k-stabilization in brane models

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Abstract

Stabilization of inter–brane distance is analyzed in 5–dimensional models with higher–order scalar kinetic terms. Equations of motion and boundary conditions for background and for scalar perturbations are presented. Conditions sufficient and (with one exception) necessary for stability are derived and discussed. It is shown that it is possible to construct stable brane configurations even without scalar potentials and cosmological constants. As a byproduct we identify a large class of non–standard boundary conditions for which the Sturm–Liouville operator is hermitian.
1 Introduction

Higher dimensional brane models belong to the most interesting recent developments in the theory of fundamental interactions. Many models have been proposed in which the space–time consists of a 5–dimensional (5D) bulk ending at two 4–dimensional (4D) branes. Usually this space–time has the structure of a warped product of a maximally symmetric 4D space–time and the one dimensional orbifold $S^1/Z_2$ with the branes located at the $Z_2$ fixed points. The Standard Model fields may propagate only on one of the branes called the visible one. Some other fields may live on the second, hidden, brane. Of course, the gravity fields can propagate in the whole 5D space–time.

Phenomenological features of such models depend on fields and interactions other than that of the Standard Model, on the warping, and on the distance between the branes. This distance must be fixed in a stable way. Such stabilization can not be achieved with only gravity propagating in the bulk. A simple mechanism of fixing the inter–brane distance was proposed by Goldberger and Wise \cite{Goldberger:1999uk}. The idea is to add a 5D scalar field with some bulk and brane potentials. If the background value of that field is not constant in the bulk, then the boundary conditions (or in another words: equations of motion at the branes) can be fulfilled only if the branes are located at appropriate points in the 5th dimension.

It is not enough to have a background solution with some fixed brane positions. It is necessary also that such a configuration is stable against all possible small perturbations. From the 4D point of view, the perturbations can be describe in terms of Kaluza–Klein (KK) towers of states. The lightest scalar KK state is usually called the radion \cite{Brustlein:2003ay}. Tachyonic character of the radion indicates instability of a given background. The problem of the radion mass, or of the stability of the inter–brane distance, was investigated by many authors \cite{Arkani-Hamed:1997pp,Buchmuller:1998av,Dimopoulos:1998kq,Ivanov:1998mp,Burgess:2001ur}. Its relation to inflation was discussed in \cite{Kawasaki:1998di}. Quite general criteria for the stability were found in \cite{Burgess:2001ur}. Generalization of such criteria for models with the Gauss–Bonnet interactions was presented in \cite{Hebecker:2003aa}.

In the present paper we will do the stability analysis for brane models with non–standard kinetic terms for the scalar field. Such non–standard kinetic terms appear for example in string theory due to the $\alpha'$– and the loop–corrections. Very interesting models with generalized scalar kinetic terms were investigated in the cosmological context. Kinetically driven inflation, called the k–inflation, was introduced in \cite{K roller:1999bc,Ch ekalov:2001am}. Models of k–essence were proposed as another approach to the cosmological constant problem \cite{Armendariz-Picon:1999rj,Aumann:2000jz}. Causality in the context of generalized kinetic terms was discussed by many authors (see e.g. \cite{Buchmuller:2002gc} and references therein).
There is a simple reason to expect that models with non–standard scalar kinetic terms may be interesting for the inter–brane distance stabilization. Their Lagrangians contain terms with more complicated, than just quadratic, dependence on the scalar derivatives. The scalar derivative with respect to the 5th coordinate is crucial for the stabilization mechanisms similar to that of Goldberger and Wise. This is analogous to the situations in cosmological models where the time derivative of the scalar field is crucial. The problem of radion stabilization in models with non–standard kinetic terms was addressed in [15] but unfortunately the authors used a method which in general is not correct and obtained incorrect results.

In addition to the bulk (non-standard) kinetic terms we will consider also analogous brane-localized ones. Many 5D models with (standard) brane kinetic terms for different bulk fields were proposed. Such localized kinetic terms were investigated for: pure gravity [17, 18, 19, 20], gauge fields [21, 22, 23, 24, 25, 26], fermions [22, 24, 25] and scalars [24, 27].

In section 2 we define our model and derive background equations of motion and boundary conditions. Analogous equations for the scalar perturbations are presented in section 3. In subsection 3.1 we show that the spectrum of those perturbations is real. We identify a large class of boundary conditions for which the Sturm-Liouville eigenvalue problem is self-adjoint. The stability conditions are obtained in section 4. They are discussed and compared to that in models with the standard kinetic terms in section 5. Finally, section 6 contains our conclusions.

2 Model and background

We consider 5D models compactified on the $S^1/Z_2$ orbifold with the standard gravitational interactions but with non–standard kinetic terms for a scalar field $\Phi$. Two 4D branes are localized at the $Z_2$ fixed points $y = y_i$. The authors of [15] integrated a Lagrangian with fields replaced by their background values. They called the result “the effective potential” and looked for the minima of such an object. Of course, in general the potential integrated in a given background is not equal to the correct effective potential. It happens to be equal in some simple cases, but this must be checked case by case by other methods, so the method of integrating the potential is practically useless. The authors of [15] claim e.g. that all models with the standard kinetic terms are unstable, what is in clear conflict with the results of many previous analyses [3, 5, 7, 8, 9, 16].
action takes the form

$$S = \int d^4x \, dy \, \sqrt{-G} \left\{ \frac{1}{2\kappa^2} R - P(\Phi, X) - V(\Phi) - \sum_{i=1}^{2} \tilde{\delta}(y - y_i) [Q^{(i)}(\Phi, X) + U^{(i)}(\Phi)] \right\}, \quad (1)$$

where

$$X = \frac{1}{2} (\nabla \Phi)^2, \quad (2)$$

and $\tilde{\delta}$ is the normalized Dirac delta satisfying $\int dy \, \sqrt{-G} \tilde{\delta}(y - y_i) = \sqrt{-G^{(i)}}$ with $G^{(i)}$ being the determinant of the metric induced on the brane localized at $y = y_i$ (we chose $y_1 < y_2$). The bulk kinetic term is given by some function $P(\Phi, X)$ depending on the derivatives of $\Phi$ through the combination $X$ and on the scalar field itself. We choose $P(\Phi, X)$ in such a way that it vanishes for $X = 0$. This way $V(\Phi)$ describes the whole scalar contribution to the action for constant $\Phi$. In addition to the bulk interactions, we consider brane localized contributions to the scalar kinetic term and to the potential: $Q^{(i)}(\Phi, X)$ and $U^{(i)}(\Phi)$, respectively.

