The influence of a line with fast diffusion and nonlocal exchange terms on Fisher-KPP propagation

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Abstract

A new model to describe biological invasion influenced by a line with fast diffusion has been introduced by H. Berestycki, J.-M. Roquejoffre and L. Rossi in 2012. The purpose of this article is to present a related model where the line of fast diffusion has a nontrivial range of influence, i.e. the exchanges between the line and the surrounding space has a nontrivial support. We show the existence of a spreading velocity depending on the diffusion on the line. An intermediate model is also discussed.

Introduction

The purpose of this study is a continuation of [9] in which was introduced, by H. Berestycki, J.-M. Roquejoffre and L. Rossi, a new model to describe biological invasions in the plane when a strong diffusion takes place on a line, given by (1).

\[
\begin{aligned}
\partial_t u - D\partial_{xx}u &= \nu v(x,0,t) - \mu u & x \in \mathbb{R}, \ t > 0 \\
\partial_t v - d\Delta v &= v(1-v) & (x,y) \in \mathbb{R} \times \mathbb{R}^*_+, \ t > 0 \\
-d\partial_y v(x,0,t) &= \mu u(x,t) - \nu v(x,0,t) & x \in \mathbb{R}, \ t > 0.
\end{aligned}
\] (1)

A two-dimensional environment (the plane $\mathbb{R}^2$) includes a line (the line $\{(x,0), \ x \in \mathbb{R}\}$) in which fast diffusion takes place while reproduction and usual diffusion only occur outside the line. For the sake of simplicity, we will refer to the plane as “the field“ and the line as “the road“, as a reference to the biological situations. The density of the population is designated by $v = v(x,y,t)$ in the field, and $u = u(x,t)$ on the road. Exchanges of populations take place between the field and the road: a fraction $\nu$ of individuals from the field at the road (i.e. $v(x,0,t)$) join the road, while a fraction $\mu$ of the population on the road join the field. The diffusion coefficient in the field is $d$, on the road $D$. Of course, the aim is to study the case $D > d$. The nonlinearity $f$ is of Fisher-KPP type, i.e. strictly concave with $f(0) = f(1) = 0$. Considering a nonnegative, compactly supported initial datum $(u_0,v_0) \neq (0,0)$, the main result of [9] was the existence of an asymptotic speed of spreading $c_*$ in the direction of the road. They also explained the dependence of $c_*$ on $D$, the coefficient of diffusion on the road. In their model, the line separates the plane in two half-planes which do not interact with each other, only with the line. Moreover,
interactions between a half-plane and the line occur only with the limit-condition in (1). That is why, in [9], the authors consider only a half-plane as the field.

New results on (1) have been recently proved. Further effects like a drift or a killing term on the road have been investigated in [8]. The case of a fractional diffusion on the road was studied and explained by the three authors and A.-C. Coulon in [3] and [10]. Models with an ignition-type nonlinearity are also studied by L. Dietrich in [11] and [12].

Our aim is to understand what happens when local interactions are replaced by integral-type interactions: exchanges of populations may happen between the road and a point of the field, not necessarily at the road. The density of individuals who jump from a point of the field to the road is represented by $y \mapsto \nu(y)$, from the road to a point of the field by $y \mapsto \mu(y)$. This is a more general model than the previous one, but interactions still only occur in one dimension, the $y$-axis. We are led to the following system:

$$
\begin{cases}
\partial_t u - D \partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y) dy & x \in \mathbb{R}, 
\end{cases}
$$

where $\overline{\mu} = \int \mu(y) dy$, the parameters $D$ and $\mu$ and $\nu$ are supposed constant positive, $\mu$ and $\nu$ are supposed nonnegative, and $f$ is a reaction term of KPP type. We can first observe that, at least formally, the system (1) is the limit of (2) when $\mu, \nu$ tend to $\delta$, the Dirac function. We will be concerned with the study of this limit in further work. In the same vein as (2), it is natural to consider the following semi-limit model

$$
\begin{cases}
\partial_t u - D \partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y) dy & x \in \mathbb{R}, 
\end{cases}
$$

where interactions from the road to the field are local whereas interactions from the field to the road are still nonlocal.

Let us describe the main results of this work. The first one concerns the stationary solutions of (2) and the convergence of the solutions to this equilibrium.

**Proposition 0.1.** under the assumptions on $f$, $\nu$, and $\mu$, then:

1. problem (2) (resp. (3)) admits a unique positive bounded stationary solution $(U_s, V_s)$, which is $x$-independent;

2. for all nonnegative and uniformly continuous initial condition $(u_0, v_0)$, the solution $(u, v)$ of (2) (resp. 3) starting from $(u_0, v_0)$ satisfies $(u(t, x), v(t, x, y)) \underset{t \to \infty}{\longrightarrow} (U_s, V_s)$ locally uniformly in $(x, y) \in \mathbb{R}^2$.

The second and principal result of this paper deals with the spreading in the $x$-direction: we show the existence of an asymptotic speed of spreading $c_*$ such that the following Theorem holds

**Theorem 0.1.** Let $(u, v)$ be a solution of (2) with a nonnegative, compactly supported initial datum $(u_0, v_0)$. Then, pointwise in $y$, we have:

- for all $c > c_*$, \( \lim_{t \to \infty} \sup_{|x| \geq t^c} (u(x, t), v(x, y, t)) = (0, 0) \);
for all $c < c_*$, $\lim_{t \to \infty} \inf_{|x| \leq ct}(u(x,t), v(x,y,t)) = (U_*, V_*)$. Because $f$ is a KPP-type reaction term, it is natural to look for positive solutions of the linearised system
\[
\begin{align*}
\partial_t u - D \partial_{xx} u &= -nu + \int \nu(y)v(t,x,y)dy & x \in \mathbb{R}, \ t > 0 \\
\partial_t v - d \Delta v &= f'(0)v + \mu(y)u(t,x) - \nu(y)v(t,x,y) & (x,y) \in \mathbb{R}^2, \ t > 0.
\end{align*}
\]
(4)

We will construct exponential travelling waves and use them to compute the asymptotic speed of spreading in the $x$-direction. Theorem 0.1 relies on the following Proposition:

**Proposition 0.2.**
1. There exists a limiting velocity $c_*$, depending on $D$ and $d$, such that for all $c > c_*$, $\exists \lambda > 0$, $\exists \phi \in H^1(\mathbb{R})$ positive such that $(t,x,y) \mapsto e^{-\lambda(x-ct)} \left( \begin{array}{c} 1 \\ \phi(y) \end{array} \right)$ is a solution of (4).

2. If $D \leq 2d$, then $c_* = c_{KPP} = 2\sqrt{df'(0)}$. If $D > 2d$, then $c_* > c_{KPP}$.

Our results and methods shed a new light on those of [9] and [8]. It is striking to find the same condition on $D$ and $d$ for the enhancement of the spreading in one direction. We also get the same kind of asymptotics with $D \to +\infty$. The stationary solutions are nontrivial and more complicated to bring out. The computation of the spreading speed $c_*$ comes from a nonlinear spectral problem, and not from an algebraic system which could be solved explicitly. It also involves some tricky arguments of differential equations.

Reaction-diffusion equations of the type
\[
\partial_t u - d \Delta u = f(u)
\]
have been introduced in the celebrated articles of Fisher [13] and Kolmogorov, Petrovsky and Piskounov [17] in 1937. The initial motivation came from population genetics. The reaction term are that of a logistic law, whose archetype is $f(u) = u(1-u)$ for the simplest example. In their works in one dimension, Kolmogorov, Petrovsky and Piskounov revealed the existence of propagation waves, together with an asymptotic speed of spreading of the dominating gene, given by $2\sqrt{df'(0)}$. The existence of an asymptotic speed of spreading was generalised in $\mathbb{R}^n$ by D. G. Aronson and H. F. Weinberger in [1] (1978). Since these pioneering works, front propagation in reaction-diffusion equations have been widely studied. Let us cite, for instance, the works of Freidlin and Gärtnert [14] for an extension to periodic media, or [18], [5] and [6] for more general domains. An overview of the subject can be found in [4].

The first section of this paper is concerned with the Cauchy problem, stationary solutions and the long time behaviour. Its conclusion is the proof of Proposition 0.1. The second section is devoted to the proof of Proposition 0.2. The third section deals with the spreading result, and the last one presents a short overview of how these results can be extended to the intermediate model (3).

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1 Stationary solutions and long time behaviour

In this section, we are concerned with the well-posedness of the system (2) combined with the initial condition

\[
\begin{align*}
    u|_{t=0} &= u_0 \in \mathbb{R} \\
    v|_{t=0} &= v_0 \in \mathbb{R}^2.
\end{align*}
\]  

(5)

We always assume that \(u_0\) and \(v_0\) are nonnegative, bounded and uniformly continuous. Our assumptions on the reaction term are of KPP-type:

\[f \in C^1([0,1]), \quad f(0) = f(1) = 0, \quad \forall s \in (0,1), \quad 0 < f(s) \leq f'(0)s.\]

We extend it to a concave and Lipschitz function, such that

\[
\lim_{s \to \infty} \frac{f(s)}{s} < -\max(\|\nu\mu - \nu\|_{L^\infty}, 2\frac{\nu}{d})
\]

, with the obvious notation \(\nu = \int \nu(y)dy\). Our assumptions on the exchange terms are the following:

- \(\mu\) is supposed to be nonnegative, continuous, and decreasing faster than an exponential function: \(\exists M > 0, \quad a > 0\) such that \(\forall y \in \mathbb{R}, \quad \mu(y) \leq M \exp(-a|y|)\).

