Chiral Symmetry Breaking in a Uniform External Magnetic Field

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Abstract

Using the nonperturbative Schwinger-Dyson equation, we show that chiral symmetry is dynamically broken in QED at weak gauge couplings when an external magnetic field is present and that chiral symmetry is restored at temperatures above \( T_c \approx \frac{\alpha}{\pi} \sqrt{2\pi |eH|} \), where \( \alpha \) is the fine structure constant and \( H \) is the magnetic field strength.
I. INTRODUCTION

Two of us have recently proposed a method to study dynamical chiral symmetry breaking in gauge theories in the presence of an external field using the Schwinger-Dyson equation approach \[1\]. It is the purpose of this paper to describe the details of the methodology, using the case of a uniform magnetic field as an example. We also show how to adopt our formalism to study finite-temperature effects.

We use quantum electrodynamics (QED) as our model gauge theory and consider chiral symmetry breaking in the presence of a constant external magnetic field. We introduce the formalism in Section II and derive the Schwinger-Dyson equation for the fermion self-energy in the quenched, ladder approximation, Eq.(34). Using an approximation suitable for weak gauge couplings, we derive in Section III an approximate gap equation, Eq.(52), from which the infrared dynamical fermion mass, Eq.(53), is obtained. Our result is consistent with that found by Gusynin, Miransky, and Shovkovy \[2\], who used a different approach. We show in the appendix how our formalism can be applied to the approach of Gusynin \textit{et al.}, and establish the existence of the Nambu-Goldstone boson of the spontaneously broken chiral symmetry.

It has been suggested in the literature \[2\] that the chiral symmetry breaking solution may find applications in the electroweak phase transition during the early evolution of the universe. To verify this possibility, it is necessary to take into account the thermal conditions present in the early universe. Our formalism makes it easier to study such finite-temperature effects. This is discussed in Section IV where we obtain an estimate of the critical temperature for chiral symmetry breaking, Eq.(70). Our result indicates that, in order for the chiral symmetry breaking solution found here to be relevant for the electroweak phase transition, an unacceptably large magnetic field must be present at the time of the phase transition. We offer our conclusions in Section V.
II. FORMALISM

Let us consider chiral symmetry breaking in QED in the presence of a static, external electromagnetic field. The Green’s function that describes the motion of a fermion with electric charge $e$ in such an external field satisfies the equation,

$$\gamma \cdot \Pi(x)G_A(x,y) + \int d^4x' M(x,x')G_A(x',y) = \delta^{(4)}(x - y),$$

(1)

where $\Pi_\mu(x) = -i\partial_\mu - eA^\text{ext}_\mu(x)$, and $M(x,x')$ is the mass operator $\hat{M}$ in the coordinate representation: $M(x,x') = \langle x|\hat{M}|x'\rangle$. As pointed out by Ritus [3], $\hat{M}$ is a scalar $\gamma$-matrix function of the $\Pi_\mu$ and the $F_{\mu\nu} = \partial_\mu A^\text{ext}_\nu - \partial_\nu A^\text{ext}_\mu$, and for constant $F_{\mu\nu}$,

$$\hat{M} = \hat{M}(\gamma^\mu\Pi_\mu, \sigma^{\mu\nu}F_{\mu\nu}, (F_{\mu\nu}\Pi)^2, \gamma_5 F_{\mu\nu}\tilde{F}^{\mu\nu}).$$

(2)

In other words, for uniform external fields, only four independent $\gamma$-matrix valued scalars can be formed out of $\Pi_\mu$ and $F_{\mu\nu}$, as listed in Eq.(2), where $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\tau}F_{\lambda\tau}$. Furthermore, all these four scalars commute with $(\gamma \cdot \Pi)^2$; consequently,

$$[\hat{M}, (\gamma \cdot \Pi)^2] = 0.$$

(3)

It follows that the mass operator will be diagonal in the basis spanned by the eigenfunctions of $(\gamma \cdot \Pi)^2$:

$$-(\gamma \cdot \Pi)^2\psi_p(x) = p^2\psi_p(x).$$

(4)

If we work in the chiral representation in which $\Sigma_3 = i\gamma_1\gamma_2$ and $\gamma_5$ are both diagonal with eigenvalues $\sigma = \pm 1$ and $\chi = \pm 1$, respectively, the eigenfunctions of $(\gamma \cdot \Pi)^2$ has the general form

$$\psi_p(x) = E_{p\sigma\chi}(x)\omega_{\sigma\chi},$$

(5)

where $\omega_{\sigma\chi}$ are bispinors which are the eigenvectors of $\Sigma_3$ and $\gamma_5$. The exact functional form of the $E_{p\sigma\chi}(x)$ will depend on the specific external field configuration. Our method is based on the use of these eigenfunctions as basis functions. This is a natural choice as
they are the wavefunctions of the asymptotic states when a uniform external field is present. The advantage of using this representation is obvious: $\hat{M}$ can now be put in terms of its eigenvalues, so the problems arising from its dependence on the operator $\Pi$ can be avoided.

We now restrict our consideration to the case of a constant magnetic field of strength $H$, the vector potential of which may be taken to be $A_\mu^{\text{ext}} = (0, 0, Hx_1, 0)$, $\mu = 0, 1, 2, 3$. Our metric has the signature $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The eigenfunctions $E_{p\sigma\chi}(x)$ are now given by

$$E_{p\sigma}(x) = N e^{i(p_0 x_0 + p_2 x_2 + p_3 x_3)} D_n(\rho),$$

where $N$ is a normalization factor and $D_n(\rho)$ are the parabolic cylinder functions \[4\] with argument $\rho = \sqrt{2|eH|(x_1 - \frac{p_2^2}{eH})}$ and indices (which are the quantum numbers of the Landau levels)

$$n = n(k, \sigma) \equiv k + \frac{eH\sigma}{2|eH|} - \frac{1}{2}, \quad n = 0, 1, 2, ...$$

Note that, in the absence of an external electric field, the eigenfunctions do not depend on $\chi$. The eigenvalue $p$ stands for the set $(p_0, p_2, p_3, k)$, where $k$ is the quantum number of the quantized squared transverse momentum:

$$-(\gamma \cdot \Pi)\psi_p(x) \equiv -(\gamma^1\Pi_1 + \gamma^2\Pi_2)^2 \psi_p(x)$$

$$= (\Pi_1^2 + \Pi_2^2 - eH\Sigma_3)\psi_p(x)$$

$$= p^2 \psi_p(x)$$

$$= 2|eH|k\psi_p(x). \quad (8)$$

Note that $(\gamma \cdot \Pi)^2 = (\gamma \cdot \Pi_\parallel)^2 + (\gamma \cdot \Pi_\perp)^2$, where $(\gamma \cdot \Pi_\parallel)^2 \equiv (\gamma^0\Pi_0 + \gamma^3\Pi_3)^2 = \Pi_0^2 - \Pi_3^2$, hence $p^2 = -p_0^2 + p_3^2 + 2|eH|k$. The allowed values for $k$ are seen from Eq.(7) to be $k = 0, 1, 2, ...$.