The terms in the action (1) containing the brane localized kinetic functions $Q^{(i)}(\Phi, X)$ must be treated with special care. Let us discuss now in some detail the meaning of an integral containing a product of $Q^{(i)}(\Phi, X)$ and the Dirac delta. Writing explicitly the arguments in one of such expressions we get

$$\int d^4x \int dy \, \sqrt{G(x, y)} \tilde{\delta}(y - y_i) Q^{(i)}(\Phi(x, y), \frac{1}{2} \left( G^{55}(x, y) \Phi'^2(x, y) + \ldots \right) ), \quad (3)$$

where prime denotes differentiation with respect to the orbifold coordinate $y$ and, in the second argument of $Q^{(i)}$, the ellipsis stand for terms in $X$ with derivatives of $\Phi$ in directions other than $y$. In brane models the derivatives with respect to the orbifold coordinate(s) are usually discontinuous at the brane positions. The derivative $\Phi'(x, y)$, being a $\mathbb{Z}_2$ odd function, is exactly zero at $y = y_i$. On the other hand, due to the brane sources, the limits $\lim_{y \to y_i^\pm} \Phi'(x, y)$ can be different from zero. The square of the scalar field

\footnote{There are two kinds of brane kinetic terms considered in the literature. Some authors assume that such terms involve derivatives with respect to all 5 coordinates (e.g. [24]-[26]) while other assume that the derivative in the orbifold direction is not present (e.g. [18]-[23]). We apply the former approach which seems to be natural when treating thin branes as limits of thick ones. Generalization of our results to the case of brane kinetic terms $Q$ which do not depend on $\partial \Phi/\partial y$ is quite straightforward.}
derivative, \( \Phi'^2(x, y) \), is even under the \( \mathbb{Z}_2 \) symmetry, and can be written as a product of \( \text{sgn}^2(y - y_i) \) and a smooth function. Usually \( \Phi'^2(x, y) \) is discontinuous at \( y = y_i \) and, strictly speaking, its integral with the Dirac delta localized at \( y_i \) is not well defined. All expressions of this kind must be regularized. Physically, such regularization corresponds to using a thick brane and taking the limit of its thickness decreasing to zero. Technically, one replaces \( \delta(y - y_i) \) and \( \text{sgn}(y - y_i) \) with some smooth functions \( \delta_\varepsilon(y - y_i) \) and \( \text{sgn}_\varepsilon(y - y_i) \) satisfying the relations \( \text{sgn}_\varepsilon'(y - y_i) = 2\delta_\varepsilon(y - y_i) \) and approaching the Dirac delta and the signum function, respectively, when \( \varepsilon \to 0 \). We calculate the integrals like (3) for regularized expressions and at the end remove the regulator taking the limit \( \varepsilon \to 0 \). Thus, we obtain for example

\[
\int d^4x \int_{y_a}^{y_b} dy \sqrt{G} \delta(y - y_i) \Phi'^{2n}(x, y) = \int d^4x \sqrt{G(i)} \lim_{y \to y_i} \frac{\Phi'^{2n}(x, y)}{2n + 1},
\]

if \( y_a < y_i < y_b \). It is not necessary to specify the direction of the limit in (4) because \( \Phi'^{2n}(x, y) \) is an even function of \( (y - y_i) \) for any integer \( n \). However, the limit itself is necessary because usually \( \Phi'^{2n}(x, y) \) is discontinuous at \( y = y_i \). In the down–stairs approach, one of the limits of integration is equal to \( y_i \) and the r.h.s. of the above equation must be multiplied by 1/2.

In this work we are interested in warped background solutions with the flat 4D foliation described by the ansatz

\[
ds^2 = a^2(y) (\eta_{\mu\nu} dx^\mu dx^\nu + dy^2),
\]

\[
\Phi = \phi(y).
\]

The bulk equations of motion for the system described by action (1) and satisfying ansatz (5–6) are given by (we use units \( \kappa = 1 \))

\[
(P_X \phi')' + 3\frac{a'}{a} P_X \phi' - a^2 (V_\Phi + P_\phi) = 0,
\]

\[
\frac{a''}{a} - 2 \left( \frac{a'}{a} \right)^2 + \frac{1}{3} P_X \phi'^2 = 0,
\]

\[
6 \left( \frac{a'}{a} \right)^2 - P_X \phi'^2 + a^2 (V + P) = 0,
\]

where the subscripts \( X \) and \( \Phi \) denote derivatives with respect to the arguments \( X \) and \( \Phi \), respectively.\(^3\)

\(^3\) It is straightforward to generalize the equations of motion to the case of any non–flat maximally symmetric 4D foliation of the 5D background. For example, for the 4D dS space–time characterized by the Hubble constant \( H \), the left hand sides of equations (8) and (9) should be modified by adding \( H^2 \) and \( -6H^2 \), respectively.
The boundary conditions for the background can be obtained from the full equations of motion resulting from (11) with the brane terms taken into account. Integrating such equations over an infinitesimal intervals containing the brane positions $y_i$ one gets

$$\lim_{y \to y_1^+(y_2^-)} a' = \pm \frac{a^2}{6} \left( U^{(i)} + \int dy \, \delta(y - y_i) Q^{(i)} \right) \bigg|_{y = y_1(y_2)},$$

(10)

$$\lim_{y \to y_1^-(y_2^+)} (P_X \phi') = \pm \frac{a^2}{2} \left( U^{(i)} + \int dy \, \delta(y - y_i) Q^{(i)} \right) \bigg|_{y = y_1(y_2)},$$

(11)

where for $y_1 < y_2$ the upper (lower) signs are to be taken for $y = y_1$ ($y = y_2$).

From now on we use the Dirac delta distribution with the usual normalization $\int dy \delta(y) = 1$ (in the upstairs approach).

The above background bulk equations of motion (7-9) and boundary conditions (10-11) reduce to the known results for the standard kinetic terms after substituting $P = X$ and $Q^{(i)} = 0$.

3 Scalar perturbations

Solving the bulk equations of motion (7-9) and the boundary conditions (10-11) one can find possible background configurations characterized by the warp factor $a(y)$ and the scalar field $\phi(y)$. Not all such background configurations are stable. To check the stability one has to consider all possible small perturbations around a given background. Instabilities occur if any of the perturbations has a tachyonic character. In this paper we concentrate on the scalar perturbations. From the 4D point of view they form an infinite Kaluza–Klein tower of scalars. The state with the lowest (4D) mass squared is called the radion. The positivity of its mass squared is a necessary condition for the stabilization of the inter–brane distance.

To find the radion mass we have to investigate the equations of motion for the scalar perturbations around the background. Using the generalized longitudinal gauge, the scalar perturbations can be written in the following way

$$ds^2 = a^2 \left[ (1 + 2 F_1) (\eta_{\mu \nu} dx^\mu dx^\nu) + (1 + 2 F_2) dy^2 \right],$$

(12)

$$\Phi = \phi + F_3,$$

(13)

where $a$ and $\phi$ are background solutions depending only on the 5-th coordinate $y$, while the perturbations $F_j$ are arbitrary (but small) functions of all
coordinates. To find the masses of the KK modes of scalar perturbations it is enough to consider equations of motion linear in $F_j$.

