obviously, the theory of fractional-order calculus develops very slowly and difficultly compared with the theory of integer-order calculus. One reason which restricts development of the theory of fractional-order calculus is that it has been lacked application background for a long time in the past. Another reason is that the theory of fractional-order calculus has been lacked effective methods and analytical tools on investigating solutions of fractional-order differential equations because many classical (effective) methods in the integer-order field can not be directly applied to fractional-order field. Since 1960s, the two issues have been greatly improved and developed. On the one hand, more and more researchers have found that the fractional-order differential models can be more accurately described complex problems in various fields such as mathematical mechanics, control theory, signal processing, aerodynamics, chemistry, biology and so forth. Especially, fractional-order calculus has been recognized as one of the best tools to describe long-memory processes, viscoelastic phenomena and behavior of anomalous diffusion in the past decades. Fractional-order differential models are interesting for science and engineering but also for pure mathematics. In the study of pure mathematics, in order to make up for the deficiency of fractional-order differential models, researchers sometimes directly transform the classical integer-order differential models into the corresponding fractional-order differential models to study them. On the other hand, there appeared many effective methods for solving nonlinear fractional-order differential equations in recent decades. The main methods include Adomian decomposition method [1,2], homotopy analysis method [3,4], invariant analysis method [5,6], fractional variational iteration method [7-9], invariant subspace method [10-12], method of fractional complex transformation [13-15] and the method of separating variables [16-18], etc.

Although some exact and approximate analytical solutions of fractional-order differential equations can be obtained by above methods, it is still far from enough to solve more com-
plex nonlinear fractional-order partial differential equations (PDEs). What’s more, we find that the method of fractional complex transformation appeared in Refs. [13-15] is based on a wrong fractional chain rule which given by Jumarie in Refs. [19-21]. In fact, Jumarie’s fractional chain rule has been verified that it is invalid in Refs. [22-24]. This means people need to redesign some new methods to specifically target more complex fractional-order PDEs such as nonlinear time-fractional PDEs. Recently, based on the separation method of variables and combined with other methods such as homogenous balanced principle, idea of invariant subspace, integral bifurcation method and dynamical system method, we introduced several new methods [24-28] for solving nonlinear time-fractional PDEs. Obviously, in these several methods, unlike the traditional separation method of variables is that we set the part of time-function as some specific special functions such as Mittag-Leffler function or power function. We call this improved separation method of variables a separation method of semi-fixed variables. Especially, in Ref. [28], used this kind of improved separation method of variables and combined with bifurcation theory of differential dynamical system (dynamical system method) [29-32], we investigated existence and dynamical property of solutions of a nonlinear time-fractional PDE. But can this method solve nonlinear time-fractional PDEs with high-order terms? In this paper, we will introduce applications on the separation method of semi-fixed variables combining with dynamical system method. As example, by using this combination method, we will investigate exact solutions, existence and dynamical properties of solution of time-fractional reaction-diffusion model with higher-order terms formed as

\[
\delta \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 [u^m]}{\partial x^2} + \kappa u - \sigma u^m, \quad (1.1)
\]

where \(0 < \alpha < 1\), \(u = u(x,t),\ t > 0,\ x \in \mathbb{R},\ m \in N^+\), the parameters \(\delta, \sigma\) are two nonzero constants and \(\kappa\) is an arbitrary constant, the sign \(\frac{\partial^{\alpha}}{\partial t^{\alpha}}\) defines a fractional-order differential operator of Caputo type \(C_0^\alpha D_t^\alpha\) or Riemann-Liouville type \(RL_0^\alpha D_t^\alpha\). A more general form of the equation (1.1) can be written as

\[
\delta \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 [F(u)]}{\partial x^2} + K(u), \quad (1.2)
\]

where \(F(u)\) and \(K(u)\) are two arbitrary and smooth functions of \(u\), the sign \(\frac{\partial^{\alpha}}{\partial t^{\alpha}}\) is fractional-order differential operator of Riemann-Liouville type or Caputo type. When \(\alpha \to 1\), the equation (1.2) becomes the following integer-order reaction-diffusion equation of Fisher-KPP
type
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 [F(u)]}{\partial x^2} + K(u) \]  
(1.3)
as model of biological population which given by Malaguti and Marcelli [33]. Especially, when \( F(u) = u^m \) and \( K(u) = \lambda u^n(1 - u) \), the equation (1.3) becomes a reaction-diffusion model with higher-order terms as follows:
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 [u^m]}{\partial x^2} + \lambda u^n(1 - u), \]  
(1.4)
which studied by Pablo and Vazquez [34]. In the conditions \( m > 1, \lambda = 1, n = 1 \), Aronson investigated sharp-type solutions of the equation (1.4) in [35].

In this paper, we will investigate exact solutions, existence and dynamical property of solutions of the equation (1.1) from mathematical point of view. Because there are distinct differences between the Riemann-Liouville operator and Caputo operator, the solutions of the equation (1.1) are completely different, and the difficulty for searching exact solutions is also different. In fact, under the two definitions of differential operator (Riemann-Liouville type and Caputo type), the fractional-order derivatives of the constant, Mittag-Leffler function and power function are different, the details which can be seen below.

\[ RL_t^\alpha c = \frac{c}{\Gamma(1 - \alpha)} t^{-\alpha}, \quad C_0^\alpha t^\alpha c = 0, \]
\[ RL_t^\alpha \theta_c = \frac{\Gamma(1 + y)}{\Gamma(1 + y - \alpha)} t^{y - \alpha}, \quad (\gamma > -1), \quad C_0^\gamma t^\gamma c = \frac{\Gamma(1 + \gamma)}{\Gamma(1 + \gamma - \alpha)} t^{\gamma - \alpha}, \quad (\gamma > 0), \]
where \( \gamma, \alpha \) are constants and \( 0 < \alpha < 1 \).

The organization of this paper is as follows: In Section 2, under the definition of fractional-order differential operator of Caputo type, we will investigate exact solutions, existence and dynamical properties of solutions of (1.1). In Section 3, under the definition of fractional-order differential operator of Riemann-Liouville type, we will investigate exact solutions, existence and dynamical properties of solutions of (1.1) in two different cases.
2 Exact solutions and dynamical properties of (1.1) under the definition of Caputo operator

Under the definition of Caputo operator, the equation (1.1) can be rewritten as

\[
\delta \left(C_0^\alpha D_t^\alpha u\right) = \frac{\partial^2 [u^m]}{\partial x^2} + \kappa u - \sigma u^m,
\]

(2.1)

where \(C_0^\alpha D_t^\alpha\) defines a fractional-order differential operator of Caputo type, parameters \(\delta\), \(\sigma\), \(\kappa\) are nonzero constants. According to the classical separation method of variables, the form of solutions of the equation (2.1) needs to be supposed as follows:

\[
u(x, t) = v(x)T(t),\]

(2.2)

where \(v(x)\) and \(T(t)\) are both undetermined functions. Here, if we directly fix the time function \(T(t)\) to a Mittag-Leffler function, then (2.2) becomes

\[
u(x, t) = v(x)E_\alpha(\lambda t^\alpha),\]

(2.3)

where the function \(v = v(x)\) is undetermined function, the \(\lambda\) is undetermined coefficient, both of them can be determined later. Thus, this improved method will become simpler and the amount of computation is greatly reduced during the process for solving the model (2.1). Obviously, this is appropriate since solutions of fractional differential equations often contain Mittag-Leffler function. We call this improved method the separation method of semi-fixed variables.