- \(\nu\) is supposed to be nonnegative, continuous and twice integrable, both in \(+\infty\) and \(-\infty\), id est

\[
\int_0^{+\infty} \int_x^{+\infty} \nu(y)dydx < +\infty.
\]  

(6)

- We suppose \(\mu, \nu \not\equiv 0\), and \(\nu(0) > 0\)

1.1 Existence, uniqueness and comparison principle

The system (2) is quite standard, in the sense that the coupling does not appear in the diffusion nor the reaction term. Anyway, well-posedness still has to be mentioned.

**Proposition 1.1.** Under the above assumptions on \(f, \mu\) and \(\nu\), the Cauchy problem (2)-(5) admits a unique nonnegative bounded solution.

Using the formalism of [16], it is easy to show that the linear part on (2) defines a sectorial operator, and that the non-linear is globally Lipschitz on \(X := C_{unif}(\mathbb{R}) \times C_{unif}(\mathbb{R}^2)\), which gives the existence and uniqueness of the solution of (2).

We can also derive the uniqueness of the solution of (2) by showing that comparison between subsolutions and supersolutions is preserved during the evolution. Moreover, the following property will also be the key point in our later study of the spreading. Throughout this article, we will call a subsolution (resp. a supersolution) a couple satisfying the system (in the classical sense) with the equal signs replaced by \(\leq\) (resp. \(\geq\)) signs, which is also continuous up to time 0.

**Proposition 1.2.** Let \((\underline{u}, \underline{v})\) and \((\overline{u}, \overline{v})\) be respectively a subsolution bounded from above and a supersolution bounded from below of (2) satisfying \(\underline{u} \leq \overline{u}\) and \(\underline{v} \leq \overline{v}\) at \(t = 0\). Then, either \(\underline{u} < \overline{u}\) and \(\underline{v} < \overline{v}\) for all \(t > 0\), or there exists \(T > 0\) such that \((\underline{u}, \underline{v}) = (\overline{u}, \overline{v}), \quad \forall t \leq T\).

Once again, the proof is quite classical. See for instance [9]. Notice that the above proposition and its proof is independent with the effective existence of solution.
1.2 Long time behaviour and stationary solutions

The main purpose of this section is to prove that any (nonnegative) solution of (2) converges locally uniformly to a unique stationary solution \((U_s, V_s)\), which is bounded, positive, \(x\)-independent, and solution of the stationary system of equations (7):

\[
\begin{aligned}
-DU''(x) &= -pU(x) + \int \nu(y)V(x,y)dy \\
-d\Delta V(x,y) &= f(V) + \mu(y)U(x) - \nu(y)V(x,y).
\end{aligned}
\tag{7}
\]

In the same way as above, we call a subsolution (resp. a supersolution) of (7) a couple satisfying the system (in the classical sense) with the equal signs replaced by \(\leq\) (resp. \(\geq\)).

The next proposition and its proof is adapted from [8].

**Proposition 1.3.** Let \((u,v)\) be the solution of (2) starting from \((u_0,v_0) \neq (0,0)\). then there exist two positive, bounded, \(x\)-independent, stationary solutions \((U_1,V_1)\) and \((U_2,V_2)\) such that

\[
\begin{aligned}
U_1 \leq \liminf_{t \to +\infty} u(x,t) &\leq \limsup_{t \to +\infty} u(x,t) \leq U_2, \\
V_1(y) \leq \liminf_{t \to +\infty} v(x,y,t) &\leq \limsup_{t \to +\infty} v(x,y,t) \leq V_2(y),
\end{aligned}
\]

locally uniformly in \((x,y) \in \mathbb{R}^2\).

**Proof.** The proof is adapted from [8]. We first need a \(L^\infty\) a priori estimate.

**A priori estimate** Considering the hypothesis on the reaction term \(f\), there exists \(K > 0\) such that

\[\forall s \geq K, \ f(s) \leq s\left(\frac{p}{\mu}(\mu(y) - \nu(y))\right), \ \forall y \in \mathbb{R}.\]

Thus, for all constant \(V \geq K, V\left(\frac{p}{\mu}, 1\right)\) is a supersolution of (2).

**Construction of \((U_1,V_1)\)** Let \(R > 0\) large enough in such a way that the first eigenvalue of the Laplace operator with Dirichlet boundary condition in \(B_R \subset \mathbb{R}^2\) is less than \(\frac{\mu(0)}{\mu}\). \(\phi_R\) is the associated eigenfunction. We extend \(\phi_R\) to 0 outside \(B_R\). \(\phi_R\) is continuous, bounded, and satisfies

\[d\Delta \phi_R \leq \frac{1}{3} f'(0) \phi_R \text{ in } \mathbb{R}^2.\]

Let us choose \(\varepsilon > 0\) such that if \(0 < x \leq \varepsilon, \ f(x) > \frac{2}{3} f'(0) x\). Then define \(M\) such that \(\forall y / |y| > M - R, \ \nu(y) \leq \frac{1}{3} f'(0)\). Since \((u_0,v_0) \neq (0,0)\) and \((0,0)\) is a solution, the comparison principle implies that \(u,v > 0, \ \forall t > 0\). Now, let us define \(\eta\) such that \(\eta \phi_R(x,|y| - M) < v(x,y,1)\) and \(\eta \|\phi_R\|_\infty \leq \varepsilon\). Define \(V(x,y) := \eta \phi_R(x,|y| - M)\), and \((0,V)\) is a subsolution of (2) which is strictly below \((u,v)\) at \(t = 1\). Let \((u_1,v_1)\) be the solution of (2) starting from \((0,V)\) at \(t = 1\); \((u_1,v_1)\) is strictly increasing in time, bounded by \(K\left(\frac{p}{\mu}, 1\right)\), and converges to a positive stationary solution \((U_1,V_1)\), satisfying

\[U_1 \leq \liminf_{t \to +\infty} u \quad \quad V_1 \leq \liminf_{t \to +\infty} v\]

locally uniformly in \((x,y) \in \mathbb{R}^2\).
It remains to show that \((U_1, V_1)\) is invariant in \(x\). For \(h \in \mathbb{R}\), let us denote \(\tau_h\) the translation by \(h\) in the \(x\)-direction: \(\tau_h w(x, y) = w(x + h, y)\). Since \(\overline{V}\) is compactly supported, there exists \(\varepsilon > 0\) such that

\[
\forall h \in (-\varepsilon, \varepsilon), \quad \tau_h \overline{V} < V_1 \text{ and } \tau_h \overline{V} < v \text{ at } t = 1.
\]

Thus, because of the \(x\)-invariance of the system \((2)\), the solution \((\tilde{u}_1, \tilde{v}_1)\) of \((2)\) starting from \((0, \tau_h \overline{V})\) at \(t = 1\) is equal to the translated \((\tau_h u_1, \tau_h v_1)\). So, \((\tilde{u}_1, \tilde{v}_1)\) converges to \((\tau_h U_1, \tau_h V_1)\). But, since \((\tilde{u}_1, \tilde{v}_1)\) is below \((U_1, V_1)\) at \(t = 1\) and \((U_1, V_1)\) is a (stationary) solution, from the comparison principle given by Proposition 1.2 we deduce \((\tilde{u}_1, \tilde{v}_1) < (U_1, V_1), \forall t > 1, \text{ and then}\)

\[
(\tau_h U_1, \tau_h V_1) \leq (U_1, V_1), \forall h \in (-\varepsilon, \varepsilon).
\]

Namely, \((U_1, V_1)\) does not depend on \(x\).

**Construction of \((U_2, V_2)\)** Let \(\overline{V} = \max(\|v_0\|, K)\) and \(\overline{U} = \max(\|u_0\|, \overline{V}^{\overline{\theta}})\). Let \((u_2, v_2)\) be the solution of \((2)\) with initial datum \((\overline{U}, \overline{V})\). From the comparison principle \((1.2), (u, v)\) is strictly below \((u_2, v_2)\), for all \(t > 0, (x, y) \in \mathbb{R}^2\). Moreover, since \((\overline{U}, \overline{V})\) is a supersolution of \((2)\), but not a solution, it is clear that \(\partial_t u_2, \partial_t v_2 \leq 0\) at \(t = 0\). Still using Proposition 1.2, it is true for all \(t \geq 0, \text{ and } u_2 \text{ and } v_2 \text{ are nonincreasing in } t, \text{ bounded from below by } (U_1, V_1)\). Thus, \((u_2, v_2)\) converges as \(t \to \infty\) to a stationary solution \((U_2, V_2)\) of \((2)\) satisfying

\[
\limsup_{t \to +\infty} u(x, t) \leq U_2 \quad \limsup_{t \to +\infty} v(x, y, t) \leq V_2(y),
\]

locally uniformly in \((x, y) \in \mathbb{R}^2\). From the construction of \((U_2, V_2)\), which is totally independent of the \(x\)-variable, it is easy to see that \((U_2, V_2)\) does not depend in \(x\). \(\square\)

**Uniqueness of the stationary solution** The previous proposition provides a theoretical proof of the existence of stationary solutions. It also means that a solution is either converging to a stationary solution, or will remain between two stationary solutions. In order to obtain a more precise description of the long time behaviour, we need the following uniqueness result.

**Proposition 1.4.** There is a unique positive, bounded, stationary solution of \((2)\), denoted \((U_s, V_s)\).

To prove the uniqueness, we first need the following intermediate lemma which is the key to all uniqueness properties in this kind of problem. The idea that a bound from below implies uniqueness appeared for the first time in [7].

