A BILINEAR PROOF OF DECOUPLING FOR THE CUBIC MOMENT CURVE

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Abstract. Using a bilinear method that is inspired by the method of efficient congruencing of Wooley [Woo16], we prove a sharp decoupling inequality for the moment curve in $\mathbb{R}^3$.

1. Introduction

For an interval $J \subset [0, 1]$, define an extension operator

$$(\mathcal{E}_J g)(x) := \int_J g(\xi) e(x \cdot \gamma(\xi)) \, d\xi$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\gamma(\xi) = (\xi_1, \xi_2^2, \xi_3^3)$ and $e(z) := e^{2\pi iz}$ for a real number $z \in \mathbb{R}$. For $\delta \in \mathbb{N}^{-1}$, let $P_\delta([0, 1])$ denote the partition of $[0, 1]$ into intervals of length $\delta$. Moreover, let $D(\delta)$ be the smallest constant such that

$$\|\mathcal{E}_{[0,1]} g\|_{L^1(\mathbb{R}^3)} \leq D(\delta) \left( \sum_{J \in P_\delta([0,1])} \|\mathcal{E}_J g\|_{L^1(\mathbb{R}^3)}^{1/4} \right)^4$$

holds for all functions $g : [0,1] \to \mathbb{C}$. From Drury [Dru85], both sides are finite at least for smooth $g$. An inequality of this form is called an $\ell^4 L^{12}$ decoupling inequality. Our goal will be to show the following result, which proves a sharp $\ell^4 L^{12}$ decoupling theorem for the moment curve $t \mapsto (t, t^2, t^3)$.

Theorem 1.1. For every $\varepsilon > 0$ and every $\delta \in \mathbb{N}^{-1}$, there exists a constant $C_\varepsilon > 0$ such that

$$D(\delta) \leq C_\varepsilon \delta^{-3/2 - \varepsilon}.$$ (1.2)

The constant $C_\varepsilon$ depends only on $\varepsilon$.

By a standard argument (see Section 4 of [BDG16]), Theorem 1.1 implies that

$$\int_{[0,1]^d} \left| \sum_{j=1}^X e(x_1j + x_2j^2 + \cdots + x_dj^{d-1}) \right|^{d(d+1)/2} \, dx_1 \cdots dx_d \leq C_\varepsilon X^{d(d+1)/2} \varepsilon,$$ (1.3)

for $d = 3$, every positive integer $X$, every $\varepsilon > 0$ and some constant $C_\varepsilon$ depending on $\varepsilon$. Indeed, (1.1) implies that if $F = \sum_{J \in P_\delta([0,1])} F_J$ where $F_J$ is supported in a $\delta^3$ neighborhood of the image of $J$ under $\gamma$, then

$$\|F\|_{L^{12}(\mathbb{R}^3)} \leq D(\delta) \left( \sum_{J \in P_\delta([0,1])} \|F_J\|_{L^{12}(\mathbb{R}^3)}^{1/4} \right),$$

which implies (1.3) for $d = 3$ upon setting $\delta = 1/X$ and

$$F_J(x) := \phi(x/X^3) e(x \cdot \gamma(j/X))$$

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for every $J = \lfloor j/X, (j + 1)/X \rfloor \in P_3([0, 1])$; here $\phi$ is a Schwartz function on $\mathbb{R}^3$ with $\phi \geq 1$ on $[0, 1]^3$, and $\hat{\phi}$ supported on the unit ball centered at the origin. Therefore, we recover the sharp Vinogradov mean value estimate in $\mathbb{R}^3$, which was first proven by Wooley [Woo16], using the method of efficient congruencing. Later, Bourgain, Demeter and Guth [BDG16] recovered (1.3) at $d = 3$ and proved it for every $d \geq 4$, by using the method of decoupling. We also refer to Wooley [Woo18] for a proof of (1.3) for every $d \geq 3$ using the method of efficient congruencing.

In order to prove (1.3) at $d = 3$, Bourgain, Demeter and Guth first proved a stronger version of the decoupling inequality (1.2). To be precise, by Minkowski’s inequality, the main result of [BDG16] gives rise to

$$
\|\mathcal{E}_{[0,1]}g\|_{L^{12}(\mathbb{R}^3)} \leq C_\varepsilon \delta^{-\varepsilon} \left( \sum_{j \in P_3([0,1])} \|\mathcal{E}_j g\|_{L^{12}(\mathbb{R}^3)}^2 \right)^{1/2},
$$

for every $\varepsilon > 0$. By Hölder’s inequality, it is not difficult to see that (1.4) implies (1.2). Moreover, by using the standard argument in Section 4 of [BDG16], (1.4) implies (1.3) at $d = 3$ just like (1.2). In other words, (1.2) and (1.4) have the same strength when deriving exponential sum estimates of the form (1.3).

The methods of efficient congruencing and decoupling use different languages: One uses the language of number theory, while the other uses purely harmonic analysis. It is a very natural and interesting question to ask whether understanding one method better could enhance our understanding of the other method. This is the goal of the current paper: We will provide a new proof of the decoupling inequality (1.2) by using a method that is inspired by the method of efficient congruencing. Unfortunately, the new argument does not fully recover the slightly stronger decoupling inequality (1.4). This will be explained later in Remark 2 in Section 4.1.

One significant difference between the proof here and the proof in [BDG16] is that the lower dimensional input for our proof comes from a sharp “small ball” $\ell^4L^4$ decoupling for the parabola rather than a sharp $\ell^2L^6$ decoupling for the parabola as in [BDG16].

The authors benefited very much from the note [HB15] written by Heath-Brown. In the note, Heath-Brown simplified Wooley’s efficient congruencing in $\mathbb{R}^3$. In the current paper, we follow the structure of [HB15]. We will also point out (in Section 2) the one-to-one correspondence between main lemmas that are used in [HB15] and those used in the current paper.

**Organization of paper.** In Section 2 we will introduce the main quantities that will play crucial roles in the later proof, list the main properties of these quantities, and prove a few of them that are simple. The two key properties (Lemma 2.6 and Lemma 2.7) will be proven in Section 4 and Section 6 respectively. After proving all these lemmas, we will use them to run an iteration argument and finish the proof of the main theorem. This step will be carried out in Section 6.

**Notation.** For a frequency interval $I$, we will use $|I|$ to denote its length. We use $P_3(I)$ to denote the partition of $I$ into intervals of length $\delta$. This implicitly assumes $|I|/\delta \in \mathbb{N}$. If $I = [0, 1]$, we usually omit $[0, 1]$ and just write $P_3$ rather than $P_3([0, 1])$. For a spatial cube $B \subset \mathbb{R}^3$, we also use $P_B(B)$ to denote the partition of $B$ into cubes of side length $R$. By $B(c, R)$, we will mean a square (or cube depending on context) centered at $c$ of side length $R$. 
Let $E > 10^3$ be a large integer. Let $T$ be a parallelepiped where $T = A[0,1]^3 + c$ for some $3 \times 3$ invertible matrix $A$ and some vector $c \in \mathbb{R}^3$. In the current paper, the columns of $A$ will be almost at right angles to each other, but can have different lengths. We write

$$w_{T,E}(x) := (1 + |A^{-1}(x - c)|)^{-E}.$$ 

for a weight that is comparable to $1$ on $T$ and decays like the (non-isotropic) distance to the power $E$ outside $T$. Also write

$$\tilde{w}_{T,E}(x) := w_{T,3E}(x) = (1 + |A^{-1}(x - c)|)^{-3E}$$

for a weight with a faster decay. One key property we will use about these weights is that, if $\{T\}$ is a collection of parallelepipeds that tiles a spatial ball $B$, then

$$\sum_T \tilde{w}_{T,E} \lesssim_E w_{B,E}, \quad (1.5)$$

with a constant that depends only on $E$. The volume of $T$ is $|T| = |\det A|$, and we write

$$\phi_{T,E}(x) := \frac{1}{|T|} (1 + |A^{-1}(x - c)|)^{-E}$$

for an $L^1$ normalized version of $w_{T,E}$, that is essentially supported on $T$.

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2. **Main quantities and their properties**

For an interval $I \subset [0,1]$, let $c_I$ denote the center of $I$. Let $\hat{T}_I$ denote the parallelepiped that is centered at the origin, of dimension $|I|^{-1} \times |I|^{-2} \times |I|^{-3}$, given by

$$\hat{T}_I := \{ x \in \mathbb{R}^3 : |x \cdot \gamma'(c_I)| \leq |I|^{-1}, |x \cdot \gamma''(c_I)| \leq |I|^{-2}, |x \cdot \gamma'''(c_I)| \leq |I|^{-3} \}.$$ 

For an extremely small number $\delta$ and $\delta^{-1} \ll \nu \ll 1$ (throughout the paper we will assume that $\delta^{-1}, \nu^{-1} \in \mathbb{N}$), define the following two bilinear decoupling constants. For $a, b \in \mathbb{N}$, let $\mathcal{M}_{1,a,b}(\delta, \nu, E)$ and $\mathcal{M}_{2,a,b}(\delta, \nu, E)$ be the best constant such that

$$\int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{\hat{T}_I,E}) (|\mathcal{E}_I g|^{10} \ast \phi_{\hat{T}_I,E}) \leq \mathcal{M}_{1,a,b}(\delta, \nu, E)^{12}$$

$$\times \left( \sum_{J \in P_1(I)} |\mathcal{E}_J g|_{L^{12}(\mathbb{R}^3)}^4 \right)^{1/2} \left( \sum_{J \in P_1(I)} |\mathcal{E}_J g|_{L^{12}(\mathbb{R}^3)}^6 \right)^{5/2} \quad (2.1)$$
For the first term, parabolic rescaling shows that it can be bounded by

\[ \int_{\mathbb{R}^3} \left( |E_I g|^4 \ast \phi^{E}_{T_I} \right) \left( |E_{I'} g|^8 \ast \phi^{E}_{T_{I'}} \right) \leq \mathcal{M}_{2,a,b}(\delta, \nu, E)^{12} \]

\[ \times \left( \sum_{J \in P_3(I)} \|E_J g\|_{L^{12}(\mathbb{R}^3)}^4 \right) \left( \sum_{J' \in P_3(I')} \|E_{J'} g\|_{L^{12}(\mathbb{R}^3)}^4 \right)^2 \]

(2.2)

hold separately, for all functions \( g : [0,1] \to \mathbb{C} \), and all pairs of intervals \( I \in P_\nu \cdot ([0,1]), I' \in P_\nu \cdot ([0,1]) \) with \( d(I, I') \geq \nu \). Note that expressions such as \( |E_I g|^2 \ast \phi^{E}_{T_I} \) above are constant (up to a \( O_E(1) \) multiplicative factor) on any \( |I|^{-1} \times |I|^{-2} \times |I|^{-3} \) parallelepiped parallel to \( I \).

