On the spin-1/2 Aharonov-Bohm problem in conical space: bound states, scattering and helicity nonconservation

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Abstract

In this work the bound state and scattering problems for a spin-1/2 particle undergone to an Aharonov-Bohm potential in a conical space in the nonrelativistic limit are considered. The presence of a \(\delta\)-function singularity, which comes from the Zeeman spin interaction with the magnetic flux tube, is addressed by the self-adjoint extension method. One of the advantages of the present approach is the determination of the self-adjoint extension parameter in terms of physics of the problem. Expressions for the energy bound states, phase-shift and \(S\) matrix are determined in terms of the self-adjoint extension parameter, which is explicitly determined in terms of the parameters of the problem. The relation between the bound state and zero modes and the failure of helicity conservation in the scattering problem and its relation with the gyromagnetic ratio \(g\) are discussed. Also, as an application, we consider the spin-1/2 Aharonov-Bohm problem in conical space plus a two-dimensional isotropic harmonic oscillator.

Keywords: Self-adjoint extension, Aharonov-Bohm effect, Bound State, Scattering, Helicity

1. Introduction

The Aharonov-Bohm (AB) effect [1] (first predicted by Ehrenberg and Siday [2]) is one of most weird results of quantum phenomena. The effect reveals that the electromagnetic potentials, rather than the electric and magnetic fields, are the fundamental quantities in quantum mechanics. The interest in this issue appears in the different contexts, such as solid-state physics [3], cosmic strings [4–14] \(\kappa\)-Poincaré-Hopf algebra [15, 16], \(\delta\)-like singularities [17–19], supersymmetry [20, 21], condensed matter [22, 23], Lorentz symmetry violation [24], quantum chromodynamics [25], general relativity [26], nanophysics [27], quantum ring [28–30], black hole [31, 32] and noncommutative theories [33, 34].

In the AB effect of spin-1/2 particles [7], besides the interaction with the magnetic potential, an additional two dimensional \(\delta\)-function appears as the mathematical description of the Zeeman interaction between the spin and the magnetic flux tube [18, 19]. This interaction is the basis of the spin-orbit coupling, which causes a splitting on the energy spectrum of atoms depending on the spin state. In Ref. [17] is argued that this \(\delta\)-function contribution to the potential can not be neglected when the system has spin, having shown that changes in the amplitude and scattering cross section are implied in this case. The presence of a \(\delta\)-function potential singularity, turns the problem more complicated to be solved. Such kind of point interaction potential can then be addressed by the self-adjoint extension approach [35]. The self-adjoint extension of symmetric operators [36] is a very powerful mathematical method and it can be applied to various systems in relativistic and nonrelativistic quantum mechanics, supersymmetric quantum mechanics and vortex-like models.

This paper extends our previous report [37] on a general physical regularization method, both in details and depth. The method has the advantage of solving problems in relativistic and nonrelativistic quantum mechanics whose Hamiltonian is singular. The description of the formalism is based on the works of Kay-Studer (KS) [38] and Bulla-Gesztesy (BG) [39], both using the self-adjoint extension method. The present method is based on the physics...
of the problem and one of his particularities is that it gives us the self-adjoint extension parameter for both bound and scattering scenarios. Recently, it has been applied for determination of bound states and scattering matrix for systems with curved surfaces [40], quantum deformations [16], and for AB-like systems [41–43]. Here, we address issues which have to do with the existence of a negative eigenvalue in the spin 1/2 AB spectrum and with the helicity nonconservation in the scattering. We also add a two-dimensional isotropic harmonic oscillator in the spin-1/2 AB problem and calculates the bound states and the self-adjoint extension parameter for this system.

The paper is organized as follows. In Sec. 2 we write the Hamiltonian of the spin-1/2 AB problem and derive the equation of motion that governs the dynamics of the particle. In Sec. 3 we present the KS and BG self-adjoint extension methods used in the formulation of the regularization method proposed here. The KS method has the advantage of yielding the self-adjoint extension parameter in terms of the physics of the problem, but it is not appropriate for dealing with scattering problems; on the other hand, the BG method is suitable to address both bound and scattering scenarios, with the disadvantage of allowing arbitrary self-adjoint extension parameters. Further, we also derive the expressions for the energy bound state, phase shift and the scattering matrix in terms of the physics of the problem. By combining the KS and BG methods, a relation between the self-adjoint extension parameter and the physical parameters of the problem is found. In Sec. 4 we apply the method for the spin-1/2 AB problem plus a two-dimensional isotropic harmonic oscillator. We derive the expression for the particle energy spectrum and analyze it in the limit case of the vanishing harmonic oscillator potential recasting the result of usual spin-1/2 AB problem in conical space. In Sec. 5 we present a brief conclusion.

2. The equation of motion

The idealized situation of a relativistic quantum particle in the presence of a cosmic string is an example of gravitational effect of topological origin, where a particle is transported along a closed curve around the cosmic string [8]. This situation corresponds to the gravitational analogue of the electromagnetic AB effect with the cosmic string replacing the flux tube [9–13]. Such effects are of purely topological origin rather than local. The bound state for the spinless AB effect around a cosmic string was addressed in [44]. The authors observed that the self-adjoint extension of the Hamiltonian of a particle moving around a shielded cosmic string gives rise to a gravitational analogue of the bound state AB effect. Here, our initial proposal is to analyze the spin-1/2 AB problem in the cosmic string spacetime with an internal magnetic field. The cosmic string background is described by the following metric in cylindrical coordinates $(t, r, \varphi, z)$:

$$ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 + dz^2,$$

(1)

with $-\infty < t < \infty$, $r \geq 0$ and $0 \leq \varphi < 2\pi$. The parameter $\alpha$ is related to the linear mass density $\tilde{m}$ of the string by $\alpha = 1 - 4\tilde{m} \hbar$ runs in the interval $(0, 1]$ and corresponds to a deficit angle $\gamma = 2\pi(1 - \alpha)$. The external gravitational field due to a cosmic string may be approximately described by a commonly called conical geometry. Usually, only the case $\alpha < 1$ is considered in cosmology, since $\alpha > 1$ corresponds to a negative mass density cosmic string. For $\alpha = 1$, the cone turns into a plane. The above metric has a cone-like singularity at $r = 0$ and the curvature tensor of this metric, considered as a distribution, is given by

$$R^{(2)}_{12} = R^{(2)}_{1} = R^{(2)}_{2} = 2\pi \frac{1 - \alpha}{\alpha} \delta^3(r),$$

(2)

where $\delta^3(r)$ is the two-dimensional $\delta$-function in flat space [45]. This implies a two-dimensional conical singularity symmetrical in the $z$-axis, which characterizes it as a linear defect.

In order to study the dynamics of the particle in a non-flat spacetime, we should include the spin connection in the differential operator and define the respective Dirac matrices in this manifold. The modified Dirac equation in the curved space reads [46] ($\hbar = c = 1$):

$$\left[i\gamma^\mu(\partial_\mu + \Gamma_\mu) - e\gamma^\nu A_\nu - M\right]\Psi = 0,$$

(3)

where $e$ is the charge, $M$ is mass of the particle, $\Psi$ is a four-component spinorial wave function, and $\Gamma_\mu$ is the spin connection given by

$$\Gamma_\mu = -\frac{1}{4} \epsilon^{(a)}(\alpha)\epsilon^{(b)}(\beta)\epsilon^{(c)}(\gamma)e_{(\alpha)(\beta)(\gamma)\mu},$$

(4)
and $\gamma^\mu = e^\mu_{(a)}(x)\gamma^{(a)}$ are the $\gamma$ matrices in the curved spacetime. We take the basis tetrad \([46–48]\),

$$
e^\mu_{(a)}(x) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \varphi & -\sin \varphi/\alpha & 0 \\
0 & \sin \varphi & \cos \varphi/\alpha & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad (5)
$$

(with $\alpha = 1$ giving the flat space-time) satisfying the condition

$$e^\mu_{(a)}e^\nu_{(b)}\eta^{(a)(b)} = g^{\mu \nu}, \quad (6)$$

with $g^{\mu \nu} = \text{diag}(−, +, +, +)$. For this conical spacetime the spin connection can be expressed by

$$\gamma^\mu \Gamma_\mu = -\frac{1}{2\alpha r} \gamma', \quad (7)$$

and

$$\gamma' = \cos \varphi \gamma^{(1)} + \sin \varphi \gamma^{(2)} = \begin{bmatrix} 0 & \sigma' \\
-\sigma' & 0 \end{bmatrix}. \quad (8)$$

Moreover the $\alpha$ matrices are now written as

$$\alpha^i = e^i_{(a)} \begin{bmatrix} 0 & \sigma^{(a)} \\
-\sigma^{(a)} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma^i \\
\sigma^i & 0 \end{bmatrix}, \quad (9)$$

where $\sigma^i = (\sigma', \sigma^\varphi, \sigma^z)$ are the Pauli matrices in cylindrical coordinates obtained from the basis tetrad (5).