Substituting (2.3) into (2.1), it yields

\[
\lambda \delta v E_\alpha(\lambda t^\alpha) = \frac{d^2 v^m}{dx^2} + \kappa v E_\alpha(\lambda t^\alpha) - \sigma v^m [E_\alpha(\lambda t^\alpha)]^m.
\]

(2.4)

The equation (2.4) can be rewritten as

\[
(\lambda \delta - \kappa) v E_\alpha(\lambda t^\alpha) = \left[ m(m-1)v^{m-2} \left( \frac{dv}{dx} \right)^2 + mv^{m-1} \frac{d^2 v}{dx^2} - \sigma v^m \right] [E_\alpha(\lambda t^\alpha)]^m.
\]

(2.5)

Letting each coefficient of Mittag-Leffler functions in (2.5) to be zero, we obtain

\[
\lambda \delta - \kappa = 0
\]

(2.6)

and

\[
m(m-1)v^{m-2} \left( \frac{dv}{dx} \right)^2 + mv^{m-1} \frac{d^2 v}{dx^2} - \sigma v^m = 0.
\]

(2.7)
Solving the equation (2.6), it yields
\[ \lambda = \frac{\kappa}{\delta}. \] (2.8)

Letting \( \frac{dv}{dx} = y \), the equation of (2.7) can be reduced to the following system
\[
\begin{cases}
\frac{dv}{dx} = y, \\
\frac{dy}{dx} = \sigma v^m - m(m-1)v^{m-2}y^2.
\end{cases}
\] (2.9)

Obviously, the planar system (2.9) is a regular system when \( m = 1 \), but it is singular system when \( m \geq 2 \). In other words, the derivative \( \frac{dy}{dx} \) is not defined at \( v = 0 \) when \( m \geq 2 \). However, \( v = 0 \) is a trivial solution of the equation (2.7) when \( m \geq 2 \). In other words, the system (2.9) is not equivalent to the equation (2.7) at \( v = 0 \) when \( m \geq 2 \). In order to obtain a completely equivalent system of the equation (2.7) no matter how the function \( v \) vary when \( m \geq 2 \), we make a scalar transformation as follows:
\[ dx = mv^{m-1}d\tau, \] (2.10)

where \( \tau \) is a parameter. Under the transformation (2.10), the singular system (2.9) can be reduced to a regular system as follows:
\[
\begin{cases}
\frac{dv}{d\tau} = mv^{m-1}y, \\
\frac{dy}{d\tau} = \sigma v^m - m(m-1)v^{m-2}y^2.
\end{cases}
\] (2.11)

Obviously, both systems (2.9) and (2.11) have same first integral as follows:
\[ y^2 = \frac{\sigma}{m^2} v^2 + hv^{2-2m}, \] (2.12)

where \( h \) is an integral constant. In order to facilitate discussion, the equation (2.12) can be rewritten as
\[ H(v, y) \equiv v^{2m-2}y^2 - \frac{\sigma}{m^2} v^{2m} = h. \] (2.13)

From (2.11) and (2.13), we easily verify that \( \frac{\partial H}{\partial y} \neq -\frac{dv}{d\tau} \), \( \frac{\partial H}{\partial v} \neq \frac{dy}{d\tau} \). Therefore, the system (2.11) is not a Hamiltonian system. For convenience on discussion, we write Jacobian matrix and Jacobian determinant of (2.11) as follows:
\[
M(v, y) = \begin{bmatrix}
m(m-1)v^{m-2}y & mv^{m-1} \\
\sigma mv^{m-1} - m(m-1)(m-2)v^{m-3}y^2 & -2m(m-1)v^{m-2}y
\end{bmatrix}, \] (2.14)
\[ J(v, y) = (-m^4 + m^3)v^{2m-4}y^2 - \sigma m^2 v^{2m-2}. \]  

(2.15)

Obviously, the system (2.11) has only one equilibrium points \( O(0, 0) \) at the \( v \)-axis. Substituting the equilibrium points \( O(0, 0) \) into (2.13) and (2.15), it yields

\[
 h_O = H(0, 0) = 0, \quad J_O = J(0, 0) = \begin{cases} -\sigma, & m = 1, \\ 0, & m \geq 2. \end{cases}
\]  

(2.16)

According to theory of planar dynamical systems [29-32] and the Lemma 1 in [28], we know that the equilibrium point \( O(0, 0) \) is center point if \( \sigma < 0 \) and it is saddle point if \( \sigma > 0 \) when \( m = 1 \). It is easy to find that Trace\( M(0, 0) \equiv 0 \) and \( J_O \equiv 0 \) when \( m \geq 2 \). But, the Poincaré index of the equilibrium point \( O(0, 0) \) is not zero in this case. Thus, we know that the equilibrium point \( O(0, 0) \) is not a center point (or saddle point) when \( m \geq 2 \); it is degenerative center point if \( \sigma < 0 \) and it is saddle point if \( \sigma > 0 \) when \( m \geq 2 \).

Obviously, the phase portraits of the system (2.9) can be constructed by using the phase portraits of the system (2.11) together the singular line \( v = 0 \). According to above information, we draw graphs of phase portraits for the system (2.9) in four cases, which can be shown in the Fig.1.

![Phase portraits](image)

(a) The case of \( m = 1, \sigma < 0 \)  
(b) The case of \( m = 1, \sigma > 0 \)
By means of phase portraits in Fig.1, we obtain some information about distribution of orbits for the system (2.9) as follows:

When \( m = 1, \sigma < 0 \) and \( h > 0 \), system (2.11) has infinite numbers of closed orbits, which can be shown in Fig.1a. When \( m = 1, \sigma > 0 \) and \( h = 0 \), system (2.11) has two orbits of line type marked by black colour, which can be shown in Fig.1b. When \( m = 1, \sigma > 0 \) and \( h < 0 \), system (2.11) has infinite numbers of hyperbolic orbits (two pairs of them marked by green colour), which can be shown in Fig.1b. When \( m = 1, \sigma > 0 \) and \( h > 0 \), system (2.11) has infinite numbers of hyperbolic orbits (four pairs of them marked by red colour), which can be shown in Fig.1b. Especially, when \( m \geq 2 \), the equilibrium point \( O(0,0) \) occurs degradation phenomenon due to the influence of the singular straight line, the orbits are divided into left and right parts by the singular line \( v = 0 \), and the original closed orbits are no longer closed. According to above information about distribution of orbits, we get two theorems in the below.

**Theorem 2.1** When \( m = 1 \), the following conclusions hold.

(i) Equation (2.7) has a family of smooth periodic solutions if \( \sigma < 0 \) and \( h > 0 \). Accordingly, equation (2.1) has a smooth solution with periodic property.

(ii) Equation (2.7) has two smooth solution of exponential function type if \( \sigma > 0 \) and \( h = 0 \). Accordingly, equation (2.1) has two unbounded solutions.

(iii) Equation (2.7) has two solutions of hyperbolic cosine function if \( \sigma > 0 \) and \( h < 0 \), it has two solutions of hyperbolic sine function if \( \sigma > 0 \) and \( h > 0 \). Accordingly, equation (2.1)
has three solutions of hyperbolic function type.

**Theorem 2.2** When \( m \geq 2 \), the following conclusions hold.

(i) Equation (2.7) has two family of non-smooth periodic solutions if \( \sigma < 0 \) and \( h > 0 \). Accordingly, equation (2.1) has two family of non-smooth solutions with periodic property.

(ii) Equation (2.7) has two unbounded solutions of exponential function type if \( \sigma > 0 \) and \( h = 0 \). Accordingly, equation (2.1) has two unbounded solutions.

(iii) Equation (2.7) has two solutions of hyperbolic cosine function if \( \sigma > 0 \) and \( h < 0 \), it has two solutions of hyperbolic sine function if \( \sigma > 0 \) and \( h > 0 \). Accordingly, equation (2.1) has three solutions of hyperbolic function type.

Different kinds of solutions mentioned above theorems satisfy \( u \to 0 \) in the condition of \( \delta \kappa < 0 \) when \( t \to +\infty \). On the contrary, they satisfy \( u \to \pm \infty \) in the condition of \( \delta \kappa > 0 \) when \( t \to +\infty \).

