Particle number density fluctuations and pressure effects on structure formation

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We provide a covariant and gauge-invariant approach to the question of how a first order pressure can be incorporated self-consistently in a cosmological scenario. The approximation is relevant, in the linear regime, to weakly self-interacting or warm dark matter models. We also derive number density fluctuations in which new modes appear because the number density fluctuations are no longer simply proportional to the density fluctuations.

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I. INTRODUCTION

According to current cosmological models, the large-scale structure, as seen today, formed from the evolution of gravitational instabilities which originated from inflationary processes in the primordial fluid. In studying the evolution of such instabilities, analytically in the fluid approximation\textsuperscript{4}, one usually assumes a dust or radiation energy momentum tensor. The object of this paper is to consider some implications of modifying these assumptions.

First we note that while the two extreme fluid states are reasonable for most of the history of the universe, there are short, but at the same time important, specific periods in which matter is in the non-relativistic regime with a tiny although non-negligible pressure. These are periods during which collisions, or small random motions (velocity dispersion) of collisionless matter, give rise to a small but non-zero matter pressure. Even though in these cases the matter pressure in the background may be negligible, small disturbances (perturbations) of the spacetime induce a tiny pressure term which can modify the evolution of the inhomogeneities. Such a scenario may also have applications when studying weakly self-interacting matter (for details see Spergel and Steinhardt \textsuperscript{[1]}) or warm dark matter (for more details of this approximation see Hogan and Dalcanton \textsuperscript{[1]}) in the linear regime. These dark matter models have been the subject of much recent work using numerical integrations in the nonlinear regime, where they have been employed to investigate ways of resolving problems with the standard CDM model on small scales (see \textsuperscript{[1]}) in galaxy formation.

Second we take the particulate nature of matter seriously. It must be born in mind that in structure formation one really needs to look at the clumping of a set of particles rather than at the evolution of energy density instabilities. Although in taking the dust assumption to be valid such a distinction is not relevant, this is not true for those periods in which matter is undergoing a transition to the non-relativistic regime and, consequently, for which the dust approximation is not accurate enough, even in the linear regime. To model the effects of the particulate nature of the matter we make use of thermodynamic arguments. An alternative Kinetic Theory approach is given in \textsuperscript{[1]}. The method we use is based on the Ellis-Bruni \textsuperscript{[3]} covariant and gauge-invariant perturbation formalism, which has proved useful in different cosmological problems \textsuperscript{[3], [3], [3], [3]}. It is one of the gauge-independent descriptions of relativistic perturbations, described in detail in the literature, see for instance \textsuperscript{[3], [3], [3], [3], [3], [3], [3], [3], [3]}. In our treatment of the fluid models we generalise the usual equations for the growth of cosmological perturbations for matter in a Friedman-Robertson-Walker (FRW) background to include first order pressure effects due to a non-relativistic component, in the radiation and in the matter dominated eras. Thus the simplifying hypothesis that the matter is dust, i.e. has vanishing pressure, is relaxed, in both cases, to include a small pressure due to collisions or thermal motions of the particles which make up the fluid.

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\textsuperscript{1}For high accuracy the full Boltzmann equations are used and these require numerical codes, which provide different insights into the process.
II. MATTER VERSUS ENERGY DENSITY PERTURBATIONS

The starting point for the Ellis-Bruni formalism is the choice of a time-like congruence $u^a$ along which typical (comoving) observers move. Once this four-velocity has been selected all the physical quantities are split into their spatial and time-like parts using the spatial projector tensor $h^{ab}$. The relevant physical quantities are defined as follows

$$
\delta a = a \frac{D_a \rho}{\rho}, \quad \theta a = a D_a \theta, \quad p_a = a \frac{D_a \rho}{\rho}, \quad \nu_a = a \frac{D_a n}{n},
$$

(1)

where $D_a \equiv h^{b} a \nabla_b$ is the spatial projection of the covariant derivative, $\theta \equiv u^a a$ the expansion, $p$ the pressure, $\rho$ the energy density and $a$ is a representative or average length scale defined by $3 \dot{a}/a = \theta$. In a FRW metric $a$ is the background scale factor and we will use the notation $3H = \theta$ in the background. The quantities $\delta_a$, $p_a$, $\theta_a$ and $\nu_a$ describe the spatial inhomogeneity of quantities which are homogeneous and isotropic in the background. Each of them is spatial (contraction with $u^a$ vanishes), covariant and gauge-invariant to first order, as all of them vanish in the FRW background \cite{6}. The scalar parts of perturbations described in (1) are obtained by taking the comoving spatial divergence, e.g., $\delta = a D^a \delta_a$.