For the specific tetrad basis used here, the spin connection is

$$\Gamma_\mu = (0, 0, \Gamma_\varphi, 0), \quad (10)$$

where the nonvanishing element given as

$$\Gamma_\varphi = i\frac{(1-\alpha)}{2} \Sigma^z, \quad (11)$$

with $\Sigma^z$ being the third component of the spin operator $\Sigma = (\Sigma^r, \Sigma^\varphi, \Sigma^z)$,

$$\Sigma^r = \begin{bmatrix} 0 & \sigma^r \\
\sigma^r & 0 \end{bmatrix}, \quad \Sigma^\varphi = \begin{bmatrix} 0 & \sigma^\varphi \\
\sigma^\varphi & 0 \end{bmatrix}, \quad \Sigma^z = \begin{bmatrix} \sigma^z & 0 \\
0 & \sigma^z \end{bmatrix}. \quad (12)$$

We are interested in the nonrelativistic limit of the Dirac equation, so it is convenient to express it in terms of a Hamiltonian formalism

$$\hat{H}\psi = \bar{E}\psi, \quad (13)$$

with

$$\hat{H} = \alpha' (-i\nabla_i - eA_i) - i\gamma^0 \gamma^\mu \Gamma_\mu + \beta M. \quad (14)$$

Exploiting the symmetry under $z$ translations, we can access the $(2+1)$-dimensional Dirac equation which follows from the decoupling of $(3+1)$-dimensional Dirac equation for the specialized case where $\partial_z = 0$ and $A_z = 0$, into two uncoupled two-component equations, such as implemented in Refs. [49–51]. The Dirac equation in $(2+1)$ dimensions reads

$$[\beta \gamma \cdot \Pi + \beta M] \psi = \bar{E}\psi, \quad (15)$$

where

$$\Pi = \frac{1}{i}(\nabla + \Gamma) - eA, \quad (16)$$

is the generalized momentum, $\psi$ is a two-component spinor, and the $(2+1)$ dimensional $\gamma$ matrices are given in terms of the Pauli matrices in cylindrical coordinates

$$\beta = \gamma^0 = \sigma^z, \quad \beta \gamma' = \sigma', \quad \beta \gamma^\varphi = s\sigma^\varphi, \quad (17)$$
where $s$ is twice the spin value, with $s = +1$ for spin “up” and $s = -1$ for spin “down”.

The magnetic flux tube in the background space described by the metric above considered is related to the magnetic field by

$$eB = e\nabla \times A = -\frac{\phi}{r} \frac{\delta(r)}{\alpha} \hat{z},$$

where $\phi = \Phi/\Phi_0$ is the flux parameter with $\Phi_0 = 2\pi\gamma/e$, and the vector potential in the Coulomb gauge is

$$eA = -\frac{\phi}{\alpha r} \hat{\phi}.$$

The choice (18) also gives the flux tube coinciding with the cosmic string and the $z$ axis.

The second order equation implied by (15) is obtained by applying the matrix operator $[M + \beta E - \gamma \cdot \Pi]_{\beta}$. The result is

$$(\hat{E}^2 - M^2)\psi = \left[\Pi^2 - es(\sigma \cdot B)\right]\psi = \left[\Pi^2 + \frac{\phi s}{\alpha}\frac{\delta(r)}{r}\right]\psi,$$

In the nonrelativistic limit

$$\hat{E} = M + E, \quad M \gg E,$$

we have the Schrödinger-Pauli equation for $\psi$

$$\left\{\frac{1}{\ell^2} \nabla^2 + \left(\frac{1 - \alpha}{2\alpha r^2} \phi \frac{\delta(r)}{r} \right) \left[\frac{\delta(r)}{r}\right] + \frac{\phi s}{\alpha} \frac{\delta(r)}{r}\right\} \psi = k^2 \psi,$$

where $k^2 = 2ME$ and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2},$$

is the Laplacian operator in the conical space.

Before we go on to a calculation of the bound states and scattering, some remarks on Hamiltonian in (22) are in order. If we do not take into account the spin, the resulting Hamiltonian, in this case, is essentially self-adjoint and positive definite [52]. Therefore, its spectrum is $\mathbb{R}^+$, it is transitonally invariant and there is no bound states. The introduction of spin changes the situation completely. The singularity at the origin due to the spin is physically equivalent to extract this single point from the plane $\mathbb{R}^2$ and in this case the translational invariance is lost together with the self-adjointness. This fact has impressive consequences in the spectrum of the system [53]. Since we are effectively excluding a portion of space accessible to the particle we must guarantee that the Hamiltonian is self-adjoint in the region of the motion, as is necessary for the generator of time evolution of the wave function. The most adequate approach for studying this scenario is the theory of self-adjoint extension of symmetric operators of von Neumann-Krein [35, 36, 54]. It yields a family of operators labeled by a real parameter. We shall see that for all values of this parameter there is an additional scattering amplitude resulting from the interaction of the spin with the magnetic flux; if the parameter is negative there is a bound state with a negative eigenvalue. The existence of a negative eigenvalue in the spectrum can be considered rather unexpected, since the actions of suggest it as a positive definite operator. However, the positivity of such an operator does not just depend on its action, but also depend on its domain. Indeed, there are several works in the literature which use the self-adjoint extensions and claim the existence of such a bound state. For example, the works of Gerbert et al. [7, 55], Jackiw [56] (in this reference an equivalence between renormalization and self-adjointness is discussed), Voropaev et al. [57], Borda et al. [58, 59], Park et al. [60, 61] and Filgueiras et al. [62, 63], to cite few. In fact, the existence of this negative eigenvalue can be proved like shown by Albeverio et al. [35, 64]. Now, we can return to our main problem.

Making use of the underlying rotational symmetry expressed by the fact that $[\hat{H}, \hat{J}_z] = 0$, where

$$\hat{J}_z = -i\hbar \sigma_z + \frac{1}{2} \sigma^z,$$

is the total angular momentum operator in the $z$-direction, we decompose the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2)$ with respect to the angular momentum $\mathcal{H} = S_{\theta} \otimes S_{\phi}$, where $S_{\theta} = L^2(\mathbb{R}^+; drd\theta)$ and $S_{\phi} = L^2(S^1, d\phi)$, with $S^1$ denoting the unit
For $m + 1/2 = \pm 1/2, \pm 3/2, \ldots$, with $m \in \mathbb{Z}$. Inserting this into equation (22), we can extract the radial equation for $f_m(r)$

$$H f_m(r) = \frac{k^2}{r} f_m(r),$$

where

$$H = H_0 + \frac{\phi s \delta(r)}{\alpha r},$$

and

$$H_0 = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m + \phi + (1 - \alpha)/2)^2}{\alpha^2 r^2}.$$

The Hamiltonian in Eq. (27) governs the quantum dynamics of a spin-1/2 charged particle in the conical spacetime, with a magnetic field $\mathbf{B}$ along the z-axis, i.e., a spin-1/2 AB problem in the conical space. We note that in the case of flat space, $\alpha = 1$ (no spin connection), we recover the radial Hamiltonian for the usual spin-1/2 AB problem in Refs. [17, 60],

$$-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m + \phi)^2}{r^2} + \phi s \frac{\delta(r)}{r}.$$

For $\alpha \in (0, 1]$ we summarize the possible physical scenarios of obtaining scattering and bound states in Table 1, based on the signal of the $\delta$ function coupling constant in (27). Since we have two possibilities for achieving bound states and scattering, we will focus our attention first on the conditions giving bound states. Afterwards, only when we study the scattering problem we will take into account the other two conditions.

### Table 1: Summary for the physical scenarios based on the signal of the $\delta$ coupling constant for $\alpha \in (0, 1)$.

| $s$ | $\phi$ | $\phi s/\alpha$ | State               |
|-----|--------|-----------------|---------------------|
| +1  | > 0    | > 0             | Scattering          |
| −1  | < 0    | > 0             | Scattering          |
| +1  | < 0    | < 0             | Bound and Scattering|
| −1  | > 0    | < 0             | Bound and Scattering|

sphere in $\mathbb{R}^2$. So, it is possible to express the eigenfunctions of the two dimensional Hamiltonian in terms of the eigenfunctions of $\hat{J}_z$:

$$\psi(r, \varphi) = \begin{pmatrix} f_m(r) e^{im\varphi} \\ g_m(r) e^{i(m+1)\varphi} \end{pmatrix},$$

with $m + 1/2 = \pm 1/2, \pm 3/2, \ldots$, with $m \in \mathbb{Z}$. Using the unitary operator $V : L^2(\mathbb{R}^*, rdr) \to L^2(\mathbb{R}^*, dr)$, given by $(V \xi)(r) = r^{1/2} \xi(r)$, the operator $H_0$ becomes

$$\hat{H}_0 = V H_0 V^{-1} = -\frac{d^2}{dr^2} + \left(\frac{(m + \phi + (1 - \alpha)/2)^2}{\alpha^2 r^2} - \frac{1}{4}\right) \frac{1}{r^2}.$$