Following Ritus \[3\], we form the eigenfunction-matrices $E_p(x) = \text{diag}(E_{p11}(x), E_{p-11}(x), E_{p1-1}(x), E_{p-1-1}(x))$. As noted above, in a pure magnetic field, references to $\chi$ are irrelevant and can be dropped, hence
\[ E_p(x) = \sum_{\sigma} E_{p\sigma}(x) \text{diag}(\delta_{\sigma_1, \delta_{\sigma-1}}, \delta_{\sigma_1}, \delta_{\sigma-1}) \]
\[ \equiv \sum_{\sigma} E_{p\sigma}(x) \Delta(\sigma). \]  

(9)

Using the orthogonal property of the parabolic cylinder functions \[5\],
\[ \int_{-\infty}^{\infty} d\rho D_{n'}(\rho) D_n(\rho) = \sqrt{2\pi n!} \delta_{nn'}, \]  

(10)

it is straightforward to establish that the \( E_p \) are orthonormal (\( \bar{E}_p \equiv \gamma^0 E^\dagger_p \gamma^0 \)):
\[ \int d^4x \bar{E}_{p'}(x) E_p(x) = (2\pi)^4 \delta^{(4)}(p - p') \]
\[ \equiv (2\pi)^4 \delta_{kk'} \delta(p_0 - p'_0) \delta(p_2 - p'_2) \delta(p_3 - p'_3) \]  

(11)

as well as complete:
\[ \oint d^4p E_p(x) \bar{E}_p(y) = (2\pi)^4 \delta^{(4)}(x - y), \quad \oint d^4p \equiv \sum_k \int dp_0 dp_2 dp_3 \]  

(12)

provided that the normalization constant in Eq.\((3)\) is taken to be \( N(n) = (4\pi|eH|)^{1/4}/\sqrt{n!} \).

Since the \( E_p \) are linear combinations of the eigenfunctions of the mass operator, they satisfy
\[ \int d^4x' M(x, x') E_p(x') = E_p(x) \bar{\Sigma}_A(\bar{p}), \]  

(13)

where \( \bar{\Sigma}_A(\bar{p}) \) represents the eigenvalue matrix of the mass operator. The \( E_p \) also satisfy the important property that
\[ \gamma \cdot \Pi E_p(x) = E_p(x) \gamma \cdot \bar{p}, \]  

(14)

where \( \bar{p}_0 = p_0, \bar{p}_1 = 0, \bar{p}_2 = -\text{sgn}(eH)\sqrt{2|eH|k}, \bar{p}_3 = p_3 \). Note that, due to the rotational symmetry about the direction of the magnetic field, the system is effectively a (2+1)-dimensional one, as is evident from the momentum \( \bar{p} \).

By using the above properties of the \( E_{p'} \)-functions, the fermion Green’s function may be expressed as
\[ G_A(x, y) = \oint \frac{d^4p}{(2\pi)^4} E_p(x) \frac{1}{\gamma \cdot \bar{p} + \bar{\Sigma}_A(\bar{p})} E_p(y), \]  

(15)
Eqs. (14) and (13) guarantee that Eq. (1) is satisfied. It follows that, in the $E_\nu$-representation,

$$G_A(p, p') \equiv \int d^4x d^4y E_p(x)G_A(x, y)E_{p'}(y) = (2\pi)^4 \delta(4)(p - p') \frac{1}{\gamma \cdot \bar{p} + \Sigma_A(\bar{p})}$$

which shows explicitly that the fermion propagator is diagonal (in momentum) in this representation. Similarly, the mass operator may be written in the $E_\nu$-representation as

$$M(p, p') = \int d^4x d^4x' E_p(x)M(x, x')E_{p'}(x') = (2\pi)^4 \delta(4)(p - p') \tilde{\Sigma}(\bar{p}).$$

We may now write down the Schwinger-Dyson (SD) equation for the fermion self-energy.

We shall work in the quenched, ladder approximation in which

The SD Eq. (20) can be simplified by performing the integrations over $x$, $x'$, $p_0''$, $p_2''$, and $p_3''$ exactly. Consider first the $x$-integrals. The $x_0$, $x_2$, and $x_3$-integration each yields a $\delta$-function, leaving

$$\int d^4x E_p(x)\gamma^\mu E_{p''}(x)e^{iq\cdot x} = (2\pi)^3 \delta^{(3)}(p'' + q - p) \sum_{\sigma, \sigma''} N(n)N(n'')$$

$$\cdot \int_{-\infty}^{\infty} dx_1 D_n(\rho)D_{n''}(\rho'')e^{iq_1x_1}\Delta^\sigma \gamma^\mu \Delta'',$$

where $\delta^{(3)}(p'' + q - p) \equiv \delta(p''_0 + q - p_0)\delta(p''_2 + q_2 - p_2)\delta(p''_3 - p_3)$, $\rho'' = \sqrt{2|eH|(x_1 - p''_1)}$, $n'' = n(k'', \sigma'')$, and $\Delta'' = \Delta(\sigma'')$. Due to the presence of the $\delta(p''_2 + q_2 - p_2)$, the remaining $x_1$-integral may be more conveniently written as
\[
\int_{-\infty}^{\infty} dx_1 e^{iq_1 x_1} D_n(\rho) D_{n''}(\rho'') = \frac{1}{\sqrt{2|eH|}} e^{\frac{i}{2eH}(p_1^2 + p_2^2)} I_{nn''}(\hat{q}_1, \hat{q}_2),
\]
(22)

where
\[
I_{nn''}(\hat{q}_1, \hat{q}_2) \equiv \int_{-\infty}^{\infty} d\eta e^{i \text{sgn}(eH)\hat{q}_1} D_n(\eta - \hat{q}_2) D_{n''}(\eta + \hat{q}_2),
\]
(23)

\[\eta \equiv \rho + \hat{q}_2, \text{ and } \hat{q}_\mu \text{ are dimensionless variables defined as}
\]
\[
\hat{q}_\mu \equiv \frac{q_\mu \sqrt{2|eH|}}{2eH}, \quad \mu = 0, 1, 2, 3.
\]
(24)

