AX-LINDEMANN-WEIERSTRASS WITH DERIVATIVES AND THE GENUS 0
FUCHSIAN GROUPS

GUY CASALE, JAMES FREITAG, AND JOEL NAGLOO

To Keiji Nishioka on his retirement.

ABSTRACT. We prove the Ax-Lindemann-Weierstrass theorem with derivatives for the uniformizing functions of genus zero Fuchsian groups of the first kind. Our proof relies on differential Galois theory, monodromy of linear differential equations, the study of algebraic and Liouvillian solutions, differential algebraic work of Nishioka towards the Painlevé irreducibility of certain Schwarzian equations, and considerable machinery from the model theory of differentially closed fields.

Our techniques allow for certain generalizations of the Ax-Lindemann-Weierstrass theorem which have interesting consequences. In particular, we apply our results to answer a question of Painlevé (1895). We also answer certain cases of the André-Pink conjecture, namely in the case of orbits of commensurators of Fuchsian groups.

1. INTRODUCTION

In this paper our central work is to prove a series of functional transcendence results for the automorphic functions $j_\Gamma$ associated with a Fuchsian group $\Gamma$ of genus 0. We will also refer to the automorphic function $j_\Gamma$ as a Hauptmodul or uniformizing function of $\Gamma$. Our general results are most easily expressed in the language of model theory and algebraic differential equations, but a special case of our functional transcendence results is what has come to be called the Ax-Lindemann-Weierstrass theorem with derivatives for $j_\Gamma$:

Theorem 1.1. Let $C(V)$ be an algebraic function field, where $V \subset C^n$ is an irreducible algebraic variety defined over $C$. Let $t_1, \ldots, t_n \in C(V)$ take values in $H$ at some $P \in V$ and are geodesically independent\footnote{We say that $t_1, \ldots, t_n$ are geodesically independent if $t_i$ is nonconstant for $i = 1, \ldots, n$ and there are no relations of the form $t_i = \gamma t_j$ for $i \neq j, i, j \in \{1, \ldots, n\}$ and $\gamma$ is an element of the commensurator of $\Gamma$.}. Then the $3n$-functions

$$j_\Gamma(t_1), j_\Gamma'(t_1), j_\Gamma''(t_1), \ldots, j_\Gamma(t_n), j_\Gamma'(t_n), j_\Gamma''(t_n)$$

(considered as functions on $V$ locally near $P$) are algebraically independent over $C(V)$.

One can also describe Theorem 1.1 in more geometric terms. Let $W \subset C^n$ be an algebraic variety which has a nonempty intersection with $H^n$. Theorem 1.1 precisely characterizes

\begin{align*}
2010 \text{ Mathematics Subject Classification.} & \quad 11F03, 12H05, 03C60. \\
 & \quad G. \text{ Casale is partially supported by Math-AMSUD project "Complex geometry and foliations". J. Freitag is partially supported by NSF grant DMS-1700095. J. Nagloo is partially supported by NSF grant DMS-1700336.} \\
^1 & \quad \text{We say that } t_1, \ldots, t_n \text{ are geodesically independent if } t_i \text{ is nonconstant for } i = 1, \ldots, n \text{ and there are no relations of the form } t_i = \gamma t_j \text{ for } i \neq j, i, j \in \{1, \ldots, n\} \text{ and } \gamma \text{ is an element of the commensurator of } \Gamma. 
\end{align*}
those varieties \( W \) whose image under the automorphic function (and derivatives) applied to each coordinate:

\[
\bar{j}_\Gamma : (t_1, \ldots, t_n) \mapsto (j_\Gamma(t_1), j'_\Gamma(t_1), j''_\Gamma(t_1), \ldots, j_\Gamma(t_n), j'_\Gamma(t_n), j''_\Gamma(t_n))
\]

is an algebraic variety. Intuitively, the function \( j_\Gamma \) is highly transcendental, so the varieties obtained in this way should be restricted to a very special class. Indeed, Theorem 1.1 says that if \( \bar{j}_\Gamma(W) \) is an algebraic variety, then \( W \) must have been defined by instances of relation of the form \( t_i = \gamma t_j \) where \( \gamma \) is an element of the commensurator of \( \Gamma \), giving a very restrictive (countable) class of complex varieties coming from the image of \( \bar{j}_\Gamma \).

As we will explain in additional detail below, our methods also allow for more general results, which are most naturally stated in the language of model theory. For instance, statements incorporating other transcendental functions on additional coordinates (such as Weierstrass \( \wp \)-functions and exponential functions on semi-abelian varieties) similar to Theorem 1.6 of [43] will follow from our general result.

Theorem 1.1 is a generalization of Theorem 1.6 of [43] and Theorem 1.1 [44], in which Pila established the special case with one group \( \Gamma = PSL_2(\mathbb{Z}) \) (in [43] without derivatives and later in [44] with derivatives). Theorem 1.1 also overlaps nontrivially with a number of recent results, which we detail next. Note that none of the following results involve the derivatives of the automorphic functions in question. Pila and Tsimerman generalized Theorem 1.6 of [43] to the uniformizing functions associated with the moduli spaces of higher dimensional abelian varieties (their result specializes to Theorem 1.6 of [43] for purposes of comparing with Theorem 1.1). In a different direction, Pila and Tsimerman [47] generalized Theorem 1.6 of [43] to an Ax-Schanuel type statement for the \( j \)-function. In [57], Ullmo and Yafaev prove an Ax-Lindemann-Weierstrass result for the uniformizing functions of cocompact shimura varieties (without derivatives), and so a statement of Theorem 1.1 without derivatives in the case that \( \Gamma \) is arithmetic and cocompact is a consequence of their work. Later, Klinger, Ullmo, and Yafaev [20] removed the assumption of cocompactness, and Gao [13] generalized the result to mixed Shimura varieties. Finally, Mok, Pila, and Tsimerman [31] have established the (more general) Ax-Schanuel theorem for the uniformizing function of a Shimura variety.

The previous Ax-Lindemann-Weierstrass (ALW for short) results discussed above employ various techniques from group theory, complex variables, and number theory, but each one also shares a common element in their approach: a model theoretic tool called \( o \)-minimality. The theory of \( o \)-minimality is a natural generalization of real algebraic geometry to include certain non-oscillatory transcendental functions. It was developed starting in the 1980s by model theorists [59], but in the early 2000s, \( o \)-minimality was connected with various aspects of number theory in part through the work of Pila-Wilkie [48] and Peterzil-Starchenko [39, 42, 41]. The common line of reasoning in the results mentioned in the previous paragraph is to embed the problem in an \( o \)-minimal context by proving that the \( \Gamma \)-automorphic function (restricted to an appropriate fundamental domain) is interpretable in \( \mathbb{R}_{\text{an,exp}} \), an \( o \)-minimal structure in which the definable sets are given by inequalities involving the algebraic functions, the exponential function, and real analytic functions restricted to bounded sets. Following this, certain \( o \)-minimal tools such as the Pila-Wilkie [48] theorem or definable versions of results from complex geometry [40] are
used to detect and characterize algebraic relations. Our approach is completely different, and does not employ the theory of o-minimality at all. Rather, our proof relies on differential Galois theory, monodromy, the study of algebraic and Liouvillian solutions to linear differential equations, differential algebraic work of Nishioka towards the Painlevé irreducibility of certain Schwarzian equations, and considerable machinery from the model theory of differentially closed fields.

Recently there has been a surge in interest around functional transcendence statements of the type in Theorem 1.1 in part due to their connection with a class of problems from number theory called special points conjectures or problems of unlikely intersections; in [13] the Ax-Lindemann-Weierstrass theorem is central to the proof of the André-Oort conjecture for $\mathbb{C}^n$. Each of the other functional transcendence results mentioned above can be applied in certain special points settings. For instance, in [9] Daw and Ren give applications of the Ax-Schanuel conjecture proved in [31]. Our functional transcendence results are no exception - we apply them to certain cases of a special points conjecture called the André-Pink conjecture, following Orr [36, 37]. Numerous variations on the conjecture are possible (depending for instance, on the definition of Hecke-orbits one takes), but we will describe the specific setup next.

The André-Pink conjecture predicts that when $W$ is an algebraic subvariety of a Shimura variety with group $\Gamma$ and $S$ is the orbit of the commutator of $\Gamma$, Comm($\Gamma$), on a point $\bar{a} = (a_1, \ldots, a_n)$, if $W \cap S$ is Zariski dense in $W$, then $W$ is has a very restrictive form, which we will refer to as $\Gamma$-special, which we describe next.

When $\gamma \in \text{Comm}(\Gamma)$, it turns out that $(j_\Gamma(t), j_\Gamma(\gamma t))$ are algebraically dependent and lie on an irreducible curve given by the vanishing of a polynomial in two variables which we will refer to as a $\Gamma$-special polynomial. The $\Gamma$-special varieties are intersections of $\Gamma$-special polynomials and relations of the form $x_i = b_i$ where $b_i$ is in the Comm($\Gamma$)-orbit of $a_i$. Orr [36, 37] proved various special cases of the conjecture (for instance, when $W$ is an algebraic curve). In [11] Freitag and Scanlon used Pila’s ALW with derivatives theorem from [41] to prove the André-Pink conjecture when $\bar{a}$ is assumed to be a transcendental point and $\Gamma$ is commensurable with $\text{PSL}_2(\mathbb{Z})$. In this paper, we generalize that result to allow for an arbitrary Fuchsian group $\Gamma$.

The central idea employed is a beautiful technique which has its origins in the work of Hrushovski [16] and Buium [6]. In order to understand intersections of algebraic varieties with an arithmetically defined set of points (e.g. torsion points on an algebraic group, Hecke orbits, etc.), replace the arithmetic set with a more uniformly defined algebraic object, the solution set of some algebraic differential differential or difference equation.

We replace our arithmetic object (the orbits of the commutators of some discrete groups, $\Gamma$) by the solutions sets of certain differential equations satisfied by the uniformizing functions $j_\Gamma$. An inherent restriction of the technique is that it generally only works for diophantine problems in function fields, hence the assumption that $\bar{a}$ is a tuple of transcendents. In pursuing our approach to the André-Pink conjecture, it becomes necessary to prove more far reaching functional transcendence results than the ALW theorem as stated above; our results are most naturally phrased in terms of the model theory of differential fields, one of the main tools we use to establish our results. One of the chief advantages of this approach is that it leads to an effective solution of our case of André-Pink, that is we are able to give
bounds on the degree of the Zariski closure of the intersection of Comm(Γ_i)-orbits with a variety V, which depend on algebro-geometric invariants of the variety V. So, for instance, if the variety V is a non-special curve (or a variety which does not contain a special curve), we can give a bound on the number of special points contained in the curve.

At the relevant sections of our paper (e.g. 6) we will give equivalent formulations (in algebro-geometric language) of the model-theoretic properties we describe next. We prove that for any Fuchsian group Γ, the set defined by the differential equation satisfied by the uniformizing function j_Γ is strongly minimal and has geometrically trivial forking geometry. This result generalizes work of [11] which covers the cases when Γ is commensurable with \( PSL_2(\mathbb{Z}) \). In particular, our work gives many new examples of geometrically trivial strongly minimal sets in differentially closed fields. This also establishes an interesting new connection between two important dividing lines on the logic and group theory: the differential equation satisfied by \( j_Γ \) is \( \aleph_0 \)-categorical if and only if the group \( Γ \) is not arithmetic. Further, we characterize all instances of nonorthogonality between these sets (each such instance comes from commensurability of two groups \( Γ_1 \) and \( Γ_2 \)). These results also have various interesting consequences related to determining the isomorphism invariants of differentially closed fields, which we will not explore further in this article.

We should also mention that this work also settles a conjecture of Painlevé [38, Page 519], concerning the irreducibility of the differential equations satisfied by \( j_Γ \) for Γ a Fuchsian group. In [34] and [35], Nishioka proved a weak form of Painlevé’s conjecture; various techniques from Nihsioka’s paper have inspired our work.

Acknowledgements. G.C and J.N take this opportunity to thank the organizers of the CIRM meeting “Algebra, Arithmetic and Combinatorics of Differential and Difference Equations” in May 2018, where this research collaboration started.

2. The basic theory

2.1. Fuchsian groups and the associated Schwarzian equations. We direct the reader to [19] and [23] for the basics on Fuchsian groups and the corresponding automorphic functions. The appendices of [60] also give a very detailed introduction to the associated Schwarzian equations.

Let \( \mathbb{H} \) be the upper half complex plane and let \( \mathbb{H} := \mathbb{H} \cup \mathbb{P}^1(\mathbb{R}) \). Recall that \( SL_2(\mathbb{R}) \) and \( PSL_2(\mathbb{R}) \) acts on \( \mathbb{H} \) (and \( \mathbb{H} \)) by linear fractional transformation: for \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{R}) \) and \( \tau \in \mathbb{H} \)

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot \tau = \frac{a\tau + b}{c\tau + d}
\]

This action yields all the orientation preserving isometries of \( \mathbb{H} \).

Let \( Γ \subset PSL_2(\mathbb{R}) \) be a fuchsian group of first kind and of genus zero. We call a point \( \tau \in \mathbb{H} \) a cusp if its stabilizer group \( Γ_\tau = \{ g \in Γ : g \cdot \tau = \tau \} \) has infinite order. On the other hand, for any point \( \tau \in \mathbb{H} \), the group \( Γ_\tau \) is finite and cyclic. A point \( \tau \in \mathbb{H} \) is said to be elliptic of order \( \ell \geq 2 \) if \( |Γ_\tau| = \ell \). By our assumptions on Γ, there are only finitely many

---

2 Irreducibility is closely related to the strong minimality of a differential equation. See [32] for a more complete discussion and history.
elliptic points. If $m_1, \ldots, m_r$ denotes the orders of the elliptic points as well as of those of the cusps (which would be $\infty$'s), then $\Gamma$ is said to have signature $(0; m_1, \ldots, m_r)$. The zero here reflects that $\Gamma$ has genus 0. The group then has the following presentation

$$\Gamma = \langle g_1, \ldots, g_r : s_1^{m_1} = \cdots = s_r^{m_r} = g_1 \cdots g_r = 1 \rangle$$

When one or more of the $m_i$'s are infinity, one simply remove the relations containing the infinite $m_i$'s in the above presentation.

**Example 2.1.** $PSL_2(\mathbb{Z})$ is a Fuchsian (triangle) group of type $(0; 2, 3, \infty)$. Recall that traditionally might consider the following generators of $SL_2(\mathbb{Z})$:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

Nonetheless, by setting $g_1 = -S, g_2 = -T^{-1}S$ and $g_3 = T$ one has that

$$SL_2(\mathbb{Z}) = \langle g_1, g_2, g_3 : g_1^2 = g_2^3 = g_1g_2g_3 = -1 \rangle.$$ 

$PSL_2(\mathbb{Z})$ is obtained from the above using the natural projection $\pi : SL_2(\mathbb{Z}) \to PSL_2(\mathbb{Z})$.