Contrary to the background equations of motion, for the perturbations we obtain non–trivial off–diagonal Einstein equations

$$2F_1 + F_2 = 0,$$

$$ (a^2 F_1)' + \frac{1}{3} a^2 P_X \phi' F_3 = 0. $$

They have to be fulfilled in order to stay in the longitudinal gauge. The diagonal Einstein equations, combined with the background equations of motion (7–9), give the third equation for the scalar perturbations:

$$\Box F_1 + 4 \frac{a'}{a} F_1' - 4 \left( \frac{a'}{a} \right)^2 F_2 + \frac{a'}{a} \left( P_X - \frac{2}{3} XP_{XX} \right) \phi' F_3 $$

$$+ \frac{1}{3} (P_X + 2XP_{XX}) \left[ \phi'^2 F_2 + \phi'' F_3 - \phi' F_3' \right] = 0, \quad (16)$$

where $\Box$ is the 4–dimensional D’Alembertian. The part of the boundary conditions linear in the scalar perturbations are quite complicated and reads

$$\pm 2 \lim_{y \rightarrow y_i^\pm} \left[ (P_X + 2XP_{XX}) F_3' \right]$$

$$+ \int_{y_i} \phi'' \left[ (P_X + 2XP_{XX}) F_3 - (P_X + 8XP_{XX} + 4X^2P_{XXX}) F_2 \right]$$

$$= \left[ aF_3 \left( U_{\Phi \Phi}^{(i)} + \int_{y_i} \delta_i Q_{\Phi \Phi}^{(i)} \right) - \frac{\Box F_3}{a} \int_{y_i} \delta_i Q_{XX}^{(i)} \right] \bigg|_{y = y_i}, \quad (17)$$

where $\delta_i = \delta(y - y_i)$. The subscript $y_i$ at the integrals indicates that the range of integration is an infinitesimal interval containing $y_i$.

The off–diagonal Einstein equations (14) and (15) can be used to express two of the perturbations introduced in the ansatz (12–13) in terms of the third one. It is convenient to eliminate $F_2$ and $F_3$ and to use the product $a^2 F_1$ as an independent perturbation. We expand it in the 4D modes as

$$a^2(y) F_1(t, \vec{x}, y) = \sum_{m^2} K_{m^2}(y) \left[ \int d^3k f_{(m^2, k)}(t) e^{i\vec{k} \cdot \vec{x}} \right], \quad (18)$$

and substitute to eqs. (16) and (17). Then, the 4D part of the bulk equation (16) takes the usual form

$$\ddot{f}_{(m^2, k)} + \left( k^2 + m^2 \right) f_{(m^2, k)} = 0. \quad (19)$$
The equation for the “shape” $K_{m^2}(y)$ of the KK mode with mass squared equal $m^2$ can be written as the Sturm–Liouville equation
\begin{equation}
-pK'_m + qK_m = m^2 rK_m, \tag{20}
\end{equation}
where $p$, $q$ and $r$ are the following functions depending on the background
\begin{equation}
p = \frac{3}{2aP_X\phi'^2}, \quad q = \frac{1}{a}, \quad r = \frac{3}{2a(P_X + 2XP_{XX})}\phi'^2 = c^2sp. \tag{21}
\end{equation}
In the last equality we have introduced a local ($y$–dependent) “speed of sound”
\begin{equation}
c^2_s = \frac{P_X}{P_X + 2XP_{XX}}. \tag{22}
\end{equation}
The boundary condition (17) in terms of $K_{m^2}$ takes the form
\begin{equation}
\left[ (b_i - c_i m^2) \frac{\partial}{\partial n} K_{m^2} - m^2 P_X K_{m^2} \right] \bigg|_{y^\pm_i} = 0, \tag{23}
\end{equation}
where from now on $y^\pm_i$ stands for $y^+_i$ or $y^-_i$. The corresponding limits have to be taken for quantities discontinuous on the branes. The $\partial/\partial n$ differentiation is in the direction of the outer normal at the boundary, i.e. $(-d/dy)$ at $y_1$ and $(+d/dy)$ at $y_2$. Quantities $b_i$ and $c_i$ are the following functions of the background solution and the brane interactions
\begin{align}
b_i &= \frac{1}{2} \left[ aU^{(i)}_{\phi\phi} \bigg|_{y=y_i} + a \int_{y_i} \delta_i Q^{(i)}_{\phi\phi} - \int_{y_i} \phi'' (P_{\Phi X} + 2XP_{\Phi XX}) \right] \\
&\pm \lim_{y \to y^\pm_i} (P_X + 2XP_{XX}) \left( \frac{\phi''}{\phi'} - \frac{a'}{a} \right), \tag{24}
\end{align}
\begin{align}
c_i &= \frac{1}{2a} \int_{y_i} \delta_i Q^{(i)}_X. \tag{25}
\end{align}
All integrals in (24) and (25) should be calculated with the same regularization as that used in (4).

The square of the radion mass is given by the lowest eigenvalue of the equation of motion (20) satisfying the boundary conditions (23). Of course, in general it is not possible to find the spectrum of the system (20)-(25) by solving it explicitly. To get some information about the smallest eigenvalue we will use methods analogous to those developed for a similar problem in [9] (where the corresponding boundary conditions have a form of (23) with $c_i = 0$). But first one has to check whether the differential equation (20)
together with the boundary conditions (23) constitute a self-adjoint system. This is a non trivial problem because conditions (23) are unusual and quite complicated. The eigenvalue \( m^2 \) of the equation of motion (20) appears multiplying both \( K_m^2 \) and its normal derivative. In the next subsection we will show that our eigenvalue problem is self-adjoint with boundary conditions even more general than (23).

### 3.1 Self-adjoint eigenvalue problem

Let us consider a differential eigenvalue problem

\[ \mathcal{O}v = \lambda v \]  

(26)

for the operator \( \mathcal{O} \) of the Sturm-Liouville type

\[ \mathcal{O}v = \frac{1}{r} \left[ - (pv')' + qv \right] . \]  

(27)

The boundary conditions on the interval \((y_1, y_2)\) have the form

\[ \left[ \sigma_1^{(i)} v + \sigma_2^{(i)} v' + \sigma_3^{(i)} (\mathcal{O}v) + \sigma_4^{(i)} (\mathcal{O}v)' \right]_{y=y_i} = 0 , \]  

(28)

where \( \sigma_j^{(i)} \) are some constants. The spectrum of our eigenvalue problem is real if \( \mathcal{O} \) is hermitian. In order to prove this one has to find such a scalar product \((\cdot, \cdot)\) for which

\[ (v, \mathcal{O}u) = (\mathcal{O}v, u) . \]  

(29)

The standard boundary conditions discussed in many mathematical textbooks have the form of (28) with \( \sigma_3^{(i)} = \sigma_4^{(i)} = 0 \). In such a case, \( \mathcal{O} \) is hermitian in the scalar product \((f, g) = \int_{y_1}^{y_2} rf g \) (for simplicity we consider real functions \( f \) and \( g \)). Let us generalize this scalar product by adding some boundary terms \[4\]

\[ (f, g) = \int_{y_1}^{y_2} rfg + \left[ \rho_1^{(i)} fg + \rho_2^{(i)} (fg)' + \rho_3^{(i)} f'g' \right]_{y_1}^{y_2}, \]  

(30)

with yet unspecified constants \( \rho_j^{(i)} \). For this scalar product we calculate

\[ (v, \mathcal{O}u) - (\mathcal{O}v, u) = \left\{ p \left[ vu' - vu' \right] + \rho_1^{(i)} [v(\mathcal{O}u) - (\mathcal{O}v)u] \right. \]  

\[ + \rho_3^{(i)} \left[ (v(\mathcal{O}u))' - ((\mathcal{O}v)u)' \right] \Big|_{y_1}^{y_2} . \]  