If we transform to the polar coordinates: \((\hat{q}_\perp \equiv \sqrt{\hat{q}_1^2 + \hat{q}_2^2}, \varphi \equiv \arctan(\hat{q}_2/\hat{q}_1))\), we can evaluate the \(I_{nn''}\) by first noting that they satisfy [3]
\[
\frac{\partial I_{nn''}(\hat{q}_\perp, \varphi)}{\partial \varphi} = i \text{sgn}(eH)(n - n'') I_{nn''}(\hat{q}_\perp, \varphi).
\]
(25)

Hence,
\[
I_{nn''}(\hat{q}_\perp, \varphi) = I_{nn''}(\hat{q}_\perp) e^{i \text{sgn}(eH)(n - n'') \varphi},
\]
(26)

where
\[
I_{nn''}(\hat{q}_\perp) \equiv \int_{-\infty}^{\infty} d\eta e^{i \text{sgn}(eH)\hat{q}_\perp} D_n(\eta) D_{n''}(\eta).
\]
(27)

Note that \(I_{nn''}(\hat{q}_\perp) = I_{n''n}(\hat{q}_\perp)\). To compute \(I_{nn''}(\hat{q}_\perp)\), we use the relation,
\[
D_n(\eta) D_{n''}(\eta) = e^{-\eta^2/4} \sum_{m=0}^{\min(n, n'')} \frac{n! n''!}{m!(n - m)! (n'' - m)!} D_{n+n''-2m}(\eta),
\]
(28)

and the Rodrigues formula,
\[
D_n(\eta) = (-1)^n e^{\eta^2/4} \frac{d^n}{d\eta^n} e^{-\eta^2/2},
\]
(29)

to secure
\[
I_{nn''}(\hat{q}_\perp) = \sqrt{2\pi} e^{-\hat{q}_\perp^2/2} J_{nn''}(\hat{q}_\perp),
\]
(30)

where
\[ J_{nn''}(\hat{q}_\perp) \equiv \sum_{m=0}^{\min(n,n'')} \frac{n!n''!}{m!(n-m)!(n''-m)!} [\text{sgn}(eH)\hat{q}_\perp]^{n+n''-2m}. \quad (31) \]

Putting all the pieces together, we have

\[
\int d^4x \tilde{E}_\mu(x) \gamma^\mu E^{\nu'}(x)e^{iq\cdot x} = (2\pi)^4 \delta^{(3)}(p'' + q - p)e^{iq_1(p''_2 + p'_2)/(2eH)} \cdot e^{-\hat{q}_\perp^2/2} \sum_{\sigma,\sigma''} \frac{1}{\sqrt{n!n''!}} e^{i\text{sgn}(eH)(n-n'')\varphi} J_{nn''}(\hat{q}_\perp) \Delta \gamma^\mu \Delta'', \quad (32)
\]

where we have used the identity, \( \gamma^0 \Delta \gamma^0 = \Delta \). Similarly, the \( x' \)-integrals yield

\[
\int d^4x' \tilde{E}_{\nu'}(x') \gamma^\nu E^{\mu'}(x')e^{-iq'\cdot x'} = (2\pi)^4 \delta^{(3)}(p'' + q' - p')e^{-iq_1(p''_2 + p'_2)/(2eH)} \cdot e^{-\hat{q}_\perp^2/2} \sum_{\sigma',\sigma''} \frac{1}{\sqrt{n'!n''''!}} e^{i\text{sgn}(eH)(n''-n')\varphi} J_{n'n''}(\hat{q}_\perp) \tilde{\Delta}'' \gamma^\nu \Delta', \quad (33)
\]

where \( n' = n(k',\sigma'), \tilde{n}'' = n(k'',\tilde{\sigma}''), \Delta' = \Delta(\sigma'), \) and \( \tilde{\Delta}'' = \Delta(\tilde{\sigma}''). \) The presence of the \( \delta \)-functions in Eqs. (32) and (33) allows easy integrations over \( p''_0, p''_2, \) and \( p''_3 \) in Eq. (24), yielding \( \delta^{(3)}(p - p') = \delta(p_0 - p'_0)\delta(p_2 - p'_2)\delta(p_3 - p'_3) \) which matches that on the left hand side of Eq. (24). The SD equation is therefore reduced to

\[
\tilde{\Sigma}_A(\hat{p}) \delta_{kk'} = i e^2(2|eH|) \sum_{k''} \sum_{\{\sigma\}} \int \frac{d^4\hat{q}}{(2\pi)^4} \frac{e^{i\text{sgn}(eH)(n-n''+\tilde{n}''-n'')\varphi}}{\sqrt{n!n''!n''''!}} \cdot e^{-\hat{q}_\perp^2} J_{nn''}(\hat{q}_\perp) J_{n''n'}(\hat{q}_\perp) \frac{1}{q^2} \left( g_{\mu\nu} - (1 - \xi) \frac{\hat{q}_\mu \hat{q}_\nu}{q^2} \right) \cdot \Delta \gamma^\mu \Delta'' \gamma^\nu \Delta', \quad (34)
\]

where the summation over \( \{\sigma\} \) means summing over \( \sigma, \sigma', \sigma'', \) and \( \tilde{\sigma}'' \), and the momentum \( \hat{p}'' \) is understood to be: \( \hat{p}''_0 = p_0 - q_0, \hat{p}''_1 = 0, \hat{p}''_2 = -\text{sgn}(eH)\sqrt{2|eH|k''}, \hat{p}''_3 = p_3 - q_3. \)
III. SOLUTION TO THE SCHWINGER-DYSON EQUATION

