ON FORM FACTORS IN $\mathcal{N} = 4$ SYM THEORY AND POLYTOPES

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Abstract

In this paper we discuss different recursion relations (BCFW and all-line shift) for the form factors of the operators from the $\mathcal{N} = 4$ SYM stress-tensor current supermultiplet $T^{AB}$ in momentum twistor space. We show that cancelations of spurious poles and equivalence between different types of recursion relations can be naturally understood using geometrical interpretation of the form factors as special limit of the volumes of polytopes in $\mathbb{CP}^4$ in close analogy with the amplitude case. We also show how different relations for the IR pole coefficients can be easily derived using momentum twistor representation. This opens an intriguing question - which of powerful on-shell methods and ideas can survive off-shell?

Keywords: Super Yang-Mills Theory, amplitudes, form factors, polytopes, twistors, superspace.
1 Introduction

In the last years tremendous progress has been achieved in understanding the structure of the S-matrix (amplitudes) of four dimensional gauge theories \[1\]. The most impressive results have been obtained in $\mathcal{N} = 4$ SYM theory (for example, see \[2\] and reference therein). New computational techniques such as different sets of recursion relations for the tree level amplitudes, the unitarity based methods for loop amplitudes were used to obtain deep insights in the structure of the $\mathcal{N} = 4$ SYM S-matrix. It is believed that this efforts will eventually lead to the complete determination of the $\mathcal{N} = 4$ SYM S-matrix in planar limit. Also, probably, some beautiful geometrical ideas and insights will be encountered along the way \[3\] \[4\] \[5\] \[6\] \[7\] \[8\].

There is another class of objects of interest in $\mathcal{N} = 4$ SYM which resembles amplitudes - the form factors. The form factors are matrix elements of the form

$$\langle p_1^{\lambda_1}, \ldots, p_n^{\lambda_n} | \mathcal{O} | 0 \rangle,$$

(1.1)

where $\mathcal{O}$ is some gauge invariant operator which acts on the vacuum of the theory and produces some state $\langle p_1^{\lambda_1}, \ldots, p_n^{\lambda_n} |$ with momenta $p_1, \ldots, p_n$ and helicities $\lambda_1, \ldots, \lambda_n$\[1\]. One can think about this object as an amplitude of the processes where classical current or field, coupled through gauge invariant operator $\mathcal{O}$, produces some quantum state $\langle p_1^{\lambda_1}, \ldots, p_n^{\lambda_n} |$.

\[1\] Note that scattering amplitudes in "all ingoing" notation can schematically be written as $\langle p_1^{\lambda_1}, \ldots, p_n^{\lambda_n} | 0 \rangle$. 

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It is interesting to study the form factors in $\mathcal{N} = 4$ SYM systematically for several reasons:

- Symmetries, such as dual conformal symmetry, play essential role in the structure of amplitudes in gauge theories. Moreover, it is expected that $\mathcal{N} = 4$ SYM is an integrable system (see [9, 10, 11, 12] and references there). Studying the form factors in integrable systems (for example, see [13] and references therein) usually can be useful for better understanding of origins and properties of symmetries in such theories. One may hope that studying the form factors in $\mathcal{N} = 4$ SYM may be useful for understanding of symmetry properties of the $\mathcal{N} = 4$ SYM S-matrix and the correlation functions.

- The form factors are intermediate objects between fully on-shell quantities such as amplitudes and fully off-shell quantities such as correlation functions (which are one of the central objects in AdS/CFT). Since the powerful computational methods have appeared recently for the amplitudes in $\mathcal{N} = 4$ SYM, it would be desirable to have some analog of them for the correlation functions [14, 15]. Understanding of the structure of the form factors and the development of computational methods will be useful in better understanding of the structure of correlation functions of multiple ($n > 2$) gauge invariant local operators in $\mathcal{N} = 4$ SYM. The latter may also be useful in understanding of "triality" relations: amplitudes, Wilson loops, correlation functions and subsequent relations for the amplitudes [16, 17].

- The form factors in $\mathcal{N} = 4$ SYM are excellent testing objects for developing and testing new computational methods which can be efficient beyond the planar sector of maximally supersymmetric gauge theories. Indeed, form factors naturally incorporate none planarity and violate some supersymmetries (at least the form factors of operators from the chiral truncation of $\mathcal{N} = 4$ SYM stress tensor supermultiplet).

The investigation of the form factors in $\mathcal{N} = 4$ SYM was first initiated in [18], almost 20 years ago. Unique investigation of the form factors of single field none gauge invariant operators (off-shell currents) was made in [19], using "perturbiner" technique.

After pause which lasted nearly decade the investigation of 1/2-BPS form factors was initiated in [20, 21]. Different on-shell methods were successfully applied to the form factors [22, 23, 24, 25]. Different multiloop results were obtained in [21, 26, 27]. Different types of regularizations and colour-kinematic duality were considered in [28, 29]. Strong coupling limit results for the form factors were obtained in [30, 31]. The form factors in theories with maximal supersymmetry in dimensions different from $D = 4$ were investigated in [32, 33].

The aim of this article is the following: we would like to apply momentum twistor representation for the form factors of $\mathcal{N} = 4$ SYM stress-tensor supermultiplet and formulate BCFW recursion relation for tree level form factors in such formulation. It is known that in the case of the amplitudes written in momentum twistor variables interesting geometrical properties and symmetries of the amplitudes are represented most clearly.
and naturally \[3, 5\]. It is interesting to know what situation will be if we are considering partially off-shell object. Which on-shell ideas and methods such as \[3, 5, 6\] can survive for partially off-shell objects?

This article is organised as follows. In section 2 we briefly discuss the general structure of the form factors of operators from \(\mathcal{N} = 4\) SYM stress-tensor supermultiplet in on-shell harmonic superspace. In section 3 we establish and solve BCFW recursion relations for tree level form factors in NMHV sector in on-shell harmonic superspace. In section 4 we discuss how to rewrite NMHV form factors in momentum twistor representation, establish BCFW recursion relations for general N\(k\)MHV form factors in momentum twistor space. In section 5 we represent a sketch of the proof of equivalence between BCFW and all-line shift (CSW) recursion relations for NMHV sector in momentum twistor space and use geometrical representation of the form factors as a special limit of the volumes of the polytopes to show that all-line shift (CSW) representation of NMHV sector is free from spurious poles. The latter would imply the spurious poles cancellation in BCFW representation as well. In appendix we give more details on the harmonic superspace construction, discuss some particular examples of the spurious poles cancellation and also discuss how relations between IR pole coefficients at one loop in NMHV sector can be naturally established in the momentum twistor representation.

2 The form factors of stress-tensor current supermultiplet in \(\mathcal{N} = 4\) SYM

In this chapter we are going to introduce essential ideas and notations regarding the general structure of the form factor of stress-tensor supermultiplet formulated in harmonic superspace.

To describe the stress-tensor supermultiplet in manifestly supersymmetric and \(SU(4)\) covariant way it is useful to consider the harmonic superspace parameterised by the set of coordinates \[34, 35\]:

\[
\mathcal{N} = 4 \text{ harmonic superspace } = \{x^{\alpha \dot{\alpha}}, \theta^{+a}, \theta^{-a'}, \bar{\theta}^{+\dot{a}}, \bar{\theta}^{-\dot{a}'}, u\}. \tag{2.2}
\]

Here \(u\) is the set of

\[
SU(4) / SU(2) \times SU(2)^{'} \times U(1)
\]

harmonic variables, \(a\) and \(a'\) are \(SU(2)\) indices, \(\pm\) corresponds to \(U(1)\) charge. \(\theta\)'s are Grassmann coordinates, \(\alpha\) and \(\dot{\alpha}\) are \(SL(2, \mathbb{C})\) indices. Hereafter we will not write some indices explicitly in all expressions when it does not lead to misunderstanding.

The stress-tensor supermultiplet will be given by

\[
T = Tr(W^{++}W^{++}) \tag{2.3}
\]
where $W^{++}(x, \theta^+, \bar{\theta}^+)$ is harmonic superfield which contains all component fields of the $\mathcal{N} = 4$ supermultiplet, which are $\phi^{AB}$ scalars (anti-symmetric in $SU(4)_R$ indices $AB$), $\psi^A, \bar{\psi}^A$ fermions and $F^{\mu\nu}$ - the gauge field strength tensor, all in the adjoint representation of $SU(N_c)$ gauge group. The details of harmonic superspace construction will be given in the appendix. Note that this superfield is on-shell in the sense that algebra of supersymmetric transformations which should leave $W^{++}$ invariant is closed only if the component fields in $W^{++}$ obey their equations of motion.

Space of on-shell states of $\mathcal{N} = 4$ supermultiplet is naturally described in manifestly supersymmetric fashion by means of on-shell momentum superspace. We are going to use its harmonic version:

$$\mathcal{N} = 4 \text{ harmonic on-shell momentum superspace} = \{\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \eta^-_a, \eta^+_a, u\}.$$ \hspace{1cm} (2.4)

Here $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}$ are $SL(2, \mathbb{C})$ commuting spinors which parameterise momenta carried by on-shell state: $p_\alpha = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ if $p^2 = 0$. All creation/annihilation operators of on-shell states, which are two physical polarizations of gluons $|g^-, g^+\rangle$, four fermions $|\Gamma^A\rangle$ with positive and four fermions $|\bar{\Gamma}^A\rangle$ with negative helicity, and three complex scalars $|\phi^{AB}\rangle$ (anti-symmetric in $SU(4)_R$ indices $AB$ ) can be combined together into one $\mathcal{N} = 4$ invariant superstate ("superwave-function") $|\Omega_i\rangle = \Omega_i|0\rangle$ ($i$ numerates momenta carried by the state):

$$|\Omega_i\rangle = \left(g^+_i + (\eta_i \Gamma_i) + \frac{1}{2!}(\eta \eta \phi_i) + \frac{1}{3!}(\varepsilon \eta \eta \hat{\Gamma}_i) + \frac{1}{4!}(\varepsilon \eta \eta \eta \bar{\Gamma}_i)\right)|0\rangle,$$ \hspace{1cm} (2.5)

where $(\ldots)$ represents contraction with respect to $SU(2) \times SU(2)' \times U(1)$ indices, $(\varepsilon \ldots)$ represents contraction with $\varepsilon_{ABCD}$ symbol. It is implemented one has to express all $SU(4)$ indices in terms of $SU(2) \times SU(2)' \times U(1)$ once using the set of harmonic variables $u$. The $n$ particle superstate $|\Omega_n\rangle$ is then given by $|\Omega_n\rangle = \prod_{i=1}^n \Omega_i|0\rangle$. Note that on-shell momentum superspace is chiral. Because of that and subtleties \cite{22, 23} with on-shell realisation of the stress tensor supermultiplet in terms of $W^{++}$ superfield it is natural to consider chiral (self dual) sector of the stress tensor supermultiplet only. This can be done by putting all $\theta$ to 0 by hand in $T$ (this often called "chiral truncation"):

$$T(x, \theta^+) = Tr(W^{++}W^{++})|_{\theta=0}. $$ \hspace{1cm} (2.6)

All operators from $T$ are constructed from fields from self dual part of the $\mathcal{N} = 4$ supermultiplet. Also it is important to mention that all component fields in $T$ are off-shell.

So we can consider the form factors of chiral truncation (self dual sector) of $\mathcal{N} = 4$ stress tensor supermultiplet $\mathcal{F}_n$:

$$\mathcal{F}_n\{\lambda, \tilde{\lambda}, \eta, x, \theta^+\} = \langle \Omega_n | T(x, \theta^+) | 0 \rangle,$$ \hspace{1cm} (2.7)
Here we are considering the colour ordered object \( F_n \). The physical form factor \( \mathcal{F}_n^{phys} \) in the planar limit should be obtained from \( \mathcal{F}_n \) as:

\[
\mathcal{F}_n^{phys}(\{\lambda, \tilde{\lambda}, \eta\}, x, \theta^+) = \sum_{\sigma \in S_n/Z_n} Tr(t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(n)}}) \mathcal{F}_n(\{\lambda, \tilde{\lambda}, \eta\}, x, \theta^+), \tag{2.8}
\]

where the sum runs over all possible none-cyclic permutations \( \sigma \) of the set \( \{\lambda, \tilde{\lambda}, \eta\} \) and the trace involves \( SU(N_c) \) \( t^a \) generators in the fundamental representation, factor \((2\pi)^{n-2}2^{n/2}\) is dropped. The normalization \( Tr(t^at^b) = 1/2 \) is used.

Let’s now consider the general Grassmann structure of \( \mathcal{F}_n \). It is convenient to perform transformation from \( \theta^+ \) and \( x \) to \( q \) and the set of axillary variables \( \{\lambda', \eta'^-; \lambda'', \eta''^-\} \), \( \lambda'\eta'^- + \lambda''\eta''^- = \gamma^- \):

\[
\hat{T}[\ldots] = \int d^4x^\alpha \ d^{-4}\theta \exp(iqx + \theta^+\gamma^-)[\ldots], \tag{2.9}
\]

\[
Z_n(\{\lambda, \tilde{\lambda}, \eta\}, \{q, \gamma^-\}) = \hat{T}[\mathcal{F}_n]. \tag{2.10}
\]

Using supersymmetry arguments (\( Z_n \) should be annihilated by an appropriate set of supercharges) one can say that in general \([22, 23]\):

\[
Z_n(\{\lambda, \tilde{\lambda}, \eta\}, \{q, \gamma^-\}) = \delta^{4}(\sum_{i=1}^{n} \lambda'^i \eta'^i - q_{aa}) \delta^{4}(q_{aa}^- + \gamma_{aa}^-) \delta^{4+4}(q_{d\alpha}^+) \mathcal{X}_n^0(\{\lambda, \tilde{\lambda}, \eta\}), \tag{2.11}
\]

\[
\mathcal{X}_n = \mathcal{X}_n^{(0)} + \mathcal{X}_n^{(4)} + \ldots + \mathcal{X}_n^{(4n-8)},
\]

where

\[
q_{d\alpha}^+ = \sum_{i=1}^{n} \lambda'^i \eta'^i, \quad q_{d\alpha}^- = \sum_{i=1}^{n} \lambda'^i \eta'^i.
\tag{2.12}
\]

Grassmann delta functions are defined as (see appendix for the whole set of definitions regarding Grassmann delta functions and their integration)

\[
\delta^{\pm 4}(q_{d\alpha}^+/a\alpha) = \prod_{a'/a, b'/b=1}^{2} e^{\alpha\beta} q_{d\alpha}'/a\beta q_{d\beta}'/b\beta. \tag{2.13}
\]

\( \mathcal{X}_n^{(4m)} \) are the homogenous \( SU(4)_R \) and \( SU(2) \times SU(2)' \times U(1) \) invariant polynomials of the order of \( 4m \). Hereafter for saving space we will use notations:

\[
\delta^8(q + \gamma) \equiv \delta^{4}(q_{aa}^- + \gamma_{aa}^-) \delta^{4+4}(q_{d\alpha}^+).
\tag{2.14}
\]

\( ^2g \rightarrow 0 \) and \( N_c \rightarrow \infty \) of \( SU(N_c) \) gauge group so that \( \lambda = g^2 N_c \) = fixed.
Assigning helicity $\lambda = +1$ to $|\Omega_i\rangle$ and $\lambda = +1/2$ to $\eta$ and $\lambda = -1/2$ to $\theta$, one sees that $F_n$ has an overall helicity $\lambda_{\Sigma} = n$, $\delta^{++}$ has $\lambda_{\Sigma} = 2$, the exponential factor has $\lambda_{\Sigma} = 0$. From this we see that $X_n^{(0)}$ has $\lambda_{\Sigma} = n - 2$, $X_n^{(4)}$ has $\lambda_{\Sigma} = n - 4$, etc. $X_n^{(0)}$, $X_n^{(4)}$ etc. are understood as analogs \[38\] of the MHV, NMHV etc. parts of the superamplitude i.e. part of the super form factor proportional to $X_n^{(0)}$ will contain component form factors with overall helicity $n - 2$ which we will call MHV form factors, part of super form factor proportional to $X_n^{(4)}$ will contain component form factors with overall helicity $n - 4$ which we will call NMHV etc. up to $X_n^{(4n-8)}$ overall helicity $2 - n$ which we will call MHV.