The general procedure to obtain evolution and constraint equations for the quantities defined in (1) starts from a suitable spatial projection of the Ricci and Bianchi identities as given in terms of the fluid variables once the field equations have been taken into account (see for example \cite{9} for a pedagogical account). Here we shall follow the notation of \cite{4}.

Restricting ourselves to a spatially flat FRW background, the equations governing the evolution of the comoving fractional density gradient $\delta_a$ and the comoving expansion gradient $\theta_a$ are (see \cite{4} equations 25,26)

$$
\dot{\delta}_a = 3wH \delta_a - (1 + w)\theta_a,
$$

(2)

$$
\dot{\theta}_a = \frac{9}{2}H^2 p_a + 2H \theta_a - a D_a D^b \dot{u}_b + \frac{3H^2}{2} (\rho_a + 3p_a) = 0,
$$

(3)

where

$$
w = \frac{p}{\rho}.
$$

(4)

Note that equation (2) omits second order terms proportional to $\sigma_{ab} \delta^b$ and $\omega_{ab} \delta^b$ where $\sigma_{ab}$ and $\omega_{ab}$ are the shear and vorticity tensors.

We will now discuss perturbations in the particle number density. This discussion is motivated by the idea that a full understanding of large scale structure formation in the framework of gravitational instabilities requires an analysis of how particles, which comprise the cosmic fluid, clump together due to their mutual attraction. Although dealing with the energy density is the usual procedure, we believe that it is desirable to consider the particle number density $n$ because, ultimately, particles are the building blocks of all the cosmic structures we see today. For dust, it does not make any difference whether one chooses $\rho$ or $n$ because no random motions are present and the internal energy vanishes identically, yielding $\rho = mn$, i.e.

$$
\delta_a = \nu_a.
$$

(5)

The situation is quite different when internal energy associated with thermal motions is introduced. The behaviour of $\nu_a$ requires a separate investigation. In \cite{4} it is shown that from the Gibbs equation

$$
Td s = d \left( \frac{2}{n} \right) + pd \left( \frac{1}{n} \right),
$$

(6)

it follows that

$$
e_a = \delta_a - (1 + w)\nu_a,
$$

(7)

where

$$
e_a = \frac{anTD_a s}{\rho},
$$

(8)
is a dimensionless entropy gradient. Perturbations which are non-dissipative (\( \dot{s} = 0 \)) and without spatial variation of the entropy (\( e_a = 0 \)) are called isentropic in [4] because they have the same entropy at all points of the spacetime. In the weaker case where no spatial variation of the entropy is allowed (\( e_a = 0 \)) the number density perturbations are related algebraically to the energy density perturbations by

\[
\nu_a = \frac{\delta_a}{1 + w}.
\]

(9)

In fact, \( e_a = 0 \) is a strong assumption which does not hold in general. Its departure from zero affects the growth of perturbations. From the particle conservation equation

\[
\dot{n} + n\theta = 0,
\]

(10)

we get an evolution equation for \( \nu_a \)

\[
\dot{\nu}_a + \nu_a \theta - \frac{\theta}{1 + w} p_a = 0.
\]

(11)

If we use the standard, but generally unphysical, equation of state \( w = \text{constant} \) then together with (2), the integral of (11) yields

\[
\nu_a = \frac{\delta_a}{1 + w} + k_a, \quad \dot{k}_a = 0.
\]

(12)

Thus, in general, when this simple equation of state is used, a new stationary (entropy) mode appears in the evolution of \( \nu_a \) which could have an impact on the structure formation at certain stages of the evolution of inhomogeneities. The problem with (12) is that it makes sense only for the very special, but widely used, cases of dust and radiation. In more general circumstances such as in self interacting dark matter or where there are collisions [1], the equations of state should involve two independent thermodynamical quantities, i.e. \( n \) and \( \rho \). To deal with the inhomogeneities that arise in non-relativistic matter with a non-vanishing pressure we may follow two different approaches.