**Proof 2.1** (i) When \( m = 1 \), \( \sigma < 0 \) and \( h > 0 \), the equation (2.12) can be rewritten as

\[
y = \pm \sqrt{h} \sqrt{1 - \left( \frac{\sigma}{h} v \right)^2}.
\] (2.17)

Obviously, each closed orbit in Fig.1a which is defined by (2.17) has two intersection points \((\pm \sqrt{-\frac{h}{\sigma}}, 0)\) with the \( v \)-axis. Taking \( (\sqrt{-\frac{h}{\sigma}}, 0) \) or \( (-\sqrt{-\frac{h}{\sigma}}, 0) \) as initial value point, then substituting (2.17) into the first equation \( \frac{dv}{dx} = y \) of (2.9) and integrating it, we get

\[
\int_{v}^{\sqrt{-\frac{h}{\sigma}}} \frac{dv}{\sqrt{1 - (\sqrt{-\frac{\sigma}{h}} v)^2}} = \pm \int_{0}^{x} \sqrt{h} dx.
\] (2.18)

Solving (2.18), we obtain a smooth periodic solution of equation (2.7) as follows:

\[
v = \sqrt{-\frac{h}{\sigma}} \cos(\sqrt{-\sigma}x).
\] (2.19)

Plugging (2.19) and (2.8) into (2.3), we obtain an exact solutions of the equation (2.1) as follows:

\[
u = \sqrt{-\frac{h}{\sigma}} \cos(\sqrt{-\sigma}x) \, E_{\alpha} \left( \frac{\kappa}{\delta} \right).
\] (2.20)

(ii) When \( m = 1 \) and \( \sigma > 0 \), \( h = 0 \), the equation (2.12) can be reduced to

\[
y = \pm \sqrt{\sigma} v.
\] (2.21)
Substituting (2.21) into the first equation $\frac{dv}{dx} = y$ of (2.9) to directly integrate it, we get two general solutions of exponential function type as follows:

$$v = ce^{\sqrt{\sigma}x}$$  \hspace{1cm} (2.22)

and

$$v = ce^{-\sqrt{\sigma}x},$$  \hspace{1cm} (2.23)

where $c$ is arbitrary constant. Respectively substituting (2.22), (2.23) and (2.8) into (2.3), we obtain two unbounded solutions of equation (2.1) as follows:

$$u = ce^{\sqrt{\sigma}x}E_\alpha\left(\frac{\kappa}{\delta}t^\alpha\right),$$  \hspace{1cm} (2.24)

$$u = ce^{-\sqrt{\sigma}x}E_\alpha\left(\frac{\kappa}{\delta}t^\alpha\right).$$  \hspace{1cm} (2.25)

(iii) When $m = 1$ and $\sigma > 0$, $h < 0$, the equation (2.12) can be reduced to

$$y = \pm \sqrt{\sigma}\sqrt{v^2 - \left(\sqrt{-\frac{h}{\sigma}}\right)^2}. \hspace{1cm} (2.26)$$

Respectively taking $\left(\sqrt{-\frac{h}{\sigma}}, 0\right)$ and $\left(-\sqrt{-\frac{h}{\sigma}}, 0\right)$ as initial points, then substituting (2.26) into the first equation $\frac{dv}{dx} = y$ of (2.9) and integrating them, we get two solutions of hyperbolic cosine function as follows:

$$v = \pm \sqrt{-\frac{h}{\sigma}}\cosh \left(\sqrt{\sigma}x\right). \hspace{1cm} (2.27)$$

Substituting (2.27) and (2.8) into (2.3), we obtain two unbounded solutions of equation (2.1) as follows:

$$u = \pm \sqrt{-\frac{h}{\sigma}}\cosh \left(\sqrt{\sigma}x\right)E_\alpha\left(\frac{\kappa}{\delta}t^\alpha\right). \hspace{1cm} (2.28)$$

When $m = 1$ and $\sigma > 0$, $h > 0$, the equation (2.12) can be reduced to

$$y = \pm \sqrt{h}\sqrt{1 + \left(\sqrt{\frac{\sigma}{h}}v\right)^2}. \hspace{1cm} (2.29)$$

Substituting (2.29) into the equation $\frac{dv}{dx} = y$ to directly integrate it, we get two general solutions as follows:

$$v = \frac{\sqrt{\sigma h}}{2\sigma c} \left[\sigma e^{-\sqrt{\sigma}x} - c^2 e^{\sqrt{\sigma}x}\right]$$ \hspace{1cm} (2.30)

and

$$v = \frac{\sqrt{\sigma h}}{2\sigma c} \left[\sigma e^{\sqrt{\sigma}x} - c^2 e^{-\sqrt{\sigma}x}\right], \hspace{1cm} (2.31)$$
where \( c \) is arbitrary constant. Especially, when \( c = \sqrt{\sigma} \), the solutions (2.30) and (2.31) become the following two types of hyperbolic sine function.

\[
v = \sqrt{\sigma} \sinh(\sqrt{\sigma}x), \quad (2.32)
\]
\[
v = -\sqrt{\sigma} \sinh(\sqrt{\sigma}x). \quad (2.33)
\]

Respectively substituting (2.32), (2.33) and (2.8) into (2.3), we obtain two unbounded solutions of equation (2.1) as follows:

\[
u = \sqrt{h} \sinh(\sqrt{\sigma}x) E_{\alpha} \left( \frac{\kappa}{\delta} t^\alpha \right), \quad (2.34)
\]
\[
u = -\sqrt{h} \sinh(\sqrt{\sigma}x) E_{\alpha} \left( \frac{\kappa}{\delta} t^\alpha \right). \quad (2.35)
\]

**Proof 2.2**

(i) When \( m \geq 2 \), \( \sigma < 0 \) and \( h > 0 \), the equation (2.12) can be rewritten as

\[
y = \pm v \sqrt{\frac{\sigma}{m^2} + hv^{-2m}}. \quad (2.36)
\]

Substituting (2.36) into the first equation \( \frac{dv}{dx} = y \) of (2.9) to integrate it, we obtain two periodic solutions of equation (2.7) as follows:

\[
v = \left[ -\frac{\sigma}{m^2 h} \sec^2(\sqrt{-\sigma}x) \right]^{-\frac{1}{2m}} \quad (2.37)
\]

and

\[
v = -\left[ -\frac{\sigma}{m^2 h} \sec^2(\sqrt{-\sigma}x) \right]^{-\frac{1}{2m}}. \quad (2.38)
\]

Respectively plugging (2.37), (2.38) and (2.8) into (2.3), we obtain two exact solutions of the equation (2.1) as follows:

\[
u = \left[ -\frac{\sigma}{m^2 h} \sec^2(\sqrt{-\sigma}x) \right]^{-\frac{1}{2m}} E_{\alpha} \left( \frac{\kappa}{\delta} t^\alpha \right) \quad (2.39)
\]

and

\[
u = -\left[ -\frac{\sigma}{m^2 h} \sec^2(\sqrt{-\sigma}x) \right]^{-\frac{1}{2m}} E_{\alpha} \left( \frac{\kappa}{\delta} t^\alpha \right). \quad (2.40)
\]

(ii) When \( m \geq 2 \), \( \sigma > 0 \) and \( h = 0 \), the equation (2.12) can be rewritten as

\[
y = \pm \frac{\sqrt{\sigma}}{m} v. \quad (2.41)
\]

Substituting (2.41) into the first equation \( \frac{dv}{dx} = y \) of (2.9) to directly integrate it, we get two general solutions of exponential function type as follows:

\[
v = ce^{\frac{\sqrt{\sigma}}{m} x} \quad (2.42)
\]
\[ v = ce^{-\frac{\sqrt{\sigma} x}{m}}, \quad (2.43) \]

where \( c \) is arbitrary constant. Respectively substituting (2.22), (2.23) and (2.8) into (2.3), we obtain two unbounded solutions of equation (2.1) as follows:

\[ u = ce^{\frac{\sqrt{\sigma} x}{m}} E_\alpha \left( \frac{\kappa}{\delta} t^\alpha \right), \quad (2.44) \]
\[ u = ce^{-\frac{\sqrt{\sigma} x}{m}} E_\alpha \left( \frac{\kappa}{\delta} t^\alpha \right). \quad (2.45) \]

(iii) When \( m \geq 2, \sigma > 0 \) and \( h < 0 \), the equation (2.12) can be rewritten as

\[ y = \pm \sqrt{-h} v \sqrt{\frac{\sigma}{m^2(-h)}} - v^{-2m} \cdot (2.46) \]

Respectively taking \((-\frac{m^2 h}{\sigma})^{\frac{1}{2m}}, 0\) and \((-\frac{m^2 h}{\sigma})^{\frac{1}{2m}}, 0\) as initial points, then substituting (2.46) into the first equation \( \frac{dv}{dx} = y \) of (2.9) and integrating them, we get two solutions of hyperbolic cosine function as follows:

\[ v = \pm \left[ -\frac{m^2 h}{\sigma} \cosh^2 \left( \sqrt{\sigma x} \right) \right]^{\frac{1}{2m}}. \quad (2.47) \]