(31)

\[4\] A simple example of a non-standard scalar product was discussed for example in [27]. It was calculated for a canonical kinetic term localized on a brane in a flat background. In our notation this corresponds to \( p = 1, q = 0, r = 1, \sigma_1 = 0, \sigma_4 = 0. \)
Introducing three additional constants $\rho_4^{(i)}, \rho_5^{(i)}, p_1^{(i)}$, at each boundary, we can rewrite the above equation in the following form:

\[
(v, \mathcal{O}u) - (\mathcal{O}v, u) = \left\{ v \left[ \rho_4^{(i)} u - p_1^{(i)} u' + \rho_1^{(i)} (\mathcal{O}u) + \rho_2^{(i)} (\mathcal{O}u)' \right] \\
- u \left[ \rho_4^{(i)} v - p_1^{(i)} v' + \rho_1^{(i)} (\mathcal{O}v) + \rho_2^{(i)} (\mathcal{O}v)' \right] \\
+ v' \left[ p_2^{(i)} u + \rho_5^{(i)} u' + \rho_2^{(i)} (\mathcal{O}u) + \rho_3^{(i)} (\mathcal{O}u)' \right] \\
- u' \left[ p_2^{(i)} v + \rho_5^{(i)} v' + \rho_2^{(i)} (\mathcal{O}v) + \rho_3^{(i)} (\mathcal{O}v)' \right] \right\}_{y_1}^{y_2}
\]

(32)

where $p_2^{(i)} = p(y_i) - p_1^{(i)}$. Our operator $\mathcal{O}$ is hermitian if the r.h.s. of the above equation vanishes for all $v$ and $u$ fulfilling the boundary conditions \[28\]. This is the case when each square bracket in \[32\] is proportional to the square bracket in \[28\]:

\[
\rho_4^{(i)} = n_1^{(i)} \sigma_1^{(i)}, \quad -p_1^{(i)} = n_1^{(i)} \sigma_2^{(i)}, \quad \rho_1^{(i)} = n_1^{(i)} \sigma_3^{(i)}, \quad \rho_2^{(i)} = n_1^{(i)} \sigma_4^{(i)},
\]

(33)

\[
p_2^{(i)} = n_2^{(i)} \sigma_1^{(i)}, \quad \rho_5^{(i)} = n_2^{(i)} \sigma_2^{(i)}, \quad p_2^{(i)} = n_2^{(i)} \sigma_3^{(i)}, \quad \rho_3^{(i)} = n_2^{(i)} \sigma_4^{(i)}.
\]

(34)

For generic values of $\sigma_j^{(i)}$ this set of linear equations can be easily solved. At each boundary there are 8 equations and 8 independent constants: $\rho_1^{(i)}, \rho_2^{(i)}, \rho_3^{(i)}, \rho_4^{(i)}, \rho_5^{(i)}, n_1^{(i)}, n_2^{(i)}, p_1^{(i)}$. In fact we are interested only in those three, $\rho_1^{(i)}, \rho_2^{(i)}, \rho_3^{(i)}$, which enter the definition of the scalar product \[30\]. The solution reads:

\[
(f, g) = \int_{y_1}^{y_2} r f g + \left[ p \left( \sigma_3^{(i)} \right)^2 f g + \sigma_3^{(i)} \sigma_4^{(i)} (f g)' + \left( \sigma_4^{(i)} \right)^2 f g' \right]_{y_1}^{y_2}.
\]

(35)

We have shown that the eigenvalue problem \[26\] with the boundary conditions \[23\] is self-adjoint. Thus, all its eigenvalues $\lambda$ are real and the eigenfunctions corresponding to different $\lambda$ are orthogonal in the scalar product \[30\].

Let us now use the above result for our k-stabilization mechanism. The boundary conditions \[23\] have the form of \[28\] with:

\[
\sigma_1^{(i)} = 0, \quad \sigma_2^{(i)} = (-1)^i b_i, \quad \sigma_3^{(i)} = -P_X(y_i), \quad \sigma_4^{(i)} = -(-1)^i c_i,
\]

(36)

with no summation over $i$. The factors of $(-1)^i$ appear because the outer normal derivative $\partial / \partial n$ was used in \[23\]. Substituting \[36\] into \[35\] we
obtain the following scalar product appropriate to show that the eigenvalue
problem \((20), (23)\) is self-adjoint:

\[
(f, g) = \int_{y_1}^{y_2} r f g + \sum_{i=1,2} \left[ p \frac{P^2_X f g + P_X c_i \frac{\partial}{\partial n} (f g) + c_i^2 \frac{\partial}{\partial n} f \frac{\partial}{\partial n} g}{P_X b_i} \right] \bigg|_{y_i} .
\] (37)

The prime at the sum symbol denotes that the boundary contributions should
be taken only for those boundaries at which \(P_X b_i \neq 0\). The reason is that for
\(b_i = 0\) and/or \(P_X = 0\) the boundary condition \((23)\) reduces to the standard
one for which \((f, g) = \int r f g\) without any boundary terms (at that boundary).

4 Stability conditions

The spectrum of the scalar perturbations in a given background is given by the
eigenvalues of the Strum–Liouville equation \((20)\) with the boundary con-
ditions \((23)\) at the branes. In the previous subsection we have shown that this
spectrum is real. The most interesting for us is the lowest eigenvalue which
we identify with the square of the radion mass. The inter–brane distance
is stable only if this mass squared is positive. In this section we will find
conditions sufficient for such stability. We will show also when the radion is
massless and identify some classes of backgrounds which are unstable.

First we check whether there is a massless mode in the KK tower of the
scalar perturbations. In such a case, the boundary condition \((23)\) at the first
brane reduces, for nonzero \(b_1\), to \(K_0'(y_1^+) = 0\) (the case with vanishing \(b_1\)
will be considered later). For \(m^2 = 0\), the solution of the bulk equation of
motion \((20)\), satisfying the boundary condition at \(y = y_1\) and normalized to
\(K_0(y_1) = 1\), can be written in quite a simple form

\[
K_0(y) = \frac{a^2(y)}{a^2(y_1)} - \frac{2a'(y)}{a^2(y)a^2(y_1)} \int_{y_1}^{y} d\tilde{y} a^3(\tilde{y}) .
\] (38)

Using the background equation of motion \((8)\), the derivative of the above
solution simplifies to

\[
K_0'(y) = \frac{2P_X(y)\phi'(y)}{3a(y)a^2(y_1)} \int_{y_1}^{y} d\tilde{y} a^3(\tilde{y}) .
\] (39)

The boundary condition at the second brane reads \(b_2 K'(y_2^-) = 0\). The in-
tegral in eq. \((39)\) is strictly positive, so this condition is fulfilled only when
the product \(b_2 P_X(y_2^-)\phi'(y_2^-)\) vanishes. Repeating the same reasoning starting
from the second brane, one gets analogous result for the first brane. Putting
both cases together, we find that for $p$ and $r$ regular in the bulk, the necessary and sufficient condition for existence of a massless mode is

$$b_1 b_2 P_X (y_1^+) P_X (y_2^-) \phi'(y_1^+) \phi'(y_2^-) = 0. \quad (40)$$

Conditions sufficient for the stability can be found in the following way. Multiplying eq. (20) with $K m^2$, integrating over the whole 5th dimension, and using the boundary conditions (23) we get

$$m^2 \int_{y_1}^{y_2} r (K m^2)^2 + \sum_i \frac{b_i}{m^2} \frac{p}{P_X} (K m^2)_i^2 \bigg|_{y_i^\pm}$$

$$= \int_{y_1}^{y_2} \left[ q (K m^2)^2 + p (K m^2)_i^2 \right] + \sum_i c_i \frac{p}{P_X} (K m^2)_i^2 \bigg|_{y_i^\pm}. \quad (41)$$