One can think \[22\] that it is still possible to describe the form factors of full stress tensor supermultiplet disregard subtleties with on-shell realisation, at least at tree level, using symmetry arguments and full $W^{++}(x, \theta^+, \bar{\theta}^+)$ superfield. To do this one has to introduce none chiral version of on-shell momentum superspace, which in our case can be obtain by performing the following Grassmann Fourier transform

$$\langle \Omega_i \rangle = \int d^{+2}\eta_i \exp(\eta_i^+ \bar{\eta}_i^-) |\Omega_i\rangle,$$

$$\langle \bar{\Omega}_i \rangle = (g_i^+ (\bar{\eta}_i^- \bar{\eta}_i^-) + \ldots + (\eta_i^- \bar{\eta}_i^-) g_i^-) |0\rangle.$$  \hspace{1cm} (2.15)$$

After that one can define form factor of full stress tensor supermultiplet

$$F_n^{full}(\{\lambda, \bar{\lambda}, \eta, \bar{\eta}\}, x, \theta^+, \bar{\theta}^+) = \langle \Omega_i | T(x, \theta^+, \bar{\theta}^+)|0\rangle.$$  \hspace{1cm} (2.16)$$

Performing $\hat{T}$ transformation from $(x, \theta^+, \bar{\theta}^+)$ to $(q, \gamma^-, \bar{\gamma}^-)$ one can obtain $Z_n^{full}$:

$$Z_n^{full}(\{\lambda, \bar{\lambda}, \eta, \bar{\eta}\}, \{q, \gamma^-, \bar{\gamma}^\prime\}) = \delta^4(\sum_{i=1}^{n} \lambda_{\alpha}^i \bar{\lambda}_{\alpha}^i - q_{\alpha\bar{\alpha}}) \delta^{-4}(q_\alpha^- + \gamma^-_{\alpha\bar{\alpha}}) \delta^{-4}(\bar{q}_\alpha^- a^\prime_{\alpha} + \bar{\gamma}_\alpha^- a^\prime_{\alpha}) \times$$

$$\times \int \prod_{k=1}^{n} d^{+2}\eta_k \exp(\eta_k^+ \bar{\eta}_k^-) \delta^{+4}(q_{a^\prime\bar{\alpha}}) X_n(\{\lambda, \bar{\lambda}, \eta\}),$$

(2.17)

where now after Fourier transformation

$$\bar{q}_{\alpha}^{- a'^\prime} = \sum_{i=1}^{n} \lambda_{\alpha}^i \bar{\eta}_i^- a'^\prime.$$  \hspace{1cm} (2.18)$$

We see that at least at tree level the form factors of full stress tensor supermultiplet up to trivial Grassmann delta function are defined by Grassmann Fourier transformed $X_n$ function, which one can compute using chiral truncated (self dual sector) stress tensor supermultiplet only \[22\]. Keeping this in mind we will focus on the self dual sector form factors.
Using BCFW recursion relations [20] one can show that for MHV sector at tree level one can obtain for \( n \) point form factor (here we drop momentum conservation delta function):

\[
Z_n^{(0)MHV} = \delta^8(q + \gamma) A_n^{(0)}, \quad A_n^{(0)} = \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}.
\]

We will use this result in the next chapter. Also for completeness let’s write down well known answers for tree level MHV \( n \) and MHV \( 3 \) amplitudes

\[
A_n^{(0)MHV} = \frac{\delta^8(q)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \quad A_3^{(0)MHV} = \frac{\hat{\delta}^4(\eta_{[23]} + \eta_{[31]} + \eta_{[12]})}{[12][23][31]},
\]

which we will also use in the next section.

### 3 BCFW and all-line shift for NMHV sector

Recursion relations for the tree level form factor were considered in the literature before. BCFW recursion for MHV sector, as was mentioned earlier, was considered in [20] for the component form factors. All-line shift (CSW) recursion for NMHV sector was considered in [22] in the on-shell momentum superspace and momentum twistor spaces. BCFW for form factors of more general 1/2-BPS operators in on shell momentum superspace were considered in [25].

In [22] it was argued that for general \([i, j]\) shift \( N^k \)MHV form factor vanishes as \( z \to \infty \), so BCFW recursion without ”boundary terms”. Let’s consider BCFW recursion for NMHV sector in on-shell momentum superspace. It is useful to recall how BCFW recursion for NMHV amplitudes works before going to form factors. It will help us to introduce important structures and make useful analogies. For adjacent \([i - 1, i]\) shift

\[
\hat{\lambda}_i = \lambda_i + z\lambda_{i-1},
\]

\[
\hat{\lambda}_{i-1} = \tilde{\lambda}_{i-1} - z\tilde{\lambda}_i,
\]

\[
\hat{\eta}_i = \eta_i + z\eta_{i-1}.
\]

there are two types of contributions in BCFW recursion in the NMHV sector, which are combined from (MHV \( \otimes \) MHV) and (NMHV \( n-1 \otimes \) MHV \( 3 \)) amplitudes. MHV\( \otimes \)MHV terms are given by so called \( R_{rst} \) 2mh functions times MHV tree level amplitude. NMHV \( n-1 \otimes \) MHV\( 3 \) term can be represented in terms of \( R_{rst} \) functions as well. \( R_{rst} \) function can be written as:

\[
R_{rst} = \frac{\langle ss - 1 \rangle \langle tt - 1 \rangle \delta^4(\Xi_{rst})}{x_{st}^2 \langle r | x_{rs} x_{ts} | s \rangle \langle r | x_{rt} x_{ts} | s - 1 \rangle \langle r | x_{rs} x_{st} | t \rangle \langle r | x_{rt} x_{st} | t - 1 \rangle},
\]

\( ^3 \otimes \) stands for summation over internal states (Grassmann integration) and substitution of corresponding \( z \) values.
Figure 1: Diagrammatical representation of the quadruple cut proportional to $R_{rst}$. The white blob is $\overline{\text{MHV}}_3$ vertex, the light-grey blob is MHV amplitude.

$$\Xi^A_{rst} = \sum_{i=t}^{r-1} \eta^A_i \langle l | x_{ts} x_{sr} | r \rangle + \sum_{i=r}^{s-1} \eta^A_i \langle l | x_{st} x_{tr} | r \rangle = (\Theta^A_{tr} | x_{ts} x_{sr} | r \rangle + (\Theta^A_{rs} | x_{st} x_{tr} | r \rangle. \quad (3.23)$$

where $x_{ij}$ and $\Theta^A_{ij}$ are dual variables, defined as $(|l\rangle \equiv \lambda_l)$

$$x_{ij} = \sum_{k=i}^{j-1} p_k, \quad \langle \Theta^A_{ij} | = \sum_{l=i}^{j-1} \eta^A_l \langle l |. \quad (3.24)$$

In the harmonic superspace formulation $\Xi^A_{rst}$ splits into $\Xi^{+a}_{rst}$ and $\Xi^{-a'}_{rst}$ as well as the Grassmann delta function $\tilde{\delta}^4 = \tilde{\delta}^{-2} \tilde{\delta}^2 + 2$ (see appendix for details). Throughout the paper we will assume that numbers of momenta $r, s, t, ...$ etc. are arranged anti clockwise for the form factors, were it is not mentioned otherwise. All sums are understood in cyclic sense, for example, if $n=6$ $s=5, t=3$ then $\sum_{s}^{t} - \sum_{t}^{1} = \sum_{6}^{5} + \sum_{1}^{3}$. For $n \leq 4$ the $R_{rst}$ function vanishes. $R_{rst}$ 2mh functions may also be obtained by quadruple cuts of one loop NMHV amplitude. In fact there is deep connection between on-shell recursion relation for tree level amplitudes and their loop level structure [6, 43, 36, 37]. $R_{rst}$ are invariants with respect to dual superconformal transformations [38] from $SU(2, 2|4)$ as well as ordinary superconformal group. Even more they are invariants with respect to full Yangian algebra [39]. In harmonic superspace formulation they also invariants with respect to $SU(2) \times SU(2) \times U(1)$. There is also interesting geometrical interpretation [5] of them which we will discuss further in details. Using this functions one can write the results of BCFW recursion for NMHV sector for the amplitudes for $[1, 2]$ shift as:

$$A_n^{(0)NMHV} = (A_{n-1}^{(0)NMHV} \otimes A_3^{(0)MHV}) + A_n^{(0)MHV} \sum_{i=4}^{n-1} R_{12i}. \quad (3.25)$$
This recursion relations can be solved in terms of $R_{rst}$ functions (note that some terms in this sum are actually equal to 0):

$$A_{n}^{(0)NMHV} = A_{n}^{(0)MHV} \left( \sum_{j=2}^{n-2} \sum_{i=j+2}^{n} R_{ji} \right). \quad (3.26)$$

It is natural to assume that NMHV sector of the form factors can be represented in terms of quadruple cut coefficients as well as in the case of the amplitudes. Quadruple cuts for NMHV sector of the form factors were studied in [24]. There are three different types of analogs of $R_{rst}$ functions for the form factors $R_{(1)rst}$, $R_{(2)rst}$ and $\tilde{R}_{rtt}$.

$$R_{(1)rst} = \frac{\langle s + 1|t + 1\rangle \delta^4 \left( \sum_{i=r+1}^{t} \eta_{i}\langle i|p_{s+1...t}p_{s+1...r+1}|r\rangle - \sum_{i=r}^{s+1} \eta_{i}\langle i|p_{s+1...t}p_{r...r+1}|r\rangle \right)}{p_{s+1...t}^2 \langle r|p_{r...s+1}p_{s+1}|t + 1\rangle \langle r|p_{r...s+1}p_{t...s+1}|t\rangle \langle r|p_{t...r}p_{s+1}|s + 1\rangle \langle r|p_{t...r}p_{s+1}|s\rangle}, \quad (3.27)$$

$$R_{(2)rst} = \frac{\langle s + 1|t + 1\rangle \delta^4 \left( \sum_{i=r+1}^{t} \eta_{i}\langle i|p_{s+1...t}p_{r+1...s}|r\rangle + \sum_{i=r}^{s+1} \eta_{i}\langle i|p_{s+1...t}p_{r...r+1}|r\rangle \right)}{p_{s...t+1}^2 \langle r|p_{r...s}p_{s...t+1}|t + 1\rangle \langle r|p_{r...s}p_{s...t-1}|t\rangle \langle r|p_{t...r+1}p_{s...t+1}|s + 1\rangle \langle r|p_{t...r+1}p_{s...t+1}|s\rangle}, \quad (3.28)$$

$$\tilde{R}_{rtt} = \frac{\langle tt + 1\rangle \delta^4 \left( \sum_{i=t+1}^{r} \eta_{i}\langle i|p_{t+1...n}p_{r...t+1}|r\rangle - \sum_{i=r}^{t+1} \eta_{i}\langle i|p_{t+1...n}p_{t...t+1}|r\rangle \right)}{q^4 \langle r|p_{r...t+1}p_{1...n}|t\rangle \langle r|p_{t...q}|t + 1\rangle \langle r|p_{t...r+1}p_{1...n}|r\rangle}, \quad (3.29)$$
where we used notations \( p_{i_1...i_n} = p_{i_1} + ... + p_{i_n} \), \( q = \sum_{i=1}^{n} p_i \). The same notations will be used hereafter. One can see that \( R^{(1)}_{rst} \), \( R^{(2)}_{rst} \) in fact coincides with \( R_{rst} \) computed in corresponding kinematics, while \( \tilde{R}^{(1)}_{rte} \) is different. Note also that due to the presence of momenta \( q \) carried by operator in momentum conservation condition for the form factors \( R^{(1)}_{rst} \), \( R^{(2)}_{rst} \) and \( \tilde{R}^{(1)}_{rte} \) can be defined (are none vanishing) starting with number of particles \( n = 3 \).

Now we are ready to return to tree level form factors. As it was stated earlier we hope that NMHV sector of the form factors can be represented in terms of quadruple cut coefficients \( R^{(1)}_{rst} \), \( R^{(2)}_{rst} \) and \( \tilde{R}^{(1)}_{rte} \). Indeed this is just the case. By explicit computation one can see that in the case of \([1,2]\) shift:

\[
Z_n^{(0)NMHV} = \left( Z_{n-1}^{(0)NMHV} \otimes A_3^{(0)MHV} \right) + Z_n^{(0)MHV} \left( \tilde{R}^{(1)}_{122} + \sum_{i=3}^{n-1} R^{(1)}_{i12} + \sum_{i=3}^{n} R^{(2)}_{i12} \right). \tag{3.30}
\]

Just as in the amplitude case coefficients \( R^{(1)}_{122} \), \( R^{(2)}_{112} \) and \( \tilde{R}^{(1)}_{122} \) are given by 2mh quadruple cuts. Let’s write several answers for some fixed \( n \). For example, for \( n = 3 \) and \( n = 4 \) one can get:

\[
Z_3^{(0)NMHV} = Z_3^{(0)MHV} \tilde{R}^{(1)}_{122}, \tag{3.31}
\]

\[
Z_4^{(0)NMHV} = Z_4^{(0)MHV} \left( \tilde{R}^{(1)}_{133} + \tilde{R}^{(1)}_{122} + R^{(1)}_{132} + R^{(2)}_{112} \right). \tag{3.32}
\]

Here \( \tilde{R}^{(1)}_{133} \) is given by \( \left( Z_3^{(0)NMHV} \otimes A_3^{(0)MHV} \right) \) term. Note also that \( R^{(2)}_{132} = 0 \) for \( n = 4 \) case. In general the result for \( \left( Z_{n-1}^{(0)NMHV} \otimes A_3^{(0)MHV} \right) \) for \( Z_n^{(0)NMHV} \) can be conve-
niently written in terms of $Z_{n-1}^{(0)NMHV}$ by introducing shift operator $S$ which shifts number of arguments of function starting with 2 by $+1$, (for example, $Sf(x_0, x_1, x_2, x_3) = f(x_0, x_1, x_3, x_6)$):

$$
\left( Z_{n-1}^{(0)NMHV} \otimes A_3^{(0)MHV} \right) \{\lambda_i, \tilde{\lambda}_i, \eta\}_{i=1}^n, \{q, \gamma\} = Z_n^{(0)MHV} \frac{S Z_{n-1}^{(0)NMHV}}{Z_{n-1}^{(0)MHV}} \{\lambda_i, \tilde{\lambda}_i, \eta\}_{i=1}^{n-1}.
$$

(3.33)

$\{q, \gamma\}$ are unshifted. This can be seen using BCFW recursion in MHV sector and representing $R$ functions as quadruple cut which is given by product of MHV$_n$ and MHV$_3$ amplitudes and form factors. Using this observation one can write answer for $Z_n^{(0)NMHV}$.

\[\text{In the amplitude case the similar procedure can be most easily formulated in terms of on-shell diagrams. In the case of the form factors one may hope that the extension of on-shell diagrams formalism also exists, but we are not going to discuss this issue here.}\]
Figure 6: BCFW diagrams contributing to the $n = 4$ case, for $[1, 2]$ shift. $B_2 = B_5 = 0$ due to the kinematical reasons.

