COPRIME MODULES AND OTHER RELATED TOPICS

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Abstract. Let R be a commutative ring with unity and let M be a unitary R-module. In this paper we study the relationships between coprime modules and other kinds of modules.

1. Introduction
Let R be a commutative ring with unity and let M be an R-module. M is called coprime module (dual notion of prime modules) if \( R \, \text{ann} \, M = \sum R \, \text{ann} \, N \) for every proper submodule N of M [7].

Note that \( R \, \text{ann} \, \sum R \, \text{ann} \, \mathcal{M} = \mathcal{N} \). Equivalently M is coprime R-module if and only if M is second, see [17] (where M second if for every \( r \in R \), the homothety \( r^* \) on M is either zero or surjective where a homothety \( r^* \) on M means \( r^* \in \text{End}(M) \) and \( r^*(x) = r \, x \) for each \( x \in M \) [24].

The main purpose of this this paper is to study the relationships between coprime R-modules and other kinds of modules such as prime modules, Noetherian, Artinian, cohopfian, hopfian, anti-hopfian, fully stable and M-coprime modules and give the necessary and(or) sufficient condition under which these concepts with coprime R-modules are equivalent.

Recall that an R-module M is called a prime module if \( R \, \text{ann} \, \mathcal{N} = \sum R \, \text{ann} \, \mathcal{M} \) for all non-zero submodule N of M [12],[9]. Equivalently M is prime if for every \( r \in R \), the homothety \( r^* \) on M is either zero or injective [21, Proposition (1.1.15)].

**Proposition (2.1)** Let M be an R-module such that \( r \, M \cap \mathcal{N} = r \, \mathcal{N} \) for each proper submodule N of M; \( r \in R \). If M is prime, then M is coprime.

**Proof:** Let \( r \in \mathcal{M} \), then \( r \, M \subseteq \mathcal{N} \). Since \( r \, M \cap \mathcal{N} = r \, \mathcal{N} \), then \( r \, M = r \, \mathcal{N} \), hence for every \( m \in M \), there exists \( n \in \mathcal{M} \) such that \( r \, m = r \, n \), so \( r \, (m - n) = 0 \), that is \( r \in \sum R \, \text{ann} \, (m - n) \). But M is prime, so \( \text{ann} R \, \text{ann} \, (m - n) = \sum R \, \text{ann} \, \mathcal{M} \) by [21, proposition (1.1.15)]. Thus \( r \in \sum R \, \text{ann} \, \mathcal{M} \) and M is coprime. ■

We notice that the condition \( r \, M \cap \mathcal{N} = r \, \mathcal{N} \) for all proper submodule N of M can not be dropped from proposition (2.1), for example Z as Z-module is prime, but not coprime, where if \( N = 2Z \), then \( 4Z \cap N \neq 4N = 8Z \).
Recall that an \( R \)-module \( M \) is said to be \textit{F-regular} if every submodule of \( M \) is pure, see [14], where a submodule \( N \) of \( M \) is called \textit{pure} if \( I \cap N = I \cap M \) for every ideal \( I \) of \( R \), and hence \( r \cdot M \cap N = r \cdot N \) for every \( r \in R \), see [14]. A ring \( R \) is regular (in sense of von Neumann) if \( I \cap J = I \cdot J \). Equivalently \( R \) is a regular ring if for any \( a \in R \) there exists \( x \in R \) such that \( a = a^2 x \).

Hence, we have the following result.

\textbf{Corollary (2.2):} Let \( M \) be a F-regular \( R \)-module. If \( M \) is prime, then \( M \) is coprime.

The following result shows that the concepts of coprime and prime are equivalent in the class of regular rings (in sense of von Neumann).

\textbf{Corollary (2.3):} Let \( R \) be a regular ring, then \( M \) is a prime \( R \)-module if and only if \( M \) is a coprime \( R \)-module.

