On Low-Dimensional Locally Compact Quantum Groups

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Abstract. Continuing our research on extensions of locally compact quantum groups, we give a classification of all cocycle matched pairs of Lie algebras in small dimensions and prove that all of them can be exponentiated to cocycle matched pairs of Lie groups. Hence, all of them give rise to locally compact quantum groups by the cocycle bicrossed product construction. We also clarify the notion of an extension of locally compact quantum groups by relating it to the concept of a closed normal quantum subgroup and the quotient construction. Finally, we describe the infinitesimal objects of locally compact quantum quantum groups with 2 and 3 generators - Hopf *-algebras and Lie bialgebras.

1 Introduction

In this paper we continue the research on extensions of locally compact (l.c.) quantum groups, initiated in [51]. The first wide class of l.c. quantum groups, namely G.I. Kac algebras, was introduced in the early sixties (see [18]) in order to explain in a symmetric way duality for l.c. groups. This class included besides usual l.c. groups and their duals also nontrivial (i.e., non-commutative and non-cocommutative) objects [19], [20]. The general Kac algebra theory was completed independently on the one hand by G.I. Kac and the second author [21] and on the other hand by M. Enock and J.-M. Schwartz (for a survey see [12]). However, this theory was not general enough to cover important new

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examples constructed starting from the eighties [3], [24], [25], [31], [41], [42], [52], [56], [60] - [66], which motivated essential efforts to get a generalization that would cover these examples and that would be as elegant and symmetric as the theory of Kac algebras. Important steps in this direction were made by S. Baaj and G. Skandalis [4], S.L. Woronowicz [58], [59], [60], T. Masuda and Y. Nakagami [34] and A. Van Daele [54]. The general theory of l.c. quantum groups was proposed by J. Kustermans and the first author [26], [27] (see [28] for an overview). Some motivations and applications of this theory can be found in the recent lecture notes [29].

The mentioned examples of l.c. quantum groups are, first of all, formulated algebraically, in terms of generators of Hopf *-algebras and commutation relations between them. Then one represents the generators as (typically, unbounded) operators on a Hilbert space and tries to give a meaning to the commutation relations as relations between these operators. There is no general approach to this nontrivial problem, and one elaborates specific methods in each specific case. Finally, it is necessary to associate an operator algebra with the above system of operators and commutation relations and to construct comultiplication, antipode and invariant weights as applications related to this algebra. This problem is even more difficult than the previous one and again one must consider separately each specific case (see the same papers). So, it would be desirable to have some general constructions of l.c. quantum groups which would allow to construct systematically concrete examples in a unified way.

One of such possibilities is offered by the cocycle bicrossed product construction. According to G.I. Kac [17], in the simplest case the needed data for this construction contains:

1. A pair of finite groups $G_1$ and $G_2$ equipped with their mutual actions on each other (as on sets) or, equivalently, $G_1$ and $G_2$ must be subgroups of a certain group $G$ such that $G_1 \cap G_2 = \{e\}$ and any $g \in G$ can be written as $g = g_1g_2$ ($g_1 \in G_1, g_2 \in G_2$) - we write briefly $G = G_1G_2$. We then say, that $G_1$ and $G_2$ form a matched pair of groups [47].

2. A pair of compatible 2-cocycles for these actions, so $G_1$ and $G_2$ must form a cocycle matched pair (in what follows we often write simply "cocycle" rather then "2-cocycle").

Then, due to [17], one can construct a finite-dimensional Kac algebra from cocycle crossed products of the algebras of functions on each of the groups $G_1$ and $G_2$ with the cocycle action of the other group, and this construction gives exactly all extensions of the above groups in the category of finite-dimensional Kac algebras.

It is tempting to similarly treat Lie groups instead of finite groups, being supported by the theory of cocycle bicrossed products and extensions of l.c.
groups developed in [51] (in fact, in [51], the general theory of cocycle bicrossed products and extensions of l.c. quantum groups was developed). But first of all it turns out that the above definition of a matched pair of groups in terms of the equality $G = G_1G_2$ does not cover all interesting examples, see [3]. Following S. Baaj and G. Skandalis, one can just require $G_1G_2$ to be an open subset of $G$ with complement of measure zero. Then, the Lie algebras $\mathfrak{g}_1$ of $G_1$ and $\mathfrak{g}_2$ of $G_2$ are Lie subalgebras of the Lie algebra $\mathfrak{g}$ of $G$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as the direct sum of vector spaces, i.e., they form a matched pair of Lie algebras [33], 8.3. Thus, to get matched pairs of Lie groups one can start with matched pairs of Lie algebras (which are easier to find) and then try to exponentiate them.

To construct in this way cocycle matched pairs of Lie groups, one has to resolve two problems. First, given a matched pair of Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$ with $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, one can always exponentiate $\mathfrak{g}$ to a connected and simply connected Lie group $G$ and then find Lie subgroups $G_1$ and $G_2$ whose Lie algebras are $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. However, such a choice of $G$ does not guarantee that $G_1G_2$ is dense in $G$, even if $\dim(G_1) = \dim(G_2) = 1$ [33], [44], [51], and it also may happen that $G_1 \cap G_2 \neq \{e\}$. So, it is necessary to pass to some non-connected Lie group $G$ with the same Lie algebra $\mathfrak{g}$ in order to find a matched pair of its subgroups $G_1$ and $G_2$ [49], [51]. Secondly, given a matched pair of Lie groups, one has to find the corresponding cocycles. We give a solution of both these problems for real Lie groups $G_1$ and $G_2$ with $\dim(G_1) = 1$, $\dim(G_2) \leq 2$ and construct essentially all possible (up to obvious redundancies) matched pairs of such Lie groups having at most 2 connected components. Then, using the machinery of cocycle bicrossed products developed in [51], we construct l.c. quantum groups which are extensions of the mentioned Lie groups. Our discussion is motivated, apart from the above work by G.I. Kac, also by the works by S. Majid [30] - [33], S. Baaj and G. Skandalis [1], [3], [44], and by the works on extensions of Hopf algebras [1], [3], [13].

The material is organized as follows. In Section 2, we recall the necessary facts of the theory of l.c. quantum groups and, following [51], the main features of the cocycle bicrossed product construction for l.c. groups in connection with the theory of extensions. In the last subsection we introduce the notion of a closed normal quantum subgroup of a l.c. quantum group and explain its relation to the theory of extensions. As we explained above, the basic notion of this theory is that of a matched pair of l.c. groups. If the groups forming a matched pair are Lie groups, we naturally have a matched pair of their Lie algebras. But the converse problem, to construct a matched pair of Lie groups from a given matched pair $(\mathfrak{g}_1, \mathfrak{g}_2)$ of Lie algebras, is much more subtle. In particular, in Section 3 we show that any matched pair with $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathbb{C}$ can be exponentiated to a matched pair of complex Lie groups, but there are simple examples of matched pairs of real and complex Lie algebras for which the exponentiation is impossible.
The study of matched pairs of Lie algebras with \( \dim \mathfrak{g}_1 = n, \dim \mathfrak{g}_2 = 1 \) in Section 4 splits in three cases. In case 1, when \( \mathfrak{g}_1 \) is an ideal in \( \mathfrak{g} \), \( G \) can be constructed as semi-direct product of connected and simply connected Lie groups corresponding to \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) (this is possible also for \( \dim \mathfrak{g}_2 > 1 \)). In case 2, when \( \mathfrak{g}_1 \) contains an ideal of codimension 1, the results of Section 3 show that for complex Lie algebras the exponentiation always exists when \( n = 1 \) and it does not exist in general if \( n \geq 2 \). For real Lie algebras we show that for \( n \leq 4 \) there always exists the exponentiation to a matched pair of Lie groups with at most two connected components, and for \( n \geq 5 \) the exponentiation does not exist in general. In the remaining case 3, for complex Lie algebras the exponentiation always exists when \( n \leq 3 \) and it does not exist in general if \( n \geq 4 \). For real Lie algebras we show that the exponentiation always exists when \( n \leq 4 \).

Section 5 is devoted to the complete classification of all matched pairs of real Lie algebras \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) when \( \dim(\mathfrak{g}_1) = 1, \dim(\mathfrak{g}_2) \leq 2 \) and to their explicit exponentiation to matched pairs of real Lie groups having at most 2 connected components. Here, we also describe the l.c. quantum groups obtained from these matched pairs by the bicrossed product construction. In Section 6, we calculate the cocycles for all the above mentioned matched pairs. Finally, Section 7 is devoted to the description of l.c. quantum groups with 2 and 3 generators and their infinitesimal objects - Hopf \( * \)-algebras and Lie bialgebras, having the structure of a cocycle bicrossed product - equivalently, those that can be obtained as extensions (we call them decomposable). At last, to complete the picture of low-dimensional l.c. quantum groups, we review the indecomposable ones and their infinitesimal objects.

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## 2 Preliminaries

**General notations** Let \( B(\mathcal{H}) \) denote the algebra of all bounded linear operators on a Hilbert space \( \mathcal{H} \), let \( \otimes \) denote the tensor product of Hilbert spaces.
or von Neumann algebras and Σ (resp., σ) the flip map on it. If \(H, K\) and \(L\) are Hilbert spaces and \(X \in B(H \otimes L)\) (resp., \(X \in B(H \otimes K)\), \(X \in B(K \otimes L)\)), we denote by \(X_{13}\) (resp., \(X_{12}\), \(X_{23}\)) the operator \((1 \otimes \Sigma^*)(X \otimes 1)(1 \otimes \Sigma)\) (resp., \(X \otimes 1, 1 \otimes X\)) defined on \(H \otimes K \otimes L\). Sometimes, when \(H = H_1 \otimes H_2\) itself is a tensor product of two Hilbert spaces, we switch from the above leg-numbering notation with respect to \(H \otimes K \otimes L\) to the one with respect to the finer tensor product \(H_1 \otimes H_2 \otimes K \otimes L\), for example, from \(X_{13}\) to \(X_{124}\). There is no confusion here, because the number of legs changes.

Given a comultiplication \(\Delta\), denote by \(\Delta^\text{op}\) the opposite comultiplication \(\sigma \Delta\). Our general reference to the modular theory of normal semi-finite faithful (n.s.f.) weights on von Neumann algebras is [45]. For any weight \(\theta\) on a von Neumann algebra \(N\), we use the notations

\[
\mathcal{M}_\theta^+ = \{x \in N^+ \mid \theta(x) < +\infty\}, \quad \mathcal{N}_\theta = \{x \in N \mid x^* x \in \mathcal{M}_\theta^+\} \quad \text{and} \quad \mathcal{M}_\theta = \text{span} \mathcal{M}_\theta^+.
\]

**L.c. quantum groups** A pair \((M, \Delta)\) is called a (von Neumann algebraic) l.c. quantum group [27] when

- \(M\) is a von Neumann algebra and \(\Delta : M \to M \otimes M\) is a normal and unital \(*\)-homomorphism satisfying the coassociativity relation : \((\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta\).
- There exist n.s.f. weights \(\varphi\) and \(\psi\) on \(M\) such that
  - \(\varphi\) is left invariant in the sense that \(\varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x)\omega(1)\) for all \(x \in \mathcal{M}_{\varphi}^+\) and \(\omega \in \mathcal{M}_{\varphi}^+\),
  - \(\psi\) is right invariant in the sense that \(\psi((\iota \otimes \omega)\Delta(x)) = \psi(x)\omega(1)\) for all \(x \in \mathcal{M}_{\psi}^+\) and \(\omega \in \mathcal{M}_{\psi}^+\).

Left and right invariant weights are unique up to a positive scalar [26], Theorem 7.14.

Represent \(M\) on the Hilbert space of a GNS-construction \((H, \iota, \Lambda)\) for the left invariant n.s.f. weight \(\varphi\) and define a unitary \(W\) on \(H \otimes H\) by

\[
W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)) \quad \text{for all } a, b \in \mathcal{N}_\varphi.
\]

Here, \(\Lambda \otimes \Lambda\) denotes the canonical GNS-map for the tensor product weight \(\varphi \otimes \varphi\). One proves that \(W\) satisfies the pentagonal equation: \(W_{12}W_{13}W_{23} = W_{23}W_{12}\), and we say that \(W\) is a multiplicative unitary. The von Neumann algebra \(M\) and the comultiplication on it can be given in terms of \(W\) respectively as

\[
M = \{(\iota \otimes \omega)(W) \mid \omega \in B(H)\}^{\text{\sigma-strong}^*}
\]

and \(\Delta(x) = W^*(1 \otimes x)W\), for all \(x \in M\). Next, the l.c. quantum group \((M, \Delta)\) has an antipode \(S\), which is the unique \(\sigma\)-strong* closed linear map from \(M\).
to $M$ satisfying $(\iota \otimes \omega)(W) \in \mathcal{D}(S)$ for all $\omega \in B(H)_*$ and $S(\iota \otimes \omega)(W) = (\iota \otimes \omega)(W^*)$ and such that the elements $(\iota \otimes \omega)(W)$ form a $\sigma$-strong$^*$ core for $S$. $S$ has a polar decomposition $S = R_{\tau_{-1/2}}$ where $R$ is an anti-automorphism of $M$ and $(\tau_t)$ is a strongly continuous one-parameter group of automorphisms of $M$. We call $R$ the unitary antipode and $(\tau_t)$ the scaling group of $(M, \Delta)$. From [26], Proposition 5.26 we know that $\sigma(R \otimes R)\Delta = \Delta R$. So $\varphi R$ is a right invariant weight on $(M, \Delta)$ and we take $\psi := \varphi R$.

Let us denote by $(\sigma_t)$ the modular automorphism group of $\varphi$. From [24], Proposition 6.8 we get the existence of a number $\nu > 0$, called the scaling constant, such that $\psi \sigma_t = \nu^{-t} \psi$ for all $t \in \mathbb{R}$. Hence, we get the existence of a unique positive, self-adjoint operator $\delta_M$ affiliated to $M$, such that $\sigma_t(\delta_M) = \nu^t \delta_M$ for all $t \in \mathbb{R}$ and $\psi = \varphi_{\delta_M}$, see [26], Definition 7.1. Formally this means that $\psi(x) = \varphi(\delta_M^{1/2} x \delta_M^{1/2})$, and for a precise definition of $\varphi_{\delta_M}$ we refer to [7]. The operator $\delta_M$ is called the modular element of $(M, \Delta)$. If $\delta_M = 1$ we call $(M, \Delta)$ unimodular. The scaling constant can be characterized as well by the relative invariance $\varphi \tau_t = \nu^{-t} \varphi$.

The dual l.c. quantum group $(\hat{M}, \hat{\Delta})$ is defined in [26], Section 8. Its von Neumann algebra $\hat{M}$ is

$$\hat{M} = \{(\omega \otimes \iota)(W) \mid \omega \in B(H)_*\}^{-\sigma\text{-strong}^*}$$

and the comultiplication $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma$ for all $x \in \hat{M}$. If we turn the predual $M_*$ into a Banach algebra with product $\omega \mu = (\omega \otimes \mu)\Delta$ and define

$$\lambda : M_* \to \hat{M} : \lambda(\omega) = (\omega \otimes \iota)(W),$$

then $\lambda$ is a homomorphism and $\lambda(M_*)$ is a $\sigma$-strong$^*$ dense subalgebra of $\hat{M}$. To construct explicitly a left invariant n.s.f. weight $\hat{\varphi}$ with GNS-construction $(H, \iota, \Lambda)$, first introduce the space

$$\mathcal{I} = \{\omega \in M_* \mid \text{there exists } \xi(\omega) \in H \text{ s.t. } \omega(x^*) = \langle \xi(\omega), \Lambda(x) \rangle \text{ when } x \in \mathcal{N}_\varphi\}.$$ 

If $\omega \in \mathcal{I}$, then such a vector $\xi(\omega)$ clearly is uniquely determined. Next, one proves that there exists a unique n.s.f. weight $\hat{\varphi}$ on $\hat{M}$ with GNS-construction $(H, \iota, \hat{\Lambda})$ such that $\lambda(\mathcal{I})$ is a core for $\hat{\Lambda}$ (when we equip $\hat{M}$ with the $\sigma$-strong$^*$ topology and $H$ with the norm topology) and such that

$$\hat{\Lambda}(\lambda(\omega)) = \xi(\omega) \quad \text{for all} \quad \omega \in \mathcal{I}.$$ 

One proves that the weight $\hat{\varphi}$ is left invariant, and the associated multiplicative unitary is denoted by $\hat{W}$. From [26], Proposition 8.16 it follows that $\hat{W} = \Sigma W^*\Sigma$.

Since $(\hat{M}, \hat{\Delta})$ is again a l.c. quantum group, we can introduce the antipode $\hat{S}$, the unitary antipode $\hat{R}$ and the scaling group $(\hat{\tau}_t)$ exactly as we did it for $(M, \Delta)$. Also, we can again construct the dual of $(M, \Delta)$, starting from the left
invariant weight $\hat{\varphi}$ with GNS-construction $(H, \iota, \hat{\Lambda})$. From [26], Theorem 8.29 we have that the bidual l.c. quantum group $(\hat{\hat{M}}, \hat{\Delta})$ is isomorphic to $(M, \Delta)$.

We denote by $(\hat{\delta}_t)$ the modular automorphism groups of the weight $\hat{\varphi}$. The modular conjugations of the weights $\varphi$ and $\hat{\varphi}$ will be denoted by $J$ and $\hat{J}$ respectively. Then it is worthwhile to mention that

$$R(x) = \hat{J}x^*\hat{J} \quad \text{for all } x \in M \quad \text{and} \quad \hat{R}(y) = \hat{J}y^*\hat{J} \quad \text{for all } y \in \hat{M}.$$  

Let us mention important special cases of l.c. quantum groups.

1. **Kac algebras** [12]. From [12], we know that $(M, \Delta)$ is a Kac algebra if and only if $(\tau_t)$ is trivial and $\sigma_t R = R \sigma_{-t}$ for all $t \in \mathbb{R}$. Now, denote by $(\sigma_t')$ the modular automorphism group of $\psi$. Because $\psi = \varphi R$ we get that $\sigma_t' R = R \sigma_{-t}$ for all $t \in \mathbb{R}$. Hence $(M, \Delta)$ is a Kac algebra if and only if $(\tau_t)$ is trivial and $\sigma' = \sigma$. From [10], we know that $\sigma_t'(x) = \delta_M^t \sigma_t(x) \delta_M^{-t}$ for all $x \in M$ and $t \in \mathbb{R}$. Hence $\sigma' = \sigma$ if and only if $\delta_M$ is affiliated to the center of $M$.

In particular, $(M, \Delta)$ is a Kac algebra if $M$ is commutative. Then $(M, \Delta)$ is generated by a usual l.c. group $G : \quad M = L^\infty(G), \quad (\Delta f)(g, h) = f(gh), \quad (S f)(g) = f(g^{-1}), \quad \varphi(f) = \int f(g) \, dg$, where $f \in L^\infty(G)$, $g, h \in G$ and we integrate with respect to the left Haar measure $dg$ on $G$. The right invariant weight $\psi$ is given by $\psi(f) = \int f(g^{-1}) \, dg$. The modular element $\delta_M$ is given by the strictly positive function $g \mapsto \delta_{G}(g)^{-1}$.

The von Neumann algebra $M = L^\infty(G)$ acts on $H = L^2(G)$ by multiplication and

$$(W_G \xi)(g, h) = \xi(g, g^{-1}h)$$

for all $\xi \in H \otimes H = L^2(G \times G)$. Then $\hat{M} = L(G)$ is the group von Neumann algebra generated by the operators $(\lambda_g)_{g \in G}$ of the left regular representation of $G$ and $\hat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$. Clearly, $\hat{\Delta}^{\text{op}} := \sigma \hat{\Delta} = \hat{\Delta}$, so $\hat{\Delta}$ is cocommutative.

b) A l.c. quantum group is called **compact** if its Haar measure is finite: $\varphi(1) < +\infty$, which is equivalent to the fact that the norm closure of $\{(i \otimes \omega) | \omega \in B(H)_s\}$ is a unital $C^*$-algebra. A l.c. quantum group $(M, \Delta)$ is called **discrete** if $(M, \hat{\Delta})$ is compact.

**Crossed and bicrossed products** An action of a l.c. quantum group $(M, \Delta)$ on a von Neumann algebra $N$ is a normal, injective and unital $*$-homomorphism $\alpha : N \to M \otimes N$ such that $(i \otimes \alpha) \alpha(x) = (\Delta \otimes \iota) \alpha(x)$ for all $x \in N$. This generalizes the definition of an action of a (separable) l.c. group $G$ on a (\(\sigma\)-finite) von Neumann algebra $N$, as a continuous map $G \to \text{Aut} \, N : s \mapsto \alpha_s$ such that $\alpha_{st} = \alpha_s \alpha_t$ for all $s, t \in G$. Indeed, putting $M = L^\infty(G)$, one can identify $M \otimes N$ with $L^\infty(G, N)$ and $M \otimes M \otimes N$ with $L^\infty(G \times G, N)$ and define the above homomorphism $\alpha$ by $(\alpha(x))(s) = \alpha_{s^{-1}}(x)$. The fixed point algebra of an action $\alpha$ is defined by $N^\alpha = \{x \in N \mid \alpha(x) = 1 \otimes x\}$. 


A cocycle for an action of a l.c. group $G$ on a commutative von Neumann algebra $N$ is a Borel map $u : G \times G \to N$ such that $\alpha_r(u(s, t)) u(r, s) = u(r, s) u(rs, t)$ nearly everywhere. Then, putting $M = L^\infty(G)$, one can define a unitary $U \in M \otimes M \otimes N$ by $U(s, t) = u(t^{-1}, s^{-1})$ satisfying

$$(\iota \otimes \iota \otimes \alpha)(U)(\Delta \otimes \iota \otimes \iota)(U) = (1 \otimes U)(\iota \otimes \Delta \otimes \iota)(U).$$

For the general definition of a cocycle action of a l.c. quantum group on an arbitrary von Neumann algebra, we refer to Definition 1.1 in [51].

The cocycle crossed product $G_{\alpha,U} \ltimes N$ is the von Neumann subalgebra of $B(L^2(G)) \otimes N$ generated by $\alpha(N)$ and $\{ (\omega \otimes \iota \otimes \iota)(\tilde{W}) \mid \omega \in L^1(G) \}$, where $\tilde{W} = (W_G \otimes 1)U^\ast$. This is a von Neumann algebraic version of the twisted $C^*$-algebraic crossed product [40]. There exists a unique action $\hat{\alpha}$ of $(\mathcal{L}(G), \hat{\Delta})$ on $G_{\alpha,U} \ltimes N$ such that

$$\hat{\alpha}(\alpha(x)) = 1 \otimes \alpha(x) \quad \text{for all } x \in N,$$

$$(\iota \otimes \hat{\alpha})(\tilde{W}) = W_{G,12} \tilde{W}_{134},$$

and for any n.s.f. weight $\theta$ on $N$, we can define the dual n.s.f. weight $\tilde{\theta}$ on $G_{\alpha,U} \ltimes N$ by the formula

$$\tilde{\theta} = \theta \alpha^{-1} (\hat{\phi} \otimes \iota \otimes \iota)\hat{\alpha}.$$

**Definition 2.1.** (see [3]) Let $G, G_1$ and $G_2$ be (separable) l.c. groups and let a homomorphism $i : G_1 \to G$ and an anti-homomorphism $j : G_2 \to G$ have closed images and be homeomorphisms onto these images. Suppose that $i(G_1) \cap j(G_2) = \{e\}$ and that the complement of $i(G_1)j(G_2)$ in $G$ has measure zero. Then we call $G_1$ and $G_2$ a matched pair of l.c. groups.

Observe that this definition of a matched pair of l.c. groups, due to Baaaj, Skandalis and the first author, is more general than the one studied in [5] and [51]. Indeed, in [5], there is given an example of a matched pair in the sense of the definition above, which does not fit in the definition of [5]. More specifically, consider the map

$$\theta : G_1 \times G_2 \to G : (g, s) \mapsto i(g)j(s),$$

which is clearly injective. In [5] and [51], the map $\theta$ is supposed to have a range $\Omega$ which is open in $G$, with complement of measure zero and such that $\theta$ is a homeomorphism of $G_1 \times G_2$ onto $\Omega$. In the example of [5], the range of $\theta$ has an empty interior. However, the following proposition holds:

**Proposition 2.2.** If, in Definition 2.1, $G$ is a Lie group, then the map

$$\theta : G_1 \times G_2 \to G : (g, s) \mapsto i(g)j(s)$$

is a Lie group.
has an open range \( \Omega \) and is a diffeomorphism of \( G_1 \times G_2 \) onto \( \Omega \), where \( G_1 \) and \( G_2 \) are Lie groups under the identification with closed subgroups of \( G \).

**Proof.** Denote by \( \mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2 \) the Lie algebras of \( G, G_1, G_2 \), respectively. Then, we have an injective homomorphism and anti-homomorphism 

\[
\begin{align*}
d_i : g_1 &\to \mathfrak{g} \quad \text{and} \quad dj : g_2 \to \mathfrak{g}.
\end{align*}
\]

Because \( i(G_1) \cap j(G_2) = \{e\} \), we get \( di(g_1) \cap dj(g_2) = \{0\} \) (otherwise, the exponential mapping produces elements in \( i(G_1) \cap j(G_2) \)). Hence, we can take a linear subspace \( \mathfrak{k} \) of \( \mathfrak{g} \) (not necessarily a Lie subalgebra) such that 

\[
\mathfrak{g} = di(g_1) \oplus dj(g_2) \oplus \mathfrak{k} \text{ as vector spaces. We first prove that } \mathfrak{k} = \{0\}.
\]

Denote by \( \exp_{\mathfrak{g}} \) the exponential mapping of \( G \) and analogously for \( \exp_{\mathfrak{g}_1,\mathfrak{g}_2} \). Take open subsets \( \mathcal{U}_1 \subset \mathfrak{g}_1, \mathcal{V} \subset \mathfrak{k} \) containing \( 0 \) such that \( \exp_{\mathfrak{g}} \) is a diffeomorphism of \( \exp(\mathcal{U}_1) \times \exp(\mathcal{V}) \to G \) onto an open subset of \( G \). Define 

\[
\rho : \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{V} \to G : \rho(v, w, z) = i(\exp_{\mathfrak{g}_1}(v)) j(\exp_{\mathfrak{g}_2}(w)) \exp_{\mathfrak{g}}(z)
\]

\[
= \exp_{\mathfrak{g}}(di(v)) \exp_{\mathfrak{g}}(dj(w)) \exp_{\mathfrak{g}}(z).
\]

Because \( \mathfrak{g} = di(g_1) \oplus dj(g_2) \oplus \mathfrak{k} \), we find that \( d\rho(0, 0, 0) \) is bijective. So, for \( \mathcal{U}_1, \mathcal{U}_2, \mathcal{V} \) small enough, \( \rho \) is a diffeomorphism onto an open subset of \( W \) of \( G \) containing \( e \) and \( \exp_{\mathfrak{g}} \) will be a diffeomorphism of \( \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{V} \) onto an open subset of \( G \) of \( \mathcal{G} \).