Substituting (2.47) and (2.8) into (2.3), we obtain two unbounded solutions of equation (2.1) as follows:

\[ u = \pm \left[ -\frac{m^2 h}{\sigma} \cosh^2 \left( \sqrt{\sigma x} \right) \right]^{\frac{1}{2m}} E_\alpha \left( \frac{\kappa}{\delta} t^\alpha \right). \quad (2.48) \]

When \( m \geq 2, \sigma > 0 \) and \( h > 0 \), the equation (2.12) can be rewritten as

\[ y = \pm \sqrt{h} v \sqrt{\frac{\sigma}{m^2 h} + v^{-2m}}. \quad (2.49) \]

Substituting (2.49) into the first equation \( \frac{dv}{dx} = y \) of (2.9) and directly integrating it, we get two solutions of hyperbolic sine function as follows:

\[ v = \pm \left[ \frac{m^2 h}{\sigma} \sinh^2 \left( \sqrt{\sigma x} \right) \right]^{\frac{1}{2m}}. \quad (2.50) \]

Substituting (2.50) and (2.8) into (2.3), we obtain two unbounded solutions of equation (2.1) as follows:

\[ u = \pm \left[ \frac{m^2 h}{\sigma} \sinh^2 \left( \sqrt{\sigma x} \right) \right]^{\frac{1}{2m}} E_\alpha \left( \frac{\kappa}{\delta} t^\alpha \right). \quad (2.51) \]

According to property of Mittag-Leffler function \( E_\alpha \left( \frac{\xi t^\alpha}{\delta} \right) \), it is easy to know that all above solutions satisfy \( u \to 0 \) in the condition of \( \delta \kappa < 0 \) when \( t \to +\infty \) and \( u \to \pm \infty \) in the
condition of \( \delta \kappa > 0 \) when \( t \to +\infty \). In order to show dynamical property of above solutions intuitively, as examples, we plot 3D-graphs of dynamical profiles of the solutions (2.20) and (2.39) respectively, which are shown in Fig. 2 and Fig.3.

\[
\begin{align*}
\text{(c) The case of } & \kappa \delta < 0 \\
\text{(d) The case of } & \kappa \delta > 0 \\
\end{align*}
\]

**Fig. 2** The 3D-graphs of profiles of the solution (2.20); parameters taken as \( m = 1, \ h = 4, \ \delta = 3, \ \kappa = \mp 1, \ \sigma = -1.5, \ \alpha = 0.5, \ t \in [0, 30], \ x \in [-7, 8]. \)

\[
\begin{align*}
\text{(c) The case of } & \kappa \delta < 0 \\
\text{(d) The case of } & \kappa \delta > 0 \\
\end{align*}
\]

**Fig. 3** The 3D-graphs of profiles of the solution (2.39); parameters taken as \( m = 4, \ h = 4, \ \delta = 3, \ \kappa = \mp 1, \ \sigma = -1.5, \ \alpha = 0.75, \ t \in [0, 10], \ x \in [-4, 4]. \)

The solutions (2.20) and (2.39) and their profiles show that both of them have periodic property, but the former is smooth and the latter is non-smooth. This is due to the system (2.9) increases a singular line \( v = 0 \) when \( m \geq 2 \).
3 Exact solutions and dynamical properties of (1.1) under the definition of Riemann-Liouville operator

In this section, under the definition of Riemann-Liouville operator, we will investigate exact solutions, existence and dynamical properties of solutions of the equation (1.1). Under the definition of Riemann-Liouville operator, the equation (1.1) becomes

\[ \delta \left( \frac{\text{RL}_0 D_t^\alpha}{\partial x^2} u \right) = \frac{\partial^2 [u^m]}{\partial x^2} + \kappa u - \sigma u^m, \]  

(3.1)

where \( \frac{\text{RL}_0 D_t^\alpha}{\partial x^2} \) defines fractional-order differential operator of Riemann-Liouville type, parameters \( \delta, \sigma \) are two nonzero constants, \( \kappa \) is an arbitrary constant.

It is well known that the Riemann-Liouville operator has singularity compared with the Caputo operator, which can be seen by the derivatives of the constant and Mittag-Leffler function given in the previous section. In fact, we can’t obtain exact solutions of the equation (3.1) in the case of \( m \geq 2 \) and \( \kappa \neq 0 \). Therefore, we will discuss the solutions of the equation (3.1) in two cases of \( m \geq 2, \kappa = 0 \) and \( m = 1, \kappa \neq 0 \).

3.1 The case of \( m \geq 2, \kappa = 0 \)

When \( m \geq 2 \) and \( \kappa = 0 \), the equation (3.1) becomes

\[ \delta \left( \frac{\text{RL}_0 D_t^\alpha}{\partial x^2} u \right) = \frac{\partial^2 [u^m]}{\partial x^2} - \sigma u^m. \]  

(3.2)

We suppose that equation (3.2) has solutions formed as follows:

\[ u = v(x)t^\gamma, \]  

(3.3)

where the function \( v = v(x) \) is undetermined function and \( \gamma \) is undetermined constant.

Substituting (3.3) into (3.2), we get

\[ \delta \frac{\Gamma(1 + \gamma)}{\Gamma(1 + \gamma - \alpha)} t^{\gamma - \alpha} v = \left[ m(m - 1)v^{m-2} \left( \frac{dv}{dx} \right)^2 + mv^{m-1} \frac{d^2 v}{dx^2} - \sigma v^m \right] t^{m\gamma}. \]  

(3.4)

According to homogenous balanced principle, in the equation (3.4) we let

\[ \gamma - \alpha = m\gamma. \]  

(3.5)

Solving (3.5), it yields

\[ \gamma = -\frac{\alpha}{m - 1} > -1. \]  

(3.6)
Write $\frac{\Gamma(1 - \frac{\alpha}{m-1})}{\Gamma(1 - \frac{\alpha}{m})} = \Omega_0$. Substituting (3.6) into (3.4) and then diving out the term $t^{-\frac{\alpha}{m-1}}$ on both sides of the equation (3.4), we get

$$\delta \Omega_0 v = m(m - 1)v^{m-2}\left(\frac{dv}{dx}\right)^2 + mv^{m-1}\frac{d^2v}{dx^2} - \sigma v^n.$$  

(3.7)

Letting $\frac{dv}{dx} = y$, the equation (3.7) can be reduced to the following singular system

$$\begin{cases}
\frac{dv}{dx} = y, \\
\frac{dy}{dx} = \frac{\delta \Omega_0 v + \sigma v^m - m(m - 1)v^{m-2}y^2}{mv^{m-1}}.
\end{cases}$$  

(3.8)

Also, the derivative $\frac{dy}{dx}$ can not be defined at $v = 0$. Similarly, we make a scalar transformation as follows:

$$dx = mv^{m-1}d\tau,$$  

(3.9)

where $\tau$ is a parameter. Under the transformation (3.9), the system (3.8) can be reduced to

$$\begin{cases}
\frac{dv}{d\tau} = mv^{m-1}y, \\
\frac{dy}{d\tau} = \delta \Omega_0 v + \sigma v^m - m(m - 1)v^{m-2}y^2.
\end{cases}$$  

(3.10)

Obviously, systems (3.8) and (3.10) have same first integral as follows:

$$y^2 = \frac{2\delta \Omega_0}{m(m + 1)}v^{3-m} + \frac{\sigma}{m^2}v^2 + hv^{2m},$$  

(3.11)

where $h$ is an integral constant. As in the section 2, we rewrite (3.11) as

$$H(v, y) \equiv \frac{-2\delta \Omega_0}{m(m + 1)}v^{m+1} - \frac{\sigma^2}{m}v^{2m} - 2m(m - 1)v^{m-2}y^2 = h.$$  

(3.12)

Write $P(v, y) = mv^{m-1}y$, $Q(v, y) = \delta \Omega_0 v + v^m\sigma - m(m - 1)v^{m-2}y^2$. Similarly, the Jacobian matrix and Jacobian determinant of system (3.10) are written as follows:

$$M(v, y) = \begin{bmatrix}
m(m - 1)v^{m-2}y & mv^{m-1} \\
\delta \Omega_0 + \sigma mv^{m-1} - m(m - 1)(m - 2)v^{m-3}y^2 & -2m(m - 1)v^{m-2}y
\end{bmatrix},$$  

(3.13)

$$J(v, y) \equiv \det M(v, y) = (-m^4 + m^3)v^{2(m-2)}y^2 - \delta \Omega_0 mv^{m-1} - \sigma m^2 v^{2(m-1)}.$$  

(3.14)

From (3.10) and (3.12), we easily verify $-\frac{\partial H}{\partial y} \neq \frac{dv}{d\tau}$, $\frac{\partial H}{\partial v} \neq \frac{dy}{d\tau}$. So, the system (3.10) is not a Hamiltonian system yet. In addition, we know that the system (3.10) has two equilibrium points $O(0, 0)$ and $A\left(\left(-\frac{\delta \Omega_0}{\sigma}\right)^{\frac{1}{m-1}}, 0\right)$ if $m$ is even number. The system (3.10) has three
equilibrium points $O(0,0)$ and $B_{1,2} \left( \pm \left( -\frac{\delta \Omega_0}{\sigma} \right)^{\frac{1}{m-1}}, 0 \right)$ if $m$ is odd number and $\frac{\delta \Omega_0}{\sigma} < 0$. The system (3.10) has only one equilibrium point $O(0,0)$ if $m$ is odd number and $\frac{\delta \Omega_0}{\sigma} > 0$.