Figure 7: Schematic representation of corresponding $R$ functions contributing to the $n = 4$ case, for $[1, 2]$ shift.

in closed form using $R_{rst}^{(1)}$, $R_{rst}^{(2)}$ and $\tilde{R}_{rtt}^{(1)}$ functions:

$$Z_{n}^{(0)NMHV} = Z_{n}^{(0)MHV} \left( \sum_{i=2}^{n-1} \tilde{R}_{1ii}^{(1)} + \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} R_{1ji}^{(1)} + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \sum_{j=i+1}^{n-1} R_{1ji}^{(2)} \right).$$  \hfill (3.34)

As by product let’s also consider different BCFW shifts. For example, for $[2, 3]$ shift, $n = 4$ one can obtain the following representation of $Z_{4}^{(0)NMHV}$:

$$Z_{4}^{(0)NMHV} = Z_{4}^{(0)MHV} \left( \tilde{R}_{244}^{(1)} + R_{243}^{(1)} + R_{213}^{(2)} + \tilde{R}_{233}^{(1)} \right).$$  \hfill (3.35)

Adding results of $[1, 2]$ and $[2, 3]$ shifts with $1/2$ coefficient we obtain representation of $Z_{4}^{(0)NMHV}$ computed as coefficient before IR pole at one loop for the NMHV form factor

13
This can be written in the following cyclic invariant form\textsuperscript{[24]}:

\begin{equation}
Z_{4}^{(0)NMHV} = Z_{4}^{(0)MHV} \frac{1}{2}(1 + \mathbb{P} + \mathbb{P}^2 + \mathbb{P}^3)(\hat{R}_{311}^{(1)} + R_{241}^{(1)}).
\end{equation}

Here we used identity $R_{413}^{(2)} = R_{241}^{(1)}$ (see appendix).

As an illustration let’s consider computation of term which gives $\hat{R}_{123}^{(1)}$ in $n = 3$, $[1, 2]$ shift case. For $n = 3$ we have only one term contributing to $Z_{3}^{(0)NMHV} = A_2$ which is given by (see fig., $A_1 = 0$ due to the kinematical reasons):

\begin{equation}
A_2 = \int d^4\eta P Z_{2}^{(0)MHV}(\hat{P}, \hat{1}) A_{3}^{(0)MHV}(-\hat{P}, \hat{2}, 3),
\end{equation}

preforming Grassmann integration and substituting $z_{13} = [13]/[23]$, $\hat{p}_{13} = p_{13} + z_{13}\lambda_1\lambda_2$.

\textsuperscript{5}$\mathbb{P}$ is permutation operator, which shifts number all arguments of function by +1, i.e. for example: $\mathbb{P}f(x_0, x_1, x_2, x_3) = f(x_1, x_2, x_3, x_0)$.
we obtain
\[
A_2 = \frac{\delta^8(q + \gamma)}{\langle \hat{1} \hat{p} \rangle \langle 3 \hat{p} \rangle \langle 2 \hat{p} \rangle^2} \int d^4 \eta \, \delta^8(\lambda g \eta^3 + \hat{\lambda}_1 \hat{\eta}_1 - \hat{\lambda}_P \hat{\eta}_P) \\
= \frac{\delta^8(q + \gamma) \delta^4([2|\hat{\rho}_{13}\rangle|1\rangle_1 + [13]/[23]|2\rangle|\hat{\rho}_{13}\rangle|3\rangle_3 + [2|\hat{\rho}_{13}\rangle|3\rangle_3)}{\langle 13 \rangle \langle 1|\hat{\rho}_{13}\rangle|2\rangle \langle 3|\hat{\rho}_{13}\rangle|2\rangle \langle 2|\hat{\rho}_{13}\rangle|2\rangle^2 \langle \eta_3 \rangle} \\
= \frac{\delta^8(q + \gamma) \delta^4([23]|\eta_1 + [31]|\eta_2 + [21]|\eta_3)}{\langle 2|\hat{p}_{13}\rangle|2\rangle^2},
\]
(3.38)

after noting that
\[
\langle \hat{2}|\hat{p}_{13}|2\rangle = \langle 2|\hat{p}_{13}|2\rangle + p_{13}^2 = q^2,
\]
(3.39)

we can write (note also that momentum \( q \) carried by operator is equal to \( q = p_{123} \))
\[
A_2 = \frac{\delta^8(q + \gamma)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \times \frac{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \delta^4([23]|\eta_1 + [31]|\eta_2 + [21]|\eta_3)}{q^4 \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \\
= \frac{\delta^8(q_{123} + \gamma)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \times \frac{\langle 23 \rangle \delta^4(\eta_2(2|q_{231}|1\rangle - \eta_1(1|q_{p_{21}}|1\rangle - \eta_3(3|q_{p_{21}}|1\rangle))}{q^4 \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \\
= \frac{Z_{(0)^{MHV}}^{3}}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \times \hat{R}_{(t122)^{(1)}}.
\]
(3.40)

Pole \( q^4 \) is canceled in this expression on the support of \( \delta^8(q_{123} + \gamma) \). This will be important for us later on. Indeed \( \delta^8(q + \gamma) = \delta^{-4}(q^- + \gamma^-) \delta^{+4}(q^+) \), \( q_i^+ = \sum_{i=1}^{3} \eta_i^+, i \lambda_i \) so
\[
\hat{\delta}^{+2}(\eta_2^+ (2|q_{231}|1\rangle - \eta_1^+ (1|q_{p_{21}}|1\rangle - \eta_3^+ (3|q_{p_{21}}|1\rangle)) = \\
\hat{\delta}^{+2}(\eta_2^+ (21|q^2 - \sum_{i=1}^{3} \eta_i^+ i|q_{p_{12}}|1\rangle) = \\
\hat{\delta}^{+2}(\eta_2^+ (21|q^2 - 0) = q^2 (21)^{\delta^{+2}(\eta_2^+)}.\]
(3.41)

Cancellation of \( q^4 \) pole will be true also for arbitrary \( n \) for \( \hat{R}_{(t122)^{(1)}} \).

Let’s briefly discuss analytical properties of the results of BCFW recursion. As an example we will consider \( n = 4 \) case. Each \( R_{(r12)}^{(1)}, R_{(r12)}^{(2)} \) and \( R_{(t122)}^{(1)} \) term is rational function of \( \lambda_i, \hat{\lambda}_i \) variables and has several poles. Some of them are physical i.e. correspond to appropriate factorisation channels \([30]\), while others are spurious and must cancel in the whole sum. The presence of spurious poles is the general feature of BCFW recursion, and its application to the form factors is no exception. So in \( n = 4 \), \([1, 2]\) shift case the list of poles is the following:
\[
R_{(132)}^{(1)} : \langle 3|q|2\rangle, \langle 3|q|4\rangle, p_{124}^2, p_{12}^2, p_{14}^2 \quad R_{(142)}^{(2)} : \langle 1|q|4\rangle, \langle 1|q|2\rangle, p_{234}^2, p_{34}^2, p_{23}^2
\]
(3.42)
\( \tilde{R}_{122}^{(1)} : \langle 3|q|2\rangle, \langle 1|q|2\rangle, p_{134}^{2}, \tilde{R}_{133}^{(1)} : \langle 1|q|4\rangle, \langle 3|q|4\rangle, p_{234}^{2}. \)  

(3.43)

Poles

\[ p_{123}^{2}, p_{124}^{2}, p_{234}^{2}, p_{132}^{2}, p_{125}^{2}, p_{341}^{2}, p_{123}^{2}. \]  

(3.44)

are physical, while

\[ \langle 1|q|2\rangle, \langle 1|q|4\rangle, \langle 3|q|2\rangle, \langle 3|q|4\rangle. \]  

(3.45)

are spurious once. The structure of \( Z_{4}^{(0)NMHV} \) suggests that spurious poles should cancel themselves between \( R \) functions (for example \( \langle 1|q|2\rangle \) should cancel between \( R_{142}^{(2)} \) and \( \tilde{R}_{122}^{(1)} \)) but it is not easy to see how it really works. Also it would be nice to observe some general pattern of such cancelations for general \( n \). There are also several related questions.

1. One can consider a different type of recursion relations for the form factors: all-line shift (CSW) \([40, 41, 42]\). Indeed one can show that under anti holomorphic all-line shift the form factors with operators from stress-tensor supermultiplet (number of fields in operator \( m = 2 \)) behave as:

\[ Z_{n}(z) \to z^{s}(\text{or better}) \text{ as } z \to \infty, \text{ with } s = \frac{2 - n + \lambda_{\Sigma}}{2}, \]  

(3.46)

(note \( 2 - n \) instead of \( n - 4 \) in the amplitude case due to the different mass dimension of the form factor) so for \( \lambda_{\Sigma} = n - 4 \) as in NMHV case recursion is valid. So one can easily obtain \([25]\):

\[ Z_{n}^{(0)NMHV} = Z_{n}^{(0)MHV} \left( \sum_{i=1}^{n} \sum_{j=i+2}^{i+1-n} R_{aij} \right), \text{ with } \lambda^{*} = 0, \eta^{A} = 0. \]  

(3.47)

Here we exchange the problem of cancellation of spurious poles to the problem of proving that the poles of the form \( \langle i|q|* \rangle \) should cancel. This cancellation will imply that the result is independent of the choice of \( \tilde{\lambda}^{*} \). Note also that representations for NMHV sector given by BCFW and all-line shift (CSW) recursions naively look rather different. It would be nice to show how one can transform one into another.

2. It would be also nice to write some simple recursion relation for the general \( N^{k}MHV \) form factor.

3. In the one loop generalised unitarity based computations (for example, see \([24]\)) one encounters different none obvious relations between \( R \) functions. It would be nice to have some simple representation for \( R \) functions where these relations becomes obvious.

These questions are not unique to the form factors and one encounters their analogs in the amplitude case as well. In the case of amplitudes they all can be answered in beautiful geometrical picture based on momentum twistor representation and the interpretation of the amplitudes as the volumes of polytopes in \( \mathbb{CP}^{4} \) in the first non trivial NMHV case.
4 Momentum twistor space representation

To use momentum twistors one has to introduce dual variables $x_i$ for momenta $p_i$:

$$p_i^{a\dot{a}} = x_i^{a\dot{a}} - x_{i-1}^{a\dot{a}}. \quad (4.48)$$

and their fermionic counterparts $q_{a\alpha,i} = \lambda_{a\alpha,i} \eta_{a\alpha,i}$, $q_{a'\alpha,i} = \lambda_{a'\alpha,i} \eta_{a'\alpha,i}^+$, and $\Theta_{a\alpha}, \Theta_{a'\alpha}$:

$$q_{a\alpha,i} = \Theta_{a\alpha,i} - \Theta_{a\alpha,i-1}, \quad (4.49)$$

$$q_{a'\alpha,i} = \Theta_{a'\alpha,i} - \Theta_{a'\alpha,i-1}. \quad (4.50)$$

This is where periodical configuration first appears [20, 22]. Indeed, we are working with a colour ordered object, so positions of momenta of external particles $p_i$ are fixed. But the operator, which carries the momentum $q$, is colour singlet and can be inserted between any pair of momenta. The same is true also for the fermionic counterpart of $q$, when we are dealing with the superspace formulation of the form factors. One can think of working with different (with respect to position where $q$ is inserted) closed contours, but it is not obvious how to combine terms defined on different contours. Infinite periodical (with period equal to $q$) configuration solves this problem. In fact we will need only $2n$ $x_i$ independent variables to describe any kinematical invariant $p_2^{a\dot{a}},...,i$ we may encounter in the case of $n$ particle form factor, at least at tree level in the MHV and NMHV sectors. The only feature that the periodical contour brings into play and one should take into account is some sort of redundancy. Everything is defined up to the shift over $k$ periods over contour, so one should "gauge fix" which periods will be used. Also the periodical configuration is very natural from AdS/CFT point of view [30]. The insertion of operator corresponds to consideration of closed string state on the string worldsheet in addition to open ones (which corresponds to particles) in the dual picture. After such insertion, $T$-duality transformation gives infinite periodical configuration with period equal to momenta carried by the closed string state. Periodical contour and hence dual variables $\Theta^-, \Theta^+$ also can be introduced to the total super momentum carried by particles $q^+, q^-$. Period will be equal to the super momenta $\gamma^+, \gamma^-$ carried by operator. Note that since $\gamma^+ = 0$, the corresponding fermionic part of contour in superspace will be closed.

Now we are ready to introduce momentum supertwistors [3, 4]. Points in dual superspace are mapped to lines in momentum twistor space $(x_i, \Theta_i) \sim Z_{i-1} \wedge Z_i$ (as usual $i$ is
the number of particle, with:

\[
Z_i^\pm\Delta = \begin{pmatrix} Z_i^M \\ \chi_{a/a',i}^\pm \end{pmatrix},
\]  

(4.51)

The fermionic part of super twistor is given by:

\[
\chi_{ai}^- = \Theta_{ai}^- \lambda_i, \quad \chi_{ai}^+ = \Theta_{ai}^+ \lambda_i.
\]

(4.52)

Note that \( \chi_{ai}^- \) part of the supertwistor belongs to the infinite periodical contour, \( \chi_{ai}^+ \) belongs to the "closed part of the fermionic contour" due to the \( \gamma^+ = 0 \) condition. Sometimes it will be convenient to consider \( \chi_{ai}^+ \) also as a part of the infinite periodical contour in the intermediate expressions, and apply \( \gamma^+ = 0 \) only in the final. Because all our expressions are polynomials in Grassamann variables a smooth limit in \( \gamma^+ \to 0 \) always exists.

The bosonic part of supertwistor is

\[
Z_i^M = \begin{pmatrix} \lambda_i^\alpha \\ \mu_i^\dot{\alpha} \end{pmatrix}, \quad \mu_i^\dot{\alpha} = x_i^{a\dot{\alpha}} \lambda_{ai},
\]

(4.53)

where \( M = 1...4 \). Corresponding objects transform under the action of dual conformal \( SU(2, 2) \) group. \( \Delta \) is multi index for \( SU(2, 2) \) indices and \( \pm a/ \pm a' \) indices of \( SU(2) \) and \( U(1) \). Standard notation for dual conformal \( SU(2, 2) \) invariant will be used:

\[
\langle i, j, k, l \rangle = \epsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D.
\]

(4.54)
In terms of components of twistors this expression can be written as (here \( \epsilon_{\dot{\alpha}\dot{\beta}}\mu^\dot{\alpha}\mu^\dot{\beta} \)).

\[
\langle i, j, k, l \rangle = \langle ij \rangle [kl] + \langle il \rangle [jk] + \langle kl \rangle [ij] + \langle k \rangle [lj] + \langle jk \rangle [il].
\] (4.55)

Hereafter we will drop indices on twistors and their components everywhere when it does not lead to misunderstanding. Due to the periodical configuration with period \( q = \sum_{i=1}^n p_i \) we will have the following relation for the momentum twistors:

\[
Z_i = (\lambda_i, x_{i\lambda_i}), \quad Z_{i+nk} = (\lambda_i, x_{i+nk\lambda_i}), \quad i = 1...n, \quad k \in \mathbb{N}.
\] (4.56)

Using \( \langle i, j, k, l \rangle \) one can write the following expressions for kinematical invariants and products of spinors:

\[
\left( \sum_{i=1}^{j-1} p_i \right)^2 = x_{ij}^2 = \langle ii + 1 \rangle \langle jj + 1 \rangle \langle i, i+1, j, j+1 \rangle,
\] (4.57)

and

\[
\langle tt + 1 \rangle \langle r|x_{rs}|s \rangle = \langle r, t + 1, s \rangle.
\] (4.58)

Because we are working with the periodical configuration one can shift simultaneously all numbers in \( \langle r, t + 1, t, s \rangle \) and \( \langle i, i+1, k, k+1 \rangle \) by \( kn, \quad k \in \mathbb{N} \) without changing the result.

