PARTIAL SPREADS IN RANDOM NETWORK CODING

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Abstract. Following the approach by R. Kötter and F. R. Kschischang, we study network codes as families of $k$-dimensional linear subspaces of a vector space $\mathbb{F}_q^n$, $q$ being a prime power and $\mathbb{F}_q$ the finite field with $q$ elements. In particular, following an idea in finite projective geometry, we introduce a class of network codes which we call partial spread codes. Partial spread codes naturally generalize spread codes. In this paper we provide an easy description of such codes in terms of matrices, discuss their maximality, and provide an efficient decoding algorithm.

0. Introduction

The topology of a network is well-modeled by a directed multigraph. Vertices without incoming edges play the role of sources and vertices without outgoing edges play the role of sinks. Vertices which are neither sources nor sinks are called nodes. The interest in network modeling is due to its several applications in technology (distributed storage, peer-to-peer networking and, in particular, wireless communications).

In \cite{1} Ahlswede, Cai, Li, and Yeung discovered that the information rate may be improved by employing coding at the nodes of a network (instead of simply routing). Moreover, Li, Cai and Yeung proved in \cite{14} that, in a multicasting situation, maximal information rate can be achieved by allowing the nodes to transmit linear combinations of the inputs they receive, provided that the size of the base field is large enough.

A turning point in the study of linear network codes was the paper \cite{12} by R. Kötter and F. R. Kschischang. The authors suggested an algebraic approach to the topic, developing a clear and rigorous mathematical setup. Interesting connections with classical projective geometry also emerged. Several other interesting papers followed the same approach, e.g., \cite{6}, \cite{7}, and \cite{13}.

In this paper, we propose and study a class of network codes, which fit within the same framework. In Section \textsuperscript{1} the algebraic approach by Kötter and Kschischang is briefly recalled. In Section \textsuperscript{2} we introduce a family of network codes which we call partial spread codes, and which generalize spread codes (see \cite{16}). Our codes have the same cardinality and distance distribution as the codes proposed in \cite{8}. The elements of our codes however are given as rowspaces of appropriate matrices in block form. The structure of this family of matrices allow us to derive properties of the code, which we discuss in Section \textsuperscript{3}. In particular, we establish the maximality of partial spread codes with respect to containment. Based on the same block matrix structure, in Section \textsuperscript{4} we are able to give an efficient decoding algorithm.

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2010 Mathematics Subject Classification. 11T71.

Key words and phrases. network code, spread code, subspace distance.
1. Preliminaries

Let $q$ be a prime power and let $\mathbb{F}_q$ denote the finite field with $q$ elements. Fix an integer $n > 1$ and let $\mathcal{P}(\mathbb{F}_q^n)$ be the projective geometry of $\mathbb{F}_q^n$ i.e., the set of all the vector subspaces of $\mathbb{F}_q^n$. Following [12], a $q$-ary network code of length $n$ is defined to be a subset $\mathcal{C} \subseteq \mathcal{P}(\mathbb{F}_q^n)$ with at least two elements. The subspace distance on $\mathcal{P}(\mathbb{F}_q^n)$ is the distance map $d : \mathcal{P}(\mathbb{F}_q^n) \times \mathcal{P}(\mathbb{F}_q^n) \to \mathbb{N}$ defined, for any $U, V \in \mathcal{P}(\mathbb{F}_q^n)$, by

$$d(U, V) := \dim(U) + \dim(V) - 2\dim(U \cap V).$$

As in classical Coding Theory, the minimum distance of a network-code $\mathcal{C} \subseteq \mathcal{P}(\mathbb{F}_q^n)$ is the integer $d(\mathcal{C}) := \min\{d(U, V) : U, V \in \mathcal{C}, U \neq V\}$. The maximum dimension of $\mathcal{C}$ is denoted and defined by $\ell(\mathcal{C}) := \max_{V \in \mathcal{C}} \dim(V)$. Let us briefly recall from [12] the framework for errors and erasures in random network coding. If $1 \leq e < n$ is an integer, then an $e$-erasure on an element $V \in \mathcal{P}(\mathbb{F}_q^n)$ such that $\dim(V) \geq e$ is the projection of $V$ onto an $e$-dimensional subspace of $V$. In other words, an $e$-erasure replaces $V$ with an $e$-dimensional subspace of $V$. A $t$-dimensional error $E$ on an element $V \in \mathcal{P}(\mathbb{F}_q^n)$ corresponds to the direct sum $V \oplus E$, where $E \in \mathcal{P}(\mathbb{F}_q^n)$, $\dim(E) = t$ and $E \cap V = \{0\}$. If $\mathcal{C} \subseteq \mathcal{P}(\mathbb{F}_q^n)$ is a network code, then an input codeword $V \in \mathcal{C}$ and its output $U \in \mathcal{P}(\mathbb{F}_q^n)$ are related by $U = \mathcal{H}_e(V) \oplus E$, where $1 \leq e \leq \dim(V)$, $\mathcal{H}_e$ is an $e$-erasure operator and $E \in \mathcal{P}(\mathbb{F}_q^n)$ the error. As usual, one can bound the number of erasures and errors that can take place such that a minimum distance decoder is guaranteed to successfully return the sent codeword.

**Theorem 1** ([12], Theorem 2). Let $\mathcal{C} \subseteq \mathcal{P}(\mathbb{F}_q^n)$ be a network-code of minimum distance $d$. Assume that an input $V \in \mathcal{C}$ and its output $U \in \mathcal{P}(\mathbb{F}_q^n)$ are related by $U = \mathcal{H}_e(V) \oplus E$, where $e \leq \ell(\mathcal{C})$, $\mathcal{H}_e$ is an $e$-erasure and $E \in \mathcal{P}(\mathbb{F}_q^n)$ is an error. Set $t := \dim(E)$. A minimum distance decoder corrects $U$ in $V$, provided that $2(t + \ell(\mathcal{C}) - e) < d$.