In this section and the next two sections (but not in the last section, Section 5), \( C_0 \) is a large absolute constant whose precise value is not important and may vary from line to line.

**Lemma 2.1** (Parabolic rescaling, cf. Lemma 1 of [HBI15]). Let \( 0 < \delta < \sigma < 1 \) be such that \( \delta/\sigma \in \mathbb{N}^{-1} \). Let \( I \) be an arbitrary interval in \([0,1]\) of length \( \sigma \). Then

\[ \|E_I g\|_{L^{12}(\mathbb{R}^3)} \leq D(\delta/\sigma) \left( \sum_{J \in P_3(I)} \|E_J g\|_{L^{12}(\mathbb{R}^3)}^4 \right)^{1/4} \]

for all \( g : [0,1] \to \mathbb{C} \).

The proof of this lemma is standard so we omit the proof (see for example [BD15 Proposition 4.1], [BD17 Proposition 7.1], or [Li17 Section 4]).

One corollary of parabolic rescaling is almost multiplicativity of \( D(\delta) \). This allows us to patch together the various integrality constraints that appear throughout our argument.

**Corollary 2.2** (Almost multiplicativity). Suppose \( \delta_1, \delta_2 \in \mathbb{N}^{-1} \), then

\[ D(\delta_1 \delta_2) \leq D(\delta_1)D(\delta_2). \]

**Lemma 2.3** (Bilinear reduction, cf. Lemma 2 of [HBI15]). If \( \delta \) and \( \nu \) were such that \( \nu \delta^{-1} \in \mathbb{N} \), then

\[ D(\delta) \leq \nu^{-1/4} D(\delta/\nu) + \nu^{-1} \mathcal{M}_{2,1,1}(\delta, \nu, E). \]

Proof. We have

\[ \|E_{[0,1]} g\|_{L^{12}(\mathbb{R}^3)} = \| \sum_{J, J' \in P_\nu} E_J g \overline{E_J' g} \|_{L^{12}(\mathbb{R}^3)}^{1/2} \]

\[ \leq \left( \sum_{J \in P_\nu} \|E_J g\|_{L^{12}(\mathbb{R}^3)}^2 \right)^{1/2} + \nu^{-1} \max_{d(J, J') \geq \nu} \|E_J g \overline{E_J' g}\|_{L^{12}(\mathbb{R}^3)}^{1/2}. \]

For the first term, parabolic rescaling shows that it can be bounded by

\[ D(\delta/\nu) \left( \sum_{J \in P_\nu} \left( \sum_{J' \in P_\nu(J)} \|E_{J'} g\|_{L^{12}(\mathbb{R}^3)}^4 \right)^{1/2} \right)^{1/2}. \]

Applying Hölder in the sum over \( J \), this is bounded by

\[ \nu^{-1/4} D(\delta/\nu) \left( \sum_{J \in P_\nu} \|E_J g\|_{L^{12}(\mathbb{R}^3)}^4 \right)^{1/4}. \]
This gives the first term of our desired result. The second term follows from the observation that
\[
\int_{\mathbb{R}^3} |E_J g|^6 |E_J' g|^6 \leq (\int_{\mathbb{R}^3} |E_J g|^4 |E_J' g|^8)^{1/2} (\int_{\mathbb{R}^3} |E_J g|^8 |E_J' g|^4)^{1/2},
\]
and the pointwise estimate
\[
|E_I g|^p \lesssim_{p,E} |E_I g|^p * \phi_{\tilde{T}_1,E} \tag{2.3}
\]
for an interval $I \subset [0,1]$ and for every $p \geq 1$. This pointwise estimate follows from writing $E_I g$ as a convolution of itself with a Schwartz function adapted to $\tilde{T}_1$, the triangle inequality, and Hölder. It follows that
\[
\max_{J,J' \in P_t, d(J,J') \geq \nu} \|E_J g E_{J'} g\|_{L^6(\mathbb{R}^3)}^{1/2} \les E \max_{J,J' \in P_t, d(J,J') \geq \nu} \left( \int_{\mathbb{R}^3} (|E_J g|^4 * \phi_{\tilde{T}_1,E}) (|E_{J'} g|^8 * \phi_{\tilde{T}_1,E}) \right)^{1/12}.
\]
In light of the definition of $M_{2,1,1}$, this completes the proof of the lemma. \hfill \square

**Lemma 2.4** (cf. Lemma 3 of [HB15]). If $a$ and $b$ are integers and $\delta$ and $\nu$ were such that $\nu^a \delta^{-1}, \nu^b \delta^{-1} \in \mathbb{N}$, then
\[
M_{2,a,b}(\delta, \nu, E) \lesssim E M_{2,b,a}(\delta, \nu, E/C_0)^{1/3} M_{1,a,b}(\delta, \nu, E/C_0)^{2/3},
\]
for some large absolute constant $C_0$.

**Proof.** The proof of Lemma 2.4 is essentially via Hölder’s and Bernstein’s inequalities.

Suppose $I \in P_{2\nu}([0,1]), I' \in P_{2\nu}([0,1])$ with $d(I,I') \geq \nu$. We first recall a version of Bernstein’s inequality. The same proof as Bernstein in Corollary 4.3 of [BD17] shows that for every $p \geq 1$, there is an absolute constant $C_0$ such that
\[
\left( \int_{\mathbb{R}^3} |E_I g(x)|^p \phi_{\tilde{T}_1,E}(x) \, dx \right)^{1/p} \lesssim_{p,E} \int_{\mathbb{R}^3} |E_I g(x)| \phi_{\tilde{T}_1,E}(x) \, dx. \tag{2.4}
\]
From (2.3), observe that
\[
\int_{\mathbb{R}^3} |E_I g|^4 \phi_{\tilde{T}_1,E} = \left( \int_{\mathbb{R}^3} |E_I g|^4 \phi_{\tilde{T}_1,E} \right)^{1/3} \left( \int_{\mathbb{R}^3} |E_I g|^8 \phi_{\tilde{T}_1,E} \right)^{2/3} \les E \left( \int_{\mathbb{R}^3} |E_I g|^4 \phi_{\tilde{T}_1,E} \right)^{1/4} \left( \int_{\mathbb{R}^3} |E_I g|^8 \phi_{\tilde{T}_1,E} \right)^{1/4}.
\]
It follows that
\[
|E_I g|^4 * \phi_{\tilde{T}_1,E} \les E \left( |E_I g|^2 * \phi_{\tilde{T}_1,E/C_0} \right) \left( |E_I g|^2 * \phi_{\tilde{T}_1,E/C_0} \right)\]
where we used $1 = \frac{3}{4} \cdot 4 \cdot \frac{1}{3} = \frac{3}{8} \cdot 4 \cdot \frac{2}{3}$. Similarly,
\[
|E_I g|^8 * \phi_{\tilde{T}_1,E/C_0} \les E \left( |E_I g|^2 * \phi_{\tilde{T}_1,E/C_0} \right) \left( |E_I g|^2 * \phi_{\tilde{T}_1,E/C_0} \right) \]
where we used $1 = \frac{3}{20} \cdot 8 \cdot \frac{5}{6} = \frac{4}{5} \cdot 8 \cdot \frac{1}{6}$. This shows
\[
\int_{\mathbb{R}^3} (|E_I g|^4 * \phi_{\tilde{T}_1,E}) (|E_I g|^8 * \phi_{\tilde{T}_1,E}) \les E \int_{\mathbb{R}^3} \left( |E_I g|^4 * \phi_{\tilde{T}_1,E/C_0} \right) \left( |E_I g|^8 * \phi_{\tilde{T}_1,E/C_0} \right) \times \left( |E_I g|^4 * \phi_{\tilde{T}_1,E/C_0} \right) \left( |E_I g|^8 * \phi_{\tilde{T}_1,E/C_0} \right). \tag{2.5}
\]
By convexity and Hölder, the last display can be bounded by
\[
\left( \int_{\mathbb{R}^3} (|E g|^2 \ast \phi_{T_1,E/C_0}) (|E g|^4 \ast \phi_{T_1,E/C_0}) \right)^{\frac{2}{3}} \times \left( \int_{\mathbb{R}^3} (|E g|^8 \ast \phi_{T_1,E/C_0}) (|E g|^4 \ast \phi_{T_1,E/C_0}) \right)^{\frac{1}{2}}.
\] (2.6)

Recalling the definitions of $M_{2,a,b}$ and $M_{1,a,b}$, this finishes the proof of the lemma.

\[\square\]

**Lemma 2.5** (cf. Lemma 4 of [HB15]). If $a$ and $b$ are integers and $\delta$ and $\nu$ were such that $\nu \delta^{-1}, \nu^{2} \delta^{-1} \in \mathbb{N}$, then
\[
M_{1,a,b}(\delta, \nu, E) \lesssim_{E} M_{2,b,a}(\delta, \nu, E/C_0)^{1/4} D(\delta)^{3/4},
\]
for some large absolute constant $C_0$.

**Proof.** Suppose $I \in P_{\nu}(\{0,1\})$, $I' \in P_{\nu}(\{0,1\})$ with $d(I, I') \geq \nu$. We start with an estimate that is similar to (2.5) and (2.6):
\[
\int_{\mathbb{R}^3} (|E g|^2 \ast \phi_{T_1,E}) (|E g|^4 \ast \phi_{T_1,E}) \lesssim_{E} \int_{\mathbb{R}^3} (|E g|^8 \ast \phi_{T_1,E}) (|E g|^4 \ast \phi_{T_1,E/C_0}) \lesssim_{E} \left( \int_{\mathbb{R}^3} |E g|^4 \ast \phi_{T_1,E/C_0} \right)^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} |E g|^4 \ast \phi_{T_1,E/C_0} \right)^{\frac{1}{3}}.
\]

Since $\phi_{T_1,E/C_0}$ is $L^1$-normalized,
\[
\int_{\mathbb{R}^3} |E g|^4 \ast \phi_{T_1,E/C_0} \lesssim \int_{\mathbb{R}^3} |E g|^4.
\]

We finish the proof by applying parabolic rescaling. \[\square\]

The proofs of the following two lemmas will be given in Sections 3 and 4.