\textbf{Proof:} Let \( M \) be a prime module, since \( R \) is regular, then every \( R \)-module is F-regular. Hence the result follows by corollary (2.2).

To prove the converse, let \( M \) be a coprime \( R \)-module, then \( \overline{R} = R / \text{ann}_R \) \( M \) is an integral domain, see [17, note (8)]. But \( R \) is a regular ring, so \( \overline{R} \) is a regular domain. Thus \( \overline{R} \) is a field, and hence by [3, Rem. and Ex. 1.1.3 (6)] \( M \) is a prime \( \overline{R} \)-module, which implies that \( M \) is a prime \( R \)-module. \( \square \)

Recall that an \( R \)-module \( M \) is called \textit{divisible} if for each non-zero divisor \( r \) of \( R \), \( r \cdot M = M \), see [22, p.32].

As another consequence of prop. (2.1), we have the following result.

\textbf{Corollary (2.4):} Let \( M \) be a prime module over an integral domain and every proper submodule of \( M \) is divisible, then \( M \) is coprime.

\textbf{Proof:} Let \( N \) be a proper submodule of \( M \). Then \( r \cdot N = N \) for each \( r \in R \), \( r \neq 0 \). Hence \( r \cdot M \cap N = r \cdot M \cap r \cdot N = r \cdot N \). Thus we have the result by proposition (2.1). \( \square \)

We notice that the condition every submodule of \( M \) is divisible can not be dropped for example:

\( Z \) as \( Z \)-module is prime, but not coprime and every submodule of \( Z \) is not divisible.

Recall that an \( R \)-module \( M \) is called \textit{semisimple} if every submodule \( N \) of \( M \) is a direct summand, see [19, p.107], where a submodule \( N \) of an \( R \)-module \( M \) is said to be \textit{direct summand} of \( M \) if and only if there exists a submodule \( K \) of \( M \) such that \( M = N \oplus K \), see [19,p.31], [6,p.61].

\textbf{Note (2.5):} A semisimple \( R \)-module need not be coprime, for example:
$Z_{6}$ as $Z$-module is semisimple and not coprime.

Also, in the class of semisimple modules the concepts coprime and prime modules are equivalent.

**Proposition (2.6):** Let $M$ be a semisimple $R$-module, then $M$ is a prime $R$-module if and only if $M$ is a coprime $R$-module.

**Proof:** Let $M$ be a prime module, to prove $M$ is a coprime module, let $N$ be a proper submodule of $M$; that is there exists a submodule $W$ of $M$ such that $N \oplus W = M$. Since $M$ is prime, then $\text{ann}_R M = \text{ann}_R W$. But $W \cong \frac{M}{N}$, so it is easy to check that $\text{ann}_R W = \frac{M}{N}$, and so $\text{ann}_R M = \frac{M}{N}$. Thus $M$ is coprime.

To prove the converse, let $N$ be a submodule of $M$, then there exists a submodule $W$ of $M$ such that $N \oplus W = M$; that is $N \cong \frac{M}{W}$ and $\text{ann}_R N = \frac{M}{W}$. Since $M$ is a coprime $R$-module, then $\text{ann}_R \frac{M}{W} = \text{ann}_R M$, which implies that $\text{ann}_R M = \text{ann}_R N$. Thus $M$ is a prime $R$-module. ■

Note that the condition of semisimple in proposition (2.6) is necessary for example: $Z_{p^\infty}$ is coprime, but it is not semisimple and not prime.

Compare the following result with proposition (1.1.6) in [21] (Let $M$ be a prime $R$-module, let $N$ be a proper submodule of $M$, if $N$ is second, then $N$ is prime).

**Proposition (2.7):** Let $M$ be a coprime $R$-module and $N$ be a proper submodule of $M$, if $N$ is a prime $R$-submodule, then $N$ is a coprime $R$-module.