As in the section 2, substituting the values of above equilibrium points into (3.12) and (3.14), we obtain

\begin{align}
h_O &= 0, \\
h_A = h_B &= -\frac{2\delta \Omega_0}{m(m+1)} \left( -\frac{\delta \Omega_0}{\sigma} \right)^{\frac{m+1}{m-1}} - \frac{\sigma}{m^2} \left( -\frac{\delta \Omega_0}{\sigma} \right)^{\frac{2m}{m-1}}, \\
J_O &= 0, \\
J_A = J_B &= \frac{(1-m)m\delta^2 \Omega_0^2}{\sigma}.
\end{align}

According to theory of planar dynamical systems [29-32] and the Lemma 1 in [28], we easily obtain some information about equilibrium points of the system (3.10) as follows:

The original point $O$ is a degenerative equilibrium point because $J_0 = 0$ and its Poincaré index does not equal zero; it is a degenerative saddle point if $\sigma > 0$ and it is a degenerative center point if $\sigma < 0$. When $m$ is even number and $\sigma < 0$, the equilibrium point $A$ is a center point; it is on the left side of the $y$-axle if $\delta \Omega_0 < 0$, it is on the right side of the $y$-axle if $\delta \Omega_0 > 0$. When $m$ is even number and $\sigma > 0$, the equilibrium point $A$ is a saddle point; it is on the left side of the $y$-axle if $\delta \Omega_0 > 0$, it is on the right side of the $y$-axle if $\delta \Omega_0 < 0$. When $m$ is odd number and $\sigma < 0$, $\delta \Omega_0 > 0$, the equilibrium points $B_{1,2}$ are two center points. When $m$ is odd number and $\sigma > 0$, $\delta \Omega_0 < 0$, the equilibrium points $B_{1,2}$ are two saddle points.

Obviously, the phase portraits of the system (3.8) can be constructed by the phase portraits of the system (3.10) together with the singular line $v = 0$. According to above information, we draw graphs of the phase portraits for the system (3.8), which can be shown in the Fig.4 and Fig.5.

(a) The case of $\sigma < 0$, $\delta \Omega_0 < 0$ (b) The case of $\sigma < 0$, $\delta \Omega_0 > 0$
Fig. 4 The graphs of phase portraits of the system (3.8) when $m$ is even number.

Fig. 5 The graphs of phase portraits of the system (3.8) when $m$ is odd number.

According to above information about distribution of orbits, we get two theorems as
follows:

**Theorem 3.1** When $m$ is even number and $m \geq 2$, $\kappa = 0$, the following conclusions hold.

(i) If $h_A < h \leq 0$ and $\sigma < 0$, then the system (3.8) has infinitely many closed orbits, thus the equation (3.7) has a family of smooth periodic solutions. Accordingly, the equation (3.2) has a family of gradually stable solutions with periodic property which satisfy $u \to 0$ when time $t \to +\infty$.

(ii) If $h > 0$ and $\sigma < 0$, then the system (3.8) has infinitely many open orbits, thus the equation (3.7) has a family of bounded solutions. Accordingly, the equation (3.2) has a family of gradually stable solutions with bounded property which satisfy $u \to 0$ when time $t \to +\infty$.

(iii) If $h = 0$ and $\sigma > 0$, then the system (3.8) has an open orbit which pass through the original point $O$, thus the equation (3.7) has a solution of hyperbolic function type. Accordingly, the equation (3.8) has an attenuated solutions which satisfy $u \to 0$ when time $t \to +\infty$.

(iv) If $h = h_A$ and $\sigma > 0$, then the system (3.8) has two heteroclinic orbits which pass through the equilibrium point $A$, thus the equation (3.7) has two (semi-kink and anti-semi-kink wave) solutions. Accordingly, the equation (3.10) has two gradually stable solutions with semi-kink and anti-semi-kink type which satisfy $u \to 0$ when time $t \to +\infty$.

**Theorem 3.2** When $m$ is odd number and $m \geq 3$, $\kappa = 0$, the following conclusions hold.

(i) If $h = 0$ and $\sigma < 0$, $\delta \Omega_0 > 0$, then the system (3.8) has two closed orbits of semilune (a large elliptical orbit is separated by the singular line $v = 0$ into two parts of the left and right), thus the equation (3.7) has two periodic solution. Accordingly, the equation (3.2) has two gradually stable solutions with periodic property which satisfy $u \to 0$ when time $t \to +\infty$.

(ii) If $h_B < h < 0$ and $\sigma < 0$, $\delta \Omega > 0$, then the system (3.8) has two family of closed orbits, thus the equation (3.7) has two family of smooth periodic solutions. Accordingly, the equation (3.2) has two family of gradually stable solutions with periodic property which satisfy $u \to 0$ when time $t \to +\infty$.

(iii) If $h = h_B$ and $\sigma > 0$, $\delta \Omega_0 < 0$, then the system (3.8) has four open orbits which pass through two equilibrium points $B_{1,2}$ respectively, thus the equation (3.7) has four (semi-kink and anti-semi-kink wave) solutions of hyperbolic function type. Accordingly, the equation
(3.8) has four gradually stable solutions which satisfy $u \to 0$ when time $t \to +\infty$.

(iv) If $h > 0$ and $\sigma < 0$, $\delta \Omega_0 < 0$, then the system (3.8) has infinitely many pairs of open orbits on both sides of the singular line $v = 0$, thus the equation (3.7) has a family of non-smooth periodic solutions. Accordingly, the equation (3.10) has a family of gradually stable solutions with periodic property which satisfy $u \to 0$ when time $t \to +\infty$.

(v) If $h = 0$ and $\sigma > 0$, $\delta \Omega_0 < 0$, then the system (3.8) has two open orbits, thus the equation (3.7) has two unbounded solutions. Accordingly, the equation (3.10) has two attenuated solutions with unbounded property which satisfy $u \to 0$ when time $t \to +\infty$.

(vi) If $h = 0$ and $\sigma > 0$, $\delta \Omega_0 > 0$, then the system (3.8) has two open orbits marked by black colour in Fig.5d, thus the equation has two smooth solutions of hyperbolic sine function, they are unbounded solutions. If $h > 0$ (or $h < 0$) and $\sigma > 0$, $\delta \Omega_0 > 0$, then the system (3.10) has three family of open orbits around the equilibrium point $O$, thus the equation (3.7) has three family of unbounded solutions. Accordingly, the equation (3.10) has three family of attenuated solutions with unbounded property which satisfy $u \to 0$ when time $t \to +\infty$.

We can only obtain exact expressions of these solutions mentioned in above theorems in some particular parameter conditions because the order number $m$ of functions is too high to obtain exact expressions of such solutions by integrating. Therefore, we only prove some of the propositions in the theorems 3.1 and 3.2.

**Proof 3.1** (i). If $m$ is even number and $h_A < h \leq 0$, $\sigma < 0$, then the equation (3.11) can be reduced to

$$y = \pm v \sqrt{\frac{\sigma}{m^2} + \frac{2\delta \Omega_0}{m(m + 1)} v^{1-m} + h v^{-2m}}. \quad (3.18)$$

Theoretically, this kinds of periodic solutions can be obtained by substituting (3.18) into (3.8), but the order number $m$ is too high to obtain exact expressions of solutions by integrating, indeed. However, in the case of $h = 0$ or $m = 2$, we can easily obtain an exact expression for this kind of periodic solution, see the next discussion.

