Existence, uniqueness, comparison theorem and stability theorem for unbounded solutions of scalar BSDEs with sub-quadratic generators

Shengjun Fan*, Ying Hu**

Abstract

We first establish the existence of an unbounded solution to a backward stochastic differential equation (BSDE) with generator $g$ allowing a general growth in the state variable $y$ and a sub-quadratic growth in the state variable $z$, like $|z|^{\alpha}$ for some $\alpha \in (1, 2)$, when the terminal condition satisfies a sub-exponential moment integrability condition like $\exp \left( \mu L^{2/\alpha^*} \right)$ for the conjugate $\alpha^*$ of $\alpha$ and a positive parameter $\mu > \mu_0$ with a certain value $\mu_0$, which is clearly weaker than the usual $\exp(\mu L)$ integrability and stronger than $L^p$ ($p > 1$) integrability. Then, we prove the uniqueness and comparison theorem for the unbounded solutions of the preceding BSDEs under the additional assumptions that the terminal conditions have sub-exponential moments of any order and the generators are convex or concave in $(y, z)$. Afterwards, we extend the uniqueness and comparison theorem to the non-convexity and non-concavity case, and establish a general stability result for the unbounded solutions of the preceding BSDEs. Finally, with these tools in hands, we derive the nonlinear Feynman-Kac formula in this context.

Keywords: Backward stochastic differential equation, Existence and uniqueness, Sub-quadratic growth, $\exp \left( \mu L^{2/\alpha^*} \right)$-integrability, Comparison theorem, Stability theorem, Feynman-Kac formula.

2010 MSC: 60H10

1. Notations and introduction

Let us fix a positive real number $T > 0$ and a positive integer $d$, and let $x \cdot y$ represent the usual scalar inner product for $x, y \in \mathbb{R}^d$. Let $(B_t)_{t \in [0, T]}$ be a standard $\mathbb{R}^d$-valued Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $(\mathcal{F}_t)_{t \in [0, T]}$ being its natural filtration augmented by all $\mathbb{P}$-null sets of $\mathcal{F}$. All the measurability with respect to processes will

*Shengjun Fan is supported by the State Scholarship Fund from the China Scholarship Council (No. 201806425013). Ying Hu is partially supported by Lebesgue center of mathematics “Investissements d’avenir” program-ANR-11-LABX-0020-01, by CAESARS-ANR-15-CE05-0024 and by MFG-ANR-16-CE40-0015-01.

*School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China. E-mail: fsj@126.com

**Univ. Rennes, CNRS, IRMAR-UMR6625, F-35000, Rennes, France. E-mail: ying.hu@univ-rennes1.fr

October 21, 2019
refer to this filtration. Let us recall that a progressively measurable scalar process \((X_t)_{t \in [0,T]}\) belongs to class (D) if the family of random variables \(\{X_\tau\}, \tau \) running all \((\mathcal{F}_t)\)-stopping times valued in \([0,T]\), is uniformly integrable.

Denote \(\mathbb{R}_+ := [0, +\infty), 1_A(x) = 1\) when \(x \in A\) otherwise 0, and \(\text{sgn}(x) := 1_{x>0} - 1_{x \leq 0}\). Let \(a \land b\) denote the minimum of two real \(a\) and \(b\), \(a^- := -(a \land 0)\) and \(a^+ := (-a)^-\). For each \(\alpha \in (1, 2)\), let \(\alpha^*\) stand for the conjugate of \(\alpha\), that is, \(1/\alpha + 1/\alpha^* = 1\) or \(\alpha^* := \alpha/\alpha - 1 > 2\).

For any real \(p \geq 1\), let \(L^p\) represent the set of (equivalent classes of) all \(\mathcal{F}_T\)-measurable scalar random variables \(\xi\) such that \(E[|\xi|^p] < +\infty\), \(L^p\) the set of (equivalent classes of) all progressively measurable scalar processes \((X_t)_{t \in [0,T]}\) such that \(\|X\|_{L^p} := \left\{ E\left[ \left( \int_0^T |X_t|^p dt \right)^{1/p} \right] \right\}^{1/p} < +\infty\), \(S^p\) the set of (equivalent classes of) all progressively measurable and continuous scalar processes \((Y_t)_{t \in [0,T]}\) such that \(\|Y\|_{S^p} := \left( E\left[ \sup_{t \in [0,T]} |Y_t|^p \right]\right)^{1/p} < +\infty\), and \(M^p\) the set of (equivalent classes of) all progressively measurable \(\mathbb{R}^d\)-valued processes \((Z_t)_{t \in [0,T]}\) such that \(\|Z\|_{M^p} := \left\{ E\left[ \left( \int_0^T |Z_t|^2 dt \right)^{p/2} \right] \right\}^{1/p} < +\infty\).

We study the following backward stochastic differential equation (BSDE for short):

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T],
\]

where \(\xi\) is an \(\mathcal{F}_T\)-measurable scalar random variable called the terminal condition, the function \(g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}\) is progressively measurable for each \((y, z)\) and continuous in \((y, z)\) called the generator, and the pair of processes \((Y_t, Z_t)_{t \in [0,T]}\) valued in \(\mathbb{R} \times \mathbb{R}^d\) is called the solution of eq. (1.1), which is progressively measurable such that \(\mathbb{P}\) - a.s., \(t \mapsto Y_t\) is continuous, \(t \mapsto Z_t\) is square-integrable, \(t \mapsto g(t, Y_t, Z_t)\) is integrable, and verifies (1.1).

In this paper, we always assume that \(\xi\) is a terminal condition and \(g\) is a generator which is continuous in \((y, z)\), and we use BSDE(\(\xi, g\)) to denote the BSDE with terminal condition \(\xi\) and generator \(g\). We consider the BSDE with generator \(g\) satisfying \(d\mathbb{P} \times dt - a.e.,\)

\[
\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad |g(\omega, t, y, z)| \leq |g(\omega, t, 0, 0)| + \beta|y| + \gamma|z|^\alpha
\]

(1.2)
with $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$. We usually say that $g$ has a linear growth in the state variable $z$ when $\alpha = 1$, a sub-linear growth in $z$ when $\alpha \in (0, 1)$, a quadratic growth in $z$ when $\alpha = 2$, a superquadratic growth in $z$ when $\alpha > 2$, and a sub-quadratic growth in $z$ when $\alpha \in (1, 2)$. Our attention focuses on the last case. Let us first recall some related results in previous four cases, which have been intensively studied. For narrative convenience, we denote $g_0 := g(\cdot, 0, 0)$.

Assume first that the generator $g$ has a linear growth in $(y, z)$, i.e., \( (1.2) \) with $\alpha = 1$ holds for $g$. It is well known that for $(\xi, g_0) \in L^p \times L^p$ with some $p > 1$, BSDE\( (\xi, g) \) admits a solution in $S^p \times M^p$, and the solution is unique when $g$ further satisfies the uniformly Lipschitz condition in $(y, z)$. Readers are referred to Pardoux and Peng [22], El Karoui et al. [12], Lepeltier and San Martin [20], Briand et al. [3] and Fan and Jiang [16] for more details. Recently, Hu and Tang [18], Buckdahn et al. [7] and Fan and Hu [15] extended this result and established the existence and uniqueness of an unbounded solution to BSDE\( (\xi, g) \) with linear-growth generator $g$ by assuming that $(\xi, g_0)$ satisfies an $L \exp (\mu \sqrt{2 \ln(1 + L)})$-integrability condition for $\mu \geq \gamma \sqrt{T}$, which is weaker than the usual $L^p$ ($p > 1$) integrability and stronger than $L \ln L$ integrability.

Secondly, assume that the generator $g$ has a linear growth in $y$ and a sub-linear growth in $z$, i.e., eq. \((1.2)\) with $\alpha \in (0, 1)$ is satisfied for $g$. It follows from Briand et al. [3] that for $(\xi, g_0) \in L^1 \times L^1$, BSDE\( (\xi, g) \) admits a solution $(Y, Z)$ such that $Y$ belongs to class (D), and the solution is unique when $g$ further satisfies the uniformly Lipschitz condition in $(y, z)$. See for example Briand and Hu [5] and Fan [13, 14] for more details on this topic.

Thirdly, from Delbaen et al. [10] it is well known that superquadratic BSDEs, i.e., eq. \((1.2)\) with $\alpha > 2$ holds for the generator $g$, are not solvable in general. Some solvability results under the Markovian setting can be founded in Delbaen et al. [10], Masiero and Richou [21], Richou [24] and Cheridito and Nam [8].

Finally, we assume that the generator $g$ has a linear growth in $y$ and a quadratic growth in $z$, i.e., eq. \((1.2)\) with $\alpha = 2$ is satisfied for $g$. It is also well known from Kobylanski [19] that if both $\xi$ and $\int_0^T |g_0| \, dt$ are bounded, then BSDE\( (\xi, g) \) admits a solution $(Y, Z)$ such that $Y$ is a bounded process and $Z \in M^2$, and the solution is unique if $g$ further satisfies the uniformly Lipschitz condition in $y$ and a locally Lipschitz condition in $z$. Readers are referred to Briand and Elie [4], Hu and Tang [17] and Fan [13] for further research on the bounded solution of quadratic BSDEs. Later, Briand and Hu [5, 6] and Delbaen et al. [11] extended this result and established the existence and uniqueness of an unbounded solution to BSDE\( (\xi, g) \) with generator $g$ having a quadratic growth in $z$ by assuming that $(\xi, g_0)$ has only $\gamma e^{3T}$-order exponential moment integrability, where the generator $g$ need to be uniformly Lipschitz with respect to the variable $y$ and convex (concave) with respect to the variable $z$ for the uniqueness.
of the solution, see also Barrieu and El Karoui [2] for more details.

In this paper, we study the existence, uniqueness, comparison theorem and stability theorem for unbounded solutions of BSDE(\(\xi, g\)) with generator \(g\) having a linear growth in \(y\) and a sub-quadratic growth in \(z\), namely, eq. (1.2) with \(\alpha \in (1, 2)\) is satisfied for \(g\). We prove that if \((\xi, g_0)\) satisfies an \(\exp(\mu L^{2/\alpha^*})\)-integrability condition for a positive parameter \(\mu > \mu_0\) with a certain value \(\mu_0\), which is clearly weaker than the usual \(\exp(\mu L)\) integrability and stronger than \(L^p (p > 1)\) integrability, then BSDE(\(\xi, g\)) admits a solution \((Y, Z)\) such that \(Y\) belongs to class (D) and \(Z \in \mathcal{M}^2\), and the solution is unique and the comparison theorem and stability theorem hold when \(g\) further satisfies a (extended) convexity or concavity condition with respect to the variables \((y, z)\), and \((\xi, g_0)\) satisfies the \(\exp(\mu L^{2/\alpha^*})\)-integrability condition for all \(\mu > 0\).

We remark that in our final results, the linear growth condition of the generator \(g\) in \(y\) is also weakened to a one-sided growth condition, see (H1”) in Remark 3.5 at the end of Section 3.

The paper is organized as follows. In the next section, we introduce the whole idea of this paper and make an important preparation (see Proposition 2.1) for the proof of the main results. In Section 3 we establish the existence result and in Section 4 we prove the uniqueness and comparison theorem under the convexity or concavity condition of the generator. Afterwards, in Section 5 we extend the uniqueness and comparison theorem to the non-convexity and non-concavity case, and in Section 6 we establish a stability result for unbounded solutions for the preceding BSDEs under general assumptions. Finally, with these tools in hands, in Section 7 we derive the nonlinear Feynman-Kac formula in this context.

2. The whole idea

For the existence of an unbounded solution to BSDE(\(\xi, g\)) with generator \(g\) satisfying (1.2) with \(\alpha \in (1, 2)\), our whole strategy is to establish some uniform a priori estimate on the first process \(Y^{m,p}\) in the solution of the usual approximated BSDEs (see the definition in (3.11) of Section 3) and to apply the localization procedure put forward initially in Briand and Hu [5]. In order to obtain the a priori estimate, the idea consists in searching for an appropriate function \(\phi(s, x)\) and applying Itô-Tanaka’s formula to \(\phi(s, |Y^{m,p}_s|)\) on the time interval \(s \in [t, \tau_m]\) with \((\mathcal{F}_t)\)-stopping time \(\tau_m\) valued in \([t, T]\) for \(t \in [0, T]\). More precisely, we need to find a positive real number \(\delta > 0\), and a positive and smooth function \(\phi(s, x) : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfying that \(\phi_x(s, x) > 0\), \(\phi_{xx}(s, x) > \delta\) and

\[
-\phi_x(s, x) (\beta x + \gamma|z|^\alpha) + \frac{1}{2} (\phi_{xx}(s, x) - \delta)|z|^2 + \phi_y(s, x) \geq 0,
\]

\((s, x, z) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}^d, (2.1)\)
where and hereafter, \( \phi_s(\cdot, \cdot) \) stands for the first-order partial derivative of \( \phi(\cdot, \cdot) \) with respect to the first variable, and \( \phi_x(\cdot, \cdot) \) and \( \phi_{xx}(\cdot, \cdot) \) respectively the first-order and second order partial derivative of \( \phi(\cdot, \cdot) \) with respect to the second variable.

