We propose an inductive approach to the representation theory of the chain of complex reflection groups \(G(m, 1, n)\). We obtain the Jucys–Murphy elements of \(G(m, 1, n)\) from the Jucys–Murphy elements of the cyclotomic Hecke algebra and study their common spectrum using representations of a degenerate cyclotomic affine Hecke algebra. We construct representations of \(G(m, 1, n)\) using a new associative algebra whose underlying vector space is the tensor product of the group ring \(\mathbb{C}G(m, 1, n)\) with a free associative algebra generated by the standard \(m\)-tableaux.

**Keywords:** group tower, Hecke algebra, reflection group, maximal commutative subalgebra, Young diagram, Young tableau

1. Introduction

Complex reflection groups generalize the Coxeter groups, and the complete list of irreducible finite complex reflection groups consists of the series of groups denoted by \(G(m, p, n)\), where \(m, p,\) and \(n\) are positive integers such that \(p\) divides \(m\), and 34 exceptional groups [1]. Analogues of the Hecke algebra and the braid group exist for all finite complex reflection groups.

The Hecke algebra of \(G(m, 1, n)\), which is denoted by \(H(m, 1, n)\) here, was introduced in [2]–[4] and is called the cyclotomic Hecke algebra. This is the Hecke algebra of type A for \(m = 1\) and of type B for \(m = 2\). The representation theory of the algebras \(H(m, 1, n)\) was developed in [2] (and in [5] for the Hecke algebra of type B); an inductive approach, à la Okounkov–Vershik [6], to the representation theory of the chain (in \(n\)) of the algebras \(H(m, 1, n)\) was suggested in [7]. Here, we are concerned with an inductive approach, in the spirit of [6], to the representation theory of the chain of the complex reflection groups \(G(m, 1, n)\).

We construct representations of \(G(m, 1, n)\) using a certain associative algebra, denoted by \(\mathcal{T}\) in what follows; as a vector space, \(\mathcal{T}\) is the tensor product of the group ring \(\mathbb{C}G(m, 1, n)\) with the free algebra generated by the standard \(m\)-tableaux of shape \(\lambda^{(m)}\) for all \(m\)-partitions \(\lambda^{(m)}\) of \(n\). The existence of the algebra \(\mathcal{T}\) is interesting in itself, and the construction of the representations presented here seems new.

The representation theory of the wreath products of finite groups by symmetric groups is well known [8] (also see [9], [10]; an inductive approach was proposed in [11]). The group \(G(m, 1, n)\) is the wreath product of the cyclic group \(C_m\) by the symmetric group \(S_n\), and its representation theory is therefore known (the representation theory was developed in [12] for the Coxeter groups of type B, i.e., for the groups \(G(2, 1, n)\)). The chain of groups \(G(m, 1, n)\) is the simplest example of a chain of wreath products \(A \wr S_n\) with simple branching. The branching of a chain of wreath products \(A \wr S_n\), where \(A\) is a finite group, is simple if and only if \(A\) is Abelian.

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Moreover, the representation theory of $G(m, 1, n)$ can be directly deduced from the representation theory of $H(m, 1, n)$ by taking the limit

$$\xi_i \rightarrow v_i, \quad i = 1, \ldots, m, \quad q \rightarrow \pm 1$$

in the formulas (see, e.g., [7]) for matrix elements of the generators of $H(m, 1, n)$; here, $\{\xi_i, i = 1, \ldots, m\}$ is the set of distinct $m$th roots of unity. The parameters $q, v_1, \ldots, v_m$ enter the definition of the cyclotomic Hecke algebra as follows: the algebra $H(m, 1, n)$ is generated by the elements $\tau, \sigma_1, \ldots, \sigma_{n-1}$ with the defining relations

$$\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n-2, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i, \quad i, j = 1, \ldots, n-1, \quad |i - j| > 1, \\
\sigma_i^2 &= (q - q^{-1}) \sigma_i + 1, \quad i = 1, \ldots, n-1, \\
\tau \sigma_1 &= \sigma_1 \tau, \\
\tau \sigma_i &= \sigma_i \tau, \quad i > 1, \\
(\tau - v_1) \cdots (\tau - v_m) &= 0.
\end{align*}$$

Taking limit (1) in defining relations (2), we obtain the standard presentation of the group $G(m, 1, n)$ (see formulas (4) and (5) in Sec. 2; the generators $\tau, \sigma_1, \ldots, \sigma_{n-1}$ of $H(m, 1, n)$ respectively tend to $t, s_1, \ldots, s_{n-1}$ in the classical limit).

It is interesting to independently develop the inductive approach to the representation theory of the groups $G(m, 1, n)$. We here emphasize that our approaches for the groups $G(m, 1, n)$ and for the algebras $H(m, 1, n)$ are parallel. In particular, the proofs of “classical” statements are conducted by the same scheme as the proofs of analogous statements in the $q$-deformed situation, which allows saving space by omitting much of the proofs. The corresponding definitions and results concerning $H(m, 1, n)$ are contained in [13]. Moreover, important objects, such as Jucys–Murphy (JM) elements and intertwiners, can be obtained by taking the classical limit; we describe the needed limit procedures. As is seen, there are certain subtleties in passing to the classical “group” situation (care should be exercised regarding the order of taking limits, and so on). As often happens, the classical situation is more complicated than the quantum situation.

The representation theory of the Hecke algebras $H(m, 1, n)$ in the inductive approach requires studying representations of the same type-A affine Hecke algebra for all $m$. It turns out that a certain version of degenerate affine cyclotomic Hecke algebras, denoted here by $A_{m,n}$, plays the role of the classical counterpart of the affine Hecke algebra $\hat{H}_n$. The algebras $A_{m,n}$ for all $m = 1, 2, \ldots$ can be obtained from the same affine Hecke algebra $\hat{H}_n$ by a certain limit procedure. The important tool in our approach is the existence of a surjective morphism $A_{m,n} \rightarrow CG(m, 1, n)$, which is the classical analogue of a surjective morphism from $\hat{H}_n$ to $H(m, 1, n)$. The arising algebras and the relations between them are shown by the diagram

$$\begin{array}{ccc}
\hat{H}_n & \rightarrow & H(m, 1, n) \\
\downarrow & & \downarrow \\
A_{m,n} & \rightarrow & CG(m, 1, n),
\end{array}$$

where the horizontal arrows are surjective morphisms and the columns are limit procedures.

We start the approach by presenting the classical JM elements of the group $G(m, 1, n)$ (we explain how they can be obtained as classical limits of certain expressions involving the JM elements of $H(m, 1, n)$; a similar process was used in [14] for Weyl groups). The JM elements were defined in [11], [15] for the wreath
product of any finite group $A$ by the symmetric group. The JM elements obtained by a limit procedure coincide with those in [11], [15] if a cyclic group is chosen as $A$. As in the nondegenerate situation, the JM elements are the main tool in our construction of the the representation theory.

We show that the JM elements of $G(m, 1, n)$ are images of the “universal” JM elements in the algebra $\mathfrak{A}_{m,n}$. On the classical level, we verify the commutativity of the double set $\{x_1, \tilde{x}_1, \ldots, x_n, \tilde{x}_n\}$ of elements in the algebra $\mathfrak{A}_{m,n}$. The algebra $\mathfrak{A}_{m,n}$ turns out to coincide with a particular case of the wreath Hecke algebra defined in [16] (also see [17]). We do not include commutativity of the set $\{x_1, \tilde{x}_1, \ldots, x_n, \tilde{x}_n\}$ in the list of defining relations of $\mathfrak{A}_{m,n}$ in contrast to the algebra in [16]; by virtue of the results proved here, the two algebras are in fact isomorphic.