It is clear that \( \theta(W_1 \times W_2) \subset W \) and \( \rho^{-1}(\theta(W_1 \times W_2)) = \mathcal{U}_1 \times \mathcal{U}_2 \times \{0\} \). As a diffeomorphism, \( \rho \) is a Borel isomorphism and so, if \( \mathfrak{k} \neq \{0\} \), \( \theta(W_1 \times W_2) \) has measure zero in \( G \). This contradicts the result of [1], saying that \( \theta \) is automatically a Borel isomorphism. Hence, \( \mathfrak{k} = \{0\} \).

But then, \( \theta : W_1 \times W_2 \to W \) is a diffeomorphism. In particular, \( W \subset \Omega \). If now \( i(y_0)j(s_0) \in \Omega \), it follows that \( i(y_0)j(s_0) \in i(y_0)\mathcal{W}j(s_0) \subset \Omega \). Hence, \( \Omega \) is open in \( G \) and \( \theta \) is a diffeomorphism of \( G_1 \times G_2 \) onto \( \Omega \), because we know that \( \theta \) is injective. \( \square \)

In [3], it is proved that \( \theta \) is automatically a Borel isomorphism, i.e. it induces an isomorphism between \( L^\infty(G_1 \times G_2) \) and \( L^\infty(G) \). Hence, this data allows to construct as follows two actions: \( \alpha \) of \( G_1 \) on \( M_2 = L^\infty(G_2) \) and \( \beta \) of \( G_1[0\mathfrak{p}] \) on \( M_1 = L^\infty(G_1) \) verifying certain compatibility relations.

Define \( \Omega \) to be the image of \( \theta \) and define the Borel isomorphism 

\[
\rho : G_1 \times G_2 \to \Omega^{-1} : (g, s) \mapsto j(s)i(g).
\]

So \( \mathcal{O} = \theta^{-1}(\Omega \cap \Omega^{-1}) \) and \( \mathcal{O}' = \rho^{-1}(\Omega \cap \Omega^{-1}) \) are Borel subsets of \( G_1 \times G_2 \), with complement of measure zero, and \( \rho^{-1}\theta \) is a Borel isomorphism of \( \mathcal{O} \) onto \( \mathcal{O}' \). For all \( (g, s) \in \mathcal{O} \) define \( \beta_s(g) \in G_1 \) and \( \alpha_s(g) \in G_2 \) such that 

\[
\rho^{-1}(\theta(g, s)) = (\beta_s(g), \alpha_s(s)).
\]

Hence we get 

\[
\begin{align*}
\rho^{-1}(\theta(g, s)) &= (\beta_s(g), \alpha_s(s)) \\
&= (i(g)j(s), i(s)j(g)) \\
&= i(g)j(s) \quad \text{for all } (g, s) \in \mathcal{O}.
\end{align*}
\]
Lemma 2.3. ([51], Lemma 4.8)

Let \((g, s) \in \mathcal{O}\) and \(h \in G_1\). Then \((hg, s) \in \mathcal{O}\) if and only if \((h, \alpha_g(s)) \in \mathcal{O}\), and in that case

\[
\alpha_{hg}(s) = \alpha_h(\alpha_g(s)) \quad \text{and} \quad \beta_s(hg) = \beta_s(h) \beta_s(g).
\]

Let \((g, s) \in \mathcal{O}\) and \(t \in G_2\). Then \((g, ts) \in \mathcal{O}\) if and only if \((\beta_s(g), t) \in \mathcal{O}\) and in that case

\[
\beta_{ts}(g) = \beta_t(\beta_s(g)) \quad \text{and} \quad \alpha_g(ts) = \alpha_g(t) \alpha_g(s).
\]

Finally, for all \(g \in G_1\) and \(s \in G_2\) we have \((g, e) \in \mathcal{O}\), \((e, s) \in \mathcal{O}\), and

\[
\alpha_g(e) = e, \quad \alpha_e(s) = s, \quad \beta_s(e) = e \quad \text{and} \quad \beta_e(g) = g.
\]

This can be viewed as a definition of a matched pair of l.c. groups in terms of mutual actions.

The cocycles for the above actions can be introduced as measurable maps \(U : G_1 \times G_1 \times G_2 \to U(1)\) and \(V : G_1 \times G_2 \times G_2 \to U(1)\), where \(U(1)\) is the unit circle in \(\mathbb{C}\), satisfying

\[
\begin{align*}
U(g, h, \alpha_g(s)) U(gh, k, s) &= U(h, k, s) U(g, hk, s), \\
V(\beta_s(g), t, r) V(g, s, rt) &= V(g, s, t) V(g, ts, r), \\
V(gh, s, t) \tilde{U}(g, h, ts) &= \tilde{U}(g, h, s) \tilde{U}(\beta_{\alpha_h(s)}(g), \beta_s(h), t) \\
V(g, \alpha_h(s), \alpha_{\beta_h(t)}) V(h, s, t)
\end{align*}
\]

nearly everywhere. Then we have a definition of a cocycle matched pair of l.c. groups.

Fixing a cocycle matched pair of l.c. groups \(G_1\) and \(G_2\), denoting \(H_i = L^2(G_i)\) \((i = 1, 2)\), \(H = H_1 \otimes H_2\) and identifying \(U\) and \(V\) with unitaries in \(M_1 \otimes M_1 \otimes M_2\) and in \(M_1 \otimes M_2 \otimes M_2\) respectively, define unitaries \(W\) and \(\tilde{W}\) on \(H \otimes H\) by

\[
\tilde{W} = (\beta \otimes \iota \otimes \iota)((W_{G_1} \otimes 1)U^\star) (\iota \otimes \iota \otimes \alpha)(V(1 \otimes \tilde{W}_{G_2})) \quad \text{and} \quad W = \Sigma \tilde{W}^\star \Sigma.
\]

On the von Neumann algebra \(M = G_1 \alpha, \iota \rtimes L^\infty(G_2)\), let us define a faithful \(*\)-homomorphism

\[
\Delta : M \to B(H \otimes H) : \Delta(z) = W^\star(1 \otimes z)W \quad (\forall z \in M)
\]

and denote by \(\varphi\) the dual weight of the canonical left invariant trace \(\varphi_2\) on \(L^\infty(G_2)\). Then, Theorem 2.13 of [51] shows that \((M, \Delta)\) is a l.c. quantum group with \(\varphi\) as a left invariant weight, which we call the cocycle bicrossed product of \(G_1\) and \(G_2\). One can also show that its scaling constant is 1. The dual l.c. quantum group is \((\bar{M}, \bar{\Delta})\), where \(\bar{M} = G_2 \alpha, \iota \rtimes L^\infty(G_1)\) and \(\bar{\Delta}(z) = W^\star(1 \otimes z)\tilde{W}\) for all \(z \in \bar{M}\).

One can get explicit formulas for the modular operators, modular conjugations of the left invariant weights, unitary antipodes, scaling groups and
modular elements of both \((M, \Delta)\) and its dual in terms of the above mutual actions, the cocycles and the modular functions \(\delta, \delta_1\) and \(\delta_2\) of the l.c. groups \(G, G_1\) and \(G_2\). In particular, one can characterize all cocycle bicrossed products of l.c. groups which are Kac algebras.

**Proposition 2.4.** The l.c. quantum group \((M, \Delta)\) is a Kac algebra if and only if
\[
\delta(i(g \beta_s(g)^{-1})) \delta_1(g^{-1} \beta_s(g)) \delta_2(\alpha_g(s)s^{-1}) = 1 \quad \text{and} \quad \\
\frac{\delta_1(\beta_s(g))}{\delta_1(g)} = \frac{\delta_2(\alpha_g(s))}{\delta_2(s)}.
\]

This proposition implies three helpful corollaries.

**Corollary 2.5.** If \(\alpha\) or \(\beta\) is trivial, \((M, \Delta)\) and \((\hat{M}, \hat{\Delta})\) are Kac algebras.

**Corollary 2.6.** If both \(\alpha\) and \(\beta\) preserve modular functions and Haar measures, then \((M, \Delta)\) and \((\hat{M}, \hat{\Delta})\) are Kac algebras.

Remark that the conditions of this corollary are fulfilled if both groups are discrete. Indeed, any discrete group is unimodular and the Haar measure is constant at an arbitrary point of such a group.

**Corollary 2.7.** If \((G_1, G_2)\) is a fixed matched pair of l.c. groups and cocycles \(U\) and \(V\) satisfy (2.1), we get a cocycle bicrossed product \((M, \Delta)\). If one of these cocycle bicrossed products is a Kac algebra, then all of them are Kac algebras.

**Proof.** The necessary and sufficient conditions for \((M, \Delta)\) to be a Kac algebra in Proposition 2.4 are independent of \(U\) and \(V\). \(\square\)

It is easy to check that the above measurable mutual actions \(\alpha_g\) and \(\beta_s\) of \(G_1\) and \(G_2\) are in fact the restrictions of the canonical continuous actions \(\tilde{\alpha}_g\) of \(G_1\) on \(G/G_1\) and \(\tilde{\beta}_s\) of \(G_2\) on \(G_2\setminus G\) (topologies on \(G_1\) and \(G_2\setminus G\) and, respectively, on \(G_2\) and \(G/G_1\), are in general different). This allows, in particular, to express the \(C^*\)-algebras of the \(C^*\)-algebraic versions of the split extension (i.e. with trivial cocycles) and its dual respectively as \(G_1 \tilde{\alpha} \ltimes C_0(G/G_1)\) and \(C_0(G_2\setminus G) \rtimes \tilde{\beta} G_2\), see [6].

**Extensions of l.c. groups** To clarify the following definition, recall that any normal \(*\)-homomorphism \(\beta : M_1 \to \hat{M}\) of l.c. quantum groups satisfying \(\Delta \beta = (\beta \otimes \beta) \Delta_1\) generates two canonical actions: \(\mu\) of \((M_1, \Delta_1)\) on \(M\) and \(\theta\) of \((M_1, \Delta_1^p)\) on \(M\) ([5], Proposition 3.1). On a formal level, this can be understood easily: the morphism \(\beta\) gives rise to a dual morphism \(\tilde{\beta} : \hat{M} \to M_1\) and \(\mu\) should be thought of as \(\mu = (\tilde{\beta} \otimes i) \Delta\), while \(\theta\) should be thought of as \(\theta = (\tilde{\beta} \otimes i) \Delta^p\).
Definition 2.8. Let $G_i$ ($i = 1, 2$) be l.c. groups and let $(M, \Delta)$ be a l.c. quantum group. We call

$$(L^\infty(G_2), \Delta_2) \overset{\alpha}{\longrightarrow} (M, \Delta) \overset{\beta}{\longrightarrow} (L(G_1), \hat{\Delta}_1)$$

a short exact sequence, if

$$\alpha : L^\infty(G_2) \to M \quad \text{and} \quad \beta : L^\infty(G_1) \to \hat{M}$$

are normal, faithful $*$-homomorphisms satisfying

$$\Delta \alpha = (\alpha \otimes \alpha) \Delta_2 \quad \text{and} \quad \hat{\Delta} \beta = (\beta \otimes \beta) \Delta_1$$

and if $\alpha(L^\infty(G_2)) = M^\theta$, where $\theta$ is the canonical action of $(L(G_1), \hat{\Delta}_1^{op})$ on $M$ generated by the morphism $\beta$. In this situation, we call $(M, \Delta)$ an extension of $G_2$ by $\hat{G}_1$.

The faithfulness of the morphisms $\alpha$ and $\beta$ reflects the exactness of the sequence in the first and third place. The formula $\alpha(L^\infty(G_2)) = M^\theta$ reflects its exactness in the second place. Given a short exact sequence as above, one can check that the dual sequence

$$(L^\infty(G_1), \Delta_1) \overset{\beta}{\longrightarrow} (\hat{M}, \hat{\Delta}) \overset{\alpha}{\longrightarrow} (L(G_2), \hat{\Delta}_2)$$

is exact as well.

Given a cocycle matched pair of l.c. groups, one can check that their cocycle bicrossed product is an extension in the sense of Definition 2.8. Moreover, it belongs to a special class of extensions, called cleft extensions ([51], Theorem 2.8). This theorem also shows that, conversely, all cleft extensions of l.c. groups (and of l.c. quantum groups) are given by the cocycle bicrossed products. This means that, whenever $(M, \Delta)$ is a cleft extension of $G_2$ by $G_1$, the pair consisting of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$ is a cocycle matched pair in the sense of [51], Definition 2.1 and $(M, \Delta)$ is isomorphic to their cocycle bicrossed product. From the results of [6], it follows that this precisely means that $(G_1, G_2)$ is a matched pair in the sense of Definition 2.1 with cocycles as in Equation (2.1).

By definition, two extensions

$$(L^\infty(G_2), \Delta_2) \overset{\alpha_a}{\longrightarrow} (M_a, \Delta_a) \overset{\beta_a}{\longrightarrow} (L(G_1), \hat{\Delta}_1) \quad \text{and} \quad (L^\infty(G_2), \Delta_2) \overset{\alpha_b}{\longrightarrow} (M_b, \Delta_b) \overset{\beta_b}{\longrightarrow} (L(G_1), \hat{\Delta}_1)$$

are called isomorphic, if there is an isomorphism $\pi : (M_a, \Delta_a) \to (M_b, \Delta_b)$ of l.c. quantum groups satisfying $\pi \alpha_a = \alpha_b$ and $\hat{\pi} \beta_a = \beta_b$, where $\hat{\pi}$ is the canonical isomorphism of $(M_a, \Delta_a)$ onto $(M_b, \Delta_b)$ associated with $\pi$. 

Given a matched pair \((G_1, G_2)\) of l.c. groups, any couple of cocycles \((\mathcal{U}, \mathcal{V})\) satisfying (2.1) generates as above a cleft extension
\[
(L^\infty(G_2), \Delta_2) \xrightarrow{\alpha} (M, \Delta) \xrightarrow{\beta} (L(G_1), \hat{\Delta}_1).
\]
The extensions given by two pairs of cocycles \((\mathcal{U}_a, \mathcal{V}_a)\) and \((\mathcal{U}_b, \mathcal{V}_b)\), are isomorphic if and only if there exists a measurable map \(R\) from \(G_1 \times G_2\) to \(U(1)\), satisfying
\[
\mathcal{U}_b(g, h, s) = \mathcal{U}_a(g, h, s) \mathcal{R}(h, s) \mathcal{R}(g, \alpha_h(s)) \mathcal{R}(gh, s)
\]
\[
\mathcal{V}_b(g, s, t) = \mathcal{V}_a(g, s, t) \mathcal{R}(\beta_s(g), t) \mathcal{R}(g, ts)
\]
almost everywhere. If this is the case, the pairs \((\mathcal{U}_a, \mathcal{V}_a)\) and \((\mathcal{U}_b, \mathcal{V}_b)\) will be called cohomologous. Then the set of equivalence classes of cohomologous pairs of cocycles \((\mathcal{U}, \mathcal{V})\) satisfying (2.2), exactly corresponds to the set \(\Gamma\) of classes of isomorphic extensions associated with \((G_1, G_2)\).

The set \(\Gamma\) can be given the structure of an abelian group by defining
\[
\pi(\mathcal{U}_a, \mathcal{V}_a) \cdot \pi(\mathcal{U}_b, \mathcal{V}_b) = \pi(\mathcal{U}_a \mathcal{U}_b, \mathcal{V}_a \mathcal{V}_b)
\]
where \(\pi(\mathcal{U}, \mathcal{V})\) denotes the equivalence class containing the pair \((\mathcal{U}, \mathcal{V})\). The group \(\Gamma\) is called the group of extensions of \((L^\infty(G_2), \Delta_2)\) by \((L(G_1), \hat{\Delta}_1)\) associated with the matched pair of l.c. groups \((G_1, G_2)\). The unit of this group corresponds to the class of cocycles cohomologous to trivial. The corresponding extension is called split extension; all other extensions are called non-trivial extensions.

Closed normal quantum subgroups Definition 2.8 is the partial case of the general definition of a short exact sequence
\[
(M_2, \Delta_2) \xrightarrow{\alpha} (M, \Delta) \xrightarrow{\beta} (\hat{M}_1, \hat{\Delta}_1),
\]
where \((M_1, \Delta_1)\), \((M_2, \Delta_2)\) and \((M, \Delta)\) are l.c. quantum groups, see Definition 3.2 in [51]. We explain the relation between this notion and the following notion of a closed normal quantum subgroup.

**Definition 2.9.** A l.c. quantum group \((M_2, \Delta_2)\) is called a closed quantum subgroup of \((M, \Delta)\) if there exists a normal, faithful \(*\)-homomorphism \(\alpha : M_2 \rightarrow M\) such that \(\Delta \alpha = (\alpha \otimes \alpha) \Delta_2\).

This definition might need some justification: in [23], J. Kustermans defines morphisms between l.c. quantum groups on the (natural) level of universal \(C^*\)-algebraic quantum groups. So, it might seem strange to require the existence of a normal morphism on the von Neumann algebra level. We claim, however, that this precisely characterizes the closedness (or properness of the injective embedding). Let us illustrate this with an example. Consider the identity map from \(\mathbb{R}_d\) with the discrete topology to \(\mathbb{R}\) with its usual topology. Dualizing,
we get a morphism $\alpha : C_0(\mathbb{R}) \to M(C_0(\mathbb{R}_d)) = C_b(\mathbb{R}_d)$ which is injective. It is clear that we want to exclude this type of morphisms. This is precisely achieved by requiring the normality (weak continuity) of the morphism. To conclude, we mention that in the case where $M_2 = L(G_2)$ and $M = L(G)$, we precisely are in the situation of an identification $\pi : G_2 \to G$ of $G_2$ with a closed subgroup of $G$ and $\alpha(\lambda_g) = \lambda_{\pi(g)}$, see Theorem 6 in [46].

Next, we define normality of a closed quantum subgroup. Recall that when $A_1$ is a Hopf subalgebra of a Hopf algebra $A$, $A_1$ is called normal if $A_1$ is invariant under the adjoint action. Using Sweedler notation, this means

$$\sum a_1 x S(a_2) \in A_1 \quad \text{for all} \quad x \in A_1, a \in A.$$ 

Recalling that $S((\iota \otimes \omega)(W)) = (\iota \otimes \omega)(W^*)$ and that

$$\Delta((\iota \otimes \omega)(W)) = (\iota \otimes \iota \otimes \omega)(W_{13}W_{23})$$

because of the pentagon equation, it is easy to verify that the operator algebraic version of normality is given as follows.

**Definition 2.10.** If $\alpha : M_2 \to M$ turns $(M_2, \Delta_2)$ into a closed quantum subgroup of the l.c. quantum group $(M, \Delta)$, we say that $(M_2, \Delta_2)$ is normal if

$$W(\alpha(M_2) \otimes 1)W^* \subset \alpha(M_2) \otimes B(H).$$

As could be expected, we now prove the bijective correspondence between closed normal quantum subgroups and short exact sequences.

**Theorem 2.11.** Suppose that $\alpha : M_2 \to M$ turns $(M_2, \Delta_2)$ into a closed normal quantum subgroup of $(M, \Delta)$. Then, there exists a unique (up to isomorphism) l.c. quantum group $(M_1, \Delta_1)$ and a unique $\beta : M_1 \to \hat{M}$ such that

$$(M_2, \Delta_2) \xrightarrow{\alpha} (M, \Delta) \xrightarrow{\beta} (\hat{M}_1, \hat{\Delta}_1)$$

is a short exact sequence.

If, conversely, we have a short exact sequence, then $\alpha : M_2 \to M$ turns $(M_2, \Delta_2)$ into a closed normal quantum subgroup of the l.c. quantum group $(M, \Delta)$.

**Proof.** Suppose first that we have a short exact sequence. Consider the coaction $\theta$ of $(M_1, \Delta_1^\text{op})$ on $M$ associated with $\beta$. By definition of exactness, we have $\alpha(M_2) = M^\theta$. Let $x \in M_2$. It suffices to prove that $(\theta \otimes \iota)(W(\alpha(x) \otimes 1)W^*) = W_{23}(1 \otimes \alpha(x) \otimes 1)W^*_{23}$. From Proposition 3.1 of [45] and with the notations introduced over there, it follows that it is sufficient to prove that $1 \otimes \alpha(x)$ commutes with $Z_1$, or equivalently, $\mu(\alpha(x)) = 1 \otimes \alpha(x)$. But,

$$\mu(\alpha(x)) = (\hat{R}_1 \otimes R)\theta(\alpha(x)) = (\hat{R}_1 \otimes R)\theta(\alpha(R_2(x))) = 1 \otimes \alpha(x).$$

This proves the most easy, second part of the theorem.
Next, suppose that we have a closed normal quantum subgroup \((M_2, \Delta_2)\) of \((M, \Delta)\). Using Proposition 3.1 from [48], the morphism \(\alpha\) generates two actions: \(\hat{\mu}\) is an action of \((M_2, \Delta_2)\) on \(\hat{M}\) and \(\hat{\theta}\) is an action of \((M_2, \Delta^{op})\) on \(\hat{M}\) and they are determined by

\[
\hat{\mu}(x) = \hat{Z}_1(1 \otimes x)\hat{Z}_1^* \quad \text{and} \quad \hat{\theta}(x) = \hat{Z}_2(1 \otimes x)\hat{Z}_2^* \quad \text{for all} \ x \in \hat{M},
\]

where

\[
\hat{Z}_1 = (\iota \otimes \alpha)(\hat{W}_2^*) \quad \text{and} \quad \hat{Z}_2 = (J_2 \otimes J)\hat{Z}_1(J_2 \otimes J).
\]

The actions \(\hat{\mu}\) and \(\hat{\theta}\) are related by the formula \(\hat{\theta}(x) = (\hat{R}_2 \otimes \hat{R})\hat{\mu}(\hat{R}(x))\) and satisfy

\[
(\hat{\mu} \otimes \iota)(\hat{W}) = (\iota \otimes \alpha)(\hat{W}_2)_{13}\hat{W}_{23} \quad \text{and} \quad (\hat{\theta} \otimes \iota)(\hat{W}) = \hat{W}_{23}(\iota \otimes \alpha)(\hat{W}_2)_{13}.
\]

Using the definition of the left invariant weight \(\hat{\varphi}\) on the dual \((\hat{M}, \hat{\Delta})\) of \((M, \Delta)\), we easily conclude that \(\hat{\varphi}\) is invariant under the action \(\hat{\mu}\) and moreover, for all \(x \in N_{\hat{\varphi}}\) and \(\omega \in \hat{M}_{2,*}\), we have \((\omega \otimes \iota)\hat{\mu}(x) \in N_{\hat{\varphi}}\) and

\[
\hat{\Lambda}((\omega \otimes \iota)\hat{\mu}(x)) = (\omega \otimes \iota)(\hat{Z}_1)\hat{\lambda}(x).
\]

From Proposition 4.3 of [48], it then follows that \(\hat{Z}_1\) is the canonical implementation of the action \(\hat{\mu}\) (in the sense of Definition 3.6 of [48]). We want to prove that \(\hat{\mu}\) is integrable (see Definition 1.4 in [48]) and we will use Theorem 5.3 of [48] to do this. So, we have to construct a normal \(^*\)-homomorphism \(\rho : \hat{M}_{2,\hat{\mu}} \times \hat{M} \to B(H)\) such that

\[
\rho(\hat{\mu}(x)) = x \quad \text{for all} \ x \in \hat{M} \quad \text{and} \quad (\iota \otimes \rho)(\hat{W}_2 \otimes 1) = \hat{Z}_1^*.
\]

We first define

\[
\hat{\rho} : \hat{M}_{2,\hat{\mu}} \times \hat{M} \to B(H \otimes H) : \hat{\rho}(z) = \mathcal{V}(\alpha \otimes \iota)(\hat{Z}_1^* z \hat{Z})\mathcal{V}^*,
\]

where \(\mathcal{V} = (J \otimes J)W(J \otimes J)\) has the properties \(\mathcal{V} \in M \otimes M'\) and \(\Delta^{op}(y) = \mathcal{V}^*(1 \otimes y)\mathcal{V}\) for all \(y \in M\). This map \(\hat{\rho}\) is well-defined for the following reasons. For \(x \in \hat{M}\), we have \(\hat{Z}_1^* \hat{\mu}(x)\hat{Z}_1 = 1 \otimes x\). We can apply \(\alpha \otimes \iota\) and because \(\mathcal{V} \in M \otimes M'\), we find that \(\hat{\rho}(\hat{\mu}(x)) = 1 \otimes x\). Next, for \(\omega \in \hat{M}_{2,*}\), we find

\[
\hat{Z}_1^*((\omega \otimes \iota)(\hat{W}_2) \otimes 1)\hat{Z}_1 = (\omega \otimes \iota \otimes \iota)(\iota \otimes \iota \otimes \alpha)(\hat{W}_{2,23}\hat{W}_{2,12}\hat{W}_{2,23}^*)
\]

\[
= (\omega \otimes \iota \otimes \iota)(\hat{W}_{2,12}(\iota \otimes \alpha)(\hat{W}_2)_{13}).
\]

Again, it is possible to apply \(\alpha \otimes \iota\) and we find

\[
(\iota \otimes \hat{\rho})(\hat{W}_2 \otimes 1) = \mathcal{V}_{23}(\iota \otimes \alpha \otimes \alpha)(\iota \otimes \Delta^{op}_{23})(\hat{W}_2)\mathcal{V}_{23}^* = \hat{Z}_{1,13}^*,
\]

because \(\alpha\) is a morphism and \(\mathcal{V}^*\) implements \(\Delta^{op}\). The \(\rho\) that we were looking for, is then obtained as \(\hat{\rho}(z) = 1 \otimes \rho(z)\) for all \(z \in M_{2,\hat{\mu}} \times \hat{M}\). Hence, \(\hat{\mu}\) is integrable.
Defining the von Neumann algebra \( M_1 := \hat{M}^\theta \), the fixed point algebra of \( \hat{\mu} \). Because \( (\iota \otimes \hat{\Delta})\hat{\mu} = (\hat{\mu} \otimes \iota)\hat{\Delta} \), it is clear that \( \hat{\Delta}(M_1) \subset M_1 \otimes \hat{M} \). We claim that also \( \hat{\Delta}(M_1) \subset \hat{M} \otimes M_1 \). For this, we will need the normality. Observe that the right leg of \( \hat{Z}_1 \) generates \( \alpha(M_2) \). Hence, by definition, \( M_1 = \hat{M} \cap \alpha(M_2)^\prime \). Because \( \hat{R}_\alpha = \alpha R_2 \) and \( R(x) = Jx^*J \), we conclude that \( \hat{J}M_1 \hat{J} = (\hat{M} \cup \alpha(M_2))^\prime \). By normality, we know that \( W(\alpha(M_2) \otimes 1)W^* \subset \alpha(M_2) \otimes \hat{M} \).

