LINEAR AND FULLY NONLINEAR ELLIPTIC EQUATIONS WITH MORREY DRIFT

N.V. KRYLOV

Abstract. We present some results concerning the solvability of linear elliptic equations in bounded domains with the main coefficients almost in VMO, the drift and the free terms in Morrey classes containing $L_d$, and bounded zeroth order coefficient. We prove that the second-order derivatives of solutions are in a local Morrey class containing $W^{2,p}_{loc}$. Actually, the exposition is given for fully nonlinear equations and encompasses the above mentioned results, which are new even if the main part of the equation is just the Laplacian.

1. Introduction

In this article we are dealing with fully nonlinear uniformly nondegenerate elliptic equations

$$H[u](x) := H(u, Du, D^2 u, x) = 0$$

in bounded domains in $\mathbb{R}^d$, that is a Euclidean space of points $x = (x^1, ..., x^d)$, where $H(u, x)$ is a function given for $x \in \mathbb{R}^d$ and $u = (u', u'')$, $u' = (u'_0, u'_1, ..., u'_d) \in \mathbb{R}^{d+1}$, and $u'' \in \mathbb{S}$, and $\mathbb{S}$ is the set of symmetric $d \times d$-matrices. We assume that the growth of $H$ with respect to $|Dv|$ is controlled by the product of $|Dv|$ and a function from a Morrey class containing $L_d$. The case when this function is in $L_d$ is treated earlier in [13]. The dependence of $H$ on $x$ is assumed to be of BMO type. Among other things we prove that there exists $d_0 \in (d/2, d)$ such that for any $p \in (d_0, d)$ the equation with prescribed continuous boundary data has a solution whose second-order derivatives are locally in a Morrey class contained in $L_p$. Our results are new even if $H$ is linear, when (1.1) becomes

$$a^{ij}(x)D_{ij} + b^i(x)D_i u + c(x)u + f(x) = 0.$$  

Naturally,

$$D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = D_i D_j, \quad Du = (D_i u), \quad D^2 u = (D_{ij} u).$$

The interest in solutions with $D^2 u$ in Morrey classes has a long history (see, for instance, [6] for an extensive list of references and the history of the
subject). The main motivation under this interest is that on the account of assuming better summability properties of the free terms in $L_p$, in terms of belonging to a Morrey class, one obtained better summability of $D^2u$ in $L_p$, in the same terms, and by using embedding theorems for Morrey classes one gets much better regularity of $Du$ than using just Sobolev embedding theorems. Roughly speaking, having already estimates of $D^2u$ in $L_p$, one tries to extend them to estimates in a Morrey class contained in $L_p$.

Our motivation is quite different. If $b = (b^i)$ in (1.2) is not in $L_d$, generally, there are no a priori estimates, no matter how smooth $u$ is, even if $a^{ij} = \delta^{ij}$.

**Example 1.1.** Take a constant $c$ and let $b = -cx/|x|^2$, $u(x) = 1 - |x|^2$, then $\Delta u + b^i D_i u = 2c - 2d$, which is zero if $c = d$. Also $u = 0$ on the boundary of $\{|x| < 1\}$ and there is no estimates of $u$ through the free term and boundary data.

In this example $b \notin L_d$, but it is in Morrey classes containing $L_d$ and we will show that, if $c$ is small enough, the equation $\Delta u + b^i D_i u = f$ is solvable even when $\Delta$ is replaced with $a^{ij} D_{ij}$, if it is a uniformly elliptic operator with $a^{ij}$ almost in VMO. In short, we can do with the drift term with summability property below $L_d$. This possibility was already exploited in the literature, see, for instance, [4] and the references therein, however with not so general $a^{ij}$.

The closest to our results the author could find in the literature are those in [6] and [16] which contain plenty of information, beyond the scope of this article, in the case of linear equations with $b$ at least in $L_d$. For instance, in [6] the power of summability $p$ of $D^2u$ can be any number in $(1, \infty)$. In our results when the linear equation appears as a particular case of fully nonlinear equation we have a restricted range of $p$, but $b$ is in a Morrey class containing $L_d$.

The authors of [2] treat fully nonlinear case by first showing the solvability in spaces with Muckenhoupt weights and from them, by using an elegant observation which first appeared in the proof of Theorem 3 in [5], derive the solvability in Morrey spaces. This approach can also work in our case, but requires developing first a theory similar to [7] of solvability in $W^{2,p}_\text{loc}$-spaces with weights and $p < d$. It is assumed in [7] that $p > d$, but after having the Aleksandrov type estimate in [8] and [15] for equation (1.2) with $f \in L_p$, $p < d$, and $b$ in a Morrey class containing $L_d$ one could mimic what was done in [7].

However, we decided to make a shortcut and separate the Morrey space theory from the weighted space one. In [2] the authors obtain global estimates in case the first order “coefficients” are bounded. Our estimates are only local, however, the author does not see much difficulties to make them global as well. By the way, specified to (1.2) the results of [2] require $a$ to be uniformly sufficiently close to a continuous function. In our case $a \in VMO$ suffices.
We follow a very natural approach from [16] based on the idea that the Morrey class estimates should more or less easily follow from the $L^p$ estimates for equations without lower order terms and then treating lower order terms as perturbations. It is important to stress that one cannot treat lower order terms even in Example 1.1 as perturbations dealing with the Sobolev space theory but it is possible to do so in the Morrey space theory mainly due to the Adams theorem 3.1 of [1]. One of main new facts which we use is the Aleksandrov type estimate obtained in [15].

The article is organized as follows. We present our main results and some examples in Section 2. Section 3 contains some auxiliary results. In Section 4 we prove interior estimates and in the final Section 5 we prove existence theorems.

We finish the introduction with some notation. Define
\[ B_r(x) = \{ y \in \mathbb{R}^d : |x - y| < r \}, \quad B_r = B_r(0). \]
For measurable $\Gamma \subset \mathbb{R}^d$ set $|\Gamma|$ to be its Lebesgue measure and when it makes sense set
\[ f_\Gamma = \int_\Gamma f \, dx = \frac{1}{|\Gamma|} \int_\Gamma f \, dx. \]

For domains $\Omega \subset \mathbb{R}^d$, $p \in [1, \infty)$, and $\mu \in (0, d/p]$, introduce Morrey’s space $E_{p,\mu}(\Omega)$ as the set of $g \in L^p(\Omega)$ such that
\[
\|g\|_{E_{p,\mu}(\Omega)} := \sup_{\rho < \infty, x \in \Omega} \rho^\mu \|g_{I_\Omega}\|_{L^p(B_{\rho}(x))} < \infty,
\]
where
\[
\|g\|_{L^p(\Gamma)} = \left( \int_\Gamma |g|^p \, dx \right)^{1/p}.
\]
Observe that for bounded $\Omega$ one can restrict $\rho$ in (1.3) to $\rho \leq \text{diam}(\Omega)$ due to $\mu \leq d/p$. Let
\[
E^2_{p,\mu}(\Omega) = \{ u : u, Du, D^2u \in E_{p,\mu}(\Omega) \}
\]
and provide $E^2_{p,\mu}(\Omega)$ with an obvious norm.

We will often, always tacitly, use the following formulas in which $u(x) = v(x/R)$:
\[
\|u\|_{L^p(B_R)} = \|v\|_{L^p(B_1)}, \quad \|u\|_{E_{p,\mu}(B_R)} = R^\mu \|v\|_{E_{p,\mu}(B_1)},
\]
\[
\|Du\|_{E_{p,\mu}(B_R)} = R^{\mu-1} \|Du\|_{E_{p,\mu}(B_1)}, \quad \|D^2u\|_{E_{p,\mu}(B_R)} = R^{\mu-2} \|v\|_{E_{p,\mu}(B_1)}.
\]

2. Main results

Fix some constants $\delta \in (0, 1]$, $K_0 \in [0, \infty)$, and a measurable function $K(x) \geq 0$ on $\mathbb{R}^d$. Let $S_\delta$ be the subset of $S$ consisting of matrices whose eigenvalues are between $\delta$ and $\delta^{-1}$. In our assumptions below we also use the parameters $\theta, \theta$, $q$, and $\hat{b}$, the values of which will be specified in the statements of our results.
Assumption 2.1. There are measurable functions \( F(u, x) = F(u'', x) \) and \( G(u, x) \) such that
\[
H = F + G.
\]
Furthermore, for all \( u'' \in S, u' \in \mathbb{R}^{d+1} \), and \( x \in \mathbb{R}^d \), we have
\[
|G(u, x)| \leq \hat{\theta}|u''| + K_0|u'| + b(x)|u'| + K(x), \quad F(0, x) \equiv 0, \quad (2.1)
\]
where \( [u'] := (u'_1, ..., u'_d) \).

Recall that Lipschitz continuous functions are almost everywhere differentiable thanks to the Rademacher theorem.

