Prior information and inference of optimality in thermodynamic processes

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Abstract
We propose a Bayesian inference rule to derive the prior distribution function for a constrained thermodynamic process with incomplete information. Based on this prior, we develop procedures to estimate the work extracted from a heat engine operating between two finite reservoirs. In particular, we find that the optimal work extractable can be inferred with very good agreement which extends to the far-from-equilibrium regime. The estimate for efficiency is shown to follow a universal behavior beyond the linear response term, \( \eta \approx \eta_c/2 + (\eta_c)^2/8 \), where \( \eta_c \) is the Carnot bound. Estimation of this feature can be ascribed to a symmetry with respect to different allowed inferences, with each assigned an equal weight. In contrast to finite-time irreversible models considered in the literature, this universality holds for a reversible model of a heat engine but with incomplete information.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Subjective probability or Bayesian inference seeks to quantify uncertainty in an inferential problem, by taking into account the prior information [1, 2]. The uncertainty may be in regard to the specific values taken by the system parameters. This uncertainty is captured by postulating a (prior) probability distribution, which expresses the degrees of our beliefs in the values taken by an uncertain parameter. The choice of an appropriate prior distribution has been a crucial issue in the Bayesian analysis, which may for a long time have hampered its development and general acceptance [3]. In recent years, however, the Bayesian methods have found applications in many different areas of research, such as astronomy [4], cosmology [5], economics [6], machine learning [7], human cognition [8] and so on. Even if the uniqueness of a prior corresponding to a given problem may be hard to establish, an appropriate one can still be motivated by using certain principles of coherence, our expectations on rational
grounds about the performance of the system, or from the symmetry contained within the problem [2, 9].

To formulate the problem in the present context, we consider a system which depends on two parameters \( T_1 \) and \( T_2 \), where each parameter lies within a specified range, \([T_{\text{L}}, T_{\text{H}}]\). Let us assume a one-to-one relation, \( T_1 = F(T_2) \), which assigns a unique value of \( T_1 \) for a given value of \( T_2 \), and vice versa. However, we shall assume ignorance of the exact values of any of the two parameters. In this paper, we make an inference, which has this subjective flavor due to our lack of complete information, to estimate the expected behavior of the above-mentioned system. This will be achieved by arriving at the prior distribution appropriate to the problem. We are able to show that for certain thermodynamic systems, the behavior inferred for the maximum work extraction process is the same as the optimal behavior of the system in the presence of complete information. This equivalence is obtained in the near-equilibrium regime, but extends beyond the linear response behavior. The present approach was first proposed in [10] and applied to quantify incomplete information in models of quantum and classical heat engines [11–13]. An interesting conclusion was that the optimal behavior of the engine is expected at certain well-known efficiencies, such as the Curzon–Ahlborn efficiency [14].

In the case of exact information, the specific values of both parameters are known. In particular, if we know the value of one parameter, say \( T_2 \), it also implies a specific value of \( T_1 \), given by \( T_1 = F(T_2) \). Now however, we assume that these values are uncertain. Still, due to the relation \( F(\cdot) \), there is essentially one uncertain parameter in the problem, so we seek to attribute prior probabilities for the likely values of either \( T_1 \), or \( T_2 \). From the Bayesian perspective, these probabilities serve to quantify the prior information i.e. the information which exists in the defined problem before any measurement or experiment.

To proceed, we make the following assumptions:

(i) the state of knowledge of an observer is the same whether the observer chooses to quantify the uncertainty in parameter \( T_1 \) or \( T_2 \). This seems plausible, because each parameter is defined within the same interval and the nature of each is identical (i.e. we assume that they represent the same physical quantity, but perhaps for different subsystems). So for each parameter, we choose the same form of the prior distribution function, \( P(T_1) \) and \( P(T_2) \), respectively.

