Abstract

Possibility of structured mixed phases at first order phase transitions is examined with taking into account of charge screening and surface effects. Hadron-quark phase transition in dense neutron star interior is considered, as concrete example.

Ref. [1] suggested presence of a wide region of mixed phase at any first order phase transitions in multi-component systems of charged particles. Existence of structured mixed phase in dense neutron star interiors would have important consequences for equation of state, also affecting neutrino emissivities [2], glitch phenomena and r modes, cf. [3]-[4]. However inhomogeneity effects of the field profiles were disregarded in these treatments. On the other hand, ref. [5] demonstrated that for the appearance of the structured mixed phase the Coulomb plus surface energy per droplet of the new phase should have a minimum, as function of the droplet radius. Corrections to the Coulomb solutions due to screening effects were disregarded. In spite of the question has been raised long ago, cf. [1]-[6], up to now there is no consistent treatment of these effects. Therefore, further study of the screening and surface effects seems to be of prime importance.

Consider the structured mixed phase consisting of two phases I and II. We suppose the lattice of droplets (phase I) each placed in the Wigner-Seitz cell (the exterior of droplets is phase II). Each droplet in the cell occupies the domain $D^I$ of volume $v^I$ separated by a sharp boundary $\partial D$ from matter in phase II (region $D^\Pi$ of volume $v^\Pi$). We exploit thermodynamic
potential (effective energy) per cell composed of a density functional \( \Omega \),

\[
\Omega = E[\rho] - \mu_1^I \int_{D^I} d\vec{r} \rho_1^I - \mu_1^I \int_{D^I} d\vec{r} \rho_1^I,
\]

(1)

\( E[\rho] \) is the energy of the cell, \( \rho = \{ \rho_1^I, \rho_1^I \} \) are densities of different particle species, \( i = 1, \ldots, N^I \) in phase I and \( i = 1, \ldots, N^{II} \) in phase II, \( N^I, N^{II} \) are total number of particle species per cell in phases I and II. Summation over the repeated Latin indices is implied. Chemical potentials \( \mu_i^I, \mu_i^{II} \) are constants, if, as we assume, each phase is in the ground state. We also assume that matter in phase I or II is in chemical equilibrium by means of the weak and strong interactions. Equations of motion, \( \frac{\delta \Omega}{\delta \rho_\alpha^i} = 0 \), render

\[
\mu_\alpha^i = \frac{\delta E[\rho]}{\delta \rho_\alpha^i}, \quad \alpha = \{ I, II \}.
\]

(2)

The energy of the cell consists of four contributions:

\[
E[\rho] = \int_{D^I} d\vec{r} \epsilon_{\text{kin+str}}^1[\rho_1^I] + \int_{D^{II}} d\vec{r} \epsilon_{\text{kin+str}}^{II}[\rho_1^{II}] + \int_{\partial D} dS \epsilon_S[\rho] + E_V.
\]

(3)

The first two contributions are the sums of the kinetic and strong-interaction energies and \( \epsilon_S[\rho] \) is the surface energy density, which depends on all the particle densities at the boundary \( \partial D \). One may approximate it, as we shall do, in terms of surface tension \( \sigma \). \( E_V \) is the Coulomb interaction energy,

\[
E_V = \frac{1}{2} \int d\vec{r} d\vec{r}' Q_i \rho_i(\vec{r}) Q_j \rho_j(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|},
\]

(4)

with \( Q_i \) being the particle charge (\( Q = -e < 0 \) for the electron).

Then equations of motion (2) can be re-written as

\[
\mu_\alpha^i = \frac{\partial \epsilon_{\text{kin+str}}^\alpha}{\partial \rho_\alpha^i} - N_{\text{ch.}}^\alpha (V^\alpha - V^0), \quad N_{\text{ch.}}^\alpha = Q_i^\alpha / e,
\]

(5)

\(^1\)In reality there is no sharp boundary between phases and all the densities and constant chemical potentials are defined in the whole space (c.f. models of atomic nuclei). Then particle densities, \( \rho_\alpha^i \), are changed continuously. The surface energy is given by integration of \( \epsilon^\alpha \) over a narrow region around surface, where \( \rho_\alpha^i \) change sharply. Our formulation is also applicable to this case.
where $V^\alpha(\vec{r})$ is the electric potential generated by the particle distributions, which can be shifted by an arbitrary constant ($V^0$) due to the gauge transformation, $V \rightarrow V - V^0$,

$$V(\vec{r}) = -\int d\vec{r}' \frac{e Q_i \rho_i(\vec{r}'')}{|\vec{r} - \vec{r}'|} + V^0 \equiv \begin{cases} V^I(\vec{r}), & \vec{r} \in D^I \\ V^{II}(\vec{r}), & \vec{r} \in D^{II} \end{cases} \quad (6)$$