As it was claimed before in the case of the amplitudes (closed contour) \( R_{rst} \) function is invariant with respect to the dual superconformal transformations from \( SU(2, 2|4) \).

Using momentum supertwistors one can see that the following combination of 5 arbitrary twistors is \( SU(2, 2|4) \) invariant:

\[
[a, b, c, d, e] = \frac{\delta^4(\langle a, b, c, d \rangle \chi_e + \text{cycl.})}{\langle a, b, c, d \rangle \langle b, c, d, e \rangle \langle c, d, e, a \rangle \langle d, e, a, b \rangle \langle e, a, b, c \rangle}.
\] (4.59)

Here \( \delta^4 = \tilde{\delta}^{-2}\tilde{\delta}^+2 \) (see appendix). The \( R_{rst} \) function is a special case of this invariant:

\[
R_{rst} = [r, s, s + 1, t, t + 1].
\] (4.60)

What about \( \tilde{R}^{(1)}_{rst}, \quad R^{(2)}_{rst} \) and \( \tilde{R}^{(1)}_{r't} \) functions for the form factors? Using momentum supertwistors defined on periodical contour one can see that the following identities hold for \( \tilde{R}^{(1)}_{1st}, \quad R^{(2)}_{1st} \):

\[
\tilde{R}^{(1)}_{1st} = [1, t, t + 1, s - n, s + 1 - n],
\] (4.61)

and

\[
R^{(2)}_{1st} = [1, t, t + 1, s, s + 1].
\] (4.62)
Figure 11: Schematic representation of relation between $R^{(1)}_{rst}$, $R^{(2)}_{rst}$ functions and corresponding $[abcde]$ momentum twistor dual superconformal invariants in the 2mh case.

Here $n$ is the number of twistors (particles) in period of the contour. As it was explained earlier, due to periodical nature of momentum twistor configuration we are considering, this form is not unique. For example, for $n = 4$ one can see that: $[5, 6, 7, 3, 4] = [1, 2, 3, -1, 0]$. We choose this particular form ("fix the gauge") because it naturally arises in $[1, 2]$ shift. It is implemented that condition $\gamma^+ = 0$ is imposed in the argument of $\hat{\delta}^4$. The case of $\tilde{R}^{(1)}_{rst}$ is special. Nevertheless it is also possible to rewrite it in terms of $[a, b, c, d, e]$ momentum twistor invariant, but with nontrivial "bosonic" coefficient:

$$\tilde{R}^{(1)}_{1tt} = c^{(n)}_t [1, t, t + 1, t - n, t + 1 - n],$$
$$c^{(n)}_t = \frac{\langle 1, t, t + 1, t - n \rangle \langle 1, t - n, t + 1 - n, t + 1 \rangle}{\langle 1, t, t + 1, 1 + n \rangle \langle t, t + 1, t - n, t + 1 - n \rangle}. \quad (4.63)$$

As an illustration how one can rewrite $R$ coefficients in momentum twistor variables let’s consider $n = 4$ case (as usual we have $q = p_{1234}$), $\tilde{R}^{(1)}_{122}$:

$$\tilde{R}^{(1)}_{122} = \frac{\langle 23 \rangle \hat{\delta}^4(X_{122})}{q^4\langle 1|p_{1234}|3 \rangle \langle 1|p_{34}|2 \rangle \langle 1|p_{34}|1 \rangle}. \quad (4.64)$$
Figure 12: Schematic representation of relation between $R^{(1)}_{rsst}$ functions and corresponding $[abcde]$ momentum twistor dual superconformal invariants, special cases.

where

$$X_{122} = -\eta_2 \langle 2 | q p_{134} | 1 \rangle + \sum_{i=1,3,4} \eta_i \langle i | q p_2 | 1 \rangle =$$

$$= - \sum_{i=1,2} \eta_i \langle i | q p_{34} | 1 \rangle + \sum_{k=3,4} \eta_k \langle k | q p_{12} | 1 \rangle.$$

(4.65)

Now using momentum twistors we can write:

$$\langle -1, -2, 2, 3 \rangle = \langle 23 \rangle \langle -1 | x_{-13} x_{2-2} | -2 \rangle = \langle 23 \rangle \langle 3 | q(q+2)|2 \rangle = \langle 23 \rangle q^2,$$

(4.66)

$$\langle 1, -1, -2, 2 \rangle = \langle 23 \rangle \langle 1 | x_{1-1} x_{-12} | 2 \rangle = \langle 23 \rangle \langle 1 | p_{34} q | 2 \rangle,$$

(4.67)

$$\langle 1, 2, 3, -1 \rangle = \langle 23 \rangle \langle 1 | x_{13} x_{3-1} | 1 \rangle = \langle 23 \rangle \langle 1 | p_{12} q | 3 \rangle,$$

(4.68)

$$\langle 1, -1, -2, -3 \rangle = \langle -1 - 2 \rangle \langle 1 | x_{-1-1} x_{-1-3} | -3 \rangle = \langle 23 \rangle \langle 1 | p_{34} q | 1 \rangle.$$

(4.69)
Figure 13: Schematic representation of relation between \( R^{(2)}_{rst} \) functions and corresponding \([abcde]\) momentum twistor dual superconformal invariants, special case.

So substituting this relations in \( \tilde{R}^{(1)}_{122} \) one can see that:

\[
\begin{align*}
\tilde{R}^{(1)}_{122} &= \frac{\langle 23 \rangle \delta_4(X_{122})}{\langle -1, -2, 2, 3 \rangle \langle 1, -1, -2, 2 \rangle \langle 1, -1, 2, 3 \rangle} \\
&= \frac{\langle 1, 2, 3, -2 \rangle \langle 1, -2, -1, 3 \rangle}{\langle 1, -2, -1, -3 \rangle \langle 2, 3, -2, -1 \rangle} \times \frac{\langle 23 \rangle \delta_4(X_{122})}{\langle 1, -1, -2, 2 \rangle \langle -1, -2, 2, 3 \rangle \langle -2, 2, 3, 1 \rangle \langle 2, 3, 1, -1 \rangle \langle 3, 1, -1, -2 \rangle}.
\end{align*}
\]

From the last expression one can conclude that

\[
\frac{\langle 1, 2, 3, -2 \rangle \langle 1, -2, -1, 3 \rangle}{\langle 1, -2, -1, -3 \rangle \langle 2, 3, -2, -1 \rangle} = c^{(4)}_2.
\]

Now, let's rewrite \( X_{122} \) in terms of the momentum supertwistors (here we suppress \( SU(2) \times SU(2)' \times U(1) \) indices, \( \langle \Theta_{ij} \rangle \equiv \Theta_{ij}, \langle i \rangle \equiv \lambda_i \)). Here we treat \( \chi^+_i \) and \( \chi^-_i \) one equal footing, and will take \( \gamma^+ \to 0 \) limit only in the final expression. One can see that on periodical contour

\[
\langle \Theta_{13} \rangle = \sum_{i=1,2} \eta_i \langle i \rangle, \quad \langle \Theta_{-11} \rangle = - \sum_{i=3,4} \eta_i \langle i \rangle,
\]

and

\[
x_{-11} = p_{34}, \quad x_{-13} = -x_{3-1} = q, \quad x_{31} = -p_{12},
\]

so

\[
X_{133} = \langle \Theta_{13} | x_{3-1} x_{-11} | 1 \rangle + \langle \Theta_{-1} | x_{-13} x_{31} | 1 \rangle.
\]
Then we can write [4]:

\[
\langle \Theta_{13} | x_{3-1} x_{-11} | 1 \rangle + \langle \Theta_{1-1} | x_{-13} x_{31} | 1 \rangle = \frac{\chi_1(2, 3, -1, -2) + \text{perm.}}{(23)^2}.
\] (4.75)

Substituting this in \( \tilde{R}^{(1)}_{122} \) we get:

\[
\tilde{R}^{(1)}_{122} = \left\{ \begin{array}{c}
\langle 1, -1, -2, 2 \rangle \langle -1, -2, 2, 3 \rangle \langle -2, 2, 3, 1 \rangle \langle 2, 3, 1, -1 \rangle \langle 3, 1, -1, -2 \rangle \\
\end{array} \right\} = \left\{ \begin{array}{c}
\langle 1, 2, 3, -2, -1 \rangle,
\end{array} \right\}
\] (4.76)

as expected. Also now we can take \( \gamma^+ \to 0 \) limit.

Using these results one can easily rewrite BCFW recursion relations in NMHV sector for the form factors in momentum supertwistors (hereafter we drop \( (0) \) subscript for simplicity):

\[
Z_{NMHV}^n(Z_{2-n}, \ldots, Z_{1+n}) = Z_{NMHV}^n(Z_{n-1}, \ldots, Z_1, Z_3, Z_4, \ldots, Z_{1+n}) +
+ \sum_{j=3}^n [1, 2, 3, j, j + 1] + \sum_{j=3}^{n-1} [1, 2, 3, j - n, j + 1 - n] + c_2^{(n)} [1, 2, 3, 2 - n, 3 - n].
\] (4.77)

As an illustration let’s write answers for or \( n = 3, 4, 5 \) in momentum supertwistor notations:

\[
\frac{Z_{NMHV}^3}{Z_{MHV}^3} = c_2^{(3)} [-1, 0, 1, 2, 3],
\] (4.78)

\[
\frac{Z_{NMHV}^4}{Z_{MHV}^4} = \left( 8c_2^{(3)} \right) [-1, 0, 1, 3, 4] + [1, 2, 3, 4, 5] + [1, 2, 3, 0, -1]
+ c_2^{(4)} [1, 2, 3, -2, -1],
\] (4.79)

\[
\frac{Z_{NMHV}^5}{Z_{MHV}^5} = \left( 8^2c_2^{(3)} \right) [-1, 0, 1, 4, 5] + [1, 3, 4, 5, 6] + [-1, 0, 1, 3, 4]
+ \left( 8c_2^{(4)} \right) [-2, -1, 1, 3, 4] + [1, 2, 3, 4, 5] + [1, 2, 3, 5, 6]
+ [1, 2, 3, -2, -1] + [1, 2, 3, -1, 0] + c_2^{(5)} [1, 2, 3, -3, -2].
\] (4.80)

As by product using these explicit expressions let’s discuss relation between the form factors with the supermomentum carried by operator equal to zero and amplitudes. In
it was observed that the following relation between the form factors and amplitudes likely holds

\[ Z_n(\{\lambda, \tilde{\lambda}, \eta\}, \{0, 0\}) = g \frac{\partial A_n(\{\lambda, \tilde{\lambda}, \eta\})}{\partial g} . \]  

(4.81)

In our momentum supertwistor notations the limit of \( q \to 0, \gamma^\pm \to 0 \) corresponds to "gluing" all periods of the contour together i.e. for the \( n \) particle case \( Z_i \to Z_{i+kn} \) for any integer \( k \) and \( i \). Taking this limit in written above answers for the form factors one can see that (remember that \([a, b, c, d, e] = 0\) if any of two arguments coincide):

\[ Z_{3}^{NMHV}|_{z_i \to z_{i+3k}} = 0, \]  

(4.82)

\[ Z_{4}^{NMHV}|_{z_i \to z_{i+4k}} = 0, \]  

(4.83)

\[ Z_{5}^{NMHV}|_{z_i \to z_{i+5k}} = 2A_{5}^{NMHV}[1, 2, 3, 4, 5] = 2A_{5}^{NMHV}, \]  

(4.84)

as one would expect because there are no 3 and 4 point NMHV amplitudes. The presence of coefficient 2 is unexpected. However one can also see that from BCFW representation of \( N^k \)MHV form factors that the coefficient will be \( 2k \) in \( N^k \)MHV sector. Note also that in our case of the super form factors this limit is well defined and can be easily taken, while in components it is singular for some particular answers and in on-shell momentum superspace \([23]\) it is not obvious at first glance how exactly these singularities cancel.

Using the momentum supertwistors one can also easily write recursion relations for \( N^k \)MHV form factor \( Z_{n}^{tree(k)} \) at tree level in full analogy with the amplitude case. Performing the following shift of momentum super twistor \([2, 45]\)

\[ \hat{Z}_2 = Z_2 + wZ_3, \]  

(4.85)

which is equivalent to \([1, 2]\) shift in the momentum superspace one can obtain the following recursion relations \((Z_{n}^{tree(0)} \equiv Z_{n}^{NMHV}, A_{n}^{tree(0)} \equiv A_{n}^{MHV})\):

\[
\frac{Z_{n}^{(k)}(\ldots, Z_{-n+2}, Z_{-n+3}, \ldots, Z_1, Z_2, Z_3, \ldots, Z_n, Z_{n+1}, \ldots)}{Z_{n}^{(0)}} = \\
= \frac{Z_{n-1}^{(k-1)}(\ldots, Z_{1-n}, \ldots, Z_1, Z_3, Z_4, \ldots, Z_{1+n}, \ldots)}{Z_{n-1}^{(0)}} \\
+ \sum_{j=3}^{n}[1, 2, 3, j, j + 1] \times \frac{A_{n_1}^{(k_1)}}{A_{n_1}^{(0)}} (Z_{1-n}, \ldots, Z_j, \ldots, Z_{j+1}, \ldots) \\
\times \frac{Z_{n_2}^{(k_2)}}{Z_{n_2}^{(0)}} (\ldots, Z_0, Z_1, Z_I, \ldots, Z_{j+1-n}) \\
+ \sum_{j=3}^{n-1}[1, 2, 3, j - n, j + 1 - n] \times \frac{Z_{n_1}^{(k_1)}}{Z_{n_1}^{(0)}} (\ldots, Z_{j-n}, Z_I, \ldots, Z_{j+1-n}) \\
\times \frac{Z_{n_2}^{(k_2)}}{Z_{n_2}^{(0)}} (\ldots, Z_{j-n}, Z_I, \ldots, Z_{j+1-n}) \\
+ c_2^{(n)}[1, 2, 3, 2 - n, 3 - n] \times \frac{Z_{2}^{(k_1)}}{Z_{2}^{(0)}} (\ldots, Z_{2-n}, Z_I, \ldots, Z_{j+1-n}) \\
\times \frac{Z_{2}^{(k_2)}}{Z_{2}^{(0)}} (\ldots, Z_{2-n}, Z_I, \ldots, Z_{j+1-n}) \\
\times \frac{A_{n}^{(k_2)}}{A_{n}^{(0)}} (Z_I, \ldots, Z_{j+1-n}).
\]