**Lemma 1.1.** Let \((U, V)\) be a positive, bounded stationary solution of \((2)\). Then there exists \(m > 0\) such that

\[
\forall (x, y) \in \mathbb{R}^2, \quad U(x) \geq m, \quad V(x, y) \geq m.
\]

**Proof.** Let \((U, V)\) be such a stationary solution.

*First step:* there exists \(M > 0\) such that

\[
m_1 = \inf \{V(x, y), |y| > M\} > 0.
\]
Let $R > 0$ large enough in such a way that the first eigenvalue of the Laplace operator with Dirichlet boundary condition in $B_R \subset \mathbb{R}^2$ is less than $\frac{f'(0)}{2M}$, $\phi_R$ the associated eigenfunction. We extend $\phi_R$ to 0 outside $B_R$. $\phi_R$ is continuous, bounded in $\mathbb{R}^2$, positive in $B_R$. Now let us define $M$ such that $\forall y / |y| > M - R$, $\nu(y) \leq \frac{1}{3} f'(0)$. As we have already seen above, there exists $\varepsilon > 0$ such that $\forall K \geq M, \langle x, y \rangle \rightarrow \varepsilon \phi_R(x, |y| - K)$ is a subsolution of (7) which is strictly below $V$ on the circles $\{|y| = K = R\}$. Then, the elliptic maximum principle yields $V(x, y) \geq \varepsilon \phi_R(x, |y| - K)$, $\forall (x, y) \in \mathbb{R}^2$. In particular, we get

$$\forall x \in \mathbb{R}, \forall |y| > M, V(x, y) \geq m_1 \geq \varepsilon \phi_R(0, 0) > 0.$$  

Second step:

$$m_2 := \inf \{V(x, y), (x, y) \in \mathbb{R}^2\} > 0.$$  

If $m_2 = m_1$, the assumption is proved. It is obvious that $m_2 \geq 0$. Let us assume by way of contradiction that $m_2 = 0$. We consider $(x_n, y_n)$ such that $V(x_n, y_n) \rightarrow 0$ with $n \rightarrow \infty$. Now, we set

$$U_n := U(. + x_n), V_n := V(. + x_n, + y_n), \mu_n := \mu(. + y_n), \nu_n := \nu(. + y_n).$$

Using the fact that $U$ and $V$ are smooth and bounded, by standard elliptic estimates (see [15] for example), there exists $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing such that $(U_{\varphi(n)})_n, (V_{\varphi(n)})_n$ converge locally uniformly to some functions $\tilde{U}, \tilde{V}$ satisfying

$$\begin{align*}
-DU''(x) &= -\pi U(x) + \int \tilde{\nu}(y) \tilde{V}(x, y) dy \\
-d\Delta \tilde{V}(x, y) &= f(\tilde{V}) + \tilde{\mu}(y) \tilde{U}(x) - \tilde{\nu}(y) \tilde{V}(x, y)
\end{align*}$$

where $\tilde{\mu}, \tilde{\nu}$ are some translated of $\mu, \nu$. Furthermore, $\tilde{V} \geq 0$ and $\tilde{V}(0, 0) = 0$. Thus in a neighbourhood of $(0, 0)$ we have

$$-d\Delta \tilde{V}(x, y) + \tilde{\nu}(y) \tilde{V}(x, y) \geq 0, \ \min(\tilde{V}) = 0.$$  

From the strong elliptic maximum principle, we deduce $\tilde{V} \equiv 0$. But by step 1 $\tilde{V}(., 2M) \geq m_1 > 0$, and we get a contradiction. Hence the result stated above, $m_2 := \inf(V) > 0$.

Third step: $U$ is also bounded from below by a positive constant. Indeed, if we set $\phi(x) = \frac{1}{B} \int \nu(y) V(x, y) dy$, $U$ is solution of

$$-U'' + \frac{\bar{\nu}}{D} U = \phi,$$

with $\phi$ continuous and $\phi \geq m_2 \|\nu\|_{L^1}$. Using $\Phi(x) = \frac{D}{2\pi} \exp(-\sqrt{\frac{\pi}{D}} |x|)$ which is the fundamental solution of (8) we get

$$U(x) = \phi * \Phi(x) \geq \|\Phi\|_{L^1} m_2 \|\nu\|_{L^1} := m_3 > 0.$$  

Now, set $m = \inf(m_1, m_2, m_3)$ and the proof is concluded. \qed
Proof of proposition 1.4 It remains now to prove the uniqueness of the stationary solution of (2). The difficulties come from the fact that it is a coupled system in an unbounded domain: for bounded domains, uniqueness was proved in [2]. Let \((U_1, V_1), (U_2, V_2)\) be two bounded, positive solutions of (7), and let us show that \((U_1, V_1) = (U_2, V_2)\). From Lemma 1.1, there exists \(m > 0\) such that \((U_i, V_i) \geq m, \ i = 1..2\). Hence, for \(T\) large enough, \(T(U_1, V_1) > (U_2, V_2)\). Let

\[ T_1 = \inf\{T, \forall T' > T, \ T'(U_1, V_1) > (U_2, V_2)\} > 0, \]

and

\[(\delta U, \delta V) = T_1(U_1, V_1) - (U_2, V_2).\]

The couple \((\delta U, \delta V)\) satisfies the following system:

\[
\begin{aligned}
-D\delta U''(x) &= -\mu\delta U(x) + \int \nu(y)\delta V(x, y) \, dy \\
-d\Delta \delta V(x, y) &= T_1 f(V_1) - f(V_2) + \mu(y)\delta U(x) - \nu(y)\delta V(x, y)
\end{aligned}
\]

and \(\inf(\delta U) = 0\) or \(\inf(\delta V) = 0\). In order to show that \((\delta U, \delta V) \equiv 0\) we have to distinguish five cases.

Case 1: there exists \((x_0, y_0) \in \mathbb{R}^2, \ \delta V(x_0, y_0) = 0\). Then, using the fact that \(f(0) = 0\) and that \(f\) is strictly concave, we can easily check that \(T_1 f(V_1) - f(V_2) \geq 0\) in a neighbourhood of \((x_0, y_0)\). Thus, because \(\delta U \geq 0\), \(\delta V\) is solution of the inequality system

\[
\begin{aligned}
-d\Delta \delta V + \nu \delta V &\geq 0 \\
\delta V &\geq 0
\end{aligned}
\]

\(\delta V(x_0, y_0) = 0\).

From the elliptic maximum principle, we infer \(\delta V \equiv 0\). Because \(\mu \neq 0\), we immediately get \(\delta U \equiv 0\). So \((U_2, V_2) = T_1(U_1, V_1)\); subtracting the two systems (7) in \((U_1, V_1)\) and \(T_1(U_1, V_1)\) yields \(T_1 f(V_1) = f(V_2)\) and \(V_1 > 0\). So \(T_1 = 1\), and \((U_2, V_2) = (U_1, V_1)\).

Case 2: there exists \(x_0\) such that \(\delta U(x_0) = 0\). In the same way we infer \(\delta U \equiv 0\). Then, \(\forall x \in \mathbb{R}, \ \int \nu \delta V = 0\). In particular, there exists \(y_0\) such that \(\delta V(x_0, y_0) = 0\), and the problem is reduced to the (solved) first case: \(T_1 = 1\), and \((U_2, V_2) = (U_1, V_1)\).

Case 3: there is a contact point for \(U\) at infinite distance. Formally, there exists \((x_n), |x_n| \to \infty\) such that \(\delta U(x_n) \to 0\) with \(n \to \infty\). We set

\[ U_i^n := U_i(., + x_n), \ V_i^n := V_i(., + x_n, .), \ i = 1, 2. \]

In the same way as above, there exist \(\tilde{U}_i, \ \tilde{V}_i\) such that, up to a subsequence, \((U_i^n, V_i^n)\) converges locally uniformly to \((\tilde{U}_i, \ \tilde{V}_i)\), and the couples \((\tilde{U}_1, \tilde{V}_1)\) and \((\tilde{U}_2, \tilde{V}_2)\) both satisfy (7) and

\[
\begin{aligned}
T_1 &= \inf\{T, \forall T' > T, \ T'(\tilde{U}_1, \tilde{V}_1) > (\tilde{U}_2, \tilde{V}_2)\}, \\
(T_1 \tilde{U}_1 - \tilde{U}_2)(0) &= 0.
\end{aligned}
\]

The problem is once again reduced to the first case, and \(T_1 = 1\).

Case 4: there is a contact point for \(V\) at infinite distance in \(x\, finite distance in y\, say \(y_0\). We use the same trick as above, the limit problem is this time reduced to the second case, and we still get \(T_1 = 1\).

Case 5: there is a contact point for \(V\) at infinite distance in \(y\). That is to say there exist \((x_n), (y_n), \) with \(|y_n| \to \infty\) such that \(\delta V(x_n, y_n) \to 0\). Once again, we set

\[ V_i^n := V_i(., + x_n, . + y_n), \ i = 1, 2. \]
Now, considering that $U_1$, $U_2$ are bounded and that $\mu, \nu \to 0$, $(V^{n}_1)_n$, $(V^{n}_2)_n$ converge locally uniformly to some functions $\tilde{V}_1$, $\tilde{V}_2$ which satisfy

\[
\begin{cases}
-d\Delta \tilde{V}_i = f(\tilde{V}_i) \\
(T_1 \tilde{V}_1 - \tilde{V}_2)(0, 0) = 0
\end{cases}
\]

and $(T_1 \tilde{V}_1 - \tilde{V}_2) \geq 0$ in a neighbourhood of $(0, 0)$. Thus, using the concavity of $f$ as in the first case, we get $T_1 = 1$.

From the five cases considered above, whatever may happen, $T_1 = 1$. Symmetrically, set

$$T_2 = \inf\{T, \forall T' > T, T'(U_2, V_2) > (U_1, V_1)\} > 0.$$ 

It is quite obvious that the previous assertions are still true, and then $T_2 = T_1 = 1$. So, $(U_1, V_1) = (U_2, V_2)$, and the proof is complete.

The proof of Proposition 0.1 is now a consequence of Propositions 1.3 and 1.4.