An approximate solution to Eq. (34) may be obtained by observing that, due to the factor $e^{-\hat{q}_\perp^2}$ in the integrand, contributions from large values of $\hat{q}_\perp$ are suppressed. Thus, by keeping only the terms with the smallest power of $\hat{q}_\perp$ in $J_{nn''}(\hat{q}_\perp)$, i.e.,

$$J_{nn''}(\hat{q}_\perp) \to \frac{\max(n,n'')}{|n-n''|!} (i \text{sgn}(eH)\hat{q}_\perp)^{|n-n''|}$$

$$\to n! \delta_{nn''}$$

(35)

and similarly for $J_{n'n''}(\hat{q}_\perp)$, the SD equation is simplified to

$$\tilde{\Sigma}_A(\tilde{p})_{\delta_{kk'}} \simeq ie^2(2|eH|) \sum_{k''} \sum_{\{\sigma\}} \delta_{nn''}\delta_{\tilde{n}'n''} \int \frac{d^4\hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_\perp^2}}{\hat{q}^2} \left( g_{\mu\nu} - (1 - \xi) \hat{q}_\mu \hat{q}_\nu \right) \Delta^{\gamma\mu} \Delta^{n''} \Delta^{\tilde{n}''} \Delta' \cdot \frac{1}{\gamma \cdot \tilde{p}'' + \Sigma_A(\tilde{p}'')} \Delta' \cdot \Sigma_A(\tilde{p}'') \left[ G_1 - (1 - \xi) W_1 \right] - \left[ 1 + Z(\tilde{p}'') \right] \left[ 1 + Z(\tilde{p}'') \right] \left[ 1 + Z(\tilde{p}'') \right] \left[ 1 + Z(\tilde{p}'') \right] \Sigma_A(\tilde{p}'')$$

(36)

The solution is expected to have the form $\tilde{\Sigma}_A(\tilde{p}) = Z(\tilde{p})_{\gamma \cdot \tilde{p} + \Sigma_A(\tilde{p})}$, where $\Sigma_A(\tilde{p})$ is the dynamically generated fermion mass and is assumed to be proportional to the unit matrix (see remarks after Eq. (48)). Eq. (36) then reads

$$[Z(\tilde{p})_{\gamma \cdot \tilde{p} + \Sigma_A(\tilde{p})}]_{\delta_{kk'}} \simeq ie^2(2|eH|) \sum_{k''} \sum_{\{\sigma\}} \delta_{nn''}\delta_{\tilde{n}'n''} \int \frac{d^4\hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_\perp^2}}{\hat{q}^2} \left( g_{\mu\nu} - (1 - \xi) \hat{q}_\mu \hat{q}_\nu \right) \Delta^{\gamma\mu} \Delta^{n''} \Delta^{\tilde{n}''} \Delta' \cdot \frac{1}{\gamma \cdot \tilde{p}'' + \Sigma_A(\tilde{p}'')} \Delta' \cdot \Sigma_A(\tilde{p}'')[G_1 - (1 - \xi) W_1] - [1 + Z(\tilde{p}'')][G_2 - (1 - \xi) W_2],$$

(37)

where

$$G_1 \equiv \Delta^{\gamma\mu} \Delta^{n''} \Delta^{\tilde{n}''} \gamma_{\mu} \Delta'',$$

$$W_1 \equiv \frac{1}{\hat{q}^2} \Delta(\gamma \cdot \hat{q}) \Delta^{n''} \Delta^{\tilde{n}''} (\gamma \cdot \hat{q}) \Delta',$$

$$G_2 \equiv \Delta^{\gamma\mu} \Delta^{n''} (\gamma \cdot \tilde{p}'') \Delta^{\tilde{n}''} \gamma_{\mu} \Delta',$$

$$W_2 \equiv \frac{1}{\hat{q}^2} \Delta(\gamma \cdot \hat{q}) \Delta^{n''} (\gamma \cdot \tilde{p}'') \Delta^{\tilde{n}''} (\gamma \cdot \hat{q}) \Delta'.$$

(38)

The matrices $G_{1,2}$ and $W_{1,2}$ may be simplified as follows. First we note that the $\Delta$-matrices may be expressed as

$$\Delta(\sigma) = \frac{1}{2} (1 + D_\sigma \Sigma_3),$$

(39)
where \( D_\sigma \equiv (\delta_{\sigma_1} - \delta_{\sigma_1}) \), \( D^2 = 1 \). They also satisfy the commutation relations,

\[
\Delta \gamma^\mu \equiv \gamma^\mu \nabla
\]

and

\[
[\Delta, \gamma^\mu] = 0 = [\nabla, \gamma^\mu],
\]

where the subscript \( \perp \) refers to the transverse components, \( \mu = 1, 2 \), the subscript \( \parallel \) refers to the longitudinal components, \( \mu = 0, 3 \), and \( \nabla \) is the complement of \( \Delta \):

\[
\nabla(\sigma) = 1 - \Delta(\sigma) = \frac{1}{2} (1 - D_\sigma \Sigma_3).
\]

Secondly, products of the \( \Delta \)-matrices may be expressed in terms of a single \( \Delta \)-matrix, e.g.,

\[
\Delta(\sigma) \Delta(\sigma'') = \delta_{\sigma \sigma''} \Delta(\sigma),
\]

and similarly for the \( \nabla \)-matrices. Using these relations, we find that, after performing the summation over the spin indices,

\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{\tilde{n}'n'} G_1 = -2\delta_{kk'} \left[ \delta_{kk''} + \delta_{k,k''-\text{sgn}(eH)} \Delta(1) + \delta_{k,k''+\text{sgn}(eH)} \Delta(-1) \right],
\]

\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{\tilde{n}'n'} W_1 \simeq -\delta_{kk'}^2 \delta_{kk''},
\]

\[
\sum_{\{\sigma\}} \delta_{nn''} \delta_{\tilde{n}''n'} G_2 = 2\delta_{kk'} \left\{ \delta_{kk''} (\gamma \cdot \bar{p}_{\perp}) + \gamma \cdot (\bar{p}_\parallel - q_\parallel) \right\}.
\]

\[
\sum_{\{\sigma\}} \delta_{nn''} \delta_{\tilde{n}''n'} W_2 \simeq \delta_{kk'}^2 \delta_{kk''} \left\{ \gamma \cdot \bar{p}_{\perp} + \frac{1}{q^2} (\gamma \cdot \bar{q}_\parallel) \right\} \left( \gamma \cdot \bar{q}_\parallel \right).
\]

where \( \gamma \cdot \bar{p}_{\perp} = \gamma^2 \bar{p}_2 \), \( \Delta(1) = \text{diag}(1, 0, 1, 0) \), and \( \Delta(-1) = \text{diag}(0, 1, 0, 1) \). In accordance with the small \( \bar{q}_\perp \) approximation used on the \( J_{nn''} \) and \( J_{\tilde{n}'n''} \), Eq.(35), terms in \( W_1 \) and \( W_2 \) that are proportional to \( \bar{q}_\perp \) have been dropped. It is satisfying that the spin summation produces the Kronecker delta, \( \delta_{kk'} \), which matches the one on the left hand side of Eq.(37).