**Proof:** Let $r \in R$ and let $r^*$ be a homothety on $N$, to prove either $r \in \text{ann}_R N$ or $r^*$ is surjective. Assume that $r \not\in \text{ann}_R N$, hence $r \not\in \text{ann}_R M$. But $M$ is a coprime $R$-module, then by [17, corollary (9)] $r M = M$.

Let $y \in N$, then $y \in M = r M$, that is there exists $m \in M$ such that $y = r m$. But $r m \in N$ and $N$ is a prime submodule, implies that $m \in N$ or $r \in [N : M]$. If $r \in [N : M]$, then $r M \subseteq N$, so $M = N$ which is a contradiction. Thus $m \in N$ and so $r^*(m) = r m = y$. Therefore $r^*$ is surjective. ■
S.Yassemi introduced the following theorem without proof, we give its proof for completeness.

**Theorem (2.8):**[24] If M is a non-zero finitely generated coprime R-module, then M is prime.

**Proof:** To prove M is prime, we shall prove \( \text{ann}_R(x) = \text{ann}_R M \) for every \( x \in M, x \neq 0 \). It is clear that \( \text{ann}_R M \subseteq \text{ann}_R(x) \). Assume that \( \text{ann}_R(x) \nsubseteq \text{ann}_R M \) for some \( x \in M, x \neq 0 \), then there exists \( r \in R \) such that \( r \in \text{ann}_R(x) \) and \( r \notin \text{ann}_R M \). Since M is coprime and \( r \notin \text{ann}_R M \), then by[13, corollary (9)] \( r M = M \). But M is a finitely generated R-module, so by[18, p.50] there exists \( r' \in R \) such that \((1 - r r')M = 0\). Thus \( x = r' r x = 0 \) which is a contradiction, so \( r \in \text{ann}_R M \) and \( \text{ann}_R(x) = \text{ann}_R M \). ■

The condition M is finitely generated can not be dropped from theorem (2.8). Consider the Z-module \( \mathbb{Z}_{p^n} \), This module is coprime and it is not finitely generated. However it is not prime because if \( N = \langle \frac{1}{p^2} + Z \rangle \), then \( p^2 Z = \text{ann}_R N \neq \text{ann}_R M = (0) \).

Recall that an R-module M is said to be **Noetherian** if every submodule of M is finitely generated, see [8, Proposition. 6.2, p.75].

The following result follows directly from (2.8).

**Corollary (2.9):** Let M be a Noetherian R-module if M is coprime, then M is prime.

**Proof:** Since M is Noetherian, then M is finitely generated. Hence the result obtained by theorem (2.8). ■

Recall that a proper submodule N of an R-module M is called **fully invariant** if for each \( f \in \text{End}_R(M) \), \( f(N) \subseteq N \). M is called **duo** if every submodule of M is fully invariant, see [25 ].

Let M be an R-module, it is well known that we can consider M as E-module, where \( E = \text{End}_R(M) \) as follows for any \( f \in E, m \in M, f(m) \in M \)

To give the next result first we need the following lemmas.

**Lemma (2.10):** If M is a **duo** R-module, then every R-submodule is E-submodule.

**Proof:** The proof is obvious. ■

**Lemma (2.11):** Let M be a **duo** R-module, if M is a prime E-module, then M is a prime R-module.
Proof: Let N be an R-submodule. To prove $\text{ann}_R M = \text{ann}_R N$, let $r \in \text{ann}_R N$, then $r N = 0$. Define $f: M \rightarrow M$ by $f(m) = r m$. Hence $f(N) = r N = 0$. But by lemma (2.10), N is an E-submodule, thus $f \in \text{ann}_E M = \text{ann}_E N$ because M is a prime E-module. Then $f(M) = r M = 0$.

Thus $r \in \text{ann}_R M$.  

Recall that an R-module is called finendo if M is finitely generated over endomorphism ring $\text{End}_R(M)$, see [15].

Corollary (2.12): Let M be a finendo duo coprime E-module, then M is a prime R-module.