Observe from Young’s inequality that

\[
\frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta} |z|^\alpha \leq c_{\alpha, \gamma} \left( \frac{\phi_x(s, x)}{\phi_{xx}(s, x) - \delta} \right)^\frac{2}{\alpha} + \frac{1}{2} |z|^2
\]

and then

\[
-\gamma \phi_x(s, x)|z|^\alpha + \frac{1}{2} (\phi_{xx}(s, x) - \delta) |z|^2
\]

\[
= (\phi_{xx}(s, x) - \delta) \left( -\frac{\gamma \phi_x(s, x)}{\phi_{xx}(s, x) - \delta} |z|^\alpha + \frac{1}{2} |z|^2 \right)
\]

\[
\geq -c_{\alpha, \gamma} \left( \frac{\phi_x(s, x)}{\phi_{xx}(s, x) - \delta} \right)^\frac{2}{\alpha} |z|^\alpha,
\]

where

\[
c_{\alpha, \gamma} := \frac{2 - \alpha}{2 - \alpha^2} \frac{\gamma^2}{\alpha^2}.
\]

It is clear that (2.1) holds if the function \( \phi(\cdot, \cdot) \) satisfies the following condition:

\[
-\beta \phi_x(s, x)x - c_{\alpha, \gamma} \left( \frac{\phi_x(s, x)}{\phi_{xx}(s, x) - \delta} \right)^\frac{2}{\alpha} + \phi_s(s, x) \geq 0, \quad (s, x) \in [0, T] \times \mathbb{R}_+.
\]

(2.2)

Now, let \( \mu_s : [0, T] \to \mathbb{R}_+ \) be a nondecreasing and continuously differentiable function with \( \mu_0 = \varepsilon \) for some \( \varepsilon > 0 \), and let

\[
k_{\alpha, \varepsilon} := \left( \frac{(1 + \varepsilon)^{\frac{2 - \alpha}{\alpha}}}{2(\alpha - 1)\varepsilon \left(1 + \varepsilon^{\frac{2 - \alpha}{\alpha}} - 1\right)} \right)^{\frac{\alpha}{2(\alpha - 1)}}.
\]

(2.3)

We choose the following function

\[
\phi(s, x; \varepsilon) := \exp \left( \mu_s(x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}} \right) = \exp \left( \mu_s(x + k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \right), \quad (s, x) \in [0, T] \times \mathbb{R}_+.
\]

(2.4)

to explicitly solve the inequality (2.2). For each \((s, x) \in [0, T] \times \mathbb{R}_+\), a simple computation gives

\[
\phi_x(s, x; \varepsilon) = \phi(s, x; \varepsilon) \frac{2(\alpha - 1)\mu_s}{\alpha(x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}}} > 0,
\]

(2.5)

\[
\phi_{xx}(s, x; \varepsilon) = \phi(s, x; \varepsilon) \frac{2(\alpha - 1)\mu_s}{\alpha(x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}}} \left[ 2(\alpha - 1)\mu_s(x + k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} - 1 + (\alpha - 1) \right] > 0
\]

(2.6)

and

\[
\phi_s(s, x; \varepsilon) = \phi(s, x; \varepsilon)(x + k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \mu_s' > 0.
\]

(2.7)

Furthermore, for each \((s, x) \in [0, T] \times \mathbb{R}_+\), in view of the fact of \( \mu_s \geq \mu_0 = \varepsilon \) and (2.3), we have

\[
2(\alpha - 1)\mu_s(x + k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \geq 2(\alpha - 1)\varepsilon (k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \geq \frac{(1 + \varepsilon)^{\frac{2 - \alpha}{\alpha}}}{(1 + \varepsilon^{\frac{2 - \alpha}{\alpha}} - 1)}
\]

(2.8)
and
\[ \phi(s, x; \varepsilon) \frac{2(\alpha - 1)^2 \mu_s}{\alpha^2 (x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}}} \geq \exp \left( \varepsilon (x + k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \right) \frac{2(\alpha - 1)^2 \varepsilon}{\alpha^2 (x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}}}. \] (2.9)

Observe that the function in the right hand side of (2.9) is positive and continuous in \( \mathbb{R}_+ \), and tends to infinity as \( x \to +\infty \). It follows that there exists a constant \( \delta_{\alpha, \varepsilon} > 0 \) depending only on \((\alpha, \varepsilon)\) such that
\[ \phi(s, x; \varepsilon) \frac{2(\alpha - 1)^2 \mu_s}{\alpha^2 (x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}}} \geq \delta_{\alpha, \varepsilon}, \quad (s, x) \in [0, T] \times \mathbb{R}_+. \] (2.10)

Combining (2.6), (2.8) and (2.10) yields that for each \((s, x) \in [0, T] \times \mathbb{R}_+\),
\[ \phi_{xx}(s, x; \varepsilon) - \delta_{\alpha, \varepsilon} \geq \phi(s, x; \varepsilon) \frac{2(\alpha - 1)^2 \mu_s}{\alpha^2 (x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}}} (1 + \varepsilon) \frac{2(\alpha - 1)^2 \mu_s}{\alpha^2 (x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}}} \]
\[ = \phi(s, x; \varepsilon) \frac{4(\alpha - 1)^2 \mu_s^2}{(1 + \varepsilon) \alpha^2 (x + k_{\alpha, \varepsilon})^{\frac{2(2\alpha - 1)}{\alpha}}}. \] (2.11)

In the sequel, we substitute (2.5), (2.7) and (2.11) into the left side of (2.2) with \( \delta = \delta_{\alpha, \varepsilon} \) to obtain that for each \((s, x) \in [0, T] \times \mathbb{R}_+\),
\[-\beta \phi_{x}(s, x; \varepsilon) x - c_{\alpha, \gamma} \left( \frac{(\phi_{x}(s, x; \varepsilon))^{\frac{2}{\alpha}}}{(\phi_{xx}(s, x; \varepsilon) - \delta_{\alpha, \varepsilon})^{\frac{2}{\alpha}}} \right) + \phi(s, x) \]
\[ \geq -\beta \frac{2(\alpha - 1)^2 \mu_s \phi(s, x; \varepsilon)(x + k_{\alpha, \varepsilon})}{\alpha (x + k_{\alpha, \varepsilon})^{\frac{2}{\alpha}}} - c_{\alpha, \gamma} \left( \frac{2(\alpha - 1)^2 \mu_s}{\alpha} \phi(s, x; \varepsilon) \right)^{\frac{2}{\alpha}} (x + k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \]
\[ + \phi(s, x; \varepsilon)(x + k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \mu_s' \]
\[ = \phi(s, x; \varepsilon)(x + k_{\alpha, \varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \left( -\frac{2(\alpha - 1)\beta}{\alpha} \mu_s - c_{\alpha, \gamma} \frac{1 + \varepsilon}{\mu_s^{\frac{2}{\alpha}}} + \mu_s' \right). \]

Thus, (2.2) holds if the function \( \mu_s, s \in [0, T] \) satisfies the following ODE:
\[ \mu_s' = \frac{2(\alpha - 1)\beta}{\alpha} \mu_s + C_{\alpha, \gamma} \frac{1 + \varepsilon}{\mu_s^{\frac{2(\alpha - 1)}{\alpha}}}, \quad s \in [0, T] \] (2.12)

with \( \mu_0 = \varepsilon \) and
\[ C_{\alpha, \gamma} := \frac{c_{\alpha, \gamma}}{(2(\alpha - 1))^{\frac{2(\alpha - 1)}{\alpha}} (\frac{2(\alpha - 1)}{\alpha})^{\frac{2}{\alpha}}} \]
\[ = -\frac{(2 - \alpha)\alpha^{\frac{\alpha}{2\alpha - 1}}}{2} \left( \frac{2(\alpha - 1)}{\alpha} \right)^{\frac{2}{\alpha}}. \]

It remains to solve ODE (2.12). We need to distinguish two different cases \( \beta = 0 \) and \( \beta > 0 \). We first consider the case of \( \beta = 0 \). After separating variables for (2.12) we integrate on the time interval \([0, s]\) to get that
\[ \frac{2 - \alpha}{\alpha} \left( \mu_s^{\frac{2}{\alpha} - \alpha^{\frac{\alpha}{2\alpha - 1}}} \right) \varepsilon^{\frac{\alpha}{2\alpha - 1}} = \left( 2 - \alpha \right) \mu_s^{\frac{2}{\alpha} - \alpha^{\frac{\alpha}{2\alpha - 1}}} \int_0^s \mu_r^{\frac{2(\alpha - 1)}{\alpha}} dr = C_{\alpha, \gamma}(1 + \varepsilon)s \]
and then
\[ \mu_s = \left( \frac{\alpha}{2 - \alpha} C_{\alpha,\gamma}(1 + \varepsilon)s + \varepsilon^{\frac{\alpha}{2 - \alpha}} \right)^{\frac{2 - \alpha}{\alpha}}, \quad s \in [0, T]. \]

For the case of \( \beta > 0 \), after separating variables for (2.12) again we integrate on interval \([0, s]\) to get that
\[ \frac{2 - \alpha}{2(\alpha - 1)\beta} \ln \left( \frac{2(2 - \alpha)\beta s^{\frac{2 - \alpha}{\alpha}}}{\alpha} + C_{\alpha,\gamma}(1 + \varepsilon) \right) \bigg|_0^s = \int_0^s \frac{2(\alpha - 1)\beta}{\alpha} \frac{\mu_r^{\frac{2 - \alpha}{\alpha}}}{\mu_r^{\frac{2 - \alpha}{\alpha}} + C_{\alpha,\gamma}(1 + \varepsilon)} \, dr = s \]

and then
\[ \mu_s = \left\{ \left( \frac{\alpha}{2 - \alpha} + \frac{C_{\alpha,\gamma}(1 + \varepsilon)}{2(\alpha - 1)\beta} \right) \exp \left( \frac{2(\alpha - 1)\beta}{2 - \alpha} s \right) - \frac{C_{\alpha,\gamma}(1 + \varepsilon)}{2(\alpha - 1)\beta} \right\}^{\frac{2 - \alpha}{\alpha}}, \quad s \in [0, T]. \]

We summarize the preceding arguments into the following proposition, which will play a crucial role in the proof of the main results of this paper later.

**Proposition 2.1.** Given \( \alpha \in (1, 2) \) and \( \beta, \gamma > 0 \). For each \((s, x) \in [0, T] \times \mathbb{R}_+ \) and \( \varepsilon > 0 \), define
\[ \tilde{\varphi}(s, x; \varepsilon) := \exp \left( \tilde{\mu}_{\alpha,\gamma,\varepsilon}(s) (x + k_{\alpha,\varepsilon})^{\frac{2}{\alpha}} \right) = \exp \left( \tilde{\mu}_{\alpha,\gamma,\varepsilon}(s) (x + k_{\alpha,\varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \right) \] (2.13)

and
\[ \tilde{\varphi}(s, x; \varepsilon) := \exp \left( \tilde{\mu}_{\alpha,\beta,\gamma,\varepsilon}(s) (x + k_{\alpha,\varepsilon})^{\frac{2}{\alpha}} \right) = \exp \left( \tilde{\mu}_{\alpha,\beta,\gamma,\varepsilon}(s) (x + k_{\alpha,\varepsilon})^{\frac{2(\alpha - 1)}{\alpha}} \right), \] (2.14)

where
\[ \tilde{\mu}_{\alpha,\gamma,\varepsilon}(s) := \left( \bar{c}_{\alpha,\gamma}(1 + \varepsilon)s + \varepsilon^{\frac{\alpha}{2 - \alpha}} \right)^{\frac{2 - \alpha}{\alpha}}, \quad k_{\alpha,\varepsilon} := \left( \frac{(1 + \varepsilon)^{\frac{2 - \alpha}{\alpha}}}{2(\alpha - 1)\varepsilon \left((1 + \varepsilon)^{\frac{2 - \alpha}{\alpha}} - 1\right)} \right) \] (2.15)

and
\[ \tilde{\mu}_{\alpha,\beta,\gamma,\varepsilon}(s) := \left\{ \left( \varepsilon^{\frac{2 - \alpha}{\alpha}} + (1 + \varepsilon)\bar{c}_{\alpha,\beta,\gamma} \right) \exp \left( \frac{2(\alpha - 1)\beta}{2 - \alpha} s \right) - (1 + \varepsilon)\bar{c}_{\alpha,\beta,\gamma} \right\}^{\frac{2 - \alpha}{\alpha}} \] (2.16)

with
\[ \bar{c}_{\alpha,\gamma} := \frac{(\alpha\gamma)^{\frac{2}{2 - \alpha}}}{2 \left( \frac{2(\alpha - 1)}{\alpha} \right)^{\frac{2(\alpha - 1)}{2 - \alpha}}} \quad \text{and} \quad \bar{c}_{\alpha,\beta,\gamma} := \frac{(2 - \alpha)(\alpha\gamma)^{\frac{2}{2 - \alpha}}}{4\alpha\beta \left( \frac{2(\alpha - 1)}{\alpha} \right)^{\frac{2(\alpha - 1)}{2 - \alpha}}}. \] (2.17)

Then, there exists a constant \( \delta_{\alpha,\varepsilon} > 0 \) depending only on \((\alpha, \varepsilon)\) such that for each \((s, x, z) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}^d\), we have
\[ -\gamma \tilde{\varphi}_x(s, x; \varepsilon) |z|^\alpha + \frac{1}{2} \left( \tilde{\varphi}_{xx}(s, x; \varepsilon) - \delta_{\alpha,\varepsilon} \right) |z|^2 + \tilde{\varphi}_s(s, x; \varepsilon) \geq 0 \] (2.18)

and
\[ -\tilde{\varphi}_x(s, x; \varepsilon) (\beta x + \gamma |z|^\alpha) + \frac{1}{2} \left( \tilde{\varphi}_{xx}(s, x; \varepsilon) - \delta_{\alpha,\varepsilon} \right) |z|^2 + \tilde{\varphi}_s(s, x; \varepsilon) \geq 0. \] (2.19)
Finally, for the uniqueness, comparison theorem and stability theorem of unbounded solutions to BSDE($\xi, g$) with generator $g$ satisfying (1.2) with $\alpha \in (1, 2)$, some stronger assumptions than those needed for the existence are required as usual. We first assume in addition that ($\xi, g_0$) has sub-exponential moments integrability of any order and the generator $g$ is convex or concave with respect to the state variables ($y, z$), which appears a natural assumption for a non-linear growth function (see e.g. Briand and Hu [6] and Delbaen et al. [11]), and then relax the convexity (concavity) assumption. The main idea is to use the $\theta$-technique developed in Briand and Hu [6] to prove these results. More specifically, in order to take advantage of the (extended) convexity condition, we will estimate $Y_1^\cdot - \theta Y_2^\cdot$, for each $\theta \in (0, 1)$, instead of estimating the difference between the processes $Y_1^\cdot$ and $Y_2^\cdot$. Moreover, it turns out that the uniform a priori estimate is also the key to solve the uniqueness, comparison theorem and stability theorem of the solutions.

3. Existence of the solution

In this section, we assume that $\xi$ is a terminal condition and $g$ is a generator which is continuous in ($y, z$), and satisfies the following assumption:

(H1) There exist three constants $\alpha \in (1, 2)$, $\beta \geq 0$ and $\gamma > 0$ such that for $\omega \in \Omega$, $t \in [0, T]$,

$$|g(\omega, t, y, z)| \leq |g(\omega, t, 0, 0)| + \beta|y| + \gamma|z|^\alpha, \quad (y, z) \in \mathbb{R} \times \mathbb{R}^d.$$  

Define the function

$$\psi(x, \mu) := \exp \left( \mu \cdot x^\alpha \right) = \exp \left( \mu \cdot x^{\frac{2(\alpha-1)}{\alpha}} \right), \quad (x, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+,$$  

and the two constants

$$\tilde{\mu}_{\alpha, \gamma, T}^0 := (\tilde{c}_{\alpha, \gamma} T)^{\frac{2-\alpha}{\alpha}} = \alpha^2 \frac{\alpha - 1}{2(\alpha - 1)^{\frac{2(\alpha-1)}{\alpha}}} \gamma^\alpha T^{\frac{2-\alpha}{\alpha}},$$

$$\tilde{\mu}_{\alpha, \beta, \gamma, T}^0 := \left\{ \tilde{c}_{\alpha, \beta, \gamma} \exp \left( \frac{2(\alpha - 1)\beta}{2 - \alpha} T \right) - \tilde{c}_{\alpha, \beta, \gamma} \right\}^{\frac{2-\alpha}{\alpha}},$$

where $\tilde{c}_{\alpha, \gamma}$ and $\tilde{c}_{\alpha, \beta, \gamma}$ are defined in (2.17). Note that $\tilde{\mu}_{\alpha, \gamma, T}^0 = \lim_{\varepsilon \to 0^+} \tilde{\mu}_{\alpha, \gamma, \varepsilon}(T)$ and $\tilde{\mu}_{\alpha, \beta, \gamma, T}^0 = \lim_{\varepsilon \to 0^+} \tilde{\mu}_{\alpha, \beta, \gamma, \varepsilon}(T)$, where $\tilde{\mu}_{\alpha, \gamma, \varepsilon}(\cdot)$ and $\tilde{\mu}_{\alpha, \beta, \gamma, \varepsilon}(\cdot)$ are defined in (2.15) and (2.16) respectively.

