A MOND PROGRAMME FROM EINSTEIN HILBERT ACTION

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Abstract

In the usual derivation of Einstein’s equation from action, the surface terms are neglected. Hawking [2] gave a derivation of the gravitational Hamiltonian keeping all surface terms. Using such surface terms Easson et.al. [3] showed that Friedmann equation could get modified and using the modified Friedmann equation they could explain the cosmic acceleration. We study the effect of surface terms on a galactic scale and find that the classical limit of the modified Friedmann equation will lead to a MOND like acceleration term.

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Two of the challenging problems in galactic dynamics and modern cosmology are the flat rotation curve of galaxies and the acceleration of the universe. According to Newton’s law of gravitation, the centrifugal acceleration \( v^2/r \) should balance the gravitational attraction \( GM(r)/r^2 \) which immediately gives \( v^2 = GM(r)/r \). The velocities of stars as well as those of galaxies in galactic clusters are not following the predicted Newtonian \( r^{-1} \) law. The velocity approximately levels off with \( r \) in the halo region. The only way to interpret this result of observation is to accept that the mass \( M(r) \) increases linearly with distance \( r \). Luminous mass distribution in the galaxy does not follow this behavior hence the hypothesis that there must be huge amounts of nonluminous matter hidden in the halo. This unseen matter is given the technical name ‘dark matter’, but so far no conclusive experimental evidence is obtained for the existence of dark matter.

Milgrom\(^1\) proposed a new way of explaining galaxy rotation curves by modifying the usual Newtonian dynamics. Such a programme is called MOdified Newtonian Dynamics (MOND). In this programme he introduced a new constant acceleration usually denoted by \( 'a_0' \), which has a value of about \( 10^{-10} m/s^2 \). The MOND approach can explain the rotation curves of galaxies as well as those of clusters to a good approximation without invoking the existence of dark matter. This theory can explain the gravitational lensing also.

In field theories, Hamiltonian can be derived from a covariant action. In general relativity the situation is a little complicated by the fact that the Einstein-Hilbert action includes surface terms and generally the surface terms are ignored in the calculation. A new development in this direction is to include surface terms in the calculations. Recently Hawking derived Hamiltonian from Einstein-Hilbert action retaining all surface terms\(^2\). With the help of such surface terms Easson et al. got a modified Friedmann equation and they were able to explain the recently observed acceleration of the universe\(^3\).

In this paper we study the classical limit of modified Friedmann equation derived by Easson et al\(^3\). We get a modified force equation and this equation is applied to a galactic scale. In the present calculation two terms are appearing in the expression for the gravitational force, one depending on the inverse of \( R \) and other depending on inverse of \( R^2 \). The inverse of \( R^2 \) term is always attractive but the first term (which depends on inverse of \( R \)) can be attractive or repulsive. We study both cases and their consequences.

We start with the covariant Lorentzian action with surface terms as introduced by
Hawking [2],

\[ I(g, \Phi) = \int_M \left[ \frac{\tilde{R}}{16\pi} + \mathcal{L}(g, \Phi) \right] + \frac{1}{8\pi} \oint K, \]  

(1)

where \( \tilde{R} \) is the Ricci scalar, \( \mathcal{L} \) is the matter Lagrangian density and \( K \) is the trace of the extrinsic curvature of the boundary, from which Friedman equation can be derived [3]:

\[ \ddot{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{3}{2\pi}H^2 + \frac{3}{4\pi}\dot{H}, \]  

(2)

where \( \dot{a} \) is the cosmological scale factor, \( H \) the Hubble constant \( \rho \) and \( P \) are cosmic density and pressure respectively. Assuming pressure to be zero for this case the above equation becomes

\[ c_1 \ddot{a} = -\frac{4\pi G}{3} \rho + c_2 \left( \frac{\dot{a}}{a} \right)^2, \]  

(3)

where \( c_1 \) and \( c_2 \) are two constants. Multiplying with \( \dot{a} \) on both sides and again with \( \dot{r} \) and assuming \( ar = R \) [4], we find

\[ \ddot{R} = -\alpha \frac{4\pi G \rho R}{3} + \beta \frac{\dot{R}^2}{R}, \]  