The representation theory of $\mathfrak{A}_{m,2}$, the simplest nontrivial degenerate cyclotomic affine Hecke algebra, carries important information about the recurrence properties of the JM elements of $G(m,1,n)$. We present the list of irreducible representations with diagonalizable $x_1, \tilde{x}_1, x_2$, and $\tilde{x}_2$ of the algebra $\mathfrak{A}_{m,2}$ and then use it to study the spectrum of the JM elements. In particular, we characterize the sets of common eigenvalues of the JM elements of $G(m,1,n)$ and then establish the relation to the $m$-partitions and the $m$-tableaux. After the representations of the group $G(m,1,n)$ are constructed, we count dimensions to verify that all irreducible representations are obtained with this approach.

The representations of $G(m,1,n)$ constructed using the algebra $\mathfrak{T}$ (the tensor product of $\mathbb{C}G(m,1,n)$ with a free associative algebra generated by the standard $m$-tableaux corresponding to $m$-partitions of $n$) are analogues for $G(m,1,n)$ of the seminormal representations of the symmetric group. We determine the $G(m,1,n)$-invariant Hermitian scalar product on the representation spaces and describe analogues of the orthogonal representations of the symmetric group.

We briefly describe the contents of this paper. In Sec. 2, we recall the standard presentation of the complex reflection groups $G(m,1,n)$ and define the JM elements of $\mathbb{C}G(m,1,n)$ using a limit procedure. In Sec. 3, we describe the degenerate cyclotomic affine Hecke algebra $\mathfrak{A}_{m,n}$ and show that the JM elements of $G(m,1,n)$ are images of the “universal” JM elements in $\mathfrak{A}_{m,n}$. In Sec. 4, we study the spectrum of the JM elements of $G(m,1,n)$ and establish the relation to the standard $m$-tableaux. In Sec. 5, we construct representations of $G(m,1,n)$ using the algebra $\mathfrak{T}$. In Sec. 6, we conclude our study of the representation theory of $G(m,1,n)$, giving the completeness result and some consequences. In the appendix, we study the intertwining operators (introduced in [16]) in the degenerate cyclotomic affine Hecke algebra, which provide certain information about the spectrum of the JM elements. We explain how to obtain these intertwining operators by taking the classical limit of appropriate intertwining operators of the nondegenerate affine Hecke algebra.

Here, the ground field is the field $\mathbb{C}$ of complex numbers. The spectrum of an operator $T$ is denoted by $\text{Spec}(T)$. For two integers $k, l \in \mathbb{Z}$, $k < l$, we let $[k,l]$ denote the set of integers $\{k,k+1,\ldots,l-1,l\}$.

2. Complex reflection group $G(m,1,n)$ and Jucys–Murphy elements

The group $G(m,1,n)$ is generated by elements $t,s_1,\ldots,s_{n-1}$ with the defining relations

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad i = 1,\ldots,n-2,$$

$$s_is_j = s_js_i, \quad i,j = 1,\ldots,n-1, \quad |i-j| > 1,$$

$$s_i^2 = 1, \quad i = 1,\ldots,n-1,$$

and

$$ts_1ts_1 = s_1ts_1t, \quad ts_i = s_it, \quad i > 1, \quad t^m = 1.$$
The group $G(m, 1, n)$ is isomorphic to the group $C_m \wr S_n$, the wreath product of the cyclic group $C_m$ with $m$ elements by the symmetric group $S_n$; its order is equal to $m^n n!$. The subgroup of the group $G(m, 1, n)$ generated by $s_1, \ldots, s_{n-1}$ is isomorphic to $S_n$.

We recall that the JM elements $J_i$ of $H(m, 1, n)$ are defined by the initial condition $J_1 = \tau$ and the recurrence relation $J_{i+1} = \sigma_i J_i \sigma_i$, $i = 1, \ldots, n - 1$. We define the classical analogues of the elements $J_i$:

$$j_i := \lim_{q \to 1} \lim_{v_i \to \xi_i} J_i,$$  \hspace{1cm} (6)

$$\tilde{j}_i := \frac{1}{m} \lim_{q \to 1} \lim_{v_i \to \xi_i} \frac{J_i^m - 1}{q - q^{-1}}.$$  \hspace{1cm} (7)

We emphasize that the order of taking limits here is important: we first take the limit with respect to the $v_i$ and then with respect to $q$. It is maybe more instructive to write (7) in the form

$$\tilde{j}_i := \frac{1}{m} \lim_{q \to 1} \lim_{v_i \to \xi_i} \frac{(J_i^m - 1)}{(q - q^{-1})}.$$  \hspace{1cm} (6)

3. Degenerate cyclotomic affine Hecke algebra

The JM elements of the cyclotomic Hecke algebra $H(m, 1, n)$ are the images of the "universal" JM elements of the affine Hecke algebra. Similarly, the elements $j_i$ and $\tilde{j}_i$ are the images of certain elements of a "universal" degenerate cyclotomic affine Hecke algebra, which we now describe.

**Definition 1.** Let $\mathfrak{A}_{m,n}$ denote the algebra generated by $\bar{s}_1, \ldots, \bar{s}_{n-1}$ and two more generators, $x_1$ and $\bar{x}_1$. We introduce the defining relations in three steps. First, there are the defining relations involving only the generators $\bar{s}_1, \ldots, \bar{s}_{n-1}$:

$$\bar{s}_i \bar{s}_{i+1} \bar{s}_i = \bar{s}_{i+1} \bar{s}_i \bar{s}_{i+1}, \quad i = 1, \ldots, n - 2,$$

$$\bar{s}_i \bar{s}_j = \bar{s}_j \bar{s}_i, \quad i, j = 1, \ldots, n - 1, \quad |i - j| > 1,$$

$$\bar{s}_i^2 = 1, \quad i = 1, \ldots, n - 1.$$  \hspace{1cm} (8)

Second, there are relations additionally involving the generator $x_1$:

$$x_1 \bar{s}_1 x_1 \bar{s}_1 = \bar{s}_1 x_1 \bar{s}_1 x_1, \quad x_1 \bar{s}_i = \bar{s}_1 x_1, \quad i > 1, \quad x_1^m = 1.$$  \hspace{1cm} (9)

Third, there are the relations additionally involving the generator $\bar{x}_1$:

$$\bar{x}_1 \left( \bar{s}_1 \bar{x}_1 \bar{s}_1 + \frac{1}{m} \sum_{p=1}^{m} x_1^p \bar{s}_1 x_1^{-p} \right) = \left( \bar{s}_1 \bar{x}_1 \bar{s}_1 + \frac{1}{m} \sum_{p=1}^{m} x_1^p \bar{s}_1 x_1^{-p} \right) \bar{x}_1,$$

$$\bar{x}_1 \bar{s}_i = \bar{s}_i \bar{x}_1, \quad i > 1, \quad \bar{x}_1 x_1 = x_1 \bar{x}_1, \quad \bar{x}_1 \bar{s}_1 x_1 \bar{s}_1 = \bar{s}_1 x_1 \bar{s}_1 \bar{x}_1.$$  \hspace{1cm} (10)

We call the algebra $\mathfrak{A}_{m,n}$ the degenerate cyclotomic affine Hecke algebra.