The following existence theorem is the main result of this section.

**Theorem 3.1.** Assume that $\xi$ is a terminal condition, $g$ is a generator which is continuous with respect to ($y, z$) and satisfies assumption (H1) with parameters $\alpha$, $\beta$ and $\gamma$, and the function $\psi(x, \mu)$ and the constants $\tilde{\mu}_{\alpha, \gamma, T}^0$ and $\tilde{\mu}_{\alpha, \beta, \gamma, T}^0$ are defined respectively in (3.1) and (3.2).
(i) Let $\beta = 0$ and $\mu_{\alpha,\gamma,\varepsilon}(\cdot)$ be defined in (2.15). If there exists a constant $\mu > \bar{\mu}_{\alpha,\gamma,T}$ such that
\[
E\left[\exp\frac{1}{\mu} \left( |\xi| + \int_0^T |g(t,0,0)|dt \right) \right] < +\infty, \quad (3.3)
\]
then BSDE($\xi,g$) admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that $(\psi(|Y_t|, \mu_{\alpha,\gamma,\varepsilon}(t)))_{t \in [0,T]}$ belongs to class (D) for some $\varepsilon > 0$ and $Z \in \mathcal{M}^2$. Moreover, for some constant $\delta_{\alpha,\varepsilon} > 0$ depending only on $(\alpha, \varepsilon)$ we have, $\mathbb{P} - a.s.,$
\[
\psi(|Y_t|, \mu_{\alpha,\gamma,\varepsilon}(t)) + \frac{\delta_{\alpha,\varepsilon}}{2} E\left[\int_t^T |Z_s|^2 ds \right] \leq C_{\mu,\alpha,\varepsilon} E\left[\psi\left( |\xi| + \int_0^T |g(s,0,0)|ds, \mu_{\alpha,\gamma,\varepsilon}(T) \right) \right] \quad (3.4)
\]
where $C_{\mu,\alpha,\varepsilon}$ is a positive constant depending only on $(\mu, \alpha, \varepsilon)$.

(ii) Let $\beta > 0$ and $\tilde{\mu}_{\alpha,\beta,\gamma,\varepsilon}(\cdot)$ be defined in (2.16). If there exists a constant $\mu > \bar{\mu}_{\alpha,\beta,\gamma,T}$ such that (3.3) holds, then BSDE($\xi,g$) admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that $(\psi(|Y_t|, \tilde{\mu}_{\alpha,\beta,\gamma,\varepsilon}(t)))_{t \in [0,T]}$ belongs to class (D) for some $\varepsilon > 0$ and $Z \in \mathcal{M}^2$. Moreover, for some $\delta_{\alpha,\varepsilon} > 0$ depending only on $(\alpha, \varepsilon)$ we have, $\mathbb{P} - a.s.,$
\[
\psi(|Y_t|, \tilde{\mu}_{\alpha,\beta,\gamma,\varepsilon}(t)) + \frac{\delta_{\alpha,\varepsilon}}{2} E\left[\int_t^T |Z_s|^2 ds \right] \leq C_{\mu,\alpha,\varepsilon} E\left[\psi\left( |\xi| + \int_0^T |g(s,0,0)|ds, \tilde{\mu}_{\alpha,\beta,\gamma,\varepsilon}(T) \right) \right] \quad (3.5)
\]
where the constant $C_{\mu,\alpha,\varepsilon}$ is the same as (i).

**Remark 3.2.** It is not very hard to check that $\bar{\mu}_{\alpha,\gamma,T}$ and $\tilde{\mu}_{\alpha,\beta,\gamma,T}$ defined in (3.2) tends respectively to $2\gamma$ and $2\gamma e^{\beta T}$ as $\alpha \to 2$, which is a direct correspondence of the known result for the quadratic growth case in Briand and Hu [5, 6]. From this point of view, the condition (3.3) in Theorem 3.1 is seemingly the reasonably weakest possible one guaranteeing the existence of the solution. However, by now we can not prove it.

In order to prove Theorem 3.1, we need the following proposition, which establishes some a priori estimate for solutions to BSDEs with bounded terminal conditions and sub-quadratic growth generators.

**Proposition 3.3.** Assume that $\xi$ is a terminal condition, $g$ is a generator which is continuous in the state variables $(y,z)$ and satisfies assumption (H1) with parameters $\alpha, \beta$ and $\gamma$, and
the functions \( \bar{\mu}_{\alpha,\gamma,\varepsilon}(s) \), \( \bar{\mu}_{\alpha,\beta,\gamma,\varepsilon}(s) \) and \( \psi(x, \mu) \) together with the constant \( k_{\alpha,\varepsilon} \) are respectively defined in (2.15), (2.16) and (3.1).

Let \( |\xi| + \int_0^T |g(t, 0, 0)| dt \) be a bounded random variable, and \( (Y_t, Z_t)_{t \in [0, T]} \) a solution to BSDE(\( \xi, g \)) such that \( Y \) is a bounded process (and \( Z \in \mathcal{M}^2 \)). Then for any \( \varepsilon > 0 \), there exists a constant \( \delta_{\alpha,\varepsilon} > 0 \) depending only on \( (\alpha, \varepsilon) \) such that \( \mathbb{P} \)-a.s., for each \( t \in [0, T] \), the inequality

\[
\psi(|Y_t|, \bar{\mu}_{\alpha,\gamma,\varepsilon}(t)) + \frac{\delta_{\alpha,\varepsilon}}{2} \mathbb{E} \left[ \int_t^T |Z_s|^2 ds \right] \mathcal{F}_t 
\leq \exp \left( \bar{\mu}_{\alpha,\gamma,\varepsilon}(T) k_{\alpha,\varepsilon} \right) \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(s, 0, 0)| ds, \bar{\mu}_{\alpha,\gamma,\varepsilon}(T) \right) \right] \mathcal{F}_t
\]

holds for \( \beta = 0 \), and the inequality

\[
\psi(|Y_t|, \bar{\mu}_{\alpha,\beta,\gamma,\varepsilon}(t)) + \frac{\delta_{\alpha,\varepsilon}}{2} \mathbb{E} \left[ \int_t^T |Z_s|^2 ds \right] \mathcal{F}_t 
\leq \exp \left( \bar{\mu}_{\alpha,\beta,\gamma,\varepsilon}(T) k_{\alpha,\varepsilon} \right) \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(s, 0, 0)| ds, \bar{\mu}_{\alpha,\beta,\gamma,\varepsilon}(T) \right) \right] \mathcal{F}_t
\]

holds for \( \beta > 0 \).

**Proof.** We first consider the case of \( \beta = 0 \). Define

\[
\bar{Y}_t := |Y_t| + \int_0^t |g(s, 0, 0)| ds \quad \text{and} \quad \bar{Z}_t := \text{sgn}(Y_t)Z_t, \quad t \in [0, T].
\]

It follows from Itô-Tanaka’s formula that

\[
\bar{Y}_t = \bar{Y}_T + \int_t^T (\text{sgn}(Y_s)g(s, Y_s, Z_s) - |g(s, 0, 0)|) ds - \int_t^T \bar{Z}_s \cdot dB_s - \int_t^T dB_s, \quad t \in [0, T],
\]

where \( L \) stands for the local time of \( Y \) at 0. Now, we fix \( \varepsilon > 0 \) and apply Itô-Tanaka’s formula to the process \( \bar{\varphi}(s, \bar{Y}_s; \varepsilon) \), where the function \( \bar{\varphi}(s, x; \varepsilon) \) is defined in (2.13), to derive, in view of assumption (H1) with \( \beta = 0 \),

\[
d\bar{\varphi}(s, \bar{Y}_s; \varepsilon) = \bar{\varphi}_x(s, \bar{Y}_s; \varepsilon) (-\text{sgn}(Y_s)g(s, Y_s, Z_s) + |g(s, 0, 0)|) ds + \bar{\varphi}_x(s, \bar{Y}_s; \varepsilon) \bar{Z}_s \cdot dB_s \\
+ \bar{\varphi}_x(s, \bar{Y}_s; \varepsilon) dL_s + \frac{1}{2} \bar{\varphi}_{xx}(s, \bar{Y}_s; \varepsilon) |Z_s|^2 ds + \bar{\varphi}_x(s, \bar{Y}_s; \varepsilon) ds \\
\geq \left[ -\gamma \bar{\varphi}_x(s, \bar{Y}_s; \varepsilon) |Z_s|^2 + \frac{1}{2} \bar{\varphi}_{xx}(s, \bar{Y}_s; \varepsilon) |Z_s|^2 + \bar{\varphi}_x(s, \bar{Y}_s; \varepsilon) \right] ds \\
+ \bar{\varphi}_x(s, \bar{Y}_s; \varepsilon) \bar{Z}_s \cdot dB_s.
\]

Thus, from (2.18) in Proposition 2.1 we know the existence of a positive constant \( \delta_{\alpha,\varepsilon} > 0 \) depending only on \( \alpha, \varepsilon \) such that

\[
d\bar{\varphi}(s, \bar{Y}_s; \varepsilon) \geq \frac{1}{2} \delta_{\alpha,\varepsilon} |Z_s|^2 ds + \bar{\varphi}_x(s, \bar{Y}_s; \varepsilon) \bar{Z}_s \cdot dB_s, \quad s \in [0, T].
\]
Let us denote, for each \( t \in [0, T] \) and each integer \( m \geq 1 \), the following stopping time

\[
\tau_m := \inf \left\{ s \in [t, T] : \int_t^s (\tilde{\varphi}_x(r, \tilde{Y}_r; \varepsilon))^2 |\tilde{Z}_r|^2 dr \geq m \right\} \wedge T
\]

with the convention \( \inf \emptyset = +\infty \). It follows from the inequality (3.8) and the definition of \( \tau_m \) that for each \( t \in [0, T] \) and \( m \geq 1 \),

\[
\tilde{\varphi}(t, \tilde{Y}_t; \varepsilon) + \frac{\delta_{a,\varepsilon} E}{2} \left[ \int_t^{\tau_m} |Z_s|^2 ds \right]_{\mathcal{F}_t} \leq E \left[ \tilde{\varphi}(\tau_m, \tilde{Y}_{\tau_m}; \varepsilon) \right]_{\mathcal{F}_t}, \quad t \in [0, T]. \tag{3.9}
\]

Furthermore, in view of the definition of \( \tau_m \) again, by sending \( m \) to infinity and using Fatou’s lemma and Lebesgue’s dominated convergence theorem in above inequality we get

\[
\tilde{\varphi}(t, \tilde{Y}_t; \varepsilon) + \frac{\delta_{a,\varepsilon} E}{2} \left[ \int_t^T |Z_s|^2 ds \right]_{\mathcal{F}_t} \leq E \left[ \tilde{\varphi}(T, \tilde{Y}_T; \varepsilon) \right]_{\mathcal{F}_t}, \quad t \in [0, T]. \tag{3.10}
\]

And, from the definitions of \( \tilde{\varphi}(s, x; \varepsilon) \) and \( \psi(x, \mu) \) with the inequality \((a + b)^\lambda \leq a^\lambda + b^\lambda\) for \( a, b \geq 0 \) and \( \lambda \in (0, 1) \), observe that for each \( x \in \mathbb{R}^+, \ t \in [0, T] \) and \( \varepsilon > 0 \),

\[
\psi(x, \tilde{\mu}_{a,\gamma,\varepsilon}(t)) \leq \tilde{\varphi}(t, x; \varepsilon) \leq \exp \left( \tilde{\mu}_{a,\gamma,\varepsilon}(t) k_{a,\varepsilon}^\frac{\varepsilon}{a} \right) \psi(x, \tilde{\mu}_{a,\gamma,\varepsilon}(t)). \tag{3.11}
\]

The desired inequality (3.6) follows immediately from (3.9) and (3.10).

Finally, in the case of \( \beta > 0 \), by a similar argument as above we can use the functions \( \tilde{\varphi}(s, x; \varepsilon) \) and \( \tilde{\mu}_{a,\beta,\gamma,\varepsilon}(t) \) defined respectively in (2.14) and (2.16) of Proposition 2.1 instead of \( \tilde{\varphi}(s, x; \varepsilon) \) and \( \tilde{\mu}_{a,\gamma,\varepsilon}(t) \), and apply (2.19) in Proposition 2.1 to get the desired inequality (3.7). The proof is then completed. \( \square \)

Remark 3.4. From the above proof, it is easy to see that in Proposition 3.3, if \(|Y|\) and \(|\xi|\) are replaced with \(Y_+^+\) and \(\xi_+\) respectively, and \((H1)\) is replaced with the following assumption \((H1')\):

\[
(H1') \quad \text{There exist three constants } \alpha \in (1, 2), \beta \geq 0 \text{ and } \gamma > 0 \text{ such that } d\mathbb{P} \times dt - a.e.,
\]

\[
g(\omega, t, y, z)1_{y>0} \leq |g(\omega, t, 0, 0)| + \beta|y| + \gamma|z|, \quad (y, z) \in \mathbb{R} \times \mathbb{R}^d,
\]

then the conclusions of Proposition 3.3 still hold for \(Y_+^+\) and \(\xi_+\), but the term \(|Z_\varepsilon|^2\) in (3.6) and (3.7) needs to be replaced with \(1_{Y_\varepsilon>0}|Z_\varepsilon|^2\). For this, in the above proof one needs to respectively use \(Y_+^+\), \(1_{Y_\varepsilon>0}\) and \(\frac{1}{2}L_\varepsilon\) instead of \(|Y|\), sgn(\(Y\)) and \(L_\varepsilon\).

Now, we can give the proof of Theorem 3.1.

The proof of Theorem 3.1. For any given positive integers \(n, p \geq 1\), set

\[
\xi^{n,p} := \xi_+^+ \wedge n - \xi^- \wedge p \quad \text{and} \quad g^{n,p}(\omega, t, y, z) := g^+ (\omega, t, y, z) \wedge n - g^- (\omega, t, y, z) \wedge p.
\]
As both the terminal condition $\xi^{n,p}$ and the generator $g^{n,p}$ are bounded and $g^{n,p}(t, y, z)$ remains to be continuous in $(y, z)$, in view of the existence result in Lepeltier and San Martin [20], the following BSDE($\xi^{n,p}, g^{n,p}$) admits a maximal bounded solution $(Y^{n,p}_t, Z^{n,p}_t)_{t \in [0,T]}$ such that $Y^{n,p}$ is a bounded process and $Z^{n,p} \in \mathcal{M}^2$:

$$
Y^{n,p}_t = \xi^{n,p} + \int_t^T g^{n,p}(s, Y^{n,p}_s, Z^{n,p}_s)ds - \int_t^T Z^{n,p}_s \cdot dB_s, \quad t \in [0, T].
$$

(3.11)

And, by virtue of the comparison theorem, $Y^{n,p}$ is nondecreasing in $n$ and non-increasing in $p$.