A natural class of network codes is obtained by considering subsets of $\mathcal{P}(\mathbb{F}_q^n)$, all of whose elements have the same dimension $1 \leq k \leq n - 1$. Such codes are called constant dimension codes. By introducing the Grassmannian variety

$$\mathcal{G}_q(k, n) := \{V \in \mathcal{P}(\mathbb{F}_q^n) : \dim(V) = k\},$$

a $q$-ary constant dimension network code of length $n$ and dimension $k$ is simply a subset $\mathcal{C} \subseteq \mathcal{G}_q(k, n)$ of at least two elements. It easily follows from the definition that any constant dimension network code has even minimum distance.

**Remark 2.** The cardinality of the Grassmannian variety $\mathcal{G}_q(k, n)$ is known to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)} = \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1}.$$

Bounds on the size of network codes have been discussed in depth by R. Kötter and F. R. Kschischang in [12]. More recently, in [8], T. Etzion and A. Vardy obtained other bounds.

**Theorem 3** (Singleton-like Bound, [12], Theorem 9). Let $\mathcal{C} \subseteq \mathcal{G}_q(k, n)$ be a network code of minimum distance $d$. Then

$$|\mathcal{C}| \leq \left\lfloor \frac{n - (d - 2)/2}{\max(k, n - k)} \right\rfloor_q.$$

The family of spread codes has been introduced in [16], and an efficient decoding algorithm for such codes has been provided in [9].

**Definition 4.** A $k$-spread of $\mathbb{F}_q^n$ is a collection of subspaces $(V_i)_{i=1}^t$ of $\mathbb{F}_q^n$ (here we take $k < n$) such that
(1) \(\dim V_i = \dim V_j = k\) for any \(i, j \in \{1, \ldots, t\}\),
(2) \(V_i \cap V_j = \{0\}\) whenever \(i \neq j\),
(3) \(F_q^n = \bigcup_{i=1}^t V_i\).

**Remark 5.** A \(k\)-spread of \(F_q^n\) exists if and only if \(k\) divides \(n\) (see [11], Corollary 4.17). From the definition we see that if \(\{V_i\}_{i=1}^t\) is a \(k\)-spread of \(F_q^n\) then \(t = (q^n - 1)/(q^k - 1)\). Being a subset of the Grassmannian \(G_q(k, n)\), a \(k\)-spread in \(F_q^n\) is a \(q\)-ary network code of length \(n\), dimension \(k\) and minimum distance \(2k\). It is easily checked that spread codes meet the Singleton-like bound (Theorem 3).

## 2. Partial spread codes

In this section we introduce a generalization of the definition of spread and a related family of network codes, whose parameters \(k\) and \(n\) can be chosen freely.

**Definition 6.** A partial \(k\)-spread of \(F_q^n\) is a subset \(C \subseteq G_q(k, n)\) such that \(U \cap V = \{0\}\) for any \(U, V \in C\) with \(U \neq V\). A partial \(k\)-spread of \(F_q^n\) with at least two elements is a \(q\)-ary network code of length \(n\), dimension \(k\) and minimum distance \(2k\). We will call such a code a partial spread code.

**Lemma 7.** Let \(C \subseteq G_q(k, n)\) be a partial spread code. Denote by \(r\) the remainder obtained dividing \(n\) by \(k\). Then

\[|C| \leq q^n - q^r \left\lfloor \frac{q^r}{q^k-1} \right\rfloor.\]

**Proof.** Since \(C\) is a set of \(k\)-dimensional vector subspaces of \(F_q^n\) with trivial pairwise intersections, we deduce \(|C| \cdot (q^k - 1) + 1 \leq q^n\). Since \(k\) divides \(n - r\), \((q^{n-r} - 1)/(q^k - 1)\) is an integer. Hence

\[|C| \leq \frac{q^n - 1}{q^k - 1} = \frac{q^r(q^{n-r} - 1)}{q^k - 1} + \frac{q^r - 1}{q^k - 1} = \frac{q^n - q^r}{q^k - 1}.\]

The bound given in Lemma 7 admits some non-trivial improvements. See [3] and [4] for details. The following lower bound for partial \(k\)-spread in \(F_q^n\) is due to A. Beutelspacher (see [2] for a non-constructive proof).

**Lemma 8.** Let \(q\) be a prime power and let \(1 \leq k < n\) be integers. Write \(n = hk + r\) with \(0 \leq r \leq k - 1\). Denote by \(\mathcal{A}_q(k, n, 2k)\) the largest possible size of a network code \(C \subseteq G_q(k, n)\) of minimum distance \(2k\). Then

\[\mathcal{A}_q(n, k, 2k) \geq (q^n - q^r)/(q^k - 1) - q^r + 1.\]

**Remark 9.** An alternative proof of Lemma 8 is given in [8], Theorem 11. For interesting discussions on the sharpness of the bound see [5] and [10].

Here we introduce a construction for partial spread codes whose size attains the lower bound of Lemma 8. Notice that the vector spaces of the partial spread are given as row spaces of appropriate easy-computable matrices.

**Lemma 10** ([15], Ch. 2.5). Let \(q\) be a prime power and let \(F_q\) be the finite field with \(q\) elements. Choose an irreducible monic polynomial \(p \in F_q[x]\) of degree \(k \geq 1\) and write \(p = \sum_{i=0}^{k-1} p_i x^i\). Define
Since following three forms:

\[ M := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{k-1} \end{bmatrix} \]

The \( \mathbb{F}_q \)-algebra \( \mathbb{F}_q[P] \) is a finite field with \( q^k \) elements.

