Numerical solution of Saint-Venant equation using Runge-Kutta fourth-order method

M Sukron\(^1\), U Habibah\(^2\) and N Hidayat\(^2\)
\(^1\)Postgraduate Program of Mathematics, University of Brawijaya, Malang, Indonesia
\(^2\)Department of Mathematics, University of Brawijaya, Malang, Indonesia

Email: syukronmath@gmail.com

Abstract. The Saint-Venant Equation (SVE) is the equation that describes the flow below a pressure surface in a fluid in unidirectional form. The SVE is in the form of partial differential equations. To solve the partial differential equations is rather complicated than the ordinary differential equations analytically and numerically. In this paper, we construct numerical schemes of the SVE by changing it into semi-discrete equation by using a finite difference in space \((x)\) such that the SVE becomes ordinary differential equations (ODEs). Furthermore we solve the semi-discrete form of SVE by using Runge-Kutta fourth-order method since this method has smaller error and higher accuracy than the others method to solve ODE.

1. Introduction

The shallow water is a fluid layer with constant density in which the horizontal scale of the flow is greater than the layer depth of the water [6]. One form of shallow water model is the Saint-Venant’s equation (SVE). The shallow water flow can be seen in the Figure 1

\[ \zeta = \zeta_b + h = H + \Delta \zeta. \]

Figure 1. The shallow water system [6].

where \(h\) is the thickness of a water column, \(H\) its mean thickness, \(\zeta\) the height of the free surface and \(\zeta_b\) is the height of the lower, rigid, surface, above some arbitrary origin, typically chosen such that the average of \(\zeta_b\) is zero. \(\Delta \zeta\) is the deviation free surface height, so we have \(\zeta = \zeta_b + h = H + \Delta \zeta\).
The SVE is used to model open-channel flow and simulated the ocean, for example to simulate tsunami and tidal waves. The Saint-Venant’s equation (SVE) is hyperbolic partial differential equations. The governing equation of SVE is

$$\frac{\partial \eta}{\partial t} + \frac{\partial (hu)}{\partial x} = 0,$$

(1)

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x} \left( h u^2 + \frac{1}{2} g \cos(\theta) h^2 \right) = g \sin(\theta) h - C_f \frac{|u|}{h^{\alpha}},$$

(2)

where $\eta$ is surface elevation, $h$ is height of water, $u$ is horizontal water velocity, $g$ is gravitational acceleration and $\theta$ is bed inclination. Meanwhile $C_f$ is bottom friction parameter that depends on the choice of constant $\alpha[2]$. 

Equation (1) and (2) are in the form of the partial differential equation. Chalfen and Niemiec (1986) investigated flood routing in the Nysa river using the SVE analytically and numerically. They used the general Preissmann scheme to get the numerical solution. Pudjaprasetya and Ribal (2009) studied numerical solution of SVE using predictor-corrector Mc Cormack method. They caould show the evolution of small amplitude monochromatic wave into roll wave. Fauzi and Wiryanto (2018) solve the SVE using predictor-corrector scheme (Adam-Bashforth for predictor and Adam-Moulton for corrector) and simulate the wave propagation on shallow water region. Pudjaprasetya and Ribal (2009) and Fauzi and Wiryanto (2018) used the method for solving ordinary differential equation (ODE) to get numerical solution of the SVE after they changed the SVE into semi-discrete form such that the SVE became the ordinary differential equation.

According to the previous research, we purpose the fourth-order Runge-Kutta method to solve the SVE numerically. We construct numerical schemes of the SVE by changing it into semi-discrete equation by using a finite difference in space ($x$) such that the SVE becomes ordinary differential equations (ODEs) [3][8]. Furthermore, we solve the semi-discrete form of SVE by using the fourth-order Runge-Kutta method since this method has smaller error and higher accuracy than the others method to solve ODE [5] and we need only one step to get the numerical solution. In this paper, we show wave propagation of the SVE in shallow water.

