The exact radiation-reaction equation
for a classical charged particle§

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Abstract

An unsolved problem of classical mechanics and classical electrodynamics is the search of the exact relativistic equations of motion for a classical charged point-particle subject to the force produced by the action of its EM self-field. The problem is related to the conjecture that for a classical charged point-particle there should exist a relativistic equation of motion (RR equation) which results both non-perturbative, in the sense that it does not rely on a perturbative expansion on the electromagnetic field generated by the charged particle and non-asymptotic, i.e., it does not depend on any infinitesimal parameter. In this paper we intend to propose a novel solution to this well known problem, and in particular to point out that the RR equation is necessarily variational. The approach is based on two key elements: 1) the adoption of the relativistic hybrid synchronous Hamilton variational principle recently pointed out (Tessarotto et al, 2006). Its basic feature is that it can be expressed in principle in terms of arbitrary ”hybrid” variables (i.e., generally non-Lagrangian and non-Hamiltonian variables); 2) the variational treatment of the EM self-field, taking into account the exact particle dynamics.

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I. INTRODUCTION

A famous (and unsolved) theoretical issue both in classical and quantum mechanics is related to the radiation reaction (RR) problem, i.e., the treatment of the dynamics of a charged particle in the presence of its EM self-field (for an introduction and background see Feynman, 1970 [1]) to be based on the construction of its relativistic RR equations of motion (RR equation). For contemporary science the search of a possible exact solution of the RR problem represents not merely an unsolved intellectual challenge, but a fundamental prerequisite for the proper formulation of all physical theories which are based on the description of relativistic dynamics of classical charged particles. These involve the consistent formulation of the relativistic kinetic theory of charged particles and of the related fluid descriptions (i.e., the relativistic magnetohydrodynamic equations obtained by means of suitable closure conditions), both essential in plasma physics and astrophysics. Despite efforts spent by many the problem of its theoretical description remains still elusive.

In classical mechanics the RR problem was first posed by Lorentz in his historical work (Lorentz, 1985 [3]; see also Abraham, 1905 [4]). Traditional approaches are based either on the RR equation due to Lorentz, Abraham and Dirac (first presented by Dirac in 1938 [5]), nowadays popularly known as the LAD equation, or the equation derived from it by Landau and Lifschitz [6] via a suitable ”reduction process”, the so-called LL equation. As recalled elsewhere [see related discussion in [7] (Ref.A)] several aspects of the RR problem - and of the LAD and LL equations - are yet to find a satisfactory formulation/solution. Common feature of all previous approaches is the adoption of an asymptotic expansion for the EM self-field (or for the corresponding EM 4-potential), rather than the exact representation of the same force-field. This, in turn, implies that such methods permit to determine - at most - only an asymptotic approximation for the (still elusive) exact equation of motion for a charged particle subject to its own EM self-field (RR equation). A side consequence of such approximations is the possible violation of basic principles of classical dynamics (for a review see for example [8] and related discussion in Ref.A).

A major critical aspect of the RR problem is, however, related to its (still missing) possible variational formulation. This is reflected by the circumstance that - as pointed out in Ref.A - all RR equations obtained so far (in particular the LAD and LL equations) are non-variational, i.e., they cannot be derived from a variational principle. This result is
clearly in contrast to the basic principles of classical mechanics. In particular it conflicts with
Hamilton’s action principle, which - under such premises (i.e., the validity of LAD and/or
LL equations) - should actually hold true only in the case of inertial motion! A major
consequence which follows is that the dynamics of point-like charged particles described by
these approximate model equations is not Hamiltonian. This is actually the reason why
in contemporary literature relativistic systems of charged particles are not considered as
Hamiltonian systems. Nevertheless, it is not clear whether this feature is only an accident,
i.e., is only due to the approximations introduced in the RR equations adopted so far or is
actually intrinsic to the nature of the RR problem. Unfortunately, a satisfactory answer to
this fundamental question has not yet been given. Another key issue is, however, related to
the condition of validity of the relativistic Hamilton variational principle \([9]\).