Because \( W(\hat{M} \otimes 1)W^* = \hat{\Delta}^{op}(M) \subset \hat{M} \otimes M \), we get \( W(\hat{J}M_1 \hat{J} \otimes 1)W^* \subset \hat{J}M_1 \hat{J} \otimes M \). Writing \( \hat{J} \otimes J \) around this equation, we find \( W^*(\hat{M}_1 \otimes 1)W \subset M_1 \otimes B(H) \). Because \( W \in \hat{M} \otimes \hat{M} \), it follows that \( W^*(\hat{M}_1 \otimes \hat{M}^\prime)W \subset M_1 \otimes B(H) \). Taking commutants, we conclude that \( M_1 \otimes 1 \subset W^*(\hat{M}_1 \otimes \hat{M})W \). Bringing the \( W \) to the other side, we have proven our claim that \( \hat{\Delta}^{op}(M_1) \subset M_1 \otimes \hat{M} \).

Defining \( \Delta_1 \) to be the restriction of \( \hat{\Delta} \) to \( M_1 \), we have found a von Neumann algebra with comultiplication \( (M_1, \Delta_1) \). In order to produce invariant weights, we first prove that \( \hat{R}(M_1) \subset M_1 \). Verifying the following equality on a slice of \( W \), we easily arrive at the formula

\[
(\iota \otimes \hat{\mu})\hat{\Delta}(x) = ((\hat{\theta} \otimes \iota)\hat{\Delta}(x))_{213} \quad \text{for all } x \in \hat{M}.
\]

Let now \( x \in M_1 \). Then \( \hat{\Delta}(x) \in M_1 \otimes M_1 \), so that

\[
\hat{\Delta}(x)_{13} = (\iota \otimes \hat{\mu})\hat{\Delta}(x) = ((\hat{\theta} \otimes \iota)\hat{\Delta}(x))_{213}.
\]

So,

\[
\hat{\Delta}(x) \in \hat{M}^\theta \otimes M_1 = \hat{R}(M_1) \otimes M_1,
\]

because of the relation between \( \hat{\mu} \) and \( \hat{\theta} \). If we regard the restriction of \( \hat{\Delta}^{op} \) as a map from \( M_1 \) to \( \hat{M} \otimes M_1 \), then it will be an action of \( (M, \hat{\Delta}^{op}) \) on \( M_1 \). But then we know that the \( \sigma \)-strong* closure of

\[
\{(\omega \otimes \iota)\hat{\Delta}^{op}(x) \mid x \in M_1, \omega \in \hat{M}_+ \}
\]

equals \( M_1 \). Combining this with Equation (2.2), we find that \( M_1 \subset \hat{R}(M_1) \). Applying \( \hat{R} \), we get the equality \( M_1 = \hat{R}(M_1) \). In particular, we also have \( M_1 = \hat{M}^\theta \).

Because the restriction of \( \hat{R} \) to \( M_1 \) will be an anti-automorphism of \( M_1 \) anti-commuting with the comultiplication \( \Delta_1 \), it now suffices to produce a left invariant weight on \( (M_1, \Delta_1) \), in order to get that \( (M_1, \Delta_1) \) is a l.c. quantum group. Choose an arbitrary n.s.f. weight \( \eta \) on \( M_1 \). Because \( \hat{\mu} \) is integrable, also \( \hat{\theta} \) is integrable and we can define an n.s.f. operator valued weight \( T \) from \( \hat{M} \) to \( M_1 = \hat{M}^\theta \) by the formula \( T(z) = (\hat{\phi}_2 \otimes \iota)\hat{\theta}(z) \) for all \( z \in \hat{M}^+ \). Defining \( \hat{\eta} = \eta T \), we get an n.s.f. weight \( \hat{\eta} \) on \( \hat{M} \). We claim that the weight \( \hat{\eta} \) is invariant under the action \( \hat{\mu} \). In fact, by verifying the next formula on slices of \( \hat{W} \), we easily get that

\[
(\iota \otimes \hat{\theta})\hat{\mu}(x) = ((\iota \otimes \hat{\theta})\hat{\theta}(x))_{213} \quad \text{for all } x \in \hat{M}.
\]
So, for all $z \in \hat{M}$, $\omega \in \hat{M}$, and $\omega' \in \hat{M}$, we get
\[ \omega'(T((\omega \otimes \iota)\tilde{\mu}(z))) = (\omega \otimes \omega')\tilde{\mu}(T(z)) = \omega(1) \omega'(T(z)), \]

because $T(z)$ belongs to the extended positive part of $M_1$. Hence, $\tilde{\eta}((\omega \otimes \iota)\tilde{\mu}(z)) = \omega(1) \tilde{\eta}(z)$, proving our claim.

Above, we already observed that $\tilde{\varphi}$ is invariant under $\tilde{\mu}$. It then follows from Lemma 3.9 in [48] that the Connes cocycle $u_t = [D\tilde{\varphi} : D\tilde{\eta}]_t$ belongs to $M_1$ for all $t \in \mathbb{R}$. From the theory of operator valued weights, we know that $\sigma^\eta_t(x) = \sigma^\eta_t(x)$ for all $x \in M_1$. Hence, $(u_t)$ is a cocycle with respect to the modular group ($\sigma^\eta$) on $M_1$. So, there exists a unique n.s.f. weight $\varphi_1$ on $M_1$ such that $[D\varphi_1 : D\eta]_t = u_t$. Define $\tilde{\varphi}_1 = \varphi_1T$. From operator valued weight theory, we get
\[ [D\tilde{\varphi}_1 : D\tilde{\eta}]_t = [D\varphi_1 : D\eta]_t = u_t = [D\tilde{\varphi} : D\tilde{\eta}]_t, \]

which yields $\tilde{\varphi}_1 = \tilde{\varphi}$. Let now $x \in \hat{M}$. Because
\[ (\iota \otimes \tilde{\theta})\hat{\Delta}(x) = ((\iota \otimes \hat{\Delta})\tilde{\theta}(x))_{213}, \]

we find, for all $\omega', \omega' \in \hat{M}$,
\[ \omega'(T((\omega \otimes \iota)\hat{\Delta}(x))) = (\omega \otimes \omega')\Delta_1(T(x)). \]

When $x \in \hat{M}$ is such that $T(x)$ is bounded, we conclude that
\[ \tilde{\varphi}_1((\omega \otimes \iota)\hat{\Delta}(x)) = \varphi_1((\omega \otimes \iota)\Delta_1(T(x))). \]

Because $\tilde{\varphi}_1 = \tilde{\varphi}$, the left hand side equals $\omega(1) \tilde{\varphi}_1(x) = \omega(1) \varphi_1(T(x))$. Hence, for all $x \in \hat{M}$ such that $T(x)$ is bounded and for all $\omega \in M_1$, we find
\[ \omega(1) \varphi_1(T(x)) = \varphi_1((\omega \otimes \iota)\Delta_1(T(x))). \]

Take an increasing net $(u_i)$ in $\hat{M}$ such that $T(u_i)$ converges increasingly to 1. Take $y \in M_1$. By lower semi-continuity, we get
\[ \omega(1) \varphi_1(y^*y) = \sup_i \omega(1) \varphi_1(T(y^*u_iy)) = \sup_i \varphi_1((\omega \otimes \iota)\Delta_1(T(y^*u_iy))) = \varphi_1((\omega \otimes \iota)\Delta_1(y^*y)). \]

Hence, $\varphi_1$ is an n.s.f. left invariant weight on $(M_1, \Delta_1)$ and the latter is a l.c. quantum group.

Define $\beta$ to be the identity map, embedding $M_1$ into $\hat{M}$. In order to obtain that
\[ (M_2, \Delta_2) \xrightarrow{\alpha} (M, \Delta) \xrightarrow{\beta} (\hat{M}, \hat{\Delta}_1) \]

is a short exact sequence, it remains to show that $\alpha(M_2) = M^\theta$, where $\theta$ is the canonical action of $(M_1, \Delta^\theta)$ on $M$, associated to $\beta$ (see Proposition 3.1 in [51]). Because $\theta = (\hat{R}_1 \otimes R)\mu R$ and $R(\alpha(M_2)) = \alpha(M_2)$, it suffices to show that $\alpha(M_2) = M^\theta$. From Proposition 3.1 in [51], we immediately deduce that
$M^\mu = M \cap \beta(M_1)'$. Above we already saw that $M_1 = \hat{M} \cap \alpha(M_2)'$. Because $\alpha(M_2)$ is a two-sided coideal of $(M, \Delta)$, it follows from Théorème 3.3 in [1] that

$$M \cap (\hat{M} \cap \alpha(M_2)')' = \alpha(M_2).$$

So, we have a short exact sequence of l.c. quantum groups.

Finally, we should prove the uniqueness of this short exact sequence up to isomorphism. Suppose that we have another short exact sequence $(M_2, \Delta_2) \xrightarrow{\alpha} (M, \Delta) \xrightarrow{\gamma} (\hat{M}_3, \hat{\Delta}_3)$.

We still have the same action $\hat{\mu}$ of $(\hat{M}_2, \hat{\Delta}_2)$ on $\hat{M}$ and a reasoning as in the previous paragraph yields that $\gamma(M_3) = M^\mu$. Hence, it follows that $\gamma$ gives an isomorphism of l.c. quantum groups between $(M_3, \Delta_3)$ and the l.c. quantum group $(M_1, \Delta_1)$ constructed above.

\[\square\]

## 3 Matched pairs of Lie groups and Lie algebras

In what follows we consider Lie groups and Lie algebras over the field $k = \mathbb{C}$ or $\mathbb{R}$.

**Definition 3.1.** We call $(G_1, G_2)$ a matched pair of Lie groups if, in Definition 2.1, $G$ is a Lie group.

Observe that it follows from Proposition 2.2 that $\theta$ is a diffeomorphism of $G_1 \times G_2$ onto the open subset $\Omega$ of $G$.

The infinitesimal form of this definition is as follows (see [33]).

**Definition 3.2.** We call $(\mathfrak{g}_1, \mathfrak{g}_2)$ a matched pair of Lie algebras, if there exists a Lie algebra $\mathfrak{g}$ with Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector spaces.

These conditions are equivalent to the existence of a left action $\triangleright : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ and a right action $\triangleleft : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$, so that $\mathfrak{g}_1$ is a left $\mathfrak{g}_2$-module and $\mathfrak{g}_2$ is a right $\mathfrak{g}_1$-module and

1. $x \triangleright [a, b] = [x \triangleright a, b] + [a, x \triangleright b] + (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a$,

2. $[x, y] \triangleleft a = [x, y \triangleleft a] + [x \triangleleft a, y] + x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a)$,

for all $a, b \in \mathfrak{g}_1$, $x, y \in \mathfrak{g}_2$. Then, for the decomposition of vector spaces above we have

$[a \oplus x, b \oplus y] = (\{a, b\} + x \triangleright b - y \triangleright a) \oplus ([x, y] + x \triangleleft b - y \triangleleft a)$

(see [33], Proposition 8.3.2).
Two matched pairs of Lie algebras, \((g_1, g_2)\) and \((g'_1, g'_2)\), are called isomorphic if there is an isomorphism of the corresponding Lie algebras \(g\) and \(g'\) sending \(g_i\) onto \(g'_i\) (\(i = 1, 2\)).

Let us explain the relation between the two notions of a matched pair.

**Proposition 3.3.** Let \((G_1, G_2)\) be a matched pair of Lie groups in the sense of Definition 3.1. If \(g\) denotes the Lie algebra of \(G\), and if \(g_1\), resp. \(g_2\), are the Lie subalgebras corresponding to the closed subgroups \(i(G_1)\), resp. \(j(G_2)\), then \((g_1, g_2)\) is a matched pair of Lie algebras.

**Proof.** The fact that \(g = g_1 \oplus g_2\) as vector spaces follows from the fact that \(\theta\) is a diffeomorphism in the neighbourhood of the unit element.

The converse problem, to construct a matched pair of Lie groups from a given matched pair \((g_1, g_2)\) of Lie algebras, is much more subtle. Indeed, one can take, of course, the connected, simply connected Lie group \(G\) of the corresponding \(g\) and find unique connected, closed subgroups \(G_1\) and \(G_2\) of \(G\) whose tangent Lie algebras are \(g_1\) and \(g_2\), respectively. However, in the proof of the following proposition, we see that \((G_1, G_2)\) is not necessarily a matched pair of Lie groups even if \(\text{dim } g_1 = \text{dim } g_2 = 1\).

**Proposition 3.4.** Every matched pair of complex Lie algebras \(g_1 = g_2 = \mathbb{C}\) can be exponentiated to a matched pair of Lie groups \((G_1, G_2)\) where \(G_1, G_2\) are either \((\mathbb{C}, +)\) or \((\mathbb{C} \setminus \{0\}, \cdot)\).

Every matched pair of real Lie algebras \(g_1 = g_2 = \mathbb{R}\) can be exponentiated to a matched pair of Lie groups \((G_1, G_2)\) where \(G_1, G_2\) are either \((\mathbb{R}, +)\) or \((\mathbb{R} \setminus \{0\}, \cdot)\).

**Proof.** Consider first the complex case. The only two-dimensional complex Lie algebras are the abelian one and the one with generators \(X, Y\) and relation \([X, Y] = Y\). If \(g\) is abelian, the mutual actions of \(g_1\) and \(g_2\) on each other are trivial and exponentiation is obviously a direct sum.

If \(g\) is generated by \([X, Y] = Y\), we either have that \(g_1\) or \(g_2\) is equal to \(\mathbb{C}Y\), in which case one of the actions is trivial and \(G\) can be constructed as a semi-direct product of the connected, simply connected Lie groups of \(g_1\) and \(g_2\), or we have that both \(g_1\) and \(g_2\) differ from \(\mathbb{C}Y\). In the latter case, there is, up to isomorphism, only one possibility, namely \(g_1 = \mathbb{C}X, g_2 = \mathbb{C}(X + Y)\).

Define on \(\mathbb{C} \setminus \{0\} \times \mathbb{C}\) the Lie group with product
\[
(t, s)(t', s') = (tt', s + ts') .
\]
Define \(G_1 = G_2 = \mathbb{C} \setminus \{0\}\) with embeddings \(i(g) = (g, 0)\) and \(j(s) = (s, s - 1)\), we indeed get a matched pair of complex Lie groups with mutual actions
\[
\alpha_g(s) = g(s - 1) + 1, \quad \beta_s(g) = \frac{sg}{g(s - 1) + 1} .
\]
(3.1)
The real case is completely analogous. \qed
Remark 3.5. The connected simply connected complex Lie group $G$ of $\mathfrak{g}$ consists of all pairs $(t, s)$ with $t, s \in \mathbb{C}$ and the product

$$(t, s)(t', s') = (t + t', s + \exp(t)s')$$

(see, for example, [14], §10.1), and its closed subgroups $G_1$ and $G_2$ corresponding to the decomposition $\mathfrak{g} = \mathbb{C}X \oplus \mathbb{C}(X + Y)$ above consist respectively of all triples of the form $(g, 0)$ and $(s, \exp(s) - 1)$ with $g, s \in \mathbb{C}$. These groups do not form a matched pair because $G_1 \cap G_2 = \{(2\pi i n, 0) | n \in \mathbb{Z}\} = Z(G)$. So, it is crucial not to take $G$ simply connected above.

Allowing $g, t, s$ above to be only real, we come to the example of a matched pair of real Lie groups from [51], Section 5.3. Here $\mathfrak{g}$ is a real Lie algebra generated by $X$ and $Y$ subject to the relation $[X, Y] = Y$ and one considers the decomposition $\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}(X + Y)$. Then, to get a matched pair of Lie groups, we consider $G$ as the variety $\mathbb{R} \setminus \{0\} \times \mathbb{R}$ with the product

$$(s, x)(t, y) = (st, x + sy)$$

and embed $G_1 = G_2 = \mathbb{R} \setminus \{0\}$ by the formulas $i(g) = (g, 0)$ and $j(s) = (s, s-1)$. Remark that here, it is impossible to take the connected component of the unity of the group of affine transformations of the real line as $G$, because it is easy to see that for its closed subgroups $G_1$ and $G_2$ corresponding to the above mentioned subalgebras, the set $G_1 G_2$ is not dense in $G$.

The next example shows that in general, for a given matched pair of Lie algebras, it is even possible that $G_1 \cap G_2 \neq \{e\}$ for any corresponding pair of Lie groups, which means that such a matched pair of Lie algebras cannot be exponentiated to a matched pair of Lie groups in the sense of Definition 3.1.

Example 3.6. Consider a family of complex Lie algebras $\mathfrak{g} = \text{span}\{X, Y, Z\}$ with $[X, Y] = Y$, $[X, Z] = \alpha Z$, $[Y, Z] = 0$, where $\alpha \in \mathbb{C} \setminus \{0\}$, and the decomposition $\mathfrak{g} = \text{span}\{X, Y\} \oplus \mathbb{C}(X + \alpha Z)$. The corresponding connected simply connected complex Lie group $H$ consists of all triples $(t, u, v)$ with $t, u, v \in \mathbb{C}$ and the product

$$(t, u, v)(t', u', v') = (t + t', u + \exp(t)u', v + \exp(\alpha t)v')$$

(see, for example, [14], §10.3), and its closed subgroups $H_1$ and $H_2$ corresponding to the decomposition above consist respectively of all triples of the form $(t, u, 0)$ and $(s, 0, \exp(\alpha s) - 1)$ with $t, u, s \in \mathbb{C}$. These groups do not form a matched pair because $H_1 \cap H_2 = \{(2\pi i n, 0, 0) | n \in \mathbb{Z}\}$.

We claim that, if $1/\alpha \notin \mathbb{Z}$ and if $G$ is any complex Lie group with Lie algebra $\mathfrak{g}$, such that $G_1, G_2$ are closed subgroups of $G$ with tangent Lie algebras $\mathfrak{g}_1$, resp. $\mathfrak{g}_2$, then $G_1 \cap G_2 \neq \{e\}$. Indeed, since the Lie group $H$ is connected and simply connected, the connected component $G^{(e)}$ of $e$ in $G$ can be identified with the quotient of $H$ by a discrete central subgroup. If $\alpha \notin \mathbb{Q}$, the center of $H$ is trivial, so that we can identify $G^{(e)}$ and $H$. Under this identification, the
connected components of \( e \) in \( G_1, G_2 \) agree with \( H_1, H_2 \). Because \( H_1 \cap H_2 \neq \{ e \} \), our claim follows. If \( \alpha = \frac{m}{n} \) for \( m, n \in \mathbb{Z} \setminus \{0\} \) mutually prime, the center of \( H \) consists of the elements \( \{(2\pi nN, 0, 0) \mid N \in \mathbb{Z}\} \). Hence, the different possible quotients of \( H \) are labeled by \( N \in \mathbb{Z} \) and are given by the triples \( (a, u, v) \in \mathbb{C}^3, a \neq 0 \) and the product

\[
(a, u, v)(a', u', v') = (aa', u + a^n u', v + a^m v').
\]

The closed subgroups corresponding to \( g_1 \) and \( g_2 \) are given by \( (a, u, 0) \) and \( (b, 0, b^{mN} - 1) \) with \( a, b, u \in \mathbb{C} \) and \( a, b \neq 0 \). The intersection of both subgroups is non-trivial whenever \( mN \neq \pm 1 \). This proves our claim.

Considering now the complex Lie algebras above as real Lie algebras with generators \( X, iX, Y, iY, Z, iZ \) and the decomposition above as a decomposition of real Lie algebras, we get a matched pair of real Lie algebras which cannot be exponentiated to a matched pair of real Lie groups.

In the remaining case \( \alpha = 1/n \) with \( n \in \mathbb{Z} \setminus \{0\} \), we can consider the Lie group \( G \) defined by Equation (3.2) with \( m = N = 1 \). Consider \( G_1 = \mathbb{C} \setminus \{0\} \times \mathbb{C} \) with \( (a, u)(a', u') = (aa', u + a^n u') \) and \( G_2 = \mathbb{C} \setminus \{0\} \). Writing \( i(a, u) = (a, u, 0) \) and \( j(v) = (v, 0, v - 1) \), we get a matched pair of Lie groups with mutual actions

\[
\alpha_{(a,u)}(v) = a(v - 1) + 1 \quad \text{and} \quad \beta_{(a,u)}(v, u) = \left(\frac{va}{a(v - 1) + 1}, \frac{u}{(a(v - 1) + 1)^n}\right).
\]

In the next section, we study more closely the exponentiation of a matched pair of Lie algebras when one of the Lie algebras has dimension 1.

## 4 Matched pairs of Lie groups and Lie algebras in dimension \( n + 1 \)

We use systematically the following terminology.

**Terminology 4.1.** A matched pair \((g_1, g_2)\), resp. \((G_1, G_2)\), is said to be of dimension \( n_1 + n_2 \) if the dimension of \( g_i \), resp. \( G_i \), is \( n_i \).

Suppose that \( g = g_1 \oplus g_2 \) is a matched pair of Lie algebras with \( \dim g_2 = 1 \). Put \( g_2 = kA \). For all \( X \in g_1 \), we define \( \beta(X) \in g_1 \) and \( \chi(X) \in k \) such that

\[
[X, A] = \beta(X) + \chi(X)A.
\]

Then, \( \beta \) and \( \chi \) are linear, and, for all \( X, Y \in g_1 \), the Jacobi identity for \( g \) gives:

\[
\chi([X, Y]) = 0, \quad \beta([X, Y]) = [X, \beta(Y)] + [\beta(X), Y] + \beta(X)\chi(Y) - \beta(Y)\chi(X).
\]

(4.1)
By induction, one verifies that
\[ \beta^n([X,Y]) = \sum_{k=0}^{n} \binom{n}{k} [\beta^k(X),\beta^{n-k}(Y)] \]
\[ + \sum_{k=1}^{n} \binom{n}{k-1} (\beta^k(X)\chi(\beta^{n-k}(Y)) - \beta^k(Y)\chi(\beta^{n-k}(X))) . \]

Hence,
\[ \chi(\beta^n([X,Y])) = n(\chi(\beta^n(X))\chi(Y) - \chi(\beta^n(Y))\chi(X)) . \tag{4.2} \]

Then, we claim that the linear forms \( \chi, \chi\beta \) and \( \chi\beta^2 \) are linearly dependent. If not, we find \( X_0, X_1, X_2 \in g_1 \), such that \( \chi(\beta^i(X_j)) = \delta_{ij} \) for \( i,j \in \{0,1,2\} \), where \( \delta_{ij} \) is the Kronecker symbol. Because \( \chi(X_1) = \chi(X_2) = 0 \), we get \( \chi(\beta^n([X_1,X_2])) = 0 \) for all \( n \). Define
\[ g_0 = \bigcap_{i=0}^{2} \ker \chi\beta^i . \]

Then, \( \beta([X_1,X_2]) \in g_0 \). Using Equation (4.1), we get
\[ \beta([X_1,X_2]) = [X_1,\beta(X_2)] + [\beta(X_1),X_2] . \tag{4.3} \]

On the other hand, using Equation (4.2), it follows that \( [X_1,\beta(X_2)] \in g_0 \), because \( \chi(\beta(X_2)) = \chi(X_1) = 0 \). Combining this with Equation (4.3), we get that \( [\beta(X_1),X_2] \in g_0 \). Nevertheless, using once again Equation (4.2), we get that \( \chi(\beta^2([\beta(X_1),X_2])) = -2 \), contradicting the fact that \( [\beta(X_1),X_2] \in g_0 \). So, we have proved that \( \chi, \chi\beta \) and \( \chi\beta^2 \) are linearly dependent.

Hence, we can separate three different possibilities.

Case 1. \( \chi = 0 \).

Case 2. \( \chi \neq 0 \) and \( \beta(\ker \chi) \subset \ker \chi \).

Case 3. \( \chi \) and \( \chi\beta \) linearly independent, and \( \beta(\ker \chi \cap \ker \chi\beta) \subset \ker \chi \cap \ker \chi\beta \).

**Case 1**

The action of \( g_1 \) on \( g_2 \) is trivial and \( g_2 \) acts on \( g_1 \) by automorphisms. To exponentiate such a matched pair it suffices to use a semi-direct product of \( k \) and the connected, simply connected Lie group \( G_1 \) of \( g_1 \).

**Case 2**

In this case \( g_0 := \ker(\chi) \) is an ideal of \( g \), on which \( g_2 \) acts as an automorphism group. There exists an \( a \in k \) such that \( \chi\beta = a\chi \), by assumption. Take \( X_0 \in g_1 \) such that \( \chi(X_0) = 1 \). Then, we get
\[ [X_0,A] = A + aX_0 + Y_0 \quad (Y_0 \in g_0) . \]
Putting $\hat{A} = A + aX_0$, we get $[X_0, \hat{A}] = \hat{A} + Y_0$. This suggests how to exponentiate $g$. We start with the connected, simply connected Lie group $G_0$ of $g_0$ and observe that, due to the action of $\hat{A}$, $k$ acts by automorphisms $(\mu_x)_{x \in k}$ on $G_0$. We exponentiate $kA + g_0$ on the space $k \times G_0$ with product 

$$(x, g)(y, h) = (x + y, gp_x(h)).$$

Next, we observe that, due to the action of $\hat{A}$, $k$ acts by automorphisms on $k \ltimes G_0$, and one would suggest to make another semi-direct product. But in this case the subgroups corresponding to $g_1$ and $kA$ do not form a matched pair of Lie groups if $a \neq 0$. The subgroups corresponding to $g_1$ and $kA$ do form a matched pair of Lie groups when $a = 0$. So, in what follows, we treat the case $a \neq 0$. Example 3.6 shows that if $k = \mathbb{C}$, the exponentiation is in general impossible if $\dim g_1 \geq 2$. So, we restrict to the case $k = \mathbb{R}$.

We first prove some general results which allow to obtain the exponentiation whenever $n \leq 4$. Afterwards, we will give an example showing that if dimension $n \geq 5$, the exponentiation is in general impossible.

Writing $\mu = \operatorname{Ad} X_0$ and $\rho = \operatorname{Ad} A$ on $g_0$, we are given $\mu$ and $\rho$, derivations of $g_0$ and an element $Y_0 \in g_0$ such that $[\mu, \rho] = \rho + \operatorname{Ad} Y_0$. Further, we have $[X_0, \hat{A}] = \hat{A} + Y_0$, $g_1 = \mathbb{R}X_0 + g_0$ and $g_2 = \mathbb{R}A$, where $A = \hat{A} - aX_0$.