**Lemma 2.6** (cf. Lemma 5 of [HB15]). Let $a$ and $b$ be integers such that $1 \leq a \leq 3b$. Suppose $\delta$ and $\nu$ were such that $\nu^{2b-1} \delta^{-1} \in \mathbb{N}$. Then
\[
M_{1,a,b}(\delta, \nu, E) \lesssim_{a,b,E} \nu^{-\frac{1}{4}(3b-a)-C_0} M_{1,3b,b}(\delta, \nu, E/C_0)
\] (2.7)
for some absolute constant $C_0$.

**Lemma 2.7** (cf. Lemma 6 of [HB15]). Let $a$ and $b$ be integers such that $1 \leq a \leq b$. Suppose $\delta$ and $\nu$ were such that $\nu^{2b-a} \delta^{-1} \in \mathbb{N}$ and $\nu \in 2^{-2^3} \cap (0, 1/1000)$. Then for every $\varepsilon > 0$,
\[
M_{2,a,b}(\delta, \nu, E) \lesssim_{\varepsilon,E} \nu^{-\frac{1}{4}(1+\varepsilon)(b-a)-C_0} M_{2,2b-a,b}(\delta, \nu, E/C_0),
\]
for some absolute constant $C_0$. 
3. The first bilinear constant \( M_{1,a,b} \)

We break the proof of (27) into the following three different lemmas.

**Lemma 3.1** (\( L^2 \) decoupling). If \( 1 \leq a \leq b \), then for any pair of frequency intervals \( I, I' \subset [0,1] \) with \( |I| = \nu^a \), \( |I'| = \nu^b \), dist\((I, I') \geq \nu \), we have

\[
\begin{align*}
\int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{I,E}^\circ) (|\mathcal{E}_{I'} g|^10 \ast \phi_{I',E}^\circ) \\
\lesssim E \sum_{J \in \mathcal{P}_{1,b}(I)} \int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{J,E/C_0}^\circ) (|\mathcal{E}_{I'} g|^10 \ast \phi_{J',E/C_0}^\circ)
\end{align*}
\]

(3.1)

for large enough \( E \) and for some absolute constant \( C_0 \).

**Lemma 3.2** (Ball inflation). If \( b \leq a \leq 2b \), then for any pair of frequency intervals \( I, I' \subset [0,1] \) with \( |I| = \nu^a \), \( |I'| = \nu^b \), dist\((I, I') \geq \nu \), we have

\[
\begin{align*}
\int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{I,E}^\circ) (|\mathcal{E}_{I'} g|^10 \ast \phi_{I',E}^\circ) \\
\lesssim E \nu^{-C_0} \sum_{J \in \mathcal{P}_{2b}(I)} \int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{J,E/C_0}^\circ) (|\mathcal{E}_{I'} g|^10 \ast \phi_{J',E/C_0}^\circ)
\end{align*}
\]

(3.2)

for large enough \( E \) and for some absolute constant \( C_0 \).

**Lemma 3.3** (Ball inflation). If \( 2b \leq a \leq 3b \), then for any pair of frequency intervals \( I, I' \subset [0,1] \) with \( |I| = \nu^a \), \( |I'| = \nu^b \), dist\((I, I') \geq \nu \), we have

\[
\begin{align*}
\int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{I,E}^\circ) (|\mathcal{E}_{I'} g|^10 \ast \phi_{I',E}^\circ) \\
\lesssim E \nu^{-C_0} \sum_{J \in \mathcal{P}_{3b}(I)} \int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{J,E/C_0}^\circ) (|\mathcal{E}_{I'} g|^10 \ast \phi_{J',E/C_0}^\circ)
\end{align*}
\]

(3.3)

for large enough \( E \) and for some absolute constant \( C_0 \).

Combining the three lemmas, we see that if \( 1 \leq a \leq 3b \) and \( \nu^b \delta^{-1} \in \mathbb{N} \), then for any pair of frequency intervals \( I, I' \subset [0,1] \) with \( |I| = \nu^a \), \( |I'| = \nu^b \), dist\((I, I') \geq \nu \), we have

\[
\begin{align*}
\int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{I,E}^\circ) (|\mathcal{E}_{I'} g|^10 \ast \phi_{I',E}^\circ) \\
\lesssim E \nu^{-C_0} \sum_{J \in \mathcal{P}_{3b}(I)} \int_{\mathbb{R}^3} (|\mathcal{E}_I g|^2 \ast \phi_{J,E/C_0}^\circ) (|\mathcal{E}_{I'} g|^10 \ast \phi_{J',E/C_0}^\circ)
\end{align*}
\]

which is further bounded by

\[
\begin{align*}
\lesssim E \nu^{-C_0} M_{1,3b,b} (\delta, \nu, E/C_0)^{12} \\
\lesssim E \nu^{-C_0} (3b-a)^{-1} \beta^{1/2} M_{1,3b,b} (\delta, \nu, E/C_0)^{12} \\
\lesssim E \nu^{-C_0} \left( \frac{\sum_{J \in \mathcal{P}_b(I)} \| \mathcal{E}_I g \|^4_{L^{12}}}{\sum_{J' \in \mathcal{P}_b(I')} \| \mathcal{E}_{I'} g \|^4_{L^{12}}} \right)^{1/2} \left( \frac{\sum_{J \in \mathcal{P}_b(I)} \| \mathcal{E}_I g \|^4_{L^{12}}}{\sum_{J' \in \mathcal{P}_b(I')} \| \mathcal{E}_{I'} g \|^4_{L^{12}}} \right)^{5/2}.
\end{align*}
\]
Proof of Lemma 3.1. Let

Therefore we write the left hand side of (3.1) as

Next, observe that

It is clear that (2.7) now follows from the definition of $M_{1,a,b}(\delta, \nu, \cdot)$.

First we prove a small technical lemma that will be used in the proof of Lemma 3.1.

Lemma 3.4. For $J \subset I \subset [0,1]$,

$$|\mathcal{E}_J g|^2 \ast \phi_{T_I} \leq_E \|\mathcal{E}_J g\|^2 \ast \phi_{T_J,E/C_0}$$

for some sufficiently large $C_0$.

Proof. First it suffices to instead show that for $J \subset I \subset [0,1]$, we have

$$\|\mathcal{E}_J g\|_{L^2(\phi_{T_I,E}^{-\infty})} \leq \|\mathcal{E}_J g\|_{L^2(\phi_{T_J,E/C_0}^{-\infty})}. \tag{3.4}$$

Suppose $|J| = 1/R'$ and $|I| = 1/R$ with $R' \geq R$. It suffices to only show the case when $I = [0,1/R]$. Since $J \subset [0,1/R]$, the angle between $T_I$ and $T_J$ is $O(1/R)$. Therefore $T_I$ is contained in a rectangle that is a $O(1)$ dilation of $T_J$ but pointing in the same direction as $T_J$. Furthermore this dilate of $T_I$ is contained in a $O(1)$ dilation of $T_J$. Thus there exists a sufficiently large absolute constant $C$ such that $T_I \subset C^2 T_J$. The same reasoning gives that for $k \geq 0$, $2^k T_I \subset C^2 T_J$ where $C$ is an absolute constant.

We first prove an unweighted version of (3.4). Fix $k \geq 0$. Then

$$\frac{1}{|2^k T_J|} \|\mathcal{E}_J g\|^2_{L^2(2^k T_I)} \leq \|\mathcal{E}_J g\|^2_{L^2(C^2 T_J)} \leq E \|\mathcal{E}_J g\|^2_{L^2(\phi_{C^2 T_J,E/100}^{-\infty})} \leq E 2^k \|\mathcal{E}_J g\|^2_{L^2(\phi_{T_J,E/100}^{-\infty})}.$$

Next, observe that

$$\phi_{T_I,E} \leq E \sum_{k \geq 0} 2^{-k(E-3)} \frac{1}{|2^k T_I|} \phi_{2^k T_I}.$$ 

Therefore

$$\|\mathcal{E}_J g\|^2_{L^2(\phi_{T_I,E}^{-\infty})} \leq \sum_{k \geq 0} 2^{-k(E-3)} \int_{\mathbb{R}^3} \|\mathcal{E}_J g\|(x)^2 \frac{1}{|2^k T_I|} \phi_{2^k T_I} \, dx \leq E \sum_{k \geq 0} 2^{-k(E-6)} \int_{\mathbb{R}^3} \|\mathcal{E}_J g\|(x)^2 \phi_{2^k T_J,E/100} \, dx \leq E \|\mathcal{E}_J g\|^2_{L^2(\phi_{T_J,E/100}^{-\infty})}.$$ 

This completes the proof of Lemma 3.4.

We now move on to the proofs of Lemmas 3.1–3.3.