In this paper we intend to analyze in detail a result which is already well-known in the
literature, namely that in its customary form the Hamilton principle does not apply for
point-particles. This is due to intrinsic divergences (in particular due to the occurrence of
an infinite EM mass) produced by the EM self-field \([2]\): as a consequence, for point-particles
the radiation-reaction effect cannot be consistently taken into account in the framework of
classical electrodynamics. For this reason in the past several authors, including Born and
Infeld, Dirac, Wheeler and Feynman (see discussion in Ref.\([1]\)), tried to modify classical
electrodynamics in an effort to eliminate all divergent contributions arising due to EM self-
interactions. This is the so-called regularization problem for point-particles, based on the
introduction of suitable modifications of Maxwell’s electrodynamics. There is an extensive
literature devoted to possible ways to achieve this goal (for a review and references on
the subject see for example \([10]\)). A possible strategy involves introducing appropriate
modifications of the EM self 4-potential. Typically this is done (see for example Rohlich
\([11]\)) by assuming that there exists a decomposition of the EM field, whereby each particle
”feels” only the action of external particles and of a suitable part of the EM self-field. While
this decomposition becomes clearly questionable for finite-size particles, its consistency with
first principles - and in particular with standard quantum mechanics - seems dubious, to say
the least. Indeed, according to Feynman’s own’s words \([1]\) up to now ”nobody ever succeeded
in making a self-consistent quantum theory out of any of the(se) modified theories”.

In our view these motivations clearly indicate that the route to the solution of the RR
problem should be based on the search of the exact relativistic RR equation, i.e., the con-
struction of a non-perturbative RR equation. In this paper we intend to propose a novel solution to the RR problem, by pointing out that it can be achieved by means of the relativistic Hamilton variational principle formulated in the framework of classical electrodynamics. The approach is based on the adoption of a synchronous variational principle and for finite-size spherical-shell charges. As a consequence, the explicit variational treatment of the retarded EM self-potential generated by the same particles is made possible. Based on the construction of the Euler-Lagrange equations stemming from the variational principle, the exact relativistic equations of motion for a charged finite-size particle immersed in prescribed EM and gravitational fields can in principle be obtained in this way.

II. VARIATIONAL DESCRIPTION OF CLASSICAL POINT-PARTICLE RELATIVISTIC DYNAMICS

From the mathematical viewpoint, one of the corner-stones of classical mechanics is the assumption that the coupled set of equations formed by the particle dynamical equations and Maxwell’s equations is variational. In other words, both the particle state and the EM field in which the particle is immersed are completely determined by means of a suitable variational principle. In relativistic classical mechanics it is well known that - consistent with Maxwell’s classical electrodynamics - this is realized by the Hamilton variational principle, to be formulated in the framework of a fully covariant description. The choice of the dynamical variables which define the particle state remains in principle arbitrary. Thus, they can always be represented by so-called ”hybrid” variables, i.e., superabundant variables which generally do not define a Lagrangian state. This implies, thanks to Darboux theorem, that it should always be possible to identify them locally with canonical variables. As a basic consequence, classical systems should be necessarily Hamiltonian, i.e., their canonical states should be extrema of the corresponding Hamiltonian action, while the corresponding particle equations of motion, i.e., the Euler-Lagrange equations provided by the same variational principle, necessarily should coincide with Hamilton’s equations of motion. Here, in particular, we intend to show that the same variational principle should hold also when: (a) the EM field is considered variational, namely the variation of the action with respect to the EM 4-potential delivers also the complete set of Maxwell’s equations; (b) more generally, when the EM field, specified via its 4-potential, is represented in terms of an arbitrary
"admissible" superposition of prescribed and variational parts. By definition in the sequel a
decomposition of the four potential of the form $A_\mu = A_\mu^{(1)} + A_\mu^{(2)}$, where $A_\mu^{(1)}$ and $A_\mu^{(2)}$ denote
'a priori' arbitrary contributions to $A_\mu$, is denoted as admissible for a prescribed variational
functional if all related contributions, appearing in the same functional, actually exist. This
means, that the considered field decomposition must be specified in such a way that both
$A_\mu^{(1)}$ and $A_\mu^{(2)}$ are defined in the whole phase-space and are summable in a suitable sense so
that all relevant integrals involving the two 4-vectors actually can be uniquely defined. Thus,
if Hamilton variational principle holds for an arbitrary choice of the EM field and arbitrary
initial conditions for a system of charged classical particles, it is expected to apply also in
the presence of the EM self-field generated by the particles themselves. In other words, the
relativistic equations of motion for charged classical particles should be variational also in
the presence of the radiation reaction (EM self-force) arising from the particles themselves.