Proof: Since M is finendo coprime E-module, then M is a finitely generated coprime E-module, hence by Theorem (2.8), M is a prime E-module, so that by lemma (2.11) M is a prime R-module.  

Recall that an R-module M is called hopfian if for every $f \in \text{End}_R(M)$ f is surjective, then f is injective, see [16].

Proposition (2.13): If M is a hopfian coprime R-module, then M is prime.

Proof: Since M is a coprime R-module, then every non-zero homothety $r^*$ is surjective. But M is a hopfian R-module, hence $r^*$ is injective. Thus M is a prime R-module, see [21, proposition (1.1.15)].  

Recall that an R-module M is said to be cohopfian if for every $f \in \text{End}_R(M)$, f is injective, then f is surjective, see [10].

The following theorem shows that cohopfian R-module is a sufficient condition for prime module to be coprime.

Proposition (2.14): Let M be a cohopfian R-module, if M is prime, then M is coprime.

Proof: Let $r \in R$, let $r^*$ be a non-zero homothety on M. Since M is prime, then by [21, proposition (1.1.15)] $r^*$ is injective. But M is cohopfian, hence $r^*$ is surjective. Thus M is a coprime R-module.  

From proposition (2.14) and Theorem (2.8) we have.

Corollary (2.15): If M is a finitely generated cohopfian R-module, then M is prime if and only if M is coprime.
Recall that an $R$-module $M$ is called Artinian if $M$ satisfies a decreasing chain conditions (dcc) on submodules of $M$.

S. Yassemi in [24] introduced the following result.

**Proposition (2.16):** Let $M$ be an Artinian $R$-module, if $M$ is prime, then $M$ is coprime.

Since every Artinian is cohopfian, then we get the following directly by (2.15).

**Corollary (2.17):** If $M$ is a finitely generated Artinian $R$-module, then $M$ is prime if and only if $M$ is coprime.

Now, we can give the following result.

**Proposition (2.18):** If $M$ is a coprime $E$-module, then $M$ is cohopfian, where $E = \text{End}_R(M)$.

**Proof:** Since $M$ is a coprime $E$-module, then by [21, corollary (9)] either $f(M) = 0$ or $f(M) = M$ for all $f \in E$, that is either $f = 0$ or surjective. Thus every injective mapping is surjective; that is $M$ is cohopfian. ■

Similarly, we have the following result.

**Proposition (2.19):** Let $M$ be a prime $E$-module, then $M$ is hopfian, where $E = \text{End}_R(M)$.

**Proof:** Let $f \in \text{End}_R(M)$ such that $f$ is surjective to prove $f$ is injective, let $m \in \ker f$, hence $f(m) = 0$. But $M$ is a prime $E$-module, then by [21, proposition (1.1.15)] (0) is a prime $E$-submodule. It follows that either $m = 0$ or $f \in \text{ann}_EM = 0$. But $f \neq 0$, so $m = 0$. Therefore $f$ is injective. ■

Recall that a non-simple $R$-module $M$ is called antihopfian if $M \cong M / N$ for all proper submodule $N$ of $M$, see [16].

**Remark (2.20):**

1. It is clear that every anti-hopfian $R$-module $M$ is a coprime $R$-module, but the converse may not be true, since the $Z$-module $Q$ is coprime, and it is not anti-hopfian because $Q \not\cong Q / Z$.

2. Every anti-hopfian $R$-module $M$ is coprime $E$-module, where $E = \text{End}_R(M)$.

**Proof:** By [4, prop.1.3.1] every $f \in E$, $f = 0$ or $f$ is surjective. Thus $f(M) = 0$ or $f(M) = M$; that is $M$ is a coprime $E$-module, by [17, corollary (9)]. ■
Recall that if an $R$-module $E$ is extension of an $R$-module $M$, then $E$ is an essential extension of $M$, if for every non-zero submodule $E'$ of $E$, $E' \cap M \neq 0$ [22, p.40].