**Notation 11.** Let \( V \) be a vector space over a field \( \mathbb{F} \) and let \( S \subseteq V \) be any subset. The vector space generated by \( S \), i.e., the smallest vector subspace of \( V \) containing \( S \), is denoted by \( \langle S \rangle \). We always have \( \dim \mathbb{F}(S) \leq |S| \).

**Lemma 12.** Let \( V \) be a finite-dimensional vector space over a field \( \mathbb{F} \). Let \( D \subseteq V \) be any subset and set \( d := \dim \mathbb{F}(D) \). Choose a finite subset \( S \subseteq D \). Then \( \dim \mathbb{F}(D \setminus S) \geq d - |S| \).

**Proof.** Since \( D = \langle D \setminus S \rangle \cup S \) we have \( \langle D \setminus S \rangle + \langle S \rangle \subseteq \langle D \setminus S \rangle \cup \langle S \rangle = \langle D \rangle \). It follows

\[ \dim \mathbb{F}(D \setminus S) + \dim \mathbb{F}(S) \geq d + \dim \mathbb{F}(D \setminus S) \cap \langle S \rangle. \]

Since \( \dim \mathbb{F}(S) \leq |S| \) we conclude \( \dim \mathbb{F}(D \setminus S) + |S| \geq d \). \( \square \)

**Theorem 13.** Let \( q \) be a prime power and let \( \mathbb{F}_q \) be the finite field with \( q \) elements. Choose integers \( 1 \leq k < n \) and write \( n = hk + r \) with \( 0 \leq r \leq k - 1 \). Assume \( h \geq 2 \). Let \( p, p' \in \mathbb{F}_q[x] \) be two irreducible monic polynomials of degree \( k \) and \( k + r \) respectively, and let \( P := M(p), P' := M(p') \) be their companion matrices. For any \( 1 \leq i \leq h - 1 \) set

\[ \mathcal{M}_i(p, p') := \{ 0_k \cdots 0_k I_k A_{i+1} \cdots A_{h-1} A(k) : A_{i+1}, \ldots, A_{h-1} \in \mathbb{F}_q[P], A \in \mathbb{F}_q[P'] \}, \]

where \( 0_k \) is the \( k \times k \) matrix with zero entries, \( I_k \) is the \( k \times k \) identity matrix, and \( A(k) \) denotes the last \( k \) rows of \( A \). The set

\[ \mathcal{C} := \bigcup_{i=1}^{h-1} \{ \text{rowsp}(M) : M \in \mathcal{M}_i(p, p') \} \cup \{ \text{rowsp}[0_k \cdots 0_k 0_{k \times r} I_k] \} \]

is a partial spread code in \( \mathbb{F}_q^n \) of dimension \( k \). In particular, the minimum distance of \( \mathcal{C} \) is \( 2k \).

**Proof.** Choose matrices \( M_1 \neq M_2 \in \mathcal{M}_i(p, p') \) and set \( V_1 := \text{rowsp}(M_1), V_2 := \text{rowsp}(M_2) \). Since by definition \( d(V_1, V_2) = 2k - 2 \dim(V_1 \cap V_2) \), we have \( d(V_1, V_2) = 2k \) if and only if

\[ \mathrm{rk} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = 2k. \]

Since \( M_1 \neq M_2 \), it is possible to find either in \( \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \), or in \( \begin{bmatrix} M_2 \\ M_1 \end{bmatrix} \), a submatrix in one of the following three forms:

\[ N_1 := \begin{bmatrix} I_k & B \\ 0_k & I_k \end{bmatrix}, \quad N_2 := \begin{bmatrix} I_k & B_1 \\ I_k & B_2 \end{bmatrix}, \quad N_3 := \begin{bmatrix} I_k & X(k) \\ I_k & Y(k) \end{bmatrix}, \]

with \( B, B_1 \neq B_2 \in \mathbb{F}_q[P] \) and \( X \neq Y \in \mathbb{F}_q[P] \). Let us compute the ranks of such matrices case by case. The rank of \( N_1 \) is easily computed as

\[ \dim \mathbb{F}_q \text{rowsp}[I_k B] + \dim \mathbb{F}_q \text{rowsp}[0_k I_k] - \dim \mathbb{F}_q (\text{rowsp}[I_k B] \cap \text{rowsp}[0_k I_k]) = 2k. \]

The rank of \( N_2 \) is equal to the rank of

\[ \begin{bmatrix} I_k & B_1 \\ 0_k & B_2 - B_1 \end{bmatrix}. \]
Since $B_1 \neq B_2$, by Lemma 10 we get that $B_2 - B_1$ is an invertible matrix, hence
\[
\det \begin{bmatrix} I_k & B_1 \\ 0_k & B_2 - B_1 \end{bmatrix} = \det(B_2 - B_1) \neq 0.
\]

It follows that $\mathrm{rk}(N_2) = 2k$. In order to study the latter case, consider the $2(k + r) \times 2(k + r)$ matrix
\[
H := \begin{bmatrix} I_{k+r} & X \\ I_{k+r} & Y \end{bmatrix}.
\]
By using the same argument as above, we get $\mathrm{rk}(H) = 2(k + r)$. Delete from $H$ the rows from one to $r$ and from $k + r + 1$ to $k + 2r$. A matrix of size $2k \times (2k + 2r)$, say $\tilde{H}$, is obtained. We observe that the rows of $\tilde{H}$ are exactly the rows of $N_3$ with $r$ extra zeroes in the beginning. In particular, $\mathrm{rk}(\tilde{H}) = \mathrm{rk}(N_3)$. By Lemma 12 we get $\mathrm{rk}(\tilde{H}) \geq 2(k + r) - 2r = 2k$ and so $\mathrm{rk}(N_3) = 2k$. To conclude the proof, take a matrix $M_1 \in \mathcal{M}(p, p')$ and set $M_2 := \begin{bmatrix} 0_k & \cdots & 0_k & 0_{k \times r} & I_k \end{bmatrix}$. It follows
\[
\mathrm{rk} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = 2k.
\]
These arguments prove that that $\mathcal{C}$ is a set of $k$-dimensional vector subspaces of $\mathbb{F}_q^n$, whose pairwise intersections are trivial. \hfill \Box