As is well known, $\Gamma$ acts on the set $C_{\Gamma}$ of its cusps and the quotient $\Gamma \setminus H_{\Gamma}$, where $H_{\Gamma} := \mathbb{H} \cup C_{\Gamma}$, is a compact Riemann surface of genus zero. The group $\Gamma$ is said to be cocompact if $C_{\Gamma} = \emptyset$. In other words, if the quotient $\Gamma \setminus H$ is already a compact space. Since $\Gamma \setminus H_{\Gamma}$ is compact and of genus zero, it is bi-rationally isomorphic to the complex points of a projective curve $X(\Gamma)$. By an automorphic function for $\Gamma$, we mean a meromorphic function $f$ on $H$ which is meromorphic at every cusp of $\Gamma$ and which is invariant under the action of $\Gamma$:

$$f(\tau) = f(\Gamma \cdot \tau) \quad \text{for all} \quad g \in \Gamma \text{ and } \tau \in \mathbb{H}.$$ 

One has that the field of automorphic functions $A_0(\Gamma)$ for $\Gamma$ (or equivalently the field of meromorphic functions of $\Gamma \setminus H_{\Gamma}$) is isomorphic to the field $\mathbb{C}(X(\Gamma))$ of rational functions on $X(\Gamma)$. By an Hauptmodul or uniformizer $j_{\Gamma}(t)$ for $\Gamma$ we mean an automorphic function for $\Gamma$ which generates $A_0(\Gamma)$ (and so $\mathbb{C}(j_{\Gamma}) \simeq \mathbb{C}(X(\Gamma))$). We will also write $j_t$ for the map taking $\Gamma \setminus H_{\Gamma}$ bijectively onto the Riemann sphere $\mathbb{P}^1(\mathbb{C})$.

It is well known that $j_t$ is not unique and that it satisfies a third order ordinary differential equation of Schwarzian type:

$$(*) \quad S_{\frac{d}{dy}}(y) + (y')^2 \cdot R_{j_{\Gamma}}(y) = 0$$

where $S_{\frac{d}{dy}}(y) = \left( \frac{y''}{y'} \right)' - \frac{1}{2} \left( \frac{y''}{y'} \right)^2$ denotes the Schwarzian derivative ($' = \frac{d}{dy}$) and $R_{j_{\Gamma}} \in \mathbb{C}(y)$ depends on the choice of $j_{\Gamma}$. Moreover, the ‘shape’ of the function $R_{j_{\Gamma}}$, depends on knowing the fundamental half domain for the $\Gamma$-action on $H$: Let us assume that it is given by a polygon $P$ with $r$ vertices $b_1, \ldots, b_r$ and whose sides are identified by pairs and having internal angles $\alpha_1, \ldots, \alpha_r \pi$. Then

$$R_{j_{\Gamma}}(y) = \frac{1}{2} \sum_{i=1}^{r} \frac{1 - \overline{a_i^2}}{(y - a_i)^2} + \sum_{i=1}^{r} \frac{A_i}{y - a_i}$$





where \( j_\Gamma(b_i) = a_i \) and the \( A_i \)'s are real numbers that do not depend on \( j_\Gamma \) and satisfy some very specific algebraic relations (cf. [60, page 142]).

**Example 2.2.** A well-known example is \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) and \( j_\Gamma \) is the classical \( j \)-function. In this case the equation is given with

\[
R_j(y) = \frac{y^2 - 1968y + 2654208}{y^2(y - 1728)^2}
\]

\( \Gamma = \text{PSL}_2(\mathbb{Z}) \) is an example of a triangle group. In Section 5 the case of the Fuchsian triangle groups is explained in more details. We also direct the reader to [4] where more examples of uniformizers - beyond those attached to triangle groups - are studied.

There is a long tradition of functional transcendence results around automorphic functions. For instance, a very weak form of our results was conjectured by Mahler, and answered by Nishioka:

**Fact 2.3 ([11]).** The Hauptmodul \( j_\Gamma \) satisfies no algebraic differential equation of order two or less over \( \mathbb{C}(t, e^{\pi it}) \), for any \( t \in \mathbb{C} \). The same is true for all \( \Gamma \)-automorphic functions.

Using the Seidenberg’s embedding theorem and the composition rule of the Schwarzian derivative, we also have

**Lemma 2.4** (cf. [11]). Let \( K \) be an abstract differential field extension of \( \mathbb{C}(t) \) generated by \( y_1, \ldots, y_n \) solutions of equation (3). Here \( \mathbb{C} \) is a finitely generated subfield of \( \mathbb{C} \). Then there are elements \( g_1, \ldots, g_n \in \text{GL}_2(\mathbb{C}) \) such that

\[
K \cong \mathbb{C}(t, j_\Gamma(g_1t), \ldots, j_\Gamma(g_nt)).
\]

**Proof.** By the Seidenberg’s embedding theorem, we may assume that \( y_1, \ldots, y_n \) are meromorphic functions on some domain \( U \) contained in \( \mathbb{H} \). Since the \( j_\Gamma \) is a non constant holomorphic function from \( \mathbb{H} \) to \( \mathbb{C} \), there are holomorphic functions \( \psi_i : U \to \mathbb{H} \) such that \( y_i(t) = j_\Gamma(\psi_i(t)) \). Repeating the arguments in [11] - using the composition rule for \( S_\Gamma(y) \) and the fact that \( j_\Gamma(\psi_i(t)) \) is a solution of the equation (2) - we get that \( S_\Gamma(\psi_i(t)) = 0 \). Hence \( \psi_i(t) = g_it \) for some \( g_i \in \text{GL}_2(\mathbb{C}) \).

**Remark 2.5.** Notice that the \( g_1, \ldots, g_n \) are not arbitrary elements of \( \text{GL}_2(\mathbb{C}) \). Indeed, since the \( y_i(t) \)'s are meromorphic on \( U \subset \mathbb{H} \), it must be that \( g_i : U \to \mathbb{H} \). Also, for each \( i \), from the inverse \( g_i^{-1} \) of \( g_i \), we have well defined solutions \( j_\Gamma(g_i^{-1}t) \) and \( j_\Gamma(g_i g_i^{-1}t) \) of (2).

In this paper, depending on the context, we will freely alternate between thinking of solutions of the Schwarzian equation (2) as points in an abstract differential field or as meromorphic functions of the form \( j_\Gamma(gt) \). The latter form will always mean that \( g \) is an element of \( \text{GL}_2(\mathbb{C}) \) that maps (a subset of) \( \mathbb{H} \) to \( \mathbb{H} \).

### 2.2. Arithmetic Fuchsian groups

We have already seen one important dividing line among those \( \Gamma \), which we consider, namely whether or not \( \Gamma \) is cocompact. Another, perhaps even more important (for our work) property that \( \Gamma \) might possess is that of arithmeticity. We will begin by reviewing some key definitions. A standard reference for this subsection is [61]. Throughout \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) is a Fuchsian group of first kind of genus zero.
Let $F$ be a field of characteristic zero and let $A$ be a quaternion algebra over $F$: a central simple algebra of dimension 4 over $F$. Since the characteristic of $F$ is zero, there are elements $i$ and $j$ in $A$ and $a, b \in F^*$ such that

$$i^2 = a, \quad j^2 = b, \quad ij = -ji,$$

and $A = F + Fi + Fj + Fij$. As customary, we use the Hilbert symbol notation $A = \left( \frac{a,b}{F} \right)$. For $\alpha = a_0 + a_1i + a_2j + a_3ij \in A$, we define its conjugation as $\overline{\alpha} = a_0 - a_1i - a_2j - a_3ij \in A$. Then, the reduced trace $tr(\alpha)$ is defined to be $\alpha + \overline{\alpha} = 2a_0 \in F$ and the reduced norm $n(\alpha)$ is defined to be $a\overline{\alpha} = a_0^2 - a_1^2a - a_2^2b + a_3^2ab \in F$.

**Example 2.6.** The $2 \times 2$ matrices over $F$, $M_2(F) = \left( \frac{1}{F} \right)$ and in this case the norm is simply the determinant.

If $F = \mathbb{R}$ or a non-Archimedean local field, then up to isomorphism, there are only two quaternion algebras: $M_2(F)$ or a division algebra. When $F$ is a number field and $v$ a place of $F$, we say that $A$ splits at $v$ if the localization $A \otimes_F F_v$ is isomorphic to $M_2(F_v)$. Here $F_v$ denote the completion of $F$ with respect to $v$. If on the other hand $A \otimes_F F_v$ is isomorphic to a division algebra, we say $A$ ramifies at $v$. It is known that the number of ramified places is finite and the discriminant of $A$ is defined as the product of the finite ramified places.

Assume now that $F$ is a totally real number field of degree $k + 1$ and we denote by $\mathbb{Z}_F$ it ring of integers. Assume further that $A$ splits at exactly one infinite place, that is,

$$A \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R}) \times \mathcal{H}^k$$

where $\mathcal{H}$ is Hamilton’s quaternion algebra $\left( \frac{-1,-1}{\mathbb{R}} \right)$. Then, up to conjugation, there is a unique embedding $\rho$ of $A$ into $M_2(\mathbb{R})$. In particular for any $\alpha \in A$, one has that $n(\alpha) = det(\rho(\alpha))$.

Let $\mathcal{O}$ be an order in $A$, namely a finitely generated $\mathbb{Z}_F$-module that is also a ring with unity containing a basis for $A$, that is $\mathcal{O} \otimes_{\mathbb{Z}_F} F \simeq A$. Denote by $\mathcal{O}^1$ the norm-one group of $\mathcal{O}$, that is $\mathcal{O}^1 = \{ \alpha \in \mathcal{O} : n(\alpha) = 1 \}$. Then the image $\rho(\mathcal{O}^1)$ of $\mathcal{O}^1$ under $\rho$ is a discrete subgroup of $SL_2(\mathbb{R})$. We denote by $\Gamma(A, \mathcal{O})$ the projection in $PSL_2(\mathbb{R})$ of the group $\rho(\mathcal{O}^1)$.

**Definition 2.7.** The group $\Gamma$ is said to be arithmetic if it is commensurable with a group of the form $\Gamma(A, \mathcal{O})$.

Perhaps the best known example of an arithmetic group is $PSL_2(\mathbb{Z})$. Recall that two groups $\Gamma_1$ and $\Gamma_2$ are commensurable, denoted by $\Gamma_1 \sim \Gamma_2$, if their intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both $\Gamma_1$ and $\Gamma_2$.

If $\Gamma$ is arithmetic, then the quotient $\Gamma \backslash \mathbb{H}_\Gamma$ is called a Shimura curve of genus 0. As well known, Shimura curves are generalizations of classical modular curves. We direct the reader to [3] and [56] where the Schwarzian equations for many examples of these curves are derived and studied.

We now look at the connection between arithmeticity of $\Gamma$ and existence of correspondences on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ whose preimage under $f_\Gamma$ is also algebraic (cf. [30] and [53]). Let $Comm(\Gamma)$ be the commensurator of $\Gamma$, namely

$$Comm(\Gamma) = \{ g \in PSL_2(\mathbb{R}) : g\Gamma g^{-1} \sim \Gamma \}.$$
By a $\text{Comm}(\Gamma)$-correspondence on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ we mean a subset of the form

$$X(\Gamma g \Gamma) = \{ j_\Gamma(\tau) \times j_\Gamma(g \cdot \tau) : \tau \in H_\Gamma \}$$

where $g \in \text{Comm}(\Gamma)$. It turns out that $X(\Gamma g \Gamma)$ is an absolutely irreducible curve and that it depends only on the coset $\Gamma g \Gamma$ and not on the choice of $g$ (cf. [53] Chapter 7). We suppose that $X(\Gamma g \Gamma)$ is given by the equation $\Psi_g(X,Y) = 0$, so that $\Psi_g(j_\Gamma, j_\Gamma(gt)) = 0$. We write $\tilde{\Gamma}$ to highlight that the equation depends on $\Gamma g \Gamma$ and not $g$. With this notation, for $g_1, g_2 \in \text{GL}_2(\mathbb{C})$ we more generally say that $j_\Gamma(g_1 t)$ and $j_\Gamma(g_2 t)$ are in $\text{Comm}(\Gamma)$-correspondence if $\Psi_g(j_\Gamma(g_1 t), j_\Gamma(g_2 t)) \neq 0$ for some $\Gamma g \Gamma$. One has the following result of Margulis:

**Fact 2.8 ([25]).** The group $\Gamma$ is arithmetic if and only if $\Gamma$ has infinite index in $\text{Comm}(\Gamma)$ and as a result there are infinitely many $\text{Comm}(\Gamma)$-correspondences.

The modular polynomials (also known as Hecke correspondences) are the classical examples (when $\Gamma = \text{PSL}_2(\mathbb{Z})$). Returning to the Schwarzian equations we see that arithmetic Fuchsian groups of genus 0 give examples of ODE’s with rich binary relations.

### 2.3. A touch of Model theory.

We end this section by saying a few words about the concepts in model theory and differential algebra that will be required in the next sections. We will then be ready to state the main results in the paper. Throughout we work in a differentially closed field of characteristic zero.

**Definition 2.9.** Let $\mathcal{Y}$ be the set defined by an ODE $y^{(n)} = f(y, y', \ldots, y^{(n-1)})$, where $f$ is rational over $\mathbb{C}$. Then $\mathcal{Y}$ is said to be strongly minimal if every definable subset is finite or co-finite. Equivalently, $\mathcal{Y}$ is strongly minimal if for any differential field extension $K$ of $\mathbb{C}$ and solution $y \in \mathcal{Y}$, $\text{tr.deg.}_K K \langle y \rangle = 0$ or $n$.

Strong minimality is fundamental to the model theoretic approach to differential algebra (cf. [32]). It is also closely related to Umemura’s notion of irreducibility of the ODE with respect to classical functions [58]. It turns out that there is a very general classification of strongly minimal sets in differentially closed fields about which we will say a few more words in Section 6.1. For now, we only mention the kind of strongly minimal set that is relevant for equation $\ast$:

**Definition 2.10.** Let $\mathcal{Y}$ strongly minimal as above. Then $\mathcal{Y}$ is **geometrically trivial** if for any differential field extension $K$ of $\mathbb{C}$, and for any distinct solutions $y_1, \ldots, y_m$, if the collection consisting of $y_1, \ldots, y_m$ together with all their derivatives $y_i^{(j)}$ is algebraically dependent over $K$ then for some $i < j$, $y_i, y_j$ together with their derivatives is algebraically dependent over $K$.

So geometric triviality limits the complexity of the structure of the algebraic relations on the definable set. However, given such a set, for the results which we pursue, much greater precision is required. Throughout for simplicity, we will say that an ODE is strongly minimal and geometrically trivial just in the case that its solution set is strongly minimal as a definable set. Our first theorem is the following:

**Theorem 2.11.** The Schwarzian equation $\ast$ for the Hauptmodul $j_\Gamma$ of a genus 0 Fuchsian group $\Gamma$ of first kind is strongly minimal and geometrically trivial.
We will give the proof in subsection 6.1. This result was previously only known for $\text{PSL}_2(\mathbb{Z})$ (the $j$-function see Example 2.2) as well as for arithmetic subgroups of $\text{PSL}_2(\mathbb{Z})$ (cf. [11]). Our proof, which handles all Schwarzian equations of genus zero Fuchsian functions at once, also is the first which does not use o-minimality. The first proof for $\text{PSL}_2(\mathbb{Z})$ (of [11]) relied on the main result of [44], where Pila employs the same strategy from [43], relying on o-minimality and counting of points of bounded height. Later, [2] also gave a proof of the special case of $\text{PSL}_2(\mathbb{Z})$ which relied on the Ax-Schanuel type results of [45], where again, an o-minimal strategy was employed.

