D-bound and Bekenstein bound for Vaidya solution surrounded by dark energy cosmological fields

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Abstract

The Bousso’s D-bound approach is investigated for particular dynamical black holes, namely the Vaidya solution surrounded by particular dark energy cosmological fields of quintessence, phantom and cosmological constant. We derive D-bound and Bekenstein bound for these solutions and compare them to show that the more dark energy cosmological fields are diluted, the more D-bound and Bekenstein bound are identified. In particular, we find that these two bounds are exactly identified for the case of cosmological constant. Therefore, the identification of D-bound and Bekenstein bound can be considered as a thermodynamical criteria by which the cosmological constant is preferred as viable dark energy cosmological field among other proposed dark energy cosmological fields.

Keywords: D-bound, Bekenstein bound, Vaidya solution.

I. INTRODUCTION

In the 70s of last century, the quantum physics of black holes started by the works of Bekenstein and Hawking. There is a general conviction that Hawking radiation and Bekenstein-Hawking entropy are the main features of a yet unknown theory of quantum gravity which will be able to unify Einstein’s general theory of relativity with quantum mechanics. In fact, the researchers who are expert on quantum gravity claim that black holes are the fundamental bricks of quantum gravity which play the same role like the atoms in quantum mechanics. In this framework, Bekenstein has found a fundamental result indicating the maximum entropy of the black hole which is allowed by quantum theory and general theory of relativity for a given mass and size. The Bekenstein bound puts an upper bound on the entropy of the system with a finite amount of energy and a given size. This bound is the maximum amount of information required to describe a system by considering its quantum properties. If the energy and size of the system is finite, the information required to describe it completely, is finite too. One of the important consequences of Bekenstein bound is in the physics of information and in computer science when it is connected with the so-called Breuermann’s Limit. It puts a maximum information-processing rate for a system with finite size and energy. Another consequence of Bekenstein bound is the derivation of the field equations of general theory of relativity. There are some investigations in trying to find some forms of the bound by considering consistency of the laws of thermodynamics with the general theory of relativity. In this framework, a generalization of the Bekenstein bound was derived by Bousso, conjecturing an entropy bound with its statistical origin which is valid in all space-times consistent with Einstein’s equation. This so-called covariant entropy bound reduces to Bekenstein bound in the system of limited self-gravity. Another attempt in this regard has been done by Bousso in considering the systems with cosmological horizon which has led to the so-called D-bound. Bousso has derived D-bound for asymptotically non-flat Schwarzschild-de Sitter black hole solution. One can look for D-bound for other solutions which are not asymptotically flat and include a cosmological apparent horizon.

Surrounded Vaidya black holes, as asymptotically non-flat solutions, show interesting results under consideration of D-bound which we intend to study them in this paper. In fact, Vaidya solution provides a non-static solution for the Einstein field equations which is a generalization of the static Schwarzschild black hole solution. This solution is dependent on dynamical mass function $m = m(u)$ by a retarded time coordinate $u$, and an ingoing/outgoing flow $\sigma(u, r)$. Because of this feature of Vaidya solution, it can be considered as a classical model for dynamical black hole which is effectively evaporating or accreting. The process of spherical symmetric gravitational collapse has also been studied by applying the Vaidya solution. On the other hand, this solution is a testing ground for the cosmic censorship conjecture, see also for other application.
In this paper, we will investigate D-bound in the framework of the surrounded Vaidya black hole solutions [16]. The organization of the paper is as follows. In section 2, we review very briefly the D-bound and Bekenstein bound. In section 3, we introduce the Vaidya black hole solution. In section 4, we derive D-bound and Bekenstein bound for surrounded Vaidya solution by cosmological constant-like field. In sections 5 and 6, the same bounds are obtained for surrounded Vaidya solution by quintessence field and phantom field, respectively. In section 7, the other fields which do not have a cosmological horizon such as dust and radiation are considered. At the end, we give a conclusion in section 8.

II. BEKENSTEIN BOUND AND D-BOUND

In this section, we study the Bekenstein bound and D-bound and summarize the important results obtained in [10] (For more details see [1, 5, 17]).

Bekenstein bound is expressed by the following statement: Isolated, stable thermodynamic systems in asymptotically flat space are constrained by universal entropy bound

\[ S_m \leq 2\pi R E, \]

where \( R \) is the radius of sphere circumscribing system and \( E \) its total energy. Bekenstein bound has been considered in two forms, empirical and logical.

- **Empirical form:** All physically reasonable, weakly gravitating matter systems, satisfy the Bekenstein bound [17, 18]. Some of the systems saturate the bound. For example, the bound is saturated by Schwarzschild black hole through \( S = \pi R^2 \) and \( R = 2E \). It seems that the Bekenstein bound is the tightest one for any physical system. There are some controversial examples which claim the violation of Bekenstein bound [19]. However, some of these counter-examples are shown not to correctly include the whole of the gravitating matter system in \( E \) and including them can restore the Bekenstein bound. The rest of counter-examples also contain controversial matter and excluding them from \( E \) can restore the Bekenstein bound [20, 21].