Formally varying eq. (5) with respect to $V^\alpha$ or $\mu^\alpha_i$ we have the matrix form relation,

$$A^\alpha_{ij} \frac{\partial \rho^\alpha_j}{\partial V^\alpha} = N^\text{ch,}^\alpha_i, \quad A^\alpha_{ij} B^\alpha_{jk} = \delta_{ik}, \quad (7)$$

where matrices $A$ and $B$ are defined as

$$A^\alpha_{ij} \equiv \frac{\delta^2 E^\alpha_{\text{kin+str}}}{\delta \rho^\alpha_i \delta \rho^\alpha_j}, \quad B^\alpha_{ij} \equiv \frac{\partial \rho^\alpha_i}{\partial \mu^\alpha_j}. \quad (8)$$

Eqs. (7), (8) reproduce gauge-invariance relation,

$$\frac{\partial \rho^\alpha_i}{\partial V^\alpha} = N^\text{ch,}^\alpha_j \frac{\partial \rho^\alpha_j}{\partial \mu^\alpha_i}, \quad (9)$$

clearly showing that constant-shift of the chemical potential is compensated by gauge transformation of $V^\alpha$. Hence chemical potential $\mu^\alpha_i$ acquires physical meaning only after fixing of the gauge of $V^\alpha$.

Applying Laplacian ($\Delta$) to the l.h.s. of eq. (6) we recover the Poisson equation ($\vec{r} \in D^\alpha$),

$$\Delta V^\alpha = 4\pi e^2 \rho^{\text{ch,}}^\alpha \equiv 4\pi e Q^\alpha_i \rho^\alpha_i. \quad (10)$$

The charge density $\rho^{\text{ch,}}^\alpha$ as a function of $V^\alpha$ is determined by equations of motion (5), so that eq. (10) is a nonlinear differential equation for $V^\alpha$. The boundary conditions are

$$V^I = V^{II}, \quad \nabla V^I = \nabla V^{II}, \quad \vec{r} \in \partial D, \quad \text{(11)}$$

where we neglected a small contribution of surface charge accumulated at the interface of the phases. We also impose condition $\nabla V^{II} = 0$ at the boundary of the Wigner-Seitz cell, which implies that each cell must be charge-neutral. Once eqs. (10) are solved giving $V^\alpha$ and the potentials are matched at the boundary, we have density distributions of particles in the domain $D^\alpha$. 

Note that there are two conservation laws relevant in neutron star matter: baryon number and charge conservation. These quantities are well defined over the whole space, not restricted to each domain. Accordingly the baryon number and charge chemical potentials ($\mu_B$ and $\mu_Q$), being linear combinations of $\mu_i^\alpha$, become constants over the whole space,

$$\mu_B^I = \mu_B^II \equiv \mu_B, \quad \mu_Q^I = \mu_Q^II \equiv \mu_Q.$$  \hspace{1cm} (12)

This fact requires two conditions for $\mu_i^\alpha$ at the boundary $\partial D$, which prescribe the conversion manner of particle species of two phases at the interface. In other words, charge and baryon number densities should be continuous across the boundary due to eq. (5). In particular, electron chemical potential is equal to the charge chemical potential ($\mu_Q = \mu_e^\alpha$) and its number density is related as

$$\rho_e^\alpha = \frac{(\mu_Q - V^\alpha + V^0)^3}{3\pi^2},$$  \hspace{1cm} (13)

from eq. (5). Note that this is a gauge-invariant quantity.

Once eq. (5) is satisfied, pressure becomes constant in each domain,

$$-P^\alpha v^\alpha = \int_{D^\alpha} d\vec{r} \left\{ \epsilon^\alpha_{\text{kin,str}}[\rho_i^\alpha] - \frac{1}{2} N_i^\alpha \rho_i^\alpha V^\alpha - \mu_i^\alpha \rho_i^\alpha \right\}.$$  \hspace{1cm} (14)

Hence, the extremum condition for $\Omega$ with respect to a modification of the boundary of arbitrary shape (under the total volume of the Wigner-Seitz cell being fixed) reads

$$P^I = P^II + \sigma \frac{dS}{du^I},$$  \hspace{1cm} (15)

$S$ is the area of the boundary $\partial D$ and $\sigma$ is surface tension. The boundary of the cell does not contribute since all the densities are continuous quantities at this point. Eq. (13) is

Each particle density is not necessarily continuous across the boundary, since it is only defined in each phase, while densities of leptons are well defined over the whole space. When particles of the same species $i$ are allocated in both domains and the conversion of particle species becomes trivial, we must further impose the relations, $\mu_i^I = \mu_i^II$, and $\rho_i^I = \rho_i^II$ at the boundary.
the pressure equilibrium condition between two phases. Thus we satisfy Gibbs conditions (13), (14) in our formalism.

