Remarks on antichains in the causality order of space-time

Stephan Foldes
Tampere University of Technology
PL 553, 33101 Tampere, Finland
sf@tut.fi
2 July 2012

Abstract

The two closely related Lorentz-invariant partial orders of space-time are distinguished with respect to their maximal chains, the existence of antichain cutsets and the possibility of grading. World lines of particles with or without mass are the maximal chains in the causality order of space-time, and antichain cutsets are the levels of the various gradings of the causality partial order. The maximal chains of the weaker, subluminal causality order need not be connected topologically, subluminal causality has no antichain cutsets and cannot be graded.

Keywords: space-time, causality, subluminal causality, partial order, world line, chain, antichain, cutset, graded poset, rank, level, space-like hypersurface, light-like vector, light-like line

1 Poset grading, levels, antichain cutsets

In a graded (ranked) partially ordered set, each level (set of all elements of the same rank) is a maximal antichain that intersects every maximal chain. If the partially ordered set satisfies certain conditions, then the converse is also true: in such posets, every antichain that intersects every maximal chain is a level under a grading of the poset (see Rival and Zaguia [RZ] and [FW]). Grading is a natural idea in discrete posets, but also in general, a surjective map \( g \) from a poset \( P \) onto a totally ordered set (chain) \( R \) may be called a grading if its restriction to each maximal chain \( C \) of \( P \) is an isomorphism from \( C \) to \( R \). A level with respect to this grading is then defined as the pre-image under \( g \) of any element of \( R \). For example, on the distributive lattice
\(\mathbb{R}^n\), the sum of vector components defines a grading \(\mathbb{R}^n \rightarrow \mathbb{R}\), which is not at all unique, but its restriction to the integer lattice yields the essentially unique grading \(\mathbb{Z}^n \rightarrow \mathbb{Z}\).

In a partially ordered set, an antichain that intersects every maximal chain is called an (antichain) cutset. Every level of a graded poset is an antichain cutset, and every antichain cutset is a maximal antichain. Converse statements do not hold generally, and even in such well-behaved posets as a Boolean lattice, maximal antichains need not be cutsets. However, extending a result contained in [RZ], it was shown in [FW] that in every discrete, strongly connected poset, every antichain cutset is a level under an essentially unique grading.

2 Causality order and space-like hypersurfaces as antichains

For each non-negative integer \(n\) and positive real number \(c\), the causality order \(\leq_c\) on \((n + 1)\)-dimensional space-time \(\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}\) is given by

\[
\mathbf{x}_t <_c \mathbf{x}'_{t'} \iff t < t' \text{ and } \frac{\|\mathbf{x}' - \mathbf{x}\|}{t' - t} \leq c
\]

Mathematical interest in the causality order is due in significant part to the Alexandrov-Zeeman Theorem (see [A1], [AO], [A-CJM], [Z]), which states that in at least \((2+1)\)-dimensional space-time, the order automorphism group of the causality order is the semi-direct product of the Poincaré group and the group of dilations (or equivalently, it is generated by Lorentz boosts, space rotations, translations, and dilations). The result does not hold in \((1+1)\)-dimensional space-time, due to the paucity of space rotations. In this case the causality poset is a distributive lattice isomorphic to the componentwise lattice order on \(\mathbb{R}^2\), but it is not a lattice in higher dimensions. In \(2+1\) and higher dimensions, maximal antichains of the causality order include space-like lines and space-like hyperplanes.

By a world line in \((n + 1)\)-dimensional space-time \(\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}\) we mean the inverse of the graph in \(\mathbb{R} \times \mathbb{R}^n\) of any Lipschitz continuous function \(f : \mathbb{R} \rightarrow \mathbb{R}^n\) with Lipschitz constant \(c > 0\), i.e. satisfying \(\|f(t) - f(t')\| \leq c|t - t'|\) for all \(t, t' \in \mathbb{R}\). In other words, a world line is a set of points \(C \subseteq \mathbb{R}^n \times \mathbb{R}\) such that for every \(t \in \mathbb{R}\) there is one and only one \(\mathbf{x} \in \mathbb{R}^n\) with \(\mathbf{x}_t = (\mathbf{x}, t) \in C\) and where \(f : t \mapsto \mathbf{x}\) is Lipschitz continuous with constant \(c\).
These serve to describe the evolution in physical space-time of particles with or without mass.

