Time reversed states in barrier tunneling
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Abstract

Tunneling, though a physical reality, is shrouded in mystery. Wave packets cannot be con-
structed under the barrier and group velocity cannot be defined. The tunneling particle can
be observed on either sides of the barrier but its properties under the barrier has never been
probed due to several problems related to quantum measurement. We show that there are
ways to bypass these problems in mesoscopic systems and one can even derive an expression
for the quantum mechanical current under the barrier. A general scheme is developed to derive
this expression for any arbitrary system. One can use mesoscopic phenomenon to subject the
expression to several theoretical and experimental cross checks. For demonstration we consider
an ideal 1D quantum ring with Aharonov-Bohm flux \( \Phi \), connected to a reservoir. It gives clear
evidence that propagation occur under the barrier resulting in a current that can be measured
non-invasively and theoretically cross checked. Time reversed states play a role but there is
no evidence of violation of causality. The evanescent states are known to be largely stable
and robust against phase fluctuations making them a possible candidate for device applications and
so formalizing current under barrier is important.

1 Introduction

Evanescent modes are one of the puzzling features of quantum mechanics wherein a particle with
energy less than the barrier height can tunnel through the barrier and this has no classical counter-
part. Although mathematical analogies can be found with electromagnetic wave-packets, physical
interpretations pose a serious challenge. It is not possible to draw any classical correspondence
because one cannot construct a wave-packet with such evanescent states and so one cannot define
propagation of the particle under the barrier. This has lead to study several aspects of states under
the barrier and one such aspect is tunneling time. Since without a wave-packet one cannot define a
group velocity and therefore addressing the problem lead to several puzzles [1]. In a situation where
a proper theory or formalism is missing making sense of experiments is difficult [2]. One can address
the problem with the help of physical clocks like Larmor clock and Wigner delay time but that in-
variably appeals to semi-classical concepts like spin precision and stationary phase approximations
[3]. Feynman path approach has also been applied [4] and different approaches give different results.
Hartman effect like phenomenon [5] show that a tunneling particle can come out of the barrier before
entering it which raises questions like if at all there is propagation under the barrier. We describe
here a situation wherein one can make a theoretical experiment to confirm current due to evanescent
modes and hence conclude propagation under the barrier. That will also be a testing ground for the
validity of physical clock like the Larmor clock. Of course one way to validate Larmor clock is to
appeal to Berger’s circuit and analyticity [6]. There is no harm in making an alternate validation by
showing that current under the barrier as a measurable quantity can be established from analyticity
of scattering matrix elements even if one can not prove its existence from quantum mechanics alone.
For this we will use the set up described in the next section. For the complexity of physical inter-
pretations we limit ourselves to the simplest system of 1D rings. The sample in the form of a 1D
quantum ring is not a loss of generality. Such 1D rings can be achieved experimentally using lateral
confinement and materials for which effective mass approximation works [7]. Besides, as shown in
the book by S. Datta [8], one can make finite thickness rings that consist of many independent 1D
channels. A finite thickness lead and ring can have multiple channels but DOS per channel is just
the 1D DOS that reflects in conductance measurements [7].
2 The set up

We will first provide a simple description of the mesoscopic set up and then discuss how this set up helps us address the problems discussed in the previous section. The mesoscopic sample is taken to be a ring pierced by a magnetic flux $\Phi$ through the center of the ring such that there is no magnetic field on the electrons confined to the ring. The ring is usually made up of gold or copper or semiconductors that can accommodate an electron gas. We are interested in the equilibrium response of this sample which in this case is response to a magnetic field. There is a reservoir shown as a 3D block which is a source of electrons at chemical potential $\mu$ and temperature $T$, making it a grand canonical system. The reservoir injects electrons to the sample or to the ring through a lead shown as region A. The ring can thus exchange electrons with the reservoir through the lead. This exchange does not result in a net current in the lead because electrons that go into the ring also come out of it and escape to the classical reservoir which is very typical of a voltage probe. The system therefore constitutes an equilibrium system where the reservoir also acts as a source of decoherence according to the Landauer-Buttiker formalism.