Proof of Lemma 3.7. Let $[\square]$ be a partition of $\mathbb{R}^3$ into cubes of side length $\nu^{-b}$. We write the left hand side of (3.1) as

$$\sum_{\square} \int_{\square} \left( |\mathcal{E}_I g|^2 \ast \phi_{T_I,E} \right) \left( |\mathcal{E}_J g|^2 \ast \phi_{T_J,E} \right).$$
We bound the above expression by
\[
\sum_{\square} \left( \sup_{x \in \square} |\mathcal{E}_{I'} g|^{10} * \phi_{\dot{I}', E} (x) \right) \int_{\square} \left( |\mathcal{E}_{I} g|^{2} * \phi_{I, E} \right). \tag{3.5}
\]

We write the latter factor as
\[
\int_{\mathbb{R}^3} \left[ \int_{\square} |\mathcal{E}_{I} g(x)|^{2} dx \right] \phi_{I, E} (y) dy,
\tag{3.6}
\]
where \( \square_{y} := \square - y \). By \( L^2 \) orthogonality (see for instance Appendix of [GZo18]), we have
\[
3E \leq \sum_{J \in P_{b}(I)} \int_{\mathbb{R}^3} \left[ \int_{\square} |\mathcal{E}_{J} g(x-y)|^{2} w_{\square, C_0} (x) dx \right] \phi_{\dot{I}', E} (y) dy \tag{3.7}
\]
\[
\leq E \sum_{J \in P_{b}(I)} \int_{\mathbb{R}^3} \left( |\mathcal{E}_{J} g|^{2} * \phi_{\dot{I}', E/C_0} \right) w_{\square, C_0} E.
\]
where in the last step we have used Lemma 3.4. This, combined with the definition of the weight \( w_{\square, C_0} E \), implies that (3.5) can be bounded by
\[
\sum_{\square} \sum_{J \in P_{b}(I)} \sum_{z \in \mathbb{Z}^2} (1 + |\gamma|)^{-C_0 E} \left( \sup_{x \in \square} |\mathcal{E}_{I} g|^{10} * \phi_{\dot{I}', E} (x) \right) \int_{\square_{z}} \left( |\mathcal{E}_{J} g|^{2} * \phi_{\dot{I}', E/C_0} \right).
\tag{3.8}
\]
In the end, we just need to observe that
\[
\sup_{x \in \square_{z}} |\mathcal{E}_{I} g|^{2} * \phi_{\dot{I}', E/C_0} (x) \leq |\gamma| E^{C_0} \inf_{x \in \square_{z}} |\mathcal{E}_{I} g|^{2} * \phi_{\dot{I}', E/C_0} (x'), \tag{3.9}
\]
and
\[
\sup_{x \in \square} |\mathcal{E}_{I} g|^{10} * \phi_{\dot{I}', E/C_0} (x) \sim E \inf_{x \in \square} |\mathcal{E}_{I} g|^{10} * \phi_{\dot{I}', E/C_0} (x) \tag{3.10}
\]
both of which follow from the definition of the weight \( \phi \). This finishes the proof of the lemma. \( \square \)

**Proof of Lemma 3.2** Suppose \( b \leq a \leq 2b \). Let \( \square \) be a spatial cube of side length \( \nu^{-2b} \). Let \( \gamma (\xi) = (\xi, \xi^2, \xi^3) \) and \( \xi_1, \xi_2 \) be the centers of the intervals \( I \) and \( I' \). For \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N} \) with \( \alpha_j \leq ja \) for \( j = 1, 2, 3 \), consider a parallelepiped
\[
\{ x \in \mathbb{R}^3 : |x \cdot \gamma' (\xi_1)| \leq \nu^{-\alpha_1}, |x \cdot \gamma'' (\xi_1)| \leq \nu^{-\alpha_2}, |x \cdot \gamma''' (\xi_1)| \leq \nu^{-\alpha_3} \}.
\]
Note that \( \mathcal{E}_{I'} g \) is morally locally constant on every translate of this parallelepiped. Tile \( \mathbb{R}^3 \) with essentially disjoint translates of this parallelepiped and let \( T_{\alpha_1, \alpha_2, \alpha_3} (I) \) be the parallelepipeds in this tiling. Similarly we tile \( \mathbb{R}^3 \) with essentially disjoint translates of the parallelepiped
\[
\{ x \in \mathbb{R}^3 : |x \cdot \gamma' (\xi_2)| \leq \nu^{-\beta_1}, |x \cdot \gamma'' (\xi_2)| \leq \nu^{-\beta_2}, |x \cdot \gamma''' (\xi_2)| \leq \nu^{-\beta_3} \}
\]
and define \( T_{\beta_1, \beta_2, \beta_3} (I') \) to be the parallelepipeds in this tiling whenever \( \beta_1, \beta_2, \beta_3 \in \mathbb{N} \) with \( \beta_j \leq j \nu \) for \( j = 1, 2, 3 \). Consider
\[
\int_{\square} \left( |\mathcal{E}_{I} g|^{2} * \phi_{\dot{I}, E} \right) \left( |\mathcal{E}_{I'} g|^{10} * \phi_{\dot{I}', E} \right), \tag{3.11}
\]

Notice that there exists $c_T, c_{T'}$, for every $T \in \mathcal{T}_{a,2b,2b}$ and every $T' \in \mathcal{T}_{b,2b,2b}$ such that on $\square$,

$$
|\mathcal{E}_I g|^2 * \phi_{T,J,E} \sim E \sum_{T \in \mathcal{T}_{a,2b,2b}} c_T^2 I_T \quad (3.11)
$$

$$
|\mathcal{E}_I g|^2 * \phi_{T',E} \sim E \sum_{T' \in \mathcal{T}_{b,2b,2b}} c_{T'}^2 I_{T'}.
$$

Therefore, (3.11) can be bounded by

$$
\int_{\square} \left( \sum_{T \in \mathcal{T}_{a,2b,2b}, T \subset 2\square} c_T^2 I_T \right) \left( \sum_{T' \in \mathcal{T}_{b,2b,2b}, T' \subset 2\square} c_{T'}^2 I_{T'} \right)
$$

(3.13)

For such $T$ and $T'$, we have a crucial geometric inequality

$$
\frac{|T \cap T'|}{|\square|} \lesssim \nu^{-2} \frac{|T|}{|\square|^2},
$$

(3.14)

because

$$
|T \cap T'| \lesssim |\{ x \in \mathbb{R}^3 : |x \cdot \gamma'(\xi_1)| \leq \nu^{-a}, |x \cdot \gamma'(\xi_2)| \leq \nu^{-b}, |x \cdot \gamma''(\xi_2)| \leq \nu^{-2b} \}|,
$$

the latter of which is comparable to

$$
\nu^{-a} \nu^{-b} \nu^{-2b} \left| \det \begin{pmatrix} 1 & 1 & 1 \\ \gamma'(\xi_1) & \gamma'(\xi_2) & \gamma'(\xi_2) \\ \gamma''(\xi_1) & \gamma''(\xi_2) & \gamma''(\xi_2) \end{pmatrix} \right|^{-1} \sim \frac{(\nu^{-a} \nu^{-2b} \nu^{-2b})(\nu^{-b} \nu^{-2b} \nu^{-2b})}{(\nu^{-2b})^3} (\xi_1 - \xi_2)^{-2}
$$

$$
\lesssim \nu^{-2} \frac{|T|}{|\square|^2}.
$$

This implies

$$
(3.13) \lesssim \frac{1}{|\square|} \left( \int_{\square} \sum_{T \in \mathcal{T}_{a,2b,2b}, T \subset 2\square} c_T^2 I_T \right) \left( \int_{\square} \sum_{T' \in \mathcal{T}_{b,2b,2b}, T' \subset 2\square} c_{T'}^2 I_{T'} \right)
$$

$$
\lesssim E \left( \int_{\square} \frac{|\mathcal{E}_I g|^2 * \phi_{T,J,E}}{\sqrt{\nu}} \right) \left( \int_{\square} \frac{|\mathcal{E}_I g|^2 * \phi_{T',E}}{\sqrt{\nu}} \right).
$$

(3.15)

By $L^2$ orthogonality and an argument that is essentially the same as that in (3.7), we have

$$
\int_{\square} |\mathcal{E}_I g|^2 * \phi_{T,J,E} \lesssim E \sum_{J \in \mathcal{P}_{a,2b}(I)} \int_{\mathbb{R}^3} |\mathcal{E}_J g|^2 * \phi_{T,J,E/C_0} u_{\square,C_0 E}.
$$

(3.16)

By the definition of the weight $u_{\square,C_0 E}$, the term (3.15) can be bounded by

$$
\sum_{J \in \mathcal{P}_{a,2b}(I)} \sum_{k \in \mathbb{Z}^3} (1 + |k|)^{-a} E \left( \int_{\square} |\mathcal{E}_J g|^2 * \phi_{T,J,E/C_0} \right) \left( \int_{\square} |\mathcal{E}_I g|^2 * \phi_{T,J,E} \right)
$$

In the end, we just need to apply (3.9), with $b$ replaced by $2b$. This finishes the proof of Lemma 3.2.

Proof of Lemma 3.3. Suppose $2b \leq a \leq 3b$. We may follow line by line the proof of Lemma 3.2 except that

- the radius $\nu^{-2b}$ of the spatial ball $\square$ replaced by $\nu^{-3b}$,
- $\mathcal{T}_{a,2b,2b}$ replaced by $\mathcal{T}_{a,3b,3b}$; and
- $\mathcal{T}_{b,2b,2b}$ replaced by $\mathcal{T}_{b,2b,2b}$.

\[ \square \]
This is because when $2b \leq a \leq 3b$, the uncertainty principle asserts that morally speaking, $|E_I g|$ is locally constant on all tubes in $T_{a,3b,2b}$, and $|E_I' g|$ is locally constant on all tubes in $T_{0,2b,3b}$. This shows that (3.12) holds with $T_{a,2b,2b}$ replaced by $T_{a,3b,2b}$, and $T_{b,2b,2b}$ replaced by $T_{0,2b,3b}$. The crucial geometric inequality (3.14) now follows since we still have

$$|T \cap T'| \leq \{ x \in \mathbb{R}^3 : |x \cdot \gamma' (\xi_1)| \leq \nu^{-a}, |x \cdot \gamma' (\xi_2)| \leq \nu^{-b}, |x \cdot \gamma'' (\xi_2)| \leq \nu^{-2b} \}$$

the latter of which is comparable to

$$\nu^{-a} \nu^{-b} \nu^{-2b} \left| \det \begin{pmatrix} \gamma' (\xi_1) \\ \gamma' (\xi_2) \\ \gamma'' (\xi_2) \end{pmatrix} \right|^{-1} \approx \frac{(\nu^{-a} \nu^{-3b} \nu^{-3b})(\nu^{-b} \nu^{-2b} \nu^{-3b})}{(\nu^{-3b})^3} (\xi_1 - \xi_2)^{-2} \leq \nu^{-2} \frac{|T||T'|}{|\square|}.$$
4.1. Small ball $\ell^4 L^4$ decoupling. Let

$$(E_1 g)(x) := \int_I g(\xi) e(\xi x_1 + \xi^2 x_2) \, d\xi$$

be the extension operator for the parabola associated to the interval $I \subset [0, 1]$.