We recall that an $R$-module is an injective hull (envelope) of an $R$-module $M$ if and only if $E$ is a minimal injective extension of $M$, where $E$ is a minimal injective extension of $M$ if:
1. $E$ is injective.
2. Whenever $E'$ is a proper submodule of $E$ contains $M$, then $E'$ is not injective, see [22, p.43].

We used the symbol $\hat{M}$ to denote an injective hull of $M$.

To give the next result, we need the following remark.

**Remark (2.21):** Let $M$ be a module over an integral domain, then $\hat{M}$ is a coprime $R$-module.

**Proof:** Since $\hat{M}$ is an injective and $R$ is an integral domain, then by [17, corollary (33)] we have $\hat{M}$ is a coprime $R$-module. ■

The following result shows that the injective hull of any coprime $R$-module is coprime.

**Corollary (2.22):** If $M$ is a coprime $R$-module, then $\hat{M}$ is a coprime $R$-module.

**Proof:** Since $M$ is a coprime $R$-module, then $\overline{R} = R / \text{ann}_{R} M$ is an integral domain. Hence $M$ is a module over an integral domain $\overline{R}$. Then by remark (2.21), $\hat{M}$ is coprime $\overline{R}$-module, and by [17, corollary (11)] we have $\hat{M}$ is a coprime $R$-module. ■

We notice that the converse of this corollary is not true in general as it is shown by the following example:

For the $Z$-module $Z$, $\hat{Z} = Q$ is a coprime $Z$-module, but $Z$ is not coprime.

The following proposition shows that the converse of corollary (22) is true under certain condition.

**Proposition (2.23):** Let $M$ be an $R$-module such that $[U : M] = [U : \hat{M}]$ for every proper submodule $U$ of $M$, if $\hat{M}$ is a coprime $R$-module, then $M$ is coprime $R$-module.
**Proof:** \( \text{ann} M \subseteq [U : M] \) for every proper submodule \( U \) of \( M \). Since \( \hat{M} \) is coprime, then \( [U : \hat{M}] = \text{ann} \hat{M} \). But by assumption \( [U : M] = [U : \hat{M}] \), that is \( \text{ann} M \subseteq \text{ann} \hat{M} \). Thus \( \text{ann} M = \text{ann} \hat{M} = [U : \hat{M}] = [U : M] \). Therefore \( M \) is coprime. ■

Also, the converse of proposition (2.22) holds under the class of modules over regular ring.

**Proposition (2.24):** Let \( R \) be a regular ring. If \( \hat{M} \) is a coprime \( R \)-module, then \( M \) is a coprime \( R \)-module.

**Proof:** Follows by [17, proposition (16)]. ■

Recall that an \( R \)-module \( M \) is called **fully-stable** if each submodule \( N \) of \( M \) is stable, where a submodule \( N \) of \( M \) is called **stable** if \( f(N) \subseteq N \) for each \( R \)-homomorphism \( f \) from \( N \) into \( M \), see [1].

A module \( M \) is called **fully pseudo-stable** (abbreviated p-stable) if each submodule of \( M \) is pseudo-stable, where a submodule \( N \) of \( M \) is said to be pseudo-stable if \( f(N) \subseteq N \) for each \( R \)-monomorphism \( f : N \rightarrow M \), see [1, Definition 2.1, ch.2].

It is clear that every fully stable is fully p-stable, see [1].

**Remarks (2.25):**

1. fully-stable module may not be coprime module, for example: for all \( n \in \mathbb{Z}^+ \), \( Z_n \) is a fully-stable \( \mathbb{Z} \)-module by [1] but \( Z_n \) is a coprime \( \mathbb{Z} \)-module if and only if \( n \) is prime.
2. A coprime module may not be fully stable as the following example shows:

\( \mathbb{Q} / \mathbb{Z} \) as \( \mathbb{Z} \)-module is coprime by [21,corollary (2.1.13)]. But by [1, Ex. 1.2 (c), ch.1] it is not fully stable.

