THE Ext-ALGEBRA OF STANDARD MODULES OVER DUAL EXTENSION ALGEBRAS

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Abstract

We exhibit an isomorphism of associative algebras between the Ext-algebra $\text{Ext}_{\Lambda}^*(\Delta, \Delta)$ of standard modules over the dual extension algebra $\Lambda$ of two directed algebras $B$ and $A$ and the dual extension algebra of the Ext-algebra $\text{Ext}_B^*(L,L)$ with $A$. There are natural $A_\infty$-structures on these Ext-algebras, and, under certain technical assumptions on $B$, we describe that on $\text{Ext}_{\Lambda}^*(\Delta, \Delta)$ completely in terms of that on $\text{Ext}_B^*(L,L)$. As an example, we compute these $A_\infty$-structures explicitly in the case where $B = A = K\mathcal{A}_n/(\text{rad}K\mathcal{A}_n)^{\ell}$. 
1. Introduction

In [CPS88], Cline, Parshall and Scott introduced the notion of highest weight category as a categorical axiomatization of structures arising in the representation theory of complex semisimple Lie algebras. Moreover, they showed that those finite-dimensional algebras whose module categories are equivalent to a highest weight category are exactly the quasi-hereditary algebras. Typical examples of quasi-hereditary algebras are hereditary algebras, algebras of global dimension two, Schur algebras and blocks of BGG category \( \mathcal{O} \). The defining feature of quasi-hereditary algebras is the existence of a particular collection of modules, called the standard modules. These are certain quotients of the indecomposable projective modules and serve as the main protagonists of the representation theory of quasi-hereditary algebras. Closely related is the associated category \( \mathcal{F}(\Delta) \), the full subcategory of the module category consisting of those modules which admit a filtration by standard modules.

An important step towards understanding \( \mathcal{F}(\Delta) \) for a general quasi-hereditary algebra was taken by Koenig, Külshammer and Ovsienko in [KKO14]. Using powerful techniques involving \( A_\infty \)-algebras and boxes, they showed that for any quasi-hereditary algebra, there is a directed box such that the category of representations of this box is equivalent to \( \mathcal{F}(\Delta) \), allowing the study of \( \mathcal{F}(\Delta) \) through the study of boxes and their representations (a box is a bimodule over an algebra together with a comultiplication and counit obeying the natural coassociativity and counitality axioms [Roi80]). The representation theory of boxes is important in its own right, notably playing a sizeable role in the proof of Drozd’s theorem on tame and wild dichotomy [Dro80].

The result by Koenig, Külshammer and Ovsienko leads us to the problem of, given a quasi-hereditary algebra, determining the corresponding box describing \( \mathcal{F}(\Delta) \). This may be done by taking the so-called “\( A_\infty \)-Koszul dual” of the algebra of extensions between standard modules, \( \text{Ext}^\ast(\Delta, \Delta) \). The first (and arguably most arduous) step in this process is to determine the \( A_\infty \)-structure on \( \text{Ext}^\ast(\Delta, \Delta) \). Loosely speaking, such a structure is meant to capture the idea of an algebra whose multiplication is not strictly associative, but only associative up to a system of higher homotopies [Sta63a, Sta63b].

Unfortunately, describing the \( A_\infty \)-structure on \( \text{Ext}^\ast(\Delta, \Delta) \) may be extremely complicated. Examples of algebras where this \( A_\infty \)-structure has been explicitly computed are few and far between, however, one family of examples may be found in [KS12]. The main goal of the present paper is to study a somewhat large class of algebras which are computationally well-behaved enough to permit an explicit description.

In [X94], Xi introduced the dual extension algebra as part of his study of BGG algebras, that is, quasi-hereditary algebras admitting a simple-preserving duality on their module category. Originally introduced as taking only one input (\( \mathcal{A}(B, A^{op}) \) with \( B = A \)), these were soon generalized and further studied in [DX94, Xi95, DX96, Xi00, Wu08, LW13, LX17]. Importantly, in [LX17], Li and Xu connected the Koszulity of the dual extension algebra \( \mathcal{A}(B, A^{op}) \) to that of \( B \) and \( A \). The dual extension algebras, as it turns out, behave well enough with respect to computations to allow the explicit description of the \( A_\infty \)-structure mentioned above.

The following is a description of the main results of the present article. Let \( B \) and \( A \) be directed algebras and let \( \Delta = \mathcal{A}(B, A^{op}) \) denote their dual extension algebra. In this case, \( \Delta \) is quasi-hereditary by a result of Xi. Let \( \Delta \) denote the direct sum of standard modules over \( \Delta \) (one from each isomorphism class) and let \( \mathcal{L} \) denote the direct sum of simple modules over \( \Delta \) (one from each isomorphism class).

(A) There is an isomorphism of graded algebras between the Ext-algebra of standard modules over the dual extension algebra \( \Delta \) and the dual extension algebra of the Ext-algebra of simple modules over \( B \) with \( A \), that is,
\[
\text{Ext}_{\Delta}^\ast(\Delta, \Delta) \cong \mathcal{A}(\text{Ext}_{\mathcal{L}}^\ast(\mathcal{L}, \mathcal{L}), A).
\]

(B) The special case of Merkulov’s construction [Mer99] given in [Lu+06] provides \( A_\infty \)-structures on \( \text{Ext}_{\mathcal{L}}^\ast(\mathcal{L}, \mathcal{L}) \) and on \( \text{Ext}_{\Delta}^\ast(\Delta, \Delta) \). Denote the higher multiplications by \( m^B_{n=1} \) and \( m^A_{n=1} \), respectively. Merkulov’s construction involves several choices and hence the higher multiplications
produced are not canonical. We show that, under certain technical assumptions on $B$, the construction may be performed on $\Ext^*_B(L,L)$ and $\Ext^*_\Lambda (\Delta, \Delta)$ in such a way that the higher multiplications $\{m^n_{\Lambda} \}_{n=1}^\infty$ are given in terms of the data produced by performing the construction on $\Ext^*_B(L,L)$.

More precisely, we prove the formulae below. For all details on the notation, and the assumptions needed, we refer to Section 6. The maps $p^B, h^B$ and $\lambda^B_{n-1}$ are obtained from performing Merkulov’s construction on $\Ext^*_B(L,L)$, while $F$ denotes the induction functor $\Lambda \otimes_B - : B\text{-mod} \rightarrow \Lambda\text{-mod}$.

Let $n \geq 2$. For any $f'_1, \ldots, f'_n \in \Hom_\Lambda (\Delta, \Delta)$ and $\varepsilon_1, \ldots, \varepsilon_n \in \Ext^*_B(L,L)$, such that $\deg \varepsilon_i \geq 1$, for all $1 \leq i \leq n$, we have the following.

(i) If there is $1 \leq i < n$ such that $f'_i \in \rad(\Delta^n_L(j), \Delta^n_L(k))$, we have $m^n_{\Lambda} (f'_n \varepsilon_n, \ldots, f'_1 \varepsilon_1) = 0$.

(ii) $m^n_{\Lambda} (f'_n \varepsilon_n, \ldots, f'_2 \varepsilon_2, \varepsilon_1) = (-1)^{n+1} f'_n \left( p^B h^B \left( \lambda^B_{n-1}(\varepsilon_{n-1}, \ldots, \varepsilon_1) \right) \right)$.

(C) Lastly, we provide formulae for the $A\infty$-multiplications on $\Ext^*_B(L,L)$ obtained from Merkulov’s construction in the case $B = K^{\Lambda^n}/(\rad K^{\Lambda^n})^\ell$ and, using (B), explicitly describe the corresponding $A\infty$-multiplications that this gives on $\Ext^*_A(B,B^\op)$ $(\Delta, \Delta)$.

This article is organized in the following way. In Section 2, we fix some notation and recall the necessary definitions, as well as Xi’s initial result on the quasi-hereditary structure of the dual extension algebra. In Section 3, we compute the space of extensions between standard modules over the dual extension algebra $\mathcal{A}(B, A^{\op})$. Section 4 is devoted to the description of the algebra structure on the Ext-algebra of standard modules over $\mathcal{A}(B, A^{\op})$ and contains the proof of (A).

In Section 5, we provide an analogue of the theorem by Li and Xu in [LX17], which states that $\mathcal{A}(B, A)$ is Koszul if and only if both $B$ and $A$ are Koszul, in terms of linear resolutions of standard modules. In Section 6, we investigate the $A\infty$-structure on $\Ext^*_\mathcal{A}(B, A^{\op}) (\Delta, \Delta)$ provided by Merkulov’s construction and precisely state and prove (B).

Finally, in Sections 7 and 8, we give an example by performing this construction for

$$B = A = K^{\Lambda^n}/(\rad K^{\Lambda^n})^\ell$$

and using the results of Section 6 to give an application to the dual extension algebra $\mathcal{A}(B, B^{\op})$. 
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2. Notation and setup

Throughout, we let \( K \) be an algebraically closed field. Letting \( Q = (Q_0, Q_1) \) be a quiver with vertex set \( \{1, \ldots, n\} \), we denote the path algebra of \( Q \) by \( KQ \). Throughout, all quivers will have vertex set \( \{1, \ldots, n\} \).

This set inherits a natural ordering relation, \( \prec \), from the natural numbers, which we fix from here on. For an arrow \( \alpha \in Q_1 \), denote by \( s(\alpha) \) and \( t(\alpha) \) the starting and terminal vertex of \( \alpha \), respectively. For vertices and arrows \( i \xrightarrow{\beta} j \) we write the composition \( \beta \) after \( \alpha \) as \( \beta \alpha \). For a path \( p = a_n \cdots a_1 \in Q \), we write \( s(p) = s(a_1) \) and \( t(p) = a_n \). Given an admissible ideal \( I \subset KQ \), we may form the corresponding quotient algebra \( B = \frac{KQ}{I} \). Let \( L_B(i) \) and \( P_B(i) \) denote the simple and indecomposable projective \( B \)-modules, respectively.

If the quiver \( Q \) is finite and acyclic, we may assume that for any arrow \( i \xrightarrow{\alpha} j \), we have \( i \prec j \) (with respect to the natural order on \( \{1, \ldots, n\} \)). We say that \( B = \frac{KQ}{I} \), where \( I \) is admissible, is directed if \( Q \) is finite, acyclic and its vertices are numbered as above. Of course, if \( B \) instead is an algebra whose quiver only has arrows in decreasing direction, we may reverse the direction of arrows to obtain a directed algebra in the above sense.

1. Definition. [XI94, XI06, LI17] Let \( B \cong \frac{KQ}{I} \) and \( A \cong \frac{KQ'}{I'} \) be algebras and let the quivers \( Q \) and \( Q' \) be such that \( Q_0 = Q'_0 \). We define the dual extension algebra \( \Lambda = \mathcal{O}(B, A) \) of \( B \) and \( A \) as \( \mathcal{O}(B, A) \cong B \mathcal{E} / J \) where \( E = (E_0, E_1) \) and \( J \) are as follows.

\[
\begin{align*}
(i) & \quad E_0 = Q_0 = Q'_0. \\
(ii) & \quad E_1 = Q_1 \cup Q'_1. \\
(iii) & \quad \text{If } I = \langle \rho_i \rangle, I' = \langle \rho'_j \rangle, \text{ then} \\
& \quad J = \langle \rho_i, \rho'_j, \alpha \beta' | \forall \alpha \in Q_1, \forall \beta' \in Q'_1 \rangle.
\end{align*}
\]

It is clear that both \( B \) and \( A \) occur in a natural way as subalgebras as well as quotients of \( \mathcal{O}(B, A) \). Importantly, this fact allows us to view modules over \( B \) and \( A \) as modules over \( \Lambda \). Letting \( p_B : \Lambda \to B \) be the natural surjection, we define the action of \( a \in \Lambda \) on a \( B \)-module \( M \) by \( a \cdot m = p_B(a) \cdot p_B^m \), for all \( m \in M \), where \( -B \) denotes the action of \( B \) on \( M \). For a \( B \)-module, this coincides with extending the action of \( B \) to an action of \( \Lambda \) by letting all arrows in the quiver of \( \Lambda \) which come from the quiver of \( A \) act as 0. Of course, a similar idea works for \( A \)-modules. Moreover, there are functors

\[
F : \Lambda \otimes_B - : B \text{-mod} \to \Lambda \text{-mod} \quad \text{and} \quad G : B \otimes \Lambda \otimes \Lambda \text{-mod} \to B \text{-mod},
\]

which will be of importance. In the above definition, the set \( \{1, \ldots, n\} \) indexes isomorphism classes of simple modules over \( B \) as well as over \( A \). Note that \( \{1, \ldots, n\} \) also indexes isomorphism classes of simple modules over \( \Lambda = \mathcal{O}(B, A) \). Consider the following quotient of the indecomposable projective \( P_B(i) \), called the standard module at \( i \).

\[
\Delta_B(i) = P_B(i) / \sum_{f : P_B(j) \to P_B(i)} \text{im} f.
\]

Here, the sum is taken over all homomorphisms \( f : P_B(j) \to P_B(i) \) such that \( i \prec j \). The algebra \( B \) is said to be quasi-hereditary the following hold.

\[
\begin{align*}
(i) & \quad \text{End}_B(\Delta_B(i)) \cong K \text{ for each } i \in \{1, \ldots, n\}. \\
(ii) & \quad \text{The indecomposable projectives } P_B(i) \text{ are filtered by standard modules, i.e., each projective } P_B(i) \text{ admits a chain of submodules} \\
& \quad 0 \subset M_0 \subset M_1 \subset \cdots \subset M_\ell = P_B(i) \text{ such that all subquotients } M_k / M_{k-1} \text{ are standard modules.}
\end{align*}
\]

2. Definition. [Kön95, BKK20, Definition 3.4] Let \( (\Lambda, \prec) \) be a quasi-hereditary algebra with \( n \) simple modules, up to isomorphism. Then, a subalgebra \( B \subset \Lambda \) is called an exact Borel subalgebra provided that
(i) $B$ also has $n$ simple modules up to isomorphism and $(B, <)$ is directed,
(ii) the functor $\Lambda \otimes_B -$ is exact, and
(iii) there are isomorphisms $\Lambda \otimes_B L_B(i) \cong \Delta_A(i)$.
If, in addition, the map $\text{Ext}^k_B(L_B(i), L_B(j)) \rightarrow \text{Ext}^k_A(\Lambda \otimes_B L_B(i), \Lambda \otimes_B L_B(j))$ induced by the functor $\Lambda \otimes_B -$ is an isomorphism for all $k \geq 1$ and $i, j \in \{1, \ldots, n\}$, $B \subset \Lambda$ is called a regular exact Borel subalgebra.

3. **Theorem.** [Xi94, Example 1.6] Let $B \cong KQ/I$ and $A \cong KQ/1'$ be directed algebras. Then $\Lambda = \mathcal{A}(B, A^{\text{op}})$ is quasi-hereditary with respect to the natural ordering on $\{1, \ldots, n\}$. Moreover, there are isomorphisms of $\Lambda$-modules $\Delta_A(i) \cong P_A^{\text{op}}(i)$ for all $i \in \{1, \ldots, n\}$.

4. **Example.** Consider the quiver $Q = 1 \overset{\alpha}{\longrightarrow} 2 \overset{\beta}{\longrightarrow} 3$ and put $B = A = KQ$. Then, the dual algebra extension $\Lambda = \mathcal{A}(B, B^{\text{op}})$ is given by the quiver

$$
\begin{array}{c}
1 \\
\alpha \\
\beta
\end{array}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\begin{array}{c}
1 \\
2 \\
3
\end{array}

subject to the relations $aa' = 0$ and $\beta \beta' = 0$. The indecomposable projective modules over $\Lambda$ have Loewy diagrams

$$
\begin{array}{c}
P_\Lambda(1): 3 \\
2 \\
1
\end{array},
\begin{array}{c}
P_\Lambda(2): 3 \\
2 \\
1
\end{array}, \quad \text{and} \quad \begin{array}{c}
P_\Lambda(3): 2 \\
1
\end{array}
$$

We see that, then, the standard modules over $\Lambda$ have Loewy diagrams

$$
\Delta_\Lambda(1) \cong L_\Lambda(1): 1, \quad \Delta_\Lambda(2): 1, \quad \text{and} \quad \Delta_\Lambda(3) \cong P_\Lambda(3): 2
$$

which, as we expect, coincide with the Loewy diagrams of the indecomposable projective modules over $B^{\text{op}}$.

3. **Extensions between standard modules**

In this section, we compute $\text{Ext}^k_A(\Delta_A(i), \Delta_A(j))$ for $B, A$ directed and $\Lambda = \mathcal{A}(B, A^{\text{op}})$. We have the following crucial lemma, which is a collection of observations found in [DX94, Lemma 1.6] and [Wu09, Lemma 2].

5. **Lemma.**

1. The functor $F$ is exact.
2. There are isomorphisms of left $\Lambda$-modules $F(L_B(i)) \cong \Delta_A(i)$ and $F(P_B(i)) \cong P_\Lambda(i)$.
3. For any $M \in B^{\text{mod}}$, $P^* \rightarrow M$ is a minimal projective resolution if and only if $F(P^* \rightarrow F(M)$ is a minimal projective resolution.
4. The functors $G \circ F$ and $\text{Id}_{B^{\text{mod}}}$ are naturally isomorphic.

It is now clear that $B \subset \mathcal{A}(B, A^{\text{op}})$ is an exact Borel subalgebra. This was already noticed by Xi in [Xi94]. However, we will see in Example 8 that $B \subset \mathcal{A}(B, A^{\text{op}})$ is, in general, not regular.

Next, we want to compute the spaces $\text{Ext}^k_A(\Delta_A(i), \Delta_A(j))$ for general $i, j \in \{1, \ldots, n\}$, by applying the functor $\text{Hom}_A(\Lambda, \Delta_A(j))$ to a minimal projective resolution of $\Delta_A(i)$. Observe that Lemma 5 implies that $P^* \rightarrow L_B(i)$ is a minimal projective resolution if and only if $F(P^* \rightarrow F(\Delta_A(i)$ is. This motivates the investigation of the spaces $\text{Hom}_A(P_\Lambda(i), \Delta_A(j)$, about which we make the following observation.
6. Proposition. Let \( \pi_i : P_A(i) \to P_A(i) \) be the natural projection. Then, there are isomorphisms of vector spaces

\[
\text{Hom}_A(P_A(i), \Delta_A(j)) \cong \text{Hom}_A(P_A(i), P_A(j)) \cong \text{Hom}_{A^\text{op}}(P_A(i), P_A(j))
\]

for all \( i, j, \in \{1, \ldots, n\} \), given by precomposition with \( \pi_i \).