**Lemma 4.2.** We have the decoupling inequality

$$\|E_{[0,1]} g\|_{L^4(w \delta^{-1}, E)} \lesssim_{\varepsilon, E} \delta^{-2} \left( \sum_{J \in P_0(\{0,1\})} \|E_J g\|_{L^4(w \delta^{-1}, E)}^4 \right)^{1/4}, \quad (4.2)$$

for every $\varepsilon_0 > 0$, every $g : [0, 1] \to \mathbb{C}$ and all balls $B_{\delta^{-1}} \subset \mathbb{R}^2$ of radius $\delta^{-1}$.

In the above estimate, the power of $\delta^{-1}$ is optimal. This can be seen by taking the function $g$ to be the indicator function of $[0, 1]$.

**Proof of Lemma 4.2.** As the constant $E$ in various weight functions is fixed in the proof of this lemma, we will drop the dependence on it and simply write $w_B$ or $\tilde{w}_B$ for a ball $B$. We will also make use of the uncertainty principle here, however this argument can be made rigorous in the same manner as in [BG11].

Let $K \ll \delta^{-1}$ be a large integer. For each $\alpha \in P_{K^{-1}}([0, 1])$ and each ball $B' \subset B_{\delta^{-1}}$ of radius $K$, define

$$c_\alpha(B') := \left( \frac{1}{|B'|} \int_{B'} |E_\alpha g|^4 \right)^{1/4}.$$ 

Notice that by the uncertainty principle, the function $|E_\alpha g|$ is essentially a constant on every ball of radius $K$. Let $\alpha^*(B')$ be the interval that maximises $\{c_\alpha(B')\}_\alpha$. If there is one $\alpha^{**}$ such that $c_{\alpha^{**}}(B') \geq c_{\alpha^*(B')}/K$ and $\text{dist}(\alpha^*, \alpha^{**}) \geq 1/K$, then

$$\|E_{[0,1]} g\|_{L^4(B')} \leq K^2 \|E_{\alpha^{**}} g\|_{L^4(B')}^{1/2} \|E_{\alpha^{**}} g\|_{L^4(B')}^{1/2} \leq K^2 \|E_{\alpha^*(B')} g\|^2 \|E_{\alpha^{**}} g\|^2_{L^4(B')}.$$

If such $\alpha^{**}$ does not exist, we have

$$\|E_{[0,1]} g\|_{L^4(B')} \lesssim \|E_{\alpha^{**}} g\|_{L^4(B')}.$$ 

We sum over all balls $B' \subset B_{\delta^{-1}}$ and obtain

$$\|E_{[0,1]} g\|_{L^4(B_{\delta^{-1}})} \lesssim \sum_{\alpha \in P_{K^{-1}}} \|E_\alpha g\|_{L^4(B_{\delta^{-1}})}^4 + C_K \left( \sum_{\text{dist}(\alpha_1, \alpha_2) \geq 1/K} \left\| \prod_{j=1}^2 |E_{\alpha_j} g|^{1/2} \right\|_{L^4(B_{\delta^{-1}})}^4 \right)^{1/4}. \quad (4.3)$$

Here $C_K$ is a large constant that depends on $K$.

We will apply a bilinear restriction estimate for parabola to the second term and obtain

$$\left\| \prod_{j=1}^2 |E_{\alpha_j} g|^{1/2} \right\|_{L^4(B_{\delta^{-1}})} \lesssim_K \|g\|_2.$$
By a localization argument, this further implies
\[ \left\| \prod_{j=1}^{2} |E_{\alpha_j}g|^{1/2} \right\|_{L^4(B_{\frac{1}{2}})} \leq K \delta^{1/2} \|E_{[0,1]}g\|_{L^2(\mathbb{R}^4)} \]
\[ \leq K \delta^{1/2} \left( \sum_{j \in P_s} \|E_Jg\|_{L^2(\mathbb{R}^4)}^2 \right)^{1/2} \] (4.4)

In the second step, we have applied $L^2$ orthogonality. By Hölder’s inequality, we obtain
\[ \|E_{[0,1]}g\|_{L^4(B_{\frac{1}{2}})}^{4} \leq \sum_{\alpha \in P_{s/16}} \|E_{\alpha}g\|_{L^4(B_{\frac{1}{2}})}^{4} \]
\[ + C_K \delta^{-1} \left( \sum_{j \in P_s} \|E_Jg\|_{L^4(\mathbb{R}^4)}^4 \right). \] (4.5)

This finishes the first step of our argument.

In the second step, we will iterate the estimate (4.5). Let $I \subset [0, 1]$ be a dyadic interval of length $\sigma \in [\sqrt{\delta}, 1/K]$. Without loss of generality, we assume that $I = [0, \sigma]$. Then
\[ \|E_{\alpha}g\|_{L^4(B_{\frac{1}{2}})}^{4} \leq \sum_{\alpha \in P_{s/2}} \|E_{\alpha}g\|_{L^4(B_{\frac{1}{2}})}^{4} \]
\[ + \sum_{B_{\frac{1}{2}} \subset I} \|E_Jg\|_{L^4(\mathbb{R}^4)}^{4}. \] (4.6)

Now we change all variables back and obtain
\[ \leq \sum_{\alpha \in P_{s/2}} \|E_{\alpha}g\|_{L^4(B_{\frac{1}{2}})}^{4} + C_K \sigma^2 \delta^{-1} \sum_{j \in P_s} \|E_Jg\|_{L^4(\mathbb{R}^4)}^{4}. \]

Notice that the frequency scale in the last sum is $\sigma^{-1} \delta$, which is larger than the desired $\delta$. We apply $L^2$ orthogonality, interpolation with a trivial bound at $L^\infty$ and obtain
\[ \|E_{\alpha}g\|_{L^4(B_{\frac{1}{2}})}^{4} \leq C_0 \sum_{\alpha \in P_{s/16}} \|E_{\alpha}g\|_{L^4(B_{\frac{1}{2}})}^{4} + C_K \delta^{-1} \sum_{j \in P_s} \|E_Jg\|_{L^4(\mathbb{R}^4)}^{4} \] (4.7)

for some possibly large constant $C_0$.

In the next step, we iterate (4.7) $M$ times, with $K^M = \delta^{-1/2}$. We obtain
\[ \|E_{[0,1]}g\|_{L^4(B_{\frac{1}{2}})}^{4} \leq C_0^M \sum_{\alpha \in P_{s/2}} \|E_{\alpha}g\|_{L^4(B_{\frac{1}{2}})}^{4} + C_K M \delta^{-1} \sum_{j \in P_s} \|E_Jg\|_{L^4(\mathbb{R}^4)}^{4}, \]
We apply $L^2$ orthogonality and interpolation with a trivial $L^\infty$ bound to the first term and obtain
\[ \|E_{[0,1]}g\|^4_{L^4(B_{\delta^{-1}})} \leq C_0 \delta^{-1} \sum_{\alpha \in I_4} \|E_{\alpha}g\|^4_{L^4(w_{\delta^{-1}})} + C_K M \delta^{-1} \sum_{j \in I_4} \|E_jg\|^4_{L^4(w_{\delta^{-1}})}. \]

By picking $K$ to be large enough, depending on $\varepsilon$, we obtain the desired estimate. Note that (1.4) is exactly where we need $\sigma \geq \sqrt{\delta}$ and is also why we cannot iterate until $K^M = \delta^{-1}$.

**Remark 1.** From the above proof, (4.2) also holds for every compact $C^2$ curve with non-zero curvature.

**Remark 2.** The use of $\ell^4$ sum on the right hand side of (4.2) determines that the current new argument that is used to prove Theorem 1.1, which is inspired by [Woo16] and [HB15], cannot be used to recover (1.4).

### 4.2. The proof of Lemma 4.1

We can assume that $a < b$ since when $a = b$, there is nothing to show. By translation invariance, we may assume that $I' = [0, \nu^b]$. Notice that $|I'| = \nu^{-b}$, therefore the function $|\mathcal{E}_{I'}g|^8$ is essentially constant on every axis-parallel slab of dimension $\nu^{-b} \times \nu^{-2b} \times \nu^{-3b}$. Here the short side of length $\nu^{-b}$ is along the $x_1$-axis and the side of medium length is along the $x_2$-axis.

Let $\square$ denote an axis-parallel rectangular box that is also a translation of $\tilde{T}_{I'}$. To estimate left hand side of (4.1), we first consider
\[ \int_\square (|\mathcal{E}_{I'}g|^4 \ast \phi_{\tilde{T}_{I'},E})(|\mathcal{E}_{I'}g|^8 \ast \phi_{\tilde{T}_{I'},E}) \quad (4.8) \]

Notice that for every $x, x' \in \square$, we have
\[ |\mathcal{E}_{I'}g|^8 \ast \phi_{\tilde{T}_{I'},E}(x) \sim_E |\mathcal{E}_{I'}g|^8 \ast \phi_{\tilde{T}_{I'},E}(x'). \]

Therefore, we bound (4.8) by
\[ \left( \sup_{x \in \square} |\mathcal{E}_{I'}g|^8 \ast \phi_{\tilde{T}_{I'},E}(x) \right) \int_\square (|\mathcal{E}_{I'}g|^4 \ast \phi_{\tilde{T}_{I'},E}). \quad (4.9) \]

We keep the former factor as is for a while and focus on the latter factor. We first write it as
\[ \int_{\mathbb{R}^3} \left[ \int_{\square_y} |\mathcal{E}_{I'}g(x)|^4 dx \right] \phi_{\tilde{T}_{I'},E}(y) dy, \quad (4.10) \]

where $\square_y := \square - y$. We will prove the following.

**Lemma 4.3.** Let $\nu \in 2^{-2n} \cap (0, 1/1000)$, $I = [d, d + \nu^a]$, with $|d| \geq \nu$, and $\Delta$ a square in the $(x_2, x_3)$-plane of side length $\nu^{-2b}$. For every fixed $x_1 \in \mathbb{R}$, there exists an absolute constant $C_0$ such that
\[ \int_{\mathbb{R}^2} |(\mathcal{E}_{I'}g)(x)|^4 w_{\Delta,E}(x_2, x_3) dx_2 dx_3 \]
\[ \leq \varepsilon E \nu^{-(1+\varepsilon)(2b-2a)} C_0 \sum_{j \in I_4} \int_{\mathbb{R}^2} |(\mathcal{E}_{I'}g)(x)|^4 w_{\Delta,E}(x_2, x_3) dx_2 dx_3 \]

for every $\varepsilon > 0$. 