(4.86)
Figure 14: Schematic representation of BCFW recursion for N^k MHV form factor Z^{(0)(k)}_n at tree level. The vertical bold black line corresponds to the form factor. The grey blob corresponds to the amplitude. MHV form factors and amplitudes are factor out.

\[ Z_I = (jj + 1) \cap (123) \quad \text{and} \quad \hat{Z}_2 = (12) \cap (0jj + 1), \quad (4.87) \]

\[ n_1 + n_2 - 2 = n, \quad k_1 + k_2 + 1 = k. \quad (4.88) \]

These relations have a very curious property that they represent ratio of Z^{(k)}_n form factor and MHV form factor in terms of polynomials of \([a, b, c, d, e]\) brackets multiplied by coefficients \(c_p^{(k)}\) which are ratios of \(\langle a, b, c, d \rangle\) dual conformal invariants. \([a, b, c, d, e]\) bracket in the case of amplitudes is dual superconformal invariant. In the case of the form factors we impose \(\gamma^+ = 0\) condition which likely will brake some of dual superconformal symmetries, but leave ordinary dual conformal symmetry intact (?). So \([a, b, c, d, e]\) bracket in the case of the form factors is dual conformal invariant and so is \(c_p^{(k)}[a, b, c, d, e]\). The only inconsistency which one can encounter is the behaviour of \(c_p^{(k)}\) with respect to little group scaling [2]. However it is easy to see that for \(n\) particle case if \(Z_i\) and \(Z_{i+nk}, k \in \mathbb{N}\) scale the same way, which is expected, then \(c_p^{(k)}\) is invariant with respect to little group scaling. One may think that ratio of N^k MHV_{n} form factor and MHV_{n} form factor at tree level is dual conformal invariant! It is immediately tempting to speculate about the

\[ 6(jj + 1) \cap (kln) = Z_j(j + 1klm) + Z_{j+1}(jklm) \]
situation at the loop level. At one loop explicit answers are available for NMHV\textsubscript{3,4}. One may think that contributions from 3\(n\) triangles will be an obstacle \cite{24}, it is unclear at the first glance how such contributions may cancel each other. This situation as well as the symmetry properties of tree level form factors require more detailed studies.

5 Spurious poles cancellation, BCFW Vs all-line shift and polytopes

So far we have formulated how to treat the form factors in momentum twistor space, obtained BCFW recursion for N\(k\)MHV\(_n\) form factor in the momentum supertwistors representation, and very briefly discussed their possible symmetry properties. The questions regarding BCFW and all-line shift (CSW) equivalence and spurious poles cancellation remained unanswered. But now we have all appropriate tools to address them.

At first, let’s try to see that BCFW and all-line shift (CSW) recursion are equivalent, at least in NMHV sector. Here we are aiming for the concrete examples rather then general proofs, and will consider mostly \(n = 3, 4\) NMHV cases.

Let’s rewrite all-line shift (CSW) results for NMHV sector in the momentum super-twistors. One can obtain \cite{22}:

\[
Z_{n}^{NMHV}/Z_{n}^{MHV} = \sum_{i=1}^{n} \sum_{j=i+2}^{i+n-1} [*\,i, i+1, j, j+1].
\] (5.89)

Here \(Z^*\) is arbitrary super twistor with components \(\lambda^* = \chi^* = 0\). One can choose \(\mu^* = \tilde{\lambda}^*\). \(\gamma^+ \to 0\) condition is implemented.

One can also think that \(Z^*\) is obtained from twistor with arbitrary components by contraction with so called infinity twistor \(I^{AB}\) \cite{4}. The presence of the infinity twistor explicitly brakes dual conformal invariance of each [*, a, b, c, d] term in all-line shift (CSW) representation of the amplitude or the form factor. In the case of amplitudes dual conformal invariance is restored in the whole sum of [*, a, b, c, d] terms. We expect a similar situation in the case of the form factors.

Note that the form of all-line shift (CSW) representation discussed here is not unique, due to the periodical nature of the contour. One can start the first sum ("fix the gauge") \(\sum_{i=1}^{n}\) from arbitrary point on contour, for example, from \(i = -1\): \(\sum_{i=-1}^{n-2} \sum_{j=i+2}^{i+n-1}\) this will lead to the same formula if one will return from momentum twistors to momentum superspace variables, as was explained earlier. It is convenient to "fix the gauge" this way in our case, i.e. start summation from point \(i = -1\):

\[
Z_{n}^{NMHV}/Z_{n}^{MHV} = \sum_{i=-1}^{n-2} \sum_{j=i+2}^{i+n-1} [*\,i, i+1, j, j+1].
\] (5.90)
Equivalently we can shift ("fix another gauge") our BCFW results by appropriate amount of periods, but we will not do so. Then for \( n = 3 \) and \( n = 4 \) one can write:

\[
Z_{3}^{NMHV} / Z_{3}^{MHV} = [*, -1, 0, 1, 2] + [* , 0, 1, 2, 3] + [* , 1, 2, 3, 4],
\]

(5.91)

and

\[
Z_{4}^{NMHV} / Z_{4}^{MHV} = ([*, -1, 0, 1, 2] + [* , -1, 0, 2, 3]) + ([*, 0, 1, 2, 3] + [* , 0, 1, 3, 4]) + \\
+ ([*, 1, 2, 3, 4] + [* , 1, 2, 4, 5]) + ([*, 2, 3, 4, 5] + [* , 2, 3, 5, 6]).
\]

(5.92)

Our next step is to show the sketch of the proof that the following equality holds:

\[
c^{(n)}_{i}[1, i, i + 1, i - n, i + 1 - n] = [*, 1, i, i + 1 + n] + [*, 1, i - n, i + 1 - n] \\
+ [*, 1, i + 1, i + 1 - n],
\]

(5.93)

\( \gamma^{+} \rightarrow 0 \) condition is implemented, \( \chi^{*} = 0 \) and \( Z^{*} \) is the result of projection by means of the infinity twistor \( I^{AB} \). One can think about it as some kind of partial fractions decomposition. Let’s proceed by iterations. For \( n = 3 \) one can verify that this equality holds by explicit comparison of coefficients before Grassmann monomials. For example,

\[
c^{(3)}_{2}\begin{bmatrix} -1, 0, 1, 2, 3 \end{bmatrix} = [*, -1, 0, 1, 2] + [* , 0, 1, 2, 3] + [* , 1, 2, 3, 4],
\]

\[
c^{(3)}_{2} = \frac{\langle -1, 1, 2, 3 \rangle \langle -1, 0, 1, 3 \rangle}{\langle -1, 0, 2, 3 \rangle \langle 1, 2, 3, 4 \rangle}.
\]

(5.94)

Note that LHS of the equality has poles \( \langle -1, 0, 1, 2 \rangle, \langle 0, 1, 2, 3 \rangle, \) and \( \langle 1, 2, 3, 4 \rangle \). The pole \( \langle -1, 0, 1, 2 \rangle \sim q^{2} \) as was explained earlier is absent. In RHS we separated these poles by introducing \( Z^{*} \) auxiliary supertwistor. In fact for \( n = 3 \) this equality is just a statement that BCFW and all-line shift (CSW) gives the same result:

\[
Z_{3,BCFW}^{NMHV} = Z_{3,CSW}^{NMHV}.
\]

(5.95)

One can also check that dependance on axillary twistor cancels in all coefficients. Then we can substitute in BCFW recursion for \( n = 4 \) in term

\[
\left( Z_{3}^{(0)NMHV} \otimes A_{3}^{(0)MHV} \right)
\]

(5.96)

\( Z_{3}^{NMHV} \) in the form \( Z_{3,BCFW}^{NMHV} \) or \( Z_{3,CSW}^{NMHV} \). Comparing two results and considering all possible \( [i, j] \) shifts we can prove the identity (5.94) for \( n = 4 \). Then we can substitute in BCFW recursion for \( n = 5 \) the results obtained for \( n = 4 \) etc.

Now one can see that substituting in BCFW formula identity (5.94) containing the axillary supertwistor \( Z^{*} \), and using 6 term identity \([3, 4]\) for the set of twistors

\[
Z^{*}, Z_{1}, Z_{a}, Z_{b}, Z_{c}, Z_{d}
\]

(5.97)
for all other \([1, a, b, c, d]\) invariants:

\[
[1, a, b, c, d] = [*, a, b, c, d] - [*, 1, a, b, c] + [*, 1, b, c, d] - [*, 1, a, b, d] + [*, 1, a, b, c],
\]

the all-line shift (CSW) formula is reproduced. Let’s illustrate this on \(n = 4\) example. Substituting

\[
[1, 2, 3, 4, 5] = [*, 2, 3, 4, 5] - [*, 1, 2, 4, 5] + [*, 1, 2, 3, 5] + [*, 1, 2, 3, 4],
\]

\[
[-1, 0, 1, 2, 3] = [*, 0, 1, 2, 3] - [*, -1, 1, 2, 3] + [*, -1, 0, 2, 3] - [*, -1, 0, 1, 3] + [*, -1, 0, 1, 2],
\]

\[
(Sc_{2}^{(3)})[-1, 0, 1, 3, 4] = [*, -1, 0, 1, 3] + [*, 0, 1, 3, 4] + [*, 1, 3, 4, 5],
\]

\[
c_{2}^{(4)}[1, 2, 3, -2, -1] = [*, -2, -1, 1, 2] + [*, -1, 1, 2, 3] + [*, 1, 2, 3, 5],
\]

in BCFW result one obtains \([*, -2, -1, 1, 2] = [*, 2, 3, 5, 6]\) for \(n = 4\)

\[
Z_{4}^{NMHV}/Z_{4}^{MHV} = [*, 2, 3, 4, 5] + [*, 1, 2, 4, 5] + [*, 1, 2, 3, 4] + [*, 0, 1, 2, 3] + [*, -1, 0, 2, 3] + [*, -1, 0, 1, 2] + [*, 0, 1, 3, 4] + [*, 2, 3, 5, 6].
\]

which is all-line shift (CSW) formula.

So far we argued how to transform BCFW representation of NMHV form factors into all-line shift (CSW) one. But what about cancelation of spurious poles? Let’s start with \(n = 4\) point example, as an illustration, how spurious pole cancels. As it was explained earlier one of spurious poles \(\langle 1|q|2\rangle\) should cancel between terms

\[
\hat{R}_{122}^{(1)} = c_{2}^{(4)}[1, 2, 3, -2, -1] \quad \text{and} \quad \hat{R}_{142}^{(2)} = [1, 2, 3, 4, 5].
\]

Let’s consider a component expression proportional to \(\chi_{5}^{-}\chi_{5}^{-}\chi_{2}^{+}\chi_{3}^{+}\). Note also that \((\chi_{2}^{+} = \chi_{3}^{+} = \chi_{4}^{+} = 0)\) coefficient before \(\chi_{5}^{-}\chi_{2}^{+}\chi_{3}^{+}\) should be equivalent to coefficient before \(\chi_{1}^{-}\chi_{2}^{+}\chi_{4}^{+}\) due to the periodical nature of the contour. Extracting corresponding components we see that (here we drop \(\mp\) subscript):

\[
[1, 2, 3, 4, 5]|_{\chi_{2}^{2}\chi_{2}} = \frac{\langle 1, 2, 3, 4 \rangle}{\langle 3, 4, 5, 2 \rangle \langle 5, 1, 2, 3 \rangle},
\]

28
and
\[ c_2^{(4)}[1, 2, 3, -2, -1] \bigg|_{\chi_1^{x-2x-1}} = \left(\mathbb{P}^4 c_2^{(4)} \right) [2, 3, 5, 6, 7] \bigg|_{\chi_2^{x+2x+3}} = \frac{(2, 5, 6, 7)}{(1, 2, 3, 5)(2, 3, 5, 6)}. \]

(5.106)

So for the form factor we have
\[ Z_{X}^{NMHV} / Z_{X}^{MHV} \bigg|_{\chi_1^{x-2x-1}} = \frac{1}{(1, 2, 3, 5)} \left( \langle 1, 2, 3, 4 \rangle \langle 2, 3, 4, 5 \rangle + \langle 2, 5, 6, 7 \rangle \right). \]

(5.107)

\[ \langle 1, 2, 3, 5 \rangle \sim \langle 1 \mid q \mid 2 \rangle, \] and we see that if the expression in the brackets vanishes as \( \langle 1, 2, 3, 5 \rangle \rightarrow 0 \), then \( \langle 1, 2, 3, 5 \rangle \) pole cancels exactly as in [5] example. Using identity for 6 twistors \( Z_1, \ldots, Z_5, Z_X \):
\[ \langle 2, 3, 1, 4 \rangle \langle 2, 3, 5, X \rangle + \langle 2, 3, 1, 5 \rangle \langle 2, 3, 4, X \rangle + \langle 2, 3, 1, X \rangle \langle 2, 3, 4, 5 \rangle = 0 \]

(5.108)

One can see that as \( \langle 1, 2, 3, 5 \rangle \rightarrow 0 \)
\[ \frac{\langle 2, 3, 1, 4 \rangle}{\langle 2, 3, 4, 5 \rangle} = \frac{\langle 2, 3, 1, X \rangle}{\langle 2, 3, 5, X \rangle}. \]

(5.109)

This identity is valid for arbitrary 6 twistors, so we can choose \( Z_X = Z_6 \). Noticing that for the periodical configuration the following identity holds
\[ \langle i + n, i + 1 + n, i + 2 + n, i + 1 + n \rangle = \langle i, i + 1, i + 2, i + 1 + n \rangle, \]

(5.110)

and using it in our case \( \langle 5, 6, 7, 2 \rangle = \langle 1, 2, 3, 6 \rangle \) one can see that indeed as \( \langle 1, 2, 3, 5 \rangle \rightarrow 0 \) expression in brackets cancels. This is a good sign, but one would like to have more general statement regarding the spurious pole cancelation.

Transforming BCFW representation into CSW we recast all BCFW spurious poles into poles containing \( Z^* \) twistor: \( \langle *, a, b, c \rangle \). We also get rid of terms with coefficients \( c_i^{(n)} \), so our answer is represented only as the sum of \( \langle *, a, b, c, d \rangle \) invariants.

In the amplitude case one can use geometrical interpretation of the amplitude as the volumes of a polytope in \( \mathbb{C}P^4 \) to show that all poles of the form \( \langle *, a, b, c \rangle \) cancel [5]. \( \langle a, b, c, d, e \rangle \) invariant is interpreted as the volume of 4-simplex in \( \mathbb{C}P^4 \) [2, 5]. The NMHV amplitude is the sum of volumes of such 4-simplices and hence can be interpreted as the volume of the polytope. 4-simplices in BCFW or all-line shift (CSW) recursion represents particular triangulation of this polytope. The poles in \( [a, b, c, d, e] \) are "brackets" of the form \( \langle a, b, c, d \rangle \) which correspond to the vertexes of the 4-simplex in the geometrical picture. Cancellation of spurious poles can be seen in this picture as "cancellation" of contribution of corresponding vertexes: 4-simplices will combine into polytope (amplitude) in such a way that the resulting polytope (amplitude) will have only such vertexes that correspond to the physical poles.