Let us consider first such models for which the background dependent bulk functions $p$, $q$ and $r$ are regular and positive while the brane parameters $b_i$ are positive and $c_i$ are non-negative. Then, the r.h.s. of (41) is positive while the l.h.s. is negative for negative $m^2$ and may be divergent for vanishing $m^2$. Thus, the condition (41) can be fulfilled only for positive $m^2$. The function $q = 1/a$ is always positive. Functions $p$ and $r$ have the same sign as $P_X$ and $(P_X + 2X P_{XX})$, respectively. They become singular if any of the functions $P_X$, $(P_X + 2X P_{XX})$ or $\phi'$ vanishes for any value of $y$. Thus, the following conditions

$$b_i > 0, \quad c_i \geq 0, \quad (42)$$

$$\forall y \in [y_1^+, y_2^-] \quad \phi'^2 (y) > 0, \quad P_X (y) > 0, \quad P_X (y) + 2X (y) P_{XX} (y) > 0, \quad (43)$$

are sufficient for the stability of the inter–brane distance (positivity of the radion mass squared). By $y \in [y_1^+, y_2^-]$ we denote the interior of the bulk, $y_1 < y < y_2$ and the limits $y \to y_1^+$ and $y \to y_2^-$.}

Showing that the above conditions are sufficient for stability was quite easy. It is much more difficult to check which conditions are necessary. We will show now that there must be at least one tachyonic mode if any of the functions, $\phi'$, $P_X$ or $(P_X + 2X P_{XX})$, vanishes anywhere in the bulk. The arguments are similar to those used in [7] and [9]. We will compare the properties of two solutions of the bulk equation of motion (20), one for $m^2 = 0$ and second for $m^2 = -M^2$ in the limit $M \to \infty$. Both solutions satisfy the boundary condition at one brane (let us first choose it to be the first one located at $y_1$).
We start with solving the bulk equation of motion (20) in the limit of large negative \( m^2 = -M^2 \). In the leading order in \( 1/M \), equation (20) has the following approximate solution

\[
K_{-M^2}(y) \approx \frac{1}{\sqrt{PC_0}} \left[ C^+ \exp \left( +M \int_{y_1}^{y} c_s \right) + C^- \exp \left( -M \int_{y_1}^{y} c_s \right) \right], \quad (44)
\]

\[
K'_{-M^2}(y) \approx M \frac{c_s}{P} \left[ C^+ \exp \left( +M \int_{y_1}^{y} c_s \right) - C^- \exp \left( -M \int_{y_1}^{y} c_s \right) \right]. \quad (45)
\]

In the same limit, the boundary condition (23) at the first brane becomes

\[
\left. (c_1 K_{-M^2} - P_X K'_{-M^2}) \right|_{y=y_1} \approx 0. \quad (46)
\]

Because of the \( M \) prefactor in (45), for any \( c_1 \neq 0 \) and large enough \( M \), the above boundary condition can be fulfilled when \( C^+ \approx C^- \). We choose \( C^\pm \) to be positive because later we will compare this solution with \( K_0 \) normalized to 1 at \( y_1 \). When \( c_1 \) and \( P_X(y_1) \) have the same sign, the boundary condition (46) can be fulfilled only when \( K_{-M^2}(y_1) \) and \( K'_{-M^2}(y_1) \) have the same sign. Thus, \( C^+ > C_- \) and the square bracket in (44) does not change its sign in the whole bulk. For very large \( M \) the first term in (44) starts to dominate over the second one even for small values of \( y - y_1 \) (it is slightly bigger even at \( y_1 \)) and away from the first brane the solution is approximated by

\[
K_{-M^2}(y) \approx C^+ \phi' \sqrt{\frac{2a}{3}} \sqrt{P_X (P_X + 2XP_{XX})} \exp \left( M \int \frac{P_X}{P_X + 2XP_{XX}} \right). \quad (47)
\]

Using this solution we can investigate models when some of the conditions in (42,43) are not fulfilled.

It is convenient to define the following function of \( m^2 \)

\[
B_2(m^2) = \left[ (b_2 - c_2 m^2) \frac{\partial}{\partial m} K_{m^2} - m^2 P_X K_{m^2} \right] \left|_{y=y_2} \right. . \quad (48)
\]

It is equal to the l.h.s. of the boundary condition (23) for \( K_{m^2} \) satisfying the bulk equation of motion and the boundary condition at the first brane, and normalized to 1 at \( y_1 \). The spectrum of the KK tower of scalar perturbations consists of those values \( m^2 \) for which \( B_2(m^2) = 0 \).

Now we check whether the positivity of \( b_i \) and \( c_i \) are necessary conditions for the stability, assuming that all the bulk conditions (43) are fulfilled. For very large negative \( m^2 \) the boundary function \( B_2 \) at the second brane is dominated by the term proportional to \( K'_{-M^2} \). From eq. (47) and the discussion before it, it follows that

\[
\text{sgn} \left[ B_2(-M^2) \right] = \text{sgn} \left[ M^2 c_2 K'_{-M^2}(y_2^-) \right] = \text{sgn} \left[ c_2 \right] . \quad (49)
\]
On the other hand, for the solution $K_0$ given by (38) and (39) we get

$$\text{sgn} [B_2(0)] = \text{sgn} [b_2K_0'(y^-_2)] = \text{sgn}[b_2],$$

(50)

where we used the fact that $K_0'$ is always positive when the inequalities (43) are fulfilled. Comparing (49) with (50), we conclude that there must be at least one negative eigenvalue when the parameters $b_2$ and $c_2$ have opposite signs. For $b_2c_2 < 0$, the function $B_2(m^2)$ has different sign for $m^2 = 0$ and for large (enough) negative $m^2$. There must be some negative $m^2$ for which $B_2$ vanishes because the solutions of (20) change continuously with $m^2$.

Repeating the above reasoning but starting from the brane at $y_2$, we obtain an analogous condition for parameters $b_1$ and $c_1$. Thus, the conditions

$$b_1c_1 \geq 0, \quad b_2c_2 \geq 0,$$

(51)

are necessary for the stability.