Note also that, due to the restriction \( n = n'' \) which arises from the small \( \bar{q}_\perp \) approximation, the summation over \( k'' \) is now restricted to only three terms: for a given \( k \), \( k'' = k \), \( k \pm 1 \).
The SD equation may now be written as

\[
Z(\bar{p})\gamma \cdot \bar{p} + \Sigma_A(\bar{p}) \\
\simeq -ie^2(2|eH|) \sum_{k''} \int \frac{d^4 \hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_-^2}}{\hat{q}_-^2} \frac{1}{[1 + Z(\bar{p}'')][\bar{p}_\parallel - q_\parallel]^2 + \Sigma_A^2(\bar{p}'')} \\
\cdot \left\{ [1 + Z(\bar{p}'')] \left[ \delta_{k''k} \left( (1 + \xi)\gamma \cdot \bar{p}_\perp - \frac{1 - \xi}{\hat{q}_-^2} \left( \gamma \cdot \hat{q}_\parallel \right) \gamma \cdot \left( \bar{p}_\parallel - q_\parallel \right) \left( \gamma \cdot \hat{q}_\parallel \right) \right) \\
+ 2 \left( \delta_{k''k, \text{sgn}(eH)\Delta(1) + \delta_{k''k, -\text{sgn}(eH)\Delta(-1)}} \gamma \cdot \left( \bar{p}_\parallel - q_\parallel \right) \right) \right] \\
+ \Sigma_A(\bar{p}'') \left[ \delta_{k''k} (1 + \xi) \right. \\
\left. + 2 \left( \delta_{k''k, \text{sgn}(eH)\Delta(1) + \delta_{k''k, -\text{sgn}(eH)\Delta(-1)}} \right) \right] \right\}. \tag{48}
\]

Recall that \(\bar{p}''^2 = (\bar{p}_\parallel - q_\parallel)^2 + 2|eH|k''\).

Eq. (48) shows that our earlier assumption of \(\Sigma_A(\bar{p})\) being proportional to the unit matrix is correct only for the \(k'' = k\) term. The reason is that \(\gamma \cdot \Pi\), and hence the mass operator \(\hat{M}\), does not commute with \(\Sigma_3\). However, if we consider only the low energy \((\bar{p}^2 \ll |eH|)\) behaviors, in particular, in the \(\bar{p}_\perp = 0 = k\) limit, the \(k'' = 0\) term will dominate and we obtain the approximate SD equation,

\[
Z(\bar{p}_\parallel)\gamma \cdot \bar{p}_\parallel + \Sigma_A(\bar{p}_\parallel) \\
\simeq -ie^2(2|eH|) \int \frac{d^4 \hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_-^2}}{\hat{q}_-^2} \frac{1}{[1 + Z(\bar{p}_\parallel - q_\parallel)][\bar{p}_\parallel - q_\parallel]^2 + \Sigma_A^2(\bar{p}_\parallel - q_\parallel)} \\
\cdot \left\{ [1 + Z(\bar{p}_\parallel - q_\parallel)] \frac{\xi - 1}{\hat{q}_-^2} \left( \gamma \cdot \hat{q}_\parallel \right) \gamma \cdot \left( \bar{p}_\parallel - q_\parallel \right) \left( \gamma \cdot \hat{q}_\parallel \right) + \Sigma_A(\bar{p}_\parallel - q_\parallel) (1 + \xi) \right\}. \tag{49}
\]

In this case \(\Sigma_A\) is proportional to the unit matrix. This approximation is equivalent to the lowest Landau level approximation employed in [2].

We see from Eq. (49) that, in the Feynman gauge \((\xi = 1)\), \(Z(\bar{p}_\parallel) = 0\) and the SD equation for the dynamically generated fermion mass becomes

\[
\Sigma_A(\bar{p}_\parallel) \simeq e^2(4|eH|) \int \frac{d^4 \hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_-^2}}{\hat{q}_-^2} \frac{\Sigma_A(\bar{p}_\parallel - q_\parallel)}{[\bar{p}_\parallel - q_\parallel]^2 + \Sigma_A^2(\bar{p}_\parallel - q_\parallel)}. \tag{50}
\]

where we have made a Wick rotation to Euclidean space: \(p_0 \rightarrow ip_4\), \(q_0 \rightarrow iq_4\). This can be turned into a differential equation for \(\Sigma_A\), as was done in Ref. [3] where the momentum dependence of the dynamical mass is discussed. Here we content ourselves with finding a solution for the infrared fermion mass scale,
\[
\Sigma_A(0) \simeq e^2(4|eH|) \int \frac{d^4\hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_\perp^2}}{\hat{q}^2} \frac{\Sigma_A(q_\parallel)}{2|eH|q_\parallel^2 + \Sigma_A^2(q_\parallel)}.
\] (51)

Since \(\Sigma_A(q_\parallel)\) is expected to diminish with increasing \(q_\parallel^2\), the integral in Eq.(51) is dominated by the contributions from small \(q_\parallel^2\). It is therefore reasonable to approximate \(\Sigma_A(q_\parallel)\) in the integrand by \(\Sigma_A(0) = m \times \mathbf{1}\), where \(m\) is the dynamical mass and \(\mathbf{1}\) is the unit matrix, thus securing the gap equation,

\[
1 \simeq e^2(4|eH|) \int \frac{d^4\hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_\perp^2}}{\hat{q}^2} \frac{1}{2|eH|q_\parallel^2 + m^2} = e^2 \frac{1}{4\pi^2|eH|} \int_0^\infty dq_\parallel^2 \int_0^\infty dq_\perp^2 \frac{e^{-\hat{q}_\perp^2}}{q_\parallel^2 + q_\perp^2} \frac{1}{2|eH|q_\parallel^2 + m^2} \simeq \frac{\alpha}{\pi}|eH| \int_0^\infty dq_\perp^2 \frac{e^{-\hat{q}_\perp^2} \ln(2|eH|q_\perp^2/m^2)}{2|eH|q_\perp^2 - m^2}.
\] (52)

The solution to Eq.(52) has the form

\[
m \simeq a \sqrt{|eH|} e^{-b \sqrt{\pi/\alpha}},
\] (53)

where \(a\) and \(b\) are positive constants of order 1. The nonperturbative nature of this result is apparent. Furthermore, according to the last equality of Eq.(52), the dominant contributions to the integral come from the region \(2|eH|q_\perp^2 \sim m^2\). Consistency with our small \(\hat{q}_\perp\) assumption requires that \(m \ll \sqrt{|eH|}\), which in turn requires that \(\alpha \ll 1\). In other words, the solution for the dynamical mass found above applies to the weak-coupling regime of QED.