First let’s see how to use Lemma [13] to finish the proof. By applying Lemma [4.3] (with $E$ replaced by $100E$) and Fubini, we can bound (4.10) by

$$
\lesssim_{rE} \nu^{-(1+\epsilon)(2b-2a)-C_0} \sum_{J \in P_{r2b-a}(I)} \int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}^3} |(E_j g)(x-y)|^4 w_{J,E}(x) \phi_{rI,E}(y) dy \right] \phi_{rI,E}(y) dy
$$

It remains to prove that

$$
\lesssim_{rE} \nu^{-(1+\epsilon)(2b-2a)-C_0} \sum_{J \in P_{r2b-a}(I)} \int_{\mathbb{R}^3} \left[ |E_j g|^4 \phi_{rI,E/C_0} \right] w_{J,E}.
$$

But the proof of this is essentially the same as in the steps [3.8], [3.9] and [3.10]: one would bound the left hand side by

$$
\sum_{J} \sum_{\kappa \in E_3} \nu^{-2-\kappa-3b} \left( \sup_{x \in \square} |E_j g|^8 \phi_{rI,E/C_0}(x) \right) \int_{\square_{\nu^{2-H}}} \left[ |E_j g|^4 \phi_{rI,E/C_0} \right] \phi_{rI,E/C_0}(x').
$$

where $\nu^{-2} \kappa := (\nu^{-2b}\kappa_1, \nu^{-2b}\kappa_2, \nu^{-2b}\kappa_3)$ for $\kappa = (\kappa_1, \kappa_2, \kappa_3)$, and use the following inequalities: we use

$$
\sup_{x \in \square_{\nu^{2-H}}} |E_j g|^4 \phi_{rI,E/C_0}(x) \lesssim_{E, \inf_{x \in \square} |E_j g|^4 \phi_{rI,E/C_0}(x')},
$$

(here we used that the side lengths of $\tilde{T}_J$ are longer than those of $\square$, which holds because $b \geq a$), and that

$$
\sup_{x \in \square} |E_j g|^8 \phi_{rI,E/C_0}(x) \sim_{E} \inf_{x \in \square} |E_j g|^8 \phi_{rI,E/C_0}(x).
$$

4.3. The proof of Lemma [4.3]. In the proof, to avoid using too many subscripts, we will use $(x, y, z)$ to stand for a point in $\mathbb{R}^3$ rather than $(x_1, x_2, x_3)$.

We begin with some preliminaries for the proof of Lemma [4.3]. For an interval $I \subset \mathbb{R}$ centered at $c$ of length $R$, we let

$$
w_{I,E}(x) := (1 + \frac{|x-c|}{R})^{-E}
$$

for $x \in \mathbb{R}$. Next for $B \subset \mathbb{R}^2$, a square centered at $c = (c_1, c_2)$ of side length $R$, define

$$
\tilde{w}_{B,E}(x, y) := (1 + \frac{|x-c_1|}{R})^{-E}(1 + \frac{|y-c_2|}{R})^{-E}
$$

and

$$
w_{B,E}(x, y) := (1 + \frac{|(x, y) - c|}{R})^{-E}.
$$

Finally we have the following lemma.
Lemma 4.4. Let $\nu \in 2^{-2^N} \cap (0, 1/1000)$ and $x$ a real number such that $\nu \leq x \leq 1/1000$. Then there exists an integer $N$ such that $N \mid \nu^{-1}$ and $\frac{1}{2N} \leq x \leq \frac{1}{N}$.

Proof of Lemma 4.4. Write $\nu = 2^{-2^a}$ for some $a \in \mathbb{N}$ sufficiently large. Since we want $N \leq \frac{1}{\nu} \leq 2N$, it suffices to choose $N$ from the set $\{2^9, 2^{10}, \ldots, 2^{2^a-1}\}$. This completes the proof of Lemma 4.4. \qed

Proof of Lemma 4.3. By Corollary 2.4 of [Li17] and the same proof as Proposition 2.11 of [Li17], it suffices to prove instead

$$
\int_{\mathbb{R}^2} |(E_I g)(x, y, z)|^4 \mathbb{1}_\Delta(y, z) \, dy \, dz 
$$

and furthermore, by shifting $y$ and $z$, it suffices to show this only in the case when $\Delta$ is centered at the origin. Let $\hat{\Delta}$ be the square centered at the origin with coordinates $(\nu^{-2b}, 0)$, $(-\nu^{-2b}, 0)$, $(0, \nu^{-2b})$, and $(0, -\nu^{-2b})$. Since $\Delta \subset \hat{\Delta}$, it suffices to show (4.11) with $\mathbb{1}_\Delta(y, z)$ on the left hand side replaced with $\mathbb{1}_{\hat{\Delta}}(y, z)$. This small reduction will make the algebra later simpler.

Expanding the left hand side gives

$$
\int_{\mathbb{R}^2} g(t) e(t) e(t^2 y + t^3 z) \, dt \, \mathbb{1}_{\hat{\Delta}}(y, z) \, dy \, dz.
$$

Rescaling $[d, d + \nu^a]$ to $[0, 1]$ shows that the above is equal to

$$
\nu^{1a} \int_{\mathbb{R}^2} g(d + \nu^a t) e((d + \nu^a t) x) \times \nu^{2a} t^2 (y + (3d + \nu^a t) z)) \, dt \, \mathbb{1}_{\hat{\Delta}}(y, z) \, dy \, dz.
$$

Before we proceed, let us first describe the idea. It will become clear why we organize different terms in the phase function as above. Notice that $\nu^a$ is an extremely small number. We will treat $\nu^a t$ as a small perturbation and end up looking at the extension operator for a (perturbed) parabola.

To make this idea precise, we make the change of variables

$$
\begin{pmatrix}
 y' \\
 z'
\end{pmatrix} = \begin{pmatrix}
 2\nu^a d & 3\nu^a d^2 \\
 \nu^{2a} & 3\nu^{2a} d \\
\end{pmatrix} \begin{pmatrix}
 y \\
 z \\
\end{pmatrix}.
$$

Denote the matrix above by $T$ and let $G(t, x) := g(d + \nu^a t) e((d + \nu^a t) x)$. Then using that $|d| \geq \nu$, (4.12) is bounded by

$$
\nu^{a-2} \int_{\mathbb{R}^2} \int_0^1 G(t, x) e(y'(t - \frac{2\nu^{2a} d^2}{3d^2} t^3) + z'(t^2 + \frac{2\nu^a}{3d} t^3)) \, dt \, \mathbb{1}_{T\hat{\Delta}}(y', z') \, dy' \, dz'.
$$

(4.13)

Ignoring the weight for the moment, we will now want a decoupling theorem for the curve $(t - \frac{2\nu^{2a} t^3}{3d^2}, t^2 + \frac{2\nu^a}{3d} t^3)$ which is a small perturbation of the parabola. But this comes from the generalization of Lemma 4.2 through Remark 4.
We split the former case into two further cases. We focus on the former case.

We have two cases, either \( \nu \leq |d| \leq 1/1000 \) or \( 1/1000 < |d| \leq 1 \). We will only focus on the former case. The latter case is slightly easier, as we have \( O(1) \) separation.

We split the former case into two further cases \( d > 0 \) and \( d < 0 \). Again we only focus on the former case \( d > 0 \). The proof for the other case is similar.

Since \( d \) is sufficiently small, the longest diagonal is created by connecting the points \( A \) and \( B \) which lies on the line \( z' = \frac{\nu}{2\eta} y' \). Let \( \theta \) be such that \( \tan \theta = \frac{\nu}{2\eta} \) and let \( R_\theta \) be the rotation matrix that rotates by an angle \( \theta \) in the counterclockwise direction. Therefore \( R_\theta^{-1} T(\Delta) \) is a parallelogram with the line connecting \( R_\theta^{-1} A \) and \( R_\theta^{-1} B \) in the \( y' \)-axis. The \( y' \)-coordinate of \( R_\theta^{-1} A \) is

\[
\nu^{-2b+a}(2d \cos \theta + \nu^a \sin \theta).
\]

and the \( z' \)-coordinate of \( R_\theta^{-1} C \) is

\[
3d \nu^{-2b+a}(-d \sin \theta + \nu^a \cos \theta).
\]

With \( x = |d| \) in Lemma 4.4, we can find an integer \( N \) such that \( N | \nu^{-2b+2a} \) (since \( 2b - 2a \geq 2 \)) and \( \frac{1}{N} \nu^{-2b+2a} \leq |d| \leq \frac{1}{N} \). Therefore \( R_\theta^{-1} T(\Delta) \) is contained in a rectangle \( \Delta' \) centered at the origin of length

\[
2\nu^{-2b+a}(2d \cos \theta + \nu^a \sin \theta) = 2\nu^{-2b+a} \sqrt{4d^2 + \nu^{2a}} \leq 6d \nu^{-2b+a} \leq \frac{6}{N} \nu^{-2b+a}
\]

and height

\[
6d \nu^{-2b+a}(-d \sin \theta + \nu^a \cos \theta) = 3d \nu^{-2b+2a} \frac{2d}{\sqrt{4d^2 + \nu^{2a}}} \leq 6d \nu^{-2b+2a} \leq \frac{6}{N} \nu^{-2b+2a}.
\]

Partition this rectangle into \( \nu^{-a} \) squares \( \square \) of side length \( \frac{1}{N} \nu^{-2b+2a} \). Thus in this case we have shown that

\[
\mathbb{1}_{T(\Delta)}(y', z') \leq \sum \mathbb{1}(R_\theta^{-1}(y', z')) \leq E \sum \mathbb{1}_{\square, 100E}(R_\theta^{-1}(y', z')) = \sum \mathbb{1}_{B(0, \frac{1}{N} \nu^{-2b+2a}, 100E)}((y', z') - R_\theta \square).
\]

where the last equality we have used that \( \mathbb{1}_{B(0, R, 100E)}(x) \) is a radial function. Therefore (4.13) is

\[
\leq E \nu^{-2a} \sum \int_{\mathbb{R}^2} \int_0^1 G(t, x) e((y(t - \frac{\nu^{2a}}{2d^2} + z'(t^2 + \frac{2\nu^a}{3d^3})) \ dt| x \times \mathbb{1}_{B(R \infty, \frac{1}{N} \nu^{-2b+2a}, 100E)}(y', z') \ dy' dz'.
\]
Applying Lemma 1.2 with Remark 1 shows that we can decouple to frequency scale \( N^{2b-2a} \), that is, the above is

\[
\leq \epsilon E N^{-(2b-2a)(1+c) - 2} \times \sum_{J \in \mathcal{P}_{N^{2b-2a}}} \nu^a \int_{\mathbb{R}^2} \int_I |G(t, x) e(y' (t - \nu^{2a}/3d^2 t^3) + z'(t^2 + 2\nu^a/3d^3)) \, dt|^4 \times w_B(R_0 E, c, \nu^{2b-2a}, 100E) (y', z') \, dy' \, dz'.
\]