It is worth mentioning that Painlevé [38, Page 519] conjectured that strong minimality (or irreducibility as he called it) would hold for the equations we consider. In [35], Nishioka could only proved a very weak form of that conjecture. Nevertheless, Nishioka’s paper contains techniques that inspired our own proof.

We have also obtained a full description of the structure of the definable sets. One can think of these results as a weak form of the Ax-Lindemann-Weierstrass Theorem with derivatives for $\Gamma$.

**Theorem 2.12.** Suppose that $\Gamma$ is arithmetic and suppose that $j_\Gamma(g_1t), \ldots, j_\Gamma(g_nt)$ are distinct solutions of the Schwarzian equation $(\star)$ that are pairwise not in $\text{Comm}(\Gamma)$-correspondence. Then the $3n$ functions $j_\Gamma(g_1t), j'_\Gamma(g_1t), j''_\Gamma(g_1t), \ldots, j_\Gamma(g_nt), j'_\Gamma(g_nt), j''_\Gamma(g_nt)$ are algebraically independent over $\mathbb{C}(t)$.

**Theorem 2.13.** Suppose that $\Gamma$ is non-arithmetic. Then there is a $k \in \mathbb{N}$ such that if $j_\Gamma(g_1t), \ldots, j_\Gamma(g_nt)$ are distinct solutions of the Schwarzian equation $(\star)$ satisfying

$$\text{tr.deg}_{\mathbb{C}(t)} \mathbb{C}(t, j_\Gamma(g_1t), \ldots, j_\Gamma(g_nt)) = 3n,$$

then for all other solutions $j_\Gamma(g_t)$, except for at most $n \cdot k$,

$$\text{tr.deg}_{\mathbb{C}(t)} \mathbb{C}(t, j_\Gamma(g_1t), \ldots, j_\Gamma(g_nt), j_\Gamma(g_t)) = 3(n + 1).$$

So, by the previous two theorems, we have that the the set defined by the Schwarzian equation $(\star)$ is $\aleph_0$-categorical if and only if the group $\Gamma$ is non-arithmetic. It was a longstanding open problem in the model theory of differential fields (recently resolved by [11]) to find a non-$\aleph_0$-categorical geometrically trivial strongly minimal set; the non-existence of such sets was part of a strategy for certain diophantine problems suggested by Hrushovski [17] see page 292]. Theorem 2.12 gives many new examples of geometrically trivial non-$\aleph_0$-categorical equations, and together with Theorem 2.13 also provides an interesting connection between categoricity and arithmetic groups. We view the following question as the next major challenge in the classification of geometrically trivial strongly minimal sets in differentially closed fields:

---

3 The ALW statement we are pursuing allows for characterizing algebraic relations between functions which don’t formally satisfy the same differential equation, but we will use to Theorems 2.12 and 2.13 to prove our most general results, which imply the pertinent version of ALW.
**Question 2.14.** Are there non-$\aleph_0$-categorical strongly minimal sets that do not arise from arithmetic Fuchsian groups?\footnote{Later in the paper, it will be clear to model theorists that by “arise from” arithmetic Fuchsian groups, we mean “are non-orthogonal to the differential equation (\textcite{14}) or one of its other fibers”. An answer to the question is of interest in part because if there were a strong classification of the geometrically trivial strongly minimal sets in differential fields, some of the strategy laid out in \textcite{17} for certain diophantine problems might be possible.}

Finally let us talk about the full Ax-Lindemann-Weierstrass Theorem with derivatives for $\Gamma$. We closely follow the description of the problem as in \textcite{44}. Let $V \subset \mathbb{C}^n$ be an irreducible algebraic variety defined over $\mathbb{C}$ such that $V \cap \mathbb{H}^n \neq \emptyset$ and $V$ projects dominantly to each of its coordinates (each coordinate function is nonconstant). Let $t_1, \ldots, t_n$ be the functions on $V$ induced by the canonical coordinate functions on $\mathbb{C}^n$. We say that $t_1, \ldots, t_n$ are $\Gamma$-geodesically independent if there are no relations of the form

$$t_i = gt_j$$

where $i \neq j$ and $g \in \text{Comm}(\Gamma)$ acts by fractional linear transformations.

**Theorem 2.15.** With the notation (and assumption $V \cap \mathbb{H}^n \neq \emptyset$) as above, suppose that $t_1, \ldots, t_n$ are $\Gamma$-geodesically independent. Then the $3n$ functions

$$j_1(t_1), j'_1(t_1), j''_1(t_1) \ldots, j_1(t_n), j'_1(t_n), j''_1(t_n)$$

(defined locally) on $V$ are algebraically independent over $\mathbb{C}(V)$.

We will prove Theorem 2.15 in section 7. Pila \textcite{44} had already proved the result for $\text{PSL}_2(\mathbb{Z})$ (see also \textcite{11} where the same is established for arithmetic subgroups of $\text{PSL}_2(\mathbb{Z})$).

### 3. A criterion for strong minimality of a general Fuchsian equation

We now aim to give a criterion that can used to show that the Schwarzian equation (\textcite{14}) is strongly minimal. This criterion is applicable to Schwarzian equations in general sense, namely to any equation of the form

\begin{align*}
S_{\frac{d}{dt}}(y) + (y')^2 \cdot R(y) &= 0. \quad (\star')
\end{align*}

So here we do not assume the rational function $R$ to necessarily correspond to some Hauptmodul. We only require that $R$ is rational over $\mathbb{C}$. By the Riccati equation attached to (\textcite{14}) we mean the equation

\begin{align*}
\frac{du}{dy} + u^2 + \frac{1}{2}R(y) &= 0. \quad (\star\star)
\end{align*}

**Condition 3.1.** The Riccati equation (\textcite{14}) has no solution in $\mathbb{C}(y)^{\text{alg}}$.

**Theorem 3.2.** Let $(K, \partial)$ be any differential field extension of $\mathbb{C}$ and let us assume that Condition 3.1 holds. If $y$ is a solution of the Schwarzian equation (\textcite{14}) we have that

$$\text{tr.deg}_K K\langle y \rangle = 0 \text{ or } 3.$$

In other words, if Condition 3.1 holds, then equation (\textcite{14}) is strongly minimal.
Proof. For contradiction let us assume that $\text{tr.deg.}_K K \langle y \rangle \neq 0$ or 3. We tackle the two cases separately.

**Case 1:** Suppose $\text{tr.deg.}_K K \langle y \rangle = 1$.

Then we have that $y', y'' \in K(y)^{\text{alg}}$. In particular $u = \frac{y''}{y} \in K(y)^{\text{alg}}$. But it is easy to check that since $y$ is a solution of (*), the following is true

$$
\left(\frac{y''}{y}\right)' \frac{1}{y'} + \frac{1}{2} \left(\frac{y''}{y}\right)^2 + R(y) = 0.
$$

Notice that $y' \not\in K^{\text{alg}}$, for otherwise $y', y'', y''' \in K^{\text{alg}}$ and so using equation $(\star')$ one has that $R(y) \in K^{\text{alg}}$. This would contradict that $y \not\in K^{\text{alg}}$. This observation allows us, using Seidenberg’s embedding theorem, to apply the chain rule and obtain

$$
du dy + \frac{1}{2} u^2 + R(y) = 0.
$$

But then $u/2$ can be seen as an algebraic solution of $(\star\star)$ in $C(y)^{\text{alg}}$ contradicting Condition 3.1.

**Case 2:** Suppose $\text{tr.deg.}_K K \langle y \rangle = 2$.

We use an idea of Nishioka [35]. Let $L = K(y)^{\text{alg}}$ and $\partial$ be the extension of the derivation of $K$ such that $\partial(y) = 0$. Since $y'$ is transcendental over $L$ we can work in the field $L \langle \langle 1/y' \rangle \rangle$ of Puiseux series in $1/y'$ over $L$. We have a well defined derivation

$$
\left(\sum a_i y'^{\lambda_i}\right)' = \sum \partial(a_i) y'^{\lambda_i} + \sum \frac{\partial a_i}{\partial y} y'^{\lambda_i+1} + \sum \lambda_i a_i y'^{\lambda_i-1} y''
$$

where $\lambda_i, i \geq 0$, are descending rational numbers with a common denominator, $a_i \in L$ and $a_0 \neq 0$. It follows that $K \langle y \rangle^{\text{alg}}$ is a differential subfield of $L \langle \langle 1/y' \rangle \rangle$. Now let $u = \frac{y'}{y}$ and by replacing in equation $(\star')$ we get

$$
\frac{u'}{y'} + \frac{1}{2} u^2 + R(y) = 0.
$$

Writing $u = \sum a_i y'^{\lambda_i}$ with $i \geq 0$ and $a_0 \neq 0$ and differentiating

$$
\frac{u'}{y'} = \sum \partial(a_i) y'^{\lambda_i-1} + \sum \frac{\partial a_i}{\partial y} y'^{\lambda_i} + \sum \lambda_i a_i y'^{\lambda_i-1} \sum a_i y'^{\lambda_i} = -\frac{1}{2} \left(\sum a_i y'^{\lambda_i}\right)^2 - R(y).
$$

Since $R(y)$ (a coefficient of $y'^{0}$) appear non-trivially in the above and since $R(y) \not\in K^{\text{alg}}$, it is easily seen that $\lambda_0 = 0$ and

$$
\frac{\partial a_0}{\partial y} + \frac{1}{2} a_0^2 + R(y) = 0.
$$

But then as before $a_0/2$ can be seen as an algebraic solution of $(\star\star)$ in $C(y)^{\text{alg}}$ contradicting Lemma 3.1.

The next section is devoted to proving that Condition 3.1 holds for Equation (*).
4. THE GENERAL PROOF OF STRONG MINIMALITY

4.1. Liouvillian solutions, algebraic solutions, and Picard-Vessiot theory.

Definition 4.1. Fix a differential field $K$ extending $\mathbb{C}(y)$ such that the derivation on $K$ extends $\frac{d}{dy}$. We say that $K$ is Liouvillian if there is a tower of field extensions $\mathbb{C}(y) \subset K_0 \subset K_1 \subset \ldots \subset K_n = K$ such that for each $i = 1, \ldots, n$, $K_i/K_{i-1}$ is generated by an element $a_i$ such that one of the following holds:

1. $a_i' \in K_{i-1}$.
2. $\frac{a_i'}{a_i} \in K_{i-1}$.
3. $a_i \in K_{i-1}^{\text{alg}}$.

In case 1, $a_i = \int f$ for some $f \in K_i$ and in case 2, $a_i = e^{\int f}$ for some $f \in K_i$. So, occasionally we will refer to these cases as integrals or exponentials of integrals.

Consider the differential equation

\begin{equation}
z'' + pz' + qz = 0 \tag{4.1}
\end{equation}

where $p, q$ are rational functions in $\mathbb{C}(y)$. Define

\begin{align*}
u &= \frac{z'}{z} + \frac{p}{2} \\
v' &= v^2 + \frac{p}{2}
\end{align*}

One can show via direct computation that

\begin{align*}
u' + u^2 + \frac{1}{2}R &= 0 \tag{4.4} \\
v'' + \frac{1}{2}Rv &= 0 \tag{4.5}
\end{align*}

where $R = -p' + 2q - \frac{p^2}{2}$.

Lemma 4.2. There is a bijective correspondence between rational (algebraic) solutions to equation (4.4) and order one subvarieties of equation (4.1) defined over $\mathbb{C}(y)$ (over $\mathbb{C}(y)^{\text{alg}}$). Any order one subvariety of equation (4.1) defined over $\mathbb{C}(y)^{\text{alg}}$ corresponds to a Liouvillian solution to equation (4.1).

Proof. Suppose that $u = f(y)$ is a rational (algebraic) solution of equation (4.4). Then $f(y) = \frac{z'}{z} + \frac{p}{2}$, and so $z' + \frac{p}{2}z - f(y)z$ defines an order one subvariety defined over $\mathbb{C}(y)$ (over $\mathbb{C}(y)^{\text{alg}}$). Conversely, an order one subvariety of equation (4.1) is, without loss of generality, given by an equation of the form $z' + g(y)z = 0$ for some $g(y) \in \mathbb{C}(y)$ (or $g(y) \in \mathbb{C}(y)^{\text{alg}}$). In this case, $u = -g(y) + \frac{p}{2}$ is a rational (algebraic) solutions to equation (4.4).

Notice that the order one subvarieties of (4.1) correspond to elements $y$ such that the logarithmic derivative of $y$ is in $\mathbb{C}(y)$ (respectively, $\mathbb{C}(y)^{\text{alg}}$), and as such are Liouvillian solutions to equation (4.1). □
Now, the verification of Condition 3.1 follows from showing that equation 4.1 has no Liouville solutions. In fact, the classification of the Liouville solutions of order two linear equations has been extensively studied, and in [22], an algorithmic solution to determining the Liouville solutions was given.

Let \( z \) be a solution to equation 4.1, and let \( v = e^{\frac{1}{2} \int p \, dz} \). It follows by direct computation that

\[
v'' + \left( b - \frac{1}{4} a^2 - \frac{1}{2} a' \right) v = 0
\]

Because previous transformation only involves scaling by a Liouvillian element, the Liouvillian solutions of equation 4.6 are in bijective correspondence with the Liouvillian solutions to equation 4.1, and so without loss of generality, we may now assume that the order two equation in which we are interested is given in the following normal form:

\[
z'' = r(y)z
\]

where \( r(y) \in \mathbb{C}(y) \).

**Theorem 4.3.** [22, page 5] With regard to the Liouvillian solutions of a second order linear differential equation with coefficients in \( \mathbb{C}(y) \), there are four mutually exclusive options:

1. The differential equation 4.7 has a solution of the form \( e^{\int w} \) where \( w \in \mathbb{C}(y) \).
2. The differential equation 4.7 has a solution of the form \( e^{\int w} \) where \( w \in \mathbb{C}(y) \) is an algebraic function of degree two over \( \mathbb{C}(y) \).
3. All of the solutions of 4.7 are algebraic over \( \mathbb{C}(y) \).
4. No solution of 4.7 are Liouvillian.

**Theorem 4.4.** [22, page 8, case 4] Let \( G \) be the Picard-Vessiot group of 4.7. If \( G = \text{SL}_2(\mathbb{C}) \) then there are no Liouvillian solutions to equation 4.7.

It follows that to establish Condition 3.1, it is sufficient to prove that the Picard-Vessiot group of the associated order two equation 4.1 is \( \text{SL}_2(\mathbb{C}) \), an issue we turn to in the next subsection.