To establish that the above solution to the SD equation for the fermion self-energy does indeed correspond to a dynamical chiral symmetry breaking solution, it is necessary to demonstrate the existence of the corresponding Nambu-Goldstone (NG) boson. One way to establish this is by studying the Bethe-Salpeter equation of the bound-state NG boson, as was done by Gusynin et al. [2], who found a solution consistent with our Eq.(53). We show in the Appendix how the same solution, Eq.(53), can be obtained from the Bethe-Salpeter equation for the NG boson, using the \(E_\rho\)-representation of the fermion propagator. This helps to justify that an external magnetic field serves as a catalyst for chiral symmetry breaking in QED.
IV. SYMMETRY BREAKING AT NONZERO TEMPERATURE

The formalism described above is very useful for studying dynamical symmetry breaking in an external field at nonzero temperatures. As expected on physical grounds [7], the long-range order of a system decreases as the temperature increases and chiral symmetry will generally be restored above a certain critical temperature. We shall see that this expectation is indeed correct for the case of chiral symmetry breaking in a magnetic field and an estimate of the critical temperature will be obtained.

To incorporate the thermal effects, we use the imaginary time and energy formalism [8] in which

$$0 \leq ix_0 \leq \beta \tag{54}$$

and

$$q_0 = 2l'\pi/(-i\beta) \quad \text{for bosons, } l' = 0, \pm 1, \pm 2, ...$$

$$p_0 = (2l + 1)\pi/(-i\beta) \quad \text{for fermions, } l = 0, \pm 1, \pm 2, ... \tag{55}$$

where $\beta = 1/T$, with the Boltzmann constant $k_B = 1$. The analysis given in Sections II and III can be repeated for the case of finite temperature by implementing the following replacements [9]:

$$\int d^4x \to \int_x \equiv \int_0^{-i\beta} dx_0 \int d^3x$$

$$\int d^4p \to \int_p \equiv \frac{2\pi}{-i\beta} \sum_l \sum_k \int dp_2 \int dp_3 \tag{56}$$

$$\hat{\delta}^{(4)}(p - p') \to \hat{\delta}_T^{(4)}(p - p') \equiv -\frac{i\beta}{2\pi} \delta_{ll'}\delta_{kk'}\delta(p_2 - p_2')\delta(p_3 - p_3')$$

and for the photon propagator

$$\int d^4q \to \frac{2\pi}{-i\beta} \sum_q \int dq_1 \int dq_2 \int dq_3 \tag{57}$$

The gap equation now reads

$$1 \simeq \frac{2\alpha}{\pi T |eH|} \int_{-\infty}^{\infty} dq_3 \int_0^{\infty} dq_1^2 e^{-q_1^2} \sum_{l'} \frac{1}{Q_2 + 4\pi^2 T^2 l'^2} \frac{1}{Q_1 + \pi^2 T^2(2l' - 1)^2} \tag{58}$$
where \( Q_1 \equiv q_3^2 + m^2 \) and \( Q_2 \equiv q_3^2 + 2|eH|q_3^2 \). Following Ref. [10], we sum the series by means of the Poisson sum formula, which states that, if \( c(\tau) \) is the Fourier transform of \( b(\omega) \),

\[
c(\tau) = \int_{-\infty}^{\infty} b(\omega) e^{-i\omega\tau} d\omega,
\]

the following identity will hold:

\[
\sum_{\nu = -\infty}^{\infty} b(\nu') = \sum_{\lambda = -\infty}^{\infty} c(2\pi \lambda).
\]

Here, taking

\[
b(\omega) = \frac{1}{Q_2 + 4\pi^2 T^2 \omega^2} \frac{1}{Q_1 + \pi^2 T^2 (2\omega - 1)^2},
\]

we have

\[
c(\tau) = \frac{1}{8\pi T^2} \frac{1}{\sqrt{Q_1 Q_2}} \int_{-\infty}^{\infty} du \exp \left[ - \left( \frac{i u}{2} + \frac{|u|}{2\pi T} \sqrt{Q_1} + \frac{|\tau - u|}{2\pi T} \sqrt{Q_2} \right) \right].
\]

After integrating over \( u \), we make use of the Poisson sum formula to rewrite the summation in Eq.(58) in terms of a more manageable series:

\[
\sum_{\nu} \frac{1}{Q_2 + 4\pi^2 T^2 \nu^2} \frac{1}{Q_1 + \pi^2 T^2 (2\nu - 1)^2} = \frac{1}{2T \sqrt{Q_1 Q_2}} \sum_{\lambda = 0}^{\infty} \left[ \left( e^{\frac{-\lambda \sqrt{Q_2}}{T}} + (-1)^\lambda e^{\frac{-\lambda \sqrt{Q_1}}{T}} \right) \frac{\sqrt{Q_1} + \sqrt{Q_2}}{(\sqrt{Q_1} + \sqrt{Q_2})^2 + \pi^2 T^2} \right.
\]

\[
\left. \left( e^{\frac{-\lambda \sqrt{Q_2}}{T}} - (-1)^\lambda e^{\frac{-\lambda \sqrt{Q_1}}{T}} \right) \frac{\sqrt{Q_1} - \sqrt{Q_2}}{(\sqrt{Q_1} - \sqrt{Q_2})^2 + \pi^2 T^2} \right],
\]

where the prime on the summation sign means that the \( \lambda = 0 \) term is counted with half weight. Substituting Eq.(63) into Eq.(58) and summing over the infinite geometric series, we obtain

\[
1 \simeq \frac{\alpha}{\pi} |eH| \int_{-\infty}^{\infty} dq_3 \int_{0}^{\infty} dq_1^2 \frac{e^{-q_1^2}}{\sqrt{Q_1 Q_2}} \left[ \left( \frac{1}{1 - e^{-\frac{\sqrt{Q_2}}{T}}} + \frac{1}{1 + e^{-\frac{\sqrt{Q_1}}{T}}} - 1 \right) \frac{\sqrt{Q_1} + \sqrt{Q_2}}{(\sqrt{Q_1} + \sqrt{Q_2})^2 + \pi^2 T^2} \right.
\]

\[
\left. + \left( \frac{1}{1 - e^{-\frac{\sqrt{Q_2}}{T}}} - \frac{1}{1 + e^{-\frac{\sqrt{Q_1}}{T}}} \right) \frac{\sqrt{Q_1} - \sqrt{Q_2}}{(\sqrt{Q_1} - \sqrt{Q_2})^2 + \pi^2 T^2} \right].
\]

13
Note that the only approximations we have made so far are the quenched, ladder approximation (Eq. (18)), the small \( q_\perp \) approximation, and keeping only the \( k'' = 0 \) terms in Eq. (13).