By standard results, the symmetric radial operator $\hat{H}_0$ is essentially self-adjoint for $|m + \phi + (1 - \alpha)/2|/\alpha > 1$. For those values of $m$ fulfilling $|m + \phi + (1 - \alpha)/2|/\alpha < 1$ it is not essentially self-adjoint, admitting an one-parameter family
of self-adjoint extensions [36]. In order to proceed to the self-adjoint extensions of \( H_0 \), we must find its deficiency subspaces, \( N_\pm \), which are defined by
\[
N_\pm = \{ \xi_\pm \in \mathcal{D}(H_0^\dagger), H_0^\dagger \xi_\pm = z_\pm \xi_\pm, \Im z_\pm \geq 0 \},
\]
with dimensions \( n_\pm = \dim N_\pm \), which are called deficiency indices of \( H_0 \) [36]. A necessary and sufficient condition for \( H_0 \) being essentially self-adjoint is that \( n_+ = n_- = 0 \). On the other hand, if \( n_+ = n_- \geq 1 \) the operator \( H_0 \) has an infinite number of self-adjoint extensions parametrized by the unitary operators \( U : N_+ \to N_- \). Therefore, according to the von Neumann-Krein theory of self-adjoint extensions, the domain of \( H_0^\dagger \) is given by
\[
\mathcal{D}(H_0^\dagger) = \mathcal{D}(H_0) \oplus N_+ \oplus N_-.
\]
One observes that even if the operator is Hermitian \( H_0^\dagger = H_0 \), its domains could be different. The self-adjoint extension approach consists, essentially, in extending the domain \( \mathcal{D}(H_0) \) to match \( \mathcal{D}(H_0^\dagger) \) in (34), turning \( H_0 \) a self-adjoint operator. We then have
\[
\mathcal{D}(H_{0,0}) = \mathcal{D}(H_0^\dagger) = \mathcal{D}(H_0) \oplus N_+ \oplus N_-.
\]
where \( H_{0,0} \) represents the self-adjoint extension of \( H_0 \) parametrized by \( \eta \in [0, 2\pi) \).

In what follows, to characterize the one parameter family of self-adjoint extension of \( H_0 \), we will use the KS [38] and the BG [39] approaches, both based on boundary conditions. In the KS approach, the boundary condition is a match of the logarithmic derivatives of the zero-energy solutions for Eq. (26) and the solutions for the problem \( H_0 \) plus self-adjoint extension. In the BG approach, the boundary condition is a mathematical limit allowing divergent solutions for the Hamiltonian (28) at isolated points, provided they remain square integrable.

### 3.1. KS method

In this section, we employ the KS approach to find the bound states for the Hamiltonian in Eq. (27). Following [38], we temporarily forget the \( \delta \)-function potential and find the boundary conditions allowed for \( H_0 \). For this intent, we substitute the problem in Eq. (26) by the eigenvalue equation for \( H_0 \),
\[
H_0 f_\rho = k_0^2 f_\rho,
\]
plus self-adjoint extensions. Here, \( f_\rho \) is labeled by the parameter \( \rho \) of the self-adjoint extension, which is related to the behavior of the wave function at the origin. In order for the \( H_0 \) to be a self-adjoint operator in \( \mathcal{D} \), its domain of definition has to be extended by the deficiency subspace, which is spanned by the solutions of the eigenvalue equation (cf. Eq. (33))
\[
H_0^\dagger f_\pm = \pm i k_0^2 f_\pm,
\]
where \( k_0 \in \mathbb{R} \) is introduced for dimensional reasons. Since \( H_0^\dagger = H_0 \), the only square integrable functions which are solutions of Eq. (37) are the modified Bessel functions of second kind,
\[
f_\pm = K_{[m+\phi+(1-\alpha)/2]/\alpha}(\sqrt{-1}k_0 r),
\]
with \( \Im \sqrt{-1} > 0 \). These functions are square integrable only in the range \( [m + \phi + (1 - \alpha)/2]/\alpha \in (-1, 1) \), for which \( H_0 \) is not self-adjoint. The dimension of such deficiency subspace is \( (n_+, n_-) = (1, 1) \). So, we have two situations for \( [m + \phi + (1 - \alpha)/2]/\alpha \), i.e.,
\[
-1 < [m + \phi + (1 - \alpha)/2]/\alpha < 0,
\]
\[
0 < [m + \phi + (1 - \alpha)/2]/\alpha < 1,
\]
and to treat these two situations simultaneously, it is more convenient to use
\[
f_\pm = K_{[m+\phi+(1-\alpha)/2]/\alpha}(\sqrt{-1}k_0 r).
\]
Thus, \( \mathcal{D}(H_{p,0}) \) in \( L^2(\mathbb{R}^+, rdr) \) is given by the set of functions [36]

\[
f_\rho(r) = f_m(r) + C \left[ K_{\mid m + \phi + (1 - \alpha)/2 \mid \alpha} (\sqrt{-ik_0} r) + e^{i \phi} K_{\mid m + \phi + (1 - \alpha)/2 \mid \alpha} (\sqrt{-ik_0} r) \right],
\]

where \( f_m(r), \) with \( f_m(0) = \int f_0 dr = 0 \) (\( f \equiv df/dr \)), is the regular wave function and the parameter \( \rho \in [0, 2\pi) \) represents a choice for the boundary condition. For each \( \rho \), we have a possible domain for \( H_0 \) and the physical situation is the factor that will determine the value of \( \rho \) [7, 40, 62, 63]. Thus, to find a fitting for \( \rho \) compatible with the physical situation, a physically motivated form for the magnetic field is preferable for the regularization of the \( \delta \)-function. This is accomplished by replacing (19) with [17, 18, 67, 68]

\[
e_A = \begin{cases} -\delta \hat{\phi}, & r > r_0 \\ 0, & r < r_0. \end{cases}
\]

This modification mathematically effects the replacement of idealized zero thickness filament by one of a finite very small radius \( r_0 \) smaller than the Compton wave length \( \lambda_C \) of the electron [58]. So one makes the replacement

\[
\frac{\delta(r)}{r} \to \frac{\delta(r - r_0)}{r_0}.
\]

Although the functional structure of \( \delta(r)/r \) and \( \delta(r - r_0)/r_0 \) are quite different, as discussed in [17], we are free to use any form of potential once that the specific details of the model (43) can be shown to be irrelevant provided that only the contribution is independent of angle and has no \( \delta \)-function contribution at the origin. It should be remarked that the \( \delta(r - r_0)/r_0 \) is one dimensional and well defined contrary to the two dimensional \( \delta(r)/r \).

Now, we are in the position to determine a fitting value for \( \rho \). To do so, we follow [38] and consider the zero-energy solutions \( f_0 \) and \( f_{\rho,0} \) for \( H \) with the regularization in (43) and \( H_0 \), respectively, i.e.,

\[
-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{[m + \phi + (1 - \alpha)/2]^2}{\alpha^2 r^2} + \frac{\phi s \delta(r - r_0)}{r_0} f_0 = 0,
\]

\[
-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{[m + \phi + (1 - \alpha)/2]^2}{\alpha^2 r^2} f_{\rho,0} = 0.
\]

The value of \( \rho \) is determined by the boundary condition

\[
\lim_{r_0 \to 0^+} r_0 \frac{f_0}{f_0_{r = r_0}} = \lim_{r_0 \to 0^+} r_0 \frac{f_{\rho,0}}{f_{\rho,0}_{r = r_0}}.
\]

The left-hand side of this equation can be achieved integrating (44) from 0 to \( r_0 \),

\[
\int_0^{r_0} \frac{1}{r} \frac{d}{dr} \left( \frac{d f_0(r)}{dr} \right) r dr = \frac{\phi s}{\alpha} \int_0^{r_0} f_0(r) \delta(r - r_0) r dr + \int_0^{r_0} \frac{[m + \phi + (1 - \alpha)/2]^2}{\alpha^2} \int_0^{r_0} f_0(r) r^2 dr.
\]

From (44), the behavior of \( f_0 \) as \( r \to 0 \) is \( f_0 \sim r^{m + \phi + (1 - \alpha)/2} \), so we find

\[
\int_0^{r_0} \frac{f_0(r)}{r} r^2 dr \approx \int_0^{r_0} r^{m + \phi + (1 - \alpha)/2} \alpha^2 dr \to 0,
\]

as \( r_0 \to 0^+ \). So, we have

\[
\lim_{r_0 \to 0^+} r_0 \frac{f_0}{f_0_{r = r_0}} = \frac{\phi s}{\alpha}.
\]