4.2. Monodromy and the PV-group. At this point, let us recall that the the Schwarzian equation (*) we focus on is given with

\[
R_{ij}(y) = \frac{1}{2} \sum_{i=1}^{r} \frac{1 - \alpha_i^2}{(y - a_i)^2} + \sum_{i=1}^{r} \frac{A_i}{y - a_i},
\]

where the \( \alpha_i \)'s, \( A_i \)'s and \( a_i \)'s are obtained from the fundamental domain for \( \Gamma \)-action on \( \mathbb{H} \). As discussed in the previous subsection, if the Riccati equation corresponding to (2)

\[
\frac{du}{dy} + u^2 + \frac{1}{2} R_{ij}(y) = 0.
\]

were to have an algebraic solution \( f \in \mathbb{C}(y)^{alg} \), then it is not hard to check that \( e^{\int f} \) is a Liouvillian solution of the linear equation

\[
\frac{d^2 z}{dy^2} + \left( \frac{1}{4} \sum_{i=1}^{r} \frac{1 - \alpha_i^2}{(y - a_i)^2} + \frac{r}{2} \sum_{i=1}^{r} \frac{A_i}{y - a_i} \right) z = 0.
\]
This equation is an example of the most general (normal) form of a Fuchsian equation of second order:

**Definition 4.5.** The linear equation \( \frac{d^2z}{dy^2} + a_1 \frac{dz}{dy} + a_2 z = 0 \), where \( a_1, a_2 \) are rational functions in \( \mathbb{C}(y) \) is called Fuchsian if all points of \( \mathbb{P}^1(\mathbb{C}) \) are regular or regular singular. In particular, if the equation is Fuchsian then the coefficients \( a_1 \) and \( a_2 \) are of the form

\[
a_i(y) = \frac{B_i(y)}{\prod_{j=1}^{s}(z - \beta_j)^i}
\]

where \( B_i(y) \) is a polynomial of degree \( \leq i(s-1) \).

**Remark 4.6.** We have already seen in the previous section how to obtain the normal form of the a second order linear equation (see equation 4.6).

As it turns out, the problem of existence of Liouvillian solutions for Fuchsian equations of second order is a classical one. We direct the reader to [15] and [52] for some historical perspectives. We will only review parts of the theory that is relevant to this paper. Our focus will be the work of Poincaré on the relationship between the monodromy group of the Fuchsian equation 4.9 and ‘its’ Fuchsian group \( \Gamma \). It is this work - partly rediscovering Schwarz’s uniformization of \( \mathbb{P}^1(\mathbb{C}) \) by the \( j \Gamma \)’s - that lead Poincaré to introduce the theory of Fuchsian groups and functions, and to attack the problem of the uniformization of other Riemman surfaces.

From now on, we assume that the equation

\[
\frac{d^2z}{dy^2} = r(y)z
\]

is Fuchsian and denote by \( S \) its set of singular points. For \( z \in \mathbb{P}^1(\mathbb{C}) \setminus S \), let \( f_1 \) and \( f_2 \) be analytic solutions in a neighborhood of \( z \). We also assume that \( f_1 \) and \( f_2 \) are a basis of solutions, i.e. that the are linearly independent over \( \mathbb{C} \). Given any \( \gamma \in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; z) \), we can analytically continue \( f_1 \) and \( f_2 \) along \( \gamma \) and obtain new solutions \( \tilde{f}_1 \) and \( \tilde{f}_2 \) of 4.10.

So there exists a matrix \( M_\gamma \in GL_2(\mathbb{C}) \) such that

\[
\begin{pmatrix}
\tilde{f}_1 \\
\tilde{f}_2
\end{pmatrix} = M_\gamma \cdot 
\begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}
\]

The mapping \( \rho : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; z) \to GL_2(\mathbb{C}) \), taking \( \gamma \mapsto M_\gamma \) is a group homomorphism called the monodromy representation. Its image \( M \) is called the monodromy group of equation 4.10. From the monodromy group, one can determine the Picard-Vessiot group of the equation:

**Fact 4.7.** [8, Chapter 7] Let \( G \) be the Picard-Vessiot group of the Fuchsian equation 4.10. Then,

1. \( G \subseteq SL_2(\mathbb{C}) \);
2. if \( M \) be its monodromy group, then \( G \) is the Zariski closure of \( M \).

Note that in particular from (1), for the Fuchsian equation 4.9 the monodromy group \( M \) is a subgroup of \( SL_2(\mathbb{C}) \). We will now explain how in the case of equation 4.9 the monodromy group \( M \) is related to the Schwarzian equation. The following well-known fact - which can be easily verified - will be needed.
Fact 4.8. Let $t(y) = j^{-1}_I(y)$ be a branch of the inverse of $y = j_I(t)$. Then $t(y)$ satisfies the following equation
\begin{equation}
S \frac{d}{dy} (t) = R_j(y).
\end{equation}
Furthermore, the functions

\begin{align*}
z_1 &= \frac{t}{(\frac{d}{dy} t)^2} \\
z_2 &= \frac{1}{(\frac{d}{dy} t)^2}
\end{align*}

form a basis of solutions of the Fuchsian equation 4.9

\begin{equation}
\frac{d^2 z}{dy^2} + \frac{1}{2} R_j(y) z = 0.
\end{equation}

Notice that in particular $\frac{z_1(y)}{z_2(y)} = t(y)$. This allows us to define from $M$ the projective monodromy of the equation 4.11. Namely, if $M_\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$ is monodromy matrix (as above), then

\begin{equation}
t^* = a t + b \\
c t + d
\end{equation}

is again a solution of the equation 4.11. The collection of matrices $\hat{M}_\gamma : t \mapsto t^*$ is called the projective monodromy group $\hat{M}$ of the equation 4.11. Of course $\hat{M}$ is the image of $M$ under the natural projection $\pi : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$. The following proposition is attributed to Poincaré in various sources but we know of no reference for a proof of it and thus reproduce it here.

Proposition 4.9. The projective monodromy group of the equation 4.11 is $\Gamma$. As a consequence, the monodromy group of the Fuchsian equation 4.9 is $\pi^{-1}(\Gamma)$.

Proof. Throughout $t(y) = j^{-1}_I(y)$ is a branch of the inverse of $j_I$ locally defined on some small domain $U$ and $\hat{M}$ is the projective monodromy group.

We have

\begin{align*}
g \in \hat{M} \setminus \{I\} & \iff gt(y) \text{ is another branch of the inverse of } j_I \text{ (defined on some larger domain } U'). \\
& \iff y = j_I(t(y)) = j_I(gt(y)) \\
& \iff g \in \Gamma \setminus \{I\}.
\end{align*}

We have used here that $j_I$ is a globally defined single-valued function. \hfill \square

Proposition 4.10. There are no Liouvillian solutions of the Fuchsian equation 4.9. Consequently, Condition 3.1 holds for the Riccati equation 4.8.

Proof. We have that $\pi^{-1}(\Gamma)$, the monodromy group of the Fuchsian equation 4.9 is Zariski dense in $SL_2(\mathbb{C})$. Hence by Fact 4.7 the Picard-Vessiot group is $G = SL_2(\mathbb{C})$. By Fact 4.4 there are no Liouvillian solutions for the equation. \hfill \square

We thus obtain the first part of Theorem 2.11; namely the Schwarzian equation (2) is strongly minimal.
5. Strong minimality for the special case of triangle groups

As before, we assume that \( \Gamma \) is a Fuchsian group of first kind and of genus zero. The group \( \Gamma \) is said to be a Fuchsian triangle group of type \( (k, l, m) \) if its signature is \( (0; k, l, m) \) (see Section\(^2\)). We will without loss of generality always assume that \( 2 \leq k \leq l \leq m \leq \infty \). We write \( \Gamma_{(k,l,m)} \) for the Fuchsian triangle group of type \( (k, l, m) \).

The fundamental domain in \( \mathbb{H} \) of \( \Gamma_{(k,l,m)} \) is the union of a hyperbolic triangle with angles \( \frac{\pi}{k}, \frac{\pi}{l}, \frac{\pi}{m} \) at the vertices \( v_k, v_l, v_m \) respectively, together with its image via hyperbolic reflection of one side connecting the vertices. Notice that since \( k, l, m \) relates to the angle of an hyperbolic triangle, if \( \Gamma_{(k,l,m)} \) is a triangle group then

\[
\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1.
\]

Also, the vertices \( v_k, v_l, v_m \) are the fixed points of the generators \( g_1, g_2 \) and \( g_3 \) respectively.

**Definition 5.1.** The function \( j_{(k,l,m)} \) will denote the (unique) Hauptmodul \( \Gamma_{(k,l,m)} \setminus \mathbb{H} \rightarrow \mathbb{P}^1(\mathbb{C}) \) sending \( v_k, v_l, v_m \) to \( 1, 0, \infty \) respectively.

With this definition (cf. [1] Chapter 5) we have that \( j_{(k,l,m)} \) satisfies the Schwarzian equation \((\ast)\) with

\[
R_{j_{(k,l,m)}}(y) = \frac{1 - l^{-2}}{y^2} + \frac{1 - k^{-2}}{(y - 1)^2} + \frac{k^{-2} + l^{-2} - m^{-2} - 1}{y(y - 1)}.
\]

Notice that with Definition \(5.1\) the Hauptmodul \( j_{(2,3,\infty)} \) for \( \text{PSL}_2(\mathbb{Z}) \) is not the classical \( j \)-function. Rather, one has that \( j = 1728j_{(2,3,\infty)} \) (see Example\(^2\)).

Finally let us mention that there is a full classification, up to \( \text{PSL}_2(\mathbb{R}) \)-conjugation, of the arithmetic triangle groups

**Fact 5.2.** Up to \( \text{PSL}_2(\mathbb{R}) \)-conjugation, there are finitely many arithmetic triangle groups; 76 cocompact and 9 non-cocompact \( [54] \). Among these, there are 19 distinct commensurability classes represented \( [55] \).

In the special case of triangle groups, proving that the Riccati equation\(^4\) has no algebraic solutions (and thus establishing the strong minimality of the associated order three nonlinear Schwarzian differential equations) can be accomplished without any appeal to Picard-Vesiot theory but instead by using classical work around the hypergeometric equation. Already, in \([35]\) see page 601, Nishioka shows that equation\(^4\) has no algebraic solutions in the case the \( \Gamma \) is a cocompact triangle group (which corresponds to the case that none of \( k, l, m \) are \( \infty \)). Hence Condition\(^3\) and thus Theorem\(^3\) holds in the case of cocompact triangle groups. We will, via a very similar argument, show the same result holds in the case that \( \Gamma \) is not cocompact. To emphasize, these results are a special case of our general result on Fuchsian groups, but we feel their inclusion is worthwhile in part because the method, which deals more directly with the order two linear equation\(^4\) and Riccati equation\(^4\) might generalize to Schwarzian equations of the form of equation\(^2\) which do not necessarily come from a group action of \( \Gamma \) on \( \mathbb{H} \). This restriction appears to be more inherent in our main approach of the previous section.
Let
\begin{align*}
\lambda &= \frac{1}{l} \\
\mu &= \frac{1}{k} \\
\nu &= \frac{1}{m}
\end{align*}
where the integers $2 \leq k \leq l \leq m \leq \infty$ are as above. We have already seen $\lambda + \mu + \nu < 1$. Now let $\alpha$, $\beta$ and $\gamma$ be any complex numbers such that, $\lambda = 1 - \gamma$, $\mu = \gamma - \alpha - \beta$, and $\nu = \alpha - \beta$.

Now, we know that the second order equation 4.1 corresponding to equation (5) with rational function 5.1 (equation (5) of [35]) is reducible if and only if one of $\alpha$, $\beta$, $\gamma - \alpha$, $\gamma - \beta$ is an integer. Since [35] covers the cocompact case, we can assume without loss of generality that $\nu = \infty$.

We are interested in the case of non-cocompact triangle groups, so we are assuming $\nu = \infty$. Thus, in the above notation, $\alpha = \beta$. Now,
\[
\alpha = 1 - \frac{1}{l} - \frac{1}{k}.
\]
So, in this case, since by the triangle requirement, $\frac{1}{l} + \frac{1}{k} < 1$, so $\alpha$ is never an integer.

Further, we have
\[
\gamma - \alpha = \frac{1 - \frac{1}{l} + \frac{1}{k}}{2}.
\]
This quantity is never an integer, since $\frac{1}{l} + \frac{1}{k} < 1$. Thus, in the non-cocompact case, we have that the corresponding equation 4.1 is always irreducible, which, by the correspondence explained in [4] implies that there are no rational solutions to equation 4.4 in this case.

Now, under the assumption of irreducibility of equation 4.1, we have that there is an algebraic (but irrational) solution of 4.4 if and only if two of $\lambda - \frac{1}{l}$, $\mu - \frac{1}{k}$, $\nu - \frac{1}{l}$ are integers [28, pages 96-100]. This is impossible for any triangle group as at most one of these is an integers as long as $\lambda + \mu + \nu < 1$.

Thus, we have shown, in a more direct way, that Condition 3.1 and thus Theorem 3.2 also holds in the case of non-cocompact triangle groups.

**Remark 5.3.** At first glance the above arguments only seem to show that the differential equations for the uniformizers $\tilde{f}_{(k,l,m)}$ are strongly minimal. However, all other uniformizers are rational functions (over $\mathbb{C}$) of the $\tilde{f}_{(k,l,m)}$’s. From this, strong minimality follows for the other equations as well.

6. **Geometric triviality and Algebraic relations**

6.1. **The classification of strongly minimal sets.** In this section we will discuss some general model-theoretic results regarding strongly minimal sets in differentially closed fields. In particular, we will explain some consequences of the (unpublished) work of Hrushovski
and Sokolovic on the classification of strongly minimal sets. We will, from these considerations, obtain geometric triviality of the Schwarzian equations satisfied by the uniformizing functions in the earlier sections. Let us denote by \( \mathcal{U} \) the differentially closed field of characteristic zero that we work in. We assume that \( \mathbb{C} \) (defined by \( y' = 0 \)) is its field of constants. Notice incidentally that \( \mathbb{C} \) is itself a strongly minimal definable set. Indeed, up to definable isomorphism, it is the only definable strongly minimal subfield of \( \mathcal{U} \).

The zero set of any irreducible order one differential polynomial in a single variable (by irreducible, we will always mean as a polynomial) is also strongly minimal. Higher order linear differential equations are never strongly minimal (one can define linear subspaces using elements of a fundamental set of solutions). For higher order non-linear equations, it seems that it is in general difficult to establish strong minimality. However, if the strong minimality of an equation is established, one can often employ a variety of model theoretic tools to establish even stronger results.

Other important examples of strongly minimal sets are given by the following

**Fact 6.1** ([7], [18]). Let \( A \) be an abelian variety defined over \( \mathcal{U} \). We identify \( A \) with its set \( A(\mathcal{U}) \) of \( \mathcal{U} \)-points. Then

1. \( A \) has a (unique) smallest Zariski-dense definable subgroup, which we denote by \( A^\sharp \).
2. If \( A \) is a simple abelian variety that does not descend to \( \mathbb{C} \), then \( A^\sharp \) is strongly minimal.

The subgroup \( A^\sharp \) is called the Manin kernel of \( A \) (cf. [26]). The trichotomy theorem, gives a classification of strongly minimal sets up to non-orthogonality, a notion we will explain following the statement of the theorem.

**Theorem 6.2** ([18]). Let \( Y \) be a strongly minimal set. Then exactly one of the following holds:

1. \( Y \) is nonorthogonal to the strongly minimal set \( \mathbb{C} \),
2. \( Y \) is nonorthogonal to \( A^\sharp \) for some simple abelian variety \( A \) over \( \mathcal{U} \) which does not descend to \( \mathbb{C} \),
3. \( Y \) is geometrically trivial.

**Definition 6.3.** Let \( Y \) and \( Z \) be strongly minimal sets. Denote by \( \pi_1 : Y \times Z \to Y \) and \( \pi_2 : Y \times Z \to Z \) the projections to \( Y \) and \( Z \) respectively. We say that \( Y \) and \( Z \) are nonorthogonal if there is some infinite definable relation \( R \subseteq Y \times Z \) such that \( \pi_1|_R \) and \( \pi_2|_R \) are finite-to-one functions.