Aside from these approximations, the finite-temperature gap equation, Eq. (64), is exact in its dependence on the coupling constant, the magnetic field, and the temperature.

We shall consider below the zero, low, and high temperature limits of Eq. (64). For \( T = 0 \), Eq. (64) is reduced to

\[
1 \simeq \frac{\alpha}{\pi} |eH| \int_{-\infty}^{\infty} dq_3 \int_{0}^{\infty} dq_\perp^2 e^{-q_\perp^2} \frac{1}{\sqrt{Q_1 Q_2} (\sqrt{Q_1} + \sqrt{Q_2})} \tag{65}
\]

which, one can easily check, is just what Eq. (52) becomes when the integration there over \( q_4 \) is done. Thus, we have recovered the \( T = 0 \) result.

For the low temperature case, we can approximate the sum over \( l' \) in Eq. (58) by the \( \lambda = 0 \) term (with half weight) on the right hand side of Eq. (63), the other terms being exponentially small. Treating the thermal effects as a perturbation, we write \( m^2(T) = m_0^2 + \delta m^2 \) with \( \delta m^2 \ll m_0^2 \), where \( m_0 \) is the fermion dynamical mass for \( T = 0 \) (i.e., the solution in Eq. (53)). At small \( T \), the difference between Eq. (58) and Eq. (65) yields

\[
\delta m^2 = -\frac{2\pi^2 T^2}{I_2} I_1 \tag{66}
\]

where

\[
I_1 = \int_{-\infty}^{\infty} dq_3 \int_{0}^{\infty} dq_\perp^2 e^{-q_\perp^2} \frac{1}{Q \sqrt{Q_2} \sqrt{Q + Q_2}} \left( \frac{1}{\sqrt{Q}} + \frac{1}{\sqrt{Q + Q_2}} \right)
\]

\[
I_2 = \int_{-\infty}^{\infty} dq_3 \int_{0}^{\infty} dq_\perp^2 e^{-q_\perp^2} \frac{1}{\sqrt{Q_1 Q_2} (\sqrt{Q_1} + \sqrt{Q_2})^3}
\]

with \( Q \equiv m_0^2 + q_3^2 \). Note that the infrared regions of \( q_3 \) and \( q_\perp^2 \) dominate the integrals, just as in the \( T = 0 \) case. More importantly, both \( I_1 \) and \( I_2 \) are finite and positive so that \( \delta m^2 \) is negative (and small, being proportional to \( T^2 \)). Thus, thermal effects tend to reduce the fermion dynamical mass, i.e., chiral symmetry tends to be restored as the temperature increases, as discussed earlier. However, we should stress that, so long as \( T \) is small, a non-zero dynamical mass will be generated in the presence of the magnetic field.
At high temperatures \( T > \sqrt{|eH|} \), the \( l' = 0 \) term dominates the sum in Eq.(58), thus yielding

\[
1 \simeq \frac{2\alpha}{\pi} T|eH| \int_{-\infty}^{\infty} dq_3 \int_0^{\infty} dq_2 \frac{e^{-\hat{q}_2^2}}{Q_2 Q_1 + \pi^2 T^2}.
\] (68)

Since the dominant contributions to the \( q_3 \)-integral come from small values of \( q_3 \), we may approximate the denominator \((Q_1 + \pi^2 T^2)\) by \([m^2(T) + \pi^2 T^2]\). After evaluating the integrals, we obtain

\[
m^2(T) + \pi^2 T^2 \simeq \alpha T \sqrt{2\pi |eH|}
\] (69)

Obviously, at weak couplings \((\alpha \ll 1)\) and at high temperatures \((T > \sqrt{|eH|})\), there is no non-negative solution for \( m^2(T) \), i.e., no chiral symmetry breaking solution. Thus, both the high temperature and the low temperature solutions indicate that chiral symmetry will be restored at high temperatures.

If we define the critical temperature for chiral symmetry breaking to be the temperature at which \( m^2(T) \) vanishes, i.e., \( m^2(T_c) = 0 \), Eq.(69) provides an estimate of the critical temperature:

\[
T_c \simeq \frac{\alpha}{\pi^2} \sqrt{2\pi |eH|} \sim 1.5 \times 10^{-2} \text{ eV} \sqrt{\frac{|H|}{10^4 \text{ gauss}}}.
\] (70)

For a magnetic field of any given strength, the critical temperature can be more exactly calculated by numerically solving the gap equation, Eq.(64), with \( Q_1 \) replaced by \( q_3^2 \).

Finally, we consider the suggestion by Gusynin et al. [2] that the chiral symmetry breaking solution found above may play a role during the electroweak phase transition in the early universe. The electroweak phase transition took place at a temperature of order 100 GeV. From Eq.(70), this requires a magnetic field of \( 10^{30} \) gauss or stronger, which is significantly larger than any estimates of the magnetic field strength at the time of the electroweak phase transition [11]. We therefore conclude that the chiral symmetry breaking solution considered here does not play any role in the electroweak phase transition.
V. CONCLUSION

We have described a formalism for studying the physics of chiral symmetry breaking in an external field via the nonperturbative Schwinger-Dyson equation. The $E_p$-representation for the fermion propagator proposed here has the advantage that the dependence on the operator $\Pi_\mu$ is removed. It also has the advantage over Schwinger’s proper time formalism for calculating finite-temperature effects.

We have applied our method to examine chiral symmetry breaking in QED in a constant external magnetic field. We find that, even when the coupling constant is small, an external magnetic field can trigger the dynamical breaking of chiral symmetry in QED, with the dynamical mass of the fermion given by Eq.(53). Our result agrees with that found in Ref. [4] which used an approach rather different from ours. The existence of the Nambu-Goldstone boson of chiral symmetry breaking is demonstrated in the Appendix by showing that the same solution (Eq.(53)) solves the Bethe-Salpeter equation for the bound-state Nambu-Goldstone boson.

We have also obtained an estimate of the critical temperature $T_c$ of the aforementioned chiral symmetry breaking, Eq.(70). Chiral symmetry is a good symmetry above this critical temperature, and it is spontaneously broken at temperatures below $T_c$. Our result renders invalid the suggestion in the literature [4] that the chiral symmetry breaking solution described above may be relevant for the electroweak phase transition in the early universe.