Proof. We prove the existence of the first isomorphism, as the existence of the second is immediate. Since \( \Delta_A(j) \) and \( P_A(j) \) are isomorphic as \( \Lambda \)-modules, it suffices to exhibit an isomorphism

\[
\text{Hom}_{A^\text{op}}(P_A(i), P_A(j)) \cong \text{Hom}_{A^\text{op}}(P_A(i), P_A(j)).
\]

The module \( P_A(i) \) has a basis given by the set \( \{q'p | s(p) = e_i, t(p) = s(q')\} \), where \( p \) is a path in \( B \) and \( q' \) is a path in \( A^\text{op} \). First of all, note that if \( q'p \) is such that \( p \neq e_i \), then \( q'p \in \ker \pi_i \). Clearly, such elements form a basis of \( \ker \pi_i \). Let \( \varphi : P_A(i) \to P_A(j) \) be a homomorphism (of \( \Lambda \)-modules). We claim that, then, \( q'p \in \ker \varphi \) so that \( \ker \pi_i \subset \ker \varphi \). Indeed, we have

\[
\varphi(q'p) = \varphi(q'pe_i) = q'p\varphi(e_i) = 0
\]

since \( \varphi(e_i) \) is an element of \( P_A(j) \) and \( p \), being a non-trivial path in \( B \), acts as 0 on this module. The fact that \( \ker \pi_i \subset \ker \varphi \) implies that the homomorphism \( \varphi \) factors uniquely through \( P_A(i) \to \ker \pi_i \cong P_A(i) \), i.e., there is a unique homomorphism \( \overline{\varphi} : P_A(i) \to P_A(j) \) making the following diagram commutative.

\[
\begin{array}{ccc}
P_A(i) & \xrightarrow{\pi_i} & P_A(i) \\
\downarrow{\varphi} & & \downarrow{\overline{\varphi}} \\
P_A^\text{op}(i) & \xrightarrow{\overline{\varphi}} & P_A^\text{op}(j)
\end{array}
\]

Next, define two maps, \( \Phi \) and \( \Psi \), as follows.

\[
\Phi : \text{Hom}_{A^\text{op}}(P_A(i), P_A(j)) \to \text{Hom}_{A^\text{op}}(P_A(i), P_A(j)), \quad q \mapsto \overline{\varphi},
\]

\[
\Psi : \text{Hom}_{A^\text{op}}(P_A(i), P_A(j)) \to \text{Hom}_{A^\text{op}}(P_A(i), P_A(j)), \quad \overline{\varphi} \mapsto \overline{\varphi}.\pi_i.
\]

From the above considerations, we know that the map \( \Phi \) is well-defined. It is clear that \( \Psi \) is well-defined, and that both maps are linear. Finally, we find that

\[
\Phi \circ \Psi(\overline{\varphi}) = \Phi(\overline{\varphi}.\pi_i) = \Phi(\overline{\varphi}) = \overline{\varphi} \quad \text{and} \quad \Psi \circ \Phi(q) = \Psi(q) = \overline{\varphi}.\pi_i = q,
\]

showing that \( \Phi \) and \( \Psi \) are mutually inverse linear isomorphisms. \( \square \)

In what follows, we will often need to consider homomorphisms between indecomposable projective \( \Lambda \)-modules. Such homomorphisms are (linear combinations of) homomorphisms which act by right multiplication with a certain path in the quiver of \( \Lambda \). More precisely, for vertices \( i \) and \( j \), and a path \( i \xrightarrow{p} j \), there is a homomorphism \( \rho_p : P_A(i) \to P_A(i) \) defined by \( x \mapsto xp \).

7. Proposition. Let \( P_B^k \) be a minimal projective resolution of \( L_B(i) \) with terms

\[
P_B^k = \bigoplus_{\ell=1}^n P_B(\ell)^{\oplus m_{\ell, k}}, \quad k \geq 0.
\]

Then, there are linear isomorphisms

\[
\text{Ext}^k_A(\Delta_A(i), \Delta_A(j)) \cong \text{Hom}_A(\Lambda \otimes_B P_B^k, \Delta_A(j)) \cong \bigoplus_{\ell=1}^n \{e_j A e_\ell \}^{\oplus m_{\ell, k}}.
\]

Proof. By Lemma\( \square \), the module \( \Delta_A(i) \) has a minimal projective resolution with terms

\[
\Lambda \otimes_B P_B^k \cong \bigoplus_{\ell=1}^n \Lambda \otimes_B P_B(\ell)^{\oplus m_{\ell, k}} \cong \bigoplus_{\ell=1}^n P_A(\ell)^{\oplus m_{\ell, k}}.
\]
We first show that \( \text{Ext}_A^k(\Delta_A(i), \Delta_A(j)) \equiv \text{Hom}_A(A \otimes_B P_B^k, \Delta_A(j)) \). Let \( \partial \) be the differential on the complex \( \text{Hom}_A(A \otimes_B P_B^k, \Delta_A(j)) \), with the convention that
\[
\partial^k : \text{Hom}_A(A \otimes_B P_B^{k-1}, \Delta_A(j)) \to \text{Hom}_A(A \otimes_B P_B^k, \Delta_A(j)).
\]
Then, we have \( \text{Ext}_A^k(\Delta_A(i), \Delta_A(j)) = \ker \partial^k / \text{im} \partial^{k+1} \). Therefore, to prove our claim, it suffices to show that \( \partial \) is the zero map in each degree. The differential \( \partial \) is given, in each degree, by a matrix whose entries are linear combinations of maps of the form \( \rho_p \circ \partial \). Therefore, it is enough to prove that every map of the form \( \rho_p \circ \partial \) is the zero map. Consider
\[
\rho_p : \text{Hom}_A(P_A(\ell), \Delta_A(j)) \to \text{Hom}_A(P_A(\ell'), \Delta_A(j)).
\]
Note that \( \rho_p : P_A(\ell') \to P_A(\ell) \) is given by right multiplication with a path \( p \) in \( B \), because it is in the image of the functor \( F \).

By the proof of Proposition \([5]\) any homomorphism \( \psi \) factors through the projection \( \pi_\ell \). For any \( x \in P_A(\ell') \), we have
\[
\pi_\ell \circ \rho_p(x) = \pi_\ell(xp) = 0
\]
because \( p \) is a path in \( B \). We conclude that \( \text{im} \rho_p \subset \ker \pi_\ell \), which implies that \( \psi \circ \rho_p = \psi \circ \pi_\ell \circ \rho_p = 0 \). So, in each degree, the matrix constituting the differential \( \partial \) has only zero entries. This finishes the proof of the first isomorphism in the statement of the proposition. For the second one, we apply Proposition \([6]\)

\[
\text{Ext}_A^k(\Delta_A(i), \Delta_A(j)) \equiv \text{Hom}_A \left( A \otimes_B P_B^k, \Delta_A(j) \right) \equiv \text{Hom}_A \left( \bigoplus_{\ell=1}^n P_A(\ell)^{\oplus m_{\ell k}}, \Delta_A(j) \right)
\]
\[
\equiv \bigoplus_{\ell=1}^n \text{Hom}_A(P_A(\ell), \Delta_A(j))^{\oplus m_{\ell k}} \equiv \bigoplus_{\ell=1}^n \text{Hom}_A(P_A(\ell), P_{A^e}(j))^{\oplus m_{\ell k}}
\]
\[
\equiv \bigoplus_{\ell=1}^n \left( e_{\ell A} P e_j \right)^{\oplus m_{\ell k}} \equiv \bigoplus_{\ell=1}^n \left( e_j A e_{\ell} \right)^{\oplus m_{\ell k}}.
\]

### 8. Example
We consider again the algebra \( A = \mathcal{O}(B, B^{op}) \) where \( B = A = K( 1 \oplus 2 \oplus 3 ) \). We wish to compare the spaces \( \text{Ext}_B^1(L_B(1), L_B(3)) \) and \( \text{Ext}_A^1(\Delta_A(1), \Delta_A(3)) \). Recall that \( \Lambda \) is given by the quiver
\[
1 \xleftarrow{a} 2 \xrightarrow{\beta} 3
\]
subject to the relations \( aa' = 0 \) and \( \beta\beta' = 0 \). We immediately see that
\[
\dim \text{Ext}_B^1(L_B(1), L_B(3)) = 0,
\]
since this dimension coincides with the number of arrows \( 1 \rightarrow 3 \) in the quiver of \( B \), and there are zero such arrows. Since we have a minimal projective resolution
\[
0 \to P_B(2) \to P_B(1) \to L_B(1),
\]
it follows from Proposition \([7]\) and Lemma \([5]\) part (3), that we have
\[
\dim \text{Ext}_A^1(\Delta_A(1), \Delta_A(3)) \equiv \text{Hom}_A(P_A(2), \Delta_A(3)),
\]
\[
\dim \text{Ext}_B^1(L_B(1), L_B(3)) \equiv \text{Hom}_B(L_B(2), \Delta_B(3)).
\]
and this space contains the map $f$ given by right multiplication by the arrow $\beta'$. This means that $\dim \text{Ext}_\Lambda(\Delta_1, \Delta_3) \geq 1$, so that $B$ is not regular.

Proposition $\textbf{7}$ sheds some further light on how $B \subset \Lambda$ fails to be a regular exact subalgebra. Under the assumption of the proposition, one can check that $\text{Ext}_B^k(L_B(i), L_B(j)) \cong \text{Hom}_B(P_B^k, L_B(j))$, meaning we get an extension for each copy of $P_B(j)$ appearing in $P_B^k$, because

$$\dim \text{Hom}_B(P_B(\ell), L_B(j)) = \begin{cases} 1, & \text{if } \ell = j, \\ 0, & \text{otherwise}. \end{cases}$$

Similarly, for the standard modules over $\Lambda$, we saw that $\text{Ext}_\Lambda^k(\Delta(i), \Delta(j)) \cong \text{Hom}_\Lambda(\Lambda \otimes B P_B^k, \Delta(j))$. This space decomposes into a direct sum of spaces of the form $\text{Hom}_\Lambda(P(\ell), \Delta(j))$. Of course, if $\ell = j$, this space contains the projection $\pi_j : P(j) \to \Delta(j)$, which is the image of the projection $P_B(j) \to L_B(j)$ under the functor $\Lambda \otimes_{B \rightarrow B}$. However, in Proposition $\textbf{6}$, we saw that the spaces $\text{Hom}_\Lambda(P(\ell), \Delta(j))$ are in general not zero for $\ell \neq j$, yielding additional extensions which are not contained in the image of the functor $\Lambda \otimes_{B \rightarrow B}$.

4. THE $\text{Ext}$-ALGEBRA OF STANDARD MODULES

Put $\Delta = \Delta_1(1) \oplus \cdots \oplus \Delta_n(1)$ and $B = L_B(1) \oplus \cdots \oplus L_B(n)$. The space $\text{Ext}_\Lambda^*(\Delta, \Delta) = \bigoplus_{k \geq 0} \text{Ext}_\Lambda^b(\Delta, \Delta)$ has a natural structure of a graded algebra (as does the space $\text{Ext}_B^*(\Lambda, \Lambda)$), with the multiplication given by Yoneda product. This section is devoted to its description. Let $P^k_\Lambda$ and $Q^k_\Lambda$ be minimal projective resolutions of $\Delta(i)$ and $\Delta(j)$, respectively. Then, $\text{Ext}_\Lambda^k(\Delta(i), \Delta(j)) \cong \text{Hom}_{\text{proj}}(P^k_\Lambda, Q^k_\Lambda)$, where $\text{proj}$ denotes the shift functor on complexes. Under this isomorphism, the Yoneda product corresponds to composition of (equivalence classes of) chain maps. We wish to find chain maps corresponding to basis elements of $\text{Ext}_\Lambda^k(\Delta(i), \Delta(j))$ under this isomorphism. Recall that $\text{Ext}_\Lambda^k(\Delta(i), \Delta(j)) \cong \text{Hom}_\Lambda(P^k_\Lambda, \Delta(j))$ by Proposition $\textbf{7}$, so a basis element of $\text{Ext}_\Lambda^k(\Delta(i), \Delta(j))$ is represented by $\varphi \in \text{Hom}_\Lambda(P^k_\Lambda, \Delta(j))$.

$$P^k_\Lambda : \cdots \to P^k_{\Lambda 1} \to P^k_\Lambda \to P^{k-1}_\Lambda \to \cdots$$

$$Q^k_\Lambda[k] : \cdots \to Q^k_\Lambda \to P(j) \to \Delta(j) \to \cdots$$

Such a chain map should have components $\varphi_k, \varphi_{k+1}, \ldots$ as indicated above. Additionally, the component $\varphi_k$ should lift the homomorphism $\varphi$, that is, $\pi_j \circ \varphi_k = \varphi$. In the above picture, the maps are matrices whose entries are homomorphisms between indecomposable projective $\Lambda$-modules, or in the case of the map $\varphi$, homomorphisms $P_\Lambda(x) \to \Delta_\Lambda(j)$. Assume that the projective module $P_\Lambda(j)$ does not appear in the direct sum decomposition of the module $P^k_\Lambda$. By Proposition $\textbf{6}$, each entry of the matrix constituting the homomorphism $\varphi : P^k_\Lambda \to \Delta_\Lambda(j)$ factors as $\rho_{\gamma'} \circ \pi_x$, where $x$ is such that $P_\Lambda(x)$ is a direct summand of $P^k_\Lambda$ and $\gamma'$ is a linear combination of paths in the quiver of $\Lambda^{op}$.

$$\begin{array}{c}
P_\Lambda(x) \\
\rho_{\gamma'}
\end{array} \begin{array}{c}
\downarrow \rho_{\gamma'} \circ \pi_x \\
\downarrow \pi_j \\
P_\Lambda(j) \end{array} \begin{array}{c}
\downarrow \pi_j \\
\Delta_\Lambda(j)
\end{array}$$

Next, we claim that the above triangle commutes. Recall that the module $P_\Lambda(x)$ has a basis made up of elements of the form $q' p$ where $p$ is a path in $B$, starting in $x$, and $q'$ is a path in $\Lambda^{op}$. If $p = e_x$, we have

$$\rho_{\gamma'} \circ \pi_x(q' p) = \rho_{\gamma'} \circ \pi_x(q') = \rho_{\gamma'}(q') = q' \gamma' = \pi_j \circ \rho_{\gamma'}(q') = \pi_j \circ \rho_{\gamma'}(q' p)$$
and, if \( p \neq e_x \), we have
\[
\rho_{y'} \circ \pi_x(q'p) = 0 = \pi_j(q'p'y') = \pi_j \circ \rho_{y'}(q'p),
\]
since \( p'y' = 0 \) according to the dual extension relation, which proves the claim. Then, taking the map \( \varphi_k \) to be the matrix having an entry \( \rho_{y'} \) whenever the matrix describing \( \varphi \) has an entry \( \rho_{y'} \circ \pi_x \), we see that the following diagram commutes.

\[
\begin{array}{ccc}
P^k_A & \xrightarrow{\varphi_k} & \Delta_A(j) \\
| \downarrow \varphi_k \downarrow \pi_j | & & | \downarrow \pi_j | \\
P_A(j) & \xrightarrow{\pi_j} & \Delta_A(j)
\end{array}
\]

Next, note that the differentials on the complexes \( P^*_A \) and \( Q^*_A \) are matrices whose entries are linear combinations of maps \( \rho_{\alpha} \), where \( \alpha \) is a path in \( B \). If \( \beta' \) is a path in \( A^{op} \), we have \( \rho_{\beta'} \circ \rho_{\alpha} = 0 \) since \( \alpha \beta' = 0 \) according to the dual extension relation. Returning to the picture

\[
\begin{array}{ccc}
P^*_A: \ldots & \xrightarrow{\partial_{k+1}} & P^k_A \\
| & \downarrow \varphi_{k+1} | & | \downarrow \varphi_k | \\
Q^*_A[k]: \ldots & \xrightarrow{\partial_{k+1}} & P_A(j) \\
| \downarrow \varphi_0 \downarrow \pi_j | & & | \downarrow \pi_j | \\
& \xrightarrow{\pi_j} & \Delta_A(j)
\end{array}
\]

our observations now imply that we may extend \( \varphi_k \) to a chain map by putting \( \varphi_{k+1} = \varphi_{k+2} = \cdots = 0 \). First, we showed that \( \pi_j \circ \varphi_k = \varphi \). Then, we checked that \( \varphi_k \circ \varphi_{k+1} = 0 \). All other squares commute trivially, meaning we have found a chain map representative of the extension given by \( \varphi \).

Note that the above construction does not work if the projective module \( P_A(j) \) appears in the direct sum decomposition of \( P^k_A \). If it does, the space \( \text{Hom}_A(P^k_A, \Delta_A(j)) \) contains the projection \( \pi_j: P_A(j) \to \Delta_A(j) \), which may be lifted to the identity homomorphism on \( P_A(j) \). Then, the collection of maps \( \varphi_k, \varphi_k+1, \ldots \) given above is no longer a chain map.

Of particular importance is the special case when \( k = 0 \), corresponding to homomorphisms, rather than proper extensions, from \( \Delta_A(i) \) to \( \Delta_A(j) \). Then, the first part of our picture looks like

\[
\begin{array}{ccc}
\ldots & \xrightarrow{P_A(i)} & \Delta_A(i) \\
| \downarrow \varphi_0 \downarrow \varphi | & & | \downarrow \varphi | \\
\ldots & \xrightarrow{P_A(j)} & \Delta_A(j)
\end{array}
\]

which ensures that our construction goes through.

9. Proposition. Let \( i, j \) and \( \ell \) be vertices such that \( i \neq j \), and assume \( k \geq 1 \). Then, the multiplication
\[
m_2: \text{Ext}^2_A(\Delta_A(i), \Delta_A(\ell)) \times \text{Hom}_A(\Delta_A(i), \Delta_A(j)) \to \text{Ext}^2_A(\Delta_A(i), \Delta_A(\ell))
\]
is the zero map.

Proof. Let \( P_A^*, Q_A^* \) and \( R_A^* \) be minimal projective resolutions of \( \Delta_A(i), \Delta_A(j) \) and \( \Delta_A(\ell) \), respectively. The fact that \( i \) and \( j \) are distinct ensures that, if \( \varphi \in \text{Hom}_A(\Delta_A(i), \Delta_A(j)) \) is a homomorphism, our construction of the chain map representing \( \varphi \), in the previous discussion, goes through. Consider the following diagram.
Since the only nonzero component of the chain map representing \( \varphi \) has codomain \( Q_0^0 \) and \( k \geq 1 \), the statement follows. 

10. Theorem. There are isomorphisms of graded algebras

\[
\text{Ext}^*_A(\Delta, \Delta) \cong \mathscr{A}(\text{Ext}^*_B(\mathbb{I}, \mathbb{I}), \text{Hom}_\Delta(\Delta, \Delta)) \cong \mathscr{A}(\text{Ext}^*_B(\mathbb{I}, \mathbb{I}), A).
\]

Proof. The algebras \( \text{Ext}^*_A(\Delta, \Delta) \) and \( \text{Ext}^*_B(\mathbb{I}, \mathbb{I}) \) are naturally graded by the degree of extensions. We extend these to gradings on \( \mathscr{A}(\text{Ext}^*_B(\mathbb{I}, \mathbb{I}), \text{Hom}_\Delta(\Delta, \Delta)) \) and on \( \mathscr{A}(\text{Ext}^*_B(\mathbb{I}, \mathbb{I}), A) \) by letting elements of \( \text{Hom}_\Delta(\Delta, \Delta) \) and \( A \) be homogeneous of degree 0. We first prove that \( \text{Ext}^*_A(\Delta, \Delta) \cong \mathscr{A}(\text{Ext}^*_B(\mathbb{I}, \mathbb{I}), \text{Hom}_\Delta(\Delta, \Delta)) \).