The Debye screening parameter is determined by the Poisson equation if one expands the charge density in $\delta V^\alpha = V^\alpha - V_i^\alpha$ around a reference value $V_i^\alpha$. Then eq. (10) renders

$$\Delta \delta V^\alpha = 4\pi e^2 \rho^{\text{ch,}\alpha}(V^\alpha = V_i^\alpha) + (\kappa^\alpha(V^\alpha = V_i^\alpha))^2 \delta V^\alpha + 2\pi e^2 \left[ \frac{\partial^2 \rho^{\text{ch,}\alpha}}{(\partial V^\alpha)^2} \right]_{V^\alpha = V_i^\alpha} (\delta V^\alpha)^2 + ...,$$

where we used eq. (3). Then we calculate contribution to the thermodynamic potential (effective energy) of the cell up to $O((\delta V^\alpha)^2)$. A proper electric field energy of the cell is

$$E_V = \int_{D^I} d\vec{r}\, \epsilon_V^I + \int_{D^II} d\vec{r}\, \epsilon_V^{II} = \int_{D^I} (\nabla V^I)^2 \frac{d\vec{r}}{8\pi e^2} + \int_{D^II} (\nabla V^{II})^2 d\vec{r},$$

that in the case of unscreened distributions is usually called the Coulomb energy. Besides the terms given by (18), there are another contributions arising from effects associated with inhomogeneity of the electric potential profile, due to implicit dependence of partial contributions to the particle densities on $V^I(V^\alpha)$. We will call them correlation terms. Then taking $\rho_i^\alpha$ as function of $V^\alpha$ we expand $\epsilon_{\text{kin+str}}^\alpha$ in $\delta V^\alpha$:

$$\epsilon_{\text{kin+str}}^\alpha[\rho] = \epsilon_{\text{kin+str}}^\alpha(\rho_i^\alpha(V^\alpha)) + \left[ \left( \mu_i^\alpha + N_i^{\text{ch,}\alpha}(V^\alpha - V^0) \right) \frac{\partial \rho_i^\alpha}{\partial V^\alpha} \right]_{V^\alpha = V_i^\alpha} \delta V^\alpha + \frac{1}{2} \left[ \left( \kappa_i^\alpha \right)^2 + \left( \mu_i^\alpha + N_i^{\text{ch,}\alpha}(V^\alpha - V^0) \right) \frac{\partial^2 \rho_i^\alpha}{(\partial V^\alpha)^2} \right]_{V^\alpha = V_i^\alpha} (\delta V^\alpha)^2 + ...$$

We used eqs. (2), (3), (4) and (8) in this derivation. The term in (15) $\propto (V^\alpha - V^0)(\delta V^\alpha)^2$ is actually of the higher order of smallness than other terms if $V^\alpha - V^0 \sim V_i^\alpha$. Using expansion

---

3 As noted in footnote 1, in realistic problem with continuous density distributions, i.e. in absence of sharp boundary, the contribution of the surface energy is absorbed into $P^\alpha$. Hence $P^I = P^{II}$ in such a more detailed treatment.
\[ -\mu_i^\alpha \rho_i^\alpha = -[\mu_i^\alpha \rho_i^\alpha]_V - \left[ \mu_i^\alpha \frac{\partial \rho_i^\alpha}{\partial V^\alpha} \right] V^\alpha \delta V^\alpha - \frac{1}{2} \left[ \mu_i^\alpha \frac{\partial^2 \rho_i^\alpha}{(\partial V^\alpha)^2} \right] (\delta V^\alpha)^2 + ... \] (20)

we obtain the corresponding correlation contribution to the thermodynamic potential \( \Omega_{\text{cor}} = \int_{D_i^I} d\vec{r} \omega_{\text{cor}}^I + \int_{D_i^II} d\vec{r} \omega_{\text{cor}}^II \):

\[ \omega_{\text{cor}}^\alpha = \epsilon_{\text{kin}+str}^\alpha(V_r^\alpha) - \mu_i^\alpha \rho_i^\alpha(V_r^\alpha) - \rho^{ch,\alpha}(V_r^\alpha)(V_r^\alpha - V^0) + \frac{(V_r^\alpha - V^0)\Delta V^\alpha}{4\pi e^2} + \frac{(\kappa^\alpha(V_r^\alpha))^2(\delta V^\alpha)^2}{8\pi e^2} + ... \] (21)

where we also used eqs. (16) and (17). In general \( V_r^I \neq V_r^II \) and they may depend on the droplet size. Their choice should provide appropriate convergence of the above expansion in \( \delta V \). Taking \( V_r^I = V_r^II = V_r = \text{const} \) we find

\[ \omega_{\text{cor}}^\alpha = \frac{(\kappa^\alpha(V_r))^2(V_r^\alpha - V_r)^2}{8\pi e^2} + \text{const}, \] (22)

and one may count the potential from the corresponding constant value.