Recall that a module \( M \) over an integral domain \( R \) is **non-torsion** if there exists \( m \in M \) such that \( \text{ann}_R (m) = 0 \). Thus a torsion free \( R \)-module is non-torsion.

However, we have the following result.

**Proposition (2.26):** Let \( M \) be a non-torsion fully p-stable (stable) over an integral domain \( R \), then \( M \) is faithful coprime.

**Proof:** By [1, Theorem 1.5, ch.3] and [1, Corollary 1.6, ch.3] we have \( M \) is divisible and so by [17, remark (29)], we have the result. ■
**Corollary (2.27):** Let M be torsion free over an integral domain R. If \( M \) is fully stable (p-stable), then \( M \) is faithful coprime.

Recall that an R-module \( M \) is called **quasi-injective** if for every submodule \( N \) of \( M \), every R-homomorphism of \( N \) into \( M \) can be extended to an R-endomorphism of \( M \), see [15].

Next, we have the following.

**Proposition (2.28):** Let \( M \) be multiplication non-torsion over a Dedekind domain \( R \), then the following statements are equivalent:

1. \( M \) is a quasi-injective \( R \)-module.
2. \( M \) is a fully-stable \( R \)-module.
3. \( M \) is an injective \( R \)-module.
4. \( M \) is a divisible \( R \)-module.
5. \( M \) is a faithful coprime \( R \)-module.

**Proof:**

(1) \( \iff \) (2) follows by [1, Corollary 2.3, ch.3].

(2) \( \iff \) (3) follows by [1, Corollary 1.8, ch.3].

(3) \( \iff \) (4) \( \iff \) (5) see [17, Proposition (35)].

(3) \( \iff \) (1). It is obvious. ■

Notice that a fully stable module need not be prime module for example:
\( \mathbb{Z}_6 \) as \( \mathbb{Z} \)-module is fully stable and not prime.

Now, we have the following.

**Theorem (2.29):** Let \( M \) be a prime fully stable (P-stable) \( R \)-module. Then \( M \) is a coprime \( R \)-module.

**Proof:** Since \( M \) is a prime \( R \)-module, then \( \overline{R} = R / \text{ann}_R M \) is an integral domain and by [20], [11] \( M \) is a torsion free \( \overline{R} \)-module. But \( M \) is a fully stable (p-stable) \( \overline{R} \)-module, so \( M \) is a fully stable (p-stable) \( \overline{R} \)-module, by [1,Prop.2.12,ch.3]. Then by corollary (2.27) \( M \) is coprime \( \overline{R} \)-module. Thus by [17,corollary (11)] \( M \) is a coprime \( R \)-module. ■

To give our next result, we need the following lemma.

**Lemma (2.30):** If \( M \) is a fully stable prime \( R \)-module, then \( M \) is cyclic.
Proof: Since M is fully stable, then by [1, Prop. 2.5, ch.3], \( R \ann M : R \ann x \] \subseteq \[ x : M \] for each \( x \) in M. But M is prime, so \( R \ann M = R \ann x \) for every \( x \in M, x \neq 0 \). Hence \( [ R \ann M : R \ann M ] = [ x : M ] \); that is \( R = [ x : M ] \), which implies that \( M = ( x ) \). ■

Corollary (2.31): Let M be a fully stable R-module. Then the following statements are equivalent.
1. M is a prime R-module.
2. M is a cyclic coprime R-module.
3. M is a simple R-module.

Proof: (1) \( \rightarrow \) (2) follows by Theorem (2.29) and lemma (2.30).
(2) \( \rightarrow \) (3) follows by [17, Rem. and Ex. 3 (6)].
(3) \( \rightarrow \) (1). It is clear. ■

Recall that a submodule N of an R-module M is called annihilator submodule if \( N = \ann M \) for some ideal I of R, see [2].