The sets \( Y \) and \( Z \) are defined over some finitely generated differential subfield \( K \) of \( \mathcal{U} \), and so for any differential field \( F \) containing \( K \), it makes sense to ask whether a relation \( R \) as above can be defined over \( F \).

**Definition 6.4.** We say that \( Y \) is weakly orthogonal to \( Z \) over \( F \) if no such relation \( R \) can be defined over \( F \).

The following facts from the model theory of differential fields are well-known (see for instance, [27]).

**Fact 6.5.** Let \( Y \) and \( Z \) be strongly minimal sets.

1. Nonorthogonality is an equivalence relation on strongly minimal sets.
Nonorthogonality classes of strongly minimal differential equations refine various basic invariants of the equations. For instance, if $Y, Z$ are nonorthogonal then $\text{order}(Y) = \text{order}(Z)$.

(3) If $Y$ and $Z$ are nonorthogonal, then they fall into the same category of Theorem 6.2.

(4) Strongly minimal sets that fall in cases (2) and (3) of Theorem 6.2 are said to be locally modular (and in case (1) the sets are non-locally modular).

(5) Orthogonality has a natural interpretation in terms of transcendence. Suppose that $Y$ and $Z$ are orthogonal strongly minimal sets defined over $K$. Let $a, b$ be solutions of $Y, Z$, respectively. Let $F$ be any differential field extending $K$. Then

$$\text{tr.deg._F}(F\langle a, b \rangle) = \text{tr.deg._F}(F\langle a \rangle) + \text{tr.deg._F}(F\langle b \rangle).$$

Conversely, if the inequality does not hold for some $a \in Y$ and $b \in Z$ over $F$, then $Y$ is not weakly orthogonal to $Z$ over $F$.

Nonorthogonality of Manin Kernels has been further classified in terms of isogeny classes of abelian varieties.

**Fact 6.6.** If $A$ and $B$ are two simple abelian varieties which do not descend to $C$, then $A^\sharp$ and $B^\sharp$ are non-orthogonal if and only if $A$ and $B$ are isogenous.

For relations $R$ which witness nonorthogonality between modular strongly minimal sets, there is an important and very general descent result:

**Fact 6.7.** [49, Corollary 2.5.5] Two strongly minimal modular sets (in particular trivial sets are modular) are nonorthogonal if and only if they are non weakly orthogonal. That is, the relation $R$ witnessing nonorthogonality of $X$ and $Y$ can be defined over the differential field generated by the parameters used in the equations defining $X$ and $Y$.

**Proposition 6.8.** Let $Y$ be a strongly minimal set of order $> 1$ and suppose that $X$ is defined over $C$. Then $Y$ is geometrically trivial.

**Proof.** First note that since $\text{order}(Y) \neq 1$, $X$ is necessarily orthogonal to the constants $C$. So by Theorem 6.2 to show that $Y$ is geometrically trivial, we only need to show that it is orthogonal to all Manin kernels. We argue by contradiction.

Suppose that $Y$ is nonorthogonal to $A^\sharp$ for some simple abelian variety $A$ over $\mathcal{U}$ which does not descend to $C$. Let us write $A = A^\sharp_{\mathcal{U}}$ to specify that $\mathcal{U}$ are the parameters from $\mathcal{U}$ that appear in the definition of $A$. Let $\overline{\mathcal{U}}$ be a $C$-conjugate of $\mathcal{U}$ which is also independent from $\mathcal{U}$ over $C$. So we now have another Abelian variety $A_{\overline{\mathcal{U}}}$ and its Manin kernel $A^\sharp_{\mathcal{U}}$. By construction, we have that $A^\sharp_{\mathcal{U}}$ is non-orthogonal to $X$. Hence $A^\sharp_{\mathcal{U}}$ is non-orthogonal to $A^\sharp_{\overline{\mathcal{U}}}$.

Fact 6.6 tells us that $A_{\overline{\mathcal{U}}}$ and $A_{\mathcal{U}}$ are isogenous and using a result of Hrushovski and Sokolovic (see Claim in the proof of Proposition 2.8 of [18]), we have that $A_{\mathcal{U}}$ and $A_{\overline{\mathcal{U}}}$ are isomorphic to an Abelian variety defined over $C$. But this contradicts the assumption that $A_{\mathcal{U}}$ does not descend to $C$. □

**Corollary 6.9.** For $\Gamma$ a Fuchsian group, equation $(\star)$ defines a geometrically trivial strongly minimal set.

We have hence established the entirety of Theorem 2.11.

---

A more general form of this argument can be found in [24] Proposition 4.3.
6.2. Transcendence and orbits of the commensurator of $\Gamma$.

**Theorem 6.10.** Let $K$ be a differential extension of $(\mathbb{C}(t), \frac{d}{dt})$ with no new constants. Let $\Gamma$ be a Fuchsian group and $j_1, j_2$ be two solutions of the equation
\[
S_\frac{d}{dt}(y) + (y')^2 \cdot R_\Gamma(y) = 0.
\]
If
\[
\text{tr.deg.}_K K(j_1, j_1', j_1'', j_2, j_2') < 6
\]
than $j_1$ or $j_2$ is algebraic over $K$ or there is a polynomial $P(y_1, y_2)$ over $\mathbb{C}$ such that $P(j_1, j_2) = 0$.

The group $\text{PSL}_2(\mathbb{C})$ acts on pairs of solution by precomposition. We will prove that the ideal of differential relation between $(j_1, j_2)$ is stable under this action.

**Proof.** From Theorem 3.2, it follows that if $\text{tr.deg.}_K K(j_1, j_1', j_1'', j_2, j_2') < 6$ then it is either 0 or 3. But it follows from Fact 2.3 and Lemma 2.4 that if both $j_1$ and $j_2$ are not algebraic over $K$ then $\text{tr.deg.}_K K(j_1, j_1', j_1'', j_2, j_2') = 3$.

Next, define $L_1 = K[y_1, \frac{1}{Q(y_1)}, y_1', y_1'', y_2', y_2'']$, where $Q$ is a denominator of $R_{j_1}$, equipped with the derivation
\[
\begin{align*}
D_1 &= \frac{\partial}{\partial t} + y_1' \frac{\partial}{\partial y_1} + y_1'' \frac{\partial}{\partial y_1'} + \left( \frac{2}{y_1''} - (y_1')^3 \frac{\partial}{\partial y_1'} \right) \frac{\partial}{\partial y_1''}, \\
H_1 &= t \frac{\partial}{\partial t} - y_1' \frac{\partial}{\partial y_1} - 2y_1'' \frac{\partial}{\partial y_1'}, \\
Y_1 &= \frac{t}{2} \frac{\partial}{\partial t} - t y_1' \frac{\partial}{\partial y_1} - (2t y_1'' + y_1') \frac{\partial}{\partial y_1'}.
\end{align*}
\]
It is easily verified that $[X_1, H_1] = X_1$, $[H_1, Y_1] = Y_1$, $[X_1, Y_1] = H_1$. Furthermore, the equalities $[D_1, X_1] = 0$, $[D_1, H_1] = D_1$, $[D_1, Y_1] = t D_1$ can be easily verified.

When $K = \mathbb{C}(t)$, the algebraic group $\text{PSL}_2(\mathbb{C})$ acts on $L_1$ by
\[
h(t, y, y', y'') = \left( h(t), y, \frac{y'}{h'(t)}, \frac{y''}{(h'(t))^2} - y' \frac{h''(t)}{(h'(t))^3} \right)
\]
where $h$ denotes the homography of the projective line associated to an element $h$ of $\text{PSL}_2(\mathbb{C})$. One gets $(h)^* D_1 = \frac{1}{h(t)} D_1$. This equality means that the set of solutions of a Schwarzian equation is stable by the action of $\text{PSL}_2(\mathbb{C})$ by precomposition. The previously given action of $\text{psl}_2(\mathbb{C})$ is the infinitesimal action of $\text{PSL}_2(\mathbb{C})$.

We "verticalize" this action by considering $X_1 = X_1 - D_1$, $H_1 = H_1 - t D_1$ and $Y_1 = Y_1 - \frac{t^2}{2} D_1$. Now $CX_1'' + CH_1'' + CY_1''$ is a realization of $\text{psl}_2(\mathbb{C})$ acting $K$-linearly and commuting with $D_1$.

Define $L_2$ in similar way.
The ideal of the polynomial differential relations between \(j_1\) and \(j_2\) over \(K\) is an ideal \(I\) in \(L = L_1 \otimes_K L_2\). This ideal is a maximal ideal stable by

\[
(6.1) \quad D^{(2)} = \frac{\partial}{\partial t} + y'_1 \frac{\partial}{\partial y_1} + y'_2 \frac{\partial}{\partial y_2} + y''_1 \frac{\partial}{\partial y'_1} + y''_2 \frac{\partial}{\partial y'_2} + \left( \frac{3}{2} y''_1 - (y'_1)^3 R_1(y_1) \right) \frac{\partial}{\partial y'_1} + \left( \frac{3}{2} y''_2 - (y'_2)^3 R_1(y_2) \right) \frac{\partial}{\partial y'_2}.
\]

This implies that \(I\) is prime and the subfield of constants of \(F = \text{Frac}(L/I)\) with respect to the derivation \(D^{(2)}\) is \(C\)³.

We claim that \(I\) is stable under the diagonal action of \(\text{psl}_2\).

The algebra \(L/I\) is an algebraic extension of \(L_1\) and of \(L_2\), and as usual, \(D_1, X_1, H_1, Y_1, D_2, X_2, H_2, Y_2\) and their "verticalization" will also denote their extensions to \(L/I\).

**Lemma 6.11.** On \(L/I\) we have \(D_1 = D^{(2)} = D_2\).

**Proof.** Restrict the derivation \(D^{(2)}\) of \(L/I\) to its subalgebra \(L_1\). The definition of \(D^{(2)}\) gives that this restriction is \(D_1\). Now, the extension to the algebraic extension \(L/I\) of \(L_1\) is unique then \(D^{(2)} = D_1\). Same argument gives \(D^{(2)} = D_2\).

So we will just write this derivation as \(D\).

**Lemma 6.12.** There exists \(a \in C\) such that, on \(L/I\), \(X_1 = X_2, H_1 = H_2 + a(X_2 - D)\) and \(Y_1 = Y_2 + a(H_2 - tD) + \frac{a^2}{2}(X_2 - D)\).

**Proof.** By geometric triviality, \(I\) is generated by \(I \cap C[y_1, \frac{1}{Q(y_1)}, y'_1, \frac{1}{Q(y'_1)}, y''_1, y_2, \frac{1}{Q(y_2)}, y'_2, \frac{1}{Q(y'_2)}, y''_2]\). Then, on \(L/I\), both \(X_1\) and \(X_2\) coincide with \(\frac{a}{2}\), this proves \(X_1 = X_2\).

The two triplets \((X_1^p, H_1^p, Y_1^p)\) and \((X_2^p, H_2^p, Y_2^p)\) are two basis of derivations of \(F = \text{Frac}(L/I)\) over \(K\). Let \(A\) be the matrix with coefficient in \(F\) such that \((X_1^p, H_1^p, Y_1^p) = (X_2^p, H_2^p, Y_2^p) A\). From the bracket with \(D\) one gets \(0 = [D, (X_1^p, H_1^p, Y_1^p)] = (X_2^p, H_2^p, Y_2^p) D(A)\). So the coefficients of \(A\) are constant.

Now the two triplets are basis of two realisations of \(\text{psl}_2(C)\) with the same structure constants, then \(A\) is an automorphism of the Lie algebra \(\text{psl}_2(C)\). All automorphisms of \(\text{psl}_2(C)\) are inner (see [51, Proposition 14.21]), thus there exists a \(g \in \text{PSL}_2(C)\) such that \(\text{Ad}(g) = A\). This automorphism fixes \(X_1\), this implies that there exists \(a \in C\) such that \(g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\).

Then

\[
(X_1^p, H_1^p, Y_1^p) = (X_2^p, H_2^p, Y_2^p) \begin{pmatrix} 1 & a^2 \\ 0 & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \\ 0 & 1 \end{pmatrix}
\]

This proves the lemma.

³This also follows from geometric triviality.
In $F$, $y_2$ is an algebraic function over $C(t,y_1,y_1',y_1'')$ satisfying $X_2(y_2) = H_2(y_2) = Y_2(y_2) = 0$. Hence one easily computes that

\begin{equation}
X_1(y_2) = 0,
\end{equation}

\begin{equation}
H_1(y_2) = -aD(y_2),
\end{equation}

\begin{equation}
Y_1(y_2) = (-at - \frac{a^2}{2})D(y_2) = (t + \frac{a}{2})H_1(y_2).
\end{equation}

We will prove that this system of partial differential equations over $C(t,y_1,y_1',y_1'')$ has an algebraic solution if and only if $a = 0$. For contradiction, assume not. We expand $y_2$ as a Puiseux series in $1/z$ with $z = \frac{y_1''}{y_1}$, that is we think of $y_2$ as being an element of $C(t,y_1,y_1')^{alg} \langle \langle \frac{1}{z} \rangle \rangle$:

$$y_2 = \sum_{\lambda \leq n} A_\lambda(t,y_1,y_1')z^\lambda.$$  

In the coordinates $t, y_1, y_1', z$, one has

- $X_1 = \frac{\partial}{\partial t}$,
- $H_1 = t \frac{\partial}{\partial t} - y_1' \frac{\partial}{\partial y_1'}$,
- $Y_1 = \frac{\partial}{\partial t} - ty_1' \frac{\partial}{\partial y_1'} - \frac{1}{y_1} \frac{\partial}{\partial z}$,
- $D = \frac{\partial}{\partial t} - y_1' \frac{\partial}{\partial y_1'} + z(y_1')^2 \frac{\partial}{\partial y_1'} - \left( \frac{1}{2}z^2 y_1' + R_\lambda(y_1)y_1' \right) \frac{\partial}{\partial z}$.

Direct computations give

\begin{equation}
X_1(y_2) = 0,
\end{equation}

\begin{equation}
H_1(y_2) = \sum_{\lambda \leq n} -y_1' \frac{\partial A_\lambda}{\partial y_1'} z^\lambda,
\end{equation}

\begin{equation}
Y_1(y_2) = \sum_{\lambda \leq n} -ty_1' \frac{\partial A_\lambda}{\partial y_1'}(z)z^\lambda - \lambda A_\lambda \frac{1}{y_1'} z^{\lambda - 1},
\end{equation}

\begin{equation}
D(y_2) = \sum_{\lambda \leq n} R_\lambda(y_1)y_1' \lambda A_\lambda z^{\lambda - 1} + y_1' \frac{\partial A_\lambda}{\partial y_1'} z^\lambda + \left( (y_1')^2 \frac{\partial A_\lambda}{\partial y_1'} - \frac{1}{2}y_1' A_\lambda \right) z^{\lambda + 1}.
\end{equation}

**Lemma 6.13.** If $y_2$ is an algebraic solution of $6.2, 6.3, 6.4$ then $H_1(y_2) = 0$.

**Proof.** If $a = 0$ there is nothing to prove. Assume it is not. From $6.4$, one gets $-ty_1' \frac{\partial A_\lambda}{\partial y_1'} = (t + \frac{a}{2}) \left( -y_1' \frac{\partial A_\lambda}{\partial y_1'} \right)$ and then $\frac{\partial A_\lambda}{\partial y_1'} = 0$. Now $6.3$ gives $(y_1')^2 \frac{\partial A_\lambda}{\partial y_1'} - \frac{1}{2}y_1' nA_n = 0$, this implies $n = 0$ and the range of $\lambda$ is $-\mathbb{N}$.