Suppose the simple \( B \)-modules \( L_B(i), L_B(j) \) and \( L_B(\ell) \) have minimal projective resolutions \( P^*_B, Q^*_B \) and \( R^*_B \), respectively. Let \( P^*_A, Q^*_A \) and \( R^*_A \) be the induced projective resolutions of \( \Delta_A(i), \Delta_A(j) \) and \( \Delta_A(\ell) \), respectively. By Proposition 7, there holds \( \text{Ext}^k_A(\Delta_A(i), \Delta_A(j)) \cong \text{Hom}_A(P^k_A, \Delta_A(j)) \). It is easy to check that, similarly, there holds \( \text{Ext}^k_B(L_B(i), L_B(j)) \cong \text{Hom}_B(P^k_B, L_B(j)) \). Lemma 5 implies that the functor \( F \) yields a (linear) map

\[
\Lambda \otimes_B : \text{Hom}_B(P^k_B, L_B(j)) \to \text{Hom}_A(P^k_A, \Delta_B(j)).
\]

As the functor \( F \) is faithful, the map is injective. Considering the diagram

\[
\begin{array}{ccc}
\text{Ext}^k_B(L_B(i), L_B(j)) & \xrightarrow{\Lambda \otimes_B} & \text{Ext}^k_A(\Delta_A(i), \Delta_B(j)) \\
\cong & & \cong \\
\text{Hom}_{\mathbb{X}(\mathbb{A})}(P^*_B, Q^*_B[k]) & \xrightarrow{F} & \text{Hom}_{\mathbb{X}(\mathbb{A})}(P^*_A, Q^*_A[k])
\end{array}
\]

we see that it may be made to commute by letting the dashed arrow represent the map which takes a chain map \( f \), to the chain map \((\id_A \otimes f)_i\). For \( f \in \text{Hom}_{\mathbb{X}(\mathbb{A})}(P^*_B, Q^*_B[k]) \) and \( g \in \text{Hom}_{\mathbb{X}(\mathbb{A})}(Q^*_B, R^*_B[k']) \) the chain map \((gf)_i \in \text{Hom}_{\mathbb{X}(\mathbb{A})}(P^*_B, R^*_B[k+k']) \) has components \( g_{i+k}f_i : P_B^i \to R_B^{i+k+k'} \). We check that \( \Lambda \otimes_B \) is compatible with composition of chain maps and thus extends to an injective homomorphism of (graded) algebras \( \Lambda \otimes_B : \text{Ext}^*_B(\mathbb{I}, \mathbb{I}) \hookrightarrow \text{Ext}^*_A(\mathbb{I}, \mathbb{I}) \).

Next, if \( \epsilon \in \Lambda \otimes_B \text{Ext}^*_B(\mathbb{I}, \mathbb{I}) \) is such that \( \deg \epsilon \geq 1 \) and \( \varphi \in \text{Hom}_A(\Delta_A(i), \Delta_A(j)) \subset \text{Ext}^*_A(\mathbb{I}, \mathbb{I}) \), then \( m_2(\epsilon, \varphi) = 0 \) by Proposition 5. This fact, together with the existence of the embeddings of algebras \( \text{Ext}^*_B(\mathbb{I}, \mathbb{I}) \hookrightarrow \text{Ext}^*_A(\mathbb{I}, \mathbb{I}) \), and \( \text{Hom}_A(\Delta_A(i), \Delta_A(j)) \), implies that there is a homomorphism of graded algebras

\[
\Phi : \mathscr{A}(\text{Ext}^*_B(\mathbb{I}, \mathbb{I}), \text{Hom}_A(\Delta_A(i), \Delta_A(j))) \to \text{Ext}^*_A(\Delta, \Delta).
\]

We claim that \( \Phi \) is surjective. Suppose that the resolution \( P^*_B \to L_B(i) \) has terms \( P^h_B = \bigoplus_{i=1}^n P_B(t)^{\sigma^{m_{i,h}}} \). Then, the resolution \( P^*_A \to \Delta_B(i) \) has terms \( P^h_A = \bigoplus_{i=1}^n P_A(t)^{\sigma^{m_{i,h}}} \). Fix an extension

\[
a \in \text{Ext}^k_A(\Delta_A(i), \Delta_A(j)) \cong \text{Hom}_A(P^k_A, \Delta_A(j)) = \bigoplus_{i=1}^n \text{Hom}_A(P_A(t), \Delta_A(j))^{\sigma^{m_{i,h}}}.
\]
where we use Proposition 7 and additivity of the Hom-functor, to obtain the chain of isomorphisms. The extension $\alpha$ is represented by a matrix, whose entries are maps $a_1^r, \ldots, a_{n,t,k}^r \in \text{Hom}_A(P_A(t), \Delta_A(j))$. Here, $a_1^r$ is the entry corresponding to the $r$th copy of $P_A(t)$ occurring in $P_A^k$. Similarly, we have

$$\text{Ext}^k_B(L_B(i), L_B(j)) = \text{Hom}_B(P_B^k, L_B(j)) \cong \bigoplus_{t=1}^{n} \text{Hom}_B(P_B(t), L_B(j))^{m_{t,k}}.$$ 

The spaces on $\text{Hom}_B(P_B(t), L_B(j))$ are zero unless $t = j$, in which case they contain exactly the projection $P_B(j) \to L_B(j)$ (up to a scalar). This means that the image of $\text{Hom}_B(P_B(t), L_B(j))$ under the functor $F$ is spanned by the projection $\pi_j : P_A(j) \to \Delta_A(j)$ if $t = j$, and 0 otherwise. Now we apply Proposition 8 to see that any $a_1^r \in \text{Hom}_A(P_A(t), \Delta_A(j))$ may be written as $a_1^r = \alpha_1^r \pi_t$ where $\pi_t : P_A(t) \to \Delta_A(t)$ is the natural projection and $\alpha_1^r \in \text{Hom}_A(\Delta_A(t), \Delta_A(j))$. This shows that the image under the functor $F$ of $\text{Ext}^k_A(L_A, L_A)$, together with $\text{Hom}_A(\Delta_A, \Delta_A)$, is enough to generate $\text{Ext}^k_A(\Delta_A, \Delta_A)$, so $\Phi$ is surjective. To finish the proof, we count dimensions. Using Propositions 9 and 7, we have

$$\dim \text{Ext}^k_A(\Delta_A, \Delta_A(j)) = \dim \text{Hom}_A(P_A^k, \Delta_A(j)) = \dim \bigoplus_{\ell=1}^{n} \text{Hom}_A(P_A(\ell)^{\otimes m_{\ell,k}}, \Delta_A(j)) =$$

$$= \dim \bigoplus_{\ell=1}^{n} \text{Hom}_A(P_A(\ell), \Delta_A(j))^{\otimes m_{\ell,k}} = \sum_{\ell=1}^{n} m_{\ell,k} \dim \text{Hom}_A(P_A(\ell), \Delta_A(j)).$$

We compare this to the degree part of $e_j \mathcal{A}(\text{Ext}^k_A(L_A, L_A), \text{Hom}_A(\Delta_A, \Delta_A)) e_i$. Such an element may only be obtained by multiplying an extension $\epsilon_k \in \mathcal{A} \otimes B \text{Ext}^k_A(L_A, L_A)$, with $\deg \epsilon_k = k$, with an element $\varphi \in \text{Hom}_A(\Delta_A, \Delta_A)$, on the left. That is, an element of degree $k$ should have the form $\varphi \epsilon_k$. For this composition to be nonzero, we must have $\epsilon_k \in \mathcal{A} \otimes B \text{Ext}^k_A(L_B(i), L_B(\ell))$ and $\varphi \in \text{Hom}_A(\Delta_A(\ell), \Delta_A(j))$ for some $\ell$. For the extensions between the simple modules, we have

$$\dim \text{Ext}^k_A(L_B(i), L_B(\ell)) = \dim \text{Hom}_B(P_B^k, L_B(\ell)) = \dim \bigoplus_{\ell=1}^{n} \text{Hom}_B(P_B(t), L_B(\ell)) =$$

$$= \sum_{\ell=1}^{n} m_{\ell,k} \dim \text{Hom}_B(P_B(t), L_B(\ell)) = m_{\ell,k} \dim \text{Hom}_B(P_B(t), L_B(\ell)) = m_{\ell,k} \dim \text{Hom}_B(P_B(t), L_B(\ell)) = m_{\ell,k}$$

since $\dim \text{Hom}_B(P_B(t), L_B(\ell)) = 1$ if $t = \ell$ and zero otherwise. Summing the possibilities over all $\ell$, we get that the degree $k$ part of $e_j \mathcal{A}(\text{Ext}^k_A(L_A, L_A), \text{Hom}_A(\Delta_A, \Delta_A)) e_i$ has dimension

$$\sum_{\ell=1}^{n} \dim \text{Ext}^k_B(L_B(i), L_B(\ell)) \cdot \dim \text{Hom}_A(\Delta_A(\ell), \Delta_A(j)) = \sum_{\ell=1}^{n} m_{\ell,k} \cdot \dim \text{Hom}_A(\Delta_A(\ell), \Delta_A(j)).$$

From this, it follows that $\dim \mathcal{A}(\text{Ext}^k_A(L_A, L_A), \text{Hom}_A(\Delta_A, \Delta_A)) = \dim \text{Ext}^k_A(\Delta_A, \Delta_A)$ so that $\Phi$ is a surjective linear map between vector spaces of the same dimension, hence an isomorphism. This proves the first isomorphism in the statement. For the second, we note that as $\Lambda$-modules, we have

$$\Delta = \Delta_A(1) \oplus \cdots \oplus \Delta_A(n) \cong P_A^{op}(1) \oplus \cdots \oplus P_A^{op}(n) \cong A^{op} e_1 \oplus \cdots \oplus A^{op} e_n \cong A^{op}$$

and since $\text{End}_A(\Lambda) \cong A$ as algebras, we get

$$\text{Ext}_A^k(\Delta, \Delta) \cong \mathcal{A}(\text{Ext}_A^k(L_A, L_A), \text{Hom}_A(\Delta, \Delta)) \cong \mathcal{A}(\text{Ext}_A^k(L, L), A).$$

5. KOSZULITY

In this section, we investigate some of the properties of $\Lambda$ as a graded algebra. Any directed algebra admits a $Z$-grading by path length, and these gradings extend to a grading on $\Lambda = \mathcal{A}(B, A^{op})$. Recall that a graded $\Lambda$-module $M$ is said to be generated in degree $i$ if $\Lambda M_i = M$. If $M$ has a projective resolution consisting of graded $\Lambda$-modules $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M$ such that $P_i$ is generated in degree $i$ for all $i \geq 0$, we say that $M$ has a linear resolution. The algebra $\Lambda$ is said to be Koszul if every simple module $L_A(t)$ has a linear resolution. We remark that in these notions, $\Lambda$ may in principle be any graded algebra. For further
details on graded algebras and modules, we refer to [GG82]. In [LX17], the authors prove the following theorem, connecting the koszulity of $\mathcal{A}(B,A^{op})$ to the koszulity of $B$ and of $A$.

11. Theorem. [LX17] Proposition 3.5] The algebra $\Lambda = \mathcal{A}(B,A^{op})$ is Koszul if and only if both $B$ and $A$ are Koszul.

This motivates the question of whether or not there is some corresponding statement concerning the existence of linear resolutions of the standard modules $\Delta_{\Lambda}(i)$.

12. Definition. Let $\Lambda$ be a graded quasi-hereditary algebra. Then, $\Lambda$ is said to be

(i) left standard Koszul if the left standard modules $\Delta_{\Lambda}(i) \in \Lambda$-mod have linear resolutions, for all $i \in \{1,\ldots,n\}$.

(ii) right standard Koszul if the right standard modules $\Delta_{\Lambda}^{op}(i) \in \Lambda^{op}$-mod have linear resolutions, for all $i \in \{1,\ldots,n\}$.

13. Theorem. The algebra $\Lambda$ is left standard Koszul if and only if $B$ is Koszul, and right standard Koszul if and only if $A$ is Koszul.

Proof. By Lemma 5, $P_B^1 \to L_B(i)$ is a minimal projective resolution if and only if $F(P_B^1) = P_B^1 \to \Delta_{\Lambda}(i)$ is a minimal projective resolution. Since the degree of the idempotents $e_j$ is zero in both the grading on $B$ and the grading on $\Lambda$, the $k$th term of $P_B^k$, the module $P_B^k$, is generated in degree $k$ if and only if the $k$th term of $P^1$, the module $P^1$, is generated in degree $k$. Therefore, $L_B(i)$ has a linear resolution if and only if $\Delta_{\Lambda}(i)$ has a linear resolution, so $B$ is Koszul if and only if $\Lambda$ is left standard Koszul.

For the second statement, note that $\Lambda$ is right standard Koszul if and only if $\Lambda^{op}$ is left standard Koszul. But, $\Lambda^{op} = \mathcal{A}(B,A^{op}) = \mathcal{A}(A,B^{op})$, which is left standard Koszul if and only if $A$ is Koszul, by the first part of the proof.

Let $C$ be some Koszul algebra and put $L = L_C(1) \oplus \cdots \oplus L_C(n)$. Then, it is well-known that the condition of $L$ having a linear resolution is equivalent to the internal and homological gradings on $\text{Ext}^*_\Lambda(L,L)$ coinciding. This phenomenon does not generalize to the case of standard koszulity with the current grading on the dual extension algebra.

14. Example. Let $Q$ be the quiver $1 \leftarrow 2$, put $B = KQ$, and consider $\Lambda = \mathcal{A}(B,B^{op})$. Then $\Lambda$ is given by the quiver

$$1 \overset{\alpha}{\leftarrow} 2$$

with the relation $\alpha \alpha' = 0$. The standard modules are

$\Delta_{\Lambda}(1) \cong L_{\Lambda}(1)$ and $\Delta_{\Lambda}(2) : \begin{array}{c} 2 \\ \alpha' \end{array} \xrightarrow{1}$

Putting $\Delta = \Delta_{\Lambda}(1) \oplus \Delta_{\Lambda}(2)$, the space $\text{Hom}_{\Lambda}(\Delta,\Delta)$ contains the inclusion $\Delta_{\Lambda}(1) \hookrightarrow \Delta_{\Lambda}(2)$, which is of homological degree 0 but internal degree 1.

5.1. Alternate grading on the dual extension algebra. To remedy the situation in the previous example, we define a new grading on the dual extension algebra $\Lambda = \mathcal{A}(B,A^{op})$. We still grade $B$ by path length, but put $\deg \alpha' = 0$ for all arrows $\alpha'$ coming from the quiver of $A^{op}$. Note that the idempotents $e_j$ are still homogeneous of degree zero. Since the proof of Theorem 13 relies only on the degree of the idempotents being zero, its conclusion remains true under the new grading on $\Lambda$.

15. Lemma. Let $\Lambda = \mathcal{A}(B,A^{op})$ be left standard Koszul and graded as above. Then, the homological and internal gradings on $\text{Ext}^*_\Lambda(\Delta,\Delta)$ coincide.
Proof. Let $P^*_A$ be a minimal projective resolution of $\Delta_A(i)$. In the proof of Proposition [7] we saw that $\text{Ext}^*_A(\Delta(i), \Delta(j)) \cong \text{Hom}_A(P^k_A, \Delta_A(j))$, where $P^k_A$ is the $k$th term of $P^*_A$. The space $\text{Hom}_A(P^k_A, \Delta_A(j))$ is isomorphic to a direct sum of spaces of the form $\text{Hom}_A(P_A(\ell), \Delta_A(j))$. By Proposition [5] there holds

$$\text{Hom}_A(P_A(\ell), \Delta_A(j)) \cong \text{Hom}_A(P_{A^\oplus}(\ell), P_{A^\oplus}(j)).$$

(i) If $\ell = j$, the space $\text{Hom}_{A^\oplus}(P_{A^\oplus}(\ell), P_{A^\oplus}(\ell))$ equals the span of the identity homomorphism on $P_{A^\oplus}(\ell)$.

(ii) If $\ell \neq j$, any homomorphism $\varphi \in \text{Hom}_{A^\oplus}(P_{A^\oplus}(\ell), P_{A^\oplus}(j))$ is of the form $\rho q'$, where $q'$ is a linear combination of paths in $A^{op}$.

In either case, such a map is homogeneous of degree 0, but when accounting for present degree shifts, we see that the internal degree is $k$.

\[ \square \]

6. $A\infty$-structure on $\text{Ext}_A^*(\Delta, \Delta)$

6.1. Background and the Koszul case. The goal of this section is to describe completely an $A\infty$-structure on $\text{Ext}_A^*(\Delta, \Delta)$, given a certain $A\infty$-structure on $\text{Ext}_A^*(L, L)$. We describe the construction given in [Lu+06], which is a special case of the construction in [Mer99], which in turn is a special case of Kadeishvili’s original construction in [Kad80].

Let $M$ be an $A$-module with minimal projective resolution $P^* \to M$. Form the algebra $\mathcal{D} := \text{End}_A(P^*, P^*)$. Note that the elements of $\mathcal{D}$ are not necessarily chain maps.

We may endow $\mathcal{D}$ with the structure of a dg-algebra by defining a differential on homogeneous maps by

$$\delta(f) := d^P f - (-1)^{|f|} f d^P,$$

where $d^P$ denotes the differential on $P^*$, and $|f|$ denotes the (homological) degree of $f$. We have the following easy observation.

16. Lemma. Let $f \in \mathcal{D}$ be homogeneous of degree 0. Then

(i) We have $\delta(f) = 0$ if and only if $f$ is a chain map, and

(ii) $f \in \text{im} \delta$ if and only if $f$ is null-homotopic.

By Lemma [18] the zeroth homology $H^0(\mathcal{D})$ describes precisely chain maps modulo homotopy, which implies that

$$H^0(\mathcal{D}) = \text{End}_{\mathcal{D}}(\Lambda(P^*)) \cong \text{Ext}^0_A(M, M) = \text{End}_A(M)$$

as graded vector spaces. Considering instead the $k$th homology, $H^k(\mathcal{D})$, we obtain $H^k(\mathcal{D}) \cong \text{Ext}^k_A(M, M)$, and, consequently, $H^*(\mathcal{D}) \cong \text{Ext}^*_A(M, M)$. We may decompose $\mathcal{D}$ as $\mathcal{D} = Z \oplus L$, where $Z$ is the graded subspace spanned by cycles and $L$ is some complement. Then, $Z$ decomposes further as $Z = H \oplus \text{im} \delta$ where $\text{im} \delta$ are the boundary maps and $H$ is some complement. Now, applying the first isomorphism theorem, we have

$$\text{Ext}^*_A(M, M) \cong H^*(\mathcal{D}) = Z / \text{im} \delta \cong H$$

and, from this, it follows that we have an injection $i : \text{Ext}^*_A(M, M) \to H \subset \mathcal{D}$, such that the image of $\text{Ext}^*_A(M, M)$ is isomorphic to a subspace of $\mathcal{D}$, which in turn is isomorphic to the homology. All this amounts to a decomposition $\mathcal{D} \cong \text{im} \delta \oplus H \oplus L$, where $H \cong \text{Ext}^*_A(M, M)$.