The right-hand side of Eq. (46) is calculated using the asymptotic representation for \( K_\nu(z) \) in the limit \( z \to 0 \), given by

\[
K_\nu(z) \sim \frac{\pi}{2 \sin(\pi\nu)} \left[ \frac{z^{\nu}}{2^{\nu} \Gamma(1 - \nu)} - \frac{z^{-\nu}}{2^{\nu} \Gamma(1 + \nu)} \right],
\]
in Eq. (41). Thus, we arrive at
\[
\lim_{\nu \to 0} \frac{f_{\nu,0}}{f_{\nu,0} | r = \nu} = \lim_{\nu \to 0} \frac{\hat{\Omega}_{\nu}(r)}{\hat{\Omega}_{\nu}(r) | r = \nu},
\]
where
\[
\hat{\Omega}_{\nu}(r) = \left( \left( \sqrt{-i k_0 r} \right)^{\nu} \left( \sqrt{-i k_0 r} \right)^{\nu+1-2/\alpha} \right) - \left( \left( \sqrt{-i k_0 r} \right)^{\nu} \left( \sqrt{-i k_0 r} \right)^{\nu+1-2/\alpha} \right) + e^{i \theta} \left( \left( \sqrt{-i k_0 r} \right)^{\nu} \left( \sqrt{-i k_0 r} \right)^{\nu+1-2/\alpha} \right)
\]
\[
= \left( \left( \sqrt{-i k_0 r} \right)^{\nu} \left( \sqrt{-i k_0 r} \right)^{\nu+1-2/\alpha} \right) + e^{i \theta} \left( \left( \sqrt{-i k_0 r} \right)^{\nu} \left( \sqrt{-i k_0 r} \right)^{\nu+1-2/\alpha} \right)
\]
which gives us the parameter \( \alpha \) in terms of the physics of the problem, i.e., the correct behavior of the wave functions when \( \nu \to 0 \).

We now determine the bound states for \( H_0 \) and using (53) the bound state for \( H \) will be determined. So, we write Eq. (36) for the bound state. In the present system the energy of a bound state has to be negative, so that \( k \) is a pure imaginary, \( k = i \kappa \), with \( \kappa = \sqrt{-2M E_0} \), where \( E_0 < 0 \) is the bound state energy. Then, with the substitution \( k \to i \kappa \) we have
\[
\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( \frac{|m + \phi + (1 - \alpha)/2|}{\alpha^2 r^2} + \kappa^2 \right) f_\nu(r) = 0,
\]
The above equation is the modified Bessel equation whose general solution is given by
\[
f_\nu(r) = K_{|m + \phi + (1 - \alpha)/2|/\alpha} \left( r \sqrt{-2M E_0} \right).
\]
Since these solutions belong to \( D(H_\nu), \) it is of the form (41) for some \( \rho \) selected from the physics of the problem. So, we substitute (55) into (41) and compute \( \lim_{\nu \to 0} \frac{f_{\rho,0}}{f_{\rho,0} | r = \rho} \) using (50). After a straightforward calculation, we have the relation
\[
\frac{\phi s + |m + \phi + (1 - \alpha)/2|}{\phi s - |m + \phi + (1 - \alpha)/2|} \Gamma(1 + |m + \phi + (1 - \alpha)/2|/2) \Gamma(1 - |m + \phi + (1 - \alpha)/2|/2) = 0,
\]
Solving the above equation for \( E_\rho \), we find the sought energy spectrum
\[
E_\rho = \frac{-2}{M \rho^2} \left( \phi s + |m + \phi + (1 - \alpha)/2| \right) \Gamma(1 + |m + \phi + (1 - \alpha)/2|/2) \Gamma(1 - |m + \phi + (1 - \alpha)/2|/2).
\]
Notice that there is no arbitrary parameter in the above equation. Moreover, to ensure that the energy is a real number, we must have
\[
\frac{\phi s + |m + \phi + (1 - \alpha)/2|}{\phi s - |m + \phi + (1 - \alpha)/2|} \Gamma(1 + |m + \phi + (1 - \alpha)/2|/2) \Gamma(1 - |m + \phi + (1 - \alpha)/2|/2) > 0.
\]
This inequality is satisfied if \(|\phi s| \geq |m + \phi + (1 - \alpha)/2| \) and due to \(|m + \phi + (1 - \alpha)/2| \leq 1\) it is sufficient to consider \(|\phi s| \geq 1\). As shown in Table 1, a necessary condition for a \( \delta \) function to generate an attractive potential, which is able to support bound states, is that the coupling constant \((\phi s/\alpha)\) must be negative. Thus, once that \( \alpha \in (0, 1]\), the existence of bound states requires
\[
\phi s \leq -1.
\]
So, it seems that we must have \( \phi s < 0 \), in such way that the flux and the spin must be antiparallel, and must have a minimum value for the \(|\phi s|\).

---

1 In Ref. [37] the expression used for the asymptotic representation of \( K_\nu(z) \) it was \( K_\nu(z) = \frac{\pi}{\sin \nu \pi} \left[ \frac{\nu}{1 + \nu} - \frac{\nu}{1 + \nu} \right] \), i.e., the signal of the second term within the brackets must be minus as in Eq. (50).
3.2. BG method

The KS approach used in the previous section gives us the energy spectrum in terms of the physics of the problem, but it is not appropriate for dealing with scattering problems. Furthermore, it selects the value for the parameter $\rho$. On the other hand, the approach in [39] is suitable to address both bound and scattering scenarios, with the disadvantage of allowing arbitrary self-adjoint extension parameters. By comparing the results of these two approaches for bound states, the self-adjoint extension parameter can be determined in terms of the physics of the problem. Here, all self-adjoint extensions $H_{0, \lambda}$ of $H_0$ are parametrized by the boundary condition at the origin [35, 39],

$$f^{(0)} = \lambda_m f^{(1)},$$

with

$$f^{(0)} = \lim_{r \to 0^+} r^{m+\phi+(1-\alpha)/2}/\alpha f_m(r),$$

$$f^{(1)} = \lim_{r \to 0^+} r^{m+\phi+(1-\alpha)/2}/\alpha \left[ f_m(r) - \frac{1}{r^{m+\phi+(1-\alpha)/2}/\alpha} \right],$$

where $\lambda_m$ is the self-adjoint extension parameter. In [35] it is shown that there is a relation between the self-adjoint extension parameter $\lambda_m$ and the parameter $\rho$ used in the previous section. The parameter $\rho$ is associated with the mapping of deficiency subspaces and extend the domain of operator to make it self-adjoint, being a mathematical parameter. The self-adjoint extension parameter $\lambda_m$ have a physical interpretation, it represents the scattering length [69] of $H_{0, \lambda_m}$ [35]. For $\lambda_m = 0$ we have the free Hamiltonian (without the $\delta$-function) with regular wave functions at the origin and for $\lambda_m \neq 0$ the boundary condition in (60) permit a $r^{-|m+\phi+(1-\alpha)/2|/\alpha}$ singularity in the wave functions at the origin.

3.2.1. Bound states

Now we use the BG approach to determine the bound states for $H$ and in the end compare with the result obtained with the KS approach. This allows us to determine the self-adjoint extension parameter in terms of the physics of the problem.

We begin by rewriting the solutions in another form. The solutions for

$$H_0 f_m(r) = k^2 f_m(r),$$

for $r \neq 0$, taking into account both cases in [39] simultaneously, can be written in terms of the confluent hypergeometric function of the first kind $\text{M}(a, b, z)$ as

$$f_m(r) = a_m e^{ikr} (2ikr)^{m+\phi+(1-\alpha)/2}/\alpha \text{M}\left(\frac{1}{2}, \frac{|m+\phi+(1-\alpha)/2|}{\alpha}, 1 + 2\frac{|m+\phi+(1-\alpha)/2|}{\alpha}, 2ikr\right)$$

$$+ b_m e^{-ikr} (2ikr)^{-|m+\phi+(1-\alpha)/2|}/\alpha \text{M}\left(\frac{1}{2}, \frac{|m+\phi+(1-\alpha)/2|}{\alpha}, 1 - 2\frac{|m+\phi+(1-\alpha)/2|}{\alpha}, 2ikr\right),$$

where $a_m, b_m$ are the coefficients of the regular and irregular solutions, respectively. By implementing Eq. (63) into the boundary condition (60), we derive the following relation between the coefficients $a_m$ and $b_m$:

$$\lambda_m a_m = (2ik)^{-2|m+\phi+(1-\alpha)/2|/\alpha} b_m \left( 1 + \frac{\lambda_m k^2}{4(1-|m+\phi+(1-\alpha)/2|/\alpha)} \lim_{r \to 0^+} r^{2-2|m+\phi+(1-\alpha)/2|/\alpha} \right).$$