Assumption 2.2. (i) The function \( F \) is Lipschitz continuous with respect to \( u'' \) and \( D_{u''}F \in S_\delta \) at all points of differentiability of \( F \) with respect to \( u'' \).

Moreover, there exists \( R_0 \in (0, 1] \) such that, if \( r \in (0, R_0), z \in \mathbb{R}^d \), then one can find a convex function \( \bar{F}(u'') = \bar{F}_{z,r}(u'') \) (independent of \( x \)) for which

(ii) We have \( \bar{F}(0) = 0 \) and \( D_{u''}F \in S_\delta \) at all points of differentiability of \( F \);

(iii) For any \( u'' \in S \) with \( |u''| = 1 \), we have
\[
\int_{B_r(z)} \sup_{\tau > 0} \tau^{-1}|F(\tau u'', x) - \bar{F}(\tau u'')| \, dx \leq \theta. \quad (2.2)
\]

Assumption 2.3. For any \( u'' \in S, x \in \mathbb{R}^d, \) and \( \zeta \in [0, 1] \)
\[
|F(\zeta u'', x)| \leq \zeta |F(u'', x)| + K(x). \quad (2.3)
\]

Remark 2.1. Assumption 2.3 is satisfied, for instance, if \( F(u, x) \) is boundedly inhomogeneous in the sense that \( |(\partial/\partial t)(tF((1/t)u'', x))| \leq K(x) \) at all points of differentiability of \( tF((1/t)u'', x) \) with respect to \( t > 0 \). Indeed, in that case, for \( t_0 \in [0, 1] \)
\[
t_0F(u'', x) = F(t_0u'', x) - \int_{t_0}^1 (\partial/\partial s)(sF((1/s)t_0u'', x)) \, ds.
\]

To finish the setting, take \( d_0 = d_0(d, \delta) \in (d/2, d) \) from [13]. In Assumption 2.4 \( q \) is a number in \([d_0, d)\).

Assumption 2.4. There exists a constant \( \hat{b} > 0 \) such that for any ball \( B \) of radius \( r \leq R_0 \) we have
\[
\|b\|_{L_q(B)} \leq \hat{b}r^{-1}. \quad (2.4)
\]

Remark 2.2. A simple argument shows that Assumption 2.4 is satisfied with any \( q < d \) in Example 1.1 in which \( b \not\in L_{d,\text{loc}} \). It is also satisfied with \( \hat{b} \) as small as one likes on the account of choosing \( R_0 \) small enough if \( b \in L_d(\mathbb{R}^d) \), or if \( b \) is just bounded, since by Hölder’s inequality \( \|b\|_{L_q(B)} \leq \|b\|_{L_d(B)} = N(d)\|b\|_{L_d(B)r^{-1}}. \)
Here is our first main result about the interior estimates when \( b \) is bounded and \( p \) is restricted only from below. In it and below \( \nu = \nu(\alpha, \mu, d, \delta, p) > 1 \) is taken from Lemma 4.3.

**Theorem 2.1.** Let \( p \in (d_0, \infty) \), \( d/p \geq \mu > \alpha > 0 \) and let Assumption 2.1 be satisfied with \( \theta(\alpha, \mu, d, \delta, p) \) from Lemma 4.6 and Assumption 2.2 be satisfied with \( \theta = \theta(\alpha, d, \delta, p) \) from Lemma 4.1. Suppose that Assumption 2.3 is satisfied and that \( \hat{b} \) is satisfied with 2.2 is satisfied with \( \alpha < \mu \) and let Assumption 2.6.

\[
\|D^2 u\|_{E_p,\mu(B_{\rho_0})} \leq N \rho^{-2} \|u\|_{E_p,\mu(B_\rho)} + N \rho^{\mu-2} \sup_{B_\rho} |u| + N \|K\|_{E_p,\mu(B_\rho)} =: I, 
\]

(2.5)

where the constants \( N \) depend only on \( K_0, R_0, \alpha, \mu, d, \delta, p \).

If \( b \) is from a Morrey space rather than being bounded we need to restrict \( p \) from above and \( \mu \) from below.

**Theorem 2.2.** Let \( p \in (d_0, d) \), \( d/p > \mu > 1 \) and suppose that Assumption 2.2 is satisfied with \( \theta = \theta(\alpha, \mu, d, \delta, p) \) for some \( \alpha < \mu \) and Assumption 2.3 is also satisfied. Let \( \rho \leq R_0 \setminus \nu^{-1}, u \in E^2_{p,\mu}(B_\rho) \cap C(\bar{B}_\rho) \) and \( H[u] = 0 \) in \( B_\rho \). Then there exist \( \tilde{\theta} = \tilde{\theta}(\alpha, \mu, d, \delta, p) > 0 \) and \( \tilde{b} = \tilde{b}(\alpha, \mu, d, \delta, p) > 0 \) such that, if Assumptions 2.1 and 2.4 are satisfied with these \( \tilde{\theta} \) and \( \tilde{b} \) and \( q = p_\mu \) \((\in (d_0, d))\), respectively, then with \( \rho_0 = \rho/3, \) we have

\[
\|D^2 u\|_{E_p,\mu(B_{\rho_0})} \leq N \rho^{-2} \|u\|_{E_p,\mu(B_\rho)} + N \rho^{\mu-2} \sup_{B_\rho} |u| + N \|K\|_{E_p,\mu(B_\rho)}, \quad (2.6)
\]

where the constants \( N \) depend only on \( R_0, \alpha, \mu, d, \delta, p \).

Observe that Remark 2.2 shows that, for \( p \in (d_0, d), \) \( d/p > \mu > 1 \), Theorem 2.2 contains Theorem 2.1.

The above results are a priori estimates. Note that in them the equation \( H[u] = 0 \) is not even assumed to be elliptic. To state existence theorems we need three more assumptions.

**Assumption 2.5.** We are given a bounded domain \( \Omega \subset \mathbb{R}^d \) such that for some constants \( \rho, \gamma > 0 \) and any \( x \in \partial \Omega \) and \( r \in (0, \rho) \) we have \( |B_r(x) \cap \Omega^c| \geq \gamma |B_r| \). We are also given a \( g \in C(\partial \Omega) \).

**Assumption 2.6.** The function \( H(u, x) \) is continuous in \( u \) for any \( x \), is Lipschitz continuous with respect to \( u'' \), and \( D_{u''}H \in S_\delta \) at all points of differentiability of \( H \) with respect to \( u'' \).

**Assumption 2.7.** For all values of the arguments,

\[
H(u', 0, x) \text{ sign } u'_0 \leq b(x) ||u'|| + K(x) \quad (\text{sign } 0 := \pm 1). \quad (2.7)
\]

Observe that \( H(u, x) \) may not be Lipschitz continuous with respect to \( u' \) or decreasing in \( u'_0 \) in contrast with conditions in very many articles on the
subject, [2] including. We are dealing with the solvability of

\[ H[u] = 0 \]  \hspace{1cm} (2.8)

in \( \Omega \) with boundary condition \( u = g \) on \( \partial \Omega \).

Here is our result concerning the solvability of (2.8) in Morrey spaces in case \( b \) is bounded.

**Theorem 2.3.** Let \( p \in (d_0, \infty) \), \( d/p \geq \mu > 0 \), and suppose that \( b \leq K_0 \) and Assumptions 2.3, 2.5, 2.6, and 2.7 are satisfied. There exist constants \( \hat{\theta}, \theta \in (0,1) \), depending only on \( d, p, \delta, \) and \( \mu \), which are, generally, smaller than \( \hat{\theta} \) from Theorem 2.1 and such that, if Assumptions 2.2 and 2.1 are satisfied with these \( \theta \) and \( \hat{\theta} \), respectively, and \( K \in E_{p,\mu,\text{loc}}(\Omega) \cap L_p(\Omega) \), then there exists \( u \in E^2_{p,\text{loc}}(\Omega) \cap C(\Omega) \) satisfying (2.8) in \( \Omega \) and such that \( u = g \) on \( \partial \Omega \). Furthermore, in \( \Omega \)

\[ |u| \leq N \|K\|_{L_p(\Omega)} + \sup_{\partial \Omega} |g|, \]  \hspace{1cm} (2.9)

where \( N \) depends only on \( p, d, \delta, K_0, R_0, \) and the diameter of \( \Omega \).

If \( b \) is from a Morrey space, the assumptions are stronger.

**Theorem 2.4.** Let \( p \in (d_0, d) \), \( d/p > \mu > 1 \), and suppose that Assumptions 2.3, 2.5, 2.6, and 2.7 are satisfied. There exist constants \( \hat{\theta}, \theta, \hat{b} \in (0,1) \), depending only on \( d, p, \delta, \) and \( \mu \), which are, generally, smaller than those from Theorem 2.2 and such that, if Assumptions 2.1, 2.2, and 2.4 are satisfied with these \( \theta \) and \( \hat{\theta} \), respectively, and \( K \in E_{p,\mu,\text{loc}}(\Omega) \cap L_p(\Omega) \), then there exists \( u \in E^2_{p,\text{loc}}(\Omega) \cap C(\Omega) \) satisfying (2.8) in \( \Omega \) and such that \( u = g \) on \( \partial \Omega \). Furthermore, in \( \Omega \)

\[ |u| \leq N \|K\|_{L_p(\Omega)} + \sup_{\partial \Omega} |g|, \]  \hspace{1cm} (2.10)

where \( N \) depends only on \( p, d, \delta, R_0, \) and the diameter of \( \Omega \).