We now assume that $\beta = 0$ and there exists a constant $\mu > \tilde{\mu}_{\alpha,\gamma,T}^0$ such that (3.3) holds. Observe that the function

$$
\tilde{\mu}_{\alpha,\gamma,\varepsilon}(t) := \left(\tilde{\epsilon}_{\alpha,\gamma}(1 + \varepsilon)t + \varepsilon^{\frac{\alpha}{2}}\right)^{2-\alpha}
$$

defined in (2.15) is strictly increasing with respect to the variables $t \in [0, T]$ and $\varepsilon > 0$, $\tilde{\mu}_{\alpha,\gamma,\varepsilon}(T) \to \tilde{\mu}_{\alpha,\gamma,0}$ when $\varepsilon \to 0^+$ and $\tilde{\mu}_{\alpha,\gamma,\varepsilon}(T) \to +\infty$ when $\varepsilon \to +\infty$. Since $\mu > \tilde{\mu}_{\alpha,\gamma,T}^0$, we can conclude that there must exist a positive $\varepsilon_0 > 0$ such that for each $t \in [0, T]$,

$$
\varepsilon_0 = \tilde{\mu}_{\alpha,\gamma,\varepsilon_0}(0) \leq \tilde{\mu}_{\alpha,\gamma,\varepsilon_0}(t) \leq \tilde{\mu}_{\alpha,\gamma,\varepsilon_0}(T) = \mu.
$$

(3.12)

Thus, we can apply (3.6) in Proposition 3.3 with $\varepsilon = \varepsilon_0$ for BSDE (3.11) to get that there exists a positive constant $\delta_{\alpha,\varepsilon_0} > 0$ depending only on $(\alpha, \varepsilon_0)$ such that $\mathbb{P} - a.s.$, for each $t \in [0, T]$ and $n, p \geq 1$,

$$
\begin{align*}
\psi (|Y^{n,p}_t|, \varepsilon_0) &\leq \psi (|Y^{n,p}_t|, \tilde{\mu}_{\alpha,\gamma,\varepsilon_0}(t)) + \frac{\delta_{\alpha,\varepsilon_0}}{2} \mathbb{E} \left[ \int_t^T |Z^{n,p}_s|^2 ds \right] F_t \\
&\leq \exp \left( \tilde{\mu}_{\alpha,\gamma,\varepsilon_0}(T) k^{\frac{2}{\alpha}} \varepsilon_0 \right) \mathbb{E} \left[ \psi \left( |\xi^{n,p}| + \int_0^T |g^{n,p}(s, 0, 0)| ds, \tilde{\mu}_{\alpha,\gamma,\varepsilon_0}(T) \right) \right] F_t \\
&\leq \exp \left( \mu k^{\frac{2}{\alpha}} \varepsilon_0 \right) \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(s, 0, 0)| ds, \mu \right) \right] F_t < +\infty.
\end{align*}
$$

(3.13)

In previous inequality, we have used (3.12) together with definitions of $\xi^{n,p}$ and $g^{n,p}$. Now, in view of assumption (H1) and the fact that, by (3.13),

$$
|Y^{n,p}_t| = \left( \frac{1}{\varepsilon_0} \ln (\psi (|Y^{n,p}_t|, \varepsilon_0)) \right)^{\frac{2}{\alpha}}
$$

$$
\leq \left( \frac{1}{\varepsilon_0} \mu k^{\frac{2}{\alpha}} \varepsilon_0 + \frac{1}{\varepsilon_0} \ln \left( \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(s, 0, 0)| ds, \mu \right) \right] F_t \right) \right)^{\frac{2}{\alpha}},
$$

we can apply the localization procedure developed initially in Briand and Hu [5] to obtain the existence of a progressively measurable process $(Z_t)_{t \in [0,T]}$ such that $d\mathbb{P} \times dt - a.e.$, $Z^{n,p}$ tends to
as \( n, p \) tends to infinity and the pair of \((Y := \inf_p \sup_n Y^{n,p}, Z)\) is a solution to BSDE(\(\xi, g\)). Moreover, we can send \( n \) and \( p \) to infinity in (3.13) and use Fatou’s lemma to get the inequality (3.4), and then \((\psi((Y_t, \bar{\mu}_{\alpha,\gamma,\varepsilon}(t)))_{t \in [0,T]}\) belongs to class (D), and \( Z \in \mathcal{M}^2 \).

Finally, in the case of \( \beta > 0 \), by a similar argument as above we can use \( \bar{\mu}_{\alpha,\beta,\gamma,\varepsilon}(t) \) defined in (2.16) of Proposition 2.1 instead of \( \bar{\mu}_{\alpha,\gamma,\varepsilon}(t) \), and apply (3.7) with \( \varepsilon = \varepsilon_0 \) in Proposition 3.3 instead of (3.6) to get the desired inequality (3.5). The theorem is then proved.

**Remark 3.5.** From the above proof, it is not very difficult to see that the sub-quadratic growth assumption \((H1)\) in Theorem 3.1 and Proposition 3.3 can be relaxed to the following one-sided sub-quadratic growth assumption, which will be used in Section 5 and Section 6,

\[(H1') \text{ There exist four real constants } \alpha \in (1,2), \beta \geq 0, \gamma > 0 \text{ and } c > 0, \text{ and a progressively measurable } \mathbb{R}_+\text{-valued process } (f_t)_{t \in [0,T]} \text{ such that } \mathbb{dP} \times \mathbb{dt} - \text{a.e.}, \text{ for each } (y, z) \in \mathbb{R} \times \mathbb{R}^d,\]

\[\text{sgn}(y)g(\omega, t, y, z) \leq f_t(\omega) + \beta|y| + \gamma|z|^\alpha \quad \text{and} \quad |g(\omega, t, y, z)| \leq f_t(\omega) + h(|y|) + c|z|^2,\]

where \( h(\cdot) \) is a nondecreasing, continuous and deterministic function with \( h(0) = 0 \).

In this case, one only needs to replace the process \(|g(t,0,0)|\) in the conditions of Theorem 3.1 and Proposition 3.3 with the process \( f_t \).

### 4. Uniqueness and comparison theorem of the solutions

In this section, we will prove the uniqueness and comparison theorem for the unbounded solutions of BSDE (1.1) with the terminal condition \( \xi \) and the generator \( g \) satisfying assumption \((H1)\) with parameters \( \alpha, \beta \) and \( \gamma \), and the following two assumptions \((H2)\) and \((H3)\):

\[(H2) \text{ } \mathbb{dP} \times \mathbb{dt} - \text{a.e.}, \text{ the generator } g \text{ is convex or concave with respect to the variables } (y, z).\]

\[(H3) \text{ The terminal condition } \xi + \int_0^T |g(t,0,0)|dt \text{ has sub-exponential moments of any order, i.e., for any } p > 0, \text{ we have}\]

\[
\mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(t,0,0)|dt, p \right) \right] = \mathbb{E} \left[ \exp \left\{ p \left( |\xi| + \int_0^T |g(t,0,0)|dt \right)^\frac{2}{\alpha} \right\} \right] < +\infty.
\]

**Theorem 4.1.** Assume that \( \xi \) is a terminal condition, \( g \) is a generator which is continuous in \((y, z)\) and satisfies assumption \((H1)\) with parameters \( \alpha, \beta \) and \( \gamma \).
If the generator and the terminal condition further satisfy assumptions (H2) and (H3), then BSDE($\xi, g$) admits a unique solution $(Y_t, Z_t)_{t \in [0,T]}$ such that $\sup_{t \in [0,T]} |Y_t|$ has sub-exponential moments of any order, i.e.,

$$\forall p > 0, \quad \mathbb{E} \left[ \psi \left( \sup_{t \in [0,T]} |Y_t|, p \right) \right] = \mathbb{E} \left[ \exp \left\{ p \left( \sup_{t \in [0,T]} |Y_t| \right)^{\frac{2}{\beta}} \right\} \right] < +\infty. \quad (4.2)$$

Furthermore, $Z. \in \mathcal{M}^p$ for all $p > 0$.

Proof. Firstly, since the generator $g$ satisfies (H1) and (4.1) holds, it follows from Theorem 3.1 together with its proof that BSDE($\xi, g$) admits a solution $(Y_t, Z_t)_{t \in [0,T]}$ such that for each $\varepsilon > 0$, $\psi(|Y_t|, \bar{\mu}_{\alpha,\gamma,\varepsilon}(t))$ belongs to class (D) for $\beta = 0$, $\psi(|Y_t|, \bar{\mu}_{\alpha,\beta,\gamma,\varepsilon}(t))$ belongs to class (D) for $\beta > 0$, and $Z. \in \mathcal{M}^2$.

Now, we show (4.2). Indeed, since (4.1) holds, by virtue of Proposition 3.3 and Theorem 3.1 together with their proofs we can conclude that for each $p > 0$, $\mathbb{P} - a.s.,$ for each $t \in [0, T],$

$$\psi(|Y_t|, p) \leq \psi(|Y_t|, \bar{\mu}_{\alpha,\gamma,p}(t)) \leq \tilde{C}_{\alpha,\gamma,p,T} \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(s, 0, 0)|ds, \bar{\mu}_{\alpha,\gamma,p}(T) \right) | \mathcal{F}_t \right]$$

holds for $\beta = 0,$ and

$$\psi(|Y_t|, p) \leq \psi(|Y_t|, \bar{\mu}_{\alpha,\beta,\gamma,p}(t)) \leq \tilde{C}_{\alpha,\beta,\gamma,p,T} \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(s, 0, 0)|ds, \bar{\mu}_{\alpha,\beta,\gamma,p}(T) \right) | \mathcal{F}_t \right]$$

holds for $\beta > 0,$ where $\bar{\mu}_{\alpha,\gamma,p}(\cdot), k_{\alpha,p}$ and $\bar{\mu}_{\alpha,\beta,\gamma,p}(\cdot)$ are respectively defined in (2.15) and (2.16),

$$\tilde{C}_{\alpha,\gamma,p,T} := \exp \left( \bar{\mu}_{\alpha,\gamma,p}(T) k_{\alpha,p} \right) \quad \text{and} \quad \tilde{C}_{\alpha,\beta,\gamma,p,T} := \exp \left( \bar{\mu}_{\alpha,\beta,\gamma,p}(T) k_{\alpha,p} \right). \quad (4.3)$$

Consequently, in the case of $\beta = 0$, for each $t \in [0, T]$ and $p > 0$, we can derive

$$\psi \left( \sup_{t \in [0,T]} |Y_t|, p \right) \leq \tilde{C}_{\alpha,\gamma,p,T} \sup_{t \in [0,T]} \left\{ \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(s, 0, 0)|ds, \bar{\mu}_{\alpha,\gamma,p}(T) \right) | \mathcal{F}_t \right] \right\}, \quad (4.4)$$

and in the case of $\beta > 0$,

$$\psi \left( \sup_{t \in [0,T]} |Y_t|, p \right) \leq \tilde{C}_{\alpha,\beta,\gamma,p,T} \sup_{t \in [0,T]} \left\{ \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |g(s, 0, 0)|ds, \bar{\mu}_{\alpha,\beta,\gamma,p}(T) \right) | \mathcal{F}_t \right] \right\}. \quad (4.5)$$

Thus, with the help of Doob’s maximal inequality on martingale, the desired inequality (4.2) follows from inequalities (4.4), (4.5) and (4.1).

In the sequel, we prove that $Z. \in \mathcal{M}^p$ for all $p > 0$. We only prove the case of $\beta = 0$, and the case of $\beta > 0$ can be proved in the same way. Let the function $\bar{\varphi}(s, x; \varepsilon)$ be defined in (2.13). In the case of $\beta = 0$, it follows from (3.8) that there exists a constant $\delta > 0$ depending only on $\alpha$ such that for each integer $m \geq 1$,

$$\frac{\delta}{2} \int_0^{\sigma_m} |Z_s|^2 ds \leq \bar{\varphi}(\sigma_m, \bar{Y}_{\sigma_m}; 1) - \bar{\varphi}(0, \bar{Y}_0; 1) + \int_0^{\sigma_m} \bar{\varphi}_x(s, \bar{Y}_s; 1) \text{sgn}(\bar{Y}_s) Z_s \cdot dB_s,$$
where \( \bar{Y}_t := |Y_t| + \int_0^t |g(s, 0, 0)|ds \) and \( \sigma_m \) is a stopping time defined by
\[
\sigma_m := \inf \left\{ s \in [0, T] : \int_0^s (\bar{\varphi}_x(r, \bar{Y}_r; 1))^2 |Z_r| dr \geq m \right\} \wedge T.
\]

Then for each real \( p > 0 \), we have
\[
\left( \int_0^{\sigma_m} |Z_s|^2 ds \right)^{\frac{p}{2}} \leq \left( \frac{4}{\delta} \right)^{\frac{p}{2}} \left[ (\bar{\varphi}(\sigma_m, \bar{Y}_{\sigma_m}; 1))^{\frac{p}{2}} + \sup_{t \in [0, T]} \left| \int_0^{t \wedge \sigma_m} \bar{\varphi}_x(s, \bar{Y}_s; 1) \text{sgn}(Y_s)Z_s \cdot dB_s \right|^{\frac{p}{2}} \right].
\]

In view of inequality \((a + b)^\lambda \leq a^\lambda + b^\lambda\) for \( a, b \geq 0 \) and \( \lambda \in (0, 1) \) and Hölder’s inequality together with (2.13), (4.1) and (4.2), we get that for each \( q > 1 \),
\[
\mathbb{E} \left[ \left( \sup_{t \in [0, T]} \bar{\varphi}(t, \bar{Y}_t; 1) \right)^q \right] \leq \left( \tilde{C}_{\alpha, \gamma, 1, T} \right)^q \mathbb{E} \left[ \psi \left( \sup_{t \in [0, T]} |Y_t| + \int_0^T |g(s, 0, 0)|ds, q\tilde{\mu}_{\alpha, \gamma, 1}(T) \right) \right] < +\infty,
\]
where \( \tilde{\mu}_{\alpha, \gamma, 1}(\cdot) \) and \( \tilde{C}_{\alpha, \gamma, 1, T} \) are respectively defined in (2.15) and (4.3). Note from (2.5) that for each \( s \in [0, T] \) and \( x \geq 0 \), we have \( \bar{\varphi}_x(s, x; 1) \leq K \bar{\varphi}(s, x; 1) \) with
\[
K := \frac{2(\alpha - 1)\tilde{\mu}_{\alpha, \gamma, 1}(T)}{\alpha k_{\alpha, 1}},
\]
where \( k_{\alpha, 1} \) is defined in (2.15). It follows from the BDG inequality that there exists a constant \( C > 0 \) depending only on \((p, \alpha)\) such that for each \( m \geq 1 \),
\[
\left( \frac{4}{\delta} \right)^{\frac{p}{2}} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^{t \wedge \sigma_m} \bar{\varphi}_x(s, \bar{Y}_s; 1) \text{sgn}(Y_s)Z_s \cdot dB_s \right|^{\frac{p}{2}} \right] \leq C \mathbb{E} \left[ \left( \sup_{t \in [0, T]} \bar{\varphi}(t, \bar{Y}_t; 1) \right)^{\frac{p}{2}} \left( \int_0^{\sigma_m} |Z_s|^2 ds \right)^{\frac{p}{2}} \right] \leq \frac{1}{2} \mathbb{E} \left[ \left( \int_0^{\sigma_m} |Z_s|^2 ds \right)^{\frac{p}{2}} \right] + \frac{C^2 K^p}{2} \mathbb{E} \left[ \left( \sup_{t \in [0, T]} \bar{\varphi}(t, \bar{Y}_t; 1) \right)^p \right].
\]
Combining the previous three inequalities yields the existence of a constant \( \bar{C} > 0 \) depending only on \((p, \alpha, \gamma, T)\) such that for each \( m \geq 1 \),
\[
\mathbb{E} \left[ \left( \int_0^{\sigma_m} |Z_s|^2 ds \right)^{\frac{p}{2}} \right] \leq \bar{C} \mathbb{E} \left[ \psi \left( \sup_{t \in [0, T]} |Y_t| + \int_0^T |g(s, 0, 0)|ds, p\tilde{\mu}_{\alpha, \gamma, 1}(T) \right) \right] < +\infty,
\]
from which the conclusion that \( Z \in \mathcal{M}^p \) for all \( p > 0 \) follows using Fatou’s lemma.