In the above equation, the coefficient of $b_m$ diverges as $\lim_{r \to 0^+} r^{2-2|m+\phi+(1-\alpha)/2|/\alpha}$, if $|m+\phi+(1-\alpha)/2|/\alpha \geq 1$. Thus, $b_m$ must be zero for $|m+\phi+(1-\alpha)/2|/\alpha \geq 1$, and the condition for the occurrence of a singular solution is $|m+\phi+(1-\alpha)/2|/\alpha < 1$. So, the presence of an irregular solution stems from the fact the operator is not self-adjoint for $|m+\phi+(1-\alpha)/2|/\alpha < 1$, and this irregular solution is associated with a self-adjoint extension of the operator $H_0$ [70, 71]. In other words, the self-adjoint extension essentially consists in including irregular solutions in $D(H_0)$, which allows to select an appropriate boundary condition for the problem.
The bound state wave function is obtained with the substitution $k \rightarrow ik$. So we have

$$f_m^b(r) = a_m e^{\alpha r} (-2kr)^{m+\phi+(1-\alpha)/2}/\alpha M \left(1 + m + \phi + (1 - \alpha)/2, 1 + 2m + \phi + (1 - \alpha)/2, -2kr\right)$$

$$+ b_m e^{\alpha r} (-2kr)^{-m+\phi+(1-\alpha)/2}/\alpha M \left(1 - m + \phi + (1 - \alpha)/2, 1 - 2m + \phi + (1 - \alpha)/2, -2kr\right).$$ (65)

In order to be a bound state $f_m^b(r)$ must vanish at large $r$, i.e., it must be normalizable. By using the asymptotic representation of $M(a, b, z)$ for $z \to \infty$,

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} z^{-\alpha - b} + \frac{\Gamma(b)}{\Gamma(b - a)} (-z)^{-a},$$ (66)

the normalizability condition yields the relation

$$\frac{b_m}{a_m} = -16^{m+\phi+(1-\alpha)/2}/\alpha \Gamma(\alpha),$$ (67)

From Eq. (64), for $|m + \phi + (1 - \alpha)/2|/\alpha < 1$ we have

$$\frac{b_m}{a_m} = \lambda_m (-2k)^{2m+\phi+(1-\alpha)/2}/\alpha.$$ (68)

Combining these two later equations, the bound state energy is determined,

$$E_b = -\frac{2}{M} \left[\frac{1}{\lambda_m} \Gamma(1 + |m + \phi + (1 - \alpha)/2|/\alpha) \Gamma(1 - |m + \phi + (1 - \alpha)/2|/\alpha) \right]^{a/2|m+\phi+(1-\alpha)/2|}.$$ (69)

This coincides with Eq. (3.13) of Ref. [60] for $\alpha = 1$, i.e., the spin-1/2 AB problem in flat space. Also, this coincides with Eq. (26) of Ref. [59] for the bound states energy for particles with an anomalous magnetic moment (with the replacement $\lambda \rightarrow 1/\lambda_m$ in that reference).

By comparing Eq. (69) with Eq. (57), we find

$$\frac{1}{\lambda_m} = -\frac{1}{\lambda_0} \left(\frac{\phi s + |m + \phi + (1 - \alpha)/2|}{\phi s - |m + \phi + (1 - \alpha)/2|}\right)$$ (70)

We have thus attained a relation between the self-adjoint extension parameter and the physical parameters of the problem. It should be mentioned that some relations involving the self-adjoint extension parameter and the $\delta$-function coupling constant were previously obtained by using Green’s function in Ref. [61] and the renormalization technique in Ref. [56], being both, however, deprived from a clear physical interpretation. Also, in Ref. [59] a relation between the self-adjoint extension parameter and the anomaly magnetic moment was found and it is commented that the dimension of the self-adjoint extension parameter is $r^{2m+\phi+(1-\alpha)/2}$ but does not show an explicit relation as found by us in Eq. (70).

In Ref. [58] the authors comment, based on a result of Aharonov and Casher [72] which states that in a cylindrical magnetic field with flux $\phi$ a charged particle with magnetic moment and gyromagnetic ratio $g = 2$ possesses $N$ ($N$ being the number of entire flux quanta in $\phi$) zero modes, i.e., normalizable states with zero binding energy, any additional attractive force which occurs for $g > 2$ turns the zero modes into bound states. This $g > 2$ value is related with the self-adjoint extension value, i.e., different values for the self-adjoint extension parameter corresponds to different values of the $g$ [58]. The explicit relation between the self-adjoint extension parameter and the $g$ will be subject of a future work.

Moreover, the bound state wave function is given by

$$f_m^b(r) = N_m K_{|m+\phi+(1-\alpha)/2|/a}(-\sqrt{-2ME_b} r),$$ (71)

where $N_m$ is a normalization constant and $E_b$ is given by (69).
3.2.2. Scattering

Once the bound energy problem has been examined, let us now analyze the AB scattering scenario. In this case, the boundary condition is again given by Eq. (60), but with the replacement \( \lambda_m \rightarrow \lambda_m' \), where \( \lambda_m' \) is the self-adjoint extension parameter for the scattering problem. In the scattering analysis it is more convenient to write the solution for Eq. (62) in terms of Bessel functions

\[
f_m(r) = c_m J_{|m+\phi+(1-\alpha)/2|/\alpha}(kr) + d_m Y_{|m+\phi+(1-\alpha)/2|/\alpha}(kr),
\]

with \( c_m \) and \( d_m \) being constants. Upon replacing \( f_m(r) \) in the boundary condition (60), we obtain

\[
\frac{c_m}{d_m} = \frac{[8k^{-|m+\phi+(1-\alpha)/2|/\alpha} - \lambda_m J_{|m+\phi+(1-\alpha)/2|/\alpha} + BDK^{-|m+\phi+(1-\alpha)/2|/\alpha} \lim_{r \to 0^+} r^{2-2|\alpha+\phi+(1-\alpha)/2|/\alpha}]}{\lambda_m J_{|m+\phi+(1-\alpha)/2|/\alpha} + BDK^{-|m+\phi+(1-\alpha)/2|/\alpha} \lim_{r \to 0^+} r^{2-2|\alpha+\phi+(1-\alpha)/2|/\alpha}},
\]

where

\[
\mathcal{A} = \frac{1}{2(|m+\phi+(1-\alpha)/2|/\alpha\Gamma(\alpha))},
\]

\[
\mathcal{B} = \frac{2^{2|m+\phi+(1-\alpha)/2|/\alpha} \Gamma(|m+\phi+(1-\alpha)/2|/\alpha)}{\pi},
\]

\[
\mathcal{C} = -\frac{\cos(\pi|m+\phi+(1-\alpha)/2|/\alpha) \Gamma(-|m+\phi+(1-\alpha)/2|/\alpha)}{\pi^{2|m+\phi+(1-\alpha)/2|/\alpha}},
\]

\[
\mathcal{D} = \frac{k^2}{4(1 - |m+\phi+(1-\alpha)/2|/\alpha)}.
\]

As in the bound state calculation, whenever \(|m+\phi+(1-\alpha)/2|/\alpha < 1\), we have \( d_m \neq 0 \); again, this means that there arises the contribution of the irregular solution \( Y_z(z) \) at the origin when the operator is not self-adjoint. Thus, for \(|m+\phi+(1-\alpha)/2|/\alpha < 1\), we obtain

\[
\frac{c_m}{d_m} = \frac{[8k^{-|m+\phi+(1-\alpha)/2|/\alpha} - \lambda_m J_{|m+\phi+(1-\alpha)/2|/\alpha}]}{\lambda_m J_{|m+\phi+(1-\alpha)/2|/\alpha}},
\]

and by substituting the values of \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) into above expression we find

\[
d_m = -\mu_m^{J_{\nu}}(k, \phi)c_m,
\]

where

\[
\mu_m^{J_{\nu}}(k, \phi) = \frac{\lambda_m J_{|m+\phi+(1-\alpha)/2|/\alpha} \Gamma(\alpha) \cos(\pi|m+\phi+(1-\alpha)/2|/\alpha) + 4^{\phi+(1-\alpha)/2/\alpha} |\Gamma(\alpha)|}{\lambda_m k^{2|m+\phi+(1-\alpha)/2|/\alpha} \Gamma(\alpha-1) \cos(\pi|m+\phi+(1-\alpha)/2|/\alpha) + 4^{\phi+(1-\alpha)/2/\alpha} |\Gamma(\alpha-1)|}.
\]

Since \( \delta \) is a short range potential, it follows that the behavior of \( f_m(r) \) for \( r \to \infty \) is given by [73]

\[
f_m(r) \sim \sqrt{\frac{2}{\pi|\alpha|}} \cos\left(kr - \frac{|m|\pi}{2} - \frac{\pi}{4} + \delta_m^{J_{\nu}}(k, \phi)\right),
\]

where \( \delta_m^{J_{\nu}}(k, \phi) \) is a scattering phase shift. The phase shift is a measure of the argument difference to the asymptotic behavior of the solution \( J_m(kr) \) of the radial free equation that is regular at the origin. By using the asymptotic behavior of \( J_z(z) \) and \( Y_z(z) \) given in [74] in (72) we obtain

\[
f_m(r) \sim c_m \sqrt{\frac{2}{\pi|\alpha|}} \left[ \cos\left(kr - \frac{|m|\pi}{2} - \frac{\pi}{4} + \theta_0\right) \right. - \left. \mu_m^{J_{\nu}}(k, \phi) \sin\left(kr - \frac{|m|\pi}{2} - \frac{\pi}{4} + \theta_0\right) \right].
\]