Let $p : \mathcal{D} \to \text{Ext}^*_A(M, M)$ be the composition given by the natural projection $p' : \mathcal{D} \to H$ followed by the map $i^{-1} : H \to \text{Ext}^*_A(M, M)$. Isolating degree parts, we have the following:

$$\text{im} \delta_{k+1} \oplus H_{k+1} \oplus L_{k+1} \oplus \text{im} \delta_{k+1} \oplus H_{k+2} \oplus L_{k+2}$$
Since \( \text{im} \partial_{k-1} \) and \( H_k \) consist of cycles and \( L_k \) is chosen to be a complement, we see that the differential \( \partial \) maps \( \text{im} \partial_{k-1} \oplus H_k \oplus L_k \) onto \( \text{im} \partial_k \) with kernel \( \text{im} \partial_{k-1} \oplus H_k \), so that \( L_k \cong \text{im} \partial_k \) via \( \partial \). This allows the definition of a map \( h \in \mathcal{D} \) as below.

\[
h|_{L_k \oplus H_k} = 0, \quad \text{and} \quad h|_{\text{im} \partial_{k-1}} = \partial_{k-1}^{-1}.
\]

Next, construct inductively a sequence of linear maps \( \lambda_n : \mathcal{D}^n \to \mathcal{D} \), such that \( \deg \lambda_n = 2 - n \), as follows.

\[
h \lambda_1 := - \text{id}_\mathcal{D}, \quad \lambda_2 := \text{composition}, \quad \lambda_n := \sum_{r+s=n} (-1)^{r+1} \lambda_2(h \lambda_r \circ h \lambda_s), \quad s \geq 1.
\]

17. Theorem. \([Lu+06; Mer99]\) Put \( m_n := \rho \lambda_n i^{\otimes n} \). Then \((\text{Ext}^*_\Lambda(M, M), m_2, m_3, \ldots)\) is an \( A_{\infty} \)-algebra. Moreover, this structure is unique up to \( A_{\infty} \)-isomorphism.

When \( M = \mathbb{L} \), there is the following nice characterization of when this process yields a trivial \( A_{\infty} \)-structure.

18. Proposition. \([Kel02; Prop. 1]\) The \( A_{\infty} \)-algebra \( \text{Ext}^*_\Lambda(M, \mathbb{L}) \) is \( A_{\infty} \)-isomorphic to an algebra with \( m_n = 0 \) for \( n \geq 3 \) if and only if \( B \) is Koszul.

Moreover, we crucially observe, that when the above construction is performed in the category of graded modules, the differential \( \partial \) is of internal degree 0. This implies that also the map \( h \) is of internal degree 0, which in turn implies that the maps \( \lambda_n \) are of internal degree 0. This allows us to establish a sort of analogue of Proposition 15 in the case where \( M = \Delta \).

19. Proposition. Let \( B \) and \( A \) be directed algebras with \( B \) Koszul. Put \( \Lambda = \mathcal{A}(B, A^{op}) \) and endow \( \Lambda \) with the grading defined prior to Lemma 15. Let \( (m_n) \) be the \( A_{\infty} \)-multiplications on \( \text{Ext}^*_\Lambda(\Delta, \Delta) \) constructed above. Then, we have \( m_n = 0 \) for \( n \neq 2 \).

Proof. By Theorem 11 \( A \) is left standard Koszul. Since the maps \( m_n \) constitute an \( A_{\infty} \)-structure, each \( m_n \) is of homological degree \( 2 - n \). Moreover, we saw that each \( m_n \) is of internal degree 0. By Lemma 15 the two gradings coincide, so that \( 2 - n = 0 \). □

6.2. The general case. Having dealt with the Koszul case, we return to the general case again. The construction used to obtain an \( A_{\infty} \)-structure on \( \text{Ext}^*_\Lambda(M, M) \) is not canonical, in the sense that there are several choices involved. Since our aim is to describe the \( A_{\infty} \)-structure on \( \text{Ext}^*_\Lambda(\Delta, \Delta) \) in terms of the \( A_{\infty} \)-structure on \( \text{Ext}^*_\Lambda(\mathbb{L}, \mathbb{L}) \), we start by investigating how the choices made when obtaining the \( A_{\infty} \)-structure on \( \text{Ext}^*_\Lambda(\Delta, \Delta) \) may be made compatible with those made to obtain the \( A_{\infty} \)-structure on \( \text{Ext}^*_\Lambda(\mathbb{L}, \mathbb{L}) \). Consider the following setup. Recall that, here, \( F \) denotes the functor \( \Lambda \otimes_B - : B\text{-mod} \to \Lambda\text{-mod} \). Let \( P_B^* \to \mathbb{L} \) be a minimal projective resolution, so that \( F(P_B^*) \to \Delta \) is a minimal projective resolution, too, by Lemma 5. Let \( P(B) \) denote the differential on \( P_B^* \). Then, we form the dg algebras \( \mathcal{D}^B = \text{End}_B(P_B^*) \), and \( \mathcal{D}^\Lambda = \text{End}_\Lambda(F(P_B^*)) \). Let \( \partial_B \) denote the differential on \( \mathcal{D}^B \) and let \( \partial^\Lambda \) denote the differential on \( \mathcal{D}^\Lambda \).

Note that we have an injective homomorphism of algebras \( \mathcal{D}^B \to \mathcal{D}^\Lambda \) given by \( f \mapsto \text{id}_\Lambda \otimes f \). Here, we slightly abuse notation and identify \( \Lambda \otimes_B L_B(i) \cong L(\Delta(i)) \) and \( \Lambda \otimes_B P_B(i) \cong \Lambda(i) \). We now perform Merkulov’s construction to obtain decompositions of graded vector spaces

\[
\mathcal{D}^B = H^B \oplus \text{im} \partial_B \oplus L^B, \quad \text{and} \quad \mathcal{D}^\Lambda = H^\Lambda \oplus \text{im} \partial^\Lambda \oplus L^\Lambda,
\]

where \( H^B \cong \text{Ext}^*_B(\mathbb{L}, \mathbb{L}) \), and \( H^\Lambda \cong \text{Ext}^*_\Lambda(\Delta, \Delta) \). Additionally, we obtain maps \( i^B, i^\Lambda, p^B, p^\Lambda, h^B, h^\Lambda, \lambda^B \) and \( \lambda^\Lambda \), so that the \( A_{\infty} \)-structures on \( \text{Ext}^*_B(\mathbb{L}, \mathbb{L}) \) and \( \text{Ext}^*_\Lambda(\Delta, \Delta) \) are given by

\[
m_n^B = p^B \lambda_n^B(i^B)^{\otimes n}, \quad \text{and} \quad m_n^\Lambda = p^\Lambda \lambda_n^\Lambda(i^\Lambda)^{\otimes n},
\]

respectively. We have the following useful observations.

20. Lemma. For any \( f \in \mathcal{D}^B \), there holds
\( \partial^B(f) = \partial^\Lambda(F(f)) \).

ii) \( \partial^B(f) = 0 \) if and only if \( \partial^\Lambda(F(f)) = 0 \).

iii) \( f \in \text{im} \partial^B \) if and only if \( F(f) \in \text{im} \partial^\Lambda \).

\textbf{Proof.} Note that \( \partial^\Lambda = F(\partial^B) = \text{id}_A \otimes \partial^B \).

i) We have
\[
F \circ \partial^B(f) = \text{id}_A \otimes (d^P f - (-1)^{|f|} f d^P) = (\text{id}_A \otimes d^P)(\text{id}_A \otimes f) - (-1)^{|f|}(\text{id}_A \otimes f)(\text{id}_A \otimes d^P) = \partial^\Lambda(F(f)),
\]

since \(|f| = |\text{id}_A \otimes f|\).

ii) Suppose \( \partial^B(f) = 0 \). Then, \( \partial^\Lambda(F(f)) = F(\partial^B(f)) = F(0) = 0 \), by i). Suppose instead that \( \partial^\Lambda(F(f)) = 0 \). By i), we have \( 0 = \partial^\Lambda(F(f)) = F(\partial^B(f)) \). Applying the functor \( G \), we have \( 0 = G \circ F(\partial^B(f)) \). By Lemma 5, \( G \circ F \) is naturally isomorphic to \( \text{Id}_{B_{-\text{mod}}} \), implying there is a commutative diagram
\[
\begin{array}{ccc}
G \circ F(P^*) & \xrightarrow{G \circ F(\partial^B(f))} & G \circ F(P^*) \\
\psi \downarrow & & \downarrow \psi \\
P^* & \xrightarrow{\partial^B(f)} & P^*
\end{array}
\]

and since \( \psi \) is an isomorphism, this implies \( \partial^B(f) = 0 \).

iii) Suppose \( f \in \text{im} \partial^B \), so that \( \partial^B(g) = f \) for some \( g \). We claim that \( \partial^\Lambda(F(g)) = F(f) \). Indeed,
\[
\partial^\Lambda(F(g)) = (\text{id}_A \otimes d^P)(\text{id}_A \otimes g - (-1)^{|g|}(\text{id}_A \otimes g)(\text{id}_A \otimes d^P) = \text{id}_A \otimes (d^P g - (-1)^{|g|}g d^P) = \partial^\Lambda(F(g)) = F(f).
\]

Suppose, instead, that \( \partial^\Lambda(g) = F(f) \). For any \( \alpha \in \partial^B \), we have a commutative diagram
\[
\begin{array}{ccc}
G \circ F(P^*) & \xrightarrow{G \circ F(\partial^B(\alpha))} & G \circ F(P^*) \\
\psi \downarrow & & \downarrow \psi \\
P^* & \xrightarrow{\alpha} & P^*
\end{array}
\]

according to Lemma 5, so that \( G \circ F(\alpha) = \psi^{-1} \alpha \psi \). We claim that \( \partial^B(\psi G(g) \psi^{-1}) = f \). Applying \( G \) to \( F(f) \), we get
\[
GF(f) = G(\partial^\Lambda(g)) = G\left( \text{id}_A \otimes d^P \right) g - (-1)^{|g|}g \left( \text{id}_A \otimes d^P \right) = G\left( \text{id}_A \otimes d^P \right) G(g) - (-1)^{|g|}G(g) \left( \text{id}_A \otimes d^P \right) = \psi^{-1} d^P \psi G(g) - (-1)^{|g|}G(g) (\psi^{-1} d^P \psi).
\]

Now, we use that \( GF(f) = \psi^{-1} f \psi \) and compose with \( \psi \) on the left and with \( \psi^{-1} \) on the right, to get
\[
f = \psi GF(f) \psi^{-1} = d^P (\psi G(g) \psi^{-1}) - (-1)^{|g|} (\psi G(g) \psi^{-1}) d^P = \partial^B(\psi G(g) \psi^{-1}).
\]

In what follows, let \( \hat{\text{rad}}(\Lambda, \Delta) \) denote the space of chain map representatives of homomorphisms, contained in \( \text{rad}(\Lambda, \Delta) \), which are of the form discussed prior to Proposition 5.

\textbf{21. Lemma.} Let \( \varepsilon \in \partial^\Lambda \) be homogeneous map of degree \( n \). Then, for any \( f' \in \hat{\text{rad}}(\Lambda, \Delta) \), the composition \( f' \circ \varepsilon \) has, at most, one non-zero component, and is given by a matrix, whose entries are maps of the form \( \rho q' \), where \( q' \) is some linear combination of paths in \( \Lambda^{op} \).
Proof. We draw the composition \( f' \circ \varepsilon \). Recall that the chain map \( f' \) may be chosen to be of the below form, according to the discussion prior to Proposition 9.

\[
\begin{array}{ccccccc}
\cdots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & \cdots \\
\downarrow \varepsilon_{n-1} & & \downarrow \varepsilon_n & & & & \downarrow \\
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\
\downarrow 0 & & \downarrow f & & & & \downarrow \\
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0
\end{array}
\]

From this picture it is clear that \( f' \circ \varepsilon \) has at most one non-zero component, namely, the homomorphism \( \hat{f} \varepsilon_n : P_n \rightarrow P_0 \). Fix a decomposition of \( P_n \) and \( P_0 \) into indecomposable projective modules. We already know that the matrix of \( \hat{f} \), with respect to the above decomposition, has entries of the form \( \rho_q \), where \( q' \) is a linear combination of paths in \( \Lambda^{op} \). The entries of the matrix of \( \varepsilon_n \), with respect to the above decomposition, are homomorphisms between indecomposable projective \( \Lambda \)-modules. There are three cases.

(i) The matrix of \( \varepsilon_n \) has an entry \( \varepsilon_n : P_A(x) \rightarrow P_A(x) \) for some vertex \( x \). Then, \( \varepsilon_n = a_1P_A(x) + bP_v \), where \( v \) is some linear combination of paths in \( \Lambda \) and \( a, b \in K \) are scalars. Note that the paths occurring in \( v \) are of the form \( u'w \), with both \( u \) and \( w' \) being non-trivial paths, in \( \Lambda^{op} \) and \( B \), respectively. The resulting entry of the matrix of the composition, \( f' \circ \varepsilon \), is

\[
\rho_q(a_1P_A(x) + bP_v) = a_1\rho_qP_A(x) + b\rho_v = a\rho_q,
\]

since \( vq = 0 \), because \( w'uq = 0 \), according to the dual extension relation.

(ii) The matrix of \( \varepsilon_n \) has an entry \( \varepsilon_n : P_A(x) \rightarrow P_A(y) \), with \( x > y \). Then, \( \varepsilon_n = \rho_u \), where \( v \) is a linear combination of paths from \( y \) to \( x \) and, again, the paths occurring in \( v \) are of the form \( u'w \), where \( w' \) and \( u \) are as in the previous case. The resulting entry of the matrix of the composition, \( f' \circ \varepsilon \), is

\[
\rho_q'\rho_u = \rho_v = \rho_v.
\]

(iii) The matrix of \( \varepsilon_n \) has an entry \( \varepsilon_n : P_A(x) \rightarrow P_A(y) \), with \( x < y \). Then, \( \varepsilon_n = \rho_{u'} \), where \( u' \) is some linear combination of paths, from \( y \) to \( x \), in \( \Lambda^{op} \). Note that the paths constituting \( u' \) are of the form \( u'w \), with both \( u \) and \( w' \) being non-trivial paths, in \( \Lambda^{op} \). The resulting entry of the matrix of the composition, \( f' \circ \varepsilon \), is \( \rho_q'\rho_{u'} = \rho_{u'} \).

\( \square \)

22. Lemma. There is an isomorphism of vector spaces \( \text{rad}(\Lambda, \Lambda) \circ F(H^B) \cong \text{rad}(\Lambda, \Lambda) \circ F(\text{Ext}_B^*(\mathbb{L}, \mathbb{L})) \).

Proof. Let \( \chi^B : H^B \rightarrow \text{Ext}_B^*(\mathbb{L}, \mathbb{L}) \) be the inverse of the isomorphism \( \iota^B : \text{Ext}_B^*(\mathbb{L}, \mathbb{L}) \rightarrow H^B \). Since

\[
F : H^B \rightarrow F(H^B) \quad \text{and} \quad F : \text{Ext}_B^*(\mathbb{L}, \mathbb{L}) \rightarrow F(\text{Ext}_B^*(\mathbb{L}, \mathbb{L}))
\]

are linear isomorphisms, we have an isomorphism of vector spaces \( F\chi^BF^{-1} : F(H^B) \rightarrow F(\text{Ext}_B^*(\mathbb{L}, \mathbb{L})) \).

For a homomorphism \( f' \in \text{rad}(\Lambda, \Lambda) \), let \( \hat{f} \) denote its chain map representative in \( \text{rad}(\Lambda, \Lambda) \). We claim that any two maps in \( \text{rad}(\Lambda, \Lambda) \circ F(H^B) \) are not homotopic. Indeed, assume that \( \hat{f} \varepsilon \) and \( \hat{g} \delta \) are homotopic. Consider the following picture.

\[
\begin{array}{ccccccc}
P_{n+1} & \overset{d_n}{\longrightarrow} & P_n & \overset{d_{n-1}}{\longrightarrow} & P_{n-1} \\
\downarrow h_n & & \downarrow & & \downarrow h_{n-1} \\
P_1 & \overset{d_1}{\longrightarrow} & P_0 & \overset{d_0}{\longrightarrow} & 0
\end{array}
\]

The entries of the matrix of the map \( d_1h_n - h_{n-1}d_{n-1} \) are of the form \( \rho_{q'p} \), where \( q'p \) is some linear combination of paths in \( \Lambda \). Since \( \delta^\lambda = \iota^\lambda \otimes \delta^B \), all paths in \( q'p \) contain non-trivial subpaths in \( B \). At the same time, by Lemma 21, the entries of the matrix of the map \( \hat{f} \varepsilon_n - \hat{g} \delta_n \) are of the form \( \rho_{v'p} \), where \( v' \)
is a linear combination of paths in $A^\partial$. Since any path containing a non-trivial subpath in $B$ is linearly independent from any path in $A^\partial$, we cannot have $d_1h_n + h_{n-1}d_{n-1} = f \varepsilon_n - g \delta_n$ unless $\hat{f} \varepsilon_n - \hat{g} \delta_n = 0$, which is equivalent to $\hat{f} \varepsilon = \hat{g} \delta$.

Now, define a linear map $\varphi : \operatorname{rad}(\Delta, \Delta) \circ F(H^B) \to \operatorname{rad}(\Delta, \Delta) \circ F(\operatorname{Ext}_B^*(L, L))$ by

$$\hat{f} \circ F(\varepsilon) \mapsto f' \circ F^{\chi}(\varepsilon) F^{-1}(\varepsilon) = f' \circ F^{\chi}(\varepsilon).$$

It is immediately clear that $\varphi$ is surjective. Moreover, $\varphi$ is injective. To see this, observe that if we have $f' \circ F^{\chi}(\varepsilon) = g' \circ F^{\chi}(\delta)$, then, the chain maps $\hat{f} \circ F(\varepsilon)$ and $\hat{g} \circ F(\delta)$ both lift the extension $f' \circ F^{\chi}(\varepsilon)$ to the homotopy category. Consequently, we must have $\hat{f} \circ F(\varepsilon) = \hat{g} \circ F(\delta)$.

With these elementary properties established, we are ready to prove the following key proposition. The inspiration for this result is Theorem 3.21 in [KM21], which is similar, albeit dealing with a more general context.