Finally, the uniqueness part is a direct consequence of the following comparison theorem—Theorem 4.2. Theorem 4.1 is then proved. \( \square \)
Let us turn to the comparison theorem of the unbounded solutions.

**Theorem 4.2.** Let $\xi$ and $\xi'$ be two terminal conditions, $g$ and $g'$ be two generators which are continuous with respect to the state variables $(y, z)$, and $(Y_t, Z_t)_{t \in [0, T]}$ and $(Y'_t, Z'_t)_{t \in [0, T]}$ be respectively a solution to BSDE$(\xi, g)$ and BSDE$(\xi', g')$ such that

$$\forall p > 0, \quad \mathbb{E} \left[ \psi \left( \sup_{t \in [0, T]} (|Y_t| + |Y'_t|) + \int_0^T (|g(t, 0, 0)| + |g'(t, 0, 0)|) \, dt \right) \right] < +\infty. \quad (4.6)$$

Assume that $\mathbb{P} - \text{a.s., } \xi \leq \xi'$. If $g$ (resp. $g'$) verifies assumptions (H1) and (H2), and

$$d\mathbb{P} \times dt - \text{a.e., } \quad g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t) \quad (\text{resp. } g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t)), \quad (4.7)$$

then $\mathbb{P} - \text{a.s., for each } t \in [0, T], \ Y_t \leq Y'_t$.

**Proof.** We first consider the case that the generator $g$ satisfies (H1) with parameters $\alpha, \beta$ and $\gamma$, and is convex in $(y, z)$, and $d\mathbb{P} \times dt - \text{a.e., } g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t)$. In order to utilize the convexity of $g$, we use the $\theta$-technique developed in Briand and Hu [6]. For each fixed $\theta \in (0, 1)$, define

$$\delta_\theta U_t := \frac{Y_t - \theta Y'_t}{1 - \theta} \quad \text{and} \quad \delta_\theta V : = \frac{Z_t - \theta Z'_t}{1 - \theta}. \quad (4.8)$$

Then the pair $(\delta_\theta U_t, \delta_\theta V)$ verifies the following BSDE:

$$d\delta_\theta U_t = \delta_\theta U_T + \int_t^T \delta_\theta g(s, \delta_\theta U_s, \delta_\theta V_s) \, ds - \int_t^T \delta_\theta V_s \, dB_s, \quad t \in [0, T], \quad (4.9)$$

where $d\mathbb{P} \times ds - \text{a.e., for each } (y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\delta_\theta g(s, y, z) := \frac{1}{1 - \theta} \left[ g(s, (1 - \theta)y + \theta Y'_s, (1 - \theta)z + \theta Z'_s) - \theta g(s, Y'_s, Z'_s) \right]$$

$$\quad + \frac{\theta}{1 - \theta} \left[ g(s, Y'_s, Z'_s) - g'(s, Y'_s, Z'_s) \right]. \quad (4.10)$$

It follows from the assumptions that $d\mathbb{P} \times ds - \text{a.e., for each } (y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\delta_\theta g(s, y, z) 1_{y > 0} \leq g(s, y, z) 1_{y > 0} \leq |g(s, 0, 0)| + \beta|y| + \gamma|z|^{\alpha}, \quad (4.11)$$

which means that the generator $\delta_\theta g$ satisfies assumption (H1') defined in Remark 3.4. Thus, in view of (4.6) and (4.11) and by virtue of Remark 3.4 together with the proof of Proposition 3.3, we can conclude for BSDE (4.9) that $\mathbb{P} - \text{a.s., for each } t \in [0, T]$, the inequality

$$\psi \left( \delta_\theta U_t^+, 1 \right) \leq \bar{C}_{\alpha, \gamma, 1, T} \mathbb{E} \left[ \psi \left( \delta_\theta U_T^+ + \int_0^T |g(s, 0, 0)| \, ds, \bar{\mu}_{\alpha, \gamma, 1}(T) \right) \right] \quad (4.12)$$

holds for $\beta = 0$, and the inequality

$$\psi \left( \delta_\theta U_t^+, 1 \right) \leq \bar{C}_{\alpha, \beta, \gamma, 1, T} \mathbb{E} \left[ \psi \left( \delta_\theta U_T^+ + \int_0^T |g(s, 0, 0)| \, ds, \bar{\mu}_{\alpha, \beta, \gamma, 1}(T) \right) \right] \quad (4.13)$$
holds for $\beta > 0$, where $\tilde{\mu}_{\alpha,\gamma,1}(\cdot)$, $\tilde{\mu}_{\alpha,\beta,\gamma,1}(\cdot)$, $\tilde{C}_{\alpha,\gamma,1,T}$ and $\tilde{C}_{\alpha,\beta,\gamma,1,T}$ are respectively defined in (2.15), (2.16) and (4.3). Moreover, in view of the fact that
\[
\delta_\theta U^+_T = \frac{(\xi - \theta \xi')^+}{1 - \theta} = \frac{|\xi - \theta \xi + \theta(\xi - \xi')|^+}{1 - \theta} \leq \xi^+,
\]
by (4.12) and (4.13) we derive that $\mathbb{P} - a.s.$, for each $t \in [0, T]$, for $\beta = 0$,
\[
(Y_t - \theta Y'_t)^+ \leq (1 - \theta) \left( \ln \left\{ \tilde{C}_{\alpha,\gamma,1,T} \mathbb{E} \left[ \psi \left( \xi^+ + \int_0^T |g(s, 0, 0)| ds, \tilde{\mu}_{\alpha,\gamma,1}(T) \right) \right] \right\} \right) \frac{\tilde{a}^*}{T},
\]
and for $\beta > 0$,
\[
(Y_t - \theta Y'_t)^+ \leq (1 - \theta) \left( \ln \left\{ \tilde{C}_{\alpha,\beta,\gamma,1,T} \mathbb{E} \left[ \psi \left( \xi^+ + \int_0^T |g(s, 0, 0)| ds, \tilde{\mu}_{\alpha,\beta,\gamma,1}(T) \right) \right] \right\} \right) \frac{\tilde{a}^*}{T}.
\]
Consequently, the desired conclusion follows by sending $\theta \to 1$ in the previous two inequalities.

For the case that the generator $g$ is concave with respect to the state variables $(y, z)$, we need to use $\theta Y_t - Y'_t$ and $\theta Z_t - Z'_t$ instead of $Y_t - \theta Y'_t$ and $Z_t - \theta Z'_t$ in (4.8) respectively. And, in this case the generator $\delta_\theta g$ in (4.10) should be replaced with
\[
\delta_\theta g(s, y, z) := \frac{1}{1 - \theta} \left[ \theta g(s, y, z) - g(s, (1 - \theta)y + \theta y, -(1 - \theta)z + \theta z) \right] + \frac{1}{1 - \theta} \left[ g(s, y, z) - g(s, (1 - \theta)y + \theta y, -(1 - \theta)z + \theta z) \right].
\]
Since $g$ is concave in $(y, z)$, we have, $d\mathbb{P} \times ds - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,
\[
g(s, (1 - \theta)y + \theta y, -(1 - \theta)z + \theta z) \geq \theta g(s, y, z) + (1 - \theta)g(t, -y, -z),
\]
and then, (4.11) needs to be replaced by
\[
\delta_\theta g(s, y, z) 1_{y > 0} \leq -g(s, -y, -z) 1_{y > 0} \leq |g(s, 0, 0)| + \beta |y| + \gamma |z|^\alpha,
\]
which means that the generator $\delta_\theta g$ still satisfies assumption (H1'). Consequently, both (4.12) and (4.13) still hold. Moreover, we use
\[
\delta_\theta U^+_T = \frac{(\theta \xi - \xi')^+}{1 - \theta} = \frac{|\theta \xi - \xi + (\xi - \xi')|^+}{1 - \theta} \leq (-\xi)^+ = \xi^-,
\]
instead of (4.14), and by virtue of (4.12) and (4.13), derive that $\mathbb{P} - a.s.$, for each $t \in [0, T]$, for $\beta = 0$,
\[
(\theta Y_t - Y'_t)^+ \leq (1 - \theta) \left( \ln \left\{ \tilde{C}_{\alpha,\gamma,1,T} \mathbb{E} \left[ \psi \left( \xi^+ + \int_0^T |g(s, 0, 0)| ds, \tilde{\mu}_{\alpha,\gamma,1}(T) \right) \right] \right\} \right) \frac{\tilde{a}^*}{T},
\]
and for $\beta > 0$,
\[
(\theta Y_t - Y'_t)^+ \leq (1 - \theta) \left( \ln \left\{ \tilde{C}_{\alpha,\beta,\gamma,1,T} \mathbb{E} \left[ \psi \left( \xi^+ + \int_0^T |g(s, 0, 0)| ds, \tilde{\mu}_{\alpha,\beta,\gamma,1}(T) \right) \right] \right\} \right) \frac{\tilde{a}^*}{T}.
\]
Thus, the desired conclusion follows by sending $\theta \to 1$ in the previous two inequalities.

Finally, in the same way as above, one can prove the desired conclusion under the conditions that the generator $g'$ satisfies assumptions (H1) and (H2), and $d\mathbb{P} \times dt - a.e., \ g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t)$. The proof of Theorem 4.2 is then complete.

Remark 4.3. Clearly, if $d\mathbb{P} \times dt - a.e.,$ for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d,$ $g(t, y, z) \leq g'(t, y, z)$, then the inequality (4.7) holds.

Remark 4.4. From the above proofs, it is not hard to verify that the assumption (H1) in Theorem 4.1 and Theorem 4.2 can also be relaxed to the weaker assumption (H1”) defined in Remark 3.5.

5. An extension to the comparison theorem

In this section, we first introduce a general non-convexity (non-convexity) assumption (H2’) on the generator $g$, and then illustrate that it is strictly weaker than the assumption (H2) provided that the assumption (H1”) or (H1) holds for $g$. Finally, we prove that Theorems 4.1 and 4.2 hold still under the weaker assumptions (H1”) and (H2’). Let us start by introducing assumption (H2’):

(H2’) There exist four real constants $\alpha \in (1, 2)$, $\beta \geq 0$, $\gamma > 0$ and $k > 0$, and a progressively measurable $\mathbb{R}_+$-valued process $(f_t)_{t \in [0, T]}$ such that $d\mathbb{P} \times dt - a.e.,$ for each $(y_i, z_i) \in \mathbb{R} \times \mathbb{R}^d, \ i = 1, 2$ and each $\theta \in (0, 1)$, it holds that

\[
1_{\{y_1 - \theta y_2 > 0\}} (g(\omega, t, y_1, z_1) - \theta g(\omega, t, y_2, z_2)) \leq (1 - \theta) (f_t(\omega) + k|y_2| + \beta |\delta y| + \gamma |\delta z|^\alpha) \tag{5.1}
\]

or

\[
-1_{\{y_1 - \theta y_2 < 0\}} (g(\omega, t, y_1, z_1) - \theta g(\omega, t, y_2, z_2)) \leq (1 - \theta) (f_t(\omega) + k|y_2| + \beta |\delta y| + \gamma |\delta z|^\alpha) \tag{5.2}
\]

where

\[
\delta y := \frac{y_1 - \theta y_2}{1 - \theta}, \quad \delta z := \frac{z_1 - \theta z_2}{1 - \theta}.
\]

One typical example of (H2’) is $g(\omega, t, y, z) := g_1(y) + g_2(y)$, where $g_1 : \mathbb{R} \to \mathbb{R}$ is convex or concave with one-sided linear growth, and $g_2 : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, i.e., $g$ is a Lipschitz perturbation of some convex (concave) function.

Another typical example of (H2’) is $\bar{g}(\omega, t, y, z) := g_3(z) + g_4(z)$, where $g_3 : \mathbb{R}^d \to \mathbb{R}$ is convex or concave with sub-quadratic growth, and $g_4 : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz function with bounded support, i.e., $\bar{g}$ is a locally Lipschitz perturbation of some convex (concave) function.
More generally, we have

**Proposition 5.1.** Assume that the generator \( g \) is continuous in \((y, z)\) and satisfies assumption \((H1')\). Then, assumption \((H2')\) holds for \( g \) if it satisfies anyone of the following conditions:

(i) \( \text{d}\mathbb{P} \times \text{d}t - \text{a.e.}, g(\omega, t, \cdot, \cdot) \) is convex or concave;

(ii) \( \text{d}\mathbb{P} \times \text{d}t - \text{a.e.}, \) for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), \( g(\omega, t, \cdot, z) \) is Lipschitz and \( g(\omega, t, y, \cdot) \) is convex or concave;

(iii) \( g(t, y, z) \equiv l(y)q(z) \), where both \( l : \mathbb{R} \to \mathbb{R} \) and \( q : \mathbb{R}^d \to \mathbb{R} \) are bounded Lipschitz functions, and the function \( q(z) \) has a bounded support.

Before giving the proof of this proposition, we first make the following important remark.

**Remark 5.2.** It is easy to verify that if for \( i = 1, 2 \), the generator \( g_i \) is continuous in \((y, z)\) and satisfies assumption \((H1')\) together with anyone of (i), (ii) and (iii) in Proposition 5.1 (with the same convexity or concavity when available), then \( g_1 + g_2 \) also satisfies assumption \((H2')\). Consequently, the generator \( g \) satisfying \((H2')\) may be not necessarily convex (concave) or Lipschitz in the variables \( y \) and \( z \), and it can have a general growth in \( y \).

**Proof of Proposition 5.1.** Given \((y_i, z_i) \in \mathbb{R} \times \mathbb{R}^d, i = 1, 2 \) and \( \theta \in (0, 1) \).

(i) Assume that \( \text{d}\mathbb{P} \times \text{d}t - \text{a.e.}, g(\omega, t, \cdot, \cdot) \) is convex. In view of \((H1')\), if \( \delta \theta y > 0 \), then

\[
g(\omega, t, y_1, z_1) = g(\omega, t, \theta y_2 + (1 - \theta)\delta \theta y, \theta z_2 + (1 - \theta)\delta \theta z) \\
\leq \theta g(\omega, t, y_2, z_2) + (1 - \theta)g(\omega, t, \delta \theta y, \delta \theta z) \\
\leq \theta g(\omega, t, y_2, z_2) + (1 - \theta)(f(\omega) + \beta|\delta \theta y| + \gamma|\delta \theta z|^\alpha).
\]

Thus, the inequality \((5.1)\) holds with \( k = 0 \). The concave case is similar.