The equation $6.4$ can be written as: $\forall k \in \mathbb{N}$,

\begin{equation}
\frac{a}{2} \frac{\partial A_{-k-1}}{\partial y_1'} = k \frac{A_{-k}}{y_1'}
\end{equation}

(6.4)
Let \( k_0 \) be the maximal integer such that for all strictly positive \( k \) smaller than \( k_0 \), \( A_{-k} = 0 \). The equality \( 6.4(0) \) gives that \( A_{-1} \) does not depend on \( y'_1 \). Then \( 6.4(1) \) is an equality between a derivative of an algebraic function in \( y'_1 \) and a rational function with a simple pole at 0. This implies that the latter is identically zero: \( k_0 \) is greater than 2.

Now if \( k_0 \) is finite then \( 6.4(k_0 - 1) \) is \( \frac{\partial A_{-k_0}}{\partial y'_1} = 0 \) and \( 6.4(k_0) \) is \( \frac{\partial A_{-k_0 - 1}}{\partial y'_1} = A_{-k_0} \). As a derivative of an algebraic function can not have simple pole, \( A_{-k_0} = 0 \) which contradicts the existence of \( k_0 \).

Then \( 6.6 \) proves the lemma. \( \square \)

If \( a \neq 0 \) the Lemma \( 6.13 \) and the equation \( 6.3 \) show that \( D(y_2) = 0 \). But the subfield of constants of \( D \) is \( C \) and \( y_2 \) is not constant. This contradicts the assumption on \( a \) and one gets \( a = 0 \).

Now, on \( F, X_1 = X_2, H_1 = H_2 \) and \( Y_1 = Y_2 \). These three derivations are linearly independent and their kernel is denoted by \( N \). Formulas for these derivations give \( y_1 \in N \) and \( y_2 \in N \).

The sequence of extensions \( C \subset N \subset F \) is such that \( \text{tr.deg.}_C N \geq 1 \), \( \text{tr.deg.}_N F \geq 3 \) and \( \text{tr.deg.}_C F = 4 \) then the transcendence degree of \( N \) over \( C \) is 1. This proves that \( I \) contains \( P \in C[y_1, y_2] \). It is not difficult to see that \( P \) generates \( I \) as a \( D \)-ideal. \( \square \)

Remark 6.14. It is not hard to see that Theorem \( 6.10 \) also holds for all general Schwarzian equations \( (s') \) provided that they are strongly minimal (and so geometrically trivial). Indeed the above proof did not use the fact that Fuchsian groups are involved. In particular, Theorem \( 6.10 \) holds if Condition \( 3.1 \) is true of the corresponding Riccati equations.

It now remains to understand the kind of polynomials \( P \in C[y_1, y_2] \) that can occur. Notice that if \( P(j_{\Gamma}(g_1t), j_{\Gamma}(g_2t)) = 0 \) gives an algebraic relation between two solutions \( j_{\Gamma}(g_1t) \) and \( j_{\Gamma}(g_2t) \), then there trivially is an algebraic relation between \( j_{\Gamma}(t) \) and \( j_{\Gamma}(g_2^{-1}g_1^{-1}) \), namely \( P(j_{\Gamma}(t), j_{\Gamma}(g_2^{-1}g_1^{-1})) = 0 \). So it suffices to characterize interalgebraicity with \( j_{\Gamma}(t) \).

Lemma 6.15. For \( g \notin \text{Comm}(\Gamma), j_{\Gamma}(t) \) is algebraically independent from \( j_{\Gamma}(gt) \) over \( C \).

Proof. Let \( g \notin \text{Comm}(\Gamma) \). For a contradiction, assume first that \( P \) is an algebraic relation over \( C \) holding between \( j_{\Gamma}(t) \) and \( j_{\Gamma}(gt) \). Then for all \( a \in H \), we have that \( P(j_{\Gamma}(a), j_{\Gamma}(ga)) = 0 \). For \( \gamma \in \Gamma \), consider the point \( b_\gamma = \gamma \cdot a \). Letting \( a = b_\gamma \), we have that \( P(j_{\Gamma}(b_\gamma), j_{\Gamma}(gb_\gamma)) = 0 \).

But, since \( j_{\Gamma}(b_\gamma) = j_{\Gamma}(a) \), we have that \( P(j_{\Gamma}(a), j_{\Gamma}(gb_\gamma)) = 0 \). Now, by the \( \Gamma \)-invariance of \( j_{\Gamma} \), we have that for any \( \gamma_1 \in \Gamma \), \( P(j_{\Gamma}(\gamma_1a), j_{\Gamma}(\gamma_1 g \gamma a)) = 0 \). But \( j_{\Gamma}(\gamma_1 a) = j_{\Gamma}(a) \), we have that

\[
P(j_{\Gamma}(a), j_{\Gamma}(\gamma_1 g \gamma a)) = 0
\]

for all \( \gamma_1, \gamma \in \Gamma \). However, \( j_{\Gamma} \) is precisely \( \Gamma \)-invariant, and for \( g \notin \text{Comm}(\Gamma) \), there are infinitely left coset representatives of \( \Gamma \) among the double coset \( \Gamma g \Gamma \). Then there are infinitely many points which \( j_{\Gamma}(\Gamma g \Gamma a) \) for which \( P(j_{\Gamma}(a), y) = 0 \) holds, contradicting the fact that \( P = 0 \) gives an algebraic relation. \( \square \)

Lemma 6.16. For \( g \in \text{Comm}(\Gamma) \), \( j_{\Gamma}(t) \) is algebraically dependent with \( j_{\Gamma}(gt) \) over \( C \).
**Definition 6.17.** By the previous result, when \( g \in \text{Comm}(\Gamma) \), there is an irreducible polynomial \( \Psi_g(x, y) \in \mathbb{C}[x, y] \) such that \( \Psi_g(j_1(t), j_1'(gt)) = 0 \). We call \( \Psi_g \) a \( \Gamma \)-special polynomial, and the zero set of such a polynomial a \( \Gamma \)-special curve.

Now from theorems 6.10 and 6.11 and lemmas 6.15 and 6.16 one gets the weak form of the Ax-Lindemann-Weierstrass theorems 2.12 and 2.13.

**Theorem 6.18.** Let \( K \) be a differential extension of \((\mathbb{C}(t), \frac{d}{dt})\) and \( j_1(g_1t), \ldots, j_1(g_nt) \) be distinct solutions of the Schwarzian equation \((*)\) that are not algebraic over \( K \) nor pairwise related by \( \Gamma \)-special polynomials. Then the \( 3n \) functions

\[
j_1(g_1t), j_1'(g_1t), j_1''(g_1t), \ldots, j_1(g_nt), j_1'(g_nt), j_1''(g_nt)
\]

are algebraically independent over \( K \).

**Proof.** For contradiction, assume that the \( 3n \) functions

\[
j_1(g_1t), j_1'(g_1t), j_1''(g_1t), \ldots, j_1(g_nt), j_1'(g_nt), j_1''(g_nt)
\]

are algebraically dependent over \( K \). Define the field \( K' \) as

\[
K' = K \langle j_1(g_1t), j_1'(g_1t), j_1''(g_1t), \ldots, j_1(g_nt), j_1'(g_nt), j_1''(g_nt) \rangle
\]

By strong minimality of equation \( (*) \), it must be that \( j_1(g_1t) \in K' \) and by geometric triviality of \( (*) \), we have that

\[
j_1(g_1t) \in K \langle j_1(g_1t) \rangle^{alg}
\]

for some \( i = 2, \ldots, n \). Using Theorem 6.10 we get that

\[
j_1(g_1t) \in \mathbb{C}(j_1(g_1t))^{alg}
\]

and so

\[
j_1(t) \in \mathbb{C}(j_1(g_1^{-1}t))^{alg}
\]

Now using Lemma 6.15 it must be the case that \( g = g_1g_1^{-1} \in \text{Comm}(\Gamma) \). So for the \( \Gamma \)-special polynomial \( \Psi_g \), we get

\[
\Psi_g(j_1(t), j_1(g_1^{-1}t)) = 0
\]

and hence

\[
\Psi_g(j_1(g_1t), j_1(g_1t)) = 0.
\]

This contradicts our assumption that \( j_1(g_1t) \) and \( j_1(g_1t) \) are not related by any \( \Gamma \)-special polynomials.

\[\square\]

**7. Orthogonality and the Ax-Lindemann-Weierstrass Theorem**

In the previous sections, we have understood the structure of the solution set of

\[
S_{\frac{d}{dt}}(y) + (y')^2 \cdot R_{j_1}(y) = 0.
\]

Define

\[
\chi_{\Gamma, \frac{d}{dt}}(y) = S_{\frac{d}{dt}}(y) + (y')^2 \cdot R_{j_1}(y).
\]

(7.1)
In this section, we consider equations of the form $\chi_{\Gamma, \frac{d}{dt}}(y) = a$ for $a$ an element in some differential field extension of $\mathbb{Q}$, and produce a similar analysis.

### 7.1. Strong minimality and algebraic relations on other fibers

First, we prove the solution set of the equation $\chi_{\Gamma, \frac{d}{dt}}(y) = a$ is strongly minimal and characterize the algebraic relations between solutions. Essentially, the analysis from [11, Section 5.1] goes adapts to this case, but for the sake of completeness, we will provide a brief explanation here.

Let $a \in K$ be an element in some differential field extension of $\mathbb{Q}$. By Seidenberg’s embedding theorem, we can, without loss of generality, assume $a = a(t)$ is given by a meromorphic function over some domain $U$, and the derivation is given by $\frac{d}{dt}$. After sufficiently shrinking the domain, there is some meromorphic function $\tilde{a}(t)$ satisfying $S_{\frac{d}{dt}}(\tilde{a}) = a$ such that

$$\chi_{\Gamma, \frac{d}{dt}}(j(\tilde{a}(t))) = a(t).$$

The following Lemma follows by the Schwarzian chain rule and is nearly identical to [11, Lemma 5.1]:

**Lemma 7.1.** Let $K$ be a differentially closed $\frac{d}{dt}$-field containing $a$. There exists $\partial \in K\frac{d}{dt}$ such that $\chi_{\Gamma, \partial}(y) = 0$.

**Proof.** The equation $S_{\frac{d}{dt}}(\tilde{a}) = a$, with unknown $\tilde{a}$, can be considered as a differential equation over $\mathbb{C}\langle a \rangle$. By Seidenberg’s theorem this field can be assumed to be a field of meromorphic function on $U \subset \mathbb{C}$ and by the usual Cauchy theorem, one can build a solution, holomorphic on $U' \subset U$.

By the differential nullstellensatz there exists $\tilde{a} \in K$ a solution of $S_{\frac{d}{dt}}(\tilde{a}) = a$. Then $\partial = \frac{1}{a}\frac{d}{dt}$.

**Theorem 7.2.** The sets $\chi_{\Gamma, \frac{d}{dt}}(y) = a$ are strongly minimal and geometrically trivial. If $a_1, \ldots, a_n$ satisfy $\chi_{\Gamma, \frac{d}{dt}}(a_i) = a$ and are dependent, then there exist $i, j \leq n$ and a $\Gamma$-special polynomial, $P$ such that $P(a_i, a_j) = 0$.

The proof of Theorem 7.2 is quite similar to that of [11] Proposition 5.2, but we include it here for completeness.

**Proof.** We first explain why $\chi_{\Gamma, \frac{d}{dt}}(y) = a$ is strongly minimal; it suffices to show that over some differentially closed field which contains the coefficients of the equation, that every differentially constructible set is finite or cofinite. Using properties of differentially closed field, on can find in $K \tilde{a}$ as above.

By Lemma 7.1, $K$ is a $\partial$-differential field and the sets $\chi_{\Gamma, \frac{d}{dt}}(y) = a$ and $\chi_{\Gamma, \partial}(y) = 0$ coincide. Now strong minimality follows by Theorem 3.2 and the fact that $\frac{d}{dt}$-differentially constructible sets are $\partial$-differentially constructible (over $K$).

Algebraic dependencies among elements of the set $\chi_{\Gamma, \frac{d}{dt}}(y) = a$ give algebraic dependencies among elements of the set $\chi_{\Gamma, \partial}(y) = 0$, and thus by Theorem 6.18 must be given by $\Gamma$-special polynomials. $\square$
The final piece of our analysis of the fibers of $\chi$ shows that there are no algebraic relations between different fibers.

**Theorem 7.3.** For $a \neq b$, the strongly minimal sets $\chi^{-1}_{\Gamma, \Phi} = a$ is orthogonal to $\chi^{-1}_{\Gamma, \Phi} = b$.

Theorem 7.3 is more general than [11] Theorem 5.4, but the proof is quite similar. We include an outline of the proof in this more general case.

**Proof.** As both $\chi^{-1}_{\Gamma, \Phi}(a)$ and $\chi^{-1}_{\Gamma, \Phi}(b)$ are strongly minimal and trivial, if $\chi^{-1}_{\Gamma, \Phi}(a) \napprox \chi^{-1}_{\Gamma, \Phi}(b)$, then there is a finite-to-finite correspondence between the sets, defined over $\mathbb{Q}(a, b)$. Regarding $a, b$ as meromorphic functions of a variable $t$ as above, let $R$ be the differential ring generated by $a(t), b(t), j(a(t)), j(b(t))$. Let $\nabla_{t, 2}$ denote the map $x \to (x, \frac{d^2}{dt^2}x)$.

Now noting once again that nonorthogonality of trivial strongly minimal sets requires no new parameters and that the third derivative of a solution to our equations is rational over the solution and its first two derivatives, we can assume the correspondence is given by $\nabla_{t, 2}^2(V)$, where $V$ is an absolutely irreducible over $R$, and gives a finite-to-finite correspondence between $A^3$ and $A^3$. Assume our ring $R$ is embedded in the ring $\mathcal{O}(U)$ of meromorphic functions on $U \subset H$. Fix $t_0 \in U$. We consider the fiber of $V$ over $t_0$, which gives a finite-to-finite correspondence between $A^3(C)$ and itself.

By [11] Claim 5.5, the set $\left(\nabla_{t, 2}^2 \chi^{-1}_{\Gamma, \Phi}(a) \times \nabla_{t, 2}^2 \chi^{-1}_{\Gamma, \Phi}(b)\right)(t_0)$ is Zariski dense in $V$.

Consider the action of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ on $H \times H$ extended to $J_2(H \times H)$. Let $\tilde{V}_{t_0}$ be a component of $(J_2(j_1) \times J_2(j_1))^{-1}(V_{t_0})$. Let $H_{t_0}$ denote the stabilizer of $\tilde{V}_{t_0}$ in $GL_2^+(\mathbb{R}) \times GL_2^+(\mathbb{R})$.