In the following we consider the hadron-quark phase transition, as an example. We suppose the lattice of spherical droplets of the radius \( R \) placed in the Wigner-Seitz cell of the radius \( R_W \). We also assume the quark matter inside the droplet, as phase I, and the hadronic matter outside in the cell, as phase II, both divided by a sharp boundary \( r = R \). The quark matter consists of \( u, d, s \) quarks and electrons and the kinetic plus strong interaction energy density is given by

\[ \epsilon_{\text{kin}+str}^I \approx \frac{3\pi^{2/3}}{4} \left( 1 + \frac{2\alpha_c}{3\pi} \right) \left[ \rho_u^{4/3} + \rho_d^{4/3} + \rho_s^{4/3} + \frac{m_s^2\rho_s^{2/3}}{\pi^{4/3}} \right] + B + \frac{(3\pi^2\rho_e)^{4/3}}{4\pi^2}, \]

where \( B \) is the bag constant, \( \alpha_c \) is the QCD coupling constant, \( m_s \) is the mass of strange quark. Last term is the kinetic energy of electrons.

The hadronic matter consists of protons, neutrons and electrons and the kinetic plus strong energy density is given by

\[ \epsilon_{\text{kin}+str}^II \approx \epsilon_n^{\text{kin}}[\rho_n] + \epsilon_p^{\text{kin}}[\rho_p] + \epsilon_{\text{pot}}[\rho_n, \rho_p] + \frac{(3\pi^2\rho_e)^{4/3}}{4\pi^2}, \] (23)

where \( \epsilon_i^{\text{kin}}[\rho_i], i = n, p \) are standard relativistic kinetic energies of nucleons, while \( \epsilon_{\text{pot}} \) is the potential energy contribution we take here in the form
\[ \epsilon_{\text{pot}}[\rho_n, \rho_p] = S_0 \frac{(\rho_n - \rho_p)^2}{\rho_0} + (\rho_n + \rho_p)\epsilon_{\text{bind}} + \frac{K_0(\rho_n + \rho_p)}{18} \left( \frac{\rho_n + \rho_p}{\rho_0} - 1 \right)^2 \]
\[ + C_{\text{sat}}(\rho_n + \rho_p) \left( \frac{\rho_n + \rho_p}{\rho_0} - 1 \right), \] (24)

\( \rho_0 \) is the nuclear density (\( \rho_0 \simeq 0.16 \text{fm}^{-3} \)) and constants \( \epsilon_{\text{bind}}, K_0, C_{\text{sat}} \) are determined to satisfy the nuclear saturation properties.

We use chemical equilibrium conditions for the reactions \( u + e \leftrightarrow s, d \leftrightarrow s, \) and \( n \leftrightarrow p + e \) in each phase,

\[ \mu_u - \mu_s + \mu_e = 0, \quad \mu_d = \mu_s, \quad \mu_n = \mu_p + \mu_e, \] (25)

and the conversion relation at the boundary,

\[ \mu_B \equiv \mu_n = 2\mu_d + \mu_u, \] (26)

which yield relations between quark and nucleon chemical potentials.\footnote{Other conversion relation \( \mu_p = 2\mu_u + \mu_d \) is then automatically satisfied.}

Using these conditions we obtain, cf. [5],

\[ \rho^{\text{ch},1} \simeq \left( 1 - \frac{2\alpha_e}{\pi} \right) \left[ \frac{2\mu_B^2 V^1}{9\pi^2} \left( 1 + O \left( \frac{V^1}{\mu_B} \right)^2 \right) \right] + \frac{\mu_B m_s^2}{6\pi^2}, \] (27)

where the electron contribution is omitted as small (\( \sim (V^1)^3 \)) and we fixed the gauge by taking \( V^0 = -\mu_Q \equiv -\mu_e. \)

Poisson equation \[ \text{(16)} \] with \( \rho^{\text{ch},1} \) from \[ \text{(27)} \] describing electric potential of the quark droplet can be solved analytically. For \( r < R \) with the boundary condition \( |V^1(r \to 0)| < \infty, \) we find

\[ V^1 = \frac{V^1_0}{\kappa^1 r} \text{sh}(\kappa^1 r) + U^1_0, \] (28)

with an arbitrary constant \( V^1_0. \) For the Debye parameter \( \kappa^1 \) and for the constant \( U^1_0 \) we obtain:
\begin{equation}
(k^I)^2 = \frac{8e^2\mu_B^2}{9\pi} \left( 1 - \frac{2\alpha_c}{\pi} \right), \quad U^I_0 \simeq -\frac{3m_s^2}{4\mu_B}. \tag{29}
\end{equation}

Note that solution (28) is independent of the reference value \( V^I_0 \) in this case, c.f. (17), since \( \rho^{\text{ch},I} \) in (27) is linear function of \( V^I \) in approximation used.