We introduce the notation $\mathbb{R}^*$ for the Lie group $\mathbb{R} \setminus \{0\}$ with multiplication and $\mathbb{R}_+^*$ for the subgroup of elements $s > 0$.

**Proposition 4.2.** The matched pair $(g_1, g_2)$ has an exponentiation with at most two connected components in the following cases:

1. $\rho$ is inner and the center of $g_0$ is trivial.
2. $g_0$ is abelian.
3. $g_0 = \langle X, Y \rangle \oplus g_0$, with $[X, Y] = Y$ and $g_0$ central in $g_0$.
4. $g_0 = \langle X, Y, Z \rangle$ with $[X, Y] = Z$ and $Z$ central.

In the proof of this proposition we use systematically the following lemma.

**Lemma 4.3.** Let $G_0$ be a Lie group with Lie algebra $g_0$, with center $z_0$. Suppose that $[g_0, g_0] \subset z_0$. Let $(\mu_s)$ be an action of $\mathbb{R}^*$ by automorphisms of $G_0$. Denote by $\mu$ the derivation of $g_0$ corresponding to $(\mu_s)_{s > 0}$ and denote by $\theta$ the involutive automorphism $\theta := d\mu_{-1}$, giving rise to a decomposition $g_0 = g_0^0 \oplus g_0^1$.

So, $\mu$ leaves $g_0^0$ invariant and we suppose that $\mu$ is invertible on $g_0$. Further, $\mu$ leaves $[g_0, g_0]$ invariant and we suppose that $\mu$ is invertible on $[g_0, g_0]$ as well.

Define $G := \mathbb{R}^* \ltimes G_0$ with product $(s, g)(t, h) = (st, g\mu_s(h))$ and Lie algebra $\mathbb{R}X_0 + g_0$, where $X_0$ denotes the canonical generator of $g$ corresponding to $\mathbb{R}^*$. 

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Let $C \in \mathfrak{g}_0$. Then, there exists a closed subgroup

$$K = \{(s, v(s)) \mid s \in \mathbb{R}^*\}$$

of $G$, with tangent Lie algebra $\mathbb{R}(X_0 + C)$, where $v : \mathbb{R}^* \to G_0$ is a smooth function.

**Proof.** Denote by $\text{Exp}_g$ the exponential mapping $\mathfrak{g} \to G$. Then, we get a smooth function $v : \mathbb{R}^* \to G_0$ such that $\text{Exp}_g((\log s)(X_0 + C)) = (s, v(s))$ for $s > 0$. Then, $v(1) = e$ and $v'(1) = C$. Further, $v(st) = v(s)\mu_s(v(t))$ for all $s, t > 0$. We say that $v$ is a $(\mu_s)$-cocycle. So, we are looking for an element $g_0 \in \mathfrak{g}_0$, so that we can define $v(-1) = g_0$. Then, $(-1, g_0) \in K$ and, for $s > 0$,

$$(-1, g_0)(s, v(s)) = (-s, g_0\mu_{-1}(v(s))), \quad (s, v(s))(-1, g_0) = (-s, v(s)\mu_s(g_0)).$$

Hence, we are done if we can find an element $g_0 \in G_0$ such that $g_0\mu_{-1}(v(s)) = v(s)\mu_s(g_0)$, because then we can define $v(-s)$ to be this expression.

Denote by $\text{Exp}_{g_0}$ the exponential mapping $\mathfrak{g}_0 \to G_0$. Then, we look for $D \in \mathfrak{g}_0$ such that

$$\mu_{-1}(v(s)) = \text{Exp}_{g_0}(-D)v(s)\text{Exp}_{g_0}(\exp((\log s)\mu)(D)). \quad (4.4)$$

We want to derive at $s = 1$. For this, observe that

$$\text{Exp}_{g_0}(-D)\text{Exp}_{g_0}(\exp((\log s)\mu)(D)) = \text{Exp}_{g_0}(-D + \exp((\log s)\mu)(D) - \frac{1}{2}[D, \exp((\log s)\mu)(D)]),$$

where we have used that $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{z}_0$. Taking the derivative at $s = 1$ of Equation (4.4), we look for $D \in \mathfrak{g}_0$ such that

$$\theta(C) = (\text{Ad} \text{Exp}_{g_0}(-D))(C) + \mu(D) - \frac{1}{2}[D, \mu(D)].$$

The equation becomes

$$\theta(C) = \exp(-\text{Ad} D)(C) + \mu(D) - \frac{1}{2}[D, \mu(D)].$$

Using once again that $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{z}_0$, we get the equation

$$\theta(C) - C = \mu(D) - [D, C] - \frac{1}{2}[D, \mu(D)].$$

Define $Y_2 = \mu^{-1}(\theta(C) - C)$, which is possible because $\mu$ is invertible on $\mathfrak{g}_0^-$. Next, define $Z_2 := \frac{1}{2}\mu^{-1}([Y_2, \theta(C) + C])$, which is possible because $\mu$ is invert-
where \( G \) have the same derivative at \( 0 \), to be proven equation \( \theta(C) + C \).

So, writing \( g_0 = \text{Exp}_{g_0}(D) \), we may conclude that the left and right hand side of the to be proven equation

\[
\mu_1^{-1}(v(s)) = g_0^{-1}v(s)\mu_s(g_0)
\]

have the same derivative at \( s = 1 \). But, both sides of the equations are \((\mu_s)_{s>0}\)-cocycles and hence, both sides are equal for all \( s > 0 \).

**Proof of Proposition 4.2.** Part 1 : \( \rho \) is inner and the center of \( g_0 \) is trivial.

Take the unique \( B \in g_0 \) such that \( \rho = \text{Ad}B \) on \( g_0 \). Take a new generator \( A = A - B \) in \( g \). Because the center of \( g_0 \) is trivial and

\[
0 = [\mu, \rho - \text{Ad}B] = \rho + \text{Ad}(Y_0 - \mu(B)) = \text{Ad}(Y_0 + B - \mu(B)),
\]

we get \( Y_0 + B - \mu(B) = 0 \) and hence \([X_0, A] = A\). Because \([A, g_0] = \{0\} \), the connected, simply connected Lie group of \( h : = \mathbb{R}A + g_0 \) is given by \( H : = \mathbb{R} \oplus G_0 \), where \( G_0 \) is the connected, simply connected Lie group of \( g_0 \). Using the derivation \( \text{Ad}X_0 \) on \( h \), we get an action \((\mu_s)\) of \( \mathbb{R}^*_+ \) on \( H \) of the form

\[
\mu_s(x, g) = (sx, \eta_s(g)), \quad \text{where } s > 0, x \in \mathbb{R}, g \in G_0,
\]

and where \((\eta_s)\) is an action of \( \mathbb{R}^*_+ \) on \( G_0 \). We easily extend \((\mu_s)\) to an action of \( \mathbb{R}^* \), defining \( \mu_s(x, g) = (sx, \eta_{|s|}(g)) \). So, we can define a Lie group \( G : = \mathbb{R}^* \rtimes H \), with Lie algebra \( g \), on the space \( \mathbb{R}^* \times \mathbb{R} \times G_0 \) with product

\[
(s, x, g)(t, y, h) = (st, x + sy, g\eta_{|s|}(h)).
\]

Define the closed subgroup \( G_1 \) consisting of the elements \((s, 0, g)\), where \( s \in \mathbb{R}^* \) and \( g \in G_0 \). The tangent Lie algebra of \( G_1 \) is precisely \( g_1 \). To define the closed subgroup \( G_2 \), recall that \( A = A - aX_0 = A - aX_0 + B \). Using the exponential mapping of \( G \), we find a smooth function \( v : \mathbb{R}^*_+ \to G_0 \) such that

\[
\{(s, \frac{1}{a}(1 - s), v(s)) \mid s \in \mathbb{R}^*_+\}
\]

is a closed subgroup of \( G \) with tangent Lie algebra \( \mathbb{R}A \). We define

\[
G_2 : = \{(s, \frac{1}{a}(1 - s), v(|s|)) \mid s \in \mathbb{R}^*\}
\]

and then, one verifies that \( G_2 \) is a closed subgroup of \( G \) with tangent Lie algebra \( \mathbb{R}A \). It is easy to see that \((G_1, G_2)\) is a matched pair of Lie groups.

**Part 2 :** \( g_0 \) is abelian.
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In this case, $\mu$ and $\rho$ are linear transformations of $g_0$ satisfying $\mu \rho - \rho \mu = \rho$. Then, for any polynomial $P$, we get

$$P(\mu)\rho = \rho P(\mu + \iota) .$$

(4.5)

Let $V$ be the complexified vector space of the real vector space $g_0$ and consider $\mu$ as a linear operator on $V$, which we still denote by $\mu$. Then, we have a direct sum decomposition

$$V = \bigoplus_{\lambda \in \mathbb{C}} E_\lambda ,$$

where $E_\lambda$ is the generalized eigenspace corresponding to $\lambda \in \mathbb{C}$. A vector $X \in V$ belongs to $E_\lambda$ if and only if $(\mu - \lambda \iota)^n X = 0$ for $n$ big enough. We also get a direct sum decomposition

$$g_0 = \bigoplus_{r \in \mathbb{R}} F_r ,$$

where $F_r$ is such that

$$\mathbb{C} F_r = \bigoplus_{\lambda, \text{Re}(\lambda) = r} E_\lambda .$$

The subspaces $F_r$ are invariant under $\mu$. Also $\rho$ extends to $V$ and using Equation (4.5), it is clear that $\rho(E_\lambda) \subset E_{\lambda+1}$. Hence, $\rho(F_r) = F_{r+1}$. Denote by $\varepsilon(r)$ the entire part of $r \in \mathbb{R}$, such that $\varepsilon(r) \in \mathbb{Z}$ and $\varepsilon(r) \leq r < \varepsilon(r) + 1$. Then, we define

$$g_0^+ = \bigoplus_{\varepsilon(r) \text{ is even}} F_r \quad \text{and} \quad g_0^- = \bigoplus_{\varepsilon(r) \text{ is odd}} F_r .$$

Defining $\theta(X) = X$ for $X \in g_0^+$ and $\theta(X) = -X$ for $X \in g_0^-$, we obtain an involution $\theta$ of $g_0$ satisfying $\theta \mu = \mu \theta$ and $\theta \rho = -\rho \theta$. Observe that $\mu$ leaves the subspaces $g_0^+$ and $g_0^-$ globally invariant and that $\mu - \lambda \iota$ is invertible on $g_0^+$ when $\varepsilon(\lambda)$ is odd, while $\mu - \lambda \iota$ is invertible on $g_0^-$ when $\varepsilon(\lambda)$ is even.

Define the Lie algebra $\mathfrak{h} := \mathbb{R} \tilde{A} + g_0$. Its connected, simply connected Lie group $H$ lives on the space $\mathbb{R} \times g_0$, with product

$$(x, X)(y, Y) = (x + y, X + \exp(x \rho)(Y)) .$$

We extend the derivation $\mu = \text{Ad} X_0$ to $\mathfrak{h}$ and observe that $\mu(\tilde{A}) = \tilde{A} + Y_0$. Defining $\theta(\tilde{A}) = -\tilde{A} + Z_0$, where

$$Z_0 = (\mu - \iota)^{-1}(\theta(Y_0) + Y_0) \in g_0^+ ,$$

one verifies that $\theta$ is an involutive automorphism of $\mathfrak{h}$ commuting with the derivation $\mu$. Putting together $\mu$ and $\theta$, we obtain an action $(\mu_\ast)$ of $\mathbb{R}^*$ on $H$ such that $\mu_\ast(x, X) = (sx, \eta_\ast(X)u(s, x))$, where $(\eta_\ast)$ is an action of $\mathbb{R}^*$ on $g_0$. We define the Lie group $G := \mathbb{R}^* \ltimes H$, which lives on the space $\mathbb{R}^* \times \mathbb{R} \times g_0$, with product

$$(x, X)(y, Y) = (x + y, X + \exp(x \rho)(Y)) .$$

We extend the derivation $\mu = \text{Ad} X_0$ to $\mathfrak{h}$ and observe that $\mu(\tilde{A}) = \tilde{A} + Y_0$. Defining $\theta(\tilde{A}) = -\tilde{A} + Z_0$, where

$$Z_0 = (\mu - \iota)^{-1}(\theta(Y_0) + Y_0) \in g_0^+ ,$$

one verifies that $\theta$ is an involutive automorphism of $\mathfrak{h}$ commuting with the derivation $\mu$. Putting together $\mu$ and $\theta$, we obtain an action $(\mu_\ast)$ of $\mathbb{R}^*$ on $H$ such that $\mu_\ast(x, X) = (sx, \eta_\ast(X)u(s, x))$, where $(\eta_\ast)$ is an action of $\mathbb{R}^*$ on $g_0$. We define the Lie group $G := \mathbb{R}^* \ltimes H$, which lives on the space $\mathbb{R}^* \times \mathbb{R} \times g_0$, with product

$$(x, X)(y, Y) = (x + y, X + \exp(x \rho)(Y)) .$$
with product
\[(s, x, X)(t, y, Y) = (st, x + sy, X + \exp(x\rho)(\eta_s(Y))u(s, y))\].

Define the closed subgroup \(G_1\) consisting of the elements \((s, 0, X)\), where \(s \in \mathbb{R}^*\) and \(X \in \mathfrak{g}_0\). The tangent Lie algebra of \(G_1\) is precisely \(\mathfrak{g}_1\). Finally, we have to find a closed subgroup \(G_2\) with tangent Lie algebra \(\mathbb{R}A = \mathbb{R}(\tilde{A} - aX_0)\), which consists of the elements \((s, \frac{1}{s}(1 - s), v(s))\), \(s \in \mathbb{R}^*\) and \(v : \mathbb{R}^* \to \mathfrak{g}_0\) a smooth function. Conjugating with the element \((1, -\frac{1}{s}, 0)\), an equivalent question is to find a closed subgroup with tangent Lie algebra \(\mathbb{R}(X_0 + Z_0)\) (for a certain \(Z_0 \in \mathfrak{g}_0\)), consisting of the elements \((s, 0, w(s))\), \(s \in \mathbb{R}^*\) and \(w : \mathbb{R}^* \to G_0\) a smooth function. This is possible applying Lemma 4.3 to the action \((\eta_s)\) of \(\mathbb{R}^*\) on \(\mathfrak{g}_0\). Then, it is clear that \((G_1, G_2)\) form a matched pair of Lie groups.

Part 3 : \(\mathfrak{g}_0 = \langle X, Y \rangle \oplus \mathfrak{z}_0\), with \([X, Y] = Y\) and \(\mathfrak{z}_0\) central.

Every derivation of \(\mathfrak{g}_0\) leaves the center \(\mathfrak{z}_0\) invariant. On the quotient Lie algebra \(\mathfrak{g}_0/\mathfrak{z}_0\) every derivation is inner. So, changing the generators \(X_0\) and \(\tilde{A}\) of \(\mathfrak{g}_0\), we may suppose that we are in the following situation:

\[[X_0, \tilde{A}] = \tilde{A} + Y_0, \quad [X_0, \mathfrak{g}_0] \subset \mathfrak{z}_0, \quad [\tilde{A}, \mathfrak{g}_0] \subset \mathfrak{z}_0,\]

and \(\mathfrak{g}_1 = \mathbb{R}X_0 + \mathfrak{g}_0, \mathfrak{g}_2 = \mathbb{R}(\tilde{A} - aX_0 + B_1)\) for a certain element \(B_1 \in \mathfrak{g}_0\).

Write \(\mu = \text{Ad}X_0\) and \(\rho = \text{Ad} \tilde{A}\) as derivations on \(\mathfrak{g}_0\). Because \(\mu\) and \(\rho\) preserve \([\mathfrak{g}_0, \mathfrak{g}_0]\), we get \([\mu(Y_1), \rho(Y_1)] = \rho(Y_1) = 0\). Further, \([Y_0, \mathfrak{g}_0] = ([\mu, \rho] - \rho)(\mathfrak{g}_0) \subset \mathfrak{z}_0\). Because \([\mathfrak{g}_0, \mathfrak{g}_0] = \mathbb{R}Y_1\), we get \([Y_0, \mathfrak{g}_0] = \{0\}\), which gives \(Y_0 \in \mathfrak{z}_0\) and so, \([\mu, \rho] = \rho\). Suppose \(\mu(X_1) = Z_1\) and \(\rho(X_1) = Z_2\). Because \([\mu, \rho] = \rho\), we get
\[
\mu(Z_2) - Z_2 = \rho(Z_1). \quad (4.6)
\]

As in part 2, we can find an involutive automorphism \(\theta\) of \(\mathfrak{z}_0\), such that \(\theta\) commutes with \(\mu\) and anti-commutes with \(\rho\) and such that \(\mu - \lambda \theta\) is invertible on \(\mathfrak{z}_0^\perp\) when \(\varepsilon(\lambda)\) is odd and invertible on \(\mathfrak{z}_0^\perp\) when \(\varepsilon(\lambda)\) is even. Defining \(Z_3 = \mu^{-1}(\theta(Z_1) - Z_1) \in \mathfrak{z}_0^\perp\), we can extend \(\theta\) to an involutive automorphism of \(\mathfrak{g}_0\), by putting \(\theta(X_1) = X_1 + Z_3\) and \(\theta(Y_1) = Y_1\). Then, \(\theta\) commutes with \(\mu\), by definition of \(\mathfrak{z}_3\), and anti-commutes with \(\rho\), because \(\rho(Z_3) = -Z_2 - \theta(Z_2)\). This last equality can be deduced as follows: because \((\mu - \nu)\rho = \rho\mu\), we get \(\rho\mu^{-1} = (\mu - \nu)^{-1}\rho\) on \(\mathfrak{z}_0^\perp\); in particular,
\[
\rho(Z_3) = (\mu - \nu)^{-1}(\theta(\rho(Z_1)) - \rho(Z_1)) = -Z_2 - \theta(Z_2),
\]
where we used Equation \((4.6)\).

Next, we define the Lie algebra \(\mathfrak{h} := \mathbb{R}\tilde{A} + \mathfrak{g}_0\). We extend \(\theta\) to an involutive automorphism of \(\mathfrak{h}\) by defining \(\theta(A) = -A + Z_4\), where \(Z_4 := (\mu - \nu)^{-1}(\theta(Y_0) + Y_0) \in \mathfrak{z}_0^\perp\). Extending also \(\mu = \text{Ad}X_0\) to \(\mathfrak{h}\) (recall that \(\mu(\tilde{A}) = \tilde{A} + Y_0\)), we observe that \(\theta\) and \(\mu\) commute.

As above, the derivation \(\rho\) gives rise to an action of \(\mathbb{R}\) on \(G_0\), where \(G_0\) lives on the space \(\mathbb{R}^2 \times \mathfrak{z}_0\) with product \((x, y, Z)(x', y', Z') = (x + x', y + y', Z + Z')\)
exp(\(x\))y', \(Z + Z'\)). Then, \(H := \mathbb{R}_\mu \ltimes G_0\) gives an exponentiation of \(\mathfrak{h}\), on which the derivation \(\mu\) and the involutive automorphism \(\theta\) are combined to produce an action \((\mu_\theta)\) \(\mathbb{R}^*\) by automorphisms of \(H\). We define \(G = \mathbb{R}^*_\mu \ltimes H\) and the product is given by \((s, x, g)(t, y, h) = (st, x + sy, \ldots)\). The precise form of the product can be written as above. The closed subgroup \(G_1\) consists again of the elements \((s, 0, g)\) for \(s \in \mathbb{R}^*\) and \(g \in G_0\). To find \(G_2\), we have to construct, again as above, the closed subgroup \(G_2\) of \(G\) with tangent Lie algebra \(\mathbb{R}(\hat{A} - aX_0 + B_1)\), where \(B_1 \in \mathfrak{g}_0\), and such that \(G_2\) consists of the elements \((s, \frac{1}{a}(1 - s), v(s))\), where \(s \in \mathbb{R}^*\). Conjuncting, the problem is reduced again to finding a closed subgroup of \(G\) with tangent Lie algebra \(\mathbb{R}(X + 0, B_2)\), for some arbitrary \(B_2 \in \mathfrak{g}_0\), consisting of the elements \((s, 0, w(s))\), \(s \in \mathbb{R}^*\). We solve this problem in the closed subgroup \(K := \mathbb{R}^*_\mu \ltimes G_0\) of \(G\), whose Lie algebra is \(\mathfrak{k} := \mathbb{R}X_0 + \mathfrak{g}_0\). Suppose that \(B_2 = x_1X_1 + y_1Y_1 \mod \mathfrak{g}_0\). If \(b \in \mathbb{R}\),

\[
\exp(\Ad(bY_1))(X_0 + B_2) = x_1X_1 + (y_1 - bx_1)Y_1 \mod \mathfrak{g}_0.
\]

If \(\text{Exp}_t\) denotes the exponential mapping from \(\mathfrak{t}\) to \(K\), we observe that conjugation by \(\text{Exp}_a(bY_1)\) for a well chosen \(b \in \mathbb{R}\), reduces the problem to either \(x_1 = 0\) or \(y_1 = 0\). Both cases are solved by Lemma 4.3, because \(\mathbb{R}X_1 + \mathfrak{g}_0\) and \(\mathbb{R}Y_1 + \mathfrak{g}_0\) are abelian. So, the proof of part 3 is done.

**Part 4:** \(\mathfrak{g}_0 = \langle X, Y, Z \rangle = Z\) and \(Z\) central.

A general derivation \(\mu\) of \(\mathfrak{g}_0\) has the form

\[
\mu(X) = x_1X + y_1Y + z_1Z\quad \mu(Y) = x_2X + y_2Y + z_2Z\quad \mu(Z) = (x_1 + y_2)Z .
\]

Perturbing \(\mu\) with \(\Ad(-z_2X + z_1Y)\), \(\mu\) is of the form

\[
\mu \left( \begin{array}{c} X \\ Y \end{array} \right) = P \left( \begin{array}{c} X \\ Y \end{array} \right) + \mu(Z) = \text{Tr}(P) Z .
\]

Because \([\mu, \rho] = \rho + \Ad Y_0\), we only have two possibilities. Either \(\rho = 0\), or \(\mu\) and \(\rho\) have after inner perturbation and a change of basis in \(\mathfrak{g}_0\) (respecting the relations of \(\mathfrak{g}_0\)), the form

\[
\mu(X) = \alpha X\quad \mu(Y) = (\alpha - 1)Y\quad \mu(Z) = (2\alpha - 1)Z, \\
\rho(X) = 0\quad \rho(Y) = X\quad \rho(Z) = 0 .
\]

Then, necessarily, \(Y_0 = \lambda Z\) for some \(\lambda \in \mathbb{R}\). We have \([X_0, \hat{A}] = \hat{A} + \lambda Z\), \(\mathfrak{g}_1 = \mathbb{R}X_0 + \mathfrak{g}_0\) and \(\mathfrak{g}_2 = \mathbb{R}(\hat{A} - aX_0 + B)\) for some \(B \in \mathfrak{g}_0\). To prove the existence of an exponentiation, we apply \(\exp(-\frac{1}{\alpha} \Ad \hat{A})\) and observe that it is an equivalent question to exponentiate \(\mathfrak{g}_1 = \mathbb{R}(X_0 + \frac{1}{\alpha} \hat{A}) + \mathfrak{g}_0\) and \(\mathfrak{g}_2 = \mathbb{R}(X_0 + C)\) for some \(C \in \mathfrak{g}_1\). Write \(C = eX + fY + gZ\). If we now replace \(X_0\) by \(X_0 + (g + 2ef)Z\), we see that none of the relations above change, because \(\rho(Z) = 0\), but only \(C\) changes to \(eX + fY - 2efZ\). So, we may suppose that \(C\) has this last form.

Define \(\mathfrak{h} := \mathbb{R}\hat{A} + \mathfrak{g}_0\) and exponentiate as \(H := \mathbb{R}_\mu \ltimes G_0\). Define \(\theta(\hat{A}) = -\hat{A}\), \(\theta(X) = pX\), \(\theta(Y) = -pY\) and \(\theta(Z) = -Z\), where \(p = \pm 1\) will be determined
later. Then, \( \theta \) is an involutive automorphism of \( h \) that commutes with the derivation \( \mu = \text{Ad} X_0 \) of \( h \). Both combine to an action \( \mu \) of \( \mathbb{R}^* \) on \( H \) and we define \( G = \mathbb{R}^* \mu \ltimes H \). The only problem left is the definition of the good closed subgroup of \( G \) with tangent Lie algebra \( \mathbb{R}(X_0 + C) \). Suppose first that \( \alpha \neq 1 \) and \( \alpha \neq \frac{1}{2} \). Then, we take \( p = 1 \) and observe that \( \mu \) is invertible on \( g_0 = \langle Y, Z \rangle \) and on \( [g_0, g_0] = \mathbb{R}Z \). So, Lemma 4.3 provides us with the needed subgroup. If \( \alpha = 1 \), we take \( p = -1 \) and we are done as well. Finally, take \( \alpha = \frac{1}{2} \) and \( p = 1 \). Define \( D = 4fY \) and observe that

\[
\mu(D) - [D, C] - \frac{1}{2}[D, \mu(D)] = -2fY + 4efZ = \theta(C) - C.
\]

Hence, it follows from the proof of Lemma 4.3 that we can define the right closed subgroup of \( G \).

Finally, we have to consider the case where \( \rho \) is trivial. This is very much similar to part 1 of this proof, but simpler.

Now we prove that at least one of the conditions of Proposition 4.2 is fulfilled when \( \dim g_0 \leq 3 \), up to one exceptional case, that we exponentiate explicitly ‘by hands’.

**Corollary 4.4.** In case 2 every real matched pair of dimension \( n + 1 \) with \( n \leq 4 \) can be exponentiated to a matched pair of real Lie groups with at most two connected components.

**Proof.** If \( \dim g_0 = 1 \) or 2, then either \( g_0 \) is abelian, or \( g_0 \) has trivial center and all derivations are inner. If \( \dim g_0 = 3 \) and \( g_0 \) is of rank 3, then \( g_0 = \mathfrak{sl}_2 \) or \( \mathfrak{su}_2 \). In both cases, every derivation is inner and the center is trivial.

When \( g_0 \) has rank 1 or rank 0, one can always apply Proposition 4.2.

Finally, there are only three real non-isomorphic 3-dimensional \( g_0 \) of rank 2 defined respectively by:

a) \([H, X] = X, [H, Y] = \alpha Y, [X, Y] = 0 (\alpha \in \mathbb{R});

b) \([H, X] = X + Y, [H, Y] = Y, [X, Y] = 0;

c) \([H, X] = rX + Y, [H, Y] = -X + rY, [X, Y] = 0 (r \in \mathbb{R}) \) (see [14]).