Now the analog of Claim 5.6 of [11] holds in our case (by the same argument): For each $\gamma \in \text{Comm}^+(\Gamma)$, there is $\delta \in \text{Comm}^+(\Gamma)$, such that $(\gamma, \delta) \in H_{t_0}$.

By the density of $\text{Comm}^+(\Gamma)$, it follows that $\tilde{H}_{t_0}$, the image of $H_{t_0}$ in $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ projects surjectively to each copy of $PSL_2(\mathbb{R})$. On the other hand, it must be the case that $\tilde{H}_{t_0}$ is a proper subgroup of $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$, by the fact that $V$ is a finite-to-finite correspondence. It follows, as in [21] that $\tilde{H}_{t_0}$ is the graph of an automorphism of $PSL_2(\mathbb{R})$. Every such automorphism is inner, and so there exists $g \in PSL_2(\mathbb{R})$ such that

$$\tilde{H}_{t_0} = \{ (\gamma, \gamma^g) : \gamma \in PSL_2(\mathbb{R}) \}.$$

Now, take a point $(x, y) \in \tilde{V}_{t_0}$. Let $g_1 \in PSL_2(\mathbb{R})$ such that $g_1 x_0 = y_0$. Let $T_x = \text{Stab}_x(PSL_2(\mathbb{R}))$. If $h \in T_x$ then $(1, h^g)(x, y) = (h, h^g)(x, y) \in \tilde{V}_{t_0}$. But, the fiber above $x$ in $V_{t_0}$ is finite. So, it must be that $T_y^g y$ is finite. So, $T_y^g \cap T_y$ has finite index in $T_y^g$. But the stabilizer of $x$ is connected, so it must be that $T_x^g \leq T_y^g$. $T_x^g = \pi(T_x^g) \leq \pi(T_y^g) = T_{y_0} = T_{x_0}^g$. But then $g T_{x_0} = g_1 T_{x_0}$. So one may take $g_1 = g$ above. So, $\pi_1(\tilde{V}_{t_0})$ is given by the action of $g$ on $H$. But $J_2(j_1 \times j_1)(V_{t_0}) = V_{t_0}$ is an algebraic variety, and it follows by Lemma 6.15 that $g$ must have been in $\text{Comm}^+(\Gamma)$. There are only countably many relations given by the graph of $g \in \text{Comm}^+(\Gamma)$, and one of these must hold on the generic fiber of $V$ over $\mathbb{C}$. But, this implies that $a = b$, a contradiction. \[\square\]

We can finally turn to the proof of the Ax-Lindemann-Weierstrass Theorem [2,15].
Proof of Theorem 2.15. Recall that $V \subset \mathbb{C}^n$ and for each $i = 1, \ldots, n$, the variety $V$ is assumed to project dominantly onto $\mathbb{A}^1$ under projection to the $i^{th}$ coordinate. Thus, the $i^{th}$ coordinate function is nonconstant, and it is possible to equip the field generated by the $i^{th}$ coordinate functions with various differential structures, which will be essential to the technique in our proof.

**Lemma 7.4.** There is a derivation $\delta$ on $C(V)$ such that for each of the coordinate functions $t_i$ for $i = 1, \ldots, n$, $\delta(t_i) \neq 0$.

**Proof.** Let $z_1, \ldots, z_k$ be a transcendence basis of $C(V)$ over $\mathbb{C}$ and $a_1, \ldots, a_k$ be $\mathbb{Q}$-linearly independent complex numbers. As $C(V)$ is an algebraic extension of $C(z_1, \ldots, z_k)$ the derivation $\delta = \sum_i a_i z_i \frac{\partial}{\partial z_i}$ extends a derivation of $C(V)$ and the field of constants in $C(V)$ is an algebraic extension of the field of constant in $C(z_1, \ldots, z_k)$. The latter is $\mathbb{C}$. As the projections of $V$ on the $i^{th}$ coordinate is dominant $\delta(t_i) \neq 0$.

The transcendence degree over $C(V)$ of the $3n$ functions

$$j_{t_1}(t_1), j_{t_1}(t_1), j_{t_1}(t_1) \ldots, j_{t_n}(t_n), j_{t_n}(t_n), j_{t_n}(t_n)$$

is identical to that of the $3n$ functions

$$j_{t_1}(t_1), \delta(j_{t_1}(t_1)), \delta^2(j_{t_1}(t_1)), \ldots, j_{t_n}(t_n), \delta(j_{t_n}(t_n)), \delta^2(j_{t_n}(t_n)).$$

Now, for any $t_i$, since $j_{t_1}(t_i)$ is not an algebraic function, it follows by strong minimality that $j_{t_1}(t_i)$ is a generic solution to a $\delta$-differential equation of the form $\chi_{t_1}(y) = a_i$ with $a_i = S_i(t_i) \in C(V)$.

If the $3n$ functions are not algebraically independent, then there exist $i, j$ such that the functions

$$j(t_i), \delta(j(t_i)), \delta^2(j(t_i)), j(t_j), \delta(j(t_j)), \delta^2(j(t_j))$$

are algebraically dependent over $K$, the $\delta$-field extension of $C(V)$ generated by $j(t_k)$ for those $k$ in some subset of $\{1, \ldots, n\} \setminus \{i, j\}$. Moreover one can choose $K$ such that $j(t_i)$ and $j(t_j)$ are not algebraic over $K$.

But then by strong minimality of the equations $\chi_{t_1}(y) = a_i$ and $\chi_{t_2}(y) = a_j$ (Theorem 7.2), there is a finite-to-finite correspondence between $\chi_{t_1}(y) = a_i$ and $\chi_{t_2}(y) = a_j$ defined over $K$. By Theorem 7.2 it must be that $a_i = a_j$ and $t_i$ and $t_j$ are $\Gamma$-geodesically dependent. A contradiction. □

7.2. **Orthogonality and commutators.** In this section, we analyze the algebraic relations between solutions of

$$S_{\#}(y) + (y')^2 \cdot R_{y''} = 0$$

$$S_{\#}(y) + (y')^2 \cdot R_{y''} = 0$$

Fix a subset of the coordinates such that there is an algebraic dependence as described above. Then there is some minimal such set. Picking $i, j$ to be any two coordinates of this minimal set, the subset is the collection of coordinates in the remainder of the minimal set.
when \( \Gamma_1 \) is not necessarily commensurable with \( \Gamma_2 \). If \( \Gamma_1 \) is commensurable with \( \Gamma_2 \), then it is well known that \( j_{\Gamma_1} \) is interalgebraic with \( j_{\Gamma_2} \) over \( \mathbb{C} \). Moreover this is not the whole story: we say that \( \Gamma_1 \) is commensurable with \( \Gamma_2 \) in wide sense if \( \Gamma_1 \) is commensurable to some conjugate of \( \Gamma_2 \). When such is the case and \( \Gamma_1 \) is commensurable with \( g^{-1} \Gamma_2 g \) then again one has that \( j_{\Gamma_1} \) is interalgebraic with \( j_{\Gamma_2} \circ g \) over \( \mathbb{C} \).

Notice that if \( \Gamma_1 \) is commensurable with \( \Gamma_2 \) in wide sense, then \( \text{Comm}(\Gamma_1) \) is conjugate to \( \text{Comm}(\Gamma_2) \).

**Theorem 7.5.** Suppose that \( \Gamma_1 \) is not commensurable with \( \Gamma_2 \) in wide sense. Then equations (7.2) and (7.3) are orthogonal. In particular, for any differential field \( K \)

\[
\text{tr.deg}_K (j_{\Gamma_1}(t_1), j_{\Gamma_1}^\prime(t_1), j_{\Gamma_2}(t_2), j_{\Gamma_2}^\prime(t_2)) = \text{tr.deg}_K (j_{\Gamma_1}(t_1), j_{\Gamma_1}^\prime(t_1)) + \text{tr.deg}_K (j_{\Gamma_2}(t_2), j_{\Gamma_2}^\prime(t_2))
\]

**Proof.** Let \( X_{\Gamma_1} \) and \( X_{\Gamma_2} \) be the set defined by equations (7.2) and (7.3) respectively. Assume for contradiction that \( X_{\Gamma_1} \not\subseteq X_{\Gamma_2} \). Since \( X_{\Gamma_1} \) and \( X_{\Gamma_2} \) are trivial strongly minimal sets, we have that nonorthogonality is witnessed over \( \mathbb{C} \) (i.e., the sets are non weakly orthogonal).

So for any solution \( y_1 \in X_{\Gamma_1} \) there is a solution \( y_2 \in X_{\Gamma_2} \) such that \( y_1 \in \mathbb{C} \langle y_2 \rangle^{\text{alg}} \). By invoking Fact 2.3 we have that \( j_{\Gamma_1}(t) \in \mathbb{C} \langle j_{\Gamma_2}(gt) \rangle^{\text{alg}} \) for some \( g \in \text{GL}_2(\mathbb{C}) \). Let us write

\[
P(j_{\Gamma_1}(t), j_{\Gamma_2}(gt), j_{\Gamma_1}^\prime(gt), j_{\Gamma_2}^\prime(gt), t) = 0
\]

for this algebraic relation over \( \mathbb{C} \). For any \( \gamma_1 \in \Gamma_1 \), using the fact that \( j_{\Gamma_1}(\gamma_1 t) = j_{\Gamma_1}(t) \), we have that

\[
P(j_{\Gamma_1}(t), j_{\Gamma_2}(g\gamma_1 t), j_{\Gamma_1}^\prime(g\gamma_1 t), j_{\Gamma_2}^\prime(g\gamma_1 t), \gamma_1 t) = 0.
\]

So this implies that for any \( \gamma_1 \in \Gamma_1 \), we get that \( j_{\Gamma_1}(t) \in \mathbb{C} \langle j_{\Gamma_2}(g\gamma_1 t) \rangle^{\text{alg}} \). In particular \( \mathbb{C} \langle j_{\Gamma_2}(gt) \rangle^{\text{alg}} = \mathbb{C} \langle j_{\Gamma_2}(g\gamma_1 t) \rangle^{\text{alg}} \) for all \( \gamma_1 \in \Gamma_1 \). By Theorem 6.18 it must be the case that \( g\gamma_1 g^{-1} \in \text{Comm}(\Gamma_2) \) for all \( \gamma_1 \in \Gamma_1 \), that is it must be that \( g\Gamma_1 g^{-1} \subseteq \text{Comm}(\Gamma_2) \).

Now, to get our contradiction, we consider three cases (without loss of generality):

1. Assume \( \Gamma_1 \) is arithmetic and \( \Gamma_2 \) is nonarithmetic. In this case, \( \chi_{\Gamma_1} \) is not \( \kappa_0 \)-categorical, while \( \chi_{\Gamma_2} \) is \( \kappa_0 \)-categorical (this follows from Theorem 6.18). This case could also be handled in a more elementary manner similar to our technique in the third case.
2. Assume that both \( \Gamma_1, \Gamma_2 \) are arithmetic groups. We have, by the above arguments, that \( g\Gamma_1 g^{-1} \) is contained in \( \text{Comm}(\Gamma_2) \). We will be done if we show that \( g\Gamma_1 g^{-1} \) and \( \Gamma_2 \) are commensurable in the strict sense. This follows by arguments of [29], see page 4, where the following fact is shown: for any two arithmetic Fuchsian groups \( G_1 \) and \( G_2 \), if \( G_1 \) is contained in the commensurator of \( G_2 \) then \( G_1 \) and \( G_2 \) are commensurable in the strict sense.
3. Assume that both \( \Gamma_1 \) and \( \Gamma_2 \) are non-arithmetic. By the above argument, we have that \( g\Gamma_1 g^{-1} \leq \text{Comm}(\Gamma_2) \) for some \( g \in \text{GL}_2(\mathbb{C}) \). By a symmetric argument, we have some \( h \in \text{GL}_2(\mathbb{C}) \) such that \( h\Gamma_2 h^{-1} \leq \text{Comm}(\Gamma_1) \). Replacing one of \( \Gamma_i \) with a suitable conjugate, we may assume that \( \Gamma_1 \leq \text{Comm}(\Gamma_2) \) and \( \Gamma_2 \leq \text{Comm}(\Gamma_1) \).

From this, we will show that \( \Gamma_1 \) and \( \Gamma_2 \) are commensurable. By Margulis’ Theorem, \( \Gamma_i \) is finite index in \( \text{Comm}(\Gamma_i) \). We need only show that \( \Gamma_2 \) is finite index in \( \text{Comm}(\Gamma_1) \).
We have that $\Gamma_1$ is contained in $\text{Comm}(\Gamma_2)$, $\Gamma_1$ contains only finitely many left coset representatives of $\Gamma_2$. Since $\Gamma_1$ is finite index in its own commensurator, the conclusion follows.

\[\square\]

8. Effective finiteness results around the André-Pink conjecture

The André-Pink conjecture predicts that when $W$ is an algebraic subvariety of a Shimura variety and $S$ is a Hecke orbit, if $W \cap S$ is Zariski dense in $W$, then $W$ is weakly special. For details, definitions, and proofs of certain special cases of the conjecture see [36, 37, 12].

In the setting of the present paper the conjecture concerns the intersection of an algebraic variety $W \subset \mathbb{A}^n$ with the image, under $j_\Gamma$, applied to each coordinate, of the orbit under $\text{Comm}(\Gamma)^n$ of some point in $\bar{a} \in \mathbb{H}$.

Given a Fuchsian group $\Gamma$ and a point $a \in \mathbb{C}$, we denote, by $\text{Iso}_\Gamma(a)$, the collection of points $b \in \mathbb{C}$ such that $P(a, b) = 0$ for some $\Gamma$-special polynomial $P$. Equivalently, for some (all) $\bar{a}, \bar{b} \in \mathbb{H}$ such that $j_\Gamma(\bar{a}) = a$ and $j_\Gamma(\bar{b}) = b$, there is $\gamma \in \text{Comm}(\Gamma)$ such that $\gamma \bar{a} = \bar{b}$.

Given a Fuchsian group $\Gamma$ and a point $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$, let $\text{Iso}_\Gamma(\bar{a})$ denote the product of the orbits of the points $a_1, \ldots, a_n$ under $\Gamma$-special polynomials, that is

$$\text{Iso}_\Gamma(\bar{a}) = \prod_{i=1}^n \text{Iso}_\Gamma(a_i).$$

We call call a polynomial $p(x_1, \ldots, x_n)$ $(\Gamma)$-$\text{-(a_1, \ldots, a_n)}$-special if

1. $p(\bar{x}) = x_i - b_i$ where $b_i \in \text{Iso}_\Gamma(a)$, or
2. For some $i, j$, $\text{Iso}_\Gamma(a_i) = \text{Iso}_\Gamma(a_j)$, and $p(\bar{x})$ is a $\text{Comm}(\Gamma)$-special polynomial in $x_i, x_j$.

An irreducible subvariety of $\mathbb{C}^n$ will be called $(\Gamma)$-$\text{-(a_1, \ldots, a_n)}$-special if it is given by a finite conjunction of $(\Gamma)$-$\text{-(a_1, \ldots, a_n)}$-special polynomials. If an irreducible variety $V$ is $(\Gamma)$-$\text{-(a_1, \ldots, a_n)}$-special, then it follows that $V$ has a Zariski dense set of points from $\text{Iso}_\Gamma(\bar{a})$.

Our first result of this section shows that the converse holds, at least when $\bar{a}$ is a tuple of transcendental numbers (perhaps with algebraic relations between them).