The electrons in the lead and inside the ring are purely described by quantum mechanics. If the ring is isolated then it will have some eigen functions and energy eigen states that can be obtained from Schrodinger equation. But once connected to the reservoir, the states in the ring will be affected. They will no longer be eigen-states of the Hamiltonian but the eigen-states will acquire some broadening due to life time related effects as the state can now leak into the reservoir. The states in the lead is unaffected by the ring because it is an ideal 1D system with a typical DOS given by $\frac{2\pi n}{h^2k}$. A part of the ring (region C in Fig. 1) has a potential $U$ that is shown as a thickened line. $U$ can be so adjusted that an electron can tunnel through this region C. This region C can be extended to the entire ring wherein the entire ring can be made into a tunneling region. We know the magnetic flux can drive an equilibrium persistent current [6] in the ring and this current can thus be a tunneling current in part of the ring or in the entire ring. This allows us to extend the

Fig. 1 The set up we want to consider in this work is shown which is a typical mesoscopic grand canonical system in equilibrium and described in detail in section 2.
theoretical consistency of quantum currents to the tunneling regime. We have shown in Eq. 13 that one has to follow a different approach under the barrier using analytic continuity that does not lead to a straightforward probabilistic interpretation of quantum mechanical currents. When the barrier is located only in a part of the ring then the persistent current in the ring can flow in a quantum state that is partly in the propagating regime (for example in the region B in Fig. 1) and partly in the evanescent regime (for example in the region C in Fig. 1). Consistency between them has to conserve the current at the point of propagating to evanescent crossover at the junction between regions B and C, there being no puzzle about the current in region B.

The primary advantage of theoretically studying such mesoscopic tunneling currents is that there are alternate theoretical formalisms in mesoscopic regime to verify the existence of currents inside the system under the barrier from propagating asymptotic states far from the barrier. That means there are theoretical cross checks for propagation under the barrier without studying the states under the barrier. Thus observations and calculation on current under the barrier can help us support or eliminate ideas. In short there are asymptotic states for the current carrying states in the ring and one can determine the current in the ring from these asymptotic states, that is again similar to the theory of the Larmor clock. Note that the derivation by Larmor clock giving the LHS of Eq. 18 does not need Schrodinger equation in particular. Now to calculate a scattering matrix element we may need an equation of motion but the derivation of LHS of Eq. 18 is insensitive to details like whether one is allowed to analytically continue the wave-function above the barrier or one should have a separate equation of motion for evanescent states. Rather one may interpret that since all that Eq. 16 and LHS of Eq. 18 cares is analyticity of scattering matrix elements, it should be legitimate to analytically continue wave-functions inside the system as long as it does not destroy the analyticity of scattering matrix elements. It has been claimed that the Larmor clock is more fundamental than Schrodinger equation as it gives a lot of new measurable quantities like the hierarchy of DOS that Schrodinger equation does not. So if there are doubts about what one gets from Schrodinger equation, we may expect to get confirmation from the theory of Larmor clocks. In the limit of a closed system it is also consistent with what one gets from Schrodinger equation. One can make the entire ring to be in the evanescent regime and the currents due to evanescent states has to be consistent with the asymptotic theory. A cartoon observer is shown in region A or lead of Fig. 1 and another cartoon observer is shown inside the ring. Observer A need not have any idea about the potential inside the ring. This observer does not know if the tunneling region extend over the entire ring or only a part of the ring. This observer only needs to know the infinitesimal change of $dU = \epsilon$ without any knowledge of $U$. He can measure the scattering phase shift (a theoretical measurement using analysis) in the wave-function in the lead and from there infer the current inside the ring from the analyticity of the scattering matrix elements. Observer A can vary incident energy and check analyticity of scattering matrix elements from Cauchy-Riemann conditions or similar criterion. The observer B in the ring can extend or shrink the tunneling region and also determine the tunneling current from the analytic continuation of the internal wave-function which is an unsettled issue in quantum mechanics. If the two measurements agree then the measurement by observer in the ring has to be correct.