All these cases (except a), \( \alpha = 1 \), which we will study separately), can be treated in a similar way. Namely, a general derivation \( \mu \) of \( g_0 \) has the following form:

a) \( \alpha \neq 1 \): \( \mu(H) = aX + bY \), \( \mu(X) = cX \), \( \mu(Y) = dY \), and it is inner if \( d = \alpha c \).

b) \( \mu(H) = xX + yY \), \( \mu(X) = aX + bY \), \( \mu(Y) = aY \), and it is inner if \( a = b \).

c) \( \mu(H) = xX + yY \), \( \mu(X) = aX + bY \), \( \mu(Y) = -bX + aY \), and it is inner if \( a = rb \).

Then, since \( \mu \) and \( \rho \) are derivations of \( g_0 \), we observe that in all cases above \([\mu, \rho] \) is inner. Hence, \( \rho = [\mu, \rho] - \text{Ad} Y_0 \) is inner. Also, the center of \( g_0 \) is always trivial.
At last, we study separately $g_0$ defined by $[H, X] = X$, $[H, Y] = Y$ and $[X, Y] = 0$. A general derivation $\mu$ of $g_0$ has the form

$$
\mu(H) = xX + yY, \quad \mu(X) = aX + bY, \quad \mu(Y) = cX + dY,
$$

which is inner if $b = c = 0$ and $a = d$. Because $[\mu, \rho] = \rho + \text{Ad} Y_0$, we conclude that there exists a $\lambda \in \mathbb{R}$ such that, on $(X, Y)$, $\mu \rho - \rho \mu = \rho + \lambda t$. It is easy to see that either $\rho = -\lambda t$, in which case $\rho$ is inner and the exponentiation exists, or that we can change the basis $X, Y$ and perturb $\mu$ and $\rho$ innerly such that

$$
\mu(H) = 0, \quad \mu(X) = X, \quad \mu(Y) = 0,
$$

$$
\rho(H) = 0, \quad \rho(X) = 0, \quad \rho(Y) = X.
$$

Our matched pair lives now in the Lie algebra $g$ with generators $X_0, \hat{A}, X, Y, Z$ with relations $\text{Ad} X_0 = \mu$, $\text{Ad} \hat{A} = \rho$ on $g_0$, $[X_0, \hat{A}] = \hat{A}$. Observe that $Y_0$ disappeared because, after the necessary inner perturbations of $\mu$ and $\rho$, we arrived at $[\mu, \rho] = \rho$. The Lie subalgebras $g_1$ and $g_2$ are $(X_0, H, X, Y)$ and $\mathbb{R}(\hat{A} - aX_0 + B)$, respectively, where $B \in g_0$ is arbitrary. We exponentiate $\mathbb{R}\hat{A} + g_0$ on the space $\mathbb{R}^4$ with product

$$(a, h, x, y)(a', h', x', y') = (a + a', h + h', x + \exp(h)x' + a \exp(h)y', y + \exp(h)y')$$

and we denote this Lie group by $H$. Write $B = aH + bX + \gamma Y$. Suppose first that $\alpha \neq 0$. Then, we make $\mathbb{R}^*$ act on $H$ by the automorphisms $\mu_\alpha(a, h, x, y) = (sa, h, s|x, \text{Sgn}(s)y)$. Take $G = \mathbb{R}^* \ltimes H$ in which we consider the closed subgroup $G_1$ of elements $(s, 0, h, x, y)$, $s \in \mathbb{R}^*, h, x, y \in \mathbb{R}$. Next, we have to find a good (as in the proof of Proposition [1]) closed subgroup $G_2$ with tangent Lie algebra $\mathbb{R}(\hat{A} - aX_0 + B)$. Conjugating with the element $(1, -\frac{1}{a}, 0, 0, 0)$, we have to find a good closed subgroup of $\mathbb{R}^* \ltimes G_0$ with tangent Lie algebra $\mathbb{R}(X_0 + \alpha' H + \beta' X + \gamma' Y)$, where $\alpha' = -\frac{\alpha}{a} \neq 0$. If $\alpha' \neq -1$, such a subgroup is given by

$$
\{(s, \alpha' \log |s|, \frac{\beta'}{\alpha' + 1}(|s|^\alpha' + 1) - 1, \frac{\gamma'}{\alpha'}(\text{Sgn}(s)|s|^\alpha' - 1)) \mid s \in \mathbb{R}^* \}.
$$

If $\alpha' = -1$, such a subgroup is given by

$$
\{(s, - \log |s|, \beta' \log |s|, \gamma'(1 - \frac{1}{s})) \mid s \in \mathbb{R}^* \}.
$$

One can easily check that $(G_1, G_2)$ is indeed a matched pair of Lie groups.

If next, $\alpha = 0$, we consider the action of $\mathbb{R}^*$ by automorphisms of $H$ given by $\mu_\alpha(a, h, x, y) = (sa, h, sx, y)$. Take $G = \mathbb{R}^* \ltimes H$ in which we consider the closed subgroup $G_1$ of elements $(s, 0, h, x, y)$, $s \in \mathbb{R}^*, h, x, y \in \mathbb{R}$. Conjugating with the element $(1, -\frac{1}{s}, 0, 0, 0)$, we have to find a good closed subgroup of $\mathbb{R}^* \ltimes G_0$ with tangent Lie algebra $\mathbb{R}(X_0 + \beta' X + \gamma' Y)$ and this is given by

$$
\{(s, 0, \beta'(s - 1), \gamma' \log |s|) \mid s \in \mathbb{R}^* \}.
$$
Again, we arrive at a matched pair of Lie groups.

As it was promised, we now present a matched pair of dimension 5 + 1 which has no exponentiation.

**Example 4.5.** Consider the 6-dimensional Lie algebra \( g \) with generators \( A, X_0, X_1, X_2, Y, Z \) and relations

\[
[X_0, A] = A + Y, \quad [X_0, X_1] = \frac{1}{2}X_1 + X_2, \quad [X_0, X_2] = \frac{1}{2}X_2,
\]

\[
[X_0, Y] = Y, \quad [X_0, Z] = \frac{3}{2}Z,
\]

\[
[A, X_1] = [A, Y] = [A, Z] = 0, \quad [A, X_2] = Z, \quad [X_1, X_2] = Y, \quad [X_1, Y] = Z, \quad [X_1, Z] = [X_2, Y] = [X_2, Z] = [Y, Z] = 0.
\]

Defining \( g_1 = \langle X_0, X_1, X_2, Y, Z \rangle \) and \( g_2 = \mathbb{R}(A + X_0) \), we have a matched pair of Lie algebras. There does not exist a matched pair of Lie groups which exponentiates the given matched pair of Lie algebras.

**Proof.** It is easy to check that \( g \) is indeed a Lie algebra: it is clear that \( g_0 := \langle X_1, X_2, Y, Z \rangle \) is a Lie algebra. Next, we write \( \rho = \text{Ad} A \) and \( \mu = \text{Ad} X_0 \) on \( g_0 \) and it is clear that \( \mu \) and \( \rho \) are derivations of \( g_0 \) satisfying \([\mu, \rho] = \rho + \text{Ad} Y\). The connected, simply connected Lie group \( G_0 \) of \( g_0 \) has \( \mathbb{R}^4 \) as an underlying space, with product

\[
(x_1, x_2, y, z)(x'_1, x'_2, y', z') = (x_1 + x_1', x_2 + x_2', y + y' + x_1x'_2, z + z' + \frac{x_1^2}{2}x_2' + x_1y').
\]

The derivation \( \rho \) gives rise to an action \((\rho_a)\) of \( \mathbb{R} \) on \( G_0 \) given by \( \rho_a(x_1, x_2, y, z) = (x_1, x_2, y + ax_2, z) \) and we can define the Lie group \( H := \mathbb{R} \cdot G_0 \) on the space \( \mathbb{R}^5 \). Finally, \( \mu \) gives rise to an action \((\mu_x)\) of \( \mathbb{R} \) on \( H \) given by

\[
\mu_x(a, x_1, x_2, y, z) = (\exp(x)a, \exp(\frac{1}{2}x)x_1, \exp(\frac{1}{2}x)(x_2 + xx_1), \exp(x)(y + xa + \frac{1}{2}xx_1^2), \exp(\frac{3}{2}x)(z + xax_1 + \frac{6}{2}xx_1^3))\).
\]

Defining \( G = \mathbb{R} \cdot H \) on the space \( \mathbb{R}^6 \), we obtain the connected, simply connected Lie group of \( g \). One observes easily that the center of \( G \) is trivial. Hence, \( G \) is the only connected Lie group with Lie algebra \( g \).

We claim the following: any automorphism \( \alpha \) of \( G \) leaves the closed normal subgroup \( G_0 \) invariant and modulo an inner perturbation by \( \text{Ad} g \) \((g \in G)\), we have \( \alpha = \iota \) on \( G/G_0 \). Indeed, let \( \alpha \) be an automorphism of \( G \). Denote \( \theta = d_\alpha \), the corresponding automorphism of \( g \). It is straightforward to check that \( g \) has only two ideals of dimension 4: \( g_0 \) and \( \langle A, X_2, Y, Z \rangle \). As a Lie algebra, the first one has rank 2 and the second one rank 1. Because \( \theta(g_0) \) is an ideal of dimension 4 of \( g \) isomorphic with \( g_0 \), we get \( \theta(g_0) = g_0 \). Then also \( \alpha(G_0) = G_0 \). Because \([X_0, A] = A \mod g_0\), a perturbation of \( \alpha \) by
Ad(x, a, 0, 0, 0, 0) for certain \( x, a \in \mathbb{R} \), leaves two possibilities for \( \theta : \theta = \iota \mod g_0 \) or \( \theta(X_0) = X_0 \mod g_0 \) and \( \theta(A) = -A \mod g_0 \). We have to prove that the second option is impossible. Hence, suppose that \( \theta(X_0) = X_0 + C \) and \( \theta(A) = -A + D \) with \( C, D \in g_0 \). If we arrive at a contradiction, then our claim is proved. Because \( \theta \) is an automorphism of \( g_0 \) it has the following form

\[
\begin{align*}
\theta(X_1) &= bX_1 + gX_2 + hY + kZ, \\
\theta(Y) &= bdY + beZ, \\
\theta(Z) &= b^2dZ,
\end{align*}
\]

for \( e, f, g, h, k \in \mathbb{R} \) and \( b, d \in \mathbb{R}^* \). Because \( \theta \) is an automorphism of \( g \), we get \( \theta \rho = (-\rho + \text{Ad } D)\theta \). Suppose that \( D = x_1X_1 + x_2X_2 + yY + zZ \). Verifying the equality on \( Y \) gives \( x_1bdZ = 0 \) and, hence, \( x_1 = 0 \). Next, we verify on \( X_2 \) and conclude that \( b^2dZ = -dZ \), which yields the required contradiction \( b^2 = -1 \).

Suppose now that \( \mathcal{G} \) is a Lie group with Lie algebra \( g \) and with two closed subgroups \( \mathcal{G}_1, \mathcal{G}_2 \) whose tangent Lie algebras are \( g_1, g_2 \), respectively and such that \( (\mathcal{G}_1, \mathcal{G}_2) \) is a matched pair of Lie groups. Because \( G \) is the only connected Lie group with Lie algebra \( g \), we can identify \( G \) and \( \mathcal{G}^{(e)} \), the connected component of \( e \) in \( \mathcal{G} \).

Because any automorphism of \( G \) leaves \( G_0 \) invariant, we find that \( G_0 \) is a normal subgroup of \( \mathcal{G} \). We can naturally identify \( \mathcal{G}/G_0^{(e)} \) and \( G/G_0 \). Define \( H = \mathcal{G}/G_0 \). We claim that for any \( \eta \in H/H^{(e)} \), there exists a unique representative \( u(\eta) \in H \) such that \( \text{Ad } u(\eta) = \iota \) on \( H^{(e)} \). Denote by \( \pi_1 \) the quotient map from \( \mathcal{G} \) to \( H \) and by \( \pi_2 \) the quotient map from \( H \) to \( H/H^{(e)} \). Let \( \eta \in H/H^{(e)} \). Take \( g \in G \) such that \( \pi_2(\pi_1(g)) = \eta \). Then, \( \text{Ad } g \) defines an automorphism of \( \mathcal{G}^{(e)} = G \). Using our claim, we can take an \( h \in G \), such that \( \text{Ad } (hg) \) is trivial on \( G/G_0 \). This means that \( \text{Ad } \pi_1(hg) \) is trivial on \( H^{(e)} \). But \( \pi_2(\pi_1(hg)) = \eta \). So, we have proven the existence of the required representative. The uniqueness is trivial because \( H^{(e)} = G/G_0 \) has trivial center. Then, the map \( \Psi : H/H^{(e)} \oplus H^{(e)} \to H : (\eta, g) \mapsto u(\eta)g \) is an isomorphism of Lie groups. (Recall that \( H/H^{(e)} \) has dimension zero and the discrete topology.)

The only closed subgroup of \( G \) with tangent Lie algebra \( g_1 \) is the group consisting of the elements \( (x, 0, 0, x, 0, 0) \). Hence, this last subgroup coincides with \( \mathcal{G}_1 \cap G \). On the other hand, the only closed subgroup of \( G \) with tangent Lie algebra \( g_2 \) consists of the elements \( (x, \exp(x) - 1, 0, 0, x \exp(x), 0) \). Hence, this last subgroup coincides with \( \mathcal{G}_2 \cap G \). Identifying \( H^{(e)} \) with the group \( K := \mathbb{R}^2 \) with product \( (x, a)(x', a') = (x + x', a + \exp(x)a') \), we get that \( K_1 := \pi_1(\mathcal{G}_1) \cap H^{(e)} = \{(x, 0) \mid x \in \mathbb{R} \} \) and \( K_2 := \pi_1(\mathcal{G}_2) \cap H^{(e)} = \{(x, \exp(x) - 1) \mid x \in \mathbb{R} \} \). Combining this with the fact that \( H \) is isomorphic with \( H/H^{(e)} \oplus H^{(e)} \) and with the fact that the only elements of \( K \) normalizing \( K_1, \text{resp. } K_2 \), belong to \( K_1, \text{resp. } K_2 \), we conclude that \( \pi_1(\mathcal{G}_i) = Q_i \oplus K_i \) for some subgroups \( Q_i \subset H/H^{(e)} \) and \( i = 1, 2 \). Because \( \mathcal{G}_1 \mathcal{G}_2 \) is dense in \( \mathcal{G} \), it follows that \( K_1 \cap K_2 \) should be dense in \( K \). This is clearly not the case. \( \Box \)
Case 3

Now we have $g_0 := \ker \chi \cap \ker \chi \beta$ and this is still an ideal of $g$. We can take $X, Y \in g_1$ such that

$$\chi(X) = 1, \quad \chi(Y) = 0, \quad \chi(\beta(X)) = 0, \quad \chi(\beta(Y)) = 1.$$  

Using Equation (4.2), we get $\chi(\beta([X,Y])) = -1$ and $\chi([X,Y]) = 0$. Hence, $[X,Y] = -Y \mod g_0$. Because $\chi(\beta(X)) = 0$ and $\chi(\beta(Y)) = 1$, we get

$$\beta(X) = aY \mod g_0, \quad \beta(Y) = X + bY \mod g_0,$$

$a, b \in k$.

Checking Equation (4.1), we arrive at $b = 0$. Since the quotient Lie algebra $g/g_0$ is 3-dimensional and of rank 3, it is isomorphic to $sl_2(k)$ \([14]\) (if $k = \mathbb{R}$, one must analyse also $su_2(\mathbb{R})$, but it has no 2-dimensional Lie subalgebras). Then, by the Levy-Maltsev theorem \([39]\), Chapter X, we can find a Lie subalgebra $\tilde{g}$ of $g$ isomorphic to $sl_2(k)$ and such that $g = \tilde{g} \oplus g_0$ as vector spaces. This means that we are always in the following situation: $sl_2(k)$, with the generators $A, X, Y$ satisfying

$$[X,A] = aY + A, \quad [Y,A] = X, \quad [X,Y] = -Y,$$

is represented by derivations of $g_0$ and in the semi-direct product $g := sl_2(k) \times g_0$ we have the matched pair $g_1 = \langle X,Y \rangle + g_0$, $g_2 = k(A+Z)$ for some element $Z \in g_0$.

We first analyse the case $k = \mathbb{R}$. If we replace $Y$ by $rY$ and $A$ by $(1/r)A$ for $r \neq 0$, then $a$ changes to $a/r^2$. So, we only have to consider three cases: one with $a > 0$, one with $a = 0$ and one with $a > 0$.

In terms of the standard generators $H = e_{11} - e_{22}$, $K = e_{12}$ and $L = e_{21}$, we realize the relations above by putting $X = \frac{1}{2}H$, $Y = L$ and $A = -2aL - \frac{1}{4}K$. This means that we consider the matched pair $g_1 = \langle H, L \rangle + g_0$, $g_2 = \mathbb{R}(K + 4aL + Z)$ for some $Z \in g_0$.

**Proposition 4.6.** If $k = \mathbb{R}$ and $a \leq 0$, the matched pair has an exponentiation. When $a < 0$, $G, G_1, G_2$ can be taken connected. When $a = 0$, we need two connected components for $G_1$.

If $k = \mathbb{R}$ and $a > 0$, the matched pair has an exponentiation in at least the following cases:

1. $g_0$ is abelian.
2. $\dim g_0 = 2$.

We can take a matched pair of Lie groups where $G, G_1$ have at most two connected components and $G_2$ has at most four connected components.

In particular, for $k = \mathbb{R}$ the exponentiation exists for all matched pairs of dimension $n + 1$, $n \leq 4$. 
Proof. We start off with the case $a \leq 0$ and we first suppose that $g_0 = \{0\}$.

An exponentiation of the case $a = 0$ is given by $F = \text{SL}_2(\mathbb{R})$ with subgroups

$$F_1 := \left\{ \begin{pmatrix} a & 0 \\ x & 1/a \end{pmatrix} \mid a \neq 0, x \in \mathbb{R} \right\}, \quad F_2 := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}.$$  \hspace{1cm} (4.7)

An exponentiation of the case $a = -\frac{1}{4}$ (as we saw above, it is sufficient to consider one value of $a < 0$) is given by $F = \text{SL}_2(\mathbb{R})$ with subgroups

$$F_1 := \left\{ \begin{pmatrix} a & 0 \\ x & 1/a \end{pmatrix} \mid a > 0, x \in \mathbb{R} \right\}, \quad F_2 := \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

For this last case, there is another exponentiation which is at least as important. We observe that $F = F_1F_2$ and the multiplication map is a diffeomorphism of $F_1 \times F_2$ onto $F$. This means that the mutual actions $\alpha_g(s)$ and $\beta_s(g)$, $g \in F_1, s \in F_2$ are everywhere defined and smooth. Identifying $F_2$ with $\mathbb{T}$, we observe that for all $g \in F_1$, $\alpha_g$ is a diffeomorphism of $\mathbb{T}$ satisfying $\alpha_g(1) = 1$.

Hence, there exists, for every $g \in F_1$ a unique diffeomorphism $\tilde{\alpha}_g$ of $\mathbb{R}$, such that $\tilde{\alpha}_g(0) = 0$ and $\tilde{\alpha}_g(t) = \alpha_g(p(t))$ for all $t \in \mathbb{R}$, where $p(t) = \cos t + i \sin t$.

Defining $\tilde{\beta}_t(g) = \beta_{p(t)}(g)$, we obtain a matched pair $(F_1, \mathbb{R})$, in which both actions are everywhere defined and smooth. Its corresponding big Lie group $F_{\text{sc}}$ is the connected, simply connected Lie group of $\mathfrak{sl}_2(\mathbb{R})$.

Suppose now that $g_0$ is arbitrary and $Z \in g_0$. First, take $a = 0$. Because any finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ can be exponentiated to $\text{SL}_2(\mathbb{R})$ (although $\text{SL}_2(\mathbb{R})$ is not simply connected, see e.g. [7], Chapitre VIII, par. 1, Théorème 2), we get an action $\mu$ of $F := \text{SL}_2(\mathbb{R})$ with automorphisms of $G_0$, the connected, simply connected Lie group of $g_0$. We define $G := F \ltimes G_0$ and we denote by $\text{Exp}_g$ its exponential mapping. Because

$$\text{Exp}_g(b(K + Z)) = \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right),$$

it is clear that, defining

$$G_1 := \{ (g,k) \mid g \in F_1, k \in G_0 \}, \quad G_2 := \{ \text{Exp}_g(b(K + Z)) \mid b \in \mathbb{R} \},$$

where $F_1$ is as in Equation (4.7), we get the required matched pair of Lie groups.

When $a = -\frac{1}{4}$, we proceed similarly, but now with the simply connected Lie group $F_{\text{sc}}$ of $\mathfrak{sl}_2(\mathbb{R})$. We get an action $\mu$ of $F_{\text{sc}}$ on $G_0$ by automorphisms and we define $G := F_{\text{sc}} \ltimes G_0$. As we explained above, we can find in $F_{\text{sc}}$ a matched pair $(F_1, F_2)$ with tangent Lie algebras $(X, Y)$ and $\mathbb{R}(K - L)$, such that $F_2$ can be identified with $\mathbb{R}$. If $\text{Exp}_g$ denotes the exponential mapping of $g$, it follows that

$$\text{Exp}_g(t(K - L + Z)) = (t, \ldots) \in F_2 \times G_0 \subset F_{\text{sc}} \ltimes G_0.$$
So, we can define in the same way as above the required matched pair of Lie groups. Observe that this argument would not work with $\text{SL}_2(\mathbb{R})$ instead of $F_{sc}$, because then $\text{Exp}(2\pi n(K - L + Z)) \in G_0 \subset G_1$ when $n \in \mathbb{Z}$ and hence, we no longer have $G_1 \cap G_2 = \{e\}$.

Next, we turn to $a > 0$ and we choose the value $a = \frac{1}{4}$. Then, $g_2 = \mathbb{R}(K + L + Z)$ for some $Z \in g_0$. When $g_0 = \{0\}$, we can exponentiate as follows. Write $F = \{T \in M_2(\mathbb{R}) \mid \det T = \pm 1\}$. Define

$$F_1 := \left\{ \begin{pmatrix} a & 0 \\ x & \alpha \end{pmatrix} \mid a > 0, s = \pm 1, x \in \mathbb{R} \right\}, \quad F_2 := \left\{ \begin{pmatrix} b & c \\ c & b \end{pmatrix} \mid b^2 - c^2 = \pm 1 \right\}.$$ (4.8)

It is an easy exercise to check that we indeed get a matched pair of Lie groups. For general $g_0$, we want to proceed as in the case $a = 0$. We first get an action $\mu$ of $\text{SL}_2(\mathbb{R})$ on $g_0$. We now run into the same kind of problems as in the proof of Proposition 4.2. We should first extend the action $\mu$ to an action of $F$, by adding an involutive automorphism of $g_0$ corresponding to the action of $\left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ on $G_0$ and next, we should find the good closed subgroup with tangent Lie algebra $\mathbb{R}(K + L + Z)$ and with elements whose first components are precisely the matrices $\left( \begin{smallmatrix} b & c \\ c & b \end{smallmatrix} \right)$ with $b^2 - c^2 = \pm 1$.

First, take $g_0$ abelian. Write $\mu_H, \mu_K$ and $\mu_L$ for the derivations of $g_0$ corresponding to the generators $H, K$ and $L$ of $\mathfrak{sl}_2(\mathbb{R})$. From (Chapitre VIII, par. 1, no. 2, Corollaire), we can write

$$g_0 = \bigoplus_{-N \leq n \leq N} E_n ,$$

where $N \in \mathbb{N}$, $n$ only takes values in $\mathbb{Z}$ and where

$$\mu_H(X) = nX \text{ for } X \in E_n , \quad \mu_K(E_n) \subset E_{n+2} , \quad \mu_L(E_n) \subset E_{n-2}$$

with $E_m = \{0\}$ if $m < -N$ or $m > N$. We exponentiate $\mu$ to a homomorphism $\mu : \text{SL}_2(\mathbb{R}) \rightarrow \text{GL}(g_0)$. If we define the involution $\theta$ of $g_0$ by putting $\theta(X) = X$ if $X \in E_n$, $n = 0, 1 \text{ mod 4}$ and $\theta(X) = -X$ if $X \in E_n$ and $n = 2, 3 \text{ mod 4}$, then, $\theta$ commutes with $\mu_H$ and anti-commutes with $\mu_K$ and $\mu_L$. So, we extend $\mu$ to the group $F$ of matrices with determinant $\pm 1$ by writing $\mu \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) = \theta$.

We define, on the space $F \times g_0$, the Lie group $G$ with product $(P, X)(Q, Y) = (PQ, X + \mu(P)Y)$ for $P, Q \in F$ and $X, Y \in g_0$. Define $G_1$ consisting of the pairs $(P, X)$ for $P \in F_1$ and $X \in g_0$, with $F_1$ as in Equation (4.8). Finally, we have to find the good closed subgroup with tangent Lie algebra $\mathbb{R}(K + L + Z)$ for $Z \in g_0$. This procedure is described in great detail in the proof of Proposition 4.2. After a well chosen conjugation, we have to find a closed subgroup of $G$ with tangent Lie algebra $\mathbb{R}(H + C)$, for some $C \in g_0$, consisting of the elements

$$\{ \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, v(a, b) \right) \mid ab = \pm 1 \} .$$
As we see from the proof of Lemma 4.3, the only point is to define $v(-1, -1)$ and $v(1, -1)$. It follows from [7], Chapitre VIII, par. 1, no. 5, Corollaire, that

$$\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X = (-1)^n X$$

for $X \in E_n$. Denote by $\rho$ this involution of $g_0$. The proof of Lemma 4.3 suggests us to define $v(-1, -1) = \mu_H^{-1}(\rho(C) - C)$ and $v(1, -1) = \mu_H^{-1}(\theta(C) - C)$. Both are well-defined, because $\rho(C) - C \in \bigoplus_{n=1 \mod 2} E_n$, $\theta(C) - C \in \bigoplus_{n=2,3 \mod 4} E_n$, and $\mu_H$ is invertible on both subspaces. Also, $v(-1, -1)$ and $v(1, -1)$ should be compatible, in the sense that

$$v(-1, -1) + \rho(v(1, -1)) = v(1, -1) + \theta(v(-1, -1)),$$

or equivalently

$$(\rho - \iota)\mu_H^{-1}(\theta - \iota)(C) = (\theta - \iota)\mu_H^{-1}(\rho - \iota)(C),$$

which is the case because the factors commute.