By comparing the above expression with Eq. (81), we have

\[
\cos\left(kr - \frac{|m|\pi}{2} - \frac{\pi}{4} + \theta_0\right) = \cos\left(kr - \frac{|m|\pi}{2} - \frac{\pi}{4} + \delta_m^{J_{\nu}}(k, \phi)\right).
\]
with $\theta_{\kappa_m}$ given as
\[ \cos \theta_{\kappa_m} = c_m, \quad \sin \theta_{\kappa_m} = \mu_{\kappa_m}^i (k, \phi). \] (84)

Therefore, Eq. (81) is satisfied if
\[ c_m = \left[ 1 + \left( \mu_{\kappa_m}^i (k, \phi) \right)^2 \right]^{-1/2}. \] (85)

Now, comparing the arguments of the cosines above, the following phase shift is achieved:
\[ \delta_s^m \delta_m(k, \phi) = \Delta_{AB}^m (\phi) + \theta_{\kappa_m}, \] (86)
where
\[ \Delta_{AB}^m (\phi) = \frac{\pi}{2} (|m| - |m + \phi|), \] (87)

is the usual phase shift of the AB scattering and
\[ \theta_{\kappa_m} = \arctan \left( \mu_{\kappa_m}^i (k, \phi) \right). \] (88)

Therefore, the scattering operator $S_{\kappa_m}^{\delta_s^m}$ (S matrix) for the self-adjoint extension is
\[ S_{\phi,m}^{\delta_s^m} = e^{2i\lambda_m^L (k, \phi)} e^{2i\theta_{\kappa_m}}, \] (89)
that is,
\[ S_{\phi,m}^{\delta_s^m} = e^{2i\lambda_m^L (\phi)} \left[ 1 + i \mu_{\kappa_m}^i (k, \phi) \right]. \] (90)

Using Eq. (80), we have
\[ S_{\phi,m}^{\lambda_m^L} = e^{2i\lambda_m^L (\phi)} \left[ \frac{\lambda_m^L}{\lambda_m^L} e^{2i\lambda_m^L (\phi)} \right]. \] (91)

Hence, for any value of the self-adjoint extension parameter $\lambda_m^L$, there is an additional scattering. If $\lambda_m^L = 0$, we achieve the corresponding result for the usual AB problem with Dirichlet boundary condition; in this case, we recover the expression for the scattering matrix found in Ref. [75], $S_{\phi,m}^{\delta_s^m} = e^{2i\lambda_m^L (\phi)}$. If we make $\lambda_m^L = \infty$, we get $S_{\phi,m}^{\delta_s^m} = e^{2i\lambda_m^L (\phi) + 2i|m + \phi + (1 - \alpha)/2|/\alpha}$.

In accordance with the general theory of scattering, the poles of the S matrix in the upper half of the complex plane [76] determine the positions of the bound states in the energy scale, Eq. (69). These poles occur when the denominator of Eq. (91) is equal to zero with the replacement $k \rightarrow ik$. So, we have
\[ \lambda_m^L (k)^{2m+\phi+(1-\alpha)/2}/(\alpha) e^{0m+\phi+(1-\alpha)/2/\alpha} + 4^{m+\phi+(1-\alpha)/2}/\alpha \Gamma^{(+)}. \] (92)

Solving this equation for $E_b$, we found
\[ E_b = -\frac{2M}{\lambda_m^L} \left[ \frac{1}{\lambda_m^L} \Gamma^{(1 + |m + \phi + (1 - \alpha)/2|/\alpha)} \right], \] (93)
for $\lambda_m^L < 0$. Hence, the poles of the scattering matrix only occur for negative values of the self-adjoint extension parameter. In this latter case, the scattering operator can be expressed in terms of the bound state energy
\[ S_{\phi,m}^{\lambda_m^L} = e^{2i\lambda_m^L (\phi)} \left[ \frac{\lambda_m^L}{\lambda_m^L} e^{2i\lambda_m^L (\phi)} - \frac{\kappa/k}{1 - \kappa/k} e^{2i\lambda_m^L (\phi)} \right]. \] (94)

By comparing Eq. (93) above with Eq. (69), we find $\lambda_m^L = \lambda_m$, with $\lambda_m$ given by Eq. (70), and the self-adjoint extension parameter for the scattering scenario being the same one as that for the bound state problem. This is a very interesting result first discussed in [37]. Thus, we also obtain the phase shift and the scattering matrix in terms of physics of the problem.
The scattering amplitude \( f_\theta(k, \varphi) \) can be obtained using the standard methods of scattering theory, namely

\[
f_\theta^a(k, \varphi) = \frac{1}{\sqrt{2\pi i k}} \sum_{m=0}^{\infty} \left( S_m^a(k, \phi) - 1 \right) e^{im\varphi} = \frac{1}{\sqrt{2\pi i k}} \left\{ \sum_{|m+\phi+(1-\alpha)/2|/\alpha \geq 1} \left( e^{2i\lambda_m^a(\phi)} - 1 \right) e^{im\varphi} + \sum_{|m+\phi+(1-\alpha)/2|/\alpha < 1} \left( e^{2i\lambda_m^a(\phi)} \frac{1 + i\mu_m^a(k, \phi)}{1 - i\mu_m^a(k, \phi)} - 1 \right) e^{im\varphi} \right\}. \tag{95}\]

For the special case of \( \alpha = 1 \) (flat space) and \( \phi = 0 \) (zero magnetic flux) we have \( f_0^a(k, \varphi) = 0 \), as it should be. In the above equation we can see that it differs from the usual AB scattering amplitude off a thin solenoid because its energy dependence. As Goldhaber [77] observed, since the only length scale in the nonrelativistic problem is set by 1/k, it follows that the scattering amplitude would be a function of the angle alone, multiplied by 1/k. This is the manifestation of the helicity conservation. So, the inevitable failure of helicity conservation expressed in Eq. (95) shows that the singularity must lead to inconsistencies if the Hamiltonian and the helicity operator, \( \hat{h} = \Sigma \cdot \Pi \), are treated as well as well-defined operators whose commutation away from the singularity implies commutation everywhere [78–81]. After separation of the variables used in (25), the helicity operator is

\[
\hat{h} = \begin{cases} 
0 & \text{if } |m + \phi + (1-\alpha)/2|/\alpha < 1, \\
-i \left( \frac{\partial}{\partial r} - s \frac{m + \phi + (1-\alpha)/2}{\alpha} \right) & \text{if } |m + \phi + (1-\alpha)/2|/\alpha \geq 1, \\
0 & \text{if } |m + \phi + (1-\alpha)/2|/\alpha = 1, 
\end{cases}
\tag{96}
\]

This operator suffers from the same issue as the Hamiltonian operator in the interval \( |m + \phi + (1-\alpha)/2|/\alpha < 1 \), i.e. it is not self-adjoint [82, 83]. Defined on a finite interval \([0, L]\), \( \hat{h} \) can be interpreted as a self-adjoint operator on functions satisfying \( \xi(L) = e^{i\theta} \xi(0) \). However, because the helicity operator must be defined on an infinite interval \([0, \infty)\), \( \hat{h} \) has no self-adjoint extension [84], and consequently need not be conserved, and the helicity can leak at the origin [77, 79].

As already commented at the end of Section 3.2.1, this result could be compared with those obtained in Ref. [59] where the self-adjoint extension parameter was obtained as a function of anomaly of the magnetic moment. In an idealized version of \( g - 2 \) experiment, change in the helicity in a magnetic field becomes a measure of the departure of the gyromagnetic ratio of the electron or muon from the Dirac value of \( 2e/2M \) [85]. For vanishing of \( g - 2 \) there could be no change of helicity even if the magnetic field were inhomogeneous on a very short length scale. So, once again, different values for the self-adjoint extension is related to different values of the \( g \).