In the following we shall stick purely to classical electrodynamics. For this purpose we
intend to adopt the classical Hamilton variational principle for relativistic particles [9]. For
definiteness, let us first consider the case of a charged point-particle immersed in an EM
field, identifying with $(r^\mu, u_\mu)$ the particle state, with $r^\mu$ and $u_\mu$ the position and velocity 4-
vectors, and with $A_\mu(r)$ the EM 4-vector potential associated to the EM field (and depending
on the 4-vector $r \equiv r^\nu$). In classical mechanics the variational functional (the Hamilton
action functional) is well-known, and can be realized either by means of asynchronous [6]
or synchronous [9] variational principles. The variational functional (action functional) is
defined in terms of the curves $r^\mu(s)$ and $u_\mu(s)$, functions of the proper time $s$ [with $s \in \mathbb{R}$],
and the EM 4-potential $A_\mu(r)$. In the case of a point-particle the following Theorem/Axiom
should hold [16]

**THM.1 - Hamilton principle for point-particles** Let us assume that: 1) the real
functions $f(s) \equiv [r^\mu(s), u_\mu(s), \chi(s)]$ and the real 4-vector $A_\mu(r)$ belong respectively to suitable
functional classes $\{f\}$ and $\{A_\mu\}$ in which end points and boundaries are kept fixed; 2) the
functional

$$
S(r^\mu, u_\mu, \chi, A_\mu) = \int_1^2 \left( m_0 c u_\mu + \frac{q}{c} A_\mu(r) \right) dr^\mu +
\int_{s_1}^{s_2} ds \chi(s) [u_\mu(s) u^\mu(s) - 1] + \frac{1}{16\pi c} \int \frac{d\Omega}{\sqrt{-g}} F^{\mu\nu} F_{\mu\nu}
$$

(Hamilton action integral) exists for all $f(s) \in \{f\}$, and $A_\mu(r) \in \{A_\mu\}$. Here, $u^\mu(s) =
g^{\mu\nu} u_\nu(s)$, while $g^{\mu\nu} = g^{\mu\nu}(r(s))$ denotes the counter-variant components of the metric tensor,
each one to be considered dependent of the generic varied curve \( r(s) \); furthermore, \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( F^{\mu \nu} = g^{\mu \alpha} g^{\nu \beta} F_{\alpha \beta} \), while \( m_o \) and \( q \) are respectively the constant rest mass and electric charge of a point particle, \( d\Omega = \sqrt{-g} dt dx dy dz \) the 4-volume element and \( ds \) the line element; 3) if \( f(s), A_\mu(r) \) are extremal curves of \( S \) (see below) the line element \( ds \) satisfies the constraint \( ds^2 = g_{\mu \nu}(r(s)) dr^\mu(s) dr^\nu(s) \). It follows that: \( T_1 \) for arbitrary independent synchronous variations \( \delta f(s) \), the synchronous variational principle

\[
\delta S = 0, \tag{2}
\]
delivers the following set of Euler-Lagrange equations for the extremal curves \( f(s) \):

\[
-d \left( m_o c u_\mu + \frac{q}{c} A_\mu \right) + \frac{q}{c} \frac{\partial}{\partial r^\mu} A_\nu dr^\nu + 2u_\alpha u_\beta \chi(s) \partial_\mu \left( g^{\alpha \beta} \right) = 0, \tag{3}
\]

\[
m_o c dr^\mu + 2\chi(s) u^\mu(s) ds = 0, \tag{4}
\]

\[
u_\mu(s) u^\mu(s) - 1 = 0, \tag{5}
\]

where the extremal value of the Lagrange multiplier \( \chi \) reads for all \( s \in \mathbb{R} \), \( \chi(s) = -\frac{m_o c}{2} \); \( T_2 \) for arbitrary variations \( \delta A_\mu \) in which \( \delta A_\mu \) is considered independent of the extremal curve \( r^\nu(s) \), Eq. (2) delivers as Euler-Lagrange equation the Maxwell’s equations for the EM 4-potential \( A_\mu(r) \):

\[
\partial_\mu F^{\mu \nu} = \frac{4\pi}{c} j^\nu, \tag{6}
\]

where the 4-vector \( j^\mu = j^\mu(r) \) in Eq. (6) reads \( j^\mu(r) = q c \int_{s_1}^{s_2} ds' u^\mu(s') \delta^{(4)}(r - r(s')) \), i.e., it is the 4-current of a point charge moving along the world-line \( r^\mu = r^\mu(s) \) with a 4-velocity \( u^\mu(s) \).