Equivalently, \( N = \ann M \ann N \).

Next, we can give the following proposition.

Proposition (2.32): If every submodule of an R-module M is an annihilator submodule and \( R \ann M \) is a prime ideal, then M is a coprime module.

Proof: Let N be a proper submodule of M. Then \( N = \ann M \) for some ideal I of R. To prove \( [ N : M ] = \ann M \), let \( r \in [ N : M ] \), then \( r M \subseteq N = \ann I \). Hence \( r I M = 0 \). Thus \( r I \subseteq \ann M \), which implies that \( r \in \ann M \) or \( I \subseteq \ann M \). If \( I \subseteq \ann M \), then \( I M = 0 \), and so \( M \subseteq \ann I = N \), so \( M = N \) which is a contradiction. Thus \( r \in \ann M \), so \( [ N : M ] = \ann M \) and M is coprime. ■

Hence, we have the following result.

Corollary (2.33): Let M be a fully stable finitely generated module over a dedekind domain with \( R \ann M \) is prime, then M is coprime.

Proof: By [1, Corollary 1.22, ch.3] every submodule N of M is an annihilator submodule. Hence by proposition (2.32), M is coprime. ■
As another consequence of (2.32), we have the following result:

**Corollary (2.34):** If M is a finitely generated multiplication R-module over a Dedekind domain and \( \text{ann}_R M \) is a prime ideal, then the following statements are equivalent.

1. M is a fully-stable R-module.
2. For every submodule N of M, N is an annihilator.
3. M is a coprime R-module.
4. M is a q-injective R-module.

**Proof:**

(1) \( \Leftrightarrow \) (2) follows by [1, Corollary 1.22,ch.3].

(2) \( \rightarrow \) (3) follows by corollary (2.32).

(3) \( \rightarrow \) (4). M is a coprime multiplication R-module, implies M is simple. Hence M is q-injective.

(4) \( \rightarrow \) (1) follows by [1, Proposition 2.1,ch.3]. ■

**Corollary (2.35):** Let M be a finitely generated prime module over Dedekind domain, then the following statements are equivalent.

1. M is a fully stable.
2. For any submodule N of M, N is an annihilator submodule.
3. M is simple, (and hence it is coprime).
4. M is a multiplication q-injective R-module.

**Proof:**

(1) \( \Leftrightarrow \) (2). See [1, Proposition 1.22,ch.3].

(2) \( \rightarrow \) (3). Since M is a prime R-module, then for every non-zero submodule N of M \( \text{ann}_R M = \text{ann}_R N \). Hence M = \( \text{ann}_R M = \text{ann}_M \text{ann}_R N = N \). Thus M is a simple R-module, and so it is coprime.

(3) \( \rightarrow \) (4) is clear.

(4) \( \rightarrow \) (1) follows by [1, Corollary 2.3,ch.3]. ■

R.Ameri, Y.Talebi and M.Maghsoomi in [12] introduced the notion of M-coprime module by the following definition.
**Definition (2.36):** A module $X$ is said to be $M$-coprime if $\text{Hom}_R(X,M) \neq 0$ and $\text{Tr}_R M (X) = \text{Tr}_R M (X/Y)$ for every submodule $Y$ of $X$ such that $\text{Hom}_R(X/Y,M) \neq 0$, where $\text{Tr}_R M (X) = \sum \{\text{Im} f; f \in \text{Hom}_R(X,M)\}$.

Recall that if $M$ is an $R$-module and $U$ is a non-empty set of $R$-modules, $M$ is said to be generated by $U$ if $M$ is a summation of submodules which are homomorphic images of modules in $U$, see [19, Definition 3.3.1, p.52].

**Examples (2.37):**

1. Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z}_6$. If $U = \{\mathbb{Z}_6/\langle 2 \rangle, \mathbb{Z}_6/\langle 3 \rangle\}$, then it is clear that $M$ is generated by $U$.