(ii) Assume that \( \text{d}\mathbb{P} \times \text{d}t - \text{a.e.}, \) for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), \( g(\omega, t, \cdot, z) \) is Lipschitz with Lipschitz constant \( \beta \), and \( g(\omega, t, y, \cdot) \) is convex. Then, noticing by \((H1')\) that \( |g(\omega, t, 0, z)| \leq f + \gamma|z|^2 \), we have

\[
g(\omega, t, y_1, z_1) - \theta g(\omega, t, y_2, z_2) \\
\leq |g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_1)| + g(\omega, t, y_2, z_1) - \theta g(\omega, t, y_2, z_2) \\
\leq \beta|y_1 - y_2| + g(\omega, t, y_2, \theta z_2 + (1 - \theta)\delta \theta z) - \theta g(\omega, t, y_2, z_2) \\
\leq \beta(|y_1 - \theta y_2| + (1 - \theta)|y_2|) + (1 - \theta)(|g(\omega, t, y_2, \delta \theta z) - g(\omega, t, 0, \delta \theta z)| + |g(\omega, t, 0, \delta \theta z)|) \\
\leq (1 - \theta)(\beta|\delta \theta y| + 2\beta|y_2| + f(\omega) + \gamma|\delta \theta z|^\alpha).
\]

Thus, \((5.1)\) holds with \( 2\beta \) instead of \( k \). The concave case is similar.
(iii) With loss of generality, we assume that the functions \( l(y) \) and \( q(z) \) have Lipschitz constants \( \beta \) and \( \gamma \) together with a same bound \( M > 0 \), and \( q(z) \equiv 0 \) when \( |z| > R \) for some \( R > 0 \). Noticing that

\[
|q(\theta z_2) - q(z_2)| = |g(\theta z_2) - q(z_2)|1_{\theta \in (0,1/2]} + |g(\theta z_2) - q(z_2)|1_{\theta \in (1/2,1]}
\]

\[
\leq (1 - \theta) \frac{2M}{1 - \theta} 1_{\theta \in (0,1/2]} + (1 - \theta)|\gamma| |z_2| 1_{|z_2| \leq 2R} 1_{\theta \in (1/2,1]}
\]

we have

\[
g(\omega, t, y_1, z_1) - \theta g(\omega, t, y_2, z_2) = l(y_1)q(z_1) - \theta l(y_2)q(z_2)
\]

\[
\leq |l(y_1) - l(y_2)||q(z_1)| + |l(y_2)||q(z_1) - \theta q(z_2)|
\]

\[
\leq M\beta|y_1 - y_2| + M(|\gamma| z_1 - \theta z_2| + |\gamma| z_2| + (1 - \theta)|q(z_2)|)
\]

\[
\leq (1 - \theta)\gamma (\beta|\delta_0 y_1| + \beta|y_2| + \gamma|\delta_0 z_1| + 5M + 2\gamma R)
\]

\[
\leq (1 - \theta)\gamma (5M + 2\gamma R + \gamma + \beta|y_2| + \beta|\delta_0 y_1| + \gamma|\delta_0 z_1|).
\]

Thus, the inequality (5.1) holds with \( M(5M + 2\gamma R + \gamma) \) instead of \( f_\gamma \), \( M\beta \) instead of \( k \) and \( \beta \), and \( M\gamma \) instead of \( \gamma \) respectively. The proposition is then proved. \( \square \)

**Remark 5.3.** (i) Letting \( y_1 = y_2 = y \) and \( z_1 = z_2 = z \) in (5.1) and (5.2) respectively yields that

\[
1_{\{y > 0\}}g(\omega, t, y, z) \leq f_t(\omega) + (\beta + k)|y| + \gamma|z|^\alpha
\]

and

\[
-1_{\{y < 0\}}g(\omega, t, y, z) \leq f_t(\omega) + (\beta + k)|y| + \gamma|z|^\alpha,
\]

whose combination implies that \( g \) has a one-sided linear growth in the state variable \( y \) and a sub-quadratic growth in the state variable \( z \).

(ii) Letting first \( z_1 = z_2 = z \) in (5.1) and (5.2) and then letting \( \theta \to 1 \) yields that

\[
1_{\{y_1 - y_2 > 0\}} (g(\omega, t, y_1, z) - g(\omega, t, y_2, z)) \leq \beta|y_1 - y_2|,
\]

which means that \( g \) satisfies the so-called monotonicity condition in \( y \).

(iii) Set \( y = \frac{y_1 - \theta y_2}{1 - \theta}, \ z = \frac{z_1 - \theta z_2}{1 - \theta}, \ \bar{y} = y_2 \) and \( \bar{z} = z_2 \). Then, (5.1) and (5.2) can be respectively rewritten as the following forms:

\[
1_{\{y > 0\}} (g(\omega, t, (1 - \theta)y + \theta \bar{y}, (1 - \theta)z + \theta \bar{z}) - \theta g(\omega, t, \bar{y}, \bar{z}))
\]

\[
\leq (1 - \theta) (f_t(\omega) + k|\bar{y}| + \beta|y| + \gamma|z|^\alpha)
\]
are the main results of this section.

**Theorem 5.4.** Let \( \xi \) and \( \xi' \) be two terminal conditions such that \( \mathbb{P} \)-a.s., \( \xi \leq \xi' \), \( g \) and \( g' \) be two generators which are continuous in \((y,z)\), \( g \) (resp. \( g' \)) verifies assumption \((H2')\) with constants \((\alpha,\beta,\gamma,k)\) and process \( f \), and \((Y_t, Z_t)_{t \in [0,T]}\) and \((Y'_t, Z'_t)_{t \in [0,T]}\) be respectively a solution to BSDE\((\xi,g)\) and BSDE\((\xi',g')\) such that

\[
\forall p > 0, \quad \mathbb{E} \left[ \psi \left( \sup_{t \in [0,T]} \left( |Y_t| + |Y'_t| \right) + \int_0^T f_t dt, \ p \right) \right] < +\infty. \tag{5.5}
\]

If \( d\mathbb{P} \times dt \) - a.e., we have

\[
g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t) \quad (\text{resp. } g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t)),
\]

then \( \mathbb{P} \)-a.s., for each \( t \in [0,T] \), \( Y_t \leq Y'_t \).

**Proof.** We only prove the case that the generator \( g \) satisfies (5.1) with constants \((\alpha,\beta,\gamma,k)\) and process \( f \), and \( d\mathbb{P} \times dt \) - a.e., \( g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t) \). For each fixed \( \theta \in (0,1) \), with the notations in (4.8), we know that the pair \((\delta_\theta U, \delta_\theta V)\) verifies BSDE (4.9) with generator \( g \) defined in (4.10). Then, in view of (5.3), it follows from the assumptions that \( d\mathbb{P} \times ds \) - a.e., for each \((y,z) \in \mathbb{R} \times \mathbb{R}^d\), we have

\[
\delta_\theta g(s,y,z)1_{y>0} \leq |f_s| + k|Y'_s| + \beta|y| + \gamma|z|^{\alpha}, \tag{5.6}
\]

which means that the generator \( \delta_\theta g \) satisfies assumption \((H1')\) with the process \(|f_s| + k|Y'_s|\) instead of \(|g(\cdot,0,0)|\). Thus, thanks to (5.5), the rest of proof runs as that in Theorem 4.2. \( \square \)

**Theorem 5.5.** Assume that \( \xi \) is a terminal condition and \( g \) is a generator which is continuous in \((y,z)\) and satisfies assumptions \((H1'')\) and \((H2')\) with constants \((\alpha,\beta,\gamma,k)\) and process \( f \).

If the terminal condition \( \xi + \int_0^T f_t dt \) has sub-exponential moments of any order, i.e.,

\[
\forall p > 0, \quad \mathbb{E} \left[ \psi \left( \xi + \int_0^T f_t dt, \ p \right) \right] = \mathbb{E} \left[ \exp \left\{ p \left( \xi + \int_0^T f_t dt \right) \right\} \right] < +\infty,
\]

then BSDE\((\xi,g)\) admits a unique solution \((Y_t, Z_t)_{t \in [0,T]}\) such that \( \sup_{t \in [0,T]} |Y_t| \) has sub-exponential moments of any order (i.e., (4.2) holds). Furthermore, \( Z \in \mathcal{M}\P \) for all \( p > 0 \).

**Proof.** In view of Remark 3.5 and Theorem 5.4, the proof runs as that in Theorem 4.1. \( \square \)
Remark 5.6. From Remark 3.5, Proposition 5.1 and Remark 5.2, it is not difficult to see that Theorems 5.4 and 5.5 respectively extend Theorems 4.2 and 4.1 to the non-convexity and non-concavity case.

6. A stability theorem of the solutions

In this section, we establish the following stability result for the unbounded solutions of BSDEs under general assumptions (H1”) and (H2’).

Theorem 6.1. Assume that $\xi$ is a terminal condition, $g$ is a generator which is continuous in $(y, z)$ and satisfies assumptions (H1”) and (H2’) with constants $(\alpha, \beta, \gamma, k)$ and process $f$, and $(Y_t, Z_t)_{t \in [0,T]}$ is the (unique) solution to BSDE$(\xi; g)$ such that $\sup_{t \in [0,T]} |Y_t|$ has sub-exponential moments of any order.

Assume also that for each $n \geq 1$, $\xi^n$ is a terminal condition, $g^n$ is a generator which is continuous in $(y, z)$ and satisfies assumptions (H1”) and (H2’) with constants $(\alpha, \beta, \gamma, k)$ and process $f^n$, and $(Y^n_t, Z^n_t)_{t \in [0,T]}$ is the (unique) solution to BSDE$(\xi^n; g^n)$ such that $\sup_{t \in [0,T]} |Y^n_t|$ has sub-exponential moments of any order.

Let us assume further that for each $p > 0$,

$$
E \left[ \exp \left\{ p \left( |\xi| + \int_0^T f(t) \, dt \right)^{\frac{\alpha}{2}} \right\} \right] + \sup_{n \geq 1} E \left[ \exp \left\{ p \left( |\xi^n| + \int_0^T f^n(t) \, dt \right)^{\frac{\alpha}{2}} \right\} \right] < +\infty. \quad (6.1)
$$

If $P$–a.s., $\xi^n \to \xi$ as $n \to \infty$ and there exists a real $q > 1$ such that

$$
\lim_{n \to \infty} E \left[ \left( \int_0^T |g^n(s, Y_s, Z_s) - g(s, Y_s, Z_s)| \, ds \right)^q \right] = 0, \quad (6.2)
$$

then for each $p > 0$, we have

$$
\lim_{n \to \infty} E \left[ \exp \left\{ p \left( \sup_{t \in [0,T]} |Y^n_t - Y_t| \right)^{\frac{\alpha}{2}} \right\} \right] = 1. \quad (6.3)
$$

And, if the function $h(\cdot)$ defined in (H1”) further satisfies that for some constant $c > 0$,

$$
h(|x|) \leq c \exp(c |x|^{\frac{\alpha}{2}}), \quad x \in \mathbb{R}, \quad (6.4)
$$

then for each $p > 0$, we have

$$
\lim_{n \to \infty} E \left[ \left( \int_0^T |Z^n_s - Z_s|^2 \, ds \right)^{\frac{p}{2}} \right] = 0. \quad (6.5)
$$
Proof. It follows from the integrability assumption (6.1) and the proof of Theorem 5.5 and Theorem 4.1 (see, in particular, inequality (4.5)) that the sequence \((Y^n_t, Z^n_t)_{t\in[0,T]}\) satisfies

\[
\forall \ p > 0, \ \sup_{n\geq 1} \mathbb{E} \left[ \exp \left\{ p \left( \sup_{t\in[0,T]} |Y^n_t| \right)^\frac{2}{p} \right\} + \left( \int_0^T |Z^n_s|^2 \, ds \right)^\frac{p}{2} \right] < +\infty. \tag{6.6}
\]

It is thus enough to prove that

\[
\sup_{t\in[0,T]} |Y^n_t - Y_t| + \int_0^T |Z^n_s - Z_s|^2 \, ds
\]

converges to 0 in probability to get the desired conclusion.

We only prove the case that \(\beta = 0\) and inequality (5.1) holds for \(g\) and \(g^n\). The other cases can be proved in the same way. For each fixed \(\theta \in (0,1)\), define

\[
\delta^n_U := \frac{Y^n - \theta Y}{1 - \theta} \quad \text{and} \quad \delta^n_V := \frac{Z^n - \theta Z}{1 - \theta}.
\]

Then the pair \((\delta^n_U, \delta^n_V)\) verifies the following BSDE:

\[
\delta^n_U_t = \delta^n_U_T + \int_t^T \delta^n g(s, \delta^n U_s, \delta^n V_s) \, ds - \int_t^T \delta^n V_s \cdot dB_s, \quad t \in [0,T], \tag{6.7}
\]

where \(d\mathbb{P} \times ds - a.e., \) for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d, \)

\[
\delta^n g(s, y, z) := \frac{1}{1 - \theta} \left( g^n(s, (1 - \theta)y + \theta Y_s, (1 - \theta)z + \theta Z_s) - \theta g^n(s, Y_s, Z_s) \right) + D^n(s) \tag{6.8}
\]

with

\[
D^n(s) := \frac{\theta}{1 - \theta} \left( g^n(s, Y_s, Z_s) - g(s, Y_s, Z_s) \right).
\]

Since (5.1) holds for \(g^n\), it follows from (6.8) and (5.3) that \(d\mathbb{P} \times ds - a.e., \) for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d, \)

\[
\delta^n g(s, y, z) 1_{y>0} \leq f^n_a + k|Y_s| + |D^n(s)| + \gamma|z|^{\alpha}. \tag{6.9}
\]

Now, let the functions \(\tilde{\varphi}(s, x; \varepsilon)\) and \(\psi(x, \mu)\) be defined respectively in (2.13) and (3.1), and denote

\[
\Delta^n_U_t := (\delta^n_U_t)^+ + \int_0^t \left( f^n_a + k|Y_s| \right) \, ds, \quad t \in [0,T].
\]

Applying Itô-Tanaka’s formula to the process \(\tilde{\varphi}(s, \Delta^n_U_s; 1)\) and using (6.7), (6.9) and (2.18) in Proposition 2.1, a similar computation to that in the proof of Proposition 3.3 yields the existence of a positive constant \(\delta > 0\) depending only on \(\alpha\) such that

\[
d\tilde{\varphi}(s, \Delta^n_U_s; 1) \geq -\tilde{\varphi}_x(s, \Delta^n_U_s; 1)|D^n(s)| \, ds + \frac{\tilde{\varphi}_x^2(s, \Delta^n_U_s; 1)}{2} \, ds
\]

\[
+ \tilde{\varphi}_x(s, \Delta^n_U_s; 1) \mathbb{1}_{\delta^n_U_s > 0} \, d\tilde{\varphi}_x U_s \cdot dB_s, \quad s \in [0,T]. \tag{6.10}
\]