Because of relations (8) and (9), we have the homomorphism

$$\iota : \mathbb{C}G(m, 1, n) \to \mathfrak{A}_{m,n}, \quad \iota(s_i) = \bar{s}_i, \quad i = 1, \ldots, n - 1, \quad \iota(t) = x_1.$$  \hspace{1cm} (11)
Let \( \pi \) be the map from the generator set \( \{ s_1, \ldots, s_{n-1}, x, \hat{x}_1 \} \) to \( \mathbb{C}G(m, 1, n) \) defined by

\[
\pi: s_i \mapsto s_i \quad i = 1, \ldots, n-1, \quad x_1 \mapsto t, \quad \hat{x}_1 \mapsto 0.
\] (12)

Clearly, \( \pi \) extends to a homomorphism, denoted here by the same symbol \( \pi \), from the algebra \( \mathfrak{A}_{m,n} \) to \( \mathbb{C}G(m, 1, n) \) (the homomorphism \( \pi \) is well defined because relations (10) are satisfied trivially for \( \hat{x}_1 \) equal to zero). Moreover, the composition \( \pi \circ \hat{i} \) leaves the generators of \( G(m, 1, n) \) invariant and is therefore the identity homomorphism of the algebra \( \mathbb{C}G(m, 1, n) \). In particular, the map \( \hat{i} \) is injective; equivalently, the subalgebra of \( \mathfrak{A}_{m,n} \) generated by the elements \( s_1, \ldots, s_{n-1} \) and \( x_1 \) is isomorphic to \( \mathbb{C}G(m, 1, n) \).

We define the “higher” elements \( x_i \) and \( \hat{x}_i \) for \( i = 2, \ldots, n \) by

\[
x_{i+1} = s_i x_i s_i, \quad \hat{x}_{i+1} = s_i \hat{x}_i s_i + \frac{1}{m} \sum_{p=1}^{m} x^p_i s_i x_i^{-p}, \quad i = 1, \ldots, n-1.
\] (13)

The first relation in (9) and the first and fourth relations in (10) can be respectively rewritten as

\[
x_1 x_2 = x_2 x_1, \quad \hat{x}_1 \hat{x}_2 = \hat{x}_2 \hat{x}_1, \quad \hat{x}_1 x_2 = x_2 \hat{x}_1.
\] (14)

Proposition 1. We have

\[
\pi(x_i) = j_i, \quad \pi(\hat{x}_i) = \hat{j}_i.
\] (15)

Because the JM elements \( J_i \) commute in the algebra \( H(m, 1, n) \), it follows from definitions (6) and (7) that the elements \( j_i, i = 1, \ldots, n \), and the elements \( \hat{j}_i, i = 1, \ldots, n \), together form a commutative set. We do not include the commutativity of the corresponding set formed by the elements \( x_i, i = 1, \ldots, n \), and the elements \( \hat{x}_i, i = 1, \ldots, n \), in the defining relations for the algebra \( \mathfrak{A}_{m,n} \): the commutativity of this set (and, therefore by Proposition 1, of its image under the morphism \( \pi \), i.e., of the set formed by the elements \( j_i, i = 1, \ldots, n, \) and \( \hat{j}_i, i = 1, \ldots, n \) follows from relations (8)-(10)), as is now seen.

Lemma 1. It follows from relations (8)-(10) that \( x_i \) and \( \hat{x}_i \) commute with \( s_k \) for \( k > i \) or \( k < i - 1 \).

Proposition 2. It follows from relations (8)-(10) that

\[
x_k x_l = x_l x_k, \quad \hat{x}_k \hat{x}_l = \hat{x}_l \hat{x}_k, \quad \hat{x}_k \hat{x}_l = \hat{x}_l \hat{x}_k
\] (16)

for all \( k, l = 1, \ldots, n \).

4. Spectrum of Jucys–Murphy elements

4.1. Representations of algebra \( \mathfrak{A}_{m,2} \). An important step in understanding the spectrum of JM elements and in constructing representations is to analyze the representations of the smallest nontrivial degenerate cyclotomic affine Hecke algebra, the algebra \( \mathfrak{A}_{m,2} \). Here, we present the list of irreducible representations of \( \mathfrak{A}_{m,2} \) with diagonalizable \( x_1, \hat{x}_1, x_2, \) and \( \hat{x}_2 \).

We consider the algebra \( \mathfrak{A}_{m,2} \) generated by \( x, y, \hat{x}, \hat{y} \), and \( s \) with the defining relations

\[
xy = yx, \quad \hat{x} \hat{y} = \hat{y} \hat{x}, \quad xx = \hat{x} x, \quad y \hat{x} = \hat{x} y,
\]

\[
y = sxs, \quad x^m = 1, \quad \hat{y} = s \hat{x}s + \frac{1}{m} \sum_{p=1}^{m} x^p s x^m s^{-p}, \quad s^2 = 1.
\] (17)
For all $i = 1, \ldots, n-1$, the subalgebra of $\mathbb{C}G(m, 1, n)$ generated by $j_i, j_{i+1}, \tilde{j}_i, \tilde{j}_{i+1}$, and $s_i$ is a quotient of $\mathfrak{A}_{m,2}$. For $m = 1$, $\mathfrak{A}_{m,2}$ is the degenerate affine Hecke algebra used in [6] for the representation theory of the symmetric groups $S_n$.

The four elements $x$, $y$, $\tilde{x}$, and $\tilde{y}$ pairwise commute (see Proposition 2). Investigating irreducible representations of $\mathfrak{A}_{m,2}$ with diagonalizable $x$, $y$, $\tilde{x}$, and $\tilde{y}$, we obtain the following classification:

- **One-dimensional representations**: The action of the generators is given by
  \[ x \mapsto a, \quad y \mapsto a, \quad \tilde{x} \mapsto \tilde{a}, \quad \tilde{y} \mapsto \tilde{a} + \epsilon, \quad s \mapsto \epsilon, \]
  where $a^m = 1$ and $\epsilon^2 = 1$.

- **Two kinds of two-dimensional representations**: In the first kind, the matrices of the generators of $\mathfrak{A}_{m,2}$ are given by
  \[ s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad y \mapsto \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \quad \tilde{x} \mapsto \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{b} \end{pmatrix}, \quad \tilde{y} \mapsto \begin{pmatrix} \tilde{b} & 0 \\ 0 & \tilde{a} \end{pmatrix}, \]
  where $a^m = b^m = 1$ and $a \neq b$. In the second kind, the matrices of the generators of $\mathfrak{A}_{m,2}$ are given by
  \[ s \mapsto \begin{pmatrix} (\tilde{b} - \tilde{a})^{-1} & 1 - (\tilde{b} - \tilde{a})^{-2} \\ 1 & - (\tilde{b} - \tilde{a})^{-1} \end{pmatrix}, \quad x, y \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \]
  \[ \tilde{x} \mapsto \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{b} \end{pmatrix}, \quad \tilde{y} \mapsto \begin{pmatrix} \tilde{b} & 0 \\ 0 & \tilde{a} \end{pmatrix}, \]
  where $a^m = 1$ and $\tilde{b} \neq \tilde{a}$. Representation (20) is irreducible if and only if $\tilde{b} \neq \tilde{a} \pm 1$.

**4.2. Classical spectrum.** We begin by constructing representations of the ring $\mathbb{C}G(m, 1, n)$ satisfying two conditions. First, the classical JM elements $j_1, \ldots, j_n$, $\tilde{j}_1, \ldots, \tilde{j}_n$ are represented by semisimple (diagonalizable) operators. Second, for each $i = 1, \ldots, n-1$, the action of the subalgebra generated by $j_i, j_{i+1}, \tilde{j}_i, \tilde{j}_{i+1}$, and $s_i$ is completely reducible. We use the name “$C$-representations” ($C$ is the first letter in “completely reducible”) for such representations. At the end of the construction, it is seen that all irreducible representations of $\mathbb{C}G(m, 1, n)$ are $C$-representations.