23. Proposition. Suppose we perform Merkulov's construction on $\mathcal{B}$, to obtain the decomposition $\mathcal{B} = H^B \oplus \im \partial^B \oplus L^B$. Then, we may perform Merkulov's construction on $\mathcal{A}$, to obtain a decomposition $\mathcal{A} = H^A \oplus \im \partial^A \oplus L^A$, such that

$$H^A = F(H^B) \oplus (\operatorname{rad}(\Delta, \Delta) \circ F(H^B)), \quad \im \partial^A = F(\im \partial^B) \oplus \partial^A(L) \quad \text{and} \quad L^A = F(L^B) \oplus L,$$

where $\operatorname{rad}(\Delta, \Delta)$ is the space consisting of chain map representatives of homomorphisms in $\operatorname{rad}(\Delta, \Delta)$, which are of the form described in the discussion prior to Proposition[3].

Proof. Suppose we have $\mathcal{B} = Z^B \oplus L^B$, where $Z^B$ is the subspace spanned by cycles and $L^B$ is some complement. Let $Z^A \subset \mathcal{A}$ be the subspace spanned by cycles. Then, there holds $\im F(Z^B) \subset Z^A$, by Lemma 20.

We claim that $\im F(L^B) \cap Z^A = \{0\}$. To see this, let $f \in \im F(L^B) \cap Z^A$. Since $f$ is in the image of $F$, we have $f = \id_A \circ g$, for some $g \in \mathcal{B}$. Then $\partial^A(\id_A \circ g) = 0$, which implies $\partial^A(g) = 0$, so that $g \in Z^B \cap L^B$, which, in turn, implies $g = 0$. Then, we also have $f = 0$. This implies that we have a decomposition

$$\mathcal{A} = Z^A \oplus (F(L^B) \oplus \hat{L}),$$

for some $\hat{L}$. Next, we claim that $\im \partial^A = F(\im \partial^B) \oplus \partial^A(\hat{L})$. To see this, we note that $\im F(\im \partial^B) \cap \partial^A(\hat{L}) = \{0\}$. Indeed, suppose $f \in \im F(\im \partial^B) \cap \partial^A(\hat{L})$. Then, we have $f = \id_A \circ \partial^B(g) = \partial^A(\id_A \circ g)$. If $f \neq 0$, we must have $\partial^A(g) \neq 0$. We may then write $g = g_1 + g_2$, with $g_1 \in L^B$ and $g_2 \in Z^B$. At the same time, we have $f = \partial^A(g_2)$ for some $g_2 \in L^A$. Now we have $f = \partial^A(g_2) = \partial^A(\id_A \circ g_1)$ with $\id_A \circ g_1 \in L^A$. Since $\partial^A$ restricts to an isomorphism on $L^A$, we get $g = \id_A \circ g_1 = 0$ since $F(L^B) \cap L = \{0\}$, and, consequently, $f = 0$. Then, we see that

$$\im \partial^A = \partial^A(\im \partial^B) \oplus \partial^A(\hat{L}) = \partial^A(\im \partial^B) \oplus \partial^A(\hat{L}) = F(\im \partial^B) \oplus \partial^A(\hat{L}),$$

where, in the last equality, we use that the intersection $F(\im \partial^B) \cap \partial^A(\hat{L})$ is zero, to conclude that the sum on the right is direct. This gives us a decomposition $\mathcal{A} = H^A \oplus F(\im \partial^B) \oplus \partial^A(\hat{L}) \oplus (F(L^B) \oplus \hat{L})$. Next, we claim that $F(H^B) \cap (F(\im \partial^B) \oplus \partial^A(\hat{L})) = \{0\}$. Let $\id_A \circ f \in F(H^B) \cap (F(\im \partial^B) \oplus \partial^A(\hat{L}))$. Then, we must have $\id_A \circ f = \id_A \circ \partial^B(g) + \partial^A(\hat{g})$ for $f \in H^B, g \in L^B$ and $\hat{g} \in \hat{L}$. But, then, we get

$$\id_A \circ f = \id_A \circ \partial^B(g) + \partial^A(\hat{g}) = \partial^A(\id_A \circ g),$$

which implies that $f \in \im \partial^B$, by Lemma 20. This means that we must have $f = 0$, since $f \in H^B \cap \im \partial^B$. This fact, in turn, implies that we may choose a complement $\hat{H}$ of $F(H^B) \cap \im \partial^A$ in $Z^A$, giving us a decomposition}
Next, we want to show that it is possible to choose \( \hat{H} = \hat{\text{rad}}(\Delta, \Delta) \circ F(H^B) \). To this end, we claim, that for \( f \in \text{rad}(\Delta, \Delta) \circ F(H^B) \), we have \( \partial^\Lambda(f) = 0 \). Consider the following picture.

\[
P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \\
\downarrow 0 \quad \downarrow f \quad \downarrow 0 \\
P_1 \xrightarrow{f} P_0
\]

Due to \( f \) having only one non-zero component, the only possibly non-zero component of the map \( \partial^\Lambda(f) \) is \( f d_{n+1} \). Since the entries of the matrix of \( f \) are right multiplications by paths in \( A^{op} \), and the entries of the matrix of \( d_{n+1} \) are right multiplications by linear combinations of paths in \( B \), we have \( \hat{f} d_{n+1} = 0 \) now follows from the fact that \( \rho_{\tilde{q}'\rho_p} = \rho p q' = 0 \), where we use that \( p q' = 0 \), according to the dual extension relation.

Next, we check that \((\hat{\text{rad}}(\Delta, \Delta) \circ F(H^B)) \cap (F(H^B) \oplus F(\text{im}\partial^B) \oplus \partial^\Lambda(\hat{L})) = \{0\} \). Let \( f = f_1 + f_2 + f_3 \) be contained in this intersection, where \( f \in \text{rad}(\Delta, \Delta) \circ F(H^B), f_1 \in F(H^B), f_2 \in F(\text{im}\partial^B) \), and \( f_3 \in \partial^\Lambda(\hat{L}) \). Since \( f \in \text{rad}(\Delta, \Delta) \circ F(H^B) \), we apply Lemma 21 to see that \( f \) has at most one non-zero component. Therefore, we compare the maps \( \hat{f} = \hat{f}_1 + \hat{f}_2 + \hat{f}_3 : P_n \to P_0 \). Decomposing \( P_n \) and \( P_0 \) into indecomposable summands, the map \( \hat{f} = \hat{f}_1 + \hat{f}_2 + \hat{f}_3 \) is given by a matrix. Considering a non-zero entry, \( \hat{f}_i \), of this matrix, we write \( f = f_1 + f_2 + f_3 \). Since \( f_1 \in F(H^B) \), we have \( f_1 = \rho p_1 \) for some linear combination of paths, \( p_1 \), in \( B \). Similarly, we have \( f_2 = \rho p_2 \), for some linear combination of paths, \( p_2 \), in \( B \). Since \( \partial^\Lambda = \text{id} \circ \partial^B \), and \( f_3 \in \partial^\Lambda(\hat{L}) \), we have \( f_3 = \rho w \), where \( w \) is a path in \( \Lambda \) containing a non-trivial subpath in \( B \). But any path in \( A^{op} \) is linearly independent from \( p_1, p_2 \) and \( w \), which implies \( f = 0 \).

This shows that a complement \( \hat{H} \) of \( F(H^B) \), in the homology \( H^\Lambda \), may be chosen so that the subspace \( \text{rad}(\Delta, \Delta) \circ F(H^B) \) is contained in \( \hat{H} \). It is now enough to show that the space \( \text{rad}(\Delta, \Delta) \circ F(H^B) \) has the “correct” dimension.

By construction, the homology \( H^\Lambda \) is isomorphic to \( \text{Ext}_A^\Lambda(\Delta, \Delta) \). In the proof of Theorem 10, we saw that \( \text{Ext}_A^\Lambda(\Delta, \Delta) \) is generated by the extensions \( F(\text{Ext}^B_0(\hat{L}, \hat{L})) \) together with the homomorphisms \( \text{rad}(\Delta, \Delta) \). If \( B \) and \( B' \) are bases of \( \text{Ext}^B_0(\hat{L}, \hat{L}) \) and of \( \text{rad}(\Delta, \Delta) \), respectively, then, the non-zero elements of the set \( \{\epsilon, f \circ \epsilon | \epsilon \in B, f \in B'\} \) form a basis of \( \text{Ext}_A^\Lambda(\Delta, \Delta) \), as shown in the proof of Theorem 10. This implies that

\[
\dim \text{Ext}_A^\Lambda(\Delta, \Delta) = \dim F(\text{Ext}^B_0(\hat{L}, \hat{L})) + \dim(\text{rad}(\Delta, \Delta) \circ F(\text{Ext}^B_0(\hat{L}, \hat{L}))) = \dim F(H^B) + \dim \hat{H},
\]

which, in turn, implies that \( \dim \hat{H} = \dim(\text{rad}(\Delta, \Delta) \circ F(\text{Ext}^B_0(\hat{L}, \hat{L}))) \). Now we appeal to Lemma 22, the statement of which is equivalent to

\[
\dim(\text{rad}(\Delta, \Delta) \circ F(\text{Ext}^B_0(\hat{L}, \hat{L}))) = \dim(\hat{\text{rad}}(\Delta, \Delta) \circ F(H^B)),
\]

completing the proof.

24. Corollary. For any \( f \in B \), there holds \( F \circ p^B(f) = p^A \circ F(f) \).

Proof. Let \( \chi^B : H^B \to \text{Ext}^B_0(\hat{L}, \hat{L}) \) be the inverse of the isomorphism \( \iota^B : \text{Ext}^B_0(\hat{L}, \hat{L}) \to H^B \). Similarly, let \( \chi^A : H^A \to \text{Ext}^A_0(\hat{L}, \hat{L}) \) be the inverse of the isomorphism \( \iota^A : \text{Ext}^A_0(\Delta, \Delta) \to H^A \). Recall that, by definition, we have \( p^B = \chi^B p^A \) and \( p^A = \chi^A p^\Lambda \), where \( p^A \) and \( p^\Lambda \) are as below. Consider the following diagram.
The left square commutes, by Proposition 23. We proceed by induction. For \( n \) we have
\[
H^\Delta = F(H^B) \oplus (\hat{\text{rad}}(\Delta, \Delta) \circ F(H^B)), \quad \text{and} \quad \text{Ext}_\Lambda^\Delta(\Delta, \Delta) = F(\text{Ext}_B^\Delta(\Lambda, \Lambda)) \oplus (\hat{\text{rad}}(\Delta, \Delta) \circ F(\text{Ext}_B^\Delta(\Lambda, \Lambda))).
\]

Since \( F : H^B \to F(H^B) \) and \( F : \text{Ext}_B^\Delta(\Lambda, \Lambda) \to F(\text{Ext}_B^\Delta(\Lambda, \Lambda)) \) are isomorphisms, we may define an isomorphism \( \psi : F(H^B) \to F(\text{Ext}_B^\Delta(\Lambda, \Lambda)) \) by \( \psi = F \chi F^{-1} \). Let \( \varphi : \hat{\text{rad}}(\Delta, \Delta) \circ F(H^B) \to \hat{\text{rad}}(\Delta, \Delta) \circ F(\text{Ext}_B^\Delta(\Lambda, \Lambda)) \) be the isomorphism constructed in the proof of Lemma 22. Define the isomorphism \( \chi : F(H^B) \oplus (\hat{\text{rad}}(\Delta, \Delta) \circ F(H^B)) \to F(\text{Ext}_B^\Delta(\Lambda, \Lambda)) \) as the matrix \( \chi = [\varphi \ 0] \), and consider the following diagram.

\[
\begin{array}{ccc}
H^B & \xrightarrow{\chi^B} & \text{Ext}_B^\Delta(\Lambda, \Lambda) \\
F(H^B) \oplus (\hat{\text{rad}}(\Delta, \Delta) \circ F(H^B)) & \xrightarrow{[\varphi \ 0]} & F(\text{Ext}_B^\Delta(\Lambda, \Lambda)) \oplus (\hat{\text{rad}}(\Delta, \Delta) \circ \text{Ext}_B^\Delta(\Lambda, \Lambda))
\end{array}
\]

Now, for any \( f \in H^B \), we have
\[
\begin{bmatrix}
\psi & 0 \\
0 & \varphi
\end{bmatrix}
\begin{bmatrix}
f
\end{bmatrix}
= \begin{bmatrix}
\psi & 0 \\
0 & \varphi
\end{bmatrix}
\begin{bmatrix}
F(f)
\end{bmatrix}
= \begin{bmatrix}
\psi F(f) & 0
\end{bmatrix}
= \begin{bmatrix}
F \chi^B(f) & 0
\end{bmatrix}
= \begin{bmatrix}
F
\end{bmatrix}
\chi^B(f).
\]

25. Corollary. For any \( f \in F^B \), there holds \( F \circ h^B(f) = h^\Lambda \circ F(f) \).

Proof. We consider the diagram

\[
\begin{array}{ccc}
H^B \oplus \text{im} \partial^B \oplus L^B & \xrightarrow{F} & F(H^B) \oplus (\hat{\text{rad}}(\Delta, \Delta) \circ F(H^B)) \oplus (\text{im} \partial^B) \oplus \partial^\Lambda(\hat{L}) \oplus (\text{im} L^B) \\
\text{im} \partial^B & \xrightarrow{h^B} & h^\Lambda \\
L^B & \xrightarrow{F} & F(L^B)
\end{array}
\]

Suppose \( f \in H^B \oplus L^B \). Then the bottom path is immediately zero, and the top path is too, because we have \( F(f) \in H^\Lambda \oplus L^\Lambda \), which implies that \( h^\Lambda F(f) = 0 \). If, instead, \( f \in \text{im} \partial^B \), put \( f = \partial^B(g) \) for some \( g \in L^B \). Then, the bottom path equals \( F(g) \). For the top path, we get \( h^\Lambda F(f) = h^\Lambda F(\partial^B(g)) = h^\Lambda \partial^\Lambda F(g) = F(g) \) since \( F(g) \in F(L^B) \).

Armed with Proposition 23 and its corollaries, we are ready to show that the \( A_\infty \)-structures on \( \text{Ext}_B^\Delta(\Lambda, \Lambda) \) and \( \text{Ext}_B^\Delta(\Lambda, \Delta) \) obtained from our process respect the embedding \( \text{Ext}_B^\Delta(\Lambda, \Lambda) \to \text{Ext}_B^\Delta(\Lambda, \Delta) \).

26. Proposition. Let \( n \geq 2 \). For any \( \epsilon_1, \ldots, \epsilon_n \in \text{Ext}_B^\Delta(\Lambda, \Lambda) \), there holds the following formula.
\[
F\left(m_n^B(\epsilon_1, \ldots, \epsilon_n)\right) = m_n^\Lambda(F(\epsilon_1), \ldots, F(\epsilon_n)).
\]

Proof. We begin by proving the corresponding statement for the maps \( \lambda_n^B \) and \( \lambda_n^\Lambda \), that is,
\[
F\left(\lambda_n^B(\epsilon_1, \ldots, \epsilon_n)\right) = \lambda_n^\Lambda(F(\epsilon_1), \ldots, F(\epsilon_n)).
\]

Note that, in the above formula, we have abused notation and identified the extensions \( \epsilon_1, \ldots, \epsilon_n \) with their chain map representatives. We proceed by induction. For \( n = 2 \), we have
\[
F(\lambda_2^B(\epsilon_1, \epsilon_2)) = F(\epsilon_1 \epsilon_2) = F(\epsilon_1)F(\epsilon_2) = \lambda_2^\Lambda(F(\epsilon_1)F(\epsilon_2)),
\]
since the maps $\lambda^B_0$ and $\lambda^A_2$ are defined as composition. The basis of the induction is thus clear. Let $n \geq 3$ and suppose the formula (1) holds for all $r, s < n$. Then, adopting the convention that $h^A F \lambda^A_1 = - F$, we have

$$\lambda^A_n(F(\epsilon_1), \ldots, F(\epsilon_n)) = \sum_{r+s=n} (-1)^{s+1} \lambda^A_2(h^A F \lambda^A_r(F(\epsilon_1), \ldots, F(\epsilon_r) \otimes h^A \lambda^A_s(F(\epsilon_{r+1}), \ldots, F(\epsilon_n)))$$

$$= \sum_{r+s=n} (-1)^{s+1} \lambda^A_2(h^A F(\lambda^B_r(\epsilon_1, \ldots, \epsilon_r)) \otimes \lambda^A_s(\epsilon_{r+1}, \ldots, \epsilon_n))$$

$$= \sum_{r+s=n} (-1)^{s+1} \lambda^A_2(F(h^B \lambda^B_r(\epsilon_1, \ldots, \epsilon_r) \otimes \lambda^B_s(\epsilon_{r+1}, \ldots, \epsilon_n))$$

$$= F \left( \sum_{r+s=n} (-1)^{s+1} \lambda^B_2(h^B \lambda^B_r(\epsilon_1, \ldots, \epsilon_r) \otimes \lambda^B_s(\epsilon_{r+1}, \ldots, \epsilon_n)) \right)$$

$$= F(\lambda^B_n(\epsilon_1, \ldots, \epsilon_n)),$$

where we additionally use Corollary 25. This fact, together with Corollary 24 then implies

$$m^A_n(F(\epsilon_1), \ldots, F(\epsilon_n)) = p^A \lambda^A_n(i^A) \otimes m(F(\epsilon_1), \ldots, F(\epsilon_n)) = p^A \lambda^A_n(F(\epsilon_1), \ldots, F(\epsilon_n)) = p^A(F(\lambda^B_n(\epsilon_1, \ldots, \epsilon_n)))$$

$$= F(p^B \lambda^B_n(\epsilon_1, \ldots, \epsilon_n)) = F(p^B \lambda^B_n \otimes m(\epsilon_1, \ldots, \epsilon_n)) = F(m^B_n(\epsilon_1, \ldots, \epsilon_n)). \tag*{\square}$$

Proposition 26 describes how the $A_\infty$-multiplications on $\text{Ext}_A^n(\Delta, \Delta)$ behave when their arguments are extensions of the form $F(\epsilon)$ for $\epsilon \in \text{Ext}_B^n(\mathbb{L}, \mathbb{L})$. But, as we know from Theorem 10, the algebra $\text{Ext}_A^n(\Delta, \Delta)$ contains extensions which are not of this form.

To be able to say something about the $A_\infty$-structure on the whole of $\text{Ext}_A^n(\Delta, \Delta)$, we make the following technical assumption. Assume in what follows, that in the decomposition $\mathcal{D}^A = H^A \oplus \im \partial^A \oplus L^A$, the space $L^A$ may be chosen in such a way that all components of any map $\epsilon \in L^B$ are radical maps, that is, the components of $\epsilon \in L^B$ are given by matrices, whose entries are of the form $\rho_p$, where $p$ is a linear combination of non-trivial paths in $B$.