23
Note from (2.5) that for each $s \in [0, T]$ and $x \geq 0$, we have $\bar{\varphi}_x(s, x; 1) \leq K \bar{\varphi}(s, x; 1)$ for some positive constant $K$ depending only on $(\alpha, \gamma, T)$. It follows from the BDG inequality, Young’s inequality, (6.1) and (6.6) that the process

$$
\left( \int_0^t \bar{\varphi}_x(s, \Delta^\theta U_s; 1) 1_{\delta^\theta U_s > 0} \delta^\theta V_s \cdot dB_s \right)_{t \in [0, T]}
$$

is a uniformly integrable martingale. Then, from (6.10) we know that

$$
\bar{\varphi}(t, \Delta^\theta U_s; 1) + \frac{\bar{\delta}}{2} \mathbb{E} \left[ \int_t^T 1_{\delta^\theta U_s > 0} |\delta^\theta V_s|^2 ds \right] \mathcal{F}_t
\leq \mathbb{E} [\bar{\varphi}(T, \Delta^\theta U_T; 1) | \mathcal{F}_t] + K \mathbb{E} \left[ \int_t^T \bar{\varphi}(s, \Delta^\theta U_s; 1) |D^\theta(s)| ds \right] \mathcal{F}_t, \; t \in [0, T].
$$

Furthermore, by the definitions of functions $\bar{\varphi}$ and $\psi$ together with $\Delta^\theta U_s$ we can conclude that there exists a positive constant $\bar{K}$ depending only on $(\alpha, \gamma, T, k)$ such that $\mathbb{P} - a.s.$,

$$
\psi ((\delta^\theta U_t)^+, 1) + \frac{\bar{\delta}}{2} \mathbb{E} \left[ \int_t^T 1_{\delta^\theta U_s > 0} |\delta^\theta V_s|^2 ds \right] \mathcal{F}_t
\leq \bar{K} \mathbb{E} \left[ \psi (\Delta^\theta U_T, \bar{K}) + \int_0^T \psi (\Delta^\theta U_s, \bar{K}) |D^\theta(s)| ds \right] \mathcal{F}_t, \; t \in [0, T].
$$

Consequently, for each $n \geq 1, \theta \in (0, 1)$ and $t \in [0, T]$, we have

$$
(Y^n_t - \theta Y^n_t)^+ \leq (1 - \theta) \left( \bar{K} \mathbb{E} \left[ \psi (\Delta^\theta U_T, \bar{K}) + \int_0^T \psi (\Delta^\theta U_s, \bar{K}) |D^\theta(s)| ds \right] \right)^{\frac{1}{\theta}}.
$$

On the other hand, for each fixed $\theta \in (0, 1)$, we define

$$
\delta^\theta U := \frac{Y^n - \theta Y^n}{1 - \theta}, \quad \delta^\theta V := \frac{Z^n - \theta Z^n}{1 - \theta}.
$$

Then the pair $(\delta^\theta U, \delta^\theta V)$ verifies the following BSDE:

$$
\delta^\theta U_t = \delta^\theta U_T + \int_t^T \delta^\theta g(s, \delta^\theta U_s, \delta^\theta V_s) ds - \int_t^T \delta^\theta V_s \cdot dB_s, \; t \in [0, T],
$$

where $d\mathbb{P} \times ds - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$
\delta^\theta g(y, z) := \frac{1}{1 - \theta} \left( g^n(s, (1 - \theta)y + \theta Y^n_s, (1 - \theta)z + \theta Z^n_s) - \theta g^n(s, Y^n_s, Z^n_s) \right) + \bar{D}^\theta(s)
$$

with

$$
\bar{D}^\theta(s) := \frac{1}{1 - \theta} \left( g(s, Y_s, Z_s) - g^n(s, Y_s, Z_s) \right).
$$

Since (5.1) holds for $g^n$, it follows that $d\mathbb{P} \times ds - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$
\delta^\theta g(s, y, z) 1_{y > 0} \leq f^n_s + k|Y^n_s| + |\bar{D}^\theta(s)| + \gamma|z|^\alpha.
$$
A similar computation as that from inequality (6.10) to inequality (6.11) yields that for each $n \geq 1$, $t \in [0,T]$ and $\theta \in (0,1)$,

$$(Y_t - \theta Y^n_t)^+ \leq (1 - \theta) \left( K \mathbb{E} \left[ \psi \left( \Delta^n_{\theta} U_T, \bar{K} \right) + \int_0^T \psi \left( \Delta^n_{\theta} U_s, \bar{K} \right) |D^n_{\theta}(s)|ds \right] \bigg| \mathcal{F}_t \right) \right)^{\frac{2}{\alpha'}}. \tag{6.12}$$

In the sequel, combining (6.11) and (6.12) together with inequalities

$$Y^n_t - Y_t \leq (Y^n_t - \theta Y_t)^+ + (1 - \theta)|Y_t| \quad \text{and} \quad Y_t - Y^n_t \leq (Y_t - \theta Y^n_t)^+ + (1 - \theta)|Y^n_t|,$$

we can deduce that for each $n \geq 1$, $\theta \in (0,1)$ and $t \in [0,T]$,

$$|Y^n_t - Y_t| \leq (1 - \theta)(|Y_t| + |Y^n_t|) + 4(1 - \theta) \left( K \mathbb{E} \left[ \psi \left( X^n(\theta) + G^n_T, \bar{K} \right) \right] \right)^{\frac{2}{\alpha'}} + \frac{4}{(1 - \theta)^{\frac{\alpha - 2}{\alpha}}} \left( \frac{K}{(1 - \theta)^{\frac{2}{\alpha}}} \mathbb{E} \left[ \psi \left( H^n(\theta), \bar{K} \right) \psi (G^n_T, \bar{K}) \right] \right)^{\frac{2}{\alpha'}} \tag{6.13}$$

where

$$X^n(\theta) := \frac{|\xi - \theta \xi^n| \vee |\xi^n - \theta \xi|}{1 - \theta}, \quad H^n(\theta) := \frac{\sup_{t \in [0,T]} (|Y_t| + |Y^n_t|)}{1 - \theta}$$

and

$$G^n_T := \int_0^T (f^n_s + k|Y^n_s| + k|Y_s|) ds.$$

Now, let us fix $\varepsilon > 0$. It follows from (6.13) and Doob’s maximal inequality on martingale that

$$\mathbb{P} \left( \sup_{t \in [0,T]} |Y^n_t - Y_t| > \varepsilon \right) \leq \frac{3(1 - \theta)}{\varepsilon} \mathbb{E} \left[ \sup_{t \in [0,T]} (|Y^n_t| + |Y_t|) \right] + \left( \frac{12(1 - \theta)}{\varepsilon} \right)^{\frac{2}{\alpha'}} K \mathbb{E} \left[ \psi \left( X^n(\theta), \bar{K} \right) \psi (G^n_T, \bar{K}) \right] + \frac{K}{(1 - \theta)^{\frac{2}{\alpha}} \varepsilon} \mathbb{E} \left[ \psi \left( H^n(\theta), \bar{K} \right) \psi (G^n_T, \bar{K}) \right] \int_0^T |g^n(s, Y_s, Z_s) - g(s, Y_s, Z_s)| ds.$$

Observe from the inequalities (6.1) and (6.6) that the sequences

$$\left( \sup_{t \in [0,T]} (|Y^n_t| + |Y_t|) \right)_{n \geq 1}, \quad (\psi (G^n_T, \bar{K}))_{n \geq 1} \quad \text{and} \quad (\psi (H^n(\theta), \bar{K}))_{n \geq 1}$$

are bounded in all $L^p$ spaces. From the previous inequality together with Hölder’s inequality we deduce that there exist a universal constant $C > 0$ and a constant $C(\theta, q) > 0$ depending
only on $\theta$ and $q$ such that
\[
\begin{align*}
\mathbb{P} \left( \sup_{t \in [0,T]} |Y^n_t - Y_t| > \varepsilon \right) & \\
& \leq \frac{3(1-\theta)}{\varepsilon} C + \left( \frac{12(1-\theta)}{\varepsilon} \right)^{\frac{\alpha}{2}} \bar{K} C \left( \mathbb{E} \left[ \psi \left( X^n(\theta), 2\bar{K} \right) \right] \right)^{1/2} \\
& \quad + \frac{\bar{K} C(\theta, q)}{(1-\theta)^{\frac{\alpha c}{c-2}}} \left( \frac{12}{\varepsilon} \right)^{\frac{\alpha}{2}} \left( \mathbb{E} \left[ \left( \int_0^T |g^n(s, Y^n_s, Z^n_s) - g(s, Y_s, Z_s)| \, ds \right)^q \right] \right)^{\frac{1}{q}}.
\end{align*}
\] (6.14)

From inequality (6.1) and the fact that $\mathbb{P} - a.s., \xi^n \to \xi$, it follows that as $n$ goes to $\infty$, $\mathbb{E} \left[ \psi \left( X^n(\theta), 2\bar{K} \right) \right]$ converges to $\mathbb{E} \left[ \psi \left( |\xi|, 2\bar{K} \right) \right]$ for each $\theta \in (0,1)$. Thus, in view of (6.2), sending first $n \to \infty$ and then $\theta \to 1$ in inequality (6.14) yields that $\sup_{t \in [0,T]} |Y^n_t - Y_t|$ converges to 0 in probability, and the conclusion (6.3) follows due to the inequality (6.6).

Finally, let (6.4) be further satisfied, and we show that (6.5) holds. In fact, by Itô’s formula we get that for each $n \geq 1$,
\[
\begin{align*}
\mathbb{E} \left[ \int_0^T |Z^n_s - Z_s|^2 \, ds \right] & \\
& \leq \mathbb{E} \left[ |\xi^n - \xi|^2 + 2 \sup_{t \in [0,T]} |Y^n_t - Y_t| \int_0^T |g^n(s, Y^n_s, Z^n_s) - g(s, Y_s, Z_s)| \, ds \right].
\end{align*}
\] (6.15)

On the other hand, it follows from (H1”) with (6.4) as well as (6.1) and (6.6) that
\[
\begin{align*}
\sup_{n \geq 1} \mathbb{E} \left[ \left( \int_0^T |g^n(s, Y^n_s, Z^n_s) - g(s, Y_s, Z_s)| \, ds \right)^2 \right] & < +\infty.
\end{align*}
\] (6.16)

Then, by virtue of Hölder’s inequality, the desired conclusion (6.5) follows from (6.15), (6.16), (6.3) and (6.6). The proof is then complete.

**Remark 6.2.** We note that if assumption (H1”) for $g$ and $g^n$ in Theorem 6.1 is respectively replaced with the stronger assumption (H1) with the process $f_t$ instead of $|g(t,0,0)|$ and $|g^n(t,0,0)|$, and assumption (6.2) is replaced with the following assumption: $d\mathbb{P} \times dt - a.e.,$ for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d, g^n(t, y, z) \to g(t, y, z),$ then the conclusions (6.3) and (6.5) of Theorem 6.1 still hold. Indeed, it is easy to check that these two assumptions together with (6.1) can imply that (6.2) holds for any $q > 1$, and that (6.4) holds for some $c > 0$.

**7. Application to sub-quadratic PDEs**

In this section, we give an application of our results concerning BSDEs to PDEs which are sub-quadratic with respect to the gradient of the solution. More precisely, we will derive the nonlinear Feynman-Kac formula for these PDEs. Let us consider the following semilinear PDE
\[
\begin{align*}
\partial_t u(t,x) + \mathcal{L} u(t,x) + g(t,x,u(t,x),\sigma^2 \nabla_x u(t,x)) = 0, \quad u(T, \cdot) = h(\cdot),
\end{align*}
\] (7.1)
where $\mathcal{L}$ is the infinitesimal generator of the diffusion solution $X^{t,x}$ to the following SDE

$$X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r)dr + \int_t^s \sigma(r, X^{t,x}_r)dB_r, \quad t \leq s \leq T, \quad \text{and} \quad X^{t,x}_t = x, \quad 0 \leq s < t. \quad (7.2)$$

The nonlinear Feynman-Kac formula consists in proving that the function defined by

$$\forall \ (t, x) \in [0, T] \times \mathbb{R}^n, \quad u(t, x) := Y^t_x,$$

where, for each $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, $(Y^{t_0,x_0}_s, Z^{t_0,x_0}_s)$ represents the solution to the BSDE

$$Y_t = h(X^{t_0,x_0}_T) + \int_t^T g(s, X^{t_0,x_0}_s, Y_s, Z_s)ds + \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (7.4)$$

is a solution, at least a viscosity solution, to PDE (7.1).

The objective of this section is to derive the above probabilistic representation for the solution to PDE (7.1) when the nonlinearity $g$ is sub-quadratic of $\alpha \in (1, 2)$ order with respect to $\nabla_x u$ and when $h$ and $g$ have a power growth of $p < 2/\alpha^*$ order with respect to $x$. Let us first recall the following definition of a viscosity solution to PDE (7.1).

**Definition 7.1.** A continuous function $u$ defined on $[0, T] \times \mathbb{R}^n$ such that $u(T, \cdot) = h(\cdot)$ is said to be a viscosity super-solution (respectively sub-solution) to PDE (7.1) if

$$\partial_t u(t_0, x_0) + \mathcal{L} u(t_0, x_0) + g(t_0, x_0, u(t_0, x_0), \sigma^* \nabla_x \varphi(t_0, x_0)) \leq 0 \quad (\text{resp.} \geq 0)$$

as soon as the function $u - \varphi$ has a local minimum (resp. maximum) at the point $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$ where $\varphi$ is a smooth function. Moreover, a viscosity solution is both a viscosity super-solution and a viscosity sub-solution.

Let us now introduce our assumptions concerning the linear part of the PDE namely the coefficients of the diffusion.

(A1) $b(t, x) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(t, x) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are continuous functions and there exists a constant $K > 0$ such that for each $(t, x, x') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$|b(t, 0)| + |\sigma(t, x)| \leq K$$

and

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq K|x - x'|.$$

Classical results on SDEs show that under the assumption (A1), for each $(t, x) \in [0, T] \times \mathbb{R}^n$ the SDE (7.2) admits a unique solution $X^{t,x}_s \in \mathcal{S}^q$ for all $q \geq 1$. And, since $\sigma$ is a bounded function, an argument in page 563 of Briand and Hu [6] has showed that for each $q \in [1, 2)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$, we have

$$\forall \lambda > 0, \quad \mathbb{E} \left[ \sup_{s \in [0, T]} \exp \left( \lambda |X^{t,x}_s|^q \right) \right] \leq C \exp(\lambda C |x|^q), \quad (7.5)$$

27
where the constant $C$ depends only on $(g, \lambda, T, K)$. Furthermore, we assume that the point sequence $\{(t_m, x_m)\}_{m=1}^{\infty}$ in the space $[0, T] \times \mathbb{R}^n$ converges to a point $(t, x) \in [0, T] \times \mathbb{R}^n$ as $m$ tends to $+\infty$. Classical results on SDEs show that

$$\forall \lambda > 0, \quad \lim_{m \to \infty} \mathbb{E} \left[ \sup_{s \in [0, T]} |X_{t_m}^{t,x_m} - X_s^{t,x}|^\lambda \right] = 0. \quad (7.6)$$

And, by a similar analysis as that in page 563 of [6] we can also deduce that for each $q \in [1, 2)$,

$$\forall \lambda > 0, \quad \mathbb{E} \left[ \sup_{m \geq 1} \sup_{s \in [0, T]} \exp (\lambda |X_{t_m}^{t,x_m}|^q) \right] \leq C \exp (\lambda C |x|^q), \quad (7.7)$$

where the constant $C$ is the same as in (7.5).