**Theorem 8.1.** Fix a complex algebraic variety $V \subset \mathbb{C}^n$, a Fuchsian group $\Gamma$, and a point $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that for all but at most one $i \in \{1, \ldots, n\}$, $a_i \notin \mathcal{Q}^{alg}$. Then $V \cap \text{Iso}_\Gamma(\bar{a})^{\text{Zar}}$ is a finite union of $(\Gamma)$-$\text{-(a_1, \ldots, a_n)}$-special varieties.

**Proof.** The (perhaps reducible) variety $V \cap \text{Iso}_\Gamma(\bar{a})^{\text{Zar}}$ consists of finitely many components $W_1, \ldots, W_k$, and so we need only show that the varieties $W_i$ are $(\Gamma)$-$\text{-(a_1, \ldots, a_n)}$-special. Working component by component, it suffices to show that for an arbitrary irreducible variety $V$, if $\text{Iso}_\Gamma(\bar{a})$ is Zariski dense in $V$, then $V$ is $(\Gamma)$-$\text{-(a_1, \ldots, a_n)}$-special.

Without loss of generality, assume that all of the coordinates of $\bar{a}$, except perhaps $a_1$, are transcendental over $\mathbb{Q}$. We also assume $a_1 \in \mathcal{Q}^{alg}$ without loss of generality - otherwise just ignore arguments about this coordinate in the proof.

Embed $\mathcal{Q}(a_2, \ldots, a_n)$ into the field of meromorphic functions on some connected subset of $\mathbb{H}$ such that $a_2, \ldots, a_n$ are non-constant.
Let $\bar{a}_2, \ldots, \bar{a}_n$ be as in the proof of Theorem 7.2—that is, $j_{\Gamma}(\bar{a}_i) = a_i$ for $i = 2, \ldots, n$. In the differential closure, $K$ of the field generated by the $a_i$ over $\mathbb{Q}$ we have, by Theorem 7.2 that

\[ \{x \in K \, | \, \chi_{\Gamma}(x) = \chi_{\Gamma}(a_i)\} = \text{Iso}_{\Gamma}(a_i), \]

so $\chi_{\Gamma}(a_i) = \chi_{\Gamma}(a_j)$ if and only if $\text{Iso}_{\Gamma}(a_i) = \text{Iso}_{\Gamma}(a_j)$.

Consider the collection of $i \in \{1, \ldots, n\}$ such that $V$ projects dominantly onto the coordinate corresponding to $x_i$. Then if $\text{Iso}_{\Gamma}(\bar{a})$ is dense in $V$, and we let $b_2, \ldots, b_n$ be a generic a collection of generic solutions of $\chi_{\Gamma}(b_i) = \chi_{\Gamma}(a_i)$ and let $b_1$ be a generic constant, we have that the tuple $\bar{b}$ is dependent over $\mathbb{C}$, but as $b_2, \ldots, b_n$ satisfy equations which are strongly minimal and trivial, it must be that two of the coordinates are nonorthogonal. But now we are done, since all instance of nonorthogonality are given by Theorem 7.2, since none of the coordinates $2, \ldots, n$ can be nonorthogonal $b_1$, a constant. \hfill $\square$

**Remark 8.2.** The assumption in Theorem 8.1 that all but at most one of the elements in the tuple $\bar{a}$ are transcendental is an inherent restriction of the method we employ, which is similar to the technique employed in various applications of differential algebra to diophantine problems. We replace a arithmetic (discrete) object by the solution to a system of differential equations. Generally speaking, the technique works when the discrete set satisfies some interesting differential equation, which one is able to understand. But the only derivation on $Q^{al}$ is the trivial one, and so such a coordinate can not. For other instances of the applying this general idea, see [14, 16, 50, 6].

It would be interesting to see if the methods here might be combined with methods solving other special cases of the conjecture (e.g. [36]) to remove the transcendence restrictions of Theorem 8.1.

**Remark 8.3.** The technique by which we prove Theorem 8.1 has natural limitations described in the previous remark, but it also has an interesting natural advantage over other techniques. Because we replace an arithmetic object, whose definition is very non-uniform, with a differential algebraic variety, results from differential algebraic geometry can be used to give effective bounds the degree of the Zariski-closure of the solutions set.

A general purpose Bezout-type theorem for algebraic differential equations (generalizing a theorem of Hrushovski and Pillay) was established in [10]. In what follows, $\tau_\ell \mathbb{A}^n$ denotes the $\ell^{th}$-prolongation space of $\mathbb{A}^n$, and for a differential field $K$, we define

\[ (X, S \setminus T)^\sharp(K) = \{a \in X(K) : (a, a', \ldots, a^{(\ell)}) \in S \setminus T(K)\}. \]

**Theorem 8.4.** Let $X$ be a closed subvariety of $\mathbb{A}^n$, with $\dim(X) = m$, and let $S, T$ be closed subvarieties (not necessarily irreducible) of $\tau_\ell \mathbb{A}^n$ for some $\ell \in \mathbb{N}$. Then the degree of the Zariski closure of $(X, S \setminus T)^\sharp(\mathbb{C})$ is at most $\deg(X)^{2m^2} \deg(S)^{2m^2 - 1}$. In particular, if $(X, S \setminus T)^\sharp(\mathbb{C})$ is a finite set, this expression bounds the number of points in that set.

Next, we aim to put our differential relations in a form such that we may apply Theorem 8.1. Recall our Schwarzian differential equation:

\[ S_{yy}(y) + (y')^2 \cdot R_{ii}(y) = 0 \]
Remark 8.6. One can also establish (by the same means as the previous proof) a version of Theorem 8.5 with one coordinate algebraic rather than transcendental (the bound is slightly better in this case). The bounds of Theorem 8.5 can also be improved (using more elaborate arguments) by applying the results of [5], a process carried out in [5] in the case that \( \Gamma \) is the modular group.
REFERENCES

[1] M. Ablowitz and A. Fokas, Complex variables: introduction and applications, Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, second edition, 2003.

[2] V. Aslanyan, Ax-Schanuel type theorems and geometry of strongly minimal sets in differentially closed fields, arXiv preprint arXiv:1606.01778 (2016).

[3] P. Bayer and A. Travesa, Uniformization of Triangle Modular Curves, Proceedings of the Primeras Jornadas de Teoria de Números, Publ. Mat. (2007), 43-106.

[4] P. Bayer and A. Travesa, Uniformizing functions for certain Shimura curves, in the case $D = 6$, Acta Arith. 126 (2007), no. 4, 315-339.

[5] G. Binyamini, Bezout-type theorems for differential fields, Compositio Mathematica 153.4 (2017): 867-888.

[6] A. Buium, Geometry of differential polynomial functions, III: moduli spaces, American Journal of Mathematics 117, no. 1 (1995): 1-73.

[7] A. Buium, Differential Algebraic Groups of Finite Dimension, Lecture Notes in Mathematics 1506, Springer 1992.

[8] T. Crespo, Z. Hajto, Algebraic Groups and Differential Galois Theory, Graduate Studies in Mathematics 122, AMS, 2011.

[9] C. Daw and J. Ren, Applications of the hyperbolic Ax-Schanuel conjecture, Compositio Mathematica 154, no. 9 (2018): 1843-1888.

[10] J. Freitag and O. Léon Sánchez, Effective uniform bounding in partial differential fields, Advances in Mathematics 288 (2016): 308-336.

[11] J. Freitag and T. Scanlon, Strong minimality and the $j$-function, J. Eur. Math. Soc. 20 (2018): 119-136.

[12] Z. Gao, A special point problem of André-Pink-Zannier in the universal family of abelian varieties, Annals of the Scuola Normale Superiore di Pisa Science, Vol. 17, No 1 (2017): 231-266.

[13] Z. Gao, Towards the Andre-Oort conjecture for mixed Shimura varieties: The Ax-Lindemann theorem and lower bounds for Galois orbits of special points, Journal für die reine und angewandte Mathematik (Crelles Journal) 2017, no. 732 (2017): 85-146.

[14] D. Ghioca, F. Hu, T. Scanlon, and U. Zannier, A variant of the Mordell-Lang conjecture, arXiv preprint arXiv:1804.10561, 2018.

[15] J. Gray, Linear differential equations and group theory from Riemann to Poincaré, Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2008.

[16] E. Hrushovski, The Mordell-Lang conjecture for function fields, Journal of the American mathematical society 9, no. 3 (1996): 667-690.

[17] E. Hrushovski, Geometric model theory, Proceedings of the International Congress of Mathematicians, Berlin, 1998 , Volume I, 2817302.

[18] E. Hrushovski and Z. Sokolovic, Strongly minimal sets in differentially closed fields, unpublished manuscript, 1994.

[19] S. Katok, Fuchsian groups, Chicago Lect. in Math., U. of Chicago Press, Chicago, 1992.

[20] B. Klingler, E. Ullmo, and A. Yafaev, The hyperbolic Ax-indemann-Weierstrass conjecture, Publications mathématiques de l'IHÉS 123.1 (2016): 333-360.

[21] E. Kolchin, Algebraic groups and algebraic dependence, American Journal of Mathematics, (1968): 1151-1164.

[22] J. Kovacic, An algorithm for solving second order linear homogeneous differential equations, J. Symbolic Comput., 2 (1986): 3-43.

[23] J. Lehner, Discontinuous groups and automorphic functions, in: Mathematical Surveys, No. VIII American Mathematical Society. Providence, R.I., 1964.

[24] O. León Sánchez and R. Moosa, Isolated types of finite rank: an abstract Dixmier-Moeglin equivalence, arXiv preprint arXiv:1712.00933 2017.

[25] G. Margulis, Discrete Subgroups of Semisimple Lie Groups, Ergebnisse der Mathematik unter ihrer Grenzgebiete 17, Springer (1990).
D. Marker, Manin kernels, in Connections between Model Theory and Algebraic and Analytic Geometry, edited by A. Macintyre, Quaderni Matematica vol. 6, Naples II, 2000.

D. Marker, Model theory of differential fields. In Model theory of fields (pp. 38-113). Association for Symbolic Logic, 1996.

M. Matsuda, Lectures on algebraic solutions of hypergeometric differential equations, Lecture notes in Math. 15, Kinokuniya, 1985.

G. Mchane, Geodesic intersections and isosial Fuchsian groups, arXiv preprint arXiv:1709.08958 (2017).

S. Mochizuki, Correspondences on hyperbolic curves, Journal of Pure and Applied Algebra, 131 (1998): 227-244.

N. Mok, J. Pila, J. Tsimerman, Ax-Schanuel for Shimura varieties, arXiv preprint arXiv:1711.02189 (2017).

J. Nagloo and A. Pillay, On Algebraic relations between solutions of a generic Painlevé equation, J. Reine Angew. Math. 726 (2017): 1-27.

K. Nishioka, A conjecture of Mahler on automorphic functions, Archiv der Mathematik, 53 (1989): 46-51.

K. Nishioka, Painlevé’s Theorem on Automorphic Functions, Manuscripta Math. 66 (1990), no. 4, 341-349.

K. Nishioka, Painlevé’s Theorem on Automorphic Functions II, Funkcialaj, 35 (1992): 597-602.

M. Orr, The André-Pink conjecture: Hecke orbits and weakly special subvarieties, Ph. D thesis, Université Paris Sud - Paris XI, 2013.

M. Orr, Families of abelian varieties with many isogenous fibers, J. reine angew. Math. 705 (2015): 211-231.

P. Painlevé, Leçons sur la théorie analytique des équations différentielles professées à Stockholm (1895) in Oeuvres de Paul Painlevé. Tome I, Éditions du Centre National de la Recherche Scientifique, Paris, 1973, 825 pp.

Y. Peterzil and S. Starchenko, Uniform definability of the Weierstrass functions and generalized tori of dimension one, Selecta Mathematica, New Series 10.4 (2005): 525-550.

Y. Peterzil and S. Starchenko, Complex analytic geometry and analytic-geometric categories, Journal für die reine und angewandte Mathematik (Crelles Journal) 2009.626 (2009): 39-74.

Y. Peterzil and S. Starchenko, Tame complex analysis and o-minimality, Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II-IV: Invited Lectures. 2010.

Y. Peterzil and S. Starchenko, Definability of restricted theta functions and families of abelian varieties, Duke Mathematical Journal 162.4 (2013): 731-765.

J. Pila, O-minimality and the André-Oort conjecture for $C^n$, Annals of mathematics (2011): 1779-1840.

J. Pila, Modular Ax-Lindemann-Weierstrass with derivatives, Notre Dame J. Formal Logic 54 (2013): 553-565.

J. Pila and J. Tsimerman, Ax-Schanuel for the $j$-function, Duke Mathematical Journal 165, no. 13 (2016): 2587-2605.

J. Pila and J. Tsimerman, Ax-Lindemann for $A_g$, Annals of mathematics (2014): 659-681.

J. Pila and J. Tsimerman, Ax-Schanuel for the $j$-function, Duke Mathematical Journal 165.13 (2016): 2587-2605.

J. Pila and A. Wilkie, The rational points of a definable set, Duke Mathematical Journal 133.3 (2006): 591-616.

A. Pillay, Geometric stability theory, No. 32. Oxford University Press, 1996.

A. Pillay, Mordell-Lang conjecture for function fields in characteristic zero, revisited, Compositio Mathematica 140, no. 1 (2004): 64-68.

A. Sagle and R. Walde, Introduction to Lie algebras and Lie groups, Academic Press, New York, 1973.

H. Saint Gervais, Uniformisation des surfaces de Riemann: retour sur un théorème centenaire, ENS Editions, Lyon, 2010.

G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, Bull. Amer. Math. Soc. 79 (1973): 514-516.

K. Takeuchi, Arithmetic triangle groups, J. Math. Soc. Japan. 29 (1977): 91-106.

K. Takeuchi, Commensurability classes of arithmetic triangle groups: Dedicated to Professor Y. Kawada on his 60th birthday, Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics, Vol.24 (1977): 201-212.
[56] F. Tu, Schwarzian differential equations associated to Shimura curves of genus zero, Pacific Journal of Mathematics 269 (2), 453-489.
[57] E. Ullmo and A. Yafaev, Hyperbolic Ax-Lindemann theorem in the cocompact case, Duke Mathematical Journal 163.2 (2014): 433-463.
[58] H. Umemura, Second proof of the irreducibility of the first differential equation of Painlevé, Nagoya Math. J. 117 (1990): 125-171.
[59] L. Van den Dries, *Tame topology and o-minimal structures*, Vol. 248. Cambridge university press, 1998.
[60] A. Venkov, *Spectral Theory of Automorphic Functions and its Applications*, Kluwer Academic Publishers, Dordrecht, 1990.
[61] M. Vigneras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Mathematics, vol. 800, Springer, Berlin, 1980.

GUY CASALE, UNIV RENNES, CNRS, IRMAR-UMR 6625, F-35000 RENNES, FRANCE
E-mail address: guy.casale@univ-rennes1.fr

JAMES FREITAG, UNIVERSITY OF ILLINOIS CHICAGO, DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, 851 S. MORGAN STREET, CHICAGO, IL, USA, 60607-7045.
E-mail address: freitag@math.uic.edu

JOEL NAGLOO, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, BRONX COMMUNITY COLLEGE, CUNY BRONX, NY 10453, USA.
E-mail address: joel.nagloo@bcc.cuny.edu