27. Lemma. In the decomposition $\mathcal{D}^A = H^A \oplus \im \partial^A \oplus L^A$, the space $L^A$ may be chosen in such a way that all components of any map in $L^A$ are given by matrices whose entries are of the form $\rho_p q$, where $p$ is a linear combination of non-trivial paths in $B$ and $q'$ is some linear combination of paths in $A^{op}$.

Proof. Consider $\epsilon \in \mathcal{D}^A$, with $\epsilon$ homogeneous of degree $n$. Recall that, if $P^* \rightarrow \mathbb{L}$ is a minimal projective resolution, then so is $F(P^*) \rightarrow \Delta$. We write $\epsilon = \alpha + \beta + \gamma$, where $\alpha \in H^A, \beta \in \im \partial^A$ and $\gamma \in L^A$. Consider the following picture.

$$\begin{array}{ccc}
F(P_{n+1}) & \rightarrow & F(P_n) \rightarrow F(P_{n-1}) \\
\epsilon : & \epsilon_{n+1} & \epsilon_n \\
F(P_1) & \rightarrow & F(P_0) \rightarrow 0
\end{array}$$

Fix decompositions of $P_n$ and $P_0$ into direct sums of indecomposable projective $B$-modules. This induces natural decompositions of $F(P_n)$ and $F(P_0)$ into direct sums of indecomposable projective $A$-modules. Assume now that the matrix of the map $\epsilon_n$ has a non-zero entry, $f$, at some fixed position, which is not a radical map. Then, we may write $f = \mu \cdot 1_{P_{\lambda}(x)} + f'$, where $\mu \in K$ is some scalar and $f' \in \rad(P_{\lambda}(x), P_{\lambda}(x))$.

Consider now instead the map $\tau \in \mathcal{D}^B$, homogeneous of degree $n$, such that the entry at the fixed position of the matrix of $\tau_n$ is equal to $\mu \cdot 1_{P_{\lambda}(x)}$ and all other entries are 0. Put $\tau_k = 0$ for $n \neq k$. 
Now, we write \( \mathcal{T} = \mathcal{T}' + \mathcal{T}' \), where \( \mathcal{T}' \in H^B, \mathcal{T} \in \mathcal{L}^B \). The maps in \( L^B \) have only radical components by assumption, and since the differential maps on \( P^\bullet \) are radical maps, the map \( \mathcal{T}' \) has only radical components. Since the map \( 1_{p_{\mathcal{T}}} \) cannot be written as a linear combination of radical maps, it follows that the map \( \mathcal{T}' \) is given by a matrix whose entry at the fixed position equals \( \mu \cdot 1_{p_{\mathcal{T}}} + \gamma \), where \( g : P_{\mathcal{T}}(x) \to P_{\mathcal{T}}(y) \) is a radical map. Since \( B \) is directed, we have \( \dim \text{End}_{B}(P_{\mathcal{T}}(x)) = 1 \), which implies \( g = 0 \).

Consider now the map \( F(\mathcal{T}) \). This is a chain map, which, by construction, is such that the map \( F(\mathcal{T})_n \) is given by a matrix whose entry at the fixed position is \( F(\mu \cdot 1_{p_{\mathcal{T}}}) = \mu \cdot 1_{p_{\mathcal{T}}(x)} \).

It follows now, that the map \( (\mathcal{T}' - F(\mathcal{T}))_n \) is given by a matrix, whose entry at the fixed position is
\[
f - \mu \cdot 1_{p_{\mathcal{T}}(x)} = \mu \cdot 1_{p_{\mathcal{T}}(x)} + f' - \mu \cdot 1_{p_{\mathcal{T}}(x)} = f'.
\]

Repeating this argument, we may write \( \mathcal{T}' = \bar{\mathcal{T}} + \mathcal{T}' \), where \( \mathcal{T}' \) has only radical components, and where \( \sum_{n=1}^{N} F(\mathcal{T}'_n) \in F(H^B) < H^A \).

Assume now, instead, that the map \( \mathcal{T}_n \) is given by a matrix whose entry at the fixed position equals \( f = \rho q' \), where \( q' \) is a linear combination of paths in \( A^\bullet \), and \( r' \) is a linear combination of non-trivial paths in \( A^\bullet \). According to the discussion prior to Proposition 9 there is a chain map \( \delta : F(P^\bullet) \to F(P^\bullet)[n] \), such that the map \( \delta_n : F(P_n) \to F(P_0) \) is given by a matrix, whose only non-zero entry is at the fixed position, where it is \( \rho_{r'} \), and with \( \delta_k = 0 \), for \( k \neq n \). Now, we argue as in the previous case, and consider the map \( (\mathcal{T}' - \delta)_n \), which is given by a matrix whose entry at the fixed position is equal to \( \rho_{q'} + \rho_{r'} - \rho_{r'} = \rho_{q'} \).

It follows that we may write \( \mathcal{T}' = \bar{\mathcal{T}} + \delta \), where \( \bar{\mathcal{T}} \) has no component given by a matrix with an entry of the form \( \rho_{q'} \), where \( q' \) is a linear combination of non-trivial paths in \( A^\bullet \), and where \( \delta \in \text{rad}(\Delta, \Delta) \subset H^A \).

Next, let \( \mathcal{D}_A \) denote the subspace of maps whose components are matrices whose entries are of the form \( \rho_{q'} \), where \( p \) is a linear combination of non-trivial paths in \( B \) and \( q' \) is some linear combination of paths in \( A^\bullet \). The argument above shows that we may write \( \mathcal{D}_A = H^A + \mathcal{D}_A \). We observe that \( \mathcal{D}_A \) is a dg subalgebra of \( \mathcal{D}_A \), since, clearly, \( \delta(\mathcal{D}_A) \subset \mathcal{D}_A \). Letting \( Z \subset \mathcal{D}_A \) denote the subspace spanned by cycles, we may write \( \mathcal{D}_A = Z \oplus \mathcal{L} \), where \( \mathcal{L} \) is some complement. We claim that \( \mathcal{D}_A = Z \oplus \mathcal{L} \). We have
\[
\mathcal{D}_A = H^A + \mathcal{D}_A = Z^A + \mathcal{D}_A = Z^A + (Z + \mathcal{L}) = Z^A + \mathcal{L},
\]

since \( H^A \subset Z^A \) and \( Z \subset Z^A \). Assume that \( f \in Z \cap \mathcal{L} \). Then, clearly, \( f \in Z \). But, since \( Z \) and \( \mathcal{L} \) are chosen to be complements, we must have \( f = 0 \).

28. **Lemma.** Let \( \mathcal{T} \in \mathcal{D}_B \) be homogeneous of degree \( n \), such that \( \mathcal{T} = F(\mathcal{T}) \) for some \( \mathcal{T} \). Let \( f' \in \text{rad}(\Delta, \Delta) \) be a chain map representative of a homomorphism \( \Delta(\mathcal{T}) \to \Delta(\mathcal{T}) \). Then, we have \( p^A(f' \mathcal{T}) = f' F(p^B(\mathcal{T})) \) as elements of \( \text{Ext}^A(\Delta, \Delta) \).

**Proof.** By definition, we have \( p^A = \chi^A \pi^A \), where the map \( \pi^A : \mathcal{D}_B \to H^A \) is the natural surjection, and the map \( \chi^A : H^A \to \text{Ext}^A(\Delta, \Delta) \) is the isomorphism defined in the proof of Corollary 24. Similarly, we have \( p^B = \chi^B \pi^B \). Consider the following diagram. Write \( \mathcal{D}_B = H^B \oplus \text{im} \mathcal{D}_B \oplus L^B \) and let \( \mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 \in \mathcal{D}_B \), where \( \mathcal{T}_1 \in H^B, \mathcal{T}_2 \in \text{im} \mathcal{D}_B \) and \( \mathcal{T}_3 \in L^B \).

\[
\begin{array}{cccc}
\mathcal{D}_B & \xrightarrow{p^B} & H^B & \xrightarrow{\pi^B + \text{Ext}^A_B(\mathcal{L}, \mathcal{L})} & F(\text{Ext}^A_B(\mathcal{L}, \mathcal{L})) \\
\mathcal{D}_A & \xrightarrow{\pi^A} & H^A & \xrightarrow{\chi^A} & \text{Ext}^A_A(\Delta, \Delta)
\end{array}
\]
For the top path, we then have
\[ f' F \chi^B \rho^B(\tau) = f' F \chi^B \rho^B(\epsilon_1 + \epsilon_2 + \epsilon_3) = f' F \chi^B(\epsilon_1). \]

Consider now the bottom path. The first two arrows correspond to the map
\[ \epsilon_1 + \epsilon_2 + \epsilon_3 \rightarrow F(\epsilon_1) + F(\epsilon_2) + F(\epsilon_3) \rightarrow f' F(\epsilon_1) + f' F(\epsilon_2) + f' F(\epsilon_3). \]

Arguing as in the proof of Lemma [21] we know that the map \( f' F(\epsilon_2) \) has at most one non-zero component, which we denote by \( \hat{\epsilon} \). Since \( \hat{\epsilon} \) is in \( \text{im} \partial B \), and the differential maps are radical maps, the map \( \epsilon_2 \) is given by a matrix whose entries are of the form \( \rho_p \), where \( p \) is a linear combination of non-trivial paths in \( B \). Moreover, since \( f \in \text{rad}(\Delta) \), the entries of the matrix of \( \hat{\epsilon} \) are of the form \( \rho_{p'} \), where \( q' \) is a linear combination of non-trivial paths in \( A \). Then, \( \hat{\epsilon} \hat{\epsilon} = 0 \), since \( \rho_q \rho_p = \rho_{pq} = 0 \), according to the dual extension relation.

Since \( \epsilon_3 \in L^B \), and we have assumed that \( L^B \) may be chosen to consist of maps with only radical maps for components, a similar argument as above shows that \( f' F(\epsilon_3) = 0 \).

This means that, in the bottom path, we have \( f' F(\epsilon) = f' F(\epsilon_1) \). Recall that we have chosen \( H^A = F(\hat{\rho}(\epsilon) \cdot \hat{\rho}(\hat{\Delta}) \cdot F(\hat{\rho}(\hat{H})) \) and \( \text{Ext}^A_1(\Delta, \Delta) = F(\text{Ext}^B_1(\epsilon, \mathbb{L})) \oplus \text{rad}(\Delta) \cdot F(\text{Ext}^B_0(\epsilon, \mathbb{L})) \).

Then, we get
\[ \chi^A \cdot \pi^A(f' F(\epsilon_1)) = \chi^A \left[ \begin{array}{c} 0 \\ f' F(\epsilon_1) \end{array} \right] = \begin{bmatrix} 0 & 0 \\ 0 & f(\epsilon_1) \end{bmatrix} = \varphi(f(\epsilon_1)) = f F \chi^B(\epsilon_1), \]

so the top path equals the bottom path, and we are done. \( \square \)

29. Lemma. Let \( \epsilon \in \mathbb{D}^A \) be a homogeneous map of degree \( n \), and let \( f \in \text{rad}(\Delta) \). Then, there holds \( h^A(f \circ \epsilon) = 0 \).

Proof. We write \( \epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3 \), where \( \epsilon_1 \in H^A, \epsilon_2 \in \text{im} \partial^A \) and \( \epsilon_3 \in L^A \). Then, we have
\[ h^A(f \epsilon) = h^A(f \epsilon_1) + h^A(f \epsilon_2) + h^A(f \epsilon_3). \]

Since we have chosen \( H^A = F(\hat{\rho}(\epsilon) \cdot \text{rad}(\Delta) \cdot F(\hat{\rho}(\hat{H})) \), we have \( \hat{\epsilon} \epsilon_1 \in H^A \), which implies that \( h^A(f \epsilon_1) = 0 \), by definition. Since the components of \( \epsilon_3 \) are matrices whose entries are of the form \( \rho_p \), where \( p \) is a linear combination of non-trivial paths in \( B \), we have \( \epsilon_3 = 0 \), by appealing to the dual extension relation, just like in the previous proof. Next, note that \( \epsilon_2 \in \text{im} \partial^A = F(\text{im} \partial^B \cdot \partial^A(\hat{L})) \), where \( \hat{L} \) is as in the statement of Proposition [23]. Since the differential maps on \( P^* \rightarrow \mathbb{L} \) are radical maps, the space \( F(\text{im} \partial^B) \) consists of maps whose components are given by matrices whose entries are of the form \( \rho_p \), where \( p \) is a linear combination of non-trivial paths in \( B \). By Lemma [27] the space \( \partial^A(\hat{L}) \) consists of maps whose components are given by matrices whose entries are of the form \( \rho_{p'} \), where \( p \) is a linear combination of non-trivial paths in \( B \). Then, a similar argument as before, using the dual extension relation, shows that \( f \epsilon_2 = 0 \). \( \square \)

Now we are ready to prove our main theorem. Note that, by linearity, the following formulae determine the \( A_{\infty} \)-structure on \( \text{Ext}^A_1(\Delta, \Delta) \) completely.

30. Theorem. Let \( n \geq 2 \). For any \( f_1', \ldots, f_n' \in \text{Hom}_A(\Delta, \Delta) \) and \( \epsilon_1, \ldots, \epsilon_n \in \text{Ext}^B_0(\epsilon, \mathbb{L}) \), such that \( \text{deg} \epsilon_i \geq 1 \), for all \( 1 \leq i \leq n \), then hold the following.

(i) If there is \( 1 \leq i \leq n - 1 \), such that \( f_i' \in \text{rad}(\Delta_A(j), \Delta_A(k)) \), we have \( m^A_1(f_1' \epsilon_n, \ldots, f_i' \epsilon_1) = 0 \).
(ii) We have
\[ m^A_1(f_n' \epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_1) = (-1)^{n+1} f_n' F \left( p^B \epsilon_n h^B(\lambda^B_{n-1}(\epsilon_n-1, \ldots, \epsilon_1)) \right). \]

Proof. We first prove the following formulae for \( \lambda^A_1 \).

\[ \left\{ \begin{array}{ll} \lambda^A_1(f_1' \epsilon_n, \ldots, f_i' \epsilon_1) = 0, & \text{ if } 1 \leq i \leq n - 1 : f_i' \in \text{rad}(\Delta_A(j), \Delta_A(k)); \\ \lambda^A_1(f_n' \epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_1) = (-1)^{n+1} f_n' \epsilon_n h^A \lambda^A_{n-1}(\epsilon_n, \ldots, \epsilon_1). \end{array} \right. \]
If \( n = 2 \), the first formula claims that
\[
\lambda_n^\mathcal{A} (f'_2 e_2, f'_1 e_1) = f'_2 e_2 f'_1 e_1 = 0.
\]
Using the proof of Proposition \[23\] we see that a chain map representative of \( f'_1 \) may be chosen so that the composition \( e_2 f'_1 \) equals the zero chain map. Assume that
\[
\lambda_k^\mathcal{A} (f'_k e_k, \ldots, f'_1 e_1) = 0, \quad \text{and} \quad \lambda_k^\mathcal{A} (f'_k e_k, e_{k-1}, \ldots, e_1) = (-1)^{k+1} f'_k e_k h^\mathcal{A} \lambda_{k-1}^\mathcal{A} (e_{k-1}, \ldots, e_1),
\]
for all \( k < n \), and assume that \( f'_i \in \text{rad}(\Delta, \Lambda) \), for some \( 2 \leq i \leq n - 1 \). Consider the sum
\[
\lambda_n^\mathcal{A} (f'_n e_n, \ldots, f'_1 e_1) = \sum_{r+s=n} (-1)^{s+1} \lambda_2^\mathcal{A} (h^\mathcal{A} \lambda_{n-1}^\mathcal{A} (f'_n e_n, \ldots, f'_{s+1} e_{s+1}) \otimes h^\mathcal{A} \lambda_s (f'_s e_s, \ldots, f'_1 e_1)), \quad s \geq 1.
\]
By the first claim of the induction hypothesis, all terms in the sum vanish, except for
\[
(-1)^{n-i+1} \lambda_n^\mathcal{A} (h^\mathcal{A} \lambda_{n-i+1}^\mathcal{A} (f'_n e_n, \ldots, f'_{i+1} e_{i+1}) \otimes h^\mathcal{A} \lambda_i^\mathcal{A} (f'_i e_i, \ldots, f'_1 e_1)).
\]
By the second claim of the induction hypothesis and Lemma \[29\] we have
\[
h^\mathcal{A} \lambda_n^\mathcal{A} (f'_n e_n, \ldots, f'_1 e_1) = (-1)^{n+1} n^\mathcal{A} (f'_n e_n h^\mathcal{A} \lambda_{n-1}^\mathcal{A} (e_{n-1}, \ldots, e_1)) = 0.
\]
Note that \( \lambda_n^\mathcal{A} (f'_n e_n, \ldots, f'_1 e_1) = 0 \) implies that \( m_n^\mathcal{A} (f'_n e_n, \ldots, f'_1 e_1) = 0 \). Now we prove the second formula. If \( n = 2 \), the second formula claims that
\[
\lambda_n^\mathcal{A} (f'_2 e_2, e_1) = -f'_2 e_2 h^\mathcal{A} \lambda_1^\mathcal{A} (e_1).
\]
This is true, as \( \lambda_1^\mathcal{A} \) is just composition, and \( h^\mathcal{A} \lambda_1^\mathcal{A} = \text{id} \), by definition. Suppose
\[
\lambda_k^\mathcal{A} (f'_k e_k, e_{k-1}, \ldots, e_1) = (-1)^{k+1} f'_k e_k h^\mathcal{A} \lambda_{k-1}^\mathcal{A} (e_{k-1}, \ldots, e_1),
\]
for all \( k < n \). Consider the sum
\[
\lambda_n^\mathcal{A} (f'_n e_n, \ldots, e_1) = \sum_{r+s=n} (-1)^{s+1} \lambda_2^\mathcal{A} (h^\mathcal{A} \lambda_{n-1}^\mathcal{A} (f'_n e_n, \ldots, e_{s+1}) \otimes h^\mathcal{A} \lambda_s (f'_s e_s, \ldots, e_1)).
\]
If \( r > 1 \), we have
\[
h^\mathcal{A} \lambda_n^\mathcal{A} (f'_n e_n, \ldots, e_{s+1}) = h^\mathcal{A} (-(-1)^{r+1} f'_n e_n h^\mathcal{A} \lambda_{n-r}^\mathcal{A} (e_{n-r-1}, \ldots, e_1)) = (-1)^{r+1} h^\mathcal{A} (f'_n e_n h^\mathcal{A} \lambda_{n-r}^\mathcal{A} (e_{n-r-1}, \ldots, e_1)) = 0,
\]
again using the induction hypothesis and Lemma \[29\]. Therefore, all terms in the sum with \( r > 1 \) vanish. It follows, then, that we have
\[
\lambda_n^\mathcal{A} (f'_n e_n, \ldots, e_1) = (-1)^{n+1} \lambda_n^\mathcal{A} (-f'_n e_n \otimes h^\mathcal{A} \lambda_{n-1}^\mathcal{A} (e_{n-1}, \ldots, e_1)) = (-1)^{n+1} f'_n e_n h^\mathcal{A} \lambda_{n-1}^\mathcal{A} (e_{n-1}, \ldots, e_1).
\]
Plugging this formula into the definition of \( m_n^\mathcal{A} \), we get
\[
m_n^\mathcal{A} (f'_n e_n, \ldots, e_1) = p^\mathcal{A} \lambda_n^\mathcal{A} (f'_n e_n, \ldots, e_1) = p^\mathcal{A} (-(-1)^{n+1} f'_n e_n h^\mathcal{A} \lambda_{n-1}^\mathcal{A} (e_{n-1}, \ldots, e_1))
\]
\[
= (-1)^{n+1} p^\mathcal{A} (f'_n e_n h^\mathcal{A} \lambda_{n-1}^\mathcal{A} (e_{n-1}, \ldots, e_1))
\]
\[
= (-1)^{n+1} p^\mathcal{A} (f'_n e_n h^\mathcal{A} (f_B^{\mathcal{A} n-1} (e_{n-1}, \ldots, e_1)))
\]
\[
= (-1)^{n+1} p^\mathcal{A} (f'_n e_n h^\mathcal{A} (f_B^{\mathcal{A} n-1} (e_{n-1}, \ldots, e_1)))
\]
\[
= (-1)^{n+1} p^\mathcal{A} (f'_n e_n h^\mathcal{A} (f_B^{\mathcal{A} n-1} (e_{n-1}, \ldots, e_1)))
\]
by applying Corollary \[24\] Corollary \[25\] and Lemma \[28\].