For phase II, expanding the charge density \( \rho^{\text{ch},II} \) around a reference value, \( \rho^{\text{ch},II} \simeq \rho_p(V^{II} = V^{II}_r) + \delta\rho_p - \rho_e(V^{II} = V^{II}_r) - \delta\rho_e \), and using eqs. (7), (8), (17) and (23) we find up to linear order
\begin{equation}
\delta\rho_p \simeq \frac{1}{C_0}(V^{II}(r) - V^{II}_r), \quad \delta\rho_e = \frac{(V^{II})^2}{\pi^2}(V^{II}(r) - V^{II}_r) \tag{30}
\end{equation}

\( C_0 = \frac{A_{22}}{|A|} \simeq \frac{\pi^2}{p_Fp_0^2 + m^2} + \frac{4S_0}{\rho_0}, \quad p_Fp = (3\pi^2\rho_p(V^{II} = V^{II}_r))^{1/3}. \)

For \( r > R \), the Poisson equation with the boundary condition \( V'|_{R_W} = 0 \) yields
\begin{equation}
V^{II} = V^{II}_0 R \left( \frac{\rho^{\text{ch},II}(r - R_W)}{r} \right)(1 - \delta) + U^{II}_0 \tag{31}
\end{equation}

with an arbitrary constant \( V^{II}_0 \), where the constant \( U^{II}_0 \) is given by
\begin{equation}
U^{II}_0 = -\frac{4\pi e^2 \rho^{\text{ch},II}(V^{II} = V^{II}_r)}{(k^{II})^2} + V^{II}_r. \tag{32}
\end{equation}

We will further drop numerically small \( \delta \) correction. We take the reference value \( V^{II}_r = V(r = R_W) \). Since size of the Wigner-Seitz cell \( R_W \) is substantially larger than \( R \), we have \( V(r = R_W) \simeq V^{\text{bulk}} \), where \( V^{\text{bulk}} \) is constant bulk solution of the Poisson equation, \( (V^{\text{bulk}})^3 = -\mu_e^3 = -3\pi^2\rho_p(V^{\text{bulk}} = -\mu_e) \), that coincides with the local charge-neutrality condition for the case of the spatially homogeneous matter. Hence we find \( U^{II}_0 \simeq V^{II}_r \simeq -\mu_e. \)

The charge screening in the external region is determined by the Debye parameter
\begin{equation}
(k^{II})^2 = \frac{4e^2\mu_e^2}{\pi} + \frac{4e^2\pi}{C_0}, \tag{33}
\end{equation}

where second term is contribution of proton screening. Taking \( \rho^{II}_B = 1.5\rho_0, \mu_e \simeq 170 \text{ MeV}, \mu_B = \mu_n \simeq 1020 \text{ MeV}, \alpha_c \simeq 0.4 \), we roughly estimate typical Debye screening lengths as \footnote{Thus solutions (28) and (31) are proved to be consistent with eq. (6) for the gauge choice, \( V^0 = -\mu_e \).}
\[ \lambda_D \equiv 1/\kappa^I \simeq 3.4/m_\pi, \text{ and } \lambda_D^H \equiv 1/\kappa^H \simeq 4.2/m_\pi, \text{ whereas one would have } \lambda_D^H \simeq 8.5/m_\pi, \text{ if the proton contribution to the screening (33) was absent (} C_0^{-1} = 0). \]

Matching of the fields yields

\[ V_0^I \equiv \left( \frac{U_0^H - U_0^I}{\alpha_0 \text{sh}(\kappa^I R) \text{sh}(\kappa^H (R_W - R)) + \text{ch}(\kappa^I R) \text{ch}(\kappa^H (R_W - R))} \right), \]

\[ V_0^H \equiv - \left( \frac{U_0^H - U_0^I}{\alpha_0 \text{sh}(\kappa^I R) \text{sh}(\kappa^H (R_W - R)) + \text{ch}(\kappa^I R) \text{ch}(\kappa^H (R_W - R))} \right), \]

where we introduced notation \( \alpha_0 = \kappa^H / \kappa^I \).

The charge in the sphere of current radius \( r < R \) is given by

\[ Q(r) = V_0^I \ r \text{ch}(\kappa^I r) \left( 1 - \frac{\text{th}(\kappa^I r)}{\kappa^I r} \right) < 0, \]

being, thereby, negative, since \( U_0^H > U_0^I \) and \( V_0^I < 0 \). This negative charge is completely screened by positive charge induced in the region \( R < r \leq R_W \).