(i) If we assume that the fluid has reached a collision-dominated equilibrium\(^2\), we learn from kinetic theory that the equation of state in the non-relativistic regime is

\[
\rho = mn + \frac{3}{2} p,
\]

(13)

which together with the energy balance equation

\[
\dot{\rho} + \theta(\rho + p) = 0,
\]

(14)

and the particle number balance equation (10) leads to

\[
\dot{p} = -\frac{5}{3} \theta p,
\]

(15)

with solution

\[
p = p_0 \left( \frac{a_0}{a} \right)^5.
\]

(16)

Also from (13) we get the equation

\[
\delta_a = \frac{mn}{\rho} \nu_a + \frac{3}{2} p_a
\]

(17)

which, together with (13) and the conservation equations, determines \( \nu_a \).

\(^2\)For an example of such a model see the paper by Spergel and Steinhardt [3] where it is pointed out that under certain circumstances the cold dark matter behaves as a collision dominated gas.
(ii) Without imposing the restriction of a collision-dominated equilibrium fluid, we could tackle the problem of a collisionless gas in which a small pressure arises because of velocity dispersion (random motions) of the particles making up the system. In [3] it was shown that by neglecting higher powers of the velocity dispersion in the Boltzmann equation, the evolution equation for pressure is given by (13). We remark that in obtaining this equation no assumption about the equation of state was made. Only the absence of collisions and the smallness of the velocity dispersion were used. Despite the evident mathematical similarities to case (i), a crucial difference emerges here: for a collisionless gas the Gibbs equation (13) does not apply and, as we shall show below, this leads to some new results.

In the next section we discuss cosmological settings where the small pressure terms introduced above are included.

III. NON-RELATIVISTIC MATTER PERTURBATIONS WITH PRESSURE

In this section we investigate the role played by a first order pressure in the non-relativistic regime on an FRW background filled, in the first case, by a dust gas and, in the second, by dust and radiation. The point is that while the idealized background can be described by dust or dust and radiation, as soon as the dust is disturbed (by velocity dispersion or collisions) a small nonzero pressure arises. This means that the pressure is a first order quantity, as it vanishes in the background. This differs from the usual approach which takes an identically zero pressure in the background and in the real (perturbed) spacetime. We relax this assumption by allowing first order corrections to pressure due either to the fluid being in a collision-dominated regime (case (i) above) or through velocity dispersion without collisions (case (ii) above). This enables us to initiate a theoretical discussion of the effects being studied mainly qualitatively or numerically in the papers [1]. One-component and two-component models will be considered in turn. We note here that the perturbations will involve perturbations of the entropy. The precise relation of the covariant description of entropy perturbations to the metric description (13) is not straightforward and will not be discussed here.

A. Dust background

For simplicity we use an Einstein-de Sitter background. As stated above we focus on perturbations for which pressure is a first order quantity. From the previous section we have that, both in case (i) and case (ii), $p$ is given by the evolution equation (15) leading to (16)

$$p = p_0 \left( \frac{a_0}{a} \right)^5. \quad (18)$$

As $p$ is now a first order quantity we get from (4) and (5),

$$\ddot{\delta}_a - 2H\dot{\delta}_a + \frac{3}{2}H^2\delta_a + \frac{a}{\rho}D_aD^2p = 0, \quad (19)$$

where the momentum balance equation

$$(\rho + p)\dot{u}_a = -D_a p, \quad (20)$$

has been used. Note that for large-scale perturbations the Laplacian term is negligible and we recover the standard evolution equation for dust. Hence, as expected, pressure plays a role only at small scales.

B. Collision dominated fluid

At this point we distinguish between the two cases discussed in section two. If we assume that the fluid is in collision-dominated equilibrium, then the Gibbs equation holds which, together with the equation of state (13), leads to

$$D_a p = 0, \quad (21)$$

3Such a dark matter model has recently been described by Hogan and Dalcanton [4]
at first order, reducing (19) to the standard equation valid for dust at any scale. Expression (21) implies that the spatial gradient of \( p \) is a gauge-invariant second order quantity since it vanishes at first order. This allows us to go beyond first order and derive a second order equation for the pressure as we will now show. A fully covariant and gauge-invariant treatment for second order perturbations is not possible at the moment. The point is that in order to construct \( n \)-order gauge-invariant variables following Ellis-Bruni, we have to ensure that they are constant (usually zero) at all orders below \( n \) [11]. In particular it is not clear how to find (if at all possible) a second-order meaningful variable for the perturbed density in such a way that it vanishes at zero and first orders. Bruni and co-workers provide a systematic method to tackle relativistic perturbations beyond the linear order, but one has to pay the price of losing the gauge-invariant character [12].