**Remark 2.3.** The fact that we need Assumption 2.4 satisfied with \( \hat{b} \) small enough for (2.10) to hold is well illustrated by Example 1.1.

**Remark 2.4.** Observe that generally there is no uniqueness in Theorems 2.3 or 2.4. For instance, in the one-dimensional case the (quasilinear) equation

\[ D^2u + \sqrt{12|Du|} = 0 \]

for \( x \in (-1, 1) \) with zero boundary data has two solutions: one is identically equal to zero and the other one is \( 1 - |x|^3 \).

Another example is given by the (semilinear) equation

\[ D^2u + 2u(1 + \sin^2 x + u^2)^{-1} = 0 \]

on \((-\pi/2, \pi/2)\) with zero boundary condition. Again there are two solutions: one is \( \cos x \) and the other one is identically equal to zero.
Example 2.1. Let $d = 3$, $f, K, b \in L_d(\Omega)$, $\alpha \in (0, 1]$. Let $w(t), t \in [0, \infty)$, be a continuously differentiable function with sufficiently small derivative. Then the equation
\[
H(Du, D^2u, x) := K(x) \wedge |D_{12}u| + K(x) \wedge |D_{23}u| + K(x) \wedge |D_{31}u|
\]
\[
+ 2\Delta u + w(|D^2u|) + b(x)|Du|\alpha - f(x) = 0
\]  
(2.11)
satisfies our assumptions and Theorem 2.4 is applicable with any $p \in (d_0, d)$, $d/p > \mu > 1$.

Observe that $H$ in (2.11) is neither convex nor concave with respect to $D^2u$. Also note that we can replace $\Delta u$ with $a^{ij}(x)D_{ij}u$ if $a(x) = (a^{ij}(x))$ is an $S_\delta$-valued VMO-function such that $a(x) \geq 2(\delta^{ij})$.

Example 2.2. Let $A$ and $B$ be some countable sets and assume that for $\alpha \in A$, $\beta \in B$, $x \in \mathbb{R}^d$, and $u' \in \mathbb{R}^{d+1}$ we are given an $S_\delta$-valued function $a^\alpha(x)$ (independent of $\beta$) and a real-valued function $b^{\alpha\beta}(u', x)$. Assume that these functions are measurable in $x$, $a^\alpha$ and $b^{\alpha\beta}$ are continuous with respect to $u'$ uniformly with respect to $\alpha, \beta, x$, and
\[
|b^{\alpha\beta}(u', x)| \leq b(x)\left|\left(u'_1, \ldots, u'_{d}\right)\right| + K(x),
\]
where $K, b \in L_d(\Omega)$. Next assume that there is an $R_0 \in (0, \infty)$ such that for any $z \in \Omega$, $r \in (0, R_0]$ one can find $\bar{a}^\alpha \in S_\delta$ (independent of $x$) such that
\[
\int_{B_r(z)} \sup_{\alpha \in A} |a^\alpha(x) - \bar{a}^\alpha| \, dx \leq \theta,
\]
where $\theta$ is sufficiently small (to accommodate Theorem 2.3).

Consider equation (2.8), where
\[
H(u, x) := \inf_{\beta \in B} \sup_{\alpha \in A} \left[ \sum_{i,j=1}^d a^{ij}_\alpha(x)u''_{ij} + b^{\alpha\beta}(u', x) \right].
\]
Define
\[
F(u'', x) := \sup_{\alpha \in A} \sum_{i,j=1}^d a^{ij}_\alpha(x)u''_{ij}.
\]
As in Example 10.1.24 of [12] one easily sees that we are in the framework of Theorem 2.4 with any $p \in (d_0, d)$, $d/p > \mu > 1$.

Example 2.3. A further specification of Example 2.2 is given by linear equations. Suppose that we are given an $S_\delta$-valued measurable function $a(x)$ and an $\mathbb{R}^d$-valued function $b(x)$ such that $b \in E_{p_\mu, \mu'}(\mathbb{R}^d)$, where $p \in (d_0, d)$, $d/p > \mu > 1$, $0 < \mu' < 1$.

Next assume that there is an $R_0 \in (0, \infty)$ such that for any ball $B \subset \mathbb{R}^d$ of radius smaller than $R_0$
\[
\int_B |a(x) - \bar{a}_B| \, dx \leq \theta.
\]
For $f \in L_d(\Omega)$ and nonnegative and bounded $c$ consider equation (1.2) in $\Omega$ with boundary condition $u = g$ on $\partial\Omega$. 

In this situation one can obviously take $F(u'', x) = a^{ij}(x)u''_{ij}$ and satisfy Assumption 2.2 with $F(u) = \tilde{a}^{ij}_{\zeta}(x)u''_{ij}$. Assumptions 2.1 (with $\hat{\theta} = 0$, $K_0 = \sup c$, $b = |b|$, $K = |f|$) and 2.7 are also satisfied. Then observe that

$$r\|b\|_{L_p(B_r(x))} \leq r^{1-\mu'}\|b\|_{E_{p,\mu'}(\mathbb{R}^d)},$$

which for small $r$ can be made as small as we like because $\mu' < 1$. Hence, Assumption 2.4 is satisfied for small $R_0$. Therefore, by Theorem 2.4, if $\theta$ is sufficiently small, depending only on $d, p, \delta$, and $\Omega, g$ satisfy Assumption 2.5, the above boundary value problem has a solution in $u \in E^2_{p,\mu',d,\delta}(\Omega) \cap C(\bar{\Omega})$. Owing to Theorem 3.2 this solution is unique.

Even this result is new. Also observe that, generally, $u \not\in W^2_{d,\delta}(\Omega)$. The main novelty in this example is that, generally, $b \not\in L_{d,\delta}(\Omega)$, and even if $a$ is constant the result was not known before.

3. Auxiliary results

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$. For $\rho > 0$ set $\Omega^\rho = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \rho\}$.

**Theorem 3.1.** Let $p > d_0$ and $f \in L_p(\Omega)$. Then there exist a constant $\theta \in (0,1]$, depending only on $d, p, \delta$, such that, if Assumptions 2.2 is satisfied with this $\theta$, then, for any $u \in W^2_{p,\delta}(\Omega) \cap C(\bar{\Omega})$ that satisfies $F[u] = f$ in $\Omega$ and $0 < \rho < \rho_{\text{int}}(\Omega) \wedge 1$, where $\rho_{\text{int}}(\Omega)$ is the interior radius of $\Omega$, we have

$$\|u\|_{W^2_p(\Omega^\rho)} \leq N\|f\|_{L_p(\Omega)} + N\rho^{-2}\|u\|_{C(\bar{\Omega})}, \quad (3.1)$$

where the constants $N$ depend only on $d, p, \delta, R_0$, and $\text{diam}(\Omega)$.

This theorem looks like a particular case of Theorem 1.1 of [13], in which lower order terms are present in the equation, but $p$ is restricted to $(d_0, d)$. The upper bound $d$ for $p$ is caused by the presence of the first order terms with the “coefficient” in $L_d$. However, if there are no lower order terms, the arguments in [13] go through for the full range $p > d_0$.

Next, introduce $L$ as the set of operators $L = a^{ij}D_{ij} + b^iD_i$ with measurable coefficients on $\mathbb{R}^d$ such that $(a^{ij})$ is $\mathcal{S}_\delta$-valued and $|b| \leq b$. Here is Theorem 1.4 of [15].

**Theorem 3.2.** Assume that, for a $R_0 \in (0, \infty)$, estimate (2.4) holds with $q = d_0$ and $\hat{b} = \hat{b}(d, \delta)$ from Theorem 1.1 of [15] for any $r \in (0, R_0]$ and ball $B$ of radius $r$. Let $u \in W^2_{d_0,\delta}(\Omega) \cap C(\bar{\Omega})$, $L \in L$. Take a function $c \geq 0$. Then on $\Omega$

$$u \leq N\|L_{d_0,\delta}(\Omega^\rho) = \sup u_+ \cdot \|L_{d_0,\delta}(\Omega^\rho) \cap C(\bar{\Omega}), \quad (3.2)$$

where $N$ depends only on $d, \delta, R_0, \text{ and the diameter of } \Omega$.

With this version of the Aleksandrov estimate at hand one can repeat what is done in [14] in case $b \in L_d$ and arrive at the following result (see
Corollary 4.12 in [14]) about the boundary behavior of solutions of linear equations.