**Proposition 2.1** World lines are precisely the maximal chains in the causality order of space-time.

**Proof.** Every world line is a chain due to the Lipschitz condition, and it is a maximal chain, as the addition of any other point would result in two points with the same time component, which would be incomparable in the causality order.

Conversely, let \( C \) be a maximal chain. Observe first that \( C \) must be topologically closed, and that no two points in \( C \) can have the same time component. Let \( T \) be the set of time components of the points in \( C \). The map \( f \) associating to each \( t \in T \) the unique \( x \in \mathbb{R} \) such that \( xt \in C \) is Lipschitz continuous with constant \( c \), consequently \( T \) is also closed in \( \mathbb{R} \). Let us show that \( T = \mathbb{R} \). If \( T \) had a largest element \( s \), then adding the point \((f(s), s + 1)\) to \( C \) would result in a larger chain, which is impossible. Therefore \( T \) is not bounded above, and similarly, \( T \) is not bounded below. If some real number \( r \) failed to belong to \( T \), then \( T \) would have a greatest element \( a \) smaller than \( r \), and a smallest element \( b \) greater than \( r \). Adding to \( C \) any internal point of the segment between \((f(a), a)\) and \((f(b), b)\) would result in a larger chain, which is again impossible. \( \square \)

The following characterizes antichain cutsets as space-like hypersurfaces, which include all space-like hyperplanes but relaxes the linearity and smoothness requirements.

**Proposition 2.2** In the causality order \( \leq \) of \((n+1)\)-dimensional space-time \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \), a set \( A \) of points constitutes an antichain cutset if and only if it is the graph of a Lipschitz-continuous function \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) with strict constant \( c^{-1} \) (i.e. \(|h(x) - h(x')| < c^{-1} \|x - x'\| \text{ for all } x, x' \in \mathbb{R}^n\)).

**Proof.** If \( A \) is an antichain cutset, then it intersects at exactly one point each world line with fixed space component, and therefore it is the graph of a map \( \mathbb{R}^n \rightarrow \mathbb{R} \). As any two points on this graph are unrelated by causality, the Lipschitz condition must hold.

For the same reason, the graph \( A \) of a Lipschitz continuous function is an antichain. There is a unique map \( g : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) such that for each \( xt \in \mathbb{R}^{n+1} \),

\[
(x, t - g(xt)) \in A
\]
This map is surjective onto $\mathbb{R}$ and it is a grading of the causality partial order for which $A$ is the level zero. \hfill \Box

**Proposition 2.3** The causality order $\leq_c$ of $(n + 1)$-dimensional space-time $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ is a gradable partial order, in which a set of points $A \subseteq \mathbb{R}^{n+1}$ is an antichain cutset if and only if it is a level set with respect to some grading.

**Proof.** Gradability of the causality order is obvious, e.g. by $xt \mapsto t$. In fact, as indicated in the proof of Proposition 2.2, different gradings can be based on different space-like hyperplanes, all of these being antichain cutsets. That proof shows that every antichain cutset is a level set of some grading, and the converse is true in all graded posets. \hfill \Box

## 3 Partial orders invariant under space isometries and dilations

For any given partial order $\leq$ on any set, another, weaker order $\leq'$ on the same set is defined by

$$ a \leq' b \iff a \leq b, \text{ and the interval } [a, b] \text{ is not a chain unless } a = b $$

Generally the original order cannot be reconstructed from this weaker order. However, as apparent in Zeeman [Z], for all $n \geq 1$ the causality order $\leq_c$ can be reconstructed from the subluminal causality order $\leq'_c$ and they have the same automorphisms. We have

$$ u \leq_c v \iff u \leq'_c v \text{ or } (\forall w \neq u, v \quad v \leq'_c w \Rightarrow u \leq'_c w) $$

Let $C$ be a world line as defined and characterized in the previous section. A light-like line can be defined as a maximal order-convex chain in the causality order. A light-like segment of $C$ is a non-singleton connected component of the intersection of $C$ with some light-like line, or equivalently, a maximal non-singleton order-convex subset of $C$. Light-like segments are necessarily closed topologically, there are at most countably infinitely many of them, at most two of which are not compact, and any two of them intersect in at most one point. Suppose that the light-like segments of $C$ are all compact and pairwise disjoint. Then each of the segments has two endpoints. Removing from $C$ all the interior points of the light-like segments, and one
of the endpoints of each segment, we remain with a subset \( C' \subseteq C \) that we call a *worldline with optical gap(s).* Note that in the removal of endpoints, the endpoint with the lower time coordinate may be removed from some segments, and the one with the higher time coordinate from others.