It is possible, however to obtain a second order gauge invariant constraint on the perturbations from the above results. Operate on equation (15) with \( a \frac{D}{\rho} \) and use the identity (26) below to obtain

\[
\dot{p}_a + p_a \frac{\dot{\rho}}{\rho} + \frac{5}{3} \theta_3 p_a + \frac{5}{3} \theta p_a = 0
\]

and then use,

\[
\dot{\delta}_a = -\theta_a + O(2)
\]

\[
\dot{\rho} = -3 \left( \frac{a}{a_0} \right) \rho + O(1)
\]

derived from equations (2) and (14) and where \( O(n) \) indicates quantities of order \( n \). Note that \( p_a \) is of order \( O(2) \).

After a change in the independent variable the equations (22) become

\[
p'_a + 2 \left( \frac{a_0}{a} \right) p_a = \frac{5p}{3\rho} \delta'_a,
\]

where a prime denotes a derivative with respect to \( a/a_0 \). According to the argument above and the fact that \( p \) and \( \delta_a \) are first order quantities, equation (23) contains only second order quantities. It is one of the set of second order equations. We treat it as a constraint on the gauge invariant pressure perturbations in the sense that a full solution to all the second order perturbation equations will have to satisfy (23) although the actual solution may be more restricted. With this in mind we proceed to solve it on a dust background. In equation (23) we use equations (16) and (29), below, for the dust background and the standard first order solution for dust

\[
\delta_a = K^+_a \left( \frac{a}{a_0} \right) + K^-_a \left( \frac{a}{a_0} \right)^{-3/2},
\]

where \( K^+_a \) and \( K^-_a \) are constants of order \( O(1) \), to obtain

\[
p_a = A_1 K^+_a \left( \frac{a}{a_0} \right) + A_2 K^-_a \left( \frac{a}{a_0} \right)^{7/2} + K^*_a \left( \frac{a}{a_0} \right)^2,
\]

\[
A_1 \equiv \frac{5p_0}{3\rho_0 a_0} \equiv A_2 a_0^{5/2}.
\]

Here \( A_1, A_2, K^+_a \) and \( K^-_a \) are constants of order \( O(1) \), so the products are of order \( O(2) \) and \( K^*_a \) are constants of order \( O(2) \). The solution (25) is a simple and exact solution giving a limitation on the evolution of a second order quantity in the Ellis-Bruni formalism. The point is that we can apply the formalism because \( D_a p \) vanishes at the zero and at the first order approximation.

C. Collision free fluid

The more interesting and physically appealing situation is that of a collisionless gas for which the foregoing arguments are not applicable. Applying the operator \( aD_a D^2 \) to (14) and using the first order identity

\[
(aD_a f) = aD_a \dot{f}.
\]

\[4\] Since \( p_a \) is second order, the first order identity gives rise to second order terms and any corrections are of higher order.
where \( f \) denotes any first order scalar or tensor quantity, we get
\[
(aD_aD^2p)' = -7 \left( \frac{a_0}{a} \right) aD_aD^2p,
\]
and hence
\[
aD_aD^2p = A_0^0 \left( \frac{a_0}{a} \right)^7, \quad A_0^0 \equiv aD_aD^2p|_{a=a_0}, \quad \dot{A}_a^0 = 0.
\]

For a dust background
\[
H = \sqrt{\frac{\rho_0}{3}} \left( \frac{a_0}{a} \right)^{3/2}, \quad \rho = \rho_0 \left( \frac{a_0}{a} \right)^3,
\]
so equation (19) reduces to
\[
\delta''_a + \left( \frac{H'}{H} + 3 \frac{a_0}{a} \right) \delta'_a - \frac{3}{2} \left( \frac{a_0}{a} \right)^2 \delta_a = \frac{A_0^0}{\rho_0 H^2} \left( \frac{a_0}{a} \right)^6,
\]
which can be solved to give
\[
\delta_a = K^+_a \left( \frac{a}{a_0} \right) + K^-_a \left( \frac{a_0}{a} \right)^{3/2} - \frac{3A^0_0}{\rho_0^2} \left( \frac{a_0}{a} \right) \delta_a
\]
where \( K^\pm_a = 0 \). Thus a new non-adiabatic mode is obtained which decays, albeit at a slower rate than the standard decaying mode for dust. Although the effect of the mode will be at small scales, as desired, the fact that it is decaying means that it is unlikely to be significant.