With these observations in hand, we can give our assumptions on the nonlinear term of the PDE, the generator $g$, and the terminal condition.

(A2) $g(t, x, y, z) : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $h(x) : \mathbb{R}^n \to \mathbb{R}$ are continuous functions and there exist three real constants $\alpha \in (1, 2)$, $p \in [1, \alpha^*)$ and $k \geq 0$ such that for each $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$, $(y', z') \in \mathbb{R} \times \mathbb{R}^d$ and $\theta \in (0, 1)$, it holds that

$$\text{sgn}(y)g(t, x, y, z) \leq k \left( 1 + |x|^p + |y| + |z|^\alpha \right),$$

$$|g(t, x, y, z)| + |h(x)| \leq k \left( 1 + |x|^p + \exp(k|y|^{1/\theta}) + |z|^2 \right)$$

and

$$1_{\{y - \theta y' > 0\}} \left( g(t, x, y, z) - \theta g(t, x, y', z') \right) \leq (1 - \theta)k \left( 1 + |x|^p + |y'| + \left| \frac{y - \theta y'}{1 - \theta} \right| + \left| \frac{z - \theta z'}{1 - \theta} \right|^{\alpha} \right)$$

or

$$-1_{\{y - \theta y' < 0\}} \left( g(t, x, y, z) - \theta g(t, x, y', z') \right) \leq (1 - \theta)k \left( 1 + |x|^p + |y'| + \left| \frac{y - \theta y'}{1 - \theta} \right| + \left| \frac{z - \theta z'}{1 - \theta} \right|^{\alpha} \right).$$

The following example shows that assumption (A2) are more general than those used in some existing literature.

**Example 7.2.** From Proposition 5.1 and Remark 5.2, it is not difficult to verify that the assumption (A2) holds for the following generator $g$ and terminal function $h$:

$$g(t, x, y, z) := 1 + |x|^p \sin |x| + y^{2m}1_{y \leq 0} + \sin y + |z|^\alpha + l(y)q(z),$$

$$h(x) := |x|^p \cos |x|, \quad (t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d,$$

where $\alpha \in (1, 2)$, $p \in [1, \alpha^*)$, $m$ is a positive integer, and the function $l(\cdot)$ and $q(\cdot)$ are defined in (iii) of Proposition 5.1. It is clear that $g$ has a power growth in the state variables $(y, z)$ and it is non-Lipschitz continuous in $y$ and non-convex (non-concave) in $z$, and that both $g$ and $h$ have a power growth in the state variable $x$ and they are not uniformly continuous in $x$. 28
In the sequel, due to \( p \in [1, \alpha^*], \) it follows from (7.5) that for each \((t_0, x_0) \in [0, T] \times \mathbb{R}^n\) and each \( \lambda > 0, \) we have
\[
\mathbb{E} \left[ \exp \left( \lambda \left( \left\| X_{T}^{t_0, x_0} \right\|^p \right)^{\frac{2}{p^*}} \right) \right] \leq \mathbb{E} \left[ \sup_{s \in [0, T]} \exp \left( \lambda \left\| X_{s}^{t_0, x_0} \right\|^\frac{2}{p^*} \right) \right] \leq \tilde{C} \exp(\lambda \tilde{C} \left\| x_0 \right\|^\frac{2}{p^*}) < +\infty \quad (7.8)
\]
and
\[
\mathbb{E} \left[ \exp \left( \lambda \left( \int_0^T \left\| X_{s}^{t_0, x_0} \right\|^p \, ds \right)^{\frac{2}{p^*}} \right) \right] \leq \mathbb{E} \left[ \sup_{s \in [0, T]} \exp \left( \lambda T^\frac{2}{p^*} \left\| X_{s}^{t_0, x_0} \right\|^\frac{2}{p^*} \right) \right] \leq \tilde{C} \exp(\lambda T^\frac{2}{p^*} \tilde{C} \left\| x_0 \right\|^\frac{2}{p^*}) < +\infty,
\]
where the constant \( \tilde{C} \) depends only on \((p, \alpha, \lambda, T, K).\) Then, the assumption (A2) together with the inequalities (7.8) and (7.9) allows us to use Theorem 5.5 to construct a unique solution, \((Y_{t_0, x_0}, Z_{t_0, x_0}),\) to the BSDE (7.4) such that
\[
\forall \lambda > 0, \quad \mathbb{E} \left[ \exp \left\{ \lambda \left( \sup_{t \in [0, T]} \left\| Y_{t}^{t_0, x_0} \right\|^\frac{2}{p^*} \right) \right\} \right] < +\infty
\]
and \(Z_{t_0, x_0} \in \mathcal{M}^q\) for all \(q \geq 1.\) Furthermore, by a classical analysis we know that \(u\) defined by the formula (7.3) is a deterministic function.

Now we can state and prove the main result of this section.

**Theorem 7.3.** Let the assumptions (A1) and (A2) hold. Then, the function \(u\) defined in (7.3) is continuous on \([0, T] \times \mathbb{R}^n\) and there exists a constant \(C > 0\) such that
\[
\forall \ (t, x) \in [0, T] \times \mathbb{R}^n, \quad |u(t, x)| \leq C(1 + |x|^p). \quad (7.10)
\]
Moreover, \(u\) is a viscosity solution to PDE (7.1).

**Proof.** Let us first show that \(u\) is a continuous function. Indeed, we assume that the point sequence \(\{(t_m, x_m)\}_{m=1}^{\infty}\) in the space \([0, T] \times \mathbb{R}^n\) converges to a point \((t, x) \in [0, T] \times \mathbb{R}^n\) as \(m\) goes to \(+\infty.\) From the continuity of function \(h\) and inequality (7.6) it follows that \(\mathbb{P} - a.s.,\)
\[
\lim_{m \to \infty} h \left( X_{T}^{t_m, x_m} \right) = h \left( X_{T}^{t, x} \right). \quad (7.11)
\]
And, since \(g\) is a continuous function and satisfies assumption (A2), by Lebesgue’s dominated convergence theorem and inequalities (7.6) and (7.7) together with the integrability condition of the process \((Y_{t_0, x_0}, Z_{t_0, x_0})\) we can derive that for each \(q > 1,\)
\[
\lim_{m \to \infty} \mathbb{E} \left[ \left( \int_0^T \left| g(s, X_{s}^{t_m, x_m}, Y_{s}^{t, x}, Z_{s}^{t, x}) - g(s, X_{s}^{t, x}, Y_{s}^{t, x}, Z_{s}^{t, x}) \right| \, ds \right)^q \right] = 0. \quad (7.12)
\]

29
Furthermore, in view of the growth condition of function $h$ and inequality (7.7), a similar argument to (7.8) and (7.9) yields that for each $\lambda > 0$,

$$
\sup_{m \geq 1} \mathbb{E} \left[ \exp \left\{ \lambda \left( |h(X_{t,m,x}^m)| + \int_0^T \left| X_{s,m,x}^m \right| \mathbb{P} \, ds \right)^{\frac{\sigma^2}{2}} \right\} \right] < +\infty. \tag{7.13}
$$

In view of (7.11)-(7.13) and (A2), using the stability theorem (Theorem 6.1) leads to $\mathbb{P} - a.s.$,

$$
\lim_{m \to \infty} \sup_{s \in [0,T]} |Y_{s,m}^t,x - Y_s^t,x| = 0,
$$

which together with the continuity of $Y_{t,x}$ with respect to the time variable yields that $u$ is a continuous function on $[0, T] \times \mathbb{R}^n$.

Secondly, in view of assumption (A2) and Remark 3.5, the inequality (7.10) follows from inequalities (7.8) and (7.9) with the estimates (3.4) and (3.5) in Theorem 3.1.

Finally, we use a double approximation procedure and a stability result to prove that the function $u$ is a viscosity solution to PDE (7.1). For each $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$ and each pair of positive integers $m$ and $l$, we define

$$
g_{m,l}^m(t, x, y, z) := \inf \left\{ g^+(t, x, y, z) + m|y - y'| + m|z - z'|, \ (y', z') \in \mathbb{Q} \times \mathbb{Q}^d \right\}
$$

$$
- \inf \left\{ g^-(t, x, y, z) + l|y - y'| + l|z - z'|, \ (y', z') \in \mathbb{Q} \times \mathbb{Q}^d \right\}.
$$

By Lepeltier and San Martin [20] it is well known that $g_{m,l}^m$ is uniformly Lipschitz continuous in $(y, z)$, $g_{m,l}^m$ converges decreasingly uniformly on compact sets to a limit $g_{m,\infty}$ as $l$ tends to $+\infty$, and $g_{m,\infty}$ converges increasingly uniformly on compact sets to the generator $g$ as $m$ tends to $+\infty$. For $(t, x) \in [0, T] \times \mathbb{R}^n$, let $(Y_{t,m,l,x}^{m,l,t,x}, Z_{t,m,l,t,x}^{m,l})$ be the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ to the BSDE with the terminal condition $h \left( X_{T}^{t,x} \right)$ and the generator $g_{m,l}^m \left( \cdot, X_{t,x}^{t,x}, \cdot, \cdot \right)$. We denote $u_{m,l}^m(t, x) := Y_{t,m,l,t,x}^{m,l}$. Then, by the classical nonlinear Feynman-Kac formula (see, e.g. El Karoui et al. [12] and Pardoux and Peng [23]), $u_{m,l}^m(\cdot, \cdot)$ is a viscosity solution to the following PDE

$$
\partial_t u(t, x) + \mathcal{L} u(t, x) + g_{m,l}^m(t, x, u(t, x), \sigma^x \nabla_x u(t, x)) = 0, \ u(T, \cdot) = h(\cdot).
$$

Moreover, by virtue of the classical comparison theorem and the stability Theorem 6.1 we can derive that $u_{m,l}^m(\cdot, \cdot)$ is decreasing and converges pointwisely to a continuous function $u_{m,\infty}^m(\cdot, \cdot)$ as $l$ tends to $+\infty$, and $u_{m,\infty}^m(\cdot, \cdot)$ is increasing and converges pointwisely to the continuous function $u(\cdot, \cdot)$ as $m$ tends to $+\infty$. Dini’s theorem implies that the convergence is also uniform on compact sets of $[0, T] \times \mathbb{R}^n$. Then, we can apply the stability theorem 1.7 in Chapter 5 of Bardi and Capuzzo-Dolcetta [1] to show that $u$ is a viscosity solution to the PDE (7.1). The proof is then complete. \qed
Remark 7.4. When the generator $g$ does not depend on the variable $y$ and is convex or concave on the variable $z$, it can be shown that the function $u$ defined by the formula (7.3) is the unique viscosity solution with following growth: $|u(t,x)| \leq C(1 + |x|^\alpha*)$. This follows from the uniqueness results in Da Lio and Ley [9] concerning Bellman-Isaacs equation.

References

[1] Bardi, M., Capuzzo-Dolcetta, I., 1997. Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations. Birkhäuser, Boston, MA.

[2] Barrieu, P., El Karoui, N., 2013. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. Ann. Probab. 41 (3B), 1831–1863.

[3] Briand, P., Delyon, B., Hu, Y., Pardoux, E., Stoica, L., 2003. $L^p$ solutions of backward stochastic differential equations. Stochastic Process. Appl. 108 (1), 109–129.

[4] Briand, P., Elie, R., 2013. A simple constructive approach to quadratic BSDEs with or without delay. Stochastic Process. Appl. 123, 2921–2939.

[5] Briand, P., Hu, Y., 2006. BSDE with quadratic growth and unbounded terminal value. Probab. Theory Related Fields 136 (4), 604–618.

[6] Briand, P., Hu, Y., 2008. Quadratic BSDEs with convex generators and unbounded terminal conditions. Probab. Theory Related Fields 141 (3), 543–567.

[7] Buckdahn, R., Hu, Y., Tang, S., 2018. Existence of solution to scalar BSDEs with $L^\exp(\mu\sqrt{2\log(1+L)})$ integrable terminal values. Electron. Commun. Probab. 23, Paper No. 59, 8pp.

[8] Cheridito, P., Nam, K., 2014. BSDEs with terminal conditions that have bounded Malliavin derivative. Journal of Functional Analysis 266, 1257–1285.

[9] Da Lio, F., Ley, O., 2006. Uniqueness results for second-order Bellman-Isaacs equations under quadratic growth assumptions and applications. SIAM J. Control Optim. 45 (1), 74–106 (electronic)

[10] Delbaen, F., Hu, Y., Bao, X., 2011. Backward SDEs with superquadratic growth. Probab. Theory Related Fields 150, 145–192.

[11] Delbaen, F., Hu, Y., Richou, A., 2011. On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. Ann. Inst. Henri Poincaré Probab. Stat. 47, 559–574.

[12] El Karoui, N., Peng, S., Quenez, M. C., 1997. Backward stochastic differential equations in finance. Math. Finance 7 (1), 1–71.

[13] Fan, S., 2016. Bounded solutions, $L^p$ ($p > 1$) solutions and $L^1$ solutions for one-dimensional BSDEs under general assumptions. Stochastic Process. Appl. 126, 1511–1552.

[14] Fan, S., 2018. Existence, uniqueness and stability of $L^1$ solutions for multidimensional BSDEs with generators of one-sided osgood type. J. Theoret. Probab. 31, 1860–1899.

[15] Fan, S., Hu, Y., 2019. Existence and uniqueness of solution to scalar BSDEs with $L^\exp(\mu\sqrt{2\log(1+L)})$ integrable terminal values: the critical case. Electron. Commun. Probab. 24, Paper No. 49, 10pp.

[16] Fan, S., Jiang, L., 2012. $L^p$ ($p > 1$) solutions for one-dimensional BSDEs with linear-growth generators. Journal of Applied Mathematics and Computing 38 (1–2), 295–304.
[17] Hu, Y., Tang, S., 2016. Multi-dimensional backward stochastic differential equations of diagonally quadratic generators. Stochastic Process. Appl. 126 (4), 1066–1086.

[18] Hu, Y., Tang, S., 2018. Existence of solution to scalar BSDEs with $L \exp \sqrt{\frac{2}{\lambda} \log(1 + L)}$-integrable terminal values. Electron. Commun. Probab. 23, Paper No. 27, 11pp.

[19] Kobylyanski, M., 2000. Backward stochastic differential equations and partial differential equations with quadratic growth. Ann. Probab. 28 (2), 558–602.

[20] Lepeltier, J.-P., San Martin, J., 1997. Backward stochastic differential equations with continuous coefficient. Statist. Probab. Lett. 32 (4), 425–430.

[21] Masiero, F., Richou, A., 2013. A note on the existence of solutions to Markovian superquadratic BSDEs with an unbounded terminal condition. Electron. J. Probab 18 (50), 1–15.

[22] Pardoux, E., Peng, S., 1990. Adapted solution of a backward stochastic differential equation. Syst. Control Lett. 14 (1), 55–61.

[23] Pardoux, E., Peng, S., 1992. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In Stochastic Partial Differential Equations and Their Applications (Charlotte, NC, 1991) 200–217. Lecture Notes in Control and Inform. Sci. 176. Springer, Berlin.

[24] Richou, A., 2012. Markovian quadratic and superquadratic BSDEs with an unbounded terminal condition. Stochastic Process. Appl. 122, 3173–3208.