Then we calculate contribution to the thermodynamic potential (effective energy) of the Wigner-Seitz cell per droplet volume. We start with the proper electric field energy term;

\[ \tilde{\epsilon}_V = \tilde{\epsilon}_V^I + \tilde{\epsilon}_V^H, \]

\[ \tilde{\epsilon}_V^I = \frac{3}{4\pi R^3} \int_0^R \frac{(\nabla V^I)^2}{8\pi e^2} 4\pi r^2 dr \]

\[ \approx \beta_0 \left( 1 + \alpha_0 \xi \text{th}(\alpha_1 \xi) \right)^2 \frac{-\frac{1}{2} \xi \text{th}^2 \xi + \frac{1}{2} \xi \text{th}^2 \xi + \frac{1}{2} \xi \text{th} \xi}{\xi^2 (\alpha_0 \xi \text{th}(\alpha_1 \xi) + 1)^2}, \]

being expressed in dimensionless units

\[ \xi = \kappa^I R, \quad \alpha_1 = \frac{\alpha_0 (1 - f^{1/3})}{f^{1/3}}, \quad f^{1/3} = \frac{R}{R_W}, \quad \beta_0 = \frac{3 (U_0^H - U_0^I)^2 (\kappa^I)^2}{8\pi e^2}, \]

where we used eqs. (18), (28), (34). With the help of eqs. (34), (35), from eq. (18) we find

\[ \tilde{\epsilon}_V^H = \frac{3}{4\pi R^3} \int_R^{R_W} \frac{(\nabla V^H)^2}{8\pi e^2} 4\pi r^2 dr \]

\[ \approx \beta_0 \left( 1 - \frac{1}{2} \xi \text{th} \xi \right)^2 \left( 1 - \frac{1}{2} \alpha_0 \alpha_1 \xi^2 (1 - \text{th}^2(\alpha_1 \xi)) + \frac{1}{2} \alpha_0 \xi \text{th}(\alpha_1 \xi) \right), \]

\[ \xi^2 (\alpha_0 \xi \text{th}(\alpha_1 \xi) + 1)^2. \]

\[ 9 \]
In order to explicitly calculate correlation terms we introduce the quantity \( \delta V^I \equiv V^I(r) + \mu_e \) for \( r < R \), thus taking \( V^I_r = -\mu_e \). Averaging (22) over the droplet volume, with the help of (28), (34), we obtain

\[
\tilde{\omega}^I_{\text{cor}} = \frac{3}{4\pi R^3} \int_0^R 4\pi r^2 dr \omega^I_{\text{cor}} \simeq \frac{\beta_0}{2\xi^3} \left( 1 + \frac{\alpha_0 \xi \text{th}(\alpha_1 \xi)}{\text{th}(\alpha_1 \xi) + 1} \right)^2 \left( \text{th} \xi - \frac{\xi}{\text{ch}^2 \xi} \right)
+ \frac{2\beta_0}{\xi^3} \left( 1 + \frac{\alpha_0 \xi \text{th}(\alpha_1 \xi)}{\text{th}(\alpha_1 \xi) + 1} \right) \left( \text{th} \xi - \xi + \frac{\beta_0}{3} \right). \quad (40)
\]

In the hadron phase introducing \( \delta V^H = V^H(r) - V^H(r = R_W) \), where we used \( V^H_r = V^H(r = R_W) \approx -\mu_e \), with the help of eqs. (22), (31), (33), (35), we obtain

\[
\tilde{\omega}^H_{\text{cor}} = \frac{3}{4\pi R^3} \int_{R_W}^R 4\pi r^2 dr \omega^H_{\text{cor}} \simeq \frac{\beta_0 \alpha_0}{6} \left( 1 - \frac{1}{\xi} \text{th} \xi \right)^2 \left[ \frac{1}{\xi} \text{th}(\alpha_1 \xi) + 3\alpha_1 (1 - \text{th}^2(\alpha_1 \xi)) - \frac{4}{\xi} \text{th}(\alpha_1 \xi) / \text{ch}(\alpha_1 \xi) \right]. \quad (41)
\]

One can see that \( \tilde{\omega}^H_{\text{cor}} \to 0 \), if \( \alpha_1 \to 0 \), and also in the case \( \alpha_0 \to 0 \). We could also use other values for \( V^I_r \) and \( V^H_r \), e.g. we have checked that using general eq. (21) with \( V^I_r = V^I(0) \) and \( V^H_r = V^H(R_W) \) leads to the very same result.