(4)

where \( \alpha \) and \( \beta \) are two constants. From the above equation we can write the equation for acceleration as,

\[ \ddot{R} = -\frac{GM}{R^2} + \beta \frac{\dot{R}^2}{R}. \]  

(5)

Assuming \( F = m\dot{R} \) and \( \dot{R} = V_e \) the expansion velocity of universe, which is added to the equation of force otherwise similar to Newton’s law. The final equation of force can be written as

\[ F = -\frac{GMm}{R^2} + \beta m \frac{V_e^2}{R}. \]  

(6)

This is similar to Newton’s law but, a new term appears which is similar to a centripetal acceleration which depends on cosmic expansion rate which will be very small. A force proportional to \( R^{-1} \) occurs in (MOND) [5]. This shows that expansion of the universe affects the normal motion with a very small force whose acceleration looks like a centripetal acceleration. But if it is repulsive there may be a possibility that the two terms in the right hand side may cancel each other and can have a force free region and the value of \( R \) for this case is given by

\[ R = \frac{\alpha GM}{\beta V_e^2}. \]  

(7)
This is contradictory to our common experience. But if it is attractive it gives a correction to
Newton’s law. In the galactic scale, the second term in equation (6) dominates and gives
constant rotational velocity
\[ V = V_\text{e}, \]  
thus explaining the flat rotation curves of \( V_\text{e}. \)

We now take the correction term as an attractive one. The correction term gives a
constant acceleration. If we calculate using \( R \) as the radius of cosmic horizon, \( R = c/H \)
then the correction term becomes \( \beta cH. \) From the studies of MOND we know the asymptotic
acceleration is \( cH. \) If \( \beta = 1 \) this gives the MOND acceleration term, then the equation can
be written as
\[ F = -\frac{\alpha GM m}{R^2} + \beta mcH. \]  
The second term gives the asymptotic acceleration \( cH \) of MOND \[5\].

The gravitational potential also gets modified in this case. The modified potential is
given by (for a test mass \( m \) )
\[ F = -\frac{\partial \Phi}{\partial R}, \]  
and is given by
\[ \Phi = -\frac{\alpha GM m}{R} + \frac{\beta m V_\text{e}^2}{2}. \]  
The potential depends on the cosmic expansion rate from which we can also calculate the
modified escape velocity which is given by,
\[ v = \sqrt{\frac{2\alpha GM}{R} - \beta V_\text{e}^2}. \]  
This equation shows that the escape velocity depends on cosmic expansion rate.

The Newtonian equation for planetary motion is
\[ F = m\left[ \frac{d^2 R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2 \right], \]  
and in the present case this equation gets modified as,
\[ m\left[ \frac{d^2 R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2 \right] = -\frac{\alpha GM m}{R^2} + \frac{\beta m V_\text{e}^2}{R}. \]  
This equation can be simplified as
\[ \frac{d^2 R}{dt^2} - R(\omega^2 + \beta \omega^2_\text{e}) = -\frac{\alpha GM}{R^2}, \]  
4
where $\omega_e$ is the angular velocity derived from cosmic expansion. If we call
\[ (\omega^2 + \beta \omega_e^2) = \omega_{\text{tot}}^2. \] (16)
the equation of planetary motion now takes the normal form
\[ \frac{d^2 R}{dt^2} - R\omega_{\text{tot}}^2 = -\alpha \frac{GM}{R^2}. \] (17)
This leads to the usual elliptical motion and is similar to Newtonian case. But here the correction appears in the angular velocity term and when the angular velocity $\omega$ is less than $\omega_e$ the second term dominates. Since it is very small its effect appears only to slowly moving distant planets.

Introduction of surface term modifies Friedman equation and in the classical limit we get a Newtonian analogue equation. If the correction term is attractive then it gives a result as in the case of MOND. If it is a repulsive term there exists a force free distance from the source in which gravitational attractive force exactly cancels the repulsive force. The modified equation leads to the law of planetary motion for normal angular velocities of the planet and gets modified considerably for very small angular velocities. The effect of second term in equation (6) dominates in galactic scale. It gives constant rotational velocities for stars in the halo region and thus explains the flat rotation curves of stars and galaxies.

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