**D. Dust-radiation background**

The novelty in this subsection is that the dynamics of the background is given by a decoupled mixture of dust and radiation. We will consider perturbations of the matter component which again will be taken as dust in the background but with a non-vanishing first order pressure in the real spacetime. This situation mimics that of the cosmological fluid around the time of the transition from the radiation to the matter dominated era with a matter component which is acquiring a major role in the dynamics. In the following \( p_r \), \( \rho_r \) and \( \rho_m \) denote the zero order pressure and energy density of radiation and matter respectively, whereas \( p_m \) is the first order pressure of matter as given by (16).

Assuming that the fluids are decoupled and share the same 4-velocity we get
\[
(\rho_r + p_r)\dot{u}_a + D_a p_r = 0, \quad \rho_m \dot{u}_a + D_a p_m = 0.
\]
From these equations and the equation of state for radiation we find that
\[
\frac{\nabla a p_r}{\rho_r + p_r} = \frac{\nabla a p_m}{\rho_m} \Rightarrow \delta_a^r = 4 p^m_a,
\]
where
\[
\delta_a^r = a \frac{\nabla a p_r}{\rho_r}, \quad p^m_a = a \frac{\nabla a p_m}{\rho_m}.
\]

\(^5\)Using the same velocity is valid when the two fluids are coupled. Here it is a simplifying assumption which enables us to provide an analytical treatment of a problem, that has only been treated numerically up to now, and we are able to obtain new results. The role of the assumption is made explicit, in particular in the comment following equation (19), where it is relevant.
Note that if we were to keep the matter component as dust both in the background and in the real spacetime, we would have ended up with \( \dot{u}_a = 0 = D_a p_r \), i.e. radiation perturbations would not have been allowed.

In order to get an evolution equation for \( \theta_a \) we start with the Raychaudhuri equation

\[
\dot{\theta} + \frac{1}{3} \theta^2 - D^a \dot{u}_a + \frac{1}{2} (\rho + 3p) = 0, \tag{36}
\]

where \( \rho = \rho_r + \rho_m \) is the total energy density and \( p = p_r + p_m \) is the total pressure. We have to be very careful in dealing with \( p \) since in the background \( p = p_r \) but when we apply the spatial derivative operator we get \( D^a p = D^a p_r + D^a p_m \). Bearing all this in mind the evolution equation for \( \theta_a \) becomes

\[
\dot{\theta}_a + 2 H \theta_a = a D_a D^b \dot{u}_b - \frac{9}{2} H^2 a \dot{u}_a (1 + w) - \frac{1}{2} (8 \rho_r p_a^m + 3 \rho_m p_a^m + \rho_m \delta_a^m),
\tag{37}
\]

where equation (14) has been used.

Before dealing with the acceleration term we recall the identity (4)

\[
D^2 (D_a f) = D_a (D^2 f) + \frac{2}{3} (\rho - 3H^2) D_a f + 2 f \text{curl} \omega_a.
\tag{38}
\]

From this identity and bearing in mind that \( p_m \) is a first order quantity, we have

\[
a D_a D^b \dot{u}_b = -a \frac{D_a D^2 p_m}{\rho_m} = -a \frac{D^2 D_a p_m}{\rho_m} = -D^2 p_a^m.
\tag{39}
\]

It is important to note that the result obtained from commuting the operators \( D_a \) and \( D^2 \) would have been different if we had used \( p_r \) instead of \( p_m \). In the latter case a first order term \( \sim \rho_c \text{curl} \omega_a \) would have arisen. This means that consistency requires

\[
\text{curl} \omega_a = 0,
\tag{40}
\]

i.e., in a radiation-matter decoupled mixture for which (a) both fluids share the same 4-velocity and (b) the matter pressure arises from perturbations so that \( p_m \) is a first order quantity, \( \text{curl} \omega_a \) vanishes at first order.