4. The spin 1/2 AB problem plus a two-dimensional harmonic oscillator

In this section, an application of our method is presented. We address the spin 1/2 AB problem in conical space plus a two dimensional harmonic oscillator. After including the harmonic oscillator (HO) potential and by using the angular momentum decomposition,

\[
\Phi(r, \varphi) = \begin{pmatrix} \chi_m(r) e^{im\varphi} \\ \zeta_m(r) e^{im+1}\varphi \end{pmatrix}, \tag{97}
\]

the radial equation for \( \chi_m(r) \) becomes

\[
H\chi_m(r) = k^2 \chi_m(r), \tag{98}
\]

where

\[
H = H_0 + M^2 \omega^2 r^2 + \frac{\phi s \delta(r)}{\alpha} r \tag{99}\]

with \( \omega \) the angular frequency, and \( H_0 \) given by (28). In order to have a more detailed analysis of this problem, we will first examine the motion of the particle considering two cases (i) excluding the \( r = 0 \) region and (ii) including the \( r = 0 \) region afterwards. At the end, we compare with some results in the literature.
4.1. Solution of the problem excluding \( r = 0 \) region

In this case, the Hamiltonian (99) does not include the delta function potential. By directly solving (98) we obtain [74]

\[
\chi_m(r) = a_m(M\omega)^{1/2+[m+\phi+(1-\alpha)/2]/2\alpha} r^{[m+\phi+(1-\alpha)/2]/\alpha} e^{-M\omega r^2/2} U\left(d, 1 + \frac{m + \phi + (1 - \alpha)/2}{\alpha}, M\omega r^2\right)
\]

\[
+ b_m(M\omega)^{1/2+[m+\phi+(1-\alpha)/2]/2\alpha} r^{[m+\phi+(1-\alpha)/2]/\alpha} e^{-M\omega r^2/2} U\left(d, 1 + \frac{m + \phi + (1 - \alpha)/2}{\alpha}, M\omega r^2\right),
\]

where

\[
d = \frac{1}{2} \left(1 + \frac{m + \phi + (1 - \alpha)/2}{\alpha}\right) - \frac{E}{2\omega}.
\]

(101)

\(U(a, b, z)\) is the confluent hypergeometric function of the second kind, and \(a_m, b_m\) are constants. However, as only \(M(a, b, z)\) is regular at the origin, it should be imposed \(b_m = 0\). Moreover, if \(d\) is 0 or a negative integer the series terminates and the hypergeometric function becomes a polynomial of degree \(n\) [74]. This condition guarantees that the confluent hypergeometric function is regular at the origin, which is essential for the treatment of the physical system since the region of interest is that around the flux tube. Therefore, the series in (100) must converge if we consider that \(d = -n, n \in \mathbb{Z}^+\), with \(\mathbb{Z}^+\) denoting the set of the nonnegative integers. This condition also guarantees the normalizability of the wave function. So, using this condition, we obtain the discrete values for the energy whose expression is given by

\[
E_n = \left(2n + 1 + \frac{[m + \phi + (1 - \alpha)/2]}{\alpha}\right)\omega, \quad n = 0, 1, 2, \ldots .
\]

(102)

The bound state wave function is given by

\[
\chi_m^b(r) = N_m (M\omega)^{1/2+[m+\phi+(1-\alpha)/2]/2\alpha} r^{[m+\phi+(1-\alpha)/2]/\alpha} e^{-M\omega r^2/2} U\left(-n, 1 + \frac{m + \phi + (1 - \alpha)/2}{\alpha}, M\omega r^2\right),
\]

(103)

with \(N_m\) a normalization constant. Notice that in Eq. (102), \([m + \phi + (1 - \alpha)/2]/\alpha\) can assume any value. However, we will see that this condition is no longer true when we include the \(\delta\) function. Next to study the motion of the particle in all space, including the \(r = 0\) region, the self-adjoint extension approach is invoked.

4.2. Solution including the \( r = 0 \) region

In this case, the dynamics includes the \(\delta\) function. So, we follow the procedure outlined in Sec. 3.1 to find the bound states for the system. Like before we need to find all the self-adjoint extension for the operator \(H_0 + M^2\omega^2 r^2\). So, we substitute the problem in Eq. (100) by

\[
[H_0 + M^2\omega^2 r^2] \chi_\theta(r) = k^2 \chi_\theta(r),
\]

(104)

plus self-adjoint extensions, with \(\chi_\theta\) labeled by a parameter \(\theta\). The solution to this equation is given in (100). However, the only square integrable function is \(U(d, 1 + [m + \phi + (1 - \alpha)/2]/\alpha, M\omega r^2)\). Then, this implies that \(a_m = 0\) in Eq. (100), and we have

\[
\chi_\theta(r) = (M\omega)^{1/2+[m+\phi+(1-\alpha)/2]/2\alpha} r^{[m+\phi+(1-\alpha)/2]/\alpha} e^{-M\omega r^2/2} U\left(d, 1 + \frac{m + \phi + (1 - \alpha)/2}{\alpha}, M\omega r^2\right).
\]

(105)

In order to guarantee that \(\chi(r) \in L^2(\mathbb{R}, r dr)\), it is advisable to study their behavior as \(r \to 0\), which implies analyzing the possible self-adjoint extensions. Now, to construct the self-adjoint extensions, we must find the deficiency subspaces,

\[
[H_0 + M^2\omega^2 r^2] \chi_\pm(r) = \pm ik_0^2 \chi_\pm(r).
\]

(106)

The solution to this equation is

\[
\chi_\pm(r) = r^{[m+\phi+(1-\alpha)/2]/\alpha} e^{-M\omega r^2/2} U\left(d_\pm, 1 + \frac{m + \phi + (1 - \alpha)/2}{\alpha}, M\omega r^2\right),
\]

(107)
where
\[ d_\pm = \frac{1}{2} \left( 1 + \frac{m + \phi + (1 - \alpha)/2}{\alpha} \right) \mp \frac{i k_0}{2 \omega} \] (108)

Now considering the asymptotic behavior of \( U(a, b, z) \) as \( z \to 0 \) [74], let us find under which condition the term,
\[ \int |\chi_z(r)|^2 r dr, \] (109)
has a finite contribution near the origin region. Using Eq. (107) we found
\[ \lim_{\alpha \to 0} |\chi_z(r)|^2 E^{1 + 2|m + \phi + (1 - \alpha)/2|/\alpha} \to |\mathcal{A}_1| E^{1 + 2|m + \phi + (1 - \alpha)/2|/\alpha} + |\mathcal{A}_2| E^{1 + 2|m + \phi + (1 - \alpha)/2|/\alpha}, \] (110)
where \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are constants. Studying Eq. (110), we see that \( \chi_z(r) \) is square-integrable only for \([m + \phi + (1 - \alpha)/2]/\alpha \in (-1, 1)\). In this case, since \( \mathcal{N}_+ \) is expanded by \( \chi_z(r) \) only, we have that its dimension is \( n_+ = 1 \). The same applies to \( \mathcal{N}_- \) and \( \chi_z(r) \), resulting in \( n_- = 1 \). Then, the Hilbert space, for both cases of Eq. (39), contains vectors of the form
\[ \chi_0(r) = \chi_m(r) + c \left\{ \rho_{m + \phi + (1 - \alpha)/2/\alpha} e^{-M \omega^2 r^2/2} U \left( d_+, 1 + \frac{|m + \phi + (1 - \alpha)/2|}{\alpha}, M \omega^2 \right) \right. \]
\[ + \left. e^{\theta} \rho_{m + \phi + (1 - \alpha)/2/\alpha} e^{-M \omega^2 r^2/2} U \left( d_-, 1 + \frac{|m + \phi + (1 - \alpha)/2|}{\alpha}, M \omega^2 \right) \right\}, \] (111)
where \( c \) is an arbitrary complex number, \( \chi_m(0) = \dot{\chi}_m(0) = 0 \) and \( \chi_m(r) \in L^2(\mathbb{R}^+, r dr) \). For a range of \( \theta \), the behavior of the wave functions (111) was addressed in [86]. The boundary condition at the origin will select the value of this parameter. The difference here is the presence of the harmonic term. However, this harmonic term does not contribute to the BG logarithmic derivative boundary condition (cf. Eq. (46)), since the integration of the harmonic vanishes as \( r_0 \to 0^+ \). After this identification, proceeding in an analogous way we did in the Section 3.1, it is found that the bound state energy is implicitly determined by the equation
\[ \frac{\Gamma(1 + 2|m + \phi + (1 - \alpha)/2|/2\alpha - E_b/2\omega)}{\Gamma(1 + 2|m + \phi + (1 - \alpha)/2|/2\alpha - E_b/2\omega)} = -\frac{1}{\rho_{m + \phi + (1 - \alpha)/2/\alpha} \rho_{m + \phi + (1 - \alpha)/2/\alpha}} \left( \frac{\phi s + |m + \phi + (1 - \alpha)/2|}{\phi s - |m + \phi + (1 - \alpha)/2|} \right) \]
\[ \times \frac{\Gamma(1 + |m + \phi + (1 - \alpha)/2|/\alpha)}{\Gamma(1 - |m + \phi + (1 - \alpha)/2|/\alpha)}. \] (112)
The above expression is too complicated to evaluate the bound state energy, but its limiting features are interesting. If we take limit \( r_0 \to 0 \) in this expression, the bound state energy are determined by the poles of the gamma functions, i.e.,
\[ -1 < \frac{m + \phi + (1 - \alpha)/2}{\alpha} < 0, \quad E_b = \left( 2n + 1 + \frac{|m + \phi + (1 - \alpha)/2|}{\alpha} \right) \omega, \] (113)
\[ 0 < \frac{m + \phi + (1 - \alpha)/2}{\alpha} < 1, \quad E_b = \left( 2n + 1 - \frac{|m + \phi + (1 - \alpha)/2|}{\alpha} \right) \omega, \] (114)
or
\[ E_b = \left( 2n + 1 \pm \frac{|m + \phi + (1 - \alpha)/2|}{\alpha} \right) \omega, \quad n = 0, 1, 2, \ldots. \] (115)
The + (−) sign refers to solutions which are regular (irregular) at the origin. This result coincide with the Eq. (1) of Ref. [87], for the special case of \( \alpha = 1 \). Another interesting case is that of vanishing HO potential. This is achieved using the asymptotic behavior of the ratio of gamma functions for \( \omega \to 0 \) [88],
\[ \frac{\Gamma(1 + 2|m + \phi + (1 - \alpha)/2|/2\alpha - E/2\omega)}{\Gamma(1 - 2|m + \phi + (1 - \alpha)/2|/2\alpha - E/2\omega)} \sim \left( \frac{E}{2\omega} \right)^{|m + \phi + (1 - \alpha)/2|/\alpha}, \] (116)
which holds for \( E < 0 \) and this is the necessary condition for the usual AB system to have a bound state. Using this limit in the Eq. (112), one finds