Lemma 3.3. Under the assumptions of Theorem 3.2 let $0 \in \partial \Omega$ and suppose that for some constants $\rho, \gamma > 0$ and any $r \in (0, \rho)$ we have $|B_r \cap \Omega^c| \geq \gamma |B_r|$. Let $w(r)$ be a concave continuous function on $[0, \infty)$ such that $w(0) = 0$ and $|u(x) - u(0)| \leq w(|x|)$ for all $x \in \partial \Omega$. Then there exists $\beta = \beta(d, \delta, \gamma) > 0$ such that, for any $L \in \mathbb{L}$ and $x \in \Omega$,

$$|u(x) - u(0)| \leq N|x|^{\beta}||Lu||_{L_{d,\theta}(\Omega)} + \omega(N|x|^{\beta/2}),$$

where $N$ depends only on $R_0, d, \delta, \gamma, \rho$, and the diameter of $\Omega$.

The details of the proof of this lemma will be presented elsewhere.

4. Interior estimates

Remark 4.1. If a function $u$ satisfies $F[u] = f$ in $B_\rho$ with $\rho \leq 1$, then $u_\rho(x) := \rho^{-2}u(\rho x)$, satisfies $F_\rho[u_\rho] = f_\rho$ in $B_1$, where $F_\rho(u'', x) = F(u'', \rho x)$, $f_\rho(x) = f(\rho x)$. It is important that if Assumption 2.2 is satisfied for the original $F$, then it is also satisfied with the same $\theta$ and $R_0$ for $F_\rho$.

Lemma 4.1. Let $p \in (d_0, \infty)$, $0 < 4\rho_1 \leq \rho_2 \leq 1$, $\alpha > 0$, $u \in W^{2}_{\rho_1,\rho_2}(B_{\rho_2}) \cap C(B_{\rho_2})$ and $f := F[u] \in L_p(B_{\rho_2})$. Then there exists a constant $\theta = \theta(\alpha, d, \delta, p) > 0$, such that, if Assumption 2.2 is satisfied with this $\theta$, then

$$\|D^2 u\|_{L_p(B_{\rho_1})} \leq N_1(\rho_2/\rho_1)^2 \|f\|_{L_p(B_{\rho_2})} + N(\rho_2/\rho_1)^{\alpha} \rho_2^{-2} \sup_{\partial B_{\rho_2}} |u - l|,$$

where the constants depend only on $R_0, d, \delta, \rho, \alpha$, and $l$ is any affine function.

Proof. Observe that to prove (4.2) it suffices to concentrate on $\alpha < 1$. In that case take $q > d/\alpha$ and use Theorem 10.1.14 of [12] to find a solution $v \in W^{2}_{\rho_1,\rho_2}(B_{\rho_2}) \cap C(B_{\rho_2})$ of the equation $F[v] = 0$ in $B_{\rho_2}$ with boundary value $v = u$ on $\partial B_{\rho_2}$. By taking into account Remark 4.1 and using scaling and estimate (3.1) we get that, if $\theta$ is chosen appropriately, then

$$\|D^2 v\|_{L_q(B_{\rho_2}/2)} \leq N(\rho_2 - \rho_2/2)^{-2} \sup_{B_{\rho_2}} |v - l| = N \rho_2^{-2} \sup_{\partial B_{\rho_2}} |v - l| = N \rho_2^{-2} \sup_{\partial B_{\rho_2}} |u - l|,$$

where $l$ is any affine function and the equalities follow from the fact that $0 = F[v] - F[0] = a^{ij}D_{ij}v$ for certain $S_\delta$-valued $(a^{ij})$. It follows by Hölder’s inequality that

$$\|D^2 v\|_{L_p(B_{\rho_1})} \leq \|D^2 v\|_{L_q(B_{\rho_2}/2)} \leq N(\rho_2/\rho_1)^{d/q} \|D^2 v\|_{L_q(B_{\rho_2}/2)} \leq N(\rho_2/\rho_1)^{d/q} \rho_2^{-2} \sup_{\partial B_{\rho_2}} |u - l|.$$
Assumption 4.1. We have

\[ D^2 v \leq N (\rho_2/\rho_1) \int B_{2r_1} \| f \|_{L^p}^2 \sup_{B_{2r_1}} |u - l|. \]  

(4.3)

Then a very particular case of (3.1) is that

\[ D^2 u \leq N \| f \|_{L^p}^2 + N \rho_1^{-2} \sup_{B_{2r_1}^1} |u - l'|, \]  

(4.4)

where \( l' \) is any affine function. Here \( u - l' = w + (v - l') \), where \( w = 0 \) on \( \partial B_{r_2} \) and \( f = F[w + v] = F[w + v] = a^{ij} D_{ij} w \) for some \( \mathbb{S}_g \)-valued \( (a^{ij}) \). It follows from [9], [3], [10] or [15] that

\[ \| w \| \leq N \rho_2^2 \| f \|_{L^p}(B_{r_2}). \]  

Hence the last term in (4.4) is dominated by

\[ \int B_{2r_1} \| f \|_{L^p}^2 + N \rho_1^{-2} \sup_{B_{2r_1}^1} |v - l'|, \]

where the second term, for an appropriate choice of \( l' \), is estimated by a constant times \( \| D^2 v \|_{L^p}(B_{2r_1}) \) owing to the Poincaré inequality. After that, to get (4.1), it only remains to refer to (4.3). Estimate (4.2) is obtained from (4.3) by the Poincaré inequality. The lemma is proved.

The following result is well known when \( p \geq d \) and \( F = \bar{F} \).

**Corollary 4.2.** If \( F \) is independent of \( x \), \( p \in (d_0, \infty) \), \( \alpha > 0 \), Assumption 2.2 is satisfied with \( \theta(\alpha, d, \delta, p) \), \( u \in W^2_{\rho, \text{loc}}(\mathbb{R}^d) \) satisfies \( F[u] = 0 \) in \( \mathbb{R}^d \) and

\[ \lim_{R \to \infty} \inf_{l \in \mathcal{L}} \sup_{B_{2R}} \| u - l \| = 0, \]

where \( \mathcal{L} \) is the set of affine functions, then \( u \in \mathcal{L} \).

**Assumption 4.1.** We have \( p \in (d_0, \infty) \), \( \mu > \alpha > 0 \) and Assumption 2.2 is satisfied with \( \theta = \theta(\alpha, d, \delta, p) \) introduced in Lemma 4.1.

**Lemma 4.3.** Suppose that Assumption 4.1 is satisfied. Let \( \rho \leq 1 \), \( u \in W^2_{\rho, \text{loc}}(B_{r}) \cap C(B_{r}) \) and \( f \) for \( F[u] \in L^p(B_{r}) \). Then there exists \( \nu = \nu(\alpha, \mu, d, \delta, p) > 1 \) such that for any \( r \leq \rho/\nu \) we have

\[ r^\mu \| D^2 u \|_{L^p(B_r)} \leq N \sup_{r \leq s \leq \rho} s^\mu \| f \|_{L^p(B_s)} \]

\[ + (1/2) \min \left( \rho^\mu \| D^2 u \|_{L^p(B_{\rho})}, N \rho^{\mu-2} \sup_{B_{\rho}} |u - l| \right), \]  

(4.5)

where \( l \) is any affine function and the constants \( N \) depend only on \( \alpha, \mu, R_0, d, \delta, p \).

**Proof.** Take the smallest \( \kappa \geq 4 \) such that \( N \kappa^{-\mu} \leq 1/2 \). Then, for \( r \leq \rho/\kappa \), define \( r_n = \kappa^n r \), \( m = \max \{ n \geq 0 : r_{n+1} \leq \rho \} \),

\[ A_n = r_n^\mu \| D^2 u \|_{L^p(B_{r_n})}, \quad B = \sup_{r \leq s \leq \rho} s^\mu \| f \|_{L^p(B_s)}. \]

For \( 0 \leq n \leq m \), \( \rho_1 = r_n, \rho_2 = r_{n+1} \) estimate (4.2) yields

\[ A_n \leq N_1 \kappa^{-\mu} B + N_2 \kappa^{-\mu} A_{n+1} \leq N_3 B + (1/2) A_{n+1}. \]
By iterating we obtain \( A_0 \leq 2N_3B + 2^{-m}A_m \) and arrive at
\[
 r^\mu \|D^2u\|_{L_p(B_r)} \leq N \sup_{r \leq s \leq \rho} s^\mu \|f\|_{L_p(B_s)} + 2^{-m}r^\mu_m \|D^2u\|_{L_p(B_{r_m})}. \tag{4.6}
\]
Here \( r_m \leq \rho/\kappa \) and \( r_m \geq \rho/\kappa^2 \). It follows that the last term in (4.6) is less than
\[
2^{-m}r^\mu_m (r_m/\rho)^{-d/p} \|D^2u\|_{L_p(B_{r_m})} \leq 2^{-m} \kappa^{-\mu+2d/p} \rho^\mu \|D^2u\|_{L_p(B_{r})}.
\]
Now define \( m_0 = m_0(\mu, d, \delta, p) \) as the smallest integer \( m \geq 1 \) satisfying the inequality \( 2^{-m} \kappa^{-\mu+2d/p} \leq 1/4 \) and set \( \nu = \kappa^{m_0} \). Then for \( r \leq \rho/\nu \) we have \( m \geq m_0 - 1 \) and \( 2^{-m} \kappa^{-\mu+2d/p} \leq 1/2 \), so that the left-hand side of (4.5) is less than the first term on the right plus one half of the first term inside the min sign. On the other hand,
\[
\|D^2u\|_{L_p(B_{r_m})} \leq N \|f\|_{L_p(B_{2r_m})} + N r^{-2}_m \sup_{B_{2r_m}} |u - l|,
\]
and the left-hand side of (4.5) is less than the first term on the right plus one half of the second term inside the min sign as well. This proves the lemma.