There remain several interesting questions which we are investigating. For instance, how do other background field configurations affect chiral symmetry breaking? (As an example, it is expected that an electric field would tend to break up the condensate and destabilize the vacuum, thus inhibiting chiral symmetry breaking [12].) Are there strong-coupling solutions of chiral symmetry breaking in an external magnetic field? What are the effects of additional four-fermion interactions? Recall that four-fermion operators play a crucial role in obtaining a consistent chiral symmetry breaking solution in quenched, ladder QED [13]. We hope to report our findings in the near future.
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APPENDIX A: NAMBU-GOLDSTONE BOSONS

We verify in this appendix that the dynamical mass found in Eq. (53) corresponds to a spontaneous chiral symmetry breaking solution by examining the Bethe-Salpeter equation of the Nambu-Goldstone (NG) boson. In the quenched, ladder approximation, the Bethe-Salpeter equation of the NG boson has the form [14]

\[ \phi(x, y; P) = -ie^2 \int d^4x' \int d^4y' G_A(x, x') \gamma^\mu \phi(x', y'; P) \gamma^\nu G_A(y', y) D_{\mu\nu}(y' - x'). \]  

(A1)

In the \( E_p \)-representation, this can be expressed as

\[ (2\pi)^4 \delta^4(p - p') \tilde{\phi}(p; P) = -ie^2 \int d^4x' \int d^4y' \int \frac{d^4p''}{(2\pi)^4} \frac{1}{\gamma_p + m} \tilde{E}_p(x) \gamma^\mu E_{p''}(x) \cdot \tilde{\phi}(p''; P) \tilde{E}_{p''}(y) \gamma^\nu E_{p'}(y) \frac{1}{\gamma_p' + m} D_{\mu\nu}(y - x), \]  

(A2)

where

\[ \phi(x, y; P) = \oint \frac{d^4p}{(2\pi)^4} E_p(x) \tilde{\phi}(p; P) \tilde{E}_p(y) \]  

(A3)

and \( m \) is the dynamically generated fermion mass.

If we introduce

\[ \chi(p; P) \equiv (\gamma \cdot \bar{p} + m) \tilde{\phi}(p; P) (\gamma \cdot \bar{p} + m), \]  

(A4)

we find

\[ (2\pi)^4 \delta^4(p - p') \chi(p; P) = -ie^2 \int d^4x' \int d^4y' \int \frac{d^4p''}{(2\pi)^4} \frac{1}{\gamma_p + m} \tilde{E}_p(x) \gamma^\mu E_{p''}(x) \cdot \tilde{\phi}(p''; P) \tilde{E}_{p''}(y) \gamma^\nu E_{p'}(y) \frac{1}{\gamma_p' + m} D_{\mu\nu}(y - x). \]  

(A5)

Note the similarity of this equation to Eq. (20). After integrating over \( x \) and \( y \) (see Eq. (32) and Eq. (33)) as well as over \( p_0'' \), \( p_1'' \) and \( p_2'' \), we obtain (in the Feynman gauge)

\[ \chi(p; P) \delta_{kk'} = -ie^2 (2|eH|) \sum_{k'} \sum_{\{a\}} \int \frac{d^4\hat{q}}{(2\pi)^4} e^{-i\text{sgn}(eH)(n - n' + \tilde{n} - \tilde{n}')\phi} \cdot \frac{\tilde{J}_{nn'}(\hat{q}) J_{\tilde{n}'\tilde{n}'}(\hat{q}) \Delta \gamma^\mu \Delta''}{\sqrt{n!n'!n''!\tilde{n}!\tilde{n}''!}} \cdot \frac{1}{\gamma \cdot \bar{p}'' + m} \chi(p''; P) \cdot \frac{1}{\gamma \cdot \bar{p}'' + m} \Delta'' \gamma_{\mu} \Delta'. \]  

(A6)
where \( \bar{p}'_0 = p_0 + q_0, \bar{p}'_1 = 0, \bar{p}'_2 = -\text{sgn}(eH)\sqrt{2|eH|k''}, \bar{p}'_3 = p_3 + q_3. \)

We are interested in the \( P = 0 \) behavior of \( \chi(p; P) \), which is expected to have the form

\[
\chi(p; 0) = A(p)\gamma_5, \tag{A7}
\]

where \( A(p) \) is a scalar function. It follows from Eq.(A6) that

\[
A(p)\delta_{kk'} = i e^2(2|eH|) \sum_{k''} \sum_{\{\sigma\}} \int \frac{d^4 \hat{q}}{(2\pi)^4} \frac{e^{-i\text{sgn}(eH)(n-n''+\tilde{n}'-n')\varphi}}{\sqrt{n!n''!\tilde{n}'!\tilde{n}!!}} \cdot \frac{e^{-\hat{q}_\perp^2/\hat{q}^2}}{\hat{q}^2} J_{nn''}(\hat{q}_\perp) J_{\tilde{n}'\tilde{n}''}(\hat{q}_\perp) G_1 A(p''), \tag{A8}
\]

with \( G_1 \) defined in Eq.(38). After taking the small \( \hat{q}_\perp \) approximation, Eq.(35), this is simplified to

\[
A(p)\delta_{kk'} = i e^2(2|eH|) \sum_{k''} \sum_{\{\sigma\}} \delta_{nn''}\delta_{\tilde{n}'\tilde{n}''} \int \frac{d^4 \hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_\perp^2/\hat{q}^2} G_1 A(p'')}{\hat{q}^2 \bar{p}'^2 + m^2}. \tag{A9}
\]

If we consider the infrared behavior of \( A(p) \), we have

\[
A(0) \simeq -i e^2(4|eH|) \int \frac{d^4 \hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_\perp^2/\hat{q}^2} A(q_\parallel)}{2|eH|\hat{q}_\parallel^2 + m^2}. \tag{A10}
\]

where we have performed the spin summation and ignored the \( k'' = 1 \) term, in accordance with the approximation used in obtaining Eq.(51). Transforming to Euclidean space and noting that the integral is dominated by contributions from small \( \hat{q}_\parallel^2 \) so that \( A(q_\parallel) \) in the integrand can be approximated by \( A(0) \), we secure

\[
1 \simeq e^2(4|eH|) \int \frac{d^4 \hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_\perp^2/\hat{q}^2}}{2|eH|\hat{q}_\parallel^2 + m^2}, \tag{A11}
\]

which is the same as the gap equation, Eq.(52). Thus, the dynamical mass found in Eq.(53) provides a consistent solution to the Bethe-Salpeter equation for the NG boson.
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and

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