2. Method
The steps in this study are shown in Figure 2.

![Figure 2](image-url)
2.1. Finite difference method

Finite difference methods are a class of numerical techniques for solving differential equations by approximating derivatives with finite differences. We can approximate function $u(x_0 + \Delta x)$ by using Taylor series. A central difference is defined as follows

$$\frac{\partial u}{\partial x} \approx \frac{u^{k+1}_j - u^{k-1}_j}{2\Delta x},$$

(3)

where $\Delta x$ is time step. The forward difference is defined as

$$\frac{\partial u}{\partial x} \approx \frac{u^{k+1}_j - u^k_j}{\Delta x}.$$  

(4)

For example, we have the partial differential equation as follows

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} + B = 0,$$

(5)

where $A$ and $B$ are constants. If we approached $x$ variable with forward difference and time kept continuously in equation 6, we yield a semi-discrete equation of ordinary differential equations [7] such that we have

$$\frac{du}{dt} + A \frac{u^{k+1}_j - u^k_j}{\Delta x} + B = 0,$$

(6)

which is can be solve numerically using the methods that can be used to solve ODEs numerically, the Euler, the Runge-Kutta, the predictor-corrector methods, and soon. In this paper, we solve the ODE using the fourth-order Runge-Kutta method.

2.2. The Runge-Kutta method

The Runge-Kutta methods are a family of implicit and explicit iterative methods, which include the well-known routine called the Euler Method, used in temporal discretization for the approximate solutions of ordinary differential equations $dv/dt = f(t, v)$. The fourth-order Runge-Kutta method [7] is

$$v^{n+1} = v^n + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4).$$

(7)

with

$$K_1 = \Delta t f^n,$$

$$K_2 = \Delta t f \left( t_n + \frac{1}{2} \Delta t, v^n + \frac{1}{2} K_1 \right),$$

$$K_3 = \Delta t f \left( t_n + \frac{1}{2} \Delta t, v^n + \frac{1}{2} K_2 \right),$$

$$K_4 = \Delta t f \left( t_n + \frac{1}{2} \Delta t, v^n + K_3 \right).$$

2.3. Consistency

Suppose the advection equation

$$\frac{\partial \Phi}{\partial t} + c \frac{\partial \Phi}{\partial x} = 0,$$

(8)
The numerical approximation of equation with finite-difference formula is

\[
\frac{\Phi_j^{n+1} - \Phi_j^n}{\Delta t} + c \frac{\Phi_j^n - \Phi_j^{n-1}}{\Delta x} = 0, \tag{9}
\]

Furthermore, about \((n \Delta t, j \Delta x)\) is expanded by the Taylor series and substituted into the equation so that is obtained

\[
\frac{\Phi_j^{n+1} - \Phi_j^n}{\Delta t} + c \frac{\Phi_j^n - \Phi_j^{n-1}}{\Delta x} = \frac{\Delta t \partial^2 \Phi}{2 \partial t^2} - c \frac{\Delta x \partial^2 \Phi}{2 \partial x^2} + ..., \tag{10}
\]

The right side of equation (10) is the truncation error of the finite-difference scheme. If the truncation error of the finite-difference scheme approaches zero as \(\Delta t \to 0\) and \(\Delta x \to 0\), the scheme is consistent. Sometimes consistency is needed to find a relation between \(\Delta t\) and \(\Delta x\), such as \(\Delta t/\Delta x \to 0\), to achieve consistency[1].

### 3. Numerical Solution

In this paper, the fourth order Runge-Kutta method are used to solve the SVE equation. The Runge-Kutta method has been succeeded simulating Burgers equation in [4]. To get solution of the governing equations by using the Runge-Kutta method, firstly we need to rewrite the SVE equation (1) and (2) as follows

\[
\frac{\partial h}{\partial t} = -\left( h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x} \right), \tag{11}
\]

\[
\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} - g \cos(\theta) \frac{\partial h}{\partial x} + g \sin(\theta) - C_f u |u|, \tag{12}
\]

since \(\partial h/\partial t = \partial(\eta - \eta_b)/\partial t\) where \(\eta_b\) the base of topography (from Figure 1, \(\zeta = \eta\)). Using several algebras, we can simplify equation (11) and (12) as follows

\[
\frac{\partial h}{\partial t} = F \left( h, u, \frac{\partial h}{\partial x}, \frac{\partial u}{\partial x} \right), \tag{13}
\]

\[
\frac{\partial u}{\partial t} = G \left( h, u, \frac{\partial h}{\partial x}, \frac{\partial u}{\partial x} \right), \tag{14}
\]

where

\[
F \left( h, u, \frac{\partial h}{\partial x}, \frac{\partial u}{\partial x} \right) = -\left( h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x} \right),
\]

and

\[
G \left( h, u, \frac{\partial h}{\partial x}, \frac{\partial u}{\partial x} \right) = -\frac{\partial u}{\partial x} - g \cos(\theta) \frac{\partial h}{\partial x} + g \sin(\theta) - C_f u |u|.
\]