If $m$ is even number and $h = 0$, $\sigma < 0$, then the equation (3.18) can be reduced to

$$y = \pm \sqrt{\frac{-\sigma}{m}} v \sqrt{-1 - \frac{2m\delta \Omega_0}{\sigma(m + 1)} v^{1-m}}. \quad (3.19)$$

Obviously, the $(v_0, 0)$ is an intersection point of the closed orbit and the $v$-axis, where $v_0 = \left( -\frac{\sigma(m+1)}{2m\delta \Omega_0} \right)^{1/m}$. Taking $(v_0, 0)$ as initial point, substituting (3.19) into the first equation
of (3.8) and then integrating it, we get
\[ \int_{v_0}^{v} \frac{dv}{\sqrt{-1 - \frac{2m\delta\Omega_0}{\sigma(m+1)}v^{1-m}}} = \pm \sqrt{-\sigma} \int_{x_0}^{x} dx. \quad (3.20) \]

Solving (3.20), we obtain a smooth periodic solution of (3.7) as follows:
\[ v = \left[ -\frac{2m\delta\Omega_0}{\sigma(m+1)} \cos^2 \left( \frac{(m-1)\sqrt{-\sigma}}{2m} x \right) \right]^{\frac{1}{m-1}}. \quad (3.21) \]

Substituting (3.21) and \( \gamma = -\frac{\alpha}{m-1} \) into (3.3), we obtain an exact solution of the equation (3.2) as follows
\[ u = \left[ -\frac{2m\delta\Omega_0}{\sigma(m+1)} \cos^2 \left( \frac{(m-1)\sqrt{-\sigma}}{2m} x \right) \right]^{\frac{1}{m-1}} t^{-\frac{\alpha}{m-1}}, \quad (3.22) \]
where \( \Omega_0 = \frac{\Gamma(1-\frac{\alpha}{m-1})}{\Gamma(1-\frac{m-1}{m-1}-\alpha)} \) and \( \alpha \neq \frac{m-1}{m} \). This implies that the equation (3.2) has not the type of solution formed as (3.3) when \( \alpha = \frac{m-1}{m} \); maybe it has other type of solution when \( \alpha = \frac{m-1}{m} \), but we can’t obtain its expression by current method at here.

If \( m = 2 \) and \( h_A < h < 0, \sigma < 0 \), then the equation (3.18) can be reduced to
\[ y = \pm \frac{\sqrt{-\sigma} \sqrt{\frac{4}{\sigma} h - \frac{4\delta\Omega_0}{3\sigma} v^3 - v^4}}{2v}. \quad (3.23) \]

From Fig.4a and Fig.4b, we can see that all closed orbits have two intersection points with the \( v \)-axis. We suppose the two intersection points expressed by \( Q_1(\phi_1, 0) \) and \( Q_2(\phi_2, 0) \) and write \( \phi_1 > \phi_2 \). In fact, \( v = \phi_1, \phi_2 \) are two real roots of the following equation
\[ -\frac{4}{\sigma} h - \frac{4\delta\Omega_0}{3\sigma} v^3 - v^4 = 0 \quad (3.24) \]
There is no doubt the equation (3.24) has other two conjugate complex roots, we write as \( v = s, \bar{s} \), which can be solved out by computer. For example, giving the values of parameters \( h = -1, \alpha = 0.75, \delta = 2, \sigma = -1 \), by using computer to solve the equation (3.24), we can obtain \( \phi_1 = -1.471190097, \phi_2 = -2.458039854, s = 0.6009274126 + 0.8631374920i, \bar{s} = 0.6009274126 - 0.8631374920i. \) Under the above assumption, the equation(3.23) can be reduced to the following form
\[ y = \pm \frac{\sqrt{-\sigma}}{2} \frac{\sqrt{(\phi_1 - v)(v - \phi_2)(v - s)(v - \bar{s})}}{v}, \quad (3.25) \]
Taking \( (\phi_2, 0) \) as initial conditions, substituting (3.25) into the first equation of (3.10) and then integrating, it yields
\[ \int_{\phi_2}^{v} \frac{dv}{\sqrt{(\phi_1 - v)(v - \phi_2)(v - s)(v - \bar{s})}} = \pm \sqrt{-\sigma} \int_{0}^{\tau} d\tau. \quad (3.26) \]
Solving (3.26), we obtain a common expression of these periodic solutions as follows:

$$v = \frac{A\phi_2 + B\phi_1}{A + B} \left[ 1 + \frac{P_1\text{cn}(\omega\tau, \hat{k})}{1 + P_2\text{cn}(\omega\tau, \hat{k})} \right], \quad (3.27)$$

where \(\text{cn}(\omega\tau, \hat{k})\) is Jacobian elliptic function, \(\tau\) is a parameter and

\[A = \sqrt{(\phi_2^2 - \frac{s + s'}{2})^2 - \frac{(s-s')^2}{4}}, \quad B = \sqrt{(\phi_2^2 - \frac{s^2}{2})^2 - \frac{(s-s')^2}{4}}.\]

Substituting (3.27) into (3.9) and then integrating it, we get

\[x = \frac{B\phi_1 + A\phi_2}{B + A} \frac{2}{P_2\omega} \left[ P_1\omega\tau + \frac{P_2 - P_1}{1 - P_2^2} \left( \Pi \left( \text{am}(\omega\tau, \hat{k}), \frac{P_2^2}{P_2^2 - 1}, \hat{k} \right) - P_2 f_1 \right) \right], \quad (3.28)\]

where \(\Pi \left( \text{am}(\omega\tau, \hat{k}), \frac{P_2^2}{P_2^2 - 1}, \hat{k} \right)\) is a normal elliptic integral function of the third kind,

\[f_1 = \sqrt{\frac{1 - P_2^2}{k^2 + k'^2 + k'^2}} \arctan \left[ \sqrt{\frac{k^2 + k'^2 + k'^2}{1 - P_2^2}} \text{sd}(\omega\tau, \hat{k}) \right], \quad \hat{k} = \sqrt{1 - k^2}.\]

Substituting (3.27),(3.28) and \(\gamma = -\alpha\) into (3.3), we obtain an exact periodic solution of the equation (3.2) as follows:

\[\begin{align*}
    u &= \left[ \frac{A\phi_2 + B\phi_1}{A + B} \left( \frac{1 + P_1\text{cn}(\sigma\tau, k)}{1 + P_2\text{cn}(\sigma\tau, k)} \right) \right] t^{-\alpha}, \\
    x &= \frac{B\phi_1 + A\phi_2}{B + A} \frac{2}{P_2\omega} \left[ P_1\sigma\tau + \frac{P_2 - P_1}{1 - P_2^2} \left( \Pi \left( \text{am}(\sigma\tau, \frac{P_2^2}{P_2^2 - 1}, k\right) - P_2 f_1 \right) \right].
\end{align*}\]

(3.29)

In order to show dynamical property of above solutions intuitively, as example, we plot 3D-graphs of dynamical profiles of the solution (3.22), which are shown in Fig. 6a and Fig.6b.

(c) The case of \(\alpha = 0.25, \ \delta\Omega_0 < 0\) \hspace{1cm} (d) The case of \(\alpha = 0.75, \ \delta\Omega_0 > 0\)

Fig. 6 The 3D-graphs of solution (3.22); \(m = 2, \ \delta = 2, \ \sigma = -1, \ t \in [0,6], \ x \in [-15,18].\)

Compared with the solution (2.20), the solution (3.22) always satisfies \(u \to 0\) when \(t \to +\infty\), but the convergence rate of the solution (3.22) is obviously not as fast as the
solution (2.20) during the process of \( t \to +\infty \). This implies that the properties of the solutions of the equation (1.1) are completely different under the definitions of two different fractional differential operators.

Here, we omit the proof of the proposition (ii) because the method and the result are very similar to the former.