   However, $M$ is not generated by $U = \{\mathbb{Z}_6/\langle 2 \rangle\}$.

2. Let $M = \mathbb{Z}_{p^e}$ as $\mathbb{Z}$-module. Let $U = \{ \mathbb{Z}_{p^e}/G \}$, where $G$ is a submodule of $\mathbb{Z}_{p^e}$. $\mathbb{Z}_{p^e} \cong \mathbb{Z}_{p^e}/G$. Thus $\mathbb{Z}_{p^e}$ is a generated by $U$.

3. Let $U = \{\mathbb{Z}_5/(0)\}$. $\mathbb{Z}_5$ is a generated by $U$.

4. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $N = \mathbb{Z}_2 \oplus \langle 0 \rangle$. $M/N \cong \mathbb{Z}_2$. $M$ is generated by $M/N$.

R. Ameri, Y. Talebi and Maghsoomi in [5] gave the following:

**Theorem (2.38):** [5] Let $M$ be an $R$-module. The following statements are equivalent.

1. $M$ is $M$-coprime.

2. $M$ is generated by every non-zero factor module of $M$.

Note that the second statement is given as a definition of "coprime module", see [23, Exc. (16), p.103].

However, we have the following.

**Proposition (2.39):** If $M$ is $M$-coprime, then $M$ is coprime $R$-module.

In order to prove this proposition, we state and prove the following lemma.

**Lemma (2.40):** If $f: X \longrightarrow Y$ is an epimorphism, where $X$ and $Y$ are two $R$-modules, then $\text{ann}_R(X) \subseteq \text{ann}_R(Y)$.

**Proof:** Clear. ■
Now, we are ready to prove proposition (2.39).

**Proof:** If M is M-coprime. Let N be a proper submodule of an R-module M, then M is generated by M / N. Thus $M = \sum_{i \in \Lambda} w_i$; $\Lambda$ is some index set, where $w_i$ is a homomorphic image of M / N. By lemma (2.40) we have for any $i \in \Lambda$ $\text{ann}(M / N) \subseteq \text{ann}w_i$. Thus $\text{ann}(M / N) \subseteq \bigcap_{i \in \Lambda} \text{ann}w_i = \text{ann}(\sum_{i \in \Lambda} w_i) = \text{ann}M$.

Thus $\text{ann}(M / N) \subseteq \text{ann}M$, that is $\text{ann}(M / N) = \text{ann}M$ and M is coprime. 

We notice that the converse of proposition (2.39) is not true in general as the following example shows:

Consider the Z-module $M = \mathbb{Q} \oplus \mathbb{Z}_{p^n}$ and $N = \mathbb{Q} \oplus (0)$ be a submodule of M. M is coprime Z-module and $M / N \cong \mathbb{Q} / \mathbb{Z} \oplus \mathbb{Q}$. Thus M is not generated by $M / N = \{ Z_{p^n} \}$.

Recall that an R-module X is said to be comonoform if every non-zero homomorphism $f : X \rightarrow X / Y$ is epimorphism, where Y is any submodule of X, see [5].

According to the above definition we have.

**Remark (2.41):** Every comonoform R-module is coprime. 

**Proof:** Since M is comonoform, then by [5, Corollary 2.9], M is M-coprime and by proposition (2.39) we have M is coprime. 

We notice that the converse of remark (2.41) may be false as the following example shows:

Let M be the Z-module $\mathbb{Q} \oplus \mathbb{Q}$, M is a coprime Z-module. If $N = \mathbb{Z} \oplus (0)$, then $M / N = \mathbb{Q} \oplus \mathbb{Q} \cong \mathbb{Q} / \mathbb{Z} \oplus \mathbb{Q}$.

Define $f : M \rightarrow M / N$ by $f(a,b) = (a + \mathbb{Z},0)$, $f$ is Z-homomorphism, but not epimorphism, so M is not comonoform.

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