By our choice of \( N \) and that \( |d| \leq 1/1000, N \geq 500 \). Furthermore, by undoing the change of variables, one controls the above by

\[
\leq \epsilon E \nu^{-(2b-2a)(1+c) - 2} \times \sum_{J \in \mathcal{P}_{\nu^{2b-a}}} \int_{\mathbb{R}^2} \int_I |(E_j g)(x, y, z)|^4 \sum_{\Box} w_B(R_0 E, c, \nu^{2b-2a}, 100E) (T(y, z)) \, dy \, dz.
\]

Since \( N \sim |d|^{-1} \) and \( |d| \geq \nu \), we can use the triangle inequality to decouple \( I \) from frequency scale \( N \nu^{2b-a} \) to scale \( \nu^{2b-a} \), losing only a factor of \( O(N^3) \). Therefore the above is

\[
\leq \epsilon E \nu^{-(2b-2a)(1+c) - 5} \times \sum_{J \in \mathcal{P}_{\nu^{2b-a}}} \int_{\mathbb{R}^2} \int_I |(E_j g)(x, y, z)|^4 \sum_{\Box} w_B(R_0 E, c, \nu^{2b-2a}, 100E) (T(y, z)) \, dy \, dz.
\]

Thus we will have proved (4.11) in the case when \( \nu \leq |d| \leq 1/1000, d > 0 \) provided we can show that

\[
\sum_{\Box} w_B(R_0 E, c, \nu^{2b-2a}, 100E) (T(y, z)) \leq E \tilde{w}_{\Delta, 10E} (y, z). \tag{4.14}
\]

We have

\[
\sum_{\Box} w_B(R_0 E, c, \nu^{2b-2a}, 100E) (T(y, z)) = \sum_{\Box} \mu_{\Box, 10E} (R_0^{-1} T(y, z)).
\]

Since \( (1 + |y|)(1 + |z|) \leq (1 + (|y, z|))^2 \), to show (4.14), it suffices to show that

\[
\sum_{\Box} \tilde{w}_{\Box, 50E} (R_0^{-1} T(y, z)) \leq E \tilde{w}_{\Delta, 10E} (y, z). \tag{4.15}
\]

Writing the centers of the \( \Box \) that partition \( \Delta' \) as \((c, 1, c, 2)\), we have

\[
\sum_{\Box} \mathbb{I}_I (c, 1, c, 2, \nu^{-2b+2a}) (y) \mathbb{I}_I (c, 2, c, 2, \nu^{-2b+2a}) (z) = \mathbb{I}_I (0, 0, \nu^{-2b+2a}) (y) \mathbb{I}_I (0, 0, \nu^{-2b+2a}) (z)
\]

where \( I(a, L) \) is the interval \([a - L/2, a + L/2]\). By the proof of Lemma 2.1 and Remark 2.3 of [L17],

\[
(\mathbb{I}_I (c, 1, c, 2, \nu^{-2b+2a}) \ast \mathbb{I}_I (0, 0, \nu^{-2b+2a}, 50E) (y)) \gtrsim E \left( \frac{1}{N} \nu^{-2b+2a} \right) w_{I(c', 1, c, 2, \nu^{-2b+2a}, 50E} (y)
\]

and similarly for the \( z \)-coordinate, the left hand side of (4.15) is

\[
\leq \epsilon E \left( \frac{1}{N} \nu^{-2b+2a} \right)^{-2} \times \left( \mathbb{I}_I^y (0, 0, \nu^{-2b+2a}) \mathbb{I}_I^z (0, 0, \nu^{-2b+2a}) \ast \mathbb{I}_I^y (0, 0, \nu^{-2b+2a}, 50E) \mathbb{I}_I^z (0, 0, \nu^{-2b+2a}, 50E) (R_0^{-1} T(y, z))
\]
where here we have used $\mathbb{1}_{I(0,R)}^y$ to be shorthand for $\mathbb{1}_{I(0,R)}(y)$, and similarly for $\mathbb{1}_{I(0,R)}^z$, $\mathbb{1}_{I(0,R)}^w$, and $\mathbb{1}_{I(0,R)}^i$. Thus it suffices to show that
\[
(w_{I(0,\nu^{-2b+a}),50E}^y w_{I(0,\nu^{-2b+2a}),50E}^z) (R^{-1}_\theta T(y, z)) \lesssim E \tilde{w}_{B(0,1),10E}(y, z).
\]
Rescaling $y$ and $z$, it is enough to prove
\[
(w_{I(0,\nu^{-2b+a}),50E}^y w_{I(0,\nu^{-2b+2a}),50E}^z) (\nu^{-2b} R^{-1}_\theta T(y, z)) \lesssim E \tilde{w}_{B(0,1),10E}(y, z).
\]
(4.16)
The left hand side of (4.16) is equal to
\[
(1 + \left| (\nu^{-a} R^{-1}_\theta T(y, z))_1 \right| -50E (1 + \left| (\nu^{-2a} R^{-1}_\theta T(y, z))_2 \right|)^{50E}.
\]
We observe that
\[
\nu^{-2a} R^{-1}_\theta T = N d \cdot \frac{2d}{\sqrt{4d^2 + \nu^{2a}}} \left( \nu^{-a} \left( 2 + \frac{2a}{2d^2} \right) \nu^{-a} \left( 3d + \frac{3}{2} \nu^{2a} d^{-1} \right) \right)
\]
\[
:= N d \cdot \frac{2d}{\sqrt{4d^2 + \nu^{2a}}} S
\]
and
\[
\frac{1}{\sqrt{5}} \leq N d \cdot \frac{2d}{\sqrt{5}} \leq N d \cdot \frac{2d}{\sqrt{4d^2 + \nu^{2a}}} \leq N d \leq 1.
\]
(4.17)
Therefore from (4.17) and that $2 \leq 2 + \frac{\nu^{2a}}{2d^2} \leq 3$,
\[
(1 + \left| (\nu^{-a} R^{-1}_\theta T(y, z))_1 \right| -50E \leq E (1 + \left| (2 + \frac{\nu^{2a}}{2d^2}) y + (3d + \frac{3}{2} \nu^{2a} d^{-1}) z \right| -50E)
\]
\[
\leq E (1 + \left| y + d \left( 6d^2 + 3\nu^{2a} \right) z \right|) -50E
\]
and
\[
(1 + \left| (\nu^{-2a} R^{-1}_\theta T(y, z))_2 \right| -50E \leq E (1 + \left| z \right|) -50E.
\]
Thus to prove (4.16), it remains to show that
\[
\frac{(1 + |y|)}{(1 + |y + d(\frac{6d^2 + 3\nu^{2a}}{4d^2 + \nu^{2a}}) z|)^5} (1 + |z|)^4
\]
is a bounded function independent of $y$, $z$, $d$, $\nu$, and $a$. To see that (4.18) is bounded, we consider the following two cases:

- Suppose $|y + d(\frac{6d^2 + 3\nu^{2a}}{4d^2 + \nu^{2a}}) z| \geq \frac{|y|}{2}$. Then (4.18) is controlled by
\[
\frac{1 + |y|}{(1 + |y|/2)^5} (1 + |z|)^4 \leq 1.
\]
- Suppose $|y + d(\frac{6d^2 + 3\nu^{2a}}{4d^2 + \nu^{2a}}) z| < \frac{|y|}{2}$. Then $|y| \leq 2|d(\frac{6d^2 + 3\nu^{2a}}{4d^2 + \nu^{2a}})| |z| \leq |z|$ and hence (4.18) is controlled by
\[
\frac{1}{(1 + |z|)^3} \leq 1.
\]
This then proves (4.16) and hence also (4.11) in the case when $\nu \leq |d| \leq 1/1000$, $d > 0$. \qed
5. The iteration

We now let $C_0$ be the largest of any $C_0$ that appears in the statements of Lemmas 2.4, 2.7 in Section 2. It will no longer vary line by line as before and will now be fixed.

**Lemma 5.1.** Let $a$ and $b$ be integers such that $1 \leq a \leq b$. Suppose $\delta$ and $\nu$ were such that $\nu^{36b^3-1} \in \mathbb{N}$ and $\nu \in 2^{-2^3} \cap (0, 1/1000)$. Then

$$M_{2,a,b}(\delta, \nu, E) \leq C_0 \nu^{\frac{1}{3}b(5+6\epsilon)a - \frac{1}{36}(7+6\epsilon)b - \frac{2}{3}C_0} \times$$

$$M_{2,b,2b-a}(\delta, \nu, E/C_0^{1/3}) M_{2,b,3b}(\delta, \nu, E/C_0^{1/6}) D(\frac{\delta}{\nu b})^{1/2}.$$

Proof. We have

$$M_{2,a,b}(\delta, \nu, E) \leq \nu^{\frac{1}{3}b(1+\epsilon)(b-a)-C_0} M_{2,b-a,b}(\delta, \nu, E/C_0)$$

$$\leq \nu^{\frac{1}{3}b(1+\epsilon)(b-a)-C_0} M_{2,b,2b-a}(\delta, \nu, E/C_0^{2/3}) M_{1,2b-a,b}(\delta, \nu, E/C_0^{2/3})$$

where here we have used Lemmas 2.4 and 2.7. Next, Lemmas 2.6 and 2.8 give that

$$M_{1,2b-a,b}(\delta, \nu, E/C_0^{2/3}) \leq \nu^{\frac{1}{3}(a+b)-C_0} M_{1,3b,b}(\delta, \nu, E/C_0^{3/4})$$

$$\leq \nu^{\frac{1}{3}(a+b)-C_0} M_{2,b,3b}(\delta, \nu, E/C_0^{1/4}) D(\frac{\delta}{\nu b})^{3/4}.$$

Inserting this estimate into (5.1) and observing that

$$-\frac{1}{6}(1 + \epsilon)(b-a) - \frac{1}{36}(a+b) = \frac{1}{36}(5+6\epsilon)a - \frac{1}{36}(7+6\epsilon)b$$

then completes the proof of Lemma 5.1. \qed

Let $\lambda \geq 0$ be the smallest real number such that $D(\delta) \leq \epsilon^{-1/4-\lambda-\epsilon}$ for all $\delta \in \mathbb{N}$. The trivial bound on $D(\delta)$ shows that $\lambda \leq 1/2$. If $\lambda = 0$, then we are done. We now assume $\lambda > 0$ and derive a contradiction.