(iii). If \( m \) is even number and \( h = 0, \sigma > 0 \), also the equation (3.11) can be reduced to
\[
y = \pm v \sqrt{\frac{2\delta \Omega_0}{m(m+1)}v^{1-m} + \frac{\sigma}{m^2}}. \tag{3.30}
\]
By using the same method, we obtain a smooth solution of hyperbolic function type for (3.7) as follows:
\[
v = \left[ -\frac{2m\delta \Omega_0}{\sigma(m+1)}\cosh^2 \left( \frac{(m-1)\sqrt{\sigma}}{2m}x \right) \right]^{\frac{1}{m-1}}. \tag{3.31}
\]
Substituting (3.31) and \( \gamma = \frac{-\alpha}{m-1} \) into (3.3), we obtain an exact solution of the equation (3.2) as follows
\[
u = \left[ -\frac{2m\delta \Omega_0}{\sigma(m+1)}\cosh^2 \left( \frac{(m-1)\sqrt{\sigma}}{2m}x \right) \right]^{\frac{1}{m-1}} t^{-\frac{\alpha}{m-1}}, \tag{3.32}
\]
where \( \Omega_0 = \frac{\Gamma(1-\frac{\alpha}{m}+\frac{1}{2})}{\Gamma(1-\frac{\alpha}{m-1}+\frac{1}{2})} \) and \( \alpha \neq \frac{m-1}{m} \).

(iv) If \( m = 2, h = h_B = \frac{\delta \Omega_0}{12\sigma^2} \) and \( \sigma > 0 \), then the equation (3.11) can be reduced to
\[
y = \pm \sqrt{\frac{1}{12\sigma^3}} \sqrt{3\sigma^4v^4 + 4\delta \Omega_0 \sigma^2v^3 + \delta^4 \Omega_0^2}. \tag{3.33}
\]
We rewrite (3.33) as follows:
\[
y = \pm \sqrt{\frac{1}{12\sigma^3}} \left( a_1 + b_1 v \right) \sqrt{a + bv + cv^2}, \tag{3.34}
\]
where \( a_1 = \Omega_0 \delta, b_1 = \sigma, a = \Omega_0^2 \delta^2, b = -2\Omega_0 \delta \sigma, c = 3\sigma^2 \). Substituting (3.34) into the first equation of (3.8) and then integrating it, we get
\[
\int \frac{vdv}{(a_1 + b_1 v)\sqrt{cv^2 + bv + a}} = \pm \sqrt{\frac{1}{12\sigma^3}} \int dx. \tag{3.35}
\]
Completely integrating (3.35) and setting the integral constant as zero, we obtain two implicit solutions of (3.7) as follows:
\[
\sqrt{2} \ln \left( \sqrt{M} + \sqrt{3\sigma v - \frac{\sqrt{3}}{3} \Omega_0 \delta} \right) - \ln \left( \frac{12\sigma^2 \Omega_0^2 \delta^2 - \Omega_0 \delta^2 \sigma^2 (8N + 2\sqrt{6M})}{N} \right) = \pm \frac{\sqrt{\sigma}}{2} x, \tag{3.36}
\]
where \( M = 3\sigma^2v^2 - 2\Omega_0\delta v + \Omega_0^2\delta^2 \), \( N = \Omega_0\delta + \sigma v \). Substituting \( \gamma = -\alpha \) into (3.3) and combining with (3.36), we get two exact solutions of (3.2) as follows:

\[
\begin{aligned}
  u &= vt^{-\alpha}, \\
  x &= \pm \sqrt{\frac{2}{\sigma}} \left[ \sqrt{2} \ln \left( \sqrt{M} + \sqrt{3\sigma v} - \frac{\sqrt{2}}{3} \Omega_0\delta \right) - \ln \left( \frac{12\sigma^2\Omega_0^2\delta^2 - \Omega_0\delta\sigma^2(8N+2\sqrt{6M})}{\Omega_0\delta + \sigma v} \right) \right].
\end{aligned}
\]  

(3.37)

The (3.37) defines two attenuated solutions which satisfy \( u \to 0 \) as time \( t \to +\infty \). In order to show dynamical property of the solution (3.37) intuitively, under parameters \( \sigma = 1 \), \( \delta = 2 \), \( \alpha = 0.25 \), \( t \in [0.1, 8] \), \( v \in [-0.2, 1] \), we plot dynamical profiles of the solution (3.37) in space \((x, t, u)\), which are shown in Fig. 7a and Fig 7b.

(a) The case where \( x \) takes "+" sign

(b) The case where \( x \) takes "−" sign

Fig. 7 The 3D-graphs of solution (3.37); \( \sigma = 1 \), \( \delta = 2 \), \( \alpha = 0.25 \), \( t \in [0.1, 8] \), \( v \in [-0.2, 1] \).

**Proof 3.2** (i) If \( m \) is odd number and \( h = 0 \), \( \sigma < 0 \), \( \delta\Omega_0 > 0 \), then the equation (3.11) can be reduced to

\[
y = \pm \sqrt{ \frac{2\delta\Omega_0}{m(m+1)} } v \sqrt{ \frac{\sigma(m+1)}{2m\delta\Omega_0} + v^{1-m} }. \]

(3.38)

Obviously, the \((v_{1,2}, 0)\) is two intersection points of the two closed orbits and the \( v \)-axis, where

\[
v_{1,2} = \pm \left( -\frac{\sigma(m+1)}{2m\delta\Omega_0} \right)^{\frac{1-m}{m}}. \]

Respectively taking \((v_1, 0)\) and \((v_2, 0)\) as initial points, substituting (3.38) into the first equation of (3.8) and then integrating it, we get

\[
\int_{v_{1,2}}^{v} \frac{dv}{v \sqrt{ \frac{\sigma(m+1)}{2m\delta\Omega_0} + v^{1-m} }} = \pm \sqrt{ \frac{2\delta\Omega_0}{m(m+1)} } \int_{0}^{x} dx.
\]

(3.39)

Solving (3.39), we obtain two periodic solutions of (3.7) as follows:

\[
v = \pm \left[ -\frac{2m\delta\Omega_0}{\sigma(m+1)} \cos^2 \left( \frac{(m-1)\sqrt{-\sigma}}{2m} x \right) \right]^{\frac{1}{m-1}}.
\]

(3.40)
Substituting (3.40) and $\gamma = \frac{-a}{m-1}$ into (3.3), we obtain two exact solutions of the equation (3.2) as follows
\[ u = \pm \left[ -\frac{2m\delta\Omega_0}{\sigma(m+1)} \cos^2 \left( \frac{(m-1)\sqrt{-\sigma}}{2m} x \right) \right]^{\frac{1}{m-1}} t^{-\frac{\alpha}{m-1}}, \tag{3.41} \]
where $\Omega_0 = \frac{\Gamma(1-\alpha_{m-1})}{\Gamma(1-\alpha_{m-1}-\alpha)}$ and $\alpha \neq \frac{m-1}{m}$.

(v) If $m$ is odd number and $h = 0$, $\sigma > 0$, $\delta\Omega_0 < 0$, then the equation (3.11) can be reduced to
\[ y = \pm \sqrt{-\frac{2\delta\Omega_0}{m(m+1)}} \sqrt{-\frac{\sigma(m+1)}{2m\delta\Omega_0} - v^{1-m}}. \tag{3.42} \]
Obviously, there is not any intersection point of the two open orbits and the $v$-axis. Directly substituting (3.42) into the first equation of (3.8), then integrating it, it yields
\[ \int \frac{dv}{\sqrt{-\frac{\sigma(m+1)}{2m\delta\Omega_0} - v^{1-m}}} = \pm \sqrt{-\frac{2\delta\Omega_0}{m(m+1)}} \int dx. \tag{3.43} \]
Solving (3.43) and setting the integral constant is zero, we obtain two solutions of hyperbolic function type for (3.7) as follows:
\[ v = \pm \left[ -\frac{2m\delta\Omega_0}{\sigma(m+1)} \cosh^2 \left( \frac{(m-1)\sqrt{\sigma}}{2m} x \right) \right]^{\frac{1}{m-1}}. \tag{3.44} \]
Substituting (3.44) and $\gamma = \frac{-a}{m-1}$ into (3.3), we obtain two exact solutions of the equation (3.2) as follows
\[ u = \pm \left[ -\frac{2m\delta\Omega_0}{\sigma(m+1)} \cosh^2 \left( \frac{(m-1)\sqrt{\sigma}}{2m} x \right) \right]^{\frac{1}{m-1}} t^{-\frac{\alpha}{m-1}}, \tag{3.45} \]
where $\Omega_0 = \frac{\Gamma(1-\alpha_{m-1})}{\Gamma(1-\alpha_{m-1}-\alpha)}$ and $\alpha \neq \frac{m-1}{m}$.