PROOF

The proof of \( T_1 \) is straightforward (see Ref. [16]). To obtain the Euler-Lagrange equation for \( \delta A_\mu \) one invokes the identities

\[
\frac{1}{16\pi} \delta \int \frac{d\Omega}{\sqrt{-g}} F^{\mu \nu} F_{\mu \nu} = -\frac{1}{4\pi} \int \frac{d\Omega}{\sqrt{-g}} \delta A_\nu \partial_\mu F^{\mu \nu}, \tag{7}
\]

\[
\delta \int_{s_1}^{s_2} ds A_\mu(r(s)) \frac{dr^\mu(s)}{ds} = \int \frac{d\Omega}{\sqrt{-g}} \int_{s_1}^{s_2} ds' \delta A_\mu(r(s')) \frac{dr^\mu(s')}{ds'} \delta^{(4)}(r(s) - r(s')), \tag{8}
\]

where \( \delta^{(4)}(r - r(\tau)) \) is the 4-dimensional Dirac delta. Requiring that \( \delta A_\mu \) is independent of the 4-vector \( r^\nu(s) \) the variational principle (2) delivers the Maxwell’s equations (6), in which the 4-current \( j^\mu(r^\nu) \) necessarily takes the form defined in THM.1.
Let us now investigate the conditions of validity of THM.1. It is important to remark that if this theorem is true for an arbitrary (suitably smooth) EM 4-potential $A_\mu(r)$, it must apply also if the EM 4-potential $A_\mu(r)$ is represented in terms on an arbitrary decomposition for the 4-potential. Thus for example, one of the two components (say $A^{(1)}_\mu$) can be in principle considered prescribed, in the sense that there results by assumption $\delta A^{(1)}_\mu \equiv 0$ for all synchronous variations such that $\delta f(s) \equiv 0$ (as an example, we may consider the trivial decomposition in which also $A^{(2)}_\mu \equiv 0$). Therefore, in particular, the Hamilton variational principle [Eq.(2)] should hold true also in the case in which the EM 4-potential takes the form

$$A_\mu = A^{(\text{self})}_\mu + A^{(\text{ext})}_\mu,$$

i.e., it is represented in terms of the EM self- and external 4-potentials, while letting $\delta A^{(\text{self})}_\mu \equiv 0$, for all variations $\delta A_\mu = \delta A^{(\text{ext})}_\mu$ and such that $\delta f(s) \equiv 0$. Since - thanks to the linearity of Maxwell’s equations - this decomposition can always be made, it should be admissible, by definition, for the variational functional $\{1\}$. On the other hand, its possible violation (i.e., in case the decomposition is not admissible) would have the fundamental consequence that the Hamilton variational principle becomes invalid for point-particles when the EM self-field of the point-particle is taken into account. In such a case the following result can be proven:

**THM.2 - Violation of THM.1 for the EM self-force of point-particles**

As a consequence of THM.1 it follows that: $C_1$) the Euler-Lagrange equations obtained by imposing an arbitrary synchronous variation $\delta A_\mu$ must hold for any decomposition $A_\mu(r) = A^{(1)}_\mu(r) + A^{(2)}_\mu(r)$ of the 4-vector $A_\mu(r)$ which is admissible for $\{1\}$, where one of the two terms, for example $A^{(1)}_\mu(r)$, is considered a prescribed function of the 4-vector $r(s) \in \{f\}$, i.e., such that $\delta A^{(1)}_\mu(r) = 0$ when $\delta f(s) \equiv 0$; $C_2$) the decomposition $\{9\}$ is not admissible for the action $\{1\}$, i.e., in terms of the 4-potentials of the EM self-field and of the external EM field, $A^{(\text{self})}_\mu$ and $A^{(\text{ext})}_\mu$.

**PROOF**

The proof of proposition $C_1$ is as follows. Let us consider and arbitrary admissible decomposition $A_\mu(r) = A^{(1)}_\mu(r) + A^{(2)}_\mu(r)$, requiring that $A^{(1)}_\mu(r)$ is a prescribed function of the extremal curve $r^\nu(s)$, so that there results identically $\delta A_\mu \equiv \delta A^{(2)}_\mu$. The variational equation for $\delta A_\mu$ remains manifestly unchanged. To prove proposition $C_2$, let us now pose the problem whether Eq. $\{9\}$ is an admissible decomposition or not. For definiteness, let us
consider the particular case of flat space-time (i.e., $\sqrt{-g} = 1$). In this case the expression of $A^{(\text{self})}_\mu$ for point-particles is well-known and coincides with the so-called Lienard-Wiechart potentials $[6]$. It is immediate to prove that $A^{(\text{self})}_\mu$ carries diverging contributions to the action functional $S$, in particular due to the integral $\int \frac{d\Omega}{\sqrt{-g}} F^{(\text{self})}\mu\nu F^{(\text{self})}_{\mu\nu}$ which is manifestly divergent. Hence the decomposition (9) is not admissible.