- **Logical form:** Bekenstein has claimed that for weakly gravitating systems, the bound is a result of generalized second law of thermodynamics (GSL) [1, 5, 22, 23]. By the Geroch process (A gedankenexperiment), the system is collapsed into a large black hole. The entropy of the system (black hole) becomes \( \Delta A/4 = 8\pi R E \). According to GSL, \( \Delta A/4 - S_m \geq 0 \) where \( S_m \) is the entropy of lost matter system before the formation of black hole. There are also some controversial arguments in that whether one can derive Bekenstein bound by Geroch process considering quantum effects [24–26]. There is no certain result coming out of these arguments.

D-bound is expressed by the following statement: D-bound is a bound on the entropy of matter systems in de-Sitter space which is shown to be closely related to the Bekenstein bound in a flat background [10]. The definition of D-bound on matter entropy in de-Sitter space is as follows. Assume an observer located within his apparent cosmological horizon corresponding to a matter system, in a universe that is asymptotically de-Sitter in the future. The observer moves relative to the matter until the matter is located at his apparent cosmological horizon. He will realize that crossing out of the matter from his apparent cosmological horizon is a thermodynamic process. The entropy of system after the matter is crossed out the cosmological horizon is

\[ S_0 = \frac{A_0}{4}, \]

where \( A_0 \) is the area of cosmological horizon given by

\[ A_0 = \pi r_0^2 = \frac{12\pi}{\Lambda}. \]

The entropy of the initial state is the sum of the matter system’s entropy \( S_m \) and a quarter of the apparent cosmological horizon

\[ S = S_m + \frac{A_c}{4}. \]

According to the generalized second law of thermodynamics, the observer concludes that the entropy increases. Thus, by comparing equations (2) and (4) we have
\[ S_m \leq \frac{1}{4}(A_0 - A_c), \] (5)

which is the D-bound on the matter system in asymptotically de-Sitter space. The D-bound has been derived by Bousso for entropy of the matter systems in de Sitter space. It is indicated that the D-bound is the same as Bekenstein bound of the system in this model. Also, Bousso has achieved the same result for arbitrary dimensions. In an another example, the D-bound entropy for the various possible black hole solutions on a 4-dimensional brane have been considered in [27]. It is found that the D-bound entropy for this solution is apparently different from that of obtained for the 4-dimensional black hole solutions. This difference is considered as the extra loss of information which comes from the extra dimension, when an extra-dimensional black hole is moved outward the observer’s cosmological horizon. The obtained results there also have been considered, by adopting the recent Bohr-like approach to black hole quantum physics for the excited black holes [27].

### III. SURROUNDED VAIDYA BLACK HOLE SOLUTION

The metric of Vaidya black hole solutions surrounded by cosmological fields introduced in [16, 28] is given by

\[ ds^2 = -\left(1 - \frac{2M(u)}{r} - \frac{N_s(u)}{r^{3\omega_s + 1}}\right) du^2 + 2\epsilon du dr + r^2 d\Omega^2, \] (6)

where \( M(u), N_s(u) \) and \( \omega_s \) are black hole dynamical mass, surrounding field characteristic parameter and equation of state parameter of the surrounding field, respectively. In contrast to the stationary spacetimes, the local definitions of the various horizons do not necessarily coincide with the location of the event horizon for dynamical black holes [29]. For such dynamical spacetimes, one is left with the question: “For which surface should one define the black hole area, surface gravity, temperature or entropy?” The canonical choice is to use the event horizon. However, there are some evidences that it is the apparent horizon, and not the event horizon, that plays the key role in the Hawking radiation [30–33], see also [34–36]. This finding has became a key point in hopes to demonstrate the Hawking radiation in the laboratory using the models of analogue gravity [37]. Therefore, we consider Bekenstein-Hawking entropy for apparent horizons associated to the metric (6) with various cosmological fields. Then, we derive D-bound and Bekenstein bound for these backgrounds.

In the following sections, we investigate the D-bound and Bekenstein bound for Vaidya black holes surrounded by various cosmological fields. Then, we compare the D-bound with Bekenstein bound to show that the more cosmological fields are diluted, the more D-bound and the Bekenstein bound are identified.

### IV. D-BOUND AND BEKENSTEIN BOUND FOR SURROUNDED VAIDYA SOLUTION BY COSMOLOGICAL CONSTANT-LIKE FIELD

#### A. D-bound

For the case of Vaidya solution with the equation of state by \( \omega_c = -1 \) [16, 28], the metric (6) becomes like the black hole with cosmological constant-like field solution

\[ ds^2 = -\left(1 - \frac{2M(u)}{r} - N_c(u)r^2\right) du^2 + 2\epsilon du dr + r^2 d\Omega^2, \] (7)

where \( N_c(u) \) is the normalization parameter for the cosmological field surrounding the black hole. Positive energy condition on the surrounding cosmological field leads to \( N_c > 0 \) [16]. The cosmological background which has negative surface gravity decreases the gravitational attraction of the black hole. This gravitational repulsion is the most favored candidates for the dark energy which is responsible for the accelerating expansion of the universe [35]. The metric (7)
indicates the non-trivial effects of the surrounding cosmological field which differs from Vaidya black hole in an empty background. The background cosmological field changes the causal structure of the Vaidya black hole in an empty space. The causal structure change of Vaidya to Vaidya-de Sitter space is similar to the causal structure change of schwarzschild to schwarzschild-de Sitter space.