Now we investigate the stability conditions for the bulk quantities (43). The solution (47) for large negative $m^2$ vanishes at a point at which $\phi'$ or $P_X$ vanishes. It must change sign there because from (20) it follows that $K$ and $K'$ can vanish at the same point only for trivial solution vanishing everywhere. Thus, for very large negative $m^2$ the function $K(y)$ vanishes close to the point where $P_X\phi'$ is zero. On the other hand, from (38) and (39) it follows that $K_0$ is positive for all $y$. So, there must be some negative $\tilde{m}^2$ for which $K_{\tilde{m}^2}$ has a zero point but is nowhere negative. It is easy to see that such a zero point must be at the second brane, $y = y_2$, and that the derivative of $K_{\tilde{m}^2}(y^-_2)$ is negative. In such a situation

$$\text{sgn} [B_2(\tilde{m}^2)] = \text{sgn} [(b_2 - c_2\tilde{m}^2)K'_{\tilde{m}^2}(y^-_2)] = -\text{sgn}[b_2],$$

(52)

where the last equality follows from the condition (51). Comparing (51) and (52) we find that there must be some negative mode with the eigenvalue $\tilde{m}^2$ satisfying $\tilde{m}^2 < \tilde{m}^2 < 0$ for which $B_2(\tilde{m}^2) = 0$. The radion is tachyonic if $\phi'$ or $P_X$ vanishes in the bulk.

The above arguments are rather complicated but the result is quite intuitive. We consider backgrounds for which $P_X\phi'$ vanishes at some $y_0 < y_2$ in the bulk. For any such background $K_0(y)$ defined in (38) is a zero mode in a model restricted to the interval $[y_1, y_0]$. It is quite natural that the KK states becomes lighter when the compact space becomes bigger. So, with a massless mode on $[y_1, y_0]$ there should be a tachyonic one on the bigger orbifold $[y_1, y_2]$.

Equation (47) can be used to show that also $(P_X + 2X P_{XX})$ should be strictly positive. If it is not, there are two possibilities depending on how
fast it approaches zero. If the integral in (47) is finite then \( K - M^2 \) vanishes because of \( (P_X + 2X P_{XX}) \) in the prefactor and a reasoning similar to that for the case of vanishing \( P_X \phi' \) may be applied to prove the existence of at least one tachyonic mode. On the other hand, a divergent integral in (47) indicates the breakdown of the perturbativity assumption. This is not surprising. Vanishing \( (P_X + 2X P_{XX}) \) corresponds to infinite speed of sound while negative \( (P_X + 2X P_{XX}) \) gives negative square of the speed of sound (for positive \( P_X \), which is anyway necessary for the stability). In both cases one should expect strong instabilities.

We showed above that the conditions (43) on the bulk quantities are not only sufficient but also necessary for the stability. We were not able to prove the same for the brane conditions (42). If one of them is fulfilled then the other has also to be fulfilled. The only possible loophole occurs when both conditions (42) are violated, namely when \( b_1 < 0 \) and \( c_1 < 0 \) or when \( b_2 < 0 \) and \( c_2 < 0 \). However, these are not very appealing possibilities. Parameters \( c_i \) are proportional to the integrals \( \int \delta_i Q_X^{(i)} \) and can be negative only for localized brane kinetic terms very different from the standard one.

## 5 Discussion

With the results presented in the two previous sections we can investigate how the stabilization of branes is influenced by the presence of non-trivial scalar kinetic terms in the bulk and/or on the branes. Such terms change the background configurations and the spectrum of the scalar perturbations. We start the discussion with the background.

Combining eqs. (8) and (9), the dynamical equation describing the change of the warp factor can be written as

\[
3 \frac{a''}{a} + a^2 (V + P) = 0.
\]

(53)

The source for the change of the warp factor \( a(y) \) is the full “matter” Lagrangian density \((V + P)\) irrespective of whether the kinetic part is standard or not. The modification of the scalar equation of motion given in (7)

\[
(P_X \phi')' + 3 \frac{a'}{a} (P_X \phi') - a^2 (V_\phi + P_\Phi) = 0,
\]

is twofold. First, similarly as in the case of the warp factor, the role of the potential in this equation is played by the full non-gravitational Lagrangian density. Second, it seems that a natural variable to describe the change of the scalar background is the product \( P_X \phi' \) and not \( \phi' \) itself. The equation of
motion for this generalized variable $P_X \phi'$ looks formally the same as that in 
the standard theory (derivative of the full Lagrangian as a source and $3a'/a$ 
as “friction”). Thus, as compared to the standard theory, for the same local 
non–gravitational energy density and the warp factor slope, the scalar field $\phi$ 
changes faster (slower) if $P_X$ is smaller (bigger) than 1. Of course this is only 
a qualitative feature and in most of the cases any quantitative corrections 
can be found only by numerical calculations.

The positions of the branes are determined by the boundary conditions. 
The modifications to the boundary conditions (10) and (11) are analogous 
to those in the bulk background equations. Namely, not only the potentials 
but the full Lagrangians localized at the branes determine the jumps of $a'$ 
while their derivatives with respect to $\Phi$ determine the jumps of $P_X \phi'$.

Usually in Randall–Sundrum type models, the warp factor changes mono-
tonically in the bulk, so its derivative has the same sign for all $y$. Thus, be- 
because of opposite overall signs in the boundary conditions (10) at two branes, 
one brane must have positive tension while the second one must have negative 
tension. To check the signs of the brane tensions in the class of models 
considered in this work we rewrite eq. (8) in the following form

$$
\left( \frac{a'}{a^2} \right)' = -\frac{1}{3a} P_X \phi'^2 . 
$$

(54)

In the previous section we showed that the stability of the model requires 
that $P_X \phi'$ is everywhere non–zero. Thus, the r.h.s. of the above equation is 
always negative. The ratio $a'/a^2$ always decreases and the warp factor $a(y)$ 
can not have a minimum in the bulk. Because of that, it is not possible to 
construct a stable model with two positive tension branes. At least one brane 
must have a negative tension:

$$
\min_i \left( U^{(i)} \big|_{y_i} + \int_{y_i} \delta_i Q^{(i)} \right) < 0 . 
$$

(55)

In all stable models $\phi(y)$ must be a monotonic function ($\phi'$ can not vanish) 
and $P_X$ can not change sign. The limit of the product $P_X \phi'$ has the same 
sign at both branes. Thus, it follows from the boundary condition (11) that

$$
\left( U^{(1)} \big|_{y_1} + \int_{y_1} \delta_1 Q^{(1)} \right) \cdot \left( U^{(2)} \big|_{y_2} + \int_{y_2} \delta_2 Q^{(2)} \right) < 0 .
$$

(56)

We turn now to the stability conditions. One of them is the positivity of 
$b_i$ parameters defined in (23). Using the background equation of motion (7)
the last term in the definition of \( b_i \) can be rewritten as

\[
\mp \lim_{y \to y_i} (P_X + 2XP_{XX}) \left( \frac{\phi''}{\phi'} - \frac{a'}{a} \right)
\]

\[
= \lim_{y \to y_i} \left[ \mp 4P_X \frac{a'}{a} \pm P_{\phi X} \phi' \mp \frac{a^2 (V_{\phi} + P_\phi)}{\phi'} \right]
\]

\[
= \lim_{y \to y_i} \left[ -4P_X \left( \frac{\partial}{\partial n} a \right) - P_{\phi X} \frac{\partial}{\partial n} \phi \mp \frac{a^2 (V_{\phi} + P_\phi)}{\partial_{\phi} \phi} \right]
\]  

where we used the outer normal derivative introduced in eq. (23). The first term in the last square bracket of the above equation gives negative (positive) contribution to the \( b \) parameter on the positive (negative) tension brane. So, positivity of \( b \) at the positive tension brane is more difficult to achieve. Stability is improved when, close to the brane(s), \( P_{\phi X} \) has the opposite sign and \( (V_{\phi} + P_\phi) \) has the same sign as the normal derivative of the scalar field \( \partial \phi / \partial n \). There is another term in the definition of \( b \) which depends on the bulk background, namely \( \int \phi'' (P_{\phi X} + 2XP_{\phi XX}) \). Its sign depends on the background and on the details of the generalized bulk kinetic function \( P \). Non–trivial \( \Phi \)-dependence of \( P \) can be, at least in some cases, used to increase the radion mass. Finally, large enough values of the second derivatives of the brane kinetic terms \( Q^{(i)} \) may be used to make \( b_i \) positive.