We let $\text{Spec} \left( \frac{j_1}{j_1}, \ldots, \frac{j_n}{j_n} \right)$ denote the set of common eigenvalues of the elements $j_1, \tilde{j}_1, \ldots, j_n, \tilde{j}_n$ in the $C$-representations:
\[
\Lambda = \left( a_1^{(A)}, \ldots, a_n^{(A)} \right)
\]
\[
\tilde{a}_1^{(A)}, \ldots, \tilde{a}_n^{(A)}
\]
belongs to $\text{Spec} \left( \frac{j_1}{j_1}, \ldots, \frac{j_n}{j_n} \right)$ if there is a vector $e_\Lambda$ in the space of a $C$-representation such that $j_i(e_\Lambda) = a_i^{(A)} e_\Lambda$ and $\tilde{j}_i(e_\Lambda) = \tilde{a}_i^{(A)} e_\Lambda$ for all $i = 1, \ldots, n$.

The elements $j_i$ and $\tilde{j}_i$ commute with $s_k$ for $k > i$ or $k < i - 1$ (see Lemma 1), which implies that the action of $s_k$ on $\text{Spec} \left( \frac{j_1}{j_1}, \ldots, \frac{j_n}{j_n} \right)$ is “local” in the sense that $s_k(e_\Lambda)$ is a linear combination of $e_{\Lambda'}$ such that $a_i^{(A)} = a_i^{(A')}$ and $\tilde{a}_i^{(A)} = \tilde{a}_i^{(A')}$ for $i \neq k, k + 1$.

We call $2 \times n$ arrays (21) strings, keeping the name “string” used for a set of common eigenvalues of the JM elements for the algebra $H(m, 1, n)$.
Proposition 3. Let
\[ \Lambda = \left( a_1, \ldots, a_i, a_{i+1}, \ldots, a_n \right) \in \text{Spec} \left( j_1, \ldots, j_n \right), \]
and let \( e_{\Lambda} \) be a corresponding vector. Then the following statements hold:

1. We have \( a_i^m = 1 \) for all \( i = 1, \ldots, n \), and if \( a_i = a_{i+1} \), then \( \tilde{a}_i \neq \tilde{a}_{i+1} \).

2. If \( a_{i+1} = a_i \) and \( \tilde{a}_{i+1} = \tilde{a}_i + \epsilon \), where \( \epsilon = \pm 1 \), then \( s_i(e_{\Lambda}) = e_{\Lambda} \).

3. If \( a_{i+1} \neq a_i \) or if \( a_{i+1} = a_i \) and \( \tilde{a}_{i+1} \neq \tilde{a}_i \pm 1 \), then
\[ \Lambda' = \left( a_1, \ldots, a_{i+1}, a_i, \ldots, a_n \right) \in \text{Spec} \left( j_1, \ldots, j_n \right). \]
Moreover, if \( a_{i+1} \neq a_i \), then the vector \( s_i(e_{\Lambda}) \) corresponds to the string \( \Lambda' \) (see matrices (19) with \( a = a_i, b = a_{i+1}, \tilde{a} = \tilde{a}_i, \text{ and } \tilde{b} = \tilde{a}_{i+1} \)). If \( a_{i+1} = a_i \) and \( \tilde{a}_{i+1} \neq \tilde{a}_i \pm 1 \), then the vector \( s_i(e_{\Lambda}) - e_{\Lambda}/( \tilde{a}_{i+1} - \tilde{a}_i ) \) corresponds to the string \( \Lambda' \) (see matrices (20) with \( a = a_i, \tilde{a} = \tilde{a}_i, \text{ and } \tilde{b} = \tilde{a}_{i+1} \)).

4.3. Classical content strings.

Definition 2. A classical content string \( \left( \frac{a_1}{\tilde{a}_1}, \ldots, \frac{a_n}{\tilde{a}_n} \right) \) is a string of columns of numbers satisfying the following three conditions:

1. We have \( \tilde{a}_1 = 0 \) and \( a_i^m = 1 \) for all \( i = 1, \ldots, n \).

2. For all \( j > 1 \), if \( \tilde{a}_j \neq 0 \), then there exists \( i < j \) such that \( a_i = a_j \) and \( \tilde{a}_i \in \{ \tilde{a}_j - 1, \tilde{a}_j + 1 \} \).

3. If \( a_j = a_k \) and \( \tilde{a}_j = \tilde{a}_k \) for \( j \) and \( k, j < k \), then there exist \( i_1, i_2 \in [j + 1, k - 1] \) such that \( a_{i_1} = a_{i_2} = a_j = a_k, \tilde{a}_{i_1} = \tilde{a}_j - 1, \text{ and } \tilde{a}_{i_2} = \tilde{a}_j + 1 \).

We let \( \text{cCont}_m(n) \) denote the set of classical content strings.

Proposition 4. Let a string of columns of numbers \( \left( \frac{a_1}{\tilde{a}_1}, \ldots, \frac{a_n}{\tilde{a}_n} \right) \) belong to the set \( \text{Spec} \left( j_1, \ldots, j_n \right) \). Then it belongs to the set \( \text{cCont}_m(n) \).

4.4. Classical content of a node in a Young m-diagram. We recall that a Young m-diagram or m-partition is an m-tuple of Young diagrams \( \lambda^{(m)} = (\lambda_1, \ldots, \lambda_m) \). The size \( |\lambda| \) of a Young diagram \( \lambda \) is the number of nodes of the diagram. By definition, the size of an m-tuple \( \lambda^{(m)} = (\lambda_1, \ldots, \lambda_m) \) is \( |\lambda^{(m)}| := |\lambda_1| + \cdots + |\lambda_m| \).

Let the size of an m-diagram be \( n \). We place the numbers \( 1, \ldots, n \) at the nodes of these diagrams. This is a Young m-tableau of shape \( \lambda^{(m)} \). The Young m-tableau is standard if the numbers in the nodes in every diagram are in ascending order along rows to the right and columns downward.

The classical content of a node in a Young diagram is \( (s - r) \) when the node is in line \( r \) and column \( s \). To extend this definition to Young m-diagrams, we must specify the diagram where the node is located in the m-diagram. Therefore, the content of a node in a Young m-diagram is a pair of numbers: the first number specifies the diagram (where the node is located) in the m-diagram, and the second number gives the content of the node in the specified diagram. To relate this information to the spectra of the JM elements, we (arbitrarily) fix a bijection between \( \{1, \ldots, m\} \) and the set of distinct mth roots of unity. Let
\( \xi_k \) be the root of unity associated with \( k \in \{1, \ldots, m\} \) by this bijection. We define the classical content of a node in the line \( r \) and column \( s \) of the \( k \)th diagram of the \( m \)-diagram as the column \( \left( \frac{\xi_k}{s-r} \right) \).

With a standard Young \( m \)-tableau of size \( n \), we associate a string
\[
\left( a_1, \ldots, a_n \right) \quad \left( \tilde{a}_1, \ldots, \tilde{a}_n \right),
\]
of columns of numbers, where \( \left( \frac{a_i}{\tilde{a}_i} \right) \) is the classical content of the node where the number \( i \) is located in the \( m \)-tableau. This association provides the bijection stated in the following proposition.

**Proposition 5.** There is a bijection between the set of standard Young \( m \)-tableaux of size \( n \) and the set \( \text{cCont}_m(n) \).

We give an example of a standard Young 2-tableau of size \( n = 10 \):

\[
\begin{array}{ccc}
1 & 2 & 4 \\
6 & 9 & 7 \\
\end{array}
\quad \begin{array}{cccc}
1 & 3 & 8 & 10 \\
5 & 4 & 2 \\
\end{array}
\]

The string associated with this standard Young 2-tableau is
\[
\left( \xi_1, \xi_1, \xi_2, \xi_1, \xi_2, \xi_1, \xi_2, \xi_1, \xi_2, 0, 1, 0, 2, -1, -1, -2, 1, 0, 2 \right),
\]
where \( \{\xi_1, \xi_2\} \) is the set of distinct square roots of unity.