To derive D-bound for the Vaidya case one needs the apparent cosmological horizon which will be described completely in this section. First, we have to find the apparent cosmological horizons of this solution. In Ref. [10], the black hole horizons and the apparent cosmological horizons are obtained for Vaidya solution surrounded by cosmological constant-like field, in details. There are black hole and apparent cosmological horizons, subject to a particular condition \( \Delta(u) = 1 - 27M^2(u)N_c(u) > 0 \), representing inner and outer horizons, respectively as \[ \frac{1}{\sqrt{N_c(u)}} - M(u) - \frac{3}{2}M^2(u)\sqrt{N_c(u)} - 4M^3(u) + O(N_c^3(u)). \] (8)

\[ r_{AH^-} = 2M(u) + 8M^3(u)N_c(u) + O(N_c^4(u)), \] (9)

The inner apparent horizon \( r_{AH^-} \) is larger than the dynamical schwarzschild radius \( r(u) = 2M(u) \) and the outer apparent cosmological horizon \( r_c = r_{AH^+} \) tends to infinity for \( N_c(u) \ll 1 \). The cosmological field and the black hole mass have positive contributions for inner apparent horizon, whereas the black hole mass has negative contribution for the outer horizon and pulls cosmological horizon back towards center of the black hole. The black hole evaporation leads to shrinking and vanishing of the inner apparent horizon while the outer horizon is tending to reach its asymptotic value \( N_c \).

Now, we apply the Bousso’s method like the one defined to some extend in the section 2. We consider an observer inside a system which is circumscribed by a sphere of radius \( r_{AH^+} \). Then, we assume that the observer moves away the matter system (black hole) until he witnesses the crossing of the matter system towards the outside of his apparent cosmological horizon with radius of \( r_{AH^+} \). The GSL claims that the entropy of final state of this apparent cosmological horizon in the absence of black hole is greater than the entropy of the initial state of this apparent cosmological horizon with black hole. The final state system circumscribing of sphere with radius \( r_0 \) has entropy \( S_0 = \frac{4\pi}{r_0^2} \), where \( A_0 \) is the area of apparent cosmological horizon in the absence of the matter system \((M(u) = 0)\) and \( r_0 = r_c(M(u) = 0) = N_c^{-\frac{1}{3}} \). The entropy of initial state system is the sum of the matter system (black hole) entropy \( S_m = S_{AH^-} \) and the entropy of cosmological field horizon \( S_{AH^+} \). We can write them as follows

\[ S_{AH^-} = \pi r_{AH^-}^2 = \pi(4M^2(u) + 32M^4(u)N_c(u)) + O(N_c^5(u)), \] (10)

\[ S_{AH^+} = \pi r_{AH^+}^2 = \pi\left(\frac{1}{N_c(u)} - 2\frac{M(u)}{\sqrt{N_c}} - 2M^2(u) - 5M^3(u)\sqrt{N_c(u)} - 16M^4(u)N_c(u) + O(N_c^5(u))\right) \] (11)

According to GSL the final entropy \( S_0 = \frac{4\pi}{r_0^2} \) is greater than initial entropy \( S_{AH^-} + S_{AH^+} \). Thus, using (10) and (11) in \( S_0 \geq S_{AH^-} + S_{AH^+} \), which is like the inequality (5), we have

\[ S_m \leq \pi \left(2\frac{M(u)}{\sqrt{N_c}} + 2M^2(u) + 5M^3(u)\sqrt{N_c(u)} + 16M^4(u)N_c(u)\right), \] (12)

which is the D-bound for surrounded Vaidya solution by cosmological constant-like field. In the limit of dilute system surrounding field \( N_c(u) \ll 1 \), the inequality (12) becomes

\[ S_m \leq 2\pi \frac{M(u)}{\sqrt{N_c}}. \] (13)

The inequality (13) puts an upper bound for the entropy of the black hole. The normalization parameter for the cosmological field \( N_c(u) \) in the limit of dilute field makes larger the upper bound for the black hole entropy but the black hole mass \( M(u) \) has opposite role. One can recognize from inequality (12) that, the first two terms in RHS cancel each others and the remaining ones are all positive or both parameters \( M \) and \( N \) have positive effects on the upper entropy of black hole, imposed by D-bound.