2 Exponential solutions of the linearised system

Looking for supersolution of the system (2) lead us to search positive solutions of the linearised system (4), hence we are looking for solutions of the form:

$$\left( \begin{array}{c} u(x, t) \\ v(x, y, t) \end{array} \right) = e^{-\lambda(x-c t)} \left( \begin{array}{c} 1 \\ \phi(y) \end{array} \right),$$

where $\lambda, c$ are positive constants, and $\phi$ is a nonnegative function in $H^1(\mathbb{R})$. The system on $(\lambda, \phi)$ reads:

\[
\begin{cases}
-D\lambda^2 + \lambda c + \mu = \int \nu(y)\phi(y)dy \\
-d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y).
\end{cases}
\]

The first equation of (10) gives the graph of a function $\lambda \mapsto \Psi_1(\lambda, c) := -D\lambda^2 + \lambda c + \mu$, which, if (9) is a solution of (4), is equal to $\int \nu(y)\phi(y)dy$.

The second equation of (10) gives, under some assumptions on $\lambda$, a unique solution $\phi = \phi(y; \lambda, c)$ in $H^1(\mathbb{R})$. To this unique solution we associate the function $\Psi_2(\lambda, c) := \int \nu(y)\phi(y)dy$. Let us denote $\Gamma_1$ the graph of $\Psi_1$ in the $(\lambda, \Psi_1(\lambda))$ plane, and $\Gamma_2$ the graph of $\Psi_2$. So, (10) amounts to the investigation of $\lambda, c > 0$ such that $\Gamma_1$ and $\Gamma_2$ intersect.

The graph of $\lambda \mapsto \Psi_1(\lambda)$ is a parabola. As we are looking for a nonnegative function $\phi$, we are interested in the positive part of the graph. The function $\lambda \mapsto \Psi_1(\lambda)$ is nonnegative for $\lambda \in [\lambda^-_1(c), \lambda^+_1(c)]$, with $\lambda^+_1(c) = \frac{c+\sqrt{c^2+4D\mu}}{2D}$. It reaches its maximum value in $\lambda = \frac{c}{2D}$, with $\Psi_1(\frac{c}{2D}) = \mu + \frac{c^2}{4D} > \mu$.

We also have

$$\Psi_1(0) = \Psi_1(\frac{c}{D}) = \mu,$$

which will be quite important later.

It may be observed that: with $D$ fixed, $(\lambda^-_1(c), \lambda^+_1(c)) \xrightarrow{c \to +\infty} (0^-, +\infty)$; $\lambda \mapsto \Psi_1(\lambda)$ is strictly concave:

$$\left. \frac{d\Psi_1}{d\lambda} \right|_{\lambda = c/D} = -c.$$

We can summarize it in fig. (1).
2.1 Study of $\Psi_2$

The study of $\Psi_2$ relies on the investigation of the solution $\phi = \phi(\lambda; c)$ of

$$
\begin{align*}
-\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) &= \mu(y) \\
\phi &\in H^1(\mathbb{R}).
\end{align*}
$$

(12)

Since $\mu$ is continuous and decays no slower than an exponential, $\mu$ belongs to $L^2(\mathbb{R})$. Since $\nu$ is nonnegative and bounded, the Lax-Milgram theorem assures us that (12) admits a unique solution if $\lambda c - d\lambda^2 - f'(0) > 0$, that is to say if $\lambda$ belongs to $]\lambda^-_2(c), \lambda^+_2(c)[$, where

$$
\lambda^\pm_2(c) = \frac{c \mp \sqrt{c^2 - c_{KPP}^2}}{2d},
$$

with

$$
c_{KPP} = 2\sqrt{df'(0)}.
$$

As in [9], the KPP-asymptotic spreading speed will have a certain importance in the study of the spreading in our model. Moreover, since $\nu, \mu$ tend to 0 with $|y| \to \infty$, an easy computation will show that, for $\lambda < \lambda^-_2$ or $\lambda > \lambda^+_2$, equation (12) cannot have a constant sign solution. Thus,

$$
\Gamma_2 \text{ exists if and only if } c > c_{KPP}.
$$

(13)

The main properties of $\Psi_2$ are the following:

**Proposition 2.1.** If $c > c_{KPP}$, then:

1. $\lambda \mapsto \Psi_2(\lambda)$ defined on $]\lambda^-_2, \lambda^+_2[$ is positive, smooth, strictly convex and symmetric with respect to the line $\{\lambda = \frac{c}{2d}\}$. With $\lambda$ fixed we also have $\frac{d}{dc}\Psi_2(\lambda; c) < 0$.

2. $\Psi_2(\lambda) \xrightarrow[\lambda \to \lambda^+_2]{} \overline{\mu}$.

3. $\frac{d\Psi_2}{d\lambda}(\lambda) \xrightarrow[\lambda \to \lambda^-_2]{} -\infty$ with $\lambda > \lambda^-_2$.

The graph $\Gamma_2$ looks like fig. (2).

Proof of the first part of proposition (2.1)
Positivity, smoothness  

For all \( \lambda \in [\lambda^-_2, \lambda^+_2] \),

\[
P(\lambda) := \lambda c - d\lambda^2 - f'(0) > 0.
\]

Consequently, \( \forall \lambda \in [\lambda^-_2, \lambda^+_2], \forall y \in \mathbb{R}, P(\lambda) + \nu(y) > 0 \). From the elliptic maximum principle, as \( \mu \) is nonnegative, we deduce that \( \phi(y) > 0, \forall y \in \mathbb{R} \). Hence, since \( \nu \) is nonnegative, we have \( \Psi_2(\lambda) = \int \phi(y; \lambda)\nu(y)dy > 0 \), and \( \Psi_2 \) is positive.

Considering that \( \lambda \mapsto P(\lambda) \) is smooth, with the uniqueness of the solution and the implicit function theorem, we see immediately that \( \lambda \mapsto \phi(y; \lambda) \) is smooth, uniformly for all compact in \( y \).

Since \( \nu \) is integrable, \( \lambda \mapsto \Psi_2(\lambda) \) is smooth.

From the symmetry of \( \lambda \mapsto P(\lambda) \) and the uniqueness of the solution, we deduce the symmetry of \( \Gamma_2 \) with respect to the line \( \{\lambda = \frac{c^2}{2d}\} \).

Monotonicity, convexity  

Denote by \( \phi_\lambda \) the derivative of \( \phi \) with respect to \( \lambda \). Then, if we differentiate (12) with respect to \( \lambda \), we can see that \( \phi_\lambda \) satisfies:

\[
-d\phi''_\lambda(y) + (P(\lambda) + \nu(y))\phi_\lambda(y) = (2d\lambda - c)\phi(y).
\]

In the same way as equation (12), (15) has a unique solution in \( H^1(\mathbb{R}) \) for all \( \lambda \in [\lambda^-_2, \lambda^+_2] \). Since \( \phi \) is positive, \( \phi_\lambda \) is of constant sign, with the sign of \( (2d\lambda - c) \). Hence we have that \( \Psi_2 \) is decreasing on \( [\lambda^-_2, \frac{c^2}{2d}] \) and increasing on \( [\frac{c^2}{2d}, \lambda^+_2] \).

Differentiating once again (15) with respect to \( \lambda \), the second derivative of \( \phi \) with respect to \( \lambda \) satisfies:

\[
-d\phi'''_{\lambda\lambda}(y) + (P(\lambda) + \nu(y))\phi_{\lambda\lambda}(y) = 2d\phi(y) + 2(2d\lambda - c)\phi_\lambda(y).
\]

In the same way, \( \phi \) is positive for all \( \lambda \in [\lambda^-_2, \lambda^+_2] \), and \( \phi_\lambda(\lambda) \) has the positivity of \( (2d\lambda - c) \). Hence the left term of equation (16) is positive, for all \( \lambda \in [\lambda^-_2, \lambda^+_2] \), and \( \Psi_2 \) is strictly convex on \( [\lambda^-_2, \lambda^+_2] \).

With the same arguments we see that \( \phi_c \), the derivative of \( \phi \) with respect to \( c \), satisfies

\[
-d\phi''_c + (P(\lambda) + \nu)\phi_c = -\lambda \phi < 0,
\]

and then we get \( \int \phi_c(y)\nu(y)dy = \frac{d}{dc}\Psi_2(\lambda; c) < 0 \).
In order to end the proof of the proposition (2.1), we need to study the behaviour of $\Psi_2$ near $\lambda^2$. Setting $\varepsilon = P(\lambda)$, it is sufficient to study the behaviour of the solution $\phi = \phi(y; \varepsilon)$ of
\[
\begin{cases}
-\phi''(y) + (\varepsilon + \nu(y))\phi(y) = \mu(y) \\
\phi \in H^1(\mathbb{R}), \quad \varepsilon > 0, \quad \varepsilon \to 0.
\end{cases}
\tag{17}
\]
The main lemma here is the following, which will evidently conclude Proposition 2.1:

**Lemma 2.1.** 1. If $\phi$ is solution of (17) then $\int_\mathbb{R} \phi(y)\nu(y)dy \to \overline{\mu}$ holds true. Moreover, $\|\phi\|_{L^\infty}$ is uniformly bounded on $\varepsilon$.

2. The derivative of $\phi$ with respect to $\varepsilon$, denoted $\phi_\varepsilon$, satisfies $\int_\mathbb{R} \phi_\varepsilon(y)\nu(y)dy \to -\infty$.

**Proof of the first part of the Lemma 2.1** Under the assumptions on $\nu$ and $\mu$, there exist $\alpha, M, m_1 > 0$ such that:

- $\nu(y) \geq \alpha 1_{[-m_1, m_1]}, \forall y \in \mathbb{R}$ (because $\nu(0) > 0$, and $\nu$ is continuous);

- $\mu(y) \leq Me^{-a|y|}, \forall y \in \mathbb{R}$ (from the exponential decay of $\mu$).