7. \( A_\infty \)-structure on the Ext-algebra of simple modules over \( K^\mathcal{A}\mathcal{A}_n/(\text{rad} K^\mathcal{A}\mathcal{A}_n) \)

Next, we want to apply Theorem \[30\] in an example. Since the theorem describes the \( A_\infty \)-multiplications on \( \text{Ext}_n^\mathcal{A} (\Delta, \Delta) \) in terms of the data constituting the \( A_\infty \)-structure on \( \text{Ext}_0^\mathcal{A} (\mathcal{L}, \mathcal{L}) \), any example requires that we first compute the \( A_\infty \)-structure on \( \text{Ext}_0^\mathcal{A} (\mathcal{L}, \mathcal{L}) \).
7.1. **Quiver and relations.** We consider the algebra $B = K \Lambda_n / (\text{rad} K \Lambda_n)^\ell$, for $\ell \geq 3$. Then, $B$ is described by the quiver $1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} n$ modulo the relations $a_i + \ell \cdots a_i = 0$. The Loewy diagrams of the indecomposable projective $B$-modules are:

$$
\begin{array}{c}
\begin{array}{c}
P_B(i) : \\
\downarrow \\
i + \ell - 1
\end{array}
\end{array}
\quad \text{if } i \leq n - \ell,
\quad \text{and}
\begin{array}{c}
P_B(i) : \\
\downarrow \\
i
\end{array}
\quad \text{if } i > n - \ell.
$$

It is easy to compute that there is a minimal projective resolution $P^* \to L_B(i)$, which has terms

$$
P^k = \begin{cases} 
P_B(i + q\ell), & \text{if } k = 2q; \\ 
P_B(i + q\ell + 1), & \text{if } k = 2q + 1. 
\end{cases}
$$

For indecomposable projective $B$-modules, $P_B(i)$ and $P_B(j)$, such that $j \geq i$, and $|i - j| < \ell$, we have $\dim \text{Hom}_B(P_B(j), P_B(i)) = 1$. This space contains scalar multiples of the map $\rho_a$, that is, right multiplication with the (unique) path $a : i \to j$ in the quiver. Denote this map by $f^a_{ij}$.

Now, we check that, in the decomposition $\mathcal{O}^B = H^B \oplus \text{im}(\partial^B \oplus L^B)$, the space $L^B$ may be chosen in such a way that any component of a map $\varepsilon \in L^B$ has components given by matrices whose entries are of the form $\rho_p$, where $p$ is a linear combination of non-trivial paths in $B$. To this end, let $\varepsilon \in \mathcal{O}^B$ be a map, homogeneous of degree $k$, such that the matrix of the map $\varepsilon_k$ has an entry, $g$, at position $(r, s)$, which is not a radical map. Since the maps $f^a_{ij}$ are radical for $i \neq j$, the map $g$ is an endomorphism of some indecomposable projective $B$-module, $P_B(x)$. Since $B$ is directed, we have $\dim \text{End}_B(P_B(x)) = 1$, so it follows that $g = \mu \cdot 1_{P_B(x)}$, for some scalar $\mu \in K$.

$$
\begin{array}{c}
P_{k+1} \xrightarrow{f^a_{ij}} P_k \xrightarrow{\varepsilon_k} P_{k-1} \\
\downarrow \\
P_1 \xrightarrow{f^a_{ij}} P_0 \xrightarrow{\varepsilon_k} 0
\end{array}
$$

Note that each module $P_k$ has exactly $n$ non-isomorphic summands, since each term in the minimal projective resolution of a single simple $B$-module is exactly one indecomposable projective $B$-module. Therefore, consider the following picture.

$$
\begin{array}{c}
P_B(y) \xrightarrow{f^a_{ij}} P_B(x) \xrightarrow{1_{P_B(x)}} \cdots \\
\downarrow \\
P_B(z) \xrightarrow{f^a_{ij}} P_B(x) \xrightarrow{1_{P_B(x)}} 0
\end{array}
$$

It is clear that, if we let the dashed arrow represent the map $f^a_{ij}$, we get a commutative square, since $f^a_{ij} = f^a_{ij} f^a_{ij}$. Now, putting the map $f^a_{ij}$ into a matrix in the appropriate position, it is clear that we can make the square

$$
\begin{array}{c}
P_{k+1} \xrightarrow{f^a_{ij}} P_k \xrightarrow{\varepsilon_k} P_{k-1} \\
\downarrow \\
P_1 \xrightarrow{f^a_{ij}} P_0 \xrightarrow{\varepsilon_k} 0
\end{array}
$$

commute. Continuing in this way, we obtain a chain map, $\delta$, such that the matrix of $\delta_k : P_k \to P_0$ has an entry $1_{P_B(x)}$ at position $(r, s)$. 
Note that this procedure does not work in general, but crucially depends on the form of the minimal projective resolution of simple $B$-modules and, indeed, on the quiver of $B$. Our procedure guarantees that the map $\varepsilon - \mu \cdot \delta$ is such that the component $(\varepsilon - \mu \cdot \delta)_k$ is given by a matrix whose entry at position $(r,s)$ is 0. Repeating our argument, it follows that we may write $\varepsilon = \varepsilon + \gamma$, where $\gamma \in H^B$ and where $\varepsilon$ has only radical components. Now, we argue exactly as in the proof of Lemma \cite{27} to see that this implies that $L^B$ may be chosen to be of the desired form.

To compute $\text{Ext}^m(B_0, L_B(j))$, we apply $\text{Hom}_B(\_ , L_B(j))$ to $P^*$, obtaining the complex:

$$0 \rightarrow \text{Hom}_B(P_B(i), L_B(j)) \rightarrow \text{Hom}_B(P_B(i+1), L_B(j)) \rightarrow \text{Hom}_B(P_B(i+\ell), L_B(j)) \rightarrow \ldots$$

We know that $\dim \text{Hom}_B(P_B(i), L_B(j)) = \delta_{ij}$, with the space $\text{Hom}_B(P_B(i), L_B(j))$ containing (up to scalar) only the projection $P_B(i) \rightarrow \text{top} P_B(i)$, if $i = j$, and the zero map otherwise. Because the differential maps on $P^*$ are radical maps, we get $\text{Ext}_B^m(L_B(i), L_B(j)) = \text{Hom}_B(P_B^n, L_B(j))$, which implies that

$$\dim \text{Ext}_B^k(L_B(i), L_B(j)) = \begin{cases} 1, & \text{if } j = i + k\ell; \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\dim \text{Ext}_B^{k+1}(L_B(i), L_B(j)) = \begin{cases} 1, & \text{if } j = i + k\ell + 1; \\ 0, & \text{otherwise}. \end{cases}$$

These extensions have chain map representatives

$$\ldots \rightarrow P_B(i + (k+1)\ell) \rightarrow P_B(i + k\ell + 1) \rightarrow P_B(i + k\ell) \rightarrow \ldots \rightarrow P_B(i)$$

$$\downarrow 1 \quad \downarrow 1 \quad \downarrow 1$$

$$\ldots \rightarrow P_B(i + (k+1)\ell) \rightarrow P_B(i + k\ell + 1) \rightarrow P_B(i + k\ell),$$

and

$$\ldots \rightarrow P_B(i + (k+1)\ell + 1) \rightarrow P_B(i + (k+1)\ell) \rightarrow P_B(i + k\ell + 1) \rightarrow P_B(i + k\ell) \rightarrow \ldots \rightarrow P_B(i)$$

$$\downarrow 1 \quad \downarrow 1$$

$$\ldots \rightarrow P_B(i + (k+1)\ell + 1) \rightarrow P_B(i + k\ell + 2) \rightarrow P_B(i + k\ell + 1),$$

respectively. Let $e_i$ be the natural basis vector of the space $\text{Ext}_B^1(L_B(i), L_B(i+1))$, and let $\delta_i$ be the natural basis vector of the space $\text{Ext}_B^2(L_B(i), L_B(i+1))$. It is then easy to check that we must have $e_i e_i \delta_i = \delta_i + 1 e_i$, as elements of $\text{Ext}_B^3(L_B(i), L_B(i+1))$. Since $\text{Ext}_B^3(L_B(i), L_B(i+2)) = 0$, and $\ell \geq 3$, we must have $e_i e_i \delta_i = 0$ for all $i$.

Let $C$ be the path algebra of the quiver $Q$, given by:

$$1 \overset{a_1}{\rightarrow} 2 \overset{a_2}{\rightarrow} \ldots \overset{\epsilon}{\rightarrow} 1 + \ell \overset{\epsilon a_1}{\rightarrow} 2 + \ell \overset{\epsilon a_2}{\rightarrow} \ldots \overset{\epsilon a_1 a_2}{\rightarrow} 1 + 2\ell \rightarrow \ldots \rightarrow n,$$

modulo the relations $a_{i+1} a_i = 0$, and $a_{i+1} \beta_i = \beta_i + 1 a_i$. The above observations imply that there exists a homomorphism of algebras $\Phi : C \rightarrow \text{Ext}_B^2(\_ , L_B(j))$, defined by $a_i \mapsto e_i$, and $\beta_i \mapsto \delta_i$.

Consider now the projective resolution $P^* \rightarrow L_B(i)$. If we let $m_{j, k}$ denote the multiplicity of $P_B(j)$ in the $k$th term $P^k$, we have $\dim \text{Ext}_B^k(L_B(i), L_B(j)) = \dim \text{Hom}_B(P_B^k, L_B(j)) = m_{j, k}$. From the form of the projective resolution of $L_B(i)$, we know that

$$P^k = \begin{cases} P_B(i + q\ell), & \text{if } k = 2q; \\ P_B(i + q\ell + 1), & \text{if } k = 2q + 1, \end{cases} \Rightarrow m_{j, k} = \begin{cases} 1, & \text{if } k = i + q\ell; \\ 1, & \text{if } k = i + q\ell + 1; \\ 0, & \text{otherwise}. \end{cases}$$

Here, $q$ must be such that $i + q\ell \leq n$ and $i + q\ell + 1 \leq n$, respectively. Next, we claim that

$$\delta_{i+(q-1)\ell} \ldots \delta_i \beta_i, \quad \text{and} \quad a_{i+q\ell} \delta_{i+(q-1)\ell} \ldots \beta_i \delta_i,$$
are non-split extensions. To see this, we look at the chosen chain map representatives of and note that they have components which are the identity homomorphism on some projective. This ensures the chain maps are not null-homotopic. Therefore, the extensions $\epsilon_i$ and $\delta_i$ generate $\text{Ext}^k_B(L,L)$, so that $\Phi$ is surjective. Finally, we do a dimension count. We have $\dim \text{Ext}^k_B(L_B(i),L_B(j)) = m_{j,k}$, which we compare to the dimension of the degree $k$ part of the space $\epsilon_j C \epsilon_i$.

In $C$, our relations imply that

$$\beta_{i+r\ell + 1} \cdots \beta_{i+1} \beta_i \beta_{i+r\ell + 1} \cdots \beta_{i+1} \beta_i = \cdots = a_{r+1}\beta_{i+r\ell + 1} \cdots \beta_{i+1} \beta_i$$

which shows that non-zero paths in $C$ have one of the two following forms.

1. The path is of the form $\beta_{i+r\ell + 1} \cdots \beta_{i+1} \beta_i$ going from $i$ to $i+r\ell$.
2. The path is of the form $a_{i+r\ell} \beta_{i+r\ell + 1} \cdots \beta_{i+1} \beta_i$ going from $i$ to $i+r\ell + 1$.

This shows that

$$\dim \text{Ext}^n_C(L_B(i),L_B(j)) = m_{j,k} = \begin{cases} 1, & \text{if } k = i + q\ell; \\ 1, & \text{if } k = i + q\ell + 1; \\ 0, & \text{otherwise} \end{cases} = \dim e_j C \epsilon_i,$$

which implies that $\dim \text{Ext}^n_B(L,L) = \dim C$, so that $\Phi$ is an isomorphism of algebras.

### 7.2. $A_\infty$-structure

In [Mad02], Madsen computed the $A_\infty$-structure on the Ext-algebra of simple modules over the path algebra of quiver

$$1 \circlearrowleft a$$

modulo the relation $a^n = 0$, $n \geq 3$ as an example. We use his computation to predict the formula for the present case. We abuse notation and suppress the indices on arrows $\alpha$ and $\beta$ when writing their concatenations. We write a basis element of $\text{Ext}^n_B(L,L)$ as $a^x \beta^y$. Here, $x \in [0,1]$, and $y$ is such that if the starting vertex of the first $\beta$ is $s$, then $s+y\ell \leq n$. Consider $m_k(a^x \beta^y, \ldots, a^x \beta^y)$ for $k \geq 3$. This produces an extension from some $L_B(s)$ to $L_B(t)$ where $t = s + \sum_{i=1}^k x_i + y_i \ell$.

We have seen above, that there are non-zero extensions from $L_B(s)$ to $L_B(t)$ if and only if

$$t-s \equiv 0 \mod \ell \quad \text{or} \quad t-s \equiv 1 \mod \ell \iff \sum_{i=1}^k x_i + \ell y_i \equiv 0 \mod \ell \quad \text{or} \quad \sum_{i=1}^k x_i + \ell y_i \equiv 1 \mod \ell.$$

Suppose $k \neq \ell$. The combined degree of the arguments of $m_k$ is $\sum_{i=1}^k x_i + 2y_i$. Since $m_k$ is of degree $2-k$, this should produce an extension of degree $2-k + \sum_{i=1}^k x_i + 2y_i$.

(i) If $\sum_{i=1}^k x_i = q\ell$, the non-split extension from $L_B(s)$ to $L_B(t)$ is of degree $2(q + \sum_{i=1}^k y_i)$. Then,

$$2(q + \sum_{i=1}^k y_i) = 2-k + q\ell + 2 \sum_{i=1}^k y_i \implies (2-\ell)q = 2-k \implies q = \frac{2-k}{2-\ell}.$$

If $k < \ell$, we get $q > 1$. This is a contradiction, since $0 \leq \sum_{i=1}^k x_i \leq k$. If $k > \ell$, we have $q < 1$, which implies $q = 0$, since $q$ is a non-negative integer. This implies the equality

$$2-k + \sum_{i=1}^k 2y_i = 2 \sum_{i=1}^k y_i,$$

which implies $2 = k$. Since $k \geq 3$, by assumption, this is a contradiction.

(ii) If $\sum_{i=1}^k x_i = q\ell + 1$, an identical argument works.
Left to consider is the case \( k = \ell \). There are three possibilities for the condition on the sum \( \sum_{i=1}^{k} x_i + \ell y_i \).

(i) We have \( x_i = 0 \) for all \( 1 \leq i \leq \ell \). The sum of degrees of the arguments is \( 2 \sum_{i=1}^{\ell} y_i \). Since \( m_{\ell} \) is of degree \( 2 - \ell \) this would yield an extension of degree \( 2 - \ell + 2 \sum_{j=1}^{\ell} y_j \) from \( L_B(s) \) to \( L_B(t) \) but the only such extension is of degree \( 2 \sum_{j=1}^{\ell} y_j \). This is contradiction since \( \ell \geq 3 \).

(ii) We have \( x_i = 1 \) for exactly one \( 1 \leq i \leq \ell \). We mimic the previous case.

(iii) We have \( x_i = 1 \) for all \( 1 \leq i \leq \ell \). Then we are dealing with an expression of the form \( m_{\ell}(a\beta^{y_1}, \ldots, a\beta^{y_\ell}) \).

We claim that \( m_{\ell}(a\beta^{y_1}, \ldots, a\beta^{y_\ell}) = \beta^{y_{\ell+1}} \ldots \beta^{y_\ell} \). Consider \( \gamma = \lambda_r(a\beta^{y_1}, \ldots, a\beta^{y_\ell}) \) for \( r < \ell \). Put \( \sigma = \sum_{i=1}^\ell y_i \).

Suppose the starting vertex of the first \( \beta \) is \( s \). Then \( \gamma \) is a chain map from the projective resolution of \( L_B(s) \) to that of \( L_B(t) \), where \( t = s + r + \ell \sigma \). We see that \( \deg \gamma = 2 - r + 2\sigma = 2(1 + \sigma) \).

Earlier we saw that

\[
P^k = \begin{cases} 
P_B(i + q\ell) & \text{if } k = 2q \\
P_B(i + q\ell + 1) & \text{if } k = 2q + 1 
\end{cases}
\]

which implies that \( P^{\deg \gamma} = P_B(s + (1 + \sigma)) \). We claim that \( \lambda_r(a\beta^{y_1}, \ldots, a\beta^{y_\ell}) \) is the chain map

\[
\cdots \to P_B(s + \ell(1 + \sigma) + 1) \to P_B(s + \ell(1 + \sigma)) \to \cdots \to P_B(s) \downarrow f \downarrow f \to \cdots \to P_B(s + r + \ell(1 + \sigma) + 1) \to P_B(s + r + \ell(1 + \sigma)) \to P_B(s + r + \ell(1 + \sigma) + 1) \downarrow f \downarrow f \to \cdots 
\]

and that it is the image under the differential of the map

\[
\cdots \to P_B(s + \ell(2 + \sigma)) \to P_B(s + \ell(1 + \sigma) + 1) \to P_B(s + \ell(1 + \sigma)) \to \cdots \to P_B(s) \downarrow \delta \downarrow f \to \cdots \to P_B(s + r + \ell(1 + \sigma) + 1) \to P_B(s + r + \ell(1 + \sigma)) \to P_B(s + r + \ell(1 + \sigma) + 1) \downarrow f \downarrow f \to \cdots 
\]

We proceed by induction. It is clear that the claim holds for \( r = 2 \), by just writing down the composition. Moreover, it is clear that \( \gamma \) is the image under the differential of a map of the form given above, which follows from the fact that if we have vertices \( a, b, c \) such that \( a \leq b \leq c \) and \( |c - a| < \ell \), then \( f^c_a = f^c_b f^b_a \).