We discretize the right-hand side of equation (13) and (14) by using central difference such that we get

\[
F^n_j = -h_j^n \frac{u_j^{n+1} - u_j^n}{\Delta x} - h_j^n \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta x}, \tag{15}
\]

\[
G^n_j = -u_j^n \frac{h_j^{n+1} - h_j^{n-1}}{2\Delta x} - g \cos(\theta) \frac{h_j^{n+1} - h_j^{n-1}}{2\Delta x} + g \sin(\theta) - C_f \frac{u_j^n |u_j^n|}{h_j^n}. \tag{16}
\]

The equation (15) and (16) substitute into equations (13) and (14) such that becomes semi-discrete differential equation so that the equations (13) and (14) can written as the system of ordinary differential
equation (ODE) as follows
\[
\frac{dh}{dt} = -h^n_{j} u^n_{j+1} - u^n_{j-1} - \frac{h^n_{j+1} - h^n_{j-1}}{2\Delta x},
\]
(17)
\[
\frac{du}{dt} = -u^n_{j} h^n_{j+1} - h^n_{j-1} + g \cos(\theta) \frac{h^n_{j+1} - h^n_{j-1}}{2\Delta x} + g \sin(\theta) - C_f \frac{|u^n_{j}|}{(h^n_{j})^\alpha}.
\]
(18)

Furthermore, the left-hand side of the equation the equations (13) and (14) are solved numerically by using the fourth-order Runge-Kutta method
\[
h^n_{j+1} = h^n_{j} + \frac{1}{6} \Delta t (k_1 + 2k_2 + 2k_3 + k_4),
\]
(19)
\[
u^n_{j+1} = u^n_{j} + \frac{1}{6} \Delta t (l_1 + 2l_2 + 2l_3 + l_4).
\]
(20)

with
\[
k_1 = F(h^n, u^n),
\]
\[
l_1 = G(h^n, u^n),
\]
\[
k_2 = F(h^n + \frac{1}{2}k_1, u^n + \frac{1}{2}l_1),
\]
\[
l_2 = G(h^n + \frac{1}{2}k_1, u^n + \frac{1}{2}l_1),
\]
\[
k_3 = F(h^n + \frac{1}{2}k_2, u^n + \frac{1}{2}l_2),
\]
\[
l_3 = G(h^n + \frac{1}{2}k_2, u^n + \frac{1}{2}l_2),
\]
\[
k_4 = F(h^n + k_3, u^n + l_3),
\]
\[
l_4 = G(h^n + k_3, u^n + l_3).
\]

The initial condition is gives by \(u(0, x) = 0\) and \(h(x, 0) = A_t \cos(k_t x - \omega t)\).

4. Numerical Consistency
Next step is to proof consistency of numerical scheme. According to the theory of consistency, the first step is we take the Taylor expansion of the function \(h\) in \(n\) and \(j\) such that we have
\[
h^n_{j+1} = h^n_{j} + \Delta t \frac{\partial h}{\partial t} + \frac{\Delta^2 t^2 h}{2} + ..., 
\]
(21)
\[
h^n_{j+1} = h^n_{j} + \Delta x \frac{\partial h}{\partial x} + \frac{\Delta^2 x^2 h}{2} + ..., 
\]
(22)
\[
h^n_{j-1} = h^n_{j} - \Delta x \frac{\partial h}{\partial x} + \frac{\Delta^2 x^2 h}{2} - ..., 
\]
(23)
\[
h^n_{j} = h.
\]
(24)
The same procedure is used for function \( u \) such that we end up
\[
\begin{align*}
  u_{j}^{n+1} &= u_{j}^{n} + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta x^{2}}{2} \frac{\partial^{2} u}{\partial t^{2}} + \ldots, \\
  u_{j+1}^{n} &= u_{j}^{n} + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^{2}}{2} \frac{\partial^{2} u}{\partial x^{2}} + \ldots, \\
  u_{j-1}^{n} &= u_{j}^{n} - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^{2}}{2} \frac{\partial^{2} u}{\partial x^{2}} - \ldots, \\
  u_{j}^{n} &= u.
\end{align*}
\]
(25)-(28)