(ii) we assign the same probabilities for those values of \( T_1 \) and \( T_2 \) which are related by the given function \( F(\cdot) \):

\[
P(T_1) \, dT_1 = P(T_2) \, dT_2.
\]

The above relation quantifies the natural assumption that in the face of incomplete knowledge, the degree of belief in a certain value of \( T_1 \) to lie in a small range \([T_1, T_1 + dT_1]\), is the same as the degree of belief in the corresponding value of \( T_2 \) to lie in a small range \([T_2, T_2 + dT_2]\), where the specific \( T_1 \) and \( T_2 \) are related via the given function \( F(\cdot) \). The task then is to solve for the function \( P \), using the prior information contained in the function \( F(\cdot) \). As pointed out above, the possibility of priors based on other reasonable-looking criteria cannot be ruled out. Still, a use of the well-known maximum entropy principle [9] is not clear here as the prior information is not in the form of constraints on average values.

Once the prior is specified, the next step is to find the estimate for \( T_i \) (\( i = 1 \) or \( 2 \)), defined as \( \overline{T}_i = \int T_i P(T_i) \, dT_i \), the average value over the prior. Usually, an estimate of any quantity \( Q \) which is a function of \( T_i \) is also calculated as average value over the prior: \( \overline{Q} = \int Q P(T_i) \, dT_i \). In this paper, we also propose and compare a different strategy which is to estimate the quantities that are functions of \( T_i \), by the replacement \( T_i \rightarrow \overline{T}_i \). Thus the estimate of \( Q \) is defined by \( \overline{Q} = Q(\overline{T}_i) \).
The paper is organized as follows. In section 2, we present the physical problem modeled by the above scenario and determine the corresponding prior. The estimates for physical quantities, such as work and efficiency, are carried out in section 3, followed by a discussion in section 4 on the results obtained and an outlook for future research.

2. The model

It is easy to visualize a physical analog of the abstract problem considered in the introduction. Consider a pair of identical thermodynamic systems, each in its own equilibrium state, at respective temperatures of \( T_+ \) and \( T_- \). Assume that \( T_+ > T_- \). Let the fundamental thermodynamic relation specify the entropy of each system as \( S \propto U^{\omega_1} \), where \( U \) is the internal energy and \( \omega_1 \) is a known constant. The constant of proportionality above may depend on the system volume, number of particles and so on. Such a relation is encountered in well-known physical systems, whose behavior in a certain range of temperatures can be approximated with \( \omega_1 \) value, for example, \( \omega_1 = 1/2 \) (ideal Fermi gas), \( \omega_1 = 3/5 \) (degenerate Bose gas) and \( \omega_1 = 3/4 \) (black-body radiation). Even the classical ideal gas can also be treated in the limit \( \omega_1 \to 0 \).

Using the basic definition, \( (\partial S/\partial U)_T = 1/T \), we get: \( U \propto T^{1/(1-\omega_1)} \). Thus the temperature dependence of entropy is given by: \( S \propto T^{\omega_1/(1-\omega_1)} \). In the following, we restrict ourselves to the case \( 0 < \omega_1 < 1 \), which implies that the systems considered have a positive heat capacity.

Now, we allow an interaction (to be specified below) between the two systems, whereby their temperatures may change and take on the values \( T_1 \) and \( T_2 \) respectively, which obey the following relation (the function \( F \)):

\[
T_1 = (T_+^{\omega_1} + T_-^{\omega_1} - T_e^{\omega_1})^{\frac{1}{\omega_1}},
\]

where \( \omega = \omega_1/(1-\omega_1) \). Physically, this dependence is obtained if we regard the interaction as a reversible process, conserving the total entropy of the composite system \[15\]. This implies \( \Delta S = \Delta S_1 + \Delta S_2 = 0 \), where \( \Delta S = S_{\text{final}} - S_{\text{initial}} \). Moreover, we assume that the systems are coupled to a work source, so that one can extract work (\( W \)) during this process, which is equal to the decrease in internal energy of the total system, \( W = -\Delta U \). The expression for work (again absorbing the constant of proportionality) is:

\[
W = (T_+^{\frac{1}{1-\omega}} + T_-^{\frac{1}{1-\omega}}) - (T_1^{\frac{1}{1-\omega}} + T_2^{\frac{1}{1-\omega}}).
\]