Below by \( \nu \) we always mean the constant from Lemma 4.3.

The following is quite natural. It looks like it first appeared in [6], which makes the author wonder how in the past people claiming that flat boundary and interior Morrey estimates lead to global estimates in smooth domains using flattening the boundary and partitions of unity. These procedures unavoidably lead to appearance of the first order terms, the way to deal with which was not exhibited before [6]. Unfortunately, the proof in [6] contains an error (see Lemma 4.2 there). We give a different proof.

**Lemma 4.4.** Let \( p \in (1, \infty), \ 0 < \mu \leq d/p, \ R \in (0, \infty), \ u \in W^2_p(B_R) \), and \( x_0 \in B_R \). Then there is a constant \( N = N(d, p, \mu) \) such that, for any \( \varepsilon \in (0, 1), \ r \leq 2R, \)
\[
 r^\mu \|I_{BR} Du\|_{L_p(B_r(x_0))} \leq N\varepsilon R \sup_{r \leq s \leq 2R} s^\mu \|I_{BR} D^2u\|_{L_p(B_s(x_0))}
 + N\varepsilon^{-1} R^{-1} \sup_{r \leq s \leq 2R} s^\mu \|I_{BR}(u - c)\|_{L_p(B_s(x_0))}, \tag{4.7}
\]
where \( c \) is any constant. In particular,
\[
\|Du\|_{E_p, \mu(B_R)} \leq N\varepsilon R \|D^2u\|_{E_p, \mu(B_R)} + N\varepsilon^{-1} R^{-1} \|u\|_{E_p, \mu(B_R)}. \tag{4.8}
\]

Proof. Scalings show that we may assume that \( R = 1 \). Obviously we may also assume that \( c = 0 \).
Then denote \( v = Du, \ w = D^2u, \ G_s = B_s(x_0) \cap C_1, \)
\[
U = \sup_{r \leq s \leq 2} s^\mu \|I_{B_1}u\|_{L_p(B_s(x_0))}, \quad W = \sup_{r \leq s \leq 2} s^\mu \|I_{B_1}D^2u\|_{L_p(B_s(x_0))},
\]
By Poincaré’s inequality, for \( r \leq s \leq 2, \)
\[
\|v - v_{G_s}\|_{L_p(G_s)} \leq N(d)s \|w\|_{L_p(G_s)} \leq Ns^{1-\mu}W.
\]
Also by interpolation inequalities, there exists a constant \( N = N(d,p) \) such that, for \( \varepsilon \in (0,1] \) and \( \varepsilon \leq s \leq 2, \)
\[
\|v - v_{G_s}\|_{L_p(G_s)} \leq 2\|v\|_{L_p(G_s)} \leq N \|w\|_{L_p(G_s)} \|u\|_{L_p(G_s)}^{1/2}
\]
\[
+Ns^{-1} \|u\|_{L_p(G_s)} \leq N \|w\|_{L_p(G_s)} \|u\|_{L_p(G_s)}^{1/2} + N\varepsilon^{-1} \|u\|_{L_p(G_s)}, \quad (4.9)
\]
which for \( 2 \geq s \geq \varepsilon \) and \( r \) yields
\[
s^\mu \|v - v_{G_s}\|_{L_p(G_s)} \leq NW^{1/2}U^{1/2} + N\varepsilon^{-1}U.
\]
Hence, for any \( \varepsilon \in (0,1] \) and \( \varepsilon \leq s \leq 2 \)
\[
s^\mu \|v - v_{G_s}\|_{L_p(G_s)} \leq N_1\varepsilon W + N_2\varepsilon^{-1}U,
\]
where \( N_1 = N_1(d,p), \ N_2 = N_2(d,p). \)
Since \( \mu \in (0,d/p], \) Campanato’s results (see, for instance, Proposition 5.4 in [11]) imply that
\[
r^\mu \|v\|_{L_p(G_r)} \leq N_3(N_1\varepsilon W + N_2\varepsilon^{-1}U) + N_3 \|v\|_{L_p(G_2)}
\]
where \( N_3 = N_3(d,p,\mu). \) We estimate the last term as in (4.9) and come to what implies (4.7). The lemma is proved.

**Lemma 4.5.** Let \( \mu \leq d/p \) and suppose that Assumptions 2.3 and 4.1 are satisfied and we have \( r \leq \rho_0 < \rho_1 \leq 1/\nu, \) such that \( \rho_1 - \rho_0 \leq \rho_0. \) Let \( u \in W_p^2(B_{\rho_1}). \) Set \( f = F[u]. \) Then
\[
r^\mu \|D^2u\|_{L_p(B_r)} \leq N\tilde{f} + N(\rho_1 - \rho_0)^{-2}\tilde{u} + N\tilde{u} + N\tilde{K}, \quad (4.10)
\]
where
\[
\tilde{f} = \sup_{r \leq s \leq \rho_1} s^\mu \|f\|_{L_p(B_s)}, \quad \tilde{u} = \sup_{r \leq s \leq \rho_1} s^\mu \|u\|_{L_p(B_s)},
\]
\[
\tilde{K} = \rho_1^{\mu-2} \sup_{B_{\rho_1}}|u|,
\]
and the constants depend only on \( \alpha, \mu, d, p, \delta, R_0. \)

Proof. Define \( \kappa = \rho_1 - \rho_0, \ r_0 = \rho_0 \) and for \( n \geq 1 \)
\[
r_n = \rho_0 + \kappa \sum_{k=1}^n 2^{-k}.
\]
Also introduce smooth \( \zeta_n(x) \) such that \( 0 \leq \zeta_n(x) \leq 1, \ \zeta_n(x) = 1 \) for \( |x| \leq r_n, \)
\( \zeta_n(x) = 0 \) for \( |x| \geq r_{n+1}, \)
\[
|D\zeta_n| \leq N(d)2^n\kappa^{-1}, \quad |D^2\zeta_n| \leq N(d)4^n\kappa^{-2}.
\]
Observe that \( r_n \leq \rho_1 = (\nu \rho_1)/\nu \) and \( \nu \rho_1 \leq 1 \), so that by Lemma 4.3
\[
W_n := \sup_{r \leq s \leq r_n} s^\mu \| D^2 u \|_{L^p(B_s)} \leq N \hat{u} + NF_n,
\]
where
\[
F_n := \sup_{r \leq s \leq \nu \rho_1} s^\mu \| F(D^2(\zeta u)) \|_{L^p(B_s)} = \sup_{r \leq s \leq r_{n+1}} s^\mu \| F(D^2(\zeta u)) \|_{L^p(B_s)},
\]
where the equality is due to Hölder’s inequality, assumption that \( \mu \leq d/p \), and the fact that \( F(D^2(\zeta u)) = 0 \) outside \( B_{r_{n+1}} \). By our assumptions (dropping the argument \( x \))
\[
|F(D^2(\zeta u))| = |F(\zeta_n D^2 u + 2 D\zeta_n \otimes Du + u D^2 \zeta_n)|
\leq \zeta_n |F(D^2 u)| + N I_{B_{r_{n+1}}} (2^{n \kappa - 1} |Du| + 4^n \kappa^{-2} |u|) + K_1 I_{B_{r_{n+1}}},
\]
which implies that
\[
F_n \leq NF + N2^n \kappa^{-1} \sup_{r \leq s \leq r_{n+1}} s^\mu \| Du \|_{L^p(B_s)} + N4^n \kappa^{-2} \hat{u} + N \bar{K}.
\]
Here for \( r \leq s \leq r_{n+1} \) by Lemma 4.4 (with \( x_0 = 0 \), \( R = r_{n+1} \))
\[
s^\mu \| Du \|_{L^p(B_s)} \leq N \varepsilon r_{n+1} W_{n+1} + N \varepsilon^{-1} r_{n+1}^{-1} \hat{u},
\]
so that for any \( \varepsilon \in (0, 1] \)
\[
F_n \leq NF + N_1 \varepsilon 2^n \kappa^{-1} r_{n+1} W_{n+1} + N (2^n \kappa^{-1} \varepsilon^{-1} r_{n+1}^{-1} + 4^n \kappa^{-2}) \hat{u} + N \bar{K}.
\]
We may assume that \( N_1 \geq 1 \), so that \( N_1 2^n \kappa^{-1} r_{n+1} \geq \rho_0/(\rho_1 - \rho_0) \geq 1 \). Therefore, now we can take \( \varepsilon \in (0, 1] \) such that
\[
N_1 \varepsilon 2^n \kappa^{-1} r_{n+1} = 1/8.
\]
Then \( \varepsilon^{-1} = N2^n \kappa^{-1} r_{n+1} \) and
\[
2^n \kappa^{-1} \varepsilon^{-1} r_{n+1}^{-1} + 4^n \kappa^{-2} \leq N4^n \kappa^{-2}.
\]
Coming back to (4.11) we get
\[
W_n \leq N(F + \hat{u}) + 8^{-1} W_{n+1} + N4^n \kappa^{-2} \hat{u} + N \bar{K}.
\]
We multiply this inequality by \( 8^{-n} \) and sum over \( n = 0, 1, \ldots \). Then we obtain
\[
\sum_{n=0}^\infty 8^{-n} W_n \leq N(F + \hat{u}) + \sum_{n=1}^\infty 8^{-n} W_n + N \kappa^{-2} \hat{u} + N \bar{K}.
\]
Canceling (finite) like terms yields \( W_0 \leq N(F + \hat{u}) + N \kappa^{-2} \hat{u} + N \bar{K} \), which implies (4.10) and proves the lemma.