(v) If $m$ is odd number and $h = 0$, $\sigma > 0$, $\delta\Omega_0 > 0$, then the equation (3.11) can be reduced to
\[ y = \pm \frac{\sqrt{\sigma}}{m} \sqrt{1 + \frac{2\delta\Omega_0}{\sigma(m+1)} v^{1-m}}. \tag{3.46} \]
Obviously, there is not any intersection point of the two open orbits and the $v$-axis. Directly substituting (3.42) into the first equation of (3.8), then integrating it, we get
\[ \int \frac{dv}{\sqrt{1 + \frac{2\delta\Omega_0}{\sigma(m+1)} v^{1-m}}} = \pm \frac{\sqrt{\sigma}}{m} \int dx. \tag{3.47} \]
Solving (3.46) and setting the integral constant is zero, we obtain two solutions of hyperbolic function type for (3.7) as follows:

\[ v = \pm \left[ \frac{2m\delta\Omega_0}{\sigma(m+1)} \sinh^2 \left( \frac{(m-1)\sqrt{\sigma}}{2m} x \right) \right]^{\frac{1}{m-1}}. \] (3.48)

Substituting (3.47) and \( \gamma = -\frac{\alpha}{m-1} \) into (3.3), we obtain two exact solutions of the equation (3.2) as follows

\[ u = \pm \left[ \frac{2m\delta\Omega_0}{\sigma(m+1)} \sinh^2 \left( \frac{(m-1)\sqrt{\sigma}}{2m} x \right) \right]^{\frac{1}{m-1}} t^{-\frac{\alpha}{m-1}}, \] (3.49)

where \( \Omega_0 = \frac{\Gamma(1-\frac{\alpha}{m-1})}{\Gamma(1-\frac{\alpha}{m-1}-\alpha)} \) and \( \alpha \neq \frac{m-1}{m} \).

When \( h \neq 0 \), we omit the proof of other propositions because the order number \( m \) of functions is too high to obtain exact expressions of solutions by integrating.

### 3.2 The case of \( m = 1 \) and \( \kappa \neq 0 \)

When \( m = 1 \) and \( \kappa \neq 0 \), the equation (3.1) becomes a linear time-fractional PDE as follows:

\[ \delta \left( \frac{RL}{0} D_t^{\alpha} u \right) = \frac{\partial^2 u}{\partial x^2} + \kappa u - \sigma u. \] (3.50)

According to traditional separation method of variables, we suppose that the equation (3.50) has solutions formed as follows:

\[ u = v(x)T(t), \] (3.51)

where \( v(x) \) and \( T(t) \) are two undetermined functions. Substituting (3.51) into (3.50), it can be reduced to

\[ \delta \left( \frac{RL}{0} D_t^{\alpha} v(x) \right) T(t) = v''(x)T(t) + \kappa v(x)T(t) - \sigma v(x)T(t), \] (3.52)

where \( v''(x) = \frac{d^2 v(x)}{dx^2} \). Separating variables, it yields

\[ \frac{\delta \left( \frac{RL}{0} D_t^{\alpha} \right) T(t)}{T(t)} = \frac{v''(x) + \kappa v(x) - \sigma v(x)}{v(x)} = \lambda, \] (3.53)

where \( \lambda \) is nonzero constant. The equation (3.53) can be rewritten as

\[ \frac{RL}{0} D_t^{\alpha} T(t) = \frac{\lambda}{\delta} T(t) \] (3.54)
and

\[ v''(x) + (\kappa - \sigma - \lambda)v(x) = 0. \tag{3.55} \]

We suppose that the initial condition \[ [^{RL}_0 D_t^{\alpha-1} T(t)]_{t=0} = C_0, \] where \( C_0 \) is arbitrary constant.

Making Laplace transformation for the equation (3.54), it yields

\[ s^\alpha T(s) - s^\alpha C_0 = \frac{\lambda}{s} T(s). \tag{3.56} \]

Solving the equation (3.56), we get

\[ T(s) = \frac{C_0}{(s^\alpha - \frac{\lambda}{s})}. \tag{3.57} \]

Making Laplace inverse transformation for the equation (3.57), it yields

\[ T(t) = C_0 t^{\alpha-1} E_{\alpha,\alpha} \left( \frac{\lambda t^\alpha}{\delta} \right). \tag{3.58} \]

It is easy to know that the equation (3.55) has three kinds of solutions as follows:

\[ v(x) = C_1 + C_2 x, \quad \text{if} \quad \lambda = \kappa - \sigma, \tag{3.59} \]

\[ v(x) = C_1 e^{-\sqrt{\kappa - \lambda - \sigma} x} + C_2 e^{\sqrt{\kappa - \lambda - \sigma} x}, \quad \text{if} \quad \lambda < \kappa - \sigma, \tag{3.60} \]

\[ v(x) = C_1 \cos(\sqrt{\lambda + \sigma - \kappa} x) + C_2 \sin(\sqrt{\lambda + \sigma - \kappa} x), \quad \text{if} \quad \lambda > \kappa - \sigma, \tag{3.61} \]

where \( C_1, C_2 \) are arbitrary constants. Thus, the equation (3.50) has three kinds of exact solutions as follows:

\[ u = (c_1 + c_2 x) t^{\alpha-1} E_{\alpha,\alpha} \left[ \frac{(\kappa - \sigma) t^\alpha}{\delta} \right], \tag{3.62} \]

\[ u = \left( c_1 e^{-\sqrt{\kappa - \lambda - \sigma} x} + c_2 e^{\sqrt{\kappa - \lambda - \sigma} x} \right) t^{\alpha-1} E_{\alpha,\alpha} \left( \frac{\lambda t^\alpha}{\delta} \right), \tag{3.63} \]

\[ u = \left[ c_1 \cos(\sqrt{\lambda + \sigma - \kappa} x) + c_2 \sin(\sqrt{\lambda + \sigma - \kappa} x) \right] t^{\alpha-1} E_{\alpha,\alpha} \left( \frac{\lambda t^\alpha}{\delta} \right), \tag{3.64} \]

where \( c_1 = C_0 C_1, \ c_2 = C_0 C_2 \) are two arbitrary constants.

In order to show dynamical property of above solutions intuitively, as example, we plot 3D-graphs of dynamical profiles of the solution (3.63), which are shown in Fig. 8a and Fig.8b.
(c) The case of $\alpha = 0.25$, $\delta = -8$

(d) The case of $\alpha = 0.75$, $\delta = 8$

Fig. 8 The 3D-graphs of solution (3.63); $c_1 = c_2 = 0.5$, $\lambda = -4$, $\sigma = 2$, $\kappa = 3$.

4 Conclutions

In this work, we improved the traditional separation method of variables, we call this improved method the separation method of semi-fixed variables. Compared with the traditional separation method of variables, this improved method will become simpler and the amount of computation is greatly reduced during the process for solving a time-fractional PDE. By using this separation method of semi-fixed variables together with dynamical system method, we successfully studied a time-fractional reaction-diffusion equations with higher-order terms under two kinds of fractional-order differential operators (Caputo differential operator and Riemann-Liouville differential operator). Different kinds of exact solutions of this time-fractional reaction-diffusion equation with higher-order terms are obtained. In addition, we find that the solutions and their dynamical properties of the equation (1.1) are completely different under the definitions of two different fractional differential operators. All solutions of the equation (1.1) can be divided into two major categories, one with power functions and another with Mittag-Leffler functions. Those solutions which contain power functions, all of them have attenuation properties and satisfy $u \to 0$ as $t \to +\infty$ due to their powers are negative. Those solutions which contain Mittag-Leffler functions, some of them have attenuation property and satisfy satisfy $u \to 0$ in the condition of $\delta \kappa < 0$ as $t \to +\infty$ and others have incremental property and satisfy $u \to \pm \infty$ in the condition of $\delta \kappa > 0$ as $t \to +\infty$. In order to show dynamical properties of these exact solutions intuitively, the 3D-graphs of dynamical profiles of some representative solutions are illustrated.
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Data Availability Statement

The authors assure that all data present within the text of the manuscript are available and reliable.

Conflict of Interest:

The authors declare that they have no conflict of interest.

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