In our dimensionless units the total quark plus hadron surface contribution to the energy per droplet volume renders

\[
\tilde{\epsilon}_S / \beta_0 = \beta_1 / \xi, \quad \beta_1 = 3\kappa^I \sigma / \beta_0, \quad (42)
\]

see (38), and we used that \( \epsilon_S = 3\sigma / R \). Coefficients \( \beta_0, \beta_1 \) are evaluated with the help of eqs. (29), (38) and (12). For the above used quantities \( \mu_e \approx 170 \text{ MeV}, \mu_n \approx 1020 \text{ MeV}, \alpha_c \approx 0.4 \text{ and } m_\pi \approx 120 \div 150 \text{ MeV} \) we estimate \( \beta_0 \approx 1.6 m_\pi^4 \). Thus, with the value \( \sigma \approx 1.3 m_\pi^3 \) we obtain \( \beta_1 \approx 0.7 \), whereas with \( \sigma \approx 10 \text{ MeV}/\text{fm}^2 \approx 0.14 m_\pi^3 \) we would get \( \beta_1 \approx 0.08 \).

Coulomb solution for the case of a tiny quark fraction volume is obtained, if we first put \( \alpha_1 \to \infty \), and then expand the terms \( \tilde{\epsilon}^I_V + \tilde{\epsilon}^H_V + \tilde{\epsilon}_S \) in \( \xi \ll 1 \). Thus, we recover the Coulomb plus surface energy per droplet volume

\[
\tilde{\epsilon}_{C,S} = \tilde{\epsilon}_C + \tilde{\epsilon}_S = \beta_0 \left( \frac{1}{45} \xi^2 + \frac{1}{9} \xi^2 + \frac{\beta_1}{\xi} \right). \quad (43)
\]
where partial contributions correspond to the terms $\tilde{\epsilon}^I_V$, $\tilde{\epsilon}^II_V$ and $\tilde{\epsilon}_S$. Both the correlation terms $\tilde{\omega}^I_{\text{cor}} \propto \xi^4$ and $\tilde{\omega}^II_{\text{cor}} \propto \xi^3$ can be dropped in the Coulomb limit, for droplets of a tiny size $\xi \ll 1$.

Function $\tilde{\epsilon}_{C,S}$ has the minimum at $\xi = \xi_m = (15\beta_1/4)^{1/3}$, corresponding to the optimal size of the unscreened droplet. Coulomb solution is reproduced only for $\xi_m \ll 1$, whereas with above estimate $\beta_1 \gtrsim 0.1$ we always get $\xi_m \sim 1$. On the other hand, for $\beta_1 \ll 0.1$ we would obtain $\kappa^I \gtrsim m_\pi$, corresponding to unrealistically small droplet size $R < 1/m_\pi$. Thus, we conclude that pure Coulomb solution is never realized within mixed phase.

In the limit $\alpha_1 \xi \gg 1$, $\xi \gg 1$, corresponding to the single large size drop, from (37), (39), (40) and (41) we find that all the terms contribute to the surface energy density ($\propto \xi^{-1}$) and, therefore, the electric field effects can be treated with the help of an effective surface tension. The full surface tension $\sigma^\text{spher}_{\text{tot}}$ then renders
\begin{equation}
\sigma^\text{spher}_{\text{tot}} = \sigma + \sigma_V = \sigma - \lambda_D^I \frac{\beta_0 \alpha_0 [\alpha_0 + 4/3]}{3(1 + \alpha_0)^2}.
\end{equation}

The first $\sigma$ term is the contribution of the strong interaction, and the second negative term is the contribution of the electric field effects, which depends largely on the values of parameters. For $\mu_e = 170$ MeV, $\mu_n = 1020$ MeV, $m_s = 150$ MeV, $\alpha_c = 0.4$, we estimate the contribution to the surface tension from the electric effects as $\sigma_V \simeq -70$ MeV/fm$^2$.