### B. Bekenstein bound

To derive Bekenstein bound (1) for surrounded Vaidya solution by cosmological constant-like field we need to know the radius of the sphere \( R \) circumscribing the system and its energy \( E \). To find the Bekenstein bound we will apply
the Bousso’s method \[10\]. For Vaidya black hole surrounded by cosmological field, the energy of the system is not well-defined, due to the lack of a suitable asymptotic region. But there exists a solution which is known as Vaidya black hole solution surrounded by cosmological field which behaves like the metric of de-Sitter space with cosmological horizon radius \(r_c\), at large distances. This solution is like the “system’s equivalent black hole” \(r_g\). The \(r_g\) for schwarzschild black hole equals the twice energy of the black hole which is the same as event horizon radius of the black hole. But, for this solution there is some delicate points, as follows. Here the \(r_g\) is the same as apparent horizon of the black hole, but it is not the same as twice energy of the black hole. Thus, the corrected \(r_g\) and cosmological horizon \(r_c\) are

\[
\begin{align*}
    r_g &= 2m = r_{AH-} = 2M(u) + 8M^3(u)N_c(u) + O(N_c^2(u)), \\
    r_c &= r_{AH+} = \frac{1}{\sqrt{N_c(u)}} - M(u) - \frac{3}{2}M^2(u)\sqrt{N_c(u)} - 4M^3(u) + O(N_c^3(u)).
\end{align*}
\]

The Bekenstein bound in the system’s equivalent black hole with its gravitational radius \(r_g\) is written as follows \[10\]

\[
S_m \leq \pi r_g R,
\]

where \(R\) is the radius of sphere which circumscribes the system which is equal to \(r_c\) here. Now, we put the equations \[14\] and \[15\] into \[16\]. Then, we have

\[
S_m \leq \pi\left(\frac{2M(u)}{\sqrt{N_c(u)}} + 5M^3(u)\sqrt{N_c(u)} - 2M^2(u) - 8M^4(u) - 8M^4(u)N_c(u) - 32M^6N_c(u) + O(N_c^2(u))\right).
\]

We see in the above inequality that for \(N_c(u) \ll 1\) only the first term in RHS is dominant which is exactly the same as D-bound. However, in inequality \[12\] for \(N_c(u) \ll 1\) the dominant terms are the first, second and third. The first and second terms cancel each other when we put \(r_0\). But the third term remains which is exactly the same as the dominant term in equation \[17\] (i.e \(S_m \leq \pi \frac{2M(u)}{\sqrt{N_c(u)}}\)). So, Bekenstein bound and D-bound \[13\] are identified for very dilute surrounding field. For the case of a little less dilute surrounding field, the Bekenstein bound \[17\] becomes

\[
S_m \leq \pi\left(\frac{2M(u)}{\sqrt{N_c(u)}} - 2M^2(u) - 8M^4(u)\right),
\]

which is a tighter bound than D-bound \(S_m \leq \pi\left(\frac{2M(u)}{\sqrt{N_c(u)}} + 2M^2(u)\right)\) derived by this less dilute approximation from \[12\].

Except for very dilute system limit, the Bekenstein bound \[17\] or surrounded Vaidya solution by cosmological constant-like field and its D-bound \[12\] are not the same. The parameters \(N_c\) and \(M(u)\) always have positive effects in the D-bound, but in Bekenstein bound they have both positive and negative effects. If negative parts dominate in Bekenstein bound to positive ones, then RHS in \[13\] becomes negative which is impossible. The requirement of a positive upper bound in Bekenstein bound puts constraint on the parameters \(N_c\) and \(M(u)\) in \[18\]. There is no such a constraint for D-bound \[12\] regarding this solution because RHS in \[12\] is always positive.

V. D-BOUND AND BEKENSTEIN BOUND FOR SURROUNDED VAIDYA BLACK HOLE BY QUINTESSENCE FIELD

A. D-bound

For the case of Vaidya solution with the equation of state \(\omega_c = -\frac{2}{3}\) \[16, 28\], the metric \[6\] becomes like the black hole surrounded by quintessence field solution,

\[
ds^2 = -\left(1 - \frac{2M(u)}{r} - N_q(u)r\right)du^2 + 2dudr + r^2d\Omega_2^2,
\]

where \(N_q(u)\) is the normalization parameter for the quintessence field surrounding the black hole. Positive energy condition on the surrounding quintessence field leads to \(N_q > 0\) \[10\]. According to the metric \[19\], it is obvious that the surrounding quintessence field has non-trivial contribution to the metric of Vaidya black hole. The background quintessence field changes the causal structure of black hole solution in comparison to that of the original Vaidya black
hole in an empty background. An almost similar effect occurs when one immerses a schwarzschild black hole in a de Sitter background which is asymptotically de-Sitter [39]. As we mentioned before for Vaidya black hole surrounded by cosmological field, the surface gravity of the black hole here is also negative and it leads to gravitational repulsion. It is a suitable candidate which is responsible for dark energy.