In terms of practical measurements, there are problems associated with measuring tunneling currents in quantum mechanics and whatever we know about quantum measurements through an entanglement of sample states with the states of the detector do not apply for evanescent modes. One can also not measure tunneling currents classically as that will require the detector to be placed under the barrier and classical detectors can not work under the barrier. These problems can be avoided in the above described mesoscopic set up. Because such currents can also cause magnetization that can be observed in a practical experiment for further validation. This magnetization can be measured without disturbing the state in the system and thus not invoking the unresolved issues.
of quantum measurement. In mesoscopic systems electrons are quantum mechanical but fields are
classical. We can measure the magnetization due to the currents classically using hall magnetometers
or squids remotely. This is why Fig. 1 shows a detector to measure magnetization, placed above the
ring. Thus currents due to evanescent modes inside a ring can be measured non-invasively without
encountering the above mentioned problems of quantum measurement. So it makes a lot of sense to
study persistent currents due to evanescent modes inside the ring in the set up of Fig. 1.

3 Theoretical treatment

In Fig. 2 we redraw the system in Fig. 1 in order to depict the parameters used and the wave-
functions in the different regions for which the details can be seen in the figure caption. Here the
reservoir is shown as an irregular thick line on the left. Region A of Fig. 1 is called region O and
the three regions in the ring are labeled I, II and III. The Schrodinger equation in 1D is

$$\frac{-\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + U(x) \psi(x) = E \psi(x)$$

The solutions of Schrodinger equation in regions O (or lead), I, II and III of Fig. 2 give the wave
functions in absence of magnetic field as

$$\psi_0 = \frac{1}{\sqrt{k}} (e^{ikx} + Re^{-ikx})$$  \hspace{1cm} (1)

$$\psi_I = \frac{1}{\sqrt{k}} (ae^{ikx_1} + be^{-ikx_1})$$  \hspace{1cm} (2)
\[
\psi_{II} = \frac{1}{\sqrt{q}}(ce^{iqx} + de^{-iqx}) 
\]
(3)

\[
\psi_{III} = \frac{1}{\sqrt{k}}(fe^{ikx} + ge^{-ikx}) 
\]
(4)

Here

\[
q = \left(\frac{2m}{\hbar^2}(E - U)\right)^{1/2} \quad \text{and} \quad k = \left(\frac{2m}{\hbar^2}E\right)^{1/2} 
\]
(5)

It can be shown [6] that the magnetic field can be included through the boundary conditions and the magnetic field results in a current in the ring.

The objective of this study is to consider the situation when \(E < U\) in Eq. 5 and 3. Then the persistent current in region II is carried by evanescent modes. In this situation \(q \rightarrow is\) which is known as analytical continuation and the expression for the current can be derived using the following scheme. Although, such an analytic continuation completely destroys the idea of a propagating wavepacket as a Hilbert space element, an analytic continuation will always satisfy the Schrodinger equation. First of all we substitute \(q \rightarrow is\) where \(s = \left(\frac{2m}{\hbar^2}(U - E)\right)^{1/2}\) in Eq. 3 and 5. \(\psi_0, \psi_I, \psi_{III}\) remain unchanged while \(\psi_{II}\) becomes

\[
\psi_{II} = \frac{1}{\sqrt{s}}(ce^{(is)x} + de^{-i(is)x}) 
\]
(6)

Single valuedness of wavefunction implies

\[
\psi_0\big|_{x=0^-} = \psi_I\big|_{x_1=0^+} = \psi_{III}\big|_{x_3=l_3^-} \\
\psi_I\big|_{x_1=l_1^-} = \psi_{II}\big|_{x_2=0^+} \\
\psi_{II}\big|_{x_2=l_2^-} = \psi_{III}\big|_{x_3=0^+} 
\]
(7)

Also from current conservation or Kirchoff’s law in one dimension one can write at the junctions \(x = 0, x_1 = l_1\) and \(x_2 = l_2\) (see Fig. 2)

\[
\sum_i \frac{d\psi_i}{dx_i} = 0 
\]
(8)