It is well-known that one may extract work until the two systems achieve a common temperature \( T_1 = T_2 = T_e \) \[15\], given by

\[
T_e = \left( \frac{T_+^{\omega_1} + T_-^{\omega_1}}{2} \right)^{\frac{1}{\omega_1}}.
\]

For the intermediate stages, we may regard \( W \) as a function of \( T_2 \) only, by substituting \( T_1 \) from equation (2) into equation (3),

\[
W = T_+^{\frac{1}{1-\omega}} + T_-^{\frac{1}{1-\omega}} - (T_+^{\omega_1} + T_-^{\omega_1} - T_e^{\omega_1})^{\frac{1}{\omega_1}} - T_2^{\frac{1}{1-\omega}}.
\]

We choose to do the following analysis in terms of the parameter \( T_2 \). The symmetry between \( T_1 \) and \( T_2 \) in equation (3), indicates that we can equivalently use \( T_1 \) as the uncertain parameter.

2.1. The prior

Now let us assume ignorance of the extent to which the process of work extraction proceeds. This implies that we lack information about the final temperatures \( T_1 \) and \( T_2 \). In view of the incomplete information, we propose a prior distribution for these temperatures, from which
we make the estimate for temperatures and other thermodynamic quantities. Based on the procedure outlined in the introduction, and incorporating equation (2) in (1), we obtain, rather straightforwardly, the normalized prior distribution as:

\[ P(T_2) = \frac{\omega T_2^{\omega - 1}}{(T_2^\omega - T_2^{-\omega})}. \]  

It is understood that \( P(T_2) \) is conditioned on the values of \( T_+, T_- \) and \( \omega \). Further, for the particular case of a classical ideal gas, we have \( \omega \to 0 \), and then the prior takes the form: \( P(x) \propto 1/x \).

Now the expected value of \( T_2 \) is:

\[ T_2 = \frac{\int_{T_-}^{T_+} T_2 P(T_2) \, dT_2}{\int_{T_-}^{T_+} P(T_2) \, dT_2}. \]  

We also choose for simplicity, \( T_+ = 1 \) and \( \theta = T_-/T_+ \). After solving the above integral, we obtain

\[ T_2 = \omega \frac{1}{1 - \theta^\omega}. \]

3. Results

3.1. Work

The estimate for the work extracted is obtained by substituting \( T_2 \) in place of \( T_2 \) in (5):

\[ \tilde{W}_p = 1 + \theta^\omega - \left( 1 + \theta \omega \right) - \left( \omega \left( 1 - \theta^\omega \right) \right)^\omega - \left( \omega \left( 1 - \theta^\omega \right) \right)^1. \]

Here subscript \( p \) refers to the power-law prior. This procedure is different from the standard way in which the work may be defined as the average value \( \int W(T_2) P(T_2) \, dT_2 \), which we will discuss later.

In the near-equilibrium regime, if we expand \( \tilde{W}_p \) about \( \theta = 1 \), we obtain

\[ \tilde{W}_p \approx \frac{1}{1 - \alpha \omega} \left( 1 - \theta \right)^2 + \frac{1 - 2\omega}{(1 - \omega)^2} \left( 1 - \theta \right)^3 + O\left( 1 - \theta \right)^4. \]

Now a reasonable estimate for work must have an upper bound, which is the optimal work (\( W_o \)) that can be extracted from this set-up, with complete information. For optimal work, the final temperatures of the two systems are given by equation (4). So we have

\[ W_o = 1 + \theta^\omega - 2 \left( \frac{1 + \theta^\omega}{2} \right)^1. \]