**Notation 14.** The partial spread code $\mathcal{C}$ defined in the statement of Theorem 13 will be denoted by $\mathcal{C}_q(k, n; p, p')$. Since, for any code $\mathcal{C} \subseteq \mathcal{G}_q(k, n)$, $\mathcal{C}^\perp \subseteq \mathcal{G}_q(n - k, n)$ is a code with the same cardinality and the same distance distribution as $\mathcal{C}$ (see [12], Section III), we always assume $1 \leq k \leq n/2$.

**Remark 15.** Partial spread codes provide a generalization of spread codes (see [16], Definition 2). Indeed, it is easily seen that spread codes are obtained by taking $r := 0$ and $p' := p$ in the statement of Theorem 13. On the other hand, partial spread codes exist also when $k$ does not divide $n$.

**Example 16.** Here we construct a partial spread code of length 7 and dimension 2 over the binary field $\mathbb{F}_2$. Let $(q, k, n) := (2, 2, 7)$ and observe that $n \equiv 1 \mod k$. Hence, in the notation of Theorem 13, $r = 1$. Take irreducible monic polynomials $p := x^2 + x + 1, p' := x^3 + x + 1 \in \mathbb{F}_2[x]$ of degree $k$ and $k + r$, respectively. The companion matrices of $p$ and $p'$ are easily computed as follows:
\[
P := \mathbf{M}(p) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad P' := \mathbf{M}(p') = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\]
As a consequence, the elements of $\mathcal{C}_2(2, 7; p, p')$ are the row spaces of all the matrices in the following forms:
\[
\begin{bmatrix} 1 & 0 & A_1 & A_{(2)} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ B_{(2)} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},
\]
where $A_1$ is any matrix in $\mathbb{F}_q[P]$ and $A_{(2)}, B_{(2)}$ denote the last two rows of any $A, B \in \mathbb{F}_q[P']$. It can be checked that $\mathcal{C}_2(2, 7; p, p')$ has $2^2 \cdot 2^3 + 2^2 + 1 = 41$ elements. The cardinality computation will be easily generalized in Proposition 17.
3. SOME PROPERTIES OF PARTIAL SPREAD CODES

In this section we discuss some relevant properties of partial spread codes introduced in Theorem [13]. In particular, Proposition [17] provides their size and Proposition [20] proves their maximality, with respect to inclusion, as collections of $k$-dimensional vector subspaces of $F_q^n$ with trivial pairwise intersections.

**Proposition 17.** Let $\mathcal{C} := \mathcal{C}_q(k, n; p, p')$ be a partial spread code. The size of $\mathcal{C}$ is given by the formula

$$|\mathcal{C}| = \frac{q^n - q^r}{q^k - 1} - q^r + 1.$$  

**Proof.** We follow the notation of Theorem [13]. Let $X, Y$ be matrices in $F_q[p']$ and assume $X_{(k)} = Y_{(k)}$. If $X \neq Y$ we have

$$\text{rk} \begin{bmatrix} I_{k+r} & X \\ I_{k+r} & Y \end{bmatrix} = 2(k + r)$$

and so, as in the proof of Theorem [13]

$$\text{rk} \begin{bmatrix} I_k & X_{(k)} \\ I_k & Y_{(k)} \end{bmatrix} = 2k,$$

which yields a contradiction. It follows that $X = Y$. Notice that the matrices in the statement of Theorem [13] are given in row-reduced echelon form, which is canonical (see [17], Chapter 2.2). As a consequence, the size of $\mathcal{C}$ is easily computed as

$$|\mathcal{C}| = 1 + q^{k+r} \sum_{i=0}^{k-2} q^{ki} = (q^n - q^r)/(q^k - 1) - q^r + 1,$$

as claimed. □

**Corollary 18.** Let $\mathcal{C} := \mathcal{C}_q(k, n; p, p')$ be a partial spread code. Denote by $\mathcal{A}_q(k, n, 2k)$ the largest possible size of a network code in $\mathcal{G}_q(k, n)$ of minimum distance $2k$. Let $r$ be the remainder obtained dividing $n$ by $k$. Then

$$\mathcal{A}_q(k, n, 2k) - |\mathcal{C}| \leq q^r - 1.$$  

**Proof.** Combine Lemma [7] and Proposition [17] □

**Remark 19.** In [8] T. Etzion and A. Vardy provide a construction of partial spread codes (see the proof of Theorem 11). Their codes have the same cardinality and minimum distance as $\mathcal{C}_q(k, n; p, p')$. The main contribution of this paper is introducing a block-matrices description of partial spread codes. Thanks to our construction, in Section 5 we are able to provide an efficient decoding algorithm for partial spread codes. In the next proposition, we discuss the maximality of partial spread codes.

**Proposition 20.** Let $\mathcal{N}_q(k, n, 2k)$ be the set of all the possible network codes $\mathcal{C} \subseteq \mathcal{G}_q(k, n)$ of minimum distance $2k$. Let $\mathcal{C} := \mathcal{C}_q(k, n; p, p')$ be a partial spread code. Then $\mathcal{C}$ is a maximal element of $\mathcal{N}_q(k, n, 2k)$ with respect to inclusion.