Quantum mechanical current is given by the expression

\[
J(k, \Phi) = \frac{e\hbar}{2mi}\left[\psi^*\nabla\psi - HC\right] 
\]
(9)

At the junction of regions I and II, current conservation implies that the sum of currents arriving and leaving the points given by \(x_1 = l_1^-\) and \(x_2 = 0^+\) is zero. Thus Eq. 9 implies (ignoring the \(\frac{e\hbar}{2mi}\) factor that can be added later)

\[
\left[\psi_I^*\nabla\psi_I - HC\right]_{x_1=l_1^-} - \left[\psi_{II}^*\nabla\psi_{II} - HC\right]_{x_2=0^+} = 0 
\]
(10)

From Eqs. 2 and 6

\[
\frac{1}{k}\left[\left(ac^{i(kx_1+\Phi)} + be^{-ikx_1}\right) - \left(ac^{i(kx_1+\Phi)} + be^{-ikx_1}\right)\right]_{x_1=l_1^-} = 0 
\]
\[- \frac{1}{8} \left( e^{i(1)} x_2 + de^{-i(1)} x_2 - i \Phi' \right)^* \frac{d}{dx_2} \left( e^{i(1)} x_2 + de^{-i(1)} x_2 - i \Phi' \right) - HC \right] \bigg|_{x_2=0^+} = 0 \quad (11)\]

Here \( \Phi' = \frac{2 \pi \Phi}{\Phi_0} \) and the fact that only some terms acquire this phase factor to account for the magnetic field through the boundary conditions has been explained in detail using Feynman path approach in Ref. [6]. From Eq. 11 it is clear that the first bracketed term give the current in region I and the second bracketed term give the current in region II. Simplifying the second bracket along with its Hermitian conjugate and reintroducing the \( \frac{\hbar}{2m} \) factor we get

\[- \frac{e}{2m} \left[ |c|^2 - c^* d e^{-i \Phi'} + d^* e^{i \Phi'} - |d|^2 - HC \right] \quad (12)\]

Subtracting the Hermitian conjugate we get current density expression in the region II as

\[ J(k, \Phi) = \frac{e}{m} \left[ d c^* e^{-i \Phi'} - c d^* e^{i \Phi'} \right] \quad (13)\]

Differential current in an energy interval \( dE \) will be

\[ dJ(k, \Phi) = \frac{e \hbar}{m} \left[ d c^* e^{-i \Phi'} - c d^* e^{i \Phi'} \right] \frac{dn}{dE} dE \]

\[ = \frac{e}{\hbar i} \left[ d c^* e^{-i \Phi'} - c d^* e^{i \Phi'} \right] dE \quad (14)\]

Similarly, current in region I where there is no potential, will be

\[ dJ(k, \Phi) = \frac{e}{\hbar} \left[ |a|^2 - |b|^2 \right] dE \quad (15)\]

Thus if \( E > U \) then the current in the regions I, II and III will be given by \( \frac{e}{\hbar i} (|a|^2 - |b|^2) \), \( \frac{e}{\hbar i} (|c|^2 - |d|^2) \) and \( \frac{e}{\hbar i} (|f|^2 - |g|^2) \), respectively, and this follows from the standard definition of quantum mechanical current even in the presence of flux \( \Phi \). Continuity of current implies that they should be all equal and that obviously comes out in our calculations. In absence of flux a, c and f are the amplitude of clockwise moving electrons while b, d and g are the amplitudes of anticlockwise electrons and they can only differ by a relative phase because none of them is preferred over the other. Flux breaks this symmetry so that \( (|a|^2 - |b|^2) \) is the difference between the number of electrons moving clockwise and those moving anticlockwise in region I and that constitutes a current. This means that when \( \Phi \) is zero then there is no current in the system as is expected for an equilibrium situation. However for \( \Phi \neq 0 \) the current is finite. It is an equilibrium current called persistent current, purely quantum mechanical in origin [6]. Eq. 14 is far more complicated to be interpreted physically. Simple decoupling in terms of clockwise moving electron probability and anticlockwise moving electron probability is not possible. Time reversed amplitudes \( c^* \) and \( d^* \) appear in the expression. Tunneling under the barrier has long been studied as an avenue for superluminality but conclusive results do not exist as discussed in the introduction. Current expression with current conservation conclusively show that time reversed states have a role to play in the propagation.