In figure 1, we plot the behavior of \( \tilde{W}_p \) and \( W_o \) versus the ratio \( \theta \) of the initial temperatures. The resemblance between the estimated and optimal work is striking. A significant fact is that if we expand \( W_o \) near equilibrium, we obtain the same expansion as in equation (10) i.e. a remarkable agreement up to third order between the estimated and the optimal work. It may be expected that the estimate would, in general, depend on the prior used. So is any other form of prior relevant for our problem? As an alternative, we consider the uniform prior. Note that the power-law form of the prior was derived by making use of the functional relation between \( T_1 \) and \( T_2 \). This relation was already used in the expression for work, equation (5), to make it a one-parameter problem. Now, somehow, if one prefers to ignore this information (assumption
Figure 1. Work as a function of the ratio of initial temperatures, $\theta$; (a) ideal classical gas ($\omega_1 \to 0$), (b) ideal Fermi gas ($\omega_1 = 1/2$), (c) degenerate Bose gas ($\omega_1 = 3/5$), and (d) black-body radiation ($\omega_1 = 3/4$). The dashed curve is for $W_o$, the thin curve is for $\tilde{W}_p$, and the thick curve is for $\tilde{W}_u$. In (b), the three curves overlap one another.

(iii) for the purpose of deriving the prior, and relies on assumption (i) only, then we may use a uniform prior. This prior incorporates minimal information about the parameter and depends only on its range. With the range being $[T_-, T_+]$, and $T_+ = 1$, the uniform prior is given as $P(T_2) = 1/(1 - \theta)$. So the corresponding estimate for temperature is: $T_2^e = (1 + \theta)/2$. The estimate for work is then

$$\tilde{W}_u = 1 + \theta \frac{1}{1+\omega_1} - \left(1 + \theta \omega_1 - \frac{1 + \theta}{2} \right)^{\frac{1}{1+\omega_1}} - \left(1 + \theta \right)^{\frac{1}{1+\omega_1}}. \quad (12)$$

In figure 1 we also present $\tilde{W}_u$ with different values of $\omega_1$. Note that in the near-equilibrium regime, the uniform prior also yields an estimate of work, approximated as equation (10). It is clear from the graphs that far from equilibrium, we may see deviations between the behavior inferred from the uniform prior and the power-law prior. In general, the results from the latter prior stay much closer to the optimal behavior as compared to the estimates from the uniform prior.

Finally, we also consider the estimates of work defined in the standard way as:

$$\langle W \rangle = \int_{T^-}^{T^+} WP(T_2) dT_2.$$  

For the power-law prior, we obtain

$$\langle W \rangle_p = 1 + \theta \frac{1}{1+\omega_1} - 2 \left( \frac{\omega_1}{1 + \omega_1} \right) \left( \frac{1 - \theta}{1 + \theta} \right)^{\frac{1}{1+\omega_1}}. \quad (13)$$
Figure 2. Work as a function of $\theta$; (a) ideal classical gas ($\omega_1 \rightarrow 0$), (b) ideal Fermi gas ($\omega_1 = 1/2$), (c) degenerate Bose gas ($\omega_1 = 3/5$), (d) black-body radiation ($\omega_1 = 3/4$). The dashed curve is for $W_o$, thin curve is for $\langle W \rangle_p$, and thick curve is for $\langle W \rangle_u$.

On the other hand, using the uniform prior, we obtain

$$\langle W \rangle_u = 1 + \theta^{\omega_1} \left[ \frac{(1 - \omega_1)(1 - \theta^{\omega_1})}{(2 - \omega_1)} - \frac{(1 + \theta^{\omega_1})^2}{(1 - \theta)} \right] \times \left[ _2F_1 \left( -1 + \frac{1}{\omega_1}, -\frac{1}{\omega_1} ; 1 \omega_1 ; 1 + \theta^{\omega_1} \right) - \theta_2 F_1 \left( -1 + \frac{1}{\omega_1}, -\frac{1}{\omega_1} ; 1 \omega_1 ; 1 + \theta^{\omega_1} \right) \right],$$

where $_2F_1(a, b; c; z)$ is the ordinary hypergeometric function [16]. The complete behavior of equations (13) and (14) is shown in figure 2.

Upon expanding equations (13) and (14) around equilibrium, we obtain

$$\langle W \rangle_p \approx \langle W \rangle_u \approx \frac{1}{(1 - \omega_1)} \left( 1 - \theta \right)^2 + \frac{1 - 2\omega_1}{(1 - \omega_1)^2} \frac{(1 - \theta)^3}{12} + O[1 - \theta]^4.$$  (15)

Here again, the averages from both priors are identical for the near-equilibrium condition. Moreover, we notice a scale factor of 2/3 between equations (15) and (10). The relative magnitudes of the estimates will be elaborated on in section 4.