**Proof.** We must prove that there is no partial $k$-spread $\mathcal{C}'$ in $F_q^n$ such that $\mathcal{C}' \supseteq \mathcal{C}$ and $|\mathcal{C}'| > |\mathcal{C}|$. Write $n = hk + r$ with $0 \leq r < k$ and $h \geq 2$ (see Notation [14]). Define the partial $k$-spread

$$\overline{\mathcal{C}} := \mathcal{C} \setminus \{\text{rowsp}\{0_k \cdots 0_k 0_{k \times r} I_k\}\}.$$  

Assume, by contradiction, that there exists a partial $k$-spread $\mathcal{C}'$ in $F_q^n$ such that $\mathcal{C}' \supseteq \overline{\mathcal{C}}$ and $|\mathcal{C}'| \geq |\overline{\mathcal{C}}| + 2$. Set $S := \bigcup \overline{\mathcal{C}} \setminus \{0\}$. By combining Theorem [13] and Proposition [17] we easily compute

$$|\overline{\mathcal{C}}| = (q^n - q^r)/(q^k - 1) - q^r, \quad |S| = (q^k - 1) \cdot |\overline{\mathcal{C}}| = q^n - q^{k+r}.$$
The set $X := \{ x \in \mathbb{F}_q^n : x_i = 0 \text{ for any } i = 1,\ldots, (h-1)k \}$ is a vector subspace of $\mathbb{F}_q^n$ of dimension $k + r$. We clearly have an inclusion $X \subseteq \mathbb{F}_q^n \setminus S$. Since
\[ |\mathbb{F}_q^n \setminus S| = q^n - (q^n - q^{k+r}) = q^{k+r}, \]
we deduce $X = \mathbb{F}_q^n \setminus S$, $\mathbb{F}_q^n = X \cup S$, with $X$ a $(k + r)$-dimensional vector subspace of $\mathbb{F}_q^n$. Since $\mathcal{C}' \supseteq \mathcal{C} \supseteq \mathcal{C}$, $|\mathcal{C}'| \geq |\mathcal{C}| + 2$ and for any $s \in S$ there exists a $V_s \in \mathcal{C}$ such that $s \in V_s$, we deduce the existence of two $k$-dimensional vector subspaces $V_1, V_2 \in \mathcal{C}$ such that $V_1 \cap V_2 = \{0\}$ and $V_1, V_2 \subseteq X$. Since $X$ is a vector subspace of $\mathbb{F}_q^n$ containing $V_1 \cup V_2$ and, by definition, $V_1 + V_2$ is the smallest vector subspace of $\mathbb{F}_q^n$ containing both $V_1$ and $V_2$, we conclude $V_1 + V_2 \subseteq X$. It follows
\[ \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2) \leq \dim(X) \]
and so $2k \leq k + r$, which is a contradiction. \hfill \Box

**Remark 21.** Proposition 20 ensures that a partial spread code $\mathcal{C}_q(k,n;p,p')$ cannot be improved (as a network code in $\mathcal{Q}_q(k,n)$ of minimum distance $2k$) by adding new codewords.

### 4. The Block Structure

Here we investigate the block structure of partial spread codes introduced in the statement of Theorem 13. This will allow us to produce an efficient decoding algorithm, which we present in the next section. The results of this section are a generalization of those contained in [9].

**Lemma 22.** Let $\mathcal{C} := \mathcal{C}_q(k,n;p,p')$ be a partial spread code and let $V \in \mathcal{C}$ be a codeword, say
\[ V := \text{rowsp} \left[ S_1 \quad \cdots \quad S_{h-1} \quad S \right], \]
where the $S_i$’s are $k \times k$ matrices and $S$ is a $k \times (k + r)$ matrix. Let $X \subseteq \mathbb{F}_q^n$ be a $t$-dimensional vector subspace given as the row space of a matrix of the form
\[ \begin{bmatrix} M_1 & \cdots & M_{h-1} & M \end{bmatrix}, \]
where the $M_i$’s are $k \times k$ matrices and $M$ is a $k \times (k + r)$ matrix.\(^1\) If $d(V,X) < k$ then $X$ decodes to $V$. Moreover, for any $1 \leq i \leq h - 1$ the following two facts are equivalent:

1. $S_i = 0_k$,
2. $\text{rk}(M_i) \leq (t-1)/2$.

**Proof.** Since the minimum distance of $\mathcal{C}$ is $2k$ (Theorem 13) and $d(V,X) < k$, the space $X$ obviously decodes to $V$. Let us prove $(1) \Rightarrow (2)$. Without loss of generality, we assume that $[S_1 \quad \cdots \quad S_{h-1} \quad S]$ is in row-reduced echelon form. Assume that for a fixed index $1 \leq i \leq h - 1$ we have $S_i = 0_k$. Since $d(V,X) < k$, we have $\dim_{\mathbb{F}_q}(V \cap X) > t/2$. By definition of $\mathcal{C}$, exactly one of the following cases occurs:

(a) there exists an index $1 \leq j \leq h - 1$ with $j \neq i$ such that $S_j = I_k$;
(b) $S_j = 0_k$ for any $1 \leq j \leq h - 1$.