## 5. Construction of representations

In this section, we describe an analogue (on the classical level) of the construction of representations of \( H(m, 1, n) \) presented in [7], [13]: we define an algebra structure on a tensor product of the algebra \( \mathbb{C}G(m, 1, n) \) with a free associative algebra generated by the standard \( m \)-tableaux corresponding to \( m \)-partitions of \( n \). Further, using the simplest one-dimensional representation of \( G(m, 1, n) \) (from the right), we build representations.

### 5.1. Product of the algebra \( \mathbb{C}G(m, 1, n) \) with a free associative algebra generated by the standard \( m \)-tableaux corresponding to \( m \)-partitions of \( n \)

Let \( \lambda^{(m)} \) be an \( m \)-partition of size \( n \). We consider the set of free generators labeled by standard \( m \)-tableaux of shape \( \lambda^{(m)} \) and let \( X_{\lambda^{(m)}} \) denote the generator corresponding to the standard \( m \)-tableau \( X_{\lambda^{(m)}} \).

We let \( \left( c(X_{\lambda^{(m)}, i}) \right) \) denote the entries of the content column of the node where \( i \) is located in the standard \( m \)-tableau \( X_{\lambda^{(m)}} \).

For any \( m \)-tableau \( X_{\lambda^{(m)}} \) and any permutation \( \pi \in S_n \), we define \( X_{\lambda^{(m)}}^{\pi} \) as the \( m \)-tableau obtained from the \( m \)-tableau \( X_{\lambda^{(m)}} \) by applying the permutation \( \pi \) to the numbers occupying the nodes of \( X_{\lambda^{(m)}} \). For example, \( X_{\lambda^{(m)}}^{s_i} \) is obtained from the \( m \)-tableau \( X_{\lambda^{(m)}} \) by exchanging the numbers \( i \) and \( i + 1 \) in the \( m \)-tableau \( X_{\lambda^{(m)}} \).

For a standard \( m \)-tableau \( X_{\lambda^{(m)}} \), the \( m \)-tableau \( X_{\lambda^{(m)}}^{\pi} \) is not necessarily standard. Concerning the generators of the free algebra, we use the symbol \( X_{\lambda^{(m)}}^{\pi} \) for the generator corresponding to the \( m \)-tableau \( X_{\lambda^{(m)}}^{\pi} \) if the \( m \)-tableau \( X_{\lambda^{(m)}}^{\pi} \) is standard. If the \( m \)-tableau \( X_{\lambda^{(m)}} \) is nonstandard, then we set \( X_{\lambda^{(m)}}^{\pi} = 0 \).

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Proposition 6. The relations

\[ (t - p(X_{\lambda(m)}[1])) \cdot \mathcal{X}_{\lambda(m)} = 0, \]  
\[ s_i \cdot \mathcal{X}_{\lambda(m)} = \mathcal{X}_{\lambda(m)}^{s_i} \cdot s_i \]  

for \( p(X_{\lambda(m)}[i]) \neq p(X_{\lambda(m)}[i + 1]) \) and

\[ \left( s_i + \frac{1}{c(X_{\lambda(m)}[i]) - c(X_{\lambda(m)}[i + 1])} \right) \cdot \mathcal{X}_{\lambda(m)} = \mathcal{X}_{\lambda(m)}^{s_i} \cdot \left( s_i + \frac{1}{c(X_{\lambda(m)}[i + 1]) - c(X_{\lambda(m)}[i])} \right) \]  

for \( p(X_{\lambda(m)}[i]) = p(X_{\lambda(m)}[i + 1]) \) are compatible with the defining relations for the generators \( t, s_1, \ldots, s_{n-1} \) of the group \( G(m, 1, n) \).

We explain the meaning of the word “compatible” in the formulation of the proposition. Let \( \mathcal{F} \) be the free associative algebra generated by \( t, s_1, \ldots, s_{n-1} \). The group \( G(m, 1, n) \) is a natural quotient of \( \mathcal{F} \). Also let \( \mathbb{C}[^\lambda] \) be the free associative algebra with the generators \( \mathcal{X}_{\lambda(m)} \) corresponding to all standard \( m \)-tableaux of shape \( \lambda(m) \) for all \( m \)-partitions \( \lambda(m) \) of \( n \).

We consider an algebra structure on the space \( \mathbb{C}[\mathcal{X}] \otimes \mathcal{F} \) for which (1) the map \( \iota_1 : x \mapsto x \otimes 1, x \in \mathbb{C}[\mathcal{X}] \), is an isomorphism of \( \mathbb{C}[\mathcal{X}] \) with its image under \( \iota_1 \); (2) the map \( \iota_2 : \phi \mapsto 1 \otimes \phi, \phi \in \mathcal{F} \), is an isomorphism of \( \mathcal{F} \) with its image under \( \iota_2 \); and (3) the formulas (22)–(24) extended by associativity ensure the rules for rewriting elements of the form \( 1 \otimes \phi(x \otimes 1) \), where \( x \in \mathbb{C}[\mathcal{X}] \) and \( \phi \in \mathcal{F} \), as elements of \( \mathbb{C}[\mathcal{X}] \otimes \mathcal{F} \).

The compatibility means that we have an induced structure of an associative algebra on the space \( \mathbb{C}[\mathcal{X}] \otimes \mathcal{C}G(m, 1, n) \). More precisely, if we multiply any defining relation of \( \mathcal{C}G(m, 1, n) \) (the relation is viewed as an element of the free algebra \( \mathcal{F} \)) from the right by a generator \( \mathcal{X}_{\lambda(m)} \) (this is an expression of the form “a relation of \( \mathcal{C}G(m, 1, n) \) times \( \mathcal{X}_{\lambda(m)} \)” and use “instructions” (22)–(24) to move all the appearing \( \mathcal{X} \) to the left (the free generator changes but the expression always remains linear in \( \mathcal{X} \)), then we obtain a linear combination of expressions of the form “\( \mathcal{X}_{\lambda(m)}^\pi, \pi \in S_n \), times a relation of \( \mathcal{C}G(m, 1, n) \).”

We let \( \mathcal{F} \) denote the obtained algebra.

It turns out that the product of any JM element by a free generator \( \mathcal{X}_{\lambda(m)} \) is proportional to \( \mathcal{X}_{\lambda(m)} \). The proportionality coefficients are determined by the content columns of nodes.

Lemma 2. It follows from relations (22)–(24) that

\[ (j_i - p(X_{\lambda(m)}[i])) \cdot \mathcal{X}_{\lambda(m)} = 0 \]  
for all \( i = 1, \ldots, n \),

\[ (j_i - c(X_{\lambda(m)}[i])) \cdot \mathcal{X}_{\lambda(m)} = 0 \]  
for all \( i = 1, \ldots, n \).