Denoting $\psi = \psi(y; \varepsilon)$ the solution of
\[
-\psi'' + (\varepsilon + \alpha 1_{[-m_1, m_1]})\psi = Me^{-a|y|},
\tag{18}
\]
$\psi$ is a supersolution for (17) and
\[
\forall \varepsilon > 0, \forall y \in \mathbb{R}, \quad 0 < \phi(y; \varepsilon) \leq \psi(y; \varepsilon).
\tag{19}
\]
We have already seen that $\forall \varepsilon > 0$, $\int_\mathbb{R} \phi''(y; \varepsilon)dy = 0$. Consequently, the assumption $\int_\mathbb{R} \phi(y)\nu(y)dy \to \overline{\mu}$ is equivalent to $\varepsilon \int_\mathbb{R} \phi(y; \varepsilon)dy \to 0$. To conclude, it remains to compute the solution $\psi$ and to show that $\varepsilon \int_\mathbb{R} \psi(y; \varepsilon)dy \to 0$. But the solution of (18) can be explicitly computed, which gives that $\|\phi(\varepsilon)\|_{L^\infty(\mathbb{R})}$ is uniformly bounded on $\varepsilon$ and that there exists $C > 0$ such that for $\varepsilon > 0$ small and $y > m_1$,
\[
\psi(y; \varepsilon) < Ce^{-\sqrt{\varepsilon}y},
\]
so
\[
\int_\mathbb{R} \psi(y; \varepsilon)dy = O\left(\frac{1}{\sqrt{\varepsilon}}\right) \quad \text{as } \varepsilon \to 0
\]
and
\[
\varepsilon \int_\mathbb{R} \psi(y; \varepsilon)dy \to 0,
\] which concludes the proof of the first statement in Lemma 2.1. Notice that we also get that there exist two constant $C_1, C_2$ not depending on $\varepsilon$ such that for all $y$ in $\mathbb{R}$, $\psi(y; \varepsilon) \leq C_1 e^{-\sqrt{\varepsilon}|y|} + C_2 e^{-a|y|}$, that will be useful later.
Let us prove the second part of Lemma (2.1). In order to prove it, we will first deal with the study of the homogeneous limit differential equation.

**Lemma 2.2.** Let us consider the scalar homogeneous equation (20):

$$-\psi'' + \nu.\psi = 0. \quad (20)$$

Under the assumptions on \( \nu \), there exist \( \phi_1, \phi_2 \) satisfying

- \( \phi_1(x) \to 0 \) as \( x \to +\infty \), and, for \( x \) large enough, \( \phi_1(x) \geq 0 \);
- \( \exists C_1, C_2 > 0 \) such that \( C_1 x \leq \phi_2(x) \leq C_2 x \) when \( x \) goes to \( +\infty \) (notation: \( \phi_2(x) = \Omega(x) \));

such that

\[
\begin{aligned}
\psi_1 &:= 1 + \phi_1 \\
\psi_2 &:= \phi_2(1 + \phi_1)
\end{aligned}
\]

is a fundamental system of solutions of (20).

**Proof.** Construction of \( \phi_1 \): let \( \psi := 1 + \phi_1 \) be a solution of (20). Thus, \( \phi_1 \) must satisfy

$$-\phi_1'' + \nu + \nu.\phi_1 = 0. \quad (21)$$

Let us show that there exists a solution of (21) which is nonnegative for \( x \) large enough and tends to 0 as \( x \) goes to \( +\infty \). Let \( M \geq 0 \) such that \( \int_{M}^{\infty} \int_{x}^{\infty} \nu(y)dydx < 1 \) which is possible thanks to the assumption (6) on \( \nu \). From a classical Banach fixed point argument, there exists a unique positive solution \( \phi_1 \) in \( C([M, +\infty[) \) satisfying \( \phi_1(x) \to 0 \) as \( x \to +\infty \).

Moreover, without loss of generality, we can only consider the case \( M = 0 \).

Construction of \( \phi_2 \): we are looking for a second solution of (20) in the form \( \psi_2 = \phi_2.\psi_1 \). Integrating the equation we get for \( x \geq 0 \):

\[
\phi_2(x) = \int_{0}^{x} \frac{dy}{(1 + \phi_1(y))^{2}},
\]

and \( \psi_2 := \phi_2(1 + \phi_1) \) is a second solution of the homogeneous equation (20). Finally, considering that \( \phi_1(x) \to 0 \) with \( x \to +\infty \), we get the desired estimate for \( \phi_2 \).

Of course, we have a similar result for \( x \to -\infty \). This lemma first allows us to give a useful lower bound of \( \phi(y; \varepsilon) \) at the limit \( \varepsilon = 0 \).

**Corollary 1.** Let \( \phi = \phi(y; \varepsilon) \) be the solution of (17). There exists \( k > 0 \) such that, \( \forall y \in \mathbb{R}, \exists \varepsilon_y, \forall \varepsilon < \varepsilon_y, \phi(y; \varepsilon) \geq k \), and this uniformly on every compact set in \( y \).

**Proof.** Since \( \mu \not\equiv 0 \) there exists a nonnegative compactly supported function \( \mu_c \not\equiv 0 \) such that \( 0 \leq \mu_c \leq \mu \). Let us now consider the (unique) solution \( \phi = \phi(y; \varepsilon) \) of

\[
\begin{aligned}
-\phi''(y) + (\varepsilon + \nu(y))\phi(y) &= \mu_c(y) \\
\phi &\in H^1(\mathbb{R}), \varepsilon > 0.
\end{aligned}
\]

(22)

From the first part of Lemma 2.1, we know that \( \exists K > 0, \forall y \in \mathbb{R}, \forall \varepsilon > 0, 0 < \phi(y; \varepsilon) \leq \phi(y; \varepsilon) < K \). Let us recall that for fixed \( y \in \mathbb{R}, \phi(y; \varepsilon) \) is increasing with \( \varepsilon \to 0 \) and
bounded by $K$. Hence there exists a positive function $\phi_0$ such that $\phi(y; \varepsilon) \xrightarrow{\varepsilon \to 0} \phi_0(y)$. Moreover, from the uniform boundedness of $\phi(\varepsilon)$ and Ascoli’s theorem, the convergence is uniform for $\phi$ and $\phi'$ in every compact set. Thus, $\phi_0$ satisfies in the classical sense

$$\begin{aligned}
-\phi''_0(y) + \nu(y)\phi_0(y) &= \mu_c(y) \\
0 &< \phi_0 \leq K.
\end{aligned}$$

As $\mu_c$ is compactly supported, for $|y|$ large enough, let us say greater than $A > 0$, $\phi_0$ is a solution of (20), that is to say, in the positive semi-axis

$$\begin{aligned}
-\phi''_0(y) + \nu(y)\phi_0(y) &= 0, \quad y > A \\
0 &< \phi_0(y) \leq K < +\infty \quad y > A.
\end{aligned}$$

Thus, there exist $\alpha^+, \beta^+$ such that

$$\forall y > A, \quad \phi_0(y) = \alpha^+(1 + \phi_1(y)) + \beta^+\phi_2(y)(1 + \phi_1(y)),$$

where $\phi_1$ and $\phi_2$ are defined in Lemma 2.2. Now considering that $\phi_1(y) = o(1)$ and $\phi_2(y) = \Theta(y)$ in $y \to +\infty$, as $\phi_0$ is bounded, $\beta^+ = 0$. Then, as $\phi_0 > 0$, $\alpha^+ > 0$. We have a similar result for $y < -A$, with $\beta^- = 0$ and $\alpha^- > 0$. Finally, define

$$k = \frac{1}{2} \min(\alpha^-, \alpha^+, \min\{\phi_0(y), y \in [-A, A]\}) > 0$$

and the proof is concluded. \(\square\)

**Proof of the second part of Lemma 2.1** Differentiating equation (17) with respect to $\varepsilon$, we get for the derivative $\phi_\varepsilon$

$$-\phi''_\varepsilon(y; \varepsilon) + (\varepsilon + \nu(y))\phi_\varepsilon(y; \varepsilon) = -\phi(y; \varepsilon). \quad (23)$$

Since $\phi$ is positive, we get that $\phi_\varepsilon$ is negative. Let us denote

$$\varphi(y) = \varphi(y; \varepsilon) := -\phi_\varepsilon(y; \varepsilon) > 0.$$

We have previously seen (in the proof of the first part of Proposition 2.1) that $\forall y \in \mathbb{R}$, $\frac{d}{d\varepsilon}\varphi(y; \varepsilon) < 0$, i.e. $\varphi$ is increasing with $\varepsilon \to 0, \varepsilon > 0$. Our purpose is to show that in a neighbourhood of 0, $\inf(\varphi(\varepsilon)) \xrightarrow{\varepsilon \to 0} +\infty$. For all $\varepsilon > 0$, define the function $\overline{\varphi} = \overline{\varphi}(y; \varepsilon)$ as the unique solution of

$$\begin{aligned}
-\overline{\varphi}''(y; \varepsilon) + (\varepsilon + \nu(y))\overline{\varphi}(y; \varepsilon) &= \min(k, \varphi(y; \varepsilon)) \\
\overline{\varphi} &\in H^1(\mathbb{R}).
\end{aligned} \quad (24)$$