In the former case, let us consider the matrix $M_{ij}$ defined by
\[ M_{ij} := \begin{bmatrix} 0_k & I_k \\ M_i & M_j \end{bmatrix}. \]

\(^1\)Notice that $t \leq k$. This assumption is not restrictive from the following point of view: the decoder can stop collecting incoming vectors as soon as it receives $k$ inputs (as an alternative, $k$ linearly independent inputs); then it can attempt to decode the collected data.
We get \( \text{rk}(M_{ij}) \leq \dim(V + X) = k + t - \dim_{\mathbb{F}_q}(V \cap X) < k + t/2 \). Assume by contradiction that \( \text{rk}(M_{ij}) > (t - 1)/2 \). By deleting the last \( k \) columns of \( M_{ij} \) (which are linearly independent and do not lie in the space generated by the first \( k \)) we easily deduce the following contradiction:

\[
k + (t - 1)/2 < \text{rk} \left[ \begin{array}{c|c}
0_k & I_k \\
M_i & M_j
\end{array} \right] < k + t/2.
\]

In the latter case, by definition of \( \mathcal{C} \), we have \( V = \text{rowsp} \left[ \begin{array}{cccc}
0_k & \cdots & 0_k & 0_{k \times r} & I_k
\end{array} \right] \). Hence

\[
k + (t - 1)/2 < \text{rk} \left[ \begin{array}{c|c}
0_k & 0_{k \times r} I_k \\
M_i & M
\end{array} \right] \leq \dim(V + X) = k + t - \dim_{\mathbb{F}_q}(V \cap X) < k + t/2,
\]

a contradiction. Now we prove (2) \( \Rightarrow \) (1). Assume \( \text{rk}(M_{ij}) \leq (t - 1)/2 \) for some index \( 1 \leq i < h - 1 \). If \( S_i \neq 0_k \) then, by definition of \( \mathcal{C} \), \( \text{rk}(S_i) = k \). Denote by \( \pi: \mathbb{F}_q^n \rightarrow \mathbb{F}_q^k \) the projection on the coordinates \( k_i + 1, k_i + 2, \ldots, k_i + t \). Since \( \text{rowsp}(S_i) = \pi(V) \) and \( \text{rk}(S_i) = k \), we get that \( \pi|_V \) is surjective. Since \( \dim_{\mathbb{F}_q}(V) = k \), it follows that \( \pi|_V \) is also injective. As a consequence,

\[
\dim_{\mathbb{F}_q}(V \cap X) = \dim_{\mathbb{F}_q}\pi(V \cap X) \leq \dim_{\mathbb{F}_q}(\pi(V) \cap \pi(X)) \leq \dim_{\mathbb{F}_q}\pi(X) = \text{rk}(M_{ij}) \leq (t - 1)/2,
\]

which contradicts the assumption that \( d(V) < k \).

\[\tag*{\Box}\]

Remark 23. Lemma \[22\] has the following useful interpretation. Assume that a partial spread code \( \mathcal{C} := \mathcal{C}_q(k, n; p, p') \) is used for random network coding and a \( t \)-dimensional vector space \( X := \text{rowsp} [M_1 \cdots M_{h-1} \ M] \) is received. Assume the existence of a (unique) codeword \( V \in \mathcal{C} \) such that \( d(V, X) < k \) (i.e., \( X \) decodes to \( V \)). If \( \text{rk}(M_{ij}) \leq (t - 1)/2 \) for any \( 1 \leq i \leq h - 1 \) then \( V = \text{rowsp} \left[ \begin{array}{cccc}
0_k & \cdots & 0_k & 0_{k \times r} & I_k
\end{array} \right] \). Otherwise, let \( i \) denote the smallest integer \( 1 \leq i \leq h - 1 \) such that \( \text{rk}(M_{ij}) > (t - 1)/2 \). Then there exist unique matrices \( A_{i+1}, \ldots, A_{h-1} \in F_q[p] \) and a unique matrix \( A \in F_q[p'] \) such that \( V = \text{rowsp} \left[ \begin{array}{cccc}
0_k & \cdots & 0_k & I_k & A_{i+1} & \cdots & A_{h-1}
\end{array} \right] \), where the identity matrix \( I_k \) is the \( i \)-th \( k \times k \) block.

Lemma 24. With the setup of Remark \[23\] assume that \( V \neq \text{rowsp} \left[ \begin{array}{cccc}
0_k & \cdots & 0_k & 0_{k \times r} & I_k
\end{array} \right] \). For any \( i + 1 \leq j \leq h - 1 \) we have

\[
d \left( \text{rowsp} [I_k \ A_j], \text{rowsp} [M_{ij}] \right) < k, \quad d \left( \text{rowsp} [I_k \ A_{(k)}], \text{rowsp} [M_{ij}] \right) < k.
\]

Proof. Fix an integer \( j \) such that \( i + 1 \leq j \leq h - 1 \) and denote by \( \pi: \mathbb{F}_q^n \rightarrow \mathbb{F}_q^{2k} \) the projection on the coordinates \( k_i + 1, k_i + 2, \ldots, k_i + t \). We have \( \pi(V) = \text{rowsp} [I_k \ A_j] \) and \( \pi(X) = \text{rowsp} [M_{ij}] \). In particular, \( \pi|_V = \pi|_X \) is injective. By the trivial inclusion of vector spaces \( \pi(V) \cap \pi(X) \subseteq \pi(V) \cap \pi(X) \) it follows \( \dim_{\mathbb{F}_q}\pi(V) \leq \dim_{\mathbb{F}_q}\pi(X) \). Hence

\[
d(\pi(V), \pi(X)) = k + \dim_{\mathbb{F}_q}\pi(V) - 2\dim_{\mathbb{F}_q}(\pi(V) \cap \pi(X)) \leq k + t - 2\dim_{\mathbb{F}_q}\pi(V \cap X) = k + t - 2\dim_{\mathbb{F}_q}(V \cap X) < d(V, X).
\]