5.2. Representations. Proposition 6 provides an effective tool for constructing representations of \( G(m, 1, n) \).

Let \( | \rangle \) be a “vacuum,” a basis vector of a one-dimensional \( G(m, 1, n) \)-module. For example, \( s_i | \rangle = | \rangle \) and \( t | \rangle = \xi_1 | \rangle \). Moving the elements \( \mathcal{X} \) to the left in the expressions \( \phi \mathcal{X}_{\lambda(m)}[i] \), \( \phi \in \mathcal{C}G(m, 1, n) \), and using the module structure, because of the compatibility, we build a representation of \( \mathcal{C}G(m, 1, n) \) on the vector space \( U_{\lambda(m)} \) with the basis \( \mathcal{X}_{\lambda(m)}[i] \). By a slight abuse of notation, we simply let \( \mathcal{X}_{\lambda(m)} \) denote \( \mathcal{X}_{\lambda(m)}[i] \). This procedure leads to the following formulas for the action of the generators \( t, s_1, \ldots, s_{n-1} \) on the basis vectors \( \mathcal{X}_{\lambda(m)} \) of \( U_{\lambda(m)} \):

\[ t : \mathcal{X}_{\lambda(m)} \mapsto p(X_{\lambda(m)}[1])\mathcal{X}_{\lambda(m)} \]  
and

\[ s_i : \mathcal{X}_{\lambda(m)} \mapsto \mathcal{X}_{\lambda(m)}^{s_i} \]
for $p(X_{\lambda(m)}|i) \neq p(X_{\lambda(m)}|i+1)$ and

$$s_i: X_{\lambda(m)} \mapsto \frac{1}{c(X_{\lambda(m)}|i+1) - c(X_{\lambda(m)}|i)} X_{\lambda(m)} +$$

$$+ \left(1 + \frac{1}{c(X_{\lambda(m)}|i+1) - c(X_{\lambda(m)}|i)} \right) X_{\lambda(m)}^{s_i}$$

for $p(X_{\lambda(m)}|i) = p(X_{\lambda(m)}|i+1)$. As before, we assume that $X_{\lambda(m)}^{s_i} = 0$ if $X_{\lambda(m)}^{s_i}$ is not a standard $m$-tableau.

**Remark 1.** The constructed representations (up to an isomorphism) are independent of the value of the generators $s_1, \ldots, s_{n-1}$ and $t$ on the vacuum $|\rangle$.

**Remark 2.** In the appendix, we study the classical intertwining operators $\tilde{u}_{i+1} := \tilde{s}_i \tilde{x}_i - \tilde{x}_i \tilde{s}_i \in \mathfrak{sym}_{m,n}$, $i = 1, \ldots, n - 1$. The image under the map $\pi$ defined in (12) of the element $\tilde{u}_{i+1}$ is $\pi(\tilde{u}_{i+1}) = s_i j_i - j_i s_i \in \mathcal{C}G(m,1,n)$, $i = 1, \ldots, n - 1$. The action of $\pi(\tilde{u}_{i+1})$ in the representation $V_{\lambda(m)}$ is

$$X_{\lambda(m)} \mapsto (c^{(i)} - c^{(i+1)} - \delta_{p^{(i)},p^{(i+1)}}) X_{\lambda(m)}^{s_i},$$

where $c^{(i)} = c(X_{\lambda(m)}|i)$, $p^{(i)} = p(X_{\lambda(m)}|i)$, $i = 1, \ldots, n$, and $\delta_{p,p'}$ is the Kronecker symbol.

**5.3. Scalar product.** The representations of $G(m,1,n)$ given by formulas (27)–(29) are analogues of the seminormal representations of the symmetric group. In this subsection, we propose analogues of the orthogonal representations of the symmetric group for $G(m,1,n)$.

Let $\lambda^{(m)}$ be an $m$-partition, and let $X_{\lambda(m)}$ and $X'_{\lambda(m)}$ be two different standard $m$-tableaux of shape $\lambda^{(m)}$. For brevity, we set $c^{(i)} = c(X_{\lambda(m)}|i)$ and $p^{(i)} = p(X_{\lambda(m)}|i)$ for all $i = 1, \ldots, n$. We define the Hermitian scalar product on the vector space $U_{\lambda(m)}$

$$\langle X_{\lambda(m)}, X'_{\lambda(m)} \rangle = 0,$$

$$\langle X_{\lambda(m)}, X_{\lambda(m)} \rangle = \prod_{\substack{j < k; \\ p^{(j)} = p^{(k)}; \\ c^{(j)} \notin \{c^{(k)}, c^{(k)} \pm 1\}}} \frac{c^{(j)} - c^{(k)} - 1}{c^{(j)} - c^{(k)}}.$$  

(32)

This scalar product is positive definite.

**Proposition 7.** Hermitian scalar product (31), (32) on $U_{\lambda(m)}$ is invariant under the action of the group $G(m,1,n)$ defined by formulas (27)–(29).

Consequently, the operators for the elements of $G(m,1,n)$ are unitary in the basis $\{\bar{X}_{\lambda(m)}\}$ where

$$\bar{X}_{\lambda(m)} := \left( \prod_{\substack{j < k; \\ p^{(j)} = p^{(k)}; \\ c^{(j)} \notin \{c^{(k)}, c^{(k)} \pm 1\}}} \left( \frac{c^{(j)} - c^{(k)} - 1}{c^{(j)} - c^{(k)}} \right)^{1/2} \right) X_{\lambda(m)}$$

for any standard $m$-tableau $X_{\lambda(m)}$ of shape $\lambda^{(m)}$. 

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6. Completeness

Using the \( C \)-representations of \( G(m, 1, n) \), we complete our study of the spectrum of the JM elements of \( \mathbb{C}G(m, 1, n) \) and of the representation theory of the chain of groups \( G(m, 1, n) \). Let \( T_n \) be the set of standard \( m \)-tableaux of size \( n \).

**Proposition 8.** There is a one-to-one correspondence between the three sets

\[
\text{Spec} \left( j_1, \ldots, j_n \atop \tilde{j}_1, \ldots, \tilde{j}_n \right), \quad \text{cCont}_m(n), \quad T_n.
\]

**Corollary 1.** The spectrum of the classical JM elements is simple in the representations \( V_{\lambda(m)} \) (corresponding to the \( m \)-partitions).

This means that for two different standard \( m \)-tableaux (not necessarily of the same shape), the elements of \( \text{Spec} \left( j_1, \ldots, j_n \atop \tilde{j}_1, \ldots, \tilde{j}_n \right) \) associated with them by Proposition 8 are different. It remains to verify that we obtain all irreducible representations of the group \( G(m, 1, n) \) with this approach. Because the sum of squares of the dimensions of the constructed representations equals the order of \( G(m, 1, n) \) (this follows from general results on products of Bratteli diagrams; see, e.g., [13]), the following proposition completes the verification.

**Proposition 9.** The representations \( V_{\lambda(m)} \) (corresponding to the \( m \)-partitions) of the group \( G(m, 1, n) \) constructed in the preceding section are irreducible and pairwise nonisomorphic.

In particular, we obtain the following conclusions from the completeness results:

- The branching rules for the chain in \( n \) of the groups \( G(m, 1, n) \) are free of multiplicities.
- The centralizer of the subalgebra \( \mathbb{C}G(m, 1, n - 1) \) in \( \mathbb{C}G(m, 1, n) \) is commutative for each \( n = 1, 2, 3, \ldots \).
- The centralizer of the subalgebra \( \mathbb{C}G(m, 1, n - 1) \) in \( \mathbb{C}G(m, 1, n) \) is generated by the center of \( \mathbb{C}G(m, 1, n - 1) \) and the JM elements \( j_n \) and \( \tilde{j}_n \).
- The subalgebra generated by the JM elements \( j_1, \ldots, j_n, \tilde{j}_1, \ldots, \tilde{j}_n \) is the maximal commutative subalgebra of \( \mathbb{C}G(m, 1, n) \).