Deriving D-bound for this case is the same as the one which we derived for Vaidya black hole solution surrounded by cosmological field in the previous section. We repeat all of the arguments there with their particular results. We have to adopt the solution that has two apparent horizons. For \( \Delta(u) = 1 - 8M(u)N_q(u) > 0 \), there is two physical inner and outer apparent horizons [10]. The locations of two apparent horizons for \( \Delta(u) > 0 \) with small quintessence normalization parameters \( N_q \ll M(u) \) are

\[
\begin{align*}
    r_{AH^-} &= 2M(u) + 4M^2(u)N_q(u) + O(N_q^2(u)), \\
    r_{AH^+} &= \frac{1}{N_q(u)} - 2M(u) - 4M^2(u)N_q(u) + O(N_q^2(u)).
\end{align*}
\]

The surrounded quintessence field has contributions both in physical inner horizon \( r_{AH^-} \) (which is larger than dynamical schwarzschild radius \( r(u) = 2M(u) \)) and physical outer horizon \( r_{AH^+} \) (cosmological) tending to infinity for \( N_q(u) \ll 1 \). The quintessence field and black hole mass have positive contributions for inner apparent horizon. The black hole mass has negative contribution for the outer horizon and it pulls cosmological horizon toward inside. The black hole evaporation leads to shrinking and vanishing of inner apparent horizon while the outer horizon is tending to its asymptotic value \( N_q^{-1} \).

Now, in this case we can apply the same methods that we used in the previous section for the black hole surrounded by cosmological field. The existence of two apparent horizons guarantees considering D-bound. Here, the entropy of the final state system is \( S_0 = \frac{4A_0}{\pi} \), where \( A_0 \) is the area of the cosmological horizon inside which there is no matter system other than the quintessence field. The initial state entropy is the sum of the black hole entropy \( S_m = S_{AH^-} \) and the cosmological horizon entropy \( S_{AH^+} \). They are given as,

\[
\begin{align*}
    S_{AH^-} &= \pi r_{AH^-}^2 \pi (4M^2(u) + 16M^3(u)N_q(u)) + O(N_q^2(u)), \\
    S_{AH^+} &= \pi r_{AH^+}^2 \pi \left( \frac{1}{N_q(u)} - \frac{4M(u)}{N_q(u)} - 4M^2(u) - 16M^3(u)N_q(u) \right) + O(N_q^2(u)).
\end{align*}
\]

According to GSL the final entropy \( S_0 = \frac{4A_0}{\pi} \) is greater than initial entropy \( S_{AH^-} + S_{AH^+} \). Thus, by using (22), (23) and \( r_0 = r_c(M(u) = 0) = N_q^{-1} \) in \( S_0 \geq S_{AH^-} + S_{AH^+} \) which is like the equation (5) we have

\[
S_m \leq \pi \left( \frac{4M(u)}{N_q(u)} + 4M^2(u) + 16M^3(u)N_q(u) \right).
\]

In the limit of \( N_q(u) \ll 1 \) the above inequality becomes

\[
S_m \leq \pi \frac{4M(u)}{N_q(u)}.
\]

The inequality (25) puts an upper bound for the entropy of the black hole. The upper bound for the black hole entropy gets large provided that the black hole mass \( M(u) \) becomes large and the normalization parameter for the quintessence field \( N_q(u) \) becomes small, in the limit of dilute field.

B. Bekenstein bound

In this case, the derivation method of Bekenstein bound [11] is the same as the method we used in the previous section for Vaidya solution surrounded by cosmological field. Regarding inequality (10), the radius \( R \) of sphere circumscribing the system and gravitational radius \( r_g \) are necessary for considering Bekenstein bound. The gravitational radius \( r_g \) here is not twice the energy of the schwarzschild black hole. It is not well defined, for the lack of asymptotic flat region, in surrounded Vaidya black hole by quintessence field. However, in this solution \( r_g \) is the location of the
apparent horizon of the black hole \( r_{AH^-} \). Also the radius \( R \) in this solution is equal to the cosmological apparent horizon \( r_c = r_{AH^+} \). They are given as follows

\[
r_g = 2m = r_{AH^-} = 2M(u) + 4M^2(u)N_q(u) + O(N_q^2(u)),
\]

\[
r_c = r_{AH^+} = \frac{1}{N_q(u)} - 2M(u) - 4M^2(u)N_q(u) + O(N_q^2(u)).
\]

Now, one can put equations (26) and (27) into equation (16) to derive Bekenstein bound as

\[
S_m \leq \pi \left( \frac{2M(u)}{N_q(u)} - 16M^3(u) N_q(u) \right) + O(N_q^2(u)).
\]

Similar to the previous case, we use \( R = r_c \) as the radius of sphere which circumscribe the system. For \( N_q(u) \ll 1 \) the dominant term in Bekenstein bound is the first term (i.e. \( S_m \leq \pi \frac{2M(u)}{N_q(u)} \)). This Bekenstein bound for surrounded Vaidya black hole by quintessence field is tighter than its D-bound (25) in the limit of dilute energy. In this limit, the normalization parameter \( N_q \), for \( N_q(u) \ll 1 \) makes larger the outer bound for Bekenstein and the mass of the black hole also does the same job for large amounts of mass. As the amount of \( N_q \) increases, the absolute values of the first and the second terms decrease and increase, respectively on the RHS of (28) and makes tighter the bound. Over all, it turns out that the Bekenstein bound here is tighter than the D-bound. In the D-bound the mass of black hole has always positive contribution on the upper bound, but in the Bekenstein bound the mass of the black hole has both positive and negative contributions. Thus, in this case the D-bound does not give the Bekenstein bound.