3.2. Efficiency

The estimation of the efficiency is subtler than that of the work discussed above. First, we observe that the input heat ($Q_1$) which is equal to the energy lost by the initially hot reservoir,
can be written in two equivalent ways:
\[ Q_1(T_1) = T_+^{1/p} - T_1^{1/p}, \]  
(16)

or in terms of \( T_2 \), as:
\[ Q_1(T_2) = T_+^{1/p} - (T_+^{1/p} + T_-^{1/p})^{1/p}. \]  
(17)

Note that \( Q_1 \), as a function, is of dissimilar form when expressed in terms of \( T_1 \) or \( T_2 \), unlike the expression for work, so the estimate of the efficiency \( (\eta = W/Q_1) \) can depend on the choice of whether \( T_1 \) or \( T_2 \) is treated as the uncertain parameter. Furthermore, from our basic assumption, the prior for \( T_1 \) is of the same form as for \( T_2 \), and so the expectation value of each is also the same. If we do the estimation by the replacement method as used in the previous section, we obtain \( Q_1(T_1) \) as well as \( Q_1(T_2) \) as two different estimates for the heat input, whereby each will lead to a different estimate for the efficiency. Recall that the estimate of work is not affected by the choice of \( T_1 \) or \( T_2 \). In the following, we do not reproduce the complete formulae, but focus on the near-equilibrium behavior.

Upon expanding the estimate for efficiency with \( T_1 \) as the uncertain parameter, for a small value of \( \eta_c = 1 - \theta \), we obtain:
\[ \eta_{p,1} \approx \eta_c + \left( \frac{4 - 5\omega_1}{1 - \omega_1} \right) \frac{\eta_c^2}{24} + \left( \frac{8 - 19\omega_1 + 9\omega_1^2}{(1 - \omega_1)^2} \right) \frac{\eta_c^3}{96} + O[\eta_c^4], \]  
(18)
\[ \eta_{u,1} \approx \eta_c + \left( \frac{2 - 3\omega_1}{1 - \omega_1} \right) \frac{\eta_c^2}{8} + \left( \frac{12 - 31\omega_1 + 17\omega_1^2}{(1 - \omega_1)^2} \right) \frac{\eta_c^3}{96} + O[\eta_c^4], \]  
(19)

where the indices \( p, u \) denote calculations with respect to the power-law and the uniform prior, respectively. Similarly, upon expanding the estimates for efficiency with \( T_2 \) as the uncertain parameter, we get:
\[ \eta_{p,2} \approx \eta_c + \left( \frac{2 - \omega_1}{1 - \omega_1} \right) \frac{\eta_c^2}{24} + \left( \frac{4 - 7\omega_1 + \omega_1^2}{(1 - \omega_1)^2} \right) \frac{\eta_c^3}{96} + O[\eta_c^4], \]  
(20)
\[ \eta_{u,2} \approx \eta_c + \frac{\omega_1}{8} \frac{\eta_c^2}{(1 - \omega_1)} + \left( \frac{5\omega_1 - 7\omega_1^2}{(1 - \omega_1)^2} \right) \frac{\eta_c^3}{96} + O[\eta_c^4]. \]  
(21)
The complete behavior of the estimates for all temperature gradients, with a chosen \( \omega_1 \) parameter, is shown in figure 3.