This current is therefore flowing in the ring and magnetizing the ring. This magnetization can be measured by the detector on top and hence can be also experimentally verified. Without resorting to a practical experiment, this current expression for evanescent modes can be subjected to several tests. First is that it satisfies current continuity at the junction of regions I and II (see Fig. 2). Secondly an experimentalist can non-invasively measure and verify the numerical values of the
current. As a third verification consider theoretically reducing the regions I and III to zero lengths, in which case the current and magnetization will be entirely due to evanescent modes. Regions I and III being reduced to zero any magnetization observed has to be purely due to current carried by evanescent modes. Such a magnetization of an entire ring carrying current due to tunneling will prove propagation under the barrier. Without waiting for an experimentalist to measure and verify this there can be an independent theoretical verification which works only in the mesoscopic regime as has been elaborated in section 2. An observer in the lead can remotely calculate this current without even knowing if the regions I and III are reduced to zero length or they have a finite length. For that we have a novel expression for current derived in [P. Mello, 1993] and [E. Akkermans et al, 1991] and stated below.

\[ dJ(E, \Phi) = \frac{c}{\pi} \frac{\partial \theta_R}{\partial \Phi} dE = \frac{e}{\hbar} \frac{\partial \theta_R}{\partial \Phi'} dE \]  

(16)

Here \( dJ(E, \Phi) \) is the differential current in an energy interval \( dE \) at a fixed flux \( \Phi \) and \( \Phi' = \frac{2\pi \Phi}{\Phi_0} \).

The reflection phase shift is defined as \( \theta_R = \text{Arctan} \left( \frac{\text{Im}(R)}{\text{Real}(R)} \right) \). The scattering phase shift \( \theta_R(E, \Phi) \) is therefore from the asymptotic wave-function in the lead far away from the ring. Observable current will be

\[ I = \int_{k_f}^{k_i} dJ(k, \Phi) \]  

(17)

Since our concern is what happens to tunneling electrons that seems to come out of the barrier before entering it, it makes sense to ask if there is density of states (DOS) under the barrier as well. Again for mesoscopic systems this too can be determined from internal wave-functions as well as asymptotic wave-function. Note that the internal wave-function in tunneling regime is not consistent with the axioms of quantum mechanics because they cannot form a wave packet, they are not elements of Hilbert space, they are at most analytic continuation of wave-functions above the barrier. As usual, the asymptotic wave-function does not care if there is a state inside the ring or not and so if it gives the same value as that determined from analytic continuation of wave-function inside the ring, it can only mean two things. First, tunneling electrons do propagate under the barrier and secondly the asymptotic formalism settles a theoretical problem in the sense that analytical continuation implies physical continuation. So this is a theoretical experiment in the sense that what cannot be concluded from axioms of quantum mechanics can be concluded from the idea of a physical clock like Larmor clock. Of course currents are a better candidate as they can also be measured in a practical experiment and they do support this point of view as will be plotted and shown below. Never the less it does not harm to get further confirmation from DOS because after all DOS is a far more valuable quantity than just the currents. It is linked to all thermodynamic properties and not just currents. Secondly current conservation ensures certain features that can be settled purely at a level of numbers without caring for a theory. DOS cannot be directly measured like one can measure current from magnetization, but is a much more complex concept connected to normalization, renormalization, regularization etc, and if there is a scope to check purely theoretical consistency with the Larmor clock, that can be exploited by checking the following Eq. 18. So now we have the DOS also leading to a current and current can be practically measured and so in our opinion worth studying in future very specifically. Currents will peak where DOS peaks and quantitative verification can be made. So we expect

\[ \frac{d\theta_R}{cdU} = \pi \rho_u^2 = \frac{m}{2\hbar^2 s} \left( |c|^2 \right) \left( 1 - e^{-2sl_2} + \frac{|d|^2}{2s} \left( e^{2sl_2} - 1 \right) + (cd^*l_2e^{i\Phi'} + dc^*l_2e^{-i\Phi'}) \right) \]  