VI. D-BOUND AND BEKENSTEIN BOUND FOR SURROUNDED VAIDYA BLACK HOLE BY PHANTOM FIELD

A. D-bound

For the case of Vaidya solution with the equation of state by \( \omega_q = -\frac{1}{3} \), the metric (8) becomes like the black hole surrounded by phantom field solution

\[
ds^2 = -\left(1 - \frac{2M(u)}{r} - N_p(u)r^3\right) du^2 + 2du dr + r^2 d\Omega_2^2,
\]

where \( N_p(u) \) is the normalization parameter for the phantom field surrounding the black hole. Positive energy condition on the surrounding cosmological field leads to \( N_p > 0 \). According to the metric (29), the surrounding phantom field has non-trivial effect on the Vaidya black hole and its causal structure. In this case, like as the previous cases, the phantom background field causes a negative surface gravity which leads to gravitational repulsion of the Vaidya black hole. This feature of phantom field with gravitational repulsion can be a candidate for dark energy (28).

In deriving D-bound for this case, we are interested in the solutions with two apparent horizons, one of them is black hole apparent horizon and the other one plays the role of cosmological horizon \( r_c \). These solutions are given for the condition \( \Delta(u) = 1 - \frac{2048}{27}M^3(u)N_p(u) > 0 \). Both of these horizons are physical (16). They are as follows

\[
r_{AH^-} = 2M(u) + 16M^4(u)N_p(u) + O(N_p^2(u)),
\]

\[
r_{AH^+} = \frac{1}{N_p(u)} - \frac{2}{3}M(u) - \frac{8}{9}M^2(u)N_p^2(u) - \frac{160}{81}M^3(u)N_p^3(u) - \frac{16}{3}M^4(u)N_p^4(u) + O(N_p^4(u)),
\]

where \( r_{AH^-} \) and \( r_{AH^+} \) represent inner and outer apparent horizon, respectively. Thus, the background phantom field creates an inner horizon larger than dynamical schwarzschild radius (apparent horizon) \( r(u) = 2M(u) \) and an outer horizon, which is cosmological apparent horizon here, blows up for \( N_p \ll 1 \). Regarding equations (30) and (31), one can realize that the phantom field makes larger the inner apparent horizon and the black hole mass makes larger the outer horizon. The black hole evaporation process shrinks the inner apparent horizon while the outer one closes to its asymptotic value \( N_p^{-3/4} \).

The existence of two apparent horizons guarantees considering D-bound. Here, the entropy of the final state system is \( S_0 = \frac{A_0}{4} \), where \( A_0 \) is the area of the cosmological horizon in the absence of matter system except the phantom.
The initial state entropy is the sum of the black hole entropy \( S_m = S_{AH^-} \) and the cosmological horizon entropy \( S_{AH^+} \). They are given as

\[
S_{AH^-} = \pi r^2_{AH^-} = \pi (4M^2(u) + 64M^5(u)N_p(u)) + O(N_p^2(u)),
\]

\[
S_{AH^+} = \pi r^2_{AH^+} = \pi \left( \frac{1}{N_p^\Delta} - \frac{4M^2(u)}{N_p^2} - \frac{4}{3}M^2(u) - \frac{224}{81}M^3(u)N_p^\frac{1}{3}(u) - \frac{1760}{243}M^4(u)N_p^\frac{2}{3}(u) - \frac{64}{3}M^5(u)N_p(u) \right) + O(N_p^4(u)).
\]

To construct the D-bound we use the generalized second law of thermodynamics. According to GLS and by using (32), (33) and \( r_0 = r_c(M(u) = 0) = N_p^{-\frac{2}{3}} \) in \( S_0 \geq S_{AH^-} + S_{AH^+} \), one can derive the D-bound for this solution as

\[
S_m \leq \pi \left( \frac{4}{3} \frac{M(u)}{N_p^\Delta} + \frac{4}{3}M^2(u) + \frac{224}{81}M^3(u)N_p^\frac{1}{3}(u) + \frac{1760}{243}M^4(u)N_p^\frac{2}{3}(u) + \frac{64}{3}M^5(u)N_p(u) \right).
\]

In the limit \( N_p \ll 1 \), the D-bound (34) becomes

\[
S_m \leq \pi \left( \frac{4}{3} \frac{M(u)}{N_p^\Delta} \right).
\]

The inequality (35) puts an upper bound for the entropy of the black hole. The upper bound for the black hole entropy becomes large for large black hole mass \( M(u) \) and small normalization parameter \( N_p(u) \) in the limit of dilute field.