The function $\overline{\varphi}$ is obviously well-defined. By its definition, the elliptic maximum principle ensures us that $0 < \overline{\varphi} \leq \varphi$, $\forall y \in \mathbb{R}, \varepsilon > 0$. We have also to notice that uniformly on every compact set in $y$, $\min(k, \varphi(y; \varepsilon)) = k$ for $\varepsilon$ small enough (consequence of corollary 1). Assume by way of contradiction that

$$\left(\min_{y \in [-1, 1]}(\overline{\varphi}(y; \varepsilon))\right)_\varepsilon$$

is bounded. \(\square\)
Let us show that it is inconsistent with the fact that \( \varphi > 0 \), \( \forall \varepsilon > 0 \). As \( \min(k, \phi(y; \varepsilon)) \) is uniformly bounded, from Harnack inequalities (see [15], Theorem 8.17 and 8.18) we know that for all \( R > 0 \), there exist \( C_1 = C_1(R), C_2 = C_2(R) \), independent of \( \varepsilon \), such that for all \( \varepsilon > 0 \),

\[
\sup_{[-R,R]} \varphi \leq C_1 \inf_{[-R,R]} (\varphi + C_2).
\]

Combining this and hypothesis (25), we get that \((\varphi(y; \varepsilon))_{\varepsilon>0}\) is increasing with \( \varepsilon \to 0 \) and uniformly in every compact set in \( y \). Using the same argument as in the proof of Corollary 1, \((\varphi(\varepsilon))_{\varepsilon}\) converges locally uniformly to some function \( \varphi_0 \) which satisfies in the classical sense

\[
\begin{cases}
-\varphi_0''(y) + \nu(y)\varphi_0(y) = k \\
\varphi_0(y) \geq 0, \forall y \in \mathbb{R}.
\end{cases}
\]

So there exist \( \alpha, \beta \in \mathbb{R} \) such that \( \varphi_0 = \alpha(1 + \phi_1) + \beta \phi_2(1 + \phi_1) + \phi_s \), where \( \phi_1, \phi_2 \) are defined in Lemma 2.2 and \( \phi_s \) is a particular solution of (26). Thus, for \( x \geq 0 \),

\[
\phi_s(y) = -k (1 + \phi_1(y))(1 + \phi_1(0)) \left( \int_0^y \int_0^y \frac{1 + \phi_1(z)}{(1 + \phi_1(t))^2} dt dz \right).
\]

Now, recall that \( \phi_1 > 0, \phi_1(y) = o(y) \) as \( y \to +\infty \). So there exists \( \gamma > 0, \phi_s(y) \underset{y \to \infty}{\sim} -\gamma \cdot y^2 \). As a result, for \( y \to \infty \),

\[
\begin{cases}
\varphi_0(y) = -\gamma \cdot y^2 + o(y^2) \underset{y \to +\infty}{\longrightarrow} -\infty \\
\varphi_0 \geq 0, \forall y \in \mathbb{R},
\end{cases}
\]

which is obviously a contradiction. So the first hypothesis (25) is false, which gives, combined with the monotonicity in \( \varepsilon \),

\[
\min_{y \in [-1,1]} (\varphi(y; \varepsilon)) \underset{\varepsilon \to 0}{\longrightarrow} +\infty,
\]

and then, as \( \nu \) is continuous and \( \nu(0) > 0 \),

\[
\int_{\mathbb{R}} \phi_s(y; \varepsilon) \nu(y) dy \underset{\varepsilon \to 0}{\longrightarrow} -\infty,
\]

and the proof of the main Lemma 2.1 is complete. \( \square \)

2.2 Intersection of \( \Gamma_1 \) and \( \Gamma_2 \), supersolution

First case: \( D > 2d \). If \( D > 2d \), we have of course \( \frac{c}{D} < \frac{c}{2d}, \forall c \geq c_{KPP} \). Thus, for \( c \) close enough to \( c_{KPP} \), \( \Gamma_2 \) does not intersect the closed convex hull of \( \Gamma_1 \). But since

\[
\frac{c}{D} \underset{c \to +\infty}{\longrightarrow} +\infty \text{ and } \lambda_2^-(c) \underset{c \to +\infty}{\longrightarrow} 0^+,
\]

there exists

\[
c_s = c_s(D) > c_{KPP}
\]

such that \( \forall c > c_s, \Gamma_1 \) and \( \Gamma_2 \) intersect, and \( \forall c < c_s, \Gamma_2 \) does not intersect the closed convex hull of \( \Gamma_1 \). Moreover, the strict concavity of \( \Gamma_1 \) and the strict convexity of \( \Gamma_2 \) allow
us to assert that for \( c = c_* \), \( \Gamma_1 \) and \( \Gamma_2 \) are tangent on \( \lambda = \lambda(c_*) \) and for \( c > c_* \), \( c \) close to \( c_* \), \( \Gamma_1 \) and \( \Gamma_2 \) intersect twice, at \( \lambda(c)^+ \) and \( \lambda(c)^- \). The different situations are illustrated in fig. (3).

When \( c \) is such that \( \lambda^- \leq \frac{c}{D} \), i.e. \( c \geq \frac{D}{2\sqrt{dD} - d^2} \), there is only one solution for \( \lambda = \lambda(c) \).

\[
\Psi_1(\lambda), \Psi_2(\lambda)
\]

\[
\Psi_1(\lambda), \Psi_2(\lambda)
\]

\[
\Psi_1(\lambda), \Psi_2(\lambda)
\]

Figure 3: Case \( D > 2d \); \( c < c_* \) (left), \( c = c_* \) (middle), \( c > c_* \), close to (right)

**Second case:** \( D = 2d \). If \( D = 2d \), then the point \( (\frac{c}{D}, \mu) \) belongs to \( \Gamma_1 \). Therefore, for all \( c > c_{KPP} \), \( \Gamma_1 \) and \( \Gamma_2 \) intersect once at \( \lambda = \lambda(c) \). We set:

\[
c_*(2d) := c_{KPP}.
\]

**Third case:** \( D < 2d \). If \( D > 2d \), we have \( \frac{c}{D} < \frac{c}{2d} \). Then, \( \forall c > c_{KPP}, \lambda^+_2(c) < \frac{c}{D} \), \( \Gamma_2 \) is strictly below \( \Gamma_1 \), and every \( c > c_{KPP} \) gives a super-solution. We set again:

\[
c_*(D) := c_{KPP}.
\]

These two last cases are outlined in fig. (4). All of this concludes the proof of Proposition 0.2. Moreover, we can assert from geometrical considerations that

\[
\frac{c_*}{D} \leq \frac{c_* - \sqrt{c_*^2 - c_{KPP}^2}}{2d} \leq c_* + \sqrt{\frac{c_*^2 + 4D\mu}{2D}}.
\]

\[(27)\]

It was proved in [9] that (27) implies that

\[
\sqrt{4\mu^2 + f'(0)^2} - 2\mu \leq \liminf_{D \to +\infty} \frac{c_*^2}{D} \leq \limsup_{D \to +\infty} \frac{c_*^2}{D} \leq f'(0).
\]

### 3 Spreading

In order to prove that solutions spread at least at speed \( c_* \), we are looking for compactly supported general stationary subsolution in the moving framework at velocity \( c < c_* \), arbitrarily close to \( c_* \). We consider the linearised system penalised by \( \delta > 0 \) in the moving framework:
The main result is here the following:

**Proposition 3.1.** Let \( c^* = c_*(D) \) be as in the previous section. Then, for \( c < c^* \) close enough to \( c^* \), there exists \( \delta > 0 \) such that (28) admits a nonnegative, compactly supported, generalised stationary subsolution \((u,v) \neq (0,0)\).

As in the previous section, we will study separately the case \( D > 2d \), which is the most interesting, and the case \( D \leq 2d \).

### 3.1 Construction of subsolutions: \( D > 2d \)

In order to keep the notation as light as possible, we will use the notation \( \tilde{f}'(0) := f'(0) - \delta \) and \( \tilde{P}(\lambda) := -d\lambda^2 + c\lambda - \tilde{f}'(0) \), because all the results will perturb for small \( \delta > 0 \). We just have to keep in mind that \( \tilde{f}'(0) < f'(0) \) and \( \delta \ll 1 \), hence \( \tilde{P}(\lambda) \approx P(\lambda) \) and \( \tilde{P}(\lambda) - P(\lambda) \ll 1 \).  

Our method is to devise a stationary solution of (28) not in \( \mathbb{R}^2 \) anymore, but in the horizontal strip \( \Omega^L = \mathbb{R} \times (-L,L) \), with \( L > 0 \) large enough. Thus, we are solving

\[
\begin{aligned}
-\partial_t u - D \partial_{xx} u + c \partial_x u &= -\overline{\mu} u + \int_{(-L,L)} \nu(y) v(t,x,y) dy \quad &x \in \mathbb{R}, t > 0 \\
-\partial_t v - d \Delta v + c \partial_x v &= (\tilde{f}'(0) - \delta)v + \mu(y) u(t,x) - \nu(y) v(t,x,y) \quad & (x,y) \in \mathbb{R}^2, t > 0 \\
V(x,L) = V(x,-L) &= 0 \quad &x \in \mathbb{R}.
\end{aligned}
\]

(29)

In a similar fashion as in the previous section, we are looking for solutions of the form

\[
\begin{pmatrix}
U(x) \\
V(x,y)
\end{pmatrix} = e^{\lambda x} \begin{pmatrix} 1 \\
\varphi(y)
\end{pmatrix},
\]

where \( \varphi \) belongs to \( H^1_0(-L,L) \). The system on \((\lambda,\varphi)\) reads:

\[
\begin{aligned}
-D \lambda^2 + \lambda c + \overline{\mu} &= \int_{(-L,L)} \nu(y) \varphi(y) dy \\
-d \varphi''(y) + (\tilde{P}(\lambda) + \nu(y)) \varphi(y) &= \mu(y) \quad \varphi(-L) = \varphi(L) = 0.
\end{aligned}
\]

(31)
The first equation of (31) gives a function \( \lambda \mapsto \Psi_1(\lambda; c) = -D\lambda^2 + \lambda c + \overline{\mu} \). The second equation of (31) gives a unique solution \( \varphi = \varphi(y; \lambda; c; L) \in H^1_\lambda(L, -L, L) \). We associate this unique solution with the function \( \Psi_2(L; c) = \int_{(-L, L)} \nu(y)\varphi(y)dy \). A solution of the form (30) exists if and only if \( \Psi_1(\lambda; c) = \Psi_2(L; c) \) for some \( \lambda, c \), that is to say if and only if \( \Gamma_1 \) and \( \Gamma_2 \) intersect (with straightforward notations). In this section, the game is to make them intersect not with real but with complex \( \lambda \).