III. VARIATIONAL DESCRIPTION FOR FINITE-SIZE CHARGES

THM.2 implies the fundamental consequence that for point-particles the variational principle (2) becomes invalid if the EM 4-potential $A^{(\text{self})}_\mu$ is properly taken into account. This is due to the divergences produced by the self-force generated by the point particle. To deal with this basic difficulty several approaches have been attempted in the past $[12, 13, 14]$ (for a summary see the discussion in Ref. $[15]$) by adopting various types of axiomatic assumptions about the nature of the singular terms. Nevertheless, their possible consistent derivation from first principles is still missing $[15]$. In the following we intend to show, however, that the validity of Hamilton variational formulation can be restored - without the introduction of any additional assumption - if the point-particles are replaced by finite-size charges, defined in such a way that the EM self-potential $A^{(\text{self})}_\mu$ remains always finite. For definiteness, let us consider as in Ref.A a finite-size spherical-shell charge carrying constant rest mass and total charge, $m_0$ and $q$. The particle charge is assumed to be uniformly distributed on a spherical shell of finite radius $\sigma > 0$, which carries the homogeneous surface charge density $\rho = q/4\pi \sigma^2$ (defined with respect to a frame locally at rest w.r. to the particle). In addition, if the gravitation self-force is ignored, the particle mass may be treated as concentrated in the center of the sphere so that the particle degree of freedom is the same as that of a point particle. One can prove that in such a case the construction of an exact relativistic RR equation can be achieved simply by identifying it with the appropriate Euler-Lagrange equation determined by the Hamilton principle. This is obtained by an appropriate generalization of THM.1. It follows by introducing in the previous definition of the action integral the formal replacements $ds \to W(r, s) \frac{d\Omega}{\sqrt{-g}}$ and $dr^\mu \to \frac{dr^\mu}{ds} W(r, s) \frac{d\Omega}{\sqrt{-g}}$, where $W(r, s)$ (the “wire function”) is generally to be identified with a suitable distribution and $r$ here
denotes the 4-vector \( r^\mu(s) \). For a generic wire-function the functional (11) becomes

\[
S(r^\mu, u_\mu, \chi, A_\mu) = \int ds \frac{d\Omega}{\sqrt{-g}} W(r, s) \left( m_c u_\mu(s) + \frac{q}{c} A_\mu(r) \right) \frac{dr^\mu(s)}{ds} +
\]

\[
+ \frac{1}{16\pi c} \int ds \frac{d\Omega}{\sqrt{-g}} F^{\mu\nu} F_{\mu\nu} + \int_{s_1}^{s_2} ds \frac{d\Omega}{\sqrt{-g}} W(r, s) \chi(s) [u_\mu(s)u^\mu(s) - 1],
\]

while - similarly - the 4-current \( j^\mu(r') \) reads
\[
j^\mu(r') = q c \int \frac{d\Omega}{\sqrt{-g}} W(r', s')u^\mu(s')\delta(4)(r - r(s')).
\]

In the case of a spherical-shell particle immersed in a Minkowsky space-time, when expressed in a reference frame locally at rest with respect to the particle (rest-frame), the wire-function \( W \) must be identified with
\[
W = \frac{\sqrt{-g}}{4\pi \sigma^2} \delta (|r - r(s)| - \sigma),
\]

where \( \sqrt{-g} = 1 \). Hence, in the rest-frame there results
\[
\int ds \frac{d\Omega}{\sqrt{-g}} W(r, s) \equiv \frac{1}{4\pi} \int ds \int dp dp^2 \int d\Sigma(n) \delta (\rho - \sigma) = \frac{1}{4\pi} \int ds \int d\Sigma(n).
\]

Here all quantities are evaluated in the rest-frame, hence \( ds = c dt \), while \( d\Sigma(n) \) is the solid angle and \( n \) is the normal unit 3-vector on the unit sphere. It follows that the appropriate generalization of Hamilton’s variational principle (again to be expressed in synchronous form [16]) requires that both \( f \equiv [r^\mu, u_\mu, \chi] \) and the 4-vector \( A_\mu \) must generally be considered as functions of \((n, s)\). Then the following theorem has the flavor of (for further details we refer to Ref.[17]):