(18)

Here \( \rho_u^2 \) is the DOS in region II under the barrier and hence the superscript \( u \), LHS giving the expression that can be obtained from asymptotic wave-function, while RHS is that obtained by
integrating the LDOS from the analytic continuation of internal wave-function. Below we will numerically check and verify this equality to authenticate that the analytic continuation of internal wave-function in region II is connected with the probability of tunneling particles. It is to be noted that the LHS, that is $\frac{d\theta}{R_{ed}U}$ as an expression of DOS do not use the tunneling wave-functions in any way and is a derivation that is independent of whether the states are tunneling or propagating in region II. It only uses the analyticity of the scattering matrix. The RHS expression consisting of 3 terms in first brackets, are completely different for evanescent modes than that for propagating modes and has to be found by integrating the absolute value of wave-function in region II with appropriate normalization constant. Interestingly, the RHS of Eq. 18 giving DOS under the barrier has some negative terms signifying there are partial states under the barrier for which counting in real numbers (measure) can yield a negative answer. But total DOS or $\rho_u^d$ is positive definite because $s = \left(\frac{2m}{\hbar^2}\left(U - E\right)\right)^{1/2}$ is positive. As a result $\left|e^{2s\ell^2} - 1\right| \geq 0$ and $\left|1 - e^{-2s\ell^2}\right| \geq 0$. Although the negative terms do not dominate over the positive terms, they do signify presence of processes back in time. However, there is no evidence that such processes can carry signal or information because for that we need propagating wave-packets.

One may further say that there are states under the barrier given by Eq. 18 consisting of three terms in first brackets. But all three terms are not responsible for carrying current. Only the last term carries current as Eq. 14 confirms this. An expression like $e^{2s\ell^2}$ on RHS can grow indefinitely as $\ell^2$ is increased and to balance that the current carrying part of DOS that is $(cd^*\ell_3 e^{i\Phi'} + dc^*\ell_2 e^{-i\Phi'})$ become fewer and fewer. Current propagates only through these partial states. So if these partial states become fewer and fewer, then the electron takes lesser and lesser time to traverse the barrier. This shows up in the Hartman effect but as these states do not constitute a propagating wave-packet they cannot transmit a signal.

4 Results and discussions

We therefore proceed to demonstrate that Eq. 14 give the same current as Eq. 16 and that the equality claimed in Eq. 18 is true. One can set up the boundary conditions in Eqs. 7 and 8 that lead to a set of simultaneous equations, that can be solved to obtain $R, a, b, c, d, f,$ and $g$ in terms of known parameters like $\ell_1, \ell_2, \ell_3$, incident wave vector $k$ and applied magnetic field through the parameters $\alpha, \beta$ and $\gamma$. In Fig. 3 we plot the current density versus $kL$ which by current conservation has to be the same throughout the ring. For evanescent regimes we use Eq. 14 and for propagating Eq. 15, respectively. Three sets of parameter values has been used and mentioned inside the figure. Note that region II has a positive constant potential of $eUL = 49$. That means for $k$ values up to $kL = 7.0$ the region II carry only evanescent states and so a vertical dotted line has been drawn at $kL = 7.0$ as a guide to the eye. The solid curve correspond to a situation wherein $l_1 = l_2 = 0$ implying the entire ring has a positive potential $eUL = 49$. Which means for $kL$ values up to $kL = 7$, the entire ring carries evanescent modes only. The inset to Fig. 3 accentuates this range from $kL = 0$ to $kL = 7.0$. Above $kL = 7$ the states become propagating. The dashed curve and dash-dotted curve correspond to situations when the states in the ring are partially propagating and partially evanescent, wherein regions I and III are propagating while region II (see Fig. 2) will have evanescent modes up to $kL = 7$. Above $kL = 7$ the entire ring will carry propagating modes only. This current density is identically the same as calculated from the alternate formalism of Eq. 16 for all these situations, which means also for the situation of our interest that is the solid curve of Fig. 3 which confirms what we wanted to show. Our expression for current in Eq. 14 must be correct as it agrees with Eq. 15 which is a formalism that does not require us to treat the evanescent modes in any special way.
Fig. 3 Plot of current density in region I as a function of wave vector $kL$ as obtained from Eq. 14 for three different systems corresponding to different lengths of regions I, II and III for the system shown in Fig. 2. Current conservation implies that the current in region II and III will be the same. Current in region II has to be obtained from Eq. 13 and it confirms this. The vertical dotted line at $kL=7$ give the crossover point for the parameters used in this figure wherein modes in region II change from evanescent to propagating. We have taken $\hbar=1$ and $2m=1$.