To finish the proof of the proposition, it remains to consider non-abelian 2-dimensional $g_0$. So, $g_0$ has generators $X, Y$ with relation $[X, Y] = Y$. The Lie algebra of derivations of $g_0$ is now isomorphic to $g_0$. Any homomorphism of $\mathfrak{sl}_2(\mathbb{R})$ into $g_0$ must be trivial, because its kernel is a non-zero ideal of $\mathfrak{sl}_2(\mathbb{R})$. So, any action of $\mathfrak{sl}_2(\mathbb{R})$ on $g_0$ is necessarily trivial and we take $G := F \oplus G_0$, where $G_0$ is the $ax + b$-group. We take $G_1 = F_1 \oplus G_0$ and for any $Z \in g_0$ we can define

$$G_2 = \left\{ \begin{pmatrix} b & c \\ c & b \end{pmatrix}, \text{Exp}_{g_0}((\log |b + c|)Z) \mid b^2 - c^2 = \pm 1 \right\}.$$

This again provides us with a matched pair of Lie groups.

**Remark 4.7.** For the case $g_0 = \{0\}$, we will give more connected exponentiations below. In the proof of the previous proposition they are not so interesting, because we will then rather have $G = \text{PSL}_2(\mathbb{R})$ and not every representation of $\text{SL}_2(\mathbb{R})$ factors through $\text{PSL}_2(\mathbb{R})$. Hence, we cannot make the right semi-direct products with $\text{PSL}_2(\mathbb{R})$ acting.

**Remark 4.8.** In case 3, for $k = \mathbb{R}$ and $n \geq 5$, there are indications that there again exist matched pairs of Lie algebras that cannot be exponentiated. Their explicit description remains however open.

Next, we analyze the case $k = \mathbb{C}$. First, let us note, that now there are only two non-isomorphic cases: with $a = 0$ and with $a \neq 0$.

**Proposition 4.9.** In case 3, any matched pair of complex Lie algebras can be exponentiated to a matched pair of connected complex Lie groups if $n \leq 3$. 

**Proof.** First, consider the case $g_0 = \{0\}$. If $a = 0$, we proceed exactly as in the case of $k = \mathbb{R}$ and just replace $\mathbb{R}$ by $\mathbb{C}$. If $a \neq 0$, we consider $g_1 = \langle H, L \rangle$ and $g_2 = \mathbb{C}(K - L)$. Define $F = \text{PSL}_2(\mathbb{C})$ and define

$$F_1 := \left\{ \begin{pmatrix} a & 0 \\ x & \frac{1}{a} \end{pmatrix} \right\} \mod \{ \pm 1 \} \mid a \neq 0, x \in \mathbb{C},$$

$$F_2 := \left\{ \begin{pmatrix} \cos z & \sin z \\ \sin z & -\cos z \end{pmatrix} \right\} \mod \{ \pm 1 \} \mid z \in \mathbb{C}. \quad (4.9)$$

Some care is needed in checking that we do get a matched pair of Lie groups. Writing the product of an element in $F_1$ and an element in $F_2$, we have to find a unique solution in $F_1, F_2$ of the equation

$$\begin{pmatrix} a \cos z \\ x \cos z - \frac{1}{a} \sin z \\ x \sin z + \frac{1}{a} \cos z \end{pmatrix} = \begin{pmatrix} u & v \\ w & r \end{pmatrix} \mod \{ \pm 1 \}$$

whenever $ur - vw = 1$. Given $u, v, w, r \in \mathbb{C}$ with $ur - vw = 1$, we proceed as follows: choose $a \in \mathbb{C}$ such that $a^2 = u^2 + v^2$ and define $\cos z = \frac{u}{a}$, $\sin z = \frac{v}{a}$ and $x = \frac{ur + vw}{a}$. Then, the required equation holds. If we choose the other square root of $u^2 + v^2$, then $a, x, \cos z$ and $\sin z$ change sign and hence, their projections mod$\{ \pm 1 \}$ do not change. Because clearly $F_1 \cap F_2 = \{ e \}$, we have a matched pair of Lie groups.

If next, $g_0 = \mathbb{C}$, the action of $\mathfrak{sl}_2(\mathbb{C})$ on $g_0$ is necessarily trivial. Our matched pair has the form $g = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$, $g_1 = \langle H, L, Z \rangle$, $g_2 = \mathbb{C}(K + 4aL + \lambda Z)$, where $Z$ is the generator of $g_0 = \mathbb{C}$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, it is clear how to exponentiate, just adding a copy of $\mathbb{C}$ to $F$ and $F_1$ above. If $\lambda \neq 0$, we change the generator $Z$ and we may suppose that $\lambda = 1$. If $a = 0$, exponentiation is again easy. If we take $a = -\frac{1}{2}$, we denote by $T$ the complex torus, consisting of the pairs $(\cos z, \sin z) \in \mathbb{C}^2$, we define $G = \text{PSL}_2(\mathbb{C}) \oplus T$ with subgroups $G_1 := F_1 \oplus T$ (with $F_1$ as in Equation (4.9)) and

$$G_2 = \left\{ \left( \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix}, (\cos z, \sin z) \right) \mid z \in \mathbb{C} \right\}.$$ 

We indeed have a matched pair of Lie groups.

We conclude with the promised counterexample in dimension $4 + 1$.

**Example 4.10.** Define $g$ to be the complex Lie algebra $g := \mathfrak{sl}_2(\mathbb{C}) \oplus g_0$, where $g_0$ has the generators $X, Y$ satisfying $[X, Y] = Y$. Use the canonical generators $H, K, L$ of $\mathfrak{sl}_2(\mathbb{C})$ and define the matched pair

$$g_1 := \langle H, L, X, Y \rangle, \quad g_2 := \mathbb{C}(K - L + Y).$$

There does not exist an exponentiation of this matched pair of Lie algebras.

**Proof.** The connected, simply connected Lie group of $g$ is given by $\text{SL}_2(\mathbb{C}) \oplus G_0$, where $G_0$ lives on the space $\mathbb{C}^2$ with product $(x, y)(x', y') = (x + x', y +$
\( \exp(x') \). Its center consists of the elements \((\pm 1, 2\pi n, 0)\), where \(n \in \mathbb{Z}\). Hence, the only connected Lie groups with Lie algebra \( g \) are \( G := SL_2(\mathbb{C}) \oplus G_0/H_N \) and \( G' := PSL_2(\mathbb{C}) \oplus G_0/H_N \), where \( H_N \) consists of the elements \((2\pi n, 0), \ n \in \mathbb{Z}\). If we take \( G := SL_2(\mathbb{C}) \oplus G_0 \), the connected closed subgroup of \( G \) with tangent Lie algebra \( g_1 \) consists of the elements

\[
\left( \begin{array}{cc} a & 0 \\ z & \frac{1}{a} \end{array} \right), \quad (x, y), \quad a \neq 0, z, x, y \in \mathbb{C}.
\]

The connected closed subgroup of \( G \) with tangent Lie algebra \( g_2 \) consists of the elements

\[
\left( \begin{array}{cc} \cos z & \sin z \\ -\sin z & \cos z \end{array} \right), (0, z) \bigg| z \in \mathbb{C}.
\]

The intersection of both subgroups is non-trivial, because it contains the elements \((1, 0, 2\pi n)\) with \(n \in \mathbb{Z}\). This intersection is not annihilated by any of the central subgroups of \( G \). So, with the same kind of reasoning as in Example 3.6, we conclude that the matched pair cannot be exponentiated.

---

5 Matched pairs of real Lie algebras of dimension 1 + 1 and 2 + 1 and their exponentiation

Next we classify, up to isomorphism, all matched pairs of real Lie algebras of dimension 1 + 1 and 2 + 1 and compute explicitly their exponentiation.

From now on, all Lie algebras and Lie groups are understood to be real.

**Theorem 5.1.** In dimension 1 + 1, there exist, up to isomorphism, the following non-isomorphic matched pairs of Lie algebras. We choose a generator \( X \) for \( g_1 \).

1. \( \chi = 0 \) and \( \beta = 0 \).
2. \( \chi = 0 \) and \( \beta(X) = X \).
3. \( \chi(X) = 1 \) and \( \beta = 0 \).
4. \( \chi(X) = 1 \) and \( \beta(X) = X \).

In dimension 2+1, there exist, up to isomorphism, the following non-isomorphic matched pairs of Lie algebras. We choose generators \( X, Y \) for \( g_1 \).

1. \( \chi = 0 \) and \( \beta = 0 \).
   1.1. \([X, Y] = 0\).
   1.2. \([X, Y] = Y\).
2. $\chi = 0$ and $\beta \neq 0$:

2.1. $[X, Y] = 0$, $\beta(X) = X$, $\beta(Y) = rY$, $-1 \leq r \leq 1$.

2.2. $[X, Y] = 0$, $\beta(X) = X + Y$, $\beta(Y) = Y$.

2.3. $[X, Y] = 0$, $\beta(X) = Y$, $\beta(Y) = 0$.

2.4. $[X, Y] = Y$, $\beta(X) = Y$, $\beta(Y) = 0$.

2.5. $[X, Y] = Y$, $\beta(X) = 0$, $\beta(Y) = Y$.

3. $\chi \neq 0$ and $\beta = 0$:

3.1. $[X, Y] = aY$, $\chi(X) = 1$, $\chi(Y) = 0$, $a \in \mathbb{R}$.

4. $\chi \neq 0$ and $\beta \neq 0$:

4.1. $[X, Y] = dY$, $\beta(X) = X + bY$, $\beta(Y) = dY$, either $d = 1$ and $b \in \mathbb{R}$, or $d \neq 1$ and $b = 0$.

4.2. $[X, Y] = dY$, $\beta(X) = Y$, $\beta(Y) = 0$, $d \in \mathbb{R}$.

4.3. $[X, Y] = -Y$, $\beta(X) = aY$, $\beta(Y) = X$, $a = 1, 0, -1$.

Every matched pair above can be exponentiated to a matched pair of Lie groups, having at most 2 connected components.

**Proof.** In dimension 1+1 the classification is obvious. If either $\beta = 0$ or $\chi = 0$, an exponentiation can be given using the semi-direct product of the corresponding connected simply connected Lie groups. In the remaining case, the exponentiation was explicitly described in Remark 3.5.

In dimension 2+1, it is again natural to separate the cases 1, 2, 3 and 4. In case 1, the classification follows from the classification of 2-dimensional Lie groups. In case 2, we observe that $\beta([X, Y]) = [\beta(X), Y] + [X, \beta(Y)]$, i.e., $\beta$ is an action. If $[X, Y] = 0$, any linear map $\beta$ defines an action. We have either $\beta$ diagonalizable (case 2.1), either $\beta$ not diagonalizable and not nilpotent (case 2.2), or $\beta$ nilpotent (case 2.3). Multiplying $A$ by a scalar, one can scale $\beta$. Hence, cases 2.2 and 2.3 cover all the non-diagonalizable $\beta$. In case 2.1, we not only scale $\beta$, but also interchange $X$ and $Y$, so that we can limit ourselves to $-1 \leq r \leq 1$.

If $[X, Y] = Y$, we see that $\beta(Y) \in \mathbb{R}Y$, and then $[\beta(X), Y] = 0$. Hence, $\beta(X) \in \mathbb{R}Y$. Replacing, if necessary, $X$ by $X - rY$ and rescaling $A$, we obtain two different cases: 2.4 and 2.5.

Next, suppose that $\chi \neq 0$ and $\beta = 0$. Then $\chi$ is a character. Take $X, Y$ such that $\chi(X) = 1$ and $\chi(Y) = 0$. Hence $[X, Y] = aY$ for some $a \in \mathbb{R}$. These are all non-isomorphic: to pass from one $a \in \mathbb{R}$ to another, we have to multiply $X$ by a scalar, but then $\chi(X) = 1$ is violated.

An exponentiation of all the above matched pairs of Lie algebras again can be given using semi-direct products of the corresponding connected simply connected Lie groups.
Finally, the general discussion before this theorem shows that in case 4, there are two special situations. First, if $\chi \beta$ is a multiple of $\chi$, we separate the cases $\chi \beta \neq 0$ and $\chi \beta = 0$. If $\chi \beta \neq 0$, we rescale $A$, so that $\chi \beta = \chi$. We can take generators $X, Y$ for $g_1$ such that $\chi(X) = 1$ and $\chi(Y) = 0$. Then, there exists $b, c \in \mathbb{R}$, such that

$$
\beta(X) = X + bY, \quad \beta(Y) = cY.
$$

Because $\chi([X, Y]) = 0$, we get $[X, Y] = dY$ for $d \in \mathbb{R}$. Checking Equation (4.1), we get $c = d$. If we replace $X$ by $X + rY$, then $b$ changes to $b + r(d - 1)$. Hence, we get two non-isomorphic families: $d = 1, b \in \mathbb{R}$ and $d \neq 1, b = 0$, as stated in case 4.1.

If $\chi \beta = 0$, an analogous reasoning gives generators $X, Y$ for $g_1$ such that $[X, Y] = dY, \chi(X) = 1, \chi(Y) = 0, \beta(X) = bY, \beta(Y) = 0$, for $b, d \in \mathbb{R}$. Because $\beta \neq 0$, we get $b \neq 0$. Rescaling $Y$, we can assume that $b = 1$. This gives case 4.2.

The existence of the exponentiation of these matched pairs has been proven in Corollary 4.4, it will be described explicitly below.

The classification in case 4.3 as well as the exponentiation follows from Proposition 4.6. To reduce the number of connected components, we modify the exponentiation slightly.

If $a > 0$, in order to exponentiate, we define

$$
G = \left\{ X \in M_2(\mathbb{R}) \mid \det(X) = \pm 1 \right\},
$$

$$
G_1 = \{(a, x) \mid a \neq 0, x \in \mathbb{R}\}, \quad (a, x)(b, y) = (ab, x + ay), \quad G_2 = (\mathbb{R} \setminus \{0\}, \cdot).
$$

Making use of the function $\text{Sq}(a) := \text{Sgn}(a) \sqrt{|a|}$, we define

$$
i(a, x) = \begin{pmatrix} \frac{1}{2} \frac{\sqrt{|a|}}{\sqrt{|a|}} & 0 \\ \frac{1}{2} \frac{\sqrt{|a|}}{\sqrt{|a|}} & \text{Sq}(a) \end{pmatrix} \mod \{\pm 1\},
$$

$$
(5.1)
$$

$$
j(s) = \begin{pmatrix} \sqrt{|s|} & \frac{1}{2} \left( \sqrt{|s|} - \frac{1}{\text{Sq}(s)} \right) \\ 0 & \frac{1}{\text{Sq}(s)} \end{pmatrix} \mod \{\pm 1\}.
$$

It is clear that the tangent Lie algebras of $i(G_1)$ and $j(G_2)$ are given by $\text{span}\{H, Y\}$ and $\mathbb{R}(H + X)$, respectively. It is not hard to check that we indeed get a matched pair of Lie groups.

If $a < 0$, the easiest way to exponentiate this matched pair goes as follows (as we saw in the proof of Proposition 4.4, there is also another way of doing so).
Define
\[ G = \text{PSL}_2(\mathbb{R}) = \frac{\text{SL}_2(\mathbb{R})}{\{\pm 1\}}, \quad G_2 = (T = \{ z \in \mathbb{C} \mid |z| = 1\}, \cdot) \]
\[ G_1 = \{(a, x) \mid a > 0, x \in \mathbb{R}\} \]

Then define
\[ i(a, x) = \left( \frac{1}{\sqrt{a}}, 0, \frac{1}{\sqrt{a}} \right) \mod \{\pm 1\}, \quad (5.2) \]
\[ j(\cos t, \sin t) = \left( \cos \frac{t}{2}, \sin \frac{t}{2}, -\sin \frac{t}{2}, \cos \frac{t}{2} \right) \mod \{\pm 1\} \]

One can check that the tangent Lie subalgebras of \( i(G_1) \) and \( j(G_2) \) agree with \( \text{span}\{H, Y\} \) and \( \mathbb{R}(X - Y) \), respectively and that we get a matched pair of Lie groups.

For any of the obtained matched pairs of Lie groups, we can now perform the bicrossed product construction in order to get a l.c. quantum group. Whenever one of the corresponding actions is trivial, we obtain a Kac algebra (see Corollary 2.5). When both actions are non-trivial, we find a lot of l.c. quantum groups which are not Kac algebras. To take a closer look at them, we need explicit forms for the corresponding mutual actions, and we use the formulas
\[ \chi(X) = X \cdot [g \mapsto \frac{d}{ds} \alpha_g(s)|_{s=0}], \quad \beta(X) = \frac{d}{ds}((d\beta_s)(X))|_{s=0}, \]
where \( X_e \) is the partial derivative in \( e \) in the direction of an arbitrary generator \( X \in g_1 \) and \( d\beta_s \) is the canonical action of \( G_2 \) on \( g_1 \) coming from \( \beta_s \).

The only case of dimension 1 + 1 with both non-trivial actions has already been presented in Remark 3.5. It is easy to check that we do not get a Kac algebra, that \( \delta_M \neq 1 \), and that the corresponding l.c. quantum group is self-dual. For the details see [51], 5.3.

In dimension 2 + 1, we analyze cases 4.1, 4.2 and 4.3.

In case 4.1, following the approach of Proposition 4.2, we define the Lie group \( G \) on the space \( \mathbb{R} \setminus \{0\} \times \mathbb{R}^2 \) with multiplication
\[ (s, x, y)(s', x', y') = (ss', x + sx', y + bu_x(s)x' + s^d y') \],
where

\[ u_d(s) = \begin{cases} 
\frac{s^d - s}{d-1}, & \text{if } d \neq 1 \\
 s \log |s|, & \text{if } d = 1
\end{cases}, \]

where \( s^d = \text{Sgn}(s)|s|^d \),

and \( G_1 \) on the space \( \mathbb{R} \setminus \{0\} \times \mathbb{R} \) with multiplication \((a, x)(a', x') = (aa', x + a^d x')\) and \( i(a, x) = (a, a - 1, x + b u_d(a)) \). Further, we put \( G_2 = \mathbb{R} \setminus \{0\} \) and \( j(s) = (s, 0, 0) \). Then, the mutual actions are given by

\[
\alpha_{(a,x)}(s) = a(s - 1) + 1, \quad \beta_s(a, x) = \left( \frac{sa}{a(s-1)+1}, \frac{x + b u_d(a) + u_d(a(s-1)+1) - u_d(as))}{(a(s-1)+1)^d} \right). \tag{5.4}
\]

One can check that the corresponding matched pair of Lie algebras is isomorphic to the initial one. Indeed, using the obvious generators, we have:

\[ [X, Y] = dY, \quad \chi(X) = 1, \quad \chi(Y) = 0, \quad \beta(X) = -X - bdY, \quad \beta(Y) = -dY. \]

The needed isomorphism is given by \( A \mapsto -A, \ Y \mapsto dY \) (if \( d \neq 0 \)) ; if \( d = 0 \), \( A \mapsto -A \) establishes an isomorphism with the special case of the initial matched pair: \( b = d = 0 \).

Because the modular functions of the groups \( G_1 \) and \( G \) are given by \( \delta_1(a, x) = |a|^{-d} \) and \( \delta(s, x, y) = |s|^{-d-1} \), we compute that the first equality of Proposition 2.4 does not hold and

\[
\delta_M(a, x, s) = |a(s-1)+1|^{d+1}, \quad \delta_{\hat{M}}(a, x, s) = \left| \frac{as}{a(s-1)+1} \right|^{d-1}. \tag{5.5}
\]

So, both the l.c. quantum group and its dual are not Kac algebras, and are non-unimodular.

In case 4.2, the Lie groups \( G \) and \( G_1 \) are defined on \( \mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^2 \) and \( \mathbb{R}^+ \setminus \{0\} \times \mathbb{R} \), respectively, with the same multiplication as in case 4.1, but with parameter \( b = 1 \). We consider \( G_2 \) to be \( \mathbb{R} \) with addition and define \( i(a, x) = (a, 0, x) \) and \( j(s) = (1, s, 0) \). Then, the mutual actions are

\[
\alpha_{(a,x)}(s) = a s, \quad \beta_s(a, x) = (a, x + u_d(a)s). \tag{5.5}
\]

One can check that the corresponding matched pair of Lie algebras coincides with the initial one, that the first equality of Proposition 2.4 holds and that \( \delta_M = 1, \quad \delta_{\hat{M}}(a, x, s) = a^{d-1} \). Hence, \((M, \Delta)\) is a unimodular Kac algebra, and \((\hat{M}, \hat{\Delta})\) is unimodular if and only if \( d = 1 \).

Finally, the exponentiations of case 4.3 with \( a > 0, < 0, = 0 \), are determined by Equations (5.1), (5.2) and (5.3), respectively. In the case \( a > 0 \), the mutual
actions are given by
\[ \alpha_{(a,x)}(s) = \frac{(x+1)s + a - x - 1}{xs + a - x}, \]
\[ \beta_s(a,x) = \left(\frac{((x+1)s + a - x - 1)(xs + a - x)}{as}, \frac{x((x+1)s + a - x - 1)}{a}\right). \] (5.6)

The corresponding matched pair of Lie algebras
\[ [X,Y] = Y, \chi(X) = -1, \chi(Y) = 0, \beta(X) = -X, \beta(Y) = 2X + Y \]
is isomorphic to the initial one: \( X \mapsto -X - \frac{Y}{T}, Y \mapsto -\frac{1}{T}, A \mapsto 2A. \)

In the case \( a < 0 \), the mutual actions are
\[ \alpha_{(a,x)}(\cos t, \sin t) = \]
\[ \frac{\left((a^2 + x^2 - 1) + (a^2 - x^2 + 1)\cos t + 2ax\sin t, 2x - 2x\cos t + 2a\sin t\right)}{(x^2 + a^2 + 1) + (-x^2 + a^2 - 1)\cos t + 2ax\sin t}, \]
\[ \beta_{(\cos t, \sin t)}(a,x) = \left(\frac{1}{2a}\left((x^2 + a^2 + 1) + (-x^2 + a^2 - 1)\cos t + 2ax\sin t\right), \frac{1}{2a}\left((x^2 - a^2 + 1)\sin t + 2ax\cos t\right)\right). \] (5.7)

Observe that these actions are everywhere defined and continuous. The corresponding matched pair of Lie algebras
\[ [X,Y] = Y, \chi(X) = -1, \chi(Y) = 0, \beta(X) = -Y, \beta(Y) = X \]
is isomorphic to the initial one: \( X \mapsto -X - \frac{Y}{T}, Y \mapsto -\frac{1}{T}, A \mapsto 2A. \)

In the case \( a = 0 \), the mutual actions are
\[ \alpha_{(a,x)}(a) = \frac{s}{a(a + xs)}, \quad \beta_s(a,x) = (|a + sx|, \text{Sgn}(a + sx)x). \] (5.8)

The corresponding matched pair of Lie algebras
\[ [X,Y] = 2Y, \chi(X) = -2, \chi(Y) = 0, \beta(X) = 0, \beta(Y) = X \]
is isomorphic to the initial one: \( X \mapsto -X - \frac{Y}{T}, Y \mapsto -\frac{1}{T}. \)

In all three cases \( a > 0, < 0, = 0 \), one verifies that the first equality of Proposition 2.4 does not hold, \( \delta_M = 1 \), while \( \delta_M \neq 1 \). In particular, we do not get Kac algebras.

6 Cocycle matched pairs of Lie groups and Lie algebras in low dimensions

So far, we have explained how to construct l.c. quantum groups which are bicrossed products of low-dimensional Lie groups without 2-cocycles. The
usage of 2-cocycles gives much more concrete examples and, what is more important, gives a more complete picture of low-dimensional l.c. quantum groups. Again, we first explain the infinitesimal picture, i.e., how 2-cocycles for matched pairs of Lie algebras look like, how they are related to the problem of extensions and then show how to exponentiate them.

The first thing that we need here is the notion of a Lie bialgebra, due to V.G. Drinfeld [10]. A Lie bialgebra is a Lie algebra $g$ equipped with a Lie bracket $[·,·]$ and a Lie cobracket $\delta$, i.e., a linear map $\delta: g \to g \otimes g$ satisfying the co-anticommutativity and the co-Jacobi identity, that is:

$$(\iota - \tau)\delta = 0, \quad (\iota + \zeta + \zeta^2)(\iota \otimes \delta)\delta = 0,$$

where

$$\tau(u \otimes v) = v \otimes u, \quad \zeta(u \otimes v \otimes w) = v \otimes w \otimes u \quad \text{(for all } u, v, w \in g)$$

are the flip maps, and these Lie bracket and cobracket are compatible in the following sense:

$$\delta[u,v] = [u, v_{[1]}] \otimes v_{[2]} + v_{[1]} \otimes [u, v_{[2]}] + [u_{[1]}, v] \otimes u_{[2]} + u_{[1]} \otimes [u_{[2]}, v].$$

Any Lie algebra (respectively, Lie coalgebra, i.e., vector space dual to a Lie algebra) is a Lie bialgebra with zero Lie cobracket (respectively, zero Lie bracket). The definition of a morphism of Lie bialgebras is obvious.

Given a pair of Lie algebras $(g_1, g_2)$, let us ask if there exists a Lie bialgebra $g_3$ such that $g_3 \twoheadrightarrow g_2 \rightarrow g_1$ is a short exact sequence in the category of Lie bialgebras. This means precisely that $g$ has a sub-bialgebra with trivial bracket, which is an ideal and such that the quotient is a Lie bialgebra with trivial cobracket.

The theory of extensions in this framework has been developed in [36] and is quite similar to the theory of extensions of l.c. groups that we have recalled above. Namely, for the existence of an extension $g$ it is necessary and sufficient that $(g_1, g_2)$ form a matched pair, and all extensions are bicrossed products with cocycles. We consider this theory as an infinitesimal version of the theory of extensions of Lie groups.

As we remember, for any matched pair of Lie algebras $(g_1, g_2)$, there are mutual actions $\triangleright: g_2 \otimes g_1 \to g_1$ and $\triangleleft: g_2 \otimes g_1 \to g_2$, compatible in a way explained in Section 3 and such that for all $a, b \in g_1$, $x, y \in g_2$ we have

$$[a \oplus x, b \oplus y] = ([a, b] + x \triangleright b - y \triangleright a) \oplus ([x, y] + x \triangleleft b - y \triangleleft a).$$

For the general definition of a pair of 2-cocycles on such a matched pair, we refer to [33, 34]. For our needs, it suffices to understand that these 2-cocycles
are linear maps
\[ U : g_1 \wedge g_1 \to g_2^*, \quad V : g_2 \wedge g_2 \to g_1^* \]
verifying certain 2-cocycle equations and compatibility equations that are infinitesimal forms of Equations (2.1). For the case of dimension \( n + 1 \), we give these equations explicitly below. Let us formulate the link between 2-cocycles on matched pairs of Lie algebras and those of Lie groups as a proposition whose proof is straightforward.