In order to prove that \( d \left( \text{rowsp} [I_k \ A_{(k)}], \text{rowsp} [M_{ij}] \right) < k \) we may notice that the same argument still works if we choose as \( \pi: \mathbb{F}_q^n \rightarrow \mathbb{F}_q^{2k + r} \) the projection on the coordinates \( k_i + 1, k_i + 2, \ldots, k_i + t, k_i + t + 1, k_i + t + 2, \ldots, k_i + h, k_i + h + 1, \ldots, k_i + h + r \).

\[\tag*{\Box}\]

Remark 25. By Lemma \[24\] when decoding a partial spread code we may restrict to one of the two cases \( n = 2k \) and \( n = 2k + r \), with \( 1 \leq r \leq k - 1 \). Moreover, the lemma allows us to parallelize the computation, reducing the decoding complexity to the case \( n = 2k + r \).
5. Decoding partial spread codes

In [12] R. Kötter and F. R. Kschischang illustrate a general network code construction and a related efficient algorithm to decode them. A more efficient algorithm to decode the same codes appears in [13]. After recalling the definition of Reed-Solomon like code, we use the results established in the previous section to adapt any decoding algorithm for such codes to partial spread codes of the form \( C_q(k, n; p, p') \).

**Definition 26.** Let \( q \) be a prime power and let \( n > 1 \) be an integer. Let \( A := \{a_1, \ldots, a_k\} \subseteq \mathbb{F}_q^n \) be a set of \( \mathbb{F}_q \)-linearly independent elements. Choose an integer \( s \leq k \) and denote by \( \mathbb{F}_q^n[x] \) the vector space of the linearized polynomial of degree at most \( s \) and coefficients in \( \mathbb{F}_q \) (see [12], Section 5.A, for details). Fix an \( \mathbb{F}_q \)-isomorphism of vector spaces \( \varphi : \mathbb{F}_q^s \rightarrow \mathbb{F}_q^n \). The Reed-Solomon like code associated to the \( 6 \)-tuple \((q, n, k, s, A, \varphi)\) is the set

\[
\mathbb{KK}_q(n, k, s, A, \varphi) := \left\{ \begin{array}{c|c}
\text{rowsp} & I_k \\
\hline
\varphi(f(a_1)) & \varphi(f(a_1)) \\
\vdots & \vdots \\
\varphi(f(a_{k-1})) & \varphi(f(a_{k-1})) \\
\varphi(f(a_k)) & \varphi(f(a_k)) \\
\end{array} \right\} : f \in \mathbb{F}_q^s[x].
\]

**Remark 27.** A Reed-Solomon like code \( \mathbb{KK}_q(n, k, s, A, \varphi) \) is a subset of the Grassmannian variety \( G_q(k, \mathbb{F}_q^n) \). As a consequence, it is a \( q \)-ary network code of length \( k + n \) and dimension \( k \). The size of such a code is given by the easy-computable formula \(|\mathbb{KK}_q(n, k, s, A, \varphi)| = q^{sn}\). See [12], Section 5.1, for a more detailed discussion.

**Lemma 28.** Let \( q \) be a prime power and let \( k \geq 1 \) be an integer. Let \( p \) an irreducible monic polynomial \( p \in \mathbb{F}_q[x] \) of degree \( k \) and let \( P := \text{M}(p) \) be its companion matrix. Choose a root \( \lambda \in \mathbb{F}_q^k \) of \( p \). Denote by \( \varphi : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n \) the \( \mathbb{F}_q \)-isomorphism defined, for any \( 0 \leq i \leq k - 1 \), by \( \varphi^{i} \rightarrow e_{i+1} \), where \( \{e_1, \ldots, e_k\} \) is the canonical basis of \( \mathbb{F}_q^k \). Let \( A \in \mathbb{F}_q[P] \) and, for any \( 1 \leq i \leq k \), let \( A^i \in \mathbb{F}_q^k \) denote the \( i \)-th row of \( A \). For any \( 1 \leq i \leq k \) we have \( \varphi^{-1}(A^i) = \lambda^{i-1} \varphi^{-1}(A^1) \). In particular, if \( f \in \mathbb{F}_q[1,x] \) is defined by \( f(x) := \varphi^{-1}(A^1)x \), then

\[
A = \begin{bmatrix}
\varphi(f(A^0)) \\
\varphi(f(A^1)) \\
\vdots \\
\varphi(f(A^{k-1})) \\
\end{bmatrix}.
\]

**Proof:** Use [9], Proposition 15, with \( n = k \). \( \square \)

**Notation 29.** In the construction of a partial spread code \( C_q(k, 2k + r, p, p') \) with \( 0 \leq r \leq k - 1 \), the companion matrix of \( p \) is never involved (see Theorem [13]). As a consequence, we write \( C_q(k, 2k + r; p') \) in this case.

**Remark 30.** By Lemma [25] in order to decode a partial spread code \( C_q(k, n; p; p') \) we may restrict to decoding partial spread codes of the form \( C_q(k, 2k + r; p) \), with \( 0 \leq r \leq k - 1 \). The case \( r = 0 \) is easily solved. Indeed, by Lemma [28] \( C_q(k, 2k; p) \backslash \{	ext{rowsp } [0_k I_k] \} \) is a Reed-Solomon like code and so we may simply proceed as in the following Algorithm [5].

**Remark 31.** In [9], a decoding procedure for \( C_q(k, h; p; p') \) spread codes which is independent of those of [12] and [13] is proposed. Lemma [24] allows us to apply the decoding algorithm from [9] to partial spread codes. This algorithm is particularly efficient in the case \( k \ll n \).
Algorithm 1 Decoding a \( \mathcal{C}_q(k, 2k; p) \) code.

**Data:** a decodable \( t \)-dimensional row space, \( X \), of a \((k \times 2k)\)-matrix \([M_1 \quad M_2]\).

**Result:** the unique \( V \in \mathcal{C}_q(k, 2k; p) \) such that \( d(V, X) < k \), given as a matrix in row-reduced echelon form whose row space is \( V \).