Substitute equation (21)-(28) into equation (13) and (14), so that we yield
\[
\frac{\partial h}{\partial t} + h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x} = \Delta t \frac{\partial^{2} h}{\partial t^{2}} + O(\Delta t)^{2} - \ldots,
\]
(29)

and
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \cos(\theta) \frac{\partial h}{\partial x} - g \sin(\theta) h + C_f \frac{u|u|}{h} = \Delta t \frac{\partial^{2} u}{\partial t^{2}} + O(\Delta t)^{2} - \ldots,
\]
(30)
on the right hand of the equation (29) and equation (30) we can get the truncation-error term. When we take \( \Delta t \to 0 \) and \( \Delta x \to 0 \), then truncation-error term tend to zero. It means that the numerical scheme is consistent.

5. Numerical Simulation
In this section, we give numerical simulation of the SVE numerical schemes. We use the values \( \Delta x = 1 \), \( \Delta t = 0.2 \) and \( g = 9.8 \) and initial condition \( h(x, 0) = A_{t} \cos(k_{t}x - \omega t) \) where \( A_{t} = 0.25 \), \( \omega = 0.5 \), \( k_{t} = 0.4556 \) and \( u(x, 0) = 0 \) with theta, we get the wave propagations as follows

![Wave propagation of the Saint-Venant equation.](image)

From Figure 3, we can see that the wave propagate at time \( x = 0 \) with initial height 0.25 to downstream far as \( x = 10 \). The figure 3 shows that the numerical solution is not representative of the SVE equation, because the change wave height in an interval of 0 to 10 is irregular. We can see that the wave is not smooth. Corner is occurred at the valley of the wave (around \( t = 8 \)).
6. Conclusion
In this paper, we have presented a numerical solution of the Saint-Venant equation (SVE). First we change the governing equation of SVE into semi-discrete scheme by using a finite difference in space \((x)\) such that the SVE becomes ordinary differential equations (ODEs). Furthermore we solve the semi-discrete form of SVE by using the fourth-order Runge-Kutta method since this method has smaller error and higher accuracy than the others method to solve ODE. We proof that the constructed numerical scheme is consistent. We also simulate the numerical scheme using artificial data, unfortunately the numerical solution of the SVE can not represent of the SVE equation since the wave is not smooth. Corner is occurred at the valley of the wave.

Acknowledgment
The author thanks to Ummu Habibah, S.Si., M.Si., Ph.D and Dr. Noor Hidayat, M.Si who have provided suggestions during this research.

References
[1] Durran D R 1999 Numerical Method for Wave Equation in Geophysical fluid dynamics (New York: Springer).
[2] Fauzi and Wiryanto L H 2018 Predictor-corrector scheme for simulating wave propagation on shallow water region IOP Conference Series: Earth and Environmental Science 162 1-8.
[3] Kurganov A, Noelle S, Petrova G 2001 Semidiscrete central-upwind schemes for hyperbolic conservation laws and Hamilton–Jacobi equations SIAM J Sci Comput 23: 707–740.
[4] Lopes M M, Domingues M O, Schneider K and Mendes O 2018 Local time-stepping for adaptive multiresolution using natural extension of Runge–Kutta methods Journal of Computational Physics 8435 1-42.
[5] Munir R 2015 Metode Numerik Revisi ke-empat Bandung: Informatika Bandung
[6] Vallis G K 2006 Atmospheric and Oceanic Fluid Dynamics (USA: Cambridge University Press).
[7] Vreugdenhil CB 1994 Numerical methods for shallow-water flow (Dordrecht: Springer Netherlands).
[8] Welahettige P, Vaagsaether K and Lie B 2018 A Solution Method for One-dimensional Shallow Water Equations Using Flux Limiter Centered Scheme for open Venturi channels The Journal of Computational Multiphase Flows 10(4) 228–238