Following the comparison with the optimal work in the previous section, it is natural to compare the above estimates with the efficiency at optimal work which behaves near equilibrium as follows:
\[ \eta_o \approx \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \left( \frac{6 - 13\omega_1 + 5\omega_1^2}{(1 - \omega_1)^2} \right) \frac{\eta_c^3}{96} + O[\eta_c^4]. \]  
(22)
It is curious that the estimates in equations (18)–(21) match with equation (22), only up to first order in \( \eta_c \). In contrast, we could estimate the optimal work even beyond the linear term (see equation (10)). The reason for this seeming discrepancy can be traced to the asymmetric nature of the function \( Q_1 \) with respect to \( T_1 \) and \( T_2 \) (equations (16) and (17)). However, each of the estimates \( Q_1(T_1) \) and \( Q_1(T_2) \) seems equally reasonable and it is hard to prefer one over the other. In such circumstances, the best estimate for \( Q_1 \) should be chosen as the equally weighted mean of the two estimates, defined as \( \tilde{Q}_1 = (Q_1(T_1) + Q_1(T_2))/2 \). This, in fact, is an application of Laplace’s ‘principle of insufficient reason’ according to which, given a set of the allowed inferences, if we have no prior information to prefer one inference over another,
Figure 3. Estimates of efficiency as a function of $\theta$, for $\omega_1 = 3/4$, compared with efficiency at optimal work (dashed line); (a) using a power-law prior; (b) using a uniform prior. In figure (a) and (b), the lower solid curve is for $\tilde{\eta}_{p,2}(\eta,2)$ and the upper solid curve is for $\eta$.

we must assign equal probabilities to each of the inferences. Indeed, if we estimate efficiency with this mean estimate of the heat input, then near equilibrium the estimates behave as:

$$\tilde{\eta}_p \approx \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \left(\frac{17 - 35\omega_1 + 11\omega_1^2}{(1 - \omega_1)^2}\right) \frac{\eta_c^3}{288} + O\left[\eta_c^4\right],$$

(23)

$$\tilde{\eta}_u \approx \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \left(\frac{3 - \omega_1 - 7\omega_1^2}{(1 - \omega_1)^2}\right) \frac{\eta_c^3}{96} + O\left[\eta_c^4\right].$$

(24)

In this manner, we are able to recover an agreement with the efficiency at optimal work near equilibrium, up to second order in $\eta_c$.

On the other hand, if our estimates of the work performed and the heat input are calculated as $\langle W \rangle$ and $\langle Q_1 \rangle$, respectively, then a corresponding estimate of efficiency may be defined as: $\eta_{av} = \langle W \rangle / \langle Q_1 \rangle$, where the expectation value is calculated over either the power-law prior or the uniform prior. In this regard, note that the power-law prior ensures that $\langle Q_1 \rangle$ has the same value irrespective of whether $T_1$ is the variable of integration or $T_2$. Thus the information about the labels of the temperatures, which could be distinguished in the function $Q_1$ as in equations (16) and (17), is lost when we evaluate the averages over the power-law prior. Then, near equilibrium the estimates behave as:

$$\eta_{av} \approx \frac{\eta_c}{3} + \frac{\eta_c^2}{9} + O\left[\eta_c^3\right].$$

(25)

Further discussion on the effect of ‘label uncertainty’ on the estimated efficiency, can be found in a recent paper [17].

4. Discussion and summary

The following remarks may help to understand the magnitudes of the estimates relative to the optimal value. It is clear that the work estimated as an average, $\langle W (T_2) \rangle$, of the possible values of work in the range $T_2 \in [T_1, T_2]$, will be less than the maximal value of work, $W_0$, inside this interval. Also, in general $W(\bar{T}_2) \leq W_0$ and the equality is obtained for $\bar{T}_2 = T_c$ only (see equation (4)). Thus either estimate of work, $W(\bar{T}_2)$ or $W(\bar{T}_2)$, is less than or equal to the optimal work. We have seen in the near-equilibrium case that the estimate can approach

1 Here, the use of the uniform prior gives agreement near equilibrium only up to first order: $\eta_{av} = \eta_c/3$. 

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the optimal or maximal value. Furthermore, as the work expression given by equation (3) has a unique maximum in the interval \([T_-, T_+]\), one can apply Jensen’s inequality for concave functions and obtain \( W(T_2) \geq W(T_2) \equiv \langle W(T_2) \rangle \). In other words, the estimate of work obtained in section 3 will be greater than or equal to the estimate obtained by standard averaging. Even so, the manner in which the estimates from an informative prior estimate the optimal behavior, as seen in figure 1, seems remarkable.