**B. Bekenstein bound**

Similar to the previous case, the gravitational radius and the outer cosmological apparent horizon in this case are obtained as

\[
r_g = 2m = r_{AH^-} = 2M(u) + 16M^4(u)N_p(u) + O(N_p^2(u)),
\]

\[
r_c = r_{AH^+} = \frac{1}{N_p^\Delta}(u) - \frac{2}{3}M(u) - \frac{8}{9}M^2(u)N_p^\frac{1}{3}(u) - \frac{160}{81}M^3(u)N_p^\frac{2}{3}(u) - \frac{16}{3}M^4(u)N_p(u) + O(N_p^2(u)).
\]

We can put equations (36) and (37) into equation (16) to derive the Bekenstein bound for \( N_p \ll 1 \) as

\[
S_m \leq \pi \left( \frac{2M(u)}{N_p^\Delta} - \frac{4}{3}M^2(u) \right).
\]

In the Bekenstein bound (16) we put \( R = r_c \) as the radius of sphere which circumscribes the system. In this case, if \( \frac{1}{N_p^\Delta(u)} < 4M(u) \) the Bekenstein bound (38) will be tighter than the D-bound (35). But, we know that \( \Delta(u) = 1 - \frac{2048}{27}M^3(u)N_p(u) > 0 \) or \( \frac{1}{N_p^\Delta(u)} > 4M(u) \) gives two real solutions which lead to two physical apparent horizons which are necessary for considering D-bound. The other amounts of \( \Delta \) (i.e \( \Delta(u) \leq 0 \)) which lead to \( \frac{1}{N_p^\Delta(u)} < 4M(u) \) cannot give two physical apparent horizons as solutions. So, the Bekenstein bound here, cannot be tighter than the D-bound. Therefore, if \( \frac{1}{N_p^\Delta(u)} > 4M(u) \) the D-bound will be tighter than the Bekenstein bound. However, for \( \frac{1}{N_p^\Delta(u)} = 4M(u) \) there is no D-bound because we have not two physical apparent horizons.
VII. D-BOUND AND BEKENSTEIN BOUND FOR SURROUNDED VAIDYA BLACK HOLE BY EMPTY BACKGROUND, DUST AND RADIATION FIELDS

Constructing D-bound for the Vaidya black hole by empty background is impossible because there is no cosmological horizon in this case and there is only one physical apparent horizon which is the black hole horizon \[16\]. With this apparent horizon of the black hole, one can only talk about Bekenstein bound and covariant entropy bound. Thus, the D-bound in this case is undefinable.

There is a same story for the case of Vaidya black hole surrounded by dust field. Because there is only one apparent horizon here without cosmological horizon \[16\], we cannot construct the D-bound in this case, too. The apparent horizon indicates the horizon of the black hole. This horizon can give us the Bekenstein bound of the system or the the covariant entropy bound. The only difference with an empty background is the larger radius of apparent horizon because of the dust field. Thus, we have larger entropy in this case in comparison with the empty background.

For the case of Vaidya solution surrounded by radiation field the D-bound is meaningless, although there are two physical apparent horizons. The metric structure for the radiation case is just similar to the Reissner-Nordström black hole, where there is no cosmological horizon \[16\]. For the case of Vaidya solution with the equation of state by \( \omega_r = \frac{1}{3} \) \[16, 28\], the metric \(6\) becomes like the black hole surrounded by radiation field solution

\[
\begin{align*}
\frac{ds^2}{r^2} = - \left( 1 - \frac{2M(u)}{r} - \frac{N_r(u)}{r^2} \right) du^2 + 2dudr + r^2d\Omega_2^2,
\end{align*}
\]

where \(N_r(u)\) is the normalization parameter for the radiation field surrounding the black hole. Positive energy condition on the surrounding radiation field leads to \(N_r < 0\). If one define \(N_r = -N_r\), the metric \(39\) becomes

\[
\begin{align*}
\frac{ds^2}{r^2} = - \left( 1 - \frac{2M(u)}{r} + \frac{N_r(u)}{r^2} \right) du^2 + 2dudr + r^2d\Omega_2^2.
\end{align*}
\]