**Proposition 6.1.** Let \((G_1, G_2)\) be a matched pair of Lie groups equipped with cocycles \( U \) and \( V \), which are differentiable around the unit elements, and let \((g_1, g_2)\) be the corresponding matched pair of Lie algebras. Defining
\[
\langle U(X,Y), A \rangle = -i(X_e \otimes Y_e \otimes A_e - Y_e \otimes X_e \otimes A_e)(U) \quad \text{and} \\
\langle V(A,B), X \rangle = -i(A_e \otimes B_e \otimes X_e - B_e \otimes A_e \otimes X_e)(V),
\]
for \( X, Y \in g_1 \) and \( A, B \in g_2 \), we get a pair of cocycles on \((g_1, g_2)\).

Here \( \langle \cdot, \cdot \rangle \) denotes the duality between \( g_i \) and \( g_i^* \) and \( X_e, Y_e, A_e, B_e \) denote the partial derivatives at \( e \) in the direction of the corresponding generator. The factor \(-i\) appears because for Lie groups \( U \) and \( V \) take values in \( \mathbb{T} \), and for real Lie algebras we consider 2-cocycles as real linear maps.

For the dimension \( n + 1 \), \( V \) is necessarily trivial (if also \( n = 1 \), then also \( U \) is trivial, so there are no non-trivial cocycles in the dimension \( 1 + 1 \)). Returning to arbitrary \( n \), we choose a generator \( A \) for \( g_2 \) and define maps \( \beta \) and \( \chi \) as above. Then \( U \) can be regarded as an antisymmetric, bilinear form on \( g_1 \), and the 2-cocycle equations of [33], [36] reduce to the equation
\[ U([X,Y], Z) + \chi(X) U(Y, Z) + \text{cyclic permutation} = 0 \quad \text{for all} \quad X, Y, Z \in g_1. \]

It is clear that these 2-cocycles \( U \) form a real vector space.

In Section 2, we defined the notion of the group of extensions for a matched pair of Lie groups \((G_1, G_2)\) using the notion of cohomologous 2-cocycles. The same can be done for a matched pair of Lie algebras [37]. In particular, for the dimension \( n + 1 \), a 2-cocycle \( U \) is called cohomologous to trivial, if there exists a linear form \( \rho \) in \( g_1^* \) such that
\[
U(X,Y) = \rho([X,Y]) + \chi(X) \rho(Y) - \chi(Y) \rho(X).
\]

Two cocycles \( U_1 \) and \( U_2 \) are called cohomologous if \( U_1 - U_2 \) is cohomologous to trivial.

The quotient space of 2-cocycles modulo 2-cocycles cohomologous to trivial, with addition as the group operation, is called the group of extensions of the matched pair \((g_1, g_2)\).

Now let us describe all 2-cocycles on the matched pairs of real Lie algebras of dimension \( 2 + 1 \).
Proposition 6.2. Referring to the classification of matched pairs of Lie algebras of dimension $2 + 1$ given in Theorem 5.1, the following holds: the group of extensions is $\mathbb{R}$ in the cases 1.1, 2.1, 2.2, 2.3, 3 ($a = -1$), 4.1 ($d = -1$), 4.2 ($d = -1$) and 4.3. The cocycles are defined by $U(X, Y) = \lambda$, for $\lambda \in \mathbb{R}$. In the other cases, the group of extensions is trivial.

Proof. Since $\dim g_1 = 2$, any antisymmetric bilinear form $U$ on $g_1$ is a cocycle. Suppose that $X, Y$ are generators of $g_1$ with $[X, Y] = dY$. A cocycle $U$ is entirely determined by $U(X, Y) = \lambda$ for $\lambda \in \mathbb{R}$. If $\chi = 0$ and $d \neq 0$, we take $\rho(Y) = \lambda/d$ and get that $U$ is cohomologous to trivial. If $\chi = 0$ and $d = 0$, it is clear that $U$ is not cohomologous to trivial if $\lambda \neq 0$.

If $\chi \neq 0$, we may suppose that $\chi(X) = 1$ and $\chi(Y) = 0$. If $d \neq -1$, we take $\rho(Y) = \lambda/(1 + d)$ and get that $U$ is cohomologous to trivial. If $d = -1$, it is clear that $U$ is not cohomologous to trivial if $\lambda \neq 0$. □

Next, we want to exponentiate a cocycle on a matched pair of real Lie algebras, i.e., to construct a measurable map $U : G_1 \times G_1 \times G_2 \to U(1)$, with values in the unit circle of $\mathbb{C}$, satisfying

$$U(g, h, \alpha_k(s)) U(hk, k, s) = U(h, k, s) U(g, hk, s),$$

$$U(g, h, s) U(\beta_{\alpha_k(s)}(g), \beta_s(h), t) = U(g, h, ts)$$

almost everywhere. Let us define a function $A(\cdot)$ by

$$U(g, h, s) = \exp(iA(g, h, s)).$$

So $A(\cdot)$ should satisfy

$$A(g, h, \alpha_k(s)) + A(gh, k, s) = A(h, k, s) + A(g, hk, s) \mod 2\pi,$$

$$A(g, h, s) + A(\beta_{\alpha_k(s)}(g), \beta_s(h), t) = A(g, h, ts) \mod 2\pi$$

almost everywhere.

Proposition 6.3. If the group of extensions of a matched pair of Lie algebras of dimension $2 + 1$ is non-trivial, there exists an exponentiation of this matched pair with cocycles. These cocycles are labeled by $\mathbb{R}$ in all the cases, except case 4.3 ($a = 0, 1$), where they are labeled by $\mathbb{Z}$.

Proof. Following [51], Section 5.5, we look for the above function $A$ in the form

$$A(g, h, s) = P \int_0^s f(\phi_r(g, h)) \, dr,$$

where $\phi_r(g, h) := (\beta_{\alpha_k(r)}(g), \beta_r(h))$ and where the function $f$ on $G_1 \times G_1$ is such that for almost all $g, h \in G_1$ the function $r \mapsto f(\phi_r(g, h))$ has a principal value integral over any interval in $\mathbb{R}$ (dr is the Haar measure on the 1-dimensional Lie group $(\mathbb{R}, +)$ or on $\mathbb{R} \setminus \{0\}$, in which case we integrate.
from 1 to s). A necessary condition to be satisfied by \( f \) is

\[
\frac{d}{dt} \alpha_k(t) \big|_{t=0} f(g, h) + f(gh, k) = f(h, k) + f(g, hk) .
\]

Finally, having found such an \( f \), we have to check if it really gives rise to a 2-cocycle.

In those cases where the actions \( \alpha \) and \( \beta \) are everywhere defined and smooth, one can check that any smooth solution of this equation gives indeed rise to a 2-cocycle (for the details see [51], Section 5.5). In this way, it is easy to find 2-cocycles in the cases 1.1, 2.1, 2.2, 2.3, 3 \((a = -1)\), and 4.2 \((d = -1)\), namely: in the cases 1.1, 2.1, 2.2 and 2.3, the action \( \alpha \) is trivial, and \( G_1 = \mathbb{R}^2 \) with addition. So, we can take \( f(x_1, x_2; y_1, y_2) = \lambda(x_1y_2 - x_2y_1) \), for any \( \lambda \in \mathbb{R} \). In the cases 3 \((a = -1)\) and 4.2 \((d = -1)\), we observe that \( G_1 = \{(a, x) \mid a > 0, x \in \mathbb{R} \} \) with \((a, x)(b, y) = (ab, x + y/a)\). Because \( \chi(X) = 1 \), the character \( y \mapsto \frac{d}{dt} \phi_y(t) \big|_{t=0} \) is given by \((a, x) \mapsto a\), and we can take \( f(a, x; b, y) = \lambda abx \log b \), for any \( \lambda \in \mathbb{R} \).

The case 4.3 \((a = 0)\) has been studied in [51], Section 5.5: \( f(a, x; b, y) = \lambda \frac{x \log b}{ab} \). Checking if we really get 2-cocycles, observe that

\[
f(\phi_r(a, x; b, y)) = \frac{\lambda x}{(b + ry)(ab + r(ay + \frac{x}{b}))} \log |c + dr| ,
\]

then

\[
P \int_{-\infty}^{\infty} f(\phi_r(a, x; b, y)) \, dr = \frac{\lambda}{2} \pi^2 \text{Sgn}\left(\frac{y}{x} (ay + \frac{x}{b})\right) .
\]

From this, it follows that we do get 2-cocycles if and only if \( \lambda = \frac{\text{Sgn}(b)}{2\pi} \), with \( n \in \mathbb{Z} \).

The same phenomenon happens in the cases 4.1 \((d = -1)\) and 4.3 \((a > 0)\): although the above principal value integral is well defined, we do not always get a 2-cocycle \( \mathcal{U} \), as explained in [51], after Proposition 5.6. In case 4.1 \((d = -1)\), we use the matched pair explicitly described in Equation (5.4) with \( d = -1 \) and \( b = 0 \). We take again \( f(a, x; b, y) = \lambda abx \log |b| \), for any \( \lambda \in \mathbb{R} \), and we can explicitly perform the integration, to obtain 2-cocycles:

\[
A(a, x; b; y) = \lambda ax (-b(s - 1) + 1) \log |b(s - 1) + 1| + bs \log |bs| - b \log |b| .
\]

On the contrary, the situation of case 4.3 \((a > 4)\) is more delicate. We use the matched pair explicitly described in Equation (5.6). We can take \( f(a, x; b, y) = \lambda \frac{x \log |a|}{ab} \) with \( \lambda \in \mathbb{R} \), and our candidate for \( A(\cdot) \) becomes:

\[
A(a, x; b; y; s) = \lambda P \int_{1}^{s} \frac{y}{yr + b - y} \log \left| \frac{(x + ay + 1)r + ab - x - ay - 1}{a((y + 1)r + b - y - 1) (yr + b - y)} \right| \, dr .
\]
Because
\[ P \int_{-\infty}^{+\infty} \frac{c}{cr + d} \log |ar + b| \, dr = \frac{\pi^2}{2} \text{Sgn}(\frac{b}{a} - \frac{d}{c}), \]
the same reasoning as in Section 5.5 of [51] implies that we do get a 2-cocycle if \( \lambda = \frac{4n\pi}{\pi} \) for \( n \in \mathbb{Z} \).

Finally, in case 4.3 \((a > 0)\), with the explicit exponentiation given in Equation (5.7), the mutual actions are defined everywhere and are smooth, but \( G_1 = T \). Taking \( f(a, x; b, y) = \lambda \frac{y}{b} \log a \) with \( \lambda \in \mathbb{R} \), it is natural to use
\[ A(a, x; b, y; \cos t, \sin t) = \lambda \int_0^t f(\phi(s, \sin s)(a, x, b, y)) \, ds. \]
To have a 2-cocycle, we need that
\[ \lambda \int_0^{2\pi} f(\phi(s, \sin s)(a, x, b, y)) \, ds = 0 \mod 2\pi. \]
Denote the left-hand side of this expression by \( I_\lambda(a, x, b, y) \). Then, one can compute that
\[ I_\lambda(a, x, b, y) = \lambda \left( H(a, x) + H(b, y) - H(ab, x + ay) \right), \]
where \( H(a, x) := -4\pi \arctan(\frac{x}{1+ax}) \). Hence, there is no \( \lambda \) which gives us a 2-cocycle.

We can, however, find 2-cocycles, using the other exponentiation of the same matched pair of Lie algebras, as explained in the proof of Proposition 4.6. We obtain a matched pair \((G_1, \mathbb{R})\), in which both actions are everywhere defined and smooth. Hence, we obtain cocycles labeled by \( \mathbb{R} \), following the procedure described in the beginning of the proof.

7 Infinitesimal objects for low-dimensional l.c. quantum groups

The case of cocycle bicrossed product l.c. quantum groups

Given a cocycle matched pair of Lie groups \((G_1, G_2)\) whose 2-cocycles \( \mathcal{U} \) and \( \mathcal{V} \) are differentiable around the unit elements, we can construct the corresponding l.c. quantum group using the cocycle bicrossed product construction.

But, in this situation, we can also construct two other intimately related algebraic structures which can be viewed as infinitesimal objects of this l.c. quantum group: a Lie bialgebra and a Hopf \(^*\)-algebra, as in [51], Section 5.2. The precise mathematical link between these three structures is not completely clear at the moment (it is tempting to consider it as a kind of a Lie theory for
our cocycle bicrossed product l.c. quantum groups). We will discuss it mainly on the level of examples.

Let us recall the construction of infinitesimal Lie bialgebras and Hopf *-algebras in the special case of dimension $n + 1$.

Let $(G_1, G_2)$ be acocycle matched pair of Lie groups with $G_2 = \mathbb{R}$ (the case $\mathbb{R} \setminus \{0\}$ is completely analogous, replacing differentials in 0 by differentials in 1) and let $U$ be a 2-cocycle differentiable around the unit elements. Denote by $\alpha_g(s)$ and $\beta_s(g)$ the corresponding mutual actions. Then the cocycle matched pair of Lie algebras is determined (see Section 4 and Proposition 6.1) by

$$\chi(X) = X \left. e^{\frac{d}{ds} \alpha_g(s)} \right|_{s=0}, \quad X \in g_1,$$

$$\beta(X) = \left. \frac{d}{ds} ((d\beta_s)(X)) \right|_{s=0}, \quad X \in g_1,$$

$$U(X,Y) = -i \left. \frac{d}{ds} ((X_e \otimes Y_e - Y_e \otimes X_e)(U(\cdot,\cdot,s))) \right|_{s=0} \tilde{A}, \quad X,Y \in g_1.$$

The infinitesimal Lie bialgebra is precisely the corresponding cocycle bicrossed product Lie bialgebra and has generators $\tilde{A} \in g_2$ and $X \in g_1$, subject to the relations

$$[\tilde{A}, X] = \chi(X) \tilde{A},$$

$$[X, Y] = [X, Y]_1 + U(X,Y) \tilde{A},$$

$$\delta(\tilde{A}) = 0,$$

$$\delta(X) = \beta(X) \wedge \tilde{A}.$$

The dual infinitesimal Lie bialgebra has generators $A$ and $\tilde{X}_i$, subject to the relations

$$[\tilde{X}_i, A] = \sum_j \beta(X_i)_j \tilde{X}_j,$$

$$[\tilde{X}_i, \tilde{X}_j] = 0,$$

$$\langle \delta(\tilde{X}_i), X \otimes Y \rangle = \langle \tilde{X}_i, [X, Y]_1 \rangle,$$

$$\delta(A) = A \wedge \left( \sum_i \chi(X_i) \tilde{X}_i \right) + \sum_{i<j} U(X_i, X_j) \tilde{X}_i \wedge \tilde{X}_j.$$

Following [51], Section 5.2 and in order to construct the infinitesimal Hopf *-algebra, which is an algebraic cocycle bicrossed product in the sense of [33], we denote by $\tilde{A}$ the function $\tilde{A}(x) = x$ on $G_2$. Then, our Hopf *-algebra has
generators $\tilde{A}$ and $\{ X \mid X \in \mathfrak{g}_1 \}$, where $\tilde{A}^* = \tilde{A}$ and $X^* = -X$, with relations

$$[\tilde{A}, X] = X_e[g \mapsto \alpha_g(s)], \quad X \in \mathfrak{g}_1,$$
$$[X, Y] = [X, Y]_1 + (X_e \otimes Y_e - Y_e \otimes X_e)(\mathcal{U}(\cdot) \cdot s), \quad X, Y \in \mathfrak{g}_1,$$

$$\Delta(\tilde{A}) = \tilde{A} \otimes 1 + 1 \otimes \tilde{A}, \quad (\tilde{A} \otimes \tilde{A} \text{ when } G_2 = \mathbb{R} \setminus \{0\}),$$
$$\Delta(X) = 1 \otimes X + \sum_i X_i \otimes (d\beta_s)(X)_i, \quad X \in \mathfrak{g}_1.$$

This needs some explanation: several of the used expressions are functions of $X$ (in the simplest case - to the algebra of polynomials in $\tilde{A}$).

Next, $(X_i)$ is a basis for $\mathfrak{g}_1$ and $(d\beta_s)(X)_i$ is the $i$-th component of $(d\beta_s)(X)$ in this basis, and is, therefore, again a function of $s \in G_2$. Finally, $[X, Y]_1$ denotes the Lie bracket in $\mathfrak{g}_1$. When $G_2 = (\mathbb{R}, +)$, co-unit and antipode are given by $\varepsilon(\tilde{A}) = \varepsilon(X) = 0$ ($X \in \mathfrak{g}_1$), $S(\tilde{A}) = -\tilde{A}$ and

$$\begin{align*}
S(X) &= -\sum_i X_i (d\beta_{-s})(X)_i, \quad X \in \mathfrak{g}_1.
\end{align*}$$

When $G_2 = (\mathbb{R} \setminus \{0\}, \cdot)$, we rather have $\varepsilon(\tilde{A}) = 1$, $\varepsilon(X) = 0$ ($X \in \mathfrak{g}_1$), $S(\tilde{A}) = \tilde{A}^{-1}$ and

$$\begin{align*}
S(X) &= -\sum_i X_i (d\beta_{1/s})(X)_i, \quad X \in \mathfrak{g}_1.
\end{align*}$$

To describe the dual infinitesimal Hopf *-algebra, suppose that we have coordinate functions $(\hat{X}_i)$ on $G_1$, dual to the basis $(X_i)$ of $\mathfrak{g}_1$. Then we can write down, with the same kind of conventions, the dual Hopf *-algebra, with generators $(\hat{X}_i)$ and $A$, such that $\hat{X}_i^* = \hat{X}_i$ and $A^* = -A$, and with relations:

$$\begin{align*}
[\hat{X}_i, A] &= \frac{d}{ds}(\hat{X}_i(\beta_s(g)))|_{s=0}, \\
[\hat{X}_i, \hat{X}_j] &= 0, \\
\Delta(\hat{X}_i)(g, h) &= \hat{X}_i(gh), \\
\Delta(A) &= 1 \otimes A + A \otimes \frac{d}{ds}(\alpha_g(s))|_{s=0} + \frac{d}{ds}\mathcal{U}(g, h, s)|_{s=0}.
\end{align*}$$

Again, we write functions of $g, h \in G_1$ and they are tacitly assumed to belong to some algebra of functions in $\hat{X}_i$. Co-unit and antipode are given by $\varepsilon(\hat{X}_i) = \hat{X}_i(\varepsilon), \varepsilon(A) = 0$, $S(\hat{X}_i)(g) = \hat{X}_i(g^{-1})$ and

$$\begin{align*}
S(A) &= -A \frac{d}{ds}(\alpha_{g^{-1}}(s))|_{s=0} - \frac{d}{ds}\mathcal{U}(g, g^{-1}, s)|_{s=0}.
\end{align*}$$

**Remark 7.1.** Observe that in order to pass from the infinitesimal Hopf *-algebra to the infinitesimal Lie bialgebra, one replaces functions on $G_2$ by
their first-order approximations, which are multiples of \( \tilde{A} \), one takes anti-auto-
adjoint generators \( i \tilde{A} \), and \( X \in g_1 \), and takes as \( \delta \) the linear part of \( i(\Delta - \Delta^{\text{op}}) \),
where \( \Delta^{\text{op}} \) denotes the opposite comultiplication.

Let us turn now to concrete examples. For the matched pair of dimension
1+1 (see Remark 3.3) we do not have cocycles. The infinitesimal Hopf \(^*\)-algebra is determined by
\[
\tilde{A} = \tilde{A}^* \,, \quad X^* = -X \,; \quad [\tilde{A}, X] = \tilde{A} - 1 \,,
\]
\[
\Delta(\tilde{A}) = \tilde{A} \otimes \tilde{A} \,, \quad \Delta(X) = X \otimes \tilde{A}^{-1} + 1 \otimes X \,.
\]
Co-unit and antipode are \( \varepsilon(\tilde{A}) = 1 \,, \varepsilon(X) = 0 \,, S(\tilde{A}) = \tilde{A}^{-1} \,, S(X) = -X \tilde{A} \).

The infinitesimal Lie bialgebra is given by
\[
[\tilde{A}, X] = \tilde{A} \,, \quad \delta(\tilde{A}) = 0 \,, \quad \delta(X) = \tilde{A} \wedge X \,.
\]
We will now make the link between the Hopf \(^*\)-algebra and the Lie bialgebra
slightly more explicit. For a smooth function \( f \) in \( \tilde{A} \) we get \( [f(\tilde{A}), X] = f'(\tilde{A})(\tilde{A} - 1) \). If we take \( A_1 := i \log \tilde{A} \) as a new generator, we observe that
\[
[A_1, X] = A_1 + \frac{i}{2} A_1^2 + \cdots \,, \quad \Delta(A_1) = A_1 \otimes 1 - 1 \otimes A_1 \,,
\]
\[
\Delta(X) = X \otimes (1 + i A_1 - \frac{1}{2} A_1^2 + \cdots) + 1 \otimes X \,.
\]
Linearizing now the above bracket and \( i(\Delta - \Delta^{\text{op}}) \), we precisely get our Lie
bialgebra.

Since this example is self-dual, the dual infinitesimal Hopf \(^*\)-algebra and
Lie bialgebra are the same as the original ones.

**Remark 7.2.** It is easy to show that the only non-trivial Lie bialgebras of
dimension 2 non-isomorphic to the above mentioned Lie bialgebra, are defined
by the relations
\[
[A, X] = X \,, \quad \delta(A) = 0 \,, \quad \delta(X) = qX \wedge A \,.
\]
On the other hand, the Hopf \(^*\)-algebras corresponding to the l.c. quantum
"ax+b"-group considered by S.L. Woronowicz and S. Zakrzewski \([32]\) and by
A. Van Daele \([55]\) are defined by the relations
\[
a^* = a \,, \quad x^* = x \,, \quad ax = \exp(iq) xa \,, \quad \Delta(a) = a \otimes a \,, \quad \Delta(x) = x \otimes a + 1 \otimes x \,.
\]
Although this quantum group cannot be obtained by the bicrossed product
construction (see \([51]\), 5.4.d), we can apply to it the same formal procedure
to associate with it a Lie bialgebra. Namely, observing that \( [\log a, x] = i x \),
we put \( A = \frac{1}{q} \log a \) and \( X = ix \). Linearizing as above, we find precisely the
previous Lie bialgebra which then can be considered as an infinitesimal Lie
bialgebra of the l.c. quantum "ax+b"-group (we mentioned already that the
exact relation between these structures is not completely clear).
Let us now list the infinitesimal Hopf *-algebras and Lie bialgebras of all the cocycle matched pairs of Lie algebras of dimension 2+1, with two non-trivial actions (referring to Theorem 5.1, case 4).

Case 4.1 has been exponentiated concretely in Equation (5.4). Taking the obvious generators $X$ and $Y$ for $\mathfrak{g}_1$ (differentiating to $a$ and $x$, respectively), satisfying $[X, Y] = dY$, and $A$ for $\mathfrak{g}_2$, we get the infinitesimal form $\chi(X) = 1$, $\chi(Y) = 0$, $\beta(X) = -X - bdY$ and $\beta(Y) = -dY$. When $d \neq -1$, we do not have non-trivial cocycles and we get the Hopf *-algebra, with $A^* = A$, $X^* = -X$, $Y^* = -Y$ and

$$[\tilde{A}, X] = \tilde{A} - 1, \quad [\tilde{A}, Y] = 0, \quad [X, Y] = dY,$$

$$\Delta(\tilde{A}) = \tilde{A} \otimes \tilde{A},$$

$$\Delta(X) = X \otimes \tilde{A}^{-1} + 1 \otimes X + bY \otimes \tilde{A}^{-d}(1 - u_d'(\tilde{A})),$$

$$\Delta(Y) = Y \otimes \tilde{A}^{-d} + 1 \otimes Y.$$

Co-unit and antipode are $\varepsilon(\tilde{A}) = 1$, $\varepsilon(X) = \varepsilon(Y) = 0$, $S(\tilde{A}) = \tilde{A}^{-1}$, $S(X) = -X \tilde{A} - bdY u_d(A), S(Y) = -Y \tilde{A}^d$. The corresponding Lie bialgebra is

$$[\tilde{A}, X] = \tilde{A}, \quad [\tilde{A}, Y] = 0, \quad [X, Y] = dY,$$

$$\delta(\tilde{A}) = 0, \quad \delta(X) = \tilde{A} \wedge (X + bdY), \quad \delta(Y) = d\tilde{A} \wedge Y.$$

When $d = -1$, we have found non-trivial cocycles, which are infinitesimally given by $U(X, Y) = -\lambda \tilde{A}$. For the Hopf *-algebra, there changes $[X, Y] = -Y - i\lambda \log \tilde{A}$, and on the Lie bialgebra level, this becomes $[X, Y] = -Y - \lambda \tilde{A}$. Co-unit and antipode remain unchanged.

The dual Hopf *-algebra for $d \neq -1$ has anti-self-adjoint generators $\hat{X}, \hat{Y}$, and a self-adjoint generator $A$, subject to the relations

$$[\hat{X}, A] = \hat{X}(1 - \tilde{X}), \quad [\hat{Y}, A] = b\hat{X}(1 - u_d'(\hat{X})) - d\hat{X}\hat{Y}, \quad [\hat{X}, \hat{Y}] = 0,$$

$$\Delta(\hat{X}) = \hat{X} \otimes \hat{X},$$

$$\Delta(\hat{Y}) = \hat{X}^d \otimes \hat{Y} + \hat{Y} \otimes 1,$$

$$\Delta(A) = A \otimes \hat{X} + 1 \otimes A.$$

Co-unit and antipode are $\varepsilon(\hat{X}) = 1$, $\varepsilon(\hat{Y}) = \varepsilon(A) = 0$, $S(\hat{X}) = \hat{X}^{-1}$, $S(\hat{Y}) = -\hat{X}^{-d}\hat{Y}, S(A) = -A\hat{X}^{-1}$. The corresponding infinitesimal Lie bialgebra is determined by

$$[\hat{X}, A] = -\hat{X}, \quad [\hat{Y}, A] = -bd\hat{X} - d\hat{Y}, \quad [\hat{X}, \hat{Y}] = 0,$$

$$\delta(\hat{X}) = 0, \quad \delta(\hat{Y}) = d\hat{X} \wedge \hat{Y}, \quad \delta(A) = A \wedge \hat{X}.$$

In the case $d = -1$ and with the same cocycle as above, we should change $\Delta(A) = A \otimes \tilde{X} + 1 \otimes A + i\lambda\hat{X} \hat{Y} \otimes \hat{X} \log \hat{X}$ in the definition of the Hopf *-
algebra and \( \delta(A) = A \wedge \hat{X} - \lambda \hat{X} \wedge \hat{Y} \) in the definition of the Lie bialgebra. For the antipode, we change \( S(A) = -A \hat{X} + i \lambda \hat{Y} \log \hat{X} \).