The second stability condition in (42) can be quite easily fulfilled. For example: \( c_i \) given by eq. (25) vanishes if there is no kinetic term localized on the \( i \)-th brane and it is positive when such localized term is not much different from the standard one \( Q^{(i)} = X \).

Models with non–standard bulk and/or brane scalar kinetic terms are quite complicated and usually only performing numerical calculations one can find the background fields and check their stability against small perturbations. Nevertheless, it seems viable that stable solutions can exist also in models without any scalar potentials or cosmological constants. The kinetic terms alone may have structure rich enough for configurations with stabilized inter–brane distance. This is similar to the situation in models proposed in [10, 11] in which inflation was realized without any scalar potential.

Let us discuss what properties the generalized kinetic terms should have in order to support stable brane configurations. Conditions on the bulk kinetic function \( P \) are rather weak. It is enough that eq. (9) can be fulfilled for some \( y_0 \) and positive values of \( \phi'^2, P_X \) and \( (P_X + 2XP_{XX}) \). Then, the dynamical equations (7) and (8) can be used to extend the solution to \( y \neq y_0 \). The bulk stability conditions (43) are fulfilled at \( y_0 \), so they are fulfilled also in some finite interval in the 5th coordinate. Any two points in this interval may be
used to locate the branes. Of course, this is possible only when the brane kinetic terms have appropriate properties.

Restrictions on the brane kinetic functions $Q^{(i)}$ are quite strong if we want the branes to be stabilized at given positions in a given background. First of all, from eq. (56) it is obvious that without brane potentials it is necessary that $Q^{(i)}$ have some non–trivial $\Phi$–dependence. In addition, it follows from (55) that at least at one of the branes the kinetic term must give a negative contribution to its tension. This does not a priori mean that the system becomes unstable. Of course, we want the energy to be bounded from below, so the kinetic term at the second brane $Q^{(2)}(\Phi, X)$ (we call “second” that brane at which the expression in (55) is minimized) should give negative value of $\int \delta_2 Q^{(2)}$ only for some range of values of its arguments. More specifically, each $Q^{(i)}$ must satisfy two equalities (10) and (11) and two inequalities (42). The values of $\int \delta_i Q^{(i)}$ and $\int \delta_i Q^{(i)}_\Phi$ necessary to fulfill the background boundary conditions depend on the details of a given background but their signs are determined by (55) and (56). All boundary and stability conditions on the brane kinetic functions may be written in the following form

\[
\int_{y_i} \delta_i Q^{(i)} = \frac{6}{a^2} \left. \frac{\partial a}{\partial n} \right|_{y_i^{\pm}}, \tag{58}
\]

\[
\int_{y_i} \delta_i Q^{(i)}_\Phi = -\frac{2P_X}{a} \left. \frac{\partial \phi}{\partial n} \right|_{y_i^{\pm}}, \tag{59}
\]

\[
\int_{y_i} \delta_i Q^{(i)}_X \geq 0, \tag{60}
\]

\[
\int_{y_i} \delta_i Q^{(i)}_{\Phi \Phi} > -\frac{\tilde{b}_i}{a(y_i)}, \tag{61}
\]

where $\tilde{b}_i$ is the r.h.s. of (24) with $Q^{(i)}$ set to zero. It is possible to fulfill all the above conditions for example with the brane kinetic functions of the form

\[
Q^{(i)} = K^{(i)}(\Phi)X + L^{(i)}(\Phi)X^2. \tag{62}
\]

The most difficult part is to satisfy simultaneously conditions (58) and (60) at the second (negative tension) brane. Using eqs. (58), (60) and (11), one

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5 One should remember that in general $Q^{(i)}$ and $\int \delta_i Q^{(i)}$ are not just proportional to each other and can have negative values for (slightly) different regions of the parameter space. This is caused by the regularization procedure discussed in section 2.
can find a lower bounds on $L^{(i)}$

$$L^{(i)} X^2 \mid_{y_i^\pm} \geq - \frac{270}{a^2} \frac{\partial a}{\partial n} \mid_{y_i^\pm},$$

(63)

which can be translated to an upper bound on $K^{(i)}$

$$K^{(i)} X \mid_{y_i^\pm} = - \frac{3}{5} L^{(i)} X^2 \mid_{y_i^\pm} + \frac{18}{a^2} \frac{\partial a}{\partial n} \mid_{y_i^\pm} \leq \frac{180}{a^2} \frac{\partial a}{\partial n} \mid_{y_i^\pm}.$$  

(64)

At the positive tension brane $L^{(2)}(\phi(y_2))$ must be positive and big enough while $K^{(2)}(\phi(y_2))$ must be negative (with value related to the value of $L^{(2)}$). Thus, without scalar potentials it is not possible to construct a stable model with a positive tension brane if the corresponding $K^{(i)}$ is always positive.

Some $\Phi$-dependence of $K^{(i)}$ and/or $L^{(i)}$ is necessary to fulfill conditions (59) and (61). The background boundary condition (59) takes the following form

$$\left[ \frac{1}{3} K^{(i)} \Phi X + \frac{1}{5} L^{(i)} \Phi X^2 \right] \mid_{y_i^\pm} = - \frac{2P\Phi}{a} \frac{\partial \phi}{\partial n} \mid_{y_i^\pm}.$$  

(65)

In all stable configurations, the r.h.s. of this equation has opposite signs on the two branes (because $\phi'$ can not change sign). So, there are no consistent brane models without potentials if all first derivatives of $K^{(i)}$ and $L^{(i)}$ have the same sign.

The stability conditions (61) for the brane kinetic functions (62) may be rewritten as

$$\lim_{y \to y_i^\pm} \left[ \frac{1}{3} K^{(i)} \Phi X + \frac{1}{5} L^{(i)} \Phi X^2 + \frac{P\Phi}{P\Phi} \left( \frac{1}{3} K^{(i)} \Phi X + \frac{1}{5} L^{(i)} \Phi X^2 \right) \right.$$  

$$- 4P\Phi \left( \frac{1}{9} K^{(i)} \Phi X + \frac{1}{15} L^{(i)} X^2 + \frac{P\Phi}{3} \left( \frac{1}{9} K^{(i)} \Phi X + \frac{1}{15} L^{(i)} X^2 \right) \right) \right.$$  

$$> \int_{y_i^\pm} \frac{\phi''(P\Phi + 2X P\Phi X)}{a}.$$  

(66)

Some of the terms on the l.h.s. of the the above expression may be negative but they can be compensated by large enough value of $K^{(i)}_\Phi \Phi X/3 + L^{(i)}_\Phi \Phi X^2/5$.