Changing the independent variables from \( t \) to \( (a/a_0) \), we get the evolution equation for \( \theta_a \),

\[
\theta'_a + 2 \theta_a \left( \frac{a_0}{a} \right) = \left( \frac{a_0}{a} \right) \left\{ \frac{9H}{2} \left( 4 + 3 \frac{\rho_m}{\rho_r} \right) p_a^m - \frac{D^2 p_a^m}{H} - \frac{1}{2H} \left[ p_a^m (8 \rho_r + 3 \rho_m) + \rho_m \delta_a^m \right] \right\}.
\tag{41}
\]

The background equations for dust plus radiation are

\[
\rho = \rho_m + \rho_r = \frac{3}{\beta a^4} (1 + a\alpha), \quad p = p_r = \frac{1}{\beta a^4}, \quad H^2 = \frac{1}{\beta a^4} (1 + a\alpha),
\tag{42}
\]

where

\[
\beta \equiv \frac{3}{\rho_0 a_0^3}, \quad \alpha \equiv \frac{\rho_0^m}{\rho_0 a_0}.
\tag{43}
\]

From the evolution equation for \( p \) for a collisionless gas

\[
\dot{p}_m = -\frac{5}{3} \theta p_m,
\tag{44}
\]

and using the equation (21) we have

\[
p_a^m = -2H p_a^m \Rightarrow \left( p_a^m \right)' = -\frac{2}{a} p_a^m,
\tag{45}
\]

which gives

\[
p_a^m = K_a^m \left( \frac{a_0}{a} \right)^2, \quad \dot{K}_a^m = 0.
\tag{46}
\]
Using similar reasoning for $D^2 p^m_a$ and with the help of the identity
\[(D^2 f) = D^2 \dot{f} - 2H D^2 f + \dot{f} D^2 \dot{u}_a,\] (47)
from (4), we get
\[D^2 p^m_a = M^m_a \left( \frac{a_0}{a} \right)^4, \quad M^m_a = 0.\] (48)

We now define the scalar parts of the perturbations by
\[\delta \equiv a D^a \delta_a, \quad Z \equiv a D^a \theta_a.\] (49)
From (46) it follows that
\[a D^a p^m_a = K \left( \frac{a_0}{a} \right)^2, \quad \dot{K} = 0,\] (50)
where $K = a D^a K^m_a$, and that
\[a D^a [a D_a (D^b \dot{u}_b)] = -M \left( \frac{a_0}{a} \right)^4, \quad \dot{M} = 0,\] (51)
where $M \equiv a D^a M^m_a$. From equations (33) and (34) it follows that
\[a D^a (a \dot{u}_a) = -K \left( \frac{a_0}{a} \right)^2.\] (52)

Operating on equation (37) with $a D_a$ gives
\[\dot{Z} + 2HZ + \frac{1}{2} \rho_m \delta = -M \left( \frac{a_0}{a} \right)^4 - K \left( \frac{a_0}{a} \right)^2 \left[ 4\rho_r + \frac{3}{2} \rho_m - \frac{9}{2} (1 + w) H^2 \right].\] (53)

For the matter scalar perturbations, $w = 0$ and equation (2) implies $Z = -\dot{\delta}$. The evolution equation for $\delta$ written in terms of the scaled independent variable
\[x = a a_0 = \frac{\rho^m_0 a}{\rho^0_0 a_0}\]
is
\[\delta'' + \frac{3x + 2}{2(x+1)} \delta' - \frac{3}{2} \frac{1}{x(x+1)} \delta = F(x),\] (54)
where the prime now denotes a derivative with respect to the new independent variable $x$ and
\[F(x) = \frac{3}{1 + x} \left[ \frac{M}{\rho^0_0 x^2} + \frac{5}{2} K \left( \frac{\rho^m_0}{\rho^0_0} \right)^2 \frac{1}{x^4} \right].\] (55)
is the inhomogeneous part. It is immediately apparent that
\[\delta_1 = 3x + 2\] (56)
is a solution to the homogeneous part of equation (54). The method of reduction of order [14] then leads to a complete solution. The second solution of the homogeneous part is
\[\delta_2 = \delta_1 \int \frac{dx}{(3x + 2)^2 x \sqrt{x + 1}},\] (57)
which can be integrated using partial fractions to obtain
\[\delta_2 = \frac{3}{2} \sqrt{x + 1} + \frac{(3x + 2)}{4} \ln \left[ \frac{\sqrt{x + 1} - 1}{\sqrt{x + 1} + 1} \right].\] (58)
The homogeneous form of equation (54) was obtained by Mészáros [15] in 1974 who obtained a solution by transforming the equation to a hypergeometric form. Groth and Peebles [16] obtained the solutions (56, 58) and gave the asymptotic behaviour

\[ \delta_2 \approx \frac{1}{2} \ln(\tau/8), \quad x \leq 1, \]
\[ \delta_2 \approx -\frac{8}{45\tau}, \quad x \geq 1, \]

where

\[ \tau = \frac{1}{\sqrt{3}} \left( \frac{\rho_m}{\rho_r} \right)^{3/2} t. \]

The results are further described in Peebles’s book [17].