We will let $\tilde{C}(\epsilon)$ be the implied constant depending on $\epsilon$ in the estimate $D(\delta) \leq \epsilon^{-1/4-\lambda-\epsilon}$ and $C(\epsilon, E)$ the implied constant depending on $\epsilon, E$ from Lemma 5.1.

**Lemma 5.2.** Let $N \geq 0$ an integer and $\delta \in \mathbb{N}$ be fixed.

Suppose the following statement is true: If $b \in \mathbb{N}$ and $\nu \in 2^{-2^3} \cap (0, 1/1000)$ is such that $\nu^{36b^3-1} \in \mathbb{N}$, then

$$M_{2,a,b}(\delta, \nu, E) \leq C_N(a, b, \epsilon, E) \delta^{-\frac{1}{3}b}\nu^{-\alpha_N a - \beta_N b - \frac{32}{3}C_0}$$

for all $a$ such that $1 \leq a \leq b$.

Then the following statement is also true: If $b \in \mathbb{N}$ and $\nu \in 2^{-2^3} \cap (0, 1/1000)$ is such that $\nu^{36b^3-1} \delta^{-1} \in \mathbb{N}$, then

$$M_{2,a,b}(\delta, \nu, E) \leq C_{N+1}(a, b, \epsilon, E) \delta^{-\frac{1}{3}b}\nu^{-\alpha_{N+1} a - \beta_{N+1} b - \frac{32}{3}C_0}$$

for all $a$ such that $1 \leq a \leq b$ where

$$\begin{pmatrix} \alpha_{N+1} \\ \beta_{N+1} \end{pmatrix} = \begin{pmatrix} -5/36 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 \\ 5/72 - \lambda/2 \end{pmatrix} \begin{pmatrix} \alpha_N \\ \beta_N \end{pmatrix}$$

and

$$C_{N+1}(a, b, \epsilon, E) := C(\epsilon, E) C_N(b, 2b - a, \epsilon, E/C_0^{1/3}) C_N(b, 3b, \epsilon, E/C_0^{1/6}) \tilde{C}(\epsilon)^{1/2}.$$
Therefore by hypothesis,

\[ \mathcal{M}_{2,b,2b-a}(\delta, \nu, E) \leq C_N(b, 2b - a, \varepsilon, E)\delta^{-\frac{1}{4} - \lambda - \varepsilon} \nu^{\frac{3b}{2}a - (\alpha_N + 2b\delta) - \frac{36}{5}C_0} \]

Next, since \( \nu^{3N+1b}\delta^{-1} = \nu^{3N(3b)\delta^{-1}} \in \mathbb{N} \) and \( 1 \leq b \leq 3b \), by hypothesis, we have

\[ \mathcal{M}_{2,b,3b}(\delta, \nu, E) \leq C_N(b, 3b, \varepsilon, E)\delta^{-\frac{1}{4} - \lambda - \varepsilon} \nu^{-((\alpha_N + 3b\delta) - \frac{36}{5}C_0}. \]

Finally we note that by our assumption on \( D(\delta) \) and since \( \nu^{3N+1b}\delta^{-1} \in \mathbb{N} \) implies \( \nu^{b\delta^{-1}} \in \mathbb{N} \), we have

\[ D(\frac{\delta}{\nu^b}) \leq \tilde{C}(\varepsilon)\delta^{-\frac{1}{4} - \lambda - \varepsilon} \nu^{b(\frac{3}{2} + \lambda)} \nu^{b\delta^{-1}}. \]

Since \( \nu^{3b\delta^{-1}} \in \mathbb{N} \), Lemma 5.1 then gives that

\[ \mathcal{M}_{2,a,b}(\delta, \nu, E) \leq C_{N+1}(a, b, \varepsilon, E)\delta^{-\frac{1}{4} - \lambda - \varepsilon} \nu^{\frac{3}{2}a - (\frac{5}{6}\beta_N + \frac{1}{5})\delta^{-1} + \frac{36}{5}\delta^{-1}\nu^{\frac{1}{2}b} + \frac{1}{2}b(\frac{3}{2} + \lambda)} \nu^{b\delta^{-1}}. \]

Rearranging the above equation and observe that the power of \( \nu^{\frac{1}{2}} \) is \( \nu^{\frac{3}{2}a + \frac{1}{2}b} \leq 1 \) then completes the proof of Lemma 5.2. \( \square \)

**Lemma 5.3.** If \( \alpha_0 = 0, \beta_0 = 0, \) and

\[
\begin{pmatrix}
\alpha_{N+1} \\
\beta_{N+1}
\end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 1/2 & 7/6 \end{pmatrix} \begin{pmatrix} \alpha_N \\
\beta_N \end{pmatrix},
\]

then

\[
\begin{pmatrix}
\alpha_N \\
\beta_N
\end{pmatrix} = \frac{(A + 2B)N}{5} \begin{pmatrix} -1 + \frac{36}{5}(1 - \frac{6}{\varepsilon})/(A + 2B)N & A \\
3 - \frac{18}{5}(1 - \frac{6}{\varepsilon})(A + 2B)N & A + \frac{12}{5}(1 - \frac{6}{\varepsilon})(A + 2B)N \end{pmatrix}.
\]

**Proof.** This is as in Section 4 of [Hill11]. Let

\[
M = \begin{pmatrix} 0 & -1/3 \\ 1/2 & 7/6 \end{pmatrix} , \quad P = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} , \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 1/6 & 0 \end{pmatrix}.
\]

Then \( M = PDP^{-1} \). Iterating (5.2) gives

\[
\begin{pmatrix}
\alpha_N \\
\beta_N
\end{pmatrix} = \frac{1}{5} \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & 6(1 - \frac{1}{6^\varepsilon}) \end{pmatrix} \begin{pmatrix} A & 0 \\ B & 1/2 & 7/6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{N+1} \\
\beta_{N+1} \end{pmatrix}.
\]

This completes the proof of Lemma 5.3. \( \square \)

We derive a contradiction. Setting \( A = -5/36 \) and \( B = 5/72 - \lambda/2 \), we observe that \( A + 2B = -\lambda \).

By trivially controlling the bilinear constant by the linear constant, if \( \delta \) and \( \nu \) are such that \( \nu^{b\delta^{-1}} \in \mathbb{N} \), then \( \mathcal{M}_{2,a,b}(\delta, \nu, E) \leq c_\varepsilon, E^\delta^{-\frac{1}{4} - \lambda - \varepsilon} \) for all \( 1 \leq a \leq b \). Setting \( \alpha_0 = 0 \) and \( \beta_0 = 0 \) and using that \( A + 2B = -\lambda \), Lemma 5.3 shows that

\[
\alpha_N + \beta_N = -\frac{\lambda N}{5}(2 + \frac{6}{5N}(1 - \frac{1}{6^\varepsilon})(\frac{25}{72\lambda} + \frac{1}{2})).
\]
Since $\lambda > 0$, we can choose an $N_0$ sufficiently large (depending on $\lambda$) such that $\alpha_{N_0} + \beta_{N_0} < -1001 - \frac{1}{100} C_0$. Lemma [5.2] then shows that if $\delta \in \mathbb{N}^{-1}$ and $\nu \in 2^{-2^n} \cap (0, 1/1000)$ are such that $\nu^{3^{N_0}} \delta^{-1} \in \mathbb{N}$, then
\[
\mathcal{M}_{2,1,1}(\delta, \nu, E) \lesssim N_0, \nu, E \delta^{-\frac{1}{24}} \lambda^{-\epsilon} \nu^{1001}.
\]
Now choose $E > 1000$ to be a sufficiently large power of $C_0$ (depending on $N_0$). Bilinear reduction (Lemma [2.5]) then shows that if $\delta \in \mathbb{N}^{-1}$ and $\nu \in 2^{-2^n} \cap (0, 1/1000)$ are such that $\nu^{3^{N_0}} \delta^{-1} \in \mathbb{N}$, we have
\[
D(\delta) \lesssim _{N_0, \nu, E} \nu^{-\frac{1}{24}} D(\nu) + \delta^{-\frac{1}{24}} \lambda^{-\epsilon} \nu^{1000} \lesssim _{N_0, \nu, E} \delta^{-\frac{1}{24}} \lambda^{-\epsilon} (\nu^{1000} + \nu^{1000}) \lesssim _{N_0, \nu, E} \delta^{-\frac{1}{4}} \lambda^{-\epsilon} \nu^{\frac{1}{4}}
\]
where the last inequality is because $\lambda \leq 1/2$. Choosing $\nu = \delta^{1/3^{N_0}}$, then shows that if $\delta$ is such that $\delta^{1/3^{N_0}} \in 2^{-2^n} \cap (0, 1/1000)$, then
\[
D(\delta) \lesssim _{N_0, \nu, E} \delta^{-\frac{1}{4}} \lambda^{-\epsilon} (1 - \frac{1}{3^{N_0}})^{-\epsilon}.
\]
Corollary [2.2] (almost multiplicity) then shows that $D(\delta) \lesssim _{N_0, \nu, E} \delta^{-\frac{1}{4}} \lambda^{-\epsilon} (1 - \frac{1}{3^{N_0}})^{-\epsilon}$ for all $\delta \in \mathbb{N}^{-1}$. This contradicts the minimality of $\lambda$. Therefore we cannot have $\lambda > 0$ and hence we must have $\lambda = 0$. This completes the proof of Theorem [3.1].

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