**Remark 3.** For every standard \( m \)-tableau \( X_{\lambda(m)} \), we define the element \( p_{X_{\lambda(m)}} \) of the ring \( \mathbb{C}G(m, 1, n) \) by the following recurrence relation with the initial condition \( p_{\emptyset} = 1 \). Let \( \alpha \) be the node occupied by the number \( n \) in \( X_{\lambda(m)} \) and \( \mu^{(m)} \) denote the \( m \)-partition of \( n - 1 \) obtained from \( \lambda^{(m)} \) by removing the node \( \alpha \). Let \( X_{\mu^{(m)}} \) be the standard \( m \)-tableau with the numbers \( 1, \ldots, n - 1 \) at the same nodes as in \( X_{\lambda(m)} \). The recurrence relation is

\[
p_{X_{\lambda(m)}} := p_{X_{\mu^{(m)}}} \prod_{\beta \in E_*^{+}(\mu^{(m)}), \atop c(\beta) \neq c(\alpha)} \frac{\tilde{j}_n - c(\beta)}{c(\alpha) - c(\beta)} \prod_{\beta \in E_*^+(\mu^{(m)}), \atop p(\beta) \neq p(\alpha)} \frac{j_n - p(\beta)}{p(\alpha) - p(\beta)},
\]

(33)

where \( \left( \frac{p(\beta)}{c(\beta)} \right) \) is the classical content of the node \( \beta \) and \( E_*^{+}(\mu^{(m)}) \) is the set of addable nodes of \( \mu^{(m)} \) (a node \( \beta \) is said to be addable for an \( m \)-partition \( \mu^{(m)} \) if the \( m \)-tuple of sets of nodes obtained from \( \mu^{(m)} \) by adding \( \beta \) is still an \( m \)-partition). By the completeness results in this section, the elements \( p_{X_{\lambda(m)}} \) form a complete set of pairwise orthogonal primitive idempotents of the algebra \( \mathbb{C}G(m, 1, n) \).

We have a well-defined homomorphism \( \mathcal{T} \to \mathbb{C}G(m, 1, n) \), which is the identity on the generators \( s_1, \ldots, s_{n-1}, t \) and sends \( X_{\lambda(m)} \) to \( p_{X_{\lambda(m)}} \) for all standard \( m \)-tableaux \( X_{\lambda(m)} \).
Appendix: Classical intertwining operators

Here, we describe the intertwining operators in the degenerate cyclotomic affine Hecke algebra $\mathfrak{A}_{m,n}$. They can be used to investigate the spectrum of the elements $\tilde{x}_i$ in different representations. In the spirit of diagram (3) in the introduction, we discuss the origin of these intertwining operators in the (nondegenerate) affine Hecke algebra $\hat{H}_n$. We also rederive the spectrum of the JM elements $\tilde{j}_i$ from the perturbation theory standpoint. The intertwining operators can be introduced in the more general context of the wreath Hecke algebra [16].

1. Propositions 3 and 4 imply the following proposition.

**Proposition 10.** We have

$$\text{Spec}(\tilde{j}_i) \subset [1 - i, i - 1]$$

(34)

for all $i = 1, \ldots, n$.

To give an alternative proof, in the spirit of [18], we introduce the elements of the algebra $\mathfrak{A}_{m,n}$

$$\tilde{u}_{i+1} := \tilde{s}_i \tilde{x}_i - \tilde{x}_i \tilde{s}_i = \tilde{s}_i (\tilde{x}_i - \tilde{x}_{i+1}) + P_{i+1}, \quad i = 1, \ldots, n - 1,$$

(35)

where we set $P_{i+1} := (1/m) \sum_{p=1}^{m} x_i^p x_{i+1}^{-p}$. The elements $\tilde{u}_i$ are the “classical intertwining” operators and satisfy

$$\begin{align*}
\tilde{u}_{i+1} x_i &= x_{i+1} \tilde{u}_{i+1}, \\
\tilde{u}_{i+1} x_{i+1} &= x_i \tilde{u}_{i+1}, \\
\tilde{u}_{i+1} x_j &= x_j \tilde{u}_{i+1}, \\
\tilde{u}_{i+1} \tilde{x}_i &= \tilde{x}_{i+1} \tilde{u}_{i+1}, \\
\tilde{u}_{i+1} \tilde{x}_{i+1} &= \tilde{x}_i \tilde{u}_{i+1}, \\
\tilde{u}_{i+1} \tilde{x}_j &= \tilde{x}_j \tilde{u}_{i+1}, \\
\tilde{u}_{i+1} \tilde{x}_{j+1} &= \tilde{x}_j \tilde{u}_{i+1}, \\
\tilde{u}_{i+1} \tilde{x}_{j+1} &= \tilde{x}_j \tilde{u}_{i+1}.
\end{align*}$$

(36)

Moreover, the $\tilde{u}_i$ satisfy the Artin relations

$$\tilde{u}_i \tilde{u}_{i+1} \tilde{u}_i = \tilde{u}_{i+1} \tilde{u}_i \tilde{u}_{i+1}.$$  

(37)

We write one more property of the $\tilde{u}_i$:

$$\tilde{u}_{i+1}^2 = - (\tilde{x}_i - \tilde{x}_{i+1})^2 + P_{i+1} = - (\tilde{x}_i - \tilde{x}_{i+1} + P_{i+1})(\tilde{x}_i - \tilde{x}_{i+1} - P_{i+1}).$$

(38)

Therefore, for a polynomial $\chi$ in one variable, we have

$$\tilde{u}_{i+1} \chi(\tilde{x}_i) \tilde{u}_{i+1} = \chi(\tilde{x}_{i+1}) \tilde{u}_{i+1}^2 = - \chi(\tilde{x}_{i+1})(\tilde{x}_i - \tilde{x}_{i+1} + P_{i+1})(\tilde{x}_i - \tilde{x}_{i+1} - P_{i+1}).$$

(39)

The elements $\tilde{x}_i$, $\tilde{x}_{i+1}$, and $P_{i+1}$ commute. In a representation $\rho$, the spectrum of $\rho(P_{i+1})$ is contained in $\{0, 1\}$. Taking the characteristic polynomial of $\rho(\tilde{x}_i)$ as $\chi$, we conclude that

$$\text{Spec}(\rho(\tilde{x}_{i+1})) \subset \text{Spec}(\rho(\tilde{x}_i)) \cup (\text{Spec}(\rho(\tilde{x}_i)) + 1) \cup (\text{Spec}(\rho(\tilde{x}_i)) - 1).$$

(40)

Realizing $\tilde{x}_i$ by $\tilde{j}_i$ in a representation of the group $G(m, 1, n)$ and taking the “initial condition” $\tilde{j}_1 = 0$ into account, we again obtain (34).

**Remark 4.** The usual degenerate affine Hecke algebra (it corresponds to $m = 1$) is distinguished in the sense that the idempotents $P_i$ become trivial, $P_i = 1$, in contrast to the degenerate cyclotomic affine Hecke algebras with $m > 1$. 

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2. Let \( \tilde{H}_n \) be the affine Hecke algebra, i.e., the algebra with the generators \( \tau, \sigma_1, \ldots, \sigma_{n-1} \) and the defining relations given by the first five lines in (2). The algebra \( H(m, 1, n) \) is the quotient of \( \tilde{H}_n \) by \( (\tau - v_1) \cdots (\tau - v_m) \). The JM operators \( y_i \) of the algebra \( \tilde{H}_n \) are defined by \( y_1 := \tau \) and \( y_{i+1} := \sigma_i y_i \sigma_i \) for \( i > 0 \).