**Lemma 4.6.** Under the assumptions of Lemma 4.5 suppose that \( b \leq K_0 \) and \( H[u] = 0 \) in \( B_{\rho_1} \). Then there exists \( \theta = \theta(\alpha, \mu, d, \delta, p) > 0 \) such that, if Assumption 2.1 is satisfied with this \( \theta \), then
\[
r^\mu \| D^2 u \|_{L^p(B_r)} \leq N(\rho_1 - \rho_0)^{-2} \hat{u} + N \hat{u} + N \bar{K},
\]
where \( N \) depends only on \( K_0, R_0, \alpha, \mu, d, \delta, p \).
Proof. We use the same notation as in Lemma 4.5 and observe that in light of Assumption 2.1,
\[ |F(D^2 u)| \leq \hat{\theta}|D^2 u| + K_0|u| + K + b|Du| =: f. \] (4.14)
By Lemma 4.5 for any \( n \geq 0 \) and \( r \leq r_n \)
\[ r^\mu \|D^2 u\|_{L_p(B_r)} \leq N\hat{f}_{n+1} + N\kappa^{-2}\bar{u} + N\bar{u} + N\bar{K}, \] (4.15)
where
\[ \hat{f}_{n+1} := \sup_{r \leq s \leq r_n+1} s^\mu \|f\|_{L_p(B_s)}. \]
Note that
\[ \sup_{r \leq s \leq r_n+1} s^\mu \|b|Du|\|_{L_p(B_s)} \leq K_0 \sup_{r \leq s \leq r_n+1} s^\mu \|Du\|_{L_p(B_s)}, \]
where in light of (4.12) the last term is dominated by
\[ N\varepsilon r_n W_{n+1} + N\varepsilon^{-1} r_n^{-1} \bar{u} \]
for any \( \varepsilon \in (0, 1) \) with \( N \) depending only on \( d, p, \mu \). Hence,
\[ W_n \leq N_1(\hat{\theta} + \varepsilon r_n + 1) W_{n+1} + N(\kappa^{-2} + \varepsilon^{-1} r_n^{-1}) \bar{u} + N\bar{u} + N\bar{K}. \]
We choose here \( \hat{\theta} \) so that \( N_1 \hat{\theta} \leq 1/4 \) and choose the largest \( \varepsilon \in (0, 1) \) such that \( N_1 \varepsilon r_n + 1 \leq 1/4 \). Observe that \( r_n^{-1} \leq \kappa^{-1} \), so that in any case \( \varepsilon^{-1} r_n^{-1} \leq N + \kappa^{-1} \) and, since \( \rho_1 < 1 \), we have \( \kappa \leq 1, \kappa^{-1} \leq \kappa^{-2} \), and
\[ W_n \leq (1/2) W_{n+1} + N\kappa^{-2} \bar{u} + N\bar{u} + N\bar{K}. \]
This allows us to finish the proof as that of Lemma 4.5. The lemma is proved.

**Proof of Theorem 2.1.** Take \( x \in B_{\rho_0} \) and \( r \leq \rho_0 \). Then \( B_r(x) \subset B_{2\rho_0}(x) \subset B_{\rho} \) and \( 2\rho_0 \leq 1/\nu \). Hence, by taking \( x \) as a new origin and setting \( \rho_1 = 2\rho_0 \), from Lemma 4.6 we infer that
\[ r^\mu \|D^2 u\|_{L_p(B_r(x))} \leq I. \]
Since \( x \in B_{\rho_0} \) and \( r \leq \rho_0 \), we also have
\[ r^\mu \|I_{B_{\rho_0}} D^2 u\|_{L_p(B_r(x))} \leq NI. \]
This inequality is trivially extended for \( r \in [\rho_0, 2\rho_0] \) and this proves the theorem.

This theorem takes care of interior estimates in the case of bounded \( b \) when \( p \) can be any number in \((d_0, \infty)\). To treat the case of \( b \) with rather poor summability properties we need some preparation in which we restrict the range of \( p \) from above. The following fact is crucial.

**Lemma 4.7.** Let \( 1 < p < q < d \) and \( R \in (0, \infty) \). Set \( \mu = q/p \) and \( q' = pq/(q-p) \). Then for any \( p' \in [1, q'] \) and \( u \in E_{p,\mu}(B_R) \), we have
\[ \|u\|_{E_{p',\mu}(B_R)} \leq N\|Du\|_{E_{p,\mu}(B_R)} + NR^{-1}\|u\|_{E_{p,\mu}(B_R)}, \] (4.16)
where the constants \( N \) depend only on \( d, p, q \).
Proof. In light of Hölder’s inequality we may assume that $p' = q'$. Scalings show that we may assume that $R = 1$. In that case consider the mapping $\Phi : \mathbb{B}_{3/2} \to \mathbb{B}_1$, $\Phi(x) = x(2/\|x\|_1 - 1)$ that preserves $B_1$, is Lipschitz continuous and has Lipschitz continuous inverse if restricted to $\mathbb{B}_{3/2} \setminus B_1$. Then, obviously, for any $v \in E_{p,\mu}(B_1)$
\[ \|w\|_{E_{p,\mu}(B_{1/2})} \leq N \|v\|_{E_{p,\mu}(B_1)}, \] (4.17)
where $N = N(d, p, q)$ and $w(x) = v(\Phi(x))$.

Now take $x \in B_1$, $\rho \leq 2$, and take $\zeta \in C^\infty_0(\mathbb{R}^d)$ such that $\zeta = 1$ on $B_1$, $\zeta = 0$ outside $B_{3/2}$, and $|\zeta| + |D\zeta| \leq N = N(d)$.

Since $\mu - 1 = \mu p/q'$ and $1/q' = 1/p - 1/(\mu p)$, by the Adams theorem 3.1 of [1] or Theorem 2.1 of [8] for $w = u(\Phi)$ we have
\[ \rho^{\mu-1} \|uI_{B_1}\|_{L_{q'}(B_{\rho}(x))} \leq N \rho^{\mu-1} \|\zeta w\|_{L_{q'}(B_{\rho}(x))} \]
\[ \leq N \|D(\zeta w)\|_{E_{p,\mu}} \leq N \|w\|_{E_{p,\mu}(B_{3/2})}. \]

It only remains to note that the last expression is less than the right-hand side of (4.16) in light of (4.17). The lemma is proved.

**Corollary 4.8.** Let $1 < p < q < d$, $R \in (0, \infty)$, and $b \in E_{q,1}(B_R)$. Set $\mu = q/p$. Then for any $u \in E^2_{p,\mu}(B_R)$, we have
\[ \|b|Du|\|_{E_{p,\mu}(B_R)} \leq N \|b\|_{E_{q,1}(B_R)} (\|D^2 u\|_{E_{p,\mu}(B_R)} + R^{-1} \|Du\|_{E_{p,\mu}(B_R)}) \]
\[ \leq N \|b\|_{E_{q,1}(B_R)} (\|D^2 u\|_{E_{p,\mu}(B_R)} + R^{-2} \|u\|_{E_{p,\mu}(B_R)}), \] (4.18)
where the constants $N$ depend only on $d, p, \mu$.

Indeed, by Hölder’s inequality
\[ \rho^{\mu} \|I_{B_R} b|Du|\|_{L_q(B_{\rho}(x))} \leq \rho \|I_{B_R} b\|_{L_q(B_{\rho}(x))} \rho^{\mu-1} \|I_{B_R} Du\|_{L_{q'}(B_{\rho}(x))}, \]
where $q' = pq/(q - p)$. This and (4.16) obviously lead to the first inequality. The second one follows from (4.8).