In previous works [10, 11], an intriguing connection between the prior probabilities and the thermal characteristics of few-level quantum engines was noticed. This provided a novel derivation for the Curzon–Ahlborn efficiency, usually associated with finite-time irreversible thermodynamic machines. In the present work, we have considered a reversible work extraction process from a heat source/sink set-up, modeled as finite thermodynamic systems. The entropy of each reservoir satisfies the following relation: \( S \propto T^\omega \). Then we assumed ignorance of the exact values of the final temperatures \( T_1 \) and \( T_2 \) of the reservoirs by the end of the considered process. The issue we face here is the estimation of the thermal characteristics for this process. Uncertain about the exact values of the parameters, we took a subjective point of view and regarded the probabilities for the different allowed values of temperatures in a Bayesian sense. Consistency would demand that different observers in possession of equivalent information assign equal probabilities. In the present context, one can either assign probabilities to \( T_1 \) or \( T_2 \), assuming in each case the same form of the distribution function. Further, prior information about the functional relation between \( T_1 \) and \( T_2 \) governing the process is taken into account. This yields an explicit formula for the prior. We have compared the results of the power-law prior with the uniform prior and shown equivalence with the optimal performance of the engine in the near-equilibrium regime, but still beyond the linear response term.

In our analysis, we have discussed two ways of estimating the physical quantities. The standard one is to define the estimates as the average value over the prior \( P(T_2) \). We have explored a different way which assigns the estimate of a derived quantity by replacing the variable \( T_2 \) by its average value \( \bar{T}_2 \). It is observed that even far from equilibrium (\( \theta \) away from unity), the latter method yields estimates of work and efficiency, which are much closer to the optimal characteristics of the process, than the estimates obtained with the standard method of averaging.

We see that irrespective of the method of estimation, the uniform prior estimates are lower and hence further away from the optimal values than the informative prior. This is reassuring as it reflects the fact that the prior information has been incorporated in a consistent way, so that utilizing more information makes us expect a higher work output, in contrast to the uniform prior which involves minimal information.

Finally, we discuss the near-equilibrium behavior of efficiency at optimal work (equation (22)). It indicates a universality up to second order in \( \eta_c \), as the first two terms are independent of the parameter \( \omega_1 \) which is a characteristic of the reservoir. In other words, for the class of reservoirs which obey \( S \propto U^\omega \), the considered work extraction process has this universal behavior of efficiency at optimal work. Such behavior as \( \eta \approx \eta_c/2 + (\eta_c)^2/8 \), has been found in many different models [18–20] where this response occurs for efficiency at the maximum power output. Typically, it arises in finite-time models where it can be attributed to a left–right symmetry on the fluxes [20] or to a dissipation which is symmetric with respect to hot and cold reservoirs [21]. However, our process of work extraction is a reversible one, and to the best of our knowledge this kind of universality for a reversible process has not previously been noted in the literature. Our analysis shows this universality can also be anticipated from a conceptually very different inference-based approach applied to reversible thermodynamic models with incomplete information. Interestingly, the universality in the present context can also be attributed to a certain symmetry, which is to assign equal weight to each allowed
inference of the estimated heat input. Without this symmetry, the universality holds up to linear response only.

In conclusion, we have proposed that certain optimal thermodynamic processes can be estimated quite accurately by a subjective inference procedure. It seems striking that upon quantifying ignorance of the thermodynamic control parameters in a process, one obtains estimates which are very close to the observed behavior, which is actually the optimal one and seen in the case of complete information. Although we have considered rather a textbook model of a heat engine, yet the inference-based approach seems to be applicable to more general situations with an analogous form of constraints and beyond the thermodynamic processes. It may be useful in estimating the optimal characteristics in a more efficient way, where the optimal solution cannot be determined in a closed form and one usually has to resort to a numerical optimization. Finally, from a fundamental perspective, the analysis presented in the paper opens up a deeper issue as to how far the theory of thermodynamics can be derived from rational considerations like an appropriate use of prior information. Further understanding on this issue will also have a bearing on the subtle relation between information and thermodynamics.

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