In Figure we demonstrate dependence of the contribution of inhomogeneous charge distributions to the total thermodynamic potential per droplet volume (solid curves), $\delta\tilde{\omega}_{\text{tot}}/\beta_0 = (\tilde{\epsilon}_V + \tilde{\omega}^I_{\text{cor}} + \tilde{\omega}^II_{\text{cor}} + \tilde{\epsilon}_S)/\beta_0$, given by the sum of partial contributions (37), (39), (40), (41) and (42), for the case of a single droplet, as function of the droplet size $\xi$ for three values of $\alpha_0$ at fixed value $\beta_1$ (each panel). The curves labeled by “C” demonstrate the Coulomb solution $\tilde{\epsilon}_{C,S}/\beta_0$, determined by eq.(43). Each Coulomb curve has a pronounced minimum at $\xi = \xi_C \propto \beta_1^{1/3}$. For $\xi > \xi_C$ the Coulomb curve shows quadratic growth deviating drastically for $\xi > 1$ from the solid curve. For $\beta_1 \lesssim 0.01$, minimum points of the Coulomb curves $\xi_C$ deviate little from the minima of the solid ones. Only for such small values of $\beta_1$ and $\xi_m$ we recover the Coulomb limit! However, one may obtain such small values of $\beta_1$.
only for tiny values of surface tension and very large values of neutron chemical potential. With increase of the latter, the Debye parameter $\kappa^I$ is also increased and the droplet radius $R = \xi_m/\kappa^I$ is proved to be essentially smaller than $1/m_\pi$. For larger values of $\beta_1$ deviation between the minima of total and Coulomb solutions is proved to be pronounced. All the solid curves converge to $1/3$. Large $\xi$ asymptotic of solid curves is $\propto 1/\xi$ being interpreted as the surface energy term, characterizing by a significantly smaller value of surface tension \cite{11} than that determined only by strong interaction. We see that for $\sigma \geq |\sigma_V|$ that corresponds to $\beta_1 > \beta_{1c}$ (according to our Figure $\beta_{1c} \simeq 0.5$) the structured mixed phase is proved to be prohibited, since necessary condition of its existence (presence of minimum in the droplet size) is not satisfied, whereas with the Coulomb solution necessary condition is always performed. Large size drops are realized within the mixed phase if $\sigma < |\sigma_V|$, $|\sigma + \sigma_V| \ll |\sigma_V|$ and within Maxwell construction if $\sigma \geq |\sigma_V|$. The “e.m.” curve shown in each panel demonstrates the contribution $\tilde{\epsilon}_{em}/\beta_0 \equiv (\tilde{\epsilon}_V + \tilde{\epsilon}_S)/\beta_0$, ignoring correlation terms for $\alpha_0 = 1$. This quantity $\tilde{\epsilon}_{em}$ is the counterpart of the Coulomb solution $\tilde{\epsilon}_{C,S}$ when the charge screening effect is taken into account. We see that the minimum at the ”e.m.” curve disappears already at $\beta_1 > 0.03$. Difference between the solid and ”e.m.” curves shows the important contribution of correlation energy in the hadron-quark structured mixed phase.

Dependence of the curves on the ratio of the screening lengths $\alpha_0$ is also rather pronounced, whereas it was completely absent for the Coulomb solution. Our calculations also show that dependences of $\delta\tilde{\omega}_{tot}/\beta_0$ on the volume fraction $f$ are very weak in the whole range of available values.

Summarizing, in discussion of possibility of presence of structured mixed phase at first order phase transitions in multi-component systems of charged particles we consistently incorporated effects of the charge screening. As an example, our formalism was applied to the hadron-quark structured mixed phase. We demonstrated that the charge screening effect should greatly modify the description of the mixed phase, changing its parameters and affecting the possibility of existence. In absence of the mixed phase our charged distributions describe the boundary layer between two separated phases existing within the double-tangent

12
(Maxwell) construction. Consideration of non-spherical droplets (rods and slabs) does not change our conclusions. Further discussions are given in another paper [8].

Acknowledgements. Research of D.N.V. at Yukawa Institute for Theoretical Physics was sponsored by the COE program of the Ministry of Education, Science, Sports and Culture of Japan. D.N.V. acknowledges this support and kind hospitality of the Yukawa Institute for Theoretical Physics. He thanks Department of Physics at Kyoto University for the warm hospitality and also acknowledges hospitality and support of GSI Darmstadt. The present research of T.T. is partially supported by the REIMEI Research Resources of Japan Atomic Energy Research Institute, and by the Japanese Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Culture, Sports, Science and Technology (11640272, 13640282). We also acknowledge N. Glendenning and D. Blaschke for valuable remarks.
REFERENCES

[1] N.K. Glendenning, Phys. Rev. D46 (1992) 1274.

[2] S. Reddy, G.F. Bertsch, M. Prakash, Phys. Lett. B475 (2000) 1.

[3] D. Blaschke, H. Grigorian, G. Pogosian, astro-ph/0008003.

[4] N.K. Glendenning, Phys. Rep. 342 (2001) 393.

[5] H. Heiselberg, C.J. Pethick, E.F. Staubo, Phys. Rev. Lett. 70 (1993) 1355.

[6] T. Norsen and S. Reddy, Phys.Rev. C63 (2001) 065804.

[7] R.G. Parr and W. Yang, Density-Functional Theory of Atoms and Molecules, (Oxford U. Press, 1989).

[8] D.N. Voskresensky, M. Yasuhira and T. Tatsumi, in preparation.
Figure: Contribution to the effective energy due to inhomogeneous charge distributions per droplet volume in case of a single droplet versus scaled droplet radius.