**Proposition 3.1** World lines with optical gaps are precisely the maximal chains in the subluminal causality order of space-time. □

Let \( xt \in \mathbb{R}^n \times \mathbb{R} \) be any forward light-like vector, *i.e.* not \( 00 \) and such that \( \|x\| = t > 0 \). Then the set \( \{0r : r \leq 0\} \cup \{xs : t < s\} \) is a worldline with optical gap, and it avoids any antichain of subluminal causality that contains \( 00 \). This shows that, in the order of subluminal causality, for any antichain there are maximal chains that avoid it. Consequently, in subluminal causality there are no antichain cutsets, and the subluminal causality poset cannot be graded.

For every positive real constant \( c \), both causality \( \leq_c \) and subluminal causality \( \leq'_c \) are invariant under *space isometries* (*i.e.* under transformations of \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \) of the form \( xt \mapsto yt \) where \( x \mapsto y \) is an isometry of Euclidean \( n \)--space), and also invariant under all space-time dilations \( xt \mapsto (rx, rt) \), \( r > 0 \). Conversely, let \( \leq \) be any partial ordering of \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \) that is invariant under space isometries and space-time dilations. The set of elements greater or equal to \( 00 \) is then a cone \( C \) invariant under space rotations (space isometries fixing \( 00 \)), and this cone determines the order \( \leq \) by \( u \leq v \iff v - u \in C \). If \( C \) is topologically closed, then it is the forward causality cone of \( \leq_c \) for some positive constant \( c \) or its negative, the backward causality cone. If \( C \) is not closed, then there can be two cases. In the first case \( C \) is the forward subluminal causality cone of \( \leq'_c \) for some positive constant \( c \) or its negative, the backward subluminal causality cone. In the second case \( C \) is \( \{xt : t > 0\} \cup \{00\} \) or its negative \( \{xt : t < 0\} \cup \{00\} \): the corresponding partial orders are the forward and backward *temporal orderings* of Galilean spacetime, and they have too many automorphisms to which no physical meaning is attributed. Clearly we have \( u \leq v \) under temporal ordering if and only if \( u \leq_c v \) under the causality ordering for all positive \( c \), or equivalently, if and only if \( u \leq'_c v \) under subluminal causality for all positive \( c \). In that sense temporal ordering is "subluminal causality at infinite speed of light".

The only partial orderings of \((n + 1)\)--dimensional spacetime that are invariant under space isometries and space dilations are causality and sublu-
minal causality with various light speed parameters $c$, the temporal ordering, and their reverse orders.

**Acknowledgements.** This work has been co-funded by Marie Curie Actions and supported by the National Development Agency (NDA) of Hungary and the Hungarian Scientific Research Fund (OTKA), within a project hosted by the University of Miskolc, Department of Analysis. The work was also completed as part of the TAMOP-4.2.1.B.- 10/2/KONV-2010-0001 project at the University of Miskolc, with support from the European Union, co-financed by the European Social Fund.

The author wishes to thank Sándor Radeleczki and Miklós Rontó for useful comments and discussions.

![EU]![Marie Curie]![NDA]![OTKA]

**References**

[Al] A.D. Alexandrov, On Lorentz transformations, Sessions Math. Seminar, Leningrad Section of the Mathematical Institute, 15 September 1949 (abstract, in Russian)

[AO] A.D. Alexandrov, V.V. Ovchinnikova, Note on the foundations of relativity theory, Vestnik Leningrad Univ. 11 (1953) 95-100 (in Russian)

[A-CJM] A.D. Alexandrov, A contribution to chronogeometry, Canadian J. Math. 19 (1967) 1119-1128

[FW] S. Foldes, R. Woodrofe, Antichain cutsets of strongly connected posets, Order (2012) DOI: 10.1007/s11083-012-9248-2, [arXiv:1109.5705v2](https://arxiv.org/abs/1109.5705)

[FW] I. Rival, N. Zaguia, Antichain cutsets, Order 1 (3) 235-247 (1985)

[Z] E.C. Zeeman, Causality implies the Lorentz group, J. Mathematical Physics 5 (1964) 490-493