**Study of \( \Gamma_1 \)**  The function \( \lambda \mapsto \Psi_1 \) is exactly the same as in the search for supersolutions. In particular, it does not depend in \( \lambda \), and we get:

functions \( \Psi \) are both analytical in \( \lambda \) at this point, and \( \frac{d}{d\lambda}\Psi_1(\lambda), \frac{d}{d\lambda}\Psi_2(\lambda) \neq 0 \), for \( (c, \lambda) \) in a neighbourhood of \( (c_*, \lambda_*^L) \). Due to the implicit function theorem, there exist \( \lambda_1(c, \beta), \lambda_2^L(c, \beta) \) defined in a neighbourhood \( V_1 \) of \( (c_*, \beta_*^L) \), analytical in \( \beta \), such that

\[
\begin{cases}
\Psi_1(\lambda_1(c, \beta); c) = \beta \quad \forall (c, \beta) \in V_1 \\
\Psi_2(\lambda_2^L(c, \beta); c) = \beta \quad \forall (c, \beta) \in V_1.
\end{cases}
\]

Then, set

\[ h^L(c, \beta) = \lambda_2^L(c, \beta) - \lambda_1(c, \beta), \quad \text{for } (c, \beta) \in V_1, \]

and we get:

\[
\begin{cases}
\partial_\beta h^L(c_*, \beta_*^L) = 0. \\
\partial_\beta h^L(c_*, \beta_*^L) := 2a > 0. \\
\partial_c h^L(c_*, \beta_*^L) := -e < 0.
\end{cases}
\]
The first point is obvious. The second comes from the fact that $\Gamma_2^L$ is concave and $\Gamma_1$ has a positive curvature at any point. The third is obvious given the first equation of (31).

Now, because we are working in a vicinity of $(c_*^L, \beta_*^L)$, set:

$$\xi := c_*^L - c, \quad \tau = \beta - \beta_*^L.$$ 

Call $b := \partial_{c\beta} h^L(c_*^L, \beta_*^L)$. From (34) and (35), we can assert that there exists a neighborhood $V_2 \subseteq V_1$ of $(c_*^L, \beta_*^L)$, there exists $\eta = \eta(\tau, \xi)$ analytical in $\tau$ in $V_2 - (c_*^L, \beta_*^L)$, vanishing at $(0, 0)$ like $|\tau|^3 + \xi^2$, such that

$$(h^L(c, \beta) = 0, (c, \beta) \in V_2) \Leftrightarrow (a\tau^2 + b\xi\tau + e\xi = \eta(\tau, \xi)).$$

Recall that $a$ and $e$ are positive, so the discriminant $\Delta = (b\xi)^2 - 4ae\xi$ is negative for $\xi > 0$ small enough. The trinomial $a\tau^2 + b\xi\tau + e\xi$ has two roots $\tau_{\pm} = \frac{-b\xi \pm i\sqrt{4ae\xi - (b\xi)^2}}{2a}$. Then, from an adaptation of Rouché's theorem (see [9]), the right handside of (36) has two roots, still called $\tau_{\pm}$, satisfying $\tau_{\pm} = \pm i\sqrt{(e/a)\xi + O(\xi)}$. Reverting to the full notation, we can see that for $c$ strictly less than and close enough to $c_*^L$, there exist $\beta, \lambda \in \mathbb{C}$, $\varphi \in H^1_0((-L, L), \mathbb{C})$ satisfying (31). Since $\beta = \Psi_1(\lambda) = -D\lambda^2 + c\lambda + \overline{\mu}$ and $\beta$ has nonzero imaginary part, $\lambda$ has also nonzero imaginary part. We can therefore write $(\lambda, \beta) = (\lambda_1 + i\lambda_2, \beta_1 + i\beta_2)$ and:

$$
\begin{pmatrix} U \\ V \end{pmatrix} = e^{(\lambda_1 + i\lambda_2)x} \begin{pmatrix} 1 \\ \varphi_1(y) + i\varphi_2(y) \end{pmatrix}
$$

with

$$\begin{cases} 
\lambda_2, \beta_2 \neq 0 \\
\int \nu(y)\varphi_1(y)dy = \beta_1 = \beta_*^L + O(c_*^L - c) \\
\int \nu(y)\varphi_2(y)dy = \beta_2 = O(\sqrt{c_*^L - c}).
\end{cases}$$

Thus :

- $\Re(U) > 0$ on $(-\frac{\pi}{2\lambda_2}, \frac{\pi}{2\lambda_2})$ and vanishes at the ends ;

- $\Re(V) > 0 \Leftrightarrow \varphi_1 \cos(\lambda_2 x) > \varphi_2 \sin(\lambda_2 x)$.

The set where $\Re(V) > 0$ is periodic of period $\frac{2\pi}{\lambda_2}$ in the direction of the road. Its connected components intersecting the strip $\mathbb{R} \times (-L, L)$ are bounded. The function $\varphi_2$ is continuous in $c$, hence the functions $y \mapsto \varphi(y; c)$ are uniformly equicontinuous for $c$ near $c_*^L$. Since $\nu(0) > 0$ and $\int \nu \varphi_2 = O(\sqrt{c_*^L - c})$, we have $\varphi_2(0) = O(\sqrt{c_*^L - c})$, and we can make one of the connected components of $\{\Re(V) > 0\}$, denoted by $F$, satisfy the property that $\{(x, 0) \in \overline{F}\} \text{ is arbitrary close to } [-\frac{\pi}{2\lambda_2}, \frac{\pi}{2\lambda_2}]$. We can now define the following functions:

$$u(x) := \begin{cases} 
\max(\Re(U(x)), 0) & \text{if } |x| \leq \frac{\pi}{2\lambda_2} \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad v(x, y) := \begin{cases} 
\max(\Re(V(x, y)), 0) & \text{if } (x, y) \in \overline{F} \\
0 & \text{otherwise} \end{cases} \quad \text{(37)}$$

The choice of $F$ implies that $(u, v)$ is a subsolution of (28).
3.2 Subsolution: case \( D \leq 2d \)
Now assume that \( 0 \leq D \leq 2d \). In the previous section, we define \( c_\epsilon(D) = c_{KPP} = 2\sqrt{df'(0)} \). Let \( c \leq c_{KPP} \). Thus, \( 4df'(0) - c^2 > 0 \). Let \( \delta \) be such that \( 0 < 2\delta < \frac{4df'(0) - c^2}{4d} = f'(0) - \frac{c^2}{4d} \). With \( \omega = \frac{\sqrt{4d(f'(0) - 2\delta) - c^2}}{2d} \), we define
\[
\phi(x) = e^{\frac{c^2}{4d}x} \cos(\omega x) \mathbf{1}_{(-\frac{\pi}{\omega}, \frac{\pi}{\omega})}.
\]
The function \( \phi \) is continuous and satisfies
\[
-d\phi'' + c\phi = (f'(0) - 2\delta)\phi \quad \text{on} \quad (-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}).
\]
Then, let us choose \( R > 0 \) such that the first eigenvalue of \(-\partial_{yy} \) in \((-R, R)\) is equal to \( \frac{c}{2} - \alpha \), and \( \psi_R \) an associated nonnegative eigenfunction in \( H_0^1(-R, R) \), where \( 0 < \alpha < \delta \). The function \( \psi_R \) satisfies
\[
-d\psi_R'' = (\delta - \alpha)\psi_R \quad \text{in} \quad (-R, R), \quad \psi_R(y) > 0, \quad \forall |y| < R, \quad \psi_R(R) = \psi_R(-R) = 0.
\]
We extend \( \psi_R \) by 0 outside \((-R, R)\). Let \( M > 0 \) such that \( \forall |y| > M - R, \nu(y) \leq \alpha \), which is possible since \( \nu(y) \to 0 \) with \( y \to \pm \infty \). The function
\[
\mathbf{V}(x, y) := \phi(x)\psi_R(|y| - M)
\]
is a solution of
\[
\begin{cases}
-d\Delta \mathbf{V} + c\partial_x \mathbf{V} = (f'(0) - \delta)\mathbf{V} - \alpha \mathbf{V} \\
x \in (-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}), \quad |y| \in (M - R, M + R),
\end{cases}
\]
vanishing on the boundary. Hence, from the choice of \( M \) and \( \alpha \), \((0, \mathbf{V})\) is a nonnegative compactly supported subsolution of (28), non identically equal to \((0, 0)\); which concludes the proof of Proposition 3.1. The proof of the main Theorem 0.1 follows as in \([9]\).