29
Our aim now is to show that the same ideas about the spurious pole cancellation can be applied to the form factors as well, with some minor, but curious, changes.

First of all, let’s explain how one can rewrite \([a, b, c, d, e]\) invariants as volumes of \(\mathbb{CP}^4\) simplexes in the case when we are dealing with harmonic superspace. Let’s introduce new fermionic variables \(X_{a+}^i\) and \(X_{a-}^i\)

\[
X_{a+}^i = \psi_i^{(-)}, \quad X_{a-}^i = \psi_i^{(+)} \tag{5.111}
\]

such that \(\psi_i^{(-)} = \psi_i^{(+)}.\) Here \((\pm)\) subscript stands to distinguish dependence of \(\psi\) and other objects on \(\chi^+\) or \(\chi^-\). Then we can introduce 5 component objects which we will treat as the set of homogeneous coordinates on \(\mathbb{CP}^4\)

\[
Z_i^{(\pm)} = (Z_i, \psi_i^{(\pm)}) - 5 \text{ component object,} \tag{5.112}
\]

and

\[
Z_0 = (0, 0, 0, 0, 1), \tag{5.113}
\]

such that

\[
\delta^{\pm 2} (\chi_a^{\pm} (b, c, d, e) + \text{cycl.}) = \frac{1}{2!} \int d^{\pm 2}X \langle a, b, c, d, e \rangle^{2(\pm)},
\]

\[
\langle a, b, c, d \rangle = \langle 0, a, b, c, d \rangle. \tag{5.114}
\]

Where

\[
\langle a, b, c, d, e \rangle^{(\pm)} = \epsilon_{q_1q_2q_3q_4q_5} Z_a^{(\pm)q_1} Z_b^{(\pm)q_2} Z_c^{(\pm)q_3} Z_d^{(\pm)q_4} Z_e^{(\pm)q_5}, \tag{5.115}
\]

\[
\langle 0, b, c, d, e \rangle^{(\pm)} = \epsilon_{q_1q_2q_3q_4q_5} Z_0^{(\pm)q_1} Z_b^{(\pm)q_2} Z_c^{(\pm)q_3} Z_d^{(\pm)q_4} Z_e^{(\pm)q_5}. \tag{5.116}
\]

Since \(\psi_i^{(-)} = \psi_i^{(+)}\) we have \(\langle a, b, c, d, e \rangle^{2(\pm)} = \langle a, b, c, d, e \rangle^{2(+)}\), so

\[
\delta^4 (\chi_a (b, c, d, e) + \text{cycl.}) = \frac{4!}{2!2!} \int d^{-2}X d^{+2}X \frac{1}{4!} \langle a, b, c, d, e \rangle^4, \tag{5.117}
\]

and we can rewrite \([a, b, c, d, e]\) in the following way (\(\int_X \equiv 4!/2!2! \int d^{-2}X d^{+2}X\)):

\[
[a, b, c, d, e] \equiv \int_X \frac{1}{4!} \langle a, b, c, d, e \rangle^4 \langle 0, a, b, c, d \rangle \langle 0, b, c, d, e \rangle \langle 0, c, d, e, a \rangle \langle 0, d, e, a, b \rangle \langle 0, e, a, b, c \rangle. \tag{5.118}
\]

Comparing this with the formula for the volume of 4-simplex in \(\mathbb{CP}^4\)

\[
Vol_4[a, b, c, d, e] = \frac{1}{4!} \langle a, b, c, d, e \rangle^4 \langle 0, a, b, c, d \rangle \langle 0, b, c, d, e \rangle \langle 0, c, d, e, a \rangle \langle 0, d, e, a, b \rangle \langle 0, e, a, b, c \rangle. \tag{5.119}
\]
we see that

\[ [a, b, c, d, e] = \int_X Vol_4[a, b, c, d, e]. \quad (5.120) \]

One can see that the NMHV amplitude is given by the sum of \( Vol_4 \). Let's also write for comparison general formula for volume of simplex in \( \mathbb{C}P^n \)

\[ Vol_n(a_1, \ldots, a_{n+1}) = \frac{1}{n!} \langle a_1, \ldots, a_{n+1} \rangle^n \langle 0, a_1, \ldots, a_n \rangle \ldots \langle 0, a_{n+1}, a_1, \ldots, a_{n-1} \rangle. \quad (5.121) \]

To get some geometrical intuition how this volume formula works let's consider \( \mathbb{C}P^2 \) case:

\[ Vol_2[a, b, c] = \frac{1}{2!} \frac{\langle a, b, c \rangle^2}{\langle 0, a, b \rangle \langle 0, b, c \rangle \langle 0, c, a \rangle}. \quad (5.122) \]

3 component objects \( Z^I_a, Z^I_b, Z^I_c, I = 1, \ldots, 3 \), which are homogeneous coordinates on \( \mathbb{C}P^2 \) define 3 lines in dual \( \mathbb{C}P^2 \) space, with coordinates \( W_I \), via conditions \( (Z^I_a W) \equiv Z^I_a W_I = 0 \). In \( \mathbb{C}P^n \) \( Z \) will define \( n - 1 \) subspace. In the \( \mathbb{C}P^2 \) case these lines, defined by \( Z^I_a, Z^I_b, Z^I_c \) intersect at points

\[
\begin{align*}
W_{1I} &= W_{(ab)I} = \epsilon_{IJK} Z^J_a Z^K_b, \\
W_{2I} &= W_{(bc)I} = \epsilon_{IJK} Z^J_b Z^K_c, \\
W_{3I} &= W_{(ca)I} = \epsilon_{IJK} Z^J_c Z^K_a.
\end{align*}
\]

(5.123)

These points are projected on a plane, defined by \( Z^I_0 \), and one can think of them as vertexes of 2d triangle (two dimensional simplex), with edges defined by \( Z^I_a, Z^I_b, Z^I_c \). \( Vol_2[a, b, c] \) is projectively defined (it is invariant under rescalings of \( Z^I \rightarrow \lambda Z^I \) or \( W_I \rightarrow \lambda W_I \), while \( Z^I_0 \) is always fixed, \( \lambda \) is some number) area of this triangle. The vertexes of this triangle are in one to one correspondence with \( \langle 0, a, b \rangle \), etc. "scalar products". In terms of \( W \)'s the \( Vol_2[a, b, c] \) is given by \( \langle (Z^I_0 W_I) = \langle 0, a, b \rangle \), etc.\)

\[ Vol_2[a, b, c] = \frac{1}{2!} \frac{\langle W_1, W_2, W_3 \rangle}{\langle Z^I_0 W_I \rangle \langle Z^I_0 W_2 \rangle \langle Z^I_0 W_3 \rangle}. \quad (5.124) \]

Using projective invariance \( W_I \rightarrow \lambda W_I \) one can always choose \( W_1, W_2, W_3 \) in the form \( W_1 = (x_1, y_1, 1), W_2 = (x_2, y_2, 1), W_3 = (x_3, y_3, 1) \). \( x_i, y_i \) are then the coordinates of vertexes of \( (a, b, c) \) triangle in the plane defined by \( Z^I_0 \).

Situation when one of the brackets in denominator (for example \( \langle 0, a, b \rangle = 0 \)) is becoming equal to 0 corresponds in general to the case when \( W_1 \) point moves to infinity so that the \( Vol_2[a, b, c] \) becomes singular (infinite).

\[ \text{\footnote{We are considering projective geometry, so if one will consider } W \text{ as points in 3 dimensional affine spaces } W, \text{ condition } (ZW) = 0, \text{ for fixed } Z \text{ defines a plane in } W. \text{ Intersection of this plane with the plane defined by } Z^I_0 \text{ gives us line, which we are talking about.}} \]
In the \( \mathbb{CP}^4 \) case, we are really interested in, \( Z \) twistors define three dimensional subspaces in dual \( \mathbb{CP}^4 \) space. Intersections of these three dimensional subspaces define vertexes of four dimensional simplex. The vertexes of this simplex are in one to one correspondence with \( \langle 0, a, b, c, d \rangle = \langle a, b, c, d \rangle \) poles.

To see how one can observe cancellation of poles (vertexes) in this geometrical picture, let’s return to \( \mathbb{CP}^2 \) example [2]. Let’s consider two triangles defined by \( Z_1, Z_2, Z_3 \) and \( Z_1, Z_3, Z_4 \). In the difference \( \text{Vol}_2[1, 2, 3] - \text{Vol}_2[1, 4, 3] \) (\( \text{Vol}_2[1, 4, 3] = -\text{Vol}_2[1, 3, 4] \)) contribution of \( \langle 0, 1, 3 \rangle \) vertex will drop out, so the difference is regular in the \( \langle 0, 1, 3 \rangle \to 0 \) limit. See fig.16.

To see this cancellation in more algebraic way, without drawing pictures, which is very convenient when we are dealing with four dimensional volumes, let’s introduce a boundary operator \( \partial \) for simplex in \( \mathbb{CP}^n \) which will give the volume of the boundary of this simplex (i.e. combination of volumes of simplexes in \( \mathbb{CP}^{n-1} \)) [2]:

\[
\partial \text{Vol}_n[1, 2, 3, ..., n] = \sum_{i=1}^{n} (-1)^{i+1} \text{Vol}_{n-1}[1, 2, ..., i - 1, i + 1, ..., n]|Z_i. \quad (5.125)
\]

One can verify that as expected \( \partial^2 = 0 \). \( \text{Vol}_{n-1}[1, 2, ..., i - 1, i + 1, ..., n]|Z_i \) is defined as projection of \( (1, 2, ..., i - 1, i + 1, ..., n) \) lines into \( n - 1 \) dimensional subspace defined by \( Z_i \). Returning to \( \mathbb{CP}^2 \) case one can see that

\[
\partial \text{Vol}_2[1, 2, 3] = \text{Vol}_1[2, 3]|Z_1 - \text{Vol}_1[1, 3]|Z_2 + \text{Vol}_1[1, 2]|Z_3,
\]

\[
\partial \text{Vol}_2[1, 3, 4] = \text{Vol}_1[4, 3]|Z_1 - \text{Vol}_1[1, 4]|Z_2 + \text{Vol}_1[1, 3]|Z_3. \quad (5.126)
\]

The boundaries (line segments) of the triangles \( \text{Vol}_1[1, 3]|Z_2 \) and \( \text{Vol}_1[1, 3]|Z_4 \) corresponding to \( \langle 013 \rangle \) vertex (pole) encounters with the opposite sign. This corresponds to the situation when such vertex is absent in the final polytope (sum of simplexes). The same will be true in the general case of the sum of the simplexes in \( \mathbb{CP}^{n-1} \).

In summary [2, 5] one can say that to figure out which vertexes (poles) will be present in the polytope combined from the set of simplexes one has to act with the boundary operator \( \partial \) on each simplex and ”cancel” all vertexes with the opposite sign ignoring \( |Z_i \) subscript. Hereafter we will drop \( |Z_i \) subscript.
Figure 16: Cancellation of (1, 3) pole in $\text{Vol}_2[1, 4, 3] - \text{Vol}_2[1, 2, 3]$.

As an example one can check that in the case of all-line shift (CSW) representation of $n = 5$ NMHV amplitude in the result of the action of the boundary operator on the individual simplexes all the poles (vertexes) of the form $(0, *, a, b, c)$ “cancel” and only physical poles of the form $(0, a, b, c, d)$ remain. This also reflects the fact that the result should be independent of the explicit choice of $\mu^*$ in $\mathcal{Z}^*$. In fact in the case of amplitudes one can see that the result is independent of the choice of all components in $\mathcal{Z}^*$ recasting the all-line shift (CSW) representation into BCFW one using 6 term identity.

Now let’s return to the form factors. Due to the presence of $\gamma^+ = 0$ condition on the periodical contour (fermionic part $\chi^i_+ \text{ of the contour is closed}) \psi_i^(-) \neq \psi_i^+(+).$ So in the case of the form factors one can write

$$[a, b, c, d, e] \equiv \int_X \frac{1}{4!} \frac{\langle a, b, c, d, e \rangle^{(-)}^2}{\langle a, b, c, d, e \rangle^{(+)}^2} \times \frac{\langle 0, a, b, c, d, e \rangle^{(-)} \langle 0, c, d, e, a \rangle \langle 0, d, e, a, b \rangle \langle 0, e, a, b, c \rangle^{(+)}^{1/2}}{\langle 0, a, b, c, d, e \rangle^{(+)}}.$$  \hspace{1cm} (5.127)

The only difference in $(\text{Vol}_4[a, b, c, d, e]^{(-)})^{1/2}$ and $(\text{Vol}_4[a, b, c, d, e]^{(+)})^{1/2}$ is fermionic components $\chi_i^+$ and $\chi_i^-$. Because it is not convenient to work with square roots of volumes one can consider auxiliary objects where $\gamma^-$ and $\gamma^+$ (hence $\chi_i^+$ and $\chi_i^-$) enters on the equal footing and the limit $\gamma^+ \to 0$ is taken only in the final result. As it was explained before this limit is not singular. If some poles cancel in the sum of [a, b, c, d, e] before $\gamma^+ \to 0$ limit they also should cancel after this limit is taken. [a, b, c, d, e] are rations of polynomials. So if in the sum of such rations of polynomials some poles of individual terms cancel, taking one coefficient to 0 in the numerators of such polynomials should not affect pole
cancellation. From this point of view NMHV form factor is not exactly $\mathbb{CP}^4$ polytope but rather its special limit ($\gamma^+ \to 0$).

Let’s now consider three point NMHV form factor (here we choose the contour periods as in \cite{22})

\[
Z_3^{NMHV} / Z_3^{MHV} = [*, 0, 1, 2, 3] + [*, 1, 2, 3, 4] + [*, 2, 3, 4, 5].
\]

(5.128)

Considering $\gamma^+ \neq 0$ let’s apply the boundary operator to the individual terms:

\[
\begin{align*}
\partial Vol_3[*, 0, 1, 2, 3] &= Vol_3[0, 1, 2, 3] - Vol_3[*, 1, 2, 3] + (Vol_3[*, 0, 2, 3] - Vol_3[*, 0, 1, 3]) \\
&+ Vol_3[*, 0, 1, 2], \\
\partial Vol_3[*, 1, 2, 3, 4] &= Vol_3[1, 2, 3, 4] - Vol_3[*, 2, 3, 4] + (Vol_3[*, 1, 3, 4] - Vol_3[*, 1, 2, 4]) \\
&+ Vol_3[*, 1, 2, 3], \\
\partial Vol_3[*, 2, 3, 4, 5] &= Vol_3[2, 3, 4, 5] - Vol_3[*, 3, 4, 5] + (Vol_3[*, 2, 4, 5] - Vol_3[*, 2, 3, 5]) \\
&+ Vol_3[*, 2, 3, 4]. \\
\end{align*}
\]

(5.129)

We see that poles corresponding to $Vol_3[*, 0, 2, 3], Vol_3[*, 0, 1, 3], Vol_3[*, 1, 3, 4], Vol_3[*, 1, 2, 4], Vol_3[*, 2, 3, 5], Vol_3[*, 2, 4, 5]$ do not cancel in such auxiliary object. All other poles cancel (Note that $Vol_3[*, 3, 4, 5]$ and $Vol_3[*, 0, 1, 2]$ corresponds to the same pole in the $n = 3$ case). We also see that

\[
Vol_3[*, 1, 3, 4] - Vol_3[*, 1, 2, 4] = \mathbb{P}(Vol_3[*, 0, 2, 3] - Vol_3[*, 0, 1, 3]),
\]

(5.130)

and

\[
Vol_3[*, 2, 4, 5] - Vol_3[*, 2, 3, 4] = \mathbb{P}^2(Vol_3[*, 0, 2, 3] - Vol_3[*, 0, 1, 3]).
\]

(5.131)

So if these poles cancel in the first term, they will cancel in other terms as well.