General intertwining operators \( U_{i+1} \), \( i = 1, \ldots, n-1 \), of the affine Hecke algebra \( \tilde{H}_n \) are defined as operators satisfying

\[
U_{i+1}y_i = y_{i+1}U_{i+1}, \quad U_{i+1}y_{i+1} = y_iU_{i+1}, \quad U_{i+1}y_k = y_kU_{i+1}, \quad k \neq i, i+1,
\]

for all \( i = 1, \ldots, n-1 \). The intertwining operators (solutions \( U_{i+1} := U_{i+1} \) of (41)) used in [18] are

\[
U_{i+1} := \sigma_i y_i - y_i \sigma_i.
\]

In addition to (41), the operators \( U_{i+1} \) satisfy the Artin relation

\[
U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1},
\]

and their squares are equal to the function of the JM elements

\[
U_{i+1}^2 = -(y_{i+1} - q^2 y_i)(y_{i+1} - q^{-2} y_i).
\]

In contrast to (38), the right-hand side of (44) does not contain anything analogous to the projector \( P_{i+1} \). We explain the appearance of projectors in the classical limit.

Formulas (6) and (7) show that the Taylor series expansions of the JM elements of the cyclotomic algebra \( H(m, 1, n) \) begin as

\[
J_i = j_i + j_i \tilde{x}_i \alpha + O(\alpha^2),
\]

where \( \alpha \) is the deformation parameter, \( q^2 = 1 + \alpha + O(\alpha^2) \). We “lift” formula (45) to the affine Hecke algebra by assuming that the Taylor series expansions of the JM elements of the affine algebra \( \tilde{H}_n \) begin as

\[
y_i = x_i + \tilde{x}_i \alpha + O(\alpha^2),
\]

where \( x_i \) and \( \tilde{x}_i \) belong to the degenerate cyclotomic affine Hecke algebra \( \mathfrak{A}_{m,n} \) (see (8)–(10)).

To take the classical limit, we take into account that the Taylor series expansion of the elements \( \sigma_i \) begins as

\[
\sigma_i = \tilde{s}_i + \alpha \frac{\alpha}{2} + O(\alpha^2).
\]

We note that by (46) and (47), the operators \( U_{i+1} \) given by (42) tend to the operators

\[
u_{i+1} := \tilde{s}_i x_i - x_i \tilde{s}_i \equiv \tilde{s}_i(x_i - x_{i+1}).
\]

The operators \( u_{i+1} \) satisfy all the relations for the operators \( \tilde{u}_{i+1} \) listed in (36), but the intertwining operators \( u_{i+1} \) do not help in understanding the spectrum of the images of the elements \( \tilde{x}_i \) in some representation.

As noted in [18], the operators \( U_{i+1} := U_{i+1} f(y_i, y_{i+1}) \), where \( f \) is an arbitrary function, are intertwining operators satisfying the Artin relation. It can be shown by induction that for any positive integer \( L \),

\[
\sigma_i y_i^L - y_i^L \sigma_i = U_{i+1} \sum_{b=0}^{L-1} y_i^b y_{i+1}^{L-1-b}.
\]
Therefore, the operators $\sigma_i y_i^L - y_i^L \sigma_i$ are intertwining operators for any positive integer $L$.

Under assumption (46), the operators $\sigma_i y_i^m - y_i^m \sigma_i \equiv \sigma_i (y_i^m - 1) - (y_i^m - 1) \sigma_i$ tend to zero as $\alpha$ tends to zero. These operators are of the order $O(\alpha)$. We set
\[
\tilde{U}_{i+1} := \frac{1}{m} \left(\frac{y_i^m - 1}{\alpha} - \frac{y_i^m - 1}{\alpha} \sigma_i\right).
\] (50)
The elements $\tilde{U}_{i+1}$ tend to $\tilde{u}_{i+1}$ as $\alpha$ tends to zero, and the following lemma allows recovering result (38) from the perturbative standpoint.

**Lemma 3.** We have
\[
\tilde{U}_{i+1}^2 = - (\tilde{x}_i - \tilde{x}_{i+1} + P_{i+1})(\tilde{x}_i - \tilde{x}_{i+1} - P_{i+1}) + O(\alpha).
\]

3. The elements $j_i$ satisfy $j_i^m = 1$; the characteristic equations for the elements $j_i$ are not interesting on the classical level. It is easy to obtain the characteristic equation for $\tilde{j}_i$ starting from a characteristic equation for $J_i$. Let $A_0$ be a semisimple operator on a vector space $V$. We consider a perturbation of $A_0$ of the form
\[
A = A_0 + A_1 \alpha + O(\alpha^2),
\] (51)
where $A_1$ is semisimple and $A_1$ commutes with $A_0$. Let $r$ be an eigenvalue of $A_0$ and $V_r$ be the corresponding eigenspace. The operator $A$ on the space $V_r$ has the form $r I + \alpha A_1$ up to the order $\alpha^2$, and its eigenvalues are $r + \alpha t_0$, where $\{t_0\}$ is the set of eigenvalues of the restriction of $A_1$ to $V_r$.

In the particular situation where $A_0 = j_i$, $A_0 = j_i$, and $A = J_i$, the spectrum of $A$, generally speaking, is a subset of $\{v_l q^{2n}, l = 1, \ldots, m, n \in [1 - i, i - 1]\}$ (see [7], [13]). We first take the limit $v_l \rightarrow \xi_l$, $l = 1, \ldots, m$. Then $\xi_l q^{2n} = \xi_l + \xi_l n + O(\alpha^2)$ (because $q^2 = 1 + \alpha + O(\alpha^2)$). Therefore, the spectrum of $\tilde{j}_i$ is a subset of $[1 - i, i - 1]$, and we obtain Proposition 10 from the perturbative standpoint.

REFERENCES

1. G. C. Shephard and J. A. Todd, *Canad. J. Math.*, 6, 274–304 (1954).
2. S. Ariki and K. Koike, *Adv. Math.*, 106, 216–243 (1994).
3. M. Broué and G. Malle, *Astérisque*, 212, 119–189 (1993).
4. I. V. Cherednik, *Duke Math. J.*, 54, 563–577 (1987).
5. P. Høegsmitt, “Representations of Hecke algebras of finite groups with BN-pairs of classical type,” Doctoral dissertation, Univ. of British Columbia, Vancouver (1974).
6. A. Okounkov and A. Vershik, *Selecta Math.*, n. s., 2, 581–605 (1996).
7. O. V. Ogievetsky and L. Poulain d’Andecy, *Modern Phys. Lett. A*, 26, 795–803 (2011); arXiv:1012.5844v1 [math-ph] (2010).
8. W. Specht, *Schriften Math. Seminar* (Berlin), 1, 1–32 (1932).
9. G. James and A. Kerber, *The Representation Theory of the Symmetric Group* (Encycl. Math. Its Appl., Vol. 16), Addison-Wesley, Reading, Mass. (1981).
10. I. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon, Oxford (1998).
11. I. A. Pushkarev, *J. Math. Sci. (New York)*, 96, 3590–3599 (1999).
12. A. Young, *Proc. London Math. Soc.* (2), 31, 273–288 (1930).
13. O. V. Ogievetsky and L. Poulain d’Andecy, “Jucys–Murphy elements and representations of cyclotomic Hecke algebras,” arXiv:1206.0612v2 [math.RT] (2012).
14. A. Ram, *Proc. London Math. Soc.*, 75, 99–133 (1997); arXiv:math.RT/9511223v1 (1995).
15. W. Wang, *Proc. London Math. Soc.*, 88, 381–404 (2004); arXiv:math.QA/0203004v4 (2002).
16. J. Wan and W. Wang, *Internat. Math. Res. Notices*, 128 (2008); arXiv:0806.0196v2 [math.RT] (2008).
17. A. Ram and A. Shepler, *Comment. Math. Helv.*, 78, 308–334 (2003); arXiv:math.GR/0209135v1 (2002).
18. A. P. Isaev and O. V. Ogievetsky, *Czechoslovak J. Phys.*, 55, 1433–1441 (2005).