**Proof of Theorem 2.2.** Set $\rho_1 = 2\rho_0$ and use $\kappa, r_n$ from the proof of Lemma 4.5. By Theorem 2.1 with $F[u] + K$ in place of $H[u]$, where $K = -F[\zeta_n u]$, we have
\[ \|D^2 u\|_{E_{p,\mu}(B_{r_n})} \leq \|D^2 (\zeta_n u)\|_{E_{p,\mu}(B_{r_n+1})} \leq J + N_1 \|F[\zeta_n u]\|_{E_{p,\mu}(B_{r_n+1})}, \] (4.19)
where $J = N \rho^{-2} \|u\|_{E_{p,\mu}(B_{\rho})} + N \rho^{\mu-2} \sup_{B_{\rho}} |u|$. In light of the arguments in the proof of Lemma 4.5 and (4.8), (4.14), and (4.18), the last term in (4.19) is dominated by
\[ \|F[u]\|_{E_{p,\mu}(B_{r_n+1})} + N 2^n \kappa^{-1} \|Du\|_{E_{p,\mu}(B_{r_n+1})} + N 4^n \kappa^{-2} \|u\|_{E_{p,\mu}(B_{\rho_1})} + \|K\|_{E_{p,\mu}(B_{\rho_1})} \]
\[ \leq N_1 (\hat{\theta} + \hat{b} + 2^n \kappa^{-1} \varepsilon) I_{n+1} + N (\hat{b} r_{n+1}^{-2} + 4^n \kappa^{-2} + 2^n \kappa^{-1} \varepsilon^{-1}) \|u\|_{E_{p,\mu}(B_{\rho_1})} + \|K\|_{E_{p,\mu}(B_{\rho_1})}. \]
for any $\varepsilon \in (0, 1]$. We now choose and fix $\hat{\theta}$ and $\hat{b}$ so that $N_{1}(\hat{\theta} + \hat{b}) \leq 1/16$ and take the largest $\varepsilon \in (0, 1]$ for which $N_{1}2^{n}\kappa^{-1}\varepsilon \leq 1/16$. Then

$$\hat{b}r_{n+1}^{-2} + 4^{n}\kappa^{-2} + 2^{n}\kappa^{-1}\varepsilon^{-1} \leq N4^{n}\kappa^{-2},$$

so that coming back to (4.19) we get

$$I_{n} \leq (1/8)I_{n+1} + N4^{n}\kappa^{-2}\|u\|_{E_{p,\mu}(B_{r_{n+1}})} + N\|K\|_{E_{p,\mu}(B_{r_{n+1}})}.$$

This allows us to finish the proof as that of Lemma 4.5. The theorem is proved.

5. Existence theorems

Here is a general result in which only a few of our assumptions are supposed to hold.

Lemma 5.1. Let $u \in W^{2}_{d_{0}, \text{loc}}(\Omega) \cap C(\bar{\Omega})$ satisfy (2.8) in $\Omega$ and be such that $u = g$ on $\partial\Omega$. Let estimate (2.4) hold with $q = d_{0}$ and $\hat{b} = b(d, \delta)$ from Theorem 1.1 of [15] for any $r \in (0, R_{0}]$ and ball $B$ of radius $r$ and let Assumptions 2.5 and 2.6 be satisfied. Suppose that Assumption 2.1 is satisfied with some $\hat{\theta}$. For $\rho > 0$ introduce constants $M_{\rho}$ such that

$$M_{\rho} \geq \|u\|_{W^{2}_{d_{0}}(\Omega_{\rho})}.$$

Then the modulus of continuity of $u$ in $\bar{\Omega}$ is dominated by a continuous function $\omega(r), r \geq 0$, such that $\omega(0) = 0$, depending only on $r, R_{0}, p, M,$ the diameter of $\Omega, \rho, \gamma$ from Assumption 2.5, $L_{d_{0}}(\Omega)$-norms of $K$ and $u$, and the modulus of continuity of $g$.

Proof. Looking at $0 = [H(u, Du, D^{2}u) - H(u, Du, 0)] + H(u, Du, 0)$ and using Assumptions 2.6 and 2.1 one sees that $a^{ij}D_{ij}u + b^{i}D_{i}u + f = 0$, where $(a^{ij})$ is $S_{d}$-valued, $|b| \leq b$, and $|f| \leq K_{0}|u| + K$. Then $|u(x_{1}) - u(x_{2})|$ for $x_{1}, x_{2}$ that are close to $\partial\Omega$ is estimated by using Lemma 3.3. If they are far, we use embedding theorems ($d_{0} > d/2$) to estimate the difference in terms of $M$. The combination of these estimates leading to the desired result is a simple exercise. The lemma is proved.

Coming closer to the proof of Theorem 2.3, observe that estimate (2.9) follows from Theorem 3.2. Indeed, the assumption of this theorem concerning (2.4) is obviously satisfied since $b \leq K_{0}$ and on the set $\Omega \cap \{u > 0\}$ we have

$$0 = H[u] = [H(u, Du, D^{2}u) - H(u, Du, 0)] + H(u, Du, 0) \leq a^{ij}D_{ij}u + b|Du| + K = a^{ij}D_{ij}u + b^{i}D_{i}u + K,$$

where $(a^{ij})$ is $S_{d}$-valued and $|b| \leq b \leq K_{0}$. Hence,

$$u \leq N\|I_{\Omega, u > 0}K\|_{L_{d_{0}}} + \sup_{\partial\Omega}u_{+}.$$

Similarly, the estimate of $-u$ is obtained.
In the following lemma \( \hat{H}(u'', x) \) is a measurable function such that it is Lipschitz continuous with respect to \( u'' \), \( (D_{ij}^* \hat{H}) \in S_\delta \) at all points of differentiability of \( \hat{H} \) and \( \hat{H}[0] \in L_{p,\text{loc}}(\mathbb{R}^d) \). Observe that \( \hat{H}(u'', \cdot) \in L_{p,\text{loc}}(\mathbb{R}^d) \) for any \( u'' \) and by Lebesgue’s theorem

\[
\lim_{r \downarrow 0} \| \hat{H}(u'', \cdot) - \hat{H}(u'', x_0) \|_{L_p(B_r(x_0))} = 0 \tag{5.1}
\]

for almost any \( x_0 \). Since \( \hat{H}(u'', x) \) is Lipschitz continuous in \( u'' \) one can choose a set of \( x_0 \) of full measure such that (5.1) holds for any \( u'' \).

**Lemma 5.2.** Let \( p \in [d_0, \infty), R \in (0, \infty), u \in W^2_p(B_R), f \in L_p(B_R) \). Then \( \hat{H}[u] \geq f \) in \( B_R \) if and only if for any \( B_r(x_0) \subset B_R \) and any \( \phi \in C^2(B_r(x_0)) \) we have in \( B_r(x_0) \) that

\[
u \leq \phi + \sup_{\partial B_r(x_0)} (u - \phi) + N r^2 \| f - \hat{H}[\phi] \|_{L_p(B_r(x_0))} \tag{5.2}
\]

with \( N \) independent of \( u, \phi, x_0, r \).

**Proof.** “only if”. We have \( f - \hat{H}[\phi] \leq H[u] - \hat{H}[\phi] = a^{ij} D_{ij}(u - \phi) \) with a \( S_\delta \)-valued \( (a^{ij}) \) and (5.2) follows from Theorem 3.2 and a scaling argument. “if”. Take \( x_0 \in B_R \) such that

\[
\phi(x) = u(x_0) + (x - x_0)^j D_j u(x_0) + (1/2)(x - x_0)^j (x - x_0)^j D_{ij} u(x_0) + o(|x - x_0|^2).
\]

By the Zygmund-Calderón theorem one can take almost any \( x_0 \in B_R \) since \( p > d/2 \). We may even restrict this \( x_0 \) to satisfy (5.1) for any \( u'' \) and satisfy

\[
\lim_{r \downarrow 0} \| f - f(x_0) \|_{L_p(B_r(x_0))} = 0.
\]

Then fix an \( \varepsilon > 0 \) and for \( r \), such that \( o(r^2) \leq \varepsilon r^2 \), in \( B_r(x_0) \) introduce

\[
\phi(x) = u(x_0) + (x - x_0)^j D_j u(x_0) + (1/2)(x - x_0)^j (x - x_0)^j D_{ij} u(x_0) + \varepsilon(2|x - x_0|^2 - r^2).
\]

We will send \( r \downarrow 0 \) and, therefore, we may concentrate on \( r \) such that \( B_r(x_0) \subset B_R \) and \( u \leq \phi \) on \( \partial B_r(x_0) \). Then (5.2) at \( x = x_0 \) yields

\[
\varepsilon r^2 \leq N r^2 \| (f - \hat{H}(D_{ij} u(x_0) + 4\varepsilon \delta_{ij}, \cdot)) \|_{L_p(B_r(x_0))},
\]

which after letting \( r \downarrow 0 \) becomes

\[
\varepsilon \leq N \| (f(x_0) - \hat{H}(D_{ij} u(x_0) + 4\varepsilon \delta_{ij}, x_0)) \|_{L_p(B_r(x_0))}.
\]

We send \( \varepsilon \downarrow 0 \) and get \( f(x_0) - \hat{H}(D_{ij} u(x_0), x_0) \leq 0 \), thus proving the lemma.