Now what will change if we take $\gamma^+ \to 0$ limit? First off all let’s note that $(0, *, 0, 2, 3) = (*, 0, 2, 3)$ and $(0, *, 0, 1, 3) = (*, 0, 1, 3)$ vertexes in fact correspond to the same pole $[|q⟩|3]$:

\[
\begin{align*}
\langle *, 0, 2, 3 \rangle &= \langle 23 | * | x_{30} | 3 \rangle = \langle 23 | * | q | 3 \rangle, \\
\langle *, 0, 1, 3 \rangle &= \langle 31 | * | x_{13} | 3 \rangle = \langle 31 | * | q | 3 \rangle.
\end{align*}
\]

(5.132)

Formulas

\[
\langle *, i - 1, i, j \rangle = \langle i - 1 | * | x_{ij} | j \rangle, \quad x_{ij} = \sum_{k=i}^{j-1} p_k,
\]

(5.133)

\[
Z_i = (\lambda_i, x_i \lambda_i), \quad Z_{i+nk} = (\lambda_i, x_{i+nk} \lambda_i), \quad i = 1...n, \ k \in \mathbb{N}.
\]

(5.134)
were used. Now let’s consider the argument of $\hat{\delta}^{+2}$ function in $[*0123]$ in more details. The argument looks like (note that $\chi^+_3 = 0$)
\[
\chi^+_3(*012) + \chi^+_0(*123) + \chi^+_1(*023) + \chi^+_2(*013).
\]
(5.135)

$\gamma^+ = 0$ corresponds to $\chi^+_0 = \chi^+_3$, so we can write the argument of delta function as
\[
\chi^+_3((*012) + (*123)) + \chi^+_1(*023) + \chi^+_2(*013).
\]
(5.136)

For $(*012)$ and $(*123)$ one can get
\[
\langle *012 \rangle + \langle *123 \rangle = [\!*|q|3]\langle 12 \rangle,
\]
(5.137)

so $[\!*|q|3]$ factors out from the delta function and one can see that $[\!*|q|3]^2 \sim \hat{\delta}^{+2}$. The poles $(*023) \sim [\!*|q|3]$ and $(*013) \sim [\!*|q|3]$ exactly cancel! This is similar to the cancellation of $q^2$ pole in $\hat{R}^{(3)}_{\mu_i}$. Note that such factorisation is possible only in $\gamma^+ \to 0$ limit. $\hat{\delta}^{-2}$ dose not factories in such a way. From geometrical point of view this means that as $[\!*|q|3] \to 0 \text{ Vol}_4[*0123]^{(-)}$ becomes singular, while $\text{Vol}_4[*0123]^{(+)} \to 0$ in the way that their product remains finite. Such cancellation of the poles is the general pattern for all $[\!*a,b,c,d]$ coefficients with $a = i$ and $b = i \pm n$ for the $n$ point form factor.

For the general $n$ the situation is the same as in $n = 3$ example and all poles containing $\mu^*$ dependance, except pairs of poles which come from $[\!*a,b,c,d]$ coefficients with $a = i$ and $b = i \pm n$, cancel already in the axillary expression with $\gamma^+ \neq 0$. Remaining pairs of poles cancel in $\gamma^+ \to 0$ limit. In appendix one can find details on $n = 4$ example.

Summing up for $Z_{nmHV}/Z_{nmHV}$ in all-line shift (CSW) representation the answer is free from poles containing $Z^*$ dependance which also imply cancellation of spurious poles in BCFW picture and independance of CSW result on the choice of $\mu^*$. This cancellation most easily can be seen geometrically, when we are representing $[\!*a,b,c,d]$ invariants as the volumes or the products of volumes of the simplexes in $\mathbb{CP}^4$. This situation is similar to the amplitude case, but there are some differences unique to the form factors due to their special Grassmann structure. The form factor is not exactly $\mathbb{CP}^4$ polytope but rather special limit ($\gamma^+ \to 0$) of such polytope.

6 Conclusion

In this article we considered different types of recursion relations for the form factors of operators from the stress tensor supermultiplet in $\mathcal{N} = 4$ SYM theory. We formulated BCFW recursion relations in momentum twistor space for general helicity configuration and considered NMHV sector in more details. Using the momentum twistor space representation we demonstrated equivalence between BCFW and all-line shift (CSW) recursion relations at least for NMHV sector and used geometrical interpretation of NMHV form factors as the volumes of polytopes to show that BCFW/CSW representations of the
form factors are free from spurious poles. Relations between logarithmical derivative of the form factor with respect to coupling constant and the amplitudes were also considered. In addition we briefly discussed how momentum twistor representation can be used to clarify relations between IR pole coefficients at one loop level. We hope that similar ideas can be used beyond NMHV sector.

The main conceptual result of this article is that ”on-shell structures and ideas” such as momentum twistor representation, Yangian momentum twistor invariant function \([abcde]\) or polytope interpretation of NMHV amplitudes still play an essential role for partially off-shell objects such as the form factors (or at least for the form factors of operators from the stress tensor supermultiplet). However several important questions still remain unanswered.

It is well known that different BCFW shifts give representations of the same amplitude which looks different at the first glance. For example, for NMHV sector six point amplitude we have for \([1,2]\) shift:

\[
\frac{A_{6}^{\text{NMHV}}}{A_{6}^{\text{MHV}}} = [1,2,3,4,5] + [1,2,3,5,6] + [1,3,4,5,6], \tag{6.138}
\]

while for \([2,3]\) shift:

\[
\frac{A_{6}^{\text{NMHV}}}{A_{6}^{\text{MHV}}} = \mathbb{P}([1,2,3,4,5] + [1,2,3,5,6] + [1,3,4,5,6])
\]

\[
= [6,1,2,3,4] + [6,1,2,4,5] + [6,2,3,4,5]. \tag{6.139}
\]

In general case equivalence between different BCFW representations can be shown using representation of the amplitude as an integral over Grassmannian and residues theorems for functions of multiple complex variables \([46]\). The case \(n = 6\) may be also seen as the manifestation of six term identity

\[
0 = [1,2,3,4,5] + [1,2,3,5,6] + [1,3,4,5,6]
\]

\[
- \mathbb{P}([1,2,3,4,5] + [1,2,3,5,6] + [1,3,4,5,6]), \tag{6.140}
\]

for \([a,b,c,d,e]\) functions, which can be interpreted as ”boundary of 5-simplex in \(\mathbb{C}P^4 = 0\)” in the polytope picture. In the case of the form factor we have similar relations between \([a,b,c,d,e]\) functions in special kinematics \((\gamma^+ = 0)\). For \([1,2]\) shift one can get:

\[
\frac{Z_{4}^{\text{NMHV}}}{Z_{4}^{\text{MHV}}} = (S_{2}^{(3)})[{}^{-1},0,1,3,4] + [1,2,3,4,5] + [1,2,3,0,-1] + c_{2}^{(4)}[1,2,3,-2,-1], \tag{6.141}
\]

while for \([2,3]\) shift:

\[
\frac{Z_{4}^{\text{NMHV}}}{Z_{4}^{\text{MHV}}} = \mathbb{P}\left((S_{2}^{(3)})[{}^{-1},0,1,3,4] + [1,2,3,4,5] + [1,2,3,0,-1] + c_{2}^{(4)}[1,2,3,-2,-1]\right). \tag{6.142}
\]
and as the consequence
\[ 0 = (S c_2^{(3)})[-1, 0, 1, 3, 4] + [1, 2, 3, 4, 5] + [1, 2, 3, 0, -1] + c_2^{(4)}[1, 2, 3, -2, -1] \]
\[ - P\left((S c_2^{(3)})[-1, 0, 1, 3, 4] + [1, 2, 3, 4, 5] + [1, 2, 3, 0, -1] + c_2^{(4)}[1, 2, 3, -2, -1]\right). \]

(6.143)

Is there any geometrical picture behind such identities (see also (5.94))?

It would be interesting to find representations for the form factors as an integral over Grassmannian [46] similar to the amplitudes\(^8\) case:

\[ A_n^{(0)(k)} = \int \frac{d^nxk C_{al}}{Vol(GL(k))} \prod_{a=1}^{k} \delta^{4|4} \left( \sum_{l=1}^{n} C_{al} \mathcal{W}_l^A \right), \]

(6.144)

or prove that such representation is impossible. Such representation is the first step in the on-shell diagram formalism [5], which may be very useful for the form factors as well as for the amplitudes. Representation of ratio of NMHV and MHV form factors as the sum of \([*, a, b, c, d]\) functions gives hope that such Grassmannian integral representation is possible.

It would be interesting to formulate recursion relations for the integrand of the form factors at loop level. The form factors of operators from the stress tensor supermultiplet naturally involve none planar contributions starting from two loops, so to formulate such recursion relations one must incorporate none planarity.

And also it would be interesting to continue the investigation of the form factors/Wilson loop duality. One can hope that results obtained in this article will be useful in mentioned above quests.

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**A \ \mathcal{N} = 4\ harmonic\ superspaces**

Standard \( \mathcal{N} = 4\) coordinate superspace is convenient to describe supermultiplets of fields or local operators. It is parameterized by the following coordinates:

\[ \mathcal{N} = 4\ coordinate\ superspace = \{ x^{\alpha \dot{\alpha}}, \theta^A, \bar{\theta}_{\dot{A}} \}, \]

(A.145)
where \( x, a \) are ordinary coordinates, which are bosonic variables and \( \theta \)'s are additional fermionic coordinates. \( A \) is \( SU(4)_R \) index, \( \alpha, \dot{\alpha} \) are Lorentz \( SL(2, C) \) indices.

The \( \mathcal{N} = 4 \) supermultiplet of fields (containing \( \phi^{AB} \) scalars, \( \psi^A_{\alpha} \), \( \bar{\psi}^A_{\dot{\alpha}} \) fermions and \( F^{\mu\nu} \) the gauge field strength tensor, all in the adjoint representation of \( SU(N_c) \) gauge group) is realised in \( \mathcal{N} = 4 \) coordinate superspace as the constrained superfield \( W^{AB}(x, \theta, \bar{\theta}) \) with the lowest component \( W^{AB}(x, 0, 0) = \phi^{AB}(x) \). \( W^{AB} \) in general is not a chiral object and satisfies several constraints: a self-duality constraint

\[
W^{AB}(x, \theta, \bar{\theta}) = W^{AB}(x, \theta, \bar{\theta}) = \frac{1}{2} \epsilon^{ABCD} W^{CD}(x, \theta, \bar{\theta}),
\]

which implies \( \phi^{AB} = \bar{\phi}^{AB} = \frac{1}{2} \epsilon^{ABCD} \phi^{CD} \) and two additional constraints

\[
D^\alpha_A W^{AB}(x, \theta, \bar{\theta}) = -\frac{2}{3} \delta^\alpha_A D^\alpha_B W^{BL}(x, \theta, \bar{\theta}),
\]

\[
\bar{D}^{\dot{\alpha}}(C W^{AB})(x, \theta, \bar{\theta}) = 0,
\]

where \( D^\alpha_A \) is a standard coordinate superspace derivative. Note that in this formulation the full \( \mathcal{N} = 4 \) supermultiplet of fields is on-shell in the sense that the algebra (more precisely the last two anti-commutators) of the generators \( Q_A^B, \bar{Q}^\dot{B}_A \) for the supersymmetric transformation of the fields in this supermultiplet

\[
\{ Q_A^B, \bar{Q}^\dot{B}_A \} = 2 \delta^B_A P_{a\dot{a}}, \quad \{ Q_A^B, Q_B^C \} = 0, \quad \{ \bar{Q}^\dot{A}_a, \bar{Q}^\dot{B}_{\dot{a}} \} = 0
\]

(A.148)

is closed only if the fields obey their equations of motion (in addition the closure of the algebra requires the compensating gauge transformation [33]).

The off-shell formulation of full \( \mathcal{N} = 4 \) supermultiplet is still unknown. But fortunately the self dual (chiral) sector of full \( \mathcal{N} = 4 \) supermultiplet can be formulated off-shell. In \( SU(4)_R \) covariant way this can be done using \( \mathcal{N} = 4 \) harmonic superspace [34, 35].

The \( \mathcal{N} = 4 \) harmonic superspace is obtained by adding additional bosonic coordinates (harmonic variables) to the \( \mathcal{N} = 4 \) coordinate superspace or on-shell momentum superspace. These additional bosonic coordinates parameterize the coset

\[
SU(4) / SU(2) \times SU(2)' \times U(1)
\]

(A.149)

and carry the \( SU(4) \) index \( A \), two copies of \( SU(2) \) indices \( a, \dot{a} \) and \( U(1) \) charge \( \pm \)

\[
(u^+_A, u^-_A) \quad \text{and c.c. once } (\bar{u}^-_a, \bar{u}^+_\dot{a}).
\]

(A.150)

Using these variables one presents all the Grassmann objects with \( SU(4)_R \) indices. Grassmann coordinates in the original \( \mathcal{N} = 4 \) coordinate superspace then can be transformed as

\[
\theta^+_{\alpha} = u^+_A \theta^A_{\alpha}, \quad \theta^-_{\dot{\alpha}} = u^-_A \theta^A_{\dot{\alpha}},
\]

(A.151)

\[\text{[9]}^{9}[* , *] \text{ denotes antisymmetrization in indices, while } ( , ) \text{ denotes symmetrization in indices.}\]

\[\text{[10]}^{10}\text{ which is } D^A_A = \partial/\partial \theta^A_{\alpha} + i \theta^{A\alpha} \partial/\partial x^{a\dot{\alpha}}.\]
\[ \tilde{\theta}_{a\dot{a}}^- = \tilde{u}_{a}^{-A} \tilde{\theta}_{a\dot{a}}, \quad \tilde{\theta}_{a\dot{a}}^+ = \tilde{u}_{a}^{+A} \tilde{\theta}_{a\dot{a}}, \quad \text{(A.152)} \]

and in the opposite direction

\[ \theta_{a}^A = \theta_{a}^{+a} \tilde{u}_{a}^{-A} + \theta_{a}^{-a'} \tilde{u}_{a}^{+A}, \quad \text{(A.153)} \]

\[ \tilde{\theta}_{a\dot{a}} = \tilde{\theta}_{a\dot{a}}^+ u_{a}^{-a'} + \tilde{\theta}_{a\dot{a}}^- u_{a}^{+a}. \quad \text{(A.154)} \]

The same is true for supercharges:

\[ Q_{a\alpha} \rightarrow (Q_{a\alpha}, \bar{Q}^+_{a\dot{\alpha}}), \quad \tilde{Q}_{\dot{a}}^A \rightarrow (\bar{Q}^A_{\dot{a}}, \tilde{Q}^+_{\dot{a}}). \quad \text{(A.155)} \]

So the \( \mathcal{N} = 4 \) harmonic superspace is parameterized with the following set of coordinates

\[ \mathcal{N} = 4 \text{ harmonic superspace } \equiv \{ x^{a\dot{a}}, \theta_{a\alpha}, \theta_{a\dot{a}}^- \alpha, \bar{\theta}_{a\dot{a}}^+ \dot{\alpha}, \bar{\theta}_{a\dot{a}}^- u \}. \quad \text{(A.156)} \]

Using \( u \) harmonic variables one can project the \( W^{AB} \) superfield as

\[ W^{AB} \rightarrow W^{AB} u^+_{A} u^+_{B} = \epsilon^{ab} W^{++}, \quad \text{(A.157)} \]

\[ W^{++} = W^{++}(x, \theta_{a\alpha}, \theta_{a\dot{a}}^- \alpha, \bar{\theta}_{a\dot{a}}^+ \dot{\alpha}), \quad \text{(A.158)} \]

where \( \epsilon^{ab} \) is an \( SU(2) \) totally antisymmetric tensor. This \( W^{++} \) superfield is \( SU(4)_R \) and \( SU(2) \times SU(2)' \times U(1) \) covariant, but carries \(+2\) \( U(1) \) charge.