1. \textbf{if} \( \text{rk}(M_1) \leq (t-1)/2 \) \textbf{then}
   
   \[ V = \text{rowsp} \left[ \begin{bmatrix} 0_k & I_k \end{bmatrix} \right] \]

2. \textbf{else}

   use a decoding algorithm for Reed-Solomon like codes on \( \mathcal{C}_q(k, 2k; p) \setminus \text{rowsp} \left[ \begin{bmatrix} 0_k & I_k \end{bmatrix} \right] \).

**end if**

Now we focus on a decoding procedure for partial spread codes of the form \( \mathcal{C}_q(k, 2k + r; p) \) with \( 1 \leq r \leq k - 1 \). To be precise, in the following Proposition 32 we construct a canonical embedding of a partial spread code \( \mathcal{C}_q(k, 2k + r; p) \) into the spread code \( \mathcal{C}_q(k + r, 2(k + r); p) \). Any decoding procedure for \( \mathcal{C}_q(k + r, 2(k + r); p) \) gives, in this way, a decoding procedure for \( \mathcal{C}_q(k, 2k + r; p) \).

**Proposition 32.** Let \( \mathcal{C} := \mathcal{C}_q(k, 2k + r; p) \) be a partial spread code with \( 1 \leq r \leq k - 1 \). Let \( X := \text{rowsp} \left[ M_1 \quad M \right] \) be a \( t \)-dimensional vector space in \( \mathbb{F}^{2n+2} \). Define the following two matrices:

\[
\begin{align*}
\overline{M}_1 & := 0_r \\ & \\
0_k & \begin{bmatrix} s \end{bmatrix} \begin{bmatrix} d \end{bmatrix} \\ & \\
\begin{bmatrix} M_1 \end{bmatrix} & \\
\end{align*}
\]

**Proof.** Let \( V := \text{rowsp} \left[ M_1 \quad M \right] \) be a \( t \)-dimensional vector space in \( \mathbb{F}^{2n+2} \). Assume the existence of a matrix \( A \in \mathbb{F}_q[P] \) such that \( d \left( \text{rowsp} \left[ I_k \quad A \right], \text{rowsp} \left[ M_1 \quad M \right] \right) \leq k + r \). Define the following two \( (k + r) \times (k + r) \)-matrices:

\[
\begin{align*}
\overline{M}_1 & := 0_r \\ & \\
0_k & \begin{bmatrix} s \end{bmatrix} \begin{bmatrix} d \end{bmatrix} \\ & \\
\begin{bmatrix} M_1 \end{bmatrix} & \\
\end{align*}
\]

We have

\[
d \left( \text{rowsp} \left[ I_k \quad A \right], \text{rowsp} \left[ \overline{M}_1 \quad \overline{M} \right] \right) \leq k + r.
\]

**Remark 33.** Proposition 32 has the following useful interpretation. Assume that a partial spread code \( \mathcal{C}_q(k, 2k + r; p) \) is given, with \( 1 \leq r \leq k - 1 \) and \( X := \text{rowsp} \left[ M_1 \quad M \right] \) is received (\( M_1 \) and \( M \) being as in the statement of the proposition). Then we may construct the matrices \( \overline{M}_1 \) and \( \overline{M} \) as described and consider the vector space \( \overline{X} := \text{rowsp} \left[ \overline{M}_1 \quad \overline{M} \right] \). The minimum distance of the (partial) spread code \( \mathcal{C}_q(k + r, 2(k + r); p) \) is \( 2(k + r) \). By Proposition 32 if \( X \) decodes to \( V := \text{rowsp} \left[ I_k \quad A \right] \) in \( \mathcal{C}_q(k, 2k + r; p) \), then \( \overline{V} \) decodes to \( \overline{V} := \text{rowsp} \left[ I_k \quad A \right] \) in \( \mathcal{C}_q(k, 2k + r; p) \). It follows that Algorithm 5 applied to \( \overline{X} \) produces \( \left[ I_k \quad A \right] \). Finally, \( V \) is the rowspace of the matrix obtained by deleting the first \( r \) rows and the first \( r \) columns of \( I_k \quad A \). This discussion leads to the following Algorithm 5.

**Remark 34.** By Proposition 32 in Algorithm 5 we may replace the use of Algorithm 5 with any other decoding algorithm for spread codes.
Algorithm 2 Decoding a $\mathcal{C}_q(k, 2k + r; p)$ code with $1 \leq r \leq k - 1$.

**Data:** a decodable $t$-dimensional row space, $X$, of a $(k \times 2k + r)$-matrix $[M_1 \ M]$.

**Result:** the unique $V \in \mathcal{C}_q(k, 2k + r; p)$ such that $d(V, X) < k$, given as a matrix in row-reduced echelon form whose row space is $V$.

if $\text{rk}(M_1) \leq (t-1)/2$ then

$V = \text{rowsp} [0_k \ 0_{k\times r} \ I_k]$.

else

construct the matrix $[\overline{M}_1 \ \overline{M}]$ as explained in Lemma 32. Then use Algorithm 5 with $\mathcal{C}_q(k + r, 2(k + r); p)$ on $[\overline{M}_1 \ \overline{M}]$. Delete the first $r$ rows and the first $r$ columns of the output.

end if

CONCLUSIONS

In this paper we provide an easy description of partial spreads over finite fields, whose interest dates back to classical problems in projective geometry. We suggest the use of partial spreads as network codes, investigating the mathematical properties due to our construction, proving their maximality, and providing a decoding algorithm for them.

ACKNOWLEDGMENT

The authors would like to thank Leo Storme for useful discussions on partial spreads in finite projective geometry.

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