Assume our claim holds for all \( 2 \leq r, s < k \). We put \( \delta = \lambda_r(a\beta^{y_1}, \ldots, a\beta^{y_{s+1}}) \) and \( \gamma = \lambda_s(a\beta^{y_1}, \ldots, a\beta^{y_\ell}) \). Let \( P^*, Q^* \) and \( R^* \) denote minimal projective resolutions of \( L_B(a), L_B(b) \) and \( L_B(c) \), respectively. We have

\[
\deg \gamma = 2(1 + \sum_{i=1}^{k} y_i) := d_\gamma \quad \text{and} \quad \deg \delta = 2(1 + \sum_{i=s+1}^{k} y_i) := d_\delta.
\]

Then \( \gamma \) is the chain map

\[
\gamma = \cdots \to P^{d_\gamma+2} \to P^{d_\gamma+1} \to P^{d_\gamma} \to \cdots \to P^0, \quad \text{and} \quad \delta = \cdots \to Q^{d_\delta+2} \to Q^{d_\delta+1} \to Q^{d_\delta} \to \cdots \to Q^0
\]

Also by assumption, we have

\[
y_{\gamma} = \cdots \to P^{d_\gamma+2} \to P^{d_\gamma+1} \to P^{d_\gamma} \to \cdots \to P^0 \quad \text{and} \quad h_\delta = \cdots \to Q^{d_\delta+2} \to Q^{d_\delta+1} \to Q^{d_\delta} \to \cdots \to Q^0
\]
and the composition of these is the map

$$\cdots \longrightarrow p^{d_i+d_j-1} \longrightarrow p^{d_i+d_j} \longrightarrow p^{d_i+d_j+1} \longrightarrow \cdots \longrightarrow p^{d_i+1} \longrightarrow p^{d_i} \longrightarrow \cdots \longrightarrow p^0$$

$$\downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f$$

$$\cdots \longrightarrow R^3 \longrightarrow R^2 \longrightarrow R^1 \longrightarrow R^0$$

which is 0. Left to consider are the cases $s = 1$ and $r = 1$. If $s = 1$ we have the composition

$$\cdots \longrightarrow p^{2y_i+3} \longrightarrow p^{2y_i+2} \longrightarrow p^{2y_i+1} \longrightarrow \cdots \longrightarrow p^{2y_i+2} \longrightarrow p^{2y_i+1} \longrightarrow \cdots \longrightarrow p^0$$

$$\downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f$$

$$\cdots \longrightarrow R^3 \longrightarrow R^2 \longrightarrow R^1$$

and if $r = 1$ we have

$$\cdots \longrightarrow p^{2y_i+3} \longrightarrow p^{2y_i+d_i+2} \longrightarrow p^{2y_i+d_i+1} \longrightarrow \cdots \longrightarrow p^{d_i+1} \longrightarrow p^{d_i} \longrightarrow \cdots \longrightarrow p^0$$

$$\downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f \quad \downarrow f$$

$$\cdots \longrightarrow R^3 \longrightarrow R^2 \longrightarrow R^1 \longrightarrow R^0$$

and adding the two maps proves the claim. Finally, we consider the case $k = \ell$. Put $\omega = \lambda_1(a\alpha \beta, \ldots , a\beta \gamma)$. Then $\omega$ is a chain map from the projective resolution of $L_B(i)$ to that of $L_B(i+t)$, where $t = (1 + \sigma)\ell$. It suffices to check that, in this case, the projective resolutions line up in the following way.

$$\cdots \longrightarrow P_B(i + t + 1) \longrightarrow P_B(i + t) \longrightarrow \cdots \longrightarrow P_B(i)$$

$$\downarrow \alpha_{i+1} \quad \downarrow \alpha_i \quad \downarrow \alpha_{i+1}$$

$$\cdots \longrightarrow P_B(i + t + 1) \longrightarrow P_B(i + t)$$

Our claim now implies that $\omega_j = \text{id}_{P_B(j)}$ for all $j \geq i + t$, so that

$$m_\ell(\alpha \beta \gamma, \ldots , a\beta \gamma) = \beta^{\sum_{i=1}^{\ell} y_i + 1}.$$  

We remark that this formula could also be obtained by applying [Tam21 Theorem 4.9].

8. APPLICATION TO THE DUAL EXTENSION ALGEBRA

Having computed the $A_\infty$-structure on $\text{Ext}^*_\Lambda(L, L)$, we turn to $\text{Ext}^*_\Lambda(D, D)$ for $\Lambda = \mathcal{A}(B, B^{\text{op}})$. We have $\text{Ext}^*_\Lambda(D, D) \cong \mathcal{A}(\text{Ext}_B^*(L, L), B)$ so $\text{Ext}^*_\Lambda(D, D)$ is isomorphic to the path algebra of the quiver

$$\begin{array}{ccccccccccc}
1 & \xleftarrow{a_1} & 2 & \xrightarrow{a_2} & \cdots & \xleftarrow{a_{n+1}} & \cdots & \xrightarrow{a_{2\ell}} & 1 + 2\ell & \xrightarrow{a_{2\ell+1}} & \cdots & \xrightarrow{a_{2\ell+1}} & n \\
\beta_1 & \cdots & \beta_2 & \cdots & \beta_{n+1} & \cdots & \beta_{2\ell} & \cdots & \beta_{2\ell+1} & \cdots & \beta_{2\ell+1} & \cdots \\
\end{array}$$

modulo the relations

$$a_{i+1}a_i = 0, \quad a_{i+\ell} \beta_i = \beta_{i+1} a_i \quad \text{and} \quad a_{i+\ell - 1} \cdots a_{i+1} a_i = 0$$

as well as the dual extension relations, $a_{i+1} a_i = 0$ and $\beta_{i+1} a_i = 0$. 

In light of Theorem 30, we consider an expression of the form \( m_n(g'_n, \varepsilon_n, \ldots, \varepsilon_1) \) where \( g'_n \in \text{Hom}_\Lambda(\Delta, \Delta) \) and \( \varepsilon_1, \ldots, \varepsilon_n \in \text{Ext}^*_B(\mathbb{L}, \mathbb{L}) \). By our theorem, there holds
\[
m_n(g'_n, \varepsilon_n, \ldots, \varepsilon_1) = (-1)^{n+1} g'_n P \left( \sum_{k=1}^n \varepsilon_n h^B A_{n-1}(\varepsilon_{n-1}, \ldots, \varepsilon_1) \right).
\]

We have non-split extensions \( \beta^k \in \text{Ext}^2_B(\Delta_B(i), \Delta_B(i + k \ell)) \) and \( \alpha^\beta \in \text{Ext}^2_B(\Delta_B(i), \Delta_B(i + k \ell + 1)) \) which are induced from the corresponding extensions between simple \( B \)-modules. These can be composed with \( g' \in \text{Hom}_\Lambda(\Delta, \Delta) \) to obtain new extensions
\[
g' \beta^k \in \text{Ext}^2_B(\Delta_B(i), \Delta_B(j)) \quad \text{and} \quad g' \alpha^\beta \in \text{Ext}^2_B(\Delta_B(i), \Delta_B(j + 1)).
\]

If \( A_{n-1}(\varepsilon_{n-1}, \ldots, \varepsilon_1) \) is zero, or if \( h^B A_{n-1}(\varepsilon_{n-1}, \ldots, \varepsilon_1) \) is zero, there is nothing to compute. We may assume \( \varepsilon_i = \alpha^\beta \gamma_i \), with \( x_i \) and \( y_i \) as before. Then, the degree of \( A_{n-1}(\varepsilon_{n-1}, \ldots, \varepsilon_1) \) is
\[
2 - (n - 1) + \sum_{i=1}^{n-1} x_i + 2y_i.
\]

We apply \( h^B \) and compose with \( \varepsilon_n = \alpha^\gamma \beta \gamma_n \) to get a chain map of degree \( 2 - n + \sum_{j=1}^n x_j + 2y_j \).

We may recycle the arguments from the computation for \( \text{Ext}^*_B(\mathbb{L}, \mathbb{L}) \) to get that \( m_n(g'_n, \varepsilon_n, \ldots, \varepsilon_1) = 0 \) unless \( n = \ell \). This works because, in the grading on \( \text{Ext}^*_B(\mathbb{L}, \mathbb{L}) \), elements of \( \text{Hom}_\Lambda(\Delta, \Delta) \) are homogeneous of degree 0, leaving the degree-based arguments unchanged. Again, there are three possible cases satisfying the requirement on the sum \( \sum_{j=1}^\ell x_j \). If \( x_j = 0 \) for all \( 1 \leq j \leq \ell \), or \( x_j = 1 \) for exactly one \( 1 \leq j \leq \ell \), we may recycle the arguments from the case of \( \text{Ext}^*_B(\mathbb{L}, \mathbb{L}) \), arriving at a contradiction. Therefore, we consider the map \( \alpha^\beta \gamma \gamma_{\ell-1}(\alpha^\beta \gamma \gamma_{\ell-1}, \ldots, \alpha^\beta \gamma) \). We put \( \gamma = \lambda_{\ell-1}(\alpha^\beta \gamma \gamma_{\ell-1}, \ldots, \alpha^\beta \gamma) \) and \( d_\gamma = \text{deg} \gamma \).

Then
\[
\gamma = \begin{array}{cccccccc}
\ldots & \rightarrow & P_{d_\gamma} & \rightarrow & P_{d_\gamma} & \rightarrow & \ldots & \rightarrow & P_0 \\
\downarrow & & \downarrow f & & \downarrow f & & \downarrow & & \downarrow \leftarrow
\end{array}, \quad \text{and} \quad h^B \gamma = \begin{array}{cccccccc}
\ldots & \rightarrow & P_{d_\gamma+1} & \rightarrow & P_{d_\gamma+2} & \rightarrow & P_{d_\gamma+1} & \rightarrow & \ldots & \rightarrow & P_0 \\
0 & & 0 & & 0 & & 0 & & \ldots & & \vdots
\end{array},
\]

which, composed with \( \alpha^\beta \gamma \gamma_{\ell-1} \), yields the map
\[
\begin{array}{cccccccc}
P_{d_\gamma} & \rightarrow & P_{d_\gamma+2} & \rightarrow & \ldots & \rightarrow & P_{d_\gamma+1} & \rightarrow & P_{d_\gamma} & \rightarrow & \ldots & \rightarrow & P_0 \\
\downarrow 0 & & \downarrow f & & \downarrow 0 & & \downarrow f & & \downarrow 0 & & \downarrow f & & \downarrow 0 \end{array}
\]

Put \( \sigma_k = \sum_{j=1}^k y_j \). Computing degrees of the involved maps and using our formula for the projective resolution of simple modules over \( B \), we see that the above picture may be more precisely given as follows.

\[
\begin{array}{cccccccc}
\ldots & \rightarrow & P_B(i + \ell(1 + \sigma_\ell + 1)) & \rightarrow & P_B(i + \ell(1 + \sigma_\ell)) & \rightarrow & \ldots & \rightarrow & P_B(i + \ell(1 + \sigma_\ell - 1)) & \rightarrow & P_B(i + \ell(1 + \sigma_\ell - 1)) \\
\downarrow 0 & & \downarrow 1 & & \downarrow 0 & & \downarrow 1 & & \downarrow 0 & & \downarrow 1 \end{array}
\]

We denote this composition by \( \Gamma \) and put \( d := \text{deg} \Gamma = 2(1 + \sigma_\ell) \). Consider the space of homogeneous maps of degree \( \text{deg} \Gamma \) from \( P^* \) to \( R^* \). Denote this space by \( V^d \). It is clear from the above picture that \( V^d \) has a basis given by maps \( b_i \) of the following form.
Suppose this basis is \{b_0, \ldots, b_n\}. We know that \(\dim \text{Ext}^d_B(L_B(i), L_B(i + \ell(1 + \sigma_i))) = 1\). This space contains the extension \(\varepsilon\) represented by the chain map

\[
\begin{array}{c}
\ldots \\ \ldots \\ \ldots \\
0 \\
\uparrow \\
1 \\
\uparrow \\
\downarrow \\
1 \\
\uparrow \\
0 \\
\downarrow \\
\end{array}
\]

and, clearly, \(\varepsilon = \sum b_i\). Note that \(\varepsilon\) is a basis for the homology of \(V^d\). Since \(\varepsilon\) is linearly independent from \(b_1, \ldots, b_n\), the set \(\{\varepsilon, b_1, \ldots, b_n\}\) is a basis of \(V^d\). In this basis, we have \(\Gamma = \varepsilon - b_1 - b_2 - \ldots\) so that the projection of \(\Gamma\) onto the homology is \(\varepsilon\). This, in turn, implies that \(p^B(\Gamma)\) is the extension \(b^{\sigma_{i+1}}\).

This yields the formula

\[
m_i(g' a^{\sigma_1}, \ldots, a^{\sigma_{i+1}}) = g' b^{\sigma_{i+1}} = g' m_i(a^{\sigma_1}, \ldots, a^{\sigma_{i+1}}).
\]

### 8.1. Computing the box.

The results from [KKO14] guarantee that there is an algebra \(B\), Morita equivalent to \(\Lambda\), which admits a regular exact Borel subalgebra \(B \subset R\). In our setup, \(B \subset \Lambda\) is only an exact Borel subalgebra. We compute \(\hat{B}\) and \(R\). We know that \(\text{Ext}_A^1(\Lambda, \Delta)\) is given by

\[
\begin{array}{c}
1 \\
\alpha_1 \\
\alpha_2 \\
\alpha_{1+i} \\
\beta_1 \\
\beta_2 \\
\beta_{1+i} \\
\end{array}
\]

modulo the relations

\[
a_{i+1}a_i = 0, \quad a_{i+1}\beta_i = \beta_{i+1}a_i \quad a_{i+1}a_i = \beta_{i+1}a_i = 0 \quad \text{and} \quad a_{i+i-1} \ldots a_{i+1}a_i = 0.
\]

Next, we briefly describe the method we use to find \(\hat{B}\) and \(R\), referring to [Kul17, Section 4.6] and to [KKO14]. We put \(L = L_\Lambda(1) \oplus \cdots \oplus L_\Lambda(n)\). For any \(n \geq 2\), we have a map

\[
m_n : (\text{Ext}_A^1(\Lambda, \Delta))^{\otimes n}_n \to \text{Ext}_A^2(\Lambda, \Delta)
\]

because \(m_n\) is of degree \(2 - n\) and the total degree of inputs is exactly \(n\). Up to natural isomorphism, this gives a dual map

\[
\mathcal{D}m_n : \mathcal{D}\text{Ext}_A^2(\Lambda, \Delta) \to (\mathcal{D}\text{Ext}_A^1(\Lambda, \Delta))^{\otimes n}.
\]

Summing these maps over all \(n \geq 2\), we obtain the map

\[
\sum \mathcal{D}m_n : \mathcal{D}\text{Ext}_A^2(\Lambda, \Delta) \to \bigoplus_{n \geq 2} (\mathcal{D}\text{Ext}_A^1(\Lambda, \Delta))^{\otimes n}
\]

Then, we obtain \(\hat{B}\) as

\[
\bigoplus_{n \geq 2} (\mathcal{D}\text{Ext}_A^1(\Lambda, \Delta))^{\otimes n}/(\text{im} \sum \mathcal{D}m_n) = \hat{B}.
\]
The extensions of degree 1 starting in $\Delta \Lambda(i)$ are

\[ a_i \in \text{Ext}^1_\Lambda(\Delta \Lambda(i), \Delta \Lambda(i+1)) \]
\[ a_{i+1} a_i \in \text{Ext}^1_\Lambda(\Delta \Lambda(i), \Delta \Lambda(i+2)) \]

\[ \vdots \]
\[ a_{i+\ell-1} \ldots a_{i+1} a_i \in \text{Ext}^1_\Lambda(\Delta \Lambda(i), \Delta \Lambda(i+\ell)). \]

Then, $\hat{B}$ has the following quiver.

1. Vertices are 1, ..., $n$.
2. For each $i$, there are arrows $i \rightarrow i+1$, $i \rightarrow i+2$, ..., $i \rightarrow i+\ell$ and no other arrows.

Since the only extensions of degree 2 are of the form $\varphi \beta$ where $\varphi \in \text{Hom}_\Lambda(\Delta, \Delta)$, and these are uniquely obtained from higher multiplications by $m_\ell(\varphi a_1, \ldots, a) = \varphi \beta$ we impose the relations $\varphi a_\ell = 0$. We draw the case $n = 5, \ell = 3$. The quiver, then, is

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (1,0) {2};
\node (3) at (2,0) {3};
\node (4) at (3,0) {4};
\node (5) at (4,0) {5};
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (5);
\draw[<->] (1) -- (4);
\draw[<->] (2) -- (5);
\end{tikzpicture}
\end{center}

with the relations

\[ a_3 a_2 a_1 = 0, \quad a_4 a_3 a_2 = 0, \quad \gamma_3 a_2 a_1 = 0. \]

The Loewy diagrams of the indecomposable projective modules are then as follows.

\begin{center}
P(5) \cong L(5), \quad P(4) : 4 \quad P(3) : 3 \quad P(2) : 4 \leftarrow 3 \leftarrow 2 \quad P(1) : 3 \leftarrow 2 \leftarrow 1 \rightarrow 4 \rightarrow 5
\end{center}

which implies that

\[ R \otimes_{\hat{B}} P_{\hat{B}}(5) = P_\Lambda(5) \]
\[ R \otimes_{\hat{B}} P_{\hat{B}}(4) = P_\Lambda(4) \]
\[ R \otimes_{\hat{B}} P_{\hat{B}}(3) = P_\Lambda(3) \oplus P_\Lambda(5) \]
\[ R \otimes_{\hat{B}} P_{\hat{B}}(2) = P_\Lambda(2) \oplus P_\Lambda(4) \oplus P_\Lambda(5)^{g_2} \]
\[ R \otimes_{\hat{B}} P_{\hat{B}}(1) = P_\Lambda(1) \oplus P_\Lambda(3) \oplus P_\Lambda(4)^{g_2} \oplus P_\Lambda(5)^{g_3} \]

so that

\[ R = \text{End}_\Lambda (P_\Lambda(1) \oplus P_\Lambda(2) \oplus P_\Lambda(3)^{g_2} \oplus P_\Lambda(4)^{g_2} \oplus P_\Lambda(5)^{g_3})^{\text{op}}. \]

If an $A_\infty$-structure is not known, it is also possible to use the results by Conde in [Con20a, Con20b] to find $R$. 

Acknowledgement

The author thanks the anonymous referee for the insightful and helpful comments which aided in improving the exposition of this work.
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