The full solution to the inhomogeneous equation (54) is given by

\[ \delta = C_1 \delta_1 + C_2 \delta_2 + \delta_1 \int \frac{1}{E \delta_1} \left( \int E \delta_1 F dx \right) dx, \]  
(59)

where \( E = \exp \left( \int (3x + 2)[2x(x + 1)]^{-1} dx \right) = x\sqrt{x + 1}. \)

The expression \( E \delta_1 F \) can be written in the form

\[ E \delta_1 F = \frac{1}{\sqrt{x + 1}} \left\{ A_0 + \frac{A_1}{x} + \frac{A_0}{3x^2} + \frac{A_1}{3x^3} \right\} \]  
(60)

where \( A_0 = 9M/\rho_0^m, \quad A_1 = \frac{45}{2} K \left( \rho_0^m/\rho_0^r \right)^2. \) From this the particular integral can be calculated term by term. After integration, the first term \( \delta_{p1} \) of the particular integral is given by

\[ \delta_{p1} = \delta_1 \int \frac{1}{E \delta_1} \left( \int \frac{A_0}{\sqrt{x + 1}} dx \right) dx = A_0 + \frac{A_0}{2} (3x + 2) \ln \left( \frac{x}{3x + 2} \right). \]

The remaining terms are more difficult to integrate. From the first term it appears that the full inhomogeneous solution will contribute new modes. Among these it will contribute a constant mode and may contribute a mode similar to the second term in the solution \( \delta_2 \) and hence modify the coefficient of the factor \((3x + 2)\) which could affect the asymptotic behaviour. The significance of this has to be viewed with caution because the remaining, still to be determined terms, may contribute further modes or cause cancellations. From the physics one is led to expect that the contribution of a small pressure perturbation will only be decaying modes.

Two other features of equation (54) are worthy of note. First, for large \( x \) the asymptotic form of the homogeneous equation is the usual equation for first order perturbations in dust. Second, even for initial conditions in which \( \delta \) and its first derivative with respect to \( x \) are zero, perturbations will arise from the influence of the function \( F(x) \). In other words small pressure effects or number density fluctuations can source density perturbations. This is possibly more significant, for the formation of structure, than the modification of the modes.

IV. CONCLUSION

For analytical as opposed to numerical modelling of the evolution of inhomogeneities in the universe, it is conventional to use a fluid approximation. At a detailed level this is at variance with reality because the matter is generally more particulate than hydrodynamics allows. Also most of the analytical literature assumes that the cosmological non-relativistic matter is dust. If we take the particulate nature into account, then at early times weak self-interactions or small random motions of collisionless matter give rise to a small but non-zero pressure in a fluid model. With this motivation we have discussed some of the implications of including a first order pressure in the matter distribution for fluid models.

We have derived the relation between the energy density perturbations and the number density perturbations for the isentropic case and we show that in the simple non-isentropic case with \( p/\rho = \text{constant} \) a new stationary mode appears. This new mode will affect the formation of structure at certain stages in the evolution of the inhomogeneity.
In the main body of the paper we derive an exact solution to the energy density perturbation equation when the matter pressure is non-zero at first order. The implications of assuming a non-zero first order matter pressure are quite deep as is illustrated in equation (25), which is an equation for second order gauge invariant quantities, and in equation (33), which together with (32) shows that if $D_a p_m$ were equal to 0, then $\dot{u}_a = 0 = D_a p_r$. This would reduce the problem to the standard case.

The complete solution (59) is different from the usual solutions used in CDM approximations and from the Mészáros solution. Having such a solution may be useful as an analytical tool for understanding some features of recent numerical work on weakly self-interacting or warm dark matter models in the non-linear regime [1]. This is a subject of further investigation.

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