The metric \(40\) behaves like a radiating charged Vaidya black hole regarding to the dynamical charge as \(Q^2 = N_r\). According to this similarity, the effective charge-like term \(\frac{N_r}{r^2}\) is considered as a positive contribution (For more details refer to the Bonnor-Vaidya solution \[40\]). Charge-like term changes the causal structures of the black hole and has effect on apparent horizons of the system. The two real apparent horizons are achieved for \(\Delta(u) = (M^2(u) - N_r(u)) > 0\) \[16\]. These physical horizons are given by

\[
\begin{align*}
r_{AH^-} &= \frac{N_r(u)}{2M(u)} + O(N_r^2), \quad \text{(41)}
\end{align*}
\]

\[
\begin{align*}
r_{AH^+} &= 2M(u) - \frac{N_r(u)}{2M(u)} + O(N_r^2), \quad \text{(42)}
\end{align*}
\]

in the dilute radiation background \(N_r(u) \ll M(u)\). One can define the covariant entropy bound for two apparent horizons. The entropy of the horizons are given as

\[
\begin{align*}
S_{AH^-} &= \pi r_{AH^-}^2 = \pi \left( \frac{N_r^2(u)}{4M^2(u)} + O(N_r^3(u)) \right), \quad \text{(43)}
\end{align*}
\]

\[
\begin{align*}
S_{AH^+} &= \pi r_{AH^+}^2 = \pi \left( 4M^2(u) - 2N_r(u) + O(N_r^2(u)) \right). \quad \text{(44)}
\end{align*}
\]

To derive the Bekenstein bound we use the inequality \[16\]. Now, one can put equation \(11\) as the gravitational radius \(r_g\) and the equation \(12\) as the radius \(R\) of sphere circumscribes the system into equation \(16\) to derive Bekenstein bound as follows

\[
S_m \leq \pi (N_r(u) - \frac{N_r^2(u)}{4M^2(u)}). \quad \text{(45)}
\]

We know from \(\Delta(u) = (M^2(u) - N_r(u)) > 0\) that \(4M^2(u) > 2N_r(u)\). Thus, RHS of the equation \(16\) is always positive and there is a Bekenstein bound for this solution. According to this bound, the effective mass of the black hole cannot tend toward zero.
VIII. CONCLUSIONS AND RESULTS

We have derived the D-bound for surrounded Vaidya solutions by cosmological fields and indicated that for some of the solutions the D-bound is the same as the Bekenstein bound in dilute systems. The results are as follows:

- D-bound for surrounded Vaidya solution by cosmological constant-like field is the same as Bekenstein bound for very dilute systems. As the background field becomes more considerable, the equality of D-bound and Bekenstein bound is more ruined. In the case of dilute cosmological constant-like field, the contribution of background field in the metric is $N_c r^2$ which leads to the equality of D-bound with Bekenstein bound.

- D-bound for surrounded Vaidya black hole by quintessence field is the same as Bekenstein bound in light background field, expect for a constant coefficient 2. Since the contribution of background field in the metric is $N_q r$, which is weaker than the case of cosmological constant like field $N_c r^2$ at $r > 1$, the D-bound does not coincide with the Bekenstein bound, even in the light background systems. The D-bound in light quintessence background field is more than the Bekenstein bound, hence the D-bound is looser than the Bekenstein bound.

- D-bound for surrounded Vaidya black hole by phantom field is the same as Bekenstein bound in light background field, expect for a constant coefficient. Since the contribution of background field in the metric is $N_p r^3$, which is stronger than the case of cosmological constant like field $N_c r^2$ at $r > 1$, the D-bound does not again coincide with the Bekenstein bound, even in the light background systems. The D-bound in light phantom background field is less than the Bekenstein bound, hence the D-bound is stronger than the Bekenstein bound.

- The D-bound for background fields like dust and radiation, and for an empty space without cosmological horizon, is undefinable.

The conclusions are as follows. The dynamical background fields, including cosmological horizons, play the role of a repulsion force like the case of a cosmological constant which manifests itself in the metric as $r^2$. For this repulsion force, the D-bound is identified with the Bekenstein bound in dilute systems. Any deviation from $r^2$ term corresponding to the quintessence and phantom fields with contributions as $r$ and $r^3$ terms having less and more repulsion forces than that of the cosmological constant leads to D-bounds looser and tighter than the Bekenstein bound, respectively. At the end, it is worth mentioning that D-bound and Bekenstein bound are the direct consequence of GSL. Therefore, we conclude that both of them should lead to the same entropy bound imposing on a certain matter system. This conclusion leads to one possible option as follows:

- Cosmological constant field viability: The cosmological constant field has a reasonable behaviour, among two other cosmological fields, namely quintessence and phantom, regarding the identification of D-bound and Bekenstein bound for light systems. It seems that by implementation of a thermodynamical criteria, namely the identification of D-bound and Bekenstein bound, on the Vaidya black hole solution surrounded by dark energy cosmological fields, one may exclude the quintessence and phantom fields and just keep the cosmological constant as the viable dark energy cosmological field for which D-bound and Bekenstein bound are exactly identified.$^2$

We intend to study and impose the same thermodynamical criteria on the other known dynamical black hole solutions surrounded by cosmological fields to explore weather the cosmological constant is preferred as the viable cosmological field in comparison to the other known cosmological fields.$^5$

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$^2$ The violation of second law of thermodynamics by quintessence and phantom fields which represents their un-physical behaviors in many ways, has been considered in $^{14,56}$. 


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