\[
E_b = -\frac{2}{Mr_0^2} \left( \frac{\phi s + |m + \phi + (1 - \alpha)/2|}{\phi s - |m + \phi + (1 - \alpha)/2|} \right) \frac{\Gamma(1 + |m + \phi + (1 - \alpha)/2|/\alpha)}{\Gamma(1 - |m + \phi + (1 - \alpha)/2|/\alpha)}^{m/\alpha},
\]

in agreement with the result obtained in Eq. (57). Thus, in the limit of vanishing harmonic oscillator, we recover the usual AB problem in conical space, as it should be.

Now we have to remark that this result contains a subtlety that must be interpreted as follows: the presence of the singularity in the problem establishes the range \(|m + \phi + (1 - \alpha)/2|/\alpha < 1\). If we ignore the singularity and impose that the wave function is regular at the origin (\(\chi_m(r) \equiv \chi_m(r) \equiv 0\)), we achieve the same spectrum of Eq. (115), but with \(|m + \phi + (1 - \alpha)/2|/\alpha\) assuming any value [89–91]. In this sense the self-adjoint extension prevents us from obtaining a spectrum incompatible with the singular nature of the Hamiltonian when we take into account the singular \(\delta\) function [92, 93]. We have to take into account that the true boundary condition is that the wave function must be square-integrable through all space, regardless it is irregular or regular at the origin [38, 93].

4.3. Determination of self-adjoint extension parameter

In this section the self-adjoint extension parameter will be determined in terms of the physics of the problem. For our intent, it is more convenient to write the solution in Eq. (100) for \(r \neq 0\) solely in terms of the confluent hypergeometric function \(M(a, b, z)\), as

\[
\chi_m(r) = a_m(M\omega)^{1/2 + |m + \phi + (1 - \alpha)/2|/2\alpha} M(d, 1 - |m + \phi + (1 - \alpha)/2|/\alpha) e^{-M\omega r^2/2} M_{\alpha}^\alpha\left(d, 1 + |m + \phi + (1 - \alpha)/2|/\alpha, M\omega r^2\right)
\]

\[
+ b_m(M\omega)^{1/2 - |m + \phi + (1 - \alpha)/2|/2\alpha} M(d, 1 - |m + \phi + (1 - \alpha)/2|/\alpha) e^{-M\omega r^2/2} M_{\alpha}^\alpha\left(d, 1 + |m + \phi + (1 - \alpha)/2|/\alpha, M\omega r^2\right),
\]

where \(a_m, b_m\) are the coefficients of the regular and singular solutions, respectively. By implementing Eq. (118) into the boundary condition (60), we derive the following relation between the coefficients \(a_m\) and \(b_m\):

\[
\lambda_m a_m(M\omega)^{|m + \phi + (1 - \alpha)/2|/\alpha} = b_m \left(1 - \frac{\lambda_m E}{4(1 - |m + \phi + (1 - \alpha)/2|/\alpha)} \lim_{r \to 0} r^{2 - 2|m + \phi + (1 - \alpha)/2|/\alpha}\right),
\]

where \(\lambda_m\) is the self-adjoint extension parameter for the spin 1/2 AB problem plus a two-dimensional HO. In the above equation, the coefficient of \(B_m\) diverges as \(\lim_{r \to 0} r^{2 - 2|m + \phi + (1 - \alpha)/2|/\alpha}\), if \(|m + \phi + (1 - \alpha)/2|/\alpha > 1\). Thus, \(b_m\) must be zero for \(|m + \phi + (1 - \alpha)/2|/\alpha > 1\), and the condition for the occurrence of a singular solution is \(|m + \phi + (1 - \alpha)/2|/\alpha < 1\). So, the presence of an irregular solution stems from the fact the operator is not self-adjoint for \(|m + \phi + (1 - \alpha)/2|/\alpha < 1\), recasting the condition of non-self-adjointness of the previews sections.

Applying the normalizability condition in the Eq. (118), yields the relation

\[
b_m = \frac{\Gamma(-\alpha) \Gamma(1/2 + |m + \phi + (1 - \alpha)/2|/2\alpha - E/2\omega)}{\Gamma(1/2 + |m + \phi + (1 - \alpha)/2|/2\alpha - E/2\omega) \Gamma(-\alpha)} a_m(M\omega)^{|m + \phi + (1 - \alpha)/2|/\alpha}.
\]

From Eq. (119), for \(|m + \phi + (1 - \alpha)/2|/\alpha < 1\) we have \(b_m = \lambda_m(M\omega)^{|m + \phi + (1 - \alpha)/2|/\alpha} a_m\) and by using Eq. (120), the bound state energy is implicitly determined by the equation

\[
\frac{\Gamma(1/2 + |m + \phi + (1 - \alpha)/2|/2\alpha - E_b/2\omega)}{\Gamma(1/2 - |m + \phi + (1 - \alpha)/2|/2\alpha - E_b/2\omega)} = \frac{\lambda_m(M\omega)^{|m + \phi + (1 - \alpha)/2|/\alpha}}{\Gamma(1 - |m + \phi + (1 - \alpha)/2|/\alpha)} \frac{\Gamma(1 + |m + \phi + (1 - \alpha)/2|/\alpha)}{\Gamma(-\alpha)}
\]

\[
\frac{1}{\lambda_m} = \frac{2}{r_0^{2|m + \phi + (1 - \alpha)/2|/\alpha}} \left(\frac{\phi s + |m + \phi + (1 - \alpha)/2|}{\phi s - |m + \phi + (1 - \alpha)/2|}\right).
\]
Then, the relation between the self-adjoint extension parameter and the physics of the problem for the usual AB has the same mathematical structure as for the AB plus HO. However, we must observe that the self-adjoint extension parameter is negative for the usual AB, confirming the restriction of negative values of the self-adjoint extension parameter made in [61], in such way we have an attractive $\delta$-function. It is a necessary condition to have a bound state in the usual AB system.

5. Conclusions

We have presented a general regularization procedure to address systems endowed with a singular Hamiltonian (due to localized fields sources or quantum confinement). Using the KS approach, the bound states were determined in terms of the physics of the problem, in a very consistent way and without any arbitrary parameter. In the sequel, we employed the BG approach. By comparing the results of these approaches, we have determined an expression for the self-adjoint extension parameter for the bound state problem, which coincides with the one for scattering problem. We have thus obtained the S matrix in terms of the physics of the problem as well. In this point, we remark that the important results of Refs. [7, 52, 60] are given in terms of an arbitrary self-adjoint extension parameter. In our work this parameter was determined in terms of the physics of the problem. The outcomes obtained by Park are a particular case of our results for a fixed value of the self-adjoint extension parameter. To our knowledge, it was not known in the literature an expression for the bound state energies for the AB with a defined self-adjoint extension parameter. In Ref. [37] this expression was presented by the first time, whose details are derived here.

To illustrate the applicability of our approach to other physical systems, we deal with the spin-$1/2$ AB problem in conical space plus a two dimensional HO. Two cases were considered: (i) without and (ii) with the inclusion of the $\delta$ function potential in the nonrelativistic Hamiltonian. Even though we have obtained an equivalent mathematical expression for both cases, it has been shown that, in (i) $|m + \phi + (1 - \alpha)/2|/\alpha$ can assume any value while in (ii) it is in the range $|m + \phi + (1 - \alpha)/2|/\alpha < 1$. In the first case, it is reasonable to impose that the wave function vanish at the origin. However, this condition does not give a correct description of the problem in the $r = 0$ region. Therefore, the energy spectrum obtained in the second case is physically acceptable. The presence of the singularity establishes that the effective angular momentum must obey the condition $|m + \phi + (1 - \alpha)/2|/\alpha < 1$ and implies that irregular solutions must be taken into account in this range. The only situation in which we can neglect the $\delta$ function potential is that one in which one looks only for topological phases. A natural extension of this work is the inclusion of the Coulomb potential, which naturally appears in two-dimensional systems, such as graphene [94] and anyons systems [95, 96]. This will be reported elsewhere.

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