Using harmonics one can project constraints \( \text{(A.147)} \) such that \[ D_{a\alpha} W^{++} = 0, \quad \text{(A.159)} \]

Thus, the superfield \( W^{++} \) contains the dependence on half of the Grassmannian variables \( \theta \)'s and \( \bar{\theta} \)'s:

\[ W^{++} = W^{++}(x, \theta_{a\alpha}, \theta_{a\dot{a}}^- \alpha, u). \quad \text{(A.160)} \]

Now one can put all \( \bar{\theta} = 0 \) in \( W^{++} \), corresponding supercharges e.c.t. and observe that all component fields in \( W^{++}(x, \theta_{a\alpha}, 0, u) \) are off-shell in a sense that remaining chiral part of SUSY algebra \( \{Q_{a\alpha}, \bar{Q}_{B\dot{\beta}} \} = 0 \) which acts on \( W^{++} \) is closed without using equation of motion for the component fields.

Chiral part \( T \) of the stress tensor supermultiplet can now be constructed simply as:

\[ T(x, \theta^{+}, u) = Tr(W^{++}W^{++})|_{\theta = 0}. \quad \text{(A.161)} \]

\[ ^{11} \text{Strictly speaking this is true only in the free theory (g = 0), in the interacting theory one has to replace } D^a_a, D^a_{\dot{a}} \text{ by their gauge covariant analogs, which contain superconnection, but the final result is the same.} \]
\( \mathcal{T} \) is the first operator in the series of so-called 1/2-BPS operators of the form \( Tr[(W^{++})^k] \). Its lowest component is

\[
\mathcal{T}(x, 0, u) = Tr(\phi^{++}\phi^{++}), \quad \phi^{++} = \frac{1}{2} \epsilon_{ab} u^+_A u^{++}_B \phi^{AB},
\]

and its highest component which is proportional to \((\theta^+)^4\) is the Lagrangian of \( \mathcal{N} = 4 \) SYM written in special (chiral) form. All components of \( \mathcal{T} \) can be found in [33]. Using supercarges one can write \( \mathcal{T} \) as:

\[
\mathcal{T}(x, \theta^+, u) = \exp(\theta^+a Q^\alpha_a) Tr(\phi^{++}\phi^{++}+). \quad (A.163)
\]

Also the lowest component \( \mathcal{T}(x, 0, u) \) commutes with the half of the chiral and anti-chiral supercharges of the theory:

\[
[\mathcal{T}(x, 0, u), Q^+_{\alpha a}] = 0, \quad [\mathcal{T}(x, 0, u), \bar{Q}^+_{\dot{\alpha} a}] = 0. \quad (A.164)
\]

These properties allow to determine the general Grassmann structure of the form factor [23].

Harmonic variables also can be used in on-shell momentum superspace to treat on-shell states of the theory on equal footing as operators from supermultiplets. Using harmonic variables one can write:

\[
\mathcal{N} = 4 \text{ harmonic on-shell momentum superspace } = \{\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \eta^{-}_a, \eta^{+}_{a'}, u\}. \quad (A.165)
\]

Here \( \lambda_\alpha \) and \( \tilde{\lambda}_{\dot{\alpha}} \) are \( SL(2,\mathbb{C}) \) spinors associated with momenta carried by massless state (particle): \( p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \quad p^2 = 0 \). Supercharges which act in this superspace can be represented in n-particle case as

\[
q^{-}_{a\alpha} = \sum_{i=1}^{n} \lambda_{\alpha, i} \eta^{-}_{a, i}, \quad q^{+}_{a\alpha} = \sum_{i=1}^{n} \lambda_{\alpha, i} \eta^{+}_{a, i};
\]

\[
q^{+\dot{\alpha}} = \sum_{i=1}^{n} \tilde{\lambda}_{\dot{\alpha}, i} \partial \eta^{-}_{a, i}, \quad q^{-\dot{\alpha}} = \sum_{i=1}^{n} \tilde{\lambda}_{\dot{\alpha}, i} \partial \eta^{+}_{a, i}. \quad (A.166)
\]

The Grassmann delta functions which one can encounter in this article are given by \((\langle ij \rangle \equiv \lambda_{\alpha, i} \lambda_{\beta, j}^\alpha):  

\[
\delta^{-4}(q^{-}_{a\alpha}) = \sum_{i,j=1}^{n} \prod_{a, b=1}^{2} \langle ij \rangle \eta^{-}_{a, i} \eta^{-}_{b, j}, \quad \delta^{+4}(q^{+}_{a\alpha}) = \sum_{i,j=1}^{n} \prod_{a', b'=1}^{2} \langle ij \rangle \eta^{+}_{a', i} \eta^{+}_{b', j}. \quad (A.168)
\]
\[ \delta^{-2}(X^{-a}) = \prod_{a=1}^{2} X^{-a}, \quad \hat{\delta}^{+2}(X^{+_a}) = \prod_{a=1}^{2} X^{+_a}. \]  

(A.169)

We also will use notations

\[ \delta^{-4} \delta^{+4} \equiv \delta^8, \quad \hat{\delta}^{-2} \hat{\delta}^{+2} \equiv \hat{\delta}^4. \]  

(A.170)

Using this delta functions one can rewrite MHV\(_n\) and \(\overline{\text{MHV}}_3\) amplitudes, \(R_{rst}\) functions etc. in the form nearly identical to the form they have in ordinary on-shell momentum superspace.

Grassmann integration measures are defined as

\[ d^{-2}\eta = \prod_{a=1}^{2} d\eta^{a}, \quad d^{+2}\eta = \prod_{a=1}^{2} d\eta^{+a}, \quad d^{-2}\eta d^{+2}\eta \equiv d^4\eta. \]  

(A.171)

in on-shell momentum superspace and

\[ d^{-4}\theta = \prod_{a,\alpha=1}^{2} d\theta^{-a}, \quad d^{+4}\theta = \prod_{a,\alpha=1}^{2} d\theta^{+a}, \]  

(A.172)

in ordinary superspace. \(\delta^{\pm4}\) functions can be represented as \(\hat{\delta}^{\pm2}\) functions using identity

\[ \delta^{\pm4}(q^{\pm}) = \langle lm \rangle^2 \hat{\delta}^{\pm2} \left( \eta^{\pm}_{i} + \sum_{i=1}^{n} \frac{\langle mi \rangle}{\langle ml \rangle} \eta^{\pm}_{i} \right) \hat{\delta}^{\pm2} \left( \eta^{\pm}_{m} + \sum_{i=1}^{n} \frac{\langle li \rangle}{\langle lm \rangle} \eta^{\pm}_{i} \right), \quad i \neq l, \ i \neq m. \]  

(A.173)

which can be integrated as usual Grassmann delta functions.

**B Spurious pole cancellation in \(A_{5}^{\text{NMHV}}(0)\) and \(Z_{4}^{\text{NMHV}}(0)\)**

Now let’s illustrate how the cancellation of the spurious poles can be seen on the example of \(\text{NMHV}_5\) amplitude. Let’s consider all-line shift (CSW) representation of \(\text{NMHV}_5\) amplitude:

\[ \frac{A_{5}^{\text{NMHV}}}{A_{5}^{\text{MHV}}} = [*, 1, 2, 3, 4] + [*, 2, 3, 4, 5] + [*, 3, 4, 5, 1] + [*, 4, 5, 1, 2] + [*, 5, 1, 2, 3]. \]  

(B.174)

Applying boundary operator to all terms in \(A_{5}^{\text{NMHV}}/A_{5}^{\text{MHV}}\) we get:

\[ \partial \text{Vol}_4[* , 1, 2, 3, 4] = \text{Vol}_3[1, 2, 3, 4] - \text{Vol}_3[* , 2, 3, 4] + \text{Vol}_3[*, 1, 3, 4] - \text{Vol}_3[*, 1, 2, 4] \]
\[ \partial V ol_4[*, 2, 3, 4, 5] = V ol_3[2, 3, 4, 5] - V ol_3[*, 3, 4, 5] + V ol_3[*, 2, 4, 5] - V ol_3[*, 2, 3, 5] \\
+ V ol_3[*, 2, 3, 4], \\
\partial V ol_4[*, 3, 4, 5, 1] = V ol_3[3, 4, 5, 1] - V ol_3[*, 1, 4, 5] + V ol_3[*, 1, 3, 5] - V ol_3[*, 1, 3, 4] \\
+ V ol_3[*, 3, 4, 5], \\
\partial V ol_4[*, 4, 5, 1, 2] = V ol_3[4, 5, 1, 2] - V ol_3[*, 1, 2, 5] + V ol_3[*, 1, 2, 4] - V ol_3[*, 2, 4, 5] \\
+ V ol_3[*, 1, 4, 5], \\
\partial V ol_4[*, 5, 1, 2, 3] = V ol_3[5, 1, 2, 3] - V ol_3[*, 1, 2, 3] + V ol_3[*, 2, 3, 5] - V ol_3[*, 1, 3, 5] \\
+ V ol_3[*, 1, 2, 5]. \\
\] (B.175)

We see that all terms containing \( Z \) cancel each other, which indicates that in the sum of all terms all spurious poles \((*, a, b, c)\) cancel.

Now let’s consider NMHV\(_4\) form factor. In all-line shift (CSW) representation it can be written as:

\[ Z_{4}^{NMHV}/Z_{4}^{MHV} = ([*, -1, 0, 1, 2] + [*, -1, 0, 2, 3]) + ([*, 0, 1, 2, 3] + [*, 0, 1, 3, 4]) + ([*, 1, 2, 3, 4] + [*, 1, 2, 4, 5]) + ([*, 2, 3, 4, 5] + [*, 2, 3, 5, 6]). \] (B.176)

Note also that equivalently one can rewrite last two terms as

\[ [*, 2, 3, 4, 5] = [*, -2, -1, 0, 1], [*, 2, 3, 5, 6] = [*, -2, -1, 1, 2]. \] (B.177)

Applying \( \partial \) to all this terms one can obtain:

\[ \partial V ol_4[*, -1, 0, 1, 2] = V ol_3[-1, 0, 1, 2] - V ol_3[0, 1, 2, *] + V ol_3[1, 2, *, -1] - V ol_3[2, *, -1, 0] \\
+ V ol_3[*, -1, 0, 1], \\
\partial V ol_4[*, -1, 0, 2, 3] = V ol_3[-1, 0, 2, 3] - V ol_3[0, 2, 3, *] + (V ol_3[2, 3, *, -1] - V ol_3[3, *, -1, 0]) \\
+ V ol_3[*, -1, 0, 2], \\
\partial V ol_4[*, 0, 1, 2, 3] = V ol_3[0, 1, 2, 3] - V ol_3[1, 2, 3, *] + V ol_3[2, 3, *, 0] - V ol_3[3, *, 0, 1] \\
+ V ol_3[*, 0, 1, 2], \\
\partial V ol_4[*, 0, 1, 3, 4] = V ol_3[0, 1, 3, 4] - V ol_3[1, 3, 4, *] + (V ol_3[3, 4, *, 0] - V ol_3[4, *, 0, 1]) \\
+ V ol_3[*, 0, 1, 3], \\
\partial V ol_4[*, 1, 2, 3, 4] = V ol_3[1, 2, 3, 4] - V ol_3[2, 3, 4, *] + V ol_3[3, 4, *, 1] - V ol_3[4, *, 1, 2] \\
+ V ol_3[*, 1, 2, 3], \\
\partial V ol_4[*, 1, 2, 4, 5] = V ol_3[1, 2, 4, 5] - V ol_3[2, 4, 5, *] + (V ol_3[4, 5, *, 1] - V ol_3[5, *, 1, 2]) \\
+ V ol_3[*, 1, 2, 4]. \]

42
\[\partial Vol_4[*,-2,-1,0,1] = Vol_3[-2,-1,0,1] - Vol_3[-1,0,1] + Vol_3[0,1,*,2] - Vol_3[1,*,2,-1] + Vol_3[*,-2,-1,0].\]

\[\partial Vol_4[*,-2,-1,1,2] = Vol_3[-2,-1,1,2] - Vol_3[-1,1,2] + (Vol_3[1,2,*,2] - Vol_3[2,*,2,-1]) + Vol_3[*,-2,-1,1].\]  

We see that poles corresponding to terms containing \(Z^*\) in the \((...)\) bracket "cancel" in \(\gamma^+ \to 0\) limit, while all other \(Z^*\) dependant poles "cancel" among themselves.

\section{IR pole coefficients relations}

In one loop generalized unitarity based calculations for the NMNV sector the following identities for \(R\) functions were used in \(n=4\) case:

\[\tilde{R}^{(1)}_{244} = \tilde{R}^{(1)}_{211}, \quad \tilde{R}^{(1)}_{144} = \tilde{R}^{(1)}_{311}, \quad R^{(2)}_{413} = R^{(1)}_{241}.\]  

(C.179)

We now want to show that they are transparent and easily derived in the momentum twistor variables.

Let’s start with \(\tilde{R}^{(1)}_{244} = \tilde{R}^{(1)}_{211}\). It is essentially trivial, it is the same \(R\) functions written using clockwise and anti clockwise conventions.

For \(\tilde{R}^{(1)}_{144}, \tilde{R}^{(1)}_{311}\), one can obtain (note that here legs are ordered clockwise )

\[\tilde{R}^{(1)}_{144} = \frac{\langle 1,2,4,-1\rangle \langle 4,-1,0,1\rangle}{\langle -1,0,3,4\rangle \langle 1,3,4,-1\rangle} [1,3,4,1,0] = [*,0,1,3,4] + [*,1,0,1,3] + [*,1,3,4,5].\]  

(C.180)

\[\tilde{R}^{(1)}_{311} = \frac{\langle 3,4,5,0\rangle \langle 5,0,1,3\rangle}{\langle 0,1,4,5\rangle \langle 3,-1,0,1\rangle} [3,4,5,0,1] = [*,0,1,3,4] + [*,1,3,4,5] + [*,3,-1,0,1].\]  

(C.181)

Indeed as expected \(\tilde{R}_{144} = \tilde{R}_{311}\).

For \(R^{(2)}_{413}, R^{(1)}_{241}\), we see that

\[R^{(2)}_{413} = [4,0,1,2,3], \quad \text{and} \quad R^{(1)}_{241} = [2,3,4,0,1],\]  

(C.182)

so \(R^{(2)}_{413} = R^{(1)}_{241}\) as expected.
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