4 The intermediate model (3)

Derivation of the semi-limit model  
Starting from the full model (2), we consider normal (i.e. integral) exchange from the field to the road but localised exchange from the road to the field. Formally, we define \( \mu_\epsilon = \frac{1}{\epsilon} \mu(\frac{y}{\epsilon}) \) and take the limit with \( \epsilon \to 0 \) of the system (38):

\[
\begin{cases}
\partial_t u - D\partial_{xx} u = -\nu u + \int \nu(y)v(t, x, y)dy & x \in \mathbb{R}, \quad t > 0 \\
\partial_t v - d\Delta v = f(v) + \mu_\epsilon(y)u(t, x) - \nu(v)v(t, x, y) & (x, y) \in \mathbb{R}^2, \quad t > 0.
\end{cases}
\]

(38)

There is no influence in the first equation (the dynamic on the road), which is the same in the limit system. Though the second equation in (38) tends to
\[
\partial_t v - d\Delta v = f(v) - \nu(v)v(t, x, y), \quad (x, y) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}, \quad t > 0.
\]

It remains to determine the limit condition between at the road. We may assume that for \( \epsilon = 0 \) \( v \) is still continuous at \( y = 0 \). Now set \( \xi = y/\epsilon \) and \( \tilde{v}(t, x, \xi) := v(t, x, y) \). The second equation in (38) becomes in the \((t, x, \xi)\)-variables
\[
\epsilon^2 (\partial_\xi \tilde{v} - d\partial_{xx} \tilde{v} - f(\tilde{v}) + \nu(\xi)\tilde{v}(t, x, \xi)) - d\partial_{\xi \xi} \tilde{v} = \epsilon \mu(\xi)u(t, x).
\]
Passing to the limit, it yields, in the \( y \)-variable:

\[-d \left( \partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-) \right) = \mu u(t, x).\]

Consequently, the formal limit system of (38) should be (3) presented in the Introduction, which is the system we will study from now. Our assumptions on \( \nu \) and \( f \) are the same as above. The investigation is similar to the one done for the model (2), and we will only develop the parts which differ.

**Comparison principle** Throughout this chapter, we will call a supersolution of (3) a couple \((\mu, \nu)\) satisfying, in the classical sense, the following system:

\[
\begin{cases}
\partial_t \mu - D \partial_{xx} \mu \geq v(x, 0, t) - \mu u + \nu(y)v(t, x, y) & x \in \mathbb{R}, \ t > 0 \\
\partial_t \nu - d \Delta \nu \geq f(v) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, \ t > 0 \\
v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, \ t > 0 \\
-d \{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} \geq \mu u(t, x) & x \in \mathbb{R}, \ t > 0,
\end{cases}
\tag{39}
\]

which is also continuous up to time 0. Similarly, we will call a subsolution of (3) a couple \((\underline{u}, \underline{v})\) satisfying (39) with the inverse inequalities (i.e. the \( \geq \) signs replaced by \( \leq \)). We now need a comparison principle in order to get monotonicity for solutions:

**Proposition 4.1.** Let \((\underline{u}, \underline{v})\) and \((\overline{u}, \overline{v})\) be respectively a subsolution bounded from above and a supersolution bounded from below of (3) satisfying \( \underline{u} \leq \overline{u} \) and \( \underline{v} \leq \overline{v} \) at \( t = 0 \). Then, either \( \underline{u} < \overline{u} \) and \( \underline{v} < \overline{v} \) for all \( t > 0 \), or there exists \( T > 0 \) such that \( (\underline{u}, \underline{v}) = (\overline{u}, \overline{v}) \), \( \forall t \leq T \).

We omit the proof.

**Long time behaviour and stationary solutions** We want to show that any (nonnegative) solution of (3) converges locally uniformly to a unique stationary solution \((U_s, V_s)\), which is bounded, positive, \( x \)-independent, and of course is solution of the stationary system of equations (40):

\[
\begin{cases}
-DU''(x) = -\overline{\mu}U + \int \nu(y)V(x, y) \\
d\Delta V(x, y) = f(V) - \nu(y)V(x, y) \\
V(x, 0^+) = V(x, 0^-) \\
-d \{\partial_y V(x, 0^+) - \partial_y V(x, 0^-)\} = \mu U(x).
\end{cases}
\tag{40}
\]

Proofs of Propositions 1.3 and 1.4 can be easily adapted to this new system. The only nontrivial point lies in the existence of an \( L^\infty \) a priori estimate. Set \( \lambda = \frac{\overline{\mu} - \overline{\nu}}{\mu} \). From conditions on the reaction term, there exists \( M_1 \) such that \( \forall s > M_1, \ f(s) < -\overline{\mu} s \cdot \lambda \). Now, set

\[M = \max(M_1, \frac{\overline{\mu}}{\mu} \|u_0\|_\infty, \|v_0\|_\infty)\]

and the couple \((\overline{U}, \overline{V})\) given by

\[\overline{V}(y) = M(1 + e^{-\lambda|y|}), \ \overline{U} = \frac{1}{\mu} \int \nu(y)\overline{V}(y)dy\]

is a supersolution of (40) which is above \((u_0, v_0)\).

The proof of the corresponding Proposition 0.1 follows easily.
Lemma 4.1. Let us show the following lemma, which will prove the well-posedness of (43):

\[
\begin{aligned}
\partial_t u - D\partial_{xx} u &= v(x,0,t) - \nu u + \nu(y)v(t,x,y) && x \in \mathbb{R}, \; t > 0 \\
\partial_t v - d\Delta v &= f'(0)v - \nu(y)v(t,x,y) && (x,y) \in \mathbb{R} \times \mathbb{R}^*, \; t > 0 \\
v(t,x,0^+) &= v(t,x,0^-), && x \in \mathbb{R}, \; t > 0 \\
-d\left\{ \partial_y v(t,x,0^+) - \partial_y v(t,x,0^-) \right\} &= \nu u(t,x) && x \in \mathbb{R}, \; t > 0,
\end{aligned}
\]

and these solutions will be looked for under the form

\[
\begin{pmatrix}
u(t,x) \\
v_1(t,x,y) \\
v_2(t,x,y)
\end{pmatrix} = e^{-\lambda(x-ct)}
\begin{pmatrix}1 \\
\phi_1(y) \\
\phi_2(y)
\end{pmatrix}
\]

for \( \lambda, c \) are positive constants and \( \phi \) is a nonnegative function in \( H^1(\mathbb{R}) \), with \( v = v_1, \phi = \phi_1 \) for \( y \geq 0 \) and \( v = v_2, \phi = \phi_2 \) for \( y \leq 0 \). The system in \( (\lambda, \phi) \) reads

\[
\begin{aligned}
-D \lambda^2 + \lambda c + \nu &= \int \nu(y)\phi(y)dy \\
-d\phi_1'(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi_1(y) &= 0 && y \geq 0. \\
-d\phi_2''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi_2(y) &= 0 && y \leq 0. \\
\phi_1(0) &= \phi_2(0) && \text{i.e. } \phi \text{ is continuous.} \\
-\phi_1'(0) + \phi_2'(0) &= \nu. 
\end{aligned}
\]

The study is exactly the same as in the previous chapter. The only point which deserves some explanation is the well-posedness of (43). For \( M > 0 \) let us consider \( \varphi_M \) the unique solution of

\[
\begin{aligned}
-d\varphi_M''(y) + (P(\lambda) + \nu(y))\varphi_M(y) &= 0 && y \in ]0, +\infty[ \\
\varphi_M(0) &= M \\
\varphi_M &\in H^1(\mathbb{R}^+).
\end{aligned}
\]

Let us show the following lemma, which will prove the well-posedness of (43):

**Lemma 4.1.**

1. \( M \mapsto \varphi_M'(0) \) is decreasing ;

2. \( \varphi_M'(0) \xrightarrow{M \to 0} 0 \);

3. \( \varphi_M'(0) \xrightarrow{M \to +\infty} -\infty \).

**Proof.** Let us consider \( M_1, M_2 \) with \( 0 < M_1 < M_2 \), \( \varphi_{M_1}, \varphi_{M_2} \) the associated solutions of (44). The elliptic maximum principle yields \( 0 < \varphi_{M_1}(y) < \varphi_{M_2}(y) \), \( \forall y \geq 0 \) and Hopf’s lemma gives \( 0 > \varphi_{M_1}'(0) > \varphi_{M_2}'(0) \), which proves the first point.

Then, if we integrate (44) we get

\[
\varphi_M'(0) = -\frac{1}{d} \int_0^\infty (P(\lambda) + \nu(y))\varphi_M(y)dy.
\]

Let us now consider \( \overline{\varphi}_M \) the (unique) solution of

\[
\begin{aligned}
-d\overline{\varphi}_M'(y) + P(\lambda)\overline{\varphi}_M(y) &= 0 && y \in ]0, +\infty[ \\
\overline{\varphi}_M(0) &= M \\
\overline{\varphi}_M &\in H^1(\mathbb{R}^+).
\end{aligned}
\]
$\varphi_M$ is a supersolution of (44). Thus, $\varphi_M(y) \geq \varphi_M(y)$, $\forall y \geq 0$. Moreover we have an explicit expression for $\varphi_M$: $\varphi_M(y) = M \exp(-\sqrt{\frac{P(\lambda)}{d}}y)$. Hence,

$$0 \leq -\varphi'_M(0) \leq \frac{M}{d} \int_0^\infty (P(\lambda) + \nu(y))e^{-\sqrt{\frac{P(\lambda)}{d}}y}dy$$

and

$$-\varphi'_M(0) \xrightarrow{M \to 0} 0$$

uniformly in $\lambda$, which proves the second point.

In the same way, the unique solution $\varphi_M$ of

$$\begin{cases}
-d\varphi''_M(y) + (P(\lambda) + ||\nu||_{\infty})\varphi_M(y) = 0 & y \in ]0, +\infty[ \\
\varphi_M(0) = M \\
\varphi_M \in H^1(\mathbb{R}^+).
\end{cases}$$

is a subsolution of (44), and $\varphi_M(y) \leq \varphi_M(y)$, $\forall y \geq 0$. Hence,

$$-\varphi'_M(0) \geq \frac{1}{d} \int_0^\infty (P(\lambda) + \nu(y))\varphi_M(y)dy$$

$$\geq \frac{M}{d} \int_0^\infty (P(\lambda) + \nu(y))e^{-\sqrt{\frac{P(\lambda)}{d}}y}dy$$

$$-\varphi'_M(0) \xrightarrow{M \to +\infty} +\infty$$

which concludes the proof of Lemma 4.1.

The corresponding Proposition 0.2 and Theorem 0.1 follows as in the previous part.
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