It is clear that it is possible to choose functions $K^{(i)}$ and $L^{(i)}$ which satisfy all the above boundary and stability conditions for a given background. So, models in which the inter–brane distance is fixed in a stable way can be constructed even without any scalar potentials or cosmological constants. The brane induced kinetic terms may have a relatively simple form $Q =$
\[ KX + LX^2 \] if the functions \( K \) and \( L \) are generic enough\footnote{Of course, one fine tuning of parameters is necessary as in all models with flat 4D foliation.}. It would be interesting to check whether any higher order kinetic terms predicted for example by string theories have an appropriate structure.

\section{Conclusions}

We considered 5D brane models with bulk and brane scalar kinetic terms generalized to some functions of \( X = (\nabla \phi)^2/2 \) and the scalar field itself. The background equations of motion and boundary conditions have structure similar to the case with standard kinetic terms. There are two kinds of modifications. First: the scalar potential is replaced by the sum of the potential and the kinetic term. Second: derivatives of the scalar field are multiplied by derivatives of the bulk kinetic term with respect to \( X \).

Stability of background configurations has been checked by analyzing the spectrum of small scalar perturbations. A given background with fixed branes positions is stable only when all the masses squared in the spectrum are positive. The bulk equation of motion determining the shape of the KK modes of such perturbations was written in the Sturm–Liouville form. The corresponding boundary conditions have rather complicated form. They may be expressed in terms of four parameters (two for each brane), \( b_i \) and \( c_i \), determined by the background and by the bulk interactions described effectively by some potentials and generalized kinetic terms. The boundary conditions depend also on the eigenvalues and this dependence is more complicated than in models with standard kinetic terms. We have shown that our eigenvalue problem is self-adjoint with those complicated boundary conditions. We identified even larger class of boundary conditions for which the Sturm-Liouville operator is hermitian.

The eigenvalue–dependence of the boundary conditions makes the stability considerations more difficult. Sufficient conditions for the stability are: \( b_i > 0 \), \( c_i \geq 0 \) at each brane and the positivity of bulk functions \( P_X \), \( (P_X + 2XP_{XX}) \) and \( \phi'^2 \) for all values of the 5th coordinate \( y \). If \( c_i \geq 0 \) then the remaining conditions are not only sufficient but also the necessary ones. This changes when any of the \( c_i \) parameters is negative. It seems that it may be possible to have stable configurations with negative both \( b_1 \) and \( c_1 \) (or \( b_2 \) and \( c_2 \)). The lowest KK mode, the radion, becomes tachyonic when any of the quantities \( b_i c_i < 0 \) or any of the quantities \( \phi'^2 \), \( P_X \) or \( (P_X + 2XP_{XX}) \) is not strictly positive.
We have shown that stable brane models may be constructed without bulk and/or brane potentials and cosmological constants. This may be achieved for example when the brane localized kinetic terms take the form \( Q^{(i)} = K^{(i)}(\Phi)X + L^{(i)}(\Phi)X^2 \). Conditions for the functions \( K^{(i)}(\Phi) \) and \( L^{(i)}(\Phi) \) have been found.

Acknowledgments

This work has been supported by a Marie Curie Transfer of Knowledge Fellowship of the European Community’s Sixth Framework Programme under contract number MTKD-CT-2005-029466 (2006-2010). The author would like to thank for the hospitality experienced at Ludwig Maximilian University and Max Planck Institute in Munich where this work has been done.

References

[1] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. 83 (1999) 4922 [arXiv:hep-ph/9907447].

[2] C. Charmousis, R. Gregory and V. A. Rubakov, Phys. Rev. D 62 (2000) 067505 [arXiv:hep-th/9912160].

[3] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, Phys. Rev. D 62, 046008 (2000) [arXiv:hep-th/9909134].

[4] T. Tanaka and X. Montes, Nucl. Phys. B 582 (2000) 259 [arXiv:hep-th/0001092].

[5] C. Csáki, M. L. Graesser and G. D. Kribs, Phys. Rev. D 63 (2001) 065002 [arXiv:hep-th/0008151].

[6] S. Mukohyama and L. Kofman, Phys. Rev. D 65, 124025 (2002) [arXiv:hep-th/0112115].

[7] J. Lesgourgues and L. Sorbo, Phys. Rev. D 69 (2004) 084010 [arXiv:hep-th/0310007].

[8] A. V. Frolov and L. Kofman, Phys. Rev. D 69 (2004) 044021 [arXiv:hep-th/0309002].

[9] D. Konikowska, M. Olechowski and M. G. Schmidt, Phys. Rev. D 73, 105018 (2006) [arXiv:hep-th/0603014].
[10] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999) [arXiv:hep-th/9904075].

[11] J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, 219 (1999) [arXiv:hep-th/9904176].

[12] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D 62, 023511 (2000) [arXiv:astro-ph/9912463].

[13] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000) [arXiv:astro-ph/0004134]; Phys. Rev. D 63, 103510 (2001) [arXiv:astro-ph/0006373].

[14] E. Babichev, V. Mukhanov and A. Vikman, JHEP 0802, 101 (2008) [arXiv:0708.0561 [hep-th]].

[15] D. Maity, S. SenGupta and S. Sur, Phys. Lett. B 643, 348 (2006) [arXiv:hep-th/0604195]; [arXiv:hep-th/0609171].
A. Dey, D. Maity and S. SenGupta, Phys. Rev. D 75, 107901 (2007) [arXiv:hep-th/0611262].

[16] L. Kofman, J. Martin and M. Peloso, Phys. Rev. D 70, 085015 (2004) [arXiv:hep-ph/0401189].

[17] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485, 208 (2000) [arXiv:hep-th/0005016].

[18] M. S. Carena, A. Delgado, J. D. Lykken, S. Pokorski, M. Quiros and C. E. M. Wagner, Nucl. Phys. B 609, 499 (2001) [arXiv:hep-ph/0102172].

[19] G. R. Dvali, G. Gabadadze, M. Kolanovic and F. Nitti, Phys. Rev. D 64, 084004 (2001) [arXiv:hep-ph/0102216].

[20] R. Bao, M. S. Carena, J. Lykken, M. Park and J. Santiago, Phys. Rev. D 73, 064026 (2006) [arXiv:hep-th/0511266].

[21] M. S. Carena, T. M. P. Tait and C. E. M. Wagner, Acta Phys. Polon. B 33, 2355 (2002) [arXiv:hep-ph/0207056].

[22] B. s. Kyae, [arXiv:hep-th/0207272].

[23] H. Davoudiasl, J. L. Hewett and T. G. Rizzo, Phys. Rev. D 68, 045002 (2003) [arXiv:hep-ph/0212279].
[24] F. del Aguila, M. Perez-Victoria and J. Santiago, JHEP 0302, 051 (2003) [arXiv:hep-th/0302023].

[25] F. del Aguila, M. Perez-Victoria and J. Santiago, Acta Phys. Polon. B 34, 5511 (2003) [arXiv:hep-ph/0310353].

[26] A. Mück, L. Nilse, A. Pilaftsis and R. Rückl, Phys. Rev. D 71, 066004 (2005) [arXiv:hep-ph/0411258].

[27] C. Csaki, J. Hubisz and P. Meade, arXiv:hep-ph/0510275