Another interesting feature to be noted from Fig. 3 is that as the region II of Fig. 2, or the region C of Fig. 1 is enlarged by making $l_1$ and $l_3$ shrink to zero keeping total length $L$ unchanged, then for energy $E < eU$ current in the ring make a gradual shift from paramagnetic to diamagnetic. This is shown in the inset to Fig. 3. When the length of region II is $l_2 = 0.5L$, then the current in the ring change from diamagnetic to paramagnetic at $kL = 6.68$ (dash-dotted curve). When $l_2$ is increased to $0.7L$ then this crossover happens at $kL = 7.47$ (dashed curve). For the solid curve, this cross over occur at $kL = 7.67$. This diamagnetic to paramagnetic oscillation in current is a property of propagating modes only, and when the entire ring is in the tunneling regime then under the barrier current can only be diamagnetic. This has to do with the fact that propagating modes have parity effect wherein alternate states alter in parity. Even states have an even parity with an even number of nodes while odd states have odd parity with odd number of nodes. Evanescent mode wave-function is not wave like in nature at all. They have no nodes and so also no parity. One may ask the question if there is an observable diamagnetic shift of the currents due to a partial or complete tunneling region inside the ring. Observable current given by Eq. 17 will be the area...
Fig. 4 Plot of measurable current in region I as a function of wave vector $kL$ as obtained from Eq. 16 for the same three different systems considered in Fig. 3. We have taken $\hbar=1$ and $2m=1$.

under the curves in Fig. 3 up to a Fermi wave-vector $k_fL$. In Fig. 4 we have plotted this integrated current as a function of $k_fL$ and at low energies we do see a diamagnetic shift up to $k_fL = 8.35$. But there is no such discernible diamagnetic shift at higher energies. But one can always have higher energy evanescent modes in the ring that show a diamagnetic shift at such energies. Such high energy evanescent modes are possible in presence of sub-bands due to lateral confinement as higher sub-bands have higher propagating thresholds. Interestingly this integrated current show roughly same magnitude of current in the evanescent regime as that above the barrier. The peak values of the current at the first peak is roughly -1 while the propagating current peak value goes to $\pm 2$ (see
This is because in presence of partial or full tunneling the current is small in magnitude but always diamagnetic resulting in a larger area under the curves. Propagating state current alters between paramagnetic and diamagnetic and has a lot of cancellation effect.

For completion we have also plotted DOS for the region II in Fig. 5. We plot the RHS and the LHS of Eq. 18 that coincide with each other for all three graphs as expected. Curves are for the same value of parameters used in Fig. 3 and the same convention is followed. Note that all three curves show that when DOS under the barrier is high then current is also high.
5 Conclusion

In summary, asymptotic theory works for current as well as DOS and has been also used to show that propagation under the barrier is a reality. Propagation under the barrier is not a well defined problem in quantum mechanics and so the asymptotic theory that goes beyond quantum mechanics, uses the idea of physical clocks and analyticity of scattering matrix elements help confirm this. This is referred to as a theoretical experiment but one can also make clean practical experiments to confirm this further. It also provides evidence that time reversed states have a role to play in propagation under the barrier. So there is a lot of merit in trying to address the question of propagation time and superluminality of tunneling particles.
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