Case 4.2 has been exponentiated in Equation (5.5). Taking again the obvious generators, we have the infinitesimal form \([X,Y] = dY, \, \beta(X) = Y, \, \beta(Y) = 0, \, \chi(X) = 1 \) and \( \chi(Y) = 0 \). When \( d \neq -1 \), there are no non-trivial cocycles. We find the Hopf \(*\)-algebra with generators \( \hat{A} = A^* \) and \( X,Y \) anti-self-adjoint, satisfying

\[
[\hat{A}, X] = \hat{A}, \quad [\hat{A}, Y] = 0, \quad [X,Y] = dY,
\]
\[
\Delta(\hat{A}) = \hat{A} \otimes \hat{A},
\]
\[
\Delta(X) = X \otimes 1 + 1 \otimes X + Y \otimes \hat{A},
\]
\[
\Delta(Y) = Y \otimes 1 + 1 \otimes Y .
\]

Co-unit and antipode are \( \varepsilon(\hat{A}) = 1, \varepsilon(X) = \varepsilon(Y) = 0 \), \( S(\hat{A}) = \hat{A}^{-1}, S(X) = -X - Y \hat{A}^{-1}, S(Y) = -Y \). The corresponding Lie bialgebra is

\[
[\hat{A}, X] = \hat{A}, \quad [\hat{A}, Y] = 0, \quad [X,Y] = dY ,
\]
\[
\delta(\hat{A}) = 0, \quad \delta(X) = Y \wedge \hat{A}, \quad \delta(Y) = 0 .
\]

When \( d = -1 \), we have found non-trivial cocycles, which are infinitesimally given by \( \mathcal{U}(X,Y) = -\lambda A \). For the Hopf \(*\)-algebra, there changes \([X,Y] = -Y - i \lambda \hat{A} \) and on the Lie bialgebra level, this becomes \([X,Y] = -Y - \lambda \hat{A} \). Co-unit and antipode remain unchanged.

The dual Hopf \(*\)-algebra for \( d \neq -1 \) has anti-self-adjoint generators \( \hat{X}, \hat{Y} \) and a self-adjoint generator \( A \), subject to the relations

\[
[X, A] = 0, \quad [Y, A] = u_d(\hat{X}), \quad [\hat{X}, \hat{Y}] = 0 ,
\]
\[
\Delta(\hat{X}) = \hat{X} \otimes \hat{X},
\]
\[
\Delta(\hat{Y}) = \hat{X}^d \otimes \hat{Y} + \hat{Y} \otimes 1 ,
\]
\[
\Delta(A) = A \otimes \hat{X} + 1 \otimes A .
\]

Co-unit and antipode are \( \varepsilon(\hat{X}) = 1, \varepsilon(\hat{Y}) = \varepsilon(A) = 0 \), \( S(\hat{X}) = \hat{X}^{-1}, S(\hat{Y}) = -\hat{X}^{-d} \hat{Y}, S(A) = -A \hat{X}^{-1} \). The corresponding infinitesimal Lie bialgebra is determined by

\[
[X, A] = 0, \quad [Y, A] = \hat{X}, \quad [\hat{X}, \hat{Y}] = 0 ,
\]
\[
\delta(\hat{X}) = 0 , \quad \delta(\hat{Y}) = dX \wedge \hat{Y}, \quad \delta(A) = A \wedge \hat{X} .
\]

In the case \( d = -1 \) and with the same cocycle as above, we should change \( \Delta(A) = A \otimes \hat{X} + 1 \otimes A + i \lambda X \hat{Y} \otimes X \log \hat{X} \) in the definition of the Hopf \(*\)-algebra and \( \delta(A) = A \wedge \hat{X} - \lambda X \wedge \hat{Y} \) in the definition of the Lie bialgebra. For the antipode, we change \( S(A) = -A \hat{X}^{-1} + i \lambda \hat{Y} \log \hat{X} \).

Case 4.3 \( (a < 0) \). For the exponentiation given in Equation (5.3), we have natural generators \( X,Y \) and \( A \), giving the infinitesimal form \([X,Y] = Y\),...
χ(X) = −1, χ(Y) = 0, β(X) = −X and β(Y) = 2X + Y. There are cocycles, infinitesimally given by \( U(X, Y) = −\lambda \tilde{A} \). The Hopf \(*\)-algebra has generators \( \tilde{A} = \tilde{A}^* \) and \( X, Y \), anti-self-adjoint, satisfying

\[
[\tilde{A}, X] = 1 − \tilde{A}, \quad [\tilde{A}, Y] = −(1 − \tilde{A})^2, \quad [X, Y] = Y − i\lambda \log \tilde{A},
\]

\[
\Delta(\tilde{A}) = \tilde{A} \otimes \tilde{A},
\]

\[
\Delta(X) = X \otimes \tilde{A}^{-1} + 1 \otimes X, \quad \Delta(Y) = Y \otimes \tilde{A} + 1 \otimes Y + X \otimes (\tilde{A} − \tilde{A}^{-1}).
\]

Co-unit and antipode are \( \varepsilon(\tilde{A}) = 1, \varepsilon(X) = \varepsilon(Y) = 0, S(\tilde{A}) = \tilde{A}^{-1}, S(X) = −X\tilde{A}, S(Y) = −Y\tilde{A}^{-1} + X(\tilde{A} − \tilde{A}^{-1}) \). The corresponding Lie bialgebra is

\[
[\tilde{A}, X] = −\tilde{A}, \quad [\tilde{A}, Y] = 0, \quad [X, Y] = Y − \lambda \tilde{A}, \quad \delta(\tilde{A}) = 0, \quad \delta(X) = \tilde{A} \wedge X, \quad \delta(Y) = (2X + Y) \wedge \tilde{A}.
\]

The dual Hopf \(*\)-algebra has self-adjoint generators \( \tilde{X}, \tilde{Y} \) and an anti-self-adjoint generator \( \tilde{A} \), subject to the relations

\[
[\tilde{X}, \tilde{A}] = 1 − \tilde{X} + 2\tilde{Y}, \quad [\tilde{Y}, \tilde{A}] = \tilde{X}^{-1}\tilde{Y}(1 + \tilde{Y}), \quad [\tilde{X}, \tilde{Y}] = 0, \quad \Delta(\tilde{X}) = \tilde{X} \otimes \tilde{X}, \quad \Delta(\tilde{Y}) = \tilde{X} \otimes \tilde{Y} + \tilde{Y} \otimes 1, \quad \Delta(\tilde{A}) = \tilde{A} \otimes \tilde{X}^{-1} + 1 \otimes \tilde{A} + i\lambda \log \tilde{X} \otimes \tilde{X}^{-1}\tilde{Y}.
\]

Co-unit and antipode are \( \varepsilon(\tilde{X}) = 1, \varepsilon(\tilde{Y}) = \varepsilon(\tilde{A}) = 0, S(\tilde{X}) = \tilde{X}^{-1}, S(\tilde{Y}) = −\tilde{X}^{-1}\tilde{Y}, S(\tilde{A}) = −\tilde{A}\tilde{X} + i\lambda \tilde{Y} \log \tilde{X} \). The corresponding Lie bialgebra is determined by the equations

\[
[\tilde{X}, \tilde{A}] = −\tilde{X} + 2\tilde{Y}, \quad [\tilde{Y}, \tilde{A}] = \tilde{Y}, \quad [\tilde{X}, \tilde{Y}] = 0, \quad \delta(\tilde{X}) = 0, \quad \delta(\tilde{Y}) = \tilde{X} \wedge \tilde{Y}, \quad \delta(\tilde{A}) = \tilde{X} \wedge \tilde{A} − \lambda \tilde{X} \wedge \tilde{Y}.
\]

For the exponentiation of the case 4.3 (\( a = 0 \)) described in Equation (5.8) with infinitesimal form \([X, Y] = 2Y, \chi(X) = −2, \chi(Y) = 0, \beta(X) = 0 \) and \( \beta(Y) = X \), there are non-trivial cocycles determined by \( U(X, Y) = −\lambda \tilde{A} \), giving rise to the following Hopf \(*\)-algebra, with a self-adjoint generator \( \tilde{A} \) and anti-self-adjoint generators \( X, Y \), subject to the relations

\[
[\tilde{A}, X] = −2\tilde{A}, \quad [\tilde{A}, Y] = −\tilde{A}^2, \quad [X, Y] = 2Y − i\lambda \tilde{A}, \quad \Delta(\tilde{A}) = \tilde{A} \otimes 1 + 1 \otimes \tilde{A}, \quad \Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(Y) = Y \otimes 1 + X \otimes \tilde{A} + 1 \otimes Y.
\]
Co-unit and antipode are $\varepsilon(\tilde{A}) = \varepsilon(X) = \varepsilon(Y) = 0, S(\tilde{A}) = -\tilde{A}, S(X) = -X, S(Y) = -Y + X\tilde{A}$. The corresponding Lie bialgebra is determined by

$$[\tilde{A}, X] = -2\tilde{A}, \quad [\tilde{A}, Y] = 0, \quad [X, Y] = 2Y - \lambda\tilde{A},$$

$$\delta(\tilde{A}) = 0, \quad \delta(X) = 0, \quad \delta(Y) = X \wedge \tilde{A}.$$ 

The dual Hopf *-algebra has self-adjoint generators $\tilde{X}, \tilde{Y}$ and an anti-self-adjoint generator $A$, subject to the relations

$$[\tilde{X}, A] = \tilde{Y}, \quad [\tilde{Y}, A] = 0, \quad [\tilde{X}, \tilde{Y}] = 0,$$

$$\Delta(\tilde{X}) = \tilde{X} \otimes \tilde{X}, \quad \Delta(\tilde{Y}) = \tilde{X} \otimes \tilde{Y} + \tilde{Y} \otimes \tilde{X}^{-1}, \quad \Delta(A) = A \otimes \tilde{X}^{-2} + 1 \otimes A + i\lambda \tilde{X}^{-1}\tilde{Y} \otimes \tilde{X}^{-2} \log \tilde{X}.$$

Co-unit and antipode are $\varepsilon(\tilde{X}) = 1, \varepsilon(\tilde{Y}) = \varepsilon(A) = 0, S(\tilde{X}) = \tilde{X}^{-1}, S(\tilde{Y}) = -\tilde{Y}, S(A) = -AX^2 + i\lambda XY \log \tilde{X}$. The corresponding Lie bialgebra is determined by

$$[X, A] = Y, \quad [Y, A] = 0, \quad [\tilde{X}, \tilde{Y}] = 0,$$

$$\delta(X) = 0, \quad \delta(Y) = 2X \wedge \tilde{Y}, \quad \delta(A) = -2A \wedge \tilde{X} + \lambda \tilde{Y} \wedge \tilde{X}.$$ 

Finally, to treat case 4.3 ($a < 0$), we use the modification of the exponentiation as in Section 5 with the usage of the universal covering Lie group. This gives $[X, Y] = Y, \chi(X) = -1, \chi(Y) = 0, \beta(X) = -Y$ and $\beta(Y) = X$. There are non-trivial cocycles, given by $\mathcal{U}(X, Y) = -\lambda\tilde{A}$. Hence, we get a Hopf *-algebra with generators $\tilde{A} = A^*, X = -X^*$ and $Y = -Y^*$, subject to the relations

$$[\tilde{A}, X] = -\sin \tilde{A}, \quad [\tilde{A}, Y] = -1 + \cos \tilde{A}, \quad [X, Y] = Y - i\lambda\tilde{A},$$

$$\Delta(\tilde{A}) = \tilde{A} \otimes 1 + 1 \otimes \tilde{A}, \quad \Delta(X) = X \otimes \cos \tilde{A} - Y \otimes \sin \tilde{A} + 1 \otimes X, \quad \Delta(Y) = X \otimes \sin \tilde{A} + Y \otimes \cos \tilde{A} + 1 \otimes Y.$$

Co-unit and antipode are $\varepsilon(\tilde{A}) = \varepsilon(X) = \varepsilon(Y) = 0, S(\tilde{A}) = -\tilde{A}, S(X) = -X \cos \tilde{A} - Y \sin \tilde{A}, S(Y) = X \sin \tilde{A} - Y \cos \tilde{A}$. The corresponding Lie bialgebra is

$$[\tilde{A}, X] = -\tilde{A}, \quad [\tilde{A}, Y] = 0, \quad [X, Y] = Y - \lambda\tilde{A},$$

$$\delta(\tilde{A}) = 0, \quad \delta(X) = \tilde{A} \wedge Y, \quad \delta(Y) = X \wedge \tilde{A}.$$
The dual Hopf $^\ast$-algebra has self-adjoint generators $\tilde{X}, \tilde{Y}$ and the generator $A = -A^\ast$, subject to the relations

$[\tilde{X}, A] = \tilde{Y}$, $[\tilde{Y}, A] = \frac{1}{2}(\tilde{X}^{-1}\tilde{Y}^2 - \tilde{X} + \tilde{X}^{-1})$, $[\tilde{X}, \tilde{Y}] = 0$,

$\Delta(\tilde{X}) = \tilde{X} \otimes \tilde{X}$, $\Delta(\tilde{Y}) = \tilde{X} \otimes \tilde{Y} + \tilde{Y} \otimes 1$, $\Delta(A) = A \otimes \tilde{X}^{-1} + 1 \otimes A - i\lambda \log \tilde{X} \otimes \tilde{X}^{-1} \tilde{Y}$.

Co-unit and antipode are $\varepsilon(\tilde{X}) = 1$, $\varepsilon(\tilde{Y}) = \varepsilon(A) = 0$, $S(\tilde{X}) = \tilde{X}^{-1}$, $S(\tilde{Y}) = -\tilde{X}^{-1}\tilde{Y}$, $S(A) = -A\tilde{X} - i\lambda \tilde{Y} \log \tilde{X}$. The corresponding Lie bialgebra is determined by

$[\tilde{X}, A] = \tilde{Y}$, $[\tilde{Y}, A] = -\tilde{X}$, $[\tilde{X}, \tilde{Y}] = 0$,

$\delta(\tilde{X}) = 0$, $\delta(\tilde{Y}) = \tilde{X} \wedge \tilde{Y}$, $\delta(A) = \tilde{X} \wedge A - \lambda \tilde{X} \wedge \tilde{Y}$.

The above list of Lie bialgebras $\mathfrak{g}$ covers all decomposable Lie bialgebras of dimension 3, i.e., those that are extensions of the form

$\mathfrak{g}_2 \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_1$.

Naturally, their duals are extensions of the form

$\mathfrak{g}_1^\ast \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_2$.

Here $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the Lie algebras forming the cocycle matched pair.

The case of indecomposable low-dimensional l.c. quantum groups

Comparing the above list with the full list of all mutually non-isomorphic Lie bialgebras $\mathfrak{g}$ of dimension 3 obtained by X. Gomez [15], one can observe that there are also indecomposable Lie bialgebras of dimension 3. Below we list them and describe the corresponding Hopf $^\ast$-algebras and, when they are known, the corresponding locally compact quantum groups.

We start with the Hopf $^\ast$-algebras $U_q(\mathfrak{su}_2)$, $U_q(\mathfrak{su}_{1,1})$, $U_q(\mathfrak{sl}_2(\mathbb{R}))$, which are real forms of the Hopf algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$, and their duals $SU_q(2)$, $SU_q(1,1)$ and $SL_q(2,\mathbb{R})$. Hopf algebraically, they are studied by several authors, see e.g. [13]. Operator algebraically, $SU_q(2)$ and its dual are studied in [52, 61, 62] as an example of a compact (dually, discrete) quantum group. $SL_q(2,\mathbb{R})$ is studied as a l.c. quantum group recently by W. Pusz and S.L. Woronowicz, see [42] for a related example. $SU_q(1,1)$ and its dual is treated as a l.c. quantum group in [24, 25, 65].
The Hopf algebra $U_q(sl_2(\mathbb{C}))$ is defined by 3 generators $a, x, y$ and the following relations ($q$ is a complex parameter):

$$ax = qx a, \quad ay = q^{-1}ya, \quad [x, y] = \frac{1}{q - q^{-1}}(a^2 - a^{-2}),$$

$$\Delta(a) = a \otimes a,$$

$$\Delta(x) = x \otimes a + a^{-1} \otimes x,$$

$$\Delta(y) = y \otimes a + a^{-1} \otimes y.$$

It is well known that there exist three different Hopf $^*$-algebra structures on $U_q(sl_2(\mathbb{C}))$: we get $U_q(su_2)$ by putting $q > 0$, $q \neq 1$, $a = a^*$ and $x = y^*$, we get $U_q(su_{1,1})$ for the same values of $q$, $a = a^*$ and $x = -y^*$ and we finally get $U_q(sl_2(\mathbb{R}))$ for $|q| = 1$, $q \neq \pm 1$, $a = a^*$, $x^* = -x$, $y^* = -y$.

One can construct the corresponding Lie bialgebras using the above mentioned formal procedure of linearization. For $U_q(su_2)$, we put $H = -\frac{1}{i \log q} \log a$, $X = i(x+y)$ and $Y = x - y$, and we arrive at the Lie bialgebra

$$[H, X] = Y, \quad [H, Y] = -X, \quad [X, Y] = \frac{8 \log q}{q - q^{-1}} H,$$

$$\delta(H) = 0, \quad \delta(X) = 2 \log q H \wedge X, \quad \delta(Y) = 2 \log q H \wedge Y.$$

For $U_q(su_{1,1})$, we put $H = -\frac{1}{i \log q} \log a$, $X = x + y$ and $Y = i(x-y)$ to obtain

$$[H, X] = -Y, \quad [H, Y] = X, \quad [X, Y] = \frac{8 \log q}{q - q^{-1}} H,$$

$$\delta(H) = 0, \quad \delta(X) = 2 \log q H \wedge X, \quad \delta(Y) = 2 \log q H \wedge Y.$$

Finally, for $U_q(sl_2)$, we put $q = \exp(ir)$, $H = -i\frac{2}{r} \log a$, $X = x$ and $Y = y$ to arrive at

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = \frac{r}{\sin r} H,$$

$$\delta(H) = 0, \quad \delta(X) = rH \wedge X, \quad \delta(Y) = rH \wedge Y.$$

For the dual Hopf $^*$-algebras, we take the following versions. $SU_q(2)$ has generators $a, a^*, b, b^*$, parameter $q > 0$ and relations

$$ab = qba, \quad ab^* = qb^*a, \quad bb^* = b^*b,$$

$$aa^* - a^*a = (q^2 - 1)b b^*, \quad aa^* + bb^* = 1,$$

$$\Delta(a) = a \otimes a - q^{-1}b \otimes b^*,$$

$$\Delta(b) = a \otimes b + b \otimes a^*.$$

Suppose $a = \exp(A)$. We first formally calculate that

$$[A, b] = \log q \ b, \quad [A^*, b] = - \log q \ b.$$
So, we put $H = A - A^*$, $X = i(b + b^*)$ and $Y = b - b^*$ to obtain the linearization

\[
[H, X] = 2 \log qX, \quad [H, Y] = 2 \log qY, \quad [X, Y] = 0, \\
\delta(H) = q^{-1}X \wedge Y, \quad \delta(X) = Y \wedge H, \quad \delta(Y) = H \wedge X.
\]

Next, we write $SL_q(2, \mathbb{R})$ with $|q| = 1$ and self-adjoint generators $a, b, c, d$, subject to the relations

\[
ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \\
b = cb, \quad [a, d] = (q - q^{-1})bc, \quad ad - qbc = 1, \\
\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \\
\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d.
\]

Again, writing $a = \exp(A), q = \exp(ir), H = iA, X = ib$ and $Y = ic$, we get the Lie bialgebra

\[
[H, X] = -rX, \quad [H, Y] = -rY, \quad [X, Y] = 0, \\
\delta(H) = X \wedge Y, \quad \delta(X) = 2H \wedge X, \quad \delta(Y) = 2Y \wedge H.
\]

Finally, we present $SU_q(1, 1)$ with $q > 0$, generators $a, b$ and relations

\[
ab = qba, \quad ab^* = qba, \quad bb^* = b^*b, \\
aa^*-a^*a = (1-q^{-2})bb^*, \quad aa^*-bb^* = 1, \\
\Delta(a) = a \otimes a + q^{-1}b \otimes b^*, \\
\Delta(b) = a \otimes b + b \otimes a^*.
\]

Following the same road as for $SU_q(2)$, we get the Lie bialgebra

\[
[H, X] = 2 \log qX, \quad [H, Y] = 2 \log qY, \quad [X, Y] = 0, \\
\delta(H) = q^{-1}Y \wedge X, \quad \delta(X) = Y \wedge H, \quad \delta(Y) = H \wedge X.
\]

The Hopf *-algebras corresponding to the l.c. quantum group of motions of the plane and its dual was considered in e.g. [22]. They are treated as l.c. quantum groups in [3], [56], [57] and [63].

Take $\mu > 0$ and consider the Hopf algebra defined by

\[
ax = \mu xa, \quad \Delta(a) = a \otimes a, \quad \Delta(x) = a \otimes x + x \otimes a^{-1}.
\]

We can put two different Hopf *-algebra structures. First, we get $U_\mu(e_2)$ by taking $a$ self-adjoint and $x$ normal. Next, we get $E_\mu(2)$ by supposing that $a$ is unitary and $x$ is normal. We linearize $U_\mu(e_2)$ by writing $H = i \log a, X = i(x + x^*)$ and $Y = x - x^*$. This gives us the Lie bialgebra

\[
[H, X] = -\log \mu Y, \quad [H, Y] = \log \mu X, \quad [X, Y] = 0, \\
\delta(H) = 0, \quad \delta(X) = 2H \wedge X, \quad \delta(Y) = 2H \wedge Y.
\]
For $E_\mu(2)$, we write $H = \log a$ (which is indeed anti-self-adjoint), $X = i(x + x^*)$ and $Y = x - x^*$, to arrive at the Lie bialgebra

$$
[H, X] = \log \mu X, \quad [H, Y] = \log \mu Y, \quad [X, Y] = 0, \\
\delta(H) = 0, \quad \delta(X) = 2Y \land H, \quad \delta(Y) = 2H \land X.
$$

Observe that the list of 3-dimensional Lie bialgebras [15] contains some more objects, and we now want to present the corresponding Hopf $^*$-algebras, which are less known. As far as we know, they have not yet been considered on the level of l.c. quantum groups.

Let $\mu \in \mathbb{C}$, $\mu \neq 0$ and let $\rho > 0$. Put $\lambda = -\frac{\log \rho}{\mu}$. Then, we can define a Hopf $^*$-algebra with relations

$$
[a, x] = \mu x, \quad xx^* = \rho x^* x, \quad a = a^*, \\
\Delta(a) = a \ot 1 + 1 \ot a, \\
\Delta(x) = x \ot \exp(\lambda a) + 1 \ot x.
$$

Co-unit and antipode are given by $\varepsilon(a) = \varepsilon(x) = 0, S(a) = -a, S(x) = -x \exp(-\lambda a)$. The specific form of $\lambda$ is needed to ensure that $\Delta$ respects the relation $xx^* = \rho x^* x$. Then, putting $H = i a$, $X = i(x + x^*)$ and $Y = x - x^*$, and observing that $X$ and $Y$ commute in a first order approximation, we get the corresponding Lie bialgebra

$$
[H, X] = -\text{Im} \mu X - \text{Re} \mu Y, \quad [H, Y] = \text{Re} \mu X - \text{Im} \mu Y, \quad [X, Y] = 0, \\
\delta(H) = 0, \quad \delta(X) = (\text{Re} \lambda X - \text{Im} \lambda Y) \land H, \quad \delta(Y) = (\text{Im} \lambda X + \text{Re} \lambda Y) \land H.
$$

One can check that $\delta$ respects the relation $[X, Y] = 0$, because $\text{Im}(\lambda \mu) = 0$. Also, one can check that this family of Lie bialgebras is self-dual, i.e., the dual of any Lie bialgebra with specific values of $\mu$ and $\rho$ belongs again to this family (but with different values of $\mu$ and $\rho$). So, the dual Hopf $^*$-algebras are of the same form as above.

Next, we take real numbers $\alpha$ and $\beta$, and we write the Hopf $^*$-algebra with self-adjoint generators $a, x, y$ and relations:

$$
[a, x] = -ix, \quad [a, y] = -i\alpha y, \quad xy = \exp(-i\alpha \beta)yx, \\
\Delta(a) = a \ot 1 + 1 \ot a, \\
\Delta(x) = x \ot \exp(\beta a) + 1 \ot x, \\
\Delta(y) = y \ot \exp(-\alpha \beta a) + 1 \ot y.
$$

Co-unit and antipode are given by $S(x) = -x \exp(-\beta a), S(y) = -y \exp(\alpha \beta a)$, $\varepsilon(a) = \varepsilon(x) = \varepsilon(y) = 0$. To linearize, we write $H = ia$, $X = ix$ and $Y = iy$. Observing again that $X$ and $Y$ commute in a first order approximation, we
obtain the corresponding Lie bialgebra

\[ [H, X] = X, \quad [H, Y] = \alpha Y, \quad [X, Y] = 0, \]
\[ \delta(H) = 0, \quad \delta(X) = \beta X \land H, \quad \delta(Y) = \alpha \beta H \land Y. \]

In the same sense as in the previous paragraph, this family of Lie bialgebras is self-dual, so the dual Hopf ∗-algebras are of the same form.

Finally, there is one isolated Lie bialgebra, which is defined by

\[ [H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H, \]
\[ \delta(H) = H \land Y, \quad \delta(X) = X \land Y, \quad \delta(Y) = 0. \]

We can write the following Hopf ∗-algebra, which appears in [9], Section 6.4.F and which has generators \( h^* = -h, \ x = x^*, \ y = y^* \) and relations

\[ [h, x] = 2x - \frac{1}{2} h^2, \quad [h, y] = 2(1 - \exp(y)), \quad [x, y] = h, \]
\[ \Delta(h) = h \otimes \exp(y) + 1 \otimes h, \]
\[ \Delta(x) = x \otimes \exp(y) + 1 \otimes x, \]
\[ \Delta(y) = y \otimes 1 + 1 \otimes y. \]

One can check that \( [x, \exp(y)] = \exp(y)h + \exp(y)(1 - \exp(y)), \) so taking \( H = h, \ X = -ix \) and \( Y = iy, \) and linearizing we get the above Lie bialgebra. For the dual Lie bialgebra, we cannot construct at the moment a corresponding Hopf ∗-algebra. The main problem to construct this exponentiation is the fact that the dual Lie bialgebra has no non-trivial Lie sub-bialgebra.

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