Similarly, or just taking \( -\hat{H}(-u'', x) \) in place of \( \hat{H}(u'', x) \), one proves the following.

**Lemma 5.3.** Let \( p \in [d_0, \infty), R \in (0, \infty), u \in W^2_p(B_R), f \in L_p(B_R) \). Then \( \hat{H}[u] \leq f \) in \( B_R \) if and only if for any \( B_r(x_0) \subset B_R \) and any \( \phi \in C^2(B_r(x_0)) \) we have in \( B_r(x_0) \) that

\[
u \geq \phi - \sup_{\partial B_r(x_0)} (u - \phi) - N r^2 \| f - \hat{H}[\phi] \|_{L_p(B_r(x_0))} \tag{5.3}
\]
with $N$ independent of $u, \phi, x_0, r$.

**Proof of Theorem 2.3.** For $n = 1, 2, \ldots$ introduce
$$H_n(u, x) = I_{K \leq n} H(u, x) + I_{K > n} F(u''', x).$$

Observe that $H_n = F + G_n$, where $G_n(u, x) = I_{K \leq n} G(u, x)$,
$$|G_n(u, x)| \leq \hat{\theta}|u'''| + K_0|u'| + I_{K \leq n} K.$$ 

Here the free term belongs to $L_{p, \text{loc}}(\mathbb{R}^d)$ for any $p > 1$. It follows from 10.1.14 of [12] (see also Remark 10.1.15 there) that for appropriate $\hat{\theta}, \theta$, depending only on $d, \delta, p$, there exists $u \in W^{2, p}_d(\Omega) \cap C(\overline{\Omega})$ such that $H_n[u_n] = 0$ in $\Omega$ and $u_n = g$ on $\partial \Omega$.

Estimate (2.9) shows that $u_n$ are uniformly bounded in $\overline{\Omega}$. Then Theorem 2.1 implies that for any $\rho > 0$ the $E_{p, \mu}^2(\Omega^\rho)$-norms of $u_n$ are bounded, provided that Assumptions 2.2 and 2.1 are satisfied with appropriate $\theta, \theta$. In particular, by Lemma 5.1 the family $\{u_n\}$ is uniformly bounded and uniformly continuous in $\Omega$. Therefore, there exists a subsequence $u_{n_k}$ and $u \in C(\overline{\Omega})$ such that $u_{n_k} \rightarrow u$ uniformly on $\Omega$.

Next, by the compactness of embeddings, for each $\rho > 0$, the family $Du_n$ is precompact in $L_p(\Omega^\rho)$. Hence, by using the Cantor diagonalization method we may assume that
\begin{equation}
\bar{K} := |Du_{n_1}| + \sum_{k=1}^{\infty} |Du_{n_{k+1}} - Du_{n_k}| \in L_p(\Omega^\rho)
\end{equation}
for any $\rho > 0$. Then $Du_{n_k}$ converge in $L_p(\Omega^\rho)$ for any $\rho > 0$ and almost everywhere in $\Omega$ to some functions which automatically coincide with $Du$. The weak limit of $D^2 u_{n_k}$ is, of course, $D^2 u$, so that $u \in E_{p, \mu, \text{loc}}^2(\Omega)$.

To prove that $H[u] = 0$ in $\Omega$, for $m = 1, 2, \ldots$ set
$$\hat{H}_m(u''', x) = \sup_{k \geq m} H_{n_k}(u_{n_k}(x), Du_{n_k}(x), u''', x).$$

Observe that
$$|\hat{H}_m(0, x)| \leq K_0 \sup_{n=1, 2, \ldots} \sup_\Omega |u_n| + \bar{K} + K \in L_{p, \text{loc}}(\Omega).$$

Also, obviously, $(Du_{n_k}'', \hat{H}_m) \in S_{\delta}$ at all points of differentiability of $\hat{H}_m$. For $k \geq m$ we have $\hat{H}_m(D^2 u_{n_k}) \geq 0$ in $\Omega$ implying by Lemma 5.2 that for any $B_r(x_0) \subset \Omega$ and any $\phi \in C^2(B_r(x_0))$ we have in $B_r(x_0)$ that
\begin{equation}
u_{n_k} \leq \phi + \sup_{\partial B_r(x_0)} (u_{n_k} - \phi) + N \eta^2 \|\hat{H}_m[\phi] - \|_{L_p(B_r(x_0))}\end{equation}
with $N$ independent of $u, \phi, x_0, r$, and $m$. The fact that $N$ is indeed independent of $m$ easily follows from the proof of Lemma 5.2. We pass to the limit as $k \rightarrow \infty$, which allows us to replace $u_{n_k}$ on the left in (5.6) with $u$.
and conclude by Lemma 5.2 that $\hat{H}_m(D^2 u) \geq 0$ in $\Omega$. This inequality on the set $\{K \leq n_m\}$ means that

$$\sup_{k \geq m} H(u_{n_k}(x), Du_{n_k}(x), D^2 u(x), x) \geq 0.$$ 

Since $Du_{n_k}(x) \to Du$ almost everywhere in $\Omega$, $u_{n_k}(x) \to u(x)$ in $\Omega$ and $H$ is a continuous function of $u'$, by setting $m \to \infty$, we conclude that $H[u] \geq 0$ in $\Omega$. Similarly, by using Lemma 5.3 one proves that $H[u] \leq 0$ in $\Omega$. The theorem is proved.

**Proof of Theorem 2.4.** Estimate (2.10) is derived as (the identical) (2.9) by using Theorem 3.2. To prove the existence, for $n = 1, 2, \ldots$ introduce

$$H_n(u, x) = H(u_0, n[u']/(n + b), u'', x).$$

Observe that $H_n = F + G_n$, where

$$|G_n(u, x)| \leq \tilde{\theta}|u''| + K_0|u'_0| + (nb/(n + b))||u'|| + K.$$

We apply Theorem 2.3 upon observing that above the coefficients of $u'$ are bounded and the free term belongs to $E_{p, \mu, \text{loc}}(\Omega) \cap L_p(\Omega)$. Then we conclude that there exists $u_n \in E_{p, \mu, \text{loc}}(\Omega) \cap C(\Omega)$ satisfying (2.8) in $\Omega$ with $H_n$ in place of $H$ and such that $u_n = g$ on $\partial \Omega$. Theorem 2.2 guarantees that for any $\rho > 0$ the $E_{p, \mu}(\Omega^n)$-norms of $u_n$ are bounded, provided that Assumptions 2.1, 2.2, and 2.4 are satisfied with appropriate $\tilde{\theta}, \theta, \tilde{b}$. After that, as in the proof of Theorem 2.3, we find a subsequence $u_{n_k}$ and $u \in W^{1,2}_{p, \mu, \text{loc}}(\Omega) \cap C(\Omega)$ such that $u_{n_k} \to u$ uniformly on $\Omega$ and $Du_{n_k} \to Du$ almost everywhere. Of course, $u \in E_{p, \mu, \text{loc}}(\Omega)$.

Furthermore, by Corollary 4.8, for any $B_r(x_0) \subset \Omega$ with $r \leq R_0$ the integrals

$$\int_{B_r(x_0)} b^p |Du_n|^p dx$$

are bounded by a constant independent of $n$. It follows that for any $p' \in [d_0, p)$

$$\lim_{k \to \infty} \int_{B_r(x_0)} b^{p'} |Du_{n_k} - Du|^{p'} dx = 0$$

and there exists a subsequence which we identify with the above one such that

$$\tilde{K} := b|Du_{n_1}| + \sum_{k=1}^{\infty} b|Du_{n_{k+1}} - Du_{n_k}| \in L_{d_1, \mu, \text{loc}}(\Omega). \quad (5.7)$$

Then we introduce $\hat{H}_m$ as in the proof of Theorem 2.3, observe that (5.5) holds with $d_0$ in place of $p$ and with the help of Lemma 5.2 conclude that $\hat{H}_m(D^2 u) \geq 0$ in $\Omega$. Since $u_{n_k}, n_k Du_{n_k}/(n_k + b) \to u, Du$ almost everywhere as $k \to \infty$ and $H$ is continuous in $u'$,

$$0 \leq \lim_{m \to \infty} \hat{H}_m(D^2 u) = \lim_{k \to \infty} H(u_{n_k}, n_k Du_{n_k}/(n_k + b), D^2 u) = H[u].$$
Similarly, by using Lemma 5.3 one proves that $H[u] \leq 0$ in $\Omega$. The theorem is proved.

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Email address: nkrylov@umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455