**THM.3 - Hamilton principle for finite-size spherical-shell charges**

The action functional appropriate in case of the wire-function (11) is taken of the form

\[
S(r^\mu, u_\mu, \chi, A_\mu) = \frac{1}{4\pi} \int_{s_1}^{s_2} ds \int d\Sigma(n) \left( m_c u_\mu(n, s) + \frac{q}{c} A_\mu(r(n, s)) \right) \frac{dr^\mu(n, s)}{ds} +
\]

\[
+ \frac{1}{16\pi c} \int \frac{d\Omega}{\sqrt{-g}} F^{\mu\nu} F_{\mu\nu} + \frac{1}{4\pi} \int_{s_1}^{s_2} ds \int d\Sigma(n, s) [g^{\mu\nu}(r(n, s))u_\mu(n, s)u_\nu(n, s) - 1],
\]

where the line element \( ds \) is required to satisfy the constraint #3 of THM.1. It follows that:

- \( T_1 \) The Euler-Lagrange equations obtained considering as independent the synchronous variations \( \delta f(n, s) \) and \( \delta A_\mu(n, s) \) yield identically Eqs.[3], [4], [5] and (12); \( T_2 \) the Euler-Lagrange equations obtained by imposing an arbitrary synchronous variation \( \delta A_\mu \) hold for any decomposition \( A_\mu(r) = A_\mu^{(1)}(r) + A_\mu^{(2)}(r) \) of the 4-vector \( A_\mu(r) \) which is admissible for (12). In particular, the decomposition [9] is admissible for the action (11), where \( A_\mu^{(self)} \) and \( A_\mu^{(ext)} \) are respectively the 4-potentials of the EM self-field and of the external EM field.

**PROOF**

The proof of proposition \( T_1 \) is similar to that given in THM.1. In particular, the Euler-Lagrange equations for \( \delta A_\mu(n, s) \) - again to be identified with Maxwell’s equations
- follow by noting that the functional \( \int ds \int d\Sigma(n)A_\mu(r(s)) \frac{d\nu(n, s)}{ds} \) can also be written as \( \int ds' \int \frac{d\Omega'}{\sqrt{-g}} \int d\Sigma(n)A_\mu(r(s')) \frac{d\nu(n, s')}{ds'} \delta^4(r - r(s')) \). Hence, the 4-current \( j^\mu(r) \) now reads

\[
j^\mu(r) = \frac{qc}{4\pi} \int ds' \int d\Sigma(n) \frac{d\nu(n, s')}{ds'} \delta^4(r - r(s')) . \tag{13}\]

For a detailed proof of THM.3 (in particular of proposition \( T_2 \)), which requires the explicit calculation of the retarded EM 4-potential \( A_\mu^{(self)} \), we refer to [17]. The main consequence of THM.3 is that the Euler-Lagrange equation obtained by means of the synchronous variation of the Hamilton action functional (12) with respect to \( \delta r^\mu(n, s) \) yields a possible realization of the exact relativistic RR equation of motion for a point-particle carrying a finite-size classical charge.

IV. CONCLUSIONS

In this paper the variational treatment of the radiation-reaction problem has been investigated. First we have analyzed the Hamilton variational principle, proving that it becomes invalid for charged point-particles if the proper form of the EM self-field prescribed by classical electrodynamics is taken into account. This conclusion (contained in THM. 1 and 2), which is consistent with the customary interpretation of the RR problem [8], is of course scarcely surprising. In fact, it is well-known that the RR equation of motion becomes invalid for point-particles (see Ref.A). However - as proven in this paper - the validity of Hamilton principle can be restored for finite-size charges which admit a finite EM self-potential \( A_\mu^{(self)} \). In particular, the result can be achieved by considering finite-size spherical-shell charges (THM.3) (this choice is consistent with the approach used in Ref.A to construct the LAD equation). Remarkable consequences which can be drawn from these conclusions include: 1) the treatment of the RR problem can be achieved via a variational formulation (Hamilton variational principle); 2) the variational formulation is made transparent by adopting a synchronous variational principle; 3) for prescribed finite-size charges in principle the exact relativistic RR equations of motion can be achieved in this way [17]. This suggests that: A) it should be possible to extend the validity of the theory to curved space-time and moreover that B) analogous conclusions should apply also for the gravitational radiation-reaction problem, i.e., in other words, the complete treatment of the full EM-gravitational self-force should be possible in the framework of a variational formulation. For these reasons, the present
results appear of primary importance for relativistic theories (such as the kinetic theory of charged particles and the gyrokinetic theory for magnetoplasmas in curved space-time [16]) and related applications in astro- and plasma physics.

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